Conformal relativity versus Brans-Dicke and superstring theories.

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(Dated: July 1, 2021)

Conformal relativity theory which is also known as Hoyle-Narlikar theory has recently been given some new interest. It is an extended relativity theory which is invariant with respect to conformal transformations of the metric.

In this paper we show how conformal relativity is related to the Brans-Dicke theory and to the low-energy-effective superstring theory. We show that conformal relativity action is equivalent to a transformed Brans-Dicke action for Brans-Dicke parameter $\omega = -\frac{3}{2}$ in contrast to a reduced (graviton-dilaton) low-energy-effective superstring action which corresponds to a Brans-Dicke action with Brans-Dicke parameter $\omega = -1$. In fact, Brans-Dicke parameter $\omega = -\frac{3}{2}$ gives a border between a standard scalar field and a ghost.

We also present basic cosmological solutions of conformal relativity in both Einstein and string frames. The Einstein limit for flat conformal cosmology solutions is unique and it is flat Minkowski space. This requires the scalar field/mass evolution instead of the scale factor evolution in order to explain cosmological redshift.

It is interesting that like in ekpyrotic/cyclic models, a possible transition through a singularity in conformal cosmology in the string frame takes place in the weak coupling regime.

PACS numbers: 98.80.Hw, 04.20.Jb, 04.50.+h, 11.25.Mj

I. INTRODUCTION

It is well-known that some of the fundamental equations of physics such as Maxwell equations or massless Dirac equation are invariant with respect to conformal transformations of the metric [1]. On the other hand, the Einstein equations and the massless Klein-Gordon equation are not invariant with respect to these transformations. However, a modification of these equations which involves both metric and scalar field (scalar-tensor gravity) may possess the property of conformal invariance. Such theories have recently been given some new interest and called conformal relativity theories. They are not exactly new since they were studied already and their fundamental version is well-known under the name of the Hoyle-Narlikar theory [2, 3, 4, 5, 6]. However, some new aspects have been added to them recently and, in particular, geometrical evolution of the universe was reinterpreted as an evolution of the mass represented by a scalar field in a flat universe [7, 8, 9]. The idea is quite interesting and it can help to resolve the problem of the dark energy in the universe [10, 11, 12, 13, 14]. Similar ideas have been developed in yet another modification of general relativity called Self Creation Cosmology [15] which also solves the dark energy problem together with a series of other cosmological problems including Pioneer spacecraft puzzle [16].

In this paper we look for the connection between conformal relativity and other alternative gravity theories such as Brans-Dicke theory [17] and their overlap with unified gauge-gravity theories superstring and M-theory [18, 19, 20]. In fact, Brans-Dicke theory (with Dirac’s Large Numbers Hypothesis as a precursor [21]), belongs to a larger class of the scalar-tensor theories of gravity [22] whose properties has been recently studied in the context of supernovae data and density perturbations [23, 24].

In Section II we discuss the basic properties of conformal invariance, present the field equations of conformal relativity and show how they behave under conformal transformations of the metric. In Section III we study the mutual relations between conformal relativity, Brans-Dicke theory and low-energy-effective superstring theories for a reduced graviton-dilaton spectrum. In Section IV we present the basic cosmological solutions of conformal relativity for Friedmann universes in both the Einstein and the string frames. In Section V we...
II. CONFORMAL RELATIVITY

Suppose that we have two spacetime manifolds $M, \tilde{M}$ with metrics $g_{\mu\nu}, \tilde{g}_{\mu\nu}$ and the same coordinates $x^\mu$. We say that the two manifolds are conformal to each other if they are related by the following conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad (II.1)$$

and the function $\Omega$ which is called a conformal factor must be a twice-differentiable function of coordinates $x^\mu$ and lie in the range $0 < \Omega < \infty$. The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system $x^\mu$ on the manifolds $M, \tilde{M}$, respectively, but they preserve the angles between vectors (in particular null vectors which define light cones) which leads to a conservation of the (global) causal structure of the manifold [26]. If we take $\Omega = \text{const.}$ we deal with the so-called scale transformations [22]. In fact, conformal transformations are localized scale transformations $\Omega = \Omega(x)$. On the other hand, the coordinate transformations $x^\mu \to \tilde{x}^\mu$ only relabel the coordinates and do not change geometry and they are entirely different from conformal transformations [5]. This is crucial since conformal transformations lead to a different physics on conformally related manifolds $M, \tilde{M}$ [22]. Since this will usually be related to a different coupling of a physical field to gravity we will be talking about different frames in which the physics is studied (see also Ref. [27] for a slightly different view).

In $D = 4$ spacetime dimensions the determinant of the metric $g = \det g_{\mu\nu}$ transform as

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g}. \quad (II.2)$$

It is obvious from (II.1) that the following relations for the inverse metrics and the spacetime intervals hold

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}, \quad (II.3)$$

$$d\tilde{s}^2 = \Omega^2 ds^2. \quad (II.4)$$

Finally, the notion of conformal flatness means that

$$\tilde{g}_{\mu\nu} \Omega^{-2}(x) = g_{\mu\nu} = \eta_{\mu\nu}, \quad (II.5)$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric.

The application of (II.1) to the Christoffel connection coefficients gives [27]

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{\Omega} \left( g^\lambda_{\rho\nu} \Omega_{\rho\mu} + g^\lambda_{\rho\mu} \Omega_{\rho\nu} - g_{\mu\nu} g^{\lambda\rho} \Omega_{\rho} \right), \quad (II.6)$$

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} - \frac{1}{\Omega} \left( \tilde{g}^\lambda_{\rho\nu} \Omega_{\rho\mu} + \tilde{g}^\lambda_{\rho\mu} \Omega_{\rho\nu} - \tilde{g}_{\mu\nu} \tilde{g}^{\lambda\rho} \Omega_{\rho} \right). \quad (II.7)$$

The Ricci tensors and Ricci scalars in the two related frames $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ transform as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \Omega^{-2} \left[ 4 \Omega_d \Omega_{\mu\sigma} - \Omega_{\sigma\rho} \Omega^{\sigma}_{\rho\nu} g_{\mu\nu} \right] - \Omega^{-1} \left[ 2 \Omega_d \mu_{\sigma\nu} + \Box \Omega g_{\mu\nu} \right], \quad (II.8)$$

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - 3 \Omega^{-2} \Omega_{\rho\sigma} \tilde{g}^{\rho\sigma} \tilde{g}_{\mu\nu} + \Omega^{-1} \left[ 2 \Omega_d \mu_{\sigma\nu} + \tilde{g}_{\mu\nu} \Box \tilde{\Omega} \right], \quad (II.9)$$

$$\tilde{R} = \Omega^{-2} \left[ R - 6 \frac{\Box \Omega}{\Omega} \right], \quad (II.10)$$

$$R = \Omega^2 \left[ \tilde{R} + 6 \frac{\Box \Omega}{\Omega} - 12 \tilde{g}_{\mu\nu} \Omega_d \Omega_{\mu\nu} \right]. \quad (II.11)$$
and the appropriate d’Alambertian operators change under (II.1) as

\[ \Box \phi = \Omega^{-2} \left( \Box + 2 g^{\mu\nu} \frac{\partial}{\partial \Omega} \phi_{,\mu} \right), \quad (\text{II.12}) \]

\[ \Box \phi = \Omega^{-2} \left( \Box - 2 g^{\mu\nu} \frac{\partial}{\partial \Omega} \phi_{,\mu} \right). \quad (\text{II.13}) \]

In these formulas the d’Alembertian \( \Box \) taken with respect to the metric \( \tilde{g}_{\mu\nu} \) is different from \( \Box \) which is taken with respect to a conformally rescaled metric \( g_{\mu\nu} \).

An important feature of the conformal transformations is that they preserve Weyl conformal curvature tensor

\[ C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{2}{D-2} \left( g_{\mu[\sigma} R_{\rho]\nu] + g_{\nu[\rho} R_{\sigma]\mu} \right) + \frac{2}{(D-1)(D-2)} R g_{\mu[\rho} g_{\sigma]\mu}, \quad (\text{II.14}) \]

which means that we have

\[ \tilde{C}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}, \quad (\text{II.15}) \]

under (II.1).

Let us remind that the vacuum Einstein-Hilbert action of general relativity reads as

\[ S_{EH} = \frac{1}{2 \kappa^2} \int d^4x \sqrt{-g} R, \quad (\text{II.16}) \]

where

\[ \kappa^2 = 8\pi G. \quad (\text{II.17}) \]

Application of the conformal transformation (II.1) together with the help of (II.10) yields (we have assumed that \( \kappa^2 = 6 \) for a while)

\[ S_{EH} = \frac{1}{2} \int d^4x \sqrt{-g} \Omega^2 \left( \frac{1}{6} R - \frac{\Box}{\Omega} \right), \quad (\text{II.18}) \]

which means that the vacuum part of the Einstein-Hilbert action is not invariant under conformal transformation (II.1), apart from the case of the global transformations of the trivial type \( \tilde{g}_{\mu\nu} = \text{const.} \times g_{\mu\nu} \).

However, a modification of the Einstein-Hilbert action (II.16) which allows for a scalar field \( \Phi \) which reads as

\[ \tilde{S} = \frac{1}{2} \int d^4x \sqrt{-g} \Phi \left( \frac{1}{6} R \Phi - \Box \Phi \right), \quad (\text{II.19}) \]

together with the appropriate redefinition of the scalar field

\[ \tilde{\Phi} = \Omega^{-1} \Phi, \quad (\text{II.20}) \]

is, in fact, conformally invariant since the conformally transformed action has the same form, i.e.,

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \Phi \left( \frac{1}{6} R \Phi - \Box \Phi \right). \quad (\text{II.21}) \]

Now, one can see that the original form of the Einstein-Hilbert action can be recovered from (II.19) (or, alternatively (II.21)) provided we assume that

\[ \kappa^2 = \frac{6}{\bar{\Phi}^2} = \frac{6}{\varphi^2} = \text{const.} \quad (\text{II.22}) \]

The action (II.19) (or similarly (II.21)) is usually represented in a different form by the application of the expression for a covariant d’Alambertian for a scalar field in general relativity

\[ \Box \Phi = \frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} \partial^{\mu} \Phi \right), \quad (\text{II.23}) \]
which after integrating out the boundary term, gives

$$\tilde{S} = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{6} \tilde{R} \tilde{\Phi}^2 + \tilde{\partial}_\mu \tilde{\Phi} \tilde{\partial}^\mu \tilde{\Phi} \right],$$

(II.24)

and the second term is just a kinetic term for the scalar field \( \tilde{\Phi} \) or \( \Phi \). The equations (II.24) are also conformally invariant since the application of the formulas (II.2), (II.10) and (II.20) together with the appropriate integration of the boundary term gives the same form of the equations

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \frac{1}{6} R \Phi^2 + \partial_\mu \Phi \partial^\mu \Phi \right].$$

(II.25)

Because of the type of non-minimal coupling of gravity to a scalar field \( \tilde{\Phi} \) or \( \Phi \) in (II.24) and (II.25) and the relation to Brans-Dicke theory (see Section (III) we say that these equations are presented in the Jordan frame [22, 28].

The conformally invariant actions (II.19) and (II.21) are the basis to derive the equations of motion via the variational principle. The resulting equations of motion are conformally invariant, too. The equations of motion for scalar fields \( \tilde{\Phi} \) and \( \Phi \) are conformally invariant

$$\left( \tilde{\Box} - \frac{1}{6} \tilde{R} \right) \tilde{\Phi} = \Omega^{-3} \left( \Box - \frac{1}{6} R \right) \Phi = 0,$$

(II.26)

and they have the structure of the Klein-Gordon equation with the mass term replaced by the curvature term [3]. The conformally invariant Einstein equations are obtained from variation of \( \tilde{S} \) with respect to the metric \( \tilde{g}_{\mu\nu} \) and read as

$$\left( \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) \frac{1}{6} \tilde{\Phi}^2 + \frac{1}{6} \left[ 4 \tilde{\Phi}_{,\mu} \tilde{\Phi}_{,\nu} - \tilde{g}_{\mu\nu} \tilde{\Phi}_{,\alpha} \tilde{\Phi}_{,\alpha} \right] + \frac{1}{3} \left[ \tilde{g}_{\mu\nu} \tilde{\Phi} \tilde{\Box} \tilde{\Phi} - \tilde{\Phi} \tilde{\Phi}_{,\mu} \tilde{\Phi}_{,\nu} \right] = 0.$$  

(II.27)

In order to prove the conformal invariance of the field equations (II.27) it is necessary to know the rule of the conformal transformations for the double covariant derivative of a scalar field, i.e.,

$$\Phi_{,\mu} = \Phi_{,\mu} - \Gamma_{\mu\rho}^{\nu} \Phi_{,\rho} = -\Omega^{-2} \partial_\mu \Omega + \Omega^{-1} \Phi_{,\mu} + 4 \Omega^{-3} \Phi_{,\mu} \Omega_{,\nu} - 2 \Omega^{-2} (\Phi_{,\mu} \Omega_{,\nu} + \Omega_{,\mu} \Phi_{,\nu}) - \Omega^{-3} \Phi g_{\mu\rho} \Omega_{,\rho} + \Omega^{-2} g_{\mu\nu} \Omega_{,\rho} \Omega^\rho,$$

(II.28)

and

$$\Phi_{,\mu \nu} = \Phi_{,\mu} \Omega_{,\nu} + \Omega \Phi_{,\mu \nu} + \frac{2}{\Omega} \Phi_{,\mu} \Omega_{,\nu} + 2 \left( \Omega_{,\mu} \Phi_{,\nu} + \Phi_{,\mu} \Omega_{,\nu} \right) - \frac{1}{\Omega} \Phi \tilde{g}_{\mu\nu} \Omega^\rho - \frac{1}{\Omega} \tilde{g}_{\mu\nu} \Phi \Omega^\rho = 0.$$  

(II.29)

(\(\Box\) means the covariant derivative with respect to \( \tilde{g}_{\mu\nu} \)).

Inserting (II.15), (II.16), (II.20) and (II.25) into (II.27) gives the same conformally invariant form of the field equations as

$$\left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \frac{1}{6} \Phi^2 + \frac{1}{6} \left[ 4 \Phi_{,\mu} \Phi_{,\nu} - g_{\mu\nu} \Phi_{,\alpha} \Phi_{,\alpha} \right] + \frac{1}{3} [g_{\mu\nu} \Phi \Box \Phi - \Phi \Phi_{,\mu \nu}] = 0.$$  

(II.30)

These are exactly the same field equations as in the Hoyle-Narlikar theory [3]. Note that the scalar field equations of motion (II.26) can be obtained by the appropriate contraction of equations (II.27) and (II.30) so that they are not independent and do not supply any additional information [31].

Another point is that the equations (II.27) or (II.30) apparently could give directly the vacuum Einstein field equations for \( \tilde{\Phi} = \varphi_0 = \sqrt{6}/\kappa = \sqrt{6}/8\pi G = \) const. (cf. Eq. (II.22)). The same is obviously true for the field equations (II.30) with the same value of \( \Phi = \varphi_0 = \sqrt{6}/\kappa = \sqrt{6}/8\pi G = \) const. However, this limit is restricted to the case of vanishing Ricci curvature \( R = 0 \) or \( \tilde{R} = 0 \) (so only flat Minkowski space limit is allowed) which can be seen from the scalar field equations of motion (II.26).

The admission of the matter part

$$S_{\text{Matter}} = \frac{1}{2} \int d^4x \sqrt{-g} L_{\text{Matter}},$$

(II.31)

into the action (II.21), with the matter energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} \left( \sqrt{-g} L_{\text{Matter}} \right),$$

(II.32)
allows to generalize the field equations (II.30) to
\[
\left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \frac{1}{6} \Phi^2 + \frac{1}{6} \left[ 4 \Phi_{,\mu} \Phi_{,\nu} - g_{\mu\nu} \Phi_{,\alpha} \Phi_{,\alpha} \right] + \frac{1}{3} \left[ g_{\mu\nu} \Phi \Box \Phi - \Phi \Phi_{,\mu\nu} \right] = T_{\mu\nu} .
\] (II.33)
These equations (II.33), after contraction, give modified field equations (II.26)
\[
\left( \Box - \frac{1}{6} R \right) \Phi = \frac{T}{\Phi} .
\] (II.34)
Note that putting \( \Phi = \phi_0 = \sqrt{\frac{6}{8\pi G}} \) into (II.33) gives Einstein field equations provided \( T = -(1/6)R\Phi^2 \).

In the conformal frame we add the matter term
\[
\tilde{S}_{\text{Matter}} = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \Omega^{-4} \tilde{\mathcal{L}}_{\text{Matter}} ,
\] (II.35)
with
\[
\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}} \frac{\partial}{\partial \tilde{g}_{\mu\nu}} \left( \sqrt{-\tilde{g}} \Omega^{-4} \tilde{\mathcal{L}}_{\text{Matter}} \right) .
\] (II.36)
Some manipulations with the help of (II.1), (II.2) and (II.3) show that
\[
\tilde{T}^{\mu\nu} = \Omega^{-6} T^{\mu\nu} ,
\] (II.37)
\[
\tilde{T} = \Omega^{-4} T .
\] (II.38)
The admission of (II.35) gives the conformally invariant field equations (II.21) as
\[
\left( \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) \frac{1}{6} \tilde{\Phi}^2 + \frac{1}{6} \left[ 4 \tilde{\Phi}_{,\mu} \tilde{\Phi}_{,\nu} - \tilde{g}_{\mu\nu} \tilde{\Phi}_{,\alpha} \tilde{\Phi}_{,\alpha} \right] + \frac{1}{3} \left[ \tilde{g}_{\mu\nu} \tilde{\Phi} \Box \tilde{\Phi} - \tilde{\Phi} \tilde{\Phi}_{,\mu\nu} \right] = \tilde{T}_{\mu\nu} ,
\] (II.39)
which after contraction give a modified equation (II.26)
\[
\left( \Box - \frac{1}{6} \tilde{R} \right) \tilde{\Phi} = \frac{\tilde{T}}{\tilde{\Phi}} .
\] (II.40)
Note again that putting \( \tilde{\Phi} = \tilde{\phi}_0 = \sqrt{\frac{6}{8\pi G}} \) into (II.39) gives the Einstein field equations, provided \( \tilde{T} = -(1/6)\tilde{R}\tilde{\Phi}^2 \). However, as we will see in a moment the trace of the energy-momentum tensor must be zero in order to conserve the energy-momentum in a conformal frame.

Let us take the perfect fluid as a source of gravity with the four-velocity \( u^\mu (u_\mu u^\mu = -1) \), the energy density \( \rho \) and the pressure \( p \), i.e.,
\[
T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} ,
\] (II.41)
and transform it into a conformally related frame
\[
\tilde{T}^{\mu\nu} = (\tilde{\rho} + \tilde{p})\tilde{u}^\mu \tilde{u}^\nu + \tilde{p}\tilde{g}^{\mu\nu} ,
\] (II.42)
where
\[
\tilde{u}^\mu = \frac{dx^\mu}{d\tilde{s}} = \frac{1}{\Omega} \frac{dx^\mu}{ds} = \Omega^{-1} u^\mu .
\] (II.43)
It is easy to note that the imposition of the conservation law in the first frame
\[
T^{\mu\nu}_{,\nu} = 0 ,
\] (II.44)
gives in the conformally related frame
\[
\tilde{T}^{\mu\nu}_{,\nu} = -\frac{\Omega^{\mu}}{\Omega} \tilde{T} .
\] (II.45)
From (II.45) it appears transparent that the conformally transformed energy-momentum tensor is conserved only if the trace of it vanishes (\( \tilde{T} = 0 \)) [23, 30]. For example, in the case of barotropic fluid with
\[
p = (\gamma - 1)\rho \quad \gamma = \text{const.},
\] (II.46)
it vanishes only for radiation \( p = (1/3)\rho \). This means that only the photons may obey the equivalence principle and this is not the case for other types of matter since with non-vanishing trace in (II.45) we deal with creation of matter process (compare with Self Creation Cosmology of Ref. [15] which has the same field equations (II.39) and (II.45), but the equation (II.40) is the same only for a vanishing curvature scalar \( \tilde{R} \)).
III. RELATION TO BRANS-DICKE AND LOW-ENERGY-EFFECTIVE SUPERSTRING THEORIES

Due to the admission of a non-minimally coupled scalar field, conformal relativity suggests its close relation to some other scalar-tensor theories of gravity such as Brans-Dicke theory and the reduced low-energy-effective superstring theories. It is widely known that Brans-Dicke theory (BD) [17] was based on the ideas of Jordan [28] and Mach [29]. According to them, the inertial masses of the elementary particles are not fundamental constants but the result of the particles’ interaction with the rest of the universe represented by some cosmic field [30]. In physical terms it can be formulated by the fact that gravitational “constant” $G$ is related to an average value of a scalar field $\Phi_{BD}$ which is not constant. Due to some estimations concerning a simple form of a covariant field equation, the size and mean mass density of the universe, the expectation value of the scalar field $\langle \Phi_{BD} \rangle \approx \frac{1}{G}$.

These assumptions led Brans and Dicke to replace $G$ by the inverse of the scalar field $\frac{1}{\Phi_{BD}}$ and to include an extra energy-momentum tensor for the scalar field. The resulting Brans-Dicke action reads as [5]

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \Phi_{BD} R - \frac{\omega}{\Phi_{BD}} \partial_{\mu} \Phi_{BD} \partial^{\mu} \Phi_{BD} \right] + S_m, \quad (III.1)$$

where $\omega$ is the Brans-Dicke parameter.

Varying the action (III.1) (with matter Lagrangian included) one gets the field equations of the Brans-Dicke theory in the form

$$\Box \Phi_{BD} = \frac{8\pi T}{3 + 2\omega}, \quad (III.2)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\Phi_{BD}} T_{\mu\nu} + \frac{\omega}{\Phi_{BD}^2} \left( \Phi_{BD,\mu} \Phi_{BD,\nu} - \frac{1}{2} g_{\mu\nu} \Phi_{BD,\rho} \Phi_{BD,\rho} \right) + \frac{1}{\Phi_{BD}} (\Phi_{BD,\mu\nu} - g_{\mu\nu} \Phi_{BD}) \cdot (III.3)$$

Independently, the conservation law for matter energy-momentum tensor

$$T^{\mu\nu} = 0 \quad (III.4)$$

may be imposed [30].

In fact, the equation (III.2) was obtained by subtracting the equation of motion of the Brans-Dicke field obtained from the action (III.1)

$$2\Phi_{BD} \Box \Phi_{BD} - \Phi_{BD,\rho} \Phi_{BD,\rho} + \frac{R}{\omega} \Phi_{BD}^2 = 0, \quad (III.5)$$

and the contracted equation (III.3). The Einstein limit of equations (III.2)-(III.3) is recovered for $\omega \to \infty$ [30].

In order to relate conformal relativity with Brans-Dicke theory we refer to conformally invariant actions (II.24) and (II.25) in Jordan frame and define

$$\frac{1}{12} \tilde{\Phi}^2 = e^{-\tilde{\phi}}, \quad (III.6)$$

$$\frac{1}{12} \tilde{\Phi}^2 = e^{-\tilde{\phi}}, \quad (III.7)$$

which gives these actions in the form

$$S = \int d^4x \sqrt{-g} e^{-\phi} \left[ \frac{1}{6} R + \frac{3}{2} \partial_{\mu} \phi \partial^{\mu} \phi \right], \quad (III.8)$$

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} e^{-\tilde{\phi}} \left[ \frac{1}{6} \tilde{R} + \frac{3}{2} \partial_{\mu} \tilde{\phi} \partial^{\mu} \tilde{\phi} \right]. \quad (III.9)$$

These actions, however, are special cases of the Brans-Dicke action which can be realized once one defines

$$\Phi_{BD} = e^{-\phi} \quad (III.10)$$
in (III.1) together with putting that \(16\pi = 1\), which gives (III.1) in the form
\[
S = \int d^4x \sqrt{-g} e^{-\phi} [R - \omega \partial_\mu \phi \partial^\mu \phi] , \tag{III.11}
\]
so that one can immediately see that (III.8) and (III.11) are the same provided that the Brans-Dicke parameter
\[
\omega = -\frac{3}{2} . \tag{III.12}
\]
On the other hand, if one takes
\[
\omega = -1 \tag{III.13}
\]
in (III.11), then one obtains the low-energy-effective superstring action for only graviton and dilaton in the spectrum \([19, 27, 36]\)
\[
S = \int d^4x \sqrt{-g} e^{-\phi} [R + \partial_\mu \phi \partial^\mu \phi] . \tag{III.14}
\]
In fact, the action (III.11) represents Brans-Dicke theory in a special frame which is known as string frame. It is because in superstring theory the coupling constant \(g_s\) is related to the vacuum expectation value of the dilaton by \([19, 20]\)
\[
g_s \propto \frac{e^\phi}{2} . \tag{III.15}
\]
Now, the field equations which are obtained by the variation of (III.11) with respect to the dilaton \(\phi\) and the graviton \(g_\mu\nu\), respectively, are \([32]\)
\[
R + \omega \partial_\mu \phi \partial^\mu \phi - 2\omega \Box \phi = 0 , \tag{III.16}
\]
\[
R_\mu\nu - \frac{1}{2}g_\mu\nu R = 8\pi e^\phi T_\mu\nu + (\omega + 1) \partial_\mu \phi \partial_\nu \phi - \left(\frac{\omega}{2} + 1\right) g_\mu_\nu \partial_\rho \phi \partial^\rho \phi + g_\mu\nu \Box \phi - \phi_\mu\nu . \tag{III.17}
\]
Following the same track as for Brans-Dicke field equations now we can contract (III.17) and use (III.16) to get a similar equation to (III.2) in string frame, i.e.,
\[
\partial_\rho \phi \partial^\rho \phi - \Box \phi = \frac{8\pi}{2\omega + 3} e^\phi T . \tag{III.18}
\]
Notice that putting \(T = 0\) in (III.18) (which is the case for dilaton-graviton theory of superstring cosmology) we have that \(\Box \phi = \partial_\rho \partial^\rho \phi\) which gives exactly the form of pre-big-bang field equations as presented in \([32]\) with an arbitrary value of \(\omega\). On the other hand, the limit \(\omega \to -\frac{3}{2}\) of the equation (III.18) is singular unless we take the trace of the energy-momentum tensor \(T\) to be zero.

In fact, it can be just obtained by the application of (III.10) into (III.2) since
\[
\Box \Phi_{BD} = e^{-\phi} (\partial_\rho \phi \partial^\rho \phi - \Box \phi) . \tag{III.19}
\]
It is interesting to note that the Ricci scalar which can be calculated from (III.10), (III.17) as
\[
R_\mu\nu = 8\pi e^\phi T_\mu\nu - \phi_\mu\nu + (\omega + 1) (\partial_\mu \phi \partial_\nu \phi - g_\mu_\nu \partial_\rho \phi \partial^\rho \phi + g_\mu\nu \Box \phi) , \tag{III.20}
\]
and for low-energy-effective superstring theory \(\omega = -1\) the whole lot of its terms vanish. However, this is not the case in conformal relativity \(\omega = -(3/2)\) for which this expression is not so simple and of course from (III.18) one sees that the trace \(T\) of the energy momentum tensor must be zero in this limit.

In the context of superstring theory (with no matter fields) one often starts looking for the exact solutions starting from the two equations \([33, 34, 35]\)
\[
R + \omega \partial_\mu \phi \partial^\mu \phi - 2\omega \Box \phi = 0 , \tag{III.21}
\]
\[
R_\mu\nu + \phi_\mu\nu - (\omega + 1) (\partial_\mu \phi \partial_\nu \phi - g_\mu_\nu \partial_\rho \phi \partial^\rho \phi + g_\mu\nu \Box \phi) = 0 . \tag{III.22}
\]
In the next section we will try to use these equations to present cosmological solutions.

Now we come to an important remark. Namely, taking the limit \(\omega = -3/2\) in Brans-Dicke field equation (III.2) is singular unless we assume that the trace of the energy-momentum tensor of matter vanishes. The
point is that, in fact, the conformal relativity field equation (II.34) is not an independent equation from (II.33), as it is the case in Brans-Dicke theory, but that it is exactly a contraction of (II.33)! Then, it may suggest that in order to get the proper limit of Brans-Dicke theory from conformal relativity one should also assume that in Brans-Dicke theory the trace of the energy-momentum of matter should vanish. As we have mentioned, vanishing of the energy-momentum for the perfect fluid requires its equation of state for radiation.

Now let us move to the problem of frames. The different frames are defined by the coupling properties of the scalar field (dilaton) to gravity in the theory. In all three cases we have discussed so far (conformal relativity, Brans-Dicke, superstrings) we deal with non-minimal coupling of the scalar field to gravity. For Brans-Dicke we call it Jordan frame and for superstrings we call it string frame. For conformal relativity we will also call it Jordan frame although we deal with non-minimal coupling in both conformally related frames (conformal invariance).

However, for all three theories one is usually interested in the question of what is going on in the Einstein frame which is defined as the frame in which the scalar field is minimally coupled to gravity.

Let us now begin with the action (III.11) which admits an arbitrary value of the parameter $\omega$. It is easy to show [33, 34, 36] that under a choice of a conformal factor

$$\Omega = e^{-\Phi/2},$$

the action (III.11) transforms into

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \left( \omega + \frac{3}{2} \right) \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi \right]$$

(III.24)

which for $\omega = -3/2$ gives exactly the Einstein-Hilbert action with no matter Lagrangian

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \tilde{R}.$$  

(III.25)

However, for low-energy-effective theory with $\omega = -1$ this is not Einstein-Hilbert action, but the Einstein gravity coupled minimally to a non-vanishing scalar field $\phi$. Notice that in terms of the $\Phi$ field defined in conformal relativity according to (III.6) the relation (III.23) reads as

$$\Omega = e^{-\Phi/2} = \frac{\Phi}{\sqrt{12}},$$

(III.26)

or, basically, $\tilde{\Phi} = \text{const.}$ in (II.20) which has already been mentioned gives the Einstein limit of conformal relativity. This means that the choice of the Einstein frame is unique in conformal relativity and requires $\Phi = \text{const.}$

The appropriate field equations which come from (III.24) are

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \tilde{T}_{\mu\nu} \equiv \left( \omega + \frac{3}{2} \right) \left[ 2 \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi - g_{\mu\nu} \tilde{\partial}_\mu \phi \tilde{\partial}^\nu \phi \right],$$

(III.27)

$$\left( \omega + \frac{3}{2} \right) \Box \phi = 0.$$  

(III.28)

From (III.27) - (III.28) it follows that $\omega = -3/2$ is exactly the border between a standard scalar field and a ghost (negative kinetic energy) which is required in order to have an expansion minimum in varying constant cosmologies [37] and to mimic the phantom field [38] which violates the weak energy condition $\tilde{\rho} + \tilde{p} < 0$. This is easily seen after calculating the energy-momentum tensor and expressing the energy density and pressure in terms of scalar field, i.e.,

$$\tilde{\rho} = \left( \omega + \frac{3}{2} \right) \phi^2,$$

(III.29)

$$\tilde{p} = \left( \omega + \frac{3}{2} \right) \phi^2.$$  

(III.30)

This also means that, formally, the energy density and pressure vanish due to a special choice of Brans-Dicke parameter $\omega$ and not necessarily due to a vanishing of the scalar field in the Einstein frame.
Obviously, a more general choice of the conformal factor which leaves the action (III.11) invariant reads as an appropriate generalization of the choice (II.20)

\[ \Omega = \frac{e^{-\frac{\phi}{2}}}{e^{\frac{\tilde{\phi}}{2}}} = \frac{\Phi}{\tilde{\Phi}}, \tag{III.31} \]

and this is exactly the conformally invariant transformation (II.20) of the scalar field in conformal relativity. Having applied (III.31) to (III.11) shows its conformal invariance, i.e.,

\[ \tilde{S} = \int d^4x \sqrt{-\tilde{g} e^{-\tilde{\phi}}} \left[ \tilde{R} + \frac{3}{2} \partial_{\mu} \tilde{\phi} \partial^{\mu} \tilde{\phi} \right]. \tag{III.32} \]

In fact it works for any value of \( \omega \). Notice that conformal invariance leads to a self-duality transformation \[ \Phi \rightarrow \tilde{\Phi}, \tag{III.33} \]
or,

\[ \tilde{\phi} \rightarrow \phi, \tag{III.34} \]
or

\[ \Omega \rightarrow \frac{1}{\Omega}. \tag{III.35} \]

A slightly different way of getting conformal invariance of the action (III.8) is the following conformal transformation \[ \Omega = e^{-\phi}, \tag{III.36} \]

which brings it to the form

\[ S = \int d^4x \sqrt{-g e^{-\phi}} \left[ \tilde{R} + \frac{3}{2} \partial_{\mu} \phi \partial^{\mu} \phi \right], \tag{III.37} \]

and it remains conformally invariant provided we replace

\[ \phi \rightarrow -\tilde{\phi}, \tag{III.38} \]

which gives

\[ S = \int d^4x \sqrt{-g e^{-\tilde{\phi}}} \left[ \tilde{R} + \frac{3}{2} \partial_{\mu} \tilde{\phi} \partial^{\mu} \tilde{\phi} \right], \tag{III.39} \]

and this is another example of duality symmetry known from superstring theory and may relate weak coupling regime with a strong coupling regime of various superstring actions \[ 39]. \]

**IV. CONFORMAL COSMOLOGY IN EINSTEIN AND STRING FRAMES**

We discuss Friedmann cosmology in the two conformally related frames as given in (II.1), i.e.,

\[ ds^2 = -dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{IV.1} \]

\[ d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2 \left( \frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right). \tag{IV.2} \]

From (II.1), (IV.1) and (IV.2) one easily see that the time coordinates and scale factors are related by

\[ d\tilde{t} = \Omega dt, \tag{IV.3} \]

\[ \tilde{a} = \Omega a, \tag{IV.4} \]
where for the full conformal invariance one has to apply the definition of conformal factor \((II.20)\). However, in the case of studying the Einstein limit we must take one of the scalar fields constant, so that the conformal factor has the form given by \((III.23)\). This requires a replacement of \((IV.3)-(IV.4)\) into

\[
\tilde{t} = e^{-\frac{\phi}{2}} dt , \quad (IV.5)
\]

\[
\tilde{a} = e^{-\frac{\phi}{2}} a , \quad (IV.6)
\]

where the quantities with tildes are in the Einstein frame while those without tildes are in the string frame. In fact, the transformations \((IV.3)\) and \((IV.5)\) are coordinate transformations rather than conformal transformations and so they are responsible for a difference in the time measurements/scales in both frames. Historically these scales were called atomic and cosmological, for example \([5, 21]\).

Imposing Friedmann metric \((IV.2)\) with in the Einstein frame the equations \((III.27)-(III.28)\) give

\[
\left( \omega + \frac{3}{2} \right) \left[ \ddot{\phi} + \frac{3}{a} \dot{a} \right] = 0 , \quad (IV.7)
\]

\[
3 \frac{\dot{a}^2 + k}{a^2} = \left( \omega + \frac{3}{2} \right) \dot{\phi}^2 , \quad (IV.8)
\]

\[
-2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = \left( \omega + \frac{3}{2} \right) \dot{\phi}^2 . \quad (IV.9)
\]

The solutions of \((IV.7)-(IV.9)\) for an arbitrary value of the parameter \(\omega \neq -3/2\) and \(k = 0\) read as

\[
\dot{a} = |\tilde{t}|^{\frac{3}{2}} , \quad (IV.10)
\]

\[
\phi = \phi_0 + \frac{1}{\sqrt{3(\omega + \frac{3}{2})}} \ln |\tilde{t}| . \quad (IV.11)
\]

On the other hand, it is clear from \((IV.7)-(IV.9)\) that for \(\omega = -3/2\) and \(k = 0\) the unique solution gives

\[
\dot{a} = 0 , \quad (IV.12)
\]

which is just a flat Minkowski universe. If we assume \(k \neq 0\), then we get from \((IV.8)\) that

\[
\dot{a} = \sqrt{-k} \tilde{t} + \tilde{t}_0 , \quad (IV.13)
\]

which is admissible only for \(k = -1\) and this solution represents Milne universe \([\tilde{5}]\) (in which there is no acceleration of the expansion since \(\dot{a} = 0\)). However, its relation to Minkowski spacetime requires coordinate transformations which involves the two time scales - a dynamical one and an atomic one \([\tilde{5}]\) which may be responsible for cosmological redshift effect. On the other hand, the solution for \(k = +1\) would be possible only if the cosmological constant was admitted - again, cosmological redshift in this Static Einstein model would be the result of a different time scaling \([\tilde{5}]\).

In the string frame we use the Friedmann metric \((IV.1)\) which imposed into the equations \((III.21)-(III.22)\) for an arbitrary value of the parameter \(\omega\) gives the following set of equations

\[
\dot{\phi} - 3 \frac{\dot{a}}{a} \phi + \dot{\phi} = 0 , \quad (IV.14)
\]

\[
-3 \frac{\dot{a}^2 + k}{a^2} = -\left( \frac{\omega}{2} + 1 \right) \phi^2 + \ddot{\phi} + \dot{\phi} , \quad (IV.15)
\]

\[
-2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = \frac{\omega}{2} \phi^2 + \frac{\dot{a}}{a} \phi . \quad (IV.16)
\]

These equations \((IV.14)-(IV.16)\) give the following solutions

\[
a(t) = |t| \frac{3(\omega + 1)^{\frac{1}{2}}(\sqrt{3} - 1)}{\sqrt{2(3\omega + 4)}} , \quad (IV.17)
\]

\[
\phi(t) = -\frac{1 \pm \sqrt{3(2\omega + 3)}}{3\omega + 4} \ln |t| , \quad (IV.18)
\]
where following pre-big-bang/ekpyrotic scenario the solutions for negative times are also admitted. From (IV.19)-(IV.20) one can first find the pre-big-bang solutions for \( \omega = -1 \) which are very well-known and read

\[
a(t) = | t |^{\frac{1}{\omega}} ,
\]

\[
\phi(t) = (\pm \sqrt{3} - 1) \ln | t | .
\]

However, the conformal relativistic solutions for \( \omega = -\frac{3}{2} \) are

\[
a(t) = | t | ,
\]

\[
\phi(t) = 2 \ln | t | ,
\]

and show that they do not allow for two branches `+′ and `−′ and so they do not seem to allow scale factor duality \[40\]

\[
a(t) \to \frac{1}{a(-t)} , \quad \phi \to \phi - 6 \ln a ,
\]

which is a cosmological consequence of string duality symmetries \[18\]. However, unlike pre-big-bang solutions (IV.19)-(IV.20) which must be regularized at Big-Bang singularity because both the curvature and the string coupling (III.15) diverge there, the solutions (IV.21)-(IV.22) do not lead to strong coupling singularity in the sense of string theory, since

\[
g_s = e^{\frac{\phi}{2}} = | t | ,
\]

and is regular for \( t = 0 \). This has an interesting analogy with the ekpyrotic/cyclic universe scenario where, in fact, the transition through Big-Bang singularity takes place in the weak coupling regime \[41\]. Note that form eqs. (IV.5)-(IV.6) we have

\[
\tilde{t} = \tilde{t}_0 + \ln | t | ,
\]

which according to (IV.21) gives

\[
\tilde{a} = 1 .
\]

This is consistent with what we have obtained in (IV.12) and reflects the fact that in the Einstein frame the limit of an expanding flat Friedmann metric is Minkowski universe.

V. DISCUSSION

We have shown that there is a simple relation between conformal relativity (Hoyle-Narlikar theory) and a reduced (graviton-dilaton) low-energy-effective superstring theory to Brans-Dicke theory. This relation shows that the former is recovered from Brans-Dicke theory if one takes \( \omega = -3/2 \) while the latter if one takes \( \omega = -1 \). This may allow to study the exact cosmological solutions of conformal relativity and its properties appealing to the well-studied properties of the Brans-Dicke theory and, in particular, to low-energy-effective superstring effective theory.

In fact, Brans-Dicke parameter \( \omega = -3/2 \) gives a border between a standard scalar field and a ghost (negative kinetic energy) which is required in order to have an expansion minimum in varying constant cosmologies \[37\] and to mimic the phantom field \[38\] which violates the weak energy condition \( \rho + p < 0 \).

The conformally transformed energy-momentum tensor is conserved in conformal relativity only if its trace vanishes. In the case of barotropic fluid it vanishes only for radiation. This means that only the photons may obey the equivalence principle and this is not the case for other types of matter, since with non-vanishing trace we deal with creation of matter process. We have compared that conformal relativity has lots of analogy with Self Creation Cosmology of Ref. \[15\] which has the same field equations provided we deal with cosmological models of vanishing curvature scalar.

We have presented basic cosmological solutions of conformal relativity in both Einstein (minimally coupled) and string (nonminimally coupled) frames. The Einstein limit of flat conformal cosmological solutions is unique and it is flat Minkowski space which requires the scalar field/mass evolution instead of the scale factor evolution to explain cosmological redshift \[9\]. An interesting observation is that like in ekpyrotic/cyclic
models, a possible transition through a singularity in conformal cosmology in the string frame takes place in the weak coupling regime.

Some other interesting points can be made about the vanishing of some of the PPN (parametrized-post-Newtonian) parameters in the theories under consideration. It appears that in conformal relativity only one of them ($\beta_1$) vanishes while for superstring-effective theory two of them vanish ($\gamma$ and $\beta_4$).

Despite its restricted generality, conformally invariant theory of gravity still seems to be attractive, for instance with respect to the problem of a unique choice of vacuum in quantum field theory in curved spaces. There is no doubt that conformally invariant theories still attract some attention in the context of superstring theory and M-theory (see e.g. [39, 42, 43]) and require further studies.

VI. ACKNOWLEDGMENTS

MPD acknowledges a support from the Polish Research Committee grant No 2PO3B 090 23.

[1] Birell N.D. and Davies P.C.W. Quantum field theory in curved space, Cambridge University Press, 1982.
[2] Hoyle, F., and Narlikar, J.V., Proc. Roy. Soc. A282 (1964), 191 ; ibid A294 (1966), 138 ; ibid A270 (1962), 334.
[3] Chernikov, N., and Tagirov, E., Ann. Inst. Henri Poincaré 9 (1968), 109.
[4] Bekenstein, J.D., Ann. Phys. (NY) 82 (1974) 535.
[5] Narlikar, J.V., Introduction to Cosmology, Jones and Bartlett Publishers, Inc. Portola Valley (1983).
[6] Penrose, R., Relativity, Groups and Topology, Gordon and Breach London (1964).
[7] Behnke, D., Blaschke, D.B., Pervushin, V.N., Proskurin, D.V., Phys. Lett. B 530 (2002), 20.
[8] Blaschke, D., et al., in: On the nature of dark energy, eds. P. Brax, J. Martin, and J.-P. Uzan, IAP Paris (2002).
[9] M.P. Dąbrowski, D. Behnke and D. Blaschke, in preparation.
[10] Weinberg, S., Rev. Mod. Phys. 61 (1989), 23.
[11] Perlmutter, S., et al., ApJ 517 (1999), 555.
[12] Riess, A.G., et al., AJ 116 (1998), 1009.
[13] Riess, A.G., et al., ApJ 560 (2001), 49.
[14] Barbera, G., Gen. Rel. Grav. 14 (1982), 117; Astroph. Space Sci. 282 (2002), 683.
[15] Anderson et al., Phys. Rev. D65 (2002), 082004.
[16] Brans, C., and Dicke, R.H., Phys. Rev. 124 (1961), 925.
[17] Polchinski J., String Theory, Cambridge University Press (1998).
[18] Dąbrowski, M.P., Ann. Phys. (Leipzig) 10 (2001), 195.
[19] Dąbrowski, M.P., String Cosmologies, University of Szczecin Press (2002).
[20] Dirac P.A.M., Proc. Roy. Soc. A165 (1938), 199.
[21] Y. Fujiw and K.-I. Maeda, The Scalar-Tensor Theory of Gravitation, Cambridge University Press (2003).
[22] Boisseau, B., Esposito-Farese, Polarski D., and Starobinsky A.A., Phys. Rev. Lett. 85 (2000), 2236.
[23] Esposito-Farese, G., and Polarski, D., Phys.Rev.D 63 (2001), 063504.
[24] Misner, C.W., Thorne, K.S., and Wheeler, J.A., Gravitation, W.H. Freeman and Company New York (1995).
[25] Hawking, S.W., Ellis, G.F.R., The large-scale structure of space-time, Cambridge Univ. Press (1999).
[26] Flanagan É.E. [gr-qc/0403063].