Geometric Methods for Invariant-Zero Cancellation in Linear Multivariable Systems: Illustrative Examples

Elena Zattoni

Abstract

This note presents some numerical examples worked out in order to show the reader how to implement, within a widely accessible computational setting, the methodology for achieving zero cancellation in linear multivariable systems discussed in [1]. The results are evaluated in light of applicability and performance of different methods available in the literature.

Index Terms

Invariant zeros, zero cancellation, minimum-phase systems, linear systems, geometric approach.

I. INTRODUCTION

The methodology for achieving zero cancellation in linear multivariable systems developed in [1] is based on the geometric characterization of the invariant zeros of a linear time-invariant multivariable system as the internal unassignable eigenvalues of the maximal output nulling controlled invariant subspace [2]. In particular, in [1] it is shown that a series of three state-space basis transformations, determined in connection with the maximal output nulling controlled invariant subspace and a friend linear map, results in a representation of the system where the structure of the minimum-phase invariant zeros is caught by a certain pair of matrices. Hence, the linear maps associated with those matrices are used to define the feedforward compensator achieving zero cancellation while retaining some special properties of the original system, such as stabilizability and right-invertibility.

In this note, three numerical examples are presented. The first example is worked out in detail, with the aim of illustrating every single step of the application of the procedure proposed in [1]. The second example is borrowed from the literature, with the purpose of comparing the results obtained with the techniques described in [1] to those provided by the approach introduced in [3]. As to the third example, it has been known for a long time that zero assignment can effectively be exploited to improve the transient response of a control system [4]. Thus, the third example is devised in order to show how a slight modification of the method discussed in [1] leads to a feedforward compensator that not only attains zero cancellation with the additional constraints

The author is with the Department of Electrical, Electronic and Information Engineering “Guglielmo Marconi”, Alma Mater Studiorum · University of Bologna, Viale Risorgimento 2, 40136 Bologna, Italy. E-mail: elena.zattoni@unibo.it
of retaining reachability and right-invertibility, but also ameliorates the system step response by eliminating the overshoot.

The computational framework consists of the Matlab files implementing the geometric approach algorithms, first appeared with [2] and now available online in an upgraded version. The variables are displayed in scaled fixed point format with five digits, although computations are made in floating point precision.

Notation: $\mathbb{R}$ stands for the set of real numbers. Matrices and linear maps are denoted by uppercase letters, like $A$. The spectrum, the image, and the kernel of $A$ are denoted by $\sigma(A)$, $\text{im} A$, and $\ker A$, respectively. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. Vector spaces and subspaces are further characterized by subscripts when the same subscripts are used to denote the matrices of the systems they refer to. The symbols $I_n$ and $O_{m \times n}$ are respectively used for the identity matrix of dimension $n$ and the $m \times n$ zero matrix (subscripts are omitted when the dimensions can be inferred from the context).

II. EXAMPLE 1

Consider the continuous-time linear time-invariant system
\begin{align}
\dot{x}(t) &= A x(t) + B u(t), \quad (1) \\
y(t) &= C x(t) + D u(t), \quad (2)
\end{align}
and assume
\[
A = \begin{bmatrix}
-0.79 & -1.89 & -1 & -1.01 & -0.2 \\
0.89 & -4.3 & -0.76 & -0.48 & -0.12 \\
0.8 & -5.57 & -3.25 & -3.01 & -1.52 \\
-1.18 & 3.41 & 0.26 & -1.03 & 0.92 \\
1.62 & -6.41 & 0.55 & -4.15 & -5.63
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
2 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
-1 & 0 & 2 & 0
\end{bmatrix}, \\
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0.89 & -2.8 & -0.76 & -0.48 & -1.12 \\
-0.29 & -0.89 & -0.25 & -1.51 & -0.2
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

The matrices of the given system satisfy the rank conditions stated in [1 Section II]. Note that the given system is reachable, since the minimal $A$-invariant subspace containing $B = \text{im} B$ or, equivalently, the reachable subspace of the pair $(A, B)$ is given by $\mathcal{R} = \min J(A, B) = \mathcal{X} = \mathbb{R}^5$. Moreover, the given system is right-invertible, since the maximal output-nulling controlled invariant subspace $\mathcal{V}^*$ and the minimal input-containing conditioned invariant subspace $\mathcal{S}^*$ are
respectively given by

\[ V^* = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ -0.6695 & 0 & 0 \\ 0.6180 & -0.5547 & 0 \\ -0.4120 & -0.8321 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad S^* = \text{im} \begin{bmatrix} 0.4862 & 0 & 0 \\ 0 & -1 & 0 \\ 0.4813 & 0 & 0 \\ -0.7293 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

and, therefore, \( V^* + S^* = X \). Furthermore, the set of the invariant zeros \( Z(A, B, C, D) \), which is the union of the sets of the minimum-phase invariant zeros \( Z_{MP}(A, B, C, D) \) and of the nonminimum-phase invariant zeros \( Z_{NMP}(A, B, C, D) \), is given by

\[ Z(A, B, C, D) = Z_{MP}(A, B, C, D) \cup Z_{NMP}(A, B, C, D) = \{-1.2509\} \cup \{0.7534\}. \]

On these conditions, the geometric methodology presented in [1] allows us to design a feed-forward compensator that cancels the minimum-phase zero \( z_{MP} = -1.2509 \), while maintaining reachability and right-invertibility in the resulting cascade system.

The first step of the geometric method requires us to pick a linear map \( F \) such that \( (A + BF)V^* \subseteq V^* \) and \( V^* \subseteq \ker (C + DF) \). A linear map \( F \) satisfying these conditions is represented by the matrix

\[ F = \begin{bmatrix} 0 & 0.7121 & 0.1166 & 1.5990 & 0.2000 \\ 0 & 1.0731 & -0.5352 & 1.3434 & 1.1200 \\ 0 & 1.9832 & -0.8226 & 2.7324 & 2.3320 \\ 0 & 0.6742 & -0.3712 & 0.7916 & -1.0000 \end{bmatrix}. \]

In order to determine the first similarity transformation \( T \), defined according to [1] Lemma 1], it is worth noting that

\[ \mathcal{R}_{V^*} = V^* \cap S^* = \text{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

Then,

\[ T = \begin{bmatrix} 0 & 0 & 0 & -0.4862 & 0 \\ 0 & 0.6695 & 0 & 0 & -1 \\ 0 & -0.6180 & -0.5547 & -0.4813 & 0 \\ 0 & 0.4120 & -0.8321 & 0.7293 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \]
Consequently, the matrices $A'_{F}$ and $C'_{F}$, defined according to [1, eqs. (7), (8)], are

$$A'_{F} = \begin{bmatrix} -1.1660 & -1.4810 & 0.9086 & -0.4116 & 3.1557 \\ 0 & 0.0040 & -0.6763 & -0.2900 & -3.0932 \\ 0 & -1.3907 & -0.5015 & -1.8603 & 0.0092 \\ 0 & 0 & 0 & -2.5481 & -2.4227 \\ 0 & 0 & 0 & -1.7707 & -4.6237 \end{bmatrix},$$

$$C'_{F} = \begin{bmatrix} 0 & 0 & 0 & -0.4862 & 0 \\ 0 & 0 & 0 & 0.8204 & 1.7269 \\ 0 & 0 & 0 & 0.2701 & 0.1779 \end{bmatrix}.$$  

According to [1, Lemma 2], the similarity transformation $T'$ is determined by solving the Sylvester equation. The relation

$$\sigma(A'_{11}) \cap \sigma(A'_{22}) = \{-1.1660\} \cap \{0.7534, -1.2509\} = \emptyset$$

ensures the existence and uniqueness of the solution of the Sylvester equation: i.e.,

$$X = \begin{bmatrix} -1.7146 & -0.3776 \end{bmatrix}.$$  

Hence,

$$T' = \begin{bmatrix} 1 & -1.7146 & -0.3776 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$  

Consequently, $A''_{F}$ and $C''_{F}$, defined according to [1, eqs. (13), (14)], are

$$A''_{F} = \begin{bmatrix} -1.1660 & 0 & 0 & -1.6113 & -2.1444 \\ 0 & 0.0040 & -0.6763 & -0.2900 & -3.0932 \\ 0 & -1.3907 & -0.5015 & -1.8603 & 0.0092 \\ 0 & 0 & 0 & -2.5481 & -2.4227 \\ 0 & 0 & 0 & -1.7707 & -4.6237 \end{bmatrix},$$

and $C''_{F} = C'_{F}$. In order to determine the similarity transformation $T''$, considered in [1, Lemma 3], the matrix

$$J' = \begin{bmatrix} J_S & J_U \end{bmatrix} = \begin{bmatrix} 0.4744 & 0.6699 \\ 0.8803 & -0.7424 \end{bmatrix},$$
is computed. Thus,

\[
T'' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0.4744 & 0.6699 & 0 & 0 \\
0 & 0.8803 & -0.7424 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

As a consequence, \(A_{F''}^{'''}\) and \(C_{F''}^{'''}\), defined according to [1, eqs. (15), (16)], are

\[
A_{F''}''' = \begin{bmatrix}
-1.1660 & 0 & 0 & -1.6113 & -2.1444 \\
0 & -1.2509 & 0 & -1.5517 & -2.4314 \\
0 & 0 & 0.7534 & 0.6658 & -2.8954 \\
0 & 0 & 0 & -2.5481 & -2.4227 \\
0 & 0 & 0 & -1.7707 & -4.6237 \\
\end{bmatrix}
\]

and \(C_{F''}''' = C_{F''}'''\). Then, one gets the resolving subspace

\[
\mathcal{V}_S = \text{im}\ V_S'''' = \text{im}\ \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\]

according to [1 Theorem 1]. With respect to the original coordinates, \(\mathcal{V}_S\) is given by

\[
\mathcal{V}_S = \text{im}\ (\bar{T} \ V_S''') = \text{im}\ \begin{bmatrix}
0 \\
0.3176 \\
-0.7815 \\
-0.5370 \\
-1.1458 \\
\end{bmatrix},
\]

where \(\bar{T} = T \ T' \ T''\). According to [1 Corollary 1], from \(A_{F''}'''\), \(F\), and \(V_S\) one also gets the matrices that point out the structure of the minimum-phase invariant zeros of the original system: i.e.,

\[
W = -1.2509, \quad L = \begin{bmatrix}
-0.9528 \\
-1.2457 \\
-2.8665 \\
1.2249 \\
\end{bmatrix}.
\]

Therefore, the matrices of the feedforward compensator, defined according to [1 eqs. (24), (25)].
are

\[
A_f = -1.2509, \quad B_f = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
C_f = \begin{bmatrix} -0.9528 \\ -1.2457 \\ -2.8665 \\ 1.2249 \end{bmatrix}, \quad D_f = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

The matrices of the cascade in Eq. (5), defined according to Eq. (32), are

\[
A_e = \begin{bmatrix} -0.79 & -1.89 & -1 & -1.01 & -0.2 \\ 0.89 & -4.3 & -0.76 & -0.48 & -0.12 \\ 0.8 & -5.57 & -3.25 & -3.01 & -1.52 \\ -1.18 & 3.41 & 0.26 & -1.03 & 0.92 \\ 1.62 & -6.41 & 0.55 & -4.15 & -5.63 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ -0.3176 \\ 0.7815 \\ 0.5370 \\ 1.1458 \end{bmatrix},
\]

\[
C_e = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.89 & -2.8 & -0.76 & -0.48 & -1.12 \\ -0.29 & -0.89 & -0.25 & -1.51 & -0.2 \end{bmatrix}, \quad D_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Note that the cascade is reachable, since \( R_e = \min J(A_e, B_e) = X \), which means that the feedforward compensator has maintained reachability. The cascade is right-invertible, since

\[
V^*_e = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ -0.6695 & 0 & 0 \\ 0.6180 & -0.5547 & 0 \\ -0.4120 & -0.8321 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad S^*_e = \text{im} \begin{bmatrix} 0.4863 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0.4949 & 0 & 0.8242 & 0 \\ -0.7202 & 0 & 0.5663 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

and, therefore, \( V^*_e + S^*_e = X \). This shows that the precompensator has kept right-invertibility. Furthermore, the set of the invariant zeros is \( Z(A_e, B_e, C_e, D_e) = \{0.7534\} \), which means that the minimum-phase invariant zero of the original system has been cancelled, while the nonminimum-phase invariant zero is still present.

### III. Example 2

Consider the continuous-time linear time-invariant system (1), (2) and assume

\[
A = \begin{bmatrix} -1 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
The matrices of the system satisfy the rank conditions stated in [1, Section II]. The system is not reachable and not right-invertible. The system has an invariant zero at \(-1\) with multiplicity equal to 3. By applying the standard procedure, one gets that the subspace \(V^*_S\) is given by

\[
V^*_S = \text{im} \, V^*_S = \text{im} \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0.3333 & 0.1667 \\
0 & 0 & -0.5
\end{bmatrix},
\]

and the structure of the minimum-phase invariant zero is given by the matrices

\[
W = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad L = \begin{bmatrix}
0 & 0.3333 & 0.1667 \\
0 & 0 & -0.5
\end{bmatrix}.
\]

In particular, the matrix \(W\) shows that the zero dynamics have a defective eigenvalue. Nonetheless, by applying the procedure described in [1], one gets the following matrices for the feedforward compensator

\[
A_f = W, \quad B_f = \begin{bmatrix}
I_3 & O_{3 \times 2}
\end{bmatrix}, \quad C_f = L, \quad D_f = \begin{bmatrix}
O_{2 \times 3} & I_2
\end{bmatrix}.
\]

The matrices of the equivalent form of the cascade are

\[
A_e = A, \quad B_e = \begin{bmatrix}
-V^*_S & B
\end{bmatrix}, \quad C_e = C, \quad D_e = O_{2 \times 5}.
\]

The system thus obtained is not reachable, is not right-invertible and does not have any invariant zeros. It is worth noting that, in comparison with the solution proposed in [3], where the feedforward compensator has dynamic order equal to 6, the solution proposed herein guarantees the feedforward compensator with the minimal dynamic order. In the specific case, the dynamic order of the feedforward compensator is 3.
IV. EXAMPLE 3

Consider the continuous-time linear time-invariant system (1), (2) and assume

\[
A = \begin{bmatrix}
-4 & 4 & 1 & 0 & 0 \\
0 & -5 & -3 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & -30 & -12.5 \\
0 & 0 & 0 & 16 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 \\
3.4640 & 0 & 0 \\
0 & 0 & 0 \\
0 & 3.5355 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
3.4640 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 3.5355
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 & 0 \\
-0.5 & 0 & 1
\end{bmatrix}.
\]

The matrices of the system satisfy the rank conditions stated in [1, Section II]. The system is reachable and right-invertible. The system has only one zero, which is minimum-phase: namely, \(z_{MP} = -0.5\). By applying the standard procedure, one gets that the subspace \(V^*_S\) is given by

\[
V^*_S = \text{im} \begin{bmatrix}
0 \\
-0.2425 \\
0.9701 \\
0.0281 \\
-0.0849
\end{bmatrix},
\]

and the structure of the minimum-phase invariant zero is given by the matrices

\[
W = -0.5, \quad L = \begin{bmatrix}
0.5251 \\
0.0497 \\
-0.4074
\end{bmatrix}.
\]

A slight modification of the feedforward compensator design, carried out according to [1, Remark 4], yields the feedforward compensator matrices

\[
A_f = -0.5, \quad B_f = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad C_f = \begin{bmatrix}
0.5251 \\
0.0497 \\
-0.4074
\end{bmatrix}, \quad D_f = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Consequently, the matrices of the equivalent form of the cascade system are

\[
A_e = \begin{bmatrix}
-4 & 4 & 1 & 0 & 0 \\
0 & -5 & -3 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & -30 & -12.5 \\
0 & 0 & 0 & 16 & 0
\end{bmatrix}, \quad B_e = \begin{bmatrix}
0 & 0 & 0 \\
0.2425 & 0 & 0 \\
-0.9701 & 0 & 0 \\
-0.0281 & 3.5355 & 1 \\
0.0849 & 0 & 1
\end{bmatrix},
\]

\[
C_e = \begin{bmatrix}
3.4640 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 3.5355
\end{bmatrix}, \quad D_e = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
The cascade system thus obtained is reachable, right-invertible and without invariant zeros. The comparison between the step response of the original system and that of the cascade shows that zero cancellation has yielded the elimination of the overshoot in the new system (Figures 1 and 2). However, as mentioned in [1, Remark 4], the selection of the control inputs has implied an increase in the relative degree from 2 to 3.

V. CONCLUSION

Three numerical examples have been presented with the aim of illustrating the geometric techniques for accomplishing zero cancellation in linear multivariable systems devised in [1]. Benefits of the proposed methodology compared with those available in the literature have been shown.

REFERENCES

[1] E. Zattoni, “Geometric methods for invariant-zero cancellation in linear multivariable systems with application to signal rejection with preview,” accepted for publication in Asian Journal of Control, 2013.
[2] G. Basile and G. Marro, Controlled and Conditioned Invariants in Linear System Theory. Englewood Cliffs, New Jersey: Prentice Hall, 1992.
[3] Y. Wan, S. Roy, and A. Saberi, “Explicit precompensator design for invariant-zero cancellation,” International Journal of Control, vol. 82, no. 5, pp. 808–811, May 2009.
[4] A. Emami-Naeini and G. F. Franklin, “Zero assignment in the multivariable robust servomechanisms,” in Proceedings of the 21st IEEE Conference on Decision and Control, Orlando, Florida, December 8–10, 1982.
Fig. 2. Example 3: Step response of the equivalent cascade system