R-matrix and Mickelsson algebras for orthosymplectic quantum groups

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Abstract

Let \( g \) be a complex orthogonal or symplectic Lie algebra and \( g' \subset g \) the Lie subalgebra of rank \( \text{rk} \ g' = \text{rk} \ g - 1 \) of the same type. We give an explicit construction of generators of the Mickelsson algebra \( Z_q(g,g') \) in terms of Chevalley generators via the R-matrix of \( U_q(g) \).

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1 Introduction

In the mathematics literature, lowering and raising operators are known as generators of step algebras, which were originally introduced by Mickelsson [1] for reductive pairs of Lie algebras, $g' \subset g$. These algebras naturally act on $g'$-singular vectors in $U(g)$-modules and are important in representation theory, [2, 3].

The general theory of step algebras for classical universal enveloping algebras was developed in [2, 4] and was extended to the special linear and orthogonal quantum groups in [5]. They admit a natural description in terms of extremal projectors, [4], introduced for classical groups in [6, 7] and extended to the quantum group case in [8]. It is known that the step algebra $Z(g, g')$ is generated by the image of the orthogonal complement $g \ominus g'$ under the extremal projector of the $g'$. Another description of lowering/rasing operators for classical groups was obtained in [9, 10, 11, 12] in an explicit form of polynomials in $g$.

A generalization of the results of [9, 10] to quantum $gl(n)$ can be found in [13]. In this special case, the lowering operators can be also conveniently expressed through "modified commutators" in the Chevalley generators of $U(g)$ with coefficients in the field of fractions of $U(h)$. Extending [11, 12] to a general quantum group is not straightforward, since there are no immediate candidates for the nilpotent triangular Lie subalgebras $g_\pm$ in $U_q(g)$. We suggest such a generalization, where the lack of $g_\pm$ is compensated by the entries of the universal R-matrix with one leg projected to the natural representation. Those entries are nicely expressed through modified commutators in the Chevalley generators turning into elements of $g_\pm$ in the quasi-classical limit. Their commutation relation with the Chevalley generators modify the classical commutation relations with $g_\pm$ in a tractable way. This enabled us to generalize the results of [9, 10, 11, 12] and construct generators of Mickelsson algebras for the non-exceptional quantum groups.

1.1 Quantized universal enveloping algebra

In this paper, $g$ is a complex simple Lie algebra of type $B$, $C$ or $D$. The case of $gl(n)$ can be easily derived from here due to the natural inclusion $U_q(gl(n)) \subset U_q(g)$, so we do not pay special attention to it. We choose a Cartan subalgebra $h \subset g$ with the canonical inner product $(,)$ on $h^*$. By $R$ we denote the root system of $g$ with a fixed subsystem of positive roots $R^+ \subset R$ and the basis of simple roots $\Pi^+ \subset R^+$. For every $\lambda \in h^*$ we denote by $h_\lambda$ its image under the isomorphism $h^* \cong h$, that is $(\lambda, \beta) = \beta(h_\lambda)$ for all $\beta \in h^*$. We put $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ for the Weyl vector.
Suppose that \( q \in \mathbb{C} \) is not a root of unity. Denote by \( U_q(\mathfrak{g}_\pm) \) the \( \mathbb{C} \)-algebra generated by \( e_{\pm \alpha}, \alpha \in \Pi^+ \), subject to the \( q \)-Serre relations

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} e_{\pm a_{ij}}^{-k} e_{\pm a_{ij}} e_{\pm a_{ij}} = 0,
\]

where \( a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \), \( i, j = 1, \ldots, n = \text{rk} \, \mathfrak{g} \), is the Cartan matrix, \( q_\alpha = q^{\frac{\alpha_\alpha}{2}} \), and

\[
\begin{bmatrix}
m \\ k
\end{bmatrix}_q = \frac{[m]_q!}{[k]_q![m-k]_q!, \quad [m]_q! = [1]_q \cdot [2]_q \cdots [m]_q.
\]

Here and further on, \([z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}\) whenever \( q^{\pm z} \) make sense.

Denote by \( U_q(\mathfrak{h}) \) the commutative \( \mathbb{C} \)-algebra generated by \( q^{\pm h_\alpha}, \alpha \in \Pi^+ \). The quantum group \( U_q(\mathfrak{g}) \) is a \( \mathbb{C} \)-algebra generated by \( U_q(\mathfrak{g}_\pm) \) and \( U_q(\mathfrak{h}) \) subject to the relations

\[
q^{h_\alpha} e_{\pm \beta} q^{-h_\alpha} = q^{\pm\alpha, \beta} e_{\pm \beta}, \quad [e_\alpha, e_\beta] = \delta_{\alpha, \beta} q^{h_\alpha} - q^{-h_\alpha} \quad q_\alpha = q^{\frac{\alpha_\alpha}{2}}.
\]

Remark that \( \mathfrak{h} \) is not contained in \( U_q(\mathfrak{g}) \), still it is convenient for us to keep reference to \( \mathfrak{h} \).

Fix the comultiplication in \( U_q(\mathfrak{g}) \) as in [14]:

\[
\Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha, \quad \Delta(e_\alpha) = e_{-\alpha} \otimes 1 + q^{-h_\alpha} \otimes e_{-\alpha},
\]

\[
\Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},
\]

for all \( \alpha \in \Pi^+ \).

The subalgebras \( U_q(\mathfrak{b}_\pm) \subset U_q(\mathfrak{g}) \) generated by \( U_q(\mathfrak{g}_\pm) \) over \( U_q(\mathfrak{h}) \) are quantized universal enveloping algebras of the Borel subalgebras \( \mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{g}_{\pm} \subset \mathfrak{g} \).

The Chevalley generators \( e_\alpha \) can be extended to a set of higher root vectors \( e_\beta \) for all \( \beta \in \mathbb{R} \). A normally ordered set of root vectors generate a Poincaré-Birkhoff-Witt (PBW) basis of \( U_q(\mathfrak{g}) \) over \( U_q(\mathfrak{h}) \), [14]. We will use \( \mathfrak{g}_{\pm} \) to denote the vector space spanned by \( \{e_{\pm \beta}\}_{\beta \in \mathbb{R}^+} \).

The universal R-matrix is an element of a certain extension of \( U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \). We heavily use the intertwining relation

\[
\mathcal{R} \Delta(x) = \Delta^{op}(x) \mathcal{R}, \quad (1.1)
\]

between the coproduct and its opposite for all \( x \in U_q(\mathfrak{g}) \). Let \( \{\varepsilon_i\}_{i=1}^n \subset \mathfrak{h}^* \) be the standard orthonormal basis and \( \{h_{\varepsilon_i}\}_{i=1}^n \) the corresponding dual basis in \( \mathfrak{h} \). The exact expression for \( \mathcal{R} \) can be extracted from [14], Theorem 8.3.9, as the ordered product

\[
\mathcal{R} = q^{\sum_{i=1}^n h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \prod_{\beta} \exp_{q^\beta} \{(1 - q_{\beta}^2)(e_\beta \otimes e_{-\beta})\} \in U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-), \quad (1.2)
\]
where $\exp_q(x) = \sum_{k=0}^{\infty} q^{\frac{k}{2}(k+1)} \frac{x^k}{k!}$. 

We use the notation $e_i = e_{a_i}$ and $f_i = e_{-a_i}$ for $a_i \in \Pi^+$, in all cases apart from $i = n$, $g = \mathfrak{so}(2n+1)$, where we set $f_n = [\frac{1}{2}] q e_{-a_n}$. The reason for this is two-fold. Firstly, the natural representation can be defined through the classical assignment on the generators, as given below. Secondly, we get rid of $q_{a_n} = q^{\frac{1}{2}}$ and can work over $\mathbb{C}[q]$, as the relations involved turn into

$$[e_n, f_n] = \frac{q^{h_{a_n} - q^{-h_{a_n}}}}{q - q^{-1}},$$

$$f_n^3 f_{n-1} - (q + 1 + q^{-1}) f_n^2 f_{n-1} f_n + (q + 1 + q^{-1}) f_n f_{n-1} f_n^2 - f_{n-1} f_n^3 = 0.$$ 

It is easy to see that the square root of $q$ disappears from the corresponding factor in the presentation (1.2).

In what follows, we regard $\mathfrak{gl}(n) \subset g$ to be the Lie subalgebra with the simple roots $\{a_i\}_{i=1}^{n-1}$ and $U_q(\mathfrak{gl}(n))$ the corresponding quantum subgroup in $U_q(g)$.

Consider the natural representation of $g$ in the vector space $\mathbb{C}^N$. We use the notation $i' = N + 1 - i$ for all integers $i = 1, \ldots, N$. The assignment

$$\pi(e_i) = e_{i,i+1} \pm e_{i'-1,i'}, \quad \pi(f_i) = e_{i+1,i} \pm e_{i',i'-1}, \quad \pi(h_{a_i}) = e_{ii} - e_{i+1,i+1} + e_{i'-1,i'-1} - e_{i'i'},$$

for $i = 1, \ldots, n - 1$, defines a direct sum of two representations of $\mathfrak{gl}(n)$ for each sign. It extends to the natural representation of the whole $g$ by

$$\pi(e_n) = e_{n,n+1} \pm e_{n'-1,n'}, \quad \pi(f_n) = e_{n+1,n} \pm e_{n',n'-1}, \quad \pi(h_{a_n}) = e_{nn} - e_{n'n'},$$

$$\pi(e_n) = e_{nn'}, \quad \pi(f_n) = e_{n'n}, \quad \pi(h_{a_n}) = 2e_{nn} - 2e_{n'n'},$$

$$\pi(e_n) = e_{n-1,n'} \pm e_{n,n'+1}, \quad \pi(f_n) = e_{n',n-1} \pm e_{n'+1,n}, \quad \pi(h_{a_n}) = e_{n-1,n-1} + e_{nn} - e_{n'n'} - e_{n'+1,n'+1},$$

respectively, for $g = \mathfrak{so}(2n+1)$, $g = \mathfrak{sp}(2n)$, and $g = \mathfrak{so}(2n)$.

Two values of the sign give equivalent representations. The choice of minus corresponds to the standard representation that preserves the bilinear form with entries $C_{ij} = \delta_{i'j}$, for $g = \mathfrak{so}(N)$, and $C_{ij} = \text{sign}(i' - i) \delta_{i'j}$, for $g = \mathfrak{sp}(N)$. However, we fix the sign to $+$ in order to simplify calculations. The above assignment also defines representations of $U_q(g)$.

## 2 $R$-matrix of non-exceptional quantum groups

Define $\hat{\mathcal{R}} = q^{-\sum_{i=1}^{n} h_{e_i} \otimes h_{e_i}} \mathcal{R}$. Denote by $\hat{R}^- = (\pi \otimes \text{id})(\hat{\mathcal{R}}) \in \text{End}(\mathbb{C}^N) \otimes U_q(g_-)$ and by $\hat{R}^+ = (\pi \otimes \text{id})(\hat{\mathcal{R}}_{21}) \in \text{End}(\mathbb{C}^N) \otimes U_q(g_+)$. In this section, we deal only with $\hat{R}^-$ and suppress the label "$-$" for simplicity, $\hat{R} = \hat{R}^-$. 

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Denote by $N_+$ the ring of all upper triangular matrices in $\text{End}(\mathbb{C}^N)$ and by $N'_+$ its ideal spanned by $e_{ij}, i < j + 1$.

**Lemma 2.1.** One has

$$\hat{R} = 1 \otimes 1 + (q^{1+\delta_1n} - q^{-1-\delta_1n}) \sum_{i=1}^n \pi(e_i) \otimes f_i \mod N'_+ \otimes U_q(g_-),$$

where $\delta_{1n}$ is present only for $g = \text{sp}(2n)$.

**Proof.** For all positive roots $\alpha, \beta$ the matrix $\pi(e_\alpha e_\beta)$ belongs to $N'_+$. Also, $\pi(e_\beta) \in N'_+$ for all $\beta \in R^+ \setminus \Pi^+$. Therefore, the only terms that contribute to $\text{Span}_{\varepsilon_i - \varepsilon_j \in \Pi^+} \{ e_{ij} \otimes U_q(g_-) \}$ are those of degree 1 from the series $\exp_{q_\alpha}(1 - q^{-2}_\alpha)(e_\alpha \otimes e_{\alpha})$ with $\alpha \in \Pi^+$.

Write $\hat{R} = \sum_{i,j=1}^N e_{ij} \otimes \hat{R}_{ij}$, where $\hat{R}_{ij} = 0$ for $i > j$. Due to the $h$-invariance of $\hat{R}$, the entry $\hat{R}_{ij} \in U_q(g_-)$ carries weight $\varepsilon_j - \varepsilon_i$.

For all $g$, we have $f_{k,k+1} = f_k = f_{k'-1,k'}$ once $k < n$ and $f_{n,n+1} = f_n = f_{n+1,n'}$ for $g = \text{so}(2n + 1)$, $f_{n-1,n'} = f_n = f_{n,n'+1} $ for $g = \text{so}(2n)$ and $f_{nn'} = [2]qf_n$ for $g = \text{sp}(2n)$. We present explicit expressions for the entries $f_{ij}$ in terms of modified commutators in Chevalley generators, $[x,y]_a = xy - ayx$, where $a$ is a scalar; we also put $\bar{q} = q^{-1}$.

**Proposition 2.2.** Suppose that $\varepsilon_i - \varepsilon_j \in R^+ \setminus \Pi^+$. Then the elements $f_{ij}$ are given by the following formulas:

For all $g$ and $i + 1 < j \leq \frac{N+1}{2}$:

$$f_{ij} = [f_{j-1}, \ldots, [f_{i+1}, f_i]_q]_q \ldots]_q, \quad f_{j'q} = [\ldots [f_i, f_{i+1}]_q, \ldots f_{j-1}]_q.$$ \hfill (2.3)

Furthermore,

- For $g = \text{so}(2n + 1)$: $f_{nn'} = (q-1)f_n^2$ and

$$f_{i,n+1} = [f_n, f_{i,n}]_q, \quad f_{n+1,i'} = [f_{n',i'}, f_n]_q, \quad i < n,$$

$$f_{ij'} = q^{\delta_{ij}}[f_{n+1,j'}, f_{i,n+1}]_q^\delta_{ij}, \quad i, j < n.$$

- For $g = \text{sp}(2n)$: $f_{nn'} = [2]_q f_n$ and

$$f_{in'} = [f_n, f_{in}]_q^2, \quad f_{nv'} = [f_{n',v}, f_n]_q^2, \quad i < n,$$

$$f_{ij'} = q^{\delta_{ij}}[f_{nj'}, f_{in}]_q^{1+s_{ij}}, \quad i, j < n.$$
• For $\mathfrak{g} = \mathfrak{so}(2n)$: $f_{nn'} = 0$ and

$$f_{in'} = [f_n, f_{i,n-1}]_q, \quad f_{nn'} = [f_{n'+1,i'}, f_n]_q, \quad i < n - 2,$$

$$f_{ji'} = q^{\delta_{ij}}[f_{n'i'}, f_{j,n}]_q, \quad i, j \leq n - 1.$$

**Proof.** The proof is a direct calculation with the use of the identity

$$(f_\alpha \otimes 1)\hat{R} - \hat{R}(f_\alpha \otimes 1) = \hat{R}(q^{-h_\alpha} \otimes f_\alpha) - (q^{h_\alpha} \otimes f_\alpha)\hat{R},$$

which follows from the intertwining axiom (1.1) for $x = f_\alpha$. This allows us to construct the elements $f_{ij}$ by induction starting from $f_\alpha, \alpha \in \Pi^+$. $\square$

For each $\alpha \in \Pi^+$, denote by $P(\alpha)$ the set of ordered pairs $l, r = 1, \ldots, N$, with $\varepsilon_l - \varepsilon_r = \alpha$. We call such pairs simple.

**Proposition 2.3.** The matrix entries $f_{ij} \in U_q(\mathfrak{g}-)$ such that $\varepsilon_i - \varepsilon_j \notin \Pi^+$ satisfy the identity

$$[e_\alpha, f_{ij}] = \sum_{(l,r) \in P(\alpha)} (f_{il} \delta_{jr} q^{h_\alpha} - q^{-h_\alpha} \delta_{il} f_{rj}),$$

for all simple positive roots $\alpha$.

**Proof.** The proof is a straightforward calculation based on the intertwining relation (1.1), which is equivalent to

$$(1 \otimes e_\alpha)\hat{R} - \hat{R}(1 \otimes e_\alpha) = \hat{R}(e_\alpha \otimes q^{h_\alpha}) - (e_\alpha \otimes q^{-h_\alpha})\hat{R},$$

for $x = e_\alpha, \alpha \in \Pi^+$. Alternatively, one can use the expressions for $f_{ij}$ from Proposition 2.2. $\square$

### 3 Mickelsson algebras

Consider the Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ corresponding to the root subsystem $R_{\mathfrak{g}'} \subset R_{\mathfrak{g}}$ generated by $\alpha_i, i > 1$, and let $\mathfrak{h}' \subset \mathfrak{g}'$ denote its Cartan subalgebra. Let the triangular decomposition $\mathfrak{g'} = \mathfrak{h}' \oplus \mathfrak{g}'_-$ be compatible with the triangular decomposition of $\mathfrak{g}$. Recall the definition of step algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ of the pair $(\mathfrak{g}, \mathfrak{g}')$. Consider the left ideal $J = U_q(\mathfrak{g})\mathfrak{g}'_+$ and its normalizer $\mathcal{N} = \{ x \in U_q(\mathfrak{g}) : e_\alpha x \subset J, \forall \alpha \in \Pi_{\mathfrak{g}'}^+ \}$. By construction, $J$ is a two-sided ideal in the algebra $\mathcal{N}$. Then $Z_q(\mathfrak{g}, \mathfrak{g}')$ is the quotient $\mathcal{N}/J$. 6
For all $\beta_i \in R_q^+/R_q^+$ let $e_{\beta_i}$ be the corresponding PBW generators and let $Z$ be the vector space spanned by $e_{-\beta_i}^{k_i} \ldots e_{-\beta_1}^{k_1} e_0^{k_0} e_{\beta_1}^{-m_1} \ldots e_{\beta_i}^{-m_i}$, were $e_0 = q^{h_\alpha_1}$, $k_i \in \mathbb{Z}_+$, and $k_0 \in \mathbb{Z}$. The PBW factorization $U_q(\mathfrak{g}) = U_q(\mathfrak{g}'_+) \mathbb{Z}_+ U_q(\mathfrak{h}') U_q(\mathfrak{g}'_+)$ gives rise to the decomposition

$$U_q(\mathfrak{g}) = \mathbb{Z}_+ U_q(\mathfrak{h}') \oplus (\mathfrak{g}'_+ U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \mathfrak{g}'_+).$$

**Proposition 3.1** ([5], Theorem 1). The projection $U_q(\mathfrak{g}) \to \mathbb{Z}_+ U_q(\mathfrak{h}')$ implements an embedding of $\mathbb{Z}_+ U_q(\mathfrak{g}, \mathfrak{g}')$ in $U_q(\mathfrak{h}')$.

**Proof.** The statement is proved in [5] for the orthogonal and special linear quantum groups but the arguments apply to symplectic groups too.

It is proved within the theory of extremal projectors that generators of $\mathbb{Z}_+ U_q(\mathfrak{g}, \mathfrak{g}')$ are labeled by the roots $\beta \in R_q \setminus R_q'$ plus $z_0 = q^{h_\alpha_1}$. We calculate them in the subsequent sections, cf. Propositions 3.5 and 3.9.

### 3.1 Lowering operators

In what follows, we extend $U_q(\mathfrak{g})$ along with its subalgebras containing $U_q(\mathfrak{h})$ over the field of fractions of $U_q(\mathfrak{h})$ and denote such an extension by hat, e.g. $\hat{U}_q(\mathfrak{g})$. In this section we calculate representatives of $\mathbb{Z}_+ U_q(\mathfrak{g}, \mathfrak{g}')$ in $\hat{U}_q(\mathfrak{b}_-)$. Set $h_i = h_{\epsilon_i} \in \mathfrak{h}$ for all $i = 1, \ldots, N$ and introduce $\eta_{ij} \in \mathfrak{h} + \mathbb{C}$ for $i, j = 1, \ldots, N$, by

$$\eta_{ij} = \rho_{ij} - h_{ij} + (\epsilon_i - \epsilon_j, \rho) - \frac{1}{2} \| \epsilon_i - \epsilon_j \|^2. \quad (3.4)$$

Here $\| \mu \|$ is the Euclidean norm on $\mathfrak{h}^*.$

**Lemma 3.2.** Suppose that $(l, r) \in P(\alpha)$ for some $\alpha \in \Pi^+$. Then

i) if $l < r < j$, then $\eta_{lj} - \eta_{rj} = h_\alpha + (\alpha, \epsilon_j - \epsilon_r),$

ii) if $i < l < r$, then $\eta_{li} - \eta_{ri} = h_\alpha + (\alpha, \epsilon_i - \epsilon_r),$

iii) $\eta_{lr} = h_\alpha.$

**Proof.** We have $(\alpha, \rho) = \frac{1}{2} \| \alpha \|^2$ for all $\alpha \in \Pi^+.$ This proves iii). Further, for $\epsilon_l - \epsilon_r = \alpha$:

$$\eta_{lj} - \eta_{rj} = h_\alpha + \frac{1}{2} \| \alpha \|^2 + \frac{1}{2} \| \epsilon_j - \epsilon_r \|^2 - \frac{1}{2} \| \epsilon_j - \epsilon_r - \alpha \|^2 = h_\alpha + (\alpha, \epsilon_j - \epsilon_r), \quad r < j,$$

$$\eta_{li} - \eta_{ri} = h_\alpha + \frac{1}{2} \| \alpha \|^2 + \frac{1}{2} \| \epsilon_i - \epsilon_r \|^2 - \frac{1}{2} \| \epsilon_i - \epsilon_r - \alpha \|^2 = h_\alpha + (\alpha, \epsilon_i - \epsilon_r), \quad i < l,$$

which proves i) and ii).
We call a strictly ascending sequence $\vec{m} = (m_1, \ldots, m_s)$ of integers a route from $m_1$ to $m_s$. We write $m < \vec{m}$ and $\vec{m} < m$ for $m \in \mathbb{Z}$ if, respectively, $m < \min \vec{m}$ and $\max \vec{m} < m$. More generally, we write $\vec{m} < \vec{k}$ if $\max \vec{m} < \min \vec{k}$. In this case, a sequence $(\vec{m}, \vec{k})$ is a route from $\min \vec{m}$ to $\max \vec{k}$.

Given a route $\vec{m} = (m_1, \ldots, m_s)$, define the product

$$f_{\vec{m}} = f_{m_1, m_2} \cdots f_{m_{s-1}, m_s} \in U_q(\mathfrak{g}^{-}).$$

Consider a free right $\hat{\Phi}(\mathfrak{h})$-module $\Phi_{1\vec{m}}$, generated by $f_{\vec{m}}$ with $1 \leq \vec{m} < j$ and define an operation $\partial_{lr} : \Phi_{1j} \to \hat{\Phi}(\mathfrak{b}^-)$ for $(l, r) \in P(\alpha)$ as follows. Assuming $1 \leq \vec{l} < l < r < \vec{\rho} < j$, set

$$\partial_{lr} f_{(\vec{e}, l)} f_{(l, r)} f_{(r, \vec{\rho})} = f_{(\vec{e}, l)} f_{(l, r)} [\eta_j - \eta_{rj}] q,$$

$$\partial_{lr} f_{(\vec{e}, l)} f_{(l, \vec{\rho})} = - f_{(\vec{e}, l)} f_{(l, \vec{\rho})} q^{-\eta_j + \eta_{rj}},$$

$$\partial_{lr} f_{(\vec{e}, r)} f_{(r, \vec{\rho})} = f_{(\vec{e}, r)} f_{(r, \vec{\rho})} q^{\eta_j - \eta_{rj}},$$

$$\partial_{lr} f_{\vec{m}} = 0, \quad l \not\in \vec{m}, \quad r \not\in \vec{m}.$$

Extend $\partial_{lr}$ to entire $\Phi_{1j}$ by $\hat{\Phi}(\mathfrak{h})$-linearity. Let $p : \Phi_{1j} \to \hat{U}(\mathfrak{g})$ denote the natural homomorphism of $\hat{\Phi}(\mathfrak{h})$-modules.

**Lemma 3.3.** For all $\alpha \in \Pi^+$ and all $x \in \Phi_{1j}$, $e_\alpha \circ p(x) = \sum_{(l, r) \in P(\alpha)} \partial_{lr} x \mod \hat{U}(\mathfrak{g}) e_\alpha$.

**Proof.** A straightforward analysis based on Proposition 2.3 and Lemma 3.2. \hfill \Box

To simplify the presentation, we suppress the symbol of projection $p$ in what follows.

Introduce elements $A^\alpha_l \in \hat{U}(\mathfrak{h})$ by

$$A^\alpha_l = \frac{q - q^{-1}_{-\alpha}}{q^{-2\eta_{lj}} - 1},$$

for $r, j \in [1, N]$ subject to $r < j$. For each simple pair $(l, r)$ we define $(l, r)$-chains as

$$f_{(\vec{e}, l)} f_{(l, r)} A^\alpha_l + f_{(\vec{e}, l)} f_{(l, r)} f_{(r, \vec{\rho})} A^\alpha_r + f_{(\vec{e}, r)} f_{(r, \vec{\rho})} A^\alpha_r, \quad f_{(\vec{e}, l)} f_{(l, r)} A^\alpha_l + f_{(\vec{e}, r)} A^\alpha_r,$$

where $1 \leq \vec{l} < l$ and $r < \vec{\rho} \leq j$. Remark that $f_{(l, r)} = \left[\frac{\alpha, \alpha}{-2}\right]_q e_{-\alpha}$, where $\alpha = \varepsilon_l - \varepsilon_r$.

**Lemma 3.4.** The operator $\partial_{lr}$ annihilates $(l, r)$-chains.

**Proof.** Applying $\partial_{lr}$ to the 3-chain in (3.6), we get

$$f_{(\vec{e}, l)} f_{(r, \vec{\rho})} (-q^{-\eta_{lj} + \eta_{rj}} A^\alpha_l + [\eta_{lj} - \eta_{rj}] q A^\alpha_l A^\alpha_r + q^{\eta_{lj} - \eta_{rj}} A^\alpha_r).$$

The factor in the brackets turns zero on substitution of 3.5.

Now apply $\partial_{lj}$ to the right expression in (3.6) and get

$$f_{(\vec{e}, l)} ([h_\alpha] q A^\alpha_l + q^{h_\alpha}) = f_{(\vec{e}, l)} \left(\frac{q^{h_\alpha} - q^{-h_\alpha}}{q^{-2\eta_{lj}} - 1} + q^{h_\alpha}\right) = f_{(\vec{e}, l)} \frac{[h_\alpha - \eta_{lj}] q}{[-\eta_{lj}] q} = 0,$$

so long as $\eta_{lj} = h_\alpha$ by Lemma 3.2. \hfill \Box
Given a route $\vec{m} = (m_1, \ldots, m_s)$, put $A^j_{\vec{m}} = A^j_{m_1} \cdots A^j_{m_s} \in \hat{U}_q(\mathfrak{h})$ (and $A^j_{\vec{m}} = 1$ for the empty route) and define
\[
A^j_{\vec{m}} = A^j_{m_1} \cdots A^j_{m_s} \in \hat{U}_q(\mathfrak{b}_-), \quad j = 2, \ldots, N,
\]
where the summation is taken over all possible $\vec{m}$ subject to the specified inequalities plus the empty route.

**Proposition 3.5.** $e_\alpha z_{-j} = 0 \mod \hat{U}_q(\mathfrak{g}) e_\alpha$ for all $\alpha \in \Pi^+_g$ and $j = 1, \ldots, N - 1$.

**Proof.** Thanks to Lemma 3.3, we can reduce consideration to the action of operators $\partial_{lr}$, with $(l, r) \in P(\alpha)$. According to the definition of $\partial_{lr}$ the summands in (3.7) that survive the action of $\partial_{lr}$ can be organized into a linear combination of $(l, r)$-chains with coefficients in $\hat{U}_q(\mathfrak{b}_-)$. By Lemma 3.4 they are killed by $\partial_{lr}$. \hfill $\Box$

The elements $z_{-i}$, $i = 1, \ldots, N - 1$, belong to the normalizer $N$ and form the set of negative generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the negative part of $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by $z_{-i}$, $i = 1, \ldots, N - 2$.

### 3.2 Raising operators

In this section we construct positive generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$, which are called raising operators. Consider an algebra automorphism $\omega: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined on the generators by $f_\alpha \leftrightarrow e_\alpha$, $q^{h_\alpha} \mapsto q^{\pm h_\alpha}$. For $i < j$, let $g_{ji}$ be the image of $f_{ij}$ under this isomorphism. The natural representation restricted to $U_q(\mathfrak{g}_\pm)$ intertwines $\omega$ and matrix transposition. Since $(\omega \otimes \omega)(\mathcal{R}) = \mathcal{R}_{21}$, the matrix $\mathcal{R}^+ = (\pi \otimes \text{id})(\mathcal{R}_{21})$ is equal to $1 \otimes 1 + (q - q^{-1}) \sum_{i<j} e_{ji} \otimes g_{ji}$.

**Lemma 3.6.** For all $\alpha \in \Pi^+_g$ and all $i > 1$, $e_\alpha g_{i1} = \sum_{(l, r) \in P(\alpha)} \delta_{il} g_{r1} \mod \hat{U}_q(\mathfrak{g}) e_\alpha$.

**Proof.** Follows from the intertwining property of the R-matrix. \hfill $\Box$

Consider the right $\hat{U}_q(\mathfrak{h})$-module $\Psi_{i1}$ freely generated by $f_{(\vec{m}, k)} g_{k1}$ with $i \leq \vec{m} < k$. We define operators $\partial_{lr}: \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$ similarly as we did it for $\Phi_{1j}$. For a simple pair $(l, r) \in P(\alpha)$, put
\[
\partial_{lr} f_{(\vec{m}, k)} g_{k1} = \begin{cases} f_{(\vec{m}, l)} g_{r1}, & l = k, \\ (\partial_{lr} f_{(\vec{m}, k)}) g_{k1}, & l \neq k, \end{cases} \quad i \leq \vec{m} < r.
\]

The Cartan factors appearing in $\partial_{lr} f_{(\vec{m}, k)}$ depend on $h_\alpha$. When pushed to the right-most position, $h_\alpha$ is shifted by $(\alpha, \varepsilon_1 - \varepsilon_r)$. We extend $\partial_{lr}$ to an action on $\Psi_{i1}$ by the requirement

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that \( \partial_r \) commutes with the right action of \( \hat{U}_q(\mathfrak{h}) \). Let \( p \) denote the natural homomorphism of \( \hat{U}_q(\mathfrak{h}) \)-modules, \( p: \Psi_{i_1} \to \hat{U}_q(\mathfrak{g}) \). One can prove the following analog of Lemma 3.3.

**Lemma 3.7.** For all \( \alpha \in \Pi_\mathfrak{g}^+ \) and all \( x \in \Psi_{i_1}, \) \( e_\alpha \circ p(x) = \sum_{(l,r) \in P(\alpha)} \partial_{lr} x \mod \hat{U}_q(\mathfrak{g}) e_\alpha. \)

**Proof.** Straightforward. \( \square \)

We suppress the symbol of projection \( p \) to simplify the formulas.

Define \( \sigma_i \) for all \( i = 1, \ldots, N \) as follows. For \( i < j \) let \( |i - j| \) be the number of simple positive roots entering \( \varepsilon_i - \varepsilon_j \). For all \( i, k = 2, \ldots, N, \) \( i < k, \) put

\[
A_k^i = \frac{q^{\eta k_1 - \eta_{11}}}{[\eta_1 - \eta_{11}]_q}, \quad B_k^i = \frac{(-1)^{|i-k|}}{[\eta_1 - \eta_{11}]_q},
\]

For each \((l, r) \in P(\alpha), \) where \( \alpha \in \Pi_\mathfrak{g}_\mathfrak{e}^+, \) define 3-chains as

\[
f(i, \vec{m}, l)g_1 B_l^i + f(i, \vec{m}, l)f(l, r)g_{lr} A_{lr} B_r^i + f(i, \vec{m}, r)g_{lr} B_r^i,
\]

(3.8)

with \( i < \vec{m} < l < r \leq N \) and

\[
f(i, \vec{l}, l)g_1 A_l^i + f(i, \vec{l}, l)f(l, r)g_{lr} A_{lr} A_r^i + f(i, \vec{l}, r)f(l, r)g_{lr} A_r^i
\]

(3.9)

with \( i < \vec{l} < l < r < \vec{p} < k \leq N. \) The 2-chains are defined as

\[
g_1 + f(i, r)g_{lr} B_r^i, \quad f(i, \vec{m}, k)g_1 + f(i, r)f(r, \vec{m}, k)g_{lr} A_r^i
\]

(3.10)

where \( r \) is such that \( \varepsilon_i - \varepsilon_r \in \Pi_\mathfrak{g}_\mathfrak{e}^+ \) and \( i < r < \vec{m} < k \leq N. \) In all cases the empty routes \( \vec{m} \) are admissible.

**Lemma 3.8.** For all \( \alpha \in \Pi_\mathfrak{g}_\mathfrak{e}^+ \) and all \((l, r) \in P(\alpha)\) the \((l, r)\)-chains are annihilated by \( \partial_{lr}. \)

**Proof.** Suppose that \( i = l \) and apply \( \partial_{lr} \) to the left 2-chain in (3.10). The result is

\[
g_1 + [h_\alpha]_q g_{lr} B_r^i = g_{lr} (1 + [h_\alpha + (\alpha, \varepsilon_1 - \varepsilon_r)]_q B_r^i) = g_{lr} (1 + [\eta_{11} - \eta_{11}]_q B_r^i) = 0,
\]

by Lemma 3.2. Applying \( \partial_{lr} \) to the right 2-chain in (3.10) we get

\[
f(i, \vec{m}, k)g_1 (-q^{-\eta_{11} + \eta_{11}} + [\eta_{11} - \eta_{11}]_q A_r^i) = 0.
\]

Now consider 3-chains. The action of \( \partial_{lr} \) on the (3.9) produces

\[-f(i, \vec{l}, l)q^{-h_\alpha} f(r, \vec{p}, k)g_{kr} A_l^i + f(i, \vec{l}, l)q_{h_\alpha} f(r, \vec{p}, k)g_{kr} A_l^i A_r^i + f(i, \vec{l}, l)q^{h_\alpha} f(r, \vec{p}, k)g_{kr} A_r^i,
\]

which turns zero since \(-q^{\eta_{11} - \eta_{11}} A_l^i + [\eta_{11} - \eta_{11}]_q A_l^i A_r^i + q^{\eta_{11} - \eta_{11}} A_r^i = 0. \) The action of \( \partial_{lr} \) on (3.8) yields

\[
f(i, \vec{m}, l)g_{lr} B_l^i + f(i, \vec{m}, l)[h_\alpha]_q g_{lr} A_l^i B_r^i + f(i, \vec{m}, l)q_{h_\alpha} g_{lr} B_r^i.
\]

This is vanishing since \( B_l^i + [\eta_{11} - \eta_{11}]_q A_l^i B_r^i + q^{\eta_{11} - \eta_{11}} B_r^i = B_l^i + [\eta_{11} - \eta_{11}]_q B_r^i = 0. \) \( \square \)
Given a route $\vec{m} = (m_1, \ldots, m_k)$ such that $i < \vec{m}$ let $A_{\vec{m}}^i$ denote the product $A_{m_1}^i \cdots A_{m_k}^i$.

Introduce elements $z_i \in \hat{U}_q(\mathfrak{g}_-)\mathfrak{g}_+$ of weight $\varepsilon_1 - \varepsilon_i$ by

$$z_{i-1} = g_{i1} + \sum_{i < \vec{m} < k \leq N} f_{(i, \vec{m}, k)} g_{k1} A_{\vec{m}}^i B_{k}^i, \quad i = 2, \ldots, N.$$ 

Again, the summation includes empty $\vec{m}$.

**Proposition 3.9.** $e_\alpha z_i = 0 \mod \hat{U}_q(\mathfrak{g})e_\alpha$, for all $\alpha \in \Pi^+_g$ and $i = 1, \ldots, N - 1$.

**Proof.** By Lemma 3.6, the vectors $g_{2'1}$ and $z_{N-1} = g_{1'1}$ are normalizing the left ideal $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, so is $z_{N-2} = g_{2'1} + f_1 g_{1'1} B_{2'1}$. Once the cases $i = 2', 1'$ are proved, we further assume $i < 2'$.

In view of Lemma 3.7, it is sufficient to show that $z_{i-1}$ is killed, modulo $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, by all $\partial_{lr}$ such that $\varepsilon_l - \varepsilon_r \in \Pi^+_g$. Observe that $z_{i-1}$ can be arranged into a linear combination of chains, which are killed by $\partial_{lr}$, as in Lemma 3.8. \QED

The elements $z_i$, $i = 1, \ldots, N - 1$, belong to the normalizer $\mathcal{N}$. They form the set of positive generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the positive part of $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by $z_i$, $i = 1, \ldots, N - 2$.

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