Homogenization of the spectral equation in one-dimension

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Abstract

The asymptotic behavior of a one-dimensional spectral problem with periodic coefficient is addressed for high frequency modes by a method of Bloch wave homogenization. The analysis leads to a spectral problem including both microscopic and macroscopic eigenmodes. Numerical simulation results are provided to corroborate the theory.

Keywords. Homogenization, Bloch waves, spectral problem, two-scale transform.

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1 Introduction

We consider the spectral problem

\[- \partial_x (a^\varepsilon \partial_x w^\varepsilon) = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon\]  

(1)

posed in an one-dimensional open bounded domain \( \Omega \subset \mathbb{R} \) with Dirichlet boundary conditions. An asymptotic analysis of this problem is carried out where \( \varepsilon > 0 \) is a parameter tending to zero and the coefficients are \( \varepsilon \)-periodic, namely \( a^\varepsilon = a(\frac{x}{\varepsilon}) \) and \( \rho^\varepsilon = \rho(\frac{x}{\varepsilon}) \) where \( a(y) \) and \( \rho(y) \) are 1-periodic in \( \mathbb{R} \). The homogenization of such spectral problem has been studied in various works providing the asymptotic behaviour of eigenvalues and eigenvectors. The low frequency part of the spectrum has been investigated in [17], [18], [25]. Then, many configurations have been analyzed, as [16] and [13] for a fluid-structure interaction, [7], [3] for neutron transport, [22], [24] for \( \rho \) which changes sign or [4] for the first high frequency eigenvalue and eigenvector for a one-dimensional non-self-adjoint problem with Neumann boundary conditions. In [6], G. Allaire and C. Conca studied the asymptotic behaviour of both the low and high frequency spectrum. In order to analyze the asymptotic behaviour of the high frequency eigenvalues, they used the Bloch wave homogenization method. It is a blend of two-scale convergence, see e.g. [1], [2], [21], and Bloch wave decomposition, see e.g. [15], [12], [14], and was previously introduced in [5] to a fluid-solid interaction problem. They have shown that the limit of the set of renormalized eigenvalues \( \varepsilon^2 \lambda^\varepsilon \) is the union of the Bloch spectrum and the boundary layer spectrum, when \( \varepsilon \) goes to 0. However, the asymptotic behaviour of the corresponding eigenvectors was not addressed. This is the goal of the present work which focuses on the Bloch spectrum of the high frequency part. Precisely, we search eigenvalues \( \lambda^\varepsilon \) such that

\[\varepsilon^2 \lambda^\varepsilon = \lambda^k_n + \varepsilon \lambda^1 + \varepsilon O(\varepsilon)\]  

(2)

where \( \lambda^k_n \) is solution of the Bloch wave spectral problem, also called the microscopic equation in this work,

\[- \partial_y (a(y) \partial_y \phi^k_n (y)) = \lambda^k_n \rho(y) \phi^k_n (y) \text{ for } n \in \mathbb{N}^*\]  

(3)

with \( k \)-quasi-periodic boundary conditions for some \( k \in \mathbb{R} \). From [6], it is known that each \( \lambda^k_n \) can be reached as a limit of a subsequence of \( \varepsilon^2 \lambda^\varepsilon \). For each \( n \in \mathbb{N}^* \) and each \( k \), \( \lambda^k_n \) is either a simple or a double eigenvalue and \( \lambda^k_n = \lambda_{-k}^n \). We pose \( I^k = \{-k, k\} \) if \( k \neq 0 \) and \( I^k = \{0\} \) otherwise. To guarantee that Bloch waves are kept in the weak limit, we apply the modulated two-scale transform \( S^\varepsilon_k \), defined in [8] from the usual two-scale transform in [20], [19], [10], [9] or [11]. Passing to the limit in the weak formulation, it is shown that \( \sum_{\sigma \in I^k} S^\varepsilon_\sigma w^\varepsilon \) is weakly converging to two-scale modes

\[g_k (x, y) = \sum_{\sigma \in I^k} \sum_{m} u^\sigma_m (x) \phi^\sigma_m (y)\]

where the second sum runs over all modes \( \phi^\sigma_m \) with the same eigenvalue \( \lambda^k_n \). Here, the modes \( \phi^\sigma_m \) are called microscopic modes. The factors \( (u^\sigma_m)_m \) are solution of the macroscopic system of first order
differential equation,
\[ \sum_m c(\sigma, n, m) \partial_x u^\sigma_m + \lambda^1 b(\sigma, n, m) u^\sigma_m = 0 \text{ in } \Omega \text{ for each } \sigma \in I^k, \]  
(4)

which boundary conditions and the constant \( c(\sigma, n, m) \) are depending on the involved microscopic modes and eigenvalues. The physical solution \( w^\varepsilon \) is then approximated by two-scale modes
\[ w^\varepsilon(x) \approx \sum_{\sigma \in I^k} \sum_m u^\sigma_m(x) \phi^\sigma_m\left(\frac{x}{\varepsilon}\right). \]  
(5)

These results are established for Neumann boundary conditions.

In fact, this method is inspired from [8] dedicated to the wave equation, except that in the latter work the two-scale transforms \( S^\varepsilon_k w^\varepsilon \) and \( S^{-\varepsilon}_k w^\varepsilon \) were analyzed separately and the macroscopic boundary conditions were lacking. Moreover, the model derivation in [8] is starting from the wave equation written as a first order system. So, for the sake of comparison, we derive the homogenized spectral equation from a first order formulation.

In addition, we report exploration results regarding approximations of physical eigenmodes by two-scale modes. First, for a given \( \varepsilon \) and each high frequency physical eigenelement \( (\lambda^\varepsilon, w^\varepsilon) \), we show how to find quadruplets \( (\lambda^k, \lambda^1, \phi^k_n, u^k_n)_{n,k} \) satisfying the approximations (2) and (5). This shows that each high frequency eigenelement can be approximated by a two-scale mode. Conversely, the high-frequency physical eigenelements can be built from the two-scale eigenelements only. Namely, for a given Bloch mode \( (\lambda^k, \phi^k_n) \), a macroscopic eigenelement \( (\lambda^1, u^k_n) \) is minimizing the error on the physical equation (1) where \( w^\varepsilon \) and \( \lambda^\varepsilon \) are replaced by their approximations (2) and (5).

This paper is organized as follows. In Section 2 we state the physical spectral equation with Dirichlet boundary conditions. In Section 3 the notations and elementary properties, which are used throughout the paper, are introduced. In Section 4 and 5 the model homogenization is derived based on the second order and first order formulations respectively. Finally, the numerical results are reported in the last section.

## 2 Statement of the problem

We consider \( \Omega = (0, \alpha) \subset \mathbb{R}^+ \) an interval, which boundary is denoted by \( \partial \Omega \), and two functions \((a^\varepsilon, \rho^\varepsilon)\) assumed to obey a prescribed profile,
\[ a^\varepsilon := a\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \rho^\varepsilon := \rho\left(\frac{x}{\varepsilon}\right), \]  
(6)

where \( \rho \in L^\infty(\mathbb{R}) \), \( a \in W^{1,\infty}(\mathbb{R}) \) are both \( Y \)-periodic where \( Y \) is an open interval. Moreover, they are required to satisfy the standard uniform positivity and ellipticity conditions:
\[ \rho^0 \leq \rho \leq \rho^1 \quad \text{and} \quad a^0 \leq a \leq a^1, \]  
(7)

for some given strictly positive \( \rho^0, \rho^1, a^0 \) and \( a^1 \).

With the operators \( P^\varepsilon = -\partial_x (a^\varepsilon \partial_x) \), the spectral problem with Dirichlet boundary conditions is
\[ P^\varepsilon w^\varepsilon = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \quad \text{in} \quad \Omega \quad \text{and} \quad w^\varepsilon = 0 \quad \text{on} \quad \partial \Omega, \]  
(8)

where as usual \( \varepsilon > 0 \) denotes a small parameter intended to go to zero.
The eigenvectors \( w^\varepsilon \in H^2(\Omega) \cap H^1_0(\Omega) \) are normalized by
\[
\|w^\varepsilon\|_{L^2(\Omega)} = \left( \int_\Omega |w^\varepsilon|^2 \, dx \right)^{\frac{1}{2}} = 1, \tag{9}
\]
and we search the eigenvalues such that
\[
\varepsilon^2 \lambda^\varepsilon = \lambda^0 + \varepsilon \lambda^1 + \varepsilon O(\varepsilon), \tag{10}
\]
where \( \lambda^0 \) is a non negative real number and \( O(\varepsilon) \) tends to zero with \( \varepsilon \). The weak formulation of the spectral problem (8) is: find \( w^\varepsilon \in H^1_0(\Omega) \) such that
\[
\int_\Omega a^\varepsilon \partial_x w^\varepsilon \partial_x v \, dx = \lambda^\varepsilon \int_\Omega \rho^\varepsilon w^\varepsilon v \, dx \text{ for all } v \in H^1_0(\Omega). \tag{11}
\]
Since \( \varepsilon^2 \lambda^\varepsilon \) is bounded, it results the uniform bound
\[
\|\varepsilon \partial_x w^\varepsilon\|_{L^2(\Omega)} \leq N_0. \tag{12}
\]

3 Notations and elementary properties

The functional space \( L^2(\Omega) \) of square integrable functions is over \( \mathbb{C} \). Let \( u = (u_i)_i \) and \( v = (v_i)_i \) be \( m \)-dimensional complex vector valued functions in \( L^2(\Omega) \), the dot product is denoted by \( u.v := \sum_i u_i v_i \) and the hermitian inner product by
\[
\int_\Omega u \cdot v \, dx = \int_\Omega u(x)\overline{v(x)} \, dx. \tag{13}
\]
The notation \( O(\varepsilon) \) refers to numbers or functions tending to zeros when \( \varepsilon \to 0 \) in a sense made precise in each case. The notations \( \partial_x u = \frac{\partial u}{\partial x} \), \( \partial_y u = \frac{\partial u}{\partial y} \) are for \( x \)- and \( y \)-derivatives of a function \( u \). The vectors \( n_\Omega, n_Y \) are the outer unit normals of \( \partial \Omega \) and \( \partial Y \).

**Bloch decomposition** We follow the definition of Bloch decomposition in [8] with \( N = 1 \), \( L = \mathbb{Z} \), and \( Y = (0,1) \), so \( \mathbb{R} = [0,1] + L \). The dual lattice is necessarily \( L^* = \mathbb{Z} \), and the equivalence class \( Y^* = \mathbb{R}/L^* \) is chosen as \( Y^* = (-1/2,1/2) \). For \( K \in \mathbb{N}^* \), considering the dual lattices \( K L = K\mathbb{Z} \) and \( L^*/K = \mathbb{Z}/K \), we pose
\[
L_K = \begin{cases} \{-\frac{K}{2}, \ldots, \frac{K}{2}-1\} & \text{if } K \text{ is even,} \\ \{-\frac{K-1}{2}, \ldots, \frac{K-1}{2}\} & \text{if } K \text{ is odd,} \end{cases}
\]
so that \( L = L_K + K L \). Posing \( L_K^* = L_K^*/K \) yields \( L^* / K = L^* + L_K^* \).

**Functional spaces of quasi-periodic functions** For any \( k \in Y^* \), we define the \( k \)-quasi-periodic \( L^2 \)-vector space over \( \mathbb{C} \) with the hermitian inner product (13) by
\[
L^2_k = \{ u \in L^2_{loc}(\mathbb{R}) \mid u(x + \ell) = u(x)e^{2i\pi k\ell} \text{ a.e. in } \mathbb{R} \text{ for all } \ell \in L \},
\]
or equivalently
\[
L^2_k = \{ u \in L^2_{loc}(\mathbb{R}) \mid \exists v \in L^2_k \text{ such that } u(x) = v(x)e^{2i\pi kx} \text{ a.e.} \},
\]
where \( L^2_k \) is the traditional notation for \( L^2_k \) in the periodic case i.e. when \( k = 0 \). Likewise, for \( s \geq 0 \) we set
\[
H^s_k := L^2_k \cap H^s_{loc}(\mathbb{R})
\]
bearing in mind that the subscript $\sharp$ would be more appropriate in the periodic case $k = 0$.

**The modulated two-scale transform** Let us assume from now that the domain $\Omega$ is the union of a finite number of entire cells of size $\varepsilon$ or equivalently that the sequence $\varepsilon$ is exactly $\varepsilon_n = \frac{2}{n}$ for $n \in \mathbb{N}^*$. Setting $C_{\varepsilon} := \{\omega_{\varepsilon} = \varepsilon l + \varepsilon Y \mid l \in L, \varepsilon l + \varepsilon Y \subset \Omega\}$ is the set of all cells of $\Omega$.

**Definition 1** For any $k \in Y^*$, the modulated two-scale transform of the function $u \in L^2(\Omega)$, $S_k^{\varepsilon} : L^2(\Omega) \to L^2(\Omega \times Y)$ is defined by

$$S_k^{\varepsilon}u(x, y) = \sum_{\omega_{\varepsilon} \in C_{\varepsilon}} u(\varepsilon l_{\omega_{\varepsilon}} + \varepsilon y) \chi_{\omega_{\varepsilon}}(x) e^{-2i\pi kl_{\omega_{\varepsilon}}},$$

where $\varepsilon l_{\omega_{\varepsilon}}$ stands for the unique node in $\varepsilon L$ of $\omega_{\varepsilon}$ and $\chi_{\omega_{\varepsilon}}$ is the characteristic function of $\omega_{\varepsilon}$.

The three following properties can be checked by using (14) and are admitted. For $u, v \in L^2(\Omega)$

$$\|S_k^{\varepsilon}u\|_{L^2(\Omega \times Y)}^2 = \int_{\Omega \times Y} |S_k^{\varepsilon}u|^2 \, dx \, dy = \sum_{\varepsilon} \int_{\omega_{\varepsilon}} |u|^2 \, dx = \|u\|_{L^2(\Omega)}^2,$$

and $S_k^{\varepsilon}(\partial_x u)(x, y) = \frac{1}{\varepsilon} \partial_y S_k^{\varepsilon}u(x, y)$ for $u \in H^1(\Omega)$.

**Remark 2** Let $k \in Y^*$ and a sequence $u^\varepsilon$ bounded in $L^2(\Omega)$ such that $S_k^{\varepsilon}u^\varepsilon$ converges to $u^k$ in $L^2(\Omega \times Y)$ weakly when $\varepsilon \to 0$, then $S_{-k}^{\varepsilon}u^\varepsilon$ converges to some $u^{-k}$ in $L^2(\Omega \times Y)$ weakly. Moreover, since $S_k^{\varepsilon}u^\varepsilon$ and $S_{-k}^{\varepsilon}u^\varepsilon$ are conjugate then $u^k$ and $u^{-k}$ are also conjugate.

The adjoint $S_k^{\varepsilon} : L^2(\Omega \times Y) \to L^2(\Omega)$ of $S_k^{\varepsilon}$ is defined by

$$\int_{\Omega} (S_k^{\varepsilon}v)(x) \cdot w(x) \, dx = \int_{\Omega \times Y} v(x, y) \cdot (S_k^{\varepsilon}w)(x, y) \, dx \, dy,$$

for all $w \in L^2(\Omega)$ and $v \in L^2(\Omega \times Y)$, and we denote by $\Re$ the operator operating on functions $v(x, y)$ defined in $\Omega \times \mathbb{R}$,

$$(\Re v)(x) = v(x, \frac{x}{\varepsilon}).$$

The next Lemma shows that $\Re$ is an approximation of $S_k^{\varepsilon}$ for $k-$quasi-periodic functions.

**Lemma 3** Let $v \in C^1(\Omega \times Y)$ a $k-$quasi-periodic function in $y$ then

$$S_k^{\varepsilon}v = \Re v + O(\varepsilon) \text{ in the } L^2(\Omega) \text{ sense.}$$

**Proof.** The proof is carried out in two steps. First the explicit expression of $S_k^{\varepsilon}v$ is derived, then the approximation is deduced.

(i) Let us prove that

$$(S_k^{\varepsilon}v)(x) = \sum_{\omega_{\varepsilon} \in C_{\varepsilon}} \varepsilon^{-1} \int_{\omega_{\varepsilon}} v \left( z, \frac{x - \varepsilon l_{\omega_{\varepsilon}}}{\varepsilon} \right) \, dz \chi_{\omega_{\varepsilon}}(x) e^{2i\pi kl_{\omega_{\varepsilon}}}.$$

From the definition of the two-scale transform with $r = \varepsilon l_{\omega_{\varepsilon}} + \varepsilon y \in \omega_{\varepsilon}$,

$$\int_{\Omega \times Y} v(x, y) \cdot (S_k^{\varepsilon}w)(x, y) \, dx \, dy = \sum_{\omega_{\varepsilon} \in C_{\varepsilon}} \int_{\Omega \times \omega_{\varepsilon}} \varepsilon^{-1} v \left( x, \frac{r - \varepsilon l_{\omega_{\varepsilon}}}{\varepsilon} \right) \cdot w(r) \chi_{\omega_{\varepsilon}}(x) e^{-2i\pi kl_{\omega_{\varepsilon}}} \, dx \, dr$$

5
or equivalently,

\[ \int_{\Omega} \sum_{\omega \in C_{\varepsilon}} \varepsilon^{-1} \int_{\omega} v \left( x, \frac{r - \varepsilon l_{\omega}}{\varepsilon} \right) dx \cdot w(r) \chi_{\omega_{\varepsilon}}(r) e^{-2i\pi kl_{\omega}} \, dr. \]

Changing the variable names and using the definition of \( S_{k}^{\varepsilon} \),

\[ \int_{\Omega} (S_{k}^{\varepsilon} v)(x) \cdot w(x) \, dx = \int_{\Omega} \sum_{\omega \in C_{\varepsilon}} \varepsilon^{-1} \int_{\omega} v \left( z, \frac{x - \varepsilon l_{\omega}}{\varepsilon} \right) dz e^{2i\pi kl_{\omega}} \cdot w(x) \chi_{\omega_{\varepsilon}}(x) \, dx. \]

This establishes the explicit expression of \( S_{k}^{\varepsilon} \).

(ii) Let us derive the expected approximation for \( v \in C^{1}(\Omega \times Y) \) and \( k \)–quasi-periodic in \( y \). Since \( \varepsilon |Y| = |\omega_{\varepsilon}| \) and

\[ v(z, y) = v(x, y) + \partial_{x} v(x, y)(z - x) + \varepsilon O(\varepsilon) \text{ in } L^{2}(\Omega) \text{ for a.e. } y \in Y \]

then

\[ (S_{k}^{\varepsilon} v)(\varepsilon l_{\omega}, \varepsilon y) = \frac{1}{|\omega_{\varepsilon}|} \int_{\omega_{\varepsilon}} v(x, y) + \partial_{x} v(x, y)(z - x) \, dz \, e^{2i\pi kl_{\omega}} + O(\varepsilon) \]

for a.e. \( y \in Y \) and all \( \omega_{\varepsilon} \in C_{\varepsilon} \). Remarking that \( z - x = (z - \varepsilon l_{\omega}) + (\varepsilon l_{\omega} - x) \) and

\[ \int_{\omega_{\varepsilon}} (z - \varepsilon l_{\omega}) \, dz = -\frac{1}{2} \varepsilon O(\varepsilon). \]

So for all \( \omega_{\varepsilon} \) and \( y \in Y \),

\[ e^{-2i\pi kl_{\omega}} |\omega_{\varepsilon}| (S_{k}^{\varepsilon} v)(\varepsilon l_{\omega}, \varepsilon y) = |\omega_{\varepsilon}| v(x, y) + \left( -\frac{1}{2} \varepsilon O(\varepsilon) + (\varepsilon^{2} y) \right) \partial_{x} v(x, y) + \varepsilon O(\varepsilon). \]

Therefore,

\[ (S_{k}^{\varepsilon} v)(x) = \sum_{\omega_{\varepsilon}} v \left( x, \frac{x}{\varepsilon} - l_{\omega_{\varepsilon}} \right) \chi_{\omega_{\varepsilon}}(x) e^{2i\pi kl_{\omega}} + O(\varepsilon). \]

Using the \( k \)–quasi-periodic of \( v \) in \( y \),

\[ (S_{k}^{\varepsilon} v)(x) = \sum_{\omega_{\varepsilon}} v \left( x, \frac{x}{\varepsilon} \right) \chi_{\omega_{\varepsilon}}(x) + O(\varepsilon) \]

in \( L^{2}(\Omega) \), hence the formula (18) follows.

In the proof, we constantly use the following consequence.

**Corollary 4** Let \( v \in C^{1}(\Omega \times Y) \) and \( k \)–quasi-periodic in \( y \), for any sequence \( u^{\varepsilon} \) bounded in \( L^{2}(\Omega) \) such that \( S_{k}^{\varepsilon} u^{\varepsilon} \) converges to \( u \) in \( L^{2}(\Omega \times Y) \) weakly when \( \varepsilon \to 0 \) then

\[ \int_{\Omega} u^{\varepsilon} \cdot R v \, dx \to \int_{\Omega \times Y} u \cdot v \, dx dy \text{ when } \varepsilon \to 0. \]

Note that for \( k = 0 \), this corresponds to the definition of two-scale convergence in [1] and [23].

**Two-scale operators** For a function \( v(x, y) \) defined in \( \Omega \times \mathbb{R} \), we pose

\[ P^{0} v = -\partial_{x} (a \partial_{x} v), \quad P^{1} v = -\partial_{x} (a \partial_{y} v) - \partial_{y} (a \partial_{x} v) \quad \text{and} \quad P^{2} v = -\partial_{y} (a \partial_{y} v), \]
so that
\[ P^c \mathfrak{R} v = \sum_{n=0}^{2} \varepsilon^{-n} \mathfrak{R} P^n v. \] (19)

**Bloch waves** For a given \( k \in Y^* \), we denote by \((\lambda_n^k, \phi_n^k)\) the Bloch wave eigenelements indexed by \( n \in \mathbb{N}^* \) that are solution to
\[ \mathcal{P}(k) : P^2 \phi_n^k = \lambda_n^k \rho \phi_n^k \text{ in } Y \text{ with } \phi_n^k \in H^2_k(Y) \text{ and } \|\phi_n^k\|_{L^2(Y)} = 1. \] (20)
The corresponding weak formulation is: find \( \phi_n^k \in H^1_k(Y) \) solution to
\[ \int_Y a \partial_y \phi_m^k \cdot \partial_y v - \lambda_n^k \rho \phi_n^k \cdot v \, dy = 0 \text{ for all } v \in H^1_k(Y). \] (21)
Since the operator \( P^2 : H^2_k(Y) \subset L^2_k(Y) \rightarrow L^2_k(Y) \) is self-adjoint, its spectra is real. Furthermore, for \( n, m \in \mathbb{N}^* \), we introduce the coefficients
\[ c(k, n, m) = \int_Y a \partial_y \phi_n^k \cdot \phi_m^k - \lambda_n^k \rho \phi_n^k \cdot a \partial_y \phi_m^k \, dy \text{ and } b(k, n, m) = \int_Y \rho \phi_m^k \cdot \phi_n^k \, dy \] (22)
and observe that the following properties hold,
\[ c(k, n, m) = c(-k, n, m), \quad c(k, m, n) = -c(k, n, m), \quad c(k, n, m) = -c(-k, m, n) \]
and
\[ b(k, n, m) = b(k, m, n), \quad b(k, n, m) = b(-k, m, n), \quad b(k, n, n) > 0. \]
In particular for \( k = 0 \), if the eigenvectors are chosen as real functions thus \( c(0, n, n) = 0 \). In the special case \( \rho = 1 \), \( b(k, n, n) = 1 \) for \( n = m \) and \( b(k, n, m) = 0 \) otherwise.

**Notation 5** For \( k \neq 0 \), \( \overline{\phi_n^k} \in H^2_k(Y) \), the conjugate of \( \phi_n^k \), is solution of \( \mathcal{P}(-k) \). We choose the numbering of eigenvectors \( \phi_n^{-k} \) so that \( \phi_n^{-k} = \overline{\phi_n^k} \) and remark that \( \lambda_n^{-k} = \lambda_n^k \).

**Remark 6** In one dimension, for \( k \in Y^* \), it is well-known that all eigenvalue \( \lambda_n^k \) are simple, except for \( k = 0 \) where they are double.

Finally, we denote
\[ I^k = \{k, -k\} \text{ if } k \in Y^* \setminus \{0\} \text{ and } I^0 = \{0\} \text{ otherwise.} \]

### 4 Homogenization of the high-frequency eigenvalue problem

For \( k \in Y^* \), we decompose
\[ \frac{\alpha k}{\varepsilon} = h^k_\varepsilon + l^k_\varepsilon \text{ with } h^k_\varepsilon = \left[ \frac{\alpha k}{\varepsilon} \right] \text{ and } l^k_\varepsilon \in [0, 1), \] (23)
and assume that the sequence of the \( \varepsilon \) is varying in a set \( E_k \subset \mathbb{R}^{+*} \) depending on \( k \) so that
\[ l^k_\varepsilon \rightarrow l^k \text{ when } \varepsilon \rightarrow 0 \text{ and } \varepsilon \in E_k \text{ with } l^k \in [0, 1). \] (24)
We note that for \( k = 0 \), \( h^k_\varepsilon = 0 \), \( l^k_\varepsilon = 0 \), so \( l^k = 0 \) and \( E_0 = \mathbb{R}^{+*} \).
4.1 Main result

The macroscopic equation is stated for each \( k \in Y^* \) and each Bloch wave eigenvalue \( \lambda_k \). For \( k \neq 0 \), we assume that \( c(\sigma, n, n) \neq 0 \) for each \( \sigma \in I_k \), so it is stated as an eigenvalue problem

\[
c(\sigma, n, n) \partial_x u_n^\sigma + \lambda^1 b(\sigma, n, n) u_n^\sigma = 0 \quad \text{in } \Omega
\]

for each \( \sigma \), with the boundary conditions

\[
\sum_{\sigma \in I^k} u_n^\sigma(x) \phi_n^\sigma(0) e^{sign(\sigma)2\pi i l_k^x} = 0 \quad \text{on } x \in \partial \Omega,
\]

where \( l_k \) is defined in (24). We observe that the first order operator \( c(k, n, n) \left( \begin{array}{cc} \partial_x & 0 \\ 0 & -\partial_x \end{array} \right) \) of this system is self-adjoint on the domain

\[
D_k = \{(u_n, v_n) \in H^1(\Omega)^2 \text{ satisfying } \text{(26)}\}
\]

so \( \lambda^1 \) is real.

For \( k = 0 \), assuming that \( \lambda_n^0 \) is a double eigenvalue corresponding to two eigenvectors \( \phi_n^0 \) and \( \phi_m^0 \), and that \( c(0, n, m) \neq 0 \), the macroscopic system states

\[
\sum_{q \in \{n, m\}} c(0, p, q) \partial_x u_q^0 + \lambda^1 b(0, p, q) u_q^0 = 0 \quad \text{in } \Omega \quad \text{for } p \in \{n, m\},
\]

with the boundary conditions

\[
\sum_{q \in \{n, m\}} u_q^0(x) \phi_q^0(0) = 0 \quad \text{on } x \in \partial \Omega.
\]

Again \( \lambda^1 \in \mathbb{R} \) since \( c(0, n, m) \left( \begin{array}{cc} 0 & \partial_x \\ -\partial_x & 0 \end{array} \right) \) is self-adjoint on

\[
D^0 = \{(u_n, u_m) \in H^1(\Omega)^2 \text{ satisfying } \text{(28)}\}.
\]

**Remark 7**

(i) If \( c(k, n, n) = 0 \) for \( k \neq 0 \) or \( c(0, p, q) = 0 \) for all \( p, q \) varying in \( \{n, m\} \), the macroscopic equations (25) or (27) are \( \lambda^1 = 0 \) or \( u = (u_n^\sigma)_{n, \sigma} = 0 \). But \( u = 0 \) is impossible since \( \|w^\varepsilon\|_{L^2(\Omega)} = 1 \) for all eigenmodes \( w^\varepsilon \). So \( \lambda^1 = 0 \) and this model does not provide any equation for \( u_n^\sigma \).

(ii) For \( k \neq 0 \), if \( \phi_k^m(0) = 0 \) then \( \phi_k^m(1) = 0 \) and \( \phi_k^m \) is a periodic solution that is a solution of \( k = 0 \). So, we consider always that \( \phi_k^m(0) \neq 0 \) for \( k \neq 0 \).

(iii) For \( k = 0 \), in case where \( \phi_n(0) = \phi_m(0) = 0 \) the boundary conditions of the macroscopic equation vanishes.

**Remark 8** This work focuses on the Bloch spectrum. To avoid eigenmodes related to the boundary spectrum, according to Proposition 7.7 in [2] we shall assume that the weak limit of \( S_k w^\varepsilon \) in \( L^2(\Omega; H^1(Y)) \) is not vanishing.

The main Theorem states as follows.
Theorem 9 For \( k \in \mathbb{Y}^{*} \), let \((\lambda^{\varepsilon}, w^{\varepsilon})\) be solution of (8) then \( \sum_{\sigma \in I^{k}} S_{\sigma}^{\varepsilon} w^{\varepsilon} \) is bounded in \( L^{2}(\Omega; H^{1}(Y)) \).

For \( \varepsilon \in E_{k} \), as in [23, 24], assuming that the weak limit of \( S_{\sigma}^{\varepsilon} w^{\varepsilon} \) in \( L^{2}(\Omega; H^{1}(Y)) \) is non-vanishing and the renormalized sequence \( \varepsilon^{2} \lambda^{\varepsilon} \) satisfies the decomposition (14), there exists \( n \in \mathbb{N}^{*} \) such that \( \lambda^{0} = \lambda_{n}^{k} \) with \( \lambda_{n}^{k} \) an eigenvalue of the Bloch wave spectrum and the limit \( g_{\varepsilon} \) of any weakly converging extracted subsequence of \( \sum_{\sigma \in I^{k}} S_{\sigma}^{\varepsilon} w^{\varepsilon} \) in \( L^{2}(\Omega; H^{1}(Y)) \) can be decomposed on the Bloch modes

\[
 g_{k}(x, y) = \sum_{\sigma \in I^{k}} u_{n}^{\sigma}(x) \phi_{n}^{\sigma}(y) \text{ for } k \neq 0 \text{ and } g_{0}(x, y) = \sum_{q \in \{n, m\}} u_{q}^{0}(x) \phi_{q}^{0}(y) \text{ otherwise } \tag{29}
\]

Moreover, \( u_{n}^{\sigma} \in H^{1}(\Omega) \) and \((u_{m}^{\sigma})_{m, \sigma}\) are solutions of the macroscopic equations (25, 26) and (27, 28).

Finally, \( u_{n}^{\sigma} = u_{m}^{-\sigma} \) are conjugate.

Thus, it follows from (29) that the physical solution \( w^{\varepsilon} \) is approximated by two-scale modes

\[
 w^{\varepsilon}(x) \approx \sum_{\sigma \in I^{k}} u_{n}^{\sigma}(x) \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) \text{ for } k \neq 0 \text{ and } w^{\varepsilon}(x) \approx \sum_{q \in \{n, m\}} u_{q}^{0}(x) \phi_{q}^{0}\left(\frac{x}{\varepsilon}\right) \text{ otherwise. } \tag{30}
\]

The boundary conditions (26) and (28) can be directly derived by replacing \( u^{\varepsilon} \) in the physical boundary condition by its approximations,

\[
 \sum_{\sigma \in I^{k}} u_{n}^{\sigma}(x) \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) = 0 \text{ for } k \neq 0 \text{ and } \sum_{q \in \{n, m\}} u_{q}^{0}(x) \phi_{q}^{0}\left(\frac{x}{\varepsilon}\right) = 0 \text{ otherwise at } x \in \partial \Omega. \tag{31}
\]

For \( k \neq 0 \), they result from

\[
 \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) = \phi_{n}^{\sigma}(0) e^{2i\pi \sigma \frac{x}{\varepsilon}} = \phi_{n}^{\sigma}(0) e^{\text{sign} (\sigma) 2i \pi x \frac{\lambda^{k} y_{k}}{\alpha}} = \phi_{n}^{\sigma}(0) e^{\text{sign} (\sigma) 2i \pi x \lambda^{k}} \text{ for } x \in \partial \Omega
\]

and the assumption \( l^{k}_{\varepsilon} \to l^{k} \). For \( k = 0 \), the conditions follow from the periodicity of \( \phi_{n}^{0} \). Furthermore, we observe that \( g_{k}(x, 0) \) and \( g_{k}(x, 1) \) are generally not vanishing except for \( k = 0 \).

Proposition 10 For \( k \in \mathbb{Y}^{*} \), \( n \in \mathbb{N}^{*} \), if the macroscopic solution \( u^{\varepsilon}_{n} \) is a non-vanishing constant, then any two-scale mode (31) is a physical eigenmode i.e. a solution to (8).

Proof. For \( k \in \mathbb{Y}^{*} \), \( n \in \mathbb{N}^{*} \), if the macroscopic solution \( u^{\varepsilon}_{n} \) is constant then \( \lambda^{1} = 0 \) and \((u_{m}^{\sigma})_{m, \sigma}\) are constant for all \( \sigma \in I^{k} \) and \( m \in \mathbb{N}^{*} \) such that \( \lambda_{m}^{\sigma} = \lambda_{n}^{\sigma} \). Now, we consider \( \rho = 1 \) and the proof is similar for \( \rho \neq 1 \). Based on Remark 14 about the macroscopic solutions in Section 4.4, \( \lambda^{1} = 0 \) is equivalent to \( \ell = \frac{2k_{\alpha}}{\varepsilon} \). From the \( \sigma \)-quasi-periodicity of \( \hat{\phi}_{n}^{\sigma} \),

\[
 \hat{\phi}_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) = \hat{\phi}_{n}^{\sigma}(0) e^{\text{sign} (\sigma) 2i \pi x \frac{\lambda^{k} y_{k}}{\alpha}} = \hat{\phi}_{n}^{\sigma}(0) e^{\text{sign} (\sigma) 2i \pi x \lambda^{k}} = \pm \hat{\phi}_{n}^{\sigma}(0),
\]

then \( \phi_{n}^{\sigma} \) is \( \alpha \)-periodic or \( \alpha \)-anti-periodic for \( \sigma \in I^{k} \). Hence \( \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) \) is a solution of the equation

\[
 \partial_{x} \left( a \left(\frac{x}{\varepsilon}\right) \partial_{x} \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) \right) = -\frac{\lambda^{\sigma}}{\varepsilon^{2}} \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) \text{ in } \Omega \tag{32}
\]

and \( \phi_{n}^{\sigma}\left(\frac{x}{\varepsilon}\right) \) is \( \alpha \)-periodic or \( \alpha \)-anti-periodic.
and \( u^\sigma_m \phi^\sigma_m \left( \frac{x}{\epsilon} \right) \) is also a solution of (32). Denote by \( w^\epsilon := \sum_{\sigma \in I^k} \sum_m u^\sigma_m \phi^\sigma_m \left( \frac{x}{\epsilon} \right) \) and observe that \( w^\epsilon \) is a solution of the equation

\[
\partial_x \left( a^\epsilon \partial_x w^\epsilon \right) = -\lambda^\epsilon w^\epsilon \quad \text{in } \Omega
\]

with the boundary conditions

\[
w^\epsilon (0) = \sum_{\sigma \in I^k} \sum_m u^\sigma_m \phi^\sigma_m (0) = 0 \quad \text{and} \quad w^\epsilon (\alpha) = \sum_{\sigma \in I^k} \sum_m u^\sigma_m \phi^\sigma_m \left( \frac{x}{\epsilon} \right) = \pm w^\epsilon (0) = 0.
\]

Finally, Proposition [10] is concluded. ■

**Remark 11** The converse is probably true, and is numerically studied in Section 6.2, i.e. for any \((\lambda^\epsilon, w^\epsilon)\) solution to (3), there exist \(k \in Y^*\), \(n \in \mathbb{N}^*\) and two complex numbers \(\xi_1\) and \(\xi_2\) such that \(\lambda^\epsilon = \lambda_n^k / \epsilon^2\) and

\[
w^\epsilon (x) = \xi_1 \phi_n^k \left( \frac{x}{\epsilon} \right) + \xi_2 \phi_n^{\bar{k}} \left( \frac{x}{\epsilon} \right) \quad \text{if } k \neq 0 \quad \text{and} \quad w^\epsilon (x) = \xi_1 \phi_0^k \left( \frac{x}{\epsilon} \right) + \xi_2 \phi_0^{\bar{k}} \left( \frac{x}{\epsilon} \right) \quad \text{otherwise}
\]

for \(\xi_1, \xi_2\) two numbers such that the boundary conditions (28), respectively (26), are satisfied for \(k = 0\), respectively for \(k \neq 0\). In the later case \(\xi_1\) and \(\xi_2\) are conjugate.

**Remark 12** (i) The case of non-constant coefficients \(u^k_n\) is used for approximations of the solution to the homogenized wave equation that may be derived from [8]. In such case \(k\) belongs to a finite subset \(L^*_K\) of \(Y^*\) made with values distant from \(1/K\) and including 0. We cannot expect that there always exists a pair \((k, n)\) such that \(u^k_n\) is a constant.

(ii) The case of non-constant coefficients \(u^k_n\) is also seen as a preparation to derive homogenized spectral problems in higher dimension where the boundary conditions constitute a more difficult problem and may require a more general solution than constant \(u^k_n\).

**Proof of Theorem 9**

**Proof.** The proof is based on Lemma [13] in Section 4.2 and on the macroscopic model derivation in Section 4.3. For a given \(k \in Y^*\), let \(w^\epsilon\) be solution of \((8)\) which is bounded in \(L^2(\Omega)\), the property (15) yields the uniform boundness of \(\|S^\epsilon_\sigma w^\epsilon\|_{L^2(\Omega \times Y)}\) for any \(\sigma \in I^k\). So there exist \(w^\sigma \in L^2(\Omega \times Y)\) such that up the extraction of a subsequence \(S^\epsilon_\sigma w^\epsilon \to w^\sigma\) in \(L^2(\Omega \times Y)\) weakly. Since \(\|S^\epsilon_\sigma (\varepsilon \partial_x w^\epsilon)\|_{L^2(\Omega \times Y)} = \|\partial_y S^\epsilon_\sigma w^\epsilon\|_{L^2(\Omega \times Y)}\) is uniformly bounded as \(\|\varepsilon \partial_x w^\epsilon\|_{L^2(\Omega)}\). Hence

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times Y} \partial_y S^\epsilon_\sigma w^\epsilon \cdot v dxdy = \lim_{\varepsilon \to 0} \int_{\Omega \times Y} -S^\epsilon_\sigma w^\epsilon \cdot \partial_y v dxdy = -\int_{\Omega \times Y} w^\sigma \cdot \partial_y v dxdy
\]

for all \(v \in L^2(\Omega; H^1_0(Y))\), and \(w^\sigma \in L^2(\Omega; H^1(Y))\) then

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times Y} \partial_y S^\epsilon_\sigma w^\epsilon \cdot v dxdy = \int_{\Omega \times Y} \partial_y w^\sigma \cdot v dxdy.
\]

Therefore \(S^\epsilon_\sigma w^\epsilon\) tends weakly to \(w^\sigma\) also in \(L^2(\Omega; H^1(Y))\). Hence, \(\sum_{\sigma \in I^k} S^\epsilon_\sigma w^\epsilon\) converges to

\[
g_k (x, y) = \sum_{\sigma \in I^k} w^\sigma (x, y).
\]
Using the decomposition (34) of $w^\sigma$ in Lemma 13, for $(\phi^\sigma_p)^{\sigma,p}$ the Bloch wave eigenmodes corresponding to $\lambda^0$,
\[
\begin{align*}
g_k(x, y) &= \sum_{\sigma \in I_k} u^\sigma_n(x) \phi^\sigma_n(y) \text{ for } k \neq 0, \\
g_0(x, y) &= \sum_{p \in \{n, m\}} u^0_p(x) \phi^0_p(y) \text{ for } k = 0.
\end{align*}
\]
Finally, $(u^\sigma_p)^{\sigma,p}$ is solution of the macroscopic problem as proved in Section 4.3. ■

4.2 Modal decomposition on the Bloch modes

**Lemma 13** For $(\lambda^\varepsilon, w^\varepsilon)$ solution of (8) and satisfying (10), for a fixed $k \in Y^*$ there exists at least a subsequence of $S_k^\varepsilon w^\varepsilon$ converging weakly towards non-vanishing function $w^k$ in $L^2(\Omega \times Y)$ when $\varepsilon$ tends to zero. If $w^k \in L^2(\Omega; H^2(Y))$ then $(\lambda^0, w^k)$ is solution of the Bloch wave equation (20) and $w^k$ admits the modal decomposition,
\[
w^k(x, y) = \sum_m u^k_m(x) \phi^k_m(y) \text{ for } u^k_m \in L^2(\Omega)
\]
where the sum is over all Bloch modes $\phi^k_m$ associated to $\lambda^0$. Moreover for $k \neq 0$ the two factors $u^k_m$ and $u^{-k}_m$ are conjugate.

**Proof.** The test functions of the weak formulation (11) are chosen as
\[
v^\varepsilon := \mathcal{R} v \in H^1_0(\Omega) \cap H^2(\Omega),
\]
with
\[
v \in H^1_0(\Omega; L^2(Y)) \cap L^2(\Omega; H^2_k(Y)) \cap H^2(\Omega; L^2_k(Y)).
\]
Applying two integrations by parts and the boundary conditions satisfied by $w^\varepsilon$ and by $\mathcal{R} v$, it remains
\[
\int_\Omega w^\varepsilon \cdot (P^\varepsilon - \lambda^\varepsilon \rho^\varepsilon) v^\varepsilon \, dx = 0.
\]
From (19) multiplied by $\varepsilon^2$ and (10),
\[
\int_\Omega w^\varepsilon \cdot \mathcal{R} ((P^2 - \lambda^0 \rho) v) \, dx = O(\varepsilon).
\]
Since $(P^2 - \lambda^0 \rho)v$ is $k$–quasi-periodic and $S_k^\varepsilon w^\varepsilon \rightarrow w^k$ in $L^2(\Omega \times Y)$ weakly, Corollary 4 allows to pass to the limit
\[
\int_{\Omega \times Y} w^k \cdot (P^2 - \lambda^0 \rho) v \, dx \, dy = 0,
\]
or equivalently
\[
\int_{\Omega \times Y} w^k \cdot \partial_y (a \partial_y v) + w^k \cdot \lambda^0 \rho v \, dx \, dy = 0.
\]
Using the assumption $w^k \in L^2(\Omega; H^2(Y))$ and applying integrations by parts,
\[
\int_{\Omega \times Y} \partial_y (a \partial_y w^k) \cdot v + w^k \cdot \lambda^0 \rho v \, dx \, dy + \int_{\Omega \times \partial Y} aw^k \cdot \partial_y v - a \partial_y w^k \cdot v \, dx \, dy = 0.
\]
Then, choosing test functions $v \in L^2(\Omega; H^1_0(Y))$ comes the strong form

$$- \partial_y (a \partial_y w^k) = \lambda^0 \rho w^k \text{ in } \Omega \times Y. \quad (39)$$

So, it remains

$$\int_{\Omega} [aw^k \cdot \partial_y v - a \partial_y w^k \cdot v]_0^1 \, dx = 0$$

for general test functions $v$, which implies that $w^k$ and $\partial_y w^k$ are $k-$quasi-periodic in the variable $y$.

As we know that $\lambda^0$ is an eigenvalue $\lambda^k_n$ of the Bloch wave spectrum, then $w^k$ is a Bloch eigenvector and is decomposed as

$$w^k(x, y) = \sum_m u_m^k(x) \phi_m^k(y) \text{ with } u_m^k \in L^2(\Omega)$$

the sum being over all Bloch modes $\phi_m^k$ associated to $\lambda^0$ where $u_m^k(x) = \int_Y w^k(x, y) \cdot \phi_m^k(y) \, dy$. For $k \neq 0$, $\phi_m^k = \overline{\phi_m^{-k}}$ and from Definition 1 of modulated two-scale transform, $S_k^\pm w^\varepsilon = \overline{S_{-k}^\mp w^\varepsilon}$ thus $u_m^k$ and $u_m^{-k}$ are conjugate i.e. $u_m^k = \overline{u_m^{-k}}$.

### 4.3 Derivation of the macroscopic equation

In the macroscopic model derivation, we distinguish between the two cases $k \neq 0$ and $k = 0$.

#### 4.3.1 Case $k \neq 0$

We consider $\lambda^0 = \lambda^k_n$ and the two conjugate eigenvectors $\phi_n^k$ and $\phi_n^{-k}$ discussed in Notation 5. We restart from the very weak formulation (37) with the test function

$$v^\varepsilon(x) := \Re(v^k + v^{-k}) \in H^1_0(\Omega) \cap H^2(\Omega). \quad (40)$$

Furthermore, we pose $v^\sigma(x, y) = \psi^\sigma(x) \phi_n^\sigma(y)$ with $\psi^\sigma \in H^2(\Omega)$ for $\sigma \in I^k$ and use the $\sigma-$quasi-periodicity of $\phi_n^\sigma$, i.e. $\phi_n^\sigma(\underline{x}) = \phi_n^\sigma(0) e^{2i\pi \underline{k} \cdot \underline{x}}$ at any $x \in \partial \Omega$. So the boundary condition in (40) is equivalent to

$$\psi^k(x) \phi_n^k(0) e^{2i\pi \underline{k} \cdot \underline{x}} + \psi^{-k}(x) \phi_n^{-k}(0) e^{-2i\pi \underline{k} \cdot \underline{x}} = 0 \text{ at any } x \in \partial \Omega.$$

Applying the relation (23),

$$\psi^k(x) \phi_n^k(0) e^{2i\pi \underline{k} \cdot \underline{x}} + \psi^{-k}(x) \phi_n^{-k}(0) e^{-2i\pi \underline{k} \cdot \underline{x}} = 0.$$

Since $x_{\underline{k} / \underline{\alpha}}^\underline{h} = 0$ at $x = 0$ and $x_{\underline{k} / \underline{\alpha}} = h_{\underline{k}}^\underline{h}$ at $x = \underline{\alpha}$ with $h_{\underline{k}}^\underline{h} \in \mathbb{Z}$ then $e^{\pm 2i\pi x_{\underline{k} / \underline{\alpha}}^\underline{h}} = 1$. From (24), $e^{\pm 2i\pi x_{\underline{k} / \underline{\alpha}}^\underline{h}} \rightarrow e^{\pm 2i\pi \underline{k} \cdot \underline{x}}$ when $\varepsilon \rightarrow 0$. Passing to the limit, the boundary conditions of the test function are

$$\psi^k(x) \phi_n^k(0) e^{2i\pi \underline{k} \cdot \underline{x}} + \psi^{-k}(x) \phi_n^{-k}(0) e^{-2i\pi \underline{k} \cdot \underline{x}} = 0 \text{ on } \partial \Omega. \quad (41)$$

From (19) multiplied by $\varepsilon$, (10) and $P^2 v^\sigma - \lambda^0 \rho v^\sigma = 0$,

$$\sum_{\sigma \in I^k} \int_{\Omega} w^\varepsilon \cdot \Re(-P^1 v^\sigma + \lambda^1 \rho v^\sigma) \, dx = O(\varepsilon). \quad (42)$$
Extracting a subsequence of $w^\varepsilon$ so that $S^\varepsilon_k w^\varepsilon$ and $S_{-k}^\varepsilon w^\varepsilon$ are converging to $w^k$ and $-w^{-k}$ in $L^2(\Omega \times Y)$ weak, since $-P^1 v^\sigma + \lambda^1 \rho v^\sigma$ is $\sigma-$quasi-periodic then Corollary 3 infers that
\[
\sum_{\sigma \in I^k} \int_{\Omega \times Y} w^\sigma \cdot (-P^1 v^\sigma + \lambda^1 \rho v^\sigma) \, dxdy = 0,
\]
i.e.
\[
\sum_{\sigma \in I^k} \int_{\Omega \times Y} w^\sigma \cdot (\partial_x (a\partial_y v^\sigma) + \partial_y (a\partial_x v^\sigma) + \lambda^1 \rho v^\sigma) \, dxdy = 0.
\]
This is the very weak form of the macroscopic equation for all test functions $v^\sigma \in H^1(\Omega; H^1_k(Y))$, reached by density, satisfying (41). Now, we derive the strong formulation. We assume that $w^\sigma \in H^1(\Omega; L^2(Y))$, since $w^\sigma \in L^2(\Omega; H^1(Y))$ after two integrations by parts,
\[
\sum_{\sigma \in I^k} \left[ \int_{\Omega \times Y} \partial_y (a\partial_x w^\sigma) \cdot v^\sigma + \partial_x (a\partial_y w^\sigma) \cdot v^\sigma + \lambda^1 \rho w^\sigma \cdot v^\sigma 
\right. 
\left. \, dxdy 
\right. 
\left. + \int_{\partial_t \times Y} w^\sigma \cdot a\partial_y v^\sigma - a\partial_y w^\sigma \cdot v^\sigma 
\right. 
\left. \, dxdy 
\right. 
\left. + \int_{\Omega \times \partial Y} w^\sigma \cdot a\partial_x v^\sigma - a\partial_x w^\sigma \cdot v^\sigma 
\right. 
\left. \, dxdy \right] = 0.
\]
From Lemma 13, $w^\sigma$ is solution to the Bloch mode equation and is decomposed as
\[
w^\sigma(x, y) = u^\sigma(x) \phi_n^\sigma(y). \tag{43}
\]
After replacement,
\[
\sum_{\sigma} \left[ \int_Y \partial_y (a\phi_n^\sigma) \cdot \phi_n^\sigma + a\partial_y \phi_n^\sigma \cdot \phi_n^\sigma \, dy \int_{\Omega} \partial_x u^\sigma \cdot \psi^\sigma \, dx + \lambda^1 \int_Y \rho \phi_n^\sigma \cdot \phi_n^\sigma \, dy \int_{\Omega} u^\sigma \cdot \psi^\sigma \, dx \right. 
\left. + \int_Y \phi_n^\sigma \cdot a\partial_y \phi_n^\sigma - a\partial_y \phi_n^\sigma \cdot \phi_n^\sigma \, dy \int_{\partial \Omega} u^\sigma \cdot \psi^\sigma \, dx 
\right. 
\left. + \int_{\partial Y} \phi_n^\sigma \cdot a\phi_n^\sigma \, dy \int_{\Omega} u^\sigma \cdot \partial_x \psi^\sigma - \partial_x u^\sigma \cdot \psi^\sigma \, dx \right] = 0.
\]
Let us recall that $b(\cdot, \cdot, \cdot)$ and $c(\cdot, \cdot, \cdot)$ have been defined in (22). For the sake of simplicity, we use $c(\sigma, n) := c(\sigma, n, n)$ and $b(\sigma, n) := b(\sigma, n, n)$ and observe that
\[
\int_{\Omega} \partial_y (a\phi_n^\sigma) \cdot \phi_n^\sigma + a\partial_y \phi_n^\sigma \cdot \phi_n^\sigma \, dy = c(\sigma, n),
\]
which results from integrations by parts and from the $\sigma-$quasi-periodicity of $\phi_n^\sigma$. So, using the $\sigma$-quasi-periodicity of $\phi_n^\sigma$, (44) can be rewritten as
\[
\sum_{\sigma} \left[ \int_{\Omega} (c(\sigma, n) \partial_x u^\sigma + \lambda^1 b(\sigma, n) u^\sigma) \cdot \psi^\sigma \, dx - c(\sigma, n) \int_{\partial \Omega} u^\sigma \cdot \psi^\sigma \, dx \right] = 0.
\]
Choosing the test function $\psi^\sigma = 0$ on $\partial \Omega$, the boundary condition (41) is satisfied and by density of $H^1_0(\Omega)$ in $L^2(\Omega)$, the internal equation satisfied by $u^\sigma$ follows,
\[
c(\sigma, n) \partial_x u^\sigma + \lambda^1 b(\sigma, n) u^\sigma = 0 \text{ in } \Omega \text{ for each } \sigma. \tag{45}
\]
Choosing general $\psi^\sigma \in H^1(\Omega)$ satisfying (41) yields the boundary conditions

$$\sum_{\sigma} c(\sigma, n)u^\sigma \psi^\sigma = 0 \text{ on } \partial \Omega. \quad (46)$$

We introduce the matrices $C_1 = \text{diag}((c(\sigma, n))_\sigma)$, $C_2 = \text{diag}((b(\sigma, n))_\sigma)$ and the vectors $u = (u^\sigma)_\sigma$, $\psi = (\psi^\sigma)_\sigma$, $\varphi = \left(\phi_n(0)e^{\text{sign}(\sigma)2\pi ikx}\right)_\sigma$ with $\sigma \in I^k$, so that (41, 45, 46) can be written on the matrix form

$$C_1 \partial_x u + \lambda^1 C_2 u = 0 \quad \text{in } \Omega,$$

and $C_1 u(x) \bar{\psi}(x) = 0$ on $\partial \Omega$ for all $\psi$ such that $\varphi(x, 0) \bar{\psi}(x) = 0$ on $\partial \Omega$.

The boundary condition is equivalent to $C_1 u(x)$ is collinear with $\varphi(x, 0)$ i.e. $\det(C_1 u(x), \varphi(x, 0)) = 0$.

Equivalently

$$\begin{cases}
  c(k, n) u^k(0) \phi_n^k(0) - c(-k, n) u^{-k}(0) \phi_n^{k*}(0) = 0, \\
  c(k, n) u^k(\alpha) \phi_n^k(0) e^{-2\pi i k \alpha} - c(-k, n) u^{-k}(\alpha) \phi_n^{k*}(0) e^{2\pi i k \alpha} = 0.
\end{cases}$$

Finally, since $c(k, n) = -c(-k, n)$ and $c(k, n)$ is assumed to do not vanish, the boundary conditions of macroscopic equation (15) are

$$u^k(x) \phi_n^k(0) e^{2\pi i k \alpha} + u^{-k}(x) \phi_n^{-k}(0) e^{-2\pi i k \alpha} = 0 \text{ at } x \in \partial \Omega.$$  

4.3.2 Case $k = 0$

In case $k = 0$, to avoid any confusion with $\lambda^0$, the upper indices $k = 0$ are removed. We denote by $\phi_n, \phi_m$ the eigenvectors associated to $\lambda^0 = \lambda_n = \lambda_m$, solutions to $P(0)$ in (20), and by $\sum_p, \sum_q$ the sums over $p$ or $q$ varying in $\{n, m\}$. We restart with a test function

$$\psi^\sigma(x) := \Re\left(\sum_p \psi_p(x) \phi_p(0)\right) \in H^1_0(\Omega) \cap H^2(\Omega) \quad (47)$$

for the very weak formulation (12). We pose $\psi_p(x) = \psi_p(x) \phi_p(y)$ with $\psi_p(x) \in H^1(\Omega)$ for $p \in \{n, m\}$. Since $\phi_p$ is periodic thus $\phi_p(\frac{x}{\epsilon}) = \phi_p(0)$ at $x \in \partial \Omega$ and the boundary condition in (47) is equivalent to

$$\sum_p \psi_p(x) \phi_p(0) = 0 \text{ at } x \in \partial \Omega.$$  

By setting $c(p, q) := c(0, p, q)$ for $p, q \in \{n, m\}$, using the expression in Lemma 13 of the weak limit $w^0$ of $S_\epsilon^w$,

$$w^0(x, y) = \sum_p u_p(x) \phi_p(y), \quad (48)$$

using the periodicity of $(\phi_p)_p$ and conducting the same calculations as for $k \neq 0$, we obtain

$$\sum_{p, q} \left[ \int_\Omega (c(p, q) \partial_x u_p + \lambda^1 b(p, q) u_q) \cdot \psi_p \, dx - \int_{\partial \Omega} c(p, q) u_q \cdot \psi_p \, dx \right] = 0.$$

With $u = (u_p)_p$, $\psi = (\psi_p)_p$, $\phi = (\phi_p)_p$ and $C_1 = (c(p, q))_{p, q}$, $C_2 = (b(p, q))_{p, q}$, the macroscopic problem turns to be

$$C_1 \partial_x u + \lambda^1 C_2 u = 0 \text{ in } \Omega, \quad (49)$$
with the boundary conditions
\[ C_1 u(x) \psi(x) = 0 \text{ on } \partial \Omega \text{ for all } \psi \text{ such that } \psi(x) \phi(0) = 0 \text{ on } \partial \Omega. \]

Equivalently, \( C_1 u(x) \) is collinear to \( \phi(0) \) on \( \partial \Omega \) or
\[
\det (C_1 u(x), \phi(0)) = 0 \text{ on } \partial \Omega. \tag{50}
\]

But \( c(p, p) = 0 \), so (50) simplifies to
\[
\begin{align*}
&\left\{ c(n, m) u_m(0) \phi_m(0) - c(m, n) u_n(0) \phi_n(0) = 0, \\
&c(n, m) u_m(\alpha) \phi_m(0) - c(m, n) u_n(\alpha) \phi_n(0) = 0.
\end{align*}
\]

Finally, since \( c(n, m) = -c(m, n) \) and \( c(n, m) \neq 0 \), the boundary conditions are
\[ u_n(x) \phi_n(0) + u_m(x) \phi_m(0) = 0 \text{ on } \partial \Omega. \]

4.4 Analytic solutions

For \( k \in Y^* \) and \( \rho = 1 \), we solve the macroscopic equations in Section 4.4.1. These solutions are used to validate the numerical results in the final Section. Moreover, in Section 4.4.2 the exact formulations of the two-scale eigenmodes are found for \( \rho = 1 \) and \( a = 1 \).

4.4.1 The case \( \rho = 1 \)

For \( k \neq 0 \) and \( b(n, n) = 1 \), the exact solutions of the macroscopic equation (25) are
\[ u_n^\sigma(x) = d^\sigma e^{-\lambda_1 c(\sigma, n)^{-1} x} \text{ for each } \sigma \in \mathcal{I}^k \]
where \( d^\sigma \) is any complex number. Applying the boundary condition (26) and assuming that \( \phi_n^k(0) \neq 0 \), the eigenvalue is
\[ \lambda_1 = \frac{c(k, n)}{\alpha} (2i\pi k - i\ell \pi) \quad \text{for } \ell \in \mathbb{Z}. \tag{51} \]

Furthermore, \( u_n^k = \overline{u_n^{-k}} \) and \( \phi_n^k(0) = \overline{\phi_n^{-k}(0)} \) then \( \text{Re}(d_k \phi_n^k(0)) = 0 \), or \( d_k \phi_n^k(0) = i\delta \) for any \( \delta \in \mathbb{R} \). Thus,
\[ d_k = \frac{i\delta}{\phi_n^k(0)} \text{ and } d^{-k} = -\frac{i\delta}{\phi^{-k}_n(0)} \text{ for any } \delta \in \mathbb{R}. \]

For \( k = 0 \), using the equalities \( c(n, n) = c(m, m) = 0 \), \( b(n, m) = b(m, n) = 0 \) and \( b(n, n) = b(m, m) = 1 \), the macroscopic equation (27) is rewritten
\[
\begin{align*}
&\left\{ c(n, m) \partial_x u_n^0 + \lambda_1 u_n^0 = 0 \text{ in } \Omega, \\
&c(m, n) \partial_x u_m^0 + \lambda_1 u_m^0 = 0 \text{ in } \Omega.
\end{align*} \tag{52}
\]

If \( \lambda_1 = 0 \), \( \partial_x u_n^0 = 0 \) and \( \partial_x u_m^0 = 0 \) in \( \Omega \), then \( u_n^0 \) and \( u_m^0 \) are independent on \( x \), equivalently, \( u_n^0 \) and \( u_m^0 \) are complex numbers.

If \( \lambda_1 \neq 0 \), the first equation gives \( u_n^0 = -\frac{c(n, m) \partial_x u_m^0}{\lambda_1} \) in \( \Omega \) and since \( c(n, m) = -c(m, n) \) then
\[ \partial_{xx} u_m^0 = -\left( \frac{\lambda_1}{c(n, m)} \right)^2 u_m^0 \tag{53} \]
and

\[ u_0^m(x) = d_1 \cos \left( \frac{\lambda^1}{c(n,m)x} \right) + d_2 \sin \left( \frac{\lambda^1}{c(n,m)x} \right) \]

for two constants for \( d_1, d_2 \in \mathbb{C} \) and \( u_0^m \) follows by its above expression. Applying the boundary condition \((25)\), if \( \phi_m^0(0) \neq 0 \),

\[ \lambda^1 = \frac{\ell \pi c(n,m)}{\alpha} \quad \text{for } \ell \in \mathbb{Z} \quad \text{and} \quad d_1 = -d_2 \frac{\phi_m^0(0)}{\phi_m^0(0)} \]  \hfill (54)

for any \( \ell \in \mathbb{Z} \) and \( d_2 \in \mathbb{C} \). If \( \phi_m^0(0) = 0 \) then \( \phi_n^0(0) = 0 \) or \( u_0^m(x) = 0 \) on \( \partial \Omega \). In the case \( \phi_m^0(0) = 0 \), the macroscopic equation is lacking of boundary conditions and their solutions are not unique, they depend on arbitrary coefficients \( d_1, d_2 \) and \( \lambda^1 \). When \( u_0^m(x) = 0 \) at \( \partial \Omega \), there is an alternative, or \( u_0^m \) is the trivial solution or

\[ \det \begin{pmatrix} -\sin \left( \frac{\lambda^1}{c(n,m)} \alpha \right) & \cos \left( \frac{\lambda^1}{c(n,m)} \alpha \right) \\ \end{pmatrix} = 0 \]

and then \( d_2 = 0, \lambda^1 = \frac{\ell \pi c(n,m)}{\alpha} \) for any \( \ell \in \mathbb{Z} \) and \( d_1 \in \mathbb{C} \).

**Remark 14** According to \((57)\) and \((54)\), \( \lambda^1 = 0 \) iff \( \ell = 2l^k \) for \( k \neq 0 \) and \( \ell = 0 \) otherwise. So, in any case small values of \( \lambda^1 \) correspond to indices \( \ell \) in a vicinity of \( 2l^k \) or to \( \frac{2k\alpha}{\varepsilon} \) when \( \varepsilon > 0 \).

### 4.4.2 The case \( \rho = 1 \)

We consider the spectral problem

\[ -\partial^2_{yy} \phi^k = \lambda^k \phi^k \quad \text{in } Y \]

with the \( k \)-quasi-periodicity conditions.

For \( k \neq 0 \), for a mapping \( m \mapsto n(m) \) from \( \mathbb{Z} \) to \( \mathbb{N} \) not detailed here, \( \lambda^{k}_{n(m)} = 4\pi^2(m+k)^2 \) and there are exactly two conjugated solutions \( \phi^\sigma_{n(m)}(y) = e^{i\text{sign}(\sigma)2i\pi(m+k)y} \) for any \( m \in \mathbb{Z} \) and \( \sigma \in I_k \). It follows that \( c(\sigma, n(m)) = \text{sign}(\sigma)4i\pi(m+k), \) \( b(\sigma, n(m)) = 1 \) and \( \lambda^1 = -4\pi^2 \alpha (2l^k - \ell)(m+k) \) for any \( \ell \in \mathbb{Z} \), so

\[ u^\sigma_{n(m)}(x) = d^\sigma e^{i\text{sign}(\sigma)\pi(2l^k-\ell)x} \]

and the resulting two-scale eigenmode is

\[ w(\sigma, x, y) = d^\sigma e^{i\text{sign}(\sigma)\pi(2l^k-\ell)x} e^{i\text{sign}(\sigma)2i\pi(n+k)y}. \]

For \( k = 0 \), for each \( \lambda^{0}_{n(m)} = (2\pi m)^2 \) there are two eigenvectors \( \phi_{n(m)}(y) = \cos(2\pi my) \) and \( \phi_{n(m)+1}(y) = \sin(2\pi my) \) so

\[ C_1 = 2m\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \phi_{n(m)}(0) \\ \phi_{n(m)+1}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

It implies that \( \lambda^1 = \frac{4m\pi^2}{\alpha} \) for any \( \ell \in \mathbb{Z} \) and

\[ u_{n(m)}(x) = d_0 \sin \left( \frac{\ell \pi x}{\alpha} \right) \quad \text{and} \quad u_{n(m)+1}(x) = d_0 \cos \left( \frac{\ell \pi x}{\alpha} \right), \]

then the two-scale eigenmode is

\[ w(x, y) = d_0 [\sin \left( \frac{\ell \pi x}{\alpha} \right) \cos(2\pi my) + \cos(\ell \pi x) \sin(2\pi my)] \quad \text{for } \ell, m \in \mathbb{Z}. \]
4.5 Neumann boundary conditions

We consider the spectral problem with Neumann boundary conditions

\[ P^\varepsilon w^\varepsilon = \lambda^\varepsilon \rho^\varepsilon w^\varepsilon \quad \text{in} \quad \Omega \quad \text{and} \quad \partial_x w^\varepsilon = 0 \quad \text{on} \quad \partial \Omega. \]

The process of homogenization and the results are similar to the case of Dirichlet boundary conditions. The microscopic problem and the internal macroscopic equation are unchanged while the boundary conditions of the latter are

\[ \sum_{\sigma \in I_k} \sum_m u^\sigma_m (x) \partial_y \phi^\sigma_m (0) e^{\text{sign}(\sigma) 2 \pi \frac{k_x}{\alpha}} = 0 \quad \text{on} \quad \partial \Omega. \]

where the cases \( k \neq 0 \) and \( k = 0 \) are not separated so a general notation is adopted for the sum over \( m \) and \( \sigma \). Their derivation follows the same steps, so we only mention the boundary condition satisfied by the test functions. They are chosen to satisfy \( \partial_x v^\varepsilon (x) = 0 \) on \( \partial \Omega \) or equivalently,

\[ \sum_{\sigma \in I_k} \sum_m \partial_x \psi^\sigma_m (x) \phi^\sigma_m \left( \frac{x}{\varepsilon} \right) + \frac{1}{\varepsilon} \psi^\sigma_m (x) \partial_y \phi^\sigma_m \left( \frac{x}{\varepsilon} \right) = 0 \quad \text{on} \quad \partial \Omega. \]

Multiplying by \( \varepsilon \),

\[ \sum_{\sigma \in I_k} \sum_m \psi^\sigma_m (x) \partial_y \phi^\sigma_m \left( \frac{x}{\varepsilon} \right) + O(\varepsilon) = 0 \quad \text{on} \quad \partial \Omega, \quad (55) \]

then using the \( \sigma \)-quasi-periodicity of \( \phi^\sigma_m \) and passing to the limit

\[ \sum_{\sigma \in I_k} \sum_m \psi^\sigma_m (x) \partial_y \phi^\sigma_m (0) e^{\text{sign}(\sigma) 2 \pi \frac{k_x}{\alpha}} = 0 \quad \text{on} \quad \partial \Omega. \]

5 Homogenization based on a first order formulation

In this section, the homogenized model is derived based on a first order formulation. The calculations are less detailed than in Section 4, only the main results and the proof principles are given.

5.1 Reformulation of the spectral problem and the main result

We start by setting

\[ U^\varepsilon = \left( \frac{\sqrt{a^\varepsilon} \partial_x w^\varepsilon}{i \sqrt{\lambda^\varepsilon}}, \sqrt{\rho^\varepsilon} w^\varepsilon \right), \quad \mu^\varepsilon = \sqrt{\lambda^\varepsilon}, \]

\[ A^\varepsilon = \begin{pmatrix} 0 & \sqrt{a^\varepsilon} \partial_x \left( \frac{1}{\sqrt{\rho^\varepsilon}} \right) \\ -\frac{1}{\sqrt{\rho^\varepsilon}} \partial_x \left( \sqrt{a^\varepsilon} \right) & 0 \end{pmatrix}, \quad n_{A^\varepsilon} = \begin{pmatrix} 0 & \sqrt{a^\varepsilon} n_\Omega \\ \sqrt{a^\varepsilon} n_\Omega & 0 \end{pmatrix} \]

with the domain of the operator \( A^\varepsilon \),

\[ D (A^\varepsilon) := \{ (\varphi, \phi) \in L^2 (\Omega) \times L^2 (\Omega) \mid \sqrt{a^\varepsilon} \varphi \in H^1 (\Omega), \phi \in H^1_0 (\Omega) \} \subset L^2 (\Omega)^2, \]

so that \( i A^\varepsilon \) is self-adjoint on \( L^2 (\Omega)^2 \) as proved in [8]. The spectral equation [8] can be recasted as a first-order system

\[ A^\varepsilon U^\varepsilon = i \mu^\varepsilon U^\varepsilon \quad \text{in} \quad \Omega \quad \text{and} \quad U^\varepsilon_2 = 0 \quad \text{on} \quad \partial \Omega, \quad (56) \]
where $U_2^\varepsilon$ is the second component of $U^\varepsilon$. We observe that $\| \sqrt{\rho^\varepsilon} w^\varepsilon \|_{L^2(\Omega)} \leq \| \sqrt{\rho^\varepsilon} \|_{L^\infty(\Omega)}$ and that $\left\| \frac{\sqrt{\rho^\varepsilon} \partial y^\varepsilon}{i\sqrt{\lambda^\varepsilon}} \right\|_{L^2(\Omega)} \leq M_0$ can be deduced from the weak formulation (11), therefore $U^\varepsilon$ is uniformly bounded,

$$\|U^\varepsilon\|_{L^2(\Omega)}^2 \leq M_1.$$  \tag{57}$$

We start our analysis from the system expressed in a distributional sense,

$$\int_\Omega U^\varepsilon \cdot (i\mu^\varepsilon - A^\varepsilon) \Psi \, dx = 0,$$  \tag{58}$$

for all admissible test functions $\Psi = (\varphi, \psi) \in H^1(\Omega) \times H^1_0(\Omega)$. We choose $\mu_0 = \sqrt{\lambda^0}$ and $\mu_1 = \frac{\lambda^1}{2\mu_0}$, so $\mu^\varepsilon$ can be decomposed as

$$\mu^\varepsilon = \frac{\mu_0}{\varepsilon} + \mu_1 + O(\varepsilon).$$  \tag{59}$$

The asymptotic spectral problem (20) is also restated as a first order system by setting

$$A_k := \begin{pmatrix} 0 & \sqrt{a^\varepsilon} \partial_y (\sqrt{\rho^\varepsilon}) \\ \frac{1}{\sqrt{\rho}} \partial_y (\sqrt{a^\varepsilon}) & 0 \end{pmatrix} \quad \text{and} \quad n_{A_k} = \frac{1}{\sqrt{\rho}} \begin{pmatrix} 0 & \sqrt{a^\varepsilon} n_Y \\ \sqrt{a^\varepsilon} n_Y & 0 \end{pmatrix},$$

and

$$e_n^k := \frac{1}{\sqrt{2}} \begin{pmatrix} -i \frac{s_n}{\sqrt{\lambda_n^k}} \sqrt{a^\varepsilon} \partial_y (\phi_n^k) \\ \sqrt{\rho} \phi_n^k \end{pmatrix} \quad \text{and} \quad \mu_n^k = s_n \sqrt{\lambda_n^k} \quad \text{for all} \quad n \in \mathbb{Z}^*,$$  \tag{60}$$

$s_n$ denoting the sign of $n$. As proved in [8], $iA_k$ is self-adjoint on the domain

$$D(A_k) := \left\{ (\varphi, \phi) \in L^2(Y)^2 \left| \sqrt{a^\varepsilon} \varphi \in H^1_k(Y), \frac{\phi}{\sqrt{\rho}} \in H^1_k(Y) \right. \right\} \subset L^2(Y)^2.$$

The Bloch wave spectral problem $\mathcal{P}(k)$ is equivalent to finding pairs $(\mu_n^k, e_n^k)$ indexed by $n \in \mathbb{Z}^*$ solution to

$$Q(k) : A_k e_n^k = i\mu_n^k e_n^k \quad \text{in} \quad Y \quad \text{with} \quad e_n^k \in H^1_k(Y)^2.$$  \tag{61}$$

The corresponding weak formulation is

$$\int_Y e_n^k \cdot (A_k - i\mu_n^k) \Psi \, dy = 0 \quad \text{for all} \quad \Psi \in D(A_k).$$  \tag{62}$$

The relation between the operator $A^\varepsilon$ and the scaled operator $A_k$ is obtained by considering any regular vector $\psi = \psi(x, y)$ depending on both space scales,

$$A^\varepsilon \left( \psi \left( x, \frac{x}{\varepsilon} \right) \right) = \left( \frac{1}{\varepsilon} A_k + B \right) \psi \left( x, \frac{x}{\varepsilon} \right),$$  \tag{63}$$

where the operator $B$ is defined as the result of the formal substitution of $x-$derivatives by $y-$derivatives in $A_k$, i.e.

$$B := \begin{pmatrix} 0 & \sqrt{a^\varepsilon} \partial_x \left( \frac{1}{\sqrt{\rho^\varepsilon}} \right) \\ \frac{1}{\sqrt{\rho}} \partial_x (\sqrt{a^\varepsilon}) & 0 \end{pmatrix}.$$
Remark 15 From now on, we shall assume that the weak limit of $S^\varepsilon_k U^\varepsilon$ in $L^2(\Omega \times Y)$ is not vanishing to avoid eigenmodes related to the boundary spectrum (see Proposition 7.7 in [6]).

Theorem 16 For $k \in Y^\ast$, let $(\mu^\varepsilon, U^\varepsilon)$ be solution of (56) then $\sum_{\sigma \in I_k} S^\varepsilon_\sigma U^\varepsilon$ is bounded in $L^2(\Omega \times Y)$. For $\varepsilon \in E_k$, assuming that the renormalized sequence $\varepsilon \mu^\varepsilon$ satisfies the decomposition (57) with $\mu_0 = \mu^k_n$ an eigenvalue of the Bloch wave spectrum, any weak limit $G_k$ of $\sum_{\sigma \in I_k} S^\varepsilon_\sigma U^\varepsilon$ in $L^2(\Omega \times Y)$ has the form

$$G_k(x, y) = \sum_{\sigma \in I_k} \sum_{m \in M^\sigma_n} u^\sigma_m(x) e^\sigma_m(y),$$

where $(u^\sigma_m)_{m, \sigma}$ are the solutions of the macroscopic equations (23-26) or (27-28).

Therefore, the physical solution $U^\varepsilon$ can be approximated by

$$U^\varepsilon(x) \approx \sum_{\sigma \in I_k} \sum_{m \in M^\sigma_n} u^\sigma_m(x) e^\sigma_m\left(\frac{x}{\varepsilon}\right).$$

Proof. For a given $k \in Y^\ast$, let $U^\varepsilon$ be solution of (56) which is bounded in $L^2(\Omega)$, the property (15) yields the boundness of $\|S^\varepsilon_\sigma U^\varepsilon\|_{L^2(\Omega \times Y)}$. So there exist $U^\sigma \in L^2(\Omega \times Y)^2$ such that, up the extraction of a subsequence, $S^\varepsilon_\sigma U^\varepsilon$ tends weakly to $U^\sigma$ in $L^2(\Omega \times Y)^2$ and hence, $\sum_{\sigma \in I_k} S^\varepsilon_\sigma U^\varepsilon$ converges to $G_k(x, y) = \sum_{\sigma \in I_k} U^\sigma(x, y)$. Using the decomposition (66) of $U^\sigma$ in the forthcoming Lemma 17

$$G_k(x, y) = \sum_{\sigma \in I_k} \sum_{m \in M^\sigma_n} u^\sigma_m(x) e^\sigma_m(y)$$

The macroscopic problem solved by the coefficients $(u^\sigma_m)_{\sigma, m}$ is derived in Section 5.2.2. ■

5.2 Model derivation

5.2.1 Modal decomposition on the Bloch modes

Lemma 17 Let a sequence $(\mu^\varepsilon, U^\varepsilon)$ be solution of (56) and satisfies (59) with $\mu_0 = \mu^k_n$ for given $n \in \mathbb{Z}^\ast$ and $k \in Y^\ast$, we extract a subsequence of $\varepsilon$, still denoted by $\varepsilon$, such that $S^\varepsilon_k U^\varepsilon$ converges weakly to $U^k$ in $L^2(\Omega \times Y)^2$. If $U^k \in D(A_k)$ then $(\mu^k_n, U^k)$ is solution of the Bloch wave equation (67) and $U^k$ admits the modal decomposition

$$U^k(x, y) = \sum_{m \in M^k_h} u^k_m(x) e^k_m(y) \text{ with } u^k_m \in L^2(\Omega).$$

Proof. For each $k \in Y^\ast$, taking $\Psi(x, y) := \theta(x)\phi(y)$ with $\theta(x) \in C^\infty_c(\Omega)$ and $\phi(y) \in C^\infty(Y)^2$ $k-$quasi-periodic in $y$, considering $\Re \Psi$ as a test functions in (58), and using (68-69),

$$\int_{\Omega} U^\varepsilon \cdot \Re \left(i \frac{\mu_0}{\varepsilon} + i \mu_1 - A_k - B\right) \Psi \; dx + O(\varepsilon) = 0.$$
and passing to the limit thanks to Corollary 4
\[ \frac{1}{|Y|} \int_{\Omega \times Y} U^k \cdot (i\mu_0 - A_k) \Psi \, dx \, dy = 0 \]
which is the weak formulation of the Bloch wave equations. If in addition \( U^k \in D(A_k) \), integrating by parts yields
\[ \frac{1}{|Y|} \int_{\Omega \times Y} (A_k - i\mu_0) U^k \cdot \Psi \, dx \, dy - \frac{1}{|Y|} \int_{\Omega \times \partial Y} U^k \cdot n_{A_k} \Psi \, dx \, dy = 0 \quad (67) \]
providing in turn the strong formulation,
\[ A_k U^k = i\mu_0 U^k \text{ in } \Omega \times Y. \quad (68) \]
Since the product of a periodic function by a \( k \)-quasi-periodic function is \( k \)-quasi-periodic then \( n_{A_k} \Psi \) is \( k \)-quasi-periodic in \( y \). Therefore, \( U^k \) is \( k \)-quasi-periodic in \( y \) and finally is a Bloch eigenvector in \( y \). By projection, it can be decomposed as
\[ U^k(x, y) = \sum_{m \in M^k_n} u^k_m(x) e^k_m(y) \text{ with } u^k_m = \frac{1}{b(k, m, m)} \int_{Y} U^k \cdot e^k_m \, dy \in L^2(\Omega). \]

5.2.2 Derivation of the macroscopic equation

The macroscopic equation is stated for each \( k \in Y^* \) and each eigenvalue \( \mu_n^k \) of the Bloch wave spectral problem \( Q(k) \). We pose
\[ \kappa(k, n, m) = -\frac{ic(k, n, m)}{2\mu_0} \text{ for } m \in M^k_n \quad (69) \]
where \( c(k, n, m) \) is defined in (22) and notice that
\[ \kappa(k, n, m) = -\kappa(-k, m, n), \quad \kappa(k, n, m) = -\kappa(-k, n, m), \quad \kappa(k, m, n) = -\kappa(k, m, n), \quad \kappa(0, n, n) = 0. \]
For the sake of simplicity, we do the proof for \( n \in \mathbb{Z}^{+} \) only and denote by \( \kappa(k, n) = \kappa(k, n, n) \) and \( \kappa(n, m) = \kappa(0, n, m) \). For general \( n \), the proof is the same but \( \phi^k_n \) is replaced by \( \phi^k_{|n|} \).

Case \( k \neq 0 \) The pairs \( (\mu_n^k, e_n^k) \) and \( (\mu_n^{-k}, e_n^{-k}) \) are the eigenmodes of the spectral equations \( Q(\pm k) \) in (61) corresponding to the eigenvalue \( \mu_0 = \mu_n^k = \mu_n^{-k} \). We pose \( \Psi^\varepsilon = \Re(\Psi^k + \Psi^{-k}) \in H^1(\Omega) \times H^1_0(\Omega) \) as a test function in the weak formulation (58), with each \( \Psi^\sigma(x, y) = \psi^\sigma(x) e^\sigma_n(y) \) where \( \psi^\sigma \in H^1(\Omega) \) and satisfies the boundary conditions,
\[ \sum_{\sigma} \psi^\sigma(x) \phi^\sigma_n \left( \frac{x}{\varepsilon} \right) = 0 \text{ on } \partial\Omega. \]
Notice that this condition is related to the second component of \( \Psi^\varepsilon \) only. Proceeding as in Section 4.3.4 yields (41). Since \( (i\mu_0 - A_n) \Psi^\varepsilon = 0 \) for all \( \sigma \), applying (58, 59), then Equation (58) yields
\[ \sum_{\sigma} \int_{\Omega} U^\varepsilon \cdot \Re(i\mu_1 - B) \Psi^\sigma \, dx + O(\varepsilon) = 0. \quad (70) \]
But \((i \mu_1 - B) \Psi^\sigma\) is \(\sigma\)-quasi-periodic so passing to the limit thanks to Corollary 4.

\[
\frac{1}{|Y|} \sum_\sigma \int_{\Omega \times Y} U^\sigma \cdot (i \mu_1 - B) \Psi^\sigma \, dx \, dy = 0.
\] (71)

From Lemma 17, \(U^\sigma\) is decomposed as

\[
U^\sigma (x, y) = u_n^\sigma (x) e_n^\sigma (y).
\]

After replacement,

\[
\sum_\sigma \int_\Omega (-i \mu_1 b(\sigma, n) u_n^\sigma \cdot \psi^\sigma + \kappa(\sigma, n) u_n^\sigma \cdot \partial_x \psi^\sigma) \, dx = 0
\]

for all \(\psi^\sigma \in H^1(\Omega)\) fulfilling (41). Moreover, if \(u_n^\sigma \in H^1(\Omega)\) it satisfies the strong form of the internal equations

\[
\kappa(\sigma, n) \partial_x u_n^\sigma - i \mu_1 b(\sigma, n) u_n^\sigma = 0 \quad \text{in} \quad \Omega \quad \text{for all} \quad \sigma \in I^k,
\] (72)

and the boundary conditions

\[
\sum_\sigma \kappa(\sigma, n) u_n^\sigma \psi^\sigma = 0 \quad \text{on} \quad \partial \Omega.
\]

Following the same calculations as in Section 4.3.1, with the matrices \(C_1 = \text{diag} (\kappa(\sigma, n))\), \(C_2 = \text{diag} (b(\sigma, n))\) and the vectors \(u = (u_n^\sigma)_\sigma\), \(\psi = (\psi^\sigma)_\sigma\), \(\varphi = \left(\phi^\sigma (0) e^{\text{sign}(\sigma) \text{sign} x \frac{\alpha}{k}}\right)_\sigma\), (72) is written on the matrix form

\[
C_1 \partial_x u = i \mu_1 C_2 u \quad \text{in} \quad \Omega,
\]

with boundary condition

\[
C_1 u (x) . \overline{\psi} (x) = 0 \quad \text{on} \quad \partial \Omega \quad \text{for all} \quad \psi \quad \text{such that} \quad \overline{\varphi} (x, 0) . \overline{\psi} (x) = 0 \quad \text{on} \quad \partial \Omega.
\]

Equivalently, \(Cu (x)\) is collinear with \(\overline{\varphi} (x, 0)\) yielding the boundary conditions

\[
u_n^k (x) \phi_n^k (0) e^{2 \pi i x \frac{\alpha}{k}} + u_n^{-k} (x) \phi_n^{-k} (0) e^{-2 \pi i x \frac{\alpha}{k}} = 0 \quad \text{on} \quad \partial \Omega
\] (73)

after remarking that \(\kappa(\sigma, n) \neq 0\). Finally, with \((62)\) and \(\lambda^1 = 2 \mu_0 \mu_1\) the macroscopic problem \((25, 26)\) is recovered.

**Case \(k = 0\)** We adopt the same simplifications of notations that in Section 4.3.2. Let \(e_n\) and \(e_m\) be the Bloch eigenmodes of \(Q(0)\) in \((61)\) regarding the double eigenvalue \(\mu_0 = \mu_n = \mu_m\). In this case \(M_n^0 = \{n, m\}\). Taking \(\Psi^\varepsilon = \sum_{p \in M_n^0} \Psi (\Psi_p) \in H^1(\Omega) \times H^1_0(\Omega)\) as a test function with \(\Psi_p (x, y) = \psi_p (x) e_p (y)\) and \(\psi_p \in H^1(\Omega)\). Due to the periodicity of \(\phi_p\), the second component of \(\Psi^\varepsilon\) satisfies the boundary conditions

\[
\sum_{p \in M_n^0} \psi_p (x) \phi_p (0) = 0 \quad \text{on} \quad \partial \Omega.
\] (74)

Following similar calculations as for the case \(k \neq 0\), the weak limit \(U^0\) of \(S^\varepsilon U^\varepsilon\) in \(L^2(\Omega \times Y)^2\) is

\[
U^0 (x, y) = \sum_{p \in M_n^0} u_p (x) e_p (y)
\]
and \( u_p \) is solution to the weak formulation

\[
\sum_{q \in \mathcal{M}^0_n} \int_{\Omega} -i \mu_1 b(p, q) u_q \cdot \psi_p + \kappa(p, q) u_q \cdot \partial_x \psi_p \, dx = 0
\]

for all \( \psi_p \in H^1(\Omega) \) with \( p \in \mathcal{M}^0_n \). If \( u_q \in H^1(\Omega) \) it is a solution to the internal equations

\[
\sum_{q \in \mathcal{M}^0_n} \kappa(p, q) \partial_x u_q - i \mu_1 b(p, q) u_q = 0 \ 	ext{in} \ \Omega \ \text{for} \ p \in \mathcal{M}^0_n,
\]

and to the boundary conditions

\[
\int_{\partial \Omega} \sum_{p,q \in \mathcal{M}^0_n} \kappa(p,q) u_q \cdot \psi_p \, dx = 0.
\]

Here, with \( C_1 = (\kappa(p,q))_{p,q}, \ C_2 = (b(p,q))_{p,q}, \ u = (u_p)_p, \ \psi = (\psi_p)_p, \ \phi = (\phi_p)_p, \)

\[
C_1 \partial_x u = i \mu_1 C_2 u \ \text{in} \ \Omega,
\]

and \( C\psi(x) \cdot \overline{\psi}(x) = 0 \) on \( \partial \Omega \) for all \( \psi \) such that \( \phi(0) \overline{\psi}(x) = 0 \) on \( \partial \Omega \).

But \( \kappa(p,p) = 0 \), therefore

\[
u_n(x) \phi_n(0) + u_m(x) \phi_m(0) = 0 \ \text{on} \ \partial \Omega.
\]

As for \( k \neq 0 \), these macroscopic equations are equivalent to \((27, 28)\).

6 Numerical simulations

We report simulations regarding comparisons of physical eigenmodes and their approximation by two-scale modes for \( \rho = 1 \). In Subsection 6.2, for each given high frequency physical eigenvalue a two-scale eigenvalue realizing a good approximation is identified. This shows that the two-scale model can actually be used as an approximation of the complete high-frequency spectra. Conversely, Subsection 6.3 addresses the modeling problem i.e. it introduces a way to generate approximations of high-frequency spectra from the two-scale model only. Finally, in 6.4 the order of convergence with respect to \( \varepsilon \) is analyzed. The next section describes the main simulation parameters.

6.1 Simulation methods and conditions

Both, the physical spectral problem and the Bloch wave spectral problem are discretized by a quadratic finite element method. The number of elements are respectively denoted \( N_{\text{phys}} \) and \( N_{\text{bloch}} \). The implementation of the \( k\)–quasi-periodic boundary condition is achieved by elimination of the last degree of freedom. More precisely, for \( n \in \{1, ..., 2N_{\text{bloch}} + 1\} \) the node indices, \( \phi_n \) a degree of freedom of \( \phi \) a Bloch eigenmode and \( \varphi_n \) the corresponding quadratic Lagrange interpolation function,

\[
\phi(y) \simeq \sum_{n=2}^{2N_{\text{bloch}}} \phi_n \varphi_n + \phi_1 \varphi_1 + \phi_{2N_{\text{bloch}}+1} \varphi_{2N_{\text{bloch}}+1}.
\]
Using the relation \( \phi(1) = e^{2i\pi k} \phi(0) \) and taking \( \varphi_1 + e^{2i\pi k} \varphi_{2N_{\text{bloch}}+1} \) as the first base function allows to eliminate \( \varphi_{2N_{\text{bloch}}+1} \),

\[
\phi(y) \simeq \sum_{n=2}^{2N_{\text{bloch}}} \phi_n \varphi_n + \phi_1 \left( \varphi_1 + e^{2i\pi k} \varphi_{2N_{\text{bloch}}+1} \right).
\]

The sets of indices considered in the simulations of high frequency physical modes and Bloch modes are denoted by \( J^\varepsilon \) and \( J^k \), the former being generally included in \((\alpha/2\varepsilon, N_{\text{phys}}/2)\). The Bloch modes are calculated for \( k \geq 0 \) only, and the other cases can be deduced by conjugation. For each Bloch eigenmode \((\lambda^k_n, \phi^k_n)\), the macroscopic solutions \((\lambda^{1,\ell}, u^\sigma_{m,\ell})_{m,\ell}\) are given in Section 4.4.1 with \( \delta = 1 \) and \( d_2 = \phi^0_m(0) \) for any \( m \) such that \( \lambda^k_m = \lambda^k_n \) and \( \ell \in \mathbb{Z} \). In fact, according to Remark 14 the index \( \ell \) should vary in \( J^k_n = \left[ \frac{2k}{\varepsilon} \right] + \{ -r, ..., r \} \), for a small integer \( r \), so that only the first macroscopic eigenmodes be taken into account. In the next discussions, we use the following notations for the two-scale approximations of the eigenvalues and eigenmodes exhibiting clearly their parameters \( \varepsilon, k, n \) and \( \ell \),

\[
\tilde{\gamma}^\varepsilon_{n,\ell} := \lambda^k_n + \varepsilon \lambda^{1,\ell} \quad \text{and} \quad \tilde{\psi}^\varepsilon_{n,\ell}(x) := \sum_{\sigma \in \ell^k} \sum_{m} u^\sigma_{m,\ell}(x) \phi_m^\sigma \left( \frac{x}{\varepsilon} \right) \quad \text{for} \quad \ell \in J^k_n, \ n \in J^k.
\]

In the simulations reported in Sections 6.2 and 5.3 only one physical problem is used, namely \( \Omega = (0,1), \ a^\varepsilon(x) = \sin(2\pi x/\varepsilon) + 2, 50 \) cells (i.e. \( \varepsilon = 1/50 \)), and \( N_{\text{phys}} = 2,000 \). Other number of cells are used in Section 6.4 for the convergence analysis. Consequently, the coefficient of the Bloch wave spectral problem is \( a(y) = \sin(2\pi y) + 2 \). The set \( Y^* \) of positive wave numbers in \( Y^* \) is discretized by \( L_{125}^+ = \{ 0, ..., 62/125 \} \) with step \( \Delta_k = 1/125 \) and \( N_{\text{bloch}} = 50 \). The subset of macroscopic eigenvalues is restricted by \( r = 15 \).

The first ten graphs \((k \mapsto \lambda^k_n)_{n=1, ..., 10}\) of Bloch eigenvalues are described in Figure 1. The graphs are symmetric about the axis \( k = 0 \) which confirms that \( \lambda^k_n = \lambda^{-k}_n \) as remarked in Notation 5. Moreover, all eigenvalues \( \lambda^k_n \) are simple for \( k \neq 0 \) and double for \( k \in \{ 0, \pm \frac{1}{2} \} \).

![Figure 1: First ten eigenvalues of the Bloch wave spectral problem.](image)

### 6.2 Approximation of physical modes by two-scale modes

We discuss the approximation of a given solution \((\lambda^\varepsilon_p, w^\varepsilon_p)\) of Equation (8) for a given value of \( \varepsilon \). From Remark 14 we expect to show numerically that there exists a suitable pair \((k, n)\) such that the equality \((\lambda^\varepsilon_p, w^\varepsilon_p) = (\gamma^\varepsilon_{n,\ell}, \psi^\varepsilon_{n,\ell})\) is exact with \((\gamma^\varepsilon_{n,\ell}, \psi^\varepsilon_{n,\ell})\) defined in (77) and \( \lambda^{1,\ell} = 0 \). Moreover,
in the perspective of Remark 12, $k$ varies in $L_{125}^{*+}$ only and approximations with $\lambda^{1,\ell} \neq 0$ are expected. Whatever if $\lambda^{1,\ell}$ vanishes or not, we expect to search approximations for both eigenvalues and eigenvectors which turns to be an multi-objective optimization problem that might be solved by a dedicated method. However, to reduce the computational cost, we propose an alternate approach consisting in minimizing the error on eigenvalues in the approximation (10),

$$er_{\text{value}}(k) = \min_{n \in \mathbb{N}, \ell \in J_k} \left| \frac{\varepsilon^2 \lambda_{e}^{n,k} - \lambda_{k,n,\ell}^{e,k}}{\varepsilon^2 \lambda_{p}^e} \right|,$$

for each $k \in L_{125}^{*+}$, and then in finding which one minimizes

$$er_{\text{vector}}(k) = \frac{\left\| w_p^e - \psi_{n_k,\ell_k}^{e,k} \right\|_{L^2(\Omega)}}{\left\| w_p^e \right\|_{L^\infty(\Omega)}},$$

the error on eigenvectors in the approximation (30) where $\ell_k, n_k$ are the optimal arguments in (78). The optimal error on eigenvectors is then

$$er_{\text{vector}} = \min_{k \in L_{125}^{*+}} er_{\text{vector}}(k).$$

Figure 2: (a) Errors for $p = 85$ and $k \in L_{125}^{*+}$. (b) Errors for a selection of $k$ s.t. $er_{\text{vector}}(k) \leq 0.2$.

Figure 2(a) shows the distributions of errors $er_{\text{value}}(k)$ and $er_{\text{vector}}(k)$ in logarithmic scale for the index $p = 85$ of physical eigenmode with respect to $k$ varying in $L_{125}^{*+}$. The minimal error is reached for $k = 0.16$, $n = 2$, $\ell = 17$, $\lambda_{n}^k = 51.1$ and $\lambda_{1,\ell}^{1} = 58.9$ yielding the errors $er_{\text{value}} = 10^{-4}$ and $er_{\text{vector}} = 4.10^{-3}$. Figure 2(b) focuses on values of $k$ such that $er_{\text{vector}}(k) \leq 0.2$. In Figure 3(a) the real (dashed line) and the imaginary (solid line) parts of the Bloch wave $\phi_{n,k}^e$ are shown when Figure 3(b) presents the real (solid line) and the imaginary (dashed-dotted line) parts of $u_{n,\ell}^{e,k}$ and also the real (dotted line) and the imaginary (dashed line) parts of $u_{n,\ell}^{-e,k}$. In addition, the physical eigenmode $w_p^e$ and the relative error vector between $w_p^e$ and $\psi_{n_k,\ell_k}^{e,k}$ are plotted in Figure 4(a) and (b).

After presenting a detailed study of the approximation of a given physical mode, i.e. for a single physical mode index $p$, we report approximation results for the list $\mathcal{J}_0 = \{40, ..., 150\} \setminus \{50\}$ of
Figure 3: (a) Bloch wave solution $\phi^k_n$. (b) Macroscopic solutions $u^k_{n,\ell}$ and $u^{-k}_{n,\ell}$.

Figure 4: (a) Physical eigenmode $w^\varepsilon_p$. (b) Relative error between between $w^\varepsilon_p$ and $\psi^\varepsilon_{n,\ell}$.

consecutive physical mode indices. The list starts at $p = 40$ corresponding to an intermediary mode between the low frequency modes approximated by the classical homogenized method and the high frequency modes considered in this paper. The index $p = 50$ is excluded from the list since the corresponding eigenvector is evanescent, and as such corresponds to an element of the boundary spectrum. The previous optimization has been applied to each $p$ yielding errors plotted in logarithm scale in Figure 5 (a). The error bounds are $er_{value} \leq 6.10^{-3}$ and $er_{vector} \leq 8.10^{-2}$.

Globally, the errors start by growing before to decrease except around $p = 100$ where they exhibit a peak that we do not explain. Figure 5 (b) reports the corresponding macroscopic eigenvalues $\lambda^{1,\ell}$. Some of them are close to pairs $(k, n)$ such that $\lambda^{1,\ell}$ vanishes as discussed in Remark 11; their relative errors on eigenvalues are in the order of $10^{-5}$. A way to answer the question in Remark 11 is to decrease the step $\Delta_k$ and see if all error decrease. A detailed presentation is made in the table below for two indices, namely $p = 66$ related to an eigenvalue in the beginning of the high frequency spectrum and $p = 102$ corresponding to one of the large errors. In both cases, the error diminishes as the step $\Delta_k$ is reduced from $8e-3$ to $3e-3$. 

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Figure 5: (a) Errors for $p$ varying in $J_0^e$. (b) Macroscopic eigenvalues.

![Figure 5](image)

Table 1: Errors for $\Delta_k = 8.0e-3$ and $3.0e-3$.

| $\Delta_k$ | $p$   | $k$      | $n$ | $\lambda^{1/2}$ | $e_r^{value}$ | $e_r^{vector}$ |
|------------|-------|----------|-----|----------------|---------------|---------------|
| 8.0e-3     | 66    | 2.16e-1  | 2   | -92            | 1.2e-3        | 1.9e-2        |
| 3.0e-3     | 66    | 3.4e-1   | 2   | 21.7           | 9.0e-5        | 5.3e-3        |
| 8.0e-3     | 102   | 4.0e-2   | 3   | -147           | 4.0e-4        | 5.8e-3        |
| 3.0e-3     | 102   | 1.5e-2   | 3   | 35.9           | 3.0e-5        | 1.4e-3        |

Figure 6 (a) is a global view of the errors in logarithm scale when $\Delta_k = 8.0e - 3$ for $90 \leq p \leq 110$. It shows that for this $k$-step a large part of the errors on eigenvalues is in the range of $1.0e-5$ i.e. almost the roundoff error. A measure of the error reduction is provided in Figure 6 (b) where the two ratios

$$E_{value} = \frac{e_r^{value}_{\Delta_k=3.0e-3}}{e_r^{value}_{\Delta_k=8.0e-3}}$$

and

$$E_{vector} = \frac{e_r^{vector}_{\Delta_k=3.0e-3}}{e_r^{vector}_{\Delta_k=8.0e-3}}$$

of error reduction are represented in logarithmic scale.

Figure 6: (a) Error of approximation for $\Delta_k = 3.0e - 3$. (b) Ratios $E_{value}$ and $E_{vector}$ of error reduction.
6.3 The modeling problem

The modeling problem is reciprocal to the previous one. It consists in fixing a period $\varepsilon$ as well as the parameters $(k, n)$ of a Bloch mode and to search if there exists $\ell \in J_{kn}^k$ such that $(\gamma_{n,\ell}^{\varepsilon,k}, \psi_{n,\ell}^{\varepsilon,k})$ is close from a physical mode or in other words if it is almost a solution to the physical spectral problem i.e. if

$$\varepsilon^2 P_{n,\ell}^{\varepsilon,k} - \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k} = O(\varepsilon) \text{ in } \Omega.$$  \hfill (80)

Posing for $\ell \in J_{kn}^k$,

$$F_{n,\ell}^{\varepsilon,k}(\ell) = \frac{\|\varepsilon^2 P_{n,\ell}^{\varepsilon,k} - \gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k}\|_{L^2(\Omega)}}{\|\gamma_{n,\ell}^{\varepsilon,k} \psi_{n,\ell}^{\varepsilon,k}\|_{L^2(\Omega)}}$$ \hfill (81)

the modeling problem relies to the minimization problem $F_{n,\ell}^{\varepsilon,k}(\ell_0) = \min_{\ell \in J_{kn}^k} F_{n,\ell}^{\varepsilon,k}(\ell)$. If the minimum is small enough, $(\gamma_{n,\ell_0}^{\varepsilon,k}, \psi_{n,\ell_0}^{\varepsilon,k})$ is close from a physical eigenelement and it is a solution to the modeling problem. A subsequent problem is to identify the corresponding physical eigenelement. This is done by minimizing the errors $e_{\text{rvalue}}$ and $e_{\text{rvector}}$ introduced in the previous section but considered as depending on the parameter $p \in J^\varepsilon$ instead of $k$. Two illustrative examples are reported in the table below, one yielding $\lambda_{1,\ell} = 0$ and the other $\lambda_{1,\ell} \neq 0$. The solution $\psi_{n,\ell}^{\varepsilon,k}$ and the relative error between $\psi_{n,\ell}^{\varepsilon,k}$ and $w_p^\varepsilon$ are reported in Figures 7 (a) and (b).

| $k$   | $n$  | $\lambda_n^k$ | $F_{n,\ell}^{\varepsilon,k}(\ell)$ | $\lambda_{1,\ell}$ | $p$  | $e_{\text{rvalue}}$ | $e_{\text{rvector}}$ |
|-------|------|----------------|------------------------------------|-------------------|-----|---------------------|---------------------|
| 1.6e-1| 2    | 5.11e1         | 8.9e-3                             | 0                 | 84  | 3.4e-5              | 2.1e-5              |
| 3.52e-1| 2  | 3.14e1         | 4.5e-2                             | -8.55             | 65  | 1.5e-2              | 4.3e-3              |

Table 2: Results for the modeling problem

Figure 7: (a) Two-scale eigenmode $\psi_{n,\ell}^{\varepsilon,k}$. (b) Relative error vector between $\psi_{n,\ell}^{\varepsilon,k}$ and $w_p^\varepsilon$.

Additional results for $k = 3.52e-1$ with $n = \{1, ..., 15\}$ are reported in Figures 8 (a) and (b) showing $\lambda_{1,\ell}$ and $\gamma_{n,\ell}^{k}$ respectively.
6.4 Order of convergence

For a given pair \(k\) and \(n \in J_k\), we investigate the order of convergence of the errors \(er_{value}\) and \(er_{vector}\) when the number of cells increases. To follow the convergence result, the sequence of periods \(\varepsilon\) is in fact a subsequence \(\varepsilon_h\) satisfying

\[
\frac{1}{\varepsilon_h} = \frac{h + l}{k} \in \mathbb{N}^*
\]

with \(l \in [0, 1)\) and for a sequence of \(h \in \mathbb{N}^*\). Table 3 summarizes the results for \(k = 0.3\), \(l = 0.6\) and \(h \in \{3, 9, 15, 21\}\).

| \(h\) | \(\varepsilon_h\) | \(er_{value}^{h,\ell}\) | \(er_{vector}^{h,\ell}\) | \(p\) |
|---|---|---|---|---|
| 3  | \(8.3e-2\) | \(4.3e-2\) | \(6.3e-3\) | 17 |
| 9  | \(3.1e-2\) | \(1.6e-2\) | \(2.4e-3\) | 45 |
| 15 | \(1.91e-2\) | \(1.0e-2\) | \(1.5e-3\) | 73 |
| 21 | \(1.4e-2\) | \(7.0e-3\) | \(1.0e-3\) | 101 |

Table 3: Errors for a decreasing subsequence \(\varepsilon_h\)

To evaluate the decay rate of the errors, we pose \(er_{value}^{h,\ell} = c_{value} (\varepsilon_h)^{q_{value}}\) and \(er_{vector}^{h,\ell} = c_{vector} (\varepsilon_h)^{q_{vector}}\), so the decay rates satisfy

\[
q_{value} = \frac{\log (er_{value}^{h,\ell}/er_{value}^{h',\ell})}{\log (\varepsilon_h/\varepsilon_{h'})} \quad \text{and} \quad q_{vector} = \frac{\log (er_{vector}^{h,\ell}/er_{vector}^{h',\ell})}{\log (\varepsilon_h/\varepsilon_{h'})}.
\]

Using successive results for \(h\) and \(h'\), yields

\[
q_{value} = \{0.988, 0.995, 0.985\} \approx 1 \quad \text{and} \quad q_{vector} = \{0.985, 0.993, 0.994\} \approx 1
\]

with coefficients

\[
c_{value} = \{0.504, 0.518, 0.497\} \approx 0.5 \quad \text{and} \quad c_{vector} = \{0.0734, 0.0755, 0.0757\} \approx 0.07.
\]
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