STABILITY OF BERNSTEIN TYPE THEOREM FOR THE MINIMAL SURFACE EQUATION

GUOSHENG JIANG, ZHEHUI WANG, AND JINTIAN ZHU

Abstract. Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$) be an unbounded convex domain. We study the minimal surface equation in $\Omega$ with boundary value given by the sum of a linear function and a bounded uniformly continuous function in $\mathbb{R}^n$. If $\Omega$ is not a half space, we prove that the solution is unique. If $\Omega$ is a half space, we prove that graphs of all solutions form a foliation of $\Omega \times \mathbb{R}$. This can be viewed as a stability type theorem for Edelen-Wang’s Bernstein type theorem in [9]. We also establish a comparison principle for the minimal surface equation in $\Omega$.

1. Introduction

In the research of minimal graphs over Euclidean space, one important result is the Bernstein theorem, which says that entire minimal graphs over $\mathbb{R}^n$ with $2 \leq n \leq 7$ are hyperplanes (refer to [4, 7, 1, 21]). For $n \geq 8$, there were non-planar entire minimal graphs constructed by Bombieri, De Giorgi and Giusti [5] and so the Bernstein theorem fails in higher dimensions. Just recently, Edelen and the second named author [9] established a Bernstein type theorem for minimal graphs over unbounded convex domains. In any dimension they can show that if the boundary of a minimal graph over an unbounded convex domain $\Omega \subseteq \mathbb{R}^n$ is contained in a hyperplane, then so is the minimal graph itself.

Here we would like to interpret the Edelen-Wang’s theorem from the view of partial differential equations, and the situation can be divided into two cases:

- If the unbounded convex domain $\Omega \subseteq \mathbb{R}^n$ is not a half space, the Edelen-Wang’s theorem can be understood as the existence and uniqueness for solutions of the minimal surface equation in $\Omega$ with a linear boundary value.
- If the unbounded convex domain $\Omega \subseteq \mathbb{R}^n$ is a half space, the uniqueness fails but all solutions form a one-parameter family, whose graphs form a foliation of $\Omega \times \mathbb{R}$.

This interpretation motivates us to further study the stability of the Edelen-Wang’s theorem. We are going to show that the existence, uniqueness and foliation structure are preserved even if the linear boundary value is perturbed by a bounded and uniformly continuous function.

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In order to state our main theorem, we introduce some necessary notation. In the rest of this paper, let \( n \geq 2 \) and \( \Omega \subseteq \mathbb{R}^n \) be an unbounded domain. Moreover, let \( l: \mathbb{R}^n \to \mathbb{R} \) be a linear function and \( \phi: \mathbb{R}^n \to \mathbb{R} \) a bounded and uniformly continuous function. In the following, we consider the minimal surface equation

\[
\mathcal{M}u := \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \Omega
\]

with the Dirichlet boundary value

\[
u = l + \phi \quad \text{on } \partial \Omega.
\]

Now our main theorem can be stated as following.

**Theorem 1.1.** The following statements are true:

1. If \( \Omega \subseteq \mathbb{R}^n \) is a convex domain but not a half space, then \((1.1)-(1.2)\) has a unique solution.
2. If \( \Omega \subseteq \mathbb{R}^n \) is a half space, then all solutions of \((1.1)-(1.2)\) form a one-parameter family and the graphs of these solutions form a foliation of \( \Omega \times \mathbb{R} \).

When \( \Omega \) is a convex domain but not a half space, the existence comes from the standard exhaustion argument while the uniqueness part turns out to be much more difficult. Before our work, the discussions on the uniqueness mainly focus on dimension two. In particular, Nitsche [19] came up with the following conjecture.

**Conjecture.** Suppose \( D \subset \mathbb{R}^2 \) is contained in a wedge with opening angle less than \( \pi \). Then, the solution of the minimal surface equation with continuous boundary value is unique.

It turns out that the original Nitsche’s conjecture is too ideal. Actually, Collin [6] provided a counterexample indicating that the uniqueness can not hold if the boundary value grows too fast. On the other hand, after further requiring the boundary value to be bounded, Hwang [12] and Mikljukov [17] confirmed Nitsche’s conjecture independently. Moreover, when \( D \subset \mathbb{R}^2 \) is the union of a compact convex subset with finitely many disjoint half strips attached to its boundary, Sa Earp and Rosenberg [20] proved the uniqueness of the solution of the minimal surface equation with bounded uniformly continuous boundary value.

For higher dimensions, Massari and Miranda [16] showed the existence of a solution of the minimal surface equation in unbounded convex domain with any continuous boundary value, but the uniqueness problem remained unsolved. Again Collin’s counterexample suggests that the uniqueness needs to be considered with boundary values satisfying controlled growth (e.g. bounded). With a further limit to bounded uniformly continuous boundary values, our work gives a partial answer to Massari-Miranda’s uniqueness
problem. Namely, we have the following immediate corollary of Theorem 1.1.

**Corollary 1.2.** If $\Omega \subsetneq \mathbb{R}^n$ is a convex domain but not a half space, then the solution of the minimal surface equation in $\Omega$ with bounded and uniformly continuous boundary value is unique.

In our proof, the uniqueness comes from the following more general comparison theorem for the minimal surface equation.

**Theorem 1.3.** Assume that $\Omega \subsetneq \mathbb{R}^n$ is a convex domain but not a half space and that $u_1$ and $u_2$ are two solutions of equation (1.1) satisfying the boundary value

$$u_i = l_i + \phi_i \text{ on } \partial \Omega, \ i = 1, 2,$$

where $\phi_1$ and $\phi_2$ are bounded and uniformly continuous functions. If $u_1 \leq u_2$ on $\partial \Omega$, then $u_1 \leq u_2$ in $\Omega$.

Based on the work in [9], we can reduce above theorem to the following special one. We say that a domain $\Omega$ satisfies the *exterior cone property* if there is an infinity cone outside $\Omega$.

**Proposition 1.4.** Let $\Omega$ be an unbounded domain with the exterior cone property. Assume that $u_1$ and $u_2$ are two solutions of equation (1.1) satisfying

$$u_i = l + \phi_i \text{ on } \partial \Omega, \ i = 1, 2,$$

and

$$\|u_i - l\|_{C^0(\bar{\Omega})} < +\infty, \ i = 1, 2,$$

where $\phi_1$ and $\phi_2$ are bounded and uniformly continuous functions with $\phi_1 \leq \phi_2$. Then $u_1 \leq u_2$.

When $\Omega$ is a half space, we can construct a family of solutions of (1.1)-(1.2) characterized by their growth rate at infinity. In fact, the graph of such a solution has a unique "approximate hyperplane" in the form of $\{x : l(x) + cx_n = 0\}$ (refer to Proposition 2.3). On the other hand, given any solution of (1.1)-(1.2), we can prove that it must belong to the family of solutions in our construction (refer to Corollary 5.3). Hence $c$ can serve as a parametrization for all solutions, and we denote by $u_c$ the solution having "approximate hyperplane" in the form of $\{x : l(x) + cx_n = 0\}$. The precise meaning of "foliation" in Theorem 1.1 is that the map

$$\Phi : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}^n_+ \times \mathbb{R}, \ (x, c) \mapsto (x, u_c(x))$$

can be shown to be a homeomorphism (refer to Proposition 5.4). Moreover, the homeomorphism can be improved to be a $C^1$-diffeomorphism after changing parametrization if $\phi$ is $C^1$ with $\|\phi\|_{C^1(\partial \mathbb{R}^n_+)} < \infty$ (refer to Proposition 5.5).
The rest of this paper is organized as follows. Section 2 contains the proof of the existence of (1.1)-(1.2) when \( \Omega \subset \mathbb{R}^n \) is an unbounded convex domain. In Section 3, we present a decay or blow up alternative for linear elliptic equations of divergence form, which will be used later. In Section 4, we prove the first part of Theorem 1.1 while the second part is proved in Section 5.

2. Existence: the exhaustion method

2.1. Construction of exhaustion domains. Here we devote to show the following proposition.

**Proposition 2.1.** If \( \Omega \subset \mathbb{R}^n \) is an unbounded convex domain, then there is an exhaustion \( \{ \Omega_k \}_{k \geq 1} \) of \( \Omega \) satisfying the following

- \( \Omega_k \) is smooth and convex;
- \( \Omega_k \subset B_k \cap \Omega \);
- \( \Omega_k \subset \Omega_{k+1} \).

**Proof.** Set \( U_k = B_k \cap \Omega \) and \( d_k(x) := \text{dist}(x, \partial U_k) \). We note that \( d_k \leq d_{k+1} \), \( U_k \) is a convex domain, and \( d_k \) is a convex function. By [2, Theorem 1], there is a smooth convex function \( f_k : U_k \to \mathbb{R} \), such that for any \( x \in U_k \),

\[
d_k(x) - \frac{1}{k^2} \leq f_k(x) \leq d_k(x).
\]

Let

\[
\Omega_k := \left\{ x \in U_k : f_k(x) > \frac{1}{k} \right\}.
\]

We note \( \Omega_k \) is a convex subset of \( U_k \), and without loss of generality we can assume it is also smooth by Sard’s theorem. Let us show \( \{ \Omega_k \}_{k \geq 1} \) forms an exhaustion of \( \Omega \). For any \( x \in \Omega \), there is a \( k \) large enough such that \( x \in U_k \) and \( d_k(x) > 2/k \). Then we have \( f_k(x) > 1/k \) and \( x \in \Omega_k \). Therefore, \( \bigcup_{k \geq 1} \Omega_k = \Omega \). Moreover, for any \( x \in \Omega_k \), we have

\[
f_{k+1}(x) \geq d_{k+1}(x) - \frac{1}{(k+1)^2} \geq d_k(x) - \frac{1}{(k+1)^2} \geq f_k(x) - \frac{1}{(k+1)^2} > \frac{1}{k+1}.
\]

As a consequence, \( x \in \Omega_{k+1} \) and then we have \( \Omega_k \subset \Omega_{k+1} \). \( \square \)

2.2. Proof of existence. Here we present the proof of the existence of (1.1)-(1.2). We gives the detail for the case when \( \Omega \subset \mathbb{R}^n \) is an unbounded convex domain but not a half space, and we omit the half space case since it is similar.

**Proposition 2.2.** If \( \Omega \subset \mathbb{R}^n \) is a convex domain but not a half space, then there is a solution \( u \) of (1.1)-(1.2) with

\[
\|u - l\|_{C^0(\Omega)} \leq \|\phi\|_{C^0(\mathbb{R}^n)}.
\]
Proof. Let \( \{\Omega_k\} \) be the exhaustion constructed in Proposition 2.1, we consider Dirichlet problems:

\[
\mathcal{M}u_k = 0 \text{ in } \Omega_k; \quad u_k = l + \phi \text{ on } \partial \Omega_k.
\]

By the maximum principle, we have for any \( k > 0 \) and \( x \in \Omega_k \),

\[
l(x) - \|\phi\|_{C^0(\mathbb{R}^n)} \leq u_k(x) \leq l(x) + \|\phi\|_{C^0(\mathbb{R}^n)}.
\]

For any \( \Omega' \subset \subset \Omega_k \), by the interior gradient estimate of the minimal surface equation (see [10, Theorem 3.2.3]), and \( \operatorname{osc} u_k \leq 2\|\phi\|_{C^0(\mathbb{R}^n)} \), we have

\[
\sup_{x \in \Omega'} |\nabla u_k(x)| \leq C(n, \|\phi\|_{C^0(\mathbb{R}^n)}, \text{dist}(\Omega', \partial \Omega)).
\]

Fixing any \( \Omega' \subset \subset \Omega \), we choose \( k_0 > 1 \) large enough such that \( \Omega' \subset \subset \Omega_{k_0} \), then \( \{u_k\}_{k \geq k_0} \) is well defined in \( \Omega' \). By the Schauder estimate to \( u_k - l \), we have

\[
\|u_k - l\|_{C^{2,\alpha}(\Omega')} < C(n, \|\phi\|_{C^0(\mathbb{R}^n)}, \text{dist}(\Omega', \partial \Omega)).
\]

Hence after pass to a subsequence, \( \{u_k\}_{k \geq 1} \) converges to a \( C^2(\Omega) \) function and we denote it by \( u \). It is obvious that \( \mathcal{M}u = 0 \) in \( \Omega \) and

\[
\|u - l\|_{C^0(\Omega)} \leq \|\phi\|_{C^0(\mathbb{R}^n)}.
\]

Fix any \( x_0 \in \partial \Omega \). For any \( \varepsilon > 0 \), there is a constant \( \delta > 0 \), such that, for any \( x \in B_\delta(x_0) \cap \overline{\Omega} \),

\[
|l(x) - l(x_0)| + |\phi(x) - \phi(x_0)| < \varepsilon,
\]

and we fix such a point \( x \).

Take \( \{x_k\}_{k \geq 1} \) with \( x_k \in \partial \Omega_k \) and \( x_k \rightarrow x_0 \) as \( k \rightarrow \infty \). Note there is \( k_1 > 1 \) large enough such that for all \( k \geq k_1 \), we have \( |x_k - x_0| < \delta \), \( x \in \Omega_k \) and

\[
|u(x) - u_k(x)| < \varepsilon.
\]

By the estimate of the modulus of continuity of solutions for the minimal surface equation in bounded domain (see [11, Theorem 3.2.3]), we can assume

\[
|u_k(x) - l(x_k) - \phi(x_k)| < \varepsilon.
\]

Then,

\[
|u(x) - l(x_0) - \phi(x_0)| \leq |u(x) - u_k(x)| + |u_k(x) - l(x_k) - \phi(x_k)| + |l(x_k) + \phi(x_k) - l(x_0) - \phi(x_0)| < 3\varepsilon.
\]

Hence, \( \lim_{x \rightarrow x_0, x \in \Omega} u(x) = l(x_0) + \phi(x_0) \).

With a similar argument we can show

**Proposition 2.3.** If \( \Omega \) is the half space \( \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\} \), then for any real constant \( c \) there is a solution \( u_c \) of (1.1) with (1.2) with

\[
\|u_c - l - cx_n\|_{C^0(\Omega)} \leq \|\phi\|_{C^0(\mathbb{R}^n)}.
\]
3. Linear Theory: Decay or Blow Up Alternative

In this section, we will consider the following equation
\begin{equation}
L u := \text{div}(A(x) \nabla u) = 0 \text{ in } \Omega
\end{equation}
with the condition
\begin{equation}
u = 0 \text{ on } \partial \Omega, \text{ and } u > 0 \text{ in } \Omega,
\end{equation}
where
\[ A(x) \in L^\infty(\Omega), \lambda I \leq A \leq \lambda^{-1} I, \text{ and } \lambda \in (0, 1) \text{ is a constant}. \]

First we mention that (3.1)-(3.2) has been studied in special cases when the domain \( \Omega \) is an infinite cone or an infinite cylinder (see [14] and [3] respectively). In both cases, it was proved that the dimension of the space consisting of all solutions is determined by the number of ends of the underlying domain \( \Omega \). This philosophy, however, remains to be an open problem when \( \Omega \) turns out to be a general unbounded domain. One cannot apply those methods from [14, 3] to general cases. Indeed, their arguments rely heavily on the scaling or translating invariance of the underlying domain \( \Omega \), which guarantees that a Harnack inequality with a uniform constant even holds around the infinity. Readers can turn to our brief discussion in Appendix A for a quick feeling.

We are more concerned with the blow up phenomenon of the solution of (3.1)-(3.2), which is closely related to our work. For solutions to the equation (3.1) in exterior domains, Moser [18] established the Hölder decay or blow up alternative on the oscillation as an application of the Harnack inequality (with a uniform constant around infinity). From the same technique, such alternative can be established for solutions to (3.1)-(3.2) (see Lemma A.5) by the application of boundary Harnack inequality when the domain \( \Omega \) is an infinite cone or an infinite cylinder.

Here we would like to deal with more general case, where the domain \( \Omega \) satisfies the exterior cone property, i.e. there is an infinite cone outside \( \Omega \). Given an unbounded domain \( \Omega \), we use \( \Omega_{ext} \) and \( (\partial \Omega)_{ext} \) to denote \( \Omega - K \) and \( \partial \Omega - K \) for some fixed compact set \( K \). The main result in this section is the following theorem.

**Theorem 3.1.** Let \( \Omega \) be an unbounded domain satisfying the exterior cone property and \( u \in W_{loc}^{2,p}(\Omega_{ext}) \cap C^0(\Omega_{ext}) \) be a solution of the equation (3.1) in \( \Omega_{ext} \) with \( u = 0 \) on \( (\partial \Omega)_{ext} \) and \( u > 0 \) in \( \Omega_{ext} \). Then the function
\[
\text{osc}(r) = \max_{\Omega \cap \partial B_r} u
\]
has a limit as \( r \to +\infty \), and the limit is either \( +\infty \) or \( 0 \).

As a preparation, we have

**Proposition 3.2.** If \( \Omega \) is an unbounded domain satisfies the exterior cone property, then any solution of (3.1)-(3.2) must be unbounded.
Figure 1. Extend domain $\Omega$ to infinite cones

**Proof.** As shown in Figure 1, we can extend the domain $\Omega$ to infinite cones $C_1$ and $C_2$ from the exterior cone property. Let us define

$$\tilde{A}(x) = \begin{cases} A(x), & x \in \Omega; \\ I, & x \in C_2 - \Omega, \end{cases}$$

and consider (3.1)-(3.2) in the infinite cone $C_2$. From Theorem A.2, we can obtain a positive solution $w$ and we would like to show the following estimate

$$m(r) := \inf_{C_1 \cap S_r} w \to +\infty, \text{ as } r \to +\infty,$$

where $S_r$ is the sphere with radius $r$ centered at the pole of cone $C_1$. Otherwise, the restriction $w|_{\partial C_1}$ is a bounded continuous function on $\partial C_1$. From the standard exhaustion method, it is not difficult to construct a bounded harmonic function $v$ in $C_2 - \hat{C}_1$ such that $v = 0$ on $\partial C_2$ and $v = w|_{\partial C_1}$ on $\partial C_1$ as well as $v \leq w$. From the uniqueness and Corollary A.6, the function $w$ is unbounded in $C_2 - \hat{C}_1$ and so $w - v$ is a positive harmonic function in $C_2 - \hat{C}_1$ with vanishing boundary value. It follows from Corollary A.3 that

$$w - v = cr^\beta \phi_1$$

for some positive constants $c$ and $\beta$. The Harnack inequality then yields $m(r) \to +\infty$ as $r \to +\infty$, which leads to a contradiction.

Now we can use $w$ as a comparison function to deduce a contradiction under the assumption that there is a bounded solution $u$ of (3.1)-(3.2). From the maximum principle it is easy to see $u \leq \epsilon w$ for any $\epsilon > 0$. However, this implies $u \equiv 0$ which is impossible. \[\square\]

Now let us prove Theorem 3.1.

**Proof for Theorem 3.1.** By the maximum principle, we see that the function $\text{osc}(r)$ is monotone when $r$ is large enough. There are two possibilities:
Case 1. \(\text{osc}(r)\) is monotone increasing when \(r \geq r_0\). Let us deuce a contradiction when \(\text{osc}(r)\) is uniformly bounded from above. As in the proof of Proposition 3.2, we can construct a comparison function \(w\) such that
\[
u \leq \text{osc}(r_0) + \epsilon w \quad \text{in} \quad \Omega - B_{r_0}
\]
for any \(\epsilon > 0\). As a result, \(\text{osc}(r) \equiv \text{osc}(r_0)\) when \(r \geq r_0\). The strong maximum principle yields that \(u\) is a constant function, which is impossible. So we have \(\text{osc}(r) \to +\infty\) as \(r \to +\infty\).

Case 2. \(\text{osc}(r)\) is monotone decreasing when \(r \geq r_0\). In this case, we still adopt the contradiction argument and assume that \(\text{osc}(r)\) converges to a positive constant as \(r\) tends to infinity. Extend the domain \(\Omega\) to an infinite cone \(C\) and define
\[
\tilde{A}(x) = \begin{cases} A(x), & x \in \Omega; \\ I, & x \in C - \Omega. \end{cases}
\]
Let \(\phi\) be a nonnegative function on \(\partial C\) with compact support which is positive somewhere. Then we can solve a bounded positive function \(v\) such that
\[
\text{div}(\tilde{A}(x)\nabla v) = 0 \quad \text{in} \quad C, \quad \text{and} \quad v = \phi \quad \text{on} \quad \partial C.
\]
It follows from the decay or blow up alternative (Lemma A.5) that
\[
\lim_{|x| \to +\infty} v(x) = 0.
\]
Up to a scaling we can assume \(u < v\) on \(\partial(\Omega - B_{r_0})\). Let us consider the function \(w = u - v\) in the domain
\[
\Omega_+ := \{u - v > 0\} \subset \Omega.
\]
Note \(\Omega_+\) must be non-empty and unbounded since \(\text{osc}(r)\) has a positive limit. And \(w\) is clearly a bounded positive function satisfying (3.1)-(3.2) in \(\Omega_+\). However, since \(\Omega_+\) satisfies the exterior cone property we know \(w\) must be unbounded by Proposition 3.2. This is a contradiction and we complete the proof. \(\square\)

4. Uniqueness

We will apply the following uniform continuity lemma and Proposition 3.2 to prove Proposition 1.4.

Lemma 4.1. If \(\Omega\) is a convex domain and \(u\) is a solution of (1.1)-(1.2) satisfying
\[
\|u - l\|_{C^0(\Omega)} < +\infty,
\]
then \(u\) is uniformly continuous in \(\Omega\).

Proof. For any \(\tau > 0\), set
\[
\eta_\tau = \frac{\tau \text{Lip} l + \|u - l\|_{C^0(\Omega)} + \|\phi\|_{C^0(\Omega)} |x|^2}{\tau^2},
\]
where \(\text{Lip} l\) is the Lipschitz constant of \(l\). It is clear that \(\eta_\tau\) is a subsolution of (1.1). Let us construct a suitable comparison function \(w\) by solving the
equation $\mathcal{M}w_\tau = 0$ in $B_1^+ := B_1 \cap \mathbb{R}_+^n$ with the Dirichlet boundary value $w_\tau = \eta_\tau$ on $\partial B_1^+$. (Actually, we need to modify $B_1^+$ to a smooth mean-convex domain but this is not much more difficult and it will not affect our argument later.) By the maximum principle, it is easy to check

$$w_\tau \geq \eta_\tau \text{ in } B_1^+.$$

Fix any point $x$ on $\partial \Omega$ and let $P$ be a supporting hyperplane at the point $x$. After translation and rotation, we can assume that $x$ is the origin and $P$ is the hyperplane \{x_n = 0\} without loss of generality. Since $\phi$ and $l$ are both uniformly continuous, for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|(l + \phi)(y) - (l + \phi)(x)| \leq \epsilon, \text{ for all } y \in \partial \Omega \text{ satisfying } |y - x| \leq \delta.$$

Consider the function $v_+ = (l + \phi)(x) + \epsilon + w_\delta$, Clearly, $v_+$ solves the minimal surface equation and $u \leq v_+$ on $\partial(B_\delta \cap \Omega)$. Therefore, it follows from the maximum principle that $u \leq v_+$ in $B_\delta \cap \Omega$. Let $v_- = (l + \phi)(x) - \epsilon - w_\delta$. The same argument leads to the fact $u \geq v_-$ in $B_\delta \cap \Omega$. Now we can take $\delta'$ small enough such that

$$|u(y) - (l + \phi)(x)| \leq 2\epsilon, \text{ for all } y \in B_{\delta'} \cap \Omega.$$

This completes the proof. \qed

**Proof of Proposition 4.4** Assume $u_1 > u_2$ at some point $x_0$. Set

$$\epsilon = (u_1(x_0) - u_2(x_0))/2,$$

$$w = (u_1 - u_2 - \epsilon), \text{ and } \Omega_\epsilon = \{w > 0\}.$$

By Lemma 4.1 we know $\text{dist}(\Omega_\epsilon, \partial \Omega)$ has a positive lower bound and we denote it by $\delta$. For any $x'$ in $\Omega_\epsilon$, we apply the interior gradient estimate for the minimal surface equation (see [10, Theorem 16.5]) to $u_i$ ($i = 1, 2$) in $B_{\delta/2}(x')$, and it yields

$$|\nabla u_i(x')| \leq \exp\left(C \frac{\text{osc}_{B_{\delta/2}} u_i}{\delta}\right) \leq \exp\left(C \frac{\|u_i - l\|_{C^0(\Omega)} + \delta \text{ Lip } l}{\delta}\right),$$

where $C$ is a uniform constant depending only on $n$. It follows that $|\nabla u_1|$ and $|\nabla u_2|$ are uniformly bounded in $\Omega_\epsilon$. Therefore, $w$ satisfies a uniformly elliptic equation in $\Omega_\epsilon$ and vanishes on $\partial \Omega_\epsilon$. We also note that $w$ is bounded due to (1.5). If $\Omega_\epsilon$ is bounded, then the maximum principle shows $w = 0$ and it is a contradiction. If $\Omega_\epsilon$ is unbounded, note that it satisfies the exterior cone condition, by Proposition 4.1 $w$ can not be bounded and it is a contradiction again. \qed

We point out that Edelen-Wang’s argument in [9] leads to the following comparison theorem.
Proposition 4.2. If $\Omega$ is a convex domain but not a half space and $u$ is a solution of (1.1) with $u \leq l$ on $\partial \Omega$, then $u \leq l$ in $\Omega$.

Proof. Suppose $X := \{x \in \Omega : u(x) - l(x) > 0\} \neq \emptyset$ and $Y \subset X$ be one of its connected component. Then $u|_Y : Y \to \mathbb{R}$ is a solution of the minimal surface equation with Dirichlet boundary value $l$. Since $Y$ is contained in a convex cone or a slab, then by Edelen and Wang [9] (actually their arguments still work for domains contained in slabs), $u|_Y = l$, which is a contradiction. Hence, $X = \emptyset$. □

This leads to the following quick corollary.

Corollary 4.3. If $\Omega \subseteq \mathbb{R}^n$ is a convex domain but not a half space and $u$ is a solution of (1.1)-(1.2), then

\begin{equation}
    l - \|\phi\|_{C^0(\partial \Omega)} \leq u \leq l + \|\phi\|_{C^0(\partial \Omega)}.
\end{equation}

Now we are ready to prove Theorem 1.3.

Proof for Theorem 1.3. Without loss of generality, we assume $l_1(0) = l_2(0) = 0$. When $l_1 = l_2$, the desired result follows directly from Proposition 1.4 and estimate (4.1). If $l_1 \neq l_2$, we may write $l_i = l_i' + c_i x_n$ for some linear function $l_i' : \mathbb{R}^{n-1} \to \mathbb{R}$ and constants $c_1 > c_2$ by choosing a suitable coordinate of $\mathbb{R}^n$. Then we divide the discussion into two cases:

Case 1. The domain $\Omega$ has all supporting plane parallel to the hyperplane $P = \{x_n = 0\}$. Since $\Omega$ is not a half space, it must be a stripe parallel to $P$. Then points in $\Omega$ has a uniform bound in $x_n$ and so we have

\[ \|u_i - l_i'\|_{C^0(\Omega)} < +\infty. \]

Clearly $u_i = l_i' + \phi_i'$ on $\partial \Omega$, where $\phi_i' = \phi_i + c_i x_n$ keeps bounded and uniformly continuous. This reduces to the case when $l_1 = l_2$.

Case 2. The domain $\Omega$ has one supporting plane $\Sigma$ not parallel to $P$.

Let us consider the domain

\[ H_+ = \left\{ x \in \mathbb{R}^n : x_n > \frac{\|\phi_1\|_{C^0(\partial \Omega)} + \|\phi_2\|_{C^0(\partial \Omega)}}{c_1 - c_2} \right\}. \]

Figure 2. The domains $\Omega$ and $H_+$.
As shown in Figure 2, there is at least one point \( x \in H_+ \) outside \( \Omega \) since \( H_+ \) crosses the hypersurface \( \Sigma \). From the fact \( u_1 \leq u_2 \) on \( \partial \Omega \) we can verify \( \partial \Omega \cap H_+ = \emptyset \). Combined with the convexity of \( H_+ \), we see that \( \Omega \cap H_+ \) is empty.

On the other hand, estimate 4.1 yields \( u_1 < u_2 \) in \( H_- = \{ x \in \mathbb{R}^n : x_n < -\| \phi_1 \|_{C^0(\partial \Omega)} + \| \phi_2 \|_{C^0(\partial \Omega)} \} \).

So we just need to show \( u_1 \leq u_2 \) in \( \Omega' = \Omega - H_- \). Since \( \partial \Omega' \) is contained in \( \partial H_- \) and \( \partial \Omega \), we know \( u_1 \leq u_2 \) on \( \partial \Omega' \). Notice that points in \( \Omega' \) are uniformly bounded in \( x_n \)-direction and \( u_i (i = 1, 2) \) are uniformly continuous on \( \partial \Omega' \). Again we can reduce to the case when \( l_1 = l_2 \). \( \square \)

**Proof for the first part in Theorem 1.1.** This case is a direct consequence of Theorem 1.3. \( \square \)

5. The foliation structure

We turn to the case when \( \Omega \) is a half space in this section. We begin with the following two propositions.

**Proposition 5.1.** If \( \Omega = \mathbb{R}^n_+ \), then for any solution \( u \) of (1.1)-(1.2), there is a unique real constant \( c \) such that for any \( x \in \mathbb{R}^n_+ \),

\[
|u(x) - l(x) - cx_n| \leq \| \phi \|_{C^0(\partial \Omega)}.
\]

**Proof.** Let

\[
\varphi = \max \{ \min \{ u + \| \phi \|_{C^0(\partial \Omega)}, l \}, u - \| \phi \|_{C^0(\partial \Omega)} \}.
\]

Then \( \varphi = l \) on \( \{ x_n = 0 \} \). Similar as Proposition 2.2, we could construct a family of solutions \( \{ v_k \} \) to the minimal surface equation in \( B_k^+ \) with \( v_k = \varphi \) on \( \partial B_k^+ \). By the maximum principle, we know

\[
u - \| \phi \|_{C^0(\partial \Omega)} \leq v_k \leq u + \| \phi \|_{C^0(\partial \Omega)} \text{ in } B_k^+.
\]

Also, \( v_k \) converges to a solution \( v \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) \) to the minimal surface equation with boundary value \( l \). Note that we have

\[
u - \| \phi \|_{C^0(\partial \Omega)} \leq v \leq u + \| \phi \|_{C^0(\partial \Omega)} \text{ in } \Omega.
\]

By the Bernstein type theorem in [9], we know \( v(x) = l(x) + cx_n \) for some constant \( c \). Hence,

\[
|u(x) - l(x) - cx_n| \leq \| \phi \|_{C^0(\partial \Omega)}.
\]

\( \square \)

**Proposition 5.2.** Suppose \( \Omega = \mathbb{R}^n_+ \) and there are two solution \( u_a \) and \( u_b \) of (1.1)-(1.2) corresponding to two constant \( a \) and \( b \) as in Proposition 5.1 respectively. If \( a > b \), then \( u_a > u_b \) in \( \Omega \); and if \( a = b \), then \( u_a = u_b \) in \( \Omega \).
Proof. Note for any $x \in \Omega$,
\[ |u_a(x) - l(x) - ax_n| \leq \|\phi\|_{C^0(\partial \Omega)} \]
and
\[ |u_b(x) - l(x) - bx_n| \leq \|\phi\|_{C^0(\partial \Omega)}. \]

Then,
\[ u_a(x) - u_b(x) > -2\|\phi\|_{C^0(\partial \Omega)} + (a - b)x_n \text{ for all } x \in \Omega. \]

If $a > b$, then $u_a > u_b$ for $x_n$ large enough. Hence
\[ \{u_a - u_b < 0\} \subset \{0 < x_n < \tau\} \]
for some constant $\tau$.

Fix any $\varepsilon > 0$ and suppose $\{u_a - u_b < -\varepsilon\}$ is not empty, then we have the following two cases:

Case 1a. $\{u_a - u_b < -\varepsilon\}$ is bounded. In this case, there is a constant $R > 0$ such that $u_a - u_b \geq 0$ on $\partial B_R^+$ and $\{u_a - u_b < -\varepsilon\} \subset B_R^+$. Note $w = u_a - u_b$ satisfies the elliptic equation $\partial_t(a_{ij}\partial_j w) = 0$ in $B_R^+$, where
\[ a_{ij} = \int_0^1 \frac{1}{\sqrt{1 + |p(t)|^2}} \left( \delta_{ij} - \frac{p_i(t)p_j(t)}{1 + |p(t)|^2} \right) dt \text{ and } p(t) = (1 - t)\nabla u_b + t\nabla u_a. \]

By the maximum principle, $w \geq 0$ in $B_R^+$, which is a contradiction.

Case 1b. $\{u_a - u_b < -\varepsilon\}$ is unbounded. In this case, $u_b - u_a - \varepsilon$ is a positive bounded solution to the elliptic equation $\partial_t(a_{ij}\partial_j w) = 0$ in unbounded domain $\{u_a - u_b < -\varepsilon\}$ with zero boundary value. Note that by Lemma 4.1, there is a $\delta > 0$ such that
\[ \{u_a - u_b < -\varepsilon\} \subset \{x_n > \delta\}. \]

By the standard interior gradient estimate (as in the proof of Proposition 1.3), we know $|\nabla u_a|$ and $|\nabla u_b|$ are uniformly bounded in $\{x_n > \delta\}$. Therefore, $a_{ij}$ is uniformly elliptic. However, Proposition 3.2 implies $u_b - u_a - \varepsilon$ is unbounded, which is a contradiction.

In conclusion, if $a > b$ then $\{u_a - u_b < -\varepsilon\}$ is empty for any $\varepsilon > 0$. So $u_a - u_b > 0$ in $\{x_n > 0\}$.

On the other hand, if $a = b$, then $u_a - u_b$ is bounded. Similar as above, we fix any $\varepsilon > 0$ and separate the proof to two cases.

Case 2a. $\{u_a - u_b < -\varepsilon\}$ is bounded. In this case, $u_a - u_b + \varepsilon$ is a solution to the uniformly elliptic equation with zero boundary value. Hence $u_a - u_b + \varepsilon = 0$ in $\{u_a - u_b < -\varepsilon\}$, which is a contradiction.

Case 2b. $\{u_a - u_b < -\varepsilon\}$ is unbounded. In this case, as in Case 1b we know Proposition 3.2 implies $u_b - u_a - \varepsilon$ is unbounded, which is a contradiction.

Case 2a and Case 2b imply $u_a - u_b \geq -\varepsilon$ for any $\varepsilon > 0$ and hence $u_a \geq u_b$. A similar argument also implies $u_b \geq u_a$. So $u_a = u_b$ when $a = b$. \hfill \Box

As an immediate consequence of Proposition 5.1 and Proposition 5.2.

Corollary 5.3. If $\Omega = \mathbb{R}^n_+$, then for any solution $u$ of (1.1) - (1.2), there is a unique constant $c$, such that $u = u_c$, where $u_c$ comes from Proposition 2.2.
We rewrite the second part of Theorem 1.1 to the following proposition.

**Proposition 5.4.** For any \( c \in \mathbb{R} \), let \( u_c \) be the solution constructed in Proposition 2.3. Then the map

\[
\Phi : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}_+^n \times \mathbb{R}, \quad (x, c) \mapsto (x, u_c(x))
\]

is a homeomorphism.

**Proof.** We first show the continuity of \( \Phi \), and it suffices to prove \( u_c(x) \) is continuous with respect to \((x, c)\). Take any sequence \((p_k, c_k) \to (p, c)\) as \( k \to \infty \), and by definition we know for any \( k \), \( u_{c_k} \) is solution of \((1.1)-(1.2)\) with estimate

\[
|u_{c_k} - l - c_k x_n| \leq \|\phi\|_{C^0(\mathbb{R}^n)}.
\]

By interior gradient estimate for the minimal surface equation and Lemma 4.1 up to a subsequence the function \( u_{c_k} \) converges to a limit function \( u' \) in \( C_{\text{loc}}^\infty(\mathbb{R}_+^n) \) and \( C^0_{\text{loc}}(\mathbb{R}_+^n) \) satisfying \((1.1)-(1.2)\). Also, \( u' \) satisfies

\[
|u' - l - c x_n| \leq \|\phi\|_{C^0(\mathbb{R}^n)}.
\]

By Proposition 5.2, \( u' = u_c \) and hence \( u_{c_k} \) converges to \( u_c \) in \( C_{\text{loc}}^0(\mathbb{R}_+^n) \). This implies \( u_{c_k}(p_k) \to u_c(p) \) as \( k \to \infty \).

Next, the bijectivity of \( \Phi \) comes from Proposition 2.3, Proposition 5.1, Proposition 5.2 and the continuity of \( \Phi \).

Finally let us prove \( \Phi \) is a closed map, that is, \( \Phi(A) \) is closed in \( \mathbb{R}_+^n \times \mathbb{R} \) for any closed set \( A \subset \mathbb{R}_+^n \times \mathbb{R} \). Suppose now \( \{(p_k, c_k)\}_{k \geq 1} \subset A \) is a sequence such that

\[
\Phi(p_k, c_k) = (p_k, u_{c_k}(p_k)) \to (p_0, q) \text{ in } \mathbb{R}_+^n \times \mathbb{R}, \text{ as } k \to \infty,
\]

and here we write \( p_k = (x_1(p_k), \ldots, x_n(p_k)) \) and \( p_0 = (x_1(p_0), \ldots, x_n(p_0)) \). Then,

\[
x_n(p_k) \to x_n(p_0) > 0 \text{ as } k \to \infty.
\]

By Proposition 5.1 we deduce that

\[
\left| \frac{u_{c_k}(p_k) - l(p_k)}{x_n(p_k)} - c_k \right| \leq \frac{\|\phi\|_{C^0(\mathbb{R}^n)}}{x_n(p_k)}
\]

for any \( k \geq 1 \). Thus, \( \{c_k\} \) is bounded in \( \mathbb{R} \). Up to a subsequence, we may assume \( \{c_k\} \) converges to \( c_0 \) as \( k \to \infty \). Then, \((x_0, c_0) \in A \) and it follows from the continuity of \( \Phi \) that

\[
q = \lim_{k \to \infty} u_{c_k}(p_k) = u_{c_0}(p_0).
\]

Hence, \( \Phi(A) \) is closed in \( \mathbb{R}_+^n \times \mathbb{R} \). \( \square \)

The graphs of functions \( u_c \) actually form a differential foliation of \( \mathbb{R}_+^n \times \mathbb{R} \) if the function \( \phi \) has better regularity. For our purpose, let us change the parametrization for the family of functions \( u_c \). Note that the restriction map

\[
\Phi_{x_0} : \mathbb{R} \to \mathbb{R}, \ c \mapsto u_c(x_0),
\]

is also a homeomorphism for any point \( x_0 \) in \( \mathbb{R}_+^n \).
Fix a point $x_0$ in $\mathbb{R}_+^n$. We define 
\[ \tilde{u}_t(x) = u_{c(t)}(x) \] with $c(t) = \Phi^{-1}(x_0(t))$ and consider the corresponding map 
\[ \tilde{\Phi} : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}_+^n \times \mathbb{R}, \ (x,t) \mapsto (x, \tilde{u}_t(x)) \).

We have the following

**Proposition 5.5.** If $\phi$ is $C^1$ on $\partial \mathbb{R}_+^n$ with $\|\phi\|_{C^1(\partial \mathbb{R}_+^n)} < +\infty$, then the map $\tilde{\Phi}$ is a $C^1$-diffeomorphism.

**Proof.** First notice that we can improve estimates for functions $u_c$ from the $C^1$-bound of $\phi$. From the gradient estimate for the minimal surface equation (see [10, 11, 13]), for any $c_* > 0$ there is a universal constant $C = C(c_*, \|\phi\|_{C^1(\partial \mathbb{R}_+^n)}, n)$ such that 
\[ |\nabla u_c(x)| \leq C, \forall c \in [-c_*, c_*] \text{ and } x \in \mathbb{R}_+^n. \]

On the other hand, given any compact subset $K$ in $\mathbb{R}_+^n$ and any nonnegative integer $k$ there is a universal constant $C_k = C_k(K, c_*, n)$ such that 
\[ |\nabla^k u_c(x)| \leq C_k, \forall c \in [-c_*, c_*] \text{ and } x \in K. \]

The rest proof will be divided into three steps.

**Step 1.** The functions $\tilde{u}_t(x)$ is differentiable with respect to $t$. We are going to show 
\[ \frac{\partial}{\partial t} \tilde{u}_t(x) = \tilde{v}_t(x), \]
where $\tilde{v}_t$ is the unique solution to the equation 
\[ \partial_t(\bar{a}_{ij,t} \partial_j \tilde{v}_t) = 0 \text{ in } \mathbb{R}_+^n \]
with the condition 
\[ \tilde{v}_t = 0 \text{ on } \partial \mathbb{R}_+^n, \text{ and } \tilde{v}_t(x_0) = 1, \]
where 
\[ \bar{a}_{ij,t} = \delta_{ij} - \frac{\partial_i \tilde{u}_t \partial_j \tilde{u}_t}{(1 + |\nabla \tilde{u}_t|^2)^{\frac{3}{2}}}. \]

The uniqueness of $\tilde{v}_t$ follows from the gradient estimate (5.1) and Theorem A.2. For any real number $\tau$, let us take 
\[ \tilde{v}_{t,\tau} = \tau^{-1}(\tilde{u}_{t+\tau} - \tilde{u}_t). \]
It suffices to show that for any sequence $\tau_k$ such that $\tau_k \to 0$ as $k \to +\infty$ the functions $\tilde{v}_{t,\tau_k}$ converges $\tilde{v}_t$. The idea is to use the uniqueness of $\tilde{v}_t$. Clearly, by Proposition 5.2 each $\tilde{v}_{t,\tau_k}$ is a positive function on $\mathbb{R}_+^n$ with value 1 at point $x_0$, which vanishes on the boundary $\partial \mathbb{R}_+^n$. It also satisfies the equation 
\[ \partial_t(\bar{a}_{ij,t,\tau_k} \partial_j \tilde{v}_{t,\tau_k}) = 0 \text{ in } \mathbb{R}_+^n, \]
where 
\[ \bar{a}_{ij,t,\tau_k} = \delta_{ij} - \int_0^1 \frac{\partial_i \bar{w}_{s,\tau_k} \partial_j \bar{w}_{s,\tau_k}}{(1 + |\nabla \bar{w}_{s,\tau_k}|^2)^{\frac{3}{2}}} \, ds. \]
with
\[ \bar{w}_{s, \tau_k} = (1 - s)\bar{u}_t + s\bar{u}_{t + \tau_k}. \]

From the interior estimate \((5.2)\) the coefficients \(\bar{a}_{ij,t,\tau_k}\) converge smoothly to \(a_{ij,t}\) in any compact subset of \(\mathbb{R}^n_+\) up to a subsequence. On the other hand, we also have good estimates for functions \(\bar{v}_{t,\tau_k}\). As a beginning, we point out that they have locally uniform \(C^0\)-bounds. To see this, we take \(\rho > 0\) large enough such that the semi-ball \(B^+_\rho = B_\rho \cap \mathbb{R}^n_+\) contains the point \(x_0\), and then construct a solution \(\bar{w}_{\tau_k}\) of equation \((5.5)\) in \(B^+_\rho\) by prescribing a Dirichlet boundary value \(\psi \in C^0(\partial B^+_\rho)\), which is positive on \(\partial B_\rho \cap \mathbb{R}^n_+\) and vanishes on \(\partial \mathbb{R}^n_+\). It is not difficult to see that \(\{\bar{w}_{\tau_k}\}_k\) is compact in \(C^0(B^+_\rho)\) and so
\[ \bar{w}_{\tau_k}(x_0) > \epsilon_0, \]
where \(\epsilon_0\) is a positive constant independent of \(k\). Then the boundary Harnack inequality (see Theorem \(\ref{thm:boundaryharnack}\)) combined with the maximum principle yields that
\[ \bar{v}_{t,\tau_k} \leq C'\epsilon_0^{-1}\|\psi\|_{C^0} \text{ in } B^+_\rho/2, \]
where \(C'\) is a positive constant independent of \(k\). From the standard elliptic PDE theory, the functions \(\bar{v}_{t,\tau_k}\) must have locally uniform up-to-boundary Hölder estimate and locally uniform \(C^l\)-estimates in \(\mathbb{R}^n_+\), so it converges to a limit function \(\bar{v}'_t\) up to a subsequence in \(C^0_{loc}(\mathbb{R}^n_+) \cap C^0_{loc}(\mathbb{R}^n_+)\), which solves \((5.3)-(5.4)\). Since \((5.3)-(5.4)\) has a unique solution, it implies \(\bar{v}'_t = \bar{v}_t\), and we obtain the desired consequence.

**Step 2.** All partial derivatives of the function \(\bar{U}(x,t) := \bar{v}_t(x)\) are continuous with respect to \((x,t)\). First we deal with \(\bar{v}_t(x)\), the partial derivative of \(\bar{U}(x,t)\) with respect to \(t\). It suffices to show that \(\bar{v}_t\) converges to \(\bar{v}_t'\) in \(C^0_{loc}(\mathbb{R}^n_+)\) as \(t' \to t\) and the proof is almost identical to that in Step 1. The only modification is that the functions \(\bar{v}_{t,\tau_k}\) need to be replaced by the functions \(\bar{v}_t\) with \(t_k \to t\) as \(k \to +\infty\), where \(\bar{v}_t\) is the unique solution of \((5.3)-(5.4)\) (after \(t\) is replaced by \(t_k\)). The argument in Step 1 goes smoothly without any difficulty in this case.

It remains to show that \(\partial_t \bar{u}_t\) is continuous with respect to \((x,t)\). With the same idea, we would like to prove that \(\bar{u}_t\) converges to \(\bar{u}_t\) in \(C^0_{loc}(\mathbb{R}^n_+)\) as \(t' \to t\). Take any sequence \(t_k \to t\) as \(k \to +\infty\). Recall that \(\bar{u}_t\) is exactly the function \(u_{c(t)}\), where \(c(t) = \Phi_{x_0}^{-1}(t)\) is continuous with respect to \(t\). Similar as the proof of the continuity part in Proposition \((5.4)\), from Proposition \((2.3)\) and Lemma \((4.4)\) up to a subsequence the function \(u_{c(t_k)}\) converges to a limit function \(u_{c(t)}\) in \(C^0_{loc}(\mathbb{R}^n_+) \cap C^0_{loc}(\mathbb{R}^n_+)\) satisfying \((1.1)-(1.2)\).

**Step 3.** \(\Phi\) is a \(C^1\)-diffeomorphism. From the definition we see
\[ \Phi(x,t) = \Phi(x,\Phi_{x_0}^{-1}(t)). \]
This yields that \(\Phi\) is a homeomorphism. According to the inverse function theory, all we need to show is that the Jacobian of the map \(\Phi\) has non-zero
determinant. It is not difficult to see
\[ J_\Phi = \begin{pmatrix} I_{n \times n} & 0 \\ \ast & \tilde{v}_t(x) \end{pmatrix}. \]
From Step 1 we know \( \tilde{v}_t(x) \) is positive in \( \mathbb{R}^n_+ \), and hence the determinant of \( J_\Phi \) is non-zero. So we complete the proof. \( \square \)

**Remark 5.6.** From our proof, if \( \phi \) is \( C^k \) on \( \partial \mathbb{R}^n_+ \) with bounded \( \| \phi \|_{C^k(\partial \mathbb{R}^n_+)} \), then \( \Phi \) is a \( C^k \)-diffeomorphism.

### Appendix A. Some preliminary results

#### A.1. Boundary Harnack inequality

First we will recall the boundary Harnack inequality from [8]. Let \( x = (x', x_n) \) and \( B'_1 \) be the unit ball in \( \mathbb{R}^{n-1} \). Let \( g : B'_1 \to \mathbb{R} \) be a Lipschitz function with Lipschitz norm \( L \) and \( g(0) = 0 \). Define the graph
\[
\Gamma = \{(x', x_n) \in \mathbb{R}^n : x_n = g(x'), x' \in B'_1 \}
\]
and the height function
\[
h_\Gamma : B'_1 \times \mathbb{R} \to \mathbb{R}, (x', x_n) \mapsto x_n - g(x').
\]
For convenience, we let
\[
C_{r, \rho} = \{ x = (x', x_n) \in B'_1 \times \mathbb{R} : |x'| \leq r, 0 < h_\Gamma(x) < \rho \}
\]
and \( C_r = C_{r,r} \).

![Figure 3. Domain \( C_1 \) above Lipschitz graph \( \Gamma \)](image)

In the following, we consider the following uniformly elliptic operator of divergence form
\[ Lu := \text{div}(A(x) \nabla u), \]
with \( A(x) \) is bounded and measurable satisfying
\[ \lambda I \leq A(x) \leq \lambda^{-1} I, \] for some constant \( \lambda \in (0, 1) \).

The boundary Harnack inequality can be stated as the following.
Theorem A.1. If $u_1$ and $u_2$ are solutions of the equation
\[ Lu = 0 \text{ in } C_1 \]
satisfying
- $u_1$ and $u_2$ vanish on $\Gamma$,
- $u_1$ and $u_2$ are positive in $C_1$,
- and $u_1(0, \frac{1}{2}) = u_2(0, \frac{1}{2}) = 1$,
then there is a universal constant $C = C(n, \lambda, L)$ such that
\[ C^{-1} \leq \frac{u_1}{u_2} \leq C \text{ in } C_1/2. \]

A.2. Positive solutions of uniformly elliptic linear equations in cones.
In the following, the operator $L$ is assumed to satisfy the same requirements in the previous subsection. The symbol $C$ means an infinite Lipschitz cone now.

A.2.1. Existence and uniqueness of the solution.

Theorem A.2. The equation $Lu = 0$ in $C$ with $u = 0$ on $\partial C$ and $u > 0$ in $C$ has a unique solution in $W^{2,p}_{\text{loc}}(C) \cap C^0(\overline{C})$ up to a scaling.

We point out that Theorem A.2 is the main theorem proved in [14] by Landis and Nadirashvili. For reader’s convenience, here we provide a different proof inspired from [3] based on the boundary Harnack inequality.

Proof of existence. Let $r_n$ be a sequence of positive real numbers with $r_n \to +\infty$ as $n \to \infty$. We let $B_n$ be the ball centered at the pole of $C$ with radius $r_n$ and $C_n$ the intersection $C \cap B_n$. It is easy to construct a continuous function $\phi_n$ on $\partial C_n$ such that $\phi$ vanishes on $\partial C \cap B_n$ and $\phi$ is positive on $C \cap \partial B_n$. Clearly, we can find a solution $u_n \in W^{2,p}(C_n) \cap C^0(\overline{C_n})$ of the equation $Lu_n = 0$ in $C_n$ with the Dirichlet boundary value $\phi_n$ (see [10] for instance). The maximum principle then yields that $u_n$ is positive in $C_n$. Fix a point $P$ in the intersection of all $C_n$. Up to a scaling we can normalize the function $u_n$ to satisfies $u_n(P) = 1$. From the interior Harnack inequality, the boundary Harnack inequality (Theorem A.1) and the $W^{2,p}$-estimate, $u_n$ converges to a nonnegative function $u$ satisfying $Lu = 0$ in $C$ with zero boundary value on $\partial C$. \qed

Proof of uniqueness. Fix a ray $\gamma$ in $C$ starting from the pole of $C$. First we are going to show the following property: if $u$ and $v$ are two solutions satisfying the hypothesis in Theorem A.1 such that $u$ equals to $v$ at the point $\gamma \cap \partial B_r$ for some $r > 0$, then there is a universal constant $C_0 = C_0(n, \lambda, C)$ such that
\[ C_0^{-1} \leq \frac{u}{v} \leq C_0 \text{ in } C. \]

To prove this, we first notice that if $u$ equals $v$ at the point $\gamma \cap \partial B_r$, then there is a universal constant $C_1 = C_1(n, \lambda, C)$ such that
\[ C_1^{-1} \leq \frac{u}{v} \leq C_1 \text{ on } \partial B_r \cap C. \]
By the maximum principle, we conclude that 
\[ C_{-1}^{-1} \leq \frac{u}{v} \leq C_1 \]
holds for any point in \( \mathcal{C} \cap B_r \). On the other hand, suppose we have the inequality
\[ C_{-1}^{-1} \leq \frac{u}{v} \leq C_1 \]
at some point in \( \mathcal{C} \cap B_s \), then
\[ (A.1) \quad C_{-2}^{-1} \leq \frac{u(P)}{v(P)} \leq C_1^2, \]
where \( P = \gamma \cap \partial B_s \). Otherwise, if \( \frac{u}{v} > C_1^2 \) at the point \( P \), then there is a constant \( C(P) \) such that \( C(P) > C_1^2 \) and \( u(P) = C(P)v(P) \). By applying previous argument to \( u \) and \( C(P)v \), we have
\[ u \geq C_{-1}^{-1}C(P)v > C_1v \]
in \( \mathcal{C} \cap B_s \), which is a contradiction. The left hand side inequality of \( (A.1) \) could be proved in the same way. Clearly, \( (A.1) \) holds for all large \( s \) and it provides a control of the ratio \( u/v \) at infinity. Applying the boundary Harnack inequality and the maximum principle once again, finally we arrive at
\[ C_{-1}^{-1} \leq \frac{u}{v} \leq C_0 \text{ in } \mathcal{C}, \]
where \( C_0 = C_1^3 \).

Now the uniqueness is almost direct. Let \( u \) be the solution contructed above and \( v \) be any other solution. From previous discussion, we see that there is a positive constant \( \epsilon \) such that \( \epsilon u \leq v \leq \epsilon^{-1}u \). Define
\[ \epsilon^* := \sup \{ \epsilon > 0 : v > \epsilon u \} \]
and we consider the function \( w := v - \epsilon^*u \). Clearly, \( w \) is a nonnegative solution of the equation \( \mathcal{L}w = 0 \) in \( \mathcal{C} \) with zero boundary value on \( \partial\mathcal{C} \). The Harnack inequality yields that \( w \) is either the zero function or a positive function in \( \mathcal{C} \). In the latter case, we can find another positive constant \( \epsilon' \) such that \( w \geq \epsilon'u \) and so \( v \geq (\epsilon^* + \epsilon')u \). This contradicts to the definition of \( \epsilon^* \) and we complete the proof. \( \square \)

As a special case of Theorem A.2, we see

**Corollary A.3.** If \( \mathcal{L} = \Delta \) is the Laplace operator and the cone \( \mathcal{C} \) has the form of \( \mathcal{C} = \{tx : t > 0, x \in S\} \) for a smooth domain \( S \) in \( S^{n-1} \), then any harmonic function in \( \mathcal{C} \) with \( u = 0 \) on \( \partial\mathcal{C} \) and \( u > 0 \) in \( \mathcal{C} \) has the form \( u = cu^\beta \phi_1 \), where \( c \) is a positive constant, \( \phi_1 \) is the first eigenfunction of \( S \) with corresponding first eigenvalue \( \lambda_1 \) and
\[ \beta = \frac{-(n-2) + \sqrt{(n-2)^2 + 4\lambda_1}}{2}. \]
A.2.2. Hölder decay or blow up alternative of the solution from the Harnack inequality.

**Lemma A.4.** Let $C_\rho = C \cap B_\rho$ and let $\mathcal{A}_r$ be the annulus $\mathcal{A}_r := \overline{C_{4r}} - \overline{C}_r$ for any $r > 0$. If $u \in W^{1,2}(\mathcal{A}_r) \cap C^0(\mathcal{A}_r)$ is a solution of $\mathcal{L}u = 0$ in $\mathcal{A}_r$ such that $u \leq 0$ on the side boundary $\partial C \cap (B_{4r} - \overline{B}_r)$, then we have

\[
\sup_{\partial B_{2r} \cap C} u^+ \leq (1 - \delta) \sup_{\partial \mathcal{A}_r} u^+
\]

for some universal constant $\delta = \delta(n, \lambda, C)$, where $u^+$ is the positive part of $u$.

**Proof.** If $u$ is non-positive on $\partial \mathcal{A}_r$, then it follows from the maximum principle that both sides of above inequality equal to zero. So we just need to deal with the case when $u$ is positive somewhere on $\partial \mathcal{A}_r$. Without no loss of generality, we can assume $u = 0$ on the side boundary $\partial C \cap (B_{4r} - \overline{B}_r)$ and $0 \leq u \leq 1$ on $\partial \mathcal{A}_r$, otherwise we can introduce $u'$ and here $u'$ is the solution to the same elliptic equation in $\mathcal{A}_r$ with the Dirichlet boundary value

\[
\left( \sup_{\partial \mathcal{A}_r} u \right)^{-1} \max \left\{ 0, u \big|_{\partial \mathcal{A}_r} \right\}.
\]

Based on these assumptions, it is easy to see $0 \leq u \leq 1$ by the maximum principle and $u = u^+$ on $\partial \mathcal{A}_r$. Without loss of generality, we only deal with the case when $r = 1$.

Let us consider the function $v = 1 - u$. Note the $L^\infty$-norm of $v$ is bounded by 1, we conclude that $v$ is bounded below by $\frac{1}{7}$ in some neighborhood of $\partial C \cap \partial B_2$ by the boundary Hölder estimate (see [15, Theorem 3.6]). Combined with the Harnack inequality, $v$ must be bounded below by a positive constant $\delta$ on $\partial B_2 \cap C$. This yields $u \leq 1 - \delta$ on $\partial B_2 \cap C$ and we complete the proof. \hfill $\square$

Based on this lemma, we have the following alternative.

**Lemma A.5.** Let $u \in W^{2,p}_{\text{loc}}(C) \cap C^0(\overline{C})$ be a solution of the equation $\mathcal{L}u = 0$ in $C$ with $u = 0$ on $\partial C - \overline{B}_1$ and $u > 0$ in $C - B_1$. Then the function

\[
\text{osc}(r) = \max_{C \cap \partial B_r} u
\]

has a limit as $r \to +\infty$, and the limit is either $+\infty$ or 0. Moreover, either $\text{osc}(r) \geq cr^\beta$ or $\text{osc}(r) \leq cr^{-\beta}$ holds for some positive constants $c$ and $\beta$.

**Proof.** It is clear that $u$ cannot be a constant function. Then it follows from the maximum principle that the function $\text{osc}(r)$ cannot have a local maximum, and so it must be monotone when $r$ is large enough. Hence the function $\text{osc}(r)$ has a limit when $r$ tends to $+\infty$.

To show the alternative for the limit of $\text{osc}(r)$, we just need to show $\text{osc}(r) \to 0$ as $r \to +\infty$ under the assumption that $\text{osc}(r)$ is bounded from above by a positive constant $C$. Fix $r_0$ to be a large positive constant. Since $u \leq C$ holds on $\partial B_{r_0} \cap C$ for all $r \geq r_0$, we obtain by Lemma A.4 that $u \leq (1 - \delta)C$ on $\partial B_{2r} \cap C$ for all $r \geq 2r_0$. Through iteration we conclude
that \( u \leq (1 - \delta)^k C \) on \( \partial B_r \cap \mathcal{C} \) for all \( r \geq 2^k r_0 \), which yields \( \text{osc}(r) \to 0 \) as \( r \to +\infty \). From above discussion, it is not difficult to deduce the inequality

\[
\text{osc}(r) \leq C \left( \frac{r}{r_0} \right)^{\log_2(1-\delta)}, \quad \text{for all } r \geq r_0.
\]

This yields \( \text{osc}(r) \leq cr^{-\beta} \) for some positive constants \( c \) and \( \beta \) in the case when \( \text{osc}(r) \to 0 \).

Now let us deal with the case when \( \text{osc}(r) \to +\infty \). At this time, \( \text{osc}(r) \) must be monotone increase when \( r \geq r_0 \) for some positive constant \( r_0 \). From Lemma A.4 we have

\[
\text{osc} \left( 4^k r_0 \right) \geq (1 - \delta)^{-k} \text{osc}(r_0).
\]

This can be used to deduce

\[
\text{osc}(r) \geq \text{osc}(r_0) \left( \frac{r}{r_0} \right)^{-\log_2(1-\delta)}, \quad \text{for all } r \geq r_0.
\]

This yields \( \text{osc}(r) \geq cr^\beta \) for some positive constants \( c \) and \( \beta \) and we complete the proof. \( \square \)

Now it is clear that

**Corollary A.6.** Let \( u \in W^{2,p}_{\text{loc}}(\mathcal{C}) \cap C^0(\mathring{\mathcal{C}}) \) be a solution of the equation

\[Lu = 0\]

in \( \mathcal{C} \) with \( u = 0 \) on \( \partial \mathcal{C} \) and \( u > 0 \) in \( \mathcal{C} \). Then \( u \) is unbounded.

**Proof.** Since \( u \) cannot be the zero function, it follows from the maximum principle and the alternative that \( \text{osc}(r) \to +\infty \) as \( r \to +\infty \). \( \square \)

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**School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE, Beijing Normal University, Beijing, 100875, China**

*Email address: gjiang@bnu.edu.cn*

**Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China**

*Email address: wangzhehui@amss.ac.cn*

**Beijing International Center for Mathematical Research, Peking University, Beijing, 100871, China**

*Email address: zhujintian@bicmr.pku.edu.cn*