Rota-Baxter operators and related structures on anti-flexible algebras

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ABSTRACT

In this paper, we first construct a graded Lie algebra which characterizes Rota-Baxter operators on an anti-flexible algebra as Maurer-Cartan elements. Next, we study infinitesimal deformations of bimodules over anti-flexible algebras. We also consider compatible Rota-Baxter operators on bimodules over anti-flexible algebras. Finally, We define \(\mathcal{ON}\)-structures which give rise to compatible Rota-Baxter operators and vice-versa.

Key words: Anti-flexible algebra, Rota-Baxter operator, Infinitesimal deformation, Nijenhuis operator, \(\mathcal{ON}\)-structure.

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1 Introduction

Flexible algebra was introduced by Oehmke \[16\] as a natural generalization of associative algebras. Let \(A\) be a vector space over a field \(K\) equipped with a bilinear product \(A \times A \to A, \quad (a, b) \mapsto ab\). We denote the associator of \(A\) as

\[(a, b, c) = (ab)c - a(bc), \quad \text{for all } a, b, c \in A.\]

\(A\) is called a flexible algebra if the following identity is satisfied

\[(a, b, a) = 0, \quad \text{or equivalently, } (ab)a = a(ba), \quad \text{for all } a, b \in A.\]

Anti-flexible algebras are a natural generalization of flexible algebras introduced by Rodabaugh in \[18\]. Rodabaugh studied anti-flexible algebras in more detail in \[19\], \[20\], \[21\].
Let $A$ be a vector space equipped with a bilinear product $(a, b) \mapsto a \cdot b$. $A$ is called an anti-flexible algebra if the following identity is satisfied

$$(a, b, c) = (c, b, a), \text{ or equivalently, } (a \cdot b) \cdot c - a \cdot (b \cdot c) = (c \cdot b) \cdot a - c \cdot (b \cdot a), \text{ for all } a, b, c \in A.$$ 

The notion of anti-flexible bialgebras was studied in [6]. Goze and Remm [10] constructed a graded Lie algebra structure on the graded space of all multilinear maps on a vector space, and studied cohomology and the deformation of anti-flexible algebras. In this paper, we construct a graded Lie algebra structure that characterizes Rota-Baxter operators on an anti-flexible algebra as Maurer-Cartan elements.

The deformation of algebraic structures began with the seminal work of Gerstenhaber [8, 9] for associative algebras, and followed by its extension to Lie algebras by Nijenhuis and Richardson [14, 15]. In general, deformation theory was developed for binary quadratic operads by Balavoine [1]. Deformations of morphisms were developed in [7, 22].

While studying the fluctuation theory in probability, the notion of Rota-Baxter operators on associative algebras was introduced by Baxter [3] in 1960. Since its introduction, it has been found many applications, including in Connes-Kreimer’s algebraic approach to the renormalization in perturbative quantum field theory [4]. Rota-Baxter operator is closely related with dendriform algebras, Lie algebras, and solution of the classical Yang-Baxter equation, see [13, 11] for more details. Rota-Baxter operators are also useful in the study of dendriform algebras operads, which give rise to the splitting of operads [2, 17]. With motivation from Poisson structures, the notion of Rota-Baxter operators on bimodules over associative algebras was introduced by Uchino [23]. Recently, the notions of compatible Rota-Baxter operators and $\mathcal{ON}$-structures was introduced by Liu, Bai, and Sheng in [12], and they proved that an $\mathcal{ON}$-structure gives rise to a hierarchy of Rota-Baxter operators, and that a solution of the strong Maurer-Cartan equation on the associative twilled algebra associated to a Rota-Baxter operator gives rise to a pair of $\mathcal{ON}$-structures which are naturally in duality. In [5], Das constructed an explicit graded Lie algebra whose Maurer-Cartan elements are Rota-Baxter operators on associative algebras and studied linear and formal deformations of a Rota-Baxter operator on an associative algebra. Our main objectives in this paper are certain operators on anti-flexible algebras. More precisely, we are interested in the notions of compatible Rota-Baxter operators and $\mathcal{ON}$-structures on anti-flexible algebras. We show that an $\mathcal{ON}$-structure gives rise to compatible Rota-Baxter operators and conversely given two compatible Rota-Baxter operators there is an $\mathcal{ON}$-structures such that this correspondence naturally in duality.

The paper is organized as follows. In Section 2, we construct the graded Lie algebra that characterizes Rota-Baxter operators on an anti-flexible algebra as Maurer-Cartan elements. In Section 3, we show that the cohomology of a Rota-Baxter operator can also be described as the Hochschild cohomology of a certain anti-flexible algebra with a
suitable bimodule. We also relate the cohomology of a Rota-Baxter operator on an anti-
flexible algebra with the cohomology of the corresponding Rota-Baxter operator on the
commutator Lie algebra. In Section 4 we study infinitesimal deformations of bimodules
over anti-flexible algebras. In Section 5 we consider compatible Rota-Baxter operators
on bimodules over anti-flexible algebras. We define ON-structures which give rise to a
hierarchy of compatible Rota-Baxter operators.

Throughout this paper, we work over the complex field \( \mathbb{K} \), and all the vector spaces
are finite-dimensional.

## 2 Rota-Baxter operators

In this section, we first recall the basics of Rota-Baxter operators on anti-flexible
algebra and their morphisms. Then, we construct a graded Lie algebra with a graded Lie
bracket whose Maurer-Cartan elements are Rota-Baxter operators on anti-flexible algebra.
This construction allows us to define cohomology for a Rota-Baxter operator.

**Definition 2.1.** ([6]) Let \( A \) be a vector space equipped with a bilinear product \( (x, y) \rightarrow x \cdot y \).
\( A \) is called an anti-flexible algebra if the following identity is satisfied
\[
(a \cdot b) \cdot c - a \cdot (b \cdot c) = (c \cdot b) \cdot a - c \cdot (b \cdot a), \text{ for all } a, b, c \in A.
\]

(2.1)

**Example 2.2.** Every associative algebra is automatically an anti-flexible algebra.

**Example 2.3.** Let \( (A, \cdot) \) be an anti-flexible algebra and \( B \) an associative algebra, then
\( A \otimes B \) is an anti-flexible algebra with product given by
\[
(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 \cdot a_2 \otimes b_1 b_2, \text{ for all } a_1, a_2 \in A, b_1, b_2 \in B.
\]

**Example 2.4.** Let \( (A, \cdot) \) and \( (B, \cdot) \) be anti-flexible algebras, then \( (A \oplus B, \cdot) \) is an anti-
flexible algebra with the operation componentwise multiplication.

**Definition 2.5.** ([6]) Let \( (A, \cdot) \) be an anti-flexible algebra and \( M \) be a vector space. Let
\( l, r : A \rightarrow gl(M) \) be two linear maps. If for any \( a, b \in A, \)
\[
l(a \cdot b) - l(a)r(b) - r(b \cdot a), \quad l(a)r(b) - r(b)l(a) = l(b)r(a) - r(a)l(b).
\]

(2.2)

Then it is called a bimodule of \( (A, \cdot) \), denoted by \( (M, l, r) \).

Given an anti-flexible algebra \( (A, \cdot) \) and a bimodule \( (M, l, r) \), the vector space \( A \oplus M \)
carries an anti-flexible algebra structure with product given by
\[
(a, m) \cdot (b, n) = (a \cdot b, l(a)n + r(b)m), \text{ for all } a, b \in A, m, n \in M.
\]

This is called the semi-direct product of \( A \) with \( M \).
Definition 2.6. (6) Rota-Baxter operator on an anti-flexible algebra \((A, \cdot)\) with respect to the bimodule \((M, l, r)\) is given by a linear map \(T : M \to A\) that satisfies
\[
T(m) \cdot T(n) = T(l(T(m)))n + r(T(n))m, \quad \forall m, n \in M. \quad (2.4)
\]

Following Uchin[23], we have the following proposition.

Proposition 2.7. A linear map \(T : M \to A\) is a Rota-Baxter operator on an anti-flexible algebra \(A\) with respect to the bimodule \((M, l, r)\) if and only if the graph
\[
\text{Gr}(T) = \{(T(m), m) | m \in M\}
\]
is a subalgebra of the semi-direct product algebra \(A \oplus M\).

Definition 2.8. Let \((A, \cdot)\) be an anti-flexible algebra. A linear map \(N : A \to A\) is said to be a Nijenhuis operator if its Nijenhuis torsion vanishes, that is,
\[
N(a) \cdot N(b) = N(Na \cdot b + a \cdot Nb - N(a \cdot b)), \quad \text{for all } a, b \in A.
\]
The operation \(\cdot_N : A \otimes A \to A\) given by
\[
a \cdot_N b = Na \cdot b + a \cdot Nb - N(a \cdot b), \quad \text{for all } a, b \in A.
\]
is an anti-flexible algebra and \(N\) is an anti-flexible algebra homomorphism from \((A, \cdot_N)\) to \((A, \cdot)\).

By direct calculations, we have

Lemma 2.9. Let \((A, \cdot)\) be an anti-flexible algebra and \(N\) be a Nijenhuis operator on \(A\). For all \(l, k \in \mathbb{K}\),
\begin{enumerate}[(i)]
\item \((A, \cdot_{N^k})\) is an anti-flexible algebra,
\item \(N^l\) is also a Nijenhuis operator on the anti-flexible algebra \((A, \cdot_{N^k})\),
\item The anti-flexible algebras \((A, (\cdot_{N^k})_{N^l})\) and \((A, (\cdot_{N^l})_{N^k})\) coincide,
\item The anti-flexible algebras \((A, (\cdot_{N^k})_{N^l})\) and \((A, (\cdot_{N^l})_{N^k})\) are compatible, that is, any linear combination of \(\cdot_{N^k}\) and \(\cdot_{N^l}\) still makes \(A\) into an anti-flexible algebra,
\item \(N^l\) is an anti-flexible algebra homomorphism from \((A, (\cdot_{N^k})_{N^l})\) to \((A, (\cdot_{N^l})_{N^k})\).
\end{enumerate}

Another characterization of a Rota-Baxter operator can be given in terms of anti-flexible-Nijenhuis operator on anti-flexible algebras.

Proposition 2.10. A linear map \(T : M \to A\) is a Rota-Baxter operator on \((A, \cdot)\) with respect to the bimodule \((M, l, r)\) if and only if \(N_T = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : A \oplus M \to A \oplus M\) is an anti-flexible-Nijenhuis operator on the semi-direct product algebra \(A \oplus M\).
Next, we recall pre-anti-flexible algebra structures which were first introduced by Bai [6], pre-anti-flexible algebras can be regarded as a natural generalization of dendriform algebras introduced by Loday [13]. On the other hand, from the point of view of operads, like dendriform algebras being the splitting of associative algebras, pre-anti-flexible algebras are the splitting of anti-flexible algebras ([2, 17]).

Definition 2.11. ([6]) Let \( A \) be a vector space with two bilinear products \( \prec, \succ \): \( A \otimes A \to A \). We call it a pre-anti-flexible algebra denoted by \( (A, \prec, \succ) \) if for any \( a, b, c \in A \), the following equations are satisfied

\[
(a \succ b) \prec c - a \succ (b \prec c) = (c \prec b) \prec a - c \prec (b \prec a),
\]

(2.5)

\[
(a \ast b) \succ c - a \succ (b \prec c) = (a \prec b) \prec c - a \prec (b \ast c),
\]

(2.6)

where \( a \ast b = a \prec b + a \succ b \).

A Rota-Baxter operator has an underlying pre-anti-flexible algebra structure [6].

Proposition 2.12. Let \( T : M \to A \) be a Rota-Baxter operator on an anti-flexible algebra \( (A, \cdot) \) with respect to the bimodule \( (M, l, r) \). Then the vector space \( M \) carries a pre-anti-flexible algebra structure with

\[
m \succ n = l(T(m))n, \quad m \prec n = r(T(n))m, \quad \text{for all } m, n \in M.
\]

Definition 2.13. A morphism of Rota-Baxter operator from \( T \) to \( T' \) consists of a pair \( (\phi, \psi) \) of an algebra morphism \( \phi : A \to B \) and a linear map \( \psi : M \to N \) satisfying

\[
T' \circ \psi = \phi \circ T,
\]

(2.7)

\[
l(\phi(a))\psi(m) = \psi(l(a)m),
\]

(2.8)

\[
r(\phi(a))\psi(m) = \psi(r(a)m),
\]

(2.9)

for any \( a \in A \) and \( m \in M \).

It is called an isomorphism if \( \phi \) and \( \psi \) are both linear isomorphisms.

The proof of the following result is straightforward and we omit the details.

Proposition 2.14. A pair of linear maps \( (\phi : A \to B, \psi : M \to N) \) is a morphism of Rota-Baxter operators from \( T \) to \( T' \) if and only if

\[
Gr((\phi, \psi)) := \{(a, m), (\phi(a), \psi(m)) | a \in A, m \in M \} \subset (A \oplus M) \oplus (B \oplus N)
\]

is a subalgebra, where \( A \oplus M \) and \( B \oplus N \) are equipped with semi-direct product algebra structures.
Proposition 2.15. Let $T$ be a Rota-Baxter operator on an anti-flexible algebra $(A, \cdot)$ with respect to a bimodule $(M, l, r)$ and $T'$ be a Rota-Baxter operator on $(B, \cdot)$ with respect to a bimodule $(N, l, r)$. If $(\phi, \psi)$ is a morphism from $T$ to $T'$, then $\psi : M \to N$ is a morphism between induced pre-anti-flexible algebra structures.

Proof. For all $m, m' \in M$, we have

$$
\psi(m \prec_M m') = \psi(\phi(T(m'))m) = r(\phi(T(m')))\psi(m) = r(T'\phi(m'))\psi(m) = \psi(m) \prec_N \psi(m').
$$

Similarly, we obtain $\psi(m \succ_M m') = \psi(m) \succ_N \psi(m').$\qed

In the sequel, we follow the result of Goze and Remm [10] and the derived bracket construction of Voronov [24] to construct an explicit graded Lie algebra whose Maurer-Cartan elements are Rota-Baxter operators. This construction is somewhat similar to Das [5] but more helpful to study deformation theory of Rota-Baxter operators.

Recall that, in [10] Goze and Remm constructed a graded Lie algebra structure on the graded space of all multilinear maps on a vector space $V$. Recall that, for each $n \geq 0$, $g^n = \text{Hom}(V^\otimes_{n+1}, V)$ and a graded Lie bracket on $\bigoplus_n g^n$ by:

$$
[f, g] = f\overline{\sigma}g - (-1)^{mn} g\overline{\sigma}f, \text{ for all } f \in g^m, g \in g^n,
$$

and $\overline{\sigma}$ is defined by

$$(f\overline{\sigma}g)(x_1, \ldots, x_{m+n+1}) = \sum_{i=1}^{m+1} \sum_{\sigma \in \Sigma_{m+n-1}} (-1)^{\epsilon(\sigma)}(-1)^{(i-1)(n-1)}f(x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}, g(x_{\sigma(i)}, \ldots, x_{\sigma(i+n-1)}, x_{\sigma(i+n)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(m+n+1)}),$$

where $\Sigma_p$ refers to the $p$-symmetric group and $\epsilon(\sigma)$ denotes the sign of $\sigma$.

Let $A$ be an anti-flexible algebra equipped with multiplication map $\mu : A \otimes A \to A$, $\mu(a, b) = a.b$. We know that $A \oplus M$ has also an anti-flexible algebra structure. Consider the graded Lie algebra structure on $g^n = \text{Hom}((A \oplus M)^\otimes_{n+1}, A \oplus M)$ associated to the direct sum vector space $V = A \oplus M$. Observe that the elements $\mu, l, r \in g^1 = \text{Hom}((A \oplus M)^\otimes_2, A \oplus M)$. Therefore, $\mu + l + r \in g^1$.

Proposition 2.16. The product $\mu$ defines a multiplication structure on $A$ and $l, r$ defines an $A$-bimodule structure on $M$ if and only if $(\mu + l + r)\overline{\sigma}(\mu + l + r) = 0$, i.e. $\mu + l + r \in g^1$ is a Maurer-Cartan element in $g$. 

6
Proof. For any $a_1, a_2, a_3 \in A$ and $m_1, m_2, m_3 \in M$, we have

\[
(\mu + l + r)\mathfrak{g}(\mu + l + r)((a_1, m_1), (a_2, m_2), (a_3, m_3))
= (\mu + l + r)((\mu + l + r)((a_1, m_1), (a_2, m_2)), (a_3, m_3))
- (\mu + l + r)(((a_1, m_1), (\mu + l + r)((a_2, m_2)), (a_3, m_3)))
- (\mu + l + r)(((a_3, m_3), (\mu + l + r)((a_2, m_2)), (a_1, m_1)))
+ (\mu + l + r)(((a_3, m_3), (\mu + l + r)((a_2, m_2)), (a_1, m_1)))
= ((a_1a_2)a_3, l(a_1a_2)m_3 + l(a_1)r(a_3)m_2 + r(a_3)r(a_2)(m_1))
- (a_1(a_2a_3), l(a_1)l(a_2)m_3 + r(a_3)l(a_1)m_2 + r(a_3)r(a_2)(m_1))
- ((a_3a_2)l, r(a_1)r(a_2)m_3 + l(a_3)r(a_1)m_2 + r(a_3)r(a_2)(m_1))
+ (a_3(a_2a_1), r(a_2a_1)m_3 + r(a_1)l(a_2)m_2 + r(a_3)r(a_2)(m_1))
= 0.
\]

This holds if and only if $\mu$ defines a multiplication structure on the anti-flexible algebra $A$ and $l, r$ define an $A$-bimodule structure on $M$. □

Consider the graded vector space

\[
C^*(M, A) := \bigoplus_{n \geq 1} C^n(M, A) = \bigoplus_{n \geq 1} \text{Hom}(\otimes M^\otimes n, A).
\]

Theorem 2.17. With the above notations, $(C^*(M, A), [[\cdot, \cdot]])$ is a graded Lie algebra, where the graded Lie bracket $[[\cdot, \cdot]] : C^m(M, A) \times C^n(M, A) \to C^{m+n}(M, A)$ is defined by

\[
[[P, P']] := (-1)^m[[\mu + l + r, P], P'],
\]

for any $P \in C^m(M, A), P' \in C^n(M, A)$.

More precisely, we have

\[
[[P, P']](v_1, \ldots, v_{m+n})
= \sum_{k=1}^{m} \sum_{\sigma \in \Sigma_{m+n}} (-1)^{(k-1)n}(-1)^{\epsilon(\sigma)} P(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)},
\quad l(P'(v_{\sigma(k)}, \ldots, v_{\sigma(k+n-1)})v_{\sigma(k+n)}, \ldots, v_{\sigma(m+n)}))
- \sum_{k=1}^{m} \sum_{\sigma \in \Sigma_{m+n}} (-1)^{kn}(-1)^{\epsilon(\sigma)} P(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)},
\quad r(P'(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+n-1)}))v_{\sigma(k)}, v_{\sigma(k+n-1)}, \ldots, v_{\sigma(m+n)}))
- \sum_{k=1}^{n} \sum_{\sigma \in \Sigma_{m+n}} (-1)^{(k+n-1)m}(-1)^{\epsilon(\sigma)} P'(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)},
\quad l(P(v_{\sigma(k)}, \ldots, v_{\sigma(k+m-1)}))v_{\sigma(k+m)}, \ldots, v_{\sigma(m+n)}))
+ \sum_{k=1}^{n} \sum_{\sigma \in \Sigma_{m+n}} (-1)^{(k+n)m}(-1)^{\epsilon(\sigma)} P'(v_{\sigma(1)}, \ldots, v_{\sigma(k-1)},
\quad r(P(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+m)}))v_{\sigma(k)}, v_{\sigma(k+m+1)}, \ldots, v_{\sigma(m+n)}))
\]
for any $P \in C^m(M, A), P' \in C^n(M, A)$. Moreover, its Maurer-Cartan elements are Rota-
Baxter operator on the anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$.

**Proof.** The graded Lie algebra $(C^*(M, A), [[\cdot, \cdot]])$ is obtained via the derived bracket [10]. In fact, the Balavoine bracket $[[\cdot, \cdot]]$ associated to the direct sum vector space $A \oplus M$ gives rise to a graded Lie algebra $(C^*(A \oplus M, A \oplus M), [[\cdot, \cdot]])$. By the above proposition, we deduce that $(C^*(A \oplus M, A \oplus M), [[\cdot, \cdot]], d = [\mu + l + r, \cdot])$ is a differential graded Lie algebra. Obviously $C^*(M, A)$ is an abelian subalgebra. Furthermore, we define the derived bracket on the graded vector space $C^*(M, A)$ by

$$[[P, P']] := (-1)^m[d(P), P'] = (-1)^m[[\mu + l + r, P], P'],$$

for any $P \in C^m(M, A), P' \in C^n(M, A)$. The derived bracket $[[\cdot, \cdot]]$ is closed on $C^*(M, A)$, which implies that $(C^*(M, A), [[\cdot, \cdot]])$ is a graded Lie algebra.

For $T \in C^1(M, A)$, we have

$$[[T, T]](u, v) = 2(Tu \cdot Tv - T(l(Tu)v) - T(r(Tv)u)).$$

Thus, $T$ is a Maurer-Cartan element (i.e. $[[T, T]] = 0$) if and only if $T$ is a Rota-Baxter operator on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. The proof is finished. □

Thus, Rota-Baxter operators can be characterized as Maurer-Cartan elements in a gLa. It follows from the above theorem that if $T$ is a Rota-Baxter operator, then $d_T := [[T, \cdot]]$ is a differential on $C^*(M, A)$ and makes the gLa $(C^*(M, A), [[\cdot, \cdot]])$ into a dgLa.

The cohomology of the cochain complex $(C^*(M, A), d_T)$ is called the cohomology of the Rota-Baxter operator $T$ on $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. We denote the corresponding cohomology groups simply by $H^*(M, A)$.

**Theorem 2.18.** Let $T$ be a Rota-Baxter operator on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. The sum $T + T'$ is a Rota-Baxter operator if and only if $T'$ is a Maurer-Cartan element of $(C^*(M, A), [[\cdot, \cdot]], d_T)$, that is,

$$[[T + T', T + T']] = 0 \iff d_T T' + \frac{1}{2}[[T', T']] = 0.$$

3 Cohomology of Rota-Baxter operators as Hochschild cohomology

In this section, we show that the cohomology of a Rota-Baxter operators on a anti-
flexible algebra can also be described as the Hochschild cohomology of a certain anti-
flexible algebra with a suitable bimodule. We also show that the cohomology of a Rota-Baxter operator on an anti-flexible algebra is related with the cohomology of the corresponding Rota-Baxter operator on the commutator Lie algebra.

Let \( T : M \to A \) be a Rota-Baxter operator on an anti-flexible algebra \((A, \cdot)\) with respect to the bimodule \((M, l, r)\). By Proposition 2.12, then the vector space \( M \) carries an anti-flexible algebra structure with the product

\[
m \star_T n = r(T(n))m + l(T(m))n, \text{ for all } m, n \in M.
\]

**Lemma 3.1.** Let \( T : M \to A \) be a Rota-Baxter operator on an anti-flexible algebra \((A, \cdot)\) with respect to the bimodule \((M, l, r)\). Define

\[
l_T : M \to gl(A), \quad l_T(m)(a) = T(m) \cdot a - T(r(a)m),
\]

\[
r_T : M \to gl(A), \quad r_T(m)(a) = a \cdot T(m) - T(l(a)m), \text{ for all } m \in M, a \in A.
\]

Then \( l_T, r_T \) defines an \( M \)-bimodule structure on \((A, \cdot)\).

**Proof.** For any \( m, n \in M \) and \( a \in A \), we have

\[
[l_T(m \star_T n) - l_T(m)l_T(n) - r_T(m)r_T(n) + r_T(n \star_T m)](a)
= l_T(m \star_T n)(a) - l_T(m)l_T(n)(a) - r_T(m)r_T(n)(a) + r_T(n \star_T m)(a)
= (T(m) \cdot T(n)) \cdot a - T(r(a)(r(T(n)m + l(T(m))n))
- T(m) \cdot (T(n) \cdot a) + T(l(T(m))(r(a)n)) + r(T(n)a)m
- (a \cdot T(n)) \cdot T(m) + T(l(aT(n))m + r(T(m))l(a)n)
+ a \cdot (T(n) \cdot T(m)) - T(l(a)l(T(n))m - l(a)r(T(m))n)
= 0.
\]

Similarly, we have

\[
l_T(m)r_T(n) - r_T(n)l_T(m) - l_T(n)r_T(m) + r_T(m)l_T(n) = 0.
\]

Then \( l_T, r_T \) defines an \( M \)-bimodule structure on \((A, \cdot)\). Hence the proof is finished. \( \square \)

By Lemma 3.1 we obtain an \( M \)-bimodule structure on the vector space \((A, \cdot)\). Therefore, we may consider the corresponding Hochschild cohomology of \( M \) with coefficients in \((A, l_T, r_T)\). More precisely, we define

\[
C^n(M, A) := \text{Hom}(M^\otimes n, A), \text{ for all } n \geq 0,
\]

and the differential is given by

\[
d_H(a)(m) = l_T(m)(a) - r_T(m)(a)
= T(m) \cdot a - T(r(a)m) - a \cdot T(m) + T(l(a)m), \text{ for all } a \in A = C^0(M, A),
\]
and
\[(d_H f)(u_1, \ldots, u_{n+1}) = T(u_1)f(u_2, \ldots, u_{n+1}) - T(r(f(u_2, \ldots, u_{n+1})u_1)\]
\[+ \sum_{i=1}^n \sum_{\sigma \in \Sigma_{n+1}} (-1)^i(-1)^{\ell(\sigma)} f(u_{\sigma(1)}, \ldots, u_{\sigma(i-1)}, r(T(u_{\sigma(i+1)})u_{\sigma(i)}) \]
\[+ l(T(u_{\sigma(i)}))u_{\sigma(i+1)}, \ldots, u_{\sigma(n+1)}) \]
\[+ (-1)^{n+1} f(u_1, \ldots, u_n) \cdot T(u_{n+1}) - (-1)^{n+1} T(l_T(u_{n+1})f(u_1, \ldots, u_n)).\]

We denote the group of \( n \)-cocycles by \( Z^n(M, A) \) and the group of \( n \)-coboundaries by \( B^n(M, A) \). The corresponding cohomology groups are defined by \( H^n(M, A) = Z^n(M, A) / B^n(M, A) \), \( n \geq 0 \).

It follows from the above definition that
\[ H^0(M, A) = \{ a \in A | d_H(a) = 0 \} \]
\[ = \{ a \in A | a \cdot T(m) - T(m) \cdot a = T(l(a)m - r(a)m), \text{ for all } m \in M \}. \]

By [6], if \( a, b \in A \), define the commutator by \([a, b]_g = a \cdot b - b \cdot a\), then it is a Lie algebra and we denote it by \((g(A), [\cdot, \cdot]_g)\). Furthermore, it is easy to check that \( H^0(M, A) \) has a Lie algebra structure induced from that of \((A, \cdot)\).

Note that a linear map \( f \in C^1(M, A) \) is closed if it satisfies
\[ T(u) \cdot f(v) + f(u) \cdot T(v) - T(l_T(u)f(v) + r_T(v)f(u)) - f(l_T(u)T(v) + r_T(v)T(u)) = 0, \]
for any \( u, v \in M \).

For a Rota-Baxter operator \( T \) on an anti-flexible algebra \((A, \cdot)\) with respect to the bimodule \((M, l, r)\), we get two coboundary operators \( d_T = [[T, \cdot]] \) and \( d_H \) on the same graded vector space \( C^\bullet(M, A) = \oplus_{n \geq 0} C^n(M, A) \).

The following proposition relates the above two coboundary operators.

**Proposition 3.2.** Let \( T : M \to A \) be a Rota-Baxter operator on an anti-flexible algebra \((A, \cdot)\) with respect to the bimodule \((M, l, r)\). Then the two coboundary operators are related by
\[ d_T f = (-1)^n d_H f, \text{ for all } f \in C^n(M, A). \]
**Proof.** For any \( f \in C^n(M,A) \) and \( u_1, \ldots, u_{n+1} \in M \), we have
\[
(d_T f)(u_1, \ldots, u_{n+1}) = \frac{1}{2} \left[ T(f) \right](u_1, \ldots, u_{n+1}) \]
\[
= T(l(f(u_2, \ldots, u_{n+1})u_{n+1}) - (-1)^n T(r(f(u_2, \ldots, u_{n+1})u_1)
\]
\[-(-1)^n \sum_{i=1}^n \sum_{\sigma \in \Sigma_{n+1}} (-1)^{i-1} (-1)^{\epsilon(\sigma)} f(u_{\sigma(1)}, \ldots, u_{\sigma(i-1)}; r(T(u_{\sigma(i+1)}u_{\sigma(i)})) + l(T(u_{\sigma(i)}))u_{\sigma(i+1)}, \ldots, u_{\sigma(n+1)}))
\]
\[+(-1)^n T(u_1)f(u_2, \ldots, u_{n+1}) - f(u_1, \ldots, u_n)T(u_{n+1})
\]
\[= (-1)^n (d_H f)(u_1, \ldots, u_{n+1}).
\]
This completes the proof. \( \square \)

**Definition 3.3.** Let \((g, [\cdot, \cdot]_g)\) be a Lie algebra and \(\rho : g \rightarrow gl(M)\) be a representation of \(g\) on a vector space \(M\). A Rota-Baxter operator on \(g\) with respect to the representation \(M\) is a linear map \(T : M \rightarrow g\) satisfying
\[
[T(m), T(n)] = \rho(Tm)(n) - \rho(Tn)(m), \text{ for all } m, n \in M.
\]

**Lemma 3.4.** Let \((M, l, r)\) be a bimodule of an anti-flexible algebra \((A, \cdot)\). Then \((M, l - r)\) is a representation of the associated Lie algebra \((g(A), [\cdot, \cdot]_g)\).

With the above notations, we have the following

**Proposition 3.5.** The collection of maps \(S_n : \text{Hom}(A^\otimes n, M) \rightarrow \text{Hom}(\wedge^n A, M)\) defined by
\[
S_n(f)(a_1, \ldots, a_n) = \sum_{\sigma \in \Sigma_n} (-1)^{\epsilon(\sigma)} f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})
\]
is a morphism from the Hochschild cochain complex of \(A\) with coefficients in the bimodule \(M\) to the Chevalley-Eilenberg complex of the commutator Lie algebra \((g(A), [\cdot, \cdot]_g)\) with coefficients in the representation \((M, l - r)\).

**Proposition 3.6.** Let \(T : M \rightarrow A\) be a Rota-Baxter operator on an anti-flexible algebra \((A, \cdot)\) with respect to the bimodule \((M, l, r)\). Then \(T\) is also a Rota-Baxter operator on the commutator Lie algebra \((g(A), [\cdot, \cdot]_g)\) with respect to the representation \((M, \rho)\).

**Proof.** For any \(m, n \in M\), we have
\[
[T(m), T(n)] = T(m) \cdot T(n) - T(n) \cdot T(m)
\]
\[
= T(r(Tn)m + l(Tm)n) - T(r(Tm)n + l(Tn)m)
\]
\[
= T((l - r)(Tm)n - (l - r)(Tn)m)
\]
\[
= T(\rho(Tm)n - \rho(Tn)m).
\]
This proves the proposition. \( \square \)
4 Infinitesimal deformations of bimodules over anti-flexible algebras

Let \((A, \cdot)\) be an anti-flexible algebra and \((M, l, r)\) a bimodule. Let \(\omega : \otimes^2 A \to A\), \(\phi : A \to gl(M)\) and \(\psi : A \to gl(M)\) be linear maps. Consider a \(t\)-parametrized family of multiplication operations and linear maps:

\[
a \cdot_t b = a \cdot b + \omega(a, b), \quad l^t(a) = l(a) + t\phi(a), \quad r^t(a) = r(a) + t\psi(a),
\]

for all \(a, b \in A\).

If \((A, \cdot_t)\) are anti-flexible algebras and \((M, l^t, r^t)\) are bimodules for all \(t \in \mathbb{K}\), we say that \((\omega, \phi, \psi)\) generates an infinitesimal deformation of the \(A\)-bimodule \(M\).

Let \(\mu_t\) denote the anti-flexible algebra structure \((A, \cdot_t)\). By Proposition 2.16, the bimodule \((M, l^t, r^t)\) over the anti-flexible algebra \((A, \cdot_t)\) is an infinitesimal deformation of the \(A\)-bimodule \(M\) if and only if

\[
(\mu_t + l^t + r^t)\Sigma(\mu_t + l^t + r^t) = 0,
\]

which is equivalent to

\[
(\omega + \phi + \psi)\Sigma(\omega + \phi + \psi) = 0,
\]

\[
(\mu_t + l^t + r^t)\Sigma(\omega + \phi + \psi) = 0.
\]

Definition 4.1. Let the bimodules \((M, l^t, r^t)\) and \((M, l'^t, r'^t)\) be two infinitesimal deformations of an \(A\)-bimodule \(M\) over anti-flexible algebras \((A, \cdot_t)\) and \((A, \cdot'_t)\) respectively. We call them equivalent if there exists \(N \in gl(g)\) and \(S \in gl(M)\) such that \((Id_A + tN, Id_M + tS)\) is a homomorphism from the bimodule \((M, l'^t, r'^t)\) to the bimodule \((M, l^t, r^t)\).

By direct calculations, the bimodule \((M, l^t, r^t)\) over the anti-flexible algebra \((A, \cdot_t)\) and the bimodule \((M, l'^t, r'^t)\) over the anti-flexible algebra \((A, \cdot'_t)\) are equivalent deformations if and only if

\[
(\omega + \phi + \psi)(a + m, b + n) - (\omega' + \phi' + \psi')(a + m, b + n) = d(N + S)(a + m, b + n),
\]

\[
(\omega' + \phi' + \psi')(N(a) + S(m), N(b) + S(n)) = 0,
\]

and

\[
(N + S)(\omega + \phi + \psi)(a + m, b + n) = (\omega' + \phi' + \psi')(a + m, N(b) + S(n))
\]

\[
+(\omega' + \phi' + \psi')(N(a) + S(m), b + n) + (\mu + l + r)(N(a) + S(m), N(b) + S(n)).
\]

Summarizing the above discussion, we have the following conclusion:
Theorem 4.2. Let the bimodule \((M, l^t, r^t)\) over the anti-flexible algebra \((A, \cdot, \cdot)\) be an infinitesimal deformation of an \(A\)-bimodule \(M\) generated by \((\omega, \phi, \psi)\). Then \(\omega + \phi + \psi \in C^2(M, A)\) is closed, i.e. \(d(\omega, \phi, \psi) = 0\). Furthermore, if two infinitesimal deformations \((M, l^t, r^t)\) and \((M, l^{t'}, r^{t'})\) over anti-flexible algebra \((A, \cdot, \cdot)\) generated by \((\omega, \phi, \psi)\) and \((\omega', \phi', \psi')\) respectively are equivalent, then \(\omega + \phi + \psi\) and \(\omega' + \phi' + \psi'\) are in the same cohomology class in \(H^2(M, A)\).

Definition 4.3. An infinitesimal deformation of an \(A\)-bimodule \(M\) is said to be trivial if it is equivalent to the \(A\)-bimodule \(M\).

One can deduce that the bimodule \((M, l^t, r^t)\) over the anti-flexible algebra \((A, \cdot, \cdot)\) is a trivial deformation if and only if for all \(a, b \in A, m, n \in M\), we have

\[
\begin{align*}
(\omega + \phi + \psi)(a + m, b + n) &= d(N + S)(a + m, b + n), \\
(N + S)(\omega + \phi + \psi)(a + m, b + n) &= (\mu + l + r)(N(a) + S(m), N(b) + S(n)).
\end{align*}
\]

Equivalently, we have

\[
\begin{align*}
\omega(a, b) &= N(a) \cdot b + a \cdot N(b) - N(a \cdot b), \\
N\omega(a, b) &= N(a) \cdot N(b), \\
\phi(a) &= l(N(a)) + l(a) \circ S - S \circ l(a), \\
l(N(a)) \circ S &= S \circ \phi(a), \\
\psi(a) &= r(N(a)) + r(a) \circ S - S \circ r(a), \\
r(N(a)) \circ S &= S \circ \psi(a).
\end{align*}
\]

It follows from Eqs.(4.1) and (4.2) that \(N\) must be a Nijenhuis operator on the anti-flexible algebra \((A, \cdot, \cdot)\). It follows from Eqs.(4.3) and (4.4) that \(N\) and \(S\) should satisfy the condition:

\[
l(N(a))S(m) = S(l(N(a))(m) + l(a)(S(m)) - S(l(a)m)), \quad \forall a \in A, m \in M. (4. 7)
\]

It follows from Eqs. (4.5) and (4.6) that \(N\) and \(S\) should also satisfy the condition:

\[
r(N(a))S(m) = S(r(N(a))m + r(a)(S(m)) - S(r(a)m)), \quad \forall a \in A, m \in M. (4. 8)
\]

Theorem 4.4. Let \((M, l, r)\) be a bimodule over an anti-flexible algebra \((A, \cdot, \cdot)\), \(N \in gl(A)\) and \(S \in gl(M)\). If \(N\) is a Nijenhuis operator on the anti-flexible algebra \((A, \cdot, \cdot)\) and if \(S\) satisfies Eqs. (4.7) and (4.8), then a deformation of the \(A\)-bimodule \(M\) can be obtained by putting

\[
\begin{align*}
\omega(a, b) &= N(a) \cdot b + a \cdot N(b) - N(a \cdot b), \\
\phi(a) &= l(N(a)) + l(a) \circ S - S \circ l(a), \\
\psi(a) &= r(N(a)) + r(a) \circ S - S \circ r(a),
\end{align*}
\]

for any \(a, b \in A\). Furthermore, this deformation is trivial.
Note that the conditions that $N$ is a Nijenhuis operator and $S$ satisfies Eqs. (4.7) and (4.8), can be expressed simply by the following result.

**Proposition 4.5.** Let $(M,l,r)$ be a bimodule over an anti-flexible algebra $(A,·)$. Then $N$ is a Nijenhuis operator on the anti-flexible algebra $(A,·)$ and $S$ satisfies Eqs. (4.7) and (4.8), if and only if $N + S$ is a Nijenhuis operator on the semidirect product anti-flexible algebra $A ⊕ M$.

**Definition 4.6.** Let $(M,l,r)$ be a bimodule over an anti-flexible algebra $(A,·)$. A pair $(N,S)$, where $N ∈ gl(A)$ and $S ∈ gl(M)$, is called a Nijenhuis structure on an $A$-bimodule $M$ if $N$ and $S^*$ generate a trivial infinitesimal deformation of the dual $A$-module $M^*$.

Note that the condition of the above definition is equivalent to the fact that $N$ is a Nijenhuis tensor on $A$ and

$$l(N(a))S(m) = S(l(N(a))m) + l(a)S^2(m) - S(l(x)S(m)),$$

$$r(N(a))S(m) = S(r(N(a))m) + r(a)S^2(m) - S(r(x)S(m)).$$

for all $a ∈ A, m ∈ M$.

**Example 4.7.** Let $N : A → A$ be a Nijenhuis operator on the anti-flexible algebra $(A,·)$. Then $(N,N^*)$ is a Nijenhuis structure on the coadjoint module $A^*$.

**Corollary 4.8.** Let $(N,S)$ be a Nijenhuis structure on a $A$-bimodule $M$, then the pairs $(N^i,S^i)$ are Nijenhuis structures on an $A$-bimodule $M$.

### 5 ON-structures on bimodules over anti-flexible algebras and compatible Rota-Baxter operators

In this final section, we show how compatible Rota-Baxter operators and ON-structures are related.

Let $T : M → A$ be a Rota-Baxter operator on an anti-flexible algebra $(A,·)$ with respect to the bimodule $(M,l,r)$. By Proposition 2.12, then the vector space $M$ carries an anti-flexible algebra structure with the product

$$m *_T n = l(T(m))n + r(T(n))m, \text{ for all } m, n ∈ M.$$  

We define the multiplication $*^S_T : M ⊗ M → M$ to be the deformed multiplication of $*_T$ by $S$, i.e.

$$m *^S_T n = S(m) *_T n + m *_T S(n) - S(m *_T n).$$

**Definition 5.1.** Let $T : M → A$ be a Rota-Baxter operator and $(N,S)$ a Nijenhuis structure on an $A$-bimodule $M$. The triple $(T,N,S)$ is called an $ON$-structure on an
A-bimodule $M$ if $T$ and $(N, S)$ satisfy the following conditions

$$N \circ T = T \circ S,$$

$$m \ast_{N \circ T} n = m \ast_T^S n, \text{ for all } m, n \in M.$$

Define two linear maps $\tilde{l}, \tilde{r} : A \rightarrow gl(M)$ as follows:

$$\tilde{l}(a) := l(N(a)) - l(a) \circ S + S \circ l(a),$$

$$\tilde{r}(a) := r(N(a)) - r(a) \circ S + S \circ r(a), \text{ for all } a \in A.$$

Then it is easy to check that $(M, \tilde{l}, \tilde{r})$ is a bimodule of $(A, \cdot)$. Furthermore, we have an anti-flexible algebra structure with the product

$$m \tilde{\ast}_T n = \tilde{l}(T(m))n + \tilde{r}(T(n))m, \text{ for all } m, n \in M.$$

Direct calculation, for any $m, n \in M$, we have $m \tilde{\ast}_T n + m \ast_T^S n = 2(m \ast_{N \circ T} n)$. Then we have the following lemma.

**Lemma 5.2.** Let $(T, N, S)$ be an an ON-structure on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. Then we have

$$m \ast_T^S n = m \tilde{\ast}_T n.$$

**Definition 5.3.** Two Rota-Baxter operators $T_1, T_2 : M \rightarrow A$ on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$ are said to be compatible if their sum $T_1 + T_2 : M \rightarrow A$ is also a Rota-Baxter operator.

**Proposition 5.4.** Let $T_1, T_2 : M \rightarrow A$ be two Rota-Baxter operators on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$ . If $T_1, T_2$ are compatible and $T_2$ is invertible then $N = T_1 \circ T_2^{-1} : A \rightarrow A$ is a Nijenhuis operator on the anti-flexible algebra $(A, \cdot)$. Conversely, if $T_1, T_2$ are both invertible and $N$ is a Nijenhuis tensor then $T_1, T_2$ are compatible.

**Proof.** Let $T_1, T_2$ be compatible and $T_2$ invertible. For any $a, b \in A$, there exists elements $m, n \in M$ such that $T_2(m) = a$ and $T_2(n) = b$. Then

$$Na \cdot Nb - N(Na \cdot b + a \cdot Nb) + N^2(a \cdot b)$$

$$= NT_2(m) \cdot NT_2(n) - N(NT_2(m) \cdot T_2(n) + T_2(m) \cdot NT_2(n)) + N^2(T_2(m) \cdot T_2(n))$$

$$= T_1(m) \cdot T_1(n) - N(T_1(m) \cdot T_2(n) + T_2(m) \cdot T_1(n)) + N^2(T_2(m) \cdot T_2(n))$$

$$= T_1(l(T_1(m))n + r(T_1(n))m) - N(T_1(l(T_2(m))n + r(T_2(n))m)$$

$$+ T_2(l(T_1(m))n + r(T_1(n))m)) + N^2(T_2(l(T_2(m))n + r(T_2(n))m))$$

$$= 0.$$
Conversely, if $N$ is a Nijenhuis tensor then for all $m, n \in M$,

$$NT_2(m) \cdot NT_2(n) = N(NT_2(m) \cdot T_2(n) + T_2(m) \cdot NT_2(n)) - N^2(T_2(m) \cdot T_2(n)).$$

This implies that

$$NT_2(l(T_1(m)))n + r(T_1(n)m) = N(T_1(m) \cdot T_2(n) + T_2(m) \cdot T_1(n)) - NT_2(l(T_2(m))n + r(T_2(n)m).$$

Since $N$ is invertible, we may apply $N^{-1}$ to both sides to get the above identity. Hence $T_1$ and $T_2$ are compatible. □

**Theorem 5.5.** Let $(T, N, S)$ be an ON-structure on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. Then

(i) $T$ is a Rota-Baxter operator on the deformed anti-flexible algebra $(A, \cdot_N)$ with respect to the bimodule $(\tilde{M}, \tilde{l}, \tilde{r})$,

(ii) $N \circ T$ is a Rota-Baxter operator on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$.

**Proof.** (i) For any $m, n \in M$, we have

$$T(m \ast_T n) = T(m \ast_T n) = T(S(m) \ast_T n + m \ast_T S(n) - S(m \ast_T n)) = T \circ S(m) \cdot_N T(n) + T(m) \cdot_N T \circ S(n) - T \circ S(m \ast_T n) = N \circ T(m) \cdot_N T(n) + T(m) \cdot_N N \circ T(n) - N(T(m) \cdot_N T(n)) = T(m) \cdot_N T(n).$$

Then $T$ is a Rota-Baxter operator on the deformed anti-flexible algebra $(A, \cdot_N)$ with respect to the bimodule $(\tilde{M}, \tilde{l}, \tilde{r})$.

(ii) By the fact that $N$ is a Nijenhuis tensor, we have

$$N \circ T(m \ast_{N \circ T} n) = N \circ T(m \ast_{T} S(n)) = N(T(m) \cdot_N T(n)) = N \circ T(m) \cdot N \circ T(n).$$

Hence $N \circ T$ is a Rota-Baxter operator on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. □

**Proposition 5.6.** Let $(T, N, S)$ be an ON-structure on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. Then $T$ and $N \circ T$ are compatible Rota-Baxter operators.

**Proof.** For any $m, n \in M$, we have

$$m \ast_{T+N \circ T} n = m \ast_T n + m \ast_{N \circ T} n = m \ast_T n + m \ast_T S(n).$$
Furthermore, we have

\[(T + N \circ T)(m \star_{T+N\circ T} n)\]
\[= T(m \star_T n) + T(m \star_S n) + (N \circ T)(m \star_T n) + (N \circ T)(m \star_S n)\]
\[= T(m \star_T n) + T(S(m) \star_T n + m \star_T S(n) - S(m \star_T n))\]
\[+ (N \circ T)(m \star_T n) + (N \circ T)(m \star_{N\circ T} n)\]
\[= T(m) \cdot T(n) + (T \circ S)(m) \cdot T(n) + T(m) \cdot (T \circ S)(n) + (N \circ T)(m) \cdot (N \circ T)(n)\]
\[= T(m) \cdot T(n) + (N \circ T)(m) \cdot T(n) + T(m) \cdot (N \circ T)(n) + (N \circ T)(m) \cdot (N \circ T)(n)\]
\[= (T + N \circ T)(m) \cdot (T + N \circ T)(n).\]

Then $T$ and $N \circ T$ are compatible Rota-Baxter operators. \hfill \Box

In the next proposition, we construct an $ON$-structure from compatible Rota-Baxter operators.

**Proposition 5.7.** Let $T_1, T_2 : M \to A$ be two compatible Rota-Baxter operators on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$. If $T_2$ is invertible then $(T_2, N = T_1 \circ T_2^{-1}, S = T_2^{-1} \circ T_1)$ is an $ON$-structure.

**Proof.** Since $T_1, T_2$ are compatible Rota-Baxter operators, we have

\[T_1(m) \cdot T_2(n) + T_2(m) \cdot T_1(n) = T_1(l(T_2(m))m + r(T_2(n))m) + T_2(l(T_1(m))n + r(T_1(n))m).\]

By replacing $T_1$ with $T_2 \circ S$ in above equation, we have

\[(T_2 \circ S)(m) \cdot T_2(n) + T_2(m) \cdot (T_2 \circ S)(n)\]
\[= (T_2 \circ S)(l(T_2(m))n + r(T_2(n))m) + T_2(l((T_2 \circ S)(m))n + r((T_2 \circ S)(n))m). \tag{5.1}\]

On the other hand, $T_2$ is a Rota-Baxter operator implies that

\[(T_2 \circ S)(m) \cdot T_2(n) + T_2(m) \cdot (T_2 \circ S)(n)\]
\[= T_2(l(T_2(S(m)))n + r(T_2(n))S(m)) + l(T_2(m))S(n) + r(T_2(S(n)))m). \tag{5.2}\]

From Eqs.(5.1) and (5.2) and using the fact that $T_2$ is invertible, we get

\[S(l(T_2(m))n + r(T_2(n))m) = l(T_2(m))S(n) + r(T_2(n))S(m).\]

By replacing $n$ by $S(n)$, we have

\[S(l(T_2(m))S(n) + r((T_2 \circ S)(n))m) = l(T_2(m))S^2(n) + r(T_2 \circ S(n))S(m). \tag{5.3}\]

As $T_1 = T_2 \circ S$ and $T_2$ are Rota-Baxter operators,

\[T_2(m \star_{T_2 \circ S} n) = (T_2 \circ S)(m) \cdot (T_2 \circ S)(n) = T_2(S(m) \star_{T_2} S(n)).\]
The invertibility of $T_2$ implies that

$$S(l((T_2 \circ S)(m))n + r((T_2 \circ S)(n))m) = l((T_2 \circ S)(m))S(n) + r((T_2 \circ S)(n))S(m). \quad (5.4)$$

From Eqs. (5.3) and (5.4) and using the fact that $T_2$ is invertible, we get

$$l((T_2 \circ S)(m))S(n) = l(T_2(m))S^2(n) + S(l((T_2 \circ S)(m))n - S(l(T_2(m)))S(n).$$

Substitute $a = T_2(m)$, using $T_2 \circ S = N \circ T_2$ and the invertibility of $T_2$,

$$l(N(a))S(n) = l(a)S^2(n) + S(l(N(a))n - S(l(a)S(n)).$$

Hence the identity Eq. (4.7) follows. Similarly, Eq. (4.8) holds. Thus, the pair $(N, S)$ is a Nijenhuis structure on an anti-flexible algebra $(A, \cdot)$ with respect to the bimodule $(M, l, r)$.

Next, observe that $N \circ T_2 = T_2 \circ S = T_1$. Moreover,

$$m \ast_{T_2} S_n - m \ast_{T_2 \circ S} n = l(T_2(m))S(n) + r(T_2(n))S(m) - S(l(T_2(m))n + r(T_2(n))m)$$

which implies that $m \ast_{T_2} S_n = m \ast_{T_2 \circ S} n$. Therefore, $(T_2, N = T_1 \circ T_2^{-1}, S = T_2^{-1} \circ T_1)$ is an ON-structure. \hfill \Box

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