AN A POSTERIORI KAM THEOREM FOR WHISKERED TORI IN HAMILTONIAN PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO SOME ILL-POSED EQUATIONS

RAFAEL DE LA LLAVE AND YANNICK SIRE

Abstract. The goal of this paper is to develop a KAM theory for tori with hyperbolic directions, which applies to Hamiltonian partial differential equations, even to some ill-posed ones.

The main result has an a-posteriori format, i.e., we show that if there is an approximate solution of an invariance equation which also satisfies some non-degeneracy conditions, then there is a true solution nearby. This allows, besides dealing with the quasi-integrable case, to validate numerical computations or formal perturbative expansions as well as to obtain quasi-periodic solutions in degenerate situations. The a-posteriori format also has other automatic consequences (smooth dependence on parameters, bootstrap of regularity, etc.). We emphasize that the non-degeneracy conditions required are just quantities evaluated on the approximate solution (no global assumptions on the system such as twist). Hence, they are readily verifiable in perturbation expansions.

The method of proof is based on an iterative method to solve a functional equation for the parameterization of the torus satisfying the invariance equations and for parametrization of directions invariant under the linearization. The iterative method does not use transformation theory or action-angle variables. It does not assume that the system is close to integrable. We do not even need that the equation under consideration admits solutions for every initial data. In this paper we present in detail the case of analytic tori when the equations are analytic in a very weak sense.

We first develop an abstract theorem. Then, we show how this abstract result applies to some concrete examples, including the scalar Boussinesq equation and the Boussinesq system so that we construct small amplitude tori for the equations, which are even in the spatial variable. Note that the equations we use as examples are ill-posed. The strategy for the abstract theorem is inspired by that in [FdlLS09b, FdlLS09a]. The main part of the paper is to study infinite dimensional analogues of dichotomies which applies even to ill-posed equations and which is stable under addition of unbounded perturbations. This requires that we assume smoothing properties. We also present very detailed bounds on the change of the splittings under perturbations.

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1. Introduction
   1.1. Some general considerations and relations with the literature

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1. Introduction

The goal of this paper is to develop a KAM theory for tori with hyperbolic directions, which applies to Hamiltonian partial differential equations, even to some ill-posed ones.
The main result, Theorem 3.5, is stated in an a-posteriori format, that is, we formulate invariance equations and show that approximate solutions that satisfy some explicit non-degeneracy conditions, lead to a true solution. This a-posteriori format leads automatically to several consequences (see Section 3.6.2) and can be used to justify numerical solutions and asymptotic expansions. We note that the results do not assume that the equations we consider define evolutions and indeed we present examples of quasi-periodic solutions in some well known ill-posed equations. See Sections 10, 11.

1.1. Some general considerations and relations with the literature. Some partial differential equations appear as models of evolution in time for Physical systems. It is natural to consider such evolutionary PDE’s as a dynamical system and try to use the methods of dynamical systems.

Adapting dynamical systems techniques to evolutionary PDE’s has to overcome several technical difficulties. For starters, since the PDE’s involve unbounded operators, the standard theory of existence, uniqueness developed for ordinary differential equations does not apply. As it is well known, by now, there are systematic ways of defining the evolution using e.g. semigroup theory [Sho97, Paz83, Gol85] and many dynamical systems techniques can be adapted in the generality of semigroups (see the pioneering work of [Hen81] and more modern treatises [Hal88, Miy92, Tem97, CFNT89, Rob01, SY02, CV02, HMO02, CM12].) Besides the analytic difficulties, adapting ODE techniques to PDE’s has to face that several geometric arguments fail to hold. For instance, symplectic structures on infinite-dimensional spaces (see for instance [CM74, Bam99]) could lack several important properties. Hence, the techniques (e.g. KAM theory) that are based on geometric properties have to overcome several difficulties specially the methods based on transformation theory [Kuk93, Kuk94, Kuk00, Kuk06, KP03]. Some recent methods based on avoiding transformation theory are [CW93, CW94, Bon99, Ber07, Cra00]. When working near an equilibrium point, one also has to face the difficulty that the action angle variables are singular (even in finite dimensions) [KP03, GK14]. In the approach of this paper, we do not use action angle variables, which present difficulties even in finite dimensional fixed points and, much more in PDE’s.

One class of evolutionary equations that has not received much systematic attention is ill-posed equations. In ill-posed equations, one cannot define the evolution for all the initial data in a certain space (an equation may be ill-posed in a space and well posed in another) or the evolution is not continuous in this space. Nevertheless, it can be argued that even if one cannot find solutions for all the initial data, one can still find interesting solutions which provide accurate descriptions of physical phenomena. Many ill-posed equations in the literature are obtained as a heuristic approximation of a more fundamental equation. The solutions of the ill-posed equation may be approximate solutions of the true equation.

For example, many long wave approximations of water waves turn out to be ill-posed (e.g. the Boussinesq equations used as examples here, see Section 10) but several special solutions (e.g. traveling waves or the quasi-periodic solutions considered in this paper)
of the long wave approximations can be constructed. These special solutions are such
that, for them, the long wave approximation is rather accurate. Hence, the solutions
obtained here for the long wave approximation provide approximate solutions of the
original water wave equation and are physically relevant.

Note that the long-wave approximations are PDE’s while the water waves problem
is a free boundary and many techniques are different, notably in numerical analysis.
Being able to validate the numerical solutions is useful.

Of course, the straightforward adaptation of ODE methods for invariant manifolds
to ill-posed equations present some challenges because some methods (e.g. graph trans-
form, index theory methods, etc.), which are very useful in ODEs, require taking ar-
bitrary initial conditions. Nevertheless, we will present rather satisfactory adaptations
of some of the methods of hyperbolic dynamical systems.

In the present paper, we are concerned with the construction of quasi-periodic mo-
tions of PDEs. The method is very general. Some concrete examples of ill-posed
equations to which the method applies will be presented in Sections 10 and 11.

The tori we consider are whiskered, that is the linearization has many hyperbolic di-
rections, indeed, as many directions as it is possible to be compatible with the preserva-
tion of the symplectic structure. There is a rich KAM theory for whiskered tori [Gra74,
Zeh76] or for lower dimensional tori will elliptic directions [Eli89, You99, LY05, Sev06].
A treatment of normally elliptic tori by methods similar to those here is in [LV11].

In PDE’s, where the phase space is infinite dimensional, the quasiperiodic solutions
are very low dimensional. Nevertheless, most of the literature in PDE is concerned
with normally elliptic tori, so that most of the small divisors come from the elliptic
normal directions. The models considered here have no elliptic normal directions. On
the other hand, the models we consider do not admit solutions for all initial conditions
and present very severe unstable terms. Hence, methods based on transformation
theory, normal forms etc. are very difficult in our case. We also deal with unbounded
perturbations.

1.2. Overview of the method. We are going to follow roughly the method de-
scribed in [FdlLS09b] and implemented in [FdlLS09a] for finite dimensional systems,
in [LdlL09, FdlLS15] for infinite dimensional systems (but whose evolution is a smooth
differential equation; the main difficulty overcome in [LdlL09] was the fact that the
equations involve delays, a new difficulty in [FdlLS15] is the spatial structure). In this
paper we overcome the difficulty that the evolution equations are PDE’s which are
perturbed by unbounded operators. Hence, we have to overcome many problems (un-
bounded operators, regularity issues and spectral theory for instance ). Some results
in KAM with unbounded perturbations by very different methods appear in [LYTI].

The method we use is based on the solution of a functional equation whose unknown
is a parameterization of the invariant torus and deiving a Newton method to solve
these equations by quadratically convergent schemes. We assume that the linearized
evolution admits an invariant splitting. In the hyperbolic directions we can use essen-
tially soft functional analysis methods. There are subtleties such that we have to deal
with unbounded perturbations and be very quantitative in the hyperbolic perturbation theory, and a center direction case, in which we have to deal with equations involving small divisors and use heavily the number theoretic properties of the equation and the symplectic geometry.

The method does not rely on methods that require the evolution for all initial data on a ball. Also, the symplectic geometry properties are used only sparingly. We certainly do no use action-angle variables. The solutions we construct are very unstable – indeed, some perturbations near them may lead to a solution of the evolution equation – but they are in some precise sense hyperbolic in the usual meaning of dynamical systems. We expect that one can define stable and unstable manifolds for them and we hope to come back to this problem. Fortunately, the analysis on the center is very similar to the analysis in the finite dimensional case. The bulk of the work is in the study of hyperbolic splittings with unbounded perturbations. We hope that the theory developed here can be used in other contexts.

Indeed, other theories of persistence of invariant splitting (having significant applications to PDE) have already been developed in [Hen81, PS99, CL95, CL96, HI11]. The main difference between Section 6 and [CL95, CL96] is that we take advantage of the smoothing properties and, hence, can deal with more singular perturbations. We also take advantage of the fact that the dynamics on the base is a rotation whereas [CL95, CL96] deal with more general dynamics. This allows us to obtain analyticity results which are false in the more general contexts considered in [CL95, CL96].

The method presented here applies even to some ill-posed equations. A fortiori, it applies also to well posed equations. Even then, it presents advantages, notably our main result has an a-posteriori format that can justify several expansions and deal with situations with weak hyperbolicity, bootstrap regularity, establish smooth dependence, etc. It also leads to efficient numerical algorithms. See Section 3.6.2. In a complementary direction, we point out that for finite dimensional problems the present methods leads to efficient algorithms (See [HdlLS12]). The case without center directions and no Hamiltonian structure has been considered in [CH15].

1.3. Organization of the paper. This paper is organized as follows: In Section 2 we present an overview of the method, describing the steps we will take, but ignoring some important precisions (e.g. domains of the operators), and proofs. In Section 3 we start developing the precise formulation of the results. We first present an abstract framework in the generality of equations defined in Banach spaces, including the abstract hypothesis. The general abstract results are stated in Section 3.6.1 and in Section 3.6.3 we discuss how to apply the results to some concrete examples. Some possible extensions are discussed in Section 3.6.2. The rest of the paper is devoted to the proof of the results following the strategy mentioned in the previous sections. One of the main technical results, which could have other applications is the persistence of hyperbolic evolutions with smoothing properties. See Section 6.
2. Overview of the method

In this section, we present a quick overview describing informally the steps of the method. We present the equations that need to be solved and the manipulations that need to be done ignoring issues such as domain of operators, estimates. These precisions will be taken up in Section 3. This section can serve as motivation for Section 3 since we use the formal manipulations to identify the issues that need to be resolved by a precise formulation.

We will discuss first abstract results, but in Sections 10 and 11 we will show that the abstract result applies to concrete examples.

One example to keep in mind and which has served as an important motivation for us is the Boussinesq equation

\[ u_{tt} = \mu u_{xxxx} + u_{xx} + (u^2)_{xx} \quad x \in \mathbb{T}, \; t \in \mathbb{R}, \; \mu > 0 \]  

In Section 11 we will also consider the Boussinesq system. Other models in the literature which fit our scheme are the Complex Ginzburg-Landau equation and the derivative Complex Ginzburg-Landau equation for values of the parameters in suitable ranges.

Remark 2.1. There are several equations called the Boussinesq equation in the literature (in Section 11 we also present the Boussinesq system), notably the Boussinesq equation for fluids under thermal buoyancy. The paper [McK81] uses the name Boussinesq equation for \[ u_{tt} = -u_{xxxx} + (u^2)_{xx} \] and shows it is integrable in some sense made precise in that paper. Note that this equation is very different from (1) because of the sign of the fourth space derivative and (less importantly), the absence of the term with the second derivative. The sign of the fourth derivative term causes that the wave propagation properties of (1) because of the sign of the fourth space derivative and (less importantly), the absence of the term with the second derivative. The sign of the fourth derivative term causes that the wave propagation properties of (1) are completely different.

Sometimes people refer to (1) with \( \mu > 0 \) as the “bad” Boussinesq equation, and call the equation with \( \mu < 0 \), the “good” Boussinesq equations. We note that the case \( \mu > 0 \) considered here is the case that appears in water waves (see [Bou72, Equation (26)]).

Remark 2.2. We note that the fourth derivative in (1) is just the next term in the long wave expansion of the water wave problem (which is not a PDE, but rather a free boundary problem). Equations similar to (1) appear in many long wave approximations for waves. See [CGNS05, Cra08] for modern discussions.

The special solutions of (1) which are in the range of validity of the long wave approximation are good approximate solutions of the water wave problem, but they are analyzable by PDE methods rather than the free boundary methods required by the original problem. [CNS11, LM09]. Note that the solutions produced here lie in the regime (low amplitude, long wave) where the equation (1) was derived, so that they provide approximate solutions to the water wave problem.
2.1. **The evolution equation.** We consider an evolutionary PDE, which we write symbolically,

\[ \frac{du}{dt} = \mathcal{X} \circ u \]

where \( \mathcal{X} \) will be a differential and possibly non-linear operator. This will, of course, require assumptions on domains etc. which we will take up in Section 3. For the moment, we will just say that \( \mathcal{X} \) is defined in a domain inside a Banach space \( X \). We will write

\[ \mathcal{X}(u) = Au + N(u) \]

where \( A \) is linear and \( N \) is a nonlinear and possibly unbounded operator.

The differential equations \( \dot{u} = Au \) will not be assumed to generate dynamical evolution for all initial conditions (we just assume that it generates forward and backward evolutions when restricted to appropriate subspaces). Of course, we will not assume that (2) defines an evolution either. Lack of solutions for all the initial conditions will not be a severe problem for us since we will only try to produce some specific solutions.

The meaning in which (2) is to hold may be taken to be the classical sense. As we will see we will take the space \( X \) to consist of very differentiable functions so that the derivatives can be taken in the elementary classical sense. As intermediate steps, we will also find useful some solutions in the *mild* sense, satisfying some integral equations formally equivalent to (2). The mild solutions require less regularity in \( X \). Again, we emphasize that the solutions we try to produce are only special solutions.

We will assume that the nonlinear operator \( N \) is “sub-dominant” with respect to the linear part. This will be formulated later in Section 3, but we anticipate that this means roughly that \( A \) is of higher order than \( N \) and that the evolution generated by \( A \) when restricted to appropriate sub-spaces gains more derivatives than the order of \( N \). We will formulate all this precisely later.

We will follow [Hen81] and formulate these effects by saying that the operator \( N \) is an analytic function from a domain \( U \subset X \) – \( X \) is a Banach space of smooth functions – to \( Y \) – a space corresponding to less smooth functions and that the evolution operators map \( Y \) back to \( X \) with some quantitative bounds.

In the applications that we present in Sections 10 and 11, the equations we consider are polynomial but the method can deal with more general nonlinearities.

2.2. **The linearized evolution equations.** Note that, in this set up we can define a linearized evolution equation around a curve \( u(t) \) in \( X \), i.e.

\[ \frac{d\xi}{dt} = D\mathcal{X} \circ u(t)\xi \equiv A\xi + D\mathcal{N}(u(t))\xi \]

\(^1\)The equations we consider are taken from the literature of approximations of water waves. In these derivations, it is customary to expand the non-linearity and keep only the lower order terms
The equations (4) are to be considered as evolution equations for $\xi$ while $u(t)$ is given and fixed. The meaning of the term $D\mathcal{N}$ could be understood if $\mathcal{N}$ is a differentiable operator from $X$ to $Y$.

Of course, when $u(t)$ is solution of the evolution equation (2), equations (4) are the variational equations for the evolution. In our case, the evolution is not assumed to exist and, much less, the variational equations are assumed to provide a description of the effect of the initial conditions on the variation. We use these equations (4) even when $u(t)$ is not a solution of the evolution equation (2) and we will show that they are indeed a tool to modify an approximate solution $u(t)$ into a true solution.

Notice that (4) is non-autonomous, linear non-homogeneous, but that the existence of solutions is not guaranteed for all the initial conditions (even if the time dependent term is omitted).

In the finite dimensional case, equations of the form (4) even when $u(t)$ is not a solution are studied when performing a Newton method to construct a solution; for example in multiple shooting. Here, we will use (4) in a similar way. We will see that (4) can be studied using that $\mathcal{A}$ is dominant and has a splitting (and that $u(t)$ is not too wild).

2.3. The invariance equation. Given a fixed $\omega \in \mathbb{R}^\ell$ that satisfies some good number theoretic properties (formulated precisely in Section 3.3), we will be seeking an embedding $K: \mathbb{T}^\ell \to X$ in such a way that

$$\mathcal{X} \circ K = DK \cdot \omega$$

(5)

Note that if (5) holds, then, for any $\theta \in \mathbb{T}^\ell$, $u(t) = K(\omega t + \theta)$ will be solution of (2). Hence, when we succeed in producing a solution of (5), we will have a $\ell$-parameter family of quasi-periodic solutions. The meaning of these parameters is the origin of the phase as is very standard in the theory of quasi-periodic functions.

2.4. Outline of the main result. The main ingredient of the main result, Theorem 3.5 is that we will assume given an approximate solution $K_0$ of (5). That is, we are given an embedding $K_0$ in such a way that

$$\mathcal{X} \circ K_0 - DK_0 \cdot \omega \equiv e$$

(6)

is small enough. We will also assume that the linearized evolution satisfies some non-degeneracy assumptions. The conclusions is that there is a true solution close to the original approximate solutions. Theorems of these form in which we start from an approximate solution and conclude the existence of a true one are often called “a posteriori” theorems.

In the concrete equations that we consider in the applications, the approximate solutions will be constructed using Lindstedt series.

The sense in which the error $e$ is small requires defining appropriate norms, which will be taken up in Section 3. The precise form of the non-degeneracy conditions will be motivated by the following discussion which specifies the steps we will perform for the Newton method for the linearized equation.
\begin{equation}
\frac{du}{dt} = D\mathcal{X} \circ K_0(\theta + \omega t)u
\end{equation}

The non-degeneracy conditions have two parts. We first assume that for each \( \theta \in \mathbb{T}^q \), the linearized equation satisfies some spectral properties. These spectral properties mean roughly that there are solutions of (7) that decrease exponentially in the future (stable solutions), others that decrease exponentially in the past (unstable solutions), and some center directions that can grow or decrease with a smaller exponential rate. The span of these three class of solutions is the whole space. We will also assume that the evolutions, when they can be defined, gain regularity.

In the ODE case, this means that the linearized equation admits an exponential trichotomy in the sense of [SS76].

In the PDE case, there are some subtleties not present in the ODE case. For instance, the vector field is not differentiable and is only defined on a dense subset.

We will not assume that (7) defines an evolution for all time and all the initial conditions. We will however assume that (7) admits a solution forward in time for initial conditions in a space (the center stable space) and backwards in time for the another space (the center unstable space). We will furthermore assume that the center stable and center unstable spaces span the whole space, and they have a finite dimensional intersection (we will also assume that they have a finite angle, which we will formulate as saying that the projections are bounded). We emphasize that we will not assume that the evolution forward of (7) can be defined outside of the center stable space nor that the backward evolution can be defined outside of the center unstable space.

Furthermore, we will assume that the evolutions defined in these spaces are smoothing. Of course, these subtleties are only present when we consider evolutions generated by unbounded operators and are not present in the ODE case.

A crucial result for us is Lemma 6.1 which shows that this structure (the trichotomy with smoothing) is stable under the addition of unbounded terms of lower order. We also present very quantitative estimates on the change of the structure under perturbations. Note that the result is also presented in an a-posteriori format so that we can use just the existence of an approximate invariant splitting.

The smoothing properties along the stable directions overcome the loss of regularity of the perturbation. Hence, we can obtain a persistence of the spaces under unbounded perturbations of lower order. A further argument shows the persistence of the smoothing properties. The result in Lemma 6.1 can be considered as a generalization of the finite dimensional result on stability of exponential dichotomies to allowing unbounded perturbations. An important consequence is that, when \( \mathcal{N}(u) \) is small enough (in an appropriate sense) we can transfer the hyperbolicity from \( \mathcal{A} \) to the approximate solution, which is the way that we construct the approximately hyperbolic solutions in the applications.

We will need to assume that in the center directions, there is some geometric structure that leads to some cancellations (sometimes called automatic reducibility). These
cancellations happen because of the symplectic structure. We note that, in our case, we only need a very weak form of symplectic structure, namely that it can be made sense of in a finite dimensional space consisting of rather smooth functions. Note that the infinitesimal perturbations do not grow in the tangent directions. The preservation of the geometric structure also implies that some of the perpendicular directions evolve not faster than linearly. Hence, the tori we consider are never normally hyperbolic and that for $\ell$-dimensional tori, the space of directions with subexponential growth is at least $2\ell$ dimensional. We will assume that the tori are as hyperbolic as possible while preserving of the symplectic structure. That is, the set of directions with subexponential growth is precisely $2\ell$ dimensional. These tori are called whiskered in the finite dimensional case.

We note that the geometric structure we need only requires to make sense as the restriction to an infinitesimal space and be preserved only in a set of directions. The geometric structure that appears naturally in applications will be given by an unbounded form and many of the deeper features of symplectic structures in finite dimensions will not be available. Hence, it is important to note that the present method does not rely much in the symplectic structure. We do not rely on transformation theory we only use some geometric identities in finite dimensional spaces to construct a good system of coordinates in finite dimensions and to show that some (finite dimensional) averages vanish. In systems without the geometric structure, the system of coordinates and the averages would require adjusting external parameters.

Remark 2.3. We note that (7) is formally the variation equation giving the derivative of the flow of the evolution equation. This interpretation is very problematic since the equations we will be interested in do not define necessarily a flow.

An important part of the effort in Section 3 consists in defining these structures in the restricted framework considered in this paper when many of the geometric operations used in the finite dimensional case are not available.

We also need to make assumptions that are analogues of the twist conditions in finite dimensions. See Definition 3.4. The twist condition we will require is just that a finite dimensional matrix is invertible. The matrix is computed explicitly on the approximate solution and does not require any global considerations on the differential equation.

2.5. Overview of the proof. The method of proof will be to show that, under the hypotheses we are making, a quasi-Newton method for equation (5) started in the initial guess, converges to a true solution. We emphasize that the unknown in equation (5) is the embedding $K$ of $T^\ell$ into a Banach space $X$. Hence, we will need to introduce families of Banach spaces of embeddings (the proof of the convergence will be patterned after the corresponding proofs [Mos66b, Zeh75]).

For simplicity, we will only consider analytic spaces of embeddings. Note that the regularity of the embedding $K$ as a function of their argument $\theta \in T^\ell$ is different from the regularity of the functions $K(\theta) \in X$. The term $K(\theta)$ will be functions of the $x$ variable. The space $X$ encodes the regularity with respect to the variable $x$. Indeed, we
will consider also other Banach spaces $Y$ consisting of functions of smaller regularity in $x$.

The Newton method consists in solving the equation

$$
\frac{d}{dt} \Delta(\theta + \omega t) - D\chi \circ K_0(\theta + \omega t) \Delta(\theta + \omega t) = -e
$$

and then, taking $K_0 + \Delta$ as an improved solution.

Clearly, (8) is a non-homogeneous version of (7). Hence, the spectral properties of (7) will play an important role in the solution of (8) by the variations of constants formula. Following [FdlLS09b, FdlLS09a], we will show that using the trichotomy, we can decompose (8) into three equations, each one of them corresponding to one of the invariant subspaces.

The equations along the stable and unstable directions can be readily solved using the variation of parameters formula also known as Duhamel formula (which holds in the generality of semigroups) since the exponential contraction and the smoothing allow us to represent the solution as a convergent integral.

The equations along the center direction, as usual, are much more delicate. We will be able to show the geometric properties to establish the automatic reducibility. That is, we will show that there is an explicit change of variables that reduces the equation along the center direction to the standard cohomology equations over rotations (up to an error which is quadratic – in the Nash-Moser sense – in the original error in the invariance equation). It is standard that we can solve these cohomology equations under Diophantine assumptions on the rotation and that we can obtain tame estimates in the standard meaning of KAM theory [Mos66b, Mos66a, Zeh75]. One geometrically delicate point is that the cohomology equations admit solutions provided that certain averages vanish. The vanishing of these averages over perturbations is related to the exactness properties of the flow. Even if this is, in principle, much more delicate in the infinite dimensional case, it will turn out to be very similar to the finite dimensional case, because we will work on the restriction to the center directions which are finite dimensional. The procedure is very similar to that in [FdlLS09b].

We will not solve the linearized equations in center direction exactly. We will solve them up to an error which is quadratic in the original error. The resulting modified Newton method, will still lead to quadratically small error in the sense of Nash-Moser theory and can be used as the basis of a quadratically convergent method.

Once we have the Newton-like step under control we need to show that the step can be iterated infinitely often and it converges to the solution of the problem.

A necessary step in the strategy is to show stability of the non-degeneracy assumptions. The stability of the twist conditions is not difficult since it amounts to the invertibility of a finite dimensional matrix, depending on the solution. The stability of spectral theory is reminiscent of the standard stability theory for trichotomies [SS76, HPS77] but it requires significant more work since we need to use the smoothing properties of the evolution semigroups to control the fact that the perturbations are unbounded. Then, we need to recover the smoothing properties to be able to solve the
cohomology equations. For this functional analysis set up, we have found very inspiring the “two spaces approach” of [Hen81] and some of the geometric constructions of [Hen81, PS99, CL95, CL96]. Since the present method is part of an iterative procedure, we will need very detailed estimates of the change.

We note that rather than presenting the main result as a persistence result, we prove an a-posteriori result showing that an approximate invariant structure implies the existence of a truly invariant one and we bound the distance between the original approximation and the truly invariant one. This, of course, implies immediately the persistence results.

3. The precise framework for the results

In this section we formalize the framework for our abstract results. As indicated above, we will present carefully the technical assumptions on domains, etc. of the operators under consideration, and the symplectic forms. We will formulate spectral non-degeneracy conditions and the twist non-degeneracy assumption.

In Section 3.6 we will state our main abstract result, Theorem 3.5. The proof will be obtained in the subsequent Sections. Then, in Sections 10 and 11 we will show how the abstract theorem applies to several examples. The abstract framework has been chosen so that the examples fit into it, so that the reader is encouraged to refer to these sections for motivation. Of course, the abstract framework has been formulated with the goal that it applies to other problems in a more or less direct manner. We leave these to the reader.

We note that the formalism we use is inspired by the two-space formalism of [Hen81]. We consider two Hilbert spaces $X$ and $Y$. The differential operators, which are unbounded from a space to itself will be very regular operators considered as operators from $X$ to $Y$. Some evolutions will have smoothing properties and map $Y$ to $X$ with good bounds.

3.1. The evolution equation. We will consider an evolution equation as in (2) and (3).

We assume

**H1** There are two complex Hilbert spaces

$$X \hookrightarrow Y,$$

with continuous embedding. The space $X$ (resp. $Y$) is endowed with the norm $\|\cdot\|_X$ (resp. $\|\cdot\|_Y$)

We denote by $\mathcal{L}(X_1, X_2)$ the space of bounded linear operators from $X_1$ to $X_2$.

We will assume furthermore that $X$ is dense in $Y$. We will assume in applications that $\mathcal{A}$ and $\mathcal{N}$ are such that they map real functions into real functions; it will be part of the conclusions that the solutions of the invariance equations we obtain are then real.

**H2** The non-linear part $\mathcal{N}$ of (3) is an analytic function from $X$ to $Y$. 

We recall that the definition of an analytic function is that it is locally defined by a norm convergent sum of multilinear operators. Since we will be considering an implicit function theorem, it suffices to consider just one small neighborhood and a single expansion in multi-linear operators. The examples in Sections 10 and 11 have nonlinearities which are just polynomials (finite sums of multilinear operators).

**Remark 3.1.** In our case, it seems that some weaker assumptions would work. It would suffice that $X \circ K(\theta)$ is analytic for any analytic embedding $K$. In many situations this is equivalent to the stronger definition [HP74, Chapter III]. In the main examples that we will consider and in other applications, the vector field $X$ is a polynomial.

**Remark 3.2.** It also seems possible that one could deal with finite differentiable problems. For the experts, we note that there are two types of KAM smoothing techniques: either smoothing only the solutions in the iterative processs (single smoothing) [Sch60, CdlL10b] or smoothing also the problems (double smoothing) [Mos66b, Zeh75]. In general, double smoothing techniques produce better differentiability in the results. On the other hand, in this case, the approximation of the problems seems fraught with difficulties (how to define smoothings in infinite dimensional spaces, also for unbounded operators). Nevertheless, single smoothing methods do not seem to have any problem. Of course, if the non-linearities have some special structure (e.g. they are obtained by composing with a non-linear function) it seems that a double smoothing could also be applied.

**Remark 3.3.** Note that the structure of $X$ assumed in (3) allows us to estimate always the errors in $Y$, even if the unknowns $K$ are in $X$.

This is somewhat surprising since the loss of derivatives from $X$ to $Y$ is that of the subdominant term $N$. We expect that the results of applying $A$ to elements in $X$ does not lay in $Y$.

Nevertheless, using the structure in (3) and the smoothing properties we will be able to show by induction that if the error is in $Y$ at one step of the iteration, we can estimate the error in subsequent steps of the iteration. Note that the new error is the error in the Taylor approximation of $X \circ (K + \Delta)$, which is the error in the Taylor approximation of $N \circ (K + \Delta)$.

Of course, we also need to ensure that the initial approximation satisfies this hypothesis. In the practical applications, we will just take a trigonometric polynomial.

**3.2. Symplectic properties.** We will need that there is some exact symplectic structure. In our method, this does not play a very important role. We just use the preservation of the symplectic structure to derive certain identities in the (finite dimensional) center directions. These are called automatic reducibility and use the exactness to show that some (finite dimensional) averages vanish (vanishing lemma) so that we can prove the result without adjusting parameters.

We will assume that there is a (exact) symplectic form in the space $X$ and that the evolution equation (2) can be written in Hamiltonian form in a suitable weak sense, which we will formulate now.
Motivated by the examples in Sections 10 and 11 and others in the literature, we will assume that the symplectic form is just a constant operator over the whole space $X$ (notice that we can identify all the tangent spaces). We will not consider the fact that the symplectic form depends on the position. Note that heuristically, the fact that the symplectic form is constant ensures $d\Omega = 0$ and, because we are considering a Banach space, Poincaré lemma would give $\Omega = d\alpha$. We will need only weak forms of these facts. General symplectic forms in infinite dimensions may present surprising phenomena not present in finite dimensions \cite{CM74, Bam99, KP03}. Fortunately, we only need very few properties in finite dimensional subspaces in a very weak sense.

**H3** There is an anti-symmetric bounded operator $\Omega : X \times X \to \mathbb{C}$ taking real values on real vectors.

The operator $\Omega$ is assumed to be non-degenerate in the sense that $\Omega(u, v) = 0 \forall v \in X$, implies $u = 0$.

$\Omega$ will be referred to as the symplectic form.

As we mentioned above, we are assuming that the symplectic form is constant.

In some of the applications, $\Omega$ could be a differential operator or the inverse of a differential operator. When $\Omega$ is a differential operator, the fact that $\Omega$ is bounded only means that we are considering a space $X$ consisting of functions with high enough regularity. The form $\Omega$ could be unbounded in $L^2$ or in spaces consisting of functions with lower regularity than the functions in $X$.

Notice that given a $C^1$ embedding $K$ of $T^\ell$ to $X$ we can define the pull-back of $\Omega$ by the customary formula

$$K^*\Omega_\theta(a, b) = \Omega(DK(\theta)a, DK(\theta)b)$$

(9) The form $K^*\Omega$ is a form on $T^\ell$. If $K$ is $C^r$ as a mapping form $T^\ell$ to $X$ (in our applications it will be analytic), the form $K^*\Omega$ will be $C^{r-1}$.

**H3.1** We will assume that $\Omega$ is exact in the sense that, for all $C^2$ embeddings $K : T^\ell \to X$ we have

$$K^*\Omega = d\alpha_K$$

(10) with $\alpha_K$ a one-form on the torus.

In the applications we will have that $\alpha_K = K^*\alpha$ for some 1-form in $X$. Note that if $\Omega$ is not constant, we will need that $\alpha$ depends on the position.

**H4** There is an analytic function $H : X \to \mathbb{C}$ such that for any $C^1$ path $\gamma : [0, 1] \to X$, we have

$$H(\gamma(1)) - H(\gamma(0)) = \int_0^1 \Omega(\dot{\gamma}(s)\gamma(s)) \, ds$$

(11) Note that **H4** is a weak form of the standard Hamilton equations $i_X\Omega = dH$. We take the Hamiltonian equations and integrate them along a path to obtain (11).
A consequence of \( H3 \) and \( H4 \) is we have that for any closed loop \( \Gamma \) with image in \( \mathbb{T}^\ell \)
\[
\int_{\Gamma} i_{\chi_0 K} K^* \Omega = 0.
\]

**Remark 3.4.** The formulation of (11) is a very weak version of the Hamilton equation. In particular, it is somewhat weaker than the formulation in [Kuk06], but on the other hand, we will assume more hyperbolicity properties than in [Kuk06].

### 3.2.1. Some remarks on the notation for the symplectic form.

The symplectic form can be written as
\[
\Omega(u, v) = \langle u, Jv \rangle_Z
\]
where \( Z \) is a Hilbert space and \( \langle \cdot, \cdot \rangle_Z \) denotes the inner product in \( Z \) and \( J \) is a (possibly unbounded) operator in \( Z \) – but bounded from \( X \) to \( Z \).

Once we have defined the operator \( J \), we can talk about the operator \( J^{-1} \) if it is defined in some domain.

The evolution equations can be written formally
\[
\frac{du}{dt} = J^{-1} \nabla H(u)
\]
where \( \nabla H \) is the gradient understood in the sense of the metric in \( Z \). In the concrete applications here, we will take \( Z = L^2 \), \( X = H^m \), \( Y = H^{m-a} \) for large enough \( m \). Of course, in well posed systems we can take \( X = Y \).

We recall that the definition of a gradient (which is a vector field) requires a metric to identify differentials with vector fields. This is true even in finite dimensions. In infinite dimensions, there are several more subtleties such as the way that the derivative is to be understood. Hence, we will not use much the gradient notation and the operator \( J \) except in Section 7, which is finite dimensional.

**Remark 3.5.** In the Physical literature (and in the traditional calculus of variations) it is very common to take \( Z \) to be always \( L^2 \), even if the functions in the space \( X \) or \( Y \) are significantly more differentiable. In some ways the space \( Z = L^2 \) is considered as fixed and the spaces \( X, Y \) are mathematical choices. So that the association of the symplectic form to a symplectic operator is always done with a different inner product \( Z \). The book [Neu10] contains a systematic treatment of the use of gradients associated to Sobolev inner products.

### 3.3. Diophantine properties.

We will consider frequencies that satisfy the standard Diophantine properties.

**Definition 3.1.** Given \( \kappa > 0 \) and \( \nu \geq \ell - 1 \), we define \( D(\kappa, \nu) \) as the set of frequency vectors \( \omega \in \mathbb{R}^\ell \) satisfying the Diophantine condition:
\[
|\omega \cdot k|^{-1} \leq \kappa |k|^{\nu}, \quad \text{for all } k \in \mathbb{Z}^\ell - \{0\}
\]
where \( |k| = |k_1| + ... + |k_\ell| \). We denote
\[
D(\nu) = \cup_{\kappa > 0} D(\kappa, \nu).
\]
It is well known that when $\nu > \ell$, the set $\mathcal{D}(\nu)$ has full Lebesgue measure.

### 3.4. Spaces of analytic mappings from the torus.

We will denote $D_\rho$ the complex strip of width $\rho$, i.e.

$$D_\rho = \{ z \in \mathbb{C}/\mathbb{Z}^\ell : |\text{Im } z_i| < \rho \ i = 1, \ldots, \ell \}.$$  

We introduce the following $C^m$-norm for $g$ with values in a Banach space $W$

$$|g|_{C^m(B),W} = \sup_{0 \leq |k| \leq m} \sup_{z \in B} ||D^k g(z)||_W.$$  

Let $\mathcal{H}$ be a Banach space and consider $A_{\rho,\mathcal{H}}$ the set of continuous functions on $\overline{D_\rho}$, analytic in $D_\rho$ with values in $\mathcal{H}$. We endow this space with the norm

$$\|u\|_{\rho,\mathcal{H}} = \sup_{z \in D_\rho} \|u(z)\|_{\mathcal{H}}.$$  

$(A_{\rho,\mathcal{H}}, \| \cdot \|_{\rho,\mathcal{H}})$ is well known to be a Banach space. Some particular cases which will be important for us are when the space $\mathcal{H}$ is a space of linear mappings (e.g. projections).

We will also need some norms for linear operators. Fix $\theta \in D_\rho$ and consider $A(\theta)$ a continuous linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$, two Banach spaces. Then we define $\|A\|_{\rho,\mathcal{H}_1,\mathcal{H}_2}$ as

$$\|A\|_{\rho,\mathcal{H}_1,\mathcal{H}_2} = \sup_{z \in D_\rho} \|A(z)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)},$$  

where $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ denotes the Banach space of linear continuous maps from $\mathcal{H}_1$ into $\mathcal{H}_2$ endowed with the supremum norm.

#### Definition 3.2.

Let $T^\ell = \mathbb{R}/\mathbb{Z}^\ell$ and $f \in L^1(T^\ell, \mathcal{H})$ where $\mathcal{H}$ is some Banach space. We denote $\text{avg } (f)$ its average on the $\ell$-dimensional torus, i.e.

$$\text{avg } (f) = \int_{T^\ell} f(\theta) \, d\theta.$$  

#### Remark 3.6.

Of course, in the previous definition, since $\mathcal{H}$ might be an infinite-dimensional space, the above integral, in principle, has to be understood as a Dunford integral. Nevertheless, since we will consider rather smooth functions, it will agree with simple approaches such as Riemann integrals.

### 3.5. Non-degeneracy assumptions.

This section is devoted to the non-degeneracy assumptions associated to approximate solutions $K$ of (6). We first deal with the spectral non degeneracy conditions. The crucial quantity is the linearization equation around a map $K$ given by

$$\frac{d\Delta}{dt} = A(\theta + \omega t)\Delta,$$

where $A(\theta) = D(X \circ K)(\theta)$ is an operator mapping $X$ into $Y$.

Roughly, we want to assume that there is a splitting of the space into directions on which the evolution can be defined either forwards or backwards and that the evolutions thus defined are smoothing. We anticipate that in Section 6, we will present
other conditions that imply Definition 3.3. We will just need to assume approximate versions of the invariance.

**Definition 3.3. Spectral non degeneracy**

We will say that an embedding \( K : D_\rho \to X \) is spectrally non degenerate if for every \( \theta \) in \( D_\rho \), we can find a splitting

\[
X = X^{s}_\theta \oplus X^{c}_\theta \oplus X^{u}_\theta
\]

with associated bounded projection \( \Pi^{s,c,u}_\theta \in \mathcal{L}(X, X) \) and where \( X^{s,c,u}_\theta \) are in such a way that:

- **SD1** The mappings \( \theta \to \Pi^{s,c,u}_\theta \) are in \( A_{\rho,\mathcal{L}(X,X)} \) (in particular, analytic).
- **SD2** The space \( X^{c}_\theta \) is finite dimensional with dimension \( 2\ell \). Furthermore the restriction of the operator \( J \) to \( X^{c}_\theta \) denoted \( J^{c}_\theta \) induces a symplectic form on \( X^{c}_\theta \) which is preserved by the evolution on \( X^{c}_\theta \) (see below).
- **SD3** We can find families of operators

\[
\begin{align*}
U^{s}_\theta(t) : Y^{s}_\theta &\to X^{s}_{\theta + \omega t}, & t > 0 \\
U^{u}_\theta(t) : Y^{u}_\theta &\to X^{u}_{\theta + \omega t}, & t < 0 \\
U^{c}_\theta(t) : Y^{c}_\theta &\to X^{c}_{\theta + \omega t}, & t \in \mathbb{R}
\end{align*}
\]

such that:

- **SD3.1** The families \( U^{s,c,u}_\theta \) are cocycles over the rotation of angle \( \omega \) (cocycles are the natural generalization of semigroups for non-autonomous systems)

\[
U^{s,c,u}_\theta(t')U^{s,c,u}_\theta(t) = U^{s,c,u}_\theta(t + t')
\]

- **SD3.2** The operators \( U^{s,c,u}_\theta \) are smoothing in the time direction where they can be defined and they satisfy assumptions in the quantitative rates. There exist \( \alpha_1, \alpha_2 \in [0, 1), \beta_1, \beta_2, \beta^+_3, \beta^-_3 > 0 \) and \( C_h > 0 \) independent of \( \theta \) such that the evolution operators are characterized by the following rate conditions:

\[
\begin{align*}
\|U^{s}_\theta(t)\|_{\rho,Y,X} &\leq C_h e^{-\beta_1 t} t^{-\alpha_1}, & t > 0, \\
\|U^{u}_\theta(t)\|_{\rho,Y,X} &\leq C_h e^{\beta_2 |t|} t^{-\alpha_2}, & t < 0, \\
\|U^{c}_\theta(t)\|_{\rho,X,X} &\leq C_h e^{\beta^+_3 t}, & t > 0 \]
\]

\[
\|U^{c}_\theta(t)\|_{\rho,X,X} \leq C_h e^{\beta^-_3 |t|}, & t < 0
\]

with \( \beta_1 > \beta^+_3 \) and \( \beta_2 > \beta^-_3 \).
The operators $U^{s,u,c}_\theta$ are fundamental solutions of the variational equations in the sense that

$$U^s_\theta(t) = \text{Id} + \int_0^t A(\theta - \omega \sigma)U^s_{\theta-\omega \sigma}(\sigma) \, d\sigma \quad t > 0$$

$$U^u_\theta(t) = \text{Id} + \int_0^t A(\theta - \omega \sigma)U^u_{\theta-\omega \sigma}(\sigma) \, d\sigma \quad t < 0$$

$$U^c_\theta(t) = \text{Id} + \int_0^t A(\theta - \omega \sigma)U^c_{\theta-\omega \sigma}(\sigma) \, d\sigma \quad t \in \mathbb{R}$$

(22)

**Remark 3.7.** Note that as consequence of the integral equations and the rate conditions (19), (20), (21) we have, using just the triangle inequality

$$||U^s_\theta(t)||_{\rho,Y,Y} \leq 1 + \int_0^t A_s e^{-\beta_1 s} ds$$

Proceeding similarly for the others, we obtain

$$||U^s_\theta(t)||_{\rho,Y,Y} \leq \tilde{C}_he^{-\beta_1 t}, \quad t > 0,$$

$$||U^u_\theta(t)||_{\rho,Y,Y} \leq \tilde{C}_he^{\beta_2 t}, \quad t < 0,$$

$$||U^c_\theta(t)||_{\rho,Y,Y} \leq \tilde{C}_he^{\beta_3 |t|}, \quad t > 0$$

(23)

$$||U^c_\theta(t)||_{\rho,Y,Y} \leq \tilde{C}_he^{\beta_3 |t|}, \quad t < 0$$

**Remark 3.8.** We are not aware of any general argument that would show that:

$$||U^s_\theta(t)||_{\rho,X,X} \leq \tilde{C}_he^{-\beta_1 t}, \quad t > 0,$$

$$||U^u_\theta(t)||_{\rho,X,X} \leq \tilde{C}_he^{\beta_2 t}, \quad t < 0,$$

$$||U^c_\theta(t)||_{\rho,X,X} \leq \tilde{C}_he^{\beta_3 t}, \quad t > 0$$

$$||U^c_\theta(t)||_{\rho,X,X} \leq \tilde{C}_he^{\beta_3 |t|}, \quad t < 0$$

(24)

follow from the other assumptions. Needless to say, we would be happy to hear about one.

One can, however, clearly have that since $||U^s_\theta(t)||_{X,X} \leq ||U^s_\theta(t)||_{Y,X}$ so that the semigroups are exponentially decreasing for large $t$.

One notable case, which happens in practice, when one can deduce (24) is when the spaces $X$ and $Y$ are Hilbert spaces. In such a case, taking Hilbert space adjoints in (3.7) we obtain:

$$U^s_\theta(t)^* = \text{Id} + \int_0^t U^s_{\theta-\omega \sigma}(\sigma)^* A(\theta - \omega \sigma)^* \, d\sigma \quad t > 0$$

and using the fact that the adjoints preserve the norm, we can easily obtain the bounds in the same way as (23).
Remark 3.9. We remark that when the equation preserves a symplectic structure, we can have without loss of generality

\[ \beta^+_3 = \beta^-_3, \quad \beta_1 = \beta_2. \]

Conversely, if (25) is satisfied, the center direction automatically preserves a symplectic structure. See Lemma 7.3.

We anticipate that the results in Section 6 on persistence of trichotomies (a fortiori dichotomies) with smoothing are developed without assuming that the equation is Hamiltonian and, hence apply also to dissipative equations. Similarly, the solutions of linearized equations in the hyperbolic directions developed in Section 5 are obtained without using the Hamiltonian structure. The Hamiltonian structure is used only to deal with the linearized equations in the center direction in Section 7.

Let us comment on the previous spectral non-degeneracy conditions.

The first observation is that, if we assume that the spaces \( X, Y \) are Sobolev spaces of high enough index (so that the functions in them are \( C^r \) for \( r \) high enough) then we have that (22) holds in a classical sense if it holds in the sense of mild solutions (the sense of integral equations). In the applications we have in mind, it is always possible to take the spaces \( X, Y \) that have arbitrarily high derivatives.

Then, (22) is just a form of

\[
\begin{align*}
\frac{d}{dt} U^s_\theta(t) &= A(\theta + \omega t)U^s_\theta(t) \quad t > 0 \\
\frac{d}{dt} U^u_\theta(t) &= A(\theta + \omega t)U^u_\theta(t) \quad t < 0 \\
\frac{d}{dt} U^c_\theta(t) &= A(\theta + \omega t)U^c_\theta(t) \quad t \in \mathbb{R}
\end{align*}
\]

Often (22) is described as saying that the derivatives in (26) are understood in the mild sense.

Making sense of the integrals in (22) is immediate after some reflection. Our conditions just require the existence of an evolution for positive and negative times on certain subspaces. The important conditions on these evolutions are the characterization of the splitting by rates (19)-(20), expressing the fact that the operators are bounded and smoothing from \( Y \) into \( X \) (recall that \( X \hookrightarrow Y \)). If the system were autonomous, such properties would hold under some spectral assumptions on the operator \( A(\theta) \) (bisectoriality or generation of strongly continuous semi-groups, see [Paz83]).

Since the spaces \( X^c_\theta \) and \( Y^c_\theta \) are finite dimensional and of the same dimension, the evolution \( U^c_\theta(t) \) can be considered as an operator from \( Y^c_\theta \) to \( Y^c_{\theta + t\omega} \).

In the finite dimensional case (or in the cases where there is a well defined evolution), property SD.1 follows from the contraction rates assumption SD.3 by a fixed point argument in spaces of analytic functions. See [Hall06]. In our case, we have not been able to adapt the finite dimensional argument, that is why we have included it as an independent assumption (even if may end up be redundant). We note that SD.1, SD.3 are clearly true when \( N \equiv 0 \) and in this paper we will show it is stable.
under perturbations, hence SD.3 will hold for all small enough \( u \). This suffices for our purposes, so we will not pursue the question of whether SD.1 can be obtained from SD.3 in general.

The fact that \( \Omega|_{X_0^c} \) is non-dengenerate (which is a part of SD.2) follows from the rate conditions SD.3 as we show in Lemma 7.3.

One situation when all the above abstract properties are satisfied is when the evolution is given just by the linear part \( \mathcal{A} \), i.e. \( \mathcal{N} \equiv 0 \). The assumptions of our set up are verified if the spectrum of \( \mathcal{A} \) is just eigenvalues of finite multiplicity and the spectrum is the union of a sector around the positive axis, another sector around the negative axis and a finite set of eigenvalues of finite multiplicity around the imaginary axis. Then, the stable space is the spectral projection over the sector in the negative real axis, the unstable space will be the spectral projection over the sector along the positive axis and the center directions will be the spectral space associated to the eigenvalues in the finite set. There are many examples of linear operators satisfying these properties.

It will be important that the main result of Section 6 is these structures persist when we add a lower order perturbation which is small enough. Indeed, we will show that if we find splittings that satisfy them approximately enough, there is true splitting nearby. This would allow to validate numerical computations, formal expansions, etc.

3.5.1. The twist condition. As it is standard in KAM theory, one has to impose another non-degeneracy assumption, namely the twist condition. This is the object of the next definition. Notice that it amounts to a finite dimensional matrix being invertible. It is identical to the conditions that were used in the finite dimensional cases \([\text{dlLGJV05, PdlLS09a}]\).

**Definition 3.4.** Denote \( N(\theta) \) the \( \ell \times \ell \) matrix such that \( N(\theta)^{-1} = DK(\theta)^\perp DK(\theta) \).

Denote \( P(\theta) = DK(\theta)N(\theta) \)

Let \( J_c \) stand for restriction of symplectic operator \( J \) to \( X_0^c \). We will show in Lemma 7.3 that the form \( \Omega_c \equiv \Omega|_{X_0^c} \) is non-degenerate so that the operator \( J_c \) is invertible.

We now define the twist matrix \( S(\theta) \) (the motivation will become apparent in Section 7, but it is identical to the definition in the finite dimensional case in \([\text{dlLGJV05, PdlLS09a}]\)). The average of the matrix

\[
(27) \quad S(\theta) = N(\theta)DK(\theta)^\perp [J_c^{-1} \partial_\omega(DK N) - AJ_c^{-1}(DK N)](\theta)
\]

is non-singular.

We note that the matrix \( S \) in (27) is a very explicit expression that can be computed out of the approximate solution of the invariant equation and the invariant bundles just taking derivatives, projections and performing algebraic operations. So that it is easy to verify in applications when we are given an approximate solution.

We will say that an embedding is non-degenerate (and we denote it \( K \in ND(\rho) \)) if it is non-degenerate in the sense of Definitions 3.3 and 3.4.
Remark 3.10. As it will become apparent in the proof, the twist condition has a very clear geometric meaning, namely that the frequency of the quasiperiodic motions changes when we change the initial conditions in a direction (conjugate to the tangent to the torus).

Note that, given an invariant torus, we can consider it as an approximate solution for similar frequencies and that the twist condition also holds.

Using the a-posteriori theorem shows that under the conditions, we have many tori with similar frequencies near to the torus.

3.5.2. Description of the iterative step. Once the two non-degeneracy conditions are met for the initial guess of the modified Newton method, the iterative step goes as follows:

(1) We project the cohomological equations with respect to the invariant splitting.
(2) We then solve the equations for the stable and unstable subspaces.
(3) We then solve the equation on the center subspace. This involves small divisor equations. We note that solving the equation in the center requires to use the exactness so that we can show that the equations are solvable.
(4) To be able to iterate we will need to show that the corrections also satisfy the non-degeneracy conditions (with only some slightly worse quantitative assumptions). This amounts to showing the stability of the spectral non-degeneracy conditions, and developing explicit estimates of the changes in the properties given the changes on the embedding.

3.6. Statement of the results.

3.6.1. General abstract results. The following Theorem 3.5 is the main result of this paper. It provides the existence of an embedding $K$ for equation (5) under some non-degeneracy conditions for the initial guess. We stress here that Theorem 3.5 is in an a posteriori format (an approximate solution satisfying nondegeneracy conditions implies the existence of a true solution close to it). As already pointed out in the papers [FdlLS09b, FdlLS09a, FdlLS15], this format allows to validate many methods that construct approximate solutions, including asymptotic expansions or numerical solutions. We also note that it has several automatic consequences presented in Section 3.6.2.

Theorem 3.5. Suppose assumptions $H_1, H_2, H_3$ are met; let $\omega \in D(\kappa, \nu)$ for some $\kappa > 0$ and $\nu \geq \ell - 1$. Assume that

- $K_0$ satisfies the non-degeneracy Conditions $5.3$ and $5.4$ for some $\rho_0 > 0$.
- We assume that the range of $K_0$ acting on a complex extension of the torus is well inside of $U$ the domain of analyticity of $N$ introduced in $H_2$. More precisely:

$$\text{dist}_X(K_0(D_{\rho}), X \setminus U) \geq r > 0$$

That is, if $x = K_0(\theta), \theta \in D_{\rho_0}$ and $\|x - y\|_X \leq \rho_0$, then $y \in U$. 
Define the initial error
\[ E_0 = \partial_\omega K_0 - X \circ K_0 \]
Then there exists a constant \( C > 0 \) depending on \( l, \nu, \rho_0, \|X\|_{C^1(B_r)}, \|DK_0\|_{\rho_0, X}, \|N_0\|_{\rho_0}, \|S_0\|_{\rho_0}, \|\Pi_{K_0(\theta)}^{c,s,u}\|_{\rho_0, Y,Y} \) such that, if \( E_0 \) satisfies the estimates
\[ C|\text{avg} (S_0)^{-1}2\kappa^4\delta^{-4\nu}\|E_0\|_{\rho_0, Y} < 1 \]
and
\[ C|\text{avg} (S_0)^{-1}2\kappa^2\delta^{-2\nu}\|E_0\|_{\rho_0, Y} < r, \]
where \( 0 < \delta \leq \min(1, \rho_0/12) \) is fixed, then there exists an embedding \( K_\infty \in ND(\rho_\infty := \rho_0 - 6\delta) \) such that
\[ \partial_\omega K_\infty(\theta) = X \circ K_\infty(\theta). \]
Furthermore, we have the estimate
\[ \|K_\infty - K_0\|_{\rho_\infty, X} \leq C|\text{avg} (S_0)^{-1}2\kappa^2\delta^{-2\nu}\|E_0\|_{\rho_0, Y}. \]
The torus \( K_\infty \) is also spectrally non degenerate in the sense of Definition 3.3 with \( \rho \) in Definition 3.3 replaced by \( \rho_\infty \) and with other constants differing from those of \( K_0 \) modifying by an amount bounded by \( C\|E_0\|_{\rho_0} \).
Furthermore, if we have two solutions \( K_1, K_2 \) satisfying (5) and spectrally nondegenerate in the sense of Definition 3.3 and that satisfy
\[ \|K_1 - K_2\|_{\rho_\infty, X} \leq C|\text{avg} (S_0)^{-1}2\kappa^2\delta^{-2\nu}\]
Then, there exists \( \sigma \in \mathbb{R}^\ell \) such that
\[ K_1(\theta) = K_2(\theta + \sigma) \]
The statement that \( K_\infty \) satisfies the Definition 3.3 is a consequence of the estimates in Section 6.
The uniqueness statement will be proved in Section 8. It is exactly the same as the one in the finite dimensional case in [FdlLS09a].

3.6.2. Some consequences of the a-posteriori format. The a-posteriori format leads immediately to several consequences. When we have systems that depend on parameters, observing that the solution for a value of the parameter is an approximate solution for similar values of the parameters, one obtains Lipschitz dependence on parameters, including the frequency. If one can obtain Lindstedt expansions in the parameters, one can obtain Taylor expansions. If the parameter ranges over \( \mathbb{R}^n \), this is the hypothesis of the converse Taylor theorem [AR67, Nel69] so that one obtains smooth dependence on parameters. In the case that the parameters range on a closed set, we obtain one of the conditions of the Whitney extension theorem. Some general treatments are [Van02, CCdlL15].

In many perturbative solutions, one gets that the twist condition is small but that the error is much smaller. Note that in the main result, we presented explicitly that the smallness conditions on the error are proportional to the square of the twist condition.
Hence, we obtain the *small twist condition*. Note also that the twist condition required is not a global condition on the map, but rather a condition that is computed on the approximate solution. Indeed, we will take advantage of this feature in the sections on applications.

The abstract theorem can be applied to several spaces. Some spaces of low regularity (e.g. $H^m$) and others with high regularity (e.g. analytic). The existence results are more powerful in the high regularity spaces and the local uniqueness is more powerful in the low regularity spaces.

Given a sufficiently regular solution, one can obtain an analytic approximate solution by truncating the Fourier series, which leads to an analytic solution, which has to be the original one. Hence, one can *bootstrap the regularity*. See [CdlL10a] for an abstract version.

3.6.3. Results for concrete equations. Consider the following one-dimensional Boussinesq equation subject to periodic boundary conditions, i.e.

\[ u_{tt} = \mu u_{xxxx} + u_{xx} + (u^2)_{xx}, \quad x \in \mathbb{T}, \ t \in \mathbb{R}. \]

Looking for solutions of the linearization of the form \( u(x, t) = e^{2\pi i (kx + \omega(k)t)} \) we obtain the eigenvalue relation

\[ \omega^2(k) = -\mu |k|^4 (2\pi)^2 + |k|^2 \]

We see that for large \( |k| \), \( \omega(k) \approx \pm 2i\pi \mu^{1/2} |k|^2 \). Hence, the Fourier modes may grow at an exponential rate and the rate is quadratic in the index of the mode. So that even analytic functions evolving under the linearized equation leave instaneously even spaces of distributions. The non-linear term does not restore the well posedness. (See Remark 10.1.) The previous equation (32) is Hamiltonian on $L^2(\mathbb{T})$. Indeed, we introduce first the skew-symmetric operator

\[
J^{-1} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}
\]

and define

\[ H_\mu(u, v) = \int_0^1 \frac{1}{2} \left\{ u^2 + v^2 - \mu (\partial_x u)^2 \right\} + \frac{1}{3} u^3. \]

Therefore, equation (32) writes

\[ \dot{z} = J^{-1} \nabla H_\mu(z), \ z = (u, v) \]

where \( \nabla \) has to be understood w.r.t. the inner product in $L^2(\mathbb{T})$. Note, however that when \( \mu \) is small enough, there are several values of \( k \), which for which \( \omega(k) \) is real. We denote by \( \omega^0 \) the vector whose components are all the real frequencies that appear

\[ \omega^0 = (\omega(k_1), \omega(k_2), \ldots, \omega(k_\ell)); \]

\[ \{k_1, \ldots k_\ell\} = \{ k \in \mathbb{Z} \mid k > 0; -\mu |k|^4 (2\pi)^2 + |k|^2 \geq 0 \} \]
We can think of $\omega^0$ as the frequency vector of the motions for very small amplitude.

Note that the equation (32) conserves the quantity $\int_0^1 \partial_t u(t, x) dx$ (called the momentum). Hence $\int_0^1 u(t, x) dx$ (the center of mass) evolves linearly in time.

We can always change to a system of coordinates in which $\int_0^1 \partial_t u(t, x) dx = 0$. Hence, in this system $\int_0^1 u(t, x) dx = cte$. By adding the constant we can assume without loss of generality that $\int_0^1 u(t, x) dx = 0$.

Hence we will assume (without loss of generality) that

\[ \int_0^1 \partial_t u(t, x) dx = 0 \]
\[ \int_0^1 u(t, x) dx = 0 \]

Remark 3.11. We emphasize that the two parts of (36) are not two independent equations. The first one is just a derivative with respect to time of the second. Even if the relation is formal, it makes sense when we are dealing with polynomial approximate solutions.

We also note that the equation (32) leaves invariant the space of functions which are symmetric around $x$ (it does not leave invariant the space of functions antisymmetric around $x$). Hence, we can consider the equation as defined on the space of general functions or in the space of symmetric functions.

\[ u(t, x) = u(t, -x) \]

The main difference between the symmetric and the general case is that center space is of different dimension.

We introduce the following Sobolev-type spaces $H^\rho,m(\mathbb{T})$ for $\rho > 0$ and $m \in \mathbb{N}$ being the space of analytic functions $f$ in $D_\rho$ such that the quantity

\[ \|f\|_{\rho,m}^2 = \sum_{k \in \mathbb{Z}} |f_k|^2 e^{4\pi \rho |k|} (|k|^{2m} + 1) \]

is finite, and where $\{f_k\}_{k \in \mathbb{Z}}$ are the Fourier coefficients of $f$. Let

\[ X = H^\rho,m(\mathbb{T}) \times H^{\rho,m-2}(\mathbb{T}) \]

for $m \geq 2$.

We state the following conjecture.

Conjecture 3.6. Consider a parameter $\mu > 0$ in (32) such that the center space has dimension $2\ell \geq 2$ and fix a Diophantine exponent $\nu > \ell$, a regularity exponent $m > 5/2$ and a positive analyticity radius $\rho_0$.

Then there exist three explicit functions $a, b_d, b_a : \mathbb{R}^+ \to \mathbb{R}^+$ such that

\[ a(s) \to 0, b_d(s), b_a(s) \to \infty, \ s \to 0 \]
in such a way that: for \( \varepsilon \) sufficiently small, denote by \( B_{a(\varepsilon)}(\omega^0) \subset \mathbb{R}^\ell \) the ball of radius \( a(\varepsilon) \) around \( \omega^0 \) and let \( \omega \in \mathcal{D}(b(\varepsilon), \nu) \cap B_{a(\varepsilon)}(\omega^0) \).

Then, there exists \( K_1 \), an analytic function from \( D_\rho \rightarrow H^{\rho,m}(\mathbb{T}) \times H^{\rho,m-2}(\mathbb{T}) \) solving \( (5) \) with frequency \( \omega \).

Furthermore,

\[
\frac{|\mathcal{D}(b(\varepsilon), \nu) \cap B_{a(\varepsilon)}(\omega^0)|}{|B_{a(\varepsilon)}(\omega^0)|} \rightarrow 1
\]

The mapping that given \( \omega \) produces \( K_1 \) is Lipschitz when \( K_1 \) are given the topology of analytic embeddings from \( D_\rho \) to \( X \) when \( \rho < \rho_0 \).

In Section 10.5 we present a complete proof of the following result.

**Theorem 3.7.** Conjecture 3.6 is true under the extra assumption that \( \ell = 1 \) which amounts to take \( \mu \in \left[ \frac{1}{8\pi}, \frac{1}{2\pi} \right) \).

Informally, following the standard Lindstedt procedure, for \( \varepsilon \) small we find families of approximate solutions up to an error which is smaller than an arbitrarily large power of \( \varepsilon \).

We can also verify that the non-degeneracy assumptions hold with a condition number which is a fixed power of \( \varepsilon \). If \( \varepsilon \) is very small one can allow frequencies with a large Diophantine constant, and obtain that the functions are analytic in a very large domain. As we will see in the proof, we can take the functions \( a, b, c, b_a \) to be just powers.

The first step of constructing very approximate solutions is accomplished for all values of \( \mu \) as in Conjecture 3.6.

To verify the non-degeneracy conditions, it suffices to compute the determinant of an explicit matrix and checking it is not zero. This is the only step we are missing to verify Conjecture 3.6. This calculation is, not very hard, but it is tedious. Of course, there may be insights that make it possible to verify it. In the present paper, we will concentrate on \( \ell = 1 \) to check this condition.

**Remark 3.12.** We expect that Theorem 3.7 can be greatly expanded (a wider range of parameters, removing the symmetry conditions) by just performing longer calculations using the Lindstedt method. We hope to come back to this problem in future work.

**Remark 3.13.** Note that the case \( \ell = 1 \), amounts to periodic orbits so that there are no small denominators. In this case, one can use simpler fixed point theorems. There are already numerical computer assisted proofs in this case \[CGL15, FGdlLL16\].

Similar results will be proved for other equations such as the Boussinesq system of water waves (see Section 11). The system under consideration is

\[
(39) \quad \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x - \mu \partial_{xxx} \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \partial_x (uv) \\ 0 \end{pmatrix}
\]

where \( t > 0 \) and \( x \in \mathbb{T} \). System (39) has a Hamiltonian structure given by:
\[ J^{-1} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \]

and

\[ H_\mu(u, v) = \int_0^1 \frac{1}{2} \left\{ u^2 + v^2 - \mu \left( \partial_x v \right)^2 \right\} + \int_0^1 uv. \]

In this case, one has to take

\[ X = H^{0, m}(\mathbb{T}) \times H^{0, m+1}(\mathbb{T}) \]

and

\[ Y = H^{0, m-1}(\mathbb{T}) \times H^{0, m}(\mathbb{T}). \]

The elementary linear analysis around the \((0, 0)\) equilibrium has been performed in [LL09]. The dispersion relation is given by

\[ \omega(k) = \pm |k| 2\pi i \sqrt{1 - 4\pi^2 \mu k^2} \quad k \in \mathbb{Z} \]  

(40)

We take the principal determination of the square root. We denote by \(\omega^0\) the vector whose components are all the real frequencies that appear

\[ \omega^0 = (\omega(k_1), \omega(k_2), \ldots, \omega(k_\ell)); \]

\[ \{k_1, \ldots, k_\ell\} = \{k \in \mathbb{Z} | k > 0; 1 - 4\pi^2 \mu k^2 \geq 0\} \]  

(41)

Similarly to the Boussinesq equation, we state

**Conjecture 3.8.** Fix a Diophantine exponent \(\nu > \ell\) and a regularity exponent \(m\) large. Then there exist three explicit functions \(a, b, d : \mathbb{R}^+ \to \mathbb{R}^+\) such that

\[ a(s) \to 0, b_d(s), b_a(s) \to \infty, \quad s \to 0 \]

in such a way that: for \(\varepsilon\) sufficiently small, denote by \(B_{a(\varepsilon)}(\omega^0) \subset \mathbb{R}^\ell\) the ball of radius \(a(\varepsilon)\) around \(\omega^0\) and let \(\omega \in \mathcal{D}(b(\varepsilon), \nu) \cap B_{a(\varepsilon)}(\omega^0)\).

Then, there exists \(K \in X\) solving (4) with frequency \(\omega\), the parametrization of the Boussinesq system for water waves [104].

Furthermore,

\[ \frac{|B_{b(\varepsilon)}(\omega^0) \cap B_{a(\varepsilon)}(\omega^0)|}{|B_{a(\varepsilon)}(\omega^0)|} \to 1 \]

**Theorem 3.9.** Conjecture (3.8) is true provided that \(\ell = 1\), i.e. \(\mu \in \left[\frac{1}{8\pi}, \frac{1}{2\pi}\right]\).

4. The linearized invariance equation

The crucial ingredient of the Newton method is to solve the linearized operator around an embedding \(K\). This is motivated because one can hope to improve the solution of (2). Notice the appearance of the linearized evolution does not have a dynamical motivation. The linearized equation does not appear as measuring the change of the evolution with respect to the initial conditions, it appears as the linearization of (2).
Let us denote
\begin{equation}
\mathcal{F}_\omega(K) = \partial_\omega K - \mathcal{X} \circ K.
\end{equation}
Clearly, the invariance equation (5) can be written concisely as \( \mathcal{F}_\omega(K) = 0 \).
We prove the following result.

**Lemma 4.1.** Consider the linearized equation
\begin{equation}
D\mathcal{F}_\omega(K)\Delta = -E.
\end{equation}
Then there exists a constant \( C \) that depends on \( \nu, l, \|DK\|_{\rho,X}, \|N\|_{\rho}, \|\Pi^{s,c,u}_\theta\|_{\rho,Y,Y}, \|(\text{avg } S)^{-1}\| \) and the hyperbolicity constants such that assuming that \( \delta \in (0, \rho/2) \) satisfies
\begin{equation}
C\kappa\delta^{-(\nu+1)}\|E\|_{\rho,Y} < 1
\end{equation}
we have

**A** There exists an approximate solution \( \Delta \) of (43), in the following sense: there exists a function \( \tilde{E}(\theta) \) such that \( \Delta \) solves exactly
\begin{equation}
D_K\mathcal{F}_\omega(K)\Delta = -E + \tilde{E}
\end{equation}
with the following estimates: for all \( \delta \in (0, \rho/2) \)
\begin{equation}
\|\tilde{E}\|_{\rho-\delta,Y} \leq C\kappa^2\delta^{-(2\nu+1)}\|E\|_{\rho}\|\mathcal{F}_\omega(K)\|_{\rho,Y}
\end{equation}
\begin{equation}
\|\Delta\|_{\rho-2\delta,X} \leq C\kappa^2\delta^{-2\nu}\|E\|_{\rho,Y},
\end{equation}
\begin{equation}
\|D\Delta\|_{\rho-2\delta,X} \leq C\kappa^2\delta^{-2\nu-1}\|E\|_{\rho,Y},
\end{equation}

**B** If \( \Delta_1 \) and \( \Delta_2 \) are approximate solutions of the linearized equation (43) in the sense of (45), then there exists \( \alpha \in \mathbb{R}^\ell \) such that for all \( \delta \in (0, \rho) \)
\begin{equation}
\|\Delta_1 - \Delta_2 - DK(\theta)\alpha\|_{\rho-\delta,X} \leq C\kappa^2\delta^{-(2\nu+1)}\|E\|_{\rho,Y}\|\mathcal{F}_\omega(K)\|_{\rho,Y}.
\end{equation}

The previous Lemma 4.1 is the cornerstone of the KAM iteration and the goal of the following sections is to prove this result. We will also need to prove that the non-degeneracy conditions are preserved under the iteration and that the constants measuring the non-degeneracy deteriorate only slightly. This will follow from the quantitative estimates developed in Section 6.

Note that (45), (46) is the main ingredient of several abstract implicit function theorems which lead to the existence of a solution. See, for example [Zeh75] or, particularly [CdLL10b] Appendix A] for implicit function theorems based on existence of approximate inverses with tame bounds.

Note also that in part (2) of Lemma 4.1 we have established some uniqueness for the solutions of the linearized equation. In Section 8 we will show how this can be used to prove rather directly the uniqueness result in Theorem 4.1.

The proof of Lemma 4.1 is based on decomposing the equation into equations along the invariant bundles assumed to exist in the hypothesis that the approximate solution satisfies Definition 3.3. In the hyperbolic directions we will roughly use the variations
of parameters formula, but we will have to deal with the fact that the perturbations are unbounded. In the center directions, we will have to use the number theory and the geometry. Fortunately, the center space is finite dimensional.

The theory of solutions of the linearized equation is developed in Sections 5 and 7 and Lemma 4.1 is obtained just putting together Lemma 5.1 and the results in Section 7.4.

We also note that the solutions of the linearized equation in the hyperbolic directions will be important in the perturbattion theory of the bundles, which is needed to show that the linearized equation can be applied repeatedly.

For coherence of the presentation, we have written together all the results requiring hyperbolic technology (the solution in the hyperbolic directions and the perturbation theory of bundles). Of course, we hope that the sections can be read independently in the order prefered by the reader.

5. Solutions of linearized equations on the stable and unstable directions

In this Section we develop the study of linearized equations of a system with splitting. See Lemma 5.1. Lemma 5.1 will be one of the ingredients in Lemma 4.1.

**Lemma 5.1.** Let $L(\theta) : X \to Y$ for fixed $\theta \in D_\rho$ be a vector field admitting an invariant splitting in the following sense: the space $X$ has an analytic family of splittings $X = X^s_\theta \oplus X^c_\theta \oplus X^u_\theta$ (We say that a splitting is analytic when the associated projections depend on $\theta$ in an analytic way) invariant in the following sense: we can find families of operators $\{U^s_\theta(t)\}_{t>0}$, $\{U^c_\theta(t)\}_{t<0}$, $\{U^u_\theta(t)\}_{t<0}$ with domains $X^s_\theta$, $X^c_\theta$, $X^u_\theta$ respectively. These families are analytic in $\theta, t$ when considered as operators satisfying

$$U^s,c,u_\theta(t)X^{s,c,u}_\theta = X^{s,c,u}_{\theta+\omega t}.$$

Let $\Pi^{s,c,u}_\theta$ the projections associated to this splitting. Assume furthermore there exist $\beta_1, \beta_2, \beta_3^\pm > 0$, $\alpha_1, \alpha_2 \in (0,1)$ and $C_h > 0$ independent of $\theta \in D_\rho$ satisfying $\beta_3^+ < \beta_1$, $\beta_3^- < \beta_2$ and such that the splitting is characterized by the following rate conditions:

$$\|U^s_\theta(t)\|_{\rho,Y,X} \leq C_h e^{-\beta_1 t \alpha_1}, \quad t > 0,$$
$$\|U^u_\theta(t)\|_{\rho,Y,X} \leq C_h e^{\beta_2 t \alpha_2}, \quad t < 0,$$
$$\|U^c_\theta(t)\|_{\rho,X,X} \leq C_h e^{\beta_3^+ t}, \quad t > 0,$$
$$\|U^c_\theta(t)\|_{\rho,X,X} \leq C_h e^{\beta_3^- t}, \quad t < 0.$$

Let $F^{s,u} \in A_\rho,Y$ taking values in $Y^s$ (resp. $Y^u$). Consider the equations

$$\partial_\omega \Delta^{u,s}(\theta) - L(\theta)\Delta^{u,s}(\theta) = F^{u,s}(\theta).$$
Then there are unique bounded solutions for (50) which are given by the following formulas:

\[(51) \Delta^s(\theta) = \int_0^{\infty} U_{\theta - \omega \tau}^s(\tau) F^s(\theta - \omega \tau) \, d\tau.\]

and

\[(52) \Delta^u(\theta) = \int_{-\infty}^{0} U_{\theta - \omega \tau}^u(\tau) F^u(\theta - \omega \tau) \, d\tau.\]

Furthermore, the following estimates hold

\[\|\Delta^{s,u}\|_{\rho,X^s,u} \leq C \|\Pi_{\rho}^{s,u}\|_{\rho,Y,Y} \|F\|_{\rho,Y} s\theta.\]

**Remark 5.1.** The assumptions of the previous Lemma are very similar to the standard setup of the theory of dichotomies, but we have to take care of the fact that the evolution operators are smoothing and the perturbations unbounded.

**Proof.** The proof is based on the integration of the equation along the characteristics by using \(\theta + \omega t\). We give the proof for the stable case, the unstable case being symmetric (for negative times). Furthermore, the proof is similar to the one in [FdlLS09a] up to some modifications of the functional spaces. Denote \(\tilde{\Delta}^s(t) = \Delta^s(\theta + \omega t)\). By the variation of parameters formula (Duhamel formula), which is valid in the mild solutions context (see [Paz83]) one has

\[(53) \tilde{\Delta}^s(t) = U_{\theta}(t)\tilde{\Delta}^s(0) + \int_0^t U_{\theta}(t - z) F^s(\theta + \omega z) \, dz.\]

Since the previous formula is valid for all \(\theta \in D_{\rho} \supset \mathbb{T}^d\) we can use it substituting \(\theta\) by \(\theta - \omega t\) and then

\[\Delta^s(\theta) = U_{\theta - \omega t}^s(t)\Delta^s(\theta - \omega t) + \int_0^t U_{\theta - \omega(t - z)}^s(t - z) F^s(\theta - \omega(t - z)) \, dz.\]

By the previous bounds on the semi-group, we have that \(U_{\theta - \omega t}(t)\Delta^s(\theta - \omega t)\) goes to 0 when \(t\) goes to \(\infty\).

Hence this leads to the following representation formula after replacing \(t - z\) by \(\tau > 0\) in the integral

\[(54) \Delta^s(\theta) = \int_0^{\infty} U_{\theta - \omega \tau}(\tau) F^s(\theta - \omega \tau) \, d\tau.\]

Furthermore, from the previous formula, one has that \(\Delta^s\) is analytic in \(\theta\).

We now estimate the integral in (54) to show that it converges and to establish bounds on it. Notice that the operator \(U_{\theta}(t)\) maps \(Y_{\theta}^s\) into \(X_{\theta}^s\) continuously and the following estimate holds for every \(\theta \in D_{\rho}\) and every \(t > 0\)

\[\|U_{\theta}(t) F^s(\theta)\|_{X_{\theta}^s} \leq \frac{C}{t^{\alpha_1}} e^{-\beta_1 t} \|F^s(\theta)\|_{Y_{\theta}^s}.\]
The exponential bound in SD3.2 ensures the convergence at infinity of the integral and the fact that $\alpha_1 \in (0, 1)$ ensures the convergence at 0 and one gets easily the desired bound.

The unstable case can be obtained by reversing the direction of time or given a direct proof which is identical to the present one. □

6. Perturbation theory of hyperbolic bundles in an infinite-dimensional framework

In this section we develop a perturbation theory for hyperbolic bundles and their smoothing properties. We consider a slightly more general framework than the one introduced in the previous sections since we hope that the results in this section could be useful for other problems (e.g. in dissipative PDE’s). In particular, we note that we only assume that the spaces $X$ and $Y$ are Banach spaces. Also, we do not need to assume that the (unbounded) vector field $\mathcal{X}$ giving the equation is Hamiltonian. In agreement with previous results, we note that we do not assume that the equations define an evolution for all initial conditions. We only assume that we can define evolutions in the future (or on the past) of the linearization in some spaces. This is obvious for the linear operator and in this section, we will show that this is persistent under small perturbations.

The theory of perturbations of bundles for evolutions in infinite dimensional spaces has a long history. See for example [Hen81, PS99]. A treatment of partial differential equations has already been considered in the literature. For example in [CL95, CL96].

Our treatment has several important differences with the above mentioned works. Among them: 1) We study the stability of smoothing properties, 2) We take advantage of the fact that the dynamics in the base is a rotation, so that we obtain results in the analytic category, which are false when the dynamics in the base is more complicated. 3) We present our main results in an a-posteriori format, which, of course, implies the standard persistence results but has other applications such as validating numerical or asymptotic results. 4) We present very quantitative estimates of the changes of the splitting and its merit figures under perturbations. This is needed for our applications since we use it as an ingredient of an iterative process and we need to show that it converges.

The main result in this Section is Lemma 6.1 which shows that the invariant splittings and their smoothing properties when we change the linearized equation. Of course, in the applications in the iterative Nash-Moser method, the change of the equation will be induced by a change in the approximate solution.

**Lemma 6.1.** Assume that $A(\theta)$ is a family of linear maps as before. Let $\tilde{A}(\theta)$ be another family such that $\|\tilde{A} - A\|_{\rho,X,Y}$ is small enough.

Then there exists a family of analytic splittings

$$X = \tilde{X}^s_\theta \oplus \tilde{X}^c_\theta \oplus \tilde{X}^u_\theta$$
which is invariant under the linearized equations
\[ \frac{d}{dt} \Delta = \tilde{A}(\theta + \omega t) \Delta \]
in the sense that the following hold
\[ \tilde{U}^{s,c,u}_\theta(t) \tilde{X}^{s,c,u}_\theta = \tilde{X}^{s,c,u}_\theta. \]

We denote \( \tilde{\Pi}^{s,c,u}_\theta \) the projections associated to this splitting. Then there exist \( \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3^+, \tilde{\beta}_3^- > 0, \tilde{\alpha}_1, \tilde{\alpha}_2 \in (0, 1) \) and \( \tilde{C}_h > 0 \) independent of \( \theta \) satisfying \( \tilde{\beta}_3 < \tilde{\beta}_1, \tilde{\beta}_3 < \tilde{\beta}_2 \) and such that the splitting is characterized by the following rate conditions:
\[
\begin{align*}
\| \tilde{U}^s_\theta(t) \|_{\rho,Y,X} &\leq \tilde{C}_h e^{-\tilde{\beta}_1 t}, \quad t > 0, \\
\| \tilde{U}^u_\theta(t) \|_{\rho,Y,X} &\leq \tilde{C}_h e^{\tilde{\beta}_2 t}, \quad t < 0, \\
\| \tilde{U}^c_\theta(t) \|_{\rho,X,X} &\leq \tilde{C}_h e^{\tilde{\beta}_3^+ t}, \quad t > 0, \\
\| \tilde{U}^c_\theta(t) \|_{\rho,X,X} &\leq \tilde{C}_h e^{\tilde{\beta}_3^- |t|}, \quad t < 0.
\end{align*}
\]

Furthermore the following estimates hold
\[
\begin{align*}
\| \tilde{\Pi}^{s,c,u}_\theta - \Pi^{s,c,u}_\theta \|_{\rho,Y,Y} &\leq C \| \tilde{A} - A \|_{\rho,X,Y}, \\
| \tilde{\beta}_i - \beta_i | &\leq C \| \tilde{A} - A \|_{\rho,X,Y}, \quad i = 1, 2, 3^\pm, \\
\tilde{\alpha}_i &= \alpha_i, \quad i = 1, 2 \\
\tilde{C}_h &= C_h.
\end{align*}
\]

\textbf{Proof.} We want to find invariant subspaces for the linearized evolution equation. We concentrate on the stable subspace, the theory for the other bundles being similar. We do so by finding a family of linear maps indexed by \( \theta \), denoted \( \mathcal{M}_\theta : X^s_\theta \to X^{cu}_\theta \equiv X^c_\theta \oplus X^u_\theta \) in such a way that the graph of \( \mathcal{M}_\theta \) is invariant under the equation. Note that since we do not assume that the equation defines a flow, the fact that we can evolve the elements in the graph in the future is an important part of the conclusions. We will also show that the family of maps depends analytically in \( \theta \).

\textbf{Step 1: Construction of the invariant splitting.}

We will consider first the case of the stable bundle. The others are done identically. We will first try to characterize the initial conditions of the linearized evolution equation that lead to a forward evolution which is a contraction. We will see that these lie in a space. We will formulate the new space as the graph of a linear function \( \mathcal{M}_\theta \) from \( X^s_\theta \) to \( X^{cu}_\theta \). We will show that if such a characterization was possible, \( \mathcal{M}_\theta \) would have to satisfy some equations. To do that, we will formulate the problem of existence of forward solutions and the invariance of the bundle as two (coupled) fixed point problems (see (65) and (66).) One fixed point problem will formulate the invariance of the space , and the other fixed point problem the existence of forward solutions. We will show that, in some appropriate spaces, these two fixed point problems can be
stumbled using the contraction mapping principle. The definition of the spaces will be somewhat elaborate since they will also encode the analytic dependence on the initial conditions, which is natural if we want to show the analytic dependence on $\theta$ of the invariant spaces.

Note that, since the main tool will be a contraction argument, it follows that the main result is an a-posteriori result. Given approximate solutions of the invariance equations (obtained e.g numerically or through formal expansions, etc.) one can find a true solution close to the approximate one. We leave to the reader the recasting of Lemma 6.1 in this style.

Now, we implement in detail the above strategy: We first derive the functional equations, then, specify the spaces.

We start by considering the linearized equation with an initial phase $\theta$. For subsequent analysis, it will be important to study the dependence on $\theta$ of the solutions. Eventually, we will show that the new invariant spaces depend analytically on $\theta$. This will translate in the geometric properties of the bundles. Consider

\begin{equation}
\frac{d}{dt} W_\theta(t) = \tilde{A}(\theta + \omega t)W_\theta(t)
\end{equation}

Note that we use the index $\theta$ to indicate that we are considering the equation with initial phase $\theta$.

We write (60) as

\begin{equation}
\frac{d}{dt} W_\theta(t) = A(\theta + \omega t)W_\theta(t) + B(\theta + \omega t)W_\theta(t)
\end{equation}

with $B = \tilde{A} - A$. Denote $\gamma = \|\tilde{A} - A\|_{\rho,X,Y} \equiv \|B\|_{\rho,X,Y}$, which we will assume to be small.

We recall that this is an equation for $W_\theta$ and that we are not assuming solutions to exist. Indeed, one of our goals is to work out conditions that ensure that forward solutions exist. Hence, we will manipulate the equation (60) to obtain some conditions.

We compute the evolution of the projections of $W_\theta(t)$ along the invariant bundles by the linearized equation when $B \equiv 0$. For $\sigma = s, c, u$ we have:

\begin{equation}
\frac{d}{dt} \left( \Pi_{\theta+\omega t}^\sigma W_\theta(t) \right) = \left( \omega \cdot \partial_\theta \Pi_{\theta+\omega t}^\sigma \right) W_\theta(t) + \Pi_{\theta+\omega t}^\sigma \left( \frac{d}{dt} W_\theta(t) \right)
\end{equation}

\begin{align*}
&= \left( \omega \cdot \partial_\theta \Pi_{\theta+\omega t}^\sigma \right) W_\theta(t) + \Pi_{\theta+\omega t}^\sigma A(\theta + \omega t) + \Pi_{\theta+\omega t}^\sigma B(\theta + \omega t)W_\theta(t) \\
&= A^\sigma(\theta + \omega t)\Pi_{\theta+\omega t}^\sigma W_\theta(t) + \Pi_{\theta+\omega t}^\sigma B(\theta + \omega t)W_\theta(t)
\end{align*}

In the last line of (62), we have used that the calculation in the first two lines of (62) is also valid when $B = 0$ and that, in that case, the invariance of the bundles under the $A$ evolution implies that all the terms appearing can be subsumed into $A^\sigma$ which only depends on the projection on the bundle.
Of course the same calculation is valid for the projections over the center-unstable (and center-stable, etc.) bundles. We denote by $\Pi_{\theta}^{cu} = \Pi_{\theta}^{c} + \Pi_{\theta}^{u}$ the projection over the center-unstable bundle. Note that $\Pi_{\theta}^{c} + \Pi_{\theta}^{cu} = \text{Id}$.

Our goal now is to try to find a subspace in which the solutions of (60) (equivalently (62)) can be defined forward in time.

We will assume that this space where solutions can be defined is given as the graph of a linear function $M_\theta$ from $X^{s}_\theta$ to $X^{c}_\theta \oplus X^{u}_\theta$. That is, we introduce the notation $W^{cu}_\theta(t) = \Pi^{cu}_\theta + \Pi^{cu}_\theta W_\theta(t)$, $W^{s}_\theta(t) = \Pi^{s}_\theta + \omega t W_\theta(t)$ and we will assume that the solutions of (60) have the form

$$W^{cu}_\theta(t) = M_{\theta + \omega t} W^{s}_\theta(t)$$

We will have to show that this linear subspace of $X$ can indeed be found and, show that it depends analytically on $\theta$. For any $T > 0$, if there were solutions of the equation satisfied by $W^{cu}_\theta$ we would have Duhamel’s formula. Then, imposing that it is in the graph:

(63) $\quad M_\theta W^{s}_\theta(0) = W^{cu}_\theta(0)$

$$= U^{u}_{\theta + \omega T}(-T)M_{\theta + \omega T} W^{s}_\theta(t)$$

$$+ \int_{0}^{T} U^{u}_{\theta + \omega T}(t - T)\Pi^{cu}_{\theta + \omega T} B(\theta + \omega t)(\text{Id} + M_{\theta + \omega t})W^{s}_\theta(t) dt.$$  

Similarly, one has

(64) $W^{s}_\theta(t) = U^{u}_{\theta + \omega T} W^{s}_\theta(0) + \int_{0}^{t} U^{u}_{\theta + \omega T}(t - \tau)\Pi^{cu}_{\theta + \omega T} B(\theta + \omega \tau)(\text{Id} + M_{\theta + \omega T})N_\theta(t) dt.$

Notice that the fact that (64) is linear implies that if its solutions are unique, then $W^{s}_\theta(t)$ depends linearly on $W^{s}_\theta(0)$ (it depends very nonlinearly on $M_\theta$). We will write $W^{s}_\theta(t) = N_\theta(t) W^{s}_\theta(0)$ where $N_\theta(t)$ is a linear operator.

We have then

$$M_\theta = U^{u}_{\theta + \omega T}(-T)M_{\theta + \omega T} N_\theta(t)$$

(65) $+ \int_{0}^{T} U^{u}_{\theta + \omega T}(t - T)\Pi^{cu}_{\theta + \omega T} B(\theta + \omega t)(\text{Id} + M_{\theta + \omega t})N_\theta(t) dt.$

Similarly, we have that (64) is implied by

(66) $N_\theta(t) = U^{u}_{\theta + \omega T}(0) + \int_{0}^{t} U^{u}_{\theta + \omega T}(t - \tau)\Pi^{cu}_{\theta + \omega T} B(\theta + \omega \tau)(\text{Id} + M_{\theta + \omega T})N_\theta(\tau).$

We can think of (65) and (66) as equations for the two unknowns $M$ and $N_\theta$ where $M$ will be a function of $\theta$ and $N$ a function of $\theta, t$.

Note that (65) and (66) are already written as fixed point equations for the operators defined by the right hand side of the equations. It seems intuitively clear that the R.H.S. of the equations will be contractions since the linear terms involve a factor $B$ which we are assuming is small. Of course, to make this intuition precise, we have to
specify appropriate Banach spaces and carry out some estimates. After the spaces are defined, the estimates are somewhat standard and straightforward. We point out that operators similar to (65) appear in the perturbation theory of hyperbolic bundles and operators similar to (66) appear in the theory of perturbations of semigroups. The integral equations are also very common in the study of neutral delay equations.

6.0.4. Definition of spaces. Let $\rho > 0$. For $\theta \in D_\rho$, we denote by $\mathcal{L}(X_\theta^s, X_\theta^{cu})$ the space of bounded linear maps from $X_\theta^s$ into $X_\theta^{cu}$. We considered it endowed with the standard supremum norm of linear operators.

Denote also by $\mathcal{L}_\rho(X^s, X^{cu})$ the space of analytic mappings from $D_\rho$ into the space of bounded linear operators in $X$ that to each $\theta \in D_\rho$, assign a linear operator in $\mathcal{L}(X^s(\theta), X^{cu}(\theta))$. We also require from the maps in $\mathcal{L}_\rho(X^s, X^{cu})$ that they extend continuously to the boundary of $D_\rho$. We endow $\mathcal{L}_\rho(X^s, X^{cu})$ with the topology of the supremum norm, which makes it into a Banach space.

We also introduce the standard $C^0([0, T], \mathcal{L}_\rho(X^s, X^{cu}))$, endowed with the supremum norm. For each $\theta \in D_\rho$, we denote $C^0_\theta([0, T], \mathcal{L}(X^s, X^c))$ the space of continuous functions which for every $t \in [0, T]$, assign a linear operator in $\mathcal{L}(X^s_{\theta+\omega t}, X^{cu}_{\theta+\omega t})$. Of course, the space is endowed with the supremum norm. For typographical reasons, we will abbreviate the above spaces to $C^0$ and $C^0_\theta$. It is a standard result that the above spaces are Banach spaces when endowed with the above norms.

6.0.5. Some elementary estimates. We denote by $\mathcal{T}_1$, $\mathcal{T}_2$ the operators given by the R. H. S. of the equations (65) and (66) respectively. For typographical reasons, we just denote $\|B\| = \sup_{\theta \in D_\rho} \|B(\theta)\|_{X, Y}$.

Using just the triangle inequality and bounds on the semi-group $U_\theta^s$, we have:

$$
\| \mathcal{T}_2(\mathcal{M}, \mathcal{N}) - \mathcal{T}_2(\tilde{\mathcal{M}}, \tilde{\mathcal{N}}) \|_{C^0} \leq C \left( (1 + \|\mathcal{M}\|_{\mathcal{L}_\rho}) \|B\| \right) \|\mathcal{N} - \tilde{\mathcal{N}}\|_{C^0} + \max(\|\mathcal{N}\|_{C^0}, \|\tilde{\mathcal{N}}\|_{C^0}) \|\mathcal{M} - \tilde{\mathcal{M}}\|_{C^0} \\
\| \mathcal{T}_1(\mathcal{M}, \mathcal{N}) - \mathcal{T}_1(\tilde{\mathcal{M}}, \tilde{\mathcal{N}}) \|_{C^0} \leq \left( C_h T^{-\alpha_1} e^{-\beta_1 T} \|\mathcal{M}\|_{\mathcal{L}_\rho} + C(1 + \|\mathcal{M}\|_{\mathcal{L}_\rho}) \|B\| \right) \|\mathcal{N} - \tilde{\mathcal{N}}\|_{C^0} + C_h T^{-\alpha_1} e^{-\beta_1 T} \|\tilde{\mathcal{M}} - \mathcal{M}\|_{\mathcal{L}_\rho} + C \|B\| \max(\|\mathcal{N}\|_{C^0}, \|\tilde{\mathcal{N}}\|_{C^0}) \|\mathcal{M} - \tilde{\mathcal{M}}\|_{C^0}
$$

Since $\|U_\theta^s\| \leq A$, we choose $\mathcal{S} = \{ (\mathcal{N}, \mathcal{M}) \leq 2A \}$. We first fix $T$ large enough so that $C T^{-\alpha_1} e^{-\beta_1 T} \leq 10^{-2}$. Then, we see that if $\|B\|$ is small enough, $(\mathcal{T}_1, \mathcal{T}_2)(\mathcal{S}) \subset \mathcal{S}$.

Furthermore, under another smallness condition in $\|B\|$, using the previous bounds, we see that $(\mathcal{T}_1, \mathcal{T}_2)$ is a contraction in $\mathcal{S}$.

Therefore, with the above choices we can get solutions of (65), (66) which are sufficient conditions to obtain a forward evolution and that the graph is invariant under this evolution.
6.0.6. Some small arguments to finish the construction of the invariant subspaces. Since we have the function \( W \) defined in all \( D_\rho \), it follows that the function \( \mathcal{W}(t) = W(\theta + \omega t) \) is defined for all time as desired. The argument also shows that for a fixed \( \theta \), the function solves the linearized equation for a short time. Of course, the argument can be done in the same way for other dichotomies running the time backwards. Hence we obtain the stability of the splitings \( X^{sc} \) and \( X^u \). The space \( X^c \) can be reconstructed as \( X^c = X^{cu} \cap X^{sc} \).

**Step 2. Estimates on the projections.** To get the bounds for the projections we use the same argument as in [FdlIL09a]. We only give the argument for the stable subspace. Let \( \mathcal{M}_\theta^{cu} \) be the linear map whose graph gives \( \tilde{X}_\theta^{cu} \).

We write
\[
\Pi_\theta^s \xi = (\xi^s, 0), \quad \tilde{\Pi}_\theta^s \xi = (\tilde{\xi}^s, \mathcal{M}_\theta^s \tilde{\xi}^s), \]
\[
\Pi_\theta^{cu} \xi = (0, \xi^{cu}), \quad \tilde{\Pi}_\theta^{cu} \xi = (\mathcal{M}_\theta^{cu} \tilde{\xi}^{cu}, \tilde{\xi}^{cu}),
\]
and then
\[
\xi^s = \tilde{\xi}^s + \mathcal{M}_\theta^{cu} \tilde{\xi}^{cu},
\]
\[
\xi^{cu} = \mathcal{M}_\theta^s \tilde{\xi}^s + \tilde{\xi}^{cu}.
\]
Since \( \mathcal{M}_\theta^s \) and \( \mathcal{M}_\theta^{cu} \) are \( O(\gamma) \) in \( \mathcal{L}(X, X) \) we can write
\[
\begin{pmatrix}
\tilde{\xi}^s \\
\tilde{\xi}^{cu}
\end{pmatrix} = \begin{pmatrix}
\text{Id} & \mathcal{M}_\theta^{cu} \\
\mathcal{M}_\theta^s \text{Id}
\end{pmatrix}^{-1}
\begin{pmatrix}
\xi^s \\
\xi^{cu}
\end{pmatrix}
\]
and then deduce that
\[
\| (\tilde{\Pi}_\theta^s - \Pi_\theta^s) \xi \|_Y \leq \| (\tilde{\xi}^s - \xi^s, \mathcal{M}_\theta^s \tilde{\xi}^s) \|_Y \leq C\gamma.
\]

**Step 3. Stability of the smoothing properties.**

In this step, we will show that the smoothing properties of the cocycles are preserved under the lower order perturbations considered before. That is, we will show that if we define the evolutions in the invariant spaces constructed in Step 1 above, they satisfy bounds of the form in SD.3 but with slightly worse parameters. To be able to apply this repeatedly, it will be important for us to develop estimates on the change of the constants as a function of the correction.

We will first study the stable case. The unstable case is studied in the same way, just reversing the direction of time. The maps \( U_\theta^s \) and \( \tilde{U}_\theta^s \) satisfy the variational equations
\[
\frac{dU_\theta^s}{dt} = A(\theta + \omega t)U_\theta^s(t)
\]
and
\[
\frac{d\tilde{U}_\theta^s}{dt} = A(\theta + \omega t)\tilde{U}_\theta^s(t).
\]
Since \((U_{\theta}^s - \tilde{U}_{\theta}^s)(0) = 0\), one has by the variation of parameters formula

\[(67) \quad \tilde{U}_{\theta}^s(t) = U_{\theta}^s(t) + \int_0^t U_{\theta}^s(t - \tau)(\tilde{A} - A)(\theta + \omega \tau) \tilde{U}_{\theta}^s(\tau) \, d\tau,\]

for \(t \geq 0\).

Let \(C_{\alpha,\beta,\rho}(X)\) be the space of continuous functions from \((0, \infty)\) into the space \(A_{\rho, L(X,X)}\) endowed with the norm

\[|||U|||_{\alpha,\beta,\rho} = \sup_{\theta \in D_\rho t > 0} |||U(\theta(t))|||_{Y,X} e^{\beta t^\alpha}.\]

We fix \(\tilde{A}, A\) and \(U_{\theta}^s\) and consider the left hand-side of \((67)\) as an operator on \(\tilde{U}_{\theta}^s\), i.e. denote

\[\mathcal{T}\tilde{U}_{\theta}^s(t) = U_{\theta}^s(t) + \int_0^t U_{\theta}^s(t - \tau)(\tilde{A} - A)(\theta + \omega \tau) \tilde{U}_{\theta}^s(\tau) \, d\tau.\]

Hence \((67)\) is just a fixed point equation. We note that the operator \(\mathcal{T}\) is affine in its argument. We write it as

\[\mathcal{T}(U_{\theta}^s) = O + \mathcal{L}(U_{\theta}^s)\]

where \(O\) is a constant vector and \(\mathcal{L}\) is a linear operator. To show that \(\mathcal{T}\) is a contraction, it suffices to estimate the norm of \(\mathcal{L}\). We have

\[|||\mathcal{L}U_1 - \mathcal{L}U_2|||_{\alpha,\beta,\rho} \leq C \gamma \left( t^\alpha e^{\beta t} \int_0^t \frac{e^{-\beta_1 (t - \tau)}}{(t - \tau)^{\alpha_1}} e^{-\beta \tau} \tau^{-\alpha} \, d\tau \right) |||U_1 - U_2|||_{\alpha,\beta,\rho}.\]

We now estimate

\[C(t) = t^\alpha e^{\beta t} \int_0^t \frac{e^{-\beta_1 (t - \tau)}}{(t - \tau)^{\alpha_1}} e^{-\beta \tau} \tau^{-\alpha} \, d\tau.\]

We have

\[C(t) = t^\alpha \int_0^t \frac{e^{(\beta - \beta_1)(t - \tau)}}{(t - \tau)^{\alpha_1}} \tau^{-\alpha} \, d\tau.\]

Changing variables, one gets

\[C(t) = t^\alpha \int_0^t \frac{e^{(\beta - \beta_1)z}}{(t - z)^{\alpha_1}} z^{-\alpha_1} \, dz.\]

We now choose \(\beta\) such that \(\beta < \beta_1\) denoting \(\beta = \beta_1 - \varepsilon\). Making the change of variables \(z = t u\) in the integral, one gets

\[C(t) = t^{1 - \alpha_1} \int_0^1 \frac{e^{-\varepsilon tu}}{(1 - u)^{\alpha_1}} u^{-\alpha_1} \, du.\]

This is clearly bounded for \(t \leq 1\) since \(\alpha \in (0, 1)\) and \(1 - \alpha_1 > 0\). We now consider the case \(t > 1\). There exists a constant \(C > \text{universal}\) such that the following estimate holds

\[e^{-t \varepsilon u} \leq \frac{C}{(1 + t \varepsilon u)^{1 - \alpha_1}}\]
for any \( t, u \geq 0 \). Therefore we estimate for \( t > 1 \)
\[
C(t) \leq C t^{1-\alpha_1} \int_0^1 \frac{du}{(1-u)^{\alpha_1}(1+\varepsilon tu)^{1-\alpha_1}},
\]
which is uniformly bounded as \( t \) goes to \( \infty \). Recalling that \( \|\mathcal{L} U_1 - \mathcal{L} U_2\|_{\rho, \alpha_1, \beta_1} \leq C \gamma \)
where \( C \) is the constant we just computed, we obtain that \( \mathcal{L} \) is a contraction in the space \( \mathcal{C}_{\alpha_1, \beta_1, \rho}(X) \) for any \( \beta < \beta_1 \) and any \( \alpha_1 \in (0,1) \) when \( \gamma \) is sufficiently small. \( \square \)

The first consequence of Proposition 6.1 is that in the iterative step the small change
of \( K \) produces a small change in the invariant splitting and in the hyperbolicity con-
stants.

**Corollary 6.2.** Assume that \( K \) satisfies the hyperbolic non-degeneracy Condition [3,3] and that \( \|K - \tilde{K}\|_{\rho, X} \) is small enough. If we denote \( \tilde{A}(\theta) = D\mathcal{X}(K) \), there exists an analytic family of splitting for \( \tilde{K} \), i.e.
\[
X = X^s_{\tilde{K}(\theta)} \oplus X^c_{\tilde{K}(\theta)} \oplus X^u_{\tilde{K}(\theta)}
\]
which is invariant under the linearized equation (15) (replacing \( K \) by \( \tilde{K} \)) in the sense that
\[
\tilde{U}\sigma(t)X_{\tilde{K}(\theta)}^\sigma = X_{\tilde{K}(\theta + \omega t)}^\sigma, \quad \sigma = s, c, u
\]
We denote \( \Pi^s_{\tilde{K}(\theta)} \), \( \Pi^c_{\tilde{K}(\theta)} \) and \( \Pi^u_{\tilde{K}(\theta)} \) the projections associated to this splitting. There exist \( \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3^+, \tilde{\beta}_3^- > 0, \tilde{\alpha}_1, \tilde{\alpha}_2 \in (0,1) \) and \( \tilde{C}_h > 0 \) independent of \( \theta \) satisfying \( \tilde{\beta}_3^+ < \tilde{\beta}_1, \tilde{\beta}_3^- < \tilde{\beta}_2 \) and such that the splitting is characterized by the following rate conditions:
\[
\|\tilde{U}^s_\theta(t)\|_{\rho, Y, X} \leq \tilde{C}_h e^{-\tilde{\beta}_1 t \frac{1}{\tilde{t}^2}}, \quad t > 0,
\]
\[
\|\tilde{U}^u_\theta(t)\|_{\rho, Y, X} \leq \tilde{C}_h e^{\tilde{\beta}_2 t \frac{1}{\tilde{t}^2}}, \quad t < 0,
\]
\[
\|\tilde{U}^c_\theta(t)\|_{\rho, X, X} \leq \tilde{C}_h e^{\tilde{\beta}_3^- t |t|}, \quad t > 0
\]
\[
\|\tilde{U}^c_\theta(t)\|_{\rho, X, X} \leq \tilde{C}_h e^{\tilde{\beta}_3^+ |t|}, \quad t < 0.
\]
Furthermore the following estimates hold
\[
\|\Pi^{s,c,u}_{\tilde{K}(\theta)} - \Pi^{s,c,u}_{\tilde{K}(\theta)}\|_{\rho, Y, X} \leq C \|K - \tilde{K}\|_{\rho, X},
\]
\[
|\tilde{\beta}_i - \beta_i| \leq C \|K - \tilde{K}\|_{\rho, X}, \quad i = 1, 2, 3,
\]
\[
|\tilde{\alpha}_i - \alpha_i| \leq C \|K - \tilde{K}\|_{\rho, X}, \quad i = 1, 2
\]
\[
\tilde{C}_h = C_h.
\]
**Proof.** We just take \( A(\theta) = D\mathcal{X}(K(\theta)) \), \( \tilde{A}(\theta) = D\mathcal{X}(\tilde{K}(\theta)) \), \( X^{s,c,u}_{\tilde{K}(\theta)} = X^{s,c,u}_{\tilde{K}(\theta)} \), \( X^{s,c,u}_{\tilde{K}(\theta)} = \tilde{X}^{s,c,u}_{\tilde{K}(\theta)} \), \( \Pi^{s,c,u}_{\tilde{K}(\theta)} = \Pi^{s,c,u}_{\tilde{K}(\theta)} \) and \( \Pi^{s,c,u}_{\tilde{K}(\theta)} = \tilde{\Pi}^{s,c,u}_{\tilde{K}(\theta)} \) in Lemma 6.1 and we use that \( \|\tilde{A}(\theta) - A(\theta)\|_{\rho, Y, X} \leq \|\mathcal{X}(\tilde{K}(\theta) - K(\theta))\|_{\rho, X} \). \( \square \)
7. Solution of the cohomology equation on the center subspace

We now come to the solution of the projected equation (43) on the center subspace. The first point which has to be noticed is that by the spectral non-degeneracy assumption the center subspace $X^c_\theta$ is finite-dimensional (with dimension $2\ell$). As a consequence, we end up with standard small divisors equations. This is in contrast with other studies of Hamiltonian partial differential equations like the Schrödinger equation for which there is an infinite number of eigenvalues on the imaginary axis (see [Bou98]) and the KAM theory is more involved. Another aspect of Definition 3.3 is that the formal symplectic structure on $X$ restricts to a standard one on the center bundle. Finally, it has to be noticed that by the finite-dimensionality assumption, all the issues related to unbounded operators become irrelevant.

We denote
\[ \Delta^c(\theta) = \Pi^c_\theta \Delta K(\theta). \]
The projected linearized equation (43) becomes
\[ \partial_\omega \Delta^c(\theta) - (D\mathcal{X}) \circ K \Delta^c(\theta) = -\Pi^c_\theta E(\theta) = -E^c(\theta). \]

We first recall a well-known result by Rüssmann (see [Rüs76a, Rüs76b, Rüs75, dllGJV05]) which allows to solve small divisor equations along characteristics.

**Proposition 7.1.** Assume that $\omega \in D_h(\kappa, \nu)$ with $\kappa > 0$ and $\nu \geq \ell - 1$ and that $\mathcal{M}$ is a finite dimensional space. Let $h : D_\rho \supset T^\ell \to \mathcal{M}$ be a real analytic function with zero average with values in $\mathcal{M}$. Then, for any $0 < \delta < \rho$ there exists a unique analytic solution $v : D_{\rho-\delta} \supset T^\ell \to \mathcal{M}$ of the linear equation
\[ \sum_{j=1}^{\ell} \omega_j \frac{\partial v}{\partial \theta_j} = h \]
having zero average. Moreover, if $h \in A_{\rho, \mathcal{M}}$ then $v$ satisfies the following estimate
\[ \|v\|_{\rho-\delta, \mathcal{M}} \leq C\kappa \delta^{-\nu} \|h\|_{\rho, \mathcal{M}}, \quad 0 < \delta < \rho. \]
The constant $C$ depends on $\nu$ and the dimension of the torus $\ell$.

As in [FdlLS09a] and [dllGJV05], we will find an explicit change of variables so that the vector-field $D\mathcal{X} \circ K \Delta^c(\theta)$ becomes a constant coefficient vector-field. Then we will be able to apply the small divisor result as stated in Proposition 7.1 to the cohomology equations (72).

7.1. Geometry of the invariant tori. As it is well known in KAM theory, in a finite dimensional framework, maximal invariant tori are Lagrangian submanifolds and whiskered tori are isotropic. In our context of an infinite dimensional phase space $X$, the picture is less clear, but nevertheless, thanks to our assumptions (which are satisfied in some models under consideration), one can produce a non trivial solution.

We prove the following lemma on the isotropic character of approximate invariant tori.
Lemma 7.2. Let $K : D_{\rho} \supset \mathbb{T}^\ell \to \mathcal{M}$, $\rho > 0$, be a real analytic mapping. Define the error in the invariance equation as

$$E(\theta) := \partial_\omega K(\theta) - \mathcal{X}(K(\theta)).$$

Let $L(\theta) = DK(\theta)^{-1} J_c DK(\theta)$ be the matrix which expresses the form $K^*\Omega$ on the torus in the canonical basis.

There exists a constant $C$ depending on $l, \nu$ and $\|DK\|_\rho$ such that

$$\|L\|_{\rho-2, X^c_\theta X^c_\theta} \leq C\kappa\delta^{-(\nu+1)}\|E\|_{\rho, Y}, \quad 0 < \delta < \rho/2.$$  

In particular, if $E = 0$ then

$$L \equiv 0$$

Proof. By assumption **H3.2** we have that there exists a one-form $\alpha_K$ on the torus $\mathbb{T}^\ell$ such that

$$K^*\Omega = d\alpha_K.$$ 

In coordinates on $\mathbb{T}^\ell$, $\alpha_K$ writes

$$\alpha_K = g_K(\theta) d\theta.$$ 

Hence one has $L(\theta) = Dg_K^T(\theta) - Dg_K(\theta)$ and the lemma follows from Cauchy estimates and Proposition 7.1 (see also [dlLGJV05]). □

7.2. Basis of the center subspace $X^c_\theta$. We introduce a suitable representation of the center subspace $X^c_\theta$. In [dlLGJV05, FdlLS09a, FdlLS09b] it is shown that the change of variables given by the following matrix

$$[DK(\theta), J_c^{-1}DK(\theta)N(\theta)].$$  

allows to transform the linearized equations in the center subspace into two cohomology equations with constant coefficients.

The argument presented in the references above works word by word here thanks to the fact that the center subspace $X^c_\theta$ is finite dimensional. We will go over the main points in Section 7.3. We will start by recalling some symplectic properties.

7.2.1. Some symplectic preliminaries. We prove the following lemma.

Lemma 7.3. The 2–form $\Omega$ which is the restriction to the center subspace is non-degenerate in the sense that $\Omega(u, v) = 0 \forall u \in X$ implies that $v = 0$.

Proof. A quick proof would follow from the fact that the symplectic form is non-degenerate at the origin. Then, because the non-degeneracy assumptions are open, it follows in a small neighborhood. The following argument gives a more global argument valid in all the center manifold.

By the non-degeneracy assumptions [3.3] there exist maps $U^{s,c,u}_\theta(t)$ generating the linearizations on $X^{s,c,u}_\theta$. These maps preserves $\Omega$. Indeed, one has: let $u(t), v(t)$ satisfy

$$\frac{du(t)}{dt} = A(\theta + \omega t) u(t)$$
and
\[
\frac{dv(t)}{dt} = A(\theta + \omega t)v(t)
\]
where \( A(\theta) = J^{-1}\nabla^2 H \circ K(\theta) \). Then
\[
\Omega(u(t), v(t)) = \Omega(u(0), v(0)).
\]
Indeed,
\[
\frac{d}{dt} \Omega(u(t), v(t)) = \Omega(\dot{u}(t), v(t)) + \Omega(u(t), \dot{v}(t)) = < J^{-1}\nabla^2 H \circ K(\theta + \omega t)u(t), Jv(t) > + < u(t), JJ^{-1}\nabla^2 H \circ K(\theta + \omega t)v(t) >
\]
since \( \nabla^2 H \circ K \) is symmetric. Hence the result.

Therefore, we have for any \( u, v \in X^{s,c,u}_\theta \)
\[
\Omega(u, v) = \Omega(U^{s,c,u}_\theta(t)u, U^{s,c,u}_\theta(t)v), \quad t \in \mathbb{R}^+, \mathbb{R}, \mathbb{R}^-.
\]
Using now the estimates in [33], we have the following: the form \( \Omega \) satisfies \( \Omega(u, v) = 0 \) in the following cases
- \( u, v \in X^s_\theta \),
- \( u, v \in X^u_\theta \),
- \( u \in X^s_\theta \cup X^u_\theta \) and \( v \in X^c_\theta \),
- \( v \in X^c_\theta \) and \( v \in X^s_\theta \cup X^u_\theta \)

This implies that the form \( \Omega \) restricted to the center bundle \( X^c_\theta \) is non degenerate and the lemma is proved.

The form \( \Omega \) is then a symplectic form since we assumed that the restriction of the form to \( X^c_\theta \) is closed. Denote by \( J_c \) the restriction of the operator \( J \) on \( X^c_\theta \). Finally we define the operator \( M(\theta) \) from \( \mathbb{R}^\ell \) into \( X^c_\theta \).

(74)
\[
M(\theta) = [DK(\theta), J_c^{-1}DK(\theta)N(\theta)].
\]

Notice that by assumption \( X^c_\theta \) is isomorphic to \( Y^c_\theta \). We emphasize on the fact that the operator \( M(\theta) \) belongs to \( X^c_\theta \). Indeed, it is clear from the equation that \( DK \) (by just differentiating) belongs to the center space and so is \( J_c^{-1}DK(\theta)N(\theta) \) by the fact that we consider the restriction \( J_c \) of \( J \) to the center.

7.3. Normalization procedure. Let \( W : D_\rho \supset \mathbb{T}^\ell \rightarrow X^c_\theta \) be such that

\[
\Delta^c(\theta) = M(\theta)W(\theta)
\]
From now on, the proof is very similar to the one in [dlLGJV05] and we just sketch the proofs. We refer the reader to [dlLGJV05] for the details. The following first lemma provides a reducibility argument for exact solutions of (5). We note that since the space \( X^c_\theta \) is finite dimensional the symplectic form needs to be defined only in a very weak sense.
Lemma 7.4. Let $K$ be a solution of
\[ \partial_\omega K(\theta) = \mathcal{X}(K(\theta)) \]
with $M$ be defined as above and $K(\mathbb{T}^\ell)$ is an isotropic manifold. Then there exists an $\ell \times \ell$-matrix $S(\theta)$ such that
\[ (75) \quad \partial_\omega M(\theta) - A(\theta)M(\theta) = M(\theta) \begin{pmatrix} 0_\ell & S(\theta) \\ 0_\ell & 0_\ell \end{pmatrix}, \]
where
\[ S(\theta) = N(\theta)DK(\theta)^\top [J_c^{-1} \partial_\omega (DKN) - A(\theta)J_c^{-1}DKN](\theta) \]
where we have denoted $A(\theta) = J_c^{-1}D(\nabla H(K))$.

Proof. By differentiating the equation, we clearly have that the first $\ell$ columns of the matrix
\[ W(\theta) = A(\theta)M(\theta) - \partial_\omega M(\theta) \]
are zero. Now write
\[ W_1(\theta) = A(\theta)J_c^{-1}DK(\theta)N(\theta) - J_c^{-1}\partial_\omega (DK(\theta)N(\theta)). \]
Easy computations show that
\[ W_1(\theta) = A(\theta)J_c^{-1}DK(\theta)N(\theta) - J_c^{-1}\partial_\omega (DK(\theta)N(\theta)) \]
\[ + J_c^{-1}DK(\theta)N(\theta)\partial_\omega (DK(\theta)^\top N(\theta)) + J_c^{-1}DK(\theta)N(\theta)DK(\theta)^\top \partial_\omega (DK(\theta))N(\theta). \]
But since $DK$ and $J_c^{-1}DK(\theta)N(\theta)$ form a basis of the center subspace, one can write
\[ W_1 = DK S + J_c^{-1}DKNT. \]
We will prove that $T = 0$, giving the form of the matrix in the lemma. Multiply the previous equation by $DK(\theta)^\top J_c$; then by the lagrangian character of $K$, we have
\[ DK(\theta)^\top J_c W_1(\theta) = T. \]
Hence using straightforward computations, we have that the second term plus the fourth term in $DK(\theta)^\top J_c W_1(\theta)$ is zero and the first term plus the third term in $DK(\theta)^\top J_c W_1(\theta)$ is equal to
\[ (DK^\top D(\nabla H(K))J_c^{-1} + \partial_\omega (DK)^\top)DKN. \]
But using the fact the symplectic form is skew-symmetric, the quantity into parenthesis is just the derivative of the equation. Hence it has to be zero.

We now check the expression of the matrix $S$. We multiply by $NDK^\top$ to have
\[ S = NDK^\top W_1 = NDK^\top (A(\theta)J_c^{-1}DK(\theta)N(\theta) - J_c^{-1}\partial_\omega (DK(\theta)N(\theta))). \]
This gives the result. \[ \square \]
The next lemma provides a generalized inverse for the operator $M$. 
Lemma 7.5. Let $K$ be a solution of (5). Then the matrix $M^\perp J_c M$ is invertible and

$$(M^\perp J_c M)^{-1} = \begin{pmatrix} N^\top D K^\top J_c^{-1} D K N & -\Id_{\ell} \\ \Id_{\ell} & 0 \end{pmatrix}.$$ 

We now establish a similar result for approximate solutions, i.e. solutions of (5) up to error $E(\theta) = \mathcal{F}_\omega(K)(\theta)$. When $K$ is just an approximate solution, we define

$$(e_1, e_2) = \partial_\omega M(\theta) - A(\theta) M(\theta) - M(\theta) \begin{pmatrix} 0_{\ell} & S(\theta) \\ 0_{\ell} & 0_{\ell} \end{pmatrix}.$$ 

Using that $\partial_\omega D K(\theta) - A(\theta) D K(\theta) = D E(\theta)$ and the definition of $S$ above mentioned give $e_1 = D E$ and $e_2 = O(\|E\|_{\rho,Y}, \|D E\|_{\rho,Y})$.

We then get

$$(\theta) = -E^c(\theta),$$

For the approximate solutions of (5), we have the following lemma.

Lemma 7.6. Assume $\omega$ is Diophantine in the sense of definition 3.1 and $\|E^c\|_{\rho,Y}^{\perp}$ small enough. Then there exist a matrix $B(\theta)$ and vectors $p_1$ and $p_2$ such that, by the change of variables $\Delta^c = M \xi$, the projected equation on the center subspace can be written

$$(78) \quad \left[ \begin{pmatrix} 0_{\ell} & S(\theta) \\ 0_{\ell} & 0_{\ell} \end{pmatrix} + B(\theta) \right] \xi(\theta) + \partial_\omega \xi(\theta) = p_1(\theta) + p_2(\theta).$$

The following estimates hold

$$(79) \quad \|p_1\|_{\rho,X^{\perp}_{\delta}} \leq C \|E^c\|_{\rho,Y^{\perp}_{\delta}},$$

$$(80) \quad \|p_2\|_{\rho-\delta,X^{\perp}_{\delta}} \leq C \kappa \delta^{-(\nu+1)} \|E^c\|_{\rho,Y^{\perp}_{\delta}}^{2}$$

and

$$(81) \quad \|B\|_{\rho-2\delta,X^{\perp}_{\delta}} \leq C \kappa \delta^{-(\nu+1)} \|E^c\|_{\rho,Y^{\perp}_{\delta}}^{\perp},$$

where $C$ depends $l$, $\nu$, $\rho$, $\|N\|_{\rho}$, $\|D K\|_{\rho,Y}$, $\|H\|_{C^2(B_\tau)}$. Furthermore the vector $p_1$ has the expression

$$(82) \quad p_1(\theta) = \begin{pmatrix} -N(\theta)^\top D K(\theta)^\top E^c(\theta) \\ D K(\theta)^\top J_c E^c(\theta) \end{pmatrix}$$

Proof. The proof follows more or less the one in [31LGJV05] with suitable adaptations due to our infinite dimensional setting. Notice however that the center subspace is finite dimensional. From the previous computations one has

$$(e_1, e_2) = \partial_\omega M(\theta) - A(\theta) M(\theta) - M(\theta) \begin{pmatrix} 0_{\ell} & S(\theta) \\ 0_{\ell} & 0_{\ell} \end{pmatrix}.$$ 

Hence we have

$$M^\perp J_c \left[ \partial_\omega M(\theta) - A(\theta) M(\theta) \right] \xi(\theta) = (M^\perp J_c M) \partial_\omega \xi = M^\perp J_c E_c.$$
Hence by the previous Lemma,

\[
(82) \left[ \begin{smallmatrix} 0 & S(\theta) \\ 0 & 0 \end{smallmatrix} \right] + (M^\perp J_c M)^{-1}(e_1, e_2) \right] \xi(\theta) + \partial_\omega \xi(\theta) = (M^\perp J_c M)^{-1}M^\perp J_c E_c.
\]

Hence denoting

\[
B(\theta) = (M^\perp J_c M)^{-1}(e_1, e_2).
\]

Then direct computations give \(p_1\) and \(p_2\) and the desired estimates.

\[
\Box
\]

7.4. Solutions to the reduced equations. We anticipate that from Lemma 7.6, the terms \(B\xi\) and \(p_2\) are quadratic in the error. Hence an approximate solution has the form \(\xi = (\xi_1, \xi_2)\) and solves

\[
\begin{align*}
S(\theta)\xi_2(\theta) - \partial_\omega \xi_1(\theta) &= -N(\theta)^T DK(\theta)^T E^c(\theta), \\
\partial_\omega \xi_2(\theta) &= DK(\theta)^T J_c E^c(\theta).
\end{align*}
\]

We prove the following result, providing a solution to equations (83).

**Proposition 7.7.** There exists a solution \((\xi_1, \xi_2)\) of (83) with the following estimates

\[
\begin{align*}
|\xi_1|_{\rho - \delta, x_0} &\leq C_1 \kappa \delta^{-\nu} ||E^c||_{\rho, x_0}, \\
|\xi_2|_{\rho - 2\delta, x_0} &\leq C_2 \kappa \delta^{-2\nu} ||E^c||_{\rho, x_0},
\end{align*}
\]

for any \(\rho \in (0, \delta/2)\) and where the constants \(C_1, C_2\) just depend on \(l, \nu, \rho, ||N||_\rho, ||DK||_{\rho, x_0}, \text{avg}(S)|^{-1}\).

**Proof.** In order to apply Prop. 7.1 one needs to study the average on the torus \(T^l\) of \(DK(\theta)^T J_c E^c(\theta)\). To do so, we first consider assumption H3.1 which gives in coordinates

\[
DK^T J_c DK = Dg^T - Dg
\]

for some function \(g\) on \(T^l\). Now taking the inner product with \(\omega\) and using the equation, one has

\[
DK^T J_c (E + X(K)) = Dg^T \cdot \omega - Dg \cdot \omega.
\]

Therefore, the average of \(DK^T J_c E\) is the sum of the average of \(Dg^T \cdot \omega - Dg \cdot \omega\) which is zero and the average of \(DK^T J_c X(K)\). Now notice that

\[
DK^T J_c X(K) = i_{X \circ K} K^* \Omega(\cdot).
\]

Hence its average is zero by assumption H4. As a consequence the average on \(T^l\) of the R.H.S. \(DK(\theta)^T J_c E^c(\theta)\) is zero. Hence an application of Prop. 7.1 gives the solvability in \(\xi_2\) with the desired bound. Since the average of \(\xi_2\) is free, one uses it and the twist condition to solve in \(\xi_1\). This gives the desired result (see [dlLGJV05] for details). \(\Box\)
8. Uniqueness statement

In this section, we prove the uniqueness part of Theorem 3.5. We assume that the embeddings $K_1$ and $K_2$ satisfy the hypotheses in Theorem 3.5 in particular $K_1$ and $K_2$ are solutions of (85). If $\tau \neq 0$ we write $K_1$ for $K_1 \circ T_\tau$ which is also a solution. Therefore $F_\omega(K_1) = F_\omega(K_2) = 0$. By Taylor’s theorem we can write

$$0 = F_\omega(K_1) - F_\omega(K_2) = D_K F_\omega(K_2)(K_1 - K_2)$$

(84)

where

$$R(K_1, K_2) = \frac{1}{2} \int_0^1 D^2 F_\omega(K_2 + t(K_1 - K_2))(K_1 - K_2)^2 \, dt.$$ 

Then, there exists $C > 0$ such that

$$\|R(K_1, K_2)\|_{\rho, Y} \leq C\|K_1 - K_2\|_{\rho, X}^2.$$

Hence we end up with the following linearized equation

$$D_K F_\omega(K_2)(K_1 - K_2) = -R(K_1, K_2).$$

(85)

We denote $\Delta = K_1 - K_2$. Projecting (85) on the center subspace with $\Pi_{K_2(\theta + \omega t)}^c$, writing $\Delta^c(\theta) = \Pi_{K_2(\theta)}^c \Delta(\theta)$ and making the change of function $\Delta^c(\theta) = M(\theta)W(\theta)$, where $M$ is defined in (74) with $K = K_2$. We now perform the same type of normalization as in Section 7 to arrive to two small divisor equations of the type

$$S(\theta)\xi_2(\theta) - \partial_\omega \xi_1(\theta) = -N(\theta) \, D\theta(\theta)^c R(0, 0, K_1, K_2)(\theta)^c,$$

$$\partial_\omega \xi_2(\theta) = DK(\theta)^c J, R(0, 0, K_1, K_2)(\theta)^c.$$

We begin by looking for $\xi_2$. We search it in the form $\xi_2 = \xi_2^1 + \text{avg} (\xi_2)$. We have

$$\|\xi_2^1\|_{\rho - \delta} \leq Ck_\delta^{-2\nu}\|K_1 - K_2\|_{\rho, X}^2.$$

The condition on the right-hand side of (86) to have zero average gives $|\text{avg} (\xi_2)| \leq Ck_\delta^{-2\nu}\|K_1 - K_2\|_{\rho, X}^2$. Then

$$\|\xi_1 - \text{avg} (\xi_1)\|_{\rho - 2\delta} \leq Ck_\delta^{-2\nu}\|K_1 - K_2\|_{\rho, X}^2$$

but $\text{avg} (\xi_1)$ is free. Then

$$\|\Delta^c - (\text{avg} (\Delta^c)_{1, 0})^c\|_{\rho - 2\delta} \leq Ck_\delta^{-2\nu}\|K_1 - K_2\|_{\rho, X}^2.$$

The next step is done in the same way as in [dILGJV05]. We quote Lemma 14 of that reference using our notation. It is basically an application of the standard implicit function theorem.

Lemma 8.1. There exists a constant $C$ such that if $C\|K_1 - K_2\|_{\rho, X} \leq 1$ then there exists an initial phase $\tau_1 \in \{\tau \in \mathbb{R}^\ell \mid |\tau| < \|K_1 - K_2\|_{\rho, X}\}$ such that

$$\text{avg} (T_2(\theta)\Pi_{K_2(\theta)}(K_1 \circ T_{\tau_1} - K_2)(\theta)) = 0.$$
The proof is based on the implicit function theorem in \( \mathbb{R}^l \).

As a consequence of Lemma 8.1, if \( \tau_1 \) is as in the statement, then \( K \circ T_{\tau_1} \) is a solution of (5) such that for all \( \delta \in (0, \rho/2) \) we have the estimate
\[
\|W\|_{\rho-2\delta,X} < C \kappa^2 \delta^{-2\nu} \|R\|_\rho^2 \leq C \kappa^2 \delta^{-2\nu} \|K_1 - K_2\|_{\rho,X}^2.
\]

This leads to, on the center subspace
\[
\|\Pi_{K_{\tau_1}}(K_1 \circ T_{\tau_1} - K_2)\|_{\rho-2\delta,X} \leq C \kappa^2 \delta^{-2\nu} \|K_1 - K_2\|_{\rho,X}^2.
\]

Furthermore, taking projections on the hyperbolic subspace, we have that \( \Delta^h = \Pi_{K_{\tau_1}}(K_1 - K_2) \) satisfies the estimate
\[
\|\Delta^h\|_{\rho-2\delta,X} < C \|R\|_{\rho,Y}.
\]

All in all, we have proven the estimate for \( K_1 \circ T_{\tau_1} \circ T_{\tau_1} - K_2 \) (up to a change in the original constants)
\[
\|K_1 \circ T_{\tau_1} - K_2\|_{\rho-2\delta,X} \leq C \kappa^2 \delta^{-2\nu} \|K_1 - K_2\|_{\rho,X}^2.
\]

We are now in position to carry out an argument based on iteration. We can take a sequence \( \{\tau_m\}_{m \geq 1} \) such that \( |\tau_1| \leq \|K_1 - K_2\|_{\rho,X} \) and
\[
|\tau_m - \tau_{m-1}| \leq \|K_1 \circ T_{\tau_{m-1}} - K_2\|_{\rho_{m-1},X}, \quad m \geq 2,
\]
and
\[
\|K_1 \circ T_{\tau_m} - K_2\|_{\rho_{m},X} \leq C \kappa^2 \delta_m^{-2\nu} \|K_1 \circ T_{\tau_{m-1}} - K_2\|_{\rho_{m-1},X}^2,
\]
where \( \delta_1 = \rho/4, \delta_{m+1} = \delta_m/2 \) for \( m \geq 1 \) and \( \rho_0 = \rho, \rho_m = \rho_0 - \sum_{k=1}^{m} \delta_k \) for \( m \geq 1 \). By induction argument we end up with
\[
\|K_1 \circ T_{\tau_m} - K_2\|_{\rho_{m},X} \leq (C \kappa^2 \delta_1^{-2\nu} 2^{2\nu} \|K_1 - K_2\|_{\rho_0,X})^{2m} 2^{-2\nu m}.
\]

Therefore, under the smallness assumptions on \( \|K_1 - K_2\|_{\rho,X} \), the sequence \( \{\tau_m\}_{m \geq 1} \) converges and one gets
\[
\|K_1 \circ T_{\tau_{\infty}} - K_2\|_{\rho/2,X} = 0.
\]

Since both \( K_1 \circ T_{\tau_{\infty}} \) and \( K_2 \) are analytic in \( D_\rho \) and coincide in \( D_{\rho/2} \) we obtain the result.

9. Nash-Moser iteration

In this section, we show that, if the initial error of the approximate invariance equation (6) is small enough the Newton procedure can be iterated infinitely many times and converges to a solution. This is somewhat standard in KAM theory given the estimates already obtained.

Let \( K_0 \) be an approximate solution of (5) (i.e. a solution of the linearized equation with error \( E_0 \)). We define the following sequence of approximate solutions
\[
K_m = K_{m-1} + \Delta K_{m-1}, \quad m \geq 1,
\]
where $\Delta K_{m-1}$ is a solution of

$$D_K\mathcal{F}_\omega(K_{m-1})\Delta K_{m-1} = -E_{m-1}$$

with $E_{m-1}(\theta) = \mathcal{F}_\omega(K_{m-1})(\theta)$. The next lemma provides that the solution at step $m$ improves the solution at step $m-1$ and the norm of the error at step $m$ is bounded in a smaller complex domain by the square of the norm of the error at step $m-1$.

**Proposition 9.1.** Assume $K_{m-1} \in ND(\rho_{m-1})$ is an approximate solution of equation (5) and that the following holds

$$r_{m-1} = \|K_{m-1} - K_0\|_{\rho_{m-1},X} < r.$$  

If $E_{m-1}$ is small enough such that Proposition 7.6 applies, i.e.

$$C\kappa\delta_{m-1}^{-\nu-1}\|E_{m-1}\|_{\rho_{m-1},Y} < 1/2$$

for some $0 < \delta_{m-1} \leq \rho_{m-1}/3$, then there exists a function $\Delta K_{m-1} \in A_{\rho_{m-1}-3\delta_{m-1},X}$ for some $0 < \delta_{m-1} < \rho_{m-1}/3$ such that

$$\|\Delta K_{m-1}\|_{\rho_{m-1}-2\delta_{m-1},X} \leq (C_{m-1}^1 + C_{m-1}^2 \kappa_2 \delta_{m-1}^{-2\nu})\|E_{m-1}\|_{\rho_{m-1},Y},$$

(88)

$$\|D\Delta K_{m-1}\|_{\rho_{m-1}-3\delta_{m-1},X} \leq (C_{m-1}^1 \delta_{m-1}^{-1} + C_{m-1}^2 \kappa_2 \delta_{m-1}^{-2\nu+1})\|E_{m-1}\|_{\rho_{m-1},Y},$$

where $C_{m-1}^1, C_{m-1}^2$ depend only on $\nu, l, |\mathcal{X}|, C_\omega, l$, $\|DK_{m-1}\|_{\rho_{m-1},X}$, $\|\Pi_{K_{m-1}(\theta)}^c\|_{\rho_{m-1},Y^c,\nu, X}$, $\|\Pi_{K_{m-1}(\theta)}^u\|_{\rho_{m-1},Y^u,\nu, X}$, and $|\text{avg}(S_{m-1})|^{-1}$. Moreover, if $K_m = K_{m-1} + \Delta K_{m-1}$ and

$$r_{m-1} + (C_{m-1}^1 + C_{m-1}^2 \kappa_2 \delta_{m-1}^{-2\nu})\|E_{m-1}\|_{\rho_{m-1},Y} < r$$

then we can redefine $C_{m-1}^1$ and $C_{m-1}^2$ and all previous quantities such that the error $E_m(\theta) = \mathcal{F}_\omega(K_m)(\theta)$ satisfies (defining $\rho_m = \rho_{m-1} - 3\delta_{m-1}$)

$$\|E_m\|_{\rho_m,Y} \leq C_{m-1}^1 \kappa_4 \delta_{m-1}^{-4\nu}\|E_{m-1}\|_{\rho_{m-1},Y}^2,$$

(89)

**Proof.** We have $\Delta K_{m-1}(\theta) = \Pi_{Y^h}^\theta \Delta K_{m-1}(\theta) + \Pi_{Y^u}^\theta \Delta K_{m-1}(\theta)$, where $\Pi_{Y^h}^\theta$ is the projection on the hyperbolic subspace and belong to $\mathcal{L}(Y^h, X)$. Estimates (46) follow from the previous two sections. The second part of estimate (46) follows from the first line of (46). Cauchy’s inequalities and the fact that the projected equations on the hyperbolic subspace are exactly solved.

Thanks to the previous proposition, one is able to obtain the convergence of the Newton method in a standard way.

The others non-degeneracy conditions can be checked in exactly the same way as described in [FdlLS09a] and we do not repeat the arguments.

**Lemma 9.2.** If $\|E_{m-1}\|_{\rho_{m-1},Y^c}$ is small enough, then
If $DK_{m-1}^\perp DK_m$ is invertible with inverse $N_{m-1}$ then

$$DK_m^\perp DK_m$$

is invertible with inverse $N_m$ and we have

$$\|N_m\|_{\rho_m} \leq \|N_{m-1}\|_{\rho_{m-1}} + C_{m-1}\kappa^2\delta_{m-1}^{-(2\nu+1)}\|E_{m-1}\|_{\rho_{m-1},Y_\delta^\perp}.$$

If $\text{avg} (S_{m-1})$ is non-singular then also $\text{avg} (S_m)$ is and we have the estimate

$$|\text{avg} (S_m)|^{-1} \leq |\text{avg} (S_{m-1})|^{-1} + C_b' \kappa^2 \delta_{m-1}^{-(2\nu+1)}\|E_{m-1}\|_{\rho_{m-1},Y_\delta^\perp}.$$

10. Construction of quasi-periodic solutions for the Boussinesq equation

This section is devoted to an application of Theorem 3.5 to a concrete equation that has appeared in the literature.

In Section 10.1, we will verify the formal hypothesis of the general Theorem 3.5. First we will verify the geometric hypothesis, choose the concrete spaces that will play the role of the abstract ones, etc. In Section 10.5, we will construct approximate solutions that satisfy the quantitative properties. By applying Theorem 3.5 to these approximate solutions, we will obtain Theorem 3.7.

10.1. Formal and geometric considerations. The Boussinesq equation has been widely studied in the context of fluid mechanics since the pioneering work [Bou72]. It is the equation (in one dimension) with periodic boundary conditions

$$(90) \quad u_{tt} = \mu u_{xxxx} + u_{xx} + (u^2)_{xx} \quad \text{on } \mathbb{T}, \ t \in \mathbb{R},$$

where $\mu > 0$ is a parameter.

We will introduce an additional parameter $\varepsilon$ which will be useful in the sequel as a mnemonic device to perform perturbation theory. Note however that it can be eliminated by rescaling the $u$, considering $v = \varepsilon u$. So that discussing small $\varepsilon$ is equivalent to discussing small amplitude equations.

The equation (90) is ill-posed in any space and one can construct initial data for which there is no existence in any finite interval of time. As we will see later, the non-linear term does not make it well posed in the spaces $X$ we will consider later.

The equation (90) is a 4th order equation in space. Since it is second order in time, it is standard to write it as a first order system

$$(91) \quad \dot{z} = \mathcal{L}_\mu z + \mathcal{N}(z),$$

where

$$\mathcal{L}_\mu = \begin{pmatrix} 0 & 1 \\ \partial_x^2 + \mu \partial_x^4 & 0 \end{pmatrix}$$

and

$$\mathcal{N}(z) = (0, \partial_x^2 u^2).$$
Notice that (91) has the structure we assumed in (3), namely that the evolution operator is the sum of a linear and constant operator and a nonlinear part, which is of lower order than the linear part.

10.2. Choice of spaces. In this section we present some choices of spaces $X,Y$ for which the operators entering in the Boussinesq equation satisfy the assumptions of Theorem 3.5. As indicated in Section 3.6.2, there are several choices and it is advantageous to follow a choice for the local uniqueness part and a different one for the existence. The spaces we consider will have one free parameter.

For $\rho > 0$ we denote:

$$D_\rho = \{ z \in \mathbb{C}/\mathbb{Z} \mid |\text{Im } z| < \rho \}$$

and denote $H^{\rho,m}(\mathbb{T})$ for $\rho > 0$ and $m \in \mathbb{N}$, the space of analytic functions $f$ in $D_\rho$ such that the quantity

$$\|f\|_{\rho,m}^2 = \sum_{k \in \mathbb{Z}} |f_k|^2 e^{4\pi \rho |k|} (|k|^{2m} + 1)$$

is finite, and where $\{f_k\}_{k \in \mathbb{Z}}$ are the Fourier coefficients of $f$. For any $\rho > 0$ and $m \in \mathbb{N}$, the space $\left(H^{\rho,m}(\mathbb{T}), \| \cdot \|_{\rho,m} \right)$ is a Hilbert space. Furthermore, this scale of Hilbert spaces $H^{\rho,m}(\mathbb{T})$ for $\rho > 0$ and $m > \frac{1}{2}$ is actually a Hilbert algebra for pointwise multiplication, i.e. for every $u,v \in H^{\rho,m}(\mathbb{T})$ there exists a constant $C$ such that

$$\|u v\|_{\rho,m} \leq C \|u\|_{\rho,m} \|v\|_{\rho,m}.$$  

Extending the definition to $\rho = 0$, $H^{0,m}(\mathbb{T})$ is the standard Sobolev space on the torus and for $\rho > 0$, $H^{\rho,m}(\mathbb{T})$ consists of analytic functions on the extended strip $D_\rho$ with some $L^2$-integrability conditions on the derivatives up to order $m$ on the strip $D_\rho$.

As already noticed, we are going to construct quasi-periodic solutions in the class of small amplitude solutions for (90).

For the system (91), it is natural to consider the space for $\rho > 0$ and $m > \frac{5}{2}$

$$X_{\rho,m} = H^{\rho,m} \times H^{\rho,m-2}.$$  

We note that $\mathcal{L}_\mu$ sends $X_{\rho,m}$ into $X_{\rho,m-2}$, but we observe that this is not really used in Theorem 3.5. By the Banach algebra property of the scale of spaces $H^{\rho,m}(\mathbb{T})$ when $m > 1/2$ and the particular form of the nonlinearity, we have the following proposition (see dlL09).

**Proposition 10.1.** The nonlinearity $\mathcal{N}$ is analytic from $X_{\rho,m}$ into $X_{\rho,m}$ when $m > 5/2$.

In the system language, it is useful to think of $\mathcal{L}_\mu$ as an operator of order 2 and of $\mathcal{N}$ as an operator of order 0.

Hence, in the present case, we can take $Y = X$ in the abstract Theorem 3.5.

**Remark 10.1.** Note that this gives a rigorous proof that the nonlinear evolution is ill-posed. If the non-linear evolution was well-posed in some of the $X_{\rho,m}$ spaces with $m > 5/2$, we could consider the nonlinear evolution as a perturbation of the linear one.
Using the usual Duhamel formula of Lipschitz perturbations of semigroups [Hen81], we could conclude that the linear evolution is well posed, which is patently false.

We will be actually considering a subspace of $X$ denoted $X_0$ consisting of functions $z(t) \in X$ such that

\begin{equation}
\int_0^1 dx z(\cdot, x) dx = 0. \tag{93}
\end{equation}

\begin{equation}
\int_0^1 dx \partial_t z(\cdot, x) = 0. \tag{94}
\end{equation}

\begin{equation}
z(\cdot, x) = z(\cdot, -x) \tag{95}
\end{equation}

At the formal level, the subspace $X_0$ is invariant under the equation of (91). In contrast with the normalizations (93) and (94) that can be enforced by a change of variables, (95) is a real restriction. It is possible to develop a theory without (95), but we will not pursue it here.

We now check that the assumptions of Theorem 3.5 are met. The main steps are to verify the formal assumptions of Theorem 3.5 and construct approximate solutions which are non degenerate.

10.3. Linearization around 0. We first study the eigenvalue problem for $U \in X, \sigma \in \mathbb{C}$

\[ \mathcal{L}_\mu U = \sigma U. \]

This leads to the eigenvalue relation

\[ \sigma^2 = -4\pi^2 k^2 + 16\pi^4 \mu k^4 = -4\pi^2 k^2 (1 - 4\pi^2 \mu k^2) \]

for $k \in \mathbb{Z}$. By symmetry, we assume that $k \geq 0$ and the spectrum follows by reflection with respect to the imaginary axis. We have the following lemma.

**Lemma 10.2.** The operator $\mathcal{L}_\mu$ has discrete spectrum in $X$. Furthermore, we have the following

- The center spectrum of $\mathcal{L}_\mu$ consists in a finite number of eigenvalues. Furthermore, the dimension of the center subspace is even.

- The hyperbolic spectrum is well separated from the center spectrum.

**Proof.** From the equation,

\[ \sigma^2 = -4\pi^2 k^2 + 16\pi^4 \mu k^4 = -4\pi^2 k^2 (1 - 4\pi^2 \mu k^2) \]

we deduce easily that the spectrum is discrete in $X$. Furthermore, 0 is not an eigenvalue since we assume $u$ to have average 0. Finally, we notice that when $0 < k^2 < \frac{1}{4\pi^2 \mu}$, one has $\sigma^2 < 0$ and since there is a finite (even) number of values in this set, this leads to the desired result. The separation of the spectrum directly follows from the discreteness of the spectrum. \qed
We then have the following set of eigenvalues
\[
\text{Spec}(\mathcal{L}_\mu) = \left\{ \pm 2\pi i |k| \sqrt{1 - 4\pi^2 \mu k^2} = \pm \sigma_k(\mu) \right\}_{k \geq 1}.
\]

The center space \(X_0^c\) is the eigenspace degenerate by the eigenfunctions corresponding to the eigenvalues \(\sigma_k(\mu)\) for which indices \(k = 1, \ldots, \ell\) we have \(1 - 4\pi^2 \mu k^2 \geq 0\). The center subspace \(X_0^c\) is spanned by the eigenvectors
\[
U_k = (u_k, v_k) = (\cos(2\pi kx), \sigma_k(\mu) \cos(2\pi kx))_{k=1,\ldots,\ell}.
\]

Any element \(U\) on the center subspace can be expressed as:
\[
U = \sum_{k=1}^\ell \alpha_k U_k,
\]
with the \(\alpha_k\) arbitrary real numbers.

10.4. Verifying the smoothing properties of the partial evolutions of the linearization around 0. We now come to the evolution operators and their smoothing properties. We have:

**Lemma 10.3.** The operator \(\mathcal{L}_\mu\) generates semi-group operators \(U^{s,u}_\theta(t)\) in positive and negative times. Furthermore, the following estimates hold
\[
\|U^s_\theta(t)\|_{X,X} \leq \frac{C}{t^{1/2}} e^{-Dt}, \quad t > 0
\]
and
\[
\|U^u_\theta(t)\|_{X,X} \leq \frac{C'}{|t|^{1/2}} e^{D't}, \quad t < 0
\]
for some constants \(C, C', D, D' > 0\).

**Proof.** The proof is given in detail in [dlL09 page 404-405]. It is based on observing that the evolution operator in the (un)stable spaces can be expressed in Fourier series. Since the norms considered are given by the Fourier terms (with different weights), it suffices to estimate the sup of the multipliers times the ratio of the weights. \(\square\)

Until now, we have considered only the linearization around the equilibrium 0 in \(X\). Of course, by the stability theory of the splittings developed in Section 6, the spectral non-degeneracy properties will be satisfied by all the approximate solutions that are small enough in the smooth norms. As we will see, our approximate solutions will be trigonometric polynomials with small coefficients.
10.5. Construction of an approximate solution. This section is devoted to the construction of an approximate non-degenerate solution for equation (91). We use a Lindstedt series argument to construct approximate solutions for all “nonresonant” values of \( \mu \). Then, we will verify the twist non-degeneracy conditions for some values of \( \mu \) only.

**Remark 10.2.** For the experts, we note that the analysis is remarkably similar to the perturbative analysis near elliptic fixed points in Hamiltonian systems. We have found useful the treatment in [Poi99, Vol 2]. More modern treatments based on transformation theory are in [Mos68, Zeh73, Dou88]. In our case, the transformation theory is more problematic, hence we take advantage of the a-posteriori format and just construct approximate solutions for the initial guess.

The following result establishes the existence (and some uniqueness which we will not use) of the Lindstedt series under appropriate non-resonance conditions.

**Lemma 10.4.** Let \( \ell \) be as before. For all \( N \geq 2 \), assume the nonresonance condition to order \( N \) given by

\[
F(k, j) \neq 0, \quad k \in \mathbb{Z}^\ell, j \in \mathbb{N}, 1 < |k| \leq N
\]

where

\[
F(k, j) \equiv \left[ (\omega_0 \cdot k)^2 - 2\pi^2 (j^2 - 2\mu j^2) \right].
\]

Then, for all \( U_1 \) depending on \( \ell \) parameters, there exist \( (\omega^1, ..., \omega^N) \in (\mathbb{R}^\ell)^N \) and \( (U_2, ..., U_N) \in (H^{p,m}(\mathbb{T}))^{N-1} \) parametrized by \( (A^1_1, ..., A^1_\ell) \in \mathbb{R}^\ell \) for any \( \rho > 0 \) such that for any \( \sigma \geq 0 \)

\[
\|(u^\varepsilon_{\leq N})_{tt} - (u^\varepsilon_{\leq N})_{xx} - \mu(u^\varepsilon_{\leq N})_{xxxx} - ((u^\varepsilon_{\leq N})^2)_{xx} - (\mu u^\varepsilon_{\leq N})_{xxxx}\|_{H^{p,m}(\mathbb{T})} \leq C\varepsilon^{N+1}
\]

for some constant \( C > 0 \) and

\[
u^\varepsilon_{\leq N}(t, x) = \sum_{k=1}^{N} \varepsilon^k U_k(\omega^\varepsilon_{\leq N} t, x)
\]

where

\[
\omega^\varepsilon_{\leq N} = \omega^0 + \sum_{k=1}^{N} \varepsilon^k \omega^k.
\]

The coefficients \( U_k \) are trigonometric polynomials and can be obtained in such a way that the projection over the kernel of

\[
\mathcal{M}_0 = (\omega^0 \cdot \partial_\theta)^2 - \partial^2_{xx} - \mu \partial^4_{xxxx}
\]

is zero. Moreover, the normalizations (96), (93) are satisfied. With such a normalization, they are unique.
Before going into the proof itself, we comment a bit on the theory of Lindstedt series. We define the hull function as

\[ u_\varepsilon(t, x) = U_\varepsilon(\omega_\varepsilon t, x) \]

where \( U_\varepsilon : \mathbb{T}^\ell \times \mathbb{T} \rightarrow \mathbb{R} \) with \( \ell = \frac{\text{dim}X_c}{2} \).

There are two versions of the theory: one assuming the symmetry condition for the solutions

\[ U_\varepsilon(\theta, \cdot) = U_\varepsilon(-\theta, \cdot) \tag{96} \]

and another one without assuming (96). For simplicity, we will assume the symmetry. We note that, thanks to the a-posteriori format of the theorem, we only need to produce an approximate solution and verify the non-degeneracy conditions.

The function \( U_\varepsilon \) and the frequency \( \omega_\varepsilon \) produce a solution of (91) if and only if they satisfy the equation

\[ (\omega_\varepsilon \cdot \partial_\theta)^2 U_\varepsilon = \partial^2_{xx} U_\varepsilon + \mu \partial^4_{xxxx} U_\varepsilon + (U_\varepsilon^2)_{xx}. \tag{97} \]

We emphasize that we are considering now that both \( U_\varepsilon \) and \( \omega_\varepsilon \) are unknowns to be determined in (97). As we will see, we will obtain \( U_\varepsilon \) and \( \omega_\varepsilon \), depending on \( \ell \) free arbitrary parameters.

Following the standard procedure of Lindstedt series, we will consider formal expansions \( U_\varepsilon \) and \( \omega_\varepsilon \) in powers of \( \varepsilon \). We will impose that finite order truncations to order \( N \) satisfy the equation (97) up to an error \( C_N |\varepsilon|^{N+1} \). Hence, the series are not meant to converge (in general they will not) but they indicate a sequence of approximate solutions that solve the equation to higher and higher order in \( \varepsilon \). We will also verify the other non-degeneracy hypothesis of Theorem 3.5.

We consider the formal sums

\[ U_\varepsilon(\theta, x) \sim \sum_{k=1}^{\infty} \varepsilon^k U_k(\theta, x) \]

\[ \omega_\varepsilon \sim \omega^0 + \sum_{k=1}^{\infty} \varepsilon^k \omega^k. \tag{98} \]

**Remark 10.3.** Notice that the sum for \( U_\varepsilon \) starts with \( \varepsilon \) since we have in mind to consider small amplitude solutions of the equation.

The meaning of formal power solutions is that we truncate these sums at order \( N \) arbitrary, \( N \geq 1 \) and consider

\[ u_\varepsilon^{[\leq N]}(\theta, x) = \sum_{k=1}^{N} \varepsilon^k U_k(\theta, x) \]

\[ \omega_\varepsilon^{[\leq N]} = \omega^0 + \sum_{k=1}^{N} \varepsilon^k \omega^k. \]
As it often happens in Lindstedt series theory, the first terms of the recursion are different from the others. In our case, the first step will allow us to choose solutions of the first step depending on $\ell$ parameters. Once these solutions are chosen, we can obtain all the other solutions in a unique way. We note that the computations are very algorithmic and subsequently can be programmed. The normalization in the last item of Lemma 10.4 is natural in Lindstedt series theory. If one changes the parameters, introducing new parameters $A_1^i = B_1^i + \varepsilon \hat{A}_i(B_1^1, \ldots, B_1^\ell; \varepsilon)$, one obtains a totally different series, which of course parametrizes the same set of solutions. In any case, we emphasize that for us the main issue is to construct an approximate solution.

**Proof.** We substitute the sums for $\omega_\varepsilon$ and $U_\varepsilon$ into (97) and identify at all orders.

**Order 1:** We get

$$(\omega_0 \cdot \partial_{\theta})^2 U_1 = \partial_{xx}^2 U_1 + \mu \partial_{xxxx}^4 U_1.$$ 

We search for solutions of the form $\cos(2\pi \omega_0^j \theta_j) \cos(2\pi j x)$ where $j \in \mathbb{N}$. Therefore the frequencies are given by the relation

$$\omega_0^j = 2\pi |j| \sqrt{1 - 4\pi^2 \mu j^2}.$$ 

We assume now that $4\pi^2 \mu j^2 \neq 1$ and $1 - 4\pi^2 \mu j^2 \geq 0$ which means that $j = 1, \ldots, \ell$ where $\ell = \lfloor \sqrt{\frac{1}{2\pi\mu}} \rfloor$.

Now, we get the frequency vector $\omega^0$, given by:

$$(\omega^0)_{j=1,\ldots,\ell} = \left(2\pi |j| \sqrt{1 - 4\pi^2 \mu j^2}\right)_{j=1,\ldots,\ell}.$$ 

All the solutions of the equation satisfying the symmetry conditions (36), (96) are given by:

$$(99) \quad U_1(\theta, x) = \sum_{j=1}^{\ell} A_1^j \cos(2\pi \omega_0^j \theta_j) \cos(2\pi j x).$$

This is the customary analysis of the linearized equations in normal modes. For future reference, we denote

$$M_0 = (\omega_0 \cdot \partial_{\theta})^2 - \partial_{xx}^2 - \mu \partial_{xxxx}^4.$$ 

We note that the operator $M_0$ is diagonal on trigonometric polynomials and we have that

$$M_0 \cos(2\pi k \cdot \theta) \cos(2\pi j x) = F(k, j) \cos(2\pi k \cdot \theta) \cos(2\pi j x)$$

where

$$F(k, j) \equiv \left[ (\omega_0 \cdot k)^2 - 2\pi^2 (j^2 - 2\mu \pi^2 j^2) \right].$$

For convenience we will make the important non-resonance condition to order $N$

$$(100) \quad F(k, j) \neq 0, \quad k \in \mathbb{Z}, j \in \mathbb{N}, 1 < |k| \leq N.$$
The nonresonance condition is very customary in the study of elliptic fixed points. It says that the basic frequencies are not a combination of each other. Note that if we fix \( \ell, k \) and \( j \) the condition \( F(k, j) = 0 \) is a polynomial equation in \( \mu \) so that it is satisfied only for a finite number of \( \mu \). This says that for the interval of \( \mu \) where \( \ell \) is constant, we may have to exclude at most a finite number of values of \( \mu \). Of course, requiring the result for all \( N \) means excluding at most a countable number of values of \( \mu \). A detailed analysis may obtain sharper conclusions on the values of \( \mu \) that need to be excluded. In the final applications, we will only consider the interval in which \( \ell = 1 \), where it is easy to see that there is no resonant value.

**Proposition 10.5.** Under the non-resonance condition, the kernel of the operator \( M_0 \) is precisely \( \omega^0 \cdot \partial_\theta \) of the span of the solutions \( U_1 \) obtained before in (99).

**Order** \( m \geq 2 \): The general equation to be solved at order \( m \) to ensure that the equation (97) is solvable to order \( m \) has the form

\[
M_0 U_m + 2(\omega^{m-1} \cdot \partial_\theta)(\omega^0 \cdot \partial_\theta) U_1 = R_m(U_1, ..., U_{m-1}, \omega^0, ..., \omega^{m-2})
\]

where \( R_m \) is polynomial in its arguments and their derivatives (up to order 4). In particular, if \( U_1, ..., U_{m-1} \) are trigonometric polynomials then so is \( R_m \). It is also easy to see that if \( U_1, ..., U_{m-1} \) have the symmetry properties (96) so does \( R_m \). Hence, using the addition formula for products of angles, we can express

\[
R_m = \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} C_{k,j}(A^0_1, ..., A^0_\ell) \cos(2\pi k \cdot \theta) \cos(2\pi j x)
\]

We inductively assume that \( U_1, ..., U_{m-1} \) are trigonometric polynomials and that \( \omega^0, ..., \omega^{m-2} \) have been found. Then, we will show that we can find \( \omega^{m-1}, U_m \) in such a way that the equation (101) is solvable. Furthermore, the solution is unique if we impose the normalization at the end of Lemma 10.4. The equation (101) can be solved by identifying the coefficients of \( \cos(2\pi k \cdot \theta) \cos(2\pi j x) \) on both sides.

Since \( R_m \) is a trigonometric polynomial, we can separate the terms into terms that are in the kernel of \( M_0 \) and terms for which the multiplier \( F(k, j) \) corresponding to \( M_0 \) is not zero. We also note that, under the non-resonance hypothesis, we have that the kernel of \( M_0 \) is precisely the functions that appear in \( U_1 \). The term \((\omega^{m-1} \cdot \partial_\theta)(\omega^0 \cdot \partial_\theta)U_1\) lies in the kernel of \( M_0 \).

Since \( M_0 \) is diagonal, the terms in the kernel of \( M_0 \) are precisely those that are not in the range of \( M_0 \). For the terms for which the multiplier \( F(k, j) \) is non zero (i.e. those terms in the range of \( M_0 \)), we can invert \( M_0 \) and, hence obtaining

\[
U_m(k, j) = \frac{C_{k,j}}{F(k, j)}.
\]

For the terms that lie in the kernel of \( M_0 \), we cannot divide by the multiplier \( F(k, j) \) but instead obtain uniquely \( \omega^{m-1} \) to solve (101). Note that this uses the non-resonance condition so that that the kernel of \( M_0 \) is precisely functions that appear in \( U_1 \).
Of course, to solve (101), we could add any function in the kernel of $\mathcal{M}_0$. Under the normalization condition, we see that the term to add is uniquely determined to be zero. The evaluation of the norm in the Lemma comes directly from the fact that we are dealing with trigonometric polynomials, hence belonging to any Sobolev space. □

10.6. Application of Theorem 3.5 to the approximate solutions. End of the proof of Theorem 3.7. Let $\omega^0$ as in Theorem 3.7 and consider $U_\varepsilon$ the function constructed in the previous section. Denote

$K_0(\theta) = \left( \begin{array}{c} U_\varepsilon(\cdot, \cdot) \\ \omega_\varepsilon \cdot \partial_{\theta} U_\varepsilon(\cdot, \cdot) \end{array} \right) \in X_0$.

We will proceed to verify the assumptions of Theorem 3.5 taking as initial conditions of the iteration the results of the Lindstedt series. This will require carrying out explicitly the calculations indicated before to order 3 and verifying that the twist condition is satisfied.

10.6.1. Smallness assumption on the error and range of $K_0$. Consider $K_0$ as above. Then Lemma 10.4 ensures directly that the smallness assumption in Theorem 3.5 are satisfied with an error smaller than $C_N|\varepsilon|^{N+1}$ for arbitrary large $N$.

Note that this is verified for all values of $\ell$.

10.6.2. Spectral non-degeneracy. We check conditions 3.3. For $\varepsilon = 0$, all the conditions in 3.3 are met by the previous discussion. In particular there exists an invariant splitting denoted

$$X_0 = X^n_0 \oplus X^s_0 \oplus X^c_0.$$  

Now, by construction of $K_0$, choosing $\varepsilon$ small enough again and using the perturbation theory of the bundles developed in section 6 (see Lemma 6.2), there exists an invariant splitting for $K_0$ for $\varepsilon$ small enough satisfying all the desired properties and this proves the spectral non-degeneracy conditions 3.3 for $K_0$, together with the suitable estimates.

Note that this is verified for all values of $\ell$.

10.6.3. Twist condition. We now check the twist condition in Definition 3.4. Pick a Diophantine frequency $\omega$ as in Theorem 3.7. Recall that the family of perturbative solutions is parameterized by $A_j^1$ for $j = 1, \ldots, \ell$, the $\ell$ parameters giving $U_1$. In the system of coordinates given by $(A_1^1, \ldots, \theta)$, the twist condition amounts to showing that

$$|\det \left( \partial_{A_1^1} \omega_i^N \right)^{-1} > T_N(\varepsilon) > 0.$$  

To verify the twist condition, we will assume that $\ell = 1$. This is the only reason why in Theorem 3.7 we are assuming $\ell = 1$. 

If we can show that $T_N(\varepsilon) > C|\varepsilon|^a$ for some positive $a, C$, ($1 \leq a < N$) then we claim that we can finish the construction. The crucial remark is that we also have $T_{\tilde{N}}(\varepsilon) \geq \tilde{C}|\varepsilon|^a$ for any $\tilde{N} > N$ since we are only adding higher order terms. As we will see $\omega^1 = 0$ so we will have to go to order 3. Let us first consider the case $m = 2$. We have that the equation at order 2 and assuming that $\ell = 1$ writes
\[ M_0 U_2 + 2(\omega^1 \cdot \partial_\theta)(\omega^0 \cdot \partial_\theta)U_1 = (U_1^2)_{xx}. \]
We have
\[ U_1^2 = A^2 \cos(2\pi \theta) \cos^2(2\pi \varepsilon). \]
It yields
\[ U_1^2 = \frac{A^2}{4} (1 + \cos(4\pi \theta))(1 + \cos(4\pi \varepsilon)) \]
and
\[ (U_1^2)_{xx} = -4\pi^2 A^2 (1 + \cos(4\pi \theta)) \cos(4\pi \varepsilon), \]
since this is not in the range, hence one has $\omega^1 = 0$. We then go to order $m = 3$ which gives the equation (taking into account that $\omega^1 = 0$)
\[ M_0 U_3 + 2(\omega^0 \cdot \partial_\theta)(\omega^2 \cdot \partial_\theta)U_1 = 2(U_1 U_2)_{xx}. \]
From the previous step, one has
\[ U_2 = -4\pi^2 A^2 \left( \frac{\cos(4\pi \varepsilon) \cos(4\pi \theta)}{F(2,2)} + \frac{\cos(4\pi \varepsilon)}{F(0,2)} \right). \]
Hence we have
\[ (U_1 U_2)_{xx} = -4\pi^2 A^4 \left( -\cos(2\pi \varepsilon) - 9 \cos(6\pi \varepsilon) \right) \left( \frac{-(\cos(2\pi \theta) + \cos(6\pi \theta))}{4F(2,2)} + \frac{\cos(2\pi \theta)}{2F(0,2)} \right). \]
Identifying according to the discussion before, one gets that $\omega^2$ is given by
\[ \omega^2 = CA^4 \frac{1}{4F(2,2)} - \frac{1}{2F(0,2)}, \]
for some constant $C$. We check now that $\frac{1}{4F(2,2)} - \frac{1}{2F(0,2)} \neq 0$. We compute
\[ F(0,2) - 2F(2,2) = -12 - 8\mu \neq 0, \]
hence $\omega^2 \neq 0$. As a consequence one has
\[ \omega_{\varepsilon^{\leq N}} = \omega^0 + \varepsilon^2 \omega^2 + h.o.t. \]
and furthermore $\omega^2 \neq 0$. Since $\omega^0$ does not depend on $A$, we have that the twist condition writes
\[ \varepsilon^2 \left( \frac{d\omega^2}{dA} \right) + h.o.t. \]
Hence, taking $\tilde{N}$ sufficiently large, we can apply Theorem 4.1 to obtain Theorem 3.7.
11. Application to the Boussinesq system

In this section, we consider the Boussinesq system of water waves. This system is even more interesting than the Boussinesq equation (see Section 10) for at least two reasons: first the system is more "singular"; second, the full power of the two spaces approach has to be used, i.e. one has to take the spaces \( X \) and \( Y \) such that \( X \neq Y \). The system writes

\[
\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x - \mu \partial_{xxx} \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \partial_x(uv) \\ 0 \end{pmatrix}
\]

where \( t > 0 \) and \( x \in \mathbb{T} \).

The elementary linear analysis around the \((0, 0)\) equilibrium can be found in [dlL09]. Recall that the eigenvalues of the linearization around 0 are given by

\[
\omega(k) = \pm |k| 2\pi i \sqrt{1 - 4\pi^2 \mu k^2} \quad k \in \mathbb{Z}
\]

The eigenvectors are given by

\[
U_j = (2\pi j \cos(2\pi \theta_j) \cos(2\pi j), \sqrt{(2\pi j)^2 - \mu(2\pi jx)^2} \sin(2\pi \theta_j) \sin(2\pi jx))
\]

for \( j = 1, \ldots, \ell \) where \( \ell \) is the smallest integer such that \( 1 - 4\pi^2 \mu k^2 \geq 0 \).

We denote by \( \omega^0 \) the vector whose components are all the real frequencies that appear

\[
\omega^0 = (\omega(k_1), \omega(k_2), \ldots, \omega(k_\ell));
\]

\[
\{k_1, \ldots, k_\ell\} = \{k \in \mathbb{Z} \mid k > 0; 1 - 4\pi^2 \mu k^2 \geq 0\}
\]

The following symmetries are preserved formally by the equation

\[
\begin{cases}
  u(t, -x) = u(-t, x) = u(t, x), \\
  v(t, -x) = v(-t, x) = -v(t, x).
\end{cases}
\]

We remind that we take

\[
X = H^{\rho,m}(\mathbb{T}) \times H^{\rho,m+1}(\mathbb{T})
\]

and

\[
Y = H^{\rho,m-1}(\mathbb{T}) \times H^{\rho,m}(\mathbb{T})
\]

We denote by \( X_0 \) the set of functions in \( X \) satisfying the symmetries (107) and also the momentum

\[
\int_0^1 u(t, x) \, dx = 0
\]

and

\[
\int_0^1 v(t, x) \, dx = 0.
\]

The previous quantities, as in the case of the Boussinesq equation, are preserved by the equation under consideration. It is proved in [dlL09] the following proposition

**Proposition 11.1.** The nonlinearity \( \mathcal{N}(u, v) = (\partial_x(uv), 0) \) is analytic (indeed a polynomial) from \( X \) to \( Y \).

Furthermore one has (see also [dlL09])
Lemma 11.2. For $t > 0$, one has
\[ \|U^u_\theta(t)\|_{Y,X} \leq \frac{C}{t^{1/2}} e^{-D t} \]
and for $t < 0$ one has
\[ \|U^v_\theta(t)\|_{Y,X} \leq \frac{C'}{|t|^{1/2}} e^{D' t} \]
for some $C, C', D, D' > 0$.

11.0.4. Approximate solution. We will not repeat the whole discussion which is very close to the one on the Boussinesq equation. Instead, we provide the necessary changes. The strategy is completely parallel to the one for the Boussinesq equation. Define two hull functions
\[
u_\varepsilon(t, x) = U_\varepsilon(\omega_\varepsilon t, x)
\]
and
\[
u_\varepsilon(t, x) = V_\varepsilon(\omega_\varepsilon t, x)
\]
Once again we consider Lindstedt series in powers of $\varepsilon$.

Similarly to the previous section, we have

Lemma 11.3. Let $\ell$ be as before. For all $N > 1$, there exists $(\omega^1, ..., \omega^N) \in (\mathbb{R}^\ell)^N$, $(U_1, ..., U_N) \in (H^{\rho,m}(\mathbb{T}))^N$ and $(V_1, ..., V_N) \in (H^{\rho,m-1}(\mathbb{T}))^N$ for some $\rho > 0$ such that
\[
\left\| \partial_t \begin{pmatrix} u_\varepsilon \\
 v_\varepsilon \end{pmatrix} - \begin{pmatrix} 0 & -\partial_x - \mu \partial_{xxx} \\
 -\partial_x & 0 \end{pmatrix} \begin{pmatrix} u_\varepsilon \\
 v_\varepsilon \end{pmatrix} + \begin{pmatrix} \partial_x (u_\varepsilon v_\varepsilon) \\
 0 \end{pmatrix} \right\|_{H^{\rho,m}(\mathbb{T}) \times H^{\rho,m-1}(\mathbb{T})} \leq C \varepsilon^{N+1}
\]
for some constant $C > 0$ and
\[
u_\varepsilon^{[\leq N]}(t, x) = \sum_{k=1}^N \varepsilon^k U_k(\omega_\varepsilon^{[\leq N]} t, x),
\]
\[
u_\varepsilon^{[\leq N]}(t, x) = \sum_{k=1}^N \varepsilon^k V_k(\omega_\varepsilon^{[\leq N]} t, x),
\]
where
\[
\omega_\varepsilon^{[\leq N]} = \omega^0 + \sum_{k=1}^N \varepsilon^k \omega^k.
\]

The solutions depend on $\ell$ arbitrary parameters, where $\ell$ is the number of the degrees of freedom of the kernel.

Proof. We develop a general theory, parallel with the one of the Boussinesq equation in the previous section. The main new difficulties is that we are dealing with systems of equations and that the linear operator is not diagonal in an obvious sense. Denote
\[
A = \begin{pmatrix} 0 & -\partial_x - \mu \partial^3_x \\
-\partial_x & 0 \end{pmatrix}
\]
At general order \( m \geq 2 \), we search for solutions of the form
\[
U_m(\theta, x) = \sum_{j, k} U_{k,j}^m \cos(2\pi k \cdot \theta) \cos(2\pi j x)
\]
and
\[
V_m(\theta, x) = \sum_{j, k} V_{k,j}^m \sin(2\pi k \cdot \theta) \sin(2\pi j x).
\]
The previous formulae come from the assumptions of symmetry of the solutions. Denoting \( W_m = (U_m, V_m) \) one has
\[
(\omega^0 \cdot \partial_\theta - A) W_m + \omega^{m-1} \cdot \partial_\theta W_1 = R_m(\omega^0, \ldots, \omega^{m-2}, W_{m-1}).
\]
It is important to notice the operator \( M_0 = (\omega^0 \cdot \partial_\theta - A) \) is not self-adjoint in \( X \) and does not act as a multiplication in an easy basis of vectors. We then need to understand the range of this operator. Its domain is spanned by
\[
\left((\cos(2\pi k \cdot \theta) \cos(2\pi j x), \sin(2\pi k \cdot \theta) \sin(2\pi j x))\right).
\]
The range is then the space of vector functions of the form of linear combinations of the basis
\[
\left((\sin(2\pi k \cdot \theta) \cos(2\pi j x), \cos(2\pi k \cdot \theta) \sin(2\pi j x))\right).
\]
**Order 1** One has
\[
(109) \quad \omega^0 \cdot \partial_\theta \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} -\partial_x V_1 - \mu \partial_x^3 V_1 \\ -\partial_x U_1 \end{pmatrix}
\]
We expand
\[
U_1 = \sum_{j=1}^\ell A_j \cos(2\pi \theta_j) \cos(2\pi j x)
\]
and
\[
V_1 = \sum_{j=1}^\ell B_j \sin(2\pi \theta_j) \sin(2\pi j x)
\]
As in the case of the Boussinesq equation, this gives directly the vector \( \omega^0 \) and one can take any \( A_j, B_j \). For convenience later, we assume
\[
A_j \neq 0, B_j \neq 0, \quad j = 1, \ldots, \ell
\]
The rest of the orders is like in the previous section on the Boussinesq equation. □

We now prove Theorem \( 3.9 \) i.e. considering the case \( \ell = 1 \). It amounts to apply the abstract theorem \( 3.9 \). As in Section \( 10 \), this is done by checking the twist condition, the rest of the proof being completely parallel. We have first
\[
W_1 = A \begin{pmatrix} \cos(2\pi \theta) \cos(2\pi x) \omega^0 2\pi \\ \sin(2\pi \theta) \sin(2\pi x) 2\pi \omega^0 \end{pmatrix}
\]
For simplicity of writing we suppress the harmless parameter \( A \).
At order 2, one has

\[ \omega^0 \cdot \partial_\theta \left( \frac{U_2}{V_2} \right) + \omega^1 \cdot \partial_\theta \left( \frac{U_1}{V_1} \right) = \left( -\partial_x V_2 - \mu \partial_x^3 V_2 + \partial_x (U_1 V_1) \right) / -\partial_x U_2 \]

Furthermore, one has (the map \( F(j, k) \) is defined as in the previous section)

\[ \partial_x (U_1 V_1) = \frac{1}{2} \sin(4 \pi \theta) \sin(4 \pi x). \]

This is never in the range of \( M_0 = \omega^0 \cdot \partial_\theta - \mathcal{A}. \) Therefore, we obtain \( \omega^1 = 0. \) Additionally, one has

\[ \mathcal{W}_2 = \frac{1}{F(2, 2)} \left( \frac{1}{8} \cos(4 \pi \theta) \cos(4 \pi x) \right) + \frac{1}{F(-2, 2)} \left( -\frac{1}{2} \cos(4 \pi \theta) \cos(4 \pi x) \right) \]

We go now to order 3. We have

\[ \mathcal{M}_0 \left( \frac{U_3}{V_3} \right) + \omega^2 \cdot \partial_\theta \left( \frac{U_1}{V_1} \right) = \left( \partial_x (U_1 V_2) + \partial_x (U_2 V_1) \right) \]

We have by lengthy but straightforward computations

\[ U_1 V_2 = \frac{1}{8 F(2, 2)} \left( \sin(6 \pi \theta) - \sin(2 \pi \theta) \right) \left( \sin(6 \pi x) - \sin(2 \pi x) \right) - \]

\[ \frac{\omega^0}{8 F(2, 2)} \left( \sin(6 \pi \theta) - \sin(2 \pi \theta) \right) \left( \sin(6 \pi x) - \sin(2 \pi x) \right) \]

Similarly

\[ U_2 V_1 = \frac{1}{8 F(2, 2)} \left( \sin(6 \pi \theta) - \sin(2 \pi \theta) \right) \left( \sin(6 \pi x) - \sin(2 \pi x) \right) + \]

\[ \frac{\omega^0}{8 F(-2, 2)} \left( \sin(6 \pi \theta) - \sin(2 \pi \theta) \right) \left( \sin(6 \pi x) - \sin(2 \pi x) \right) \]

Hence one has

\[ \partial_x (U_1 V_2) + \partial_x (U_2 V_1) \]

\[ = \frac{18}{F(-2, 2)} - \frac{\omega^2}{F(2, 2)} + \frac{1}{F(2, 2)} - \frac{1}{F(-2, 2)} \left( 2 \pi \sin(2 \pi \theta) \cos(2 \pi x) \right) + R(\theta, x) \]

where \( R(\theta, x) \) is a trigonometric polynomial involving higher order frequencies. Since the coefficient

\[ \frac{\pi}{4} \left( \frac{1}{F(-2, 2)} - \frac{\omega^0}{F(2, 2)} + \frac{1}{F(-2, 2)} - \frac{1}{F(2, 2)} \right) \]

is non-zero only on a finite number of values of \( \mu, \) one deduces that \( \omega^2 \) is nonzero, hence the twist condition. The rest of the proof follows.
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School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332
E-mail address: rafael.delallave@math.gatech.edu

Johns Hopkins University, Krieger Hall, Baltimore, USA
E-mail address: sire@math.jhu.edu