Research article

Anti-periodic dynamics on high-order inertial Hopfield neural networks involving time-varying delays

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Abstract: Taking into accounting time-varying delays and anti-periodic environments, this paper deals with the global convergence dynamics on a class of anti-periodic high-order inertial Hopfield neural networks. First of all, with the help of Lyapunov function method, we prove that the global solutions are exponentially attractive to each other. Secondly, by using analytical techniques in uniform convergence functions sequence, the existence of the anti-periodic solution and its global exponential stability are established. Finally, two examples are arranged to illustrate the effectiveness and feasibility of the obtained results.

Keywords: high-order inertial neural networks; anti-periodic solution; global exponential stability; time-varying delay

Mathematics Subject Classification: 34C25, 34K13, 34K25

1. Introduction

Due to the engineering backgrounds and strong biological significance, Babcock and Westervelt \([1, 2]\) introduced an inertial term into the traditional multidirectional associative memory neural networks, and established a class of second order delay differential equations, which was called as the famous delayed inertial networks model. Arising from problems in different applied sciences such as mathematical physics, control theory, biology in different situations, nonlinear vibration, mechanics, electromagnetic theory and other related fields, the periodic oscillation is an important qualitative property of nonlinear differential equations \([3–9]\). Consequently, assuming that the activation functions are bounded and employing reduced-order variable substitution which convert
functions with period and \(q\) propose a novel approach involving di

differential inequality techniques coupled with Lyapunov function method to demonstrate the existence and global exponential stability of anti-periodic solutions for system (1.1). Particularly, our results are new and supplement some corresponding ones of the existing literature [19–28]. In a nutshell, the contributions of this paper can be summarized as follows. 1) A class of anti-periodic high-order inertial Hopfield neural networks involving time-varying delays are proposed; 2) Under some appropriate anti-periodic assumptions, all solutions and their derivatives
in the proposed neural networks model are guaranteed to converge to the anti-periodic solution and its derivative, respectively; 3) Numerical results including comparisons are presented to verify the obtained theoretical results.

The remaining parts of this paper are organized as follows. In Section 2, we make some preparations. In Section 3, the existence and the global exponential stability of the anti-periodic solution are stated and demonstrated. Section 4 shows numerical examples. Conclusions are drawn in Section 5.

2. Preliminaries

To study the existence and uniqueness of anti-periodic solutions to system (1.1), we first require the following assumptions and some key lemmas:

Assumptions:

\((F_1)\) For \(i, j, l \in D\), \(A_j(u), B_j(u), Q_j(u)\) are all non-decreasing functions with \(A_j(0) = B_j(0) = Q_j(0) = 0\), and there are nonnegative constants \(L_j^A, L_j^B, L_j^O\) and \(M_j^O\) such that

\[
|A_j(u) - A_j(v)| \leq L_j^A |u - v|, \quad |B_j(u) - B_j(v)| \leq L_j^B |u - v|, \quad |Q_j(u) - Q_j(v)| \leq L_j^O |u - v|,
\]

\[
|Q_j(u)| \leq M_j^O, \quad \bar{c}_{ij}(t + T)A_j(u) = -\bar{c}_{ij}(t)A_j(-u), \quad \bar{d}_{ij}(t + T)B_j(u) = -\bar{d}_{ij}(t)B_j(-u),
\]

and

\[
\theta_{ij}(t + T)Q_j(u)Q_i(v) = -\theta_{ij}(t)Q_j(-u)Q_i(-v),
\]

for all \(u, v \in \mathbb{R}\).

\((F_2)\) There are constants \(\beta_i > 0\) and \(\alpha_i \geq 0, \gamma_i \geq 0\) obeying

\[
E_i(t) < 0, \quad 4E_i(t)G_i(t) > H_i^2(t), \quad \forall t \in \mathbb{R}, \quad i \in D,
\]

where

\[
E_i(t) = \alpha_i \gamma_i - \bar{a}_i(t)\alpha_i^2 + \frac{1}{2} \alpha_i^2 \sum_{j=1}^{n} (|\bar{c}_{ij}(t)L_j^A| + |\bar{d}_{ij}(t)L_j^B|)
\]

\[
+ \frac{1}{2} \alpha_i^2 \sum_{j=1}^{n} \sum_{l=1}^{n} |\theta_{ij}(t)|(M_j^OL_j^A + L_j^O M_j^O),
\]

\[
G_i(t) = -\bar{b}_i(t)\alpha_i \gamma_i + \frac{1}{2} \sum_{j=1}^{n} (|\bar{c}_{ij}(t)L_j^A| + |\bar{d}_{ij}(t)L_j^B|)\alpha_i \gamma_i
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} |\bar{c}_{ij}(t)L_j^A + \bar{d}_{ij}(t)L_j^B| \frac{1}{1 - \theta_{ij}}
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} (|\bar{c}_{ij}(t)L_j^A + \bar{d}_{ij}(t)L_j^B| \frac{1}{1 - \theta_{ij}})\alpha_i \gamma_i
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha \gamma_i| |\theta_{ij}(t)|(M_j^OL_j^A + L_j^O M_j^O)
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha^2 \gamma_i \theta_{ij}(t)M_j^OL_j^A + L_j^O M_j^O \frac{1}{1 - \theta_{ij}}
\]

\[
+ \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha^2 \gamma_i \theta_{ij}(t)M_j^OL_j^A + L_j^O M_j^O \frac{1}{1 - \theta_{ij}}.
\]
\[ H_i(t) = \beta_i + \gamma_i^2 - \tilde{a}_i(t)\alpha_i\gamma_i - \tilde{b}_i(t)\alpha_i^2, \quad q^*_{ij} = \sup_{t \in \mathbb{R}} q^*_{ij}(t), \]

\[ \dot{q}^*_{ij} = \sup_{t \in \mathbb{R}} \eta^*_{ij}(t), \quad \dot{\xi}^*_{ij} = \sup_{t \in \mathbb{R}} \xi^*_{ij}(t), \quad \dot{q}^*_{ij} = \sup_{t \in \mathbb{R}} q_{ij}(t), \quad \eta^*_{ij} = \sup_{t \in \mathbb{R}} \eta_{ij}(t), \]

\[ \xi^*_{ij} = \sup_{t \in \mathbb{R}} \xi_{ij}(t), \quad c^*_{ij} = \sup_{t \in \mathbb{R}} |c_{ij}(t)|, \quad d^*_{ij} = \sup_{t \in \mathbb{R}} |d_{ij}(t)|, \forall i, j, l \in D. \]

\((F_3)\) For \(i, j, l \in D, q_{ij}, \eta_{ij}, \text{and} \xi_{ij}\) are continuously differentiable, \(q^*_{ij}(t) = \dot{q}_i(t) < 1, \eta^*_{ij}(t) = \dot{\eta}_i(t) < 1 \text{ and } \xi^*_{ij}(t) = \dot{\xi}_i(t) < 1 \text{ for all } t \in \mathbb{R}.

We will adopt the following notations:

\[ \theta^*_{ij} = \max_{t \in [0, T]} |\theta_{ij}(t)|, \forall i, j, l \in D. \]

**Remark 2.1.** Since \((1.1)\) can be converted into the first order functional differential equations. In view of \((F_1)\) and \((29), p176, Theorem 5.4\), one can see that all solutions of \((1.1)\) and \((1.2)\) exist on \([0, +\infty)\).

**Lemma 2.1.** Under \((F_1), (F_2)\) and \((F_3)\), label \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))\) and \(y(t) = (y_1(t), y_2(t), \ldots, y_n(t))\) as two solutions of system \((1.1)\) satisfying

\[ x_i(s) = \varphi_i^x(s), \quad x'_i(s) = \psi_i^x(s), \quad y_i(s) = \varphi_i^y(s), \quad y'_i(s) = \psi_i^y(s), \quad (2.2) \]

where \(-\tau_i \leq s \leq 0, \forall i \in D, \varphi_i^x, \psi_i^x, \varphi_i^y, \psi_i^y \in C([-\tau_i, 0], \mathbb{R}). \) Then, there are two positive constants \(\lambda\) and \(M = M(\varphi^x, \psi^x, \varphi^y, \psi^y)\) such that

\[ |x_i(t) - y_i(t)| \leq Me^{-\lambda t}, \quad |x'_i(t) - y'_i(t)| \leq Me^{-\lambda t}, \text{ for all } t \geq 0, \forall i \in D. \]

**Proof.** Denote \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))\) and \(y(t) = (y_1(t), y_2(t), \ldots, y_n(t))\) as two solutions of \((1.1)\) and \((1.2)\). Let \(w_i(t) = y_i(t) - x_i(t)\), then

\[ w'_i(t) = -\tilde{a}_i(t)w_i(t) - \tilde{b}_i(t)w_i(t) + \sum_{j=1}^n \xi_i(t)\tilde{A}_j(w_j(t)) + \sum_{j=1}^n \xi_i(t)B_j(w_j(t) - q_{ij}(t)) \]

\[ + \sum_{j=1}^n \sum_{l=1}^n \theta_{ij}(t)\left[Q_j(x_j(t) - \eta_{ij}(t))(y_i(t) - \xi_{ij}(t)) - Q_j(y_j(t) - \eta_{ij}(t)) \right. \]

\[ \times [Q_j(x_j(t) - \eta_{ij}(t)) + \sum_{j=1}^n \sum_{l=1}^n (Q_j(x_j(t) - \eta_{ij}(t)))Q_i(x_i(t) - \xi_{ij}(t)) \]

\[ - Q_j(x_j(t) - \eta_{ij}(t))\bar{Q}_i(x_i(t) - \xi_{ij}(t))], \quad (2.3) \]

where \(i, j \in D, \tilde{A}_j(w_j(t)) = A_j(y_j(t)) - A_j(x_j(t)) \) and

\[ \tilde{B}_j(w_j(t) - q_{ij}(t)) = B_j(y_j(t) - q_{ij}(t)) - B_j(x_j(t) - q_{ij}(t)). \]

According to \((F_2)\) and the periodicity in \((1.1)\), one can select a constant \(\lambda > 0\) such that

\[ E^*_i(t) < 0, \quad 4E^*_i(t)G^*_i(t) > (H^*_i(t))^2, \forall t \in \mathbb{R}, \quad (2.4) \]
Define the Lyapunov function by setting

$$
E^i_t(t) = \lambda \alpha t^2 + \alpha_i \gamma_i - \bar{a}_i(t) \alpha_i^2 + \frac{1}{2} \alpha_i^2 \sum_{j=1}^n (|\bar{c}_{ij}(t)|L_i^A + |\bar{d}_{ij}(t)|L_i^B)
$$

$$
+ \frac{1}{2} \alpha_i^2 \sum_{j=1}^n \sum_{l=1}^n |\theta_{ij}(t)| (M_i^Q L_i^Q + L_i^B M_i^Q),
$$

$$
G^i_t(t) = -\bar{b}_i(t) \alpha_i \gamma_i + \lambda \beta_i + \lambda \gamma_i^2 + \frac{1}{2} \sum_{j=1}^n (|\bar{c}_{ij}(t)|L_i^A + |\bar{d}_{ij}(t)|L_i^B) \alpha_i \gamma_i
$$

$$
+ \frac{1}{2} \sum_{j=1}^n \alpha_j^2 (|\bar{e}_{ij}(t)|L_i^A + \bar{d}_{jj}^+ L_i^B \frac{1}{1 - \bar{q}_{ij}} e^{2\lambda l_{2j}})
$$

$$
+ \frac{1}{2} \sum_{j=1}^n (|\bar{e}_{ij}(t)|L_i^A + \bar{d}_{jj}^+ L_i^B \frac{1}{1 - \bar{q}_{ij}} e^{2\lambda l_{2j}}) \alpha_j \gamma_j
$$

$$
+ \frac{1}{2} \sum_{j=1}^n \sum_{l=1}^n (\alpha_j^2 + |\alpha_j \gamma_j|) \theta_{jl}^+ M_j^Q L_j^Q e^{2\lambda l_{2j}} \frac{1}{1 - \xi_{lj}}
$$

$$
+ \frac{1}{2} \sum_{j=1}^n (\alpha_j^2 + |\alpha_j \gamma_j|) \theta_{jl}^+ M_j^Q L_j^Q e^{2\lambda l_{2j}} \frac{1}{1 - \xi_{lj}}, \quad i, j \in D.
$$

Define the Lyapunov function by setting

$$
K(t) = \frac{1}{2} \left[ \sum_{i=1}^n \beta_i w_i^2(t) e^{2\lambda t} + \frac{1}{2} \sum_{i=1}^n (\alpha_i w_i'(t) + \gamma_i w_i(t))^2 e^{2\lambda t} \right]
$$

$$
+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\alpha_i^2 \bar{d}_{ij} + |\alpha_i \gamma_i| \bar{d}_{ij}^+ e^{2\lambda l_{2j}} L_j^B w_j(t) \int_{t-q_{ij}(t)}^t \frac{1}{1 - \bar{q}_{ij}} e^{2\lambda s} ds
$$

$$
+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ M_j^Q L_j^Q e^{2\lambda l_{2j}}
\int_{t-\xi_{lj}(t)}^t \frac{1}{1 - \bar{\xi}_{lj}} e^{2\lambda s} ds
$$

$$
+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ M_j^Q L_j^Q e^{2\lambda l_{2j}}
\int_{t-\eta_{lj}(t)}^t \frac{1}{1 - \bar{\eta}_{lj}} e^{2\lambda s} ds.
$$

Straightforward computation yields that

$$
K'(t) = 2 \lambda \left[ \frac{1}{2} \sum_{i=1}^n \beta_i w_i^2(t) e^{2\lambda t} + \frac{1}{2} \sum_{i=1}^n (\alpha_i w_i'(t) + \gamma_i w_i(t))^2 e^{2\lambda t} \right]
$$

$$
+ \sum_{i=1}^n \beta_i w_i(t) w_i'(t) e^{2\lambda t} + \sum_{i=1}^n (\alpha_i w_i'(t) + \gamma_i w_i(t))(\alpha_i w_i''(t) + \gamma_i w_i'(t)) e^{2\lambda t}
$$

$$
+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\alpha_i^2 \bar{d}_{ij} + |\alpha_i \gamma_i| \bar{d}_{ij}^+ e^{2\lambda l_{2j}} \frac{1}{1 - \bar{q}_{ij}} L_j^B
\int_{t-q_{ij}(t)}^t \frac{1}{1 - \bar{q}_{ij}} e^{2\lambda s} ds
$$

$$
\times [w_j^2(t) e^{2\lambda t} - w_j^2(t - q_{ij}(t)) e^{2\lambda (t-q_{ij}(t))} (1 - \bar{q}_{ij}(t))]
$$

$$
+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ M_j^Q L_j^Q e^{2\lambda l_{2j}} \frac{1}{1 - \xi_{lj}}
\int_{t-\xi_{lj}(t)}^t \frac{1}{1 - \bar{\xi}_{lj}} e^{2\lambda s} ds
$$

$$
\times [w_j^2(t) e^{2\lambda t} - w_j^2(t - \xi_{lj}(t)) e^{2\lambda (t-\xi_{lj}(t))} (1 - \bar{\xi}_{lj}(t))].
$$
\[
\begin{align*}
&\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_j^2 + |\alpha_j \gamma_i|) \theta_{ij}^l M_j^Q L_j^Q e^{2 \lambda j_i} \frac{1}{1 - \eta_{ji}^-} \\
&\times [w_i^2(t)e^{2\mu} - w_i^2(t - \eta_{ji}(t))e^{2\mu(1-\eta_{ji}(t))}(1 - \eta_{ji}^-)] \\
&= 2\lambda [\frac{1}{2} \sum_{i=1}^{n} \beta_i w_i w_i(e^{2\mu} + \frac{1}{2} \sum_{i=1}^{n} (\alpha_i w_i^2(t) + \gamma_i w_i(t))^2 e^{2\mu}] \\
&+ \sum_{i=1}^{n} (\beta_i + \gamma_i^2) w_i(t)w_i(t)e^{2\mu} + \sum_{i=1}^{n} \alpha_i (\alpha_i w_i^2(t) + \gamma_i w_i(t)) e^{2\mu} \\
&\times [-\tilde{a}_i(t) w_i'(t) - \tilde{b}_i(t) w_i(t) + \sum_{j=1}^{n} \tilde{c}_{ij}(t) \tilde{A}_j(w_j(t)) + \sum_{j=1}^{n} \tilde{d}_{ij}(t) \tilde{B}_j(w_j(t - q_{ij}(t))) \\
&+ \sum_{j=1}^{n} \theta_{ij}(t)(Q_j(y_j(t - \eta_{ij}(t)))Q_j(y_j(t - \xi_{ij}(t))) \\
&- Q_j(y_j(t - \eta_{ij}(t)))Q_j(x_i(t - \xi_{ij}(t))) + Q_j(y_j(t - \eta_{ij}(t)))Q_j(x_i(t - \xi_{ij}(t))) \\
&- Q_j(x_i(t - \eta_{ij}(t)))Q_j(x_i(t - \xi_{ij}(t)))] + \sum_{i=1}^{n} \alpha_i \gamma_i (w_i(t))^2 e^{2\mu} \\
&\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 d_{ij}^l + |\alpha_i \gamma_i| d_{ij}^l) e^{2\lambda j_i} L_j^B \\
&\times [w_i^2(t) \frac{1}{1 - \tilde{q}_{ij}^+} e^{2\mu} - w_i^2(t - q_{ij}(t)) e^{2\mu(1-\tilde{q}_{ij}^+)} \frac{1}{1 - \tilde{q}_{ij}^+}] \\
&+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{ij}^l M_j^Q L_j^Q e^{2 \lambda j_i} \\
&\times [w_i^2(t) \frac{1}{1 - \tilde{\xi}_{ij}^+} e^{2\mu} - w_i^2(t - \tilde{\xi}_{ij}(t)) e^{2\mu(1-\tilde{\xi}_{ij}^+)} \frac{1}{1 - \tilde{\xi}_{ij}^+}] \\
&+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{ij}^l M_j^Q L_j^Q e^{2 \lambda j_i} \\
&\times [w_i^2(t) \frac{1}{1 - \tilde{\eta}_{ij}^-} e^{2\mu} - w_i^2(t - \eta_{ij}(t)) e^{2\mu(1-\tilde{\eta}_{ij}^-)} \frac{1}{1 - \tilde{\eta}_{ij}^-}] \\
&\leq e^{2\mu} \left\{ \sum_{i=1}^{n} (\beta_i + \gamma_i^2 + 2\lambda \alpha_i \gamma_i - \tilde{a}_i(t) \alpha_i \gamma_i - \tilde{b}_i(t) \alpha_i^2) w_i(t) w_i'(t) \right. \\
&\left. + \sum_{i=1}^{n} (\lambda \alpha_i^2 + \lambda \alpha_i \gamma_i - \tilde{a}_i(t) \alpha_i^2)(w_i(t))^2 - \sum_{i=1}^{n} (\tilde{b}_i(t) \alpha_i \gamma_i - \lambda \beta_i - \lambda \gamma_i^2) w_i^2(t) \right\} \\
&\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 d_{ij}^l + |\alpha_i \gamma_i| d_{ij}^l) \frac{1}{1 - \tilde{q}_{ij}^+} e^{2\lambda j_i} L_j^B w_i^2(t) \\
&\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{ij}^l M_j^Q L_j^Q e^{2 \lambda j_i} \frac{1}{1 - \tilde{\xi}_{ij}^+}
\end{align*}
\]

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\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_j^2 + |\alpha_j \gamma_i|) \theta_{jl}^+ L_i^O M_j^O e^{2 \lambda_l} w_j^2(t) \frac{1}{1 - \bar{q}_{jl}}
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 \bar{d}_{ij}^+ + |\alpha_i \gamma_i| \bar{d}_{ij}^+) L_i^B w_j^2(t - q_{ij}(t))
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ M_j^O L_i^O w_l^2(t - \xi_{ij}(t))
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ L_i^O M_l^O w_l^2(t - \eta_{jl}(t))
+ \sum_{i=1}^{n} (\alpha_i^2 |w_i^2(t)| + |\alpha_i \gamma_i||w_i(t)||\bar{c}_{ij}(t)||\bar{A}_j(w_j(t))|
+ \sum_{i=1}^{n} (\alpha_i^2 |w_i^2(t)| + |\alpha_i \gamma_i||w_i(t)|||\bar{B}_j(w_j(t) - q_{ij}(t))||)
+ \sum_{i=1}^{n} (\alpha_i^2 |w_i^2(t)| + |\alpha_i \gamma_i||w_i(t)|)
\times \sum_{j=1}^{n} \sum_{l=1}^{n} [\theta_{jl}(t)](M_j^O L_i^O |w_j(t - \xi_{ij}(t))| + L_j^O |w_j(t - \eta_{jl}(t))|M_i^O)]
= e^{2 \lambda_l} \sum_{i=1}^{n} (\beta_i + \gamma_i^2 + 2 \lambda \alpha_i \gamma_i - \bar{a}_i(t) \alpha_i \gamma_i - \bar{b}_i(t) \alpha_i^2) w_i(t) w_i^2(t)
+ \sum_{i=1}^{n} (\alpha \alpha_i^2 + \alpha_i \gamma_i - \bar{a}_i(t) \alpha_i^2)(w_i^2(t))^2
+ \sum_{i=1}^{n} [-\bar{b}_i(t) \alpha \gamma_i + \lambda \beta_i + \lambda \gamma_i^2 + \frac{1}{2} \sum_{j=1}^{n} (\alpha_j^2 \bar{d}_{ji}^+ + |\alpha_j \gamma_j| \bar{d}_{ji}^+) \frac{1}{1 - \bar{q}_{ji}^+} e^{2 \lambda_{ji}^+} L_i^B
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ M_j^O L_i^O e^{2 \lambda_{jl}^+} \frac{1}{1 - \bar{q}_{jl}^+}]
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ L_i^O M_j^O e^{2 \lambda_{jl}^+} \frac{1}{1 - \bar{q}_{jl}} w_l^2(t)
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 \bar{d}_{ij}^+ + |\alpha_i \gamma_i| \bar{d}_{ij}^+) L_i^B w_j^2(t - q_{ij}(t))
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ M_j^O L_i^O w_l^2(t - \xi_{ij}(t))
- \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i \gamma_i|) \theta_{jl}^+ L_i^O M_l^O w_l^2(t - \eta_{jl}(t))
\]
\[
\begin{aligned}
&+ \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 |w_i'(t)| + |\alpha_i \gamma_i||w_i(t)||\bar{c}_{ij}(t)||\bar{A}_j(w_j(t))|
\end{aligned}
\]
\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 |w_i'(t)| + |\alpha_i \gamma_i||w_i(t)||\bar{d}_{ij}(t)||\bar{B}_j(w_j(t) - q_{ij}(t)))|
\]
\[
+ \sum_{i=1}^{n} (\alpha_i^2 |w_i'(t)| + |\alpha_i \gamma_i||w_i(t)||\sum_{j=1}^{n} |\theta_{ij}(t)|
\]
\[
\times (M_j^Q L_j^Q |w_i(t) - \xi_{ij}(t)| + L_j^Q |w_j(t) - \eta_{ij}(t)||M_j^Q), \quad \forall t \in [0, +\infty).
\]

(2.5)

It follows from (F1) and \(PQ \leq \frac{1}{2}(P^2 + Q^2)(P, Q \in \mathbb{R})\) that

\[
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 |w_i'(t)| + |\alpha_i \gamma_i||w_i(t)||\bar{c}_{ij}(t)||\bar{A}_j(w_j(t))| &
\leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^2 |\bar{c}_{ij}(t)||L_j^A ((w_i'(t))^2 + w_j^2(t))
\end{aligned}
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i \gamma_i||\bar{c}_{ij}(t)||L_j^A (w_i^2(t) + w_j^2(t))
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^2 |\bar{c}_{ij}(t)||L_j^A (w_i'(t))^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (|\alpha_i \gamma_i||\bar{c}_{ij}(t)||L_j^A + \alpha_j^2 |\bar{c}_{ji}(t)||L_i^A + |\alpha_j \gamma_j||\bar{c}_{ji}(t)||L_i^A)w_j^2(t),
\]

\[
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i^2 |w_i'(t)| + |\alpha_i \gamma_i||w_i(t)||\bar{d}_{ij}(t)||\bar{B}_j(w_j(t) - q_{ij}(t))) &
\leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^2 |\bar{d}_{ij}(t)||L_j^B ((w_i'(t))^2 + w_j^2(t - q_{ij}(t)))
\end{aligned}
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i \gamma_i||\bar{d}_{ij}(t)||L_j^B (w_i^2(t) + w_j^2(t - q_{ij}(t)))
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^2 |\bar{d}_{ij}(t)||L_j^B (w_i'(t))^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (|\alpha_i \gamma_i||\bar{d}_{ij}(t)||L_j^B + \alpha_j^2 |\bar{d}_{ji}(t)||L_i^B)w_j^2(t - q_{ij}(t))
\]
\]

and

\[
\sum_{i=1}^{n} (\alpha_i^2 |w_i'(t)| + |\alpha_i \gamma_i||w_i(t)|)
\]
\begin{align*}
\times & \sum_{j=1}^{n} \sum_{l=1}^{n} |\theta_{ij}(t)(M_j^Q L_1^Q w_j(t - \xi_{ijl}(t))) + L_j^Q w_j(t - \eta_{ijl}(t))| M_i^Q \\
 \leq & \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^2 |\theta_{ij}(t)| M_j^Q L_1^Q ((w_i'(t))^2 + w_i^2(t - \xi_{ijl}(t))) \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i \gamma_i| |\theta_{ijl}(t)| M_j^Q L_1^Q ((w_i(t))^2 + w_i^2(t - \xi_{ijl}(t))) \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i \gamma_i| |\theta_{ijl}(t)| L_j^Q M_i^Q ((w_i(t))^2 + w_i^2(t - \eta_{ijl}(t))) \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i \gamma_i| |\theta_{ijl}(t)| L_j^Q M_i^Q ((w_i(t))^2 + w_i^2(t - \eta_{ijl}(t))) \\
 = & \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 |\theta_{ij}(t)|(M_j^Q L_1^Q + L_j^Q M_i^Q)(w_i'(t))^2 \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i \gamma_i| |\theta_{ijl}(t)|(M_j^Q L_1^Q + L_j^Q M_i^Q)(w_i(t))^2 \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 + |\alpha_i \gamma_i| |\theta_{ijl}(t)| M_j^Q L_1^Q w_i^2(t - \xi_{ijl}(t)) \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 + |\alpha_i \gamma_i| |\theta_{ijl}(t)| L_j^Q M_i^Q w_i^2(t - \eta_{ijl}(t)) \\
 &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 |\theta_{ij}(t)|(M_j^Q L_1^Q + L_j^Q M_i^Q)(w_i'(t))^2 \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i \gamma_i| |\theta_{ijl}(t)|(M_j^Q L_1^Q + L_j^Q M_i^Q)(w_i(t))^2 \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 + |\alpha_i \gamma_i| |\theta_{ijl}(t)| M_j^Q L_1^Q w_i^2(t - \xi_{ijl}(t)) \\
 & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 + |\alpha_i \gamma_i| |\theta_{ijl}(t)| L_j^Q M_i^Q w_i^2(t - \eta_{ijl}(t)),
\end{align*}

which, together with (2.4) and (2.5), entails that

\[ K'(t) \leq e^{2.1t} \left[ \sum_{i=1}^{n} (\beta_i + \gamma_i^2 + 2 \lambda \alpha_i \gamma_i - \bar{a}_i(t) \alpha_i \gamma_i - \bar{b}_i(t) \alpha_i^2) w_i(t) w_i'(t) \\
+ \sum_{i=1}^{n} [\lambda \alpha_i^2 + \alpha_i \gamma_i - \bar{a}_i(t) \alpha_i^2 + \frac{1}{2} \alpha_i^2 \sum_{j=1}^{n} (|\bar{c}_{ij}(t)| L_j^A + |\bar{d}_{ij}(t)| L_j^B) \\
+ \frac{1}{2} \alpha_i^2 \sum_{j=1}^{n} |\theta_{ijl}(t)|(M_j^Q L_1^Q + L_j^Q M_i^Q))(w_i'(t))^2 \right]. \]
3. Anti-periodicity of system (1.1)

Now, we set out the main result of this paper as follows.
Theorem 3.1. Under assumptions (F1)–(F3), system (1.1) possesses a global exponential stable T-anti-periodic solution.

Proof. Denote \( \kappa(t) = (\kappa_1(t), \kappa_2(t), \cdots, \kappa_n(t)) \) be a solution of system (1.1) satisfying:

\[
\kappa_i(s) = \varphi_i^*(s), \quad \kappa'_i(s) = \psi_i^*(s), \quad t - \tau_i \leq s \leq 0, \quad \varphi_i^*, \psi_i^* \in C([-\tau_i, 0], \mathbb{R}), \quad i \in D. \tag{3.1}
\]

With the aid of (F1), one can see that

\[
\begin{align*}
\bar{a}_i(t + T) &= \bar{a}_i(t), \quad \bar{b}_i(t + T) = \bar{b}_i(t), \quad q_{ij}(t + T) = q_{ij}(t), \\
\eta_{ij}(t + T) &= \eta_{ij}(t), \quad \xi_{ij}(t + T) = \xi_{ij}(t), \quad J_i(t + T) = -J_i(t), \\
(-1)^m \bar{c}_{ij}(t + (m + 1)T) A_j(\kappa_j(t + (m + 1)T)) &= \bar{c}_{ij}(t) A_j((-1)^m \kappa_j(t + (m + 1)T)) \\
(-1)^m \bar{d}_{ij}(t + (m + 1)T) B_j(\kappa_j(t + (m + 1)T) - q_{ij}(t)) &= \bar{d}_{ij}(t) B_j((-1)^m \kappa_j(t + (m + 1)T) - q_{ij}(t)),
\end{align*}
\]

and

\[

t \in \mathbb{R} \text{ and } i, j, l \in D.
\]

Consequently, for any nonnegative integer \( m, \)

\[
((-1)^m \kappa_i(t + (m + 1)T))' = -\bar{a}_i(t)((-1)^m \kappa_i(t + (m + 1)T))' - \bar{b}_i(t)((-1)^m \kappa_i(t + (m + 1)T)) + \sum_{j=1}^{n} \bar{c}_{ij}(t) A_j((-1)^m \kappa_j(t + (m + 1)T)) + \sum_{j=1}^{n} \bar{d}_{ij}(t) B_j((-1)^m \kappa_j(t + (m + 1)T) - q_{ij}(t)) + \sum_{j=1}^{n} \sum_{l=1}^{n} \theta_{ijkl}(t) \\
\times Q_j((-1)^m \kappa_j(t + (m + 1)T) - \eta_{ij}(t))) \\
\times Q_l((-1)^m \kappa_l(t + (m + 1)T) - \xi_{ij}(t))) \\
+ J_i(t), \quad \text{for all } i \in D, t + (m + 1)T \geq 0. \tag{3.2}
\]

Clearly, \((-1)^m \kappa(t + (m + 1)T) (t + (m + 1)T \geq 0)\) satisfies (1.1), and \( v(t) = -\kappa(t + T) \) is a solution of system (1.1) involving initial values:

\[
\varphi_i^*(s) = -\kappa_i(s + T), \quad \psi_i^*(s) = -\kappa'_i(s + T), \quad \text{for all } s \in [-\tau_i, 0], \quad i \in D.
\]

Thus, with the aid of Lemma 2.1, we can pick a constant \( M = M(\varphi^*, \psi^*, \varphi'^*, \psi'^*) \) satisfying

\[
|\kappa_i(t) - v_i(t)| \leq Me^{-\lambda t}, \quad |\kappa'_i(t) - v'_i(t)| \leq Me^{-\lambda t}, \quad \text{for all } t \geq 0, \quad i \in D.
\]

Hence,

\[
\begin{align*}
|(-1)^p \kappa_i(t + pT) - ((-1)^{p+1} \kappa_i(t + (p + 1)T))| &= |\kappa_i(t + pT) - v_i(t + pT)| \leq Me^{-\lambda(t+pT)}, \\
|((-1)^p \kappa_i(t + pT))' - ((-1)^{p+1} \kappa_i(t + (p + 1)T))'| &= |\kappa'_i(t + pT) - v'_i(t + pT)| \leq Me^{-\lambda(t+pT)},
\end{align*}
\]

\[ \forall i \in D, \quad t + pT \geq 0. \tag{3.3} \]
Remark 3.1. For inertial neural networks without high-order terms respectively, suppose
\[ \kappa(t + (m + 1)T) = \kappa(t) + \sum_{p=0}^{m} [(-1)^{p+1} \kappa(t + (p + 1)T) - (-1)^p \kappa(t + pT)] (i \in D) \]
and
\[ ((-1)^{m+1} \kappa(t + (m + 1)T))' = \kappa'(t) + \sum_{p=0}^{m} [((-1)^{p+1} \kappa(t + (p + 1)T))' - ((-1)^p \kappa(t + pT))'] (i \in D). \]

Therefore, (3.3) suggests that there exists a continuous differentiable function \( y(t) = (y_1(t), y_2(t), \cdots, y_n(t)) \) such that \( \{(-1)^m \kappa(t + mT)\}_{m \geq 1} \) and \( \{(((-1)^m \kappa(t + mT))'\}_{m \geq 1} \) are uniformly convergent to \( y(t) \) and \( y'(t) \) on any compact set of \( \mathbb{R} \), respectively.

Moreover,
\[ y(t + T) = \lim_{m \to +\infty} (-1)^m \kappa(t + T + mT) = -\lim_{(m+1) \to +\infty} (-1)^{m+1} \kappa(t + (m + 1)T) = -y(t) \]

involves that \( y(t) \) is \( T \)-anti-periodic on \( \mathbb{R} \). It follows from \( (F_1)-(F_3) \) and the continuity on (3.2) that \( \{\kappa''(t + (m + 1)T)\}_{m \geq 1} \) uniformly converges to a continuous function on any compact set of \( \mathbb{R} \). Furthermore, for any compact set of \( \mathbb{R} \), setting \( m \to +\infty \), we obtain
\[ y''_i(t) = -\bar{a}_i(t)y'_i(t) - \bar{b}_i(t)y_i(t) + \sum_{j=1}^{n} \bar{c}_{ij}(t)A_j(y_j(t)) + \sum_{j=1}^{n} \bar{d}_{ij}(t)B_j(y_j(t - q_{ij}(t))) + \sum_{j=1}^{n} \sum_{l=1}^{n} \theta_{ijl}(t)Q_j(y_j(t - \eta_{ijl}(t)))Q_l(y_l(t - \xi_{ijl}(t))) + J_i(t), \ i \in D, \]

which involves that \( y(t) \) is a \( T \)-anti-periodic solution of (1.1). Again from Lemma 2.1, we gain that \( y(t) \) is globally exponentially stable. This finishes the proof of Theorem 3.1.

Remark 3.1. For inertial neural networks without high-order terms respectively, suppose
\[ |\bar{a}_i - \bar{b}_i| < 2, A_i \text{ and } B_i \text{ are bounded}, \ i \in D, \quad (3.4) \]
and
\[ |\bar{a}_i - \bar{b}_i + 1| < 1, \ i \in D, \quad (3.5) \]

the authors gained the existence and stability on periodic solutions in [10, 11] and anti-periodic solutions in [12]. Moreover, the reduced-order method was crucial in [10–12] when anti-periodicity and periodicity of second-order inertial neural networks were considered. However, (3.4) and (3.5) have been abandoned in Theorem 3.1 and the reduced-order method has been substituted in this paper. Therefore, our results on anti-periodicity of high-order inertial Hopfield neural networks are new and supplemental in nature.
4. Examples and numerical simulations

Example 4.1. Let $n = 2$, and consider a class of high-order inertial Hopfield neural networks in the form of

\[
\begin{align*}
x_1''(t) &= -14.92x_1'(t) - 27.89x_1(t) + 2.28\sin(t)A_1(x_1(t)) + 2.19\cos(t)A_2(x_2(t)) \\
&\quad\quad -0.84\cos(2t)B_1(x_1(t - 0.2 \sin^2 t)) + 2.41\cos(2t)B_2(x_2(t - 0.3 \sin^2 t)) \\
&\quad\quad +4\sin(2t)Q_1(x_1(t - 0.4 \sin^2 t))Q_2(x_2(t - 0.5 \sin^2 t)) + 55 \sin t, \\
x_2''(t) &= -15.11x_2'(t) - 31.05x_2(t) - 1.88\sin(t)A_1(x_1(t)) \\
&\quad\quad -2.33\cos(t)A_2(x_2(t)) \\
&\quad\quad -2.18\sin(2t)B_1(x_1(t - 0.2 \cos^2 t)) + 3.18\cos(2t)B_2(x_2(t - 0.3 \cos^2 t)) \\
&\quad\quad +3.8\sin(2t)Q_1(x_1(t - 0.4 \cos^2 t))Q_2(x_2(t - 0.5 \cos^2 t)) + 48 \sin t,
\end{align*}
\]

where $t \geq 0$, $A_1(u) = A_2(u) = \frac{1}{35}|u|$, $B_1(u) = B_2(u) = \frac{1}{48}u$, $Q_1(u) = Q_2(u) = \frac{1}{35} \arctan u$.

Using a direct calculation, one can check that (4.1) satisfies (2.4) and $(F_1) - (F_3)$. Applying Theorem 3.1, it is obvious that system (4.1) has a globally exponentially stable $\pi$-anti-periodic solution. Simulations reflect that the theoretical anti-periodicity is in sympathy with the numerically observed behavior (Figures 1 and 2).

![Figure 1](image-url)

**Figure 1.** Numerical solutions $x(t)$ to system (4.1) with initial values: $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s)) \equiv (-1, 3, 0, 0), (-2, 1, 0, 0), (2, -3, 0, 0), s \in [-5, 0]$. 

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conclusions in [10–82] and the references cited therein can not be applied to show the anti-periodic
[10, 11] only considered the polynomial power stability of some proportional time-delay systems, but
never been touched by using the non-reduced order method. Manifestly, the assumptions (3.4) and
the anti-periodicity on high-order inertial Hopfield neural networks involving time-varying delays has
π
and there exists a
Remark 4.1.
From the figures 1–4, one can see that the solution is similar to sinusoidal oscillation,
by Theorem 3.1, one can find that all solutions of networks (4.2) are convergent to a
solution (See Figures 3 and 4).
Example 4.2. Regard the following high-order inertial Hopfield neural networks involving
time-varying delays and coefficients:
\[
\begin{align*}
    x''_1(t) &= -(14 + 0.9|\sin t|)x'_1(t) - (27 + 0.8|\cos t|)x_1(t) + 2.28(\sin t)A_1(x_1(t)) \\
    &\quad + 0.29(\cos t)A_2(x_2(t)) \\
    &\quad - 0.84(\cos 2t)B_1(x_1(t - 0.2 \sin^2 t)) + 2.41(\cos 2t)B_2(x_2(t - 0.3 \sin^2 t)) \\
    &\quad + 4(\sin t)Q_1(x_1(t - 0.4 \sin^2 t))Q_2(x_2(t - 0.5 \sin^2 t)) + 100 \sin t, \\
    x''_2(t) &= -(15 + 0.1|\cos t|)x'_2(t) - (31 + 0.1|\sin t|)x_2(t) - 1.88(\sin t)A_1(x_1(t)) \\
    &\quad - 2.33(\cos t)A_2(x_2(t)) \\
    &\quad - 3.18(\sin 2t)B_1(x_1(t - 0.2 \cos^2 t)) + 3.18(\cos 2t)B_2(x_2(t - 0.3 \cos^2 t)) \\
    &\quad + 3.8(\sin t)Q_1(x_1(t - 0.4 \cos^2 t))Q_2(x_2(t - 0.5 \cos^2 t)) + 200 \sin t,
\end{align*}
\]
where \( t \geq 0 \), \( A_1(u) = A_2(u) = \frac{1}{32}|u|, B_1(u) = B_2(u) = \frac{1}{88}|u|, Q_1(u) = Q_2(u) = \frac{1}{110}(|x + 1| - |x - 1|) \). Then,
by Theorem 3.1, one can find that all solutions of networks (4.2) are convergent to a \( \pi \)-anti-periodic
solution (See Figures 3 and 4).

Remark 4.1. From the figures 1–4, one can see that the solution is similar to sinusoidal oscillation,
and there exists a \( \pi \)-anti-periodic solution satisfying \( x(t + \pi) = -x(t) \). To the author’s knowledge,
the anti-periodicity on high-order inertial Hopfield neural networks involving time-varying delays has
never been touched by using the non-reduced order method. Manifestly, the assumptions (3.4) and
(3.5) adopted in [10, 11] are invalid in systems (4.1) and (4.2). In addition, the most recently papers
[10, 11] only considered the polynomial power stability of some proportional time-delay systems, but
not involved the exponential power stability of the addressed systems. And the results in [35–82] have
not touched on the anti-periodicity of inertial neural networks. This entails that the corresponding
conclusions in [10–82] and the references cited therein can not be applied to show the anti-periodic
convergence for systems (4.1) and (4.2).
5. Conclusion

In this paper, abandoning the reduced order method, we apply inequality techniques and Lyapunov function method to establish the existence and global exponential stability of anti-periodic solutions for a class of high-order inertial Hopfield neural networks involving time-varying delays and anti-periodic environments. The obtained results are essentially new and complement some recently published results. The method proposed in this article furnishes a possible approach for studying anti-periodic on other types high-order inertial neural networks such as shunting inhibitory cellular neural networks, BAM neural networks, Cohen-Grossberg neural networks and so on.
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Conflict of interest

The authors confirm that they have no conflict of interest.

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