Electromagnetic Properties of Kerr-anti-de Sitter Black Holes

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Abstract

We examine the electromagnetic properties of Kerr-anti-de Sitter (Kerr-AdS) black holes in four and higher spacetime dimensions. Assuming that the black holes may carry a test electric charge we show that the Killing one-form which represents the difference between the timelike generators in the spacetime and in the reference background can be used as a potential one-form for the associated electromagnetic field. In four dimensions the potential one-form and the Kerr-AdS metric with properly re-scaled mass parameter solve the Einstein-Maxwell equations, thereby resulting in the familiar Kerr-Newman-AdS solution. We solve the quartic equation governing the location of the event horizons of the Kerr-Newman-AdS black holes and present closed analytic expressions for the radii of the horizons. We also compute the gyromagnetic ratio for these black holes and show that it corresponds to $g = 2$ just as for ordinary black holes in asymptotically flat spacetime. Next, we compute the gyromagnetic ratio for the Kerr-AdS black holes with a single angular momentum and with a test electric charge in all higher dimensions. The gyromagnetic ratio crucially depends on the dimensionless ratio of the rotation parameter to the curvature radius of the AdS background. At the critical limit, when the boundary Einstein universe is rotating at the speed of light, it tends to $g = 2$ irrespective of the spacetime dimension. Finally, we consider the case of a five dimensional Kerr-AdS black hole with two angular momenta and show that it possesses two distinct gyromagnetic ratios in accordance with its two orthogonal 2-planes of rotation. In the special case of two equal angular momenta, the two gyromagnetic ratios merge into one leading to $g = 4$ at the maximum angular velocities of rotation.
I. INTRODUCTION

Black holes remain one of the most important objects of study in four and higher dimensions. We begin with higher-dimensional black holes in asymptotically flat spacetimes. Of particular interest is the role of the black holes in string/M-theory, since they may serve as theoretical laboratories to test the novel predictions of the theory. The intriguing example of this is the statistical-mechanical explanation of the Bekenstein-Hawking entropy for certain supersymmetric black holes in five dimensions [1]. Developments have revealed new unexpected features of black holes in higher dimensions. It turns out that higher dimensions endow black holes with different horizon topologies, whereas in four dimensions the horizon is uniquely determined by the topology of a two-sphere [2]. Some basic properties of black holes in four dimensions such as their stability and uniqueness properties also change in higher dimensions.

The simplest metrics for black holes with spherical horizon topology in all higher dimensions were found by Tangherlini many years ago [3]. These metrics generalize the familiar nonrotating Schwarzschild and Reissner-Nordström solutions of four-dimensional general relativity. The rotational dynamics of these black holes was first explored by Myers and Perry, who discovered the general rotating black hole solutions in arbitrary dimensions [4]. The Myers-Perry solution is not unique, unlike its four-dimensional counterpart, the Kerr solution. There exists a rotating black ring solution with the horizon topology of $S^2 \times S^1$ which may have the same mass and spin as the Myers-Perry solution in five dimensions [5] (see also Ref. [6]). In this sense, the five-dimensional spacetime harbors a relative of the Myers-Perry black hole in the form of a "donut-shaped" rotating black hole. The different properties of black holes and black rings as well as new exact solutions have been discussed in [7]-[12].

Higher-dimensional black holes have a rich phenomenology. This became fully transparent after the advent of Large Extra Dimension Scenarios [13]. These scenarios are built on the idea that our observable universe is a slice, a braneworld, in a higher-dimensional spacetime. The braneworld black holes are in general higher-dimensional objects as they must carry the imprint of the extra dimensions [14]-[16]. The Large Extra Dimension Scenarios open up the possibility of directly probing TeV-scale mini black holes in high-energy collisions [17]. For other related works on classical and quantum properties of black holes in braneworld
Developments in string/M-theory have also greatly stimulated the study of black hole solutions in anti-de Sitter space. The striking examples of this come from the AdS/CFT correspondence between a weakly coupled gravity system in an AdS background and a strongly coupled conformal field theory (CFT) on its boundary [25]. It is well known that the familiar Schwarzschild solution in AdS space describes a simple nonrotating AdS black hole. The most important feature of the black hole is that it has a minimum critical temperature determined by the curvature radius of the AdS background. This means that there must be a thermal phase transition between AdS space and Schwarzschild-AdS space at a fixed temperature: At low temperatures thermal radiation in the AdS space is in stable equilibrium, while at temperatures higher than the critical value there is no stable equilibrium configuration without a black hole [26]. The Hawking-Page transition was interpreted by Witten [27] in terms of a transition between the confining and deconfining phases of the corresponding conformal field theory. As an interesting application of the AdS/CFT correspondence, this result has been extensively discussed in the context of both static and rotating AdS black holes in various spacetime dimensions (see Refs. [28]-[32]). It is also worth to note stability properties of AdS black holes in four dimensions. It has been shown that large Kerr-AdS black holes are always classically stable, while small Kerr-AdS black holes become unstable with respect to scalar and gravitational perturbations via the superradiant amplification mechanism [33, 34].

The authors of Ref. [29] have studied the relationship between Kerr-AdS black holes in the bulk and conformal field theory living on a boundary Einstein universe. Probing the AdS/CFT correspondence in the critical limit, in which the rotation of the boundary Einstein universe occurs at the speed of light, they found that the generic thermodynamic features of the conformal field theory agree with those of the black holes in the bulk. In general, the clearest description of the boundary conformal field theory in a rotating Einstein universe imprinted by a Kerr-AdS black hole in the bulk is a very complicated and subtle question. Exploring the critical limit where the boundary Einstein universe rotates at the speed of light makes a significant simplification as it incorporates generic features of both bulk and boundary theories. There are subtleties even with the definition of the total mass and angular velocities of the Kerr-AdS black holes. In Ref. [35], it has been argued that one must evaluate the mass and angular velocities relative to a frame which is nonrotating.
at infinity. Only these quantities define the most important characteristics of the Kerr-AdS black holes relevant for their CFT duals and satisfy the first law of thermodynamics. A more detailed analysis has led to a general equivalence between the bulk and the boundary thermodynamic variables [36, 37], thereby clarifying the lack of unanimity in many previous cases [38].

In the light of these developments, we address further important properties of the black holes in AdS space, namely the electromagnetic properties of the Kerr-AdS black holes carrying a test electric charge. In order to construct the associated solution of the Maxwell field equations, we use the difference between the timelike generators in the Kerr-AdS metric and in the reference background. The latter is taken to be a rotating Einstein universe. We also compute the magnetic dipole moments and the gyromagnetic ratios for the rotating charged AdS black holes. We consider the following cases: Kerr-Newman-AdS black holes in four dimensions, Kerr-AdS black holes with a single angular momentum in all higher dimensions and Kerr-AdS black holes with two angular momenta in five dimensions. The basic result for the gyromagnetic ratio of the Kerr-AdS black holes with a single angular momentum is given in [39]. Here we present full details of this result.

In classical electrodynamics the gyromagnetic ratio $g$ relates the magnetic dipole moment of a charged rotating body to its total angular momentum and $g = 1$ for a constant ratio of the charge to mass density. However, quantum electrodynamics predicts, up to radiative corrections, that $g = 2$ for charged fermions, like electrons and muons. It has been shown that at the tree-level, $g = 2$ is the natural value of the gyromagnetic ratio for elementary particles of arbitrary spin [40]. The exact result $g = 2$ is related to unbroken supersymmetry and the factor $g - 2$ is considered to be a measure of supersymmetry-breaking (SUSY-breaking) effects (see Ref. [41] for a recent review). From a classical point of view, it is also clear that a rotating charged black hole must have a magnetic dipole moment. However, the magnetic dipole moment is not an independent quantity but is determined by the mass, angular momentum and the electric charge of the black hole. It is long known that, unlike a uniformly charged rotating body, the gyromagnetic ratio for a rotating and charged black hole in general relativity is equal to 2, the same value as for an electron in the Dirac theory [42]. In further developments, this remarkable fact has been confirmed in many cases of Einstein-Maxwell fields in four dimensions [43, 44].

In recent works [8, 9], the gyromagnetic ratio was studied in higher dimensions for asym-
totically flat Myers-Perry black holes carrying a test electric charge as well as for arbitrary values of the electric charge in the limit of slow rotation. A detailed numerical treatment of the problem was given in Refs. [11]. It should be noted that, unlike four dimensions, the value of the gyromagnetic ratio is not universal in higher dimensions. For a five dimensional Myers-Perry black hole with a test electric charge the gyromagnetic ratio was found to be \( g = 3 \). Earlier, the same value of the gyromagnetic ratio was found for a supersymmetric rotating black hole in five dimensions [45]. The gyromagnetic ratio of black rings was studied in [46]. On the other hand, it is known that for black holes in five-dimensional Kaluza-Klein theory the \( g \)-factor approaches unity in the ultrarelativistic limit [47]. This value is the natural \( g \)-factor for massive states in the Kaluza-Klein theory [48]. It is also known that for Kaluza-Klein black holes in ten-dimensional supergravity \( g = 1 \) [49], while some \( p \)-brane solutions in higher dimensions have a gyromagnetic ratio that corresponds to \( g = 2 \) [50].

The present paper is organized as follows. In Sec. II we begin with a brief description of the basic properties of the Kerr-AdS metric in four dimensions. We give the definition of the angular velocity for \textit{locally nonrotating} observers in this spacetime and obtain the expressions for the mass, angular momentum and angular velocity that are consistent with the first law of thermodynamics. We put a test electric charge on the black hole and construct the corresponding solution of the Maxwell equations in the Kerr-AdS background. In Sec. III we assume that the black hole may have an arbitrary electric charge. In this case, by an appropriate re-scaling of the mass parameter one can pass from the Kerr-AdS solution to the Kerr-Newman-AdS one in which the electromagnetic field is still given by the expressions found in Sec. II within the “test-charge” approach. Next, we solve the quartic equation governing the location of the event horizons of the Kerr-Newman-AdS black holes and present closed analytic expressions for the radii of the horizons. We also compute the gyromagnetic ratio for these black holes and show that it corresponds to the value \( g = 2 \) irrespective of the AdS nature of the spacetime. The electromagnetic properties of rotating Kerr-AdS black holes with a single angular momentum in all higher dimensions are studied in Sec. IV. Here we extend the test-charge approach to higher dimensions and find the potential one-form for the electromagnetic field generated by a test electric charge of the black holes. We also compute the magnetic dipole moment and the gyromagnetic ratio for these black holes using thermodynamically consistent expressions for the mass and angular momentum. The value of the gyromagnetic ratio crucially depends on the dimensionless ratio
of the rotation parameter to the curvature radius of the AdS spacetime. In the critical limit in which the boundary Einstein universe is rotating at the speed of light, the gyromagnetic ratio approaches \( g = 2 \) regardless of the spacetime dimension. It is known that at the critical limit of rotation, the Kerr-AdS black holes are related to SUSY configurations \[51\]. Thus, it follows from our result that a supersymmetric black hole in an AdS background must have the gyromagnetic ratio corresponding to \( g = 2 \). Finally, in Sec. V we consider a general five-dimensional Kerr-AdS black hole with two independent angular momenta and with a test electric charge. We obtain the precise expressions for the angular velocities of locally nonrotating observers and re-derive the expressions for the mass and angular momenta. We construct the potential one-qform that describes the test electromagnetic field of the black holes. Here we also define a natural orthonormal frame in which the electromagnetic field two-form takes its simplest form. This is a generalization of the familiar Carter frame in four-dimensional Kerr-Newman spacetime. We show that a five dimensional charged Kerr-AdS black hole possesses two distinct gyromagnetic ratios. In the special case of two equal angular momenta, the two gyromagnetic ratios merge into one which tends to \( g = 4 \) for the maximally rotating boundary of the spacetime.

II. KERR-ADS BLACK HOLES IN FOUR DIMENSIONS

The exact solution of the Einstein field equations with a cosmological constant that describes rotating black holes in four-dimensional spacetime with asymptotic (anti)-de Sitter behavior was found in \[52\]. The corresponding spacetime metric in the Boyer-Lindquist coordinates has the form

\[
ds^2 = -\frac{\Delta_r}{\Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi} \, d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_{\theta}} d\theta^2 + \frac{\Delta_{\theta} \sin^2 \theta}{\Sigma} \left( a \, dt - \frac{r^2 + a^2}{\Xi} \, d\phi \right)^2 ,
\]

(1)

where

\[
\Delta_r = \left( r^2 + a^2 \right) \left( 1 + \frac{r^2}{l^2} \right) - 2Mr , \quad \Sigma = r^2 + a^2 \cos^2 \theta ,
\]

\[
\Delta_{\theta} = 1 - \frac{a^2}{l^2} \cos^2 \theta , \quad \Xi = 1 - \frac{a^2}{l^2} . \quad (2)
\]

The parameters \( M \) and \( a \) are related to the mass and angular momentum of the black hole, \( l \) is the curvature radius determined by the negative cosmological constant \( \Lambda = -3l^{-2} \). The
metric determinant is given by
\[ \sqrt{-g} = \frac{\Sigma \sin \theta}{\Xi}. \]  
(3)

Clearly, the rotation parameter \( a \) must satisfy the relation \( a^2 < l^2 \), but when approaching the critical limit \( a^2 = l^2 \), the metric becomes singular. In this limit the boundary of AdS spacetime, which is a three-dimensional Einstein universe, rotates at the speed of light.

The stationarity and rotational symmetry properties of the metric (1) imply the existence of two commuting Killing vector fields
\[ \xi(t) = \frac{\partial}{\partial t}, \quad \xi(\phi) = \frac{\partial}{\partial \phi}. \]  
(4)

The various scalar products of these Killing vectors can be expressed through the metric components as follows
\[ \xi(t) \cdot \xi(t) = g_{tt} = -1 + 2 M r \frac{r^2 + a^2 \sin^2 \theta}{\Sigma} - \frac{r^2}{l^2}, \]
\[ \xi(t) \cdot \xi(\phi) = g_{t\phi} = a \frac{\sin^2 \theta}{\Xi} \left( \frac{r^2 + a^2}{l^2} - \frac{2 M r}{\Sigma} \right), \]  
(5)
\[ \xi(\phi) \cdot \xi(\phi) = g_{\phi\phi} = \frac{\sin^2 \theta}{\Xi^2} \left[ \left( r^2 + a^2 \right) \Delta + \frac{2 M r a^2 \sin^2 \theta}{\Sigma} \right]. \]

Another important feature of the Kerr-AdS spacetime becomes transparent when one introduces a family of locally nonrotating observers that move on orbits with constant \( r \) and \( \theta \) and with a four-velocity \( u^\mu \) such that \( u \cdot \xi(\phi) = 0 \). The coordinate angular velocity of these observers is given by
\[ \Omega = - \frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a \Xi \left[ (r^2 + a^2) \Delta - \Delta r \right]}{\Gamma}, \]  
(6)
where
\[ \Gamma = \left( r^2 + a^2 \right)^2 \Delta - \Delta r a^2 \sin^2 \theta. \]  
(7)

It is easy to see that, in contrast to the case of an ordinary Kerr black hole in asymptotically flat spacetime, the angular velocity does not vanish at spatial infinity. Instead, we have the expression
\[ \Omega_\infty = -\frac{a}{l^2}. \]  
(8)
This means that the Kerr-AdS metric (1) is given in a coordinate system which is rotating at spatial infinity. When approaching the horizon of the black hole, \( r \to r_+ (\Delta_r = 0) \), the
angular velocity in (6) tends to its constant value
\[ \Omega_H = \frac{a \Xi}{r_+^2 + a^2} \]  
which can be thought of as the angular velocity of the black hole. This is confirmed by the fact that the co-rotating Killing vector field
\[ \chi = \xi(t) + \Omega_H \xi(\phi) \]  
becomes null at the surface \( \Delta_r = 0 \), i.e. it is tangent to the null surface of the horizon. Clearly, one can also define the angular velocity of the black hole with respect to a frame that is static at infinity. We have
\[ \omega_H = \Omega_H - \Omega_\infty = \frac{a}{r_+^2 + a^2} \left( 1 + \frac{r_+^2}{l^2} \right) . \]  
It turns out that this angular velocity is the most important characteristic of the rotating AdS black holes in the sense that it enters their consistent thermodynamics [31]. On the other hand, it is easy to show that this angular velocity coincides with that of the boundary Einstein universe [29], thereby providing the relevant basis for a CFT dual of the bulk Kerr-AdS black hole.

Next, we calculate the mass and angular momentum of the metric (1). As is known, for black holes in asymptotically flat spacetime these quantities are unambiguously determined using the Komar approach [53]. However, the Komar approach must be used with care for rotating AdS black holes, since the integral for the mass gives a divergent result. Therefore, in order to find a physically meaningful result for the mass one needs to perform a “background subtraction” that may require care as well. With all this in mind, we have the Komar integrals
\[ M' = -\frac{1}{8\pi} \oint *d(\delta \hat{\xi}(t)) , \quad J' = \frac{1}{16\pi} \oint *d(\delta \hat{\xi}(\phi)) , \]  
where the * operator denotes the Hodge dual and the Killing one-form \( \hat{\xi} = \xi_\mu dx^\mu \) is associated with the Killing vectors in (4). In performing the above integrals one must integrate the differences \( \delta \hat{\xi} \) between the Killing isometries of the spacetime under consideration and its reference background. For the reference background, we use the solution (11) with vanishing mass parameter \( (M = 0) \). We note that this procedure does not contribute to the result for the angular momentum as the calculation of the angular momentum is unambiguous in the Komar method.

Using the asymptotic expansions of the integrands in equation (12)
\[ \delta \xi_{r}^{\tau r} = \frac{M}{r^2} + \mathcal{O} \left( \frac{1}{r^4} \right), \]

\[ \delta \xi_{(\phi)}^{r r} = -\frac{3aM \sin^2 \theta}{r^2 \Xi} + \mathcal{O} \left( \frac{1}{r^4} \right), \]  \hspace{1cm} (13)

we perform the integration over a 2-sphere at \( r \to \infty \). This gives

\[ \mathcal{M}' = \frac{M}{\Xi}, \quad J' = \frac{aM}{\Xi^2}, \]  \hspace{1cm} (14)

where \( J' \) is the actual angular momentum of the Kerr-AdS metric. It agrees with the angular momentum obtained earlier in the literature using different approaches \[54, 55\]. However, the expression for \( \mathcal{M}' \) can not be regarded as the actual mass, since it does not satisfy the first law of thermodynamics. This fact was first emphasized in \[35, 36\]. The reason for this is the salient feature of the Kerr-AdS metric (1) that leads to a nonvanishing drag at spatial infinity, see Eq. (8). This fact also means that the timelike Killing vector in (4), that is used in the calculation of the mass, is indeed rotating at infinity. Therefore one must calculate the mass with respect to a new timelike Killing vector which is nonrotating at infinity. This Killing vector is

\[ \partial_t - \frac{a}{l^2} \partial_{\phi}. \]  \hspace{1cm} (15)

It is easy to show that it indeed has the vanishing scalar twist for \( M = 0 \). The calculation of the mass becomes transparent when employing, for instance, the superpotential technique of Katz, Bičák and Lynden-Bell \[56\]. Adapting it to our case we have the integral

\[ K = -\frac{1}{16\pi} \oint *d(\delta \hat{\xi}) - \frac{1}{8\pi} \oint *d(\delta S), \]  \hspace{1cm} (16)

where

\[ S = \frac{1}{2} \xi_{[\mu} k_{\nu]} dx^\mu \wedge dx^\nu, \quad k^\mu = g^{\mu \nu} \delta \Gamma^\lambda_{\nu \lambda} - g^{\alpha \beta} \delta \Gamma^\mu_{\alpha \beta}. \]  \hspace{1cm} (17)

We note that \( \delta \Gamma^\lambda_{\alpha \rho} \) stands for the difference between the Christoffel symbols of the Kerr-AdS spacetime and those of its reference background. The difference between the metric determinants \( \delta g = 0 \). Having performed straightforward evaluation of the above integral with respect to the Killing vector (15) and over a 2-sphere at infinity we obtain the relation

\[ M' = K[\partial_t - \frac{a}{l^2} \partial_{\phi}] = K[\partial_t] - \frac{a}{l^2} K[\partial_{\phi}] \]  \hspace{1cm} (18)
with

\[ K [\partial_t] = \mathcal{M}', \quad K [\partial_{\phi}] = -J', \]

so that the mass of the Kerr-AdS metric (1) is given by

\[ M' = \mathcal{M}' + \frac{a}{l^2} J' = \frac{M}{\Xi^2}. \tag{19} \]

It is important to note that with the angular momentum in (14), only this mass satisfies the first law of thermodynamics [35].

\textbf{A. Electric Charge}

We shall now assume that the Kerr-AdS black holes described above may also possess a small test electric charge. The effect of the electromagnetic field of this charge on the spacetime geometry can be neglected and the spacetime can still be well described by the unperturbed metric (1). In the asymptotically flat case, the associated solution of the source-free Maxwell equations is constructed using the well-known fact that for Ricci-flat metrics a Killing one-form is closed and co-closed [8, 57]. This implies that the Killing one-form can be used as a potential one-form for a test Maxwell field in the Ricci-flat metrics. However, the Kerr-AdS metric under consideration is not Ricci-flat. Fortunately, one can still use the Killing isometries to describe the electromagnetic field of a Kerr-AdS black hole. Namely, it is straightforward to show that the Killing one-form \( \delta \xi_{(t)} \) that represents the difference between the timelike isometries of the Kerr-AdS metric and those of its reference background can be used as a potential one-form for the electromagnetic field. The Killing one-form \( \xi_{(t)} \) is associated with the timelike Killing vector in (4) and, as before, we use the metric (1) with \( M = 0 \) as a reference background. We seek a potential one-form

\[ A = \alpha \delta \xi_{(t)}, \tag{20} \]

where the constant parameter \( \alpha \) is determined from examining the Gauss flux

\[ Q' = \frac{1}{4\pi} \oint \ast F \tag{21} \]

which yields

\[ \alpha = -\frac{Q}{2M}. \tag{22} \]
The desired potential one-form is thus given by

\[ A = -\frac{Q r}{\Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right), \tag{23} \]

where the parameter \( Q \) is related to the electric charge of the black hole by

\[ Q' = \frac{Q}{\Xi}. \tag{24} \]

The associated electromagnetic field two-form is given by

\[ F = \frac{Q (\Sigma - 2r^2)}{\Sigma^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right) \wedge dr + \frac{Q ra \sin 2\theta}{\Sigma^2} \left( a dt - \frac{r^2 + a^2}{\Xi} d\phi \right) \wedge d\theta. \tag{25} \]

For some further purposes, it is also useful to calculate the nonzero contravariant components of the electromagnetic field tensor. We find that

\[ F^{01} = -\frac{Q (r^2 + a^2)}{\Sigma^3} (\Sigma - 2r^2), \quad F^{02} = -\frac{Q a^2 r \sin 2\theta}{\Sigma^3}, \]

\[ F^{13} = \frac{Q a (\Sigma - 2r^2)}{\Sigma^3} \Xi, \quad F^{23} = \frac{2Q ar}{\Sigma^3} \Xi \cot \theta. \tag{26} \]

We recall once again that we have employed here the test-charge approximation that led to the potential one-form given in (23). In the next section we consider the arbitrary values of the electric charge.

### III. KERR-NEWMAN-ADS BLACK HOLES IN FOUR DIMENSIONS

Let us now consider a rotating AdS black hole with an arbitrary electric charge. In this case we need to solve the coupled system of the Einstein-Maxwell equations to construct the spacetime metric with associated electromagnetic fields. It is straightforward to verify that the metric (1) and the potential one-form (23) solve the Einstein-Maxwell equations

\[ R_{\mu \nu} = 2 \left( F_{\mu \lambda} F^{\nu \lambda} - \frac{1}{4} \delta_{\mu}^{\nu} F_{\alpha \beta} F^{\alpha \beta} \right) - 3l^{-2} \delta_{\mu}^{\nu}, \tag{27} \]

\[ \partial_{\nu} (\sqrt{-g} F^{\mu \nu}) = 0, \]

provided that the mass parameter of the metric is re-scaled as

\[ M \rightarrow M - \frac{Q^2}{2r}. \tag{28} \]
Thus, the metric \( (1) \) with the new function
\[
\Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{l^2}\right) - 2Mr + Q^2
\] (29)
goes over into the familiar Kerr-Newman-anti-de Sitter solution \([52]\) in which the electromagnetic field is given by the same potential one-form as \([23]\). Furthermore, it is easy to show that all the salient features of the Kerr-AdS metric given in equations (6) - (11) remain valid for the Kerr-Newman-AdS solution as well.

Next, we describe the horizons of the Kerr-Newman-AdS black holes. The radii of the horizons are determined by the real roots of the equation
\[
\Delta_r = 0.
\] (30)
This is a quartic equation which in general has four roots satisfying Vieta’s relations
\[
\begin{align*}
    r_1 + r_2 + r_3 + r_4 &= 0, \\
    r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= a^2 + l^2, \\
    r_1r_2r_3 + r_1r_2r_4 + r_2r_3r_4 + r_1r_3r_4 &= 2Ml^2, \\
    r_1r_2r_3r_4 &= l^2 \left(a^2 + Q^2\right).
\end{align*}
\] (31)
From these relations we observe that the quartic equation must allow two real and a pair of complex conjugate roots. The largest of the real roots \( r_1 = r_+ \) corresponds to the radius of the black hole’s outer event horizon, while the other real root \( r_2 = r_- \) represents the radius of the inner Cauchy horizon. The real solutions of equation (30) can be written in a compact form using a real root \( u \) of its resolvent cubic equation. For the real root of the resolvent equation we find
\[
u = \frac{l^2 + a^2}{3} + \frac{l^{4/3} \left(M_{1e}^2 - M_{2e}^2\right)^{2/3}}{(2N^2 - M_{1e}^2 - M_{2e}^2)^{1/3}} + l^{4/3} \left(2N^2 - M_{1e}^2 - M_{2e}^2\right)^{1/3},
\] (32)
where we have introduced two extreme mass parameters
\[
M_{1e} = (l/\sqrt{54}) \sqrt{\zeta + \eta^3}, \quad M_{2e} = (l/\sqrt{54}) \sqrt{\zeta - \eta^3}.
\] (33)
Here
\[
\zeta = \left(1 + \frac{a^2}{l^2}\right) \left[\frac{36 (a^2 + Q^2)}{l^2} - \left(1 + \frac{a^2}{l^2}\right)^2\right], \quad \eta = \left[\left(1 + \frac{a^2}{l^2}\right)^2 + \frac{12 (a^2 + Q^2)}{l^2}\right]^{1/2}.
\] (34)
and
\[ N^2 = M^2 + \sqrt{(M^2 - M_{1e}^2)(M^2 - M_{2e}^2)}. \]  
(35)

It is easy to show that only the mass parameter \( M_{1e} \) has a definite physical meaning. For vanishing cosmological constant, \( l \to \infty \), one finds that \( M_{1e}^2 \to a^2 + Q^2 \), while \( M_{2e}^2 \to -\infty \). The expression for the extreme mass \( M_{1e} \) in (33) agrees with that given in Ref. [31]. It is clear that the black hole mass parameter \( M \) must satisfy the relation
\[ M \geq M_{1e}, \]  
(36)
where the equality corresponds to an extreme black hole. The horizons are located at the radii
\[ r_+ = \frac{1}{2}(X + Y), \quad r_- = \frac{1}{2}(X - Y), \]  
(37)
where
\[ X = \sqrt{u - l^2 - a^2}, \quad Y = \sqrt{-u - l^2 - a^2 + \frac{4Ml^2}{X}}. \]  
(38)

For \( a = 0 \) and \( Q = 0 \), the inner horizon disappears, \( r_\pm = 0 \), and \( r_+ \) represents the event horizon of a Schwarzschild-AdS black hole [58]. Expanding the above expressions in powers of \( 1/l \) with \( M/l \ll 1 \), we find
\[ r_+ = \tilde{r}_+ - \frac{\tilde{r}_+^2}{2l^2} \frac{2M\tilde{r}_+ - Q^2}{\tilde{r}_+ - M} + O\left(\frac{1}{l^4}\right), \]  
(39)
\[ r_- = \tilde{r}_- - \frac{\tilde{r}_-^2}{2l^2} \frac{2M\tilde{r}_- - Q^2}{\tilde{r}_- - M} + O\left(\frac{1}{l^4}\right), \]  
(40)
where
\[ \tilde{r}_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \]

We see that the location of the event horizon lies in the range \( r_- < r_+ < \tilde{r}_+ \).

It is also worth emphasizing that for \( M = M_{1e} \) the outer and inner horizons merge to form an extreme black hole located at the radius
\[ r_{eh} = \frac{l}{\sqrt{6}} \left( \eta - 1 - \frac{a^2}{l^2} \right)^{1/2}. \]  
(41)

In the critical limit of rotation in which \( a^2 = l^2 \) and for \( Q = 0 \), from equations (33) and (41) we obtain that
\[ \tilde{r}_{eh} = \frac{3}{8} M = \frac{l}{\sqrt{3}}. \]  
(42)
This gives the limiting size for the horizon of the extreme black hole. For $a^2 < l^2$ we find that $r_{eh} < \tilde{r}_{eh}$. In the critical limit $a^2 = l^2$, the angular velocity in (11) becomes

$$\omega_H = \frac{1}{l}$$

regardless of the horizon size. As we have mentioned above, this is the same as the angular velocity of the Einstein universe on the boundary of AdS spacetime. Thus, the boundary is rotating at the speed of light ($v = \omega l \rightarrow 1$) when $a^2 \rightarrow l^2$.

A. Gyromagnetic Ratio

An important characteristic of the Kerr-Newman-AdS black hole is its gyromagnetic ratio. We recall that one of the remarkable facts about a Kerr-Newman black hole in asymptotically flat spacetime is that it can be assigned a gyromagnetic ratio $g = 2$, just as an electron in the Dirac theory \[42\]. The parameter $g$ is defined as a constant of proportionality in the equation for the magnetic dipole moment

$$\mu = g \frac{Q J}{2 M},$$

where $M$ is the mass, $J$ is the angular momentum and $Q$ is the electric charge of the Kerr-Newman black hole. Here we wish to calculate the value of the gyromagnetic ratio when the black hole has an asymptotic AdS behavior. We begin with the associated magnetic dipole moment. The most direct way to determine it is to examine the asymptotic behavior of the magnetic field generated by a Kerr-Newman-AdS black hole. For this purpose, it is useful to introduce an orthonormal tetrad frame which is given by the basis one-forms

$$e^0 = \left( \frac{\Delta r}{\Sigma} \right)^{1/2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right),$$

$$e^3 = \left( \frac{\Delta \theta}{\Sigma} \right)^{1/2} \sin \theta \left( a dt - \frac{r^2 + a^2}{\Xi} d\phi \right),$$

$$e^1 = \left( \frac{\Sigma}{\Delta r} \right)^{1/2} dr, \quad e^2 = \left( \frac{\Xi}{\Sigma} \right)^{1/2} d\theta.$$

The remarkable property of this frame is that an observer at rest in it measures only the radial components of the electric and magnetic fields \[42\]. Writing the electromagnetic two-form \[25\] in this frame and using the relation \[24\], we find the following asymptotic
expansions for the radial fields

\[ E_\varphi = \frac{Q'\Xi}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \]  

\[ B_\varphi = \frac{2Q'a\Xi}{r^3} \cos\theta + \mathcal{O}\left(\frac{1}{r^5}\right). \]

It is easy to check that the Gaussian flux of the radial electric field gives the correct value for the electric charge of the black hole, see Eq. (24). The second equation for the dominant behavior of the radial magnetic field shows that the black hole can be assigned a magnetic dipole moment given by

\[ \mu' = Q'a = \frac{\mu}{\Xi}, \]  

where \( \mu = Qa \) is the magnetic dipole moment parameter. Defining now the \( g \)-factor in terms of the actual mass, angular momentum and electric charge, we have a relation similar to (44); that is,

\[ \mu' = g Q' J' \frac{2M'}{M'}. \]

It follows that the Kerr-Newman-AdS black holes must have a gyromagnetic ratio corresponding to \( g = 2 \) just as the usual Kerr-Newman black holes in asymptotically flat spacetime. We recall that for Kerr-Newman black holes in de Sitter space one must take \( l^2 \to -l^2 \), which does not affect the value of the gyromagnetic ratio.

IV. HIGHER DIMENSIONAL CHARGED KERR-ADS BLACK HOLES

The higher dimensional generalization of the Kerr-Newman-AdS solution has not yet been found. However, one can still examine some electromagnetic properties of rotating AdS black holes in higher dimensions employing the test-charge approach described in Sec. II. We therefore consider a weakly charged black hole and use the spacetime geometry described by the higher-dimensional Kerr-AdS solution found in Refs. \[29, 59\]. We first focus on the Kerr-AdS metric with a single angular momentum in \( N + 1 \) dimensions with \( N \geq 3 \). In the Boyer-Lindquist type coordinates it is given by
\[
\begin{align*}
    ds^2 &= -\frac{\Delta_r}{\Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 \\
    &\quad + \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left( a dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2 + r^2 \cos^2 \theta d\Omega_{N-3}^2.
\end{align*}
\] (50)

This metric satisfies the Einstein field equations with the cosmological term, \( R_{\mu\nu} = -Nl^{-2} g_{\mu\nu} \). The curvature radius \( l \) of the AdS space is related to the negative cosmological constant by \( \Lambda = -\frac{1}{2} N(N - 1) l^{-2} \), the metric functions are the same as in (2) except for

\[
    \Delta_r = \left( r^2 + a^2 \right) \left( 1 + \frac{r^2}{l^2} \right) - mr^{4-N},
\] (51)

and

\[
    d\Omega_{N-3}^2 = d\chi_1^2 + \sin^2 \chi_1 \left( d\chi_2^2 + \sin^2 \chi_2 \left( ... d\chi_{N-3}^2 ... \right) \right),
\] (52)

which is the metric on a unit \((N-3)\)-sphere. Here \( m \) is the mass parameter that reduces to \( 2M \) for \( N = 3 \). For the determinant of the metric (50) we find

\[
    \sqrt{-g} = \frac{\Sigma \sin \theta}{\Xi} \sqrt{\gamma} r^{N-3} \cos^{N-3} \theta,
\] (53)

where \( \gamma \) stands for the determinant of the metric (52). Clearly, the metric (50) admits the same timelike and rotational Killing vectors as those given in (4). In a similar way to (6), one can define for the locally nonrotating observers an angular velocity that at the horizon of the black hole reduces to

\[
    \Omega_H = \frac{a \Xi}{r_+^2 + a^2},
\] (54)

where \( r_+ \) is the radius of the horizon, i.e. the largest root of equation \( \Delta_r = 0 \).

Next, we evaluate the mass and angular momentum of the metric (50). Following the four-dimensional case in Sec. II, we first employ the Komar integrals which in \( N + 1 \) dimensions have the form

\[
    \mathcal{M}' = -\frac{1}{16\pi} \frac{N-1}{N-2} \oint *d(\delta \hat{\xi}(\ell)) , \quad J' = \frac{1}{16\pi} \oint *d(\delta \hat{\xi}(\phi)) .
\] (55)

We recall that in performing the above integrals one must again integrate the differences \( \delta \hat{\xi} \) between the Killing isometries of the metric under consideration and its reference background. The latter is given by the metric (50) with vanishing mass parameter \((m = 0)\).
Substituting into these integrals the asymptotic expansions
\[ \delta \xi^{t,r}_{(t)} = \frac{m(N-2)}{2r^{N-1}} + O\left(\frac{1}{r^{N+1}}\right), \]
\[ \delta \xi^{t,r}_{(\phi)} = -\frac{amN}{\Xi} \frac{\sin^2 \theta}{2r^{N-1}} + O\left(\frac{1}{r^{N+1}}\right). \] (56)
and performing integration over a \((N-1)\)-sphere at \(r \to \infty\) we obtain
\[ \mathcal{M}' = \frac{m(N-1)A_{N-1}}{16\pi \Xi}, \quad J' = \frac{amA_{N-1}}{8\pi \Xi^2}, \] (57)
where \(A_{N-1} = 2^{N/2} \frac{\pi^{N/2}}{\Gamma(N/2)}\) is the area of a unit \((N-1)\)-sphere. We note that the obtained value for the angular momentum can be regarded as the actual angular momentum for the Kerr-AdS black holes. This is unambiguously confirmed in the framework of various methods [29, 35, 60]. However, with this angular momentum the obtained mass does not satisfy the first law of thermodynamics. As in the four-dimensional case, the consistent mass can be found if one uses a new timelike Killing vector instead of \(\partial_t\) that is nonrotating at \(m = 0\). We take the new Killing vector in the same form as that given in (15) and again employ the superpotential technique of Katz, Bičák and Lynden-Bell [56]. A straightforward calculation of the integral (16) over a \((N-1)\)-sphere at infinity shows that the relation in (18) holds in the higher-dimensional case as well. Thus, for the actual mass we have the relation
\[ M' = \mathcal{M}' + \frac{a}{l^2} J'. \] (58)
Finally, we find that the mass and angular momentum of the Kerr-AdS metric are given by the expressions
\[ M' = \frac{m'A_{N-1}}{16\pi} \left[2 + (N-3) \Xi\right], \quad J' = \frac{j'A_{N-1}}{8\pi}, \] (59)
where we have defined the specific mass and angular momentum as
\[ m' = \frac{m}{\Xi^2}, \quad j' = \frac{am}{\Xi^2}, \] (60)
which are reminiscent of the corresponding relations for the mass and angular momentum of the Kerr-AdS metric in four dimensions. We note that the expression for the mass in (59) agrees with that appearing in [35] when adapting the latter to the case of a single angular momentum.
We turn now to the description of the electromagnetic field generated by a test electric charge of a higher-dimensional Kerr-AdS black hole. By the same arguments as those given in Sec. II, one can construct the vector potential of the electromagnetic field using the difference between the timelike generators in the metric (50) and in its reference background. This gives rise to the potential one-form

\[ A = - \frac{Q r^{4-N}}{(N-2) \Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right). \]  

(61)

The parameter \( Q \) is related to the electric charge of the black hole by Gauss’s law in \( N + 1 \) dimensions

\[ Q' = \frac{1}{A_{N-1}} \oint \star F. \]  

(62)

For the electric charge, we find the same relation as that given in (24); that is,

\[ Q' = \frac{Q}{\Xi}. \]  

(63)

The basis one-forms for the metric (50) can be chosen as

\[ e^0 = \left( \frac{\Sigma}{\Delta r} \right)^{1/2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right), \]

\[ e^1 = \left( \frac{\Sigma}{\Delta r} \right)^{1/2} dr, \quad e^2 = \left( \frac{\Sigma}{\Delta \theta} \right)^{1/2} d\theta, \]  

(64)

\[ e^3 = \left( \frac{\Delta \theta}{\Sigma} \right)^{1/2} \sin \theta \left( a dt - \frac{r^2 + a^2}{\Xi} d\phi \right), \]

\[ e^4 = r \cos \theta d\chi_1, \quad e^5 = r \cos \theta \sin \chi_1 d\chi_2 \]

and so on. As in the four-dimensional case, the significance of these basis one-forms is that they define a natural orthonormal frame in which the electromagnetic field tensor takes its simplest form. It is straightforward to show that the electromagnetic field two-form written in terms of basis one-forms (64) is given by

\[ F = - \frac{Q r^{3-N}}{(N-2) \Sigma^2} \left\{ \left[ (N-2) \Sigma - 2 a^2 \cos^2 \theta \right] e^0 \wedge e^1 - 2 a r \cos \theta e^3 \wedge e^2 \right\}, \]  

(65)

which involves the radial component of the electric field and only one component of the magnetic field. The dominant behavior of these fields at spatial infinity is given by
Again, the Gauss flux of the radial electric field confirms the value of the electric charge in (63). We see that the magnetic field is determined by the quantity
\[ \mu' = \frac{J'Q'}{m'} = \frac{aQ}{\Xi}, \]
which can be thought of as the magnetic dipole moment of the black hole. Using the expressions in (59) we can rewrite the magnetic dipole moment in the form
\[ \mu' = \left[ 2 + (N - 3) \Xi \right] \frac{J'Q'}{2M'}. \]
Comparing now this expression with that given in (49) we read off the value of the gyromagnetic ratio
\[ g = 2 + (N - 3) \Xi \]
for the Kerr-AdS black holes carrying a test electric charge and a single angular momentum in all higher dimensions. For \( N = 3 \), this expression shows that \( g = 2 \) is a universal feature of four dimensions. For vanishing cosmological constant, \( \Lambda \to \infty \), it recovers the value of the gyromagnetic ratio found for weakly charged Myers-Perry metrics [8, 9]. However, the most striking feature of the expression (70) appears in the critical limit \( \Xi \to 0 \), in which the boundary Einstein universe rotates at the speed of light. We see that \( g \to 2 \) irrespective of the spacetime dimension. It is interesting to note that in a recent work, Ref. [51], it has been argued that at the critical limit of rotation the Kerr-AdS black holes are related to SUSY configurations. That is, a supersymmetric black hole in an AdS background has the same angular velocity with respect to a frame that is nonrotating at infinity as that given in (43). One may therefore conclude that the supersymmetric black hole must have the gyromagnetic ratio \( g = 2 \).

A. Alternative calculations

The value of the gyromagnetic ratio found above can be proved by alternative calculations using a different approach. For this purpose, we define the twist of a Killing one-form that
is associated with the timelike Killing vector $\partial_t$ of the metric \( (50) \). This is given by the \((N - 2)\)-form

$$\hat{\omega}_{N-2} = \frac{1}{(N - 2)} \left( \hat{\xi}_t \wedge d\hat{\xi}_t \right) .$$

(71)

Physically, this quantity measures the failure of the Killing vector to be hypersurface orthogonal. For the metric \( (50) \), we find

$$\hat{\omega}_{N-2} = \frac{am \cos^{N-3} \theta}{N - 2} \left\{ \frac{(N - 2)\Sigma - 2a^2 \cos^2 \theta}{\Sigma^2} + \frac{2r^{N-2}}{ml^2} \right\} \sin \theta d\theta + 2 \cos \theta \left( \frac{r}{\Sigma^2} - \frac{r^{N-3}}{ml^2} \right) dr \wedge d\Sigma_{N-3} .$$

(72)

where

$$d\Sigma_{N-3} = \frac{\sqrt{\gamma}}{(N - 3)!} \epsilon_{i_1 i_2 \ldots i_{N-3}} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{N-3}} .$$

(73)

It is straightforward to show that the twist form is closed, \( d\hat{\omega}_{N-2} = 0 \), implying the existence (locally) of the twist potential \((N - 3)\)-form

$$\hat{\Omega}_{N-3} = \frac{a \cos^{N-2} \theta}{N - 2} \left( \frac{m}{\Sigma} + \frac{2}{l^2} \frac{r^{N-2}}{N - 2} \right) d\Sigma_{N-3} .$$

(74)

We see that this quantity is not zero for vanishing mass parameter \( m \rightarrow 0 \), reflecting the fact that the associated background spacetime is indeed rotating at infinity. Performing in (74) the background subtraction, we obtain the “physical” twist

$$\delta\hat{\Omega}_{N-3} = \frac{a}{\Sigma} \frac{m \cos^{N-2} \theta}{N - 2} d\Sigma_{N-3} .$$

(75)

Next, we define the magnetic field \((N - 2)\)-form

$$\hat{B}_{N-2} = i_{\hat{\xi}_t} \star F = \star \left( \hat{\xi}_t \wedge F \right) ,$$

(76)

which in the Kerr-AdS spacetime under consideration has the form

$$\hat{B}_{N-2} = \frac{aQ \cos^{N-3} \theta}{\Sigma^2} \left\{ \left( (N - 2)\Sigma - 2a^2 \cos^2 \theta \right) \sin \theta d\theta + 2r \cos \theta dr \right\} \wedge d\Sigma_{N-3} .$$

(77)

It is easy to verify that one can also introduce the magnetic potential \((N - 3)\)-form by the equation

$$\hat{B}_{N-2} = -d\varphi_{N-3} ,$$

(78)

where

$$\varphi_{N-3} = \frac{aQ \cos^{N-2} \theta}{\Sigma} \frac{m}{N - 2} d\Sigma_{N-3} .$$

(79)
From a comparison of this expression with (75), it follows that the asymptotic behavior of the magnetic potential \((N - 3)\)-form determines the magnetic moment parameter \(\mu = Qa\), just as the twist potential \((N - 3)\)-form determines the specific angular momentum parameter \(j = am\). This fact can also be expressed in the form

\[
\varphi_{N-3} = \frac{Q}{m} \delta \Omega_{N-3},
\]

which is equivalent to the relation (68). This proves the value of the gyromagnetic ratio in (70).

V. GENERAL KERR-ADS BLACK HOLES IN FIVE DIMENSIONS

The general metric for Kerr-AdS black holes with two independent rotation parameters in five dimensions was first given in [29]. The simplest form of the metric is given by

\[
ds^2 = -\frac{\Delta_r}{r^2 \Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right)^2 + \Sigma \left( \frac{r^2}{\Delta_r} dr^2 + \frac{d\theta^2}{\Delta_\theta} \right)
+ \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left( a dt - \frac{r^2 + a^2}{\Xi_a} d\phi \right)^2 + \frac{\Delta_\theta \cos^2 \theta}{\Sigma} \left( b dt - \frac{r^2 + b^2}{\Xi_b} d\psi \right)^2
+ \frac{1 + r^2 l^{-2}}{r^2 \Sigma} \left( a b dt - \frac{b(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi - \frac{a(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\psi \right)^2,
\] (81)

where

\[
\Delta_r = (r^2 + a^2) (r^2 + b^2) \left( 1 + \frac{r^2}{l^2} \right) - m r^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta,
\]

\[
\Delta_\theta = 1 - \frac{a^2}{l^2} \cos^2 \theta - \frac{b^2}{l^2} \sin^2 \theta, \quad \Xi_a = 1 - \frac{a^2}{l^2}, \quad \Xi_b = 1 - \frac{b^2}{l^2}
\] (82)

and \(a\) and \(b\) are two independent rotation parameters. The metric determinant is given by

\[
\sqrt{-g} = \frac{r \Sigma \sin \theta \cos \theta}{\Xi_a \Xi_b}.
\]

(83)

It is also clear that the metric admits three commuting Killing vector fields

\[
\xi_{(t)} = \frac{\partial}{\partial t}, \quad \xi_{(\phi)} = \frac{\partial}{\partial \phi}, \quad \xi_{(\psi)} = \frac{\partial}{\partial \psi}.
\]

(84)
which reflect stationarity and bi-azimuthal symmetry of this spacetime. The various scalar
products of these Killing vectors are given by

\[ \xi(t) \cdot \xi(t) = g_{tt} = -1 + \frac{m}{\Sigma} - \frac{r^2 + a^2 \sin^2 \theta + b^2 \cos^2 \theta}{l^2}, \]

\[ \xi(\phi) \cdot \xi(\phi) = g_{\phi\phi} = \frac{\sin^2 \theta}{\Xi_a} \left[ (r^2 + a^2) \Xi_a + \frac{m a^2 \sin^2 \theta}{\Sigma} \right], \]

\[ \xi(\psi) \cdot \xi(\psi) = g_{\psi\psi} = \frac{\cos^2 \theta}{\Xi_b} \left[ (r^2 + b^2) \Xi_b + \frac{m b^2 \cos^2 \theta}{\Sigma} \right], \quad (85) \]

\[ \xi(t) \cdot \xi(\phi) = g_{t\phi} = -\frac{a \sin^2 \theta}{\Xi_a} \left( \frac{m}{\Sigma} - \frac{r^2 + a^2}{l^2} \right), \]

\[ \xi(t) \cdot \xi(\psi) = g_{t\psi} = -\frac{b \cos^2 \theta}{\Xi_b} \left( \frac{m}{\Sigma} - \frac{r^2 + b^2}{l^2} \right), \]

\[ \xi(\phi) \cdot \xi(\psi) = g_{\phi\psi} = \frac{m a b \sin^2 \theta \cos^2 \theta}{\Sigma \Xi_a \Xi_b}. \]

It proves useful to define a family of locally nonrotating observers. We recall that in four
dimensions a locally nonrotating observer has a vector of four-velocity that is orthogonal
to the axial Killing vector. Similarly, in the five-dimensional Kerr-AdS metric we can also
define a unit vector of five-velocity for a locally nonrotating observer. It is given by

\[ u^\mu = u^\mu(r, \theta) = C \left( \xi_t^{\mu} + \Omega_a \xi_{\phi}^{\mu} + \Omega_b \xi_{\psi}^{\mu} \right), \quad (86) \]

where the parameter \( C \) is fixed by the normalization condition \( u^2 = -1 \). By analogy, we
require that \( u \cdot \xi(\phi) = 0 \) and \( u \cdot \xi(\psi) = 0 \); that is,

\[ g_{t\phi} u^t + g_{\phi\phi} u^\phi + g_{\phi\psi} u^\psi = 0, \quad (87) \]

\[ g_{t\psi} u^t + g_{\psi\phi} u^\phi + g_{\psi\psi} u^\psi = 0. \quad (88) \]

These equations determine the coordinate angular velocities of the observers. Straightforward
calculations show that they are given by

\[ \Omega_a = \frac{u^\phi}{u^t} = \frac{g_{t\psi} g_{\phi\psi} - g_{t\phi} g_{\psi\psi}}{g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2} = \frac{a \Xi_a \left[ m (r^2 + b^2) \Delta_\theta - l^2 \Delta_\gamma \Xi_a \right]}{m (r^2 + a^2) (r^2 + b^2) \Delta_\theta + \Delta_\gamma \Xi_a \Xi_b}, \quad (89) \]

\[ \Omega_b = \frac{u^\psi}{u^t} = \frac{g_{t\phi} g_{\phi\psi} - g_{t\psi} g_{\phi\phi}}{g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2} = \frac{b \Xi_b \left[ m (r^2 + a^2) \Delta_\theta - l^2 \Delta_\gamma \Xi_a \right]}{m (r^2 + a^2) (r^2 + b^2) \Delta_\theta + \Delta_\gamma \Xi_a \Xi_b}. \quad (90) \]
For vanishing cosmological constant, \( l \to \infty \), these expressions reduce to those obtained in [8]. Far from the black hole we have

\[
\Omega_a = -\frac{a}{l^2} + \frac{am}{r^4} \frac{\Delta_\theta}{\Xi_a \Xi_b} + \mathcal{O} \left( \frac{1}{r^6} \right), \quad \Omega_b = -\frac{b}{l^2} + \frac{bm}{r^4} \frac{\Delta_\theta}{\Xi_a \Xi_b} + \mathcal{O} \left( \frac{1}{r^6} \right). \tag{91}
\]

We see that the angular velocities of the locally nonrotating observers in both \( \phi \) and \( \psi \) 2-planes of rotation do not vanish at spatial infinity; the bi-dragging of inertial frames occurs at spatial infinity as well. When approaching the black hole horizon, \( \Delta_r \to 0 \), it follows from equations (89) and (90) that

\[
\Omega_a^{(+)} = \frac{a \Xi_a}{r^2_+ + a^2}, \quad \Omega_b^{(+)} = \frac{b \Xi_b}{r^2_+ + b^2}, \tag{92}
\]

where \( r_+ \) is the radius of the outer event horizon. These quantities can be interpreted as angular velocities of the event horizon in two independent orthogonal 2-planes of rotation [29]. One can also verify that the Killing vector

\[
\chi = \xi_t + \Omega_{a(+)} \xi_\phi + \Omega_{b(+)} \xi_\psi, \tag{93}
\]

which is a linear combination of the Killing vectors in (84), correctly describes the isometry properties of the horizon geometry. It becomes tangent to the null surface of the event horizon, thereby showing that the Killing horizon is the same as the event horizon of the five-dimensional Kerr-AdS metric.

The Komar mass and angular momenta of the metric (81) are obtained by employing the integrals (55) in the five-dimensional case. We find the expressions

\[
\mathcal{M}' = \frac{3\pi m}{8 \Xi_a \Xi_b}, \quad J'_a = \frac{\pi am}{4 \Xi_a^2 \Xi_b}, \quad J'_b = \frac{\pi bm}{4 \Xi_a \Xi_b^2}. \tag{94}
\]

These are easily verified by using the asymptotic expansions

\[
\delta \xi_t^{(\phi)} = \frac{m}{r^3} + \mathcal{O} \left( \frac{1}{r^5} \right),
\]

\[
\delta \xi_t^{(\psi)} = -\frac{2am \sin^2 \theta}{\Xi_a} \frac{1}{r^3} + \mathcal{O} \left( \frac{1}{r^5} \right),
\]

\[
\delta \xi_t^{(\psi)} = -\frac{2bm \cos^2 \theta}{\Xi_b} \frac{1}{r^3} + \mathcal{O} \left( \frac{1}{r^5} \right) \tag{95}
\]

and performing integration over a 3-sphere at spatial infinity. As in the previous cases, the Komar expressions for the angular momenta are unambiguous. Again, with these angular
momenta the expression for the mass does not agree with the first law of thermodynamics \[35\]. The agreement is achieved for a mass that is associated with the timelike Killing vector

\[ \partial_t - \frac{a}{l^2} \partial_\phi - \frac{b}{l^2} \partial_\psi , \]

which has a vanishing twist in the \( m = 0 \) reference background. Evaluating the integrals in \[16\] with respect to this Killing vector, we obtain the expression

\[ M' = K \left[ \partial_t - \frac{a}{l^2} \partial_\phi - \frac{b}{l^2} \partial_\psi \right] = K [\partial_t] - \frac{a}{l^2} K [\partial_\phi] - \frac{b}{l^2} K [\partial_\psi] , \]

where

\[ K [\partial_t] = M' , \quad K [\partial_\phi] = -J'_a , \quad K [\partial_\psi] = -J'_b . \]

Using the expressions in \[94\), we obtain the desired actual mass of the five dimensional Kerr-AdS metric

\[ M' = \frac{\pi m (2 \Xi_a + 2 \Xi_b - \Xi_a \Xi_b)}{8 \Xi_a^2 \Xi_b^2} . \]

This formula is in agreement with that obtained in \[35\] from purely thermodynamic considerations.

We shall now describe the electromagnetic field that is generated by a test electric charge of the Kerr-AdS black hole. Again, the corresponding vector potential can be constructed using the timelike isometries of the metric \(50\) and those of its \( m = 0 \) reference background. Using the expressions \(20\) and \(62\) in the case under consideration, we find that the potential one-form is given by

\[ A = -\frac{Q}{2 \Sigma} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right) , \]

where \( Q \) is determined by the electric charge of the black hole through the relation

\[ Q' = \frac{Q}{\Xi_a \Xi_b} . \]

For the electromagnetic field two-form we find

\[ F = -\frac{Qr}{\Sigma^2} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right) \wedge dr + \frac{Qa \sin 2\theta}{2\Sigma^2} \left( a dt - \frac{r^2 + a^2}{\Xi_a} d\phi \right) \wedge d\theta - \frac{Qb \sin 2\theta}{2\Sigma^2} \left( b dt - \frac{r^2 + b^2}{\Xi_b} d\psi \right) \wedge d\theta . \]
For some purposes, it is also useful to know the nonzero contravariant components of the electromagnetic field tensor. They are given by

\[
F^{tr} = \frac{Q (r^2 + a^2) (r^2 + b^2)}{r \Sigma^3}, \quad F^{t\theta} = -\frac{Q (a^2 - b^2) \sin 2\theta}{2 \Sigma^3},
\]

\[
F^{r\phi} = -\frac{Q a \Xi_a (r^2 + b^2)}{r \Sigma^3}, \quad F^{r\psi} = -\frac{Q b \Xi_b (r^2 + a^2)}{r \Sigma^3},
\]

\[
F^{\theta\phi} = \frac{Q a \Xi_a \cot \theta}{\Sigma^3}, \quad F^{\theta\psi} = -\frac{Q b \Xi_b \tan \theta}{\Sigma^3}.
\]

It is clear that a five-dimensional charged Kerr-AdS black hole must have two independent magnetic dipole moments due to its rotations in two independent orthogonal 2-planes. To determine the value of these magnetic dipole moments, we first need to generalize the Carter frame given in (45) to include the case of the five-dimensional metric (81). Rather lengthy calculations show that the desired frame is given by the basis one-forms

\[
e^0 = \frac{1}{r} \left( \frac{\Delta r}{\Sigma} \right)^{1/2} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right),
\]

\[
e^1 = r \left( \frac{\Sigma}{\Delta r} \right)^{1/2} dr, \quad e^2 = \left( \frac{\Sigma}{\Delta \theta} \right)^{1/2} d\theta,
\]

\[
e^3 = \frac{\sin \theta}{r \Sigma^{1/2}} \left\{ \left[ B - b^2 \cos^2 \theta \left( 1 + \frac{r^2}{l^2} \right) \frac{A}{Z} \right] \left( a dt - \frac{r^2 + a^2}{\Xi_a} d\phi \right)
\]

\[
+ ab \cos^2 \theta \left( 1 + \frac{r^2}{l^2} \right) \frac{A}{Z} \left( b dt - \frac{r^2 + b^2}{\Xi_b} d\psi \right) \right\},
\]

\[
e^4 = \frac{\cos \theta}{r \Sigma^{1/2}} \left\{ \left[ A - a^2 \sin^2 \theta \left( 1 + \frac{r^2}{l^2} \right) \frac{B}{Z} \right] \left( b dt - \frac{r^2 + b^2}{\Xi_b} d\psi \right)
\]

\[
+ ab \sin^2 \theta \left( 1 + \frac{r^2}{l^2} \right) \frac{B}{Z} \left( a dt - \frac{r^2 + a^2}{\Xi_a} d\phi \right) \right\},
\]

where

\[
A = \left[ \Delta \theta r^2 + a^2 \left( 1 + \frac{r^2}{l^2} \right) \right]^{1/2}, \quad B = \left[ \Delta \theta r^2 + b^2 \left( 1 + \frac{r^2}{l^2} \right) \right]^{1/2},
\]

\[
Z = r \Sigma^{1/2} \Delta \theta^{1/2} + AB.
\]

In this frame, the electromagnetic field two-form (101) becomes

\[
F = -\frac{Q r}{\Sigma^2} e^0 \wedge e^1 + \frac{Q a \cos \theta}{\Sigma^2} f e^3 \wedge e^2 - \frac{Q b \sin \theta}{\Sigma^2} h e^4 \wedge e^2,
\]
where

\[ f = \frac{r \Delta \theta^{1/2} A + \Sigma^{1/2} B}{Z}, \quad h = \frac{r \Delta \theta^{1/2} B + \Sigma^{1/2} A}{Z}. \]  \hspace{1cm} (106)

In the asymptotic region, \( r \to \infty \), and for \( a = 0 \) or \( b = 0 \) as well as in the special case \( a = b \), the functions \( f \) and \( h \) tend to unity, while for arbitrary values of the rotation parameters they depend on the angle \( \theta \) and the dimensionless ratios of the rotation parameters to the curvature radius of the AdS background. Thus, from the asymptotic behavior of the electromagnetic field two-form in (105), it follows that the Kerr-AdS black hole can be assigned two distinct magnetic dipole moments

\[ \mu'_{(a)} = \frac{Q_a}{\Xi_a \Xi_b}, \quad \mu'_{(b)} = \frac{Q_b}{\Xi_a \Xi_b}, \]  \hspace{1cm} (107)

where we have used relation (100). For vanishing cosmological constant these expressions are in agreement with those obtained in [8, 9]. Using expressions in (94) and (98), it is easy to see that the magnetic dipole moments are expressed in terms of the mass and angular momenta of the black hole as follows

\[ \mu'_{(a)} = \left( 2 - \Xi_a + 2 \frac{\Xi_a}{\Xi_b} \right) \frac{J'_a Q'_a}{2 M'}, \quad \mu'_{(b)} = \left( 2 - \Xi_b + 2 \frac{\Xi_b}{\Xi_a} \right) \frac{J'_b Q'_b}{2 M'}. \]  \hspace{1cm} (108)

Comparing these expressions with the definition of the gyromagnetic ratio in (49), we find the two distinct gyromagnetic ratios

\[ g_{(a)} = 2 - \Xi_a + 2 \frac{\Xi_a}{\Xi_b}, \quad g_{(b)} = 2 - \Xi_b + 2 \frac{\Xi_b}{\Xi_a}. \]  \hspace{1cm} (109)

in accordance with two independent rotations of the Kerr-AdS black hole in five dimensions. First of all, we note that for zero cosmological constant, \( \Xi_a \to 1 \) and \( \Xi_b \to 1 \), the two gyromagnetic ratios merge into one value \( g = 3 \) [8]. For a single angular momentum, \( a = 0 \) or \( b = 0 \), we have the gyromagnetic ratio corresponding to that given by (70). In the special case with two equal angular momenta, the gyromagnetic ratios again merge leading to the value

\[ g = 4 - \Xi. \]  \hspace{1cm} (110)

It is interesting to note that in the critical limit \( \Xi \to 0 \), where the Einstein universe rotates at the speed of light, the gyromagnetic ratio tends \( g = 4 \), in contrast to the case of a single angular momentum for which \( g \to 2 \) (see Sec. IV).
VI. CONCLUSION

In this paper we have discussed the electromagnetic properties of rotating charged black holes in anti-de Sitter spacetime in four and higher dimensions focusing basically on test electromagnetic fields. The test field approach has enabled us to employ an elegant way of constructing the associated solutions of the source-free Maxwell field equations. Namely, we have shown that the difference between the timelike generators in the Kerr-AdS spacetime and in the corresponding reference background can be taken as a potential one-form for the electromagnetic field. In four dimensions, the test potential one-form is the same as that describing the full electromagnetic field of the familiar Kerr-Newman-AdS black hole. We have also solved the quartic equation determining the positions of horizons in the Kerr-Newman-AdS metric and found analytic formulas for the radii of the horizons. The gyromagnetic ratio for these black holes turns out to be $g = 2$, the same value as for asymptotically flat black holes.

Turning to the case of Kerr-AdS black holes carrying a single angular momentum and a test electric charge in all higher dimensions, we have re-derived the expressions for the mass and angular momentum that are consistent with the first law of thermodynamics. Exploring then the asymptotic behavior of the electromagnetic fields of these black holes we have determined their gyromagnetic ratio. The use of thermodynamically consistent expressions for the mass and angular momentum has led to the value of the gyromagnetic ratio that crucially depends on the dimensionless ratio of the rotation parameter to the curvature radius of the AdS background. The striking feature of this dependence appears for maximally rotating black holes for which the gyromagnetic ratio approaches $g = 2$ regardless of the spacetime dimension. In this case the boundary of the AdS spacetime is rotating at the speed of light.

Finally, we have examined a general five-dimensional Kerr-AdS black hole involving two independent angular momenta. We have given the precise expressions for the angular velocities of locally nonrotating observers in the spacetime and re-derived its thermodynamically consistent mass and angular momenta. Moreover, we have derived the potential one-form that describes the electromagnetic field generated by a test electric charge of the black hole. For the five-dimensional Kerr-AdS metric we have also defined a natural orthonormal frame in which the electromagnetic field two-form takes its simplest form. This is a generalization
of the familiar Carter frame in four-dimensional Kerr-Newman metric. As expected, a five-dimensional charged Kerr-AdS black hole possesses two distinct magnetic dipole moments due to its rotation in two orthogonal 2-planes. We have shown that the black hole has two distinct gyromagnetic ratios, unlike its asymptotically flat counterpart for which the $g$-factor is always equal to 3. The gyromagnetic ratios merge in the special case of two equal angular momenta. Furthermore, in this case the gyromagnetic ratio tends to $g = 4$ for the maximum values of the angular velocities, in contrast to the single angular momentum case where $g = 2$.

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