Perturbation theory of the space-time non-commutative real scalar field theories

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Abstract

The perturbative framework of the space-time non-commutative real scalar field theory is formulated, based on the unitary S-matrix. Unitarity of the S-matrix is explicitly checked order by order using the Heisenberg picture of Lagrangian formalism of the second quantized operators, with the emphasis of the so-called minimal realization of the time-ordering step function and of the importance of the $\star$-time ordering. The Feynman rule is established and is presented using $\phi^4$ scalar field theory. It is shown that the divergence structure of space-time non-commutative theory is the same as the one of space-space non-commutative theory, while there is no UV-IR mixing problem in this space-time non-commutative theory.
1 Introduction

Non-commutative field theory (NCFT) [1] is the field theory defined on the non-commutative (NC) coordinates. We will consider NC coordinates which obey

$$[x^\mu, x^\nu] = i \theta^{\mu \nu} \tag{1.1}$$

where $\theta^{\mu \nu}$ is an antisymmetric c-number. Space non-commutative theory (SS NC) involves only the space non-commuting coordinates ($\theta^{0 \nu} = 0$), whereas space-time non-commutative theory (STNC) contains the non-commuting time ($\theta^{01} \neq 0$).

The non-commuting nature of the coordinates is naturally adapted to the operator formalism for the first quantization of the theory. However, the operator formalism is not convenient for second quantized field theory. Fortunately, there exists another formalism suited for NCFT based on the Weyl’s idea [2]: NCFT is constructed using the $\star$-product of fields but space-time coordinates are commuting. The $\star$-product encodes all the non-commuting nature of the theory and fixes the ordering ambiguity of non-commuting coordinates. We adopt the Moyal product [3] as the $\star$-product representations,

$$f \star g (x) = e^{i \theta^{\alpha \beta} \partial_\alpha \partial_\beta} f(x) g(y) \big|_{y=x} \tag{1.2}$$

with $a \wedge b \equiv \theta^{\mu \nu} a_\mu b_\nu$. In terms of the $\star$-product, the non-commuting nature of the coordinates in (1.1) is written as

$$[x^\mu ; x^\nu] \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i \theta^{\mu \nu}$$

and now the coordinates itself $x^\mu$ is commuting each other. The merit of the Moyal product is that it does not change the kinetic term of the action even after introducing the $\star$-product, and allows the conventional perturbation with respect to the free theory [4].

NCFT is a non-local field theory and the theory behaves very differently in many respects. Lorentz symmetry is usually broken even though there are some attempt to cure this [5]. Especially, STNC is known to have micro-causality problem [6] and unitarity problem [7] due to the infinite number of time-derivatives.

Among these problems, it is proposed in [8,9,10] that the unitarity problem can be avoided if one uses the careful time-ordering in the S-matrix. However, each proposal has different aspects, which needs to be distinguished from each other. The proposal by [8] is the first attempt to solve the unitarity problem and pointed out the unitarity problem is not inherent to the theory but due to the formalism of the theory. The proposal provides the lowest order S-matrix, which needs higher order derivative correction to make the proper S-matrix. After this correction, however, it turns out [9] that the time-ordering should be done before the $\star$-operation in contrast to their proposal. The proposal by [10] is called the time-ordered perturbation theory but is pointed out in [11] that the gauge invariance may not be respected when applied to a gauge theory.

Our proposal in [9] is critically different from the other two in the sense of the time-ordering. The time-ordering is done in terms of the so-called minimal realization and the time-ordering should be performed before the $\star$-operation. The purpose of this paper is...
to clarify and justify the time-ordering in the S-matrix of STNC QFT proposed in [9] and is to construct the systematic formalism of the perturbation theory.

In section 2, S-matrix is explicitly constructed using the Lagrangian of the second quantized operator in the Heisenberg picture following Yang and Feldman [12]. Even though the unitary transformation at finite time could not be found, there exists the unitary S-matrix. The unitarity of the S-matrix is explicitly proven order by order in the coupling constant. In section 3, Feynman rule is established and perturbation theory is formulated using the real scalar $\phi^4$ theory. Section 4 is the conclusion and outlook.

2 S-matrix for scalar STNC field theory

2.1 $\star$-operation and interaction Lagrangian

The Lagrangian of a real scalar STNC field theory consists of the free part and interacting part. Using the starred notation,

$$\phi^p = \phi \star \phi \star \cdots \star \phi$$

the interaction Lagrangian in $D-1$ dimensional space is given as

$$L_I(t) = \int d^{D-1}x \mathcal{L}_I(\phi_\star(x)), \quad \mathcal{L}_I(\phi_\star(x)) = -\frac{g}{p!} \phi^p(x)$$

(2.1)

where $g$ is a coupling constant and the integration is done over the $D-1$ space dimension.

It is convenient to introduce a $\star$-operator $\mathcal{F}_x$:

$$\mathcal{F}_x(\phi^p(x)) \equiv \phi(x) \star \phi(x) \star \cdots \star \phi(x) = \phi^p(x).$$

(2.2)

The notation $\bar{x}$ is to put down the explicit argument which is to be starred. The interaction Lagrangian density $\mathcal{L}_I(\phi_\star(x))$ is related through the $\mathcal{F}_x$ with an un-starred interaction density $\mathcal{V}(\phi(x))$:

$$\mathcal{L}_I(\phi_\star(x)) = \mathcal{F}_x(\mathcal{V}(\phi(x)))$$

(2.3)

where

$$\mathcal{V}(\phi(x)) \equiv -\frac{g}{p!} \phi^p(x).$$

(2.4)

It is worth to mention that the un-starred quantity uniquely defines the starred quantity

$$A(x) \star B(x) = \mathcal{F}_x(A(x)B(x)).$$

However, the inverse is not true since, even if $A(x)B(x) = B(x)A(x)$, the starred one is not;

$$\mathcal{F}_x(A(x)B(x)) \neq \mathcal{F}_x(B(x)A(x)),$$

(2.5)

because of the non-commuting nature of the star-product. This ordering ambiguity is to be treated carefully. A composite $\star$-operator can be also defined

$$\mathcal{F}_{xy} \equiv \mathcal{F}_x \mathcal{F}_y,$$

(2.6)

which is commutative

$$\mathcal{F}_{xy} = \mathcal{F}_{yx}.$$
2.2 Out-field

The field at an arbitrary time is obtained from the field equation

\[(\Box + m^2) \phi(x) = \xi_*(\phi(x))\] 

(2.8)

where \(\xi_*\) is the functional of fields, derived from the interaction Lagrangian (2.1),

\[\xi_*(\phi(x)) \equiv \frac{\delta}{\delta\phi(x)} \int dt L_1(t)\]

\[= -\frac{g}{p!} \int d^Dy \left( \delta(x-y) \overset{\dagger}{\phi}_*^{-1}(y) + \phi(y) \overset{\dagger}{\phi}_*^{-1}(y) \delta(x-y) \phi(y) + \cdots \right).\] 

(2.9)

Introducing a compact notation for the symmetrization of \(n\) distinctive quantities,

\[\{ \prod_{i=1}^{n} A_i \}^s \equiv \sum_{s(1,2,\cdots,n)} \prod_{i=1}^{n} A_{s(i)},\]

where \(s(1,2,\cdots,n)\) is the permutation of \(1,2,\cdots,n\), we may put \(\xi_*\) as

\[\xi_*(\phi(x)) = -\frac{g}{p!} \int d^Dy \mathcal{F}_y \left\{ \delta(x-y) \phi(y) \right\}_{s(y)}\] 

(2.10)

where the subscript \(s(y)\) refers to the symmetrization of operators with argument \(y\).

The solution of Eq. (2.8) is given as

\[\phi(x) = \phi_{\text{in}}(x) + \int d^Dy \Delta_{\text{ret}}(x-y) \xi_*(\phi(y))\]

\[= \phi_{\text{out}}(x) + \int d^Dy \Delta_{\text{adv}}(x-y) \xi_*(\phi(y)).\] 

(2.11)

Here \(\Delta_{\text{ret}}(x)\) (\(\Delta_{\text{adv}}(x)\)) denotes the retarded (advanced) Green’s function,

\[\Delta_{\text{ret}}(x) = -\theta(x^0) \Delta(x), \quad \Delta_{\text{adv}}(x) = \theta(-x^0) \Delta(x)\] 

(2.12)

and \(\Delta(x)\) is the free commutator function,

\[[\phi_0(x), \phi_0(0)] = i\Delta(x).\] 

(2.13)

Employing the delta-function identity,

\[\int d^Dy \int d^Dz A(x-y) \delta(y-z) B(z) = \int d^Dy \left( A(x-y) \overset{\dagger}{B}(y) \right)\]

(2.14)

for arbitrary function of \(A(x-y)\) and an operator \(B(y)\), we may put the retarded or advanced Green’s function of (2.11) inside the star-operation:

\[\phi(x) = \phi_{\text{in}}(x) + \int d^Dy \mathcal{F}_y \left\{ -\frac{g}{p!} \Delta_{\text{ret}}(x-y) \phi(y) \right\}_{s(y)}\]

\[= \phi_{\text{out}}(x) + \int d^Dy \mathcal{F}_y \left\{ -\frac{g}{p!} \Delta_{\text{adv}}(x-y) \phi(y) \right\}_{s(y)}.\] 

(2.15)
This gives the relation between the out-field and the in-field,
\[ \phi_{\text{out}}(x) = \phi_{\text{in}}(x) + \int d^D y \mathcal{F}_y \left\{ \frac{g}{p!} \Delta(x - y) \phi_{p-1}^0(y) \right\}_{s(y)}. \] (2.16)

Therefore, the out-field is written iteratively in terms of the in-field if one uses the relation in (2.15): Putting \( \phi \) as \( \phi = \phi_0 + \phi_1 + \phi_2 \cdots \) where \( \phi_n \) represents the order of \( g^n \) contribution. A few explicit forms of \( \phi_n \)'s are given as
\[
\phi_0(x) = \phi_{\text{in}}(x) \\
\phi_1(x) = \int d^D y \Delta_{\text{ret}}(x - y) \xi_*(\phi_0(y)) \\
\phi_2(x) = \int d^D y \mathcal{F}_y \left\{ - \frac{g}{p!} \Delta_{\text{ret}}(x - y) \phi_{p-1}^0(y) \right\}_{s(y)} \\
\phi_3(x) = \int d^D y \mathcal{F}_y \left\{ - \frac{g}{p!} \Delta_{\text{ret}}(x - y) \left( \phi_{p-2}^0(y) \phi_1(y) + \phi_{p-3}^0(y) \phi_2^0(y) \right) \right\}_{s(y)} \\
\phi_n(x) = \int d^D y \mathcal{F}_y \left\{ \sum_{q_1 + \cdots + q_{p-1} = n-1} - \frac{g}{p!} \Delta_{\text{ret}}(x - y) \phi_{q_1}(y) \cdots \phi_{q_{p-1}}(y) \right\}_{s(y)}. \] (2.17)

For later use, we put the explicit form of the out-field as
\[ \phi_{\text{out}} = \sum_{i=0}^{\infty} \varphi_i(x), \]
where \( \varphi_0 = \phi_0 \) and for \( n \geq 1 \)
\[ \varphi_n(x) = \int d^D y \mathcal{F}_y \left\{ \sum_{q_1 + \cdots + q_{p-1} = n-1} \frac{g}{p!} \Delta(x - y) \phi_{q_1}(y) \cdots \phi_{q_{p-1}}(y) \right\}_{s(y)}. \] (2.18)

### 2.3 S-matrix

The S-matrix relates the out-field with the in-field:
\[ \phi_{\text{out}} = S^\dagger \phi_{\text{in}} S. \] (2.19)

With the notation \( S = e^{i\delta} \), the out-field would be written as
\[ \phi_{\text{out}} = \phi_{\text{in}} + [\phi_{\text{in}}, i\delta] + \frac{1}{2} [[\phi_{\text{in}}, i\delta], i\delta] + \cdots. \] (2.20)

The first order term in \( g \) should be written as
\[ [\phi_{\text{in}}, i\delta] = \varphi_1(x) = \int d^D y \mathcal{F}_y \left\{ \frac{g}{p!} \Delta(x - y) \phi_{p-1}^0(y) \right\}_{s(y)}. \] (2.21)
and determines the phase $\delta$ to the first order in $g$,
\[
\delta = \int d^D y \, F_y \left( -\frac{g}{p!} \phi_0^p(y) \right) + O(g^2) = \int d^D y \, L_i \left( \phi_{0i}(y) \right) + O(g^2).
\] (2.22)

Higher order solutions require the time-ordering as in the ordinary field theory. However, the time-ordering needs a special care in the $\star$-product and a consistent unitary S-matrix is proposed in [9] as
\[
S = \sum_{n=0}^{\infty} i^n A_n
\] (2.23)
where $A_n$ is the order of $g^n$ with $A_0 = 1$:
\[
A_n = \int \cdots \int d^D x_1 \cdots d^D x_n \, F_{1 \cdots n} \left( \theta_{12 \cdots n} \, V \left( \phi_{01}(x_1) \right) \cdots V \left( \phi_{0n}(x_n) \right) \right)
\] (2.24)
where we use the composite version of $\star$-operation
\[
F_{12 \cdots n} \equiv F_{x_1} F_{x_2} \cdots F_{x_n},
\]
whose operation is independent of the permutation of operators.

The time-ordering is given in terms of the step function,
\[
\theta_{12 \cdots n} \equiv \theta(t_1 - t_2) \theta(t_2 - t_3) \cdots \theta(t_{n-1} - t_n).
\]
The ambiguity of the time-ordering is due to the point splitting ambiguity of the arguments in $\theta(x^0 - y^0)$. For example, one might have
\[
F_{xy} \left( \theta(x^0 - y^0) \phi^p(x) \phi^q(y) \right) \neq F_{xy} \left( \theta(x^0 - y^0) \phi^p(x) \theta(x^0 - y^0) \phi^q(y) \right),
\] (2.25)
depending on how one splits the coordinates to define the proper $\star$-product. We fix this ambiguity by using the so-called minimal realization of the step-function in the $\star$-operation: The minimal realization of $\star$-operation is to change the step function $\theta(x^0 - y^0)$ to $\theta(x^0 - y^0)$ and is to use the step function only once;
\[
F_{xy} \left( \theta(x^0 - y^0) \phi^p(x) \phi^q(y) \right)
= F_{xy} \left( \theta(x^0 - y^0) \phi(x_1) \cdots \phi(x_i) \phi(y_1) \cdots \phi(y_j), \quad x_i = x, y_j = y \right).
\] (2.26)

The split coordinates of $\theta(x^0 - y^0)$ is to be assigned a posteriori as the argument of the spectral function $\Delta(x^0 - y^0)$ which connects two vertices. And even in the presence of many spectral functions we have only one step function,
\[
\theta(x^0 - y^0) \prod_{a,b} \Delta(x_a - y_b) \rightarrow \theta(x^0 - y^0) \prod_{a,b} \Delta(x_a - y_b)
\] (2.27)
where $i (j)$ is just one of indices among $a$'s ($b$'s). The minimal realization assumption sounds ad hoc, but is necessary to prove the relation $\phi_{out} = S^i \phi_{in} S$ in section 2.4. This
minimal realization of the step-function is the crucial difference from the recipe given in the time-ordered perturbation theory given in [10].

Introducing $\ast$-time-ordering $T_\ast$ as

$$ T_\ast \{ A(t_1)A(t_2) \} = \mathcal{F}_{12} \left( \theta_{12} A(t_1)A(t_2) + \theta_{21} A(t_2)A(t_1) \right). \quad (2.28) $$

we may put the S-matrix in a compact form as

$$ S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^D x_1 \cdots \int d^D x_n T_\ast \left( \mathcal{V}(\phi_0(x_1)) \cdots \mathcal{V}(\phi_0(x_n)) \right) $$

$$ \equiv T_\ast \exp \left( i \int d^D x \mathcal{V}(\phi_0(x)) \right). \quad (2.29) $$

### 2.4 S-matrix and in- and out-field

In this section, we check that $S \phi_{in}(x)S^\dagger$ reproduces the correct out-field given in (2.18). For this purpose, we evaluate the out field using the S-matrix definition (2.23) and denote it as $\Phi_{out}(x)$:

$$ \Phi_{out}(x) \equiv S \phi_{in}(x)S = \sum_{n=0}^{\infty} \Phi_{(n)}(x) \quad (2.30) $$

where $\Phi_{(n)}$ is the out-field term of order $g^n$. Each order is given as

$$ \Phi_{(0)}(x) = \phi_0(x), \quad \Phi_{(n)}(x) = \sum_{\ell,m} (-i)^{\ell+m} A^\dagger_{\ell} \phi_0(x) A_m. \quad (2.31) $$

This result is to be compared with the out-field obtained from the equation of motion, $\varphi_n$ in (2.18). The evaluation at the order of $g$ is given as

$$ \Phi_{(1)} = i \left( \phi_0(x) A_1 - A^\dagger_1 \phi_0(x) \right) $$

$$ = i \int d^D y \mathcal{F}_y \left( \left[ \phi_0(x), \mathcal{V}(\phi_0(y)) \right] \right) $$

$$ = - \left( - \frac{g}{p!} \right) \int d^D y \mathcal{F}_y \left( \left\{ \Delta(x-y) \phi_0^{p-1}(y) \right\}_{x(y)} \right) $$

$$ = - \int d^D y \Delta(x-y) \xi_\ast(\phi_0(y)) = \varphi_1. \quad (2.32) $$

At the order of $g^2$, the out-field is given as

$$ \Phi_{(2)} = i^2 \left( \phi_0(x) A_2 - A^\dagger_1 \phi_0 A_1 + A^\dagger_2 \phi_0(x) \right) $$

$$ = i^2 \int d^D y_1 d^D y_2 \mathcal{F}_{12} \left( \theta_{12} \phi_0(x) \mathcal{V}(\phi_0(y_1))\mathcal{V}(\phi_0(y_2)) - \mathcal{V}_1(\phi_0(y_1))\phi_0(\mathcal{V}(\phi_0(y_2))) + \theta_{12} \mathcal{V}(\phi_0(y_2))\mathcal{V}(\phi_0(y_1))\phi_0(x) \right) $$

$$ = i^2 \int d^D y_1 d^D y_2 \mathcal{F}_{12} \left( \theta_{12} \left[ \left[ \phi_0(x), \mathcal{V}(\phi_0(y_1)) \right], \mathcal{V}(\phi_0(y_2)) \right] \right). \quad (2.33) $$
We note that all the quantum operators inside the \( \star \)-operation are properly ordered except the time-ordering step function. This ordering ambiguity of the step function will be settled using the minimal realization.

First note that the commutator \([ [\phi_0(x), \phi_0^p(y_1)], \phi_0^p(y_2) ]\) is given as

\[
[ [\phi_0(x), \phi_0^p(y_1)], \phi_0^p(y_2) ] = i \left[ \{ [\Delta(x-y_1), \phi_0^{p-1}(y_1)]_{s_1}, \phi_0^p(y_2) \} 
= i \left\{ \Delta(x-y_1) \phi_0^{p-2}(y_1) \{ [\phi_0(y_1), \phi_0^p(y_2)] \} \right\}_{s_1} 
= - \left\{ \Delta(x-y_1) \phi_0^{p-2}(y_1) \{ [\Delta(y_1-y_2), \phi_0^{p-1}(y_2)] \} \right\}_{s_2} \right\}_{s_1} .
\]

\(s_1\) (\(s_2\)) refers to the symmetrization with respect to \(y_1\) \(y_2\) fields and functions. Therefore, at each step there is no position ambiguity for this operators and functions. Next, the presence of the time-ordering step function \(\theta_{12}\) changes the commutator function into the retarded Green’s function,

\[
\theta_{12} [[\phi_0(x), \phi_0^p(y_1)], \phi_0^p(y_2)] = \left\{ \Delta(x-y_1) \phi_0^{p-2}(y_1) \{ [\Delta_{\text{ret}}(y_1-y_2), \phi_0^{p-1}(y_2)] \} \right\}_{s_2} \right\}_{s_1} .
\]

Here we used the minimal realization since we put the step function as the specific position corresponding to the spectral function, \(\Delta(y_1-y_2)\).

Finally, the \( \star \)-operation on the \(s_2\) symmetrized part will give \(\phi_1(y_1)\) in (2.17):

\[
\int d^Dy_2 \mathcal{F}_2 \left( \theta_{12} [[\phi_0(x), \phi_0^p(y_1)], \mathcal{V}(\phi_0(y_2))] \right) \\
= \left\{ - \frac{g}{p!} \right\} \Delta(x-y_1) \phi_0^{p-2}(y_1) \int d^Dy_2 \mathcal{F}_2 \left( \{ [\Delta_{\text{ret}}(y_1-y_2), \phi_0^{p-1}(y_2)] \} \right)_{s_2} \right\}_{s_1} \\
= \left\{ \Delta(x-y_1) \phi_0^{p-2}(y_1) \phi_1(y_1) \right\}_{s_1} .
\]

Therefore, the out-field \(\Phi_{(2)}(x)\) reduces to \(\varphi_{(2)}(x)\) in (2.18):

\[
\Phi_{(2)}(x) = - \left\{ - \frac{g}{p!} \right\} \int dy \mathcal{F}_y \left( \Delta(x-y) \{ \phi_0^{p-2}(y) \phi_1(y) \} \right) = \varphi_{(2)}(x) .
\]

At the order of \(g^3\) the out-field is given as

\[
\Phi_{(3)} = i^3 \left( \phi_0(x) A_3 - A_3^\dagger \phi_0(x) A_2 + A_2^\dagger \phi_0(x) A_1 - A_1^\dagger \phi_0(x) \right) \\
= i^3 \int \int d^Dy_1 d^Dy_2 d^Dy_3 \\
\times \mathcal{F}_{123} \left( \theta_{123} \phi_0(x) \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) - \theta_{23} \mathcal{V}(y_1) \phi_0(x) \mathcal{V}(y_2) \mathcal{V}(y_3) \\
+ \theta_{12} \mathcal{V}(y_2) \phi_0(x) \mathcal{V}(y_3) - \theta_{123} \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \phi_0(x) \right) \\
= i^3 \int \int d^Dy_1 d^Dy_2 d^Dy_3 \mathcal{F}_{123} \left( \theta_{123} \left[ \left[ \phi_0(x), \mathcal{V}(y_1) \right], \mathcal{V}(y_2) \right], \mathcal{V}(y_3) \right) .
\]
Evaluation of the commutator $[[[\phi_0(x), \mathcal{V}(y_1)], \mathcal{V}(y_2)], \mathcal{V}(y_3)]$ is done in a few steps:

$$
[[[\phi_0(x), \phi_0^p(y_1)], \phi_0^p(y_2)], \phi_0^p(y_3)]
= i \left\{ \{\Delta(x-y_1)\phi_0(y_1)^{p-1}\}, \phi_0^p(y_2) \right\}_{s_1}, \phi_0^p(y_3)
= i \left\{ [\Delta(x-y_1)\phi_0(y_1)^{p-1}, \phi_0^p(y_2)] \right\}_{s_1}, \phi_0^p(y_3)
= i^2 \left\{ \Delta(x-y_1)\phi_0^{p-2}(y_1) \left\{ \Delta(y_1-y_2)\phi_0^{p-1}(y_2) \right\}_{s_2}, \phi_0^p(y_3) \right\}_{s_1}
= i^2 \left\{ [\Delta(x-y_1)\phi_0^{p-2}(y_1)\Delta(y_1-y_2)\phi_0^{p-1}(y_2)], \phi_0^p(y_3) \right\}_{s_1}
= i^2 \left\{ \Delta(x-y_1)\phi_0^{p-2}(y_1) \left\{ \Delta(y_1-y_2)\phi_0^{p-1}(y_2) \right\}_{s_2}, \phi_0^p(y_3) \right\}_{s_1}
+ i^2 \left\{ \Delta(x-y_1)\phi_0^{p-2}(y_1) \left\{ \{\Delta(y_1-y_2)\phi_0^{p-1}(y_2) \right\}_{s_2}, \phi_0^p(y_3) \right\}_{s_1}
= 2i^3 \left\{ \Delta(x-y_1)\phi_0^{p-3}(y_1) \left\{ \Delta(y_1-y_3)\phi_0^{p-1}(y_3) \right\}_{s_3}, \Delta(y_1-y_2)\phi_0^{p-1}(y_2) \right\}_{s_2}
+ i^3 \left\{ \Delta(x-y_1)\phi_0^{p-2}(y_1) \left\{ \Delta(y_1-y_2)\phi_0^{p-2}(y_2) \right\}_{s_2}, \Delta(y_2-y_3)\phi_0^{p-1}(y_3) \right\}_{s_3}, \phi_0^p(y_3) \right\}_{s_1}
$$

where in the last identity, the factor 2 comes from the symmetry of $y_2$ and $y_3$ in the symmetrization. Next, the time-ordered step-function is evaluated as

$$
\theta_{123} \left\{ [[\phi_0(x), \phi_0^p(y_1)], \phi_0^p(y_2)], \phi_0^p(y_3) \right\}
= i^3 (\theta_{123} + \theta_{132}) \left\{ \Delta(x-y_1)\phi_0^{p-3}(y_1) \left\{ \Delta(y_1-y_3)\phi_0^{p-1}(y_3) \right\}_{s_3}, \Delta(y_1-y_2)\phi_0^{p-1}(y_2) \right\}_{s_2}
+ i^3 \theta_{123} \left\{ \Delta(x-y_1)\phi_0^{p-2}(y_1) \left\{ \Delta(y_1-y_2)\phi_0^{p-2}(y_2) \right\}_{s_2}, \Delta(y_2-y_3)\phi_0^{p-1}(y_3) \right\}_{s_3}, \phi_0^p(y_3) \right\}_{s_1}
= i^3 \left\{ \Delta(x-y_1)\phi_0^{p-3}(y_1) \left\{ \Delta_{\text{ret}}(y_1-y_3)\phi_0^{p-1}(y_3) \right\}_{s_3}, \Delta_{\text{ret}}(y_1-y_2)\phi_0^{p-1}(y_2) \right\}_{s_2}
+ i^3 \left\{ \Delta(x-y_1)\phi_0^{p-2}(y_1) \left\{ \Delta_{\text{ret}}(y_1-y_2)\phi_0^{p-2}(y_2) \right\}_{s_2}, \Delta_{\text{ret}}(y_2-y_3)\phi_0^{p-1}(y_3) \right\}_{s_3}, \phi_0^p(y_3) \right\}_{s_1}
$$

where we use the symmetric property of $y_2$ and $y_3$ in the first identity and the step-function identity in the last identity,

$$
\theta_{123} + \theta_{132} = \theta_{12} \theta_{13}.
$$

Again the minimal realization is used to get the retarded Green’s function. Using the definition of $\phi_1(y)$ and $\phi_2(y)$ in (2.17), and after applying the $\ast$-product we have the out-field of order $g^3$ as

$$
\Phi_{(3)} = \frac{g}{p!} \int dy \mathcal{F}_y \left\{ \Delta(x-y)\phi_0^{p-2}(y)\phi_2(y) \right\}_s + \left\{ \Delta(x-y)\phi_0^{p-3}(y)\phi_1^{2}(y) \right\}_s = \varphi_{(3)} \quad (2.41)
$$
which is the out-field given in (2.18).

Higher order proof goes similarly. We provide up to the order of $g^4$ since non-local Yukawa theory \[13\] gives non-trivial result at this order. The out-field at the order of $g^4$ is given as

$$
\Phi_{(4)}(x) = \left(\phi_0(x) A_4 - A_1^\dagger \phi_0(x) A_3 + A_2^\dagger \phi_0(x) A_2 - A_3^\dagger \phi_0(x) A_1 + A_4^\dagger \phi_0(x)\right)
$$

$$
= \int \int \int d^D y_1 d^D y_2 d^D y_3 d^D y_4 \mathcal{F}_{1234} \left(\theta_{1234} \phi_0(x) \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4)\right.
$$

$$
- \theta_{234} \mathcal{V}(y_1) \phi_0(x) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) + \theta_{123} \mathcal{V}(y_1) \phi_0(x) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4)
$$

$$
- \theta_{123} \mathcal{V}(y_1) \mathcal{V}(y_2) \phi_0(x) \mathcal{V}(y_3) \mathcal{V}(y_4) + \theta_{1234} \mathcal{V}(y_1) \mathcal{V}(y_3) \mathcal{V}(y_2) \mathcal{V}(y_4) \phi_0(x)
$$

$$
= \left[ \left[ \left[ \left[ \phi_0(x), \mathcal{V}(y_1) \right], \mathcal{V}(y_2) \right], \mathcal{V}(y_3) \right], \mathcal{V}(y_4) \right] \right]. \quad (2.42)
$$

Evaluation of the commutator $\left[ \left[ \left[ \phi_0(x), \mathcal{V}(y_1) \right], \mathcal{V}(y_2) \right], \mathcal{V}(y_3) \right], \mathcal{V}(y_4) \right]$ can be done in a few steps:

$$
\left[ \left[ \left[ \phi_0(x), \phi_0^p(y_1) \right], \phi_0^p(y_2) \right], \phi_0^p(y_3) \right], \phi_0^p(y_4) \right]
$$

$$
= i \left\{ \left[ \left[ \left[ \Delta(x - y_1), \phi_0^{p-1}(y_1) \right], \phi_0(y_2) \right], \phi_0(y_3) \right], \phi_0(y_4) \right\}_{s_1}
$$

$$
= i^2 \left\{ \left[ \left[ \left[ \Delta(x - y_1), \phi_0^{p-2}(y_1) \right], \Delta(y_1 - y_2) \phi_0^{p-1}(y_2) \right], \phi_0(y_3) \right], \phi_0(y_4) \right\}_{s_2}
$$

$$
= i^3 \left\{ \left[ \left[ \left[ \left[ \Delta(x - y_1), \phi_0^{p-3}(y_1) \right], \Delta(y_1 - y_3) \phi_0^{p-1}(y_3) \Delta(y_1 - y_2), \phi_0(y_4) \right), \phi_0^p(y_4) \right], \phi_0(y_4) \right] \right\}_{s_3}
$$

$$
= 6 \left\{ \Delta(x - y_1) \phi_0^{p-4}(y_1) \left[ \Delta(y_1 - y_4) \phi_0^{p-1}(y_4) \right] \right\}_{s_4} \left\{ \Delta(y_1 - y_3) \phi_0^{p-1}(y_3) \right\}_{s_3} \left\{ \Delta(y_1 - y_2) \phi_0^{p-1}(y_2) \right\}_{s_2}
$$

$$
+ \left\{ \Delta(x - y_1) \phi_0^{p-3}(y_1) \left[ \Delta(y_1 - y_3) \phi_0^{p-2}(y_3) \right] \right\}_{s_4} \left\{ \Delta(y_1 - y_4) \phi_0^{p-1}(y_4) \right\}_{s_3} \left\{ \Delta(y_1 - y_2) \phi_0^{p-1}(y_2) \right\}_{s_2}
$$

$$
+ \left\{ \Delta(x - y_1) \phi_0^{p-3}(y_1) \left[ \Delta(y_1 - y_4) \phi_0^{p-2}(y_4) \right] \right\}_{s_4} \left\{ \Delta(y_1 - y_3) \phi_0^{p-2}(y_3) \right\}_{s_3} \left\{ \Delta(y_1 - y_2) \phi_0^{p-1}(y_2) \right\}_{s_2}
$$

$$
+ \left\{ \Delta(x - y_1) \phi_0^{p-3}(y_1) \left[ \Delta(y_1 - y_4) \phi_0^{p-3}(y_4) \right] \right\}_{s_4} \left\{ \Delta(y_1 - y_3) \phi_0^{p-2}(y_3) \right\}_{s_3} \left\{ \Delta(y_1 - y_2) \phi_0^{p-2}(y_2) \right\}_{s_2}
$$

$$
+ 2 \left\{ \Delta(x - y_1) \phi_0^{p-2}(y_1) \left[ \Delta(y_1 - y_2) \phi_0^{p-3}(y_2) \right] \right\}_{s_4} \left\{ \Delta(y_1 - y_4) \phi_0^{p-2}(y_4) \right\}_{s_3} \left\{ \Delta(y_2 - y_3) \phi_0^{p-1}(y_3) \right\}_{s_2}
$$

$$
+ \left\{ \Delta(x - y_1) \phi_0^{p-2}(y_1) \left[ \Delta(y_1 - y_2) \phi_0^{p-2}(y_2) \right] \right\}_{s_4} \left\{ \Delta(y_1 - y_4) \phi_0^{p-2}(y_4) \right\}_{s_3} \left\{ \Delta(y_3 - y_4) \phi_0^{p-1}(y_4) \right\}_{s_1}.
$$
As demonstrated in the above derivation, the minimal realization and the unitarity condition is given as

\[ \theta_{1234} + (234 \text{ permutation}) = \theta_{12}(\theta_{234} + \theta_{243}) + \theta_{13}(\theta_{324} + \theta_{342}) + \theta_{14}(\theta_{423} + \theta_{432}) \]
\[ = \theta_{12}\theta_{23}\theta_{24} + \theta_{13}\theta_{35}\theta_{34} + \theta_{14}\theta_{42}\theta_{43} = \theta_{12}\theta_{13}\theta_{14} \]
\[ \theta_{1234} + \theta_{1324} + \theta_{1342} = (\theta_{12}\theta_{13} - \theta_{13}\theta_{32})\theta_{34} + \theta_{1324} + \theta_{1342} \]
\[ = \theta_{12}\theta_{13}\theta_{34} - \theta_{132}\theta_{34} + \theta_{1324} + \theta_{1342} = \theta_{12}\theta_{13}\theta_{34} \]
\[ \theta_{1234} + \theta_{1243} = \theta_{12}\theta_{23}\theta_{24} , \]

we may put the commutator with the time-ordering as

\[ \theta_{1234} \left[ \left[ \left[ \phi_0(x), \phi_0^\dagger(y_1) \right], \phi_0^\dagger(y_2) \right], \phi_0(y_3) \right] , \phi_0(y_4) \right] = - \int d^D y \Delta(x - y) \left\{ \delta^D(y - y_1) \phi_0^{p-4}(y_1) \right\} \]
\[ \times \left( \left\{ \Delta_{\text{ret}}(y_1 - y_4)\phi_0^{p-1}(y_1) \right\}_{s_4} \left\{ \Delta_{\text{ret}}(y_1 - y_3)\phi_0^{p-1}(y_3) \right\}_{s_3} \left\{ \Delta_{\text{ret}}(y_1 - y_2)\phi_0^{p-1}(y_2) \right\}_{s_2} \right) + \phi_0(y_1) \left( \left\{ \Delta_{\text{ret}}(y_1 - y_3)\phi_0^{p-2}(y_3) \right\}_{s_4} \left\{ \Delta_{\text{ret}}(y_1 - y_4)\phi_0^{p-1}(y_4) \right\}_{s_3} \left\{ \Delta_{\text{ret}}(y_1 - y_2)\phi_0^{p-1}(y_2) \right\}_{s_2} \right) + \phi_0^2(y_1) \left( \left\{ \Delta_{\text{ret}}(y_1 - y_2)\phi_0^{p-3}(y_2) \right\}_{s_4} \left\{ \Delta_{\text{ret}}(y_2 - y_4)\phi_0^{p-1}(y_4) \right\}_{s_3} \left\{ \Delta_{\text{ret}}(y_2 - y_3)\phi_0^{p-1}(y_3) \right\}_{s_2} \right) + \phi_0^3(y_1) \left( \left\{ \Delta_{\text{ret}}(y_1 - y_2)\phi_0^{p-2}(y_2) \right\}_{s_4} \left\{ \Delta_{\text{ret}}(y_2 - y_3)\phi_0^{p-2}(y_3) \right\}_{s_3} \left\{ \Delta_{\text{ret}}(y_3 - y_4)\phi_0^{p-1}(y_4) \right\}_{s_2} \right) \right) . \]

Using the identities of \( \phi_n(y) \) and after applying the \( \ast \)-product we have the out-field of order \( g^4 \) as

\[ \Phi_{(4)} = \frac{g}{p!} \int d^D y \mathcal{F}_y \left( \Delta(x - y) \left( \phi_0^{p-4}(y)\phi_1^\dagger(y) + \phi_0^{p-3}(y)\phi_1(y)\phi_2(y) + \phi_0^{p-2}(y)\phi_3(y) \right) \right)_{s(y)} = \varphi_{(4)} . \]

(2.43)

As demonstrated in the above derivation, the minimal realization and the \( \ast \)-time ordering are enough for proving that the S-matrix connects the in- and out-field correctly to all orders of perturbation.

### 2.5 Unitarity of S-matrix

In this section, we provide a proof that this S-matrix is unitary,

\[ SS^\dagger = S^\dagger S = 1 . \]

(2.44)

To do this we use the S-matrix in Eq. (2.23) and (2.24), the coupling constant expanded version of S-matrix, and evaluate \( SS^\dagger \) order by order in \( g \). We remark that the product of \( SS^\dagger \) is not the \( \ast \)-product but is the ordinary product since S-matrix does not depend on coordinates explicitly.

The unitarity at the order of \( g \) is trivially satisfied since \( A_1^\dagger = A_1 \). At the order of \( g^2 \), the unitarity condition is given as

\[ A_2 + A_2^\dagger = A_1^\dagger A_1 . \]

(2.45)
The proof goes as follows:

\[
\text{LHS} = A_2 + A_1^\dagger \\
= \int \int d^Dy_1 \, d^Dy_2 \, \mathcal{F}_{12} \left( \theta_{12} \left( \mathcal{V}(\phi_0(y_1)) \mathcal{V}(\phi_0(y_2)) + \mathcal{V}(\phi_0(y_2)) \mathcal{V}(\phi_0(y_1)) \right) \right) \\
= \int \int d^Dy_1 \, d^Dy_2 \, \mathcal{F}_{12} \left( \left( \theta_{12} + \theta_{21} \right) \mathcal{V}(\phi_0(y_1)) \mathcal{V}(\phi_0(y_2)) \right) \\
= \int d^Dy_1 \, \mathcal{F}_1 \left( \mathcal{V}(\phi_0(y_1)) \right) \int d^Dy_2 \, \mathcal{F}_1 \left( \mathcal{V}(\phi_0(y_2)) \right),
\]

\[
\text{RHS} = A^\dagger_1 A_1 \\
= \int d^Dy_1 \, \mathcal{F}_1 \left( \mathcal{V}(\phi_0(t_1)) \right) \int d^Dy_2 \, \mathcal{F}_1 \left( \mathcal{V}(\phi_0(t_2)) \right)
\]

and therefore, \( LHS = RHS \). (Note that the \( \dagger \) operation is applied to the fields \( \phi_0 \)'s not the time-ordering or \( * \)-operation). Here we use the change of variables to get the third line and the identity \( \theta_{12} + \theta_{21} = 1 \). It should be noted that this step-function identity always holds even when the coordinates are split as far as the split coordinates are concerned:

\[
\theta(x_i - y_j) + \theta(y_j - x_i) = 1.
\]

This is the reason why the unitarity holds without using the minimal realization.

At the order of \( g^3 \), the unitarity condition is given as

\[
A_3 - A_3^\dagger = A_1^\dagger A_2 - A_2^\dagger A_1.
\]

The proof goes as follows:

\[
\text{LHS} = A_3 - A_3^\dagger \\
= \int \int \int d^Dy_1 \, d^Dy_2 \, d^Dy_3 \, \mathcal{F}_{123} \left( \theta_{123} \left( \mathcal{V}(\phi_0(y_1)) \mathcal{V}(\phi_0(y_2)) \mathcal{V}(\phi_0(y_3)) \right) \\
- \mathcal{V}(\phi_0(y_3)) \mathcal{V}(\phi_0(y_2)) \mathcal{V}(\phi_0(y_1)) \right) \\
= \int \int \int d^Dy_1 \, d^Dy_2 \, d^Dy_3 \, \mathcal{F}_{123} \left( \theta_{123} - \theta_{321} \right) \mathcal{V}(\phi_0(y_1)) \mathcal{V}(\phi_0(y_2)) \mathcal{V}(\phi_0(y_3)) \\
= \int \int \int d^Dy_1 \, d^Dy_2 \, d^Dy_3 \, \mathcal{F}_{123} \left( \theta_{23} - \theta_{21} \right) \mathcal{V}(\phi_0(y_1)) \mathcal{V}(\phi_0(y_2)) \mathcal{V}(\phi_0(y_3)) \\
= \int \int \int d^Dy_1 \, d^Dy_2 \, d^Dy_3 \, \left( \mathcal{F}_1(\mathcal{V}(y_1)) \mathcal{F}_23(\theta_{23} \mathcal{V}(y_2) \mathcal{V}(y_3)) \\
- \mathcal{F}_{12}(\theta_{21} \mathcal{V}(y_1) \mathcal{V}(y_2)) \mathcal{F}_3(\mathcal{V}(y_3)) \right) \right)
\]

\[
\text{RHS} = A^\dagger_1 A_2 - A^\dagger_2 A_1 \\
= \int \int \int d^Dy_1 \, d^Dy_2 \, d^Dy_3 \, \left( \mathcal{F}_1(\mathcal{V}(y_1)) \mathcal{F}_23(\theta_{23} \mathcal{V}(y_2) \mathcal{V}(y_3)) \\
- \mathcal{F}_{12}(\theta_{12} \mathcal{V}(y_2) \mathcal{V}(y_1)) \mathcal{F}_3(\mathcal{V}(y_3)) \right),
\]
where we use the identity
\[ \theta_{123} - \theta_{321} = (1 - \theta_{21})\theta_{23} - \theta_{32}\theta_{21} = \theta_{23} - \theta_{21}. \]  \hfill (2.50)

Comparing with both sides, we have LHS = RHS.

At the order of \( g^4 \), the unitarity condition is given as
\[ A_4 + A_4^\dagger = A_4^\dagger A_3 - A_2 A_2 + A_3 A_1. \]  \hfill (2.51)

The proof goes as follows:
\[
\text{LHS} = A_4 + A_4^\dagger \\
= \int d^D y_1 \cdots \int d^D y_4 \mathcal{F}_{1234} \left( \theta_{1234} \left( \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) \\
+ \mathcal{V}(y_4) \mathcal{V}(y_3) \mathcal{V}(y_2) \mathcal{V}(y_1) \right) \right) \\
= \int d^D y_1 \cdots \int d^D y_4 \mathcal{F}_{1234} \left( (\theta_{1234} + \theta_{4321}) \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) \right). \]  \hfill (2.52)

Using the identities,
\[
\theta_{1234} = \theta_{234} - (\theta_{2134} + \theta_{2341} + \theta_{234}) = \theta_{234} - (\theta_{2134} + \theta_{23}\theta_{21}\theta_{34}) \\
\theta_{4321} = \theta_{321} - (\theta_{3421} + \theta_{3241} + \theta_{324}) = \theta_{321} - (\theta_{3421} + \theta_{32}\theta_{21}\theta_{24}) \\
\theta_{2134} + \theta_{23}\theta_{21}\theta_{34} = \theta_{34}(\theta_{213} + \theta_{231}) = \theta_{34}\theta_{21}\theta_{23} \\
\theta_{3421} + \theta_{32}\theta_{21}\theta_{24} = \theta_{21}(\theta_{342} + \theta_{324}) = \theta_{21}\theta_{34}\theta_{32},
\]
we have
\[
\text{LHS} = \int d^D y_1 \cdots \int d^D y_4 \mathcal{F}_{1234} \left( (\theta_{234} - \theta_{21}\theta_{34} + \theta_{32}) \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) \right). \]  \hfill (2.53)

On the other hand,
\[
\text{RHS} = A_4^\dagger A_3 - A_2 A_2 + A_3 A_1 \\
= \int d^D y_1 \cdots \int d^D y_4 \left( \mathcal{F}_1(\mathcal{V}(y_1)) \mathcal{F}_{234} \left( \theta_{234} \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) \right) \\
- \mathcal{F}_{12}(\theta_{12} \mathcal{V}(y_2) \mathcal{V}(y_1)) \mathcal{F}_{34} \left( \theta_{34} \mathcal{V}(y_3) \mathcal{V}(y_4) \right) \right) \\
+ \mathcal{F}_{123} \left( \theta_{123} \mathcal{V}(y_3) \mathcal{V}(y_2) \mathcal{V}(y_1) \right) \mathcal{F}_4(\mathcal{V}(y_4)) \\
= \int d^D y_1 \cdots \int d^D y_4 \mathcal{F}_{1234} \left( (\theta_{234} - \theta_{21}\theta_{34} + \theta_{32}) \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) \right). \]  \hfill (2.54)

Comparing with both sides, we have LHS = RHS.

One can confirm that the higher order proof goes similarly with the ordinary perturbation case. In this proof, only the time-ordering matters irrespective of the \( \star \)-operation. One may put the time-ordering out-side of the star-operation as far as the unitarity is concerned. However, the time-ordering outside the star-operation does not fulfill
the correct in- and out-field relation. It is remarked that the time-ordering outside the \( \star \)-operation is different from the time-ordering inside the \( \star \)-operation up to higher derivatives. It is like the contact terms in the ordinary gauge theory.

To see this we give an explicit expression for this difference up to order of \( g^3 \). At the order of \( g^1 \), there is no distinction between two since there is no time ordering. At the order of \( g^2 \), let us denote the ordinary time-ordered one as \( a_2 \), which puts the time-ordering outside the \( \star \)-operation:

\[
a_2 = \int \int dy_1 dy_2 \theta_{12} F_{12} (V(t_1) V(t_2)). \tag{2.55}
\]

\( a_2 \) satisfies the relation:

\[
a_2 + a_2^\dagger = A_2^2.
\]

The difference is denoted as \( c_2 \):

\[
ic_2 = A_2 - a_2, \quad c_2 = c_2^\dagger, \tag{2.56}
\]

which is given as

\[
ic_2 &= -\frac{1}{2} \int \int d^D y_1 d^D y_2 \left( \theta_{12} F_{12} - F_{12} \theta_{12} \right) \left( \mathcal{V}(y_1) \mathcal{V}(y_2) + \mathcal{V}(y_1) \mathcal{V}(y_2) \right) \\
&\quad -\frac{1}{2} \int \int d^D y_1 d^D y_2 \left( \theta_{12} F_{12} - F_{12} \theta_{12} \right) \left( [\mathcal{V}(y_1), \mathcal{V}(y_2)] \right) \\
&\quad + \frac{1}{2} \int \int d^D y_1 d^D y_2 \left( (\theta_{12} + \theta_{21}) F_{12} - F_{12} (\theta_{12} + \theta_{21}) \right) \left( \mathcal{V}(y_1) \mathcal{V}(y_2) \right) \\
&\quad -\frac{1}{2} \int \int d^D y_1 d^D y_2 \left( \theta_{12} F_{12} - F_{12} \theta_{12} \right) \left( [\mathcal{V}(y_1), \mathcal{V}(y_2)] \right) \\
&\quad = -\frac{1}{2} \int \int d^D y_1 d^D y_2 \left( \theta_{12} F_{12} - F_{12} \theta_{12} \right) \left( [\mathcal{V}(y_1), \mathcal{V}(y_2)] \right), \tag{2.57}
\]

where we use the identity \( \theta_{12} + \theta_{21} = 1 \). This is the source of higher derivative terms to the lowest order, which is to be supplemented by the S-matrix proposed in [14]. If one evaluates the commutator of the step function and the \( \star \)-product, this leaves us with the time derivatives of the fields and of the spectral functions.

For the order of \( g^3 \), we have

\[
A_3 = a_3 + ic_2 A_1 + c_3. \tag{2.58}
\]

\( a_3 \) is the ordinary time-ordered one:

\[
a_3 = \int \int \int d^D y_1 d^D y_2 d^D y_3 \theta_{123} F_{123} (\mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3)). \tag{2.59}
\]

Using the identity,

\[
\int \int \int d^D y_1 d^D y_2 d^D y_3 \theta_{123} \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3)
\]
\[
\begin{aligned}
&= \frac{1}{6} \int \int \int d^D y_1 d^D y_2 d^D y_3 \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \\
&\quad + \frac{1}{3} \int \int \int d^D y_1 d^D y_2 d^D y_3 (\theta_{123} + \theta_{132}) [\mathcal{V}(y_1), [\mathcal{V}(y_2), \mathcal{V}(y_3)]] \\
&\quad + \frac{1}{2} \int \int \int d^D y_1 d^D y_2 d^D y_3 \theta_{12}[\mathcal{V}(y_1), \mathcal{V}(y_2)], \mathcal{V}(y_3)]
\end{aligned}
\]

we have

\[
c_3 = -\frac{1}{3} \int \int d^D y_1 d^D y_2 d^D y_3 (\theta_{123} \mathcal{F}_{123} - \mathcal{F}_{123} \theta_{123}) \left( [\mathcal{V}_1, [\mathcal{V}_2, \mathcal{V}_3]] + [\mathcal{V}_2, [\mathcal{V}_1, \mathcal{V}_3]] \right),
\]

with \( c_3 = c_3^\dagger \).

### 3 Feynman rule in Momentum-space

In this section we illustrate the perturbation approach to the STNC field theory in the momentum-space. The momentum space calculation will be complementary to the coordinate space representation described in section 2. The minimal realization of the time-ordering is to be properly represented. For definiteness, we consider \( \phi^4 \) theory,

\[
L_I(t) = -\frac{\lambda}{4!} \int d^{D-1}x \phi^4(x).
\]

Two-point function is represented in terms of the positive spectral function \( \Delta_+(x) \) instead of Feynman propagator,

\[
\Delta_+(x) = \langle 0 | \phi_{in}(x) \phi_{in}(0) | 0 \rangle = \int \frac{d^Dk}{(2\pi)^D} e^{-ikx} \tilde{\Delta}_+(k)
\]

where \( \tilde{\Delta}_+(k) \) is the Fourier transform of the free spectral function,

\[
\tilde{\Delta}_+(k) = \quad k_\rightarrow = 2\pi \delta(k^2 - m^2) \theta(k^0)
\]

where we specify the arrow to denote the momentum flow.

In addition, we need a “time-ordered” spectral function \( \Delta_R(x) \) to describe the time ordering effect.

\[
\Delta_R(x) = \theta(x^0)\Delta_+(x) = \int \frac{d^Dk}{(2\pi)^D} e^{-ikx} \tilde{\Delta}_R(k)
\]

\[
\tilde{\Delta}_R(k) = \quad \frac{i}{2\omega(k)} \frac{1}{(k_0 - \omega_k + i\epsilon)} = \quad k_\rightarrow
\]

with \( \omega_k = \sqrt{k^2 + m^2} \). \( \tilde{\Delta}_R(k) \) is represented as a triangled arrow to emphasize the ordering effect.
It is noted that the time-ordered two-point function $\Delta_R(x)$ in (3.3) is not confused with the retarded Green’s function $\Delta_{ret}(x)$ given in (2.12): Each has a different pole structure. The Feynman propagator is given in terms of the time-ordered spectral function,

$$i\Delta_F(x) = \Delta_R(x) + \Delta_R(-x).$$

The four-point vertex is given as

$$-i(2\pi)^d \delta^d(p_1 + p_2 + p_3 + p_4) \Gamma_4(k_1, k_2, k_3, k_4)$$

and its lowest order diagram is given as

$$-i\Gamma^{(0)}_4(k_1, k_2, k_3, k_4) = -i\lambda v(p_1, p_2, p_3, p_4)$$

where

$$v(p_1, p_2, p_3, p_4) = \frac{1}{3} \left( \cos \left( \frac{p_1 \wedge p_2}{2} \right) \cos \left( \frac{p_3 \wedge p_4}{2} \right) \right. \left. + \cos \left( \frac{p_1 \wedge p_3}{2} \right) \cos \left( \frac{p_2 \wedge p_4}{2} \right) + \cos \left( \frac{p_1 \wedge p_4}{2} \right) \cos \left( \frac{p_2 \wedge p_3}{2} \right) \right).$$

The vertex function $v$ is permutationally symmetric in the external momentum indices and is insensitive to the sign of the momenta;

$$v(p_1, p_2, p_3, p_4) = v(\pm p_1, \pm p_2, \pm p_3, \pm p_4) = v(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}, p_{\sigma(4)})$$

where $\sigma(i)$ is the permutation operation.

The Feynman rule for this theory is summarized as follows.

1. Each vertex is assigned as $-i\lambda v(p_1, p_2, p_3, p_4)$ where $p_i$’s are incoming momenta of four legs and its total momentum vanishes.
2. The legs are either external legs or are connected to other vertices, making internal lines.
3. Each vertex is numbered so that the internal lines are assigned with arrows. The arrows point from high-numbered vertex to low-numbered one.
4. Among the arrows, only one arrow between two adjacent vertices is assigned as triangled one and the total number of the triangled arrows should be $n - 1$ for the $n$ connected vertices. This assignment is due to the minimal realization of the time-ordered step function.
5. The diagrams with the same topology with arrows are identified and the numbering of vertices is ignored. As a result the number of diagrams are reduced from the original $n!$ diagrams.
6. The distinctive Feynman diagrams are multiplied with the symmetric factors.
7. The momentum flows along the arrows. The arrowed internal line with momentum $k$ is assigned as $\hat{\Delta}_+(k)$ and the triangled arrowed internal line as $\hat{\Delta}_R(k)$.

Note that the rule (5) originates from the time-ordering. To understand this, we provide a few examples. Let us denote the three vertex diagram $a\rightarrow b\rightarrow c$ as the numbered
vertex \((a, b, c)\), where as many as internal lines between vertices may exist. When two diagrams numbered as \((3, 1, 2)\) and \((2, 1, 3)\) are combined,

\[
\theta_{123}((3, 1, 2) + (2, 1, 3)) = (\theta_{123} + \theta_{132})(3, 1, 2) = (\theta_{123} + \theta_{132})(2, 1, 3),
\]

one may use the step function identity \(\theta_{123} + \theta_{132} = \theta_{12} \theta_{13}\), and rearrange the time-ordering so that the ordering is directly relevant for the diagram \((213)\) or \((312)\):

\[
\theta_{123}((3, 1, 2) + (2, 1, 3)) = \theta_{12} \theta_{13} (2, 1, 3) = \theta_{12} \theta_{13} (3, 1, 2).
\]

This reduces the two numbered diagrams into the one distinct Feynman diagram with the arrow topology; \(\rightarrow o \leftarrow o\).

Consider a four-point vertex diagram denoted as \(\{abcd\} \equiv b \rightarrow c \rightarrow d\).

6 diagrams numbered as \(\{ab1c\}\) with \(a, b,\) and \(c\) the permutations of 234 will be reduced to a Feynman diagram \(\rightarrow o \leftarrow o\), since \(\theta_{1234} + \text{permutations of 234} = \theta_{12} \theta_{13} \theta_{14}\).

Three diagrams \((2134)\), \((3124)\) and \((4123)\) with \(\{abcd\} \equiv a \rightarrow b \rightarrow c \rightarrow d\) are reduced to a Feynman diagram \(\rightarrow o \leftarrow o \leftarrow o\) since \(\theta_{1234} + \theta_{1324} + \theta_{1342} = \theta_{12} \theta_{13} \theta_{34}\). The reduction of the numbered diagrams to an arrowed one is very general in momentum space and hence, the rule (5) follows.

### 3.1 Self-energy:

Self energy is defined as

\[
-i \Sigma(1)(p_1, p_2) = \langle -p_1 \rightarrow S - 1 \leftarrow p_2 \rangle_c
\]

where \(\langle \cdots \rangle_c\) refers to the amputated one-particle irreducible function. In perturbation, we use the one particle state representation with momentum \(p\), \(\langle p \left| \phi_{\text{in}}(x) \right| 0 \rangle = Ne^{ipx}\) with \(N = 1\) as a proper normalization constant.

The one loop contribution to the self-energy comes from the first term of S-matrix, \(A_1\) in \(\bigcirc\): \(\langle -p_2 \left| iA_1 \right| p_1 \rangle_c\)

\[
-i \Sigma(1)(p_1, p_2) = \frac{1}{2} p_1 \begin{array}{c}
\bigcirc
\end{array} p_2
\]

\[
= -\frac{i\lambda}{2} \int_k \tilde{\Delta}(k) v(p_1, p_1, k, k)
\]

where \(\frac{1}{2}\) is the symmetric factor and \(\int_k\) is the abbreviated notation for the momentum integration:

\[
\int_k \equiv \int \frac{d^Dk}{(2\pi)^D}.
\]
This one-loop contribution can be written as
\[ \Sigma_{(1)}(p_1, p_2) = \frac{\lambda}{2} \left( \frac{2}{3} \int k \frac{i}{k^2 - m^2 + i\epsilon} + \int k \bar{\Delta}_+(k) \cos(p_1 \wedge k) \right) . \] (3.5)

Here we use the identity (see Appendix),
\[ \int k \bar{\Delta}_+(k) = \int k \frac{i}{k^2 - m^2 + i\epsilon} . \] (3.6)

The first term in (3.5) is UV-divergent when \( D \geq 2 \) and can be absorbed into the mass renormalization. Note that the factor 2/3 is different for the commuting case.

The second term is the non-planar contribution. One may put the integration for even \( D \) (see Appendix) as
\[ \int k \bar{\Delta}_R(k) \bar{\Delta}_+(k) \cos(p_1 \wedge k) = \int_{0}^{\infty} \frac{d\alpha}{(4\pi\alpha)^{D/2}} e^{-\alpha m^2 - \frac{\alpha p^2}{m^2}} = \frac{m^{(D-2)/2}}{(2\pi)^{D/2}} (p \circ p) \frac{2-D}{4} K_{D-2} (m\sqrt{p \circ p}) , \] (3.7)

where \( p \circ k = p^\mu \theta_{\mu\nu} \theta_{\nu\rho} k_\rho \) and the \( K_v(x) \) is the modified Bessel function. Therefore, the second term is finite as far as \( \theta \) and mass do not vanish. The feature that non-planar diagram is finite is very general in SSNC QFT [14]. The same conclusion applies to STNC QFT also. Furthermore, it should be noted that unlike in SSNC QFT, there is no UV-IR mixing since \( p \circ p \geq m^2 \) when \( p \) is on-shell [15]. This feature is not changed even if \( \Sigma_{(1)} \) is included in a higher loop graph since \( \Sigma_{(1)} \) is connected through the two-point function given in (3.2) which maintains the on-shell condition due to the delta-function, and therefore, the loop-diagrams do not present any UV-IR mixing problem.

The two-loop contribution comes from the terms: \( -p_2 |i^2 A_2| p_1 \). \n\[ -i \Sigma_{(2)}(p_1, p_2) = \Sigma_{(2a)}(p_1, p_2) + \Sigma_{(2b)}(p_1, p_2) . \]

The first contribution is given as
\[ -i \Sigma_{(2a)}(p_1, p_2) = \frac{\lambda^2}{4} \int k \bar{\Delta}_R(k) \bar{\Delta}_+(k) v(p_1, p_2, k, k) v(k, k, \ell, \ell) . \]

Using the identity (see Appendix),
\[ \int k \bar{\Delta}_R(k) \bar{\Delta}_+(k) = -\frac{1}{2} \int k \frac{1}{(k^2 - m^2 + i\epsilon)^2} , \] (3.8)

we may put this as
\[ -i \Sigma_{(2a)}(p_1, p_2) = \frac{\lambda^2}{4} \left( \frac{4}{9} \right) \int k \frac{i}{(k^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)^2} . \]
\[-\frac{\lambda^2}{2} \int_{k,\ell} \tilde{\Delta}_+(\ell) \tilde{\Delta}_R(k) \tilde{\Delta}_+(-k) \left( \frac{1}{9} \right) \left( 2 \cos(k \wedge p_1) + 2 \cos(k \wedge (p_1 - \ell)) + \frac{1}{2} \cos(k \wedge (p + \ell)) \right).\]

The first term is the planar diagram contribution and is divergent with the factor reduced to 4/9. The divergence is absorbed in the mass and coupling constant renormalization. The second term is the non-planar contribution and is again UV-IR finite.

The second contribution of the two loop is given as

\[-i\Sigma_{(2b)}(p_1, p_2) = \frac{1}{3!} \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) + \frac{1}{3!} \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right).\]

(3.9)

\[- \frac{\lambda^2}{6} \int_{k,\ell,q} \tilde{\Delta}_+(k) \tilde{\Delta}_R(\ell) \tilde{\Delta}_+(-q) v(p_1, k, \ell, q)^2 \times \left( (2\pi)^D \delta(q + k + \ell + p_1) + p_1 \leftrightarrow -p_1 \right).\]

Using the identity,

\[-i \int_{k,\ell,q} \tilde{\Delta}_+(k) \tilde{\Delta}_R(\ell) \tilde{\Delta}_+(-q) \left( (2\pi)^D \delta(q + k + \ell + p_1) + p_1 \leftrightarrow -p_1 \right)\]

we may put \((3.9)\) as

\[-i\Sigma_{(2b)}(p_1, p_2) = \frac{i\lambda^2}{6} \left( \frac{1}{3!} \right) \int_{k,\ell,q} \left( (2\pi)^D \delta(q + k + \ell + p_1) \right) \]

\[- \frac{\lambda^2}{6} \int_{k,\ell,q} \tilde{\Delta}_+(k) \tilde{\Delta}_R(\ell) \tilde{\Delta}_+(-q) \left( (2\pi)^D \delta(q + k + \ell + p_1) + p_1 \leftrightarrow -p_1 \right) \]

\[\times \left( \frac{1}{9} \right) \left( \frac{1}{2} \cos(k + p) \wedge (p + q) + \frac{1}{2} \cos(k + p) \wedge (p + \ell) + \frac{1}{2} \cos(k + p) \wedge (k - q) \right)\]

\[+ \frac{1}{2} \cos(k + p) \wedge (p - k) + \frac{1}{2} \cos(k + p) \wedge (q - \ell) + \frac{1}{2} \cos(k + p) \wedge (p - q) \]

\[+ \frac{1}{2} \cos(q + p) \wedge (k - \ell) + \frac{1}{2} \cos(p + \ell) \wedge (p - \ell) + \frac{1}{2} \cos(p + \ell) \wedge (q - k) \].

The first term is the planar contribution and is divergent while the rest is the non-planar contribution and is UV-IR finite.

### 3.2 Four point function

In this section, we provide a few diagramatic examples corresponding to the four point function. The one loop correction to the four point function is given as

\[-i\Gamma_4^{(1)} = \frac{1}{2} \left( \begin{array}{c} p_2 \\ p_1 \end{array} \right) + \left( p_2 \leftrightarrow p_3 \right) + \left( p_2 \leftrightarrow p_4 \right)\]

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where each has 2-arrowed diagrams,

\[
\begin{align*}
&\phantom{=} + k_2 \\
&\phantom{=} + k_2
\end{align*}
\]

= \int_{k_1,k_2} \left( v(p_1, p_2, k_1, k_2) v(k_1, k_2, p_3, p_4) \tilde{\Delta}_R(k_1) \tilde{\Delta}_+(k_2) (2\pi)^D \delta^D(p_1 + p_2 + k_1 + k_2) \\
&\phantom{=} + (p_1, p_2) \leftrightarrow (p_3, p_4) \right).
\]

There are four types of two loop diagrams,

\[
\begin{align*}
\begin{array}{c}
\frac{1}{4} \quad p_1 \\
\frac{1}{4} \quad p_1
\end{array}
\end{align*}
\]

and their crossed channels $p_2 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_4$.

The first diagram in (3.10) has the 4 distinct arrowed diagrams, whose contributions are given as

\[
\begin{align*}
&\phantom{=} + k_2 \\
&\phantom{=} + k_2
\end{align*}
\]
\[-i \int_{k_1, k_2, \ell_1, \ell_2} v(p_1, p_2, k_1, k_2)v(k_1, k_2, \ell_1, \ell_2)v(\ell_1, \ell_2, p_3, p_4) \tilde{\Delta}_R(k_1) \tilde{\Delta}_R(\ell_1) \tilde{\Delta}_+(k_2) \tilde{\Delta}_+(\ell_2) \]

\[
 \times (2\pi)^D \left( \delta^D(p + k) + \delta^D(p - k) \right) \left( \delta^D(k - \ell) + \delta^D(k + \ell) \right)
\]

where \( p = p_1 + p_2 = p_3 + p_4, k = k_1 + k_2 \) and \( \ell = \ell_1 + \ell_2 \).

The second diagram in (3.10) also has the 6 distinct diagrams,

\[
= -i \int_{k_1, k_2, \ell_1, \ell_2} v(p_1, p_2, k_1, k_2)v(k_1, k_2, \ell_1, \ell_2)v(\ell_1, \ell_2, p_3, p_4) \tilde{\Delta}_+(\ell) \tilde{\Delta}_R(k_1)
\]

\[
\times (2\pi)^D \left( \left( \tilde{\Delta}_R(k_1) \tilde{\Delta}_+(k_2) + \tilde{\Delta}_+(k_1) \tilde{\Delta}_R(k_2) \right) \left( \delta^D(p + k) + \delta^D(p - k) \right) \right.
\]

\[
\left. + \tilde{\Delta}_+(k_1) \tilde{\Delta}_R(k_2) \left( \delta^D(p - k_1 + k_2) + \delta^D(p - k_1 - k_2) \right) \right).
\]

The third diagram in (3.10) has 6 distinct diagrams,
4 Conclusion and outlook

The unitary S-matrix has been constructed in space-time non-commutative field theory by introducing a proper treatment of the time-ordering, the so-called minimal realization of the time-ordering and \( \star \)-time ordering. Based on this unitary S-matrix, the Feynman rule is established for the perturbation of STNC real scalar field theory. We note that our time-ordering differs from the one suggested or conjectured in [8, 10]; their S-matrix can be unitary but will not guarantee the Heisenberg equation of motion as requested in the Yang-Feldmann approach [12].

Loop calculations of the STNC theory demonstrate that the divergent structure is the same as in the SSNC theory, which comes from the planar diagrams. The non-planar diagrams are finite as in the SSNC real scalar field theory and remarkably, there is no UV/IR mixing problem in the STNC result.

The perturbation theory is not limited to the real scalar theory. One may generalize this formalism to complex scalar field theory, fermionic theory, and gauge theory. Especially, the gauge theory possesses derivative interaction and needs further care such as in time-ordering and gauge symmetry. The details of which are in preparation and will be published elsewhere [16].

Finally, it is noted that the formalism is considered so far in terms of the Lagrangian formalism of the second quantized operators in the Heisenberg picture. The Hamiltonian formalism is not easy to obtain [17] and there lacks the path-integral formalism. The path
integral formalism is necessary to accommodate the non-abelian gauge theory. Currently, finding the path-integral approach of the theory looks a very challenging problem to solve.

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Appendix

In this appendix, we provide some identities useful for loop correlations.

1. Identity of Eq. (3.6): The left hand side of (3.6) becomes

\[
LHS = \int_k \Delta_+(k) = \int \frac{d^Dk}{(2\pi)^D} 2\pi \delta(k^2 - m^2) \theta(k_0) \\
= \int \frac{d^Dk}{(2\pi)^D} 2\pi \delta \left(k_0^2 - \omega_k^2\right) \theta(k_0) \\
= \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{1}{2\omega_k},
\]

where we use the following identity

\[
\delta(y^2 - a^2) = |2a|^{-1} [\delta(y - a) + \delta(y + a)].
\]

The right hand side of (3.6) is given as;

\[
RHS = i \int_{\frac{1}{k^2 - m^2 + i\epsilon}} \frac{1}{k^2 - m^2 + i\epsilon} = i \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \int \frac{dk_0}{2\pi} \frac{1}{(k_0 - \omega_k + i\delta)(k_0 + \omega_k - i\delta)}.
\]

To evaluate this, one may use contour integral over \(k_0\). The contour integral on the upper half-plane has the contribution from the positive imaginary pole:

\[
RHS = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{1}{2\omega_k}.
\]

Hence, (3.6) follows:

\[
\int_k \Delta_+ = i \int_k \frac{1}{k^2 - m^2 + i\epsilon}.
\]

2. Identity of (3.7): We can derive this identity by taking advantage of the delta function in \(\Delta_+(k)\).

\[
LHS = \int_k \Delta_+(k) \cos(p \wedge k) = \int_k 2\pi \delta(k^2 - m^2) \theta(k_0) \cos(p_0 k^{(N)} \theta - p^{(N)} k_0 \theta)
\]
\[
\int \theta(k_0) 2\pi \delta(k^2 - m^2) \cos(p_0 k^{(N)} \theta) \cos(p^{(N)} k_0 \theta) \\
= \frac{1}{2} \int \cos(p_0 k^{(N)} \theta) \cos(p^{(N)} k_0 \theta) \\
= \frac{1}{2} \int \cos(p_0 k^{(N)} \theta) \cos(p^{(N)} k_0 \theta) \\
= \frac{1}{2} \int \cos(p_0 k^{(N)} \theta) \cos(p^{(N)} k_0 \theta)
\]

where we use the symmetry of spatial component of \( k \): \( k \to -k \) in the second line. \( p^{(N)} \) or \( k^{(N)} \) refers to the non-commuting component of the spatial momentum. Using the integral representation of the delta-function we have

\[
LHS = \frac{1}{2} \int \cos(p_0 k^{(N)} \theta) \cos(p^{(N)} k_0 \theta) \int_{-\infty}^{\infty} d\alpha e^{i\alpha(k^2 - m^2)}
\]

\[
= \frac{1}{2} \int \cos(p_0 k^{(N)} \theta) \cos(p^{(N)} k_0 \theta) \int_{-\infty}^{\infty} d\alpha e^{i\alpha(k^2 - m^2)} + c.c.
\]

\[
= \frac{1}{2} \int 0 \int_{k_0} e^{i\alpha k_0} \cos(p^{(N)} k_0 \theta) \int_{k} e^{-i\alpha k} \cos(p^{(N)} k_0 \theta).
\]

\( k_0 \) integration becomes

\[
\int \frac{dk_0}{2\pi} e^{i\alpha k_0} \cos(p^{(N)} k_0 \theta) = e^{-i\frac{(p^{(N)} \theta)^2}{4\alpha}} \int \frac{dk_0}{2\pi} e^{i\alpha k_0} = e^{-i\frac{(p^{(N)} \theta)^2}{4\alpha}} e^{i\pi/4} \sqrt{4\pi\alpha}
\]

where we shift \( k_0 \) by \( k_0 \pm p^{(N)} \theta/(2\alpha) \) in the first identity, and rotate \( k_0 \) by \( \frac{e^{i\pi/4}}{\sqrt{4\alpha}} k_0 \) in the last identity to evaluate the Gaussian integral. Likewise, we have

\[
\int \frac{D-1}{(2\pi)^{D-1}} e^{-i\alpha k^2} \cos(p_0 k^{(N)} \theta) = e^{i\frac{(p_0 \theta)^2}{4\alpha}} \int \frac{D-1}{(2\pi)^{D-1}} e^{-i\alpha k^2} = e^{i\frac{(p_0 \theta)^2}{4\alpha}} \left( \frac{e^{-i\pi/4}}{\sqrt{4\pi\alpha}} \right)^{D-1}
\]

after shifting \( k^{(N)} \) by \( k^{(N)} \pm p_0 \theta/(2\alpha) \), and rotating \( k \) by \( \frac{e^{i\pi/4}}{\sqrt{4\alpha}} k \). Hence, we have

\[
LHS = \frac{1}{2} \int_0^\infty d\alpha \left( \frac{e^{-i\pi(D-2)/4}}{4\pi\alpha} \right)^{D/2} e^{-i\alpha m^2 + \frac{p \cdot p}{4\alpha}} + c.c.
\]

where \( p \cdot p \equiv (p_0 \theta)^2 - (\vec{p} \theta)^2 \). Finally, one more rotation of \( \alpha \) to \( e^{-i\pi/4} \alpha \) gives the desired result;

\[
LHS = \frac{1}{2} \int_0^\infty d\alpha \left( \frac{e^{-\alpha m^2 - \frac{p \cdot p}{4\alpha}}}{4\pi\alpha} \right)^{D/2} + c.c. = \int_0^\infty d\alpha \left( \frac{e^{-\alpha m^2 - \frac{p \cdot p}{4\alpha}}}{4\pi\alpha} \right)^{D/2} e^{-\alpha m^2 - \frac{p \cdot p}{4\alpha}}.
\]

Therefore, we have

\[
\int_k \Delta_+(k) \cos(p \wedge k) = \int_0^\infty \frac{d\alpha}{(4\pi\alpha)^{D/2}} e^{-\alpha m^2 - \frac{p \cdot p}{4\alpha}}.
\]

3. Identity of (3.2): Left hand side is integrated over \( k_0 \) using the delta-function:

\[
LHS = \int_k \Delta_+(-k) \Delta_R(k)
\]

\[
= \int \frac{d^D k}{(2\pi)^D} 2\pi \delta(k^2 - m^2) \theta(-k_0) \frac{i}{2\omega_k k_0 - \omega_k + i\epsilon}
\]
\[ = -i \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{(2\omega_k)^3}. \]

One integrates over \( k_0 \) in the right hand side by the contour integral:

\[
\int \frac{1}{k^2 - m^2 + i\epsilon)^2} = i \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{dk_0} \left( \frac{1}{k_0 - \omega_k + i\epsilon} \right)^2 \bigg|_{k_0=-\omega_k} = 2i \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{(2\omega_k)^3}.
\]

Hence the identity follows:

\[
\int_k \Delta_+(-k) \Delta_R(k) = -\frac{1}{2} \int_k \frac{1}{(k^2 - m^2 + i\epsilon)^2}.
\]

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