On the similarity solutions for a steady MHD equation

Jean-David Hoernel

To cite this version:

Jean-David Hoernel. On the similarity solutions for a steady MHD equation. 2006. hal-00111283
On the similarity solutions for a steady MHD equation

Jean-David HOERNEL †,*

† Department of Mathematics, Technion-Israel Institute of Technology
Amado Bld., Haifa, 32000 ISRAEL
E-mail: j-d.hoernel@wanadoo.fr

Abstract

In this paper, we investigate the similarity solutions for a steady laminar incompressible boundary layer equations governing the magnetohydrodynamic (MHD) flow near the forward stagnation point of two-dimensional and axisymmetric bodies. This leads to the study of a boundary value problem involving a third order autonomous ordinary differential equation. Our main results are the existence, uniqueness and nonexistence for concave or convex solutions.

1 Introduction

Boundary layer flow of an electrically conducting fluid over moving surfaces emerges in a large variety of industrial and technological applications. It has been investigated by many researchers, Wu [1] has studied the effects of suction or injection on a steady two-dimensional MHD boundary layer flow on a flat plate, Takhar et al. [2] studied a MHD asymmetric flow over a semi-infinite moving surface and numerically obtained the solutions. An analysis of heat and mass transfer characteristics in an electrically conducting fluid over a linearly stretching sheet with variable wall temperature was investigated by Vajravelu and Rollins [3]. In [4] Muhapatra and Gupta treated the steady two-dimensional stagnation-point flow of an incompressible viscous electrically conducting fluid towards a stretching surface, the flow being permeated by a uniform transverse magnetic field. For more details see also [5], [6], [7], [8] and the references therein.

Motivated by the above works, we aim here to give analytical results about the third order non-linear autonomous differential equation

\[ f''' + \frac{m+1}{2} f'' + m(1 - f'^2) + M(1 - f') = 0 \quad \text{on } [0, \infty) \]  

accompanied by the boundary conditions

\[ f(0) = a, \quad f'(0) = b, \quad f'(\infty) = 1 \]  

where \( a, b, m, M \in \mathbb{R} \) and \( f'(\infty) := \lim_{t \to \infty} f'(t) \). Equation (1) is very interesting because it contains many known equations as particular cases. Let us give some examples.

Setting \( M = 0 \) in (1), leads to the well-known Falkner-Skan equation (see [9], [10], [11] and the references therein). While the case \( M = -m \) reduces (1) to equation that arises when considering the mixed

MSC: 34B15, 34C11, 76D10
PACS: 47.35.Tv, 47.65.d, 47.15.Cb
Key words and phrases: Boundary layer, similarity solution, third order nonlinear differential equation, boundary value problem, MHD.

* The author thanks the Department of Mathematics of the Technion for supporting his researches through a Postdoctoral Fellowship in the frame of the RTN “Fronts-Singularities”.

1
convection in a fluid saturated porous medium near a semi-infinite vertical flat plate with prescribed temperature studied by many authors, we refer the reader to [12], [13], [14], [15] and the references therein. The case \( M = m = 0 \) is refereed to the Blasius equation introduced in [16] and studied by several authors (see for example [17], [18], [19]). Recently, the case \( m = -1 \) have been studied in [20], the authors show existence of “pseudo-similarity” solution, provided that the plate is permeable with suction. Mention may be made also to [21], where the authors show existence of an infinite number of similarity solutions for the case of a non-Newtonian fluid. More recently, some results have been obtained by Brighi and Hoernel [22], about the more general equation

\[
 f''' + f f'' + g(f') = 0 \quad \text{on} \quad [0, \infty)
\]

with the boundary conditions

\[
 f(0) = \alpha, \quad f'(0) = \beta, \quad f'(\infty) = \lambda
\]

where \( \alpha, \beta, \lambda \in \mathbb{R} \) and \( g \) is a given function. Guided by the analysis of [22] we shall prove that problem (1)-(2) admits a unique concave or a unique convex solution for \( m > -1 \) according to the values of \( M \). We give also non-existence results for \( m \in \mathbb{R} \) and related values of \( M \).

2 Flow analysis

Let us suppose that an electrically conducting fluid (with electrical conductivity \( \sigma \)) in the presence of a transverse magnetic field \( B(x) \) is flowing past a flat plate stretched with a power-law velocity. According to [20], [23], [24], such phenomenon is described by the following equations

\[
 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
 u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e u_{ex} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B^2(x)}{\rho} (u_e - u).
\]

Here, the induced magnetic field is neglected. In a cartesian system of co-ordinates \((O, x, y)\), the variables \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions respectively. We will denote by \( u_e(x) = \gamma x^m, \gamma > 0 \) the external velocity, \( B(x) = B_0 x^{\frac{m-1}{2}} \) the applied magnetic field, \( m \) the power-law velocity exponent, \( \rho \) the fluid density and \( \nu \) the kinematic viscosity.

The boundary conditions for problem (3)-(6) are

\[
 u(x, 0) = u_w(x) = \alpha x^m, \quad v(x, 0) = v_w(x) = \beta x^{\frac{m-1}{2}}, \quad u(x, \infty) = u_e(x)
\]

where \( u_w(x) \) and \( v_w(x) \) are the stretching and the suction (or injection) velocity respectively and \( \alpha, \beta \) are constants. Recall that \( \alpha > 0 \) is referred to the suction, \( \alpha < 0 \) for the injection and \( \alpha = 0 \) for the impermeable plate.

A little inspection shows that equations (3) and (4) accompanied by conditions (7) admit a similarity solution. Therefore, we introduce the dimensional stream function \( \psi \) in the usual way to get the following equation

\[
 \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = u_e u_{ex} + \nu \frac{\partial^3 \psi}{\partial y^3} + \frac{\sigma B^2(x)}{\rho} (u_e - u).
\]

The boundary conditions become

\[
 \frac{\partial \psi}{\partial y}(x, 0) = \alpha x^m, \quad \frac{\partial \psi}{\partial x}(x, 0) = -\beta x^{\frac{m-1}{2}}, \quad \frac{\partial \psi}{\partial y}(x, \infty) = \gamma x^m.
\]
and substituting in equations (8) and (9) we get the following boundary value problem

\[
\begin{aligned}
f''(t) + \frac{m+1}{2} f f'' + m(1-f^2) + M(1-f') = 0, \\
f(0) = a, \quad f'(0) = b, \quad f'(\infty) = 1
\end{aligned}
\]  

(10)

where \( a = \frac{2\beta}{(m+1)\nu}, \ b = \frac{\alpha}{\gamma} \) and \( M = \frac{\sigma B^2}{\gamma p} > 0 \) is the Hartmann number and the prime is for differentiating with respect to \( t \).

3 Various results

First, we give the following

**Remark 1** Let \( b = 1 \), then the function \( f(t) = t + a \) is a solution of the problem (1)-(2) for any values of \( m \) and \( M \) in \( \mathbb{R} \). We cannot say much about the uniqueness of the previous solution, but if \( g \) is another solution with \( g'(0) = \gamma > 0 \) then, since \( g'(0) = g'(<0) = 1 \) there exists \( t_0 > 0 \) such that \( g'(t_0) > 1 \), \( g''(t_0) = 0 \) and \( g'''(t_0) \leq 0 \). However, from (1) we obtain that for \( m > 0 \) and \( M > 0 \), \( g'''(t_0) = -m(1-g'^2(t_0)) - M(1-g'(t_0)) > 0 \) and thus a contradiction.

Suppose now that \( f \) verifies the equation (1) only. We will now establish some estimations for the possible extremals of \( f' \).

**Proposition 3.1** Let \( f \) be a solution of the equation (1) and \( t_0 \) be a minimum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \geq 0 \)), if it exists. For such a point \( t_0 \) we have the following possibilities, according to the values of \( m \) and \( M \).

- **For \( m < 0 \)**
  - if \( M < -2m \), then \(-1 - \frac{M}{m} \leq f'(t_0) \leq 1\),
  - if \( M = -2m \), then \( f'(t_0) = 1 \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \leq 1 - \frac{M}{m} \).

- **For \( m = 0 \)**
  - if \( M < 0 \), then \( f'(t_0) \leq 1 \),
  - if \( M > 0 \), then \( 1 \leq f'(t_0) \).

- **For \( m > 0 \)**
  - if \( M < -2m \), then \( f'(t_0) \leq 1 \) or \(-1 - \frac{M}{m} \leq f'(t_0) \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \) or \( f'(t_0) \leq 1 - \frac{M}{m} \).

**Proof.** Let \( t_0 \) be a minimum of \( f' \) with \( f \) a solution of (1). Using the equation (1) and the fact that \( f''(t_0) = 0 \), we obtain that

\[
f'''(t_0) + m(1-f^2(t_0)) + M(1-f'(t_0)) = 0.
\]

Setting \( p(x) = m(1-x^2) + M(1-x) \), we have that \( f'''(t_0) \geq 0 \) leads to \( g(f'(t_0)) \leq 0 \) and the results follows. Let us remark that in both cases \( m = M = 0 \) and \( m > 0 \), \( M = -2m \) we cannot deduce anything about \( f'(t_0) \).

**Proposition 3.2** Let \( f \) be a solution of the equation (1) and \( t_0 \) be a maximum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \leq 0 \)), if it exists. For such a point \( t_0 \) we have the following possibilities, according to the values of \( m \) and \( M \).

- **For \( m < 0 \)**
  - if \( M < -2m \), then \( f'(t_0) \leq -1 - \frac{M}{m} \) or \( f'(t_0) \geq 1 \),
  - if \( M > -2m \), then \( f'(t_0) \leq 1 \) or \( f'(t_0) \geq -1 - \frac{M}{m} \).

3
\begin{itemize}
  \item For \( m = 0 \) \(- \) if \( M < 0 \), then \( f'(t_0) \geq 1 \),
  - if \( M > 0 \), then \( f'(t_0) \leq 1 \).
  \item For \( m > 0 \) \(- \) if \( M < -2m \), then \( 1 \leq f'(t_0) \leq -1 - \frac{M}{m} \),
  - if \( M = -2m \), then \( f'(t_0) = 1 \),
  - if \( M > -2m \), then \( -1 - \frac{M}{m} \leq f'(t_0) \leq 1 \).
\end{itemize}

**Proof.** We proceed as in the previous Proposition, but this time, with the condition \( g(f'(t_0)) \geq 0 \). Let us remark that in both of cases \( m < 0 \), \( M = -2m \) and \( m = M = 0 \) we cannot deduce anything about \( f'(t_0) \).

We will now use the two previous Propositions to deduce results about the possible extremals for \( f' \) with \( f \) a solution of the problem (1)-(2).

**Theorem 1** Let \( f \) be a solution of the problem (1)-(2). \( t_0 \) be a minimum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \geq 0 \)), if it exists, and \( t_1 \) be a maximum for \( f' \) (i.e. \( f''(t_1) = 0 \) and \( f'''(t_1) \leq 0 \)), if it exists. For such points \( t_0 \) and \( t_1 \), we have the following possibilities for the values of \( f' \).

\begin{itemize}
  \item For \( m < 0 \) \(- \) if \( M < -2m \), then \(-1 - \frac{M}{m} \leq f'(t_0) \leq 1 \leq f'(t_1) \),
  - if \( M = -2m \), then \( f'(t_0) = 1 \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \leq -1 - \frac{M}{m} \leq f'(t_1) \).
  \item For \( m = 0 \) \(- \) if \( M < 0 \), then \( f'(t_0) \leq 1 \leq f'(t_1) \),
  - if \( M > 0 \), then \( f' \) cannot vanish.
  \item For \( m > 0 \) \(- \) if \( M < -2m \), then \( f'(t_0) \leq 1 \leq f'(t_1) \leq -1 - \frac{M}{m} \),
  - if \( M = -2m \), then \( f'(t_0) = 1 \),
  - if \( M > -2m \), then \( f'(t_0) \leq -1 - \frac{M}{m} \leq f'(t_1) \leq 1 \).
\end{itemize}

**Proof.** Taking into account the fact that \( f' \to 1 \) for large \( t \) and combining Proposition 1 and Proposition 2 lead to the results.

**Remark 2** A consequence of the previous Theorem is that, for \( m = 0 \) and \( M > 0 \) all the solutions of the problem (1)-(2) have to be concave or convex everywhere.

\section{The concave and convex solutions}

In this section we will first prove that, under some hypotheses, the problem (1)-(2) admits a unique concave solution or a unique convex solution for \( m > -1 \). Then, we will give some nonexistence results about the concave or convex solutions for \( m \in \mathbb{R} \) according to the values of \( M \). To this aim, we will use the fact that, if \( f \) is a solution of the problem (1)-(2), then the function \( h \) defined by

\[
f(t) = \sqrt{\frac{2}{m+1}} h \left( \sqrt{\frac{m+1}{2}} t \right)
\]

with \( m > -1 \), is a solution of the equation

\[
h''' + hh'' + g(h') = 0
\]

on \([0, \infty)\), with the boundary conditions

\[
h(0) = \sqrt{\frac{m+1}{2}} a, \quad h'(0) = b, \quad h'(\infty) = 1
\]

and where

\[
g(x) = \frac{2m}{m+1} (1-x^2) + \frac{2M}{m+1} (1-x).
\]

In the remainder of this section we will made intensive use of the results found in the paper [22] by Brighi and Hoernel.
Let $m$ be a concave solution of the problem (1)-(2). We then have that $\frac{4m-1}{m+1}$ is negative for all $m > -1$ and $f(t) > 0$ for $t$ large enough because $f'(t) \to 1$ as $t \to \infty$. Using the fact that $\frac{m+1}{2}f'' > 0$ near infinity, we obtain from (3) that
\[
f''' = -m(1-f'^2) - M(1-f')
\]
near infinity. As the polynomial function $-m(1-x^2) - M(1-x)$ is negative for all $x$ in $[1, \infty]$ if $m \leq -1$ and $M \leq -2m$, we get that $f''' < 0$ near infinity because $f' > 1$ everywhere. This is a contradiction, so concave solutions cannot exist in this case. Consider now $m > -1$ and $h$ a solution of the problem (12)-(13). Let us define the function $\hat{g}$ by $\hat{g}(x) = g(x) - x^3 + x$, a simple calculation leads to
\[
\hat{g}(x) = \frac{1}{m+1} \left( -(3m+1)x^2 + (m+1-2M)x + 2(m+M) \right).
\]
Then, the Theorem 2 from [22] tells us that problem (12)-(13) admits no concave solutions for $a \leq 0$ if $\forall x \in [1, b]$, $\hat{g}(x) \geq 0$ and $-a + \max_{x \in [1, b]} \hat{g}(x) > 0$. These conditions lead to the results for problem (1)-(2) with $m > -1$.

The results from Theorem 2 and Theorem 3 are summarized in the Figure 1 in which the plane $(m, M)$ contains three disjoint regions $A$, $B$, and $C$ are defined as
- $A$: Existence of a unique concave solution for $m > -1$, $b > 1$ and $a \in \mathbb{R}$,
- $B$: No concave solutions for $m > -1$, $b > 1$ and $a \leq 0$,
- $C$: No concave solutions for $m \leq -1$, $b > 1$ and $a \in \mathbb{R}$.
4.2 Convex solutions

We will now give existence, uniqueness and non-existence results for the convex solutions of the problem (1)-(2).

**Theorem 4** Let $a \in \mathbb{R}$ and $0 \leq b < 1$. Then, there exists a unique convex solution of the problem (1)-(2) in the following cases
- $-1 < m < 0$ and $M \geq -2m$,
- $m \geq 0$ and $M > -m(b + 1)$.
Moreover, there exists $l > a$ such that $\lim_{t \to \infty} \{ f(t) - (t + l) \} = 0$ and for all $t \geq 0$ we have $t + a \leq f(t) \leq t + l$.

**Proof.** We proceed the same way as for Theorem 2, but with the condition that $g(x) > 0$ for all $x$ in $[b, 1)$. We conclude by using first the Theorem 3 from [22], then the Proposition 2 from [22].

**Theorem 5** Let $0 \leq b < 1$. Then, there are no convex solutions of the problem (1)-(2) in the following cases
- $a \in \mathbb{R}$, $m \leq -1$ and $M \leq -m(b + 1)$,
- $a \leq 0$, $-1 < m < -\frac{1}{3}$ and $M \leq -\frac{(3m + 1)b + 2m}{2}$,
- $a \leq 0$, $m = -\frac{1}{3}$ and $M = \frac{1}{3}$,
- $a < 0$, $m = -\frac{1}{3}$ and $M > \frac{1}{3}$,
- $a \leq 0$, $m > -\frac{1}{3}$ and $M \leq -\frac{5m + 1}{2}$.

**Proof.** For $m \geq -1$ and $a \leq 0$, the proof is the same as the previous one, but this time we need that $\forall x \in [b, 1], \hat{g}(x) \leq 0$ and $-a + \max_{x \in [b, 1]} \hat{g}(x) > 0$, according to the Theorem 4 from [22]. Consider now $m \leq -1$, $a \in \mathbb{R}$ and let $f$ be a convex solution of the problem (1)-(2). We have that $b \leq f' < 1$, $f'' > 0$, $f''' < 0$ everywhere and that $f(t) > 0$ for $t$ large enough because $f'(t) \to 1$ as $t \to \infty$. According to equation (1), we have that

$$f''' = \frac{-m + 1}{2} f f'' - m(1 - f'^2) - M(1 - f')$$

with $-\frac{m + 1}{2} f f'' > 0$ near infinity. As the polynomial function $-m(1 - x^2) - M(1 - x)$ is positive for all $x$ in $[b, 1]$ if $m \leq -1$ and $M \leq -m(b + 1)$, we get that $f''' > 0$ near infinity because $b \leq f' < 1$. This is a contradiction, thus convex solutions cannot exist in this case.

The results from Theorem 4 and Theorem 5 are summarized in the Figure 2 in which the plane $(m, M)$ contains three disjoint regions A, B and C that corresponds to
A: Existence of a unique convex solution for $m > -1$, $0 \leq b < 1$ and $a \in \mathbb{R}$,
B: No convex solutions for $m > -1$, $0 \leq b < 1$ and $a \leq 0$,
C: No convex solutions for $m \leq -1$, $0 \leq b < 1$ and $a \in \mathbb{R}$.

5 Conclusion

In this paper, we have shown the existence of a unique concave or a unique convex solution of the problem (1)-(2) for $m > -1$, according to the values of $M$. We also have obtained nonexistence results for $m \in \mathbb{R}$ and related values of $M$, as well as some clues about the possible behavior of $f'$. This paper is a first work on this problem, there is still much left to do because of its complexity. Notice that the case $M = -2m$ plays a particular role in the problem (1)-(2), because it is the only one for which we are able to predict the possible changes of concavity for $f$. Its study will be the subject of a forthcoming paper.

Acknowledgement

The author would like to thank Prof. B. Brighi for his many advices and for introducing him to the similarity solutions family of problems.

References

[1] Y. K. Wu, Magnetohydrodynamic boundary layer control with suction or injection, J. Appl. Phys., 44 (1973), 2166–2171.
[2] H.S. Takhar, A.A. Raptis, A.A. Perdikis, MHD asymmetric flow past a semi-infinite moving plate, Acta Mech., 65 (1987) 278–290.
[3] K. Vajravelu, D. Rollins, Hydromagnetic flow in an oscillating channel, J. Math. Phys. Sci., 31 (1997), 11–24.
[4] T. R. Muhapatra, A.S. Gupta, Magnetohydrodynamic stagnation-point flow towards a stretching sheet, Acta Mech., 152 (2001) 191–196.
[5] A. Chakrabarti, A.S. Gupta, Hydromagnetic flow and heat transfer over a stretching sheet, Q. Appl. Maths., 37 (1979) 73–78.
[6] M. Kumari, G. Nath, Conjugate MHD flow past a flat plate, Acta Mech., 106 (1994) 215–220.
[7] I. Pop, T-Y. Na, A Note of MHD flow over a stretching permeable surface, Mechanics Research Communications, 25 (1998) 263–269.

[8] H. S. Takhar, M. A. Ali, A. S. Gupta, Stability of magnetohydrodynamic flow over a stretching sheet, Liquid Metal Hydrodynamics (Lielpeteris & Moreau ed.), Kluwer Academic Publishers, Dordrecht, 1989 pp. 465–471.

[9] V. M. Falkner, S. W. Skan, Solutions of the boundary layer equations, Phil. Mag., 12 (1931) 865–896.

[10] W. A. Coppel, On a differential equation of boundary layer theory, Phil. Trans. Roy. Soc. London, Ser. A 253 (1960) 101–136.

[11] M. Guedda, Z. Hammouch, On similarity and pseudo-similarity solutions of Falkner-Skan problems, Fluid Dynamic Research, 38 (2006) 211–223.

[12] E. H. Aly, L. Elliott, D. B. Ingham, Mixed convection boundary-layer flow over a vertical surface embedded in a porous medium, Eur. J. Mech. B Fluids 22 (2003) 529–543.

[13] B. Brighi, J.-D. Hoernel, On the concave and convex solution of mixed convection boundary layer approximation in a porous medium, Appl. Math. Lett. 19 (2006) 69–74.

[14] M. Guedda, Multiple solutions of mixed convection boundary-layer approximations in a porous medium, Appl. Math. Lett. 19 (2006) 63–68.

[15] M. Kumari, H. S. Takhar, G. Nath, Mixed convection flow over a vertical wedge embedded in a highly porous medium, Heat Mass Transfer 37 (2000) 139–146.

[16] H. Blasius, Grenzchichten in Flussigkeiten mit kleiner Reibung, Z. math. Phys. 56 (1908) 1–37.

[17] Z. Belhachmi, B. Brighi, K. Taous, On the concave solutions of the Blasius equation, Acta Math. Univ. Comenian, 69, (2) (2000), 199–214.

[18] B. Brighi, A. Fruchard, T. Sari, On the Blasius problem, Preprint.

[19] W. R. Utz, Existence of solutions of a generalized Blasius equation, J. Math. Anal. Appl. 66 (1978) 55-59.

[20] M. Amkadni, A. Azzouzi, Z. Hammouch, On the exact solutions of laminar MHD flow over a stretching flat plate, Comm. in Nonl. Sci. and Num. Sim., In Press.

[21] M. Guedda, Z. Hammouch, J. D. Hoernel, Analytical and numerical results for a MHD power-law non-Newtonian fluid flow, In preparation.

[22] B. Brighi, J. -D. Hoernel, On general similarity boundary layer equation, Submitted for publication.

[23] P. S. Lawrence, B. N. Rao, Effect of pressure gradient on MHD boundary layer over a flat plate, Acta Mech., 113 (1995) 1-7.

[24] J. A. Shercliff, A textbook of magnetohydrodynamics, Pergamon Press 1965.