Speed-ups to isothermality: Enhanced quantum heat engines through control of the system-bath coupling

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Isothermal transformations are minimally dissipative but slow processes, as the system needs to remain close to thermal equilibrium along the protocol. Here, we show that smoothly modifying the system-bath interaction can significantly speed up such transformations. In particular, we construct protocols where the overall dissipation $W_{\text{diss}}$ decays with the total time $\tau_{\text{tot}}$ of the protocol as $W_{\text{diss}} \propto \tau_{\text{tot}}^{-2\alpha - 1}$, where each value $\alpha > 0$ can be obtained by a suitable modification of the interaction, whereas $\alpha = 0$ corresponds to a standard isothermal process with the system-bath interaction remaining constant. Considering heat engines based on such speed-ups, we show that the corresponding efficiency at maximum power interpolates between the Curzon-Ahlborn efficiency for $\alpha = 0$ and the Carnot efficiency for $\alpha \to \infty$. We confirm our analytical results with two numerical examples where $\alpha = 1/2$, namely the time-dependent Caldeira-Leggett and resonant-level models, with strong system-environment correlations taken fully into account.

I. INTRODUCTION

Isothermal transformations play a fundamental role in thermodynamics, being the building block of optimal processes such as the Carnot engine [1]. In principle, however, they are infinitesimally slow: this means in practice that the total time of the process needs to be much larger than the timescale of thermalization, $\tau_{\text{eq}}$, over which the system of interest equilibrates with its thermal environment. These processes can then be sped up by increasing the system-environment coupling, which naturally reduces $\tau_{\text{eq}}$. However, modifying the interaction also induces additional dissipation (see Refs. [2–5] for recent discussions), which prohibits the non-physical possibility of performing an isothermal process arbitrarily quickly. Given this non-trivial trade-off, the goal of this article is to develop quantum-thermodynamic protocols that smoothly modify the system-bath interaction in order to speed up an isothermal process while keeping the overall dissipation constant. This enables us to increase the power of finite-time heat engines without compromising their efficiency, a well-known challenge in thermodynamics [6–9].

The idea of speeding up different thermodynamic processes by external control has received a lot of attention in the last years. Particularly relevant are shortcuts to adiabaticity (STA), which speed up unitary (and hence closed-system) evolutions [10], making them suited to improve the adiabatic part of thermodynamic cycles [11–13]. For open quantum systems, speed-ups of the evolution to a particular target state [14–17], such as an equilibration or thermalization process [18–20], have also been developed. For classical systems, such equilibration speed-ups (the so-called Engineered Swift Equilibration [21–23]) have been experimentally tested [24, 25]. Furthermore, these ideas have been extended to full isothermal classical processes, so that the state remains in the desired Gibbs distribution along the whole process [26–28]. These ideas have also been recently applied to the optimisation of a finite-time Carnot cycle [29], Otto engines [30, 31], and refrigerators [32]. In general, such speed-ups are possible by adding a time-dependent term to the Hamiltonian which, in the presence of a thermal bath, leads to a new source of dissipation. Indeed, speed-ups of equilibration and thermalization generally come with an extra work cost [20, 23, 24, 32] (see also the discussion in [34] for thermodynamic computing).

Here, our aim is to design speed-ups to isothermal processes which do not come at the price of higher dissipation or work cost. As a consequence, our speed-ups to isothermality (STI) can be readily used to maximize the power of finite-time Carnot engines [30, 35–37] while keeping their efficiency constant. Due to the extra control of the system-bath interaction we find that the dissipation $W_{\text{diss}}$ for optimal STI can asymptotically decay as

$$W_{\text{diss}} \propto \frac{1}{\tau_{\text{tot}}^{2\alpha + 1}},$$

where $\tau_{\text{tot}}$ is the total time of the process, and different $\alpha > 0$ can be obtained by a suitable STI. In particular, we provide two explicit examples where $\alpha = 1/2$. The decay in Eq. (1) can substantially outperform the standard scaling for large $\tau_{\text{tot}}$: $W_{\text{diss}} \propto \tau_{\text{tot}}^{-1}$, commonly found in protocols where no control on the system-bath interaction is possible [6–8, 38–40]. Furthermore, we show how the scaling in Eq. (1) leads to a new family of efficiencies at maximum power that interpolate between the Curzon-Ahlborn efficiency (for $\alpha = 0$) and the Carnot efficiency (for $\alpha \to \infty$).

These results are obtained by working in the regime of slow driving (i.e. the timescale of the driving is slower than the time-dependent equilibration timescale), and by assuming that the (time-dependent) coupling $g$ remains
weak but non-negligible along the whole process. In particular, we assume that the timescale of thermalization $\tau_{eq}(g)$ scales as $\tau_{eq}(g) \propto 1/g^2$, where $g$ is the strength of the interaction. This scaling is expected in Markovian open quantum systems, where our findings are expected to be most relevant. However, we will also argue that some of our findings also apply beyond weak coupling, in particular for strongly correlated systems that thermalize.

The above heuristic approach is supported by explicit calculations for two specific yet illustrative models. Firstly, we study quantum Brownian motion, where a quantum harmonic oscillator with time-dependent frequency interacts with a time-dependent coupling to a large (but finite) set of quantum harmonic oscillators [42–48]. By exploiting the quadratic nature of the Hamiltonian we simulate the full unitary dynamics, which allows us to account for all sources of dissipation, including the one induced by changing the interaction strength. Secondly, we consider the resonant-level model [44–48], where a single fermionic level with a time-dependent energy couples to an infinite bath of fermionic modes via a time-dependent interaction. We employ a quantum Langevin approach to derive analytical results general.

Consider a driven Hamiltonian

$$H(t) = H^{(S)}(t) + g(t)V + H^{(B)}$$  \hspace{1cm} (2)

where $H^{(S)}(t)$ is the Hamiltonian of the system S, on which one has experimental control, while $H^{(B)}$ is the Hamiltonian of the bath B and V is the interaction between the two, whose strength is governed by the parameter $g$. The whole information of system and bath together (SB) is contained in the density matrix $\rho$.

Consider a transformation between an initial Hamiltonian $H(0) = H^i$ and final one $H(\tau_{tot}) = H^f$. Without loss of generality we can normalise the parameter $t$ to the unit interval by introducing the compact notation $X_t \equiv X(t\tau_{tot})$ with $t \in [0, 1]$, $\tau_{tot}$ the duration of the process under consideration and $X = H, H^{(S)}, \rho$ etc. The average work associated to this transformation is given by the expression

$$W = \int_0^1 dt \text{Tr} \left( \rho_t \dot{H}_t \right),$$  \hspace{1cm} (3)

where $\rho_t$ describes the instantaneous state of SB.

Suppose first that the integrand is well described by the equilibrium value at all times, i.e. $\text{Tr} \left( \rho_t \dot{H}_t \right) = \text{Tr} \left( \rho^0_t \dot{H}_t \right)$ with

$$\rho^0_t = e^{-\beta H_t},$$  \hspace{1cm} (4)

and $Z = \text{Tr}(e^{-\beta H_t})$. It follows that

$$W = \int_0^1 dt \text{Tr} \left( \rho^0_t \dot{H}_t \right) = \frac{1}{\beta} \ln \frac{Z_i}{Z_f} =: \Delta F,$$  \hspace{1cm} (5)

where $Z$ is the partition function, $Z_{i/f} = \text{Tr}(e^{-\beta H^i/f})$, and $F = -\frac{1}{\beta} \log(Z)$ is the free energy of SB. Eq. (5) is fulfilled in the limit $\tau_{tot} \to \infty$ and when the driven observables $H$ thermalize (as expected for local observables). Note that the quantities in Eq. (5) depend on the Hamiltonian $H$ of the system and bath together, in general.

In the slow driving limit, i.e. for large but finite $\tau_{tot}$, Eq. (5) no longer holds as some work is dissipated into the bath. In order to quantify the dissipated work, one introduces $W_{\text{diss}} = W - \Delta F \geq 0$, which tends to zero as $\tau_{tot} \to \infty$. Expanding $W_{\text{diss}}$ in powers of $1/\tau_{tot}$, one obtains

$$W_{\text{diss}} = \frac{1}{\tau_{tot}} \int_0^1 dt G_{\rho^0_t} \left( \dot{H}_t, \dot{H}_t \right) + O \left( \frac{1}{\tau_{tot}} \right)$$  \hspace{1cm} (6)

where $G_{\rho^0_t}$ is a bilinear form evaluated at equilibrium $\rho^0_t$. The form $G_{\rho^0_t}$ was previously studied in different contexts. It was obtained through linear-response theory [8, 49, 51], by master equation approaches [10, 52–54], or directly from the partition function [55, 56]. For clarity of the exposition, here we focus on the latter, but our (heuristic) arguments can be extended to more general $G_{\rho^0_t}$ (see Appendix A). Furthermore, for this work it is enough to consider time-dependent Hamiltonians satisfying $\dot{H}_t = \lambda_t \dot{H}_0$, where $\dot{H}_0$ is some (time-independent) observable and $\lambda_t$ is the control parameter. In this case, we can write [53, 55, 56]

$$W_{\text{diss}} = \frac{\tau_{eq}}{\tau_{tot}} \int_0^1 dt \lambda_t^2 \text{cov}_{\rho^0_t} \left( \dot{H}_t, \ddot{H}_t \right) + O \left( \frac{\tau_{eq}^2}{\tau_{tot}^2} \right),$$  \hspace{1cm} (7)

where $\tau_{eq}$ is the time-scale of relaxation (associated to $\text{Tr}(\rho_t \dot{H})$), and

$$\text{cov}_{\rho^0_t} \left( \dot{H}_t, \ddot{H}_t \right) = \frac{1}{\beta^2} \frac{\partial^2 \ln Z}{\partial \lambda^2},$$  \hspace{1cm} (8)
which can be expressed in terms of the generalised covariance defined as
\[
\text{cov} \rho_i^{th} (A, B) = \text{Tr} \left( A \int_0^1 ds \left( \rho_i^{th} \right)^{1-s} (B - \text{Tr}(\rho_i^{th} B) I) \right) \left( \rho_i^{th} \right)^s
\]
Eq. (8) (and hence Eq. (7)) gives the standard notion of a thermodynamic metric commonly used to describe dissipative systems near equilibrium [55, 56], and it provides us with a simple analytical form which depends on a single timescale \( \tau_{eq} \).

In this work we are interested in modifications of the system-bath interaction strength, assuming initially weak coupling. In this regime, we can expand around \( g = 0 \), corresponding to replacing the thermal state of the interacting system \( \rho_i^{th} \) by the non interacting one \( \rho_0^{th} \). In particular, for \( \text{cov} \rho_i^{th} (A, A) \) we have
\[
\text{cov} \rho_i^{th} (A, A) = c_A^{(0)} + c_A^{(1)} g + c_A^{(2)} g^2 + ..., \tag{10}
\]
where we note that a similar expansion can be performed for the more general \( G_i^{th} \) in Eq. (6). Due to weak coupling between \( S \) and \( B \), we also assume that the thermalization time \( \tau_{eq}(g) \) is related to the strength of the interaction \( g \) introduced in Eq. (2) via
\[
\tau_{eq}(g) \propto \frac{1}{g^2}, \tag{11}
\]
which is expected in common dissipative evolutions [57].

Given Eqs. (7) and (11), it is clear that the dissipated work \( W_{diss} \) may be reduced by increasing \( g \) and hence decreasing the thermalization timescale \( \tau_{eq}(g) \). However, any modification of the Hamiltonian will require additional work to be performed, leading to a non-trivial trade-off between speed and dissipated work. In what follows we develop strategies to optimally modulate \( g \) in order to speed up the process while keeping the overall dissipation constant.

### III. SPEED-UPS TO ISOThERMALITY

Consider thermodynamic protocols governed by the time-dependent Hamiltonian,
\[
H_t = H_1^{(S)} + g_t V + H^{(B)}.
\tag{12}
\]
Note that the time dependence in (12) affects both the Hamiltonian of the system \( H_1^{(S)} \) and the interaction strength \( g_t \). We focus on protocols comprising the following three steps:

1. The interaction between system and bath is increased from \( g_0 \) to \( g_1 \) in a time \( \tau_{on} \), keeping the system Hamiltonian constant.

2. An isothermal transformation \( H_0^{(S)} \rightarrow H_1^{(S)} \) is performed in a time \( \tau_{iso} \), while the interaction strength is kept constant at \( g_1 \).

3. The interaction between the system and the bath is reduced to the initial value \( g_0 \) in a time \( \tau_{off} \), again holding \( H^{(S)} \) constant.

In Fig. 1 we give a schematic representation of the thermodynamic protocol for different coupling strengths. For simplicity we assume \( \tau_{off} = \tau_{on} \). Since both \( \tau_{on} \) and \( \tau_{iso} \) are finite, work is dissipated during each step of the protocol. We call \( W_{diss}^{\text{on}} \), \( W_{diss}^{\text{iso}} \) and \( W_{diss}^{\text{off}} \) the dissipated work in steps 1, 2 and 3, respectively. The total dissipation reads \( W_{diss} = W_{diss}^{\text{on}} + W_{diss}^{\text{iso}} + W_{diss}^{\text{off}} \) and the total duration of the protocol is given by
\[
\tau_{rot} = \tau_{on} + \tau_{iso} + \tau_{off} = 2\tau_{on} + \tau_{iso}.
\tag{13}
\]
Our goal is to optimize \( \tau_{on} \), \( \tau_{iso} \) and the interaction strength \( g_t \) such that \( \tau_{rot} \) is reduced and the dissipated work \( W_{diss} \) stays approximately constant.

### A. Steps 1 and 3: Taming the dissipation when the interaction is increased or decreased

We consider a family of protocols where the system-bath interaction strength changes polynomially in time according to
\[
g_t = g_0 + (g_1 - g_0)t^\alpha, \tag{14}
\]
for \( \alpha > 1 \) and \( t \in (0, 1) \). Throughout this section, we take \( g_1 = kg_0 \), with \( g_0 > 0 \). The assumption of a small, non-zero initial interaction strength is technically necessary to ensure that we remain in the slow-driving regime, i.e. \( g_t \tau_{eq} (g_t) \ll 1 \) even for small \( t \).

In order to quantify the dissipation during the transformation we make use of an expansion analogous to Eq. (7),
\[
W_{diss}^{\text{on}} = \frac{\beta}{\tau_{on}} \int_0^1 dt \tau_{eq}(g_t) \dot{g}_t^2 \text{cov} \rho_i^{th} (V, V), \tag{15}
\]
where we have introduced a time-dependent equilibration time-scale \( \tau_{eq}(g_t) \). Furthermore, through Eq. (11) we have

\[
\tau_{eq}(g_t) = \frac{\tau_{eq}(g_0)}{(1 + (k - 1)t\omega)^2}.
\]

(16)

In order to evaluate Eq. (15), we approximate the covariance as in (10). We assume that \( \tau_{eq}^{(1)} = 0 \), since \( \text{Tr}(V^0_{\rho(t)}) = 0 \) holds exactly in a broad class of relevant open quantum systems, such as the examples discussed in Sec. V. Let us now consider two cases separately: keeping only the lowest-order term \( (\alpha^{(0)}_g) \) or retaining also the second-order one \( (\alpha^{(2)}_g) \).

1. Zeroth order

In this case, by replacing \( \text{cov}_{\rho^h}(V,V) \) by \( \alpha^{(0)}_g \) in (15) we obtain

\[
W^{(1)}_{diss} = \frac{\beta g_0^2 \tau_{eq}(g_0) \alpha^{(0)}_g}{\tau_{on}} F^{(1)}(\alpha, k) \tag{17}
\]

with

\[
F^{(1)}(\alpha, k) = \int_0^1 dt \frac{\alpha^2(1 - k)^2 t^{2(\alpha - 1)}}{(1 + (k - 1)t\omega)^2},
\]

(18)

an integral that admits a solution in terms of the incomplete beta function. For large \( k \), we can approximate \( F(\alpha, k) \) as

\[
F^{(1)}(\alpha, k) \approx \pi(\alpha - 1) \sin^{-1} \left( \frac{\pi}{\alpha} \right) k^{\frac{3}{2}}
\]

(19)

with \( \alpha > 1 \), whereas \( F^{(1)}(1, k) = (k - 1)^2/2k \).

Now, let us introduce a time-scale \( \tau_{on}^{weak} \), corresponding to the time of turning on the interaction for \( k = 1 \). Then, note that by choosing,

\[
\tau_{on} = F^{(1)}(\alpha, k) \tau_{on}^{weak},
\]

(20)

the dissipation (17) becomes independent of \( k \). Namely, Eq. (20) indicates how to scale up \( \tau_{on} \) with \( k \) in such a way that the dissipation stays constant at leading order in \( g \) when the interaction is increased.

2. Second order

One can also consider a more conservative choice than the one in Eq. (20) by accounting for \( \alpha^{(2)}_g \) in Eq. (10). The dissipation \( W^{(2)}_{diss} \) induced by the second order term corresponds to

\[
W^{(2)}_{diss} = \frac{\beta g_0^4 \tau_{eq}(g_0) \alpha^{(2)}_g}{\tau_{on}} F^{(2)}(\alpha, k) \tag{21}
\]

with \( F^{(2)}(\alpha, k) = (k - 1)^2 \alpha^2/(2\alpha - 1) \). So, for large \( k \), we can assume for simplicity,

\[
F^{(2)}(\alpha, k) \approx \frac{\alpha^2}{2\alpha - 1} k^2.
\]

Thus, by taking

\[
\tau_{on} = F^{(2)}(\alpha, k) \tau_{on}^{weak},
\]

(23)

we ensure that \( W^{(2)}_{diss} \) is independent of \( k \). Note that this choice is more conservative since \( F^{(2)}(\alpha, k) \geq F^{(1)}(\alpha, k) \) for \( k \geq 1 \).

3. Higher orders

In principle, one can extend the previous considerations to find more conservative choices of \( \tau_{on} \) as a function of \( k \) by accounting for higher orders in Eq. (10). However, for stronger couplings a more useful approach is to use the fact that \( \text{cov}_{\rho^h}(V,V) \leq 2||V||^2 \), in order to bound the (exact) dissipation (15) as

\[
W_{diss} \leq \frac{\beta g_0^2 \tau_{eq}(g_0)}{\tau_{on}} 2||V||^2 F^{(1)}(\alpha, k). \tag{24}
\]

Hence, in models where \( g_0^2||V||^2 \) is finite (and possibly small), it appears plausible that the choice (20) is in fact already sufficient to keep the dissipation controlled (note that with (20) the upper bound becomes independent of \( k \)). Importantly, the bound (24) also works for strongly correlated and non-Markovian systems, suggesting that our considerations in principle also apply beyond weak coupling for strongly correlated systems that thermalize (41). In particular, the bound (24) is tight for finite-dimensional and locally interacting systems, such as fermionic or spin models, where \( ||V|| \) is of the order of the system-bath boundary and independent of the size of the bath. In such cases, the scaling of the equilibration time might differ from Eq. (11), but our framework can be easily adapted to account for that.

4. Discussion

In this section, we showed how to scale up \( \tau_{on} \) with \( k \) to ensure that \( W_{diss} \) does not increase as we increase the interaction. We followed two complementary approaches. First, taking a perturbative expansion of \( W_{diss} \) for weak coupling, we derived two possible choices: Eq. (20) and Eq. (23). The former ensures stays \( W_{diss} \) stays constant at leading order in the expansion (zeroth order), whereas the latter ensures that \( W_{diss} \) does not increase with \( k \) up to second order in \( g \). Secondly, we showed that one can also upper bound \( W_{diss} \) by a \( k \)-independent bound by combining (24) and (20), a bound which holds at arbitrary strong coupling (i.e. large \( g \)) as long as \( ||V|| \) is finite. In Sec. V we will test these choices for fermionic and bosonic baths (see Figs. 7), showing that these generic considerations work well in relevant physical models.
B. Step 2: Isothermal part of the process

Now we focus on the isothermal part of the protocol. The protocol consists of modifying the Hamiltonian of the system $H_{\text{is}}^{(k)}$ whilst keeping the coupling strength $g$ constant. Analogously to the previous section, we introduce $\tau_{\text{iso}}^{\text{weak}}$ as the time spent to perform the isothermal part of the protocol for $k = 1$, i.e., in absence of modulations of the interaction. By assuming the scaling in Eq. (11) and by using the expansion in Eq. (7), we can choose the time $\tau_{\text{iso}}$ of the isothermal process for $k > 1$ as

$$\tau_{\text{iso}} = \frac{\tau_{\text{iso}}^{\text{weak}}}{k^2}$$  \hfill (25)

in order to keep the dissipation constant for any value of $k$. Similar to the previous section, this is strictly valid at leading order in $g_0$, i.e. when keeping only the first term in $\langle \hat{H}_\alpha \rangle$. This appears justified in the dissipative systems we consider in this work (see Sec. V).

C. Full protocol

Collecting all the considerations above, we have devised choices of $\tau_{\text{on}}$ and $\tau_{\text{iso}}$ as a function of $k$ which guarantee an overall constant dissipation. The total time of the protocol reads,

$$\tau_{\text{tot}} = 2F^{(1)}(\alpha, k)\tau_{\text{on}}^{\text{weak}} + \frac{\tau_{\text{iso}}^{\text{weak}}}{k^2},$$ \hfill (26)

where $F^{(1)}(\alpha, k)$ is given by either Eq. (18) or Eq. (22), depending on the specific model (see Sec. V for examples). In Fig. 2 we show the behaviour of $\tau_{\text{tot}}$ for different values of the dimensionless ratio $\mathcal{T} = \tau_{\text{iso}}^{\text{weak}} / \tau_{\text{on}}^{\text{weak}}$, and for both choices (18) and (22). Note that for large $\mathcal{T}$, as in realistic situations (normally the isothermal process is much longer than the time spent switching the system-bath interaction on and off), we obtain substantial improvements on the time of protocol. Our proposal hence provides a way of substantially speeding up isothermal processes through control of the system-bath interaction, which crucially does not come at the price of increased dissipation or work cost.

D. Optimal protocols and decay of dissipation

Above, we designed a family of protocols in which the dissipation remains constant, while the total time of the process can be adjusted as a function of $k$ (see also Fig. 2). Let us now minimize the expression in Eq. (26) to find the fastest isothermal process for a given dissipation.

We first consider the zeroth-order expansion from Sec. III A 1. To obtain an analytical expression, we use the large-$k$ approximation in Eq. (19) to obtain

$$\tau_{\text{tot}} = 2D_\alpha k\frac{\tau_{\text{on}}^{\text{weak}}}{\tau_{\text{on}}^{\text{weak}} + \tau_{\text{iso}}^{\text{weak}}/k^2},$$ \hfill (27)

where $D_\alpha = \alpha(\alpha - 1) \sin^{-1}(\pi/\alpha)$. This expression can be minimized with respect to $k$, yielding

$$k = \left(\frac{\alpha \tau_{\text{iso}}^{\text{weak}}}{D_\alpha \tau_{\text{on}}^{\text{weak}}} \right)^{\frac{\alpha}{\pi + \pi}},$$ \hfill (28)

and the corresponding minimal time

$$\tau_{\text{tot}} = C_\alpha \tau_{\text{iso}}^{\text{weak}} \left(\frac{\tau_{\text{iso}}^{\text{weak}}}{\tau_{\text{on}}^{\text{weak}}} \right)^{\frac{2\alpha}{\pi + \pi}},$$ \hfill (29)

where $C_\alpha$ is the constant

$$C_\alpha = (2\alpha + 1) \left(\frac{D_\alpha}{\alpha} \right)^{\frac{2\alpha}{\pi + \pi}}.$$ \hfill (31)

For a standard isothermal process at $k = 1$, in which the interaction is not modified, at leading order in $1/\tau_{\text{iso}}^{\text{weak}}$ the dissipated work can be expressed as

$$W_{\text{diss}}^{\text{weak}} = \sum_{\tau_{\text{iso}}^{\text{weak}}}$$ \hfill (30)

where $\Sigma > 0$ can be obtained from the integral expression in Eq. (7). By construction, the family of protocols in Eq. (29) will dissipate the same $W_{\text{diss}}^{\text{weak}} = W_{\text{diss}}^{\text{weak}}$. If we combine this observation with Eq. (29) and (30), we obtain that

$$W_{\text{diss}} = \Sigma C_\alpha^{2\alpha+1} \left(\frac{\tau_{\text{iso}}^{\text{weak}}}{\tau_{\text{iso}}^{\text{weak}}} \right)^{2\alpha+1},$$ \hfill (31)
with \( \alpha > 0 \). For constant \( \tau_{\text{weak}} \), the dissipation decays as \( \tau_{\text{tot}}^{(2\alpha+1)} \) in the total time \( \tau_{\text{tot}} \) of the process, which can greatly outperform the standard decay in Eq. (30).

Naively, the decay in Eq. (31) may suggest that one can make the dissipation arbitrary small simply by increasing \( \alpha \). This is not the case, however, due to the contribution of the constant \( C_{a} \), which diverges exponentially as \( \alpha \) increases. As a consequence one can show that for any \( \tau \) there exists an optimal \( \alpha \), which scales logarithmically in \( \tau \). Hence, one needs exponentially long protocols in order to choose larger \( \alpha \).

Next, we discuss the case where \( \tau_{\text{on}} \) is scaled as in Eq. (22), in order to account for contributions to the dissipated work at second order in \( \gamma \). Using the large-\( k \) approximation in Eq. (22), the total time now reads as

\[
\tau_{\text{tot}} = 2B_{\alpha} \tau_{\text{on}}^{\text{weak}} k^2 + \tau_{\text{iso}}^{\text{weak}}/k^2, \tag{32}
\]

where \( B_{\alpha} = \alpha/(2\alpha - 1) \). Following the same steps as before, we find that the total time is minimized when we choose \( k^4 = \tau_{\text{iso}}^{\text{weak}}/(2B_{\alpha} \tau_{\text{on}}^{\text{weak}}) \), yielding \( \tau_{\text{tot}} = \sqrt{8B_{\alpha} \tau_{\text{iso}}^{\text{weak}}/\tau_{\text{on}}^{\text{weak}}} \). This leads to a decay of the efficiency given by

\[
W_{\text{diss}} = \frac{8\alpha^2 \Sigma_{\gamma}^{\text{weak}}}{(2\alpha - 1)\tau_{\text{tot}}^2}. \tag{33}
\]

Therefore, for a fixed \( \tau_{\text{on}}^{\text{weak}} \), the dissipation decays with the total time as \( \tau_{\text{tot}}^{-2} \), in contrast to the standard decay (30).

**IV. EFFICIENCY AT MAXIMUM POWER THROUGH SPEED-UPS TO ISOTHERMALITY**

In this section, we study the implications of optimal shortcuts to isothermality for thermodynamic cycles. In Sec. III, we presented different possible choices for speeding up an isothermal process. We discussed two possible scenarios arising from perturbation theory around weak coupling. Here, we will use a general form for the decay of the dissipated work which encompasses all regimes considered in Sec. III.

Let us assume that the dissipation decays as

\[
W_{\text{diss}} = \frac{\Sigma_{\gamma}}{\tau_{\text{tot}}^{\gamma}}, \tag{34}
\]

with \( \gamma \geq 1 \) and \( \tau_{\text{tot}} \) is the time of the process. For the optimal shortcuts to isothermality, we have that \( \gamma = 2\alpha + 1 \) with \( \alpha > 0 \) and \( \Sigma_{\gamma} = \Sigma C_{a}^{2\alpha+1}(\tau_{\text{on}}^{\text{weak}})^{2\alpha} \), whereas for the more conservative choice in Eq. (22) we have \( \gamma = 2 \) and \( \Sigma_{\gamma} \) given in Eq. (33).

We consider a finite-time Carnot-like cycle between two thermal baths at different temperatures \( T_{h} \) and \( T_{c} \). Furthermore, when the (finite-time) isothermal part of the cycle is carried out, we assume a decay as in Eq. (34). Using \( Q + W = \Delta E_{S} \), the heat exchanged between the system and each of the two thermal baths reads

\[
Q_{c} = T_{c} \left( -\Delta S - \frac{\Sigma_{\gamma}}{\tau_{c}} + ... \right)
\]

\[
Q_{h} = T_{h} \left( \Delta S - \frac{\Sigma_{\gamma}}{\tau_{h}} + ... \right) \tag{35}
\]

where \( \tau_{c,h} \) are the times of the isothermal processes (with the cold, hot bath, respectively), and we have assumed a symmetric cycle such that the constants \( \Sigma_{\gamma} \) are equal for each isothermal process [35][37]. The efficiency of the engine is given by,

\[
\eta = 1 + \frac{Q_{c}}{Q_{h}} \tag{36}
\]

whereas the power reads,

\[
P = \frac{Q_{c} + Q_{h}}{\tau_{h} + \tau_{c}}. \tag{37}
\]

In the case of \( \gamma = 1 \), i.e. \( W_{\text{diss}} \propto \tau_{\text{tot}}^{-1} \), the efficiency at maximum power \( \eta^{*} \) is given by the Curzon-Ahlborn efficiency [6][34]. We want to compute \( \eta^{*} \) for a generic value of \( \gamma \). The maximum power is obtained by imposing the two conditions:

\[
\frac{\partial P}{\partial \tau_{c}} = 0, \quad \frac{\partial P}{\partial \tau_{h}} = 0. \tag{38}
\]

The system has a unique real and positive solution for \( \tau_{c,h} \) given by:

\[
\tau_{c} = \frac{\theta - \tau_{c}}{\theta + 1} \left( \frac{\Sigma_{\gamma} (\gamma + 1) \theta (\theta - \tau_{c})^{\gamma + 1}}{\Delta S (1 - \theta)} \right)^{\frac{1}{\gamma}}
\]

\[
\tau_{h} = \theta + 1 \tau_{c}, \tag{39}
\]

where we used the notations \( \theta \equiv T_{c}/T_{h} \). The corresponding efficiency at maximum power reads

\[
\eta_{C}^{*} = 1 - \left( \frac{(\gamma + 1) \theta (\theta - \tau_{c})^{\gamma + 1}}{\Delta S (1 - \theta)} + \gamma + \theta \right) \tag{40}
\]

which depends only on the ratio of temperatures \( \theta \) and \( \gamma \). This formula has two interesting limits: for \( \gamma \rightarrow 1 \), one obtains the Curzon-Ahlborn efficiency \( \eta_{C}^{*} = 1 - \sqrt{\theta} \equiv \eta_{CA} \), while for \( \gamma \rightarrow \infty \) we regain the Carnot efficiency \( \eta_{C}^{*} = 1 - \theta \equiv \eta_{C} \). The efficiency at maximum power interpolates between these two regimes as \( \gamma \) varies, as illustrated in Fig. 3. If we expand Eq. (40) around \( \theta \rightarrow 1 \) (i.e. \( \eta_{C} \rightarrow 0 \)), we obtain

\[
\eta_{C}^{*} = \frac{\gamma}{\gamma + 1} \eta_{C} + \frac{\gamma}{2(\gamma + 1)^{2}} \eta_{C}^{2} + \mathcal{O}(\eta_{C}^{3}). \tag{41}
\]

The expansion in Eq. (41) neatly shows how \( \eta_{C}^{*} \) approaches \( \eta_{C} \) as \( \gamma \) increases. Notice that for the optimal STI that we defined in the previous section, the time of the process in Eq. (39) is proportional to \( C_{r} \), and hence tends to infinity as \( \eta_{C}^{*} \rightarrow \eta_{C} \), preventing the possibility of achieving a Carnot cycle with finite power.
FIG. 3. Efficiency at maximum power. The figure shows $\eta^*_\gamma$ for $\gamma = 1, 2, 4, 8, \infty$; with $\gamma = 1$ and $\gamma = \infty$ corresponding to Curzon-Ahlborn ($\eta_{CA}$) and Carnot efficiency ($\eta_C$), respectively.

V. NUMERICAL RESULTS

In the previous sections we have combined heuristic and rigorous arguments and shown that the time of an isothermal process in Eq. (26) can be considerably reduced by suitably modifying the coupling between system and bath. The goal of this section is to illustrate these considerations for exactly solvable models. Specifically, we consider two complementary examples: a bosonic environment described by the Caldeira-Leggett model and a fermionic bath described by the resonant-level model. The quadratic nature of their corresponding Hamiltonians allows us to efficiently simulate large or infinite bath sizes such that our statements about thermalization remain meaningful.

With the Caldeira-Leggett model, we study a problem with bosonic degrees of freedom using exact calculations but with a finite, discretized bath. In this context, we quantitatively demonstrate that the heuristic assumptions underlying our analytical results hold to an excellent approximation, even with relatively fast driving. Then we move to a resonant-level model with fermionic degrees of freedom, which is analyzed using an approximate analytical approach. This allows us to study the slow-driving regime in order to demonstrate a genuine speed-up for isothermal processes at strong coupling.

A. The Caldeira-Leggett model

We start by illustrating our results with the Caldeira-Leggett (CL) model (42), (43), (57), prototypical example of a quantum Brownian motion. The CL model describes a Brownian quantum particle of mass $m$ in a harmonic potential. The full Hamiltonian consists of four terms,

$$H = H^{(S)} + H^{(B)} + V + H^{(R)}.$$  \hfill (42)

where the Hamiltonian of the system $S$ reads

$$H^{(S)} = \frac{1}{2} \left( m \omega_S^2 x^2 + \frac{p^2}{m} \right),$$  \hfill (43)

where $x$ and $p$ are the position and momentum operators; the Hamiltonian of the bath $B$ is

$$H^{(B)} = \frac{1}{2} \sum_n \left( \frac{p_n^2}{m_n} + m_n \omega_n^2 x_n^2 \right),$$  \hfill (44)

where $\omega_n = \frac{n}{N} (\omega_{\text{max}} - \omega_{\text{min}}) + \omega_{\text{min}}$ are the frequencies of the modes in the bath, and we defined $\omega_{\text{max}} = 2 \omega_S$ and $\omega_{\text{min}} = \omega_S/N$. The interaction $V$ between the system and the bath is defined as

$$V = x \sum_n \gamma_n x_n,$$  \hfill (45)

where $\gamma_n$ are the coupling constant between system and bath. In the remainder we will assume all the masses $m, m_n = 1$, and that the couplings satisfy: $\gamma_n = g\omega_n \sqrt{2\omega_{\text{max}}/(\pi N \omega_S)}$, which tends to an Ohmic spectral density with hard cutoffs in the continuum limit ($N \to \infty$), as shown in Appendix G of [3]. The last term $H^{(R)}$ in Eq. (42) is a renormalization term which ensures the positivity of $H$,

$$H^{(R)} = x^2 \sum_n \frac{\gamma_n^2}{m_n \omega_n^2},$$  \hfill (46)

which may be absorbed within $H^{(S)}$.

The CL Hamiltonian in Eq. (42) is quadratic. This enables us to diagonalize it efficiently and to describe the time-evolved state by covariance matrices (of size $2N \times 2N$ for systems composed of $N$ particles), allowing us to reach large but finite baths. Thus, the dynamics induced by the CL Hamiltonian in Eq. (42) can be simulated without making any assumption on the coupling strength $g$ (see e.g. [58] for details).

1. Thermalization in the CL model.

We first study the dependence of the thermalization time on $g$ for observables on the system. In the simulation, we take as an initial state the thermal state of the non-interacting Hamiltonian $\rho^{\text{th}}_S(H(t = 0)) = \rho^{\text{th}}_S(H^S) \otimes \rho^{\text{th}}_B(H^B)$, and then perform a quench to a finite interaction strength $g = k g_0$, and consider the corresponding relaxation to the new equilibrium state (the number of particles in the bath is $N = 300$, and $g_0 = 0.1 \cdot \omega_S$).

In particular, in Fig. 4, we compute the relative entropy $S(\rho \parallel \sigma) = \text{Tr}(\rho (\log \rho - \log \sigma))$ between the
FIG. 4. Relaxation to the thermal state. We compute the relative entropy $S(p_{S}(t) \parallel p_{S}^{\text{th}}(t))$ between the reduced state $p_{S}(t) = \text{Tr}_{B}(p(t))$ of the total time-dependent density matrix and thermal state of the system $p_{S}^{\text{th}}(t)$. For a wide range of values of $k$ the time evolved state approaches the thermal equilibrium exponentially $S(p_{S}(t) \parallel p_{S}^{\text{th}}(t)) \sim e^{-t/\tau_{eq}(g)}$. Inset: We extrapolate the decay of the relaxation time with a power law $\tau_{eq}(g) \sim \tau_{eq}(g_{0})k^{-\nu}$. The optimal parameters correspond to $\nu = 2.06$ and $\tau_{eq}(g_{0}) = 58.5$.

FIG. 5. Thermalization of the potential. Moving average of the expectation value of the potential as a function of time for different coupling strengths. The value of $\langle \Delta V(t) \rangle$ decays exponentially as a function of time. The slope becomes increasingly steeper for stronger coupling $g_{\text{max}}(k)$, i.e. the thermalization is faster. Inset: We extrapolate the decay of the relaxation time with a power law $\log(\tau_{eq}(g)) = \log(\tau_{eq}(g_{0})) - \nu_{V}\log(k)$. The optimal parameters correspond to $\nu_{V} = 1.97$ and $\tau_{eq}(g_{0}) = 113$.

marginal of the time evolved state $p_{S}(t) = \text{Tr}_{B}(p(t))$ and the thermal state of the system $p_{S}^{\text{th}}(t)$. The relative entropy decays exponentially in time

$$S(p_{S}(t) \parallel p_{S}^{\text{th}}(t)) \sim e^{-t/\tau_{eq}(g)},$$

where $\tau_{eq}(g)$ is the relaxation time-scale for a given coupling strength $g$. As we expect, the slope becomes increasingly steeper for stronger couplings. In order to understand the behaviour of $\tau_{eq}(g)$ as a function of $g$, we assume a power law decay $\tau_{eq}(g) = c(0)g^{-\nu}$, where $\tau_{eq}(g_{0})$ corresponds to the relaxation time for $k = 1$ and $\nu$ quantifies the scaling with interaction strength. In Fig. 4 (inset) we fit the function $\log(\tau_{eq}(g)) = \log(\tau_{eq}(g_{0})) - \nu\log(k)$ with a straight line, which confirms the scaling predicted by Eq. (11) with $\nu \approx 2$ even for rather large coupling strengths up to $g/\omega_{S} \approx 0.5$.

Similarly, we need to verify that the interaction energy thermalizes and satisfies Eq. (11). This is shown in Fig. 5, where we plot $\Delta V = \langle V(t) \rangle - \langle V_{eq} \rangle$ for different values of the coupling $g_{\text{max}}(k)$, and where $\langle V(t) \rangle$ is the exact value of the interaction energy for the unitary-evolved state and $V_{eq}$ is its thermal equilibrium value (with respect to the global thermal state). By performing an extrapolation as the one of Fig. 5 we confirm the scaling in Eq. (10) up to the relatively large interaction strength of $g/\omega_{S} \approx 0.5$.

2. Generalised covariance.

In Fig. 6 we show the behavior of the covariance from Eq. (9) for the relevant quantities $H(S)$ and $V$ as a function of the interaction strength $g$. One observes that $\text{cov}(H(S), H(S))$ stays essentially constant, which means that only $c^{(0)}$ in Eq. (10) contributes, hence also justifying Eq. (24). On the other hand, $\text{cov}(V, V)$ does vary with $g$, suggesting that higher order terms in the expansion in the expansion (10) can play a role.

3. Shortcuts to isothermality.

In order to confirm the intuition given by the generalised covariance, as a last step we simulate the full thermodynamic protocol and compute the associated dissipation. First, in Fig. 7 (a) we show the total dissipation $W_{\text{diss}}$ for increasing and decreasing the interaction between system and bath according to Eq. (23) (in order to account for higher order corrections in $g$). One observes that $W_{\text{diss}}$ either decreases or stays constant with $k$, as expected from our analytic reasoning. Furthermore, in Fig. 7 (b) we show the total dissipated work for the full thermodynamic protocol depicted in Fig. 1 and with $\tau_{\text{eq}}(g_{0}) = 133$. As we increase the interaction strength, the dissipation remains constant or drops close to zero, and as shown in Fig. 2 the time substantially decreases. Hence, we have obtained the desired speed-ups. Regarding the timescales
of the process, notice that the times shown in Fig. 7 are comparable to the thermalization times in Fig. 4 and 5.

B. The resonant-level model

In this section, we benchmark our predictions using the analytically tractable resonant-level model. Specifically, the system of interest comprises a single distinguished fermionic mode coupled to an infinite collection of reservoir modes, also fermionic. The total system-bath Hamiltonian reads as $H = H^{(S)} + H^{(B)} + H^{(SB)}$, with

$$H^{(S)} = \varepsilon(t) a^\dagger a, \quad (48)$$
$$H^{(B)} = \sum_k \omega_k b^\dagger_k b_k, \quad (49)$$
$$H^{(SB)} = g(t) \sum_k \gamma_k \left( a^\dagger b_k + b^\dagger_k a \right). \quad (50)$$

Here, $a$ annihilates a fermion with time-dependent energy $\varepsilon(t)$, while $b_k$ annihilates a fermion in the bath with energy $\omega_k$. The relevant bath properties are characterised by the spectral density $\mathcal{J}(\omega) = 2\pi \sum_k \gamma_k^2 \delta(\omega - \omega_k)$, where we take

$$\mathcal{J}(\omega) = \Gamma \Theta(\Lambda - |\omega|), \quad (51)$$

with $\Gamma$ a characteristic dissipation rate, $\Lambda$ a high-frequency cutoff and $\Theta(z)$ the Heaviside step function.

1. Solution for the dynamics

Exact solutions for the resonant-level model have recently been presented in the context of a debate regarding heat in strongly coupled open quantum systems, with particular emphasis on the wide-band limit $\Lambda \to \infty$ [44–46]. Note, however, that the system-bath interaction energy is proportional to $\Lambda$, and thus formally divergent in this limit (this can be seen easily using the reaction-coordinate representation [50], for example). We thus take $\Lambda$ to be finite but much larger than all other energy scales.

Under this assumption, we use a quantum Langevin approach to solve for the open-system evolution, detailed in Appendix A. Our approximate analysis requires that the dynamics proceeds much more slowly than the inverse cutoff scale $\Lambda^{-1}$, but otherwise allows for arbitrary driving protocols and strong system-bath coupling. Taking a factorised system-bath density matrix at the initial time, $\rho(0) = \rho^{(S)}(0) \rho^{(B)}(0)$, we find the level occupation $n(t) = \langle a^\dagger a \rangle$ and the system-bath correlations $c(t) = \sum_k \gamma_k \langle a^\dagger b_k + \text{h.c.} \rangle$ to be given by

$$n(t) = \frac{1}{2} + |K(t, 0)|^2 n(0)$$
$$- \frac{1}{2} \int_0^t ds \int_0^t ds' K(t, s) g(s) \phi(s - s') g(s') K^\dagger(t, s')$$
$$c(t) = \text{Im} \int_0^t ds K^\dagger(t, s) \phi(t - s) g(s). \quad (53)$$

These expressions are written in terms of the propagator $K(t, t') = \exp \left[ \int_{t'}^t ds \left( -i\varepsilon(s) - \frac{\Gamma}{2} g(s)^2 \right) \right]$ and the noise correlation function

$$\phi(t) = \frac{\Gamma}{i\beta} \left[ \frac{1}{\sinh(\pi t/\beta)} - \frac{\cos(\Lambda t)}{\pi t/\beta} \right]. \quad (55)$$

Note that the second, cutoff-dependent term is essential to regulate the divergence of the integrand in Eq. (53) as $s \to t$, but plays essentially no role in Eq. (52) for large $\Lambda$.

It follows immediately from Eqs. (54) and (55) that the relaxation timescale is given by $\gamma_{eq} \sim 1/\Gamma g^2$, in agreement with Eq. (1), even though the evolution is non-Markovian, in general. Note also that, since Eq. (55) contains one propagator while Eq. (54) includes two factors of $K(t, t')$, the relaxation timescale of the interaction energy is twice as long as that of $n(t)$. This is in accordance with the relaxation behaviour of the CL model shown in Figs. 2 and 5.

2. Dissipated work

We now compute the dissipated work during an isothermal protocol, during which the level energy $\varepsilon(t)$ is linearly ramped from an initial to a final value $\varepsilon_i \to \varepsilon_f$, while interacting with the bath. At the start and end of the protocol, the system-bath interaction energy is switched on according to Eq. (14) with $\alpha = 1$, $g_0 = 0$ and $g_1 = k$, and switched off via the reverse procedure.
As a consequence, our proposal allows for increasing the time of the isothermal process). This confirms that control over the system-bath interaction can indeed reduce the time taken by an isothermal process without incurring additional dissipation (c.f. Fig. 2 showing the time of the isothermal process).

VI. CONCLUSIONS

We have put forward the idea of a speed-up to isothermality (STI), where an isothermal process is sped up by smoothly increasing (and decreasing) the system-bath interaction. This leads to faster isothermal processes while keeping the overall thermodynamic dissipation constant. As a consequence, our proposal allows for increasing the power of a finite-time Carnot cycles [6, 7, 35–38] without compromising their efficiency.

The proposed STI are based upon two main assumptions:

1. We work in the slow-driving (or adiabatic) regime, allowing for an expansion of the dissipation as in Eq. (6).

2. We assume that the time-scale of thermalization satisfies \( \tau_{\text{eq}} \propto g^{-2} \), where \( g \) quantifies the strength of the system-bath interaction. This behaviour, which is expected in Markovian open system dynamics [57], has been confirmed in our models of interest (bosonic and fermionic bath) even for reasonably large coupling strengths.

These assumptions hold in a wide range of relevant open quantum systems. Furthermore, we have also argued that some of our ideas can be extended beyond weak system-reservoir coupling to strongly correlated many-body systems that thermalize [41] (see, for example, the discussion after Eq. (24)). In this case, the relation \( \tau_{\text{eq}} \propto g^{-2} \) might change, but it is worth pointing out that the developed framework is rather flexible and that a different dependence can be easily accounted for. This will naturally change the qualitative scaling of the speed-ups but not the essence of the method, namely to speed-up isothermal processes without increasing the dissipation.

Under these assumptions, we have shown that STI can decrease the time of a given isothermal process by several orders of magnitude, see Eq. (26) and Fig. 2. This leads to faster decays of the dissipation with time (Sec. 1.1) and higher efficiencies at maximum power of finite-time Carnot engines (Sec. IV). These new thermodynamic behaviours have been illustrated for fermionic and bosonic baths, where we computed exactly the system-bath evolution at strong coupling in order to account for all.
sources of dissipation, and to control all the different assumptions underlying STI.

An interesting future direction is to combine these ideas with open systems techniques to deal with strong, time-dependent coupling such as the reaction-coordinate mapping [2, 5, 59–62], or more sophisticated tensor-network methods [63–65]. Another interesting direction is to characterise the work fluctuations due to such STI, which have been characterised in e.g. STA [66], and other tradeoffs between thermodynamic cost and time enhancements [67, 68]. Indeed, because STIs allow for accessing larger energy scales, one expects that they shall generate higher work fluctuations [66]. In this sense, we note that one expects a competing effect in the work fluctuations generated by a STI: because we are accessing stronger coupling and hence larger energy scales, one expects stronger fluctuations; however, for a fixed time, STI allow for decreasing dissipation, and in the quasistatic regime the minimisation of dissipation comes together with the minimisation of fluctuations, at least for commuting protocols [54]. A further interesting possibility is to combine these considerations with geometric optimal paths [8, 52, 51].

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Appendix A: Extensions to more general dissipations

In this section we consider the case in which the dissipation is of the form as in Eq. (B4):

\[ W_{\text{diss}} = \frac{1}{\tau} \int_0^1 dt \rho^{th}_t (\dot{H}_t, \dot{H}_t) + O \left( \frac{\tau^2}{\tau^2} \right). \]  

(A1)

where \( \rho^{th}_t \) is a bilinear form which depends only on the base point \( \rho^{th}_t \). This expression is generic, and it arises in the expansion of the entropy production rate \( \dot{\sigma}_t \) in the quasi-static limit.

\[ \dot{\sigma}_t = G_{\rho^{th}}^{\text{th}} (\dot{H}_t, \dot{H}_t) + O \left( \frac{\tau^2}{\tau^2} \right). \]  

(A2)

In particular, if the dynamics is described by the time dependent Liouvillian equation:

\[ \rho = L_t [\rho], \]  

(A3)

where \( L_t \) has for every \( t \) only one thermal steady state and, moreover, the real part of all its eigenvalues is negative (this two conditions are sufficient to ensure thermalisation), then the integrand in Eq. (A1) is given at first order by:

\[ G_{\rho^{th}}^{\text{th}} (\dot{H}_t, \dot{H}_t) = -\beta \text{Tr} \left[ \dot{H}_t L_t^+ \left[ \rho^{th}_0 (H_t) | \dot{H}_t ] \right] \right], \]  

(A4)

where we defined the two operators:

\[ L_t^+ [A] := \int_0^\infty d\nu \ e^{\nu L_t} (\rho^{th}_0 (H_t)) \text{Tr} [A - A]. \]  

(A6)

Before going on, it should be noticed that the operator \( \mathbb{J}_\omega \) is related to the generalised covariance through the equality:

\[ \text{cov} (A, B) = \text{Tr} [A \mathbb{J}_\omega B]. \]  

(A7)

Moreover, carrying out the integral in Eq. (A6) in the eigenbasis of \( L_t \) shows that the eigenvalues of \( L_t^+ \) are directly connected with the different thermalisation time-scales in the system. In particular, in the case in which all the observables thermalises at the same rate, Eq. (A4) reduces to Eq. (7).

Considering again the simplified case in which the derivative of the Hamiltonian is given by \( H = \lambda t X \), we have the chain of inequalities:

\[ |W_{\text{diss}}| = \frac{\beta}{\tau} \left( \int_0^1 dt \lambda_t^2 \text{Tr} [X L_t^+ \left[ \rho^{th}_0 (H_t) | X ] \right] \right) \leq \frac{\beta}{\tau} \sup_{t \in [0,1]} \text{cov}_{\rho^{th}} (X, X) \int_0^1 dt \lambda_t^2 \tau_{\text{g}(t)}, \]  

(A8)

where we indicate with \( \tau_{\text{g}(t)} \) the biggest eigenvalue of \( L_t^+ \). Since during the turning on and off procedure we want...
to keep track of the dependence of the thermalisation time-scale on the interaction strength, we keep this term inside the integral. This expression should be compared with Eq. (15) and (17) above.

As a final remark, the bound in Eq. (17) on the covariance can be improved to [69]:

$$\sup_{t \in [0,1]} \rangle \rho_t^k (F, F) \leq 2 \sup_{t \in [0,1]} \left( (V^2) \rho_t^k - (V^2)^2 \right). \quad (A9)$$

This quantity is expected to be finite even in the limit in which $||V|| \rightarrow \infty$.

**Appendix B: Solution of the resonant-level model**

In this appendix, we detail our approach to solve the resonant-level model described in Sec. V B. Starting from the Hamiltonian given in Eqs. (48)–(50), we derive the Heisenberg equations

$$\dot{a}(t) = -i \varepsilon(t)a(t) - ig(t) \sum_k \gamma_k b_k(t), \quad (B1)$$
$$\dot{b}_k(t) = -i \omega_k b_k(t) - i g(t) \gamma_k a(t). \quad (B2)$$

The second equation can be formally solved to give

$$\sum_k \gamma_k b_k(t) = \xi(t) - i \int_{t_0}^t dt' \chi(t - t') g(t') a(t'), \quad (B3)$$

where we defined the noise operator

$$\xi(t) = \sum_k \gamma_k e^{-i \omega_k (t - t_0)} b_k(t_0), \quad (B4)$$

whose Gaussian statistics with respect to the initial state define the memory kernel $\chi(t - t') = \langle \{ \xi(t), \xi(t') \} \rangle$ and the noise correlation function $\phi(t - t') = \langle \{ \xi(t), \xi(t') \} \rangle$. These are given explicitly by

$$\chi(t) = \int \frac{d\omega}{2\pi} e^{-i \omega t} \tilde{J}(\omega), \quad (B5)$$
$$\phi(t) = \int \frac{d\omega}{2\pi} e^{-i \omega t} \tilde{J}(\omega) \tanh(\beta(\omega - \mu))/2, \quad (B6)$$

where the spectral density is defined by Eq. (51). Note that here, for completeness, we allow for a finite chemical potential $\mu$. In the wide-band limit $\Lambda \rightarrow \infty$, the chemical potential can be set to zero without loss of generality by simply redefining all energies relative to $\mu$, which justifies our choice of $\mu = 0$ in the main text.

To obtain a tractable description, we approximate the memory kernel as

$$\chi(t) = \frac{\Gamma \sin(\Lambda t)}{\pi t} \approx \Gamma \delta(t). \quad (B7)$$

This is an exact equality (in the distributional sense) in the limit $\Lambda \rightarrow \infty$, and is a good approximation for finite $\Lambda$ so long as slowly varying functions and large times relative to the cut-off scale $\Lambda^{-1}$ are considered. The noise correlation function is approximated as

$$\phi(t) = \frac{\Gamma}{2} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{\pi} e^{-i \omega t} \tanh(\beta(\omega - \mu))/2$$
$$\approx \frac{\Gamma}{2} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} e^{-i \omega t} \tanh(\beta(\omega - \mu))/2$$
$$- \frac{\Gamma}{2} \int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi} e^{-i \omega t}$$
$$= \frac{\Gamma}{i\beta} \left[ e^{-i \omega t} \right] \frac{\cos(\Lambda t)}{\sinh(\pi t/\beta)}. \quad (B8)$$

On the second line, the integration domain is partitioned into three parts, and the approximation tanh$(z) \approx \pm 1$ for $z \gg 1$ is made. The first integral is essentially the Fourier transform of tanh$(z)$, which is calculated by a standard contour integration, resulting in a geometric sum over Matsubara frequencies that evaluates to the first term in Eq. (B8). The remaining two integrals yield the second term in Eq. (B8) with the help of the Sokhotski–Plemelj theorem. Note that this second term regulates the $1/t$ divergence as $t \rightarrow 0$ but is negligible (in the distributional sense) for time scales $t \gg \Lambda^{-1}$. It can be shown that, within these approximations, the fluctuation-dissipation relation $\tilde{\phi}(\omega) = \tilde{\chi}(\omega) \tanh(\beta(\omega - \mu))/2$ between the Fourier components of the memory kernel $\chi(\omega)$ and the noise spectrum $\phi(\omega)$ holds for all $|\omega| < \Lambda$.

As a consequence of Eq. (B7), Eq. (B1) reduces to a time-local differential equation

$$\dot{a}(t) = \left( -i \varepsilon(t) a(t) - \frac{\Gamma}{2} g(t)^2 \right) a(t) - i g(t) \xi(t), \quad (B9)$$

which can be easily solved to find

$$a(t) = K(t, t_0) a(t_0) - i \int_{t_0}^t dt' K(t, t') g(t') \xi(t'), \quad (B10)$$

where the propagator is given by Eq. (54). Combining this with Eqs. (53), (57) and (58), and the fact that $\langle a(t_0) b(t_0) \rangle = 0$ for a factorized initial condition at $t_0 = 0$, we deduce Eqs. (52) and (53).

Our analysis relies on two approximations, given by Eqs. (B7) and (B8). The former assumes that the dynamics is much slower than $\Lambda^{-1}$, while the latter requires that the temperature and chemical potential are much smaller than $\Lambda$. In particular, we require that $\varepsilon(t), \Gamma g(t)^2, \beta^{-1}, |\mu|$ and $|\mu \pm \beta^{-1}|$ are all much smaller than $\Lambda$. 