AFFINE CONNECTIONS AND GAUSS-BONNET THEOREMS IN THE HEISENBERG GROUP

YONG WANG

Abstract. In this paper, we compute sub-Riemannian limits of Gaussian curvature associated to two kinds of Schouten-Van Kampen affine connections and the adapted connection for a Euclidean $C^2$-smooth surface in the Heisenberg group away from characteristic points and signed geodesic curvature associated to two kinds of Schouten-Van Kampen affine connections and the adapted connection for Euclidean $C^2$-smooth curves on surfaces. We get Gauss-Bonnet theorems associated to two kinds of Schouten-Van Kampen affine connections in the Heisenberg group.

1. Introduction

In [5], Gaussian curvature for non-horizontal surfaces in sub-Riemannian Heisenberg space $\mathbb{H}^1$ was defined and a Gauss-Bonnet theorem was proved. In [1],[2], Balogh-Tyson-Vecchi used a Riemannian approximation scheme to define a notion of intrinsic Gaussian curvature for a Euclidean $C^2$-smooth surface in the Heisenberg group $\mathbb{H}^1$ away from characteristic points, and a notion of intrinsic signed geodesic curvature for Euclidean $C^2$-smooth curves on surfaces. These results were then used to prove a Heisenberg version of the Gauss-Bonnet theorem. In [8], Veloso verified that Gaussian curvature of surfaces and normal curvature of curves in surfaces introduced by [5] and by [1] to prove Gauss-Bonnet theorems in Heisenberg space $\mathbb{H}^1$ were unequal and he applied the same formalism of [5] to get the curvatures of [1]. With the obtained formulas, it is possible to prove the Gauss-Bonnet theorem in [1] as a straightforward application of the Stokes theorem. In [9], we proved Gauss-Bonnet theorems for the affine group and the group of rigid motions of the Minkowski plane. In [10], we obtained Gauss-Bonnet theorems for BCV spaces and the twisted Heisenberg group.

In [6], Klatt proved a Gauss-Bonnet theorem associated to a metric connection (see Proposition 5.2 in [6]). When a Riemannian manifold has a splitting tangent bundle, we can define a Schouten-Van Kampen affine connection which is a metric connection. In [3], [7], Schouten-Van Kampen affine connections on foliations and almost (para) contact...
manifolds were studied. In [3], in order to prove a Gauss-Bonnet theorem in the Heisenberg group, the adapted connection was introduced. The adapted connection is a metric connection. Motivated by above works, it is interesting to study Gauss-Bonnet theorems associated to Schouten-Van Kampen affine connections and the adapted connection in the Heisenberg group. Let $K_{\Sigma, \nabla^1, \infty}$ and $k_{\gamma_i, \Sigma}^{\infty, \nabla^1, s}$ be the intrinsic Gauss curvature associated to the second kind of Schouten-Van Kampen affine connections and the intrinsic signed geodesic curvature associated to the second kind of Schouten-Van Kampen affine connections. Our main theorem (Theorem 3.9) in this paper is as following: (see Section 3 for related definitions)

**Theorem 1.1.** Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface with finitely many boundary components $(\partial \Sigma)_i, i \in \{1, \cdots, n\}$, given by Euclidean $C^2$-smooth regular and closed curves $\gamma_{i}: [0, 2\pi] \rightarrow (\partial \Sigma)_i$. Suppose that the characteristic set $C(\Sigma)$ satisfies $\mathcal{H}^1(C(\Sigma)) = 0$ and that $|\nabla_{H} u|_{H}^{-1}$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma)$, then

$$\int_{\Sigma} K_{\Sigma, \nabla^1, \infty} d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k_{\gamma_i, \Sigma}^{\infty, \nabla^1, s} ds = 0. \tag{1.1}$$

In Section 2, we compute sub-Riemannian limits of Gaussian curvature associated to the first kind of Schouten-Van Kampen affine connections for a Euclidean $C^2$-smooth surface in the Heisenberg group away from characteristic points and signed geodesic curvature associated to the first kind of Schouten-Van Kampen affine connections for Euclidean $C^2$-smooth curves on surfaces. We get the Gauss-Bonnet theorem associated to the first kind of Schouten-Van Kampen affine connections in the Heisenberg group. In Section 3, we compute sub-Riemannian limits of Gaussian curvature associated to the second kind of Schouten-Van Kampen affine connections for a Euclidean $C^2$-smooth surface in the Heisenberg group away from characteristic points and signed geodesic curvature associated to the second kind of Schouten-Van Kampen affine connections for Euclidean $C^2$-smooth curves on surfaces. We get the Gauss-Bonnet theorem associated to the second kind of Schouten-Van Kampen affine connections in the Heisenberg group. In Section 4, we compute sub-Riemannian limits of Gaussian curvature associated to the adapted connection for a Euclidean $C^2$-smooth surface in the Heisenberg group away from characteristic points and signed geodesic curvature associated to the adapted connection for Euclidean $C^2$-smooth curves on surfaces.

2. **THE GAUSS-BONNET THEOREM ASSOCIATED TO THE FIRST KIND OF SCHOUTEN-VAN KAMPEN AFFINE CONNECTIONS IN THE HEISENBERG GROUP**

Firstly we introduce some notations on the Heisenberg group. Let $\mathbb{H}$ be the Heisenberg group $\mathbb{R}^3$ where the non-commutative group law is given by

$$(a, b, c) \ast (x, y, z) = (a + x, b + y, c + z - \frac{1}{2}(xb - ya)).$$
Let
\begin{equation}
X_1 = \partial_{x_1} - \frac{x_2}{2} \partial_{x_3}, \quad X_2 = \partial_{x_2} + \frac{x_1}{2} \partial_{x_3}, \quad X_3 = \partial_{x_3},
\end{equation}
and span\{X_1, X_2, X_3\} = T\mathbb{H}. Let H = span\{X_1, X_2\} be the horizontal distribution on \mathbb{H}. Let \omega_1 = dx_1, \omega_2 = dx_2, \omega_3 = \omega = dx_3 + \frac{1}{2}(x_2dx_1 - x_1dx_2). For the constant L > 0, let \(g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L\omega \otimes \omega\) be the Riemannian metric on \mathbb{H}. Then \(X_1, X_2, \tilde{X}_3 := L^{-\frac{1}{2}}X_3\) are orthonormal basis on \(T\mathbb{H}\) with respect to \(g_L\). We have
\begin{equation}
[X_1, X_2] = X_3, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = 0.
\end{equation}
Let \(\nabla^L\) be the Levi-Civita connection on \(\mathbb{H}\) with respect to \(g_L\). By Lemma 2.8 in [1], we have

**Lemma 2.1.** Let \(\mathbb{H}\) be the Heisenberg group, then
\begin{equation}
\nabla^L_{X_j} X_j = 0, \quad 1 \leq j \leq 3, \quad \nabla^L_{X_1} X_2 = \frac{1}{2} X_3, \quad \nabla^L_{X_2} X_1 = -\frac{1}{2} X_3,
\end{equation}
\begin{equation}
\nabla^L_{X_1} X_3 = -\frac{L}{2} X_2, \quad \nabla^L_{X_2} X_1 = -\frac{L}{2} X_2,
\end{equation}
\begin{equation}
\nabla^L_{X_2} X_3 = \nabla^L_{X_3} X_2 = \frac{L}{2} X_1.
\end{equation}

Let \(H^\perp = \text{span}\{X_3\}\) and \(P : T\mathbb{H} \to H\) and \(P^\perp : T\mathbb{H} \to H^\perp\) be the projections. We define the first kind of Schouten-Van Kampen affine connections in the Heisenberg group:
\begin{equation}
\nabla_X Y = P\nabla^L_X P Y + P^\perp \nabla^L_X P^\perp Y.
\end{equation}

By Definition 3.1 in [1], we have

**Definition 2.2.** Let \(\gamma : [a, b] \to (\mathbb{H}, g_L)\) be a Euclidean C^1-smooth curve. We say that \(\gamma\) is regular if \(\dot{\gamma} \neq 0\) for every \(t \in [a, b]\). Moreover we say that \(\gamma(t)\) is a horizontal point of \(\gamma\) if
\[\omega(\dot{\gamma}(t)) = \frac{\dot{\gamma}_2(t)}{\gamma_1(t)} - \dot{\gamma}_3(t) = 0.\]

**Definition 2.3.** Let \(\gamma : [a, b] \to (\mathbb{H}, g_L)\) be a Euclidean C^2-smooth regular curve in the Riemannian manifold \((\mathbb{H}, g_L)\). The curvature \(k_{\gamma, \nabla}^L\) associated to \(\nabla\) of \(\gamma\) at \(\gamma(t)\) is defined as
\begin{equation}
k_{\gamma, \nabla}^L := \sqrt{\frac{||\nabla_2 \dot{\gamma}}{||\ddot{\gamma}||^2_L} - \frac{\langle \nabla_2 \dot{\gamma}, \dot{\gamma} \rangle^2_L}{||\ddot{\gamma}||^4_L}}.
\end{equation}

By (2.3) and (2.4), we have

**Lemma 2.4.** Let \(\mathbb{H}\) be the Heisenberg group, then
\begin{equation}
\nabla_{X_3} X_1 = -\frac{L}{2} X_2, \quad \nabla_{X_3} X_2 = \frac{L}{2} X_1, \quad \nabla_{X_j} X_k = 0, \quad \text{for other } X_j, X_k.
\end{equation}
Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, then
\begin{equation}
\dot{\gamma}(t) = \dot{\gamma}_1 X_1 + \dot{\gamma}_2 X_2 + \omega(\dot{\gamma}(t)) X_3.
\end{equation}

By (2.6) and (2.7), we have
\begin{equation}
\nabla \dot{\gamma} = \left[ \frac{\ddot{\gamma}_1 + \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2}}{2} \right] X_1 + \left[ \frac{\ddot{\gamma}_2 - \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2}}{2} \right] X_2 + \frac{\dd t}{\dd t} (\omega(\dot{\gamma}(t))) X_3.
\end{equation}

Similar to Lemma 2.4 in [9], we have

**Lemma 2.5.** Let $\gamma : [a, b] \to (H, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(H, g_L)$. Then
\begin{equation}
k^L_{\nabla}(\gamma) = \left\{ \left[ \frac{\ddot{\gamma}_1 + \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2}}{2} \right]^2 + \left[ \frac{\ddot{\gamma}_2 - \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2}}{2} \right]^2 + L \left[ \frac{\dd t}{\dd t} (\omega(\dot{\gamma}(t))) \right]^2 \right\} \cdot \left[ \frac{\dd^2 + \dd^2 + L(\omega(\dot{\gamma}(t)))^2}{2} \right]^{-2}
\end{equation}

In particular, if $\gamma(t)$ is a horizontal point of $\gamma$,
\begin{equation}
k^L_{\nabla}(\gamma) = \left\{ \left[ \frac{\ddot{\gamma}_1 + \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2}}{2} \right]^2 + L \left[ \frac{\dd t}{\dd t} (\omega(\dot{\gamma}(t))) \right]^2 \right\} \cdot \left[ \frac{\dd^2 + \dd^2 + L(\omega(\dot{\gamma}(t)))^2}{2} \right]^{-2}
\end{equation}

**Definition 2.6.** Let $\gamma : [a, b] \to (H, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(H, g_L)$. We define the intrinsic curvature associated to the connection $\nabla$ at $\gamma(t)$ to be
\begin{equation}
k^\infty_{\nabla}(\gamma) := \lim_{L \to +\infty} k^L_{\gamma},
\end{equation}

if the limit exists.

We introduce the following notation: for continuous functions $f_1, f_2 : (0, +\infty) \to \mathbb{R}$,
\begin{equation}
f_1(L) \sim f_2(L), \text{ as } L \to +\infty \iff \lim_{L \to +\infty} \frac{f_1(L)}{f_2(L)} = 1.
\end{equation}

Similar to Lemma 2.6 in [9], we have
Lemma 2.7. Let \( \gamma : [a, b] \to (\mathbb{H}, g_L) \) be a Euclidean \( C^2 \)-smooth regular curve in the Riemannian manifold \((\mathbb{H}, g_L)\). Then

\[
k^\infty_\gamma = \frac{\sqrt{\gamma_1^2 + \gamma_2^2}}{2|\omega(\gamma(t))|}, \text{ if } \omega(\gamma(t)) \neq 0,
\]

\[
k^\infty_\gamma = \frac{|\gamma_1\dot{\gamma}_2 - \gamma_2\dot{\gamma}_1|}{(\gamma_1^2 + \gamma_2^2)^{3/2}}, \text{ if } \omega(\gamma(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\gamma(t))) = 0,
\]

\[
\lim_{L \to +\infty} \frac{k^L_\gamma}{\sqrt{L}} = \frac{\frac{d}{dt}(\omega(\gamma(t)))}{\gamma_1^2 + \gamma_2^2}, \text{ if } \omega(\gamma(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\gamma(t))) \neq 0.
\]

We will say that a surface \( \Sigma \subset (\mathbb{H}, g_L) \) is regular if \( \Sigma \) is a Euclidean \( C^2 \)-smooth compact and oriented surface. In particular we will assume that there exists a Euclidean \( C^2 \)-smooth function \( u : \mathbb{H} \to \mathbb{R} \) such that

\[
\Sigma = \{(x_1, x_2, x_3) \in \mathbb{G} : u(x_1, x_2, x_3) = 0\}
\]

and \( u, \partial_x + u_x \partial_{x_1} + u_{x_2} \partial_{x_2} + u_{x_3} \partial_{x_3} \neq 0 \). Let \( \nabla_H u = X_1(u)X_1 + X_2(u)X_2 \). A point \( x \in \Sigma \) is called characteristic if \( \nabla_H u(x) = 0 \). We define the characteristic set \( C(\Sigma) := \{x \in \Sigma | \nabla_H u(x) = 0\} \). Our computations will be local and away from characteristic points of \( \Sigma \). Let us define first

\[
p := X_1u, \quad q := X_2u, \quad \text{and } r := \tilde{X}_3u.
\]

We then define

\[
2.15 \quad l := \sqrt{p^2 + q^2}, \quad l_L := \sqrt{p^2 + q^2 + r^2}, \quad \bar{p} := \frac{p}{l},
\]

\[
\bar{q} := \frac{q}{l}, \quad \bar{p}_L := \frac{p}{l_L}, \quad \bar{q}_L := \frac{q}{l_L}, \quad \bar{r}_L := \frac{r}{l_L}.
\]

In particular, \( \bar{p}_L^2 + \bar{q}_L^2 = 1 \). These functions are well defined at every non-characteristic point. Let

\[
2.16 \quad v_L = \bar{p} X_1 + \bar{q} X_2 + \bar{r} \tilde{X}_3, \quad e_1 = \bar{q} X_1 - \bar{p} X_2, \quad e_2 = \bar{r} \bar{p} X_1 + \bar{r} \bar{q} X_2 - \frac{l}{l_L} \tilde{X}_3,
\]

then \( v_L \) is the Riemannian unit normal vector to \( \Sigma \) and \( e_1, e_2 \) are the orthonormal basis of \( \Sigma \). On \( T \Sigma \) we define a linear transformation \( J_L : T \Sigma \to T \Sigma \) such that

\[
2.17 \quad J_L(e_1) := e_2; \quad J_L(e_2) := -e_1.
\]

For every \( U, V \in T \Sigma \), we define \( \nabla^U_U \pi V = \pi \nabla_U V \) where \( \pi : T \mathbb{H} \to T \Sigma \) is the projection. Then \( \nabla^\Sigma \) is the metric connection on \( \Sigma \) with respect to the metric \( g_L \). By (2.8),(2.16), we have

\[
2.18 \quad \nabla^\Sigma_{\dot{\gamma}} \dot{\gamma} = \langle \nabla^\Sigma_{\dot{\gamma}} \dot{\gamma}, e_1 \rangle_L e_1 + \langle \nabla^\Sigma_{\dot{\gamma}} \dot{\gamma}, e_2 \rangle_L e_2,
\]
we have
\[ \nabla_{\dot{\gamma}} \dot{\gamma} = \left\{ \begin{array}{l} \dot{\gamma}_1 + L\omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_2}{2} - \bar{p} \left[ \dot{\gamma}_2 - L\omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2} \right] \\ + \left\{ \frac{rL}{p} \left[ \dot{\gamma}_1 + L\omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_2}{2} \right] \\ + \frac{rL}{q} \left[ \dot{\gamma}_2 - L\omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{2} \right] - \frac{l}{L} L^\frac{1}{2} \frac{d}{dt}(\omega(\dot{\gamma}(t))) \right\} \right\} e_1. \]  

**Definition 2.8.** Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. The geodesic curvature associated to $\nabla$, $k_{\gamma,\Sigma}^{L,\nabla}$ of $\gamma$ at $\gamma(t)$ is defined as
\[ k_{\gamma,\Sigma}^{L,\nabla} := \sqrt{\frac{||\nabla_{\dot{\gamma}}^2 \dot{\gamma}||_{\Sigma,L}^4 - \langle \nabla_{\dot{\gamma}}^2 \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma,L}^6}{||\dot{\gamma}||_{\Sigma,L}^4}}. \]  

**Definition 2.9.** Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. We define the intrinsic geodesic curvature associated to $\nabla$, $k_{\gamma,\Sigma}^\infty,\nabla$ of $\gamma$ at $\gamma(t)$ to be
\[ k_{\gamma,\Sigma}^\infty,\nabla := \lim_{L \to +\infty} k_{\gamma,\Sigma}^{L,\nabla}, \] if the limit exists.

Similar to Lemma 3.3 in [9], we have

**Lemma 2.10.** Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then
\[ k_{\gamma,\Sigma}^\infty,\nabla = \frac{|\bar{q}\dot{\gamma}_1 + \bar{q}^2|}{2|\omega(\dot{\gamma}(t))|}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0, \]
\[ k_{\gamma,\Sigma}^\infty,\nabla = 0 \text{ if } \omega(\dot{\gamma}(t)) = 0, \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \]
\[ \lim_{L \to +\infty} \frac{k_{\gamma,\Sigma}^{L,\nabla}}{\sqrt{L}} = \frac{\frac{d}{dt}(\omega(\dot{\gamma}(t)))}{(\bar{q}\dot{\gamma}_1 - \bar{q}\dot{\gamma}_2)^2}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0. \]

**Definition 2.11.** Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. The signed geodesic curvature associated to $\nabla$, $k_{\gamma,\Sigma}^{L,\nabla,s}$ of $\gamma$ at $\gamma(t)$ is defined as
\[ k_{\gamma,\Sigma}^{L,\nabla,s} := \frac{\langle \nabla_{\dot{\gamma}}^2 \dot{\gamma}, J_L(\dot{\gamma}) \rangle_{\Sigma,L}}{||\dot{\gamma}||_{\Sigma,L}^2}. \]
Definition 2.12. Let \( \Sigma \subset (\mathbb{H}, g_L) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. We define the intrinsic geodesic curvature associated to \( \nabla \), \( k_{\gamma, \Sigma}^{\infty, \nabla, s} \) of \( \gamma \) at the non-characteristic point \( \gamma(t) \) to be
\[
k_{\gamma, \Sigma}^{\infty, \nabla, s} := \lim_{L \to \infty} k_{\gamma, \Sigma}^{L, \nabla, s},
\]
if the limit exists.

Similar to Lemma 3.6 in [9], we have

Lemma 2.13. Let \( \Sigma \subset (\mathbb{H}, g_L) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. Then
\[
k_{\gamma, \Sigma}^{\infty, \nabla, s} = \frac{\mathcal{P}\gamma_1 + \mathcal{P}\gamma_2}{2|\omega(\dot{\gamma}(t))|}, \text{ if } \omega(\dot{\gamma}(t)) \neq 0,
\]
\[
k_{\gamma, \Sigma}^{\infty, \nabla, s} = 0 \text{ if } \omega(\dot{\gamma}(t)) = 0, \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0,
\]
\[
\lim_{L \to \infty} \frac{k_{\gamma, \Sigma}^{L, \nabla, s}}{\sqrt{L}} = \frac{(-\mathcal{P}\gamma_1 + \mathcal{P}\gamma_2) \frac{d}{dt}(\omega(\dot{\gamma}(t)))}{|\mathcal{P}\gamma_1 - \mathcal{P}\gamma_2|^3}, \text{ if } \omega(\dot{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.
\]

In the following, we compute the sub-Riemannian limit of the Gaussian curvature associated to \( \nabla \) of surfaces in the Heisenberg group. We define the second fundamental form associated to \( \nabla \), \( II^{\nabla, L} \) of the embedding of \( \Sigma \) into \((\mathbb{H}, g_L)\):

\[
II^{\nabla, L} = \left( \begin{array}{ccc}
\langle \nabla_{e_1}v_L, e_1 \rangle_L, & \langle \nabla_{e_1}v_L, e_2 \rangle_L \\
\langle \nabla_{e_2}v_L, e_1 \rangle_L, & \langle \nabla_{e_2}v_L, e_2 \rangle_L
\end{array} \right).
\]

Similarly to Theorem 4.3 in [4], we have

Theorem 2.14. The second fundamental form \( II^{\nabla, L} \) of the embedding of \( \Sigma \) into \((\mathbb{H}, g_L)\) is given by

\[
II^{\nabla, L} = \left( \begin{array}{ccc}
h_{11}, & h_{12} \\
h_{21}, & h_{22}
\end{array} \right),
\]

where
\[
h_{11} = \frac{l}{l_L} [X_1(\mathcal{P}) + X_2(\mathcal{Q})], \quad h_{12} = -\frac{l}{l} \langle e_1, \nabla_H(\mathcal{P}L) \rangle_L,
\]
\[
h_{21} = -\frac{l}{l} \langle e_1, \nabla_H(\mathcal{Q}L) \rangle_L - \frac{\sqrt{L}}{2} - \frac{\sqrt{L}}{2} r^2_L,
\]
\[
h_{22} = -\frac{l^2}{l_L} \langle e_2, \nabla_H(\mathcal{Q}L) \rangle_L + \mathcal{X}_3(\mathcal{P}L).
\]
Proof. By Theorem 4.3 in [4] and Lemma 2.4, we have
\begin{equation}
\langle \nabla_{e_1} v_L, e_1 \rangle_L = \langle \nabla_{e_1}^L v_L, e_1 \rangle_L, \quad \langle \nabla_{e_1} v_L, e_2 \rangle_L = \langle \nabla_{e_1}^L v_L, e_2 \rangle_L + \frac{\sqrt{L}}{2},
\end{equation}
\begin{equation}
\langle \nabla_{e_2} v_L, e_1 \rangle_L = \langle \nabla_{e_2}^L v_L, e_1 \rangle_L - \frac{\sqrt{L}}{2} r_L^2, \quad \langle \nabla_{e_2} v_L, e_2 \rangle_L = \langle \nabla_{e_2}^L v_L, e_2 \rangle_L.
\end{equation}

By Theorem 4.3 in [4] and (2.28), we get this theorem. \qed

The mean curvature associated to \( \nabla, H_{\nabla,L} \) of \( \Sigma \) is defined by
\[ H_{\nabla,L} := \text{tr}(II_{\nabla,L}). \]

Define the curvature of a connection \( \nabla \) by
\begin{equation}
R(X,Y)Z = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.
\end{equation}

Let
\begin{equation}
K^{\Sigma,\nabla}(e_1, e_2) = -\langle R^\Sigma(e_1, e_2)e_1, e_2 \rangle_{\Sigma,L}, \quad K^{\nabla}(e_1, e_2) = -\langle R(e_1, e_2)e_1, e_2 \rangle_L.
\end{equation}

By the Gauss equation (in fact the Gauss equation holds for any metric connections), we have
\begin{equation}
K^{\Sigma,\nabla}(e_1, e_2) = K^{\nabla}(e_1, e_2) + \text{det}(II_{\nabla,L}).
\end{equation}

Similar to Proposition 3.8 in [9], we have

**Proposition 2.15.** Away from characteristic points, the horizontal mean curvature associated to \( \nabla, H_{\nabla,\infty} \) of \( \Sigma \subset \mathbb{H} \) is given by
\begin{equation}
H_{\nabla,\infty} = \lim_{L \to +\infty} H_{\nabla,L} = X_1(\bar{p}) + X_2(\bar{q}).
\end{equation}

By Lemma 2.4 and (2.29), we have

**Lemma 2.16.** Let \( \mathbb{H} \) be the Heisenberg group, then
\[ R(X_1, X_2)X_1 = \frac{L}{2} X_2, \quad R(X_1, X_2)X_2 = -\frac{L}{2} X_1, \quad R(X_i, X_j)X_k = 0, \text{ for other } i, j, k. \]

**Proposition 2.17.** Away from characteristic points, we have
\begin{equation}
K^{\Sigma,\nabla}(e_1, e_2) \to K^{\Sigma,\nabla,\infty} + O\left( \frac{1}{\sqrt{L}} \right), \text{ as } L \to +\infty,
\end{equation}
where
\begin{equation}
K^{\Sigma,\nabla,\infty} := -\frac{1}{2} \langle e_1, \nabla_H \left( \frac{X_3 u}{|\nabla_H u|} \right) \rangle - \frac{(X_3 u)^2}{2(p^2 + q^2)}.
\end{equation}
Proof. By Lemma 2.16 and similar to (3.33) and (3.34) in [9], we have
\[
K^\nabla (e_1, e_2) = -\frac{L}{2} r_L^2. 
\]
By Theorem 2.14, (2.31) and (2.35), similar to Proposition 3.10 in [9], we can obtain this proposition. □

Let us first consider the case of a regular curve \( \gamma : [a, b] \to (\mathbb{H}, g_L) \). We define the Riemannian length measure
\[
ds_L = ||\dot{\gamma}\||_L dt.
\]
By [1], we have
\[
\frac{1}{\sqrt{L}} ds_L \to ds := |\omega(\dot{\gamma}(t))| dt \quad \text{as } L \to +\infty.
\]
(2.36)
\[
\frac{1}{\sqrt{L}} e_1^* \wedge e_2^* \to d\sigma_{\Sigma} := p\omega_2 \wedge \omega_3 - q\omega_1 \wedge \omega_3 \quad \text{as } L \to +\infty,
\]
where \( e_1^*, e_2^* \) are the dual basis of \( e_1, e_2 \). We recall the local Gauss-Bonnet theorem for the metric connection (see Proposition 5.2 in [6]).

**Theorem 2.18.** Let \( \Sigma \) be an oriented compact two-dimensional manifold with many boundary components \( (\partial \Sigma)_i, i \in \{1, \cdots, n\} \), given by Euclidean \( C^2 \)-smooth regular and closed curves \( \gamma_i : [0, 2\pi] \to (\partial \Sigma)_i \). Let \( \nabla \) be a metric connection and \( K^\nabla \) be the Gauss curvature associated to \( \nabla \) and \( k^s_{\gamma_i} \) be the signed geodesic curvature associated to \( \nabla \), then
\[
\int_{\Sigma} K^\nabla d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k^s_{\gamma_i} ds = 2\pi \chi(M).
\]

By Lemma 2.13 and Proposition 2.17 and Theorem 2.18, similar to the proof of Theorem 1.1 in [1], we have

**Theorem 2.19.** Let \( \Sigma \subset (\mathbb{H}, g_L) \) be a regular surface with finitely many boundary components \( (\partial \Sigma)_i, i \in \{1, \cdots, n\} \), given by Euclidean \( C^2 \)-smooth regular and closed curves \( \gamma_i : [0, 2\pi] \to (\partial \Sigma)_i \). Suppose that the characteristic set \( C(\Sigma) \) satisfies \( H^1(C(\Sigma)) = 0 \) and that \( ||\nabla_h u||^1_H \) is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set \( C(\Sigma) \), then
\[
\int_{\Sigma} K^{\Sigma, \nabla^\omega^\infty} d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k^{\infty, \nabla^\omega^s}_{\gamma_i, \Sigma} ds = 0.
\]

By Lemma 2.13 and (2.34), we note that Theorem 2.19 is the same as Theorem 1.1 in [1] up to the scaler \( \frac{1}{2} \).

3. **The Gauss-Bonnet theorem associated to the second kind of Schouten-van Kampen affine connections in the Heisenberg group**

Let \( \widehat{H} = \text{span}\{X_2, X_3\} \) and \( \widehat{H}^\perp = \text{span}\{X_1\} \) and \( \overline{P} : \widehat{\mathbb{H}} \to \widehat{H} \) and \( \overline{P}^\perp : \widehat{\mathbb{H}} \to \widehat{H}^\perp \) be the projections. We define the second kind of Schouten-van Kampen affine connections in the Heisenberg group:
\[
\nabla^1_X Y = \overline{P} \nabla^L_X \overline{P} Y + \overline{P}^\perp \nabla^L_X \overline{P}^\perp Y.
\]
By Lemma 2.1 and (3.1), we have

**Lemma 3.1.** Let $\mathbb{H}$ be the Heisenberg group, then

\[
\nabla^1_{X_1}X_2 = \frac{1}{2}X_3, \quad \nabla^1_{X_1}X_3 = -\frac{L}{2}X_2, \quad \nabla^1_{X_j}X_k = 0, \quad \text{for other} \quad X_j, X_k.
\]

Similar to the definition 2.3, we can define the curvature $k^L_{\gamma, \nabla^1}$ associated to $\nabla^1$ of $\gamma$ at $\gamma(t)$. By Lemma 3.1 and (2.7), we have

\[
\nabla^1_{\dot{\gamma}} \dot{\gamma} = \frac{\dot{\gamma}_1X_1}{2} + \frac{L}{2} \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) + \frac{1}{2} \dot{\gamma}_1 \dot{\gamma}_2 \right] X_3.
\]

**Lemma 3.2.** Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$. Then

\[
k^L_{\gamma, \nabla^1} = \left\{ \frac{\dot{\gamma}_1^2}{2} + \frac{\dot{\gamma}_2^2}{2} + L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) + \frac{1}{2} \dot{\gamma}_1 \dot{\gamma}_2 \right] \right\}
\cdot \left[ \left( \frac{\dot{\gamma}_1^2}{2} + \frac{\dot{\gamma}_2^2}{2} + L(\omega(\dot{\gamma}(t))) \right) \right]^{-2}
\cdot \left\{ \frac{\dot{\gamma}_1 \dot{\gamma}_2}{2} + L(\omega(\dot{\gamma}(t))) \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) + \frac{1}{2} \dot{\gamma}_1 \dot{\gamma}_2 \right] \right\}^{\frac{3}{2}}.
\]

In particular, if $\gamma(t)$ is a horizontal point of $\gamma$,

\[
k^L_{\gamma, \nabla^1} = \left\{ \frac{\dot{\gamma}_1^2}{2} + \frac{\dot{\gamma}_2^2}{2} + L \left[ \frac{d}{dt}(\omega(\dot{\gamma}(t))) + \frac{1}{2} \dot{\gamma}_1 \dot{\gamma}_2 \right] \right\} \cdot \left[ \frac{\dot{\gamma}_1^2}{2} + \frac{\dot{\gamma}_2^2}{2} \right]^{-2}
\cdot \left\{ \dot{\gamma}_1 \dot{\gamma}_2 \right\}^{\frac{1}{2}} \cdot \left[ \frac{\dot{\gamma}_1^2}{2} + \frac{\dot{\gamma}_2^2}{2} \right]^{-3} \right\}^{\frac{1}{2}}.
\]

Similar to the definition 2.6, we can define the intrinsic curvature associated to the connection $\nabla^1$, $k^\infty_{\gamma, \nabla^1}$ of $\gamma$ at $\gamma(t)$. Similar to the lemma 2.7, we have

**Lemma 3.3.** Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$. Then

\[
k^\infty_{\gamma, \nabla^1} = \frac{|\dot{\gamma}_1|}{2|\omega(\dot{\gamma}(t))|}, \quad \text{if} \quad \omega(\dot{\gamma}(t)) \neq 0.
\]
\textbf{Lemma 3.4.} Let \( \Sigma \subset (\mathbb{H}, g_\Sigma) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. Then

\begin{equation}
\tag{3.11}
k_{\gamma_{\Sigma}}^{\infty, \nabla^1} = \frac{|\overline{p}\gamma|}{2|\omega(\gamma(t))|}, \text{ if } \omega(\gamma(t)) \neq 0,
\end{equation}

\begin{equation}
\tag{3.12}
k_{\gamma_{\Sigma}}^{\infty, \nabla^1} = 0 \text{ if } \omega(\gamma(t)) = 0, \text{ and } \frac{d}{dt}(\omega(\gamma(t))) + \frac{1}{2} \gamma_1 \gamma_2 = 0,
\end{equation}

\begin{equation}
\tag{3.13}
\lim_{L \to +\infty} \frac{k_{\gamma_{\Sigma}}^{L, \nabla^1}}{\sqrt{L}} = \frac{|d\omega(\gamma(t)) + \frac{1}{2} \gamma_1 \gamma_2|}{(\overline{\gamma}_1 - \overline{p}\gamma_2)^2}, \text{ if } \omega(\gamma(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\gamma(t))) + \frac{1}{2} \gamma_1 \gamma_2 \neq 0.
\end{equation}

Similar to the definitions 2.11 and 2.12, we can define the signed geodesic curvature associated to \( \nabla^1, k_{\gamma_{\Sigma}}^{L, \nabla^1, s} \) of \( \gamma \) at \( \gamma(t) \) and the intrinsic geodesic curvature associated to \( \nabla^1, k_{\gamma_{\Sigma}}^{\infty, \nabla^1, s} \) of \( \gamma \). Similar to Lemma 2.10, we have

\textbf{Lemma 3.5.} Let \( \Sigma \subset (\mathbb{H}, g_\Sigma) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. Then

\begin{equation}
\tag{3.13}
k_{\gamma_{\Sigma}}^{\infty, \nabla^1, s} = \frac{\overline{p}\gamma_1}{2|\omega(\gamma(t))|}, \text{ if } \omega(\gamma(t)) \neq 0,
\end{equation}

\begin{equation}
\tag{3.14}
k_{\gamma_{\Sigma}}^{\infty, \nabla^1, s} = 0 \text{ if } \omega(\gamma(t)) = 0, \text{ and } \frac{d}{dt}(\omega(\gamma(t))) + \frac{1}{2} \gamma_1 \gamma_2 = 0,
\end{equation}

\begin{equation}
\tag{3.15}
\lim_{L \to +\infty} \frac{k_{\gamma_{\Sigma}}^{L, \nabla^1, s}}{\sqrt{L}} = \frac{|d\omega(\gamma(t)) + \frac{1}{2} \gamma_1 \gamma_2|}{(\overline{\gamma}_1 - \overline{p}\gamma_2)^2}, \text{ if } \omega(\gamma(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\gamma(t))) + \frac{1}{2} \gamma_1 \gamma_2 \neq 0.
\end{equation}
\( k_{\gamma, \Sigma}^{L, \nabla^1, s} \frac{\sqrt{L}}{\sqrt{L}} = (\frac{d}{dt}(\omega(\gamma(t))) + \frac{1}{2} \dot{\gamma}_1 \dot{\gamma}_2), \)

if \( \omega(\gamma(t)) = 0 \) and \( \frac{d}{dt}(\omega(\gamma(t))) + \frac{1}{2} \dot{\gamma}_1 \dot{\gamma}_2 \neq 0. \)

Similar to (2.26), we can define the second fundamental form associated to \( \nabla^1, II^{\nabla^1, L} \) of the embedding of \( \Sigma \) into \((\mathbb{H}, g_L)\). Similarly to Theorem 2.14, we have

**Theorem 3.6.** The second fundamental form \( II^{\nabla^1, L} \) of the embedding of \( \Sigma \) into \((\mathbb{H}, g_L)\) is given by

\[
II^{\nabla^1, L} = \begin{pmatrix} h_{11}^1 & h_{12}^1 \\ h_{21}^1 & h_{22}^1 \end{pmatrix},
\]

where

\[
h_{11}^1 = \frac{l}{l^2} [X_1(\overline{\varphi}) + X_2(\overline{\varphi})] + \frac{\sqrt{L} t \overline{p} q \overline{r}}{2},
\]

\[
h_{12}^1 = -\frac{l}{l^2} \langle e_1, \nabla H(\overline{r}_L) \rangle_L - \frac{1}{2} \overline{r}_L \overline{q}_L - \frac{l}{2l^2} \overline{q}_L \sqrt{L},
\]

\[
h_{21}^1 = -\frac{l}{l^2} \langle e_1, \nabla H(\overline{r}_L) \rangle_L - \frac{\sqrt{L} l^2}{2} \overline{r}^3 - \frac{l}{2l^2} \overline{r}^2 q_l,
\]

\[
h_{22}^2 = -\frac{l^2}{l^2} \langle e_2, \nabla H(\overline{r}_L) \rangle_L - \frac{\sqrt{L} l^2}{2} \overline{r}^3 - \frac{l}{2l^2} \overline{r}^2 q_l.
\]

Similar to (2.29) and (2.30), we can define \( R^1(X, Y)Z, K^{\Sigma, \nabla^1}(e_1, e_2) \) and \( K^{\nabla^1}(e_1, e_2) \) (2.31) is correct for \( \nabla^1 \). Similar to Proposition 2.15, we have

**Proposition 3.7.** Away from characteristic points, the horizontal mean curvature associated to \( \nabla^1, \mathcal{H}_{\nabla^1, \infty} \) of \( \Sigma \subset \mathbb{H} \) is given by

\[
\mathcal{H}_{\nabla^1, \infty} = \lim_{L \to \infty} \mathcal{H}_{\nabla^1, L} = X_1(\overline{p}) + X_2(\overline{q}).
\]

By Lemma 3.1, we have \( R^1(X_i, X_j)X_k = 0 \) for any \( i, j, k \). Similar to Proposition 2.17, we have

**Proposition 3.8.** Away from characteristic points, we have

\[
K^{\Sigma, \nabla^1}(e_1, e_2) \to K^{\Sigma, \nabla^1, \infty} + O(\frac{1}{\sqrt{L}}), \text{ as } L \to +\infty,
\]

where

\[
K^{\Sigma, \nabla^1, \infty} := -\frac{\overline{p} \overline{q}(X_3(u))}{2\sqrt{p^2 + q^2}} [X_1(\overline{p}) + X_2(\overline{q})] - \frac{\overline{q}^2}{2} \left[ \langle e_1, \nabla H(\overline{X}_3(u)) \rangle + \frac{(X_3(u))^2}{p^2 + q^2} \right].
\]

By Lemma 3.5 and Proposition 3.8, similar to Theorem 2.19, we have
Theorem 3.9. Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface with finitely many boundary components $(\partial \Sigma)_i, i \in \{1, \ldots, n\}$, given by Euclidean $C^2$-smooth regular and closed curves $\gamma_i : [0, 2\pi] \to (\partial \Sigma)_i$. Suppose that the characteristic set $C(S)$ satisfies $H^1(C(S)) = 0$ and that $||\nabla H u||_H$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set $C(S)$, then

$$\int_{\Sigma} K_{\Sigma} \nabla \Sigma, \nabla_1, \infty d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k_{\gamma_i, \Sigma} \nabla \gamma_i, s \nabla \Sigma, ds = 0.$$  

(3.19)

We note that Theorem 3.9 is different from Theorem 1.1 in [1].

4. The sub-Riemannian limit and the adapted connection

We define the adapted connection $\nabla^2$ on the Heisenberg group by $\nabla^2 X_j X_k = 0$ for any $j, k$. Then $\nabla^2$ is a metric connection. Similar to the definition 2.3, we can define the curvature $k_{\gamma, \nabla^2}$ associated to $\nabla^2$ of $\gamma$ at $\gamma(t)$. We have

$$\nabla^2 \dot{\gamma} = \dot{\gamma}_1 X_1 + \dot{\gamma}_2 X_2 + \frac{d}{dt}(\omega(\gamma(t))) X_3.$$  

(4.1)

Similar to Lemma 2.5, we have

Lemma 4.1. Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$. Then

$$k_{\gamma, \nabla^2} = \left\{ \begin{array}{c} \ddot{\gamma}_1^2 + \ddot{\gamma}_2^2 + L \left[ \frac{d}{dt}(\omega(\gamma(t))) \right]^2 \\ \cdot \left[ \ddot{\gamma}_1^2 + \ddot{\gamma}_2^2 + L(\omega(\gamma(t)))^2 \right]^{-2} \\ - \left( \dot{\gamma}_1 \ddot{\gamma}_1 + \dot{\gamma}_2 \ddot{\gamma}_2 + L(\gamma(t)) \frac{d}{dt}(\omega(\gamma(t))) \right) \left[ \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + L(\omega(\gamma(t))) \right]^{-3} \end{array} \right\}^{\frac{1}{2}}.$$  

(4.2)

In particular, if $\gamma(t)$ is a horizontal point of $\gamma$,

$$k_{\gamma, \nabla^2} = \left\{ \begin{array}{c} \ddot{\gamma}_1^2 + \ddot{\gamma}_2^2 + L \left[ \frac{d}{dt}(\omega(\gamma(t))) \right]^2 \\ \cdot \left[ \ddot{\gamma}_1^2 + \ddot{\gamma}_2^2 \right]^{-2} \\ - \left( \dot{\gamma}_1 \ddot{\gamma}_1 + \dot{\gamma}_2 \ddot{\gamma}_2 \right) \right\}^{\frac{1}{2}}.$$  

(4.3)

Similar to the definition 2.6, we can define the intrinsic curvature associated to the connection $\nabla^2, k_{\gamma, \nabla^2}$ of $\gamma$ at $\gamma(t)$. Similar to the lemma 2.7, we have
Lemma 4.2. Let $\gamma : [a, b] \to (\mathbb{H}, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(\mathbb{H}, g_L)$. Then

\begin{equation}
 k_{\gamma}^{\infty, \nabla^2} = 0, \text{ if } \omega(\hat{\gamma}(t)) \neq 0,
\end{equation}

\begin{equation}
 k_{\gamma}^{\infty, \nabla^2} = \frac{\abs{\hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_2 \hat{\gamma}_1}}{(\hat{\gamma}_1^2 + \hat{\gamma}_2^2)^{\frac{3}{2}}}, \text{ if } \omega(\hat{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\hat{\gamma}(t))) = 0,
\end{equation}

\begin{equation}
 \lim_{L \to +\infty} \frac{k_{L, \nabla^2}}{\sqrt{L}} = \frac{\abs{\frac{d}{dt}(\omega(\hat{\gamma}(t)))}}{\hat{\gamma}_1^2 + \hat{\gamma}_2^2}, \text{ if } \omega(\hat{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\hat{\gamma}(t))) \neq 0.
\end{equation}

Similar to (3.10), we have

\[ \nabla_{\hat{\gamma}}^2 \gamma = \{ \overline{\nabla}_1 - \overline{\nabla}_2 \} e_1 + \left\{ \frac{l}{L} \overline{\nabla}_1 \overline{\nabla}_1 + \frac{l}{L} \overline{\nabla}_2 \overline{\nabla}_2 - \frac{l}{L} L^2 \frac{d}{dt}(\omega(\hat{\gamma}(t))) \right\} e_2. \]

Similar to Definitions 2.8 and 2.9, we can define the geodesic curvature associated to $\nabla^2$, $k_{\gamma, \Sigma}^{\nabla^2}$ of $\gamma$ at $\gamma(t)$ and the intrinsic geodesic curvature associated to $\nabla^2$, $k_{\gamma, \Sigma}^{\infty, \nabla^2}$ of $\gamma$ at $\gamma(t)$. Similar to Lemma 2.10, we have

Lemma 4.3. Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then

\begin{equation}
 k_{\gamma, \Sigma}^{\infty, \nabla^2} = 0, \text{ if } \omega(\hat{\gamma}(t)) \neq 0,
\end{equation}

\begin{equation}
 k_{\gamma, \Sigma}^{\infty, \nabla^2} = 0 \text{ if } \omega(\hat{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\hat{\gamma}(t))) = 0,
\end{equation}

\begin{equation}
 \lim_{L \to +\infty} \frac{k_{L, \nabla^2, s}}{\sqrt{L}} = \frac{\abs{\frac{d}{dt}(\omega(\hat{\gamma}(t)))}}{(\overline{\gamma}_1^2 + \overline{\gamma}_2^2)^{\frac{3}{2}}}, \text{ if } \omega(\hat{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\hat{\gamma}(t))) \neq 0.
\end{equation}

Similar to the definitions 2.11 and 2.12, we can define the signed geodesic curvature associated to $\nabla^2$, $k_{\gamma, \Sigma}^{L, \nabla^2, s}$ of $\gamma$ at $\gamma(t)$ and the intrinsic signed geodesic curvature associated to $\nabla^2$, $k_{\gamma, \Sigma}^{\infty, \nabla^2, s}$ of $\gamma$. Similar to Lemma 2.13, we have

Lemma 4.4. Let $\Sigma \subset (\mathbb{H}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then

\begin{equation}
 k_{\gamma, \Sigma}^{\infty, \nabla^2, s} = 0, \text{ if } \omega(\hat{\gamma}(t)) \neq 0,
\end{equation}

\begin{equation}
 k_{\gamma, \Sigma}^{\infty, \nabla^2, s} = 0 \text{ if } \omega(\hat{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\hat{\gamma}(t))) = 0,
\end{equation}

\begin{equation}
 \lim_{L \to +\infty} \frac{k_{L, \nabla^2, s}}{\sqrt{L}} = \frac{\abs{-\overline{\gamma}_1 + \overline{\gamma}_2} \frac{d}{dt}(\omega(\hat{\gamma}(t)))}{(\overline{\gamma}_1 - \overline{\gamma}_2)^3}, \text{ if } \omega(\hat{\gamma}(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\hat{\gamma}(t))) \neq 0.
\end{equation}
Similar to (2.26), we can define the second fundamental form associated to $\nabla^2, II^{\nabla^2,L}$ of the embedding of $\Sigma$ into $(\mathbb{H}, g_L)$. Similarly to Theorem 2.14, we have

**Theorem 4.5.** The second fundamental form $II^{\nabla^2,L}$ of the embedding of $\Sigma$ into $(\mathbb{H}, g_L)$ is given by

\[
II^{\nabla^2,L} = \begin{pmatrix}
h_{11}^2 & h_{12}^2 \\
h_{21}^2 & h_{22}^2
\end{pmatrix},
\]

where

\[
h_{11}^2 = \frac{l}{L} [X_1(\mathfrak{p}) + X_2(\overline{q})], \quad h_{12}^2 = -\frac{l}{L} \langle e_1, \nabla_H(\overline{r}_L) \rangle_L,
\]
\[
h_{21}^2 = -\frac{l}{L} \langle e_1, \nabla_H(\overline{r}_L) \rangle_L - \frac{\sqrt{L}}{2} \frac{l^2}{L^2} - \frac{\sqrt{L}}{2} \frac{r_L^2}{L},
\]
\[
h_{22}^2 = -\frac{l^2}{L} \langle e_2, \nabla_H(\overline{r}_L) \rangle_L + \tilde{X}_3(\overline{r}_L).
\]

Similar to Propositions 2.15 and 2.17, we have

**Proposition 4.6.** Away from characteristic points, we have

\[
H_{\nabla^2,\infty} = \lim_{L \to +\infty} H_{\nabla^2,L} = X_1(\mathfrak{p}) + X_2(\overline{q}).
\]

\[
K^{\Sigma,\nabla^2}(e_1, e_2) \to K^{\Sigma,\nabla^2,\infty} = 0, \text{ as } L \to +\infty.
\]

5. Acknowledgements

The author was supported in part by NSFC No.11771070.

**References**

[1] Z. Balogh, J. Tyson, E. Vecchi, *Intrinsic curvature of curves and surfaces and a Gauss-Bonnet theorem in the Heisenberg group*, Math. Z. **287** (2017), no. 1-2, 1-38.

[2] Z. Balogh, J. Tyson, E. Vecchi, *Correction to: Intrinsic curvature of curves and surfaces and a Gauss-Bonnet theorem in the Heisenberg group*, Math. Z. **296** (2020), no. 1-2, 875-876.

[3] A. Bejancu, *Schouten-Van Kampen and Vranceanu connections on foliated manifolds*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **52** (2006), no. 1, 37-60.

[4] L. Capogna, D. Danielli, S. Pauls, J. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Progress in Mathematics, 259. Birkhauser Verlag, Basel, 2007. xvi+223 pp.

[5] M. Diniz, M. Veloso, *Gauss-Bonnet theorem in sub-Riemannian Heisenberg space*, J. Dyn. Control Syst. **22**(4) (2016), 807-820.

[6] B. Klatt, *The Euler class from a general connection relative to a metric*, arXiv:2101.06791.

[7] Z. Olszak, *The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure*, Publ. Inst. Math. (Beograd) (N.S.) **94**(108) (2013), 31-42.
[8] M. Veloso, *Rotation surfaces of constant Gaussian curvature as Riemannian approximation scheme in sub-Riemannian Heisenberg space* $\mathbb{H}^1$, arxiv:1909.13341.

[9] Y. Wang, S. Wei, *Gauss-Bonnet theorems in the affine group and the group of rigid motions of the Minkowski plane*, arXiv:1912.00302. To appear in Sci. China Math.

[10] Y. Wang, S. Wei, *Gauss-Bonnet theorems in the BCV spaces and the twisted Heisenberg group*, Results Math. **75** (2020), no. 3, Paper No. 126, 21 pp.

**Yong Wang**
School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
Email: wangy581@nenu.edu.cn