DIFFUSION BOUND FOR THE NONLINEAR ANDERSON MODEL

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Abstract. In this paper we prove the power-law in time upper bound for the diffusion of a 1D discrete nonlinear Anderson model. We remove completely the decaying condition restricted on the nonlinearity of Bourgain-Wang (Ann. of Math. Stud. 163: 21–42, 2007.). This gives a resolution to the problem of Bourgain (Illinois J. Math. 50: 183–188, 2006.) on diffusion bound for nonlinear disordered systems. The proof uses a novel “norm” based on tame property of the Hamiltonian.

1. Introduction and main results

Consider first the linear random Schrödinger equation on $\mathbb{Z}$

$$i\dot{q}_j + \epsilon(q_{j-1} + q_{j+1}) + v_j q_j = 0,$$

where $V = (v_j)_{j \in \mathbb{Z}}$ is a family of identical independent distributed (i.i.d.) random variables with uniform distribution on $[0, 1]$ and $\epsilon > 0$ is the coupling for describing the strength of random disorder. This so-called 1D Anderson model was originally introduced by Anderson [And58] in the context of wave propagation of non-interacting quantum particles through random disordered media. Since this seminal work, a great deal of attention has been paid to dynamical and spectral properties of this model both in physics and mathematics community. Of particular importance is the phenomenon of dynamical localization (DL for short), i.e., for all fast decaying initial state $q(0) = \{q_j(0)\}_{j \in \mathbb{Z}}$, the absence of diffusion holds:

$$\sup_{t \in \mathbb{R}} \sum_{j \in \mathbb{Z}} j^2 |(e^{-itH_0}q(0))_j|^2 < \infty,$$

where $H_0$ is the time independent part of (1.1) defined by

$$(H_0 q)_j = \epsilon(q_{j-1} + q_{j+1}) + v_j q_j.$$

The DL for (1.1) has been proved for a.e. $V \in [0, 1]^\mathbb{Z}$ in [Aiz94, dRJLS96, GDB98, JZ19] via different methods. We would also like to mention that considerable progress has been made towards localization for Anderson type models in higher dimensions based on works of [FSS3, AM93].

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A natural question is what can be said about diffusion in the nonlinear models? Precisely, let us consider the nonlinear perturbation of \((1.1)\)

\[
i\ddot{q}_j + \epsilon(q_{j-1} + q_{j+1}) + v_j q_j + \lambda_j |q_j|^2 q_j = 0,
\]

where \((\lambda_j)_{j \in \mathbb{Z}} \in \mathbb{R}\). We call \((1.2)\) the \textit{nonlinear Anderson model}. This type of models has important applications in a variety of physical systems, especially the Bose-Einstein Condensation [DGPS99], and has been extensively studied. We are interested in the bound on the following diffusion

\[
D(t, q(0)) = \sum_{j \in \mathbb{Z}} j^2 |q_j(t)|^2,
\]

where \(q(t) = \{q_j(t)\}_{j \in \mathbb{Z}}\) is the solution of \((1.2)\) with initial datum \(q(0) \in \ell^2(\mathbb{Z})\). We say \((1.2)\) has \textit{nonlinear dynamical localization} if

\[
\sup_{t \in \mathbb{R}} D(t, q(0)) < \infty
\]

holds for \textit{all} fast decaying initial state \(q(0)\). The problem for establishing nonlinear DL remains largely open and only partial results were obtained. Suppose \(\lambda_j = \delta > 0\) (which corresponds to the usual nonlinear Schrödinger equation). The first progress on nonlinear DL was made by Fröhlich-Spencer-Wayne [FSW86] in case \(\epsilon = 0\) and \(0 < \delta \ll 1\): By using KAM iterations, they constructed full dimensional KAM tori (or almost-periodic solutions) with action variables satisfying \(|q_j(0)| \leq e^{-|j|^C} (C > 1)\). This means that there exists \textit{sup-exponential} decaying initial state such \((1.4)\) holds. This result was enhanced later by Pöschel [P60] to \(|q_j(0)| \leq e^{-(\ln |j|)^C} (C > 1)\). Along this line and for \(0 < \epsilon, \delta \ll 1\), Bourgain-Wang [BW08] got the existence of finitely dimensional KAM tori (or quasi-periodic solutions) for some discrete nonlinear random Schrödinger equations in higher dimensions. They made use of powerful Green’s function estimate methods of Bourgain-Goldstein-Schlag [BGS02] in quasi-periodic operators theory.

Another aspect is to work in \textit{finite} time scales in case \(\lambda_j = \delta \ll 1\). If the initial state is polynomially localized, by using Birkhoff normal form method, Benettin-Fröhlich-Giorgilli [BFG88] showed that the \(\ell^2\)-norm of \(q(t)\) remains localized in very long (but finite) time for some \(dD\) lattice nonlinear oscillation equations with i.i.d. Gaussian random potential. In a significant work [WZ09], Wang-Zhang proved the first \textit{long-time} nonlinear Anderson localization (inspired by RAGE Theorem) with roughly initial state (i.e. \(q(0) \in \ell^2(\mathbb{Z})\)) in time scale \(t \leq (\epsilon + \delta)^{-A}, A > 1\). In this theorem, they required actually both high disorder and weak nonlinearity (i.e., \(\epsilon + \delta \ll 1\)). Fishman-Krivolapov-Soffer [FKS08] obtained the \textit{exponentially} DL (i.e., with \(|j|^2\) being replaced by the exponential bound in \((1.3)\)) of time scale \(t \leq \delta^{-2}\) for arbitrary \(\epsilon > 0\). Subsequently, some results of [FKS08] have been improved to \(t \leq \delta^{-A}\) for any \(A \geq 2\) [FKS09] by the same authors, but the proof is partly rigorous. Very recently, Cong-Shi-Zhang [CSZ20] proved
nonlinear Anderson localization in time scale \( t \leq e^{\frac{\ln(\epsilon + \delta)}{|\ln(\epsilon + \delta)|}} \). As a result, they confirmed a conjecture of Wang-Zhang [WZ09] in long-time scale.

In 2006, Bourgain [Bou06] revisited the dynamical transport of the nonlinear Anderson model and raised the following problem on diffusion bound:

**Problem 1.1 (cf. [Bou06]).** What may be said about the growth of \( D(t, q(0)) \) in general (i.e., for \( \lambda_j = \delta > 0 \))?

Regarding this problem, Bourgain-Wang showed in the elegant work [BW07] that if \( \lambda_j \) satisfies the decaying condition

\[
|\lambda_j| \leq \frac{\epsilon}{(1 + |j|)\tau} \text{ for some } \tau > 0,
\]

then for any \( q(0) \in H^1 = \left\{ \{q_j\}_{j \in \mathbb{Z}} : \sum_{j \in \mathbb{Z}} j^2 |q_j|^2 < \infty \right\} \), the following diffusion bound holds for all \( \kappa > 0 \), \( 0 < \epsilon < \epsilon_0(\tau, \kappa) \ll 1 \) and a.e. \( V \):

\[
D(t, q(0)) \leq t^\kappa \text{ as } t \to \infty.
\]

The proof used the Birkhoff normal form type arguments and the decaying condition (1.5) plays an essential role there.

In this paper we consider (1.2) with non-decaying nonlinearity, i.e., \( \lambda_j = \delta > 0 \) for all \( j \in \mathbb{Z} \). We will show that for \( q(0) \in H^1 \), \( \kappa > 0 \) and \( 0 < \epsilon + \delta < \epsilon_0(\kappa) \ll 1 \), the following holds for a.e. \( V \):

\[
D(t, q(0)) \leq t^\kappa \text{ as } t \to \infty.
\]

More precisely, we study

\[
i q_j = \epsilon(q_{j-1} + q_{j+1}) + v_j q_j + \delta |q_j|^2 q_j,
\]

where \( \epsilon, \delta > 0 \) and \( V = (v_j)_{j \in \mathbb{Z}} \) is a family of i.i.d. random variables with uniform distribution on \([0, 1]\). We have the following main result.

**Theorem 1.2.** Consider (1.6) and fix \( \kappa > 0 \). Then there exists \( \varepsilon_0 = \varepsilon_0(\kappa) > 0 \) (depending only on \( \kappa \)) such that for \( 0 < \epsilon + \delta < \varepsilon_0 \) the following holds true: Assuming \( q(0) = \{q_j(0)\}_{j \in \mathbb{Z}} \) satisfies

\[
\sum_{j \in \mathbb{Z}} j^2 |q_j(0)|^2 < \infty,
\]

then for a.e. \( V \in [0, 1]^\mathbb{Z} \) the solution \( q(t) = \{q_j(t)\}_{j \in \mathbb{Z}} \) of (1.6) with initial datum \( q(0) \) obeys

\[
\sum_{j \in \mathbb{Z}} j^2 |q_j(t)|^2 < t^\kappa \text{ as } t \to +\infty.
\]

**Remark 1.1.**

- We remove completely the decaying condition (1.5) of Bourgain-Wang [BW07].
• Our result gives a resolution to the Problem 1.1 of Bourgain [Bou06]
mentioned above. Of course, nonlinear dynamical localization for
(1.6) remains open, and the bound (1.4) is unknown even for very
fast decaying initial state $q(0)$ (for example, $q(0) = \delta_0$).
• Motivated by results of Bourgain [Bou99] and Wang [Wan08], it is
reasonable to expect for some $C > 0$
$$D(t, q(0)) \leq (\ln t)^C \text{ as } t \to \infty.$$  
• In case $\delta = 0$, it is well-known that (cf. [JZ19]) DL holds for all
coupling $\epsilon > 0$. Thus it is interesting to study diffusion bound for
(1.6) for arbitrary $\epsilon > 0$.
• We also hope that the power-law diffusion bound can be extended
to nonlinear Anderson model on $\mathbb{Z}^d$ for $d \geq 2$.

About the proof: The main scheme of our proof is adapted from Bourgain-
Wang [BW07] which uses Birkhoff normal form type transformations to
construct barriers centered at some $\pm j_0, j_0 > 1$ of width $\sim \ln j_0$, where the
terms responsible for diffusion are small enough. As mentioned earlier, the
decaying condition (1.5) is crucial in Bourgain-Wang’s arguments. In fact,
Bourgain-Wang emphasized in [BW07] that: “Here we only want to point
out that condition (2.1.2) (i.e., (1.5) in the present) plays an essential role
in the construction. It enables us to work only with monomials of bounded
degrees. The small divisors arising in the process (cf. (2.3.13)) are then
controlled by shifting in $V$”. However, in the absence of decaying condition
(1.5), Bourgain-Wang’s iterations scheme becomes invalid. The main new
ingredient in our approach here is that we can make use of some kind of
tame property of the Hamiltonian which was first introduced by Bamus-
Grébert [BG06] in dealing with polynomial long time stability result for a
class of nonlinear Hamiltonian PDEs. Such tame property originates from
the so-call tame inequality
$$\| p * q \|_{H^s} \leq C(s) (\| p \|_{H^s} \| q \|_{H^1} + \| p \|_{H^1} \| q \|_{H^s}) \text{ for some } C(s) > 0, \quad (1.7)$$
where
$$p, q \in \mathbb{C}^\mathbb{Z}, \ (p * q)_j = \sum_i p_{j-i} q_i, \ \| q \|_{H^s}^2 := \sum_{j \in \mathbb{Z}} |j|^{2s} |q_j|^2.$$  
Considering (1.7) and assuming that $q = \{q_j\}_{j \in \mathbb{Z}}$ with $q_j = 0$ when $|j| \leq j_0$,
then one has
$$\| q * q \|_{H^s} \leq C(s) j_0^{-s+1} \| q \|_{H^s}^2. \quad (1.8)$$  
According to (1.8), we can introduce a new “norm” (see Definition 2.1 in
the following) which is key to construct barriers around $\pm j_0$. Based on this new
“norm”, it suffices to eliminate monomials of bounded degree. Besides the
above technical improvements, we also develop a more tractable iteration
scheme. This reformulation may have applications in localization problems
for other nonlinear disordered models.
The rest of the paper is organised as follow. Some important facts on Hamiltonian dynamics, such as the tame “norm”, Poisson bracket, symplectic transformation and non-resonant conditions are presented in §2. The Birkhoff normal form type theorem is proved in §3. The estimate on the probability when handling the small divisors can be found in §4. The proof of our main theorem is finished in §5.

2. Structure of the Hamiltonian

We recast (1.6) as a Hamiltonian equation

\[ i \dot{q}_j = 2 \frac{\partial H}{\partial \bar{q}_j}, \]

with

\[ H(q, \bar{q}) = \frac{1}{2} \left( \sum_{j \in \mathbb{Z}} |v_j q_j|^2 + \epsilon \sum_{j \in \mathbb{Z}} (\bar{q}_j q_{j+1} + q_j \bar{q}_{j+1}) + \frac{1}{2} \sum_{j \in \mathbb{Z}} |q_j|^4 \right). \] (2.1)

In order to prove the main result, we need to control the time derivative of truncated sum of higher modes

\[ \frac{d}{dt} \sum_{|j| > j_0} |q_j(t)|^2. \] (2.2)

In what follows, we will deal extensively with monomials in \( q_j \). So we first introduce some notation. Rewrite any monomial in the form \( \prod_{j \in \mathbb{Z}} q_j^{n_j} \bar{q}_j^{n_j'} \). Let \( n = (n_j, n_j')_{j \in \mathbb{Z}} \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z} \). We define

\[
\text{supp } n = \{ j \in \mathbb{Z} : n_j \neq 0 \text{ or } n_j' \neq 0 \},
\]

\[
\Delta(n) = \text{diam(supp } n) = \sup_{j, j' \in \text{supp } n} |j - j'|,
\]

\[
|n| = \sum_{j \in \text{supp } n} (n_j + n_j').
\]

If \( n_j = n_j' \) for all \( j \in \text{supp } n \), then the monomial is called resonant. Otherwise, it is called non-resonant. Note that non-resonant monomials contribute to the truncated sum in (2.2), while resonant ones do not. Moreover, we define the resonant set as

\[
\mathcal{N} = \left\{ n \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z} : n_j = n_j' \text{ for } \forall j \in \mathbb{Z} \right\}. \] (2.3)

Given any large \( j_0 \in \mathbb{N} \) (s.t. \( j_0 \gg 100(\ln j_0)^2 \)) and \( N \in \mathbb{N} \), define

\[
A(j_0, N) = \{ j \in \mathbb{Z} : |j| - j_0 | \leq N \}.
\]

**Definition 2.1.** Given a Hamiltonian

\[ H(q, \bar{q}) = \sum_{n \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z}} H(n) \prod_{j \in \text{supp } n} q_j^{n_j} \bar{q}_j^{n_j'}. \]
then for any $0 < \tau < 1/100$ we define
\[ \|H(n)\|_{\tau,j} = j_0^{2-|n|}\tau|H(n)|, \quad \|H(n)\|_{L,\tau,j}^C = \sup_{j \in \mathbb{Z}} j_0^{(2-|n|)\tau}\partial_{v_j} H(n), \]
where $V = (v_j)_{j \in \mathbb{Z}}$ is the parameter. For convenience, we introduce the notation
\[ |||H(n)|||_{\tau,j} = \|H(n)\|_{\tau,j} + \|H(n)\|_{L,\tau,j}^C, \quad |||H(n)|||_{\tau,j} = \sup_{n \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z}} |||H(n)|||_{\tau,j}. \]

**Remark 2.1.** The basic idea in our paper is that observing the so-called tame property of the nonlinearity permits us to introduce the norm $|||\cdot|||_{\tau,j}$. Then the nonlinear terms will satisfy some kind of “decaying” properties (see (3.2) for the details), which guarantees the validity of normal form iterations. However, note that
\[ \|H(n)\|_{\tau,j} \leq \epsilon \Rightarrow |H(n)| \leq \epsilon j_0^{(|n|-2)}. \]
This means actually that there is some growth of $H(n)$ if $|n| \geq 3$. At this stage, we will use tame property of the nonlinearity again to overcome this growth problem (see (5.5)) when estimating the time derivative of truncated sum of higher modes (2.2).

**Definition 2.2.**

Given $H(q, \bar{q}) = \sum_{n \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z}} H(n) \prod_{j \in \mathbb{Z}} q_j^{n_j} \bar{q}_j^{n'_j}$, $G(q, \bar{q}) = \sum_{m \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z}} G(m) \prod_{j \in \mathbb{Z}} q_j^{m_j} \bar{q}_j^{m'_j}$, the Poisson bracket of $H$ and $G$ is defined by
\[ \{H, G\} = i \sum_{n,m \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z}} H(n)G(m) \sum_{k \in \mathbb{Z}} (n_km_k' - n'_km_k)q_k^{n_k + m_k - 1} \bar{q}_k^{n'_k + m'_k - 1} \times \left( \prod_{j \neq k} q_j^{n_j + m_j} \bar{q}_j^{n'_j + m'_j} \right). \]

We also introduce the non-resonant conditions on the frequency.

**Definition 2.3 (Non-resonant conditions).** Given $\gamma > 0$, we say the frequency $V = (v_j)_{j \in \mathbb{Z}}$ is $\gamma$-non-resonant if for any $0 \neq k \in \mathbb{Z}^\mathbb{Z}$,
\[ \sum_{j \in \mathbb{Z}} k_j v_j \geq \frac{\gamma}{\Delta^2(k)|k|\Delta(k)+2}. \] (2.4)

3. **The Birkhoff Normal Form**

We now construct the symplectic transformation $\Gamma$ (by a finite-step induction) in the spirit of Birkhoff normal form.
3.1. **The First Step Induction.** At the first step, i.e. \( s = 1 \) (see (2.1)), we let \( \delta = \epsilon \) and

\[
H_1 = \frac{1}{2} \left( \sum_{j \in \mathbb{Z}} v_j |q_j|^2 + \epsilon \sum_{j \in \mathbb{Z}} (\bar{q}_j q_{j+1} + q_j \bar{q}_{j+1}) + \frac{\epsilon}{2} \sum_{j \in \mathbb{Z}} |q_j|^4 \right)
\]

(3.1)

which can be rewritten as

\[
H_1 = D_1 + Z_1 + R_1
\]

\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} v_{1j} |q_j|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} Z_1(n) \prod_{\text{supp } n} |q_j^n| \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_1(n) \prod_{\text{supp } n} q_j^n \\bar{q}_j^n,
\]

where

\[ v_{1j} = v_j, \quad Z_1(n) = \frac{\epsilon}{4}, \quad R_1(n) = \frac{\epsilon}{2}. \]

Note that we have \(|n| = 4, \Delta(n) = 0\) in \(Z_1(n)\), and \(|n| = 2, \Delta(n) = 1\) in \(R_1(n)\). Recalling Definition 2.1, then we have

\[
|||Z_1(n)|||_{\tau,j_0} \leq \frac{1}{4} j_0^{-2\tau} \epsilon = \frac{1}{4} j_0^{(2-|n|)\tau} \epsilon \max(\Delta(n), 1),
\]

(3.2)

\[
|||R_1(n)|||_{\tau,j_0} \leq \frac{\epsilon}{2} = \frac{1}{2} j_0^{(2-|n|)\tau} \epsilon \max(\Delta(n), 1),
\]

(3.3)

\[
|||R_1|||_{\tau,j_0} \leq \epsilon.
\]

(3.4)

**Lemma 3.1.** Let \( V_1 = (v_{1j})_{j \in \mathbb{Z}} \) satisfy the \( \gamma \)-non-resonant conditions (2.4) with \( \gamma = \epsilon^{1/100} \). Then there exists a change of variables \( \Gamma_1 = X_{F_1}^1 \) (time-1 flow of \( F_1 \)) such that

\[
H_2 = H_1 \circ X_{F_1}^1
\]

\[
= D_2 + Z_2 + R_2
\]

\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} v_{2j} |q_j|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} Z_2(n) \prod_{\text{supp } n} |q_j^n| \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_2(n) \prod_{\text{supp } n} q_j^n \\bar{q}_j^n.
\]

Moreover, one has

\[
|||F_1|||_{\tau,j_0} \leq \epsilon^{9/10},
\]

(3.5)

\[
|||Z_2(n)|||_{\tau,j_0} \leq \frac{4}{9} (2-|n|)\tau \epsilon^{4} \max(\Delta(n), 1),
\]

(3.6)

\[
|||R_2(n)|||_{\tau,j_0} \leq \frac{4}{9} (2-|n|)\tau \epsilon^{4} \max(\Delta(n), 1),
\]

(3.7)

\[
|||R_2|||_{\tau,j_0} \leq \epsilon_2,
\]

(3.8)
where

$$
\epsilon_2 = \epsilon \left(\epsilon^{1/2} + j_0^{-\tau/2}\right),
$$

$$
\mathcal{R}_2 = \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_2(n) \prod_{\text{supp } n \cap A(j_0, N_2) \neq \emptyset} q_j^n \bar{q}_j^n,
$$

$$
N_2 = \ln j_0 - 20 \frac{\ln \epsilon_2}{\ln \epsilon}.
$$

**Proof.** Applying the Birkhoff normal form theory yields that $F_1$ satisfies the homological equation

$$
L V_1 F_1 = \mathcal{R}_1,
$$

(3.9)

where

$$
L V_1 : H \mapsto L V_1 H = i \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \left(\sum_{j \in \mathbb{Z}} (n_j - n_j') v_{1j}\right) H(n) \prod_{\text{supp } n} q_j^n \bar{q}_j^n
$$

and

$$
\mathcal{R}_1 = \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_1(n) \prod_{\text{supp } n \cap A(j_0, N_1) \neq \emptyset} q_j^n \bar{q}_j^n,
$$

$$
N_1 = \ln j_0 - 20.
$$

Unless $n \in \mathcal{N}$ (see (2.3)), one has $F_1(n) = \frac{R_1(n)}{\sum_{j \in \mathbb{Z}} (n_j - n_j') v_{1j}}$. Note that frequency $V_1$ satisfies $\gamma$-non-resonant conditions (2.4) with $\gamma = \epsilon^{1/100}$. Then we have

$$
|F_1(n)| \leq |R_1(n)| \left(\epsilon^{-1/100} \Delta^2(n) |n| \Delta(n)^{2+2}\right).
$$

(3.10)

By (3.3), we obtain

$$
\|F_1(n)\|_{\tau, j_0} \leq \frac{1}{2} j_0^{\|2-|n|\|} \epsilon^\frac{\alpha}{\alpha} \max\{\Delta(n), 1\},
$$

(3.11)

where we use $\epsilon$ is small enough. On the other hand, for any $\tilde{j} \in \mathbb{Z}$ we have

$$
\partial_{v_{1j}} F_1(n) = \frac{\partial_{v_{1j}} R_1(n)}{i \sum_{j \in \mathbb{Z}} (n_j - n_j') v_{1j}} - \frac{R_1(n)}{i \left(\sum_{j \in \mathbb{Z}} (n_j - n_j') v_{1j}\right)^2} \partial_{v_{1j}} \left(\sum_{j \in \mathbb{Z}} (n_j - n_j') v_{1j}\right).
$$

Then following the proof of (3.11), one has

$$
\|F_1(n)\|_{\tau, j_0} \leq \frac{1}{2} j_0^{\|2-|n|\|} \epsilon^\frac{\alpha}{\alpha} \max\{\Delta(n), 1\}.
$$

(3.12)

Combining (3.11) and (3.12) yields the proof of (3.5).
Next, using Taylor’s formula yields

$$H_2 = H_1 \circ X_{F_1}^1$$

$$= D_1 + \{D_1, F_1\} + \frac{1}{2}\{\{D_1, F_1\}, F_1\} + \cdots$$

$$+ Z_1 + \{Z_1, F_1\} + \frac{1}{2}\{\{Z_1, F_1\}, F_1\} + \cdots$$

$$+ \mathcal{R}_1 + \{\mathcal{R}_1, F_1\} + \frac{1}{2}\{\{\mathcal{R}_1, F_1\}, F_1\} + \cdots$$

$$+ (R_1 - \mathcal{R}_1) + \{R_1 - \mathcal{R}_1, F_1\} + \frac{1}{2}\{\{R_1 - \mathcal{R}_1, F_1\}, F_1\} + \cdots$$

$$= D_2 + Z_2 + R_2$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}} v_{2j} |q_j|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2 \atop n \in \mathcal{X}, |n| \geq 4} Z_2(n) \prod_{\supp n} |q_j^n|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2 \atop n \notin \mathcal{N}^2} R_2(n) \prod_{\supp n} q_j^n q_j^n'.$$

Now we will estimate \(||Z_2(n)||_{r,j_0}, ||R_2(n)||_{r,j_0}|| and \(||\mathcal{R}_2||_{r,j_0}||, respectively. For a Hamiltonian \(G\), let \(G^{(s)} := \{G^{(s-1)}, F_1\}\) for \(s \geq 1\) with \(G^{(0)} = G\).

**Step 1:** Estimate \(||(\mathcal{R}_1 \circ X_{F_1}^1 - \mathcal{R}_1) (n)||_{r,j_0}||. First, we have

$$\mathcal{R}_1 \circ X_{F_1}^1 = \mathcal{R}_1 + \sum_{s \geq 1} \frac{\mathcal{R}_1^{(s)}}{s!}.$$  

**Step 1.1:** Estimate \(||\mathcal{R}_1^{(1)}(n)||_{r,j_0}||. Note that \(\mathcal{R}_1^{(1)} = \{\mathcal{R}_1, F_1\}\) and the coefficients of \(\mathcal{R}_1^{(1)}\) are given by

$$\mathcal{R}_1^{(1)}(n) = i \sum_{k \in \mathbb{Z}} \left( \sum_{l,m \in \mathbb{N}^2 \times \mathbb{N}^2} R_1(l) F_1(m) (l_k m_k' - l'_{k} m_k) \right)$$  

(3.13)

with the sum \(\sum_{l,m \in \mathbb{N}^2 \times \mathbb{N}^2}\) being taken over

\(n_j = l_j + m_j - 1, \quad n_j' = l_j' + m_j' - 1\) if \(j = k,\)

\(n_j = l_j + m_j, \quad n_j' = l_j' + m_j'\) if \(j \neq k.\)

Fixing \(l, m \in \mathbb{N}^2 \times \mathbb{N}^2,\) one obtains

$$\sum_{k \in \mathbb{Z}} |R_1(l) F_1(m) (l_k m_k' - l'_{k} m_k)| \leq |l| \cdot |m| \cdot |R_1(l)| \cdot |F_1(m)|$$

$$\leq 2^{|l|+|m|} |R_1(l)| \cdot |F_1(m)|.$$  

(3.14)
Obviously, the number of realizations of a fixed monomial \( \prod_{n} q_{j_{1}}^{n_{1}} q_{j_{2}}^{n_{2}} \) in \( \{R, F\} \) is bounded by
\[
2^{\tau_{l}} (\Delta(l) \wedge \Delta(m)) .
\]
In view of (3.14), one gets
\[
|\{R, F\}(n)| \leq 4 \cdot 4^{\tau_{l}} (\Delta(l) \wedge \Delta(m)) \cdot |R_{1}(l)| \cdot |F_{1}(m)|
\]
by using \( |n| = |l| + |m| - 2 \). Hence,
\[
j_{0}^{(2-|n|)\tau} |\{R, F\}(n)| \leq 4 \cdot 4^{\tau_{l}} (\Delta(l) \wedge \Delta(m)) \left( j_{0}^{(2-|l|)\tau} |R_{1}(l)| \right) \left( j_{0}^{(2-|m|)\tau} |F_{1}(m)| \right)
\]
by using \( |l| = |n| + |m| - 2 \) again. Therefore using (3.3) and (3.11), one has
\[
\|\{R, F\}(n)\|_{\tau_{j_{0}}} \leq 4 \cdot 4^{\tau_{l}} (\Delta(l) \wedge \Delta(m)) \cdot \left( \frac{1}{2} j_{0}^{(2-|l|)\tau} \epsilon_{\max(\Delta(l), 1)} \right)
\]
\[
\times \left( \frac{1}{2} j_{0}^{(2-|m|)\tau} \epsilon_{\max(\Delta(m), 1)} \right)
\]
\[
\leq \epsilon_{1/100} \cdot j_{0}^{(2-|n|)\tau} \epsilon_{\max(\Delta(n), 2)}^{\frac{1}{4}}
\]
(3.15)
where for \( \epsilon \ll 1 \) and \( \Delta(n) \leq \Delta(l) + \Delta(m) \). Similarly, we can obtain
\[
\|\{R, F\}(n)\|_{\tau_{j_{0}}} \leq \epsilon_{1/100} \cdot j_{0}^{(2-|n|)\tau} \epsilon_{\max(\Delta(n), 2)}^{\frac{1}{4}}.
\]
(3.16)

**Step 1.2:** Estimate \( \|\{R_{1}^{(2)}(n)\|_{\tau_{j_{0}}} \). Following the proof of (3.15), we obtain
\[
\|\{R_{1}^{(2)}(n)\|_{\tau_{j_{0}}} = \|\{R_{1}^{(1)}, F_{1}\}(n)\|_{\tau_{j_{0}}} \leq 4 \cdot 4^{\tau_{l}} (\Delta(l) \wedge \Delta(m)) \cdot \|\{R_{1}^{(1)}(l)\|_{\tau_{j_{0}}} \cdot \|F_{1}(m)\|_{\tau_{j_{0}}}
\]
\[
\leq \epsilon_{1/100} \cdot j_{0}^{(2-|n|)\tau} \epsilon_{\max(\Delta(n), 3)}^{\frac{1}{4}}
\]
(3.17)
where we use (3.15) and \( \epsilon \ll 1 \). Similarly, we obtain
\[
\|\{R_{1}^{(2)}(n)\|_{\tau_{j_{0}}} \leq \epsilon_{1/100} \cdot j_{0}^{(2-|n|)\tau} \epsilon_{\max(\Delta(n), 3)}^{\frac{1}{4}}.
\]
(3.17)

**Step 1.3:** Estimate \( \|\{R_{1}^{(s)}(n)\|_{\tau_{j_{0}}} \) for \( s \geq 3 \). By induction and following the proof of (3.17), one has
\[
\|\{R_{1}^{(s)}(n)\|_{\tau_{j_{0}}} \leq \epsilon_{1/100} \cdot j_{0}^{(2-|n|)\tau} \epsilon_{\max(\Delta(n), s)}^{\frac{1}{4}}.
\]
(3.18)
In view of (3.16), (3.17) and (3.18), one has
\[
\|\{R_{1} \circ X_{F_{1}}, \mathcal{R}_{1}\}(n)\|_{\tau_{j_{0}}} \leq j_{0}^{(2-|n|)\tau} \epsilon_{\max(\Delta(n), 2)}^{\frac{1}{4}}.
\]
(3.19)
Step 2: Estimate $\|(Z_1 \circ X_{F_1}^1 - Z_1)(n)\|_{\nu,j_0}$. Following the proof of (3.19) and using (3.2), one has

$$\|(Z_1 \circ X_{F_1}^1 - Z_1)(n)\|_{\nu,j_0} \leq j_0^{\frac{1}{2}(2-|n|)^\tau} \epsilon_{s-1}^{\frac{1}{2}} \epsilon_{s-1} \max\{\Delta(n),2\}. \quad (3.20)$$

Step 3: Estimate $\|(R_1 - \mathcal{R}_1) \circ X_{F_1}^1 - (R_1 - \mathcal{R}_1)(n)\|_{\nu,j_0}$. Following the proof of (3.19) and using (3.3), one has

$$\|(R_1 - \mathcal{R}_1) \circ X_{F_1}^1 - (R_1 - \mathcal{R}_1)(n)\|_{\nu,j_0} \leq j_0^{\frac{1}{2}(2-|n|)^\tau} \epsilon_{s-1}^{\frac{1}{2}} \epsilon_{s-1} \max\{\Delta(n),2\}. \quad (3.21)$$

In view of (3.19), (3.20) and (3.21), we obtain (3.6) and (3.7) respectively. Finally, (3.8) also follows from (3.19), (3.20) and (3.21). □

3.2. The General Step Induction. Next, we perform the general step of induction. For $s \geq 2$, let

$$\epsilon_s = \frac{3}{2} \epsilon_{s-1} + j_0^{-\tau} \epsilon_{s-1}(s \geq 2), \quad \epsilon_1 = \epsilon, \quad (3.22)$$

$$\delta_s = \prod_{j=1}^{s-1} \left(1 - \frac{1}{5} \cdot \frac{1}{j^2}\right)(s \geq 2), \quad \delta_1 = 1, \quad (3.23)$$

$$N_s = \ln j_0 - 20 \ln \frac{\epsilon_s}{\ln \epsilon}. \quad (3.24)$$

Lemma 3.2 (Iterative Lemma). Consider the Hamiltonian

$$H_s = D_s + Z_s + R_s$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}} v_j |q_j|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} Z_s(n) \prod_{n \in \mathbb{N}^2 \times \mathbb{N}^2} |q_j^n|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_s(n) \prod_{n \in \mathbb{N}^2 \times \mathbb{N}^2} q_j^n q_j^n. \quad (3.25)$$

Let $V_s = (v_j)_{j \in \mathbb{Z}}$ satisfy the $\gamma$-non-resonant conditions (2.4) with $\gamma = \epsilon_s^{1/100}$ and assume that

$$\|Z_s(n)\|_{\nu,j_0} \leq j_0^{\delta_s-1(2-|n|)^\tau} \epsilon_{s-1}^{\frac{1}{2}} \epsilon_{s-1} \max\{\Delta(n),1\}, \quad (3.24)$$

$$\|R_s(n)\|_{\nu,j_0} \leq j_0^{\delta_s-1(2-|n|)^\tau} \epsilon_{s-1}^{\frac{1}{2}} \epsilon_{s-1} \max\{\Delta(n),1\}, \quad (3.25)$$

$$\|R_s\|_{\nu,j_0} \leq \epsilon_s, \quad (3.26)$$

where

$$R_s = \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_s(n) \prod_{n \in \mathbb{N}^2 \times \mathbb{N}^2} q_j^n q_j^n. \quad (3.27)$$
Then there exists a change of variables $\Gamma_s = X_{F_s}^1$ such that

$$H_{s+1} = H_s \circ X_{F_s}^1$$

$$= D_{s+1} + Z_{s+1} + R_{s+1}$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}} v_{(s+1)j} |q_j|^2 + \sum_{n \in \mathbb{N}^2 \times \mathbb{Z}} Z_{s+1}(n) \prod_{\text{supp } n} |q_{nj}|^2$$

$$+ \sum_{n \in \mathbb{N}^2 \times \mathbb{Z}} R_{s+1}(n) \prod_{\text{supp } n} q_{nj} q_{nj}'.$$

Moreover, one has

$$|||F_s|||_{\tau,j_0} \leq \epsilon_s^{9/10},$$

(3.28)

$$|||Z_{s+1}(n)|||_{\tau,j_0} \leq J_0^\delta_s(2-|n|)\tau \delta_s \max\{\Delta(n),1\},$$

(3.29)

$$|||R_{s+1}(n)|||_{\tau,j_0} \leq J_0^\delta_s(2-|n|)\tau \delta_s \max\{\Delta(n),1\},$$

(3.30)

$$|||R_{s+1}|||_{\tau,j_0} \leq \epsilon_{s+1},$$

(3.31)

where

$$R_{s+1} = \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_{s+1}(n) \prod_{\text{supp } n \cap A(j_0,N_{s+1}) \neq \emptyset} q_{nj} q_{nj}'.$$

(3.32)

Proof. As done before, we know that $F_s$ will satisfy the homological equation

$$L_{V_s} F_s = \tilde{R}_s; \tilde{R}_s = \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} R_s(n) \prod_{\Delta(n) \leq \frac{10 \ln \epsilon_s + 4}{\epsilon_s |n| \in \mathbb{N}^2 \times \mathbb{N}^2}} q_{nj} q_{nj}'.$$

(3.33)

By the direct computations, one has

$$F_s(n) = \sum_{j \in \mathbb{Z}} \frac{R_s(n)}{\sum_{j \in \mathbb{Z}} (n_j-n'_{j})w_{nj}}, \text{ unless } n \in \mathbb{N}^2.$$

Since the frequency $V_s$ satisfies the $\gamma$-non-resonant conditions (2.4) with $\gamma = \epsilon_s^{1/100}$, we get $|F_s(n)| \leq |R_s(n)| \epsilon_s^{-1/100} \Delta^2(n)|n| \Delta(n)+2$. Hence we have

$$||F_s(n)||_{\tau,j_0} \leq \epsilon_s^{-1/100} \Delta^2(n)|n| \Delta(n)+2 \epsilon_s \leq \frac{1}{2} \epsilon_s^{9/10},$$

(3.34)

$$||F_s(n)||_{\tau,j_0} \leq \left( \epsilon_s^{-1/100} \Delta^2(n)|n| \Delta(n)+2 \right)^{\delta_s-1(2-|n|)\tau} \epsilon_s \delta_s \max\{\Delta,1\}$$

$$\leq J_0^{\delta_s-1}\epsilon_s^{9/10},$$

$$||F_s(n)||_{\tau,j_0} \leq \epsilon_s^{-1/100} \Delta^2(n)|n| \Delta(n)+2 \epsilon_s \leq \frac{1}{2} \epsilon_s^{9/10},$$

(3.35)

Finally, combining (3.34) and (3.35) implies (3.28).
Next, using Taylor’s formula yields
\[ H_{s+1} = H_s \circ X_{F_s}^1 \]
\[ = D_s + \{D_s, F_s\} + \frac{1}{2}\{\{D_s, F_s\}, F_s\} + \cdots \]
\[ + Z_s + \{Z_s, F_s\} + \frac{1}{2}\{\{Z_s, F_s\}, F_s\} + \cdots \]
\[ + \tilde{R}_s + \{\tilde{R}_s, F_s\} + \frac{1}{2}\{\{\tilde{R}_s, F_s\}, F_s\} + \cdots \]
\[ + (R_s - \tilde{R}_s) + \{R_s - \tilde{R}_s, F_s\} + \frac{1}{2}\{\{R_s - \tilde{R}_s, F_s\}, F_s\} + \cdots \]
\[ + (R_s - R_s) + \{R_s - R_s, F_s\} + \frac{1}{2}\{\{R_s - R_s, F_s\}, F_s\} + \cdots \]
\[ = D_{s+1} + Z_{s+1} + R_{s+1} \]
\[ = \frac{1}{2} \sum_{j \in Z} v_{(s+1),j}|q_j|^2 + \sum_{n \in \mathbb{N} \times \mathbb{N}, n \notin N, |n| \geq 4} Z_{s+1}(n) \prod_{\supp n} |q^{n_j}_{j}|^2 \]
\[ + \sum_{n \in \mathbb{N} \times \mathbb{N}, n \notin N} R_{s+1}(n) \prod_{\supp n} q^{n_j}_{j} q^{n_j'}_{j}. \]

Now we will estimate \( |||Z_{s+1}(n)|||_{\tau,j_0}, \)
\[ |||R_{s+1}(n)|||_{\tau,j_0} \]
and \( |||R_{s+1}|||_{\tau,j_0}, \) respectively. We have the following cases:

**Step 1:** Estimate \( ||(Z_s \circ X_{F_s}^1 - Z_s)(n)|||_{\tau,j_0}. \) First, we have
\[ Z_s \circ X_{F_s}^1 - Z_s = \sum_{j \geq 1}^{1\over 2} Z_{(j)}^2. \]

**Step 1.1:** Estimate \( ||Z_{s}^{(1)}(n)|||_{\tau,j_0}. \) Following the proof of (3.15) and using (3.28), (3.34) and (3.35), one has
\[ ||\{Z_s, F_s\}(n)|||_{\tau,j_0} \leq 4 \cdot 4^{|n|} (\Delta(l) \wedge \Delta(m)) \left(\frac{\delta_{s-1}(2-|l|)}{j_0^0} \epsilon^{\delta_{s-1} \max\{\Delta(l),1\}}\right) \]
\[ \times \left(\frac{\delta_{s-1}(2-|m|)}{j_0^0} \epsilon^{\sqrt{\delta_{s-1} \max\{\Delta(m),1\}}}\right) \]
\[ \leq \epsilon^{1/100} \frac{\delta_{s}(2-|n|)}{j_0^0} \epsilon^{\delta_{s} \max\{\Delta(n),2\}}. \]

Similarly, we obtain
\[ ||\{Z_s, F_s\}(n)|||_{\tau,j_0} \leq \epsilon^{1/100} \frac{\delta_{s}(2-|n|)}{j_0^0} \epsilon^{\delta_{s} \max\{\Delta(n),2\}}. \]

**Step 1.2:** Estimate \( ||Z_{s}^{(2)}(n)|||_{\tau,j_0}. \) Using (3.37) shows
\[ ||Z_{s}^{(2)}(n)|||_{\tau,j_0} \leq \epsilon^{1/100} \frac{\delta_{s}(2-|n|)}{j_0^0} \epsilon^{\delta_{s} \max\{\Delta(n),3\}}. \]
Step 1.3: Estimate $\left|\left|Z_s^{(j)}(n)\right|\right|_{\tau,j_0}$, $j \geq 3$. By induction and following the proof of (3.38), one has
\[
\left|\left|Z_s^{(j)}(n)\right|\right|_{\tau,j_0} \leq j_0^{1/100} \varepsilon_0^{2-\left|n\right|}\epsilon^\delta \max\{\Delta(n),1\}.
\] (3.39)

In view of (3.37), (3.38) and (3.39), one has
\[
\left|\left|(Z_s \circ X_{F_s}^{-1} - Z_s)(n)\right|\right|_{\tau,j_0} \leq j_0^{\delta(2-\left|n\right|)}\epsilon^\delta \max\{\Delta(n),2\}.
\] (3.40)

Step 2: Estimate $\left|\left|(R_s \circ X_{F_s}^{-1} - R_s)(n)\right|\right|_{\tau,j_0}$. Following the proof of (3.40), one has
\[
\left|\left|(R_s \circ X_{F_s}^{-1} - R_s)(n)\right|\right|_{\tau,j_0} \leq j_0^{\delta(2-\left|n\right|)}\epsilon^\delta \max\{\Delta(n),2\}.
\] (3.41)

Therefore, in view of (3.40) and (3.41), we finish the proof of (3.28) and (3.29).

To show (3.31), it suffices to prove
\[
\left|\left|R_s - \bar{R}_s\right|\right|_{\tau,j_0} \leq \epsilon_{s+1},
\]
which follows from (3.29) and the conditions that $\Delta(n) > 10 \ln e / \ln \epsilon$ or $|n| > 10 \kappa$.

Finally, we present the main theorem of this section.

3.3. Birkhoff Normal Form Theorem.

Theorem 3.3 (Birkhoff Normal Form). Consider the Hamiltonian (3.1) and assume that the frequency $V = (v_j)_{j \in \mathbb{Z}}$ satisfies the $\gamma$-non-resonant conditions (2.4) with $\gamma = \epsilon^{1/100}$. Then there exists $\varepsilon_0 = \varepsilon_0(\kappa) > 0$ such that, for any $0 < \epsilon < \varepsilon_0$ and any large enough $j_0 \in \mathbb{N}$ there exists a symplectic transformation $\Gamma = \Gamma_1 \circ \cdots \circ \Gamma_M$ with $M \sim \ln \ln j_0$ such that
\[
\tilde{H} = H_1 \circ \Gamma = \tilde{D} + \tilde{Z} + \tilde{R}
\]
\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} v_j |q_j|^2 + \sum_{n \in \mathbb{Z}^2} \tilde{Z}(n) \prod_{n \in \mathbb{N}, |n| \geq 4} \sum_{n \in \mathbb{N}} \|q_j^n\|^2 + \sum_{n \in \mathbb{N}^2, n \notin \mathbb{N}, |n| \geq 4} \tilde{R}(n) \prod_{n \in \mathbb{N}} q_j^n q_j^n,
\]
with
\[
\left|\left|\tilde{Z}(n)\right|\right|_{\tau,j_0} \leq j_0^{\delta(2-\left|n\right|)}\epsilon^\delta \max\{\Delta(n),1\},
\] (3.42)
\[
\left|\left|\tilde{R}(n)\right|\right|_{\tau,j_0} \leq j_0^{\delta(2-\left|n\right|)}\epsilon^\delta \max\{\Delta(n),1\},
\] (3.43)
\[
\left|\left|\tilde{R}\right|\right|_{\tau,j_0} \leq j_0^{-3/\kappa},
\] (3.44)
where
\[ \tilde{R} = \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \tilde{R}(n) \prod_{\text{supp } n \cap A(j_0, \frac{1}{2} \ln j_0) \neq \emptyset} q_j^n q_j'^n. \tag{3.45} \]

Proof. First, note that the Hamiltonian (3.1) satisfies all assumptions (3.24)–(3.26) with \( s = 1 \), which follows from (3.2)–(3.4), respectively.

Then using Iterative Lemma (i.e. Lemma 3.2), one can find a symplectic transformation \( \Gamma = \Gamma_1 \circ \cdots \circ \Gamma_M \) such that \( \tilde{H} := H_{M+1} = H_1 \circ \Gamma \), which satisfies (3.28)–(3.30) with \( s = M + 1 \). In view of (3.23), one has
\[ \delta_M \geq \frac{1}{2}. \tag{3.46} \]
By (3.22) and \( M \sim \ln \ln j_0 \), we obtain
\[ \epsilon_{M+1} \leq j_0^{-3/\kappa}. \tag{3.47} \]
and
\[ N_M = \ln j_0 - 20 \frac{\ln \epsilon_M}{\ln \epsilon} \geq \ln j_0 - C \frac{\ln j_0}{|\ln \epsilon|} \geq \frac{1}{2} \ln j_0. \tag{3.48} \]
In view of (3.46)–(3.48), we finish the proof of (3.42)–(3.45). \( \square \)

4. Estimate on the measure

Given \( n \in \mathbb{N}^2 \times \mathbb{N}^2 \), let
\[ j_+(n) = \max \{ j \in \mathbb{Z} : n_j - n_{j'} \neq 0 \}. \]
The non-resonant set contains
\[ \mathcal{S} = \bigcap_{|k| - j_0| < \ln j_0} \bigcap_{s=1, \ldots, M} \bigcap_{j_+(n) = k} \mathcal{R}_s(n) \tag{4.1} \]
where the resonant set \( \mathcal{R}_s(n) \) is given by
\[ \mathcal{R}_s(n) = \left\{ V = (v_j)_{j \in \mathbb{Z}} : \left| \sum_{j \in \mathbb{Z}} (n_j - n_{j'}) v_{sj} \right| < \frac{1/100}{\Delta^2(n)|n|^{\Delta(n)+2}} \right\} \]
with
\[ \Delta(n) \leq 10 \frac{\ln \epsilon_s}{\ln \epsilon}, \ |n| \leq \frac{10}{\kappa}. \tag{4.2} \]
When \( s = 1 \), it is easy to see that
\[ \text{mes}(\mathcal{R}_1(n)) \leq \frac{\epsilon^{1/100}}{\Delta^2(n)|n|^{\Delta(n)+2}}. \]
Then we have for $\epsilon \leq \epsilon(\kappa) \ll 1$,

$$\text{mes}\left(\bigcup_{j_+(n)=k \atop n \text{ satisfies (4.2)}} \mathcal{R}_1(n)\right) \leq \epsilon^{1/100} \sum_{j_+(n)=k \atop n \text{ satisfies (4.2)}} \frac{1}{\Delta^2(n)|n|\Delta(n)+2} \leq \epsilon^{1/200}. \quad (4.3)$$

When $2 \leq s \leq M$, if we consider $V_s = (v_{sj})_{j \in \mathbb{Z}}$ as parameters, then following the proof of (4.3) we can show that

$$\text{mes}\left(\bigcup_{j_+(n)=k \atop n \text{ satisfies (4.2)}} \mathcal{R}_s(n)\right) \leq \epsilon_s^{1/125}. \quad (4.4)$$

In view of (4.3) and (4.4), we have

$$\text{mes}\left(\bigcap_{s=1,\ldots,M} \bigcap_{j_+(n)=k \atop n \text{ satisfies (4.2)}} \mathcal{R}_s^c(n)\right) \geq 1 - \left(\epsilon^{1/200} + \sum_{s=2}^{M} \epsilon_s^{1/125}\right) \geq 1 - \epsilon^{1/1000}. \quad (4.5)$$

Combining (4.5) and (4.1) yields

$$\text{mes}(\mathcal{S}) \geq \left(1 - \epsilon^{1/1000}\right)^{5\ln j_0} \geq j_0^{-5\epsilon^{1/1000}}. \quad (4.6)$$

Now we will convert the estimate (4.6) to estimate in $V = (v_j)_{j \in \mathbb{Z}}$. Let $W^{(s)} = \left(W_j^{(s)}\right)_{j \in \mathbb{Z}}$ with $W_j^{(s)} = v_{sj} - v_{1j}$. On one hand, we have

$$v_{sj} - v_{1j} = 0 \quad (4.7)$$

unless

$$||j| - j_0| \leq \ln j_0. \quad (4.8)$$

On the other hand,

$$\sup_l \left|\frac{\partial W_j^{(s)}}{\partial v_l}\right| \leq \sum_{j=1}^{s} \epsilon_j \leq \sqrt{\epsilon}. \quad (4.9)$$

Using (4.7), (4.8) and (4.9) implies that

$$j_0^{-\sqrt{\epsilon}} < (1 - \sqrt{\epsilon})^{\ln j_0} < \left|\text{det} \frac{\partial V_s}{\partial V}\right| < (1 + \sqrt{\epsilon})^{\ln j_0} < j_0^{\sqrt{\epsilon}}. \quad (4.10)$$

In view of (4.6) and (4.10), one has

$$\text{mes}(\mathcal{S}) > j_0^{-\sqrt{\epsilon}} j_0^{-5\epsilon^{1/1000}} > j_0^{-6\epsilon^{1/1000}}. \quad (4.11)$$

Note that $\mathcal{S}$ defined in (4.1) corresponds to a rare event for a fixed $j_0$. To circumvent that, we use ideas of Bourgain-Wang [BW07], i.e., we allow $j_0$...
to vary in a dyadic interval \([\bar{\jmath}_0, 2\bar{\jmath}_0]\) \((\bar{\jmath}_0 \gg 1)\) taking into account that the restriction in \((4.11)\) only relates to \(v_{\jmath} \big|_{|\jmath| - \bar{\jmath}_0| \leq 2\ln \bar{\jmath}_0}\) in view of \((4.8)\). Using independence, we then obtain that with probability at least
\[
1 - \left(1 - j_0^{-6e^{1/1000}}\right) e^{\ln j_0} > 1 - e^{-\sqrt{j_0}},
\]
the condition in \((4.1)\) holds for some \(\jmath_0 \in [\bar{\jmath}_0, 2\bar{\jmath}_0]\). We finish the measure estimate of frequencies.

5. The proof of main theorem

Now we are in a position to prove Theorem 1.2.

Proof. In view of Theorem 3.3, \(H\) is symplectically transformed into \(\tilde{H}\):
\[
\tilde{H} = \frac{1}{2} \sum_{j \in \mathbb{Z}} \tilde{v}_j |q_j|^2 + \sum_{\substack{n \in \mathbb{N}^2 \times \mathbb{N}^2 \ni n \| n \| \geq 4}} \tilde{Z}(n) \prod_{\supp n} \tilde{q}_j^n \tilde{q}_j^{n_j}
\]
\[
+ \sum_{\substack{n \in \mathbb{N}^2 \times \mathbb{N}^2 \ni n \| n \| \geq 4}} \tilde{R}(n) \prod_{\supp n} \tilde{q}_j^n \tilde{q}_j^{n_j}.
\]

In view of \((3.45)\), we can write \(\tilde{R} = \tilde{R} + \left(\tilde{R} - \tilde{R}\right)\), where \(\tilde{R}\) satisfies \((3.44)\).

Assume
\[
\sum_{\jmath \in \mathbb{Z}} j^2 |q_j(0)|^2 < \infty.
\]

We want to bound the diffusion norm
\[
\sum_{\jmath \in \mathbb{Z}} j^2 |q_j(t)|^2 \tag{5.1}
\]
in terms of \(t\) as \(t \to \infty\). The coordinates \(q(t) = \{q_j(t)\}_{j \in \mathbb{Z}}\) satisfy
\[
i \tilde{q} = 2 \frac{\partial \tilde{H}}{\partial \tilde{q}}.
\]

Denote the new coordinates in \(\tilde{H}\) by \(\tilde{q}\) to avoid confusion. We estimate \((5.1)\) via the truncated sum \(\sum_{|k| > \jmath_0} |\tilde{q}_k(t)|^2\). Then one has
\[
\frac{d}{dt} \left[ \sum_{|k| > \jmath_0} |\tilde{q}_k(t)|^2 \right] = 4\mathfrak{H} \sum_{|k| > \jmath_0} \tilde{q}_k(t) \cdot \frac{\partial \tilde{H}}{\partial \tilde{q}_k}
\]
\[
\sim \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \tilde{R}(n) \left( \sum_{|k| > \jmath_0} (n_k - n'_k) \prod_{\supp n} \tilde{q}_j^n \tilde{q}_j^{n_j} \right). \tag{5.2}
\]
Now we will analyze \((5.2)\) further.
In view of (3.43), we may assume that the terms in $\tilde{R} - \tilde{R}$ satisfy $\Delta(n) \leq \frac{10 \ln j_0}{\kappa \ln \epsilon}$. For otherwise, one has $\left\| \tilde{R}(n) \right\|_{r, j_0} \leq j_0^{-3/\kappa}$. Next, if $\supp n \cap (-\infty, -j_0] \cup [j_0, \infty) \neq \emptyset$, then

$$\supp n \subset \left( -\infty, -j_0 - \frac{10 \ln j_0}{\kappa \ln \epsilon} \right] \cup \left[ j_0 + \frac{10 \ln j_0}{\kappa \ln \epsilon}, \infty \right).$$

Further, if $\left\| \tilde{R}(n) \right\|_{r, j_0} \geq j_0^{-3/\kappa}$, then (3.44) implies that

$$\supp n \subset \left( -\infty, -j_0 - \frac{1}{2} \ln j_0 \right] \cup \left[ j_0 + \frac{1}{2} \ln j_0, \infty \right) \subset (-\infty, -j_0) \cup (j_0, \infty).$$

The last set in (5.3) is precisely the set that is summed over in (5.2). We have $\sum_{|k| > j_0} (n_k - n_k') = 0$. In conclusion, only terms where $\left\| \tilde{R}(n) \right\|_{r, j_0} < j_0^{-3/\kappa}$ contribute to (5.2). Hence one has

$$\left| \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \tilde{R}(n) \left( \sum_{|k| > j_0} (n_k - n_k') \prod_{\supp n} \tilde{q}_j^{n_j'} q_j^{n_j} \right) \right|$$

$$\leq j_0^{-3/\kappa} \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \left| j_0^{(|n|-2)\tau} \left( \sum_{|k| > j_0} (n_k - n_k') \prod_{\supp n} \tilde{q}_j^{n_j'} q_j^{n_j} \right) \right|. \quad (5.4)$$

Note that the sum in (5.4) satisfies

$$\supp n \subset (-\infty, -j_0 - \ln j_0] \cup [j_0 + \ln j_0, \infty) = B(j_0).$$

Then one has

$$\sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \left| j_0^{(|n|-2)\tau} \left( \sum_{|k| > j_0} (n_k - n_k') \prod_{\supp n} \tilde{q}_j^{n_j'} q_j^{n_j} \right) \right|$$

$$\leq \frac{1}{2} j_0^{-2\tau} \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \left| j_0^{2|n|\tau} \prod_{\supp n} \tilde{q}_j^{n_j'} q_j^{n_j} \right|$$

$$\leq \frac{1}{2} j_0^{-2\tau} \sum_{t \geq 2} \sum_{n \in \mathbb{N}^2 \times \mathbb{N}^2} \left| \prod_{\supp n} \left( j^{2\tau} \tilde{q}_j \right)^{n_j} \left( j^{2\tau} q_j \right)^{n_j'} \right|$$

$$\leq \frac{1}{2} j_0^{-2\tau} \sum_{t \geq 2} \left( \sum_{j \in B(j_0)} j^{2\tau} \tilde{q}_j \right)^l$$

$$\leq \frac{1}{2} j_0^{-2\tau} \sum_{t \geq 2} \left( \frac{C(\tau) \sum_{j \in \mathbb{N}^2} (1 + |j|)^{-1 - 2\tau}}{j_0^{1/2 - \tau}} \right)^l \left( \sum_{j \in \mathbb{Z}} |j|^2 \tilde{q}_j^2 \right)^{l/2}. \quad (5.5)$$
In view of (5.2), (5.4) and (5.5), one has
\[
\frac{d}{dt} \left[ \sum_{|k| > j_0} |\tilde{q}_k(t)|^2 \right] \leq j_0^{-3/\kappa}.
\]
Finally, following the proof of (2.5.36) in [BW07], we have
\[
\sum_{j \in \mathbb{Z}} j^2 |q_j(t)|^2 < \kappa
\]
for a.e. $V$. 

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