Fluctuation induced forces in the presence of mobile carrier drift

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Abstract

A small polarizable object (an atom, molecule or nanoparticle), placed above a medium with flowing dc current in it, is considered. It is shown that the dc current can have a strong effect on the force exerted on the particle. The Casimir-Lifshitz force, well studied in the absence of current, gets modified due to drifting mobile carriers in the medium. Furthermore, a force in the lateral direction appears. This force is a non-monotonic function of the drift velocity and its maximal value is comparable with the Casimir-Lifshitz force. If the temperatures of the medium and the particle are different, this lateral force can be directed along the current (drag) or in the opposite direction (anti-drag).
I Introduction

All bodies are surrounded by a fluctuating electromagnetic field, due to the random motion of charges inside a body. When a second body is placed in the vicinity of the first one, fluctuation-induced (Casimir-Lifshitz) forces appear between the bodies. These forces are of great relevance in chemistry, nanotechnology and biology [1,2]. Much of the recent work on the fluctuation-induced forces, as well as on the related phenomena of near field heat transfer and the noncontact friction, deals with systems out of equilibrium (for some reviews see [3–9]). One should distinguish among several out-of-equilibrium situations:

(i) Different parts of the system have different temperatures but there is no relative motion between those parts (a hot body embedded into the cold environment is the simplest example of such situation [9–11]). Under such conditions the Casimir-Lifshitz forces will be modified, as compared to their equilibrium value [3–8,12–17].

(ii) Different parts of the system are in relative motion. For instance, two macroscopic plates, separated by a vacuum gap, move one on top of the other. Another example is an atom (or a nanoparticle) moving above a macroscopic plate. Relative motion between bodies affects the Casimir-Lifshitz forces and, in particular, leads to dissipation and noncontact friction. This kind of problems was considered by many authors (3–8,18–23 and references therein), with rather controversial results (see Ref. [7] for various contradictions and inconsistencies in the literature).

(iii) There is no relative motion between parts of the system but some of the parts are subjected to a dc electric current [24–28]. The simplest example is to consider a semiconducting plate, with a dc current flowing in it, and ask how this current affects fluctuations of the electromagnetic field inside and outside the plate. This problem has been considered in [24,25]. In the present paper we further elaborate on electromagnetic field fluctuations in the presence of carrier drift and, in particular, calculate the fluctuation force acting on an atom, or a nanoparticle placed above a sample with a dc current in it.

Let us stress that setups (ii) and (iii) are quite different- a fact not sufficiently appreciated in the literature. For one thing, the dc current in the sample produces a stationary (time independent) magnetic field which affects the atomic spectrum and, if inhomogeneous, exerts a force on the atom as a whole. More importantly, the fluctuation-induced forces in the two setups are not the same. The point is that in setup (iii) the mobile carriers are in motion (in the laboratory frame) while the lattice is fixed. Therefore the spontaneous fluctuations originating in the sub-system of the mobile carriers will be Doppler shifted with respect to those residing in the lattice. Moreover, in the presence of drift it is generally not even possible to assign a definite temperature to the mobile carriers, which makes the existing theory of the fluctuation-induced forces inapplicable. The purpose of this work is to study the effect of carrier drift on the
fluctuation forces exerted on a small polarizable object (an atom, molecule or a nanoparticle).

The organization of the paper is as follows: In Section II we define the model and discuss how the fluctuational electrodynamics (Rytov’s theory) should be modified in the presence of mobile carrier drift. Section III is devoted to the properties of the fluctuating field, outside the sample with drifting carriers. In Section IV a particle is introduced, above the surface of a medium with drifting carriers, and the forces acting on the particle (both in the lateral and in the normal direction) are calculated. Various specific examples are presented in Section V and the conclusions are summarized in Section VI.

II Fluctuational Electrodynamics in the Presence of Carrier Drift

We consider a conducting medium, e.g., a semiconductor, containing mobile carriers with charge $e$, effective mass $m$ and equilibrium concentration $n_0$. When a dc voltage is applied to the sample, the carriers acquire some drift velocity $\vec{v}_0$, so that there is a steady state dc current $\vec{j}_0 = en_0 \vec{v}_0$. On top of this stationary drift there are fluctuations of the carrier and current density which cause fluctuations of the electric field. We designate the fluctuating part of these quantities as $\vec{n}(\vec{r},t)$, $\vec{j}(\vec{r},t)$ and $\vec{E}(\vec{r},t)$, respectively. Thus, for instance, the total current density is $\vec{j}_0 + \vec{j}(\vec{r},t)$. The fluctuating part $\vec{E}(\vec{r},t)$ of the electric field is of particular interest because, unlike $n$ and $\vec{j}$, it exists also outside the sample and exerts forces on nearby objects. It should be emphasized that $\vec{j}(\vec{r},t)$ accounts only for the motion of the mobile carriers. In addition, there are fluctuating polarization currents due to the lattice. We briefly recapitulate the main equations of the theory, following with some modifications Ref [25].

The relation between $\vec{j}$ and $\vec{E}$, in the frequency-wavevector domain is

$$j_\alpha (\omega, \vec{k}) = \sigma_{\alpha\beta} (\omega, \vec{k}) E_\beta (\omega, \vec{k}),$$

where summation over $\beta$ is implied. The conductivity tensor $\sigma_{\alpha\beta}$ is defined with respect to the non-equilibrium steady state, i.e., it connects quantities fluctuating on top of the stationary current flow. That is why, even for an intrinsically isotropic medium, $\sigma_{\alpha\beta}$ is a tensor depending not only on $\omega$ but also on $\vec{k}$. The dependence on $\vec{k}$ occurs because the fluctuations are carried away by the flow, thus producing a non-local response (spatial dispersion). Adding the conduction current, Eq (1), to the fluctuating polarization current of the lattice yields the fluctuating displacement

$$D_\alpha (\omega, \vec{k}) = \epsilon_L (\omega) E_\alpha (\omega, \vec{k}) + i \frac{4\pi}{\omega} \sigma_{\alpha\beta} (\omega, \vec{k}) E_\beta (\omega, \vec{k}) \equiv \epsilon_{\alpha\beta} (\omega, \vec{k}) E_\beta (\omega, \vec{k}),$$

where $\epsilon_L$ is the lattice dielectric function which can depend on $\omega$ but not on $\vec{k}$. Eq (2) defines the dielectric tensor $\epsilon_{\alpha\beta} (\omega, \vec{k})$ which controls the dynamics of
electrical fluctuations in the medium. The form of $\epsilon_{\alpha\beta}(\omega, \vec{k})$ depends on the specific system or model. We assume here Drude model, with drift, which is a special case of the more general hydrodynamic model (see Eq(11) of [25] with the thermal pressure term neglected):

$$\epsilon_{\alpha\beta}(\omega, \vec{k}) = (\epsilon'_L + i\epsilon''_L) \delta_{\alpha\beta} - \frac{\omega^2_p}{\omega - \vec{k} \cdot \vec{v}_0 + i\nu} \left( \delta_{\alpha\beta} + \frac{\nu_0 \alpha \beta}{\omega - \vec{k} \cdot \vec{v}_0} \right),$$

where $\nu$ is the collision frequency of the mobile carriers, $\omega^2_p = \frac{4\pi e^2 n_0}{m}$, and $\epsilon_L$ has been separated into the real and imaginary parts.

In our dealing with fluctuations we use Rytov’s method in which random Langevin sources are introduced into the Maxwell equations, similarly to what is done in the theory of Brownian motion. These random sources play the role of "external" currents and charges in the Maxwell equations and, if their correlation functions are known, one can compute the correlation function for various components of the electromagnetic field. We shall be interested in fluctuational phenomena close to the surface of the sample and neglect the retardation effects. In this limit the electromagnetic field is rotationless, $\vec{E}(\vec{r}, t) = -\nabla \Phi(\vec{r}, t)$, and Rytov’s fluctuational electrodynamics reduces to the Poisson equation supplemented by the Langevin sources. In the bulk of the sample this equation is

$$k^2 \epsilon(\omega, \vec{k}) \Phi(\omega, \vec{k}) = 4\pi \rho_r(\omega, \vec{k}),$$

where $\Phi(\omega, \vec{k})$, $\rho_r(\omega, \vec{k})$ are the Fourier transforms of the potential $\Phi(\vec{r}, t)$ and of the random Langevin sources $\rho_r(\vec{r}, t)$, and

$$\epsilon(\omega, \vec{k}) = \frac{k_\alpha k_\beta}{k^2} \epsilon_{\alpha\beta}(\omega, \vec{k}) = \epsilon'_L(\omega) + i\epsilon''_L(\omega) - \frac{\omega^2_p}{\omega - \vec{k} \cdot \vec{v}_0 + i\nu} \left( \omega - \vec{k} \cdot \vec{v}_0 \right).$$

This expression has a simple interpretation. The dielectric function $\epsilon(\omega, \vec{k})$ relates the displacement and the field in a longitudinal wave. If the wave propagates in the direction of flow, $\vec{k} \parallel \vec{v}_0$, then there is a Doppler shift of the wave frequency. There is no such shift if the propagation direction is perpendicular to $\vec{v}_0$. Note that only the plasma component of $\epsilon(\omega, \vec{k})$ undergoes the Doppler shift, while the lattice component remains the same as in equilibrium.

For a system at equilibrium ($\vec{j}_0 = 0$) the correlation function of the random sources is determined by the fluctuation-dissipation theorem [10,11]:

$$\langle \rho_r(\omega, \vec{k}) \rho^*_r(\omega', \vec{k}') \rangle = 2\pi \delta(\omega - \omega') \langle \rho_r(\vec{k}) \rho^*_r(\vec{k}^\prime) \rangle = (2\pi)^4 \delta(\omega - \omega') \delta(\vec{k} - \vec{k}^\prime) \langle \rho_r(\rho^*_r)_{\omega\vec{k}} \rangle_{\omega\vec{k}},$$

with

$$\langle \rho_r(\rho^*_r)_{\omega\vec{k}} \rangle_{\omega\vec{k}} = \frac{\hbar k^2}{4\pi} \epsilon''(\omega) \coth \frac{\hbar \omega}{2T}.$$
where $\langle \cdots \rangle$ denotes thermal and quantum average, $T$ is the temperature of the system and $\epsilon''(\omega)$ is the imaginary part of its dielectric function [Eq (5) with $(\vec{v}_0 = 0)$. Eq (6) defines the spectral densities $\langle \rho_r (\vec{k}) \rho^*_r (\vec{k'}) \rangle_\omega$, $\langle \rho_r \rho^*_r \rangle_{\omega \vec{k}}$, and Eq (7) contains the essence of the fluctuation-dissipation theorem. Strictly speaking, $\rho_r, \vec{j}(\vec{r}, t), \vec{E}(\vec{r}, t),$ etc., should be understood as quantum-mechanical operators and various correlation functions should be properly antisymmetrized. These changes, however, would be only "cosmetic" and would not affect the final results. The point is that in the RHS of Eq (7) the correct quantum mechanical spectral density is given. With this caveat, Rytov’s theory becomes essentially classical [30].

Since our system is out of equilibrium ($\vec{j}_0 \neq 0$), there is no general prescription for writing down the correlator of the random sources $\rho_r (\omega, \vec{k})$. However, under some conditions, it is possible to do so. For instance, if dissipation in the system of the mobile carriers can be neglected (the limit of collisionless plasma), then the random sources originate only in the lattice. Since the latter is in equilibrium, at some temperature $T_L$, one can use the fluctuation-dissipation theorem so that [25,31,32]

$$\langle \rho_r (\vec{k}) \rho^*_r (\vec{k'}) \rangle_\omega = \frac{\hbar k^2}{4\pi} (2\pi)^3 \delta (\vec{k} - \vec{k'}) \epsilon''_L (\omega) \coth \frac{\hbar \omega}{2T_L}$$  \hspace{1cm} (8)

In this case the spontaneous creation of the fluctuations is not sensitive to the drift of the mobile carriers. Only the subsequent dynamics of the fluctuations, which is governed by the full dielectric function, [Eq (5) with $\nu = 0$], is affected by the drift.

One can envisage the opposite situation when the lattice is “noiseless”, i.e., $\epsilon''_L = 0$, and the fluctuating sources reside only in the plasma. This requires, of course, $\nu$ being different from zero. Since the electron plasma is out of equilibrium, it cannot be generally described by a well defined temperature. In some cases, however, the plasma is in a state of internal equilibrium, in spite of its overall drift. This happens, for instance, at low drift velocities, when the electron distribution is close to Fermi-Dirac (or Boltzmann), with the temperature of the lattice. The more interesting example is the case of large drift velocities when, due to strong mutual interactions, the electronic system undergoes rapid internal thermalization, with a temperature higher than that of the lattice ("hot electrons"). If the condition of internal equilibrium with some temperature $T_{el}$ is satisfied, then one can still use the fluctuation-dissipation theorem. In this case the creation of fluctuations is controlled by the imaginary part of the last term in Eq (5). Denoting this term by $\epsilon_{el}(\omega)$, where $\omega = \omega - \vec{k} \cdot \vec{v}_0$, we can write the fluctuation-dissipation theorem as

$$\langle \rho_r (\vec{k}) \rho^*_r (\vec{k'}) \rangle_\omega = \frac{\hbar k^2}{4\pi} (2\pi)^3 \delta (\vec{k} - \vec{k'}) \epsilon''_{el} (\omega) \coth \frac{\hbar \omega}{2T_{el}}$$ \hspace{1cm} (9)

The important difference between the Eqs (8) and (9), besides the trivial replacement of $\epsilon''_L, T_L$ by $\epsilon''_{el}, T_{el}$, is that the frequency $\omega$ appears in Eq (9), i.e,
e., the spontaneous creation of the fluctuations is now affected by the drift: the frequency of the fluctuations, as measured in the laboratory frame, is Doppler shifted. Let us note in this context that the total dielectric function, Eq (5), even for $\epsilon''_L = 0$, is not a function of $\omega_-$ only. This is because $\epsilon'_L$ depends on the "bare" frequency $\omega$, not on the Doppler shifted $\omega_-$. Therefore even for a noiseless lattice the problem of fluctuations in the presence of carrier drift is not equivalent to that for a moving sample. Only if one makes the additional assumption that $\epsilon'_L = \text{const}$ do the two problems become equivalent (provided that the drifting electrons are in an internal equilibrium, which is in itself a rather strong assumption).

Later, when considering the phenomena near a planar surface of the medium, we shall need Eq (4) in a somewhat different form. Assuming that the velocity vector $\vec{v}_0$ is in the $(x,y)$-plane, i.e., $\varepsilon (\omega, \vec{k})$ does not depend on $k_z$, we can transform Eq (4) back to space, in the $z$ direction, obtaining

$$\varepsilon (\omega, \vec{q}) \left( \frac{\partial^2}{\partial z^2} + q^2 \right) \Phi (\omega, \vec{q}, z) = 4\pi \rho_\Gamma (\omega, \vec{q}, z),$$

(10)

where $\vec{q} = (k_x, k_y)$ denotes the transverse (in-plane) wave vector and $\Phi (\omega, \vec{q}, z)$ is the Fourier transform of $\Phi (x, y, z, t)$ with respect to time and the $x, y$-coordinates (the same for $\rho_\Gamma$). This $(\omega, \vec{q}, z)$-representation is convenient for handling the planar geometry. The spectral densities of the random sources, Eqs (7,8,9), should be also transformed to the $(\omega, \vec{q}, z)$-representation. For instance, Eq (8) becomes

$$\langle \rho_\Gamma (\vec{q}, z) \rho_\Gamma^* (\vec{q}', z') \rangle = \frac{\hbar}{4\pi} (2\pi)^2 \delta (\vec{q} - \vec{q}') \left[ q^2 \delta (z - z') + \frac{\partial^2}{\partial z \partial z'} \delta (z - z') \right]$$

$$\times \epsilon''_L (\omega) \coth \frac{\hbar \omega}{2T_L},$$

(11)

and similarly for the other spectral densities.

### III Fluctuations of the Electric Potential Near the Surface

We consider a medium occupying half space ($z < 0$) while the other half ($z > 0$) is vacuum. The random charge sources $\rho_\Gamma (\vec{r}, t)$ produce evanescent electric fields near the surface (in addition to the radiation which we do not consider within our quasi-stationary, non-retarded approximation) and we are interested in various correlation functions for the potential and field. We shall consider three different setups, see Fig 1. Although case (a) has been studied long ago [9][11][33][34] and case (b) is simply related to (a), we discuss briefly also these two cases. The correlation functions obtained in this section will serve as building blocks in calculation of the fluctuation-induced forces in the next section.
Figure 1: The three setups: (a) The medium is at rest (in the laboratory frame), at equilibrium. (b) The medium moves with a constant velocity \( \vec{v}_0 \), along the \( x \)-axis. (c) The medium is at rest but a dc current with density \( \vec{j}_0 = e n_0 \vec{v}_0 \) is flowing in the medium.

Case (a): The equilibrium dielectric function is (Eq (5) with \( \vec{v}_0 = 0 \))

\[
\epsilon(\omega) = \epsilon'_L(\omega) + ie'_L(\omega) - \frac{\omega_p^2}{(\omega + i\nu)\omega},
\]

and we have to solve Eq (10) with this expression for \( \epsilon(\omega, \vec{q}) \). If one defines the Green’s function

\[
\epsilon(\omega) \left( -\frac{\partial^2}{\partial z^2} + q^2 \right) g(z, z_1, \vec{q}, \omega) = 4\pi \delta(z - z_1)
\]

then the formal solution of Eq (10) is

\[
\Phi(\omega, \vec{q}, z) = \int_{-\infty}^{0} dz_1 g(z, z_1, \vec{q}, \omega) \rho_r(\omega, \vec{q}, z_1).
\]

The solution of Eq (13), with the source inside the medium \( (z_1 < 0) \), the observation point outside \( (z > 0) \) and the standard boundary conditions for \( \Phi \) and
its normal derivative at \( z = 0 \), is

\[
g(z, z_1, \vec{q}, \omega) = \frac{4\pi}{q} \frac{1}{\epsilon(\omega) + 1} e^{-q(z-z_1)} \quad (15)
\]

Since in equilibrium the lattice temperature \( T_L \) is the same as the electron temperature, we can leave \( T_L \) to denote the equilibrium temperature of the sample. Then, using (14), (15) and (11) (with \( \epsilon' \) temperature, we can leave \( T_L \) to denote the equilibrium temperature of the sample. Then, using (14), (15) and (11) (with \( \epsilon''(\omega) \) instead of \( \epsilon''_L(\omega) \) ) one obtains after some algebra:

\[
\left\langle \Phi(\omega, \vec{q}, z) \Phi^*(\omega', \vec{q}', z') \right\rangle = (2\pi)^3 \delta(\omega - \omega') \delta(\vec{q} - \vec{q}') \left\langle \Phi(z) \Phi^*(z') \right\rangle_{\omega \vec{q}} \quad (16)
\]

with

\[
\left\langle \Phi(z) \Phi^*(z') \right\rangle_{\omega \vec{q}} = 4\pi \hbar \frac{\epsilon''(\omega)}{|\epsilon(\omega) + 1|^2} \coth \left( \frac{\hbar \omega}{2T_L} \right) \frac{1}{q} e^{-q(z+z')} \quad (17)
\]

Expression (17) factorizes into the \( \omega \)-dependent and \( q \)-dependent parts. The Fourier transform from \( \vec{q} = (k_x, k_y) \) to \( \vec{\rho} = (x, y) \) immediately yields

\[
\left\langle \Phi(x, y, z) \Phi^*(x', y', z') \right\rangle_{\omega \vec{q}} = \frac{2\hbar}{\pi} \frac{\epsilon''(\omega)}{|\epsilon(\omega) + 1|^2} \coth \left( \frac{\hbar \omega}{2T_L} \right) \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}. \quad (18)
\]

The correlation function \( \left\langle \Phi(x, y, z) \Phi^*(x', y', z', \omega') \right\rangle \) is obtained from (18) by multiplying it by the factor \( 2\pi \delta(\omega - \omega') \). This is the general rule, for any pair of fluctuating variables, and it follows from the stationary character of the fluctuations.

Correlation functions for various components of the electric field can be obtained from Eq (18) by differentiation. Some examples can be found in the above cited literature. For instance, differentiating Eq (18) with respect to \( x \) and \( x' \) and setting at the end \( x' = x \), one obtains

\[
\left\langle E^2_{x} \right\rangle_{\omega} = \frac{\hbar}{4\pi^3} \frac{\epsilon''(\omega)}{|\epsilon(\omega) + 1|^2} \coth \left( \frac{\hbar \omega}{2T_L} \right), \quad (19)
\]

i.e., when the surface is approached, the energy density increases as \( \frac{1}{z^2} \) - the well known rule. This rule breaks down, of course, for sufficiently small \( z \), either because of spatial dispersion effects or simply because the macroscopic theory becomes inapplicable at atomic distances.

Case (b): The medium is now moving, in the laboratory frame, with velocity \( v_0 \) in the \( x \) direction. In the frame moving with the medium all the relations derived above for case (a) remain of course valid, if the coordinates and frequency refer to that frame. It is immediate to translate the results to the laboratory frame. If we denote by \( F(x-x', y-y', z, z', t-t') \) some correlation function at equilibrium (i.e., in the rest frame of the sample), then the correlation function in the laboratory frame is simply \( F_{lab}(x-x', y-y', z, z', t-t') = F(x-v_0 t - x' + v_0 t', y - y', z, z', t-t') \). This relation holds in the non-relativistic limit, \( v_0 \ll c \), assumed in the present work, and it implies that the Fourier transform...
$F_{\text{lab}}(\omega, k_x, k_y, z, z')$ is obtained from $F(\omega, k_x, k_y, z, z')$ by replacing $\omega$ with $\omega - k_x v_0 \equiv \omega_-$.

For instance the spectral density $\langle \Phi(z)\Phi^* (z') \rangle_{\omega \bar{q}}$ for a moving sample, as viewed from the laboratory frame, is given by the same expression as in the right-hand-side of Eq. (17) but with $\omega_-$ instead of $\omega$. Note, though, that this replacement results in a complicated, non-factorizable function of $k_x, k_y$ and $\omega$, and no simple expression in real space, comparable to Eq (18), can be obtained. We will not elaborate on this case further but move on to

Case (c): Here the sample is at rest but the electron subsystem moves with respect to the lattice with velocity $v_0$, producing a dc current density $j_0 = e n_0 v_0$. We do not specify the model for the lattice but just describe it by the lattice constant $\epsilon_L = \epsilon_1'(\omega) + i \epsilon''_L(\omega)$. The subsystem of the mobile carriers is described by the Drude model with drift, Eq (5). We will discuss in some detail the $\nu \to 0$ limit of this model and then briefly remark on the opposite limit, $\epsilon''_L(\omega) \to 0$.

In the limit of small collision frequency $\nu$ (the collisionless plasma model) the Langevin sources originating in the lattice dominate over those in the electronic subsystem. Taking the latter as “noiseless” and assuming the lattice in equilibrium, we can study the fluctuations using Eqs. (10), (11) with

$$\epsilon(\omega, k) = \epsilon'_L(\omega) + i \epsilon''_L(\omega) - \frac{\omega^2}{\omega_-} \equiv \epsilon_1(\omega, k_x). \quad (20)$$

This model has been considered in [25]. An unnecessary approximation was introduced there at an early stage of the calculation. Here we present a somewhat different approach.

In fact, in $(\bar{q}, \bar{z})$ - representation (i.e., Fourier transform in the $(x, y)$ - plane but not in the $z$ - direction) the calculation is straightforward and almost identical to case (a). The only difference is that the dynamics of the fluctuations is now controlled by the $(k_x)$-dependent dielectric function in Eq (20), so that instead of (17) we have

$$\langle \Phi (z) \Phi^* (z') \rangle_{\omega \bar{q}} = 4\pi \hbar \frac{\epsilon''_L(\omega)}{\epsilon_1(\omega, k_x) + 1} \coth \left( \frac{\hbar \omega}{2 T_L} \right) \frac{1}{q} e^{-q(z+z')} \quad \quad (21)$$

and the desired spectral density is

$$\langle \Phi (x, y, z) \Phi^* (x', y', z') \rangle_{\omega} = \int \int \frac{dk_x dk_y}{(2\pi)^2} e^{ik_x(x-x') + ik_y(y-y')} \langle \Phi (z) \Phi^* (z') \rangle_{\omega \bar{q}}. \quad \quad (22)$$

Because of the $(k_x)$ dependence of $\epsilon_1$ the integrand in (22) does not factorize, as it did in case (a), and no simple analytical expression can be obtained. For small drift velocities one can expand $\epsilon_1(\omega, k_x)$ in powers of $(k_x v_0^2)$. The first power does not contribute to (22) due to symmetry. The second power contributes, e.g., to the quantity $\langle E^2 \rangle_{\omega}$ in Eq (19), a term proportional to $(v_0^2/\bar{z}^5)$. This follows from a simple power counting: an extra factor $k_x^2$ in the integrand contributes an extra term $(1/\bar{z}^2)$ upon integration over $k_x$.

It is worthwhile to mention an interesting qualitative effect due to the drift. In equilibrium the spectral density, Eq (19), has a sharp maximum at the frequency of the surface plasmon $\omega_{sp} = \omega_p / \sqrt{\epsilon''_L + 1}$, when the factor $|\epsilon(\omega) + 1|$
becomes close to zero \[9\]. In the presence of drift we have \(\epsilon_1(\omega, k_x)\) instead of \(\epsilon(\omega)\), i.e., surface plasmons acquire dispersion and, upon integration over \(k_x\), the peak in \(\langle E_x^2 \rangle_\omega\) gets broadened.

Let us write down a useful spectral function which will be needed later:

\[
\langle \vec{E}(x, y, z) \cdot \vec{E}^*(x', y', z') \rangle_\omega = 8\pi \bar{\hbar} \epsilon''_L(\omega) \coth(\bar{\hbar} \omega / 2 T_L) \int \int \frac{dk_x dk_y}{(2\pi)^2} \frac{q}{|\epsilon_1(\omega, k_x) + 1|^2}
\]

This result is obtained from Eq. \(22\) [with Eq \(21\) inserted] by differentiating with respect to the pairs of variables \((x, x')\), \((y, y')\), \((z, z')\) and adding the corresponding expressions.

This concludes our discussion of the \(\nu \to 0\) limit, when the lattice is the only source of noise. In the opposite limit, \(\epsilon''_L(\omega) \to 0\), spontaneous fluctuations occur only in the electron plasma. We do not repeat the above calculations for this case but only note that the appropriate dielectric function now is

\[
\epsilon_2(\omega, k_x) = \epsilon'_L(\omega) - \frac{\omega_p^2}{\omega_- (\omega_- + i\nu)}
\]

and the appropriate spectral density for the spontaneous random sources is given in Eq. \(9\). It is then straightforward to make the necessary replacements in Eq \(21\).

### IV Fluctuation - Induced Forces

Consider an electric dipole, with dipole moment \(\vec{p}\), subjected to a space and time-dependent electromagnetic field \(\vec{E}(\vec{r}, t), \vec{B}(\vec{r}, t)\). The size of the dipole is assumed to be much smaller than the characteristic wavelength of the field (a “point dipole”). The dipole can rotate or vibrate but it does not move as a whole, i.e., it can be assigned a fixed position \(\vec{r}_0\) and an arbitrary time dependence \(\vec{p}(t)\). Under such conditions the dipole experiences an electric force \((\vec{p} \cdot \nabla) \vec{E}(\vec{r}, t)\) (\(\vec{r}\) is set equal to \(\vec{r}_0\) after differentiation) and the Lorenz magnetic force \((q/c) (\vec{v}_+ - \vec{v}_-) \times \vec{B}(\vec{r}_0, t)\), where \(\vec{v}_+ = d\vec{r}_+ / dt\) is the velocity of the positive charge \(q\) of the dipole (and similarly for \(\vec{v}_-\)). Thus, the magnetic force can be written as \(\frac{1}{c} \frac{d}{dt} \vec{p} \times \vec{B}\). Using the vector identity \((\vec{p} \cdot \nabla) \vec{E} = \text{grad} (\vec{p} \cdot \vec{E}) + \frac{1}{c} \frac{d}{dt} (\vec{p} \times \vec{B})\), one can write the total force as \(35\)

\[
\vec{f} = \text{grad} (\vec{p} \cdot \vec{E}) + \frac{1}{c} \frac{d}{dt} (\vec{p} \times \vec{B})
\]

In our problem the dipole moment and the fields are fluctuating quantities and, since the fluctuations are stationary, the last term in Eq \(25\) disappears upon averaging. We are left with the gradient term

\[
\langle \vec{f} \rangle \equiv \vec{F} = \text{grad} \left\langle \vec{p}(\vec{r}_0, t) \cdot \vec{E}(\vec{r}, t) \right\rangle
\]
where, again, setting $\vec{r} = \vec{r}_0$ after differentiation is implied. This equation holds also for a dipole in motion and it serves as the starting point for calculating the fluctuation-induced forces [5,8].

We now consider an atom, or a nanoparticle, or any entity with polarizability $\alpha(\omega)$ and size smaller than the relevant wavelength of the electromagnetic field (we use below the generic term “particle”). We allow for the particle temperature $T_p$ to be different from the sample temperature $T_L$. The particle is placed at a distance $z_0$ above the sample surface (see Fig. 2 for a schematic setup).

Figure 2: A small particle is placed at the point $\vec{r}_0 = (x_0, y_0, z_0)$ above the sample surface. Both the sample and the particle are at rest but a dc current with density $\vec{j}_0 = en_0 \vec{v}_0$ is flowing in the sample, in the $x$- direction. The temperature of the sample, i.e., of its lattice, is $T_L$. The particle temperature is $T_p$.

The force acting on the particle consists of two parts:

(i) The fluctuating field emerging from the sample induces a dipole moment in the particle. This emerging field, which is just the field considered in the previous section, is often called “free” or “spontaneous” and will be designated as $\vec{E}_s(\vec{r}, t)$. Interaction of this field with the dipole moment $\vec{p}_i(\vec{r}_0, t)$ induced in the particle is responsible for the first part, $\vec{F}_1$, of the force.

(ii) The particle itself induces a fluctuating electric field in the environment, due to the spontaneous fluctuations of its dipole moment. We denote the latter by $\vec{p}_s(\vec{r}_0, t)$ and the corresponding field by $\vec{E}_i(\vec{r}, t)$. This field acts back on the particle, giving the second part, $\vec{F}_2$, of the force.

Thus, Eq (26) splits into two parts, containing respectively $\vec{p}_i \cdot \vec{E}_s$ and $\vec{p}_s \cdot \vec{E}_i$. 

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Furthermore, since the particle polarizability \( \alpha (\omega) \) is frequency dependent, one has to rewrite Eq (26) in frequency domain [36]:

\[
\vec{F} (\vec{r}_0) = \vec{F}_1 + \vec{F}_2 = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \alpha (\omega) \nabla_{\vec{r}} \left( \vec{E} (\vec{r}_0) \cdot \vec{E}^* (\vec{r}) \right) + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \nabla_{\vec{r}} \left( \vec{p}^*_s (\vec{r}_0) \cdot \vec{E}_i (\vec{r}) \right) \omega ,
\]

(27)

where \( \vec{p}^*_s (\vec{r}_0, \omega) = \alpha (\omega) \vec{E}^*_s (\vec{r}_0, \omega) \) has been used.

In the rest of this section we specialize to the case \( \nu \to 0 \). Then the spectral density in the first term of Eq (27) is given, in somewhat different notations, in Eq (23), so that \( \vec{F}_1 \) is obtained immediately, after applying \( \nabla_{\vec{r}} \) and setting \( \vec{r} = \vec{r}_0 \) at the end. It is useful to split the integral over \( \omega \) into two pieces: from \(-\infty \) to \( 0 \) and from \( 0 \) to \(+\infty \). Switching the sign of the integration variables in the first piece, and using the conditions \( \epsilon_L (\omega) = \epsilon'_L (\omega) \), \( \alpha (\omega) = \alpha^* (\omega) \), \( \epsilon_1 (\omega, k_x) = \epsilon'_1 (\omega, k_x) \), we finally obtain the following expressions for the \( x \) and \( z \)-component of \( \vec{F}_1 \) (the \( y \)-component is zero):

\[
F_{1x} (z_0) = \frac{2\hbar}{\pi^2} \int_0^{\infty} d\omega \alpha'' (\omega) \epsilon''_L (\omega) \coth \left( \frac{\hbar \omega}{2T_L} \right) \int_{-\infty}^{\infty} dk_x dk_y \frac{q k_x}{|\epsilon_1 (\omega, k_x) + 1|^2} e^{-2qz_0},
\]

(28)

\[
F_{1z} (z_0) = -\frac{2\hbar}{\pi^2} \int_0^{\infty} d\omega \alpha' (\omega) \epsilon'_L (\omega) \coth \left( \frac{\hbar \omega}{2T_L} \right) \int_{-\infty}^{\infty} dk_x dk_y \frac{q^2}{|\epsilon_1 (\omega, k_x) + 1|^2} e^{-2qz_0},
\]

(29)

where \( \epsilon_1 \) is defined in Eq (20), \( \alpha' (\omega), \alpha'' (\omega) \) are the real and imaginary parts of \( \alpha (\omega) \), and \( q = \sqrt{k_x^2 + k_y^2} \). Since both the particle and the sample are at rest, it is natural that \( \alpha \) and \( \epsilon_L \) depend on \( \omega \) but not on \( k_x \) (no Doppler shift). The Doppler shifted frequency \( \omega = \omega - k_x v_0 \) enters only into the electronic part of \( \epsilon_1 \) which controls the dynamics of the fluctuations.

We now turn to the second term in Eq (27). First, one needs to compute the field \( \vec{E}_i (\vec{r}, \omega) \) induced by the spontaneous fluctuations of the particle dipole moment \( \vec{p}^*_s (\vec{r}_0, \omega) \). To this end we introduce the Green’s function \( G (\vec{r}, \vec{r}_0, \omega) \) as a solution of the Poisson equation for a unit charge at point \( \vec{r}_0 \). Then the electric potential created at point \( \vec{r} \) by the “point dipole” \( \vec{p}^*_s \) and the electric field of that dipole are given by

\[
\Phi_i (\vec{r}, \omega) = \vec{p}^*_s (\vec{r}_0, \omega) \cdot \nabla_{\vec{r}_0} G (\vec{r}, \vec{r}_0, \omega) \quad \text{and} \quad \vec{E}_i (\vec{r}, \omega) = -\nabla_{\vec{r}} \Phi_i (\vec{r}, \omega),
\]

(30)

so that

\[
\langle \vec{p}^*_s (\vec{r}_0) \cdot \vec{E}_i (\vec{r}) \rangle = - \langle p^*_{s\alpha} (\vec{r}_0) p_{s\beta} (\vec{r}_0) \rangle \omega \frac{\partial^2}{\partial r_\alpha \partial r_\beta} G (\vec{r}, \vec{r}_0, \omega).
\]

(31)

where \( \alpha, \beta \) labels the components and summation over indices is implied. The Green’s function can be written as

\[
G (\vec{r}, \vec{r}_0, \omega) = \int \frac{d^2 q}{(2\pi)^2} e^{iq(\vec{r}-\vec{r}_0)} g (z, z_0, \vec{q}, \omega)
\]

(32)
where \( g(z, z_0, \vec{q}, \omega) \) satisfies

\[
\epsilon_1(\omega, k_x) \left( -\frac{\partial^2}{\partial z^2} + q^2 \right) g(z, z_0, \vec{q}, \omega) = 4\pi \delta(z - z_0)
\]  

(33)

This is essentially the same as Eq (13), with \( \epsilon(\omega, k_x) \) instead of \( \epsilon(\omega) \), but now we have both the source and the observation point above the sample surface, i.e., \( z, z_0 > 0 \).

There is an apparent difficulty here, namely: \( G(\vec{r}, \vec{r_0}, \omega) \), being a response to a point source, is singular at \( \vec{r} = \vec{r_0} \), and so is \( g(z, z_0, \vec{q}, \omega) \) for \( z = z_0 \). Since in Eq (27) we have to further differentiate the expression in Eq (31) with respect to \( \vec{r} \) and then set \( \vec{r} = \vec{r_0} \), we end up with a meaningless singular expression. This problem is well known in the theory of Casimir - Lifshitz forces (see, e.g., [3]) and the remedy is to make the standard subtraction of the vacuum Green’s function \( G_0(\vec{r}, \vec{r_0}, \omega) \). Thus, the physical Green’s function is

\[
\hat{G}(\vec{r}, \vec{r_0}, \omega) = G(\vec{r}, \vec{r_0}, \omega) - G_0(\vec{r}, \vec{r_0}, \omega)
\]

or the Fourier transformed

\[
\hat{g}(z, z_0, \vec{q}, \omega) = g(z, z_0, \vec{q}, \omega) - g_0(z, z_0, \vec{q}, \omega),
\]

where \( g_0 \) is obtained from \( g \) by replacing \( \epsilon(\omega, k_x) \) by unity. The result is:

\[
\hat{g}(z, z_0, \vec{q}, \omega) = -\frac{2\pi}{q} e^{-q(z+z_0)} \Gamma_1(\omega, k_x), \quad \Gamma_1(\omega, k_x) = \frac{\epsilon_1(\omega, k_x) - 1}{\epsilon_1(\omega, k_x) + 1}
\]  

(34)

The last piece of information that we need to complete the calculation is the expression for the spectral density [37]

\[
\langle p^*_{\alpha}(\vec{r_0}) p_{\beta}(\vec{r_0}) \rangle \omega = \delta_{\alpha\beta} \frac{\hbar \alpha''(\omega)}{2T_p} \coth \left( \frac{\hbar \omega}{2T_p} \right)
\]  

(35)

where an isotropic particle, with temperature \( T_p \), has been assumed. Putting all pieces together, and using \( \Gamma_1(-\omega, -k_x) = \Gamma_1^*(\omega, k_x) \) we obtain the following expression for the components of \( \vec{F}_2 \):

\[
F_{2x}(z_0) = -\frac{\hbar}{\pi^2} \int_0^\infty d\omega \alpha''(\omega) \coth \left( \frac{\hbar \omega}{2T_p} \right) \int_{-\infty}^{\infty} dk_x dk_y \Gamma_1''(\omega, k_x) q k_x e^{-2qz_0}
\]  

(36)

\[
F_{2z}(z_0) = -\frac{\hbar}{\pi^2} \int_0^\infty d\omega \alpha''(\omega) \coth \left( \frac{\hbar \omega}{2T_p} \right) \int_{-\infty}^{\infty} dk_x dk_y \Gamma_1'(\omega, k_x) q^2 e^{-2qz_0}
\]  

(37)

where \( \Gamma_1'(\omega, k_x) \) and \( \Gamma_1''(\omega, k_x) \) are the real and imaginary parts of \( \Gamma_1(\omega, k_x) \), Eq (34). The total force acting on the particle is the sum of \( \vec{F}_1 \) and \( \vec{F}_2 \). In the next section we consider some specific examples.
V  Fluctuation - Induced Forces: Summary, Discussion and Examples

Let us summarize our general results for the fluctuation-induced forces, acting on a particle in the presence of drifting mobile carriers in the medium. The medium is described by the Drude model with drift, Eq (5), and two opposite limits were considered:

Model 1: Collisionless plasma ($\nu \to 0$): the dielectric function of the model is $\epsilon_1(\omega, k_x)$, Eq (20).

Model 2: Non-absorptive, "noiseless" lattice ($\epsilon''_L(\omega) \to 0$): the dielectric function of the model is $\epsilon_2(\omega, k_x)$, Eq (24).

The main reason for introducing these two limits, rather than dealing with the general model, Eq (5), is that the frequency of the fluctuation sources originating in the drifting plasma is Doppler shifted with respect to those originating in the lattice. Thus, while it is easy to write down the spectral density of the sources for the general case (this is just the sum of Eq (8) and Eq (9)), it would make the resulting expressions for the forces even more cumbersome and more difficult to analyze. We therefore prefer to clarify the basic physics of the problem using the two limiting models.

Let us first return to "Model 1". Since $\epsilon''_L(\omega) = \epsilon''_L(\omega, k_x)$, we have

$$\frac{\epsilon''_L(\omega)}{\epsilon_1(\omega, k_x)+1} = \frac{1}{2} \text{Im}[\epsilon_1(\omega, k_x) - 1] = \frac{1}{2} \Gamma''_1(\omega, k_x).$$

This identity enables one to write Eqs (28, 29) in terms of $\Gamma''_1(\omega, k_x)$. Adding to Eqs (28, 29) their counterparts in Eqs (36, 37) gives the final expressions for the components of the total force in "Model 1."

$$F_x(z_0) = \frac{\hbar}{\pi} \int_0^\infty d\omega \alpha''(\omega) \left[ \text{coth} \left( \frac{\hbar\omega}{2T_L} \right) - \text{coth} \left( \frac{\hbar\omega}{2T_p} \right) \right] \int_{-\infty}^{\infty} dk_x dk_y \Gamma''_1(\omega, k_x) q k_x e^{-2qz_0}. \tag{39}$$

$$F_z(z_0) = -\frac{\hbar}{\pi} \int_0^\infty d\omega \int_{-\infty}^{\infty} dk_x dk_y [\alpha'(\omega) \text{coth} \left( \frac{\hbar\omega}{2T_L} \right) \Gamma''_1(\omega, k_x) + \alpha''(\omega) \text{coth} \left( \frac{\hbar\omega}{2T_p} \right) \Gamma'_1(\omega, k_x)] q^2 e^{-2qz_0}. \tag{40}$$

Let us clarify a bit these expressions, starting with Eq (40). This normal-to-surface component is the generalization of the standard, equilibrium Lifshitz force between a particle and medium [37]. The generalization includes the effect of carrier drift in the medium and it allows for different temperatures of the medium and the particle. The first part of the force, proportional to $\text{coth}(\hbar\omega/2T_L)$, is due to the fluctuating field in the medium acting on the particle. The particle itself is "passive", hence $\alpha'(\omega)$. In the second part, proportional to $\text{coth}(\hbar\omega/2T_p)$, the fluctuating field originates in the particle and, after being "reflected" from the medium, acts back on the particle. Here the medium is passive, hence $\Gamma'_1$. Note that both coth-functions have in their argument the unshifted frequency $\omega$. This is because in "Model 1" the spontaneous
fluctuating sources of the medium reside in the lattice, which is at rest in the laboratory system (the particle is at rest as well). The Doppler shifted frequency $\omega_-$ appears only in $\Gamma_1$, which contains information on the effect of the drift on fluctuation dynamics.

The structure of Eq (39) is different. This equation describes the dissipative "drag" force, due to the current flow in the medium. For this force to exist both $\alpha''(\omega)$ and $\Gamma''_1$ (i.e., $\epsilon''_L$) must differ from zero. However, the "active" part of the system can be distinguished from the "passive" one by looking at the argument of the coth. The first term in Eq (39), proportional to $\coth \left( \frac{\hbar \omega}{2 T_L} \right)$, is due to the random sources in the medium, i.e., the medium is the emitter while the particle is the absorber (and vice versa for the second term).

To switch to "Model 2" the following replacements are required in Eqs (39,40): The lattice temperature $T_L$ is replaced by the temperature of the electron plasma $T_{el}$ and $\Gamma_1$ is changed to $\Gamma_2$, which is defined in Eq (34), with subscript 2 instead of 1. Furthermore, since the spontaneous sources in the medium now originate in the drifting plasma, the frequency in the argument of the corresponding cosh-function should be Doppler shifted. Thus, the counterparts of the Eqs (39,40) for "Model 2" read as:

$$F_x(z_0) = \frac{\hbar}{\pi} \int_0^\infty d\omega \alpha''(\omega) \int_{-\infty}^\infty dk_x dk_y \left[ \coth \left( \frac{\hbar \omega_+}{2 T_{el}} \right) - \coth \left( \frac{\hbar \omega_-}{2 T_{el}} \right) \right] \Gamma''_1(\omega, k_x) q k_x e^{-2qz_0}. \tag{41}$$

$$F_z(z_0) = -\frac{\hbar}{\pi} \int_0^\infty d\omega \int_{-\infty}^\infty dk_x dk_y \left[ \alpha'(\omega) \coth \left( \frac{\hbar \omega_+}{2 T_{el}} \right) + \alpha''(\omega) \coth \left( \frac{\hbar \omega_-}{2 T_{el}} \right) \right] \Gamma''_2(\omega, k_x) q^2 e^{-2qz_0}. \tag{42}$$

The above expressions for the forces resemble those obtained in the literature for the problem of non-contact friction, experienced by a particle moving above a medium at rest [4,7,8] (or, alternatively, the problem of "drag" exerted on the particle by a moving medium). Our problem, however, is different and so are the results. Since in our setup the plasma component is moving with respect to the lattice, the dielectric function governing dynamics of the fluctuations is in general a complicated function of $\omega$ and $k_x$. In addition, the frequency dependence of the random sources in the drifting plasma is different (Doppler shifted) with respect to those in the stationary lattice. Due to these factors the results are sensitive to the details of the model and can be quite diverse. For instance, in "Model 1" there are no "drag" at all, if the temperature of the sample and the particle are equal, see Eq (39) with $T_L = T_p$. This is because, as has been mentioned above, in "Model 1" the random sources, both in the medium and in the particle, are at rest. Only the dielectric function $\epsilon_1(\omega, k_x)$ is affected by the drift. Therefore the situation is the same as in equilibrium, but with a modified, $k_x$-dependent dielectric function of the medium.

To obtain specific results we need an explicit expression for the particle susceptibility $\alpha(\omega)$. We shall use the most simple, "generic" expression applicable
to a two-level system:

$$\alpha (\omega) = \frac{\alpha (0) \omega_0^2}{\omega_0^2 - \omega^2 - i\omega\eta}.$$  (43)

where \(\omega_0\) is the resonance frequency of the excitation and \(\eta\) is the decay rate. This expression is valid for an atom or a molecule when a single excitation is of importance. It is also applicable to a metallic or semiconducting (spherical) particle, in which case \(\alpha (0)\) is equal to the cube of the radius of the sphere and \(\omega_0 = \tilde{\omega}_p/\sqrt{3}\) is the frequency of the localized surface plasmon \[38\]. (Here \(\tilde{\omega}_p\) is the plasma frequency of the material of the particle). For a dielectric nanoparticle one may have some phonon mode or a phonon-polariton, instead of a plasmon. The value of \(\omega_0\) depends on the nature of the particle and can vary over a few orders of magnitude, say, between \(10^{12}\) and \(10^{16}\) sec\(^{-1}\).

In the weak dissipation (small \(\eta\)) limit the imaginary part of \(\alpha (\omega)\) is often approximated as

$$\alpha'' (\omega) = \frac{\alpha (0) \omega_0^2 \omega\eta}{(\omega_0^2 - \omega^2)^2 + (\omega\eta)^2} \Rightarrow \frac{\pi}{2} \alpha (0) \omega_0 \delta (\omega - \omega_0), \quad (\omega > 0).$$  (44)

However, one should keep in mind that, when \(\alpha'' (\omega)\) is integrated with some function of frequency \(f(\omega)\), the "\(\delta\)-function approximation" is valid only if \(f(\omega_0)\) is not negligibly small. Otherwise the integral will be dominated not by the peak of the Lorenzian in Eq (44) but by some other region of frequencies where \(f(\omega)\) is significant (albeit the Lorenzian is small). Below we shall encounter a situation where the integral is dominated by small frequencies and, correspondingly, the low-frequency expansion

$$\alpha'' (\omega) = \frac{\alpha (0) \omega\eta}{\omega_0^2}$$  (45)

will be used.

The same remark applies to \(\Gamma'' (\omega, k_x)\), defined in Eq (38) (and similarly for \(\Gamma'_2 (\omega, k_x)\), with \(\epsilon_2 (\omega, k_x)\) instead of \(\epsilon_1 (\omega, k_x)\)). In the small dissipation limit, the general expression can be approximated, in some cases, by the \(\delta\)-function

$$\Gamma'' (\omega_-) = \pi (1 - C) \delta (\omega_-^2 - \omega_{sp}^2), \quad \omega_{sp} = \frac{\omega_p}{\sqrt{\epsilon'_L + 1}}, \quad C \equiv \frac{(\epsilon'_L - 1)}{(\epsilon'_L + 1)}.$$  (46)

It is assumed here that the relevant frequencies are far from the resonant frequencies of the lattice and \(C\) can be treated as a constant, hence \(\omega\) and \(k_x\) are combined into a single argument \(\omega_-\). In the \(\delta\)-function approximation there is no difference between \(\Gamma_1\) and \(\Gamma_2\), so that the subscript has been removed. Note that Eq (46) does not explicitly contain \(\epsilon'_L\) or \(\nu\) although some dissipation, albeit infinitely small, is essential. Since, however, in reality the dissipation is finite, the \(\delta\)-function approximation has its limitations and, in particular, below we shall need the small frequency approximation

$$\Gamma''_2 (\omega_-) = \frac{2\nu \omega_-}{\omega_p^2}.$$  (47)
We are now in a position to work out some examples of the drift effect on fluctuation induced forces. The most interesting effect is the appearance of the aforementioned drag force.

V.1 Drag force in "Model 1"

In this model the drag force on a particle appears only if \( T_p \) and \( T_L \) are different, see Eq (59). Note that if \( T_p \) and \( T_L \) are reversed, the force changes sign, i.e., drag (force in the direction of the current) turns into "anti-drag" (force in the opposite direction) \([39]\). Let us calculate the force \( F_x \) using the \( \delta \)-approximation for \( \alpha''(\omega) \), Eq (44). This is justified because the integral is dominated by the peak of the Lorenzian. The \( \delta \)-function takes care of the integral over \( \omega \) in Eq (39), and we have to address the integral over \( k_x, k_y \), with \( \Gamma''(\omega, k_x) \). The latter quantity is defined in (38). Since \( \epsilon''_L \) is a small number, \( \Gamma''(\omega, k_x) \) has a sharp maximum when \( \epsilon''_L(\omega, k_x) + 1 = 0 \). This happens at \( k_x = k^\pm_x = (\omega_0 \pm \omega_{sp})/v_0 \). One can try to approximate the Lorenzian function in Eq (38) by the \( \delta \)-function, Eq (46), thus obtaining

\[
\Gamma''(\omega, k_x) = \frac{\pi}{\epsilon''_L + 1} \frac{\omega_{sp}}{v_0} [\delta(k_x - k^+_x) + \delta(k_x - k^-_x)].
\] (48)

In order to see how good is this approximation one must keep in mind that, due to the exponential factor, the integrand in Eq (39) has a sharp cutoff at \( k_x \sim 1/z_0 \), hence the \( \delta \)-approximation will be justified only if at least one of the roots \( k^\pm_x \) is below the cutoff—otherwise the contribution from the peak of \( \Gamma''(\omega, k_x) \) is exponentially small (we assume here that both roots are positive). The \( \delta \)-approximation is always justified for sufficiently large \( v_0 \) but the precise criterion depends on the values of \( \omega_{sp}, \omega_0 \) and \( z_0 \). For an atom \( \omega_0 \) is typically much larger than \( \omega_{sp} \) of the semiconducting medium but for a large molecule or a nanoparticle (dielectric or semiconducting) the two frequencies can be of the same order. We assume that \( \omega_0 \) is few times larger than \( \omega_{sp} \) and obtain the condition \( \omega_0 z_0/v_0 \ll 1 \) for the validity of the \( \delta \)-approximation. The force \( F_x \) is then estimated from (39) as

\[
F_x \sim \frac{\hbar \alpha(0)}{\epsilon''_L + 1} \frac{\omega_{sp}}{v_0 z_0} \left[ \frac{\omega_0}{v_0 z_0} \right]^2 \left[ \coth \left( \frac{\hbar \omega_0}{2 T_L} \right) - \coth \left( \frac{\hbar \omega_0}{2 T_p} \right) \right], \quad (v_0 \gg \omega_0 z_0).
\] (49)

In this regime \( F_x \) drops as \( v_0^{-2} \) under increase of the drift velocity. It achieves its maximum value for \( v_0 \sim \omega_0 z_0 \), at which point the \( \delta \)-approximation breaks down. For a hot medium, \( (\hbar \omega_0/2 T_L) \ll 1 \), and a "cold particle", \( (\hbar \omega_0/2 T_p) \gg 1 \), this maximum value is of the order of \( \alpha(0) \omega_{sp} T_L/\omega_0 z_0^2 (\epsilon''_L + 1) \) which is comparable with the equilibrium Casimir-Lifshitz force.

In the opposite case of small drift velocities the \( \delta \)-approximation breaks down and the integral is dominated by small \( k_x \). \( \Gamma''(\omega_0 - k_x v_0) \) should then be expanded near the point \( \omega_0 \) with respect to \( k_x v_0 \). The expansion has a linear term, unless \( \omega_{sp} = \omega_0 \) when the first correction is quadratic in \( v_0 \). We assume to be well away from this point, taking \( \omega_0 \) few times larger than \( \omega_{sp} \). The first
contribution to $F_x$ comes then from the linear term which, after substitution into (39) and integration over $k_x, k_y$, yields

$$F_x = \frac{3\hbar v_0 \alpha(0)}{z_0^3} \frac{\epsilon_L'}{(\epsilon_L' + 1)^2} \left( \frac{\omega_{sp}}{\omega_0} \right)^2 \left[ \coth \left( \frac{\hbar \omega_0}{2T_L} \right) - \coth \left( \frac{\hbar \omega_0}{2T_p} \right) \right], \quad (v_0 \ll \omega_0 z_0).$$

(50)

Note that the two expressions, Eqs (49) and (50), do not match at $v_0 \sim \omega_0 z_0$, which means that there is an intermediate region where $F_x$ sharply increases, interpolating between the small and large velocity limits. The overall behavior of $F_x$, as a function of $v_0$, is schematically sketched in Fig. 3. If one takes $\omega_0 \sim 10^{12} \text{sec}^{-1}$ and $z_0 \sim 10 \text{nm}$, then the critical drift velocity for which the maximal value of the force is reached, is $v_{0c} \sim 10^6 \text{cm/sec}$. This is comparable to a typical saturation velocity in semiconductors.

Figure 3: A qualitative plot of the drag force in "Model 1", as a function of $v_0$. The force is given in units of its maximal value, see text. For small $v_0$ the behavior is linear. The large-$v_0$ asymptotic is $1/v_0^2$.

V.2 Drag force in "Model 2"

The expression for the force is given in Eq (41). Due to the Doppler shifted frequency in the argument of the first coth, the force exists also when the medium and the particle have equal temperatures, $T_L = T_p \equiv T$, and we concentrate on this case. The most interesting limit is $T \to 0$ ("quantum drag"). In this limit the difference between the two coth functions in Eq (41) is equal to $-2$ for $0 < \omega < k_x v_0$ and it is zero otherwise (recall that $\omega$ must be positive). Thus, Eq (41) reduces to

$$F_x(z_0) = -\frac{2\hbar^2}{\pi^2} \int_{-\infty}^{\infty} dk_x dk_y \int_0^{k_x v_0} d\omega \alpha''(\omega) \Gamma''_2(\omega) q k_x e^{-2qz_0}.$$  

(51)
One might be tempted to approximate the Lorenzian $\alpha''(\omega)$ by the $\delta$-function in Eq (44). This, however, is possible only if the upper limit in the integral over $\omega$ is larger than the center $\omega_0$ of the Lorenzian peak. Since the relevant values of $k_x$ are smaller than $1/\omega_0$, we arrive again to the parameter $(v_0/\omega_0 z_0) \equiv \gamma$. Only if this parameter is large can one justify the "$\delta$-approximation".

For small value of $\gamma$ the upper limit in the integral over frequency is smaller than $\omega_0$ and one should use the low-frequency expressions for $\alpha''(\omega)$ and $\Gamma''_2(\omega)$, Eqs (45) and (47) respectively. Note that in the integration region over the frequency, $\omega_-$ is negative, and so is $\Gamma''_2(\omega_-)$. The integral over $\omega$ is proportional to $v_0^3$ and the force is of the order of

$$F_x \sim \frac{\hbar \alpha(0) \nu v_0^3}{\omega_0^2 \omega_p^2} \quad (v_0 \ll \omega_0 z_0).$$

(52)

For large $\gamma$ the main contribution to the integral over frequencies in Eq (51) comes from high frequencies, $\omega \simeq \omega_0$, and the $\delta$-approximation for $\alpha''(\omega)$ is valid. Thus, first one integrates over $\omega$ and then over $k_x$, using the $\delta$-approximation for $\Gamma''_2(\omega_0 - k_x v_0)$. Due to the restriction $\omega_0 < k_x v_0$, $\Gamma''_2(\omega_0 - k_x v_0)$ is now negative and its argument has only one root, namely, $k_+ = (\omega_0 + \omega_{sp})/v_0$. Assuming again that $\omega_0$ is few times larger than $\omega_{sp}$, we arrive to a simple estimate

$$F_x(z_0) \sim \frac{\hbar \alpha(0)}{\epsilon_L^2 + 1} \omega_{sp} \left( \frac{\omega_0}{v_0 z_0} \right)^2, \quad (v_0 \gg \omega_0 z_0).$$

(53)

The fundamental difference between this expression and its counterpart in "Model 1", Eq (49), is that Eq (53) was derived in the zero-temperature limit, when Eq (49) (as generally for equal temperatures of the particle and the medium) is zero. The qualitative behavior of $F_x$ in Eq (53), as a function of $v_0$, is similar to that shown in Fig.3, although the initial slope is less steep (proportional to $v_0^3$ instead of being linear). The maximal value of the force, $F_x(z_0) \sim \hbar \alpha(0) \omega_{sp}/z_0^2$, is achieved for $v_0 \sim \omega_0 z_0$ (in this estimate we take $\omega_0$ to be few times larger than $\omega_{sp}$ and assume $\epsilon_L' \sim 1$). This force is of the same magnitude as the usual Casimir-Lifshitz attraction force between a particle and a medium, in equilibrium.

V.3 Effect of drift on $F_z$

Unlike the lateral force $F_x$, the normal (Casimir-Lifshitz) force exists already in the equilibrium. This force, however, is affected by the drift of the mobile carriers. To concentrate exclusively on the effect of drift we take $T_L = T_p \equiv T$ and consider "Model 1", Eq (40). In this case the integral over frequencies can be reduced to a Matsubara sum, in spite of the fact that the system is not in equilibrium. Indeed, $\text{coth} (\hbar \omega/2T)$ becomes a common factor for both terms in Eq (40) and they can be combined into an expression containing $Im [\alpha(\omega) \Gamma_1(\omega, k_x)]$ which results in a Matsubara sum

$$F_z(z_0) = -\frac{\hbar}{\pi^2} \frac{2\pi T}{\hbar} Re \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y \alpha(i\zeta_n) \Gamma_1(i\zeta_n, k_x) q^2 e^{-2qz_0},$$

(54)
where \( \zeta_n = 2\pi T_n/h \) \((n = 0, 1, \ldots)\) and the prime on \( \sum \) indicates that the \( n = 0 \) term should be taken with a factor \( 1/2 \). In equilibrium \( \alpha (i\zeta_n) \) and \( \Gamma_1 (i\zeta_n) \) are real so that the sign "Re" in front of the sum becomes redundant. In the presence of drift, however, \( \Gamma_1 \) acquires \( k_x \)-dependence and becomes complex on the imaginary frequency axis. Neglecting small dissipation, we have

\[
\alpha (i\zeta) = \frac{\alpha (0) \omega_0^2}{\omega_0^2 + \zeta^2}, \quad \Gamma (i\zeta, k_x) = \frac{C((i\zeta - k_x v_0)^2 - \omega_{sp}^2)}{(i\zeta - k_x v_0)^2 - \omega_{sp}^2},
\]

(55)

where again \( C \), which is generally a function of frequency, is treated here as a constant.

In the low-\( T \) limit, i.e., \( T \ll \omega_0, \omega_{sp} \) (these are frequency scales on which \( \alpha \) and \( \Gamma \) changes significantly), the sum can be replaced by an integral according to the rule

\[
\sum'_{n} \alpha (i\zeta_n) \Gamma_1 (i\zeta_n, k_x) = \frac{\hbar}{\pi^2} T \int_0^\infty d\zeta \alpha (i\zeta) \Gamma_1 (i\zeta, k_x), \quad \text{i.e.,}
\]

\[
F_z (z_0) = -\frac{3\hbar}{4\pi z_0} Re \int_0^\infty d\zeta \alpha (i\zeta) \int_{-\infty}^\infty dk_x dk_y \Gamma_1 (i\zeta, k_x) q^2 e^{-2q z_0}.
\]

(56)

One can compute the correction to the force, due to carrier drift, by expanding \( \Gamma_1 (i\zeta_n, k_x) \) in powers of \( v_0 \). The zero-order term corresponds to equilibrium, when \( \Gamma_1 (i\zeta_n, k_x) \) does not depend on \( k_x \) and

\[
F_z^0 (z_0) = -\frac{3\hbar}{4\pi z_0} \int_0^\infty d\zeta \alpha (i\zeta) \Gamma_1 (i\zeta, 0) = -\frac{3\hbar}{8\pi z_0} \alpha (0) \omega_0 \omega_{sp} v_0.
\]

(57)

For \( C = 0 \) this coincides with the well known expression for the attraction force between a "two-level atom" and a collisionless plasma \([37]\). The coefficient \( C \) in Eq (57) accounts for the effect of the lattice. The first order correction, i.e., the one linear in \( k_x v_0 \), does not contribute to the force. The second order correction

\[
\Delta \Gamma (i\zeta, k_x) = (1 - C) \frac{\omega_{sp}^2}{\omega_{sp}^2 + \zeta^2} \frac{(k_x v_0)^2 \omega_{sp}^2 - 3\zeta^2}{\omega_{sp}^2 + \zeta^2},
\]

(58)

contributes to Eq (56) the term

\[
\Delta F_z (z_0) = -\frac{15}{16z_0^2 h\alpha (0)} \frac{\omega_0 \omega_{sp} v_0^3}{(\omega_0 + \omega_{sp})^3}.
\]

(59)

Assuming that \( C \) is not close to 1 and taking, as before, \( \omega_0 \) to be few times larger than \( \omega_{sp} \), one recovers the same condition \( v_0 \ll \omega_0 z_0 \) for the validity of the expansion.

Eq (59) can be derived directly from Eq (40), without using the Matsubara representation, although the latter is more flexible when it comes to non-negligible dissipation and arbitrary temperatures. The transformation of the expression in Eq (40) to the Matsubara sum was possible because in "Model 1" (and for \( T_L = T_p \) the spontaneous fluctuation sources are in equilibrium.
at the same temperature, only the (noiseless) plasma component is in motion. This is not the case for "Model 2", where the sources originating in the moving plasma are Doppler shifted, so one has to work directly with the expression in Eq (42). For small drift velocities and in the weak dissipation limit, when the \( \delta \)-approximation for \( \alpha'' \) and \( \Gamma''_2 \) can be used, the calculation is quite straightforward and will not be pursued here. Instead, we briefly discuss the case when the sample and the particle have different temperatures but there is no drift. Then, for negligible dissipation, Eqs (40) and (42) become identical and the result is the same for either model:

\[
F_z(z_0) = -\frac{3\hbar}{8z_0^4}\frac{\alpha(0)\omega_0}{\omega_0^2 - \omega_{sp}^2} \left[ \frac{\hbar\omega_{sp}}{2T_L} (1 - C) \coth\left(\frac{\hbar\omega_{sp}}{2T_L}\right) + (C\omega_0^2 - \omega_{sp}^2) \coth\left(\frac{\hbar\omega_0}{2T_p}\right) \right].
\]

This is a slight generalization of the result obtained in [16] where \( C = 0 \), i.e., \( \epsilon'_L = 1 \). This latter case is appropriate for the free electron gas model, while the expression \( (60) \) includes the effect of the underlying lattice. The constant \( C \) can vary between 0 and 1, and for a typical semiconductor, in broad intervals of frequencies, it can be few tenths or even close to 1, so its effect is quite significant. The \( (C\omega_0^2) \)-term in \( (60) \) can become the dominant one. For instance, taking the low temperature limit, i.e., replacing the \( \coth \) factors by 1, and assuming \( \omega_{sp} \ll \omega_0 \), one obtains \( F_z = -\left(3\hbar/8z_0^4\right)\alpha(0)\omega_0 \). This should be compared with \( F_z = -\left(3\hbar/8z_0^4\right)\alpha(0)\omega_{sp} \) for the electron gas model under the same conditions. The interesting feature, pointed out already in [16], is that, depending on the parameters of the model, the force can be either repulsive or attractive.

VI Conclusion

We have studied the fluctuation-induced forces acting on a small polarizable neutral particle (atom, molecule or a nanoparticle), located close to the surface of a conducting medium. It is shown that presence of a dc current (i.e., the mobile carrier drift) in the medium can have a significant effect on the forces. In particular, there appears a lateral force which can be in the direction of the current (drag) or in the opposite direction (anti-drag). This phenomenon is distinct from the well studied Coulomb drag [40], when current in a conductor induces a current (or voltage) in a nearby conductor. In our case the force is exerted on a small polarizable object, with a well defined excitation, at some frequency \( \omega_0 \). This can be the resonant frequency of an atom or the frequency of a localized surface plasmon of a nanoparticle. The resulting drag force is a non-monotonic function of the carrier drift velocity \( v_0 \) and it reaches a maximal value at \( v_0 \) of the order of \( \omega_0z_0 \). The maximal value of the force is not small, in the sense that it is comparable to the normal (Casimir-Lifshitz) force in equilibrium.

Formulas for the forces, obtained in the present work, resemble those which appear in the theory of non-contact friction (item (ii) in the Introduction). The two problems, however, are different. In our problem both the particle and
the sample are at rest, in the laboratory frame, only the mobile charge carriers are drifting. Our results depend on whether the random spontaneous sources reside in the lattice or in the electron plasma (Models 1 and 2, respectively). If there is no dissipation in the lattice, \( \epsilon''_L(\omega) = 0 \) (Model 2) and \( \epsilon'_L(\omega) = \text{const} \), then the dielectric function of the medium (lattice + plasma) is a function of \( \omega - k_x v_0 \) only and, since the random sources are located in the drifting plasma, the situation becomes as close as possible to the case of a medium moving as a whole. However to make the analogy complete one needs an additional strong requirement, namely, that the electrons in the drifting plasma could be considered as being in an internal equilibrium, with some effective temperature \( T_{el} \). Otherwise one cannot use Rytov’s theory for correlation functions of the random sources.

We limited our considerations to the simplest models and conditions and did not attempt possible generalizations and extensions, like treating the general case (Eq 5 with both \( \epsilon''_L \) and \( \nu \) finite), or going beyond weak dissipation limit, or including the retardation effects. Finally, let us stress that the high drift velocities, needed to make the discussed effects visible, can be achieved only in materials with low carrier density, like semiconductors, ionic conductors or other types of "bad metals".

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