Approximation of the Filter Equation for Multiple Timescale, Correlated, Nonlinear Systems

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Abstract

This paper considers the approximation of the continuous time filtering equation for the case of a multiple timescale (slow-intermediate, and fast scales) that may have correlation between the slow-intermediate process and the observation process. The signal process is considered fully coupled, taking values in \(\mathbb{R}^m \times \mathbb{R}^n\) and without periodicity assumptions on coefficients. It is proved that in the weak topology, the solution of the filtering equation converges in probability to a solution of a lower dimensional averaged filtering equation in the limit of large timescale separation. The method of proof uses the perturbed test function approach (method of corrector) to handle the intermediate timescale in showing tightness and characterization of limits. The correctors are solutions of Poisson equations.

1 Introduction

The aim of this paper is to prove a convergence result for the continuous time filtering equation of a multiple timescale and correlated nonlinear system to a lower dimensional filtering equation. Specifically, consider the coupled system of stochastic differential equations (SDEs),

\[
\begin{align*}
    dX_t^\epsilon &= \left[b(X_t^\epsilon, Z_t^\epsilon) + \frac{1}{\epsilon}b_I(X_t^\epsilon, Z_t^\epsilon)\right]dt + \sigma(X_t^\epsilon, Z_t^\epsilon)dW_t, \\
    dZ_t^\epsilon &= \frac{1}{\epsilon^2}f(X_t^\epsilon, Z_t^\epsilon)dt + \frac{1}{\epsilon}g(X_t^\epsilon, Z_t^\epsilon)dV_t,
\end{align*}
\]

(1.1)

and denote the infinitesimal generator of \((X^\epsilon, Z^\epsilon)\) as \(G^\epsilon\). \((X^\epsilon, Z^\epsilon)\) is known as the signal process and \(\epsilon \in (0,1)\) is a timescale parameter such that \(Z^\epsilon\) is a fast process and \(X^\epsilon\) is a slow process. Note that even the equation for \(X^\epsilon\) possesses an intermediate timescale due to the \(\frac{1}{\epsilon^2}b_I\) drift coefficient. In filtering theory, we consider the signal process to be non-observable, and instead have indirect measurements of \((X^\epsilon, Z^\epsilon)\) via the noisy observation process

\[
dY_t^\epsilon = h(X_t^\epsilon, Z_t^\epsilon)dt + \alpha dW_t + \gamma dU_t.
\]

With \(W, V, U\) independent Brownian motions, \(\alpha \neq 0\) indicates correlation between the slow signal \(X^\epsilon\) and the observation process \(Y^\epsilon\). The goal in filtering theory is then to calculate the conditional distribution of \((X^\epsilon, Z^\epsilon)\) given the observation history generated from \(Y^\epsilon\), which we denote by \(\pi^\epsilon\). At each time \(t > 0\), \(\pi_t^\epsilon\) is a random probability measure on the space \(\mathbb{R}^m \times \mathbb{R}^n\) and acts on test functions \(\varphi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}\) by integration \(\pi_t^\epsilon(\varphi) = \int \varphi(x, z)\pi_t^\epsilon(dx, dz)\).

The motivating question of this paper then comes from the known result that if for every fixed \(x\), the solution \(Z^x\) of

\[
dZ_t^x = f(x, Z_t^x)dt + g(x, Z_t^x)dV_t,
\]

...
is ergodic with stationary distribution \( \mu_\infty(x) \), then under appropriate assumptions, the process \( X^\epsilon \) converges in distribution to a Markov process \( X^0 \) with infinitesimal generator \( G^0 \) in the limit as \( \epsilon \to 0 \) \([PSV76; PV03; KY05]\). Therefore, if we are only interested in statistics of \( X^\epsilon \) (i.e. estimation of test functions \( \varphi: \mathbb{R}^m \to \mathbb{R} \)), then it would be computationally advantageous to know if \( \pi^{\epsilon, x} \to \pi^0 \) converges weakly to a lower dimensional filtering equation; \( \pi^0 \) being a random probability measure for each time \( t \) on \( \mathbb{R}^m \) and \( \pi^{\epsilon, x} \) being the \( x \)-marginal of \( \pi^\epsilon \).

Filtering theory has widespread applications in many fields including various disciplines of engineering for decision and control systems, the geosciences, weather and climate prediction. In many of these fields, it is not uncommon to have physics based models with multiple timescales as seen in Eq. 1.1, and also have the case were estimation of the slow process is solely of interest; for example the estimation of the ocean temperature, which is necessary for climate prediction, but the ocean model may also be coupled to a fast atmospheric model. Knowing that mathematically \( \pi^{\epsilon, x} \to \pi^0 \) in the limit as \( \epsilon \to 0 \), enables practitioners to devise more efficient methods for estimation of the slow process without great loss of accuracy (see for instance \([PNY11; KH12; BH14; Yeo+20]\)).

There are several papers providing results for \( \pi^{\epsilon, x} \to \pi^0 \) (or the associated unnormalized conditional measure or density versions) on variations of the multiple timescale filtering problem. In \([PSN10]\), \((X^\epsilon, Z^\epsilon)\) is a two dimensional process with no drift in the fast component, no intermediate scale, and no correlation. The authors made use of a representation of the slow component by a time-changed Brownian motion under a suitable measure to yield weak convergence of the filter. Homogenization of the nonlinear filter was studied in \([BB86]\) and \([Ich04]\) by way of asymptotic analysis on a dual representation of the nonlinear filtering equation. In these papers, the coefficients of the signal processes are assumed to be periodic. The approach in \([Ich04]\) is novel as the first application of backward stochastic differential equations for homogenization of Zakai-type stochastic partial differential equations (SPDEs).

Convergence of the filter for a random ordinary differential equation with intermediate timescale and perturbed by a fast Markov process was investigated in \([LH03]\). A two timescale problem with correlation between the slow process and observation process, but where the slow dispersion coefficient does not depend on the fast process, is investigated in \([Qia19]\). The main result is that the filter converges in \( L^1 \) sense to the lower dimensional filter. An energy method approach is used in \([ZR19]\) to show that the probability density of the reduced nonlinear filtering problem approximates the original problem when the signal process has constant diffusion coefficients, periodic drift coefficients and the observation process is only dependent on the slow process.

Convergence of the nonlinear filter is shown in a very general setting in \([KLS97]\), based on convergence in total variation distance of the law of \((X^\epsilon, Y^\epsilon)\). In the examples of \([KLS97]\), the diffusion coefficient is not allowed to depend on the fast component.

In contrast to other papers on the convergence of the nonlinear filter for the multiple timescale problem, Imkeller et al. \([Imk+13]\) showed a quantitative rate of convergence of \( \epsilon \) for the system in Eq. 1.1, but without intermediate timescale nor correlation of the slow process with the observation process. This is accomplished using a suitable asymptotic expansion of the dual of the Zakai equation and then harnessing a probabilistic representation of the SPDEs in terms of backward doubly stochastic differential equations. This result is then extended to the case of correlation between the slow signal and observation process in \([BNP20]\), with the same rate of convergence.

We lastly mention the work of nonlinear filter approximation given in \([Kus90, Chapter 6]\), which is most similar to the approach used in this paper. In \([Kus90, Chapter 6]\), a two timescale jump-diffusion process is considered, but with no correlation between signal and observation process. The difference of the actual unnormalized conditional measure and the reduced conditional measure converges to zero in distribution. Standard results then yield convergence in probability of the fixed time marginals. The method of proof is by averaging the coefficients of the SDEs for the unnormalized filters and showing that the limits of both filters satisfy the same SDE, which possess a unique solution.

In this paper, we address the broader multiple timescale correlation filtering problem and therefore have to modify the approach by \([Kus90, Chapter 6]\) to handle the intermediate scaling term and the correlation. This is the first paper that the authors are aware of that handles the problem of filter convergence for correlated slow-fast systems with intermediate timescale forcing. For this we make use of the perturbed test function approach where the correctors are solutions of Poisson equations. We make use of the sharp results on existence, regularity and growth of the transition densities and semigroups associated with the process.
$Z^x$ and Poisson equations for the corrector terms in [PV03]. The main result of the paper is the following:

**Theorem (Main Result)**

Recall that $\pi^{x,\epsilon}$ is the $x$-marginal of the conditional distribution $\pi^x$ and $\pi^0$ is the conditional distribution for the averaged filter equation (see for instance Eqs. 2.3, 2.6, and 2.8). Then under the assumptions stated in Theorem 2.1, $\pi^{x,\epsilon} \to \pi^0$ in probability.

The paper proceeds as follows: Section 2 provides the problem statement in greater detail and states the main theorem with full assumptions. We also introduce the unnormalized variants of $\pi^x$, $\rho^x$, which we will denote as $\rho^0$ and $\rho^0$ in this section. Similar to the notation $\pi^{x,\epsilon}$, $\rho^{x,\epsilon}$ will denote the $x$-marginal of $\rho^x$. In Section 3 we give preliminary estimates which are needed for the main results. Section 4 provides the existence of weak limits of the probability measure induced by the signed measured-valued process $\rho^{x,\epsilon} - \rho^0$, as well as the characterization of this process and the uniqueness of its limit. At the end of Section 4, the main result for convergence of $\rho^{x,\epsilon} - \rho^0$ is stated alongside a lemma that proves the convergence of $\pi^{x,\epsilon} - \pi^0$.

# 2 Problem Statement

In this section, we provide the full problem statement, some notation and the main result. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ supporting a $(w + v + u)$-dimensional $\mathcal{F}_t$-adapted Brownian motion $(W, V, U)$. We will work with the following system of SDEs,

\[
dX^\epsilon_t = \left[ b(X^\epsilon_t, Z^\epsilon_t) + \frac{1}{\epsilon} b_1(X^\epsilon_t, Z^\epsilon_t) \right] dt + \sigma(X^\epsilon_t, Z^\epsilon_t) dW_t,
\]

\[
dZ^\epsilon_t = \frac{1}{\epsilon^2} f(X^\epsilon_t, Z^\epsilon_t) dt + \frac{1}{\epsilon} g(X^\epsilon_t, Z^\epsilon_t) dV_t,
\]

\[
dY^\epsilon_t = h(X^\epsilon_t, Z^\epsilon_t) dt + \alpha dW_t + \gamma dU_t, \quad Y^\epsilon_0 = 0 \in \mathbb{R}^d,
\]

where $b, b_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $\sigma : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^w$, $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^v$ and $h : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^d$ are Borel measurable functions. The initial distribution of $(X, Z)$ is denoted by $\mathbb{Q}(X^0, Z^0)$ and is assumed independent of the $(W, V, U)$ Brownian motion. $\mathbb{Q}(X^0, Z^0)$ is also assumed to have finite moments for all orders. In Eq. 2.1, $0 < \epsilon \ll 1$, is a timescale separation parameter. We consider the case where $\alpha \in \mathbb{R}^{d \times w}, \gamma \in \mathbb{R}^{d \times u}$, and assume the following to be true

\[
K \equiv \alpha \alpha^* + \gamma \gamma^* > 0, \quad \gamma \gamma^* > 0.
\]

This implies the existence of a unique $\mathbb{R}^{d \times d} \ni \kappa > 0$ of lower triangular form, such that $K = \kappa \kappa^*$. Hence there exists a unique $\kappa^{-1}$, such that we can define an auxiliary observation process

\[
Y_t^{x,\kappa} = \int_0^t \kappa^{-1} dY_s^\epsilon = \int_0^t \kappa^{-1} h(X_s^\epsilon, Z_s^\epsilon) ds + B_t, \quad Y_0^{x,\kappa} = 0 \in \mathbb{R}^d,
\]

where

\[
B_t = \kappa^{-1} (\alpha dW_t + \gamma dU_t),
\]

is a standard $d$-dimensional Brownian motion under $\mathbb{Q}$.

We are interested in the convergence of the normalized filter, $\pi^\epsilon$, the conditional distribution of the signal given the observation filtration, to an averaged form. In particular, for any test function $\varphi \in C^{\infty}_0(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ and time $t \in [0, T]$, the normalized filter can be characterized as

\[
\pi^\epsilon_t(\varphi) = \mathbb{E}_\mathbb{Q} \left[ \varphi(X^\epsilon_t, Z^\epsilon_t) \mid Y^\epsilon_t \right],
\]

where $Y^\epsilon_t \equiv \sigma(\{Y^\epsilon_s \mid s \in [0, t]\}) \lor \mathcal{N}$, the $\sigma$-algebra generated by the observation process over the interval $[0, t]$, joined with $\mathcal{N}$, the $\mathbb{Q}$ negligible sets.
Because the filtrations generated by $\mathcal{F}_t$ and $\mathcal{F}_{t,x}$ are equivalent, from the point of view of $\pi^\epsilon$ we can use either. Hence, let us redefine the sensor function $h \leftarrow \kappa^{-1}h$, and $\alpha \leftarrow \kappa^{-1}\alpha$, $\gamma \leftarrow \kappa^{-1}\gamma$, so that the observation process can be redefined as

$$dY^\epsilon_t = h(X^\epsilon_t, Z^\epsilon_t)dt + dB_t, \quad Y^\epsilon_0 = 0 \in \mathbb{R}^d,$$

where $B = \alpha W + \gamma U$ is a standard Brownian motion under $\mathbb{Q}$ and still correlated with $W$.

In Eq. \ref{eq:2.1}, we identify the infinitesimal generators of the SDEs as follows,

$$G_S(x, z) = \sum_{i=1}^{m} b_i(x, z) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} (\sigma \sigma^*)_{ij}(x, z) \frac{\partial^2}{\partial x_i \partial x_j},$$

$$G_I(x, z) = \sum_{i=1}^{m} b_{1,i}(x, z) \frac{\partial}{\partial x_i},$$

$$G_F(x, z) = \sum_{i=1}^{n} f_i(x, z) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^{n} (g g^*)_{ij}(x, z) \frac{\partial^2}{\partial z_i \partial z_j},$$

$$G_S^\epsilon = \frac{1}{\epsilon} G_I + G_S,$$

$$G^\epsilon = \frac{1}{\epsilon^2} G_F + \frac{1}{\epsilon} G_I + G_S.$$

The Kushner-Stratonovich equation for the time evolution of the filter $\pi^\epsilon$, acting on a test function $\varphi \in C^2_b(\mathbb{R}^m \times \mathbb{R}; \mathbb{R})$, is

$$\pi^\epsilon_t(\varphi) = \pi^0_t(\varphi) + \int_0^t \pi^\epsilon_s(G^\epsilon \varphi) ds + \int_0^t \left( \pi^\epsilon_s(\varphi h + \alpha \sigma \nabla_x \varphi) - \pi^\epsilon_s(\varphi) \pi^\epsilon_s(h) \right) ds,$$

$$\pi^0_t(\varphi) = \mathbb{E}_Q [\varphi(X^0_t, Z^0_t)].$$

When we are interested in estimating test functions of $X^\epsilon$ only, i.e. $\varphi \in C^2_b(\mathbb{R}^m; \mathbb{R})$, we consider the $x$-marginal of $\pi^\epsilon$,

$$\pi^\epsilon_t(x)(\varphi) = \int \varphi(x) \pi^\epsilon_t(dx, dz).$$

\subsection{2.1 Homogenization of Stochastic Differential Equations}

The theory of homogenization of SDEs shows that if the processes $Z^{\epsilon,x}$,

$$dZ^{\epsilon,x}_t = \frac{1}{\epsilon^2} f(x, Z^{\epsilon,x}_t) dt + \frac{1}{\epsilon} g(x, Z^{\epsilon,x}_t) dV_t,$$

is ergodic with stationary distribution $\mu_\infty(x)$, then under appropriate conditions, in the limit $\epsilon \to 0$ the process $X^\epsilon$ converges in distribution to a Markov process $X^0$ with infinitesimal generator,

$$G^\dagger = \overline{G}_S + \overline{G},$$

where

$$\overline{G}_S(x) = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

$$b(x) \equiv \int_{\mathbb{R}^n} b(x, z) \mu_\infty(dz; x),$$

$$a(x) \equiv \frac{1}{2} \int_{\mathbb{R}^n} a(x, z) \mu_\infty(dz; x).$$
\[ a = \sigma \sigma^*, \] and
\[
\bar{G}(x) = \sum_{i=1}^{m} \bar{b}_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} \tilde{a}_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},
\]
\[
\bar{b}(x) = \int_{\mathbb{R}^n} (\nabla_x G_F^{-1}(-b_1)) b_1(x, z) \mu_{\infty}(dz; x),
\]
\[
\bar{a}(x) = \int_{\mathbb{R}^n} (b_1 \otimes G_F^{-1}(-b_1)) (x, z) + (G_F^{-1}(-b_1) \otimes b_1) (x, z) \mu_{\infty}(dz; x),
\]
where \( G_F^{-1}(-b_1) \) is the solution of a Poisson equation.

We define the averaged filter \( \pi^0 \), a probability measure-valued process satisfying the following evolution equation,
\[
\begin{align*}
\pi^0_t(\varphi) &= \pi^0_0(\varphi) + \int_0^t \pi^0_s(\varphi^1) ds + \int_0^t \langle \pi^0_s(\varphi \bar{h} + \alpha \sigma^* \nabla_x \varphi) - \pi^0_s(\varphi) \pi^0_s(\bar{H}), dY^\epsilon_s - \pi^0_s(\bar{H}) ds \rangle, \\
\pi^0(\varphi) &= \mathbb{E}_Q \left[ \varphi(X_0^0) \right].
\end{align*}
\]  

(2.8)

The definitions of \( \bar{h}, \sigma \) are
\[
\bar{h}(x) = \int_{\mathbb{R}^n} h(x, z) \mu_{\infty}(dz; x), \quad \sigma(x) = \int_{\mathbb{R}^n} \sigma(x, z) \mu_{\infty}(dz; x).
\]

### 2.2 Notation and Main Theorem

Before stating the main result of the paper, we set and provide a few definitions and assumptions that will be used throughout the paper. We will use \( \mathbb{N}_0 \) to denote \( \{0, 1, 2, \ldots\} \) and \( \mathbb{N} \) for \( \{1, 2, \ldots\} \). Let \( H_f \) denote the assumption that there exists a constant \( C > 0 \), exponent \( \alpha > 0 \) and an \( R > 0 \) such that for all \( |z| > R \),
\[
\sup_{x \in \mathbb{R}^m} \langle f(x, z), z \rangle \leq -C|z|^{\alpha}.
\]

(\( H_f \))

\( H_f \) is a recurrence condition, which provides the existence of a stationary distribution, \( \mu_{\infty}(x) \), for the process \( Z^x \). Let \( H_g \) denote the assumption that there are \( 0 < \lambda \leq \Lambda < \infty \), such that for any \( (x, z) \in \mathbb{R}^m \times \mathbb{R}^n \),
\[
\lambda I \preceq gg^*(x, z) \preceq \Lambda I,
\]

(\( H_g \))

where \( \preceq \) is the order relation in the sense of positive semi-definite matrices. \( H_g \) is a uniform ellipticity condition, which provides the uniqueness of the stationary distribution. We will say that a function \( \theta : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R} \) is centered with respect to \( \mu_{\infty}(x) \), if for each \( x \)
\[
\int \theta(x, z) \mu_{\infty}(dz; x) = 0, \quad \forall x \in \mathbb{R}^m.
\]

For brevity in the results to follow, let us denote \( H^{i,j+\alpha} \) to specify the regularity and boundedness of \( f \) and \( gg^* \) as follows,
\[
f \in C_b^{i,j+\alpha}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}), \quad gg^* \in C_b^{i,j+\alpha}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^{n \times n}), \quad i, j \in \mathbb{N}, \quad \alpha \in (0, 1),
\]

(\( H^{i,j+\alpha} \))

where \( \varphi(x, z) \in C_b^{i,j+\alpha} \) denotes that \( \varphi \) has \( i \) bounded derivatives in the \( x \)-component, \( j \) bounded derivatives in the \( z \)-component, and all derivatives \( \partial_x^i \partial_z^{j'} \varphi \) for \( 0 \leq i' \leq i \), \( 0 \leq j' \leq j \) are \( \alpha \)-Hölder continuous in \( z \) uniformly in \( x \).

We use the notation \( k = (k_1, \ldots, k_m) \in \mathbb{N}_0^m \) for a multiindex with order \( |k| = k_1 + \ldots + k_m \) and define the differential operator
\[
D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_m^{k_m}}.
\]
Theorem 2.1
Assume that f and g satisfy H_f and H_g, that b_t is centered with respect to μ_∞(x) for each x and that \( Q_{(X_0, Z_0)} \) has finite moments of every order. Additionally, assume either a. high regularity conditions or b. low regularity with uniform ellipticity:

a. \( H^{1.2+\alpha}_f \) holds for \( \alpha \in (0, 1) \); for each z, \( b(\cdot, z), \sigma(\cdot, z) \in C^3 \), and \( b_t(\cdot, z) \in C^2 \); that \( b \) and \( b_t \) are Lipschitz in \( z \), and \( \sigma \) is globally Lipschitz in \( z \); that \( b, b_t, \sigma \) satisfy the growth conditions

\[
|b(x, z)| + |b_t(x, z)| + |\sigma \sigma^*(x, z)| \leq C(1 + |z|)^\beta,
\]

\[
\sum_{|k|=1}^2 |D_x^k b(x, z)| + |D_x^k \sigma \sigma^*(x, z)| \leq C(1 + |z|)^q,
\]

\[
\sum_{|k|=1}^3 |D_x^k b_t(x, z)| \leq C(1 + |z|)^q,
\]

for some \( \beta < -2 \) and \( q > 0 \); that \( h \) is bounded in \((x, z), \) \( h(\cdot, z) \in C^3 \) for each \( z \), and \( h \) is globally Lipschitz in \( z \).

b. \( \bar{a} + \bar{a} \geq 0 \) uniformly in \( x \); \( H^{2.2+\alpha}_f \) holds for \( \alpha \in (0, 1) \); for each \( z \), \( b(\cdot, z), b_t(\cdot, z), \sigma(\cdot, z) \in C^2 \); that \( b \) and \( b_t \) are Lipschitz in \( z \), and \( \sigma \) is globally Lipschitz in \( z \); that \( b, b_t, \sigma \) satisfy the growth conditions

\[
|b(x, z)| + |b_t(x, z)| + |\sigma \sigma^*(x, z)| \leq C(1 + |z|)^\beta,
\]

\[
\sum_{|k|=1}^2 |D_x^k b(x, z)| + |D_x^k \sigma \sigma^*(x, z)| + |D_x^k \sigma^*(x, z)| \leq C(1 + |z|)^q,
\]

for some \( \beta < -2 \) and \( q > 0 \); \( h \) is bounded in \((x, z), \) that \( h \) is globally Lipschitz in \((x, z)\). If \( a > 0 \), which implies \( a + \bar{a} \geq 0 \), then the Lipschitz condition in \( z \) for \( b, b_t \) can be relaxed to \( \alpha \)-Hölder continuity.

Then there exists a metric \( d \) on \( C([0, T]; P(\mathbb{R}^m)) \), the space of continuous processes from \([0, T]\) to the space of probability measures on \( \mathbb{R}^m \), that generates the topology of weak convergence, such that \( \pi_{\epsilon, x} \to \pi_0 \) in probability.

Proof. From Theorem 4.1 and Lemma 4.5 we get \( \pi_{\epsilon, x} - \pi_0 \to 0 \) as \( \epsilon \to 0 \). \( \pi_{\epsilon, x} \) and \( \pi_0 \) are random variables in the space \( C([0, T]; P(\mathbb{R}^m)) \). We define a continuous bounded metric \( d \) on this space that generates the topology of weak convergence as follows,

\[
d(\mu, \nu) = 1 \wedge (\sup_{0 \leq t \leq T} d(\mu_t, \nu_t)),
\]

where we assume that \( d \) is a translation invariant metric that generates the topology of weak convergence on \( P(\mathbb{R}^m) \). Then \( d \) inherits the translation invariant property from \( \tilde{d} \), and from weak convergence of \( \pi_{\epsilon, x} - \pi_0 \) in the space of signed measures, we have

\[
\lim_{\epsilon \to 0} \mathbb{E}_Q [d(\pi_{\epsilon, x}, \pi_0)] = \lim_{\epsilon \to 0} \mathbb{E}_Q [d(\pi_{\epsilon, x} - \pi_0, 0)] = 0.
\]

And therefore we retrieve convergence in probability,

\[
\lim_{\epsilon \to 0} \mathbb{Q} \left( d(\pi_{\epsilon, x}, \pi_0) \geq \delta \right) \leq \frac{1}{\delta} \lim_{\epsilon \to 0} \mathbb{E}_Q [d(\pi_{\epsilon, x}, \pi_0)] = 0, \quad \text{for each} \ \delta > 0.
\]

It remains to show that there is a translation invariant metric \( \tilde{d} \) that generates the weak topology on \( P(\mathbb{R}^m) \). For this, we can borrow the argument from [Imk+13, Corollary 6.9, p.2322].

\[\square\]
2.3 Change of Probability Measure and Zakai Equation

To prove the main result, the analysis which we perform will actually be concerned with unnormalized conditional measures that are defined using change of probability measure transformations. The new collection of measures on the filtered probability space are denoted by \( (\mathbb{P}^\epsilon) \). For any fixed \( \epsilon \), \( \mathbb{P}^\epsilon \) and \( \mathbb{Q} \) will be mutually absolutely continuous with Radon-Nikodym derivatives

\[
D_t^\epsilon = \left. \frac{d\mathbb{P}^\epsilon}{d\mathbb{Q}} \right|_{\mathcal{F}_t} = \exp \left( -\int_0^t \langle h(X_s^\epsilon, Z_s^\epsilon), dB_s \rangle - \frac{1}{2} \int_0^t |h(X_s^\epsilon, Z_s^\epsilon)|^2 \, ds \right),
\]

\[
\tilde{D}_t^\epsilon = (D_t^\epsilon)^{-1} = \left. \frac{d\mathbb{Q}}{d\mathbb{P}^\epsilon} \right|_{\mathcal{F}_t} = \exp \left( \int_0^t \langle h(X_s^\epsilon, Z_s^\epsilon), dY_s^\epsilon \rangle - \frac{1}{2} \int_0^t |h(X_s^\epsilon, Z_s^\epsilon)|^2 \, ds \right).
\]

Then by Girsanov’s theorem, under \( \mathbb{P}^\epsilon \) the process \( Y^\epsilon \) is a Brownian motion. For a fixed test function \( \varphi \in C_b^2(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}) \) and time \( t \in [0, T] \), we characterize the unnormalized conditional measure \( \rho_t^\epsilon \) as,

\[
\rho_t^\epsilon(\varphi) = \mathbb{E}_{\mathbb{P}^\epsilon} \left[ \varphi(X_t^\epsilon, Z_t^\epsilon) \tilde{D}_t^\epsilon \mid \mathcal{Y}_t^\epsilon \right],
\]

and its relation to \( \pi_t^\epsilon \) through the Kallianpur-Striebel formula,

\[
\pi_t^\epsilon(\varphi) = \frac{\mathbb{E}_{\mathbb{P}^\epsilon} \left[ \varphi(X_t^\epsilon, Z_t^\epsilon) \tilde{D}_t^\epsilon \mid \mathcal{Y}_t^\epsilon \right]}{\mathbb{E}_{\mathbb{P}^\epsilon} \left[ \tilde{D}_t^\epsilon \mid \mathcal{Y}_t^\epsilon \right]} = \frac{\rho_t^\epsilon(\varphi)}{\rho_t^\epsilon(1)}, \quad \forall t \in [0, T], \quad \mathbb{Q}, \mathbb{P}^\epsilon\text{-a.s.}
\]

The action of \( \rho^\epsilon \) on test functions \( \varphi \in C_b^2(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}) \) gives the Zakai evolution equation,

\[
\begin{align*}
\rho_t^\epsilon(\varphi) &= \rho_0^\epsilon(\varphi) + \int_0^t \rho_s^\epsilon(\varphi h + \alpha \sigma^* \nabla_x \varphi, dY_s^\epsilon), \\
\rho_0^\epsilon(\varphi) &= \mathbb{E}_\mathbb{Q} \left[ \varphi(X_0^\epsilon, Z_0^\epsilon) \right].
\end{align*}
\]

(2.9)

When \( \varphi \in C_b^2(\mathbb{R}^m; \mathbb{R}) \), we consider the \( x \)-marginal,

\[
\rho_t^{\epsilon, x}(\varphi) = \int \varphi(x) \rho_t^\epsilon(dx, dz),
\]

which is related to \( \pi_t^{\epsilon, x} \) through the Kallianpur-Striebel formula,

\[
\pi_t^{\epsilon, x}(\varphi) = \frac{\rho_t^{\epsilon, x}(\varphi)}{\rho_t^{\epsilon, x}(1)}, \quad \forall t \in [0, T], \quad \mathbb{Q}, \mathbb{P}^\epsilon\text{-a.s.}
\]

This is easy to see since \( \rho^\epsilon(1) = \rho^{\epsilon, x}(1) \).

We next define the averaged unnormalized filter \( \rho^0 \) as the solution of the following evolution equation,

\[
\begin{align*}
\rho_t^0(\varphi) &= \rho_0^0(\varphi) + \int_0^t \rho_s^0(\varphi h + \alpha \sigma^* \nabla_x \varphi, dY_s^\epsilon), \\
\rho_0^0(\varphi) &= \mathbb{E}_\mathbb{Q} \left[ \varphi(X_0^0) \right],
\end{align*}
\]

(2.10)

where \( \varphi \in C_b^2(\mathbb{R}^m; \mathbb{R}) \). And then by the Kallianpur-Striebel formula we relate the averaged (normalized) filter \( \pi^0 \) to the unnormalized variant,

\[
\pi_t^0(\varphi) = \frac{\rho_t^0(\varphi)}{\rho_t^0(1)}, \quad \forall t \in [0, T], \quad \mathbb{Q}, \mathbb{P}^\epsilon\text{-a.s.}
\]

The uniqueness of \( \rho^0 \) follows from the same assumptions and proof to be given in Lemma 4.4.

We will later show in Lemma 4.5 that under appropriate assumptions, weak convergence of \( \rho^{\epsilon, x} - \rho^0 \) to zero will imply weak convergence of \( \pi^{\epsilon, x} - \pi^0 \) to zero, and therefore we can focus on showing convergence of the unnormalized difference for the main analysis.
2.3.1 Representation of the Averaged Unnormalized Conditional Distribution

Just as $\sigma^0$ is not the filter for the averaged system, $\rho^0$ is not the unnormalized conditional distribution for the averaged system, and therefore a representation of this measure acting on $\varphi \in C_b^2$ test functions as conditional expectation requires a bit more work. But such a representation will be necessary for the computation of some estimates.

To get such a representation, we introduce a signal process $X^0$ to be a diffusion process with infinitesimal generator $\mathcal{G}^\dagger$. Therefore, consider the following SDE,

$$
\begin{align*}
    dX_t^0 &= \left[\hat{b}(X_t^0) + \hat{b}(X_0^0)\right] dt + \tilde{a}^{1/2}(X_t^0)d\tilde{W}_t + (\tilde{a}(X_t^0) - \sigma \sigma^*(X_t^0))^{1/2}d\tilde{W}_t + \sigma(X_t^0)dW_t, \\
    X_0^0 &\sim Q_{X_0^0}.
\end{align*}
$$

(2.11)

Here $\tilde{W}$ and $\hat{W}$ are new $m$-dimensional independent Brownian motions, independent of $(V, W, U)$ under $Q$ as well as independent of the initial condition $Q_{X_0^0}$. The Cholesky factor $(\tilde{a}(X_t^0) - \sigma \sigma^*(X_t^0))^{1/2}$ exists, since from an application of Jensen’s inequality, one can show that $Q_{X_0^0}$ as well as independent of the initial condition $Q_{X_0^0}$. This will be true if $\sigma(\cdot, z) \in C_b^2$ for each $z$, which is assumed in Theorem 2.1.

Remark. An interesting observation regarding Eq. 2.11, is that we may have $\sigma = 0$, and this implies that the SDE for the averaged filter may have no correlation at all, or less correlation than the original system.

We now define the process

$$
\hat{D}_t^0 = \exp \left( \int_0^t \langle \tilde{h}(X_s^0), dY_s^\epsilon \rangle - \frac{1}{2} \int_0^t |\tilde{h}(X_s^0)|^2 ds \right),
$$

which is used to give the representation of $\rho^0$ on $C_b^2$ test functions as follows,

$$
\rho^0_t(\varphi) = \mathbb{E}_{\rho^0} \left[ \varphi(X_t^0) \hat{D}_t^0 \mid Y_t^\epsilon \right].
$$

3 Preliminary Estimates

In this section we provide several preliminary estimates that will be needed for the main analysis. Some additional comments regarding notation are first introduced and then some assumptions are defined.

The relation $a \lesssim b$ will indicate that $a \leq C b$ for a constant $C > 0$ that is independent of $a$ and $b$, but that may depend on parameters that are not critical for the bound being computed. We will use the notation $T^{\epsilon, z}$ for the semigroup of $Z^{\epsilon}$, and denote processes with $Z^{\epsilon, x}(z, \cdot)$ to represent the process $Z^{\epsilon, x}$ started at time $s$ at $z \in \mathbb{R}^n$. We will say that a function $\theta(x, z)$ is centered with respect to $\mu_\infty$ (the family of invariant measures parameterized by $x \in \mathbb{R}^m$) if

$$
\int_{\mathbb{R}^n} \theta(x, z) \mu(dz; x) = 0, \quad \forall x \in \mathbb{R}^m.
$$

Let $H_L$ denote the assumption that for each $K > 0$, there exists a constant $C_K$ such that for all $x, x' \in \mathbb{R}^m, |z| \leq K$:

$$
|b(x, z) - b(x', z)| + |b_l(x, z) - b_l(x', z)| + |\sigma(x, z) - \sigma(x', z)| \leq C_K |x - x'|.
$$

(H$_L$)

Let $H_P$ denote the assumption that there exists $K, \alpha, p_1, p_2 > 0$ such that for all $(x, z) \in \mathbb{R}^m \times \mathbb{R}^n$:

$$
|b(x, z)| \leq K (1 + |x|)(1 + |z|^{p_1}),
$$

$$
|\sigma(x, z)| = \sqrt{\text{Tr}(\sigma \sigma^*(x, z))} \leq K (1 + |x|^{1/2})(1 + |z|^{p_2}).
$$

(H$_P$)

Note that from $H_P$ we have $|\sigma(x, z)| \lesssim 1 + |x| + |z|^{2p_2}$ and hence implies a linear growth in $x$ and polynomial growth in $z$. Also from $H_P$, $|\sigma^*(x, z)| \lesssim (1 + |x|^2 + |z|^{4p_2})$.  


Let $H_I$ denote the assumption that for some $K, p > 0$, $b_I$ satisfies the following growth condition,
\[
\sup_{|\alpha| \leq 2} \sum_{x \in \mathbb{R}^m} |D^\alpha_x b_I(x, z)| \leq K (1 + |z|^p).
\]

\[ (H_I) \]

The next result is from [PV03, p.1172] and provides the result that $X^\epsilon \to X^0$ in the limit $\epsilon \to 0$.

**Theorem 3.1**
Let $(X^\epsilon, Z^\epsilon)$ satisfy the stochastic differential equations of Eq. 1.1 with initial conditions $(X^\epsilon_0, Z^\epsilon_0) = (x, z) \in \mathbb{R}^m \times \mathbb{R}^n$ for each $\epsilon \in (0, 1)$. Assume $H_f$, $H_g$, $H^{2,2+\alpha}$ for $\alpha \in (0, 1)$, $H_L$, and $H_P$. Let $b_I \in C^{2,\alpha}$ satisfy $H_I$ and be centered with respect to $\mu_\infty$. Then for any $T > 0$, the process $X^\epsilon$ converges weakly in the limit $\epsilon \to 0$, to the Markov process $X^0$ with generator $G^1$.

**Proof.** See remarks in Section 5 and [PV03, p.1172].

\[ \square \]

### 3.1 Estimates with the Fast Semigroup

**Lemma 3.1**
Assume $H^{k,l}$, with $k \in \mathbb{N}_0, l \in \mathbb{N}$, and let $\theta \in C^{k,j}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ for $j \leq l$ satisfy for some $C, p > 0$
\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq j} |D^\alpha_x D^{\beta}_z \theta(x, z)| \leq C(1 + |x|^p + |z|^p).
\]

Then
\[
(t, x, z) \mapsto T_t^{F,x}(\theta(x, \cdot))(z) \in C^{0,k,j}(\mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})
\]
and there exist $C_1, p_1 > 0$, such that for all $(t, x, z) \in [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^n$
\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq j} |D^\alpha_x D^{\beta}_z T_t^{F,x}(\theta(x, \cdot))(z)| \leq C_1 e^{C_1 t}(1 + |x|^{p_1} + |z|^{p_1}).
\]

If the bound on the derivatives of $\theta$ can be chosen uniformly in $x$, that is,
\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq j} \sup_x |D^\alpha_x D^{\beta}_z \theta(x, z)| \leq C(1 + |z|^p),
\]
then the bound on the derivatives of $T_t^{F,x}(\theta(x, \cdot))(z)$ is also uniform in $x$,
\[
\sum_{|\alpha| \leq k} \sum_{|\beta| \leq j} \sup_x |D^\alpha_x D^{\beta}_z T_t^{F,x}(\theta(x, \cdot))(z)| \leq C_1 e^{C_1 t}(1 + |z|^{p_1}).
\]

**Proof.** The proposition is a slight generalization of [Imk+13, Proposition 5.1]. The proof is the same as in [Imk+13, Proposition 5.1].

\[ \square \]

**Lemma 3.2**
Assume $H_f$, $H_g$ and $H^{k,2+\alpha}$ for $\alpha \in (0, 1)$, and $k \in \mathbb{N}_0$. Let $\theta \in C^{k,0}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R})$ satisfy for some $C, p > 0$,\n\[
\sum_{|\gamma| \leq k} \sup_x |D^\gamma_x \theta(x, z)| \leq C(1 + |z|^p).
\]

Then
\[
x \mapsto \mu_\infty(\theta; x)(x') = \int_{\mathbb{R}^n} \theta(x', z) \mu_\infty(dz; x) = \int_{\mathbb{R}^n} \theta(x', z) p_\infty(z; x) dz \in C^k_b(\mathbb{R}^n; \mathbb{R}).
\]

**Proof.** The proposition is a slight generalization of a part of [Imk+13, Proposition 5.2] and the proof follows the same argument as given there.

\[ \square \]
Lemma 3.3
Assume $H_f$, $H_g$, and $H^{k,2+\alpha}$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}_0$. Let $j \in \{0,1\}$, and $\theta \in C^{k,j+\alpha(1-j)}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R})$ satisfy the growth condition,

$$\sum_{|\gamma| \leq k} \sum_{|\beta| \leq j} \sup_x |D_\gamma D^\beta \theta(x,z)| \leq C(1 + |z|^p),$$

for some $C,p>0$. Assume additionally that $\theta$ satisfies the centering condition,

$$\int_{\mathbb{R}^n} \theta(x,z) \mu_\infty(dz;x) = 0, \quad \forall x \in \mathbb{R}^m.$$

Then

$$(x,z) \mapsto \int_0^\infty T_t^F \theta(x,\cdot)(z) dt \in C^k(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}),$$

and for every $q>0$ there exist $C',q'>0$, such that,

$$\sum_{|\gamma| \leq k} \sum_{|\beta| \leq j} \int_0^\infty \sup_x |D_\gamma D^\beta T_t^F \theta(x,\cdot)(z)|^q dt \leq C'(1 + |z|^{q'}).$$

Proof. The proposition is a slight generalization of a part of [Imk+13, Proposition 5.2] and the proof follows the same argument as given there. \qed

3.2 Estimates on SDE Solutions

Lemma 3.4
Assume $f$ is bounded and that $f$ and $gg^*$ are Hölder continuous in $z$ uniformly in $x$ for some uniform constant. Assume that the conditions $H_f$ and $H_g$ hold. Then for any $p>0$ there exists $C_p>0$ such that

$$\sup_{(t,x,z) \in [0,\infty) \times (0,1) \times \mathbb{R}^m} \mathbb{E} \left[ |Z_t^z|^p \mid (X_0^z, Z_0^z) = (x,z) \right] \leq 1 + |z|^p.$$

Proof. The proposition is a slight generalization of a part of [Imk+13, Proposition 5.3] and the proof follows the same argument as given there. \qed

Lemma 3.5
Assume the conditions $H_f$, $H_g$ and $H^{2,2+\alpha}$ for some $\alpha \in (0,1)$; that $b,\sigma$ are bounded for all $(x,z)$; and that $b_1 \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}^n; \mathbb{R}^m)$ that satisfies the centering condition,

$$\int_{\mathbb{R}^m} b_1(x,z) \mu_\infty(dz;x) = 0,$$

where $\mu_\infty(x)$ is the unique stationary distribution for the process $Z^z$, and that for some $C,q_1>0$, it has the following growth condition,

$$\sum_{|\alpha| \leq 2} \sum_{|\beta| \leq 1} \sup_x |D_\alpha D^\beta b_1(x,z)| \leq C(1 + |z|^{q_1}).$$

Then for every $p \geq 2$ there exists $q>0$ such that for $0 \leq r < t < \infty$

$$\mathbb{E} \left[ \frac{1}{t-r} \int_r^t b_1(X_s^z, Z_s^z) ds \right]^p \leq c^p(1 + |z|^q) + (t-r)^{p-1}(1 + c^p) \int_r^t 1 + \mathbb{E}|Z_s^z|^q ds$$

$$+ (t-r)^{(p/2)-1}(1 + c^p) \int_r^t 1 + \mathbb{E}|Z_s^z|^q ds.$$
Proof. We start by considering the solution of the following backward partial differential equation,

\[-\partial_s \psi_t(x, z) = \frac{1}{\epsilon^2} G_F \psi_t(x, z) + \frac{1}{\epsilon} b_t(x, z), \quad \psi_t(x, z) = 0.\]

The solution of which is given by a Feynman-Kac representation,

\[
\psi_t(x, z) = \mathbb{E} \int_t^T \frac{1}{\epsilon} b_t(x, Z_t^{x,z}) dt + \frac{1}{\epsilon} \int_t^T T_t^{F,x} (b_t(x, \cdot)) (z) dt = \epsilon \int_0^{(t-s)/\epsilon^2} T_s^{F,x} (b_t(x, \cdot)) (z) du,
\]

where \( T_t^{F,x} \) is the semigroup associated with the process \( Z_t^{x} \) and a change of time has been used in the last equality relation. Then since \( b_t \) satisfies the conditions of Lemma 3.3, we have

\[
\sum_{|\alpha| \leq 2} \sum_{|\beta| \leq 1} \sup_{s \in [0,t]} |D^\alpha_x D^\beta_z \psi_t(x, z)|^p \lesssim \epsilon^p (1 + |z|^{q_2}),
\]

for some \( q_2 > 0 \). Applying Itô's formula to \( \psi_t(x, z) \) gives,

\[
0 = \psi_t(x, z) + \frac{1}{\epsilon^2} \int_t^T G_F \psi_t(x, Z_s^{x,z}) ds + \frac{1}{\epsilon} \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) g(X_s^{x,z}) dV_s
\]

\[
+ \int_t^T G_s^F \psi_t(x, Z_s^{x,z}) ds + \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) \sigma(x, Z_s^{x,z}) dW_s
\]

\[-\frac{1}{\epsilon^2} \int_t^T G_F \psi_t(x, Z_s^{x,z}) ds - \frac{1}{\epsilon} \int_t^T b_t(x, Z_s^{x,z}) ds.
\]

Eliminating terms and rearranging simplifies to

\[
\frac{1}{\epsilon} \int_t^T b_t(X_s^x, Z_s^z) ds = \psi_t(x, z) + \frac{1}{\epsilon^2} \int_t^T G_s^F \psi_t(x, Z_s^{x,z}) ds + \frac{1}{\epsilon} \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) b_t(X_s^x, Z_s^z) ds
\]

\[
+ \frac{1}{\epsilon^2} \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) g(X_s^{x,z}) dV_s + \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) \sigma(x, Z_s^{x,z}) dW_s.
\]

The first term will contribute,

\[
\mathbb{E} |\psi_t(x, z)|^p \leq \sup_{s \in [0,t]} |\psi_s(x, z)|^p \lesssim \epsilon^p (1 + |z|^{q_2}).
\]

From the boundedness of \( b \) and \( \sigma \) we have for the second term

\[
\mathbb{E} \left| \int_t^T G_s^F \psi_t(x, Z_s^{x,z}) ds \right|^p \lesssim \int_t^T \mathbb{E} |\psi_t(x, Z_s^{x,z})|^p ds\]

\[\lesssim (t-r)^{p-1} |b|^p 1 + \mathbb{E} |Z_s^{x,z}|^{q_2} ds.
\]

For the third term,

\[
|\nabla_x \psi_t(x, Z_s^{x,z}) b_t(X_s^x, Z_s^z)|^p \lesssim |\nabla_x \psi_t(x, Z_s^{x,z})|^p |b_t(X_s^x, Z_s^z)|^p \lesssim \epsilon^p (1 + |Z_s^{x,z}|^{q_2}) (1 + |Z_s^{x,z}|^q)^p
\]

and therefore

\[
\mathbb{E} \frac{1}{\epsilon} \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) b_t(X_s^x, Z_s^z) ds \lesssim (t-r)^{p-1} \int_t^T \mathbb{E} |Z_s^{x,z}|^{q_1} ds.
\]

The stochastic integrals follow in a similar manner after application of the Burkholder-Davis-Gundy (BDG) inequality, and the boundedness of \( \sigma \) and \( g \),

\[
\mathbb{E} \left| \int_t^T \nabla_x \psi_t(x, Z_s^{x,z}) \sigma(x, Z_s^{x,z}) dW_s \right|^p \leq C_p \mathbb{E} \left( \sup_{s \in [r,t]} \left| \int_r^s \nabla_x \psi_t(x, Z_s^{x,z}) \sigma(x, Z_s^{x,z}) dW_s \right| \right)^p
\]

\[\leq C_p \mathbb{E} \left( \int_t^T |\nabla_x \psi_t(x, Z_s^{x,z}) \sigma(x, Z_s^{x,z})|^2 ds \right)^{p/2}
\]

\[\leq C_p (t-r)^{(p/2)-1} \epsilon^p |\sigma|^p \int_t^T 1 + \mathbb{E} |Z_s^{x,z}|^{q_2} ds.
\]
The bound for the other stochastic integral follows in the same manner,
\[
\mathbb{E}\left[\frac{1}{\epsilon} \int_r^t \nabla \psi_s(X_s', Z_s')g(X_s', Z_s')dV_s\right]^p \leq C_p(t-r)^{(p/2)-1}\|g\|_\infty^p \int_r^t 1 + \mathbb{E}|Z_s'|^q ds.
\]
Collecting all the terms, now yields the desired result. \qed

**Lemma 3.6**
Assume the same setup as Lemma 3.5, then for every \( p \geq 2 \) there exists \( q > 0 \) such that for \( T > 0 \)
\[
\sup_{(t, s) \in [0, T] \times (0, 1]} \mathbb{E}\left[|X_t'|^p \mid (X_0', Z_0') = (x, z)\right] \leq 1 + |x|^p + |z|^q.
\]

**Proof.** Since \( b \) and \( \sigma \) are bounded, we get,
\[
\mathbb{E}|X_t'|^p \leq 1 + \mathbb{E}|X_0'|^p + \mathbb{E}\left[\int_0^t \frac{1}{\epsilon} b_t(X_s', Z_s')ds\right]^p.
\]
Using the result of Lemma 3.5 for the moment of the intermediate scale forcing and then Lemma 3.4 for the moment of the fast process, we get
\[
\mathbb{E}|X_t'|^p \leq \mathbb{E}|X_0'|^p + \epsilon^p(1 + \mathbb{E}|Z_0'|^p) + (1 + \epsilon^p)\int_0^t 1 + \mathbb{E}|Z_s'|^q ds \leq \mathbb{E}|X_0'|^p + (1 + \epsilon^p)(1 + \mathbb{E}|Z_0'|^q).
\]
Repeating the proof, but conditioning on \((X_0', Z_0') = (x, z)\) gives the desired result. \qed

**Lemma 3.7**
Assume the same setup as Lemma 3.5, then for \( |t - s| \leq 1 \) and \( p \geq 2 \), there exists a \( q \geq 0 \) such that
\[
\mathbb{E}\left[|X_t' - X_s'|^p \mid (X_0', Z_0') = (x, z)\right] \leq \epsilon^p(1 + |z|^q) + (t - s)^{p/2}(1 + \epsilon^p)(1 + |z|^q).
\]

**Proof.** Without loss of generality, assume \( s < t \).
\[
\mathbb{E}\left[|X_t' - X_s'|^p \right] \leq (t - s)^{p-1} \int_s^t \mathbb{E}|b(X_u', Z_u')|^p du + \mathbb{E}\left[\frac{1}{\epsilon} \int_s^t b(X_u', Z_u')du\right]^p + \mathbb{E}\left[\int_s^t \sigma(X_u', Z_u')dW_u\right]^p
\]
\[
\leq (t - s)^p|b|_p^p + (t - s)^{p/2}\|\sigma\|_\infty^p + \mathbb{E}\left[\frac{1}{\epsilon} \int_s^t b(X_u', Z_u')du\right]^p.
\]
Now using Lemma 3.5, Lemma 3.4, we have
\[
\mathbb{E}\left[|X_t' - X_s'|^p \right] \leq (t - s)^p + (t - s)^{p/2} + \epsilon^p(1 + \mathbb{E}|Z_0'|^q) + (t - s)^p(1 + \epsilon^p)(1 + \mathbb{E}|Z_0'|^q) + (t - s)^{p/2}(1 + \epsilon^p)(1 + \mathbb{E}|Z_0'|^q)
\]
\[
\leq \epsilon^p(1 + \mathbb{E}|Z_0'|^q) + (t - s)^{p/2}(1 + \epsilon^p)(1 + \mathbb{E}|Z_0'|^q)
\]
Repeating the proof, but conditioning on \((X_0', Z_0') = (x, z)\) gives the desired result. \qed

**Lemma 3.8**
Assume \( h \) is bounded, then for \( p \geq 2 \) and \( T > 0 \),
\[
\sup_{\epsilon \in (0, 1]} \mathbb{E}_{\mathcal{F}_p}\left|\tilde{D}_t^\epsilon\right|^p < \infty \quad \text{and} \quad \sup_{t \leq T} \mathbb{E}_{\mathcal{F}_p}\left|\tilde{D}_t^\epsilon\right|^p < \infty.
\]
Further, for \( |t - s| < 1 \), we have
\[
\sup_{\epsilon \in (0, 1]} \mathbb{E}_{\mathcal{F}_p}\left|\tilde{D}_t^\epsilon - \tilde{D}_s^\epsilon\right|^p \leq C_p(t - s)^{p/2}\|h\|_\infty^p < \infty.
\]
Proof. We have that $\tilde{D}_t^\epsilon$ satisfies
\[ \tilde{D}_t^\epsilon = 1 + \int_0^t \tilde{D}_s^\epsilon(h(X_s^\epsilon, Z_s^\epsilon), dY_s^\epsilon). \]
Using the boundedness of $h$, the first result now follows from an application of BDG and Grönwall’s lemma. The same proof applies for the moment bound of $\tilde{D}^0$. The bound for the increment now follows,
\[ \mathbb{E}_\mathbb{P} \left| \tilde{D}_t^\epsilon - \tilde{D}_s^\epsilon \right|^p = \mathbb{E}_\mathbb{P} \left| \int_s^t \tilde{D}_u^\epsilon(h(X_u^\epsilon, Z_u^\epsilon), dY_u^\epsilon) \right|^p \leq C_p(t-s)^{(p/2)-1} \int_s^t \mathbb{E}_\mathbb{P} \left[ \left| \tilde{D}_u^\epsilon \right|^p |h(X_u^\epsilon, Z_u^\epsilon)|^p \right] du \lesssim C_p(t-s)^{p/2} |h|_\infty^p < \infty. \]

3.3 Estimates using the Poisson Equation

Theorem 3.2

Consider the Poisson equation,
\[ \mathcal{G}u(x, z) = -\psi(x, z), \]
where $x \in \mathbb{R}^m$ is a parameter, $\mathcal{G}$ is the generator
\[ \mathcal{G}(x, z) = \sum_{i=1}^n f_i(x, z) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^n (g g^*)_{ij}(x, z) \frac{\partial^2}{\partial z_i \partial z_j}, \]
and $\psi \in C^{k,\alpha}$ for $k \geq 1$ and $\alpha > 0$. Assume that $f, g$ satisfy the assumptions of $H_f$, $H_g$ and $H^{1,2+\alpha}$. Let $\mu_\infty(x)$ be the unique stationary distribution of $Z^\epsilon$ for each fixed $x \in \mathbb{R}^m$, and assume that $\psi$ is centered for each $x \in \mathbb{R}^m$,
\[ \int_{\mathbb{R}^n} \psi(x, z) \mu_\infty(dz; x) = 0. \]
Further, assume the growth conditions
\[ |\psi(x, z)| \leq C_0(1 + |z|)^\beta, \]
\[ \sum_{|j|=1}^k |D^j_2 \psi(x, z)| \leq C_1(x)(1 + |z|^q), \]
for some $\beta < -2$ and $q > 0$. Then the solution of the Poisson equation, belongs to the Sobolev space $\cap_{p \in (1, \infty)} W^2_{p, \text{loc}}$, is unique up to an additive constant such that for any $x$ the centering condition
\[ \int_{\mathbb{R}^n} u(x, z) \mu_\infty(dz; x) = 0 \]
holds, the solution satisfies $u(\cdot, z) \in C^k$ for any $z$, and the following holds true for some $q', q''$ and some constants $C_2, C_3(x),$
\[ |u(x, z)| \leq C_2, \]
\[ \sum_{|j|=1}^k |D^j_2 u(x, z)| \leq C_3(x)(1 + |z|^{q'}), \]
\[ |\nabla_x \nabla_z u(x, z)| \leq C_3(x)(1 + |z|^{q''}). \]
Proof. This is a combination of Proposition 1 and Theorem 3 from [PV03], but restricted for our needs. □

Lemma 3.9
Assume $H_f$, $H_g$ and $H^{k,2+\alpha}$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}_0$. Let $b, \sigma, h \in C^{k,0}$ satisfy for some $C, p > 0$,

$$\sum_{|\gamma| \leq k} \sup_z (|D^\gamma_b(x,z)| + |D^\gamma\sigma(x,z)| + |D^\gamma_h(x,z)|) \leq C(1 + |z|^p).$$

Then $\tilde{b}, \tilde{\sigma}, \tilde{a}, \tilde{h} \in C^k_b$.

Proof. The result follows from Lemma 3.2. □

Lemma 3.10
Assume $H_f$, $H_g$ and $H^{1,2+\alpha}$ for $\alpha \in (0,1)$. Let $b_1 \in C^{j,\alpha}$, for $j \in \mathbb{N}$, be centered with respect to $\mu_\infty$ with the growth condition

$$|b_1(x,z)| \leq C_0(1 + |z|)^\beta,$$

$$\sum_{|\gamma| = 1}^{j} |D^\gamma b_1(x,z)| \leq C(1 + |z|^q),$$

for $\beta < -2$. Then $\tilde{a} \in C^j_b$ and $\tilde{h} \in C^{j-1}_b$.

Proof. The result for $\tilde{a}$ follows from the fact that the assumptions on $b_1$ give $\mathcal{G}_F^{-1}(-b_1)(\cdot,z) \in C^j_b$ for each $z$ by Theorem 3.2 and the rest then follows from Lemma 3.2. For $\tilde{h}$ we again have from Theorem 3.2 that $|\nabla_x \mathcal{G}_F^{-1}(-b_1)| \lesssim (1 + |z|^q)$ and $\nabla_x \mathcal{G}_F^{-1}(-b_1)(\cdot,z) \in C^{j-1}_b$ for each $z$, and therefore we can use Lemma 3.2 to get the desired result. □

3.4 Estimates of the Unnormalized Conditional Distribution

Lemma 3.11
If $\rho^0$ satisfies Eq. 2.10, $\tilde{h}$ is bounded, and $\varphi \in C^2_b(\mathbb{R}^m; \mathbb{R})$, then for $p \geq 2$,

$$\mathbb{E}_Q \sup_{t \leq T} |\rho^0_t(\varphi)|^p < \infty.$$ 

If $\rho^e$ satisfies Eq. 2.9 and $h$ is bounded, then for $p \geq 2$,

$$\sup_{t \in [0,1]} \mathbb{E}_Q \sup_{t \leq T} |\rho^e_t(1)|^p < \infty.$$ 

Proof. For the first result, since $\varphi$ is bounded we have $|\rho^0_t(\varphi)| \leq |\varphi|_\infty \rho^0_t(1)$ and therefore we aim to show $\mathbb{E}_Q \sup_{t \leq T} |\rho^0_t(1)|^p < \infty$. Applying $\mathbb{E}_Q \sup_{t \leq T} |\cdot|^p$ to the evolution equation for $\rho^0_t(1)$ gives

$$\mathbb{E}_Q \sup_{t \leq T} |\rho^0_t(1)|^p \lesssim \mathbb{E}_Q |\rho^0_0(1)|^p + \mathbb{E}_Q \sup_{t \leq T} \left| \int_0^t \langle \rho^0_s(\varphi \tilde{h}), dY^x_s \rangle \right|^p$$

$$\lesssim 1 + T^{(p/2)-1} \int_0^T \mathbb{E}_Q |\rho^0_s(\varphi \tilde{h})|^p ds,$$

where in the last step we applied BDG, Hölder’s inequality, and Fubini. Using the boundedness of $\varphi \tilde{h}$ and Lemma 3.8, the integrand is bounded by

$$\mathbb{E}_Q |\rho^0_s(\varphi \tilde{h})|^p \leq \mathbb{E}_Q \left[ \mathbb{E}_{\tilde{h}} \left[ |\varphi \tilde{h}(X^0_s) \tilde{D}^0_s|^p \right] \right]$$

$$= \mathbb{E}_{\tilde{h}} \left[ \tilde{D}^0_s \mathbb{E}_{\tilde{h}} \left[ |\varphi \tilde{h}(X^0_s) \tilde{D}^0_s|^p \right] \right] \leq \frac{1}{2} \mathbb{E}_{\tilde{h}} \left( \tilde{D}^0_s \right)^2 + \frac{1}{2} |\varphi \tilde{h}|_{\infty}^2 \mathbb{E}_{\tilde{h}} \left( \tilde{D}^0_s \right)^{2p} < \infty.$$ 

The second result follows in the same manner. □
4 Existence, Characterization and Uniqueness of Weak Limits

Let $S(\mathbb{R}^m)$ be the space of finite signed Borel measures on $\mathbb{R}^m$ with the weak topology induced by $C_b(\mathbb{R}^m; \mathbb{R})$, and $C([0,T]; S(\mathbb{R}^m))$ the space of continuous paths with values in $S(\mathbb{R}^m)$ endowed with the topology of uniform convergence. For each $\epsilon$, we denote the $C([0,T]; S(\mathbb{R}^m))$-valued random variable $\zeta^\epsilon \equiv \rho^{x-} - \rho^0$, the difference of the $x$-marginal and averaged filter. With this notation, now define the $\epsilon$-parameterized family of Borel probability measures $(P^\epsilon)$ on $C([0,T]; S(\mathbb{R}^m))$, to be those induced by $(\zeta^\epsilon)$,

$$P^\epsilon(\cdot) = \mathbb{Q} \left( (\zeta^\epsilon)^{-1}(\cdot) \right).$$

We will need to prove a uniform concentration condition\footnote{This is sometimes referred to as compact confinement condition in the literature.} of the collection $(P^\epsilon)$ for the proof of the existence of weak limits of $(\zeta^\epsilon)$. In the context of our problem, the uniform concentration condition is the following:

**Definition 4.1 (Uniform Concentration Condition).** $(P^\epsilon)$ is said to satisfy the uniform concentration condition if for each $\delta > 0$, there exists a compact set $K_\delta \subset S(\mathbb{R}^m)$ such that

$$P^\epsilon(C([0,T]; K_\delta)) \geq 1 - \delta, \quad \forall \epsilon. \quad (4.1)$$

We now prove a lemma that provides a sufficient condition for the uniform concentration condition.

**Lemma 4.1**

*The uniform concentration condition holds if for some $p > 0$ and continuous $M : \mathbb{R}^m \to (0, \infty)$ with $\lim_{|x| \to \infty} M(x) = \infty$, we have

$$\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} (|\zeta_t^\epsilon| (M))^p < \infty. \quad (4.2)$$

Here $|\zeta_t^\epsilon|$ is the total variation measure of $\zeta_t^\epsilon$.*

*Proof. We first show that for $C > 0$, the set

$$K = \{ \mu \in S(\mathbb{R}^m) \mid |\mu|(M) \leq C \}$$

is tight. Given $\delta > 0$, choose $R > 0$ large enough such that $\inf_{|x| \geq R} M(x) \geq C/\delta$. Then denoting $A_\delta = B(0, R) \subset \mathbb{R}^m$, the closed ball centered at the origin with radius $R$, we have that for any $\mu \in K$

$$|\mu|(A_\delta) = |\mu|(1_{|x| \geq R}) \leq |\mu| \left( \left( \frac{\delta}{C} M \right) 1_{|x| \geq R} \right) \leq \frac{\delta}{C} |\mu|(M) \leq \frac{\delta}{C} C = \delta.$$ 

This shows that $K$ is tight. Moreover, since $M$ is bounded from below by some $m > 0$, we have

$$\sup_{\mu \in K} |\mu|(1) \leq \frac{1}{m} \sup_{\mu \in K} |\mu|(M) \leq \frac{C}{m},$$

and therefore $K$ is bounded in total variation norm. Since $S(\mathbb{R}^m)$ with the weak topology induced by $C_b(\mathbb{R}^m; \mathbb{R})$ is Polish, we have by Prokhorov’s theorem that $K$ is relatively compact. Further, by Fatou’s lemma for weak convergence, $K$ is also closed and therefore compact.

Given $\delta > 0$, choose $C > 0$ large enough so that

$$\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} (|\zeta_t^\epsilon| (M))^p \leq \delta.$$

Defining our compact set $K_\delta = \{ \mu \in S(\mathbb{R}^m) \mid |\mu|(M) \leq C \}$, we have

$$\mathbb{Q} (\zeta^\epsilon \notin C([0,T]; K_\delta)) \leq \mathbb{Q} \left( \sup_{t \leq T} |\zeta_t^\epsilon|(M) > C \right) \leq \frac{\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} (|\zeta_t^\epsilon| (M))^p}{C^p} \leq \delta.$$
The next result uses Lemma 4.1 to prove that \( (P^\epsilon) \) is tight.

**Lemma 4.2**
Assume that \( f \) and \( g \) satisfy the assumptions of \( H_f \), \( H_g \) and \( H^{2,3+\alpha} \) for \( \alpha \in (0,1) \), and that \( b,\sigma,h \) are bounded. Let \( b_1 \in C^{2,\alpha} \), be centered with respect to \( \mu_\infty(x) \) for each \( x \), and satisfy the growth conditions

\[
|b_1(x,z)| \leq C(1+|z|)^\beta,
\]

\[
\sum_{|i|=1}^2 |D_2^ib_1(x,z)| \leq C(1+|z|^q),
\]

for some \( \beta < -2 \) and \( q > 0 \). Assume that \( Q_{(X_0^\epsilon,x_0^\epsilon)} \) has finite moments of every order. Then the \( \epsilon \)-parameterized family of Borel probability measures \( (P^\epsilon) \) is tight.

**Proof.** To prove the statement, we follow criteria provided in [Jak86, Theorem 3.1, p.276], which gives conditions for a family of Borel probability measures on \( D([0,T];E) \), càdlàg path space with \( E \) a completely regular topological space with metrizable compacts, to be tight. \( C([0,T];S(\mathbb{R}^m)) \) is viewed as a subset of \( D([0,T];S(\mathbb{R}^m)) \), and \( S(\mathbb{R}^m) \) with the weak topology induced by \( C_b^2(\mathbb{R}^m;\mathbb{R}) \) is Polish and therefore a completely regular topological space with metrizable compacts.

Specifically, let \( \mathbb{F} \) be the natural injection of \( C_b^2(\mathbb{R}^m;\mathbb{R}) \) into its double dual. This collection satisfies criteria for [Jak86, Theorem 3.1, p.276], i.e., it is a collection of continuous functions that separate points in \( S(\mathbb{R}^m) \), and is closed under addition (i.e., \( f,g \in \mathbb{F} \), then \( f+g \in \mathbb{F} \)). Then to each \( f \in \mathbb{F} \) associate a map \( \tilde{f} \in \mathbb{F} \), characterized as follows,

\[
\tilde{f} : C([0,T];S(\mathbb{R}^m)) \rightarrow C([0,T];\mathbb{R}) \quad \mu \mapsto f \circ \mu.
\]

The conditions for tightness by [Jak86, Theorem 3.1, p.276] then states that \( (P^\epsilon) \) is tight if and only if the following two conditions are satisfied:

(i) For each \( \delta > 0 \) there is a compact set \( K_\delta \subset S(\mathbb{R}^m) \) such that

\[
P^\epsilon(C([0,T];K_\delta)) > 1 - \delta, \quad \forall \epsilon
\]

(ii) The family \( (P^\epsilon) \) is \( \mathbb{F} \)-weakly tight, i.e., for each \( f \in \mathbb{F} \) the family \( (P^\epsilon \circ (\tilde{f}^{-1})) \) of probability measures on \( C([0,T];\mathbb{R}) \) is tight.

The proof of (i) will follow from Lemma 4.1, which provides a sufficient condition for the uniform concentration condition. To prove the lemma, we define \( M(x) = (1+|x|^2)^{1/2} \), let \( p \geq 2 \) and check that the following condition holds,

\[
\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} \|\zeta^\epsilon_t(M)\|_p^p < \infty. \tag{4.2}
\]

Note that \( M(x) \) satisfies the conditions for Lemma 4.1 and has bounded first and second order derivatives,

\[
\left| \frac{\partial}{\partial x_i} M(x) \right| = \left| \frac{x_i}{M(x)} \right| \lesssim 1,
\]

\[
\left| \frac{\partial^2}{\partial x_i \partial x_j} M(x) \right| = \left| \frac{x_i x_j}{M(x)^2} + \delta_{ij} \right| \lesssim \frac{1}{M(x)} \lesssim 1.
\]

Directly estimating, we have

\[
\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} \|\zeta^\epsilon_t(M)\|_p^p = \sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} (\rho^\epsilon_t \cdot \sigma^\epsilon_t M(x)^3 + \rho^\epsilon_t \cdot \sigma^\epsilon_t M(x)) \lesssim \sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} \|\rho^\epsilon_t \cdot \sigma^\epsilon_t M(x)^3\|_p^p + \mathbb{E}_Q \sup_{t \leq T} \|\rho^\epsilon_t M(x)\|_p^p.
\]
Dealing with each term separately, we first address \( \sup_{t \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} (\rho^{\epsilon}_{\epsilon}(M))^p \). To handle the singular term (the intermediate drift) in the slow process, we perturb \( M \) by a corrector term. Define the perturbed test function,

\[
M'(x, z) = M(x) + \epsilon \chi(x, z),
\]

with \( \chi(x, z) \) the solution of the Poisson equation,

\[
\mathcal{G}_r \chi = -\mathcal{G}_I M.
\]

The Poisson equation is well-posed with the right hand side satisfying the conditions of Theorem 3.2 (recall that \( b_1 \) has the correct decay in the \( z \) variable), and therefore the regularity and bounds of \( \chi \) come from Theorem 3.2. Specifically,

\[
|\chi(x, z)| \lesssim 1,
\]

\[
\sum_{|i|=1}^2 |D^i \chi(x, z)| \lesssim (1 + |z|^q'),
\]

for some \( q' > 0 \). Using the identity \( \rho^{\epsilon}_{\epsilon}(M') = \rho^{\epsilon}_{\epsilon}(M) + \epsilon \rho^{\epsilon}_{\epsilon}(\chi) \), we have the representation

\[
\rho^{\epsilon}_{\epsilon}(M) = -\epsilon \rho^{\epsilon}_{\epsilon}(\chi) + \rho^{\epsilon}_{\epsilon}(M') + \int_0^t \rho^{\epsilon}_{\epsilon}(\mathcal{G}_s M) \, ds + \int_0^t \rho^{\epsilon}_{\epsilon}(\mathcal{G}_I \chi) \, ds + \epsilon \int_0^t \rho^{\epsilon}_{\epsilon}(\mathcal{G}_S \chi) \, ds
\]

\[
+ \int_0^t \left( \langle \rho^{\epsilon}_{\epsilon}(\chi h + \alpha \sigma^* \nabla_x M), dY^*_x \rangle + \epsilon \int_0^t \langle \rho^{\epsilon}_{\epsilon}(\chi h + \alpha \sigma^* \nabla_x \chi), dY^*_x \rangle \right).
\]

And therefore,

\[
\mathbb{E}_Q \sup_{t \leq T} |\rho^{\epsilon}_{\epsilon}(M)|^p \lesssim \epsilon^p \mathbb{E}_Q \sup_{t \leq T} |\rho^{\epsilon}_{\epsilon}(\chi)|^p + \mathbb{E}_Q |\rho^{\epsilon}_{\epsilon}(M')|^p
\]

\[
+ \mathbb{E}_Q \int_0^T |\rho^{\epsilon}_{\epsilon}(\mathcal{G}_s M)|^p \, ds + \mathbb{E}_Q \int_0^T |\rho^{\epsilon}_{\epsilon}(\mathcal{G}_I \chi)|^p \, ds + \epsilon^p \mathbb{E}_Q \int_0^T |\rho^{\epsilon}_{\epsilon}(\mathcal{G}_S \chi)|^p \, ds
\]

\[
+ \mathbb{E}_Q \sup_{t \leq T} \int_0^t \left( \langle \rho^{\epsilon}_{\epsilon}(\alpha \sigma^* \nabla_x M), dY^*_x \rangle \right)^p + \mathbb{E}_Q \sup_{t \leq T} \int_0^t \langle \rho^{\epsilon}_{\epsilon}(\alpha \sigma^* \nabla_x \chi), dY^*_x \rangle \right)^p.
\]

By the boundedness of \( \chi \) and application of Lemma 3.11, the first term of Eq. 4.3 is

\[
\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} |\rho^{\epsilon}_{\epsilon}(\chi)|^p \lesssim \sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} \mathbb{E}_Q |\rho^{\epsilon}_{\epsilon}(M')|^p\lesssim \sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} |\rho^{\epsilon}_{\epsilon}(1)|^p < \infty.
\]

For the second term of Eq. 4.3, we have

\[
\mathbb{E}_Q |\rho^{\epsilon}_{\epsilon}(M')|^p = \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ M'(X_0', Z_0) \mathcal{D}_0 Y_0^X \right] \right]^p = \mathbb{E}_Q \left[ \mathbb{E}_Q \left[ M'(X_0', Z_0) \right] \right]^p
\]

\[
\leq \int |M'(x, z)|^p \mathbb{Q}_{(X_0', Z_0)}(dx, dz) \lesssim \int M(x)^p + \epsilon^p |\chi(x, z)|^p \mathbb{Q}_{(X_0', Z_0)}(dx, dz),
\]

and therefore

\[
\sup_{\epsilon \in (0,1]} \mathbb{E}_Q |\rho^{\epsilon}_{\epsilon}(M')|^p < \infty.
\]

Where we used the boundedness of \( \chi \) and the fact that we have finite moments of every order for \( \mathbb{Q}_{(X_0', Z_0)} \).

From the boundedness of \( b, \sigma, D^2 \mathcal{G} M \), for \( |\gamma| \in \{1, 2\} \), Lemma 3.8 and 3.11, we have for the third term of Eq. 4.3

\[
\sup_{\epsilon \in (0,1]} \mathbb{E}_Q \int_0^T |\rho^{\epsilon}_{\epsilon}(\mathcal{G}_S M)|^p \, ds \leq |\mathcal{G}_S M|_{\infty}^p \sup_{\epsilon \in (0,1]} \mathbb{E}_Q \sup_{t \leq T} |\rho^{\epsilon}_{\epsilon}(1)|^p ds < \infty.
\]
For the fourth term of Eq. 4.3

\[ \mathbb{E}_Q \int_0^T |\rho_s^\varepsilon(\mathcal{G}_t^\varepsilon(x))|^p \, ds \lesssim \int_0^T \mathbb{E}_Q \left[ (\tilde{D}_s^\varepsilon)^2 \right]^{1/2} \mathbb{E}_p \left[ |\mathcal{G}_t^\varepsilon(x, Z_s^\varepsilon) \tilde{D}_s^\varepsilon|^{2p} \right]^{1/2} \, ds \]

\[ \leq \int_0^T \mathbb{E}_p \left[ (\tilde{D}_s^\varepsilon)^2 \right]^{1/2} \mathbb{E}_p \left[ (\tilde{D}_s^\varepsilon)^4 \right]^{1/4} \mathbb{E}_p \left[ |\mathbb{G}_\varepsilon(X_s^\varepsilon, Z_s^\varepsilon)|^{4p} \right]^{1/4} \, ds. \]

And we have

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_p\left[ |\mathcal{G}_\varepsilon(X_s^\varepsilon, Z_s^\varepsilon)|^{4p} \right] \lesssim \sup_{\varepsilon \in (0,1]} \mathbb{E}_p \left[ 1 + |Z_s^\varepsilon|^q \right] \lesssim 1 + \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \left[ |Z_0^\varepsilon|^q \right] < \infty, \quad (4.4) \]

for some \( q > 0 \), and therefore

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \int_0^T |\rho_s^\varepsilon(\mathcal{G}_t^\varepsilon(x))|^p \, ds \lesssim \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \int_0^T \mathbb{E}_p \left[ |\mathcal{G}_t^\varepsilon(X_s^\varepsilon, Z_s^\varepsilon)|^{4p} \right]^{1/4} \, ds < \infty. \]

By the same arguments as for the fourth term, we have that the fifth term of Eq. 4.3 is bounded uniformly in \( \varepsilon \),

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \int_0^T |\rho_s^\varepsilon(\mathcal{G}_t^\varepsilon(x))|^p \, ds < \infty. \]

The first stochastic integral, sixth term of Eq. 4.3, is handled with application of BDG, Hölder’s inequality, and Fubini on the first line and then change of measure, Cauchy-Schwarz, and Hölder’s inequality to give

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \| \int_0^t \langle \rho_s^\varepsilon(Mh), dY_s^\varepsilon \rangle \|_p \lesssim \sup_{\varepsilon \in (0,1]} \int_0^T \mathbb{E}_Q \| \rho_s^\varepsilon(Mh) \|^p \, ds \]

\[ \lesssim \sup_{\varepsilon \in (0,1]} \int_0^T \mathbb{E}_p \left[ (\tilde{D}_s^\varepsilon)^2 \right]^{1/2} \mathbb{E}_p \left[ |\rho_s^\varepsilon(Mh)|^{2p} \right]^{1/2} \, ds. \]

And by further application of Cauchy-Schwarz, Hölder’s inequality, boundedness of \( h \), Lemma 3.6 for some \( q > 0 \), and finite moments of all orders for \( \mathcal{Q}_\varepsilon(x_0, Z_0^\varepsilon) \), we get

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_p \left[ |\rho_s^\varepsilon(Mh)|^{2p} \right]^{1/2} \lesssim \sup_{\varepsilon \in (0,1]} \mathbb{E}_p \left[ (\tilde{D}_s^\varepsilon)^4 \right]^{1/4} \mathbb{E}_p \left[ |M(X_s^\varepsilon)|^{4p} \right]^{1/4} \]

\[ \lesssim \sup_{\varepsilon \in (0,1]} \left[ (1 + |X_s^\varepsilon|^2)^{4p} \right]^{1/4} \lesssim \sup_{\varepsilon \in (0,1]} (1 + \mathbb{E}_Q \left[ |X_s^\varepsilon|^4 \right])^{1/4} \]

\[ \lesssim \sup_{\varepsilon \in (0,1]} (1 + \mathbb{E}_Q \left[ |X_0^\varepsilon|^4 + (1 + \varepsilon^4p)(1 + |Z_0^\varepsilon|^q) \right])^{1/4} < \infty. \]

Because \( \alpha \sigma^\dagger \nabla_x M \) is bounded and by Lemma 3.11, the seventh term of Eq. 4.3 is bounded uniformly in \( \varepsilon \),

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \sup_{\varepsilon \leq T} \left| \int_0^t \langle \rho_s^\varepsilon(\alpha \sigma^\dagger \nabla_x M), dY_s^\varepsilon \rangle \right|^p \lesssim \int_0^T \mathbb{E}_Q \left| \rho_s^\varepsilon(\alpha \sigma^\dagger \nabla_x M) \right|^p \, ds < \infty. \]

Similarly using the boundedness of \( \chi h \), the eighth term of Eq. 4.3 is bounded uniformly in \( \varepsilon \),

\[ \sup_{\varepsilon \in (0,1]} \mathbb{E}_Q \sup_{\varepsilon \leq T} \left| \int_0^t \langle \rho_s^\varepsilon(\chi h), dY_s^\varepsilon \rangle \right|^p < \infty. \]

Making use of the boundedness of \( \sigma \), the polynomial growth of \( \nabla_x \chi \) in \( z \), and the finite moments of all orders
This completes the calculation that sup, \( E_Q \sup_{t \leq T} |\rho_t^\sigma(M)|^p < \infty \). We now show that \( E_Q \sup_{t \leq T} |\rho_t^\sigma(M)|^p < \infty \). We have,

\[
E_Q \sup_{t \leq T} |\rho_t^\sigma(M)|^p \lesssim E_Q |\rho_0^\sigma(M)|^p + \int_0^T E_Q |\rho_0^\sigma(\mathbb{G}^1 M)|^p ds \\
+ E_Q \sup_{t \leq T} \left| \int_0^t \langle \rho_s^0(M \tilde{h}), dY_s^\gamma \rangle \right|^p + E_Q \sup_{t \leq T} \left| \int_0^t \langle \rho_s^0(\alpha \sigma \nabla_x M), dY_s^\gamma \rangle \right|^p.
\]

By Lemma 3.9 and 3.10, we have that \( \tilde{h}, \tilde{\sigma}, \tilde{h}, \tilde{a} \) and \( \tilde{b} \) are all bounded functions. Therefore, similar arguments as for \( \rho' \) show that the right side of Eq. 4.5 is finite.

We now prove (ii), the \( \mathbb{F} \)-weak tightness condition. Let \( \varphi \in C^2_b(\mathbb{R}^m; \mathbb{R}) \). There are two conditions that must be checked for each fixed \( \varphi \in \mathbb{F} \). The first one is a boundedness condition,

\[
\lim_{N \to \infty} \sup_{\varepsilon} \mathbb{Q}(|\zeta^\varepsilon_\sigma(\varphi)| \geq N) = 0,
\]

which is trivially satisfied since \( \zeta^\varepsilon_\sigma(\varphi) = E_Q [\varphi(X^\varepsilon_\sigma)] - E_Q [\varphi(X^\varepsilon_0)] = 0 \). The second one is an equicontinuity condition–for each \( \varepsilon, \theta > 0 \) and \( \eta > 0 \) there are \( \Delta > 0 \) and \( j < \infty \) such that

\[
\mathbb{Q} \left( \sup_{|t-s| \leq \Delta} |\zeta^\varepsilon_\sigma(\varphi) - \zeta^\varepsilon_\sigma(\varphi)| \geq \delta \right) \leq \eta, \quad \forall i \geq j.
\]

A sufficient condition for Eq. 4.6 is the following–there are \( \mu, \beta, \gamma > 0 \) and \( K < \infty \) such that

\[
E_Q |\zeta^\varepsilon_\sigma(\varphi) - \zeta^\varepsilon_\sigma(\varphi)|^p \leq K |t-s|^{1+\beta} + \varepsilon^\gamma, \quad \forall i.
\]

We now show that Eq. 4.7 is true. Let \( \mu = 4, \varepsilon > 0 \) and \( \varphi^\varepsilon = \varphi + \epsilon \chi \), where \( \chi \) solves the Poisson equation,

\[
\mathbb{G} \chi = -\mathbb{G}_F \varphi.
\]

From \( \zeta^\varepsilon_\sigma(\varphi) = \rho^\varepsilon_\sigma(\varphi) - \rho^\varepsilon_0(\varphi) \) and \( \rho^\varepsilon_\gamma(\varphi') = \rho^\varepsilon_\gamma(\varphi) + \rho^\varepsilon_\sigma(\epsilon \chi) \) we have

\[
|\zeta^\varepsilon_\sigma(\varphi) - \zeta^\varepsilon_\sigma(\varphi)|^4 \lesssim |\rho^\varepsilon_\gamma(\varphi') - \rho^\varepsilon_0(\varphi')|^4 + |\rho^\varepsilon_\gamma(\varphi) - \rho^\varepsilon_0(\varphi)|^4 + |\rho^\varepsilon_\sigma(\epsilon \chi) - \rho^\varepsilon_0(\epsilon \chi)|^4.
\]

Working on the first term in this inequality,

\[
E_Q |\rho^\varepsilon_\gamma(\varphi') - \rho^\varepsilon_0(\varphi')|^4 \lesssim E_Q \left| \int_s^t \rho^\varepsilon_\gamma(\mathbb{G} \varphi + \mathbb{G}_F \chi + \epsilon \mathbb{G} \chi) du \right|^4 + E_Q \left| \int_s^t \langle \rho^\varepsilon_\gamma(\varphi' h + \alpha \sigma \nabla_x \varphi'), dY^\gamma_\gamma \rangle \right|^4 \\
\lesssim (t-s)^3 \int_s^t E_Q |\rho^\varepsilon_\gamma(\mathbb{G} \varphi + \mathbb{G}_F \chi + \epsilon \mathbb{G} \chi)|^4 du + (t-s) \int_s^t E_Q |\rho^\varepsilon_\gamma(\varphi' h + \alpha \sigma \nabla_x \varphi')|^4 du.
\]

First term of Eq. 4.9,

\[
\int_s^t E_Q |\rho^\varepsilon_\gamma(\mathbb{G} \varphi + \mathbb{G}_F \chi + \epsilon \mathbb{G} \chi)|^4 du \lesssim \int_s^t E_Q |\rho^\varepsilon_\gamma(\mathbb{G} \varphi)|^4 + E_Q |\rho^\varepsilon_\gamma(\mathbb{G}_F \chi)|^4 + E_Q |\rho^\varepsilon_\gamma(\epsilon \mathbb{G} \chi)|^4 du.
\]

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By the boundedness of \( b, \sigma, D_x^k \varphi \) for \(|k| \leq 2\) and Lemma 3.8, we have the first term bounded by

\[
\int_s^t \mathbb{E}_Q |\rho_u^*(G_S\varphi)|^4 \, du \lesssim (t - s).
\]

By the boundedness of \( b, \sigma \), the polynomial growth of \( b, D_x^k \varphi \) in \( z \), for \(|k| \leq 2\) and Lemma 3.8, we have by the same arguments as Eq. 4.4

\[
\int_s^t \mathbb{E}_Q |\rho_u^*(G_1\varphi)|^4 \, du + \int_s^t \mathbb{E}_Q |\rho_u^*(\epsilon G_S\chi)|^4 \, du \lesssim (t - s) + \epsilon^4(t - s) = (1 + \epsilon^4)(t - s).
\]

And therefore we have,

\[
\int_s^t \mathbb{E}_Q |\rho_u^*(G_S\varphi + G_1\chi + \epsilon G_S\chi)|^4 \, du \lesssim (1 + \epsilon^4)(t - s).
\]

The second term of Eq. 4.9 is bounded as follows,

\[
\int_s^t \mathbb{E}_Q |\rho_u^*(\varphi^* h + \alpha \sigma^* \nabla_x \varphi^*)|^4 \, du \lesssim \int_s^t \mathbb{E}_Q |\rho_u^*(\varphi h)|^4 + \mathbb{E}_Q |\rho_u^*(\epsilon \chi h)|^4 \, du + \int_s^t \mathbb{E}_Q |\rho_u^*(\alpha \sigma^* \nabla_x \varphi)|^4 \, du + \mathbb{E}_Q |\rho_u^*(\epsilon \alpha \sigma^* \nabla_x \chi)|^4 \, du
\]

\[
\lesssim (1 + \epsilon^4)(t - s) + \mathbb{E}_Q |\rho_u^*(\epsilon \sigma^* \nabla_x \chi)|^4 \, du,
\]

by boundedness of \( h, \sigma, \chi, D_x^k \varphi \) for \(|k| \leq 2\), and Lemma 3.8. The last term is bounded by \( \lesssim \epsilon^4(t - s) \) due to boundedness of \( \sigma \) and polynomial growth of \( \nabla_x \chi \) in \( z \) and finite moments of all orders for \( Q_{\{X_0^\gamma, Z_0^\gamma\}} \), and therefore

\[
\int_s^t \mathbb{E}_Q |\rho_u^*(\varphi^* h + \alpha \sigma^* \nabla_x \varphi^*)|^4 \, du \lesssim (1 + \epsilon^4)(t - s).
\]

The expectation of the second term in Eq. 4.8 is,

\[
\mathbb{E}_Q |\rho_0^*(\varphi) - \rho_s^*(\varphi)|^4 \lesssim \mathbb{E}_Q \left| \int_s^t \rho_u^0(G^1\varphi) \, du \right|^4 + \mathbb{E}_Q \left| \int_s^t (\rho_u^0(\varphi h + \alpha \sigma^* \nabla_x \varphi), dY^\gamma_u) \right|^4
\]

\[
\lesssim (t - s)^3 \int_s^t \mathbb{E}_Q |\rho_u^0(G^1\varphi)|^4 \, du + (t - s) \int_s^t \mathbb{E}_Q |\rho_u^0(\varphi h + \alpha \sigma^* \nabla_x \varphi)|^4 \, du
\]

\[
\lesssim (t - s)^4 + (t - s)^2,
\]

where we use the boundedness of \( \bar{h}, \bar{\sigma} \) and the coefficients of \( G^1 \), which is a result of Lemma 3.9 and 3.10, and the boundedness of \( D_x^k \varphi \) for \( k \leq 2 \).

Using the fact that \( \chi \) is bounded and Lemma 3.11, the last term of Eq. 4.8 is bounded by

\[
\mathbb{E}_Q |\rho_t^*(\epsilon \chi) - \rho_s^*(\epsilon \chi)|^4 \lesssim \epsilon^4 \mathbb{E}_Q [\rho_t^*(1)^4 + \rho_s^*(1)^4] \lesssim \epsilon^4.
\]

Lastly, since we are interested in the case \((t - s) < 1\), we satisfy Eq. 4.7 with \( \mu = 4, \beta = 1 \) and \( \gamma = 4 \).

Now that \( (P^\epsilon) \) has been shown to be tight, we need to show that this collection is weakly relatively compact and therefore any subsequence will converge to a weak limit. The fact that \( (P^\epsilon) \) is weakly relatively compact is given by the following corollary.

**Corollary 4.1**

*Every subsequence of the \( \epsilon \)-parameterized family of probability measures \( (P^\epsilon) \) induced on path space \( C([0, T]; S(\mathbb{R}^m)) \) by \( \zeta^\epsilon \), has a weak limit.*
Proof. From Lemma 4.2, \((P^r)\) is tight. Because \(C([0, T]; S(\mathbb{R}^m))\) is Hausdorff, since it is metrizable by [Jak86, Proposition 1.6iii, p.267] because \(S(\mathbb{R}^m)\) is Polish, this implies that \((P^r)\) is relatively compact (see for instance [KX95, Theorem 2.2.1, p.56]). Therefore each sequence of \((P^r)\) has a convergent subsequence. 

Given a subsequence of \((\epsilon)\), we now let \(\zeta\) be the limit point and characterize this limit point in the next lemma.

**Lemma 4.3**

Assume that \(f, g\) satisfy the assumptions of \(H_f\), \(H_g\) and \(H^{2,2+\alpha}\). Let \(b, b_1, a \in C^{2,\alpha}\) and satisfy the growth conditions

\[
|b(x, z)| + |b_1(x, z)| + |a(x, z)| \leq C(1 + |z|)^\beta,
\]

\[
\sum_{|k|=1}^2 |D^k b_1(x, z)| + |D^k b_1(x, z)| + |D^k a(x, z)| \leq C(1 + |z|)^q,
\]

for some \(\beta < -2\) and \(q > 0\). Let \(b_1\) be centered with respect to \(\mu_\infty(x)\) for each \(x\). Assume \(h\) is bounded and globally Lipschitz in \((x, z)\). Let \(\sigma\) be globally Lipschitz in \(z\). And assume that \(\mathbb{Q}(X, Z_0)\) has finite moments of every order. Then any limit point \(\zeta\) of \((\zeta')\) satisfies the equation,

\[
\zeta_t(\varphi) = \int_0^t \zeta_s(G^1 \varphi)ds + \int_0^t \langle \zeta_s(\varphi \bar{\chi} + \alpha \sigma^* \nabla_x \varphi), dY_s \rangle, \quad \zeta_0(\varphi) = 0, \quad \mathbb{Q}\text{-a.s. uniformly in } t \in [0, T].
\]

**Proof.** With an abuse of notation, let \(\epsilon\) be an element of the subsequence \((\epsilon)\), assume \(\varphi \in C_0^1(\mathbb{R}^m; \mathbb{R})\), and consider the perturbed test function,

\[
\varphi'(x, z) = \varphi(x) + \epsilon \chi(x, z) + \epsilon^2 \tilde{\chi}(x, z),
\]

where \(\chi\) and \(\tilde{\chi}\) solve the Poisson equations,

\[
G_F \chi = -G_t \varphi, \quad G_F \tilde{\chi} = -(G_S - \bar{G}_S) \varphi - (G_t \chi - \bar{G}_t \chi).
\]

From Theorem 3.2, we have

\[
|\chi(x, z)| + |\tilde{\chi}(x, z)| \lesssim 1,
\]

\[
\sum_{|i|=1}^2 |D^i \chi(x, z)| + |D^i \tilde{\chi}(x, z)| \lesssim (1 + |z|^{q'})
\]

for some \(q' > 0\). Because \(\rho_t^x(\varphi) = \rho_t^x(\varphi') - \rho_t^x(\epsilon \chi) - \rho_t^x(\epsilon^2 \tilde{\chi})\), we have

\[
\zeta_t^x(\varphi) = -\rho_t^x(\epsilon \chi) - \rho_t^x(\epsilon^2 \tilde{\chi}) + \rho_t^x(\varphi') = \int_0^t \rho_t^x(G^1 \varphi')ds - \int_0^t \rho_t^0(G^1 \varphi)ds + \int_0^t \langle \rho_t^x(\varphi' h + \alpha \sigma^* \nabla_x \varphi'), dY_s \rangle - \int_0^t \langle \rho_t^0(\varphi \tilde{\chi} + \alpha \sigma^* \nabla_x \varphi'), dY_s \rangle. \tag{4.10}
\]

When expanded, the Lebesgue integral for \(\rho_t^x(G^1 \varphi')\) becomes,

\[
\int_0^t \rho_t^x(G^1 \varphi')ds = \int_0^t \rho_t^x(G_S \varphi)ds + \int_0^t \rho_t^x(G_t \chi)ds + \epsilon \int_0^t \rho_t^x(G_S \chi)ds + \epsilon \int_0^t \rho_t^x(G_t \tilde{\chi})ds + \epsilon^2 \int_0^t \rho_t^x(G_S \tilde{\chi})ds.
\]

The term \(\rho_t^0(\varphi') - \rho_t^0(\varphi)\) is,

\[
\rho_t^0(\varphi') - \rho_t^0(\varphi) = \rho_t^0(\varphi) + \rho_t^0(\epsilon \chi) + \rho_t^0(\epsilon^2 \tilde{\chi}) - \rho_t^0(\varphi)
= \rho_t^0(\epsilon \chi) + \rho_t^0(\epsilon^2 \tilde{\chi}).
\]
And we group all terms of first order in $\epsilon$ involving $\chi$ into $O_{\chi}(\epsilon)$,

$$O_{\chi}(\epsilon) = -\rho'_t(\epsilon^2 \chi) + \rho_0(\epsilon^2 \chi) + \epsilon \int_0^t \rho_\kappa^s(\partial \chi \epsilon) ds + \epsilon \int_0^t \langle \rho_\kappa^s(\chi \epsilon^2 + \alpha \sigma \cdot \nabla \chi), dY_s^\kappa \rangle.$$

Similarly, let the terms of first and second order in $\epsilon$ involving $\bar{\chi}$ be grouped into $O_{\bar{\chi}}(\epsilon)$,

$$O_{\bar{\chi}}(\epsilon) = -\rho'_t(\epsilon^2 \bar{\chi}) + \rho_0(\epsilon^2 \bar{\chi}) + \epsilon \int_0^t \rho_\kappa^s(\partial \bar{\chi} \epsilon) ds + \epsilon^2 \int_0^t \rho_\kappa^s(\partial \bar{\chi} \epsilon) ds + \epsilon^2 \int_0^t \langle \rho_\kappa^s(\bar{\chi} \epsilon^2 + \alpha \sigma \cdot \nabla \bar{\chi}), dY_s^\kappa \rangle.$$

Eq. 4.10 now becomes,

$$\zeta'_t(\phi) = \int_0^t \zeta'_s(\partial \chi \epsilon) ds + \int_0^t \rho_\kappa^s(\partial \bar{\chi} \epsilon) ds + O_{\chi}(\epsilon) + O_{\bar{\chi}}(\epsilon)$$

$$+ \int_0^t \langle \rho_\kappa^s(\phi \epsilon^2 + \alpha \sigma \cdot \nabla \phi), dY_s^\kappa \rangle - \int_0^t \langle \rho_\kappa^s(\phi \bar{\chi} + \alpha \sigma \cdot \nabla \phi), dY_s^\kappa \rangle. \tag{4.11}$$

Next, consider the equivalence of the following terms

$$\int_0^t \rho_\kappa^s(\partial \chi \epsilon) ds = \int_0^t \rho_\kappa^s(\partial \phi) ds.$$

This follows since $\nabla^2 \phi = \nabla \cdot \nabla \phi$ is symmetric, and therefore,

$$(\nabla^2 \phi G_F^{-1}(-b_1), b_1) = \sum_{i,j=1}^m \frac{\partial^2 \phi}{\partial x_i \partial x_j} G_F^{-1}(-b_1), b_1_{ij}$$

$$= \sum_{i,j=1}^m \frac{\partial^2 \phi}{\partial x_i \partial x_j} \left( \frac{1}{2} (G_F^{-1}(-b_1) \otimes b_1)_{ij} + \frac{1}{2} (b_1 \otimes G_F^{-1}(-b_1))_{ij} \right),$$

which leads to the following,

$$\bar{G}_\chi(x) = \int_{\mathbb{R}^n} \langle \nabla \phi, b_1 \rangle(x, z) \mu(x, z) dz$$

$$= \langle \nabla \phi, b_1 \rangle \int_{\mathbb{R}^n} \mu(x, z) dz = \langle \nabla \phi, b_1 \rangle.$$
Similarly, from the boundedness of $b, \sigma, a, h$, the growth conditions on $D^k_x \tilde{X}$ for $|k| \leq 2$ and $b_1$, and the finite moments of all orders for $\mathbb{Q}_t(x,z)$, we have that

$$\lim_{\epsilon \to 0} \mathbb{E}_\mathbb{Q} \sup_{t \leq T} |O_{X(t)}(\epsilon)|^2 = 0.$$ 

Our focus now shifts to showing that

$$\lim_{\epsilon \to 0} \mathbb{E}_\mathbb{Q} \sup_{t \leq T} \left| \int_0^t \langle \rho_{cs}(h - \overline{h}), dY_s^\epsilon \rangle \right|^2 = 0, \quad \text{and} \quad \lim_{\epsilon \to 0} \mathbb{E}_\mathbb{Q} \sup_{t \leq T} \left| \int_0^t \langle \rho_{cs}(\alpha(\sigma - \overline{\sigma})^s \nabla_x \varphi), dY_s^\epsilon \rangle \right|^2 = 0. \quad (4.13)$$

Let $\psi_h(x,z) \equiv \varphi(h - \overline{h})(x,z)$ and $\psi_\sigma(x,z) \equiv \alpha(\sigma - \overline{\sigma})^s \nabla_x \varphi(x,z)$. Because $\psi_h$ and $\psi_\sigma$ are both centered with respect to $\mu_\infty(x)$ for each $x$, globally Lipschitz in $(x,z)$, and bounded in $(x,z)$, we let $\psi$ represent either $\psi_h$ or $\psi_\sigma$ and perform the same analysis for both.

Applying BDG to either term in Eq. 4.13, we get

$$\mathbb{E}_\mathbb{Q} \sup_{t \leq T} \left| \int_0^t \langle \rho_{cs}(\psi), dY_s^\epsilon \rangle \right|^2 \lesssim \mathbb{E}_\mathbb{Q} \int_0^T |\rho_{cs}(\psi)|^2 \, ds = \mathbb{E}_\mathbb{Q} \int_0^T \mathbb{E}_{\mathbb{P}_\epsilon} \left[ |\psi(X_s^\epsilon, Z_s^\epsilon)\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] = 2 \, ds. \quad (4.14)$$

We now follow the argument by Kushner [Kus90, Chapter 6] to partition the domain of the time integral into intervals of length at most $0 < \delta \ll 1$, where $\delta = \delta(\epsilon)$ will later be chosen as a function of $\epsilon$. Let $N = \left[ \frac{T}{\delta} \right] \in \mathbb{N}$ such that $T \approx N\delta + O(\delta)$. Then we have,

$$\mathbb{E}_\mathbb{Q} \int_0^T \mathbb{E}_{\mathbb{P}_\epsilon} \left[ |\psi(X_s^\epsilon, Z_s^\epsilon)\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] = \mathbb{E}_\mathbb{Q} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}_{\mathbb{P}_\epsilon} \left[ |\psi(X_s^\epsilon, Z_s^\epsilon)\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] \, ds + \mathbb{E}_\mathbb{Q} \int_{N\delta}^T \mathbb{E}_{\mathbb{P}_\epsilon} \left[ |\psi(X_s^\epsilon, Z_s^\epsilon)\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] \, ds. \quad (4.15)$$

with $t_{i+1} - t_i = \delta, \forall i$. We now consider a single time integral from $[t_i, t_{i+1}]$. For simplicity and clarity, let us use the notation $[t, t + \delta]$ instead. The analysis for the remainder term, over the interval $[N\delta, T]$, will follow from the same arguments.

We introduce terms to the conditional expectation with arguments $X_t^\epsilon$ and $\tilde{D}_t^\epsilon$ fixed at the initial time of the integral over $[t, t + \delta]$, to get,

$$\mathbb{E}_{\mathbb{P}_\epsilon} \left[ |\psi(X_s^\epsilon, Z_s^\epsilon)\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] \lesssim \mathbb{E}_{\mathbb{P}_\epsilon} \left[ (\psi(X_s^\epsilon, Z_s^\epsilon) - \psi(X_t^\epsilon, Z_t^\epsilon))\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right]^2 \quad (4.16)$$

The first term on the right-hand side of Eq. 4.16, by way of Jensen’s inequality and Fubini on the first line, change of measure, Cauchy-Schwarz, Jensen’s inequality, and the tower property of conditional expectation on the second line, contributes

$$\mathbb{E}_\mathbb{Q} \int_t^{t+\delta} \mathbb{E}_{\mathbb{P}_\epsilon} \left[ (\psi(X_s^\epsilon, Z_s^\epsilon) - \psi(X_t^\epsilon, Z_t^\epsilon))\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] \, ds \leq \mathbb{E}_\mathbb{Q} \mathbb{E}_{\mathbb{P}_\epsilon} \left[ (\psi(X_s^\epsilon, Z_s^\epsilon) - \psi(X_t^\epsilon, Z_t^\epsilon))\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right] \, ds$$

$$\leq \int_t^{t+\delta} \mathbb{E}_{\mathbb{P}_\epsilon} \left[ \tilde{D}_s \right]^{1/2} \mathbb{E}_{\mathbb{P}_\epsilon} \left[ (\psi(X_s^\epsilon, Z_s^\epsilon) - \psi(X_t^\epsilon, Z_t^\epsilon))\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right]^{1/2} \, ds.

By Lemma 3.8, $\mathbb{E}_{\mathbb{P}_\epsilon} \left[ \tilde{D}_s \right]^{1/2} < \infty$, and by application of Cauchy-Schwarz and then the Lipschitz property of $\psi$, we get

$$\mathbb{E}_{\mathbb{P}_\epsilon} \left[ (\psi(X_s^\epsilon, Z_s^\epsilon) - \psi(X_t^\epsilon, Z_t^\epsilon))\tilde{D}_s \rangle \mathcal{Y}_s^\epsilon \right]^{1/2} \leq \mathbb{E}_{\mathbb{P}_\epsilon} \left[ \psi(X_s^\epsilon, Z_s^\epsilon) - \psi(X_t^\epsilon, Z_t^\epsilon) \right]^{1/2} \mathbb{E}_{\mathbb{P}_\epsilon} \left[ \tilde{D}_s \right]^{1/2}$$

$$\lesssim \mathbb{E}_{\mathbb{P}_\epsilon} \left[ X_s^\epsilon - X_t^\epsilon \right]^{1/2}.$$
ψ} is globally Lipschitz in x, since each of the components of ψ are either globally Lipschitz in x or have a bounded derivative in x. Lemma 3.7 gives
\begin{align*}
\mathbb{E}^x_\tau \left[ |X^*_\tau - X^*_t| \right]^{1/4} \lesssim (\epsilon^8(1 + \mathbb{E}^x |Z_0^q|) + \delta^4(1 + \epsilon^8)(1 + \mathbb{E} |[\delta Z_0^q]|))^{1/4},
\end{align*}
for some q ≥ 0, and therefore by the finite moments of Q_{(X_0^z; z_t)}, the first term of Eq. 4.16 is bounded by,
\begin{align*}
\mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau) - \psi(X^*_t, Z^*_t)) \hat{D}_t^| \mathcal{Y}_s^x \right] \right|^2 ds \lesssim \delta (\epsilon^8 + \delta^4(1 + \epsilon^8))^{1/4}.
\end{align*}
(4.17)

The second term of Eq. 4.16 similarly contributes,
\begin{align*}
\mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)(\hat{D}_s - \hat{D}_t^|) \right] \mathcal{Y}_s^x \right|^2 ds \leq \int_t^{t+\delta} \mathbb{E}_Q \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)(\hat{D}_s - \hat{D}_t^|) \right] \mathcal{Y}_s^x \right] ds \\
\leq \int_t^{t+\delta} \mathbb{E}^x_\tau \left[ (\hat{D}_s^|)^{1/2} \right] \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)(\hat{D}_s - \hat{D}_t^|) \right] \mathcal{Y}_s^x \right]^{1/2} ds.
\end{align*}

We now use the boundedness of ψ to get
\begin{align*}
\mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)(\hat{D}_s - \hat{D}_t^|) \right] \mathcal{Y}_s^x \right]^{1/2} \lesssim |\psi|^2_{\infty} \mathbb{E}^x_\tau \left[ (\hat{D}_s - \hat{D}_t^|) \right]^{1/2}.
\end{align*}

Lemma 3.8 gives \( \mathbb{E}^x_\tau \left[ (\hat{D}_s - \hat{D}_t^|) \right]^{1/2} \lesssim \delta \) and therefore
\begin{align*}
\mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)(\hat{D}_s - \hat{D}_t^|) \right] \mathcal{Y}_s^x \right|^2 ds \lesssim \delta^2.
\end{align*}
(4.18)

Recall the last term in Eq. 4.16,
\begin{align*}
\mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)\hat{D}_t^|) \right] \mathcal{Y}_s^x \right|^2 ds.
\end{align*}

We first consider adding and subtracting the following term within the conditional expectation,
\begin{align*}
(\psi(X^*_\tau, \hat{Z}^{X^*_\tau}_t)) \hat{D}_t^|,
\end{align*}
where \( \hat{Z}^{X^*_\tau}_t \) is the process satisfying Eq. 2.7, but with fixed random initial condition \( x = X^*_t \). Then we have
\begin{align*}
\mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau)\hat{D}_t^|) \right] \mathcal{Y}_s^x \right|^2 ds \lesssim \mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, \hat{Z}^{X^*_\tau}_t)\hat{D}_t^|) \right] \mathcal{Y}_s^x \right|^2 ds \\
+ \mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau) - (\psi(X^*_\tau, \hat{Z}^{X^*_\tau}_t)) \right] \hat{D}_t^| \right] \mathcal{Y}_s^x \right|^2 ds. 
\end{align*}
(4.19)

Concentrating on the second term of Eq. 4.19,
\begin{align*}
\mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau) - (\psi(X^*_\tau, \hat{Z}^{X^*_\tau}_t)) \right] \hat{D}_t^| \right] \mathcal{Y}_s^x \right|^2 ds & \lesssim \mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(X^*_\tau, Z^*_\tau) - (\psi(X^*_\tau, \hat{Z}^{X^*_\tau}_t)) \right] \mathcal{Y}_s^x \right|^8 ds \\
& \lesssim \mathbb{E}_Q \int_t^{t+\delta} \left| \mathbb{E}^x_\tau \left[ (\psi(x, Z^{x; t,x,z}_s) - (\psi(x, \hat{Z}^{x; t,x,z}_s)) \right] \mathcal{Y}_s^x \right|^8 ds \approx (x, z) = (X^*_t, Z^*_t) \right]^{1/4} ds.
\end{align*}
From the global Lipschitz property of $\psi$ in the $z$ component, we have the following estimate,
\[
\mathbb{E}_p \left[ \left| \psi(x, Z^c_s(t,x,z)) - \psi(x, \tilde{Z}_s^c(x,z)) \right|^8 \right] \lesssim \mathbb{E}_p \left[ \left| Z^c_s(t,x,z) - \tilde{Z}_s^c(x,z) \right|^8 \right].
\]

In what follows, we use the notation $(X^c_s(t,x), Z^c_s(t,z))$ for the pair process realized by $Z^c_s(t,x,z)$. Similarly, we use $X^c_s(t,x,z)$ when we must make clear that we are referring to the first entry of the pair $(X^c_s(t,x), Z^c_s(t,z))$ which satisfies Eq. 1.1. The previous inequality is then bounded as follows,
\[
\mathbb{E}_p \left[ \left| Z^c_s(t,x,z) - \tilde{Z}_s^c(x,z) \right|^8 \right] \lesssim \\
\delta^7 \epsilon^{16} \int_t^{t+\delta} \mathbb{E}_p \left[ f(X^c_s(t,x), Z^c_s(t,z)) - f(x, \tilde{Z}_s^c(x,z)) \right]^8 \, ds + \delta^3 \epsilon^8 \int_t^{t+\delta} \mathbb{E}_p \left[ g(X^c_s(t,x), Z^c_s(t,z)) - g(x, \tilde{Z}_s^c(x,z)) \right]^8 \, ds \\
\lesssim \delta^7 \epsilon^{16} \int_t^{t+\delta} \mathbb{E}_p \left[ f(X^c_s(t,x), Z^c_s(t,z)) - f(x, Z^c_s(t,z)) \right]^8 + \mathbb{E}_p \left[ f(x, Z^c_s(t,x,z)) - f(x, \tilde{Z}_s^c(x,z)) \right]^8 \, ds \\
+ \delta^3 \epsilon^8 \int_t^{t+\delta} \mathbb{E}_p \left[ g(X^c_s(t,x), Z^c_s(t,z)) - g(x, Z^c_s(t,z)) \right]^8 + \mathbb{E}_p \left[ g(x, Z^c_s(t,x,z)) - g(x, \tilde{Z}_s^c(x,z)) \right]^8 \, ds \\
\lesssim \delta^3 \epsilon^8 \left( \frac{\delta^4}{\epsilon^6} |\nabla_x f|^{\infty} + |\nabla_x g|^{\infty} \right) \int_t^{t+\delta} \mathbb{E}_p \left[ X^c_s(t,x,z) - x \right]^8 \, ds \\
+ \delta^3 \epsilon^8 \left( \frac{\delta^4}{\epsilon^6} |\nabla_x f|^{\infty} + |\nabla_x g|^{\infty} \right) \int_t^{t+\delta} \mathbb{E}_p \left[ Z^c_s(t,x,z) - \tilde{Z}_s^c(x,z) \right]^8 \, ds.
\]

From Lemma 3.7, for some $q \geq 0$, we get
\[
\int_t^{t+\delta} \mathbb{E}_p \left[ X^c_s(t,x,z) - x \right]^8 \, ds \lesssim \delta \epsilon^8 (1 + |z|^q) + \delta^5 (1 + \epsilon^8)(1 + |z|^q).
\]

Let
\[
\eta(\epsilon, \delta) \equiv \left( \frac{\delta^8}{\epsilon^{10}} + \frac{\delta^4}{\epsilon^6} \right).
\]

Therefore Grönwall gives us
\[
\mathbb{E}_p \left[ \left| Z^c_s(t,x,z) - \tilde{Z}_s^c(x,z) \right|^8 \right] \lesssim \eta(\epsilon, \delta) \left( \epsilon^8 + \delta^4 (1 + \epsilon^8) \right) \exp(\eta(\epsilon, \delta))(1 + |z|^q).
\]

For further brevity, let us define
\[
\mathcal{F}(\epsilon, \delta) \equiv \eta(\epsilon, \delta) \left( \epsilon^8 + \delta^4 (1 + \epsilon^8) \right) \exp(\eta(\epsilon, \delta)).
\]

Therefore the second term in Eq. 4.19 is bounded by
\[
\mathbb{E}_Q \int_t^{t+\delta} \mathbb{E}_p \left[ \left( \psi(X^c_t, Z^c_s) - \psi(X^c_t, \tilde{Z}_s^cX^c_t) \right) \tilde{\mathcal{D}}_t \left| Y^c_s \right|^2 \right] \, ds \lesssim \int_t^{t+\delta} \mathbb{E}_p \left[ \mathcal{F}(\epsilon, \delta)(1 + |Z^c_t|^q) \right]^{1/4} \, ds \\
\lesssim \delta \mathcal{F}(\epsilon, \delta)^{1/4} (1 + \mathbb{E}_p \left[ |Z^c_0|^q \right]^{1/4} \lesssim \delta \mathcal{F}(\epsilon, \delta)^{1/4}. \tag{4.20}
\]

For the first term on the right hand side of Eq. 4.19, we condition the centering term on a larger filtration $\mathcal{H} = Y^c_t \vee F^X_t \vee F^Z_t$, and then use the fact that $\sigma(\tilde{Z}^c_sX^c_t) \vee Y^c_t \vee F^X_t \vee F^Z_t$ is independent of $\sigma(Y^c_t - Y^c_t, r \in [t, s])$ under $\mathbb{P}$ and that $(X^c_t, \tilde{Z}^c_sX^c_t)$ is Markov in the larger filtration $Y^c_t \vee F^X_t \vee F^Z_t$ to yield,
\[
\mathbb{E}_Q \int_t^{t+\delta} \mathbb{E}_p \left[ \left( \psi(X^c_t, Z^c_s) - \psi(X^c_t, \tilde{Z}_s^cX^c_t) \right) \tilde{\mathcal{D}}_t \right] \mathcal{H} \tilde{\mathcal{D}}_t \left| Y^c_s \right|^2 \, ds \\
= \mathbb{E}_Q \int_t^{t+\delta} \mathbb{E}_p \left[ \left( \psi(X^c_t, \tilde{Z}_s^cX^c_t) \right) \mathcal{H} \tilde{\mathcal{D}}_t \right] \left| Y^c_s \right|^2 \, ds.
Applications of Jensen’s inequality, Cauchy-Schwarz, the tower property, Lemma 3.8 and 3.3 then give the estimate,

\[
\mathbb{E}_Q \int_t^{t+\delta} \mathbb{E}_P \left[ \mathbb{E}_{x} \left[ \psi(x, \tilde{Z}^x_{s}, \gamma dB_s) \right] \right] dx = (X^t, Z^t) \mathbb{D}_{t} \left[ Y^x_{s} \right] \right\|^2 ds
\]

\[
\lesssim \int_t^{t+\delta} \mathbb{E}_P \left[ \mathbb{E}_{x} \left[ \psi(x, \tilde{Z}^x_{s}, \gamma dB_s) \right] \right] \lesssim \int_t^{t+\delta} \mathbb{E}_P \left[ \left| T^{F,x}_{(s-t)/\epsilon^2}(\psi(X^t, \cdot))(Z^t_s) \right| \right]^{1/4} ds = \int_t^{t+\delta} \mathbb{E}_P \left[ \left| T^{F,x}_{(s-t)/\epsilon^2}(\psi(X^t, \cdot))(Z^t_s) \right| \right]^{1/4} ds
\]

\[
= \epsilon^2 \int_0^{1/\epsilon^2} \mathbb{E}_P \left[ \left| T^{F,x}_{s}(\psi(X^t, \cdot))(Z^t_s) \right| \right]^{1/4} du \lesssim \epsilon^2 \int_0^{1/\epsilon^2} \mathbb{E}_P \left[ \left| T^{F,x}_{s}(\psi(X^t, \cdot))(Z^t_s) \right| \right]^{1/4} du
\]

\[
\lesssim \epsilon^2 \left( 1 + \mathbb{E}_P \left[ \left| Z^t_s \right|^q \right] \right)^{1/4} \lesssim \epsilon^2 \left( 1 + \mathbb{E}_P \left[ \left| Z^t_s \right|^q \right] \right)^{1/4} \lesssim \epsilon^2. \tag{4.21}
\]

Collecting all our bounds for Eq. 4.14, that is Eqs. 4.17, 4.18, 4.20, and 4.21, and accounting for the discretization of the time integral into \( N \) segments, which results in \( T/\delta \) times the estimates, we have

\[
\mathbb{E}_Q \sup_{t \leq T} \left| \int_0^t \langle \rho^s_{x}(\psi, dY^s_{x}) \right\|^2 \lesssim \left( \epsilon^8 + \delta^4 (1 + \epsilon^8) \right)^{1/4} + \delta + \mathbb{F}(\epsilon, \delta)^{1/2} + \frac{\epsilon^2}{\delta}. \tag{4.22}
\]

If we choose \( \delta(\epsilon) = \epsilon^2 (-\ln \epsilon)^p \) with \( p \in (0, 1/8) \), then \( \lim_{\epsilon \to 0^+} \delta(\epsilon) = 0 \) and \( \mathbb{F}(\epsilon, \delta) \to 0 \) (see Lemma A.1), which completes the proof.

Lemma 4.4

Under either of the assumptions:

a. the coefficients of \( \mathcal{G}^1 \) and \( \mathcal{H}, \sigma \) are \( C_0^{2+\alpha} \), for some \( \alpha \in (0, 1) \), or

b. \( a + \tilde{a} > 0 \) uniformly in \( x \) and the coefficients of \( \mathcal{G}^1 \) and \( \mathcal{H}, \sigma \) are \( C_0^\alpha \), for some \( \alpha \in (0, 1) \),

the finite signed Borel measure-valued process \( \zeta \), has the unique solution \( \zeta_t = 0 \), \( Q \)-a.s. \( \forall t \in [0, T] \).

Proof. Our objective is simply to show that

\[
\zeta_t(\varphi) = \int_0^t \zeta_s(\mathcal{G}^1 \varphi)ds + \int_0^t \langle \zeta_s(\varphi \tilde{H} + \alpha \mathbf{S}^\star \nabla \varphi), dY_s \rangle, \quad \zeta_0(\varphi) = 0,
\]

is a Zakai equation, since uniqueness then follows from [Roz91, Theorem 3.1, p.454].

Let \( X^0 \) be the diffusion process with infinitesimal generator \( \mathcal{G}^1 \). In particular, consider the following system of equations,

\[
\begin{align*}
    dX^t = [\tilde{h}(X^0_t) + \tilde{b}(X^0_t)] dt + \tilde{a}^{1/2}(X^0_t) d\tilde{W}_t + (\tilde{a}(X^0_t) = \sigma \mathbf{S}^\star (X^0_t))^{1/2} d\tilde{W}_t + \sigma(X^0_t) dW_t \\
    dY_t = \tilde{h}(X^0_t) dt + \alpha dW_t + \gamma dB_t,
\end{align*}
\]

where \( \alpha dW_t + \gamma dB_t \) is a standard Brownian motion, \( \tilde{W}, \tilde{W}, W, B \) are independent standard Brownian motions under \( Q \). This system of equations yield a Zakai equation of the desired form after the change of measure given by \( D_x = \exp(-\int_0^t \tilde{h}(X^0_u), \alpha dW_u + \gamma dB_u - \frac{1}{2} \int_0^t |\tilde{h}(X^0_u)|^2 ds) \) is performed.

Theorem 4.1

Assume that \( f \) and \( g \) satisfy \( H_f \) and \( H_g \), that \( b_t \) is centered with respect to \( \mu_{\infty}(x) \) for each \( x \) and that \( \mathbb{Q}_{(X^0_t, Y^0_t)} \) has finite moments of every order. Additionally, assume either:

\[ \sigma \]
a. $H^{3,2+\alpha}$ holds for $\alpha \in (0,1)$; for each $z$, $b(\cdot,z),\sigma(\cdot,z) \in C^{3}$, and $b_{1}(\cdot,z) \in C^{4}$; that $b$ and $b_{1}$ are Lipschitz in $z$, and $\sigma$ is globally Lipschitz in $z$; that $b,b_{1},\sigma$ satisfy the growth conditions

$$|b(x,z)| + |b_{1}(x,z)| + |\sigma\sigma^{\ast}(x,z)| \leq C(1 + |z|)^{\beta},$$

$$\sum_{|k|=1}^{2} |D_{x}^{k}b(x,z)| + |D_{x}^{k}\sigma\sigma^{\ast}(x,z)| \leq C(1 + |z|^{q}),$$

$$\sum_{|k|=1}^{3} |D_{x}^{k}b_{1}(x,z)| \leq C(1 + |z|^{q}),$$

for some $\beta < -2$ and $q > 0$; that $h$ is bounded in $(x,z)$, $h(\cdot,z) \in C^{3}$ for each $z$, and $h$ is globally Lipschitz in $z$.

b. $\tilde{a} + \tilde{a} > 0$ uniformly in $z$; $H^{2,2+\alpha}$ holds for $\alpha \in (0,1)$; for each $z$, $b(\cdot,z),b_{1}(\cdot,z),\sigma(\cdot,z) \in C^{2}$; that $b$ and $b_{1}$ are Lipschitz in $z$, and $\sigma$ is globally Lipschitz in $z$; that $b,b_{1},\sigma$ satisfy the growth conditions

$$|b(x,z)| + |b_{1}(x,z)| + |\sigma\sigma^{\ast}(x,z)| \leq C(1 + |z|)^{\beta},$$

$$\sum_{|k|=1}^{2} |D_{x}^{k}b(x,z)| + |D_{x}^{k}b_{1}(x,z)| + |D_{x}^{k}\sigma\sigma^{\ast}(x,z)| \leq C(1 + |z|^{q}),$$

for some $\beta < -2$ and $q > 0$; $h$ is bounded in $(x,z)$, that $h$ is globally Lipschitz in $(x,z)$. If $a > 0$, which implies $\tilde{a} + \tilde{a} > 0$, then the Lipschitz condition in $z$ for $b,b_{1}$ can be relaxed to $\alpha$-Hölder continuity.

Then $\zeta^{\epsilon} = \rho^{\epsilon,x} - \rho^{0} \Rightarrow 0$ as $\epsilon \to 0$.

Proof. This follows from Corollary 4.1—the existence of weak limits of the probability measures induced on path space by $\zeta^{\epsilon}$, Lemma 4.3—the characterization of the limit points, and Lemma 4.4 on the uniqueness of the limiting evolution equation.

Lemma 4.5

Let $\rho^{\epsilon}$ be a solution of Eq. 2.9 and $\rho^{0}$ a solution of Eq. 2.10. Assume that $h, \tilde{h}$ and the coefficients of $\mathcal{G}^{\dagger}$ are bounded. If $\rho^{\epsilon,x} - \rho^{0} \Rightarrow 0$ as $\epsilon \to 0$, then $\pi^{\epsilon,x} - \pi^{0} \Rightarrow 0$.

Proof. Let $\varphi \in C^{0}(\mathbb{R}^{m};\mathbb{R})$ and $t \in [0,T]$, then

$$(\pi^{\epsilon,x} - \pi^{0})(\tau)_{t}(\varphi) = \frac{\rho^{\epsilon,x}_{t}(\varphi)}{\rho^{\epsilon,x}_{t}(1)} - \frac{\rho^{0}_{t}(\varphi)}{\rho^{\epsilon,x}_{t}(1)} = \frac{(\rho^{\epsilon,x}(\cdot) - \rho^{0}(\cdot))_{t}(\varphi)}{\rho^{\epsilon,x}_{t}(1)} + \frac{\pi^{0}_{t}(\varphi) - (\rho^{0}(\cdot) - \rho^{\epsilon,x}(\cdot))_{t}(1)}{\rho^{\epsilon,x}_{t}(1)}.$$  

The weak convergence of $(\pi^{\epsilon,x} - \pi^{0})_{t}$ now follows from the estimate

$$\lim_{\delta \to 0} \inf_{\epsilon > 0} \mathbb{Q} \left( \inf_{t \leq T} \rho^{\epsilon,x}_{t}(1) > \delta \right) = 1,$$

and the fact that $\varphi$ is bounded and $\pi^{0}_{t}$ is almost surely equal to a probability measure. \qed

5 Remark on Conditions for the Fast Semigroup

The necessary conditions in this paper are sometimes at odds with Theorems 2 and 3 from [PV03, p.1171], which are used in this paper for a number of propositions and theorems listed below. Specifically, in [PV03], the condition in Theorems 2 and 3 are given as $H^{1,2+\alpha}$ (there are actually two scenarios to consider, but in this paper we only consider one of them, which is the one just quoted). In particular, only one continuous derivative in the $x$-component is ever needed in the coefficients $f$ and $g$ to be able to take $k \geq 1$ derivatives of the new function under the semigroup $T^{F,x}(\varphi)$, where $\varphi \in C^{k}$ for instance. Because the Poisson solution of [PV03, Theorem 3, p.1171] is proven based on Theorem 2, the same condition of $H^{1,2+\alpha}$ shows up there, even if $k \geq 1$ derivatives of the Poisson solution are desired. A counter example to why this condition is insufficient is given next.
5.1 Counter Example

Let \( g(x, z) = g(x) \) depend only on \( x \) and let \( f(x, z) = -z \). Then the fast process is

\[
dZ_t^x = -Z_t^x dt + g(x) dB_t,
\]

and therefore \( Z_t^x \) is an Ornstein-Uhlenbeck process (in particular Gaussian), and hence satisfies the recurrence condition for [PV03, p.1171]. We can choose \( g \) to satisfy the uniform ellipticity condition as well, assume this to be true. If \( Z_0^x = z \), then

\[
Z_t^x = e^{-t} z + \int_0^t e^{-(t-s)} g(x) dB_s \sim \mathcal{N} \left( e^{-t} z, \frac{g(x)^2}{2} (1 - e^{-2t}) \right),
\]

and thus the transition density at time \( t \) in \( z \), having started from \( (x, z') \) at the initial time is

\[
p_t(z, z'; x) = r \left( \frac{g(x)^2}{2} (1 - e^{-2t}), z' - e^{-t} z \right),
\]

where \( r(s, y) \) is the Gaussian density with variance \( s \), evaluated in \( y \). Consider now the test function \( \psi(x, z) = \cos(z) \), which is infinitely smooth in \( x \) (and in \( z \)). Note that for \( Y \sim \mathcal{N}(\mu, g^2) \) we have

\[
E[\cos(Y)] = \frac{1}{2} E[e^{iY} + e^{-iY}] = \frac{1}{2} \left( e^{i\mu} + e^{-i\mu} - \frac{1}{2} g^2 \right) = e^{-\frac{1}{2} g^2} \cos(\mu),
\]

and therefore the semigroup (notation from [PV03, p.1171]) is

\[
p_t(z, \psi; x) = E_z[\cos(Z_t^x)] = \exp \left( -\frac{1}{2} \frac{g(x)^2}{2} (1 - e^{-2t}) \right) \cos(e^{-t} z).
\]

If \( g^2 \notin C^2 \), then this function is not \( C^2 \) in \( x \).

5.2 List of Changes

The condition should be \( H^{k,2+\alpha} \), and we use this condition instead of the one given in [PV03]. The difference in the requirements of various propositions and theorems are subtle, but listed here for reference:

(i) Theorem 3.1, \( H^{1,2+\alpha} \) has become \( H^{2,2+\alpha} \).
(ii) Theorem 3.2, \( H^{1,2+\alpha} \) has become \( H^{k,2+\alpha} \).
(iii) Lemma 3.10, \( H^{1,2+\alpha} \) has become \( H^{j,2+\alpha} \).
(iv) Theorem 4.1, for a.) \( H^{3,2+\alpha} \) has become \( H^{4,2+\alpha} \). This was a result of needing the third derivative in \( x \) of \( \tilde{b} \), which required the fourth derivative in \( x \) of the Poisson solution \( G_{F^{-1}}(b_1) \).

A final remark, is that Lemma 3.3 is not affected by this, because there we are also using the density result of [PV03, Theorem 1, p.1170], which is correct and requires stronger conditions than [PV03, Theorem 2, p.1171].

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\section{Appendix}

\textbf{Lemma A.1} \hspace{1cm}
Let $p \in (0, 1/8)$, $\delta(\epsilon) = \epsilon^2(-\ln \epsilon)^p$, then

$$
\lim_{\epsilon \to 0^+} \left( \frac{\delta^8}{\epsilon^{16}} + \frac{\delta^4}{\epsilon^8} \right) (\epsilon^8 + \delta^4(1 + \epsilon^8)) \exp \left( \frac{\delta^8}{\epsilon^{16}} + \frac{\delta^4}{\epsilon^8} \right) = 0.
$$

\textbf{Proof.} We first expand the expression with the choice of $\delta(\epsilon)$ to get,

$$
(((-\ln \epsilon)^8 + (-\ln \epsilon)^4p) \epsilon^8 + \epsilon^8(-\ln \epsilon)^4p + \epsilon^{16}(-\ln \epsilon)^4p) \exp ((-\ln \epsilon)^8p + (-\ln \epsilon)^4p).
$$

Expanding and distributing the terms, we identify the term that would be most limiting for convergence to zero,

$$
(-\ln \epsilon)^{12p} \epsilon^8 \exp ((-\ln \epsilon)^{8p} + (-\ln \epsilon)^{4p}) \lesssim \epsilon^7 \exp (2(-\ln \epsilon)^{8p}).
$$

Since $8p < 1$, for all sufficiently small $\epsilon > 0$ we have,

$$
\exp (2(-\ln \epsilon)^{8p}) \leq \exp (-2\ln \epsilon) = \epsilon^{-2},
$$

and therefore

$$
\lim_{\epsilon \to 0^+} \epsilon^7 \exp (2(-\ln \epsilon)^{8p}) \leq \lim_{\epsilon \to 0^+} \epsilon^5 = 0.
$$

\qed

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