Abstract

Cut problems form one of the most fundamental classes of problems in algorithmic graph theory. For instance, the minimum cut, the minimum $s$-$t$ cut, the minimum multiway cut, and the minimum $k$-way cut are some of the commonly encountered cut problems. Many of these problems have been extensively studied over several decades. In this paper, we initiate the algorithmic study of some cut problems in high dimensions.

The first problem we study, namely, Topological Hitting Set (THS), is defined as follows: Given a nontrivial $r$-cycle $\zeta$ in a simplicial complex $K$, find a set $S$ of $r$-dimensional simplices of minimum cardinality so that $S$ meets every cycle homologous to $\zeta$. Our main result is that this problem admits a polynomial time solution on triangulations of closed surfaces. Interestingly, the optimal solution is given in terms of the cocycles of the surface. For general complexes, we show that THS is $W[1]$-hard with respect to the solution size $k$. On the positive side, we show that THS admits an FPT algorithm with respect to $k + d$, where $d$ is the maximum degree of the Hasse graph of the complex $K$.

We also define a problem called Boundary Nontrivialization (BNT): Given a bounding $r$-cycle $\zeta$ in a simplicial complex $K$, find a set $S$ of $(r+1)$-dimensional simplices of minimum cardinality so that the removal of $S$ from $K$ makes $\zeta$ non-bounding. We show that BNT is $W[1]$-hard with respect to the solution size as the parameter, and has an $O(\log n)$-approximation FPT algorithm for $(r+1)$-dimensional complexes with the $(r+1)$-th Betti number $\beta_{r+1}$ as the parameter. Finally, we provide randomized (approximation) FPT algorithms for the global variants of THS and BNT.

1 Introduction

A graph cut is a partition of the vertices of a graph into two disjoint subsets. The set of edges that have one vertex lying in each of the two subsets determines a so-called cut-set. Typically, the objective function to optimize involves the size of the cut-set. Graph cuts have a ubiquitous presence in theoretical computer science. Cuts are also related to the spectra of the adjacency matrix of the graph leading to a beautiful mathematical theory [14]. Cuts have also found many real-world applications in clustering, shape matching, image segmentation and smoothing, and energy minimization problems in computer vision.

Cut problems are related to flow problems in graphs due to the duality between cuts and flows. In fact, the max-flow min-cut theorem tells us that the maximum value of flow between a vertex $s$ and vertex $t$ equals the value of the minimum cut that separates $s$ and $t$. Figure 1 shows an example of an $s$-$t$ cut on an undirected graph.

Incidentally, graphs happen to be 1-dimensional simplicial complexes. And some of the
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Figure 1 The dashed vertical line shows a minimum $s$-$t$ cut in the graph.

Figure 2 The complex $L_1$ consists of two disjoint triangulated spheres. We do not show the entire triangulation, only the four triangles of interest. The boundary of interest is the equator of the larger sphere on the right.

Figure 3 The complex $L_2$ consists of a triangulation of the intersection of two spheres. As before, we do not show the entire triangulation, only the four triangles of interest. The boundary of interest is the circle of intersection of the two spheres.
cut problems have a natural homological interpretation. For instance, consider the following problem: What is the minimum number of edges you need to remove from a graph so that the vertices \{s, t\} do not form a bounding 0-cycle of a 1-chain over \(\mathbb{Z}_2\) in the resulting graph? Since we have an s-t cut if and only if there are no paths connecting s and t, it is easy to check that this problem is equivalent to finding the minimum s-t cut on graphs! It is natural to ask the analogous question for complexes of higher dimension. In particular, the question we ask, namely Boundary Nontrivialization, is the following one: Given a bounding \(\mathbb{Z}_2\) r-cycle \(\zeta\) in a simplicial complex \(K\), find a set \(S\) of \((r+1)\)-dimensional simplices of minimum cardinality so that the removal of \(S\) from \(K\) makes \(\zeta\) nontrivial.

For instance, consider the two complexes \(L_1\) and \(L_2\) shown in Figures 2 and 3, respectively. For complex \(L_1\) shown in Figure 2, let the equator \(e\) of the sphere on the right be the bounding 1-cycle that we want to make nontrivial. Both hemispheres are bounded by the equator. So, the two highlighted triangles from the right sphere of the complex \(L_1\) constitute the optimal solution for Boundary Nontrivialization. That is, removing these two triangles makes \(e\) a nontrivial 1-cycle. For complex \(L_2\) shown in Figure 3 the circle of intersection of the two spheres is the bounding 1-cycle of interest denoted by \(b\). Removing all the four highlighted triangles from complex \(L_2\) makes \(b\) a nontrivial 1-cycle. This also happens to be the optimal solution for making \(b\) nontrivial.

Complementary to the question of removing the minimal number of \(r+1\)-simplices in order to make a bounding cycle nontrivial, is the problem of removing the minimum number of \(r\)-simplices from a complex so that an entire homology class is destroyed. More formally, the problem Topological Hitting Set can be described as follows: given a nontrivial \(\mathbb{Z}_2\) \(r\)-cycle \(\zeta\) in a simplicial complex \(K\), find a set \(S\) of \(r\)-dimensional simplices of minimum cardinality so that \(S\) meets every cycle homologous to \(\zeta\).

Topological Hitting Set on graphs can be described as follows: Suppose we are given a graph \(G\) with \(k\) components. Let \(C\) be one of the components of \(G\). Then, \(\beta_0(G) = |k|\), and each component determines a 0-cycle. So the question of Topological Hitting Set is to determine the minimum number of vertices you need to remove so that \(C\) is not a component anymore. The answer is trivial! One needs to remove all the vertices in \(C\). For example in Figure 4, \(C_2\) ceases to be a component if and only if all four vertices in \(C_2\) are removed. It is worth noting that it is the unidimensionality of graphs that makes the problem trivial. What is more, even the ‘cut’ aspect of the problem is not immediately visible for graphs.

In contrast, for higher-dimensional complexes, the problem has a distinct cut flavor. For instance, consider the planar complex shown in Figure 5. The minimum number of edges that need to be removed so that every cycle homologous to \(\zeta\) is destroyed is three. In Figure 5, an optimal set of edges is shown in red. Note that the edges happen to be in a ‘thin’ portion of the complex, justifying our standpoint that (along with Boundary Nontrivialization) this problem can also be seen as a high dimensional cut problem.

In this work, we undertake an algorithmic study of the two high-dimensional cut problems: Boundary Nontrivialization and Topological Hitting Set.

1.1 Related work

Duval et al. [21] study the vector spaces and integer lattices of cuts and flows associated to CW complexes and their relationships to group invariants. Ghrist and Krishnan [26] prove a topological version of the max-flow min-cut theorem for directed networks using methods from sheaf theory. Then, there is also a long line of work on cuts in surface embedded graphs [23][30][10][11], which is algorithmic in spirit and is loosely related to our work.
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Figure 4 Graph $G$ with three components

Figure 5 The figure shows two cycles that belong to $[\zeta]$ in green. Note that any cycle in $[\zeta]$ must pass through at least one of the three red edges. Thus, the set of red edges constitutes an optimal solution for Topological Hitting Set on this planar complex.
There is a growing body of work on parameterized complexity in topology \cite{1,4,5,6,28,30,31,35}, and much of this paper can be characterized as such.

During the preparation of this article, we became aware of a recent paper by Maxwell and Nayyeri \cite{32} that studies problems similar to the ones we define but from a completely different point of view. While our focus was on surfaces and parameterized complexity, the main focus of their work was to find out the extent to which the conceptual and the algorithmic framework of max-flow min-cut duality generalizes to the case of simplicial complexes. While we focus only on cuts, they study both cuts and flows.

We summarize the main results of Maxwell and Nayyeri \cite{32} as we understand them: They define a topological max-flow and a topological min-cut problem, and also a combinatorial min-cut problem. They show that unlike in the case of graphs, computing maximum integral flows and combinatorial cuts on simplicial complexes is $\text{NP}$-hard. Moreover, they describe conditions under which the linear program gives the optimal value of a combinatorial cut, and also provide a generalization of the Ford-Fulkerson algorithm to the case of simplicial complexes. Their definition of combinatorial cut coincides with our definition of $\text{Boundary Nontrivialization}$, except for some important differences: they are interested in real coefficients and co-dimension one cycles, whereas we work with $\mathbb{Z}_2$ coefficients and cycles of all dimensions. We implore the reader to look up their interesting results \cite{32}.

We note that while their paper is in the same spirit as ours, their focus is quite different from ours, and there is very little overlap in terms of hardness or algorithmic results. In particular, they show $\text{NP}$-hardness for combinatorial cuts with real coefficients, and we show $\text{NP}$-hardness and $\text{W[1]}$-hardness for the same problem with $\mathbb{Z}_2$ coefficients.

2 Summary of results

Surfaces. Our first result, expounded in Section 5 is the following: \textsc{Topological Hitting Set} admits a polynomial-time algorithm on triangulations of closed surfaces. At the heart of our proof lies an appealing characterization of the optimal solutions in terms of the cocycles of the surface, which is of independent interest. Specifically, we show that a minimal solution set is necessarily a nontrivial cocycle. Further, we show that the following are equivalent: 1. A connected cocycle $\eta$ is a feasible set for the input cycle $\zeta$. 2. Every cycle in $[\zeta]$ intersects a connected cocycle $\eta$ in an odd number of edges. 3. One of the cycles in $[\zeta]$ intersects a connected cocycle $\eta$ in an odd number of edges.

In particular, this allows us to identify the nontrivial cocycles that are solutions based on a parity-based property. Having this characterization at hand, we proceed to characterize cohomology classes that are solutions. Eventually, we arrive at a very simple 3-step algorithm for \textsc{Topological Hitting Set} on surfaces.

We remark that $\text{Boundary Nontrivialization}$ is trivial for surfaces. In fact, it is easy to check that for some boundary $b$ and a 2-chain $\zeta$, if $\partial \zeta = b$, then removing any one of the triangles that appears in the chain $\zeta$ makes $b$ nontrivial.

$\text{W[1]}$-hardness and $\text{NP}$-hardness. For general complexes, in Section 6.1 we show that \textsc{Topological Hitting Set} is $\text{W[1]}$-hard with respect to the solution size $k$ as the parameter, (and hence, it is also $\text{NP}$-hard). The proof is based on a reduction from the $k$-\textsc{Multicolored Clique} problem. Here, the reduction shows the essence of hardness: its description is short, but its proof exposes various “behaviors” that we find interesting. In particular, the forward direction requires a nontrivial parity based argument, while the reverse direction shows how to “trace” a solution through the complex.
In addition, in Section 6.2 we show that Boundary Nontrivialization is also W[1]-hard with respect to the solution size \( k \) as a parameter. The principles of this reduction follow the lines of the reduction for Topological Hitting Set, though, here, both the description and the proof of the reduction are more involved because of subdivisions that help avoid some unhelpful incidences.

**Fixed-parameter tractability.** On the positive side, in Section 7.1, we show that Topological Hitting Set admits an FPT algorithm with respect to \( k + \Delta \), where \( \Delta \) is the maximum degree of the Hasse graph of the complex \( K \). Here, the main insight is that a minimal solution must be connected. Having this insight at hand, the algorithm follows: If we search across the geodesic ball of every \( r \)-simplex in the complex \( K \), we will find a solution.

In contrast, we observe that Boundary Nontrivialization does not admit this property because minimal solutions can be disconnected. This motivates the search of another parameter that makes the problem tractable. Exploiting the set-cover like structure of the problem, in Section 7.2, we show that Boundary Nontrivialization with bounding \( r \)-cycles as input has an \( O(\log n) \)-approximation FPT algorithm with \( \beta_{r+1} \) (the Betti number) as the parameter, when the input complex \( K \) is \((r+1)\)-dimensional. It is worth noting that Boundary Nontrivialization is W[1]-hard even for \((r+1)\)-dimensional complexes with solution size as the parameter since the hardness gadget used in Section 6.2 is \((r+1)\)-dimensional.

By exploiting the vector space structure of the homology groups and the boundary groups, in Sections 7.1.1 and 7.2.1, we provide a randomized FPT algorithm for Global Topological Hitting Set and a randomized FPT approximation algorithm for Global Boundary Nontrivialization respectively.

3 Preliminaries

3.1 Simplicial complexes

A \( k \)-simplex \( \sigma \) is the convex hull of a set \( V \) of \((k+1)\) affinely independent points in the Euclidean space of dimension \( d \geq k \). We call \( k \) the dimension of \( \sigma \). Any nonempty subset of \( V \) also spans a simplex, which we call a face of \( \sigma \). A simplex \( \sigma \) is said to be a coface of a simplex \( \tau \) if and only if \( \tau \) is face of \( \sigma \). We say that \( \sigma \) is a facet of \( \tau \), and \( \tau \) a cofacet of \( \sigma \), if \( \sigma \) is a face of \( \tau \) with \( \dim \sigma = \dim \tau - 1 \). We denote a facet-cofacet pair by \( \sigma \prec \tau \). A simplicial complex \( K \) is a collection of simplices that satisfies the following conditions:

- any face of a simplex in \( K \) also belongs to \( K \), and
- the intersection of two simplices \( \sigma_1, \sigma_2 \in K \) is either empty or a face of both \( \sigma_1 \) and \( \sigma_2 \).

An abstract simplicial complex \( K \) on a set of vertices \( V \) is a collection of subsets of \( V \) that is closed under inclusion. The elements of \( K \) are called its simplices. An abstract simplicial complex \( K \) is said to be a subcomplex of \( K \) if every simplex of \( L \) belongs to \( K \).

The collection of vertex sets of simplices in a geometric simplicial complex forms an abstract simplicial complex. On the other hand, an abstract simplicial complex \( K \) has a geometric realization \( |K| \) obtained by embedding the points in \( V \) in general position in a high-dimensional Euclidean space. Then, the complex \( |K| \) is defined as \( \bigcup_{\sigma \in K} |\sigma| \), where \( |\sigma| \) denotes the span of points in \( \sigma \). It is not very difficult to show that any two geometric realizations of an abstract simplicial complex are homeomorphic. Hence, going forward, we do not distinguish between abstract and geometric simplicial complexes.

The star of a vertex \( v \) of complex \( K \), written \( \text{star}_K(v) \), is the subcomplex consisting of all faces of \( K \) containing \( v \), together with their faces.
Let $V$ be the vertex set of $K$, $W$ be the vertex set of $L$, and $\phi$ a map from $V$ to $W$. If for every simplex $\{v_0, v_1, \ldots, v_r\} \in K$, the vertices $\{\phi(v_0), \phi(v_1), \ldots, \phi(v_r)\}$ span a simplex in $L$, then the $\phi$ induces a map, say $f$, from $K$ to $L$. The induced map $f : K \to L$, is said to be simplicial.

We will denote by $K^{(p)}$ the set of $p$-dimensional simplices in $K$, and $n_p$ the number of $p$-dimensional simplices in $K$. The complex induced by $K^{(p)}$ is called the $p$-dimensional skeleton of $K$, and is denoted by $K_p$. Given a simplicial complex $K$, we denote by the $H_p$, the Hasse graph of $K$, which is simply the graph that has a node for every simplex of the complex, and an edge for every facet-cofacet pair. Given a triangulated closed surface $K$, we denote by $D_K$, the dual graph of $K$, which is simply the graph that has a node for every 2-simplex and an edge connecting two nodes if the corresponding 2-simplices are incident on a common edge in the complex. The stellar subdivision of a simplex (or a polytope) is the complex formed by taking a cone over its boundary.

**Notation 1.** We use $[m]$ to denote the set $\{1, 2, \ldots, m\}$ for any $m \in \mathbb{N}$.

### 3.2 Homology and cohomology

In this work, we restrict our attention to simplicial homology with $\mathbb{Z}_2$ coefficients. For a general introduction to algebraic topology, we refer the reader to [27]. Below we give a brief description of homology over $\mathbb{Z}_2$.

Let $K$ be a connected simplicial complex. We consider formal sums of simplices with $\mathbb{Z}_2$ coefficients, that is, sums of the form $\sum_{\sigma \in K(p)} a_\sigma \sigma$, where each $a_\sigma \in \{0, 1\}$. The expression $\sum_{\sigma \in K(p)} a_\sigma \sigma$ is called a $p$-chain. Since chains can be added to each other, they form an Abelian group, denoted by $C_p(K)$. Since we consider formal sums with coefficients coming from $\mathbb{Z}_2$, which is a field, $C_p(K)$, in this case, is a vector space of dimension $n_p$ over $\mathbb{Z}_2$.

The $p$-simplices in $K$ form a (natural) basis for $C_p(K)$. This establishes a natural one-to-one correspondence between elements of $C_p(K)$ and subsets of $K^{(p)}$, and we will freely make use of this identification. The boundary of a $p$-simplex is a $(p-1)$-chain that corresponds to the set of its $(p-1)$-faces. This map can be linearly extended from $p$-simplices to $p$-chains, where the boundary of a chain is the $\mathbb{Z}_2$-sum of the boundaries of its elements. The resulting boundary homomorphism is denoted by $\partial_p : C_p(K) \to C_{p-1}(K)$. A chain $\zeta \in C_p(K)$ is called a $p$-cycle if $\partial_p \zeta = 0$, that is, $\zeta \in \ker \partial_p$. The group of $p$-dimensional cycles is denoted by $Z_p(K)$. As before, since we are working with $\mathbb{Z}_2$ coefficients, $Z_p(K)$ is a vector space over $\mathbb{Z}_2$.

A chain $\eta \in C_p(K)$ is said to be a $p$-boundary if $\eta = \partial_{p+1} \chi$ for some chain $\chi \in C_{p+1}(K)$, that is, $\eta \in \im \partial_{p+1}$. The vector space of $p$-dimensional boundaries is denoted by $B_p(K)$.

In our case, $B_p(K)$ is also a vector space, and in fact a subspace of $C_p(K)$. Thus, we can consider the quotient space $H_p(K) = Z_p(K)/B_p(K)$. The elements of the vector space $H_p(K)$, known as the $p$-th homology of $K$, are equivalence classes of $p$-cycles, called homology classes where $p$-cycles are said to be homologous if their $\mathbb{Z}_2$-difference is a $p$-boundary. For a $p$-cycle $\zeta$, its corresponding homology class is denoted by $[\zeta]$. Bases of $B_p(K)$, $Z_p(K)$ and $H_p(K)$ are called boundary bases, cycle bases, and homology bases, respectively. The dimension of the $p$-th homology of $K$ is called the $p$-th Betti number of $K$, denoted by $\beta_p(K)$.

Using the natural bases for $C_p(K)$ and $C_{p-1}(K)$, the matrix $[\partial_p \sigma_1 \partial_p \sigma_2 \cdots \partial_p \sigma_{n_p}]$ whose column vectors are boundaries of $p$-simplices is called the $p$-th boundary matrix. Abusing notation, we also denote the $p$-th boundary matrix by $\partial_p$.

The dual vector space of $C_p(K)$ (the vector space of linear maps $C_p(K) \to \mathbb{Z}_2$) is called the space of cochain, denoted by $C^p(K) = \text{Hom}(C_p(K), \mathbb{Z}_2)$. Again, there is a natural basis corresponding to the $p$-simplices of $K$, with a $p$-simplex $\sigma$ corresponding to the linear map $\eta$ with values $\eta(\sigma) = 1$ and $\eta(\rho) = 0$ for every other $p$-simplex $\rho \neq \sigma$. The adjoint map to
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the boundary map \( \partial_{p+1} : C_{p+1}(K) \to C_p(K) \) is the co-boundary map \( \delta_p : C^p(K) \to C^{p+1}(K) \). Similarly to chains and boundary maps, we may define subspaces of cocycles \( Z^p(K) = \ker \delta_p \) and coboundaries \( B^p(K) = \im \partial_{p+1} \subseteq Z^p(K) \), and form their quotient \( H^p(K) = Z^p(K)/B^p(K) \), which is the cohomology of \( K \). Again, for a \( p \)-cocycle \( \eta \), the corresponding cohomology class is denoted by \([\eta]\). The natural pairing of chains and cochains \( C_p(K) \times C^p(K) \to \mathbb{Z}\), \((\zeta, \eta) \mapsto \eta(\zeta)\) induces a well-defined isomorphism \( H^p(K) \times H_p(K) \to \mathbb{Z} \), \([\zeta],[\eta] \mapsto \eta(\zeta)\), identifying cohomology as the vector space dual to homology up to a natural isomorphism.

A set of \( p \)-cycles \( \{\zeta_1, \ldots, \zeta_g\} \) is called a homology cycle basis if the set of classes \( \{[\zeta_1], \ldots, [\zeta_g]\} \) forms a homology basis. For brevity, we abuse notation by using the term \((p\text{-th})\) homology basis for \( \{\zeta_1, \ldots, \zeta_g\} \). Similarly, a set of \( p \)-cocycles \( \{\eta_1, \ldots, \eta_g\} \) is called a \((p\text{-th})\) cohomology cocycle basis if the set of classes \( \{[\eta_1], \ldots, [\eta_g]\} \) forms a cohomology basis.

Assigning non-negative weights to the edges of \( K \), the weight of a cycle is the sum of the weights of its edges, and the weight of a homology basis is the sum of the weights of the basis elements. We call the problem of computing a minimum weight basis of \( H_1(K) \) the minimum homology basis problem. Similarly, we call the problem of computing a minimum weight basis of \( H^1(K) \), the minimum cohomology basis problem.

**Notation 2.** Since there is a 1-to-1 correspondence between the \( p \)-chains of a complex \( K \) and the subsets of \( K^{(p)} \), we abuse notation by writing \( \partial C \) in place of \( \partial(\sum_{\sigma \in C} \sigma) \), for \( C \subseteq K^{(p)} \). Likewise, for \( p \)-cochains \( \delta(\sum_{\tau \in C'} \tau) \), we often write \( \delta C' \).

We also abuse notation in the other direction. That is, we treat chains and cochains as sets. For instance, sometimes we say that a (co)chain \( \gamma \) intersects a (co)chain \( \zeta \), when we actually mean that the corresponding sets of simplices of the respective (co)chains intersect. Also, we say that a simplex \( \sigma \in \zeta \), when indeed the simplex \( \sigma \) belongs to the set associated to \( \zeta \).

### 3.3 Parameterized complexity

Let \( \Pi \) be an \( \text{NP} \)-hard problem. In the framework of Parameterized Complexity, each instance of \( \Pi \) is associated with a parameter \( k \). Here, the goal is to confine the combinatorial explosion in the running time of an algorithm for \( \Pi \) to depend only on \( k \). Formally, we say that \( \Pi \) is fixed-parameter tractable (\( \text{FPT} \)) if any instance \((I,k)\) of \( \Pi \) is solvable in time \( f(k) \cdot |I|^{O(1)} \), where \( f \) is an arbitrary computable function of \( k \).

A weaker request is that for every fixed \( k \), the problem \( \Pi \) would be solvable in polynomial time. Formally, we say that \( \Pi \) is slice-wise polynomial (\( \text{XP} \)) if any instance \((I,k)\) of \( \Pi \) is solvable in time \( f(k) \cdot |I|^{q(k)} \), where \( f \) and \( q \) are arbitrary computable functions of \( k \). In other words, for a fixed \( k \), \( \Pi \) has a polynomial time algorithm, and we refer to such an algorithm as an \( \text{XP} \) algorithm for \( \Pi \). Nowadays, Parameterized Complexity supplies a rich toolkit to design \( \text{FPT} \) and \( \text{XP} \) algorithms [16][20][24].

Parameterized Complexity also provides methods to show that a problem is unlikely to be \( \text{FPT} \). The main technique is the one of parameterized reductions analogous to those employed in classical complexity. Here, the concept of \( \text{W} \)-hardness replaces the one of \( \text{NP} \)-hardness, and for reductions we need not only construct an equivalent instance in \( \text{FPT} \) time, but also ensure that the size of the parameter in the new instance depends only on the size of the parameter in the original one.

**Definition 1 (Parameterized Reduction).** Let \( \Pi \) and \( \Pi' \) be two parameterized problems. A parameterized reduction from \( \Pi \) to \( \Pi' \) is an algorithm that, given an instance \((I,k)\) of \( \Pi \), outputs an instance \((I',k')\) of \( \Pi' \) such that:

- \((I,k)\) is a yes-instance of \( \Pi \) if and only if \((I',k')\) is a yes-instance of \( \Pi' \).
\[ k' \leq g(k) \text{ for some computable function } g. \]

The running time is \( f(k) \cdot |\Pi|^{O(1)} \text{ for some computable function } f. \)

If there exists such a reduction transforming a problem known to be \( \text{W[1]} \)-hard to another problem \( \Pi \), then the problem \( \Pi \) is \( \text{W[1]} \)-hard as well. Central \( \text{W[1]} \)-hard problems include, for example, deciding whether a nondeterministic single-tape Turing machine accepts within \( k \) steps, \text{Clique} parameterized by solution size, and \text{Independent Set} parameterized by solution size. To show that a problem \( \Pi \) is not \( \text{XP} \) unless \( \text{P} = \text{NP} \), it is sufficient to show that there exists a fixed \( k \) such that \( \Pi \) is \( \text{NP} \)-hard. If the problem \( \Pi \) is in \( \text{NP} \) for a fixed \( k \) then it is said to be para-\( \text{NP} \)-hard.

Now, suppose that the parameter \( k \) does not depend on the sought solution size, but it is a structural parameter. Then, we say that a minimization (maximization) problem \( \Pi \) admits a \( c \)-approximation \( \text{FPT} \) (with respect to \( k \)) if it admits a \( f(k) \cdot |I|^{O(1)} \)-time algorithm that, given an instance \( (I,k) \) of \( \Pi \), outputs a solution for \( (I,k) \) that is larger (smaller) than the optimal solution for \( (I,k) \) by a factor of at most \( c \). When the parameter \( k \) does depend on the sought solution size, the notion of a \( c \)-approximation \( \text{FPT} \) algorithm is defined as well, but this definition is slightly more complicated and is not required in this paper. For more information on Parameterized Complexity, we refer the reader to recent books such as [16,20,24].

## 4 Problem definitions

In this section, we define the two key problems of interest, namely, \text{Topological Hitting Set} and \text{Boundary Nontrivialization} along with their global variants, namely, \text{Global Topological Hitting Set} and \text{Global Boundary Nontrivialization}, respectively. Also, we observe that all four problems lie in \( \text{NP} \) and in \( \text{XP} \) with respect to the solution size as the parameter.

### 4.1 Topological Hitting Set

**Problem 1 (Topological Hitting Set).**

**Instance:** Given a \( d \)-dimensional simplicial complex \( K \), a natural number \( k \), a natural number \( r < d \) and a non-bounding cycle \( \zeta \in Z_r(K) \).

**Parameter:** \( k \).

**Question:** Does there exists a set \( S \) of \( r \)-dimensional simplices with \( |S| \leq k \) such that \( S \) meets every cycle homologous to \( \zeta \)?

Let \( K_C \) denote the complex obtained from \( K \) upon removal of the set of \( r \)-simplices \( C \) along with all the cofaces of the simplices in \( C \). In particular, the homology class \([\zeta] \) does not survive in \( K_C \).

Let \( \{\alpha_i\} \) for \( i \in [\beta_r(K_C)] \) be a homology basis for \( K_C \). The inclusion map \( i : K_C \hookrightarrow K \) induces a map \( i : Z_r(K_C) \rightarrow Z_r(K) \) and also a map \( \tilde{i} : H_r(K_C) \rightarrow H_r(K) \). Let \( \tilde{\alpha}_i = i(\alpha_i) \). Let \( A \) denote the matrix with nontrivial \( r \)-cycles \( \tilde{\alpha}_i \) as its columns. Let \( M \) denote the matrix \([A | \partial_{r+1}(K)]\) and \( C(M) \) the column space of \( M \). The following lemma ensures polynomial time verification for the decision variant of \text{Topological Hitting Set}.

\begin{itemize}
  \item \textbf{Lemma 2.} \( \zeta \notin \text{column space of } M \) if and only if \( S \) meets every cycle homologous to \( \zeta \).
\end{itemize}
Proof. $(\Longrightarrow)$ Let $\rho$ be a cycle homologous to $\zeta$ such that $\mathcal{S}$ does not meet $\rho$. Then, $\rho \in C(M)$ since it survives in $K_C$. The claim follows from observing that $\zeta$ is homologous to $\rho$.

$(\Longleftarrow)$ Suppose that $\mathcal{S}$ meets every cycle that is homologous to $\zeta$. Thus, at least one simplex is removed from every cycle homologous to $\zeta$. Then, a cycle homologous to $\zeta$ (in $K$) is not present in $K_C$. The claim follows.

Lemma 2 provides an easy way to check if a set constitutes a feasible solution.

Theorem 3. Checking if a set $\mathcal{S}$ is a feasible solution to Topological Hitting Set amounts to solving a linear system of equations, and can be done in $O(n^\omega)$ time, where $\omega$ is the exponent of matrix multiplication, and $n$ is the size of the complex.

Corollary 4. Topological Hitting Set is in $NP$, and is in $XP$ with respect to the solution size $k$ as the parameter.

We now define the global variant of Topological Hitting Set.

Problem 2 (Global Topological Hitting Set).

**Instance:** Given a $d$-dimensional simplicial complex $K$, a natural number $k$, a natural number $r < d$.

**Parameter:** $k$.

**Question:** Does there exist a set $\mathcal{S}$ of $r$-dimensional simplices with $|\mathcal{S}| \leq k$ such that the induced map on homology $\tilde{i} : H_r(K_\mathcal{S}) \rightarrow H_r(K)$ is non-surjective?

For a complex $L$, let $H_r(L)$ denote an $r$-th homology basis of $L$. It is well-known that such a basis can always be computed in polynomial time.

Theorem 5. Global Topological Hitting Set is in $NP$, and in $XP$ with respect to the solution size $k$ as the parameter.

Proof. $\mathcal{S}$ is a solution for Global Topological Hitting Set if and only if one of the two conditions is satisfied:

- $\beta_r(K_\mathcal{S}) < \beta_r(K)$ or
- $\beta_r(K_\mathcal{S}) \geq \beta_r(K)$ and in the column rank profile of the matrix $[\partial_{r+1}(K) | H_r(K_\mathcal{S})]$, of the last $\beta_r(K_\mathcal{S})$ columns, exactly $\beta_r(K)$ columns are nonzero.

The two conditions can be verified in polynomial time, proving the claim.

4.2 Boundary Nontrivialization

Problem 3 (Boundary Nontrivialization).

**Instance:** Given a $d$-dimensional simplicial complex $K$, a natural number $k$, a natural number $r < d$ and a bounding cycle $\zeta \in B_r(K)$.

**Parameter:** $k$.

**Question:** Does there exist a set $\mathcal{S}$ of $r + 1$-dimensional simplices with $|\mathcal{S}| \leq k$ such that removal of $\mathcal{S}$ from the $K$ makes $\zeta$ non-bounding?

Theorem 6. Boundary Nontrivialization is in $NP$, and is in $XP$ with respect to the solution size $k$ as the parameter.

Proof. A set $\mathcal{S}$ is a solution if and only if the system of equations $\partial_{r+1}(K_\mathcal{S}) \cdot x = \zeta$ has no solution, which can be checked in polynomial time.
The global variant of **Boundary Nontrivialization** can be described as follows.

**Problem 4 (Global Boundary Nontrivialization).**

**Instance:** Given a $d$-dimensional simplicial complex $K$, a natural number $k$, a natural number $r < d$ and a bounding cycle $\zeta \in B_r(K)$.

**Parameter:** $k$.

**Question:** Does there exists a set $S$ of $r + 1$-dimensional simplices with $|S| \leq k$ such that the column space of $\partial_{r+1}(K_S)$ is a strictly smaller subspace of the column space of $\partial_{r+1}(K)$?

**Theorem 7.** **Global Boundary Nontrivialization** is in $NP$, and is in $XP$ with respect to the solution size $k$ as the parameter.

**Proof.** It is easy to check in polynomial time if the column space of $\partial_{r+1}(K_S)$ is a strictly smaller subspace of $\partial_{r+1}(K)$.

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**5 Topological Hitting Set on surfaces**

In this section we describe a polynomial time algorithm for **Topological Hitting Set** on surfaces. Let $\zeta$ be a nontrivial 1-cycle in a triangulated closed surface $K$. The algorithm for surfaces has a very simple high-level description as detailed in Algorithm 1.

**Notation 3.** Note that if we evaluate the $r$-cocycle $\eta$ at an $r$-cycle $\zeta$, then by linearity,

$$\eta(\zeta) = \eta\left(\sum_{\sigma_i \in \zeta} \sigma_i \right) = \sum_{\sigma_i \in \zeta} \eta(\sigma_i).$$

Because of $\mathbb{Z}_2$ addition, $\eta(\zeta)$ is either 0 or 1.

**Algorithm 1** The algorithm for **Topological Hitting Set** on surfaces with input cycle $\zeta$

1. Find the optimal cohomology basis of $K$ with unit weights on edges.
2. Arrange the cocycles in the basis in ascending order of weight.
3. Pick the smallest weight cocycle $\eta$ with $\eta(\zeta) = 1$.

In what follows, we will establish a series of structural results about the solution set for **Topological Hitting Set** on surfaces in order to prove the correctness of Algorithm 1.

We begin with a few definitions.

**Definition 8 (Connected cocycles).** A cocycle $\eta$ is said to be connected if it induces a connected component in the dual graph, else we say that it is disconnected.

In Lemma 11, we show that a minimal solution is, in fact, a connected cocycle. Since cocycles can be potential solutions for **Topological Hitting Set**, we make the following definitions.

**Definition 9.** We say that a cocycle $\eta$ is said to be a feasible set if every cycle $\zeta' \in [\zeta]$ meets $\eta$ in an edge. A cocycle that is not a feasible set, is said to be an infeasible set.

Next, we provide a useful characterization of cocycles that constitute feasible sets.

**Lemma 10.** If there exists a cycle in $[\zeta]$ that intersects a connected cocycle $\eta$ in an odd number of edges, then every cycle in $[\zeta]$ intersects $\eta$ in an odd number of edges.
The complexity of high-dimensional cuts

Proof. Suppose that a cycle $\gamma \in [\zeta]$ intersects $\eta$ in an odd number of edges. Let $\gamma' = \gamma + \partial \sigma$. We claim that $\gamma'$ intersects $\eta$ in an odd number of edges. We have four cases to consider.

1. The simplex boundary $\partial \sigma$ is not incident on $\eta$. Then, the homologous cycle obtained by addition of $\partial \sigma$ maintains odd incidence.
2. The simplex boundary $\partial \sigma$ intersects $\eta$ in two edges and both edges also belong to $\gamma$. Then, addition of $\partial \sigma$ to $\gamma$ reduces the number of incident edges on $\eta$ by two, and the number stays odd.
3. The simplex boundary $\partial \sigma$ intersects $\eta$ in two edges and none of the edges belong to $\gamma$. Then, addition of $\partial \sigma$ to $\gamma$ reduces the number of incident edges on $\eta$ by two, and the number stays odd.
4. The simplex boundary $\partial \sigma$ intersects $\eta$ in two edges one of which belongs to $\gamma$. Then, upon addition of $\partial \sigma$ to $\gamma$, the incident edge is exchanged with the non-incident one, and the incidence number stays the same.

Figure 6 illustrates the four cases. Any cycle in $[\zeta]$ can be obtained by adding simplex boundaries $\sum_i \partial \sigma_i$ to $\gamma$. So, applying the four cases inductively, we see that every cycle in $[\zeta]$ has odd incidence on $\eta$.

Note that any connected cocycle $\eta$ induces a cycle graph which is a subgraph of the dual graph $D_K$ of the surface $K$. We denote the cycle graph by $C_\eta$.

Lemma 11. A minimal solution set is a cocycle $\eta$ that induces a cycle subgraph $C_\eta$ in the dual graph $D_K$.

Proof. Let $e_1$ be an edge in the minimal solution set $S$. Let $\sigma$ be a 2-simplex incident on $e_1$, and let $e_2$ and $e_3$ be the other two edges incident on $\sigma$. Let $\gamma \in [\zeta]$ be a cycle with $e_1$ as the unique edge incident on $S$. We know that such a cycle exists because of minimality of $S$. Then, there exists a cycle $\gamma' = \gamma + \partial \sigma$ with $e_2$ and $e_3$ incident on it. Since $\gamma$ and $\gamma'$ differ only by a boundary $\partial \sigma$, using the fact that $e_1$ is the unique edge incident on $S$, either $e_2$ or $e_3$ must be incident on $S$. Without loss of generality, assume that $e_2$ is incident on $S$. Now, consider the 2-simplex $\tau \neq \sigma$ incident on $e_2$. Using the same argument as before, and proceeding by induction, we obtain a sequence of edges in $S$ starting from $e_1$, each connected by a 2-simplex. Then, there must exist a sequence starting at $e_1$ and ending at an edge $e'$ such that both $e'$ and $e_1$ are incident on a common cofacet $\rho \neq \tau$, for if this is not the case, then we can find a a cycle $\gamma' \in [\zeta]$ which is not incident on $S$. The sequence of edges from $e_1$ to $e'$ forms a cocycle, say $\eta$, where $\eta \subset S$.

Targeting a contradiction, assume that $\eta \neq S$. By Lemma 63, $S$ induces a connected subgraph in the Hasse graph, which implies that there exists an edge $e' \in S \setminus \eta$ and a 2-simplex $\tau$ such that $e' \prec \tau$ and the other two edges $e_1, e_2 \prec \tau$ belong to $\eta$. Let $\gamma$ be a cycle with $e_1$ as the unique edge from $S$ incident on it, and let $\gamma'$ be a cycle with $e'$ as the unique edge from $S$ incident on it. Let $B$ be the set of boundaries of 2-simplices added to $\gamma$ in order to obtain $\gamma'$ from $\gamma$. Since $\gamma$ intersects $\eta$ in an odd number of edges, using Lemma 10 any cycle homologous to $\gamma$ will also intersect $\eta$ in an odd number of edges. Hence, $\gamma'$ is incident on at least one of the edges of $\eta$. But this contradicts the existence of $\gamma'$ since by assumption, $\gamma'$ is not incident on $\eta$. Therefore, an edge $e' \in S \setminus \eta$ that shares a cofacet incident on two of the edges of $\eta$ does not exist.

Finally, using Lemma 63 from Section 7.1, for any edge $f' \in S \setminus \eta$, and a cofacet $\rho$ of $f'$, there must be a path in the dual graph from $\rho$ to a 2-simplex $\sigma$ incident on two of the edges of $\eta$. But for such a path to exist, there must exist an edge $e' \in S \setminus \eta$ and a 2-simplex $\tau$ such that $e' \prec \tau$ and the other two edges $e_1, e_2 \prec \tau$ belong to $\eta$. But we showed that such an edge
Figure 6 In this figure, we illustrate the four cases discussed in Lemma 10. Let $\zeta$ be the input cycle and $\gamma \in [\zeta]$. The dotted edges in the topmost figure belong to $\gamma$. Let $\eta$ be a cocycle that meets $\gamma$ in an odd number of edges. The edges of $\eta$ in each of the 5 figures are shown in pink. The part of $\gamma$ that does not intersect $\eta$ is shown in black and the part of $\gamma$ that intersects $\eta$ is shown in pink. As shown in the topmost figure, $\gamma$ intersects $\eta$ in three (pink-dotted) edges. In the four cases labelled (I-IV), $\gamma_i = \gamma + \partial \sigma_i$, where $i \in [4]$. As before, for every $i \in [4]$, the edges of $\gamma_i$ that intersect the edges of $\eta$ are shown as pink-dotted edges and the edges of $\gamma_i$ that do not intersect $\eta$ are shown as black-dotted edges. Note that in each of the four cases, the number of pink-dotted edges is odd.
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A

B

C

D

E
e

F

τ
e

′
e

1

G

H

I

J

K

L

M

O

f

1

P

f

2

Q

f

′

Figure 7 The cocycle \( \eta \) is shown in red. \( e_1, e_2, e', f' \) and \( \tau \) are as in Lemma 11. Note that there exists a path from \( e' \) to \( f' \) in the dual graph.

\( e' \) does not exist. So, the set \( S \setminus \eta \) is empty. Since \( \eta = S \), the claim follows. Please refer to Figure 7 for an example. Note that the final part of the argument is specific to surfaces. ◁

Lemma 12. A trivial cocycle is not a minimal solution set.

Proof. Let \( \eta \) be a trivial cocycle. Then, \( \eta = \delta(S) \), where \( S \) is a collection of points. Let \( e \) be an edge of \( \eta \). By the assumption on minimality of the solution set, there exists a cycle \( \zeta' \in [\zeta] \) such that \( e \) is the only edge of \( \eta \) incident on \( \zeta' \). One of the vertices of \( e \), say \( v_1 \), belongs to \( S \). We write the cycle \( \zeta' \) as a sequence of vertices \( v_1, v_2, \ldots, v_q = v_1 \), for some \( q \), such that an edge connects subsequent vertices, the sequence starts and ends at \( v_1 \), and the edge \( \{v_{q-1}, v_1\} = e \). Then, the path from \( v_1 \) to \( v_{q-1} \) must pass through a vertex \( v' \) such that \( v' \in S \). But this is only possible if \( \zeta' \) also contains an edge of \( \eta \) other than \( e \), which in turn, contradicts the minimality of the solution set. Hence, a trivial cocycle is not a minimal solution set. See Figure 8 for an example. ◁

Lemmas 11 and 12 combine to give the following theorem.

Theorem 13. A minimal solution set is a nontrivial cocycle.

Lemma 14. If a connected cocycle \( \eta \) intersects a cycle \( \zeta_0 \in [\zeta] \) in \( m \) edges, then there exists another cycle \( \gamma \in [\zeta] \) such that \( \gamma \) also intersects \( \eta \) in \( m \) edges, and the intersection of \( \gamma \) and \( \eta \) induces a connected component in the dual graph.

Proof. To begin with, note that the intersection of \( \zeta_0 \in [\zeta] \) with \( \eta \) induces a (possibly disconnected) subgraph of the cycle graph \( C_{\eta} \), which we denote by \( C_{\eta}^{\zeta_0} \). Both \( C_{\eta} \) and \( C_{\eta}^{\zeta_0} \) are subgraphs of the dual graph \( D_K \). Let \( C_1, \ldots, C_k \) be the \( k \) connected components of \( C_{\eta}^{\zeta_0} \). If \( k = 1 \), then the lemma is already satisfied. So without loss of generality, assume \( k > 1 \). We say that a component \( C_i \) is a neighbor of a component \( C_j \) if there exists a vertex in \( C_i \) that has a path to a vertex in \( C_j \) that does not intersect the edges of \( C_{\eta}^{\zeta_0} \). It is easy to check that every component of \( C_{\eta}^{\zeta_0} \) has exactly two (possibly non-distinct) neighbors. Choose any two neighboring components \( C_i \) and \( C_j \) of \( C_{\eta}^{\zeta_0} \). Let \( v \in C_i \) and \( v' \in C_j \) be two vertices that have a simple path \( P \) with vertices \( v = u_1, u_2, \ldots, u_\ell = v' \) such that \( P \) does not intersect the
The edges in purple and red together form a trivial cocycle $\eta$ given by $\delta(A + C + E + O + Q + G + M + K + I)$. If $\eta$ forms a minimal solution set, then it intersects a cycle $\zeta' \in [\zeta]$ in a unique edge, say, edge DC. But any path that passes through DC must pass through another edge of $\eta$. In this example, DC-CQ-QM-MN is one such path.

edges of $C_\eta$. Every vertex $u_t$ corresponds to a simplex $\sigma_t$ in $K$. Now, adding the simplex boundaries $\sum_{t=2}^k \partial \sigma_t$ to $\zeta_0$ gives rise to a cycle homologous to $\zeta_0$ such that $C_i$ has one vertex (and one edge) more and $C_j$ has one vertex (and one edge) less. We repeat this process inductively until all edges are ‘transported’ from $C_j$ to $C_i$ and $C_j$ becomes empty. That is, the new cycle $\zeta_1$ we obtain is homologous to $\zeta_0$ and has $k - 1$ components. We denote the subgraph induced by intersection of $\zeta_1$ and $\eta$ by $C_{\zeta_1}^\eta$.

We apply the same procedure to $C_{\zeta_1}^\eta$ as above and get a a cycle $\zeta_2 \in [\zeta]$ whose induced subgraph $C_{\zeta_2}^\eta$ has $k - 2$ components. Proceeding inductively, we finally obtain a cycle $\gamma = \zeta_{k-1} \in [\zeta]$ whose induced subgraph $C_{\gamma}^\eta$ is a connected subgraph of the dual graph. Moreover, by design, the total number of edges in every induced graph $C_{\zeta_i}^\eta$ for $i \in [0, k-1]$ is $m$.

Lemma 15. The following are equivalent.

(a.) A connected cocycle $\eta$ is a feasible set for the input cycle $\zeta$.
(b.) Every cycle in $[\zeta]$ intersects a connected cocycle $\eta$ in an odd number of edges.
(c.) There exists a cycle in $[\zeta]$ that intersects a connected cocycle $\eta$ in an odd number of edges.

Proof. (a.) $\implies$ (b.) Assume that there exists a cycle $\xi \in [\zeta]$ that intersects $\eta$ in an even number of edges. The intersection of $\xi$ with $\eta$ induces a (possibly disconnected) subgraph of $C_\eta$, which we denote by $C_\eta^\xi$. Using Lemma 14, there exists another cycle $\gamma$ homologous to $\xi$ such that $\gamma$ intersects $\eta$ in the same number of edges, and the intersection of $\gamma$ and $\eta$ induces a connected graph in the dual graph, which we denote by $C_\gamma^\eta$. There are two cases:
Figure 9. In this figure, we provide an illustrative example of the cycle modification technique from Lemma 14. As in Figure 6, the edges of a cycle that intersect the edges of the cocycle $\eta$ are shown as pink-dotted edges and the edges of a cycle that do not intersect $\eta$ are shown as black-dotted edges. The cocycle $\eta$ is shown in pink. The intersection of $\zeta_0$ and $\eta$ induces a disconnected subgraph $C_{\zeta_0 \eta}$ of the dual graph with several components, only three of which are shown, namely, $C_1$, $C_2$ and $C_3$. Here, $C_1$ and $C_2$ are neighbors, and $C_2$ and $C_3$ are neighbors. The path from $C_1$ to $C_2$ in $D_K$ consists of simplices $\sigma_1, \sigma_2, \sigma_3$ (which are vertices in the dual graph). So we add $\partial \sigma_2 + \partial \sigma_3$ to $\zeta_0$ to obtain $\zeta'$. Next, we add $\partial \sigma_3 + \partial \sigma_4$ to $\zeta'$ to obtain $\zeta''$. Finally, we add $\partial \sigma_4 + \partial \sigma_5$ to $\zeta''$ to obtain $\zeta_1$. The number of connected components of $C_{\zeta_0}$ is one less than the number of connected components of $C_{\zeta_1}$. This was achieved by ‘transporting’ edges in $C_2$ to $C_1$. 
Figure 10: In this figure, we provide an illustrative example of the cycle modification scheme in ((a.) \implies (b.)) from Lemma 15. As in Figures 6 and 9, the edges of a cycle that intersect the edges of the cocycle $\eta$ are shown as pink-dotted edges and the edges of a cycle that do not intersect $\eta$ are shown as black-dotted edges. The cocycle $\eta$ is shown in pink. The intersection of $\gamma$ and $\eta$ induces a connected graph in the dual graph, namely $C_{\gamma\eta}$. The number of edges in $C_{\gamma\eta}$ is even. Let $\gamma' = \gamma + \partial \sigma_2 + \partial \sigma_4 + \partial \sigma_6$. The cycle $\gamma'$ does not intersect $\eta$.

Case 1: $C_{\gamma\eta}$ is not identical to $C_{\eta}$ (the cycle graph induced by the entire cocycle $\eta$), and $C_{\gamma\eta}$ is a path graph $P$ with vertices $u_1, u_2, \ldots, u_{m+1}$, where each vertex $u_i$ is distinct and corresponds to a simplex $\sigma_i$ in $K$.

Case 2: $C_{\gamma\eta}$ is the same as the entire cycle graph $C_{\eta}$ with vertices $u_1, u_2, \ldots, u_m, u_{m+1}$, where $u_{m+1} = u_1$.

In either case, upon adding the simplex boundaries $\sum_{i=1}^{m/2} \partial \sigma_{2i}$ to $\gamma$, we obtain a cycle that has an empty intersection with $\eta$. That is, there exists a cycle in $[\zeta]$ which does not meet $\eta$ in any of its edges. In other words, $\eta$ is an infeasible set.

(b.) $\implies$ (a.) This is true by the definition of a feasible set.
(c.) $\implies$ (b.) This is the content of Lemma 10.
(b.) $\implies$ (c.) This is trivially true.

This completes the proof.

Using Notation 3, Lemma 15 can be written as follows.

Lemma 16. The following are equivalent.
(a.) A connected cocycle $\eta$ is a feasible set for the input cycle $\zeta$.
(b.) For a connected cocycle $\eta$, and any cycle $\zeta' \in [\zeta]$, $\eta(\zeta') = 1$.
(c.) For a connected cocycle $\eta$, there exists a cycle $\zeta' \in [\zeta]$ such that $\eta(\zeta') = 1$. 

\[\Box\]
Lemma 17. A connected cocycle \( \eta \) is a feasible set if and only if a connected cocycle cohomologous to it is a feasible set.

Proof. A cocycle \( \eta' \) cohomologous to \( \eta \) can be written as \( \eta' = \eta + \delta(S) \), where \( S \) is a collection of vertices. Then, by linearity,

\[
\eta'(\zeta) = \eta(\zeta) + \delta(S)(\zeta) = \eta(\zeta) + \sum_{v \in S} \delta(v)(\zeta).
\]

\( \delta(v) \) is a connected trivial cocycle. So, using Lemma 12 and \( \neg((c. \implies (a.)) \) in Lemma 16, \( \delta(v)(\zeta) \) is 0 for every \( v \in S \). Hence, \( \eta'(\zeta) = 1 \) if and only if \( \eta(\zeta) = 1 \). So, the claim follows from \( ((c. \implies (a.)) \) in Lemma 16.

Next, we prove an important generalization of Lemma 17.

Lemma 18. Let \( k > 1 \) be an integer. Let \( \eta_i \) for \( i \in [k] \) be connected cocycles. On the one hand, if \( \eta_i \) for \( i \in [k] \) are infeasible sets for the input cycle \( \zeta \), then any cocycle \( \vartheta \) cohomologous to \( \sum_{i=1}^{k} \eta_i \) is an infeasible set. On the other hand, if \( \eta_i \) is a feasible set, and \( \eta_i \) for \( i \in [k-1] \) are infeasible sets for the input cycle \( \zeta \), then any cocycle \( \vartheta \) cohomologous to \( \sum_{i=1}^{k} \eta_i \) is a feasible set.

Proof. A cocycle \( \vartheta \) cohomologous to \( \sum_{i=1}^{k} \eta_i \) can be written as \( \sum_{i=1}^{k} \eta_i + \delta(S) \) where \( S \) is a collection of vertices. Then, by linearity,

\[
\vartheta(\zeta) = \sum_{i=1}^{k} \eta_i(\zeta) + \delta(S)(\zeta) = \sum_{i=1}^{k} \eta_i(\zeta) + \sum_{v \in S} \delta(v)(\zeta).
\]

Using Lemmas 12 and 16, \( \delta(v)(\zeta) = 0 \) for every \( v \in S \), and \( \eta_i(\zeta) = 0 \) for \( i \in [k-1] \). Hence, \( \vartheta(\zeta) = 1 \) if and only if \( \eta_k(\zeta) = 1 \). So, the claim follows from Lemma 16.

Remark 19 (Computing optimal (co)homology basis for surfaces). For simplicial complexes with \( n \) vertices, \( m \) edges and \( N \) simplices in total, we recall some of the known results from literature. For the special case when the input complex is an surface, Erickson and Whittlesey \[22\] devised a \( O(N^2 \log N + gN^2 + g^2N) \)-time algorithm for computing an optimal homology basis. Borradale et al. \[2\] improved on this result by providing a \( O((h + c)^3 n \log n + n^3) \)-time algorithm for the same problem. Here \( c \) denotes the number of boundary components, and \( h \) denotes the genus of the surface. Dlotko \[2\] generalized the algorithm from \[22\] for computing an optimal cohomology basis for surfaces. For general complexes, Dey et al. \[17\] and \[18\], Chen and Freedman \[12\], Busaryev et al. \[7\], and Rathod \[34\] provided progressively faster algorithms for computing an optimal homology basis.

Although we expect this to be fairly well known, for the sake of completeness, we describe an algorithm for computing minimum cohomology basis of a triangulated surface that uses the minimum homology basis algorithm as a subroutine.

Lemma 20. The minimum cohomology basis problem on surfaces can be solved in the same time as the minimum homology basis problem on surfaces.

Proof. Let \( K \) be a surface with a weight function \( w \) on its edges. Let \( \hat{K} \) be the dual cell complex of \( K \). Then, to every edge \( e \) of \( \hat{K} \) there is a unique corresponding edge \( \hat{e} \) in \( \hat{K} \). We now define a weight function on the edges of \( \hat{K} \) in the obvious way: \( w(\hat{e}) = w(e) \). Let \( \hat{K}' \) be
the simplicial complex obtained from the stellar subdivision of each of the 2-cells of $\hat{K}$. The weight function on edges of $\hat{K}$ is extended to a weight function on edges of $K'$ by assigning weight $\infty$ to every newly added edge during the stellar subdivision. Such a complex $K'$ can be computed in linear time. It is easy to check that the cocycles of $K$ are in one-to-one correspondence with the cycles of $\hat{K}$, and the cycles of $\hat{K}$ are in one-to-one correspondence with finite weight cycles of $K'$. Moreover, if $\eta$ is a cocycle of $K$, and if $\hat{\eta}$ and $\hat{\eta}'$ are the corresponding cycles in $\hat{K}$ and $K'$, respectively, then $w(\eta) = w(\hat{\eta}) = w(\hat{\eta}')$. Hence, computing a minimum homology basis for $K'$ gives a minimum cohomology basis for $K$. ◀

Theorem 21. Algorithm [4] provides a polynomial time algorithm for computing an optimal solution for Topological Hitting Set on surfaces.

Proof. Let $\{\nu_i \mid i \in [m]\}$ be an optimal cohomology basis for $K$. Then, by Theorem [13] any optimal solution set is a cocycle. So, we can let $k$ be the smallest integer for which a cocycle cohomologous to some cocycle in the span of $\{\nu_i \mid i \in [k]\}$ is a feasible solution set. Because the algorithm confirms that each $\nu_i, i \in [k-1]$ is an infeasible set, by Lemma [18] any connected cocycle cohomologous to $\sum_{i=1}^{k-1} \nu_i$ is an infeasible set. On the other hand, since there exists a feasible set $\theta = \sum_{j \in [k-1]} \nu_i + \nu_k + \beta$ where $j_i \in [k-1]$, and $\beta$ is a coboundary, by Lemma [18] $\nu_k = \sum_{j \in [k-1]} \nu_i + \beta + \theta$ is also a feasible set. Because $\{\nu_i \mid i \in [m]\}$ is an optimal cohomology basis, $\nu_k$ is, in fact, a minimal solution set.

From Lemma [20] we know that Step-1 of Algorithm [4] can be computed in polynomial time. Step-2 can be implemented by a simple sorting algorithm. Finally, Step-3 can be executed in linear time. ◀

Remark 22. The algorithmic results in this section motivate several questions: To what extent can this machinery be extended from surfaces to general complexes?

1. Are the optimal solutions sets for Topological Hitting Set nontrivial cocycles for general complexes? To the best of our knowledge, this question is open.
2. Can the optimal solution sets for Topological Hitting Set be computed efficiently for general complexes? We answer this question in the negative in Section 6.1 by showing that for general complexes Topological Hitting Set is $NP$-hard and $W[1]$-hard. Intriguingly, for the gadgets used in the reduction the optimal solution sets for Topological Hitting Set are cocycles! So they do not provide (a family of) counterexamples for the first question.
3. We believe that it should be possible to dualize the hardness results of Chen and Freedman [13] to show that computing an optimal cohomology basis for general complexes is $NP$-hard. So, in general, knowing that the optimal solutions sets are cocycles is not enough to guarantee tractability. One also needs an efficient algorithm for computing an optimal cohomology basis.

6 $W[1]$-hardness results

In this section, we obtain $W[1]$-hardness results for Topological Hitting Set and Boundary Nontrivialization with respect to the solution size $k$ as the parameter via parameterized reductions from $k$-MULTICOLORED CLIQUE. We begin this section by recalling some common notions from graph theory.

A $k$-clique in a graph $G$ is a complete subgraph of $G$ with $k$ vertices. Next, a $k$-coloring of a graph $G$ is an assignment of one of $k$ possible colors to every vertex of $G$ (that is, a
vertex coloring) such that no two vertices that share an edge receive the same color. A graph \( G \) equipped with a \( k \)-coloring is called a \( k \)-colored graph. Then, a multicolored \( k \)-clique in a colored graph is a \( k \)-clique with a \( k \)-coloring. \( k \)-MULTICOLORED CLIQUE asks for the existence of a multicolored \( k \)-clique in a \( k \)-colored graph \( G \). We remark that reducing from \( k \)-MULTICOLORED CLIQUE is a highly effective tool for showing \( \mathsf{W[1]} \)-hardness \cite{1}. Formally, \( k \)-MULTICOLORED CLIQUE is defined as follows:

**Problem 5 (\( k \)-MULTICOLORED CLIQUE).**

**Instance:** Given a graph \( G = (V, E) \), and a vertex coloring \( c : V \rightarrow [k] \).

**Parameter:** \( k \).

**Question:** Does there exist a multicolored \( k \)-clique \( H \) in \( G \)?

▶ **Theorem 23** (Fellows et al. \cite{1}). \( k \)-MULTICOLORED CLIQUE is \( \mathsf{W[1]} \)-complete.

### 6.1 \( \mathsf{W[1]} \)-hardness for Topological Hitting Set

For \( i \in [k] \), the subset of vertices of color \( i \) is denoted by \( V_i \). Clearly, the vertex coloring \( c \) induces a partition on \( V \):

\[
V = \bigcup_{i=1}^{k} V_i, \quad \text{and} \quad V_i \cap V_j = \emptyset \text{ for all } i, j \in [k].
\]

We now provide a parameterized reduction from \( k \)-MULTICOLORED CLIQUE to TOPOLOGICAL HITTING SET. For \( r = |V| - 1 \), we define an \((r + 1)\)-dimensional complex \( K(G) \) associated to the given colored graph \( G \) as follows.

**Vertices.**

The set of vertices of \( K(G) \) contains the disjoint union of the vertices \( V \) in the graph \( G \), the set of colors \([k]\), and an additional dummy vertex \( d \). Altogether, we have \( r + k + 2 \) vertices in \( K(G) \) so far. In what follows, further vertices are added to \( K(G) \).

**Simplices.**

Below, we describe the simplices that constitute the complex \( K(G) \).

**The cycle \( \zeta \).** First, add the \( r \)-simplex \( V \) corresponding to vertex set \( V \) of the graph \( G \). Next, add the \( r \)-simplices \((V \setminus \{u\}) \cup \{d\}\) for every \( u \in V \). The collection of these \( r + 2 \) simplices of dimension \( r \) forms a nontrivial \( r \)-cycle \( \zeta \).

**The simplices in \( X_1 \).** \( X_1 = \{\sigma_i | i \in [k]\} \).

- For every color \( i \in [k] \),
  - add an \((r + 1)\)-simplex \( \sigma_i = V \cup \{i\} \).

▶ **Definition 24** (Admissible and undesirable facets of \( \sigma_i \)). A facet \((V \setminus \{v\}) \cup \{i\}\) of \( \sigma_i \) is said to be undesirable if and only if \( v \notin V_i \). All other facets of \( \sigma_i \) are deemed admissible. In particular, \( V \) is admissible.
Figure 11 The figure shows some of the attachments in complex $K(G)$. In particular, $\alpha_i^v$ is the common face of $\tau_{v,i,j}$ and $\sigma_i$, $\alpha_j^u$ is the common face of $\tau_{v,j,i}$ and $\sigma_j$, and $\beta_{v,u}^{v,u}$ is the common face of $\tau_{v,i,j}$ and $\tau_{u,j,i}$. The dashed lines indicate identifications along facets. The set of $r$-simplices supported by the vertices $V \cup \{d\}$ forms a nontrivial $r$-cycle in $K(G)$. 
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The idea here is that including an admissible simplex of the form \((V \setminus \{v\}) \cup \{i\}\) in \(\mathcal{S}\) is akin to picking the vertex \(v\) of color \(i\) for constructing the colorful clique. Including undesirable simplices in the solution will be made prohibitively expensive as the coloring specified by undesirable simplices is incompatible with the coloring \(c\) that the graph \(G\) comes equipped with.

**The simplices in \(\mathcal{X}_2\).** \(\mathcal{X}_2 = \{\tau^r_{i,j} \mid i \in [k], \ v \in V_i, \ j \in [k] \setminus \{i\}\}.\)

- For every color \(i \in [k],\)
- for every vertex \(v\) in \(V_i\) and every color \(j \in [k] \setminus \{i\},\)
- add an \((r + 1)\)-simplex \(\tau^r_{i,j} = (V \setminus \{v\}) \cup \{i, j\}\).

**Definition 25 (Admissible and undesirable facets of \(\tau^r_{i,j}\)).** The admissible facets of \(\tau^r_{i,j}\) are:
- \((V \setminus \{v, u\}) \cup \{i, j\}\) with \(u \in V_j\) and \(\{u, v\} \in E,\)
- \((V \setminus \{v\}) \cup \{i\},\)
A facet of \(\tau^r_{i,j}\) that is not admissible is undesirable.

The intuition here is that picking an admissible facet of the form \((V \setminus \{v, u\}) \cup \{i, j\}\) is akin to picking the edge \(\{u, v\}\) of color \(\{i, j\}\) for constructing the colorful clique, whereas the admissible facet \((V \setminus \{v\}) \cup \{i\}\) is common with \(\sigma_i\). Including undesirable simplices in the solution will be made prohibitively expensive (as explained later). Undesirable simplices of \(\tau^r_{i,j}\) correspond either to coloring that is incompatible with \(c\) or with edges that are not even present in \(E\).

**Undesirable and inadmissible simplices.** The undesirability of certain \(r\)-simplices is implemented in the gadget as follows: Let \(m = n^3\). Then, to every undesirable \(r\)-simplex \(\omega = \{v_1, v_2, \ldots, v_{r+1}\}\), associate \(m\) new vertices \(\mathcal{U}^\omega = \{u^1_i, u^2_i, \ldots, u^m_i\}\). Now introduce \(m\) new \(r + 1\)-simplices

\[\Psi^\omega = \{\mu_i(\omega) = \{v_1, v_2, \ldots, v_{r+1}, u^r_i \mid i \in [m]\}\}

that are cofacets of \(\omega\). See Figure 12 for an illustrative example.

**Definition 26 (Set of inadmissible simplices associated to an undesirable simplex \(\omega\)).** The set of \(r\)-simplices in \(\{\{\text{facets of } \mu_i(\omega) \mid i \in [m]\}\}\) is denoted by \(\langle \omega \rangle\). The simplices in the set \(\langle \omega \rangle\) are said to be inadmissible. In particular, \(\omega\) itself is inadmissible.

Further, note that the set of vertices in \(\mathcal{U}^\omega\) and \(r\)-simplices in \(\Psi^\omega\) are unique to \(\omega\). As we observe later, introducing these new simplices makes inclusion of \(\omega\) in the solution set prohibitively expensive. Denote by \(\mathcal{Y}\) the set of all \(r + 1\)-simplices added in this step.

This completes the construction of complex \(K(G)\). It is easy to check that the inadmissible and admissible simplices of \(K(G)\) partition the set of \(r\)-simplices of \(K(G)\).

**Notation 4.** The admissible facets \((V \setminus \{v\}) \cup \{i\}\) and \((V \setminus \{v, u\}) \cup \{i, j\}\) are denoted by \(\alpha^r_{i,j}\) and \(\beta^r_{i,j}\), respectively. For every vertex \(v \in V\) of color \(i\), there is a facet \(\alpha^r_i\). For every edge \(\{u, v\} \in E\), there is a facet \(\beta^r_{i,j}\), where \(i\) is the color of \(v\) and \(j\) is the color of \(u\).

**Remark 27 (Meaning of superscripts and subscripts of simplices).** A simple mnemonic for remembering the meaning of the notation for simplices is as follows: the indices in the subscript are the included colors, and the vertices in the superscript indicate the vertices
For every undesirable simplex $\omega = \{v_1, v_2, \ldots, v_{r+1}\}$ $m$ new vertices $U^\omega = \{u^\omega_1, u^\omega_2, \ldots, u^\omega_m\}$ are added to $K(G)$. Moreover, $m$ new $r+1$-simplices $\Upsilon^\omega = \{\mu_\omega(i) = \{v_1, v_2, \ldots, v_{r+1}, u^\omega_i\} | i \in [m]\}$, where $\omega \prec \mu_\omega(i)$ for every $i \in [m]$ are also added to $K(G)$. The facets of $\mu_\omega(i)$ for every $i \in [m]$ are the inadmissible simplices associated to $\omega$ and denoted by $[\omega]$.

excluded from $V$. For instance, $\beta^{v,u}_{i,j}$ is the full simplex on the vertex set $(V \setminus \{v, u\}) \cup \{i, j\}$. In this case, colors $i$ and $j$ are included and vertices $u$ and $v$ are excluded. The same notational rule applies for $\alpha^v_i$, $\sigma_i$ and $\tau^{v}_{i,j}$.

**Remark 28 (Correspondence between colors and vertices in $\alpha^v_i$, $\beta^{v,u}_{i,j}$ and $\tau^{v}_{i,j}$).** In our notation, the first color corresponds to the first vertex, the second color to the second vertex, and so on. For instance,

- In $\alpha^v_i$, vertex $v$ is of color $i$.
- In $\beta^{v,u}_{i,j}$, $v$ is of color $i$ and $u$ is of color $j$.
- In $\tau^{v}_{i,j}$, $v$ is of color $i$ and the vertex associated to color $j$ is not specified. It is, in fact, chosen through a facet $\beta^{v,u}_{i,j} \prec \tau^{v}_{i,j}$.

### Choice of parameter.

Let $(k+\binom{k}{2}+1 = \binom{k+1}{2}+1)$ be the parameter for TOPOLOGICAL HITTING SET on the complex $K(G)$.

**Remark 29 (Size of $K(G)$).** We note that every subset of vertices of $G$ is a simplex in $K(G)$. However, $K(G)$ is represented implicitly, and the simplices of dimensions other than $r$ and $r + 1$ are not used in the reduction. Thus, although $K(G)$ as a simplicial complex is exponential in the size of $G$, the reduction itself is polynomial in the size of $G$ because the number of $r$ and $r + 1$ dimensional simplices of $K(G)$ are polynomial in size of $G$, even after inadmissible simplices.

**Lemma 30.** If there exists a multicolored $k$-clique $H = (V_H, E_H)$ of $G$, then there exists a topological hitting set $S$ for $\zeta$ consisting of $\binom{k+1}{2}+1$ $r$-simplices.

**Proof.** We construct a set $S$ of $r$-simplices that mimics the graphical structure of $H$ as follows:

$$S_\alpha = \{ \alpha^v_i | v \in V_i \cap V_H \}$$
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\[ S_B = \{ \beta_{ij}^{v,u} \mid v \in V_i, u \in V_j, \{i, j\} \in E_H \} \]

First, set \( S = \{ V \} \cup S_a \cup S_b \). Next, note that every cycle \( \zeta' \in [\zeta] \) can be expressed as

\[ \zeta' = \zeta + \sum_{\mu_i \in \chi'} \partial \mu_i + \sum_{\mu_j \in \chi'} \partial \mu_j \]

for some \( \chi' \subset \chi_1 \cup \chi_2 \) and \( \chi' \subset \chi' \). Let \( \chi'_1 = \chi' \cap \chi_1 \), and \( \chi'_2 = \chi' \cap \chi_2 \). Now, we claim that removing \( S \) from \( K(G) \) destroys every cycle \( \zeta' \in [\zeta] \). We show this by establishing that the coefficient in every \( \zeta' \in [\zeta] \) of at least one of the simplices of \( S \) is 1. In other words, \( S \cap \zeta \neq \emptyset \) for every \( \zeta' \in [\zeta] \).

**Case 1:** \( \chi'_1 \neq \emptyset \). Then, \( V \in S \) has coefficient 1 in cycle \( \zeta' \). This is because simplices in \( \chi' \) are not incident on \( V \), and \( V \in \zeta \).

**Case 2:** \( \chi'_1 \neq \emptyset \), \( \chi'_2 = \emptyset \). Then, the cycle \( \zeta' \) can be written as

\[ \zeta' = \zeta + \sum_{\sigma_j \in \chi'_1} \partial \sigma_j + \sum_{\mu_j \in \chi'} \partial \mu_j \]

Then, every \( \alpha_j^v \in S \) for \( \sigma_j \in \chi'_1 \) and \( v \in V_H \cap V_j \) has coefficient 1 in cycle \( \zeta' \). This is because \( \alpha_j^v \in \partial \sigma_j \) for every \( \sigma_j \in \chi'_1 \), but \( \alpha_j^v \notin \zeta \) and \( \alpha_j^v \notin \partial \mu_j \) for any \( \mu_j \in \chi' \).

**Case 3:** \( \chi'_1 \neq \emptyset \), \( \chi'_2 \neq \emptyset \). This case is identical to Case 1, because \( V \in S \) has coefficient 1 in cycle \( \zeta' \).

**Case 4:** \( \chi'_1 \neq \emptyset \), \( \chi'_2 \neq \emptyset \). If every simplex \( \tau_{p,q}^{v,u} \in \chi'_2 \) is such that \( v \in V_p \setminus V_H \), then this case becomes identical to Case 2. So we will assume without loss of generality that the set \( \chi'_2 = \{ \tau_{p,q}^{v,u} \mid p, q \in [k], v \in V_p \cap V_H, \tau_{p,q}^{v,u} \in \chi'_2 \} \) is empty. For some \( \{u, v\} \in E_H \) and \( u \in V_H \), if \( \tau_{p,q}^{v,u} \in \chi'_2 \) and \( \tau_{u,v}^{p,q} \notin \chi'_2 \), then the coefficient of \( \beta_{p,q}^{v,u} \in S \) is \( \zeta' \) because the only two \( (r+1) \)-simplices incident on \( \beta_{p,q}^{v,u} \) are \( \tau_{p,q}^{v,u} \) and \( \tau_{u,v}^{p,q} \). So, without loss of generality assume that the symmetric simplex \( \tau_{u,v}^{p,q} \) is also in \( \chi'_2 \). In other words, \( |\chi'_2| \) is even. Note that for every \( \tau_{p,q}^{v,u} \in \chi'_2 \), exactly one facet of \( \tau_{p,q}^{v,u} \) lies in \( S_a \), namely \( \alpha_j^v \).

Hence the cardinality of the multiset \( T = \{ \partial \tau_{p,q}^{v,u} \cap S_a \mid \tau_{p,q}^{v,u} \in \chi'_2 \} \) is even. Let \( \sigma_i \in \mathcal{E} \) if and only if the cardinality of the set \( \{ \tau_{i,j}^{v,u} \in \chi'_2 \mid \alpha_i^v \in \partial \tau_{i,j}^{v,u} \cap S_a \} \) is even, and \( \sigma_i \in \mathcal{O} \) if and only if the cardinality of the set \( \{ \tau_{i,j}^{v,u} \in \chi'_2 \mid \alpha_i^v \in \partial \tau_{i,j}^{v,u} \cap S_a \} \) is odd. It is easy to check that \( \chi'_1 \subseteq \mathcal{O} \cup \mathcal{E} \). Note that since \( |\mathcal{T}| = |\mathcal{O}| + |\mathcal{E}|, |\mathcal{O}| \) must be even.

Now, if \( \sigma_i \in \mathcal{E} \cap \chi'_1 \), then the coefficient of \( \alpha_i^v \in \mathcal{S} \) in \( \zeta' \) is 1 because the only \( (r+1) \)-simplices incident on \( \alpha_i^v \) are \( \{ \tau_{i,j}^{v,u} \in \chi'_2 \mid \alpha_i^v \in \partial \tau_{i,j}^{v,u} \cap S_a \} \) and \( \{ \tau_{i,j}^{v,u} \in \chi'_2 \mid \alpha_i^v \in \partial \tau_{i,j}^{v,u} \cap \mathcal{O} \} \) is even when \( \sigma_i \in \mathcal{E} \). So, without loss of generality assume that \( \mathcal{E} \cap \chi'_1 \) is empty. That is, we assume that \( \chi'_1 \subseteq \mathcal{O} \). But if \( \sigma \in \mathcal{O} \setminus \chi'_1 \), then the coefficient of \( \alpha_i^v \in \mathcal{S} \) in \( \zeta' \) is 1 because in that case the only \( (r+1) \)-simplices incident with \( \alpha_i^v \) will be \( \{ \tau_{i,j}^{v,u} \in \chi'_2 \mid \alpha_i^v \in \partial \tau_{i,j}^{v,u} \} \) which has odd cardinality. So, we assume that \( \mathcal{O} = \chi'_1 \). But if \( \mathcal{O} = \chi'_1 \), then \( V \in \mathcal{S} \) has coefficient 1 in \( \zeta' \) because \( \mathcal{O} \) is even and \( V \in \zeta \). This completes the proof. Please see Figure [13] for the final part of the argument.

The next few lemmas provide a method to extract a multi-colored \( k \)-clique from \( G \) given a solution set \( \mathcal{R} \) for TOPOLOGICAL HITTING SET on \( K(G) \).

**Lemma 31.** If there exists a cycle \( \zeta' \in [\zeta] \) such that only the inadmissible simplices of \( \mathcal{R} \) have coefficient 1 in \( \zeta' \), then the size of \( \mathcal{R} \) is at least \( m \).

**Proof.** We consider two cases.
Like in Figure 13, the dashed lines indicate identifications along facets. Additionally, in this figure, $\tau_{v,i,j}$ and $\tau_{u,j,i}$ belong to $X''_2$, $\sigma_i$ and $\sigma_j$ belong to $X'_1$, and $\alpha_{v,i}$ and $\alpha_{u,j}$ belong to $S_\alpha$. $T$ accounts for all the incidences of the boundaries of simplices in $X''_2$ on simplices in $S_\alpha$. The final part of the argument in Case 4 of Lemma 30 is depicted here. In Case 4 of Lemma 30, we use the fact that $|T|$ is even. Also, if $|T|$ is even and if the coefficient of all the simplices of $S \setminus \{V\}$ have coefficient 0 in some cycle $\zeta' \in \zeta$, then $X' = O$, and $O$ has even cardinality. But if this is so, then the coefficient of $V$ in $\zeta'$ is $(1 + |O|) \mod 2 = 1$. 
Case 1: A cycle $\zeta'$ has a unique inadmissible simplex $\nu$ with coefficient 1 in $R$.

Suppose that $\nu = \{v_1, v_2, \ldots, v_{r+1}\}$ is, in fact, an undesirable simplex. Assume that $\nu$ is the unique simplex in $R$ with coefficient 1 in $\zeta'$. Let $\mu_i = \{v_1, v_2, \ldots, v_{r+1}, u_i\}$, $i \in [m]$ be the inadmissible simplices in $[\nu]$. Then, a simplex in $R$ will have coefficient 1 in the cycle $\zeta'_i = \zeta' + \partial \mu_i$ only if one of the simplices in $\partial \mu_i \setminus \{\nu\}$ for every $i \in [m]$ belongs to $R$. Since $\partial \mu_i \setminus \{\nu\}$ for $i \in [m]$ are disjoint sets, the size of $R$ is at least $m$.

Next, suppose $\nu = \{v_1, v_2, \ldots, v_{r+1}\} \in [\omega]$, where $\omega$ is an undesirable simplex, and $\nu \neq \omega$. Then, $\omega$ is a facet of $\mu_j(\omega)$ for some $\mu_j(\omega) = \{v_1, v_2, \ldots, v_{r+1}, u_j\}$, $j \in [m]$. Since $\zeta' \in [\zeta]$ is a cycle, all simplices in $\partial \mu_j(\omega)$ must have coefficient 1 in $\zeta'$. Then, a simplex in $R$ will have coefficient 1 in each of the cycles $\zeta'_i = \zeta' + \partial \mu_i(\omega) + \partial \mu_j(\omega)$, $i \in [m] \setminus \{j\}$ only if one of the simplices in $\partial \mu_i(\omega) \setminus \{\nu\}$ for each $i$ belongs to $R$, where $\mu_i(\omega) = \{v_1, v_2, \ldots, v_{r+1}, u_i\}$.

Also, the cycle $\zeta'' = \zeta + \partial \mu_j(\omega)$ has a simplex in $R$ with coefficient 1 only if $\omega \in R$. Hence, in both cases, the size of $R$ is at least $m$.

Case 2: A cycle $\zeta'$ has multiple inadmissible simplices with coefficient 1 in $R$.

More generally, suppose there exist more than one inadmissible simplices in $R$ with coefficient 1 in $\zeta'$, for some cycle $\zeta' \in [\zeta]$. For an undesirable simplex $\omega$, we say that $[\omega]$ belongs to $\zeta'$ if it there exists a simplex in $[\omega]$ that has coefficient 1 in $\zeta'$. Let $J$ be an indexing set for the undesirable simplices of $K(G)$ whose classes belong to $\zeta'$. That is, for all $j \in J$, $[\omega^j]$ belongs to $\zeta'$. Define the sets $P$ and $Q$ as follows.

$$P = \{\mu_i(\omega^j) \mid j \in J, i \in [m], \text{ a simplex in } \partial \mu_i(\omega^j) \setminus \{\omega^j\} \text{ belongs to } \zeta' \text{ and } R\}.$$  

and

$$Q = \{\mu_k(\omega^j) \mid j \in J, k \in [m], \mu_k(\omega^j) \not\in P\}.$$  

Define $\zeta^J$ as follows.

$$\zeta^J = \zeta' + \sum_{\mu_i(\omega^j) \in P} \partial \mu_i(\omega^j) + \sum_{\mu_k(\omega^j) \in Q} \partial \mu_k(\omega^j).$$  

for all $\zeta^J \subset Q$.

Then, a simplex in $R$ will have coefficient 1 in each of the cycles $\zeta^J$ if and only if one of the r-simplices in $\partial \mu_k(\omega^j)$ for every $\mu_k(\omega^j) \in Q$ belongs to $R$. Clearly, $|P + Q| \geq m$, proving the claim.

\[\Box\]

\textbf{Lemma 32.} Let $R$ be a solution set for \textsc{Topological Hitting Set} on complex $K(G)$ such that $|R| \leq \binom{k+1}{2} + 1$. Then,

(1.) $V \in R$.

(2.) For every $v_i$, there is at least one simplex $\alpha_i^v$ (with $v \in V_i$) that is included in $R$.

(3.) For every unordered pair $(i, j)$, where $i, j \in [k]$, there exists a simplex $\beta_i^v,u$ for some $v, u$ that is included in $R$.

(4.) $|R| = |A_R| = \binom{k+1}{2} + 1$, where $A_R$ denotes the set of admissible simplices of $R$.

\textbf{Proof.} If $\zeta' \in [\zeta]$ is such that $\zeta' \cap A_R = \emptyset$, then we are forced to include some simplex $\omega \in \zeta'$ in $R$ such that $\omega$ is inadmissible. In that case, Lemma 31 applies and we are forced to include at least $m$ simplices. But, if we include more than $m$ facets in $R$, we exceed the budget of $\binom{k+1}{2} + 1$. So, going forward, we assume that at least one simplex in $A_R$ has coefficient 1 for every $\zeta' \in [\zeta]$.

Note that if at least one simplex in $A_R$ has coefficient 1 for every cycle in $[\zeta]$, then we do not need inadmissible simplices in $R$. We now prove the four statements of the lemma.
(1.) Since $V$ is the only admissible facet of $\zeta$, it must be included.

(2.) Since $\zeta' = \zeta + \partial \sigma_i$ is a cycle homologous to $\zeta$, and the coefficient of $V$ in $\zeta'$ is zero, the admissible simplices in $\zeta'$ are given by the set $\{\alpha_{v}^i \mid v \in V_i\}$. One of the simplices in this set must be included in $R$ for each $i$, for $R$ to be a solution set.

(3.) Note that for a fixed $i$ and $j \in [k] \setminus \{i\}$ unless some admissible facet $\beta_{i,j}^{v,u}$ for some $v, u$ is included in $R$, the coefficient of all admissible simplices in $\zeta' = \zeta + \partial \sigma_i + \sum_{v \in V_i} \partial \alpha_{r}^{v}$ will be zero. The claim follows.

(4.) This follows from the first three parts of the lemma. By (1.) we must include $V$ in $R$, by (2.) we must include at least $k \alpha$ faces in $R$, and by (3.), we must include at least $(\binom{k}{2})$ faces in $R$. Since $(\binom{k}{2}) + k + 1 = (\binom{k+1}{2}) + 1$, the claim follows. ▲

Lemma 33. If $|R| = (\binom{k+1}{2}) + 1$, then one can obtain a $k$-clique $H$ of $G$ from $R$.

Proof. If $|R| = (\binom{k+1}{2}) + 1$, then using (4.), $|R| = |\mathcal{A}_G|$. Therefore, $R$ consists entirely of admissible simplices. We now provide four conditions that characterize a solution of size $(\binom{k+1}{2}) + 1$.

As noted in Section 6.1 (1.), $V$ is part of any solution set. Using Section 6.1 (2.), for $R$ to be a solution set, at least one facet (other than $V$) of $\sigma_i$ for every $i$ must belong to $R$.

**Condition 1.** For every $i \in [k]$, the only facet of $\sigma_i$ (other than $V$) that belongs $R$ is an admissible simplex $\alpha_i^v$, for some $v \in V_i$.

Now, $\alpha_i^v$ is incident on $k - 1$ simplices, namely, $\tau_{i,j}^v$ for all $j \neq i$. Using Section 6.1 (3.), for $R$ to be a solution, we must include in $R$ at least one admissible facet $\beta_{i,j}^{u,w}$ of $\tau_{i,j}^u$ (for all $j \neq i$).

**Condition 2.** For every $v \in V_i$, such that $\alpha_i^v \in R$, and every $j \in [k] \setminus \{i\}$, the only facet of $\tau_{i,j}^u$ (other than $\alpha_i^v$) that belongs $R$ is an admissible simplex $\beta_{i,j}^{u,w}$, for some $u \in V_j$.

Note that $\beta_{i,j}^{u,w}$ is also incident on both $\tau_{j,i}^v$ and $\tau_{i,j}^u$. Then, since there must exist an admissible simplex in $R$ with coefficient 1 in $\zeta' = \zeta + \partial \sigma_i + \partial \tau_{i,j}^v + \partial \tau_{i,j}^u$, it is necessary that at least one facet of $\tau_{i,j}^u$ other than $\beta_{i,j}^{u,w}$ is included in $R$. That is, $\alpha_i^v$ must be included in $R$.

Repeating the same argument for every $\sigma_i$ and every $\tau_{i,j}^v$, it is easy to check that the only way to construct $R$ without exceeding the budget of $(\binom{k+1}{2}) + 1$ is by making these choices consistent. Thus, we obtain two additional conditions.

**Condition 3.** For every $i$ and $v$ such that $\alpha_i^v \in R$, and every $j \in [k] \setminus \{i\}$, if $\alpha_i^v$ is in $R$, and $\beta_{i,j}^{u,w}$ is in $R$, then $\alpha_i^v$ is in $R$.

**Condition 4.** For every $i$ and $v$ if $\alpha_i^v \not\in R$ (from choices made for $\sigma_i$’s in Condition 1.), then for every $j \in [k] \setminus \{i\}$, the facet $\beta_{i,j}^{u,w}$ of $\tau_{i,j}^u$ in not included in $R$.

The fact that such a set $R$ is indeed a solution set follows the same argument as in Claim 30. It is clear that such a solution set $R$ satisfies Conditions 1-4 if and only if $|R| = (\binom{k+1}{2}) + 1$. If any of the conditions are not satisfied, then either we are forced to choose more than one vertices per color, or we have that the choice of vertices $u, v$ in $\beta_{i,j}^{u,w}$ that is included in $R$ as per Section 6.1 (3.) for pairs $(i,j)$ and $(j,i)$ is inconsistent. In both cases, $|R| \geq (\binom{k+1}{2}) + 1$.

Finally, the graph $H$ is constructed from $R$ by first including one vertex $v$ per color $i$ for every $\alpha_i^v \in R$, and the edges $\{u, v\}$ for every simplex $\beta_{i,j}^{u,w} \in R$. ▲

Lemma 30 and Lemma 33 together provide a parameterized reduction from $k$-MULTICOLORED CLIQUE to TOPOLOGICAL HITTING SET. Using Theorem 23, we obtain the following result.

Theorem 34. **TOPOLOGICAL HITTING SET** is $W[1]$-hard.
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6.2 W[1]-hardness for Boundary Nontrivialization

Next, we provide a parameterized reduction from \textit{k-Multicolored Clique} to \textit{Boundary Nontrivialization}. This reduction bears some similarities with reduction from \textit{k-Multicolored Clique} to \textit{Topological Hitting Set}. So towards the end, we skip some of the details that are common to both the reductions.

Recalling some notation from Section 6.1

In Section 6.1, given a \( k \)-colored graph \( G = (V, E) \), the \((r + 1)\)-dimensional complex \( \hat{K}(G) \) was built out of types of \((r + 1)\)-simplices, namely, \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \):

\[
\mathcal{X}_1 = \{ \sigma_i \mid i \in [k] \},
\]

where \( \sigma_i = V \cup \{ i \} \), and

\[
\mathcal{X}_2 = \{ \tau_{i,j}^v \mid i \in [k], v \in V_i, j \in [k] \setminus \{ i \} \},
\]

where \( \tau_{i,j}^v = (V \setminus v) \cup \{ i, j \} \).

Furthermore, recall from Section 6.1, that for every \( i \in [k] \), if \( v \in V_i \), then

\[
\alpha_i^v = (V \setminus \{ v \}) \cup \{ i \}
\]

is an \textit{admissible} facet of \( \sigma_i \) and \( \tau_{i,j}^v \) respectively.

Also, for every \( i \in [k] \), and \( j \in [k] \setminus \{ i \} \), if \( v \in V_i, u \in V_j \), and \( \{ u, v \} \in E \), then

\[
\beta_{i,j}^{v,u} = (V \setminus \{ v, u \}) \cup \{ i, j \}
\]

is an \textit{admissible} facet of \( \tau_{i,j}^v \) and \( \tau_{j,i}^u \) respectively.

Overview of the reduction for Boundary Nontrivialization

In this section, given a \( k \)-colored graph \( G = (V, E) \), an \( r \)-dimensional complex \( L(G) \) is constructed, where \( r = |V| - 1 \). Here, we provide an overview of the construction.

Let \( \sigma_i = V \cup \{ i \} \) as in Section 6.1 and let \( \partial \sigma_i \) be the \( r \)-complex \( \partial \sigma_i \setminus \{ V \} \). The complex \( \hat{\Delta} \sigma_i \) is formed from \( \partial \sigma_i \) by the so-called S-subdivision of some of the faces of \( \partial \sigma_i \). The S-subdivision of a simplex is described in Section 6.2.1. The construction of \( \hat{\Delta} \sigma_i \) from \( \partial \sigma_i \) is described in Algorithm 3. The lexicographically highest simplex of an S-subdivided face of \( \partial \sigma_i \) is a \textit{distinguished simplex} in \( \hat{\Delta} \sigma_i \). The simplices in \( \partial \sigma_i \) that are not distinguished are called \textit{undesirable}. We wish to exclude undesirable simplices from solutions of small size. To implement the undesirability of simplices, we add further simplices to \( \hat{\Delta} \sigma_i \). The newly added simplices and the undesirable simplices are together called \textit{inadmissible simplices} of \( \hat{\Delta} \sigma_i \). That completes the high-level description of \( \hat{\Delta} \sigma_i \). Next, a subcomplex \( \hat{Z}_1 \) is built out of the union of subcomplexes \( \hat{\Delta} \sigma_i \). That is,

\[
\hat{Z}_1 = \bigcup_{i \in [k]} \hat{\Delta} \sigma_i.
\]

Let \( T = |V| \cdot (k - 1) \). It is easy to check that \( |\mathcal{X}_2| = T \). Let \( t \in [T] \) be an indexing variable such that there is a unique \( t \) that corresponds to a triple \((i, j, v)\), where \( i \in [k], j \in [k] \setminus \{ i \} \) and \( v \in V_i \). Now, for every \( t \in [T] \), add a \textit{new} set of vertices \( V_t \). Here the vertex set \( V_t \) is in one-to-one correspondence with the vertex set \((V \setminus \{ v \}) \cup \{ i, j \}\). Let \( \tau_{i,j}^v \) be the full simplex on the vertex set \( V_t \). The complex \( \Delta \tau_{i,j}^v \) is formed from \( \partial \tau_{i,j}^v \) following an S-subdivision of
some of the faces of $\partial \tau_{i,j}^v$. The construction of $\Delta \tau_{i,j}^v$ from $\partial \tau_{i,j}^v$ is described in Algorithm 4. The distinguished, undesirable, and inadmissible simplices of $\Delta \tau_{i,j}^v$ are built in a manner analogous to the distinguished, undesirable and inadmissible simplices of $\Delta \sigma_i$. For further details, please refer to Section 6.2.2. Next, a subcomplex $Z_2$ is built out of the union of subcomplexes $\Delta \tau_{i,j}^v$. That is,

$$Z_2 = \bigcup_{i \in [k]} \bigcup_{j \in [k] \setminus \{i\}} \bigcup_{v \in V_i} \Delta \tau_{i,j}^v.$$

The complex $L(G)$ is obtained from $G$ by identifying the distinguished faces of $Z_1 \cup Z_2$ as per the procedure described in Algorithm 5. The distinguished faces upon identifications are called the admissible simplices of $L(G)$. In what should remind the reader of the notation used in Section 6.1, for every $i \in [k]$, and $v \in V_i$, there is an admissible simplex denoted by $\alpha_i^v$ that belongs to $L(G)$. And for every $i \in [k]$, $j \in [k] \setminus \{i\}$ with $v \in V_i$, $u \in V_j$ and $\{u, v\} \in E$, there is an admissible simplex $\beta_{i,j}^u$ that belongs to $L(G)$. The admissible simplices of $L(G)$ encode the connectivity and coloring information of $G$. Analogous to the construction of the complex $K(G)$ described in Section 6.1, in complex $L(G)$, $\alpha_i^v$ is an admissible $r$-simplex belonging to the simplicial manifolds $\hat{\Delta} \sigma_i$ and $\Delta \tau_{i,j}^v$, whereas in the case of $\beta_{i,j}^u$ of is an admissible $r$-simplex belonging to the simplicial manifolds $\Delta \tau_{i,j}^v$ and $\Delta \tau_{i,j}^v$. We ask the reader to compare Figures 11 and 16.

**Remark 35.** Note that we use the notation $\hat{\partial} \sigma_i$ and $\hat{\Delta} \sigma_i$ for the complexes associated to $\sigma_i$, and $\partial \tau_{i,j}^v$ and $\Delta \tau_{i,j}^v$ for the complexes associated to $\tau_{i,j}^v$. This disparity in notation (that is the use of $\hat{}$ for $\sigma_i$) is to remind the reader that in the case of $\hat{\partial} \sigma_i$, a face is deleted from simplex boundary of $\sigma_i$, whereas in the case of $\partial \tau_{i,j}^v$, the full simplex boundary is used.

### 6.2.1 $S$-subdivisions of simplices

Next, we recall a lemma from Munkres [33, Lemma 3.2] that will be used to provide a guarantee that the complex described in Section 6.2.2 is, in fact, a simplicial complex.

**Lemma 36** (Munkres, [33, Lemma 3.2]). Let $L$ be a finite set of labels. Let $K$ be a simplicial complex defined on a set of vertices $V$. Also, let $f : V \rightarrow L$ be a surjective map associating to each vertex of $K$ a label from $L$. The labeling $f$ extends to a simplicial map $g : K \rightarrow K_f$ where $K_f$ has vertex set $V$ and is obtained from $K$ by identifying vertices with the same label.

If for all pairs $v, w \in V$, $f(v) = f(w)$ implies that their stars $\text{star}_K(v)$ and $\text{star}_K(w)$ are vertex disjoint, then, for all faces $\eta, \omega \in K$ we have that

- $\eta$ and $g(\eta)$ have the same dimension, and
- $g(\eta) = g(\omega)$ implies that either $\eta = \omega$ or $\eta$ and $\omega$ are vertex disjoint in $K$.

Lemma 36 provides a way of gluing faces of a simplicial complex by a simplicial quotient map obtained from vertex identifications. In particular, Lemma 36 provides conditions under which the gluing does not create unwanted identifications, and the resulting complex thus obtained is also a simplicial complex. Now, we describe a special kind of subdivision, which we call an $S$-subdivision of a $d$-simplex, with a later application of Lemma 36 in mind.
Let $\nu$ be an $d$-simplex, and let $U$ be the vertex set of $\nu$ equipped with an ordering $\succ \nu$. We construct a complex $C_{\nu}$ obtained from a subdivision of $\nu$ such that an $r$-simplex $\Omega \in C_{\nu}$ has the following property: for every vertex $v \in \Omega$, $(\text{star}_{C_{\nu}} v) \cap U = \emptyset$. The construction of the complex $C_{\nu}$ is described in Algorithm 2.

\begin{algorithm}
procedure S-SUBDIVIDE($\nu, \succ \nu$)
1: Let $U$ denote the vertices of $\nu$;
2: Let $C_0 \leftarrow \{\nu\}$; $\Omega_1 \leftarrow \nu$; $U_0 \leftarrow U$;
3: for $i = 1$ to $2(d+1)$ do
4: Perform a stellar subdivision of $\Omega_i$ to obtain $C_i$ from $C_{i-1}$;
5: Let $v_i$ be the new vertex introduced during the stellar subdivision;
6: $U_i \leftarrow U_{i-1} \cup \{v_i\}$;
7: Extend $\succ \nu$ as follows: Set $v_i \succ \nu v$ for all $v \in U_{i-1}$;
8: Let $\Omega_{i+1}$ be the lexicographically highest $d$-simplex of $C_i$;
9: end for
10: $C_{\nu} \leftarrow C_i$;
11: return $C_{\nu}, \Omega_{2(d+1)+1}$;
end procedure
\end{algorithm}

Please refer to Figure 14 for an illustrative example. In Figure 14, $U = \{A, B, C\}$ and $\Omega = \{G, H, I\}$, and the stars of $G$, $H$ and $I$ do not intersect $U$.

\begin{remark}
The total number of $d$-simplices in $C_i$ for $i \in [0, d+1]$ are $2i \cdot d + 1$. So $C_{\nu}$ has $2(d+1) + 1$ $d$-simplices. Also, by construction, $\Omega_{2(d+1)+1}$ is the lexicographically highest $d$-simplex of $C_{\nu}$.
\end{remark}

\begin{lemma}
For every $i \in [2(d+1)+1]$, $\Omega_i$ is the full simplex on the $d+1$ lexicographically highest vertices of $C_{i-1}$.
\end{lemma}

\begin{proof}
This is trivially true for $i = 1$ as $C_0$ has only $d + 1$ vertices. Suppose that the statement of the lemma holds true for all $i \in [j]$ for some $j > 1$. Let $\{u_0, u_1, \ldots, u_d\}$ be the vertices of $\Omega_j$ where $u_k \succ \nu u_{k-1}$ for $k \in [d]$. Then, by construction, $\Omega_{j+1} = \{u_1, \ldots, u_i, v_j\}$, which coincides with the set of lexicographically highest vertices of $C_j$.
\end{proof}

\begin{lemma}
Let $v_i$ be the vertex introduced during the $i$-th iteration of the algorithm. If $\{u, v_i\}$ is an edge in $C_{\nu}$, then $\nu$ has two types.
\begin{itemize}
\item (type-1) $v_i \succ \nu u$, or
\item (type-2) $u \succ \nu v_i$ and there are at most $d$ vertices $w_j$, $j \in [d]$ such that $u \succ \nu w_j \succ \nu v_i$.
\end{itemize}
\end{lemma}

\begin{proof}
In complex $C_i$, $v_i$ has degree $d + 1$. In particular, denoting the vertices of $\Omega_i$ by $\{u_0, u_1, \ldots, u_d\}$, the edges $\{v_i, u_k\}$ for $k \in [0, d]$ belong to $C_i$. So all edges of $C_i$ incident on $v_i$ are of type-1.

Moreover, for every $i' > i$, every vertex $v' \neq v_i$ of $\Omega_{i'}$ satisfies $v' \succ \nu v_i$ by Lemma 38. Hence, the newly added edges in $C_{\nu}$ for $i' > i$ that are incident on $v_i$ are of type-2. Again, using Lemma 38 inductively, there can be at most $(d+1)$ such vertices $v'$ in the final complex $C_{\nu}$.
\end{proof}

\begin{proposition}
Let $v$ be a vertex of $\Omega_{2(d+1)+1}$. Then, $(\text{star}_{C_{\nu}} v) \cap U = \emptyset$.
\end{proposition}
Figure 14 The figure shows a specific subdivision of the 2-simplex \(ABC\) defined on \(d + 1 = 3\) vertices. The vertices that are higher in the alphabetical order are also higher with respect to the ordering \(\succ\). For instance, \(G \succ D \succ A\). The above triangulation is obtained as follows: First, the simplex \(ABC\) is stellar subdivided by the introduction of the vertex \(D\). Since \(BCD\) is the lexicographically highest simplex, it is the only one that is stellar subdivided by the introduction of the vertex \(E\). Then, \(CDE\) which is the lexicographically highest simplex is subdivided by the introduction of the vertex \(F\), and so on. Note that at each step the lexicographically highest vertices always span a simplex, and that simplex is the one that is subdivided. For instance, after the first subdivision, \(BCD\) is a simplex, after the second subdivision \(CDE\) is a simplex, after the third subdivision \(DEF\) is a simplex, and so on. The process stops after \(2(d + 1)\) subdivisions. In this case, we perform six subdivisions. The total number of \(d\)-simplices introduced is \(2d(d + 1) + 1\), which in this case is 13. Note that the vertices \(A, B\) and \(C\) do not lie in the respective stars of the vertices of the highlighted triangle \(GHI\).
The complexity of high-dimensional cuts

Proof. The complex $\mathcal{C}_\nu$ has $3(d+1)$ vertices totally ordered by $\succ_\nu$. By Lemma 38, the vertices of $\Omega_{2(d+1)}$ are the highest $d+1$ vertices ordered by $\succ_\nu$. By construction, the vertices in $U$ are the lowest $d+1$ vertices ordered by $\succ_\nu$.

By Lemma 39 the vertices of $\Omega_{2(d+1)}$ do not have any edges in common with vertices in $U$. The claim follows.

6.2.2 Description of the reduction

We now give a detailed description of the reduction. As before, associated to a $k$-colored graph $G = (V,E)$, we define an $r$-dimensional complex $L(G)$ as follows.

Vertices.

Let $V^r = V \cup \{k\}$, and $r = |V|-1$. Then, $|V^r| = r + k + 1$. Include the vertex set $V^r$ in $L(G)$. In what follows, we add further vertices to $L(G)$.

Ordering relation $\succ_{V^r}$ on vertices of $L(G)$.

We now impose the following ordering relation on $V^r$. Enumerate the vertices of $G$ according to a fixed total order $V = \{v_1, v_2, \ldots, v_{r+1}\}$. For every color $i \in [k]$ and $j \in [r+1]$, we have $i \succ_{V^r} v_j$. For $i_2 \geq i_1$, we have $i_2 \succ_{V^r} i_1$, and for $j_2 \geq j_1$, we have $v_{j_2} \succ_{V^r} v_{j_1}$.

Remark 41 (Implementing undesirability). The undesirability of certain $r$-simplices is implemented in the gadget as follows: Let $m = n^3$. Then, to every undesirable $r$-simplex $\omega = \{v_1, v_2, \ldots, v_{r+1}\}$, associate $m$ new vertices $U^\omega = \{u^\omega_1, u^\omega_2, \ldots, u^\omega_m\}$. For every $\ell \in [m]$, let $\mu_\ell = \{v_1, v_2, \ldots, v_{r+1}, u^\omega_\ell\}$. Now introduce $m(r+1)$ new $r$-simplices

$T^\omega = \{\text{facets of } \mu_\ell \setminus \{\omega\} \mid \ell \in [m]\}$.

Note that for any two undesirable simplices $\omega_1$ and $\omega_2$ we have, $U^{\omega_1} \cap U^{\omega_2} = \emptyset$ and $T^{\omega_1} \cap T^{\omega_2} = \emptyset$. As observed later, introducing these new simplices makes inclusion of $\omega$ in the solution set prohibitively expensive. Please refer to Figure 15 for an illustrative example. For undesirable simplices, we denote the set of $r$-simplices in $T^\omega \cup \omega$ by $[\omega]$. For admissible simplices, $[\omega] = \omega$.

Gadgets.

The complex $L(G)$ is constructed by gluing the distinguished faces of two types of gadgets. Next, we describe these two types of gadgets.

Gadgets of type-1.

The construction of gadgets of type-1 is explained in detail in the pseudocode of Algorithm 3. Below, we provide a high-level sketch.

First, we describe the subroutine SUBDIVIDEDDELTA1. In this subroutine, given an index $i \in [k]$, we begin our construction with the complex $\hat{\partial}\sigma_i = \{\text{facets of } \sigma_i\} \setminus \{V\}$, where as in Section 6.1 $\sigma_i = V \cup \{i\}$. The vertices of $\hat{\partial}\sigma_i$ inherit an order from $\succ_{V^r}$.

Definition 42 (Pre-admissible and non-pre-admissible simplices of $\hat{\partial}\sigma_i$). For every $v \in V_i$, the simplices $u_v^\sigma = (V \setminus \{v\}) \cup \{i\}$ are called the pre-admissible simplices of $\hat{\partial}\sigma_i$, and all other simplices of $\hat{\partial}\sigma_i$ are called non-pre-admissible.
Figure 15 For every undesirable simplex $\omega = \{v_1, v_2, \ldots, v_{r+1}\}$ $m$ new vertices $U^\omega = \{u_1^\omega, u_2^\omega, \ldots, u_m^\omega\}$ are added to $L(G)$. Furthermore, $m(r+1)$ new $r$-simplices $Y^\omega = \{\{\text{facets of } u_i\} \setminus \{\omega\} \mid i \in [m]\}$ are added to $L(G)$. The $r$-simplices in $Y^\omega \cup \omega$ are denoted by $[\omega]$.

The procedure $S\text{-Subdivide}$ described in Section 6.2.1 takes an $r$-simplex $\nu$ as input and returns a subdivision of $\nu$ along with the (lexicographically highest) distinguished simplex from within the subdivided simplex. For every pre-admissible simplex $a \nu_i$, its subdivision is denoted by $C a \nu_i$, and the distinguished simplex of $C a \nu_i$ is denoted by $\alpha a \nu_i(\sigma_i)$. The complex $C$ is formed by taking the union of the subdivided pre-admissible simplices.

Let $A$ denote the collection of distinguished simplices in $C$, and let $W$ denote the set of non-pre-admissible $r$-simplices of $\partial \sigma_i$. Since there are $V_i$ pre-admissible simplices for color $i$, $|A| = |V_i|$. Finally, the complex $\Delta \sigma_i$ is formed by taking the union of the non-pre-admissible simplices, namely $W$, with the collection of subdivisions of the pre-admissible simplices, namely $C$. We end the description of $S\text{-SubdivideDelta1}$ with one last definition.

Definition 43 (Undesirable simplices of $\Delta \sigma_i$). At the end of the procedure $S\text{-SubdivideDelta1}$, the simplices in $\Delta \sigma_i \setminus A$ are called the undesirable simplices of $\Delta \sigma_i$.

In procedure $\text{TYPEZ1}$, the complex $Z_1$ is constructed. To begin with, the subroutine $S\text{-SubdivideDelta1}$ is invoked for every $i \in [k]$, which returns the complex $\Delta \sigma_i$ along with its distinguished simplices $A_{\sigma_i}$. Next, we add further simplices to $\Delta \sigma_i$ in order to implement undesirability of simplices as per Remark 41. As per the notation used in $\text{TYPEZ1}$, $\Delta \sigma_i \setminus A_{\sigma_i}$ are the undesirable simplices of $\Delta \sigma_i$. Then, to every undesirable simplex $\omega \in \Delta \sigma_i \setminus A_{\sigma_i}$, we add $(r+1)m$ simplices $\Gamma^\omega$ to $\Delta \sigma_i$, completing the construction of $\Delta \sigma_i$. The complex $Z_1$ is then given by the union of all simplices in $\Delta \sigma_i$ for every $i$. We end the description of $\text{TYPEZ1}$ with a definition.

Definition 44 (Inadmissible simplices of $\Delta \sigma_i$). At the end of the procedure $\text{TYPEZ1}$, the simplices in $\Delta \sigma_i \setminus A_{\sigma_i}$ are the inadmissible simplices of $\Delta \sigma_i$.

Gadgets of type-2.

We now provide a high-level description of gadgets of type-2, the pseudocode of which is provided in Algorithm 4. The type-2 gadgets are indexed by $t$. 
Figure 16: The figure depicts the construction of $L(G)$ via identifications of various gadgets as described in Algorithm 5. In particular, the dashed red lines show identifications of the (red) congruent faces of type-1 gadgets (shown in green) and type-2 gadgets (shown in blue). The dashed black line shows identifications of the (red) congruent faces of two distinct type-2 gadgets. Note that some of the red dashed lines are only partially drawn. The red faces are the lexicographically highest distinguished faces obtained by S-subdivisions described in Section 6.2.1. The construction of the type-1 green gadgets is described in Algorithm 3 and the construction of the type-2 blue gadgets is described in Algorithm 4. Note that the full simplex with vertex set $V$ (or its subdivision) does not appear as a simplex in any of the type-1 gadgets. In fact the type-1 gadget are $r$-manifolds with $\partial V$ as their common boundary, and the dashed purple lines depict precisely that. The type-2 gadgets are topological $r$-spheres.
Algorithm 3 Construction of complex $\Delta \sigma_i$

1: procedure Subdivide$\Delta \sigma_i(1)$
2: $\sigma_i \leftarrow V \cup \{i\}$;
3: Let $\partial \sigma_i$ be the $r$-complex $\partial \sigma_i \setminus \{V\}$;
4: The vertex set $U$ of $\partial \sigma_i$ is ordered by $\succ$, obtained by restricting $\succ_V$ to $U$;
5: Every $r$-simplex $a^v_i = (V \setminus \{v\}) \cup \{i\}$ of $\partial \sigma_i$ with $v \in V$ is deemed pre-admissible;
6: Let $W$ denote the set of $r$-simplices of $\partial \sigma_i$ that are not pre-admissible;
7: $A \leftarrow \emptyset$; $C \leftarrow \emptyset$;
8: for each pre-admissible simplex $a^v_i$ of $\partial \sigma_i$ do
9: $C^v_i, \alpha^v_i(\sigma_i) \leftarrow S$-Subdivide$(a^v_i, \succ)$;
10: $A \leftarrow A \cup \{\alpha^v_i(\sigma_i)\}$;
11: $C \leftarrow C \cup C^v_i$;
12: end for
13: $\hat{\Delta} \sigma_i \leftarrow C \cup W$;
14: return $\hat{\Delta} \sigma_i, A$; $\triangleright$ The simplices in $\hat{\Delta} \sigma_i \setminus A$ are undesirable.
15: end procedure

16: procedure TypeZ$^1$
17: $Z_1 \leftarrow \emptyset$;
18: for $i = 1$ to $k$ do
19: $\hat{\Delta} \sigma_i, A_{\sigma_i} \leftarrow Subdivide$\Delta$1(1)$ ;
20: The simplices in $A_{\sigma_i}$ are the distinguished simplices of $\hat{\Delta} \sigma_i$;
21: The simplices in $\hat{\Delta} \sigma_i \setminus A_{\sigma_i}$ are deemed the undesirable simplices of $\hat{\Delta} \sigma_i$;
22: for every undesirable simplex $\omega$ in $\hat{\Delta} \sigma_i$ do $\triangleright$ as described in Remark 41
23: Add $(r + 1)m$ simplices $\Upsilon^\omega$ to $\hat{\Delta} \sigma_i$;
24: end for $\triangleright$ The simplices in $\hat{\Delta} \sigma_i \setminus A_{\sigma_i}$ are inadmissible.
25: $Z_1 \leftarrow Z_1 \cup \hat{\Delta} \sigma_i$;
26: end for
27: end procedure
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First, we describe the subroutine \textsc{SubdivideDelta2}. In this subroutine, given a vertex \( v \in V_i \) a color \( j \neq i \), and an index \( t \), we introduce a vertex set \( V_t \) whose vertices are in one-to-one correspondence with the vertices \( (V \setminus \{v\}) \cup \{i, j\} \). Let \( \tau^v_{i,j} \) be the full \((r+1)\)-simplex on \( V_t \), and \( \partial \tau^v_{i,j} \) be the complex induced by the facets of \( \tau^v_{i,j} \). The vertices of \( \partial \tau^v_{i,j} \) are ordered according to the same rules as \( \succ \).

\textbf{Definition 45} (Pre-admissible and non-pre-admissible simplices of \( \partial \tau^v_{i,j} \)). The simplices \( a^v_i = V_i \setminus \{j\} \) for every \( v \in V_t \) and the simplices \( b^v_{i,j} = V_t \setminus \{u_t\} \), where \( u \in V_j \) and \( \{u, v\} \in E \) are also said to be the pre-admissible simplices of \( \partial \tau^v_{i,j} \). All other \( r \)-simplices of \( \partial \tau^v_{i,j} \) are deemed non-pre-admissible.

We invoke the procedure \textsc{S-Subdivide} described in Section 6.2.1 to subdivide the pre-admissible simplices of \( \partial \tau^v_{i,j} \). The subdivision of a pre-admissible simplex \( b^v_{i,j} \) is denoted by \( C_{i,j}^v \) and the distinguished simplex of \( C_{i,j}^v \) is denoted by \( \alpha_{i,j}^v(\tau^v_{i,j}) \). The subdivision of a pre-admissible simplex \( a^v_i(\tau^v_{i,j}) \) is denoted by \( C^v_i \) and the distinguished simplex of \( C^v_i \) is denoted by \( \alpha^v_i \). The complex \( C \) is formed by taking the union of the subdivided pre-admissible simplices. Furthermore, the collection of all the distinguished simplices of the subdivided pre-admissible simplices is denoted by \( A \). It is easy to check that, \(|A| = k \). Finally, the complex \( \Delta \tau^{v}_{i,j} \) is formed by taking the union of the non-pre-admissible simplices, namely \( W \), with the collection of subdivisions of the pre-admissible simplices, namely \( C \). We conclude the description of \textsc{SubdivideDelta2} with a definition.

\textbf{Definition 46} (Undesirable simplices of \( \Delta \tau^{v}_{i,j} \)). At the end of procedure \textsc{SubdivideDelta2}, the simplices in \( \Delta \tau^{v}_{i,j} \setminus A \) are said to be the undesirable simplices of \( \Delta \tau^{v}_{i,j} \).

In the procedure \textsc{TypeZ2}, the complex \( Z_2 \) is constructed. To do this, the subroutine \textsc{SubdivideDelta2} is invoked for every color \( i \), every vertex \( v \) in \( V_i \), and every color \( j \) where \( j \neq i \), which returns the complex \( \Delta \tau^{v}_{i,j} \), along with its set of distinguished simplices \( A_{\tau^{v}_{i,j}} \). Next, we add further simplices to \( \Delta \tau^{v}_{i,j} \), for every \( v \in V_i \) and \( j \in [k] \setminus \{i\} \), in order to implement undesirability of simplices as per Remark [I]. We start the construction with the undesirable simplices of \( \Delta \tau^{v}_{i,j} \), namely \( \Delta \tau^{v}_{i,j} \setminus A_{\tau^{v}_{i,j}} \). To every undesirable simplex \( \omega \in \Delta \tau^{v}_{i,j} \setminus A_{\tau^{v}_{i,j}} \), we add \((r + 1)m \) simplices \( \Upsilon^\omega \) to \( \Delta \tau^{v}_{i,j} \), completing the construction of \( \Delta \tau^{v}_{i,j} \). The complex \( Z_2 \) is then given by the union of all simplices in \( \Delta \tau^{v}_{i,j} \) for every \( i \in [k] \), every vertex \( v \in V_i \) and every \( j \in [k] \setminus \{i\} \). We conclude the description of \textsc{TypeZ2} with a definition.

\textbf{Definition 47} (Inadmissible simplices of \( \Delta \tau^{v}_{i,j} \)). At the end of procedure \textsc{TypeZ2}, the simplices in \( \Delta \tau^{v}_{i,j} \setminus A_{\tau^{v}_{i,j}} \) are the inadmissible simplices of \( \Delta \tau^{v}_{i,j} \).

\textbf{Attachments.}

Let \( K' = Z_1 \cup Z_2 \). Then, complex \( L(G) \) is formed from \( K' \) after making the attachments described in Algorithm [5].
Algorithm 4 Construction of complex $\Delta_{v,i,j}$

1: procedure Subdivide$Delta\_2(v,i,j,t)$
2:   Let $V_t \leftarrow \emptyset$;
3:   for every $u \in V \setminus v$ do
4:     Add a vertex $u_t$ to $V_t$;
5:   end for
6:   $V_t \leftarrow V_t \cup \{i_t,j_t\}$;
7:   Let $\tau_{i,j}^v$ be the full $(r+1)$-simplex on $V_t$;
8:   Let $\partial \tau_{i,j}^v$ be the $r$-complex induced by the facets of $\tau_{i,j}^v$;
9:   The vertices $V_t$ of $\partial \tau_{i,j}^v$ are in a natural one-to-one correspondence to a subset of vertices in $V'$. The ordering $\succ_t$ on $V_t$ is defined using the same rules as for $\succ_{V'}$;
10:   The $r$-simplex $a_{i,j}^v = V_t \setminus \{j_t\}$ is deemed pre-admissible;
11:   The $r$-simplices $\{b_{v,u}^{i,j} | b_{v,u}^{i,j} = V_t \setminus \{u_t\}, u \in V_j \text{ and } \{u,v\} \in E\}$ are also deemed pre-admissible;
12:   Let $W$ denote the set of $r$-simplices of $\partial \tau_{i,j}^v$ that are not pre-admissible;
13:   $C_v, \alpha_{i,j}^v(\tau_{i,j}^v) \leftarrow S\text{-subdivide}(a_{i,j}^v, \succ_t)$;
14:   $A \leftarrow \{\alpha_{i,j}^v(\tau_{i,j}^v)\}; \ C \leftarrow C_v^i$;
15:   for each pre-admissible simplex $b_{v,u}^{i,j}$ of $\partial \tau_{i,j}^v$ do
16:     $C_{i,j}^{v,u}, \beta_{i,j}^{v,u}(\tau_{i,j}^v) \leftarrow S\text{-subdivide}(b_{v,u}^{i,j}, \succ_t)$;
17:     $A \leftarrow A \cup \{\beta_{i,j}^{v,u}(\tau_{i,j}^v)\}; \ C \leftarrow C \cup C_{i,j}^{v,u}$;
19:   end for
21: return $\Delta_{v,i,j}, A$; \quad \triangleright The simplices in $\Delta_{v,i,j} \setminus A$ are undesirable.
22: end procedure

23: procedure TypeZ2
24:   $Z_2 \leftarrow \emptyset; \ t = 0;
25: for \ i = 1 \ to \ k \ do
26:   for every vertex $v$ in $V_i$, and a color $j \in [k] \setminus \{i\}$ do
27:     $t = t + 1$;
28:     $\Delta_{v,i,j}, A_{\tau_{i,j}} \leftarrow S\text{-subdivide}(v,i,j,t)$ ;
29:     The simplices in $A_{\tau_{i,j}}$ are the distinguished simplices of $\Delta_{v,i,j}$;
30:     The simplices in $\Delta_{v,i,j} \setminus A_{\tau_{i,j}}$ are deemed the undesirable simplices of $\Delta_{v,i,j}$;
31:     for every undesirable simplex $\omega$ in $\Delta_{v,i,j}$ do \quad \triangleright as described in Remark 41
32:       Add another $(r+1)m$ simplices $\Upsilon^{\omega}$ to $\Delta_{v,i,j}$;
33:     end for \quad \triangleright The simplices in $\Delta_{v,i,j} \setminus A_{\tau_{i,j}}$ are inadmissible.
34:   $Z_2 \leftarrow Z_2 \cup \Delta_{v,i,j}$;
35: end for
37: end procedure
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Algorithm 5 Construction of complex $L(G)$

1: for every $v \in V_i$ and $j \neq i$ do
2:   Identify the $r$-simplices: $\alpha_i^v(\tau_{i,j}^v) \sim \alpha_i^v(\sigma_i)$, where the identifications of vertices are consistent with respective lexicographic orderings. Denote the identified simplex by $\alpha_i^v$;
3: end for
4: for every edge $\{u,v\} \in E$ with $v \in V_i$ and $u \in V_j$ do
5:   Identify the $r$-simplices: $\beta_{i,j}^{u,v}(\tau_{i,j}^{u,v}) \sim \beta_{i,j}^{u,v}(\tau_{i,j}^{u,v})$, by respecting the respective lexicographic orderings. Denote the identified simplex by $\beta_{i,j}^{u,v}$;
6: end for

Definition 48 (Admissible simplices of $L(G)$). The simplices $\alpha_i^v$ for every $i \in [j]$ and $v \in V_i$, and the simplices $\beta_{i,j}^{u,v}$ for every edge $\{u,v\} \in E$ with $v \in V_i$ and $u \in V_j$ are said to be the admissible simplices of $L(G)$.

Proposition 49. The complex $L(G)$ formed from identifying vertices in $K'$ is a simplicial complex.

Proof. This follows immediately from Lemma 30 and Proposition 10.

This completes the construction of complex $L(G)$. Please refer to Figure 19 for a schematic illustration.

Choice of input boundary.

For the abstract simplex $V$, let $\partial V$ denote the set of facets of $V$. Note that although $V \notin L(G)$, every facet of $V$ is in $L(G)$. In fact, the complex $\hat{\Delta} \sigma_i$ for every $i$, is a simplicial $r$-manifold with $\partial V$ as its boundary. We choose $\partial V$ as our input boundary that we want to make nontrivial.

Choice of parameter.

Let $\left(k + \left(\frac{k}{2}\right)\right)\left(k+1\right)$ be the parameter for BOUNDARY NONTRIVIALIZATION on the complex $L(G)$.

Proposition 50. If there exists a $k$-clique $H$ of $G$ such that every vertex of $H$ has a different color, then a set of $\left(\frac{k+1}{2}\right)$ $r$-simplices in $L(G)$ meets every chain $\xi$ with $\partial \xi = \partial V$.

Proof. As in Section 6.1 we construct a set $S$ of $r$-simplices that mimics the graphical structure of $H$ as follows:

$$S_\alpha = \{ \alpha_i^v \mid v \in V_i \cap V_H \}$$
$$S_\beta = \{ \beta_{i,j}^{u,v} \mid v \in V_i, u \in V_j, \{i,j\} \in E_H \}$$

Set $S = S_\alpha \cup S_\beta$.

Now we want to show that at least one element from the solution set $S$ has coefficient 1 in every chain $\xi$ that satisfies $\partial \xi = \partial V$. Thus, we aim to show that removing $S$ from $L(G)$ makes $\partial V$ nontrivial. Before we proceed, we introduce some notations and definitions. To begin with let $A$ denote the set of all admissible simplices in $L(G)$ (described in Definition 18).

Notation 5. For an $r$-simplex $\omega$, let $[\omega]_{\xi}$ denote the simplices of $[\omega]$ in $\xi$.

Definition 51 (Type-1 gadget belonging to chain $\xi$). If there exists an $r$-simplex $\omega \in \hat{\Delta} \sigma_i \setminus A$ such that $\partial([\omega]_{\xi}) = 1$, then we say that $\sigma_i$ belongs to $\xi$. 

Definition 52 (Type-2 gadget belonging to chain $\xi$). If there exists an $r$-simplex $\omega \in \Delta \tau_{i,j} \setminus A$ such that $\partial([\omega]_{\xi}) = 1$, then we say that $\tau_{i,j}$ belongs to $\xi$.

Before we can finish the proof of Proposition 50, we need a few auxiliary lemmas. For the lemmas that follow, we let $\xi$ be a chain that satisfies $\partial \xi = \partial V$.

Figure 17 The figure depicts the gadget $\hat{\Delta} \sigma_i$ in a simplistic manner, that is, without the full triangulation and without the inadmissible simplices. Also, in this figure, $\sigma_i$ belongs to $\xi$. That is, there exists an $r$-simplex $\omega \in \hat{\Delta} \sigma_i \setminus A$ such that $\partial([\omega]_{\xi}) = 1$. The simplices of $\hat{\Delta} \sigma_i$, that lie in $A$ are shown in red. Then, according to Lemma 53, the boundary of the part of the complex in green equals the boundary of triangles in red (i.e., the black edges) + $\partial V$ (shown in purple).

Lemma 53. If $\sigma_i$ belongs to $\xi$, then $\partial \left( \left( \hat{\Delta} \sigma_i \setminus A \right) \cap \xi \right) = \partial V + \partial \left( \hat{\Delta} \sigma_i \cap A \right)$.

Proof. Please refer to Figure 17 for an illustration of the statement of the lemma. Since $\sigma_i$ belongs to $\xi$, there exists an $r$-simplex $\omega_1 \in \hat{\Delta} \sigma_i \setminus A$ such that $\partial([\omega_1]_{\xi}) \neq 0$. This implies that there exists a facet $\varsigma$ of $\omega_1$ such that $\varsigma \in \partial([\omega_1]_{\xi})$ and $\varsigma \notin \partial \xi$. Hence, there must be an $r$-simplex $\omega_2$ with $\varsigma$ as a facet such that $\omega_2 \in \hat{\Delta} \sigma_i \setminus A$, $\partial([\omega_2]_{\xi}) \neq 0$ and $\varsigma$ vanishes in $\partial([\omega_1]_{\xi} + [\omega_2]_{\xi})$. Repeating the argument above, we inductively add classes $[\omega_j]_{\xi}$, where $\omega_j \in \hat{\Delta} \sigma_i \setminus A$ such that $\partial([\omega_j]_{\xi}) \neq 0$. Note that by construction, $\bigcup_j [\omega_j]_{\xi} = \left( \hat{\Delta} \sigma_i \setminus A \right) \cap \xi$, where $j$ indexes the simplices in $\hat{\Delta} \sigma_i \setminus A$. Clearly, the induction stops when $\partial(\bigcup_j [\omega_j]_{\xi}) = \partial V + \partial \left( \hat{\Delta} \sigma_i \cap A \right)$.

Figure 18 The figure is a simplistic depiction of gadget $\Delta \tau_{i,j}$. In particular, the full triangulation and the the inadmissible simplices of $\Delta \tau_{i,j}$ are not shown. In this figure, $\tau_{i,j}$ belongs to $\xi$. That is, there exists an $r$-simplex $\omega \in \Delta \tau_{i,j} \setminus A$ such that $\partial([\omega]_{\xi}) = 1$. The simplices of $\Delta \tau_{i,j}$ that lie in $A$ are shown in red. Then, according to Lemma 54 the boundary of the part of the complex in blue equals the boundary of triangles in red (i.e., the black edges).
Lemma 54. If \( \tau_{i,j}^v \) belongs to \( \xi \), then \( \partial((\Delta \tau_{i,j}^v \setminus A) \cap \xi) = \partial((\Delta \tau_{i,j}^v \cap A) \cap \xi) \).

**Proof.** The argument is identical to the proof of Lemma 53. Please refer to Figure 18 for an illustration of the statement of the lemma.

Lemma 55. The cardinality of the set \( \{i \in [k] | \sigma_i \) belongs to \( \xi \} \) is odd.

**Proof.** Please see (the bottom portion of) Figure 19 for an illustration of the statement of the lemma. First, note that by construction, \( \partial V \cap \partial(S) = \emptyset \). Then, using Lemma 53, we have \( \partial V \subseteq \partial((\Delta \sigma_i \setminus A) \cap \xi) \) for every \( \sigma_i \) that belongs to \( \xi \). Since \( \partial V \) only occurs in the boundaries of type-1 gadgets and \( \partial\xi = \partial V \), the cardinality of the set \( \{i \in [k] | \sigma_i \) belongs to \( \xi \} \) must be odd.

Lemma 56. If \( \partial \xi \cap S = \emptyset \), and if \( \sigma_i \) belongs to \( \xi \) for some \( i \in [k] \), then the cardinality of

\[
I = \{ \tau_{i,j}^v | v \in V \cap V_H, j \in [k] \setminus \{i\} \text{ and } \tau_{i,j}^v \text{ belongs to } \xi \}
\]

is odd. On the other hand, if \( \sigma_i \) does not belong to \( \xi \), then \( I \) is even.

**Proof.** Please see (the middle portion of) Figure 19 for an illustration of the statement of the lemma.

**Case 1:** \( \sigma_i \) belongs to \( \xi \).

Since \( H \) is a multicolored clique, for color \( i \), there exists a vertex \( v \in V \cap V_H \). Hence, by construction, \( \alpha_i^v \in S \). Moreover, \( \alpha_i^v \) is the only \( r \)-simplex that is common to \( \hat{\Delta} \sigma_i \) and \( \Delta \tau_{i,j}^v \) for every \( \tau_{i,j}^v \in I \). Note that \( S \subseteq A \), and simplices in \( A \) have disjoint boundaries.

Since \( \sigma_i \) belongs to \( \xi \), we obtain

\[
\partial \alpha_i^v \subset \partial((\hat{\Delta} \sigma_i \setminus S) \cap \partial((\hat{\Delta} \sigma_i \setminus A) \cap \xi) = \partial((\hat{\Delta} \sigma_i \setminus A) \cap \xi).
\]

where the last equality uses Lemma 53.

Moreover for every \( \tau_{i,j}^v \in I \), we obtain

\[
\partial \alpha_i^v \subset \partial((\Delta \tau_{i,j}^v \setminus S) \cap \partial((\Delta \tau_{i,j}^v \setminus A) \cap \xi) = \partial((\Delta \tau_{i,j}^v \setminus A) \cap \xi),
\]

where the last equality uses Lemma 54.

For \( j \in [k] \setminus \{i\} \) such that \( \tau_{i,j}^v \notin I \),

\[
\partial ((\Delta \tau_{i,j}^v \setminus A) \cap \xi) = 0,
\]

which is a simple consequence of Definition 52.

Using the assumption \( \partial \xi \cap S = \emptyset \), and Equations (1), (2), we get

\[
((I + 1) \mod 2) \cdot \partial \alpha_i^v \subset \partial \xi.
\]

Since \( \partial \xi = \partial V \), and \( \alpha_i^v \cap \partial V = \emptyset \), \( I + 1 \) should be even, proving the first claim.

**Case 2:** \( \sigma_i \) does not belong to \( \xi \).

In this case,

\[
\partial \alpha_i^v \subset \partial((\hat{\Delta} \sigma_i \setminus S) \cap \partial ((\hat{\Delta} \sigma_i \setminus A) \cap \xi) \cap \frac{\partial((\hat{\Delta} \sigma_i \setminus A) \cap \xi)}{2}.
\]
Figure 19 As in the case of Figure 16, this figure also depicts some of the gadgets of $L(G)$. However, we depict only those gadgets that belong to some chain $\xi$. See Definitions 51 and 52 for what it means for a gadget to belong to a chain. The dashed red lines show identifications of the (red) congruent faces of type-1 gadgets that belong to $\xi$ to the (red) congruent faces of type-2 gadgets that belong to $\xi$. The dashed black line shows identifications along the (red) congruent faces of two distinct type-2 gadgets that belong to $\xi$.

The odd count of purple dashed lines is the content of the Lemma 55. The odd count of each group of red dashed lines in the middle is the content of Lemma 56. Hence, for every purple dashed line, there is a group of red dashed lines of odd cardinality. On the one hand, since an odd sum of odd numbers is odd, by Lemmas 55 and 56 the total number of red dashed lines should be odd. On the other hand, Lemma 57 says that the cardinality of the red dashed lines (counted from above) is even. The main idea of Proposition 50 which uses Lemmas 55–57 and a proof by contradiction is that an odd sum of odd numbers cannot be even.
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where the last non-inclusion follows from \( \partial \left( \Delta \sigma_i \setminus A \right) \cap \xi = 0 \) (as a simple consequence of Definition 51). But for every \( \tau_{i,j}^v \in I \), we still have

\[
\partial \sigma_i \subset \partial \left( \Delta \tau_{i,j}^v \cap S \right) \subset \partial \left( \Delta \tau_{i,j}^v \cap A \right) = \partial \left( \left( \Delta \tau_{i,j}^v \setminus A \right) \cap \xi \right),
\]

which gives

\[
(I \mod 2) \cdot \partial \sigma_i \subset \partial \xi.
\]

Since \( \partial \xi = \partial V \) and \( \sigma_i \cap \partial V = \emptyset \), \( I \) should be even, proving the second claim.

\begin{lemma}
Assuming \( \partial \xi \cap S = \emptyset \), we define the set \( P \) as

\[
P = \{(i,j) \mid \sigma_i \text{ and } \tau_{i,j}^v \text{ for some } v \in V_i \cap V_H \text{ belong to } \xi \}.
\]

Then, \(|P|\) is even.
\end{lemma}

\begin{proof}
Please see (the top portion of) Figure 19 for an illustration of the statement of the lemma. First, we define \( P' \) as follows.

\[
P' = \{(i,j) \mid \tau_{i,j}^v \text{ for some } v \in V_i \cap V_H \text{ belongs to } \xi \}.
\]

Suppose \((i,j) \in P' \) for some \( i \in [k] \), and \( j \neq i \). Since \( H \) is a multicolored clique, for color \( i \), there exists a vertex \( v \in V_i \cap V_H \). Also, there exists a vertex \( u \in V_j \cap V_H \) and an edge \( \{u, v\} \in E_H \). By construction of \( S, \beta_i^{v,u} \in S \). Once again, we will use the facts: 1. \( S \subseteq A \), and 2. the simplices in \( A \) have disjoint boundaries.

For every \( \tau_{i,j}^v \in I \), we obtain

\[
\partial \beta_i^{v,u} \subset \partial \left( \Delta \tau_{i,j}^v \cap S \right) \subset \partial \left( \Delta \tau_{i,j}^v \cap A \right) = \partial \left( \left( \Delta \tau_{i,j}^v \setminus A \right) \cap \xi \right),
\]

where the last equality uses Lemma 54.

Moreover, by construction, \( \beta_i^{v,u} \) belongs to only two gadgets of \( L(G) \): \( \Delta \tau_{j,i}^u \) and \( \Delta \tau_{i,j}^v \). Using \( \partial \xi = \partial V \) and \( \partial V \cap \partial \beta_i^{v,u} = \emptyset \), we deduce that \( \partial \beta_i^{v,u} \not\subset \partial \xi \). Then, using \( \partial \xi \cap S = \emptyset \), we have

\[
\partial \beta_i^{v,u} \subset \partial \left( \left( \Delta \tau_{j,i}^u \setminus A \right) \cap \xi \right)
\]

(5)

But this forces \( \tau_{j,i}^u \) to belong to \( \xi \), and hence the pair \((j, i)\) belongs to \( P' \). Therefore, using Equations 4 and 5, \( P'' \) is of even cardinality. Now, define \( P'' \) as follows.

\[
P'' = \{(i,j) \mid \tau_{i,j}^v \text{ for some } v \in V_i \cap V_H \text{ belongs to } \xi, \text{ and } \sigma_i \text{ does not belong to } \xi \}.
\]

By inductively applying Case 2 of Lemma 56 we deduce that \( P'' \) is of even cardinality. Finally, \( P = P' - P'' \). Hence, \( P \) is of even cardinality.

Now, observe that if the conditions of Lemmas 55 and 57 are simultaneously satisfied, then we reach a contradiction. This is because using Lemmas 55 and 56 \(|P|\) is an odd set of odd numbers and hence odd, whereas according to Lemma 57 \(|P|\) is even. So if the chain \( \xi \) has \( \partial V \) as its boundary, then the assumption \( \partial \xi \cap S = \emptyset \) cannot be satisfied. This concludes the proof of Proposition 50.

\begin{lemma}
If there exists a chain \( \xi' \) with \( \partial \xi' = \partial V \) such that only the inadmissible simplices of \( R \) have coefficient 1 in \( \xi' \), then the size of \( R \) is at least \( m \).
\end{lemma}
Proof. We skip the proof since it is identical to the proof of Lemma 31.

Lemma 59. Let $\mathcal{R}$ be a solution set for Boundary Nontrivialization on complex $\mathbb{L}(G)$. Then,

1. For every $\hat{\Delta} \sigma_i$, there is at least one facet $\alpha_v^i$ with $v \in V_i$ that is included in $\mathcal{R}$.
2. For every unordered pair $(i, j)$, where $i, j \in [k]$, there exists a simplex $\beta_{i,j}^v$ for some $v, u$ that is included in $\mathcal{R}$.
3. If $|\mathcal{R}| \leq \binom{k+1}{2}$, then $|\mathcal{R}| = |A_\mathcal{R}| = \binom{k+1}{2}$, where $A_\mathcal{R}$ denotes the set of admissible simplices of $\mathcal{R}$.

Proof. The proof is analogous to the proof of Section 6.1. We repeat it here for the sake of clarity and completeness.

Let $A_\mathcal{R}$ denote the set of admissible simplices of $\mathcal{R}$. If $\xi$ is such that $\partial \xi = \partial V$ and $\xi \cap A_\mathcal{R} = \emptyset$, then we are forced to include inadmissible simplices. In that case, Lemma 58 applies, and $\mathcal{R}$ is of cardinality at least $m = n^3$. But, if we include a total of (more than) $n^3$ facets in $\mathcal{R}$, we exceed the budget of $\binom{k+1}{2}$. So, going forward, we assume that at least one simplex in $A_\mathcal{R}$ has coefficient 1 in every chain $\xi$, where $\xi = \partial V$.

Note that if at least one simplex from $A_\mathcal{R}$ has coefficient 1 in every chain $\xi$ with $\xi = \partial V$, then we do not need simplices that are inadmissible in $\mathcal{R}$. Next, we prove the three claims in the lemma.

1. Let $\xi = \hat{\Delta} \sigma_i$, for some $i \in [k]$. Then, $\partial(\hat{\Delta} \sigma_i) = \partial V$. So if we do not include an admissible simplex $\alpha_v^i$ for some $v \in V_i$ in $\mathcal{R}$, then we would be forced to include some inadmissible simplices of $\hat{\Delta} \sigma_i$.
2. Next, for some fixed $i$ and $j \in [k] \setminus \{i\}$, let $\xi = \hat{\Delta} \sigma_i + \sum_{v \in V_i} (\Delta \tau_{i,j}^v)$. Then, $\partial(\hat{\Delta} \sigma_i) + \sum_{v \in V_i} \partial(\Delta \tau_{i,j}^v) = \partial V$. So unless some admissible facet $\beta_{i,j}^v$ for some $v, u$ is included in $\mathcal{R}$, the coefficient of all admissible simplices in $\xi$ will be zero, and we would be forced to include inadmissible simplices, which according to Lemma 58 is prohibitively expensive.
3. The third claim follows immediately from the first two.

Lemma 60. If $|\mathcal{R}| = |A_\mathcal{R}| = \binom{k+1}{2}$, then one can obtain a $k$-clique $H$ of $G$ from $\mathcal{R}$.

Proof. Structurally the proof is identical to Lemma 33. The roles of $\sigma_i$ and $\tau_{i,j}^v$ are played by $\hat{\Delta} \sigma_i$ and $\Delta \tau_{i,j}^v$, respectively. Moreover, there is a difference of 1 in the cardinality of solution set $\mathcal{R}$, because for Topological Hitting Set, we need to remove $V$ whereas the simplex $V$ is not a part of the complex $\mathbb{L}(G)$ in Boundary Nontrivialization.

Proposition 50 and Lemma 60 together provide a parameterized reduction from $k$-Multicolored Clique to Boundary Nontrivialization. Using Theorem 23, we obtain the following result.

Theorem 61. Boundary Nontrivialization is $\mathbf{W}[1]$-hard.

7 FPT algorithms

7.1 FPT algorithm for Topological Hitting Set

In Section 6.1, we showed that Topological Hitting Set is $\mathbf{W}[1]$-hard with the solution size $k$ as the parameter. This motivates the search of other meaningful parameters that make the problem tractable. With that in mind, in this section, we prove an important structural
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property about the connectivity of the minimal solution sets for Topological Hitting Set. First, we start with a definition.

Definition 62 (Induced subgraphs in Hasse graphs). Given a d-dimensional complex \( K \) with Hasse graph \( H_K \), and a set \( S \) of \( r \)-simplices for some \( r < d \), the subgraph of \( H_K \) induced by \( S \) is the union of \( S \) with the set of \((r+1)\)-dimensional simplices incident on \( S \).

Lemma 63. Given a d-dimensional complex \( K \), a minimal solution of Topological Hitting Set for a non-bounding cycle \( \zeta \in Z_r(K) \) for some \( r < d \) induces a connected subgraph of \( H_K \).

Proof. Let \( H_S \) be the subgraph of the Hasse graph \( H_K \) induced by a minimum topological hitting set \( S \) of a non-bounding cycle \( \zeta \in Z_r(K) \). Targeting a contradiction, assume there exist two components \( C_1 \) and \( C_2 \) such that \( C_1 \) and \( C_2 \) have no edges in common. Note that we do not assume that \( C_1 \) and \( C_2 \) are connected components, merely that they are components that do not share an edge. Since \( S \) is minimal, there exists a cycle \( \phi \in [\zeta] \) that is incident on an \( r \)-simplex in \( C_1 \) but not on any \( r \)-simplices in \( C_2 \), and a cycle \( \psi \in [\zeta] \) that is incident on an \( r \)-simplex in \( C_2 \) but not on any \( r \)-simplices in \( C_1 \). Then, \( \phi = \psi + \partial b \), for some \((r+1)\)-chain \( b \). Let \( b' \) be an \((r+1)\)-chain obtained from \( b \) by removing exactly those \((r+1)\)-simplices that are incident on \( C_1 \). Now, let \( \phi' = \psi + \partial b' \). By construction, \( \phi' \) is not incident on \( C_1 \). Also, because \( C_1 \) and \( C_2 \) are disconnected, the simplices removed from \( b \) to obtain \( b' \) are not incident on \( C_2 \). Hence, \( \phi' \) is not incident on \( C_2 \). In other words, \( \phi' \in [\zeta] \) does not meet \( S \), and \( S \) is not a hitting set, a contradiction. Therefore, the induced subgraph of \( S \) is connected. □

Note that the path from any \( r \)-simplex to a neighboring \( r \)-simplex in the Hasse graph is of size 2. So it follows from Lemma 63 that any minimal solution of size at most \( k \) lies in some geodesic ball of radius \( 2k \) of some \( r \)-simplex in the Hasse graph. In particular, if we search across the geodesic ball of every \( r \)-simplex in the complex \( K \), we will find a solution if one exists. So, if we choose \( k + \Delta \), where \( \Delta \) is the maximum degree of the Hasse graph, the search becomes tractable. In fact, we can even count the number of minimal solutions. We remark that the degree \( \Delta \) of the Hasse graph \( H_K \) is bounded when the dimension of the complex is bounded and the number of incident cofacets on every simplex is bounded.

Algorithm 6 FPT Algorithm for Topological Hitting Set with \( k + \Delta \) as the parameter

1: \( \min \leftarrow |K| \); SOL = \( K \);
2: for each \( r \)-simplex \( \tau \) of \( K \) do
3: \hspace{1em} Consider the set \( S_\tau \) of all simplices within the graph distance \( 2k \) (in \( H_K \)) of \( \tau \).
4: \hspace{1em} if a connected subset \( S \subseteq S_\tau \) with \( |S| \leq k \) is a hitting set of \( \zeta \) and \( |S| < \min \) then
5: \hspace{2em} \( \min = |S| \); SOL = \( S \);
6: \hspace{1em} end if
7: end for
8: if \( \min < k \) then return SOL;
9: end if

Correctness.

The correctness of the algorithm immediately follows from Lemma 63.
Complexity.

Note that in Line 4 of Algorithm 6, we need to enumerate only the connected subsets $S$ of cardinality less than or equal to $k$. We use Lemmas 54 and 55 by Fomin and Villanger [25] that provide very good bounds for enumerating connected subgraphs of graphs. First, we introduce some notation.

- **Notation 6.** The neighborhood of a vertex $v$ is denoted by $\text{nbd}(v) = \{u \in V : u, v \in E\}$, whereas the neighborhood of a vertex set $S \subseteq V$ is set to be $\text{nbd}(S) = \bigcup_{v \in S} N(v) \setminus S$.

- **Lemma 64 (Lemma 3.1).** Let $G = (V, E)$ be a graph. For every $v \in V$, and $b, d \geq 0$, the number of connected vertex subsets $C \subseteq V$ such that

  1. $v \in B$,
  2. $|B| = b + 1$, and
  3. $|\text{nbd}(B)| = d$

  is at most $\binom{b+d}{b}$.

- **Lemma 65 (Lemma 3.2).** All connected vertex sets of size $b+1$ with $f$ neighbors of an $n$-vertex graph $G$ can be enumerated in time $O(n^2 \cdot b \cdot (b + d) \cdot \binom{b+d}{b})$ by making use of polynomial space.

  In Algorithm 6, $b = O(k)$ and $d = O(k\Delta)$. Therefore,

  $$\binom{b+d}{b} = \left(\frac{O(k\Delta)}{O(k)}\right) \leq (k\Delta)^{O(k)} = 2^{O(k \log(k\Delta))}. \tag{6}$$

  Hence, by Lemma 65, for a single $r$-simplex, the number of connected sets enumerated in Line 4 is $O(n^2 \cdot O(k) \cdot O(k\Delta) \cdot 2^{O(k \log(k\Delta))}) = O(n^5 \cdot 2^{O(k \log(k\Delta))})$ time. Since we do this for every $r$-simplex $\tau$ in $\mathbf{K}$, the total time in enumerating all candidate sets in Lines 4-6 is at most $O(n^6 \cdot 2^{O(k \log(k\Delta))})$. Using Theorem 3, one can check if the set is a feasible solution in time $O(n^\omega)$, where $\omega$ is the exponent of matrix multiplication. Hence, the algorithm runs in $O(n^6 + \omega \cdot 2^{O(k \log(k\Delta))})$ time, which is fixed parameter tractable in $k + \Delta$.

- **Theorem 66.** **Topological Hitting Set** admits an FPT algorithm with respect to the parameter $k + \Delta$, where $\Delta$ is the maximum degree of the Hasse graph and $k$ is the solution size. The algorithm runs in $O(n^{6+\omega} \cdot 2^{O(k \log(k\Delta))})$ time.

### 7.1.1 Randomized FPT algorithm for Global Topological Hitting Set

Ostensibly, **Global Topological Hitting Set** looks a lot harder than **Topological Hitting Set**. However, this is not really the case. Fortunately, we can exploit the vector space structure of homology to design a randomized algorithm for **Global Topological Hitting Set** that uses the deterministic FPT algorithm for **Topological Hitting Set** as a subroutine.

**Algorithm 7** Randomized FPT Algorithm for **Global Topological Hitting Set** with $k + \Delta$ as the parameter

1. Find the $r$-th homology basis of $\mathbf{K}$. Denote the basis by $\mathbf{B}$. Here, $|\mathbf{B}| = \beta_r(\mathbf{K})$.
2. Arrange the cycles in $\mathbf{B}$ in a matrix. Denote the matrix by $\mathbf{B}$.
3. Let $\mathbf{x}$ be a uniformly distributed random binary vector of dimension $\beta_r(\mathbf{K})$.
4. With $\mathbf{B} \cdot \mathbf{x}$ as the input cycle, and $k + \Delta$ as the parameter, invoke Algorithm 6.
**Proposition 67.** The probability that a minimal topological hitting set of the cycle $B \cdot x$ is the optimal solution to **Global Topological Hitting Set** is at least $1/2$.

**Proof.** Note that the total number of nontrivial $r$-th homology classes of $K$ is $2^{\beta_r(K)}$. Let $S$ be an optimal solution to **Global Topological Hitting Set**. Then, because of the vector space structure of homology groups, the total number of nontrivial homology classes of $S$ is at most $2^{\beta_r(K)} - 1$. In other words, $S$ is a topological hitting set of at least $2^{\beta_r(K)} - 1$ nontrivial classes. Let $C$ be the set of $r$-th homology classes for which $S$ is a topological hitting set. Then, the probability that a uniformly random homology class chosen by $B \cdot x$ belongs to $C$ is at least $2^{\beta_{r+1}(K)} = \frac{1}{2}$. ◀

From the proposition above, the following corollary follows immediately.

**Corollary 68.** Algorithm 7 is a randomized FPT algorithm for **Global Topological Hitting Set** with $k + \Delta$ as the parameter.

### 7.2 FPT approximation algorithm for Boundary Nontrivialization

It turns out that we do not have a connectivity lemma analogous to Lemma 63 for **Boundary Nontrivialization**. For instance, consider the triangulation of a sphere as the input complex $K$, and let the boundary that needs to be made nontrivial be the equator of the sphere. Then, the two triangles at the north pole and the south pole constitute an optimal solution for **Boundary Nontrivialization**, as the removal of these triangles makes the boundary nontrivial. Clearly, the solution set consisting of these two triangles is not connected. Please refer to Figure 2 from Section 1. So it is not clear if there is an FPT algorithm for **Boundary Nontrivialization** with $k + \Delta$ as the parameter.

This motivates the search of another parameter that makes the problem tractable. To this end, we first make a few elementary observations.

**Lemma 69.** If there are two $(r+1)$-chains $\xi$ and $\xi'$ with $b$ as a boundary, then their sum is an $(r+1)$-cycle. Also, if an $(r+1)$-chain $\xi$ has $b$ as a boundary, and $\zeta$ is an $r+1$-cycle, then $\xi + \zeta$ has $b$ as a boundary.

**Proof.** If $\partial \xi = b$ and $\partial \xi' = b$, then we have $\partial(\xi + \xi') = 0$.

Next, if $\partial \xi = b$ and $\partial \zeta = 0$, then we have $\partial(\xi + \zeta) = b$. ◀

In other words, the number of chains that have $b$ as a boundary is precisely $2^{\dim Z_{r+1}(K)}$. When the complex is $(r+1)$-dimensional, $Z_{r+1}(K) = H_{r+1}(K)$. So, for $(r+1)$-dimensional complexes we provide an FPT approximation algorithm with $\beta_{r+1}(K)$ as a parameter. Let $B$ be the $(r+1)$-th boundary matrix of complex $K$. The algorithm can be described as follows.
Algorithm 8 FPT approximation algorithm for Boundary Nontrivialization

1: \( B' \leftarrow B; \quad X = \{\}\);  
2: \( \text{while } B' \cdot x = \zeta \text{ has a solution} \) do  
3: \( X \leftarrow X \cup \{x\} \).  
4: \( Y \leftarrow \) the set of all chains generated from odd linear combinations of elements in \( X \).  
5: \( \text{Let } Y \text{ denote the matrix with the chains of } Y \text{ as its columns.} \)  
6: \( \text{Let } R \text{ be the collection of row indices of } Y \text{ and } C \text{ be the collection of column indices.} \)  
7: \( \text{In the natural way, interpret } R \text{ as a collection of sets, } C \text{ as a collection of elements, and } Y \text{ as the incidence matrix between sets and elements.} \)  
8: \( \text{Solve the Set Cover problem approximately for the instance described above using the greedy method [36, Chapter 2.1].} \)  
9: \( \text{Let } S \subseteq R \text{ be the approximate solution for the setcover problem.} \)  
10: \( \text{Let } B' \text{ be the matrix formed by deleting from } B \text{ the columns specified by the } r+1\text{-simplices in } S. \)  
11: \( \text{end while} \)  
12: \( \text{Return } S. \)

\[\blacktriangleright\text{Lemma 70. The algorithm terminates in } O(2^\beta \beta n \cdot \min(n, 2^\beta)) \text{ time, where } \beta = \beta_{r+1}(K) + 1 \text{ and } n \text{ is the number of simplices in } K.\]

\[\blacktriangleright\text{Proof. First, we note that the algorithm terminates. This is because in each iteration of the while loop we add a vector } x \text{ that is linearly independent to the vectors in set } X. \text{ By Lemma 69, the number of iterations is bounded by } \beta = \beta_{r+1}(K) + 1. \text{ With an appropriate choice of data structures, Lines 4 and 5 can be executed in } O(2^\beta n) \text{ time. Note that the resulting matrix } Y \text{ has } O(2^\beta) \text{ columns and } O(n) \text{ rows.}\]

\[\text{The most expensive step in the while loop is Line 8. A simple implementation of the greedy approximation algorithm for Set Cover runs in } O(2^\beta n \cdot \min(n, 2^\beta)) \text{ time [15, Chapter 35.3].}\]

\[\blacktriangleright\text{Lemma 71. When the while loop terminates } S \text{ covers every chain whose boundary is } \zeta. \text{ The set } S \text{ returned at Line 11 provides an } O(\log n)\text{-factor approximation to Boundary Nontrivialization.} \]

\[\blacktriangleright\text{Proof. This follows from the fact that after deleting some columns of } B \text{ specified by set } S, \text{ if the loop terminates, then the new matrix } B' \text{ has no solution to the equation } B' \cdot x = \zeta. \text{ The algorithm provides an } O(\log n) \text{ factor approximation because we use the approximation algorithm for Set Cover as a subroutine in Line 8.}\]

\[\blacktriangleright\text{Lemma 70 and Lemma 71 combine to give the following theorem.} \]

\[\blacktriangleright\text{Theorem 72. Boundary Nontrivialization has an } O(\log n)\text{-factor FPT approximation algorithm that takes bounding } r\text{-cycles as input on } (r+1)\text{-dimensional complexes, and runs in } O(2^\beta \beta n \cdot \min(n, 2^\beta)) \text{ time, where } \beta = \beta_{r+1}(K) + 1 \text{ and } n \text{ is the number of simplices in } K.\]

7.2.1 Randomized FPT approximation algorithm for Global Boundary Nontrivialization

As in the case of Global Topological Hitting Set in Section 7.1.1 we now exploit the vector space structure of the boundary group to design a randomized algorithm for Global
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Boundary Nontrivialization that uses the deterministic FPT approximation algorithm for Boundary Nontrivialization as a subroutine.

Algorithm 9 Randomized FPT approximation algorithm for Global Topological Hitting

1: Find a basis $\mathcal{B}$ for the column space of $\partial_{r+1}(K)$.
2: Arrange the bounding cycles from $\mathcal{B}$ in a matrix $\mathbf{B}$.
3: Let $\mathbf{x}$ be a uniformly distributed random binary vector of dimension $|\mathcal{B}|$.
4: With $\mathbf{B} \cdot \mathbf{x}$ at the input boundary, and $\beta$ as the parameter, invoke Algorithm 8.

Proposition 73. Let $\mathcal{R}$ be a minimal set of simplices whose removal from $K$ makes $B \cdot \mathbf{x}$ nontrivial. The probability that $\mathcal{R}$ is the optimal solution to Global Boundary Nontrivialization is at least $\frac{1}{2}$.

Proof. The total number of elements in the range of $\mathcal{B}$ is $2^{|\mathcal{B}|}$. Let $\mathcal{S}$ be a optimal solution to Global Boundary Nontrivialization. Suppose that $\mathbf{c}$ is a bounding cycle that is made nontrivial by removal of $\mathcal{S}$ from $K$. Then, $\mathbf{c} \notin \text{im}(\partial_{r+1}(K_S))$. Suppose that $\mathbf{c}'$ is a bounding cycle that continues to be trivial following removal of $\mathcal{S}$ from $K$. That is, $\mathbf{c}' = \partial_{r+1}(K_S) \cdot \mathbf{x}$ for some $\mathbf{x}$. Then, $\mathbf{c} + \mathbf{c}' \notin \text{im}(\partial_{r+1}(K_S))$, for otherwise, there would exist a vector $\mathbf{y}$ such that $\mathbf{c} + \mathbf{c}' = \partial_{r+1}(K_S) \cdot \mathbf{y}$, which gives $\mathbf{c} = \partial_{r+1}(K_S) \cdot (\mathbf{x} + \mathbf{y})$, a contradiction.

Now, assume that none of the bounding cycles in the basis $\mathcal{B}$ are made nontrivial by the removal of $\mathcal{S}$ from $K$. But that implies that any linear combination of cycles in $\mathcal{B}$ also belongs to $\text{im}(\partial_{r+1}(K_S))$. This contradicts the existence of $\mathbf{c}$. So there exists at least one bounding cycle $\mathbf{b} \in \mathcal{B}$ which is made nontrivial by the removal of $\mathcal{S}$. Let $\mathcal{B}_S$ be the subset of cycles in $\mathcal{B}$ that are made nontrivial by the removal of $\mathcal{S}$. Then, we have two cases:

Case 1: Suppose $\mathbf{b}$ is the only cycle in $\mathcal{B}_S$.

Now, let $\mathbf{z}$ be any cycle that lies in the span of $\mathcal{B} \setminus \{\mathbf{b}\}$. By the argument above, $\mathbf{b} + \mathbf{z}$ is also made nontrivial by the removal of $\mathcal{S}$. So, the total number of bounding cycles that are made nontrivial by the removal of $K_S$ is $2^{|\mathcal{B}|} - 1$.

Case 2: Suppose $\mathcal{B}_S \setminus \{\mathbf{b}\}$ is nonempty.

Then, one obtains a new set of vectors $\mathcal{B}'$ from $\mathcal{B}$ as follows: For every $\mathbf{a} \neq \mathbf{b} \in \mathcal{B}$ such that $\mathbf{a}$ is made nontrivial by the removal of $\mathcal{S}$ and $\mathbf{a} + \mathbf{b}$ is in $\text{im}(\partial_{r+1}(K_S))$, replace $\mathbf{a}$ by $\mathbf{a} + \mathbf{b}$. It is easy to check that $\mathcal{B}'$ is also a basis for the column space of $\partial_{r+1}(K)$. Moreover, if $\mathbf{z}'$ is any cycle that lies in the span of $\mathcal{B}' \setminus \{\mathbf{b}\}$, then $\mathbf{b} + \mathbf{z}'$ is also made nontrivial by the removal of $\mathcal{S}$. So, the total number of bounding cycles that are made nontrivial by the removal of $K_S$ is at least $2^{|\mathcal{B}'|} - 1$.

From the above analysis, we conclude that there are at least $2^{|\mathcal{B}|} - 1$ bounding $r$-cycles that are made nontrivial by the removal of $\mathcal{S}$. Let $\mathcal{C}$ be the set of bounding $r$-cycles for which $\mathcal{S}$ is a Boundary Nontrivialization solution. Then, the probability that a uniformly random bounding cycle chosen by $\mathbf{B} \cdot \mathbf{x}$ belongs to $\mathcal{C}$ is at least $\frac{2^{|\mathcal{B}|} - 1}{2^{|\mathcal{B}|}} = \frac{1}{2}$.

From the proposition above, we obtain the following corollary immediately.

Corollary 74. Global Boundary Nontrivialization has an $O(\log n)$-factor randomized FPT approximation algorithm for $r$-th homology on $(r + 1)$-dimensional complexes, with $\beta$ as the parameter. The algorithm runs in $O(2^\beta \beta n \cdot \min(n, 2^\beta))$ time.
8 Conclusion and Discussion

In this paper, we devise a polynomial time algorithm for Topological Hitting Set on closed surfaces. We believe that our algorithm should also easily generalize to surfaces with boundary.

Moreover, we show how certain cut problems generalize naturally from graphs to simplicial complexes, motivating a complexity theoretic study of these problems. For future work, it remains to be shown that Global Topological Hitting Set and Global Boundary Nontrivialization are also \( W[1] \)-hard. We believe that the \( W[1] \)-hardness reductions for Topological Hitting Set and Boundary Nontrivialization can be extended to establish hardness results for the global variants. Finally, a theoretical future direction of our work is to investigate how (the global variants of) Topological Hitting Set and Boundary Nontrivialization may be used to study high dimensional expansion in simplicial complexes [19,29].

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