The Light-Cone Field Theory Paradigm for Spontaneous Symmetry Breaking

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1 Introduction

In the first part of this lecture I will give an introduction to light-cone field theory, focussing on the “zero mode problem”. In the second part I discuss $\phi^4$-theory in 1+1 dimensions. I will show how the dynamics of the zero modes can give rise to spontaneous symmetry breaking in spite of the trivial vacuum structure on the light-cone.
2 Review of Light-Cone Field Theory

2.1 Motivation

One of the major outstanding problems in physics is how to calculate observable processes in strongly interacting field theories like QCD and electroweak theory. In particular we do not know, how to calculate from first principles the hadronic spectrum, structure functions, fragmentation functions, weak decay amplitudes and nuclear structure. For now the two most promising attempts to tackle strongly interacting field theories are lattice calculations and light-cone field theory. This lecture is devoted to the light-cone approach. In the 60’s Fubini, Furlan and Weinberg [2] showed that in a Poincaré invariant theory calculations may be simpler in an “infinite momentum frame”, i.e. in a frame moving with $v \to c$ relative to the centre of mass. Weinberg showed that the singularities for $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \to \infty$ cancel in the physical observables. The net effect (apart from a singular scale factor) is to transform to light-cone coordinates

$$x^+ := \frac{1}{\sqrt{2}}(x^0 + x^3) \quad x^- := \frac{1}{\sqrt{2}}(x^0 - x^3) \quad x_\perp := (x^1, x^2)$$

with $x^+$ regarded as the (light-cone) time and $x^-, x^1$ and $x^2$ regarded as spatial coordinates. This interpretation is crucial as the Hamiltonian formalism doesn’t treat space and time in a symmetric way.

Note that strictly speaking the transformation to light-cone coordinates is not a Lorentz transform. Yet it is generally believed, but not trivial (and in fact not yet proven for the non-perturbative case), that the quantum theory based on light-cone coordinates is equivalent to the theory in ordinary coordinates.

2.2 Quantization on the Light-Cone

We are used to quantizing a theory by fixing commutation relations at equal times. But, as Dirac [3] pointed out in 1949, there are other hypersurfaces

Many of the topics discussed in this section are reviewed in [1].

I take the (+,-,-,-) metric and $x^+$ (rather than $x_+ = x^-$ as in [4]) as the time coordinate.
in Minkowsky space that can be used for quantizing a theory. Leutwyler and Stern [4] give a complete list and in particular discuss light-cone quantization. The general procedure is described below:

1. Choose the hypersurface that you want to use for quantization (e.g. \( x^+ = 0 \))
2. Identify a complete set of independent dynamical variables
3. Set up the (hypersurface) commutation rules for the complete set
4. Identify the 10 generators of the Poincaré group in terms of your variables and check if they satisfy the Poincaré algebra

2.2.1 Zero Modes

Let us now look at this procedure in more detail for the chosen hypersurface being \( x^+ = 0 \). If massless particles are involved we can not specify the boundary conditions for the classical dynamical evolution as we can see in the example of the massless Klein-Gordon equation in 1+1 dimensions

\[
\frac{\partial^2}{\partial t^2} \phi - \frac{\partial^2}{\partial x^2} \phi = 0 .
\]

Rewritten in light-cone coordinates

\[
\frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} \phi = 0
\]

we can immediately read off the most general classical solution

\[
\phi(x^+, x^-) = f(x^+) + g(x^-)
\]

with arbitrary \( f \) and \( g \).

If we try to fix the boundary conditions at \( x^+ = 0 \), we get almost no information about \( f(x^+) \) and in addition are not allowed to set up conditions for the \( x^+ \)-derivative for more than one point on the surface. This is why mathematicians warn us to fix the boundary conditions on characteristics (here the nullplanes are characteristics of the Klein-Gordon equation).

\[\text{W.l.g. we can assume } \int dx^- g(x^-) = 0 .\]
I denote the (spatial) Fourier modes of $\phi$ as

$$a(p^+, x^+) := \int dx^- e^{ip^+x^-} \phi(x^+, x^-).$$

The zero mode $a(p^+ = 0)$ corresponds to $f$. Hence about half of the massless degrees of freedom reside in the zero mode. They correspond to propagation along the $x^-$-axis.

As we will see, the zero mode will give rise to spontaneous symmetry breaking in the $\phi^4$ quantum theory.

### 2.2.2 QCD

To make you familiar with the procedure of finding the dynamical variables I briefly discuss QCD. First we write down the usual Lagrangian (suppressing colour and flavour)

$$\mathcal{L} = \bar{\psi}(i\partial + gA)\psi - \frac{1}{2} \text{tr}F_{\mu\nu}F^{\mu\nu}.$$

To identify the dynamical variables, we note that the projection operators fulfill

$$P^+ + P^- = 1 \quad \gamma^+ P^- = 0 \quad \gamma^- P^+ = 0.$$

So $\psi^- := P^- \psi$ doesn’t appear in combination with an $x^+$-derivative. Hence only $\psi^+ := P^+ \psi$ is a dynamical variable. So in the gauge $A_- = 0$ the only dynamical variables are $\psi^+, A_1$ and $A_2$. $\psi^-$ and $A_\perp$ have to be eliminated from the Lagrangian (i.e. expressed in terms of the dynamical variables) via constraint equations.

Now we demand canonical commutation relations for the dynamical variables and their conjugate momenta. Actually the canonical commutation relations in light-cone coordinates contain some unusual features. One has

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5If $x^-$ is confined to an interval, we can speak of discrete modes and $a(p^+ = 0) = f * \text{interval length}$.

6Please don’t confuse them with energy and momentum that are denoted by the same symbols.

7I don’t want to discuss the subtle questions arising from the zero mode of $A_-$ in light-cone quantization [14].
to derive the commutators from the Dirac-Bergmann prescription or the Schwinger action principle. For this I refer to the literature [7].

Then we can construct the energy momentum tensor $T^\mu\nu$ and the angular momentum desity $M^{\mu\nu\lambda}$ in terms of our variables. Integration of the $\mu = +$ components over $\int dx^- dx_\perp$ yields the 10 generators of the Poincaré group $P^\nu$ and $J^{\nu\lambda}$. For consistency one has to check if the generators yield the correct (Poincaré algebra) commutation relations.

### 2.2.3 Kinematical Generators

A major difference between quantization in ordinary coordinates and on the light-cone is, that different generators of the Poincaré group become simple. In the equal-time formalism the generators of (spatial) rotations ($J_{12}$, $J_{23}$, $J_{31}$) and shifts ($P_1$, $P_2$, $P_3$) leave the quantization surface invariant. Hence, as they only connect dynamical variables at one time (i.e. on the surface), they don’t contain any dynamics or interaction dependent parts and the interaction does not appear in the canonical commutation relations. The boost generators are interaction dependent. Hence usually approximations in the ordinary-coordinate-formalism will violate the boost invariance.

On the light-cone the kinematical generators that leave the quantization surface invariant are (different) shift operators ($P_-, P_1, P_2$), rotations along the z-axis ($J_{12}$), longitudinal boost ($J_{+-} = J_{30}$) and two linear combinations of the remaining boosts and rotations ($J_{-1}, J_{-2}$). Note that even the number of kinematical generators has changed\(^8\) to 7. Here spatial rotations are dependent on the dynamics. Hence often approximations on the light-cone violate rotation invariance.

Note that the dynamics, i.e. evolution in the variable $x^+$, is generated by $H = P^- = P_+$ rather than $P^0$ that would do the job in ordinary coordinates.

### 2.3 Simplicity of the light-cone vacuum

It is generally believed that the true vacuum of QCD in ordinary coordinates $(P^0|\text{vac} >= 0)$ is very much different from the perturbative vacuum state $(P^0_{\text{free}}|PT\text{vac} >= 0)$. The complicated vacuum is believed to give rise to confinement and spontaneous symmetry breaking. In general the Hamiltonian

\(^8\)In the light-cone formulation the number of kinematical generators is maximum.
(a normal ordered product of free field creation and destruction operators\textsuperscript{9}) contains terms solely made out of creation operators with the sum of their momenta being zero. So the the perturbative vacuum can not be an eigenstate of the full Hamiltonian (i.e. the true vacuum being non-trivial) and free fields must have positive and negative momenta. In ordinary coordinates we are used to left and right movers.

On the other hand the light-cone dispersion relation doesn’t allow for positive energy ($P^-$) states with negative longitudinal momentum ($P^+$). Hence there are no terms in the Hamiltonian purely made out of creation operators. Hence on the light-cone the perturbative vacuum is always an eigenstate of the full Hamiltonian (with zero eigenvalue).

Where have the broken phases and non-perturbative features gone? The zero (momentum) modes are obviously not properly accounted for in the above discussion. However they are not independent quantum degrees of freedom, but rather are determined by a constraint equation in terms of the other degrees of freedom. Now the non-trivial physics resides in the constraint equation. This will be the topic of the next section.

\section{Specific Example: $\phi^4$ in 1+1 Dimensions}

\subsection{General Theory}

In this section we study the $\phi^4$-theory in 1+1 dimensions as the simplest field theory which shows the mechanism of spontaneous symmetry breaking.

The ($\phi^4$)$_{1+1}$ theory is believed to have a second-order phase transition [6] for some critical coupling parameter $\lambda_{\text{critical}} > 0$. For $\lambda > \lambda_{\text{critical}}$, $\phi(x)$ has a nonvanishing vacuum expectation value and we are interested in understanding how this can be compatible with the trivial light-cone vacuum structure. We will see that the existence of a nontrivial constraint equation for the zero mode replaces the difficulties related to the vacuum polarisation. With certain approximations we will see how this constraint equation leads to a phase transition and an nonvanishing vacuum expectation value in a certain domain of the coupling.

\textsuperscript{9}Note that by definition a perturbative vacuum is annihilated by the destruction operators and hence single free field operators has a vanishing perturbative-vacuum expectation value.
In light-cone coordinates the bare $(\phi^4)_{1+1}$Lagrangian is

$$\mathcal{L} = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4$$

(1)

We put the system in a box (an interval) of length $d$ in $x^-$ direction and impose periodic boundary conditions.

$$\phi(x) = \frac{1}{\sqrt{d}} \sum_{n=-\infty}^{\infty} q_n(x^+e^{ik_n^+x^-}, \quad k_n^+ = \frac{2\pi n}{d}$$

(2)

If we try to quantize canonically we get $p_n = ik_n q_n$. So the momenta are connected to the coordinates and we have to use the Dirac-Bergmann prescription to quantize the theory. Since this is thoroughly discussed in literature [7], [8], we will skip this topic here.

In particular, we get $p_0 = 0$. So $q_0$ has no canonical momentum and the equation of motion for $q_0$ appears as a constraint equation: $\frac{\partial L}{\partial q_0} = 0$ (if it is applied on a physical state). As a result we get the creation and annihilation operators $a_n = \sqrt{4\pi|n|}q_n$, $n \neq 0$ with $a_{-n} = a_n^\dagger$ and the standard commutation relations $[a_n, a_{m}^\dagger] = \delta_{n,m}$. With the useful notation $|0\rangle := 1$ we have

$$\phi(x^+, x^-) = \sum_{n=-\infty}^{\infty} \frac{a_n(x^+)e^{ik_n^+x^-}}{\sqrt{4\pi|n|}}.$$  

(3)

For $a_0 = \sqrt{4\pi}q_0$ we get the constraint equation indicated above. It can be rewritten as $\frac{\partial P^-}{\partial q_0} = 0$ where $P^-$ is the light-cone Hamiltonian. This constraint equation can easily be motivated starting from the equation of motion

$$\partial_+ \partial_- \phi = -\mu^2 \phi - \frac{\lambda}{3!} \phi^3.$$  

(4)

Due to the periodic boundary conditions we get $\int_{-\frac{d}{2}}^{\frac{d}{2}} dx^- \partial_- (\partial_+ \phi) = 0$ [9]. Thus the constraint is the integral of the potential.

$$0 = \int_{-\frac{d}{2}}^{\frac{d}{2}} dx^- \mu^2 \phi + \frac{\lambda}{3!} \phi^3.$$  

(5)

If we separate the zero mode $\phi = \phi_0 + \varphi$ we get

8
0 = \mu^2 \phi_0 + \frac{\lambda}{3!d} \int_{-d/2}^{d/2} dx^- (\phi_0 + \varphi)^3. \quad (6)

This is a complicated operator equation for \( a_0 \) in terms of all the other modes in the theory. It refers to the complicated structure of the theory and due to this equation we can get a non-zero expectation value \( \langle 0 | a_0 | 0 \rangle \neq 0 \) of \( \phi(x) \) in spite of a trivial vacuum.

Spontaneous symmetry breaking is described in equal time quantisation by multiple vacua. The choice of the vacuum defines the theory and the vacuum structure is complicated. On the light-cone the vacuum is trivial but we get a complicated constraint equation for the zero mode \( a_0 \). This equation has in general multiple solutions and the choice of a specific one defines the theory.

In the next subsections we will derive the light-cone Hamiltonian and look in more detail at the constraint equation. We will solve it approximately and show how, in principle, spontaneous symmetry breaking can arise.

### 3.2 The light-cone Hamiltonian

Now we can calculate the Hamiltonian \( P^- \) for the light-cone \((\phi^4)_{1+1}\)-theory.

\[
P^- = \left( \frac{\lambda d}{96\pi^2} \right) \left( \frac{g}{2} \sum_{n=-\infty}^{\infty} \frac{a_n a_{-n}}{|n|} + \frac{1}{4!} \sum_{k, \ell, m, n} \delta_{k+\ell+m+n, 0} \frac{\delta_{k, \ell, m, n}}{\sqrt{|k||l||m||n|}} a_k a_\ell a_m a_n \right) \quad \text{where } g = \frac{24\pi \mu^2}{\lambda}, \quad |0| := 1 \quad (7)
\]

Now we have to care for the divergences in the theory. In a two-dimensions the only divergencies are tadpoles which can be removed by normal ordering. But it is not clear how to normal order the terms involving \( a_0 \) since \( a_0 \) itself has a complicated operator structure. For very general arguments [10] we use symmetric ordering for these terms. Since \( \langle 0 | a_0 | 0 \rangle \) may be different from zero, terms of the form \( a_n a_0 a_n^\dagger, a_0 a_n a_0 a_n^\dagger, \ldots \) may still give additional divergencies and require the further subtraction

\[
\frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|} (-6a_0^2) \quad (8)
\]
in Eq. (9). If we are looking to the one mode approximation, as we will do in the next subsection, this Hamiltonian is finite. However, including all modes, some divergencies may still remain.

So \( P^- \) has the form

\[
P^- = \left( \frac{\lambda d}{96 \pi^2} \right) \left( \frac{g}{2} \sum_{n \neq 0} \frac{a_n a_{-n}}{|n|} + \frac{1}{4} \sum_{k, \ell, m, n} \frac{\delta_{k+\ell+m+n,0}}{\sqrt{|k||l||m||n|}} : a_k a_\ell a_m a_n : ight) 
\]

\[
+ \frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|} \left( a_0^2 a_n a_{-n} + a_n a_{-n} a_0^2 + a_n a_0^2 a_{-n} + a_n a_0 a_{-n} a_0 
\right.
\]

\[
+ a_0 a_n a_0 a_{-n} + a_0 a_n a_{-n} a_0 - 6a_0^2 \big) + \frac{g}{2} a_0^2 + \frac{1}{4} a_0^4 + \frac{1}{4} 
\]

\[
\cdot \sum_{k, \ell, m \neq 0} \frac{\delta_{k+\ell+m,0}}{\sqrt{|k||l||m|}} \left( a_0 a_k a_\ell a_m + a_k a_0 a_\ell a_m + a_k a_\ell a_0 a_m + a_k a_\ell a_m a_0 \right) \right)
\]

(9)

where we have separated out all the \( a_0 \) terms.

### 3.3 Solving the Constraint Equation [11]

The constraint equation is now given by \( \frac{\partial P^-}{\partial a_0} = 0 \):

\[
0 = ga_0 + a_0^3 + \sum_{n \neq 0} \frac{1}{|n|} \left( a_0 a_n a_{-n} + a_n a_{-n} a_0 + a_n a_0 a_{-n} - 3a_0 \right) 
\]

\[
+ \sum_{k, \ell, m \neq 0} \frac{\delta_{k+\ell+m,0}}{\sqrt{|k||l||m|}} a_k a_\ell a_m \right)
\]

(10)

To initially solve this equation we have to make the approximations given in 1. below, and we only look for solution of the type described in 2.

1. Since the spontaneous symmetry breaking is a low energy effect it is reasonable to assume that the lowest energy mode will give the most important contribution to \( \langle 0 | a_0 | 0 \rangle \). Therefore we will truncate the constraint equation to one mode \( a_1 = a \).
2. The higher lying levels should not be affected very strongly by the mechanism of spontaneous symmetry breaking. We only look for a solution of this type. Formally this type of solution satisfies (16) below.

If we restrict the constraint equation to one mode we get:

\[
0 = ga_0 + a_0^3 + 2a_0a^\dagger a + 2a^\dagger a a_0 + a^\dagger a_0 a + a a_0 a^\dagger - a_0.
\] (11)

The solution will preserve particle number because of conserves momentum.

We are particularly interested into the vacuum expectation value

\[
\langle 0|\phi|0 \rangle = \langle 0|a_0|0 \rangle = \frac{f_0}{\sqrt{4\pi}}.
\] (12)

\(a_0\) must a function of \(N = a^\dagger a\) thus,

\[
a_0 = \sum_{k=0}^{\infty} b_k N^k.
\] (13)

This implies that \(a_0\) is diagonal and it can be written as

\[
a_0 = \sum_{k=0}^{\infty} f_k |k\rangle \langle k|, \quad f_k := \langle k|a_0|k \rangle.
\] (14)

Substituting this into the constraint equation and sandwiching it between Fock states we get the following non linear finite difference equation:

\[
0 = g f_n + f_n^3 + (4n - 1) f_n + (n + 1) f_{n+1} + n f_{n-1}
\] (15)

Since \(\langle N|a_0|N \rangle = 0\) for \(\lambda = 0\) and because of the condition 2 mentioned above we get \(\langle N|a_0|N \rangle \to 0\) for \(N \to \infty\) and all \(\lambda\). So we search for a solution with the property

\[
\lim_{n \to \infty} f_n = 0
\] (16)

Therefore we study the large \(n\) behavior of our equation and drop the \(f_n^3\) term:

\[
f_{n+1} + 4f_n + f_{n-1} = 0
\] (17)

which has the asymptotic behavior \(f_n \propto c^n\), with \(c^2 + 4c + 1 = 0\) or \(c = -2 \pm \sqrt{3}\) because of (13) we have to reject the \((-2 - \sqrt{3})^n\) solution. As we
will see shortly this is only possible for \( g \leq g_{\text{critical}} \) (except for the trivial solution \( f_n \equiv 0 \), which does always exist).

To calculate the critical point we start from a small \( f_0 \), since we are looking for solutions close to the trivial one. So we still can drop the \( f_3^n \) term. We are left with the linear equation \((f_{-1} = 0 = f_{-2})\)

\[
(4n + g - 1) f_n + (n + 1) f_{n+1} + nf_{n-1} = 0
\]

(18)

To solve this equation we introduce the generating function

\[
F(z) = \sum_{n=0}^{\infty} f_n z^n.
\]

(19)

Our difference equation for \( f_n \) gives us a differential equation for \( F(z) \):

\[
\left( z^2 + 4z + 1 \right) F'(z) + (z + g - 1) F(z) = 0
\]

(20)

This equation can easily be solved and the solution is

\[
F(z) = F(0) \left( \frac{z + 2 - \sqrt{3}}{2 - \sqrt{3}} \right)^{-\frac{\sqrt{3} - 3 + g}{2\sqrt{3}}} \left( \frac{z + 2 + \sqrt{3}}{2 + \sqrt{3}} \right)^{-\frac{\sqrt{3} + 3 - g}{2\sqrt{3}}}
\]

(21)

If \( f_n \) goes asymptotically like \((-2 + \sqrt{3})^n\) then the radius of convergence for \( F(z) \) is \( R = \frac{1}{|2 + \sqrt{3}|} = 2 + \sqrt{3} \) else it is \( R = \frac{1}{|2 - \sqrt{3}|} = 2 - \sqrt{3} \).

So we can reformulate our condition \( f_n \rightarrow 0, n \rightarrow \infty \) as follows: \( F(z) \) has no singularity in the unit disc of the complex plane. From (21) we read off that \( F(z) \) has a singularity at \( z = -2 + \sqrt{3} \) unless \(-\frac{\sqrt{3} - 3 + g}{2\sqrt{3}}\) is a non-negative integer.

\[
-\frac{\sqrt{3} - 3 + g}{2\sqrt{3}} = k \quad k = 0, 1, 2, 3, \ldots
\]

\[
\Leftrightarrow \quad g = 3 - \sqrt{3} - 2\sqrt{3}k
\]

(22)

The only solution with positive \( g \) is

\[
g_{\text{critical}} = 3 - \sqrt{3} = \frac{24\pi \mu^2}{\lambda_{\text{critical}}}
\]

(23)
or
\[
\frac{\lambda_{\text{critical}}}{\mu^2} = 4\pi \left( 3 + \sqrt{3} \right) \approx 59.5 .
\] (24)

This value agrees well with the numerical result for the equal time theory [12]
\[
\frac{\lambda_{\text{critical}}}{\mu^2} \approx 30 - 60 .
\] (25)

This agreement should nevertheless not be overestimated since the numeric value for \( \frac{\lambda_{\text{critical}}}{\mu^2} \) does not take higher modes into account. Moreover the calculations with higher modes seem to diverge logarithmically which reflects the fact that the renormalisation is still missing. A more complete discussion including higher modes is given in [13]

To explore whether or not \( g_{\text{critical}} \) is an isolated point or the beginning of a continuum of critical couplings we have to look at the full constraint equation including the \( f^3_n \) term.

### 3.3.1 The \( \delta \)-Expansion

A powerful analytical method to linearize non-linear difference equations is the \( \delta \)-expansion. We rewrite
\[
f^3_n = f_n^{1+2\delta} \approx f_n \left( 1 + \delta \ln f^2_n \right) ,
g = g^{(0)} + \delta g^{(1)} + \ldots , \quad f_n = f_n^{(0)} + \delta f_n^{(1)} + \ldots
\] (26)

and solve the difference equation as an expansion in \( \delta \) [11]. The result for the first order and \( \delta = 1 \) is
\[
g_{\text{critical}} = \left( 2 - \sqrt{3} \right) \left( 1 + \frac{1}{\sqrt{3}} \ln \left( 2 + \sqrt{3} \right) \right) - \ln f_0^2 .
\] (27)

This is quite a good approximation to the exact result for \( g < 1 \). It is shown as the dashed line in Fig. 1.

### 3.3.2 Numerical Solution

The difference equation can easily be calculated. To find the stable solution \( (f_n \to 0) \) we truncate the series by setting \( f_{N+1} = 0 \) for some large \( N(\approx 10) \).
We get a system of equations
\[
0 = (g_{\text{critical}} - 1) f_0 + f_0^3 + f_1 \\
0 = (g_{\text{critical}} + 3) f_1 + f_1^3 + 2f_2 + f_0 \\
: \\
0 = (g_{\text{critical}} + 4N - 1) f_N + f_N^3 + N f_{N-1} 
\]
and solve it numerically for \(f_0\) as a function of \(g_{\text{critical}}\). The result is indicated by the solid line in Figs. 1 and 2. At \(\lambda = \lambda_{\text{critical}}\) a second order phase transition can be seen. The domain of negative \(g\) may be of interest in the Higgs model which starts from a negative mass \(\mu^2\). The structure of solutions seems to be more involved in this domain, Fig 2.

### 3.4 Calculation of the Eigenvalues of \(P^-\)

We can also study the eigenvalues of the light-cone Hamiltonian \(P^-\) truncated to the one mode problem.

Since \(P^-\) conserves momentum it is diagonal in the number operator \(N\) so that the energy eigenstates are eigenstates of \(N\).

\[
E_n = \langle n | H | n \rangle = \frac{3}{2} n (n - 1) + n g - \frac{f_n^4}{4} - \frac{2n + 1}{4} f_n^2 + \frac{n + 1}{4} f_{n+1}^2 + \frac{n}{4} f_{n-1}^2 
\]

(29)

The result for the first three eigenvalues is shown in Fig. 3 by the solid curve. The dashed lines show the result for \(a_0 = 0\). For \(g > g_{\text{critical}}\) the two lines coincide but at \(g = g_{\text{critical}}\) there is a phase transition and the energy of the first excited state decreases as \(g\) is decreased. For large \(N\) this effect is very small as we would expect from our approximations. We note that the mass gap is a minimum at the critical point.

### 4 Work in Progress

The next step is to take several modes into account and this is discussed in detail in reference [13]. There is still an outstanding problem of operator ordering and renormalisation that has to be solved to get the complete solution.
5 Conclusions

• light-cone field theories may show the feature of spontaneous symmetry breaking inspite of their trivial vacuum structure. This effect results from the complicated properties of the zero modes in the theory. The zero mode is connected to all the other modes in the theory by the constraint equation.

• For 1 + 1-dimensional $\phi^4$-theory the solution of the one mode approximation gives the explicite result $\lambda_{\text{critical}} = 4\pi(3+\sqrt{3})\mu^2$ which lies close to the numerical result for equal time theories. At this point a second order phase transition takes place.

• For $\lambda < \lambda_{\text{critical}}$ the theory has the same spectrum as it would have without the zero mode. For $\lambda > \lambda_{\text{critical}}$ the energy of the first excited state is poositive with a minimum at $g_{\text{critical}}$. The $E_n$ for large $n$ are nearly unaffected.

• In general $a_0$ gives rise to an infinite number of new interactions in the effective Hamiltonian even if $\lambda < \lambda_{\text{critical}}$. These interactions are determined by the constraint equation. See [13] for more details.

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7 Figure Captions

**Figure 1.** $g = 24\pi\mu^2/\lambda$ vs. $f_0 = \sqrt{4\pi\langle 0 | \phi(x) | 0 \rangle}$. The solid curve obtained from the numerical solution of (28) with $N = 10$. The dashed curve is the critical curve obtained from the first-order $\delta$-expansion.

**Figure 2.** The critical curve obtained numerically shows an interesting behavior in the negative $g$ domain.
Figure 3. The lowest three energy eigenvalues as a function of $g$ from the numerical solution of (28) with $N = 10$. The dashed line is the symmetric solution $f_0 = 0$ and the solid line is the solution with $f_0 \neq 0$ for $g < g_{\text{critical}}$. 