Introduction

The theory of toric varieties is based on the fact that each toric variety of dimension $r$ has an associated fan, i.e., a finite set of rational polyhedral cones in the $r$-dimensional real space.

The theory of intersection homologies was introduced by Goresky and MacPherson in [GM1] and [GM2]. The intersection homologies are obtained by special complexes of sheaves on the variety, which are called the intersection complexes.

The intersection complex of a given variety has a variation depend on a sequence of integers which is called a perversity [GM2, §2]. It is known that the complex with the middle perversity is most important for normal complex varieties. The

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decomposition theorem and strong Lefschetz theorem for the intersection homologies with the middle perversity were proved in [BBD].

Since these theories applied for toric varieties are used in the combinatorial problems (cf. [S2]), it was expected that these theory restricted to toric varieties are described and proved by an combinatorial method in terms of the associated fans (cf. [O3]).

In [I3], we introduced the additive category $GEM(\Delta)$ of graded exterior modules on a finite fan $\Delta$. We defined and constructed the intersection complex $ic_p(\Delta)^\bullet$ as a finite complex in this category for each perversity $p$. We defined a natural functor from the category of finite complexes in $GEM(\Delta)$ to that of complexes of sheaves on the toric variety associated to the fan. It was shown that the intersection complex of the toric variety is obtained by applying this functor to the intersection complex on the fan with the corresponding perversity. The intersection homologies of the toric varieties are also calculated by the complex $\Gamma(ic_p(\Delta))^\bullet$, where $\Gamma$ is an additive functor from $GEM(\Delta)$ to an abelian category of graded modules over an exterior algebra.

In this article, we work only on the fan and the category $CGEM(\Delta)$ of finite complexes in $GEM(\Delta)$.

By considering the barycentric subdivision algebra of a finite fan $\Delta$, we define an object $SdP(\Delta)^\bullet$ of $CGEM(\Delta)$, and we show that the intersection complex of the fan is obtained as a quotient of $SdP(\Delta)^\bullet$ for every perversity.

For a barycentric subdivision of a finite fan, we prove in §2 a decomposition theorem of the intersection complex with the middle perversity.

In §3, we study the case of simplicial fans. We show that the intersection complexes of a simplicial fan defined for any perversities between the top and the bottom perversities are mutually quasi-isomorphic. Some known vanishing theorems on the cohomologies of toric varieties associated to a simplicial fan are proved in terms of
the category CGEM(Δ).

In §4, we get the first and the second diagonal theorems as a consequence of the
decomposition theorem in §2 and the results for simplicial fans in §3.

Let \( H^i(\Gamma(ic_p(\Delta))^\bullet) \) be the homogeneous degree \( j \) part of the \( i \)-th cohomology of
\( \Gamma(ic_p(\Delta))^\bullet \). Then this is a finite dimensional \( \mathbb{Q} \)-vector space and
\( H^i(\Gamma(ic_p(\Delta))^\bullet) \) = \( \{0\} \) for \( (i, j) \not\in [0, r] \times [-r, 0] \). The first diagonal theorem says that, if \( \Delta \) is a complete
fan and \( p \) is the middle perversity, then this vector space is zero unless \( i = j + r \).

The second diagonal theorem is for a cone of dimension \( r \). The strong Lefschets
theorem used by Stanley [S1] in the proof of \( g \)-conjecture on the number of faces
of a simplicial convex polytope can be replaced by this diagonal theorem (cf.[O3, Cor.4.5]).

**Notation**

In the first part [I3] of this series of papers, the coboundary maps of complexes are
denoted by \( d \) or \( \partial \), and called “\( d \)-complexes” and “\( \partial \)-complexes”, respectively. Where
\( \partial \) was used to represent the exterior derivatives of logarithmic de Rham complexes
and their direct sums.

In this paper, we need not use \( \partial \)-complexes since we avoid to work on varieties.
Hence the \( d \)-complexes are simply called complexes in this article.

The notation \( E^\bullet \) means that \( E \) is a complex and its component of degree \( i \) is
\( E^i \). When we apply to it a functor \( F \) to a category of complexes, we usually denote
by \( F(E)^\bullet \) the resulting complex. When \( E^\bullet \) is a complex in an abelian category, its
\( i \)-the cohomology is denoted by \( H^i(E^\bullet) \).

Neither derived categories nor derived functors appear in this article.
1 Barycentric subdivision algebras and resolutions

Let $N$ be a free $\mathbb{Z}$-module of a fixed finite rank $r \geq 0$. By cones in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, we always mean strongly convex rational polyhedral cones (cf. [O1, Chap.1,1.1]). We denote by $\mathbf{0}$ the trivial cone $\{0\}$. For a cone $\sigma$, we set $r_{\sigma} := \dim \sigma$ and $N(\sigma) := N \cap (\sigma + (\sigma))$. Since $\sigma$ is a rational cone, $N(\sigma)$ is a free $\mathbb{Z}$-module of rank $r_{\sigma}$.

For each cone $\sigma$ in $N_{\mathbb{R}}$, we set $det(\sigma) := \bigwedge^{r_{\sigma}} N(\sigma) \simeq \mathbb{Z}$. (1.1)

For cones $\sigma, \tau$ in $N_{\mathbb{R}}$ with $\sigma \prec \tau$ and $r_{\tau} - r_{\sigma} = 1$, we define the incidence isomorphism $q'_{\sigma/\tau} : det(\sigma) \rightarrow det(\tau)$ as follows.

By the short exact sequence

$$0 \rightarrow N(\sigma) \rightarrow N(\tau) \rightarrow N(\tau)/N(\sigma) \rightarrow 0$$

of free $\mathbb{Z}$-modules, we get an isomorphism

$$(N(\tau)/N(\sigma)) \otimes det(\sigma) \simeq det(\tau).$$

Since $\sigma$ is a codimension one face of $\tau$, the image of $\tau$ in $N(\tau)_{\mathbb{R}}/N(\sigma)_{\mathbb{R}}$ is a half line with the edge $0$. We define the isomorphism

$$det(\sigma) \simeq (N(\tau)/N(\sigma)) \otimes det(\sigma)$$

by the isomorphism $\mathbb{Z} \rightarrow N(\tau)/N(\sigma)$ such that 1 is mapped into the image of $\tau$ in $N(\tau)_{\mathbb{R}}/N(\sigma)_{\mathbb{R}}$. The isomorphism $q'_{\sigma/\tau}$ is defined to be the composite of (1.4) and (1.3). Namely, if the class $\bar{a}$ of $a \in N \cap \tau$ in $N(\tau)/N(\sigma)$ is a generator, then $q'_{\sigma/\tau}(w) = a \wedge w$ for $w \in det(\sigma)$.

We get the following lemma (cf. [I1, Lem.1.4]).
Lemma 1.1 Let $\sigma, \rho$ be cones in $N_\mathbb{R}$ with $\sigma \prec \rho$ and $r_\rho - r_\sigma = 2$. Then there exist exactly two cones $\tau$ with $\sigma \prec \tau \prec \rho$ and $r_\tau = r_\sigma + 1$. Let $\tau_1, \tau_2$ be the two cones. Then the equality

$$q'_{\tau_1/\rho} \cdot q'_{\sigma/\tau_1} + q'_{\tau_2/\rho} \cdot q'_{\sigma/\tau_2} = 0$$

(1.5) holds.

Proof. The first assertion is a consequence of the fact that any two-dimensional cone has exactly two edges (cf. [11, Prop.1.3]).

By definition, both $q'_{\tau_1/\rho} \cdot q'_{\sigma/\tau_1}$ and $q'_{\tau_2/\rho} \cdot q'_{\sigma/\tau_2}$ are isomorphisms $\det(\sigma) \cong \det(\rho)$ of the free $\mathbb{Z}$-modules of rank one. Hence it is sufficient to show that these two isomorphisms have mutually opposite signs. Take elements $a \in (\tau_1 \setminus \sigma) \cap N$, $b \in (\tau_2 \setminus \sigma) \cap N$ and $x \in \det(\sigma) \setminus \{0\}$. Then $(q'_{\tau_1/\rho} \cdot q'_{\sigma/\tau_1})(x)$ has the same sign with $b \wedge a \wedge x$, while $(q'_{\tau_2/\rho} \cdot q'_{\sigma/\tau_2})(x)$ has the same sign with $a \wedge b \wedge x$. Hence they have distinct signs. q.e.d.

Let $\sigma$ and $\tau$ be nonsingular cones generated by $\{n_1, \ldots, n_s\}$ and $\{n_1, \ldots, n_{s+1}\}$, respectively, for a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of $N$ and for an integr $0 \leq s < r$ (cf. [11, Thm.1.10]). Then, it is easy to see that

$$q'_{\sigma/\tau}(n_1 \wedge \cdots \wedge n_s) = n_{s+1} \wedge n_1 \wedge \cdots \wedge n_s = (-1)^sn_1 \wedge \cdots \wedge n_s \wedge n_{s+1}.$$  

(1.6)

If $\sigma$ and $\tau$ are simplicial cones generated by $\{n_1, \ldots, n_s\}$ and $\{n_1, \ldots, n_{s+1}\}$, respectively, for a $\mathbb{Q}$-basis $\{n_1, \ldots, n_r\}$ of $N_\mathbb{Q}$, then

$$q'_{\sigma/\tau}(n_1 \wedge \cdots \wedge n_s) = a \cdot n_{s+1} \wedge n_1 \wedge \cdots \wedge n_s$$  

(1.7)

for a positive rational number $a$, where we denote by the same symbol $q'_{\sigma/\tau}$ its extension to $\det(\sigma) \otimes \mathbb{Q} \rightarrow \det(\tau) \otimes \mathbb{Q}$.

For a cone $\pi$ in $N_\mathbb{R}$, we denote by $F(\pi)$ the set of faces of $\pi$. The zero cone $0$ and $\pi$ itself are elements of $F(\pi)$. For cones $\eta, \pi$ of $N_\mathbb{R}$ with $\eta \prec \pi$, we define the
“closed interval”

\[ F[\eta, \pi] := \{ \sigma \in F(\pi) : \eta \prec \sigma \} \]

(1.8)

and the “open interval”

\[ F(\eta, \pi) := F[\eta, \pi] \setminus \{ \eta, \pi \}. \]

(1.9)

We also use the notation

\[ F[\eta, \pi] := F[\eta, \pi] \setminus \{ \pi \}. \]

(1.10)

Let \( \Delta \) be a finite fan \([\Omega, 1.1]\). We call a subset \( \Phi \subset \Delta \) star closed if \( \sigma \in \Phi \) and \( \sigma \prec \rho \in \Delta \) imply \( \rho \in \Phi \). We call \( \Phi \) locally star closed if \( \sigma, \rho \in \Phi \) and \( \sigma \prec \rho \) imply \( F[\sigma, \rho] \subset \Phi \).

Let \( \Phi \) be a locally star closed subset of a finite fan \( \Delta \). We define a finite complex \( E(\Phi, \mathbb{Z})^\bullet \) of free \( \mathbb{Z} \)-modules as follows.

For each integer \( i \), we set

\[ E(\Phi, \mathbb{Z})^i := \bigoplus_{\sigma \in \Phi(i)} \det(\sigma), \]

(1.11)

where \( \Phi(i) := \{ \sigma \in \Phi : r_\sigma = i \} \). For \( \sigma \in \Phi(i) \) and \( \tau \in \Phi(i+1) \), the \((\sigma, \tau)\)-component of the coboundary map

\[ d^i : E(\Phi, \mathbb{Z})^i \to E(\Phi, \mathbb{Z})^{i+1} \]

is defined to be \( q_{\sigma/\tau}' \) if \( \sigma \prec \tau \) and the zero map otherwise. We have \( d^{i+1} \cdot d^i = 0 \) for every \( i \) by Lemma \([\Omega]\).

Here we explain the relation of this complex with the similar complex \( C^\bullet(\Phi, \mathbb{Z}) \) in \([\Omega]\).

Let \( M \) be the dual \( \mathbb{Z} \)-module of \( N \) and let \( M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R} \). For a cone \( \sigma \) in \( N_\mathbb{R} \), we set \( \sigma^\perp := \{ x \in M_\mathbb{R} : \langle x, a \rangle = 0, \forall a \in \sigma \} \), where \( \langle \ , \ \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R} \) is the natural pairing.
For a cone $\sigma$ in $N_{\mathbb{R}}$, we set $M[\sigma] := M \cap \sigma^\perp$, which is a free $\mathbb{Z}$-module of rank $r - r_{\sigma}$. We set $Z(\sigma) := \wedge^{r-r_{\sigma}} M[\sigma]$.

For cones $\sigma, \tau$ with $\sigma \prec \tau$ and $r_{\tau} = r_{\sigma} + 1$, the isomorphism

$$q_{\sigma/\tau} : Z(\sigma) \rightarrow Z(\tau) \quad (1.13)$$

is defined as follows. Let $p := r - r_{\sigma}$. Then $M[\sigma]$ and $M[\tau]$ are free $\mathbb{Z}$-modules of rank $p$ and $p - 1$, respectively. We take an element $n_1$ in $N$ such that the homomorphism $\langle , n_1 \rangle : M[\sigma] \rightarrow \mathbb{Z}$ is zero on the submodule $M[\tau]$ and maps $M[\sigma] \cap \tau^\vee$ onto $\{ c \in \mathbb{Z} ; c \geq 0 \}$. Then we define

$$q_{\sigma/\tau}(m_1 \wedge \cdots \wedge m_p) := \langle m_1, n_1 \rangle (m_2 \wedge \cdots \wedge m_p) \quad (1.14)$$

for $m_1 \in M[\sigma]$ and $m_2, \ldots, m_p \in M[\tau]$. This definition does not depend on the choice of $n_1$.

$C^i(\Phi, \mathbb{Z})$ was defined to be the direct sum $\bigoplus_{\sigma \in \Phi(i)} Z(\sigma)$ and the $(\sigma, \tau)$-component of $d^i : C^i(\Phi, \mathbb{Z}) \rightarrow C^{i+1}(\Phi, \mathbb{Z})$ was defined to be $q_{\sigma/\tau}$ for each $(\sigma, \tau)$ with $\sigma \prec \tau$.

For each cone $\sigma$, we define an identification $\det(\sigma) = Z(\sigma) \otimes \wedge^r N$ as follows. Let $\{ n_1, \cdots, n_r \}$ and $\{ m_1, \cdots, m_r \}$ be mutually dual $\mathbb{Z}$-basis of $N$ and $M$ such that $M[\sigma]$ is generated by $\{ m_1, \cdots, m_p \}$ and $N(\sigma)$ is generated by $\{ n_{p+1}, \cdots, n_r \}$. Then we identify the generator $(m_p \wedge \cdots \wedge m_1) \otimes (n_1 \wedge \cdots \wedge n_r)$ of $Z(\sigma) \otimes \wedge^r N$ with the generator $n_{p+1} \wedge \cdots \wedge n_r$ of $\det(\sigma)$.

It is easy to see that $q'_{\sigma/\tau}$ and $q_{\sigma/\tau}$ are compatible with respect to this identification. In particular, we have $E(\Phi, \mathbb{Z})^* = C^*(\Phi, \mathbb{Z}) \otimes \wedge^r N$.

By this observation, we interpret some elementary vanishing lemmas in [I1] and [I2] to the following lemma.

Topologically, these are due to the contractability of convex sets.
Lemma 1.2 (1) Let $\rho$ be a cone of $N_\mathbb{R}$. If $\rho \neq 0$, then all the cohomologies of the complex $E(F(\rho), \mathbb{Z})^\bullet$ are zero.

(2) Let $\Delta$ be a finite fan such that the support $\rho := |\Delta|$ is a cone. Let $\eta$ be an element of $\Delta$ and let $\Phi := \{\sigma \in \Delta ; \eta \prec \sigma, \sigma \cap \text{rel.\,int} \rho \neq \emptyset\}$, where rel.\,int $\rho$ is the interior of $\rho$ in $N(\rho)_{\mathbb{R}}$. Then

$$H^i(E(\Phi, \mathbb{Z})^\bullet) = \begin{cases} 0 & \text{for } i \neq r_\rho \\ \det(\rho) & \text{for } i = r_\rho \end{cases}. \quad (1.15)$$

Proof. (1) is a special case of [I1, Prop.2.3] as it is mentioned in the article after the proposition.

For (2), let $N' := N/N(\eta)$ and let $C$ be the image of $\rho$ in $N'_{\mathbb{R}}$. Then $C$ is a rational polyhedral cone which is not necessary strongly convex. Set $\Delta(\eta\prec) := \{\sigma ; \sigma \in \Delta, \eta \prec \sigma\}$. For each $\sigma \in \Delta(\eta\prec)$, let $\sigma[\eta]$ be the image of $\sigma$ in $N'_{\mathbb{R}}$. Then $\Delta[\eta] := \{\sigma[\eta] ; \sigma \in \Delta(\eta\prec)\}$ is a fan of $N'_{\mathbb{R}}$ with the support $C$. We set $\Psi := \{\sigma[\eta] \in \Delta[\eta] ; \sigma[\eta] \cap \text{rel.\,int} C \neq \emptyset\}$. Since $\sigma[\eta] \cap \text{rel.\,int} C \neq \emptyset$ if and only if $\sigma \in \Phi$, the complex $E(\Phi, \mathbb{Z})^\bullet$ is naturally isomorphic to $E(\Psi, \mathbb{Z})[-r_\eta]^\bullet \otimes \det(\eta)$. Then the lemma is a consequence of [I2, Lem.1.6] applied for $\pi = C$ and $\Sigma = \Delta[\eta]$. Note that $C^i(\Phi, \mathbb{Z}_{1,0})$ in [I2] is equal to $C^{i+r}(\Phi, \mathbb{Z})$ for every $i$ and the coboundary map is equal to that of $C^*\Phi, \mathbb{Z})$. Hence $H^i(C^*(\Phi, \mathbb{Z}))$ is equal to $H^{i+r}(C^*(\Phi, \mathbb{Z}))$. (In the statement of [I2, Lem.1.6], “$\pi \subset |\Sigma| \pi + (\pi)”$ is a misprint of “$\pi \subset |\Sigma| \subset \pi + (\pi)$".)

q.e.d.

As in [I3, §1], we denote by $A$ the exterior algebra $\bigwedge^\bullet N_\mathbb{Q}$ and define the grading of $A$ by $A_i := \bigwedge^{-i} N_\mathbb{Q}$ for $i \in \mathbb{Z}$, where the indexes are written as subscripts. For a cone $\rho$, the subalgebra $\bigwedge^\bullet N(\rho)_\mathbb{Q}$ of $A$ is denoted by $A(\rho)$.

For a graded $\mathbb{Q}$-subalgebra $C \subset A$, we denote by $\text{GM}(C)$ the abelian category of finitely generated left $A$-modules. We denote by $\text{CGM}(C)$ the abelian category of finite complexes in $\text{GM}(C)$. We do not take the quotient by homotopy equivalences in the definition of morphisms in $\text{CGM}(C)$ in order to keep the explicitness of the
theory. We write the complex degree by a superscript and the graded module degree by a subscript.

Let $\Delta$ be a finite fan as before.

We consider the additive categories $\text{GEM}(\Delta)$ and $\text{CGEM}(\Delta)$ as in [I3]. Namely, an object $L^\bullet \in \text{CGEM}(\Delta)$ has the following structure.

1. $L$ is a finite dimensional $\mathbb{Q}$-vector space with the decomposition
   \[ L = \bigoplus_{\sigma \in \Delta} \bigoplus_{i,j \in \mathbb{Z}} L(\sigma)^i_j. \] (1.16)

2. For each $\sigma \in \Delta$ and for each $i \in \mathbb{Z}$, $L(\sigma)^i := \bigoplus_{j \in \mathbb{Z}} L(\sigma)^i_j$ is in the category $\text{GM}(A(\sigma))$.

3. For each $\sigma \in \Delta$, $L(\sigma)^\bullet := \bigoplus_{i \in \mathbb{Z}} L(\sigma)^i$ is in the category $\text{CGM}(A(\sigma))$. The coboundary map is denoted by $d_L(\sigma/\tau)$.

4. For each pair $(\sigma, \tau)$ of distinct cones in $\Delta$ with $\sigma \prec \tau$ and for each $i \in \mathbb{Z}$, a homomorphism $d_L^i(\sigma/\tau) : L(\sigma)^i \to L(\tau)[1]^i = L(\tau)^{i+1}$ in $\text{GM}(A(\sigma))$ is given, where we consider $L(\tau)[1]^i \in \text{GM}(A(\sigma))$ by the inclusion $A(\sigma) \subset A(\tau)$.

5. For each pair $(\sigma, \rho)$ of cones in $\Delta$ with $\sigma \prec \rho$ and for each integer $i$, the equality
   \[ \sum_{\tau \in F[\sigma, \rho]} d_L^{i+1}(\tau/\rho) \cdot d_L^i(\sigma/\tau) = 0 \] (1.17)
   holds.

We call $\dim_{\mathbb{Q}} L$ the total dimension of $L^\bullet$.

Note that the homomorphism $d_L(\sigma/\tau) : L(\sigma)^\bullet \to L(\tau)[1]^\bullet$ in (4) is not a homomorphism of complexes in general, i.e., it may not commute with the coboundary maps. However, the condition (5) implies that this is a homomorphism of complexes if $r_\tau - r_\sigma = 1$, since then $F[\sigma, \rho] = \{\sigma, \rho\}$.

A homomorphism $f : L^\bullet \to K^\bullet$ in $\text{CGEM}(\Delta)$ consists of $\{f^i : L^i \to K^i ; i \in \mathbb{Z}\}$ which is compatible with $d_L$ and $d_K$. A homomorphism $f$ is said to be unmixed if $f(\sigma/\tau) : L(\sigma)^\bullet \to K(\tau)^\bullet$ is a zero map whenever $\sigma \neq \tau$. When $f$ is unmixed, the
homomorphism of complexes \( f(\sigma/\sigma) : L(\sigma)^\bullet \to K(\sigma)^\bullet \) is denoted simply by \( f(\sigma) \) for each \( \sigma \in \Delta \). An unmixed homomorphism has the kernel and the cokernel in \( \text{CGEM}(\Delta) \).

For a finite fan \( \Delta \), an objects \( P(\Delta)^\bullet \in \text{CGEM}(\Delta) \) is defined as follows.

For each \( \sigma \in \Delta \), we define \( P(\Delta)(\sigma)^i := (\det(\sigma) \otimes A(\sigma))[-r_{\sigma}]^i \), i.e.,

\[
P(\Delta)(\sigma)^i := \begin{cases} 
\det(\sigma) \otimes A(\sigma) & \text{if } i = r_{\sigma} \\
0 & \text{if } i \neq r_{\sigma} 
\end{cases} .
\] (1.18)

For \( \sigma \in \Delta(i) \) and \( \rho \in \Delta(i+1) \) with \( \sigma \prec \rho \), the \( A(\sigma) \)-homomorphism \( d^i(\sigma/\rho) : P(\Delta)(\sigma)^i \to P(\Delta)(\rho)^{i+1} \) is defined by the homomorphism

\[
d^i_{\sigma/\rho} \otimes \lambda_{\sigma/\rho} : \det(\sigma) \otimes A(\sigma) \longrightarrow \det(\rho) \otimes A(\rho) ,
\] (1.19)

where \( \lambda_{\sigma/\rho} : A(\sigma) \to A(\rho) \) is the natural inclusion map. The equality (1.17) follows from Lemma [1.1].

As we see later, there exists an unmixed surjection \( P(\Delta)^\bullet \to \text{ic}_t(\Delta)^\bullet \) to the intersection complex [3, Thm.2.9] for the top perversity \( t \). However, for a general perversity, such a description of the intersection complex in terms of \( P(\Delta)^\bullet \) is not possible. We make it possible by replacing \( P(\Delta)^\bullet \) by its \textit{barycentric resolution} \( \text{SdP}(\Delta)^\bullet \).

For a finite set \( \Phi \) of nontrivial cones of \( N_R \), we define the \textit{barycentric subdivision al}gebra \( B(\Phi) \) as follows.

We take an indeterminate \( y(\sigma) \) for each \( \sigma \in \Phi \), and denote by \( W(\Phi) \) the \( \mathbb{Q} \)-vector space with the basis \( \{ y(\sigma) ; \sigma \in \Phi \} \). The graded \( \mathbb{Q} \)-algebra \( B(\Phi) \) is defined to be the quotient of the exterior \( \mathbb{Q} \)-algebra \( \wedge^* W(\Phi) \) by the two-sided ideal generated by

\[
\{ y(\sigma) \land y(\rho) ; \sigma, \rho \in \Phi \text{ such that } \sigma \not\prec \rho \text{ and } \rho \not\prec \sigma \} .
\] (1.20)

The grading \( B(\Phi) = \bigoplus_{i=0}^{\infty} B(\Phi)^i \) is induced by that of \( \wedge^* W(\Phi) \). Note that the indexes of \( B(\Phi) \) are given as superscripts. We denote also by \( y(\sigma) \) its image in \( B(\Phi)^1 \). We denote by \( u \cdot v \) or \( uv \) the multiplicityon of \( u, v \) in \( B(\Phi) \).
We denote by $S_d(\Phi)$ the set of sequences $(\sigma_1, \ldots, \sigma_k)$ of distinct cones $\sigma_1, \ldots, \sigma_k$ with $\sigma_1 \prec \cdots \prec \sigma_k$ including the case of length zero. For $\alpha = (\sigma_1, \ldots, \sigma_k) \in S_d(\Phi)$, we define $\max(\alpha) := \sigma_k$ if $k > 0$ and $\max(\alpha) := 0$ if $k = 0$.

For each nonnegative integer $k$, we denote by $S_d^k(\Phi)$ the subset of $S_d(\Phi)$ consisting of the sequences of length $k$. By the assumption $0 \not\in \Phi$, we have $S_d^k(\Phi) = \emptyset$ for $k > r$.

For each $\alpha = (\sigma_1, \ldots, \sigma_k)$, we set $z(\alpha) := y(\sigma_1) \cdots y(\sigma_k) \in B(\Phi)^k$, where we understand $z(\alpha) = 1$ if $k = 0$.

The following lemma is clear by the definition of $B(\Phi)$.

**Lemma 1.3** For each integer $0 \leq i \leq r$, the $\mathbb{Q}$-vector space $B(\Phi)^i$ has the basis

$$\{z(\alpha) ; \alpha \in S_d^i(\Phi)\},$$

while $B(\Phi)^i = \{0\}$ for $i > r$.

Set $Y(\Phi) := \sum_{\sigma \in \Phi} y(\sigma) \in B(\Phi)^1$. By defining the coboundary map $d^i : B(\Phi)^i \to B(\Phi)^{i+1}$ to be the multiplication of $Y(\Phi)$ to the left, we regard $B(\Phi)^\bullet$ as a complex of $\mathbb{Q}$-vector spaces.

Let $\Delta$ be a finite fan of $N_\mathbb{R}$. We consider the barycentric subdivision algebra $B(\Delta \setminus \{0\})$. For each $\rho \in \Delta \setminus \{0\}$, $B(F(0, \rho))$ is a graded subalgebra of $B(\Delta \setminus \{0\})$.

We set $B(\rho) := B(F(0, \rho))y(\rho) \subset B(\Delta \setminus \{0\})$ for each $\rho \in \Delta \setminus \{0\}$. By Lemma 1.3, $B(\rho)$ is a $\mathbb{Q}$-vector space with the basis $\{z(\alpha)y(\rho) ; \alpha \in S_d(F(0, \rho))\}$. We regard $B(\rho)^\bullet$ a complex by defining $B(\rho)^i := B(F(0, \rho))^{i-1}y(\rho)$ for $i \in \mathbb{Z}$ and defining the coboundary map to be the multiplication of $Y(F(0, \rho))$ to the left. Note that $B(\rho)^i \neq \{0\}$ only for $1 \leq i \leq r_\rho$, where $r_\rho := \dim \rho$. For the zero cone $0$, let $B(0)^\bullet$ be the complex defined by $B(0)^0 := \mathbb{Q}$ and $B(0)^i := \{0\}$ for $i \neq 0$.

For each $\beta \in S_d(\Delta \setminus \{0\})$, $z(\beta)$ is in $B(\rho)$ if and only if $\max(\beta) = \rho$. If $\rho \neq 0$, then $\max(\beta) = \rho$ means that $z(\beta) = z(\alpha)y(\rho)$ for some $\alpha \in S_d(F(0, \rho))$. Hence we
get the decomposition
\[ B(\Delta \{0\}) = \bigoplus_{\rho \in \Delta} B(\rho) \quad (1.22) \]
as a \(\mathbb{Q}\)-vector space by Lemma 1.3. However this is not a direct sum of the complexes.

We introduce a decreasing filtration \(\{ F^k(B(\Delta \{0\})) \}\) by
\[ F^k(B(\Delta \{0\})) := \bigoplus_{i=k}^{r} \bigoplus_{\rho \in \Delta(i)} B(\rho) \quad (1.23) \]
Then it is easy to see that each \(F^k(B(\Delta \{0\}))\) is a subcomplex of \(B(\Delta \{0\})\) for each \(k\), and
\[ F^k(B(\Delta \{0\}))^*/F^{k+1}(B(\Delta \{0\}))^* = \bigoplus_{\rho \in \Delta(k)} B(\rho)^* \quad (1.24) \]
as complexes.

For \(\rho, \mu \in \Delta\), with \(\rho \prec \mu\) and \(\rho \neq \mu\), we define a \(\mathbb{Q}\)-linear map
\[ \varphi_{\rho/\mu} : B(\rho) \to B(\mu) \quad (1.25) \]
to be the multiplication of \(y(\mu)\) to the left, i.e., \(w \mapsto y(\mu)w\). If \(r_\mu - r_\rho = 1\), then \(\varphi_{\rho/\mu}\) is a homomorphism \(B(\rho)^* \to B(\mu)[1]^*\) of complexes, however it is not the case if \(r_\mu - r_\rho > 1\).

We consider the complex of \(A(\rho)\)-modules
\[ (B(\rho) \otimes_\mathbb{Q} A(\rho))^* := B(\rho)^* \otimes_\mathbb{Q} A(\rho) \quad (1.26) \]
By definition, \(B(\rho)^i \otimes_\mathbb{Q} A(\rho)_j \neq \{0\}\) only for \(1 \leq i \leq r_\rho\) and \(-r_\rho \leq j \leq 0\), if \(\rho \neq 0\).

The barycentric resolution \(\text{SdP}(\Delta)^* \in \text{CGEM}(\Delta)\) of \(\text{P}(\Delta)^*\) is defined as follows.

For each \(\rho \in \Delta\), we set
\[ \text{SdP}(\Delta)(\rho)^* := (B(\rho) \otimes_\mathbb{Q} A(\rho))^* \quad (1.27) \]
In particular,
\[ \text{SdP}(\Delta)^i = \bigoplus_{\alpha \in \text{Sd}_{i-1}(F(0, \rho))} (\mathbb{Z}_z(\alpha)y(\rho)) \otimes A(\rho) \quad (1.28) \]
For each $1 \leq i \leq r_\rho$ if $\rho \neq 0$, while $\text{SdP}(\Delta)(0)^0 = \mathbb{Q}$ and $\text{SdP}(\Delta)(0)^i = \{0\}$ for $i \neq 0$.

For $\rho, \mu \in \Delta$ with $\rho < \mu$ and $\rho \neq \mu$, the $(\rho, \mu)$-component

$$d_{\text{SdP}(\Delta)}(\rho/\mu) : \text{SdP}(\Delta)(\rho) \rightarrow \text{SdP}(\Delta)(\mu)[1]$$

of the coboundary map of $\text{SdP}(\Delta)^*$ is defined to be the tensor product of $\varphi_{\rho/\mu} : B(\rho) \rightarrow B(\mu)$ defined at (1.25) and the natural inclusion map $A(\rho) \rightarrow A(\mu)$. Note that this is not a homomorphism of complexes, if $r_\mu - r_\rho > 1$.

In order to check the equality (1.17) for $\text{SdP}(\Delta)^*$, it is sufficient to show the equality

$$\sum_{\tau \in F[0,\rho]} \varphi_{\tau/\mu} \cdot \varphi_{\rho/\tau} = 0, \quad (1.30)$$

where $\varphi_{\rho/\rho}$ and $\varphi_{\mu/\mu}$ are the multiplication of $Y(F(0,\rho))$ and $Y(F(0,\mu))$ to the left, respectively. This equality follows from the equality

$$\left( y(\mu)Y(F(0,\rho)) + Y(F(0,\mu))y(\mu) + \sum_{\tau \in F(\rho,\mu)} y(\mu)y(\tau) \right) y(\rho) = 0 \quad (1.31)$$

in $B(\Delta \setminus \{0\})$ which is checked easily.

For each $\rho \in \Delta$, we define a covariant functor

$$i^*_\rho : \text{CGEM}(\Delta) \rightarrow \text{CGM}(A(\rho)) \quad (1.32)$$

as follows. For $L^* \in \text{CGEM}(\Delta)$ and for each $i \in \mathbb{Z}$, we set

$$i^*_\rho(L)^i := \bigoplus_{\sigma \in F[0,\rho]} L(\sigma)^i_{A(\rho)} \quad (1.33)$$

For $\sigma, \tau \in F[0,\rho]$ with $\sigma \prec \tau$, the $(\sigma, \tau)$-component of $d^i : i^*_\rho(L)^i \rightarrow i^*_\rho(L)^{i+1}$ is defined to be the $A(\rho)$-homomorphism induced by $d^i_L(\sigma/\tau)$.

Recall that a similar functor $i^*_{\rho}$ was defined in [13, §2]. We get the definition of $i^*_{\rho}$ by replacing $F[0,\rho]$ in the definition of $i^*_\rho$ by $F[0,\rho] = F(\rho)$. 

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For \( L^\bullet \in \text{CGEM}(\Delta) \) and \( \rho \in \Delta \), we get an exact sequence

\[
0 \rightarrow L(\rho)^\bullet \rightarrow i_\rho^*(L)^\bullet \rightarrow i_\rho^\circ(L)^\bullet \rightarrow 0 \tag{1.34}
\]

in \( \text{CGM}(A(\rho)) \). Since \( i_\rho^*(L)^i = L(\rho)^i \oplus i_\rho^\circ(L)^i \) for each integer \( i \), \( i_\rho^*(L)^\bullet \) is equal to the mapping cone of a homomorphism

\[
\phi(L, \rho) : i_\rho^\circ(L)^\bullet \rightarrow L(\rho)[1]^\bullet \tag{1.35}
\]

of complexes. The restriction of \( \phi(L, \rho)^i \) to the component \( L(\sigma)^i_{A(\rho)} \) is the \( A(\rho) \)-homomorphism induced by \( d_L^i(\sigma/\rho) \) for each \( \sigma \).

Let \( f : L^\bullet \rightarrow K^\bullet \) be a homomorphism in \( \text{CGEM}(\Delta) \). For \( \rho \in \Delta \), consider the following diagram

\[
\begin{array}{ccc}
i_\rho^\circ(L)^\bullet & \overset{i_\rho^\circ(f)}{\longrightarrow} & i_\rho^\circ(K)^\bullet \\
\phi(L, \rho) \downarrow & & \downarrow \phi(K, \rho) \hspace{1cm} .
\end{array}
\tag{1.36}
\]

Let \( \{u^i \ ; \ i \in \mathbb{Z}\} \) be the collection of \( A(\rho) \)-homomorphisms \( u^i : i_\rho^\circ(L)^i \rightarrow K(\rho)^i = K(\rho)[1]^{i-1} \) induced by \( \{f^i(\sigma/\rho) \ ; \ \sigma \in F[0, \rho]\} \). Then restriction of the equality \( d_K \cdot f = f \cdot d_L \) to \( i_\rho^\circ(L)^i \) implies

\[
\phi(K, \rho)^i \cdot i_\rho^\circ(f)^i + d_K^i(K)[1]^i \cdot u^i = f(\rho)[1]^i \cdot \phi(L, \rho)^i + u^{i+1} \cdot d_L^i(L) .
\tag{1.37}
\]

Hence the difference \( \phi(K, \rho)^i \cdot i_\rho^\circ(f)^i - f(\rho)[1]^i \cdot \phi(L, \rho)^i \) is homotopy equivalent to the zero map. In particular, the diagram (1.36) induces a commutative diagram of cohomologies. If \( f \) is unmixed, then all \( u^i \)'s are zero and the diagram (1.36) is commutative.

Let \( L^\bullet \) be an object of \( \text{CGEM}(\Delta) \). \( K^\bullet \in \text{CGEM}(\Delta) \) with a homomorphism \( f : K^\bullet \rightarrow L^\bullet \) is said to be a subcomplex if \( f \) is unmixed and \( f(\sigma) : K(\sigma)^\bullet \rightarrow L(\sigma)^\bullet \) is an inclusion map of complexes for every \( \sigma \in \Delta \). By a homogeneous element of \( L^\bullet \), we mean an element of \( L(\sigma)^i_j \) for some \( \sigma \in \Delta \) and \( i, j \in \mathbb{Z} \). For a set \( S \) of
homogeneous elements of $L^\bullet$, there exists a unique subcomplex $\langle S \rangle^\bullet$ of $L^\bullet$ generated by $S$. The complex $\langle S \rangle^\bullet$ is described inductively as follows.

For $\sigma \in \Delta$ and $i, j \in \mathbb{Z}$, let $S(\sigma, i, j) := S \cap L(\sigma)^i_j$. We denote by $\langle S(\sigma) \rangle^\bullet$ the $A(\sigma)$-subcomplex of $L(\sigma)^\bullet$ generated by $S(\sigma) := \bigcup_{i,j \in \mathbb{Z}} S(\sigma, i, j)$ for each $\sigma \in \Delta$.

We set $\langle S \rangle^\bullet(0) := \langle S(0) \rangle^\bullet$. Let $\rho$ be in $\Delta \setminus \{0\}$, and assume that we already know $\langle S(\sigma) \rangle^\bullet$ for $\sigma \in F[0, \rho)$. Then $\langle S(\rho) \rangle^\bullet$ is the $A(\rho)$-subcomplex of $L(\rho)^\bullet$ given by

$$\langle S(\rho) \rangle^\bullet = \phi(L, \rho)(i_{\rho}^*(\langle S(\sigma) \rangle^\bullet)) + \langle S(\rho) \rangle^\bullet.$$  (1.38)

Let $p$ be a perversity of $\Delta$, i.e., $p$ is a map $\Delta \setminus \{0\} \to \mathbb{Z}$.

We denote by $k_p(\Delta)^\bullet$ the subcomplex of $\text{SdP}(\Delta)^\bullet$ generated by

$$\bigcup_{\sigma \in \Delta \setminus \{0\}} \bigcup_{i+j \leq p(\sigma)} \text{SdP}(\Delta)(\sigma)^i_j.$$  (1.39)

Note that if we set this set $S$, then $\langle S(\sigma) \rangle^\bullet$ is equal to the gradual truncation $\widetilde{\text{gt}}_{\leq p(\sigma)} \text{SdP}(\Delta)(\sigma)^\bullet$ (cf. [I3, §1]).

**Lemma 1.4** Let $\Delta$ be a finite fan. For each $\rho \in \Delta \setminus \{0\}$, the homomorphism

$$\phi(\text{SdP}(\Delta), \rho) : i_{\rho}^*(\text{SdP}(\Delta))^\bullet \longrightarrow \text{SdP}(\Delta)(\rho)[1]^\bullet$$  (1.40)

is an isomorphism.

**Proof.** Since $\text{SdP}(\Delta)^i = \{0\}$ for $i < 0$ and $\text{SdP}(\Delta)(\rho)^0 = \{0\}$ for $\rho \neq 0$, it is sufficient to show that $\phi(\text{SdP}(\Delta), \rho)^i$ is an isomorphism for $i \geq 0$. For $i = 0$, we have $i_{\rho}^*(\text{SdP}(\Delta))^0 = \mathbb{Z} \otimes A(\rho)$ and $\text{SdP}(\Delta)(\rho)^1 = \mathbb{Z}y(\rho) \otimes A(\rho)$, and $\phi(\text{SdP}(\Delta), \rho)^0$ is the isomorphism given by $1 \otimes 1 \mapsto y(\rho) \otimes 1$. Assume $i > 0$. The free $A(\rho)$-module $\text{SdP}(\Delta)(\rho)^{i+1} = B(\rho)^{i+1} \otimes \mathbb{Q} A(\rho)$, has the basis $\{z(\alpha)y(\rho) ; \alpha \in \text{Sd}_{i}(F(0, \rho))\}$ which is decomposed to the disjoint union

$$\bigcup_{\sigma \in F(0, \rho)} \{z(\beta)y(\sigma)y(\rho) ; \beta \in \text{Sd}_{i-1}(F(0, \sigma))\}.$$  (1.41)
On the other hand, the component $\text{SdP}(\Delta)(\sigma)^i_{A(\rho)}$ of $i^\rho_*(\text{SdP}(\Delta))^i$ has the basis

$$\{z(\beta)y(\sigma) ; \beta \in \text{Sd}_{i-1}(F(0,\sigma))\} \quad (1.42)$$

for each $\sigma \in F(0,\rho)$. Since $\phi(\text{SdP}(\Delta),\rho)^i(z(\beta)y(\sigma)) = (-1)^i z(\beta)y(\sigma)(y(\rho))$ for each $z(\beta)y(\sigma)$, $\phi(\text{SdP}(\Delta),\rho)^i$ induces an isomorphism from $\text{SdP}(\Delta)(\sigma)^i_{A(\rho)}$ to the submodule of $\text{SdP}(\Delta)(\rho)^{i+1}$ generated by $\{z(\beta)y(\sigma)(y(\rho)) ; \beta \in \text{Sd}_{i-1}(F(0,\sigma))\}$. We are done, since $\phi(\text{SdP}(\Delta),\rho)^i$ is the direct sum of these isomorphisms for $\sigma \in F(0,\rho)$.

$q.e.d.$

We define an unmixed homomorphism

$$\psi_\Delta : \text{SdP}(\Delta)^* \longrightarrow \text{P}(\Delta)^* \quad (1.43)$$

of complexes in $\text{CGEM}(\Delta)$ as follows.

We define a homomorphism

$$\psi_\Delta(\rho) : \text{SdP}(\Delta)(\rho)^* \longrightarrow \text{P}(\Delta)(\rho)^* \quad (1.44)$$

in $\text{CGM}(A(\rho))$ for each $\rho \in \Delta$. Let $k := r_\rho$. Then $\psi_\Delta(\rho)^i := 0$ for $i \neq k$ since $\text{P}(\Delta)(\rho)^i = \{0\}$. For $\alpha \in \text{Sd}_{k-1}(F(0,\rho))$, we define $a := \psi_\Delta(\rho)^k(z(\alpha)y(\rho))$ to be the generator of $\text{det}(\rho) \subset \text{P}(\Delta)(\rho)^k$ which is determined by the orientation of the sequence $\alpha = (\sigma_1, \cdots, \sigma_{k-1})$, i.e., if we take $a_i \in N \cap \text{rel. int} \sigma_i$ for $i = 1, \cdots, k-1$ and $a_k \in N \cap \text{rel. int} \rho$, then $a$ is the generator of $\text{det}(\rho)$ which has the same sign with $a_1 \wedge \cdots \wedge a_k \in \text{det}(\rho)$. The commutativity of $\psi_\Delta$ and the coboundary maps is checked easily. The only one nontrivial commutativity is of the component for $\text{SdP}(\Delta)(\rho)^{k-1}$ and $\text{P}(\Delta)(\mu)^k$ with $\mu = \rho$. For each $\alpha \in \text{Sd}_{k-2}(F(0,\rho))$, there exist exactly two $\beta_1, \beta_2 \in \text{Sd}_{k-1}(F(0,\rho))$ which contains $\alpha$ as a subsequence. We get the commutativity, since $\beta_1$ and $\beta_2$ have mutually opposite orientations and hence the equality

$$\psi_\Delta(\rho)^k(z(\beta_1)y(\rho)) + \psi_\Delta(\rho)^k(z(\beta_2)y(\rho)) = 0 \quad (1.45)$$
holds.

By the definition, $\psi_\Delta(\rho)$ is surjective. The following lemma implies that the kernel is generated by

$$\bigcup_{i=1}^{r_{\rho}-1} \text{SdP}(\Delta)(\rho)^i$$

as a subcomplex for each $\rho \in \Delta \setminus \{0\}$.

**Lemma 1.5** The above unmixed homomorphism $\psi_\Delta$ is a quasi-isomorphism in $\text{CGEM}(\Delta)$.

**Proof.** Since $\text{SdP}(\Delta)(0) = P(\Delta)(0) = Q$, $\psi_\Delta(0)$ is a quasi-isomorphism. Let $\Phi$ be a maximal subfan of $\Delta$ such that the restriction of $\psi_\Delta$ to $\Phi$ is a quasi-isomorphism. Suppose $\Phi \neq \Delta$, and let $\rho$ be a minimal element of $\Delta \setminus \Phi$.

Since $\psi_\Delta$ is unmixed, we get a commutative diagram

$$
\begin{array}{ccc}
i^\rho_p(\text{SdP}(\Delta))^* & \xrightarrow{i^\rho_p \psi_\Delta} & i^\rho_p(\text{P}(\Delta))^* \\
\phi(\text{SdP}(\Delta), \rho) \downarrow & & \downarrow \phi(\text{P}(\Delta), \rho) \\
\text{SdP}(\Delta)(\rho)[1]^* & \xrightarrow{\psi_\Delta(\rho)[1]} & \text{P}(\Delta)(\rho)[1]^* \\
\end{array}
$$

(1.47)

By Lemma 1.4, $\phi(\text{SdP}(\Delta), \rho)$ is an isomorphism, while $i^\rho_p \psi_\Delta$ is a quasi-isomorphism since it depends only on the restriction of $\psi_\Delta$ to $\Phi$.

Since the mapping cone $i^\rho_p(\text{P}(\Delta))^*$ of $\phi(\text{P}(\Delta), \rho)$ is equal to $E(F(\rho), \mathbb{Z})^* \otimes A(\rho)$, it has trivial cohomologies by Lemma 1.2 (1). Hence $\phi(\text{P}(\Delta), \rho)$ is also a quasi-isomorphism.

By the commutative diagram, $\psi_\Delta(\rho)$ is also a quasi-isomorphism. This contradicts the maximality of $\Phi$.

$q.e.d.$

**Lemma 1.6** For $L^* \in \text{CGEM}(\Delta)$ and a homomorphism $f_0 : \text{SdP}(\Delta)(0)^* \rightarrow L(0)^*$, there exists a unique unmixed homomorphism $f : \text{SdP}(\Delta)^* \rightarrow L^*$ with $f(0) = f_0$. 

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Proof. We prove the lemma by induction on the number of cones in $\Delta$.

If $\Delta = \{0\}$, then the assertion is clear.

Assume that $\Delta \neq \{0\}$ and $\pi$ is a maximal element of $\Delta$. Set $\Delta' := \Delta \setminus \{\pi\}$ and assume that $f' : \text{SdP}(\Delta')^* \to (L|\Delta')^*$ is the unique extension of $f_0$. Let $f_1 : \text{SdP}(\Delta)(\pi)^* \to L(\pi)^*$ be a homomorphism in CGM($A(\pi)$). Then $f'$ is extended to an unmixed homomorphism $f : \text{SdP}(\Delta)^* \to L^*$ by $f(\pi) := f_1$ if and only if the diagram

$$
\begin{array}{ccc}
\text{id}(\text{SdP}(\Delta))^* & \xrightarrow{i^0_\pi f'} & i^0_\pi(L)^* \\
\phi(\text{SdP}(\Delta), \pi) \downarrow & & \downarrow \phi(L, \pi) \\
\text{SdP}(\Delta)(\pi)[1]^* & \xrightarrow{f_1[1]} & L(\pi)[1]^*
\end{array}
$$

is commutative. Since $\phi(\text{SdP}(\Delta), \pi)$ is an isomorphism by Lemma 1.4, such a $f_1$ exists uniquely.

$\text{q.e.d.}$

**Theorem 1.7** Let $\mathbf{p}$ be a perversity of $\Delta$. Let

$$
\varphi(\Delta, \mathbf{p}) : \text{SdP}(\Delta)^* \longrightarrow \text{ic}_p(\Delta)^*
$$

be the unmixed homomorphism obtained by extending the identity $\text{SdP}(\Delta)(0) = \text{ic}_p(\Delta)(0) = \mathbb{Q}$. Then $\varphi(\Delta, \mathbf{p})$ is surjective and the kernel is equal to $k_p(\Delta)^*$.

Proof. We prove the theorem by induction. Namely, let $\Phi$ be the maximal subfan of $\Delta$ such that $\varphi(\Delta, \mathbf{p})(\sigma)$ is surjective and the kernel is equal to $k_p(\Delta)(\sigma)^*$ for $\sigma \in \Phi$. Since $k_p(\Delta)(0) = \{0\}$, we have $0 \in \Phi$. Suppose $\Phi \neq \Delta$ and let $\rho$ be a minimal element of $\Delta \setminus \Phi$. Since $\varphi(\Delta, \mathbf{p})$ is unmixed, we get the following commutative diagram.

$$
\begin{array}{cccc}
0 & \rightarrow & i^0_p(k_p(\Delta))^* & \rightarrow & i^0_p(\text{SdP}(\Delta))^* & \rightarrow & i^0_p(\text{ic}_p(\Delta))^* & \rightarrow & 0 \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \quad & & \\
0 & \rightarrow & k_p(\Delta)(\rho)[1]^* & \longrightarrow & \text{SdP}(\Delta)(\rho)[1]^* & \longrightarrow & \text{ic}_p(\Delta)(\rho)[1]^* & \rightarrow & 0
\end{array}
$$

$$
\tag{1.50}
$$
The upper line of this diagram is exact by the assumption. Among the vertical homomorphisms, $\phi_2$ is an isomorphism by Lemma 1.4 and $\phi_3$ is the natural surjection

$$i^\circ(\text{iC}_p(\Delta))^* \to \text{gt}^{\leq p(\rho)}(i^\circ(\text{iC}_p(\Delta)))^* = \text{iC}_p(\Delta)(\rho)[1]^*$$

by the construction of $\text{iC}_p(\Delta)^*$ [13, Thm 2.9]. Hence $v$ is surjective, while $u$ is an inclusion map.

Since there exists an exact sequence

$$0 \to \text{gt}_{\leq p(\rho)-1}(i^\circ(\text{iC}_p(\Delta)))^* \to i^\circ(\text{iC}_p(\Delta))^* \to \text{gt}^{\geq p(\rho)}(i^\circ(\text{iC}_p(\Delta)))^* \to 0 \ ,$$

(1.52)

Ker $\phi_3$ is equal to $\text{gt}_{\leq p(\rho)-1}(i^\circ(\text{iC}_p(\Delta)))^*$. Since $w$ is surjective, this is equal to the image of $\text{gt}_{\leq p(\rho)-1}(i^\circ(\text{SdP}(\Delta)))^*$ by $w$. Hence

$$w^{-1}(\text{Ker } \phi_3) = i^\circ(k_p(\Delta))^* + \text{gt}_{\leq p(\rho)-1}(i^\circ(\text{SdP}(\Delta)))^* .$$

(1.53)

Since

$$\text{gt}_{\leq p(\rho)-1}(\text{SdP}(\Delta)(\rho))[1]^* = (\text{gt}_{\leq p(\rho)}(\text{SdP}(\Delta)(\rho)))[1]^* ,$$

(1.54)

Ker $v = \phi_2(w^{-1}(\text{Ker } \phi_3))$ is equal to the sum of Im $\phi_1$ and $(\text{gt}_{\leq p(\rho)}(\text{SdP}(\Delta)(\rho)))[1]^*$ in $\text{SdP}(\Delta)(\rho)[1]^*$. On the other hand, $k_p(\Delta)(\rho)^*$ is the subcomplex of $\text{SdP}(\Delta)(\rho)^*$ generated by Im $\phi_1$ and $\bigcup_{i+j \leq p(\rho)} \text{SdP}(\Delta)(\rho)^j_i$ (cf. (1.38)). Since the subcomplex of $\text{SdP}(\Delta)(\rho)^*$ generated by the last set is $\text{gt}_{\leq p(\rho)}(\text{SdP}(\Delta)(\rho))^*$, the lower line is also exact. This contradicts the maximality of $\Phi$, and we conclude $\Phi = \Delta$. q.e.d.

For a finite fan $\Delta$, the covariant additive functor $\Gamma : \text{GEM}(\Delta) \to \text{GM}(A)$ is defined as follows.

For $L \in \text{GEM}(\Delta)$, we set

$$\Gamma(L) := \bigoplus_{\sigma \in \Delta} L(\sigma)_A .$$

(1.55)
Let $f : L \to K$ be a homomorphism in $\text{GEM} (\Delta)$. For $\sigma, \tau \in \Delta$, the $(\sigma, \tau)$-component of the homomorphism $\Gamma(f) : \Gamma(L) \to \Gamma(K)$ is defined to be the $A$-homomorphism $L(\sigma)_A \to K(\tau)_A$ induced by $f(\sigma/\tau)$ and is the zero map otherwise.

Let $L^\bullet$ be in $\text{CGEM}(\Delta)$. Then $\Gamma(L)^\bullet$ is in $\text{CGM}(A)$, i.e., is a finite complex of graded $A$-modules. For each integer $q$, the homogeneous component $\Gamma(L)^q_\bullet$ is a complex of $\mathbb{Q}$-vector spaces.

Since $A(\rho)_A = A$ for all $\rho \in \Delta$, we have

$$\Gamma(SdP(\Delta)^i) = B(\Delta \setminus \{0\})^i \otimes A$$

for each $i \in \mathbb{Z}$. By comparing the definitions of the coboundary maps, we get the following lemma.

**Lemma 1.8** For any finite fan $\Delta$, $\Gamma(SdP(\Delta))^\bullet$ is canonically isomorphic to $B(\Delta \setminus \{0\})^\bullet \otimes \mathbb{Q} A$ as a complex of $A$-modules.

The *top perversity* $t$ is defined by $t(\sigma) := r_\sigma - 1$ for every nontrivial cone $\sigma$, while the *bottom perversity* $b$ is defined by $b := -t$. We use this notation for all finite fans.

**Lemma 1.9** Let $\Delta$ be a finite fan. Then the unmixed surjection $\varphi(\Delta, t) : SdP(\Delta)^\bullet \to ic_t(\Delta)^\bullet$ induces an unmixed surjection $\varphi : P(\Delta)^\bullet \to ic_t(\Delta)^\bullet$ such that

$$\text{Ker } \varphi(\sigma)^{rs} = \det(\sigma) \otimes (N(\sigma)A(\sigma))$$

for every $\sigma \in \Delta$.

Note that $N(\sigma)A(\sigma)$ is a two-sided maximal ideal of $A(\sigma)$.

**Proof.** The kernel of the homomorphism $\varphi(\Delta, t)$ is equal to $k_t(\Delta)^\bullet$ by Theorem 1.7. By definition, this subcomplex is generated by the union of

$$\bigcup_{i+j < r_\sigma} SdP(\Delta)(\sigma)_{ij}^i$$

for every $\sigma \in \Delta$. 

By comparing the definitions of the coboundary maps, we get the following lemma.

**Lemma 1.8** For any finite fan $\Delta$, $\Gamma(SdP(\Delta))^\bullet$ is canonically isomorphic to $B(\Delta \setminus \{0\})^\bullet \otimes \mathbb{Q} A$ as a complex of $A$-modules.

The *top perversity* $t$ is defined by $t(\sigma) := r_\sigma - 1$ for every nontrivial cone $\sigma$, while the *bottom perversity* $b$ is defined by $b := -t$. We use this notation for all finite fans.

**Lemma 1.9** Let $\Delta$ be a finite fan. Then the unmixed surjection $\varphi(\Delta, t) : SdP(\Delta)^\bullet \to ic_t(\Delta)^\bullet$ induces an unmixed surjection $\varphi : P(\Delta)^\bullet \to ic_t(\Delta)^\bullet$ such that

$$\text{Ker } \varphi(\sigma)^{rs} = \det(\sigma) \otimes (N(\sigma)A(\sigma))$$

for every $\sigma \in \Delta$.

Note that $N(\sigma)A(\sigma)$ is a two-sided maximal ideal of $A(\sigma)$.

**Proof.** The kernel of the homomorphism $\varphi(\Delta, t)$ is equal to $k_t(\Delta)^\bullet$ by Theorem 1.7. By definition, this subcomplex is generated by the union of

$$\bigcup_{i+j < r_\sigma} SdP(\Delta)(\sigma)_{ij}^i$$

for every $\sigma \in \Delta$. 

By comparing the definitions of the coboundary maps, we get the following lemma.

**Lemma 1.8** For any finite fan $\Delta$, $\Gamma(SdP(\Delta))^\bullet$ is canonically isomorphic to $B(\Delta \setminus \{0\})^\bullet \otimes \mathbb{Q} A$ as a complex of $A$-modules.
for all \( \sigma \in \Delta \setminus \{0\} \). Since \( \text{SdP}(\Delta)(\sigma)_j^i \) is nonzero only for \( 1 \leq i \leq r_\sigma, -r_\sigma \leq j \leq 0 \), the condition \( i + j < r_\sigma \) means all \((i, j)\) except for \((r_\sigma, 0)\). On the other hand, the kernel of \( \psi(\sigma) : \text{SdP}(\Delta)(\sigma)^* \to \text{P}(\Delta)(\sigma)^* \) is generated as a subcomplex by

\[
\bigcup_{i=1}^{r_\sigma-1} \text{SdP}(\Delta)(\sigma)^i
\]

for each \( \sigma \in \Delta \setminus \{0\} \) as we mentioned before Lemma 1.5. Hence we get the surjection \( \varphi \). We get the lemma since

\[
\text{Ker} \varphi(\sigma)^{r_\sigma} \simeq \text{Ker} \varphi(\Delta, t)(\sigma)^{r_\sigma} / \text{Ker} \psi(\sigma)^{r_\sigma}
\]

and this is equal to

\[
\bigoplus_{j=-r_\sigma}^{-1} \psi(\sigma)(\text{SdP}(\Delta)(\sigma)^j)^{r_\sigma} = \bigoplus_{j=-r_\sigma}^{-1} \text{P}(\Delta)(\sigma)^j
\]

as a graded \( \mathbb{Q} \)-vector space. This is equal to \( \text{det}(\sigma) \otimes (N(\sigma)A(\sigma)) \). \( \text{q.e.d.} \)

We denote by \( \bar{A}(\sigma) \) the \( A(\sigma) \)-module \( A(\sigma)/N(\sigma)A(\sigma) \) of length one. By this lemma, we identify \( \text{ic}(\Delta)^* \) with the quotient complex of \( \text{P}(\Delta)^* \) by \( \text{Ker} \varphi \). Namely we have

\[
\text{ic}(\Delta)(\sigma)^i = \begin{cases} 
\text{det}(\sigma) \otimes \bar{A}(\sigma) & \text{if } i = r_\sigma \\
\{0\} & \text{if } i \neq r_\sigma
\end{cases}
\]

for every \( \sigma \in \Delta \).

**Lemma 1.10** Let \( \Delta \) be a finite fan. Then \( \text{H}^p(\Gamma(\text{ic}(\Delta))^*)_q = \{0\} \) for \( p, q \in \mathbb{Z} \) with \( p > q + r \).

**Proof.** Since \( \bar{A}(\sigma)_A = \wedge^*(N_A/N(\sigma)_A) \) and \( \dim_q N_A/N(\sigma)_A = r - r_\sigma \),

\[
\Gamma(\text{ic}(\Delta))^p_q = \bigoplus_{\sigma \in \Delta(p)} \text{det}(\sigma) \otimes (\bar{A}(\sigma)_A)_q = \{0\}
\]

for \( q < -(r - p) \), i.e, for \( p > q + r \). Hence, for the component of degree \( q \) of the complex \( \Gamma(\text{ic}(\Delta))^* \), the cohomologies vanish for \( p > q + r \). \( \text{q.e.d.} \)
2 The intersection complex of the middle perversity

Let $\Delta$ be a finite fan of $N_R$. The middle perversity $m : \Delta \setminus \{0\} \to Z$ is defined by $m(\sigma) := 0$ for every $\sigma$.

We denote the intersection complex $\text{ic}_m(\Delta)^\bullet$ simply by $\text{ic}(\Delta)^\bullet$. Since $-m = m$, the dual $D(\text{ic}(\Delta))^\bullet$ is quasi-isomorphic to $\text{ic}(\Delta)^\bullet$ by [3, Cor.2.12].

A finite fan $\Delta$ of $N_R$ is said to be a lifted complete fan, if it is a lifting of a complete fan of an $(r-1)$-dimensional space (cf. [3, §2]). In particular, $\Delta$ is a lifted complete fan if the support $|\Delta|$ is equal to the boundary of an $r$-dimensional cone.

**Theorem 2.1** Let $\Delta$ be a lifted complete fan of $N_R$. Then for any integers $p, q \in Z$, the equality

$$\dim Q H^p(\Gamma(\text{ic}(\Delta))^\bullet) = \dim Q H^{r-1-p}(\Gamma(\text{ic}(\Delta))^\bullet)$$

holds.

**Proof.** Since $D(\text{ic}(\Delta))^\bullet$ is quasi-isomorphic to $\text{ic}(\Delta)^\bullet$ by [3, Cor.2.12], this is a consequence of [3, Prop.2.8].

**q.e.d.**

**Corollary 2.2** Let $\pi$ be an $r$-dimensional cone. Then, for $\Delta := F(\pi) \setminus \{\pi\}$, the equality (2.1) folds for any integers $p, q \in Z$.

Let $\Delta$ and $\Delta'$ be finite fans of $N_R$. Then $\Delta'$ is said to be a subdivision of $\Delta$ if $|\Delta| = |\Delta'|$ and, for every $\sigma \in \Delta'$ there exists $\rho \in \Delta$ with $\sigma \subset \rho$ (cf. [2, Cor.1.16]). If $\Delta'$ is a subdivision of $\Delta$, then there exists a unique map $f : \Delta' \to \Delta$ such that $\sigma \cap \text{rel.int} f(\sigma) \neq \emptyset$ for each $\sigma \in \Delta'$. Actually, $f(\sigma)$ is defined to be the minimal cone in $\Delta$ which contains $\sigma$. The map $f$ is also called a subdivision.

Let $f : \Delta' \to \Delta$ be a subdivision. For $\rho \in \Delta$, we denote by $f^{-1}(\rho)$ the subset $f^{-1}(\{\rho\})$ of $\Delta'$. Clearly, $f^{-1}(\rho)$ is a locally star closed subset of $\Delta'$.
For $L \in \text{GEM}(\Delta')$, we define the direct image $f_* L \in \text{GEM}(\Delta)$ by
\[
(f_* L)(\rho) := \bigoplus_{\sigma \in f^{-1}(\rho)} L(\sigma)_{A(\rho)}
\]
for each $\rho \in \Delta$.

For a homomorphism $g : L \to K$ in $\text{GEM}(\Delta')$, the homomorphism $f_*(g) : f_* L \to f_* K$ is defined as follows.

For $\rho, \mu \in \Delta$ with $\rho \prec \mu$ and $\sigma \in f^{-1}(\rho)$, $\tau \in f^{-1}(\mu)$, the $(\sigma, \tau)$-component of $f_*(g)(\rho/\mu)$ is the $A(\rho)$-homomorphism $L(\sigma)_{A(\rho)} \to K(\tau)_{A(\mu)}$ induced by the $A(\sigma)$-homomorphism $g(\sigma/\tau) : L(\sigma) \to K(\tau)$ if $\sigma \prec \tau$, otherwise it is defined to be the zero map.

It is easy to see that $f_*$ is a covariant additive functor from $\text{GEM}(\Delta')$ to $\text{GEM}(\Delta)$. We denote also by $f_*$ the induced functor from $\text{CGEM}(\Delta')$ to $\text{CGEM}(\Delta)$.

A sequence of homomorphisms $0 \to L \to K \to J \to 0$ in $\text{GEM}(\Delta)$ is said to be a short exact sequence if $0 \to L(\sigma) \to K(\sigma) \to J(\sigma) \to 0$ is exact for every $\sigma \in \Delta$.

**Lemma 2.3** Let $f : \Delta' \to \Delta$ be a subdivision of a finite fan.

1. If $g : L^\bullet \to K^\bullet$ is a quasi-isomorphism in $\text{CGEM}(\Delta')$, then direct image $f_*(g) : f_* L^\bullet \to f_* K^\bullet$ is also a quasi-isomorphism.

2. If $0 \to L \xrightarrow{g} K \xrightarrow{h} J \to 0$ is a short exact sequence in $\text{GEM}(\Delta')$, then $0 \to f_* L \xrightarrow{f_* g} f_* K \xrightarrow{f_* h} f_* J \to 0$ is a short exact sequence in $\text{GEM}(\Delta)$.

**Proof.** (1) Let $\Phi$ be a maximal subfan of $\Delta'$ such that $f_*(g|\Phi) : f_* (L|\Phi)^\bullet \to f_* (K|\Phi)^\bullet$ is a quasi-isomorphism. Suppose $\Phi \neq \Delta'$ and let $\tau$ be a minimal element of $\Delta' \setminus \Phi$. Let $\Phi' := \Phi \cup \{\tau\}$ and $\rho := f(\tau)$. We get a commutative diagram
\[
\begin{array}{ccc}
0 & \to & L(\tau)_{A(\rho)}^\bullet \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
0 & \to & K(\tau)_{A(\rho)}^\bullet
\end{array}
\]
\[
\begin{array}{ccc}
f_* (L|\Phi')^\bullet & \to & f_* (L|\Phi)^\bullet \\
\downarrow \phi_3 & & \downarrow \phi_3 \\
f_* (K|\Phi')^\bullet & \to & f_* (K|\Phi)^\bullet
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & f_* (L|\Phi')^\bullet \\
\downarrow \phi_3 & & \downarrow \phi_3 \\
0 & \to & f_* (K|\Phi')^\bullet \\
\downarrow \phi_3 & & \downarrow \phi_3 \\
0 & \to & f_* (L|\Phi)^\bullet \\
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & f_* (K|\Phi)^\bullet
\end{array}
\]

(2.3)
Among the vertical homomorphisms, $\phi_1$ is a quasi-isomorphism since $g(\tau/\tau) : L(\tau)^* \to K(\tau)^*$ is quasi-isomorphic and $A(\rho)$ is a free $A(\tau)$-module, while $\phi_3$ is quasi-isomorphic by the assumption. Hence $\phi_2 = f_*(g|\Phi')(\rho)$ is also quasi-isomorphic. Since $f_*(L|\Phi')(\mu) = f_*(L|\Phi)(\mu)$ for $\mu \neq \rho$, $f_*(g|\Phi')$ is a quasi-isomorphism in CGEM($\Delta$). This contradicts the maximality of $\Phi$. Hence $\Phi = \Delta'$.

(2) Let $\Phi$ be a maximal subfan of $\Delta'$ such that

$$0 \to f_*(L|\Phi) \xrightarrow{f_*(g)} f_*(K|\Phi) \xrightarrow{f_*(h)} f_*(J|\Phi) \to 0 \quad (2.4)$$

is a short exact sequence. Suppose $\Phi \neq \Delta'$, and let $\tau$ be a minimal element of $\Delta' \setminus \Phi$ and $\rho := f(\tau)$. Let $\Phi' := \Phi \cup \{\tau\}$. It is sufficient to show that

$$0 \to f_*(L|\Phi') \xrightarrow{f_*(g)} f_*(K|\Phi') \xrightarrow{f_*(h)} f_*(J|\Phi') \to 0 \quad (2.5)$$

is a short exact sequence, since it contradicts the maximality of $\Phi$. It is enough to check it for $\rho$. Since $0 \to L(\tau)|_{A(\rho)} \to K(\tau)|_{A(\rho)} \to J(\tau)|_{A(\rho)} \to 0$ is exact by the assumption, the exactness of

$$0 \to f_*(L|\Phi')|_{(\rho)} \xrightarrow{f_*(g)} f_*(K|\Phi')|_{(\rho)} \xrightarrow{f_*(h)} f_*(J|\Phi')|_{(\rho)} \to 0 \quad (2.6)$$

follows from the nine lemma of the homology algebra. \[q.e.d.\]

Although GEM($\Delta$) and CGEM($\Delta$) are not in general abelian categories, we say that the functor $f_*$ is exact in the sense that the property (2) of the above lemma holds. The dualizing functor $D$ defined in [R, §2] is an exact contravariant functor from CGEM($\Delta$) to itself in this sense. Actually, for $L^* \in$ CGEM($\Delta$) and $\rho \in \Delta$, $D(L)(\rho)^*$ is defined by the combination of exact functors $i_{\rho}^*$ and $D_{\rho}$ (cf.[R, §2]).

Let $\Delta$ be a finite fan. For each $\rho \in \Delta \setminus \{0\}$, we take a rational point $a(\rho)$ in the relative interior of $\rho$. For each $\alpha = (\rho_1, \ldots, \rho_k) \in Sd(\Delta \setminus \{0\})$, let $c(\alpha)$ be the simplicial cone $R_0a(\rho_1) + \cdots + R_0a(\rho_k)$ if $k > 0$, and let $c(\alpha) := 0$ if $k = 0$. Then

$$\Sigma := \{c(\alpha) \mid \alpha \in Sd(\Delta \setminus \{0\})\} \quad (2.7)$$
is a simplicial subdivision of $\Delta$.

We call $\Sigma$ a **barycentric subdivision** of $\Delta$. Clearly, it is not unique for $\Delta$ since it depends on the choice of the set $\{a(\rho) : \rho \in \Delta \setminus \{0\}\}$.

Let $f : \Sigma \to \Delta$ be a barycentric subdivision of $\Delta$. For $\alpha \in \text{Sd}(\Delta \setminus \{0\})$, $f(c(\alpha)) = \max(\alpha)$ by the definition. We define an unmixed homomorphism $\lambda_\Delta : \text{SdP}(\Delta)^\bullet \to f_\ast \text{P}(\Sigma)^\bullet$ as follows.

For each $\alpha \in \text{Sd}(\Delta \setminus \{0\})$, let $c(\alpha)$ be the corresponding cone in $\Sigma$ as above. For each $\rho \in \Delta$ and integer $i$, we have

$$\text{SdP}(\Delta)(\rho)^i = \bigoplus_{\beta \in \text{Sd}_{i-1}(F(0, \rho))} \mathbb{Z}z(\beta)y(\rho) \otimes A(\rho)$$

(2.8)

and

$$f_\ast \text{P}(\Sigma)(\rho)^i = \bigoplus_{\sigma \in f^{-1}(\rho)(i)} \text{det}(\sigma) \otimes A(\rho)$$

$$= \bigoplus_{\beta \in \text{Sd}_{i-1}(F(0, \rho))} \text{det}(c(\beta) + c(\rho)) \otimes A(\rho),$$

(2.9)

where $c(\rho)$ is the one-dimensional cone generated by $a(\rho)$.

For each $\beta = (\sigma_1, \cdots, \sigma_{i-1}) \in \text{Sd}_{i-1}(F(0, \rho))$, let $z'(\beta, \rho)$ be the generator of $\text{det}(c(\beta) + c(\rho)) \simeq \mathbb{Z}$ which has the same sign with $a(\sigma_1) \wedge \cdots \wedge a(\sigma_{i-1}) \wedge a(\rho)$ in $\text{det}(c(\beta) + c(\rho)) \otimes \mathbb{Q}$. We define $\lambda_\Delta(\rho)^i$ to be the isomorphism given by $z(\beta)y(\rho) \otimes 1 \mapsto z'(\beta, \rho) \otimes 1$ for all $\beta \in \text{Sd}_{i-1}(F(0, \rho))$.

It is easy to check the compatibility with the coboundary maps. Since $\lambda_\Delta(\rho)^i$'s are isomorphic, we get the following lemma.

**Lemma 2.4** Let $f : \Sigma \to \Delta$ be a barycentric subdivision of $\Delta$. Then the above unmixed homomorphism $\lambda_\Delta : \text{SdP}(\Delta)^\bullet \to f_\ast \text{P}(\Sigma)^\bullet$ is an isomorphism.

Let $\psi_\Sigma : \text{SdP}(\Sigma)^\bullet \to \text{P}(\Sigma)^\bullet$ be the natural unmixed quasi-isomorphism. We define an unmixed homomorphism

$$\phi_{\Sigma/\Delta} : f_\ast \text{SdP}(\Sigma)^\bullet \longrightarrow \text{SdP}(\Delta)^\bullet$$

(2.10)

by $\phi_{\Sigma/\Delta} := \lambda_\Delta^{-1} \cdot f_\ast (\psi_\Sigma)$.
Lemma 2.5 Let $f : \Sigma \to \Delta$ be a barycentric subdivision and let $p$ and $q$ be perversities of $\Delta$ and $\Sigma$, respectively. Then $\phi_{\Sigma/\Delta}(f_{*}k_{q}(\Sigma))$ is contained in $k_{p}(\Delta)$ if $q(\sigma) \leq p(f(\sigma))$ for every $\sigma \in \Sigma \setminus \{0\}$. In this case, $\phi_{\Sigma/\Delta}$ induces a natural homomorphism

$$f_{*}ic_{q}(\Sigma)^{\bullet} \longrightarrow ic_{p}(\Delta)^{\bullet}. \quad (2.11)$$

Proof. By definition, $k_{q}(\Sigma)$ is the subcomplex of $SdP(\Sigma)^{\bullet}$ generated by

$$S := \bigcup_{\sigma \in \Sigma \setminus \{0\}} \bigcup_{i+j \leq q(\sigma)} SdP(\Sigma)(\sigma)^{i}_{j}. \quad (2.12)$$

Hence it is sufficient to show that

$$\phi_{\Sigma/\Delta}(f(\sigma))(SdP(\Sigma)(\sigma)^{i}_{j}) \subset k_{p}(\Delta)(f(\sigma)) \quad (2.13)$$

for all $\sigma \in \Sigma \setminus \{0\}$ and $i, j$ with $i+j \leq q(\sigma)$.

If $i < r_{\sigma}$, then $SdP(\Sigma)(\sigma)^{i}_{j}$ is mapped to zero. Hence the inclusion is obvious in this case.

We consider the case $i = r_{\sigma}$. Let $\sigma \in \Sigma$ and let $\alpha = (\sigma_{1}, \ldots, \sigma_{r_{\sigma}})$ be the corresponding element of $Sd_{r_{\sigma}}(\Delta \setminus \{0\})$. We set $\rho := f(\sigma) = \sigma_{r_{\sigma}}$. Recall that $SdP(\Sigma)(\sigma)^{r_{\sigma}}$ is the free $A(\sigma)$-module with the basis $\{z(\beta)y(\sigma) ; \beta \in Sd_{r_{\sigma}-1}(F(0, \sigma))\}$. Since $\phi_{\Sigma/\Delta}(\rho)(z(\beta)y(\sigma) \otimes 1) = \pm z(\alpha) \otimes 1$ for every $\beta \in Sd_{r_{\sigma}-1}(F(0, \sigma))$,

$$\phi_{\Sigma/\Delta}(\rho)(Zz(\beta)y(\sigma) \otimes A(\sigma)^{j}) = Zz(\alpha) \otimes A(\sigma)^{j} \subset Zz(\alpha) \otimes A(\rho)^{j} \quad (2.14)$$

for every $\beta \in Sd_{r_{\sigma}-1}(F(0, \sigma))$ and $j \leq q(\sigma) - r_{\sigma}$. Hence the image $\phi_{\Sigma/\Delta}(\rho)(S(\sigma))$ is equal to $\bigcup_{j \leq q(\sigma) - r_{\sigma}} Zz(\alpha) \otimes A(\sigma)^{j}$ and is contained in $k_{p}(\Delta)(\rho)^{r_{\sigma}}$ if $q(\sigma) \leq p(\rho)$.

q.e.d.

Clearly, the condition of the above lemma is satisfied if $p$ and $q$ are the middle perversities. In particular we get a natural unmixed homomorphism

$$\delta_{\Sigma/\Delta} : f_{*}ic(\Sigma)^{\bullet} \longrightarrow ic(\Delta)^{\bullet} \quad (2.15)$$

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for a barycentric subdivision \( f : \Sigma \to \Delta \).

Let \( f : \Delta' \to \Delta \) be a subdivision and let \( L^\bullet \) be an object of \( \text{CGEM}(\Delta') \). We define an unmixed homomorphism

\[
\kappa(f, L) : f_* D(L)^\bullet \longrightarrow D(f_* L)^\bullet \quad (2.16)
\]
as follows.

For \( \rho \in \Delta \) and \( i \in \mathbb{Z} \), we have

\[
f_* D(L)(\rho)^i = \bigoplus_{\tau \in f^{-1}(\rho)} D(L)(\tau)^i_{A(\rho)} \quad (2.17)
\]

while

\[
D(f_* L)(\rho)^i = \bigoplus_{\eta \in F(\rho)} \det(\rho) \otimes D_\eta(f_* L(\eta)^{r_\tau - i})_{A(\rho)} \quad (2.19)
\]

For \( \tau \in f^{-1}(\rho) \) and \( \sigma \in F(\tau) \), the restriction of

\[
\kappa(f, L)(\rho)^i : f_* D(L)(\rho)^i \longrightarrow D(f_* L)(\rho)^i \quad (2.22)
\]
to the component \( \det(\tau) \otimes D_\sigma(L(\sigma)^{r_\tau - i})_{A(\rho)} \) is defined to be the zero map if \( r_\tau < r_\rho \).

If \( r_\tau = r_\rho \), then \( N(\tau) = N(\rho) \) and \( \det(\tau) = \det(\rho) \). In this case, the component is defined to be the identity map to \( \det(\rho) \otimes D_\sigma(L(\sigma)^{r_\rho - i})_{A(\rho)} \).

The commutativity of the diagram

\[
\begin{array}{ccc}
D(f_* L)(\rho)^i & \xrightarrow{\partial(\rho/\rho')} & D(f_* L)(\rho')^{i+1} \\
\kappa(f, L)(\rho)^i & \downarrow & \kappa(f, L)(\rho')^{i+1} \\
D(f_* L)(\rho)^i & \xrightarrow{\partial(\rho/\rho')} & D(f_* L)(\rho')^{i+1}
\end{array}
\quad (2.23)
\]
is checked by the definitions. The only one nontrivial case is the commutativity for the component \( \det(\tau) \otimes D_\sigma(L(\sigma)^{r-\iota})_{A(\rho)} \) of \( f_* D(L(\rho)^i) \) with \( r_\tau = r_\rho - 1 \). Since \( \kappa(f,L)(\rho)^i \) is a zero map on this component, we have to show that the composite \( \kappa(f,L)(\rho)^{i+1} \cdot d_{f,D(L)}(\rho/\rho') \) is zero on it. Since \( r_\tau = r_\rho - 1 \), there exist exactly two cones \( \tau_1, \tau_2 \) in \( f^{-1}(\rho) \) with \( \tau \prec \tau_1, \tau_2 \) and \( r_{\tau_1} = r_{\tau_2} = r_\rho \). We have \( q'_{\tau/\tau_1} + q'_{\tau/\tau_2} = 0 \) under the identification \( \det(\tau_1) = \det(\tau_2) = \det(\rho) \). Hence the composite is zero on this component.

Thus we know that \( \kappa(f,L) \) is an unmixed homomorphism in \( \text{CGEM}(\Delta) \).

**Proposition 2.6** Let \( f : \Delta' \to \Delta \) be a subdivision and let \( L^* \) be an object of \( \text{CGEM}(\Delta') \). Then the unmixed homomorphism

\[
\kappa(f,L) : f_* D(L)^* \longrightarrow D(f_* L)^* \quad (2.24)
\]

is quasi-isomorphic.

**Proof.** We prove the proposition by induction on the total dimension of \( L^* \).

The assertion is trivially true if \( \dim_Q L = 0 \). We assume that \( \dim_Q L > 0 \).

Let \( \sigma \) be a maximal element of \( \Delta' \) such that \( L(\sigma)^* \) is nontrivial. Take the maximal integer \( p \) such that \( L(\sigma)^p \neq \{0\} \) and the minimal integer \( q \) such that \( L(\sigma)^q \neq \{0\} \).

We define an object \( E_{\sigma,p,q}^* \) of \( \text{CGEM}(\Delta') \) as follows.

We define \( E_{\sigma,p,q}^*(\tau)^* := \{0\} \) for \( \tau \in \Delta' \) with \( \tau \neq \sigma \) and \( E_{\sigma,p,q}^i := \{0\} \) for \( i \neq p \).

The graded \( A(\sigma) \)-module \( E_{\sigma,p,q}(\sigma)^p \) is defined to be \( \bar{A}(\sigma)(-q) \), i.e.,

\[
E_{\sigma,p,q}(\sigma)^p := \begin{cases} Q & \text{if } j = q \\ \{0\} & \text{if } j \neq q \end{cases} \quad (2.25)
\]

By taking a nontrivial \( A(\sigma) \)-homomorphism \( g_0 : E_{\sigma,p,q}(\sigma)^p \to L(\sigma)^p \), we get an unmixed homomorphism \( g : E_{\sigma,p,q}^* \to L^* \) such that \( g(\sigma)^p = g_0 \). Let \( K^* \) be the
cokernel of \( g \). Then we get a commutative diagram

\[
\begin{array}{c}
0 \to f_\ast D(K)^\bullet \to f_\ast D(L)^\bullet \to f_\ast D(E_{\sigma,p,q})^\bullet \to 0 \\
\downarrow \kappa(f, K) \quad \downarrow \kappa(f, L) \quad \downarrow \kappa(f, E_{\sigma,p,q}) \\
0 \to D(f_\ast K)^\bullet \to D(f_\ast L)^\bullet \to D(f_\ast E_{\sigma,p,q})^\bullet \to 0
\end{array}
\]

(2.26)

Since the functors \( f_\ast \) and \( D \) are exact, the two horizontal lines in the diagram are short exact sequences. Since \( \dim_Q K = \dim_Q L - 1 \), \( \kappa(f, K) \) is a quasi-isomorphism by the induction assumption. Hence it is sufficient to show that \( \kappa(f, E_{\sigma,p,q}) \) is quasi-isomorphic.

Let \( \rho \) be an element of \( \Delta \). If \( f(\sigma) \) is not in \( F(\rho) \), then both \( f_\ast D(E_{\sigma,p,q})(\rho)^\bullet \) and \( D(f_\ast E_{\sigma,p,q})(\rho)^\bullet \) are zero. We assume \( \eta := f(\sigma) \in F(\rho) \). Set

\[
\Phi := \{ \tau \in \Delta' ; \sigma < \tau \in f^{-1}(\rho) \} .
\]

(2.27)

By \((2.18)\), we have

\[
f_\ast D(E_{\sigma,p,q})(\rho)^i = \bigoplus_{\tau \in \Phi(p+i)} \det(\tau) \otimes \bar{A}(\sigma)_{A(\rho)} .
\]

(2.28)

Hence we know

\[
H^i(f_\ast D(E_{\sigma,p,q})(\rho)^\bullet) \simeq H^{p+i}(E(\Phi, \mathbb{Z})^\bullet) \otimes \bar{A}(\sigma)_{A(\rho)}
\]

(2.29)

for \( i \in \mathbb{Z} \). On the other hand, \( D(f_\ast E_{\sigma,p,q})(\rho)^i \) is zero if \( i \neq r_\rho - p \), while

\[
D(f_\ast E_{\sigma,p,q})(\rho)^{r_\rho - p} = \det(\rho) \otimes \bar{A}(\sigma)_{A(\rho)}
\]

(2.30)

by \((2.19)\).

The cohomologies \( H^i(E(\Phi, \mathbb{Z})^\bullet) \) are zero for \( i \neq r_\rho \) and \( H^{r_\rho}(E(\Phi, \mathbb{Z})^\bullet) = \det(\rho) \) by Lemma \([1.2], (2)\). We know \( H^i(f_\ast D(E_{\sigma,p,q})(\rho)^\bullet) \) is zero for \( i \neq r_\rho - p \) and isomorphic to \( \det(\rho) \otimes \bar{A}(\sigma)_{A(\rho)} \) for \( i = r_\rho - p \). Since \( \kappa(f, E_{\sigma,p,q})(\rho)^{r_\rho - p} \) is surjective, \( \kappa(f, E_{\sigma,p,q})(\rho) \) is quasi-isomorphic.

q.e.d.

The intersection complex has the following irreducibility.
Lemma 2.7 Let $\Delta$ be a finite fan, and $p$ a perversity of it. Let $L_1^\bullet$ and $L_2^\bullet$ be objects of $\text{CGEM}(\Delta)$ which are quasi-isomorphic to $\text{ic}_p(\Delta)$. If $h : L_1^\bullet \to L_2^\bullet$ is a homomorphism such that $h(0/0) : L_1(0)^\bullet \to L_2(0)^\bullet$ is a quasi-isomorphism, then $h$ is a quasi-isomorphism.

Proof. Suppose that $h$ is not quasi-isomorphic. Let $\rho$ be a minimal element of $\Delta$ such that $h(\rho/\rho) : L_1(\rho)^\bullet \to L_2(\rho)^\bullet$ is not quasi-isomorphic. By the assumption, $\rho$ is not equal to 0. By the minimality of $\rho$, the homomorphism

$$i_\rho^*(h) : i_\rho^*(L_1)^\bullet \to i_\rho^*(L_2)^\bullet \quad (2.31)$$

in $\text{CGM}(A(\rho))$ is a quasi-isomorphism.

For each of integer $i$, we have a commutative diagram

$$
\begin{array}{ccc}
H^i(i_\rho^*(L_1)^\bullet) & \xrightarrow{u^i} & H^i(i_\rho^*(L_2)^\bullet) \\
\phi_1 \downarrow & & \downarrow \phi_2 \\
H^i(L_1(\rho)[1]^\bullet) & \xrightarrow{v^i} & H^i(L_2(\rho)[1]^\bullet)
\end{array}
$$

where $u^i$ and $v^i$ are the $A(\rho)$-homomorphism induced by $i_\rho^*(h)$ and $h(\rho/\rho)$, respectively. Since $i_\rho^*(h)$ is quasi-isomorphic, $u^i$ is an isomorphism. Since $L_2^\bullet$ is quasi-isomorphic to $\text{ic}_p(\Delta)^\bullet$, $\phi_2$ is surjective by the construction of intersection complexes [3, Thm.2.9]. Hence $v^i$ is also surjective. Since $H^i(L_1(\rho)[1]^\bullet)$ and $H^i(L_2(\rho)[1]^\bullet)$ are finite dimensional $\mathbb{Q}$-vector spaces of same dimension, $v^i$ is an isomorphism for each $i$, i.e., $h(\rho/\rho)$ is a quasi-isomorphism. This contradicts the assumption. q.e.d.

Since $\text{ic}_p(\Delta)(0)^0 = \mathbb{Q}$ and $\text{ic}_p(\Delta)(0)^i = \{0\}$ for $i \neq 0$, $h(0/0)$ in the above lemma is quasi-isomorphic if and only if the induced homomorphism $H^0(L_1(0)^\bullet) \to H^0(L_2(0)^\bullet)$ is a nonzero map.

Let $f : \Delta' \to \Delta$ be a subdivision and let $L^\bullet$ be in $\text{CGEM}(\Delta')$. Then, by the definition of the functor $f_*$, the complex $\Gamma(f_*L)^\bullet$ is canonically isomorphic to $\Gamma(L)^\bullet$. In particular, if $f : \Sigma \to \Delta$ is a barycentric subdivision, then $\delta_{\Sigma/\Delta}$ in (2.14) induces
a homomorphism
\[ \Gamma(\delta_{\Sigma/\Delta}) : \Gamma(\text{ic}(\Sigma))^\bullet \longrightarrow \Gamma(\text{ic}(\Delta))^\bullet \] (2.33)
in CGM(A).

**Theorem 2.8 (Decomposition theorem)** Let \( \Delta \) be a finite fan and \( \Sigma \) a barycentric subdivision of \( \Delta \). Then, for each integer \( p \), the homomorphism of \( A \)-modules
\[ H^p(\Gamma(\text{ic}(\Sigma))^\bullet) \longrightarrow H^p(\Gamma(\text{ic}(\Delta))^\bullet) \] (2.34)
induced by \( \Gamma(\delta_{\Sigma/\Delta}) \) is a split surjection, i.e., is surjective and the kernel is a direct summand as an \( A \)-module. In particular,
\[ \dim_Q H^p(\Gamma(\text{ic}(\Delta))^\bullet)_q \leq \dim_Q H^p(\Gamma(\text{ic}(\Sigma))^\bullet)_q \] (2.35)
for any integers \( p, q \).

**Proof.** By applying the contravariant functor \( D \) to the homomorphism (2.15), we get a homomorphism
\[ D(\delta_{\Sigma/\Delta}) : D(\text{ic}(\Delta))^\bullet \longrightarrow D(f_* \text{ic}(\Sigma))^\bullet. \] (2.36)
Since \( \delta_{\Sigma/\Delta}(0) \) is an isomorphism, so is \( D(\delta_{\Sigma/\Delta})(0) \).

Since \( D(\text{ic}(\Sigma))^\bullet \) is quasi-isomorphic to \( \text{ic}(\Sigma)^\bullet \) by [I3, Cor. 2.12], \( f_* D(\text{ic}(\Sigma))^\bullet \) and \( f_* \text{ic}(\Sigma)^\bullet \) are also quasi-isomorphic by Lemma 2.3(1). By Proposition 2.6, we get a quasi-isomorphism
\[ \kappa(f, \text{ic}(\Sigma)) : f_* D(\text{ic}(\Sigma))^\bullet \longrightarrow D(f_* \text{ic}(\Sigma))^\bullet. \] (2.37)
Hence \( D(f_* \text{ic}(\Sigma))^\bullet \) is quasi-isomorphic to \( f_* \text{ic}(\Sigma)^\bullet \).

On the other hand, \( D(\text{ic}(\Delta))^\bullet \) is quasi-isomorphic to \( \text{ic}(\Delta)^\bullet \) by [I3, Cor. 2.12].

By applying [I3, Lem 2.16] for the homomorphism \( D(\delta_{\Sigma/\Delta}) \), we get \( L^\bullet \) in CGEM(\( \Delta \)), a quasi-isomorphism \( g : L^\bullet \rightarrow \text{ic}(\Delta)^\bullet \) and a homomorphism \( h : L^\bullet \rightarrow f_* \text{ic}(\Sigma)^\bullet \) such
that the homomorphisms of cohomologies induced by $h$ is compatible with those induced by $D(\delta_{\Sigma/\Delta})$.

Note that all homomorphisms in $\text{CGEM}(\Delta)$ appeared here are quasi-isomorphic on $0 \in \Delta$. Since $L^\bullet$ and $\text{ic}(\Delta)^\bullet$ are quasi-isomorphic, the composite $\delta_{\Sigma/\Delta} \cdot h : L^\bullet \to \text{ic}(\Delta)^\bullet$ is a quasi-isomorphism by Lemma 2.7.

Hence, for each integer $i$, the homomorphism (2.34) is surjective and $H^p(\Gamma(\text{ic}(\Sigma))^\bullet)$ is the direct sum of the kernel and the image of the homomorphism $H^p(L^\bullet) \to H^p(\Gamma(\text{ic}(\Sigma))^\bullet)$ induced by $h$. q.e.d.

3 Intersection complexes for simplicial fans

Lemma 3.1 Let $\pi$ be a simplicial cone. Then

$$H^i(\text{ic}_t(F(\pi))(\pi)^\bullet)_j \simeq \begin{cases} \mathbb{Q} & \text{if } (i,j) = (r_{\pi}, 0) \\ \{0\} & \text{if } (i,j) \neq (r_{\pi}, 0) \end{cases}$$ \hspace{1cm} (3.1)

On the other hand,

$$H^i(\text{ic}_t(F(\pi))^\bullet)_j \simeq \begin{cases} \mathbb{Q} & \text{if } (i,j) = (0, -r_{\pi}) \\ \{0\} & \text{if } (i,j) \neq (0, -r_{\pi}) \end{cases}$$ \hspace{1cm} (3.2)

Proof. The first assertion follows from the description (1.62).

Let $s := r_{\pi}$ and $\{x_1, \ldots, x_s\} \subset N_{\mathbb{Q}}$ be a minimal generator of the simplicial cone $\pi$. For each element $\sigma \in F(\pi)$, there exists a unique subset $\{i_1, \ldots, i_p\}$ of $\{1, \ldots, s\}$ with $i_1 < \cdots < i_p$ such that $\{x_{i_1}, \ldots, x_{i_p}\}$ generates $\sigma$. We denote $x(\sigma) := x_{i_1} \wedge \cdots \wedge x_{i_p}$.

For $\rho \in F(\pi)$, we have a description

$$\text{ic}_t(F(\pi))(\rho)_{\Lambda(\pi)}^i = \det(\rho) \otimes \bigwedge^*(N(\pi)_{\mathbb{Q}}/N(\rho)_{\mathbb{Q}}) = \bigoplus_{\sigma \in F(\rho')} \det(\rho) \otimes \mathbb{Q}x(\sigma),$$ \hspace{1cm} (3.3)

where $\rho'$ is the complementary cone of $\rho$ in $\pi$, i.e., the unique cone in $F(\pi)$ with $\rho \cap \rho' = 0$ and $\rho + \rho' = \pi$. Hence $\text{ic}_t(F(\pi))(\rho)_{\Lambda(\pi)}^i$ has the component $\det(\rho) \otimes \mathbb{Q}x(\sigma)$.
if and only if $\rho \in F(\sigma')$ for the complementary cone $\sigma'$ of $\sigma$. Note that $x(\sigma)$ is a homogeneous element of degree $-r_\sigma$ and $\sigma'$ is of dimension $r_\pi - r_\sigma$.

By this observation, we have

$$
(i^*_\pi \text{ic}_t(F(\pi))^\bullet)_j \simeq \bigoplus_{\sigma \in F(\pi)(-j)} E(F(\sigma'), \mathbb{Z})^\bullet \otimes \mathbb{Q}x(\sigma)
$$

for each integer $j$. By Lemma 1.2 (1), this complex of $\mathbb{Q}$-vector spaces has trivial cohomologies if $r_\pi + j > 0$ since $\dim \sigma' = r_\pi + j$ for $\sigma \in F(\pi)(-j)$.

For $j = -r_\pi$, $F(\pi)(-j) = \{\pi\}$ and $\pi' = 0$. Hence $H^0(i^*_\pi \text{ic}_t(F(\pi))^\bullet)_{-r_\pi} \simeq \mathbb{Q}$ and $H^i(i^*_\pi \text{ic}_t(F(\pi))^\bullet)_{-r_\pi} = \{0\}$ for $i \neq 0$.

We get the lemma since the complexes are nontrivial only for $-r_\pi \leq j \leq 0$.

$q.e.d.$

**Theorem 3.2** Let $\Delta$ be a simplicial finite fan. Then, for any perversity $p$ with $b \leq p \leq t$, $\text{ic}_p(\Delta)^\bullet$ is quasi-isomorphic to $\text{ic}_t(\Delta)^\bullet$.

**Proof.** By [I3, Thm.2.9], it suffices to show that $H^i(\text{ic}_p(\Delta)(\rho))^\bullet_j = \{0\}$ if $i + j \leq p(\rho)$ and $H^i(i^*_\pi \text{ic}_t(\Delta)^\bullet)^\bullet_j = \{0\}$ if $i + j \geq p(\rho)$ for $\rho \in \Delta \setminus \{0\}$. Since $\text{ic}_p(F(\rho))^\bullet$ is the restriction of $\text{ic}_p(\Delta)^\bullet$ to $F(\rho)$ and $-r_\rho + 1 \leq p(\rho) \leq r_\rho - 1$ by the assumption, this is a consequence of Lemma 3.1.

The following theorem is equivalent to [O2, Prop.4.1].

**Theorem 3.3** Let $\Delta$ be a simplicial complete fan. Then $H^p(\Gamma(\text{ic}_t(\Delta))^\bullet)_q = \{0\}$ for any integers $p, q \in \mathbb{Z}$ with $p \neq q + r$.

**Proof.** By Lemma 1.10, we have $H^p(\Gamma(\text{ic}_t(\Delta))^\bullet)_q = \{0\}$ for $p > q + r$. By the duality for complete fans [I3, Prop.2.5], we have $H^p(\Gamma(D(\text{ic}_t(\Delta))^\bullet))_q = \{0\}$ for $p, q \in \mathbb{Z}$ with $(r-p) > (-r-q)+r$, i.e., with $p < q+r$. $D(\text{ic}_t(\Delta))^\bullet$ is quasi-isomorphic to $\text{ic}_b(\Delta)^\bullet$ by [I3, Cor.2.12]. We get the theorem, since $\text{ic}_b(\Delta)^\bullet$ is quasi-isomorphic to $\text{ic}_t(\Delta)^\bullet$ by Theorem 3.2.

$q.e.d.$
Let \( f : \Delta' \to \Delta \) be an arbitrary subdivision of a finite fan. We define an unmixed homomorphism

\[
\tilde{\delta}_{\Delta'/\Delta} : f_* \text{ic}_t(\Delta')^\bullet \to \text{ic}_t(\Delta)^\bullet
\]

(3.5) in CGEM(\( \Delta \)) as follows.

Note that

\[
f_* \text{ic}_t(\Delta')(\rho) = \bigoplus_{\sigma \in f^{-1}(\rho)(i)} \det(\sigma) \otimes (\tilde{A}(\sigma))_{A(\rho)}
\]

(3.6) for each integer \( i \), while

\[
\text{ic}_t(\Delta)(\rho)^i = \begin{cases} 
\det(\rho) \otimes \tilde{A}(\rho) & \text{if } i = r_{\rho} \\
0 & \text{if } i \neq r_{\rho}
\end{cases}
\]

(3.7)

For each \( \sigma \in f^{-1}(\rho) \), the restriction to the component \( \det(\sigma) \otimes (\tilde{A}(\sigma))_{A(\rho)} \) of the homomorphism

\[
\tilde{\delta}_{\Delta'/\Delta}(\rho)^i : f_* \text{ic}_t(\Delta')(\rho)^i \to \text{ic}_t(\Delta)(\rho)^i
\]

(3.8) is defined to be zero if \( i = r_{\sigma} < r_{\rho} \). If \( r_{\sigma} = r_{\rho} \), then \( \det(\sigma) = \det(\rho) \) and \( A(\sigma) = A(\rho) \), and the component is defined to be the identity map to \( \det(\rho) \otimes \tilde{A}(\rho) \).

Let \( \Delta' \) be a free \( \mathbb{Z} \)-module of rank \( r' \) and let \( f_0 : N \to N' \) be a surjective homomorphism. Fans \( \Delta \) of \( N_R \) and \( \Phi \) of \( N'_R \) are said to be compatible with \( f_0 \), if \( f_0(\sigma) \) is contained in a cone of \( \Phi \) for every \( \sigma \in \Delta \), where we denote also by \( f_0 \) the linear map \( N_R \to N'_R \).

We say a map \( f : \Delta \to \Phi \) a morphism of fans if such \( f_0 \) is given and \( f(\sigma) \) is the minimal cone of \( \Phi \) which contains \( f_0(\sigma) \) for every \( \sigma \in \Delta \). The direct image functor

\[
f_* : \text{GEM}(\Delta) \longrightarrow \text{GEM}(\Phi)
\]

(3.9) is defined as follows.

Let \( L \) be in \( \text{GEM}(\Delta) \). For each \( \rho \in \Phi \), we set \( f^{-1}(\rho) := f^{-1}(\{\rho\}) \) similarly as in the case of subdivisions. If \( \sigma \in f^{-1}(\rho) \), then \( f_0 \) induces a homomorphism
\(N(\sigma) \rightarrow N'(\rho)\) and a ring homomorphism \(A(\sigma) \rightarrow A'(\rho)\). We set
\[
(f_*L)(\rho) := \bigoplus_{\sigma \in f^{-1}(\rho)} L(\sigma)_{A'(\rho)},
\]
where \(L(\sigma)_{A'(\rho)} := L(\sigma) \otimes_{A(\sigma)} A'(\rho)\). Let \(g : L \rightarrow K\) be a homomorphism in \(\text{GEM}(\Delta)\).

For \(\rho, \mu \in \Phi\) with \(\rho \prec \mu\) and for \(\sigma \in f^{-1}(\rho)\) and \(\tau \in f^{-1}(\mu)\), the \((\sigma, \tau)\)-component of the homomorphism \(f_*(g(\rho/\mu)) : (f_*L)(\rho) \rightarrow (f_*K)(\mu)\) is the map induced by \(g(\sigma/\tau)\) if \(\sigma \prec \tau\) and is the zero map otherwise.

Let \(\Delta\) and \(\Phi\) be complete fans of \(\mathbb{N}_R\) and \(\mathbb{N}'_R\), respectively, and let \(f : \Delta \rightarrow \Phi\) be a morphism of fans associated with \(f_0 : N \rightarrow N'\). We take a generator \(w\) of \(\bigwedge^{r-r'}\ker f_0 \cong \mathbb{Z}\). We define an unmixed homomorphism
\[
\bar{\delta}_{\Delta/\Phi} : f_*\text{ic}_t(\Delta)^* \rightarrow \text{ic}_t(\Phi)[-r + r']^* \tag{3.11}
\]
as follows.

Let \(\rho\) be in \(\Phi\). For each \(\sigma \in f^{-1}(\rho)\), the \(\sigma\)-component of the homomorphism
\[
\bar{\delta}_{\Delta/\Phi}(\rho)^i : f_*\text{ic}_t(\Delta)(\rho)^i \rightarrow \text{ic}_t(\Phi)[-r + r'](\rho)^i \tag{3.12}
\]
is defined to be zero if \(i \neq r_\rho + r - r'\) since then
\[
\text{ic}_t(\Phi)[-r + r'](\rho)^i = \text{ic}_t(\Phi)(\rho)^i = \{0\}. \tag{3.13}
\]
If \(i = r_\rho + r - r'\), then
\[
f_*\text{ic}_t(\Delta)(\rho)^i = \bigoplus_{\sigma \in f^{-1}(\rho)(r_\rho + r - r')} \det(\sigma) \otimes \tilde{A}(\sigma)_{A'(\rho)} \tag{3.14}
\]
and
\[
\text{ic}_t(\Phi)[-r + r'](\rho)^i = \det(\rho) \otimes \tilde{A}'(\rho). \tag{3.15}
\]
If \(\sigma \in f^{-1}(\rho)(r_\rho + r - r')\), then \(r_\sigma - r_\rho = r - r'\) and \(\text{Ker} f_0 \subset N(\sigma)\). Hence \(w \in \bigwedge^{r-r'} N(\sigma)\). The \(\sigma\)-component of the homomorphism is defined to be the tensor product of the isomorphism \(q^{w}_{\sigma/\rho} : \det(\sigma) \rightarrow \det(\rho)\) which sends \(w \wedge a\) to
\[(\Lambda^r f_0)(a) \text{ for } a \in \Lambda^r N(\sigma) \text{ and the natural isomorphism } \bar{A}(\sigma)_{A'(\rho)} \simeq \bar{A}'(\rho), \text{ where} \]
\[\Lambda^r f_0 : \Lambda^r N(\sigma) \to \text{det}(\rho) \text{ is the natural map induced by } f_0.\]

Let \(\eta\) be an element of a finite fan \(\Delta\) of \(N_R\) and let \(N'\) be the quotient free \(\mathbb{Z}\)-module \(N/N(\eta)\) of rank \(r' := r - r_\eta\). We set \(\Delta(\eta<) := \{\sigma \in \Delta : \eta < \sigma\}\) as before. For each \(\sigma \in \Delta(\eta<)\), let \(\sigma[\eta]\) be the image of \(\sigma\) in the quotient space \(N'_R = N_R/N(\eta)_R\). Then
\[
\Delta[\eta] := \{\sigma[\eta] : \sigma \in \Delta(\eta<)\} \tag{3.16}
\]
is a finite fan of \(N'_R\). For \(\tau \in \Delta[\eta]\), the notations \(N'(\tau) \subset N'\) and \(A'(\tau) \subset A' := \Lambda^{r-r_\eta} N'_Q\) are defined similarly for \(N'\) as we defined for \(N\).

For \(\sigma \in \Delta(\eta<)\), \(A'(\sigma[\eta])\) is a quotient ring of \(A(\sigma)\) by the two-sided ideal generated by \(N(\eta)\). Hence the category \(\text{GM}(A'(\sigma[\eta]))\) of finitely generated graded \(A'(\sigma[\eta])\)-modules is a full subcategory of \(\text{GM}(A(\sigma))\).

We define the categories \(\text{GEM}(\Delta[\eta])\) and \(\text{CGEM}(\Delta[\eta])\), similarly. We denote by \(\epsilon_\eta\) the natural functor
\[\epsilon_\eta : \text{GEM}(\Delta[\eta]) \longrightarrow \text{GEM}(\Delta) \tag{3.17}\]
defined by \(\epsilon_\eta(L)(\sigma) := L(\sigma[\eta])\) if \(\sigma \in \Delta(\eta<)\) and \(\epsilon_\eta(L)(\sigma) := \{0\}\) otherwise. For \(f : L \to K\) in \(\text{GEM}(\Delta[\eta])\), \(\epsilon_\eta(f) : \epsilon_\eta(L) \to \epsilon_\eta(K)\) is defined by \(\epsilon_\eta(f)(\sigma/\tau) := f(\sigma[\eta]/\tau[\eta])\) if \(\sigma \in \Delta(\eta<)\) and \(\epsilon_\eta(f)(\sigma/\tau) := 0\) otherwise.

We take a generator \(w \in \text{det}(\eta) = \text{Ker } f_0 \simeq \mathbb{Z}\). We define an isomorphism
\[h(\Delta, \eta, w) : \text{ic}_t(\Delta(\eta<))^* \simeq \epsilon_\eta(\text{ic}_t(\Delta[\eta]))[-r_\eta]^* \tag{3.18}\]
as follows.

For each \(\sigma \in \Delta(\eta<)\), we have
\[\text{ic}_t(\Delta(\eta<))(\sigma)^{r_\sigma} = \text{det}(\sigma) \otimes \bar{A}(\sigma), \tag{3.19}\]
and
\[\epsilon_\eta(\text{ic}_t(\Delta[\eta]))[-r_\eta](\sigma)^{r_\sigma} = \text{det}(\sigma[\eta]) \otimes \bar{A}'(\sigma[\eta]) \tag{3.20}\]
by \((\ref{1.62})\), while
\[
\mathrm{ic}_t(\Delta(\eta \prec))(\sigma)^i = \epsilon_{\eta}(\mathrm{ic}_t(\Delta[\eta]))[-r_{\eta}]^{\cdot i} = \{0\}
\] (3.21)
for \(i \neq r_{\sigma}\). Since \(\bar{A}(\sigma) = A(\sigma)/N(\sigma)A(\sigma)\) and \(\bar{A}'(\sigma[\eta]) = A'(\sigma[\eta])/N'(\sigma[\eta])A'(\sigma[\eta])\)
and since the kernel of the surjection \(A(\sigma) \to A'(\sigma[\eta])\) is \(N(\eta)A(\sigma) \subset N(\sigma)A(\sigma)\), \(\bar{A}'(\sigma[\eta])\) is naturally isomorphic to \(\bar{A}(\sigma)\) as an \(A(\sigma)\)-module. We define an isomorphism
\[
h(\Delta, \eta, w)(\sigma)^{r_{\sigma}} : \mathrm{ic}_t(\Delta(\eta \prec))(\sigma)^{r_{\sigma}} \to \epsilon_{\eta}(\mathrm{ic}_t(\Delta[\eta]))[-1](\sigma)^{r_{\sigma}}
\] (3.22)
to be the tensor product of \(q_{\eta}^{w}/\sigma[\eta]\) and this isomorphism. The commutativity with the coboundary maps is checked easily. When \(r_{\eta} = 1\), we take the primitive element of \(\eta \cap N\) as \(w\) and denote the isomorphism simply by \(h(\Delta, \eta, w)\).

**Lemma 3.4** Let \(\eta\) be a cone of a complete fan \(\Delta\). We define \(N', f_0 : N'_{\mathbb{R}} \to \Delta'_{\mathbb{R}}\) and \(\Delta[\eta]\) as above. If \(f_0\) induces a morphism \(f : \Delta \to \Delta[\eta]\) of fans, then the homomorphism
\[
H^i(\Gamma(\mathrm{ic}_t(\Delta(\eta \prec)))) \to H^i(\Gamma(\mathrm{ic}_t(\Delta)))
\] (3.23)
induced by the inclusion map \(\mathrm{ic}_t(\Delta(\eta \prec)) \to \mathrm{ic}_t(\Delta)\) is injective for every \(i \in \mathbb{Z}\).

**Proof.** We take a generator \(w\) of \(\det(\eta)\). By definitions, the composite of homomorphisms
\[
\Gamma(\mathrm{ic}_t(\Delta(\eta \prec))) \xrightarrow{\phi} \Gamma(\mathrm{ic}_t(\Delta)) \xrightarrow{\Gamma(f_\ast(\mathrm{ic}_t(\Delta)))} \Gamma(\mathrm{ic}_t(\Delta[\eta])[-r_{\eta}])
\] (3.24)
induced by the homomorphisms \(\mathrm{ic}_t(\Delta(\eta \prec)) \to \mathrm{ic}_t(\Delta)\) and \(\delta_{\Delta/\Delta[\eta]} : f_\ast(\mathrm{ic}_t(\Delta)) \to \mathrm{ic}_t(\Delta[\eta])[-r_{\eta}]\) is equal to the homomorphism
\[
\Gamma(\mathrm{ic}_t(\Delta(\eta \prec))) \xrightarrow{\phi} \Gamma(\mathrm{ic}_t(\Delta[\eta])[-r_{\eta}])
\] (3.25)
induced by \(h(\Delta, \eta, w)\). Since \(h(\Delta, \eta, w)\) is an isomorphism, the homomorphisms of cohomologies induced by \(\phi\) in (3.24) are injective. \(\ q.e.d.\)
Theorem 3.5 Let $\Delta$ be a simplicial complete fan and let $\eta$ be an element of $\Delta$.
Then the homomorphism
\[
H^i(\Gamma(ic_t(\Delta(\eta))))^* \to H^i(\Gamma(ic_t(\Delta))^*)
\] (3.26)
induced by the inclusion map $ic_t(\Delta(\eta))^* \to ic_t(\Delta)^*$ is injective for every $i \in \mathbb{Z}$.

Proof. Let $g : L^* \to K^*$ be a quasi-isomorphism in CGEM($\Delta$). Then we get a commutative diagram
\[
\begin{array}{ccc}
H^i(\Gamma(L|\Delta(\eta)))^* & \xrightarrow{\phi_1} & H^i(\Gamma(L)^*) \\
g_1 \downarrow & & \downarrow g_2 \\
H^i(\Gamma(K|\Delta(\eta)))^* & \xrightarrow{\phi_2} & H^i(\Gamma(K)^*)
\end{array}
\] (3.27)
for every $i \in \mathbb{Z}$. Since $g$ is a quasi-isomorphism, $g_1$ and $g_2$ are isomorphisms. Hence $\phi_1$ is injective if and only if $\phi_2$ is injective. This implies that, for the proof of the theorem, it is sufficient to show the injectivity of the homomorphisms
\[
H^i(\Gamma(L|\Delta(\eta)))^* \to H^i(\Gamma(L)^*)
\] (3.28)
for $i \in \mathbb{Z}$ for an $L^*$ which is quasi-isomorphic to $ic_t(\Delta)^*$.

Let $N^:\Delta := N/N(\eta)$ and $f_0 : N^R \to N^R_\eta$ be the natural surjection. We take a simplicial subdivision $u : \Delta' \to \Delta$ such that $\eta \in \Delta'$, $\Delta'(\eta) = \Delta(\eta)$ and $f_0$ induces a morphism $f : \Delta' \to \Delta'[[\eta]]$.

Then $u_* ic_t(\Delta'(\eta))^*$ is equal to $ic_t(\Delta(\eta))^*$. Since $\Gamma(u_* ic_t(\Delta'))^* = \Gamma(ic_t(\Delta'))^*$, the homomorphism
\[
H^i(\Gamma(ic_t(\Delta(\eta)))^*) = H^i(\Gamma(u_* ic_t(\Delta'(\eta))))^* \to H^i(\Gamma(u_* ic_t(\Delta'))^*)
\] (3.29)
is injective for every $i \in \mathbb{Z}$ by Lemma 3.4.

By applying the dualizing functor to the homomorphism
\[
\delta_{\Delta'/\Delta} : u_* ic_t(\Delta')^* \to ic_t(\Delta)^*,
\] (3.30)

we get a homomorphism

\[ D(\delta_{\Delta'/\Delta}) : D(\text{ic}_t(\Delta))^\bullet \to D(u_* \text{ic}_t(\Delta'))^\bullet. \]  

(3.31)

By [I3, Cor.2.12] and Theorem 3.2, \( D(\text{ic}_t(\Delta'))^\bullet \) is quasi-isomorphic to \( \text{ic}_t(\Delta')^\bullet \) in CGEM(\( \Delta' \)). Hence, by Lemma 2.3 (1) and Proposition 2.6, \( K^\bullet := D(u_* \text{ic}_t(\Delta'))^\bullet \) is quasi-isomorphic to \( u_* \text{ic}_t(\Delta')^\bullet \). By the injectivity of (3.29), the homomorphism

\[ H^i(\Gamma(K|\Delta(\eta<)))^\bullet \to H^i(\Gamma(K)^*) \]  

(3.32)

is injective for every \( i \in \mathbb{Z} \).

On the other hand, \( L^\bullet := D(\text{ic}_t(\Delta))^\bullet \) is quasi-isomorphic to \( \text{ic}_t(\Delta)^\bullet \) by [I3, Cor.2.12] and Theorem 3.2. Hence the theorem is equivalent to the injectivity of the homomorphism

\[ H^i(\Gamma(L|\Delta(\eta<)))^\bullet \to H^i(\Gamma(L)^*) \]  

(3.33)

for \( i \in \mathbb{Z} \). Let \( \Phi := \bigcup_{\sigma \in \Delta(\eta<)} F(\sigma) \). Then \( \Phi \) is a common subfan of \( \Delta \) and \( \Delta' \) which contains \( \Delta(\eta<) \). In particular, \( (u_* \text{ic}_t(\Delta')|\Phi)^\bullet = (\text{ic}_t(\Delta)|\Phi)^\bullet \). Hence, by the definition of \( D \), we have \( (L|\Phi)^\bullet = (K|\Phi)^\bullet \) and hence \( (L|\Delta(\eta<))^\bullet = (K|\Delta(\eta<))^\bullet \).

We get a commutative diagram

\[
\begin{array}{ccc}
H^i(\Gamma(L|\Delta(\eta<))^\bullet) & \to & H^i(\Gamma(L)^*) \\
\downarrow & & \downarrow \\
H^i(\Gamma(K|\Delta(\eta<))^\bullet) & \to & H^i(\Gamma(K)^*)
\end{array}
\]  

(3.34)

The injectivity of (3.33) follows from that of (3.32).

q.e.d.

The following theorem is essentially equal to [O2, Thm.4.2]. We write here the proof in our notation in order to show that we need not use the corresponding toric varieties.
Theorem 3.6  Let $\tilde{\Delta}$ be a complete simplicial fan of $N_R$ and $\gamma$ a one-dimensional cone in $\tilde{\Delta}$.

Let $\Delta$ be the subfan $\tilde{\Delta} \setminus \tilde{\Delta}(\gamma \prec)$ of $\tilde{\Delta}$. Then

$$H^p(\Gamma(\text{ic}_t(\Delta))^\bullet)_q = \{0\}$$ (3.35)

for all $p \neq q + r$.

Proof. Since $\Delta(\gamma \prec)$ is star closed in $\tilde{\Delta}$, there exists a short sequence

$$0 \rightarrow \Gamma(\text{ic}_t(\tilde{\Delta}(\gamma \prec))^\bullet) \rightarrow \Gamma(\text{ic}_t(\tilde{\Delta}))^\bullet \rightarrow \Gamma(\text{ic}_t(\Delta))^\bullet \rightarrow 0.$$ (3.36)

We consider the homogeneous degree $q$-part

$$\rightarrow H^{p-1}(\Gamma(\text{ic}_t(\tilde{\Delta}))^\bullet)_q \rightarrow H^p(\Gamma(\text{ic}_t(\Delta))^\bullet)_q$$

$$\rightarrow H^p(\Gamma(\text{ic}_t(\tilde{\Delta}(\gamma \prec)))^\bullet)_q \rightarrow H^p(\Gamma(\text{ic}_t(\tilde{\Delta}))^\bullet)_q \rightarrow H^p(\Gamma(\text{ic}_t(\Delta))^\bullet)_q$$

$$\rightarrow H^{p+1}(\Gamma(\text{ic}_t(\tilde{\Delta}(\gamma \prec)))^\bullet)_q \rightarrow$$ (3.37)

of the long exact sequence obtained by (3.36) for each integer $q$.

Since $\tilde{\Delta}$ is a simplicial complete fan, so is the fan $\tilde{\Delta} [\gamma]$ of the $(r-1)$-dimensional space $N_R'$.

By the isomorphism

$$h(\tilde{\Delta}, \gamma) : \text{ic}_t(\tilde{\Delta}(\gamma \prec))^\bullet \simeq \epsilon_\gamma(\text{ic}_t(\tilde{\Delta}[\gamma]))[-1]^\bullet,$$ (3.38)

we have an isomorphism

$$\Gamma(\text{ic}_t(\tilde{\Delta}(\gamma \prec)))^\bullet \simeq \Gamma(\text{ic}_t(\tilde{\Delta}[\gamma]))[-1]^\bullet,$$ (3.39)

of graded $A$-modules.

By Theorem 3.3 applied for $\tilde{\Delta}[\gamma]$, we have

$$H^p(\Gamma(\text{ic}_t(\tilde{\Delta}[\gamma]))[-1]^\bullet)_q = H^{p-1}(\Gamma(\text{ic}_t(\tilde{\Delta}[\gamma]))^\bullet)_q = \{0\}$$ (3.40)
for \( p - 1 \neq q + r - 1 \), i.e., for \( p \neq q + r \). Hence \( H^p(\Gamma(ic_t(\tilde{\Delta}(\gamma\prec)))^\bullet)_q = \{0\} \) for \( p \neq q + r \).

Since \( \tilde{\Delta} \) is a simplicial complete fan of \( N_R \), we have

\[
H^p(\Gamma(ic_t(\tilde{\Delta}))^\bullet)_q = \{0\}
\]  

for \( p \neq q + r \) by Theorem 3.3.

If \( p \neq q + r - 1, q + r \), then we have \( H^p(\Gamma(ic_t(\Delta))^\bullet)_q = \{0\} \) by the long exact sequence. We have \( H^{q+r-1}(\Gamma(ic_t(\Delta))^\bullet)_q = \{0\} \) for every \( q \), since the homomorphism

\[
H^{q+r}(\Gamma(ic_t(\tilde{\Delta}(\gamma\prec)))^\bullet) \to H^{q+r}(\Gamma(ic_t(\Delta))^\bullet)
\]

is injective by Theorem 3.3

\[ q.e.d. \]

**Lemma 3.7** Under the same assumption as the above theorem, the homomorphism

\[
H^{q+r}(\Gamma(ic_t(\tilde{\Delta}))^\bullet)_q \to H^{q+r}(\Gamma(ic_t(\Delta))^\bullet)_q
\]

induced by the natural homomorphism \( ic_t(\tilde{\Delta})^\bullet \to ic_t(\Delta)^\bullet \) is surjective.

**Proof.** This lemma follows from the long exact sequence (3.37) in the proof of the theorem, since \( H^{q+r+1}(\Gamma(ic_t(\tilde{\Delta}(\gamma\prec)))^\bullet)_q = \{0\} \).

\[ q.e.d. \]

### 4 The diagonal theorems for a complete fan and a cone

In this section, we prove two diagonal theorems.

**Theorem 4.1 (The first diagonal theorem)** Let \( \Delta \) be a complete fan of \( N_R \).

Then

\[
H^p(\Gamma(ic(\Delta))^\bullet)_q = \{0\}
\]

for \( p, q \in \mathbb{Z} \) with \( p \neq q + r \).
**Proof.** Let $\Sigma \rightarrow \Delta$ be a barycentric subdivision. Since $\Sigma$ is a simplicial fan, $\text{ic}(\Sigma)^\bullet$ is quasi-isomorphic to $\text{ic}_t(\Sigma)^\bullet$ by Theorem 3.2. Since $\Delta$ is complete, so is $\Sigma$. Hence $H^p(\Gamma(\text{ic}(\Sigma)^\bullet))_q = \{0\}$ for $p \neq q + r$ by Theorem 3.3. We get the theorem, since

$$\dim_Q H^p(\Gamma(\text{ic}(\Delta)^\bullet))_q \leq \dim_Q H^p(\Gamma(\text{ic}(\Sigma)^\bullet))_q$$

(4.2) for any $p, q$ by Theorem 2.8.

q.e.d.

**Theorem 4.2** Let $\Delta$ be a finite fan which may not be complete. Assume that there exists a complete fan $\tilde{\Sigma}$ and a one-dimensional cone $\gamma \in \tilde{\Sigma}$ such that $\Sigma := \tilde{\Sigma} \setminus \tilde{\Sigma}(\gamma \prec)$ is a barycentric subdivision of $\Delta$. Then

$$H^p(\Gamma(\text{ic}(\Delta)^\bullet))_q = 0$$

(4.3) for $p \neq q + r$.

**Proof.** By Theorem 3.6, we have $H^p(\Gamma(\text{ic}(\Sigma)^\bullet))_q = 0$ for $p \neq q + r$. Then the lemma is a consequence of Theorem 2.8. q.e.d.

**Theorem 4.3 (The second diagonal theorem)** Let $\pi \subset N_\mathbb{R}$ be a cone of dimension $r$. Then

$$H^p(\Gamma(\text{ic}(F(\pi)\setminus\{\pi\})^\bullet))_q = \{0\}$$

(4.4) unless $p + q \geq 0, p \neq q + r - 1$ or $p + q \leq -1, p \neq q + r$.

**Proof.** Since $0 \leq r_\sigma \leq r - 1$ for $\sigma \in F(\pi)\setminus\{\pi\}$, $\Gamma(\text{ic}(F(\pi)\setminus\{\pi\}))^p_q = \{0\}$ unless $0 \leq p \leq r - 1$ and $-r \leq q \leq 0$ by [I3, Prop.2.11]. In order to prove the theorem, it is sufficient to show the vanishing (4.4) for $p, q$ with $p \neq q + r - 1$ and $p + q \geq 0$, since then Theorem 2.13 implies the vanishing (4.4) for $p, q$ with $p \neq q + r$ and $p + q \leq -1$.
We denote simply by $\text{ic}(\pi)^\bullet$ the complex $\text{ic}(F(\pi))(\pi)^\bullet$ in $\text{CGM}(A)$. Since
\[
\text{ic}(\pi)^\bullet = \text{gt}^{\geq 1}(i_\pi^*(\text{ic}(F(\pi))))[-1]^\bullet \quad (4.5)
\]
\[
= \text{gt}^{\geq 1}(\Gamma(\text{ic}(F(\{\pi\}))))[-1]^\bullet \quad (4.6)
\]
\[
= (\text{gt}^{\geq 0} \Gamma(\text{ic}(F(\{\pi\})))[{-1}]^\bullet, \quad (4.7)
\]
we have
\[
\text{H}^p(\Gamma(\text{ic}(F(\{\pi\})))^\bullet_q = \text{H}^{p+1}(\text{ic}(\pi)^\bullet)_q \quad (4.8)
\]
for $p + q \geq 0$. Hence it is sufficient to show that $\text{H}^p(\text{ic}(\pi)^\bullet)_q = 0$ for $p \neq q + r$.

We take a one-dimensional cone $\gamma$ of $N_\mathbb{R}$ such that $-\gamma$ intersects the interior of $\pi$. We set
\[
\Delta := (F(\pi) \setminus \{\pi\}) \cup \{\gamma + \sigma ; \sigma \in F(\pi) \setminus \{\pi\} \} \quad (4.9)
\]
and
\[
\tilde{\Delta} := \Delta \cup \{\pi\} \quad (4.10)
\]
Then $\tilde{\Delta}$ is a complete fan.

Let $\tilde{\Sigma}$ be a barycentric subdivision of $\tilde{\Delta}$ and let $\eta \in \tilde{\Sigma}$ be the one-dimensional cone which intersects the interior of $\pi$. Then $\Sigma := \tilde{\Sigma} \setminus \tilde{\Sigma}(\eta-\langle) \text{ is a barycentric subdivision of } \Delta$.

By Theorems 4.1 and 4.2, $\text{H}^p(\Gamma(\text{ic}(\tilde{\Delta})))^\bullet_q$ and $\text{H}^p(\Gamma(\text{ic}(\Delta)))^\bullet_q$ are zero unless $p = q + r$.

By the exact sequence
\[
0 \rightarrow \text{ic}^*(\pi) \rightarrow \Gamma(\text{ic}(\tilde{\Delta}))^\bullet \rightarrow \Gamma(\text{ic}(\Delta))^\bullet \rightarrow 0 \quad (4.11)
\]
we get the exact sequence
\[
0 \rightarrow \text{H}^{q+r}(\text{ic}^*(\pi))_q \rightarrow \text{H}^{q+r}(\Gamma(\text{ic}(\tilde{\Delta})))^\bullet_q \rightarrow \text{H}^{q+r}(\Gamma(\text{ic}(\Delta)))^\bullet_q \rightarrow 0 \quad (4.12)
\]
while we have $H^p(\text{ic}^*(\pi))_q = \{0\}$ for $p \neq q + r, q + r + 1$. Hence it is sufficient to show the surjectivity of $\varphi$ for each $q \in \mathbb{Z}$.

Then we get a commutative diagram of canonical homomorphisms

$$
\begin{array}{ccc}
H^{q+r}(\Gamma(\text{ic}(\tilde{\Sigma}))^\bullet)_q & \xrightarrow{\varphi'} & H^{q+r}(\Gamma(\text{ic}(\Sigma))^\bullet)_q \\
\downarrow \psi & & \downarrow \psi \\
H^{q+r}(\Gamma(\text{ic}(\tilde{\Delta}))^\bullet)_q & \xrightarrow{\varphi} & H^{q+r}(\Gamma(\text{ic}(\Delta))^\bullet)_q
\end{array}
$$

(4.13)

Since $\tilde{\Sigma}$ is simplicial, $\text{ic}(\tilde{\Sigma})^\bullet$ is quasi-isomorphic to $\text{ic}_t(\tilde{\Sigma})^\bullet$. Hence $\varphi'$ in the diagram is surjective by Lemma 3.7. Since $\Sigma$ is a barycentric subdivision of $\Delta$, $\psi$ is also surjective by Theorem 2.8. Hence $\varphi$ is surjective for every $q$. q.e.d.

Note that the theorem says that the cohomologies may not zero only for $(p, q) = (i - 1, i - r)$ for $r/2 < i \leq r$ or $(p, q) = (i, i - r)$ for $0 \leq i < r/2$.

Here we comment the relation with the results of [O3].

If the cone $\pi$ is of dimension $r$ and $F(\pi)\{\pi\}$ is a simplicial fan, then $\text{ic}(F(\pi)\{\pi\})^\bullet$ is quasi-isomorphic to $\text{ic}_t(F(\pi)\{\pi\})^\bullet$ by Theorem 3.2. Hence the second diagonal theorem implies that

$$
H^p(\Gamma(\text{ic}_t(F(\pi)\{\pi\}))^\bullet)_q = \{0\}
$$

(4.14)

unless $p + q \geq 0, p \neq q + r - 1$ or $p + q \leq -1, p \neq q + r$.

Let $\Pi$ be a simplicial complete fan of $N_\mathbb{R}$. Suppose a continuous map $h : N_\mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions.

(1) $h$ is linear on each cone $\sigma \in \Pi$.

(2) $h$ has rational values on $N_\mathbb{Q}$.

(3) $h$ is strictly convex with respect to the fan $\Pi$, i.e., $h(x) + h(y) \geq h(x + y)$

and the equality holds only if $x, y$ are in a common cone of $\Pi$.

We set $N' := N \oplus \mathbb{Z}$ and $\tilde{\sigma} := \{(x, h(x)) ; s \in \sigma\} \subset N'_\mathbb{R}$ for each $\sigma \in \Pi$, and define

$$
\tilde{\Pi} := \{\tilde{\sigma} ; \sigma \in \Pi\}
$$

(4.15)
Then $\tilde{\Pi}$ is a simplicial fan of the $(r+1)$-dimensional space $N'_R$ and $\tilde{\Pi} = F(\pi) \setminus \{\pi\}$ for the $(r+1)$-dimensional cone $\pi := \{(x, y) ; x \in N_R, y \geq h(x)\} \subset N'_R$.

We see that the complex $C^\bullet(\tilde{\Pi}, \tilde{G}_\ell)$ defined in [O3] is isomorphic to $i_\pi(C^\bullet(\tilde{\Pi}, \tilde{G}_\ell))$ for each $0 \leq \ell \leq r - 1$. Hence by the second diagonal theorem, we have

$$H^p(\tilde{\Pi}, \tilde{G}_\ell) := H^p(C^\bullet(\tilde{\Pi}, \tilde{G}_\ell)) = \{0\}$$

except for $(p, \ell) = (i - 1, i)$ for $(r+1)/2 < i \leq r + 1$ or $(p, \ell) = (i, i)$ for $0 \leq i < (r+1)/2$. In particular, it is zero for $(p, \ell) = (i, i)$ with $r/2 < i$ and $(p, \ell) = (i - 1, i)$ with $i < r/2 + 1$ since $r$ and $i$ are integers.

Hence we get the equivalent conditions of [O3, Cor.4.5] for a (rational) fan $\Pi$ and a rational strictly convex function $h$. Note that these conditions for the rational case is a consequence of the strong Lefschetz theorem (cf.[O3]).

However irrational fans are also treated in [O3], our theory can not be applied for irrational case.

**Corollary 4.4** Let $\rho$ be a nontrivial cone in $N_R$. If $i + j \geq 0$ and $i \neq j + r_{\rho} - 1$ or if $i + j \leq -1$ and $i \neq j + r_{\rho}$, then $H^i(i_\rho^*(ic(F(\rho)\setminus\{\rho\}))^\bullet)_{j} = \{0\}$.

**Proof.** The cone $\rho$ is of maximal dimension in the real space $N(\rho)_R$ of dimension $r_{\rho}$. Since the functor $i_\rho^*$ for the fan $F(\rho)$ is equal to $\Gamma$ of this space, this is a consequence of Theorem 4.3. q.e.d.

**Corollary 4.5** Let $\rho$ be in a finite fan $\Delta$. Then we have $H^i(ic(\Delta)(\rho)^\bullet)_{j} = \{0\}$ unless $i + j \geq 1$, $i = j + r_{\rho}$, while $H^i(i_\rho^*(ic(\Delta))^\bullet)_{j} = \{0\}$ unless $i + j \leq -1$, $i = j + r_{\rho}$.

**Proof.** By the construction of $ic(\Delta)^\bullet$ in [3, Thm.2.9], $ic(\Delta)(\rho)^\bullet$ is isomorphic to $gt_{\geq 1}(i_\rho^*(ic(F(\rho)\setminus\{\rho\}))[-1])^\bullet$ in CGM($A(\rho)$), and $i_\rho^*(ic(\Delta))^\bullet$ is quasi-isomorphic to $gt_{\leq -1}(i_\rho^*(ic(F(\rho)\setminus\{\rho\})))^\bullet$. Hence this is a consequence of Corollary 4.4. q.e.d.
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