A HIGHER BOLTZMANN DISTRIBUTION

MICHAEL J. CATANZARO, VLADIMIR Y. CHERNYAK, AND JOHN R. KLEIN

Abstract. We characterize the classical Boltzmann distribution as the unique solution to a certain combinatorial Hodge theory problem in homological degree zero on a finite graph. By substituting for the graph a CW complex $X$ and a choice of degree $d \leq \dim X$, we define by direct analogy a higher dimensional Boltzmann distribution $\rho^B$ as a certain real-valued cellular $(d-1)$-cycle. We then give an explicit formula for $\rho^B$.

We explain how these ideas relate to the Higher Kirchhoff Network Theorem of [CCK]. We also give an improved version of the Higher Matrix-Tree Theorems of [CCK].

Contents

1. Introduction 1
2. Spanning co-trees 12
3. The proof of Theorem A 15
4. Tree-co-tree duality 18
5. The proof of Theorem C 20
6. Appendix: Kirchhoff-Boltzmann unification 23
References 24

1. Introduction

1.1. Motivation. Physics and chemistry are rife with processes in which some quantity varies with time in a complicated, irregular way. The evolution of such a system is often modeled by a so-called master equation, which takes the form $\dot{p} = Hp$. Here, $p$ is a time dependent distribution on the states of the system and $H$, the master operator,
governs time evolution. The stationary distribution of a master equation is known as the Boltzmann distribution (or Gibbs measure).

For example, the classical Boltzmann distribution from thermodynamics governs how the system's states are populated, depending on their energy. Specifically, the probability of the system to be in state \( j \) with energy \( E_j \) is proportional to \( e^{-\beta E_j} \), where \( \beta = \frac{1}{k_B T} \); \( T \) is the temperature and \( k_B \) is the Boltzmann constant. The Boltzmann distribution is then the normalized distribution

\[
\rho^B = \frac{\sum_j e^{-\beta E_j} j}{\sum_j e^{-\beta E_j}}.
\]

A Markov process with discrete state space is conveniently described by a state diagram. The latter is a directed graph \( \Gamma \) whose vertices label the states and whose edges label the transition probabilities between states. Each directed edge \( e \) of \( \Gamma \) is equipped with a transition rate \( k_e \).

A certain class of processes is obtained by the following procedure: start with a connected undirected graph \( X \) (i.e., a CW complex of dimension one) with vertex set \( X_0 \) and edge set \( X_1 \). Then \( \Gamma \) will be the double of \( X \), i.e., the directed graph having the same vertices, where each edge of \( X \) is now replaced by a pair of opposing directed edges. Each directed edge of \( \Gamma \) is given by a pair \( (i, \alpha) \), where \( i \) is a vertex of \( X \) and \( \alpha \) is an edge of \( X \) that is incident to \( i \). The transition rates of the process are given as follows: choose a real number \( \beta > 0 \) (inverse temperature) and functions \( E: X_0 \to \mathbb{R} \) and \( W: X_1 \to \mathbb{R} \). Then the transition rate across \( (i, \alpha) \) is given by

\[
k_{i,\alpha} := e^{\beta(E_i - W_\alpha)}.
\]

When the transition rates are written in this way, the process is said be in Arrhenius form.

The process just described is not completely general since \( \Gamma \) is a double. In fact, the above is an example of a process that is in detailed balance (or time reversible) in the sense that there exists a distribution \( \pi: \Gamma_0 \to \mathbb{R}_+ \) such that

\[
\pi(i) k_{i,\alpha} = \pi(j) k_{j,\alpha}.
\]

for any edge \( \alpha \) of \( X \) with joining a pair of vertices \( i \) and \( j \) (see e.g., [Ke, ch. 1]; in our example, take \( \pi(i) = e^{-\beta E_i} \)). The number \( \pi(i) k_{i,\alpha} \) is the probability flux across \( (i, \alpha) \). Equation (1) says that the pair of states \( i \) and \( j \) are in equilibrium along \( \alpha \). It is not difficult to show that every process (with discrete time and finite state space) in detailed balance can be written in Arrhenius form.
If $X$ is finite, the Boltzmann distribution arises as the normalized vector

$$\frac{1}{Z} \sum_{i \in X_0} \pi(i) i, \quad Z := \sum_{i \in X_0} \pi(i).$$

Consequently, the Boltzmann distribution is a normalized equilibrium distribution for the process.

The higher Boltzmann distribution developed in this paper is associated with a Markov process on a CW complex which is a natural generalization of the process we described above on a graph. With the intention of providing motivation, the process is described below. However, we will not investigate its properties here as it is not within the current scope.

Let $X$ be a finite connected CW complex, let $d \leq \dim X$ be a fixed positive integer, and let $X_k$ denote the set of $k$-cells. With respect to a mild technical assumption, we will show how associate a Markov process. A state of the process will be an integer-valued cellular $(d-1)$-cycle $\zeta$ on $X$ in a fixed homology class. Roughly, a directed edge from $\zeta$ to another state $\zeta'$ is given by an “elementary homology” in the sense that

$$\zeta' = \zeta + u \partial e,$$

where $u$ is an integer and $e$ is a $d$-cell that is incident to $\zeta$ at a specified $(d-1)$-cell $f$. We also require that $e$ not be incident to $\zeta'$ at $f$.

When $d = \dim X = 1$, it will turn out that the process coincides with the one constructed above. Before giving the details, we explain the technical assumption on $X$, which amounts a homological condition that will eventually be removed as it is not required for the results of this paper:

**Definition 1.1.** The CW complex $X$ is $d$-pseudo-regular if for every $d$-cell $e \in X_d$ and any $(d-1)$-cell $f \in X_{d-1}$, we have

$$\langle \partial e, f \rangle \in \{0, \pm 1\},$$

where the above denotes the coefficient of $f$ in the boundary of $e$. If $X$ is regular then it is $d$-pseudo-regular for all $d$. In particular any connected polytopal complex [Z, chap. 5] or any connected finite simplicial complex is $d$-pseudo-regular. If $d = \dim X = 1$, then $X$ is automatically $d$-pseudo-regular.

Assuming $X$ is $d$-pseudo-regular, we can now describe the process. Consider the oriented graph in which a vertex is given by an integer $(d-1)$-cycle $\zeta \in Z_{d-1}(X; \mathbb{Z})$. A directed edge from a vertex $\zeta$ to a vertex $\zeta'$ is given by a pair

$$(f, e)$$
in which
- \( e \in X_d \) is a \( d \)-cell and \( f \in X_{d-1} \) is a \((d - 1)\)-cell;
- \( \langle \zeta, f \rangle \neq 0 \neq \langle \partial e, f \rangle \);
- \( \langle \zeta', f \rangle = 0 \).

Note that the above conditions imply
\[
\zeta' = \zeta - \langle \zeta, f \rangle \langle \partial e, f \rangle \partial e,
\]
so \( \zeta' \) is obtained from \( \zeta \) by means of an elementary homology. We denote the directed edge by \((f, e, \zeta)\).

We now fix a base vertex
\[
\hat{x} \in Z_{d-1}(X; \mathbb{Z}),
\]
such that the homology class \([\hat{x}] \in H_{d-1}(X; \mathbb{Z})\) is non-trivial. Consider the subgraph generated by \(\hat{x}\) in the sense that a vertex \(\zeta\) lies in this subgraph if there exists a finite sequence of directed edges \(\hat{x} = \zeta_0 \rightarrow \zeta_1 \rightarrow \cdots \rightarrow \zeta_k = \zeta\), i.e., a finite directed path from \(\hat{x}\) to \(\zeta\). An edge belongs to this subgraph if it occurs in such a path. We denote this subgraph by
\[
\Gamma_{X,\hat{x}}.
\]

**Definition 1.2.** The *cycle incidence graph* of \((X, \hat{x})\) is the directed graph \(\Gamma_{X,\hat{x}}\).

**Example 1.3.** Suppose \(X\) has dimension one and \(\hat{x} := i\) is any vertex of \(X\). In this situation \(\Gamma_{X,\hat{x}}\) coincides with the double of \(X\).

**Remark 1.4.** If \(\dim X > 1\) then the cycle-incidence graph is usually not a double, so the process won’t be detailed balance. Furthermore, \(\Gamma_{X,\hat{x}}\) typically has infinitely many vertices.

**Example 1.5.** Let \(X\) be the two-dimensional torus with regular cell structure indicated by the following picture:

```
|   |   |
|---|---|
| e1| e2|
|---|---|
| e3| e4|
```

Then \(X\) is obtained from the picture by identifying the opposite sides of the outer square. Consequently, there are 4 two-cells \((e_1, \ldots, e_4)\), 8 one-cells and 4 zero-cells. Our initial one-cycle \(\zeta_0 = \hat{x}\) is the meridian...
given by the red segment. In this case the cycle incidence graph has infinitely many vertices. To see this start at $\hat{x}$ and jump across $e_1$. Modulo a choice of orientations for the cells, the jump results in the cycle given by $\zeta_1 = \hat{x} + \partial e_1$, and the latter is incident to the $e_2$. Jumping across $e_2$ results in the cycle given by $\zeta_2 = \hat{x} + \partial e_1 - \partial e_2$ which is incident to $e_1$. Moreover $\zeta_2$ is distinct from $\hat{x}$. Iterating this procedure by jumping first across $e_1$ and then across $e_2$, results in an infinite number of distinct vertices $\zeta_0, \zeta_1, \zeta_2, \ldots$ (with $\zeta_{2k+1} = \hat{x} + (2k+1)\partial e_1 - (2k)\partial e_2$ and $\zeta_{2k} = \hat{x} + (2k)(\partial e_1 - \partial e_2)$) in the cycle incidence graph of $(X, \hat{x})$.

To complete the description of the process, we must label the edges of $\Gamma_{X, \hat{x}}$ with transition rates. This is done as follows: choose functions $E: X_{d-1} \to \mathbb{R}$ and $W: X_d \to \mathbb{R}$ and an inverse temperature $\beta > 0$. Then the transition rate of the directed edge $(f, e, \zeta)$ is given by

$$k_{f, e, \zeta} : = e^{\beta(W_f - E_e)}.$$  \hspace{1cm} (2)

The description of the process is now complete.

Summarizing, as the initial $(d-1)$-cycle evolves with respect to the process, it jumps from cycle to cycle by adding/subtracting the boundaries of $d$-cells, analogous to a particle ‘jumping’ over an edge on a graph. That is, the evolving cycle does not change its homology class, and so each homology class will have its own distinct dynamics.

One can interpret the higher Boltzmann distribution defined in this paper as the long-time limit of the average distribution within each homology class. By “average,” we mean the average over the stochastic process in the sense of probability theory. This will be pinned-down in a subsequent paper.

1.2. The combinatorial Hodge problem. We now turn to the problem of formulating the higher Boltzmann distribution in terms of combinatorial Hodge theory. Let $X$ be a finite CW complex and let $d \leq \dim X$ be as above. Henceforth, we do not require $X$ to be $d$-pseudo-regular. Let $C_\ast(X; \mathbb{R})$ be the cellular chain complex of $X$ with real coefficients. Each $C_j(X; \mathbb{R})$ is equipped with a standard inner product $\langle -, - \rangle$ which is determined by declaring $X_j$ to be an orthonormal basis.

Definition 1.6. A scalar structure on $X$ consists of functions

$$E_j : X_j \to \mathbb{R}, \quad j = 0, 1, \ldots$$

When $j = d$ we typically write $E := E_{d-1}$ and $W := E_d$. 
Let $\beta > 0$ be given. Then the function $E_j$ equips $C_j(X; \mathbb{R})$ with a modified inner product
\[
\langle x, y \rangle_{E_j} := e^{\beta E_j(x)} \langle x, y \rangle_{x, y \in X_j}.
\]
In our next formulation, we only make use of the function $E = E_{d-1}$. Define the formal adjoint
\[
\partial^*_E : C_{d-1}(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R})
\]
using the standard inner product on $C_d(X; \mathbb{R})$ and the modified inner product on $C_{d-1}(X; \mathbb{R})$, i.e., $\partial^*_E = \partial^* e^{\beta E}$, where $\partial^*$ is the formal adjoint in the standard inner product structures.

**Definition 1.7.** Let $X$ and $E$ be as above. The combinatorial Hodge problem in degree $d-1$ is the following: given $x \in H_{d-1}(X; \mathbb{R})$, find an explicit formula for the unique cycle $\rho \in Z_{d-1}(X; \mathbb{R})$ such that

- $\rho$ represents $x$, and
- $\rho$ is co-closed, i.e., $\partial^*_E \rho = 0$.

The condition that $\rho$ be co-closed can be re-stated as the assertion that $\rho$ should be orthogonal to any boundary with respect to the modified inner product:
\[
\langle \alpha, \partial^*_E \rho \rangle = \langle \partial \alpha, \rho \rangle_{E} = 0,
\]
for any $\alpha \in X_d$.

**Remark 1.8.** The combinatorial Hodge problem for $(X, E)$ is equivalent to finding an orthogonal splitting of the quotient homomorphism $\rho$ appearing in the short exact sequence
\[
0 \rightarrow B_{d-1}(X; \mathbb{R}) \rightarrow Z_{d-1}(X; \mathbb{R}) \xrightarrow{\partial^*_E} H_{d-1}(X; \mathbb{R}) \rightarrow 0,
\]
with respect to the modified inner product on $Z_{d-1}(X; \mathbb{R}) \subset C_{d-1}(X; \mathbb{R})$.

The original Hodge problem asks to find a unique harmonic representative for any cohomology class on a compact, orientable Riemannian manifold. By relaxing the hypotheses to a connected CW complex, we are able to write down an explicit formula. Our solution to the combinatorial Hodge problem will involve a summation over spanning cotrees. The latter are certain subcomplexes of $X$ of dimension $d-1$ which are a higher dimensional analog of the vertices of a graph. They are homologically dual to the higher dimensional spanning trees of $[CCK]$ and hence their name. In the following definition, $\beta_k(X) = \dim H_k(X; \mathbb{Q})$ denotes the $k$th Betti number of $X$, and $X^{(k)}$ denotes the $k$-skeleton of $X$. 
Definition 1.9. A \((d - 1)\)-dimensional spanning co-tree for \(X\) is a subcomplex \(L \subset X\) such that

(a) The inclusion \(i_L: L \subset X\) induces an isomorphism
\[
i_{L*}: H_{d-1}(L; \mathbb{Q}) \xrightarrow{\cong} H_{d-1}(X; \mathbb{Q})\
\]

(b) \(\beta_{d-2}(L) = \beta_{d-2}(X)\);

(c) \(X^{(d-2)} \subset L \subset X^{(d-1)}\).

Clearly, the number of spanning co-trees is finite.

Remark 1.10. Equivalently, conditions (a)-(c) are equivalent to conditions \((a'),(b)\) and \((c)\), where

\((a')\) The relative homology group \(H_{d-1}(X, L; \mathbb{Q})\) is trivial.

Remark 1.11. We are indebted to a referee for pointing out that our notion of spanning co-tree is equivalent to that of a relatively acyclic complex in [DKM2] as well as to the complement of a cobase in [L, p. 156].

Spanning co-trees come packaged with auxiliary data that will be used for obtaining the desired splitting. Observe that the projection \(Z_{d-1}(L; \mathbb{Z}) \to H_{d-1}(L; \mathbb{Z})\) is an isomorphism since \(L\) has no \(d\)-cells. Let \(\phi_L\) be the composite
\[
(3) \quad \phi_L: Z_{d-1}(L; \mathbb{Z}) \xrightarrow{\cong} H_{d-1}(L; \mathbb{Z}) \xrightarrow{i_L*} H_{d-1}(X; \mathbb{Z})
\]

Then \(\phi_L\) becomes an isomorphism after tensoring with the rational numbers by the defining properties of \(L\). We invert \(\phi_L\) rationally to obtain a homomorphism of rational vector spaces
\[
(4) \quad \psi_L: H_{d-1}(X; \mathbb{Q}) \xrightarrow{\phi_L \otimes \mathbb{Q}^{-1}} Z_{d-1}(L; \mathbb{Q}) \xrightarrow{i_L*} Z_{d-1}(X; \mathbb{Q})
\]

Since \(H_{d-1}(X, L; \mathbb{Q})\) is trivial, the group \(H_{d-1}(X, L; \mathbb{Z})\) is finite. Let
\[
(5) \quad a_L := |H_{d-1}(X, L; \mathbb{Z})|
\]

denote its order.

We define the weight of \(L\) to be the real number
\[
(6) \quad \tau_L = \tau_L(E) := a_L^2 \prod_{b \in L_{d-1}} e^{-\beta E_b}
\]

With respect to the above definitions, we can state the solution to the combinatorial Hodge problem:
\textbf{Theorem A} (Boltzmann Splitting Formula). Let \((X, E)\) and \(x \in H_{d-1}(X; \mathbb{R})\) be as above. Then the solution to the combinatorial Hodge problem is given by \(\rho = \Psi(x)\), in which \(\Psi: H_{d-1}(X; \mathbb{R}) \to Z_{d-1}(X; \mathbb{R})\) is the homomorphism

\[
\nabla \sum_{L} \tau_{L} \psi_{L},
\]

where the sum runs over all \((d-1)\)-dimensional spanning co-trees \(L\) and \(\nabla = \sum_{L} \tau_{L}\).

Theorem A enables us to define the higher Boltzmann distribution:

\textbf{Definition 1.12}. Let \((X, E)\) be as above and let \(x \in H_{d-1}(X; \mathbb{Z})\) be an integer homology class. The \textit{higher Boltzmann distribution} at \(x\) is the real \((d-1)\)-cycle

\[
\rho^{B} := \nabla \sum_{L} \tau_{L} \psi_{L}(\bar{x}) \in Z_{d-1}(X; \mathbb{R}),
\]

where \(\bar{x} \in H_{d-1}(X; \mathbb{Q})\) is the image of \(x\) with respect to the homomorphism \(H_{d-1}(X; \mathbb{Z}) \to H_{d-1}(X; \mathbb{Q})\).

\textit{Remark 1.13}. The classical Boltzmann distribution is a special case of Definition 1.12: let \(X\) be finite connected graph and take \(d = 1\). Then the spanning co-trees of \(X\) are given by the vertices. We take \(x \in H_{0}(X; \mathbb{Z}) \cong \mathbb{Z}\) to be the canonical generator (given by choosing a vertex of \(X\); the generator is independent of this choice). For a vertex \(L = j\) the normalized weight is given by

\[
\nabla^{-1} \tau_{L} = Z^{-1} e^{-\beta E_{j}}, \quad Z = \sum_{i \in X_{0}} e^{-\beta E_{i}},
\]

since \(\phi_{L}\) is an integral isomorphism. Then \(\psi_{L}(\bar{x}) = j\) and the assertion follows.

\textit{Remark 1.14}. When \(E = 0\), the coefficients \(\tau_{L}\) are rational numbers and the map \(\Psi\) is defined as a homomorphism of rational vector spaces \(H_{d-1}(X; \mathbb{Q}) \to Z_{d-1}(X; \mathbb{Q})\). If we further assume that \(x\) is a rational homology class, then the solution to the combinatorial Hodge problem gives an explicit expression for the Harmonic “forms” with respect to the combinatorial Laplacian \(- (\partial \partial^{*} + \partial^{*} \partial)\): \(C_{d-1}(X; \mathbb{Q}) \to C_{d-1}(X; \mathbb{Q})\).

\textit{Remark 1.15}. The proof we give of Theorem A is an application of the theory of generalized inverses to the quotient map \(p: Z_{d-1}(X; \mathbb{R}) \to H_{d-1}(X; \mathbb{R})\) (cf. [M], [P], [BG]). In the late 1980s a summation formula was given for the Moore-Penrose pseudo-inverse [Be], [BT]; this is the formula we make use of. When writing the current paper, we also came
to realize that by applying the summation formula to split the inclusion map

\[ Z_d(X; \mathbb{R}) \xrightarrow{\subset} C_d(X; \mathbb{R}), \]

one gets another proof of our higher dimensional analog of Kirchhoff’s theorem on electrical networks [CCK] (see Remark 3.6 below).

Remark 1.16. The main application of Theorem A will appear in the first author’s Ph. D. thesis [C].

Remark 1.17. It is tempting to speculate whether a result like Theorem A holds in the case of a Riemannian manifold. For this, one needs notion of spanning co-tree adapted to the space of differential forms. Then the sum appearing in the Theorem A would presumably be replaced by a convergent infinite series of operators indexed over the set of spanning co-trees. This would give an explicit solution to the main result of classical Hodge theory.

1.3. Asymptotic behavior. Given \((X, E)\) and \(d\) as above, with \(E\) suitably generic, it turns out that sum of Theorem A is asymptotic as a function of \(\beta\) to the term with highest weight. To explain this, let \(E^\ast := E^*_{d-1}(X)\) denote the set of \((d-1)\)-dimensional spanning co-trees of \(X\). Let

\[ E_{F^*} : F^* \rightarrow \mathbb{R} \]

be the functional given by

\[ L \mapsto \sum_{b \in L_{d-1}} E_b. \]

A scalar function \(E : X_{d-1} \rightarrow \mathbb{R}\) is said to be non-degenerate if it is one-to-one. This is clearly a generic condition.

As spanning co-trees are a matroid basis [L, p. 156], a greedy algorithm shows that if \(E\) is non-degenerate then the function \(E_{F^*}\) possesses a unique minimum \(L^\mu\).\(^1\)

As the parameter \(\beta\) tends to \(\infty\), it is easy to check that the operator \(\Psi\) appearing in Theorem A is asymptotic to \(\psi_{L^\mu}\) when we consider these as vector-valued functions with components indexed over \(F^*\) (see the proof of [CKS, lem. 3.6]). More precisely, we have

**Corollary B.** Let \((X, E)\) and \(x \in H_{d-1}(X; \mathbb{Z})\) be as above. Assume in addition that \(E : X_{d-1} \rightarrow \mathbb{R}\) is non-degenerate. Then we have

\[ \lim_{\beta \to \infty} \frac{\tau_L}{\nabla} = \begin{cases} 0 & L \neq L^\mu; \\ 1 & L = L^\mu. \end{cases} \]

\(^1\)See e.g., [O, §1.8]. A previous draft of the paper had stronger assumptions on \(E\). We are grateful to a referee for pointing out to us that non-degeneracy suffices.
In particular, \( \rho^B(x) \) is asymptotic in \( \beta \) to the rational \((d - 1)\)-cycle \( \psi_{L^0}(x) \). Consequently, in the low temperature limit\(^2\) the higher Boltzmann distribution rationally quantizes.

1.4. **Improved higher matrix-tree theorems.** Let \( X \) be a finite connected CW complex equipped with scalar structure \( E_* \). Fix \( d \leq \dim X \). As above, we set \( E = E_{d-1} \) and \( W = E_d \).

**Definition 1.18.** Let \( \partial^*_{E,W} : C_{d-1}(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R}) \) be the linear transformation given by

\[
e^{-\beta W} \partial^* e^{\beta E},
\]

where

- \( \partial^* : C_{d-1}(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R}) \) is the formal adjoint to the boundary operator with respect to the standard inner product;
- \( e^{-\beta W} : C_d(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R}) \) is given by \( b \mapsto e^{-\beta W} b \) for \( b \in X_d \);
- \( e^{\beta E} : C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R}) \) is given by \( f \mapsto e^{\beta E} f \) for \( f \in X_{d-1} \).

The (restricted) biased Laplacian is the operator

\[
\mathcal{L}^{E,W} \ := \ \partial \partial^*_{E,W} : B_{d-1}(X; \mathbb{R}) \rightarrow B_{d-1}(X; \mathbb{R}).
\]

**Remark 1.19.** The operator \( \partial^*_{E,W} \) is just the formal adjoint to the boundary operator \( \partial \) with respect to the modified inner products \( \langle -, - \rangle_E \) and \( \langle -, - \rangle_W \).

The case \( E = 0 \) was considered in [CCK] (in this instance we simplify notation and write \( \mathcal{L}^W \) for \( \mathcal{L}^{E,W} \); similarly, if \( W = 0 \) we write \( \mathcal{L}^E \)).

Recall from [CCK, defn. 1.2] that a subcomplex \( T \subset X \) is a spanning tree (in dimension \( d \leq \dim X \)) if

- \( H_d(T) = 0 \),
- \( \beta_{d-1}(T) = \beta_{d-1}(X) \), and
- \( X^{(d-1)} \subset T \subset X^{(d)} \).

**Remark 1.20.** The above definition of spanning tree is a slight generalization of the notion given in [CCK] in that we do not assume \( d = \dim X \). The definition given here is a matter of convenience only and doesn’t change any results of that paper since the spanning trees in dimension \( d \) of \( X \) are precisely the spanning trees of \( X^{(d)} \). When \( \dim X = d = 1 \), the definition is equivalent to the usual notion of spanning tree of a graph.

For a finite complex \( Y \) and a choice of degree \( d \leq \dim Y \), let

\[
\theta_Y \in \mathbb{N},
\]

\(^2\)The limit \( \beta \rightarrow \infty \) is called the low temperature limit (cf. [CCK]).
be the order of the torsion subgroup of $H_{d-1}(Y;\mathbb{Z})$. Define the weight of a spanning tree $T$ to be
\begin{equation}
    w_T = w_T(W) := \theta_T^2 \prod_{b \in T} e^{-\beta W_b}.
\end{equation}

**Theorem C** (Improved Higher Weighted Matrix-Tree Theorem). For a finite connected CW complex $X$ and a degree $d \leq \dim X$, we have
\begin{equation}
    \det L^{E,W} = \frac{1}{\theta_X^2} (\sum_L \tau_L)(\sum_T w_T),
\end{equation}
where the first sum is indexed over all $(d-1)$-dimensional spanning co-trees, the second sum is indexed over all $d$-dimensional spanning trees, with $w_T$ is as in (8) and $\tau_L$ as in (6).

**Remark 1.21.** Theorem C effectively places spanning co-trees on the same footing as spanning trees. The special case $E = 0$ recovers [CCK, thm. C], where the number $\mu_X$ appearing there is identified here with $\sum_L \tau_L$.

It is worth singling out the special case $E = 0 = W$:

**Corollary D** (Improved Higher Matrix-Tree Theorem).
\begin{equation}
    \det L = \frac{1}{\theta_X^2} (\sum_L a_L^2)(\sum_T \theta_T^2),
\end{equation}
where $L = \partial\partial^*: B_{d-1}(X;\mathbb{R}) \to B_{d-1}(X,\mathbb{R})$, $a_L$ is as in (5) and $\theta_T$ is as in (7).

**Remark 1.22.** If the Betti numbers $\beta_j(X)$ are trivial for $j = d-1, d-2$, then Corollary D reduces to the main result of [DKM1]. The reduction follows directly from the identity
\[ a_L = \frac{\theta_{d-1}(X)\theta_{d-2}(L)}{\theta_{d-2}(X)}, \]
where $\theta_j(Y)$ denotes the order of the torsion subgroup of $H_j(Y;\mathbb{Z})$ (note: $\theta_{d-1}(Y)$ is $\theta_Y$ appearing in (7)). The identity is an easy consequence of the short exact sequence of finite abelian groups
\[ 0 \to H_{d-1}(X) \to H_{d-1}(X,L) \to H_{d-2}(L) \to H_{d-2}(X) \to 0, \]
where homology is taken with integer coefficients. Here, we have used the fact that $H_{d-1}(L) = 0$, since $\beta_{d-1}(L) = \beta_{d-1}(X) = 0$ and $\dim L \leq d-1$.

**Outline.** In §2 we develop the elementary properties of spanning co-trees. §3 contains the proof of the Boltzmann Splitting Formula (Theorem A). In §4 we introduce “tree-co-tree” duality which describes
a bijection between the spanning trees in a chain complex with the spanning co-trees in the dual cochain complex. In §5 we prove the Improved Higher Weighted Matrix-Tree Theorem (Theorem C). In the appendix (§6) we prove a summation formula for the pseudo-inverse to the boundary operator which unifies both the Kirchhoff projection and Boltzmann splitting formulas.

Acknowledgements. We are indebted to a referee for valuable suggestions leading to a vastly improved version of the paper. Much of our writing was done while the last author was visiting the University of Copenhagen. He is indebted to Lars Hesselholt for providing him with support from the Niels Bohr Professorship.

This material is based upon work supported by the National Science Foundation Grant CHE-1111350 and the Simons Foundation Collaboration Grant 317496.

2. Spanning co-trees

If $F$ is a field, recall that a $k$-chain $c \in C_k(X; F)$ is any $F$-linear combination of $k$-cells. If $b \in X_k$ is a $k$-cell, we write $\langle c, b \rangle$ for the coefficient of $b$ appearing in $c$. If $\langle c, b \rangle \neq 0$, we say that $b$ appears in $c$.

**Definition 2.1.** A $k$-cell $b \in X_k$ is said to be **essential** if there exists a $k$-cycle $z \in Z_k(X; \mathbb{Q})$ such that $\langle z, b \rangle \neq 0$.

**Lemma 2.2 ([CCK, lem. 2.2]).** Adding or removing an essential $d$-cell from $X$ increases or decreases $\beta_d(X)$ by one, respectively, and fixes $\beta_{d-1}(X)$.

**Lemma 2.3.** $X$ has a spanning co-tree.

**Proof.** The homomorphism $H_{d-1}(X^{(d-1)}; \mathbb{Q}) \to H_{d-1}(X; \mathbb{Q})$ is surjective with kernel $K_1 := B_{d-1}(X; \mathbb{Q})$. Set $Y^1 := X^{(d-1)}$. Suppose that $c \in B_{d-1}(X; \mathbb{Q})$ is nontrivial. Let $b$ be a $(d-1)$-cell of $X$ that appears in $c$. Let $Y^2$ be the result of removing $b$ from $X^{(d-1)}$. The homomorphism $H_{d-1}(Y^2; \mathbb{Q}) \to H_{d-1}(X; \mathbb{Q})$ is surjective; let $K^2$ be its kernel. Then the rank of $K^2$ is strictly less than that of $K^1$ by Lemma 2.2. Furthermore, $\beta_{d-2}(Y^2) = \beta_{d-2}(X)$. By iterating (with $Y^2$ replacing $Y^1$, etc.) we eventually obtain a subcomplex $Y^k \subset X^{(d-1)}$ such that $H_{d-1}(Y^k; \mathbb{Q}) \to H_{d-1}(X; \mathbb{Q})$ is an isomorphism. Then $Y^k$ is a spanning co-tree. \qed

**Proposition 2.4.** Let $F$ be a field of characteristic zero. Let $L \subset X$ be a $(d-1)$-dimensional subcomplex that contains $X^{(d-2)}$. Then $L$ is a spanning co-tree if and only if the composition

$$C_{d-1}(L; F) \to C_{d-1}(X; F) \to C_{d-1}(X; F)/B_{d-1}(X; F)$$
is an isomorphism.

Proof. Clearly, it suffices to prove the assertion when \( \mathbb{F} = \mathbb{Q} \). Suppose \( L \) is such that (9) is an isomorphism. Consider the following commutative diagram:

\[
\begin{array}{ccc}
Z_{d-1}(L; \mathbb{Q}) & \xrightarrow{i_L} & Z_{d-1}(X; \mathbb{Q}) & \xrightarrow{p} & H_{d-1}(X; \mathbb{Q}) \\
\downarrow{k} & & \downarrow{=} & & \downarrow{j} \\
C_{d-1}(L; \mathbb{Q}) & \xrightarrow{a} & C_{d-1}(X; \mathbb{Q}) & \xrightarrow{\pi} & C_{d-1}(X; \mathbb{Q})/B_{d-1}(X; \mathbb{Q}).
\end{array}
\]

The left square is a pullback and the right square is a pushout. By assumption, the bottom composite is an isomorphism, so the top composite is also an isomorphism. Therefore, \( i_{L*} : H_{d-1}(L; \mathbb{Q}) \rightarrow H_{d-1}(X; \mathbb{Q}) \) is an isomorphism. The remaining two conditions of Definition 1.9 are easily verified. Consequently, \( L \) is a spanning co-tree.

For the converse, let \( x \in C_{d-1}(L; \mathbb{Q}) \) be such that \( (\pi \circ a)(x) = 0 \). Then \( a(x) \in B_{d-1}(X; \mathbb{Q}) \subset Z_{d-1}(X; \mathbb{Q}) \). Since the left square is a pullback, we infer that \( x \in Z_{d-1}(L; \mathbb{Q}) \). But \( p \circ i_L \) is an isomorphism, and \( j \) is injective, so \( x = 0 \). This establishes the injectivity of (9).

For surjectivity, let \( z \in C_{d-1}(X; \mathbb{Q})/B_{d-1}(X; \mathbb{Q}) \). Lift this to any element \( y \in C_{d-1}(X; \mathbb{Q}) \). Then \( \partial(y) \in C_{d-2}(L; \mathbb{Q}) = C_{d-2}(X; \mathbb{Q}) \) lies in \( Z_{d-2}(L; \mathbb{Q}) \) since \( \partial^2 = 0 \). Furthermore, the pushforward of the homology class \( [\partial(y)] \in H_{d-2}(L; \mathbb{Q}) \) in \( H_{d-2}(X; \mathbb{Q}) \) is trivial, since \( H_{d-2}(L; \mathbb{Q}) \cong H_{d-2}(X; \mathbb{Q}) \). It follows that \( \partial(y) \) lies in \( B_{d-2}(X; \mathbb{Q}) = B_{d-2}(L; \mathbb{Q}) \). Hence, \( \partial(y) = \partial(x) \) for some \( x \in C_{d-1}(L; \mathbb{Q}) \). Then \( a(x) - y \) lies in \( Z_{d-1}(X; \mathbb{Q}) \), and since \( L \) is a spanning co-tree, there exists \( x' \in Z_{d-1}(L; \mathbb{Q}) \) so that \( \pi(a(x) - y) = (j \circ p \circ i_L)(x') \). But \( z = \pi(y) \), so

\[
z = \pi(y) = \pi(a(x)) = j(p(i_L(x')) = \pi(a(x - k(x'))).
\]

We conclude that (9) is surjective. \( \square \)

Remark 2.5. A referee has pointed out to us that Lemma 2.3 as well as Proposition 2.4 admit alternative proofs using matroids.

Lemma 2.6. Let \( \mathbb{F} \) be a field. Then a splitting of the quotient homomorphism \( C_{d-1}(X; \mathbb{F}) \rightarrow C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F}) \) induces by restriction a splitting of the quotient homomorphism \( Z_{d-1}(X; \mathbb{F}) \rightarrow H_{d-1}(X; \mathbb{F}) \).

Proof. Consider the following commutative diagram, with exact rows.

\[
\begin{array}{cccc}
0 & \longrightarrow & B_{d-1}(X; \mathbb{F}) & \longrightarrow & Z_{d-1}(X; \mathbb{F}) & \longrightarrow & H_{d-1}(X; \mathbb{F}) & \longrightarrow & 0 \\
\downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
0 & \longrightarrow & B_{d-1}(X; \mathbb{F}) & \longrightarrow & C_{d-1}(X; \mathbb{F}) & \longrightarrow & C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F}) & \longrightarrow & 0.
\end{array}
\]
Since \( H_{d-1}(X; \mathbb{F}) \subset C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F}) \), we can restrict the given splitting to get a map \( H_{d-1}(X; \mathbb{F}) \to C_{d-1}(X; \mathbb{F}) \). A simple diagram chase shows that this map factors through \( Z_{d-1}(X; \mathbb{F}) \). \( \square \)

**Notation 2.7.** For \( i \leq j \) and \( Y \subset X \) be a subcomplex. Set \( Y_{j,i} = Y^{(j)}/Y^{(i)} \).

Then \( Y_{j,i} \subset X_{j,i} \) is a subcomplex of dimension \( \leq j \).

**Corollary 2.8.** Assume \( d \geq 2 \). Then the operation \( L \mapsto L_{d,d-2} \) defines a bijection between the spanning co-trees of \( X \) and the spanning co-trees of \( X_{d,d-2} \). Furthermore, \( a_{L_{d,d-2}} = a_L \).

**Proof.** By definition \( C_{d-1}(X_{d,d-2}; \mathbb{F}) = Z_{d-1}(X; \mathbb{F}) \), so the diagram

\[
C_{d-1}(L; \mathbb{F}) \to C_{d-1}(X; \mathbb{F}) \to C_{d-1}(X; \mathbb{F})/B_{d-1}(X; \mathbb{F})
\]

coincides with the diagram

\[
Z_{d-1}(L_{d,d-2}; \mathbb{F}) \to Z_{d-1}(X_{d,d-2}; \mathbb{F}) \to H_{d-1}(X_{d,d-2}; \mathbb{F})
\]

It follows that the first part amounts to a restatement of Proposition 2.4.

To prove the second part, use the homotopy pushout diagram

\[
\begin{array}{ccc}
L & \to & X \\
\downarrow & & \downarrow \\
L_{d,d-2} & \to & X_{d,d-2}
\end{array}
\]

and the long exact sequences in homology associated with the horizontal maps. Using the five-lemma we infer that the homomorphism \( H_*(X, L) \to H_*(X_{d,d-2}, L_{d,d-2}) \) is an isomorphism in all degrees. \( \square \)

**Remark 2.9.** Corollary 2.8 reduces the proof of Theorem A to the special case of CW complexes \( X \) having trivial \((d-2)\)-skeleton (if \( d \leq 1 \), triviality is automatic). For such complexes the number \( a_L \) coincides with the order of cokernel of the homomorphism

\[
C_{d-1}(L; \mathbb{Z}) \to C_{d-1}(X; \mathbb{Z})/B_{d-1}(X; \mathbb{Z})
\]

The triviality of the \((d-2)\)-skeleton implies that the displayed map coincides with the homomorphism \( H_{d-1}(L; \mathbb{Z}) \to H_{d-1}(X; \mathbb{Z}) \).

Let \( H_{d-1}(X; \mathbb{Z})_{\text{tor}} \subset H_{d-1}(X; \mathbb{Z}) \) be the torsion subgroup. Let \( b_L \) denote the order of the cokernel of the composite map

\[
H_{d-1}(L; \mathbb{Z}) \to H_{d-1}(X; \mathbb{Z}) \to H_{d-1}(X; \mathbb{Z})/H_{d-1}(X; \mathbb{Z})_{\text{tor}}
\]
Then (10) is a monomorphism of finitely generated free abelian groups with finite cokernel. Up to sign, the determinant of (10) is well-defined and coincides with $b_L$ (see [CCK, prop. 6.63]). Furthermore,

$$a_L = \theta_X b_L,$$

where $\theta_X$ is the order of $H_{d-1}(X; \mathbb{Z})_{tor}$.

To complete the proof of Theorem A, we will construct a splitting of the map $C_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})/B_{d-1}(X; \mathbb{R})$ that will give the relevant summation formula. For this, we shall appeal to the theory of generalized inverses.

3. The proof of Theorem A

3.1. Generalized Inverses. The theory of generalized inverses was developed to study linear systems $Ax = b$ for which $A^{-1}$ does not exist. Let $A$ be an $m \times n$ matrix over $\mathbb{R}$, and let $b \in \mathbb{R}^m$ be given. Consider the linear system $Ax = b$. In general, such systems need not have a (unique) solution. One way to study the system is to attempt to minimize the norm of the residual vector $Ax - b$. Among all such $x$ for which the norm of $Ax - b$ is minimizing, we impose the additional constraint that the norm of $x$ is minimizing. This is called a least squares problem.

Remark 3.1. When $A$ is surjective the residual vector having minimum norm is the zero vector. In this case the least squares problem reduces to the problem of finding a solution of $Ax = b$ such that the norm of $x$ is minimized.

The Moore-Penrose pseudo-inverse $A^+$ gives a preferred solution to the least squares problem. If $b$ lies in the image of $A$, then a solution to $Ax = b$ exists and the Moore-Penrose solution $A^+ b$ will be a solution having the smallest norm. Furthermore, the matrix $A^+$ exists and is unique [P], [BG, p. 109].

The operation $A \mapsto A^+$ satisfies the identities

$$A^+ = A^*(AA^*)^+ = (A^*A)^+ A^*,$$

where $A^*$ is the transpose of $A$ (cf. [BG, chap. 1.6, ex. 18(d)]). In particular, when $A$ is surjective, we obtain the formula

$$A^+ = A^*(AA^*)^{-1}.$$

3This is slightly more general than the usual formulation. The classical least squares problem assumes that $A$ is injective. We will be primarily concerned here with the case when $A$ is surjective.
Remark 3.2. If $A$ is surjective, then one may drop the requirement that the target of $A$ is based. That is, suppose more generally that $A: \mathbb{R}^n \to V$ is a surjective linear transformation where $V$ is not necessarily based. Then the least squares problem as well as the formula (13) make sense if we use the formal adjoint $A^*: V^* \to (\mathbb{R}^n)^* = \mathbb{R}^n$ in place of the transpose. Similarly, if $A$ is injective, we may drop the requirement that the source of $A$ is based.

We will need a weighted version of the least squares problem. For this, we weight the standard basis elements $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$ by means of a positive functional $\mu: \{e_i\}_{i=1}^n \to \mathbb{R}_+$. Then $\mu$ defines a modified inner product $\langle -,-\rangle_\mu$ on $\mathbb{R}^n$, determined by $\langle e_i,e_j\rangle_\mu := \mu(e_i)\delta_{ij}$. The weighted least squares problem is to minimize $|Ax - b|$ such that $|x|_\mu$ is also minimized. Again, the solution $x = A^+b$ exists and is unique, where now $A^+$ is the weighted version of the Moore-Penrose pseudo-inverse.

Assume now that $A$ has rank $m$, i.e., $A$ is surjective. For a subset $S \subset \{1,\ldots,n\}$ of cardinality $m$, let $A_S$ be the $m \times m$ submatrix whose rows correspond to indices in $S$:

$$(A_S)_{ij} := A_{ij}, \quad \text{for } i = 1,\ldots,m, \ j \in S.$$ We will consider only those $S$ such that $A_S$ is invertible. Let $i_S: \mathbb{R}^m \to \mathbb{R}^n$ denote the inclusion given by the rows corresponding to $S$. Set

$$t_S := \det(A_S)^2 \prod_{i \in S} \frac{1}{\mu(e_i)},$$

and set $\nabla := \sum_S t_S$. We can now state the summation formula for $A^+$ in the case of surjective $A$.

**Theorem 3.3** (cf. [Be, Thm 1], [BT, th. 2.1]). Let $A$ be an $m \times n$ matrix of rank $m$ defined over $\mathbb{R}$. Then the weighted Moore-Penrose pseudo-inverse of $A$ is given by

$$A^+ = \frac{1}{\nabla} \sum_S t_S i_S(A_S)^{-1},$$

where the sum is taken over all indices $S \subset \{1,2,\ldots,n\}$ such that $A_S$ is invertible.

**Remark 3.4.** Theorem 2.1 of [BT] concerns the case when $A$ has rank $n$, i.e., when $A$ is injective with arbitrary weights. The main result of [Be] applies to general $A$ in the unweighted case $\mu = 1$. When $A$ is surjective, it is straightforward to deduce the weighted case from the unweighted one by the following transformation: replace $A$ by $\hat{A} = AM^{-1}$, where $M$ is the diagonal matrix having entries $\sqrt{\mu(e_i)}$ and...
replace \( x \) by \( \hat{x} = Mx \). This converts the weighted least squares problem \((Ax = b, \mu)\) to an equivalent unweighted problem \((\hat{A}\hat{x} = b, 1)\). The formula displayed in Theorem 3.3 is easily deduced from this, and we will omit the argument.

**Remark 3.5.** Suppose that \( A: \mathbb{R}^n \rightarrow V \) is a surjective linear transformation in which \( V \) is not necessarily based (cf. Remark 3.2). Furthermore, suppose that \( H \subset V \) is a lattice, i.e., a finitely generated abelian subgroup such that the induced map \( H \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow V \) is an isomorphism. Then a choice of basis for \( H \) determines one for \( V \) and Theorem 3.3 applies. Moreover, the formula is invariant with respect to base changes for \( H \), since the numbers \( t_S \) are squares of determinants. Consequently, Theorem 3.3 is really a statement about surjective linear transformations \( A: \mathbb{R}^n \rightarrow V \) for vector spaces \( V \) that come equipped with a preferred lattice \( H \).

**Proof of Theorem A.** By Remark 2.9 there is no loss in assuming that \( X \) has trivial \((d-2)\)-skeleton. By Remark 1.8 and Lemma 2.6, it suffices to produce a splitting of the quotient homomorphism \( \pi: C_{d-1}(X; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R})/B_{d-1}(X; \mathbb{R}) \). Here we use the weighted basis of \( C_{d-1}(X; \mathbb{R}) \) defined by the cells and the weighting given by \( b \mapsto e^{\beta E_b} \). Use the lattice \( H \subset C_{d-1}(X; \mathbb{R})/B_{d-1}(X; \mathbb{R}) \) given by the image of the homomorphism \( C_{d-1}(X; \mathbb{Z})/B_{d-1}(X; \mathbb{Z}) \rightarrow C_{d-1}(X; \mathbb{R})/B_{d-1}(X; \mathbb{R}) \).

Applying Theorem 3.3 and Remark 3.5 to \( \pi \) gives a splitting, written as a sum over subsets \( S \) of the basis elements of \( C_{d-1}(X; \mathbb{R}) \). By Proposition 2.4, the collection of these subsets are in bijection with the set of spanning co-trees. The inclusion \( i_S \) corresponds to the inclusion \( C_{d-1}(L; \mathbb{R}) \rightarrow C_{d-1}(X; \mathbb{R}) \) and \( \phi_L \) corresponds to \( A_S \). Then the determinant of \( A_S \) is given up to sign by \( \tau'_L := \tau_L/\theta_X \) (cf. Remark 2.9) and the prefactor is given by the reciprocal of \( \nabla' := \sum_L \tau'_L \). Hence, the desired splitting is given by

\[
\frac{1}{\nabla'} \sum \tau'_L \psi_L = \frac{1}{\nabla} \sum \tau_L \psi_L. \tag*{□}
\]

**Remark 3.6.** If we fix a function \( W: X_d \rightarrow \mathbb{R} \), we may instead apply [Be, Thm 1] to the inclusion map \( q: Z_d(X; \mathbb{R}) \rightarrow C_d(X; \mathbb{R}) \). This produces an orthogonal splitting \( C_d(X; \mathbb{R}) \rightarrow Z_d(X; \mathbb{R}) \) to \( q \) in the modified inner product on \( C_d(X; \mathbb{R}) \). The splitting is written as a sum indexed over the set of spanning trees as in [CCK]. In fact, this gives quick alternative proofs to Theorem A and Addendum B in [CCK].
4. Tree-co-tree duality

Let $X$ be a finite connected CW-complex and fix a natural number $d \leq \dim X$. We let $(C_\ast(X), \partial)$ and $(C^\ast(X), \delta)$ denote the cellular chain and cochain complexes of $X$ over $\mathbb{Z}$. The set $X_d$ determines preferred isomorphism $C^d(X) \cong C_d(X)$. With this identification, the coboundary operator $\delta : C^{d-1}(X) \to C^d(X)$ corresponds to $\partial^* : C_{d-1}(X) \to C_d(X)$, the formal adjoint to $\partial$ (here, coefficients can be taken in any commutative ring).

Denote by $F_d(X)$ and $F_{d-1}^\ast(X)$ the sets of $d$-dimensional spanning trees and spanning co-trees, respectively, of $X$. For $i \leq j$, recall the subquotient $X_{j,i} = X(j)/X(i)$.

**Lemma 4.1.** There are preferred bijections

$$F_d(X) \cong F_d(X_{d,d-2}) \quad \text{and} \quad F_{d-1}^\ast(X) \cong F_{d-1}^\ast(X_{d,d-2}).$$

**Proof.** The second bijection is merely a restatement the first part of Corollary 2.8. For $T \in F_d(X)$ and $L \in F_{d-1}^\ast(X)$, the assignments

$$T \mapsto T_{d,d-2} \quad \text{and} \quad L \mapsto L_{d,d-2}$$

define maps $F_d(X) \to F_d(X_{d,d-2})$ and $F_{d-1}^\ast(X) \to F_{d-1}^\ast(X_{d,d-2})$. Inverse maps are defined as follows: given a spanning tree $T'$ for $X_{d,d-2}$, we form a complex $T$ by attaching the set of $d$-cells appearing in $T'$ to $X^{(d-1)}$. It is straightforward to verify that $T$ is a $d$-dimensional spanning tree for $X$.

Similarly, if $L'$ is a spanning co-tree for $X_{d,d-2}$, then the complex $L$ given by attaching the $(d-1)$-cells of $L'$ to $X^{(d-2)}$ gives a $d$-dimensional spanning co-tree. \hfill \Box

For $T \in F_d(X)$, let $\theta_T$ be the order of the torsion subgroup of $H_{d-1}(T; \mathbb{Z})$ (cf. [CCK, p. 3]) and for $L \in F_{d-1}^\ast(X)$, recall that $a_L$ is the order of $H_{d-1}(X, L; \mathbb{Z})$.

**Lemma 4.2.** For $T \in F_d(X)$ and $L \in F_{d-1}^\ast(X)$ we have

$$\theta_T = \theta_{T_{d,d-2}} \quad \text{and} \quad a_L = a_{L_{d,d-2}}.$$

**Proof.** The first part follows immediately from the exactness of the sequence

$$0 \to H_{d-1}(T; \mathbb{Z}) \to H_{d-1}(T_{d,d-2}; \mathbb{Z}) \to H_{d-2}(T^{(d-2)}; \mathbb{Z})$$

and the fact that $H_{d-2}(T^{(d-2)}; \mathbb{Z})$ is free abelian. The second part is just a restatement of the second part of Corollary 2.8. \hfill \Box
Remark 4.3. In view of these lemmas, all the relevant properties of $d$-dimensional spanning trees and $(d - 1)$-dimensional spanning co-trees depend only on the boundary operator $\partial : C_d(X) \to C_{d-1}(X)$ and are therefore reduced to properties of two-stage chain complexes of finitely generated free abelian groups, i.e., to statements in linear algebra over $\mathbb{Z}$. We will now make this precise.

For a finite set $S$, let $\mathbb{Z}^S$ be the free abelian group with basis set $S$.

**Definition 4.4.** For finite sets $P$ and $Q$, set $A = \mathbb{Z}^P$ and $B = \mathbb{Z}^Q$. Let $\partial : A \to B$ be a homomorphism. A **spanning tree** of $\partial$ consists of a subset $T \subset P$ such that the composition

$$\mathbb{Z}^T \xrightarrow{\subseteq} A \xrightarrow{\partial} B$$

is an isomorphism modulo torsion. The set of spanning trees of $\partial$ is denoted by $F(\partial)$.

Similarly, a **spanning co-tree** of $\partial$ is a subset $L \subset Q$ such that the composition

$$(14) \quad \mathbb{Z}^L \subset B \to B/\partial(A)$$

is an isomorphism modulo torsion. The set of spanning co-trees of $\partial$ is denoted by $F^*(\partial)$.

For an abelian group $U$ let $U^* := \text{hom}_\mathbb{Z}(U, \mathbb{Z})$. This defines an contravariant endo-functor on abelian groups. Let $\partial^* : B^* \to A^*$ be the homomorphism induced by $\partial$. For a subset $S \subset P$, let $S^\perp = P \setminus S$ denote its complement. The proof of the following is a straightforward exercise left to the reader.

**Lemma 4.5 (Tree-Co-tree Duality).** The operation $T \mapsto T^\perp$ induces a bijection $F(\partial) \cong F^*(\partial^*)$.

**4.1. Finite chain complexes.** A $\mathbb{Z}$-graded chain complex $C_*$ over $\mathbb{Z}$ is **finite** if it is degree-wise finitely generated and free and has finitely many non-zero terms. If $C_*$ is finite then so is its linear dual

$$DC_* = \text{hom}(C_-, \mathbb{Z}).$$

If we fix $d \in \mathbb{Z}$, then the notion of $d$-dimensional spanning tree and co-tree is defined in this context. Let $F_d(C_*)$ be the set of $d$-dimensional spanning trees of $C_*$. Similarly, we let $F^*_d(C_*)$ be the set of $d$-dimensional spanning co-trees of $C_*$. The following is an immediate consequence of Lemma 4.5 combined with Remark 4.3.

**Corollary 4.6 (Chain Tree-Co-tree Duality).** Assume $C_*$ is finite. Then for $d \in \mathbb{Z}$, there is a preferred bijection $F_d(C_*) \cong F^*_d(DC_*)$. 
Remark 4.7. It is not difficult to give a spectrum-level version of Corollary 4.6. When $X$ is a finite CW spectrum, the spectrum $DX$ (corresponding to the linear dual of a chain complex) is the Spanier-Whitehead dual of $X$, i.e., the function spectrum $F(X, S^0)$.

Corollary 4.6 may be related to some of the results of [MMRW].

5. The proof of Theorem C

Given a scalar structure on $X$ and a dimension $d \leq \dim X$, one has a pair of biased Laplacians defined by the commutative diagrams

$$
\begin{array}{ccc}
\pi_1: & B_{d-1}(X; \mathbb{R}) & \xrightarrow{\partial^* e E} & B^d(X; \mathbb{R}) \\
\downarrow\phi_{e^{-\beta W}} & & & \downarrow\phi_{e^{-\beta W}} \\
\pi_2: & \pi_1^{-1} & \xrightarrow{\partial e^{-\beta W}} & \pi_1^{-1} \\
\end{array}
$$

Observe that

$$(\mathcal{L}_{E,W}^*)^* = \mathcal{L}_{W,E}^*.$$  

The operators $\mathcal{L}_{E,W}$ and $\mathcal{L}_{W,E}^*$ are invertible and have the same determinant. To avoid notational clutter, when $E$ and $W$ are understood we simplify notation and set $\mathcal{L} := \mathcal{L}_{E,W}$ and $\mathcal{L}^* := \mathcal{L}_{W,E}^*$.  

Recall that the goal is to exhibit a decomposition of the determinant of $\mathcal{L}$ as a sum over trees and co-trees:

$$\det(\mathcal{L}) = \det(\mathcal{L}^*) = \frac{1}{\varphi_X} \sum_{L,T} \tau_L \omega_T.$$  

We start with a weak version that establishes (16) up to a factor that does not depend on either $E$ or $W$. We consider $\mathcal{L}$ as a function of the pair $(E, W)$ taking values in the space of linear operators on $B_{d-1}(X; \mathbb{R})$. We then measure the variation of $\ln \det(\mathcal{L})$ in $W$ assuming $E$ is held fixed. This gives

$$d \ln \det(\mathcal{L}) = -\text{Tr}(\partial d We^{-W} \partial^* e E \mathcal{L}^{-1})$$

$$= -\text{Tr}(d We^{-W} \partial^* e E \mathcal{L}^{-1} \partial)$$

$$= -\text{Tr}(dW A),$$

where

$$A := e^{-W} \partial^* e E \mathcal{L}^{-1}: B_{d-1}(X; \mathbb{R}) \to C_d(X; \mathbb{R})$$

(compare [CCK, eqns. (6),(7)]). It is straightforward to check that $A$ is the pseudo-inverse for $\partial : C_d(X) \to B_{d-1}(X)$. Thus we can follow,
mutatis mutandis, the proof of [CCK, prop. 4.2] (which is the $E = 0$ particular case of $\mathcal{L}$) to arrive at

$$
\det(\mathcal{L}) = \det(\mathcal{L}^*) = \bar{\gamma}(E) \sum_T w_T(W).
$$

where the prefactor $\bar{\gamma}(E)$ is independent of $W$.

We next make use of duality, i.e., $\det(\mathcal{L}) = \det(\mathcal{L}^*) = \partial^* e^E \partial e^{-W}$, so that variation over $E$, with $W$ held fixed, is equivalent to the case considered above. Consequently,

$$
\det(\mathcal{L}) = \det(\mathcal{L}^*) = \bar{\gamma}^*(W) \sum_L \tau_L(E),
$$

where the prefactor $\bar{\gamma}^*(W)$ is independent of $E$.

Setting (19) and (20) equal, we immediately obtain the following weak form of Theorem C:

$$
\det(\mathcal{L}) = \det(\mathcal{L}^*) = \gamma(\sum_L \tau_L)(\sum_T w_T),
$$

in which the prefactor $\gamma$ is a constant independent of $E$ and $W$. The proof of Theorem C is complete once we establish the following claim.

**Claim.** $\gamma = 1/\theta^2_X$.

To prove the claim, by the argument of [CCK, §6] it is enough to restrict to the case when $X$ is a spanning tree of dimension $d$. Furthermore, by Lemmas 4.1 and 4.2 we can assume that $X = X_{d,d-2} = X^{(d)}/X^{(d-2)}$. Since $\gamma$ is independent of $E$ and $W$ we can further assume that $E = 0 = W$. In this instance, by [CCK, cor. D] it will suffice to establish the identity

$$
\mu_X = \sum_{L \in F^*(X)} a_L^2,
$$

where $\mu_X$ denotes the square covolume of the lattice $B_{d-1}(X; \mathbb{Z}) \subset B_{d-1}(X; \mathbb{R})$.

With respect to our hypotheses, the cellular chain complex of $X$ is determined by the (rationally injective) homomorphism $\partial: A \to B$ of finitely generated free abelian groups. By slight abuse of notation, denote this chain complex by $X$ and let $Y$ be the dual chain complex given by $\partial^*: B^* \to A^*$, where $A^* = \text{hom}(A, \mathbb{Z})$. Then

$$
\begin{align*}
\det(\mathcal{L}) &= \det(\mathcal{L}^*) \\
&= \det(\partial^* \partial) \\
&= \frac{\mu_X}{\theta_Y} \sum_{T \in F(Y)} \theta_T^2.
\end{align*}
$$
where the last equality follows from [CCK, cor. D]. In the above, we have implicitly identified $\partial^*$ with the transpose of $\partial$ using the preferred bases. The number $\mu_Y$ is the square of the covolume of the lattice defined by the image $\partial^*(B^*) \subset \partial^*(B^*) \otimes_{\mathbb{Z}} \mathbb{R}$ where the latter term is given an inner product by declaring it to be an isometric subspace of $A^* \otimes_{\mathbb{Z}} \mathbb{R} = \text{hom}(A, \mathbb{R})$. By Lemma 4.5, $\mathcal{F}(Y) = \mathcal{F}^*(X)$.

Given a spanning co-tree $L \in \mathcal{F}^*(\partial)$, let $a_L$ denote the order of the cokernel of $L \to B/\partial(A)$. For a subgroup $T \subset A$ let $\theta_T$ denote the order of the torsion subgroup of the cokernel of the composition $T \to A \to B$. Note that these definitions coincide with the ones we gave in the case of CW complexes with trivial $(d-2)$-skeleton.

The identity (22) (and hence the claim) is now an immediate consequence of the following:

**Lemma 5.1.** With respect to the above assumptions, we have

1. $\det(L) = \mu_X$;
2. $\theta_Y = \theta_X$;
3. $\mu_Y = \theta_Y^2$;
4. $a_L = \theta_{(L^\perp)^*}$.

**Proof.** Statement (1) is a special case of [CCK, cor. D]).

Let $C$ be the cokernel $\partial$. Then we have a short exact sequence $0 \to A \to B \to C \to 0$. Applying $\text{hom}(-, \mathbb{Z})$ gives a short exact sequence

$$0 \to C^* \to B^* \to A^* \to \text{ext}(C, \mathbb{Z}) \to 0.$$  

Since $C$ is finitely generated, the group $\text{ext}(C, \mathbb{Z})$ is a torsion group which is (non-canonically) isomorphic to the torsion subgroup of $C$. This gives (2).

Statement (3) follows from the discussion after definitions 6.2 and 6.5 of [CCK] as applied to the real isomorphism $\partial^*(B^*) \to A^*$. Lastly, (4) is easily deduced from the fact that the cokernels of $\mathbb{Z}L \to B/\partial(A)$ and $A \to \mathbb{Z}L^\perp$ are canonically isomorphic. \qed

This establishes the claim and completes the proof of Theorem C.

**Remark 5.2.** There is different proof of the claim that makes use of the “low-temperature limit” method of [CCK, §5] (with respect to $E$ and $W$) as applied to equation (21), with the sum on the right-hand side dominated by one term. However, the proof we have given here has the advantage of being shorter and less technical.
6. Appendix: Kirchhoff-Boltzmann unification

For any fixed dimension \(d\), the boundary operator \(\partial : C_d(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})\) factors as

\[
C_d(X; \mathbb{R}) \xrightarrow{\partial} B_{d-1}(X; \mathbb{R}) \xrightarrow{i} C_{d-1}(X; \mathbb{R})
\]

in which the first map is surjection and the second is an injection.

With respect to these inner product structures, one infers that the pseudo-inverse of \(\partial : C_d(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})\) is given by the composition

\[
C_{d-1}(X; \mathbb{R}) \xrightarrow{i^+} B_{d-1}(X; \mathbb{R}) \xrightarrow{p^+} C_d(X; \mathbb{R})
\]

consisting of the pseudo-inverses of \(i\) and \(p\). (cf. [BG, p. 48, ex. 17]). Here, \(p^+\) is the pseudo-inverse of \(p\) with respect to the modified inner product structure on \(C_d(X; \mathbb{R})\) (cf. Remarks 3.2 and 3.5). Similarly, \(i^+\) is the pseudo-inverse for \(i\) defined using the modified inner product on \(C_{d-1}(X; \mathbb{R})\).

Therefore, all three pseudo-inverse formulas will follow from one, e.g., from the formula for the surjection, since the formula for an injection can be obtained by taking transposes and applying duality (Lemma 4.5), whereas the general pseudo-inverse is obtained by taking the composition.

For a tree \(T \in \mathcal{F}_d(X)\), the composition

\[
C_d(T; \mathbb{R}) \xrightarrow{\alpha_T} C_d(X; \mathbb{R}) \xrightarrow{\partial} B_{d-1}(X; \mathbb{R})
\]

is an isomorphism. Let \(\alpha_T\) denote its inverse. Let \(\varphi_T : B_{d-1}(X; \mathbb{R}) \to C_d(X; \mathbb{R})\) be the composition

\[
B_{d-1}(X; \mathbb{R}) \xrightarrow{\varphi_T} C_d(X; \mathbb{R})
\]

Then using Theorem 3.3 one has

\[
p^+_W = \frac{1}{\Delta_W} \sum_T w_T \varphi_T, \quad \Delta_W := \sum_T w_T,
\]

where the sum is over spanning trees \(T \in \mathcal{F}_d(X)\).

Similarly, the pseudo-inverse of the inclusion \(i : B_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})\) is obtained as follows: for a co-tree \(L \in \mathcal{F}_{d-1}(X)\) the composition

\[
B_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})/C_{d-1}(L; \mathbb{R})
\]

is an isomorphism; let \(\beta_L\) denote its inverse. Let \(\zeta_L : C_{d-1}(X; \mathbb{R}) \to B_{d-1}(X; \mathbb{R})\) be the composition

\[
C_{d-1}(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R})/C_{d-1}(L; \mathbb{R}) \xrightarrow{\beta_L} B_{d-1}(X; \mathbb{R})
\]
where the first map is vector space projection.

Then

\[
i_E^+ = \frac{1}{\nabla E} \sum_L \tau_L \zeta_L, \quad \nabla E = \sum_L \tau_L,
\]

where the sum is over spanning co-trees \( L \in \mathcal{F}^{*}_{d-1}(X) \).

Combining (24) and (25) with the pseudo-inverse composition property we arrive at a formula which encompasses both the Boltzmann splitting formula (Theorem A) and the higher Kirchhoff projection formula of [CCK, thm. A]:

**Theorem 6.1** (Kirchhoff-Boltzmann Projection Formula). The pseudo-inverse of the boundary operator \( \partial: C_d(X; \mathbb{R}) \to C_{d-1}(X; \mathbb{R}) \) with respect to the modified inner products defined by \( E \) and \( W \) is given by

\[
\partial_{E,W}^+ = \frac{1}{\Delta_W \nabla_E} \sum_{L,T} \tau_L w_T \sigma_{L,T},
\]

where \( \sigma_{L,T} = \varphi_T \circ \zeta_L: C_{d-1}(X; \mathbb{R}) \to C_d(X; \mathbb{R}) \) and the sum is indexed over \( L \in \mathcal{F}^{*}_{d-1}(X) \) and \( T \in \mathcal{F}_d(X) \).

**Remark 6.2.** Theorem 6.1 gives a concrete expression for the average current in the case of periodic stochastic driving (in the adiabatic limit) [CKS]. Let \( \gamma \) be a smooth 1-dimensional cycle in the vector space of parameters \((E, W)\) and let \( \hat{x} \in \mathcal{Z}_{d-1}(X; \mathbb{Z}) \) be a \((d-1)\)-cycle. Let \( x = [\hat{x}] \in H_{d-1}(X; \mathbb{Z}) \) be the associated homology class. Then the average current of \((\gamma, [x])\) is defined to be the \(d\)-cycle

\[
q := \int_\gamma A d \rho^B \in Z_d(X; \mathbb{R}),
\]

where \( A = A(E, W) = e^{-W} \partial^* e^E \mathcal{L}_{E,W}^{-1} \) is the operator of equation (18) and \( \rho^B \) is the higher Boltzmann distribution of \( x \) (see [CKS] if \( d = \dim X = 1 \) and [C] for \( d > 1 \)). Then, after some straightforward algebraic manipulation using Theorem 6.1, we find

\[
q = \sum_{L,T} \sigma_{L,T}(\hat{x}) \int_\alpha d \varphi_T^+ \wedge d \varphi_L^-,
\]

where \( \varphi_T^+ = \varphi_T^+(W) := w_T/\Delta_T, \varphi_L^- = \varphi_L^-(E) := \tau_L/\nabla_L \) and \( \alpha \) is any smooth 2-dimensional chain in the space of parameters satisfying \( \partial \alpha = \gamma \).

**References**

[BG] Ben-Israel, A., Greville, T. N. E.: Generalized Inverses, Second Ed. Springer-Verlag, New York, 2003.
[BT] Ben-Tal A., Teboulle, M.: A Geometric Property of the Least Squares Solution of Linear Equations. *Linear Algebra Appl.* **139**, 165–170 (1990).

[Be] Berg, L.: Three Results in Connection with Inverse Matrices. *Linear Algebra Appl.* **84**, 63–77 (1986).

[C] Catanzaro, M. J.: Ph. D. Thesis, Wayne State University, 2016.

[CCK] Catanzaro, M. J., Chernyak, V. Y., Klein, J. R.: Kirchhoff’s theorems in higher dimensions and Reidemeister torsion. *Homology Homotopy Appl.* **17**, 165–189 (2015).

[CKS] Chernyak, V. Y., Klein, J. R., Sinitsyn N. A.: Algebraic Topology and the Quantization of Fluctuating Currents. *Adv. Math.* **244**, 791–822 (2013).

[DKM1] Duval, A. M., Klivans, C. J., Martin, J. L.: Cellular spanning trees and Laplacians of cubical complexes. (English summary) Adv. in Appl. Math. **46**, 247–274 (2011).

[DKM2] Duval, A. M., Klivans, C. J., Martin, J. L.: Cuts and flows of cell complexes. *J. Algebraic Combin.* **41**, 969–999 (2015).

[Ke] Kelly, F. P.: Reversibility and Stochastic Networks. Wiley Series in Probability and Mathematical Statistics. 1979.

[L] Lyons, R.: Random complexes and $\ell^2$-Betti numbers. *J. Topol. Anal.* **1**, 153–175 (2009).

[MMRW] Martin, J. L., Maxwell, M. Reiner, V.; Wilson, S. O.: Pseudodeterminants and perfect square spanning tree counts. *J. Comb.* **6**, 295–2015.

[M] Moore, E.H.: On the reciprocal of the general algebraic matrix. *Bull. Amer. Math. Soc.* **26**, 394–395 (1920).

[O] Oxley, J.G.: Matroid theory. Oxford University Press, 1992.

[P] Penrose, R.: A generalized inverse for matrices. *Proc. Camb. Phil. Soc.* **51**, 406–413 (1955).

[Z] Ziegler, G. M. Lectures on polytopes. Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995.

Dept. of Mathematics, Wayne State University, Detroit, MI 48202

E-mail address: mike@math.wayne.edu

Dept. of Chemistry, Wayne State University, Detroit, MI 48202

E-mail address: chernyak@chem.wayne.edu

Dept. of Mathematics, Wayne State University, Detroit, MI 48202

E-mail address: klein@math.wayne.edu