Causal State Feedback Representation for Linear Quadratic Optimal Control Problems of Singular Volterra Integral Equations

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Abstract. This paper is concerned with a linear quadratic optimal control for a class of singular Volterra integral equations. Under proper convexity conditions, optimal control uniquely exists, and it could be characterized via Fréchet derivative of the quadratic functional in a Hilbert space or via maximum principle type necessary conditions. However, these (equivalent) characterizations have a shortcoming that the current value of the optimal control depends on the future values of the optimal state. Practically, this is not feasible. The main purpose of this paper is to obtain a causal state feedback representation of the optimal control.

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1 Introduction.

Consider the following controlled singular linear Volterra integral equation:

\begin{equation}
X(t) = \varphi(t) + \int_{0}^{t} \frac{A(t,s)X(s) + B(t,s)u(s)}{(t-s)^{1-\beta}} ds, \quad \text{a.e.} \ t \in [0, T].
\end{equation}

In the above, \(T > 0\) is a fixed finite time horizon, \(\varphi(\cdot)\) is a given map, called the free term of the state equation, \(X(\cdot)\) is called the state trajectory taking values in the Euclidean space \(\mathbb{R}^n\), \(u(\cdot)\) is called the control taking values in the Euclidean space \(\mathbb{R}^m\), \(A(\cdot, \cdot)\) and \(B(\cdot, \cdot)\) are called the coefficients, taking values in \(\mathbb{R}^{n \times n}\) and \(\mathbb{R}^{m \times m}\), respectively, and \(\beta > 0\).

We denote \(\mathcal{X} = L^2(0, T; \mathbb{R}^n)\), \(\mathcal{U} = L^2(0, T; \mathbb{R}^m)\). Under some mild conditions, for any control \(u(\cdot) \in \mathcal{U}\), the state equation (1.1) admits a unique solution \(X(\cdot) \in \mathcal{X}\). To measure the performance of the control, we introduce the following quadratic cost functional

\begin{equation}
J(u(\cdot)) = \int_{0}^{T} \left( \langle Q(t)X(t), X(t) \rangle + 2\langle S(t)X(t), u(t) \rangle + \langle R(t)u(t), u(t) \rangle \\
+ 2\langle g(t), X(t) \rangle + 2\langle \rho(t), u(t) \rangle \right) dt + \langle GX(T), X(T) \rangle + 2\langle g, X(T) \rangle,
\end{equation}

where \(Q(\cdot) \in L^\infty(0, T; \mathbb{S}^n)\), \(S(\cdot) \in L^\infty(0, T; \mathbb{R}^{m \times n})\), \(R(\cdot) \in L^\infty(0, T; \mathbb{S}^m)\), \(q(\cdot) \in \mathcal{X}\), \(\rho(\cdot) \in \mathcal{U}\), \(G \in \mathbb{S}^n\), \(g \in \mathbb{R}^n\), with \(\mathbb{S}^k\) being the set of all \((k \times k)\) symmetric (real) matrices. Our optimal control problem can be stated as follows.

Problem (P). Find a control \(\bar{u}(\cdot) \in \mathcal{U}\) such that

\begin{equation}
J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} J(u(\cdot)).
\end{equation}

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Any $\bar{u}(\cdot)$ satisfying (1.3) is called an open-loop optimal control of Problem (P), the corresponding state $\bar{X}(\cdot)$ is called an open-loop optimal state and $(\bar{X}(\cdot), \bar{u}(\cdot))$ is called an open-loop optimal pair.

Memory exists in many application problems, heat transfer, population growth, disease spread, to mention a few. Volterra integral equations can be used to describe some dynamics involving memories. Study of optimal control problems for Volterra integral equations can be traced back to the works of Vinokurov in the later 1960s [45], followed by the works of Angell [4], Kamien-Muller [28], Medhin [35], Carlson [17], Burnap-Kazemi [14], and some recent work by de la Vega [20], Belbas [7, 8], and Bonnans–de la Vega–Dupuis [11]. All of the above-mentioned works are concerned with non-singular Volterra integral equations which exclude the case of (1.1) with $\beta \in (0, 1)$. On the other hand, in the past several decades, fractional (order) differential equations have attracted quite a few researchers’ attention due to some very interesting applications in physics, chemistry, engineering, population dynamics, finance and other sciences; See Oldham–Spanier [38] for some early examples of diffusion processes, Torvik–Bagley [44], Caputo [15], and Caputo–Mainardi [16] for modeling of the mechanical properties of materials, Benson [9] for the advection and the dispersion of solutes in natural porous or fractured media, Chern [18], Diethelm–Freed [23] for the modeling behavior of viscoelastic and viscoplastic materials under external influences, Scalas–Gorenflo–Mainardi [41] for the mathematical models in finance, Das–Gupta [19], Denirici–Unal–Özalp [21], Arafia–Rida–Khalil [5], Diethelm [22] for some population and epidemic models, Metzler et al. [36] for the relaxation in filled polymer networks, and Okyere et al. [37] for a SIR model with constant population. An extensive survey on fractional differential equations can be found in the book by Kilbas–Srivastava–Trujillo [31]. In the recent years, optimal control problems have been studied for fractional differential equations by a number of authors. We mention the works of Agrawal [1, 2], Agrawal–Defterli–Baleanu [3], Bourdin [12], Frederico–Torres [24], Hasan–Tangpong–Agrawal [27] and Kamocki [29, 30], Gomoyunov [25], Koenig [32].

It turns out that fractional differential equations (of the order no more than 1), in the sense of Riemann–Liouville or in the sense of Caputo, are equivalent to Volterra integral equations with the integrand being singular along $s = t$, and the free term $\varphi(\cdot)$ being possibly discontinuous (blowing up) at $t = 0$ (See [33] for some details). More precisely, in the linear case, the corresponding controlled state equation of form (1.1) could have the free term look like the following:

$$\varphi(t) = \frac{c}{t^{1-\beta}} \ (\text{or} \ c),$$

for some constant $c \in \mathbb{R}$. In [33], a class of controlled nonlinear singular Volterra integral equations was considered. Well-posedness of the state equation and some regularity of the state trajectory were established, and a Pontryagin type maximum principle for optimal controls was proved.

On the other hand, Pritchard–Yan [40] considered the quadratic optimal control problems for the following controlled linear Volterra integral equations in a Hilbert space $H$:

$$y(t) = f(t) + \int_0^t F(t, \tau)u(\tau)d\tau, \quad t \in [0, T].$$

It was assumed in [40] that $f(\cdot) \in C([0, T]; H)$ and $F : \bar{\Delta} \rightarrow \mathcal{L}(U; H)$ is strongly continuous in the sense that for each $u \in U$, $F(\cdot, u) \in C(\bar{\Delta}; H)$. Here, $U$ is another Hilbert space and $\bar{\Delta}$ is the closure of the following set

$$\Delta = \{(t, s) \in [0, T]^2 \mid 0 \leq s < t \leq T\}.$$

Thus, in particular, the following holds

$$\|F(t, \cdot)\|_{\mathcal{L}(U; H)} < \infty, \quad t \in [0, T].$$

This excludes our state equation (1.1) which has a singular kernel. We will see later that when the variation of constants formula is applied, our state process will have a similar representation as (1.5), but with both the free term $f(\cdot)$ and the operator $F(\cdot, \cdot)$ being not necessarily continuous.

Practically, if an optimal control exists, one expects that the optimal could have a state feedback representation which is non-anticipating. In the case of state equation being an ordinary differential equation
(or a partial differential equation, a stochastic differential equation), such kind of representation can be obtained, under some proper conditions, via a solution to a Riccati differential equation. Pandolfi [39] derived an optimal feedback control for a Volterra integro-differential equation by using the corresponding Riccati equation. That could be done because the state equation in [39] was of a special form which has the semigroup property and thus one could use semigroup representation to derive a theory of Riccati equation in a standard way. But the general Volterra equation does not have a semigroup evolutionary property. For the controlled linear Volterra integral equation of form (1.5), a so-called projection causality approach was introduced in [40]. The optimal control could be represented as a so-called linear causal feedback (see later for a precise definition) of the state trajectory with the feedback operator being determined by a solution to a Fredholm integral equation.

In this paper, we will carry out some careful analysis on the state equation, and pay special attention to certain continuity of the state trajectory since in the quadratic cost functional, the terminal value \( X(T) \) of the state trajectory is involved. Also, we make it clear that the condition \( \beta > \frac{1}{2} \) should be assumed in order the LQ problem is well-formulated. By a standard method for minimization of a quadratic functional in Hilbert space, we obtain a characterization of the (open-loop) optimal control in some abstract form. On the other hand, by variational method, in the spirit of maximum principle, we may obtain another characterization of the open-loop optimal control. We will show that these two characterizations are equivalent. However, from those characterizations, the open-loop optimal control is not non-anticipating in the sense that in determining the value \( \bar{u}(t) \) of the open-loop optimal control \( \bar{u}t(\cdot) \) at time \( t \), some future information \( \{\dot{X}(s) \mid s \in [t,T]\} \) of the optimal state trajectory has to be used. This is not practically realizable. In the classical LQ problem of differential equations, one could get a closed-loop representation of the open-loop optimal control via the solution to a Riccati equation. However, for general integral equations, such an approach is not working. In fact, the problem we considered in this paper is a nonlocal problem. Thus, we could not obtain a closed-loop optimal control whose current value only depends on the current state value, by using the standard method of ODE (or PDE, Volterra integro-differential equation in [39]) in terms of the Riccati equation. Inspired by [40], we will try to obtain a causal state feedback representation for the open-loop optimal control in the following sense: The current value \( \bar{u}(t) \) of the open-loop optimal control \( \bar{u}(\cdot) \) is written in terms of the current optimal state value \( \dot{X}(t) \), as well as a causal trajectory \( \dot{X}_t(\cdot) \) and an auxiliary trajectory \( \dot{X}^a(t) \), via a family of Fredholm integral equations which essentially plays a role of Riccati equation in the classical LQ problems. It is worthy of pointing out that \( \dot{X}_t(\cdot) \) and \( \dot{X}^a(\cdot) \) are not involved in calculating the cost functional, but they are used to represent the open-loop optimal control. Although the main idea comes from [40], our modified version of the method is more direct which reveals the essence of the problem more clearly. In the proof of [40], they introduce an abstract operator to establish the interrelations between the state trajectory and the causal trajectory. In this paper, we do not need to introduce the similar abstract operator, but give a more direct proof. Furthermore, the trajectory \( \dot{X}_t(T) \) in [40] may raise the doubt about the causality. In this paper, we can avoid this doubt by introducing the auxiliary trajectory \( \dot{X}^a(\cdot) \).

The rest of the paper is organized as follows. In section 2, we carry out some analysis for the state equation. Section 3 is devoted to the open-loop optimal control and its characterizations. Causal projection as well as abstract form of causal state feedback representation of the open-loop optimal control is presented in Section 4. We introduce a family of Fredholm integral equations in Section 5, which makes the representation obtained in Section 4 more practically accessible. In section 6, we briefly present a possible numerical scheme which is applicable to solve the Fredholm integral equation obtained in Section 5.

## 2 Preliminary Results.

In this section, we will present some preliminary results which will be useful later. Let us recall \( \Delta \) defined by (1.6). Note that the “diagonal line” \( \{(t,t) \mid t \in [0,T]\} \) is not contained in \( \Delta \). Thus if \( (t,s) \mapsto f(t,s) \) is continuous on \( \Delta \), \( f(\cdot,\cdot) \) is allowed to be unbounded as \( |t-s| \to 0 \). Throughout this paper, we denote \( t_1 \lor t_2 = \max\{t_1, t_2\} \) and \( t_1 \land t_2 = \min\{t_1, t_2\} \), for any \( t_1, t_2 \in \mathbb{R} \). The characteristic function of any set \( E \) is denoted by \( 1_E(\cdot) \). For any set \( E \subseteq \mathbb{R} \) and a function \( \varphi : E \to \mathbb{R} \), we extend it to be zero in \( \mathbb{R} \setminus E \). We call a strictly increasing continuous function \( \omega(\cdot) : [0,\infty) \to [0,\infty) \) a modulus of continuity if \( \omega(0) = 0 \). Also, \( K \) will be a generic constant which could be different from line to line.
Let us recall the Young’s inequality for convolution (Theorem 3.9.4 in [10]).

**Lemma 2.1.** Let $p, q, r \in [1, +\infty]$ satisfy $\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}$. Then for any $f(\cdot) \in L^p(\mathbb{R}^n)$, $g(\cdot) \in L^r(\mathbb{R}^n)$,

$$
(2.1) \quad \|f(\cdot) * g(\cdot)\|_{L^q(\mathbb{R}^n)} \leq \|f(\cdot)\|_{L^p(\mathbb{R}^n)} \|g(\cdot)\|_{L^r(\mathbb{R}^n)}.
$$

From the above lemma, we have the following corollary which is a refinement of that found in [33].

**Corollary 2.2.** Let $\theta : \Delta \to \mathbb{R}^n$ and $\theta_0 : [0, T] \to \mathbb{R}$ be measurable such that

$$
(2.2) \quad |\theta(t, \tau)| \leq \theta_0(\tau), \quad \text{a.e. } (t, \tau) \in \Delta.
$$

For any $s \in [0, T)$, define

$$
\eta(t, s) = \int_s^t \frac{\theta(t, \tau)}{(t - \tau)^{1 - \beta}} d\tau, \quad t \in (s, T].
$$

(i) Let $\beta \in (0, 1)$, $1 \leq r < \frac{1}{1 - \beta}$, $p = \frac{1}{1 - \beta} + \frac{1}{2}$, $q \in [1, \infty]$, and $\theta_0(\cdot) \in L^q(s, T)$. Then

$$
(2.3) \quad \|\eta(\cdot, s)\|_{L^p(s, T; \mathbb{R}^n)} \leq \left( \frac{(T - s)^{1 - r(1 - \beta)}}{1 - r(1 - \beta)} \right)^\frac{1}{r} \|\theta_0(\cdot)\|_{L^q(s, T)}.
$$

In particular, with $q = 2$,

$$
(2.4) \quad \|\eta(\cdot, s)\|_{L^p(s, T; \mathbb{R}^n)} \leq \frac{(T - s)^{\beta - \frac{1}{2}}}{\beta} \|\theta_0(\cdot)\|_{L^2(s, T)}.
$$

Further, with both $p = q = 2$,

$$
(2.5) \quad \|\eta(\cdot, s)\|_{L^2(s, T; \mathbb{R}^n)} \leq \frac{(T - s)^{\beta - \frac{1}{2}}}{\sqrt{2\beta - 1}} \|\theta_0(\cdot)\|_{L^2(s, T)}.
$$

(ii) Let there exist a $\delta_0 \in (0, \frac{T - s}{2})$, and a modulus of continuity $\omega$ such that $\forall t \in [T - 2\delta_0, T]$,

$$
(2.7) \quad \|\theta(t, \tau) - \theta(T, \tau)\| \leq \omega(T - t) \theta_1(\tau), \quad \text{a.e. } \tau \in [s, t),
$$

for some $\theta_0(\cdot), \theta_1(\cdot) \in L^q(s, T), \ q \in (1, \infty), \ 1 > \beta > \frac{1}{2}$. Then $\eta(\cdot, s)$ is continuous at $T$.

**Proof.** (i) By Lemma 2.1 with $f(\tau) = \theta_0(\tau)\mathbf{1}_{[s, T]}(\tau)$ and $g(\tau) = \frac{1}{t - s}\mathbf{1}_{(0, T - s)}(\tau)$, we can obtain our conclusion. The rest is clear.

(ii) Pick any $\delta \in (0, \delta_0)$. For any $t \in (T - \delta, T)$, we look at the following, assuming first that $q \in (1, \infty)$ and setting $\kappa = (1 - \beta)\frac{1}{q - 1} < 1$ (since $\beta > \frac{1}{2}$):

$$
\begin{align*}
|\eta(T, s) - \eta(t, s)| &= \left| \int_s^T \frac{\theta(T, \tau)}{(T - \tau)^{1 - \beta}} d\tau - \int_s^t \frac{\theta(t, \tau)}{(t - \tau)^{1 - \beta}} d\tau \right| \\
&\leq \int_s^t \left| \frac{\theta(t, \tau)}{(t - \tau)^{1 - \beta}} - \frac{1}{(t - \tau)^{1 - \beta}} \right| d\tau + \int_t^T \frac{|\theta(t, \tau) - \theta(T, \tau)|}{(T - \tau)^{1 - \beta}} d\tau \\
&\quad + \int_{t - \delta}^t \frac{\theta_0(\tau)}{(t - \tau)^{1 - \beta}} d\tau + \int_{T - \delta}^T \frac{\theta_0(\tau)}{(T - \tau)^{1 - \beta}} d\tau \\
&\leq (T - t)^{1 - \beta} \int_s^t \frac{\theta_0(\tau)}{(T - \tau)^{1 - \beta}} d\tau + \int_{t - \delta}^t \frac{\theta_0(\tau)}{(t - \tau)^{1 - \beta}} d\tau + \omega(T - t) \int_{t - \delta}^t \frac{\theta_1(\tau)}{(T - \tau)^{1 - \beta}} d\tau \\
&\quad + \|\theta_0(\cdot)\|_{L^q(s, T)} \left( \int_{t - \delta}^t \frac{d\tau}{(t - \tau)^\kappa} \right)^{\frac{1 - \kappa}{1 - \kappa}} + \|\theta_0(\cdot)\|_{L^q(s, T)} \left( \int_{T - \delta}^T \frac{d\tau}{(T - \tau)^\kappa} \right)^{\frac{1 - \kappa}{1 - \kappa}} \\
&\leq (T - t)^{1 - \beta} \frac{\theta_0(\cdot)}{\delta^{2(1 - \beta)}} + \omega(T - t) \left( \frac{(T - s)^{1 - \kappa}}{1 - \kappa} \right)^{\frac{1}{1 - \kappa}} \|\theta_1(\cdot)\|_{L^q(s, T)}
\end{align*}
$$
+\|\theta_0(\cdot)\|_{L^q(s,T)} \left[ \left( \frac{\delta^{1-\kappa}}{1-\kappa} \right)^{\frac{1}{q}} + \left( \frac{(T-t+\delta)^{1-\kappa}}{1-\kappa} \right)^{\frac{1}{q}} \right].

Hence, for any \( \varepsilon > 0 \), we first take \( \delta > 0 \) sufficiently small so that

\[
\|\theta_0(\cdot)\|_{L^q(s,T)} \left[ \left( \frac{\delta^{1-\kappa}}{1-\kappa} \right)^{\frac{1}{q}} + \left( \frac{(2\delta)^{1-\kappa}}{1-\kappa} \right)^{\frac{1}{q}} \right] < \frac{\varepsilon}{\bar{\omega}}.
\]

Since the modulus of continuity \( \omega(\cdot) \) is continuous and \( \omega(0) = 0 \), we can take \( \delta \in (0,\delta) \) even smaller so that

\[
\frac{\delta^{1-\beta}}{\delta^{2(1-\beta)}} \|\theta_0(\cdot)\|_{L^q(s,T)} + \omega(\delta) \left( \frac{(T-s)^{1-\kappa}}{1-\kappa} \right) \frac{\delta^\beta}{\beta} \|\theta_1(\cdot)\|_{L^\infty(s,T)} < \frac{\varepsilon}{\bar{\omega}}.
\]

Combining the above, we see that \( \eta(\cdot, s) \) is continuous at \( T \).

In the case \( q = \infty \), we have

\[
|\eta(T, s) - \eta(t, s)| = \left| \int_s^T \frac{\theta(T, \tau)}{(T-\tau)^{1-\beta}} d\tau - \int_s^t \frac{\theta(t, \tau)}{(t-\tau)^{1-\beta}} d\tau \right| \\
\leq \frac{(T-t)^{1-\beta}}{\delta^{2(1-\beta)}} \|\theta_0(\cdot)\|_{L^\infty(s,T)} + \omega(\delta) \left( \frac{(T-s)^{1-\kappa}}{1-\kappa} \right) \frac{\delta^\beta}{\beta} \|\theta_1(\cdot)\|_{L^\infty(s,T)} \\
+ \|\theta_0(\cdot)\|_{L^\infty(s,T)} \left[ \left( \frac{T-s}{T-t+\delta} \right)^{\frac{1}{\beta}} + \frac{(T-t+\delta)^{\frac{1}{\beta}}}{\beta} \right].
\]

Then, similar to the above, we obtain the continuity of \( \eta(\cdot, s) \) at \( T \).

We now look at the following linear Volterra integral equation

\[
X(t) = \xi(t) + \int_0^t A(t, s)X(s) \frac{ds}{(t-s)^{1-\beta}}, \quad \text{a.e. } t \in [0, T].
\]

Note that (1.1) is a case of the above with

\[
\xi(t) = \varphi(t) + \int_0^t B(t, s)u(s) \frac{ds}{(t-s)^{1-\beta}}, \quad \text{a.e. } t \in [0, T].
\]

Before going further, we introduce the following assumption for the coefficients of (1.1).

(A1) The coefficients \( A(\cdot, \cdot) \in L^\infty(\Delta; \mathbb{R}^{n \times n}) \) and \( B(\cdot, \cdot) \in L^\infty(\Delta; \mathbb{R}^{n \times m}) \). The free term \( \varphi(\cdot) \in \mathcal{X} \).

For convenience, throughout the paper, we assume that

\[
|A(t, s)| \leq \|A\|_\infty, \quad |B(t, s)| \leq \|B\|_\infty, \quad \forall (t, s) \in \Delta.
\]

Then, under (A1), for any \( u(\cdot) \in \mathcal{U} \), by Corollary 2.2, (i), we have that if \( \beta \in (0, 1) \),

\[
\left( \int_0^T \left( \int_0^t \frac{B(t, s)u(s)}{(t-s)^{1-\beta}} ds \right)^2 dt \right)^{\frac{1}{2}} \leq \|B\|_\infty \left[ \int_0^T \left( \int_0^t \frac{|u(s)|}{(t-s)^{1-\beta}} ds \right)^2 dt \right]^{\frac{1}{2}} \leq \|B\|_\infty \left( \frac{T^\beta}{\beta} \right) \|u(\cdot)\|_\mathcal{U}.
\]

Consequently, for the free term \( \xi(\cdot) \) defined by (2.9), one has

\[
\|\xi(\cdot)\|_\mathcal{X} \leq \|\varphi(\cdot)\|_\mathcal{X} + \frac{T^\beta}{\beta} \|B\|_\infty \|u(\cdot)\|_\mathcal{U}.
\]

The following gives the well-posedness of (2.8) as well as its variation of constants formula.

**Theorem 2.3.** Let (A1) hold. Then for any \( \xi(\cdot) \in \mathcal{X} \), (2.8) admits a unique solution \( X(\cdot) \in \mathcal{X} \). Moreover, there exists a measurable function \( \Phi(\cdot, \cdot) : \Delta \to \mathbb{R}^{n \times n} \) satisfying that for any \( s \in [0, T] \),

\[
\Phi(t, s) = \frac{A(t, s)}{(t-s)^{1-\beta}} + \int_s^t \frac{A(t, \tau)\Phi(\tau, s)}{(t-\tau)^{1-\beta}} d\tau, \quad t \in (s, T],
\]

\[
\Phi(t, s) = \frac{A(t, s)}{(t-s)^{1-\beta}} + \int_s^t \frac{A(t, \tau)\Phi(\tau, s)}{(t-\tau)^{1-\beta}} d\tau, \quad t \in (s, T],
\]
such that for some constant $K > 0$,
\begin{equation}
|\Phi(t, s)| \leq \frac{K}{(t-s)^{1-\beta}}, \quad (t, s) \in \Delta,
\end{equation}
and the solution $X(\cdot)$ to (2.8) can be represented by the following variation of constants formula:
\begin{equation}
X(t) = \xi(t) + \int_0^t \Phi(t, s)\xi(s)ds, \quad \text{a.e. } t \in [0, T],
\end{equation}
with the following estimate:
\begin{equation}
\|X(\cdot)\|_{X} \leq K\|\xi(\cdot)\|_{X}.
\end{equation}
Moreover, the function $\Phi(\cdot, \cdot)$ also satisfies that for any $s \in [0, T)$,
\begin{equation}
\Phi(t, s) = \frac{A(t, s)}{(t-s)^{1-\beta}} + \int_s^t \frac{A(t, \tau)A(\tau, s)}{(\tau-s)^{1-\beta}}d\tau, \quad t \in (s, T].
\end{equation}

Proof. First of all, by a standard contraction mapping argument, making use of Corollary 2.2, (i), we see that for any $\xi(\cdot) \in \mathcal{X}$, (2.8) admits a unique solution $X(\cdot) \in \mathcal{X}$.

Next, by (A1), if $\Phi(\cdot, \cdot)$ is a solution of (2.12), then
\begin{equation}
|\Phi(t, s)| \leq \frac{\|A\|_{\infty}}{(t-s)^{1-\beta}} + \int_s^t \frac{\|A\|_{\infty}|\Phi(\tau, s)|}{(\tau-s)^{1-\beta}}d\tau, \quad (t, s) \in \Delta.
\end{equation}
Thus, by Gronwall’s inequality, we have
\begin{equation}
|\Phi(t, s)| \leq \frac{\|A\|_{\infty}}{(t-s)^{1-\beta}} + K\int_s^t \frac{\|A\|_{\infty}^2 B(\beta, \beta)}{(\tau-s)^{1-2\beta}}d\tau = \frac{\|A\|_{\infty}}{(t-s)^{1-\beta}} + K\|A\|_{\infty}^2 \frac{B(\beta, \beta)(t-s)^{\beta}}{(t-s)^{1-\beta}} \leq \frac{K}{(t-s)^{1-\beta}}, \quad (t, s) \in \Delta.
\end{equation}
This proves (2.13). In the above, $B(\cdot, \cdot)$ is the Beta function, and recall that $K$ stands for a generic constant which could be different from line to line.

Now, we inductively define the following sequence of measurable functions:
\begin{equation}
F_1(t, s) = \frac{A(t, s)}{(t-s)^{1-\beta}}, \quad F_{k+1}(t, s) = \int_s^t F_1(t, \tau)F_k(\tau, s)d\tau, \quad k = 1, 2, 3, \ldots, \quad (t, s) \in \Delta.
\end{equation}
Then
\begin{equation}
|F_1(t, s)| \leq \frac{\|A\|_{\infty}}{(t-s)^{1-\beta}}, \quad (t, s) \in \Delta,
\end{equation}
\begin{equation}
|F_2(t, s)| \leq \int_s^t |F_1(t, \tau)F_1(\tau, s)|d\tau \leq \|A\|_{\infty}^2 \int_s^t \frac{d\tau}{(t-\tau)^{1-\beta}(\tau-s)^{1-\beta}} \leq \frac{\|A\|_{\infty}^2 B(\beta, \beta)}{(t-s)^{1-2\beta}}, \quad (t, s) \in \Delta.
\end{equation}
By induction, we can show that
\begin{equation}
|F_k(t, s)| \leq \int_s^t |F_1(t, \tau)F_{k-1}(\tau, s)|d\tau \leq \frac{\|A\|_{\infty}^k}{(t-s)^{1-k\beta}} \prod_{j=1}^{k-1} B(\beta, j\beta), \quad k \geq 1, \quad (t, s) \in \Delta.
\end{equation}
According to [43] (p.102), Gamma function $\Gamma(\cdot)$ admits the following asymptotic expansion:
\begin{equation}
\Gamma(z + 1) = \sqrt{2\pi z}z^ze^{-z}(1 + R(z)), \quad z \gg 1; \quad R(z) = \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \cdots.
\end{equation}
Thus, for \( j \) large enough, we have

\[
\mathcal{B}(\beta, j \beta) = \frac{\Gamma(\beta)\Gamma(j \beta)}{\Gamma(j \beta + \beta)} = \frac{\Gamma(\beta)\sqrt{2\pi(j \beta - 1)(j \beta - 1)j^{\beta-1}e^{-(j \beta - 1)}}}{\sqrt{2\pi(j \beta + \beta - 1)(j \beta + \beta - 1)j^{\beta+\beta-1}e^{-(j \beta + \beta - 1)}}(1 + R(j \beta - 1))}
\]

\[
= \Gamma(\beta)\left(\frac{j \beta - 1}{j \beta + \beta - 1}\right)^{j \beta - 1}(1 + \tilde{R}(j))
\]

\[
\leq \frac{\Gamma(\beta)j^\beta}{\beta^j} \left(1 + \tilde{R}(j)\right) \frac{1}{j^\beta} \leq \frac{\Gamma(\beta)j^{2\beta+1}}{\beta^j},
\]

for some \( \tilde{R}(j) \to 0 \) as \( j \to \infty \). Consequently, there exists a \( k_0 \) such that for \( k > k_0 \),

\[
\left| F_k(t, s) \right| \leq \frac{\|A\|_\infty k^{-1} \left( \prod_{j=1}^{k_0-1} B(\beta, j \beta) \right) \Gamma(\beta)j^{2\beta+1}}{\beta^j} \frac{1}{\left( (k_0 - 1)! \right)^\beta} \leq \frac{K_0K^k}{(t-s)^{1-k_0} \left( (k_0 - 1)! \right)^\beta},
\]

for some constants \( K_0, K > 0 \). Then, for any \( \delta > 0 \), series \( \sum_{k=1}^{\infty} F_k(t, s) \) is uniformly and absolutely convergent for \( (t, s) \in \Delta_\delta \) with

\[
\Delta_\delta = \{(t, s) \in \Delta \mid t - s \geq \delta\}.
\]

Now we define

\[
\Phi(t, s) = \sum_{k=1}^{\infty} F_k(t, s), \quad (t, s) \in \Delta,
\]

which is measurable (since each \( F_k(\cdot, \cdot) \) is measurable) and bounded on each \( \Delta_\delta, \delta > 0 \). We can easily check that the above defined \( \Phi(\cdot, \cdot) \) is the unique solution of (2.12), and therefore, estimate (2.13) holds. Further, for any \( \xi(\cdot) \in L^2(0, T; \mathbb{R}^n) \), similar to (2.10), we see that

\[
X(t) = \xi(t) + \int_0^t \Phi(t, s)\xi(s)ds, \quad \text{a.e. } t \in [0, T],
\]

is well-defined as an element in \( L^2(0, T; \mathbb{R}^n) \). In addition, for such defined \( X(\cdot) \), one has

\[
\int_0^t A(t, s)X(s)ds = \int_0^t \frac{A(t, s)}{(t-s)^{1-\beta}} \left[ \xi(s) + \int_s^t \Phi(s, \tau)\xi(\tau)d\tau \right]ds
\]

\[
= \int_0^t \frac{A(t, s)\xi(s)}{(t-s)^{1-\beta}}ds + \int_0^t \int_s^t \Phi(t, \tau)\Phi(\tau, s)\xi(\tau)d\tau dsd\tau
\]

\[
= \int_0^t \left( \frac{A(t, s)}{(t-s)^{1-\beta}} + \int_s^t \frac{A(t, \tau)\Phi(\tau, s)}{(t-\tau)^{1-\beta}}ds \right)\xi(s)ds = \int_0^t \Phi(t, s)\xi(s)ds = X(t) - \xi(t), \quad \text{a.e. } t \in [0, T].
\]

This proves (2.14). Making use of (2.13), and similar to (2.10), we obtain

\[
\|X(\cdot)\|_x \leq \|\xi(\cdot)\|_x + \left( \int_0^T \left( \int_0^t \Phi(t, s)\xi(s)ds \right)^2 dt \right)^{\frac{1}{2}}
\]

\[
\leq \|\xi(\cdot)\|_x + \left( \int_0^T \left( \int_0^t \frac{\left| \xi(s) \right|}{(t-s)^{1-\beta}}ds \right)^2 dt \right)^{\frac{1}{2}} \leq \|\xi(\cdot)\|_x + K\|\xi(\cdot)\|_x.
\]

Finally, we prove (2.16). To this end, we make the following observation:

\[
F_3(t, s) = \int_s^t F_1(t, \tau)F_2(\tau, s)d\tau = \int_s^t F_1(t, \tau)\int_s^\tau F_1(\tau, r)F_1(r, s)d\tau dr
\]

\[
= \int_s^t \int_t^\tau F_1(t, \tau)F_1(\tau, r)F_1(r, s)d\tau dr = \int_s^t F_2(t, r)F_1(r, s)dr, \quad (t, s) \in \Delta.
\]
Hence, by induction, we see that (comparing with \((2.17)\))

\[
F_{k+1}(t, s) = \int_s^t F_k(t, r) F_1(r, s) \, dr, \quad (t, s) \in \Delta, \ k \geq 1.
\]

Then we see that \((2.16)\) holds. This completes the proof. \(\square\)

According to the above theorem, for state equation \((1.1)\), we have the following representation of the state process \(X(\cdot)\) in terms of the control \(u(\cdot)\) and the free term \(\varphi(\cdot)\):

\begin{equation}
X(t) = \varphi(t) + \int_0^t \frac{B(t, s)u(s)}{(t-s)^{1-\beta}} ds + \int_0^t \Phi(t, s) \left[ \varphi(s) + \int_s^t \frac{B(s, \tau)u(\tau)}{(s-\tau)^{1-\beta}} d\tau \right] ds
\end{equation}

\begin{align}
&= \varphi(t) + \int_0^t \Phi(t, s)\varphi(s) ds + \int_0^t \frac{B(t, s)u(s)}{(t-s)^{1-\beta}} ds + \int_0^t \int_s^t \frac{\Phi(t, s)B(s, \tau)u(\tau)}{(s-\tau)^{1-\beta}} d\tau ds d\tau \\
&= \varphi(t) + \int_0^t \Phi(t, s)\varphi(s) ds + \int_0^t \left( \frac{B(t, s)}{(t-s)^{1-\beta}} + \int_s^t \frac{\Phi(t, \tau)B(\tau, s)}{(\tau-s)^{1-\beta}} d\tau \right) u(s) ds
\end{align}

\begin{equation}
\equiv \psi(t) + \int_0^t \Psi(t, s)u(s) ds,
\end{equation}

where

\begin{align}
\psi(t) &= \varphi(t) + \int_0^t \Phi(t, s)\varphi(s) ds, \quad \text{a.e. } t \in [0, T], \\
\Psi(t, s) &= \frac{B(t, s)}{(t-s)^{1-\beta}} + \int_s^t \frac{\Phi(t, \tau)B(\tau, s)}{(\tau-s)^{1-\beta}} d\tau, \quad (t, s) \in \Delta.
\end{align}

Clearly, \(\Psi : \Delta \rightarrow \mathbb{R}^{n \times m}\), and

\begin{equation}
|\Psi(t, s)| \leq \frac{\|B\|_\infty}{(t-s)^{1-\beta}} + \|B\|_\infty \int_s^t \frac{|\Phi(t, \tau)|}{(\tau-s)^{1-\beta}} d\tau \\
\leq \frac{K}{(t-s)^{1-\beta}} + K \int_s^t \frac{1}{(\tau-s)^{1-\beta}} d\tau = \frac{K}{(t-s)^{1-\beta}}, \quad (t, s) \in \Delta.
\end{equation}

Moreover, noting \(\xi(\cdot)\) defined by \((2.9)\) and the estimate \((2.11)\),

\begin{equation}
\|X(t)\|_X \leq K\|\xi(t)\|_X \leq K\left( \|\varphi(\cdot)\|_X + \|u(\cdot)\|_W \right).
\end{equation}

We call \((2.19)\) the variation of constants formula for the state \(X(\cdot)\). From the above, we see that under \((A1)\), for any control \(u(\cdot) \in \mathcal{U}\), the state equation \((1.1)\) is well-posed in \(\mathcal{X}\). Thus, the running cost in \((1.2)\) is well-defined. However, the terminal cost is still not necessarily defined. We need the state process \(X(\cdot)\) to be continuous at \(t = T\). To achieve this, we need a little more assumption which we now introduce.

\((A2)\) Let \((A1)\) hold, and in addition, there exists a modulus of continuity \(\omega(\cdot)\) and some \(\delta_0 \in (0, T]\),

\[|A(T, s) - A(t, s)| + |B(T, s) - B(t, s)| + |\varphi(T) - \varphi(t)| \leq \omega(T-t), \quad t \in [T - \delta_0, T], \quad (t, s) \in \Delta.\]

\((A3)\) \(\beta > \frac{1}{\alpha}\).

We have the following result.

**Theorem 2.4.** (i) Let \((A2)\)–\((A3)\) hold. Then for any \(s \in [0, T)\), \(t \mapsto \Phi(t, s)\) is continuous at \(t = T\).

(ii) Let \((A2)\)–\((A3)\) hold. Then for any control \(u(\cdot) \in \mathcal{U}\), the corresponding state process \(X(\cdot)\) is continuous at \(t = T\).
Proof. (i) By Corollary 2.2 (ii), we can get (i).

(ii) Now, we let (A2)–(A3) hold. Then for any \( u(\cdot) \in \mathcal{U} \),

\[
|X(t)| \leq |\psi(t)| + \int_0^t |\Psi(t,s)\phi(s)|ds \leq |\psi(t)| + \int_0^t |\Phi(t,s)\phi(s)|ds + \left( \int_0^t |\Psi(t,s)|^2 ds \right)^{\frac{1}{2}} \|u(\cdot)\|_{\mathcal{U}}
\]

\[
\leq |\psi(t)| + \left( \int_0^t |\Phi(t,s)|^2 ds \right)^{\frac{1}{2}} \|\phi(\cdot)\|_{X} + K \left( \int_0^t ds \right)^{\frac{1}{2}} \|\phi(\cdot)\|_{X} + K t^{1-\frac{1}{2}} \frac{\|u(\cdot)\|_{\mathcal{U}}}{\sqrt{2\beta - 1}}
\]

\[
\leq |\psi(t)| + K t^{\frac{\beta - 1}{2}} \frac{\|\phi(\cdot)\|_{X}}{\beta} + K t^{\frac{\beta - 1}{2}} \frac{\|u(\cdot)\|_{\mathcal{U}}}{\sqrt{2\beta - 1}} \|\phi(\cdot)\|_{X} \leq |\psi(t)| + K \left( \|\phi(\cdot)\|_{X} + \|u(\cdot)\|_{\mathcal{U}} \right).
\]

Thus, \( X(t) \) is defined at all points where \( \phi(t) \) is defined. To obtain the continuity of \( X(\cdot) \) at \( t = T \), since \( \phi(\cdot) \) is continuous at \( T \), it suffices to obtain the continuity of the following expression at \( t = T \):

\[
\int_0^t A(t, \tau)X(\tau) + B(t, \tau)u(\tau) \frac{d\tau}{(t - \tau)^{1-\beta}}.
\]

Since

\[
|A(t, \tau)X(\tau) + B(t, \tau)u(\tau)| \leq K \left( |X(\tau)| + |u(\tau)| \right), \quad (t, \tau) \in \Delta,
\]

\[
|A(T, \tau)X(\tau) + B(T, \tau)u(\tau) - A(t, \tau)X(\tau) - B(t, \tau)u(\tau)| \leq \omega(T - t) \left( |X(\tau)| + |u(\tau)| \right),
\]

\[
t \in [T - \delta_0, T], \quad (t, \tau) \in \Delta,
\]

with \( |X(\cdot)| + |u(\cdot)| \in L^2(0, T) \), by Corollary 2.2 (ii), we obtain the continuity.

\[\square\]

Note that if \( \beta \leq \frac{1}{2} \), then in general, we do not expect to have a continuity of the state process at \( t = T \) for some control \( u(\cdot) \in \mathcal{U} \). The following example illustrates this.

**Example 2.5.** Let \( T = 1 \), \( A(\cdot, \cdot) = 0 \), \( B(\cdot, \cdot) = 1 \), and \( \phi(\cdot) = 0 \). Then we have

\[
X(t) = \int_0^t \frac{u(s)}{(t - s)^{1-\beta}} ds, \quad t \in [0, 1].
\]

Let

\[
u(s) = \frac{1_{[\frac{1}{2}, 1)}(s)}{(1-s)^{\frac{1}{2}} \log(1-s)}, \quad s \in [0, 1].\]

Then,

\[
\int_0^1 |u(s)|^2 ds = \int_0^1 \frac{ds}{(1-s)^{\frac{1}{2}} \log(1-s)} = \frac{1}{\log 2}
\]

Thus, \( u(\cdot) \in \mathcal{U} \). However,

\[
X(1) = \int_0^1 \frac{u(s)}{(1-s)^{1-\beta}} ds = \int_0^1 \frac{ds}{(1-s)^{\frac{1}{2} - \beta} \log(1-s)} = \infty, \quad \forall \beta \leq \frac{1}{2}.
\]

Having the above result, we see that under (A2)–(A3), for any \( u(\cdot) \in \mathcal{U} \), the cost functional is well-defined, and therefore, Problem (P) is well-formulated.

### 3 Open-loop Optimal Control.

In this section, we will present the unique existence of open-loop optimal control and its characterizations. Let us first introduce the following operators:

\[
(\Theta u)(t) = \int_0^t \Psi(t,s)u(s)ds, \quad u \in \mathcal{U}, \quad t \in [0, T],
\]

\[
\Theta_T u = \int_0^T \Psi(T,s)u(s)ds = (\Theta u)(T), \quad u \in \mathcal{U}.
\]
Then
(3.2) \[ X(t) = \psi(t) + (\Theta u)(t), \quad t \in [0, T]; \quad X(T) = \psi(T) + \Theta_T u. \]

From (2.21), we see that (only need \( \beta \in (0, 1) \)) for any \( u(\cdot) \in \mathcal{U} \),

(3.3) \[ \|\Theta u\|_X = \left( \int_0^T \left| \int_0^t \Psi(t, s) u(s) ds \right|^2 dt \right)^{\frac{1}{2}} \leq K \left[ \int_0^T \left( \int_0^t \frac{|u(s)|}{(t-s)^{1-\beta}} ds \right)^2 dt \right]^{\frac{1}{2}} \leq K\|u(\cdot)\|_\mathcal{U}, \]

and (noting \( \beta \in (\frac{1}{2}, 1) \))

(3.4) \[ |\Theta_T u| = \left| \int_0^T \Psi(T, s) u(s) ds \right| \leq K \int_0^T \frac{|u(s)|}{(T-s)^{1-\beta}} ds \leq K\|u(\cdot)\|_\mathcal{U}. \]

Thus, \( \Theta \in \mathcal{L}(\mathcal{U}; \mathcal{X}) \) and \( \Theta_T \in \mathcal{L}(\mathcal{U}; \mathcal{R}^n) \). Consequently, their adjoint operators \( \Theta^* \in \mathcal{L}(\mathcal{X}; \mathcal{U}) \) and \( \Theta^*_T \in \mathcal{L}(\mathcal{R}^n; \mathcal{U}) \) are well-defined. Let us identify them as follows. For any \( X(\cdot) \in \mathcal{X} \),

\( \langle X(\cdot), (\Theta u)(\cdot) \rangle_X = \int_0^T \langle X(t), (\Theta u)(t) \rangle dt = \int_0^T \langle X(t), \int_0^t \Psi(t, s) u(s) ds \rangle dt \]

\[ = \int_0^T \int_0^t \langle X(t), \Psi(t, s) u(s) \rangle ds dt = \int_0^T \int_s^T \langle X(t), \Psi(t, s) u(s) \rangle ds dt \]

\[ = \int_s^T \int_0^T \Psi(t, s)^T X(t) dt, u(s) ds = \left( \int_s^T \Psi(t, s)^T X(t) dt, u(s) \right)_\mathcal{W} = \langle (\Theta^* X)(\cdot), u(\cdot) \rangle_\mathcal{W}. \]

This gives
(3.5) \[ (\Theta^* X)(s) = \int_s^T \Psi(t, s)^T X(t) dt, \quad X(\cdot) \in \mathcal{X}. \]

From Corollary 2.2, (i), we see that (only need \( \beta \in (0, 1) \))

\[ \left( \int_0^T \left| \int_s^T \Psi(t, s)^T X(t) dt \right|^2 ds \right)^{\frac{1}{2}} \leq K \left[ \int_0^T \left( \int_s^T \frac{|X(t)|}{(T-s)^{1-\beta}} dt \right)^2 ds \right]^{\frac{1}{2}} \leq K\|X(\cdot)\|_X. \]

Likewise (noting \( \beta \in (\frac{1}{2}, 1) \)),

(3.6) \[ (\Theta^* T x)(s) = \Psi(T, s)^T x, \quad \forall x \in \mathcal{R}^n, \]

with
\[ \left( \int_0^T \left| (\Theta^* T x)(s) \right|^2 ds \right)^{\frac{1}{2}} \leq K|x| \left( \int_0^T \frac{ds}{(T-s)^{2(1-\beta)}} \right)^{\frac{1}{2}} = K \left( \frac{2\beta-1}{2\beta-1} \right)^{\frac{1}{2}} |x|. \]

**Remark 3.1.** By (2.21), noting \( \beta > \frac{1}{2} \), we see that for \( X(\cdot) \in \mathcal{X} \),

\[ \| (\Theta^* X)(s) \| \leq \int_s^T \| \Psi(t, s) \| X(t) |dt \leq K \int_s^T \frac{|X(t)|}{(T-s)^{1-\beta}} dt \leq K \frac{(T-s)^{-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}} \| X(\cdot) \|_X, \quad s \in [0, T], \]

which is bounded. But,

\[ \| (\Theta^*_T x)(s) \| \leq \frac{K|x|}{(T-s)^{1-\beta}}, \quad s \in [0, T), \]

which may be unbounded. This is different from the case that (1.7) holds as in [40]. Consequently, we need to modify the technique used there so that it works for us.

Now, we would like to represent the cost functional. We will use \( Q \) to denote the bounded operator from \( \mathcal{X} \) to itself induced by \( Q(\cdot) \), and so on. Then

\( J(u(\cdot)) = \langle Q X, X \rangle + 2\langle SX, u \rangle + 2\langle Ru, u \rangle + 2\langle q, X \rangle + 2\langle \rho, u \rangle + \langle G X(T), X(T) \rangle + 2\langle g, X(T) \rangle \)

\[ = \langle Q(\psi + \Theta u), \psi + \Theta u \rangle + 2\langle S(\psi + \Theta u), u \rangle + \langle Ru, u \rangle + 2\langle q, \psi + \Theta u \rangle + 2\langle \rho, u \rangle \]

\[ + \langle G(\psi(T) + \Theta_T u), \psi(T) + \Theta_T u \rangle + 2\langle g, \psi(T) + \Theta_T u \rangle = \langle \Lambda u, u \rangle + 2\langle \lambda_1, u \rangle + \lambda_0, \]

(3.7)
where
\[
\begin{align*}
\Lambda &= \Theta^*Q\Theta + S\Theta + \Theta^*S^* + R + \Theta^*_T G\Theta_T, \\
\lambda_1 &= \Theta^*Q\psi + S\psi + \Theta^*q + \rho + \Theta^*_T G\psi(T) + \Theta^*_T g, \\
\lambda_0 &= (Q\psi, \psi) + 2(q, \psi) + (G\psi(T), \psi(T)) + 2(g, \psi(T)).
\end{align*}
\]

The above shows that our Problem (P) can be regarded as an optimization of the functional \(J(u(\cdot))\) on the Hilbert space \(\mathcal{H}\). In order the functional \(u(\cdot) \mapsto J(u(\cdot))\) to be bounded from below, it is necessary that \(\Lambda \geq 0\). In what follows, we want \(J(u(\cdot))\) to admit a unique minimum. To guarantee this, we may assume the following stronger condition:

\[
(3.8) \quad \Lambda \geq \delta,
\]

for some \(\delta > 0\). Since it is not the main theme of this paper to discuss the conditions under which (3.8) holds, we are satisfied to assume proper conditions to guarantee (3.8). For simplicity, we introduce the following standard assumption:

(A4) Let

\[
\begin{align*}
Q(\cdot) &\in L^\infty(0, T; S^n), \quad S(\cdot) \in L^\infty(0, T; \mathbb{R}^{m \times n}), \quad R(\cdot) \in L^\infty(0, T; S^m), \\
q(\cdot) &\in L^2(0, T; \mathbb{R}^n), \quad \rho(\cdot) \in L^2(0, T; \mathbb{R}^m), \quad G \in S^n, \quad g \in \mathbb{R}^n,
\end{align*}
\]

and the following holds:

\[
(3.9) \quad R(t) \geq \delta, \quad Q(t) - S(t)^\top R(t)^{-1} S(t) \geq 0, \quad G \geq 0, \quad t \in [0, T],
\]

for some \(\delta > 0\).

Now, we have the following result for Problem (P).

**Theorem 3.2.** Suppose that (A2)–(A4) hold. Then Problem (P) admits a unique open-loop optimal pair \((\bar{X}(\cdot), \bar{u}(\cdot))\) \(\in \mathcal{X} \times \mathcal{U}\). Moreover, the following relation is satisfied:

\[
\bar{u}(t) = -R(t)^{-1} \left\{ \bar{\Psi}(T, t)^\top (G\bar{X}(T) + g) + S(t)\bar{X}(t) + \rho(t) \\
+ \int_t^T \Psi(s, t)^\top (Q(s)\bar{X}(s) + S(s)^\top \bar{u}(s) + q(s)) ds \right\}, \quad \text{a.e.} \ t \in [0, T].
\]

**Proof.** Under (A4), one has (3.8). Thus, from the representation (3.7) of the cost functional, we see that \(u(\cdot) \mapsto J(u(\cdot))\) admits a unique minimum \(\bar{u}(\cdot)\) which is given by the solution to the following:

\[
0 = \Lambda \bar{u} + \lambda_1 = \left( \Theta^*Q\Theta + S\Theta + \Theta^*S^* + R + \Theta^*_T G\Theta_T \right) \bar{u} + \Theta^*Q\psi + S\psi + \Theta^*q + \rho + \Theta^*_T G\psi(T) + \Theta^*_T g \\
= \Theta^*(Q(\psi + u)) + S(\psi + u) + \Theta^*S^*u + Ru + \Theta^*_T G(\psi(T) + \Theta \bar{u}) + \Theta^*q + \rho + \Theta^*_T g \\
= \Theta^*_T [G \bar{X}(T) + g] + \Theta^*(Q \bar{X} + S^\top \bar{u} + q) + R \bar{u} + S \bar{X} + \rho.
\]

Thus,

\[
\bar{u} = -R^{-1} \left( \Theta^*_T [G \bar{X}(T) + g] + S \bar{X} + \rho + \Theta^*(Q \bar{X} + S^\top \bar{u} + q) \right),
\]

which is the same as (3.10).

Let us take a closer look at the above result. Note that

\[
\Psi(s, t) = \frac{B(s, \tau)}{(s - t)^{1 - \beta}} + \int_t^s \frac{B(t, \tau)\Phi(s, \tau)}{(\tau - t)^{1 - \beta}} d\tau, \quad (s, t) \in \Delta.
\]

Thus,

\[
\begin{align*}
\Psi(T, t)^\top (G\bar{X}(T) + g) + &\int_t^T \Psi(s, t)^\top [Q(s)\bar{X}(s) + S(s)^\top \bar{u}(s) + q(s)] ds \\
= &\frac{B(T, t)^\top [G \bar{X}(T) + g]}{(T - t)^{1 - \beta}} + \int_t^T \frac{B(s, t)^\top \Phi(s, T)^\top [G \bar{X}(T) + g]}{(s - t)^{1 - \beta}} ds \\
+ &\int_t^T \frac{B(s, t)^\top [Q(s)\bar{X}(s) + S(s)^\top \bar{u}(s) + q(s)]}{(s - t)^{1 - \beta}} ds + \int_t^T \Phi(s, T)^\top [Q(s)\bar{X}(s) + S(s)^\top \bar{u}(s) + q(s)] ds.
\end{align*}
\]
Now, we let $Y$ be the solution to the following adjoint equation:

$$Y(t) = Q(t)X(t) + S(t)^T\bar{u}(t) + q(t) + \int_t^T A(s,t)^T[G\dot{X}(T) + g]ds$$

(3.13) $\bar{u}(t) = -R(t)^{-1}\left[\int_t^T B(s,t)^T\Phi(T,s)^T[G\dot{X}(T) + g] + Q(s)\bar{X}(s) + S(s)^T\bar{u}(s) + q(s)ds + \int_s^T \Phi(\tau,s)^T [Q(\tau)\bar{X}(\tau) + S(\tau)^T\bar{u}(\tau) + q(\tau)]d\tau\right]ds,$ \quad a.e. $t \in [0,T].$

Consequently, we have the following relation for the optimal control $\bar{u}(t)$:

$$\bar{u}(t) = -R(t)^{-1}\left[\int_t^T B(s,t)^T\Phi(T,s)^T[G\dot{X}(T) + g] + Q(s)\bar{X}(s) + S(s)^T\bar{u}(s) + q(s)ds + \int_s^T \Phi(\tau,s)^T [Q(\tau)\bar{X}(\tau) + S(\tau)^T\bar{u}(\tau) + q(\tau)]d\tau\right]ds$$

(3.12)

with $\bar{X}(\cdot)$ being the optimal state trajectory.

On the other hand, we know that optimal control can also be characterized by the variational method. The following is the result for Problem (P) from a different angle.

**Theorem 3.3.** Let (A2)–(A4) hold. Then Problem (P) admits a unique open-loop optimal pair $(\bar{X}(\cdot), \bar{u}(\cdot))$ such that

$$\bar{u}(t) = -R(t)^{-1}\left[\int_t^T B(s,t)^T\Phi(T,s)^T[G\dot{X}(T) + g] + Q(s)\bar{X}(s) + S(s)^T\bar{u}(s) + q(s)ds + \int_s^T \Phi(\tau,s)^T [Q(\tau)\bar{X}(\tau) + S(\tau)^T\bar{u}(\tau) + q(\tau)]d\tau\right]ds,$$ \quad a.e. $t \in [0,T],$

where $Y(\cdot)$ is the solution to the following adjoint equation:

$$Y(t) = Q(t)X(t) + S(t)^T\bar{u}(t) + q(t) + \int_t^T A(s,t)^T[G\dot{X}(T) + g] + \int_t^T A(s,t)^TY(s)ds,$$ \quad a.e. $t \in [0,T].$

**Proof.** Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be the optimal pair. Then for any $u(\cdot) \in \mathcal{U}$, we have

$$0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left[ J(\bar{u}(\cdot)) + \varepsilon u(\cdot)) - J(\bar{u}(\cdot)) \right]$$

$$= \int_0^T \left( \langle Q(t)\dot{X}(t) + S(t)^T\bar{u}(t) + q(t), X(t) \rangle + \langle S(t)\dot{X}(t) + R(t)\bar{u}(t) + \rho(t), u(t) \rangle \right) dt + \langle GX(T) + g, X(T) \rangle,$$

where

$$X(t) = \int_0^t \frac{A(t,s)X(s) + B(t,s)u(s)}{(t-s)^{1-\beta}}ds,$$ \quad $t \in [0,T].$

Thus,

$$\langle GX(T) + g, X(T) \rangle = \int_0^T \left( \langle A(T,t)^T[G\dot{X}(T) + g], X(t) \rangle + \langle B(T,t)^T[G\dot{X}(T) + g], u(t) \rangle \right) dt.$$

This yields

$$0 = \int_0^T \left( \langle Q(t)\dot{X}(t) + S(t)^T\bar{u}(t) + q(t), X(t) \rangle + \langle S(t)\dot{X}(t) + R(t)\bar{u}(t) + \rho(t), u(t) \rangle + \int_0^t A(s,t)^TY(s)ds \right) dt.$$

Now, we let $Y(\cdot)$ be the solution to the following:

$$Y(t) = \gamma(t) + \int_t^T A(s,t)^TY(s)ds,$$ \quad a.e. $t \in [0,T],$$\quad a.e.$
with
\[
\gamma(t) = Q(t)\dot{X}(t) + S(t)^\top \ddot{u}(t) + q(t) + \frac{A(T,t)^\top [G\dot{X}(T) + g]}{(T-t)^{1-\beta}}, \quad \text{a.e. } t \in [0,T].
\]

Then for any \( u \in \mathcal{W} \),
\[
0 = \int_0^T \left( (Y(t) - \int_t^T A(s,t)^\top Y(s) \, ds, X(t)) + \langle S(t)\dot{X}(t) + R(t)\ddot{u}(t) + \rho(t) + \frac{B(T,t)^\top [G\dot{X}(T) + g]}{(T-t)^{1-\beta}}, u(t) \rangle \right) dt
\]
\[
= \int_0^T \left( (Y(t), X(t))dt - \int_0^T (Y(s), \int_s^t A(s,t)X(t) \, ds) \right) - \int_0^T \langle S(t)\dot{X}(t) + R(t)\ddot{u}(t) + \rho(t) + \frac{B(T,t)^\top [G\dot{X}(T) + g]}{(T-t)^{1-\beta}}, u(t) \rangle dt
\]
\[
= \int_0^T \left( (Y(t), \int_t^T B(s,t)^\top Y(s) \, ds + S(t)\dot{X}(t) + R(t)\ddot{u}(t) + \rho(t) + \frac{B(T,t)^\top [G\dot{X}(T) + g]}{(T-t)^{1-\beta}}, u(t) \rangle \right) dt
\]
Hence,
\[
\int_t^T B(s,t)^\top Y(s) \, ds + S(t)\dot{X}(t) + R(t)\ddot{u}(t) + \rho(t) + \frac{B(T,t)^\top [G\dot{X}(T) + g]}{(T-t)^{1-\beta}}, u(t) \rangle dt = 0, \quad \text{a.e. } t \in [0,T].
\]

Then (3.13) follows.

The above is actually the Pontryagin type maximum principle. Comparing (3.12) and (3.13), we see that they coincide if the following is true:
\[
Y(s) = \Phi(T,s)^\top [G\dot{X}(T) + g] + Q(s)\dot{X}(s) + S(s)^\top \ddot{u}(s) + q(s)
\]
\[
+ \int_s^T \Phi(\tau,s)^\top \left[ Q(\tau)\dot{X}(\tau) + R(\tau)\ddot{u}(\tau) + q(\tau) \right] d\tau, \quad \text{a.e. } s \in [0,T].
\]

This can be shown as follows. By (2.16), we have
\[
\Phi(t,s)^\top = \frac{A(t,s)^\top}{(t-s)^{1-\beta}} + \int_s^t \frac{A(\tau,s)^\top \Phi(t,\tau)^\top}{(\tau-s)^{1-\beta}} d\tau, \quad \text{a.e. } t \in [s,T].
\]

Denote
\[
z(t) = Q(t)\dot{X}(t) + S(t)^\top \ddot{u}(t) + q(t), \quad \text{a.e. } t \in [0,T]; \quad \zeta = G\dot{X}(T) + g.
\]

Then, we have
\[
Y(t) = z(t) + \frac{A(T,t)^\top \zeta}{(T-t)^{1-\beta}} + \int_t^T \frac{A(s,t)^\top Y(s)}{(s-t)^{1-\beta}} ds, \quad \text{a.e. } t \in [0,T],
\]
and we need to check:
\[
Y(t) = z(t) + \Phi(T,t)^\top \zeta + \int_t^T \Phi(s,t)^\top z(s) ds, \quad \text{a.e. } t \in [0,T].
\]

This can be checked as follows:
\[
\int_t^T A(s,t)^\top Y(s) \, ds = \int_t^T A(s,t)^\top \left[ z(s) + \Phi(T,s)^\top \zeta + \int_s^T \Phi(\tau,s)^\top z(\tau) d\tau \right] ds
\]
\[
= \int_t^T A(s,t)^\top z(s) \, ds + \int_t^T \frac{A(s,t)^\top \Phi(T,s)^\top \zeta}{(s-t)^{1-\beta}} ds + \int_t^T \left( \int_s^t \frac{A(\tau,t)^\top \Phi(s,\tau)^\top}{(\tau-t)^{1-\beta}} d\tau \right) z(s) ds
\]
\[
= \int_t^T \Phi(s,t)^\top z(s) ds + \int_t^T \frac{A(s,t)^\top \Phi(T,s)^\top \zeta}{(s-t)^{1-\beta}} ds
\]
\[
= Y(t) - z(t) - \left( \Phi(T,t)^\top - \int_t^T \frac{A(s,t)^\top \Phi(T,s)^\top}{(s-t)^{1-\beta}} ds \right) \zeta = Y(t) - z(t) - \frac{A(T,t)^\top \zeta}{(T-t)^{1-\beta}}, \quad \text{a.e. } t \in [0,T].
\]
Hence, (3.12) and (3.13) are equivalent.

4 Causal State Feedback Representation.

In the relation (3.12) (or (3.13)) for the open-loop optimal control, the current-time value $\bar{u}(t)$ of the optimal control $u(\cdot)$ is given in terms of the future-time values $\{\bar{X}(s) \mid s \in [t, T]\}$ of the corresponding optimal state trajectory $\bar{X}(\cdot)$. Practically, this is not realizable. Thus, our next goal is to seek a causal state feedback representation of optimal control, by which we mean that the value $\bar{u}(t)$ of $u(\cdot)$ at time $t$ can be written in terms of $\{\bar{X}(s) \mid s \in [0, t]\}$ and $\bar{X}(\cdot)$ for some non-anticipating auxiliary process $\bar{X}(\cdot)$. Recall that for standard LQ problems of differential equations, optimal control could admit a closed-loop representation by means of differential Riccati equations. Here, we borrow the idea from [40], but with more straightforward approach which more naturally reveals the essence of the problem.

Since our cost functional contains the cross term of the state and control, as well as a linear term in the control, we would like to make a reduction first. Let

$$u(t) = v(t) - R(t)^{-1}[S(t)X(t) + \rho(t)], \quad \text{a.e. } t \in [0, T].$$

Then, the state equation becomes

$$X(t) = \bar{\varphi}(t) + \int_0^t \frac{\bar{A}(t, s)X(s) + B(t, s)v(s)}{(t-s)^{1-\beta}}ds, \quad \text{a.e. } t \in [0, T],$$

where

$$\bar{A}(t, s) = A(t, s) - B(t, s)S(s)^{-1}S(s), \quad \bar{\varphi}(t) = \varphi(t) - \int_0^t \frac{B(t, s)S(s)^{-1}\rho(s)}{(t-s)^{1-\beta}}ds, \quad \text{a.e. } (t, s) \in \Delta.$$

Also, the running cost rate becomes

$$\langle QX, X \rangle + 2\langle SX, u \rangle + 2\langle Ru, u \rangle + 2\langle q, X \rangle + 2\langle \rho, u \rangle$$

$$= \langle (Q - S^TR^{-1}S)X, X \rangle + 2\langle q, v \rangle + 2\langle q - S^TR^{-1}\rho, X \rangle - \langle R^{-1}\rho, \rho \rangle$$

$$= \langle \tilde{Q}X, X \rangle + 2\langle \tilde{q}, X \rangle - \langle R^{-1}\rho, \rho \rangle.$$

Thus, if we define

$$\tilde{J}(v(\cdot)) = \int_0^T \left( \langle \tilde{Q}(t)X(t), X(t) \rangle + \langle R(t)v(t), v(t) \rangle + 2\langle \tilde{q}(t), X(t) \rangle \right)dt + \langle G(X(T), X(T)) + 2\langle g, X(T) \rangle,$$

then the corresponding optimal control problem is equivalent to the original one. Namely, $(\bar{X}(\cdot), \bar{u}(\cdot))$ is the open-loop optimal pair of the original problem if and only if $(\bar{X}(\cdot), \bar{v}(\cdot))$ is the optimal pair of the reduced problem with $\bar{u}(\cdot)$ and $\bar{v}(\cdot)$ being related by the following:

$$u(t) = \bar{v}(t) - R(t)^{-1}[S(t)\bar{X}(t) + \rho(t)], \quad \text{a.e. } t \in [0, T].$$

Therefore, if the optimal control $\bar{v}(\cdot)$ of the reduced problem has a causal state feedback representation, then so is $\bar{u}(\cdot)$. Hence, for simplicity, we introduce the following hypothesis.

(A5) Let (A2)–(A4) hold with

$$S(t) = 0, \quad \rho(t) = 0, \quad t \in [0, T].$$

Under (A5), we have

$$\left\{ \begin{array}{l}
\Lambda = \Theta^*Q\Theta + \Theta^*_G\Theta R + R, \\
\lambda_1 = \Theta^*Q\psi + \Theta^*_q + \Theta^*_G\psi(T) + \Theta^*_g, \\
\lambda_0 = \langle Q\psi, \psi \rangle + 2\langle q, \psi \rangle + \langle G\psi(T), \psi(T) \rangle + 2\langle g, \psi(T) \rangle,
\end{array} \right.$$
and the open-loop optimal control is given by

$$
\bar{u} = -R^{-1}\left(\Theta^*Q\bar{X} + \Theta^*_T\bar{G}\bar{X}(T) + g\right) = -R^{-1}[\Theta^*Q\bar{X} + \Theta^*_T\bar{G}\bar{X}(T)] - R^{-1}[\Theta^*q + \Theta^*_Tg],
$$

or more precisely,

$$
\bar{u}(t) = -R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top Q(\tau)\bar{X}(\tau)d\tau + \Psi(T,t)^\top G\bar{X}(T)\right]
- R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top g(\tau)d\tau + \Psi(T,t)^\top g\right], \quad \text{a.e. } t \in [0,T].
$$

(4.7)

In the above, \(\Psi(\cdot, \cdot)\), which is defined by (2.20), characterizes the control system and \(Q(\cdot), q(\cdot), G\) and \(g\) are all known a priori. The only unrealistic terms are \(\{X(\tau) \mid \tau \in [t,T]\}\) and \(\bar{X}(T)\) on the right-hand side of the above, since at the time \(t\) of determining the value \(\bar{u}(t)\) of \(\bar{u}(\cdot)\), these are not available.

The idea of getting a feasible representation of the optimal control is to introduce the following simple decomposition for the state trajectory: For any \(\sigma \in [0,T)\),

$$
\bar{X}(t) = \psi(t) + \int_0^t \Psi(t,s)\bar{u}(s)1_{[0,\sigma]}(s)ds + \int_0^t \Psi(t,s)\bar{u}(s)1_{[\sigma,T]}(s)ds
$$

$$
= \psi(t) + \int_0^{t\land \sigma} \Psi(t,s)\bar{u}(s)ds + \int_{t\land \sigma}^t \Psi(t,s)\bar{u}(s)ds \equiv \bar{X}_\sigma(t) + \int_{t\land \sigma}^t \Psi(t,s)\bar{u}(s)ds, \quad \text{a.e. } t \in [0,T].
$$

Also,

$$
\bar{X}(T) = \psi(T) + \int_0^\sigma \Psi(T,s)\bar{u}(s)ds + \int_\sigma^T \Psi(T,s)\bar{u}(s)ds \equiv \bar{X}^\sigma(\sigma) + \int_\sigma^T \Psi(T,s)\bar{u}(s)ds.
$$

Here, \(\bar{X}_\sigma(t)\) and \(\bar{X}^\sigma(\sigma)\) do not use the information of \(\bar{u}(\cdot)\) beyond \(\sigma\). With such a decomposition, we can rewrite (4.7) as follows:

$$
\bar{u}(t) = -R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top Q(\tau)\left(\bar{X}_\sigma(\tau) + \int_{t\land \sigma}^\tau \Psi(\tau,s)\bar{u}(s)ds\right)d\tau
+ \Psi(T,t)^\top G\left(\bar{X}^\sigma(\sigma) + \int_\sigma^T \Psi(T,s)\bar{u}(s)ds\right)\right]
- R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top g(\tau)d\tau + \Psi(T,t)^\top g\right], \quad \text{a.e. } t \in [0,T].
$$

(4.8)

By letting \(\sigma = t\) in the above, we obtain

$$
\bar{u}(t) = -R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top Q(\tau)\bar{X}_t(\tau)d\tau + \Psi(T,t)^\top G\bar{X}^t(t)\right]
- R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top Q(\tau)\int_{t\land \tau}^\tau \Psi(\tau,s)\bar{u}(s)dsd\tau + \Psi(T,t)^\top G\int_t^T \Psi(T,s)\bar{u}(s)ds\right]
- R(t)^{-1}\left[\int_t^T \Psi(\tau,t)^\top g(\tau)d\tau + \Psi(T,t)^\top g\right], \quad \text{a.e. } t \in [0,T].
$$

(4.9)

According to the definition of \((\bar{X}_t(\cdot), \bar{X}^t(\cdot))\), no information of \(\bar{u}(\cdot)\) beyond \(t\) is used, or they are non-anticipating. Therefore, our goal is to rewrite the last two terms on the right-hand side to be non-anticipating.

To achieve our goal, let us introduce a family of projection operators \(\Pi_\sigma : \mathcal{W} \rightarrow \mathcal{W}\) by the following:

$$
[\Pi_\sigma u(\cdot)](t) = 1_{[0,\sigma]}(t)u(t), \quad t \in [0,T],
$$

(4.10)

with \(\sigma \in [0,T)\) being the parameter. Clearly,

$$
[(I - \Pi_\sigma)u(\cdot)](t) = 1_{[\sigma,T]}(t)u(t), \quad t \in [0,T].
$$

(4.11)
Both \( \Pi_\sigma \) and \( I - \Pi_\sigma \) are idempotents on \( \mathcal{U} \). Moreover, for any \( M(\cdot) \in L^\infty(0, T; \mathbb{R}^{k \times m}) \),
\[
M(t)[\Pi_\sigma u(\cdot)](t) = M(t)1_{[0,\sigma]}(t)u(t) = 1_{[0,\sigma]}(t)M(t)u(t) = [\Pi_\sigma M(\cdot)u(\cdot)](t), \quad t \in [0, T].
\]
This means that \( \Pi_\sigma \) commutes with multiplication operators. We shall call \( \Pi_\sigma \) a causal projection. Now, let \( \mathcal{U}_\sigma = \mathcal{B}(I - \Pi_\sigma) = (I - \Pi_\sigma)\mathcal{U} \),
which is a closed subspace of \( \mathcal{U} \), and define a parameterized operator \( \Lambda_\sigma \in \mathcal{L}(\mathcal{U}_\sigma) \) by
\[
\Lambda_\sigma = (I - \Pi_\sigma)\Lambda|_{\mathcal{U}_\sigma}.
\]
Clearly, \( (I - \Pi_\sigma)\Lambda(I - \Pi_\sigma) \) is a natural extension of \( \Lambda_\sigma \), with the value being 0 on \( \mathcal{U}_{\sigma_-} \). We have the following simple lemma.

**Lemma 4.1.** Suppose that (A5) holds. Then for any given \( \sigma \in [0, T) \), the operator \( \Lambda_\sigma \in \mathcal{L}(\mathcal{U}_\sigma) \) is self-adjoint and positive definite on \( \mathcal{U}_\sigma \). Moreover
\[
\Lambda_\sigma^{-1} = (I - \Pi_\sigma)\Lambda^{-1}|_{\mathcal{U}_\sigma}; \quad \|(\Lambda_\sigma)^{-1}\|_{\mathcal{L}(\mathcal{U}_\sigma)} \leq \frac{1}{\delta}, \quad \forall \sigma \in [0, T).
\]

**Proof.** For any \( u, v \in \mathcal{U}_\sigma \), we have
\[
\langle \Lambda_\sigma u, v \rangle_{\mathcal{U}_\sigma} = \langle \Lambda u, v \rangle_{\mathcal{U}} = \langle (I - \Pi_\sigma)\Lambda u, v \rangle_{\mathcal{U}} = \langle \Lambda_\sigma (I - \Pi_\sigma)u, v \rangle_{\mathcal{U}} = \langle \Lambda_\sigma u, v \rangle_{\mathcal{U}},
\]
Thus, \( \Lambda_\sigma \) is self-adjoint on \( \mathcal{U}_\sigma \). Moreover, there exists a constant \( \delta > 0 \) such that for any \( u \in \mathcal{U}_\sigma \),
\[
\langle \Lambda_\sigma u, u \rangle_{\mathcal{U}_\sigma} = \langle \Lambda(I - \Pi_\sigma)u, (I - \Pi_\sigma)u \rangle_{\mathcal{U}} \geq \delta \| (I - \Pi_\sigma)u \|^2_{\mathcal{U}} = \delta \| u \|^2_{\mathcal{U}_\sigma}.
\]
Consequently, \( \Lambda_\sigma \) is boundedly invertible. Applying the above to \( (\Lambda_\sigma)^{-1}u \), one has
\[
\delta \|(\Lambda_\sigma)^{-1}u\|^2_{\mathcal{U}_\sigma} \leq \langle \Lambda_\sigma (\Lambda_\sigma)^{-1}u, (\Lambda_\sigma)^{-1}u \rangle_{\mathcal{U}_\sigma} = \langle u, (\Lambda_\sigma)^{-1}u \rangle_{\mathcal{U}_\sigma} \leq \| u \|_{\mathcal{U}_\sigma} \| (\Lambda_\sigma)^{-1}u \|_{\mathcal{U}_\sigma}.
\]
Then the second estimate in (4.13) follows. \(\square\)

From the above, we see that
\[
\Lambda_\sigma(I - \Pi_\sigma) = (I - \Pi_\sigma)\Lambda(I - \Pi_\sigma), \quad \Lambda_\sigma^{-1}(I - \Pi_\sigma) = (I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma),
\]
\[
(I - \Pi_\sigma)\Lambda(I - \Pi_\sigma)^{-1}(I - \Pi_\sigma) = I - \Pi_\sigma, \quad (I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma)\Lambda(I - \Pi_\sigma) = I - \Pi_\sigma.
\]
Now, we introduce the following auxiliary trajectory:
\[
X^\sigma(t) = \psi(T) + \int_0^t \Psi(T, s)u(s)ds, \quad t \in [0, T],
\]
which catches the anticipating information of the free term \( \psi(T) \) and the dynamic system represented by \( \Psi(T, \cdot) \) which are assumed to be a priori known\(^1\). Note that no anticipating information of the control is involved. At the same time, for each \( \sigma \in [0, T) \), we introduce the following Causal trajectory:
\[
X_\sigma(t) = \psi(t) + \int_0^t \Psi(t, s)(\Pi_\sigma u)(s)ds = \psi(t) + \int_0^{t\wedge \sigma} \Psi(t, s)u(s)ds, \quad t \in [0, T].
\]
This trajectory truncates the control up to time moment \( \sigma \) which is allowed to be smaller than \( t \). It is clear that
\[
X_\sigma(t) = \begin{cases} X(t), & t \in [0, \sigma], \\ \psi(t) + \int_0^{\sigma} \Psi(t, s)u(s)ds, & t \in [\sigma, T]; \end{cases}
\]
\(^1\)In time-varying LQ problems, the differential Riccati equation is solved on the whole time interval and all information of the system and the cost functional through the coefficients and the weights are allowed to be used. The situation here is similar.
Thus, the open-loop optimal control can be written as follows (see (4.20))

\[ \tilde{u}(t) = -R(t)^{-1} \left[ I - (\Lambda - R)(I - \Pi_{\sigma})A^{-1}(I - \Pi_{\sigma}) \right] \left[ \left( \Theta^* Q \tilde{X}_{\sigma}(\cdot) \right)(t) + \left[ \Theta^*_T G \tilde{X}^a(\cdot) \right](t) \right] \]

\[ -R(t)^{-1} \left[ I - (\Lambda - R)(I - \Pi_{\sigma})A^{-1}(I - \Pi_{\sigma}) \right] \left[ \left( \Theta^* q(\cdot) \right)(t) + \left[ \Theta^*_T g \right](t) \right], \quad \text{a.e. } t \in [0, T], \]

where \( \tilde{X}_{\sigma}(\cdot) \) and \( \tilde{X}^a(\cdot) \) are the causal and auxiliary trajectories corresponding to the optimal pair \( (\tilde{X}(\cdot), \tilde{u}(\cdot)) \).

**Proof.** By the definition of \( X_{\sigma}(\cdot) \) and \( X^a(\cdot) \), we have the following: for any \( \sigma \in [0, T) \),

\[ X(t) = \psi(t) + \int_0^t \Psi(t, s) \left\{ (\Pi_{\sigma} u)(s) + [(I - \Pi_{\sigma}) u](s) \right\} ds = X_{\sigma}(t) + [\Theta(I - \Pi_{\sigma}) u](t), \quad t \in [0, T], \]

Thus, the open-loop optimal control can be written as follows (see (4.46)):

\[ \tilde{u} = -R^{-1}(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} + \Theta(I - \Pi_{\sigma}) \tilde{u} \right) - R^{-1}(\Theta^* q + \Theta^*_T g) \]

\[ = -R^{-1}(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) - R^{-1}(\Lambda - R)(I - \Pi_{\sigma}) \tilde{u} - R^{-1}(\Theta^* q + \Theta^*_T g). \]

This leads to

\[ \left[ I + R^{-1}(\Lambda - R)(I - \Pi_{\sigma}) \right] \tilde{u} = -R^{-1}(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) - R^{-1}(\Theta^* q + \Theta^*_T g), \]

which is equivalent to

\[ \Lambda(I - \Pi_{\sigma}) + R \Pi_{\sigma} \tilde{u} = -\left( \Theta^* Q, \Theta^*_T G \right) \left( \tilde{X}_{\sigma} \right) - \left( \Theta^* q + \Theta^*_T g \right). \]

Applying \( (I - \Pi_{\sigma}) \) to the above gives

\[ \Lambda_{\sigma}(I - \Pi_{\sigma}) \tilde{u} = (I - \Pi_{\sigma}) \Lambda(I - \Pi_{\sigma}) \tilde{u} = -(I - \Pi_{\sigma})(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) - (I - \Pi_{\sigma})(\Theta^* q + \Theta^*_T g). \]

Thus,

\[ (I - \Pi_{\sigma}) \tilde{u} = -\Lambda_{\sigma}^{-1}(I - \Pi_{\sigma})(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) - \Lambda_{\sigma}^{-1}(I - \Pi_{\sigma})(\Theta^* q + \Theta^*_T g). \]

Substituting the above into (4.21), one obtains

\[ \tilde{u} = -R^{-1}(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) - R^{-1}(\Theta^* q + \Theta^*_T g) \]

\[ = -R^{-1}(\Lambda - R) \left[ -\Lambda_{\sigma}^{-1}(I - \Pi_{\sigma})(\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) - \Lambda_{\sigma}^{-1}(I - \Pi_{\sigma})(\Theta^* q + \Theta^*_T g) \right] \]

\[ = -R^{-1} \left[ I - (\Lambda - R) \Lambda_{\sigma}^{-1}(I - \Pi_{\sigma}) \right] \left[ (\Theta^* Q, \Theta^*_T G) \left( \tilde{X}_{\sigma} \right) + (\Theta^* q + \Theta^*_T g) \right]. \]

Setting \( \sigma = t \), we obtain our conclusion, making use of (4.14). \( \square \)

The above gives a causal state feedback representation for the open-loop optimal control in an abstract form. The appearance of \( \Lambda^{-1} \) makes the result hard to use since \( \Lambda \) is a complicated nonlocal operator. Hence, our next goal is to make the representation more explicitly accessible. We will achieve this goal in next section.
5 Representation via Fredholm Integral Equations.

Based on the result given in Theorem 4.2, we will now focus on the further manipulation of the abstract operator $\Lambda^{-1}$. We want to convert it into another feedback gain operator which can be accessed in a computational manner. To this aim, for each $\sigma \in [0, T)$, we define $M_\sigma : [0, T] \times [0, T] \to \mathbb{R}^{m \times m}$ by the following:

\[
M_\sigma(t, s) = -R^{-1}(t)\left\{ \left[ I - (A - R)(I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma) \right] \left( \Theta^*Q, \Theta^*G \right) \left( \Psi(\cdot, s) \right) \right\}(t), \quad (t, s) \in [0, T] \times [0, T].
\]

Note that $\Psi(\cdot, \cdot)$ is extended to be zero in $([0, T] \times [0, T]) \setminus \Delta$. We would like to find an equation for $M_\sigma(\cdot, \cdot)$, which is easier to be used.

**Lemma 5.1.** For any $\sigma \in [0, T)$, the following Fredholm integral equation

\[
R(t)M_\sigma(t, s) = -\int_\sigma^T \left( \int_0^T \Psi(\tau, t)^\top Q(\tau)\Psi(\tau, \xi)d\tau + \Psi(T, t)^\top G\Psi(T, \xi) \right)M_\sigma(\xi, s)d\xi
\]

\[
-\int_t^\sigma \Psi(\tau, t)^\top Q(\tau)\Psi(\tau, s)d\tau - \Psi(T, t)^\top G\Psi(T, s), \quad (t, s) \in [0, T] \times [0, T],
\]

admits a unique solution $M_\sigma(\cdot, \cdot)$, which is given by the expression in (5.1).

**Proof.** Note that (by (4.14)) and the fact that $R\Pi_\sigma = \Pi_\sigma R$,

\[
(I - \Pi_\sigma)R^{-1}[I - (A - R)(I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma)]
\]

\[
= (I - \Pi_\sigma)R^{-1}[I - \Pi_\sigma - (I - \Pi_\sigma)(A - R)(I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma)]
\]

\[
= (I - \Pi_\sigma)R^{-1}[I - \Pi_\sigma]R(I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma) = (I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma).
\]

Thus,

\[
M_\sigma(t, s) = -R^{-1}(t)\left\{ \left[ I - (A - R)(I - \Pi_\sigma)\Lambda^{-1}(I - \Pi_\sigma) \right] \left( \Theta^*Q, \Theta^*G \right) \left( \Psi(\cdot, s) \right) \right\}(t)
\]

\[
= -R^{-1}(t)\left\{ \left( \Theta^*Q, \Theta^*G \right) \left( \Psi(\cdot, s) \right) \right\}(t)
\]

\[
= -R^{-1}(t)\left\{ \left( \Theta^*Q\Psi(\cdot, s) \right)(t) + [\Theta^*G\Psi(T, s)](t) \right\}
\]

\[
- R^{-1}(t)\left( \Theta^*Q\Theta\Psi(\cdot, \cdot) \right)(t) \right\}(t), \quad (t, s) \in [0, T] \times [0, T].
\]

This is equivalent to the following:

\[
M_\sigma(t, s) = -R^{-1}(t)\left( \int_t^T \Psi(\tau, t)^\top Q(\tau)\Psi(\tau, s)d\tau + \Psi(T, t)^\top G\Psi(T, s) \right)
\]

\[
- R^{-1}(t)\int_t^T \Psi(\tau, t)^\top Q(\tau)\int_0^\tau \Psi(\tau, \xi)1_{[\sigma, T]}(\xi)M_\sigma(\xi, s)d\xi d\tau
\]

\[
- R^{-1}(t)\Psi(T, t)^\top G\int_0^T \Psi(T, \xi)1_{[\sigma, T]}(\xi)M_\sigma(\xi, s)d\xi
\]

\[
= -R^{-1}(t)\int_\sigma^T \left( \int_0^T \Psi(\tau, t)^\top Q(\tau)\Psi(\tau, s)d\tau + \Psi(T, t)^\top G\Psi(T, s) \right)M_\sigma(\xi, s)d\xi
\]

\[
- R^{-1}(t)\left( \int_0^T \Psi(\tau, t)^\top Q(\tau)\Psi(T, s)d\tau + \Psi(T, t)^\top G\Psi(T, s) \right), \quad (t, s) \in [0, T] \times [0, T].
\]
This means that $M_\sigma(\cdot, \cdot)$ is a solution to the Fredholm equation (5.2). Now, for the uniqueness, it suffices to show that if
\[
R(t)M_\sigma(t, s) + \int_\sigma^T \left( \int_\sigma^\tau \Psi(\tau, t)^\top Q(\tau, \xi)\Psi(\tau, \xi) \, d\tau + \Psi(T, t)^\top G\Psi(T, \xi) \right) M_\sigma(\xi, s) \, d\xi = 0,
\]
then $M_\sigma(t, s) = 0$, $(t, s) \in [0, T] \times [0, T]$. Note that
\[
0 = R(t)M_\sigma(t, s) + \left( (\Lambda - R)(I - \Pi_\sigma)M_\sigma(\cdot, s) \right)(t)
= \left( [R\Pi_\sigma + \Lambda(I - \Pi_\sigma)]M_\sigma(\cdot, s) \right)(t), \quad (t, s) \in [0, T] \times [0, T].
\]
Applying $(I - \Pi_\sigma)$ to the above, one has
\[
0 = \left( (I - \Pi_\sigma)\Lambda(I - \Pi_\sigma)M_\sigma(\cdot, s) \right)(t) = \left[ \Lambda_\sigma(I - \Pi_\sigma)M_\sigma(\cdot, s) \right](t), \quad (t, s) \in [0, T] \times [0, T].
\]
By the invertibility of $\Lambda_\sigma$, one has $\left( (I - \Pi_\sigma)M_\sigma(\cdot, s) \right)(t) = 0$, $(t, s) \in [0, T] \times [0, T]$. Consequently, $R(t)M_\sigma(t, s) = 0$, $(t, s) \in [0, T] \times [0, T]$. Hence, it follows from the invertibility of $R$ that $M_\sigma(t, s) = 0$, $(t, s) \in [0, T] \times [0, T]$ completing the proof.

We now prove the following theorem.

**Theorem 5.2.** Let $(A5)$ hold. Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be the open-loop optimal pair and $(\bar{X}_\sigma(\cdot), \bar{X}_\sigma^a(\cdot))$ be the corresponding truncation and auxiliary trajectories. Then the open-loop optimal control $\bar{u}(\cdot)$ admits the following representation:

\[
\bar{u}(t) = -R(t)^{-1} \left( \left[ \Theta^* Q \bar{X}_t(\cdot) \right](t) + \left[ \Theta_\tau^* G \bar{X}^a(t) \right](t) \right) - R(t)^{-1} \left( \left[ \Theta^* q(\cdot) \right](t) + \left[ \Theta_\tau^* g \right](t) \right)
= -\int_t^T M_t(t, s)R(s)^{-1} \left( \left[ \Theta^* Q \bar{X}_s(\cdot) \right](s) + \left[ \Theta_\tau^* G \bar{X}^a(t) \right](s) \right) \, ds
- \int_t^T M_t(t, s)R(s)^{-1} \left( \left[ \Theta^* q(\cdot) \right](s) + \left[ \Theta_\tau^* g \right](s) \right) \, ds, \quad \text{a.e. } t \in [0, T],
\]

where $M_\sigma(\cdot, \cdot)$ is the unique solution of Fredholm equation (5.2).

**Proof.** First, similar to (5.3), we have
\[
\left( I - \Pi_\sigma \right) R^{-1} - \left( I - \Pi_\sigma \right) \Lambda^{-1} \left( I - \Pi_\sigma \right) \left( \Lambda - R \right) R^{-1} \left( I - \Pi_\sigma \right)
= \left( I - \Pi_\sigma \right) - \left( I - \Pi_\sigma \right) \Lambda^{-1} \left( I - \Pi_\sigma \right) \left( \Lambda - \Pi_\sigma - R \right) R^{-1} \left( I - \Pi_\sigma \right)
= \left( I - \Pi_\sigma \right) - \left( I - \Pi_\sigma \right) + \left( I - \Pi_\sigma \right) \Lambda^{-1} \left( I - \Pi_\sigma \right) R \right) R^{-1} \left( I - \Pi_\sigma \right) = \left( I - \Pi_\sigma \right) \Lambda^{-1} \left( I - \Pi_\sigma \right).
\]
Let us denote
\[
\Gamma(t) = \left( \Theta^* Q, \Theta_\tau^* G \right) \left( \bar{X}_t(\cdot) \right)(t) + \left( \Theta^*, \Theta_\tau^* \right) \left( q(\cdot) \right)(t), \quad \text{a.e. } t \in [0, T].
\]
Then,
\[
\bar{u}(t) = -R(t)^{-1} \left[ I - (\Lambda - R)(I - \Pi_\sigma) \Lambda^{-1} (I - \Pi_\sigma) \right] \Gamma(t)
= -R^{-1} \Gamma(t) + R^{-1} (\Lambda - R) \left( I - \Pi_\sigma \right) R^{-1} - (I - \Pi_\sigma) \Lambda^{-1} (I - \Pi_\sigma) (\Lambda - R) R^{-1} \left( I - \Pi_\sigma \right) \Gamma(t)
= -R^{-1} \Gamma(t) + R^{-1} \left( \Theta^* Q, \Theta_\tau^* G \right) \left( \Theta \right) \left( I - \Pi_\sigma \right) R^{-1} \Gamma(t)
= -R^{-1} \Gamma(t) + R^{-1} \left[ I - (\Lambda - R)(I - \Pi_\sigma) \Lambda^{-1} (I - \Pi_\sigma) \right] \left( \Theta^* Q, \Theta_\tau^* G \right) \left( \Theta \right) \left( I - \Pi_\sigma \right) R^{-1} \Gamma(t)
\]
\[
= -R^{-1} \Gamma(t) + R^{-1} \left[ I - (\Lambda - R)(I - \Pi_\sigma) \Lambda^{-1} (I - \Pi_\sigma) \right] \left( \Theta^* Q, \Theta_\tau^* G \right) \left( \Theta \right) \left( I - \Pi_\sigma \right) R^{-1} \Gamma(t).
\]
This completes the proof. □

Now, we return to the general case, i.e., \( S(\cdot) \) and \( \rho(\cdot) \) are not necessarily zero. In this case, we summarize the result as follows:

\[
\begin{align*}
\tilde{A}(t, s) &= A(t) - B(t, s)R(s)^{-1}S(s), \quad \tilde{q}(t) = \varphi(t) - \int_t^T \frac{B(t, s)R(s)^{-1}\rho(s)}{(s - t)^{1-\beta}} ds, \\
\tilde{Q}(s) &= Q(s) - [S(s)]^\top R(s)^{-1}S(s), \quad \tilde{q}(s) = q(s) - [S(s)]^\top R(s)^{-1}\rho(s), \\
\tilde{\Phi}(t, s) &= \tilde{A}(t, s) + \int_s^t \frac{\tilde{A}(t, \tau)\tilde{\Phi}(\tau, s)}{(s - \tau)^{1-\beta}} d\tau, \\
\tilde{\Psi}(t, s) &= \frac{\tilde{B}(t, s)}{(t - s)^{1-\beta}} + \int_s^t \frac{\tilde{B}(t, \tau)\tilde{\Psi}(\tau, s)}{(s - \tau)^{1-\beta}} d\tau, \quad \tilde{\psi}(t) = \tilde{\varphi}(t) + \int_0^t \tilde{\Phi}(t, s)\tilde{\psi}(s) ds, \\
(\tilde{\Theta}v)(t) &= \int_0^T \tilde{\Psi}(t, s)v(s) ds, \quad v \in \mathcal{V}, \ t \in [0, T], \quad \tilde{\Theta}_TV = \int_0^T \tilde{\Psi}(T, s)v(s) ds = (\tilde{\Theta}v)(T), \quad v \in \mathcal{V}.
\end{align*}
\]

The truncation and auxiliary trajectories are defined by

\[
\begin{align*}
X_\sigma(t) &= \tilde{\psi}(t) + \int_0^{t/\sigma} \tilde{\Psi}(t, s)\left(u(s) + R(s)^{-1}\left[S(s)X(s) + \rho(s)\right]\right) ds, \\
X^a(t) &= \tilde{\psi}(T) + \int_0^T \tilde{\Psi}(T, s)\left(u(s) + R(s)^{-1}\left[S(s)X(s) + \rho(s)\right]\right) ds, \quad t \in [0, T].
\end{align*}
\]

The corresponding Fredholm equation reads

\[
R(t)\tilde{M}_\sigma(t, s) = -\int_0^T \left( \int_{\nu \in \mathcal{V}} \tilde{\Psi}(\tau, t)^\top \tilde{Q}(\tau, \xi) d\tau + \tilde{\Psi}(T, t)^\top G\tilde{\Psi}(T, \xi) \right) \tilde{M}_\sigma(\xi, s) d\xi
- \int_0^T \tilde{\Psi}(\tau, t)^\top \tilde{Q}(\tau, s) d\tau + \tilde{\Psi}(T, t)^\top G\tilde{\Psi}(T, s), \quad (t, s) \in [0, T] \times [0, T].
\]

Then we can state the following result whose proof is clear.

**Theorem 5.3.** Let (A2)–(A4) hold. Let \( (\tilde{X}(\cdot), \tilde{u}(\cdot)) \) be the open-loop optimal pair of Problem (P). Then the open-loop optimal control \( \tilde{u}(\cdot) \) admits the following causal state feedback representation:

\[
\tilde{u}(t) = -R(t)^{-1}\left[S(\tilde{X}(t)) + \rho(t)\right] - R(t)^{-1}\left[\tilde{\Theta}^*\tilde{Q}\hat{X}_t(t) + \tilde{\Theta}_T^*G\hat{X}^a(t)\right] (t)
- \int_0^T \tilde{M}_t(s)R(s)^{-1}\left[\tilde{\Theta}^*\tilde{Q}\hat{X}_t(s) + \tilde{\Theta}_T^*G\hat{X}^a(s)\right] (s) ds
- \int_0^T \tilde{M}_t(s)R(s)^{-1}\left[\tilde{\Theta}^*\tilde{q}(s) + \tilde{\Theta}_T^*g(s)\right] ds
- R(t)^{-1}\left[\tilde{\Theta}^*\tilde{q}(t) + \tilde{\Theta}_T^*g(t)\right], \quad a.e. \ t \in [0, T],
\]

where \( \tilde{\Theta}, \tilde{\Theta}_T, \tilde{M}_\sigma(\cdot, \cdot), \tilde{Q}(\cdot), \) and \( \tilde{q}(\cdot) \) are given by (5.6) and (5.8), \( X_\sigma(\cdot) \) and \( \hat{X}^a(\cdot) \) are defined by (5.7) with \( u(\cdot) \) being replaced by \( \tilde{u}(\cdot) \).
Since the general Volterra integral equation does not have a semigroup evolutionary property, the direct feedback implementation of the optimal control in terms of the actual trajectory $X(\tau)$, $\tau \in [0, T]$, is not possible, because the future information of the function $\psi$ is not counted. In view of this, the causal stated feedback representation is the best that can be hoped for. Because the auxiliary trajectory depends on $\hat{\psi}(T)$, one might also call the above semi-causal state feedback representation as in [40].

6 An Iteration Scheme for the Fredholm Integral Equation.

In this section, we will briefly present a possible numerical scheme which is applicable to solve Fredholm integral equation (5.2). This will, in principle, make the approach presented in the previous sections practically feasible.

During the period 1960–1990, there has been much work on developing and analyzing numerical methods for solving linear Fredholm integral equations of the second kind. The Galerkin and collocation methods are the well-established numerical methods (see [6]). Also, it is known that both the iterated Galerkin and the iterated collocation methods exhibit a higher order of convergence than the Galerkin method and collocation methods, respectively (see, for example [42]). Long–Nelakanti [34] proposed an efficient iteration algorithm having much higher order of convergence, while they need less additional computational efforts for the implementation. Making use of the similar idea of [34], we aim to obtain an efficient iteration scheme for the Fredholm integral equation (5.2). Let us make this more precise now. To this end, we denote

$$K(t, \xi) = -R(t)^{-1}\left( \int_{t \vee \xi}^{T} \Psi(t, \tau)Q(\tau)\Psi(t, \xi)d\tau + \Psi(T, t)^{\top}G\Psi(T, \xi) \right), \quad (t, \xi) \in [0, T] \times [0, T],$$

and

$$f(t, s) = -R(t)^{-1}\left( \int_{s}^{T} \Psi(t, \tau)Q(\tau)\Psi(s, \tau)d\tau + \Psi(T, s)^{\top}G\Psi(T, s) \right), \quad (t, s) \in [0, T] \times [0, T].$$

Next, we denote $\mathcal{M} = L^{2}(0, T; \mathbb{R}^{m \times m})$, and for any $\sigma \in [0, T)$, define the integral operator $\mathcal{K}_{\sigma} : \mathcal{M} \to \mathcal{M}$ by the following:

$$\mathcal{K}_{\sigma}(\eta)(t) = \int_{\sigma}^{T} K(t, \xi)(\eta)(\xi)d\xi, \quad t \in [0, T], \quad \forall \eta \in \mathcal{M}.$$  

Then, for each $\sigma \in [0, T)$, (5.2) can be reconsidered as the following: for each $s \in [0, T]$,

$$(I - \mathcal{K}_{\sigma})M_{\sigma}(t, s) = f(t, s), \quad t \in [0, T].$$

We introduce a partition $\pi : 0 = s_{0} < s_{1} < s_{2} < \cdots < s_{N} = T$ of $[0, T]$. For each $s_{i}$, $0 \leq i \leq N$, we consider the following equation:

$$(6.1) \quad (I - \mathcal{K}_{\sigma})M_{\sigma}(t, s_{i}) = f(t, s_{i}), \quad t \in [0, T].$$

By assumption (A3), we can easily obtain that the integral operator $\mathcal{K}_{\sigma}$ is a compact linear operator on $\mathcal{M}$. Indeed,

$$\int_{0}^{T} \int_{0}^{T} \left| K(t, \xi) \right|^{2} d\xi dt \leq K \int_{0}^{T} \int_{0}^{T} \left| \int_{t \vee \xi}^{T} \Psi(t, \tau)Q(\tau)\Psi(t, \xi)d\tau \right|^{2} d\xi dt + K \int_{0}^{T} \int_{0}^{T} \left| \Psi(T, t)^{\top}G\Psi(T, \xi) \right|^{2} d\xi dt \leq K \int_{0}^{T} \int_{0}^{T} \left[ \int_{t}^{T} \left| \Psi(t, \tau) \right|^{2} d\tau \right]^{2} \left[ \int_{t}^{T} \left| \Psi(t, \xi) \right|^{2} d\tau \right] d\tau dt + K \int_{0}^{T} \int_{0}^{T} \frac{1}{(T - t)^{2(1 - \beta)}} \frac{1}{(T - \xi)^{2(1 - \beta)}} d\xi dt \leq K \int_{0}^{T} \int_{0}^{T} \left[ \int_{t}^{T} \frac{1}{(T - t)^{2(1 - \beta)}} d\tau \right] \left[ \int_{t}^{T} \frac{1}{(T - \xi)^{2(1 - \beta)}} d\tau \right] d\xi dt + K \int_{0}^{T} \frac{1}{(T - t)^{2(1 - \beta)}} dt < \infty,$$

which implies that $\mathcal{K}(\cdot, \cdot) \in L^{2}((0, T) \times (0, T); \mathbb{R}^{m \times m})$. Then, it is easy to see that $\mathcal{K}_{\sigma}$ is compact on $\mathcal{M}$ (see, for example Theorem 6.12 in [13]). For each $s_{i}$, $0 \leq i \leq N$, the Fredholm alternative theorem then
guarantees the existence of a unique solution of (6.1) in \( \mathcal{M} \) (see, for example Theorem 1.3.1 in [6]). Let \( \{ \mathcal{M}_n : n \geq 1 \} \) be a sequence of increasing finite dimensional subspaces of \( \mathcal{M} \). Let \( P_n : \mathcal{M} \to \mathcal{M}_n \) be the orthogonal projection operator (see, for example Section 3.1.2 in [6]). Then, for each \( s_i, 0 \leq i \leq N \), the Galerkin approximation is the solution of

\[
M_{\sigma n}(t, s_i) = P_n f(t, s_i) + P_n \mathcal{K}_\sigma M_{\sigma n}(t, s_i), \quad t \in [0, T].
\]

We can show that \( P_n y \to y \), as \( n \to \infty \), for all \( y \in \mathcal{M} \) (see, for example Section 3.3.1 in [6]). Then, it follows from the compactness of \( \mathcal{K}_\sigma \) that

\[
\lim_{n \to \infty} \| \mathcal{K}_\sigma - P_n \mathcal{K}_\sigma \|_{\mathcal{L}(\mathcal{M})} = 0,
\]

(see, for example Lemma 3.1.2 in [6] or [42]). Then, by (6.3), one has that \((I - P_n \mathcal{K}_\sigma)^{-1}\) exist and uniformly bounded for sufficiently large \( n \), and the approximation scheme is uniquely solvable (see, for example Theorem 3.1.1 in [6]).

For each \( s_i, 0 \leq i \leq N \), the iterated Galerkin approximation (see, [42]) may be defined by

\[
\tilde{M}_{\sigma n}(t, s_i) = f(t, s_i) + \mathcal{K}_\sigma M_{\sigma n}(t, s_i), \quad t \in [0, T].
\]

Applying \( P_n \) to both side of (6.4), we have \( P_n \tilde{M}_{\sigma n} = M_{\sigma n} \), and hence for each \( s_i, 0 \leq i \leq N \), it holds that

\[
\tilde{M}_{\sigma n}(t, s_i) = f(t, s_i) + \mathcal{K}_\sigma P_n \tilde{M}_{\sigma n}(t, s_i), \quad t \in [0, T].
\]

One can show that the iterated Galerkin scheme (6.4) can converge more rapidly than the rate achieved by the approximation (6.2) (see, [26] and [42]). Further, Long–Nelakanti [34] proposed a more efficient iteration algorithm which even has much higher order of convergence than the iterated Galerkin scheme. The iteration algorithm is as follows: for each \( s_i, 0 \leq i \leq N \), set \( \tilde{M}_{\sigma n}^{(0)}(t, s_i) = \tilde{M}_{\sigma n}(t, s_i), t \in [0, T], \) then for \( k = 0, 1, \ldots \),

\begin{align*}
\text{step 1 :} & \quad \tilde{M}_{\sigma n}^{(k)}(t, s_i) = f(t, s_i) + \mathcal{K}_\sigma \tilde{M}_{\sigma n}^{(k)}(t, s_i), \quad t \in [0, T], \\
\text{step 2 :} & \quad \tilde{M}_{\sigma n}^{(k)}(t, s_i) = f(t, s_i) + \mathcal{K}_\sigma \tilde{M}_{\sigma n}^{(k)}(t, s_i), \quad t \in [0, T], \\
\text{step 3 :} & \quad g_n^{(k)}(t, s_i) = \tilde{M}_{\sigma n}^{(k)}(t, s_i) - \tilde{M}_{\sigma n}^{(k)}(t, s_i), \quad t \in [0, T], \\
\text{step 4 :} & \quad \text{for each } s_i, \ 0 \leq i \leq N, \text{ seeking a unknown function } e_n^{(k)}(t, s_i), t \in [0, T] \text{ by solving the equation } \\
& \quad (I - P_n \mathcal{K}_\sigma) e_n^{(k)}(t, s_i) = P_n g_n^{(k)}(t, s_i), \quad t \in [0, T], \\
\text{step 5 :} & \quad M_{\sigma n}^{(k+1)}(t, s_i) = \mathcal{K}_\sigma e_n^{(k)}(t, s_i) + \tilde{M}_{\sigma n}^{(k)}(t, s_i), \quad t \in [0, T].
\end{align*}

By the superconvergence rates for every step of iteration, we obtain that for each \( k = 0, 1, \ldots \), for each \( s_i, 0 \leq i < N \),

\[
M_{\sigma n}^{(k+1)}(\cdot, s_i) \to M_\sigma(\cdot, s_i) \text{ in } \mathcal{M}, \text{ as } n \to \infty.
\]

For more details, see [34]. Thus, on a subsequence, still denoted in the same way, for each \( n = 0, 1, \ldots \),

\[
\| M_{\sigma n}^{(k+1)}(\cdot, s_i) - M_\sigma(\cdot, s_i) \|_{\mathcal{M}} \leq \frac{1}{n}.
\]

Now, for \( k = 0, 1, \ldots \), let

\[
M_{\sigma n}^N(t, s) = \sum_{i=1}^{N} \left[ \frac{s_i - s}{s_i - s_{i-1}} M_{\sigma n}^{(k+1)}(t, s_{i-1}) + \frac{s - s_{i-1}}{s_i - s_{i-1}} M_{\sigma n}^{(k+1)}(t, s_i) \right] 1_{[s_{i-1}, s_i)}(s), \ (t, s) \in [0, T] \times [0, T].
\]

Then, for \( k = 0, 1, \ldots \),

\[
\begin{align*}
&| M_{\sigma n}^N(t, s) - M_{\sigma}(t, s) | \\
\leq & \sum_{i=1}^{N} \left[ \frac{s_i - s}{s_i - s_{i-1}} \left( | M_{\sigma n}^{(k+1)}(t, s_{i-1}) - M_{\sigma}(t, s_{i-1}) | + | M_{\sigma}(t, s_{i-1}) - M_{\sigma}(t, s) | \right) \\
&+ \frac{s - s_{i-1}}{s_i - s_{i-1}} \left( | M_{\sigma n}^{(k+1)}(t, s_i) - M_{\sigma}(t, s_i) | + | M_{\sigma}(t, s_i) - M_{\sigma}(t, s) | \right) \right] 1_{[s_{i-1}, s_i)}(s)
\end{align*}
\]
\[
\leq \sum_{i=1}^{N} \left[ \frac{s_i - s}{s_i - s_{i-1}} |M_\sigma(t, s_{i-1}) - M_\sigma(t, s)| + \frac{s - s_{i-1}}{s_i - s_{i-1}} |M_\sigma(t, s_i) - M_\sigma(t, s)| \right] I_{[s_{i-1}, s_i]}(s) \\
+ K \sum_{i=1}^{N} \left[ |M_{\sigma n}^{(k+1)}(t, s_{i-1}) - M_\sigma(t, s_{i-1})| + |M_{\sigma n}^{(k+1)}(t, s_i) - M_\sigma(t, s_i)| \right] I_{[s_{i-1}, s_i]}(s), \quad (t, s) \in [0, T] \times [0, T].
\]

Hence, for \( k = 0, 1, \ldots \), for each \( s \in [0, T) \),
\[
\|M_{\sigma n}^{N}(t, s) - M_\sigma(t, s)\|_{\mathcal{M}} \\
\leq K \sum_{i=1}^{N} \left[ \frac{s_i - s}{s_i - s_{i-1}} \|M_\sigma(t, s_{i-1}) - M_\sigma(t, s)\|_{\mathcal{M}} + \frac{s - s_{i-1}}{s_i - s_{i-1}} \|M_\sigma(t, s_i) - M_\sigma(t, s)\|_{\mathcal{M}} \right] I_{[s_{i-1}, s_i]}(s) \\
+ K \sum_{i=1}^{N} \left[ \|M_{\sigma n}^{(k+1)}(t, s_{i-1}) - M_\sigma(t, s_{i-1})\|_{\mathcal{M}} + \|M_{\sigma n}^{(k+1)}(t, s_i) - M_\sigma(t, s_i)\|_{\mathcal{M}} \right] I_{[s_{i-1}, s_i]}(s).
\]

We need a little more assumption which we now introduce.

\textbf{(A6)} There exists a modulus of continuity \( \omega(\cdot) \) such that
\[
|B(t, s') - B(t, s)| \leq \omega(s' - s), \quad (t, s'), (t, s) \in \Delta.
\]

Then, pick any \( s_0 \in [0, T) \), making use of the similar argument of the proof for Corollary 2.2, (ii), we obtain that \( \Psi(t, s) \) is continuous in \( L^2(0, T; \mathbb{R}^{n \times m}) \) at \( s_0 \), and \( \Psi(T, s) \) is continuous at \( s_0 \). Since \( -R^{-1} \left[ I - (A - R)A_\sigma^{-1}(I - \Pi_\sigma) \right] \Theta^*Q \) is a bounded linear operator from \( L^2(0, T; \mathbb{R}^{n \times m}) \) to \( \mathcal{M} \) and \( -R^{-1} \left[ I - (A - R)A_\sigma^{-1}(I - \Pi_\sigma) \right] \Theta^*G \) is a bounded linear operator from \( \mathbb{R}^{n \times m} \) to \( \mathcal{M} \), it holds that \( M_\sigma(t, s) \) is continuous in \( \mathcal{M} \) at \( s_0 \in [0, T) \).

Consequently, for \( k = 0, 1, \ldots \), for each \( s \in [0, T) \), we let \( N \to \infty \) and \( n \to \infty \). Then, by (6.5) and (6.6), we obtain that \( M_{\sigma n}^{N}(t, s) \to M_\sigma(t, s) \) in \( \mathcal{M} \). Thus, we obtain a feasible numerical scheme which can be used to solve Fredholm integral equation (5.2).

7 Concluding Remarks.

In this paper, we have studied an optimal control problem, with the state equation being a linear Volterra integral equation having a singular kernel. The cost functional is a form of quadratic plus linear terms of the state and the control. Under proper conditions, the open-loop optimal control uniquely exists. However, normally, the open-loop optimal control is not of non-anticipating form. Our main goal is to obtain a causal state feedback representation of the open-loop optimal control. In doing that, we have introduced a Fredholm integral equation which plays a role of Riccati equation in the standard LQ problems for ODE systems. To make our result practically feasible (in principle), we have briefly presented a possible numerical scheme for computing the solution to the Fredholm integral equation.

References

[1] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, Nonlinear Dyn., 38 (2004), pp. 323-337.
[2] O. P. Agrawal, A formulation and numerical scheme for fractional optimal control problems, J. Vib. Control, 14 (2008), pp. 1291–1299.
[3] O. P. Agrawal, O. Defterli, and D. Baleanu, Fractional optimal control problems with several state and control variables, J. Vib. Control, 16 (2010), pp. 1967–1976.
[4] T. S. Angell, On the optimal control of systems governed by nonlinear integral equations, J. Optim. Theory Appl., 19 (1976), pp. 29-45.
A. A. M. Arafa, S. Z. Rida, and M. Khalil, *Solutions of fractional order model of childhood diseases with constant vaccination strategy*, Math. Sci. Lett., 1 (2012), pp. 17–23.

K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.

S. A. Belbas, *A new method for optimal control of Volterra integral equations*, Appl. Math. Comput., 189 (2007), pp. 1902–1915.

S. A. Belbas, *A reduction method for optimal control of Volterra integral equations*, Appl. Math. Comput., 197 (2008), pp. 880–890.

D. A. Benson, *The Fractional Advection-Dispersion Equation: Development and Application*, Ph.D. thesis, University of Nevada at Reno, Reno, NV, 1998.

V. I. Bogachev, *Measure Theory*, I, Springer, New York, 2007.

J. F. Bonnans, C. de la Vega, and X. Dupuis, *First- and second-order optimality conditions for optimal control problems of state constrained integral equations*, J. Optim. Theory Appl., 159 (2013), pp. 1–40.

L. Bourdin, *A Class of Fractional Optimal Control Problems and Fractional Pontryagin’s Systems. Existence of a Fractional Noether’s Theorem*, preprint, arXiv:1203.1422v1, 2012.

H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

C. Burnap and M. A. Kazemi, *Optimal control of a system governed by nonlinear Volterra integral equations with delay*, IMA J. Math. Control Inform., 16 (1999), pp. 73–89.

M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophys. J. Roy. Astron. Soc., 13 (1967), pp. 529–539.

M. Caputo and F. Mainardi, *A new dissipation model based on memory mechanism*, Pure Appl. Geophys., 91 (1971), pp. 134–147.

D. A. Carlson, *An elementary proof of the maximum principle for optimal control problems governed by a Volterra integral equation*, J. Optim. Theory Appl., 54 (1987), pp. 43–61.

J. T. Chern, *Finite Element Modeling of Viscoelastic Materials on the Theory of Fractional Calculus*, Ph.D. thesis, Pennsylvania State University, State College, PA, 1993.

S. Das and P. K. Gupta, *A mathematical model on fractional Lotka-Volterra equations*, J. Theoret Biol., 277 (2011), pp. 1–6.

C. de la Vega, *Necessary conditions for optimal terminal time control problems governed by a Volterra integral equation*, J. Optim. Theory Appl., 130 (2006), pp. 79–93.

E. Demirci, A. Unal, and N. Özlü, *A fractional order SEIR model with density dependent death rate*, Hacet. J. Math. Stat., 40 (2011), pp. 287–295.

K. Diethelm, *A fractional calculus based model for the simulation of an outbreak of dengue fever*, Nonlinear Dyn., 71 (2013), pp. 613–619.

K. Diethelm and A. D. Freed, *On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity*, in Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties, F. Keil, W. Mackens, H. Voß, and J. Werther, eds., Springer, Heidelberg, 1999, pp. 217–224.

G. S. F. Frederico and D. F. M. Torres, *Fractional conservation laws in optimal control theory*, Nonlinear Dyn., 53 (2008), pp. 215–222.
[25] M. I. Gomoyunov, **Dynamic programming principle and Hamilton-Jacobi-Bellman equations for fractional-order systems**, SIAM J. Control Optim., 58 (2020), pp. 3185–3211.

[26] I. G. Graham, S. Joe, and I. H. Sloan, **Iterated Galerkin versus iterated collocation for integral equations of the second kind**, IMA J. Numer. Anal., 5 (1985), pp. 355–369.

[27] M. M. Hasan, X. W. Tangpong, and O. P. Agrawal, **Fractional optimal control of distributed systems in spherical and cylindrical coordinates**, J. Vib. Control, 18 (2011), pp. 1506–1525.

[28] M. I. Kamien and E. Muller, **Optimal control with integral state equations**, Rev. Econ. Stud., 43 (1976), pp. 469–473.

[29] R. Kamocki, **On the existence of optimal solutions to fractional optimal control problems**, Appl. Math. Comput., 235 (2014), pp. 94–104.

[30] R. Kamocki, **Pontryagin maximum principle for fractional ordinary optimal control problems**, Math. Methods Appl. Sci., 37 (2014), pp. 1668–1686.

[31] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, **Theory and Applications of Fractional Differential Equations**, North-Holland, Amsterdam, 2006.

[32] A. Koenig, **Lack of null-controllability for the fractional heat equation and related equations**, SIAM J. Control Optim., 58 (2020), pp. 3130–3160.

[33] P. Lin and J. Yong, **Controlled singular Volterra integral equations and Pontryagin maximum principle**, SIAM J. Control Optim., 58 (2020), pp. 136–164.

[34] G. Long and G. Nelakanti, **Iteration methods for Fredholm integral equations of the second kind**, Comput. Math. Appl., 53 (2007), pp. 886–894.

[35] N. G. Medhin, **Optimal process governed by integral equations**, J. Math. Anal. Appl., 120 (1986), pp. 1–12.

[36] R. Metzler, W. Schick, H. G. Kilian, and T. F. Nonnenmacher, **Relaxation in filled polymers: a fractional calculus approach**, J. Chem. Phys., 103 (1995), pp. 7180–7186.

[37] E. Okyere, F. T. Oduro, S. K. Amponsah, I. K. Dontwi, and N. K. Frempong, **Fractional order SIR model with constant population**, British J. Math. Comput. Sci., 14 (2016), pp. 1–12.

[38] K. B. Oldham and J. Spanier, **The Fractional Calculus**, Academic Press, New York, 1974.

[39] L. Pandolfi, **The quadratic regulator problem and the Riccati equation for a process governed by a linear Volterra integrodifferential equations**, IEEE Trans. Automat. Control, 63 (2018), pp. 1517–1522.

[40] A. J. Pritchard and Y. You, **Causal feedback optimal control for Volterra integral equations**, SIAM J. Control Optim., 34 (1996), pp. 1874–1890.

[41] E. Scalas, R. Gorenflo, and F. Mainardi, **Uncoupled continuous-time random walks: analytic solution and limiting behaviour of the master equation**, Phys. Rev. E(3), 69 (2004), 011107.

[42] I. H. Sloan, **Four variants of the Galerkin method for integral equations of the second kind**, IMA J. Numer. Anal., 4 (1984), pp. 9–17.

[43] M. R. Spiegel, **Mathematical Handbook of Formulas and Tables**, Schaum’s Outline Series, McGraw-Hill, New York, 1969.

[44] P. J. Torvik and R. L. Bagley, **On the appearance of the fractional derivative in the behavior of real materials**, J. Appl. Mech., 51 (1984), pp. 294–298.

[45] V. R. Vinokurov, **Optimal control of processes described by integra equations, parts I, II, and III**, SIAM J. Control, 7 (1969), pp. 324–355; comments by L. W. Neustadt and J. Warga, SIAM J. Control, 8 (1970), 572.