The Loss of fidelity due to quantum leakage for Josephson charge qubits

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Abstract
In this paper we calculate the loss of fidelity due to quantum leakage for the Josephson charge qubit (JCQ) in virtue of the Mathieu functions. It is shown that for an present typical parameters of JCQ $E_J/E_{ch} \sim 0.02$, the loss of the fidelity per elementary operation is about $10^{-4}$ which satisfy the DiVincenzo's low decoherence criterion. By appropriately improving the design of the Josephson junction, namely, decreasing $E_J/E_{ch}$ to $\sim 0.01$, the loss of fidelity per elementary operation can decrease to $10^{-5}$ even smaller.

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1 Introduction
A quantum computer can perform certain tasks which no classical one is able to do in acceptable times [1]. A quantum bit (qubit) is a quantum system with two levels which will be a cell for storing and processing information in a future quantum computer. So it is a vital task to find out the physical realizations of the qubits. In last years, physicists have proposed several qubit models which are based on ion traps [2], QED systems [3], nuclear spins of large numbers of identical molecules [4], quantum dots [5], Josephson junction [6] and so on. Because solid state qubits can be embedded in electronic circuits as well as scaled up to a large numbers, they are taken as a particularly promising candidates [7] of qubits for quantum computation. The Josephson junction qubit is one of this kinds of models.

Decoherence, one of the most difficult problems to be dealt with in quantum computation exists in all of the qubit models. In general, decoherence comes from the interaction of the qubits and their environment. But for some qubit models, for example, quantum dots and Josephson junctions, the decoherence also results from intrinsic source of error, such as quantum leakage [8] [9]. The quantum leakage is this kind of process that the system working in the computational Hilbert space leaks out to higher states. The leakage exists in many quantum processes [10] [11]. It is attracted a particular attention in the implementations of qubits and quantum gates for quantum computation [5] because they must satisfy the DiVincenzo’s checklist five criteria [12] one of which is low decoherence (so that error correction techniques may be used in a fault-tolerant manner)—an approximate benchmark is the loss of fidelity no more than $10^{-4}$ per elementary quantum gate operation.

In [5], Fazio et al. investigated the leakage and fidelity of the Josephson charge qubit (JCQ) operations. Where the eigenvalues and eigenstates of the JCQ Hamiltonian are obtained through diagonalizing the Hamiltonian. In fact, they can be obtained in virtue of the perturbation theory of quantum mechanics [17]. In particular, as pointed in [5] the eigenvalues and eigenstates can be obtained by solving the eigen-equation of the JCQ Hamiltonian which in fact is the Mathieu equation [14]. The eigenvalues and eigenstates of the JCQ Hamiltonian correspond to the characteristic values and
characteristic states of the Mathieu equation. In this paper we will use the well researched Mathieu functions investigating the loss of fidelity for the single JCQ.

2 Dynamics of the JCQ

The single JCQ was first introduced by Shnirman et al. [6]. Since then, much interest has been attracted into this topic. The simplest JCQ can be designed as Fig.1 (refer to Fig.1 of Ref. [7]).

Ignoring the resistance in the circuit, from the Josephson relations
\[ I = I_c \sin \varphi, \quad \frac{\Phi_0}{2\pi} \frac{d\varphi}{dt} = U_J, \]  
and the current conservation we have
\[ C_J \frac{\Phi_0}{2\pi} \frac{d^2\varphi}{dt^2} + I_c \sin \varphi = C_g \dot{U}_g, \]  
where \( I_c \) is the critical current of the Josephson junction, \( \Phi_0 = \pi \hbar/e \) is a magnetic flux quantum, \( \varphi \) is the gauge invariant phase of the superconducting junction, \( U_J \) is the voltage of the Josephson junction and \( U_g \) is the gate voltage. Due to \( U_J + U_g = V_g \), Eq. (2) can be written as
\[ C_J \left( \frac{\Phi_0}{2\pi} \right)^2 \frac{d^2\varphi}{dt^2} + \frac{\Phi_0 I_c}{2\pi} \sin \varphi = 0, \]  
where \( C = (C_J + C_g) \). Thus, we can obtain the Lagrangian of the JCQ as
\[ \mathcal{L} = \frac{1}{2} C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}^2 + \frac{\Phi_0 I_c}{2\pi} \cos \varphi. \]  
The Euler-Lagrange equation can be used to check that the Lagrangian produces the correct classical equations of motion,
\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \]  
By definition, the conjugate variable to \( \varphi \) is,
\[ p = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}. \]  
On the other hand, the charge on the Josephson junction capacitance and the gate capacitance are
\[ q = C_J U_J = C_J \frac{\Phi_0}{2\pi} \varphi \equiv n_2 e, \]  
\[ q_g = C_g U_g = -C_g \frac{\Phi_0}{2\pi} \varphi \equiv n_g 2e, \]  
where \( n \) is the Cooper pairs pass through the Josephson junction, and \( n_g \) is the number of two-unit charge \( 2e \) on the gate capacitance. From Eqs. (6) and (7) we have
\[ p = \frac{\Phi_0}{2\pi} (q - q_g) = C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi} \]  
\[ = \frac{\Phi_0}{2\pi} 2e (n - n_g). \]  
So
\[ \dot{\varphi} = \frac{2\pi}{\Phi_0} \frac{2e}{C} (n - n_g). \]  
Now we can construct the Hamiltonian via Legendre transformation,
\[ \mathcal{H} = \dot{\varphi} p - \mathcal{L} = C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}^2 - \frac{\Phi_0 I_c}{2\pi} \cos \varphi \]  
\[ = \frac{2e^2}{C} (n - n_g)^2 - \frac{\Phi_0 I_c}{2\pi} \cos \varphi. \]  
Setting \( E_{ch} = \frac{2e^2}{C} \), \( E_J = \frac{\Phi_0 I_c}{2\pi} \), we have
\[ \mathcal{H} = E_{ch} (n - n_g)^2 - E_J \cos \varphi. \]  
So we have \([p, \varphi] = -i\hbar\) and \([n, \varphi] = -i\hbar/\Phi_0 e\). As Ref. [8] to shorten notations we use units
It has been pointed that the periodic boundary conditions for an integer \( x \) and the \( \pi \)-anti-periodic Mathieu eigenfunctions \( ce_{2n} \) for a half-integer \( x \). So when one suddenly switch the offset charge from idle point to the degeneracy point \( n_g = \frac{1}{2} \) (or another half-integer), we can obtain the eigenvalues (leave over the first five terms) and the eigenfunctions (leave over the first three terms) of the Hamiltonian \( \mathcal{H} \) as

\[
E_1^e = E_{ch} \left( 1 + v - \frac{v^2}{8} - \frac{v^3}{64} - \frac{v^4}{1536} - \cdots \right), \\
E_1^o = E_{ch} \left( 1 - v - \frac{v^2}{8} + \frac{v^3}{64} - \frac{v^4}{1536} + \cdots \right),
\]

Eq. (15) is the canonical form of the Mathieu equation [13], its characteristic functions called Mathieu functions. The Mathieu functions were introduced by Mathieu [13] when analyzing the movements of membranes of elliptical shape. Since then the characters of the Mathieu functions have been investigated by Mathieu and others [14]. In recent years, the functions have been attracted much attention because they have some applications in many fields of physics [15]. The Mathieu equation has the well known periodic solutions

\[
\begin{align*}
&ce_{2n} (\varphi, v) \quad \text{even solutions with period} \pi \\
&\text{with eigenvalues} \ a_{2n} (v), \\
&\text{se}_{2n+2} (\varphi, v) \quad \text{odd solutions with period} \pi \\
&\text{with eigenvalues} \ b_{2n+2} (v), \\
&ce_{2n+1} (\varphi, v) \quad \text{even solutions with period} 2\pi \\
&\text{with eigenvalues} \ a_{2n+1} (v), \\
&\text{se}_{2n+1} (\varphi, v) \quad \text{odd solutions with period} 2\pi \\
&\text{with eigenvalues} \ b_{2n+1} (v). 
\end{align*}
\]

It has been pointed that the periodic boundary conditions \( \psi_n (\varphi = 0) = \psi_n (\varphi = \pi) \) singles out only the 2\( \pi \) periodic Mathieu eigenfunctions \( ce_{2n} \) for an integer \( x \) and the \( \pi \)-anti-periodic Mathieu eigenfunctions \( ce_{2n+1}, se_{2n+1} \) for a half-integer \( x \). So when one suddenly switch the offset charge from idle point to the degeneracy point \( n_g = \frac{1}{2} \) (or another half-integer), we can obtain the eigenvalues (leave over the first five terms) and the eigenfunctions (leave over the first three terms) of the Hamiltonian \( \mathcal{H} \) as

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E_1^e = E_{ch} \left( 1 + v - \frac{v^2}{8} - \frac{v^3}{64} - \frac{v^4}{1536} - \cdots \right), \\
E_1^o = E_{ch} \left( 1 - v - \frac{v^2}{8} + \frac{v^3}{64} - \frac{v^4}{1536} + \cdots \right),
\]

Enlightened by [18], we have

\[
\begin{align*}
| \psi^e_1 \rangle &= \sqrt{\frac{32}{64 + v^2 + \delta}} \left[ |0\rangle + \frac{v}{8} (|1\rangle + |2\rangle) + \cdots \right], \\
| \psi^o_1 \rangle &= \sqrt{\frac{32}{64 + v^2 + \delta}} \left[ |0\rangle - \frac{v}{8} (|1\rangle - |2\rangle) + \cdots \right],
\end{align*}
\]

Here, \( \delta \) denotes the higher-order effects of \( v^2 \). When \( E_J \ll E_{ch} \), we can set \( \delta \to 0 \). So after a time \( t \), the initial state \( |\beta_0 \rangle = \cos \theta |0\rangle + \sin \theta |1\rangle \) in the system becomes

\[
|\Psi\rangle_R = U_R |\beta_0 \rangle = \sum_{j=e,o} e^{-iE_j^e t} |\psi^e_j \rangle \langle \psi^e_j | \beta_0 \rangle + e^{-iE_j^o t} |\psi^o_j \rangle \langle \psi^o_j | \beta_0 \rangle.
\]

It is shown that by using the Mathieu functions we can obtain more exact results of the eigenvalues and eigenstates of \( \mathcal{H} \) than previous researches. In particular, the eigenvalues and eigenstates can approximate to a arbitrary higher order of the Mathieu functions can be obtained.
On the other hand, because the Josephson energy $E_J$ is much smaller than the charging energy $E_{ch}$, and both of them are smaller than the superconducting energy gap $\Delta$, the Hamiltonian Eq. (11) can be parameterized by the number of the Cooper pairs $n$ through the junction as

$$
\mathcal{H} = \sum_n \left\{ E_{ch} (n - n_g)^2 |n\rangle \langle n| - \frac{1}{2} E_J [2 (|n\rangle \langle n+1| + |n+1\rangle \langle n|)] \right\} \quad (21)
$$

When $n_g$ is modulated to a half-integer, say $n_g = 1/2$ and the charging energies of two adjacent states are closed each other, the Josephson tunneling mixes them strongly. Then, the system can be reduced to a two-state system (qubit) because all other charge states have much higher energy and they can be neglected, the Hamiltonian is approximately reads

$$
\mathcal{H}_I = E_{ch} \left(n - \frac{1}{2}\right)^2 \sigma_z - \frac{1}{2} E_J \sigma_x. \quad (22)
$$

This is an ideal Hamiltonian of the qubit. By choosing the reference point of the energy at $E_0 = E_{ch}/4$, the Hamiltonian can deduce to

$$
\mathcal{H}_I = -\frac{E_J}{2} \sigma_x, \quad (23)
$$

which has the eigenvalues and eigenstates as

$$
E_0 = \frac{E_J}{2}, \quad |\varphi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle),
$$

$$
E_1 = \frac{E_J}{2}, \quad |\varphi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \quad (24)
$$

So after a time $t$ the initial state $|\beta_0\rangle$ becomes

$$
|\Psi\rangle_I = U_I |\beta_0\rangle = \sum_{i=1,2} e^{-iE_i t/2} |\varphi_i\rangle \langle \varphi_i | \beta_0\rangle \\
= e^{iE_1 t/2} |\varphi_0\rangle \langle \varphi_0 | \beta_0\rangle + e^{-iE_1 t/2} |\varphi_1\rangle \langle \varphi_1 | \beta_0\rangle. \quad (25)
$$

It is shown that $|\Psi\rangle_R$ and $|\Psi\rangle_I$ are different. The difference derives from the quantum leakage. In the following, we shall investigate the loss of fidelity due to the quantum leakage for the JCQ.

### 3 Leakage and Fidelity of JCQ

The leakage of the JCQ is in fact the probability of initial state $|\beta_0\rangle$ leaks out to higher states after some time in the practical system. It can be defined as

$$
L = \max_{i \neq 0,1} \sum_{i=0,1} R \langle \Psi | \Pi_i | \Psi \rangle_R
$$

$$
= 1 - \min_{i=0,1} \sum_{i=0,1} R \langle \Psi | \Pi_i | \Psi \rangle_R \quad (26)
$$

where $\Pi_0 = |0\rangle \langle 0|$, $\Pi_1 = |1\rangle \langle 1|$, $\Pi_i = |i\rangle \langle i|$ are project operators; $|\Psi\rangle_R$ is the finally state. The loss of fidelity is the probability by measuring the state $|\Psi\rangle_R$ with the project operators $\Pi_0 = |0\rangle \langle 0|$ and $\Pi_1 = |1\rangle \langle 1|$. By use of Eq. (20) we have

$$
\sum_{i=0,1} R \langle \Psi | \Pi_i | \Psi \rangle_R = \sum_{i=0,1} \langle \beta_0 | U_R^{|i\rangle} \Pi_i U_R | \beta_0\rangle
$$

$$
= \langle \psi^\alpha_1 | \psi^\alpha_1 \rangle \langle \beta_0 | \psi^\alpha_1 \rangle^2 + \langle \psi^\alpha_0 | \psi^\alpha_0 \rangle \langle \beta_0 | \psi^\alpha_0 \rangle^2 \quad (27)
$$

Because of

$$
\langle \psi^\alpha_i | \beta_0\rangle = \sqrt{\frac{32}{64 + v^2 + \delta}} (\cos \theta + \sin \theta),
$$

$$
\langle \psi^\alpha_0 | \beta_0\rangle = \sqrt{\frac{32}{64 + v^2 + \delta}} (\cos \theta - \sin \theta),
$$

$$
\langle \psi^\alpha_i | \psi^\alpha_i \rangle = \langle \psi^\alpha_1 | \psi^\alpha_1 \rangle = \frac{64}{64 + v^2 + \delta},
$$

$$
\langle \psi^\alpha_0 | \psi^\alpha_0 \rangle = 0, \quad (28)
$$

we have

$$
L = 1 - \left( \frac{64}{64 + v^2 + \delta} \right)^2 \approx 1 - \left( \frac{64}{64 + v^2} \right)^2. \quad (29)
$$

On the other hand, in general, the fidelity is defined as $F(\rho, \sigma) = tr(\sqrt{\rho^2 \sigma \rho^2})$ for two arbitrary states $\rho$ and $\sigma$, and $F(|\psi\rangle, |\varphi\rangle) = \langle \varphi | \psi \rangle$ for two pure states $|\psi\rangle$ and $|\varphi\rangle$. By using $|\Psi\rangle_I$ and $|\Psi\rangle_R$ we can straightforwardly calculate the loss of fidelity, it is also

$$
L = 1 - |I \langle \Psi | \Pi | \Psi \rangle_R|^2 \approx 1 - \left( \frac{64}{64 + v^2} \right)^2. \quad (30)
$$

It means that if we do not consider the interaction of the qubit and the environment, the loss of fidelity
is just the quantum leakage $L$. So the relation of the fidelity and leakage is $F = 1 - L$.

For present typical parameters of Josephson junction $E_J/E_{ch} = 0.02$ \cite{8}, we can obtain the fidelity $F = 0.9999875000$ (calculated by using our formula), which is agreement with $F = 0.9999823223$ (calculated by using the Eq.(7) of Ref.\cite{3}) very well. It can be easily seen that the loss of fidelity due to the leakage will be decreased by an appropriate choice of the device parameters. For example, the fidelity will increase to $F = 0.9999968750$ for $E_J/E_{ch} = 0.01$, and to $F = 0.9999992188$ for $E_J/E_{ch} = 0.005$, which shows that the loss of the fidelity is about $10^{-6}$ per elementary gate operation as $E_J/E_{ch} \sim 0.005$. According to DiVincenzo’s low decoherence criterion the loss of fidelity is tolerable. From the subsection II we know $E_{ch} = \frac{2\pi}{C}$, $E_J = \frac{\Phi_0 I_c}{2\pi}$, and we use units where $\hbar = 1$. So $\Phi_0 = \pi \hbar/e = \pi$ in the units. Therefore $E_{ch} = \frac{\pi}{C}$, $E_J = \frac{L}{2}$. To decrease $E_J/E_{ch}$ one should decrease the critical current $I_c$ OR the total capacitance $C$. However, in Ref.\cite{20} we investigate the short-time decoherence of the JCQ, where the increasing of the critical current $I_c$ AND decreasing the total capacitance $C$ are needed for decreasing the decoherence derived from the interaction of the system and its environment. From the analysis of the two papers we know that to decrease the decoherence not only from quantum leakage but from the environment one may improve the design of the JCQ through increasing the critical current $I_c$ AND decreasing the total capacitance $C$. But the design should keep $E_J/E_{ch} \leq 0.005 \sim 0.02$.

4 Conclusions

In this paper we investigated the loss of fidelity due to the quantum leakage for JCQ. Our researches are based on a lot of results of Mathieu functions. It is shown that our results agree with previous corresponding investigations very well. However, our work can expand to higher order approximation easily because of the well researched Mathieu functions. In particular, our results provide a feedback on how to improve the design of the JCQ for quantum computation. It is shown that decreasing the critical current and decreasing the Josephson capacitance and gate capacitance can decrease the decoherence from the quantum leakage. However, in order to decrease the total decoherence one may improve the design by increasing the critical current $I_c$ AND decreasing the total capacitance $C$. But the design should keep $E_J/E_{ch} \leq 0.005 \sim 0.02$. So we think that it is necessary to develop the technology of increasing the Josephson critical current and decreasing the capacitances in the small Josephson junctions in order to make the JCQ suitable for quantum computation.

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