A Review on The Sixth Painlevé Equation
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Abstract

For the Painlevé 6 transcendents, we provide a unitary description of the critical behaviours, the connection formulae, their complete tabulation, and the asymptotic distribution of poles close to a critical point.

MSC: 34M55 (Painlevé and other special functions)

Contents

The isomonodromy deformation method provides a unitary description of the critical behaviours of the solutions of the Painlevé 6 equation (PVI), their connection formulae and the asymptotic distribution of poles close to a critical point. The paper is a review of these results. I have also included explanations and connections among the above subjects which I have never had the chance to write in other papers.

– Introduction: We introduce the Painlevé 6 equation, a non linear ODE which plays a central role in contemporary mathematics, and defines new non-linear special functions, called Painlevé Transcendents. We motivate the analysis of the critical behaviors of PVI transcendents close to the singular points of the equation (critical points), and the analysis of the connection problem.

– Section 2: We review the first general class of solution whose critical behaviors was discovered by Jimbo, who solved the connection problem by means of the method of monodromy preserving deformations. Then, we review the general scheme of this method.

– Section 3: We go deeper into the application of the method of monodromy preserving deformations. First, we give further details on the procedure which provides the class of solutions due to Jimbo, then we review a matching procedure which produces critical behaviors not contained in the former class. We explain how the symmetries of PVI allow to further enlarge the class of known transcendents. Finally, we collect all the critical behaviors known today in a complete table, in exactly the same way as the classical special functions were tabulated in the XIX and XX centuries.

– Section 4: We review the distribution of the movable poles of the Painlevé 6 transcendents close to a singular point of the equation.

– Section 5: We describe a method of local analysis, due to Shimomura, which provides the critical behaviors on spiral shaped domains contained in the universal covering of the punctured neighbourhood of a critical point. We also review the Elliptic representation of the transcendents, which yields the same results of Shimomura’s approach.

– Section 6: We review how the isomonodromy deformation approach can be extended to the whole universal covering of the punctured neighborhood of a critical point, providing a comprehensive picture of the critical behaviours on this universal covering, including the position of the movable poles.
1 Introduction

1.1 Background

The Painlevé 6 equation, denoted PVI or PVI_{\alpha,\beta,\gamma,\delta}, is the non linear differential equation

\[
\frac{d^2 y}{dx^2} = \frac{1}{2} \left[ \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right] \left( \frac{dy}{dx} \right)^2 - \left[ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right] \frac{dy}{dx} \\
+ \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right], \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}
\]

This is the sixth and last of the non linear ODEs discovered by Painlevé [38] and Gambier [14], who classified all the second order ordinary differential equations of the type

\[
\frac{d^2 y}{dx^2} = \mathcal{R} \left( x, y, \frac{dy}{dx} \right)
\]

where $\mathcal{R}$ is rational in $\frac{dy}{dx}$, $x$ and $y$, such that the branch points and essential singularities depend only on the equation (Painlevé Property). The essential singularities and branch points are called critical points; for PVI they are $0, 1, \infty$. The behaviour of a solution close to a critical point is called critical behaviour.\(^1\) A solution of the sixth Painlevé equation can be analytically continued to a meromorphic function on the universal covering of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The six equations, discovered at the beginning of the XX century, appeared to be irreducible to already known equations solvable in terms of classical functions. This fact has been rigorously proved only recently [44], as we discuss below. In the last decades the Painlevé equations have emerged as one of the central objects in pure mathematics and mathematical physics, with applications in a variety of problems, such as number theory, theory of analytic varieties (like Frobenius structures), random matrix theory, orthogonal polynomials, non linear evolutionary PDEs, combinatorial problems, etc. The properties of the classical (linear) transcendental functions have been organised and tabulated in various classical handbooks. A comparable organisation and tabulation of the properties of the Painlevé functions is now needed. Today we are able to write an essentially complete table of the critical behaviours, with full expansions, for the Painlevé 6 functions, and the corresponding connection formulae [22].

1.2 Solving PVI...

We need to solve the non linear differential equation PVI. What does it mean that we know a solution? The functions which solve PVI are generically ”new transcendental functions”, called Painlevé Transcendents. Umemura proved the irreducibility of them to classical functions [44] [45] [46], namely functions given in terms of a finite iteration of permissible operations applied to rational functions. These operations are the derivation, rational combination (sum, product, quotient), algebraic combinations (the expression is a root of a polynomial whose coefficients are rational functions), contour integrals and quadratures, solution of a linear homogeneous differential equation whose coefficients are rational functions, solution of an algebraic differential equation of the first order whose coefficients are rational functions, composition with abelian functions (the expression is $\varphi(f_1(x), ..., f_n(x))$, where $f_1, ..., f_n$ are rational functions, and $\varphi : \mathbb{C}^n/\Gamma \rightarrow \mathbb{C}$ is meromorphic, $\Gamma$ is a lattice). The elementary transcendental functions are

\(^1\)This differs from the terminology of singularity theory, where a critical point is a zero of the first derivative of a function.
classical functions, because they are the algebraic functions, or a function which is obtained from an algebraic function by integration (like the exponential, the trigonometric and hyperbolic functions), or the inverse of such an integral (like the logarithm, the elliptic functions, etc). Umemura proved that the general solution of a Painlevé equation is not a classical function. H. Watanabe [47] applied the argument to PVI, and showed that a solution of PVI is either algebraic, or solves a Riccati equation (one-parameter family of classical solutions), or it is not a classical function. All the algebraic solutions were classified in [10] when $\beta = \gamma = 0$, $\delta = \frac{1}{2}$, and then in [32] for the general PVI.

Since a Painlevé transcendent is in general not classical, we require the following minimal knowledge:

i) The knowledge of the explicit critical behaviour (or the asymptotic expansion) of a transcendent at the critical points $x = 0, 1, \infty$. We symbolise the critical behaviour of a branch $y(x)$, defined for $-\pi < \arg x < \pi$ and $-\pi < \arg(1-x) < \pi$, as follows:

$$y(x) = y_u(x, c_1^u, c_2^u), \quad \text{when } x \to u, \quad u \in \{0, 1, \infty\},$$

where $c_1^u, c_2^u$ are the integration constants.

ii) The knowledge of the explicit connection formulae among couples of integration constants at different critical points, as follows. A branch $y(x)$ has a critical behaviour of type (1) at a critical point $x = u$, and another behaviour of type (1) at another critical point $x = v$, $u \neq v \in \{0, 1, \infty\}$:

$$y(x) = \begin{cases} 
y_u(x, c_1^u, c_2^u), & x \to u \\
y_v(x, c_1^v, c_2^v), & x \to v
\end{cases}$$

The connection problem consists in the computation of the explicit formulae

$$\begin{cases} c_1^v = c_1^u (c_1^v, c_2^v) \\
c_2^v = c_2^u (c_1^v, c_2^v)
\end{cases}, \quad \text{and the inverse} \quad \begin{cases} c_1^u = c_1^v (c_1^u, c_2^v) \\
c_2^u = c_2^v (c_1^u, c_2^v)
\end{cases}$$

These are called the connection formulae in closed form.

Knowledge of i) and ii) is precisely what is meant by "solving" a Painlevé equation in the review book [12], page 8. Above, we used the word explicit. An explicit expression is a classical functions of its arguments. Knowledge of i) and ii) allows to use Painlevé transcendents in applications. As already mentioned, today we are able to write an essentially complete table of the critical behaviours (with their full expansions) for the Painlevé 6 transcendents, and the corresponding connection formulae [22]. We will review this in section 3.5.

The analysis of the critical behaviours and asymptotic/formal expansions are one side of the research on the Painlevé 6 functions. Other perspectives, such as the classification of classical solutions [47], rational solutions [35] and algebraic solutions [10] [32] will not be discussed here.

2 The Isomonodromy Deformation Approach

The properties of some Painlevé transcendents have been known since the beginning of the XX century, but the first general result concerning the critical behaviour of a two parameter class of solutions of PVI is due to Jimbo in [27].
2.1 The Critical Behaviors in Jimbo’s work (I)

A transcendent in the class obtained by Jimbo has critical behaviours [27]:

\[
y(x) = \begin{cases} 
  a_0 x^{1-\sigma_0} \left( 1 + O(x^\delta) \right), & |x| < r \\
  1 + a_1 (1-x)^{1-\sigma_1} \left( 1 + O((1-x)^\delta) \right), & |x-1| < r \\
  a_\infty x^{\sigma_01} \left( 1 + O(x^{-\delta}) \right), & |1/x| < r 
\end{cases}
\]  

where \( \delta \) is a small positive number, \( r > 0 \) is a sufficiently small radius, \( a_0, a_1, a_\infty \neq 0 \) and

\[0 \leq \Re \sigma_0 < 1, \quad 0 \leq \Re \sigma_1 < 1, \quad 0 \leq \Re \sigma_{01} < 1, \quad \sigma_{0x}, \sigma_{x1}, \sigma_{01} \neq 0 \]

are complex integration constants. The behaviors (3) hold when \( x \) converges to the critical points inside a sector with vertex on the corresponding critical point. The angular width of the sector is arbitrary, and if increased the radius \( r \) decreases. For angular width \( 2\pi \), (3) represents a branch of a Painlevé transcendent. The two independent integration constants are any of the three couples \((a_0, \sigma_{0x}), (a_1, \sigma_{x1}), (a_\infty, \sigma_{01})\). The relation among them is given by connection formulae (2), where the \( c \)'s represent the \( a \)'s and \( \sigma \)'s.

The connection problem for (3) was solved in [27], when \( \alpha, \beta, \gamma, \delta \) are generic (we refer to [27] for a precise definition of generic), using the isomonodromy deformations theory. Jimbo considered an isomonodromic \( 2 \times 2 \) Fuchsian system

\[\frac{d\Psi}{d\lambda} = A(x, \lambda) \Psi, \quad A(x, \lambda) := \begin{bmatrix} A_0(x) & A_1(x) \\ -\lambda^{-1} & \lambda^{-1} \end{bmatrix}, \quad \lambda \in \mathbb{C}.\]  

with matrix coefficients \( A_i(x) \) \((i = 0, x, 1)\). The isomonodromic deformation approach was developed in generality for the six Painlevé equations in [28] [29] [30], but for Fuchsian systems it goes back to Schlesinger’s work [42]. The system (5) is by definition isomonodromic if there exists a fundamental \( 2 \times 2 \) matrix solution \( \Psi(\lambda, x) \) whose monodromy is independent of \( x \). In other words, (5) must be one of the two systems of a Lax Pair (note that this can be rephrased by saying that we have an integrable structure). The compatibility condition of the Lax Pair is the Schlesinger equations

\[\frac{dA_0}{dx} = \frac{[A_x, A_0]}{x}, \quad \frac{dA_1}{dx} = \frac{[A_1, A_x]}{1-x}, \quad \frac{dA_x}{dx} = \frac{[A_x, A_0]}{x} + \frac{[A_1, A_x]}{1-x}.\]

If the matrix coefficients of (5) satisfy special conditions, the Schlesinger equations are equivalent to the sixth Painlevé equation, as established in [29]. These conditions are:

\[A_0 + A_1 + A_x = -\frac{\theta_\infty}{2} \sigma_3, \quad \theta_\infty \neq 0. \quad \text{Eigenvalues } (A_i) = \pm \frac{1}{2} \theta_i, \quad i = 0, 1, x;\]

where the \( \theta_\mu \)'s, \( \mu = 0, x, 1, \infty \), are defined in terms of the coefficients of \( \text{PVI}_{\alpha, \beta, \gamma, \delta} \) as follows

\[\alpha = \frac{1}{2} (\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2} \theta_1^2, \quad \gamma = \frac{1}{2} \theta_x^2, \quad \left( \frac{1}{2} - \delta \right) = \frac{1}{2} \theta_x^2, \quad \theta_\infty \neq 0\]

Under the above conditions, the matrices \( A_i(x) \) are given by explicit algebraic formulæ in terms of \( y(x) \). In particular,

\[A_{12}(x, \lambda) = \frac{g(x)(\lambda - y(x))}{\lambda(\lambda - 1)(\lambda - x)},\]
where $g(x)$ is a certain algebraic function of $x$. Therefore

$$y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}$$  \hspace{1cm} (8)

The local behaviours (3) are obtained substituting into (8) the critical behaviours at $x = 0$ of a class of solutions of the Schlesinger equations (see section 3.2). The connection problem is then solved because the parameters $(a_0, \sigma_{0x})$, $(a_1, \sigma_{x1})$ and $(a, \sigma_{01})$ can be expressed as functions of the monodromy data of the Fuchsian system. We will come back to this point in sections 2.2 and 3.2.

**Remark:** When $\Re \sigma_{ij} = 0$, there are three leading terms of the same order in $y(x)$. For example, at $x = 0$:

$$y(x) = a_0 x^{1-\sigma_{0x}} + \frac{4A^2}{a_0} x^{1+\sigma_{0x}} + Bx + O(x^2),$$

where $A = A(\sigma_{0x})$ and $B = B(\sigma_{0x})$ are

$$\begin{align*}
A^2 &= \frac{(\sigma_{0x}^2-(\theta_0-\theta_2)^2)(\sigma_{0x}^2-(\theta_0+\theta_2)^2)}{4\sigma_{0x}^4}, \\
B &= \frac{\theta_0^2-\theta_2^2+\sigma_{0x}^2}{2\sigma_{0x}^2}.
\end{align*}$$  \hspace{1cm} (9)

We can rewrite the above as

$$y(x) = x \left\{ A \sin(i\sigma_{0x} \ln x + \phi) + B \right\} + O(x^2), \quad a_0 = \frac{A}{2t} e^{i\phi}. \hspace{1cm} (10)$$

The last formula defines $\phi$ in terms of $a_0$, and the freedom $A \rightarrow -A$ corresponds to the freedom $\phi \rightarrow \phi + \pi$, while $a_0$ is fixed. It may be convenient to write $\sigma_{0x} = -2i\nu$, $\nu \in \mathbb{R}$, in such a way that

$$y(x) = x \left\{ A \sin(2\nu \ln x + \phi) - \frac{\theta_0^2-\theta_2^2-4\nu^2}{8\nu^2} \right\} + O(x^2), \quad x \rightarrow 0. \hspace{1cm} (11)$$

In the same way as above, if $\Re \sigma_{x1} = 0$, then (3) at $x = 1$ is replaced by

$$y(x) = 1 - (1 - x) \left\{ A_1 \sin(i\sigma_{x1} \ln(1-x) + \phi_1) + B_1 \right\} + O((1-x)^2), \hspace{1cm} (12)$$

for suitable $A_1$ and $B_1$. If $\Re \sigma_{01} = 0$, then (3) at $x = \infty$ is replaced (for suitable $A_{\infty}$ and $B_{\infty}$) by

$$y(x) = A_{\infty} \sin(i\sigma_{01} \ln(\frac{1}{x}) + \phi_{\infty}) + B_{\infty} + O(\frac{1}{x}). \hspace{1cm} (13)$$

### 2.2 Method of Monodromy Preserving Deformations

The paper [27] opened the way to further research on PVI, based on the method of monodromy preserving deformations. The power of this method is precisely that it allows to solve the connection problem. Its general scheme is as follows.

First, we define the monodromy data of the system (5). In the “$\lambda$-plane” $\mathbb{C} \setminus \{0, x, 1\}$ we fix a base point $\lambda_0$ and three loops, which are numbered in order 1, 2, 3 according to a counter-clockwise order referred to $\lambda_0$. We choose $0, x, 1$ to be the order 1, 2, 3. We denote the loops by $\gamma_0, \gamma_x, \gamma_1$. See figure 1. The loop at infinity will be $\gamma_{\infty} = \gamma_0 \gamma_x \gamma_1$. When $\lambda$ goes around a small loop around $\lambda = i$, $i = 0, x, 1$, the fundamental solution transforms like $\Psi \rightarrow \Psi M_i$, where $M_0, M_x, M_1$ are the monodromy matrices w.r.t. the base of loops. Let $(\theta_0, \theta_1, \theta_x, \theta_{\infty}) \in \mathbb{C}^4$ be
fixed by PVI, up to the equivalence \( \theta_k \mapsto -\theta_k, \ k = 0, x, 1 \), and \( \theta_\infty \mapsto 2 - \theta_\infty \). Denote \( \sim \) the equivalence and let the quotient be

\[
\Theta := \{(\theta_0, \theta_1, \theta_x, \theta_\infty) \in \mathbb{C}^4 \mid \theta_\infty \neq 0\}/\sim.
\]

Let \( M_\infty := M_1 M_x M_0 \) be the monodromy at \( \lambda = \infty \), and consider the set of triples of (invertible) monodromy matrices with unit determinant, defined up to conjugation \( M_i \mapsto CM_i C^{-1} \) (\( i = 0, x, 1 \)) by an invertible matrix \( C \), namely

\[
M := \{(M_0, M_x, M_1) \in \text{SL}(2, \mathbb{C}) \mid \text{Tr} M_\mu = 2 \cos \pi \theta_\mu, \ \mu = 0, 1, x, \infty\}/\text{conjugation}
\]

**Definition:** The *monodromy data* of the class of Fuchsian systems (5), with the basis of loops ordered as figure 1, is the set \( \mathcal{M} := \Theta \cup M \).

When we fix branch cuts in the \( x \) plane, for example \(-\pi < \arg x < \pi \) and \(-\pi < \arg(1-x) < \pi \), then to every branch \( y(x) \), a system (5) is associated, and so is a point in \( \mathcal{M} \). Conversely, to a point in \( \mathcal{M} \) a system or a family of systems (5) is associated through a Riemann-Hilbert problem [1], and so is either one branch \( y(x) \) or a family of branches \( y(x) \). Let

\[
f : \{y(x) \text{ branch}\} \to \mathcal{M}
\]

be the map from the set \( \{y(x) \text{ branch}\} \) of all the branches of all equations PVI\( _{\alpha,\beta,\gamma,\delta} \), \( (\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4 \), onto \( \mathcal{M} \), associating to a branch the corresponding monodromy data. This map gives a way of classifying the branches of PVI transcendents and their critical behaviours in terms of monodromy data. It is through this map that we can in principle answer the question wether the class of critical behaviours known today is complete. For if the image through \( f \) of this class is the whole of \( \mathcal{M} \), then we are sure that the class contains all the possible critical behaviours. \( f \) is generically injective, according to the following

**Proposition 1** Let the order of loops be fixed. The map (14) is injective (one-to-one) when restricted to \( f^{-1}(\Theta \cup \{(M_0, M_x, M_1) \in M \mid M_\mu \neq I, \ \forall \mu = 0, x, 1, \infty\}) \) [19].

![Figure 1: The ordered basis of loops](image-url)
Let (1) be the critical behaviour for \( x \to u, u \in \{0, 1, \infty\} \), depending on two integration constants \( c_1^u, c_2^u \). The map \( f \) can be made explicit because the monodromy matrices can be computed as explicit functions of \( c_1^u, c_2^u \). Conversely, \( c_1^u, c_2^u \) can be expressed in terms of the associated monodromy data, provided that \( f \) is restricted as in Proposition 1, so that it is injective. In order to make \( f \) explicit, we need parameters (coordinates) to identify a point of \( M \). Let

\[
p_{ij} := \text{Tr}(M_i M_j), \quad j = 0, x, 1; \quad p_{\mu} := \text{Tr}M_{\mu} = 2 \cos \pi \theta_{\mu}, \quad \mu = 0, x, 1, \infty.
\]

Observe that \( p_{ij} = p_{ji} \). These are seven invariant functions (w.r.t. conjugation and \( \sim \)) defined on \( M \). They satisfy the relation

\[
p_0^2 + p_{01}^2 + p_{x1}^2 + p_{0x} p_{01} p_{x1} - (p_0 p_x + p_1 p_{\infty})(p_0 x - (p_0 p_1 + p_1 p_{\infty}) p_0 - (p_x p_1 + p_0 p_{\infty}) p_x + \]

\[
+ p_0^2 + p_{x1}^2 + p_{0x}^2 + p_0 p_x p_1 p_{\infty} - 4 = 0
\]

in agreement with the dimension of \( M \). This relation is due to Jimbo [27] (it follows from the trace of the relation \( M_1 M_x M_0 = M_{\infty} \)), but was named Fricke cubic in [3]. The geometry of \( M \), where (15) is an affine cubic, was studied in [25]. The \( p_{ij} \) are the local coordinates on \( M \) we are looking for, according to the following proposition:

**Proposition 2** \( p_0, p_x, p_1, p_{\infty}; p_{0x}, p_{x1}, p_{1x} \) are coordinates on the subset of \( M \) where the group generated by \( M_0, M_x, M_1 \) is irreducible [25].

We come back to the problem of the explicit form of \( f \). Suppose that Propositions 1 and 2 hold for a given PVI \( \alpha, \beta, \gamma, \delta \). The \( p_{\mu} \)'s \( \mu = 0, x, 1, \infty \) are fixed by \( \alpha, \beta, \gamma, \delta \) and only two parameters \( p_{ij} \)'s are independent. They may be thought as the integration constants for the associated branch \( y(x) \) with critical behaviour (1). The asymptotic techniques of the method of monodromy preserving deformations provide explicit *parametric formulae*

\[
\begin{cases}
  c_1^u = c_1^u(p_0, p_x, p_1, p_{\infty}, p_{0x}, p_{x1}, p_{01}) \\
  c_2^u = c_2^u(p_0, p_x, p_1, p_{\infty}, p_{0x}, p_{x1}, p_{01})
\end{cases}
\]

and the inverse formulae

\[
\begin{align*}
  p_{0x} &= p_{0x}(c_1^u, c_2^u, p_0, p_x, p_1, p_{\infty}) \\
  p_{x1} &= p_{x1}(c_1^u, c_2^u, p_0, p_x, p_1, p_{\infty}) \\
  p_{01} &= p_{01}(c_1^u, c_2^u, p_0, p_x, p_1, p_{\infty})
\end{align*}
\]

Explicit means that the formulae are classical functions of their arguments, as we have already explained. Three pairs of different parametric formulae of type (16) can be written at \( x = 0, 1 \) and \( \infty \) respectively, in terms of the same monodromy data, namely for the same branch \( y(x) \). Conversely, the monodromy data associated to a given \( y(x) \) can be written as in (17) in three ways, namely in terms of the three couples of integration constants at \( x = 0, 1 \) and \( \infty \) respectively. Thus, we say that the formulae (16) and (17) are *parametric connection formulae*. The *connection formulae in closed form* (2) are obtained form the parametric form, combining (16) and (17) at two critical points \( x = u \) and \( v \).

An example of the above parametric formulae is the relation, established in [27], among the *monodromy exponents* \( \sigma_{0x}, \sigma_{x1} \) and \( \sigma_{01} \) of (3) and the \( p_{ij} \)'s:

\[
2 \cos \pi \sigma_{0x} = p_{0x}, \quad 2 \cos \pi \sigma_{x1} = p_{x1}, \quad 2 \cos \pi \sigma_{01} = p_{01}
\]
It is clear from the above that the restriction (4) implies that the image through $f$ of the branches with behaviour (3) and (10) (and similar behaviours like (10) at $x = 1, \infty$) is contained in a wide subspace of $\mathcal{M}$, characterised by

$$p_{0x}, p_{x1}, p_{01} \not\in (-\infty, -2] \cup \{2\}.$$

In particular, for solutions (10) we have $p_{0x} > 2$, while $p_{x1} > 2$, $p_{01} > 2$ for (12) and (13) respectively. Similarly, there are explicit formulae

$$a_i = a_i(p_0, p_x, p_1, p_{0x}, p_{x1}, p_{01}) \equiv a_i(\theta_0, \theta_x, \theta_1, \theta_{\infty}, \sigma, p_{x1}, p_{01}), \quad i = 0, 1, \infty, \quad (19)$$

which the reader can find in [27], or more explicitly in [3] (they are collected in [22] as well).

### 3 Finding the Critical Behaviours

In section 2, we have described a wide class of critical behaviours (3) and (10). In this section, we explain in more details how they were derived. Then, we discuss the matching method, which allows to find further critical behaviours. We also review the symmetries of PVI, which, applied to the critical behaviours obtained by Jimbo method and the matching method, produce further critical behaviours. As a final result, we obtain essentially all the possible critical behaviours, with their full expansion. They can be tabulated like the classical special functions, as we will see in section 3.5.

#### 3.1 Preamble: Symmetries

It is possible to study critical behaviours and parametric connection formulae at one critical point, say $x = 0$, and transfer the results to the others. This is because PVI admits symmetry transformations, called Okamoto’s bi-rational transformations, introduced in [37], of the form

$$y'(x) = \frac{P(x, y(x), \frac{dy(x)}{dx})}{Q(x, y(x), \frac{dy(x)}{dx})}, \quad x' = \frac{p(x)}{q(x)}, \quad (\theta_0, \theta_x, \theta_1, \theta_{\infty}) \mapsto (\theta'_0, \theta'_x, \theta'_1, \theta'_{\infty}) \quad (20)$$

such that $y(x)$ satisfies (PVI) with coefficients $\theta_0, \theta_x, \theta_1, \theta_{\infty}$ and variable $x$, if and only if $y'(x')$ satisfies (PVI) with coefficients $\theta'_0, \theta'_x, \theta'_1, \theta'_{\infty}$ and variable $x'$. The functions $P, Q$ are polynomials; $p, q$ are linear; the transformation of the $\theta_i$’s is an element of a linear representation of the Weyl Group of the root system $D_4$, or the group of Shift $l_j : v = (v_1, v_2, v_3, v_4) \mapsto v + e_j, \ j = 1, 2, 3, 4$, where $e_1 = (1, 0, 0, 0), \ ..., \ e_4 = (0, 0, 0, 1)$, or the Permutation group. The latter acts as follows:

$$\theta'_0 = \theta_1, \quad \theta'_x = \theta_x, \quad \theta'_1 = \theta_0, \quad \theta'_{\infty} = \theta_{\infty}; \quad y'(x') = 1 - y(x), \quad x' = 1 - x. \quad (21)$$

$$\theta'_0 = \theta_1, \quad \theta'_x = \theta_x; \quad \theta'_0 = \theta_0, \quad \theta'_1 = \theta_{\infty}; \quad y'(x') = \frac{1}{x}y(x), \quad x' = \frac{1}{x}. \quad (22)$$

$$\theta'_0 = \theta_{\infty} - 1, \quad \theta'_x = \theta_1, \quad \theta'_1 = \theta_{x}, \quad \theta'_{\infty} = \theta_0 + 1; \quad y'(x') = \frac{x}{y(x)}, \quad x = x'. \quad (23)$$

(21) and (22) produce critical behaviours at $x = 1, \infty$ respectively, from behaviours at $x = 0$. The transformation (23) generates new behaviours from known ones at a given critical point.
The effect of the bi-rational transformation on the monodromy data $p_{ij}$ is described in [27], [10], [19], and more generally in [11] (see also [36]). In particular, we have:

For (21): \[ p'_{01} = -p_{01} - p_{0x}p_{x1} + p_{\infty}p_x + p_1p_0, \quad p'_{0x} = p_{x1} \quad p'_{x1} = p_{0x}. \quad (24) \]

For (22): \[ p'_{0x} = -p_{01} - p_{0x}p_{x1} + p_{\infty}p_x + p_0p_1, \quad p'_{01} = p_{0x}, \quad p'_{1x} = p_{1x}. \quad (25) \]

For (23): \[ p'_{0x} = -p_{0x}, \quad p'_{01} = -p_{01}, \quad p'_{x1} = p_{x1}. \quad (26) \]

### 3.2 The Critical Behaviours in Jimbo’s work (II)

The behaviours (3) (or (10), (12) and (13)) follow from a result on a class of solutions of the Schlesinger equations, established in Lemma 2.4.8 (page 262) of [41], stating the following.

Let $A_0^0$, $A_x^0$, $A_1^0$ be constant matrices satisfying (6). Let $\Lambda := A_0^0 + A_x^0$ and let its eigenvalues $\pm \sigma_{0x}/2$ be given (they will be precisely the monodromy exponents in (3)). Suppose that $|\Re \sigma_{0x}| < 1$ and $\Lambda \neq 0$. Then, for any $\vartheta > 0$ there exists a small $r = r(\vartheta)$ such that the Schlesinger equations admit a unique solution defined for $0 < |x| < r$, $|\arg(x)| < \vartheta$, with the properties

\[ A_1(x) = A_1^0 + O(|x|^\delta), \quad x^{-\Lambda}A_0(x) \quad x^\Lambda = A_0^0 + O(|x|^\delta), \quad (27) \]

where $\delta \leq 1 - |\Re \sigma_{0x}|$ is a positive small number. Note that the following limit exists:

\[ \Lambda = \lim_{x \to 0} (A_0(x) + A_x(x)) \neq 0, \]

It is not difficult to find the general parametrization of $A_0^0$, $A_x^0$, $A_1^0$ satisfying (6) and $A_0^0 + A_x^0 = \Lambda$. An extra parameter $a_0 \neq 0$ will appear, together with $\sigma_{0x}$. Then, the behaviour (3) at $x = 0$ (or (10), if $\Re \sigma_{0x} = 0$), is obtained by the substitution of the critical behaviors (27) into (8). The behaviours at $x = 1, \infty$ can then be computed by means of (21) and (22). Note that one can substitute the condition $|\Re \sigma_{0x}| < 1$ with the equivalent

\[ 0 \leq \Re \sigma_{0x} < 1. \]

Jimbo obtained the connection formulae (16) and (17) (the role of $(c_1^0, c_2^0)$ is played by $(\sigma_{0x}, a_0)$), by reducing the system (5) to two simpler systems, whose monodromy matrices are exactly computable. He proved that the following limits, constructed from an isomonodromic fundamental solution $\Psi(\lambda, x)$ of (5), exist

\[ \Psi_{OUT}(\lambda) := \lim_{x \to 0} \Psi(\lambda, x), \quad \Psi_{IN}(\lambda) := \lim_{x \to 0} \Psi(x\lambda, x), \]

and satisfy

\[ \frac{d\Psi_{OUT}}{d\lambda} = \left[ \frac{\Lambda}{\lambda} + \frac{A_0^0}{\lambda - 1} \right] \Psi_{OUT}, \quad \frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0^0}{\lambda} + \frac{A_x^0}{\lambda - x} \right] \Psi_{IN}. \quad (28) \]

The notation $\Psi_{IN}$ and $\Psi_{OUT}$ is not in the original paper [27]; we will explain the reason for it below. It follows from the isomonodromy property that $\hat{M}_1$ coincides with $M_{1\text{OUT}}$ of $\Psi_{OUT}$, while $M_0$ and $M_x$ coincide with $M_{0\text{IN}}$ and $M_{x\text{IN}}$ of $\Psi_{IN}$ respectively. Systems (28) are equivalent to Gauss hyper-geometric equations and the monodromy matrices $M_{1\text{OUT}}$, $M_{0\text{IN}}$ and $M_{x\text{IN}}$ can be computed from the connection formulae for the hyper-geometric functions. Their matrix elements contain $\sigma_{0x}$ and $a_0$ as parameters. Then, (16) and (17) for (3) follow by taking the traces of the products $M_iM_j$, $i, j \in 0, x, 1$. We refer to the original works [27] and [3] (and the collection/tabulation [22]) for the resulting formulae (18) and (19).
3.3 Matching

The critical behaviours (3) and (10) are based on the restriction (4), which is a restriction on \( \Lambda \). The question arises whether the limit procedure of section 3.2 can be generalised when \( \Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and when

\[
\lim_{x \to 0} (A_0(x) + A_x(x)) = 0
\]

We also would like to study cases where \( A_0(x) \) and \( A_x(x) \), taken separately, diverge faster than a power \( x^{-|\sigma|/2} \), \( 0 < |\sigma| < 1 \), for example, cases when \( xA_0(x) \to \) constant matrix, for \( x \to 0 \). These behaviors substituted into (8) will produce critical behaviours different from (3).

The above question, together with the search for a constructive procedure which produces critical behaviours of \( A_0(x) \), \( A_x(x) \) and \( A_1(x) \) with minimal assumptions, was the starting point for the matching procedure developed in [18]. Originally, such procedure was suggested by Its and Novokshenov in [24], for the second and third Painlevé equations. This approach was further developed and used by Kapaev, Kitaev, Andreev, and Vartanian (see for example [2]). We finally implemented it for PVI in [18] (I thank A. Kitaev for introducing me to the method).

We briefly explain the construction. Let us divide the \( \lambda \)-plane into two domains. The “outside” domain is defined for \( |\lambda| \geq |x|^{\delta_{OUT}} \), \( \delta_{OUT} > 0 \). In this domain, (5) can be approximated by:

\[
\frac{d\Psi_{OUT}}{d\lambda} = \begin{bmatrix} A_0 + A_x \\ \frac{x A_x}{\lambda} + \sum_{n=0}^{N_{OUT}} \left( \frac{x}{\lambda} \right)^n + \frac{A_1}{\lambda - 1} \end{bmatrix} \Psi_{OUT},
\]

(29)

The “inside” domain is defined for \( \lambda \) comparable with \( x \), namely \( |\lambda| \leq |x|^{\delta_{IN}} \), \( \delta_{IN} > 0 \), and (5) can be approximated by:

\[
\frac{d\Psi_{IN}}{d\lambda} = \begin{bmatrix} A_0 \\ \frac{x A_x}{\lambda - x} - \sum_{n=0}^{N_{IN}} \lambda^n \end{bmatrix} \Psi_{IN},
\]

(30)

where \( N_{IN}, N_{OUT} \) are suitable integers. The leading term of \( y(x) \) is obtained by requiring that the fundamental matrix solutions \( \Psi_{OUT}(\lambda, x) \), \( \Psi_{IN}(\lambda, x) \) match in the region of overlap, provided this is not empty:

\[
\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \quad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}, \quad x \to 0
\]

(31)

This relation is to be intended in the sense that the leading terms of the local behaviour of \( \Psi_{OUT} \) and \( \Psi_{IN} \) for \( x \to 0 \) must be equal (note: \( \delta_{IN} \leq \delta_{OUT} \)). In this procedure, also the reduced systems (29) and (30) are isomonodromic. The matching condition (31) produces the \( x \)-leading terms for \( x \to 0 \) of \( \Psi_{OUT}(\lambda, x) \) and \( \Psi_{IN}(\lambda, x) \), and then of \( A_0(x) \), \( A_1(x) \) and \( A_x(x) \) (recall that \( A = d\Psi/d\lambda \cdot \Psi^{-1} \)). Finally, it produces the leading term(s) of \( y(x) \) in (8).

The Fuchsian systems (28) represent the simplest case, occurring when \( 0 \leq \Re \sigma_{0x} < 1 \), but the procedure in general involves non Fuchsian reductions. As a result, new critical behaviours, different from (3) and (10), are produced through (8). For example:

– The (basic) logarithmic behaviours when \( \sigma_{0x} = 0 \):

\[
y(x) = x \left[ \frac{\theta_0^2 - \theta_x^2}{4} (\ln x + a)^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2} \right] + O(x^2 \ln^3 x), \quad \text{if} \quad \theta_0^2 \neq \theta_x^2,
\]

(32)

\[
y(x) = x(a \pm \theta_0 \ln x) + O(x^2 \ln^2 x), \quad \text{if} \quad \theta_0^2 = \theta_x^2,
\]

(33)
where $a$ is an integration constant. This solution is discussed in [27] as well, as a limit of (3) for $\sigma_{0x} \to 0$.

- The (basic) Taylor expansions: we will write them in the table of section 3.5.

In order to compute the parametric connection formulae, we compute the monodromy data of (5) as follows. Once the matching $\Psi_{OUT} \leftrightarrow \Psi_{IN}$ in (31) has been completed, we match $\Psi_{OUT}$ with a fundamental solution $\Psi$ of (5) at $\lambda = \infty$, and we match $\Psi_{IN}$ with the same $\Psi$ in another region of the $\lambda$-plane, typically around $\lambda = 0$ or $x$. If this matching is realised, $M_1$ coincides with $M^{OUT}_1$ of $\Psi_{OUT}$, while $M_0$ and $M_x$ coincide with $M^{IN}_0$ and $M^{IN}_x$ of $\Psi_{IN}$. Their entries contain the integration constants of the critical behavior of $y(x)$. Thus, (16) and (17) follow by taking the traces of the products $M_i M_j$, $i, j \in 0, x, 1$. Since we need to compute the monodromy matrices $M^{OUT}_1$, $M^{IN}_0$ and $M^{IN}_x$ exactly, we should be able to solve the reduced "IN" and "OUT" systems in terms of linear special functions, or find that they are the linear systems associated to other Painlevé equations for which we already know the monodromy. In [18] and [19] the monodromy matrices associated to the basic Taylor and logarithmic solutions are computed. The Taylor solutions and the associated monodromy are computed also in [31].

### 3.4 Other behaviours generated by symmetries

The birational transformations (21) and (22) generate the critical behaviours at $x = 1$ and $\infty$. For example, when applied to (32), they provide the behaviours:

$$y(x) \sim 1 - (1 - x) \left\{ \frac{\theta_1^2 - \theta_0^2}{\theta_1^2 - \theta_x^2} + \frac{\theta_x^2 - \theta_0^2}{4}(\ln(1 - x) + a_1)^2 \right\}, \quad x \to 1,$$

$$y(x) \sim \frac{\theta_0^2}{\theta_0^2 - \theta_1^2} + \frac{\theta_1^2 - \theta_0^2}{4} \left[ \ln \frac{1}{x} + a_\infty \right]^2, \quad x \to \infty.$$

The symmetry (23) produces new behaviours at a given critical point starting from known ones. For example, when applied to (32) at $x = 0$, it provides the branch with the behaviour

$$y(x) = \frac{4}{(\theta_1^2 - (\theta_\infty - 1)^2)^2 \ln^2 x} \left[ 1 - \frac{2a}{\ln x} + \left( \frac{1}{\ln^2 x} \right) \right], \quad x \to 0$$

When applied to (11), it generates a branch with behaviour

$$y(x) = \frac{1}{A \sin(2\nu \ln x + \phi) + B + O(x)}, \quad x \to 0 \quad (34)$$

where $\nu \in \mathbb{R}\{0\}$ and $\phi \in \mathbb{C}$ are integration constants,

$$A = \left[ \left( \frac{(\theta_\infty - 1)^2}{4\nu^2} + \frac{(\theta_\infty - 1)^2 - \theta_1^2 - 4\nu^2}{8\nu^2} \right) \right]^{1/2}, \quad B = -\frac{(\theta_\infty - 1)^2 - \theta_1 - 4\nu^2}{8\nu^2},$$

and $p_{0x} = 2 \cos(\pi(1 + 2\nu)) < -2$. There are similar behaviours at $x = 1$ and $\infty$, namely

$$y(x) = 1 - \frac{1}{A_1 \sin(2\nu_1 \ln(1 - x) + \phi_1) + B_1 + O(1 - x)}, \quad x \to 1; \quad p_{x1} < -2, \quad (35)$$

$$y(x) = \frac{x}{A_\infty \sin(2\nu_\infty \ln(1/x) + \phi_\infty) + B_\infty + O(1/x)}, \quad x \to \infty; \quad p_{01} < -2. \quad (36)$$

with suitable $A$ and $B$. 


3.5 Tabulation

The critical behaviors admit full expansions (convergent, asymptotic or formal). For example, the procedure to compute the expansion of (3) at $x = 0$ is given in [20]. It involves a recursive computation, based on the substitution into PVI, which increases in complication with the order of the expansion (a conjectural closed form for the coefficients at all orders of the $\tau$ function associated to (3) can be found in [15]). The Taylor expansions can be obtained by a recursive computation based on the substitution into PVI. The same can be done starting from the behaviours (32) and (33), to generate two full "series expansions" at $x = 0$ whose coefficients are polynomials of $\log x$.

The birational transformations associated to the Weyl group, the Shifts and Permutations applied to the full expansion of the solutions obtained in subsections 3.2 and 3.3, generate a variety of critical behaviours, whose image through $f$ is essentially the whole of $\mathcal{M}$ [22]. The systematic tabulation of these behaviours is possible. In the paper [22], we provided the tables of the critical behaviours with full expansion at $x = 0$, 1 and $\infty$, and the parametric connection formulae. Below, we reproduce the table of [22], only at $x = 0$, and in a simplified form. The branches in the table may be classified according to their behaviour:

- Complex power behaviours: they are expanded in powers of $x^{n+m\lambda}$, for some $n,m \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$. In this case, $|y(x)|$ may vanish, converge to a constant or diverge when $x \to 0$. They include (3) and (10).

- Inverse oscillatory behaviours: $y(x)$ oscillates without vanishing when $x \to 0$, and may have poles in a sector centred at $x = 0$ (these poles are reviewed in section 4).

- Taylor series.

- Logarithmic behaviours, namely series expansions with coefficients which are polynomials of $\ln x$. $y(x) \to$ constant as $x \to 0$, where the constant may be zero or not.

- Inverse logarithmic behaviours, expanded as formal series of $(\ln x)^{-1}$. $y(x) = O(1/\ln x)$, or $O(1/\ln^2 x)$, as $x \to 0$.

For any $\vartheta > 0$ there exists $r(\vartheta)$ small enough (decreasing function of $\vartheta$) such that the expansion of the complex power behaviors converges for $|\arg(x)| < \vartheta$ and $0 < |x| < r(\vartheta)$. If $\Re\sigma = 0$, there is also an additional constraint $\arg(x) > \varphi_0$, for a suitable $\varphi_0$ fixed by the integration constants. Also the expansion of the denominator of the inverse oscillatory behaviours is convergent under the same conditions $|\arg(x)| < \vartheta$, for any choice of a $\vartheta > 0$, and $0 < |x| < r(\vartheta)$, plus the additional constraint $\arg(x) < \varphi_0$, for a suitable $\varphi_0$ (see section 4 and 5). When logarithms appear, no proof of convergence is known yet. Convergence of the Taylor expansions is studied in [31].

In the table, $\sigma, \phi, \nu$ and $a$ denote integration constants. The coefficients $c_{nm}$, $d_{nm}$ and $b_n$ are rational functions of $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ and $\sqrt{1-2\delta}$. The coefficients $b_n(a)$ are rational functions of $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}, \sqrt{1-2\delta}$ and $a$. $P_n(\ln x, a)$ are polynomials in $\ln x$ with coefficients which are rational functions of $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}, \sqrt{1-2\delta}$ and $a$. These coefficients can be recursively computed (essentially, by substitution into PVI).
### Complex power behaviours

| $y(x) = \sum_{n=1}^{\infty} x^n \sum_{m=-n}^{n} c_{nm}(ax^\sigma)^m$ | Integ. const. | Other Conditions |
|---|---|---|
| This is (3) and (11). | $\sigma$ | $0 \leq \Re \sigma < 1$, $2 \cos \pi \sigma = p_{0x}$, $p_{0x} \not\in (-\infty, -2] \cup \{2\}$ |
| $y(x) = \frac{\sqrt{\pi+(-)^k\sqrt{\gamma}}}{\sqrt{\alpha}} + \sum_{n=1}^{\infty} x^n \sum_{m=0}^{n} d_{nm}(ax^\rho)^m$ | $a$ | $\alpha \neq 0$. $p_{0x} = 2 \cos \pi \rho \neq \pm 2$. $\rho + 1 = (\sqrt{2\alpha} \pm \sqrt{2\gamma})\text{sgn}(\sqrt{2\alpha} \pm \sqrt{2\gamma})$ |
| $y(x) = \frac{1}{a} x^{-\omega} (1 + \sum_{n=1}^{\infty} x^n \sum_{m=0}^{n} d_{nm}(ax^\omega)^m)$ | $a$ | $\alpha = 0$. $\omega = \sqrt{2\gamma} \text{sgn}(\Re \sqrt{2\gamma})$, $\Re \omega > 0$ |

### Inverse Oscillatory Behaviours

| $y(x) = \left[ \sum_{n=0}^{\infty} x^n \sum_{m=-n-1}^{n+1} c_{nm}(e^{i\phi}x^{2\nu})^m \right]^{-1}$ | Integration constants | Other |
|---|---|---|
| $= \left[ A \sin(2\nu \ln x + \phi) + B + O(x) \right]^{-1}$ | $\nu$ | $\nu \in \Re \setminus \{0\}$, $-2 \cosh 2\pi \nu = p_{0x} < -2$. |
| $A = -\sqrt{\frac{\alpha \nu + 2\nu + \gamma - \alpha}{2\nu^2}}$, $B = \frac{2\nu^2 + \gamma - \alpha}{4\nu^2}$ | $\phi$ | |
| $y(x) = \left[ \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n+1} c_{n+1,m}(ax^{-2i\nu})^m \right]^{-1}$ | $a$ | $2i\nu = \pm(\sqrt{2\alpha} \pm \sqrt{2\gamma}) \in i\Re \setminus \{0\}$ |
| Taylor expansions | Int. const. | Other Conditions |
|-------------------|-------------|-----------------|
| $y(x) = \frac{\sqrt{\sqrt{-2\gamma}}}{\sqrt{-2\beta} + (-)^k \sqrt{1 - 2\delta}} x + \sum_{n=2}^{\infty} b_n x^n$ | | $\sqrt{-2\beta} \pm \sqrt{1 - 2\delta} \notin \mathbb{Z}$. $p_{0x} \neq \pm 2$. |
| $y(x) = \sum_{n=1}^{n=|N|} b_n x^n + ax |N| + 1 + \sum_{n=|N|+2}^{\infty} b_n(a) x^n$ | $a$ | $\sqrt{-2\beta} \pm \sqrt{1 - 2\delta} = N \neq 0$ $p_{0x} = 2 \cos \pi N = \pm 2$. |
| $y(x) = ax + a(a - 1) \left( \gamma - \alpha - \frac{1}{2} \right) x^2 + \sum_{n=3}^{\infty} b_n(a) x^n$ | $a$ | $2\beta = 2\delta - 1 = 0.$ $p_{0x} = 2$. |
| $y(x) = \frac{\sqrt{\alpha + (-)^k \sqrt{\gamma}}}{\sqrt{\alpha}} + \sum_{n=1}^{\infty} b_n x^n$, Basic Taylor | | $\alpha \neq 0$, $\sqrt{2\alpha} + (-)^k \sqrt{2\gamma} \notin \mathbb{Z}$. $p_{0x} \neq \pm 2$. |
| $y(x) = \sum_{n=0}^{n=|N|-1} b_n x^n + ax |N| + \sum_{n=|N|+1}^{\infty} b_n(a) x^n$ | $a$ | $\sqrt{2\alpha} \pm \sqrt{2\gamma} = N \neq 0$ $p_{0x} = -2 \cos \pi N = \pm 2$. |
| $y(x) = a + (1 - a)(\delta - \beta)x + \sum_{n=2}^{\infty} b_n(a) x^n$ | $a$ | $\alpha = \gamma = 0$. $p_{0x} = -2$. |
### Logarithmic behaviours

| $y(x)$ | Int. const. | Other Conditions |
|--------|-------------|------------------|
| $y(x) = \sum_{n=1}^{[N]} b_n x^n + \left( a + b_{[N]+1} \ln x \right) x^{[N]+1} + \sum_{n=[N]+2}^{\infty} P_n(\ln x; a)x^n, \ N \neq 0.$ | $a$ | $\sqrt{-2\beta} \pm \sqrt{1 - 2\delta} = N$ |
| $y(x) = \left( a \pm \sqrt{-2\beta \ln x} \right) x + \sum_{n=2}^{\infty} P_n(\ln x; a)x^n, \ N = 0.$ | $a$ | $p_{0x} = 2 \cos \pi N = \pm 2.$ |

### Inverse logarithmic behaviours

| $y(x)$ | Int. const. | Other Conditions |
|--------|-------------|------------------|
| $y(x) = \left\{ a \pm \sqrt{2\alpha \ln x} + \sum_{n=1}^{\infty} P_n(\ln x; a)x^n \right\}^{-1}$ | $a$ | $\alpha = \gamma \neq 0.$ |
| $y(x) = \frac{1}{\sqrt{2\alpha \ln x}} \left[ 1 - \frac{\alpha}{2\sqrt{2\alpha \ln x}} + O\left( \frac{1}{\ln^2 x} \right) \right]$ | $a$ | $p_{0x} = -2.$ |

### 3.6 The Parametric Connection Formulae (16) and (17)

The parametric connection formulae are quite long to write, so we refer to [22], which collects the formulae computed in [27], [3], [16], [17], [18], [19], [20]. Just to give the taste of them, we give a very simple example. Consider the following solution with Taylor expansion at $x = 0$ (namely, $u = 0$):

$$y(x) = ax + a(a - 1) \left( \gamma - \alpha - \frac{1}{2} \right) x^2 + O(x^3).$$

The formulae (16) are

$$a = \frac{2 \cos \pi \sqrt{2\gamma} - p_{01}}{4 \cos \left( \frac{\pi}{2} \sqrt{2\alpha + \sqrt{2\gamma}} \right) \cos \left( \frac{\pi}{2} \sqrt{2\alpha - \sqrt{2\gamma}} \right)}.$$
and the (17) are
\[ p_{0x} = 2, \]
\[ p_{01} = 2 \cos \pi \sqrt{2\gamma} - 4a \cos \left( \frac{\pi}{2} \left[ \sqrt{2\alpha} + \sqrt{2\gamma} \right] \right) \cos \left( \frac{\pi}{2} \left[ \sqrt{2\alpha} - \sqrt{2\gamma} \right] \right), \]
\[ p_{x1} = 2 \cos \pi \sqrt{2\gamma} + 4(a - 1) \cos \left( \frac{\pi}{2} \left[ \sqrt{2\alpha} + \sqrt{2\gamma} \right] \right) \cos \left( \frac{\pi}{2} \left[ \sqrt{2\alpha} - \sqrt{2\gamma} \right] \right). \]

In the above, \( p_\mu \) (or \( \theta_\mu \), \( \mu = 0, x, 1, \infty \)) are re-expressed in terms of the coefficients of \( \text{PV}_\alpha\beta\gamma\delta \).

### 3.7 Other Approaches to obtain Critical Behaviours

Two alternative approaches to the local analysis of PVI provide the critical behaviors, with full expansions.

The first is a local analysis of integral equations associated to PVI, due to Shimomura [43], [26]. It provides behaviours of type (3) and (10), on the universal covering of a punctured neighbourhood of the critical point. The elliptic representation of PVI gives the same results [16] [17]. We will come back to this point in Section 5.

The second approach is the method of power geometry developed by Bruno in [5] and [6]. This method is a refinement of the method of Newton Polygons. In a series of papers summarised in the review [7], Bruno and Goryuchkina construct all the (possibly formal) expansions that can be obtained by means of this method. They are classified in a way which resents of the method. In [22] we proved that they coincide with those in the table reviewed in section 3.5.

We remark that the local analysis does not provide the connection formulae, for which we need to go back to the isomonodromy deformation method.

### 4 Poles

The position of the poles is known for classical functions, but for \( \text{PV}_\alpha\beta\gamma\delta \) we still do not know the global distribution of the poles, except for special choices of \( \alpha, \beta, \gamma, \delta \). For example, in [4], the pole distribution for \( \text{PV}_1^{1\ 8\ 1\ 8\ 1\ 8\ 3\ 8} \) (the Hitchin equation [23]) is determined on the whole universal covering of \( \mathbb{C}\setminus\{0, 1, \infty\} \). A formula for an infinite series of poles is given in terms of Theta-functions. The poles are distributed along lines which are spirals at a small scale around the critical points, and more complicated lines on the whole universal covering. A birational Okamoto’s transformation transforms \( \text{PV}_1^{1\ 8\ 1\ 8\ 1\ 8\ 3\ 8} \) into \( \text{PV}_0^{0000\ 4} \), known as Picard equation [40].

Though the global distribution of the poles is limited to the above examples, we know the asymptotic distribution of the poles close to a critical point. The existence of poles close to the critical point \( x = 0 \) is due to the structure of the branches (34), because the denominator may vanish when \( x \) approaches zero. The same may happen at \( x = 1 \) and \( x = \infty \) for (35) and (36) respectively. To understand the asymptotic distribution of these poles, let us concentrate on the critical point \( x = 0 \), being the others analogous.

The branch (34) is obtained through (23) from the branch (10) = (11). As discussed in section 3.2, (3) and (10) hold when \( 0 < |x| < r \) and \( |\arg x| < \vartheta \), where \( \vartheta > 0 \) is a positive number (it can be chosen arbitrarily), and \( r = r(\vartheta) \) is small and decreasing when \( \vartheta \) increases. The symmetry (23) does not affect the condition \( |\arg x| < \vartheta \), \( 0 < |x| < r \), therefore (34) holds under the same condition as well. Since \( \arg x \) remains bounded, \( x \) may tend to zero along a radial path, while spiral paths are not allowed. With this restriction, most solutions of PVI have
no poles in a sufficiently small neighbourhood of \( x = 0 \), except precisely the class of solutions (34). Let us denote a solution (34) with \( y = y(x, \nu, \phi) \), and re-write:

\[
y(x, \nu, \phi) = \frac{1}{y_1(x) + g(x)},
\]

\[
y_1(x) := A \sin(2\nu \ln x + \phi) + B, \quad g(x) = O(x),
\]

It follows that \( y(x, \nu, \phi) \) has poles in a neighbourhood of \( x = 0 \), coinciding with the zeros of \( y_1(x) + g(x) \). They are asymptotically close to the zeros of \( y_1(x) \) when \( x \to 0 \), because \( y(x, \nu, \phi)^{-1} \sim y_1(x) \). Direct computation shows that \( y_1(x) \) has two infinite sequences of zeros, distributed along two rays converging to \( x = 0 \) (see figure 2).

The full expansion of (34) is in the table of section 3.5. It is convenient to write it in reciprocal form:

\[
\frac{1}{y(x, \nu, \phi)} = y_1(x) + xy_2(x) + x^2y_3(x) + \ldots = \sum_{n=1}^{\infty} x^{n-1}y_n(x) \tag{37}
\]

where:

\[
y_n(x) = \sum_{m=-n}^{n} A_{nm}(\nu, \alpha, \beta, \gamma, \delta) e^{im\phi} x^{2im\nu}, \quad 0 \leq \Re\phi \leq \pi.
\]  

The \( A_{nm}(\nu, \alpha, \beta, \gamma, \delta) \)'s are algebraic functions of \( \nu, \alpha, \beta, \gamma, \delta \). Their explicit form is recursively computable by the procedure of [20]. The series (37) can be formally computed for any \( x, \nu \) where \( r, \epsilon > 0 \).

\[
\text{Theorem 1} \quad \text{Let } \nu > 0 \text{ and let } y(x, \nu, \phi) \text{ be (34). There are two sequences } \{x_k(1)\}_{k \in \mathbb{Z}} \text{ and } \{x_k(2)\}_{k \in \mathbb{Z}} \text{ of zeros of } y_1(x):
\]

\[
x_k(j) = \exp \left\{ -\frac{\phi}{2\nu} - \frac{i}{2\nu} \ln \left[ (-)^j \sqrt{\frac{A_{10}^2 - A_{11}^2}{4A_{11}^2}} - \frac{A_{10}}{2A_{11}} \right] - \frac{k\pi}{\nu} \right\}, \quad k \in \mathbb{Z}, \quad j = 1, 2 \quad (40)
\]

The argument of \( \ln[ \ ] \) is fixed, being other choices absorbed into \( \frac{k\pi}{\nu} \). Let \( k_0 \in \mathbb{N} \) be sufficiently big in order to have \( |x_k(j)| < r(\theta) \), \( j = 1, 2 \), in such a way that (34) holds. There exists \( K \) sufficiently big such that for every \( k \geq \max\{K, k_0\} \), and every \( j = 1, 2 \), \( y(x, \nu, \phi) \) has a pole \( \xi_k(j) \) lying in a neighbourhood of \( x_k(j) \), with the asymptotic representation

\[
\xi_k(j) = x_k(j) - \frac{1}{2} x_k(j)^2 + \sum_{N=3}^{\infty} \Delta_N(j) x_k(j)^N, \quad k \to +\infty, \quad x_k(j) \to 0. \tag{41}
\]

The coefficients \( \Delta_N(j) \in \mathbb{C} \) are certain numbers independent of \( k \) that can be computed from the coefficients \( A_{nm} \) of (38). There are no other poles for \( |x| \) sufficiently small.

---

2 Note that, in general, for a series like (37), one expects convergence for \( 0 < |x| < r \) and \( |x^{n+2im\nu} e^{im\phi}| < \epsilon_{nm} \), where \( r, \epsilon_{nm} > 0 \) are sufficiently small. Thus

\[
\ln|x| - \Im\phi + \max \left\{ -\ln r, \sup_{m > 0, n \geq 1} \left\lfloor \frac{\ln \epsilon_{nm}}{m} \right\rfloor \right\} < 2\nu \arg x - \Im\phi + \min \left\{ \ln r, \inf_{m < 0, n \geq 1} \left\lfloor \frac{\ln \epsilon_{nm}}{m} \right\rfloor \right\}.
\]
The series (41), computed making use of (37) and (38), is at least asymptotic. We can actually prove its convergence in some cases, for example (see [21]) when the equation is $PVI_{000\frac{1}{2}}$, the already mentioned Picard equation. The position of the poles depends on $\nu$ and $\phi$, therefore it depends on the monodromy data.

**Remark:** Let $\vartheta = \pi$. The zeros $x_k(j)$ and the poles may be outside the range $|\arg x| < \pi$, depending on $\Im \phi$. However, the analytic continuation of the branch $y(x, \nu, \phi)$ corresponding to the loop $x \mapsto xe^{2\pi i}$ is [21]:

$$y(x, \nu, \phi) \mapsto y(x, \nu, \phi + 2\pi i \nu) = y(xe^{2\pi i}, \nu, \phi), \quad -\pi \leq \arg x < \pi.$$  \hfill (42)

The shift $\phi \mapsto \phi + 2\pi i \nu$ changes the imaginary exponent of the $x_k(1)$'s by $-2\pi i$. This implies that, by a sufficient number of loops, we can always find a branch with at least one of the two sequences of poles in the range $-\pi \leq \arg x < \pi$.

**Remark:** (40) implies that

$$2\nu \arg x_k(j) = -\Im \phi - \ln \left((-)^j \sqrt{\frac{A_{10}^2}{4A_{11}^2} - \frac{A_{1,1-1}}{A_{11}} - \frac{A_{10}}{2A_{11}}} \right).$$

Moreover, we know that the poles exist if $2\nu \arg x \geq -\Im \phi + \ln r$. Thus, we obtain an upper estimate of the radius of convergence of (37):

$$r \leq \min \left\{ e^{-\ln \sqrt{\frac{A_{10}^2}{4A_{11}^2} - \frac{A_{1,1-1}}{A_{11}} + \frac{A_{10}}{2A_{11}}}}, e^{-\ln \sqrt{\frac{A_{10}^2}{4A_{11}^2} - \frac{A_{1,1-1}}{A_{11}} + \frac{A_{10}}{2A_{11}}} \right\}.$$

**Example:** If $\alpha = \frac{9}{2}$, $\beta = \gamma = 0$ and $\delta = \frac{1}{2}$, then (37) is a branch associated to the quantum cohomology of $CP^2$ [8], [9]. As it is proved in [21], the branch is identified by the following
integration constants

\[ \nu = \frac{2 \ln G}{\pi} = 0.30634... \]

where \( G = \frac{1 + \sqrt{5}}{2} \) is the golden ratio (I thank M. Mazzocco for this remark), and

\[ \phi = i \ln \left\{ -\pi^2 \frac{(G^4 + 1)^2}{(G^2 + 1)^2} 2^{16i\nu} \frac{(1 - 2i\nu)^2 \nu^2}{(1 + 2i\nu)^2} \frac{\Gamma(1 - 2i\nu)^4}{\Gamma(1 - i\nu)^8} \right\}. \]  

(43)

The first approximation \( y_1(x) \) has two infinite sequences of zeros accumulating at \( x = 0 \) along the negative imaginary axis, because \( \Im \phi = \pi^2/2 \):

\[ x_k(j) = -i \exp \left\{ -\frac{\Re \phi}{2\nu} - \frac{2(j - 1)}{\nu} \left| \arccos \frac{3}{\sqrt{4\nu^2 + 9}} \right| \right\} \exp \left\{ -\frac{k\pi}{\nu} \right\}, \quad k \in \mathbb{N}, \quad j = 1, 2. \]  

(44)

The first coefficients in the asymptotic expansion (41) are

\[ \Delta_3(1) = \frac{176\nu^4 + 185 + 352\nu^2}{1024(\nu^2 + 1)^2} = 0.1792..., \quad \Delta_3(2) = \frac{401 + 176\nu^4 + 352\nu^2}{1024(\nu^2 + 1)^2} = 0.3555... \]

\[ \Delta_4(1) = \frac{-57 + 48\nu^4 + 96\nu^2}{1024(\nu^2 + 1)^2} = -0.05422..., \quad \Delta_4(2) = \frac{-273 + 48\nu^4 + 96\nu^2}{1024(\nu^2 + 1)^2} = -0.2305... \]

5 Critical Behaviors on the Universal Covering of a Critical Point

Let us consider again the critical point \( x = 0 \). The expansions of the complex power behaviours and the inverse oscillatory behaviours, in the table of section 3.5, converge in suitable domains. This is a consequence of the fact that such expansions (basically the expansions of (3), (10) and (34)), coincide with the convergent expansions obtained by a different local analysis of PVI, which we are going to discuss in subsections 5.1 and 5.2.

5.1 Shimomura Representation

The approach to the local analysis of critical behaviours, due to S. Shimomura [43] [26], consists in rewriting PVI as a system of two differential equations of the first order, and then as a system of two integral equations. The latter is solved by successive approximations and the solutions are obtained in the form of convergent expansions, as stated in the following theorem:

**Theorem 2** Let \( \widetilde{C}_0 \) be the universal covering of a punctured neighbourhood of \( x = 0 \). For any complex number \( a \neq 0 \) and for any \( \sigma \notin (-\infty, 0] \cup [1, +\infty) \) there is a sufficiently small \( r \) such that PVI\(_{\alpha,\beta,\gamma,\delta} \) has a holomorphic solution in the domain

\[ D_s(r; \sigma, a) = \{ x \in \widetilde{C}_0 \mid |x| < r, \ |ax^{-1-\sigma}| < 4r, \ |a^{-1}x^\sigma| < r/4 \} \]

with the following representation:

\[ y(x, \sigma, a) = \frac{1}{\cosh^2 \left( \frac{\sigma - 1}{2} \ln x - \frac{1}{2} \ln a + \ln 2 + \frac{\nu(x, \sigma, a)}{2} \right)}, \]  

(45)
where
\[ v(x, \sigma, a) = \sum_{n \geq 1} a_n(\sigma)x^n + \sum_{n \geq 0, \, m \geq 1} b_{nm}(\sigma)x^n(ax^{1-\sigma})^m + \sum_{n \geq 0, \, m \geq 1} c_{nm}(\sigma)x^n(a^{-1}x^{\sigma})^m, \] (46)
is convergent (and holomorphic) in \( \mathcal{D}_s(r; \sigma, a) \). Here \( a_n(\sigma), b_{nm}(\sigma), c_{nm}(\sigma) \) are certain rational functions of \( \sigma \).

The denominator of (45) does not vanish in \( \mathcal{D}_s(r; \sigma, a) \). The domain \( \mathcal{D}_s(r; \sigma, a) \) is an open domain in the plane \((\ln |x|, \arg(x))\):
\[ |x| < r, \quad \Re \sigma \ln |x| - \ln |a| - \ln \frac{r}{4} < \Im \sigma \arg(x) < (\Re \sigma - 1) \ln |x| - \ln |a| + \ln(4r). \] (47)

If \( \Im \sigma = 0 \), the domain is simply \( \mathcal{D}_0(r) := \{ x \in \mathbb{C} \} \) such that \( |x| < r \}. \) We can compute the critical behaviour of (45) for \( x \to 0 \) along a regular path contained in \( \mathcal{D}_s(r; \sigma, a) \), connecting a point \( x_0 \in \mathcal{D}_s(r; \sigma, a) \) to \( x = 0 \). Let us consider the one parameter (\( \Sigma \)) family of paths
\[ |x| < r, \quad \arg x = \arg x_0 + \frac{\Re \sigma - \Sigma}{\Im \sigma}(\ln |x| - \ln |x_0|), \quad 0 \leq \Sigma \leq 1 \] (48)
or any regular path if \( \Im \sigma = 0 \). If \( 0 \leq \Sigma < 1 \), then (45) behaves as follows
\[ y(x, \sigma, a) = ax^{1-\sigma}(1 + O(x^{1-\sigma} + x^{\sigma})), \quad \text{if} \ 0 < \Sigma < 1, \] (49)
or
\[ y(x, \sigma, a) = x \sin^2 \left( \frac{i \sigma}{2} \ln x + \psi(x) \right) + O(x^2), \quad \text{if} \ \Sigma = 0. \] (50)

The function \( \psi(x) \) has a convergent expansion (following from \( v(x) \)) of type
\[ \psi(x) = \sum_{n \geq 0} \psi_n(\sigma, a)x^{n\sigma}, \]
for certain \( \psi_n(\sigma, a) \)'s. It oscillates without vanishing when \( x \to 0 \) along the path with \( \Sigma = 0 \). It is clear that (49) coincides with (3) when \( 0 \leq \Re \sigma < 1 \) and \( x \to 0 \) along a radial path, with the identification \( \sigma_{0x} = \sigma \). On the other hand, the behaviour (50) can be rewritten as
\[ y(x, \sigma, a) = x \left\{ A \sin(i \sigma \ln x + \phi) + B \right\} + O(x^2), \quad \Sigma = 0. \] (51)

with the same coefficients \( A \) and \( B \) of (10). This follows from the fact that we can write (see [20])
\[ A \sin(i \sigma \ln x + \phi) + B = -2A \sin^2 \left( \frac{i \sigma}{2} \ln x + \frac{\phi}{2} - \frac{\pi}{4} \right) + A + B = \]
\[ = \sin^2 \left( \frac{i \sigma}{2} \ln x + \sum_{n \geq 0} \psi_n x^{n\sigma} \right) \quad \text{for suitable} \ \psi_n. \]

Thus, (50) coincides with (10) when \( \Re \sigma = 0 \) and \( x \to 0 \) along a radial path. When \( \Sigma = 1 \) we obtain the critical behaviour
\[ y(x, \sigma, a) = \frac{1}{\sin^2 \left( \frac{i \sigma}{2} \ln x + \psi(x) \right) + O(x)}, \quad \Sigma = 1. \] (52)
The function $\psi(x)$ has a convergent expansion

$$\psi(x) = \sum_{n \geq 0} \psi_n(\sigma, a)x^{n(1-\sigma)},$$

for certain $\psi_n(\sigma, a)$, which does not vanish along the path with $\Sigma = 1$. By the same argument above, $y(x, \sigma, a)$ coincides with $^3$

$$y(x) = \frac{1}{A \sin(i(1-\sigma) \ln x + \phi) + B + O(x)}, \quad x \to 0. \quad (53)$$

If $\Re \sigma = 1$, then (53) coincides with (34) (with $\sigma = 1 + 2i\nu, \nu \in \mathbb{R}$), and the convergence $x \to 0$ is along a radial path with $\arg x = \arg x_0$.

It is crucial to observe that from (45) and (46) we can compute the full expansions convergent on the domains $D_s(r; \sigma, a)$, of (49), (50) and of the denominator of (52). The one to one correspondence between a branch of a transcendent and a point of the space of monodromy data - when holds - implies that only one transcendent has the given critical behavior. As a result of this, the convergence of the full expansion corresponding to the behaviours (3), (10) and (34) is proved.

Shimomura’s result applies to any value of $\sigma_{0x}$, except for values in $(-\infty, 0] \cup [1, +\infty)$, provided that the convergence $x \to 0$ occurs along spirals whenever $\Re \sigma_{0x} < 0$ or $\Re \sigma_{0x} > 0$. Therefore, it is a generalization of (3), (10) and (34), which hold only for $0 \leq \Re \sigma_{0x} \leq 1$.

Finally, we recall that the above results have been used in section 4, where the domain (39) is the particular case of $D_s(r; \sigma, a)$ when $\sigma = 1 + 2i\nu, \nu \in \mathbb{R}$, that the denominator of (34) = (53) does not vanish in (39), and the poles are outside this domain.

### 5.2 The Elliptic Representation

The elliptic representation was introduced by P. Painlevé in [39] and R. Fuchs in [13]. Let $\wp(z; \omega_1, \omega_2)$ be the Weierstrass Elliptic function of the independent variable $z \in \mathbb{P}^1$, with half-periods $\omega_1, \omega_2$. Let us consider the following $x$ dependent half periods

$$\omega_1(x) := \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right), \quad \omega_2(x) := i\frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; 1 - x\right),$$

where $F$ are hypergeometric functions. A PVI transcendent can be represented as follows [13]:

$$y(x) = \wp\left(\frac{u(x)}{2}; \omega_1(x), \omega_2(x)\right) + \frac{1 + x}{3},$$

where $u(x)$ solves a non linear ODE of the 2nd order, equivalent to PVI. The algebraico-geometrical properties of the elliptic representations where studied in [33]. The analytic properties of the function $u(x)$ in the simplest case $\alpha = \beta = \gamma = 1 - 2\delta = 0$ are easily studied, because the equation for $u(x)$ becomes a Gauss hypergeometric equation. Accordingly, the function $u(x)$ is a linear combination of $\omega_1$ and $\omega_2$. This case was well known to Picard [40], and we have already mentioned it.

The local (at $x = 0$) analytic properties of $u(x)$ were studied in the general case in [16] and [17], as follows. Let $\nu_1$ and $\nu_2$ be complex numbers. Let us consider the domains

$$D(r; \nu_1, \nu_2) := \left\{ x \in \mathbb{C}_0 \text{ such that } |x| < r, \left| e^{\frac{i\pi}{16} - \frac{1}{16}\nu_1^2} x^{1-\nu_2} \right| < r, \left| e^{\frac{i\pi}{16} - \frac{1}{16}\nu_2^2} x^{1-\nu_1} \right| < r \right\}, \quad (54)$$

$$A^2 = \frac{[(1-\sigma)^2 - (\theta_{\infty} - 1 - \theta_1)^2] [(1-\sigma)^2 - (\theta_{\infty} - 1 + \theta_1)^2]}{4(1-\sigma)^4}, \quad B = \frac{(\theta_{\infty} - 1)^2 - \theta_1^2 + (1-\sigma)^2}{2(1-\sigma)^2}.$$
and the expansion:
\[
v(x, \nu_1, \nu_2) := \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi \nu_1} \left( \frac{x}{16} \right)^{1-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi \nu_1} \left( \frac{x}{16} \right)^{\nu_2} \right]^m.
\]  
(56)

The second order non linear ODE for \( u(x) \) can be re-casted into a system of two integral equations and solved by successive approximations in exactly the same way as in Shimomura’s works [43]. As a result, the following theorem holds [17]

**Theorem 3** For any complex \( \nu_1, \nu_2 \) such that \( 3\nu_2 \neq 0 \) there exist a positive number \( r < 1 \) and a transcendent
\[
y(x, \nu_1, \nu_2) = \varphi \left( \nu_1 \omega_1(x) + \nu_2 \omega_2(x) + v(x, \nu_1, \nu_2); \ \omega_1(x), \omega_2(x) \right) + \frac{1 + x}{3}.
\]  
(57)

such that \( v(x, \nu_1, \nu_2) \) is a holomorphic function on \( \mathcal{D}(r; \nu_1, \nu_2) \), with convergent expansion (56). The coefficients \( a_n, b_{nm} \) and \( c_{nm} \) are certain rational functions of \( \nu_2 \). For any complex \( \nu_1 \) and real \( \nu_2 \) such that \( 0 < \nu_2 < 1 \), there exist a positive number \( r < 1 \) and a transcendent (57) such that \( v(x; \nu_1, \nu_2) \) is holomorphic with convergent expansion (56) in \( \mathcal{D}_0(r) \). If \( 1 < \nu_2 < 2 \) the same result holds with \( v = v(x; -\nu_1, 2 - \nu_2) \) in \( \mathcal{D}_0(r) \).

Combining the expansion (56), the expansion at \( x = 0 \) of \( \omega_1(x) \) and \( \omega_2(x) \) and the Fourier expansion of the \( \varphi \) function, we obtain the critical behaviours of \( y(x, \nu_1, \nu_2) \) when \( x \to 0 \) along a regular path inside the domain \( \mathcal{D}(r; \nu_1, \nu_2) \) or \( \mathcal{D}_0(r) \). We do not rewrite the behaviors here, because they coincide with (49), (50) and (52), with the identification
\[
\sigma = 1 - \nu_2, \quad (\text{or } \sigma = \nu_2 - 1 \text{ when } 1 < \nu_2 < 2).
\]

Also this result proves - as in section 5.1 - the convergence of the full expansion corresponding to the behaviours (3), (10) and (34)

6 Isomonodromy Deformation Approach on the Universal Covering of a Critical Point

A local analysis similar to that of section 5.1 can be also applied to the Schlesinger equations associated to PVI. In this way, the result of lemma 2.4.8 of [41], which yields the critical behaviours (27) and thus (3) and (10), turns out to hold when \( x \to 0 \) in the domains \( \mathcal{D}_a(r; \sigma, a) \) of section 5.1, for \( \sigma \notin (-\infty, 0] \cup [1, +\infty) \). The proof of this fact can be found in [16]. The critical behaviors of \( A_0(x), A_x(x) \) and \( A_1(x) \) obtained in this way provide (49) and (51) by substitution into (8), and then (53) by a transformation (23).

As a consequence, the monodromy preserving deformation approach works for the transcendents in the Shimomura and the Elliptic representations. For Shimomura transcendents the formula providing the monodromy exponent \( \sigma \) at \( x = 0 \) in (45) is again
\[
2 \cos \pi \sigma = p_{0x}, \quad \sigma \notin (-\infty, 0] \cup [1, +\infty).
\]  
(58)

In the Elliptic representation, the analogous formula is [17]
\[
2 \cos \pi \nu_2 = -p_{0x}, \quad \nu_2 \notin (-\infty, 0] \cup \{1\} \cup [2, +\infty),
\]  
(59)
Figure 3: The union (dashed region) of some of the domains $\mathcal{D}_s(r^\pm_N, \sigma^\pm_N, a(\sigma^\pm_N))$.

The constant $a$ in (45) is given by a formula which coincides with $a_0$ in (19). In the following, we rewrite it in the simple form

$$ a = a(\sigma). $$

6.1 Extension of the Domains

The isomonodromy approach on the domains $\mathcal{D}_s(r; \sigma, a)$ has an important consequence: for a given PVI and given monodromy data $p_{0x}, p_{x1}, p_{01}$, let us consider (45). If $0 \leq \Re \sigma < 1$, then $y(x, a, \sigma)$ coincides with a branch (3) or (10) at $x = 0$ (Jimbo’s result with $a = a_0$, $\sigma = \sigma_{0x}$). If we choose another solution of (58), namely

$$ \sigma^\pm_N := \pm \sigma + 2N, \quad N \in \mathbb{Z} $$

then the branch associated to the monodromy data $p_{0x}, p_{x1}, p_{01}$ extends to the universal covering of a punctured neighborhood of $x = 0$ as a PVI-transcendent with Shimomura representations

$$ y(x, \sigma^\pm_N, a(\sigma^\pm_N)) = \frac{1}{\cosh^2(\frac{\sigma^\pm_N-1}{2} \ln x - \frac{1}{2} \ln a(\sigma^\pm_N) + \ln 2 + \nu(x))}, $$

on the different domains $\mathcal{D}_s(r_N; \sigma^\pm_N, a(\sigma^\pm_N))$, with different monodromy exponents $\sigma^\pm_N$ and different $a = a(\sigma^\pm_N)$. See figure 3. Note that $r = r_N$.

In the Elliptic representation, the same fact is related to the periodicity and parity of the \( \wp \) function, which allow to change $(\nu_1, \nu_2)$ to $(\nu_1, \nu_2 + 2N)$ and to $(-\nu_1, 2 - \nu_2 - 2N)$, where $N \in \mathbb{Z}$. The integration constant $\nu_2$ is expressed by (59), which is invariant for $\nu_2 \mapsto \nu_2 + 2N$.
Figure 4: The domains $D_1(r; \nu_1, \nu_2 + 2N) := D(r; \nu_1, \nu_2 + 2N)$, $D_2(r; \nu_1, \nu_2 + 2N) := D(r; -\nu_1, 2 - \nu_2 - 2N)$ and $D_1(r; \nu_1, \nu_2 + 2[N + 1])$, $D_2(r; \nu_1, \nu_2 + 2[N + 1])$ for arbitrarily fixed values of $\nu_1$, $\nu_2$, $N$. They are represented in the plane $(\ln |x|, \Im \nu_2 \arg x + [\pi \Im \nu_1 + (\Re \nu_2 + 2N) \ln 16])$. A PVI transcendent can be represented on the union of the domains, with different $\nu_1$ and $v(x)$ on each domain.
Figure 5: The domains $D_s(r, \sigma, a(\sigma))$ and $D_s(r; -\sigma, a(-\sigma))$, in the $(\ln|x|, \Im\sigma \arg x)$ plane. The slopes of the boundaries are indicated. In the particular case in figure, $\Re\sigma = 0$. $q = \ln|\sigma^2|/4|A|^2$.

and $\nu_2 \to 2 - \nu_2 - 2N$. On the other hand $\nu_1$ - like $a$ - depends on the monodromy data by an explicit parametric formula (see [16] [17])

$$\nu_1 = \nu_1(\nu_2, p_{x1}, p_{01}),$$

(60)

Now, let us consider a transcendent (57) of Theorem 3 and do the substitution $\nu_2 \to \nu_2 + 2N$ or $\nu_2 \to 2 - \nu_2 - 2N$. Provided $r = r_N$ is suitably small, the theorem still holds. Provided that also we change $\nu_1 = \nu_1(\nu_2, p_{x1}, p_{01}) \to \nu_1(\nu_2 + 2N, p_{x1}, p_{01})$ or $\nu_1(2 - \nu_2 - 2N, p_{x1}, p_{01})$, we obtain different representations of the same transcendent on the domains $D(r_N; \nu_1, \nu_2 + 2N)$ and the domains $D(r_N, -\nu_1, 2 - \nu_2 - 2N)$. We draw the domains in the $(\ln|x|, \Im\nu_2 \arg x)$-plane, in figure 4.

6.2 The Final Picture on the Universal Covering of a Critical Point

We can further improve the description of the critical behaviours of a transcendent on the universal covering of the punctured neighbourhood of $x = 0$. Let us study what happens when we pass from $D_s(r; \sigma, a(\sigma))$ to its neighbouring domains $D_s(r; -\sigma, a(-\sigma))$ and $D_s(r, -\sigma + 2, a(-\sigma + 2))$.

To start, consider the two Shimomura representations $y(x, \sigma, a(\sigma))$ and $y(x, -\sigma, a(-\sigma))$ of a given transcendent, defined on neighbouring domains $D_s(r; \sigma, a(\sigma))$ and $D_s(r; -\sigma, a(-\sigma))$ respectively. From the parametric connection formulae for $a = a(\sigma)$ one can prove that

$$a(\sigma) a(-\sigma) = \frac{4\sigma^2}{A^2(\sigma^2)}$$

(61)

where $A^2(\sigma^2)$ is (9). The above implies that $D_s(r; -\sigma, a(-\sigma))$ is

$$(\Re\sigma + 1) \ln|x| + \{2 \ln 2 - \ln |a| + \ln \frac{|\sigma^2|}{4|A|^2}\} - \ln r < \Im\sigma \arg x <$$
Figure 6: The union $D_s(r; \sigma, a(\sigma)) \cup \{\text{separating region}\} \cup D_s(r; -\sigma, a(-\sigma))$. Along the paths with $\Sigma = 0$ the behaviour of $y(x)$ is (62)=(63)=(51). This figure corresponds to the case of figure 5.

Figure 7: The domains $D_s(r, \sigma, a(\sigma))$ and $D_s(r, -\sigma + 2, a(-\sigma + 2))$. The slopes of the boundaries are indicated. This figure corresponds to the case of figure 5.
Figure 8: The union $D_s(r, \sigma, a(\sigma)) \cup \{\text{separating region}\} \cup D_s(r, -\sigma + 2, a(-\sigma + 2))$, and the paths for $\Sigma = 1$, along which the behaviour of $y(x)$ is (53). This figure corresponds to the case of figure 7.

Figure 9: The possible location of the poles (indicated by dots). This figure refers to the case of figure 3. Note that in this particular figure $\Re \sigma = 0$, therefore there are definitively no poles if $x \to 0$ with $\arg(x)$ bounded, which instead exist when $\Re \sigma = 1$. 
\[ < \Re \sigma \ln |x| + \{2 \ln 2 - \ln |a| + \ln \frac{|\sigma^2|}{4|A|^2}\} + \ln r, \]

while, by definition, \( D_s(r; \sigma, a(\sigma)) \) is

\[ \Re \sigma \ln |x| + \{2 \ln 2 - \ln |a|\} - \ln r < \Im \arg x < (\Re \sigma - 1) \ln |x| + \{2 \ln 2 - \ln |a|\} + \ln r, \]

The two domains do not in general intersect, as in figure 5. The critical behaviours inside the domains, along paths with \( \Sigma = 0 \), are of type (51):

\[
y(x, \sigma, a(\sigma)) = x\{A^+(\sigma^2)\sin(i\sigma \ln x + \phi(\sigma)) + B(\sigma^2)\} + O(x^2) \tag{62}
\]
when \( x_0 \in D_s(r, \sigma, a(\sigma)) \), and

\[
y(x, -\sigma, a(-\sigma)) = x\{A^-(\sigma^2)\sin(-i\sigma \ln x + \phi(-\sigma)) + B(\sigma^2)\} + O(x^2) \tag{63}
\]
when \( x_0 \in D_s(r; -\sigma, a(-\sigma)) \). Here \( A^\pm(\sigma^2) \) are the roots of \( A^2(\sigma^2) \) in (9), and \( B(\sigma^2) \) too is given in (9). Also, \( a(\sigma) = Ae^{i\phi(\sigma)}/2i \). Due to the arbitrary sign we can take \( A^-(\sigma^2) = -A^+(\sigma^2) \), so that formula (61) implies \( \phi(-\sigma) = -\phi(\sigma) \). It follows that (62) and (63) are formally the same when \( \Sigma = 0 \), though on different domains. On the other hand, if \( x_0 \) belongs to the region separating the two domains, the only path (48) which does not enter into one of the domains when \( x \to 0 \), is precisely a path with \( \Sigma = 0 \). In [16] we proved that (62) and (63) hold also in the separating region. Thus, the behaviour (51) extends to \( D_s(r; \sigma, a(\sigma)) \cup \{\text{separating region}\} \cup D_s(r; -\sigma, a(-\sigma)) \). This is represented in figure 6.

Remark: The fact that domains \( D_s(r; \sigma, a(\sigma)) \) and \( D_s(r; -\sigma, a(-\sigma)) \) do not in general intersect means that the series (46) of \( v(s, \sigma, a(\sigma)) \) and \( v(x, -\sigma, a(-\sigma)) \) do not converge on the region that separates \( D_s(r; \sigma, a(\sigma)) \) and \( D_s(r; -\sigma, a(-\sigma)) \).

The above construction can be repeated for the neighbouring domains \( D_s(r, \sigma, a(\sigma)) \) and \( D_s(r, -\sigma + 2, a(-\sigma + 2)) \) of figure 7, making use of the symmetry (23) applied to (51). It transforms \( p_{0x} \leftrightarrow -p_{0x} \), therefore \( \sigma \mapsto 1 - \sigma \) and (51) into (53), to be understood as defined on the union \( D_s(r, \sigma, a(\sigma)) \cup \{\text{separating region}\} \cup D_s(r, -\sigma + 2, a(-\sigma + 2)) \), as in figure 8. The denominator does not vanish in \( D_s(r, \sigma, a(\sigma)) \) and \( D_s(r, -\sigma + 2, a(-\sigma + 2)) \), where it has convergent expansion. It may vanish in the separating region. Thus, the poles of (53) possibly lie in the separating region. The zeros of the leading term \( A\sin(i(1 - \sigma) \ln x + \phi) + B \) are computed in a form similar to (40). If we write the formal expansion of the denominator we formally compute the equivalent of (41).

If we consider the union in figure 3 of the domains \( D_s(r_0^\pm, \sigma_N^\pm, a(\sigma_N^\pm)) \) (up to a fixed “big” \( N \)) and apply the above considerations to neighbouring domains, we obtain the final picture: a transcendent defined on the universal covering of a punctured neighbourhood of \( x = 0 \) behaves as follows:

When \( x \to 0 \) along lines of slope \( \Re \sigma_N^+ \) in the \( (\ln |x|, \Im \arg x) \) plane, the behaviour is of type (51), with \( \sigma \mapsto \sigma_N^+ \).

When \( x \to 0 \) along lies with slope \( \Re \sigma_N^+ - 1 \) the behaviour is of type (53), with \( \sigma \mapsto \sigma_N^+ \).

When \( x \to 0 \) with other slopes, the behaviour is of type (49), with \( \sigma \mapsto \sigma_N^\pm \).

Asymptotically, the poles possibly lie along lines with slopes \( \Re \sigma + 2l - 1, l \in \mathbb{Z} \), in the regions separating the domains, as in figure 9. In the particular case \( \Re \sigma = 1 \) and \( l = 0 \), the poles are described in section 4.

The results at \( x = 0 \) are transferred to \( x = 1 \) and \( x = \infty \) by the symmetries (21) and (22).
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