ADELIC PATH INTEGRALS FOR QUADRATIC LAGRANGIANS

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Abstract

Feynman’s path integral in adelic quantum mechanics is considered. The propagator $K(x'';t'';x',t')$ for one-dimensional adelic systems with quadratic Lagrangians is analytically evaluated. Obtained exact general formula has the form which is invariant under interchange of the number fields $\mathbb{R}$ and $\mathbb{Q}_p$.

1. Introduction

For simplicity, we consider one-dimensional systems, but many issues can be easily generalized to higher dimensions.

It is well known that dynamical evolution of any quantum-mechanical system, described by a wave function $\Psi(x,t)$, is given by

$$\Psi(x'',t'') = \int K(x'',t'';x',t')\Psi(x',t')dx', \quad (1.1)$$

where $K(x'',t'';x',t')$ is the kernel of the corresponding unitary integral operator acting as follows:

$$\Psi(t'') = U(t'',t')\Psi(t'). \quad (1.2)$$

$K(x'',t'';x',t')$ is also called Green’s function, or the quantum-mechanical propagator, and the probability amplitude to go a particle from a space-time point $(x',t')$ to the other point $(x'',t'')$. Starting from (1.1) one can easily derive the following three general properties:

$$\int K(x'',t'';x,t)K(x,t;x',t')dx = K(x'',t'';x',t'), \quad (1.3)$$

$$\int K(x'',t'';x',t')K(y,t'';x',t')dx' = \delta(x'' - y), \quad (1.4)$$

$$K(x'',t'';x',t') = \lim_{t'' \to t'''} K(x'',t''';x',t') = \delta(x'' - x'). \quad (1.5)$$
Since all information on quantum dynamics can be deduced from the propagator $K(x'', t''; x', t')$ it can be regarded as the basic ingredient of quantum theory. In Feynman’s formulation \([1]\) of quantum mechanics, $K(x'', t''; x', t')$ was postulated to be the path integral

$$K(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \exp \left( \frac{2\pi i}{\hbar} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) Dq,$$

(1.6)

where $x'' = q(t'')$ and $x' = q(t')$, and $\hbar$ is the Planck constant.

In its original form, the path integral (1.6) is the limit of the corresponding multiple integral of $n$ variables $q_i = q(t_i), \quad (i = 1, 2, \ldots, n)$, when $n \to \infty$. For all its history, the path integral has been a subject of permanent interest in theoretical and mathematical physics. At present days (see, e.g. \([2]\)), it is one of the most profound and promising approaches to foundations of quantum theory (in particular, quantum field theory and superstring theory). Feynman’s path integral is a natural instrument in formulation and investigation of $p$-adic and adelic quantum mechanics.

Adelic quantum mechanics, we are interested in, contains complex-valued functions of real and all $p$-adic arguments. There is not the corresponding Schrödinger equation, but Feynman’s path integral method is quite appropriate. Feynman’s path integral for $p$-adic propagator $K_p(x'', t''; x', t')$ \([3]\), where $K_p$ is complex-valued and $x'', x', t'', t'$ are $p$-adic variables, is a direct $p$-adic generalization of (1.6), i.e.

$$K_p(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_p \left( -\frac{1}{\hbar} \int_{t'}^{t''} L(\dot{q}, q, t) dt \right) Dq,$$

(1.7)

where $\chi_p(a)$ is $p$-adic additive character. The Planck constant $\hbar$ in (1.6) and (1.7) is the same rational number. We consider integral $\int_{t'}^{t''} L(\dot{q}, q, t) dt$ as the difference of antiderivative (without pseudoconstants) of $L(\dot{q}, q, t)$ in final ($t''$) and initial ($t'$) times. $Dq = \prod_{i=1}^{n} dq(t_i)$, where $dq(t_i)$ is the $p$-adic Haar measure. Thus, $p$-adic path integral is the limit of the multiple Haar integral when $n \to \infty$. To calculate (1.7) in this way one has to introduce some order on $t \in Q_p$, and it is successfully done in Ref. \([3]\). On previous investigations of $p$-adic path integral one can see \([3, 7]\). Our main task here is a derivation of the exact result for $p$-adic (1.7) and the corresponding adelic Feynman path integral for the general case of Lagrangians $L(\dot{q}, q, t)$, which are quadratic polynomials in $\dot{q}$ and $q$, without making time discretization. In fact, we will use the general requirements (in particular, (1.3) and (1.4)) which any (ordinary, $p$-adic and adelic) path integral must satisfy. In some parts of evaluation, we exploit methods inspired by Ref. \([8]\).

Adelic path integral may be regarded as an infinite product of the ordinary one and $p$-adic path integrals for all primes $p$. Formal definition of adelic functional integral, with some of its basic properties, will be presented in Section 5.

Since 1987, there have been many publications (for a review, see, e.g. \([4, 10, 11]\)) on possible applications of $p$-adic numbers and adeles in modern theoretical and mathematical physics. The first successful employment of $p$-adic numbers was in string theory. In Volovich’s article \([12]\), a hypothesis on the existence of nonarchimedean geometry at the Planck scale was proposed and $p$-adic string theory was initiated. In particular, it was
proposed generalization of the standard Veneziano and Virasoro-Shapiro amplitudes with complex-valued multiplicative characters over various number fields, and $p$-adic valued Veneziano amplitude was constructed by means of interpolation. Using $p$-adic Veneziano amplitude as the Gel’fand-Graev [13] beta function, Freund and Witten obtained [14] an attractive adelic formula, which states that the product of the standard crossing symmetric Veneziano amplitude and all its $p$-adic counterparts equals unit. Such approach gives a possibility to consider some ordinary string amplitudes as an infinite product of their inverse $p$-adic analogues. Many $p$-adic aspects of string theory and nonarchimedean geometry of the space-time at the Planck scale have been of the significant interest during the last fifteen years.

For a systematic investigation of $p$-adic quantum dynamics, two kinds of $p$-adic quantum mechanics are formulated: with complex-valued and $p$-adic valued wave functions of $p$-adic variables (for a review, see [3, 9] and [11], respectively). This paper is related to the first kind, which can be presented as a triple

$$ (L_2(\mathbb{Q}_p), W, U(t)), \quad (1.8) $$

where $L_2(\mathbb{Q}_p)$ is the Hilbert space on $\mathbb{Q}_p$, $W$ denotes the Weyl quantization procedure, and $U(t)$ is the unitary representation of an evolution operator on $L_2(\mathbb{Q}_p)$. In our approach, $U(t)$ is naturally realized by the Feynman path integral method. In order to connect $p$-adic with standard quantum mechanics, adelic quantum mechanics is formulated [4]. Within adelic quantum mechanics a few basic physical systems, including some minisuperspace cosmological models, have been successfully considered. As a result of $p$-adic effects in the adelic approach, a space-time discretness at the Planck scale is obtained. Adelic path integral plays a central role and provides an extension of possible quantum trajectories over real space to all $p$-adic spaces. There has been also application of $p$-adic numbers in the investigation of spin glasses, Brownian motion, stochastic processes, information systems, hierarchy structures and some other problems related to very complex dynamical systems.

Before to proceed with path integral, let us give a short review of some basic properties of $p$-adic numbers, adeles, and analyses over them, which provide a minimum of necessary mathematical background. It will also contain mathematical concepts to present a motivation for employment of $p$-adic numbers and adeles in physics.

2. $p$-Adic and Adelic Analyses, and Motivations for Their Use in Physics

There are physical and mathematical reasons to start with the field of rational numbers $\mathbb{Q}$. From physical point of view, numerical results of all experiments and observations are some rational numbers, i.e. they belong to $\mathbb{Q}$. From algebraic point of view, $\mathbb{Q}$ is the simplest number field of characteristic 0. Recall that any $0 \neq x \in \mathbb{Q}$ can be presented as infinite expansions into the two essentially different forms:

$$ x = \sum_{k=n}^{-\infty} a_k 10^k, \quad a_k = 0, 1, \cdots, 9, \quad a_n \neq 0, \quad (2.1) $$
which is the ordinary one to the base 10, and the other one to the base $p$ ($p$ is any prime number)

$$x = \sum_{k=m}^{+\infty} b_k p^k, \quad b_k = 0, 1, \ldots, p - 1, \quad b_m \neq 0,$$

(2.2)

where $n$ and $m$ are some integers which depend on $x$. The above representations (2.1) and (2.2) exhibit the usual repetition of digits, however the expansions are in the mutually opposite directions. The series (2.1) and (2.2) are convergent with respect to the metrics induced by the usual absolute value $| \cdot |_\infty$ and $p$-adic absolute value (or $p$-adic norm) $| \cdot |_p$, respectively. Note that these valuations exhaust all possible inequivalent non-trivial norms on $Q$. Allowing all possible combinations for digits, one obtains standard representation of real and $p$-adic numbers in the form (2.1) and (2.2), respectively. Thus, the field of real numbers $R$ and the fields of $p$-adic numbers $Q_p$ exhaust all number fields which may be obtained by completion of $Q$, and which contain $Q$ as a dense subfield. Since $p$-adic norm of any term in (2.2) is $|b_k p^k|_p = p^{-k}$ if $b_k \neq 0$, geometry of $p$-adic numbers is the nonarchimedean one, i.e. strong triangle inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$ holds and $|x|_p = p^{-n}$. $R$ and $Q_p$ have many distinct algebraic and geometric properties.

There is no natural ordering on $Q_p$. However one can introduce a linear order on $Q_p$ in the following way: $x < y$ if $|x|_p < |y|_p$, or if $|x|_p = |y|_p$ then there exists such index $r \geq 0$ that digits satisfy $x_m = y_m, x_{m+1} = y_{m+1}, \ldots, x_{m+r-1} = y_{m+r-1}, x_{m+r} < y_{m+r}$. Here, $x_k$ and $y_k$ are digits related to $x$ and $y$ in expansion (2.2). This ordering is very useful in time discretization and calculation of $p$-adic functional integral as a limit of the $n$-multiple Haar integral when $n \to \infty$.

There are mainly two kinds of analysis on $Q_p$ which are of interest for physics, and they are based on two different mappings: $Q_p \to Q_p$ and $Q_p \to C$, where $C$ is the field of ordinary complex numbers. We use both of these analyses, in classical and quantum $p$-adic models, respectively.

Elementary $p$-adic valued functions and their derivatives are defined by the same series as in the real case, but their regions of convergence are determined by means of $p$-adic norm. As a definite $p$-adic valued integral of an analytic function $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ we take difference of the corresponding antiderivative in end points, i.e.

$$\int_a^b f(x) = \sum_{n=0}^{\infty} \frac{f_n}{n+1} \left( b^{n+1} - a^{n+1} \right).$$

Usual complex-valued functions of $p$-adic variable are: (i) an additive character $\chi_p(x) = \exp 2 \pi i \{x\}_p$, where $\{x\}_p$ is the fractional part of $x \in Q_p$, (ii) a multiplicative character $\pi_s(x) = |x|^s_p$, where $s \in C$, and (iii) locally constant functions with compact support, like $\Omega(|x|_p)$, where

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1. \end{cases}$$

(2.3)

There is well defined Haar measure and integration, and we use the Gauss integral

$$\int_{Q_p} \chi_p(\alpha x^2 + \beta x) dx = \lambda_p(\alpha) |2\alpha|^{-\frac{3}{2}} \chi_p \left( -\frac{\beta^2}{4\alpha} \right), \quad \alpha \neq 0.$$ 

(2.4)
The arithmetic function $\lambda_p(x)$ in (2.4) is a map $\lambda_p : \mathbb{Q}_p^* \to \mathbb{C}$ defined as follows [9]:

$$
\lambda_p(x) = \begin{cases} 
1, & m = 2j, \quad p \neq 2, \\
\left(\frac{x}{p}\right), & m = 2j + 1, \quad p \equiv 1 \pmod{4}, \\
i \left(\frac{x}{p}\right), & m = 2j + 1, \quad p \equiv 3 \pmod{4},
\end{cases}
$$

(2.5)

$$
\lambda_2(x) = \begin{cases} 
\sqrt[4]{1 + (-1)^{x_m+1}i}, & m = 2j, \\
\sqrt[4]{(-1)^{x_m+1+x_m+2}[1 + (-1)^{x_m+1}i]}, & m = 2j + 1,
\end{cases}
$$

(2.6)

where $x$ is presented in the form (2.2), $j \in \mathbb{Z}$, $(\frac{x}{p})$ is the Legendre symbol and $\mathbb{Q}_p = \mathbb{Q}_p \setminus \{0\}$. It is often enough to use properties:

$$
\lambda_p(a^2x) = \lambda_p(x), \quad \lambda_p(x)\lambda_p(-x) = 1, \quad \lambda_p(x)\lambda_p(y) = \lambda_p(x + y)\lambda_p(x^{-1} + y^{-1}),
$$

$$
|\lambda_p(x)|_{\infty} = 1, \quad a \neq 0.
$$

(2.7)

For a more complete information on $p$-adic numbers and nonarchimedean analysis one can see [9, 13, 15].

It is worth noting that the real analogue of the Gauss integral (2.4) has the same form, \textit{i.e.}

$$
\int_{\mathbb{Q}_\infty} \chi_\infty(\alpha x^2 + \beta x)dx = \lambda_\infty(\alpha)|2\alpha|^{-\frac{1}{2}}\chi_\infty\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0,
$$

(2.8)

where $\mathbb{Q}_\infty \equiv \mathbb{R}$ and $\chi_\infty(x) = \exp(-2\pi ix)$ is additive character in the real case. Function $\lambda_\infty(x)$ is defined as

$$
\lambda_\infty(x) = \sqrt{\text{sign } x} = \exp\left(-\frac{\pi}{4}\right)\sqrt{\text{sign } x}, \quad x \in \mathbb{R}^*.
$$

(2.9)

and exhibits the same properties (2.7).

Real and $p$-adic numbers are unified in the form of adeles. An adele $x$ [13] is an infinite sequence

$$
x = (x_\infty, x_2, \cdots, x_p, \cdots),
$$

(2.10)

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $S$ of primes $p$ one has $x_p \in \mathbb{Z}_p$, where $\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\}$ is the ring of $p$-adic integers. Componentwise addition and multiplication are natural operations on the ring of adeles $\mathcal{A}$, which can be regarded as

$$
\mathcal{A} = \bigcup_S \mathcal{A}(S), \quad \mathcal{A}(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \not\in S} \mathbb{Z}_p.
$$

(2.11)

$\mathcal{A}$ is also locally compact topological space.

There are also two kinds of analysis over topological ring of adeles $\mathcal{A}$, which are generalizations of the corresponding analyses over $\mathbb{R}$ and $\mathbb{Q}_p$. The first one is related to
mapping $A \to A$ and the other one to $A \to C$. In complex-valued adelic analysis it is worth mentioning an additive character

$$\chi(x) = \chi_\infty(x_\infty) \prod_p \chi_p(x_p),$$  \hfill (2.12)

a multiplicative character

$$|x|^s = |x_\infty|^s \prod_p |x_p|^s, \quad s \in \mathbb{C},$$  \hfill (2.13)

and elementary functions of the form

$$\phi(x) = \phi_\infty(x_\infty) \prod_{p \in S} \phi_p(x_p) \prod_{p \not\in S} \Omega(|x_p|_p),$$  \hfill (2.14)

where $\phi_\infty(x_\infty)$ is an infinitely differentiable function on $\mathbb{R}$ such that $|x_\infty|^n \phi_\infty(x_\infty) \to 0$ as $|x_\infty| \to \infty$ for any $n \in \{0, 1, 2, \cdots\}$, and $\phi_p(x_p)$ is a locally constant function with compact support. All finite linear combinations of elementary functions (2.14) make up the set $S(A)$ of the Schwartz-Bruhat adelic functions. The Fourier transform of $\phi(x) \in S(A)$, which maps $S(A)$ onto $A$, is

$$\tilde{\phi}(y) = \int_A \phi(x)\chi(xy)dx,$$  \hfill (2.15)

where $\chi(xy)$ is defined by (2.12) and $dx = dx_\infty dx_2 dx_3 \cdots$ is the Haar measure on $A$.

One can define the Hilbert space on $A$, which we will denote by $L^2(A)$. It contains infinitely many complex-valued functions of adelic argument (for example, $\Psi_1(x), \Psi_2(x), \cdots$) with scalar product

$$(\Psi_1, \Psi_2) = \int_A \Psi_1(x)\Psi_2(x)dx$$

and norm

$$||\Psi|| = (\Psi, \Psi)^{\frac{1}{2}} < \infty,$$

where $dx$ is the Haar measure on $A$. A basis of $L^2(A)$ may be given by the set of orthonormal eigenfunctions in spectral problem of the evolution operator $U(t)$, where $t \in A$. Such eigenfunctions have the form

$$\psi_{S,\alpha}(x,t) = \psi_n^{(\infty)}(x_\infty, t_\infty) \prod_{p \in S} \psi_{\alpha_p}^{(p)}(x_p, t_p) \prod_{p \not\in S} \Omega(|x_p|_p),$$  \hfill (2.16)

where $\psi_n^{(\infty)}$ and $\psi_{\alpha_p}^{(p)}$ are eigenfunctions in ordinary and $p$-adic cases, respectively. $\Omega(|x_p|_p)$ is defined by (2.3) and presents a vector invariant under transformation of $U_p(t_p)$ evolution operator. Adelic quantum mechanics \cite{4} may be regarded as a triple

$$(L^2(A), W(z), U(t)),$$  \hfill (2.17)

where $W(z)$ and $U(t)$ are unitary representations of the Heisenberg-Weyl group and evolution operator on $L^2(A)$, respectively.
Since \( \mathbb{Q} \) is dense not only in \( \mathbb{R} \) but also in \( \mathbb{Q}_p \) there is a sense to ask a question: Why real (and complex) numbers are so good in description of usual physical phenomena, and, is there any aspect of physical reality which cannot be described without \( p \)-adic numbers? As adeles contain real and \( p \)-adic numbers, and consequently archimedean and nonarchimedean geometries, it has been natural to formulate adelic quantum mechanics, which provides a suitable framework for a systematic investigation of the above question. Uncertainty in space measurements

\[
\Delta x \Delta y \geq \ell_0^2 = \frac{\hbar G}{c^3} \sim 10^{-66}\text{cm}^2,
\]

which comes from ordinary quantum gravity considerations based on real numbers, restricts application of \( \mathbb{R} \) and gives rise for employment of \( \mathbb{Q}_p \) at the Planck scale. It is also reasonable to expect that fundamental physical laws are invariant under interchange of \( \mathbb{R} \) and \( \mathbb{Q}_p \)\[16\].

3. Quadratic Lagrangians and Related Classical Actions

A general quadratic Lagrangian can be written as follows:

\[
L(\dot{q}, q, t) = \frac{1}{2} \frac{\partial^2 L_0}{\partial \dot{q}^2} \dot{q}^2 + \frac{1}{2} \frac{\partial^2 L_0}{\partial \dot{q} \partial q} \dot{q} q + \frac{1}{2} \frac{\partial^2 L_0}{\partial q^2} q^2 + \frac{\partial L_0}{\partial \dot{q}} \dot{q} + \frac{\partial L_0}{\partial q} q + L_0,
\]

where index 0 denotes that the Taylor expansion of \( L(\dot{q}, q, t) \) is around \( \dot{q} = q = 0 \). In fact, we want to consider the corresponding adelic Lagrangian, i.e. an adelic collection of Lagrangians of the same form (3.1) which differ only by their valuations \( v = \infty, 2, 3, \ldots \). In this Section we present some results valid simultaneously for real as well as for \( p \)-adic classical mechanics. Coefficients in the expansion (3.1) will be regarded as analytic functions of the time \( t \), where \( \frac{\partial^2 L_0}{\partial q^2} |_{t=0} \neq 0 \), and which power series have the same rational coefficients in the real and all \( p \)-adic cases.

The Euler-Lagrange equation of motion is

\[
\frac{\partial^2 L_0}{\partial q^2} \ddot{q} + \frac{d}{dt} \left( \frac{\partial^2 L_0}{\partial q \partial \dot{q}} \right) \dot{q} + \left[ \frac{d}{dt} \left( \frac{\partial^2 L_0}{\partial q \partial \dot{q}} \right) - \frac{\partial^2 L_0}{\partial q^2} \right] q = \frac{\partial L_0}{\partial \dot{q}} - \frac{d}{dt} \left( \frac{\partial L_0}{\partial q} \right),
\]

A general solution of (3.2) describes classical trajectory

\[
q = x(t) = C_1 f_1(t) + C_2 f_2(t) + \xi(t),
\]

where \( f_1(t) \) and \( f_2(t) \) are two linearly independent solutions of the corresponding homogeneous equation, and \( \xi(t) \) is a particular solution of the complete equation (3.2). Imposing the boundary conditions \( x' = x(t') \) and \( x'' = x(t'') \), constants of integration \( C_1 \) and \( C_2 \) become:

\[
C_1 \equiv C_1(t'', t') = \frac{(x' - \xi')(f''_2 - (x'' - \xi'')f''_1)}{f''_2 f'_1 - f''_1 f''_2},
\]

\[
C_2 \equiv C_2(t'', t') = \frac{(x'' - \xi'')(f'_1 - (x' - \xi')f'')}{f''_2 f'_1 - f''_1 f''_2},
\]

17
where \( f'_1 = f_1(t') \), \( f''_2 = f_2(t'') \) and \( \xi' = \xi(t') \), \( \xi'' = \xi(t'') \). Note that not all boundary values \( x'' \) and \( x' \) are possible. Namely, if \( f''_2 f'_1 - f'_2 f''_1 = 0 \) then must be \( x'' = x' - \xi' \). Such situation is typical for the case of periodic solutions \( f_1(t) \) and \( f_2(t) \).

Using the equation of motion \((3.2)\), the Lagrangian \((3.1)\) can be rewritten as

\[
L(\dot{x}, x, t) = \frac{1}{2} \frac{d}{dt} \left[ \frac{\partial^2 L_0}{\partial x'^2} \dot{x}^2 + \frac{\partial^2 L_0}{\partial x \partial x'} \dot{x} x' + \frac{\partial L_0}{\partial x} x \right] + \frac{1}{2} \left( \frac{\partial L_0}{\partial \dot{x}} \dot{x} + \frac{\partial L_0}{\partial x} x \right) + L_0
\]  

\[
(3.6)
\]

where \( x(t) \) is related to the classical trajectory \((3.3)\). Since \( C_1(t'', t') \) and \( C_2(t'', t') \) are linear in \( x'' \) and \( x' \), the corresponding classical action \( \tilde{S}(x'', t''; x', t') = \int_{t'}^{t''} L(\dot{x}, x, t) dt \) is quadratic in \( x'' \) and \( x' \). Performing explicit integration for the first term of \((3.6)\), and taking into account \((3.4)\) and \((3.5)\), we get

\[
\tilde{S}(x'', t''; x', t') = \frac{1}{2} \frac{\partial^2 \bar{S}_0}{\partial x''} x''^2 + \frac{1}{2} \frac{\partial^2 \bar{S}_0}{\partial x' \partial x''} x'' x' + \frac{1}{2} \frac{\partial^2 \bar{S}_0}{\partial x' \partial x'} x'^2 + \frac{\partial \bar{S}_0}{\partial x''} x'' + \frac{\partial \bar{S}_0}{\partial x'} x' + \bar{S}_0,
\]

\[
(3.7)
\]

where:

\[
\frac{\partial^2 \bar{S}_0}{\partial x''} = \frac{\partial^2 L_0''}{\partial x'^2} \left( f''_2 f'_1 - f''_1 f'_2 \right) + \frac{\partial^2 L_0''}{\partial x' \partial x'} \Delta(t''', t'),
\]

\[
(3.8)
\]

\[
\frac{\partial \bar{S}_0}{\partial x''} = \frac{\partial L_0''}{\partial x'} \left( f''_2 f'_1 - f''_1 f'_2 \right),
\]

\[
(3.9)
\]

\[
\frac{\partial \bar{S}_0}{\partial x'} = \frac{\partial L_0'\prime}{\partial x''} \left( f''_2 f'_1 - f''_1 f'_2 \right),
\]

\[
(3.10)
\]

\[
\frac{\partial^2 \bar{S}_0}{\partial x' \partial x'} = \frac{\partial^2 L_0'}{\partial x^2} \left( f''_2 f'_1 - f''_1 f'_2 \right) + \frac{\partial^2 L_0'}{\partial x' \partial x'} \Delta(t''', t'),
\]

\[
(3.11)
\]

\[
\frac{\partial \bar{S}_0}{\partial x'} = \frac{\partial L_0'}{\partial x''} \left( f''_2 f'_1 - f''_1 f'_2 \right),
\]

\[
(3.12)
\]

\[
\bar{S}_0 = \frac{1}{2} \frac{\xi'' f'_2 - \xi' f''_2}{\Delta(t''', t')} \int_{t'}^{t''} \left( \frac{\partial L_0}{\partial x} f_1 + \frac{\partial L_0}{\partial x} f_2 \right) dt + \frac{1}{2} \frac{\xi'' f'_1 - \xi' f''_1}{\Delta(t''', t')} \int_{t'}^{t''} \left( \frac{\partial L_0}{\partial x} f_2 + \frac{\partial L_0}{\partial x} f_2 \right) dt
\]

\[
(3.13)
\]
The expression (3.7) is presented in the form of the Taylor expansion of the classical action around \( x'' = x' = 0 \), where coefficient functions depend on \( t'' \) and \( t' \). In the above expressions we denoted \( L''_0 = L_0(t'') \), \( L'_0 = L_0(t') \), and

\[
\Delta(t'', t') = f''_2 f'_1 - f'_2 f''_1. \tag{3.14}
\]

Thus, to Lagrangian \( L(\dot{q}, q, t) \), which is at most quadratic in \( \dot{q} \) and \( q \), corresponds classical action \( \bar{S}(x'', t''; x', t') \) at most quadratic in \( x'' \) and \( x' \).

It is worth noting that solutions \( f_2(t) \) and \( \xi(t) \) can be expressed by means of solution \( f_1(x) \) in the following way:

\[
f_2(t) = f_1(t) \int_{t_0}^{t} \frac{d\tau}{f'_1(\tau) \frac{\partial L_0(\tau)}{\partial q^2}}, \tag{3.15}
\]

\[
\xi(t) = f_1(t) \int_{t_0}^{t} \frac{d\tau}{f'_1(\tau) \frac{\partial L_0(\tau)}{\partial q^2}} \int_{\tau_0}^{\tau} \left[ \frac{\partial L_0(\eta)}{\partial q} - \frac{d}{d\eta} \left( \frac{\partial L_0(\eta)}{\partial q} \right) \right] f_1(\eta) d\eta, \tag{3.16}
\]

where \( t_0 \) and \( \tau_0 \) are already incorporated into constants \( C_1 \) and \( C_2 \) in (3.3)-(3.5).

In the next section we will use also quadratic Lagrangian of the form

\[
L(\dot{y}, y, t) = \frac{1}{2} \frac{\partial^2 L_0}{\partial q^2} y^2 + \frac{\partial^2 L_0}{\partial \dot{q} \partial q} \dot{y} y + \frac{1}{2} \frac{\partial^2 L_0}{\partial \dot{q}^2} y^2, \tag{3.17}
\]

where coefficients in the expansion are those of (3.1).

The corresponding Euler-Lagrange equation of motion is

\[
\frac{\partial^2 L_0}{\partial \dot{q}^2} \ddot{y} + \frac{d}{dt} \left( \frac{\partial^2 L_0}{\partial q \partial \dot{q}} \right) \dot{y} + \left[ \frac{d}{dt} \left( \frac{\partial^2 L_0}{\partial \dot{q} \partial q} \right) - \frac{\partial^2 L_0}{\partial \dot{q}^2} \right] y = 0 \tag{3.18}
\]

with general solution

\[
y(t) = D_1 f_1(t) + D_2 f_2(t), \tag{3.19}
\]

where \( f_1(t) \) and \( f_2(t) \) remain to be the same previous two solutions. Denoting \( y' = y(t') \) and \( y'' = y(t'') \), constants of integration \( D_1 \) and \( D_2 \) are:

\[
D_1 \equiv D_1(t'', t') = \frac{y' f''_2 - y'' f'_2}{f''_2 f'_1 - f''_1 f'_2} \tag{3.20}
\]

\[
D_2 \equiv D_2(t'', t') = \frac{y'' f'_1 - y' f''_1}{f''_2 f'_1 - f''_1 f'_2}. \tag{3.21}
\]

Using the equation of motion (3.18), the Lagrangian (3.17) becomes

\[
L(\dot{y}, y, t) = \frac{1}{2} \frac{d}{dt} \left[ \frac{\partial^2 L_0}{\partial q^2} y^2 + \frac{\partial^2 L_0}{\partial \dot{q} \partial q} \dot{y} y \right], \tag{3.22}
\]
where $y(t)$ is now related to (3.19). Since $D_1(t'', t')$ and $D_2(t'', t')$ are linear in $y''$ and $y'$, the corresponding classical action $\tilde{S}(y'', t''; y', t') = \int_{t'}^{t''} L(y, y') dt$ is quadratic in $y''$ and $y'$. An analogous integration to the previous one gives

$$
\tilde{S}(y'', t''; y', t') = \frac{1}{2} \frac{\partial^2 \tilde{S}_0}{\partial y''^2} y''^2 + \frac{\partial^2 \tilde{S}_0}{\partial y'' \partial y'} y'' y' + \frac{1}{2} \frac{\partial^2 \tilde{S}_0}{\partial y' y'^2} y'^2
$$

(3.23)

where:

$$
\frac{\partial^2 \tilde{S}_0}{\partial y''^2} = \frac{\partial^2 L''_0}{\partial y^2} \frac{\dot{j}_{2}'' f_1' - \dot{j}_{1}'' f_2'}{\Delta(t'', t')} + \frac{\partial^2 L''_0}{\partial y \partial y'} \frac{\dot{j}_{2}'' f_1' - \dot{j}_{1}'' f_2'}{\Delta(t'', t')},
$$

(3.24)

$$
\frac{\partial^2 \tilde{S}_0}{\partial y'' \partial y'} = \frac{1}{2} \frac{\partial^2 L''_0}{\partial y^2} \frac{\dot{f}_{2}'' f_1' - \dot{f}_{1}'' f_2'}{\Delta(t', t'')} + \frac{1}{2} \frac{\partial^2 L''_0}{\partial y^2} \frac{\dot{f}_{2}'' f_1' - \dot{f}_{1}'' f_2'}{\Delta(t', t'')},
$$

(3.25)

$$
\frac{\partial^2 \tilde{S}_0}{\partial y y'^2} = -\frac{\partial^2 L''_0}{\partial y^2} \frac{\dot{f}_{2}'' - \dot{f}_{1}'' f_2'}{(f_{2}' f_1' - f_{1}' f_2') (f_{2}' f_1' - f_{1}' f_2')}
$$

(3.26)

In virtue of (3.15) one has

$$
\dot{f}_2(t) f_1(t) - \dot{f}_1(t) f_2(t) = \left( \frac{\partial^2 L_0(t)}{\partial q^2} \right)^{-1}.
$$

(3.27)

Using (3.27) one can show the following useful formulae:

$$
\frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y'', t''; y', t') = - \frac{1}{f_{2}' f_1' - f_{1}' f_2'},
$$

(3.28)

$$
\frac{\partial^2}{\partial y y'^2} \tilde{S}_0(y'', t''; y, t) + \frac{\partial^2}{\partial y^2} \tilde{S}_0(y, t; y', t') = \frac{f_{2}' f_1' - f_{1}' f_2'}{(f_{2}' f_1' - f_{1}' f_2') (f_{2}' f_1' - f_{1}' f_2')}
$$

(3.29)

4. Evaluation of $p$-Adic and Ordinary Path Integrals

Quantum fluctuations lead to deformations of classical trajectory and any quantum history may be presented as $q(t) = x(t) + y(t)$, where $y' = y(t') = 0$ and $y'' = y(t'') = 0$. The corresponding Taylor expansion of $S[q]$ around classical path $x(t)$ is

$$
S[q] = S[x + y] = S[x] + \frac{1}{2!} \delta^2 S[x] = S[x] + \frac{1}{2} \int_{t'}^{t''} \left( y \frac{\partial}{\partial y} + y \frac{\partial}{\partial q} \right)^2 L(q, q, t) dt,
$$

(4.1)

where we used $\delta S[x] = 0$. For any $v = \infty, 2, 3, \ldots$, we can write

$$
\mathcal{K}_v(x'', t''; x', t') = \int \chi_v \left( -\frac{1}{h} S[x + y] \right) D y,
$$

(4.2)
or in the more explicit form,
\[
K_v(x'', t''; x', t') = \chi_v \left( -\frac{1}{h} \bar{S}(x'', t''; x', t') \right)
\times \int_{y'' - 0}^{y''} \chi_v \left( -\frac{1}{2h} \int_{y''}^{y'} \left( y \frac{\partial}{\partial q} + y \frac{\partial}{\partial q} \right)^2 L(q, q, t) dt \right) \overline{Dy},
\]
where we used \( y'' = y' = 0 \) and \( \bar{S}[x] = \bar{S}(x'', t''; x', t') \).

From (4.3) it follows that \( K_v(x'', t''; x', t') \) has the form
\[
K_v(x'', t''; x', t') = N_v(t'', t') \chi_v \left( -\frac{1}{h} \bar{S}(x'', t''; x', t') \right),
\]
where \( N_v(t'', t') \) does not depend on end points \( x'' \) and \( x' \).

To calculate \( N_v(t'', t') \), let us note that (4.3) can be rewritten as
\[
K_v(x'', t''; x', t') = \chi_v \left( -\frac{1}{h} \bar{S}(x'', t''; x', t') \right) K_v(0, t''; 0, t'),
\]
where \( K_v(0, t''; 0, t') = K_v(y'', t''; y', t') |_{y'' = y' = 0} \) and
\[
K_v(y'', t''; y', t') = \int_{y''}^{y'} \chi_v \left( -\frac{1}{h} \int_{y''}^{y'} \left[ \frac{1}{2} \frac{\partial^2 L_0}{\partial q^2} y^2 + \frac{\partial^2 \bar{S}_0}{\partial q \partial q} y' y + \frac{1}{2} \frac{\partial^2 \bar{S}_0}{\partial q^2} y^2 \right] \right) \overline{Dy}. \tag{4.6}
\]

According to (4.4) and (4.5) one has
\[
N_v(t'', t') = K_v(y'', t''; y', t') |_{y'' = y' = 0}, \tag{4.7}
\]
where
\[
K_v(y'', t''; y', t') = N_v(t'', t') \chi_v \left( -\frac{1}{h} \left[ \frac{1}{2} \frac{\partial^2 \bar{S}_0}{\partial q^2} y''^2 + \frac{\partial^2 \bar{S}_0}{\partial q \partial q} y' y' + \frac{1}{2} \frac{\partial^2 \bar{S}_0}{\partial q^2} y'^2 \right] \right). \tag{4.8}
\]

Let us employ now (1.3) and (1.4) to find a suitable expression for \( N_v(t'', t') \). The unitary condition (1.4) now reads:
\[
\int_{Q_v} K_v(y'', t''; y', t') K_v(y, t''; y', t') dy' = \delta_v(y'' - y). \tag{4.9}
\]
Substituting \( K_v(y'', t''; y', t') \) from (4.8) into (4.9), we obtain
\[
|N_v(t'', t')|^2 \int_{Q_v} \chi_v \left( \frac{1}{2h} \frac{\partial^2 \bar{S}_0}{\partial q^2} (y''^2 - y^2) + \frac{1}{2h} \frac{\partial^2 \bar{S}_0}{\partial q^2} (y'' - y) y' \right) dy' = \delta_v(y'' - y). \tag{4.10}
\]
Using the properties of \( \delta_v \)-functions, in particular: \( \int_{Q_v} \chi_v(ax) dx = \delta_v(a) \) and \( \delta_v(ax) = |a|^{-1} \delta_v(x) \), we have
\[
|N_v(t'', t')|^2 \int_{Q_v} \chi_v \left( \frac{1}{2h} \frac{\partial^2 \bar{S}_0}{\partial q^2} (y''^2 - y^2) \right) \left[ \frac{1}{h} \frac{\partial^2 \bar{S}_0}{\partial q^2} \right]^{-1} \delta_v(y'' - y) = \delta_v(y'' - y). \tag{4.11}
\]
Performing integration in (4.11) over variable \( y \), one obtains
\[
|N_v(t'', t')|_\infty = \left| \frac{1}{h} \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y'', t''; y', t') \right|_v^{\frac{1}{2}}. \tag{4.12}
\]

We have now
\[
N_v(t'', t') = \left| \frac{1}{h} \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y'', t''; y', t') \right|_v^{\frac{1}{2}} A_v(t'', t'), \tag{4.13}
\]
where \( |A_v(t'', t')|_\infty = 1 \). To use the condition (1.3), it has to be rewritten as
\[
\int_{Q_v} K_v(y'', t''; y, t) K_v(y, t; y', t') dy = K_v(y'', t''; y', t'). \tag{4.14}
\]
Inserting (4.8) into (4.14), where \( N_v(t'', t') \) has the form (4.13), we get conditions:
\[
\left| \frac{1}{h} \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y'', t''; y, t) \right|_v^{\frac{1}{2}} \left| \frac{1}{h} \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y, t; y', t') \right|_v^{\frac{1}{2}} |2\alpha|^{-\frac{1}{2}}
\]
\[
\times \left( -\frac{1}{2h} \left[ \frac{\partial^2}{\partial y''^2} \tilde{S}_0(y'', t''; y, t)y'' + \frac{\partial^2}{\partial y'^2} \tilde{S}_0(y, t; y', t') \right] \right)
\]
\[
\propto \left( \frac{1}{2h} \left[ \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y'', t''; y, t)y'' + \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y, t; y', t')y' \right]^2 \right)
\]
\[
\times \chi_v \left( -\frac{1}{h} \left[ \frac{1}{2} y''^2 \frac{\partial^2}{\partial y'^2} + y'' y' \frac{\partial^2}{\partial y'' \partial y'} + \frac{1}{2} y'^2 \frac{\partial^2}{\partial y''^2} \right] \tilde{S}_0(y'', t''; y', t') \right), \tag{4.16}
\]

where
\[
A_v(t'', t) A_v(t', t') \lambda_v(\alpha) = A_v(t'', t'), \tag{4.17}
\]
and
\[
\alpha = -\frac{1}{2h} \left[ \frac{\partial^2}{\partial y'' \partial y'} \tilde{S}_0(y'', t''; y, t) + \frac{\partial^2}{\partial y''^2} \tilde{S}_0(y, t; y', t') \right]. \tag{4.18}
\]

The condition (4.15) is satisfied due to the property (3.29). The equation (4.16) follows as the equality between arguments of additive characters, and such equality of arguments is a consequence of the formulae (3.24)-(3.29).

Solution of the equation (4.17) gives us an explicit expression for \( A_v(t'', t') \). We analyse (4.17) separately for real and \( p \)-adic cases. Using definition (2.9) of the \( \lambda_\infty \)-function in the form \( \lambda_\infty(x) = \exp \left( -\frac{i\pi}{4} \text{sign } x \right) \), and analogue of the first property in (2.7), one can easily show that
\[
\lambda_\infty \left( \frac{ab}{c} \right) = \lambda_\infty \left( \frac{c}{ab} \right) = \frac{\lambda_\infty(-c)}{\lambda_\infty(-a) \lambda_\infty(-b)}. \tag{4.19}
\]

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Using the property (3.29) and then (4.19), we find that
\[
A_\infty(t'', t') = \lambda_\infty \left( -\frac{1}{2h} \frac{\partial^2}{\partial y'' \partial y'} \bar{S}_0(y'', t''; y', t') \right).
\] (4.20)

In order to find \(A_p(t'', t')\) let us recall that we consider quadratic Lagrangian (3.17) with coefficients which are analytic functions of time \(t\), and the corresponding analytic classical solutions \(f_1(t)\) and \(f_2(t)\) of the equation of motion (3.18). Let these solutions be:
\[
f_1(t) = a_0 + a_1 t + a_2 t^2 + \cdots,
\] (4.21)
\[
f_2(t) = b_1 t + b_2 t^2 + \cdots,
\] (4.22)
with the region of convergence which does not exceed the disk \(|t|_p < |2|_p\). According to (4.21) and (4.22) we have:
\[
f''_2 f'_1 - f''_1 f'_2 = a_0 b_1 (t'' - t') + a_0 b_2 (t''^2 - t'^2) + \cdots, \tag{4.23}
\]
\[
f''_2 f_1 - f''_1 f_2 = a_0 b_1 (t'' - t) + a_0 b_2 (t''^2 - t^2) + \cdots, \tag{4.24}
\]
\[
f_2 f'_1 - f_1 f'_2 = a_0 b_1 (t - t') + a_0 b_2 (t^2 - t'^2) + \cdots. \tag{4.25}
\]
Note that \(\lambda_p(x)\), defined by (2.5) and (2.6), depend only on the first term in expansion of \(x\) if \(p \neq 2\), and the second and third terms if \(p = 2\). Hence, it is enough to take into account the first two terms in expansions (4.23)-(4.25) when their left-hand sides occur to be arguments of functions \(\lambda_p\). For example, we have
\[
\lambda_p \left( -\frac{1}{2h} [f''_2 f_1 - f''_1 f_2 + f_2 f'_1 - f_1 f'_2] \right) = \lambda_p \left( -\frac{1}{2h} [a_0 b_1 (t'' - t') + a_0 b_2 (t''^2 - t'^2)] \right)
\]
\[
= \lambda_p \left( -\frac{1}{2h} \frac{\partial^2}{\partial y'' \partial y'} \bar{S}_0(y'', t''; y', t') \right). \tag{4.26}
\]
Using expressions (3.28)-(3.29), properties (2.7) of \(\lambda_p\) functions, and the above expansions of \(f_1(t)\) and \(f_2(t)\), we get
\[
A_p(t'', t') = \lambda_p \left( -\frac{1}{2h} \frac{\partial^2}{\partial y'' \partial y'} \bar{S}_0(y'', t''; y', t') \right).
\] (4.27)

Since \(\bar{S}_0\), and terms linear in \(x''\) and \(x'\) of the classical action (3.7) do not affect function \(N_v(t'', t')\), results obtained for pure quadratic action \(\bar{S}_0(y'', t''; y', t')\) (3.23) are also valid for general quadratic case \(\bar{S}_0(x'', t''; x', t')\) (3.7).

In virtue of the above evaluation one can formulate the following

**Theorem.** The \(v\)-adic kernel \(\mathcal{K}_v(x'', t''; x', t')\) of the unitary evolution operator, defined by (1.1) and evaluated as the Feynman path integral, for quadratic Lagrangians (3.1) (and consequently, for quadratic classical actions (3.7)) has the form
\[
\mathcal{K}_v(x'', t''; x', t') = \lambda_v \left( -\frac{1}{2h} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}_0(x'', t''; x', t') \right) \left( \frac{1}{h} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}_0(x'', t''; x', t') \right)_v.
\]
extends to the general case (3.1). The property (1.5) follows from (1.4) are satisfied for the reduced Lagrangian (3.17), and in the analogous way the proof show that this expression satisfies (1.3)-(1.5). In fact, it is already shown that (1.3) and (1.5).

Proof. The formula (4.28) is a result of the above analytic evaluation, and one has to show that this expression satisfies (1.3)-(1.5). In fact, it is already shown that (1.3) and (1.4) are satisfied for the reduced Lagrangian (3.17), and in the analogous way the proof extends to the general case (3.1). The property (1.5) follows from

\[ \lim_{t' \to t''} \lambda_v \left( -\frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right) \left| \frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} \right|^\frac{1}{2} \chi_v \left[ \frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x'' \partial x'} (x''^2 - 2x''x' + x'^2) \right] = \delta_v(x'' - x'). \]

(4.29)

Starting from (4.28) and using definition (2.9) for \( \lambda_\infty \)-function one can rederive well-known result in ordinary quantum mechanics:

\[ K_\infty(x'', t''; x', t') = \left( \frac{i}{h} \frac{\partial^2}{\partial x'' \partial x'} S_0(x'', t''; x', t') \right)^\frac{1}{2} \exp \left( \frac{2\pi i}{h} \bar{S}(x'', t''; x', t') \right). \]

(4.30)

5. Adelic Path Integrals

In order to introduce adelic path integrals, let us start with analogue of (1.1) related to eigenfunctions in adelic quantum mechanics, i.e.

\[ \psi_{S, \alpha}(x'', t'') = \int A K_A(x'', t''; x', t') \psi_{S, \alpha}(x', t')dx', \]

(5.1)

where \( \psi_{S, \alpha}(x, t) \) has the form (2.16), and adelic propagator \( K_A(x'', t''; x', t') \) does not depend on \( S \). Since the equation (5.1) must be valid for any set \( S \) of primes \( p \), and adelic eigenstate is an infinite product of real and \( p \)-adic eigenfunctions, it is natural to consider adelic propagator in the following form:

\[ K_A(x'', t''; x', t') = K_\infty(x''_\infty, t''_\infty; x'_\infty, t'_\infty) \prod_p K_p(x''_p, t''_p; x'_p, t'_p), \]

(5.2)

where \( K_\infty(x''_\infty, t''_\infty; x'_\infty, t'_\infty) \) and \( K_p(x''_p, t''_p; x'_p, t'_p) \) are propagators in ordinary and \( p \)-adic quantum mechanics, respectively.

From (5.2) we see that one can introduce adelic path integral as an infinite product of ordinary and \( p \)-adic path integrals for all primes \( p \). Intuitively, we regard adelic Feynman’s functional integrals as path integrals on adelic spaces. Now we can consider (5.2) in the form of path integrals, and let us symbolically write the left-hand side as

\[ K_A(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_A \left( -\frac{1}{h} \bar{S}_A[q] \right) D_Aq, \]

(5.3)

where \( \chi_A(x) \) is adelic additive character (2.12), \( S_A[q] \) and \( D_Aq \) are adelic action and the Haar measure, respectively. For practical considerations, we define adelic path integral in the form

\[ K_A(x'', t''; x', t') = \prod_v \int_{x_v', t_v'}^{x_v'', t_v''} \chi_v \left( -\frac{1}{h} \int_{t_v'}^{t_v''} L(q_v, t_v, t_v)dt_v \right) D_q_v, \]

(5.4)
where index $v = \infty, 2, 3, \ldots, p, \ldots$ denotes real and all $p$-adic cases. As an adelic Lagrangian one understands infinite sequence

$$L_A(\dot{q}, q, t) = (L(\dot{q}_\infty, q_\infty, t_\infty), L(\dot{q}_2, q_2, t_2), L(\dot{q}_3, q_3, t_3), \ldots, L(\dot{q}_p, q_p, t_p), \ldots), \quad (5.5)$$

where $|L(\dot{q}_p, q_p, t_p)|_p \leq 1$ for all primes $p$ but a finite set $S$ of them. Consequently, an adelic quadratic Lagrangian looks like (5.5), where each element $L(\dot{q}_v, q_v, t_v)$ has the same form (3.1). Note that to escape an abundance of indices, we often omit some of them in the cases when they are implicitly understood.

Taking into account results obtained in the previous section, we can write adelic path integral for quadratic Lagrangians (and consequently, quadratic classical actions) as

$$K_A(x''; x', t') = \prod_v \lambda_v \left(-\frac{1}{2\hbar} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}_0(x''; x', t') \right) \left| \frac{1}{\hbar} \frac{\partial^2}{\partial x'' \partial x'} \bar{S}_0(x''; x', t') \right|_v^{\frac{1}{2}}$$

$$\times \chi_v \left(-\frac{1}{\hbar} \bar{S}(x''; x', t') \right). \quad (5.6)$$

Note that vacuum state $\Omega(|x_p|_p)$ transforms as

$$\Omega(|x''_p|_p) = \int_{Q_p} K_p(x''_p, t''_p; x'_p, t'_p) \Omega(|x'_p|_p) dx'_p = \int_{Z_p} K_p(x''_p, t''_p; x'_p, t'_p) dx'_p. \quad (5.7)$$

As a consequence of (5.7) one has

$$\int_{Z_p} K_p(x''_p, t''_p; x_p, t_p)K_p(x_p, t_p; x'_p, t'_p)dx_p = K_p(x''_p, t''_p; x'_p, t'_p), \quad (5.8)$$

which may be regarded as an additional condition on $p$-adic path integrals in adelic quantum mechanics for all but a finite number of primes $p$. Conditions (5.7) and (5.8) impose a restriction on a dynamical system to be adelic (see [17]). It is practically a restriction on time $t_p$ to have consistent adelic time $t$.

6. Concluding Remarks

Evaluating the method of Feynman’s functional integral simultaneously on real and $p$-adic one-dimensional spaces, in the previous sections we derived general expressions for propagators $K(x'', t''; x', t')$ in ordinary, $p$-adic and adelic quantum mechanics. Especially, it has been done for Lagrangians $L(\dot{q}, q, t)$ which are polynomials at most the second degree in dynamical variables $\dot{q}$ and $q$.

It is worth pointing out that the formalism of ordinary and $p$-adic path integrals can be regarded as the same at different levels of evaluation, and the obtained results have the same form. In fact, this property of number field invariance has to be natural for general mathematical methods in physics and fundamental physical laws (cf. [16]). In the present case it is mainly a consequence of the form invariance of the Gauss integral under interchange of $\mathbf{R}$ and $\mathbf{Q}_p$ (see (2.4) and (2.8)).
The idea to derive expression (4.27) in the above way was proposed by one of the authors (B.D.) in contribution to the Bogolyubov conference [18]. Some aspects of p-adic path integral for quadratic actions are considered earlier in Ref. [7].

The above results, obtained for one-dimensinal systems, are an appropriate starting point for a generalization to two-dimensional [19] and higher-dimensional cases. These results can be also exploited to construct and investigate a new approach to the Brownian motion with p-adic effects.

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