N-n-digit interrelations between the sets within the $R^2$ plane generated by quasi-rotation of $R^3$ space

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Abstract. The background of description of the method for geometric objects rotation around the curved axes in the form of curves of the second order is being considered. The above-described method got the name of “quasi-rotation”. The algorithm of geometric construction of the two-two-digit interrelation on the basis of the circular axis of symmetry is provided. The described interrelation is analogous to the mirror symmetry regarding a straight linear axis. The method is suitable for construction of some well-known algebraic curves. The analogies between rotation and quasi-rotation are determined and described. The formula of the same plane preimage-to-image interrelation, generated by the quasi-rotation around the curved axis of the second-order curve is provided. The presented formula allows to derive the equations that effectively describe images of the given sets. The method for geometric construction of the images of geometrical objects lying within the plane of the second-order curve at their quasi-rotation for a given angle regarding the curve is described. The obtained results describe geometrical basics for the quasi-rotation. They serve as the basis for shape-forming of algebraic surfaces of high orders, and can be utilized as the theoretical basis for the computer-aided automated geometric projection.

1. Introduction
The method for rotation of points around curved axes of the second order has been previously described [1, 2]. The above method is developed for generating cyclic surfaces models. A torus, Dupin cyclides as well as a number of new cyclic surfaces are constructed by means of rotation of the circle around the conics. The described models are presented in several projections and obtained using the methods of descriptive geometry. In the previous report [2] an assumption is made on the order of the derived surfaces, and the equations are derived to describe the circle, created by the rotation of the point around an ellipse. Another report [3] provides a mathematical description of various cases of positioning of the point at its rotation around the elliptical axis. The above works report and describe a rotation of the point, lying within the plane of the curved axis. Thus, this report considers at length a number of the problems, overlooked in the previous works [1, 2, 3].

2. Formulation of the problem
The method for geometric objects rotation around the curved axes in the form of curves of the second order is henceforth termed as “quasi-rotation”. Such term is primarily based on the following argument: there is a reasonable analogy between the rotation and the “quasi-rotation”. The first principal step is to determine the order of preimage-to-image interrelation, generated by quasi-rotation. The interrelation formula will allow to analytically predict the order of the geometric objects, generated via the quasi-rotation. It is required to determine a possibility of reversed, image-to-
preimage interrelation. Furthermore, it is required to consider a possibility for quasi-rotation of the point that does not belong to the axis’ plane.

3. Theory
Let us consider the rotation of the point around a straight axis in 3D space. For any given point \(A \in \mathbb{R}^3\) there is an apparatus of “rotational transformation” \(RT_i\), which includes following: the plane of rotation \(\alpha (A \in \alpha, \alpha \perp i)\), axis of rotation \(i\), the centre of rotation \(O \ (O = \alpha \cap i)\) and constant value \(r = |OA|\).

![Figure 1](image-url)

**Figure 1.** Schema of the apparatus of the rotation of the point \(A\) around axis \(i\).

The result of the rotation of the point \(A\) around the \(i\) axis is the circle \(k\). Any line \((i)\) in 3D space \((\mathbb{R}^3)\) determines the bundle of the planes \(\{\Delta\}\). Within any given plane \(\Delta\), \(RT_i\) transformation induces linear transformation of symmetry around \(i\) axis. As the result, the \(RT_i\) transformation splits into \(\infty\) of \(AS_i\) (i.e., “axis symmetry”) linear transformations within the bundle of the planes \(\{\Delta\}\). As shown on figure 1, point \(A^{180}\) is symmetric to point \(A\) via the symmetry axis \(i\). In the generalized case for any given plane \(\Delta^\varphi\) any point \(A^{\varphi+180°}\) is symmetric to point \(A^\varphi\) via the symmetry axis \(i\). Thus, \(RT_i \to \{AS_i\}^\infty; AS_{m} \to RT_i, m = 1, 2, \ldots, \infty\).

The transformation formula for \(AS_i\) transformation of any given line \(y = f(x)\) that belongs to the axis’ plane within Cartesian system is as follows:

\[
AS_i\{f(x)\} = \frac{(4k^2 + (k^2 - 1)^2)f(f(x)) - 2(k^2 + k)x - 2(k^2 + 1)b}{k^4 - 1} \tag{1}
\]

where \(k\) and \(b\) are the coefficients in the equation for the axis \(i\) \(y = kx + b\). The geometric interpretation of the formula (1) is shown in figure 2. To any given point \(A\) that belongs to the \(f(x)\) graph, there is an interrelated point \(A' = AS_i(A)\), which belongs to the \(AS_i\{f(x)\} = f_{AS_i}(x)\) graph, such that \(AA'\) line that is perpendicular to the axis \(i\) (i.e., \(AA' \perp i\)), intersects the latter at the point \(O \ (AA' \cap i = O)\) \(\cap |AO| = |A'O|\). The value of \(f(f(x))\) in formula (1) is the solution of the equations system for the variable \(y\):

\[
\begin{cases}
y = \left(\frac{k^2 - 1}{2k}\right)x(y) + x\left(k - \frac{k^2 - 1}{2k}\right) + b \\
y = f(x) \quad \rightarrow \quad x(y)
\end{cases} \tag{2}
\]
where \( f(x) \) is the preimage’s function \( f(f(x)) = y \), i.e., \( f(f(x)) \) is the value of \( y \), which corresponds to the value of the function \( f_{AS}(x) \). For example, position \( x' \) corresponds to the value of \( f(f(x')) = y'_A \), and \( f_{AS}(x') = y'_A \). Equation (2) describes the line \( HA \), which is symmetric to \( HA' \) via axis \( i \).

\[ y = f(f(x)) = \cdots = y'_A \]

**Figure 2.** Symmetric transformation of the \( f(x) \) line via axis \( i \).

From \( AS_{im} \to RT_i, m = 1, 2, \ldots, \infty \), it follows that \( AS_{im} \{ f(x) \} \to RT_i \{ f(x) \}, m = 1, 2, \ldots, \infty \). Thus, formula (1) is transitioning to the formula of \( RT_i \) transformation.

Let us consider a symmetry operation via the circular axis \( AS_i \) (c-circle). The quadratic transformations of the plane are described in the earlier reports [4, 5]. For example, the operation of inversion is also the symmetry transformation via the circle. Let us describe the interrelation \( AS_i \) based on the following statement: line \( i \) is the circle with a focus \( F_\infty \) in the ideal point. If the focus \( F_\infty \) is off the ideal point (i.e., \( F_\infty \to F \)), then the line \( i \) becomes a circle \( i_c \) (i.e., \( i \to i_c \)). In this case the symmetric transformation via axis \( i \) becomes a symmetric transformation via circular axis \( i_c \) (i.e., \( AS_i \to AS_{ic} \)). Thus, since to any given point \( A \), there are two interrelated \( A' \) and \( A'' \) points as shown on figure 3, derived by the symmetric interrelation via the circular axis \( i_c \), the interrelation \( AS_{ic} \) cannot be termed as transformation, but only as an interrelation [6]. Thus, the symmetry via the curves of the second order should be termed “quasi-symmetry”.

**Figure 3.** Point A symmetry via circle \( i \).

The \( FA \) line intersects circle \( i_c \) at the two points \( O' \) and \( O'' \) \((FA \cap i = O', O'')\). Then to any given point \( A \), there are two interrelated points \( A' \) and \( A'' \) such as \(|O'A'| = |O'A|, |O''A| = |O''A'|\).

\[ AS_{ic} (A) \to \begin{cases} AS_i (A) \\ AS_i' (A) \end{cases} = A', A'' \]  \hspace{1cm} (3)

From the interrelation rules \( A \to (A', A'') \) \( AS_{ic} \), it follows that interrelation equation for \( AS_{ic} \), where \( y \to AS_{ic}[f(x)] \) can be written as:

\[ (y + f(f(x)))^2 (x^2 + y^2) = 4r^2y^2 \]  \hspace{1cm} (4)
In order to find \( f(f(x)) \) in the equation (4), one must solve the equation of the fourth order. However, if the preimage for the interrelation \( AS_\text{ic} \) is a straight line that is parallel to axis \( x \), then \( f(f(x)) = \text{Const} \) for \( x \in \mathbb{R} \).

Identically to the case of the symmetry via the straight line axis \( AS_{\text{im}} \rightarrow RT_i \), \( m = 1, 2, ..., \infty \), the quasi-symmetry via the circular axis is the basis for the quasi-rotation, i.e., \( AS_{\text{icm}} \rightarrow QRT_{ic} \), \( m = 1, 2, ..., \infty \) (\( Q \)-quasi). The quasi-rotation of point \( A \) around the circular axis \( i_c \) results into two circles \( k' \) and \( k'' \).

For a random point \( A \in \mathbb{R}^3 \) there is an apparatus of \( QRT_{ic} \) quasi-rotation, which includes: rotation plane \( \alpha \), \( (A \in \alpha, F \in \alpha, \alpha \perp i_c) \), two centers of rotation \( O', O'' \), \( (O', O'' = \alpha \cap i_c) \) and constant values of \( r' = |O'A| \), \( r'' = |O''A| \). From the statement - \( AS_{\text{icm}} \rightarrow QRT_{ic} \), \( m = 1, 2, ..., \infty \), it follows that \( AS_{\text{icm}} \{f(x)\} \rightarrow QRT_{ic} \{f(x)\}, m = 1, 2, ..., \infty \). Thus, equation (4) becomes a \( QRT_{ic} \) interrelation formula.

**Figure 4.** Quasi-rotation apparatus schema for the plane \( \Delta \).

Any circle \( (i_c) \) in 3D space (\( \mathbb{R}^3 \)) determines the bundle of the incidental circular cones \( \{\Delta_c\} \). The bundle of the described cones \( \{\Delta_c\} \) is analogous to the planes \( \{\Delta\} \) bundle that are interrelated to the plane \( \Delta \) (figure 1). That is, any of the described cones \( \{\Delta_c^0\} \) is a conically wrapped plane \( \Delta^0 \).

For example, figure 4 displays a cone \( \Delta^0 \) and the points within its surface. On any \( \Delta_c^0 \) cone’s surface the interrelation \( QRT_{ic} \) induces symmetry interrelation via \( i_c \). As a result the \( QRT_{ic} \) interrelation splits within the cones bundle \( \{\Delta_c\} \) into \( \infty \) quasi-symmetric” interrelations \( AS_{ic} \). The problem of the quasi-rotation of the off-plane (i.e., axis’ plane) point is resolved via finding and determining of the single cone from within \( \{\Delta_c\} \) cones bundle, which contains the given point.

Analogous to the building of the rotation planes around the rotation axis, the quasi-rotation surfaces are also built via quasi-rotation. The principles of building of the surfaces, based on the nonlinear transformations, are described in detail elsewhere [7]. In the generalized case these are bilayered surfaces. The layers of the aforementioned surfaces tangentially intersect along the original generatrix, and can genuinely intersect. The resulting quasi-rotation surfaces are formed by the set of the circles, therefore they belong to the class of the cyclic surfaces. A surface, generated by the circle \( l \) quasi-rotation around circle \( i_c \) is shown on figure 5, and generated by the constructive geometric modelling [8].
4. Results

In this study we have derived quasi-symmetry interrelation equation (4). This formula allows to derive the equation for the function that describes the interrelated image, generated from the line by quasi-symmetric interrelation operation via circular axis. If the preimage for the $A_S$ interrelation operation is the straight line that is parallel to the axis $x$, then $f(f(x)) = \text{Const}$ for $x \in \mathbb{R}$.

Let us consider a specific example: preimage is $y = 2$ straight line. Then the interrelated image, as determined by equation (4) is an equation that describes a Nicomede conchoid:

$$ (y + 2)^2 (x^2 + y^2) = (2r)^2 y^2 $$

(5)

Nicomede conchoid is a plane algebraic curve of the fourth order, which is described in detail elsewhere [9,10,11].
Equation (4) is also utilized for deriving of the formula for $f(f(x))$ in the generalized form. For example, any given $x'$ corresponds to both $f(f(x'))$ and $f_{ik}(x')$ values. The value of $f(f(x'))$ corresponds to the points of its intersection with the image of the line, perpendicular to $OX$ ($x = x'$). This argumentation is analogous to the background for the equation (2).

Conchoid is built by the means of line $s$ transformation via focal point $F$ that does not belong to this line, such that the original conchoid’s preimage line is also its asymptote: an example of the conchoid, built from the preimage line $s$, is shown on figure 6.

Considering the derivation of equation (5) within the logical framework of this study we observe that the straight line symmetric interrelation via circle is the principally alternative method for conchoid generation. That is, any Nicomede conchoid has a straight line preimage, and corresponding $AS_c$ interrelation circular axis with a centre at the focal point $F$ and radius $r = 1/2|OP|$.  

5. Discussion

Our results indicate that $QRT_c$ interrelation generates a set of the spatial curves within $R^3$ space. A particular case for these curves is the Nicomede conchoid. The entirety of these curves set forms the quasi-rotation surface. This surface forms two layers that tangentially intersect along the original generatrix. Quasi-rotation surfaces are also formed by the quasi-rotation of any sets around a circular axis, including those not lying in the axis plane. The $AS_i$ transformation formula can be used to describe other types of symmetry, since the $AS_i$ transformation is an integral part of $AS_c$, as well as other symmetry interrelations. This is determined by formula (3). A quasi-symmetry interrelation formula for circular axis, from which it follows that the $AS_i$ transformation is of the fourth order. $AS_c$ interrelation gives two images from one preimage. Should one apply $AS_c$ interrelation to the derived images, then they will become preimages and will give two images each. The preimage will become an integral part of the secondary use of $AS_c$. Evidently, $AS_c$ interrelation is two-two-digited.

6. Conclusions

This study presents analogies between rotation and quasi-rotation, which indicate that rotation is a special case of quasi-rotation. The established analogies between $RT_i$ and $QRT_c$ allow the use of quasi-rotation to create geometric objects. Quasi-rotation, as a method of surfaces-generation, can be used in computer-aided geometric modelling [12]. Using quasi-rotation, it is possible to model cyclic surfaces of complex shapes. At the same time, it is not difficult to create models of quasi-rotation surfaces - it is necessary to specify the generatrix and curvilinear axis, and then apply the $QRT_c$ interrelation algorithm. Just as rotation solids are used in almost all spheres of human activity, the solids formed with quasi-rotation surfaces, as well as their compartments, can be used to develop forms of mechanical parts, architectural structures, vehicle bodies, and elements of container vessel and similar systems. Higher-order surfaces are used in construction and engineering [13], to create favourable conditions for catalytic processes in chemical technology [14] and in other spheres of human creative activity [15]. A thorough study of the properties of quasi-rotation solids will help to predict their application areas and create aforementioned solids with predefined properties. The results of the work are obtained on the basis of the principle of a rational combination of analytical formalism and geometric constructivism, described in [16, 17].

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