BETWEEN THE VON NEUMANN INEQUALITY AND THE CROUZEIX CONJECTURE.

PATRYK PAGACZ, PAWEŁ PIETRZYCKI, AND MICHAŁ WOJTYLAK

Abstract. A new concept of a deformed numerical range $W_q(T)$, where $T$ is a bounded linear operator or a matrix and $q \in [0, 2)$ is a parameter, is introduced. Each $W_q(T)$ is a closed convex set that contains the spectrum of $T$. Furthermore, $W_q(T)$ is decreasing with respect to $q$ and $W_1(T)$ is the numerical range. It is also shown that $W_q(T)$ is contained in the closed unit disc if and only if $T$ has a $2/(2-q)$ unitary dilation in the sense of Nagy-Foiaș. Spectral constants of $W_qT$ are investigated.

1. Introduction

The celebrated von Neumann inequality states that if $T$ is a bounded operator on a complex Hilbert space $H$ and $p$ is a polynomial then
\begin{equation}
\|p(T)\| \leq \sup_{\|T\| \leq \mathbb{D}} |p|,
\end{equation}
where $\mathbb{D}$ stands for the open unit disc. The second seminal result of interest is the following one
\begin{equation}
\|p(T)\| \leq \Psi_1(T) \sup_{W(T)} |p|,
\end{equation}
where by $W(T)$ we denote the numerical range of $T$ i.e.
\[ W(T) = \{ \langle Th, h \rangle : h \in H \}. \]
The constant $\Psi_1(T)$ on the right hand side was initially prove to exist in $\textit{[7]}$. Crouzeix in $\textit{[5]}$ established that $\Psi_1(T) \leq 11.08$ and conjectured that $\Psi_1(T) \leq 2$ for any bounded operator $T$. The conjecture is true for $2 \times 2$ matrices (see $\textit{[4]}$) and a simple $2 \times 2$ matrix example show that the constant 2 is the best possible. Up to now the proof of Crouzeix conjecture is know only for some special cases (see $\textit{[3, 9]}$). The current best estimate $\Psi_1(T) \leq 1 + \sqrt{2}$ was obtained by M. Crouzeix and C. Palencia in $\textit{[6]}$, see also $\textit{[15]}$.

Usually one expresses the inequality (1) by saying that the disc of radius $\|T\|$ is a 1-spectral set, analogously the numerical range is a $1 + \sqrt{2}$ spectral set (cf. (2)). The goal of the present paper is to construct intermediate spectral sets. For this aim we define the deformed numerical range $W_q(T)$ as the closed convex hull of
\[ \{ \xi_q(h) \langle Th, h \rangle : \|h\| = 1 \}, \quad q \in [0, 1], \]

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where
\[
\xi_q(h) = \begin{cases} 
\frac{1}{2} \left( q + \sqrt{q^2 |\langle Th, h \rangle|^2 - 4(q-1)\|Th\|^2} \right) & : \langle Th, h \rangle \neq 0 \\
0 & : \langle Th, h \rangle = 0.
\end{cases}
\]

The definition for \( q \in [1, 2) \) is more technical, see Section 2 for details. Note that \( W_1(T) \) is the usual numerical range. Our main results concerning these sets are the following. In Theorem 2 we show that the spectrum of \( T \) is contained in \( W_q(T) \), in Theorem 15 we show that \( W_q(T) \) is monotone and continuous in the Hausdorff metric with respect to \( q \in [0, 1) \).

Furthermore, we show that there exists a constant \( \Psi_q(T) \) such that
\[
\|p(T)\| \leq \Psi_q(T) \sup_{W_q(T)} |p|,
\]
and that \( q \mapsto \Psi_q(T) \) is continuous with respect to \( q \), see Theorem 22.

Another direction of research is to analyse spectral constants connected with simple regions containing the numerical range. E.g., it is known since \([14]\) that for any polynomial \( p \)
\[
\|p(T)\| \leq 2 \sup_{\nu(T) \in \overline{\mathbb{D}}} |p|,
\]
where \( \nu(T) \) stands for the spectral radius of \( T \), i.e. \( \nu(T) := \sup_{z \in W(T)} |z| \). In fact, the above inequality holds with any disc containing \( W(T) \) in place of \( \nu(T) \overline{\mathbb{D}} \). Other results in this direction were obtained recently in \([2, 10, 11]\). In our paper we show that
\[
\|p(T)\| \leq \frac{2}{2 - q} \sup_{\nu_q(T) \in \overline{\mathbb{D}}} |p|,
\]
where \( \nu_q(T) \) stands for the deformed spectral radius of \( T \), i.e. \( \nu_q(T) := \sup_{z \in W_q(T)} |z| \) (cf. \([1]\)). This constitutes a continuous passage between (1) and (3).

The paper is organized as follows. In Section 2 we define the deformed numerical range and show its basic properties together with the inclusion \( \sigma(T) \subseteq W_q(T) \) in the matrix case. Then, in Section 3 we analyse simple cases and examples. Section 4 is devoted to the infinite dimensional case, we analyse quasinilpotent operators and complete the proof of \( \sigma(T) \subseteq W_q(T) \) in the operator case. In Section 5 we show a link to the Nagy-Foiaș dilation theory and with its use we prove (4). Moreover, (4) finds his measure interpretation in Theorem 14. In Section 6 we show the announced monotonicity and continuity of \( W_q(T) \) with respect to \( q \). This is applied in Section 7 to analyse the spectral constants \( \Psi_q(T) \).

The following notation will be used. The fields of real and complex numbers are denoted by \( \mathbb{R} \) and \( \mathbb{C} \), respectively. All Hilbert spaces considered in this paper are assumed to be complex, \( \langle f, g \rangle \) and \( \|f\| \) stands for the inner product and the corresponding norm, respectively. We denote by \( \sigma(T) \) the spectrum of \( T \) and by \( \sigma_p(T) \) and \( \sigma_{ap}(T) \) we mean the point spectrum (eigenvalues) and approximative spectrum of \( T \), respectively. Furthermore, \( r(T) := \sup_{z \in \sigma(T)} |z| \) stands for the spectral radius of \( T \). By \( \overline{V} \), \( \text{Int} V \) and \( \partial V \) we mean the closure, the interior and the boundary of \( V \subseteq \mathbb{C} \). We will use without further notice the fact that one may interchange the order of taking the convex hull and the closure, i.e. \( \text{conv} \overline{V} = \text{conv} \overline{V} \), for any bounded subset \( V \) of the complex plain. As it was already used, \( r \overline{\mathbb{D}} \) is an open disc centred at origin with radius \( r \).
2. Deformed numerical range: definition and basic properties

Let us begin with defining the main objects. For \( q \in [0, 2) \) and \( h \in \mathcal{H} \) with \( \|h\| = 1 \) we set
\[
\Delta_q(h) := q^2 |\langle Th, h \rangle|^2 - 4(q - 1) \|Th\|^2
\]
and if \( \Delta_q(h) > 0 \) then
\[
\xi_q(h) := \begin{cases} 
\frac{1}{2} \left( q + \sqrt{\frac{\Delta_q(h)}{|\langle Th, h \rangle|}} \right) : \langle Th, h \rangle \neq 0 \\
0 : \langle Th, h \rangle = 0
\end{cases}
\tag{5}
\]
later on we denote the domain of the function \( \xi_q \) as
\[
\text{dom}(\xi_q) := \{ h \in \mathcal{H} : \Delta_q(h) > 0, \|h\| = 1 \}.
\]
Note that if \( Th = 0 \) then \( h \in \text{dom}(\xi_q) \) for \( q \in [0, 2) \), furthermore, if \( \langle Th, h \rangle = 0 \) and \( Th \neq 0 \) then \( h \in \text{dom}(\xi_q) \) precisely for \( q \in [0, 1] \). Such vectors \( h \) will require separate treatment in the course of the paper, especially in Theorems 2 and 9. We now list the basic properties of the functions \( \xi_q \) and \( \Delta_q \).

Proposition 1. For any bounded operator \( T \) on a Hilbert space the following holds.

(i) If \( 0 \leq q_1 \leq q_2 < 2 \) then \( \text{dom}(\xi_{q_2}) \subseteq \text{dom}(\xi_{q_1}) \) and \( \xi_{q_2}(h) \leq \xi_{q_1}(h) \), for \( h \in \text{dom}(\xi_{q_2}) \).

(ii) If \( 0 \leq q_1 \leq q_2 < 2 \) and \( h \in \text{dom}(\xi_{q_1}) \) with \( \langle Th, h \rangle \neq 0 \), then \( \xi_{q_2}(h) = \xi_{q_1}(h) \) if and only if \( h \) is an eigenvector of \( T \).

(iii) \( \text{dom}(\xi_q) = \{ h \in \mathcal{H} : \|h\| = 1 \} \) for \( q \in [0, 1] \).

(iv) \( \xi_0(h) = \frac{\|Th\|}{\|h\|} \), \( \xi_1(h) = 1 \) for \( h \in \mathcal{H} \) with \( \|h\| = 1 \), \( \langle Th, h \rangle \neq 0 \).

(v) If \( Th = \lambda h \) with some \( \lambda \in \mathbb{C} \) and \( \|h\| = 1 \) then \( h \in \text{dom}(\xi_q) \) and \( \xi_q(h) = 1 \) for \( q \in (0, 2) \).

(vi) If \( Th \neq \lambda h \) for all \( \lambda \in \mathbb{C} \) and \( \|h\| = 1 \) then \( h \notin \text{dom}(\xi_q) \) for \( q \in (q_0, 2) \) for some \( q_0 \in [1, 2) \).

Proof. The first part of statement [i] follows from the fact that for a fixed \( h \) with \( \|h\| = 1 \) we have
\[
\frac{d\Delta_q(h)}{dq} = 2(q |\langle Th, h \rangle|^2 - 2 \|Th\|^2) \leq 0, \quad q \in [0, 2).
\]
Now let \( h \in \text{dom}(\xi_{q_2}) \) and let \( \langle Th, h \rangle \neq 0 \). An elementary calculation shows that
\[
\frac{d\xi_q(h)}{dq} \leq 0, \quad q \in [0, q_2),
\]
and the equality holds if and only if \( |\langle Th, h \rangle| = \|Th\| \). This shows the second part of [i] and [ii].

Statements [iii] and [iv] are obvious. To see [v] note that \( \Delta_q(h) = |\lambda|^2(2 - q) > 0 \) for \( q \in [0, 2) \) and \( \|h\| = 1 \). Let us now show [vi]. Take \( h \) as in the statement, then \( |\langle Th, h \rangle| < \|Th\| \) and by elementary expression
\[
\Delta_q(h) = (2 - q)^2 |\langle Th, h \rangle|^2 - 4(q - 1)(\|Th\|^2 - |\langle Th, h \rangle|^2)
\tag{6}
\]
we have \( \Delta_q(h) \leq 0 \) for \( q \in [q_0, 2) \) for some \( q_0 \in [1, 2) \).

Further for \( q \in [0, 2) \) we define the deformed numerical range of a nonzero operator \( T \) as
\[
W_q(T) = \text{conv} \{ \xi_q(h) |\langle Th, h \rangle : h \in \text{dom}(\xi_q) \} \tag{7}
\]
and the deformed numerical radius as
\[
\nu_q(T) = \sup_{z \in W_q(T)} |z|.
\]

Theorem 2 below lists some basic properties of the deformed numerical range.

**Theorem 2.** For a bounded operator \( T \) on a Hilbert space \( \mathcal{H} \) and \( q \in [0, 2) \) the following holds.

(i) \( W_q(U^*TU) = W_q(T) \) for any unitary operator \( U \) on \( \mathcal{H} \);
(ii) \( W_q(\alpha T) = \alpha W_q(T) \) for any \( \alpha \in \mathbb{C} \);
(iii) if \( \mathcal{K} \) is a subspace of \( \mathcal{H} \) invariant for \( T \), then \( W_q(T|_{\mathcal{K}}) \subseteq W_q(T) \);
(iv) \( W_q(T) \subseteq \|T\| \overline{B} \);
(v) \( \sigma(T) \subseteq W_q(T) \).

Statements (i), (ii) and (iii) are elementary, (iv) follows directly from statements (i) and (v) of Proposition 1. Also it follows directly from Proposition 1(v) that the eigenvalues are contained in \( W_q(T) \) for any \( q \in [0, 2) \), hence (v) is showed if \( T \) is a matrix. The proof in the operator case will be completed in Section 4 and requires some additional preparation concerning quasinilpotent operators. Now let us study the simplest examples and instances.

**Remark 3.** In addition to Theorem 2 note that, except the case \( q = 1 \), \( W_q(T + \alpha I) \) is in general not equal to \( W_q(T) + \alpha \). This can be seen in various ways, we present here a general reason in case when \( T \) is a matrix and \( q \in [0, 1] \). Note that for a matrix \( T \) the deformed numerical range \( W_q(T) \) is contained in the closed right half-plane if and only if \( W_1(T) = W(T) \) has this property, hence for a fixed \( q \in [0, 1] \) the set \( W_q(T) \) is contained in the closed right half-plane if and only if \( T + T^* \geq 0 \). Furthermore, \( W_q(T) \) is compact and convex. Hence, if \( W_q(T + \alpha I) = W_q(T) + \alpha \) for any matrix \( T \) and any \( \alpha \in \mathbb{C} \), then, by [12] Theorem 1.4.2, \( W_q(T) = W(T) \), which clearly is a contradiction with the definition of \( W_q(T) \), see e.g. Theorem 15 below.

3. Examples

In this section we will deal with \( 2 \times 2 \) matrices.

**Example 4.** Figure 1 shows the set \( \{\xi_q(h) \langle Th, h \rangle : h \in \text{dom}(\xi_q)\} \) for \( q \leq 1 \), recall that the closure of the convex hull of this set is, by definition, the deformed numerical range. Figure 2 shows the set \( \{\xi_q(h) \langle Th, h \rangle : h \in \text{dom}(\xi_q)\} \) for \( q > 1 \).

Note that in many instances the plotted set itself is not convex and for \( q > 1 \) is not even connect. Moreover, the shape of \( W_q(T) \) depends on where zero lies. One can check that \( W(T) = [a, b]c \) for some \( a, b \in \mathbb{R}, c \in \mathbb{C} \) if and only if \( W_q(T) = [a', b']c \) for some \( a', b' \in \mathbb{R}, c \in \mathbb{C} \).

But the simple example \( T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) shows that even if \( W(T) = [a, b] \), for \( a, b \in \mathbb{C} \), the range \( W_q(T) \) can have nonempty interior.

The next example, due to its importance and length of the argument, is presented as a proposition.

**Proposition 5.** If \( T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2,2} \), then
\[ W_q(T) = (2 - q)\overline{D}. \]
Figure 1. The numerical plot of \( \{ \xi_q(h) \langle Th, h \rangle : h \in \text{dom}(\xi_q) \} \) for \( q = 0 \) (blue circles) \( q = 0.5 \) (red crosses) \( q = 1 \) (numerical range, black dots).

\[
T = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
T = \begin{bmatrix} i & 0 \\ 1 & 1 \end{bmatrix}
\]

\[
T = \begin{bmatrix} -0.1 & 1 \\ 0 & 1 \end{bmatrix}
\]
Figure 2. The numerical plot of \( \{ \xi_q(h) \langle Th, h \rangle : h \in \text{dom}(\xi_q) \} \) for \( q = 1 \) (numerical range, blue) \( q = 1.5 \) (orange) \( q = 1.9 \) (black).

\[
T = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
T = \begin{bmatrix} i & 0 \\ 1 & 1 \end{bmatrix}
\]

\[
T = \begin{bmatrix} -0.1 & 1 \\ 0 & 1 \end{bmatrix}
\]
Proof. For \( q = 1 \) the result is known, fix \( q \neq 1 \). Let \( h = [x \ y]^\top \in \mathbb{C}^2 \), so that
\[
\langle Th, h \rangle = 2y\bar{x} \quad \text{and} \quad \|Th\| = |2x|.
\]
The deformed numerical range of operator \( T \) has the following form
\[
W_q(T) = \left\{ \frac{x y}{|y|} (q|y| + \sqrt{q^2|y|^2 - 4(q - 1)}) : \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = 1, \ q^2|y|^2 - 4(q - 1) \geq 0 \right\}.
\]
Observe that \( W_q(T) \) is circular and since it is also convex, it is a disc centred at the origin.
We prove now that its radius \( \nu_q(T) \) equals \( 2 - q \). Applying (8) we obtain
\[
\nu_q(T) = \sup_{\|h\|=1, \Delta_q(h)>0} \xi_q(h) |\langle Th, h \rangle| = \sup_{|x|^2+|y|^2=1, q^2|y|^2-4(q-1)>0} |x| (q|y| + \sqrt{q^2|y|^2 - 4(q - 1)}).
\]
It is a matter of elementary calculation that for \( q \neq 1 \) we have \( \nu_q(T) \leq 2 - q \).
Setting
\[
y = \sqrt{\frac{q^2}{2(q^2 - 2q + 2)}}, \quad x = \sqrt{\frac{(q - 2)^2}{2(q^2 - 2q + 2)}}, \quad h = \begin{bmatrix} x \\ y \end{bmatrix}
\]
we see that in fact \( \nu_q(T) = 2 - q \).
\[\square\]

4. The deformed numerical range of an operator

First, let us consider the case of a quasinilpotent operator, interesting for itself and needed later on in the proof of the inclusion \( \sigma(T) \subseteq W_q(T) \).

Proposition 6. For any bounded and quasinilpotent but not nilpotent operator \( T \) on a Hilbert space \( \mathcal{H} \) there exist a sequence \( \{\mathbf{h}_k\}_{k=0}^\infty \subset \mathcal{H} \) such that \( \|\mathbf{h}_k\| = 1 \), \( k \in \mathbb{N} \) and
\[
|\langle T\mathbf{h}_k, \mathbf{h}_k \rangle| \to 1, \quad k \to \infty.
\]
In consequence, for any \( q \in [0, 2) \), dom(\( \xi_q \)) is nonempty, 0 is an accumulation point of \( W_q(T) \) and \( \nu_q(T) > 0 \).

Proof. Fix \( h \in \mathcal{H} \setminus \{0\} \) and define a function \( f \) by
\[
f(z) := \sum_{n=0}^\infty z^n T^n h, \quad z \in \mathbb{C}.
\]
Since \( T \) is quasinilpotent, we infer from the root test [16, page 199] that \( f \) is an entire \( \mathcal{H} \)-valued function. Observe that
\[
Tf(z) = \sum_{n=0}^\infty z^n T^{n+1} h = \frac{1}{z} (f(z) - h).
\]
Note that since \( T \) is not nilpotent, \( f \) is not constant and \( f(z) \neq 0 \) implies \( Tf(z) \neq 0 \). Hence,
\[
\frac{|\langle Tf(z), f(z) \rangle|}{\|Tf(z)\| \|f(z)\|} = \frac{\|f(z)\|^2 - \langle h, f(z) \rangle}{\|f(z) - h\| \|f(z)\|}.
\]
By \cite[Theorem 3.32]{[17]} there exist a sequence \(\{z_k\}_{k=0}^{\infty}\) such that \(\lim_{k\to\infty} \|f(z_k)\| = \infty\), which gives \(\|f(z_k)\|/\|f(z_k) - h\| \to 1\) and

\[
(12) \quad \left| \frac{Tf(z_k)}{\|Tf(z_k)\|} - \frac{f(z_k)}{\|f(z_k)\|} \right| \to 1, \quad (k \to \infty).
\]

Setting \(h_k = f(z_k)/\|f(z_k)\|\) finishes the proof of \((10)\).

Note that by \((6)\) for a fixed \(q \in [0,2)\) there exists \(k_0\) such that \(\Delta_q(h_k) > 0\) for \(k > k_0\). For those \(k\) we define \(w_k := \xi_q(h_k) (Th_k, h_k) \in W_q(T)\). Note that \(w_k \neq 0\) as \(\xi_q(h_k) > q/2\) and

\[
(13) \quad z_k \langle Th_k, h_k \rangle = \frac{\|f(z_k)\|^2 - \langle h_k, f(z_k) \rangle}{\|f(z_k)\|^2} \to 1, \quad (k \to \infty).
\]

To finish the proof we need to prove that \(w_k \to 0\) \((k \to \infty)\), in the light of \((13)\) and since \(|z_k| \to \infty\) it is enough to show that \(\xi_q(h_k)\) is bounded. Observe that

\[
\sqrt{\frac{\Delta_q(h_k)}{\|Th_k, h_k\|}} = \sqrt{\frac{\Delta_q(h_k)}{\|Tf(z_k)\|}} \cdot \frac{\|f(z_k) - h\|}{\|f(z_k)\| |z_k| \|Th_k, h_k\|}.
\]

Note that the first factor on the right hand side converges to \(q^2 - 4(q - 1)\) by \((10)\) and the second to 1 by \((13)\), which finishes the proof. \(\square\)

**Remark 7.** Formula \((12)\) shows that for a quasinilpotent, but not nilpotent operator some point on the unit circle belongs to the closure of the normalised numerical range, which extends Proposition 7 of \cite{[8]}, see also \cite[Proposition 1.2]{[18]}.

We are able now to complete the proof of the inclusion \(\sigma(T) \subseteq W_q(T), q \in [0,2)\), showed so far in the finite dimensional case.

**Proof of Theorem 2\([\text{[17]}]\).** First we show that \(\sigma_{ap}(T) \setminus \{0\} \subseteq W_q(T)\). Take \(\lambda \in \sigma_{ap}(T) \setminus \{0\}\). Then there exists a sequence \(\{h_n\}\) of unit vectors in \(H\) such that \(\|Th_n - \lambda h_n\| \to 0\). This implies that \(\langle Th_n, h_n \rangle \to \lambda, \|Th_n\| \to |\lambda|\) and consequently \(h_n \in \text{dom}(\xi_q)\), for \(n\) large enough, and \(\xi_q(h_n) \to 1\). Hence, \(\lambda \in W_q(T)\).

The proof now splits into several cases.

Case 1: \(0 \notin \sigma(T)\). Recall that \(\partial(\sigma(T)) \subseteq \sigma_{ap}(T)\), see e.g. \cite[Theorem 2.5]{[13]}. Consequently, \(\sigma(T) \subseteq \text{conv}(\partial(\sigma(T))) \subseteq \text{conv}(\sigma_{ap}(T)) = \text{conv}(\sigma_{ap}(T) \setminus \{0\}) \subseteq W_q(T)\).

Case 2: \(0 \in \text{Int}(\sigma(T))\). Then

\(\sigma(T) \subseteq \text{conv}(\partial(\sigma(T))) = \text{conv}(\partial(\sigma(T) \setminus \{0\})) \subseteq \text{conv}(\sigma_{ap}(T) \setminus \{0\}) \subseteq W_q(T)\).

Case 3: \(0\) is an isolated point of \(\sigma(T)\). Then, by taking the Riesz projection and applying \cite{[13]} and Case 1 we see that \(\sigma(T) \setminus \{0\}\) is contained in \(W_q(T)\). Hence, the proof of Case 3 reduces to considering \(\sigma(T) = \{0\}\). If \(0\) is an eigenvalue then we use \((6)\). If \(\sigma_p(T) = \emptyset\) then \(T\) is a quasinilpotent, but not nilpotent operator. By Proposition \cite{[6]} we have, in particular, that \(0 \in W_q(T)\).

Case 4: \(0 \in \partial(\sigma(T))\) and is a non-isolated point of \(\sigma(T)\). Then, by compactness of \(\sigma(T)\), \(0\) is a non-isolated point of \(\partial(\sigma(T))\). In consequence,

\[
\sigma(T) \subseteq \text{conv}(\partial(\sigma(T))) = \text{conv}(\partial(\sigma(T) \setminus \{0\})) = \text{conv}(\partial(\sigma(T) \setminus \{0\})) \subseteq \text{conv}(\sigma_{ap}(T) \setminus \{0\}) \subseteq W_q(T).
\]

\(\square\)
5. Connection with the classes $C_\rho$ of power bounded operators

In this section we will deal with the deformed numerical radius $\nu_q(T) = \sup_{z \in W_q(T)} |z|$. First we show the connection with theory of class $C_\rho$ of power bounded operators, introduced by Sz.-Nagy and Foiaş in [19], completed in [20]. It was shown therein that the operator $T \in \mathcal{B}(\mathcal{H})$ satisfies the following condition

$$\nu_q(T) = \frac{2}{2 - q}, \quad q = 2 - 2\rho^{-1}, \quad q \in [0, 2), \quad \rho \in [1, +\infty).$$

while for dilation theory the number $\rho$ is more natural, for technical purposes in the present paper it was much more convenient to use the parameter $q$. We adapt the condition (14) therefore.

Lemma 8. For $\rho$ and $q$ as in (14) condition (14) is equivalent to the following.

(14) $\phi_h(t) := 1 - q |\langle Th, h \rangle| t + (q - 1) \|Th\|^2 t^2 \geq 0, \quad t \in [0, 1], \quad h \in \text{dom}(\xi_q)$.

Proof. First note that for $h \in \ker T$ both conditions (I) and (I) are trivial. The implication (I) follows now by setting, for each $h \in \text{dom}(\xi_q) \setminus \ker T$, $z = wt$ with $|w| = 1$ and $\text{Re} \ \langle wTh, h \rangle = |\langle Th, h \rangle|$. To see the converse, consider first the case $q \in [0, 1)$. Then $\text{dom}(\xi_q)$ equals the whole unit sphere in $\mathcal{H}$. Setting $t = |z|$ and using the inequality $\text{Re} \ \langle zTh, h \rangle \leq t|\langle Th, h \rangle|$ we get that (I) implies (I) for $q \in [0, 1)$. The case $q = 1$ is trivial. Now let $q \in (1, 2)$. Observe first that for unit $h \in \mathcal{H} \setminus \text{dom}(\xi_q)$ with $Th \neq 0$ the inequality $\phi_h(t) \geq 0$ is automatically satisfied on $[0, 1]$, as $\phi_h(t)$ is in such case a quadratic polynomial with the positive leading coefficient and at most one real root. In consequence, $\phi_h(t) \geq 0$ on $[0, 1]$ for all unit $\mathcal{H}$, which, again by setting $t = |z|$, is equivalent to (I). \hfill \Box

It is well known that $T \in C_2$ if and only if $W(T) \subseteq \overline{D}$. We present the following generalisation.

Theorem 9. Let $T$ be a bounded nonzero operator on a Hilbert space. Then $T$ has a $\rho$ dilation, i.e. $T \in C_\rho$, if and only if the deformed numerical range $W_q(T)$ is contained in the closed unit disc, where $\rho$ and $q$ are as in (14).

Proof. Assume first that $W_q(T) \subseteq \overline{D}$, we show that (I) is also satisfied. The cases $q = 0, 1$ are known. Let $q \in (0, 1)$, we fix $h \in \text{dom}(\xi_q) = \{h \in \mathcal{H} : \|h\| = 1\}$. If $Th = 0$ then trivially $\phi_h(t) > 0$, so we assume $Th \neq 0$. Then, $\phi_h(t)$ is a quadratic polynomial with the negative leading coefficient, $\phi_h(0) = 1$ and two different real roots

$$x_{\pm} = \frac{q |\langle Th, h \rangle| \pm \sqrt{\Delta_q(h)}}{2 \|Th\|^2 (q - 1)}.

Consider first the case $\langle Th, h \rangle \neq 0$. Note that

$$x_+ < 0 < 1 \leq x_- = |\xi_q(h) \langle Th, h \rangle|^{-1},$$
where the last inequality follows by assumption that $W_q(T) \subseteq \overline{\mathbb{B}}$. Hence, $\phi_h(t) \geq 0$ for $t \in [0, 1]$.

Now take unit $h$ with $Th \neq 0$, $\langle Th, h \rangle = 0$. As $T \neq 0$, the set $\{h \in \mathcal{H} : \langle Th, h \rangle = 0\}$ has an empty interior. Indeed, fix $h \in \{h \in \mathcal{H} : \langle Th, h \rangle = 0\}$ and suppose that there exist $r > 0$ such that

$$\langle T(h + g), h + g \rangle = 0, \quad \|g\| < r$$

then also

$$\langle T(h + tg), h + tg \rangle = 0, \quad \|g\| < r, t \in [0, 1).$$

Since $\langle Th, h \rangle = 0$, we have

$$\langle T(h + tg), h + tg \rangle = \langle Th, h \rangle + \langle Th, tg \rangle + \langle T(tg), h \rangle + \langle T(tg), tg \rangle = \langle Th, tg \rangle + \langle T(tg), h \rangle + \langle T(tg), tg \rangle = 0$$

for $\|g\| < r$, $t \in [0, 1)$. This implies that

$$\langle T, h \rangle + \langle Tg, h \rangle + \langle Tg, tg \rangle = 0, \quad \|g\| < r, t \in [0, 1).$$

Letting $t \to 0$, we get

$$\langle Tg, h \rangle = 0, \quad \|g\| < r.$$

This, combined with (16), gives $\langle Tg, g \rangle = 0$, $\|g\| < r$, which yields $T = 0$. We are led to a contradiction. Hence, there exists a sequence of unit vectors $h_n$ with $\langle Th_n, h_n \rangle \neq 0$ converging to $h$. Note that $\phi_{h_n}(t)$ converges to $\phi_h(t)$ pointwise in $t$, which shows that $\phi_h(t) \geq 0$ for $t \in [0, 1]$. Summarising, we have so far showed that (L) holds for $q \in [0, 1]$.

Now let $q \in (1, 2)$, we fix $h \in \text{dom}(\xi_q)$. If $Th = 0$ then trivially $\phi_h(t) > 0$, so we assume $Th \neq 0$. Then $\phi_h(t)$ is a quadratic polynomial with the positive leading coefficient, $\phi_h(0) = 1$ and two (possibly equal) real roots given by (15). Since $h \in \text{dom}(\xi_q)$ we have $\langle Th, h \rangle \neq 0$ and consequently

$$1 \leq x_- = |\xi_q(h) \langle Th, h \rangle|^{-1} < x_+.$$

This shows that $\phi_h(t) \geq 0$ on $[0, 1]$, i.e. (L) is satisfied.

Assume now that $T \in C_\rho$, i.e. (L) is satisfied. It is enough to show that $|\xi_q(h) \langle Th, h \rangle| \leq 1$ for $h \in \text{dom}(\xi_q)$. Let us fix $h \in \text{dom}(\xi_q)$, as $0 \in \overline{\mathbb{B}}$ we may assume that $\langle Th, h \rangle \neq 0$. The cases $q = 0, 1$ are known, let now $q \in (0, 1)$. Then $\phi_h(t)$ is a quadratic polynomial with the negative leading coefficient, $\phi_h(0) = 1$ and two different real roots (15). Note that

$$x_+ < x_\pi = |\xi_q(h) \langle Th, h \rangle|^{-1},$$

and so as (L) is assumed we have that $x_- \geq 1$.

Let now $q \in (1, 2)$, then $\phi_h(t)$ is a quadratic polynomial with the positive leading coefficient, $\phi_h(0) = 1$ and two different roots, as $\Delta_q(h) > 0$ by assumption that $h \in \text{dom}(\xi_q)$. Hence, $x_- < x_\pi$ and $x_\pi > 0$ and (L) implies that $x_- \geq 1$.

Immediately we get the following:

**Corollary 10.** We have that

$$\nu_q(T) = \inf \{ t > 0 : t^{-\frac{1}{q}} T \in C_\rho \}, \quad \rho = \frac{2}{2 - q} \in [1, +\infty).$$

Furthermore, $\nu_q^{-1}(T) \in C_\rho$. 


Remark 11. The equation (17) above was remarked without proof in [1] in the following, slightly different form, namely:

\[ \inf \{ t > 0 : t^{-1}T \in C^\rho \} = \sup \{ \xi_q(h) \langle Th, h \rangle : \|h\| = 1, \ \Delta_q(h) \geq 0 \}, \quad q \in [0,2). \]

As the proof does not seem to be clear (especially, for \( q \in (1, 2) \) it is not clear if the set on the right hand side of (18) is equal to \( W_q(T) \)) we have decided to show a complete proof of Theorem 9.

We also get some basic properties of \( \nu_q(T) \).

Corollary 12. The following holds for any bounded linear operator \( T \neq 0 \) on a Hilbert space \( \mathcal{H} \):

(i) the function \([0, 2) \ni q \mapsto \nu_q(T)\) is nonincreasing;
(ii) \( \nu(T) \leq \nu_q(T) \leq \|T\| \) for \( q \in [0, 1] \);
(iii) \( r(T) \leq \nu_q(T) \leq r(T) \) for \( q \in [1, 2] \);
(iv) \( \nu_0(T) = \|T\| \);
(v) \( \nu_q(T) \to r(T) \) with \( q \to 2 \).

Proof. By Corollary (10) and by monotonicity of the classes \( C^\rho \) with respect to \( \rho \) (c.f. [20]), statement (i) follows now easily.

Statement (ii) follows directly from (i) and Theorem 2(v). Statement (iii) follows directly from (i) and Theorem 2(iv).

Statement (iv) is obvious. To see (v) let us fix \( h_n \in \mathcal{H} \) and \( q_n \to 2 \) such that \( \|h_n\| = 1 \) and \( \xi_{q_n}(h_n) = (Th_n, h_n) \to \lim \nu_q(T) \). Since \( \Delta_{q_n}(h_n) > 0 \), one can get \( \|Th_n, h_n\| \to 0 \).

Thus

\[ \lim_{n \to \infty} \xi_{q_n}(h_n) = (Th_n, h_n) \leq \lim_{n \to \infty} \frac{1}{2} (q_n |(Th_n, h_n)| + (2 - q_n)\|Th_n\|) \leq r(T). \]

We are able now to show that the disc with radius \( \nu_q(T) \) is a \( 2/(2-q) \) spectral set.

Theorem 13. For any bounded operator \( T \) in a Hilbert space and for any polynomial \( p \) we have

\[ \|p(T)\| \leq \frac{2}{2 - q} \sup_{\nu_q(T) \in \mathbb{B}} |p|, \quad q \in [0, 2). \]

Note that for \( q = 1 \) we get (3) and for \( q = 0 \) we get the von Neumann inequality (1).

Proof. We fix \( q \in [0, 2) \). By Proposition 2(i) we have \( \nu_q(\nu_q^{-1}(T)T) \leq 1 \), therefore, by Theorem 9 \( \nu_q^{-1}(T)T \) is of class \( C^\rho \) with \( \rho = \frac{2}{2 - q} \). Hence, one has the inequality

\[ \|p(\nu_q^{-1}(T))\| \leq \frac{2}{2 - q} \|p(U)\| = \frac{2}{2 - q} \sup |p|, \]

where \( U \) is an \( \rho \)–unitary dilation of \( \nu_q^{-1}(T) \) and \( p \) is any polynomial. Substituting \( p(\nu_q(T)z) \) for \( p(z) \) we get the claim.

We show one more connection with the dilation theory. In [6] and [7] two operator-valued measures on a region \( \Omega \subseteq \mathbb{C} \) defined by a contour \( \sigma(s) \) were introduced:

\[ \mu_0(\sigma, T) = \frac{1}{2\pi} (\sigma I - T^*)^{-1}(\sigma I - T^*)(\sigma I - T)^{-1}\frac{1}{\sigma} ds, \]

\[ \mu_0(\sigma, T) = \frac{1}{2\pi} (\sigma I - T^*)^{-1}(\sigma I - T^*)(\sigma I - T)^{-1}\frac{1}{\sigma} ds, \]
and
\[
(21) \quad \mu_1(\sigma, T) = \frac{1}{2\pi} (\omega(\sigma I - T)^{-1} + \bar{\omega}(\sigma I - T^* )^{-1}) ds
\]
where \( \omega = \omega(s) = \frac{d\sigma(s)}{|d\sigma(s)|} \) is the unit outward normal vector. First of these measures has values in the set of selfadjoint operators only if \( \Omega \) is a circle with center at zero. It was used in [7] in a new proof of the von Neumann theorem. The second of these measures has values in the set of selfadjoint operators and was used to show that the numerical range is a spectral set, with a remarkable improvement of the absolute constant to \( (1 + \sqrt{2}) \) in [6]. The following theorem shows a connection between these two measures and the deformed numerical radius.

**Theorem 14.** Let \( T \) be a bounded operator in a Hilbert space \( \mathcal{H} \). Then
\[
\nu_q(T) = \inf \left\{ r \geq r(T) : \mu_{\frac{q}{2-q}}(z, T) \geq 0 \text{ for } z \in r\partial \mathbb{D}, \quad q \in [0, 1], \right\}
\]
where the Hermitian operator valued measure \( \mu(z, T) \) is defined as
\[
\mu(z, T) := t\mu_1(z, T) + (1 - t)\mu_0(z, T), \quad t \in \mathbb{R}.
\]

**Proof.** First we show that \( \mu_{\frac{q}{2-q}}(z, T) \geq 0 \) on \( \nu_q(T)\mathbb{D} \), which shows the inequality ‘\( \geq \)’ as \( \nu_q(T) \geq r(T) \) by Proposition 12. For this aim let \( r = \nu_q(T) \) and note that the operator \( r^{-1}T \) is of class \( C_{\frac{q}{2-q}} \) by Theorem 9. Hence, condition \( \{I_0\} \) for \( r^{-1}T \) is satisfied, in particular setting there \( z := \bar{\omega} \in r\partial \mathbb{D} \) we obtain
\[
\|h\|^2 - qr^{-1} \text{Re} \langle \bar{\omega}Th, h \rangle + (q - 1)r^{-2} \|Th\|^2 \geq 0, \quad h \in \mathcal{H},
\]
which is equivalent to
\[
\langle (2r - q(\bar{\omega}T + \omega T^*) + (q - 1)2r^{-1}T^*Th, h \rangle \geq 0, \quad h \in \mathcal{H},
\]
and consequently
\[
\langle (q(\omega(r\bar{\omega} - T^*) + \bar{\omega}(r\omega - T)) + 2(1 - q)(r^{-2} - T^*T^{\frac{1}{r}})h, h \rangle \geq 0, \quad h \in \mathcal{H},
\]
Replacing \( h \) by \( (zI - T)^{-1}h \) we get
\[
\mu_t(z, T) \geq 0, \quad \text{for } z = r\omega \in r\partial \mathbb{D},
\]
where \( t = \frac{q}{2-q} \).

Now let \( \mu_t \geq 0 \) on \( r\partial \mathbb{D} \) for some \( r \geq r(T) \). Then \( r^{-1}T \) satisfies clearly condition \( \{I_0\} \). By analogous arguments as before for \( t = \frac{q}{2-q} \) we have
\[
(22) \quad \|h\|^2 - qr^{-1} \text{Re} \langle zTh, h \rangle + (q - 1)r^{-2} \|zTh\|^2 \geq 0, \quad h \in \mathcal{H},
\]
where \( |z| = 1 \). Since \( q \in [0, 1] \), the above function is superharmonic. Thus the inequality \( (22) \) holds for \( |z| \leq 1 \). In other words \( r^{-1}T \) satisfies \( \{I_0\} \) with \( \rho = \frac{2}{2-q} \). In consequence, \( r^{-1}T \) is of class \( C_{\rho} \) and \( r^{-1} \leq \nu_q(T)^{-1} \), by Corollary 10.
6. Monotonicity and continuity of the deformed numerical range

Next we turn to the questions of monotonicity and continuity of the sets $W_q(T)$ with respect to the parameter $q$. The latter will be understood with respect to the Hausdorff distance on complex plane

$$d_H(E,F) := \max \left\{ \sup_{e \in E} \text{dist}(e,F), \sup_{f \in F} \text{dist}(f,E) \right\} = \max \left\{ \sup_{e \in E} \inf_{f \in F} |e-f|, \sup_{f \in F} \inf_{e \in E} |f-e| \right\},$$

where $E,F$ are compact subsets on complex plane. We formulate now the main result on monotonicity and continuity of the deformed numerical range $\mathcal{W}$. The latter will be understood with respect to the Hausdorff distance on complex plane.

**Theorem 15.** For a bounded operator $T$ on a Hilbert space the following holds.

(i) If $0 \leq q_1 \leq q_2 < 2$ and $0 \in W_{q_1}(T)$ then $W_{q_2}(T) \subseteq W_{q_1}(T)$.
(ii) If $0 \leq q_1 \leq q_2 < 2$ and $0 \in \text{Int} W_{q_2}(T)$ then $W_{q_1}(T) \subseteq W_{q_1}(T)$.
(iii) If $\mathcal{H}$ is finite dimensional, $0 \leq q_1 \leq q_2 < 1$ and $0 \in \text{Int} W(T)$ then $W_{q_2}(T) \subseteq W_{q_1}(T)$.
(iv) The function $W : q \mapsto W_q(T)$ is continuous on $[0,1]$ with respect to the Hausdorff metric.

**Remark 16.** First observe that Example 4 shows that the assumptions on location of the zero in (i), (ii), (iii) are indispensable. Later on, in Section 7, we will assume that $0 \in \text{Int} W_1(T)$, which guarantees the monotonicity for all $q \in [0,2)$.

For the proof of the theorem we need three lemmas, the first of which is well known.

**Lemma 17.** The convex hull operator $V \mapsto \text{conv} V$, acting on the family of compact subsets of $\mathbb{C}$, satisfies a Lipschitz condition $d_H(\text{conv}(V_1), \text{conv}(V_2)) \leq d_H(V_1, V_2)$.

**Proof.** Let $V_i$ ($i = 1,2$) be compact subsets of $\mathbb{C}$ and let $\tilde{x} \in \text{conv} V_1$. Then there exist $x_1, x_2, \ldots, x_n \in V_1$ and $t_1, t_2, \ldots, t_n \in [0,1]$ such that $t_1 + \cdots + t_n = 1$ and $\tilde{x} = t_1 x_1 + \cdots + t_n x_n$ (it is enough to take $n = 3$ by the Carathéodory theorem). Therefore,

$$\text{dist}(\tilde{x}, \text{conv}(V_2)) \leq \text{dist}(\tilde{x}, \{t_1 y_1 + \cdots + t_n y_n : y_1, \ldots, y_n \in V_2\}) \leq t_1 \text{dist}(x_1, V_2) + \cdots + t_n \text{dist}(x_n, V_2) \leq \sup_{x \in V_1} \text{dist}(x, V_2) \leq d_H(V_1, V_2).$$

Thus $\sup_{x \in \text{conv}(V_1)} \text{dist}(x, \text{conv}(V_2)) \leq \sup_{x \in V_1} \text{dist}(x, V_2)$. Reversing the roles of $V_1$ and $V_2$ completes the proof. $\square$

For subsequent reasonings we need to define the following auxiliary sets. Let $T$ be a bounded operator on a Hilbert space, for $0 \leq q_1 \leq q_2 < 2$ we set

$$V_{q_1,q_2}(T) = \{ \xi_{q_1}(h) \langle Th, h \rangle : h \in \text{dom}(\xi_{q_2}) \}.$$

Note that the definition is correct as $\text{dom}(\xi_{q_2}) \subseteq \text{dom}(\xi_{q_1})$ by Proposition 11.

**Lemma 18.** For $q_0 \in [0,2)$ the mapping

$$V_{q_0} : [0,q_0) \ni q \mapsto V_{q,q_0}(T)$$

is continuous with respect to the Hausdorff metric.
Proof. Fix $q_0 \in [0, 2)$ and take $q_1, q_2 \in [0, q_0)$ with $|q_1 - q_2| \leq \varepsilon$ with some $\varepsilon > 0$. By Theorem 13 we may assume that $||T|| \leq 1$, so that $||Th|| \leq 1$ and $|\langle Th, h \rangle | \leq 1$ for $h \in \text{dom}(\xi_0)$. In this setting it is clear that

$$\text{dist}(\xi_0(h) \langle Th, h \rangle, V_{q_j,q_0}(T)) \leq |\xi_{q_j}(h) - \xi_{q_2}(h)|, \quad h \in \text{dom}(\xi_0), \ i, j = 1, 2.$$ 

This implies the following inequality for the Hausdorff metric

$$d_H(V_{q_1,q_0}(T), V_{q_2,q_0}(T)) \leq \sup_{h \in \text{dom}(\xi_0)} |\xi_{q_1}(h) - \xi_{q_2}(h)| \leq \sup_{h \in \text{dom}(\xi_0)} \frac{1}{2} \left( \varepsilon + |\sqrt{\Delta_{q_1}(h)} - \sqrt{\Delta_{q_2}(h)}| \right).$$

Observe that if $h \in \text{dom}(\xi_0)$ is such that $\sqrt{\Delta_{q_1}(h)} + \sqrt{\Delta_{q_2}(h)} \leq \sqrt{\varepsilon}$ then

$$|\sqrt{\Delta_{q_1}(h)} - \sqrt{\Delta_{q_2}(h)}| \leq \sqrt{\varepsilon},$$

and if $h \in \text{dom}(\xi_0)$ is such that $\sqrt{\Delta_{q_1}(h)} + \sqrt{\Delta_{q_2}(h)} > \sqrt{\varepsilon}$ then

$$|\sqrt{\Delta_{q_1}(h)} - \sqrt{\Delta_{q_2}(h)}| \leq \frac{|\Delta_{q_1}(h) - \Delta_{q_2}(h)|}{\sqrt{\varepsilon}} \leq \frac{|q_1^2 - q_2^2| + 4|q_1 - q_2|}{\sqrt{\varepsilon}} \leq 8\sqrt{\varepsilon}.$$ 

Taking together (24), (25) and (26) and the fact that $\varepsilon > 0$ was arbitrary we get the claim. 

Lemma 19. If $0 \leq q_1 < q_2 < q_3 < 2$ and $0 \in \text{conv} \ V_{q_1,q_3}$ then $\text{conv} \ V_{q_2,q_3} \subseteq \text{conv} \ V_{q_1,q_3}$.

Proof. It is enough to show that any $\lambda_0$ of the form $\lambda_0 = \xi_{q_2}(h_0) \langle Th_0, h_0 \rangle$ with $h_0 \in \text{dom}(\xi_{q_1})$ belongs to $\text{conv} \ V_{q_1,q_3}(T)$. If $\lambda_0 = 0$ then trivially $\lambda_0 \in V_{q_1,q_3}(T)$, so we assume that $\lambda_0 \neq 0$ and hence $\langle Th_0, h_0 \rangle \neq 0$ and $\xi_{q_1}(h_0) \geq \xi_{q_2}(h_0) \geq q_2/2 > 0$. Note that $\lambda_1 = \xi_{q_1}(h_0) \langle Th_0, h_0 \rangle \in V_{q_1,q_3}(T)$ and $\lambda_0 = \frac{\xi_{q_2}(h_0)}{\xi_{q_1}(h_0)} \lambda_1$. As we assumed that $0 \in \text{conv} \ V_{q_1,q_3}(T)$, we get $\lambda_0 \in \text{conv} \ V_{q_1,q_3}(T)$.

Proof of Theorem 13. The proof follows the same lines as the proof of Lemma 19 with $\text{conv} \ V_{q_1,q_3}(T)$ replaced by $W_{q_i}(T), i = 1, 2$ and $h_0 \in \text{dom}(\xi_{q_2})$ (however, the statement itself is not a direct consequence of Lemma 19).

Assume that $0 \in \text{Int} \ W_{q_2}(T)$. We show that $0 \in \overline{W_{q_1}(T)}$, which will finish the proof. Consider the set

$$Q := \{ q \in [0, q_2) : 0 \in \text{Int}(\text{conv} \ V_{q_1,q_2}(T)) \}.$$ 

As the mapping $V_{q_2} : [0, q_2) \ni q \mapsto \text{conv} \ V_{q_1,q_2}(T)$ is, by Lemmas 17 and 18, continuous, the set $Q$ is nonempty and open in $[0, 2)$. Furthermore, by Lemma 19 we see that if $q \in Q$ then $[0, q_2) \subseteq Q$. Hence, to show that $Q$ is closed in $[0, 2)$ it is enough to take a decreasing sequence $Q \ni r_n \downarrow q$ and show that $q \in Q$. Applying once again Lemma 19 we have that $\text{conv} \ V_{r_n,q_2} \subseteq \text{conv} \ V_{r_{n+1},q_2}$ and therefore, the distance of 0 to $\partial(\text{conv} \ V_{r_n,q_2})$ is bounded from below by some $\varepsilon > 0$. By continuity of $V_{q_2}$ we get $0 \in \text{conv} \ V_{q_1,q_2}$, which shows $Q = [0, q_2)$. To finish the proof note that since $q_1 \in Q$, we have $0 \in \text{conv} \ V_{q_1,q_2} \subseteq W_{q_1}(T)$.

In view of (i) and (ii) it is enough to show that $W_{q_1}(T) \neq W_{q_2}(T)$. Take $\lambda_2 \in \partial W_{q_2}(T) \setminus \sigma(T)$. Then $\lambda_2 = \xi_{q_2}(T) \langle Th, h \rangle$ for some $h \in H$. By Proposition (iii) we have $\xi_{q_1}(h) > \xi_{q_2}(h)$. Hence, $\lambda_1 = \xi_{q_1}(h) \langle Th, h \rangle \notin W_{q_2}(T)$, as otherwise $\lambda_2$ would lie in the interior of $W_{q_2}(T)$ due to $0 \in \text{Int} \ W(T) \subseteq W_{q_2}(T)$. Clearly, $\lambda_1 \in W_{q_1}(T)$. 

(iv) The statement follows directly from Lemmas 17, 18 and the fact that $\text{conv} V_{q,1} = W_q$ for $q \in [0, 1]$. □

7. Spectral constants of the deformed numerical range

Following Crouzeix [4] we define

$$\Psi_{\Omega}(T) := \sup \left\{ \| f(A) \| : f \in \mathcal{H}^\infty(\Omega), \| f \|_{L^\infty(\Omega)} \leq 1 \right\},$$

where $\Omega$ is an open, non empty, convex subset of $\mathbb{C}$ and $\mathcal{H}^\infty(\Omega)$ denotes the Hardy space. Note that $\Psi_{\Omega}(T) < +\infty$, provided that $\sigma(T) \subseteq \Omega$, due to the Cauchy integral formula.

Furthermore, we define

$$\Psi_q(T) := \sup \left\{ \| f(A) \| : f \text{ polynomial}, \| f \|_{L^\infty(W_q(T))} \leq 1 \right\}, \quad q \in [0, 2).$$

The following result was shown as [4, Lemma 2.2] for $q = 1$, we show that it is true for all $q \in [0, 1]$ under the additional assumption that $0 \in \text{Int} W_q(T)$. Note that while for $q = 1$ this assumption may be simply omitted, due to the law $W(T + \alpha I) = W(T) + \alpha$, $\alpha \in \mathbb{C}$, we cannot drop this assumption for $q < 1$.

Lemma 20. For any matrix $T$ with $0 \in \text{Int} W(T)$ and any $q \in [0, 1]$ we have

$$\Psi_q(T) = \sup \{ \Psi_q(T) : \Omega \supseteq W_q(T), \Omega \text{ open} \}.$$

Proof. Let $q \in [0, 1]$. It is enough to show that each eigenvalue on the boundary of $W_q(T)$ is on the boundary of $W(T) = W_1(T)$, then the proof follows the same lines as in [4, Lemma 2.2]. Let $\lambda \in \partial W_q(T)$ be an eigenvalue of $T$. Clearly, $\lambda \in W(T)$ and due to $0 \in \text{Int} W(T)$ we have

$$t\lambda \in W(T), \quad t \in [0, 1].$$

On the other hand, $t\lambda \notin W_q(T)$ for $t > 1$, as $\lambda$ is on the boundary of $W_q(T)$, $0$ is in the interior of $W_q(T)$ and $W_q(T)$ is convex. Hence, by Theorem 15 (iv), we have that $t\lambda \notin W(T)$ for $t > 1$, which together with (27) implies that $\lambda$ is on the boundary of $W(T)$. □

Remark 21. For the sake of completeness let us mention that it is also possible to show that for finite dimensional $\mathcal{H}$ the statement of Lemma 20 holds for $q \in [0, 1 + \varepsilon]$, where $\varepsilon > 0$ depends on $T$. The proof is, however, rather technical and the result will not be used later on.

We are able now to show the main result of this section.

Theorem 22. Let $T$ be a bounded operator on a Hilbert space with $0 \in \text{Int} W(T)$, then the function

$$[0, 1] \ni q \mapsto \Psi_q(T)$$

is continuous and increasing. If, additionally, $\mathcal{H}$ is finite dimensional, then the function in (28) is either constantly equal to $1$ or strictly increasing.

Proof. The proof is based on Lemma 2.2 from [4]. It was shown therein that for fixed $T$ the function $\Omega \mapsto \Psi_{\Omega}(T)$ is decreasing and continuous with respect to the Hausdorff metric, note that this part of the proof did not use the fact that $\mathcal{H}$ is finite dimensional. Combining this information with Lemma 20 and Theorem 15 (iv) we get the desired continuity in (28).
Monotonicity in (28) results directly from Theorem 15(ii). To prove that the function in question is strictly increasing if $\Psi_1(T) > 1$ we use Theorem 15(iii). In the light of this result, combined again with Lemma 20, it is enough to remark that $\Psi_{\Omega_2}(T) < \Psi_{\Omega_1}(T)$ if $\Omega_2 \subset \subset \Omega_1$ and $\Psi_1(T) > 1$, see again [4], Lemma 2.2. 

Note that in general the function $T \mapsto \Psi_q(T)$ is not continuous:

Example 23. Let $T = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$, then $\Psi_1(T) = 2$ for $\alpha \neq 0$, and $\Psi_1(T) = 1$ for $\alpha = 0$.

Remark 24. If the condition $0 \in \text{Int}W(T)$ is not satisfied, it is tempting to replace $T$ by $T - \alpha I$, where $\alpha$ is chosen appropriately, e.g., it is the barycentre of the spectrum of $T$. However, one should be careful with such manipulations, as $W_q(T)$ is not translatable, see Remark 3 and Example 4. Hence, the corresponding spectral constants $\Psi_q(T)$ and $\Psi_q(T - \alpha I)$ may differ.

We are also able now to complete the analysis from Proposition 5.

Corollary 25. For $T = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ we have

$$\Psi_q(T) = \frac{2}{2 - q}, \quad q \in [0, 2).$$

Proof. Take any $q \in [0, 2)$. By Proposition 5 we have that $W_q(T) = (2 - q)\mathbb{D}$. Considering the polynomial $p(z) = z$ we get $\Psi_q(T) = \frac{2}{2 - q}$. On the other hand Theorem 13 gives us the opposite inequality. 

Recall that the Crouzeix conjecture says that $\Psi_1(T) \leq 2$ for any bounded operator $T$, (equivalently: for any matrix $T$, see [5]). Note the following corollary from our considerations above.

Corollary 26. The Crouzeix conjecture does not hold if and only if there exists a matrix $T$ with $0 \in \text{Int}W(T)$ and $q \in [0, 1)$ such that $\Psi_q(T) = 2$.

Proof. Suppose that there exists a matrix $T$ with $\Psi_1(T) > 2$. Observe that $W(T)$ has a nonempty interior, otherwise $T$ is an affine transformation of a Hermitian matrix and $\Psi_1(T) = 1$, contradiction. Due to $\Psi_1(T + \alpha I) = \Psi_1(T)$ for $\alpha \in \mathbb{C}$, one can find a matrix $T$ with $0 \in \text{Int}W(T)$ and $\Psi_1(T) > 2$. Application of Theorem 22 finishes the proof of the forward implication. The converse implication follows directly from the last statement of Theorem 22. 

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Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Krakow, Poland

E-mail address: \{patryk.pagacz, pawel.pietrzycki, michal.wojtylak\}@im.uj.edu.pl