Abstract

It is shown that a trace invariant projection map, i.e. a positive unital idempotent map, of a finite dimensional $C^*$-algebra into itself is non-decomposable if and only if it is atomic, or equivalently not the sum of a 2-positive and a 2-copositive map. In particular projections onto spin factors of dimension greater than 6 are atomic.

Introduction

The classification theory for positive linear maps between $C^*$-algebras has been progressing slowly since Stinepring [6] introduced the class of completely positive maps in 1955. Even in the simple case when both $C^*$-algebras are the complex $3 \times 3$ matrices $M_3(\mathbb{C})$, the classification problem is still open. So far main emphasis has been on completely positive maps, copositive maps, i.e. those which are the composition of a completely positive map and an anti-automorphism, or more generally, k-positive maps, which are maps \( \phi \) such that \( \phi \otimes \tau_k \) is positive, where \( \tau_k \) is the identity on \( M_k(\mathbb{C}) \), and the corresponding k-copositive maps. These maps are reasonably well understood, see e.g. [5]. Furthermore maps which are sums of completely positive maps and copositive maps, called decomposable maps, and those which are not, called nondecomposable maps, have attracted much attention.

A more general class of which we have a reasonably good understanding is the cone of \((k,m)\)-decomposable maps, which are those which are the sum of a k-positive and an m-copositive map, including the case when one of them is the zero map. They are all contained in the cone of \((2,2)\)-decomposable maps. It is thus natural to consider maps which are not \((2,2)\)-decomposable, called atomic in the literature. It was shown by Ha [3] that the natural generalizations of the Choi map [1] are atomic. In the present paper we shall show that trace preserving projection maps are atomic if and only if they are nondecomposable. In [7] it was shown that the projection maps onto spin factors of dimension greater than 6 are nondecomposable, hence we obtain atomic maps in arbitrary large dimensions. We shall at the end apply the above results to give a sufficient condition for a positive map to be be atomic, in terms of algebraic properties of its fixed point set.
1 Atomic projection maps

If $B$ is a unital C$^*$-algebra a positive linear map $P: B \to B$ is called a projection map if $P(1) = 1$ and $P^2 = P$. The theory of such maps is intimately related to the theory of Jordan algebras of self-adjoint operators under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ for $a, b \in B_{sa}$, the self-adjoint operators in $B$. See [4] for the theory of Jordan algebras of self-adjoint operators. In fact, if $P$ is faithful, so $P(a) \neq 0$ for $a > 0$, then $A = P(B_{sa})$ is a Jordan subalgebra of $B_{sa}$. [2]. Furthermore, by [7] $P$ is decomposable if and only if $A$ is a reversible Jordan algebra, i.e. $A$ is closed under symmetric products $a_1a_2...a_n + a_na_n_1...a_1$, with $a_i \in A$. In the present section we shall sharpen the above result by replacing decomposable by $(2,2)$-decomposable maps. For simplicity we shall assume the C$^*$-algebra $B$ is acting on a finite dimensional Hilbert space $H$ and denote the usual trace on $B(H)$ by $\text{Tr}$.

**Theorem 1** Let $B$ be a C$^*$-algebra acting on a finite dimensional Hilbert space. Let $P: B \to B$ be a trace preserving unital projection map. Then we have:

(i) $P$ is $(2,2)$-decomposable if and only if $P$ is decomposable.

(ii) $P$ is atomic if and only if $A = P(B_{sa})$ is a nonreversible Jordan algebra.

The proof will be divided into some lemmas. We first recall that a spin system in $B(H)$ is a set $(s_i)_{i \in I}$ of symmetries, i.e. self-adjoint unitaries in $B(H)$, such that $s_i \circ s_j = 0$ for $i \neq j$. The real linear span of $1$ and the $s_i$ is a Jordan algebra called a spin factor, see [4] Chapter 6. If $H$ is finite dimensional there is a canonical positive trace preserving projection map $P$ of $B(H)$ onto $A + iA$, see [2]. For simplicity we often write $P: B(H) \to A$.

**Lemma 2** Let $A \subset B(H)$ be a spin factor. Let $e_1, ..., e_k$ be nonzero minimal projections in the commutant of $A$ with sum $1$. Let

$$P_i: e_iB(H)e_i \to Ae_i$$

be the canonical projection map, and let $P: B(H) \to A$ be the canonical projection map. Let $\alpha_i: Ae_i \to A$ by $\alpha(\alpha e_i) = a$. Then $\alpha_i$ is an isomorphism, and

$$P(a) = \sum \text{Tr}(e_i)\alpha_i(P_i(e_iae_i)).$$

Hence $P$ is decomposable if and only if each $P_i$ is decomposable.

**Proof.** Since $A$ is a simple Jordan algebra $\alpha_i$ is an isomorphism. Let $(s_j)$ be a spin system generating $A$. Then $(s_j e_i)$ is a spin system generating $Ae_i$. Let $c_i = \text{Tr}(e_i)^{-1}$. Then $c_i\text{Tr}$ is the tracial state on $e_iB(H)e_i$, and the canonical projection $P_i$ being the orthogonal projection of $e_iB(H)e_i$ with respect to the Hilbert Schmidt structure onto the subspace generated by $Ae_i$ is given by

$$P_i(e_iae_i) = c_i\text{Tr}(e_iae_i)e_i + c_i \sum_j \text{Tr}(ae_isjesje_i)e_i = c_iP(e_iae_i)e_i.$$

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Thus we have
\[ \alpha_i(P_i(e_i a e_i)) = \alpha_i(c_i P(e_i a e_i)e_i) = c_i P(e_i a e_i). \]

Let \( E : B(H) \to \bigoplus e_i B(H) e_i \), given by \( E(x) = \sum e_i x e_i \). Then \( E \) is completely positive, and by uniqueness of \( P \) as the trace invariant projection of \( B(H) \) onto \( A \), \( P \) is the restriction
\[ P = (P|\bigoplus e_i B(H) e_i) \circ E. \]

Thus, if \( a \in B(H) \) then
\[ P(a) = \sum_i P(e_i a e_i) = \sum_i c_i^{-1} \alpha_i P_i(e_i a e_i) = \sum_i \text{Tr}(e_i) \alpha_i (P_i(e_i a e_i)) \]

Since \( E \) and \( \alpha_i \) are completely positive, \( P \) is decomposable if and only if each \( P_i \) is decomposable. The proof is complete.

In the case when \( A \) is a spin factor we have reduced the proof of the theorem to the case when \( A \) is irreducible, \( C^*(A) = B(H) \), or equivalently the commutant of \( A \) is the scalars.

**Lemma 3** Let \( R_2 \) denote the real symmetric \( 2 \times 2 \) matrices. Let \( \phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \) be a positive map such that if \( \iota_{R_2} \) denotes the identity map on \( R_2 \), then \( \iota_{R_2} \geq \mu \phi \) for some \( \mu \geq 0 \). Then \( \phi = \lambda \iota_{R_2} \) for some \( \lambda \geq 0 \).

**Proof.** Let \( e \) be a 1-dimensional projection in \( R_2 \). Then \( e = \iota_{R_2}(e) \geq \mu \phi(e) \geq 0 \), so \( \phi(e) = \alpha e, \alpha \geq 0 \). Similarly \( \phi(1-e) = \beta(1-e) \), since \( 1-e \) is also 1-dimensional. Thus \( \phi(1) \) belongs to the maximal abelian subalgebra of \( R_2 \) generated by \( e \). This holds for all minimal projections \( e \) in \( R_2 \), which is possible only if \( \phi(1) = \lambda 1 \) for some \( \lambda \geq 0 \). Since \( 1 = \alpha 1 + \beta(1-e) \), \( \alpha = \beta = \lambda \), i.e., \( \phi(e) = \lambda e \) for all minimal projections \( e \in R_2 \), so that \( \phi = \lambda \iota_{R_2} \). The proof is complete.

**Lemma 4** Let \( A \subseteq B(H) \) be a spin factor and \( P \) the canonical projection of \( B(H) \) onto \( A \). Suppose \( \phi : B(H) \to B(H) \) is a positive linear map such that \( \phi \leq \mu \phi \) for some \( \mu > 0 \). Then the restriction \( \phi|A = \lambda \iota_A \) for some \( \lambda \geq 0 \), where \( \iota_A \) is the identity map on \( A \).

**Proof.** Let \( s_i \) and \( s_j, i \neq j \), belong to the spin system generating \( A \), and let \( F_{i,j} \) denote the canonical trace preserving conditional expectation of \( B(H) \) onto the \( C^* \)-algebra \( C^*(s_i, s_j) \) generated by \( s_i \) and \( s_j \). Since \( \text{span}\{1, s_i, s_j\} \approx R_2 \), by Lemma 3 the restriction \( F_{i,j} \circ \phi|_{R_2} = \lambda \iota_{R_2} \) for some \( \lambda \geq 0 \). Thus we have \( F_{i,j} \circ \phi(1) = \lambda 1 \), and \( F_{i,j} \circ \phi(s_i) = \lambda s_i \) and similarly for \( s_j \). Since this holds for all \( j \neq i \), it follows that \( F_{i,j} \circ \phi(1) = \lambda 1 \), and \( F_{i,j} \circ \phi(s_i) = \lambda s_i \) for all \( i \neq j \). Thus \( \phi(1) = \lambda 1 + a_{ij} \) with \( a_{ij} \) orthogonal to \( C^*(s_i, s_j) \) in the Hilbert Schmidt structure. But then \( a_{ij} = a_{kl} = a \) for all \( i, j, k, l \), and so \( a \) is orthogonal to \( A^2 = \{xy : x, y \in A\} \). Similarly \( \phi(s_i) = \lambda s_i + b \) with \( b \) orthogonal to \( A^2 \). Scaling \( \phi \) we may assume
\[ \phi(1) = 1 + a, \phi(s_i) = s_i + b_i, a, b_i \perp A^2. \]
Let $s = s_i$ and $e = e_i = \frac{1}{2}(1 + s_i) = \frac{1}{2}(1 + s)$. Then $e$ is a projection in $A$. Since $\phi \leq \mu P$, $\phi(e) \leq \mu e$. Since $f = \frac{1}{2}(1 - s) = 1 - e$ is a projection orthogonal to $e$, we have $\phi(e)f = 0$, and similarly $\phi(f)e = 0$. Calculating we get, using that $s^2 = 1$

\[ 0 = (\phi(1 + s))(1 - s) = ((1 + a) + (s + b))(1 - s) = a + b - as - bs, \]

\[ 0 = (\phi(1 - s))(1 + s) = ((1 + a) - (s + b))(1 + s) = a + b + as - bs. \]

Adding these two equations we get $0 = a - bs$, hence $\phi(1) = 1 + bs$, and $\phi(s) = s + b = (1 + bs)s = \phi(1)s$. Thus the product of the two self-adjoint operators $\phi(1)$ and $s$ is self-adjoint, hence they commute. Since $s = s_i$ was an arbitrary symmetry in the spin system spanning $A$, $\phi(1)$ belongs to the commutant of $A$. But $A$ was assumed to be irreducible, so $bs = 0 = b$, so that $\phi(1) = 1$, and $\phi(s_i) = s_i$ for all $i$. This completes the proof of the lemma.

**Lemma 5**: Let $A$ be an irreducible spin factor acting on the finite dimensional Hilbert space $H$. Let $\phi, \psi: B(H) \to B(H)$ be unital maps with $\phi$ 2-positive and $\psi$ 2-copositive such that their restrictions to $A$ are the identity map. Let $s, t$ be distinct symmetries in the spin system spanning $A$. Then

\[ \phi(st) = st, \psi(st) = ts. \]

**Proof.** Let $x = s + it$. Since $st = -ts$ we have

\[ x^*x = 2(1 + ist), xx^* = 2(1 - ist). \]

Hence $x^*x + xx^* = 4 \cdot 1 \in A$. Since $\phi$ is 2-positive it satisfies the Schwarz inequality \[ \Pi, \text{Cor. 2.8}, \] hence $\phi(x^*x) \geq \phi(x)^*\phi(x) = x^*x$, and $\phi(xx^*) \geq \phi(x)\phi(x)^* = xx^*$. Thus

\[ 0 = \phi(x^*x + xx^*) - 4 \cdot 1 = \phi(x^*x) + \phi(xx^*) - 4 \cdot 1 \geq x^*x + xx^* - 4 \cdot 1 = 0, \]

Hence $\phi(x^*x) = x^*x$, and $\phi(xx^*) = xx^*$. In particular $\phi(st) = st$.

Since $\psi$ is 2-copositive, $\psi(x^*x) \geq \psi(x)\psi(x)^* = xx^*$, so by the above argument applied to $\psi$ we get $\psi(st) = ts$, completing the proof.

**Lemma 6**: Let $\phi, \psi: B(H) \to B(H)$ be unital maps with $\phi$ 2-positive and $\psi$ 2-copositive. Let $a \in B(H)$. Then we have:

(i) If $\phi(aa^*) = \phi(a)\phi(a)^*$, then $\phi(ab) = \phi(a)\phi(b)$ $\forall b \in B(H)$.

(ii) If $\psi(a^*a) = \psi(a)\psi(a)^*$, then $\psi(ba) = \psi(a)\psi(b)$ $\forall b \in B(H)$.

**Proof.** Let

\[ \langle x, y \rangle = \phi(xy^*) - \phi(x)\phi(y)^*, x, y \in B(H). \]

Then $\langle, \rangle$ is an operator valued sesquilinear form such that for all states $\omega$ on $B(H)$, $\langle x, y \rangle_{\omega} = \omega(\langle x, y \rangle)$ is a sesquilinear form. Thus by the Cauchy - Schwarz inequality

\[ |\langle x, y \rangle_{\omega}| \leq \omega(\phi(xx^*) - \phi(x)\phi(x)^*)^{\frac{1}{2}}\omega(\phi(yy^*) - \phi(y)\phi(y)^*)^{\frac{1}{2}}. \]
Hence if \( a \) is as in the statement of (i), then \( \langle a, b \rangle = 0 \) for all \( \omega \), hence \( \langle a, b \rangle \geq 0 \), i.e. \( \phi(ab^*) = \phi(a)\phi(b^*) \) for all \( b \in B(H) \), proving (i).

(ii) In this case we consider the operator valued sesquilinear form
\[
\langle x, y \rangle = \psi(xy^*) - \psi(y^*)\psi(y),
\]
and we use the same arguments as in (i). The proof is complete.

Let \( V_k \) denote the spin factor generated by a spin system consisting of \( k \) symmetries, defined as in [4], section 6.2.

**Lemma 7** Let \( P \) be the canonical projection of \( B(H) \) onto the spin factor \( A \). Suppose \( P \) is \((2,2)\)-decomposable. Then \( P \) is decomposable, and \( A \) is one of the spin factors \( V_2, V_3, V_5 \).

**Proof.** By Lemma 2 we may assume the \( C^* \)-algebra generated by \( A, C^*(A) = B(H) \). Suppose \( P = \phi + \psi \) with \( \phi \) 2-positive and \( \psi \) 2-copositive. By Lemma 4 we can replace \( \phi \) and \( \psi \) by maps which are the identity map on \( A \), and assume \( P = \lambda \phi + (1 - \lambda)\psi, 0 \leq \lambda \leq 1 \). Then by Lemma 5 if \( s_k \neq s_j \) are symmetries in the spin system spanning \( A \) then \( \phi(s_k s_j) = s_k s_j \), and \( \psi(s_k s_j) = s_j s_k \). From the proof of Lemma 5 if \( x = s_k + is_j \), then \( \phi(xx^*) = xx^* \), and \( \psi(xx^*) = x^*x \), so by Lemma 6
\[
\phi(xx^*b) = \phi(xx^*)\phi(b) = xx^*\phi(b) \quad \forall b \in B(H).
\]
Now the monomials \( s_{i_1} s_{i_2} \ldots s_{i_n} \) span \( C^*(A) = B(H) \) linearly. For such monomials we have, since the above equation holds in particular for \( s_k s_j \),
\[
\phi(s_{i_1} s_{i_2} \ldots s_{i_n}) = s_{i_1} s_{i_2} \phi(s_{i_3} \ldots s_{i_n}) = s_{i_1} s_{i_2} s_{i_3} s_{i_4} \phi(s_{i_5} \ldots s_{i_n}) = \ldots = s_{i_1} s_{i_2} \ldots s_{i_n}.
\]
Thus \( \phi \) is the identity map. Similarly we have
\[
\psi(s_{i_1} s_{i_2} \ldots s_{i_n}) = s_{i_n} s_{i_{n-1}} \ldots s_{i_1},
\]
so that \( \psi \) is an anti-automorphism of order 2, which is the identity on \( A \). In particular \( \phi \) is completely positive, and \( \psi \) is copositive. It follows that \( P \) is decomposable. It then follows from [7], Corollary 7.3, that \( A \) is reversible, hence by [3], Theorem 6.2.5, that \( A \) is one of the spin factors \( V_2, V_3, V_5 \). The proof is complete.

**Proof of Theorem 1.** Let \( B \) be a \( C^* \)-algebra acting on the finite dimensional Hilbert space \( H \), and let \( P \) be a unital trace preserving positive projection map of \( B \) into itself. Let \( A = P(B_{sa}) \). By [2] \( A \) is a Jordan subalgebra of \( B_{sa} \). Composing \( P \) with the trace invariant conditional expectation \( E \) of \( B(H) \) onto \( B \) we may assume that \( P \) is a projection of \( B(H) \) onto \( A \). This does not alter the conclusion of the theorem since \( E \) is completely positive, see e.g. [8], Proposition
9.3. Let $Z$ denote the center of $A$, see [4], 2.5.1, which is the self-adjoint part of an abelian $C^*$-algebra. Let $p$ be a minimal projection in $Z$. Then by [4], Proposition 5.2.17, the center of $pA = pAp = Zp = \mathbb{R}p$, so $pAp$ is a Jordan factor, also called a JW-factor. It follows that $A = \bigoplus_j Ap_j$, with $p_j$ a minimal central projection in $A$, and $Ap_j$ is a Jordan factor.

By [4], section 6.3, a Jordan factor $C$ is either a spin factor or is reversible. In the latter case a projection onto $C$ is necessarily decomposable, [7], Corollary 7.4, hence it remains to consider the case when $Ap_j$ is a spin factor. But then Lemma 7 implies that the projection is decomposable if and only if it is $(2,2)$-decomposable, which proves part (i) of the theorem. Part (ii) is immediate from part (i) by again applying [7]. The proof is complete.

A closer look at the proof shows that we have proved a slightly more general result. Instead of assuming the projection map $P$ is the sum of a $2$-positive map $\phi$ and a $2$-copositive map $\psi$, we could have assumed $\phi$ to satisfy the Schwarz inequality $\phi(x^*x) \geq \phi(x)^*\phi(x)$ and $\psi$ the inequality $\psi(x^*x) \geq \psi(x)^*\psi(x)$.

Theorem 1 has a natural generalization to positive maps.

If $B$ is a finite dimensional $C^*$-algebra and $\phi : B \rightarrow B$ is a positive unital map, then the fixed point set $B_\phi = \{ a \in B : \phi(a) = a \}$ has a natural structure as a Jordan algebra, see [2]. Furthermore, if there exists a faithful $\phi$-invariant state on $B$, then there exists a faithful $\phi$-invariant positive projection map $P_\phi : B \rightarrow B_\phi$ making $(B_\phi)_{sa}$ a Jordan subalgebra of $B_{sa}$. We then have the following extension of Theorem 1.

**Theorem 8** Let $B$ be $C^*$-algebra acting on a finite dimensional Hilbert space. Suppose $\phi : B \rightarrow B$ is a trace preserving unital positive map. Let $P_\phi : B \rightarrow B_\phi$ be the $\phi$-invariant positive projection of $B$ onto $B_\phi$. If $\phi$ is $(2,2)$-decomposable then $P_\phi$ is decomposable, and $(B_\phi)_{sa}$ is a reversible Jordan algebra. In particular if $(B_\phi)_{sa}$ is nonreversible, then $\phi$ is atomic.

**Proof.** The projection map $P_\phi$ is a weak limit of averages $\phi_n = \frac{1}{n} \sum_0^{n-1} \phi_i$. If $\phi$ is $(2,2)$-decomposable, so is $\phi_n$, say $\phi_n = \alpha_n + \beta_n$ with $\alpha_n$ $2$-positive, and $\beta_n$ $2$-copositive. Now a subnet of $(\phi_n)$ converges weakly to $P_\phi$. If $\alpha$ and $\beta$ are corresponding weak limit points of $\alpha_n$ and $\beta_n$ then $\alpha$ is $2$-positive, and $\beta$ is $2$-copositive. Thus $P_\phi$ is $(2,2)$-decomposable, so by Theorem 1 $P_\phi$ is decomposable, because $\phi$ was assumed to be trace preserving, hence so is $P_\phi$, and therefore $P_\phi$ is faithful. By [7], Corollary 7.3, $(B_\phi)_{sa}$ is reversible. The proof is complete.

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