Crossover dark soliton dynamics in ultracold one-dimensional Bose gases

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Ultracold confined one-dimensional atomic gases are predicted to support dark soliton solutions arising from a nonlinear Schrödinger equation of suitable nonlinearity. In weakly-interacting (high density) gases, the nonlinearity is cubic, whereas an approximate model for describing the behaviour of strongly-interacting (low density) gases is one characterized by a quintic nonlinearity. We use an approximate analytical expression for the form of the nonlinearity in the intermediate regimes to show that, near the crossover between the two different regimes, the soliton is predicted and numerically confirmed to oscillate at a frequency of $\sqrt{2/3}\Omega$, where $\Omega$ is the harmonic trap frequency.

I. INTRODUCTION

Dark solitons (DSs), the most fundamental nonlinear excitations of the one-dimensional defocusing nonlinear Schrödinger (NLS) equation, have been studied in a broad range of physical systems. Apart from the theoretical work, experimental studies on DSs include their observation either as temporal pulses in optical fibers [1] or as spatial structures in bulk media and waveguides [2] (see also [3] for a review), the excitation of a nonpropagating kink in a parametrically-driven shallow liquid [3], standing DSs in a discrete mechanical system [5], high-frequency DSs in thin magnetic films [6], and so on.

Recently, DSs have attracted much attention in the physics of atomic Bose-Einstein condensates (BECs) [7], where dark matter-wave solitons have also been observed experimentally [8]. Dark solitons in BECs are known to be more robust in one-dimensional (1D) geometries and at very low temperatures (a regime which is currently experimentally accessible [9]). For that reason, the majority of theoretical studies on DSs in BECs have for simplicity been performed in the framework of the one-dimensional (1D) cubic NLS equation, which in this context is referred to as the Gross-Pitaevskii equation; the latter, is the commonly adopted mean-field theoretical model describing ultracold weakly-interacting Bose gases in the absence of thermal or quantum fluctuations. Many of the above mentioned theoretical studies have been devoted to the analysis of the dynamical properties of moving DSs, such as their oscillations [11, 11, 12, 13, 14] and sound emission [11, 11, 12, 13] in the presence of the external trapping potential. On the other hand, and in the same context of the 1D Bose systems, DSs have also been studied in the framework of a quintic NLS equation [14, 15, 16, 17], a long-wavelength model which has been proposed [14] for the opposite limit of strong interatomic coupling [18, 19]; in this case, the collisional properties of the bosonic atoms are significantly modified, with the interacting bosonic gas behaving like a system of free fermions [20]. Such, so-called, Tonks-Girardeau (TG) gases have recently been observed experimentally as well [21].

Regarding the dynamical features of the moving DSs in trapped 1D Bose gases, the above works revealed that, in the absence of other dissipative mechanisms, the soliton oscillates in the trap with a frequency which differs between the weakly and strongly interacting regimes. In particular, in the presence of a harmonic confining potential of frequency $\Omega$, the study of the cubic (quintic) NLS predicts such an oscillation frequency to be $\alpha_0\Omega$ where $\alpha = 1/\sqrt{3}$ ($\alpha = 1$) for the weakly [10, 11, 12, 13, 14] (strongly [14, 16]) interacting Bose gas. The latter result for the strongly interacting case is identical to the corresponding one obtained by a full many-body calculation [22].

The transition between the weakly and strongly interacting regimes is usually characterized by a single parameter, denoted by $\gamma$, quantifying the ratio of the average interaction energy to the kinetic energy calculated with mean field theory [10]. This parameter varies smoothly as the interatomic coupling is increased from values $\gamma \ll 1$ (weakly interacting regime), to $\gamma \gg 1$ (strongly interacting regime); thus, an approximate “crossover regime” can be identified around $\gamma \sim O(1)$, as also attained experimentally [22]. The value of $\gamma$ can be controlled experimentally by various independent parameters, such as scattering length, transverse confinement, density, or even modification of the effective mass of the system.

Motivated by the investigation of the DS dynamics in the two limiting regimes, in the present work we obtain analytical results, which are confirmed by numerical simulations, for the oscillation frequency of a DS in the crossover regime for a purely 1D system. In particular, we show that for $\gamma \sim O(1)$ the DS oscillates with a frequency $\sqrt{2/3}\Omega$, which is higher than the one ($\sqrt{1/2}\Omega$) pertaining to the weakly interacting Bose gas. Such an increase in the soliton oscillation frequency could serve as an additional diagnostic test for the deviation from pure bosonic mean field behaviour.

The paper is organized as follows: In section II we present the generalized NLS model proper and find its ground state, as well as its linear (sound waves) and nonlinear (dark solitons) excitations. Section III is devoted to the analytical derivation of the DS oscillation...
II. THE MODEL AND ITS ANALYTICAL CONSIDERATION

A. The generalized NLS equation

Importantly for our present analysis, the parameter \(|\gamma|\) separating the regimes of weak and strong interatomic coupling can be re-expressed, for a given system configuration, in terms of the inverse ratio of the system density to some critical density. This enables us to perform a unified analysis of all regimes by means of a NLS equation with a generalized nonlinearity, for the parameter \(|\psi|\) connected to the density \(n = |\psi|^2\) of an ultracold confined 1D Bose gas. This equation takes the form

\[
i\hbar \frac{\partial \psi}{\partial t} = \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + \Phi(n) \right] \psi, \quad (1)
\]

where \(m\) is the atomic mass and \(V_{\text{ext}}(x) = (1/2)m\omega_x^2 x^2\) is the external trapping potential (\(\omega_x\) being the axial confining frequency).

The exact form of the nonlinearity \(\Phi(n)\) valid in both limiting regimes and the crossover region is well-known in the homogeneous hydrodynamic limit. While the functional dependence of \(\Phi(n)\) on \(|\gamma|\) (and its analytical asymptotics) are known \(^{24}\), its precise values in the “crossover region” can only be evaluated numerically. Such intermediate values have been tabulated by Dunjko et al. in \(^{18}\), and subsequently discussed by various authors in the local density approximation, see e.g. \(^{26, 27}\). Since we are interested in this “crossover region”, we should thus use an approximate expression for the nonlinearity, which captures both limiting regimes exactly and provides a good approximation for intermediate values. At the same time, however, we are constrained in our present work by the need for a relatively simple expression which will enable rather involved analytical work to be carried out. For our purpose, it is thus sufficient to use a somewhat simplified generalized nonlinearity \(\Phi(n)\) of the form

\[
\Phi(n) = \frac{\pi^2 \hbar^2}{2m} \frac{n^2}{1 + n/n_c}, \quad (2)
\]

which nonetheless corresponds to a fairly good analytical approximation \(^{26, 27}\) to the exact nonlinearity \(^{18}\). In this notation, the critical density \(n_c\) approximately marking the crossover region is given by \(n_c = 4/(\pi^2 a_{1D})\), where \(a_{1D}\) is the effective 1D “scattering length”. This is defined in terms of the usual three-dimensional s-wave scattering length \(a_{3D}\) via \(a_{1D} = l_\perp^2/a_{3D}\), where \(l_\perp = \sqrt{\hbar/m\omega_\perp}\) is the harmonic oscillator length in the transverse direction.

Equation (1) is well-documented in the high-density limit \(n \gg n_c\) (corresponding to \(\gamma \ll 1\)). In this case, the parameter \(|\psi|\) describes the “wavefunction” of a weakly-interacting 1D Bose gas under harmonic confinement, for which Eq. (1) reduces to the 1D Gross-Pitaevskii equation,

\[
i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + \frac{4\pi\hbar^2}{m} \left( \frac{a_{3D}}{2\pi l_\perp^2} \right) |\psi|^2 \right] \psi. \quad (3)
\]

In the opposite limit of strong interatomic coupling, which, rather counter-intuitively, corresponds to the low density limit \(n \ll n_c\), the parameter \(|\psi|\) satisfies the following quintic NLS equation,

\[
i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + \frac{\pi^2 \hbar^2}{2m} |\psi|^4 \right] \psi, \quad (4)
\]

which has been derived by various different methods \(^{13, 28}\). While the validity of Eq. (1) to discuss coherence properties of strongly-interacting 1D Bose gases has been questioned \(^{29}\), the corresponding hydrodynamic equations for the density \(n\) and the phase \(|\psi|\) arising from this equation under the Madelung transformation \(\psi = \sqrt{n} \exp(i \phi)\) are well-documented in the context of the local density approximation \(^{18, 29}\). An equation of the form (1), which explicitly includes the so-called “quantum pressure term” \((\hbar^2/2m)|\partial^2 \psi/\partial x^2|\), should however only be valid for density variations which occur on a lengthscale which is larger than the Fermi healing length \(\xi \equiv 1/(\pi n_p)\), where \(n_p = |\psi(0)|^2\) is the peak density of the gas at the trap center. To avoid such potential complications, the physical analysis presented in this paper is only concerned with the limit of shallow DSSs, for which there is only a very slow density variation within the Fermi healing length. Measuring the variables \(x\) and \(t\), and the density \(|\psi|^2\), in units of the Fermi healing length \(\xi\), the time \(\tau_0 = \hbar/m\xi^2\), and the density \(\sqrt{2n_p}\), respectively, Eq. (1) can be rewritten in the following dimensionless form,

\[
iu_t = -\frac{1}{2} u_{xx} + V(x)u + g(n)u, \quad (5)
\]

where the subscripts denote partial differentiation. In view of the above scalings, the normalized confining potential becomes \(V(x) = (1/2)(\xi/l_x)^4 x^2\), where \(l_x = \sqrt{\hbar/m\omega_x}\) is the harmonic oscillator length in the axial direction. As the parameter \(\xi/l_x\) is apparently small, \(|\gamma|\) is conveniently defined to be the small parameter \(\epsilon \equiv \Omega^{-2/3}(\xi/l)^{4/3}\) where \(\Omega\) is a parameter of order \(O(1)\), that will be used in the perturbation analysis to follow. This way, the external potential is actually a function of the slow variable \(X = \epsilon^{3/2} x\), and has the form \(V(X) = (1/2)\Omega^2 X^2\), where \(\Omega\) expresses the trap frequency. Finally, the nonlinearity function \(g(n)\) (with \(n = |u|^2\) being the normalized density) becomes,

\[
g(n) = \frac{n^2}{1 + n/n_c}, \quad (6)
\]

where \(n_c = 2\sqrt{2}/\pi^2 a_{1D} n_p\).

The weakly and strongly interacting limits discussed above respectively arise when the dimensionless parameter \(\gamma = 2/(na_{1D})\) \(^{19}\) obeys \(\gamma \ll 1\) or \(\gamma \gg 1\). In our
present notation, \( n/n_c = (\pi^2/2)^{-1} \), so that our approximate crossover region actually corresponds to a value \( \gamma = 2/\pi^2 \approx 0.2 \). For such a relatively small value of \( \gamma \), it is still reasonable to use Eqs. 11-20 [or Eqs. 15-20] to describe this physical system.

Studies of DSSs based on generalized 1D NLS equations first appeared in the literature for homogeneous systems to deal with saturable nonlinearities appearing in the context of nonlinear optics [3]; in this case, the nonlinearity becomes cubic at low densities, rather than at high densities which occurs for ultracold pure 1D Bose gases. More recently, DSSs in generalized 1D NLS equations were also considered in the BEC context, but only as effective theories for weakly-interacting elongated 3D condensates. These equations contained either a cubic-quintic nonlinearity with constant coefficients [30], or a generalized non-polynomial nonlinearity depending explicitly on the trap aspect ratio [31, 32]. The aim of our work is thus restricted to the pure 1D regime, with the precise experimental conditions needed for such a crossover discussed in detail in [18, 33].

Below we apply the reductive perturbation method (see, e.g., [34, 35]), valid in the limit of shallow solitons, to analytically obtain the soliton oscillation frequency close to the critical density \( n_c \), which approximately marks the crossover between the regimes of weak and strong interatomic coupling.

B. Ground state, linear and nonlinear excitations

Starting off from the generalized NLS Eq. 3, we use the Madelung transformation, \( u = \sqrt{|n|}\exp(i\phi) \), to obtain the following set of hydrodynamic equations,

\[
\begin{align*}
n_t + (n\phi_x)_x &= 0, \\
\phi_t + g(n) + &\left(\frac{1}{2}n^2 - \frac{1}{2}n^{-1/2}(n^{1/2})_{xx} + V(X)\right) = 0.
\end{align*}
\]

The above equations are similar to the ones that have been employed to discuss the crossover from TG to BEC regime [18, 22]. The ground state of the system can be obtained upon assuming that the atomic velocity \( v \equiv \phi_x = 0 \) (i.e., no flow in the system) and \( \phi_t = -\mu_0 \) (dimensionless chemical potential). Then, as Eq. 7 implies that \( n = n_0 \) is time-independent in the ground state, we assume that \( n_0 = n_0(X) \). Thus, to leading order in \( \epsilon \) [i.e., to \( O(1) \)] Eq. 8 yields,

\[
g(n_0) = \mu_0 - V(X),
\]

in the region where \( \mu_0 > V(X) \) and \( n_0 = 0 \) outside. Equation 9 determines the density profile in the so-called Thomas-Fermi (TF) approximation; note that for the typical case of the harmonic trap, e.g., \( V(X) = (1/2)\Omega^2X^2 \), Eq. 9 recovers the well-known result that the density profile is parabolic, with \( n_0(X) = n_c^{-1}[\mu_0 - V(X)] \), in the regime \( n \gg n_c \), and elliptic, with \( n_0(X) = \sqrt{\mu_0 - V(X)} \), in the regime \( n \ll n_c \). Also, it is noticed that from Eq. 9, and for the harmonic trap under consideration, the axial size of the gas is 2L, where \( L = \sqrt{2\mu_0}/\Omega \) is the TF radius.

We now consider the propagation of small-amplitude linear excitations (e.g., sound waves) of the ground state, by seeking solutions of Eqs. (7)-(8) of the form \( n = n_0(X) + \tilde{n}(x,t) \) and \( \phi = -\mu_0t + \phi_0(x,t) \), where the functions \( \tilde{n} \) and \( \phi_0 \) describe the linear excitations. Inserting this ansatz into Eqs. (7)-(8), to order \( O(1) \) we recover the TF approximation, while to order \( O(\epsilon) \) we obtain a system of linear equations for the linear excitations. Assuming plane wave solutions of this system, i.e., \( (\tilde{n}, \phi_0) \sim \exp[i(kx - \omega t)] \), we readily obtain the dispersion relation \( \omega^2 = g_0\mu_0k^2 + k^4/4 \), where \( g_0 \equiv (dg/dn)|_{n=n_0} \). This dispersion relation has the form of a Bogoliubov-type excitation spectrum, but with the excitation frequency \( \omega \) being a function of the slow variable \( X \). The speed of sound is local, due to the presence of the external potential, and is given by

\[
C = \sqrt{g_0\mu_0},
\]

Note that Eq. 10 shows that the speed of sound is given by \( C = \sqrt{n/n_c} \) for \( n \gg n_c \), and \( C = \sqrt{2\mu_0} \) in the opposite regime \( n \ll n_c \).

Next we analyze the evolution of the nonlinear excitations on top of the ground state, employing the reductive perturbation method [34, 35] (see also [11] for relevant studies in Bose gases). As the system of Eqs. (7)-(8) is inhomogeneous, we introduce a new slow time-variable \( T = \epsilon^{1/2}(t - \int_0^t C^{-1}(x')dx') \), and the following asymptotic expansions for the density \( n \) and phase \( \phi \),

\[
n = n_0(X) + \epsilon n_1(X,T) + \epsilon^2 n_2(X,T) + \cdots,
\]

\[
\phi = -\mu_0t + \epsilon^{1/2}\phi_1(X,T) + \epsilon^{3/2}\phi_2(X,T) + \cdots.
\]

Substituting the expansions into Eqs. (7)-(8), we obtain the following results: First, to order \( O(1) \), Eq. (8) leads to the TF approximation [see Eq. 9]. Then, to the first-order of approximation in \( \epsilon \) [i.e., to orders \( O(\epsilon) \) and \( O(\epsilon^{3/2}) \)], Eqs. (8) and (7) yield the equation,

\[
\phi_1(X,T) = -\tilde{g}_0(X) \int_0^T n_1(X,T')dT',
\]

connecting the unknown functions \( n_1 \) and \( \phi_1 \). Finally, to the second order of approximation [to order \( O(\epsilon^2) \) and \( O(\epsilon^{3/2}) \)], Eqs. (7) and (8) lead to the following nonlinear evolution equation for \( n_1 \),

\[
\frac{d}{dX} n_1(X,T) - \frac{(3\tilde{g}_0 + n_0\tilde{g}_0)}{2C^3} n_1(X,T) + \frac{1}{8C^5} n_{1TTT} = -\frac{d}{dX} \ln[(C\tilde{g}_0)^{1/2}] n_1(X,T),
\]

where \( \tilde{g}_0 \equiv (d^2g/dn^2)|_{n=n_0} \). Equation 13 has the form of a Korteweg-deVries (KdV) equation with variable coefficients, which has been used in the past to describe
shallow water-waves over variable depth, or ion-acoustic solitons in inhomogeneous plasmas [33]. Moreover, such KdV equations have recently been used to analyze the dynamics of DSs in Bose gases both in the weakly-interacting [11] and the strongly-interacting [16] regimes.

As the inhomogeneity-induced dynamics of the KdV solitons has been studied analytically in the past [36], we may employ these results to analyze the coherent evolution of DSs in the Bose gas under consideration. Thus, introducing the transformations $\chi = \int (8C)^{-1/2} dX$ and $n_1 = (3/2)(3\delta_0 + n\delta) - 1/2 C^2 v(\chi, T)$, we first put Eq. (13) into the form,

$$v_\chi - 6uv_T + v_{TTT} = \lambda(\chi)v, \quad (14)$$

where $\lambda(\chi) = (d/d\chi)\ln \left[ n_{0}^{3/4} \left( 3\delta_0 + n\delta \right) \right]$. In the case $\lambda = 0$, i.e., for a homogeneous gas with $n_0(X) = n_\sigma = \text{const.}$, Eq. (14) is the completely integrable KdV equation, which possesses a single-soliton solution of the following form [37],

$$v = -2\kappa^2 \text{sech}^2 Z, \quad Z = \kappa \left[ T - \zeta(\chi) \right], \quad (15)$$

where $\zeta(\chi) = 4\kappa^2 \chi + \delta_0$ is the soliton center (with $d\zeta/d\chi = 4\kappa^2$ being the soliton velocity in the $T$-reference frame), while $\kappa$ and $\delta_0$ are arbitrary constants presenting the soliton’s amplitude (as well as inverse temporal width) and initial position respectively. Equation (14) describes a density notch on the background density $n_\sigma$, with a phase jump across it [see Eq. (12), which implies that $\phi_1 \sim \tanh Z$] and, thus, it represents an approximate DS solution of Eq. (13).

On the other hand, in the general case of the inhomogeneous gas [i.e., in the presence of $V(\chi)$], soliton dynamics can still be studied analytically, provided that the right-hand side of Eq. (14) can be treated as a small perturbation. As $\lambda(\chi)$ is apparently proportional to the density gradient, such a perturbative study is relevant in regions of small density gradients (e.g., near the trap center for a harmonic trapping potential), which is consistent with the use of the local density approximation. In this case, employing the adiabatic perturbation theory for solitons [38], we may seek for the soliton solution of Eq. (14) in the form of Eq. (13), but with the soliton parameters $\kappa$ and $\zeta$ being now unknown functions of $\chi$.

The respective evolution equations for the soliton’s amplitude and center can be solved analytically [36] and the results, expressed in terms of the slow variable $X$, read:

$$\kappa(X) = \kappa(0) \left( \frac{\sigma(X)}{\sigma(0)} \right)^{2/3}, \quad (16)$$

$$\zeta(X) = \frac{1}{2} \kappa^2(0) \left( \frac{\sigma(X)}{\sigma(0)} \right)^{4/3} \int_0^X C^{-5} dX' \left( \frac{\sigma(X')}{\sigma(0)} \right)^{2/3} dX'$$

$$+ \frac{1}{2\kappa(0)} \left[ 1 - \left( \frac{\sigma(X)}{\sigma(0)} \right)^{-2/3} \right], \quad (17)$$

where $\kappa(0)$ and $\sigma(0)$ are the values of the respective functions at $X = 0$.

Note that the above procedure is general, and does not rely on the ratio of the parameter $n/n_c$, although it has been shown to yield the correct results in both limits $n \gg n_c$ [11] and $n \ll n_c$ [16]. In this work, we use the above general results for the evolution of the soliton parameters, to derive the equation of motion of the DS and find its oscillation frequency in the “nonlinearity crossover regime” $n \approx n_c$.

### III. SOLITON OSCILLATION FREQUENCY

Confining ourselves to the “crossover regime” $n \approx n_c$, we may use a Taylor expansion of the function $g(n)$ around $n = n_c$, namely $g(n) \approx g(n_c) + g(n_c)(n - n_c) + \cdots$, to derive the approximate expression $g_0 \approx n_c(3\delta_0 - n_c)/4$; the latter, along with the TF approximation [see Eq. (9)], leads to the following density profile,

$$n_0 = \frac{4\mu}{3\delta} \left[ 1 + \frac{1}{4} \delta^2 - \left( \frac{X}{L} \right)^2 \right], \quad (18)$$

where $\delta \equiv n_c/\sqrt{\mu_0}$ (it is reminded that $L = \sqrt{2\mu_0/\Omega}$ defines the axial size of the gas). Based on Eq. (18), it is now possible to derive the equation of motion of the DS as follows. First, we find the soliton phase, which, to order $O(\epsilon^{3/2})$, reads,

$$Z = \epsilon^{1/2} \kappa(X) \left[ t - \int \frac{dX'}{C(X')} - \frac{3}{4\sqrt{13}} \frac{g_0^2(0)}{\mu_0^2 \Omega^2} \frac{X}{L} \right]. \quad (19)$$

Then, looking along the characteristic lines of soliton motion, it is possible to show that the position of the soliton satisfies the following equation of motion,

$$\frac{dX}{dt} = \frac{C}{1 + \epsilon(9\sqrt{3}/2)k^2(0)(4\mu + \delta^2)^{-3/2}C}. \quad (20)$$

For sufficiently small $\epsilon$ the second term in the denominator can be neglected; in this case, Eq. (20) shows that the velocity of sufficiently shallow DSs is approximately the same as the speed of sound given by Eq. (10), i.e., $dX/dt \approx \sqrt{\mu_0 n_0}$. Thus, Eq. (20) can be approximated by the separable differential equation,

$$\frac{n_0 + n_c}{n_0 \sqrt{2n_c + n_0 n_c}} dX = dt, \quad (21)$$

which can readily be integrated. In particular, taking into account Eq. (18), we find that Eq. (21) leads to the following result:

$$\frac{3}{4\sqrt{13}} h(\delta) \ln w(X) + \arcsin \left( \frac{X}{L} \right) = \left( \frac{\Omega}{\sqrt{\frac{2}{3}}} \right) t, \quad (22)$$

where $h(\delta) = \delta (1 + \frac{1}{4} \delta^2)^{-1/2}$,

$$w(X) = \frac{\sqrt{L^2 - X^2} + \sqrt{X^2} h(\delta) X}{\sqrt{L^2 - X^2} - \frac{\sqrt{X^2}}{2} h(\delta) X}, \quad (23)$$
and $\tilde{L} = L\sqrt{1 + \frac{2}{3} \delta^2}$. It can be seen that in the regions of small density gradient where $X/L$ is sufficiently small [which is consistent with the assumption that the perturbation $\sim \lambda(n)$ in the KdV Eq. (11) is weak], and for sufficiently small values of the parameter $\delta$, the first term on the left-hand side of Eq. (22) can safely be neglected. Then, it is readily seen that Eq. (22) is reduced to the following equation,

$$X = L \sin \left( \frac{2}{3} \Omega t \right). \quad (24)$$

Equation (24) demonstrates that a shallow DS displays an oscillatory motion in the harmonic trap $V(X) = (1/2)\Omega^2 X^2$ in the “nonlinearity crossover regime” $n \approx n_c$ of Eq. (2), with an oscillation frequency $\Omega_{osc}$ given by

$$\Omega_{osc} = \Omega \sqrt{\frac{2}{3}}. \quad (25)$$

This finding has also been verified through appropriately crafted numerical experiments. As a typical example, we show, in particular, the evolution of a shallow gray soliton (originally located at the origin) with initial speed $C = 0.9$ on the background of a potential $V(x) = \Omega^2 x^2/2$, with $\Omega = 0.0707$; $n_c$ was chosen to be 1 in this case. Figure 1 shows the evolution of the space–time contour plot of the reduced density (the ground state density minus the actual density) for the NLS equation, along side our theoretical prediction of Eq. (24). It is clear that during the first period, where the solitary wave does not significantly interact with the emitted “radiation”, the agreement between the theoretical prediction and the numerical results is very good. Subsequently, as expected given the above interaction, this agreement deteriorates. The approximate initial condition is produced by imposing a NLS gray soliton on top of the ground state of the system for the nonlinearity of interest. Finally, we note in passing that our analytical considerations are, strictly speaking, valid away from the turning points (where the wave speed vanishes). However, our numerical results, as well as alternative approaches such as the ones of [13] and [14], illustrate that the range of validity of our results is, in fact, wider than what may be expected based on the mathematical limitations of the method.

Note that the established values of the oscillation frequencies in the two limiting regimes of interatomic coupling can also be found in the framework of the presented analysis, upon utilizing Eq. (21) in the relevant limits $n/n_c \gg 1$ and $n/n_c \ll 1$ and using the respective density profiles. In the limit $n \gg n_c$ corresponding, e.g. to a weakly-interacting Bose-Einstein condensate, the oscillation frequency is $\Omega_{osc} = \Omega/\sqrt{2}$ (or $\Omega\sqrt{2}/4$), whereas, in the opposite regime $n \ll n_c$ it has recently been found [14, 16, 22] that the oscillation frequency is $\Omega_{osc} = \Omega$ (or $\Omega\sqrt{2}/2$). The oscillation frequency in Eq. (25) is thus predicted to lie between the above mentioned limiting cases. This suggests a continuous change in the predicted soliton oscillation frequency from the regime $n \gg n_c$ to $n \ll n_c$.

Assuming that the temperature is low enough for the soliton to perform many oscillations before decaying due to additional dissipative mechanisms excluded from the NLS equation, an observation of change in the oscillation frequency can be treated as a possible diagnostic tool of the system being in a particular interaction regime. A relevant experiment might thus be to create a Bose gas of a certain unknown interaction strength, and phase imprint a DS in such a system. Measurement of the oscillation frequency of this soliton will then provide important information on the system parameter regime, with any deviation from the oscillation frequency of $\Omega/\sqrt{2}$ denoting a regime of sufficiently strong correlations which, in turn, indicate that the weak-interaction model is no longer an adequate description of the system. Alternatively, one could create a dark soliton in a weakly-interacting 1D BEC, and gradually increase the effective interaction strength, as done in some experiments. In this case, one should be able to observe a gradual monotonic increase of the oscillation frequency, even though the longitudinal confinement remains unaffected.

**FIG. 1**: Spatio-temporal evolution of the reduced condensate density (the ground state density minus the actual density) for the NLS equation with $g(n) = n^2/(1+n)$ and $V(x) = \Omega^2 x^2/2$, with $\Omega = 0.0707$. The dashed line shows the theoretical prediction which agrees well with the full numerical result over (at least) the first oscillation period.

**IV. CONCLUSIONS**

In summary, we have discussed the dynamics of both linear and nonlinear excitations within a generalized nonlinear Schrödinger equation motivated by considerations of the behaviour of ultracold atomic 1D Bose gases. The considered model differs from relevant ones appearing in the context of optics in that the linear behaviour in the density dominates at high densities, with quadratic
behaviour in the density dominating at lower densities. Within this generalised model, discussed here for the first time in relation to dark solitons in 1D ultracold Bose gases, we have studied the dynamics of dark solitons in the “crossover regime” in the effective atomic interaction strength, which is approximately marked by a critical density. Our main conclusion, stemming from analytical considerations and confirmed by numerics, is that the soliton oscillation frequency in this regime lies between the known values arising in the two limiting cases of weakly and strongly interacting gases, indicating a continuous change between these two regimes. Finally, we note that although motivated by a particular physical system, our model and analysis are quite general and are not restricted to the details of this particular system.

Acknowledgments

It is a pleasure to acknowledge discussions with Luis Santos and Joachim Brand on the form of the generalised nonlinearity, and with Vladimir Konotop regarding the applicability of the analytical approach. The work of D.J.F. was partially supported by the Special Research Account of the University of Athens. PGK gratefully acknowledges support from NSF-DMS-0204585, NSF-DMS-0505663 and NSF-CAREER.

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