THE BGMN CONJECTURE VIA STABLE PAIRS

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In memory of M. S. Narasimhan

Abstract. Let \( C \) be a smooth projective curve of genus \( g \geq 2 \) and let \( N \) be the moduli space of stable rank 2 vector bundles on \( C \) and fixed determinant of odd degree. We construct a semi-orthogonal decomposition of \( D^b(N) \) conjectured by Narasimhan and by Belmans, Galkin and Mukhopadhyay. It has two blocks for each \( i \)-th symmetric power of \( C \) for \( i = 0, \ldots, g-2 \) and one block for the \( (g-1) \)-st symmetric power. We conjecture that the subcategory generated by our blocks has a trivial semi-orthogonal complement, proving the full BGMN conjecture. Our proof is based on an analysis of wall-crossing between moduli spaces of stable pairs, combining classical vector bundles techniques with the method of windows.

1. Introduction

Let \( C \) be a smooth complex projective curve of genus \( g \geq 2 \). Let \( N = \mathcal{M}_C(2, \Lambda) \) be the moduli space of stable vector bundles on \( C \) of rank 2 and fixed determinant \( \Lambda \) of odd degree. It is a smooth Fano variety of index 2, let \( \theta \) be an ample generator of \( \text{Pic} N = \mathbb{Z} \).

Theorem 1.1. \( D^b(N) \) has a semi-orthogonal decomposition \( \langle \mathcal{P}, \mathcal{A} \rangle \), where

\[
\mathcal{A} = \langle \theta^* \otimes \mathcal{G}_0, \ (\theta^*)^2 \otimes \mathcal{G}_2, \ (\theta^*)^3 \otimes \mathcal{G}_4, \ (\theta^*)^4 \otimes \mathcal{G}_6, \ \ldots, \\
\theta^* \otimes \mathcal{G}_1, \ (\theta^*)^2 \otimes \mathcal{G}_3, \ (\theta^*)^3 \otimes \mathcal{G}_5, \ (\theta^*)^4 \otimes \mathcal{G}_7, \ \ldots, \\
\mathcal{G}_0, \ \theta^* \otimes \mathcal{G}_2, \ (\theta^*)^2 \otimes \mathcal{G}_4, \ (\theta^*)^3 \otimes \mathcal{G}_6, \ \ldots, \\
\mathcal{G}_1, \ \theta^* \otimes \mathcal{G}_3, \ (\theta^*)^2 \otimes \mathcal{G}_5, \ (\theta^*)^3 \otimes \mathcal{G}_7, \ \ldots \rangle
\]

(1.1)

Each subcategory \( \mathcal{G}_i \cong D^b(\text{Sym}^i C) \) is embedded in \( D^b(N) \) by a fully faithful Fourier-Mukai functor with kernel given by the \( i \)-th tensor bundle \( \mathcal{E}^\otimes_i \) (see Section 3) of the Poincaré bundle \( \mathcal{E} \) on \( C \times N \) normalized so that \( \text{det} \mathcal{E}_x \cong \theta \) for every \( x \in C \). There are two blocks isomorphic to \( D^b(\text{Sym}^i C) \) for each \( i = 0, \ldots, g-2 \) and one block isomorphic to \( D^b(\text{Sym}^{g-1} C) \), which appears on the 1st or 2nd line of (1.1), depending on parity of \( g \).

The blocks appearing in (1.1) cannot be further decomposed \[\text{Lin21}\], but there is some flexibility in their order, most importantly blocks within each of the four lines are mutually orthogonal (see Corollary 9.8). This is remarkably compatible with the results of Muñoz \[\text{Muñ99a, Muñ99b, Muñ01}\] (cf. \[\text{BGM21} \text{ Prop. 6.4.2}\]), that the operator of the quantum multiplication
by $c_1(N)$ on the quantum cohomology $QH^\bullet(N)$ has eigenvalues $8\lambda$, where
\[
\lambda = (1-g), (2-g)\sqrt{-1}, (3-g), \ldots, (g-3), (g-2)\sqrt{-1}, (g-1)
\]
and the eigenspace of $8\lambda$ is isomorphic to $H^\bullet(\text{Sym}^{g-1-|\lambda|} C)$. The four lines of (1.1) correspond to the four coordinate rays of the complex plane. There are many other results, e.g. [DB02, Lee18], on cohomology and motivic decomposition of $N$ compatible with (1.1). This provides an ample evidence towards the expectation that $P = 0$. We hope to address this question in the future. We also hope that our methods will be useful in studying properties of analogous Fourier–Mukai functors for moduli spaces of vector bundles of higher rank on curves and for moduli spaces of sheaves with 1-dimensional support on K3 surfaces.

Partial results towards Theorem 1.1 have appeared in the literature. The case $g = 2$ is a classical theorem of Bondal and Orlov [BO95, Theorem 2.9], who also proved that $P = 0$ in that case. Fonarev and Kuznetsov [FK18] proved that $D^b(C) \leftrightarrow D^b(N)$ if $C$ is a hyperelliptic curve using an explicit description of $N$ due to Desale and Ramanan [DR76]. They also proved that $D^b(C) \leftrightarrow D^b(N)$ for a general curve $C$ by a deformation argument. Narasimhan proved that $D^b(C) \leftrightarrow D^b(N)$ for all curves [Nar17, Nar18] using Hecke correspondences. He also showed that one can add the line bundles $O$ and $\theta^*$ to $D^b(C)$ to start a semi-orthogonal decomposition of $D^b(N)$.

In [BM19], Belmans and Mukhopadhyay work with the moduli space $M_C(r, \Lambda)$ of vector bundles of rank $r$ and determinant $\Lambda$, where $r \geq 2$ and $(r, \deg \Lambda) = 1$. They show that there is a fully faithful embedding $D^b(C) \hookrightarrow D^b(M_C(r, \Lambda))$ provided the genus is sufficiently high. Moreover, they use this embedding to find the start of a semi-orthogonal decomposition of $D^b(M_C(r, \Lambda))$ of the form $\theta^*, D^b(C), O, \theta^* \otimes D^b(C)$, this way extending the decomposition on $N = M_C(2, \Lambda)$ present in [Nar18]. Belmans, Galkin and Mukhopadhyay have conjectured, independently of Narasimhan, that $D^b(N)$ should have a semi-orthogonal decomposition with blocks $D^b(\text{Sym}^i C)$ (see [Bel18, Lee18]), and have collected additional evidence towards this conjecture in [BGM21]. Lee and Narasimhan [LN21] proved using Hecke correspondences that, if $C$ is non-hyperelliptic and $g \geq 16$, there is a fully faithful functor $D^b(\text{Sym}^2 C) \hookrightarrow D^b(N)$ whose image is left semi-orthogonal to the copy of $D^b(C)$ obtained earlier. They also introduced tensor bundles $E^{\otimes i}$ of the Poincaré bundle (see Section 2), which we discovered independently. If $D \in \text{Sym}^i C$ is a reduced sum of points $x_1 + \ldots + x_i$, the fiber $(E^{\otimes i})_D$ is a vector bundle on $N$ isomorphic to the tensor product $E_{x_1} \otimes \ldots \otimes E_{x_1}$. If the points have multiplicities, $(E^{\otimes i})_D$ is a deformation of the tensor product over $\mathbb{A}^1$ (see Corollary 2.10).

Instead of using Hecke correspondences (although they do make a guest appearance in Section 6), we prove Theorem 1.1 by analyzing Fourier–Mukai functors given by tensor bundles $E^{\otimes i}$ of the universal bundle $F$ on the moduli space of stable pairs $(E, \phi)$, where $E$ is a rank-two vector bundle on $C$ with fixed odd determinant line bundle of degree $d$ and $\phi \in H^0(E)$ is a non-zero
section. Stability condition on these spaces depends on a parameter, and we use extensively results of Thaddeus [Tha94] on wall-crossing. If $d = 2g - 1$ then there is a well-known diagram of flips

\begin{equation}
\begin{array}{c}
\tilde{M}_2 \\
M_1 \\
M_0 \\
\end{array} \quad \begin{array}{c}
\tilde{M}_3 \\
M_2 \\
\vdots \\
M_{g-1} \\
N \\
\end{array}
\end{equation}

where $M_0 = \mathbb{P}^{3g-3}$, $M_1$ is the blow up of $M_0$ in $C$, the rational map $M_{i-1} \dashrightarrow M_i$ is a standard flip of projective bundles over $\text{Sym}^i C$, and $\xi : M_{g-1} \to N$ is a birational Abel–Jacobi map with fiber $\mathbb{P} H^0(E)$ over a stable vector bundle $E$. Accordingly, $D^b(M_i)$ has a semi-orthogonal decomposition into $D^b(M_{i-1})$ and several blocks equivalent to $D^b(\text{Sym}^i C)$ with torsion supports (see Proposition 3.15 or [BFR22]). While these decompositions do not descend to $N$ and are not associated with the universal bundle, they are useful. Philosophically, tensor bundles on $\text{Sym}^i C \times N$ should be related to exterior powers of the tautological bundle of the universal bundle, which appear in the Koszul complex of the tautological section that vanishes on the flipped locus. One can try to connect two Fourier–Mukai functors via mutations. In practice, this Koszul complex is difficult to analyze except for $M_1$ (see Section 5) and we followed a less direct strategy towards proving Theorem 1.1.

In order to prove semi-orthogonality in (1.1) and full faithfulness of the Fourier–Mukai functors via the Bondal–Orlov criterion, we had to investigate coherent cohomology for a large class of vector bundles. The main difficulty in this kind of analysis is to find a priori numerical bounds on the class of acyclic vector bundles to get the induction going.

**Definition 1.2.** For an object $F$ in the derived category of a scheme $M$, we say that $F$ is $\Gamma$-acyclic if $R\Gamma(F) = 0$. That is, for us $\Gamma$-acyclicity will mean vanishing of all cohomology groups, including $H^0(F)$. Other authors have used the term immaculate for this property (cf. [ABKW20]).

Theorem 1.1 then requires the proof of $\Gamma$-acyclicity for several vector bundles. It is worth emphasizing that the moduli space $N$ depends on the complex structure of the curve $C$ by a classical theorem of Mumford and Newstead [MN68] later extended by Narasimhan and Ramanan [NR75]. The uniform shape of Theorem 1.1 is thus a surprisingly strong statement about coherent cohomology of vector bundles on $N$ that does not involve any conditions of the Brill–Noether type. Our approach utilizes the method of windows into derived categories of GIT quotients of Teleman, Halpern–Leistner, and Ballard–Favero–Katzarkov [Tel00,HL15,BFK19] to systematically predict behavior of coherent cohomology under wall-crossing. This dramatically reduces otherwise unwieldy cohomological computations to a few key cases,
which can be analyzed using other techniques. Rather unexpectedly, one of the difficult ingredients in the proof is acyclicity of certain line bundles (see Section 6). While cohomology of line bundles on the space of stable pairs was extensively studied in [Tha94] in order to prove the Verlinde formula, the line bundle that we need is outside of the scope of that paper.

Analogous recent applications of windows to moduli spaces include the proof of the Manin–Orlov conjecture on $\bar{M}_{0,n}$ by Castravet and Tevelev [CT20a, CT20b, CT20c] and analysis of Bott vanishing on GIT quotients by Torres [Tor20].

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2. Tensor vector bundles

Let $C$ be a smooth projective curve over $\mathbb{C}$. For integers $\alpha \geq 1$ and $1 \leq j \leq \alpha$, let $\pi_j : C^\alpha \to C$ be the $j$-th projection and $\tau : C^\alpha \to \text{Sym}^\alpha C$ the categorical $S_\alpha$-quotient, where $S_\alpha$ is the symmetric group. Since $C^\alpha$ is Cohen–Macaulay (in fact smooth), $\text{Sym}^\alpha C$ is smooth, and $\tau$ is equi-dimenional, we conclude that $\tau$ is flat by miracle flatness. Therefore, any base change $\tau : C^\alpha \times M \to \text{Sym}^\alpha C \times M$ is also a finite and flat categorical $S_\alpha$-quotient, where $M$ is any scheme over $\mathbb{C}$. The constructions in this section are functorial in $M$. In the following sections, $M$ will be one of the moduli spaces we consider.

Notation 2.1. For an $S_\alpha$-equivariant vector bundle $\mathcal{E}$ on $C^\alpha \times M$, we will denote by $\tau_\alpha^S \mathcal{E}$ the $S_\alpha$-invariant part of the pushforward $\tau_* \mathcal{E}$.

Lemma 2.2. Let $\mathcal{E}$ be an $S_\alpha$-equivariant locally free sheaf on $C^\alpha \times M$. Then $\tau_* \mathcal{E}$ and $\tau_\alpha^S \mathcal{E}$ are locally free sheaves on $\text{Sym}^\alpha C \times M$.

Proof. The scheme $C^\alpha \times M$ is covered by $S_\alpha$-equivariant affine charts $\text{Spec} R$ and $\tau^*$ is given by the inclusion of invariants $R^{S_\alpha} \subset R$. Since $R$ is a finitely generated and flat $R^G$-module, it is also a projective $R^G$-module. Let $E = H^0(\text{Spec} R, \mathcal{E})$. Since $E$ is a projective $R$-module, it is a direct summand of $R^s$ for some $s$. It follows that $E$ is a projective $R^G$-module, i.e. $\tau_* \mathcal{E}$ is locally free. Since $E^{S_\alpha}$ is a direct summand of $E$ as an $R^G$-module, it is also a projective $R^G$-module. Therefore, $\tau_\alpha^{S_\alpha} \mathcal{E}$ is a locally free sheaf as well. □

Definition 2.3. For any vector bundle $\mathcal{F}$ on $C \times M$, we define the following tensor vector bundles on $\text{Sym}^\alpha C \times M$,

$$\mathcal{F}^\otimes = \tau_\alpha^{S_\alpha} \left( \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \right) \quad \text{and} \quad \mathcal{F}^{\otimes \alpha} = \tau_\alpha^{S_\alpha} \left( \bigotimes_{j=1}^{\alpha} \pi_j^* \mathcal{F} \otimes \text{sgn} \right),$$

where $S_\alpha$ acts on $C^\alpha$ and also permutes the factors of the corresponding vector bundle on $C^\alpha$. Here sgn is the sign representation of $S_\alpha$. 
Lemma 2.4. The formation of tensor vector bundles is functorial in $M$, that is, given a morphism $f : M' \to M$ and its base changes $C \times M' \to C \times M$ and $\text{Sym}^\alpha C \times M' \to \text{Sym}^\alpha C \times M$, which we also denote by $f$, we have

$$f^*(\mathcal{F}^\oplus\tau) = (f^*\mathcal{F})^\oplus\tau \quad \text{and} \quad f^*(\mathcal{F}^\ominus\tau) = (f^*\mathcal{F})^\ominus\tau.$$  

Proof. Since $\tau$ is flat, this follows from cohomology and base change. \hfill \square

Remark 2.5. Similarly, one can define bundles $\mathcal{F}^\oplus\tau = \tau^0_s\left(\bigoplus_{j=1}^\alpha S_j^*\mathcal{F}\right)$ and $\mathcal{F}^\ominus\tau = \tau^0_s\left(\bigoplus_{j=1}^\alpha S_j^*\mathcal{F} \otimes \text{sgn}\right)$.

Remark 2.6. Since $\mathcal{O}_{C^\alpha \times M} \otimes \text{sgn}$ is the equivariant dualizing sheaf for $\tau$ (see [KS15, Lemma 5.8]), by Grothendieck duality we have $(\mathcal{F}^\oplus\tau)^! = \mathcal{F}^\ominus\tau$.

For a divisor $D \in \text{Sym}^\alpha C$ and a vector bundle $\mathcal{G}$ on $\text{Sym}^\alpha C \times M$, let us denote by $\mathcal{G}_D = \mathcal{G}_{\{D\} \times M}$. We usually view $\mathcal{G}_D$ as a vector bundle on $M$.

Lemma 2.7. If $D = \sum \alpha_k x_k$ with $x_k \neq x_l$ for $k \neq l$, then we have

$$(\mathcal{F}^\oplus\tau)_D = \bigotimes (\mathcal{F}^\oplus\tau)_{\alpha_k x_k}, \quad (\mathcal{F}^\ominus\tau)_D = \bigotimes (\mathcal{F}^\ominus\tau)_{\alpha_k x_k}.$$  

Proof. Indeed, the quotient $\tau : C^\alpha \to \text{Sym}^\alpha C$ is étale-locally near $D \in \text{Sym}^\alpha C$ isomorphic to the product of quotients $\prod C^{\alpha_k} \to \prod \text{Sym}^{\alpha_k} C$. Moreover, $\text{sgn}$ restricts to the tensor product of sign representations of $\prod S_{\alpha_k}$. \hfill \square

Consider the non-reduced scheme $\mathbb{D}_\alpha = \text{Spec} \mathbb{C}[t]/t^\alpha$, with maps $pt \xrightarrow{\iota} \mathbb{D}_\alpha \xrightarrow{\rho} \mathbb{C}$ given by the obvious pullbacks $\mathbb{C} \xrightarrow{\rho} \mathbb{C}[t]/t^\alpha \xrightarrow{\iota} \mathbb{C}$. We will write $\iota$ and $\rho$ for the base changes to $M$ of these morphisms, that is, $M \xrightarrow{\iota} \mathbb{D}_\alpha \times M \xrightarrow{\rho} M$. For a locally free sheaf $\mathcal{F}$ on $\mathbb{D}_\alpha \times M$, we denote by $\mathcal{F}_0 = \iota^*\mathcal{F}$ its restriction to $M$.

Definition 2.8. For two vector bundles $\mathcal{F}, \mathcal{G}$ on a scheme $M$, we will say that $\mathcal{F}$ is a deformation of $\mathcal{G}$ over $\mathbb{A}^1$ if there is a coherent sheaf $\tilde{\mathcal{F}}$ on $M \times \mathbb{A}^1$, flat over $\mathbb{A}^1$, with $\tilde{\mathcal{F}}|_t \cong \mathcal{F}$ for $t \neq 0$, while $\tilde{\mathcal{F}}|_0 \cong \mathcal{G}$.

Lemma 2.9. Every locally free sheaf $\mathcal{F}$ on $\mathbb{D}_\alpha \times M$ is a deformation of $\rho^*\mathcal{F}_0$ over $\mathbb{A}^1$. In particular, $\rho_*\mathcal{F}$ is a deformation of $\mathcal{F}^\oplus_{\mathbb{A}^1}$ over $\mathbb{A}^1$.

Proof. Let $\lambda : A^1 \times \mathbb{D}_\alpha \to \mathbb{D}_\alpha$ be the map defined by its pullback $\lambda^* : t \mapsto ts$, and also denote by $\lambda$ its base change to $M$. We claim that the locally free sheaf $\lambda^*\mathcal{F}$ gives the required deformation. Indeed, the restriction of $\lambda^*\mathcal{F}$ to $\{s_0\} \in A^1_0$ is the pullback of $\mathcal{F}$ along the composition $b_{s_0} = \lambda \circ j_{s_0}$

$$\mathbb{D}_\alpha \times M \xrightarrow{j_{s_0}} \mathbb{D}_\alpha \times A^1_0 \times M \xrightarrow{\lambda} \mathbb{D}_\alpha \times M$$

determined by its pullback $b_{s_0}^* : t \mapsto s_0 t$. When $s_0 \neq 0$, $b_{s_0}^*\mathcal{F} \cong \mathcal{F}$. On the other hand, when $s_0 = 0$, the map $b_0$ factors as the composition

$$\mathbb{D}_\alpha \times M \xrightarrow{b_0} \mathbb{D}_\alpha \xrightarrow{\iota} \mathbb{D}_\alpha \times M$$

do so $b_0^*\mathcal{F} = \rho^*\iota^*\mathcal{F} = \rho^*\mathcal{F}_0$, as desired. The last statement follows from projection formula and the fact that $\rho_*\rho^*\mathcal{O}_M \cong \mathcal{O}_{\mathbb{A}^1}$. \hfill \square
Suppose $D = \alpha x$ is a fat point, i.e. a divisor given by a single point $x$ with multiplicity $\alpha$, and let $t$ be a local parameter on $C$ at $x$. Note that the notation $\mathcal{O}_D$ is unfortunately ambiguous, because it can denote both the structure sheaf of the subscheme $D \subset C$ and the skyscraper sheaf of the point $\{D\} \in \text{Sym}^\alpha C$. When confusion is possible, we denote the latter sheaf by $\mathcal{O}_{\{D\}}$. Then

$$
\tau^*\mathcal{O}_{\{D\}} \cong \mathbb{C}[t_1, \ldots, t_\alpha]/(\sigma_1, \ldots, \sigma_\alpha)
$$

is the so-called covariant algebra, where $\sigma_1, \ldots, \sigma_\alpha$ are the elementary symmetric functions in variables $t_j = \pi_j(t)$. Call $\mathbb{B}_\alpha = \text{Spec } \tau^*\mathcal{O}_{\{D\}}$. By the Newton formulas, $t_j^\alpha = 0$ for every $j = 1, \ldots, \alpha$, and in particular, every map $\pi_j : \mathbb{B}_\alpha \rightarrow C$ factors through $\mathbb{D}_\alpha$. By abuse of notation, we have a diagram of morphisms

\[
\mathbb{B}_\alpha \times M \xrightarrow{\pi_j} \mathbb{D}_\alpha \times M \xrightarrow{q} C \times M \xrightarrow{\tau} M
\]

**Corollary 2.10.** Let $D = x_1 + \ldots + x_\alpha$ (possibly with repetitions). Then both $(\mathcal{F}^{\mathbb{E}_0})_D$ and $(\mathcal{F}^{\mathbb{E}_0})_D$ are deformations of $F_{x_1} \otimes \ldots \otimes F_{x_\alpha}$ over $\mathbb{A}^1$.

**Proof.** By (2.1), it suffices to consider the case when $D = \alpha x$. Using the notation as in the diagram (2.3), the restriction $(\mathcal{F}^{\mathbb{E}_0})_D$ can be written as $\tau^*_S \left( \bigotimes \pi_j^* q^* F \right)$, by flatness of $\tau$. The construction of Lemma 2.9 commutes with the $S_\alpha$-action, so $\tau^*_S \left( \bigotimes \pi_j^* q^* F \right)$ is a deformation of $\tau^*_{\mathbb{E}_0} \left( \bigotimes \pi_j^* \rho^* F_x \right)$ over $\mathbb{A}^1$, since $(q^* F)_0 = F_x = F_{\{x\} \times M}$. Note $\pi_j^* \rho^* = \tau^*$, so using the projection formula, we get that $(\mathcal{F}^{\mathbb{E}_0})_D$ is a deformation of $\left( \bigotimes_{j=1}^\alpha F_x \right) \otimes \tau^*_{\mathbb{E}_0}(\mathcal{O}_{\mathbb{B}_\alpha \times M})$, and similarly, $(\mathcal{F}^{\mathbb{E}_0})_D$ is a deformation of $\left( \bigotimes_{j=1}^\alpha F_x \right) \otimes \tau^*_{\mathbb{E}_0}(\mathcal{O}_{\mathbb{B}_\alpha \times M} \otimes \text{sgn})$. By flatness of the quotient $C^{\alpha} \rightarrow \text{Sym}^{\alpha} C$, the covariant algebra $\mathcal{O}_{\mathbb{B}_\alpha}$ is the regular representation $\mathbb{C}[S_\alpha]$ of $S_\alpha$. It follows that it contains the trivial and the sign representations each with multiplicity 1, and therefore $\tau^*_{\mathbb{E}_0}(\mathcal{O}_{\mathbb{B}_\alpha \times M}) = \tau^*_{\mathbb{E}_0}(\mathcal{O}_{\mathbb{B}_\alpha \times M} \otimes \text{sgn}) = \mathcal{O}_M$. This concludes the proof. \qed

**Remark 2.11.** If we have a $G$-action on $M$ and a $G$-equivariant bundle $\mathcal{F}$, then the deformation of Corollary 2.10 can also be chosen to be $G$-equivariant, i.e. given by a $G$-equivariant bundle on $\mathbb{A}^1 \times M$. This is because the deformation in the proof of Lemma 2.9 can be chosen to be $G$-equivariant.

**Definition 2.12.** A vector bundle $\mathcal{F}$ on a scheme $M$ is said to be a stable deformation of a vector bundle $\mathcal{G}$ over $\mathbb{A}^1$ if there is some vector bundle $\mathcal{K}$ such that $\mathcal{F} \oplus \mathcal{K}$ is a deformation of a direct sum $\mathcal{G}^{\oplus r}$ for some $r > 0$. 
Proposition 2.13. Let $D = x + \tilde{D}$. Then the vector bundle $(\mathcal{F}^{\mathbb{G}_m})_D$ is a stable deformation of the vector bundle $\mathcal{F}_x \otimes (\mathcal{F}^{\mathbb{G}(\alpha-1)})_{\tilde{D}}$ over $\mathbb{A}^1$.

Proof. By Lemma 2.7, it suffices to consider the case $D = \alpha x$. Let $W_\alpha = \mathbb{C}^\alpha$ be the tautological representation of $S_\alpha$, which splits as a sum of the trivial and the standard representations, $W_\alpha = \mathbb{C} \oplus V_\alpha$. For any $S_\alpha$-equivariant vector bundle $\mathcal{E}$ on $\mathbb{B}_\alpha \times M$, we have

$$\tau_\alpha^S_\alpha(\mathcal{E} \otimes W_\alpha) = \tau_\alpha^S_\alpha(\mathcal{E}) \oplus \tau_\alpha^S_\alpha(\mathcal{E} \otimes V_\alpha).$$

On the other hand, we have $W_\alpha = \mathbb{C}[S_\alpha/S_{\alpha-1}]$, where $S_{\alpha-1} \hookrightarrow S_\alpha$ is the inclusion given by fixing the $\alpha$-th element. Then, by Frobenius reciprocity, $\tau_\alpha^S_\alpha(\mathcal{E} \otimes W_\alpha) = \tau_\alpha^S_{\alpha-1}(\mathcal{E}) = \rho_\alpha \circ (\pi_\alpha)^{S_{\alpha-1}}(\mathcal{E})$, where $\pi_\alpha$ is the $\alpha$-th projection.

By Lemma 2.9 this bundle is a deformation of $((\pi_\alpha)^{S_{\alpha-1}}\mathcal{E})_0$ over $\mathbb{A}^1$. Now let $\mathcal{E}$ be $\bigotimes j q^* F$. Then $\tau_\alpha^S_\alpha(\mathcal{E})$ is precisely $(\mathcal{F}^{\mathbb{G}_m})_D$ and, by projection formula,

$$((\pi_\alpha)^{S_{\alpha-1}}\mathcal{E})_0 = \mathcal{F}_x \otimes \left( (\pi_\alpha)^{S_{\alpha-1}} \left( \bigotimes_{j=1}^{\alpha-1} \pi_j q^* F \right) \right)_0$$

$$= \mathcal{F}_x \otimes (\pi_\alpha)^{S_{\alpha-1}} \left( \bigotimes_{j=1}^{\alpha-1} (\pi_j q^* F) | t_\alpha = 0 \right) = \mathcal{F}_x \otimes \left( \mathcal{F}^{\mathbb{G}(\alpha-1)} \right)_{(\alpha-1)x}$$

since the subscheme $(t_\alpha = 0) \subset \mathbb{B}_\alpha$ is isomorphic to $\mathbb{B}_{\alpha-1}$ and the restriction of $\pi_\alpha$ to it is isomorphic to the quotient $\tau$ (for the group $S_{\alpha-1}$).

Remark 2.14. We will use stable deformations for semi-continuity arguments. If $\mathcal{F}$ is a stable deformation of $\mathcal{G}$, $M$ is proper and $H^p(\mathcal{G}) = 0$, then, by the semi-continuity theorem, $H^p(\mathcal{F}) = 0$, too. In particular, if $\mathcal{G}$ is $\Gamma$-acyclic, then so is $\mathcal{F}$.

Remark 2.15. Let $D = x_1 + \tilde{D}$, $\tilde{D} = x_2 + \ldots + x_\alpha$ (possibly with repetitions). Suppose $M$ is proper. Since $(\mathcal{F}^{\mathbb{G}_m})_D$ and $\mathcal{F}_x \otimes (\mathcal{F}^{\mathbb{G}(\alpha-1)})_{\tilde{D}}$ are both deformations of $\mathcal{F}_x \otimes \ldots \otimes \mathcal{F}_x$ over $\mathbb{A}^1$ by Corollary 2.10, they have the same Euler characteristic. Combining this with Remark 2.14 if $H^p(\mathcal{F}_x \otimes (\mathcal{F}^{\mathbb{G}(\alpha-1)})_{\tilde{D}}) = 0$ for $p > 0$ then both $H^p((\mathcal{F}^{\mathbb{G}_m})_D) = 0$ for $p > 0$ and $H^0((\mathcal{F}^{\mathbb{G}_m})_D) = H^0(\mathcal{F}_x \otimes (\mathcal{F}^{\mathbb{G}(\alpha-1)})_{\tilde{D}})$. The same results hold for $(\mathcal{F}^{\mathbb{G}_m})_D$ and $\mathcal{F}_x \otimes (\mathcal{F}^{\mathbb{G}(\alpha-1)})_{\tilde{D}}$.

3. Wall-crossing on moduli spaces of stable pairs

Let $C$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. In [Tha94], Thaddeus studies moduli spaces of pairs $(E, \phi)$, where $E$ is a rank-two vector bundle on $C$ with fixed determinant line bundle $\Lambda$ and $\phi \in H^0(E)$ is a non-zero section. We use these results extensively and so, for ease of reference, try to follow the notation in [Tha94] as closely as possible. We always assume
that \( d = \deg E > 0 \). For a given choice of a parameter \( \sigma \in \mathbb{Q} \) the following stability condition is imposed: for every line subbundle \( L \subset E \), one must have

\[
\deg L \leq \begin{cases} \frac{d}{2} - \sigma & \text{if } \phi \in H^0(L), \\ \frac{d}{2} + \sigma & \text{if } \phi \notin H^0(L). \end{cases}
\]

Semi-stable pairs exist whenever \( \sigma \in (0, d/2] \) \cite{Tha94, 1.3}, which we will assume. The next lemma follows the ideas of \cite{Tha94, 2.1}:

Lemma 3.1. For a given line bundle \( \Lambda \) of degree \( d \), the moduli stack \( \mathcal{M}_\sigma(\Lambda) \) of semi-stable pairs is a smooth algebraic stack.

Proof. \( \mathcal{M}_\sigma(\Lambda) \) is a fiber of the morphism \( \mathcal{M}_d(\Lambda) \to \text{Pic}^d(C) \), \((E, \phi) \mapsto \det E\), from the stack of semi-stable pairs \((E, \phi)\), where \( E \) is a degree \( d \) vector bundle. We first show that \( \mathcal{M}_d(\Lambda) \) is smooth. Obstructions to deformations of a morphism of sheaves \( \phi \) from a fixed source \( O_C \) to a varying target \( E \) lie in \( \text{Ext}^1(O_C \to E, E) \). The truncation exact triangle of the complex \([O_C \to E]\) yields an exact sequence

\[
\text{Ext}^1(E, E) \to \text{Ext}^1(O_C, E) \to \text{Ext}^1([O_C \to E], E) \to 0.
\]

We claim that the first map is surjective, so obstructions vanish. By Serre duality, it suffices to prove injectivity of the map of sheaves \( E^*(K_C) \to E^* \otimes E(K_C) \) and this follows from \( \phi \neq 0 \) (cf. the proof of \cite{Tha94, 2.1}). Next we consider obstructions to deformations of \((E, \phi)\) fixing the determinant, which amounts to studying the map \( \text{Ext}^1(E, E)_0 \to \text{Ext}^1(O_C, E) \), where \( \text{Ext}^1(E, E)_0 \) denotes traceless endomorphisms. However, this map is also surjective because the Serre-dual map is induced by the map of sheaves \( E^*(K_C) \to \text{End}(E, E)_0(K_C) \), which is still injective - a non-zero scalar matrix cannot have rank 1.

The moduli space \( M_\sigma(\Lambda) \) of \( S \)-equivalence classes of stable pairs exists as a projective variety and, in the case there is no strictly semi-stable locus, it is smooth, isomorphic to the stack \( \mathcal{M}_\sigma(\Lambda) \) and carries a universal bundle \( F \) with a universal section \( \tilde{\phi} : O_C \times M_\sigma(\Lambda) \to F \). A salient point is that stable pairs, unlike stable vector bundles, don't have non-trivial automorphisms \cite{Tha94, 1.6}.

The spaces \( M_\sigma(\Lambda) \) can be obtained as GIT quotients as follows. Let \( \chi = \chi(E) = d + 2 - 2g \). If \( d \gg 0 \), a bundle \( E \) is generated by global sections, and \( \chi = h^0(E) \). Then \( M_\sigma(\Lambda) \) is a GIT quotient of \( U \times \mathbb{P}^\chi \) by \( SL_\chi \), where \( U \subset \text{Quot} \) is the locally closed subscheme of the Quot scheme \cite{Gro95} corresponding to locally free quotients \( O_C^\chi \to E \) inducing an isomorphism \( s : \mathbb{C}^\chi \to H^0(E) \) and such that \( \Lambda^2 E = \Lambda \). The isomorphism \( s \) maps a map \( \Lambda^2 \mathbb{C}^\chi \to H^0(\Lambda) \), and we get an inclusion \( U \times \mathbb{P}^\chi \to \mathbb{P} \text{Hom} \times \mathbb{P}^\chi \), where we write \( \mathbb{P} \text{Hom} \) for \( \mathbb{P} \text{Hom}(\Lambda^2 \mathbb{C}^\chi, H^0(\Lambda)) \), and a quotient \( s : O_C^\chi \to E \) on the left is sent to the induced map in the first coordinate. Then \( M_\sigma(\Lambda) \)
can be seen as the GIT quotient of a closed subset of $\mathbb{P} \text{Hom} \times \mathbb{P} \mathbb{C}^\chi$ by $SL_\chi$, where the linearization is given by $O(\chi + 2\sigma, 4\sigma)$.

For arbitrary $d$, we pick any effective divisor $D$ on $C$ with $\deg D \gg 0$, and $M_\sigma(\Lambda)$ can be seen as the closed subset of $M_\sigma(\Lambda(2D))$ consisting of pairs $(E, \phi)$ such that $\phi|_D = 0$. This way, $M_\sigma(\Lambda)$ is a GIT quotient by $SL_\chi'$, with $\chi' = d + 2 - 2g + 2\deg D$, of the closed subset $X \subset U' \times \mathbb{P} \mathbb{C}^{\chi'}$ determined by the condition that $\phi$ vanishes along $D$ [Tha94 §1]. Regardless of the GIT, the embedding $M_\sigma(\Lambda) \subset M_\sigma(\Lambda(2D))$ will play an important role.

**Remark 3.2.** Scalar matrices in $SL_\chi'$ act trivially on $U \times \mathbb{P} \mathbb{C}^{\chi'}$, so the action factors through the quotient $SL_\chi' \rightarrow PGL_\chi'$. If we replace $O(\chi' + 2\sigma, 4\sigma)$ by its $\chi'$-th power, this line bundle carries a $PGL_\chi'$-linearization and $M_\sigma(\Lambda)$ can also be written as a GIT quotient $X / PGL_\chi'$. Moreover, the moduli stack $M_\sigma(\Lambda)$ is isomorphic to the corresponding GIT quotient stack $[X^{ss}/PGL_\chi']$.

For fixed $\Lambda$, the spaces $M_\sigma(\Lambda)$ are all GIT quotients of the same scheme, with different stability conditions. The GIT walls occur when $\sigma \in d/2 + \mathbb{Z}$, and for $0 \leq i \leq v = [(d-1)/2]$ we have different GIT chambers with moduli spaces $M_0, M_1, \ldots, M_v$, where $M_i = M_i(\Lambda) = M_\sigma(\Lambda)$ for $\sigma \in (\max(0,d/2-i-1),d/2-i)$. These $M_i$ are smooth projective rational varieties of dimension $d + g - 2$. Indeed, $M_0 = \mathbb{P} H^1(C, \Lambda^{-1})$ is a projective space, $M_1$ is a blow-up of $M_0$ along a copy of $C$ embedded by the complete linear system of $\omega_C \otimes \Lambda$, and the remaining ones are small modifications of $M_1$. More precisely, for each $0 \leq i \leq v = [(d-1)/2]$ there are projective bundles $\mathbb{P} W_i^+$ and $\mathbb{P} W_i^-$ over the symmetric product $\text{Sym}^i C$, of (projective) ranks $d + g - 2i - 2$, $i - 1$, respectively, with embeddings $\mathbb{P} W_i^+ \hookrightarrow M_i$ and $\mathbb{P} W_i^- \hookrightarrow M_{i-1}$, and such that $\mathbb{P} W_i^+$ parametrizes the pairs $(E, \phi)$ appearing in $M_i$ but not in $M_{i-1}$, while $\mathbb{P} W_i^-$ parametrizes those appearing in $M_{i-1}$ but not in $M_i$.

We have a diagram of flips (3.1), where $\tilde{M}_i$ is the blow-up of $M_{i-1}$ along $\mathbb{P} W_i^-$ and also the blow-up of $M_i$ along $\mathbb{P} W_i^+$. Here $N$ is the moduli space of ordinary slope-semistable vector bundles as in the Introduction and the map $M_\sigma \rightarrow N$ is an “Abel-Jacobi” map with fiber $\mathbb{P} H^0(C, E)$ over a vector bundle $E$. If $d \geq 2g - 1$ the Abel-Jacobi map is surjective, and if $d = 2g - 1$ it is a birational morphism (see [Tha94 §3] for details).

\begin{align*}
(3.1) \quad M_1 & \leftarrow \tilde{M}_2 & \tilde{M}_3 & \cdots & \tilde{M}_v \\
 & \downarrow & & & \downarrow \\
 M_0 & \leftarrow M_2 & \cdots & M_v & \rightarrow N
\end{align*}

**Notation 3.3.** By abuse of notation, we will sometimes write $M_i(d)$ to denote the moduli space $M_i = M_i(\Lambda)$, where $d = \deg \Lambda$.

**Notation 3.4.** In what follows, $v$ will always denote $[(d-1)/2]$.
The Picard group of $M_1 = \text{Bl}_C M_0$ is generated by a hyperplane section $H$ in $M_0 = \mathbb{P}^{d+g-2}$ and the exceptional divisor $E_1$ of the morphism $M_1 \rightarrow M_0$. Since the maps $M_i \rightarrow M_{i+1}$ are small birational modifications for each $i \geq 1$, there are natural isomorphisms $\text{Pic} M_1 \cong \text{Pic} M_i$, $i \geq 1$. The following notation is taken from \cite{Tha94} §5.

**Definition 3.5.** For each $m, n$, we denote the line bundle $\mathcal{O}_{M_i}((m+n)H - nE_1)$ by $\mathcal{O}_i(m, n)$, while $\mathcal{O}_i(m, n)$ will denote the image of $\mathcal{O}_{M_i}(m, n)$ under the isomorphism $\text{Pic} M_1 \cong \text{Pic} M_i$.

**Remark 3.6.** By \cite{Tha94} 5.3, the ample cone of $M_i$ is bounded by $\mathcal{O}_i(1, i - 1)$ and $\mathcal{O}_i(1, i)$ for $0 < i < v$, while the ample cone of $M_v$ is bounded below by $\mathcal{O}_v(1, v - 1)$ and contains the cone bounded on the other side by $\mathcal{O}_v(2, d - 2)$. For arbitrary $m, n$, we denote the line bundle $\mathcal{O}_i(m, n)$ by $\mathcal{O}_i(m, n)$.

**Remark 3.7.** For any effective divisor $D$ on $C$ of deg $D = \alpha$, we have a closed immersion $M_{1-\alpha}(\Lambda(-2D)) \hookrightarrow M_i(\Lambda)$, as the locus of pairs $(E, \phi)$ where the section $\phi$ vanishes along $D$ \cite{Tha94} 1.9. The restriction of $\mathcal{O}_i(m, n)$ to $M_{1-\alpha}(\Lambda(-2D))$ is $\mathcal{O}_{i-\alpha}(m, n - ma)$ \cite{Tha94} 5.7.

Suppose $d \gg 0$. Then the universal bundle $F$ on $M_i$ is the descent from the equivariant vector bundle $F(1)$ on $X \times C \subset U \times \mathbb{P}C \times C$, where $\mathcal{O}_X \rightarrow F$ is the universal quotient bundle over $U \times C$, and the universal section $\phi$ descends from the universal section of $F(1)$ \cite{Tha94} 1.19. Let $\pi : C \times M_i \rightarrow M_i$ be the projection. For every $i \geq 1$, the determinant of cohomology line bundle $\det \pi_i F$ (cf. \cite{KM76}) descends from $\mathcal{O}(0, \chi)$ on $\mathbb{P} \text{Hom} \times \mathbb{P}C$ \cite{Tha94} 5.4 and 5.14. On $M_1$, $\det \pi_1 F$ corresponds to $\mathcal{O}_{M_1}((g-d-1)H - (g-d)E_1) = \mathcal{O}_1(-1, g-d)$.

For arbitrary $d$, consider an embedding $i : M_i \hookrightarrow M' = M_{\sigma}(\Lambda(2D))$, deg $D \gg 0$, as above, and let $F'$ be the universal bundle on $M'$. Then we have a short exact sequence \cite{Tha94} 1.19

$$0 \rightarrow F \rightarrow i^* F' \rightarrow i^* F'|_{D \times M_i} \rightarrow 0.$$  

In particular, $F$ is the descent from an object on $X \times C \subset U' \times \mathbb{P}C' \times C$. The same is true for $\det \pi_i F$ and $\Lambda^2 F_x$.

**Lemma 3.8.** $F_x \cong i^* F'_x$ for every $x \in C$.

**Proof.** We tensor \cite{3.2} with $\mathcal{O} \{x\} \times M_i$, which gives an exact sequence

$$0 \rightarrow \text{Tor}_1^{C \times M_i}(i^* F'|_{D \times M_i}, \mathcal{O} \{x\} \times M_i) \rightarrow F_x \rightarrow i^* F'_x \rightarrow$$

$$\rightarrow i^* F'|_{D \times M_i} \otimes_{C \times M_i} \mathcal{O} \{x\} \times M_i \rightarrow 0.$$  

If $x \notin D$ then $\text{Tor}_1^{C \times M_i}(i^* F'|_{D \times M_i}, \mathcal{O} \{x\} \times M_i) = i^* F'|_{D \times M_i} \otimes \mathcal{O} \{x\} \times M_i = 0 = F_x \cong i^* F'_x$. If $x \in D$ then $\text{Tor}_1^{C \times M_i}(i^* F'|_{D \times M_i}, \mathcal{O} \{x\} \times M_i) \cong \mathcal{O}_D \otimes \mathcal{O}_{\{x\}} \cong \mathcal{O}_x$, and the sequence splits into two isomorphisms, $F_x \cong i^* F'_x$ and $i^* F'_x \cong i^* F'_x$. □
Definition 3.9. We introduce important line bundles
\[ \psi^{-1} := \det \pi_1 F = \mathcal{O}_i(-1, g - d), \]
\[ \Lambda_M := \Lambda^2 F_x = \mathcal{O}_i(0, -1), \]
\[ \zeta := \psi \otimes \Lambda^d_M = \mathcal{O}_i(1, g - 1) \]
and
\[ \theta := \psi^2 \otimes \Lambda^b_M = \mathcal{O}_i(2, d - 2), \]
where \( \chi = d + 2 - 2g \) (cf. [Nar17, Proposition 2.1]).

Lemma 3.10. For a point \( x \in C \) and every \( i \geq 1 \), we have exact sequences
\[ 0 \to \Lambda^{-1}_M \to F^\vee_x \to \mathcal{O}_{M_i(\Lambda)} \to \mathcal{O}_{M_i-1(\Lambda(-2x))} \to 0 \]
and
\[ 0 \to \mathcal{O}_{M_i(\Lambda)} \to F_x \to \Lambda_M \to \Lambda_M|_{M_i-1(\Lambda(-2x))} \to 0. \]

Proof. By Remark 3.7, the zero locus of the section \( \phi_x \) of \( F_x \) is smooth and has codimension 2. Therefore, the Koszul complex and the dual Koszul complex of \( (F_x, \phi_x) \) are exact.

Definition 3.11. Let \( M = M_i(\Lambda) \) be a moduli space in the interior of a GIT chamber, as above, and let \( F \) be the universal bundle on \( C \times M \). We apply the constructions of Section 2 to \( F \). In particular, for a divisor \( D \in \text{Sym}^a C \), we will denote
\[ G_D = \left( F^{\otimes a} \right)_D \quad \text{and} \quad G_D^\vee = \left( \left( F^{\otimes a} \right)^\vee \right)_D. \]

Consider again the diagram (3.1). The wall between two consecutive chambers \( M_i-1 \) and \( M_i \) occurs at \( \sigma = d/2 - i \). The birational transformation \( M_i-1 \dashrightarrow M_i \) is an isomorphism outside of the loci \( \mathbb{P}W_i^- \subset M_i-1 \), \( \mathbb{P}W_i^+ \subset M_i \), where \( W_i^- \) and \( W_i^+ \) are vector bundles over the symmetric product \( \text{Sym}^i C \) of rank \( i \) and \( d + g - 1 - 2i \), respectively. We have a diagram
\[ \begin{array}{ccc}
\tilde{M} & \xleftarrow{\sigma-\epsilon} & M_{i-1} = M_{\sigma+\epsilon} \\
& \xrightarrow{\sigma-\epsilon} & M_{i} = M_{\sigma} \\
& \xleftarrow{\sigma+\epsilon} & \end{array} \]
where \( \tilde{M} \) is both the blow-up of \( M_{\sigma+\epsilon} = M_{i-1} \) along \( \mathbb{P}W_i^- \) and the blow-up of \( M_{\sigma-\epsilon} = M_i \) along \( \mathbb{P}W_i^+ \). The variety \( M_{\sigma} \) is singular, obtained from the contraction to \( \text{Sym}^i C \) of the exceptional locus \( \mathbb{P}W_i^- \times_{\text{Sym}^i C} \mathbb{P}W_i^+ \subset \tilde{M} \).

When \( d \gg 0 \), \( M_{\sigma-\epsilon}(\Lambda) \) and \( M_{\sigma}(\Lambda) \) are obtained as GIT quotients of \( U \times \mathbb{P}C^\chi \), with \( \chi = d + 2 - 2g \). When \( d \) is arbitrary, take an effective divisor \( D' \) of large degree, so that \( M_{\sigma} \hookrightarrow M'_{\sigma} := M_{\sigma}(\Lambda(2D')) \), where \( M'_{\sigma} \) is a GIT quotient with a semi-stable locus \( X' \subset U' \times \mathbb{P}C^{\chi'} \), \( \chi' = d + 2 - 2g + 2\deg D' \). The spaces \( M_{\sigma+\epsilon}(\Lambda) \) and \( M_{\sigma}(\Lambda) \) are then GIT quotients by \( SL_{\chi'} \) of a closed subset of \( U' \times \mathbb{P}C^{\chi'} \) determined by the condition that in the pair
(E', φ'), the section φ' vanishes along D'. If we call L±, L0 the corresponding linearizations, we can write X ⊂ X', the semi-stable locus of L0, as the union X = Xss(L±) ∪ Xss(L−) ∪ Z, where the locus Z = Xss(L+) ∩ Xss(L−) corresponds to pairs (E', φ'), such that E' splits as

\[ E' = L' ⊕ K', \]

with \( \deg L' = i + \deg D', \deg K' = d - i + \deg D', \) and φ' ∈ \( H^0(L') \) vanishes along D' (see [Tha94, 1.4]). The map \( O_C^{\psi'} \rightarrow E' \) is then given by a block-diagonal matrix \( (O_C^a \rightarrow L') ⊕ (O_C^b → K') \), where \( a = h^0(L') \), \( b = h^0(K') \) and \( a + b = h^0(L' ⊕ K') = χ' \). The strictly semi-stable locus \( X^ss(L_0) = X^u(L_+) \cap X^u(L_-) \) consists of the orbits whose closure intersects Z (cf. [Pot16, Remark 7.4]).

Using techniques from [HL15] and [BFK19], we compare the derived categories on either side of the wall \( M_\sigma \). We write \( M_{σ±} = X // _{L±} PGL' \) (cf. Remark 3.2) and take Kempf–Ness stratifications of the unstable loci \( X^u(L_±) \) with strata \( S_± \) determined by pairs \((Z^i, λ_±)\), where \( λ_±(t) = λ_±^j \) are one-parameter subgroups and \( Z^i \) is the fixed locus of \( λ_±^j \) (see [HL15] §2.1 for details).

**Remark 3.12.** From the discussion above, it follows that in this case the KN stratification of the unstable locus in \( X \) with respect to \( L_± \) has only one stratum \( S_± \), consisting of the vector bundle \( W_± \) over the locus \( Z \) of split bundles described above. In the notation of [HL15] §2, the stratum \( S_± \) is determined by the pair \((Z, λ)\), where \( λ = λ_± = G_m \) is the stabilizer of \( Z \), acting on a split bundle \( E' = L' ⊕ K' \) by \((θ^i, t^−a)\).

**Remark 3.13.** Let \( Z \) be the stack \([Z/LL]\), where \( LL \) is the Levi subgroup, i.e. the centralizer of \( λ \) in \( PGL' \). We have a short exact sequence of groups

\[ 1 \rightarrow C_m \rightarrow LL \rightarrow PGL_a \times PGL_b \rightarrow 1 \]

with \( C_m = λ \) acting on \( Z \) trivially and \( [Z/PGL_a \times PGL_b] \cong Sym^i C \). Indeed, the action of \( PGL_a \times PGL_b \) on \( Z \) is free, and each orbit is determined by a divisor \( D ∈ Sym^i C \), where \( D + D' \) is the zero locus of the section \( φ' ∈ H^0(L') \). Therefore \( Z \cong [Sym^i C/G_m] \), with the trivial action of \( G_m \).

For σ = d/2 − i with 1 < i ≤ v, \( M_{σ±} \) (ε > 0) is isomorphic to the corresponding quotient stack, since the action of \( PGL' \) is free on the stable locus by [Tha94, 1.6]. Let \( η_± = \text{weight}_{λ_±} \det N^\vee_{S_±/X} \). For any choice of an integer \( w \), \( D^b(M_{σ±}) \) is equivalent to the window subcategory \( C^w_\omega \subset D^b([X/PGL']) \) determined by objects having \( η_± \)-weights in the range \( \{ w, w + η_± \} \) for the unique stratum \( S_± \) (see [HL15] Theorem 2.10). If \( \text{weight}_{λ_±} \omega_∥ Z = η_− - η_+ > 0 \), we get an embedding \( D^b(M_{σ±}) \subset D^b(M_{σ−}) \) (see [HL15] Proposition 4.5) and the Remark following it.

**Lemma 3.14.** In the wall-crossing between the spaces \( M_{σ−}(Λ) = M_{i−1} \) and \( M_{σ−}(Λ) = M_i \), the window has width \( η_+ = i, η_− = d + g − 1 − 2i \) after the standard rescaling (see the proof).
Proof. We use the notation as in the discussion above, with \( M_\sigma \hookrightarrow M'_\sigma := M_\sigma(\Lambda(2D')) \), \( D' \) effective with deg \( D' \gg 0 \). For \( \mathcal{L}_\pm \), there is no strictly semi-stable locus and in fact \( PGL_{c'} \) acts freely on the semi-stable locus \cite[1.6]{Tha94}, so \( M_{i-1} = M_{\sigma+\varepsilon}(\Lambda) = X/\mathcal{L}_+SL_{\chi'} \) and \( M_i = M_{\sigma-\varepsilon}(\Lambda) = X/\mathcal{L}_-SL_{\chi'} \) are isomorphic to the quotient stacks \([X^{ss}(\mathcal{L}_+)/PGL_{c'}] \) (cf. Remark \ref{rem:quotient}). By Lemma \ref{lem:quotient} both \([X/PGL_{c'}] \) and \([X'/PGL_{c'}] \) are smooth quotient stacks of dimension \( d+g-2 \) and \( d+g-2+2\text{deg}D' \), respectively, and thus \( X \) and \( X' \) are both smooth and \( X \subset X' \) is a local complete intersection cut out precisely by the \( 2\text{deg}D' \) conditions imposed by the vanishing of a section along \( D' \).

Recall that the unique KN stratum of \( X^u(\mathcal{L}_\pm) \) is determined by \((Z, \lambda) \) (cf. Remark \ref{rem:quotient}), where for a pair \((E', \phi') \in Z, \) the bundle \( E' = L' \oplus K' \) is acted on by \( \lambda = G_m \) by \((t^b, t^{-a}) \). By \cite[Lemma 7.6]{Pot16} and its proof, the \( \lambda \)-weights of \( \mathcal{N}_{c'}^0/X \) on \( Z \) are all \( \pm(a+b)_{c'} = \pm \chi' \) or 0. Then the weights of \( \mathcal{N}_{c'}^0/X \) are all \( \pm \chi' \), and \( \eta_{\pm} = \text{weight}_{\lambda_{\pm}} \det \mathcal{N}_{c'}^0/X \big|_Z \) is just the codimension of \( S_{\pm} \subset X \). Since \( S_{\pm} \) is the bundle \( W_i \) on \( Z \), we have \( \text{codim}(S_{\pm} \subset X) = \text{rk} W_i \), so that \( \eta_+ = i\chi' \) and \( \eta_- = (d+g-1-2i)\chi' \). We get the formula above after the standard rescaling of all weights by \( 1/\chi' \).

Using this we obtain the following result.

Proposition \ref{prop:semi-orthogonal}. For \( 1 \leq i \leq \frac{d+g-1}{3} \) (resp., \( i \geq \frac{d+g-1}{3} \)) there is an admissible embedding \( D^b(M_{i-1}) \hookrightarrow D^b(M_i) \) (resp., \( D^b(M_i) \hookrightarrow D^b(M_{i-1}) \)).

When \( 1 < i \leq \frac{d+g-1}{3} \), the admissible embedding can be chosen to be the window subcategory \( G_i^+ \subset D^b(M_i) \) determined by the range of weights \([0, i) \subset [0, d+g-1-2i) \) (cf. \cite{HL15}) and moreover there is a semi-orthogonal decomposition
\begin{equation}
D^b(M_i) = \langle D^b(M_{i-1}), D^b(\text{Sym}^i C), \ldots, D^b(\text{Sym}^i C) \rangle
\end{equation}
with \( \mu = d+g-3i-1 \) copies of \( D^b(\text{Sym}^i C) \) given by the fully faithful images of functors \( Rj_*\left( L\pi^* (\cdot) \otimes O_\pi(l) \right) : D^b(\text{Sym} C) \rightarrow D^b(M_i) \) for \( l = 0, \ldots, \mu-1, \) where \( \pi : \mathbb{P}W_i^+ \rightarrow \text{Sym}^i C \) is the projection and \( j : \mathbb{P}W_i^+ \hookrightarrow M_i \) the inclusion.

The semi-orthogonal decomposition \eqref{eq:semi-orthogonal} follows from \cite{BFR22}, as the birational transformation between \( M_{i-1} \) and \( M_i \) is a standard flip of projective bundles over \( \text{Sym}^i C \). Here we provide an alternative proof for this case. We also note that \cite[Corollary 8.1]{Pot16} shows the admissible embeddings \( D^b(M_{i-1}) \hookrightarrow D^b(M_i) \) when \( i \) is in the specified range.

As explained in the introduction, Proposition \ref{prop:semi-orthogonal} does not provide a semi-orthogonal decomposition with Fourier-Mukai functors associated with Poincaré bundles and it is not used in our paper. However, we find this result relevant.

Proof. If \( i = 1 \), this follows from Orlov’s blow-up formula \cite{Ori92}. Let \( i > 1 \). From Lemma \ref{lem:rescaling} weight \( \omega_X | Z = \eta_- = \eta_+ = (d+g-1-3i) \). By \cite{HL15}
Theorem 3.18. The objects of the form $F_x, \Lambda_M, \psi, \zeta, G_D$ on both $M_{i-1}$ and $M_i$ are the descents of objects $\tilde{F}_x, \tilde{\Lambda}_M, \tilde{\psi}, \tilde{\zeta}, \tilde{G}_D$ on $D^b([X/PGL_{\chi}'])$.

Proof.

Indeed, the first equality follows directly from [HL15, Theorem 3.29] applied on $M_{\sigma+\epsilon}$, while the second is the same theorem applied on $M_{\sigma-\epsilon}$, using the fact that $\text{weight}_{\lambda-} B|_Z - \text{weight}_{\lambda-} A|_Z = -(\text{weight}_{\lambda} B|_Z - \text{weight}_{\lambda} A|_Z)$. 

We finish this section with the computation of all weights that we need in order to construct the semi-orthogonal decompositions.

Theorem 3.17. Let $\sigma = d/2 - i$, $1 < i < v$. If $A, B$ are objects in $D^b([X/PGL_{\chi}'])$, with $\lambda = \lambda_{+}-$weights satisfying the inequalities

$$1 + 2i - d - g < \text{weight}_{\lambda} B|_Z - \text{weight}_{\lambda} A|_Z < i$$

then $R\text{Hom}_{M_{\sigma+i}}(A, B) = R\text{Hom}_{M_{\sigma-i}}(A, B)$. In particular, if $1 + 2i - d - g < \text{weight}_{\lambda} B|_Z < i$ then $R\Gamma_{M_{i-1}}(B) = R\Gamma_{M_i}(B)$.

Proof. By Lemma 3.14 (3.7) is equivalent to inequalities

$$-\eta_+ < \text{weight}_{\lambda} B|_Z - \text{weight}_{\lambda} A|_Z < \eta_+,$$

so the Quantization Theorem [HL15, Theorem 3.29] implies that

$$R\text{Hom}_{M_{\sigma+i}}(A, B) = R\text{Hom}_{[X/PGL_{\chi}']}(A, B) = R\text{Hom}_{M_{\sigma-i}}(A, B).$$

Indeed, the first equality follows directly from [HL15, Theorem 3.29] applied on $M_{\sigma+i}$, while the second is the same theorem applied on $M_{\sigma-i}$, using the fact that $\text{weight}_{\lambda-} B|_Z - \text{weight}_{\lambda-} A|_Z = -(\text{weight}_{\lambda} B|_Z - \text{weight}_{\lambda} A|_Z)$. 

Consider an object $G$ in $D^b([X/PGL_{\chi}'])$ descending to some objects on $D^b(M_{i-1})$ and $D^b(M_i)$. We can use windows to determine when such object can “cross the wall”. Namely, if the weights of $G$ are in the required range, cohomology groups will be the same on either side. By abuse of notation, we often denote in the same way both the object on $D^b([X/PGL_{\chi}'])$ and the objects it descends to in $M_{\sigma_{\pm\epsilon}}(\Lambda)$. 

Corollary 3.16. If $d \leq 2g - 1$, then $D^b(M_{i-1}) \subset D^b(M_i)$ for any $1 \leq i \leq v$.

Proof. In this case $i \leq (d-1)/2 \leq g - 1$, so the inequality $i < (d+g-1)/3$ holds for every $i$. 

We finish this section with the computation of all weights that we need in order to construct the semi-orthogonal decompositions.

Theorem 3.18. The objects of the form $F_x, \Lambda_M, \psi, \zeta, G_D$ on both $M_{i-1}$ and $M_i$ are the descents of objects $\tilde{F}_x, \tilde{\Lambda}_M, \tilde{\psi}, \tilde{\zeta}, \tilde{G}_D$ on $D^b([X/PGL_{\chi}'])$.
that, after the standard rescaling (see the proof of Lemma 3.14), have \( \lambda \)-
weights
\[
\text{weights}_\lambda \tilde{F}_x|_Z = \{0, -1\}
\]
\[\text{weight}_\lambda \tilde{\Lambda}_M|_Z = -1\]
\[\text{weight}_\lambda \tilde{\psi}|_Z = d + 1 - g - i\]
\[\text{weight}_\lambda \tilde{\zeta}|_Z = g - i\]
\[\text{weights}_\lambda \tilde{G}_D|_Z = \{0, -1, \ldots, -\deg D\}.
\]

\textbf{Proof.} Let \( \sigma = d/2 - i \) and embed \( \iota : M_\sigma(\Lambda) \to M'_\sigma = M_\sigma(\Lambda(2D')) \) for an
effective divisor \( D' \), \( \deg D' > 0 \), as usual. Recall that the universal bundle
\( F' \) on \( C \times M'_\sigma \) is the descent of \( F'(1) \) on \( C \times X' \subset C \times U' \times \mathbb{P} \mathcal{C}' \), where
\( F' \) is the universal family on \( C \times U' \) [Tha94 1.19]. Let us compute the \( \lambda \)-
weights of \( F'_x(1) \) on the \( \sigma \)-strictly semi-stable locus, for a point \( x \in C \). The
fiber of \( F'_x \) over \( L' \oplus K' \) is \( L'_x \oplus K'_x \), which is acted on with weights \( b \) in the
first component and \( -a \) in the second. Since the \( \lambda \)-weight of \( \mathcal{O}_{\mathbb{P} \mathcal{C}'}(1) \) over
the section \( (\phi', 0) \) is \( -b \), the weights of \( F'_x(1) \) are 0 and \( -a - b = -\chi' \). By
Lemma 3.8 we have \( F_x \cong \iota^* F'_x \). Hence, \( F_x \) also is the descent of an object
with weights 0 and \( -\chi' \).

The bundle \( \det \pi_! F' \) descends from \( \det \pi_! F'(1) \). On the fiber of \( \pi_! F' \) over
\( L' \oplus K' \), \( \lambda \) acts on \( H^0(L') \oplus H^0(K') \) with weights \( b \) and \( -a \), with
multiplicities \( h^0(L') = a \) and \( h^0(K') = b \), respectively. Taking tensor product with
\( \mathcal{O}_{\mathbb{P} \mathcal{C}'}(1) \) shifts each weight by \( -b \), and then taking the determin-
ant we get \( \text{weight}_\lambda \det \pi_! F'(1)|_{Z'} = 0 \cdot a + (-a - b) \cdot b = -b \chi' \). For
\( \det F'_x \), which is the descent of \( \det F'_x(1) \), we see that \( \lambda \) acts with weights
\( b, -a \) on \( L'_x \oplus K'_x \), and then shifting by \( -b \) and taking determinants we get
\( \text{weight}_\lambda \det F'_x(1)|_{Z'} = -a - b = -\chi' \).

Now for the universal bundle \( F \) on \( C \times M_{\sigma \pm \epsilon}(\Lambda) \), we use the short exact
sequence [3.2]. From this we see that \( \Lambda_M = \det F_x \cong \det F'_x \) is the descent of an object with \( \lambda \)-weight equal to \( -\chi' \). Also, since \( \det \pi_! F'|_{D' \times M_{\sigma \pm \epsilon}} = \det \bigoplus_{x \in D'} F'_x = (\det F'_x)^{\deg D'} \), we obtain that
\( \psi^{-1} = \det \pi_! F = \det \pi_! F' \otimes (\det F'_x)^{-\deg D'} \) is the descent of an object with \( \lambda \)-weight equal to \( -b \chi' + \deg D' \chi' \). Recall \( \deg L' = i + \deg D' \), \( \deg K' = d - i + \deg D' \) (see
the discussion before Remark 3.12), so by Riemann-Roch \( b = h^0(K') = d - i + \deg D' + 1 - g \) and the weight of \( \psi \) is \( -\chi' (\deg D' - b) = \chi'(d + 1 - g - i) \). As for
\( \zeta = \psi \otimes \Lambda_M^{d-2g+1} \), the weights must be \( (d+1-g-i-(d-2g+1)) \chi' = (g-i) \chi' \).
Rescaling everything by \( 1/\chi' \) as in Lemma 3.14 we get the weights as in the
statement.

Finally, we consider \( G_D \). Let \( D = x_1 + \ldots + x_\alpha \). Since by construction
tensor bundles are functorial in \( M \), the bundle \( G_D \) is the descent of a vector
bundle \( (\mathcal{E}^{\alpha})_D \) on \( X \), where \( M = X // SL_{\chi'} \) and \( \mathcal{E} \) descends to \( F \). By Lemma
2.10 \( (\mathcal{E}^{\alpha})_D \) is a deformation of \( \mathcal{E}_{x_1} \otimes \ldots \otimes \mathcal{E}_{x_\alpha} \), and the deformation can
be chosen to be \( SL_{\chi'} \)-equivariant (see Remark 2.11). Therefore, \( (\mathcal{E}^{\alpha})_D \) has
the same weights as the tensor product $E_{x_1} \otimes \ldots \otimes E_{x_\alpha}$, i.e. $0, -1, \ldots, -\alpha$
(after rescaling).

**Remark 3.19.** Observe that $O_i(1, 0) = \psi \otimes \Lambda_M^{d-g}$ and $O_i(0, 1) = \Lambda_M^{-1}$, so we can use the previous theorem to see that in general, a line bundle $O_i(m, n)$ is the descent on both $M_{i-1}$ and $M_i$ of an object having $\lambda$-weight $m(1-i) + n$
on the strictly semi-stable locus of the wall.

4. **Acyclic vector bundles on $M_1$ — easy cases**

In order to prove Theorem 1.1 we will first construct fully faithful functors $\Phi_\alpha : D^b(Sym^\alpha C) \to D^b(M_i)$ for $1 \leq \alpha \leq i$ and show that, after suitable twists, the essential images of these functors are semi-orthogonal to each other in the required way (see Theorem 9.2, Definition 9.3 and Theorem 9.6 below). By means of Bondal-Orlov’s criterion [BO95], this reduces to the computation of $R\Gamma$ for a large class of vector bundles on $M_1$. In particular, we will need to prove $\Gamma$-acyclicity for several of these vector bundles.

**Theorem 4.1.** Let $D$, $D'$ be effective divisors on $C$. Let $d > 2$ and $1 \leq i \leq v$. Suppose

$$\deg D - g < t < d - \deg D' - i - 1.$$  

Then $R\Gamma_{M_1(d)}(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^i \otimes \zeta^{-1}) = 0$.

We start with a lemma.

**Lemma 4.2.** $R\Gamma_{M_1(d)}(O_{M_1(d)}(-kH + lE_1)) = 0$ for $0 < k \leq d + g - 2$ and $0 \leq l \leq d + g - 4$. In particular, $R\Gamma_{M_1(d)}(\Lambda_M^i) = 0$ for $t = 1, \ldots, d + g - 4$.

**Proof.** Consider the short exact sequence

$$0 \to O_{M_1(d)} \to O_{M_1(d)}(E_1) \to O_{\pi}(-1) \to 0,$$

where $E_1 = PW_1^+$ and $\pi : E_1 \to C$ is the $\mathbb{P}^r$-bundle, $r = d + g - 4$. $O_{M_1(d)}(-kH)$ is $\Gamma$-acyclic provided $0 < k \leq d + g - 2 = \dim M_1(d)$. Then twisting by $O_{M_1(d)}(-kH)$ and taking a long exact sequence in cohomology gives $\Gamma$-acyclicity of $O_{M_1(d)}(-kH + E_1)$ for such $k$. Similarly, twisting by powers of $O_{M_1(d)}(E_1)$ and using induction, we get that $R\Gamma_{M_1(d)}(O_{M_1(d)}(-kH + lE_1)) = 0$ as well, since $O_{\pi}(-l)$ is $\Gamma$-acyclic for $0 < l \leq d + g - 4$.  

Next we prove Theorem 4.1 for the case $i = 1$.

**Lemma 4.3.** The statement of Theorem 4.1 holds for $i = 1$.

**Proof.** Let $\alpha = \deg D$, $\beta = \deg D'$. We are given that $\alpha - g < t < d - \beta - 2$. We do induction on $\alpha + \beta$. If $\alpha = \beta = 0$, we have to check that $\Lambda_M^i \otimes \zeta^{-1} = -(t + g)H + (g + t - 1)E_1$ is $\Gamma$-acyclic on $M_1(d)$. By Lemma 4.2 this holds provided $0 < t + g \leq d + g - 2$ and $0 \leq g + t - 1 \leq d + g - 4$, which is true by hypothesis.

If $\alpha > 0$, we can write $D = x + \tilde{D}$ for some point $x \in C$. Then by Proposition 2.13 $G_D^\vee$ is a stable deformation of $F_x^\vee \otimes G_D^\vee$ over $\mathbb{A}^1$, so it...
This proves the theorem. Consider the exact sequence (3.3) from Lemma 3.10
\[ 0 \to \Lambda_M^{-1} \to F_y \to O_{M_1(d)} \to O_{M_0(d-2)} \to 0, \]
and twist it by \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \). The restriction of \( F_y \) to \( M_0(d-2) = \mathbb{P}^r \), \( r = d + g - 4 \), is equal to \( O_{\mathbb{P}^r} \oplus O_{\mathbb{P}^r}(-1) \) (see the proof of [Tha94, 3.2], with \( i = 1 \)) so, by Corollary 2.10, we have that \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \bigg|_{M_0(d-2)} \) is a deformation over \( \mathbb{A}^1 \) of a sum of bundles \( \bigoplus O_{\mathbb{P}^r}(s_j) \oplus O_{\mathbb{P}^r}(1 - t - g) \), with \(-\beta \leq s_j \leq \alpha - 1\). These are all \( \Gamma \)-acyclic on \( \mathbb{P}^{d+g-4} \), since by hypothesis
\[ (4.2) \quad \alpha - t - g < 0, \quad -\beta + 1 - t - g \geq -(d + g - 4). \]

Then \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \bigg|_{M_0(d-2)} \) is \( \Gamma \)-acyclic too, by semi-continuity.

The other two terms from the sequence (3.3) are \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \) and \( G_D^\vee \otimes G_D' \otimes \Lambda_M^{t-1} \otimes \zeta^{-1} \). Observe they both satisfy the inequalities of the hypothesis, so by induction they are \( \Gamma \)-acyclic on \( M_1(d) \), and we conclude that so is \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \), as desired.

Similarly, if \( \beta > 0 \) we write \( D' = y + \tilde{D}' \) and use the exact sequence
\[ 0 \to O_{M_1(d)} \to F_y \to \Lambda_M \to \Lambda_M \big|_{M_0(d-2)} \to 0, \]
twisted with \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \). The resulting term on the right is a deformation over \( \mathbb{A}^1 \) of a sum \( \bigoplus O_{\mathbb{P}^r}(s_j) \otimes O_{\mathbb{P}^r}(-t - g) \), with \(-\beta + 1 \leq s_j \leq \alpha\), and it is again \( \Gamma \)-acyclic by the same inequalities (4.2). Finally, the remaining two terms \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \) and \( G_D^\vee \otimes G_D' \otimes \Lambda_M^{t-1} \otimes \zeta^{-1} \) are \( \Gamma \)-acyclic by induction, and we conclude that \( R\Gamma_{M_1(d)}(G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1}) = 0. \)

**Proof of Theorem 4.1.** Let \( \alpha = \deg D \) and \( \beta = \deg D' \). We do induction on \( i \). If \( i = 1 \), this is Lemma 4.3. Suppose \( i > 1 \). By Lemma 4.3, we still have
\[ R\Gamma_{M_1(d)}(G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1}) = 0. \]

Let \( 1 < j \leq i \) and consider the wall-crossing between \( M_{j-1} \) and \( M_j \). Here, the bundle \( G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} \) descends from an object with weights
\[ \text{weights}_{\lambda} G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1} = \{-\beta - t + j - g, \ldots, \alpha - t + j - g\} \]
(see Theorem 3.18). Our hypothesis guarantees that \( \alpha - t + j - g < j = \eta_+ \) and \( -\beta + 1 + j - g > 1 + 2j - d - g = -\eta_- \), that is, all these weights live in the range \((-\eta_-, \eta_+)\) in the respective wall-crossings. By Theorem 3.17, this implies
\[ R\Gamma_{M_1(d)}(G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1}) = R\Gamma_{M_1(d)}(G_D^\vee \otimes G_D' \otimes \Lambda_M^1 \otimes \zeta^{-1}) = 0. \]
This proves the theorem. □
5. A FULLY FAITHFUL EMBEDDING $D^b(C) \subset D^b(M_1)$

The following Theorem 5.1 is a special case of Theorem 9.2 and will be needed for our proof of the latter. Namely, the result of Theorem 5.1 will be used in Sections 7 and 8, which are necessary for Theorem 9.2. While Theorem 5.1 could be avoided by including it as a step of a more complicated inductive proof, we find it more convenient to prove it first, both to make the inductions less cumbersome and to introduce some ideas that will help understand the general picture.

We assume that $v \geq 1$, i.e. $d \geq 3$. As before, let $E_1 \subset M_1$ be the exceptional locus of the blow-up $M_1 \to M_0$ along $C \subset M_0$. By Orlov’s blow-up formula [Orl92], we have a fully faithful functor $\Psi : D^b(C) \hookrightarrow D^b(M_1)$, corresponding to the Fourier-Mukai transform given by $O_Z(E_1)$, where $Z = C \times_C E_1$. Now consider the Fourier-Mukai transform

$$\Phi F = Rp_*(Lq^*(\cdot \otimes L) F) : D^b(C) \to D^b(M_1)$$

determined by the universal bundle $F$ on $C \times M_1$.

**Theorem 5.1.** The functor $\Phi F$ is fully faithful.

We need a few constructions and lemmas first. Observe that $Z = C \times_C E_1$ is supported precisely on the zero locus of the universal section $\tilde{\phi} : O_{C \times M_1} \to F$. Indeed, pairs $(E, \phi)$ in $P W_1^+ = E_1$ parametrize extensions

$$0 \to O_C(x) \to E \to \Lambda(-x) \to 0$$

with the canonical section $\phi \in H^0(C, O_C(x))$ vanishing on $x \in C$ [Tha94 3.2], and in fact $\tilde{\phi}$ has no zeros outside this locus, since $M_1 \setminus E_1$ consists of extensions $0 \to O_C \to E \to \Lambda \to 0$ together with a (constant) section $\phi \in H^0(C, O_C)$ [Tha94 3.1]. Since $Z$ has codimension 2, we have a Koszul resolution

$$\Lambda^2 F^\vee \to F^\vee \xrightarrow{\tilde{\phi}} O_{C \times M_1} \xrightarrow{\sim} O_Z. \tag{5.1}$$

**Lemma 5.2.** $R\Gamma_{M_1}(\Lambda^1_M) = 0$.

**Proof.** Recall $\Lambda^1_M = O_{M_1}(H - E_1)$. Since the embedding

$$C \xrightarrow{|\omega_C \otimes \Lambda|} M_0 = \mathbb{P}^{d+g-2}$$

is given by a complete linear system [Tha94 3.4], the image of $C$ is not contained in any hyperplane and thus $H^0(M_1, O_{M_1}(H - E_1)) = 0$. Now use the exact sequence

$$0 \to O_{M_1}(H - E_1) \to O_{M_1}(H) \to O_{E_1}(H) \to 0. \tag{5.2}$$

Observe that $H^p(M_1, O_{E_1}(H)) = H^p(C, \omega_C \otimes \Lambda)$, because $C \to M_0$ is given by $|\omega_C \otimes \Lambda|$. Since $\deg \omega_C \otimes \Lambda > \deg \omega_C$, we have $H^1(C, \omega_C \otimes \Lambda) = 0$. On
the other hand, \( H^0(M_1, \mathcal{O}_{M_1}(H)) = \mathbb{C}^{d+g-1} \) and \( H^{>0}(M_1, \mathcal{O}_{M_1}(H)) = 0 \). Taking a long exact sequence in cohomology from (5.2) we get

\[
0 \to H^0(M_1, \mathcal{O}_{M_1}(H)) \to H^0(C, \omega_C \otimes \Lambda) \to H^1(M_1, \Lambda^{-1}_M) \to 0
\]

and \( H^p(M_1, \Lambda^{-1}_M) = 0 \) for \( p \neq 1 \). By Riemann-Roch, \( H^0(C, \omega_C \otimes \Lambda) = \mathbb{C}^{d+g-1} \), so the first map in (5.3) is an isomorphism and \( H^1(M_1, \Lambda^{-1}_M) = 0 \) as well.

**Lemma 5.3.** Let \( x \in C \). Then \( R\Gamma_{M_1}(F^\vee_x) = 0 \), while \( R\Gamma_{M_1}(F_x) = \mathbb{C} \), with \( H^0(M_1, F_x) = \mathbb{C} \) given by restriction of the universal section \( \phi \) of \( F \) to \( \{x\} \times M_1 \).

**Proof.** Consider the resolution (5.1) and restrict to \( \{x\} \times M_1 \) to get

\[
[\Lambda^{-1}_M \to F^\vee_x \to \mathcal{O}_{M_1}] \sim \mathcal{O}_{\mathbb{P}^r_x}
\]

where \( \mathbb{P}^r_x = M_0(\Lambda(-2x)) \) is the fiber over \( x \in C \subset M_0 \) along the blow-up \( \pi : M_1 \to M_0 \). We twist by \( \Lambda_M = \mathcal{O}_{M_1}(E_1 - H) \) to get

\[
[\mathcal{O}_{M_1} \xrightarrow{\phi} F_x \to \Lambda_M] \sim \mathcal{O}_{\mathbb{P}^r_x}(-1),
\]

using that \( F^\vee_x \otimes \Lambda_M = F_x \) and that \( \mathcal{O}_{M_1}(H) \) restricts trivially to \( \mathcal{O}_{\mathbb{P}^r} \) (see Lemma 3.10 for a generalization of (5.4) and (5.5)). It is well-known that \( R\Gamma(\mathcal{O}_{\mathbb{P}^r_x}(-1)) = 0 \). By Lemma 4.2, we also have \( R\Gamma(\Lambda_M) = 0 \). Hence, by (5.5), \( \phi \) induces an isomorphism \( R\Gamma(\mathcal{O}_{M_1}) \cong R\Gamma(F_x) \). As \( M_1 \) is a blow up of a projective space along a smooth center, we get \( R\Gamma(F_x) \cong R\Gamma(\mathcal{O}_{M_1}) \cong \mathbb{C} \), with \( H^0(M_1, F_x) = \mathbb{C} \) given by restriction of \( \phi \) to \( \{x\} \times M_1 \).

To show that \( R\Gamma_{M_1}(F^\vee_x) = 0 \), we apply \( R\Gamma \) to (5.4). We know that \( R\Gamma_{M_1}(\mathcal{O}_{M_1}) = R\Gamma_{\mathbb{P}^r}(\mathcal{O}_{\mathbb{P}^r}) = \mathbb{C} \), while \( R\Gamma_{M_1}(\Lambda^{-1}_M) = 0 \) by Lemma 5.2. Then the claim will be proved if we show that \( H^0(M_1, F^\vee_x) = 0 \), as this would also imply that \( H^p(M_1, F^\vee_x) \) vanishes for \( p > 0 \). Any global section \( s \in H^0(M_1, F^\vee_x) \), composed with \( F^\vee_x \to \mathcal{O}_{M_1} \) gives a constant section \( \mathcal{O}_{M_1} \to \mathcal{O}_{M_1} \) vanishing along the locus \( Z \), hence identically 0. By exactness of \( 0 \to \Lambda^{-1}_M \to F^\vee_x \to \mathcal{O}_{M_1} \), the section \( s : \mathcal{O}_{M_1} \to F^\vee_x \) must lift to a section \( \mathcal{O}_{M_1} \to \Lambda^{-1}_M \). But \( \Lambda^{-1}_M \) has no global sections by Lemma 5.2.

**Proof of Theorem 5.1.** By Bondal-Orlov’s criterion [BO95], in order to show full faithfulness of \( \Phi_F \) we only need to consider the sheaves \( \Phi_F(\mathcal{O}_x) = F_x \) for closed points \( x \in C \). On the other hand, consider the functor \( \Psi \) from Orlov’s blow-up formula, with Fourier-Mukai kernel \( \mathcal{O}_Z(E_1) \), \( Z = C \times_C E_1 \). We can compute \( \Psi(\mathcal{O}_x) = \Phi_{\mathcal{O}_Z(E_1)}(\mathcal{O}_x) \) for a point \( x \in C \) using (5.1) as follows. As before, let \( \mathbb{P}^r_x = M_0(\Lambda(-2x)) \) denote the fiber over \( x \in C \subset M_0 \) along the blow-up. The fact that \( \mathcal{O}_{M_1}(H) \) restricts trivially to this fiber implies that both \( \Lambda_M \) and \( \mathcal{O}_{M_1}(E_1) \) restrict to \( \mathcal{O}_{\mathbb{P}^r}(-1) \) there. Now we restrict (5.1) to \( \{x\} \times M_1 \) and twist it by \( \Lambda_M \) to get \( \Phi_{\mathcal{O}_Z(E_1)}(\mathcal{O}_x) \cong [\mathcal{O}_{M_1} \to F_x \to \Lambda_M] \cong \mathcal{O}_{\mathbb{P}^r}(-1) \).
Since we already know that $\Psi$ is fully faithful, we have

$\text{Hom}_{D^b(M_1)}(\Psi(O_x), \Psi(O_y))[k]) = \begin{cases} 0 & \text{if } x \neq y \\ 0 & \text{if } x = y \text{ and } k \neq 0,1 \\ \mathbb{C} & \text{if } x = y \text{ and } k = 0,1. \end{cases}$

But $R\text{Hom}_{D^b(M_1)}(\Psi(O_x), \Psi(O_y)) \cong R\Gamma \circ R\mathcal{H}om(\Psi(O_x), \Psi(O_y))$ can also be obtained as follows: take $R\mathcal{H}om(\Psi(O_x), \Psi(O_y)) \sim \Psi(O_x) \otimes \Lambda \Psi(O_y)$ as an inner tensor product obtained from the double complex

\[
\begin{array}{c}
\mathcal{O}_{M_1} \rightarrow F_x^\vee \otimes \Lambda M \rightarrow \Lambda M \\
\Lambda_{-1}^M \otimes F_y \rightarrow F_x^\vee \otimes F_y \rightarrow F_y \\
\Lambda_{-1}^M \rightarrow F_x^\vee \rightarrow \mathcal{O}_{M_1}
\end{array}
\]

which produces the total complex

\[
\begin{bmatrix}
\Lambda_{-1}^M \rightarrow F_x^\vee \oplus F_y^\vee \rightarrow \mathcal{O}_{M_1}^\oplus \oplus (F_x^\vee \otimes F_y) \rightarrow F_x \oplus F_y \rightarrow \Lambda M
\end{bmatrix}
\]

\[
\cong \Psi(O_x)^\vee \otimes \Lambda \Psi(O_y),
\]

again using $F_x = F_x^\vee \otimes \Lambda M$. Recall that our descriptions of $\Psi(O_y)$ and $\Psi(O_x)^\vee$ were obtained from the Koszul resolution \([5.5]\) and its dual. In particular, the maps $\mathcal{O}_{M_1} \rightarrow F_x^\vee \otimes \Lambda M = F_x$ and $\mathcal{O}_{M_1} \rightarrow F_y$ appearing in \([5.7]\) correspond to the restriction of the universal section $\hat{\phi}$ to $\{x\} \times M_1$ and $\{y\} \times M_1$, respectively.

The hypercohomology $R\Gamma$ of \([5.8]\) can be computed by taking the spectral sequence with first page $E_{1}^{p,q} = H^q(X, F^p) \Rightarrow H^{p+q}(X, F^\bullet)$. On the other hand, we know that $R\Gamma$ of this complex is given by \([5.6]\). We will combine these to show that

\[
R\Gamma(F_x^\vee \otimes F_y) = \begin{cases} 0 & \text{if } x \neq y \\ \mathbb{C} \oplus \mathbb{C}[-1] & \text{if } x = y. \end{cases}
\]

By Lemma \([4.2]\) $R\Gamma(\Lambda M) = 0$, and by Lemma \([5.2]\) $R\Gamma(\Lambda_{-1}^M) = 0$. Also, Lemma \([5.3]\) computes hypercohomology of both $F_x$ and $F_x^\vee$. Summing up, applying $R\Gamma$ to \([5.8]\) yields a spectral sequence $E_1^{p,q}$ of the form

\[
\begin{array}{c}
\vdots \rightarrow \vdots \rightarrow H^1(F_x^\vee \otimes F_y) \rightarrow 0 \rightarrow 0 \\
0 \rightarrow 0 \rightarrow H^0(\mathcal{O}_{M_1}) \oplus H^0(F_x^\vee \otimes F_y) \rightarrow H^0(F_x) \oplus H^0(F_y) \rightarrow 0,
\end{array}
\]

where the map $H^0(\mathcal{O}_{M_1}) \oplus H^0(F_x^\vee \otimes F_y) \rightarrow H^0(F_x) \oplus H^0(F_y)$ is the isomorphism $\mathbb{C}^2 \cong \mathbb{C}^2$ given by the universal section in each coordinate, by Lemma \([5.3]\) and the
discussion above. Since this spectral sequence converges to (5.6), we obtain (5.9).

6. Acyclicity of powers of $\Lambda_M$

The goal of the present section is to prove the following generalization of Lemma 5.2.

**Theorem 6.1.** Suppose $2 < d \leq 2g + 1$ and $1 \leq k \leq l \leq v$. Then

$$R\Gamma_{M_l(d)}(\Lambda_M^{-k}) = 0.$$ 

Γ-acyclicity of these negative powers of $\Lambda_M$ will be crucial for the cohomology computations in the upcoming sections.

**Lemma 6.2.** Under the assumptions of Theorem 6.1, $H^0(M_l(d), \Lambda_M^{-k}) = 0$.

**Proof.** Since $M_l$ is isomorphic to $M_1$ in codimension 1, it suffices to prove that $H^0(M_1, \Lambda_M^{-k}) = H^0(M_1, kH - kE_1) = 0$. Recall that $M_1$ is the blow-up of $\mathbb{P}^r$ in $C$ embedded by a complete linear system of $K_C + \Lambda$, $r = d + g - 2$, $E_1$ is the exceptional divisor and $H$ is a hyperplane divisor. The claim is that there is no hypersurface $D \subset \mathbb{P}^r$ of degree $k$ that vanishes along $C$ with multiplicity $\geq k$. We argue by contradiction. Choose $r + 1$ points $p_1, \ldots, p_{r+1} \in C$ in linearly general position. Then $D$ vanishes at these points with multiplicity $\geq k$. Let $R$ be a rational normal curve passing through $p_1, \ldots, p_{r+1}$. Let $\tilde{R}$ and $\tilde{D}$ be the proper transforms of $R$ and $D$ in $\text{Bl}_{p_1, \ldots, p_{r+1}} \mathbb{P}^r$. Then $\tilde{D} \cdot \tilde{R} \leq kr - k(r + 1) < 0$. It follows that $R \subset D$. But we can choose $R$ passing through a general point of $\mathbb{P}^r$, which is a contradiction.

**Lemma 6.3.** Under the assumptions of Theorem 6.1, if $R\Gamma_{M_k(l)}(\Lambda_M^{-k}) = 0$, then $R\Gamma_{M_l(d)}(\Lambda_M^{-k}) = 0$.

**Proof.** By Theorem 3.18 in the wall between $M_{l-1}$ and $M_l$, $\Lambda_M^{-k}$ descends from an object of weight $k$, with $-\eta_- < k < \eta_+$ when $k < l \leq v$, that is, $1 + 2l - d - g < k < l$ for $l$ in that range. This way, $0 = R\Gamma_{M_k(l)}(\Lambda_M^{-k}) = R\Gamma_{M_l(d)}(\Lambda_M^{-k})$ for $l \geq k$ by Theorem 3.17.

**Definition 6.4.** For $0 \leq \alpha \leq i$, we introduce the following loci:

- $E_i^\alpha := \{(E, s) \mid Z(s) \subset C \text{ has degree } \geq \alpha \} \subset M_i$,
- $\mathcal{D}_i^\alpha := \{(D, E, s) \mid s|_D = 0\} \subset \text{Sym}^\alpha C \times M_i$,
- $R_i^\alpha := \{(D, E, s) \mid s|_D = 0 \text{ and } Z(s) \text{ has degree } \geq \alpha + 1\} \subset \mathcal{D}_i^\alpha$,

where $Z(s)$ denotes the zero locus subscheme of the section $s$.

Note that $E_i^i$ is precisely $\mathbb{P}W_i^+$, while $E_i^1 = E_i$ is the proper transform of $E_1$ under the birational equivalence given by (3.1). Recall $\mathcal{O}(E_i) = \mathcal{O}_i(1, -1)$ according to Definition 3.5. For a divisor $D \in \text{Sym}^\alpha C$, we observe that the fiber $(\mathcal{D}_i^\alpha)_D$ along the projection $\text{Sym}^\alpha C \times M_i \rightarrow \text{Sym}^\alpha C$
is isomorphic to $M_{i-\alpha}(\Lambda(-2D))$, see Remark 3.7 or [Tha94, 1.9]. Similarly, $(R^\alpha_i)_D \cong E_{i-\alpha}(\Lambda(-2D))$. In particular, $D^\alpha_i$ is smooth, and the projection

$$R^\alpha_i \rightarrow D^\alpha_i \quad (6.1)$$

is the normalization morphism.

**Lemma 6.5.** We have the following commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & \nu_* O_{D^\alpha_i}(-R^\alpha_i) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{I}_{E^\alpha_{i+1}} \\
\uparrow & & \uparrow \\
0 & \rightarrow & \mathcal{I}_{E^\alpha_{i+1}} \rightarrow \mathcal{O}_{E^\alpha_i} \rightarrow \mathcal{O}_{E^\alpha_{i+1}} & \rightarrow 0
\end{array}
$$

where $\mathcal{I}_{E^\alpha_{i+1}} \cong \nu_* O_{D^\alpha_i}(-R^\alpha_i)$ is the conductor sheaf of the normalization (6.1) and $R^\alpha_i$ (resp. $E^\alpha_{i+1}$) is a conductor subscheme in $D^\alpha_i$ (resp. $E^\alpha_{i}$).

**Proof.** From the flipping diagram (3.1), $E^\alpha_i \subset M^\alpha_i$ is the projective bundle $\mathbb{P} W^\alpha_{\alpha+i}$ and $E^\alpha_{i+1} \subset M^\alpha_{i+1}$ is isomorphic to $E^\alpha_{\alpha+1}$ away from $E^\alpha_{\alpha+1} \cong \mathbb{P} W^\alpha_{\alpha+1}$.

**Claim 6.6.** $E^\alpha_{i+1}$ has a multicross singularity generically along $E^\alpha_{i+1}$ (concretely, this means that a general section of $E^\alpha_{\alpha+1}$ that intersects $E^\alpha_{\alpha+1}$ in a point is étale locally isomorphic to the union of coordinate axes in $A^\alpha_{i+1}$).

Given the claim, and since multicross singularities are semi-normal [LV81], $E^\alpha_{i+1}$ has semi-normal singularities in codimension 1. For $i > \alpha+1$, $E^\alpha_i$ is isomorphic to $E^\alpha_{\alpha+1}$ in codimension 2, and so also has semi-normal singularities in codimension 1. Next we argue by induction on $\alpha$ that $E^\alpha_i$ is semi-normal and Cohen–Macaulay with reduced conductor subschemes $E^\alpha_{i+1} \subset E^\alpha_i$ and $R^\alpha_i \subset D^\alpha_i$, and in particular that we have a commutative diagram (6.2).

Indeed, $E^\alpha_{i} \subset M^\alpha_i$ is Cohen–Macaulay as a hypersurface in a smooth variety. Suppose $E^\alpha_i$ is Cohen–Macaulay. Since it is semi-normal in codimension 1 by the above, it is semi-normal everywhere [GT80, 2.7]. Therefore, its conductor subschemes in $E^\alpha_i$ and $D^\alpha_i$ are both reduced [Tra70, 1.3] and all of their associated primes have height 1 in $E^\alpha_i$ and $D^\alpha_i$, respectively [GT80, 7.4]. It follows that these conductor subschemes are equal to $E^\alpha_{i+1}$ and $R^\alpha_i$, respectively. Finally, $R^\alpha_i \subset D^\alpha_i$ is Cohen–Macaulay as a hypersurface in a smooth variety and therefore $E^\alpha_{i+1} \subset E^\alpha_i$ is also Cohen–Macaulay [Rob78, 2.2], and we can proceed with induction.

It remains to prove the claim. We analyze the flipping diagram (3.1) between the spaces $M^\alpha_i$ and $M^\alpha_{i+1}$, where $M^\alpha_i$ contains projective bundles $\mathbb{P} W^\alpha_{\alpha+1}$ (over $\text{Sym}^{\alpha+1} C$) and $\mathbb{P} W^\alpha_{\alpha} \cong E^\alpha_i$ (over $\text{Sym}^\alpha C$) of dimensions $2\alpha+1$ and $d+g-2-\alpha$, respectively. What is their intersection over a point...
$D' \in \text{Sym}^{\alpha+1} C$, for simplicity a reduced sum of points? By [Tha94, 3.3], $\mathbb{P}W_{\alpha+1}^-$ parametrizes pairs $(E, \phi)$ that appear in extensions

$$0 \to L \to E \to \Lambda \otimes L^{-1} \to 0$$

with $\text{deg} \ L = d - \alpha - 1$ and $\phi \notin H^0(L)$. Projecting $\phi$ to $\Lambda \otimes L^{-1}$ gives a non-zero vector $\gamma \in H^0(\Lambda \otimes L^{-1})$ so that $\Lambda \otimes L^{-1} = \mathcal{O}(D')$, where $\text{deg} \ D' = \alpha + 1$ (this gives the map from $\mathbb{P}W_{\alpha+1}^-$ to $\text{Sym}^{\alpha+1} C$). Moreover, at $D'$ the section lifts to a section of $\mathcal{O}_{D'} \otimes L \cong \mathcal{O}_{D'} \otimes \Lambda(-D')$, and this vector $p \in H^0(\mathcal{O}_{D'} \otimes \Lambda(-D'))$ (determined uniquely up to a scalar) determines $(E, \phi)$ uniquely [Tha94, 3.3].

The same pair $(E, \phi)$ belongs to $\mathbb{P}W_{\alpha}^+$ if it can be given by an extension

$$0 \to \mathcal{O}(D) \to E \to \Lambda(-D) \to 0$$

with $\phi \in H^0(\mathcal{O}(D))$ and $\text{deg} \ D = \alpha$ [Tha94, 3.2]. Since $\phi$ vanishes at $D$ and its image in $\mathcal{O}(D')$ vanishes at $D'$, we have $D' \subset D$. Since we assume that $D'$ is a reduced divisor, there are exactly $\alpha + 1$ choices for $D$. Since $p$ cannot vanish at points of $D \subset D'$, for each of these choices of $D$, there is exactly one vector $p \in H^0(\mathcal{O}_{D'} \otimes \Lambda(-D'))$ (up to a multiple) that works. Moreover, in this way we get a basis of $H^0(\mathcal{O}_{D'} \otimes \Lambda(-D')) \cong \mathbb{C}^{\alpha+1}$. It follows that, over $D' \subset \text{Sym}^{\alpha+1} C$, $\mathbb{P}W_{\alpha}^-$ and $\mathbb{P}W_{\alpha}^+ \cong E_{\alpha}^\alpha$ intersect in $\alpha + 1$ reduced points which form a basis of the projective space $(\mathbb{P}W_{\alpha+1}^+)_{D'} \cong \mathbb{P}^\alpha$.

The strict transform of $\mathbb{P}W_{\alpha+1}^+$ in $M_{\alpha+1}$ is $E_{\alpha+1}^\alpha$, which contains the bundle $\mathbb{P}W_{\alpha+1}^+$ of dimension $d + g - 3 - \alpha$ (the flipped locus). After the flip, linearly independent intersection points in $(\mathbb{P}W_{\alpha+1}^-)_{D'} \cap \mathbb{P}W_{\alpha}^+$ become linearly independent normal directions of branches of $E_{\alpha+1}^\alpha$ along $\mathbb{P}W_{\alpha+1}^+$, i.e. $E_{\alpha}^\alpha$ has a multicross singularity in codimension 1, as claimed. We illustrate the geometry of $M_{\alpha}$, $M_{\alpha+1}$ and the common resolution $\tilde{M}_{\alpha+1}$ in Figure 1.

**Corollary 6.7.** If the claim of Theorem 6.1 is proved for $1 \leq k = l \leq i - 1$, then, for $1 \leq \alpha \leq i - 1$, $R\Gamma_M(E_i(1, i-1)) \cong R\Gamma_M(E_{i+1}(1, i-1))$ via $R\Gamma(\beta)$.

**Proof.** Twisting by $\mathcal{O}_i(1, i-1)$ and applying $R\Gamma$ to the bottom sequence in (6.2), we see that it suffices to show $I_{E_{i+1}^+}(1, i-1) \cong \nu_* \mathcal{O}_{D_i^0}(\mathcal{R}_i)(1, i-1)$ is $\Gamma$-acyclic. But $\nu$ is a finite map, so this is equivalent to $\Gamma$-acyclicity of $\mathcal{O}_{D_i^0}(\mathcal{R}_i)(1, i-1)$. Using the Leray spectral sequence for the fibration $p : D_i^0 \to \text{Sym}^\alpha C$, it suffices to prove that $R\Gamma(\mathcal{O}_{D_i^0}(\mathcal{R}_i)(1, i-1)) = 0$. Under the isomorphism $(D_i^0)_D \cong M_{i-\alpha}(\Lambda(-2D))$, $R_i^0 \subset \text{Sym}^\alpha C \times M_i$ restricts to $E_{\alpha}^\alpha$ on $M_{i-\alpha}(\Lambda(-2D))$, while $\mathcal{O}_i(m, n)$ on $M_i(\Lambda)$ restricts to $\mathcal{O}(m, n - ma)$ on $M_{i-\alpha}(\Lambda(-2D))$ (cf. Remark 3.7). Therefore,

$$R\Gamma_M(\mathcal{O}_{D_i^0}(\mathcal{R}_i)(1, i-1)) = R\Gamma_{M_{i-\alpha}(d-2\alpha)}(\Lambda_{M}^{\alpha-i})$$

which is zero by hypothesis. □

**Lemma 6.8.** Suppose $d \leq 2g + 1$. Then for $1 \leq i \leq d + 1 - g$, $i \leq v$ we have $H^p(M_i, \mathcal{O}_i(1, i-1)) = 0$ for any $p > 0$. □
Proof. First, we see that there is some \( k \geq i \) such that \( k \leq v \) and \( \mathcal{O}_{M_k}(1, i - 1) \otimes \omega_{M_k}^{-1} \) is big and nef on \( M_k(d) \). Recall \( \omega_{M_k} = \mathcal{O}_{M_k}(-3, 4 - d - g) \) (see [Tha94, 6.1]), so \( \mathcal{O}_{M_k}(1, i - 1) \otimes \omega_{M_k}^{-1} = \mathcal{O}_{M_k}(4, d + g + i - 5) \) and all we need to check is that this is in the cone bounded below by \( \mathcal{O}_{M_k}(1, i - 1) \) and above by \( \mathcal{O}_{M_k}(2, d - 2) \) (cf. Remark 3.6). This is equivalent to \( i - 1 \leq d + g + i - 5 \leq \frac{d - 2}{4} \). The inequality on the left is equivalent to \( 3i \leq d + g - 1 \), which is guaranteed by the fact that \( i \leq v = \lfloor (d - 1)/2 \rfloor \) and \( d \leq 2g + 1 \). The other inequality is equivalent to \( i \leq d + 1 - g \), which is given as a hypothesis. Therefore, there is some \( k \geq i, k \leq v \) such that \( \mathcal{O}_{M_k}(1, i - 1) \otimes \omega_{M_k}^{-1} \) is big and nef. By the Kawamata–Viehweg vanishing theorem, \( H^p(M_k, \mathcal{O}_k(1, i - 1)) = 0 \) for \( p > 0 \).

Now, we claim that in fact

\[
R\Gamma_{M_l}(\mathcal{O}_l(1, i - 1)) = R\Gamma_{M_{l+1}}(\mathcal{O}_{l+1}(1, i - 1)) = \ldots = R\Gamma_{M_k}(\mathcal{O}_k(1, i - 1)).
\]

Indeed, in the wall-crossing between \( M_{l-1} \) and \( M_l \), there are windows of width \( \eta_+ = l \) and \( \eta_- = d + g - 1 - 2l \) and \( \mathcal{O}_l(1, i - 1), \mathcal{O}_{l-1}(1, i - 1) \) both descend from the same object, that has \( \lambda \)-weight \( i - l \) (see Proposition 3.18 and Remark 3.19). By Theorem 3.17 we will have \( R\Gamma_{M_{l-1}}(\mathcal{O}_{l-1}(1, i - 1)) = R\Gamma_{M_l}(\mathcal{O}_l(1, i - 1)) \) whenever

\[
1 + 2l - d - g < i - l < l.
\]

But (6.4) holds for any \( i < l \leq k \), because then \( i < 2l \), while \( 3l \leq 3(d-1)/2 < i + d + g - 1 \) provided \( d \leq 2g + 1 \). Therefore, (6.3) holds and in particular \( H^p(M_l, \mathcal{O}_l(1, i - 1)) = 0 \) for \( p > 0 \). \( \square \)
Remark 6.9. Suppose that \( d \leq 2g + 1 \). Then (6.4) holds for \( l \in (i/2, v] \), and the same reasoning shows that \( R\Gamma_M(\mathcal{O}_i(1, i - 1)) = R\Gamma_M(\mathcal{O}_i(1, i - 1)) \) for every \([i/2] \leq l \leq v\). In particular, under the same hypotheses of Lemma 6.8, \( \mathcal{O}_i(1, i - 1) \) has no higher cohomology whenever \( [i/2] \leq l \leq v\).

Definition 6.10. Let \( L_i \) be the line bundle on \( \text{Sym}^i C \) defined by
\[
L_i = \det^{-1} \pi_! \Lambda(-\Delta) \otimes \det^{-1} \pi_! \mathcal{O}(\Delta),
\]
where \( \Delta \subset \text{Sym}^i C \times C \) is the universal divisor, cf. [Tha94, 6.5]. To emphasize the degree \( d \), sometimes we denote this line bundle by \( L_i(d) \).

Lemma 6.11. \( H^p(\text{Sym}^i C, L_i(d)) = 0 \) if \( p > 0, 1 \leq i \leq d - g \).

Proof. By [Tha94, 7.5] (see also [Mac62]), and mixing notation for line bundles and divisors,
\[
L_i(d) = (d - 2i)\eta + 2\sigma \quad \text{and} \quad K_{\text{Sym}^i C} = (g - i - 1)\eta + \sigma,
\]
where \( \eta = p_0 + \text{Sym}^{i-1} C \subset \text{Sym}^i C \) is an ample divisor and \( \sigma \subset \text{Sym}^i C \) is a pull-back of a theta-divisor via the Abel–Jacobi map, in particular \( \sigma \) is nef. It follows that \( L_i(d) - K_{\text{Sym}^i C} = (d - i - g + 1)\eta + \sigma \) is ample if \( i \leq d - g \) and the result follows by Kodaira vanishing theorem.

Lemma 6.12. Suppose \( i + g \leq d \leq 2g + 1 \). Then \( \chi(M_i(d), \mathcal{O}_i(1, i - 1)) = \chi(\text{Sym}^i C, L_i(d)) \).

Proof. Since \( i \leq d - g \), we can use [Tha94, 6.2] together with [Tha94, 7.8] to compute \( \chi(\mathcal{O}_i(1, i - 1)) = \)
\[
= \text{Res}_{t = 0} \left\{ \frac{(1 + t^3)^{2i-d-1}(1 - t^2)^{2d+1-2i-2g}}{t^i+1(1 - t)^{d+g-1}} (1 - 5(1 - t)t^2 - t^5)^{g} dt \right\}
\]
(6.7)
\[
= \text{Res}_{t = 0} \left\{ \frac{(1 + t)^{2d+1-2i-2g}(t^2 + 3t + 1)^{g}(t - 1)}{t^i+1(1 + t + t^2)^{d+1-2i}} dt \right\}.
\]
On the other hand, we use Hirzebruch–Riemann–Roch theorem to compute, using the formulas
\[
\text{ch}(L_i) = e^{(d-2)\eta + \sigma}, \quad \text{td}(\text{Sym}^i C) = \left( \frac{\eta}{1 - e^{-\eta}} \right)^{i-g+1} \exp \left( \frac{\sigma}{e^{\eta} - 1} - \frac{\sigma}{\eta} \right)
\]
(see [Tha94, §7]) and notation from the proof of Lemma 6.11 that
\[
\chi(L_i) = \text{Res}_{\eta = 0} \left\{ \frac{e^{\eta(d-2i)}}{(1 - e^{-\eta})^{i-g+1}} \left( 2 + \frac{1}{e^{\eta} - 1} \right)^{g} d\eta \right\}.
\]

If we let \( u(\eta) = e^{\eta} - 1 \), then \( u \) is biholomorphic near \( \eta = 0 \), with \( u(0) = 0 \), \( u'(0) = 1 \), so we can do a change of variables \( u = e^{\eta} - 1, \, du = e^{\eta} d\eta \) to obtain
\[
\chi(L_i) = \text{Res}_{u = 0} \left\{ \frac{(1 + u)^{d-i-g}(2u + 1)^g}{u^{i+1}} du \right\}.
\]
Next, we apply an ad hoc change of variables
\[ u = \frac{t}{t^2 + t + 1}, \quad du = \frac{1 - t^2}{(t^2 + t + 1)^2} dt \]
to \((6.8)\) and we get precisely \((6.7)\) after some algebraic manipulations. \(\square\)

For what follows we need some geometric constructions. Fix a point \(p_0 \in C\) and consider a subvariety \(M_{i-1}(d-1) \subseteq M_i(d+1)\) of codimension 2 as in Remark \([3.7]\) with \(D = p_0\). Let \(B\) be the blow-up of \(M_i(d+1)\) in \(M_{i-1}(d-1)\) with exceptional divisor \(E\).

Consider the \(\mathbb{P}^1\)-bundle \(\mathbb{P} F_{p_0}\) over \(M_i(d+1)\) that parametrizes triples \((E, \phi, l)\), where \(\phi\) is a non-zero section of \(E\) and \(l \subseteq E_{p_0}\) is a line, subject to the usual stability condition (see Section \([3]\) that for every line subbundle \(L \subseteq E\) one must have

\[(6.9) \quad \deg L \leq \begin{cases} i + \frac{1}{2} & \text{if } \phi \in H^0(L), \\ d - i + \frac{1}{2} & \text{if } \phi \notin H^0(L). \end{cases} \]

**Lemma 6.13.** With the notation as above, the blow-up \(B\) of \(M_i(d+1)\) in \(M_{i-1}(d-1)\) is isomorphic to the following locus:

\[ Z = \{(E, \phi, l) : \phi(p_0) \in l\} \subseteq \mathbb{P} F_{p_0}. \]

**Proof.** Indeed, the projection of \(Z\) onto \(M_i(d + 1)\) is clearly an isomorphism outside of \(M_{i-1}(d-1)\), since the latter is precisely the locus where \(\phi(p_0) = 0\). Over \(M_{i-1}(d-1)\), the fiber of this projection is \(\mathbb{P}^1\). By the universal property of the blow-up, it suffices to check that \(Z\) is the blow-up of \(M_i(d+1)\) in \(M_{i-1}(d-1)\) locally near \((E, \phi) \in M_{i-1}(d-1)\), where we can trivialize \(F_{p_0} \cong \mathcal{O} \oplus \mathcal{O}\). Its universal section can be written as \(s = (a, b)\), where \(a, b \in \mathcal{O}\) is a regular sequence (its vanishing locus is \(M_{i-1}(d-1)\) locally near \((E, \phi)\)). Then \(Z\) is locally given by the equation \(ay - bx = 0\), where \([x : y]\) are homogeneous coordinates of the \(\mathbb{P}^1\)-bundle \(\mathbb{P} F_{p_0}\) given by the trivialization \(F_{p_0} \cong \mathcal{O} \oplus \mathcal{O}\). Thus \(Z\) is indeed isomorphic to the blow-up \(B\). \(\square\)

Now we can prove the main result of this section.

**Proof of Theorem 6.1.** We proceed by induction on \(k\). Suppose \(R \Gamma_{M_i}(\Lambda^{-k})\) is zero for \(1 \leq k < i\). By Lemma \([6.3]\) it suffices to show that \(R \Gamma_{M_i}(\Lambda^{-i}) = 0\). Twist the tautological short exact sequence for \(E_i \subseteq M_i\) by \(\mathcal{O}_i(1, i - 1)\) to get

\[ 0 \to \Lambda^{-i}_{M_i} \to \mathcal{O}_i(1, i - 1) \to \mathcal{O}_{E_i}(1, i - 1) \to 0. \]

It suffices to prove that \(R \Gamma_{M_i}(\mathcal{O}_i(1, i - 1)) \cong R \Gamma_{E_i}(\mathcal{O}_{E_i}(1, i - 1))\) via \(R \gamma\). By the induction hypothesis, we can apply Corollary \([6.7]\) to see that

\[ R \Gamma(\mathcal{O}_{E_i}(1, i - 1)) \cong \ldots \cong R \Gamma(\mathcal{O}_{E_i}(1, i - 1)) = R \Gamma(\mathcal{O}_{\mathbb{P}W_i^+}(1, i - 1)). \]

But \(\mathcal{O}_{\mathbb{P}W_i^+}(1, i - 1)\) restricts trivially to each fiber of \(\mathbb{P}W_i^+\) and is a pull-back of \(L_i\) \((6.5)\) on \(\text{Sym}^1 C\) \([\text{Tha94}, \S 5]\), which implies \(R \Gamma(\mathcal{O}_{\mathbb{P}W_i^+}(1, i - 1)) \cong\).
$R\Gamma(\text{Sym}^i C, L_i)$. Therefore, it suffices to show that

$$R\Gamma_{M_i(d)}(O_i(1, i - 1)) \cong R\Gamma_{\text{Sym}^i C}(L_i(d))$$  \hfill (6.10)

via the composition of functors as above.

Claim 6.14. If $d \geq i + g$, then \(6.10\) holds.

Proof. In this case $H^p(M_i, O_i(1, i - 1)) = H^p(\text{Sym}^i C, L_i) = 0$ for $p > 0$
by Lemmas 6.8 and 6.11. Using this together with the fact that $\Lambda^{-i}_M = O_i(0, i)$ has
no global sections by Lemma 6.2, it suffices to prove that $h^0(M_i, O_i(1, i - 1)) = h^0(\text{Sym}^i C, L_i)$ or, equivalently, that $\chi(M_i, O_i(1, i - 1)) = \chi(\text{Sym}^i C, L_i)$. Thus, Lemma 6.12 proves the Claim. \qed

We now proceed by a downward induction on $d$, starting with any $d$ such that $d \geq i + g$. For such $d$, we have the result by the Claim above.

Next we perform a step of the downward induction assuming the theorem holds for degree $d + 1$. As above, we fix a point $p_0 \in C$ and consider the subvariety $M_{i-1}(d-1) \subset M_i(d+1)$ of codimension 2 determined by Remark 3.7.

Let $\mathcal{I} \subset \mathcal{O}_{M_i(d+1)}$ be its ideal sheaf. As in the proof of Lemma 6.11, we denote the divisor $p_0 + \text{Sym}^{-i-1} C \subset \text{Sym}^i C$ by $\eta$ and, by abuse of notation, we denote its pull-back to the projective bundle $\mathbb{P} W^+_i$, which as a locus in $M_{i-1}(d-1)$ is nothing but $\mathbb{P} W^+_{i-1}$, by $\eta$ as well. To summarize, we have a commutative diagram of sheaves on $M_i(d + 1)$ with exact rows, where we suppress closed embeddings from notation.

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathcal{I} & \rightarrow & \mathcal{O}_{M_i(d+1)} & \rightarrow & \mathcal{O}_{M_{i-1}(d-1)} & \rightarrow & 0 \\
& & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \rightarrow & \mathcal{O}_{\mathbb{P} W^+_i}(-\eta) & \rightarrow & \mathcal{O}_{\mathbb{P} W^+_i} & \rightarrow & \mathcal{O}_{\mathbb{P} W^+_{i-1}} & \rightarrow & 0
\end{array}
\]  \hfill (6.11)

We tensor (6.11) with $\mathcal{O}(1, i - 1)$. Recall that the restriction of $\mathcal{O}(1, i - 1)$ to $M_{i-1}(d-1)$ is $\mathcal{O}(1, i - 2)$, to $\mathbb{P} W^+_i$ is the pull-back of $L_i(d+1)$ from $\text{Sym}^i C$, and to $\mathbb{P} W^+_{i-1}$ is the pull-back of $L_{i-1}(d-1)$ from $\text{Sym}^{i-1} C$. By inductive hypothesis and Claim 6.14, it follows that the arrows $\beta$ and $\gamma$ in (6.11) give an isomorphism in cohomology after tensoring with $\mathcal{O}(1, i - 1)$.

By the 5-lemma, we conclude that we have an isomorphism

$$R\Gamma(\mathcal{I}(1, i - 1)) \cong R\Gamma(\mathcal{O}_{\mathbb{P} W^+_i}(-\eta)(1, i - 1)).$$  \hfill (6.12)

Note that $\mathcal{O}_{\mathbb{P} W^+_i}(-\eta)(1, i - 1)$ is precisely the pull-back of $L_i(d)$ from $\text{Sym}^i C$ to the projective bundle (see (6.6)). On the other hand, (6.12) gives

$$R\Gamma_B(\mathcal{O}_{B}(1, i - 1)(-\mathcal{E})) \cong R\Gamma_{\text{Sym}^i C}(L_i(d)),$$

where $B$ is the blow-up of $M_i(d + 1)$ in $M_{i-1}(d - 1)$ and $\mathcal{E}$ its exceptional divisor.

Recall that the goal is to prove (6.10). We can do one extra simplification. Let $\sigma = \frac{d}{2} - i$ be the slope on the wall between the moduli spaces $M_i(d)$ and $M_{i-1}(d)$ and let $M_\sigma(d)$ be the corresponding (singular) GIT quotient. The
birational morphism \( M_ι(d) \to M_σ(d) \) contracts the projective bundle \( \mathbb{P} W_ι^+ \) to its base \( \text{Sym}^ι C \), and in particular proving (6.10) is equivalent to proving that

\[
(6.14) \quad RΓ_{M_σ(d)}(\mathcal{O}_i(1, i - 1)) \cong RΓ_{\text{Sym}^ι C}(L_i(d))
\]

by projection formula and Boutot theorem [Bou87]. To show how (6.13) implies (6.14), we need a geometric construction, a variant of the Hecke correspondence, relating \( B \) to \( M_σ(d) \).

By Lemma 6.13, \( B \) carries a family of parabolic (at \( p_0 \in C \)) rank 2 vector bundles \( E \) with a section \( φ \). The parabolic line at \( p_0 \) defines a quotient \( E \to \mathcal{O}_{p_0} \), and we define a rank 2 vector bundle \( E' \) as an elementary transformation, by the formula

\[
(6.15) \quad 0 \to E' \to E \to \mathcal{O}_{p_0} \to 0.
\]

Our condition \( φ(p_0) \in l \) implies that the section \( φ \) lifts to a section \( φ' \) of \( E' \). Elementary transformation is well-known to be a functorial construction [NR78], in fact we claim that \( (E', φ') \) is a \( σ \)-semistable pair, i.e. we have a morphism

\[
h : B \to M_σ(d), \quad (E, φ, l) \mapsto (E', φ').
\]

Indeed, we need to check that

\[
\deg L' \leq \begin{cases} 
i & \text{if } φ' \in H^0(L'), \\
 d - i & \text{if } φ' \notin H^0(L'). 
\end{cases}
\]

for every line subbundle \( L' \subset E' \), which follows from (6.9) applied to \( L' \).

By the Kollár vanishing theorem [Kol86] Theorem 7.1], \( Rh_* \mathcal{O}_B = \mathcal{O}_{M_σ(d)} \). Indeed, \( B \) is smooth, \( M_σ(d) \) has rational singularities and a general geometric fiber of \( h \) is isomorphic to \( \mathbb{P}^1 \) (given by extensions (6.15) with fixed \( E' \)).

By projection formula, (6.13) implies (6.14) if we can show that

\[
h^* \mathcal{O}_i(1, i - 1) \cong \mathcal{O}_B(1, i - 1)(-\mathcal{E}).
\]

Outside of \( \mathcal{E} \) and for any \( q \in C \), the bundle \( F_q \) over the stack of the \( σ \)-semistable pairs (resp. its determinant \( Λ' \)), pulls back to the bundle \( F_q \) over \( B \setminus \mathcal{E} \) (resp. its determinant \( Λ \)), by (6.15). On the other hand, the divisor \( E'_ι \) of \( σ \)-semistable stable pairs \( (E', φ') \) such that \( φ' \) has a zero, pulls back to the analogous divisor \( E_ι \) of \( B \setminus \mathcal{E} \), because the section \( φ \) of \( E \) is the same as the section \( φ' \) of \( E' \). Since \( E \) and \( Λ \) generate the Picard group of \( B \setminus \mathcal{E} \), it follows that \( h^* \mathcal{O}_i(1, i - 1) \cong \mathcal{O}_B(1, i - 1)(-c\mathcal{E}) \) for some integer \( c \). It remains to show that \( c = 1 \). To this end, we re-examine the diagram (6.11). Note that the proper transform \( \overline{P} \) of \( \mathbb{P} W_ι^+ \) in \( B \) is isomorphic to its blow-up in \( \mathbb{P} W_ι^+ \), which is a Cartier divisor \( η \). Therefore, \( \overline{P} \cong \mathbb{P} W_ι^+ \). However, the restriction \( h^* \mathcal{O}_i(1, i - 1)|_\overline{P} \) is isomorphic to the pull-back of \( L_ι^l \) from \( \text{Sym}^ι C \), while the restriction \( \mathcal{O}_B(1, i - 1)|_\overline{P} \) is isomorphic to the pull-back of \( L_ι(d + 1) \). Since \( L_ι(d) \cong L_ι(d + 1)(-η) \), and \( \mathcal{E} \) restricts to \( \overline{P} \) as \( η \), the claim follows. \( \square \)
7. Acyclic vector bundles on $M_i$ – hard cases

The main goal of the present section is to prove the following results.

**Theorem 7.1.** Suppose $2 < d \leq 2g - 1$ and $1 \leq i \leq v$. Let $D = x_1 + \ldots + x_\alpha$, $D' = y_1 + \ldots + y_\beta$ (possibly with repetitions) of degrees $\alpha$, $\beta \leq d + g - 2i - 1$, and let $t$ be an integer satisfying

\[
\deg D - i - 1 < t < d + g - 2i - 1 - \deg D'.
\]

We have:

(a) If $t \not\in [0, \deg D]$, then $R\Gamma_{M_i}(d)((\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes G_D' \otimes \Lambda_M^i) = 0$.

(b) If $\deg D \not\in [t, t + \deg D']$, then $R\Gamma_{M_i}(d)(G_D'' \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^i) = 0$.

In particular, $R\Gamma_{M_i}(d)(G_D'' \otimes G_D' \otimes \Lambda_M^i) = 0$ in either of these cases.

**Proposition 7.2.** Suppose $0 < d \leq 2g - 1$ and $0 \leq i \leq v$. Let $D = x_1 + \ldots + x_\alpha$ (possibly with repetitions), with $\alpha = \deg D < d + g - 2i - 1$. Then

\[
R\Gamma_{M_i} \left( \bigotimes_{k=1}^{\alpha} F_{x_k} \right) = R\Gamma_{M_i}(G_D) = R\Gamma_{M_i}(G_D'' \otimes \Lambda_M^i) = \mathbb{C}.
\]

Moreover, the unique (up to a scalar) global section of these bundles vanishes precisely along the union of codimension $2$ loci $M_{i-1}(\Lambda(-2x_k))$, for $k \in \{1, \ldots, \alpha\}$.

These computations will allow us to verify both the Bondal-Orlov conditions for the fully faithful embeddings of $D^b(\text{Sym}^\alpha C)$ into $D^b(M_i)$, for $\alpha \leq i$, as well as the vanishings needed in order to show semi-orthogonality between the corresponding subcategories of $D^b(M_i)$ thus defined.

**Remark 7.3.** If $\alpha = 1$ then $F_{x_1} = G_D = G_D'' \otimes \Lambda_M$. If $\alpha > 1$ then the precise relationship between vector bundles $G_D$ and $G_D'' \otimes \Lambda_M^i$ is not clear, however both are deformations of $\bigotimes_{k=1}^{\alpha} F_{x_k}$ by Corollary 2.10 see also the proof of Lemma 7.5.

We will prove both Theorem 7.1 and Proposition 7.2 simultaneously, by a combined induction on the degrees of the divisors.

**Lemma 7.4.** With the notation as in Theorem 7.1 let $0 < d \leq 2g - 1$, $i = 0$ and suppose the stronger inequality $\deg D < d + g - 1 - \deg D'$ holds. Then $R\Gamma_{M_0}(d)(G_D'' \otimes G_D' \otimes \Lambda_M^i) = R\Gamma_{M_0}(d)((\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^i) = 0$.

**Proof.** The vector bundle $G_D'' \otimes G_D' \otimes \Lambda_M^i |_{M_0}$ is a deformation over $\Lambda_1$ of $(\bigotimes_{k=1}^{\alpha} F_{x_k}^{\vee}) \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda_M^i |_{M_0}$, which has the form $\bigoplus \mathcal{O}_{\mathbb{P}^{d+g-2}}(s_j - t)$ on $M_0 = \mathbb{P}^{d+g-2}$, where $-\beta \leq s_j \leq \alpha$. By hypothesis, $\alpha - t < 0$ and $-\beta - t \geq -(d + g - 2)$, so this bundle is $\Gamma$-acyclic.

**Lemma 7.5.** Suppose $2 < d \leq 2g - 1$ and $1 \leq i \leq v$. Let $D$ be an effective divisor on $C$ and suppose that $\deg D \leq d + g - 2i-1$. Assume the statement
of Proposition 7.2 holds for all effective divisors \( \tilde{D} \) with \( \deg \tilde{D} < \deg D \).

Then

\[
R\Gamma_M(d)(G^\vee_D \otimes \Lambda^{\deg D - 1}_M) = R\Gamma_M(d)(G_D \otimes \Lambda^{-1}_M) = 0.
\]

**Remark 7.6.** Note that (7.3) is a special case of Theorem 7.1.

**Proof.** Note that for \( 1 < j \leq i \), the bundles \( G^\vee_D \otimes \Lambda^{\deg D - 1}_M \) and \( G_D \otimes \Lambda^{-1}_M \) descend from objects with weights within \([−\deg D + 1, 1]\), where \( 1 < j \) and \( −\deg D + 1 > 1 + 2j - d - g \) by hypothesis. By Theorem 3.17, it suffices to show (7.3) when \( i = 1 \). We write \( D = \alpha_1 x_1 + \ldots + \alpha_s x_s \) with \( x_k \neq x_j \).

If \( \deg D = 0 \) then we are done by Theorem 6.1 and if, say, \( \alpha_1 = 1 \) then we are done by induction on \( \deg D \) by writing \( D = D + x_1 \) and using the exact sequences obtained from Lemma 3.10.

\[
0 \to G^\vee_D \otimes \Lambda^{\deg D - 1}_M \to G^\vee_D \otimes \Lambda^{\deg D - 1}_M \to G^\vee_D \otimes \Lambda^{\deg \tilde{D} - 1}_M \to G^\vee_D \otimes \Lambda^{\deg \tilde{D} - 1}_M |_{M_0} \to 0
\]

and

\[
0 \to G_D \otimes \Lambda^{-1}_M \to G_D \otimes \Lambda^{-1}_M \to G_D \otimes G_D |_{M_0} \to 0,
\]

where \( M_0 = M_0(\Lambda(−2x_1)) \). By Proposition 7.2 (applied to \( \tilde{D} \)), the last two terms in each sequence have vanishing higher cohomology and \( H^0 = \mathbb{C} \) with a global section that does not vanish along \( M_0(\Lambda(−2x_1)) \). Thus

\[
R\Gamma_M(d)(G^\vee_D \otimes \Lambda^{\deg D - 1}_M) = R\Gamma_M(d)(G_D \otimes \Lambda^{-1}_M) = 0
\]

by the hypercohomology spectral sequence \( E_2^{p,q} = H^q(X, F^p) \). So we can assume that \( \alpha_k > 1 \) for all \( k \). By the same reasoning as above, using the sequences (7.4)–(7.5) and the same hypercohomology spectral sequence, we conclude that \( F_{x_1} \otimes G^\vee_D \otimes \Lambda^{\deg D - 1}_M \) and \( F_{x_1} \otimes G_D \otimes \Lambda^{-1}_M \) (and thus their stable deformations \( G^\vee_D \otimes \Lambda^{\deg D - 1}_M \) and \( G_D \otimes \Lambda^{-1}_M \), see Remark 2.15), have the following cohomology: \( h^p = 0 \) for \( p \geq 2 \) and \( h^0 = h^1 \). So it suffices to show that

\[
H^0(M_1(d), G_D \otimes \Lambda^{-1}_M) = H^0(M_1(d), G^\vee_D \otimes \Lambda^{\deg D - 1}_M) = 0.
\]

We recall the construction of \( G_D \) from the proof of Corollary 2.10 adapted to our case when \( D \) is not necessarily a fat point. Let \( M = M_1(d) \).

Let \( \mathbb{B}_\alpha = \text{Spec} \mathbb{C}((\xi_1, \ldots, \xi_s)_{(\sigma_1, \ldots, \sigma_0)} \) be the spectrum of the covariant algebra. Write the indexing set \( \{1, \ldots, \alpha\} \) as a disjoint union of sets \( A_k \) of cardinality \( \alpha_k \) for \( k = 1, \ldots, s \). For every \( j \in A_k \), we have a diagram of morphisms as in (2.3).

\[
\begin{align*}
\mathbb{B}_{\alpha_1} \times \ldots \times \mathbb{B}_{\alpha_s} \times M & \xrightarrow{\pi_j} \mathbb{D}_{\alpha_k} \times M \xrightarrow{q_k} C \times M \\
\tau & \circlearrowleft \quad \rho \quad \circlearrowright
\end{align*}
\]
We let \( F_k = q^* F \), where \( F \) is the universal bundle, and therefore \( G_D = \pi_* S_{\alpha_1} \times \cdots S_{\alpha_s} (\bigotimes \pi_j^* F_k) \) and \( G_D^\vee \otimes \Lambda^D = \pi_* S_{\alpha_1} \times \cdots S_{\alpha_s} (\bigotimes \pi_j^* F_k \otimes \text{sgn}) \).

Thus (7.6) is equivalent to the following: \( \Lambda^{-1}_M \otimes \bigotimes \pi_j^* F_k \) does not have invariant or skew-invariant global sections (with respect to each factor of \( S_{\alpha_1} \times \cdots S_{\alpha_s} \)).

The restriction of \( \Lambda^{-1}_M \otimes \bigotimes \pi_j^* F_i \) to the special fiber \( M \) is \( \Lambda^{-1}_M \otimes \bigotimes F^{\otimes \alpha_k} \).

While the group \( S_{\alpha_1} \times \cdots S_{\alpha_s} \) acts trivially on the special fiber, the action on the vector bundle is still non-trivial (the action permutes tensor factors within each block).

**Claim 7.7.** As a representation of \( S_{\alpha_1} \times \cdots S_{\alpha_s} \), the space \( H^0(M, \Lambda^{-1}_M \otimes \bigotimes F^{\otimes \alpha_k}_{x_k}) \) is isomorphic to the direct sum \( V_{\alpha_1} \oplus \cdots V_{\alpha_s} \) of irreducible representations, where each \( V_{\alpha} \) is the standard \( (\alpha - 1) \)-dimensional irreducible representation of \( S_{\alpha} \) and the other factors \( S_{\alpha_k} \), \( k \neq k \), act on \( V_{\alpha_k} \) trivially. If we realize the representation \( V_{\alpha_k} \) as \( \{ \sum a_j e_j \mid \sum a_j = 0 \} \subset \mathbb{C}^{\alpha_k} \) then the vector \( e_{j'} - e_{j''} \in V_{\alpha_k} \) corresponds to the global section \( s_{j' j''} \) of \( \Lambda^{-1}_M \otimes \bigotimes F^{\otimes \alpha_k}_{x_k} \) that can be written as a tensor product of universal sections \( s_k \) of \( F_{x_k} \) in positions \( j \neq j', j'' \) and the section of \( \Lambda^{-1}_M \otimes F_{x_k} \otimes F_{x_k} \) (in positions \( j', j'' \)) given by wedging (recall that \( \Lambda_M \) is the determinant of \( F_{x_k} \)).

**Proof of the Claim.** The sections \( s_{j' j''} \) satisfy the same linear relations as difference vectors \( e_{j'} - e_{j''} \), namely that \( s_{j_1 j_2} + s_{j_2 j_3} + \cdots + s_{j_{r-1} j_r} + s_{j_r j_1} = 0 \) for \( j_1, \ldots, j_r \in A_k \). Indeed, choose a basis \( \{ f_1, f_2 \} \) in a fiber of the rank 2 bundle \( F_{x_k} \) so that the universal section is equal to \( f_2 \) and the determinant is given by \( f_1 \wedge f_2 \). After reordering of \( j_1, \ldots, j_r \), we have

\[
s_{12} + s_{23} + \cdots + s_{r1} = (f_1 \otimes f_2) \otimes f_2 \otimes \cdots \otimes f_2 - (f_2 \otimes f_1) \otimes f_2 \otimes \cdots \otimes f_2 + f_2 \otimes (f_1 \otimes f_2) \otimes \cdots \otimes f_2 - f_2 \otimes (f_2 \otimes f_1) \otimes \cdots \otimes f_2 + \cdots = 0.
\]

Let \( j_k \in A_k \) be the first index of this subset for \( k = 1, \ldots, s \). It suffices to prove that the sections \( s_{j_k j \in A_k} \) for \( k = 1, \ldots, s \) and \( j \in A_k \setminus \{ j_k \} \), form a basis of \( H^0(M, \Lambda^{-1}_M \otimes \bigotimes F^{\otimes \alpha_k}_{x_k}) \). We prove this by induction on \( \alpha \). Let \( \tilde{F} = F^{\otimes \alpha_1} \otimes \cdots \otimes F^{\otimes (\alpha_s - 1)}_{x_s} \). We have the usual exact sequence obtained from Lemma 3.10:

\[
0 \to \Lambda^{-1}_M \otimes \tilde{F} \to \Lambda^{-1}_M \otimes \tilde{F} \otimes F_{x_s} \to \tilde{F} \to \tilde{F} \mid M_0 \to 0,
\]

where \( M_0 = M_0(\Lambda(-2x_s)) \). By Proposition 7.2, the last two terms have vanishing higher cohomology and \( H^0 = \mathbb{C} \). If \( \alpha \) is the first index in its group \( A_s \), i.e. if \( \alpha = j_k \), then the global section of \( \tilde{F} \) does not vanish along \( M_0 \) and therefore \( H^0(\Lambda^{-1}_M \otimes \tilde{F}) = H^0(\Lambda^{-1}_M \otimes \tilde{F} \otimes F_{x_s}) \) by the corresponding hypercohomology spectral sequence, and the basis stays the same. On the other hand, if \( \alpha \neq j_s \) then the global section of \( \tilde{F} \) (the tensor product of universal sections) vanishes along \( M_0 \) inducing the zero map \( H^0(\tilde{F}) \to H^0(\tilde{F} \mid M_0) \). Moreover, the section \( s_{j_k \alpha} \in H^0(\Lambda^{-1}_M \otimes \tilde{F} \otimes F_{x_s}) \) maps onto the
global section of $\tilde{F}$. Thus the claim also follows from the hypercohomology spectral sequence.

If $\alpha_k > 2$ then $V_{\alpha_k}$ is neither trivial nor the sign representation of $S_{\alpha_k}$. On the other hand, if $\alpha_k = 2$ but $\alpha > 2$ then $V_{\alpha_k}$ is the sign representation of $S_{\alpha_k}$ and the trivial representation for other factors $S_{\alpha_l}$, $l \neq k$. Therefore, in either of these cases, no invariant or skew-invariant global sections of $\Lambda_M^{-1} \otimes \otimes \pi_j^* F_k$ can specialize to $V_{\alpha_k}$, in particular the lemma is proved in these cases. It follows that the only case that we need to consider is $\alpha = \alpha_1 = 2$. Let $x = x_1$. Let $\nu : \mathbb{B}_2 \times M \to M$ be the degree 2 projection map. It suffices to show that the (skew-symmetric) section $s$ in either of these cases, no invariant or skew-invariant global sections of $\alpha$ in these cases. It follows that the only case that we need to consider is $1 + 2 g - 2 i - 1$. Then $\Phi_F : D^b(C) \to D^b(M)$ be the Fourier–Mukai transform with kernel $F$. Then $\tilde{F} = \Phi_F(O_{2x})$ and since $\Phi_F$ is fully faithful (Theorem 5.1), $\tilde{F}$ is indecomposable. This gives a contradiction.

Lemma 7.8. Suppose Theorem 7.1 holds for all divisors of deg $D$, deg $D' < a < d + g - 2i - 1$. Then Proposition 7.2 holds for all divisors with degree $\alpha \leq a$.

Proof. When $i = 0$, $F_{x_k} = O_{\mathbb{P}^r} \oplus O_{\mathbb{P}^r}(-1)$ on $M_0 = \mathbb{P}^r$, $r = d + g - 2$, and $\bigotimes F_{x_k}$ splits as a sum of line bundles $\bigoplus O_{\mathbb{P}^r}(s_j)$, where $-\alpha \leq s_j \leq 0$ and exactly one of the summands is $O_{\mathbb{P}^r}$. Since $\alpha \leq d + g - 2$, $R^i \Gamma_{M_k}(\bigotimes_{k=1}^n F_{x_k}) = \mathbb{C}$ in this case. Since $G_D$ and $G_D' \otimes \Lambda^D_M$ are deformations of $\bigotimes_{k=1}^n F_{x_k}$ over $\mathbb{A}^1$, we have (7.2) by semi-continuity and equality of the Euler characteristic.

Let $i \geq 1$. We see that, using Theorem 3.17, it suffices to prove (7.2) on $M_1(d)$. In fact, by Theorem 3.18, $\bigotimes_{k=1}^n F_{x_k}$ descends from an object with weights within $[-\alpha, 0]$, all of which live in the window $1 + 2j - d - g, j$ for $1 < j \leq i$, since $1 + 2j - d - g \leq 1 + 2i - d - g < -\alpha$ by hypothesis. This way we get $R^i \Gamma_{M_k}(\bigotimes_{k=1}^n F_{x_k}) = R^i \Gamma_{M_k}(\bigotimes_{k=1}^n F_{x_k})$. Similarly, $R^i \Gamma_{M_k}(G_D) = R^i \Gamma_{M_k}(G_D) \otimes \Lambda_M^D$ and $R^i \Gamma_{M_k}(G_D' \otimes \Lambda_M^D) = R^i \Gamma_{M_k}(G_D' \otimes \Lambda_M^D)$.
Hence, we take $i = 1$ and $\alpha < d + g - 3$. In this case $d > 2$. Let us show the result for $\bigotimes F_{x_k}$ first. We do induction on $\alpha$. If $D = 0$, the result is trivial. Otherwise, use the sequence (3.4) on $E_{x_k}$ to obtain an exact sequence

$$0 \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M \to \bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M \big|_{M_0(d-2)} \to 0.$$  

Of these terms, we get $R \Gamma_{M_i(d)}(\bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M) = 0$ from the vanishing in Theorem 7.1, which by hypothesis holds in this case. Also, we have $R \Gamma_{M_0(d-2)}(\bigotimes_{k=1}^{\alpha-1} F_{x_k} \otimes \Lambda_M) = 0$ from Lemma 7.4 given that $t = 1 \neq 0$ and $0 < 1 < d + g - 3 - (\alpha - 1)$. Using the hypercohomology spectral sequence $E_{1,0} = H^g(X,F^p)$ and induction, we obtain

$$R \Gamma_{M_i} \left( \bigotimes_{k=1}^{\alpha} F_{x_k} \right) = R \Gamma_{M_i} \left( \bigotimes_{k=1}^{\alpha-1} F_{x_k} \right) = \mathbb{C}.$$  

Finally, by Corollary 2.10 both $G_D$ and $G_D^\vee \otimes \Lambda_M^{\text{deg}D}$ are deformations over $\mathbb{A}^1$ of $\bigotimes_{k=1}^{\alpha} F_{x_k}$, so we have (7.2) by semi-continuity and equality of the Euler characteristic. It also follows that the global section of $G_D$ (resp., $G_D^\vee \otimes \Lambda_M^{\text{deg}D}$) is a deformation of the global section of $\bigotimes_{k=1}^{\alpha} F_{x_k}$ over $\mathbb{A}^1$, which does not vanish outside of the union of loci $M_{i-1}(\Lambda(-2\pi))$ for $k = 1, \ldots, \alpha$. On the other hand, tautological sections of these bundles (the descent of the tensor product of tautological sections of $\bigotimes \pi_j^* F_k$ (resp., this tensor product tensored with the sign representation) for $G_D$ (resp., $G_D^\vee \otimes \Lambda_M^{\text{deg}D}$) vanish precisely along these loci. \hfill \square

Now we can prove Theorem 7.1. Observe that by Lemma 7.8 this will also complete the proof of Proposition 7.2.

Proof of Theorem 7.1. We consider cases (a) and (b) separately.

The case $t \notin [0, \deg D')$. We first suppose $D = 0$ and do induction on $\deg D'$. If $D = D' = 0$, we need to show that for $t \neq 0$ with $-i - 1 < t < d + g - 2i - 1$ we have $R \Gamma_{M_{i-1}}(\Lambda_M^t) = 0$. If $t > 0$, Lemma 4.2 ensures $R \Gamma_{M_{i-1}}(\Lambda_M^t) = 0$, since $i \geq 1$ and so $t \leq d + g - 4$. But also for every $1 < j \leq i$ we have $1 + 2j - d - g < -t < 0 < j$, that is, the weight of $\Lambda_M^t$ lives in the window between $M_{j-1}$ and $M_j$, so we conclude $R \Gamma_{M_{i-1}}(\Lambda_M^t) = R \Gamma_{M_{i-1}}(\Lambda_M^t) = 0$ by Theorem 3.17. Suppose now $t < 0$, so that $-i \leq t < 0$. By Theorem 6.1, $R \Gamma_{M_{i-1}}(\Lambda_M^t) = 0$.

Let $D = 0$ and $\deg D' \geq 1$. Assume the result holds for divisors $\tilde{D}'$ with $\deg \tilde{D}' < \deg D'$. In this case (7.3) holds for $D'$ by Lemma 7.8 and Lemma 7.5 since $\deg D' \leq d + g - 2i - 1 \leq d + g - 3$. We need to show that this implies $R \Gamma_{M_{i-1}}(G_{D'} \otimes \Lambda_M^{t'}) = 0$ for $-i - 1 < t < d + g - 2i - 1 - \deg D'$ and $t \neq 0$. If $t = -1$, then $R \Gamma_{M_{i-1}}(G_{D'} \otimes \Lambda_M^{t'}) = 0$ by (7.3). If $t \neq -1$, we write $D' = \tilde{D}' + y$ and use the fact that $G_{D'}$ is a stable deformation...
of $F_y \otimes G_{D'}$, over $\mathbb{A}^1$ (see Proposition 2.13). If we take the sequence (3.4)
twisted by $G_{D'} \otimes \Lambda^i_M$, we get an exact sequence

$$0 \rightarrow G_{D'} \otimes \Lambda^i_M \rightarrow F_y \otimes G_{D'} \otimes \Lambda^i_M \rightarrow G_{D'} \otimes \Lambda^{i+1}_M \rightarrow G_{D'} \otimes \Lambda^{i+1}_{M+1} \big|_{M+1} \rightarrow 0.$$ 

Observe that this is an acyclic chain complex involving $F_y \otimes G_{D'} \otimes \Lambda^i_M$ and where the remaining three terms satisfy the corresponding inequalities from (7.1): $-i-1 < t < d+g-2i-1-\deg D'$, $-i-1 < t+1 < d+g-2i-1-\deg D'$, $-i-1 < t+1 < d+2+g-2(i-1)-1-\deg D'$. Notice that the inequality $\deg D' \leq (d-2)+g-2(i-1)-1$ is preserved too. Given that $t \neq -1$, we have both $t$, $t+1 \neq 0$ so by induction we see that $R\Gamma_{M_i(d)}(G_{D'} \otimes \Lambda^1_M) = R\Gamma_{M_i(d)}(G_{D'} \otimes \Lambda^{i+1}_M) = 0$. On the other hand, we obtain $R\Gamma_{M_{i+1}(d-2)}(G_{D'} \otimes \Lambda^{i+1}_M) = 0$ either by induction if $i > 1$, or from Lemma 7.4 if $i = 1$. Therefore we get the desired vanishing from the corresponding hypercohomology spectral sequence and semi-continuity.

Next we do induction on $\alpha = \deg D$. If $\alpha \geq 1$, we write $D = \tilde{D} + x_\alpha$ and take the sequence (3.3) with $F^\vee_{x_\alpha}$ twisted by $(\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^i_M$. This way we get an exact sequence involving $(\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^i_M$, and where the remaining terms are $(\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^{i+1}_M$ and $(\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^i_M$ on $M_i(d)$, and $(\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^i_M$ on $M_{i+1}(d-2)$. All three still satisfy the inequalities (7.1): $\deg \tilde{D} - i-1 < t-1 < d+g-2i-1-\deg D'$, $\deg \tilde{D} - i-1 < t < d+g-2i-1-\deg D'$, $\deg \tilde{D} - (i-1)-1 < t < d-2+g-2(i-1)-1-\deg D'$. Further, $t$, $t-1 \notin [0, \deg D]$ so by induction $R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^{i+1}_M) = R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^i_M) = 0$, while $R\Gamma_{M_{i+1}(d-2)}((\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^{i+1}_M) = 0$ either by induction when $i > 1$ or by Lemma 7.4 when $i = 1$ (observe that when $i = 1$ we must have $t > \deg \tilde{D}$). By looking at the corresponding hypercohomology spectral sequence we obtain the vanishing $R\Gamma_{M_i(d)}((\bigotimes_{k=1}^{\alpha-1} F^\vee_{x_k}) \otimes G_{D'} \otimes \Lambda^i_M) = 0$.

The case $\deg D \notin [t, t+\deg D']$. The proof is similar. If $D = D' = 0$, then $t \neq 0$ and the vanishing has already been shown above. Let $D' = 0$ and $\deg D \geq 1$. If $t = \deg D-1$, the vanishing follows from induction and Lemma 7.5. If $t \neq \deg D-1$, we write $D = \tilde{D} + x$. Twisting (3.3) by $G^\vee_D \otimes \Lambda^i_M$ yields an exact sequence involving $F^\vee_D \otimes G^\vee_D \otimes \Lambda^i_M$ and where the remaining terms are $G^\vee_D \otimes \Lambda^{i-1}_M$ and $G^\vee_D \otimes \Lambda^i_M$ on $M_i(d)$ and $G^\vee_D \otimes \Lambda^i_M$ on $M_{i+1}(d-2)$. Inequalities (7.1) are preserved in all of them. Further, $t \neq \deg D-1$ implies that both $\deg D \neq t$, $t-1$, so by induction and Lemma 7.4 they are all $\Gamma$-acyclic, and semi-continuity provides $R\Gamma_{M_i(d)}(G^\vee_D \otimes \Lambda^i_M) = 0$.

Finally, we do induction on $\beta = \deg D' \geq 1$. We write $D' = \tilde{D}' + y_\beta$ and use (3.4) twisted by $G^\vee_D \otimes (\bigotimes_{k=1}^{\beta+1} F_{y_k}) \otimes \Lambda^i_M$. The resulting terms other than $G^\vee_D \otimes (\bigotimes_{k=1}^{\beta} F_{y_k}) \otimes \Lambda^i_M$ all satisfy (7.1), and the fact that $\deg D \notin [t, t+\deg D'] = [t, t+1+\deg D']$, together with Lemma 7.4 when $i = 1$, guarantee they are all $\Gamma$-acyclic. Again, the hypercohomology
spectral sequence yields the required vanishing. The final statement is a direct consequence of Corollary 2.10 and semi-continuity.

8. Computation of $R \text{Hom}(G_D, G_{D'})$

Now we will compute some of the Ext functors between $G_D$ and $G_{D'}$, which will be needed in the proof of our semi-orthogonal decomposition.

**Proposition 8.1.** Let $d \leq 2g - 1$ and $0 \leq i \leq v$. If $D, D'$ are effective divisors on $C$ satisfying $\deg D \leq i$ and $\deg D' < d + g - 2i - 1$, then $H^p(M_i, G_D^\vee \otimes G_{D'}) = 0$ for $p > \deg D$.

**Proof.** We do induction on $i$. If $i = 0$ then $D$ must be zero and the result follows from Proposition 7.2. Let $i \geq 1$. In this case $d > 2$. Now we do induction on $\alpha = \deg D$. If $D = 0$ the result follows again from Proposition 7.2. Suppose $\deg D \geq 1$ and write $D = \tilde{D} + x$. By Lemma 3.10, the complex (8.1)

$$G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{-1} \rightarrow F'_x \otimes G_D^\vee \otimes G_{D'} \rightarrow G_D^\vee \otimes G_{D'} \rightarrow G_D^\vee \otimes G_D' \left|_{M_i-1(d-2)} \right.$$  

is acyclic. Since $\deg \tilde{D} \leq i - 1$, we can apply induction on the last two terms of (8.1), so

$$H^p(M_i(d), G_D^\vee \otimes G_{D'}) = H^p(M_{i-1}(d-2), G_D^\vee \otimes G_{D'}) = 0 \quad \forall p \geq \alpha.$$  

On the other hand, the fact that $\deg D' < d + g - 2i$ guarantees that the inequalities $\deg \tilde{D} - i - 1 < -1 < d + g - 2i - 1 - \deg D'$ are satisfied, which implies that $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{-1}$ is $\Gamma$-acyclic by Theorem 7.1. Therefore the hypercohomology spectral sequence $E^{p,q}_1 = H^q(X, F^p)$ of (8.1) has the following shape on its first page:

$$
\begin{array}{cccc}
& & & \\
\vdots & \vdots & \vdots & \\
H^0_{M_i(d)}(F'_x \otimes G_D^\vee \otimes G_{D'}) & \rightarrow & 0 & \rightarrow 0 \\
H^0_{M_i(d)}(F'_x \otimes G_D^\vee \otimes G_{D'}) & \rightarrow & 0 & \rightarrow 0 \\
H^0_{M_i(d)}(F'_x \otimes G_D^\vee \otimes G_{D'}) & \rightarrow & H^0_{M_i(d)}(G_D^\vee \otimes G_{D'}) & \rightarrow \cdots \\
& \vdots & \vdots & \\
& & & \\
\end{array}
$$

Since (8.1) is a $\Gamma$-acyclic complex, we must have

$$H^p(F'_x \otimes G_D^\vee \otimes G_{D'}) = 0 \quad \forall p \geq \alpha.$$  

Finally, we use the fact from Proposition 2.13 that $G_D^\vee \otimes G_{D'}$ is a stable deformation over $\mathbb{A}^1$ of $F'_x \otimes G_D^\vee \otimes G_{D'}$, and thus cohomology vanishing of the latter implies vanishing on the former. This completes the proof. □
Using the previous results we can show that $G_D^\vee \otimes G_D$ has exactly one nontrivial global section. We need a lemma first.

**Lemma 8.2.** Let $d \leq 2g - 1$ and let $D, D'$ be two effective divisors on $C$ of $\deg D = \alpha \leq i$, $\deg D' < d + g - 2i - 1$. Write $D = x_1 + \ldots + x_\alpha$, in arbitrary order and possibly with repetitions. Then for every $k \leq \alpha$ we have $h^0(M_i(d), (\bigotimes_{j=1}^k F_{x_j}^\vee) \otimes G_{D'}) \leq 1$.

**Proof.** If $i = 0$ then $\alpha = k = 0$ and this is given by Proposition 7.2. Let $i \geq 1$, so $d > 2$. We do induction on $k$. If $k = 0$, this still follows from Proposition 7.2. Otherwise, we use Lemma 3.10 to get an acyclic complex

$$0 \rightarrow \bigotimes_{j=1}^{k-1} F_{x_j}^\vee \otimes G_{D'} \otimes \Lambda_{M_i}^{-1} \rightarrow \bigotimes_{j=1}^k F_{x_j}^\vee \otimes G_{D'} \rightarrow$$

$$\rightarrow \bigotimes_{j=1}^{k-1} F_{x_j}^\vee \otimes G_{D'} \rightarrow \bigotimes_{j=1}^{k-1} F_{x_j}^\vee \otimes G_{D'} \rightarrow 0$$

where $M_{i-1} = M_{i-1}(\Lambda(-2x_k))$. The first term can be proved to be $\Gamma$-acyclic using Theorem 7.1. Indeed, here $t = -1 \notin \{0, k - 1\}$ and the inequalities $(k - 1) - i - 1 < -1 < d + g - 2i - 1 - \deg D'$ are satisfied since $k \leq \alpha \leq i$ and $\deg D' < d + g - 2i$. On the other hand, $h^0(M_i(d), (\bigotimes_{j=1}^{k-1} F_{x_j}^\vee) \otimes G_{D'}) \leq 1$ by induction. Therefore, taking the hypercohomology spectral sequence $E_1^{p,q} = H^q(X, F^p)$ of the $\Gamma$-acyclic complex above, we conclude that $h^0(M_i(d), (\bigotimes_{j=1}^k F_{x_j}^\vee) \otimes G_{D'}) \leq 1$ as well.

**Corollary 8.3.** Suppose $d \leq 2g - 1$ and let $0 \leq i \leq v$. If $\deg D \leq i$, then $\operatorname{Hom}_{M_i(d)}(G_D, G_D) = \mathbb{C}$.

**Proof.** We have $\dim \operatorname{Hom}_{M_i(d)}(G_D, G_D) = h^0(G_D^\vee \otimes G_D)$, which by Corollary 2.10 and semi-continuity, is at most $h^0(M_i(d), (\bigotimes_{j=1}^{\deg D} F_{x_j}^\vee) \otimes G_{D'})$. By Lemma 8.2, this dimension is at most 1, since by hypothesis $\deg D \leq i < d + g - 2i$. On the other hand, the identity provides a nontrivial map $G_D \rightarrow G_D$, so $\dim \operatorname{Hom}_{M_i(d)}(G_D, G_D)$ must be exactly 1.

Next we need to investigate $R\operatorname{Hom}(G_D, G_{D'})$ between different divisors. We want to obtain $\Gamma$-acyclicity of $G_D^\vee \otimes G_{D'}$, for which we need some preliminary computations.

**Lemma 8.4.** Let $2 < d \leq 2g - 1$ and let $D = \alpha_1 x_1$, $D' = \alpha_2 x_2 + \ldots + \alpha_s x_s$ be two effective divisors with $\deg D + \deg D' \leq d + g - 3$. Suppose $x_1 \notin D'$ and every point $x_k$ in the support of $D'$ appears with multiplicity $\alpha_k \geq 2$, $k = 2, \ldots, s$. Then $H^0(M_1(d), G_D^\vee \otimes G_{D'} \otimes \Lambda_{M_i}^{\alpha_i - 1}) = 0$.

**Proof.** We use the notation of Lemma 7.3, in particular we set $\alpha = \alpha_1 + \deg D'$ and partition the indexing set $\{1, \ldots, \alpha\}$ into a disjoint union of sets $A_k$ of cardinality $\alpha_k$ for $k = 1, \ldots, s$. Call $M = M_1(d)$. We will use
the diagram of morphisms (7.7). By definition of tensor vector bundles, our claim is equivalent to the following: an \((S_{\alpha_1} \times \ldots \times S_{\alpha_s})\)-equivariant

vector bundle \(\Lambda_M^{-1} \otimes \bigotimes \pi_j^* F_k\) on \(B_{\alpha_1} \times \ldots \times B_{\alpha_s} \times M\) does not have a section which is skew-invariant with respect to \(S_{\alpha_1}\) and invariant with respect to

\(S_{\alpha_2} \times \ldots \times S_{\alpha_s}\).

By Claim 7.7, \(\Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k}\) and \(H^0(M, \Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k})\), as a representation of \(S_{\alpha_1} \times \ldots \times S_{\alpha_s}\), is isomorphic to a direct sum \(V_{\alpha_1} \oplus \ldots \oplus V_{\alpha_s}\) of distinct irreducible representations. Each \(V_{\alpha_i}\) is the standard \((\alpha_k - 1)\)-dimensional irreducible representation of \(S_{\alpha_k}\) and the other factors \(S_{\alpha_l}\), \(l \neq k\), act on it trivially. The only occurrence of the required character (sign representation for \(S_{\alpha_l}\), trivial representation for \(S_{\alpha_2} \times \ldots \times S_{\alpha_s}\)) is \(V_{\alpha_1}\). Moreover, we need \(\alpha_1 = 2\).

According to Claim 7.7, the corresponding global section of \(\Lambda_M^{-1} \otimes \bigotimes F_{x_k}^{\otimes \alpha_k}\) is \(s_{12}\), which is a tensor product of universal sections \(s_k\) of \(F_{x_k}\) for \(k > 1\) and the section of \(\Lambda_M^{-1} \otimes F_{x_1} \otimes F_{x_1}\) given by wedging. We need to show that \(s_{12}\) does not deform to a section of \(\Lambda_M^{-1} \otimes \bigotimes \pi_j^* F_i\) which is skew-invariant with respect to \(S_{\alpha_1}\) and invariant with respect to \(S_{\alpha_2} \times \ldots \times S_{\alpha_s}\). Equivalently, since \(\Lambda_M^{-1} \otimes \pi_j^* F_1 \cong \pi_j^* F_1\), we claim that the morphism \(\mu = \text{Id} \otimes S \in \text{Hom}(F_{x_1}, F_{x_1} \otimes \bigotimes_{k>1} F_{x_k}^{\otimes \alpha_k})\), where \(S\) is a tensor product of universal sections, does not deform to a morphism \(\tilde{\mu} \in \text{Hom}(\pi_1^* F_1, \pi_1^* F_1 \otimes \bigotimes_{k>1} \pi_j^* F_k)\) invariant with respect to \(S_{\alpha_2} \times \ldots \times S_{\alpha_s}\). We argue by contradiction and suppose that \(\tilde{\mu}\) exists. It will be easier to reformulate the problem slightly.

Let \(\nu : B_{\alpha_1} \times \ldots \times B_{\alpha_s} \times M \to B_{\alpha_1} \times \ldots \times B_{\alpha_s} \times M\) be a degree 2 projection map. Let \(p' : \mathcal{Z}' \to B_{\alpha_2} \times \ldots \times B_{\alpha_s} \times M\) be the product of \(\mathbb{P}\)-bundles \(\prod_{k \geq 2} \mathbb{P}(\pi_j^* F_k)\) and let \(p : \mathcal{Z} \to B_{\alpha_1} \times \ldots \times B_{\alpha_s} \times M\) be its pullback via \(\nu\). Let \(\mathcal{L}'\) (resp. \(\mathcal{L}\)) be the line bundle \(\mathcal{O}(1, \ldots, 1)\) on \(\mathcal{Z}'\) (resp. its pullback to \(\mathcal{Z}\)), so that \(R\nu_* \mathcal{L} = \bigotimes_{k>1} \pi_j^* F_k\). By the projection formula applied to \(p\), it suffices to show that the morphism \(\mu = \text{Id} \otimes S \in \text{Hom}(F_{x_1}, F_{x_1} \otimes \mathcal{L})\) on the special fiber \(Z\) of \(\mathcal{Z}\) does not deform to a \((S_{\alpha_2} \times \ldots \times S_{\alpha_s})\)-equivariant morphism \(\tilde{\mu} \in \text{Hom}(\pi_1^* F_1, \pi_1^* F_1 \otimes \mathcal{L})\), where \(S \in H^0(Z, \mathcal{L})\) is induced by the universal section of \(\bigotimes_{k>1} F_{x_k}^{\otimes \alpha_k}\). We argue by contradiction and suppose that \(\tilde{\mu}\) exists. Note that \(\mu\) deforms to an obvious morphism \(\tilde{\mu} = \text{Id} \otimes \hat{S} \in \text{Hom}(\pi_1^* F_1, \pi_1^* F_1 \otimes \mathcal{L})\), where \(\hat{S} \in H^0(Z, \mathcal{L})\) is induced by the universal section of \(\bigotimes_{k>1} \pi_j^* F_k\).

In the diagram (7.7), let \(F_1 = \rho_4 q_4^* F\). Note that \(B_2 \cong \mathbb{D}_2 \cong \mathbb{C}[\epsilon]/(\epsilon^2)\). So \(F_1\) is a rank 4 vector bundle on \(M\), which is an extension of \(F_{x_1}\) by \(F_{x_1}\). We denote the subbundle by \(F'_1\) and the quotient bundle by \(F''_1\). A vector bundle \(\hat{F}_1 := \nu_4 \pi_4^* F_1 \cong \nu_4 \pi_4^* F_1\) is a pull-back of \(F_1\) via \(\beta : \mathcal{Z}' \to M\). Thus it is also an extension of \(\beta^* F'_{x_1}\) by \(\beta^* F''_{x_1}\). Pushing forward \(\hat{\mu}\) to \(\mathcal{Z}'\) by \(\nu\) gives morphisms of rank 4 vector bundles, which we denote by the same letters, \(\hat{\mu}, \tilde{\mu} : \hat{F}_1 \to \hat{F}_1 \otimes \mathcal{L}'\). Arguing as in the proof of Lemma 7.5, the difference of morphisms \(\hat{\mu} - \tilde{\mu}\) factors through a morphism of sheaves \(\hat{F}_1 \to \beta^* \beta^* F'_{x_1} \otimes \mathcal{L}'\) which is an injection on \(\beta^* F'_{x_1}\). In
particular, this morphism gives a \((S_{\alpha_2} \times \ldots \times S_{\alpha_s})\)-equivariant splitting \(\bar{F}_1 \cong \beta^* F'_{x_1} \oplus \beta^* F''_{x_1}\). It follows that \(\dim \text{Hom}_{z'}(\beta^* F_{x_1}, \bar{F}_1) \geq 2\). So, \(\dim \text{Hom}_M(F_{x_1}, \bar{F}_1) \geq 2\). Indeed, \(\beta\) factors into a product of projective bundles \(p'\), which has connected fibers, and the map \(B_{\alpha_2} \times \ldots \times \alpha_s \times M \to M\), which is a categorical quotient for the action of \(S_{\alpha_2} \times \ldots \times S_{\alpha_s}\). Thus \(\beta_{*} \equiv_{z'} \bigotimes \mathcal{O}_{z'} = \mathcal{O}_M\) and the claim follows by projection formula: \(\text{Hom}_{z'}(\beta^* F_{x_1}, \bar{F}_1) = \text{Hom}_M(F_{x_1}, \bar{F}_1)\). However, let \(\Phi_F : D^b(C) \to D^b(M)\) be the Fourier–Mukai transform with kernel \(F\). Then \(F_{x_1} = \Phi_F(O_{x_1})\) and \(\bar{F}_1 = \Phi_F(O_{x_1})\). Since \(\Phi_F\) is fully faithful (Theorem 5.1), \(\text{Hom}_M(F_{x_1}, \bar{F}_1) = \text{Hom}_C(O_{x_1}, O_{x_1}) = \mathbb{C}\), which is a contradiction.

\textbf{Lemma 8.5.} Suppose \(d \leq 2g - 1\). Let \(D, D'\) be effective divisors on \(C\) with \(D = ax \) and \(x \notin D'\).

(a) Let \(1 \leq i \leq v\). If \(\alpha + \deg D' \leq d + g - 2i - 1\), then \(R\Gamma_M(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{\alpha - 1}) = 0\).

(b) Let \(0 \leq i \leq v\). If \(\alpha + \deg D' < d + g - 2i - 1\), then \(R\Gamma_M(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha) = \mathbb{C}\).

Moreover, the unique (up to a scalar) global section of \(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha\) vanishes precisely along the union of codimension 2 loci \(M_0(\Lambda(-2x))\) and \(M_0(\Lambda(-2y))\) for \(y \in \text{supp}(D')\).

\textbf{Proof.} We first show the second claim, for which we use the fact that \(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha\) is a deformation over \(\mathbb{A}^1\) of \((F^\vee_x)^{\otimes \alpha} \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k} \otimes \Lambda_M^\alpha \cong F^\otimes x \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k}\), where \(D' = \sum y_k\). By Proposition 7.2, we see that \(R\Gamma_M(G_D^\vee \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k}) = \mathbb{C}\), so by semi-continuity and equality of the Euler characteristic, we must have \(R\Gamma_M(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha) = \mathbb{C}\) as well.

Furthermore, the global section of \(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha\) is a deformation of the global section of \(F^\otimes x \otimes \bigotimes_{k=1}^{\deg D'} F_{y_k}\) over \(\mathbb{A}^1\), which does not vanish outside of the union of loci \(M_0(\Lambda(-2x))\) and \(M_0(\Lambda(-2y))\). On the other hand, the tautological section of this bundle vanishes precisely along these loci.

The proof of the other claim is similar to the proof of Lemma 7.5. First, notice that by Theorem 3.17 it suffices to prove it for \(i = 1\) since, in each wall-crossing between \(M_{j-1}\) and \(M_j\), the weights of \(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{\alpha - 1}\) are inside \([- \deg D' - \alpha + 1, 1]\), and all of them live in the window \((1+2j-d-g, j)\) for \(1 < j < i\) by the inequality given in the hypothesis. We let \(i = 1\) and do induction on \(\deg D'\). If \(D' = 0\), then we are done by Lemma 7.5. If \(\deg D' \geq 1\), write \(D' = D' + y\), \(y \neq x\) and use the fact that \(G_{D'}\) is a stable deformation of \(G_{D'} \otimes F_y\) over \(\mathbb{A}^1\). From Lemma 3.10, we get an acyclic complex

\[(8.2) \ G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{\alpha - 1} \to F \to G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha \to G_D^\vee \otimes G_{D'} \otimes \Lambda_M^\alpha\big|_{M_0}\]
where $\mathcal{F} = G_D^\vee \otimes G_{D'}^\vee \otimes F_y \otimes \Lambda_M^1$ and $M_0 = M_0(\Lambda(-2y))$. By induction, the first term in (8.2) is $\Gamma$-acyclic, while $R\Gamma_{M_t}(G_D^\vee \otimes G_{D'}^\vee \otimes \Lambda_M^t) = R\Gamma_{M_0}(\Lambda(-2y)) = \mathbb{C}$ and moreover, the unique section vanishes precisely along $M_0(\Lambda(-2y))$, for $y_k \in \text{supp}(D')$. If one of the points in $D'$ appears with multiplicity one, say $\text{mult}_x(D') = 1$, then the map between $H^0$ of the last two terms in (8.2) is an isomorphism $\mathbb{C} \cong \mathbb{C}$. In this case, we conclude $G_D^\vee \otimes G_{D'}^\vee \otimes F_y \otimes \Lambda_M^{-1}$ is $\Gamma$-acyclic by the hypercohomology spectral sequence $E_{1,0}^{pa} = H^q(X, \mathcal{F}^p)$ of (8.2). So we can assume every point in $D'$ has multiplicity at least 2. In this case, (8.2) and semi-continuity still show that $G_D^\vee \otimes G_{D'}^\vee \otimes F_y \otimes \Lambda_M^{-1}$, and therefore $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{-1}$, have vanishing cohomology $H^p$ for $p \geq 2$, while $h^0 = h^1$. The fact that $h^0 = h^1$ on $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{-1}$ follows by equality of the Euler characteristic (cf. Remark 2.15). Thus, it suffices to show that a bundle of the form $G_{\alpha x} \otimes G_{D'} \otimes \Lambda_M^{-1}$ on $M_1$ has no global sections, where $x \notin D'$ and every point of $D'$ has multiplicity at least 2. But this is precisely the content of Lemma 8.4.

\begin{lemma}
Suppose $2 < d \leq 2g - 1$ and $1 \leq i \leq v$. Let $D, D'$ be effective divisors with $D = \alpha x$ and $t + \text{mult}_x(D') \leq \alpha - 1$. If

\begin{equation}
\alpha - i - 1 < t < d + g - 2i - 1 - \deg D'
\end{equation}

then $R\Gamma_{M_t}(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^t) = 0$.

\end{lemma}

\begin{proof}
Let us first suppose $\text{mult}_x(D') = 0$, that is, $x \notin D'$. We do induction on $\deg D'$. If $D' = 0$, we are done by Theorem 7.1 since the inequalities guarantee that $\alpha \neq t$. Suppose $\deg D' \geq 1$. If $t = \alpha - 1$, then from Lemma 8.5 we have $R\Gamma_{M_t}(G_D^\vee \otimes G_{D'} \otimes \Lambda_M^{-1}) = 0$. Therefore, we can assume $t \neq \alpha - 1$. As usual, we write $D' = \tilde{D}' + y$ and use the sequence (3.4) twisted by $G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^t$ to obtain an acyclic complex

\begin{equation}
G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^t \to \mathcal{F} \to G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^{t+1} \to G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^{t+1}_{M_{i-1}}
\end{equation}

where $\mathcal{F} = G_D^\vee \otimes F_y \otimes G_{\tilde{D}'} \otimes \Lambda_M^t$ and $M_{i-1} = M_{i-1}(d-2)$. All the terms in this sequence satisfy (8.3) and moreover $t + 1 \leq \alpha - 1$, so by induction (and by Lemma 7.4 if $i = 1$) we conclude $G_D^\vee \otimes F_y \otimes G_{\tilde{D}'} \otimes \Lambda_M^t$ is $\Gamma$-acyclic, and so is $G_D^\vee \otimes G_{D'} \otimes \Lambda_M^t$ by semi-continuity.

Finally, we do induction on $\text{mult}_x(D')$. Assume $\text{mult}_x(D') \geq 1$, so we can write $D' = \tilde{D}' + x$. Similarly, twisting (3.4) by $G_D^\vee \otimes G_{\tilde{D}'} \otimes \Lambda_M^t$ we obtain an acyclic complex like (8.4), but with $F_x$ in place of $F_y$. Now $\text{mult}_x(\tilde{D}') = \text{mult}_x(D') - 1$, so all the terms in this complex satisfy the hypotheses of the lemma, and using induction and Lemma 7.4 we conclude $R\Gamma_{M_t}(G_D^\vee \otimes F_x \otimes G_{\tilde{D}'} \otimes \Lambda_M^t)$, from the corresponding hypercohomology spectral sequence. The statement of the lemma follows from semi-continuity.

\end{proof}
Theorem 8.7. Suppose $2 < d \leq 2g - 1$ and $1 \leq i \leq v$. Let $D, D'$ be effective divisors on $C$, with $D \not\leq D'$ and satisfying $\deg D \leq i$ and $\deg D' < d + g - 2i - 1$. Then $R\Gamma_{M_i}(G^\vee_D \otimes G_{D'}) = 0$.

Proof. We do induction on $\deg D$. If $\deg D = 1$, then we have $D = x$ and $\text{mult}_x(D') = 0$, so the result follows from Lemma 8.6 with $\alpha = 1$ and $t = 0$.

Let $\deg D > 1$, and so $i > 1$ as well. Since $D \not\leq D'$, there is a point $x \in D$ with $\text{mult}_x(D) = \alpha$, $\text{mult}_x(D') \leq \alpha - 1$. If $\text{supp}(D) = \{x\}$, then $D = \alpha x$ is a fat point and the result follows from Lemma 8.6. Otherwise, we can find a point $y \neq x$ such that $\tilde{D} = D - y$ is effective. From $(3.3)$, we get an exact sequence

$$0 \to G^\vee_D \otimes G_{D'} \otimes \Lambda_M^{-1} \to F_y \otimes G^\vee_D \otimes G_{D'} \to$$

$$\to G^\vee_D \otimes G_{D'} \to G^\vee_D \otimes G_{D'} \bigg|_{M_i(d-2)} \to 0.$$  

By induction, $R\Gamma_{M_i(d)}(G^\vee_D \otimes G_{D'}) = R\Gamma_{M_i(d-2)}(G^\vee_D \otimes G_{D'}) = 0$. On the other hand, the term $G^\vee_D \otimes G_{D'} \otimes \Lambda_M^{-1}$ satisfies the inequalities (7.1) with $t = -1 \notin [0, \deg \tilde{D}]$, so by Theorem 7.1 it is $\Gamma$-acyclic. As usual, the result follows from the hypercohomology spectral sequence and semi-continuity. \qed

9. Proof of the semi-orthogonal decomposition

Throughout this section we fix $d = \deg \Lambda = 2g - 1$, so that $v = (d-1)/2 = g - 1$. We are interested in the moduli spaces $M_i = M_i(\Lambda)$, where $i$ will always be assumed to satisfy $1 \leq i \leq g - 1$. Note that when $d = 2g - 1$, the canonical bundle is $\omega_{M_i} = \mathcal{O}_i(-3, 3-3g) = \Lambda_M^{-1} \otimes \zeta^{-1} \otimes \theta^{-1}$ (see [Tha94, 6.1] and Definition 3.9).

Definition 9.1. For $1 \leq \alpha \leq i$, let $\Phi^\alpha_i : D^b(\text{Sym}^\alpha C) \to D^b(M_i(\Lambda))$ be the Fourier-Mukai functor determined by $F^{2\alpha} \in D^b(\text{Sym}^\alpha C \times M_i(\Lambda))$, where $F$ is the universal bundle on $C \times M_i(\Lambda)$ (see Definition 2.3 for $F^{2\alpha}$).

We have already proved in Theorem 5.1 that $\Phi^1_i = \Phi_F$ is fully faithful. Now we prove a generalization of that result.

Theorem 9.2. For $1 \leq i \leq g - 1$, $1 \leq \alpha \leq i$, $\Phi^\alpha_i$ is a fully faithful functor.

Proof. By Bondal-Orlov’s criterion [BO95], we only need to consider the images of skyscraper sheaves, $\Phi^\alpha_i(\mathcal{O}_{\{D\}}) = G_D$. Namely, we need to show that for two divisors $D, D' \in \text{Sym}^\alpha C$ we have

$$(9.1) \quad R^k\Gamma_{M_i(\Lambda)}(G^\vee_D \otimes G_{D'}) = \begin{cases} 0 & \text{if } D \neq D' \text{ or } k < 0 \text{ or } k > \alpha \\ \mathbb{C} & \text{if } k = 0 \text{ and } D = D'. \end{cases}$$

If $D = D'$ then $(9.1)$ follows directly from Proposition 8.1 and Corollary 8.3. Now let $D \neq D'$ be different divisors of degree $\alpha \leq i$. Notice $i \leq (d-1)/2 = g - 1$, so the inequality $\alpha \leq d + g - 2i - 2$ holds. Therefore,
in this case (9.1) follows from Theorem 8.7. We conclude that $\Phi^i_\alpha$ is a fully faithful functor.

By abuse of notation, we will denote the essential image $\Phi^i_\alpha(\text{Sym}^\alpha C)$ simply by $\Phi^i_\alpha$, which by Theorem 9.2 is an admissible subcategory of $D^b(M_i)$ equivalent to $D^b(\text{Sym}^\alpha C)$. Similarly, we will denote by $\Phi^0_\alpha$ the full triangulated subcategory generated by $\mathcal{O}_{M_i}$, which is an admissible subcategory equivalent to $D^b(\text{Sym}^\alpha C)$, since $M_i$ is a rational variety, and can be described as the image of the (derived) pullback functor from a point, $\Phi^0_\alpha = q^*$, $q : M_i \to \text{pt} = \text{Sym}^0 C$.

**Definition 9.3.** For even and odd values of $\alpha$, we define the following full triangulated subcategories of $D^b(M_i)$:

- $A_{2k} := \Phi^i_{2k} \otimes \Lambda^{-k}_M \otimes \theta^{-1}$, \quad $0 \leq 2k \leq i$
- $B_{2k} := \Phi^i_{2k} \otimes \Lambda^{-k}_M$, \quad $0 \leq 2k \leq i$
- $C_{2k+1} := \Phi^i_{2k+1} \otimes \Lambda^{-k}_M \otimes \zeta \otimes \theta^{-1}$, \quad $0 \leq 2k + 1 \leq i$
- $D_{2k+1} := \Phi^i_{2k+1} \otimes \Lambda^{-k}_M \otimes \zeta$, \quad $0 \leq 2k + 1 \leq i$.

Each of these subcategories is equivalent to some $D^b(\text{Sym}^\alpha C)$. These four families of subcategories constitute the building blocks of our semi-orthogonal decomposition on $D^b(M_i)$. We will see that different subcategories of the form $A_{2k}$ are orthogonal to each other, and the same is true for subcategories within the other three blocks. We need the following lemma.

**Lemma 9.4.** Let $D_1, D_2$ be admissible subcategories of a triangulated category $D$ and $\Omega_1, \Omega_2$ spanning classes [Huy06, 3.2] of $D_1, D_2$.

If $\text{Hom}_D(A, B[k]) = 0$ for every $A \in \Omega_1$, $B \in \Omega_2$ and $k \in \mathbb{Z}$, then $\text{Hom}_D(F, G) = 0$ for every $F \in D_1, G \in D_2$.

**Proof.** We need to show that $D_1 \subset \perp D_2$ or, equivalently, $D_2 \subset D_1^\perp$.

First we see that $\Omega_1 \subset \perp D_2$. Let $A \in \Omega_1$. Since $D = \langle D_2, \perp D_2 \rangle$, we can fit $A$ in a exact triangle $D \to A \to D' \to D[1]$ where $D \in \perp D_2$ and $D' \in D_2$. Applying $\text{Hom}(\cdot, B)$ for $B \in \Omega_2$ we get a long exact sequence where $\text{Hom}(D, B[k]) = 0$ by definition and $\text{Hom}(A, B[k]) = 0$ by hypothesis. Therefore $\text{Hom}(D', B[k]) = 0$ for every $i$ and every $B \in \Omega_2$, so $D' \cong 0$ since $\Omega_2$ is a spanning class of $D_2$. As a consequence, $A \cong D \in \perp D_2$.

Now let $G \in D_2$. Similarly, there is an exact triangle $D \to G \to D' \to D[1]$ with $D \in D_1, D' \in D_1^\perp$. Applying $\text{Hom}(A, \cdot)$ with $A \in \Omega_1$ we now see that $\text{Hom}(A, D[k]) = \text{Hom}(A, G[k]) = 0$ by the previous discussion and therefore $D' \cong 0$. This implies $G \cong D \in D_1^\perp$, as desired.

**Proposition 9.5.** Let $k \neq l$ and $0 \leq 2k, 2l \leq i$. Then

$\text{Hom}_{D^b(M_i)}(A_{2k}, A_{2l}) = 0, \quad \text{Hom}_{D^b(M_i)}(B_{2k}, B_{2l}) = 0$. 
Similarly, if \( k \neq l \) and \( 0 \leq 2k + 1, 2l + 1 \leq i \), we have
\[
\text{Hom}_{D^b(M_i)}(C_{2k+1}, C_{2l+1}) = 0, \quad \text{Hom}_{D^b(M_i)}(D_{2k+1}, D_{2l+1}) = 0.
\]

**Proof.** Let us first show orthogonality between subcategories of the form \( \mathcal{A}_2k, \mathcal{A}_l, k \neq l \), as well as orthogonality between those of the form \( \mathcal{B}_{2k}, \mathcal{B}_l, k \neq l \). Since skyscraper sheaves \( \mathcal{O}_{(D)} \) of closed points \( D \in \text{Sym}^\alpha C \) are a spanning class of \( D^b(\text{Sym}^\alpha C) \) (see [Huy06, Prop. 3.17]), Lemma 9.4 says that (semi-)orthogonality can be checked on closed points. That is, it suffices to show that for \( D \in \text{Sym}^{2k} C, D' \in \text{Sym}^{2l} C, \) with \( 0 \leq 2k, 2l \leq i \leq g - 1, k \neq l \), we have \( R\Gamma_{M_i}(G_D^\vee \otimes G_{D'} \otimes \Lambda_{M_i}^{k-l}) = 0 \). But this follows from Theorem 7.1 Indeed, the inequalities
\[
2k - i - 1 < k - l < d + g - 2i - 1 - 2l
\]
are equivalent to \( k + l < i + 1 \) and \( k + l + 2i < d - 1 + g \), which are guaranteed by the fact that \( k + l < i \leq (d - 1)/2 < g \) in this case. Also, if \( k < l \) we have \( k - l \notin [0, 2k] \), while if \( k > l \) we have \( 2k \notin [k - l, k + l] \). Notice that all divisors involved have degree \( \leq g - 1 < d + g - 2i - 1 \). This proves the first two orthogonality statements.

Similarly, in order to prove orthogonality between subcategories \( C_{2k+1}, C_{2l+1}, k \neq l \), as well as between \( D_{2k+1}, D_{2l+1}, k \neq l \), we need to prove that for \( D \in \text{Sym}^{2k+1} C, D' \in \text{Sym}^{2l+1} C, \) with \( 0 \leq 2k, 2l + 1 \leq i \leq g - 1, k \neq l \), we must have
\[
R\Gamma_{M_i}(G_D^\vee \otimes G_{D'} \otimes \Lambda_{M_i}^{k-l}) = 0.
\]
Again, this can be proved using Theorem 7.1 the inequalities
\[
2k + 1 - i - 1 < k - l < d + g - 1 - 2i - (2l + 1)
\]
are equivalent to \( k + l < i \) and \( k + l + 2i < d + g - 2 \), both of which follow from the fact that \( k + l + 1 < i \leq (d - 1)/2 < g \) in this case. Similarly, if \( k < l \) then \( k - l \notin [0, 2k + 1] \), while if \( k > l \) \( 2k + 1 \notin [k - l, k + l + 1] \). This proves the required vanishing. \( \square \)

**Theorem 9.6.** Let \( d = 2g - 1 \) and \( 1 \leq i \leq g - 1 \). On \( D^b(M_i) \), we have a semi-orthogonal list of admissible subcategories arranged in four blocks
\[
(9.2) \quad \mathcal{A}, \mathcal{C}, \mathcal{B}, \mathcal{D}
\]
where
\[
\begin{align*}
\mathcal{A} &= \mathcal{A}_0, \mathcal{A}_2, \mathcal{A}_4, \ldots & \text{for } 0 \leq 2k \leq i \\
\mathcal{C} &= \mathcal{C}_1, \mathcal{C}_3, \mathcal{C}_5, \ldots & \text{for } 1 \leq 2k + 1 \leq i \\
\mathcal{B} &= \mathcal{B}_0, \mathcal{B}_2, \mathcal{B}_4, \ldots & \text{for } 0 \leq 2k \leq \min(i, g - 2) \\
\mathcal{D} &= \mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_5, \ldots & \text{for } 1 \leq 2k + 1 \leq \min(i, g - 2),
\end{align*}
\]
as given in Definition 9.3. Within each of the four blocks in (9.2), the corresponding admissible subcategories can be rearranged in any order.
Proof. All of these are admissible subcategories of $D^b(M_I)$ by Theorem \ref{thm:adm} and we have already shown in Prop. \ref{prop:ortho} that, within each of the four blocks in (9.2), the corresponding subcategories are orthogonal to each other from both sides. It remains to prove semi-orthogonality between different blocks.

Step 1. Between $\mathcal{A}$ and $\mathcal{C}$: we show that $\text{Hom}_{D^b(M_I)}(\mathcal{C}_{2k+1}, \mathcal{A}_{2l}) = 0$. By Lemma \ref{lem:ortho}, this amounts to showing that

$$R\Gamma_M(G^\vee_D \otimes G'_{D'} \otimes \Lambda^{k-i-l} \otimes \zeta^{-1}) = 0$$

for $D \in \text{Sym}^{2k+1}C$, $D' \in \text{Sym}^{2l}C$, with $0 \leq 2k+1, 2l \leq (d-1)/2 = g-1$. We can apply Theorem \ref{thm:ortho} since the inequalities

$$2k+1 - g < k - l < d - 2l - i - 1$$

are equivalent to $k + l < g - 1$ and $k + l + i < d - 1$, which hold in this case as $k + l < (d-1)/2 = g-1$. This gives the corresponding semi-orthogonality.

Step 2. Between $\mathcal{A}$ and $\mathcal{B}$: let us show $\text{Hom}_{D^b(M_I)}(\mathcal{B}_{2k}, \mathcal{A}_{2l}) = 0$. Again by Lemma \ref{lem:ortho}, we need to show $R\Gamma_M(G^\vee_D \otimes G'_{D'} \otimes \Lambda^{k-l} \otimes \theta^{-1}) = 0$ when $D \in \text{Sym}^{2k}C$, $D' \in \text{Sym}^{2l}C$, $0 \leq 2k, 2l \leq (d-1)/2 = g-1$ and $2k \leq g - 2$. By Serre duality, given that $\omega_M = \Lambda^{-1} \otimes \zeta^{-1} \otimes \theta^{-1}$, this is equivalent to showing that $G^\vee_{D'} \otimes G_D \otimes \Lambda^{l-k-1} \otimes \zeta^{-1}$ is $\Gamma$-acyclic on $M_I$ under the conditions above. This is given by Theorem \ref{thm:acyclic} because

$$2l - g < l - k - 1 < d - 2k - i - 1$$

is equivalent to $l + k < g - 1$ and $l + k + i < d$, and these inequalities hold since $l + k + i \leq 2i \leq d - 1$ and $2l + 2k \leq g - 1 + g - 2$ in this case.

Step 3. Between $\mathcal{A}$ and $\mathcal{D}$. For $\text{Hom}_{D^b(M_I)}(\mathcal{D}_{2k+1}, \mathcal{A}_{2l})$, we need to show that $R\Gamma_M(G^\vee_D \otimes G'_{D'} \otimes \Lambda^{k-l} \otimes \theta^{-1}) = 0$ whenever $D \in \text{Sym}^{2k+1}C$, $D' \in \text{Sym}^{2l}C$, $0 \leq 2l, 2k+1 \leq i \leq (d-1)/2 = g-1$. Again by Serre duality, this is equivalent to $\Gamma$-acyclicity of $G^\vee_{D'} \otimes G_D \otimes \Lambda^{l-k-1}$. We check that this is given by Theorem \ref{thm:acyclic} if

$$2l - i - 1 < l - k - 1 < d + g - 2i - 1 - (2k + 1)$$

are equivalent to $k + l < i$ and $l + k + 2i < d + g - 1$. The former follows from $2l, 2k+1 \leq i$ and the latter follows from $l + k < i < g$ and $2i \leq d - 1$. We also check that, if $k \geq l$, then $l - k - 1 \notin [0, 2l]$, while if $k < l$, we have $2l \notin [l - k - 1, l + k]$. This proves the required $\Gamma$-acyclicity.

Step 4. Next we show semi-orthogonality between $\mathcal{C}$ and $\mathcal{B}$. This amounts to $\Gamma$-acyclicity of $G^\vee_D \otimes G'_{D'} \otimes \Lambda^{k-l} \otimes \zeta \otimes \theta^{-1} = G^\vee_D \otimes G'_{D'} \otimes \Lambda^{k-l-1} \otimes \zeta^{-1}$ (cf. Definition \ref{def:acyclic}) for $D \in \text{Sym}^{2k}C$, $D' \in \text{Sym}^{2l+1}C$, where $0 \leq 2k, 2l + 1 \leq i \leq (d-1)/2 = g-1$. We check that Theorem \ref{thm:acyclic} can be applied in this case:

$$2k - g < k - l - 1 < d - (2l + 1) - i - 1$$
is equivalent to \( k + l < g - 1 \) and \( k + l + i < d - 1 \), both of which hold in our case. This proves \( \text{Hom}_{D^b(M_i)}(B_{2k}, C_{2l+1}) = 0 \).

**Step 5.** To show that \( \text{Hom}_{D^b(M_i)}(D_{2k+1}, C_{2l+1}) = 0 \), we need to check that \( G_D^x \otimes G_{D'} \otimes \Lambda_M^{k-l} \otimes \zeta^{-1} \) is \( \Gamma \)-acyclic on \( M_i \), where \( D \in \text{Sym}^{2k+1} C \), \( D' \in \text{Sym}^{2l+1} C \), \( 1 \leq 2k+1, 2l+1 \leq i \leq (d-1)/2 = g-1 \) and \( 2k+1 \leq g-2 \). By Serre duality, this is equivalent to \( \Gamma \)-acyclicity of \( G_D^x \otimes G_{D'} \otimes \Lambda_M^{l-k} \otimes \zeta^{-1} \) and this follows from Theorem 9.4 since

\[
2l + 1 - g < l - k - 1 < d - (2k + 1) - i - 1
\]
is equivalent to \( l + k + 1 < g - 1 \) and \( l + k + i < d - 1 \), both of which hold given the conditions above.

**Step 6.** Finally, we show semi-orthogonality between blocks from \( B \) and \( D \). We need to show that if \( D \in \text{Sym}^{2k+1} C \), \( D' \in \text{Sym}^{2l} C \), \( 0 \leq 2k+1, 2l \leq i \leq (d-1)/2 = g-1 \), we have \( R\Gamma_{M_i}(G_D^x \otimes G_{D'} \otimes \Lambda_M^{k-l} \otimes \zeta^{-1}) = 0 \). We can use Theorem 9.4 since

\[
2k + 1 - g < k - l < d - 2l - i - 1
\]
is equivalent to the inequalities \( k + l < g - 1 \) and \( k + l + i < d - 1 \), again both of which hold in our situation. We conclude \( \text{Hom}_{D^b(M_i)}(D_{2k+1}, B_{2l}) = 0 \).

This completes the proof of the theorem. \( \square \)

**Remark 9.7.** On \( D^b(M_{g-1}) \), this defines a semi-orthogonal list of admissible subcategories \( A_0, A_1, \ldots, C_1, C_3, \ldots, B_0, B_2, \ldots, D_1, D_3, \ldots \) where we have two copies of \( D^b(\text{Sym}^\alpha C) \) for \( 0 \leq \alpha \leq g - 2 \) and one copy of \( D^b(\text{Sym}^{g-1} C) \). We have chosen \( D^b(\text{Sym}^{g-1} C) \) to appear in the block \( A \) when \( g - 1 \) is even and in \( C \) when \( g - 1 \) is odd, but in fact any other choice of even and odd blocks would be valid too. Indeed, a similar computation in the proof of Theorem 9.6 still gives the required semi-orthogonals.

Now let \( i = g - 1 \), and call \( \xi : M_{g-1} \to N \) the last map in (3.1), where \( N = M_{C(2, \Lambda)} \) is the space of stable rank-two vector bundles of odd degree. The Picard group of \( N \) is generated by an ample line bundle \( \theta_N \), such that \( \xi^* \theta_N = \theta \) [Tha94, 5.8, 5.9]. Let \( E \) be the universal bundle on \( C \times N \), normalized so that \( \text{det} \pi E = O_N \) and \( \text{det} E_x = \theta_N \) (cf. [Nar17]). Then we have the following corollary.

**Corollary 9.8.** Let \( E \) be the Poincaré bundle of the moduli space \( N = M_{C(2, \Lambda)} \) over a curve of genus \( \geq 3 \), normalized as above. For \( i = 0, \ldots, g - 1 \), let \( G_i \subset D^b(N) \) be the essential image of the Fourier–Mukai functor with kernel \( E^{2xi} \). Then

\[
\begin{align*}
\theta_N^* & \otimes G_0, \quad (\theta_N^*)^2 \otimes G_2, \quad (\theta_N^*)^3 \otimes G_4, \quad (\theta_N^*)^4 \otimes G_6, \quad \ldots \\
\theta_N^* & \otimes G_1, \quad (\theta_N^*)^2 \otimes G_3, \quad (\theta_N^*)^3 \otimes G_5, \quad (\theta_N^*)^4 \otimes G_7, \quad \ldots \\
G_0, & \quad \theta_N^* \otimes G_2, \quad (\theta_N^*)^2 \otimes G_4, \quad (\theta_N^*)^3 \otimes G_6, \quad \ldots \\
G_1, & \quad \theta_N^* \otimes G_3, \quad (\theta_N^*)^2 \otimes G_5, \quad (\theta_N^*)^3 \otimes G_7, \quad \ldots
\end{align*}
\]
is a semi-orthogonal sequence of admissible subcategories of $D^b(N)$. There are two blocks isomorphic to $D^b(\text{Sym}^i C)$ for each $i = 0, \ldots, g - 2$ and one block isomorphic to $D^b(\text{Sym}^{g-1} C)$. Within each of the four lines in (9.3), the corresponding admissible subcategories can be rearranged in any order.

Proof. Observe that $\xi^*$ is fully faithful. Indeed, since $\xi$ is a projective birational morphism with $N$ normal, we have $\xi_*(O_{M_{g-1}}) = O_N$ [Tha94, 5.12] and then by adjointness

$$\text{Hom}_{D^b(N)}(A, B) = \text{Hom}_{D^b(N)}(A, \xi_* B) = \text{Hom}_{D^b(N)}(A, B).$$

The pullback $\xi^*(E)$ is a vector bundle on $C \times M_{g-1}$ whose fiber over each point $x \times (E, \phi) \in C \times M_{g-1}$ is exactly the fiber $E_x$. Thus, it has to coincide with the universal bundle $F$ up to twist by a line bundle on $M_{g-1}$, so that $\xi^* E = F \otimes L$. Then $\xi^* \det E_x = \Lambda_M \otimes L^2$, which by the normalization chosen must be $\xi^* \theta_N = \theta$, so $L = \zeta$. Thus $\xi^*(E) = F \otimes \zeta$ and the result follows from Theorem 9.6. □

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