Initial boundary value problem on a half line for the MKdV equation

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1 Introduction

The inverse scattering transform method (ISM) is an effective tool for studying nonlinear integrable equations [1]. It allows one to construct large classes of exact solutions and to investigate in detail the Cauchy problem including the writing down explicit time asymptotics. As for the mixed problem for the same equations, here the success of the method is not so impressive. For instance, the ISM applied to the initial boundary value problem in its general formulation is not sufficiently effective (see., for example, [2]), except some special kinds of boundary conditions (see. [3, 4, 5, 6]). The problem of searching boundary conditions consistent with the integrability property of the equation given as well as the problem of finding proper modifications of analytical integration procedure for the corresponding boundary value problems is undoubtedly important.

Notice that in all known cases of initial boundary value problems studied up by means of ISM the integration algorithm is based on reduction to the Cauchy problem. One prolongs the initial value by means of the Bäcklund transform’s ordinary differential equations. This way is available if the equation is invariant under a reflection type transformation $x \rightarrow -x$ only.

In our articles [7], [8] a symmetry test was worked out to verify whether the boundary value problem is consistent with integrability or not. Utilizing this test in [8] and [9] some examples of boundary conditions were found for the Korteweg-de Vries type equations. Evidently these equations don’t admit any $x$ reflection. So to integrate the corresponding initial boundary

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value problems the continuation algorithm mentioned above is not suitable and in the present paper an alternative approach is proposed. The essence of our approach consists of complementing the conjugation contour of usual associated Riemann problem by adding new branches. As an illustrative example we consider the following initial boundary value problem for the Modified KdV equation

\[ u_t = u_{xxx} + 6\varepsilon u_x u^2, \quad \varepsilon = \pm 1, \quad (1) \]
\[ u|_{x=0} = 0, \quad u_x|_{x=0} = 0, \quad (2) \]
\[ u|_{t=0} = u_0(x), \quad u_0(x)|_{x \to +\infty} \to 0, \quad (3) \]

in the quadrant \( x \geq 0, t \geq 0 \). The consistency condition \( u_0(0) = u_{0x}(0) = 0 \) of the initial value with the boundary conditions is supposed to be valid at the corner point \( x = 0, t = 0 \), and besides the initial value \( u_0(x) \) is required to vanish fast enough. The problem under consideration is well posed. Notice that the correctness of mixed problems like (1)-(3) for linear and quasi-linear third order equations is a well studied question (see, for instance, [10]). It is worthwhile to say that the similar problem on the left semi-line \( x < 0 \) is not correctly posed – in this case one should put only one boundary condition at the point \( x = 0 \). This fact implies immediately that the problem (1)-(3) can not be reduced to the Cauchy problem on a real axis when the initial value \( u_0(x) \) is chosen arbitrary.

Additionally we put rather severe requirement upon the initial data meaning the absence of discrete eigenvalues of the auxiliary linear system of equations (see conjectures i)-iv) in section 5 below) thus keeping out of our attention an important question on soliton-like solutions to the boundary value problem (1)-(3). In this connection mention the recent article [11], were an explicit description of algebro-geometric solutions to the KdV equation is given satisfying integrable boundary conditions.

Discuss briefly the content of the paper. In the second section we consider a linear analogue of the problem (1)-(3) and construct its explicit solution by the Wiener-Hopf method. In third one the Lax representation is formulated
for the initial boundary value problem on a half-line. In fourth – the direct scattering problem is considered for the associated linear system of equations. On a half-line solution of this problem is not unique in contrast to the case of the whole line \([12]\). Some freedom arises even in choosing the conjugation contour – here it does not coincide with the continuous spectra of the \(x\)-equation. In §5 it is proved that the scattering matrix defines uniquely the coefficients of equation. The inverse scattering problem is reduced to the Riemann problem posed on six rays dividing the complex plane to six equal sectors. In §6 the time evolution of scattering data is defined and it is shown that potential recovered in inverse problem solves the initial boundary value problem (1)-(3). In the last seventh section the solvability of the Riemann problem on system of rays is discussed.

2 Solution of mixed problem in linear case

This section is an auxiliary one, its aim is to explain with a simple illustrative example the main parts of the algorithm proposed and in such a way make easier the reading of the next sections.

Consider a linear version of the problem (1)-(3):

\[
\begin{align*}
    u_t &= u_{xxx}, \\
    u\big|_{x=0} &= 0, \\
    u_x\big|_{x=0} &= 0, \\
    u\big|_{t=0} &= q(x), \\
    q(x)|_{x\to+\infty} &\to 0.
\end{align*}
\]

Apply to the equation (3) the Fourier transform instead of Laplace one as it used to be. Set \(v(\xi, t) = \int_0^\infty u(x, t) \exp(i\xi x)dx\), then apparently the function \(v(\xi, t)\) solves the equation

\[
v_t(\xi, t) = u_{xx}(0, t) + i\xi^3 v(\xi, t),
\]

which contains an unknown function \(u_{xx}(0, t)\). In order to exclude the uncertainty arisen let us do a change of variables in the equation putting \(h(\xi, t) = v(\xi, t) - v(\omega \xi, t)\), where \(\omega = e^{\frac{2\pi}{3}}\) is a cubic root of the unity. The
function $h(\xi, t)$ satisfies much more simple equation $h_t = i\xi^3 h$ which is solved trivially. Now it is necessary to invert the change of variables. To this end as well as in order to solve the scattering problem below we need in the following geometric figures. Define on the complex plane six congruent sectors $I_1, I_2, ... I_6$, by dividing the complex plane by six rays $\ell_j$, ended at the origin: $\ell_j = \{\xi, \arg(\xi) = \frac{\pi}{3}(j - 1)\}$ enumerated in such a way that $I_j$ is a sector located between rays $\ell_j \ell_{j+1}$ (see picture below)

![Picture. Conjugation contour.]

Note that by construction the function $h(\xi, t)$ for $t = 0$ is analytical in sector $I_1$ and vanishes at infinity. These properties are valid also for $t > 0$, but violated for $t < 0$. One can check directly that the function searched is represented for $t > 0$ as a sum of three Cauchy type integrals

$$2\pi iv(\xi) = \int_{\ell_1} \frac{h(s)ds}{s - \xi} + \int_{\ell_4} \frac{h(\omega^2 s)ds}{s - \xi} + \int_{\ell_5} \frac{h(\omega s)ds}{s - \xi}.$$  

Here the integration is taken along rays $\ell_j$ in direction from the origin to the infinity. The condition $h(0) = 0$ guarantees the continuity of the function (8) in the closed upper half-plane.
The formula (8) is simplified essentially when $\xi \in I_1 \cup I_2$. Really for such $\xi$ integration along the ray $\ell_4$ can be replaced by integration along $\ell_3$, and afterwards by means of the change of variables $\nu = \xi^3$ one can reduce sum of three integrals in (8) to a integral along $\ell_1$:

$$2\pi iv(\xi) = \int_{\ell_1} \frac{h(\nu^{1/3}) d\nu}{\nu - \xi^3}, \quad \xi \in I_1 \cup I_2.$$ 

For $\xi \in I_3$ the function $v(\xi)$ is found from: $v(\xi) = v(\omega^2 \xi) - h(\omega^2 \xi)$. Applying the inverse Fourier transform to $v(\xi, t)$ one finds the solution searched $u(x, t)$ to the problem (4)-(6).

### 3 Lax representation for the problem on a half-line

Consider for convenience a system of equations in partial derivatives containing equation (1) as a particular case

$$u_t = u_{xxx} - 6u_x uu^+, \quad u^+_t = u^+_{xxx} - 6u^+_x uu^+. \quad (9)$$

Put in accordance with (2) and (3):

$$u|_{x=0} = u_x|_{x=0} = u^+|_{x=0} = u^+_x|_{x=0} = 0 \quad (10)$$

and

$$u(x, 0) = u_0(x) \to 0, \quad u^+(x, 0) = u^+_0(x) \to 0 \quad \text{for} \quad x \to \infty. \quad (11)$$

All our constructions below are consistent with the involution $u^+ = -\varepsilon u$ reducing the problem (9)-(11) to the problem (4)-(3). The system of equations (7) is a compatibility condition of the following over determined system of equations

$$y_x = (-i\xi \sigma_3 + U)y, \quad (12)$$

$$y_t = (4i\xi^3 \sigma_3 - 4\xi^2 U - \xi V_1 - V_0)y, \quad (13)$$

where

$$V_1 = \begin{pmatrix} -2i uu^+ & -2u_x \\ -2u_x^+ & 2i uu^+ \end{pmatrix}, \quad U = \begin{pmatrix} 0 & iu \\ -iu^+ & 0 \end{pmatrix},$$
\[
V_0 = \begin{pmatrix}
-uu_x^+ + u^+u_x & -iu_{xx} + 2iu^2u^+ \\
u_{xx}^+ - 2i uu^2 & uu_x^+ - u^+u_x
\end{pmatrix}.
\]

Along the line \(x = 0\) by means of the boundary conditions (10) equation (13) takes the form

\[
y_t = (4i\xi^3\sigma_3 - U_{xx})y.
\]

Changing parameter \(\nu = -4\xi^3\) in the last equation one gets again Zakharov-Shabat equation of the form (12).

Thus three linear systems of equations (12), (13) and (14) provide the Lax representation for the initial boundary value problem (9), (10) and (11).

4 Direct scattering on a half-line

Using the fact that the potential \(U(x, 0)\) vanishes rapidly enough one can construct a matrix valued solution \(y(x, \xi)\) of the auxiliary system (12), by prescribing for \(x \to +\infty\) the following asymptotically trigonometric behavior

\[
y_+(x, \xi) \to \exp(-i\xi x\sigma_3),
\]

valid for any real \(\xi\). As it is known such a matrix valued solution always exists and its columns \(y^{(1,2)}(x, \xi)\) define two vector valued functions \(\psi(x, \xi) = y^1(x, \xi)e^{i\xi x}\) and \(\phi(x, \xi) = y^2(x, \xi)e^{-i\xi x}\), admitting analytic continuations with respect to the parameter \(\xi\) from the real axis onto the lower and, respectively, upper half-planes. Moreover, the matrix valued function

\[
(\psi(x, \xi), \phi(x, \xi))
\]

tends to the unit matrix when \(R \ni \xi \to \infty\). The matrix \(T(\xi) = (\psi(\xi), \phi(\xi))\), compound by columns \(\psi(\xi) = \psi(0, \xi)\) and \(\phi(\xi) = \phi(0, \xi)\) is called scattering matrix of the system (12) on a half line.

Since to solve completely the direct scattering problem means to find a pair of linearly independent piece-wise analytic vector valued solutions to the system of equations (12) which should be defined at all points of the \(\xi\)-plane except may be a set of measure equal to zero (see, for instance, [12]), hence we
need in one more solution. In our case the variable $x$ takes only non-negative values, so we have a freedom in choosing the second analytic solution. Utilize this freedom in the following way. Compound matrix valued functions $\Psi_j(\xi)$, analytic in sectors $I_j$, with the help of columns of the scattering matrix $T(\xi)$ by putting dawn

$$
\begin{align*}
\Psi_1(\xi) &= (\psi(\omega^2 \xi), \phi(\xi)), \\
\Psi_2(\xi) &= \Psi_3(\xi) = (\psi(\omega \xi), \phi(\xi)), \\
\Psi_4(\xi) &= \Psi_5(\xi) = (\psi(\xi), \phi(\omega^2 \xi)), \\
\Psi_6(\xi) &= (\psi(\xi), \phi(\omega \xi)).
\end{align*}
$$

(15)

Complete the discussion on the direct scattering problem defining the solution $y(x, \xi)$ of the linear problem (12), by giving up its value at the point $x = 0$,

$$
y(0, \xi) = \Psi_j(\xi), \quad \xi \in I_j.
$$

One can prove that the function $\Psi(x, \xi) = \Psi_j(x, \xi)$, given as

$$
\Psi_j(x, \xi) = y(x, \xi)e^{i\xi x \sigma_3}, \quad \xi \in I_j,
$$

is a piece-wise analytic function, defined on the whole complex $\xi$-plane. In passing up from a sector $I_j$ to the sector $I_{j+1}$ the $\psi$-function have a jump:

$$
\Psi_{j+1}(x, \xi) = \Psi_j(x, \xi)R_j(x, \xi).
$$

(16)

The system of correlations (16) can be considered as the Riemann problem on rays $\{\ell_j\}$. Solutions to the problem (16) satisfy some additional constraint posed at the infinity $\Psi_j(x, \infty) = 1$. Besides the following asymptotic representation can be shown to be valid

$$
\Psi_j(x, \xi) = 1 + \sum_{k=1}^{N} \Psi_{jk}(x)\xi^{-k} + O(\xi^{-N-1}), \quad I_j \ni \xi \to \infty,
$$

(17)

if the coefficients $u_0(x), u_0^+(x)$ of system of equations (12) are smooth, rapidly decreasing functions. It can easily be checked that the conjugation matrices depend on the variable $x$ in an explicit way

$$
R_j(x, \xi) = e^{-i\xi x \sigma_3}R_j(0, \xi)e^{i\xi x \sigma_3}.
$$

(18)
5 Inverse scattering problem

In the inverse scattering the problem is considered to recover the potential $U(x)$ of the system (12) for a scattering matrix given. Remind (see above) that the scattering matrix is described by a pair of vector functions $\psi(\xi), \phi(\xi)$, analytical in lower and, respectively, upper half-planes. Coordinates of these vectors are connected by equation $\psi_1(\xi)\phi_2(\xi) - \psi_2(\xi)\phi_1(\xi) = 1$ for $\xi \in \mathbb{R}$.

Because of the analytical properties of entries this equation allows one to express functions $\psi_1(\xi), \phi_2(\xi)$ in terms of $\psi_2(\xi)$ and $\phi_1(\xi)$ by well known formulae (see [1]), if zeros of functions $\psi_1(\xi), \phi_2(\xi)$, located in domains $Im \xi < 0$ and $Im \xi > 0$, respectively, are listed a priori, and $1 + \psi_2(\xi)\phi_1(\xi) \neq 0$ for $Im \xi = 0$.

Note that the scattering matrix of the half-line problem defines uniquely the potential of system of equations (12) in contrast to the case of whole line. Let us formulate a precise statement.

**Proposition 1.** Let $T(\xi)$ and $\tilde{T}(\xi)$ are scattering matrices, corresponding the potentials $U(x), \tilde{U}(x)$. Then the equation $T(\xi) = \tilde{T}(\xi)$ implies the equation $U(x) = \tilde{U}(x)$.

**Scheme of proof.** Suppose that $T(\xi) = \tilde{T}(\xi)$. It means that the Riemann problem (16) has two different solutions $\Psi_{j+1}(x, \xi) = \Psi_{j}(x, \xi)R_{j}(x, \xi)$ and $\tilde{\Psi}_{j+1}(x, \xi) = \tilde{\Psi}_{j}(x, \xi)\tilde{R}_{j}(x, \xi)$. Let $f(x, \xi)$ be a ratio of these two solutions

$$f(x, \xi) = \tilde{\Psi}_{j}(x, \xi)\Psi_{j}^{-1}(x, \xi) = \tilde{\Psi}_{j+1}(x, \xi)\Psi_{j+1}^{-1}(x, \xi).$$

Evidently function $f(x, \xi)$ is bounded for large values of $\xi$, satisfies a linear equation with respect to $x$: $f_x = i\xi[f, \sigma_3] + uf\sigma_2 - \tilde{u}\sigma_2f$, besides $f(0, \xi) = 1$. Consequently, it is analytical on whole complex plane and according to the Liouville theorem it doesn’t depend on $\xi$. It follows from the linear equation on $f$ that $[f, \sigma_3] = 0$. Thence, $f_x = 0$ and therefore $f(x, \xi) \equiv 1$, that means that $U(x) = \tilde{U}(x)$.

Exclude from our consideration some degenerate cases in the inverse scattering problem by imposing the following additional requirements:

$$i) \quad 1 + \psi_2(\xi)\phi_1(\xi) \neq 0 \quad \text{for} \quad \xi \in \mathbb{R},$$
\( ii) \quad \log(1 + \psi_2(\xi_1(\xi)))|_{\xi=-\infty}^{\xi=+\infty} = 0, \\
\( iii) \quad \langle \phi(\xi), \psi(\omega^2 \xi) \rangle \neq 0 \quad \text{for} \quad \xi \in \bar{I}_1 \cup \bar{I}_2, \\
\( vi) \quad \langle \phi(\xi), \psi(\omega \xi) \rangle \neq 0 \quad \text{for} \quad \xi \in \bar{I}_2 \cup \bar{I}_3, \\

where the skew-scalar product \( \langle X, Y \rangle \) between two-dimensional vectors \( X \) and \( Y \) is understood as the determinant of matrix: \( \langle X, Y \rangle = \det(X, Y) \) and the bar over a letter denotes closure of a multitude.

**Proposition 2.** Assume we are given a pair of functions \( \phi_1(\xi), \psi_2(\xi) \) such that

\[
\phi_1(\xi) = \int_0^{+\infty} \exp(i \xi x) f_1(x) dx, \quad \psi_2(\xi) = \int_0^{+\infty} \exp(-i \xi x) f_1(x) dx,
\]

where \( f_{1,2}(x) \) vanish fast enough together with all derivatives. Recover the matrix \( T(\xi) = (\psi(\xi), \phi(\xi)) \), choosing the first coordinate of the vector \( \psi \) and the second coordinate of the vector \( \phi \) as Cauchy type integrals

\[
\psi_1(\xi) = \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log(1 + \psi_2(s) \phi_1(s))}{s - \xi} ds\right), \quad \text{Im} \xi < 0,
\]
\[
\phi_2(\xi) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log(1 + \psi_2(s) \phi_1(s))}{s - \xi} ds\right), \quad \text{Im} \xi > 0.
\]

Let the all conditions i)-vi) are valid. Then matrix \( T(\xi) \) is a scattering matrix of the problem on a half-line if and only if for all \( x \geq 0 \) a non-degenerate solution \( \det \Psi_j \neq 0 \) for all \( \xi \in \bar{I}_j \) exists to the Riemann problem (16) with the conjugation matrices defined as

\[
R_1(0, \xi) = \frac{1}{A_{22}(\xi)} \begin{pmatrix} -A_{11}(\omega^2 \xi) & 0 \\ -A_{21}(\xi) & A_{22}(\xi) \end{pmatrix},
\]
\[
R_3(0, \xi) = \frac{1}{A_{11}(\omega^2 \xi)} \begin{pmatrix} 1 & A_{12}(\omega^2 \xi) \\ A_{21}(\omega^2 \xi) & -A_{22}(\omega^2 \xi) \end{pmatrix},
\]
\[
R_5(0, \xi) = \frac{1}{A_{11}(\omega \xi)} \begin{pmatrix} A_{11}(\omega \xi) & A_{12}(\omega \xi) \\ 0 & -A_{22}(\omega \xi) \end{pmatrix},
\]
\[
R_6(0, \xi) = \frac{1}{A_{22}(\omega \xi)} \begin{pmatrix} -A_{11}(\xi) & -A_{12}(\xi) \\ -A_{21}(\omega \xi) & -1 \end{pmatrix},
\]

\( R_2 = R_4 = 1, \)
where

\[ A_{11}(\xi) = \langle \psi(\omega^2 \xi), \phi(\omega \xi) \rangle, \quad A_{12}(\xi) = \langle \phi(\xi), \phi(\omega \xi) \rangle, \]
\[ A_{21}(\xi) = \langle \psi(\omega^2 \xi), \psi(\omega \xi) \rangle, \quad A_{22}(\xi) = \langle \phi(\xi), \psi(\omega^2 \xi) \rangle. \]

**Scheme of proof.** The necessity of the conditions (16), (18) was proven in the previous section. Prove now the sufficiency. Define the conjugation matrices \( R_j \) for the scattering matrix \( T(\xi) = (\psi(\xi), \phi(\xi)) \) given as it is pointed out in the proposition. Suppose that for all \( x \geq 0 \) the Riemann problem (16), (18) is uniquely solvable. Find the potential \( U(x) \) of the system (12). To this end differentiate equation (16) with respect to \( x \) and then transform it to the form

\[ \Psi_{j+1,x} \Psi_j^{-1} - i\xi \Psi_{j+1} \sigma_3 \Psi_j^{-1} = \Psi_j x \Psi_j^{-1} - i\xi \Psi_j \sigma_3 \Psi_j^{-1} = g. \]

Applying now the Liouville theorem and asymptotic representation (17) it is easy to get that \( g = -i\xi \sigma_3 + U \), i.e.

\[ \Psi_{j,x} = -i\xi [\sigma_3, \Psi_j] + U \Psi_j. \]  

(19)

**Remark to the proposition 2.** The functions \( A_{ij} \) and conjugation matrices \( R_j \) admit some analytical properties. For instance, \( A_{11} \) is analytic in the domain \( I_1 \cup I_6 \), \( A_{12} \) – in the sector \( I_1 \), \( A_{21} \) – in the sector \( I_2 \) and, lastly, \( A_{22} \) is analytic in \( I_1 \cup I_2 \). Conjugation matrices \( R_1 \) and \( R_5 \) are analytic, respectively, in \( I_2 \) and \( I_5 \). The matrices \( R_3 \) and \( R_6 \) are defined, generally speaking, only on corresponding contours \( \ell_4 \) and \( \ell_1 \).

6 Time dynamics and boundary conditions

The scattering matrix evaluates in time in a very complicated way – it is the main difficulty in investigating the initial boundary value problem. One can get some impressions about the problem by looking at the equation (7) on the function \( v(\xi, t) \) in the linear case. On the other hand side the conjugation matrices \( R_j \), which are some rational functions of the elements of the scattering
matrix, depend on $t$ explicitly. It is more convenient to postulate time dynamics in the inverse scattering problem than to find it up in the direct problem. Let the conjugation matrices $R_j(x, \xi, t)$ of the Riemann problem (16), (18) evaluate in time by means of the following equations $R_{j,t} = 4i\xi^3[\sigma_3, R_j]$, so that

$$R_j(x, \xi, t) = e^{4i\xi^3t\sigma_3}R_j(x, \xi)e^{-4i\xi^3t\sigma_3}. \quad (20)$$

Differentiate the equation (16) with respect to $t$, and then replacing by means of (20) write it down in the form

$$\Psi_{j+1,t}\Psi_j^{-1} + 4i\xi^3\Psi_{j+1}\sigma_3\Psi_j^{-1} = \Psi_{j,t}\Psi_j^{-1} + 4i\xi^3\Psi_j\sigma_3\Psi_j^{-1} = h. \quad (21)$$

Applying the Liouville theorem to the function $h$ and afterwards substituting the asymptotic representation (17) one can find that $h$ takes the form $4i\xi^3\sigma_3 - 4\xi^2U - \xi V_1 - V_0$ (see formula (13)). Consequently, $\Psi_j$ solves the equation

$$\Psi_{j,t} = 4i\xi^3[\sigma_3, \Psi_j] - (4\xi^2U + \xi V_1 + V_0)\Psi_j. \quad (21)$$

As it follows from equations (19), (21) the function

$$y(x, \xi, t) = \Psi_j(x, \xi, t)e^{(4\xi^3t-\xi x)i\sigma_3}$$

is a solution to the over-determined system of equations (12), (13), hence the potential $U(x, t)$, found by solving the Riemann problem contains as a component a solution to the system of nonlinear equations (9), moreover, the initial conditions are automatically valid (11). Check the validity of the boundary conditions (10). To this end it is enough to show that at the point $x = 0$ the system of equations (13) is invariant under rotation $\xi \to \omega \xi$, i.e. not only the functions $\Psi_j(\xi, t)$, but also functions $\Phi_j(\xi, t) = \Psi_{j+2}(\omega \xi, t)$ are solutions to the system (21). For values of the index $j$ which are greater than six the function $\Psi_j$ is defined periodically on $j$ with period 6. By construction for any $j$ the function $\Phi_j(\xi, t)$ is analytic in the sector $I_j$, as well as, the function $m_j(\xi, t) = \Psi_j^{-1}(\xi, t)\Phi_j(\xi, t)$ (requirements i)-iv), are supposed to still valid, therefore $\det \Psi_j \neq 0$ for $\xi \in \bar{I}_j$. As a result of this definition one gets a new Riemann problem similar (16),

$$\Phi_{j+1}(\xi) = \Phi_j(\xi)r_j(\xi), \quad r_j(\xi) = R_j(\omega \xi), \quad (22)$$

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with conjugation matrices \( r_j(\xi) = m_j(\xi)^{-1} R_j(\xi) m_{j+1}(\xi) \). When \( t = 0 \) it can directly be shown that matrices \( m_1, m_3, m_5 \) are upper-triangular, and matrices \( m_2, m_4, m_6 \) are lower-triangular. By beans of this triangular structure the functions

\[
\tilde{m}_j(\xi, t) = e^{4i\xi^3\sigma_3 t} m_j(\xi, 0) e^{-4i\xi^3\sigma_3 t}
\]

are also analytic in the corresponding sectors \( I_j \). But it means that functions \( \tilde{\Phi}_j(\xi, t) = \Psi_j(\xi, t) \tilde{m}_j(\xi, t) \) are also solutions to the same Riemann problem (22), which is uniquely solvable, hence the functions \( \tilde{\Phi}_j(\xi, t) \) and \( \Phi_j(\xi, t) \) should coincide, consequently, \( \tilde{m} \equiv m \). Therefore the product \( (\Phi_j e^{4i\xi^3\sigma_3 t}) \times (\Psi_j e^{4i\xi^3\sigma_3 t})^{-1} \) is equal to \( m(\xi, 0) \), or in other words, it doesn’t depend on \( t \), i.e. functions \( y(\xi, t) = \Psi_j e^{4i\xi^3\sigma_3 t} \) and \( \tilde{y}(\xi, t) = \Phi_j e^{4i\xi^3\sigma_3 t} \) solve one and the same equation of the form (13). By construction one has \( \tilde{y}(\xi, t) = y(\omega \xi, t) \), consequently, at the point \( x = 0 \) equation (13) is not changed under rotation \( \xi \rightarrow \omega \xi \), but then it is of the form (14). It is equivalent to the validity of the boundary condition (10).

The conjugation matrices \( R_1 \) and \( R_5 \) as triangular ones (see proposition 2 and remark to it) remain analytical in the corresponding sectors \( I_2 \) and \( I_5 \) for all \( t > 0 \).

7 On solvability of the Riemann problem on a system of rays

The Riemann problem on analytical conjugation on a system of rays which the inverse scattering is reduced to is not sufficiently well studied in spite of some close formulations has been met earlier in the connection with the Painlevé equations [14]. But in our case the Riemann problem has inner symmetries which allow one to reduce it to the well known version of the problem posed on the real axis by transformations different for \( x = 0 \) and for \( x > 0 \).

Let us begin with the case \( x = 0 \), \( t \geq 0 \). Note first that functions \( \Psi_1(\xi, t) \) and \( \Psi_6(\xi, t) \) are analytic not only in the corresponding sectors \( I_1 \) and \( I_6 \) but
also in larger domains such as $I_1 \cup I_2$ and $I_5 \cup I_6$, respectively, moreover they are related to each other by an involution $\Psi_6(\xi, t) = \Psi_1(\omega \xi, t)$. Define one more function by setting $\Psi_{ex}(\xi, t) = \Psi_1(\omega^2 \xi, t)$. These three functions constitute a solution to the following Riemann problem

$$
\Psi_1 = \Psi_6 M_6, \quad \Psi_6 = \Psi_{ex} M_{ex}, \quad \Psi_{ex} = \Psi_1 M_1
$$

(23)
on a system of three rays $\ell_1, \ell_3, \ell_5$ and in addition satisfy one and the same system of equations of the form (21). Hence the conjugation matrices $M_1, M_6, M_{ex}$ evaluate in time $t$ as follows

$$
M(\xi, t) = e^{4i\xi^3\sigma_3 t} M(\xi, 0) e^{-4i\xi^3\sigma_3 t}.
$$

(24)

It can be checked that the Riemann problems (16), (20) and (23), (24) are equivalent: solution to one of them can easily be transformed into a solution to the other one by a simple explicit way. Let us given, for instance, a solution to the problem (23). Then functions $\Psi_2, \Psi_3, \Psi_4, \Psi_5$ which are the solution to the problem (16) are expressed as

$$
\Psi_2 = \Psi_1 R_1, \quad \Psi_3 = \Psi_{ex} N_3, \quad \Psi_4 = \Psi_{ex} N_4, \quad \Psi_5 = \Psi_6 R_5^{-1},
$$

where $N_3$ and $N_4$ are upper- and lower-triangular matrices. It is easy to check when $t = 0$ that $N_3$ and $N_4$ are analytic in $I_3$ and $I_4$ and therefore, by means of the equations $N_{3,4}(\xi, t) = e^{4i\xi^3\sigma_3 t} N_{3,4}(\xi, 0) e^{-4i\xi^3\sigma_3 t}$ and triangular structure they preserve their analytical properties for $t > 0$ also.

Change variables $\xi = \nu^{1/3}$ in the problem (23). Evidently if $\xi$ varies in the complex plane then $\nu$ varies in three sheets Riemannien surface however in all these sheets the function defined as $\Psi(\nu, t) = \Psi_1(\xi, t)$ takes one and the same values, i.e. it has no branch points, but fulfills a jump when crosses the positive half-line $\nu > 0$. Under the change of variables the Riemann problem on three rays is brought to the problem on real axis

$$
\Phi(\nu, t) \Psi(\nu, t) = C(\nu, t), \quad \nu \in \mathbb{R},
$$

(25)

where $\Phi(\nu, t) = -A_{22}(\omega \xi)\Psi_6^{-1}(\xi, t)$ and the conjugation matrix is chosen as $C(\nu, t) = -A_{22}(\omega \xi)$ for $\nu \leq 0$, $(\xi \in \ell_2)$ $C(\nu, t) = -A_{22}(\omega \xi)R_6(0, \xi, t)$
for $\nu > 0$ ($\xi \in \ell_2$). Thus in order to solve the problem (16) it is enough to solve the problem (25). Now it becomes clear that the problem (16) is solvable if functions $\psi_2(\xi)$, $\phi_1(\xi)$ are close to zero. This follows from the famous theorem about factorisation on a neighbourhood of the unity (see, for instance, [13]). The positivity of the conjugation matrix also provides the unique solvability of the problem (25). Assume that the reduction constraint $u^+(x,t) = -\varepsilon u(x,t)$, $\varepsilon = \pm 1$ is imposed on the initial boundary value problem (9), (10), (11). Then columns $\psi$ and $\phi$ of the scattering matrix $T$ satisfy the involution $\phi_2(\xi) = \psi_1^*(\xi^*)$, $\phi_1(\xi) = -\varepsilon \psi_2^*(\xi^*)$, where the star over a letter denotes the complex conjugation. In this case the conjugation matrix of the problem (25) owing to the constraints i)-iv) from §5 is positively defined. Actually, the reduction condition gives $A_{22}^s(\omega \xi) = A_{22}(\omega \xi)$ for $\xi \in \ell_2$. Moreover, $A_{22}(0) = A_{22}(\infty) = -1$, $A_{22}(\xi) \neq 0$, hence, $A_{22}(\xi) < 0$ along the ray $\ell_2$ such that $C(\nu,t) > 0$ for $\nu \leq 0$. Compute now the principal minors of the matrix $C(\nu,t)$: $\det_1 C(\nu,t) = A_{11}(\xi)$, $\det_2 C(\nu,t) = \det C(\nu,t) = |A_{22}(\xi)|^2$ for $\nu \geq 0$. By means of the constraint mentioned both are positive. Hence $C(\nu,t)$ is positively defined and, besides, it is continuous in the Hölder sense that provides the unique solvability of the problem (25).

Consider now the case $0 < x < \infty$, $t = \text{const} > 0$, supposing that the Riemann problem (19), (17), (18), (20) is already solved for $x = 0$, $t > 0$. It means really that the scattering matrix $T(\xi,t) = (\psi(\xi,t),\phi(\xi,t))$ of the problem on the half-line $x \geq 0$ is known for all $t > 0$ and we come to the problem of recovering of the potential of the system (12) for a given scattering matrix. The only solution of this problem (on the uniqueness see above proposition 1) can be found in two ways: by means of the Riemann problem (19), (17), (18) on a system of rays and by the canonical method based on the same problem on the real axis [12]:

$$\hat{\phi}(x,\xi,t) = \hat{\psi}(x,\xi,t) \begin{pmatrix} 1 & \phi_1(\xi,t) \exp(-2i\xi x) \\ -\psi_2(\xi,t) \exp(2i\xi x) & 1 \end{pmatrix},$$

$$\hat{\phi}(x,\infty,t) = 1. \quad (26)$$

For $x \geq 0$ and for $t \geq 0$ the Riemann problems mentioned are equivalent
unless the conditions i)-iv) are violated. Solutions to these two problems are related to each other by equations
\[ \hat{\phi}(x, \xi, t) = \psi_1(x, \xi, t)m_1(x, \xi, t), \quad \hat{\phi}(x, \xi, t) = \psi_3(x, \xi, t)m_3(x, \xi, t), \]
\[ \hat{\psi}(x, \xi, t) = \psi_5(x, \xi, t)m_5(x, \xi, t), \quad \hat{\psi}(x, \xi, t) = \psi_6(x, \xi, t)m_6(x, \xi, t), \]
where \( m_j(x, \xi, t) \) are triangular matrix-valued piece-wise analytic functions. As an example we give an explicit representation of the matrix \( m_3(x, \xi, t) \):
\[ m_3(x, \xi, t) = \Psi_3^{-1}\hat{\phi} = \frac{1}{A_{11}(\omega^2\xi)} \begin{pmatrix} \phi_2(\xi, t) & 0 \\ -\psi_2(\omega\xi, t)e^{2ix\xi} & A_{11}(\omega^2\xi) \end{pmatrix}. \]
It is well known (see, for instance, [12]) that under reduction \( u^+(x, t) = -\varepsilon u(x, t), \ \varepsilon = \pm 1 \) and the conditions i)-ii) the Riemann problem (26) is uniquely solvable.

It is worth to add that in the case \( \varepsilon = -1 \) the conditions i)-iv) don’t impose any constraint on the initial value of the initial boundary value problem (1)-(3), they are always valid.

References

[1] V.E.Zakharov, S.V.Manakov, S.P.Novikov, L.P.Pitaevskij. Soliton theory. Inverse scattering method. : Nauka, 1980 (in Russian).

[2] A.S.Fokas, A.R.Its. TMF. 1992. V.92. no 3, P.386-403.

[3] M.J.Ablowitz, H.J.Segur. Math. Phys. 1975. V. 16. P. 1054.

[4] E.K.Sklyanin. Funct. anal. i priloj. 1987. V. 21. no 2. P. 86-87 (in Russian).

[5] V.O.Tarasov. Inverse Problems. 1991. V. 7. P. 435.

[6] I.T.Habibullin. Teoret. matem. fizika, 1991. V. 86. no 1. P. 43-51 (in Russian).

[7] I.T.Habibullin. Phys. Letts. A. 1993. V. 178. P. 369.
[8] B. Gürel, M. Gürses, I. Habibullin. J. Math. Phys. 1995. V. 36. no 12. P. 6809.

[9] V. Adler, B. Gürel, M. Gürses, I. Habibullin. J. Phys. A: Math. Gen. 1997. V. 30, P. 3505-3513.

[10] M. D. Ramazanov. Matem. sbornik. 1964. V. 64. no 2. P. 234. An Ton Bui. J. Diff. Eq. 1977. V. 25. no 3. P. 288. A. V. Faminskij. Trudy MMO. 1988. V. 51. P. 54 (in Russian).

[11] V. E. Adler, I. T. Habibullin, A. B. Shabat. TMF. 1997. V. 110. no 1. P. 98-113.

[12] A. B. Shabat. Differen. Uravnen. 1979. V. 15. no 10. P. 1824-1835.

[13] Z. Prössdorf. Einige Klassen singulärer Gleichungen. Akademie-Verlag-Berlin, 1974.

[14] M. Jimbo, T. Miwa. Physica D. 1981. V. 2. P. 407-448.