The Extension Structure of 2D Massive Current Algebras

J. Laartz *

Department of Mathematics, Harvard University
One Oxford Street, Cambridge, MA 02138 / USA

Abstract
The extension structure of the 2-dimensional current algebra of non-linear sigma models is analysed by introducing Kostant Sternberg \((L,M)\) systems. It is found that the algebra obeys a two step extension by abelian ideals. The second step is a non-split extension of a representation of the quotient of the algebra by the first step of the extension. The cocycle which appears is analysed.

*Permanent address : Universität Freiburg, Fakultät für Physik, Hermann-Herder Strasse 3, D-7800 Freiburg i.Br. / Germany
1 Introduction

Two-dimensional sigma models whose target space is a symmetric space are known to be integrable [4]. They are classically conformal invariant but after quantization the conformal invariance is broken and the theory becomes massive. Recently we have discovered a new current algebra structure for two-dimensional non-linear sigma models whose target space is a homogenous space [5]. It was found that the algebra closes after introducing a scalar field with values in the second symmetric tensor power of the corresponding Lie algebra. In [3] we used this result to analyse the integrable structure of sigma models on symmetric spaces which belong to the class of non-ultra-local models. We derived a new Lie-Poisson structure differing from the Yang-Baxter structure obtained in the ultra-local case.

There is a growing interest in massive current algebras, because massive theories whose ultraviolet fixed points are conformal theories can be seen as integrable pertubations of conformal field theories [1]. In the analysis of the quantum structure of massive current algebras the conserved non-local charges have been used in [2] to give an algebraic (non pertubative) definition of a subclass of massive current algebras.

Another way to achieve a better understanding of the quantized structure of integrable massive theories would be to construct unitary representations of the associated classical current algebra. But for all of these 'Ans"atze' a better understanding of the algebraic structure of this type of algebras is needed.

In this note we will deepen the analysis of the algebraic structure of the massive current algebra derived in [5] (with special emphasis on the extension structure). In the second part we will briefly review the results obtained in [5]. In the third part we will go back to an general interpretation of current algebras in terms of certain modules called $(L, M)$--modules derived by Kostant and Sternberg in the late 60's while looking at Lie-superalgebras (partially published in [7]). We will give a short explanation of this construction and then find an interpretation of the current algebra derived in [5] in terms of a $(L, M)$--module. Using this interpretation we will analyse the extension structure of the current algebra in part four. We will see that the appearing Schwinger term has an interpretation (on the level of representations of the quotient of the whole module by the first extension step) as a cocycle of the second step of an abelian extension of this module.

2 The current algebra of non-linear sigma models on homogenous spaces

We begin by briefly reviewing the results on the current algebra structure of general non-linear sigma models defined on Riemannian homogenous spaces
\( M = G/H \), with action

\[
S = \frac{1}{2} \int d^2x \, g_{ij}(\varphi) \partial^\mu \varphi^i \partial_\mu \varphi^j ,
\]

(1)
derived in \[5\]. Here \( \phi \) is the basic field with values in \( M \) and \( g_{ij} \) the metric on \( M \). Technically, we require that \( M \) is the quotient space of some connected Lie group \( G \), with Lie algebra \( g \), modulo some compact subgroup \( H \subset G \), with Lie algebra \( h \subset g \). Then \( G \) acts on \( M \) by isometries and the \( G \) invariance of the action (1) leads to a conserved Noether current \( j_\mu \), with values in \( g^* \) (the dual of \( g \)). The Poisson algebra of these currents can be written in a closed form after introducing a scalar field \( j \) with values in the second symmetric tensor power \( S^2(g^*) \) of \( g^* \). To do so it is convenient to introduce a basis \( (T^a) \) of \( g \), with structure constants \( f^c_{ab} \) defined by \( [T_a, T_b] = f^c_{ab} T_c \), together with the corresponding dual basis \( (T^a) \) of \( g^* \), and to expand \( j_\mu \) and \( j \) into components:

\[
j_\mu = j_{\mu,a} T^a , \quad j = j_{ab} T^a \vee T^b .
\]

With this notation the, the massive current algebra takes the form

\[
\{ j_{0,a}(x), j_{0,b}(y) \} = - f^c_{ab} j_{0,c}(x) \delta(x-y) , \quad (2)
\]

\[
\{ j_{0,a}(x), j_{1,b}(y) \} = - f^c_{ab} j_{1,c}(x) \delta(x-y) + j_{ab}(y) \delta'(x-y) , \quad (3)
\]

\[
\{ j_{0,a}(x), j_{bc}(y) \} = - (f^d_{ab} j_{cd}(x) + f^d_{ac} j_{bd}(x)) \delta(x-y) , \quad (4)
\]

\[
\{ j_{1,a}(x), j_{1,b}(y) \} = 0 , \quad (5)
\]

\[
\{ j_{ab}(x), j_{cd}(y) \} = 0 . \quad (6)
\]

\[
\{ j_{1,a}(x), j_{bc}(y) \} = 0 , \quad (7)
\]

3 The massive current algebra as an example for a Kostant Sternberg \((L, M)\)–system

3.1 Kostant Sternberg \((L, M)\)–systems

We begin with a short summary of the notion of \((L, M)\)–systems \[4,5\]. We like to thank S. Sternberg for pointing out this interesting structure to us and for his unpublished notes \[6\] on which this brief summary is based.

Let \( L \) be a Lie algebra and let \( M \) be a commutative ring. Let \( L \) act on \( M \) such that \( L \) is a module over \( M \) such that the module structure is consistent with the representation of \( L \) on \( M \) and the adjoint representation. That is, the two linear maps :

\[
L \otimes M \to M , \quad X \otimes f \mapsto Xf \quad (8)
\]

\[
L \otimes M \to L , \quad X \otimes f \mapsto fX \quad (9)
\]
(with \( X \in L, f \in M \)), have to satisfy the following conditions:

\( L \) acts as derivation on \( M \),

\[ X(fg) = (Xf)g + f(Xg) \]  

(10)

the action of \( L \) on \( M \) gives a representation of \( L \) on \( M \),

\[ [X,Y](f) = X(Yf) - Y(Xf) \]  

(11)

\( L \) has to be a module over \( M \),

\[ (fg)X = f(gX) \]  

(12)

the module structure is consistent with the action (representation),

\[ f(Xg) = (fX)g \]  

(13)

the module structure is consistent with the adjoint representation,

\[ [X, fY] = (Xf)Y + f[X, Y] \]  

(14)

For examples let us start with a closer look at the easiest case the exterior algebra \( A = \bigwedge_M L \) of \( L \) as module over \( M \) and considering only the zeroth and first order in the exterior product. \( A^0 = M \) and \( A^1 = L \) clearly fulfill the above condition. Furthermore we have an additional algebraic structure, that of an anti-Poisson algebra.

Generally an anti-Poisson algebra is a Lie super-algebra \( A = A_0 \oplus A_1 \) obeying:

1. \( \{A_1, A_1\} \subset A_1 \)
2. \( \{A_1, A_0\} \subset A_0 \)
3. \( \{A_0, A_0\} \subset A_1 \)

(opposite parity as in the Poisson algebra case)

That is: we have a \( Z \) and \( \bar{Z} \) graded algebra with compatible \( Z \) and \( \bar{Z} \) grading:

\[ A_0 = A^0 \oplus A^2 \oplus \cdots \]
\[ A_1 = A^1 \oplus A^3 \oplus \cdots \]

in our example \( A^0 \) is, by construction, commutative, so we remain with:

\[ \{A^1, A^1\} \subset A^1 \]  

(15)

\[ \{A^1, A^0\} \subset A^0 \]  

(16)

The definition of \( \{.,.\} \) can be extended to all of \( A \) (if \( A \) is freely generated by \( A^0 \) and \( A^1 \)), so that \( A \) itself is an anti-Poisson-algebra.

Another example is the space of smooth sections of the exterior bundle of the tangent bundle of a differentiable manifold \( N \), \( \Gamma(\bigwedge T(N)) \). If we take \( A^0 \) to be the ring of smooth functions on \( N \) and \( A^1 \) the Lie algebra of smooth vector
fields on \( N \), we have again an \((L, M)\)–system and the exterior algebra is an anti-Poisson algebra. Furthermore the tensor product of two anti-Poisson algebras is again an anti-Poisson algebra and in the case of a commutative \( A_0 \), by identifying the zeroth order term in the tensor product with \( M \) and the first order term with \( L \) the tensor product of two \((L, M)\)-systems is again an \((L, M)\)-system. We will return to this construction later in more detail.

The crucial observation of Kostant and Sternberg while looking at local current commutation relations in the late 60’s was that these relations form an anti-Poisson algebra, which itself could be written in terms of a tensor product of two anti-Poisson algebras where one factor is \( \Gamma(\bigwedge T(N)) \) and the second a finite dimensional \((L, M)\)-system. The great advantage by interpreting local current algebras in terms of \((L, M)\)-systems lies in the fact that one gets a depuzzeling into an (easy to handle, while always the same) infinite dimensional part and a finite dimensional part characterizing the current algebra. Furthermore the \( M_1 \) part also takes care of the distribution as it plays the role of test functions.

To be more concrete let us write this tensor product explicitly:

\[
A_0 = M = M_1 \otimes M_2 = C^\infty(N) \otimes M_2, \quad (17)
\]

\[
A_1 = L = M_1 \otimes L_2 \oplus L_1 \otimes M_2 = C^\infty(N) \otimes L_2 \oplus \mathcal{X}(N) \otimes M_2, \quad (18)
\]

As the resulting \((L, M)\)-system is again an anti-Poisson algebra it is useful to write down a multiplication table for the terms of degree zero and one by using the defining relations (10 – 14) and (15,16):

\[
(f \otimes \alpha) \cdot (g \otimes \beta) = fg \otimes \alpha \beta \quad (19)
\]

\[
(f \otimes \alpha) \cdot (X \otimes \beta + g \otimes \xi) = fX \otimes \alpha \beta + fg \otimes \alpha \xi \quad (20)
\]

\[
(X \otimes \alpha + f \otimes \xi) \cdot (g \otimes \beta) = Xf \otimes \alpha \beta + fg \otimes \xi \alpha \quad (21)
\]

\[
[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes \alpha \beta \quad (22)
\]

\[
[f \otimes \xi, g \otimes \zeta] = fg \otimes [\xi, \zeta] \quad (23)
\]

\[
[X \otimes \alpha, f \otimes \xi] = Xf \otimes \alpha \xi - fX \otimes \xi \alpha \quad (24)
\]

where \( f, g \in M_1, \ X, Y \in L_1, \ \alpha, \beta \in M_2 \) and \( \zeta, \xi \in L_2 \).

### 3.2 The current algebra of the non-linear sigma model

For the current algebra of the two dimensional non-linear sigma model on Riemannian symmetric spaces \( K \) we identify the infinite dimensional \((L_1, M_1)\) system with \( L_1 = \mathcal{X}(S^1) \) and \( M_1 = C^\infty(S^1) \). The finite dimensional system \((L_2, M_2)\) is given by \( L_2 = g \times S^2(g) \). (In this semidirect product \( g \) acts on \( S^2(g) \) in the adjoint representation) and \( M_2 = g \). The multiplication in \( M_2 \) is trivial (to get a clear distinction between the two different roles played by \( g \) we denote \( M_2 = g \) by \( \tilde{g} \) The two linear maps (10-11) are given by:
\( L_2 \otimes M_2 \to M_2 \), \( (\xi, \alpha) \mapsto [\xi, \alpha] \), \( (\tau, \alpha) \mapsto 0 \).

The representation of \( L_2 \) on \( M_2 \) is the adjoint representation.

\[ L_2 \otimes M_2 \to L_2, \]
\[ (\xi, \alpha) \mapsto \alpha \lor \xi, \]
\[ (\tau, \alpha) \mapsto 0, \]

so \( M_2 \) just shifts in \( L_2 \). \( (\xi \in g \subset L_2, \tau \in S^2(g) \subset L_2, \alpha \in g = M_2) \).

We can now write the three generators of the current algebra (2 – 6) as symbols for certain classes of elements in the \((L, M)\)-system.

1. \( < j_0 : f \otimes \xi > \)
2. \( < j_1 : X \otimes \alpha > \)
3. \( < j : f \otimes \tau > \)

For the current algebra in this version we derive from (19 – 24) by using (25) and (26):

\[ [< j_0 : f \otimes \xi ^+ >, < j_0 : g \otimes \zeta >] = < j_0 : fg \otimes [\xi, \zeta] > \tag{27} \]
\[ [< j_0 : f \otimes \xi >, < j_1 : X \otimes \alpha >] = < j_1 : fX \otimes [\xi, \alpha] > - \]
\[ - < j : Xf \otimes (\alpha \lor \xi) > \tag{28} \]

\[ [< j_0 : f \otimes \xi >, < j : g \otimes \tau >] = < j : fg \otimes [\xi, \tau] > . \tag{29} \]

4 The Extension Structure

To further analyse this algebraic structure let us write the tensor product of the two \((L, M)\)-systems explicitly:

\[ M = M_1 \otimes M_2 = C^\infty(\Sigma) \otimes \hat{g} \]
\[ L = L_1 \otimes M_2 \oplus M_1 \otimes L_2 = \mathcal{X}(S^1) \otimes \hat{g} \otimes C^\infty(S^1) \otimes g \times S^2 g \]

Let us begin with a few observations:

- Because of (28), we notice that there is a ‘twist’ in the sum of (31), that is, the sum is a semidirect sum.
- Because of (27), \( M_1 \otimes g \) is a submodule.
• Furthermore we see that we have an extension of \( M_1 \otimes g \) by the abelian ideal \( L_1 \otimes M_2 \oplus M_1 \otimes S^2(g) \).

Actually this is not the whole story, as a result of (28) there is a two step extension by abelian ideals. Namely:

\[
\begin{align*}
M_1 \otimes S^2 g \\
\cap \\
L_1 \otimes M_2 \oplus M_1 \otimes S^2 g \\
\cap \\
L_1 \otimes M_2 \oplus M_1 \otimes L_2 .
\end{align*}
\]  

(32)

So we have the following structure:

1. The quotient of \( L \) by the first extension is \( M_1 \otimes g \).

2. The quotient of the first extension by the second is \( L_1 \otimes M_2 \). So \( L_1 \otimes M_2 \) itself is not an ideal but rather it is the cokernel of the second extension.

3. Relations (27 - 29) can be read as a representation of \( M_1 \otimes g \) on \( L \), so that in particular we have a representation on the extensions.

To make the above observations more concrete, we look at the short exact sequence determining the first step of the extension:

\[
0 \to V' \to V \to V'' \to 0 ,
\]

where

- \( V' = L_1 \otimes M_2 \oplus M_1 \otimes S^2 g \),
- \( V = L_1 \otimes M_2 \oplus M_1 \otimes L_2 \),
- \( V'' = M_1 \otimes g \).

The first step of the extension is charcterized by the corresponding long exact sequence of cohomologies of \( M_1 \otimes g = \Xi \) (as a Lie algebra) with values in the corresponding submodules:

\[
\to H^1(\Xi, V'') \to H^2(\Xi, V') \to H^2(\Xi, V) \to H^2(\Xi, V'') \to .
\]

Because of (27) the cohomology class of \( H^2(\Xi, V) \) is zero, so that we have a split extension, i.e. no cocycle appears.

The second step of the extension is less trivial. To analyse it, recall the observation that the relations (27 - 29) can be read as representaions of \( M_1 \otimes g \) on the subsequent representation spaces. So we will look at the short exact sequence of representations of \( M_1 \otimes g \):

\[
0 \to W' \to W \to W'' \to 0
\]

where
• $W' = M_1 \otimes S^2 g$
• $W = L_1 \otimes M_2 \oplus M_1 \otimes S^2 g$
• $W'' = L_1 \otimes M_2$

We want to calculate the extensions of $W''$ by $W'$ as representations of $\Xi$, which are described by the functor $\text{Ext}$. So we should analyse $\text{Ext}^1_\Xi(W'',W')$. If we look again at (28) we see that this tells us that we have a splitting (now a splitting of vector spaces) of the short exact sequence by $X \otimes \alpha$ and the cocycle should be given by “$Xf \otimes \alpha \vee \xi$”. Let us calculate this cocycle explicitly.

To derive the cocycle condition we look at the given splitting of the short exact sequence:

$$0 \to W' \to W \to W'' \to 0 \leftarrow$$

$\phi$ (split)

To be a splitting $\phi$ has to be an algebra homomorphism,

$$g\phi(W'') = \phi(gW'') + C_g(W'') .$$

This means that from

$$[g, g']\nu = g(g'\nu) - g'(g\nu)$$

we derive the cocycle condition by evaluating this for $\nu = \phi(W'')$ and $g, g' \in \Xi$,

$$[g, g'](W' + \phi(W'')) - g(g'(W' + \phi(W'')) + g'(gW' + \phi(W'')) = 0 . \quad (33)$$

The cocycle condition reads:

$$C_{[g, g']}(W'') + gC_{g'}(W'') - C_{g'}(gW'') + C_g(g'W'') - g'C_g(W'') = 0 . \quad (34)$$

This means for $g = f \otimes \xi$, $s = X \otimes \alpha \in W''$, and $C_g(s) = Xf \otimes \xi \vee \alpha$:

$$X (fg) \otimes [\xi, \eta] \vee \alpha - (fXg + gXf) \otimes [\xi, \eta] \vee \alpha = 0 .$$

Which is exactly the requirement that $L_1$ acts by derivation on $M_1$ (but preserving the tensor product structure). As this is a cocycle in the tensor product we can ask whether the two factors are cocycles seperately. Therefore we look at the two components of the map from

$$\Xi \times W'' \to W'$$

$$(M_1 \otimes g) \times (L_1 \otimes M_2) \to (M_1 \otimes S^2(g))$$

$$(X \otimes \xi, (X \otimes \alpha)) \to Xf \otimes (\xi \vee \alpha)$$

independently. We first look at $M_1 \otimes L_1 \to M_1$, given by $(f, X) \mapsto Xf$ We would like to know whether this defines a cocycle. So we analyse the map
\[ M_1 \mapsto M_1 + L_1 \]
given by
\[ \langle f, g + X \rangle = fg + fX + Xf. \]

Using (34) for this homomorphism one obtains:
\[ X(ff') - f'Xf - fXf' = 0. \]

This is the Leibniz rule for \( L_1 \), so \( Xf \) is indeed a cocycle.

For the second part \( g \otimes M_2 \rightarrow S^2(g) \), given by \( (\xi, \alpha) \mapsto \xi \vee \alpha \), we look at the map
\[ g \mapsto M_2 + S^2(g) \]
defined by
\[ \langle \xi, \sigma + \alpha \rangle = \text{ad}_\xi \sigma + \text{ad}_\xi \alpha + \xi \vee \alpha \]
using again (34) to obtain the cocycle condition we have
\[ \langle [\xi, \xi'], \sigma + \alpha \rangle - \langle \xi, \langle \xi', \sigma + \alpha \rangle \rangle + \langle \xi', \langle \xi, \sigma + \alpha \rangle \rangle = [\xi, \xi'] \neq 0. \]

So this is not a cocycle.

Let us summarize the result on the cocycle structure: the tensor product of the above maps defines a 1-cocycle, which comes from a 1-cocycle defined by the action of the vector fields on the ring of functions over a manifold. This cocycle is converted into the 1-cocycle of the tensor map with the help of a homomorphism defined by the second map.

**Remarks:**

1. This structure gives us information about the functor \( \text{Ext}^1_\Xi \), because we see now that the interesting contribution to \( \text{Ext} \) comes from one of the associated cup products:
   \[ \text{Ext}^1_1(L_1, M_1) \otimes \text{Hom}_g g \otimes M_2, S^2(g)) \rightarrow \text{Ext}^1_1(L_1 \otimes M_2, M_1 \otimes S^2(g)). \]
   This cup product describes exactly the tensor product of the cocycles defined by the first map with \( g \)-equivariant homomorphisms \( g \otimes M_2 \rightarrow S^2(g) \) which are 1-cocycles of the tensor product of maps.

2. The above construction is also possible for arbitrary representation spaces of \( g \subset L_2 \). By this we mean that it is possible to replace \( S^2g \) and \( M_2 \) by other representation spaces.

3. One would expect to get more information by using the fact that we are only interested in the \( M \)-invariant part of \( \text{Ext}^1 \), that is, constructing a double complex and looking at the long exact sequence:
   \[ 0 \rightarrow H^1(M, \text{Hom}_\Xi) \rightarrow \text{Ext}^1(M, \Xi) \rightarrow (\text{Ext}^1_\Xi)^{M_{inv}} \rightarrow H^2(M, \text{Hom}_\Xi) \rightarrow \ldots. \]
   But because of the trivial multiplication in \( M_2 \), the \( M \) invariance of the construction gives no further restriction on the appearing cocycle.
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References

[1] A. Belavin, A. Polyakov, A. Zamolodchikov, Infinite dimensional symmetry in two-dimensional quantum field theory, Nucl. Phys. B241 (1984) 333

[2] D. Bernard, Hidden Yangians in 2D Massive Current Algebras, Commun. Math. Phys. 137 (1991) 191

[3] M. Bordemann, M. Forger, J. Laartz, U. Schäper, The Lie-Poisson Structure of Integrable Classical Non-Linear Sigma Models, Freiburg preprint THEP 91/11

[4] H. Eichenherr, M. Forger, On the Dual Symmetry of the Non-Linear Sigma Models, Nucl. Phys. B155 (1979) 381

[5] M. Forger, J. Laartz, U. Schäper, Current Algebra of Classical Non-Linear Sigma Models, Freiburg preprint THEP 91/10, to appear in Commun. Math. Phys. 145 (1992)

[6] B. Kostant, S. Sternberg, Anti-Poisson Algebras and Current Algebras, Unpublished notes

[7] Y. Ne’eman, Commutateurs de Courants locaux et termes a gradient dans une algèbre de Lie, Conference donnée a Strasbourg CNRS du 3 juin 1971