Guiding and confining of light in a two-dimensional synthetic space using electric fields

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Synthetic dimensions provide a promising platform for photonic quantum simulations. Manipulating the flow of photons in these dimensions requires an electric field. However, photons do not have charge and do not directly interact with electric fields. Therefore, alternative approaches are needed to realize electric fields in photonics. One approach is to use engineered gauge fields that can mimic the effect of electric fields and produce the same dynamical behavior. Here, we demonstrate such an electric field for photons propagating in a two-dimensional synthetic space. We achieve this using a linearly time-varying gauge field generated by direction-dependent phase modulations. We show that the generated electric field leads to Bloch oscillations and the revival of the state after a certain number of steps dependent on the field strength. We measure the probability of the revival and demonstrate good agreement between the observed values and the theoretically predicted results. Furthermore, by applying a nonuniform electric field, we show the possibility of waveguiding photons. Ultimately, our results open up new opportunities for manipulating the propagation of photons with potential applications in photonic quantum simulations.

Photons are promising candidates for implementing quantum simulations due to their wave characteristics that exhibit strong interference effects. Recently, numerous complicated quantum simulations have been performed using photonic systems by molding the flow of light in real space [1–4]. However, these systems are extremely difficult to scale and reconfigure. Synthetic dimensions provide a promising alternative approach for photonic quantum simulations in a scalable and resource efficient way without requiring complex photonic circuits [5–18]. One powerful technique to implement a synthetic space is through time-multiplexing [19–24], which can scale to a higher number of dimensions efficiently. But significant challenges remain to fully control the evolution of photons in synthetic spaces.

One important challenge is that photons do not directly interact with electromagnetic fields because of their lack of charge. This limitation makes it difficult for photons to simulate the complex dynamical behavior of electrons or atoms. But significant progress in the past decade has led to the development of techniques to engineer magnetic and electric fields for photons [25–28]. In particular, realization of synthetic magnetic fields has led to the exploration of topological physics in photonic systems [29], and measurement of associated topological invariants [18, 30, 31]. Similarly, various ways to engineer electric fields have been reported in real dimensional photonic circuits [32–34]. One outcome of applying a constant electric field in periodic systems is the generation of state revivals known as Bloch oscillations, originally predicted in electronic systems [35–37]. Bloch oscillations have been observed in photonic systems such as coupled optical waveguide arrays [38–44] and one-dimensional quantum walks of photons [45, 46]. Bloch oscillations in one dimension have also been explored using frequency as a synthetic dimension [47–49]. However, the extension of electric fields to two-dimensional synthetic spaces has not yet been explored.

Here we demonstrate an electric field for photons in a two-dimensional synthetic space. We use time-multiplexing as a versatile platform to create the synthetic space and a time-varying gauge field to create the electric field. Under the application of a constant electric field, we show that photons return to the original state after a certain number of steps, thus demonstrating Bloch oscillations. Furthermore, by generating a spatially nonuniform electric field, we realize a synthetic quantum well, which guides photons without the use of a bandgap.

In our time-multiplexed photonic quantum walk, the
quantum walker state space is mapped into time delays of optical pulses. The experimental setup is a closed-loop fiber architecture composed of two beam splitters with their ports connected to fibers of different lengths mapping the $\pm x$ and $\pm y$ directions to different time delays. Full details of the experimental setup are explained in the Supplemental Material [50]. One complete propagation of an optical pulse around the loop is equivalent to the hopping of the walker to one of the four possible corners in the synthetic space (Fig. 1). Semiconductor optical amplifiers as well as polarization controllers are used in the setup to compensate for the losses and polarization changes that the optical pulses experience in each round trip, respectively. The quantum walk distribution at each time step is studied via two photodetectors analyzing two channels that we refer to as the up and down channels (See Fig. S1). A single incident laser pulse that is injected into the up channel initializes the quantum walk evolution from the origin in the synthetic space.

In this setup, we use electro-optic modulators that can introduce desired phase shifts to pulses moving to the right or left directions to generate the synthetic gauge field. Specifically, here we implement a linearly time-varying gauge field ($\vec{A} = -E t \hat{x}$, in which $t$ denotes the time step and $\hat{x}$ is the unit vector in the $x$ direction) that leads to the generation of a constant electric field ($\vec{E} = -\partial \vec{A} / \partial t$). To implement this gauge field, a phase modulation needs to be applied that varies with the time step. Figure 1 depicts the synthetic lattice with the required phase modulation criteria describing the amount of phase that the walker accumulates in hopping to the four possible corners at time step $n$.

This method of generating an electric field is distinguished from previously used approaches in discrete-time quantum walks, which have been based on position-dependent but time-independent gauge fields [51–53]. In these approaches, an effective linear electric potential $V = -E x$ is implemented, which leads to the generation of electric fields based on $\vec{E} = -\nabla V$. In order to create such a gauge field in discrete-time quantum walks, the unitary operation in each time step must have an extra term relative to the standard quantum walk evolution operator $U_0$ as $U_\phi = e^{i\phi x} U_0$. In contrast, the current approach does not require any coordinate-dependent unitary operation to generate an electric field. This is of particular interest for time-multiplexed quantum walks, as it relaxes the need for any variation of phase modulations during each time step. The equivalence of these approaches can be understood in terms of a gauge transformation [54]. The similarity between the approach used and the conventional coordinate-based method of implementing an electric field in a two-dimensional quantum walk [53] is described in the Supplemental Material [50]. We show that the total phase accumulated in some sample closed loops that start from the origin and return to it in both pictures are the same. In fact, this similarity holds for any closed path starting from the origin and ending at it.

The application of the electric field in our two-dimensional discrete-time quantum walk will lead to Bloch oscillations and the revival of the quantum state. This can be intuitively explained through the band diagram structure of the system. As we show in the Supplemental Material [50], for a phase modulation with a fractional value of $\phi$ as $\phi = 2\pi/q$, the band structure has $2q$ bands. The analytical expressions for the pseudo energy band structure under such a phase modulation are

![Figure 2](image)
Figure 3. (a) Experimental observations and (b) theoretical predictions of the evolution of the quantum walk distribution under time-dependent phase modulation for the case of $\phi = \pi/4$. The columns from left to right show the distributions at time steps of 0, 2, 4, 6, and 8, respectively. In these plots, all the distributions are normalized to their maximum.

Given by:

$$E_{n, \pm k_x, k_y} = \frac{2n\pi \pm}{q}$$

$$\frac{1}{q} \arccos \left( \cos \left( \frac{\pi q}{2} \right) (\sin^q k_y - 1) - \cos \left( qk_x + \frac{\pi q}{2} \right) \sin^q k_y \right)$$

where $k_x$ and $k_y$ are the momentum wave vectors in inverse synthetic space and $n \in Z$. This expression for $q = 1$ returns to the form of $E_\pm = \pm \arccos (\sin (k_x) \sin (k_y))$, which represents the band structure for the quantum walk under no effective applied gauge field [55]. Figure 2 shows the band diagrams for three different values of the phase modulations. By increasing $q$, the band structure becomes flatter and the corresponding group velocities tend toward zero. As has been demonstrated in one dimensional quantum walks, this will lead to the revival of the quantum walk with high probability [50]. This band flattening also occurs in the two-dimensional Floquet quantum walks considered here, which will lead to the return of the quantum walker toward the origin after $q$ steps. In our system, the application of an electric field in the $x$ direction will lead to the revival of the quantum walk not only in the $x$ direction but also in the $y$ direction (see Supplemental Material [50]).

To experimentally demonstrate Bloch oscillations in our 2D time-multiplexed quantum walk caused by the applied electric field, we investigate the evolution of the quantum walk distribution at different time steps. In order to measure the quantum walk distribution, we measure the power of the optical pulses received by the photodetectors at different time delays for each time step. Figure 3a shows the experimentally measured quantum walk distributions at different time steps. This figure demonstrates state revival under the application of the time-varying gauge field due to the uniform electric field generated. The phase strength of the applied electric field in this case is $\phi = \pi/4$. As this figure shows, after 8 steps, the quantum walker returns to the origin with high probability. The experimental results are in good agreement with the corresponding theoretical predictions shown in Fig. 3b.

To quantify the effect of the gauge field on the revival of the quantum state, we measure the probability of the walker returning to the origin (revival probability $P_U(0,0)$) as a function of the number of steps taken (Fig. 4). We measure this probability for different time-varying phase modulations after the appropriate number of time steps ($2\pi/\phi$ steps) needed for the revival to happen. As shown in Fig. 4, the revival probability increases with increasing number of steps. Additionally, this figure indicates that the experimental results are in good agreement with the theoretical predictions. In the Supplemental Material [50], we demonstrate that by decreasing the phase modulation, $\phi$, the revival probability increases and tends toward unity. Using these measured probability distributions, we can also calculate the statistical averages of the distribution at different time steps. Specifically, we measured and analyzed the quadratic means as well as the norm ones of the $x$ and $y$ coordinates at different time steps (Figs. S2 and S3). As these results show, the quadratic means as well as the norm ones of $x$ and $y$ tend toward the local minimum values after $2\pi/\phi$ steps. The variational behaviors of these quantities with the time step confirm the revival effect in both the $x$ and

(a) Exp. (Up Channel)

(b) Theory (Up Channel)
the field pattern has led to the confinement of the quantum walk in the $y$ ribbon. We note that this confinement is not induced by a bandgap and is therefore physically distinct from the confinement in conventional crystal heterostructures. None of the three regions shown in Fig. 5 supports a bandgap, thus demonstrating the confinement is directly induced by the gauge field itself. These results demonstrate the possibility of using a nonuniform electric field in order to guide the path of the quantum walk in a desired fashion.

In conclusion, we studied the time evolution of a quantum random walk under a time-varying gauge field in a two-dimensional synthetic space. Using a linearly time-varying phase modulation, an electric field acting on photonic quantum walkers can be created. Our findings demonstrate that under the influence of such an electric field a complete revival caused by Bloch oscillations happens in two-dimensional quantum walks. This revival becomes more accurate as we increase the number of steps. Moreover, we demonstrated that a discontinuous electric field could impose confinement on the evolution of a quantum random walk, even when there is no bandgap. While we demonstrated the Bloch oscillation for a quantum walk initiated with classical coherent laser pulses, the same physics holds at the single photon level. The obtained results can be extended to the investigation of the effect of dynamic localization [56–62]. Our demonstration of an electric field for photons in time-multiplexed synthetic lattices will have potential applications in photonic quantum simulations, for example, multi-photon interference and Boson sampling in the time-domain [63], and the realization of photonic lattices with strong nonlinearities mediated via artificial atoms like quantum dots [64, 65].

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Figure 5. Confinement of the quantum walker through the application of a discontinuous electric field. (a) Schematic describing the phase modulation pattern in the synthetic space that leads to a zero electric field for $-2 < y < 2$ and a non-zero electric field outside this range. (b) Experimental observations and (c) theoretical predictions of the evolution of the quantum walk distribution under the discontinuous electric field. The columns from left to right show the distributions at time steps of 1, 5, and 9, respectively. All the distributions are normalized to their maximum. (d) Experimentally measured and theoretically predicted quadratic means of $x$ and $y$ as a function of the time step. The error bars in the measurements are smaller than the size of the plotted data points.

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Supplemental Material: Guiding and confining of light in a two-dimensional synthetic space using electric fields

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We explain the experimental details of this study in the first section of this supplemental material. In this section, we also present the measurement results as well as theoretical predictions for the variation of the quadratic means and norm ones of the quantum walk distributions as functions of the time step. In the second section, we review the theoretical analysis of one-dimensional quantum walks and explain the possibility of observing Bloch oscillations using time-dependent gauge fields in such quantum walks. In the next section, we provide the corresponding theoretical analysis for the case of two-dimensional quantum walks.

I. EXPERIMENTAL DETAILS:

Figure S1 shows the schematic of the experimental setup, which is composed of similar elements as used in our previous work [S1]. However, the phase modulation patterns used in this study are different from those used earlier. We used a laser diode made by Bookham technology (LC25W5172BA-J34) operating at the C-band of the telecom wavelength (1550 nm) for pulse generation. By modulating this laser, pulses with a power of $10^{-3}$ mW and pulse duration of 2 ns were generated, which were sent into the setup with a repetition rate of 10 $\mu$s. These pulses initiate the quantum walk in our synthetic two-dimensional space. The initial pulse after passing through the first 50/50 beam splitter continues its path through either the 1 m or 2 m fiber length. This leads to different delays that can be conceptually regarded as “choosing” the left or right direction in the $x$ movement by the quantum walker. When reaching the next beam splitter, the pulse chooses either the 130 m or 118 m fiber spool. This determines whether the pulse has gone in the up or down direction in the $y$ movement. The $x$ and $y$ coordinate of the pulses are encoded based on the delay in reaching the detectors. We used 90/10 beam splitters to out-couple 10% of the light for detection purposes. In addition, two phase modulators made by Cybel, LLC (MPZ-LN-10-P-P-FA-FA) were used to apply opposite phases to the right and left moving pulses (i.e., $\pm x$ direction).

We used semiconductor optical amplifiers made by Thorlabs (SOA1117S) to compensate for the loss in the setup.

Figure S1. Schematic of the experimental setup for the two-dimensional quantum random walk. PD: photodetector, BPF: band-pass filter, SOA: semiconductor optical amplifier, EDFA: Erbium-doped fiber amplifier, EOM: Electro-optic modulator and PC: polarization controller.
by amplifying the pulses without ruining their phase coherence. These amplifiers are only turned on for a fraction
of a repetition period at each cycle to avoid over amplifying the background noises. Moreover, in order to filter out
background noises, we used narrow band-pass filters (< 0.3nm) directly after the semiconductor optical amplifiers.
However, the ratio of the optical pulse power at different positions of the synthetic space relative to the noise degrades
with increasing time steps due to the diffusion and the added noise. We therefore limited our measurement of the
quantum walk distribution to 10 steps because of the observed degradation of the signal-to-noise ratio and the finite
ratio of the time delays corresponding to the x and y movements. The distribution of the quantum walk at every time
step was determined by measuring the power of the pulses present in each step. We also used polarization controllers
in our setup in order to compensate for the polarization changes along the optical fibers.

We measured the quantum walk distribution at different time steps under the application of the proposed time-
changing gauge fields with different phases ($\phi = 0 [\text{deg}]$, $\phi = 90 [\text{deg}]$, $\phi = 60 [\text{deg}]$, $\phi = 45 [\text{deg}]$, and $\phi = 36 [\text{deg}]$).

Using the measured probability distributions, we calculated the quadratic means and norm ones of the x and y
of the quantum walk distribution. We plotted the variation of these quantities by the time step in Figs. S2 and S3. These quantities reach their local minima after around $2\pi/\phi$ time steps depending on the value of $\phi$ used. This behavior is due to the state revival and the trapping of the quantum walk distribution close to the origin after around $2\pi/\phi$ time steps.

We can also calculate the theoretically predicted values for the quadratic means and norm ones as functions of the
time step. As Figs. S2 and S3 show, the measured results are in good agreement with the theoretical predictions.

Based on the measured results, we can also calculate the quantum walk distribution probability at the origin for
different applied phases. Using these probabilities, we plotted the revival probability in the main text as a function
of the required steps to reach the revival. Indeed, smaller phases lead to higher probability of the revival, as has been

Figure S2. Theoretical values for the quadratic means of (a) y and (c) x as a function of the time step for different gauge field
strengths. Experimental values for the quadratic means of (b) y and (d) x as a function of the time step for different gauge
field strengths.
Figure S3. Theoretical values for the norm ones of (a) $y$ and (c) $x$ as a function of the time step for different gauge field strengths. Experimental values for the norm ones of (b) $y$ and (d) $x$ as a function of the time step for different gauge field strengths.

discussed in the main text.
II. THEORETICAL ANALYSIS FOR ONE-DIMENSIONAL QUANTUM RANDOM WALKS:

In this section, we consider the evolution of pulses (representing the quantum walkers) in one-dimensional quantum random walks. For the theoretical analysis of the quantum walk, we can assume that all the losses in the setup are fully compensated by the amplifiers. Therefore, we can consider an ideal setup as shown in Fig. S4.

Two pulses just before the first beam splitter, \[ \begin{bmatrix} U_x^{(n)} \\ D_x^{(n)} \end{bmatrix} \], will produce the output pulses as (movement in the \( x \) direction):

\[
\begin{bmatrix}
U_{x+1}^{(n+1)} \\
D_{x-1}^{(n+1)}
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
\ e^{-i\phi_{n,x}} & 0 \\
0 & e^{i\phi_{n,x}}
\end{bmatrix} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
U_x^{(n)} \\
D_x^{(n)}
\end{bmatrix}.
\]

Therefore, \[ \begin{bmatrix} U_x^{(n)} \\ D_x^{(n)} \end{bmatrix} \] will produce the following pulses after traversing the beam splitter:

\[
U_{x+1}^{(n+1)} = \frac{e^{-i\phi_{n,x}}}{\sqrt{2}} \left( U_x^{(n)} - D_x^{(n)} \right)
\]
\[
D_{x-1}^{(n+1)} = \frac{e^{i\phi_{n,x}}}{\sqrt{2}} \left( U_x^{(n)} + D_x^{(n)} \right)
\]

Alternatively, based on the above results \[ \begin{bmatrix} U_x^{(n+1)} \\ D_x^{(n+1)} \end{bmatrix} \] can be produced from other pulses as:

\[
U_{x}^{(n+1)} = \frac{e^{-i\phi_{n,x-1}}}{\sqrt{2}} \left( U_{x-1}^{(n)} - D_{x-1}^{(n)} \right)
\]
\[
D_{x}^{(n+1)} = \frac{e^{i\phi_{n,x+1}}}{\sqrt{2}} \left( U_{x+1}^{(n)} + D_{x+1}^{(n)} \right)
\]

By defining \( S_x^{(n)} = U_x^{(n)} + D_x^{(n)} \) and \( P_x^{(n)} = U_x^{(n)} - D_x^{(n)} \), the obtained equations can be written as:

\[
S_{x+1}^{(n+1)} = \frac{e^{i\phi_{n,x+1}}}{\sqrt{2}} S_{x+1}^{(n)} + \frac{e^{-i\phi_{n,x-1}}}{\sqrt{2}} P_{x-1}^{(n)}
\]
\[
P_{x+1}^{(n+1)} = -\frac{e^{i\phi_{n,x+1}}}{\sqrt{2}} S_{x+1}^{(n)} + \frac{e^{-i\phi_{n,x-1}}}{\sqrt{2}} P_{x-1}^{(n)}
\]
By defining $s^{(n)}_{k_x}$ and $p^{(n)}_{k_x}$ as Fourier transforms of $S^{(n)}_x$ and $P^{(n)}_x$, we have:

\[
\begin{bmatrix}
S^{(n+1)}_x \\
P^{(n+1)}_x
\end{bmatrix} = \frac{1}{\sqrt{2}} 
\begin{bmatrix}
e^{i\phi_{n,x+1}} & e^{-i\phi_{n,x-1}} \\
-e^{i\phi_{n,x+1}} & e^{-i\phi_{n,x-1}}
\end{bmatrix} 
\begin{bmatrix}
S^{(n)}_x \\
P^{(n)}_x
\end{bmatrix}
\]

This equation can be used to solve the Fourier transforms as functions of the time step. In the following subsections, we solve this equation for two cases of no phase modulation as well as time dependent linear phase modulation.

### A. Zero phase modulation:

For the case of no applied phase, we have:

\[
\begin{bmatrix}
s^{(n+1)}_{k_x} \\
p^{(n+1)}_{k_x}
\end{bmatrix} = \frac{1}{2\pi} 
\begin{bmatrix}
\int k_x s^{(n)}_{k_x} e^{ik_x x} dk_x \\
\int k_x p^{(n)}_{k_x} e^{ik_x x} dk_x
\end{bmatrix}
\]

Note that the evolution matrix has the determinant of 1. Based on this matrix, the effective Hamiltonian, $H_{eff} = i \log(U)$, is given by:

\[
H = \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
sin(k_x) & -ie^{-ik_x} \\
-ie^{ik_x} & -\sin(k_x)
\end{array} \right) \frac{\text{arccos}\left(\cos(k_x)/\sqrt{2}\right)}{\sin\left(\text{arccos}\left(\cos(k_x)/\sqrt{2}\right)\right)}
\]

The obtained Hamiltonian is hermitian and its eigenvalues are given by:

\[
E_{\pm} = \pm \text{arccos}\left(\cos(k_x)/\sqrt{2}\right)
\]

The evolution after $n$ steps is given by:

\[
\begin{bmatrix}
s^{(n)}_{k_x} \\
p^{(n)}_{k_x}
\end{bmatrix} = \frac{1}{\sqrt{2^n}} \begin{bmatrix}
e^{ik_x} & e^{-ik_x} \\
-e^{ik_x} & e^{-ik_x}
\end{bmatrix} \begin{bmatrix}
s^{(0)}_{k_x} \\
p^{(0)}_{k_x}
\end{bmatrix}
\]

Assuming that the whole evolution is caused by a single pulse at the origin ($U^{(0)}_x = \delta(x)$, $D^{(0)}_x = 0$ which is equivalent to $S^{(0)}_x = \delta(x)$, $P^{(0)}_x = \delta(x)$), we have:

\[
\begin{bmatrix}
s^{(0)}_{k_x} \\
p^{(0)}_{k_x}
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
Therefore, for the down channel pulses, $d_{k_x}^{(n)} = 0.5 \left( s_{k_x}^{(n)} - p_{k_x}^{(n)} \right)$, we have:

$$d_{k_x}^{(n)} = \frac{e^{ik_x} \sin \left( n \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right)}{\sqrt{2} \sin \left( \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right)}$$

All the moments, such as the distribution variances and averages ($\langle x^2 \rangle_D$ and $\langle x \rangle_D$) as functions of the time step $n$, can be obtained from the above expression.

Based on the fact that:

$$d_{k_x}^{(n)} = \sum_x D_x^{(n)} e^{-ik_xx}$$

Then:

$$d_{l_x}^{(n)*} d_{k_x}^{(n)} = \sum_p \sum_x e^{-i(k_xx - l_xp)} D_p^{(n)*} D_x^{(n)}$$

Therefore, the following equations can be obtained straightforwardly:

$$P_D = \sum_x \left| D_x^{(n)} \right|^2 = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} \left| d_{k_x}^{(n)} \right|^2 dk_x$$

$$= \frac{1}{4\pi} \int_{k_x=0}^{2\pi} \sin^2 \left( n \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right) dk_x$$

$$\langle x \rangle_D = \sum_x x \left| D_x^{(n)} \right|^2 = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} d_{k_x}^{(n)*} \left( i \frac{d}{dk_x} \right) d_{k_x}^{(n)} dk_x = 0$$

$$\langle x^2 \rangle_D = \sum_x x^2 \left| D_x^{(n)} \right|^2 = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} d_{k_x}^{(n)*} \left( i \frac{d}{dk_x} \right)^2 d_{k_x}^{(n)} dk_x$$

In the limit of large $n$, we have:

$$P_D \to \frac{1}{8\pi} \int_{k_x=0}^{2\pi} \frac{1}{1 - \cos^2 \left( k_x \right)/2} dk_x = \frac{1}{2\sqrt{2}}$$

$$\langle x \rangle_D \to 0$$

$$\langle x^2 \rangle_D \to \frac{n^2}{16\pi} \int_{k_x=0}^{2\pi} \frac{\sin^2 \left( k_x \right)}{(1 - \cos^2 \left( k_x \right)/2)^2} dk_x = \frac{n^2}{8\sqrt{2}}$$

These results prove that the spatial quadratic mean of the quantum walk distribution varies linearly with the time step. Note that these average values are normalized relative to the total power in the up and down channels. However, they can also be normalized relative to the total power present in the corresponding channel. Since the probabilities of $P_D$ and $P_U$ tend toward constant values, by the latter normalization the asymptotic behavior of the quadratic mean remains linear relative to the time step.

Similarly, we can investigate the up channel:
\[ u_{k_x}^{(n)} = 0.5 \left( s_{k_x}^{(n)} + p_{k_x}^{(n)} \right) = \cos \left( n \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right) - i \sin \left( k_x \right) \sin \left( n \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right) / \sqrt{2} \right. \]

\[ P_U = \sum_{x} \left| U_x^{(n)} \right|^2 = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} \left| U_{k_x}^{(n)} \right|^2 dk_x \]

\[ = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} \cos^2 \left( n \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right) dk_x \]

\[ + \frac{1}{4\pi} \int_{k_x=0}^{2\pi} \frac{\sin^2 \left( n \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right)}{\sin^2 \left( \arccos \left( \cos \left( k_x \right) / \sqrt{2} \right) \right)} dk_x \]

\[ \langle x \rangle_U = \sum_x x \left| U_x^{(n)} \right|^2 = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} u_{k_x}^{(n)*} \left( i \frac{d}{dk_x} \right) u_{k_x}^{(n)} dk_x \]

\[ \langle x^2 \rangle_U = \sum_x x^2 \left| U_x^{(n)} \right|^2 = \frac{1}{2\pi} \int_{k_x=0}^{2\pi} u_{k_x}^{(n)*} \left( i \frac{d}{dk_x} \right)^2 u_{k_x}^{(n)} dk_x \]

In the limit of large \( n \), we have:

\[ P_U \rightarrow \frac{1}{2} + \frac{1}{8\pi} \int_{k_x=0}^{2\pi} \frac{\sin^2 \left( k_x \right)}{1 - \cos^2 \left( k_x \right)/2} dk_x = 1 - \frac{1}{2\sqrt{2}} \]

\[ \langle x \rangle_U \rightarrow \left( 1 - \frac{\sqrt{2}}{2} \right) n \]

\[ \langle x^2 \rangle_U \rightarrow \frac{n^2}{8\pi} \int_{k_x=0}^{2\pi} \frac{\sin^2 \left( k_x \right) \left( \sin^2 \left( k_x \right) + \frac{1}{2} \right)}{1 - \cos^2 \left( k_x \right)/2} \frac{dk_x}{2} = \left( 1 - \frac{9}{8\sqrt{2}} \right) n^2 \]

### B. Time-dependent phase modulation:

For the case of a time-dependent but coordinate-independent phase modulation, we have:

\[ \left[ \begin{array}{c}
   s_{k_x}^{(n+1)} \\
   p_{k_x}^{(n+1)}
\end{array} \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc}
   e^{i(k_x+\phi_n)} & e^{-i(k_x+\phi_n)} \\
   -e^{i(k_x+\phi_n)} & e^{-i(k_x+\phi_n)}
\end{array} \right] \left[ \begin{array}{c}
   s_{k_x}^{(n)} \\
   p_{k_x}^{(n)}
\end{array} \right] \]

We consider phase modulations that are linearly varying with the time step, such that \( \phi_n = n\phi \). Therefore, by defining \( w_{k_x}^{(n)} = s_{k_x-n\phi}^{(n)} \) and \( v_{k_x}^{(n)} = p_{k_x-n\phi}^{(n)} \):

\[ \left[ \begin{array}{c}
   w_{k_x+\phi}^{(n+1)} \\
   v_{k_x+\phi}^{(n+1)}
\end{array} \right] = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc}
   e^{ik_x} & e^{-ik_x} \\
   -e^{ik_x} & e^{-ik_x}
\end{array} \right] \left[ \begin{array}{c}
   w_{k_x}^{(n)} \\
   v_{k_x}^{(n)}
\end{array} \right] \]

The evolution matrix can be obtained for rational values of phase modulations, \( \phi/2\pi = p/q \). We then can obtain the pseudo-energy band diagrams from the evolution matrix. In the following, we obtain the eigen-energies for any rational value of phase modulation, straightforwardly.
Figure S5. The quantum walk probability distribution in the (a) up and (b) down channels at different time steps under no applied gauge field. The quantum walk probability distribution in the (c) up and (d) down channels at different time steps under a time-varying gauge field with a phase of $\phi = \pi/6$. Under the presence of a nonzero phase, the walker returns back to the origin after a fixed number of steps, which is determined by the applied phase.

By defining $r_{k_x}^{(n)} = \begin{bmatrix} w_{k_x}^{(n)} & v_{k_x}^{(n)} \end{bmatrix}^T$ and

$$M_{k_x} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{ik_x} & e^{-ik_x} \\ -e^{ik_x} & e^{-ik_x} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{ik_x} & 0 \\ 0 & e^{-ik_x} \end{bmatrix},$$

we have:

$$r_{k_x}^{(n+1)} = M_{k_x} r_{k_x}^{(n)}$$

After $q$ steps, we have:

$$r_{k_x}^{(n+q)} = M_{k_x+(q-1)\phi} \cdots M_{k_x} r_{k_x}^{(n)}$$

Therefore, by defining $Y_{k_x}^{(i)} = M_{k_x+(i-1)\phi} \cdots M_{k_x}$, the following holds:

$$M_{k_x} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{ik_x} & e^{-ik_x} \\ -e^{ik_x} & e^{-ik_x} \end{bmatrix}$$
Since $M$ and $Y$ are unitary matrices, the following equations hold for the eigen-energies:

$$Y_{k_x}^{(q)}(n) r_{k_x}^{(n)} = e^{i q E} r_{k_x}^{(n)}$$

$$Y_{k_x}^{(q)\dagger}(n) r_{k_x}^{(n)} = e^{-i q E} r_{k_x}^{(n)}$$

Consequently, we have:

$$\frac{1}{2} \left( Y_{k_x}^{(q)} + Y_{k_x}^{(q)\dagger} \right) r_{k_x}^{(n)} = \cos(qE) r_{k_x}^{(n)}$$

It can be verified that not only for $\phi = 2\pi/q$ but also for any $\phi = 2\pi p/q$, with $q$ and $p$ being relatively prime, the following holds:

$$\frac{1}{2} \left( Y_{k_x}^{(q)} + Y_{k_x}^{(q)\dagger} \right) = \left( \left( \cos \frac{\pi q}{2} - \cos( q k_x ) \right) \frac{(-1)^q}{\sqrt{2^q}} - \cos \left( \frac{\pi q}{2} \right) \right) I$$

Therefore, the eigen-energies are given by

$$E_{n,\pm,k_x} = \frac{2 \pi n}{q} \pm \frac{1}{q} \arccos \left( \left( \cos \frac{\pi q}{2} - \cos( q k_x ) \right) \frac{(-1)^q}{\sqrt{2^q}} - \cos \left( \frac{\pi q}{2} \right) \right),$$

which explicitly expresses the eigen-energies for any of $2q$ bands.

Based on this analytical expression for the allowed energies, it can be verified that the bandgap does not exist for any rational value of phase modulation.

Since we are interested in the evolution of the quantum walk after $q$ steps, we obtain the following for any $k_x = \frac{\pi}{2} + \frac{2 \pi m}{q}$ with $m \in \mathbb{Z}$.
Figure S7. (a) The total probability of the quantum walker to be in the up channel and (b) down channel as a function of the time step for different values of phase modulations.

\[
q E_{n, \pm, k_x} = \pm \arccos \left[ \cos \left( \frac{\pi q}{2} \right) - \cos (q k_x) \right] \frac{(-1)^q}{\sqrt{2^q}} - \cos \left( \frac{\pi q}{2} \right)
\]

which corresponds to:

\[
e^{-iqE_{n, \pm, k_x}} = \begin{cases}
-1 & q = 4g + 1 \\
+i & q = 4g + 2 \\
1 & q = 4g + 3
\end{cases}
\]

We can express any amplitude \( \psi \) representing the up channel \( U(x, n) \) or the down channel \( D(x, n) \) in terms of the initial conditions as:

\[
\psi(k_x, n) = \sum_{j=1}^{2q} e^{-iE_{j, k_x} n} A_{j, k_x}
\]

Therefore, based on the above expressions, such an amplitude is expressed after \( q \) steps for even values of \( q \) via:

\[
\psi(k_x, q) = s \sum_{j=1}^{2q} e^{ik_x x} A_{j, k_x} = s\psi(k_x, 0),
\]

in which

\[
s = \begin{cases}
-1 & q = 4g + 1 \\
+1 & q = 4g + 2
\end{cases}
\]

This equality holds for any \( k_x = \frac{x}{T} + \frac{2\pi m}{T} \) with \( m \in \mathbb{Z} \) and for other values of \( k_x \) it holds by approximation. However, with the increase of \( q \), the approximation becomes more and more accurate. Therefore, the following approximation holds, and it becomes exact in the limit of \( q \to \infty \):
$\psi(x, q) \equiv s\psi(x, 0)$.

This proves that revival happens after $q$ steps ($|\psi(x, q)| \equiv |\psi(x, 0)|$) for even values of $q$ and it becomes more accurate with the increase of $q$. The variations of the quantum walk probability distribution as a function of the time step under no applied gauge field as well as a linearly time varying gauge field with $\phi = \pi/6$ are shown in Fig. S5. This figure clearly shows the revival caused by Bloch oscillations under the time varying gauge field.

Since we start with a unity pulse at the origin in the up channel, we can plot the probability amplitude of the quantum walker to be at the origin after $q$ steps. We have plotted this amplitude in Fig. S6a. This figure confirms the fact that with the increase of $q$, the quantum walker indeed becomes more localized at the origin and the corresponding probability tends toward unity in the limit of $q \to \infty$. The corresponding normalized probability of the quantum walker to return to the origin after $q$ steps is also shown in Fig. S6b.

We can also consider the total probability of the quantum walker to be in the up channel or down channel as a function of the time step. We have plotted these probabilities in Fig. S7 for different values of phase modulations. As we expect, the probability of the quantum walker to be in the up channel is close to one after $2\pi/\phi$ time steps. It is interesting to note that the minimum value of the probability of the quantum walker to be in the up channel is not zero and instead it is close to $1/2\sqrt{2}$. The probability in the up channel reaches this value after around $\pi/\phi$ time steps.
Figures S8a and S8b summarize the numerical results for the variation of $\sqrt{\langle x^2 \rangle_D}$ and $\sqrt{\langle x^2 \rangle_U}$ with the time step for different values of $\phi$. The obtained results show that for $1 \ll n \ll \pi/\phi$, $\sqrt{\langle x^2 \rangle_D}$ varies as $n/\sqrt{8\sqrt{2}}$ as one would expect from the zero-phase modulation case. Moreover, the numerical results show that for small values of $\phi$, $\sqrt{\langle x^2 \rangle_D}$ reaches the maximum value at $n \approx \pi/\phi$. These results also show that $\langle x^2 \rangle_D$ tends toward zero at $n \approx 2\pi/\phi$, which is consistent with the above analysis. In addition to the quadratic means, we can also look at the variations in the norm ones of the quantum walk distributions. Figures S9a and S9b summarize the numerical results for the variation of $\langle |x| \rangle_D$ and $\langle |x| \rangle_U$ with the time step for different values of $\phi$. 
III. THEORETICAL ANALYSIS FOR TWO-DIMENSIONAL QUANTUM RANDOM WALKS:

In this section, we generalize the concepts that are presented in the previous section to two-dimensional quantum random walks. By assuming that all the losses in the setup are fully compensated by the amplifiers and all the polarization changes are compensated by polarization controllers, the setup can be represented as an ideal setup, as shown in Fig. S10. We have demonstrated in our earlier work how pulses will propagate under arbitrary phase modulation patterns [S1]. By representing the pulses that exist at time step $n$ by $U_{x,y}^{(n)}$ and $D_{x,y}^{(n)}$ in the up and down channels, respectively, we have shown the evolution of the pulses obey the following equations:

$$U_{x,y}^{(n+1)} = \frac{e^{i\phi_{n,x,y}-1}}{2} \left( U_{x+1,y}^{(n)} - D_{x+1,y}^{(n)} \right) - \frac{e^{-i\phi_{n,x,y}-1}}{2} \left( U_{x-1,y}^{(n)} + D_{x-1,y}^{(n)} \right)$$

$$D_{x,y}^{(n+1)} = \frac{e^{i\phi_{n,x,y}+1}}{2} \left( U_{x+1,y}^{(n)} - D_{x+1,y}^{(n)} \right) + \frac{e^{-i\phi_{n,x,y}+1}}{2} \left( U_{x-1,y}^{(n)} + D_{x-1,y}^{(n)} \right)$$

By defining $S_{x,y}^{(n)} = U_{x,y}^{(n)} + D_{x,y}^{(n)}$ and $P_{x,y}^{(n)} = U_{x,y}^{(n)} - D_{x,y}^{(n)}$, the obtained equations can be written as:

$$\begin{bmatrix} S_{x,y}^{(n+1)} \\ P_{x,y}^{(n+1)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-i\phi_{n,x,y}+1} & e^{i\phi_{n,x,y}+1} \\ -e^{-i\phi_{n,x,y}+1} & -e^{i\phi_{n,x,y}+1} \end{bmatrix} \begin{bmatrix} S_{x,y}^{(n)} \\ P_{x,y}^{(n)} \end{bmatrix} + \begin{bmatrix} e^{-i\phi_{n,x,y}+1} & e^{i\phi_{n,x,y}+1} \\ -e^{-i\phi_{n,x,y}+1} & -e^{i\phi_{n,x,y}+1} \end{bmatrix} \begin{bmatrix} S_{x-1,y}^{(n+1)} \\ P_{x-1,y}^{(n+1)} \end{bmatrix}$$

By defining $s_{k_x,k_y}^{(n)}$ and $p_{k_x,k_y}^{(n)}$ as Fourier transforms of $S_{x,y}^{(n)}$ and $P_{x,y}^{(n)}$, we have:

$$\begin{bmatrix} s_{k_x,k_y}^{(n)} \\ p_{k_x,k_y}^{(n)} \end{bmatrix} = \frac{1}{4\pi^2} \begin{bmatrix} \int \int k_x,k_y s_{k_x,k_y}^{(n)} e^{ik_x x + ik_y y} dk_x dk_y \\ \int \int k_x,k_y p_{k_x,k_y}^{(n)} e^{ik_x x + ik_y y} dk_x dk_y \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} \int \int k_x,k_y s_{k_x,k_y}^{(n+1)} e^{ik_x x + ik_y y} dk_x dk_y \\ \int \int k_x,k_y p_{k_x,k_y}^{(n+1)} e^{ik_x x + ik_y y} dk_x dk_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-i\phi_{n,x,y}+1} & e^{i\phi_{n,x,y}+1} \\ -e^{-i\phi_{n,x,y}+1} & -e^{i\phi_{n,x,y}+1} \end{bmatrix} \begin{bmatrix} \int \int k_x,k_y s_{k_x,k_y}^{(n)} e^{ik_x x + ik_y y} dk_x dk_y \\ \int \int k_x,k_y p_{k_x,k_y}^{(n)} e^{ik_x x + ik_y y} dk_x dk_y \end{bmatrix} \times \begin{bmatrix} e^{-i\phi_{n,x,y}+1} & e^{i\phi_{n,x,y}+1} \\ -e^{-i\phi_{n,x,y}+1} & -e^{i\phi_{n,x,y}+1} \end{bmatrix} \begin{bmatrix} \int \int k_x,k_y s_{k_x,k_y}^{(n)+1} e^{ik_x x + ik_y y} dk_x dk_y \\ \int \int k_x,k_y p_{k_x,k_y}^{(n)+1} e^{ik_x x + ik_y y} dk_x dk_y \end{bmatrix}$$
This equation can be used to solve the Fourier transforms as functions of the time step. The evolution for the case of no phase modulation has already been investigated [SI]. Here we focus on the case in which we apply a phase modulation that varies linearly with the time step.

\[ \frac{d}{dt} \begin{bmatrix} s_{k_x,k_y}^{(n+1)} \\ p_{k_x,k_y}^{(n+1)} \end{bmatrix} = \begin{bmatrix} ie^{-ik_x+\phi_n} \sin (k_y) & e^{ik_x+\phi_n} \cos (k_y) \\ -e^{-ik_x+\phi_n} \cos (k_y) & -ie^{ik_x+\phi_n} \sin (k_y) \end{bmatrix} \begin{bmatrix} s_{k_x,k_y}^{(n)} \\ p_{k_x,k_y}^{(n)} \end{bmatrix} \]

Therefore, by defining \( w_{k_x,k_y}^{(n)} = s_{k_x-n\phi,k_y}^{(n)} \) and \( v_{k_x,k_y}^{(n)} = p_{k_x-n\phi,k_y}^{(n)} \):

\[ \begin{bmatrix} w_{k_x+\phi,k_y}^{(n+1)} \\ v_{k_x+\phi,k_y}^{(n+1)} \end{bmatrix} = \begin{bmatrix} ie^{-ik_x} \sin (k_y) & e^{ik_x} \cos (k_y) \\ -e^{-ik_x} \cos (k_y) & -ie^{ik_x} \sin (k_y) \end{bmatrix} \begin{bmatrix} w_{k_x,k_y}^{(n)} \\ v_{k_x,k_y}^{(n)} \end{bmatrix} \]

Using this equation, we can obtain a propagation matrix for rational values of phase modulations, \( \phi/2\pi = p/q \). In the following, we obtain the eigen-energies for any rational value of phase modulation, straightforwardly.

By defining \( r_{k_x,k_y}^{(n)} = \begin{bmatrix} w_{k_x,k_y}^{(n)} \\ v_{k_x,k_y}^{(n)} \end{bmatrix} \) and

\[ M_{k_x,k_y} = \begin{bmatrix} ie^{-ik_x} \sin (k_y) & e^{ik_x} \cos (k_y) \\ -e^{-ik_x} \cos (k_y) & -ie^{ik_x} \sin (k_y) \end{bmatrix} \]

we have:

\[ r_{k_x,k_y}^{(n+1)} = M_{k_x,k_y} r_{k_x,k_y}^{(n)} \]

After \( q \) steps, we have:

\[ r_{k_x,k_y}^{(n+q)} = M_{k_x,(q-1)\phi,k_y} \cdots M_{k_x,\phi,k_y} r_{k_x,k_y}^{(n)} \]

Therefore, by defining \( Y_{k_x,k_y}^{(i)} = M_{k_x+(i-1)\phi,k_y} \cdots M_{k_x,\phi,k_y} \), the following holds:

\[ r_{k_x,k_y}^{(n+q)} = Y_{k_x,k_y}^{(q)} r_{k_x,k_y}^{(n)} \]

Since \( M \) and \( Y \) are unitary matrices, the following equations hold for the eigen-energies:

\[ Y_{k_x,k_y}^{(q)} r_{k_x,k_y}^{(n)} = e^{iqE} r_{k_x,k_y}^{(n)} \]

\[ Y_{k_x,k_y}^{(q)^\dagger} r_{k_x,k_y}^{(n)} = e^{-iqE} r_{k_x,k_y}^{(n)} \]

Consequently, we have:

\[ \frac{1}{2} \left( Y_{k_x,k_y}^{(q)} + Y_{k_x,k_y}^{(q)^\dagger} \right) r_{k_x,k_y}^{(n)} = \cos (qE) r_{k_x,k_y}^{(n)} \]
Figure S11. (a) The probability amplitude of the quantum walker to be at the origin in the up channel after $q$ steps as a function of $q$. (b) The normalized probability of the quantum walker to return to the origin in the up channel after $q$ steps as a function of $q$.

It can be verified that not only for $\phi = 2\pi/q$ but also for any $\phi = 2\pi p/q$, with $q$ and $p$ being relatively prime, the following holds:

$$\frac{1}{2} \left( Y^{(q)}_{k_x,k_y} + Y^{(q)\dagger}_{k_x,k_y} \right) = \left( \left( \cos \left( \frac{\pi q}{2} \right) - \cos \left( qk_x + \frac{\pi q}{2} \right) \right) \sin^q (k_y) - \cos \left( \frac{\pi q}{2} \right) \right) I$$

Therefore, the eigen-energies are given by

$$E_{n,\pm,k_x,k_y} = \frac{2n\pi}{q} \pm \frac{1}{q} \arccos \left[ \left( \cos \left( \frac{\pi q}{2} \right) - \cos \left( qk_x + \frac{\pi q}{2} \right) \right) \sin^q (k_y) - \cos \left( \frac{\pi q}{2} \right) \right],$$

which explicitly expresses the eigen-energies for any of $2q$ bands.

Based on this analytical expression for the allowed energies, it can be verified that the bandgap does not exist for any rational value of phase modulation. The energy band diagrams for $\phi = \pi/2$, $\phi = \pi/3$, and $\phi = \pi/4$ have been plotted in Fig. 2 of the main text.

Since we are interested in the evolution of the quantum walk after $q$ steps, we obtain the following for any $k_x = \frac{2\pi m}{q}$ with $m \in \mathbb{Z}$:

$$qE_{n,\pm,k_x,k_y} = \pm \arccos \left[ \left( \cos \left( \frac{\pi q}{2} \right) - \cos \left( qk_x + \frac{\pi q}{2} \right) \right) \sin^q (k_y) - \cos \left( \frac{\pi q}{2} \right) \right]$$

which corresponds to:

$$e^{-iqE_{n,\pm,k_x,k_y}} = \begin{cases} 
-1 & q = 4g \\
i & q = 4g + 1 \\
1 & q = 4g + 2 \\
i & q = 4g + 3 
\end{cases}$$

We can express any amplitude $\psi$ representing the up channel $U(x,y,n)$ or the down channel $D(x,y,n)$ in terms of the initial conditions as:
\[
\psi (k_x, y, n) = \sum_{j=1}^{2q} e^{-iE_j k_x k_y} n e^{i(k_y y)} A_{j, k_x, k_y} dk_y
\]

Therefore, based on the above expressions, such an amplitude is expressed after \(q\) steps for even values of \(q\) via:

\[
\psi (k_x, y, q) = s \sum_{j=1}^{2q} e^{i(k_y y) A_{j, k_x, k_y} dk_y} = s \psi (k_x, y, 0),
\]

in which

\[
s = \begin{cases} 
-1 : & q = 4g \\
+1 : & q = 4g + 2 
\end{cases}
\]

This equality holds for any \(k_x = \frac{2\pi m}{q}\) with \(m \in \mathbb{Z}\) and for other values of \(k_x\) it holds by approximation. However, with the increase of \(q\), the approximation becomes more and more accurate. Therefore, the following approximation holds, and it becomes exact in the limit of \(q \to \infty\):

\[
\psi (x, y, q) \cong s \psi (x, y, 0).
\]

This proves that revival (caused by Bloch oscillations) happens after \(q\) steps (|\(\psi (x, y, q)\)\| \cong |\(\psi (x, y, 0)\)|) for even values of \(q\) and it becomes more accurate with the increase of \(q\).

Since we start with a unity pulse at the origin in the up channel, we can plot the probability amplitude of the quantum walker to be at the origin after \(q\) steps. We have plotted this amplitude in Fig. S11a. This figure confirms the fact that with the increase of \(q\), the quantum walker indeed becomes more trapped at the origin and the corresponding probability tends toward unity in the limit of \(q \to \infty\). The corresponding normalized probability of the quantum walker to return to the origin after \(q\) steps is also shown in Fig. S11b.

We can also consider the total probability of the quantum walker to be in the up channel or down channel as a function of the time step. We have plotted these probabilities in Fig. S12 for different values of phase modulations. As we expect, the probability of the quantum walker to be in the up channel is close to one after \(2\pi/\phi\) time steps. The minimum value of the probability of the quantum walker to be in the up channel is not zero and instead it is close to \(1/\pi\). The probability in the up channel reaches this value after around \(\pi/\phi\) time steps.

We can also analyze the transient variation of the quantum walk. For this purpose, the obtained propagation equation for \(w_{k_x, k_y}^{(n)}\) and \(v_{k_x, k_y}^{(n)}\) can be transformed to the following equation:
Figure S13. The quadratic mean of (a) \(y\) and (b) \(x\) for the down channel as a function of the time step for different phase modulation values. The quadratic mean of (c) \(y\) and (d) \(x\) for the up channel as a function of the time step for different phase modulation values.

\[
\begin{bmatrix}
w^{(n+1)}_{k_x,k_y} + v^{(n+1)}_{k_x,k_y} \\
w^{(n+1)}_{k_x,k_y} - v^{(n+1)}_{k_x,k_y}
\end{bmatrix}
\begin{bmatrix}
i e^{-ik_y} \sin(k_x - \phi) & -e^{-ik_y} \cos(k_x - \phi) \\
i e^{ik_y} \cos(k_x - \phi) & -i e^{ik_y} \sin(k_x - \phi)
\end{bmatrix}
\begin{bmatrix}
w^{(n)}_{k_x-\phi,k_y} + v^{(n)}_{k_x-\phi,k_y} \\
w^{(n)}_{k_x-\phi,k_y} - v^{(n)}_{k_x-\phi,k_y}
\end{bmatrix}
\]

Focusing on the down channel, we know that at time step \(n\), we have:

\[
d^{(n)}_{k_x,k_y} = 0.5 \left(s^{(n)}_{k_x,k_y} - p^{(n)}_{k_x,k_y}\right) = 0.5 \left(w^{(n)}_{k_x+n\phi,k_y} - v^{(n)}_{k_x+n\phi,k_y}\right)
\]

From this expression, we can obtain the following equation for \(\langle y^2 \rangle_D\):

\[
\langle y^2 \rangle_D = \sum_{x,y} y^2 \left| D^{(n)}_{x,y} \right|^2
= \frac{1}{4\pi^2} \int_{k_y=0}^{2\pi} \int_{k_x=0}^{2\pi} \left(\frac{d}{dk_y} \right)^2 d^{(n)}_{k_x,k_y} dk_x dk_y
= \frac{1}{16\pi^2} \int_{k_y=0}^{2\pi} \int_{k_x=0}^{2\pi} \left(w^{(n)}_{k_x,k_y} - v^{(n)}_{k_x,k_y}\right)^* \left(\frac{d}{dk_y} \right)^2 \left(w^{(n)}_{k_x,k_y} - v^{(n)}_{k_x,k_y}\right) dk_x dk_y
\]
Therefore, it is easy to verify that the expansion of \( 0.5 \left( w_{k_x,y}^{(n)} - v_{k_x,y}^{(n)} \right) \) has terms proportional to \( e^{ik_y n} \), \( e^{ik_y (n-1)} \), ..., and \( e^{ik_y (-n+2)} \). The first and last term represent the farthest \( y \) position from the \( x \) axis after \( n \) steps. By obtaining the amplitudes of these two terms based on the above equation, we can expand \( 0.5 \left( w_{k_x,y}^{(n)} - v_{k_x,y}^{(n)} \right) \) as:
Figure S15. (a) The accumulated phases based on time-varying gauge fields implemented in this work along four sample closed paths. (b) The accumulated phases using a coordinate-dependent unitary operation along four similar closed paths.

\[ 0.5 \left( \langle u_{k_x,k_y}^{(n)} \rangle - \langle v_{k_x,k_y}^{(n)} \rangle \right) = (-i)^{n-1} e^{i n k_y} \cos(k_x - n \phi) \prod_{p=1}^{n-1} \sin(k_x - p\phi) \]

\[ + i^{n-1} e^{-i(n-2)k_y} \cos(k_x - \phi) \prod_{p=2}^{n} \sin(k_x - p\phi) \]

\[ + \sum_{y=-n+3}^{n-1} a_y (k_x, \phi, n) e^{i k_y y} \]

Which means that

\[ P_D (y = -n) = P_D (y = n - 2) \]

\[ = \frac{1}{2\pi} \int_{k_x = 0}^{2\pi} \cos^2 (k_x) \prod_{p=1}^{n-1} \sin^2 (k_x - p\phi) \, dk_x. \]

These expressions show the existence of zeros in the integrand at \( k_x = p\phi \) for \( p = 1 \) to \( n - 1 \). For \( n \sim \frac{2\pi}{\phi} \), these zeros spread across the entire region of \( k_x = 0 \) to \( k_x = 2\pi \). Especially, for the small values of \( \phi \), the presence of these zeros causes that \( P_D (y = -n) \) and \( P_D (y = n - 2) \) tend to zero at \( n \sim \frac{2\pi}{\phi} \). For other values of \( y \neq 0 \), similar expressions can be obtained for \( P_D (y) \), which they also become small for \( \phi = \frac{2\pi}{n} \). This analysis predicts that \( \langle y^2 \rangle_D \) tends toward zero for \( n = 2\pi/\phi \).

Figures S13a and S13c summarize the numerical results for the variation of \( \sqrt{\langle y^2 \rangle_D} \) and \( \sqrt{\langle y^2 \rangle_U} \) with the time step for different values of \( \phi \). The obtained results show that for \( 1 \ll n \ll \pi/\phi \), \( \sqrt{\langle y^2 \rangle_D} \) varies as \( n/\sqrt{6\pi} \), as one would expect from the zero phase modulation case. Moreover, the numerical results show that for small values of \( \phi \), \( \sqrt{\langle y^2 \rangle_D} \) reaches the maximum value at \( n \cong \pi/\phi \). These results also show that \( \langle y^2 \rangle_0 \) tends toward zero at \( n \cong 2\pi/\phi \), which is consistent with the above analysis. The effect of phase modulation on the quadratic mean of \( x \) can be investigated as well. Numerical results as depicted in Figs. S13b and S13d show that the quadratic mean of \( x \) also decreases relative to the zero-phase modulation case. However, the effect of phase modulation is more intense on decreasing the quadratic mean of \( y \) as compared with the quadratic mean of \( x \).

In addition to quadratic means, we can also look at the variations of the norm ones of the quantum walk distributions. Figures S14a and S14c summarize the numerical results for the variation of \( \langle |y| \rangle_D \) and \( \langle |y| \rangle_U \) with the time step for different values of \( \phi \). The corresponding results for the \( x \) direction are also shown in Figs. S14b and S14d.
B. Creation of electric fields using various gauge fields:

In addition to the time-dependent approach, an electric field can be implemented through the use of a coordinate-dependent gauge field as well. In the latter case, instead of using a time-dependent gauge field, a phase modulation is applied that is not direction dependent and instead depends on the coordinate linearly. In this approach, an effective linear electric potential \( V = -E_x \) is implemented that will lead to the generation of an electric field based on \( \mathcal{E} = -\nabla V \). The unitary operation in each time step has an extra term relative to the standard quantum walk evolution operator \( U_0 \) as \( U_\phi = e^{i\phi x} U_0 \). We can compare the effect of such a gauge field with the time-dependent gauge field considered in this work. In Fig. S15, we have considered four sample closed paths that start from the origin and return to it, and have calculated the total phase accumulated in them. As this figure shows, the net amount of phase in both approaches are similar, showing that indeed the time-varying gauge field will induce a similar phase accumulation in these closed paths as a conventional electric field in the \( x \) direction. This similarity holds for any closed path starting from the origin and ending at it.

[S1] H. Chalabi, S. Barik, S. Mittal, T. E. Murphy, M. Hafezi, and E. Waks, Physical Review Letters 123, 150503 (2019).