EXTENSIONS OF RAMANUJAN’S RECIPROCITY THEOREM AND THE ANDREWS–ASKEY INTEGRAL

ZHI-GUO LIU

Dedicated to the memory of my parents

Abstract. Ramanujan’s reciprocity theorem may be considered as a three-variable extension of Jacobi’s triple product identity. Using the method of \( q \)-partial differential equations, we extend Ramanujan’s reciprocity theorem to a seven-variable reciprocity formula. Using the same method, the Andrews–Askey integral formula is extended to a \( q \)-integral formula which has seven parameters with base \( q \).

1. Introduction and preliminaries

Throughout the paper we assume, unless otherwise stated, that \( |q| < 1 \) and use the standard product notation

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad \text{and} \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).
\]

The \( q \)-binomial coefficients are the \( q \)-analogs of the binomial coefficients, which are defined by

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.
\]

If \( n \) is an integer or \( \infty \), the multiple \( q \)-shifted factorials are defined as

\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \ldots (a_m; q)_n.
\]

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The Jacobi triple product identity is stated in the following proposition (see, for example [11, p. 1] and [14, p. 15]).

**Proposition 1.1.** For \( x \neq 0 \), we have the triple product identity

\[
(q, x, q/x; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n.
\]

This identity is among the most important identities in mathematics, which has many interesting applications in number theory, combinatorics, analysis, algebra and mathematical physics. Some amazing extensions of this identity have been made by various authors. Ramanujan’s \( 1\psi_1 \) summation formula and the Bailey \( 6\psi_6 \) summation formula both contain this identity as a special case, and may be considered as two important extensions of this identity, and these two extensions have wider applications than Jacobi’s triple product identity.

Ramanujan’s reciprocity theorem and the Andrews–Askey integral formula may also be regarded as two extensions of Jacobi’s triple product identity. In this paper we will use the method of \( q \)-partial differential equations to extend Ramanujan’s reciprocity theorem and Andrews–Askey integral formula to two more general \( q \)-formulae.

For simplicity, in this paper we use \( \Delta(u, v) \) to denote the theta function

\[
(1.1) \quad v(q, u/v, qv/u; q)_{\infty}.
\]

As usual, the basic hypergeometric series or \( q \)-hypergeometric series \( \phi_s \) is defined by

\[
\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \vspace{.1cm} \\ b_1, \ldots, b_s \end{array} \mid q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} (q, b_1, \ldots, b_s; q) \left( \frac{(-1)^n q^{n(n-1)/2}}{1+q} \right)^{s-r} z^n.
\]

Now we introduce the definition of the Thomae–Jackson \( q \)-integral in \( q \)-calculus, which was introduced by Thomae [28] and Jackson [17].

**Definition 1.2.** Given a function \( f(x) \), the Thomae–Jackson \( q \)-integral of \( f(x) \) on \([a, b]\) is defined by

\[
\int_{a}^{b} f(x) dq_x = (1 - q) \sum_{n=0}^{\infty} [b f(bq^n) - a f(aq^n)] q^n.
\]

If the function \( f(x) \) is continuous on \([a, b]\), then, one can deduce that

\[
\lim_{q \to 1} \int_{a}^{b} f(x) dq_x = \int_{a}^{b} f(x) dx.
\]

In his lost notebook [26, p.40], Ramanujan stated the following beautiful reciprocity theorem without proof. This formula may be considered
as a three-variable extension of Jacobi’s triple product identity. This result, now known as Ramanujan’s reciprocity theorem, was first proved by Andrews in 1981 in his important paper [5]. For another proof, see Berndt et al. [9].

**Proposition 1.3.** If \( uv \neq 0 \) and \( av \neq q^{-m}, au \neq q^{-m}, m = 0, 1, 2, \ldots \), then, we have

\[
\frac{(q, v/u, u/v; q)_\infty}{(au, av; q)_\infty} = (1 - v/u) \sum_{n=0}^\infty (-1)^n \frac{q^n(v/u)^n}{(av; q)_{n+1}} + (1 - u/v) \sum_{n=0}^\infty (-1)^n \frac{q^n(u/v)^n}{(au; q)_{n+1}}.
\]

Andrews [5, Theorem 1] also derived a four-variable reciprocity theorem by using many summation and transformation formulae for basic hypergeometric series.

Inspired by the work of Andrews, in 2003, Liu [20, Theorem 6] proved the following five-variable reciprocity formula by using the \( q \)-exponential operator to Ramanujan’s \( _2 \psi_1 \) summation.

**Proposition 1.4.** For \( \max\{|au|, |av|, |bu|, |bv|, |cu|, |cv|\} < 1 \) and \( uv \neq 0 \), then, we have

\[
v \sum_{n=0}^\infty \frac{(q/bu, acuv; q)_{n+1}}{(av, cv; q)_{n+1}}(bv)^n - u \sum_{n=0}^\infty \frac{(q/bv, acuv; q)_{n+1}}{(au, cu; q)_{n+1}}(bu)^n = \Delta(u, v)(abuv, acuv, bcuv; q)_\infty.
\]

Ramanujan’s reciprocity formula is the special case \( b = c = 0 \) of Proposition 1.4.

Using a limiting case of Watson’s \( q \)-analog of Whipple’s theorem, Kang [18, Theorem 1.2] found the following equivalent form of Proposition 1.4.

**Proposition 1.5.** We have the reciprocity formula

\[
v \sum_{n=0}^\infty (1 - q^{2n+1}v/u) \frac{(q/au, q/bu, q/cu; q)_n}{(av, cv; q)_{n+1}} q^{n(n-1)/2}(-abcuv^2)^n
\]

\[
- u \sum_{n=0}^\infty (1 - q^{2n+1}u/v) \frac{(q/av, q/bv, q/cv; q)_n}{(au, cu; q)_{n+1}} q^{n(n-1)/2}(-abcuv^2)^n
\]

\[
= (v - u) \frac{(q, qv/u, qu/v, abuv, bcuv, acuv; q)_\infty}{(au, av, bu, bv, cu, cv; q)_\infty}.
\]
By splitting Bailey’s $6\psi_6$ summation formula into two unilateral basic hypergeometric series, Chu and Zhang [13, Theorem 5] find the following six-variable reciprocity formula, and a similar formula is also obtained by Ma [24, Theorem 1.4].

**Proposition 1.6.** We have the reciprocity formula

$$
\begin{align*}
&v \sum_{n=0}^{\infty} \left( 1 - q^{2n+1} v/u \right) \frac{(q/au, q/bu, q/cu, q/dw; q)_n}{(av, bv, cv, dv; )_{n+1}} (abcdv^2/q^n) \\
&- u \sum_{n=0}^{\infty} \left( 1 - q^{2n+1} u/v \right) \frac{(q/au, q/bu, q/cv, q/dv; q)_n}{(au, bu, cu, dv; )_{n+1}} (abcdv^2/q^n) \\
&= (v - u) \frac{(q, qv/u, qu/v, abvw, acuw, aduv, bcuv, bdvw, cduv; q)_{\infty}}{(au, av, bu, bv, cu, cv, du, dv, abcdv^2/q; q)_{\infty}}.
\end{align*}
$$

Employing Bailey’s $6\psi_6$ summation formula and a formula of Milne [25, Theorem 1.7], Wei et al. [31] derived some multi-variable generalizations of Ramanujan’s reciprocity formula. In particular, they get the following seven-variable reciprocity formula [31, Corollary 4].

**Proposition 1.7.** For $|cde/abq^n| < 1$, there holds the seven-variable generalization of Ramanujan’s reciprocity formula:

$$
\rho'(a, b; c, d, e, f, n) - \rho'(b, a; c, d, e, f, n) = \frac{1}{b} \frac{(q, qa/b, qb/a, c, d, e, cd/ab, ce/ab, de/ab; q)_{\infty}}{(-qa, -qb, -c/a, -e/b, -d/a, -d/b, -e/a, e/b, -c/b, dfe/abq; q)_{\infty}}
$$

$$
\times \frac{f^q q^n (qf/e, ef/ab, q)_{n}}{ab(-f/a, -f/b; q)_{n+1}} \psi_3 \left( q^{-n}, q/e, qab/cde; q, f, q^2ab/cde; q, q \right),
$$

where

$$
\rho'(a, b; c, d, e, f, n) = \frac{1}{b} \sum_{k=0}^{\infty} \left( 1 - \frac{aq^{2k+1}}{b} \right) \frac{(-1/b; q)_{k+1}}{(-qa; q)_k}
$$

$$
\times \frac{(-qa/c, -qa/d, -qa/e, -qa/f, -q^{1+n} f/b; q)_k}{(-c/b, -d/b, -e/b, -f/b, -a/f; q^{n+1}; q)_{k+1}} \left( \frac{cde}{abq^n} \right)^k.
$$

In this paper we will give a completely new extension of Ramanujan’s reciprocity formula. For simplicity, we now introduce the notation $\rho$ in the following definition.
Definition 1.8. We use the notation $\rho(a,b,c,d,r,u,v)$ to denote the double $q$-series

$$v \sum_{n=0}^{\infty} \frac{(q/du,acuv,bcuv;q)_n(dv)^n}{(av,bv,cv;q)_{n+1}} \times _3\phi_2\left(\frac{q^{n+1},uv^{n+1}/r,q/cu}{avq^{n+1},bvq^{n+1}};q,\frac{abcruv}{q}\right).$$

Our generalization of Ramanujan’s reciprocity formula is the following seven-variable reciprocity formula.

Theorem 1.9. If $\rho$ is defined as in Definition 1.8 with $uv \neq 0$ and

$$\max\{|au|,|av|,|bu|,|bv|,|cu|,|cv|,|du|,|dv|,|abr/d|,|abcruv/q|\} < 1,$$

then, we have the following seven-variable reciprocity formula:

$$\rho(a,b,c,d,r,u,v) - \rho(a,b,c,d,r,v,u) = \Delta(u,v)(acuv,aduv,bcuv,bduv,cduv,abr/d;q)_\infty \times _3\phi_2\left(\frac{du,dv,duv/r}{aduv,bduv};q,\frac{abr}{d}\right).$$

Setting $b = 0$ in Theorem 1.9, we immediately obtain Proposition 1.4.

Remark 1.10. Both Theorem 1.9 and the formula in Proposition 1.7 have seven parameters. However, in Proposition 1.7, $n$ is restricted to be a non-negative integer, and not a true free parameter as in Theorem 1.9. In addition, the $\phi_2$ series in Theorem 1.9 is a non-terminating series while the $\psi_3$ series in Proposition 1.7 is a terminating series. Finally, the left-hand side of the equation in Theorem 1.9 is a double series but in Proposition 1.7 the equation is a single series. Hence, Theorem 1.9 is different from Proposition 1.7.

In 1981, Andrews and Askey [6] established the following interesting $q$-beta integral formula using Ramanujan $1\psi_1$ summation, which has four parameters $a,b,u,v$ with base $q$. This $q$-integral formula, now is known as the Andrews–Askey integral.

Proposition 1.11. If $\max\{|au|,|bu|,|av|,|bv|\} < 1$ and $uv \neq 0$, then, we have

$$\int_u^v (qx/u,qx/v;q)_\infty d_qx = \frac{(1-q)\Delta(u,v)(abuv;q)_\infty}{(au,ab,av,bv;q)_\infty}.$$

Subsequently, in 1982, Al–Salam and Verma [2] found that Sears’ nonterminating extension of the $q$-Saalschütz summation can be rewritten in the following simple form, see also [14, page 52].
Proposition 1.12. If \( \max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|\} < 1 \) and \( uv \neq 0 \), then, we have
\[
\int_u^v \frac{(qx/u, qx/v, abcuvx; q)_\infty}{(ax, bx, cx; q)_\infty} \, d_qx = \frac{(1 - q)\Delta(u, v)(abuvw, acuvw, bcuvw; q)_\infty}{(au, bu, cu, av, bv, cv; q)_\infty}.
\]

We call this \( q \)-integral formula the Al–Salam–Verma integral formula. When \( c = 0 \), this \( q \)-integral formula reduces to the Andrews–Askey integral in Proposition 1.11.

The Andrews–Askey integral formula or the Al–Salam–Verma integral formula has been extended to the following \( q \)-integral formula [22, Proposition 13.8] with six parameters. Extensions of the Andrews–Askey integral involving the terminating \( q \)-series have been discussed by [29] and [12].

Proposition 1.13. If \( a, b, c, d, u, v, r \) are complex numbers such that \( \max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|, |abr/c|\} < 1 \) and \( uv \neq 0 \), then, we have the following \( q \)-integral formula:
\[
\int_u^v \frac{(qx/u, qx/v, abrx; q)_\infty}{(ax, bx, cx; q)_\infty} \, d_qx = \frac{(1 - q)\Delta(u, v)(acduvx, abrx; q)_\infty}{(au, av, bu, bv, cu, cv; q)_\infty} \times 3\phi_2\left(\frac{cu, cv, cuv/r}{acuv, bcuv; q, \frac{abr}{c}}\right).
\]

In this paper we will extend the Andrews–Askey integral formula or the Al–Salam–Verma integral formula to the following integral formula, which has seven parameters \( a, b, c, d, r, u, v \) with base \( q \).

Theorem 1.14. If \( a, b, c, d, u, v, r \) are complex numbers such that \( uv \neq 0 \) and \( \max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|, |abr/c|\} < 1 \), then, we have
\[
\int_u^v \frac{(qx/u, qx/v, acduvx, abrx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} \times 3\phi_2\left(\frac{ar, ax, cx}{acduvx, abrx; q, bdvu} \, d_qx = \frac{(1 - q)\Delta(u, v)(acduvx, abrx; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv; q)_\infty} \times 3\phi_2\left(\frac{cu, cv, cuv/r}{acuv, bcuv; q, \frac{abr}{c}}\right).
\]

When \( d = 0 \), the \( 3\phi_2 \) series in the integrand reduces to 1, and in the same time Theorem 1.14 becomes Proposition 1.13. So Theorem 1.14 is really an extension of Proposition 1.13.

The remainder of this paper is organized as follows. In Section 2, we will introduce some basic facts in \( q \)-differential calculus. Section 3
is devoted to the proof of Theorem 1.9, and Theorem 1.14 is proved in Section 4. Some applications of Theorems 1.9 and 1.14 are discussed in Section 5.

2. Some facts in \( q \)-differential calculus

In this section we introduce some basic concepts in \( q \)-differential calculus.

**Definition 2.1.** For any function \( f(x) \) of one variable, the \( q \)-derivative of \( f(x) \) with respect to \( x \), is defined as

\[
D_{q,x}\{f(x)\} = \frac{f(x) - f(qx)}{x},
\]

and we further define \( D_{q,x}^0\{f\} = f \) and \( D_{q,x}^n\{f\} = D_{q,x}\{D_{q,x}^{n-1}\{f\}\} \).

The \( q \)-derivative was first introduced by L. Schendel [27] in 1877 and then by F. H. Jackson [16] in 1908, which is a \( q \)-analog of the ordinary derivative. The definition of the \( q \)-partial derivative can be found in [22].

**Definition 2.2.** A \( q \)-partial derivative of a function of several variables is its \( q \)-derivative with respect to one of those variables, regarding other variables as constants. The \( q \)-partial derivative of a function \( f \) with respect to the variable \( x \) is denoted by \( \partial_{q,x}\{f\} \).

**Definition 2.3.** A \( q \)-partial differential equation is an equation that contains unknown multivariable functions and their \( q \)-partial derivatives.

The homogeneous Rogers–Szegő polynomials play an important role in the theory of orthogonal polynomials, which are defined by [21, 22]

\[
h_n(a, b|q) = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}.
\]

By multiplying two copies of the \( q \)-binomial theorem (see, for example [14, p. 8, Eq. (1.3.2)]), one can find that [21, 22]

\[
\sum_{n=0}^{\infty} h_n(a, b|q)\frac{t^n}{(q; q)_n} = \frac{1}{(at, bt; q)_\infty}, \quad |at| < 1, |bt| < 1.
\]

It turn out that the \( q \)-partial differential equations is an important subject of study, we started the study of this subject in [22] and [23]. The following very useful expansion theorem for \( q \)-series can be found in [22, Proposition 1.6].
Theorem 2.4. If \( f(x, y) \) is a two-variable analytic function at \((0, 0) \in \mathbb{C}^2\), then, \( f \) can be expanded in terms of \( h_n(x, y|q) \) if and only if \( f \) satisfies the \( q \)-partial differential equation \( \partial_{q,x}\{f\} = \partial_{q,y}\{f\} \).

One of the most important formulae for the Rogers–Szegő polynomials is the following \( q \)-Mehler formula, which can be derived easily from Theorem 2.4, see [22, pp.219–220] for details.

Proposition 2.5. For \( \max\{|as|, |at|, |bs|, |bt|\} < 1 \), we have
\[
\sum_{n=0}^{\infty} h_n(a, b|q)h_n(s, t|q) \frac{z^n}{(q; q)_n} = \frac{(abst^2; q)_\infty}{(as, at, bs, bt; q)_\infty}.
\]

Definition 2.6. For \( x = \cos \theta \), we define \( h(x; a) \) and \( h(x; a_1, a_2, \ldots, a_m) \) as follows:
\[
h(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty = \prod_{k=0}^{\infty} (1 - 2q^k ax + q^{2k}a^2),
\]
\[
h(x; a_1, a_2, \ldots, a_m) = h(x; a_1)h(x; a_2) \cdots h(x; a_m).
\]

The continuous \( q \)-Hermite polynomials \( H_n(\cos \theta|q) \) is defined as
\[
(2.3) \quad H_n(\cos \theta|q) = \sum_{k=0}^{n} \binom{n}{k} e^{i(n-2k)\theta}.
\]

Using the definition of the homogeneous Rogers–Szegő polynomials defined in (2.1), it is easily seen that
\[
(2.4) \quad H_n(\cos \theta|q) = h_n(e^{-i\theta}, e^{i\theta}|q).
\]

Putting \( a = e^{-i\theta} \) and \( b = e^{i\theta} \) in (2.2), one can find the following proposition.

Proposition 2.7. For \( |t| < 1 \), we have
\[
(2.5) \quad \sum_{n=0}^{\infty} H_n(\cos \theta|q) \frac{t^n}{(q; q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}.
\]

In order to prove Theorems 1.9 and 1.14, we need the following proposition.

Proposition 2.8. The function \( L(a, b, u, v, s, t) \) satisfies the \( q \)-partial differential equation \( \partial_{q,a}\{L\} = \partial_{q,b}\{L\} \), where \( L(a, b, u, v, s, t) \) is defined by
\[
\frac{(av, bv, abstu/v|q)_\infty}{(as, at, au, bs, bt, bu|q)_\infty} \phi_2 \left( \frac{v/s, v/t, v/u}{av, bv} ; q, \frac{abstu}{v} \right).
\]
Proof. It is easily seen that using \( L(a, b, u, v, s, t) \) we can rewrite the formula in Proposition 1.13 in the form
\[
L(a, b, u, v, s, t) = \frac{(v/s, v/t; q)_\infty}{(1 - q)\Delta(s, t)(au, bu; q)\infty} \int_s^t \frac{(qx/s, qx/t, abux; q)\infty}{(ax, bx, vx/st; q)\infty} d_qx.
\]
Noting the definition of the \( q \)-partial derivatives and using a direct computation, we easily find that
\[
\partial_{q,a}\{L\} = \partial_{q,b}\{L\} = \frac{(v/s, v/t; q)_\infty}{(1 - q)\Delta(s, t)} \int_s^t \frac{(x + u - aux - bux) (qx/s, qx/t, abuxq; q)\infty}{(au, bu, ax, bx, vx/st; q)\infty} d_qx,
\]
which indicates that Proposition 2.8 holds.

3. The proof of Theorems 1.9

3.1. Some inequalities for \( q \)-series.

Proposition 3.1. If \( k \) is a nonnegative integer or \( \infty \), \( a \) and \( b \) are two nonnegative numbers such that \( 0 \leq b \leq 1 \), then, we have
\[
(-ab; q)_k \leq (-a; q)\infty.
\]
If we further assume that \( 0 \leq a \leq 1 \), then, we have
\[
(ab; q)_k \geq (a; q)\infty.
\]

Proof. Keeping the fact \( 0 < q < 1 \) in mind, we find that for any \( 0 \leq j \leq k - 1 \),
\[
1 + abq^j \leq 1 + aq^j.
\]
Multiplying these inequalities together, we deduce that
\[
(-ab; q)_k \leq (-a; q)_k.
\]
Since \((-aq^k; q)\infty \geq 1\), we multiply \((-aq^k; q)\infty \) to the right-hand side of the above inequality to arrive at the first inequality in the proposition. In the same way we can prove the second inequality. This completes the proof of Proposition 3.1.

Proposition 3.2. If \( \max\{|b_1|, |b_2|, \ldots, |b_r|, |x|\} < 1 \) and \( n \) is a nonnegative integer, then, we have
\[
|_{r+1}\phi_r\left(\frac{a, a_1q^n, \ldots, a_rq^n}{b_1q^n, \ldots, b_rq^n}; q, x\right) | \leq \frac{(-|ax|, -|a_1|, \ldots, -|a_r|; q)\infty}{(|x|, |b_1|, \ldots, |b_r|; q)\infty}.
\]
Proof. keeping $0 < q < 1$ in mind, using the triangle inequality and proposition 3.1, we find that for $j \in \{1, 2, \ldots, r\}$,

$$|(a_jq^n; q)_k| \leq \prod_{l=0}^{k-1}(|1 + |a_j|q^l|) \leq \prod_{l=0}^{\infty}(|1 + |a_j|q^l|) = (-|a_j|; q)_\infty,$$

and

$$|(b_jq^n; q)_k| \geq \prod_{l=0}^{k-1}(|1 - |b_j|q^l|) \geq \prod_{l=0}^{\infty}(|1 - |b_j|q^l|) = (|b_j|; q)_\infty.$$

It follows that

$$\left|\frac{(a, a_1q^n, \ldots, a_rq^n; q)_k x^k}{(q, b_1q^n, \ldots, b_rq^n; q)_k}\right| \leq \frac{(-|a_1|, \ldots, -|a_r|)_\infty (-|a|; q)_k x^k}{(|b_1|, \ldots, |b_r|)_\infty (q; q)_k}.$$

Using this inequality and the triangle inequality, we conclude that

$$\left|\frac{r+1^{\phi_r} \left(a, a_1q^n, \ldots, a_rq^n; q, x\right)}{b_1q^n, \ldots, b_rq^n; q, x}\right| \leq \frac{(-|a_1|, \ldots, -|a_r|)_\infty (-|a|; q)_k x^k}{(|b_1|, \ldots, |b_r|)_\infty (q; q)_k}.$$

Applying the $q$-binomial theorem to the right-hand side of the above inequality, we complete the proof of Proposition 3.2.

It should be pointed out that Wang [30, Theorem 1.1] has obtained a similar inequality.

3.2. In order to prove Theorem 1.9, we need the following lemma.

**Lemma 3.3.** The series in the left-hand side of the equation in Theorem 1.9 represents a two-variable analytic function of $a$ and $b$, which is analytic at $(0, 0) \in \mathbb{C}^2$.

**Proof.** The proof can be divided into two cases according to $rcd \neq 0$ and $rcd = 0$. We only prove the $rcd \neq 0$ case and the $rcd = 0$ case can be proved similarly.

For the sake of simplicity, we will use the $C_n$ and $D_n$ to denote

$$C_n(a, b, c, d, u, v) := v^{q(du, acuv, bcuv; q)_n(du)^n} (av, bv, cv; q)_{n+1},$$

$$D_n(a, b, c, r, u, v) := \phi_2 \left(q^{n+1}, vq^{n+1}/r, q/cu \frac{abcvu}{avq^{n+1}, bvq^{n+1}} q\right).$$
Using these notations we can write the left-hand side of the equation in Theorem 1.9 as

\begin{equation}
\sum_{n=0}^{\infty} C_n(a, b, c, d, u, v) D_n(a, b, c, r, u, v)
- \sum_{n=0}^{\infty} C_n(a, b, c, d, v, u) D_n(a, b, c, r, v, u).
\end{equation}

Now we will show that this series converges to a two-variable analytic function of \(a\) and \(b\) at \((0, 0) \in \mathbb{C}^2\). It is obvious that the second summation in the above equation can be obtained from the first summation by interchanging \(u\) and \(v\), so we only need consider the first summation

\begin{equation}
\sum_{n=0}^{\infty} C_n(a, b, c, d, u, v) D_n(a, b, c, r, u, v).
\end{equation}

Without loss of generality, we may assume that \(\max\{|a|, |b|\} < 1\) and in order to simplify the discussion, we also assume temporary that \(r c d \neq 0\). Using Proposition 3.1 and some simple calculation, we conclude that

\[
|C_n(a, b, c, d, u, v)| \leq \frac{|v|(-|1/du|, -|v|, -|v/q|; q)_\infty|dv|^n}{(|v|, |v|, |cv/q|; q)_\infty}.
\]

Appealing to Proposition 3.2 and a direct computation, we deduce that

\[
|D_n(a, b, c, r, u, v)| \leq \frac{(-|abr v|, -q, -|v/r|; q)_\infty}{(|v|, |v|, |q/cu/q|; q)_\infty} \leq \frac{(-|rv|, -q, -|v/r|; q)_\infty}{(|v|, |v|, |q/cu/q|; q)_\infty}.
\]

Applying the triangular inequality and these two inequalities, we have

\[
\left| \sum_{n=0}^{\infty} C_n(a, b, c, d, u, v) D_n(a, b, c, r, u, v) \right|
\leq \sum_{n=0}^{\infty} |C_n(a, b, c, d, u, v)| D_n(a, b, c, r, u, v)
\leq |v|(-q, -|rv|, -|v/r|, -|1/du|; q)_\infty \sum_{n=0}^{\infty} |dv|^n
\leq \frac{|v|(-q, -|rv|, -|v/r|, -|1/du|; q)_\infty}{(1 - |dv|(|v| q)_\infty^4(|cv|, |q/cu/q|; q)_\infty)}.
\]

This shows that the series in (3.2) converges absolutely and uniformly for \(\max\{|a|, |b|\} < 1\). It is easily seen that every term of this series is
analytic at \((a, b) = (0, 0) \in \mathbb{C}^2\), thus this series converges to a two-variable analytic function of \(a\) and \(b\) which is analytic at \((0, 0) \in \mathbb{C}^2\).

\[\square\]

3.3. The proof of Theorems 1.9. Now we begin to prove Theorem 1.9 by using Theorem 2.4 Proposition 2.8 and Lemma 3.3.

**Proof.** Using the definition of \(L\) in Proposition 2.8, we deduce that

\[
3\phi_2 \left( \frac{du, dv, duv/\overline{r}}{aduv, bduv}; \frac{abr}{d} \right) = \frac{(ar, br, au, bu, av, bv; q)}{(aduv, bduv, abr/d; q)} L(a, b, u, dv, v, r),
\]

\[
3\phi_2 \left( \frac{q^{n+1}, vq^{n+1}/r, q/cu}{avq^{n+1}, bvq^{n+1}}; \frac{abr}{q} \right) = \frac{(av, bv, ar, br, acuvq^n, bcvq^n; q)}{(avq^{n+1}, bvq^{n+1}, abcruv/q; q)} L(a, b, r, vq^{n+1}, v, cuvq^n).\]

Using these two equations we can rewrite Theorem 1.9 in the form

\[
\sum_{n=0}^{\infty} \frac{(q/du; q)_n (dv)^n}{(cv; q)_{n+1}} L(a, b, r, vq^{n+1}, v, cuvq^n) - u \sum_{n=0}^{\infty} \frac{(q/dv; q)_n (du)^n}{(cv; q)_{n+1}} L(a, b, r, uq^{n+1}, u, cuvq^n) = \Delta(u, v) \frac{(cduv; q)_{\infty}}{(cu, cv, du, dv; q)_{\infty}} L(a, b, u, dv, v, r).
\]

(3.3)

If we use \(f(a, b)\) to denote the left-hand side of this equation, then, from Lemma 3.3 we know that \(f(a, b)\) is analytic at \((0, 0) \in \mathbb{C}^2\). Using Proposition 2.8, we easily see that \(f(a, b)\) satisfies the \(q\)-partial differential equation \(\partial_{q,a}\{f\} = \partial_{q,b}\{f\}\). Thus by Theorem 2.4, there exists a sequence \(\{\alpha_n\}\) independent of \(a\) and \(b\) such that

\[f(a, b) = \sum_{n=0}^{\infty} \alpha_n h_n(a, b).\]

Putting \(b = 0\) in this equation and using the fact \(h_n(a, 0) = a^n\), we obtain

\[f(a, 0) = \sum_{n=0}^{\infty} \alpha_n a^n.\]
Noting the definition of \( f(a,b) \) and using Theorem 1.4, we find that

\[
(\alpha, \beta; q) \infty f(a,0) = v \sum_{n=0}^{\infty} (q/du, \alpha; q)_n (du)^n - u \sum_{n=0}^{\infty} (q/dv, \beta; q)_n (dv)^n
= \Delta(u, v)(\alpha; q) \infty
\]

If we use \( g(a,b) \) to denote the right-hand side of (3.3), then, \( g(a,b) \) is analytic at \((0,0) \in \mathbb{C}^2\), and satisfies the \( q \)-partial differential equation \( \partial_{q,a}\{g\} = \partial_{q,b}\{g\} \). Thus, by Theorem 2.4, there exists a sequence \( \{\beta_n\} \) independent of \( a \) and \( b \) such that

\[
g(a,b) = \sum_{n=0}^{\infty} \beta_n (a,b|q).
\]

Putting \( b = 0 \) in this equation, using \( h_n(a,0|q) = a^n \), and noting the definition of \( g(a,b) \), we find that

\[
g(a,0) = \sum_{n=0}^{\infty} \beta_n a^n = \frac{\Delta(u,v)(\alpha; q) \infty}{(\alpha; q) \infty}.
\]

It follows that

\[
\sum_{n=0}^{\infty} \alpha_n a^n = \sum_{n=0}^{\infty} \beta_n a^n.
\]

Thus we have \( \alpha_n = \beta_n \), which implies that \( f(a,b) = g(a,b) \). Hence we have proved Theorem 1.9 for \(|a| \) and \(|b| \) sufficiently small. By using analytic continuation, this completes the proof of Theorem 1.9.

---

4. The proof of Theorem 1.14

Recall the Sears \( 3\phi_2 \) transformation formula (see, for example [20, Theorem 3])

\[
3\phi_2 \left( \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} ; q, \frac{b_1 b_2}{a_1 a_2 a_3} \right) = \left( \begin{array}{c} b_2/a_3, b_1 b_2/a_1 a_2 \\ b_2, b_1 b_2/a_1 a_2 a_3 \end{array} ; q \right) \infty 3\phi_2 \left( \begin{array}{c} b_1/a_1, b_1/a_2, a_3 \\ b_1, b_1 b_2/a_1 a_2 \end{array} ; q, \frac{b_2}{a_3} \right).
\]
Using the Sears $3\phi_2$ transformation formula, we easily conclude that
\[
3\phi_2\left(\frac{ax, ar, cx}{acduvx, abrx; q, bduv}\right) = \frac{(abr/c, bcduvx; q)_\infty}{(bduv, abrx; q)_\infty} 3\phi_2\left(\frac{cduv, cduvx/r, cx}{acduvx, bcduvx; q, abr/c}\right).
\]
This transformation formula shows that Theorem 1.14 is equivalent to the following formula:

\[
\int_u^v \frac{(qx/u, qx/v, acduvx, bcduvx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} \times 3\phi_2\left(\frac{cduv, cduvx/r, cx}{acduvx, bcduvx; q, abr/c}\right) d_qx
= \frac{(1 - q)\Delta(u, v)(acuw, aduv, bcuv, bduv, cduv; q)_\infty}{(au, av, bu, bv, cu, cv, du, dv; q)_\infty} \times 3\phi_2\left(\frac{cu, cv, cuv/r}{acuw, bcuv; q, abr/c}\right).
\]

In order to prove the identity (4.1), we need to prove the following lemma.

**Lemma 4.1.** The $q$-integral in (4.1) is a two-variable analytic function of $a$ and $b$, which is analytic at $(0, 0) \in \mathbb{C}^2$.

**Proof.** For the sake of convenience, we define the compact notation $A_n$ and $B_n$ by

\[
A_n(a, b, c, d, r, u, v) := 3\phi_2\left(\frac{cduv, cdvu^2q^n/r, cuq^n}{acdvu^2q^n, bcdvu^2q^n; q, abr/c}\right),
\]
\[
B_n(a, b, c, d, r, u, v) := u(1 - q)\left(\frac{q^{n+1}, q^{n+1}u/v, acdvu^2q^n, bcdvu^2q^n; q)_\infty}{(auq^n, buq^n, cuq^n, duq^n; q)_\infty}\right).
\]

Using the definition of $q$-integral, we find that the left-hand side of the equation in (4.1) can be written as

\[
\sum_{n=0}^{\infty} A_n(a, b, c, d, r, v, u)B_n(a, b, c, d, r, v, u)q^n
- \sum_{n=0}^{\infty} A_n(a, b, c, d, r, u, v)B_n(a, b, c, d, r, u, v)q^n.
\]

Next we will prove that this series converges to a two-variable analytic function of $a$ and $d$ at $(0, 0) \in \mathbb{C}^2$. It is obvious that the first
summation can be obtained from the second one by interchanging \(u\) and \(v\). Thus we only need consider the second summation, namely, 

\[
\sum_{n=0}^{\infty} A_n(a, b, c, d, r, u, v)B_n(a, b, c, d, r, u, v)q^n.
\]

We divide our proof into two cases according \(cr \neq 0\) and \(cr = 0\). We only prove the \(cr \neq 0\) case and the \(cr = 0\) case can be proved in the same way. Without loss of generality, we can assume that 

\[
\max\{|a|, |b|, |d|\} < 1 \text{ and } 0 < |c|, |r|, |u|, |v| < 1.
\]

Using Propositions 3.1 and 3.2 and doing some simple calculations, we find that 

\[
|A_n(a, b, c, d, r, u, v)| \leq \frac{(-|abduvr|, -|cdvu^2/r|, -|cu|; q)_{\infty}}{(|abr/c|, |acdvu^2|, |bcdvu^2|; q)_{\infty}} \leq \frac{(-1, -|r|, -|1/r|; q)_{\infty}}{(|r/c|, |u|, |u|; q)_{\infty}}.
\]

Making use of Proposition 3.1 and some elementary calculations, we deduce that 

\[
|B_n(a, b, c, d, r, u, v)| \leq \frac{(-1; q)_{\infty}^3 (-|qu/v|; q)_{\infty}}{(|u|; q)_{\infty}^4}.
\]

Using the triangular inequality and the above two inequalities, we conclude that 

\[
\left| \sum_{n=0}^{\infty} A_n(a, b, c, d, r, u, v)B_n(a, b, c, d, r, u, v)q^n \right| \\
\leq \sum_{n=0}^{\infty} |A_n(a, b, c, d, r, u, v)||B_n(a, b, c, d, r, u, v)|q^n \\
\leq \frac{(-1; q)_{\infty}^4 (-|qu/v|, -|r|, -|1/r|; q)_{\infty}}{(1 - q)(|u|; q)_{\infty}^4(|r/c|; q)_{\infty}}.
\]

This indicates that the series in (4.3) converges absolutely and uniformly for \(|a| < 1\). It is easily to see that every term of this series is a analytic at \(a = 0\). Thus this series converges to an analytic function of \(a\), which is analytic at \(a = 0\). By symmetry, this function is also a analytic at \(b = 0\). Hence the series in (4.2) converges to a two-variable analytic function of \(a\) and \(b\) at \((a, b) = (0, 0) \in \mathbb{C}^2\).

Interchanging \(u\) and \(v\) in (4.3) we immediately find that the first series in (4.2) is also analytic at \((a, b) = (0, 0) \in \mathbb{C}^2\). In summary, the left-hand side of the equation in (4.1) is a two-variable analytic function of \(a\) and \(b\), which is analytic at \((a, b) = (0, 0) \in \mathbb{C}^2\). \(\Box\)
Now we begin to prove 4.1 by using Lemma 4.1, Proposition 2.8 and Theorem 2.4.

Proof. Using the definition of $L$ in Proposition 2.8 and some simple computations, we find that

$$L(a, b, duv, cdvu, x, r) = \frac{(acduv, bdcuv, abr/c; q)_\infty}{(ax, bx, ar, br, aduv, bdvu; q)_\infty} \times 3\phi_2\left(\frac{cduv, cdvu/r, cx}{acduv, bcdvu}; q, \frac{abr}{c}\right).$$

$$L(a, b, r, cuv, u, v) = \frac{(acuv, bcuv, abr/c; q)_\infty}{(au, av, bu, bv, ar, br; q)_\infty} \times 3\phi_2\left(\frac{cu, cv, cuv/r}{acuv, bcuv}; q, \frac{abr}{c}\right).$$

Using these two equations we can rewrite (4.1) in the form

$$\int_u^v \frac{(qx/u, qx/v; q)_\infty}{(cx, dx; q)_\infty} L(a, b, duv, cdvu, x, r) d_qx$$

$$= \frac{(1 - q)\Delta(u, v)(cduv; q)_\infty}{(cu, cv, du, dv; q)_\infty} L(a, b, r, cuv, u, v).$$

If we use $f(a, b)$ to denote the left-hand side of (4.4), then, $f(a, b)$ is analytic at (0, 0) $\in \mathbb{C}^2$, and satisfies the $q$-partial differential equation $\partial_{q,a}\{f\} = \partial_{q,b}\{f\}$. Thus, by Theorem 2.4, there exists a sequence $\{\alpha_n\}$ independent of $a$ and $b$ such that

$$f(a, b) = \sum_{n=0}^\infty \alpha_n h_n(a, b|q).$$

Putting $b = 0$ in this equation, using $h_n(a, 0|q) = a^n$, and noting the definition of $f(a, b)$, we find that

$$f(a, 0) = \frac{1}{(ar, aduv; q)_\infty} \int_u^v \frac{(qx/u, qx/v, acduv; q)_\infty}{(ax, cx, dx; q)_\infty} d_qx = \sum_{n=0}^\infty \alpha_n a^n.$$

Applying the Al–Salam–Verma integral to the $q$-integral in this equation, we deduce that

$$\sum_{n=0}^\infty \alpha_n a^n = \frac{(1 - q)\Delta(u, v)(acuv, cdvu; q)_\infty}{(au, av, ar, cu, dv, du; q)_\infty}.$$
\[ \partial_{q,a}\{g\} = \partial_{q,b}\{g\}. \]

Thus, by Theorem 2.4, there exists a sequence \( \{\beta_n\} \) independent of \( a \) and \( b \) such that

\[ g(a, b) = \sum_{n=0}^{\infty} \beta_n h_n(a, b|q). \]

Putting \( b = 0 \) in this equation, using \( h_n(a, 0|q) = a^n \), and noting the definition of \( g(a, b) \), we find that

\[ g(a, 0) = \sum_{n=0}^{\infty} \beta_n a^n = \frac{(1 - q) \Delta(u, v)(acuv, cduv, q)_{\infty}}{(au, av, ar, cu, cv, du, dv, q)_{\infty}}. \]

Comparing this equation with (4.5), we find that \( \alpha_n = \beta_n \), which implies that \( f(a, b) = g(a, b) \). This shows that the identity in (4.1) holds. Thus we have proved Theorem 1.14 for \( |a| \) and \( |b| \) sufficiently small. By using analytic continuation, we complete the proof of Theorem 1.14.

5. Further Discussions and Applications

5.1. A six-variable reciprocity formula.

Proposition 5.1. If \( \rho \) is defined as in Definition 1.8 with \( uv \neq 0 \) and

\[ \max\{|au|, |av|, |bu|, |bv|, |cu|, |cv|, |du|, |dv|, |abcdu^2v^2/q|\} < 1, \]

then, we have the following six-variable reciprocity formula:

\[ \rho(a, b, c, d, duv, u, v) = \rho(a, b, c, d, duv, v, u) \]

\[ = \frac{\Delta(u, v)(abuv, acuv, aduv, bcuv, bduv, cduv, q)_{\infty}}{(au, av, bu, bv, cu, cv, du, dv, \frac{abcdu^2v^2}{q}; q)_{\infty}}. \]

Proof. Upon taking \( r = duv \) in Theorem 1.9 and upon noting that \( (1; q)_k = \delta_{0,k} \), the \( 3\phi_2 \) series in the theorem reduces to 1, and thus we complete the proof of Proposition 5.1.

5.2. Some limiting cases of Theorem 1.4. Next we will discuss some limiting cases of Theorem 1.4.
Proposition 5.2. For $a \neq q^{-m}, c \neq q^{-m}, d \neq q^{-m}, m = 0, 1, 2, \ldots$, we have
\[
\sum_{n=0}^{\infty} \frac{(q/d, ac; q)_n d^n}{(a, c; q)_{n+1}} 
\times \left(n + 1 + \sum_{k=0}^{n} \frac{aq^k}{1 - aq^k} + \sum_{k=0}^{n} \frac{cq^k}{1 - cq^k} - \sum_{k=1}^{n} \frac{q^k}{d - q^k}\right) 
= \frac{(q; q)_{\infty}^3(ac, ad, cd; q)_{\infty}}{(a, c, d; q)_{\infty}^2}.
\]

Proof. Keeping in mind that $\Delta(u, v) = (v - u)(q, qu/v, qv/u; q)_{\infty}$, dividing both sides of the equation in Theorem 1.4 by $v - u$, then letting $v \to u$ in the resulting equation, using L'Hôpital's rule and simplifying, we deduce that
\[
\sum_{n=0}^{\infty} \frac{(q/du, acu^2; q)_n (du)^n}{(au, cu; q)_{n+1}} 
\times \left(n + 1 + \sum_{k=0}^{n} \frac{auq^k}{1 - auq^k} + \sum_{k=0}^{n} \frac{cuq^k}{1 - cuq^k} - \sum_{k=1}^{n} \frac{q^k}{du - q^k}\right) 
= \frac{(q; q)_{\infty}^3(acu^2, adu^2, cdu^2; q)_{\infty}}{(au, cu, du; q)_{\infty}^2}.
\]

Replacing $(au, bu, cu)$ by $(a, b, c)$, we complete the proof of proposition 5.2.

Setting $a = c = 0$ in proposition 5.2, we immediately obtain the following proposition.

Proposition 5.3. For $d \neq q^{-m}, m = 0, 1, 2, \ldots$, we have
\[
\sum_{n=0}^{\infty} (q/d; q)_n d^n \left(n + 1 - \sum_{k=1}^{n} \frac{q^k}{d - q^k}\right) = \frac{(q; q)_{\infty}^3}{(d; q)_{\infty}^2}.
\]

Putting $d = 0$ in Proposition 5.3, we immediately obtain the following identity of Jacobi [10, p. 14]:
\[
(q; q)_{\infty}^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1)q^{n(n+1)/2}.
\]

Letting $d \to q$ Proposition 5.3, we obtain the Euler identity (see, for example [7, p. 280])
\[
1 - \sum_{n=1}^{\infty} (q; q)_{n-1} q^n = (q; q)_{\infty}.
\]
Taking \( d = -q \) in Proposition 5.3, we are led to the identity

\[
1 + \sum_{n=1}^{\infty} (-q)^n (-q; q)_{n-1} \left( 2n + 3 + 2 \sum_{k=1}^{n-1} \frac{q^k}{1+q^k} \right) = \frac{(q; q)_{\infty}^3}{(-q; q)_{\infty}^2}.
\]

**Proposition 5.4.** For \( a \neq q^m, m = \pm 1, \pm 2, \ldots \), we have the Lambert series formula

\[
\frac{(q; q)_\infty^4}{(qa, q/a; q)_\infty^2} = 1 + (1-a)^2 \sum_{n=1}^{\infty} \frac{n(q/a)^n}{1-aq^n} + (1-1/a)^2 \sum_{n=1}^{\infty} \frac{n(qa)^n}{1-q^n/a}.
\]

**Proof.** Setting \( c = 0 \) and \( ad = q \) in proposition 5.2, we conclude that

\[
(q; q)_\infty^4 = \sum_{n=0}^{\infty} \frac{(q/a)^n}{1-aq^n} \left( n + \frac{1}{1-aq^n} \right) = \frac{1}{(1-a)^2} + \sum_{n=1}^{\infty} \frac{n(q/a)^n}{1-aq^n} + \sum_{n=1}^{\infty} \frac{(q/a)^n}{1-aq^n}.
\]

By a direct calculation, we can find the following elementary identity:

\[
\sum_{n=1}^{\infty} \frac{(q/a)^n}{(1-aq^n)^2} = a^{-2} \sum_{n=1}^{\infty} \frac{n(qa)^n}{1-q^n/a}.
\]

Substituting this equation into (5.1) and then multiplying both sides of the resulting equation by \( (1-a)^2 \), we complete the proof of Proposition 5.4.

Taking \( a = -1 \) Proposition 5.4, we immediately conclude that

\[
\frac{(q; q)_\infty^4}{(-q; q)_\infty^4} = 1 + 8 \sum_{n=1}^{\infty} \frac{n(-q)^n}{1+q^n}.
\]

Replacing \( q \) by \(-q \) in this equation, we can obtain Jacobi’s four-square identity (see, for example [10, p. 61])

\[
\left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 = 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 32 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}}.
\]
Writing \( q \) by \( q^2 \) and then setting \( a = q \) in the first identity in (5.1), we deduce that

\[
\frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} = \sum_{n=0}^{\infty} \frac{nq^n}{1 - q^{2n+1}} + \sum_{n=0}^{\infty} \frac{q^n}{(1 - q^{2n+1})^2} = \sum_{n=0}^{\infty} \frac{nq^n}{1 - q^{2n+1}} + \sum_{n=0}^{\infty} \frac{(n + 1)q^n}{1 - q^{2n+1}} = \sum_{n=0}^{\infty} \frac{(2n + 1)q^n}{1 - q^{2n+1}},
\]

which is equivalent to the Legendre four triangular numbers identity (see, for example [10, p. 72])

\[
\left( \sum_{n=0}^{\infty} q^{(n+1)/2} \right)^4 = \sum_{n=0}^{\infty} \frac{(2n + 1)q^n}{1 - q^{2n+1}}.
\]

Setting \( c = 0 \) and then taking \( abuv = q \) in Proposition 1.4, we can obtain the following very interesting Lambert series identity.

**Proposition 5.5.** If \( au \neq q^{-m} \) and \( av \neq q^{-m} \), \( m=0, 1, 2, \ldots \), then, we have the Lambert series identity

\[
v \sum_{n=0}^{\infty} \frac{(q/au)^n}{1 - avq^n} - u \sum_{n=0}^{\infty} \frac{(q/av)^n}{1 - auq^n} = \frac{v(q, q, u/v, qv/u; q)_{\infty}}{(au, av, q/au, q/av; q)_{\infty}}.
\]

**5.3. A special of Theorem 1.14.**

**Proposition 5.6.** If \( a, b, c, d, u, v, r \) are complex numbers such that \( \max\{|au|, |bu|, |cu|, |av|, |bv|, |cv|\} < 1 \) and \( uv \neq 0 \), then, we have

\[
\int_u^v \frac{(qx/u, qx/v, abcvx, acdwx; q)_{\infty}}{(ax, bx, cx, dx; q)_{\infty}} \times _3\phi_2 \left( \begin{array}{c} acuv, ax, cx \\ abcvx, acdwx \end{array}; q, bduv \right) dqx = \frac{(1 - q)\Delta(u, v)(abuv, acuv, aduv, bcuv, cduv; q)_{\infty}}{(au, av, bu, bv, cu, cv, du, dv; q)_{\infty}}.
\]

**Proof.** Setting \( r = cuv \) in Theorem 1.14, the \( _3\phi_2 \) series on the right-hand side of the equation in Theorem 1.14 reduces to 1. This completes the proof of Proposition 5.6. \( \square \)
5.4. A new proof of the Askey–Wilson integral. Next we will use Proposition 5.6 to give a simple proof of the Askey–Wilson integral formula [8].

**Theorem 5.7.** If \( \max\{|a|, |b|, |c|, |d|\} < 1 \), then, we have

\[
\int_0^\pi \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} d\theta = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}. 
\]

**Proof.** Noting the definition of \( \Delta(u, v) \) in (1.1) and using a simple computation, we easily find that

\[
(e^{i\theta} - e^{-i\theta}) \Delta(e^{i\theta}, e^{-i\theta}) = (q; q)_\infty h(\cos 2\theta; 1).
\]

Keeping this in mind and replacing \((u, v)\) by \((e^{i\theta}, e^{-i\theta})\) in Proposition 5.6, we deduce that

\[
(e^{i\theta} - e^{-i\theta}) \int_{e^{i\theta}}^{e^{-i\theta}} (qx e^{i\theta}, qxe^{-i\theta}; q)_\infty I(x) d_q x = (1 - q)(q, ab, ac, ad, bc, cd; q)_\infty h(\cos 2\theta; 1),
\]

where

\[
I(x) = \frac{(abcx, acdx; q)_\infty}{(ax, bx, cx, dx; q)_\infty} \phi_2 \left( \frac{ac, ax, cx}{acdx, abcx} q, bd \right).
\]

It is easily seen that \( I(x) \) is analytic near \( x = 0 \). Thus, there exists a sequence \( \{a_k\}_{k=0}^\infty \) independent of \( x \) such that

\[
I(x) = a_0 + \sum_{k=1}^\infty a_k x^k.
\]

By setting \( x = 0 \) in the above equation and using the \( q \)-binomial theorem, we conclude that

\[
a_0 = \sum_{n=0}^\infty \frac{(ac; q)_n (bd)^n}{(q; q)_n} = \frac{(abcd; q)_\infty}{(bd; q)_\infty}.
\]

Using the definition of the \( q \)-integral, we find that the left-hand side of (5.2) equals

\[
(1 - q)(1 - e^{-2i\theta}) \sum_{n=0}^\infty (q^{n+1}; q)_\infty (q^{n+1} e^{-2i\theta}; q)_\infty I(q^n e^{-i\theta}) q^n
\]

\[+ (1 - q)(1 - e^{2i\theta}) \sum_{n=0}^\infty (q^{n+1}; q)_\infty (q^{n+1} e^{2i\theta}; q)_\infty I(q^n e^{i\theta}) q^n.\]

Inspecting the first series in the equation, we see that this series can be expanded in terms of the negative powers of \( \{e^{-k\theta}\}_{k=0}^\infty \), and the
constant term of the Fourier expansion of this series is \((1 - q)a_0\). Thus, there exists a sequence \(\{\alpha_k\}_{k=1}^{\infty}\) independent of \(\theta\) such that the first series equals
\[
(1 - q)a_0 + \sum_{k=1}^{\infty} \alpha_k e^{-ik\theta}.
\]
Replacing \(\theta\) by \(-\theta\), we immediately find that the second series is equal to
\[
(1 - q)a_0 + \sum_{k=1}^{\infty} \alpha_k e^{ik\theta}.
\]
Combining the above two expressions together, we arrive at
\[
(e^{i\theta} - e^{-i\theta}) \int_{e^{i\theta}}^{e^{-i\theta}} (qxe^{i\theta}, qxe^{-i\theta}; q)_{\infty} I(x) dx
\]
\[
= 2(1 - q)a_0 + 2 \sum_{k=1}^{\infty} \alpha_k \cos k\theta.
\]
Comparing this equation with (5.2), we are led to the Fourier series expansion
\[
2(1 - q)a_0 + 2 \sum_{k=1}^{\infty} \alpha_k \cos k\theta
\]
\[
= \frac{(1 - q)(q, ab, ac, ad, bc, cd; q)_{\infty} h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)}.
\]
On integrating the above equation over \([-\pi, \pi]\) and using the fact
\[
\int_{-\pi}^{\pi} (\cos k\theta) d\theta = 2\pi \delta_{k,0},
\]
and noting that the integrand is an even function of \(\theta\), we deduce that
\[
\int_{0}^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b, c, d)} d\theta
\]
\[
= \frac{2\pi a_0}{(q, ab, ac, ad, bc, cd; q)_{\infty}}.
\]
Substituting the value of \(a_0\) in (5.4) into this equation, we complete the proof of Theorem 5.7.

\[\square\]

Remark 5.8. One of the most important properties of the \(q\)-Hermite polynomials is that they satisfy the following orthogonality relation [1, 4]:
\[
\int_{0}^{\pi} H_m(\cos \theta|q) H_n(\cos \theta|q) h(\cos 2\theta; 1) d\theta = 2\pi (q; q)_{\infty} \delta_{m,n}/(q; q)_{\infty}.
\]
This orthogonality relation has been used by several authors to evaluate the Askey–Wilson integral and other related $q$-beta integrals (see, for example [3, 15, 19]).

Conversely, we will use a special case of the Askey–Wilson integral formula to give a new proof of the the orthogonality relation for the $q$-Hermite polynomials, since we have just evaluated the Askey–Wilson integral without using the orthogonality relation for the $q$-Hermite polynomials. The proof is as follows:

Putting $c = d = 0$ in the Askey–Wilson integral formula, we immediately deduce that

\[ \int_0^{\pi} \frac{h(\cos 2\theta; 1)}{h(\cos \theta; a, b)} d\theta = \frac{2\pi}{(q, ab; q)_\infty}. \]  

(5.5)

On multiplying two copies of the identity in (2.5), we find that for $\max\{|a|, |b|\} < 1$,

\[ \sum_{m,n=0}^\infty H_m(\cos \theta|q) H_n(\cos \theta|q) \frac{a^m b^n}{(q; q)_m(q; q)_n} = \frac{1}{h(\cos \theta; a, b)}. \]

Using Proposition 2.7, we can easily show that the above double series converges uniformly for $\max\{|a|, |b|\} < 1$ on $0 \leq \theta \leq \pi$.

Substituting this series into the left-hand side of (5.5) and then integrating term by term, and applying the $q$-binomial theorem to the right-hand side of (5.5), we find that

\[ \sum_{m,n=0}^\infty (ab)^n \int_0^{\pi} H_m(\cos \theta|q) H_n(\cos \theta|q) h(\cos 2\theta; 1) d\theta \]

\[ = \frac{2\pi}{(q; q)_\infty} \sum_{n=0}^\infty (ab)^n. \]

Equating the coefficients of $a^m b^n$ on both sides of this equation, we arrive at the orthogonality relation for the $q$-Hermite polynomials.

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Department of Mathematics and Shanghai Key Laboratory of PMMP,
East China Normal University, 500 Dongchuan Road, Shanghai 200241,
P. R. China

E-mail address: zgliu@math.ecnu.edu.cn; liuzg@hotmail.com