It is well known that superselection rules restrict the allowed operations in quantum mechanics. Therefore, it is natural to suggest that the models used by Mayers to prove the impossibility of bit commitment and by Kitaev for the impossibility of coin flipping are inadequate. Here, we show that to the contrary the models originally used by Kitaev and Mayers for their respective impossibility proofs are valid even in the context of superselection rules.

Superselection rules are associated with conserved quantities such as the number of fermions. Let $H_F$ be the span of $|0\rangle^F$ and $|1\rangle^F$, a 0 fermion state and a 1 fermion state, respectively. Let $H_B$ be the polarisation state space of a photon. It may appear impossible to execute the swap gate $S$ on $H_F \otimes H_B$ which executes $|x\rangle^F|y\rangle^B \leftrightarrow |y\rangle^F|x\rangle^B$ because the fermion number is not preserved when $x \neq y$. The basic idea to unvalidate this argument can be understood with an analogy with a common phenomena which also appears to violate a conservation law. Consider a photon that is reflected by a mirror. The dynamic of the photon, if considered separately, is essentially a unitary transformation which violates the preservation of momentum. Of course, momentum is preserved because the mirror absorbs the difference in momentum. Yet the mirror does not get entangled with the photon. The mirror can be considered as a catalyst for a unitary transformation that would otherwise be impossible because it violates a conservation law. In a similar way, we will show that by using extra degrees of freedom as a catalyst, we can execute the swap operation on $H_F \otimes H_B$, an operation that would otherwise be impossible. This swap operation share a similarity with the reflection of a photon on a mirror. However, this swap operation is more powerful because, as we will see, it implies that all operations on $H_F \otimes H_E$, where $H_E$ is any extra system, are allowed despite superselection rules.

There is a subtle point here. The state of these extra degrees of freedom must itself be a superposition of states with different fermion numbers. The existence of such initial states in nature would not violate any conservation law by itself. For example, even tough momentum is a conserved quantity, a superposition of states with different momentum is a common phenomena. Nevertheless, one might ask if such states really exist in the case of all conservation laws. It turns out that, if these initial states are not allowed, the original proof that bit commitment is impossible can easily be generalized. Therefore, in both cases, the conclusion is essentially the same. In this paper, we focus on the case where initial states of the form $(1/\sqrt{2})(|0\rangle^F + |1\rangle^F)$ are allowed, but the easier case where such a superposition is not allowed will be considered also.

There is a similarity between the constraint imposed by the fermion number conservation superselection rule and the one imposed by the use of classical information in quantum protocols. In both cases, it was believed for some times that the constraint could perhaps be used to build a bit commitment protocol. In both cases, it turned out to be useless. Before the general impossibility result, we knew that bit commitment was impossible if we accept the model where classical information can be manipulated as if it was quantum information. Of course, researchers seriously looked for a bit commitment protocol that makes use of the additional constraint associated with classical information. The proof that this additional constraint cannot help is not so complicated but yet it took six months and many unsuccessful attempts before the author realised that classical information was actually useless. In the case of the fermion number conservation superselection rule, a few researchers also attempted to make use of the corresponding additional constraint, but it took only a few weeks in this case to realize that the additional constraint is useless, and this is the subject of the current paper.

To our knowledge, this is the first analysis of superselection rules in quantum cryptography. It is interesting to see that our analysis is inspired by a very simple phenomena, the reflexion of a photon in a mirror. We will not refer to this simple phenomena anymore, but anyone who knows why the mirror is not entangled with the photon should appreciate some connection. In the first section, we describe the model which includes superselection rules. In the second section, we show that, if arbitrarily small errors are ignored, this model implies no restriction on the possible operations on a given qbit.

The Model

We assume that the Hilbert space $H^{(all)}$ for the protocol is the span of a set of orthogonal states

$$|(q_x)_{x \in \mathcal{M}}^{(all)}\rangle = \otimes_{x \in \mathcal{M}} |q_x\rangle(x),$$
where $\mathcal{M}$ is the set of possible modes and, for every $x \in \mathcal{M}$, $q_x \in \{0, 1, 2, \ldots, q_x^{\text{max}}\}$ where $q_x^{\text{max}} = 1$ for fermion modes and $q_x^{\text{max}}$ depends on the protocol for boson modes. The state space $H^{(\text{all})}$ is simply the ordinary tensor product of the state spaces $H_x$ associated with the modes $x \in \mathcal{M}$. A superselection rule does not change the state space $H^{(\text{all})}$, but restricts the possible operations on this space.

Superselection rules. A superselection rule $R$ requires that, for a subset $\mathcal{M}_R \subseteq \mathcal{M}$, the quantity $Q_R = \sum_{x \in \mathcal{M}_R} q_x$ is preserved by every allowed transformation. More precisely, the allowed unitary transformations are block diagonal and each block is associated with a given value for $Q_R$. We assume that every participant has access to a countably infinite number of extra modes $x \in \mathcal{M}_R$ which are normally not used in the protocol (i.e., normally, $q_x = 0$ for these modes). For example, these extra modes can lie in some unused locations on the participant’s side.

Local transformations. We assume that $\mathcal{M}$ is partitioned between Alice and Bob in two subsets of modes $\mathcal{M}^A$ and $\mathcal{M}^B$. We denote $H^A$ the span of the orthogonal states
\[
(q_x)_{x \in \mathcal{M}^A}^{(A)} = \otimes_{x \in \mathcal{M}^A} | q_x \rangle^{(x)}.
\]
Similarly, we denote $H^B$ the span of the orthogonal states
\[
(q_x)_{x \in \mathcal{M}^B}^{(B)} = \otimes_{x \in \mathcal{M}^B} | q_x \rangle^{(x)}.
\]
We have $H^{\text{all}} = H^A \otimes H^B$. Locally, Alice can only execute unitary transformations of the form $U^A \otimes I$ which moreover respect the superselection rules. Locally, Bob can only execute unitary transformations of the form $I \otimes U^B$ which moreover respect the superselection rules.

Measurements. Alice has access to a countably infinite set of “free” modes $\mathcal{M}_f^A \subseteq \mathcal{M}^A - \mathcal{M}_R$. Similarly, Bob has access to a countably infinite set of “free” modes $\mathcal{M}_f^B \subseteq \mathcal{M}^B - \mathcal{M}_R$. We denote $\mathcal{M}_F = \mathcal{M}_f^A \cup \mathcal{M}_f^B$. Without loss of generality, we assume that only the Hilbert spaces $H_f^A = \otimes_{x \in \mathcal{M}_F} H_x$ and $H_f^B = \otimes_{x \in \mathcal{M}_F} H_x$ can be measured by Alice and Bob, respectively. However, there is no restriction on these measurements.

Communication. Alice communicates information to Bob by transferring control over a mode $x \in \mathcal{M}^A$ to Bob. Before the communication, we have $x \in \mathcal{M}^A$. After the communication, we have $x \in \mathcal{M}^B$. Note that the sets $\mathcal{M}^A$ and $\mathcal{M}^B$ can change during the execution of the protocol, but the sets $\mathcal{M}_R$ and $\mathcal{M}_F$ are fixed.

Why there is no restriction on the possible operations.

It may appear that a superselection rule $R$ restricts the possible operations on every state space $H_x$ associated with a mode $x \in \mathcal{M}_R$. However, in fact, the restriction only applies to the tensor product $\otimes_x \mathcal{M}_R H_x$, not to each state space $H_x$ individually. We will show that, if a participant has access to as many extra unused modes in $\mathcal{M}_R$ as needed, the superselection rule $R$ imposes essentially no restriction on the operations that can be executed by this participant on a given state space $H_x$ with $x \in \mathcal{M}_R$. We will only do the case where $H_x$ is a qubit, but the generalisation to higher dimensions is not difficult.

Consider the most general qubit state $\alpha |0\rangle^{(x)} + \beta |1\rangle^{(x)}$ for a mode $x$. We will show how the participant can swap this state with the state of a free mode $y$ initially in the state $|0\rangle^{(y)}$. The swapping will only be approximative, but it will be arbitrarily close to perfect. The participant uses $n - 1$ extra modes $\vec{y} = y_1, \ldots, y_{n-1}$ and prepares the state $(1/\sqrt{n}) (\sum_{j=0}^{n-1} |j\rangle^{(y)})$ where $|j\rangle^{(y)} = [1, \ldots, 1, 0, \ldots, 0]^{(y)}$ contains exactly $j$ modes $y_1, \ldots, y_j$ with $q_{y_j} = 1$. The overall state is
\[
(1/\sqrt{n}) (\alpha |0\rangle^{(x)} + \beta |1\rangle^{(x)}) \otimes (\sum_{j=0}^{n-1} |j\rangle^{(y)}) \otimes |0\rangle^{(z)}
\]
which can be rewritten as
\[
(1/\sqrt{n}) (\alpha |0, 0, 0\rangle^{(x, \vec{y}, z)} + \beta |1, n-1, 0\rangle^{(x, \vec{y}, z)} + \sum_{j=1}^{n-1} \alpha |0, j, 0\rangle^{(x, \vec{y}, z)} + \beta |1, j-1, 0\rangle^{(x, \vec{y}, z)}).
\]
Now, each term $\alpha |0, j, 0\rangle^{(x, \vec{y}, z)} + \beta |1, j-1, 0\rangle^{(x, \vec{y}, z)}$ in the sum have $Q_R = j$. Any operation that preserves $Q_R$ is allowed. In particular, the operation $S$ on $H_x \otimes H_{\vec{y}}$ that executes the swapping $|0, j\rangle^{(x, \vec{y})} \mapsto |1, j-1\rangle^{(x, \vec{y})}$, for $j = 1, \ldots, n-1$, is allowed. For $1 \leq Q_R \leq n-1$, the participant can execute the following. First, he executes a CNOT with the mode $x$ as the source and the free mode $z$ as the target. For $Q_R = j$, we obtain the component
\[
\alpha |0, j, 0\rangle^{(x, \vec{y}, z)} + \beta |1, j-1, 1\rangle^{(x, \vec{y}, z)}.
\]
Next, conditioned on the free mode $z$, he executes the swapping $S$. For $Q_R = j$, we obtain the component
\[
\alpha |0, j, 0\rangle^{(x, \vec{y}, z)} + \beta |0, j, 1\rangle^{(x, \vec{y}, z)} = |0, j\rangle^{(x, \vec{y})} \otimes (\alpha |0\rangle^{(z)} + \beta |1\rangle^{(z)}).
\]
If we sum all the components, the resulting state is
\[
(1/\sqrt{n}) (\alpha |0, 0, 0\rangle^{(x, \vec{y}, z)} + \beta |1, n-1, 0\rangle^{(x, \vec{y}, z)} + |0\rangle^{(x)} \otimes (\sum_{j=1}^{n-1} |j\rangle^{(y)}) \otimes (\alpha |0\rangle^{(z)} + \beta |1\rangle^{(z)}).
\]
Note that we have $Pr(Q_R = 0) = \alpha^2/n$, $Pr(Q_R = n) = \beta^2/n$ and $Pr(Q_R = j) = 1/n$, for $j = 1, \ldots, n-1$, as expected. No conservation law is violated. Fortunately, this state is arbitrarily close to the state
\[
(1/\sqrt{n-1}) |0\rangle^{(x)} \otimes (\sum_{j=1}^{n-1} |j\rangle^{(y)}) \otimes (\alpha |0\rangle^{(z)} + \beta |1\rangle^{(z)}).
\]
which corresponds to a swapping of the two modes $x$ and $z$. This state corresponds to the distribution of probability $\Pr(Q_R = j) = 1/(n-1)$, $j = 1, \ldots, n-1$, which is arbitrarily close to the actual distribution of probability associated with the true final state. So, the participant has essentially executed a swap operation $S_{x \leftrightarrow z}$ between the two modes $x$ and $z$, despite the fact that $z$ is a free mode and $x$ is apparently restricted by a superselection rule.

Let $H_E$ be the state space of some extra system. It is not difficult to see that, given such a swap operation, one can execute the most general transformation $U$ on $H_x \otimes H_E$. First, one executes the swap operation $S_{x \leftrightarrow z}$ on $H_x \otimes H_z$. Second, one executes $U$ on $H_z \otimes H_E$ in the same way it would have been executed on $H_x \otimes H_E$. Finally, one executes the inverse swap operation $S_{x \leftrightarrow z}^\dagger$ on $H_x \otimes H_z$.

**The easier case**

Now, we consider the easier case in which an initial superposition of states with different values for the conserved quantity is not allowed. Note that, on Alice’s side, a mixture of different values for $Q_R^A$ can be purified with a state that has a fixed value for $Q_R^A$, the maximum of the possible values for $Q_R^A$ in the mixture. The samething is true on Bob’s side. Without loss of generality, we can restrict the analysis to protocols in which the initial state on both sides is such a purification instead of a mixture. The attack will use the purification, but this is fine because a cheater against a non purified protocol can himself create such a purification in place of the mixture. This is essentially the same purification technique as in the original proof which ignored superselection rules. The difference is that here, in addition, we make sure that all states in the initial superposition have the same value for the conserved quantity. Let $Q_R^A$ and $Q_R^B$ be the fixed value for the conserved quantity on Alice’s side and Bob’s side, respectively. Because the protocol is known by both parties, we have that $Q_R^A$ and $Q_R^B$ and thus $Q_R = Q_R^A + Q_R^B$ is known by both parties. The fact that bit commitment is impossible in the model where $Q_R$ is fixed and known by both parties was proven independently by Kitaev \cite{Kitaev} and the author. The two proofs are completely different. Here we present the author’s proof. We assume that the protocol is perfectly concealing and show that Alice can swap bit 0 to bit 1. The generalization to the inexact case is not difficult.

After the commit phase, the overall state of the protocol associated with an honest commitment of bit $w \in \{0, 1\}$ can be written as $\Psi(w) = \sum_{k=0}^{n-1} \rho^B_k(w)$ where

$$\psi_k(w) = |\Lambda|^{-1/2} \sum_{A \in \Lambda} \phi^A_{\Lambda,k}(w) \otimes \phi^B_{\Lambda,k}(w),$$

and $\phi^A_{\Lambda,k}(w)$ and $\phi^B_{\Lambda,k}(w)$ are states with $Q_R^A = k$ and $Q_R^B = Q_R - k$, respectively. Both Alice and Bob can measure $k$. Because the protocol is concealing, we have that $p_k(0) = p_k(1)$ for every $k$ because otherwise $k$ provides information about $w$. Let $\rho^B_k(w) = \Tr_A(\psi_k(w)\langle \psi_k(w) |)$ be the residual density matrix on Bob’s side associated with a given $k$ and $w$. Since Bob can measure $k$ and then try to distinguish $\rho^B_k(0)$ and $\rho^B_k(1)$ to obtain information about $w$, we have that $\rho^B_k(0) = \rho^B_k(1)$, for every $k$. The impossibility of bit commitment applies to each $k$ individually. Therefore, conditioned on $k$, Alice can execute a unitary transformation on her side which maps every $\psi_k(0)$ exactly into $\psi_k(1)$. Because the different $\psi_k(w)$ are in a superposition, the relative phase between these states is important. This is not a problem because Alice has control over this relative phase and she knows exactly the states $\psi_k(w)$, including their relative phase. This concludes the proof.

**Discussion**

We have shown that a charge conservation superselection rule imposes no constraint on the allowed operations on every single qubit. Therefore, even though neither Kitaev or Mayers were aware of this fact at the time \cite{Kitaev}, the models that they used in their respective impossibility proofs are actually valid even in the context of such a superselection rule. In this way, we have addressed the specific concern mentioned by Popescu \cite{Popescu}. It should be possible to generalize this result to superselection rules that are based on more general conservation laws. An impossibility proof is no more general than the model used. This is not at all a new understanding for the author. In computational complexity, many models were considered and only then Church and Turing proposed their general thesis that all reasonable models are equivalent for the purpose of computation. The lesson that we learned here is that we should be careful in quantum cryptography before we propose this kind of thesis for quantum protocols.

On the experimental side, in view of our result, one may ask whether or not it is experimentally difficult to swap a fermion mode and the polarization of a photon. This question is irrelevant if we are interested in unconditional security, but nevertheless it is an interesting question in itself. We suspect that it is difficult and this may have lead some people to the wrong conclusion that it is fundamentally prohibited by a conservation law. The required state $\langle 1/\sqrt{n} \rangle \sum_{j=0}^{n-1} |j \rangle^{(0)}$ can in principle be prepared with a non zero probability. One simply creates the state $\otimes_{j=0}^{n-1} (1/\sqrt{2}) (|0 \rangle^{(j)}) + |1 \rangle^{(j)})$ and then do a projection on the desired state. The probability of success is small but more efficient algorithms can most likely be designed. In any case, in the context of unconditional security, efficiency is not an issue. However, it is not clear whether or not the individual states $\langle 1/\sqrt{2} \rangle (|0 \rangle^{(j)} + |1 \rangle^{(j)})$ are available in nature. Moreover, a fully functional quantum computer might not help
because it might not be able to manipulate systems that are restricted by the superselection rule.

[1] S. Popescu, “Multi-party Entanglement”, Workshop on Quantum Information Processing, fall semester 2002, MSRI, Berkeley. No proceedings. The point was made at the end of the talk. A video of the presentation is available on the MSRI web site (www.msri.org).

[2] A. Kitaev, J. Smolin, U. Vazirani (personal communication). Workshop on Quantum Information Processing, fall semester 2002, MSRI, Berkeley.

[3] D. Mayers, “Unconditionally secure quantum bit commitment is impossible”, Phys. Rev. Lett. 78 (1997) pp. 3414–3417.

[4] A. Kitaev, “Quantum Coin Flipping”, Workshop on Quantum Information Processing, fall semester 2002, MSRI, Berkeley. No proceedings. A video of the presentation is available on the MSRI web site (www.msri.org).

[5] D. Mayers, “The Trouble With Quantum Bit Commitment”, \url{http://xxx.lanl.gov/abs/quant-ph/9603015}, March 1996.

[6] H.K. Lo and H.F. Chau, “Is quantum Bit Commitment Really Possible?”, Physical Review Letters, vol. 78, no 17, April 1997, pp. 3410–3413.