Harsh thresholds for constraint satisfaction problems
and homomorphisms

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Abstract

We determine under which conditions certain natural models of random constraint satisfaction problems have sharp thresholds of satisfiability. These models include graph and hypergraph homomorphism, the \((d,k,t)\)-model, and binary constraint satisfaction problems with domain size 3.
1 Introduction

Random 3-SAT and its generalizations have been studied intensively for the past decade or so (see eg. [1, 5, 8, 17, 25, 2, 7, 13, 15, 34, 9]). One of the most interesting things about these models, and arguably the main reason that most people study them, is that many of them exhibit what is called a \textit{sharp threshold of satisfiability} \footnote{Defined formally below.}, a critical clause-density at which the random problem suddenly moves from being almost surely\footnote{Defined formally below.} satisfiable to almost surely unsatisfiable. Most of the work on these problems is, at least implicitly, an attempt to determine the precise locations of their thresholds. At this point, these locations are known only for a handful of the problems, such as [2, 7, 13, 15, 34, 9]. Just proving the existence of a sharp threshold for random 3-SAT was considered a major breakthrough by Friedgut\cite{17}. The vast majority of these generalizations appear to have sharp thresholds, but there are exceptions which are said to have coarse thresholds\footnote{Defined formally below.}.

The ultimate goal of the present line of enquiry is to determine precisely which of these models have sharp thresholds, but this appears to be quite difficult; in Section 2 we show that it is at least as difficult as determining the location of the threshold for 3-colourability, something that has been sought after for more than 50 years (see eg. [14, 28]). A more fundamental goal is to obtain a better understanding of what can cause some problems to have coarse thresholds rather than sharp ones.

Molloy\cite{27} and independently Creignou and Daude\cite{10} introduced a wide family of models for random constraint satisfaction problems which includes 3-SAT and many of its generalizations. This permits us to study them under a common umbrella, rather than one-at-a-time. Molloy determined precisely which models from this family have any threshold at all (\cite{10} provides the same result for those models with domain\footnote{Defined formally below.} size 2). But he left open the much more important question of which models have sharp thresholds. In this paper, we begin to address this question. We answer it for two of the most natural subfamilies - the so-called \((d, k, t)\)-family\footnote{Defined formally below.} (Theorem 2), and the family of graph and hypergraph homomorphism problems (Theorem 4). We also shed light on the more fundamental problem by determining the only properties that can cause a coarse threshold in binary constraint satisfaction problems with domain size 3.

The standard example of a problem with a coarse threshold is 2-colourability. Here, there is a coarse threshold precisely because unsatisfiability (i.e. non-2-colourability) can be caused only by the presence of odd cycles. Roughly speaking, Friedgut’s theorem\cite{17} implies that a problem exhibits a coarse threshold iff unsatisfiability is \textit{approximately} equivalent to having one of a set of unicyclic\footnote{Defined formally below.} subproblems. It is not hard to see that if there are unsatisfiable unicyclic instances of a problem then that problem exhibits a coarse threshold (or exhibits no threshold at all). This makes it quite natural to pose the following rule-of-thumb:

\textbf{Hypothesis A:} \textit{If a random model from the family in \cite{27} is such that: (a) it exhibits a threshold, and (b) every unicyclic instance is satisfiable, then that threshold is sharp.}

However, reality is not that simple. \cite{27} presents a counterexample to Hypothesis A; others are presented in this paper. Nevertheless, the hypothesis holds for certain subfamilies of models. Creignou and Daude\cite{10} conjectured that Hypothesis A holds for problems with domain-size
two; this was proven by Istrate [23] and independently Creignou and Daude [11] proved it for the case where the model is symmetric. Theorems 2 and 4 in this paper show that Hypothesis A holds for the \((d, k, t)\)-models and for homomorphism problems.

In general, coarse thresholds can be caused by much more subtle and insidious reasons than unsatisfiable unicyclic instances. In this paper we begin to understand some of these reasons by focusing on the case where the constraint size is two and the domain size is three (a natural next step after the well-understood domain-size-two case). In this paper, we identify a particular subtle cause, and show that this and unsatisfiable unicyclic instances are the only things that can cause a coarse threshold (Theorem 14). If we permit either greater domain sizes or greater constraint sizes then this is no longer true - there are other possible causes.

1.1 The random models

In our setting, the variables of a constraint satisfaction problem (CSP) all have the same domain of permissible values, \(\{1, ..., d\}\), and all constraints will have size \(k\), for some fixed integers \(d, k\). Given a \(k\)-tuple of variables, \((x_1, ..., x_k)\), a restriction on \((x_1, ..., x_k)\) is a \(k\)-tuple of values \(R = (\delta_1, ..., \delta_k)\) where each \(1 \leq \delta_i \leq d\). For each \(k\)-tuple \((x_1, ..., x_k)\), the set of restrictions on that \(k\)-tuple is called a constraint. The empty constraint is the constraint which contains no restrictions. We say that an assignment of values to the variables of a constraint \(C\) satisfies \(C\) if that assignment is not one of the restrictions in \(C\). An assignment of values to all variables in a CSP satisfies that CSP if every constraint is simultaneously satisfied. A CSP is satisfiable if it has such a satisfying assignment.

It will be convenient to consider a set of canonical variables \(X_1, ..., X_k\) which are used only to describe the “pattern” of a constraint. These canonical variables are not variables of the actual CSP. For any \(d, k\) there are \(d^k\) possible restrictions and \(2^{dk}\) possible constraints over the \(k\) canonical variables. We denote this set of constraints as \(C^{d,k}\). For our random model, one begins by specifying a particular probability distribution, \(P\) over \(C^{d,k}\). We use \(\text{supp}(P)\) to denote the support of \(P\); i.e. the set of constraints \(C\) with \(P(C) > 0\). Different choices of \(P\) give rise to different instances of the model.

We now define our random models. The “\(G_{n,M}\)” model, where the number of constraints is fixed to be \(M\), is the most common. But in this paper, it will be much more convenient to focus on the “\(G_{n,p}\)” model where each \(k\)-tuple of variables is chosen independently with probability \(p = c/n^{k-1}\) to receive a constraint. The two models are, in most respects, equivalent when \(M = (c/k!)n\). In particular, it is straightforward to show that one exhibits a sharp threshold iff the other does.

The \(CSP_{n,p}(P)\) Model: Specify \(n, p\) and \(P\) (typically \(p = c/n^{k-1}\) for some constant \(c\); note that \(P\) implicitly specifies \(d, k\)). First choose a random \(k\)-uniform hypergraph on \(n\) variables where each of the \(\binom{n}{k}\) potential hyperedges is selected with probability \(p\). Next, for each hyperedge \(e\), we choose a constraint on the \(k\) variables of \(e\) as follows: we take a random permutation from the \(k\) variables onto \(\{X_1, ..., X_k\}\) and then we select a random constraint according to \(P\) and map it onto the \(k\) variables.

A property holds \(\text{almost surely} \) (a.s) if the limit as \(n \to \infty\) of it holding is 1. We say that
\(CSP_{n,p}(\mathcal{P})\) has a \textit{sharp threshold of satisfiability} if there is some \(c = c(n) > 0\) such that for every \(\epsilon > 0\), if \(p = (1 - \epsilon)c/n^{k-1}\) then \(CSP_{n,p}(\mathcal{P})\) is a.s. satisfiable and if \(p = (1 + \epsilon)c/n^{k-1}\) then \(CSP_{n,p}(\mathcal{P})\) is a.s. unsatisfiable. This is often abbreviated to just \textit{sharp threshold}. We say that \(CSP_{n,p}(\mathcal{P})\) has a \textit{coarse threshold} if for all \(c\) in some interval \(c_1(n) < c < c_2(n)\), \(CSP_{n,p}(\mathcal{P})\) is neither a.s. satisfiable nor a.s. unsatisfiable. If \(CSP_{n,p}(\mathcal{P})\) has neither a sharp nor a coarse threshold, then it is easy to see that it must either be a.s. satisfiable for all \(c > 0\) or a.s. unsatisfiable for all \(c > 0\).

Each \(k\)-tuple of vertices can have at most one constraint in \(CSP_{n,p}(\mathcal{P})\). When applying Friedgut’s theorem, it will be convenient to relax this condition, and allow \(k\)-tuples to possibly receive multiple constraints. Thus up to \(k! \times |\text{supp}(\mathcal{P})|\) constraints can appear on a \(k\)-tuple of variables.

\textbf{The \( \tilde{CSP}_{n,p}(\mathcal{P})\) Model:} Specify \(n, p\) and \(\mathcal{P}\). For each of the \(n(n-1)...(n-k+1)\) ordered \(k\)-tuples of variables and each constraint \(C \in \text{supp}(\mathcal{P})\), we assign \(C\) to the ordered \(k\)-tuple with probability \(\mathcal{P}(C) \times p/k!\).

Note that the expected total number of constraints is the same under each model. Furthermore, it is easy to calculate that the probability of at least one \(k\)-tuple receiving more than one constraint in \(\tilde{CSP}_{n,p}(\mathcal{P})\) is for \(k \geq 3\), \(o(1)\) and for \(k = 2\), an absolute constant \(0 < \alpha < 1\). It follows that if a property holds a.s. in \(\tilde{CSP}_{n,p}(\mathcal{P})\) then it holds a.s. in \(CSP_{n,p}(\mathcal{P})\). As a corollary, we have:

\textbf{Lemma 1} If \(\tilde{CSP}_{n,p}(\mathcal{P})\) has a sharp threshold then so does \(CSP_{n,p}(\mathcal{P})\). The reverse is true for \(k \geq 3\).

So for the remainder of the paper, whenever we wish to prove that \(CSP_{n,p}(\mathcal{P})\) has a sharp threshold, we will work in the \(\tilde{CSP}_{n,p}(\mathcal{P})\) model.

We often focus on the \textit{constraint hypergraph} of a CSP; i.e. the hypergraph whose vertices are the variables and whose edges are the tuples of variables that have constraints. A \textit{tree-CSP} is a CSP whose constraint hypergraph is a hypertree. A CSP is \textit{unicyclic} if its constraint hypergraph is unicyclic; i.e. has exactly one cycle. (Hypertree and cycle are defined below).

We close this subsection with some hypergraph definitions. A hypergraph consists of a set of vertices and a set of \textit{hyperedges}, where each hyperedge is a collection of vertices. If every hyperedge has size exactly \(k\) then the hypergraph is \textit{k-uniform}. In a \textit{simple} hypergraph, no vertex appears twice in any one hyperedge, and no two edges are identical. So, for example, the constraint hypergraph of \(CSP_{n,p}(\mathcal{P})\) is simple, but the constraint hypergraph of \(\tilde{CSP}_{n,p}(\mathcal{P})\) may have multiple edges. Neither model permits multiple copies of a vertex in a single edge, but such edges are possible when we discuss hypergraph homomorphism problems. The edge \((v, v, ..., v)\) is called a \textit{loop}.

A walk \(P\) of size \(r\) is a sequence of \(r\) hyperedges and \(r + 1\) vertices \((v_0, e_1, v_1, e_2, v_2, ..., e_r, v_r)\) such that \(e_i\) contains both \(v_{i-1}\) and \(v_i\). A walk is a \textit{path} if the \(v_i\) are distinct. A walk is a \textit{cycle} of size \(r\) if for \(i = 1, ..., r\) the \(v_i\) and \(e_i\) are distinct, and \(v_0 = v_r\). The \textit{distance} from a vertex \(u\) to a vertex \(v\) is the minimum \(r\) such that there exists a walk of length \(r\), \((v, e_1, v_1, ..., e_r, u)\); the distance of a vertex from itself is defined to be 0. The distance from a vertex \(v\) to a set of
vertices is the minimum distance from \( v \) to any vertex in the set. A hypergraph is a \textit{hypertree} if it has no cycles and it is connected.

By \textit{contracting} two vertices \( u \) and \( v \) into a new vertex \( w \), we mean (i) adding a new vertex \( w \) to the set of the vertices, (ii) replacing \( u \) and \( v \) in every hyperedge by \( w \), and (iii) removing \( u \) and \( v \).

### 1.2 Two special families

Perhaps the most natural choice for \( \mathcal{P} \) is the distribution obtained by selecting each of the \( d^k \) possible restrictions independently with probability \( 1/d^k \). However, as noted in [4], every such choice of \( \mathcal{P} \) yields a model that is a.s. unsatisfiable for any non-trivial choice of \( p \). So this is a rather uninteresting family of models, particularly as far as the study of thresholds goes.

The next most natural choice for \( \mathcal{P} \) is to fix \( t \), the number of restrictions per clause, and to make every constraint with exactly \( t \) restrictions equally likely. (Note that for \( d = 2, t = 1 \) this yields random \( k \)-SAT.) This is often called the \((d,k,t)\)-model and has received a great deal of study, both from a theoretical perspective [26, 30] and from experimentalists (see [18] for a survey of many such studies). In [4] it is shown that when \( t \geq d^k-1 \), this model is problematic in the same way as the previously mentioned one, as it is a.s. unsatisfiable even for values of \( p = o(1/n^{k-1}) \) (i.e. when the number of constraints is \( o(n) \)). However, it was proven in [18] that for every \( 1 \leq t < d^k-1 \), the \((d,k,t)\)-model does not have that problem. One of the main contributions of this paper is to show that in this case the model exhibits a sharp threshold:

**Theorem 2** For every \( d, k \geq 2 \) and every \( 1 \leq t < d^k-1 \), the \((d,k,t)\)-model has a sharp threshold.

From a different perspective, it is quite natural to consider the case where every constraint is identical, i.e. \( |\supp(\mathcal{P})| = 1 \). It is not hard to see that every such problem is equivalent to a hypergraph homomorphism problem, as defined below:

For two \( k \)-uniform hypergraphs, \( G, H \), a \textit{homomorphism} from \( G \) to \( H \) is a mapping \( h \) from \( V(G) \) to \( V(H) \) such that for each edge \((v_1, v_2, \ldots, v_k)\) of \( G \), \((h(v_1), h(v_2), \ldots, h(v_k))\) is an edge of \( H \). We say that \( G \) is \textit{homomorphic} to \( H \), if there exists such a homomorphism. When \( k = 2 \) and \( H \) is the complete graph with no loops, we are simply asking whether \( G \) has a \( d \)-colouring. Homomorphisms are an important generalization of graph colouring (see, eg. [21]). They are often also referred to as \( H \)-colourings (eg. [22, 20]).

Suppose that \( H \) is a fixed hypergraph, and \( G \) is a random hypergraph on \( n \) vertices where each of the \( \left( \begin{array}{c} n \\ k \end{array} \right) \) potential hyperedges is selected with probability \( p \). Set \( d \) to be equal to the number of vertices in \( H \) and define a constraint \( C \) with domain size \( d \) and constraint size \( k \) by saying that \( C \) permits \( X_1 = \delta_1, \ldots, X_k = \delta_k \) iff \((\delta_1, \ldots, \delta_k)\) is a hyperedge of \( H \). Treat each vertex of \( G \) as a variable with domain \( \{1, \ldots, d\} \) and assign \( C \) to each hyperedge of \( G \). We call this the \textit{H-homomorphism problem}.

Thus we have an instance of \( CSP_{n,p}(\mathcal{P}) \) where \( C \) is the only constraint in \( \supp(\mathcal{P}) \) and furthermore \( C \) is symmetric under permutations of the canonical variables; in other words, all
constraints are identical even under permutations of variables. It is easy to see that every such \( P \) corresponds to a homomorphism problem; just take \( H \) to be the hypergraph where \((\delta_1, \ldots, \delta_k)\) is a hyperedge iff \( C \) permits \( X_1 = \delta_1, \ldots, X_k = \delta_k \). Note that here a hyperedge in \( H \) may contain multiple copies of a vertex.

Thus, these \( H \)-homomorphism problems are not only important as a fundamental graph problem, but also because they form a very natural subclass of our family of random CSP models. In this paper, we prove that Hypothesis A holds for every connected \( H \).

It is easy to see that if \( H \) has a loop \((\delta, \delta, \ldots, \delta)\) then every hypergraph is trivially homomorphic to \( H \) (just map every vertex to \( \delta \)); so the \( H \)-homomorphism problem has no threshold at all. The other trivial case is where \( H \) has no hyperedges at all and so no non-trivial hypergraph has an \( H \)-homomorphism.

**Lemma 3** Suppose that \( H \) is a nontrivial hypergraph with no loops. We have the following:

1. For \( k \geq 3 \), every unicyclic hypergraph is homomorphic to a single hyperedge, and hence to \( H \).
2. For \( k = 2 \): if the triangle is homomorphic to \( H \), then so is every unicyclic graph; and the triangle is homomorphic to \( H \) iff \( H \) contains a triangle.

**Proof.** To prove part (1), let \((v_0, e_1, v_1, e_2, v_2, \ldots, e_r, v_0)\) be the unique cycle of the hypergraph, and let \((w_0, \ldots, w_{k-1})\) be a single hyperedge. Define \( h(v_i) = w_{i \mod 2} \), for every \( 0 \leq i \leq r - 2 \) and \( h(v_{r-1}) = w_2 \). It is easy to see that one can extend \( h \) to a homomorphism from the unicyclic hypergraph to \( H \).

Part (2) easily follows from the easy and well-known fact that every cycle is homomorphic to the triangle, and the triangle is not homomorphic to any cycle of size greater than 3.

From Lemma 3 we conclude that proving that Hypothesis A holds whenever \( H \) is connected and undirected is equivalent to proving:

**Theorem 4** If \( H \) is a connected undirected loopless hypergraph with at least one edge, then the \( H \)-homomorphism problem has a sharp threshold iff (a) \( k \geq 3 \) or (b) \( k = 2 \) and \( H \) contains a triangle.

We do not have a strong feeling as to whether the “connected” condition is necessary here; we discuss the possibility of extending Theorem 4 to disconnected graphs in Section 3.

### 1.3 Tools

Our main tool is distilled from Friedgut’s main theorem in [17]. Friedgut reported to us[19] that his proof can be adapted to the setting of this paper. To provide Friedgut’s theorem for CSP’s in its full power instead of being restricted to the unsatisfiability property, we consider, as Friedgut did, every monotone property where a property \( A \) is called *monotone* if it is preserved under
constraint addition. A property $A$ on CSP’s is called *monotone symmetric* if it is monotone and invariant under CSP automorphisms. For a property $A$, $A_n$ denotes the restriction of $A$ on CSP’s with exactly $n$ variables. Roughly speaking, Friedgut’s theorem says that for a value of $p$ that is “within” the coarse threshold, there is a constant sized instance $M$ such that $\tau < \Pr[ M \subseteq \hat{\text{CSP}}_{n,p}(\mathcal{P})] < 1 - \tau$ for some constant $\tau$ which does not depend on $n$, and adding $M$ to our random CSP boosts the probability of being in $A$ by at least $2\alpha > 0$, whereas adding a linear number of new random constraints only boosts it by at most $\alpha$. First, we must formalize what we mean by “adding $M$”. Given two CSP’s $M, F$ where $M$ has $r$ variables, and $F$ has at least $r$ variables, we define $F \oplus M$ to be the CSP obtained by choosing a random $r$-tuple of variables in $F$ and then adding $M$ on those $r$ variables. Now we can state Friedgut’s theorem formally:

**Theorem 5** Let $A = \{A_i\}$ be a series of monotone symmetric properties in $\hat{\text{CSP}}_{n,p}(\mathcal{P})$ with a coarse threshold. There exist, $p = p(n), \tau, \alpha, \epsilon > 0$, a CSP $M$ whose constraints are chosen from $\text{supp}(\mathcal{P})$ such that for an infinite number of $n$:

(a) $\alpha < \Pr[ \hat{\text{CSP}}_{n,p}(\mathcal{P}) \in A ] < 1 - 3\alpha$.

(b) $\tau < \Pr[ M \subseteq \hat{\text{CSP}}_{n,p}(\mathcal{P}) ] < 1 - \tau$.

(c) $\Pr[ \hat{\text{CSP}}_{n,p}(\mathcal{P}) \oplus M \in A ] > 1 - \alpha$.

(d) $\Pr[ \hat{\text{CSP}}_{n,p(1+\epsilon)}(\mathcal{P}) \in A ] < 1 - 2\alpha$.

When as in our setting $p(n) = c(n)/n^{k-1}$, Theorem 5(b) implies that $M$ is a unicycle CSP. So we obtain the following corollary which is our main tool in this paper.

**Corollary 6** For any $\mathcal{P}$, if $\hat{\text{CSP}}_{n,p}(\mathcal{P})$ has a coarse threshold of satisfiability then there exist $p = p(n), \alpha, \epsilon > 0$, and a unicyclic CSP $M$ on a constant number of variables whose constraints are chosen from $\text{supp}(\mathcal{P})$ such that:

(a) $\alpha < \Pr[ \hat{\text{CSP}}_{n,p}(\mathcal{P}) \text{ is unsatisfiable} ] < 1 - 3\alpha$.

(b) $\Pr[ \hat{\text{CSP}}_{n,p(1+\epsilon)}(\mathcal{P}) \text{ is unsatisfiable} ] < 1 - 2\alpha$.

(c) $\Pr[ \hat{\text{CSP}}_{n,p}(\mathcal{P}) \oplus M \text{ is unsatisfiable} ] > 1 - \alpha$.

Our next tool proves some properties for local parts of a random CSP.

**Lemma 7** Suppose that $p < cn^{1-k}$ for some positive constant $c$, and let $G$ be an instance of $\hat{\text{CSP}}_{n,p}(\mathcal{P})$. Choose a set $T$ of $t$ random variables. Then for every $\epsilon > 0$, and integer $r > 0$ there exists an integer $L(c, t, r, \epsilon)$ such that with the probability of at least $1 - \epsilon$:

(i) No constraint of $G$ contains more than one variable of $T$. 

6
(ii): $G$ induces a forest on the set of the variables that are of distance at most $r$ from $T$.

(iii): There are at most $L$ variables that are of distance at most $r$ from $T$.

**Proof.** Let $E_1$, $E_2$, and $E_3$ denote the events (i), (ii), and (iii) respectively. Trivially

$$\Pr[E_1] \geq 1 - \sum_{i=2}^{k} n^{k-i} \binom{t}{i} k! p = 1 - o(1).$$

(1)

The expected number of the cycles of size at most $2r$ which contain at least one variable in $T$ is at most $t \sum_{i=2}^{2r} n^{k-i-1} p^i$. Thus

$$\Pr[E_2] \geq 1 - t \sum_{i=2}^{2r} n^{k-i-1} p^i \geq 1 - \frac{2tr(1+c)^{2r}}{n} = 1 - o(1).$$

(2)

The expected number of the variables in a distance of at most $r$ from $T$ is at most $t \sum_{i=1}^{r} n^{k-i} p^i$. So by Chebychev's inequality, for sufficiently large $L$:

$$\Pr[E_3] \geq 1 - \frac{t \sum_{i=1}^{r} n^{k-i} p^i}{L} \geq 1 - \frac{\epsilon}{2}.$$

(3)

The lemma follows from (1), (2), and (3).

Our third tool is easily proven with a straightforward first moment calculation and concentration argument (via eg. the second moment method or Talagrand’s inequality).

**Lemma 8** Let $T$ be a tree-CSP whose constraints are in $\text{supp}(P)$. There exists $z = o(n^{1-k})$ such that a.s. $\text{CSP}_{n,z}(P)$ contains $T$ as a sub-CSP.

## 2 Difficulty

The ultimate goal of this research is to characterize all distributions $P$ for which $\text{CSP}_{n,p}(P)$ exhibits a sharp threshold. However, the following example indicates that this is very difficult, even for binary CSP’s (the case where $k = 2$). In particular, it is at least as difficult as determining the location of the 3-colourability threshold, a heavily pursued open problem. (The existence of that threshold was proven in [3]; see [28] for a recent survey and see [6, 16] for the best current bounds on its location.)

We set $d = 5$ and $k = 2$, and define two constraints by listing their pairs of forbidden values:

- $C_1 = \{(1,1),(2,2)\} \cup \{(1,2) \times \{3,4,5\}\} \cup (\{3,4,5\} \times \{1,2\})$,
- $C_2 = \{(3,3),(4,4),(5,5)\} \cup (\{1,2\} \times \{3,4,5\}) \cup (\{3,4,5\} \times \{1,2\})$.

Note that each constraint forces the endpoints of every edge to take values that are either both in $\{1,2\}$ or both in $\{3,4,5\}$. A $C_1$ constraint says that they have to be different values if they are both in $\{1,2\}$. A $C_2$ constraint says that they have to be different values if they are both in $\{3,4,5\}$.

We let $C_1$ occur with probability $q$ and $C_2$ occur with probability $1 - q$ in $\mathcal{P}$. Set $c(q) = (1 - q)/q$. 

7
Fact 9  (a) If $G_{n,p=c(q)/n}$ is a.s. 3-colourable, then $CSP_{n,p}(P)$ has a sharp threshold.

(b) If there is some $\epsilon > 0$ such that $G_{n,p=(c(q)-\epsilon)/n}$ is a.s. not 3-colourable, then $CSP_{n,p}(P)$ has a coarse threshold.

Thus, determining the type of the threshold for all such models $CSP_{n,p}(P)$ requires the knowledge of for which values of $c$, $G(n,\frac{\Delta}{n})$ is a.s. 3-colourable, and for which values it is a.s. not 3-colourable.

Proof  Choose our $CSP_{n,p}(P)$ by first taking $G_{n,p=c/n}$ and then setting each edge to be $C_1$ with probability $q$ and $C_2$ otherwise. Let $G_1, G_2$ be the subgraphs formed by the edges chosen to be $C_1, C_2$ respectively. If $c < 1$ then all components of $G_{n,p=c/n}$ are trees or unicycles and the CSP is trivially satisfiable. So we can focus on the range $c > 1$ and we let $T$ denote the giant component of $G_{n,p=c/n}$. Note that the variables of $T$ must either all take values from $\{1, 2\}$ or all take values from $\{3, 4, 5\}$.

Case 1: $c > \frac{1}{q}$. Then $G_1$ is equivalent to $G_{n,p=c_1/n}$ for some $c_1 > 1$ and it follows easily that a.s. $G_1$ contains a giant component which is not 2-colourable. This giant component is a subgraph of $T$ and so the variables of $T$ must all take values from $\{3, 4, 5\}$. If follows that the CSP is satisfiable iff $G_2$ is 3-colourable. Note that $G_2$ is equivalent to $G_{n,p=c_2/n}$ for some $c_2 > c(q)$.

Case 2: $c < \frac{1}{q}$. Then $G_1$ is equivalent to $G_{n,p=c_1/n}$ for some $c_1 < 1$ and $G_2$ is equivalent to $G_{n,p=c_2/n}$ for some $c_2 < c(q)$. If $G_2$ is a.s. 3-colourable then the CSP is a.s. satisfiable. If $G_2$ is a.s. not 3-colourable then the CSP is satisfiable iff $T$ is 2-colourable; i.e., if $G_2$ does not have an odd cycle lying within $T$. It is easy to see that this occurs with probability between $\zeta$ and $1 - \zeta$ for some $\zeta > 0$; i.e. that the CSP is neither a.s. satisfiable nor a.s. unsatisfiable.

Fact 9 now follows. If $G_{n,p=c(q)/n}$ is a.s. 3-colourable, then $CSP_{n,p}(P)$ has a sharp threshold which lies somewhere above $\frac{1}{q}$. If there is some $\epsilon > 0$ such that $G_{n,p=(c(q)-\epsilon)/n}$ is a.s. not 3-colourable, then $CSP_{n,p}(P)$ has a coarse threshold running from $\frac{1}{q} - \delta$ to $\frac{1}{q}$ for some $\delta > 0$. □

3 Homomorphisms

In this section, we prove our theorem concerning $H$-homomorphisms. Let $G_{n,p}^k$ denote the random $k$-uniform hypergraph on $n$ vertices where each $k$-tuple is present as a hyperedge with probability $p$.

Proof of Theorem 4 We begin with the case $k \geq 3$. Let $H$ be some $k$-uniform hypergraph, and assume that $H$ has a coarse threshold. Let $M, p, \alpha, \epsilon$ be as guaranteed by Corollary 6. In this setting, $M$ is a unicycle $k$-uniform hypergraph, such that adding $M$ to $G_{n,p}^k$ boosts the probability of not having a homomorphism to $H$ by at least $2\alpha$.

Consider $G = G_{n,p}^k \oplus M$. Let $M^+$ be the subgraph of $G$ consisting of all hyperedges that contain at least one vertex of $M$ (and, of course all vertices in those hyperedges); in other words, $M^+$ is the subhypergraph induced by the vertices of $M$ and all their neighbours. Lemma 7 implies that there is some constant $L$ such that with probability at least $1 - \frac{\alpha}{2}$: $M^+$ is unicyclic and has at most $L$ vertices, and no hyperedge of $G_{n,p}^k$ contains more than one vertex of $M$. 8
Since $M$ is unicyclic, by Lemma 3 there exists a homomorphism $h$ from $M$ to a single edge, say $(v_1, \ldots, v_k)$. Let $h_i$ be the set of the vertices in $M$ that are mapped by $h$ to $v_i$. Obtain the hypergraph $G'$ from $G$ by (i) removing all edges in $M$; (ii) contracting all of the vertices in $h_i$ into one single new vertex $u_i$, for each $1 \leq i \leq k$; (iii) adding the single hyperedge $(u_1, \ldots, u_k)$.

Suppose that $h'$ is a homomorphism from $G'$ to $H$. Then a mapping from the vertices of $G$ to the vertices of $H$ which maps every vertex $v$ in $G - M$ to $h'(v)$, and every vertex in $h_i$ to $h'(u_i)$ is a homomorphism from $G$ to $H$. Thus, if $G'$ is homomorphic to $H$ then so is $G$.

Let $T$ be the hypertree defined as follows: $T$ has a hyperedge $(t_1, \ldots, t_k)$, and each $t_i$ lies in $L$ other hyperedges. Only $t_1, \ldots, t_k$ lie in more than one edge of $T$. Thus, $T$ has $k + k(k - 1)L$ vertices and $kL + 1$ hyperedges. Note that in $G'$ the subgraph induced by all edges containing $(u_1, \ldots, u_k)$ form a subtree of $T$. It follows that $G_{n, p}^k \oplus T$ is at least as likely to be non-homomorphic to $H$ as $G'$ is, so:

\[ \Pr[G_{n, p}^k \oplus M \text{ is not homomorphic to } H] \leq \Pr[G_{n, p}^k \oplus T \text{ is not homomorphic to } H] + \frac{\alpha}{2}. \]

By Lemma 8, increasing $p$ by an additional $\epsilon p$ a.s. results in the addition of a copy of $T$. Thus:

\[ \Pr[G_{n, p}^k \oplus T \text{ is not homomorphic to } H] \leq \Pr[G_{n, p(1+\epsilon)}^k \text{ is not homomorphic to } H] \]

which yields a contradiction to Corollary 6(b).

This proves the case where $k \geq 3$, so we now turn to the case $k = 2$. If $H$ contains no triangle, then $K_3$ is not homomorphic to $H$. Thus, $K_3$ forms a unicyclic unsatisfiable CSP using the $H$-colouring constraints and so we do not have a sharp threshold. So we will focus on graphs $H$ that contain a triangle. Our proof follows along the same lines as the case $k \geq 3$, but is complicated a bit since we can no longer assume that $M$ is homomorphic to a single edge. We only highlight the differences.

Define $M^+$ to be the subgraph of $G = G_{n, p}^k \oplus M$ induced by all vertices within distance $r = |V(H)| + |V(M)| + 3$ of the unique cycle of $M$. By Lemma 7 there is some constant $L$ such that with probability at least $1 - \frac{\alpha}{2}$: $M^+$ is unicyclic and has at most $L$ vertices.

Define $U$ to be the set of vertices of $G$ that are of distance exactly $r = |V(H)| + |V(M)| + 3$ from the unique cycle of $M$. Consider any vertex $u \in H$. By Lemma 10 below, if $M^+$ is unicyclic then there is a homomorphism from $M^+$ to $H$ such that all vertices in $U$ are mapped to $u$.

Obtain the graph $G'$ from $G$ by (i) removing all of the vertices of distance less than $r$ from the unique cycle of $M$, and (ii) contracting $U$ into a single new vertex $u$. Suppose that $h'$ is a homomorphism from $G'$ to $H$. Then by the previous paragraph, $h'$ can be extended to a homomorphism from $G$ to $H$ where each vertex $v \in V(G') - u$ is mapped to $h'(v)$, and every vertex in $U$ is mapped to $h'(u)$. Thus, if $G'$ is homomorphic to $H$ then so is $G$.

Let $T$ be the tree which consists of a vertex adjacent to $L$ leaves. Since the degree of $u$ in $G'$ is at most $L$, and using the fact that all vertices of $M$ are deleted when forming $G'$ (here is where we require $r > |M|$), the rest now follows as in the $k \geq 3$ case. \[ \square \]

**Lemma 10** Let $H$ be a connected graph which contains a triangle. Let $u$ be a vertex of $H$ and $M$ be a unicyclic graph with unique cycle $C$. Denote the vertices of $M$ in a distance of exactly
$r \geq |V(H)| + 3$ from $C$ by $U$. There is a homomorphism from $M$ to $H$ such that all vertices in $U$ are mapped to $u$.

**Proof.** Let $h$ be a homomorphism from $C$ to the triangle $(v_1, v_2, v_3)$ of $H$. Observe that for $i = 1, 2, 3$, there exist walks $(v_i =) v_{i,0}, \ldots, v_{i,r} (= u)$ of length exactly $r$ in $H$. Let $w$ be a vertex in $M$ in the distance of $j \leq r$ from $C$, and $w'$ be the vertex of $C$ which has the distance $j$ from $w$. Extend $h$ by assigning $h(w) = v_{ij}$ where $h(w') = v_i$. Observe that $h$ is a partial homomorphism from $M$ to $H$ which maps every vertex in $U$ to $u$. Trivially $h$ can be extended to a homomorphism from $M$ to $H$.

We close this section by discussing the possibilities of extending Theorem 4 to the case where $H$ is disconnected. We will focus on graphs, i.e. the $k = 2$ case. We can show that if Hypothesis A does not hold for the $H$-homomorphism problem for every graph $H$, then there must be a counterexample with two components: a triangle and a graph $H_1$ that is triangle-free and not 3-colourable. First note that every cycle is homomorphic to a triangle, and a triangle is not homomorphic to any triangle-free graph. So the condition of Hypothesis A is equivalent to $H$ containing a triangle. On the other hand $H$ contains a triangle-free component because being homomorphic to $H$ is equivalent to being homomorphic to at least one of the components $H_i$ of $H$, and so there is some component $H_i$, such that the $H_i$-homomorphism problem has a coarse threshold. Let $H^1$ be the subgraph of $H$ which consists of all triangles-free components and $H^2$ be the remaining components of $H$. It is easy to see that $H$ remains a counter-example if we add some edges to $H^1$ without creating any triangle and we substitute $H^2$ with a single triangle.

So the question of whether there is any graph $H$ for which the $H$-homomorphism problem violates hypothesis A is equivalent to the following:

**Question 11** Is there any triangle-free graph $H$ with $\chi(H) > 3$ such that for some values of $n$ and some $c > c(n)$, $G_{n,p=c/n}$ is not a.s. non-$H$-homomorphic, where $c(n)$ is the threshold value of 3-colorability?

### 3.1 Directed Graphs

Here, we provide an example of a directed graph $H$ for which

1. every unicyclic digraph has a homomorphism to $H$, and
2. the $H$-homomorphism problem under the $\text{CSP}_{n,p}(P)$ model has a coarse threshold.

Unfortunately, this does not exhibit a coarse threshold under the $\text{CSP}_{n,p}(P)$ model, so the question of whether Hypothesis A holds for all $H$-homomorphism problems for directed hypergraphs $H$ is still open.

For a directed graph $D$, let $\tilde{D}$ denote the undirected graph that is obtained from $D$ by ignoring the directions on the edges. We define $D_{n,p}$ to be the random digraph on $n$ vertices where each of the $n(n-1)$ potential directed edges is present with probability $p$. Thus $D_{n,p}$
possible constraining both $uv$ and $vu$ for some pair of vertices $v, u$, i.e. a 2-cycle; in fact, if $p = c/n$ for a constant $c$, then it is straightforward to show that the probability that $D$ contains at least one 2-cycle is $\zeta + o(1)$ for some constant $\zeta = \zeta(c) < 1$.

$H$ consists of a specific digraph $H_1$, defined below, and a pair of vertices $u_1, u_2$, where the edges $u_1, u_2$ and $u_2, u_1$ are both present. $H_1$ has the following properties:

(i): every unicyclic digraph which does not contain a 2-cycle a.s. has a homomorphism to $H_1$ and

(ii): $D_{n,p=c/n}$ is not a.s. non-$H_1$-homomorphic for some $c_1 > 1/2$.

It is easy to see that any unicyclic digraph, whose cycle is a 2-cycle, has a homomorphism to the 2-cycle. By (i), every other unicyclic digraph has a homomorphism to $H_1$. Thus, every unicyclic digraph has an $H$-homomorphism, as claimed. We will show that, for every $1/2 < c < c_1$, $D_{n,p=c/n}$ is neither a.s. $H$-homomorphic nor a.s. non-$H$-homomorphic. Thus, we have a coarse threshold. Condition (ii) above shows the latter, so we just need to prove the former.

If $c > 1/2$, then the graph $\tilde{D}$ for $D = D_{n,p}$, a.s. has a giant component, as proven by Karp [?]. It is not hard to see that a.s. if $D$ has a 2-cycle in the giant component of $\tilde{D}$, then there is no $H$-homomorphism: That 2-cycle must be mapped onto $u_1, u_2$. Since $H$ has no edge incident to $\{u_1, u_2\}$, any vertex that can be reached in $\tilde{D}$ from that 2-cycle must also be mapped onto $u_1$ or $u_2$. So the entire giant component must be mapped onto $\{u_1, u_2\}$. As that component has an odd cycle in $\tilde{D}$, and that odd cycle cannot be mapped onto a 2-cycle. It is easy to show that the probability that $\tilde{D}$ has a 2-cycle in its giant component is at least some positive constant. Therefore, $D$ is not a.s. $H$-homomorphic.

It remains only to prove the existence of some $H_1$ satisfying (i), (ii). We choose $H_1$ to be a tournament (i.e. for every pair of vertices, exactly one of the possible edges between them is present) which contains every directed graph on $k_0$ vertices as a subgraph where $k_0$ is a constant defined below.

For an undirected graph $G$ which does not contain any multiple edges, the oriented chromatic number $\chi_o$ of $G$ is the minimum number $k$ such that every directed graph $D$ satisfying $\tilde{D} = G$ is homomorphic to a directed graph $H$ with at most $k$ vertices. The acyclic chromatic number of a graph $G$ is the least integer $k$ for which there is a proper coloring of the vertices of $G$ with $k$ colors in such a way that every cycle of $G$ contains at least 3 different colors. It was proved in [31] that if the acyclic chromatic number of a graph $G$ is bounded by $k$, then its oriented chromatic number is bounded by $k.2^{k-1}$. When $D = D_{n,p}$ every edge is present in $\tilde{D}$ with probability $2p - p^2$ and independent of the other edges. Thus Lemma 12 below together with the result of [31] imply that taking $k_0 = 6 \times 2^5$ and $c_1 = c/2$, $H_1$ satisfies (ii), where $c$ is the constant which is obtained from Lemma 12.

**Lemma 12** There exists $c > 1$ such that a.s. the acyclic chromatic of $G_{n,p=c/n}$ is at most 5.

**Proof.** Let $G = G_{n,p}$. A pendant path in $G$ is a path in which no vertices other than the endpoints lie in any edge of the graph off the path. It is known that there exists $c > 1$ such
that a.s. after removing the internal vertices of pendent paths of length at least 4 from \(G\) every component is either a tree or it is unicyclic. One can use 3 colors to color the vertices in these components and then use 2 other colors to color the removed vertices such that every cycle in \(G\) is colored by at least 3 colors.

\[\text{Lemma 7 implies that there exists some constant } L \text{ such that, defining } E_1 \text{ to be the event that every } v_i \text{ has at most } L \text{ neighbours}, \Pr(E_1) \geq 1 - \frac{1}{2}. \text{ Suppose that } E_1 \text{ holds. Consider a particular } v_i \text{ and expose the } \ell \leq L \text{ edges containing it, } e_1, ..., e_\ell \text{ and the corresponding constraints } C_1, ..., C_\ell. \text{ For each } C_j, \text{ let } C'_j \text{ be the } (k - 1)\text{-variable constraint obtained by restricting } v_i \text{ to be } a_i; \text{ i.e. a } (k - 1)\text{-tuple of values for } C'_j \text{ iff } C_j \text{ permits that same } (k - 1)\text{-tuple along with } v_i = a_i. \text{ Since } C_j \text{ has at most } t \text{ restrictions, } C'_j \text{ has at most } t \text{ restrictions.}

4 The \((d, k, t)\)-model

\[\text{Proof of Theorem 2} \text{ Suppose that the } (d, k, t)\text{-model exhibits a coarse threshold. Then consider } p, \alpha, \epsilon \text{ and } M \text{ as guaranteed by Corollary 6. It is easy to verify that, since } t < d^{k-1} \text{ and } M \text{ is unicyclic, } M \text{ is satisfiable. Suppose that } V(M) = u_1, ..., u_r \text{ and let } a_i \text{ be the value of } u_i \text{ in some particular satisfying assignment } A \text{ of } M. \text{ Given a CSP } F \text{ on at least } r \text{ variables, we define } F \oplus A \text{ to be the CSP formed by choosing a random ordered } r\text{-tuple of variables } v_1, ..., v_r \text{ in } F \text{ and for each } 1 \leq i \leq r, \text{ forcing } v_i \text{ to take the value } a_i \text{ by adding a one-variable constraint on } v_i. \text{ Clearly the probability that } C\bar{S}P_{n,p}(\mathcal{P}) \oplus A \text{ is unsatisfiable is at least as high as the probability that } C\bar{S}P_{n,p}(\mathcal{P}) \oplus M \text{ is unsatisfiable.}

\text{Lemma 7 implies that there exists some constant } L \text{ such that, defining } E_1 \text{ to be the event that every } v_i \text{ has at most } L \text{ neighbours, } \Pr(E_1) \geq 1 - \frac{1}{2}. \text{ Suppose that } E_1 \text{ holds. Consider a particular } v_i \text{ and expose the } \ell \leq L \text{ edges containing it, } e_1, ..., e_{\ell} \text{ and the corresponding constraints } C_1, ..., C_\ell. \text{ For each } C_j, \text{ let } C'_j \text{ be the } (k - 1)\text{-variable constraint obtained by restricting } v_i \text{ to be } a_i; \text{ i.e. a } (k - 1)\text{-tuple of values for } C'_j \text{ iff } C_j \text{ permits that same } (k - 1)\text{-tuple along with } v_i = a_i. \text{ Since } C_j \text{ has at most } t \text{ restrictions, } C'_j \text{ has at most } t \text{ restrictions.}

\text{Let } G \text{ be a random CSP formed as follows: start with a random } C\bar{S}P_{n,p}(\mathcal{P}) \text{ and then for each of the } \binom{d^{k-1}}{t} \text{ possible constraints on } k - 1 \text{ variables and with } t \text{ restrictions, choose } rL \text{ random ordered } (k - 1)\text{-tuples of variables and place that constraint on them. The probability that } G \text{ is unsatisfiable is at least as high as the probability that } C\bar{S}P_{n,p}(\mathcal{P}) \oplus A \text{ is unsatisfiable as each batch of } L \text{ copies of every } (d, k - 1, t)\text{-constraint is at least as restrictive as forcing } v_i = a_i. \text{ Thus, adding those } rL \text{ constraints boosts the probability of unsatisfiability by at least } 2\alpha. \text{ We say that a canonical set of } \binom{d^{k-1}}{t}rL \text{ ordered } (k - 1)\text{-tuples is bad if adding the constraints to that set results in an unsatisfiable CSP. So, consider the following random experiment: pick a random } C\bar{S}P_{n,p}(\mathcal{P}) \text{ and then pick } \binom{d^{k-1}}{t}rL \text{ ordered } (k - 1)\text{-tuples of the variables. The probability that we pick a bad set is at least } 2\alpha. \text{ Since } \binom{d^{k-1}}{t}rL = O(1), \text{ a simple first moment calculation shows that a.s. the choice of } (k - 1)\text{-tuples will be vertex disjoint. Thus, the probability of picking a bad set is at least } 2\alpha - o(1) \text{ even if we condition on the } (k - 1)\text{-tuples being vertex-disjoint.}

\text{Define } T \text{ by: (i) taking the hypergraph consisting of a vertex } v \text{ lying in } rL\binom{d^k}{t} \text{ edges where no other vertex lies in more than one of the edges, and (ii) placing each of the } \binom{d^k}{t} \text{ possible } (d, k, t)\text{-constraints on } rL \text{ of the edges. For each } 1 \leq \delta \leq d, \text{ let } T_\delta \text{ denote the collection of } (k - 1)\text{-tuples obtained by removing } v \text{ from every edge containing a constraint in which every restriction has } v = \delta; \text{ note that } |T_\delta| = \binom{d^{k-1}}{t}rL. \text{ By Lemma 8, there is some } \zeta = o(n^{1-k}) \text{ such that } C\bar{S}P_{n,p=\zeta}(\mathcal{P}) \text{ a.s. contains a copy of } T. \text{ Consider adding that copy of } T \text{ to } C\bar{S}P_{n,p}(\mathcal{P}). \text{ The probability that for each } 1 \leq \delta \leq d, T_\delta \text{ is a bad set is at least } (2\alpha - o(1))^{d}.\]
Note that if every $T_δ$ is a bad set, then the resulting CSP is unsatisfiable because setting $v = δ$ requires the set of $(k - 1)$-constraints on $T_δ$ to be enforced. Thus, the probability that $CSP_{n,p=p+xz}(P)$ is unsatisfiable is at least $(2α - o(1))^d$. By considering adding $x$ copies of $T$, we see that the probability that $CSP_{n,p=p+xz}(P)$ is satisfiable is at most $(1 - (2α - o(1))^d)^x$ which is less than $α$ for some sufficiently large constant $x$. Since $z = o(n^{1-k})$, this implies that $Pr(CSP_{n,(1+ε)p}(P)$ is unsatisfiable) $> 1 - α$ which contradicts Corollary 6(b). □

5 Binary CSP’s with domain size 3

Recall that Istrate[23] (see also Creignou and Daude[11]) has proven that when the domain size $d = 2$, then Hypothesis A holds; i.e. if every unicyclic CSP is satisfiable, then $CSP_{n,p}(P)$ has a sharp threshold. This result does not extend to $d = 3$. Consider the following example, with $d = 3, k = 2$:

**Example 13** We have two constraints. $C_1$ says that either both variables are equal to 1, or neither is equal to 1. $C_2$ says that the variables cannot both have the same value. $P(C_1) = \frac{2}{3}, P(C_2) = \frac{1}{3}$.

Observe that every unicyclic CSP that uses only constraints $C_1, C_2$ is satisfiable.

Consider any $\frac{3}{2} < c < 3$. Thus, a.s. the sub-CSP formed by the $C_1$ constraints has a giant component, and the sub-CSP formed by the $C_2$ constraints does not. We will show that $CSP_{n,p}(P)$ is neither a.s. satisfiable nor a.s. unsatisfiable.

To see that it is not a.s. unsatisfiable, note that the subgraph induced by the $C_2$ constraints is 2-colourable with probability at least some positive constant. This follows from the well known fact that the for $c < 1$ the random graph $G(n, \frac{c}{n})$ is 2-colourable with probability at least some positive constant. If it is 2-colourable, then we can satisfy all the $C_2$ constraints by assigning every variable either 2 or 3; this will not violate any $C_1$ constraints.

To see that it is not a.s. satisfiable, note that subgraph formed by the $C_1$ constraints has a giant component $T$. So either every variable in $T$ is assigned 1 or none of them are. A.s. at least one $C_2$ constraint has both variables in $T$, and so they cannot both be assigned 1. Thus, a.s. no variables in $T$ can be assigned 1. This implies that if the $C_2$ constraints form an odd cycle using variables of $T$ then the CSP is not satisfiable; that event occurs with probability at least some positive constant.

The main result of this section, is that when $d = 3$ and $k = 2$, if Hypothesis A fails on some model, then it has to fail for the same reason as it failed for Example 13.

Consider a CSP $F$ where every constraint is on 2 variables. Suppose there is some constraint on variables $v, u$ which implies that if $v$ is assigned $δ$ then $u$ must be assigned $γ$; we say that $v : δ$ forces $u : γ$ and denote this by $v : i → u : j$. Moreover if there is a sequence of variables $v_1, \ldots, v_r$ and values $δ_1, \ldots, δ_r$ such that $v_i : δ_i → v_{i+1} : δ_{i+1}$ for $i = 1, \ldots, r - 1$ then we say that $v_1 : δ_1$ forces $v_r : δ_r$.
Theorem 14 Consider some \( P \) with \( d = 2, k = 3 \) such that every unicyclic CSP formed from \( \text{supp}(P) \) is satisfiable. \( \text{CSP}_{n,p}(P) \) has a coarse threshold iff there exists a unicyclic CSP \( M \) formed from \( \text{supp}(P) \), a value \( 1 \leq \delta \leq 3, p = p(n), \varepsilon > 0, z > 0, b > 0 \) such that:

(a) \( \varepsilon < \Pr(\text{CSP}_{n,p}(P) \text{ is satisfiable }) < 1 - \varepsilon; \)

(b) \( M \) cannot be satisfied using only the two values other than \( \delta; \)

(c) \( \text{CSP}_{n,p}(P) \) a.s. has at least \( zn \) variables \( v \) such that \( v : \delta \rightarrow u : \delta \) for at least \( bn \) variables \( u \).

Thus, this explains the only ways that a model with \( d = 3, k = 2 \) can have a coarse threshold. We remark that this theorem does not extend to \( d = 4, k = 2 \) nor \( d = 3, k = 3 \); in both cases there are other causes for a coarse threshold.

Proof We leave it to the reader to verify, using similar reasoning to that for Example 13 that conditions (a,b,c) imply a coarse threshold. The only difference here is that the set of variables \( u \) in (c) could change for different choices of \( v \), and it is important to note that for each value \( \delta \), there is some constraint in \( \text{supp}(P) \) that forbids both variables from receiving \( \delta \) as otherwise \( \text{CSP}_{n,p}(P) \) is trivially satisfiable by setting every variable equal to \( \delta \). We will give a proof of the other direction. The case where some domain values are bad (as defined in [27]) is easily disposed of, so we assume that there are no such values.

For each variable \( v \) and each \( 1 \leq \delta \leq 3, 1 \leq \gamma \leq 3 \) we define \( F_{\delta,\gamma}(v) \) to be the set of variables \( u \) such that \( v : \delta \rightarrow u : \gamma \), and we define \( F_\delta(v) = \cup_{1 \leq \gamma \leq 3} F_{\delta,\gamma}(v) \). We can expose \( F_\delta(v) \) by using a simple breadth-first search from \( v \). This allows us to analyze the distribution of the size of \( F_\delta(v) \) and \( F_{\delta,\gamma}(v) \) using a standard branching-process analysis (see eg. Chapter 5 of [24]). We say that \( F_{\delta,\gamma} \) percolates if there are constants \( \zeta, \beta > 0 \) such that \( \Pr(|F_{\delta,\gamma}(v)| \geq \beta n) \geq \zeta \). It is straightforward to prove:

Claim 1: If \( F_{\delta,\gamma} \) does not percolate, then for every \( \xi > 0 \) there is a constant \( L \) such that \( \Pr(|F_{\delta,\gamma}(v)| \leq L) > 1 - \xi. \)

Claim 2: If \( F_{\delta,\gamma} \) percolates, then there are constants \( z, b > 0 \) such that a.s. there are at least \( zn \) variables \( v \) with \( |F_{\delta,\gamma}(v)| \geq \beta n. \)

Claim 2 yields that \( F_{\delta,\gamma} \) percolates for at most one value \( \delta \): Suppose that \( F_{\delta,\delta} \) and \( F_{\gamma,\gamma} \) both percolate. We want to show that in this case Corollary 6(b) fails. To this end, we obtain an instance of \( \text{CSP}_{n,p(1+\varepsilon)}(P) \) as follows. First, we consider \( P_0 \), an instance of \( \text{CSP}_{n,p}(P) \). Then \( P_i \) is obtained by taking the union of \( P_{i-1} \) and an instance of \( \text{CSP}_{n,\frac{p}{2}}(P) \), for \( i = 1, 2, 3. \)

Let \( u, v, \) and \( w \) be variables such that in \( P_0, u : \delta \) forces both \( v : \delta \) and \( w : \delta \). If the restriction \( (\delta, \delta) \) is added on \( v \) and \( w \), then \( u \) cannot be assigned \( \delta \). Since there are constraints in \( \text{supp}(P) \) with the restriction \( (\delta, \delta) \) and constraints with the restriction \( (\gamma, \gamma) \) (see [27]), we can conclude that in \( P_1 \), a.s. there are two sets of variables of size \( \Theta(n) \), \( A \) and \( B \) such that \( \delta \) cannot be assigned to any variable in \( A \), and \( \gamma \) cannot be assigned to any variable in \( B \).

Since \( F_{\delta,\delta} \) percolates, there is a value \( \sigma \) and a constraint in \( \text{supp}(P) \) such that if that constraint is applied on \( v_1, v_2 \), then \( v_1 : \sigma \rightarrow v_2 : \delta. \) Let \( u \) be a variable in \( A \), and \( v \) be an arbitrary variable. If there is a constraint on \( u \) and \( v \) which justifies that \( v : \sigma \rightarrow u : \delta \), then \( \sigma \) cannot be
assigned to $v$. Now we can conclude that in $P_2$, almost surely, there is a set $C$ of size $\Theta(n)$ of variables such that $\sigma$ cannot be assigned to any variable in $C$ for the reason mentioned above. Every variable in $A$ or $B$ is in $C$ with a positive probability and independent of the other variables. So $C_1 = C \cap A$ and $C_2 = C \cap B$ are both of size $\Theta(n)$, almost surely. Without loss of generality suppose that $\delta \neq \sigma$ (otherwise we would assume $\gamma \neq \sigma$). Both of the values $\delta$ and $\sigma$ cannot be assigned to any variables in $C_1$. So there is only one value which can be assigned to these variables. But then in $P_3$ a.s. there is a constraint on two variables in $C_1$ which forbids them to be assigned this value simultaneously. So $\overline{CSP}_{n,p(1+\epsilon)}(P)$ is not satisfiable a.s. which contradicts Corollary 6(b).

It is less straightforward, but not very difficult, to prove:

Claim 3: If $F_{\delta,\gamma}$ percolates for any pair $\delta, \gamma$ then either (i) $F_{\delta,\delta}$ percolates or (ii) there is some $\mu$ such that $F_{\mu,\mu}$ percolates and there is a sequence of constraints in $\text{supp}(P)$ through which $v: \delta \rightarrow u: \mu$.

Suppose that $\overline{CSP}_{n,p}(P)$ has a coarse threshold and consider $M, \epsilon, \alpha, p = p(n)$ from Corollary 6.

Claim 4: There is a value $\delta$ such that (i) that every satisfying assignment of $M$ must use $\delta$ on at least one variable and (ii) $F_{\delta,\gamma}$ percolates for at least one value $\gamma$.

If $F_{\delta,\delta}$ percolates then this satisfies Theorem 14. Otherwise, applying Claim 3, $F_{\mu,\mu}$ percolates and there is a sequence of constraints so that $v: \delta \rightarrow u: \mu$. Attaching that sequence to every variable of $M$ yields a unicyclic CSP $M'$ for which every satisfying assignment must use $\mu$ on at least one variable. Thus $M', \mu$ satisfy Theorem 14.

Suppose that Claim 4 does not hold, and consider any satisfying assignment $A$ of $M$ in which every value $\delta$ used is such that $F_{\delta,\delta}$ does not percolate. Suppose that $M$ has $r$ variables $x_1, ..., x_i$ and that $A$ assigns $a_i$ to $x_i$. Recall from Section 4 that $CSP_{n,p}(P) \oplus A$ is formed by taking $CSP_{n,p}(P)$ and then choosing $r$ random variables $v_1, ..., v_i$ and adding one-variable constraints that force $v_i$ to take $a_i$. Clearly $\Pr(CSP_{n,p}(P) \oplus A$ is unsatisfiable) $\geq \Pr(CSP_{n,p}(P) \oplus M$ is unsatisfiable).

Expose $F = \bigcup_{i=1}^r F_{a_i}(v_i)$, and $U$, the set of variables outside of $F$ that lie in a constraint with a variable in $F$. Since none of the $F_{a_i}$ percolate, Claim 2 allows us to show that there is some $L$ such that with probability at least $1 - \alpha/2$, $|U| < L$. Since adding $M$ to $\overline{CSP}_{n,p}(P)$ increases the probability of unsatisfiability by at least $2\alpha$, it must be that the probability that $\overline{CSP}_{n,p}(P)$ is satisfiable, $|U| \leq L$ and $\overline{CSP}_{n,p}(P) \oplus A$ is unsatisfiable is at least $3\alpha/2$.

Suppose that $\overline{CSP}_{n,p}(P)$ is satisfiable and $|U| \leq L$. Consider some $u \in U$ sharing a constraint with $w \in F$ where $A$ forces $w$ to take the value $\mu$. Let $\Omega = \Omega(u)$ be the set of values which can be assigned to $u$ which, in conjunction with assigning $\mu$ to $w$ do not violate their constraint. We know that $|\Omega| \neq 0$ since otherwise $\mu$ is a bad value. We know that $|\Omega| \neq 1$ since otherwise $u \in F$. So $|\Omega(u)| \geq 2$ for each $u \in U$. Suppose that $u_1, ..., u_\ell$ are the variables in $U$ with $|\Omega| = 2$, and let $\delta_i$ be the value not in $\Omega(u_i)$. Consider taking a random CSP formed as follows: first take a $\overline{CSP}_{n,p}(P)$ and then choose $\ell$ random variables $u_1, ..., u_\ell$ and force $u_i$ to not take value $\delta_i$ using a one-variable constraint. We have proved is that the one-variable constraints boost the probability of unsatisfiability by $3\alpha/2$. 

15
At this point, we can complete the proof using the argument from the Achlioptas-Friedgut proof[3] that $d$-colourability has a sharp threshold for $d \geq 3$. In that paper, the starting point was to note that since every constraint is a colourability constraint, fixing an assignment on $M$ at worst forbids one colour from each neighbour of $M$. This put them in the same position that we are in now, and so the rest of our proof is the same as theirs.

We close this section by noting why this proof can not be extended to general $d$. The problem is that possibly some of the variables in $U$ would have their domain sizes reduced by two instead of one. The argument in [3] cannot handle that possibility.

6 Future Directions

There is clearly much work still to be done along these lines of research. The big problem still remains - determine precisely which models from [27] have a sharp threshold. Of course, Section 2 indicates that this may be overly ambitious. But lowering our sights only slightly, we can try to determine all possible causes for coarse thresholds, i.e. continue the course started in Section 5. An important subgoal would be to do this for binary CSP’s, i.e. the case where $k = 2$. Another reasonable goal to pursue would be to cover the $d = 3$ case.

As far as more specific classes of models go, one should try to extend the work in Section 3 and examine whether Hypothesis A holds for $H$-homomorphism problems when $H$ is a directed hypergraph. Such homomorphism problems are equivalent to CSP’s in which every constraint is identical under some permutation of the variables. And of course, it would be good to determine whether the “connected” condition can be removed from Theorem 4.

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