Blow up criterion for incompressible nematic liquid crystal flows

Yinghui Zhang\(^1\) Zhong Tan\(^2\) and Guochun Wu\(^3\) *

\(^1\)Department of Mathematics, Hunan Institute of Science and Technology, Yueyang 414006, China
\(^2,3\)School of Mathematical Sciences, Xiamen University, Fujian 361005, China

Abstract

In this paper, we consider the short time classical solution to a simplified hydrodynamic flow modeling incompressible, nematic liquid crystal materials in dimension three. We establish a criterion for possible breakdown of such solutions at a finite time. More precisely, if \((u, d)\) is smooth up to time \(T\) provided that

\[
\int_0^T \left\| \nabla \times u(t, \cdot) \right\|_{BMO(\mathbb{R}^3)} + \left\| \nabla d(t, \cdot) \right\|_{L^8(\mathbb{R}^3)}^8 dt < \infty.
\]

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1 Introduction

We consider the following hydrodynamic system modeling the flow of liquid crystal materials in dimension three (see \([2,3,8,10]\)) and references therein):

\[
\begin{align*}
    u_t + u \cdot \nabla u - \nu \Delta u + \nabla P &= -\nabla d \cdot \Delta d, \\
    \partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d, \\
    \nabla \cdot u &= 0 \quad |d| = 1,
\end{align*}
\]

for \((t, x) \in [0, +\infty) \times \mathbb{R}^3\). Here \(u : \mathbb{R}^3 \to \mathbb{R}^3\) represents the velocity field of the incompressible viscous fluid, \(\nu > 0\) is the kinematic viscosity, \(P : \mathbb{R}^3 \to \mathbb{R}\) represents the pressure function, \(d : \mathbb{R}^3 \to \mathbb{S}^2\) represents the macroscopic average of the nematic liquid crystal orientation field, \(\nabla \cdot\) and \(\Delta\) denote the divergence operator and the laplace operator respectively. We are interested in the Cauchy problem (1.1) with the initial value

\[(u(x, 0), d(x, 0)) = (u_0(x), d_0(x))\]  

\(*\)Corresponding author: Guochun Wu, Email address: guochunwu@126.com
satisfying the following compatibility condition:

\[ \nabla \cdot u_0(x) = 0, \quad |d_0(x)| = 1, \quad \lim_{|x| \to \infty} d_0(x) = a \in \mathbb{S}^2, \quad (1.3) \]

where \( a \) is a given unit vector.

The above system is a simplified version of the Ericksen-Leslie model, which reduces to the Ossen-Frank model in the static case, for the hydrodynamics of nematic liquid crystals developed during the period of 1958 through 1968 [2,3,8]. It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow field \( u(x,t) \), the macroscopic description of the microscopic orientation configurations \( d(x,t) \) of rod-like liquid crystals. Roughly speaking, the system (1.1) is a coupling between the non-homogeneous Navier-Stokes equation and the transported flow harmonic maps. Due to the physical importance and mathematical challenges, the study on nematic liquid crystals has attracted many physicists and mathematicians. The mathematical analysis of the liquid crystal flows was initiated by Lin [9], Lin and Liu in [10,11]. For any bounded smooth domain in \( \mathbb{R}^2 \), Lin, Lin and Wang [12] have proved the global existence of Leray-Hopf type weak solutions to system (1.1) which are smooth everywhere except on finitely many time slices (see [4] for the whole space). The uniqueness of weak solutions in two dimension was studied by [13,15]. Recently, Hong and Xin [5] studied the global existence for general Ericksen-Leslie system in dimension two. However, the global existence of weak solutions to the incompressible nematic liquid crystal flow equation (1.1) in three dimension with large initial data is still an outstanding open question.

The local well-posedness of the Cauchy problem of system (1.1) is rather standard (see [4,6,12]). At present, there is no global-in-time existence theory for classical solutions to system (1.1). In this paper, we will consider the short time classical solution to (1.1)-(1.2) and some criterion that characterizes the first finite singular time. Motivated by the famous work [1], Huang and Wang [6] have obtained a BKM type blow-up criterion. However, the techniques involved in this paper are much different from [6], which we believe that the result may have its own interest. Our main results are formulated as the following theorem:

**Theorem 1.1.** For \( u_0 \in H^s(\mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \) and \( d_0 - a \in H^{s+1}(\mathbb{R}^3) \) with \( |d_0| = 1 \) for \( s \geq 3 \). Suppose that \((u,d)\) is a smooth solution to the system (1.1)-(1.2), then for given \( T > 0 \), \((u,d)\) is smooth up to time \( T \) provided that

\[ \int_0^T \left( \| \nabla \times u(t, \cdot) \|_{BMO(\mathbb{R}^3)} + \| \nabla d(t, \cdot) \|_{L^8(\mathbb{R}^3)}^8 \right) dt < \infty. \quad (1.4) \]

**Notations.** We denote by \( L^p \), \( W^{m,p} \) the usual Lebesgue and Sobolev spaces on \( \mathbb{R}^3 \) and \( H^m = W^{m,2} \), with norms \( \| \cdot \|_{L^p}, \| \cdot \|_{W^{m,p}} \) and \( \| \cdot \|_{H^m} \) respectively. For the sake of conciseness, we do not distinguish functional space when scalar-valued or vector-valued functions are involved. We denote \( \nabla = \partial_x = (\partial_1, \partial_2, \partial_3) \), where \( \partial_i = \partial_{x_i}, \nabla_i = \partial_i \) and
put $\partial^l_x f = \nabla^l f = \nabla(\nabla^{l-1} f)$. We assume $C$ be a positive generic constant throughout this paper that may vary at different places and the integration domain $\mathbb{R}^3$ will be always omitted without any ambiguity. Finally, $\langle \cdot, \cdot \rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$.

**Remark 1.1.** It is standard that the condition (1.3) is preserved by the flow. In fact, first notice that the divergence free of the velocity field $u$ can be justified by the initial assumption that $\nabla \cdot u = 0$. Indeed, this can be easily and formally observed by take $\nabla \cdot$ to the momentum equation. Moreover, applying the maximum principle to the equation for $|d|^2$, one also can easily see that $|d| = 1$ under the initial assumption that $|d_0| = 1$.

## 2 Proof of Theorem 1.1

We prove our theorem in this section. Without loss of generality, we assume $\nu = 1$. The first bright idea to reduce many complicated computations lies in that we just need to do the lowest order and highest order energy estimates for the solutions. This is motivated by the following observation:

$$\|f\|_{H^k}^2 \leq C \|\langle f, \nabla^k f \rangle\|_{L^2}^2, \quad \forall f \in H^k. \quad (2.1)$$

This inequality (2.1) can be easily proved by combing Young’s inequality and Gagliardo-Nirenberg’s inequality.

$$\|\nabla^i f\|_{L^p} \leq C(p) \|f\|_{L^q}^{\alpha} \|\nabla^k f\|_{L^r}^{1-\alpha}, \quad \forall f \in H^k \quad (2.2)$$

where $\frac{1}{p} - \frac{i}{3} = \frac{1}{q} \alpha + (\frac{1}{r} - \frac{k}{3})(1 - \alpha)$ with $i \leq k$.

In order to prove Theorem 1.1, we also need the following logarithmic Sobolev’s which is proved in [7] and is an improved version of that in [1].

**Lemma 2.1.** For $p > n$, the following logarithmic Sobolev’s embedding theorem holds for all divergence free vector fields:

$$\|\nabla f\|_{L^\infty} \leq C[1 + \|f\|_{L^2} + \|\nabla \times f\|_{BMO} \ln(1 + \|f\|_{W^{2,p}})]. \quad (2.3)$$

Now we are in a position to prove our Theorem 1.1.

**Proof of Theorem 1.1** First of all, for classical solutions to (1.1)-(1.2), one has the following basic energy law:

$$\|u(t, \cdot)\|_{L^2}^2 + \|\nabla d(t, \cdot)\|_{L^2}^2 + \int_0^t (\|\nabla u(s, \cdot)\|_{L^2}^2 + \|\Delta d(s, \cdot) + |\nabla d|^2 d(s, \cdot)\|_{L^2}^2)ds = \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2, \quad \forall t > 0. \quad (2.4)$$

Let’s concentrate on the case $s = 3$. For each multi-index $\alpha$ with $|\alpha| \leq 3$, by applying $\partial^\alpha_x$ to (1.1a) and $\partial^{\alpha+1}_x$ to (1.1b), multiplying them by $\partial_x^\alpha u$, $\partial_x^{\alpha+1}d$ respectively
and then integrating them over $\mathbb{R}^3$, we have

$$
\frac{1}{2}\frac{d}{dt}\|\partial_x^{\alpha}(u, \nabla d)\|_{L^2}^2 + \|\nabla u, \partial_x^{\alpha}(\nabla u, \Delta d)\|_{L^2}^2
- \langle \partial_x^{\alpha}(u \cdot \nabla u), \partial_x^{\alpha}u \rangle
- \langle \partial_x^{\alpha+1}(u \cdot \nabla d, \partial_x^{\alpha+1}d) \rangle + \langle \partial_x^{\alpha+1}(|\nabla d|^2 d), \partial_x^{\alpha+1}d \rangle \tag{2.5}
$$

where $I_{[\alpha,i]}$ are the corresponding terms in the above equation which will be estimated as follows. Now for $|\alpha| = 1$ in (2.5), integrating by parts and using the divergence free condition $\nabla \cdot u = 0$, we have

$$
|I_{1,1}| = |\langle \partial_x^{1}(u \cdot \nabla u), \partial_x^{1}u \rangle| = |\langle \partial_x^{1}u \cdot \nabla u, \partial_x^{1}u \rangle| \leq C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^2}^2.
$$

Combining Cauchy’s inequality, Sobolev’s inequality and the fact $|\nabla d|^2 = -d \cdot \triangle d$ (since $|d| = 1$) gives

$$
|I_{1,2}| = |\langle \partial_x^{1}(\Delta d \cdot \nabla d), \partial_x^{1}u \rangle| = |\langle \Delta d \cdot \nabla d, \partial_x^{2}u \rangle|
\leq C\|\Delta d\|_{L^6}^3 + \frac{1}{8}\|\nabla^2 u\|_{L^2}^2
\leq C\|\nabla d\|_{L^4}^4\|\nabla \Delta d\|_{L^2}^2 + \frac{1}{8}\|\nabla^2 u\|_{L^2}^2.
\leq C\|\nabla d\|_{L^4}^4 + \frac{1}{8}(\|\nabla \Delta d\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).
$$

Similarly,

$$
|I_{1,3}| = |\langle \partial_x^{2}u \cdot \nabla d + \partial_x u \cdot \nabla \partial_x d, \partial_x^{2}d \rangle| \leq C\|\nabla u\|_{L^\infty}\|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^3 + \frac{1}{8}(\|\nabla \Delta d\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2),
\leq C\|\nabla d\|_{L^4}^4 + \frac{1}{8}\|\nabla \Delta d\|_{L^2}^2.
$$

Taking the above estimates in (2.5) for $|\alpha| = 1$, we arrive at

$$
\frac{d}{dt}\|\nabla d\|_{L^4}^4 \leq C\|\nabla u\|_{L^\infty}\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2. \tag{2.6}
$$

Next we derive an estimate for $\|\nabla d\|_{L^4}^4$. Differentiating (1.1b) with respect to $x$, we have

$$
\partial_x d_{t} - \triangle \partial_x d = \partial_x (|\nabla d|^2 d - u \cdot \nabla d). \tag{2.7}
$$

We multiply (2.7) by $|\nabla d|^2 \partial_x d$ and integrate over $\mathbb{R}^3$ to obtain that

$$
\frac{1}{4}\frac{d}{dt}\|\nabla d\|_{L^4}^4 + \|\nabla d\|_{L^2}^2 + \frac{1}{2}\|\partial_x (|\nabla d|^2 d)\|_{L^2}^2
= \langle |\nabla d|^2 \partial_x d, \partial_x (|\nabla d|^2 d - u \cdot \nabla d)\rangle
= \|\nabla d\|_{L^6}^2 + \langle (|\nabla d|^2 \partial_x d, 2\nabla d_{t} \partial_x d - \partial_x u \cdot \nabla d)\rangle
\leq C\|\nabla d\|_{L^6}^2 + \frac{1}{8}\|\nabla^3 d\|_{L_2}^2. \tag{2.8}
$$
We add (2.8) to (2.6) to obtain

$$\frac{d}{dt} \left( \| \nabla u, \nabla^2 d \|^2_{L^2} + \| \nabla d \|^4_{L^4} \right) + \| (\nabla^2 u, \nabla^3 d, |\nabla d||\nabla^2 d|) \|^2_{L^2} \leq C(\| \nabla u \|^\infty + \| \nabla d \|^8_{L^4}) \left( \| (\nabla u, \nabla^2 d) \|^2_{L^2} + \| \nabla d \|^4_{L^4} \right),$$

which gives that

$$\| (\nabla u, \nabla^2 d)(t, \cdot) \|^2_{L^2} + \| \nabla d(t, \cdot) \|^4_{L^4} + \int_{t_0}^t \| (\nabla^2 u, \nabla^3 d, |\nabla d||\nabla^2 d|) \|^2_{L^2} ds \leq C e^{C \int_{t_0}^t (\| \nabla u \|^\infty + \| \nabla d \|^8_{L^4}) ds} \left( \| (\nabla u, \nabla^2 d)(t, \cdot) \|^2_{L^2} + \| \nabla d(t, \cdot) \|^4_{L^4} \right).$$

(2.9)

Noting the condition (1.4) in Theorem 1.1, thus for any fixed small constant $\varepsilon > 0$, there exists $T_*(\varepsilon) < T$ such that

$$\int_{T_*}^T (\| \nabla \times u \|_{BMO} + \| \nabla d \|^8_{L^4}) ds \leq \varepsilon.$$

(2.10)

Define the temporal energy functional:

$$H(t) = \sup_{T_* \leq s \leq t} (\| \nabla^3 u(s, \cdot) \|^2_{L^2} + \| \nabla^4 d(s, \cdot) \|^2_{L^2}).$$

By (2.1), (2.3), (2.4), (2.9) and (2.10), we arrive at

$$\| (\nabla u, \nabla^2 d)(t, \cdot) \|^2_{L^2} + \| \nabla d(t, \cdot) \|^4_{L^4} + \int_{t_0}^t \| (\nabla^2 u, \nabla^3 d, |\nabla d||\nabla^2 d|) \|^2_{L^2} ds \leq C_* e^{C_0 (1 + H(t))^{C_*}} \| \nabla u \|^3_{L^2} \| \nabla^2 d \|^4_{L^4}, \quad \forall T_* \leq t < T,$$

(2.11)

where $C_0$ depends on $\|(u_0, \nabla d_0)\|^2_{L^2}$ and $C_*$ depends on $\|(\nabla u, \nabla^2 d)\|^2_{L^2} + \| \nabla d \|^4_{L^4}$.

Next for $|\alpha| = 3$. For $I_{3,1}$, we need to use the following Moser-type inequality (see [14, p. 43]):

$$\| D^s(fg) \|_{L^2} \leq C(\| g \|_{L^\infty} \| \nabla^s f \|_{L^2} + \| f \|_{L^\infty} \| \nabla^s g \|_{L^2}).$$

(2.12)

Thus we have

$$|I_{3,1}| = \| \partial_x^3 \text{div}(u \otimes u) \|_{L^2} \| \partial_x^3 u \|_{L^2} \leq C \| \nabla^3 (u \otimes u) \|_{L^2}^2 + \frac{1}{16} \| \nabla^4 u \|^2_{L^2} \leq C \| u \|^8_{L^\infty} \| \nabla^3 u \|^2_{L^2} + \frac{1}{16} \| \nabla^4 u \|^2_{L^2} \leq C \left( \| \nabla u \|^2_{L^2} \| \nabla^4 u \|^2_{L^2} \right)^2 + \frac{1}{16} \| \nabla^4 u \|^2_{L^2} \leq C \| \nabla u \|_{L^2}^4 + \frac{1}{8} \| \nabla^4 u \|^2_{L^2}.$$

(2.13)

For $I_{3,2}$, we apply (2.12) and the fact

$$\Delta d \cdot \nabla d = \nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I)$$

to obtain that

$$|I_{3,2}| = \| \partial_x^3 \nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I) \|_{L^2} \| \partial_x^4 u \|_{L^2} \leq C \| \nabla^3 \nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I) \|_{L^2}^2 + \frac{1}{16} \| \nabla^4 u \|^2_{L^2} \leq C \| |\nabla^2 d|^2 \|_{L^2} \| \nabla^5 u \|^2_{L^2} \leq C \| \nabla^2 d \|^4_{L^2} + \frac{1}{8} \| \nabla^4 u \|^2_{L^2}.$$

(2.14)
Similar as the proof of (2.14), $I_{3,3}, I_{3,4}$ can be bounded as follow:

$$
|I_{3,3}| = |\langle \partial_3^2 (u \cdot \nabla d), \partial_3^2 d \rangle | \\
\leq C \| \partial_3^2 (u \cdot \nabla d) \|^2_{L^2} + \frac{1}{16} \| \nabla^5 d \|^2_{L^2} \\
\leq C (\| \nabla^3 d \|^2_{L^2} + \| \nabla^4 u \|^2_{L^2}) + \frac{1}{16} \| \nabla^4 u \|^2_{L^2} \\
\leq C \{ \| \nabla^2 d \|^2_{L^2} \| \nabla^5 d \|^2_{L^2} \| \nabla u \|^2_{L^2} \| \nabla^4 u \|^2_{L^2} \\
+ \| \nabla u \|^2_{L^2} \| \nabla^4 u \|^2_{L^2} \| \nabla^2 d \|^2_{L^2} \| \nabla^5 d \|^2_{L^2} \}^2 + \frac{1}{16} \| \nabla^4 u \|^2_{L^2} \\
\leq C (\| \nabla^2 d \|^4_{L^2} + \| \nabla u \|^4_{L^2}) + \frac{1}{8} (\| \nabla^4 u \|^2_{L^2} + \| \nabla^5 d \|^2_{L^2}),
$$

(2.15)

and

$$
|I_{3,4}| = \langle \partial_3^3 (|\nabla d|^2 d), \partial_3^2 d \rangle | \\
\leq C \| \partial_3^3 (|\nabla d|^2 d) \|^2_{L^2} + \frac{1}{32} \| \nabla^5 d \|^2_{L^2} \\
\leq C (\| \nabla^3 \nabla d \|^2_{L^2} + \| \nabla d \|^2_{L^2}) + \frac{1}{32} \| \nabla^5 d \|^2_{L^2} \\
\leq C (\| \nabla d \|^2_{L^2} \| \nabla^4 d \|^2_{L^2} + \| \Delta d \|^2_{L^2} \| \nabla^3 d \|^2_{L^2}) + \frac{1}{32} \| \nabla^5 d \|^2_{L^2} \\
\leq C (\| \nabla^2 d \|^2_{L^2} + \| \nabla^3 \nabla d \|^2_{L^2} + \| \nabla^2 d \|^2_{L^2} \| \nabla^5 d \|^2_{L^2})^2 + \frac{1}{16} \| \nabla^5 d \|^2_{L^2} \\
\leq C \| \nabla^2 d \|^4_{L^2} + \frac{1}{8} \| \nabla^5 d \|^2_{L^2}.
$$

(2.16)

Putting (2.13)-(2.16) into (2.5) for $|\alpha| = 3$ and by (2.11), we arrive at

$$
\frac{d}{dt} (\| \nabla^3 u \|^2_{L^2} + \| \nabla^4 d \|^2_{L^2}) + \| \nabla^4 u \|^2_{L^2} + \| \nabla^5 d \|^2_{L^2} \\
\leq C (\| \nabla^2 d \|^4_{L^2} + \| \nabla u \|^4_{L^2}) \leq C_* (1 + H(t))^{C_0 \varepsilon}
$$

for all $T_* \leq t < T$. Integrating the above inequality with respect to time from $T_*$ to $t \in [T_*, T)$, we obtain

$$
1 + \| \nabla^3 u(t, \cdot) \|^2_{L^2} + \| \nabla^4 d(t, \cdot) \|^2_{L^2} \\
\leq 1 + \| \nabla^3 u(T_*, \cdot) \|^2_{L^2} + \| \nabla^4 d(T_*, \cdot) \|^2_{L^2} + C_*(1 + H(t))^{C_0 \varepsilon}.
$$

(2.17)

If we choose $\varepsilon$ small such that $C_0 \varepsilon < \frac{1}{2}$, then (2.17) implies

$$
1 + H(t) \leq C (1 + \| \nabla^3 u(T_*, \cdot) \|^2_{L^2} + \| \nabla^4 d(T_*, \cdot) \|^2_{L^2}),
$$

for all $T_* \leq t < T$. Since the right hand side of above inequality is independent of $t$ for $T_* \leq t < T$, we conclude that the above inequality is valid for $t = T$ which means that $u(T, \cdot) \in H^3(\mathbb{R}^3)$ and $d(T, \cdot) - a \in H^4(\mathbb{R}^3)$. Thus the proof of Theorem 1.1 is completed.

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