Graded Limits of Minimal Affinizations in Type $D^*$

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Received October 30, 2013, in final form April 14, 2014; Published online April 20, 2014
http://dx.doi.org/10.3842/SIGMA.2014.047

Abstract. We study the graded limits of minimal affinizations over a quantum loop algebra of type $D^*$ in the regular case. We show that the graded limits are isomorphic to multiple generalizations of Demazure modules, and also give their defining relations. As a corollary we obtain a character formula for the minimal affinizations in terms of Demazure operators, and a multiplicity formula for a special class of the minimal affinizations.

Key words: minimal affinizations; quantum affine algebras; current algebras

2010 Mathematics Subject Classification: 17B37; 17B10

1 Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra, $L_\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the associated loop algebra, and $U_q(L_\mathfrak{g})$ the quantum loop algebra. In [1], Chari introduced an important class of finite-dimensional simple $U_q(L_\mathfrak{g})$-modules called minimal affinizations. For a simple $U_q(\mathfrak{g})$-module $V$, we say a simple $U_q(L_\mathfrak{g})$-module $\hat{V}$ is an affinization of $V$ if the highest weight of $\hat{V}$ is equal to that of $V$. One can define a partial ordering on the equivalence classes (the isomorphism classes as a $U_q(\mathfrak{g})$-module) of affinizations of $V$, and modules belonging to minimal classes are called minimal affinizations (a precise definition is given in Section 2.6). For example, a Kirillov–Reshetikhin module is a minimal affinization whose highest weight is a multiple of a fundamental weight. Minimal affinizations have been the subjects of many articles in the recent years. See [7, 10, 12, 18, 19, 21] for instance. For the original motivations of considering minimal affinizations, see [1, Introduction]. Given a minimal affinization, one can consider its classical limit. By restricting it to the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ and taking a pull-back, a graded $\mathfrak{g}[t]$-module called graded limit is obtained. Graded limits are quite important for the study of minimal affinizations since the $U_q(\mathfrak{g})$-module structure of a minimal affinization is completely determined by the $U(\mathfrak{g})$-module structure of its graded limit.

Graded limits of minimal affinizations were first studied in [2, 5] in the case of Kirillov–Reshetikhin modules, and subsequently the general ones were studied in [18]. In that paper, Moura presented several conjectures for the graded limits of minimal affinizations in general types, and partially proved them. Graded limits of minimal affinizations in type $ABC$ were further studied in [21]. In that paper the author proved that the graded limit of a minimal affinization in these types is isomorphic to a certain $\mathfrak{g}[t]$-module $D(w_o \xi_1, \ldots, w_o \xi_n)$. Here $w_o$ is the longest element of the Weyl group of $\mathfrak{g}$, $\xi_j$ are certain weights of the affine Lie algebra $\hat{\mathfrak{g}}$ which are $\mathfrak{g}$-dominant, and $D(w_o \xi_1, \ldots, w_o \xi_n)$ is a $\mathfrak{g}[t]$-submodule of a tensor product of simple highest weight $\hat{\mathfrak{g}}$-modules, which is generated by the tensor product $v_{w_o \xi_1} \otimes \cdots \otimes v_{w_o \xi_n}$ of the extremal weight vectors with weights $w_o \xi_j$. As a corollary of this, a character formula for minimal affinizations was given in terms of Demazure operators. In addition, the defining relations of graded limits conjectured in [18] were also proved.

*This paper is a contribution to the Special Issue on New Directions in Lie Theory. The full collection is available at http://www.emis.de/journals/SIGMA/LieTheory2014.html
In type $ABC$ a minimal affinization with a fixed highest weight is unique up to equivalence, and the graded limit of a minimal affinization depends only on the equivalence class. As a consequence, the module $D(w_i \xi_1, \ldots, w_n \xi_n)$ can be determined from the highest weight only. (If the highest weight is $\lambda = \sum_{1 \leq i \leq n} \lambda_i \varpi_i$ where $\varpi_i$ are fundamental weights, then $\xi_j$ are roughly equal to $\lambda_i (\varpi_i + a_i \Lambda_0)$ where $\Lambda_0$ is the fundamental weight of $\hat{g}$ associated with the distinguished node 0, and $a_i = 1$ if the simple root $\alpha_i$ is long and $a_i = 1/2$ otherwise. For the more precise statement, see \cite{21}.)

In contrast to this, in type $D$ there are nonequivalent minimal affinizations with the same highest weights. It was proved in \cite{10}, however, that even in type $D$ if the given highest weight satisfies some mild condition (see Section 2.6), then there are at most 3 equivalence classes of minimal affinizations with the given highest weight. We say a minimal affinization is regular if its highest weight satisfies this condition. The purpose of this paper is to study the graded limits of regular minimal affinizations of type $D$ using the methods in \cite{21}.

In the sequel we assume that $\hat{g}$ is of type $D_n$. Let $\pi$ be Drinfeld polynomials and assume that the simple $U_q(\mathfrak{g})$-module $L_q(\pi)$ associated with $\pi$ is a regular minimal affinization. Then in a certain way we can associate with $L_q(\pi)$ a vertex $s \in \{1, n - 1, n\}$ of the Dynkin diagram of $\mathfrak{g}$ (see Section 2.6). In the case where the number of equivalence classes are exactly 3, this $s$ parameterizes the equivalence class of $L_q(\pi)$. In this paper we show that there exists a sequence $\xi_1^{(s)}, \ldots, \xi_n^{(s)}$ of $\hat{g}$-dominant $\hat{g}$-weights such that the graded limit $L(\pi)$ of $L_q(\pi)$ is isomorphic to $D(w_1 \xi_1^{(s)}, \ldots, w_n \xi_n^{(s)})$ (Theorem 3.1). Here $\xi_j^{(s)}$ depends not only on the highest weight of $L_q(\pi)$ but also $s$, and the correspondence is less straightforward compared with the case of type $ABC$ (see Section 3.1 for the precise statement). As a consequence, we give a character formula for $L_q(\pi)$ in terms of Demazure operators (Corollary 3.5). We also prove the defining relations of the graded limits $L(\pi)$ conjectured in \cite{18} (Theorem 3.2), which also depends not only on the highest weight but also $s$.

Recently Sam proved in \cite{22} some combinatorial identity in type $BCD$, and gave a multiplicity formula for minimal affinizations in type $BC$ using the identity and results in \cite{4} and \cite{21}. By applying the identity of type $D$ to our results, we also obtain a similar multiplicity formula for a special class of minimal affinizations in type $D$, which gives multiplicities in terms of the simple Lie algebra of type $C$ (Corollary 3.8).

The proofs of most results are similar to those in \cite{21} and are in some respects even simpler since the type $D$ is simply laced. For example we do not need the theory of $q$-characters, which was essentially needed in \textit{loc. cit}.

The organization of the paper is as follows. In Section 2, we give preliminary definitions and basic results. In particular, we recall the definition of the modules $D(\xi_1, \ldots, \xi_p)$, the classification of regular minimal affinizations of type $D$, and the definition of graded limits. In Section 3 we state Theorems 3.1 and 3.2, and discuss some of their corollaries. The proofs of Theorems 3.1 and 3.2 is given in Section 4.

2 Preliminaries

2.1 Simple Lie algebra of type $D_n$

Let $\hat{I} = \{0, 1, \ldots, n\}$ and $\hat{C} = (c_{ij})_{i,j \in \hat{I}}$ be the Cartan matrix of type $D_n^{(1)}$ whose Dynkin diagram is as follows:

\begin{equation}
\begin{array}{cccccccc}
& & & & & & & \\
& & & & \circ & & & \\
& \circ & & & & \circ & & \\
1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 \\
& \circ & & & & \circ & & \\
& & \circ & & & & \circ & \\
& & & & \circ & & \\
\end{array}
\end{equation}

(2.1)
Let $J \subseteq \hat{T}$ be a subset. In this paper, by abuse of notation, we sometimes denote by $J$ the subdiagram of $(2.1)$ whose vertices are $J$.

Let $I = \hat{I} \setminus \{0\}$, $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of type $D_n$, and $\mathfrak{g}$ the complex simple Lie algebra associated with $C$. Let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{b}$ a Borel subalgebra containing $\mathfrak{h}$. Denote by $\Delta$ the root system and by $\Delta_+$ the set of positive roots, and let $\theta \in \Delta_+$ be the highest root.

Let $\alpha_i$ and $\omega_i$ ($i \in I$) be the simple roots and fundamental weights respectively, and set $\omega_0 = 0$ for convenience. Let $P$ be the weight lattice and $P^+$ the set of dominant integral weights. Let $W$ denote the Weyl group with simple reflections $s_i$ ($i \in I$), and $w_0 \in W$ the longest element.

For each $\alpha \in \Delta$ denote by $\mathfrak{g}_\alpha$ the corresponding root space, and fix nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ and $\alpha^\vee \in \mathfrak{h}$ such that

$$[e_\alpha, f_\alpha] = \alpha^\vee, \quad [\alpha^\vee, e_\alpha] = 2e_\alpha, \quad [\alpha^\vee, f_\alpha] = -2f_\alpha.$$ We also use the notation $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$ for $i \in I$. Set $n_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_{\pm \alpha}$. For a subset $J \subseteq I$, denote by $\mathfrak{g}_J \subseteq \mathfrak{g}$ the semisimple Lie subalgebra corresponding to $J$, and let $\mathfrak{h}_J = \sum_{i \in J} \mathfrak{C} \alpha_i^\vee \subseteq \mathfrak{h}$.

### 2.2 Affine Lie algebra of type $D_n^{(1)}$

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the affine Lie algebra with Cartan matrix $\widehat{C}$, where $K$ is the canonical central element and $d$ is the degree operator. Naturally $\mathfrak{g}$ is regarded as a Lie subalgebra of $\widehat{\mathfrak{g}}$. Define a Cartan subalgebra $\widehat{\mathfrak{h}}$ and a Borel subalgebra $\widehat{\mathfrak{b}}$ as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus n_+ \oplus \mathfrak{g} \otimes t\mathbb{C}[t].$$

Set $\widehat{n}_+ = n_+ \oplus \mathfrak{g} \otimes t\mathbb{C}[t]$. We often consider $\mathfrak{h}^*$ as a subspace of $\widehat{\mathfrak{h}}^*$ by setting $\langle K, \lambda \rangle = \langle d, \lambda \rangle = 0$ for $\lambda \in \mathfrak{h}^*$. Let $\widehat{\Delta}$ be the root system of $\widehat{\mathfrak{g}}$, $\widehat{\Delta}_+$ the set of positive roots, $\widehat{\Delta}^\text{re}$ the set of real roots, and $\widehat{\Delta}^\text{re} = \widehat{\Delta}_+ \cap \widehat{\Delta}^\text{re}$. Set $\alpha_0 = \delta - \theta$, $e_0 = f_0 = e_\theta = f_\theta \otimes t$, $f_0 = e_\theta \otimes t^{-1}$ and $\alpha_0^\vee = K - \theta^\vee$.

Denote by $\Lambda_0 \in \widehat{\mathfrak{h}}^*$ the unique element satisfying $\langle K, \Lambda_0 \rangle = 1$ and $\langle \mathfrak{h}, \Lambda_0 \rangle = \langle d, \Lambda_0 \rangle = 0$, and define $\widehat{P}, \widehat{P}^+ \subseteq \widehat{\mathfrak{h}}^*$ by

$$\widehat{P} = P \oplus \mathbb{Z}\Lambda_0 \oplus \mathbb{C}\delta \quad \text{and} \quad \widehat{P}^+ = \{ \xi \in \widehat{P} \mid \langle \alpha_i^\vee, \xi \rangle \geq 0 \text{ for all } i \in \hat{I} \}.$$

Let $\widehat{W}$ denote the Weyl group of $\widehat{\mathfrak{g}}$ with simple reflections $s_i$ ($i \in \hat{T}$). We regard $W$ naturally as a subgroup of $\widehat{W}$. Let $\ell : \widehat{W} \to \mathbb{Z}_{\geq 0}$ be the length function. Let $(\ , \ )$ be the unique non-degenerate $\widehat{W}$-invariant symmetric bilinear form on $\mathfrak{h}^*$ satisfying

$$(\alpha, \alpha) = 2 \quad \text{for} \quad \alpha \in \widehat{\Delta}^\text{re}, \quad (\mathfrak{h}^*, \delta) = (\mathfrak{h}^*, \Lambda_0) = (\Lambda_0, \Lambda_0) = 0 \quad \text{and} \quad (\delta, \Lambda_0) = 1.$$

Let $\Sigma$ be the group of Dynkin diagram automorphisms of $\widehat{\mathfrak{g}}$, which naturally acts on $\widehat{\mathfrak{h}}^*$ and $\widehat{\mathfrak{g}}$, and $\widehat{W}$ the subgroup of $\text{GL}(\widehat{\mathfrak{h}}^*)$ generated by $\widehat{W}$ and $\Sigma$. Note that we have $\widehat{W} = \Sigma \times \widehat{W}$. The length function $\ell$ is extended on $\widehat{W}$ by setting $\ell(\tau w) = \ell(w)$ for $\tau \in \Sigma$, $w \in \widehat{W}$.

Denote by $V(\lambda)$ for $\lambda \in P^+$ the simple $\mathfrak{g}$-module with highest weight $\lambda$, and by $\widehat{V}(\Lambda)$ for $\Lambda \in \widehat{P}^+$ the simple highest weight $\mathfrak{g}$-module with highest weight $\Lambda$. For a finite-dimensional semisimple $\mathfrak{h}$-module (resp. $\mathfrak{h}$-module) $M$ we denote by $\text{ch}_h M \in \mathbb{Z}[\mathfrak{h}^*]$ (resp. $\text{ch}_\widehat{h} M \in \mathbb{Z}[\widehat{\mathfrak{h}}^*]$) its character with respect to $\mathfrak{h}$ (resp. $\widehat{\mathfrak{h}}$). We will omit the subscript $\mathfrak{h}$ or $\widehat{\mathfrak{h}}$ when it is obvious from the context.
2.3 Loop algebras and current algebras

Given a Lie algebra \( \mathfrak{a} \), its \textit{loop algebra} \( \mathfrak{L} \mathfrak{a} \) is defined as the tensor product \( \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}] \) with the Lie algebra structure given by \([x \otimes f, y \otimes g] = [x, y] \otimes f g\). Let \( \mathfrak{a}[t] \) and \( t^k \mathfrak{a}[t] \) for \( k \in \mathbb{Z}_{>0} \) denote the Lie subalgebras \( \mathfrak{a} \otimes \mathbb{C}[t] \) and \( \mathfrak{a} \otimes t^k \mathbb{C}[t] \) respectively. The Lie algebra \( \mathfrak{a}[t] \) is called the \textit{current algebra} associated with \( \mathfrak{a} \).

For \( \alpha \in \mathbb{C}^\times \), let \( \text{ev}_{\alpha} : \mathfrak{L} \mathfrak{g} \to \mathfrak{g} \) denote the \textit{evaluation map} defined by \( \text{ev}_{\alpha}(x \otimes f) = f(\alpha)x \), and let \( V(\lambda, a) \) for \( \lambda \in \mathbb{P}^+ \) be the \textit{evaluation module} which is the simple \( \mathfrak{L} \mathfrak{g} \)-module defined by the pull-back of \( V(\lambda) \) with respect to \( \text{ev}_{\alpha} \). An evaluation module for \( \mathfrak{g}[t] \) is defined similarly and is denoted by \( V(\lambda, a) \) (\( \lambda \in \mathbb{P}^+, \ a \in \mathbb{C} \)).

2.4 \( \hat{\mathfrak{b}} \)-submodules \( D(\xi_1, \ldots, \xi_p) \)

Let \( \xi_1, \ldots, \xi_p \) be a sequence of elements belonging to the Weyl group orbits \( \tilde{W}(\check{\mathbb{P}}^+) \) of dominant integral weights of \( \hat{\mathfrak{g}} \). We define a \( \hat{\mathfrak{b}} \)-module \( D(\xi_1, \ldots, \xi_p) \) as follows. For each \( 1 \leq j \leq p \) let \( \Lambda^j \in \check{\mathbb{P}}^+ \) be the unique element satisfying \( \xi_j \in \tilde{W} \Lambda^j \), and take a nonzero vector \( v_{\xi_j} \) in the 1-dimensional weight space \( \tilde{V}(\Lambda^j)_{\xi_j} \). Then define

\[
D(\xi_1, \ldots, \xi_p) = U(\hat{\mathfrak{b}})(v_{\xi_1} \otimes \cdots \otimes v_{\xi_p}) \subseteq \tilde{V}(\Lambda^1) \otimes \cdots \otimes \tilde{V}(\Lambda^p).
\]

If \( (\alpha_i, \xi_j) \leq 0 \) for all \( i \in I \) and \( 1 \leq j \leq p \), then \( D(\xi_1, \ldots, \xi_p) \) can be regarded as a \( \mathfrak{g}[t] \otimes \mathbb{C}K \otimes \mathbb{C}d \)-module and in particular a \( \hat{\mathfrak{g}}[t] \)-module.

Some of \( D(\xi_1, \ldots, \xi_p) \) are realized in a different way. To introduce this, we prepare some notation. Assume that \( V \) is a \( \hat{\mathfrak{g}} \)-module and \( D \) is a \( \hat{\mathfrak{b}} \)-submodule of \( V \). For \( \tau \in \Sigma \), we denote by \( F_\tau V \) the pull-back \((\tau^{-1})^*V\) with respect to the Lie algebra automorphism \( \tau^{-1} \) on \( \hat{\mathfrak{g}} \), and define a \( \hat{\mathfrak{b}} \)-submodule \( F_\tau D \subseteq F_\tau V \) in the obvious way. It is easily proved that

\[
F_\tau D(\xi_1, \ldots, \xi_p) \cong D(\tau \xi_1, \ldots, \tau \xi_p).
\]

For \( i \in \hat{I} \) let \( \hat{\mathfrak{p}}_i \) denote the parabolic subalgebra \( \hat{\mathfrak{b}} \oplus \mathfrak{C}f_i \subseteq \hat{\mathfrak{g}} \), and set \( F_i D = U(\hat{\mathfrak{p}}_i)D \subseteq V \) to be the \( \hat{\mathfrak{p}}_i \)-submodule generated by \( D \). Finally we define \( F_w D \) for \( w \in \tilde{W} \) as follows: let \( \tau \in \Sigma \) and \( w' \in \tilde{W} \) be the elements such that \( w = \tau w' \), and choose a reduced expression \( w' = s_{i_1} \cdots s_{i_k} \). Then we set

\[
F_w D = F_\tau F_{i_1} \cdots F_{i_k} D \subseteq F_\tau V.
\]

**Proposition 2.1** ([21, Proposition 2.7]). Let \( \Lambda^1, \ldots, \Lambda^p \) be a sequence of elements of \( \check{\mathbb{P}}^+ \), and \( w_1, \ldots, w_p \) a sequence of elements of \( \tilde{W} \). We write \( w_{[r,s]} = w_r w_{r+1} \cdots w_s \) for \( r \leq s \), and assume that \( \ell(w_{[1,p]}) = \sum_{j=1}^p \ell(w_j) \). Then we have

\[
D(w_{[1,1]} \Lambda^1, w_{[1,2]} \Lambda^2, \ldots, w_{[1,p-1]} \Lambda^{p-1}, w_{[1,p]} \Lambda^p) \\
\cong F_{w_{r_1}}(D(\Lambda^1) \otimes F_{w_2}(D(\Lambda^2) \otimes \cdots \otimes F_{w_{p-1}}(D(\Lambda^{p-1}) \otimes F_{w_p}D(\Lambda^p)) \cdots)). \tag{2.2}
\]

Let \( D_i \) for \( i \in \hat{I} \) be a linear operator on \( \mathbb{Z}[\hat{P}] \) defined by

\[
D_i(f) = \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}},
\]

where \( e^\lambda \) (\( \lambda \in \hat{P} \)) are the generators of \( \mathbb{Z}[\hat{P}] \). For \( w \in \tilde{W} \) with a reduced expression \( w = s_{i_1} \cdots s_{i_k} \), we set \( D_w = D_{i_1} \cdots D_{i_k} \). If \( w \in \tilde{W} \) and \( w = \tau w' \) (\( \tau \in \Sigma \), \( w' \in \tilde{W} \)), we set \( D_w = \tau D_{w'} \). The operator \( D_w \) is called a \textit{Demazure operator}. The character of the right-hand side of (2.2) is expressed using Demazure operators by [16, Theorem 5], and as a consequence we have the following (see also [21, Corollary 2.8]).
Proposition 2.2. Let $\Lambda^j \in \tilde{P}^+$ and $w_j \in \tilde{W}$ (1 \leq j \leq p) be as in Proposition 2.1. Then we have

$$\text{ch}_h D(w_{[1,1]}^{\Lambda^1}, w_{[1,2]}^{\Lambda^2}, \ldots, w_{[1,p-1]}^{\Lambda^{p-1}}, w_{[1,p]}^{\Lambda^p}) = D_w(e^{\Lambda^1} \cdot D_{w_2}(e^{\Lambda^2} \cdot \ldots \cdot D_{w_{p-1}}(e^{\Lambda^{p-1}} \cdot D_{w_p}(e^{\Lambda^p})) \ldots)).$$

2.5 Quantum loop algebras and their representations

The quantum loop algebra $U_q(L\mathfrak{g})$ is a $\mathbb{C}(q)$-algebra generated by $x^\pm_{i,r}$ and $h_{i,m}$ ($i \in I$, $r \in \mathbb{Z}$, $m \in \mathbb{Z} \setminus \{0\}$) subject to certain relations (see, e.g., [6, Section 12.2]). $U_q(L\mathfrak{g})$ has a Hopf algebra structure [6, 17]. In particular if $V$ and $W$ are $U_q(L\mathfrak{g})$-modules then $V \otimes W$ and $V^*$ are also $U_q(L\mathfrak{g})$-modules, and we have $(V \otimes W)^* \cong W^* \otimes V^*$.

Denote by $U_q(L\mathfrak{n}_\pm)$ and $U_q(L\mathfrak{h})$ the subalgebras of $U_q(L\mathfrak{g})$ generated by $\{x^\pm_{i,r} | i \in I, r \in \mathbb{Z}\}$ and $\{k^\pm_{i}, h_{i,m} | i \in I, m \in \mathbb{Z} \setminus \{0\}\}$ respectively. Denote by $U_q(\mathfrak{g})$ the subalgebra generated by $\{x^\pm_{i,r}, k^\pm_{i}, h_{i,m} | i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}\}$.

We recall basic results on finite-dimensional $U_q(\mathfrak{g})$- and $U_q(L\mathfrak{g})$-modules. Note that in the present paper we assume that $\mathfrak{g}$ is of type $D$, and when $\mathfrak{g}$ is non-simply laced some of indeterminates $q$ appearing below should be replaced by $q_i = q^{d_i}$ with suitable $d_i \in \mathbb{Z}_{>0}$.

A $U_q(\mathfrak{g})$-module (or $U_q(L\mathfrak{g})$-module) $V$ is said to be of type 1 if $V$ satisfies

$$V = \bigoplus_{\lambda \in \tilde{P}} V_\lambda, \quad V_\lambda = \{v \in V | k_i v = q^{|\alpha_i|} \lambda v\}.$$  

In this article we will only consider modules of type 1. For a finite-dimensional module $V$ of type 1, we set $\text{ch} V = \sum_{\lambda \in \tilde{P}} e^\lambda \dim V_\lambda \in \mathbb{Z}[\tilde{P}]$. For $\lambda \in \tilde{P}^+$ we denote by $V_\lambda(\mathfrak{g})$ the finite-dimensional simple $U_q(\mathfrak{g})$-module of type 1 with highest weight $\lambda$. The category of finite-dimensional $U_q(\mathfrak{g})$-modules of type 1 is semisimple, and every simple object is isomorphic to $V_\lambda(\mathfrak{g})$ for some $\lambda \in \tilde{P}^+$.

We say that a $U_q(L\mathfrak{g})$-module $V$ is highest $\ell$-weight with highest $\ell$-weight vector $v$ and highest $\ell$-weight $(\gamma_i^+(u), \gamma_i^-(u))_{i \in I} \in (\mathbb{C}(q)[[u]] \times \mathbb{C}(q)[[u^{-1}]])^I$ if $v$ satisfies $U_q(L\mathfrak{g})v = V$, $x^\pm_{i,r} v = 0$ for all $i \in I$, $r \in \mathbb{Z}$, and $\phi_i^+(u)v = \gamma_i^+(u)v$ for all $i \in I$. Here $\phi_i^+(u) \in U_q(L\mathfrak{h})[[u^{\pm 1}]]$ are defined as follows:

$$\phi_i^+(u) = k^\pm_{i} \exp \left( \pm (q - q^{-1}) \sum_{r=1}^{\infty} h_{i,r} u^\pm r \right).$$

Theorem 2.3 ([8]).

(i) If $V$ is a finite-dimensional simple $U_q(L\mathfrak{g})$-module of type 1, then $V$ is highest $\ell$-weight, and its highest $\ell$-weight $(\gamma_i^+(u), \gamma_i^-(u))_{i \in I}$ satisfies

$$\gamma_i^+(u) = q^{	ext{deg} \pi_i(u)} \left( \frac{\pi_i(q^{-1}u)}{\pi_i(qu)} \right)$$

for some polynomials $\pi_i(u) \in \mathbb{C}(q)[u]$ whose constant terms are 1. Here $()^\pm$ denote the expansions at $u = 0$ and $u = \infty$ respectively.

(ii) Conversely, for every $I$-tuple of polynomials $\pi = (\pi_1(u), \ldots, \pi_n(u))$ such that $\pi_i(0) = 1$, there exists a unique (up to isomorphism) finite-dimensional simple highest $\ell$-weight $U_q(L\mathfrak{g})$-module of type 1 with highest $\ell$-weight $(\gamma_i^+(u), \gamma_i^-(u))_{i \in I}$ satisfying (2.3).
The $I$-tuple of polynomials $\pi = (\pi_1(u), \ldots, \pi_n(u))$ are called Drinfeld polynomials, and we will say by abuse of terminology that the highest $\ell$-weight of $V$ is $\pi$ if the highest $\ell$-weight $(\gamma_\ell^+(u), \gamma_\ell^-(u))_{i \in I}$ of $V$ satisfies (2.3). We denote by $L_q(\pi)$ the finite-dimensional simple $U_q(\mathfrak{g})$-module of type 1 with highest $\ell$-weight $\pi$, and by $v_\pi$ a highest $\ell$-weight vector of $L_q(\pi)$.

Let $i \mapsto i$ be the bijection $I \to I$ determined by $\alpha_i = -w_0(\alpha_i)$.

**Lemma 2.4** ([6]). For any Drinfeld polynomials $\pi$ we have

$$L_q(\pi)^* \cong L_q(\pi^*)$$

as $U_q(\mathfrak{g})$-modules, where $\pi^* = (\pi_i(q^{-h_i'}u))_{i \in I}$ and $h_i'$ is the dual Coxeter number.

### 2.6 Minimal affinizations

For an $I$-tuple of polynomials $\pi = (\pi_i(u))_{i \in I}$, set $\text{wt}(\pi) = \sum_{i \in I} \omega_i \deg \pi_i \in P^+$.

**Definition 2.5** ([1]). Let $\lambda \in P^+$.

(i) A simple finite-dimensional $U_q(\mathfrak{g})$-module $L_q(\pi)$ is said to be an affinization of $V_q(\lambda)$ if $\text{wt}(\pi) = \lambda$.

(ii) Affinizations $V$ and $W$ of $V_q(\lambda)$ are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. We denote by $[V]$ the equivalence class of $V$.

If $V$ is an affinization of $V_q(\lambda)$, as a $U_q(\mathfrak{g})$-module we have

$$V \cong V_q(\lambda) \oplus \bigoplus_{\mu \leq \lambda} V_q(\mu)^{m_\mu(V)}$$

with some $m_\mu(V) \in \mathbb{Z}_{\geq 0}$. Let $V$ and $W$ be affinizations of $V_q(\lambda)$, and define $m_\mu(V)$, $m_\mu(W)$ as above. We write $[V] \leq [W]$ if for all $\mu \in P^+$, either of the following holds:

(i) $m_\mu(V) \leq m_\mu(W)$, or

(ii) there exists some $\nu \succ \mu$ such that $m_{\nu}(V) < m_{\nu}(W)$.

Then $\leq$ defines a partial ordering on the set of equivalence classes of affinizations of $V_q(\lambda)$ [1, Proposition 3.7].

**Definition 2.6** ([1]). We say an affinization $V$ of $V_q(\lambda)$ is minimal if $[V]$ is minimal in the set of equivalence classes of affinizations of $V_q(\lambda)$ with respect to this ordering.

For $i \in I$, $a \in \mathbb{C}(q)^{\times}$ and $m \in \mathbb{Z}_{>0}$, define an $I$-tuple of polynomials $\pi^{(i)}_{m,a}$ by

$$(\pi^{(i)}_{m,a})_j(u) = \begin{cases} (1 - aq^{-m+1}u)(1 - aq^{-m+3}u) \cdots (1 - aq^{m-1}u), & j = i, \\ 1, & j \neq i. \end{cases}$$

We set $\pi^{(i)}_{0,a} = (1,1,\ldots,1)$ for every $i \in I$ and $a \in \mathbb{C}(q)^{\times}$. The simple modules $L_q(\pi^{(i)}_{m,a})$ are called Kirillov–Reshetikhin modules.

Let us recall the classification of minimal affinizations in the regular case of type $D$, which was given in [10]. (Similar results also hold in type $E$. See [7] for type $ABCFG$, in which minimal affinizations are unique up to equivalence.) For that, we fix several notation. Set $S = \{1, n-1, n\} \subseteq I$ and define the subsets $I_s \subseteq I$ ($s \in S$) by $I_1 = \{1, 2, \ldots, n-3\}$, $I_{n-1} = \{n-1\}$, $I_n = \{n\}$. Note that $I_s$ is the connected component of the subdiagram $I \setminus \{n-2\}$ containing $s$, and $I \setminus I_s$ is the maximal type $A$ subdiagram of $I$ not containing $s$. For $s \in S$, $\varepsilon \in \{\pm\}$, $\lambda = \sum_{i \in I} \lambda_i \omega_i \in P^+$ and $a \in \mathbb{C}(q)^{\times}$, define Drinfeld polynomials $\pi^{(i)}_{\varepsilon}(\lambda, a)$ as follows:
Remark 2.7. The Drinfeld polynomials $\pi^\varepsilon_i(\lambda, a) = \prod_{i \in I} \pi^{(i)}_{\lambda_i, a_i}$ (the product being defined component-wise) with $a_1 = a$ and

$$a_i = \begin{cases} 
\frac{\varepsilon(\lambda_{i+1} + \sum_{1 \leq j < i} \lambda_j + \lambda_i + i - 1)}{a}, & 2 \leq i \leq n - 2, \\
\frac{\varepsilon(\lambda_{n-1} + \sum_{1 \leq j < n-1} \lambda_j + \lambda_{n-1} + n - 2)}{a}, & i = n - 1, n.
\end{cases}$$

Remark 2.9. Call a minimal affinization is regular if its highest weight is regular.

(ii) If $\lambda \in P^+$, then there exist exactly three equivalence classes of minimal affinizations of $V_q(\lambda)$, for each $s \in S$

$$\{ L_q(\pi^\varepsilon_i(\lambda, a)) \mid \varepsilon \in \{\pm\}, a \in \mathbb{C}(q)^\times \}$$

with $r \in S \setminus \{s\}$ (here the choice of $r$ is irrelevant since $\pi^\varepsilon_r(\lambda, a) = \pi^\varepsilon_{r'}(\lambda, a)$ holds for $r, r' \in S \setminus \{s\}$).

(ii) If $\lambda \in P^+$, then there exist exactly three equivalence classes of minimal affinizations of $V_q(\lambda)$, for each $s \in S$

$$\{ L_q(\pi^\varepsilon_i(\lambda, a)) \mid \varepsilon \in \{\pm\}, a \in \mathbb{C}(q)^\times \}$$

forms an equivalence class.

We call $\lambda \in P^+$ regular if $\lambda$ satisfies one of the assumptions of (i) or (ii) in Theorem 2.8. We call a minimal affinization is regular if its highest weight is regular.

Remark 2.9 ([9]). In the remaining case when $\lambda \in P^+$, then the number of equivalence classes of minimal affinizations increases unboundedly with $\lambda$, and the classification of minimal affinizations has not been given except for the type $D_4$. 

\begin{itemize}
  \item When $s = 1$, set $\pi^\varepsilon_1(\lambda, a) = \prod_{i \in I} \pi^{(i)}_{\lambda_i, a_i}$ (the product being defined component-wise) with $a_1 = a$ and
  $$a_i = \begin{cases} 
\frac{\varepsilon(\lambda_{i+1} + \sum_{1 \leq j < i} \lambda_j + \lambda_i + i - 1)}{a}, & 2 \leq i \leq n - 2, \\
\frac{\varepsilon(\lambda_{n-1} + \sum_{1 \leq j < n-1} \lambda_j + \lambda_{n-1} + n - 2)}{a}, & i = n - 1, n.
\end{cases}$$
\end{itemize}
2.7 Classical limits and graded limits

Let \( A = \mathbb{C}[q, q^{-1}] \) be the ring of Laurent polynomials with complex coefficients, and denote by \( U_A(L_0) \) the \( A \)-subalgebra of \( U_q(L_0) \) generated by \( \{ k_i^{\pm 1}, (x_{i,r})^k/k! \}_{i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{>0}} \), where we set \([k]_q = (q^k - q^{-k})/(q - q^{-1})\) and \([k]_q! = [k][k-1]_q \cdots [1]_q\). Define \( U_A(g) \subseteq U_q(g) \) in a similar way. We define \( C \)-algebras \( U_1(L_0) \) and \( U_1(g) \) by

\[
U_1(L_0) = \mathbb{C} \otimes_A U_A(L_0) \quad \text{and} \quad U_1(g) = \mathbb{C} \otimes_A U_A(g),
\]

where we identify \( \mathbb{C} \) with \( A/\langle q - 1 \rangle \). Then the following \( C \)-algebra isomorphisms are known to hold \([17, 6, \text{Proposition 9.3.10}]\):

\[
U(L_0) \cong U_1(L_0)/\langle k_i - 1 | i \in I \rangle_{U_1(L_0)}, \quad U(g) \cong U_1(g)/\langle k_i - 1 | i \in I \rangle_{U_1(g)}, \quad (2.4)
\]

where \( \langle k_i - 1 | i \in I \rangle_{U_1(L_0)} \) denotes the two-sided ideal of \( U_1(L_0) \) generated by \( \{ k_i - 1 | i \in I \} \), and \( \langle k_i - 1 | i \in I \rangle_{U_1(g)} \) is defined similarly.

Let \( \pi = (\pi_1(u), \ldots, \pi_n(u)) \) be Drinfeld polynomials, and assume that there exists \( b \in \mathbb{C}^\times \) such that all the roots of \( \pi_i(u) \)'s are contained in the set \( bq^Z \) (it is known that in order to describe the category of finite-dimensional \( U_q(L_0) \)-modules, it is essentially enough to consider representations attached to such families of Drinfeld polynomials. For example, see \([13, \text{Section 3.7}]\)). Note that \( \pi_s^\pm(\lambda, a) \) satisfies this assumptions when \( a \in \mathbb{C}^\times q^Z \). Let \( L_A(\pi) \) be the \( U_A(L_0) \)-submodule of \( L_q(\pi) \) generated by a highest \( \ell \)-weight vector \( v_\pi \). Then by the isomorphism \((2.4)\),

\[
\bar{L}_q(\pi) = \mathbb{C} \otimes_A L_A(\pi)
\]

becomes a finite-dimensional \( L_0 \)-module, which is called the classical limit of \( L_q(\pi) \).

Define a Lie algebra automorphism \( \varphi_b : g[t] \to g[t] \) by

\[
\varphi_b(x \otimes f(t)) = x \otimes f(t - b) \quad \text{for} \quad x \in g, f \in \mathbb{C}[t].
\]

We consider \( \bar{L}_q(\pi) \) as a \( g[t] \)-module by restriction, and define a \( g[t] \)-module \( L(\pi) \) by the pullback \( \varphi_b^\#(\bar{L}_q(\pi)) \). We call \( L(\pi) \) the graded limit of \( L_q(\pi) \). In fact, at least when \( \pi = \pi_s^\pm(\lambda, a) \), it turns out later from our main theorems that \( L(\pi) \) is a graded \( g[t] \)-module, which justifies the name “graded limit”. Set \( \bar{v}_\pi = 1 \otimes_A v_\pi \in L(\pi) \), which generates \( L(\pi) \) as a \( g[t] \)-module. The following properties of graded limits are easily proved from the construction (see \([2]\)).

**Lemma 2.10.** Assume \( \text{wt}(\pi) = \lambda \).

(i) There exists a surjective \( g[t] \)-module homomorphism from \( L(\pi) \) to \( V(\lambda, 0) \) mapping \( \bar{v}_\pi \) to a highest weight vector.

(ii) The vector \( \bar{v}_\pi \) satisfies the relations

\[
\begin{align*}
n_+ [t] \bar{v}_\pi &= 0, \\
(h \otimes t^k) \bar{v}_\pi &= \delta_{k0} (h, \lambda) \bar{v}_\pi & \text{for} \quad h \in h, k \geq 0, \quad \text{and} \\
\ell^i \bar{v}_\pi &= 0 & \text{for} \quad i \in I.
\end{align*}
\]

(iii) We have

\[
\text{ch} \ L_q(\pi) = \text{ch} \ L(\pi).
\]

(iv) For every \( \mu \in P^+ \) we have

\[
[L_q(\pi) : V_q(\mu)] = [L(\pi) : V(\mu)],
\]

where the left- and right-hand sides are the multiplicities as a \( U_q(g) \)-module and \( g \)-module, respectively.
3 Main theorems and corollaries

Throughout this section, we fix \( \lambda = \sum_{i \in \mathbb{I}} \lambda_i \varpi_i \in P^+, \varepsilon \in \{\pm\} \) and \( \alpha \in \mathbb{C}^* q^\mathbb{Z} \), and abbreviate \( \pi_s = \pi_s^\varepsilon(\lambda, a) \) for \( s \in S = \{1, n - 1, n\} \).

### 3.1 Main theorems

Denote by \( \tau_{0,1} \in \Sigma \) (resp. \( \tau_{n-1,n} \in \Sigma \)) the diagram automorphism interchanging the nodes 0 and 1 (resp. \( n - 1 \) and \( n \)). We will not use other elements of \( \Sigma \) in the sequel.

For \( s \in S \) and \( 1 \leq j \leq n \), define \( \xi_j^{(s)} = \xi_j^{(s)}(\lambda) \in \hat{P} \) as follows:

- When \( s = 1 \), let \( m, m' \) be such that \( \{m, m'\} = \{n - 1, n\} \), \( \lambda_m = \max\{\lambda_{n-1}, \lambda_n\} \) and \( \lambda_{m'} = \min\{\lambda_{n-1}, \lambda_n\} \), and define
  \[
  \xi_j^{(1)} = \begin{cases} 
  \lambda_j(\varpi_j + \Lambda_0), & 1 \leq j \leq n - 2, \\
  \lambda_{m'}(\varpi_{n-1} + \varpi_n + \Lambda_0), & j = n - 1, \\
  (\lambda_m - \lambda_{m'})(\varpi_m + \Lambda_0), & j = n.
  \end{cases}
  \]

- When \( s = n \), set
  \[
  \ell = \begin{cases} 
  0, & \text{if } \sum_{i=1}^{n-3} \lambda_i < \lambda_{n-1}, \\
  \max\left\{1 \leq j \leq n - 3 \mid \sum_{i=j}^{n-3} \lambda_i \geq \lambda_{n-1}\right\}, & \text{otherwise},
  \end{cases}
  \]
  and \( \bar{\lambda} = \lambda_{n-1} - \sum_{i=\ell+1}^{n-3} \lambda_i \). Then define
  \[
  \xi_j^{(n)} = \begin{cases} 
  \lambda_j(\varpi_j + \Lambda_0), & 1 \leq j \leq \ell, \ j = n - 2, n, \\
  \lambda_j(\varpi_j + \Lambda_0) + \bar{\lambda} \varpi_{n-1}, & j = \ell, \\
  \lambda_j(\varpi_j + \varpi_{n-1} + \Lambda_0) + \delta_{m0}\delta_{j1}\bar{\lambda}(\varpi_{n-1} + \Lambda_0), & \ell < j < n - 2, \\
  0, & j = n - 1.
  \end{cases}
  \]

- When \( s = n - 1 \), set \( \xi_j^{(n-1)}(\lambda) = \tau_{n-1,n}(\xi_j^{(n)}(\tau_{n-1,n}\lambda)) \).

Note that we have \( \lambda \equiv \sum_{1 \leq j \leq n} \xi_j^{(s)} \mod \mathbb{Z}\Lambda_0 + \mathbb{Q}\delta \) for all \( s \in S \).

**Theorem 3.1.** The graded limit \( L(\pi_s) \) is isomorphic to \( D(w_0\xi_1^{(s)}, \ldots, w_0\xi_n^{(s)}) \) as a \( \mathfrak{g}[t] \)-module.

For \( \alpha = \sum_{i \in \mathbb{I}} n_i \alpha_i \in \Delta_+ \), set \( \supp(\alpha) = \{i \in I \mid n_i > 0\} \subseteq I \). We define a subset \( \Delta_+^{(s)} \subseteq \Delta_+ \) for \( s \in S \) by

\[
\Delta_+^{(s)} = \bigcup_{r \in S \setminus \{s\}} \{\alpha \in \Delta_+ \mid \supp(\alpha) \subseteq I \setminus I_r\}.
\]

Note that if \( \alpha \in \Delta_+^{(s)} \), then the coefficient of \( \alpha_i \) in \( \alpha \) is 0 or 1 for all \( i \in I \).
Lemma 2.10(iii), the following holds.

\begin{align*}
n_+\{t\}v &= 0, \\
(h \otimes t^k)v &= \delta_{k0}\langle h, \lambda \rangle v \quad \text{for } h \in \mathfrak{h}, \; k \geq 0, \\
f^{\lambda_i}_{i+1}v &= 0 \quad \text{for } i \in I \quad \text{and} \quad (f_\alpha \otimes t)v &= 0 \quad \text{for } \alpha \in \Delta_+^{(s)}.
\end{align*}

We prove Theorems 3.1 and 3.2 in Section 4.

Remark 3.3. The defining relations of Theorem 3.2 were conjectured in [18, Section 5.11], and proved there for \(\mathfrak{g}\) of type \(D_4\). Let \(I' = I_\delta \cup \{n-2\}\), \(\lambda'_{i'} = \sum_{i \in I'} \lambda_i \omega_i\), and \(\lambda_{I' \setminus I'} = \lambda - \lambda'_{I'}\). In loc. cit., the author also conjectured that the graded limit \(L(\pi, s)\) is isomorphic to the \(\mathfrak{g}[t]\)-submodule of \(L(\pi_s(\lambda_{l'}, a)) \otimes L(\pi_s(\lambda_{I' \setminus I'}, a))\)

generated by the tensor product of highest weight vectors. This is easily deduced from Theorem 3.1.

3.2 Corollaries

The module \(D(w_o \xi_{1}^{(s)}, \ldots, w_o \xi_{n}^{(s)})\) in Theorem 3.1 has another realization introduced in Section 2.4. Define \(\sigma \in \tilde{W}\) by

\[\sigma = s_0 s_1 \cdots s_{n-1} s\]

The proof of the following lemma is straightforward.

Lemma 3.4.

(i) For \(0 \leq j \leq n\), we have

\[\sigma(\omega_j + \Lambda_0) \equiv \begin{cases} \\
\omega_{j+1} + \Lambda_0, & 0 \leq j \leq n-3, \\
\omega_{n-1} + \omega_n + \Lambda_0, & j = n-2, \\
\omega_{n-1} + \omega_1 + \Lambda_0, & j = n-1, \\
\omega_{n-1} + \Lambda_0, & j = n, \\
\end{cases} \mod \mathbb{Q}\delta,\]

and \(\sigma(\omega_{n-1}) \equiv \omega_{n-1} \mod \mathbb{Q}\delta\).

(ii) We have \(\ell(w_o \sigma^{n-1}) = \ell(w_o) + (n-1)\ell(\sigma)\).

Assume \(s \neq n - 1\) for a while. For \(1 \leq j \leq n - 1\) define \(\Lambda_j^{(s)} = \sigma^{-j} \xi_j^{(s)}\), and set \(\Lambda_n^{(s)} = \xi_n^{(s)}\).

The following assertions are easily checked using Lemma 3.4(i):

\[\Lambda_j^{(1)} \equiv \begin{cases} \\
\lambda_j \Lambda_0, & 1 \leq j \leq n-2, \\
\lambda_j \Lambda_0, & j = n-1, \\
0, & j = n, \end{cases} \mod \mathbb{Q}\delta, \quad \text{and}\]

\[\Lambda_j^{(n)} \equiv \begin{cases} \\
\lambda_j \Lambda_0, & 1 \leq j \leq \ell, \\
\lambda_j \Lambda_0 + \omega_{n-1}, & j = \ell + 1, \\
\lambda_j (\omega_{n-1} + \lambda_0) + \delta_0 \delta_1 \lambda (\omega_{n} + \Lambda_0), & \ell < j < n-2, \end{cases} \mod \mathbb{Q}\delta.\]

In particular, each \(\Lambda_j^{(s)}\) belongs to \(\tilde{P}^+\). We obtain the following chain of isomorphisms from Theorem 3.1, Lemma 3.4(ii), and Proposition 2.1:

\[L(\pi, s) \cong D(w_o \xi_{1}^{(s)}, \ldots, w_o \xi_{n}^{(s)}) \cong D(w_o \xi_{n}^{(s)}, w_o \xi_{1}^{(s)}, \ldots, w_o \xi_{n-1}^{(s)}) \cong F_{w_o} (D(\Lambda_n^{(s)}) \otimes F_{\sigma} (D(\Lambda_{n-2}^{(s)}) \otimes \cdots \otimes F_{\sigma} (D(\Lambda_0^{(s)}) \otimes F_{\sigma} (D(\Lambda_{-1}^{(s)}) \otimes \cdots))))\] (3.1)

where the second isomorphism obviously holds by definition. Hence by Proposition 2.2 and Lemma 2.10(iii), the following holds.
Corollary 3.5. If $s \in \{1, n\}$, we have
\[
\text{ch } L_q(\pi_s) = \mathcal{D}_{w_0} (e^{\Lambda_0^{(s)}} \cdot \mathcal{D}_0 (e^{\Lambda_1^{(s)}} \cdot \cdots \cdot \mathcal{D}_0 (e^{\Lambda_{n-2}^{(s)}} \cdot \mathcal{D}_0 (e^{\Lambda_{n-1}^{(s)}}))) \cdots)) |_{e^{\Lambda_0} = e^0 = 1}.
\]

Let $\lambda = \tau_{n-1, n} \lambda$, and set $\pi_n = \pi_n' (\lambda', a)$. It is easily seen from Theorem 3.1 that
\[
\text{ch } L_q(\pi_{n-1}) = \tau_{n-1, n} \text{ch } L_q(\pi_n').
\]

Hence we also obtain the character in the case $s = n - 1$.

Remark 3.6.

(i) It is possible to use other elements of $\tilde{W}$ in the expression of $\text{ch } L_q(\pi_s)$. That is, if $w_j \in \tilde{W}$ $(1 \leq j \leq n - 1)$ satisfy $w_{[1,j]} A_j^{(s)} = \xi_j^{(s)}$ and $\ell(w_{[1,n-1]}) = \ell(w_0) + \sum_{j=1}^{n-1} \ell(w_j)$ (here we set $w_{[1,j]} = w_1 w_2 \cdots w_j$), then it follows that
\[
\text{ch } L_q(\pi_s) = \mathcal{D}_{w_0} (e^{\Lambda_0^{(s)}} \cdot \mathcal{D}_{w_1} (e^{\Lambda_1^{(s)}} \cdot \cdots \cdot \mathcal{D}_{w_{n-2}} (e^{\Lambda_{n-2}^{(s)}} \cdot \mathcal{D}_{w_{n-1}} (e^{\Lambda_{n-1}^{(s)}}))) \cdots)) |_{e^{\Lambda_0} = e^0 = 1}.
\]

For example $w_j = s_{j-1} s_{j-2} \cdots s_1 \tau_{0,1}$ satisfy the above conditions when $s = 1$. Our choice is made so that the results are stated in a uniform way.

(ii) The right-hand side of the isomorphism (3.1) has a crystal analog, and using this we can express the multiplicities of $L_q(\pi_s)$ in terms of crystal bases. For the details, see [21, Corollary 4.11].

Our next result is a formula for multiplicities of simple finite-dimensional $U_q(\mathfrak{g})$-modules in $L_q(\pi_1)$ which can be deduced from our Theorem 3.2 and the results of [4] and [22]. For that, we prepare a lemma.

Lemma 3.7. Assume that $V$ is a cyclic finite-dimensional $\mathfrak{g}[t]$-module generated by a $\mathfrak{h}$-weight vector $v$, and $n_+ [t] \oplus \mathfrak{h}[t]$ acts trivially on $v$. Let $\mu \in P^+$, and $W$ be the $\mathfrak{g}[t]$-submodule of $V \otimes V(\mu, 0)$ generated by $v \otimes v_\mu$, where $v_\mu$ denotes a highest weight vector. Then for every $\nu \in P^+$, we have
\[
[W : V(\nu + \mu)] = [V : V(\nu)],
\]
where $[\cdot :]$ denotes the multiplicity as a $\mathfrak{g}$-module.

Proof. Note that
\[
[W : V(\nu + \mu)] = \dim \{ w \in W_{\nu + \mu} | n_+ w = 0 \}.
\]

Since
\[
W = U(n_-[t]) (v \otimes v_\mu) = U(n_-) (U(t n_-[t]) v \otimes v_\mu)
\]
and $W$ is a finite-dimensional $\mathfrak{g}$-module, we see that
\[
\{ w \in W_{\nu + \mu} | n_+ w = 0 \} = \{ w \in (U(t n_-[t]) v \otimes v_\mu)_{\nu + \mu} | n_+ w = 0 \} = \{ w \in (U(n_-[t]) v_\nu | n_+ w = 0 \} \otimes v_\mu = \{ w \in (U(n_-[t]) v_\nu | n_+ w = 0 \} \otimes v_\mu.
\]

Hence the assertion follows from (3.2).
Let $\mathfrak{sp}_{2n-2}$ be the simple Lie algebra of type $C_{n-1}$, and denote by $P_{\mathfrak{sp}}$ its weight lattice and by $\varpi^{sp}_i$ $(1 \leq i \leq n-1)$ its fundamental weights. We assume that $\varpi^{sp}_i$ are labeled as [15, Section 4.8]. Define a map $\iota : P^+ \to P^+_{\mathfrak{sp}}$ by

$$
\iota \left( \sum_{1 \leq i \leq n} \mu_i \varpi_i \right) = \sum_{1 \leq i \leq n-2} \mu_i \varpi^{sp}_i + \min \{ \mu_{n-1}, \mu_n \} \varpi_{n-1}.
$$

**Corollary 3.8.** For every $\mu \in P^+$, we have

$$
[L_q(\pi_1) : V_q(\mu)] = \begin{cases} 
  [S_\iota(\lambda)(V^{sp}(\varpi^{sp}_1)) : V^{sp}(\iota(\mu))] & \text{if } \mu - \mu_{n-1} = \lambda_n - \lambda_{n-1}, \\
  0 & \text{otherwise.}
\end{cases}
$$

Here $S_\nu$ ($\nu \in P^+_{\mathfrak{sp}}$) denotes the Schur functor (see [22, Section 1]) with respect to the partition

$$
\left( \sum_{j=1}^{n-1} \nu_j, \sum_{j=2}^{n-1} \nu_j, \ldots, \nu_{n-1} \right),
$$

and $V^{sp}(\nu)$ denotes the simple $\mathfrak{sp}_{2n-2}$-module with highest weight $\nu$.

**Proof.** It suffices to show that the right-hand side is equal to $[L(\pi_1) : V(\mu)]$ by Lemma 2.10(iv). Note that Theorem 3.2 and [4, Theorem 1] imply that the graded limit $L(\pi_1)$ is isomorphic to the $\mathfrak{g}[t]$-module "$P(\lambda, 0)^{F(\lambda, \Psi)}$" in the notation of [4], where we set $\Psi = \{ \alpha \in \Delta_+ \mid (\alpha, \varpi_1 + \varpi_n) = 2 \}$. Then in the case $\lambda_{n-1} = \lambda_n$, our assertion is a consequence of [4, Theorem 2] and [22, Theorem 1].

Let us assume $\lambda_{n-1} \neq \lambda_n$, and set $\lambda' = \lambda_m - \lambda_{m'}$. We have

$$
L(\pi_1) \cong D(w_0 \xi^{(1)}_1, \ldots, w_0 \xi^{(1)}_n)
$$

by Theorem 3.1, and we easily see that $D(w_0 \xi^{(1)}_1) \cong V(\lambda' \varpi_m, 0)$ holds. Hence by applying Lemma 3.7 with $V = D(w_0 \xi^{(1)}_1, \ldots, w_0 \xi^{(1)}_n)$ and $\mu = \lambda' \varpi_m$, we have for every $\nu \in P^+$ that

$$
[D(w_0 \xi^{(1)}_1, \ldots, w_0 \xi^{(1)}_n) : V(\nu + \lambda' \varpi_m)] = [D(w_0 \xi^{(1)}_1, \ldots, w_0 \xi^{(1)}_n) : V(\nu)],
$$

and the right-hand side is equal to $[L(\pi_1(\lambda - \lambda' \varpi_m, a)) : V(\nu)]$ by Theorem 3.1. Hence the assertion is deduced from the case $\lambda_{n-1} = \lambda_n$. The proof is complete. 

**4 Proofs of main theorems**

Note that the theorems for $s = n-1$ and $s = n$ are equivalent because of the existence of diagram automorphism of $\mathfrak{g}$ interchanging $n-1$ and $n$. Therefore, throughout this section we assume that $s \neq n-1$, and prove the theorems only for the case $s = n$. Similarly as the previous section we fix $\lambda = \sum_{i \in I} \lambda_i \varpi_i \in P^+$, $\varepsilon \in \{ \pm \}$ and $a \in \mathbb{C}^\times q^\mathbb{Z}$, and write $\pi_s = \pi_s^\varepsilon(\lambda, a)$.

Let $M_s(\lambda)$ denote the module defined in Theorem 3.2. We shall prove the existence of three surjective $\mathfrak{g}[t]$-module homomorphisms. More precisely, we prove $M_s(\lambda) \to L(\pi_s)$ in Section 4.1, $D(w_0 \xi^{(s)}_1, \ldots, w_0 \xi^{(s)}_n) \to M_s(\lambda)$ in Section 4.2, and $L(\pi_s) \to D(w_0 \xi^{(s)}_1, \ldots, w_0 \xi^{(s)}_n)$ in Section 4.3. Then both Theorems 3.1 and 3.2 immediately follow from them.

**4.1 Proof for $M_s(\lambda) \to L(\pi_s)$**

Though the proof is similar to that in [18], we will give it for completeness.

Let $v_{\pi_s}$ be a highest $t$-weight vector of $L_q(\pi_s)$, and set $\tilde{v}_{\pi_s} = 1 \otimes v_{\pi_s} \in L(\pi_s)$. In order to prove $M_s(\lambda) \to L(\pi_s)$, it is enough to show the relations

$$
(f_{\alpha} \otimes t) \tilde{v}_{\pi_s} = 0 \quad \text{for } \alpha \in \Delta^{(s)}_+,
$$

(4.1)

since the other relations hold by Lemma 2.10(ii).
Let $r \in S \setminus \{s\}$ and $J = I \setminus I_r$. Then the subalgebra $U_q(L_{q,J}) \subseteq U_q(L_q)$ is a quantum loop algebra of type $A$. By [10, Lemma 2.3], the $U_q(L_{q,J})$-submodule of $L_q(\pi_s)$ generated by $v_{\pi_s}$ is isomorphic to the simple $U_q(L_{q,J})$-module with highest $\ell$-weight $((\pi_s)_j(u))_{i \in J}$. Denote this $U_q(L_{q,J})$-submodule by $L'_q$. Then we see from Remark 2.7 and [10, Theorem 3.1] that $L'_q$ is also simple as a $U_q(g_J)$-module. From this and the construction of graded limits, it follows that the $L_{q,J}$-submodule

$$L' = U(L_{q,J})\tilde{v}_{\pi_s} \subseteq L(\pi_s)$$

is simple as a $g_J$-module. Hence the restriction of the surjective homomorphism $L(\pi_s) \rightarrow V(\lambda,0)$ in Lemma 2.10(i) to $L'$ is an isomorphism, which obviously implies $(g_J \otimes t)\tilde{v}_{\pi_s} = 0$. Now the relations (4.1) obviously follow from the definition of $\Delta_+$. The proof is complete.

### 4.2 Proof for $D(w_0\xi_1^{(s)}, \ldots, w_0\xi_n^{(s)}) \Rightarrow M_s(\lambda)$

Throughout this subsection, we assume that $s \in \{1,n\}$ is fixed. Note that some notation appearing below may depend on $s$ though it is not written explicitly.

Let us prepare several notation. For $1 \leq p \leq q \leq n$, set

$$\alpha_{p,q} = \begin{cases} \alpha_p + \alpha_{p+1} + \cdots + \alpha_q, & q \leq n - 1, \\ \alpha_p + \alpha_{p+1} + \cdots + \alpha_{n-2} + \alpha_n, & q = n. \end{cases}$$

Note that

$$\Delta_+ = \{\alpha_{p,q} \mid p \leq q, (p,q) \neq (n-1,n)\} \sqcup \{\alpha_{p,n} + \alpha_{q,n-1} \mid p < q < n\}.$$

Set $\sigma_i = s_is_{i+1} \cdots s_{n-1} \in \widehat{W}$ for $1 \leq i \leq n$ and $\sigma_0 = \tau_{0,1}\tau_{n-1,n}\sigma_1 = \sigma$. For $0 \leq i \leq n$ and $1 \leq j \leq n-1$, define $\rho_{i,j} : \hat{\Delta} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\rho_{i,j}(\alpha) = \sum_{k=j}^{n-1} \max \{0, - (\alpha, \sigma_i\sigma^k \Lambda_k^{(s)})\}.$$ 

When $j < n - 1$, we have

$$\rho_{n,j}(\alpha) = \rho_{0,j+1}(\alpha) + \max \{0, - (\alpha, \Lambda_j^{(s)})\} = \rho_{0,j+1}(\alpha) \quad \text{for} \quad \alpha \in \hat{\Delta}_+^{re}$$

since $\Lambda_j^{(s)} \in \hat{P}^+$. 

**Lemma 4.1.** Let $1 \leq i \leq n$ and $1 \leq j \leq n - 1$, and assume that $\alpha = \beta + k\delta \in \hat{\Delta}_+^{re}$ satisfies $\rho_{i,j}(\alpha) > 0$.

(i) If $i = n$, we have

$$\beta \in \{-\alpha_{p,n-1} \mid p < n\} \sqcup \{-(\alpha_{p,n} + \alpha_{q,n-1}) \mid p < q < n\}.$$ 

(ii) If $1 \leq i \leq n - 1$, we have

$$\beta \in \{\alpha_{i,q} \mid i \leq q < n\} \sqcup \{-\alpha_{p,i-1} \mid p < i\} \sqcup \{-\alpha_{p,n} \mid p \neq i\} \sqcup \{-(\alpha_{p,n} + \alpha_{q,n-1}) \mid p < q < n, p \neq i, q \neq i\}.$$
Proof. In both the cases $s = 1$ and $s = n$, it follows from Lemma 3.4(i) that
\[
\sigma^{k-j} \Lambda^{(s)}_k \in \sum_{0 \leq p \leq n} Z \geq 0 (w_p + \Lambda_0) + \sum_{1 \leq p \leq n} Z \geq 0 (w_p + w_{n-1} + \Lambda_0)
\]
holds for every $1 \leq j \leq k \leq n - 1$. Hence $\rho_{n,j}(\alpha) > 0$ implies $\beta \in -\Delta_+$ and $k \geq 1$, and if $\beta = -\sum_{i \in I} t_i \alpha_i$ then we have
\[
t_p + t_{n-1} > k \geq 1 \quad \text{for some} \quad 1 \leq p \leq n.
\]
This immediately implies the assertion (i). Note that $\rho_{i,j}(\alpha) = \rho_{i+1,j}(s_i \alpha)$ holds for $1 \leq i \leq n - 1$ by the definition of $\rho_{i,j}$. Hence $\rho_{i,j}(\alpha) > 0$ implies that we have either $\alpha = \alpha_i$ or $\alpha = s_i \gamma$ for some $\gamma \in \tilde{\Delta}^{re}_+$ such that $\rho_{i+1,j}(\gamma) > 0$. From this, the assertion (ii) is easily proved by the descending induction on $i$. \hfill \blacksquare

For $0 \leq i \leq n$ and $1 \leq j \leq n - 1$, set
\[
D(i,j) = D(\sigma_i \Lambda^{(s)}_j, \sigma_i \sigma \Lambda^{(s)}_{j+1}, \ldots, \sigma_i \sigma^{n-j} \Lambda^{(s)}_{n-1}), \quad \text{and} \quad v(i,j) = v_{\sigma_i \Lambda^{(s)}_j} \otimes v_{\sigma_i \sigma \Lambda^{(s)}_{j+1}} \otimes \cdots \otimes v_{\sigma_i \sigma^{n-j} \Lambda^{(s)}_{n-1}} \in D(i,j).
\]
For $\alpha = \beta + k \delta \in \tilde{\Delta}^{re}$ with $\beta \in \Delta$ and $k \in \mathbb{Z}$, denote by $x_{\alpha} \in \tilde{\mathfrak{g}}$ the vector $e_\beta \otimes t^k$. For $i \in \tilde{I}$, define a Lie subalgebra $\tilde{\mathfrak{n}}_i$ of $\tilde{\mathfrak{n}}_+$ by
\[
\tilde{\mathfrak{n}}_i = \bigoplus_{\alpha \in \tilde{\Delta}^{re} \setminus \{\alpha_i\}} \mathbb{C} x_{\alpha} \oplus t \mathfrak{h}[t].
\]
We shall determine the generators of the annihilators $\text{Ann}_{U(\tilde{\mathfrak{n}}_+)} v(i,j)$ inductively, along the lines of [21, Section 5.1]. For that, we need the following lemma which is proved in [14, Section 3] (see also [21, Lemma 5.3]).

Lemma 4.2. Let $V$ be an integrable $\tilde{\mathfrak{g}}$-module, $T$ a finite-dimensional $\tilde{\mathfrak{h}}$-submodule of $V$, $i \in \tilde{I}$ and $\xi \in \tilde{P}$ such that $(\alpha_i, \xi) \geq 0$. Assume that the following conditions hold:

(i) $T$ is generated by a $\tilde{\mathfrak{h}}$-weight vector $v \in T_\xi$ satisfying $e_i v = 0$.
(ii) There is an $\text{ad}(e_i)$-invariant left $U(\tilde{\mathfrak{n}}_i)$-ideal $\mathcal{I}$ such that
\[
\text{Ann}_{U(\tilde{\mathfrak{n}}_+)} v = U(\tilde{\mathfrak{n}}_+) e_i + U(\tilde{\mathfrak{n}}_+) \mathcal{I}.
\]
(iii) We have $\text{ch}_{\tilde{\mathfrak{h}}} F_i T = D_i \text{ch}_{\tilde{\mathfrak{h}}} T$.

Let $v' = f_{i, (\alpha_i, \xi)} v$. Then we have
\[
\text{Ann}_{U(\tilde{\mathfrak{n}}_+)} v' = U(\tilde{\mathfrak{n}}_+) e_i^{(\alpha_i, \xi)+1} + U(\tilde{\mathfrak{n}}_+) r_i(\mathcal{I}),
\]
where $r_i$ denotes the algebra automorphism of $U(\tilde{\mathfrak{g}})$ corresponding to the reflection $s_i$.

Proposition 4.3. The following assertion $(A_{i,j})$ holds for every $0 \leq i \leq n$ and $1 \leq j \leq n - 1$

\[
(A_{i,j}) \quad \text{Ann}_{U(\tilde{\mathfrak{n}}_+)} v(i,j) = U(\tilde{\mathfrak{n}}_+) \left( \sum_{\alpha \in \tilde{\Delta}^{re}_+} \mathbb{C} x_{\alpha}^{\rho_{i,j}(\alpha)+1} + t \mathfrak{h}[t] \right).
\]
It remains to show that \( A \) which is the weight of \( v \). In addition, we have \( \rho \). We will prove the assertion by the descending induction on \( (i,j) \). Therefore, it suffices to show the ad\((e_i)\) invariance of the left \( U(\hat{n}_i)\)-ideal

\[
\mathcal{I}_{i,j} = U(\hat{n}_i) \left( \sum_{\alpha \in \hat{\Delta}_+^{re} \setminus \{\alpha_i\} \left[ \alpha \right]^{\rho_{i,j}(\alpha)+1} + t\mathfrak{h}[t] \right)
\]

by Lemma 4.2 (note that the condition (iii) holds by Proposition 2.2). Since \( \rho_{i,j}(\alpha_i) = 0 \) holds by Lemma 4.1, we have

\[
\left[ e_{i-1}, t\mathfrak{h}[t] \right] = e_{i-1} \otimes t\mathbb{C}[t] \subseteq \mathcal{I}_{i,j}.
\]

Hence it is enough to verify that

\[
\left[ e_{i-1}, x_{\alpha}^{\rho_{i,j}(\alpha)} \right] \in \mathcal{I}_{i,j}
\]

for every \( \alpha \in \hat{\Delta}_+^{re} \setminus \{\alpha_i\} \). If \( \alpha = -\alpha_i + k\delta \) \( (k > 0) \), then (4.3) follows from \( t\mathfrak{h}[t] \otimes e_{i-1} \otimes t\mathbb{C}[t] \subseteq \mathcal{I}_{i,j} \). Hence we may assume that \( \alpha \) satisfies \( \left[ [e_{i-1}, x_{\alpha}], x_{\alpha} \right] = 0 \). If \( [e_{i-1}, x_{\alpha}] = 0 \), (4.3) is obvious, and otherwise we have

\[
\left[ e_{i-1}, x_{\alpha}^{\rho_{i,j}(\alpha)} \right] \in \mathbb{C} x_{\alpha}^{\rho_{i,j}(\alpha)} x_{\alpha + \alpha_i-1}.
\]

It is directly checked from Lemma 4.1 that if \( \beta \in \hat{\Delta}_+^{re} \) satisfies \( \beta - \alpha_i \in \hat{\Delta}_+^{re} \), then \( \rho_{i,j}(\beta) = 0 \). Hence we have \( \rho_{i,j}(\alpha + \alpha_i-1) = 0 \), and (4.3) follows. The proof is complete.

In the sequel we write \( \rho = \rho_{0,1} \) for brevity. Note that we have

\[
\rho(\alpha) = \sum_{1 \leq j \leq n-1} \max \{0, -\langle \alpha, \xi_j^{(s)} \rangle\}.
\]

The following assertions are proved from the definition of \( \xi_j^{(s)} \)'s by a direct calculation.

(i) Assume that \( s = 1 \).
(a) For $\beta + k\delta \in \hat{\Delta}^\text{re}_+$, $\rho(\beta + k\delta) = 0$ holds unless
\[-\beta \in \{\alpha_{p,n} + \alpha_{q,n-1} \mid p < q < n\} \quad \text{and} \quad k = 1.
\]
(b) We have
\[
\rho(- (\alpha_{p,n} + \alpha_{q,n-1}) + \delta) = \sum_{j=q}^{n-2} \lambda_j + \lambda_{m'}.
\]

(ii) Assume that $s = n$.

(a) For $\beta + k\delta \in \hat{\Delta}^\text{re}_+$, $\rho(\beta + k\delta) = 0$ holds unless
\[-\beta \in \{\alpha_{p,n-1} \mid p \leq n - 3\} \quad \text{and} \quad k = 1, \quad \text{or} \quad -\beta \in \{\alpha_{p,n} + \alpha_{q,n-1} \mid p < q < n\} \quad \text{and} \quad k = 1, 2.
\]
(b) We have
\[
\rho(-\alpha_{p,n-1} + \delta) = \min \left\{ \sum_{j=p}^{n-3} \lambda_j, \lambda_{n-1} \right\}, \quad \text{and}
\]
\[
\rho(- (\alpha_{p,n} + \alpha_{q,n-1}) + k\delta) = \begin{cases} 
\min \left\{ \sum_{j=p}^{n-3} \lambda_j, \lambda_{n-1} \right\} + \sum_{j=q}^{n-2} \lambda_j, & \text{if } k = 1, \\
\min \left\{ \sum_{j=q}^{n-3} \lambda_j, \lambda_{n-1} \right\}, & \text{if } k = 2.
\end{cases}
\]

Set $D = D(\xi_{n}^{(s)}, \xi_{1}^{(s)}, \ldots, \xi_{n-1}^{(s)})$ and $v_D = v_{\xi_{n}^{(s)}} \otimes v(0,1) \in D$. By Proposition 4.3, we have
\[
\text{Ann}_U(\hat{n}_+)v_D = \text{Ann}_U(\hat{n}_+)v(0,1) = U(\hat{n}_+) \left( \sum_{\alpha \in \hat{\Delta}^\text{re}_+} C_{x_\alpha^{(\rho(\alpha)+1)}} + t\theta[t] \right).
\]

Let $v_M \in M_\lambda(\lambda)$ denote the generator in the definition. The proof of the following lemma is elementary (see, e.g., [20, Lemma 4.5]).

**Lemma 4.4.** If $\beta, \gamma \in \hat{\Delta}^\text{re}_+$ satisfy $(\beta, \gamma) = -1$, $x_{\beta}^{b+1}v_M = 0$ and $x_{\gamma}^{c+1}v_M = 0$ with $b, c \in \mathbb{Z}_{\geq 0}$, then $x_{\beta+\gamma}^{b+c+1}v_M = 0$ holds.

**Lemma 4.5.** There exists a $(\mathfrak{h} \oplus \hat{n}_+)$-module homomorphism from $D$ to $M_\lambda(\lambda)$ mapping $v_D$ to $v_M$.

**Proof.** Since $t\theta[t]v_M = 0$ and the $\mathfrak{h}$-weights of both $v_D$ and $v_M$ are $\lambda$, it suffices to show $x_\alpha^{\rho(\alpha)+1}v_M = 0$ for all $\alpha \in \hat{\Delta}^\text{re}_+$.

First we consider the case $s = 1$. The assertion for $\alpha = - (\alpha_{p,n} + \alpha_{q,n-1}) + \delta$ with $p < q < n$ is proved by applying Lemma 4.4 with $\beta = -\alpha_{p,m} + \delta$ and $\gamma = -\alpha_{q,m'}$. The assertion for remaining $\alpha \in \hat{\Delta}^\text{re}_+$ is easily proved from the defining relations of $M_\lambda(\lambda)$.

Next we consider the case $s = n$. If $\sum_{j=p}^{n-3} \lambda_j \geq \lambda_{n-1}$, the assertion for $\alpha = -\alpha_{p,n-1} + \delta$ is proved by applying the lemma with $\beta = -\alpha_{p,n-2} + \delta$, $\gamma = -\alpha_{n-1}$. Otherwise it is proved by applying the lemma with $\beta = -\alpha_{p,n-3}$, $\gamma = -\alpha_{n-2,n-1} + \delta$. The assertion for $\alpha = - (\alpha_{p,n} + \alpha_{q,n-1}) + \delta$ is similarly proved, and then the assertion for $\alpha = - (\alpha_{p,n} + \alpha_{q,n-1}) + \delta$ is verified by applying the lemma with $\beta = - (\alpha_{p,n} + \alpha_{n-1}) + \delta$ and $\gamma = -\alpha_{q,n-2}$. Finally the assertion for $\alpha = - (\alpha_{p,n} + \alpha_{q,n-1}) + 2\delta$ is shown by applying the lemma with $\beta = -\alpha_{p,n} + \delta$ and $\gamma = -\alpha_{q,n-1} + \delta$. The assertion for remaining $\alpha \in \hat{\Delta}^\text{re}_+$ is easily proved from the defining relations of $M_\lambda(\lambda)$.
Now we can prove the existence of a surjective $\mathfrak{g}[t]$-module homomorphism

$$D(w_0\xi_n^{(s)}, w_0\xi_{n-1}^{(s)}, \ldots, w_0\xi_1^{(s)}) \to M_\lambda(\lambda)$$

by exactly the same arguments with [21, two paragraphs below Lemma 5.2] from Lemma 4.5. Since $D(w_0\xi_n^{(s)}, w_0\xi_{n-1}^{(s)}, \ldots, w_0\xi_1^{(s)}) \cong D(w_0\xi_1^{(s)}, \ldots, w_0\xi_n^{(s)})$ holds by definition, the proof is complete.

### 4.3 Proof for $L(\pi_s) \to D(w_0\xi_1^{(s)}, \ldots, w_0\xi_n^{(s)})$

We shall prove the assertion in the case $\varepsilon = +$ (the case $\varepsilon = -$ is similarly proved). For $i \in I$, set

$$p_i = \begin{cases} n - 2 - i, & i \leq n - 2, \\ 1, & i = n - 1, n, \end{cases}$$

which is the distance between the nodes $i$ and $n - 2$ in the Dynkin diagram. We need the following lemma that is proved from [3].

**Lemma 4.6.**

(i) Let $i_1, \ldots, i_p \in I$, $b_1, \ldots, b_p \in \mathbb{C}(q)^\times$ and $l_1, \ldots, l_p \in \mathbb{Z}_{>0}$, and assume that

$$b_r \notin q^{-2}q^{-l_1}q^{l_1-i_1} + \cdots + q^{l_{r-1}} - p_i - p_{i+1} \quad \text{for all } r < s.$$  

Then the submodule of $L_q(\pi^{(i_1)}_{l_1, b_1}) \otimes \cdots \otimes L_q(\pi^{(i_p)}_{l_p, b_p})$ generated by the tensor product of highest $\ell$-weight vectors is isomorphic to $L_q(\bigotimes_{k=1}^p \pi^{(i_k)}_{l_k, b_k})$.

(ii) If $i, j \in I$, $b \in \mathbb{C}(q)^\times$, $l \in \mathbb{Z}_{>0}$ and $-|p_i - p_j| \leq k \leq |p_i - p_j|$, then $L_q(\pi^{(i)}_{l, b}) \otimes L_q(\pi^{(j)}_{l, b})$ is simple.

**Proof.** (i) For $r < s$, it is directly checked that $L_q(\pi^{(i_s)}_{l_s, b_s}) \otimes L_q(\pi^{(i_r)}_{l_r, b_r})$ satisfies the condition of [3, Corollary 6.2], which assures that the module is generated by the tensor product of highest $\ell$-weight vectors. Hence

$$L_q(\pi^{(i_p)}_{l_p, b_p}) \otimes L_q(\pi^{(i_{p-1})}_{l_{p-1}, b_{p-1}}) \otimes \cdots \otimes L_q(\pi^{(i_1)}_{l_1, b_1})$$

is also generated by the tensor product of highest $\ell$-weight vectors (see [3, sentences above Corollary 6.2]). Now the assertion (i) follows by dualizing the statement and applying Lemma 2.4 (the bijection $i \mapsto i$ in the lemma is $\tau_{n-1, n}$). (ii) We see from the above argument that $L_q(\pi^{(i)}_{l, b}) \otimes L_q(\pi^{(j)}_{l, b})$ is both cyclic and cocyclic, and hence simple. 

**Lemma 4.7.** Let $i \in I \setminus \{n - 2, n - 1\}$, $b \in \mathbb{C} \times q^2$ and $l \in \mathbb{Z}_{>0}$. The graded limit $L(\pi^{(i)}_{l, b}(n-1)) \pi^{(n-1)}_{l, b}q^{bq^{l+1} - bq^{l-2}}-1$ is isomorphic to $D(lw_0(\varpi_i + \varpi_{n-1} + \Lambda_0))$ as a $\mathfrak{g}[t]$-module.

**Proof.** Note that

$$\pi^{(i)}_{l, b} \pi^{(n-1)}_{l, b}q^{n-1} = \begin{cases} \pi^{(i)}_{l, b}(l(w_i + w_{n-1}), bq^{l(n-1)(i+1)}) & \text{if } i \leq n - 3, \\ \pi^{(i)}_{l, b}(l(w_i + w_{n-1}), bq^{-l-2}) & \text{if } i = n. \end{cases}$$

Hence by Sections 4.1 and 4.2, there exists a surjective homomorphism

$$D(lw_0(\varpi_i + \varpi_{n-1} + \Lambda_0)) \to L(\pi^{(i)}_{l, b} \pi^{(n-1)}_{l, b}q^{n-1})$$
Therefore it suffices to show the equality of the dimensions. By Lemma 4.6(ii), we have

\[ L_q(\pi_{i,b}^{(i)} \pi_{i,bq^{n-1}}^{(n-1)}) \cong L_q(\pi_{i,b}^{(i)}) \otimes L_q(\pi_{i,bq^{n-1}}^{(n-1)}), \]

which implies

\[
\dim_{\mathbb{C}} L(\pi_{i,b}^{(i)} \pi_{i,bq^{n-1}}^{(n-1)}) = \dim_{\mathbb{C}(q)} L_q(\pi_{i,b}^{(i)} \pi_{i,bq^{n-1}}^{(n-1)}) \\
= \dim_{\mathbb{C}(q)} L_q(\pi_{i,b}^{(i)}) \cdot \dim_{\mathbb{C}(q)} L_q(\pi_{i,bq^{n-1}}^{(n-1)}) = \dim_{\mathbb{C}} L(\pi_{i,b}^{(i)}) \cdot \dim_{\mathbb{C}} L(\pi_{i,bq^{n-1}}^{(n-1)}) \\
= \dim_{\mathbb{C}} D(lw_0(\varpi_i + \Lambda_0)) \cdot \dim_{\mathbb{C}} D(lw_0(\varpi_{n-1} + \Lambda_0)),
\]

where the last equality follows from [5, Proposition 5.1.3]. On the other hand, we have

\[
\dim_{\mathbb{C}} D(lw_0(\varpi_i + \varpi_{n-1} + \Lambda_0)) = \dim_{\mathbb{C}} D((lw_0(\varpi_i + \Lambda_0)) \cdot \dim_{\mathbb{C}} D(lw_0(\varpi_{n-1} + \Lambda_0))
\]
by [11, Theorem 1]. Hence the assertion is proved. \(\blacksquare\)

Now let us begin the proof of the assertion \(L(\pi_s) \to D(w_0 \xi_1^{(s)}, \ldots, w_0 \xi_n^{(s)})\). First we prove this in the case \(s = n\). Let \((a_i)_{i \in I}\) be the sequence in the definition of \(\pi_n = \pi_n^+(\lambda, a)\), and define \(U_q(Lg)\)-modules \(L_q[j]\) for \(1 \leq j \leq n - 2\) and \(j = n\) by

\[
L_q[j] = \begin{cases} 
L_q(\pi_{j,a_j}^{(j)}), & \text{if } j < \ell, \quad j = n - 2, n, \\
L_q(\pi_{j-\lambda,a_j}^{(j)} \pi_{\lambda,a_j}^{(j)} \pi_{\lambda,a_j}^{(j)}), & \text{if } j = \ell, \\
L_q(\pi_{j,a_j}^{(j)} \pi_{\lambda,a_j}^{(j)} \pi_{\lambda,a_j}^{(j)}), & \text{if } j = 1, \\
L_q(\pi_{j,a_j}^{(j)} \pi_{\lambda,a_j}^{(j)} \pi_{\lambda,a_j}^{(j)}), & \text{otherwise}.
\end{cases}
\]

There exists an injective \(U_q(Lg)\)-module homomorphism

\[ L_q(\pi_n) \to L_q[1] \otimes \cdots \otimes L_q[n-2] \otimes L_q[n] \]
by Lemma 4.6, and this induces a \(U_A(Lg)\)-module homomorphism

\[ L_A(\pi_n) \to L_A[1] \otimes \cdots \otimes L_A[n-2] \otimes L_A[n], \]

where we set

\[
L_A[j] = \begin{cases} 
L_A(\pi), & \text{if } L_q[j] = L_q(\pi), \\
L_A(\pi) \otimes L_A(\pi^2), & \text{if } L_q[j] = L_q(\pi^1) \otimes L_q(\pi^2).
\end{cases}
\]

Applying \(\mathbb{C} \otimes_{\mathbb{A}} -\) and taking the pull-back with respect to the automorphism \(\varphi_\delta\), we obtain a \(g[l]\)-module homomorphism \(L(\pi_n) \to \bigotimes J L[j]\), where \(L[j]\) denotes the graded limit or the tensor product of the two graded limits. Note that, by construction, this homomorphism maps a highest weight vector of \(L(\pi_n)\) to the tensor product of highest weight vectors. By Lemma 4.7 and [5, Proposition 5.1.3], we have

\[
L[j] \cong \begin{cases} 
D((\lambda_{\ell} - \hat{\lambda}) w_0(\varpi_\ell + \Lambda_0)), & \text{if } j = \ell, \\
D(\lambda w_0(\varpi_{n-1} + \Lambda_0)) \otimes D(\lambda w_0(\varpi_1 + \varpi_{n-1} + \Lambda_0)), & \text{if } \ell = 0, \quad j = 1, \\
D(w_0 \xi_j^{(n)}), & \text{otherwise}.
\end{cases}
\]

Hence in order to complete the proof, it suffices to show that

\[
D((\lambda_{\ell} - \hat{\lambda}) w_0(\varpi_\ell + \Lambda_0), \lambda w_0(\varpi_{n-1} + \varpi_{n-1} + \Lambda_0)) \cong D(w_0 \xi_1^{(n)}), \quad \text{and}
\]

\[
D(\lambda w_0(\varpi_{n-1} + \Lambda_0), \lambda w_0(\varpi_1 + \varpi_{n-1} + \Lambda_0)) \cong D(w_0 \xi_1^{(n)}) \quad \text{when } \ell = 0.
\]
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The first isomorphism follows since we have

\[
D(w_0 e^{(n)}_\ell) \cong F_{w_0 \sigma_\ell}(\Lambda_0^{(n)}) \cong F_{w_0 \sigma_\ell}(D((\lambda_\ell - \bar{\lambda}) \Lambda_0) \otimes D(\bar{\lambda} (\varpi_{n-1} + \Lambda_0))) \\
\cong D((\lambda_\ell - \bar{\lambda}) w_0(\varpi_\ell + \Lambda_0), \bar{\lambda} w_0(\varpi_\ell + \varpi_{n-1} + \Lambda_0))
\]

by Proposition 2.1, and the second one is also proved similarly. The proof for $s = n$ is complete.

The case $s = 1$ can be proved by a similar (and simpler) argument in which we replace the definition of $L_q[i]$ given in (4.4) by the following:

\[
L_q[i] = \begin{cases} 
L_q(\pi_{\lambda, a_j}), & j \leq n-2, \\
L_q(\pi_{\lambda_{m'}, a_{m'}, \lambda_{m'}}, \pi_{\lambda(m)}), & j = n-1, \\
L_q(\pi_{\lambda_{m'-\lambda_{m'}, a_{m'}, q_{\lambda_{m'}}}}), & j = n. 
\end{cases}
\]

Acknowledgements

The author would like to thank Steven V. Sam for informing him of the results in [22]. This work was supported by JSPS Grant-in-Aid for Young Scientists (B) No. 25800006, and by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

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