Axiomatic quantum field theory. Jet formalism

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Abstract. Jet formalism provides the adequate mathematical formulation of classical field theory reviewed in hep-th/0612182v1. A formulation of QFT compatible with this classical one is discussed. We are based on the fact that an algebra of Euclidean quantum fields is graded commutative, and there are homomorphisms of the graded commutative algebra of classical fields to this algebra. As a result, any variational symmetry of a classical Lagrangian yields the identities which Euclidean Green functions of quantum fields satisfy.
Its inductive limit is called either a global algebra (in the case of von Neumann algebras) or a quasilocal algebra (for a net of $C^*$-algebras). This construction is extended to non-Minkowski spaces, e.g., globally hyperbolic spacetimes [20, 21, 60].

We follow a different formulation of axiomatic QFT where quantum field algebras are tensor algebras. Let $Q$ be a nuclear space (see Appendix C). Let us consider the direct limit

$$A_Q = \hat{\otimes} Q = \mathbb{C} \oplus Q \oplus Q\hat{\otimes} Q \oplus \cdots \hat{\otimes}^n Q \oplus \cdots$$

(1)

of the vector spaces $\hat{\otimes}^n Q = \mathbb{C} \oplus Q \oplus Q\hat{\otimes} Q \cdots \oplus Q\hat{\otimes}^n Q$, where $\hat{\otimes}$ is the topological tensor product with respect to the Grothendieck’s topology (which coincides with the $\varepsilon$-topology on the tensor product of nuclear spaces [57]). The space (1) is provided with the inductive limit topology, the finest topology such that the morphisms $\hat{\otimes}^n Q \to \hat{\otimes} Q$ are continuous and, moreover, are imbeddings [74]. A convex subset $V$ of $\hat{\otimes} Q$ is a neighborhood of the origin in this topology iff $V \cap \hat{\otimes}^n Q$ is so in $\hat{\otimes}^n Q$. Furthermore, one can show that $A_Q$ (1) is a unital nuclear barreled LF-algebra [11] (see Appendix B). The LF-property implies that a linear form $f$ on $A_Q$ is continuous iff the restriction of $f$ to each $\hat{\otimes}^n Q$ is so [74].

If a continuous conjugation $^*$ is defined on $Q$, the algebra $A_Q$ is involutive with respect to the operation

$$^*(q_1 \otimes \cdots \otimes q_n) = q_n^* \otimes \cdots q_1^*$$

(2)

on $Q\hat{\otimes}^n$ extended by continuity and linearity to $Q\hat{\otimes}^n$. One can show that $A_Q$ is a $b^*$-algebra as follows. Since $Q$ is a nuclear space, there is a family $\| \cdot \|_k, k \in \mathbb{N}_+$, of continuous norms on $Q$. Let $Q_k$ denote the completion of $Q$ with respect to the norm $\| \cdot \|_k$. Then the tensor algebra $\otimes Q_k$ is a $C^*$-algebra and $A_Q$ (1) is the projective limit of these $C^*$-algebras with respect to morphisms $\otimes Q_{k+1} \to \otimes Q_k$ [45].

Since $A_Q$ is a nuclear barreled $b$-algebra, one can apply to it the following variant of the GNS representation theorem. Let $A$ be a unital nuclear barreled $b^*$-algebra and $f$ a positive form on $A$. There exists a unique cyclic representation $\pi_f$ of $A$ in a Hilbert space by operators on a common invariant domain $D$ [45]. This domain can be topologized to conform a rigged Hilbert space such that all the operators representing $A$ are continuous on $D$.

In axiomatic QFT, one usually choose $Q$ the Schwartz space of functions of rapid decrease (see Appendix D). For the sake of simplicity, we here restrict our consideration to real scalar fields. One associates to them the Borchers algebra

$$A = \mathbb{R} \oplus RS^4 \oplus RS^8 \oplus \cdots,$$

(3)

where $RS^{4k}$ is the nuclear space of smooth real functions of rapid decrease on $\mathbb{R}^{4k}$ [17, 43]. It is the real subspace of the space $S(\mathbb{R}^{4k})$ of smooth complex functions of rapid decrease on $\mathbb{R}^{4k}$. Its topological dual is the space $S'(\mathbb{R}^{4k})$ of tempered distributions (generalized functions). Since the subset $\hat{\otimes}s(\mathbb{R}^4)$ is dense in $S(\mathbb{R}^{4k})$, we henceforth identify $A$ with the tensor algebra $A_{RS^4}$ (1). Then any continuous positive form on the Borchers algebra $A$ (3)
is represented by a collection of tempered distributions \( \{W_k \in S'(\mathbb{R}^{4k})\} \) such that
\[
f(\psi_k) = \int W_k(x_1, \ldots, x_k) \psi_k(x_1, \ldots, x_k) d^4x_1 \cdots d^4x_k, \quad \psi_k \in RS^{4k}.
\] (4)

In many cases, the \( k \)-point distributions \( W_k, \quad k > 2 \), are expressed into the two-point ones \( W_2 \) due to the Wick theorem relations
\[
W_k(x_1, \ldots, x_q) = \sum W_2(x_{i_1}, x_{i_2}) \cdots W_2(x_{i_{n-1}}, x_{i_k}),
\]
where the sum runs through all partitions of the set \( 1, \ldots, k \) in ordered pairs \( (i_1 < i_2), \ldots, (i_{k-1} < i_k) \).

For instance, the states of scalar quantum fields in the Minkowski space (see Appendix E for the case of free scalar fields) are described by the Wightman functions \( W_n \subset S'(\mathbb{R}^{4k}) \) in the Minkowski space which obey the Garding–Wightman axioms of axiomatic field theory [16, 69, 76, 78]. Let us mention the Poincaré covariance axiom, the spectrum condition and the locality condition. In particular, the Poincaré covariance condition implies the translation invariance and the Lorentz covariance of Wightman functions. Due to the translation invariance of Wightman functions \( W_k \), there exist tempered distributions \( w_k \in S'(\mathbb{R}^{4k-4}) \), also called Wightman functions, such that
\[
W_k(x_1, \ldots, x_k) = w_k(x_1 - x_2, \ldots, x_{k-1} - x_k).
\] (5)

Note that Lorentz covariant tempered distributions for one argument only are well described [16, 79].

In order to modify Wightman’s theory, one studies different classes of distributions which Wightman functions belong to [71, 73]. To involve odd fields, one considers superdistributions as continuous mappings of a certain space of superfunctions to a nuclear graded commutative algebra [52].

A problem is that there are still no interacting models of the Wightman axioms. In QFT, quantum fields created at some instant and annihilated at another one are described by complete Green functions. They are given by the chronological functionals
\[
f^c(\psi_k) = \int W^c_k(x_1, \ldots, x_k) \psi_k(x_1, \ldots, x_k) d^4x_1 \cdots d^4x_k, \quad \psi_k \in RS^{4k},
\] (6)
\[
W^c_k(x_1, \ldots, x_k) = \sum_{(i_1 \ldots i_k)} \theta(x^0_{i_1} - x^0_{i_2}) \cdots \theta(x^0_{i_{k-1}} - x^0_{i_k}) W_k(x_1, \ldots, x_k),
\] (7)
where \( W_k \in S'(\mathbb{R}^{4k}) \) are tempered distributions, \( \theta \) is the Heaviside function, and the sum runs through all permutations \( (i_1 \ldots i_k) \) of the tuple of numbers \( 1, \ldots, k \) [15]. A problem is that the functionals \( W^c_k \) (7) need not be tempered distributions (see Appendix D). For instance, \( W^c_1 \in S'(\mathbb{R}) \) iff \( W_1 \in S'(\mathbb{R}_\infty) \), where \( \mathbb{R}_\infty \) is the compactification of \( \mathbb{R} \) by means of the point \( \{+\infty\} = \{-\infty\} \) [16]. Moreover, the chronological forms are not positive. Therefore, they do not provide states of the Borchers algebra \( A_{RS^4} \) in general.

At the same time, the chronological forms (7) come from the Wick rotation of Euclidean states of the Borchers algebra [61, 62, 64] (see Appendix F). As is well known,
the Wick rotation enables one to compute the Feynman diagrams of perturbative QFT by means of Euclidean propagators. Let us suppose that it is not a technical trick, but quantum fields in an interaction zone are really Euclidean. It should be emphasized that the above mentioned Euclidean states differ from the well-known Schwinger functions in the Osterwalder–Shrädinger Euclidean QFT [16, 56, 70, 69, 78]. The Schwinger functions are the Laplace transform of Wightman functions, but not chronological forms (see Appendix G). Note that the Euclidean counterpart of time ordered correlation functions is also considered in the Euclidean quantum field theory, but not by means of the Wick rotation [40]. Usually, the Wick rotation in scalar field theory is studied. There is a problem of describing the Wick rotation on a curved space-time [47] and in spinor geometry [50]. To solve this problem, a complex space-time can be called into play [29].

Since the chronological forms (7) are symmetric, the Euclidean states of the tensor algebra $A_{RS^4}$ can be obtained as states of the corresponding commutative tensor algebra $B_{RS^4}$ [61, 62]. Therefore, let $\Phi$ be a nuclear space and $B_\Phi$ a commutative tensor algebra of $\Phi$. Provided with the direct sum topology, $B_\Phi$ becomes a topological involutive algebra. It coincides with the enveloping algebra of the Lie algebra of the additive Lie group $T(\Phi)$ of translations in $\Phi$. Therefore, one can obtain the states of the algebra $B_\Phi$ by constructing cyclic strongly continuous unitary representations of the nuclear Abelian group $T(\Phi)$ (see Appendix H). Such a representation is characterized by a continuous positive-definite generating function $Z$ on $\Phi$. By virtue of the Bochner theorem [14, 33], this function is the Fourier transform

$$Z(\phi) = \int \exp[i\langle \phi, w \rangle] d\mu(w)$$

of a positive measure $\mu$ of total mass 1 on the topological dual $\Phi'$ of $\Phi$. Then the above mentioned representation $\pi$ of $T(\Phi)$ can be given by the operators

$$\hat{\phi}u(w) = \exp[i\langle \phi, w \rangle]u(w)$$

in the Hilbert space $L^2_{C}(\Phi', \mu)$ of the equivalence classes of square $\mu$-integrable complex functions $u(w)$ on $\Phi'$. The cyclic vector $\theta$ of this representation is the $\mu$-equivalence class $\theta \approx \mu 1$ of the constant function $u(w) = 1$.

Conversely, every positive measure $\mu$ of total mass 1 on the topological dual $\Phi'$ of $\Phi$ defines the cyclic strongly continuous unitary representation (9) of the group $T(\Phi)$. One can show that distinct generating functions $Z$ and $Z'$ characterize equivalent representations $T_Z$ and $T_{Z'}$ (9) of $T(\Phi)$ in the Hilbert spaces $L^2_{c}(\Phi', \mu)$ and $L^2_{c}(\Phi', \mu')$ iff they are the Fourier transform of equivalent measures on $\Phi'$.

If the function $\alpha \rightarrow Z(\alpha \phi)$ on $\mathbb{R}$ is analytic at 0 for each $\phi \in \Phi$, a state $f$ of $B_\Phi$ is given by the expression

$$f_k(\phi_1 \cdots \phi_k) = i^{-k} \frac{\partial}{\partial \alpha^1} \cdots \frac{\partial}{\partial \alpha^k} Z(\alpha^\prime \phi_i)|_{\alpha^\prime = 0} = \int \langle \phi_1, w \rangle \cdots \langle \phi_k, w \rangle d\mu(w).$$

Then one can think of $Z$ (8) as being a generating functional of complete Euclidean Green functions $f_k$ (10).
For instance, free Euclidean fields are described by Gaussian states. Their generating functionals are of the form
\[ Z(\phi) = \exp(-\frac{1}{2} M(\phi, \phi)), \]  
where \( M(\phi, \phi) \) is a positive-definite Hermitian bilinear form on \( \Phi \) continuous in each variable. In this case, the forms \( f_k \) obey the Wick theorem relations where \( f_1 = 0 \) and \( f_2 = M(\phi, \phi') \). The generating function (11) is the Fourier transform of some Gaussian measure on \( \Phi' \). In particular, let \( \Phi = RS^4 \) and \( f \) be a Gaussian state of \( B_{RS^4} \) such that the covariance form \( M \) is represented by a distribution \( M(x, x') \in S'(\mathbb{R}^8) \) which is the Green function of some positive-definite elliptic operator \( L_x = -\Delta_x + m^2 \), \( M(x, x') = \int (p^2 + m^2)^{-1} \exp(-ip(x - x'))d_4x \), where \( p^2 \) is the Euclidean scalar.

A problem is that a measure \( \mu \) in the generating functional \( Z(\phi) \) fail to be written in an explicit form. The familiar expression
\[ \mu = \exp(-\int L(\phi)d^4x) \prod_x [d\phi(x)] \]
used in perturbative QFT fails to be a true measure. Note that there is no (translationally-invariant) Lebesgue measure on infinite-dimensional vector space as a rule (see [75] for an example of such a measure). Here, we are not concerned with different formulations of functional integrals in QFT [23, 30, 36, 46, 51, 67], but follow perturbative Euclidean QFT. This is phrased in terms of symbolic functional integrals and provide Euclidean Green functions in the Feynman diagram technique.

Let us consider a Lagrangian system of even and odd fields on \( X = \mathbb{R}^n \) which is described by the DGA \( \mathcal{P}^* \) with the basis \( \{ y^a \} \) (see Appendix A). Let \( L \in \mathcal{P}^{0,n} \) be its Lagrangian which is assumed to be nondegenerate. If an original Lagrangian is degenerate, one follows the BV prequantization procedure in order to obtain a nondegenerate gauge-fixing BRST extended Lagrangian, depending on original fields and ghosts [9, 10, 32, 37]. We suppose that \( L \) is a Lagrangian of Euclidean fields on \( \mathbb{R}^n \). Let us quantize this Lagrangian system on the framework of perturbative QFT. Since the generating functional in perturbative QFT depends on the action functional one usually replaces horizontal densities, depending on jets, with local functionals evaluated for the jet prolongations of sections of \( Y \to X \) of compact support [2, 6, 18, 49]. Note that such functionals, in turn, define differential forms on functional spaces [24, 31]. In a different way, we are based on the fact that an algebra of Euclidean quantum fields is graded commutative, and there are homomorphisms of the graded commutative algebra \( \mathcal{P}^0 \) of classical fields to this algebra [9, 65].

Let \( \Phi \) be the graded complex vector space whose basis is the basis \( \{ y^a \} \) for the DGA \( \mathcal{P}^* \). Let us consider the tensor product
\[ \Phi = Q \otimes S'(\mathbb{R}^n) \]  
5
of the graded vector space $\mathcal{Q}$ and the space $S'({\mathbb{R}^n})$ of tempered distributions on $\mathbb{R}^n$. One can think of elements of $\Phi$ (12) as being $\mathcal{Q}$-valued distributions on $\mathbb{R}^n$. Let $T(\mathbb{R}^n) \subset S'({\mathbb{R}^n})$ be a subspace of functions $\exp\{ipx\}$, $p \in \mathbb{R}_n$, which are generalized eigenvectors of translations in $\mathbb{R}^n$ acting on $S(\mathbb{R}^n)$. We denote $\phi^a_p = y^a \otimes \exp\{ipx\}$. Then any element $\phi$ of $\Phi$ can be written in the form

$$\phi(x') = y^a \otimes \phi_a(x') = \int \phi_a(p) \phi^a_p d_n p,$$

where $\phi_a(p) \in S'(\mathbb{R}_n)$ are the Fourier transforms of $\phi_a(-x')$. For instance, there are the $\mathcal{Q}$-valued distributions

$$\phi^a(x') = \int \phi^a_p e^{-ipx} d_n p = y^a \otimes \delta(x-x'),$$

$$\phi^a_{x\Lambda}(x') = \int (-i)^k p_{\lambda_1} \ldots p_{\lambda_k} \phi^a_{p} e^{-ipx} d_n p.$$ (15)

In the framework of perturbative Euclidean QFT, we associate to a nondegenerate Lagrangian system $(\mathcal{P}^*, L)$ the graded commutative tensor algebra $B_\Phi$ generated by elements of the graded vector space $\Phi$ (12) and the following state $\langle \cdot \rangle$ of $B_\Phi$. For any $x \in X$, there is a homomorphism

$$\gamma_x : s^\Lambda_{a_1 \ldots a_r} y^a_{\Lambda_1} \ldots y^a_{\Lambda_r} \mapsto s^\Lambda_{a_1 \ldots a_r} (x) \phi^a_{x\Lambda_1} \ldots \phi^a_{x\Lambda_r}, \quad s^\Lambda_{a_1 \ldots a_r} \in C^\infty(X),$$

of the algebra $\mathcal{P}^0$ of classical fields to the algebra $B_\Phi$ which sends the generating elements $y^a_{\Lambda} \in \mathcal{P}^0$ to the elements $\phi^a_{x\Lambda} \in B_\Phi$, and replaces coefficient functions $s$ of elements of $\mathcal{P}^0$ with their values $s(x)$ at a point $x$. Then the above mentioned state $\langle \cdot \rangle$ of $B_\Phi$ is defined by symbolic functional integrals

$$\langle \phi_1 \cdots \phi_k \rangle = \frac{1}{\mathcal{N}} \int \phi_1 \cdots \phi_k \exp\{- \int \mathcal{L}(\phi^a_p) d^n x \} \prod_p [d\phi^a_p],$$

(17)

$$\mathcal{N} = \int \exp\{- \int \mathcal{L}(\phi^a_p) d^n x \} \prod_p [d\phi^a_p],$$

(18)

$$\mathcal{L}(\phi^a_p) = \mathcal{L}(\phi^a_{x\Lambda}) = \mathcal{L}(x, \gamma_x(y^a_{\Lambda})),$$

(19)

where $\phi_i$ and $\gamma_x(y^a_{\Lambda}) = \phi^a_{x\Lambda}$ are given by the formulas (13) and (15), respectively. The forms (17) are expressed both into the forms

$$\langle \phi^a_{p_1} \cdots \phi^a_{p_k} \rangle = \frac{1}{\mathcal{N}} \int \phi^a_{p_1} \cdots \phi^a_{p_k} \exp\{- \int \mathcal{L}(\phi^a_p) d^n x \} \prod_p [d\phi^a_p],$$

(20)

and the forms

$$\langle \phi^a_{x_1} \cdots \phi^a_{x_k} \rangle = \frac{1}{\mathcal{N}} \int \phi^a_{x_1} \cdots \phi^a_{x_k} \exp\{- \int \mathcal{L}(\phi^a_{x\Lambda}) d^n x \} \prod_x [d\phi^a_x],$$

(21)

$$\mathcal{N} = \int \exp\{- \int \mathcal{L}(\phi^a_{x\Lambda}) d^n x \} \prod_x [d\phi^a_x],$$

$$\mathcal{L}(\phi^a_{x\Lambda}) = \mathcal{L}(x, \gamma_x(s^a_{\Lambda})).$$
which provide Euclidean Green functions. As was mentioned above, the term \( \prod_p [d\phi_p^a] \) in the formulas (17) – (18) fail to be a true measure on \( T(\mathbb{R}^n) \) because the Lebesgue measure on infinite-dimensional vector spaces need not exist. Nevertheless, treated as generalization of Berezin’s finite-dimensional integrals [12], the functional integrals (20) and (21) restart Euclidean Green functions in the Feynman diagram technique. Certainly, these Green functions are singular, unless regularization and renormalization techniques are involved.

Since a graded derivation \( \vartheta \) (38) of the algebra \( P^0 \) is a \( C^\infty(X) \)-linear morphism over \( \text{Id}_X \), it induces the graded derivation

\[
\hat{\vartheta}_x = \gamma_x \circ \vartheta \circ \gamma_x^{-1} : \phi_{x\Lambda}^a \rightarrow (x, y_\Lambda^a) \rightarrow \hat{\vartheta}_\Lambda^a(x, y_\Lambda^b) = \hat{\vartheta}_\Lambda^a(\phi_{x\Sigma}^b)
\]

of the range \( \gamma_x(P^0) \subset B_{\Phi} \) of the homomorphism \( \gamma_x \) (16) for each \( x \in \mathbb{R}^n \). The maps \( \hat{\vartheta}_x \) (22) yield the maps

\[
\hat{\vartheta}_p : \phi_p^a = \int \phi_x^a e^{ipx} d^n x \rightarrow \int \hat{\vartheta}_x(\phi_x^a) e^{ipx} d^n x = \int \hat{\vartheta}_x(\phi_x^b) e^{ipx} d^n x = \hat{\vartheta}_p^a, \quad p \in \mathbb{R}^n,
\]

and, as a consequence, the graded derivation

\[
\hat{\vartheta}(\phi) = \int \phi_a(p) \hat{\vartheta}(\phi_p^a) d_a p = \int \phi_a(p) \hat{\vartheta}_p^a d_a p
\]

of the algebra \( B_{\Phi} \). It can be written in the symbolic form

\[
\hat{\vartheta} = \int u_a^p \frac{\partial}{\partial \phi_p^a} d_a p, \quad \frac{\partial \phi_p^b}{\partial \phi_p^a} = \delta_a^b \delta(p' - p), \quad (23)
\]

\[
\hat{\vartheta} = \int u_a^x \frac{\partial}{\partial \phi_x^a} d^n x, \quad \frac{\partial \phi_x^b}{\partial \phi_x^a} = \delta_a^b \delta_{x', \lambda_1} \cdots \delta_{x', \lambda_k} \delta(x' - x).
\]

Let \( \alpha \) be an odd element, and let us consider the automorphism

\[
\hat{U} = \exp\{\alpha \hat{\vartheta}\} = \text{Id} + \alpha \hat{\vartheta}
\]

of the algebra \( B_{\Phi} \) which can provide a change of variables depending on \( \alpha \) as a parameter in the functional integrals (20) and (21) [12]. This automorphism yields a new state \( \langle \cdot \rangle' \) of \( B_{\Phi} \) given by the equalities

\[
\langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \rangle = \langle \hat{U}(\phi_{x_1}^{a_1}) \cdots \hat{U}(\phi_{x_k}^{a_k}) \rangle' = \frac{1}{N'} \int \hat{U}(\phi_{x_1}^{a_1}) \cdots \hat{U}(\phi_{x_k}^{a_k}) \exp\{ - \int \mathcal{L}(\hat{U}(\phi_{x\Lambda}^a)) d^n x \} \prod_x [d\hat{U}(\phi_x^a)],
\]

\[
N' = \int \exp\{ - \int \mathcal{L}(\hat{U}(\phi_{x\Lambda}^a)) d^n x \} \prod_x [d\hat{U}(\phi_x^a)],
\]

\[
\langle \phi_1 \cdots \phi_k \rangle = \langle \hat{U}(\phi_1) \cdots \hat{U}(\phi_k) \rangle' = \frac{1}{N'} \int \hat{U}(\phi_1) \cdots \hat{U}(\phi_k) \exp\{ - \int \mathcal{L}_{GF}(\hat{U}(\phi_p^a)) d^n x \} \prod_p [d\hat{U}(\phi_p^a)],
\]

\[
N' = \int \exp\{ - \int \mathcal{L}_{GF}(\hat{U}(\phi_p^a)) d^n x \} \prod_p [d\hat{U}(\phi_p^a)].
\]
Let us apply these relations to the Green functions (20) and (21).

It follows from the decomposition (39) that
\[
\int \mathcal{L}(\hat{U}(\phi^a_{x\Lambda}))d^n x = \int (\mathcal{L}(\phi^a_{x\Lambda}) + \alpha \hat{E}_x^a \mathcal{E}_{xa})d^n x,
\]
where \( \mathcal{E}_{xa} = \gamma_x(\mathcal{E}_a) \) are the variational derivatives. It is a property of symbolic functional integrals that
\[
\prod_x [d\hat{U}(\phi^a_x)] = (1 + \alpha \int \frac{\partial \hat{E}_x}{\partial \phi^a_x} d^n x) \prod_x [d\phi^a_x] = (1 + \alpha \text{Sp}(\tilde{\vartheta})) \prod_x [d\phi^a_x].
\]

Then the equalities (26) – (27) result in the identities
\[
\langle \hat{\vartheta}(\phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k}) \rangle + \langle \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \text{Sp}(\tilde{\vartheta} - \int \hat{E}_x \mathcal{E}_{xa} d^n x) \rangle -
\]
\[
\langle \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \rangle \text{Sp}(\tilde{\vartheta}) - \int \hat{E}_x \mathcal{E}_{xa} d^n x = 0
\]
for complete Euclidean Green functions (21) and the similar identities for the Green functions \( \langle \phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k} \rangle \) (20).

If \( \tilde{\vartheta} \) is a variational symmetry of \( \mathcal{L} \), the identities (29) are the Ward identities
\[
\langle \hat{\vartheta}(\phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k}) \rangle + \langle \phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k} \text{Sp}(\tilde{\vartheta}) \rangle - \langle \phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k} \rangle = 0,
\]
\[
\sum_{i=1}^k (-1)^{[a_i] + \cdots + [a_{i-1}]} \langle \phi^{a_{i+1}}_{p_{i+1}} \cdots \phi^{a_k}_{p_k} \hat{\vartheta} \rangle - \langle \phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k} \rangle = 0,
\]
\[
\langle \hat{\vartheta}(\phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k}) \rangle + \langle \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \text{Sp}(\tilde{\vartheta}) \rangle - \langle \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \rangle = 0,
\]
\[
\sum_{i=1}^k (-1)^{[a_i] + \cdots + [a_{i-1}]} \langle \phi^{a_{i+1}}_{x_{i+1}} \cdots \phi^{a_k}_{x_k} \hat{\vartheta} \rangle - \langle \phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k} \rangle = 0,
\]
generalizing the Ward (Slavnov–Taylor) identities in gauge theory [19, 32, 38, 49]. A glance at the expressions (30) – (31) shows that these Ward identities generally contain anomaly because the measure terms of symbolic functional integrals need not be \( \hat{\vartheta} \)-invariant. If \( \text{Sp}(\tilde{\vartheta}) \) is either a finite or infinite number, the Ward identities
\[
\langle \hat{\vartheta}(\phi^{a_1}_{p_1} \cdots \phi^{a_k}_{p_k}) \rangle = \sum_{i=1}^k (-1)^{[a_i] + \cdots + [a_{i-1}]} \langle \phi^{a_{i+1}}_{p_{i+1}} \cdots \phi^{a_k}_{p_k} \hat{\vartheta} \rangle = 0,
\]
\[
\langle \hat{\vartheta}(\phi^{a_1}_{x_1} \cdots \phi^{a_k}_{x_k}) \rangle = \sum_{i=1}^k (-1)^{[a_i] + \cdots + [a_{i-1}]} \langle \phi^{a_{i+1}}_{x_{i+1}} \cdots \phi^{a_k}_{x_k} \hat{\vartheta} \rangle = 0
\]
are free of this anomaly.

If \( \vartheta = c^a \partial_a \), \( c^a \) =const, the identities (29) take the form

\[
\sum_{r=1}^{k} (-1)[a_1]+[a_2]+[a_r]\langle \phi_{x_1}^{a_1} \cdots \phi_{x_{r-1}}^{a_{r-1}} \delta_1^{a_r} \phi_{x_{r+1}}^{a_{r+1}} \cdots \phi_{x_k}^{a_k} \rangle - \\
\langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \int \mathcal{E}_{x_1} ^a d^n x \rangle + \langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \rangle (\int \hat{E}_{x_1} ^a d^n x) = 0. 
\]

(34)

One can think of them as being equations for complete Euclidean Green functions, but they are not an Euclidean variant of the well-known Schwinger–Dyson equations [15]. For instance, they identically hold if a Lagrangian \( L \) is quadratic.

Clearly, the expressions (29), (34) are singular, unless one follows regularization and renormalization procedures, which however can induce additional anomaly terms.

**Appendix A. Classical field theory in jet formalism**

We consider a Lagrangian field system on \( X = \mathbb{R}^n \), coordinatized by \( (x^\lambda) \). Such a Lagrangian system is algebraically described in terms of the following differential graded algebra (henceforth GDA) \( \mathcal{P}^* \) [6, 7, 34].

Let \( Y = Y_0 \oplus Y_1 \rightarrow X \) be a graded vector bundle coordinatized by \( (x^\lambda, y^a) \). Finite order jet manifolds \( J^rY \), \( r = 1, \ldots \), of \( Y \rightarrow X \) are also vector bundles over \( X \) coordinatized by \( (x^\lambda, y^a, y^a_\Lambda) \), \( |\Lambda| = k \leq r \), where \( \Lambda = (\lambda_1, \ldots, \lambda_k) \) denote symmetric multi-indices. The index \( r = 0 \) conventionally stands for \( Y \). For each \( r = 0, \ldots \), we consider a graded manifold \( (X, \mathcal{A}_{J^rY_1}) \), whose body is \( X \) and the algebra of graded functions consists of sections of the exterior bundle

\[
\wedge (J^rY_1)^* = \mathbb{R} \oplus (J^rY_1)^* \oplus \mathbb{R} \oplus (J^rY_1)^* \oplus \cdots,
\]

where \( (J^rY_1)^* \) is the dual of a vector bundle \( J^rY_1 \rightarrow X \). The global basis for \( (X, \mathcal{A}_{J^rY_1}) \) is \( \{x^\lambda, y^a_\Lambda\}, |\Lambda| = 0, \ldots, r \). Let us consider the graded commutative \( \mathcal{C}^\infty(X) \)-algebra \( \mathcal{P}^0 \) generated by its elements \( y^a_\Lambda \), treated as prequantum even and odd fields and their jets. The symbol \( [a] = [y^a] = [y^a_\Lambda] \) stands for their Grassmann parity.

Let \( \mathfrak{d}\mathcal{P}^0 \) be the Lie superalgebra of graded derivations of the \( \mathbb{R} \)-algebra \( \mathcal{P}^0 \), i.e.,

\[
u(f f') = \nu(f) f' + (-1)^{|a||f|} f \nu(f'), \quad f, f' \in \mathcal{P}^0, \quad \nu \in \mathfrak{d}\mathcal{P}^0.
\]

Its elements take the form

\[
u = \nu^\lambda \partial_\lambda + \sum_{0 \leq |\Lambda|} u^a_\Lambda \partial_a^\Lambda, \quad \nu^\lambda, u^a_\Lambda \in \mathcal{P}^0.
\]

(35)

With the Lie superalgebra \( \mathfrak{d}\mathcal{P}^0 \), one can construct the minimal Chevalley–Eilenberg differential calculus

\[
0 \rightarrow \mathbb{R} \rightarrow \mathcal{P}^0 \xrightarrow{d} \mathcal{P}^1 \xrightarrow{d} \cdots \mathcal{P}^2 \xrightarrow{d} \cdots.
\]

(36)
over the \( \mathbb{R} \)-algebra \( P^0 \). It is the above mentioned DGA \( P^* \) with the basis \( \{ y^a \} \). Its elements \( \sigma \in P^k \) are graded \( P^0 \)-linear \( k \)-forms

\[
\sigma = \sum_{a_1, \ldots, a_r} \sigma^{a_1, \ldots, a_r} \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_r} \wedge dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k}
\]
on \( \partial P^0 \) with values in \( P^0 \). The graded exterior product \( \wedge \) and the graded exterior differential, obey the relations

\[
\sigma \wedge \sigma' = (-1)^{|\sigma||\sigma'| + |\sigma'||\sigma|} \sigma' \wedge \sigma,
\]

\[
d(\sigma \wedge \sigma') = d\sigma \wedge \sigma' + (-1)^{|\sigma|} \sigma \wedge d\sigma',
\]

where \(|.|\) denotes the form degree. By \( \mathcal{O}^*X \) is denoted the graded differential algebra of exterior forms on \( X \). There is the natural monomorphism \( \mathcal{O}^*X \to P^* \).

Given a graded derivation \( u \) (35) of the \( \mathbb{R} \)-algebra \( P^0 \), the interior product \( u|\sigma \) and the Lie derivative \( L_u \sigma, \sigma \in P^* \), obey the relations

\[
u| (\sigma \wedge \sigma') = (u|\sigma) \wedge \sigma' + (-1)^{|\sigma|+|u||\sigma|} \sigma \wedge (u|\sigma'),
\]

\[
\mathbf{L}_u \sigma = u|d\sigma + d(u|\sigma), \quad \mathbf{L}_u (\sigma \wedge \sigma') = \mathbf{L}_u (\sigma) \wedge \sigma' + (-1)^{|u||\sigma|} \sigma \wedge \mathbf{L}_u (\sigma').
\]

The DGA \( P^* \) is decomposed into \( P^0 \)-modules \( P^{k,r} \) of \( k \)-contact and \( r \)-horizontal graded forms

\[
\sigma = \sum_{0 \leq |\Lambda|} \sigma^{\Lambda_1, \ldots, \Lambda_k}_a \theta^{a_1}_{\Lambda_1} \wedge \cdots \wedge \theta^{a_k}_{\Lambda_k} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}, \quad \theta_a^{\Lambda} = dy_{\Lambda} - y_a^\Lambda dx^\Lambda.
\]

Accordingly, the graded exterior differential on \( P^* \) falls into the sum \( d = d_V + d_H \) of the vertical and total differentials where \( d_H \sigma = dx^\Lambda \wedge d_\Lambda \sigma \). The differentials \( d_H \) and \( d_V \) and the graded variational operator \( \delta \) split the DGA \( P^* \) into the graded variational bicomplex [6, 7, 34]. One can think of even elements

\[
L = \mathcal{L}(x^\Lambda, y_a^\Lambda)dx^a, \quad \delta L = dy^a \wedge \epsilon_a dx^a = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} dy^a \wedge d_\Lambda (\partial_a^\Lambda L)dx^a
\]
of the differential algebra \( P^* \) as being a graded Lagrangian and its Euler–Lagrange operator, respectively.

A graded derivation \( u \) (35) is called contact if the Lie derivative \( \mathbf{L}_u \) preserves the ideal of contact graded forms of the DGA \( P^* \). Here, we restrict our consideration to vertical contact graded derivations, vanishing on \( \mathcal{O}^*X \). Such a derivation takes the form

\[
\vartheta = v^a \partial_a + \sum_{0 < |\Lambda|} d_\Lambda v^a \partial_a^\Lambda.
\]

It is the jet prolongation of its first summand \( v = v^a \partial_a \). The Lie derivative \( \mathbf{L}_\vartheta L \) of a Lagrangian \( L \) (37) along a vertical contact graded derivation \( \vartheta \) (38) admits the decomposition

\[
\mathbf{L}_\vartheta L = v|\delta L + d_H \sigma.
\]
One says that an odd vertical contact graded derivation $\vartheta$ (38) is a variational supersymmetry of a Lagrangian $L$ if the Lie derivative $L_\vartheta L$ is $d_H$-exact.

**Appendix B. Unbounded operators**

Recall that by an operator in a Hilbert space $E$ is meant a linear morphism $a$ of a dense subspace $D(a)$ of $E$ to $E$. The $D(a)$ is called a domain of an operator $a$. One says that an operator $b$ on $D(b)$ is an extension of an operator $a$ on $D(a)$ if $D(a) \subset D(b)$ and $b|_{D(a)} = a$. For the sake of brevity, we will write $a \subset b$. An operator $a$ is said to be bounded on $D(a)$ if there exists a real number $r$ such that $\|ae\| \leq r\|e\|$, $e \in D(a)$. If otherwise, it is called unbounded. Any bounded operator on a domain $D(a)$ is uniquely extended to a bounded and continuous operator everywhere on $E$.

An operator $a$ on a domain $D(a)$ is called closed if the condition that a sequence $\{e_i\} \subset D(a)$ converges to $e \in E$ and that the sequence $\{ae_i\}$ does to $e' \in E$ implies that $e \in D(a)$ and $e' = ae$. An operator $a$ on a domain $D(a)$ is called closable if it can be extended to a closed operator. The closure of a closable operator $a$ is defined as the minimal closed extension of $a$.

Operators $a$ and $b$ in $E$ are called adjoint if $\langle ae|e'\rangle = \langle e|be'\rangle$, $e \in D(a)$, $e' \in D(b)$. Any operator $a$ has a maximal adjoint operator $a^*$, which is closed. An operator $a$ is called symmetric if it is adjoint to itself, i.e., $a \subset a^*$. Hence, a symmetric operator is closable. At the same time, the maximal adjoint operator $a^*$ of a symmetric operator $a$ need not be symmetric. A symmetric operator $a$ is called self-adjoint if $a = a^*$, and it is called essentially self-adjoint if $\pi = a^* = \pi^*$ (see here follow the terminology of [58]). For bounded operators, the notions of symmetric, self-adjoint and essentially self-adjoint operators coincide.

Let $E$ be a Hilbert space. The pair $(B,D)$ of a dense subspace $D$ of $E$ and a unital subalgebra $B \subset B(E)$ of (unbounded) operators in $E$ is called the $Op^*$-algebra ($O^*$-algebra in the terminology of [68]) on the domain $D$ if, whenever $b \in B$, we have: (i) $D(b) = D$ and $bD \subset D$, (ii) $D \subset D(b^*)$, (iii) $b^*|_D \subset D$ [43, 58]. The algebra $B$ is provided with the involution $b \mapsto b^+ = b^*|_D$, and its elements are closable. It is important that one can associate to an $Op^*$-algebra the von Neumann algebra which is the weak bicommutant $B''$, where $B' = \{T \in B(E) | \langle ae,T^*e'\rangle, a \in B, e,e' \in D\}$.

A representation $\pi(A)$ of an involutive algebra $A$ in a Hilbert space $E$ is an $Op^*$-algebra if there exists a dense subspace $D(\pi) \subset E$ such that $D(\pi) = D(\pi(a))$ for all $a \in A$. If a representation $\pi$ is Hermitian, i.e., $\pi(a^*) \subset \pi(a)^*$ for all $a \in A$, then $\pi(A)$ is an $Op^*$-algebra. In this case, one also considers the representations

$$\pi: a \rightarrow \pi(a) := \pi(a)|_{D(\pi)}, \quad D(\pi) = \bigcap_{a \in A} D(\pi(a)),$$

$$\pi^*: a \rightarrow \pi^*(a) := \pi(a^*)|_{D(\pi^*)}, \quad D(\pi^*) = \bigcap_{a \in A} D(\pi(a)^*),$$

called the closure of a representation $\pi$ and an adjoint representation, respectively. There are the representation extensions $\pi \subset \pi \subset \pi^*$, where $\pi_1 \subset \pi_2$ means $D(\pi_1) \subset D(\pi_2)$. The
representation \( \pi \) is Hermitian, while \( \pi^* = \pi^* \). A Hermitian representation \( \pi(A) \) is said to be closed if \( \pi = \pi \), and it is self-adjoint if \( \pi = \pi^* \). Herewith, a representation \( \pi(A) \) is closed (resp. self-adjoint) if one of operators of \( \pi(A) \) is closed (resp. self-adjoint).

The representation domain \( D(\pi) \) is endowed with the graph-topology. It is generated by the neighborhoods of the origin

\[
U(M, \varepsilon) = \{ x \in D(\pi) : \sum_{a \in M} \|\pi(a)x\| < \varepsilon \},
\]

where \( M \) is a finite subset of elements of \( A \). All operators of \( \pi(A) \) are continuous with respect to this topology. Let us note that the graph-topology is finer than the relative topology on \( D(\pi) \subset E \), unless all operators \( \pi(a), a \in A \), are bounded [68]. Let \( \overline{N}^g \) denote the closure of a subset \( N \subset D(\pi) \) with respect to the graph-topology. An element \( \theta \in D(\pi) \) is called strongly cyclic (cyclic in the terminology of [68]) if

\[
D(\pi) \subset (\pi(A)\theta)^g.
\]

In application to QFT, the following class of involutive algebras should be mentioned. Let \( A \) be a locally convex topological involutive algebra whose topology is defined by a set of multiplicative seminorms \( p_\iota \) which satisfy the condition \( p_\iota(a^*a) = p_\iota(a)^2, a \in A \). It is called a \( b^* \)-algebra. A unital \( b^* \)-algebra as like as a \( C^* \)-algebra is regular and symmetric, i.e., any element \( (1 + a^*a), a \in A \), is invertible and, moreover, \( (1 + a^*a)^{-1} \) is bounded [4, 45]. The \( b^* \)-algebras are related to \( C^* \)-algebras as follows. Any \( b^* \)-algebra is the Hausdorff projective limit of a family of \( C^* \)-algebras, and vice versa [45]. In particular, every \( C^* \)-algebra \( A \) is a barreled \( b^* \)-algebra, i.e., every absorbing balanced closed subset is a neighborhood of the origin of \( A \).

**Appendix C. Nuclear spaces**

Physical applications of Hilbert spaces are limited by the fact that the dual of a Hilbert space \( E \) is anti-isomorphic to \( E \). The construction of a rigged Hilbert space describes the dual pairs \( (E, E^*) \) where \( E^* \) is larger than \( E \) [33, 57].

Let a complex vector space \( E \) have a countable set of non-degenerate Hermitian forms \( \langle .| . \rangle_k, k \in \mathbb{N}_+ \), such that

\[
\langle e|e \rangle_1 \leq \cdots \leq \langle e|e \rangle_k \leq \cdots
\]

for all \( e \in E \). The family of norms

\[
\| . \|_k = \langle .| . \rangle_k^{1/2}, \quad k \in \mathbb{N}_+,
\]

yields a Hausdorff topology on \( E \). The space \( E \) is called a countably Hilbert space if it is complete with respect to this topology. For instance, every Hilbert space is a countably Hilbert space where all Hermitian forms \( \langle .| . \rangle_k \) coincide. Let \( E_k \) denote the completion of \( E \) with respect to the norm \( \| . \|_k \) (40). There is the chain of injections

\[
E_1 \supset E_2 \supset \cdots \supset E_k \supset \cdots
\]
together with the homeomorphism \( E = \bigcap_k E_k \). The dual spaces form the increasing chain

\[
E_1' \subset E_2' \subset \cdots \subset E_k' \subset \cdots,
\]

and \( E' = \bigcup_k E_k' \). The dual \( E' \) of \( E \) can be provided with the weak* and strong topologies (we follow the terminology of [59]). One can show that a countably Hilbert space is reflexive.

Given a countably Hilbert space \( E \) and \( m \leq n \), let \( T^m_n \) be a prolongation of the map

\[
E_n \supset E \ni e \mapsto e \in E \subset E_m
\]
to the continuous map of \( E_n \) onto a dense subset of \( E_m \). A countably Hilbert space \( E \) is called a nuclear space if, for any \( m \), there exists \( n \) such that \( T^m_n \) is a nuclear map, i.e.,

\[
T^m_n(e) = \sum_i \lambda_i \langle e|e^i_n \rangle e^i_m,
\]

where: (i) \( \{e^i_n\} \) and \( \{e^i_m\} \) are bases for the Hilbert spaces \( E_n \) and \( E_m \), respectively, (ii) \( \lambda_i \geq 0 \), (iii) the series \( \sum \lambda_i \) converges.

An important property of nuclear spaces is that they are perfect, i.e., every bounded closed set in a nuclear space is compact. It follows immediately that a Banach (and Hilbert) space is not nuclear, unless it is finite-dimensional. Since a nuclear space is perfect, it is separable, and the weak* and strong topologies (and, consequently, all topologies of uniform convergence) on a nuclear space \( E \) and its dual \( E' \) coincide.

Let \( E \) be a nuclear space, provided with still another non-degenerate Hermitian form \( \langle .|\cdot \rangle \) which is separately continuous, i.e., continuous with respect to each argument. It follows that there exist numbers \( M \) and \( m \) such that \( \langle e|e \rangle \leq M\|e\|_m \), \( e \in E \). Let \( \tilde{E} \) denote the completion of \( E \) with respect to this form. There are the injections

\[
E \subset \tilde{E} \subset E',
\]

where \( E \) is a dense subset of \( \tilde{E} \) and \( \tilde{E} \) is a dense subset of \( E' \), equipped with the weak* topology. The triple (43) is called the rigged Hilbert space. Furthermore, bearing in mind the chain of Hilbert spaces (41) and that of their duals (42), one can convert the triple (43) into the chain of spaces

\[
E \subset \cdots \subset E_k \subset \cdots \subset E_1 \subset \tilde{E} \subset E'_1 \subset \cdots \subset E'_k \subset \cdots \subset E'.
\]

**Appendix D. Generalized functions**

By generalized functions were initially meant the Schwartz and tempered distributions [16, 33].

We further follow the standard multi-index notation

\[
D^\alpha = \frac{\partial^{\vert\alpha\vert}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},
\]

13
where \( x^{\alpha_i} \) are Cartesian coordinates on \( \mathbb{R}^n \) and \( \alpha \) is an ordered \( r \)-tuple of non-negative integers \((\alpha_1, \ldots, \alpha_r)\) and \( |\alpha| = \alpha_1 + \cdots + \alpha_r \).

Let \( \mathcal{D}(\mathbb{R}^n) \) be the space of smooth complex functions on \( \mathbb{R}^n \) of compact support. It is provided with the topology determined by the seminorms

\[
p_{\{\phi_\alpha\}}(f) = \sup_x |\sum_\alpha \phi_\alpha(x) D^\alpha f(x)|, \tag{45}
\]

where \( \{\phi_\alpha\} \) are collections of smooth functions such that, on any compact subset of \( \mathbb{R}^n \), only a finite number of these functions differ from zero. With this topology, \( \mathcal{D}(\mathbb{R}^n) \) is a complete locally convex nuclear space. Its topological dual \( \mathcal{D}'(\mathbb{R}^n) \) is called the space of Schwartz distributions on \( \mathbb{R}^n \) [16, 44, 57].

The space \( \mathcal{D}(\mathbb{R}^n) \) is a subspace of another nuclear space, whose elements are smooth functions of rapid decrease. These are complex smooth functions \( \psi(x) \) on \( \mathbb{R}^n \) such that the quantities

\[
\|\psi\|_{k,m} = \max_{|\alpha| \leq k} (1 + x^2)^m \sup_x |D^\alpha \psi(x)| \tag{46}
\]

are finite for all \( k, m \in \mathbb{N} \). The space \( S(\mathbb{R}^n) \) of these functions (called the Schwartz space) is a nuclear space with respect to the topology determined by seminorms (46) [44, 57]. Its dual \( S'(\mathbb{R}^n) \) is the space of tempered distributions. The corresponding contraction form is written as

\[
\langle \psi, h \rangle = \int \psi(x) h(x) d^n x, \quad \psi \in S(\mathbb{R}^n), \quad h \in S'(\mathbb{R}^n).
\]

The space \( S(\mathbb{R}^n) \) is provided with the non-degenerate separately continuous Hermitian form

\[
\langle \psi | h \rangle = \int \overline{\psi(x)} h(x) d^n x. \tag{47}
\]

The completion of \( S(\mathbb{R}^n) \) with respect to this form is the space \( L^2_C(\mathbb{R}^n) \) of square integrable complex functions on \( \mathbb{R}^n \). We have the rigged Hilbert space

\[
S(\mathbb{R}^n) \subset L^2_C(\mathbb{R}^n) \subset S'(\mathbb{R}^n).
\]

Let \( \mathbb{R}_n \) be the dual of \( \mathbb{R}^n \) and \( (p_\alpha) \) coordinates on \( \mathbb{R}_n \). It is important that the Fourier transform

\[
\psi^F(p) = \int \psi(x) e^{ipx} d^n x, \quad px = p_\alpha x^\alpha, \tag{48}
\]

\[
\psi(x) = \int \psi^F(p) e^{-ipx} d_n p, \quad d_n p = (2\pi)^{-n} d^n p, \tag{49}
\]

defines an isomorphism between the spaces \( S(\mathbb{R}^n) \) and \( S(\mathbb{R}_n) \). The Fourier transform of tempered (and Schwartz) distributions \( h \) is defined by the condition

\[
\int h(x) \psi(x) d^n x = \int h^F(p) \psi^F(-p) d_n p,
\]

14
written in the form (48) – (49). It provides an isomorphism between spaces of tempered distributions \( S'(\mathbb{R}^n) \) and \( S'(\mathbb{R}_n) \).

Distributions are also elements of Sobolev spaces [1, 53]. Given a domain \( U \subset \mathbb{R}^n \), let \( L^p(U) \), \( 1 \leq p < \infty \), be the vector space of all measurable real functions on \( U \) such that

\[
\int_U |f(x)|^p d^n x < \infty.
\]

It is a Banach space with respect to the norm

\[
\|f\|_p = \left\{ \int_U |f(x)|^p d^n x \right\}^{1/p}.
\]

Of course, functions are identified in the space if they are equal almost everywhere in \( U \). Sobolev spaces are defined over an arbitrary domain \( U \subset \mathbb{R} \) as the following vector subspaces of spaces \( L^p(U) \). Let \( C^k(U) \) be the space of \( k \)-times differentiable function on \( U \). Let us define the functional

\[
\|f\|_{k,p} = \left\{ \sum_{|\alpha| \leq k} \|D^\alpha f\|_p^p \right\}^{1/p}, \quad k = 0, 1, \ldots,
\]

for any \( f \in C^k(U) \) on \( U \) for which the right side of (50) makes sense. The Sobolev space \( \mathcal{H}^{k,p}(U) \) is defined as the completion of the set

\[
\{ f \in C^k(U) : \|f\|_{k,p} < \infty \}
\]

with respect to the norm (50). It is important that the subset of smooth elements of \( \mathcal{H}^{k,p}(U) \) is dense in \( \mathcal{H}^{k,p}(U) \). Conversely, the Sobolev imbedding theorem states that \( \mathcal{H}^{k,p}(\mathbb{R}^n) \subset C^l(\mathbb{R}^n) \) if \( k - n/p > l \). In particular, \( \mathcal{H}^k \subset C^n(\mathbb{R}^n) \) if \( k > n/2 \). The Sobolev space \( \mathcal{H}^{k,p}(U) \) is a separable Banach space. It is reflexive, unless \( p = 1 \). In physical applications, one usually deals with Sobolev spaces of \( p = 2 \). Let us denote \( \mathcal{H}^k = \mathcal{H}^{k,2}(\mathbb{R}^n) \). It is a separable Hilbert space with respect to the Hermitian form

\[
\langle f | f' \rangle = \sum_{0 \leq |\alpha| \leq k} \int D^\alpha f D^\alpha f' d^n x.
\]

By a Sobolev space is also meant the closure \( W_0^{k,p}(U) \) of a set of smooth functions of compact support in \( \mathcal{H}^{k,p}(U) \). In particular, \( W_0^{k,p}(\mathbb{R}^n) = H^{k,p}(\mathbb{R}^n) \), but such an isomorphism need not hold for an arbitrary open subset \( U \subset \mathbb{R}^n \).

The derivatives \( D^\alpha \) are extended to elements \( f \) of \( \mathcal{H}^{k,p}(U) \) which are not differentiable functions as distributional derivatives. Namely, \( D^\alpha f \) is defined as a locally integrable function on \( U \) such that

\[
\int_U f D^\alpha f' d^n x = (-1)^{|\alpha|} \int_U D^\alpha f f' d^n x.
\]
for any smooth function $f'$ on $U$ of compact support. It follows that such a derivative is a Schwartz distribution on $U$.

The notion of a Sobolev space is extended an arbitrary real $k$. The Sobolev space $H^{k,p}(U)$, $k \in \mathbb{R}$, consists of those functions and Schwartz distributions $f$ for which the norm

$$\|f\|_{k,p} = \left\{ \int |\hat{f}(\xi)(1 + \xi^2)^{k/2}|^p d\xi \right\}^{1/p},$$

where $\hat{f}$ is the Fourier transform of $f$, is finite. If $k$ is a non-negative integer, this definition is equivalent to the above mentioned one. In particular, one can show that $H^{-k}$, $k > 0$, is the dual of $H^k$ so that the elements of $H^{-k}$ are distributions. Namely, there are the inclusions

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \cdots \subset H^k \subset \cdots \subset H^0 = L^0(\mathbb{R}^n) \subset H^{-1} \subset \cdots \subset H^{-k} \subset \cdots \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

Dealing with Schwartz and tempered distributions, one meets a problem that their multiplication $hh'$ is not defined, unless they are smooth functions $h, h' \in C^\infty(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. For instance, let $\theta \in \mathcal{S}'(\mathbb{R}^n)$ be the Heaviside function ($\theta(x) = 0$ if $x < 0$, $\theta(x) = 1$ if $x > 0$). We have

$$\theta^2 = \theta. \tag{51}$$

Then by differentiation

$$2\theta' \theta = \theta'. \tag{52}$$

Multiplication by $\theta$ results in $2\theta' \theta = \theta' \theta'$ and, consequently, $2\theta' = \theta'$, where $\theta' = \delta_0 \in \mathcal{S}'(\mathbb{R}^n)$ is the Dirac delta-function. This absurd result is either a consequence of the multiplication rule (51) or the differentiation one (52). Keeping the operation of differentiating distributions, one can not use the classical product of functions for their multiplication as distributions.

To overcome this difficulty, one enlarge the space $\mathcal{D}'(\mathbb{R}^n)$ of Schwartz distributions to an algebra $\mathcal{G}(\mathbb{R}^n)$, which is the quotient of a subalgebra of moderate elements of the ring of smooth functions on $\mathcal{D}(\mathbb{R}^n) [25, 26]$. Elements of $\mathcal{G}(\mathbb{R}^n)$ are called nonlinear (or Colombeau) generalized functions. Let $\odot$ denote the multiplication in $\mathcal{G}(\mathbb{R}^n)$. There is the canonical monomorphism

$$C^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n). \tag{53}$$

Moreover, $C^\infty(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$ is a monomorphism of algebras, i.e., $ff' = f \odot f'$, $f, f' \in C^\infty(\mathbb{R}^n)$. However, this property is not extended to continuous functions. For instance, $x|x| \not= x \odot |x|$. If a certain condition holds, one can associate to an element of $\phi \in \mathcal{G}(\mathbb{R}^n)$ a distribution $\gamma(\phi) \in \mathcal{D}'(\mathbb{R}^n)$, which is sui generis a projection of $\phi$ on $\mathcal{D}'(\mathbb{R}^n)$. For instance, any distribution $h \in \mathcal{D}'(\mathbb{R}^n)$ admits an associated distribution which is $h$ itself. At the same time, the generalized function $\delta_0 \odot \delta_0$ has no associated distribution. If $f$ and $f'$ are continuous
functions, their product \( f \odot f' \) in \( G(\mathbb{R}^n) \) admits an associated distribution which is their product \( ff' \) in \( C^0(\mathbb{R}^n) \), i.e., \( \gamma(f \odot f') = ff' \). If \( f \in C^\infty(\mathbb{R}^n) \) and \( h \in \mathcal{D}'(\mathbb{R}^n) \), the product \( f \odot h \) possesses an associated distribution which the classical product \( fh \) of the distribution theory.

The canonical monomorphism (53) is lost when passing from \( \mathbb{R}^n \) to an arbitrary manifold. A version of the Colombeau algebra on manifolds has been studied [39] and applied to different geometric constructions [48, 72]. Some particular (\( G^\infty \)-regular and \( \mathcal{R} \)-regular) classes of nonlinear generalized functions are considered [27, 54] and their different generalizations (e.g., generalized Sobolev algebras) are suggested [13].

Appendix E. Free scalar fields

The following states of the Borchers algebra \( A_{RS^4} \) describe free real scalar fields of mass \( m \) [62].

Let us provide the nuclear space \( RS^4 \) with the positive complex bilinear form

\[
(\psi|\psi') = \frac{2}{i} \int \psi(x)D^-(x - y)\psi'(y)d^4xd^4y = \int \psi^F(-\omega, -\vec{p})\psi'^F(\omega, \vec{p}) \frac{d^4p}{\omega},
\]

(54)

\[D^-(x) = i(2\pi)^{-3} \int \exp[-ipx]\theta(p_0)\delta(p^2 - m^2)d^4p,\]

where \( D^-(x) \) is the negative frequency part of the Pauli–Jordan function, \( p^2 \) is the Minkowski square, and

\[\omega = (\vec{p}^2 + m^2)^{1/2}.\]

Since the function \( \psi(x) \) is real, its Fourier transform satisfies the equality \( \psi^F(p) = \overline{\psi^F(-p)} \). The bilinear form (54) is degenerate because the Pauli–Jordan function \( D^-(x) \) obeys the mass shell equation

\[(\Box + m^2)D^-(x) = 0.\]

It takes nonzero values only at elements \( \psi^F \in RS^4 \) which are not zero on the mass shell \( p^2 = m^2 \). Therefore, let us consider the quotient space \( \gamma : RS^4 \to RS^4/J \), where \( J = \{ \psi \in RS^4 : (\psi|\psi) = 0 \} \) is the kernel of the square form (54). The map \( \gamma \) assigns to each element \( \psi \in RS^4 \) with the Fourier transform \( \psi^F(p_0, \vec{p}) \in RS^4 \) the couple of functions \( (\psi^F(\omega, \vec{D}), \psi^F(-\omega, \vec{p})) \). Let us equip the factor space \( RS^4/J \) with the real bilinear form

\[(\gamma\psi|\gamma\psi')_L = \text{Re}(\psi|\psi') =
\]

\[\frac{1}{2} \int \left[ \psi^F(-\omega, -\vec{p})\psi'^F(\omega, \vec{p}) + \psi^F(\omega, -\vec{p})\psi'^F(-\omega, \vec{p}) \right] \frac{d^4p}{\omega}.\]

Then it is decomposed into the direct sum \( RS^4/J = L^+ \oplus L^- \) of the subspaces

\[L^\pm = \{ \psi^F_\pm(\omega, \vec{p}) = \frac{1}{2}(\psi^F(\omega, \vec{p}) \pm \psi^F(-\omega, \vec{p})) \}.\]
which are mutually orthogonal with respect to the bilinear form (55). There exist continuous isometric morphisms

$$
\gamma_+ : \psi_+^F(\omega, \overrightarrow{p}) \mapsto q^F(\overrightarrow{p}) = \omega^{-1/2} \psi_+^F(\omega, \overrightarrow{p}),
$$

$$
\gamma_- : \psi_-^F(\omega, \overrightarrow{p}) \mapsto q^F(\overrightarrow{p}) = -i\omega^{-1/2} \psi_-^F(\omega, \overrightarrow{p})
$$

of spaces $L^+$ and $L^-$ to the nuclear space $RS^3$ endowed with the nondegenerate separately continuous Hermitian form

$$
\langle q | q' \rangle = \int q^F(\overrightarrow{p}) q'^F(\overrightarrow{p}) d_3 p.
$$

(56)

It should be emphasized that the images $\gamma_+(L^+)$ and $\gamma_-(L^-)$ in $RS^3$ are not orthogonal with respect to the scalar form (56). Combining $\gamma$ and $\gamma_{\pm}$, we obtain the continuous morphisms $\tau_{\pm} : RS^4 \rightarrow RS^3$ given by the expressions

$$
\tau_+(\psi) = \gamma_+(\gamma\psi) = \frac{1}{2\omega^{1/2}} \int \left[ \psi^F(\omega, \overrightarrow{p}) + \psi^F(-\omega, \overrightarrow{p}) \right] \exp[-i \overrightarrow{p} \cdot \overrightarrow{x}] d_3 p,
$$

$$
\tau_-(\psi) = \gamma_-(\gamma\psi) = \frac{1}{2i\omega^{1/2}} \int \left[ \psi^F(\omega, \overrightarrow{p}) - \psi^F(-\omega, \overrightarrow{p}) \right] \exp[-i \overrightarrow{p} \cdot \overrightarrow{x}] d_3 p.
$$

Now let us consider the CCR algebra

$$
g(RS^3) = \{ (\phi(q), \pi(q), I), q \in RS^3 \}
$$

(57)

modeled over the nuclear space $RS^3$, which is equipped with the Hermitian form (56). Using the morphisms $\tau_{\pm}$, let us define the map

$$
RS^4 \ni \psi \mapsto \phi(\tau_+(\psi)) - \pi(\tau_-(\psi)) \in g(RS^3).
$$

With this map, any representation of the nuclear CCR algebra $g(RS^3)$ induces a state

$$
f(\psi^1 \cdots \psi^n) = \langle \phi(\tau_+(\psi^1)) + \pi(\tau_-(\psi^1)) | \cdots | \phi(\tau_+(\psi^n)) + \pi(\tau_-(\psi^n)) \rangle
$$

(58)

on the Borchers algebra $A_{RS^4}$ of scalar fields. Furthermore, one can justify that the corresponding distributions $W_n$ fulfill the mass shell equation and that the following commutation relation holds:

$$
W_2(x, y) - W_2(y, x) = -iD(x - y),
$$

where

$$
D(x) = i(2\pi)^{-3} \int \exp[-ipx] (\theta(p_0) - \theta(-p_0)) \delta(p^2 - m^2) d^4 p,
$$

is the Pauli–Jordan commutation function. Thus, the states (58) describe real scalar fields of mass $m$. For instance, the Fock representation of the CCR algebra $g(RS^3)$ define the
state $f_F$ (58) which satisfies the Wick theorem relations where $f_2$ is given by the Wightman function

$$W_2(x, y) = \frac{1}{i}D^-(x - y).$$

(59)

Thus, the state $f_F$ describe standard quantum free scalar fields of mass $m$.

**Appendix F. Wick rotation**

In order to describe the Wick rotation of Euclidean states, we start with the basic formulas of the Fourier–Laplace (henceforth FL) transformation [16]. It is defined on Schwartz distributions, but we focus on the tempered ones.

Let $\mathbb{R}^n_+$ and $\mathbb{R}^n_+\overline{\cup}$ further denote the subset of points of $\mathbb{R}^n$ with strictly positive Cartesian coordinates and its closure, respectively. Let $f \in S'(\mathbb{R}^n)$ be a tempered distribution and $\Gamma(f)$ the convex subset of points $q \in \mathbb{R}^n$ such that

$$e^{-qx}f(x) \in S'(\mathbb{R}^n).$$

(60)

In particular, $0 \in \Gamma(f)$. Let $\text{Int} \Gamma(f)$ and $\partial \Gamma(f)$ denote the interior and boundary of $\Gamma(f)$, respectively. The FL transform of a tempered distribution $f \in S'(\mathbb{R}^n)$ is defined as the tempered distribution

$$f^{FL}(p + iq) = (e^{-qx}f(x))^F(p) = \int f(x)e^{i(p+iq)x}d^n x \in S'(\mathbb{R}^n),$$

(61)

which is the Fourier transform of the distribution (60) depending on $q$ as parameters. One can think of the FL transform (61) as being the Fourier transform with respect to the complex arguments $k = p + iq$.

If $\text{Int} \Gamma(f) \neq \emptyset$, the FL transform $f^{FL}(k)$ is a holomorphic function $h(k)$ of complex arguments $k = p + iq$ on the open tube $\mathbb{R}^n + i\text{Int} \Gamma(f) \subset \mathbb{C}^n$ over $\text{Int} \Gamma(f)$. Moreover, for any compact subset $Q \subset \text{Int} \Gamma(f)$, there exist strictly positive numbers $A$ and $m$, depending on $Q$ and $f$, such that

$$|f^{FL}(p + iq)| \leq A(1 + |p|)^m, \quad p \in \mathbb{R}^n, \quad q \in Q.$$  

(62)

The evaluation (62) is equivalent to the fact that the function $h(p + iq)$ defines a family of tempered distributions $h_q(p) \in S'(\mathbb{R}^n)$ of the variables $p$ depending continuously on parameters $q \in S$. If $0 \in \text{Int} \Gamma(f)$, then

$$f^{FL}(p + i0) = \lim_{q \to 0} f^{FL}(p + iq)$$

which coincides with the Fourier transform $f^F(p)$ of $f$. The case of $0 \not\in \text{Int} \Gamma(f)$ is more intricate. Let $S$ be a convex domain in $\mathbb{R}^n$ such that $0 \in \partial S$, and let $h(p + iq)$ be a holomorphic function on the tube $\mathbb{R}^n + iS$ which defines a family of tempered distributions $h_q(p) \in S'(\mathbb{R}^n)$, depending on parameters $q$. One says that $h(p + iq)$ has a generalized boundary value.
\[ h_0(p) \in S'(\mathbb{R}_n) \text{ if, for any frustum } K^r \subset S \cup \{0\} \text{ of the cone } K \subset \mathbb{R}_n \text{ (i.e., } K^r = \{ q \in K : |q| \leq r \}) \), one has\]

\[ h_0(\psi(p)) = \lim_{|q| \to 0, q \in K^r \setminus \{0\}} h_q(\psi(p)) \]

for all functions \( \psi \in S(\mathbb{R}_n) \) of rapid decrease. Then the following holds [16].

Let \( f \in S'(\mathbb{R}^n) \), \( \text{Int } \Gamma(f) \neq \emptyset \) and \( 0 \notin \text{Int } \Gamma(f) \). A generalized boundary value of the FL transform \( f^{FL}(k) \) in \( S'(\mathbb{R}_n) \) exists and coincides with the Fourier transform \( f^F(p) \) of the distribution \( f \).

Let us apply this result to the following important case. The support of a tempered distribution \( f \) is defined as the complement of the maximal open subset \( U \) where \( f \) vanishes, i.e., \( f(\psi) = 0 \) for all \( \psi \in S(\mathbb{R}^n) \) of support in \( U \). Let \( f \in S'(\mathbb{R}^n) \) be of support in \( \mathbb{R}^n_+ \). Then \( \mathbb{R}^n_+ \subset \Gamma(f) \), and the FL transform \( f^{FL} \) is a holomorphic function on the tube over \( \mathbb{R}^n_+ \), while its generalized boundary value in \( S'(\mathbb{R}_n) \) is given by the equality

\[ h_0(\psi(p)) = \lim_{|q| \to 0, q \in \mathbb{R}^n_+} f^{FL}_q(\psi(p)) = f^F(\psi(p)), \quad \forall \psi \in S(\mathbb{R}_n). \]

Conversely, one can restore a tempered distribution \( f \) of support in \( \mathbb{R}^n_+ \) from its FL transform \( h(k) = f^{FL}(k) \) even if this function is known only on \( i\mathbb{R}^n_+ \). Indeed, the formulas

\[ \tilde{h}(\phi) = \int_{\mathbb{R}^n_+} h(iq)\phi(q)d_nq = \int_{\mathbb{R}^n_+} d_nq \int e^{-qx}f(x)\phi(q)d^nxd_nx = \]

\[ \int_{\mathbb{R}^n_+} f(x)\hat{\phi}(x)d^nxd_nx, \quad \phi \in S(\mathbb{R}^n_+), \]

\[ \hat{\phi}(x) = \int_{\mathbb{R}^n_+} e^{-qx}\phi(q)d_nq, \quad x \in \mathbb{R}^n_+, \quad \hat{\phi} \in S(\mathbb{R}^n_+), \]

define a linear continuous functional \( \tilde{h}(q) = h(0) \) on the space \( S(\mathbb{R}^n_+) \). It is called the Laplace transform \( f^L(q) = f^{FL}(iq) \) of a tempered distribution \( f \). The image of the space \( S(\mathbb{R}^n_+) \) with respect to the mapping \( \phi(q) \mapsto \hat{\phi}(x) \) (64) is dense in \( S(\mathbb{R}^n_+) \). Then the family of seminorms \( ||\phi||_{k,m} = ||\hat{\phi}||_{k,m} \), where \( ||.||_{k,m} \) are seminorms (46) on \( S(\mathbb{R}^n) \), determines the new coarse topology on \( S(\mathbb{R}^n_+) \) such that the functional (63) remains continuous with respect to this topology. Then the following is proved [16].

The mappings (63) and (64) provide one-to-one correspondence between the Laplace transforms \( f^L(q) = f^{FL}(iq) \) of tempered distributions \( f \in S'(\mathbb{R}_n^+) \) and the elements of \( S'(\mathbb{R}_n^+) \) which are continuous with respect to the coarse topology on \( S(\mathbb{R}^n_+) \).

With this correspondence, the above mentioned Wick rotation of Green’s functions of Euclidean quantum fields to causal forms in the Minkowski space is described as follows.

Let us denote by \( X \) the space \( \mathbb{R}^4 \) associated to the real subspace of \( \mathbb{C}^4 \) and by \( Y \) the space \( \mathbb{R}^4 \), coordinated by \((y^0, y^{1,2,3})\) and associated to the subspace \( \mathcal{Y} \) of \( \mathbb{C}^4 \) whose points possess the coordinates \((iy^0, y^{1,2,3})\). If \( X \) is the Minkowski space, then one can think of \( Y \) as being
its Euclidean partner. Since $X$ and $Y$ in $\mathbb{C}^4$ have the same spatial subspace, we further omit the dependence on spatial coordinates. Therefore, let us consider the complex plane $\mathbb{C}^1 = X \oplus iZ$ of the time $x$ and the Euclidean time $z$ and the complex plane $\mathbb{C}_1 = P \oplus iQ$ of the associated momentum coordinates $p$ and $q$.

Let $W(q) \in S'(Q)$ be a tempered distribution such that

$$W = W_+ + W_-, \quad W_+ \in S'(\mathbb{Q}_+), \quad W_- \in S'(\mathbb{Q}_-).$$

For instance, $W(q)$ is an ordinary function at 0. For every test function $\psi_+ \in S(X_+)$, we have

$$\frac{1}{2\pi} \int_{\mathbb{Q}_+} W(q) \hat{\psi}_+(q) dq = \frac{1}{2\pi} \int dx \int_{X_+} [W(q) \exp(-q x) \psi_+(x)] =$$

$$\frac{1}{(2\pi)^2} \int_{\mathbb{Q}_+} dq \int dp \int dx [W(q) \psi_+^F(p) \exp(-ip x - q x)] =$$

$$\frac{-i}{(2\pi)^2} \int_{\mathbb{Q}_+} dq \int dp [W(q) \frac{\psi_+^F(p)}{p - iq}] = \frac{1}{2\pi} \int_{\mathbb{Q}_+} W(q) \psi_+^{FL}(iq) dq,$$

due to the fact that the FL transform $\psi_+^{FL}(p + iq)$ of the function $\psi_+ \in S(X_+) \subset S'(X_+)$ exists and that it is holomorphic on the tube $P + iQ_+, Q_+$. Moreover, $\psi_+^{FL}(p + i0) = \psi_+^F(p)$, and the function $\hat{\psi}_+(q) = \psi_+^{FL}(iq)$ can be regarded as the Wick rotation of the test function $\psi_+(x)$. The equality (66) can be brought into the form

$$\frac{1}{2\pi} \int_{\mathbb{Q}_+} W(q) \hat{\psi}_+(q) dq = \int_{X_+} \hat{W}_+(x) \psi_+(x) dx,$$

$$\hat{W}_+(x) = \frac{1}{2\pi} \int_{\mathbb{Q}_+} \exp(-q x) W(q) dq, \quad x \in X_+.$$

It associates to a distribution $W(q) \in S'(Q)$ the distribution $\hat{W}_+(x) \in S'(X_+)$, continuous with respect to the coarsen topology on $S(X_+)$.

For every test function $\psi_- \in S(X_-)$, the similar relations

$$\frac{1}{2\pi} \int_{\mathbb{Q}_-} W(q) \hat{\psi}_-(q) dq = \int_{X_-} \hat{W}_-(x) \psi_-(x) dx,$$

$$\hat{W}_-(x) = \frac{1}{2\pi} \int_{\mathbb{Q}_-} \exp(-q x) W(q) dq, \quad x \in X_-,$$

hold. Combining (67) and (68), we obtain

$$\frac{1}{2\pi} \int_{\mathbb{Q}} W(q) \hat{\psi}(q) dq = \int_{X} \hat{W}(x) \psi(x) dx, \quad \hat{\psi} = \hat{\psi}_+ + \hat{\psi}_-, \quad \psi = \psi_+ + \psi_-,$$
where $\hat{W}(x)$ is a linear functional on functions $\psi \in S(X)$, which together with all derivatives vanish at $x = 0$. One can think of $\hat{W}(x)$ as being the Wick rotation of the distribution (65).

In particular, let a tempered distribution

$$M(\phi_1, \phi_2) = \int W_2(x_1, x_2)\phi_1(x_1)\phi_2(x_2)d^n x_1 d^n x_2.$$  \hfill(70)

be the Green’s function of some positive elliptic differential operator $E$, i.e.,

$$E y_1 W_2(y_1, y_2) = \delta(y_1 - y_2),$$

where $\delta$ is Dirac’s $\delta$-function. Then the distribution $W_2$ reads

$$W_2(y_1, y_2) = w(y_1 - y_2),$$  \hfill(71)

and we obtain the form

$$F_2(\phi_1 \phi_2) = M(\phi_1, \phi_2) = \int w(y_1 - y_2)\phi_1(y_1)\phi_2(y_2)d^4 y_1 d^4 y_2 =$$

$$\int w(y)\phi_1(y_1)\phi_2(y_1 - y)d^4 y d^4 y_1 = \int w(y)\varphi(y)d^4 y = \int w^F(q)\varphi^F(-q)d_4 q,$$

$$y = y_1 - y_2, \quad \varphi(y) = \int \phi_1(y_1)\phi_2(y_1 - y)d^4 y_1.$$  \hfill(72)

For instance, if $E_{y_1} = -\Delta_{y_1} + m^2$, where $\Delta$ is the Laplacian, then

$$w(y_1 - y_2) = \frac{\exp(-iq(y_1 - y_2))}{q^2 + m^2}d_4 q,$$  \hfill(72)

where $q^2$ is the Euclidean square, is the propagator of a massive Euclidean scalar field. Let the Fourier transform $w^F$ of the distribution $w$ (71) satisfy the condition (65). Then its Wick rotation (69) is the functional

$$\hat{w}(x) = \theta(x) \int_{\mathcal{Q}_+} w^F(q)\exp(-qx)dq + \theta(-x) \int_{\mathcal{Q}_-} w^F(q)\exp(-qx)dq$$

on scalar fields in the Minkowski space. For instance, let $w(y)$ be the Euclidean propagator (72) of a massive scalar field. Then due to the analyticity of

$$w^F(q) = (q^2 + m^2)^{-1}$$

on the domain $\text{Im} \ q \cdot \text{Re} \ q > 0$, one can show that $\hat{w}(x) = -iD^c(x)$ where $D^c(x)$ is a familiar causal Green’s function.
Appendix G. Schwinger functions

Let us show the difference between Schwinger functions in Osterwalder–Shrader Euclidean QFT and the Euclidean states of the Borchers algebra.

As was mentioned above, the Wightman functions obey the spectrum condition, which implies that the Fourier transform $w_F^n$ of the distributions $w_n$ (5) is of support in the closed forward light cone $\overrightarrow{V}_+$ in the momentum Minkowski space $\mathbb{R}_4$. It follows that the Wightman function $w_n$ is a generalized boundary value in $S'((\mathbb{R}^{4n-4})$ of the function $(w_F^n)^{FL}$, which is the FL transform of the function $w_F^n$ with respect to variables $p_i^0$ and which is holomorphic on the tube $((\mathbb{R}^4 + iV_-)^{n-1} \subset \mathbb{C}^4)$. Accordingly, $W_n(x_1, \ldots, x_n)$ is a generalized boundary value in $S''((\mathbb{R}^{4n}))$ of a function $W_n(z_1, \ldots, z_n)$, holomorphic on the tube

$$\{ z_i : \text{Im} (z_{i+1} - z_i) \in V_-, \ \text{Re} z_i \in \mathbb{R}^4 \}.$$

In accordance with the Lorentz covariance, the Wightman functions admit an analytic continuation onto a wider domain in $\mathbb{C}^4$, called the extended forward tube. Furthermore, the locality condition implies that they are symmetric on this domain.

Let $X$ and $Y$ be the Minkowski subspace and its Euclidean partner in $\mathbb{C}^4$, respectively. Let us consider the subset $\tilde{Y}_n^\# \subset Y \subset \mathbb{C}^{4n}$ which consists of the points $(z_1, \ldots, z_n)$ such that $z_i \neq z_j$. It belongs to the domain of analyticity of the Wightman function $W_n(z_1, \ldots, z_n)$, whose restriction to $\tilde{Y}_n^\#$ defines the symmetric function

$$S_n(y_1, \ldots, y_n) = W_n(z_1, \ldots, z_n), \quad z_i = (iy_0^i, y_1^i, y_2^i, y_3^i),$$

on $Y_\#$. It is called the Schwinger function. On the domain $Y_n^<$ of points $(y_1, \ldots, y_n)$ such that $0 < y_1^0 < \cdots < y_n^0$, the Schwinger function takes the form

$$S_n(y_1, \ldots, y_n) = s_n(y_1 - y_2, \ldots, y_{n-1} - y_n), \quad (73)$$

where $s_n$ is an element of the space $S'(Y_n^{n-1})$ which is continuous with respect to the coarsen topology on $S(Y_n^{n-1})$. Consequently, by virtue of the formula (63), the Schwinger function $s_n (73)$ can be represented as

$$s_n(y_1 - y_2, \ldots, y_{n-1} - y_n) =$$

$$\int \exp[p^0_j(y^0_j - y^0_{j+1}) - i \sum_{k=1}^3 p^k_j(y^k_j - y^k_{j+1})]w_n^F(p^1, \ldots, p^n)d^4p^1 \cdots d^4p^{n-1}, \quad (74)$$

where $w_n^F \in S'((\mathbb{R}^n_+)$ is the Fourier transform of the Wightman function $w_n$, seen as an element of $S'(\mathbb{R}^n_+)$ of support in the subset $p_0^i \geq 0$. The formula (74) enables one to restore the Wightman functions on the Minkowski from the Schwinger functions on the Euclidean space [69, 78].

23
Appendix H. Representations of nuclear Lie groups

The typical construction is the following [77]. Let \((Q, \mu)\) be a localizable measurable space, where \(\mu\) is quasi-invariant under a transformation group \(G\). There is the unitary representation

\[
G \ni g \mapsto T_L(g)u(q) = \left(\frac{d\mu(gq)}{d\mu(q)}\right)^{1/2}u(g^{-1}q), \quad u \in L^2(Q, \mu),
\]

(75)
of \(G\) in the Hilbert space of \(L^2(Q, \mu)\) of quadratically \(\mu\)-integrable complex-valued functions on \(Q\). The group \(G\) is equipped with the weakest topology such that the representation (75) is strongly continuous. If \(Q = G\) is a locally compact group and \(\mu\) is the left Haar measure, this topology is weaker than the original one.

The main mathematical results are concerned with integral representations of continuous positive-definite functions on commutative topological groups [33, 77]. We mention the following one. Let \(Q\) be a real nuclear space and \(Z\) a continuous positive-definite function on \(Q\) i.e.

\[
Z(q_i - q_j)\bar{\alpha}^i\alpha^j \geq 0, \quad Z(0) = 1,
\]

for any \(n\) elements \(q_1, \ldots, q_n\) of \(Q\) and any \(n\) complex numbers \(\alpha^1, \ldots, \alpha^n\). In accordance with the Bochner theorem, any such a function is the the Fourier transform

\[
Z(q) = \int_{Q'} \exp[i\langle q, w \rangle]d\mu_Z(w),
\]

(76)
of some probability measure \(\mu_Z\) on the topological dual \(Q'\) of \(Q\), and vice versa [14, 33]. A continuous positive-definite function \(Z\) plays the role similar to a reproducing kernel on a locally compact group [3, 35]. One can think of \(Q\) as being the group space of the Abelian Lie group \(G_Q\). We have the strongly continuous unitary representation of \(G_Q\) in \(L^2(Q', \mu_Z)\) by operators

\[
\hat{q} : u(w) \mapsto F_q(w)u(w), \quad F_q(w) = \exp[i\langle q, w \rangle].
\]

(77)
The Hilbert space of the representation (77) is described as follows. For every element \(q\), the function \(Z\) (76) defines the continuous function \(Z_q(q') = Z(q' - q)\) of \(q' \in Q\). Let us consider finite linear combinations

\[
v = \sum_{i=1}^N v^iZ_{\phi_i}, \quad v^i \in \mathbb{C},
\]

for all elements of \(Q\) [55]. They constitute a pre-Hilbert space \(Q_Z\) equipped with the Hermitian form

\[
(v | v') = Z(q_i - q_j)\overline{v^j}v^i.
\]

This form is separating, and the corresponding completion of \(Q_Z\) is a Hilbert space \(H_Z\).
Let us consider the isometry

$$\rho : G_Z \ni Z \mapsto F_{q-q'}(w) \in L^2(Q', \mu_Z),$$

which is extended to the Hilbert space $H_Z$. Then $\rho(H_Z)$ carries out the irreducible cyclic representation (77) of the group $G_Q$. The cyclic element $\theta_Z$ is represented by the class of $\mu_Z$-equivalent functions $u(w) = 1$ on $Q'$. Nonequivalent measures $\mu_Z$ and $\mu_Z'$ imply different cyclic elements $\theta_K$ and $\theta_K'$ and nonequivalent representations (77).

If a measure $\mu_Z$ on $Q'$ is quasi-invariant under translations $w \mapsto w + w_q$, where elements $w_q \in Q \subset Q'$, by definition, satisfy the relation $\langle w_q, q' \rangle = (q', q)_Q$ for all $q' \in Q$. In this case, we have the representation of these translations by the displacement operators $T_L(q)$ (75).

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