One-loop Effective Potential for a Fixed Charged Self-interacting Bosonic Model at Finite Temperature with its Related Multiplicative Anomaly

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Abstract

The one-loop partition function for a charged self-interacting Bose gas at finite temperature in D-dimensional spacetime is evaluated within a path integral approach making use of zeta-function regularization. For D even, a new additional vacuum term —overlooked in all previous treatments and coming from the multiplicative anomaly related to functional determinants— is found and its dependence on the mass and chemical potential is obtained. The presence of the new term is shown to be crucial for having the factorization invariance of the regularized partition function. In the non interacting case, the relativistic Bose-Einstein condensation is revisited. By means of a suitable charge renormalization, for \(D = 4\) the symmetry breaking phase is shown to be unaffected by the new term, which, however, gives actually rise to a non vanishing new contribution in the unbroken phase.

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I. INTRODUCTION

The one-loop finite temperature effective potential with a non vanishing chemical potential has been considered quite often during the last years. The case of a free relativistic charged bosonic field was investigated in Refs. [1–4], while the self-interacting charged scalar field has been studied in [5,6]. Recently, the issue has been reconsidered and critically analyzed in Ref. [7], where use has been made of zeta-function regularization in the calculations and a different approach for including the chemical potential has been proposed.

We would like to recall the importance of zeta-function regularization, introduced in [8–10], as a powerful tool to deal with the ambiguities (ultraviolet divergences) present in relativistic quantum field theories (see for example [11,12]). It permits to give a meaning—in the sense of analytic continuation—to the determinant of a differential operator which, as the product of its eigenvalues, would be formally divergent. For the sake of simplicity we shall here restrict ourselves to scalar fields. In the case of a neutral scalar field, we recall that the one-loop Euclidean partition function, regularised by means of zeta-function techniques, reads [10]

\[ \ln Z = -\frac{1}{2} \ln \det \frac{L_D}{M^2} = \frac{1}{2} \zeta'(0|L_D) + \frac{1}{2} \zeta(0|L_D) \ln M^2, \]

where \( \zeta(s|L_D) = \text{Tr} L_D^{-s} \) is the zeta function related to \( L_D \), a second order elliptic differential operator, \( \zeta'(0|L_D) \) its derivative with respect to \( s \), and \( M^2 \) is a renormalization scale mass. The fact is used here that the analytically continued zeta-function is generically regular at \( s = 0 \), and thus its derivative is well defined.

In this paper, we shall take full advantage of a rigorous use of this regularization in investigating the finite temperature effects in presence of a non vanishing chemical potential. In order to illustrate the new issue we shall be dealing with, we start with the partition function for a free charged field in \( \mathbb{R}^D \), described by two real components \( \phi_i \). The Euclidean action is

\[ S = \int dx^D \left[ \phi_i \left( -\Delta_D + m^2 \right) \phi_i \right], \]

where \( \phi^2 = \phi_k \phi_k \) is \( O(2) \) invariant.

As is well known, in order to study finite temperature effects, one observes that the grand canonical partition function may be written under the form (we shall call this method of including the chemical potential, as discussed in Ref. [7], method I)

\[ Z_\beta(\mu) = \text{Tr} e^{-\beta(H - \mu Q)}, \]

in which \( H \) is the Hamiltonian, \( Q \) the conserved charge operator and \( \beta \) the inverse of the equilibrium temperature. Making use of a path integral representation and integrating over the momenta, one arrives at the following recipe [1,2,5,7,13]: (i) compactify the imaginary time \( \tau \) in the interval \( [0, \beta] \), (ii) assume a periodic boundary condition in \( \beta \), and (iii) include the chemical potential \( \mu \) by adding to the action the term

\[ i\mu \epsilon_{ik} \phi_i \partial_\tau \phi_k - \frac{1}{2} e^2 \mu^2 \phi^2, \]
where $e$ is the elementary charge. As a result, the partition function reads

$$Z_\beta(\mu) = \int_{\phi(\tau) = \phi(\tau + \beta)} d\phi_i e^{-\frac{1}{\beta} \int_0^\beta d\tau \int d^N x \phi_i A_{ij} \phi_j},$$

where $D = N + 1$ and $A$ is given by

$$A_{ij} = L_{ij} + 2e\mu \epsilon_{ij} \sqrt{L_\tau},$$

with

$$L_{ij} = (L_\tau + L_N - e^2 \mu^2) \delta_{ij}, \quad L_N = -\Delta_N + m^2,$$

in which $\Delta_N$ is the Laplace operator on $\mathbb{R}^N$ (continuous spectrum $\vec{k}^2$) and $L_\tau = -\partial^2_\tau$ (discrete spectrum $\omega_n^2 = \frac{4\pi^2 n^2}{\beta^2}$), the Laplace operator on $S^1$. Thus, one is actually dealing with a matrix-valued elliptic differential non self-adjoint operator acting on scalar fields in $S^1 \times \mathbb{R}^N$. In this case, the partition function may be written as

$$\ln Z_\beta(\mu) = -\frac{1}{2} \ln \det \left| \frac{A_{ik}}{M^2} \right| = -\frac{1}{2} \ln \det \left[ \frac{L_+ L_-}{M^2 M^2} \right],$$

where

$$L_\pm = L_\tau + L_N + e^2 \mu^2 \pm 2e\mu(L_N)^{\frac{1}{2}}.$$  

Another possible factorization (see, for example, [12]) is

$$K_\pm = L_N + L_\tau - e^2 \mu^2 \pm 2ie\mu(L_\tau)^{\frac{1}{2}}.$$  

Of course we have $L_+ L_- = K_+ K_-$ and, in both cases, one is dealing with a couple of pseudo-differential operators ($\Psi$DOs), $L_+$ and $L_-$ being also formally self-adjoint.

As is clear, in any case the product of two elliptic $\Psi$DOs, say $A_+$ and $A_-$, appears. It is known that, in general, the zeta-function regularized determinants do not satisfy the relation $\det(AB) = \det A \det B$. In fact, in general, there appears the so-called multiplicative anomaly [14,15]. In terms of $F(A, B) \equiv \det(AB)/(\det A \det B)$ [15], it is defined as:

$$a_D(A, B) = \ln F(A, B) = \ln \det(AB) - \ln \det(A) - \ln \det(B),$$

in which the determinants of the two elliptic operators, $A$ and $B$, are assumed to be defined (i.e. regularized) by means of the zeta function [3]. Thus, the partition function, chosen a factorization $A_+ A_-$, is given by

$$\ln Z_\beta(A_+, A_-) = -\frac{1}{2} \ln \det \left| \frac{A_{ik}}{M^2} \right| = -\frac{1}{2} \ln \det \left[ \frac{A_+ A_-}{M^2 M^2} \right]$$

$$= \frac{1}{2} \zeta'(0|A_+) + \frac{1}{2} \zeta'(0|A_-) + \frac{1}{2} \ln M^2 \left[ \zeta(0|A_+) + \zeta(0|A_-) \right] - \frac{1}{2} a_D(A_+, A_-).$$

Here one can see the crucial role of the multiplicative anomaly: as, in the factorization, different operators may enter, the multiplicative anomaly is necessary in order to have the
same regularized partition function in both cases, namely $\ln Z_\beta(A_+, A_-) = \ln Z_\beta(B_+ B_-)$, which is an obvious physical requirement. (What it is very easy to see in general is that $\det A_+ \neq \det B_+ \det B_-$.)

With regards to the multiplicative anomaly, an important result is that it can be expressed by means of the non-commutative residue associated with a classical $\Psi$DO, known as the Wodzicki residue \cite{16}.

In the limit of zero temperature and vanishing chemical potential, the two elliptic $\Psi$DOs $L_+$ and $L_-$ reduce to Laplace-type second order differential operators with constant potential terms defined in $\mathbb{R}^4$. In this case, the multiplicative anomaly has been computed in Ref. \cite{17}.

One could argue that the presence of a multiplicative anomaly is strictly linked with the zeta-function regularization employed and, thus, that it might be an artifact of it. In the Appendix we will prove that the multiplicative anomaly is present indeed in a large class of regularizations of functional determinants appearing in the one-loop effective action.

The contents of the paper are the following. In Sec. 2 we reconsider the free case making use of zeta-function regularization and we carry out a comparison of the two possible factorizations. In Sec. 3, the self-interacting case is presented in the one-loop approximation. In Sec. 4 we briefly introduce the Wodzicki residue and present a proof of the Wodzicki formula expressing the multiplicative anomaly in terms of the corresponding non-commutative residue of a suitable classical $\Psi$DO. In Sec. 5, the results of Wodzicki are used in the computation of the multiplicative anomaly for the interacting $O(2)$ model. In Sec. 6, the Bose-Einstein condensation phenomenon is discussed in the non-interacting case. Some final remarks are presented in the concluding section. In the Appendix, the presence of the multiplicative anomaly in a very large class of functional determinant regularizations is proven.

II. CHEMICAL POTENTIAL IN THE NON-INTERACTING CASE REVISITED

We shall treat here the non-interacting bosonic model at finite temperature with a non-vanishing chemical potential. This situation has been extensively investigated in the past. We will reconsider it here making rigorous use of the zeta-function regularization procedure, what will allow us to clarify some subtle points overlooked in previous studies.

Again, it is convenient to work in $S^1 \times \mathbb{R}^N$, with $D = N + 1$ in order to exploit the role of the dimension of the space-time. As discussed in the Introduction, here one has to deal with the functional determinant of the product of the two operators

$$L_\pm = L_\tau + \left(\sqrt{L_N} \pm \epsilon \mu \right)^2,$$

while a second factorization is obtained in terms of the operators

$$K_\pm = L_N + \left(\sqrt{L_\tau} \pm i\epsilon \mu \right)^2,$$

with $L_N = -\Delta_N + m^2$.

Let us start with the latter one, since it renders calculations easier. The zeta function can be defined, for a sufficiently large real part of $s$, by means of the Mellin transform of the related heat operator trace, namely
\[
\zeta(s|K_{\pm}) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{Tr} \exp(-tK_{\pm}),
\]

where \(\text{Tr} \exp -tK_{\pm}\) is given by
\[
\text{Tr} e^{-tK_{\pm}} = \text{Tr} e^{-tL_N} \sum_n e^{-t(\omega_n \pm i\mu)^2}.
\]

For notational simplicity, we write
\[
\text{Tr} e^{-tL_N} = \sum_i e^{-t\lambda_i} = \frac{V_N}{(4\pi t)^{\frac{N}{2}}} e^{-tm^2},
\]

with \(\lambda_i\) the eigenvalues of the operator \(L_N\), namely \(\vec{k}^2 + m^2\). Thus, via Mellin transform, one has the standard result
\[
\zeta(z|L_N) = \frac{V_N}{(4\pi)^{\frac{N}{2}}} \frac{\Gamma(z - \frac{N}{2})}{\Gamma(z)} m^{N-2z}.
\]

The Poisson-Jacobi resummation formula gives
\[
\sum_n e^{-t(\omega_n \pm i\mu)^2} = \frac{\beta}{(4\pi t)^{\frac{1}{2}}} \left(1 + 2 \sum_{n=1}^\infty \cosh(n\mu\beta)e^{-\frac{n^2\beta^2}{4t}}\right).
\]

As a result, even though \(K_+ \neq K_-\), one gets
\[
\text{Tr} e^{-tK_+} = \text{Tr} e^{-tK_-}
\]
\[
= \text{Tr} e^{-tL_N} \frac{\beta}{(4\pi t)^{1/2}} \left(1 + 2 \sum_{n=1}^\infty \cosh(n\mu\beta)e^{-\frac{n^2\beta^2}{4t}}\right),
\]

and \(\zeta(s|K_+) = \zeta(s|K_-)\). Therefore, making use of Eq. (14), Eq. (15), Eq. (18) and of the well known integral representation of the MacDonald function \(K_s(x),\)
\[
K_s(\sqrt{bx}) = \frac{1}{2} \left(\frac{x}{2\sqrt{b}}\right)^s \int_0^\infty e^{-bt - \frac{x^2}{4t}} t^{-s-1} dt,
\]

one has
\[
\frac{1}{2} \left[\zeta(s|K_+) + \zeta(s|K_-)\right] = \frac{\beta V_N}{(4\pi)^{\frac{N}{2}}} \frac{\Gamma(s - \frac{D}{2})}{\Gamma(s)} m^{D-2s}
\]
\[
+ \frac{\beta^{3/2-s}}{\sqrt{\pi}} \frac{\Gamma(s)}{\Gamma(s)} \sum_{n=1}^\infty \cosh(n\mu\beta)(n\beta)^{s-\frac{D}{2}} \sum_i \lambda_i^{\frac{D-2s}{2}} K_{s-\frac{D}{2}}(n\beta\sqrt{\lambda_i}).
\]

The second term in this equation represents the statistical sum contribution, which depends non-trivially on the temperature and chemical potential and can be calculated in arbitrary dimension. Near \(s = 0\) it can be written as \(sS(\beta, \mu) + O(s^2)\), with
\[
S(\beta, \mu) = -\sum_i \left[\ln \left(1 - e^{-\beta(\sqrt{\lambda_i} + \epsilon\mu)}\right) + \ln \left(1 - e^{-\beta(\sqrt{\lambda_i} - \epsilon\mu)}\right)\right].
\]
Taking Eq. (16) into account, this contribution can also be rewritten as follows
\begin{equation}
S(\beta, \mu) = \frac{4V_N\beta}{(4\pi)^{D/2}} \sum_{n=1}^{\infty} \cosh(ne\mu\beta)(n\beta)^{-\frac{D}{2}}(2m)^{\frac{D}{2}}K_D(n\beta m) .
\end{equation}

It should be noted that this statistical sum contribution is a series involving MacDonald functions. This series is always convergent in any \( D > 1 \) space-time dimension and for any \( \beta \) and \( e|\mu| \leq m \), even in the critical limit \( e|\mu_c| = m \), \( m \) being the lowest eigenvalue in the spectrum of \( L_N \). This follows from the asymptotics of \( K_\nu(z) \) for large \( z \):
\begin{equation}
K_\nu(z) \simeq \left( \frac{\pi}{2z} \right)^{\frac{\nu}{2}} e^{-z} \left[ 1 + O\left( \frac{1}{z} \right) \right].
\end{equation}

Coming back to Eq. (21), the first term (the vacuum contribution) has to be considered, as usual, for \( D \) odd and \( D \) even separately. For \( D \) odd, we have
\begin{equation}
\zeta(0|K_+) + \zeta(0|K_-) = 0 ,
\end{equation}
and
\begin{equation}
\frac{1}{2} (\zeta'(0|K_+) + \zeta'(0|K_-)) = \frac{\beta V_N}{(4\pi)^{D/2}} \Gamma\left(-\frac{D}{2}\right)m^D + S(\beta, \mu) .
\end{equation}

For \( D \) even, we shall restrict ourselves to the two cases \( D = 2 \) and \( D = 4 \). We obtain
\begin{align*}
\zeta(0|K_+) \ln M^2 + \zeta'(0|K_+) &= -\frac{\beta V_1 m^2}{4\pi} \left( \ln \frac{m^2}{M^2} - 1 \right) + S(\beta, \mu) , \quad D = 2 , \\
\zeta(0|K_+) \ln M^2 + \zeta'(0|K_+) &= -\frac{\beta V_3 m^4}{32\pi^2} \left( \ln \frac{m^2}{M^2} - \frac{3}{2} \right) + S(\beta, \mu) , \quad D = 4 .
\end{align*}

Let us now consider the first factorization.
\begin{equation}
\text{Tr} e^{-tL_\pm} = \text{Tr} e^{-t(\sqrt{L_N} + e\mu)^2} \sum_n e^{-t\omega_n^2} ,
\end{equation}
with
\begin{equation}
\text{Tr} e^{-t(\sqrt{L_N} + e\mu)^2} = \sum_i e^{-t(\sqrt{\lambda_i} + e\mu)^2} .
\end{equation}

Again, Poisson resummation gives
\begin{equation}
\text{Tr} e^{-tL_\pm} = \text{Tr} e^{-t(\sqrt{L_N} + e\mu)^2} \frac{\beta}{(4\pi t)^{1/2}} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\beta^2} \right) ,
\end{equation}
and, as a consequence,
\begin{align*}
\zeta(s|L_\pm) &= \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(s - \frac{1}{2}|(\sqrt{L_N} + e\mu)^2) \\
&- 2s \sum_i \ln \left( 1 - e^{-\beta(\sqrt{\lambda_i} + e\mu)} \right) + O(s^2) .
\end{align*}
As a result, for \( D \) respectively. It should be noted that one gets \( \ln Z \) sector depends explicitly on the chemical potential

\[
\zeta(z|\sqrt{L_N} + e\mu)^2 + \zeta(z|\sqrt{L_N} - e\mu)^2 = 2\zeta(z|L_N)
\]

and this leads to

\[
\frac{1}{2} \left( \zeta(s|L_+) + \zeta(s|L_-) \right) = \frac{\beta V_N m^{D-2s}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(s - \frac{D}{2})}{\Gamma(s)} \right. \\
+ \sum_{r=1}^{\infty} \frac{\Gamma(2s + 2r - 1)\Gamma(s - \frac{1}{2})}{\Gamma(s)\Gamma(s - \frac{1}{2} + r)\Gamma(2s - 1)} \left( \frac{e\mu}{m} \right)^{2r} \Gamma(s + r - \frac{D}{2}) \left\} \\
- sS(\beta, \mu) + O(s^2) .
\]

As a result, for \( D \) odd, taking the derivative with respect to \( s \) and the limit \( s \to 0 \), the two factorizations give

\[
\ln Z_{\beta,\mu}(L_+, L_-) = \frac{\beta V_N m^D}{(4\pi)^{\frac{D}{2}}} \Gamma(-\frac{D}{2}) + S(\beta, \mu) - \frac{1}{2} a_D(L_+, L_-) ,
\]

and

\[
\ln Z_{\beta,\mu}(K_+, K_-) = \frac{\beta V_N m^D}{(4\pi)^{\frac{D}{2}}} \Gamma(-\frac{D}{2}) + S(\beta, \mu) - \frac{1}{2} a_D(K_+, K_-) ,
\]

respectively. It should be noted that one gets \( \ln Z_{\beta,\mu}(L_+, L_-) = \ln Z_{\beta,\mu}(K_+, K_-) \) and thus the standard textbook result as soon as one is able to prove that the two multiplicative anomalies are vanishing for \( D \) odd. We can anticipate that this indeed happens.

For \( D \) even, the situation is different because, within the first factorization, the vacuum sector depends explicitly on the chemical potential \( \mu \). In fact, for example, one has for \( D = 2 \)

\[
\frac{1}{2} \left\{ \zeta'(0|L_+) + \zeta'(0|L_-) + [\zeta(0|L_+) + \zeta(0|L_+)] \ln M^2 \right\} = \frac{\beta V_1}{\pi} \left[ m^2 (\ln \frac{m^2}{M^2} - 1) \right] + \frac{\beta V_1}{2\pi} e^2 \mu^2 + S(\beta, \mu) ,
\]

and for \( D = 4 \)

\[
\frac{1}{2} \left\{ \zeta'(0|L_+) + \zeta'(0|L_-) + [\zeta(0|L_+) + \zeta(0|L_+)] \ln M^2 \right\} = \frac{\beta V_3}{32\pi^2} \left[ m^4 (\ln \frac{m^2}{M^2} - 3/2) \right] + \frac{\beta V_3}{8\pi^2} 2 \left[ \frac{e^4 \mu^4}{3} - e^2 \mu^2 m^2 \right] + S(\beta, \mu) ,
\]

As a result, for \( D = 2 \) the two factorizations give

\[
\ln Z_{\beta,\mu}(L_+, L_-) = \frac{\beta V_1 m^2}{4\pi} (\ln \frac{m^2}{M^2} - 1) + S(\beta, \mu) + \frac{V_1 \beta}{2\pi} e^2 \mu^2 - \frac{1}{2} a_2(L_+, L_-) ,
\]

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\[
\ln Z_{\beta,\mu}(K_+, K_-) = \frac{\beta V_1 m^2}{4\pi} (\ln \frac{m^2}{M^2} - 1) + S(\beta, \mu) - \frac{1}{2} a_2(K_+, K_-),
\]
respectively, while for \( D = 4 \) one has
\[
\ln Z_{\beta,\mu}(L_+, L_-) = \frac{\beta V_3}{32\pi^2} \left[ m^4 (\ln \frac{m^2}{M^2} - 3/2) \right] + \frac{\beta V_3}{8\pi^2} \left( \frac{e^4 \mu^4}{3} - e^2 \mu^2 m^2 \right) + S(\beta, \mu) - \frac{1}{2} a_4(L_+, L_-),
\]
and
\[
\ln Z_{\beta,\mu}(K_+, K_-) = \frac{\beta V_3}{32\pi^2} \left[ m^4 (\ln \frac{m^2}{M^2} - 3/2) \right] + S(\beta, \mu) - \frac{1}{2} a_4(K_+, K_-).
\]
In general for \( D \) even, one gets
\[
\ln Z_{\beta,\mu}(L_+, L_-) = -\beta V_N E_V(m, M) + \beta V_N E_D(m, \mu) + S(\beta, \mu) - \frac{1}{2} a_D(L_+, L_-),
\]
and
\[
\ln Z_{\beta,\mu}(K_+, K_-) = -\beta V_N E_V(m, M) + S(\beta, \mu) - \frac{1}{2} a_D(K_+, K_-),
\]
where \( E_V(m, M) \) is the naive vacuum energy density, \( D = 2Q \) and
\[
E_D(m, \mu) = \sum_{r=1}^{Q} c_{Q,r} (e\mu)^{2r} m^{D-2r}.
\]
Here the \( c_{Q,r} \) are computable coefficients, given by
\[
c_{Q,r} = \frac{\Gamma(2r - 1)}{\Gamma(2r + 1)} \frac{2^{1-N}}{\pi^\frac{N}{2}} \frac{(-1)^{Q-r}}{\Gamma(r - \frac{1}{2}) (Q - r)!}.
\]
In the cases \( D = 2, D = 4 \) and \( D = 6 \) one obtains
\[
E_2(m, \mu) = \frac{1}{2\pi} e^2 \mu^2, \quad E_4(m, \mu) = -\frac{1}{8\pi^2} \left[ e^2 \mu^2 \left( m^2 - \frac{e^2 \mu^2}{3} \right) \right],
\]
\[
E_6(m, \mu) = -\frac{1}{16\pi^3} \left[ e^2 \mu^2 \left( -\frac{1}{4} m^4 + \frac{e^2 \mu^2 m^2}{6} - \frac{2 e^4 \mu^4}{45} \right) \right].
\]
III. THE INTERACTING CASE IN THE ONE-LOOP APPROXIMATION

In this section, we will review the interacting charged boson model and we will compute, within the one-loop approximation, the corresponding operators appearing in the two possible factorizations of the partition function.

The Euclidean action for a self-interacting charged field, again described by two real components \( \phi_i \), is

\[
S = \int dx^D \left[ \phi_i \left( -\Delta_D + m^2 \right) \phi_i + \lambda_D (\phi^2)^{D-2} \right],
\]

where \( \phi^2 = \phi_k \phi_k \) is \( O(2) \) invariant and \( \lambda_D = \frac{\lambda}{D!} \) is a dimensionless coupling constant. In the important case \( D = 4 \) one has

\[
S = \int dx^4 \left[ \phi_i \left( -\Delta + m^2 \right) \phi_i + \frac{\lambda}{4!} (\phi^2)^2 \right].
\]

Finite temperature effects with non vanishing chemical potential are accounted for in the following one-loop partition function

\[
Z_\beta(\mu) = e^{-S_0} \int_{\phi(\tau)=\phi(\tau+\beta)} [d\phi_i] e^{-\int_0^\beta d\tau \int d^N x \phi_i A_{ij} \phi_j},
\]

where

\[
S_0 = \beta V_N \left[ \frac{1}{2} (m^2 - e^2 \mu^2) \Phi^2 + \lambda_D (\Phi^2)^{D-2} \right],
\]

\( \Phi \) being the background field, assumed to be constant, and \( A \) the Euclidean small disturbances operator, given by

\[
A_{ij} = L_{ij} + 2 e \mu \epsilon_{ij} \sqrt{L_T} + \frac{8 D \lambda_D}{(D-2)^2} (\Phi^2)^{\frac{D-2}{D-2}} \Phi_i \Phi_j,
\]

with

\[
L_{ij} = \left( L_T + L_N - e^2 \mu^2 + \frac{2 D \lambda_D}{D-2} (\Phi^2)^{\frac{D-2}{D-2}} \right) \delta_{ij}.
\]

Again, one is dealing with a matrix-valued, elliptic, differential, non self-adjoint operator acting on scalar fields in \( S^1 \times \mathbb{R}^N \). As a consequence, the partition function reads

\[
\ln Z_\beta(\mu) = -\frac{1}{2} \ln \det \left[ \frac{A_{ik}}{M^2} \right] = -\frac{1}{2} \ln \det \left[ \frac{L_+ L_-}{M^2} \right]
\]

where

\[
L_\pm = L_T + L_N + e^2 \mu^2 + h_D \pm \left[ \frac{4}{D^2} h_D^2 + 4 e^2 \mu^2 (L_N + h_D) \right]^{\frac{1}{2}},
\]

with
\[ h_D = \frac{2D^2 \lambda_D}{(D-2)^2} (\Phi^2)^{n-2}, \quad h_4 = \frac{\lambda}{3} \Phi^2. \]  

(56)

The other factorization is
\[ K_\pm = L_N + L_\tau - e^2 \mu^2 + h_D \pm \left( \frac{4}{D^2} h_D^2 - 4e^2 \mu^2 L_\tau \right)^{1/2}, \]  

(57)

and, again, we have \( L_+ L_- = K_+ K_- \) and we have found the couples of \( \Psi DOs \) which enter in the multiplicative anomaly. The necessity of a general formula for computing the multiplicative anomalies is clear. This issue will be discussed in the next section.

### IV. THE WODZICKI RESIDUE AND THE MULTIPLICATIVE ANOMALY FORMULA

For the reader’s convenience, we will review in this section the necessary information concerning the Wodzicki residue [16] (see [14] and the references to Wodzicki quoted therein) that will be used in the rest of the paper. Let us consider a \( D \)-dimensional smooth compact manifold without boundary \( M_D \) and a (classical) \( \Psi DO \), \( A \), acting on sections of vector bundles on \( M_D \). To any classical \( \Psi DO \), \( A \), it corresponds a complete symbol \( A(x, k) = e^{-ikx} A e^{ikx} \), such that, modulo infinitely smoothing operators, one has
\[
(Af)(x) \sim \int_{\mathbb{R}^D} \frac{dk}{(2\pi)^D} \int_{\mathbb{R}^D} dy \, e^{i(x-y)k} A(x, k)f(y).
\]  

(58)

The complete symbol admits an asymptotic expansion for \(|k| \to \infty\), given by the series
\[
A(x, k) \sim \sum_j A_{a-j}(x, k),
\]  

(59)

and the coefficients (their number is infinite) fulfill the homogeneity property \( A_{a-j}(x, tk) = t^{a-j} A_{a-j}(x, k) \), for \( t > 0 \). The number \( a \) is called the order of \( A \). For example, in the case of differential operators, the complete symbol can be obtained by the substitution \( \partial_\mu \to ik_\mu \) and it has a finite number of coefficients and the series stops at \( A_a(x, k) \), being a polynomial in \( k \) of order \( a \).

Now let us introduce the notion of non-commutative residue of a classical \( \Psi DO \) \( A \) of order \( a \). If \( P \) is an elliptic operator of order \( p > a \), according to Wodzicki one can construct the \( \Psi DO \) \( P_A(u) \), \( \text{ord} P_A(u) = p \)
\[
P_A(u) = P + uA,
\]  

(60)

and its related zeta-function
\[
\zeta(s | P_A(u)) = \text{Tr} (P + uA)^{-s}.
\]  

(61)

This zeta-function has a meromorphic analytical continuation, which can be determined by the standard method starting from the short \( t \) asymptotics of \( \text{Tr} \exp -tP_A(u) \). Since \( P_A(u) \) is a \( \Psi DO \) of order \( p \), the heat-kernel asymptotics reads [14].
\[ \text{Tr} \ e^{-tP_A(u)} \simeq \sum_{j=0}^{\infty} \alpha_j(u) t^{\frac{j}{p}} + \sum_{k=1}^{\infty} \beta_k(u) t^k \ln t. \]  

(62)

It should be noted the presence of logarithmic terms in this asymptotic expansion, absent if one is dealing with a differential elliptic operator. Taking the derivative with respect to the parameter \( u \), one also gets the meromorphic structure of \( \lim_{u \to 0} \frac{d}{du} \zeta(s|P_A(u)) \). By definition, the non-commutative residue of \( A \) is

\[ \text{res}(A) = p \text{Res} \left[ \lim_{u \to 0} \frac{d}{du} \zeta(s|P_A(u)) \right]_{s=-1}, \]  

(63)

where \( \text{Res} \) is the usual Cauchy residue. It is possible to show that this definition is independent of the elliptic operator \( P \) and that the trace of the operator \( AP^{-s} \) exists and admits a meromorphic continuation to the whole complex plane, with a simple pole at \( s = 0 \). Its Cauchy residue at \( s = 0 \) is proportional to non-commutative residue of \( A \):

\[ \text{res}(A) = p \text{Res}_{s=0} \text{Tr} \ (AP^{-s}). \]  

(64)

Strictly related to the latter result is the following one, involving the short-\( t \) asymptotic expansion

\[ \text{Tr} \ (Ae^{-tP}) \simeq \sum_j c_j t^{\frac{j}{p}-1} - \frac{\text{res}(A)}{p} \ln t + O(t \ln t). \]  

(65)

Thus, the Wodzicki residue of the \( \Psi \)DO \( A \) can be read off from the above asymptotic expansion, selecting the coefficient proportional to \( \ln t \). Furthermore, it is possible to show that \( \text{res}(A) \) is linear with respect to \( A \) and that it possesses the important property of being the unique trace on the algebra of the classical \( \Psi \)DOs, namely, one has \( \text{res}(AB) = \text{res}(BA) \).

Wodzicki has also obtained a local form of the non-commutative residue, which has the fundamental consequence of characterizing it through a scalar density. This density can be integrated to yield the Wodzicki residue, namely

\[ \text{res}(A) = \int_{M_D} \frac{dx}{(2\pi)^D} \int_{|k|=1} A_{-D}(x,k) dk. \]  

(66)

Here the homogeneous component \( A_{-D}(x,k) \) of order \( -D \) of the complete symbol appears. The above result leads to \( \text{res}(A) = 0 \) when \( A \) is an elliptic differential operator.

Now let us discuss the multiplicative anomaly formula, due to Wodzicki. A more general expression has been derived in \[13\]. Consider two invertible, self-adjoint, elliptic \( \Psi \)DOs, \( A \) and \( B \), on \( M_D \). If we assume that they commute, then the following equality (the Wodzicki multiplicative formula) holds \[14\]

\[ a(A,B) = \frac{\text{res} \left[ (\ln(A^b B^{-a}))^2 \right]}{2ab(a+b)} = a(B,A), \]  

(67)

where \( a > 0 \) and \( b > 0 \) are the orders of \( A \) and \( B \), respectively.
A sketch of the proof is presented in what follows. Recall that if $B$ is an elliptic operator of order $b > q$, according to Wodzicki, one has the following property for the non-commutative residue related to the $\Psi$DO $Q$ (Eq. (64) is just a consequence of it): in a neighborhood of $z = 0$, it holds
\[
\text{Tr} \left( QB^{-z} \right) = \frac{\text{res}(Q)}{z b} + C \frac{\text{res}(Q)}{b} + r_Q(B) + O(z) .
\] (68)

Here $C$ is the Euler-Mascheroni constant and $r_Q(B)$ a coefficient depending also on $B$. As a consequence
\[
\text{Tr} \left[ Q \left( B^{-\gamma_1 z} - B^{-\gamma_2 z} \right) \right] = \frac{(\gamma_2 - \gamma_1)}{z b \gamma_1 \gamma_2} \text{res}(Q) + O(z) ,
\] (69)
in which and $\gamma_1$, $\gamma_2$ are positive real numbers.

Then it follows that, if $\eta$ is a $\Psi$DO of zero order and $B$ a $\Psi$DO of positive order $b$, and $x$ a positive real number, then, in a neighborhood of $s = 0$, one has $\eta^{-xs} = 1 - xs \ln \eta + O(s^2)$ and
\[
s \text{Tr} \left[ \ln \eta \eta^{-xs} \left( B^{-\gamma_1 s} - B^{-\gamma_2 s} \right) \right] = \frac{(\gamma_2 - \gamma_1) \text{res}(\ln \eta)}{\gamma_1 \gamma_2 b} - s x \frac{(\gamma_2 - \gamma_1) \text{res}((\ln \eta)^2)}{\gamma_2 \gamma_1 b} + O(s^2) .
\] (70)

As a consequence,
\[
\lim_{s \to 0} \partial_s \left[ s \text{Tr} \left[ \ln \eta \eta^{-xs} \left( B^{-\gamma_1 s} - B^{-\gamma_2 s} \right) \right] \right] = -x(\gamma_2 - \gamma_1) \frac{\text{res}((\ln \eta)^2)}{\gamma_2 \gamma_1 b} .
\] (71)

Consider now two invertible, commuting, elliptic, self-adjoint operators $A$ and $B$ on $M_D$, with $a$ and $b$ being the orders of $A$ and $B$, respectively. Within the zeta-function definition of the determinants, consider the quantity
\[
F(A, B) = \frac{\det(AB)}{\det(A)(\det B)} = e^{a(A,B)} .
\] (72)

Introduce then the family of $\Psi$DOs
\[
A(x) = \eta^x B^\frac{a}{b} , \quad \eta = A^b B^{-a} ,
\] (73)
and define the function
\[
F(A(x), B) = \frac{\det(A(x)B)}{\det(A(x))(\det B)} .
\] (74)

We get
\[
F(A(0), B) = \frac{\det B^{\frac{2a+b}{b}}}{(\det B^\frac{a}{b})(\det B)} = 1 , \quad F(A(\frac{1}{b})), B) = \frac{\det(AB)}{\det(A)(\det B)} = F(A, B) .
\] (75)

As a consequence, one is led to deal with the following expression for the anomaly
\[ a(A(x), B) = \ln F(A(x), B) = \lim_{s \to 0} \partial_s \left[ \text{Tr} (A(x)B)^{-s} - \text{Tr} A(x)^{-s} - \text{Tr} B^{-s} \right]. \] 

(76)

This quantity has the properties: \( a(A(0), B) = 0 \) and \( a(A(\frac{1}{b}), B) = a(A, B) \).

The next step is to compute the first derivative of \( a(A(x), B) \) with respect to \( x \), the result being

\[ \partial_x a(A(x), B) = \lim_{s \to 0} \partial_s \left[ \ln \eta^{-xs} \left( B^{-s\frac{a+b}{b}} - B^{-s\frac{a}{b}} \right) \right]. \] 

(77)

Making use of Eq. (71), one obtains

\[ \partial_x a(A(x), B) = x \frac{b}{a(a+b)} \text{res}[(\ln \eta)^2]. \] 

(78)

And, finally, performing the integration with respect to \( x \), from 0 to \( 1/b \), one gets Wodzicki’s formula for the multiplicative anomaly, namely

\[ a(A, B) = a(B, A) = \frac{\text{res}[(\ln(A^bB^{-a}))^2]}{2ab(a+b)}. \] 

(79)

It should be noted that \( a(A, B) \) depends on a classical \( \Psi DO \) of zero order. Thus, it is independent on the renormalization scale \( M \) appearing in the path integral.

We conclude this section by observing that the Wodzicki formula is also valid for \( \Psi DOs \) formally non self-adjoint, provided they are complex functions of self-adjoint elliptic operators. An example is given by \( K_\pm \).

V. THE MULTIPLICATIVE ANOMALY FOR THE INTERACTING \( O(2) \) MODEL

In this section we come back to the problem of the computation of the multiplicative anomaly in the model considered in Sec. 3. Putting \( h_D = 0 \), which is proportional to the coupling constant, we also get the results valid in the free case. Strictly speaking, the results of the last section are valid for a compact manifold, but in the case of \( \mathbb{R}^D \), or in the finite temperature case \( S_1 \times \mathbb{R}^{D-1} \), the divergence is trivial, being contained in the spatial volume factor.

Since the multiplicative anomaly depends on the regularization of the ultraviolet divergences (in our approach this is equivalent to the asymptotic behavior of the eigenvalues at infinity and to the related divergence of the functional determinant) it follows that, due to the fact that the Minkowski space-time is ultrastatic \( (g_{00} = -1) \), its dependence on the temperature is simply proportional to \( \beta \), as we will see.

First, let us consider the self-adjoint factorization. Now the order of \( L_\pm \) is 2. Then, the Wodzicki formula gives

\[ a_D(L_+, L_-) = \frac{1}{8} \text{res}[(\ln(L_+L_-^{-1}))^2]. \] 

(80)

There is no ordering problem because we are dealing with commuting operators. We have to construct the complete symbol \( A(x, k) \) of the classical \( \Psi DO \) of zero order \([\ln(L_+L_-^{-1})]^2\). The complete symbol reads
\[ A(x, k) = \left[ \ln \left( k^2 + m^2 + e^2 \mu^2 + h_D + \left( \frac{4}{D^2} h_D^2 + 4 e^2 \mu^2 \left( \bar{k}^2 + m^2 + h_D \right) \right)^{\frac{1}{2}} \right) 
- \ln \left( k^2 + m^2 + e^2 \mu^2 + h_D - \left( \frac{4}{D^2} h_D^2 + 4 e^2 \mu^2 \left( \bar{k}^2 + m^2 + h_D \right) \right)^{\frac{1}{2}} \right) \right]^2, \quad (81) \]

where \( k^2 = \bar{k}^2 + k_+^2 \). Let us denote its asymptotics for large \( |k| \) as

\[ A(x, k) \equiv \sum_j A_{-j}(k). \quad (82) \]

According to Wodzicki, we have to select the component \( A_{-D} \) and use the multiplicative anomaly formula, namely

\[ a_D(L_+, L_-) = \frac{1}{8} \int_{M_D} \frac{dx}{(2\pi)^D} \int_{|k|=1} A_{-D}(k) dk = \frac{1}{8} \beta V_N \int_{|k|=1} A_{-D}(k) dk. \quad (83) \]

For large \( k^2 \) and \( k_+^2 \), due to the \( k^2 \) dependence, it follows that the homogeneous components with odd indices are vanishing. As a consequence, from Eq. (83) one immediately gets the following: for \( D \) odd, the multiplicative anomaly vanishes. This result is consistent with the general theorem contained in [15].

For \( D \) even, the anomaly is present and the asymptotic expansion, from which one can easily read off the even homogeneous components, is

\[ A(x, k) = \frac{1}{(k^2)^2} \left[ 16 e^2 \mu^2 \bar{k}^2 - 32(m^2 + e^2 \mu^2 + h_D) \frac{e^2 \mu^2 \bar{k}^2}{k^2} + h_D^2 + 16 e^2 \mu^2 (m^2 + h_D) \right. 
+ \left. \frac{128}{3} \frac{e^4 \mu^4 (\bar{k}^2)^2}{(k^2)^2} + O \left( \frac{1}{k^2} \right) \right]. \quad (84) \]

Thus, the first non vanishing even components are

\[ A_{-2}(x, k) = \frac{16 e^2 \mu^2 \bar{k}^2}{(k^2)^2}, \]

\[ A_{-4}(x, k) = \frac{1}{(k^2)^2} \left[ -32(m^2 + e^2 \mu^2 + h_D) \frac{e^2 \mu^2 \bar{k}^2}{k^2} + h_D^2 + 16 e^2 \mu^2 (m^2 + h_D) \right. 
+ \left. \frac{128}{3} \frac{e^4 \mu^4 (\bar{k}^2)^2}{(k^2)^2} \right]. \quad (85) \]

It follows that

\[ a_2(L_+, L_-) = \frac{\beta V_1}{2\pi} e^2 \mu^2, \quad (86) \]

and
\begin{align*}
a_4(L_+, L_-) &= \frac{\beta V_3}{8\pi^2} \left[ e^2 \mu^2 \left( \frac{e^2 \mu^2}{3} - m^2 - h_D \right) + \frac{h_D^2}{8} \right]. \tag{87}
\end{align*}

If \( e = 0 \), we recover the results of Ref. \[17\], in particular the absence of the multiplicative anomaly for \( D = 2 \). In the free case \( h_D = 0 \) and we have

\begin{align*}
a_4(L_+, L_-) &= \frac{\beta V_3}{8\pi^2} \left[ e^2 \mu^2 \left( \frac{e^2 \mu^2}{3} - m^2 \right) \right]. \tag{88}
\end{align*}

Let us consider the other factorization. The complete symbol reads now

\begin{align*}
A(x, k) &= \left[ \ln \left( k^2 + m^2 - e^2 \mu^2 + h_D + \left( \frac{4}{D^2} h_D^2 - 4 e^2 \mu^2 k^2_\tau \right)^{\frac{1}{2}} \right) \right. \\
&\quad - \left. \ln \left( k^2 + m^2 - e^2 \mu^2 + h_D - \left( \frac{4}{D^2} h_D^2 - 4 e^2 \mu^2 k^2_\tau \right)^{\frac{1}{2}} \right) \right]^2, \tag{89}
\end{align*}

For large \( k^2 \) and \( k^2_\tau \), due to the \( k^2 \) dependence, it follows again that the homogeneous components with odd indices are vanishing and, for \( D \) odd, the multiplicative anomaly is absent as in the other case.

For \( D \) even, the first non vanishing even components are

\begin{align*}
A_{-2}(x, k) &= -\frac{16 e^2 \mu^2 k^2_\tau}{(k^2)^2}, \\
A_{-4}(x, k) &= \frac{1}{(k^2)^2} \left[ 32 (m^2 + h_D - e^2 \mu^2) \frac{e^2 \mu^2 k^2_\tau}{k^2} + h_D^2 + \frac{128 e^4 \mu^4 (k^2_\tau)^2}{3} \right]. \tag{90}
\end{align*}

A straightforward calculation gives

\begin{align*}
a_2(K_+, K_-) &= -\frac{\beta V_4}{2\pi} e^2 \mu^2 \quad \tag{91}
\end{align*}

and

\begin{align*}
a_4(K_+, K_-) &= \frac{\beta V_3}{8\pi^2} \left[ e^2 \mu^2 (m^2 + h_D - \frac{e^2 \mu^2}{3}) + \frac{h_D^2}{8} \right]. \tag{92}
\end{align*}

Again, for \( e \rightarrow 0 \) we get the result of \[17\].

Some remarks are in order. According to the results of Sec. 2 and this section, it seems quite natural to make the conjecture that, in the free case and for \( D \) even, one has

\begin{align*}
a_D(K_+, K_-) = -a_D(L_+, L_-). \quad \tag{93}
\end{align*}

We have proved this conjecture for \( D = 2 \) and \( D = 4 \), with an explicit computation, namely

\begin{align*}
a_2(K_+, K_-) &= -\frac{\beta V_4}{2\pi} e^2 \mu^2, \quad a_4(K_+, K_-) &= \frac{\beta V_3}{8\pi^2} \left[ e^2 \mu^2 (m^2 - \frac{e^2 \mu^2}{3}) \right]. \tag{94}
\end{align*}
Thus, in the free case, we also have
\[
\ln Z_\beta(L_+, L_-) = \ln Z_\beta(K_+, K_-),
\] (95)

On the other hand, assuming Eq. (93) and Eq. (93) one obtains a general expression for the multiplicative anomaly, namely
\[
a_D(K_+, K_-) = -a_D(L_+, L_-) = -\beta V_N \mathcal{E}_D(m, \mu),
\] (96)

where \(\mathcal{E}_D(m, \mu)\) is given by Eq. (45). For instance, for \(D = 6\), we have
\[
a_6(K_+, K_-) = \frac{\beta V_5}{16\pi^3} \left[ e^2 \mu^2 \left( -\frac{1}{4} m^4 + \frac{e^2 \mu^2 m^2}{6} - \frac{2e^4 \mu^4}{45} \right) \right].
\] (97)

As a consequence, in the free case, one obtains
\[
\ln Z_{\beta,\mu}(K_+, K_-) = -\beta V_N \mathcal{E}_V(m, M) + V_N S(\beta, \mu) + \frac{1}{2}\beta V_N \mathcal{E}_D(m, \mu),
\] (98)

where
\[
\mathcal{E}_V = \frac{1}{V_N} E_V, \quad S(\beta, \mu) = \frac{1}{V_N} S(\beta, \mu).
\] (99)

VI. THE FREE CHARGED BOSONIC MODEL AT FINITE TEMPERATURE

The presence of this new anomaly term in the grand canonical partition requires a re-analysis of the statistical mechanical behaviour. This is better accomplished by the use of the effective potential formalism at finite temperature and charge. Part of the material of this section has appeared before (see for example, [5] and the references quoted therein). We shall keep working in arbitrary spacetime dimensions, returning to the physically important 4D case at the end of this section.

In order to study the effective action in the presence of a non vanishing mean field, it is necessary to introduce the one-loop grand partition function in the presence of external (constant) sources \(J_i\) and the chemical potential \(\mu\). This is done with the addition of the coupling term \(\int dx^D (J_i \phi_i)\) in the action. As the action is quadratic in the field degrees of freedom, a trivial Gaussian integration gives
\[
Z_\beta(\mu, J_i) = \exp \left[ \beta V_N \frac{J_i J_i}{2(m^2 - e^2 \mu^2)} \right] Z_\beta(\mu),
\] (100)

where \(V_N\) is the spatial volume. In terms of generating functionals per unit volume
\[
V_\beta(\mu, J_i) = -\frac{1}{\beta V_N} \ln Z_\beta(\mu, J_i), \quad V_6(\beta, \mu) = -\frac{1}{\beta V_N} \ln Z_\beta(\mu),
\] (101)
we have
\[ V(\beta, \mu, J_i) = V_0(\beta, \mu) - \frac{J_i J_i}{2(m^2 - e^2 \mu^2)}. \]  

(102)

which is nothing but the thermodynamics potential of the non interacting boson gas. The effective potential in presence of external sources is the Legendre transform of the generating functional per unit volume (that is to say, the thermodynamics free energy density), and reads

\[ F(\beta, \rho, \Phi_i) = V(\beta, \mu, J_i) + \mu \rho + J_i \Phi_i, \]  

(103)

where \( \mu \) and \( J_i \) have to be expressed as functions of the charge density, \( \rho \), and the mean field \( \Phi_i \), by solving the defining equations

\[ \rho = -\frac{\partial V_\beta}{\partial \mu} = \frac{\langle Q \rangle}{V_N}, \quad \Phi_i = -\frac{\partial V_\beta}{\partial J_i} = \frac{1}{V_N} \int dX^N < \phi_i >, \]  

(104)

where \( V_\beta \) is a shortened notation for \( V(\beta, \mu, J_i) \). As a consequence, with \( x = |\Phi| \), eliminating the external sources, one has

\[ F(\beta, \rho, x) = \rho - \frac{\partial V_0(\beta, \mu)}{\partial \mu} + \frac{1}{2} \left( m^2 + e^2 \mu^2 \right) x^2, \]  

(105)

\[ \rho = -\frac{\partial V_0(\beta, \mu)}{\partial \mu} + e^2 \mu x^2, \]  

(106)

where \( \mu \) is assumed to be a function \( \mu = \mu(\beta, \rho, x) \), obtained by solving Eq. (106) in the unknown \( \mu \).

The possible equilibrium states correspond to minima of the effective potential with respect to \( x \), at fixed temperature and charge density. Making use of Eq. (103) and Eq. (106), one arrives at

\[ \frac{\partial F}{\partial x} = x(m^2 - e^2 \mu^2), \quad \frac{\partial^2 F}{\partial x^2} = (m^2 - e^2 \mu^2) - 2x e^2 \mu \frac{\partial \mu}{\partial x}. \]  

(107)

Requiring the first derivative to vanish gives either:

(i) The unbroken phase, with solution \( x = 0 \). From the expression of the second derivative this will be a minimum provided \( e \mu < m \), and the related free energy density will be

\[ F_\beta = \min_x F(\beta, \rho, x) = V_0(\beta, \mu) - \mu \frac{\partial V_0(\beta, \mu)}{\partial \mu} = V_0(\beta, \mu) + \mu \rho, \]  

(108)

\[ \rho = -\frac{\partial V_0(\beta, \mu)}{\partial \mu}. \]  

(ii) The symmetry breaking solution

\[ e \mu = \pm m, \]  

(109)

here derived as an extremal property of the effective potential. In this case \( |\Phi| > 0 \), and assuming \( e \mu = m \), one has
\[ F_\beta = \min_x F(\beta, \rho, x) = V_0(\beta, e\mu = m) + \frac{m}{e}\rho, \]

\[ emx^2 = \rho + \left. \frac{\partial V_0(\beta, \mu)}{\partial \mu} \right|_{e\mu=m}. \]  

(110)

In this symmetry breaking phase, one has the phenomenon of Bose-Einstein condensation (BEC). From the last equation and the fact that \( V_0(\beta, \mu) \) is a monotonically decreasing function of \( \beta \), it follows that \( x^2 = \Phi_i \Phi_i \) is non-vanishing when \( \beta > \beta_c \), the inverse of the critical temperature of the BEC transition, as given by the condition

\[ \rho = -\left. \frac{\partial V_0(\beta_c, \mu)}{\partial \mu} \right|_{e\mu=m}, \]  

(111)

in agreement with the recent criterium for the relativistic Bose-Einstein condensation proposed in [20,21]. For \( \beta \leq \beta_c \), on the other hand, \( |\Phi| = 0 \) and the \( U(1) \)-symmetry is restored.

Now in the free case we know \( V_\beta \). In the unbroken phase, \( x = 0, e|\mu| < m, \beta < \beta_c \), and including the anomaly term, Eq. (45), with \( 2Q = D \), one has

\[ F_\beta = E_V - \frac{4m^2}{(2\pi)^2} \sum_{r=1}^{\infty} \left( \frac{1}{\beta r} \right) \frac{D}{2} - 1 \cosh(rp\beta)K_{p}(rm\beta) + \mu\rho + \frac{1}{2} \sum_{r=1}^{Q} c_{Q,r} (e\mu)^{2r}m^{2Q-2r}, \]

\[ \rho = \frac{4em^2}{(2\pi)^2} \sum_{r=1}^{\infty} \left( \frac{1}{\beta r} \right) \frac{D}{2} - 1 \sinh(rp\beta)K_{p}(rm\beta) - e \sum_{r=1}^{Q} c_{Q,r} (e\mu)^{2r-1}m^{2Q-2r}. \]  

(112)

The last term in both equations accounts for the anomaly.

In the broken phase, \( e|\mu| = m, \beta > \beta_c \), one obtains

\[ F_\beta = E_V - \frac{4m^2}{(2\pi)^2} \sum_{r=1}^{\infty} \left( \frac{1}{\beta r} \right) \frac{D}{2} - 1 \cosh(rp\beta)K_{p}(rm\beta) + \frac{1}{2} \sum_{r=1}^{Q} c_{Q,r} m^{2Q} + \frac{m}{e}\rho, \]

\[ \rho = \frac{4em^2}{(2\pi)^2} \sum_{r=1}^{\infty} \left( \frac{1}{\beta r} \right) \frac{D}{2} - 1 \sinh(rp\beta)K_{p}(rm\beta) - e \sum_{r=1}^{Q} rc_{Q,r} m^{2Q-1} + emx^2. \]  

(113)

In this case, if we formally take the zero temperature limit \( \beta \to \infty \), we get

\[ \rho \to -e \sum_{r=1}^{Q} rc_{Q,r} m^{2Q-1} + emx^2. \]  

(114)

Now observe that the anomaly contribution leads to some problems, since it does not have a definite sign. For example, for any fixed \( \rho = 0, x^2 \) in (117) could be made negative when the anomaly contribution is negative. We thus arrive at a contradiction, as happens, for example, for \( D = 6 \).

At this point, however, we should notice that our discussion has involved “unrenormalized” quantities only, regularized by using the zeta-function procedure. In particular, the charge operator has not been properly defined, and the above result indicates that this operator was not normal ordered relative to the Minkowski vacuum. Now the charge operator
appears in the Hamiltonian multiplied by $\mu$. Therefore any normal ordering ambiguity in
the charge operator gives rise to an ambiguity in the effective action, which must be a linear
homogeneous function of $\mu$, namely $K\mu$, with a freely disposable constant $K$. We have
therefore the freedom to define the generating functional $V(\beta, \mu, J_i)$ up to the linear term
$K\mu$. This will not change the effective action, the Legendre transform being unaffected by
linear terms. We then introduce the renormalized generating functional

$$ V^R = V + \mu K = \mathcal{E}_V + \frac{1}{2} \mathcal{E}_D(m, \mu) - \frac{S(\beta, \mu)}{\beta} - \frac{J_i J_i}{2(m^2 - \epsilon^2 \mu^2)} + \mu K, $$

(115)

where $\mathcal{E}_D(m, \mu)$ is the anomaly, given by Eq. (13) and $S(\beta, \mu)$ was defined in Eq. (23). So
we have also a finite renormalization of the charge density:

$$ \rho^R = -\frac{\partial V^R}{\partial \mu} + e\mu x^2 = \rho - K. $$

(116)

Since the effective potential is the same, the extremals are unchanged and we may rewrite
the free energy in the symmetry breaking phase as

$$ F_\beta = \mathcal{E}_V - \frac{1}{\beta} S(\beta, e\mu = m) + \frac{m}{e} \rho^R + \frac{m}{e} K + \frac{1}{2} \sum_{r=1}^Q c_{Q,r} m^{2Q}, $$

$$ \rho^R = -\frac{1}{\beta} \frac{\partial S(\beta, \mu)}{\partial \mu} |_{e\mu = m} - K - \frac{e}{m} \sum_{r=1}^Q r c_{Q,r} m^{2Q} + em x^2. $$

(117)

On the other hand, in the unbroken symmetric phase we have,

$$ F_\beta = \mathcal{E}_V - \frac{1}{\beta} S(\beta, \mu) + \mu \rho^R + \mu K + \frac{1}{2} \sum_{r=1}^Q c_{Q,r} (e\mu)^{2r} m^{2Q-2r}, $$

$$ \rho^R = -\frac{1}{\beta} \frac{\partial S(\beta, \mu)}{\partial \mu} - K - e \sum_{r=1}^Q r c_{Q,r} (e\mu)^{2r-1} m^{2Q-2r}. $$

(118)

The constant $K$ can now be fixed by imposing normalization conditions. By doing this
we see how the spacetime dimension affects renormalization and how special the case $D = 4$
is. We do so just by demanding that, at zero temperature and charge density, the symmetry be unbroken, which sounds as a very natural normalization condition. Using (117), this fixes $K$ to be

$$ K = -e \sum_{r=1}^Q r c_{Q,r} m^{2Q-1}. $$

(119)

As a result, in the broken phase ($|e\mu| = m$) one obtains

$$ F_\beta = \mathcal{E}_V - \frac{1}{\beta} S(\beta, e\mu = m) + \frac{m}{e} \rho^R + \frac{1}{2} \sum_{r=1}^Q c_{Q,r} (1 - 2r) m^{2Q}, $$

$$ \rho^R = -\frac{1}{\beta} \frac{\partial S(\beta, \mu)}{\partial \mu} |_{e\mu = m} + em x^2, $$

(120)
while, in the symmetric phase, one gets
\[ F_\beta = \mathcal{E}_V - \frac{1}{\beta} S(\beta, \mu) + \mu \rho^R - e \mu \sum_{r=1}^Q rC_{Q,r} m^{D-1} + \frac{1}{2} \sum_{r=1}^Q c_{Q,r} (e \mu)^{2r} m^{2Q-2r}, \]
\[ \rho^R = -\frac{1}{\beta} \frac{\partial S(\beta, \mu)}{\partial \mu} + e \sum_{r=1}^Q rC_{Q,r} m^{2Q-1} - e \sum_{r=1}^Q c_{Q,r} (e \mu)^{2r-1} m^{2Q-2r}. \] (121)

Note that in this symmetric phase, one can now take the limit \( e \to 0 \) to reach the usual expression of the free energy density for an uncharged boson gas. For general \( D = 2Q \), however, the free energy receives contributions from the anomaly term in both phases. On the other hand, with the above charge renormalization, the critical temperature is given implicitly by (choosing \( \mu > 0 \))
\[ \rho^R + \frac{1}{\beta} \frac{\partial S(\beta_c, \mu)}{\partial \mu} \bigg|_{e \mu = m} = 0. \] (122)

which is the usual condition for the critical temperature [20,21], and it remains unaffected by the anomaly. It is important to recognize that the normalization condition just discussed is not the only one that is possible. Another physically natural condition would be to demand that at zero coupling, the free energy reduces to the free energy of uncharged bosons. This then fixes \( K \) to be
\[ K = -\frac{e}{2} \sum_{r=1}^Q c_{Q,r} m^{2Q-1}. \] (123)

Now it is the free energy in the broken phase that remains unaffected, while \( \beta_c \) changes.

Let us consider now the physically important case \( D = 4 \). Then, from the expression (17) for the anomaly, we have in addition the remarkable identity
\[ \sum_{r=1}^2 rC_{2,r} = \frac{1}{2} \sum_{r=1}^2 C_{2,r} = \frac{1}{24\pi^2}. \] (124)

This shows that the two normalization conditions just discussed are actually equivalent and fix a unique value \( K = -\frac{em^3}{24\pi^2} \). As a result, in the symmetry breaking phase and for \( D = 4 \), we obtain
\[ F_\beta = \mathcal{E}_V - \frac{1}{\beta} S(\beta, m) + \frac{m}{e} \rho^R, \]
\[ ex^2 = \frac{1}{m} \left( \rho^R + \frac{1}{\beta} \frac{\partial S(\beta, \mu)}{\partial \mu} \bigg|_{e \mu = m} \right). \] (125)

which means that the broken phase is totally unaffected by the anomaly. On the other hand, in the symmetric phase \( \beta < \beta_c \), \( x = 0 \) and \( e \mu < m \), the result is
\[ F_\beta = \mathcal{E}_V - \frac{1}{\beta} S(\beta, \mu) + \mu \rho^R - \frac{\mu m^3}{12\pi^2} + \frac{1}{8\pi^2} \left( e^2 \mu^2 m^2 - \frac{1}{3} e^4 \mu^4 \right), \]
\[ \rho^R = -\frac{1}{\beta} \frac{\partial S(\beta, \mu)}{\partial \mu} - \frac{1}{8\pi^2} \left( 2e^2 \mu m^2 - \frac{4}{3} e^4 \mu^3 \right) + \frac{em^3}{12\pi^2}. \] (126)
We observe that the anomaly term (the term in square brackets in (126)) is present in the free energy above $T_C$. Moreover, it is again vanishing as $e \to 0$, and the correct expression of the free energy density for the uncharged boson gas is recovered. Although the multiplicative anomaly term itself may be not negligible, as a numerical study of it shows, the renormalization used in order to preserve the vacuum structure has rendered the anomaly term quite harmless near $T_C$ and below. On the other hand, at ultra relativistic temperatures, $T \gg m$, it is negligible in comparison with the Planckian $T^4$-term, since the anomaly term increases with mass precisely as $m^4$ (however this holds whenever the charge density is much less than $T^3 e^{-6}$, in order to neglect the Coulomb interaction). Our conclusion is that it gives a relevant correction at intermediate temperatures, say of the order of $T \simeq m$, which are relativistic for any known massive elementary particle.

VII. CONCLUDING REMARKS

In this paper some deep results of the zeta-function regularization procedure have been employed in order to study rigorously the one-loop effective potential for a fixed charged self-interacting scalar field at finite temperature. The chemical potential has been introduced according to the path integral approach [1,5] making use of method I, as discussed recently in [7]. Method I leads to the presence of a multiplicative anomaly term, whose existence had been completely overlooked in the literature. Using powerful mathematical expressions, this anomaly term can be computed exactly. Its explicit form has been obtained here for $D = 2$ and $D = 4$ (while a simple program yields it for any desired value of $D$.) We have shown it to be vanishing for $D$ odd and also its fundamental importance in getting factorization invariance (in the operational sense) of the regularized one-loop effective action. On the other hand, assuming factorization invariance, we have obtained, in the free case, a general expression for the multiplicative anomaly, valid for any even $D$.

The existence of this new contribution has led us to revisit and discuss the non-interacting case in detail. In particular, we have reexamined the spontaneous symmetry breaking issue and the related relativistic Bose-Einstein condensation phenomenon. A renormalization of the charge density has been introduced in arbitrary even dimension and has led to the usual expression for the critical temperature. In particular, in the broken phase, we have shown that only in the physically important case $D = 4$ the new contribution due to the multiplicative anomaly may be absorbed in the charge renormalization process. However, in the symmetric unbroken phase, the multiplicative anomaly term gives a non vanishing contribution, which has been overlooked in previous investigations. Although it is non leading in the ultra high temperature regime, nevertheless it can give relevant correction at intermediate temperatures of order $T \simeq m$, $m$ being the mass of the charged boson.

In the interacting case, even in at the one-loop approximation, it is quite difficult — within the first factorization — to deal with the closed expression for the zeta-functions. In fact, for $D = 4$, we have

$$\zeta(s|L_{\pm}) = \frac{\beta \Gamma(s - \frac{1}{2})}{2 \sqrt{\pi} \Gamma(s)} \zeta(s - \frac{1}{2}|L_3 + e^2 \mu^2 + h \pm \sqrt{\frac{h^2}{4} + 4e^2 \mu^2 (L_3 + h)}}$$
\[ -2s \operatorname{Tr} \ln \left( 1 - \exp \beta \sqrt{L_3 + e^2 \mu^2 + h \pm \sqrt{\frac{h^2}{4} + 4e^2 \mu^2 (L_3 + h)}} \right) + O(s^2), \quad (127) \]

while, within the second factorization, one has difficulties with the sum over the Matsubara frequencies \( \omega_n \). For example, again for \( D = 4 \), one should deal with

\[ \zeta(s|K_{\pm}) = \frac{V_3}{\Gamma(s)} \int_0^\infty dt s^{1-\frac{t}{4\pi}} \frac{e^{-t(m^2+h)}}{\sqrt{t}} \sum_n \exp \left[ -t \left( \frac{\omega_n^2 - e^2 \mu^2 \pm \sqrt{\frac{h^2}{4} - 4e^2 \mu^2 \omega_n^2}}{4} \right) \right]. \quad (128) \]

As a consequence—as far as the factorization invariance of the partition function is concerned—the relevance of the multiplicative anomaly (evaluated in Sec. 4) appears manifest, in the interacting case. The issue will require further investigation.

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**VIII. APPENDIX**

In this Appendix we will show that the multiplicative anomaly is present also in a large class of regularizations of functional determinants appearing in the one-loop effective action. They are often used in the literature and can be called “generalized proper-time regularizations”, since the prototype among them was introduced by Schwinger [18]. We will show that the presence of the multiplicative anomaly stems from the obvious—but crucial—fact that the finite part of all these regularization contains, unavoidably, the zeta-function regularization contribution \( \zeta'(0|L) \).

If we work in the Euclidean formalism, the class of regularization we shall be dealing with is defined by [12,19]

\[
\ln \det \left( \frac{L}{M^2} \right) (g_\varepsilon(t)) = - \int_0^\infty dt \, t^{-1} g_\varepsilon(t) \operatorname{Tr} e^{-t \frac{L}{M^2}},
\]

where \( g_\varepsilon(t) \) is a suitable regularizing function, necessary in order to control, for \( \varepsilon > 0 \), the (ultraviolet) divergences for small \( t \) in the integral and \( M^2 \) is the mass renormalization parameter which renders \( t \) adimensional. \( L \) is an elliptic operator. We note that the zeta-function regularization and the related Dowker-Critchley one [3] belong to this class. They correspond to the choices

\[
g_\varepsilon^1(t) = \frac{d}{d\varepsilon} \left( \frac{t^\varepsilon}{\Gamma(\varepsilon)} \right), \quad g_\varepsilon^2(t) = t^\varepsilon.
\]

(130)
As a consequence, the first regularization, as is well known, gives a finite result \( \zeta'(\varepsilon|L) \), after an analytical continuation in \( \varepsilon \), when the parameter goes to zero, while the second one gives (\( C \) being again the Euler-Mascheroni constant)

\[
\ln \det \left( \frac{L}{M^2} \right) (g^2_\varepsilon(t)) = -\Gamma(\varepsilon)\zeta(\varepsilon|L) - \frac{1}{\varepsilon} \zeta(0|L) + C\zeta(0|L) + O(\varepsilon). \tag{131}
\]

In this case, besides the divergent contribution, the finite part contains the zeta-function regularization result.

Let us consider another class of proper-time regularizations such that the functions \( g_\varepsilon(t) \) admit the Mellin transform

\[
\hat{g}_\varepsilon(s) = \int_0^\infty dt \: t^{s-1} g_\varepsilon(t). \tag{132}
\]

Three popular examples are the ultraviolet cutoff regularization \( g^3_\varepsilon(t) = \theta(t - \varepsilon) \)

\[
\ln \det \left( \frac{L}{M^2} \right) (g^3_\varepsilon(t)) = -\int_0^\infty dt \: t^{-1} \text{Tr} \: e^{-t \frac{L}{M^2}}, \quad \hat{g}^3_\varepsilon(s) = -\frac{1}{s} \varepsilon^s, \tag{133}
\]

the point-splitting regularization \( g^4_\varepsilon(t) = e^{-\frac{t}{\varepsilon}} \)

\[
\ln \det \left( \frac{L}{M^2} \right) (g^4_\varepsilon(t)) = -\int_0^\infty dt \: t^{-1} e^{-\frac{t}{\varepsilon}} \text{Tr} \: e^{-t \frac{L}{M^2}}, \quad \hat{g}^4_\varepsilon(s) = \Gamma(-s)\varepsilon^s, \tag{134}
\]

and Pauli-Villars regularization, which in our formalism may be expressed as

\[
g^5_\varepsilon(t) = \left(1 - e^{-t\Phi} \right)^{D-1}. \tag{135}
\]

Thus

\[
\ln \det \left( \frac{L}{M^2} \right) (g^5_\varepsilon(t)) = -\int_0^\infty dt \: t^{-1} \left(1 - e^{-t\Phi} \right)^{D-1} \text{Tr} \: e^{-t \frac{L}{M^2}}, \tag{136}
\]

with

\[
\hat{g}^5_\varepsilon(s) = \frac{1}{s} + \sum_j c_j(\varepsilon) \frac{1}{D - 1 + j + s}, \tag{137}
\]

where \( c_j(\varepsilon) \) are constants which diverge in the limit \( \varepsilon \to 0 \). Making use of the Parceval-Mellin identity, we may rewrite Eq. (129) in terms of a complex integral involving the Mellin transform of \( \text{Tr} \: e^{-t \frac{L}{M^2}} \), namely \( \zeta(z|L)M^{2z} \) and \( g_\varepsilon(t) \), i.e.

\[
\ln \det \left( \frac{L}{M^2} \right) (g_\varepsilon(t)) = -\frac{1}{2\pi i} \int_{\text{Re}z > D/2} dz \: \Gamma(z) \zeta(z|L)M^{2z} \hat{g}_\varepsilon(-z). \tag{138}
\]

Now the meromorphic properties of \( \zeta(z|L) \) are known and we have

\[
\zeta(z|L) = \frac{1}{\Gamma(z)} \left( \sum_r \frac{A_r}{z + r - D/2} + J(z) \right), \tag{139}
\]

23
where $A_r$ are the Seeley-De Witt coefficients (computable) and $J(z)$ is the analytical part (normally unknown). We also have $\zeta(0|L) = A_{D/2}$, with $A_{D/2} = 0$ when $D$ is odd, since we are working in manifolds without boundary. Shifting the vertical contour to the left, assuming that the Mellin transform $\hat{g}_\varepsilon(-z)$ is regular for $\Re z > 0$ and has a simple pole at $z = 0$ given, with residue 1 (see, for example, Eqs. (133) and (134)), the residue theorem yields

$$\ln \det \left( \frac{L}{M^2} \right) (g_\varepsilon(t)) = -\zeta'(0|L) + \zeta(0|L) \left( \ln M^2 + C \right)$$

$$- \sum_{r \neq D/2} A_r \hat{g}_\varepsilon(r - D/2) M^{D-z} - \zeta(0|L) \ln \varepsilon + O(\varepsilon).$$

(140)

As a consequence, the finite part, coming from the singularity at $z = 0$ contains $\zeta'(0|L)$ and the multiplicative anomaly is also present in this large class of regularizations. The sum over $r$ contains the ultraviolet divergences, controlled by $\varepsilon$. In the one-loop approximation, these divergent terms are removed by the corresponding counterterms.
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