Supergravity Vacua Today

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Abstract

We review the definition of (maximally supersymmetric) vacuum in supergravity theories, the currently known vacua in arbitrary dimensions and how the associated supersymmetry algebras can be found.

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Introduction

The vacuum is the most important state of any QFT. It can be defined as the state with lowest energy and maximal symmetry and it determines the kinematics of the theory due to the fundamental relation between symmetries, conserved quantities, quantum numbers and spectrum.

In theories of gravity, the vacuum also determines the zero point of the energy. Usually, it is associated to a classical solution of maximal symmetry which can be used as the arena on which other field theories can be defined. The basic example is Minkowski spacetime.

Some theories (in particular candidates to “Theory of Everything”) admit more than one vacuum and the problem of the vacuum selection (one of whose manifestations is the so-called “moduli problem”) is the most pressing and interesting one.

In this talk we are going to focus on vacua of supergravity (SUGRA) theories. These theories (we do not know if all of them) represent the low-energy effective limit of a Superstring Theory and, at the same time, can be considered as simply GR coupled to specific matter fields

\[
\text{GR + matter} \longrightarrow \text{SUGRA} \quad \text{[low energy]} \quad \text{Superstrings}
\]

SUGRA vacua are, to certain approximation, superstring vacua and provide new interesting GR solutions. In this talk, after a brief introduction to SUGRA theories (Section 1) we review the known vacua of SUGRA theories and explain how to find the associated supersymmetry algebras (Section 2). Finally, we review the very few known general results on supersymmetric solutions of SUGRA theories (Section 3).

1 SUGRA Theories

1.1 Supersymmetry

Supersymmetry (SUSY) is the ultimate symmetry allowed by the S matrix \( I \). It interchanges matter (fermions) with radiation (bosons, the carriers of interactions), implying a higher level of unification. In QFT, it interchanges bosonic fields \( B \) (tensors) with fermionic (anticommuting) fields \( F \) (spinors). This implies that it is generated by spinorial, anticommuting \( \text{supercharges} Q^\alpha \), and the infinitesimal parameters of the field transformations are anticommuting spinors \( \epsilon^\alpha \). By dimensional arguments \( [4] \), SUSY transformations
are always of the form
\[ \begin{align*}
\delta \epsilon B & \sim \bar{\epsilon} F, \\
\delta \epsilon F & \sim \partial \epsilon + B \epsilon.
\end{align*} \tag{1.1} \]

The simplest superalgebra includes the Poincaré algebra \( \{P_a, M_{ab}\} \) as bosonic (or even subalgebra and has the additional (anti-) commutators
\[ \{Q^\alpha, Q^\beta\} = i (\gamma^a C^{-1})^{\alpha\beta} P_a, \quad [Q^\alpha, M_{ab}] = \frac{1}{2} (\gamma_{ab})^{\alpha\beta} Q^\beta. \tag{1.2} \]

SUSY is, therefore a spacetime symmetry. In fact, it takes its simplest form as a symmetry of superspace.

### 1.2 From Supersymmetry to SUGRA

GR can be viewed\(^4\) as a gauge theory of the Poincaré group (see e.g. [3]). The gauge potential \( A_\mu \) has one component for each generator
\[ \{M_{ab}, P_a\} \longrightarrow A_\mu \equiv \frac{1}{2} \omega_\mu^{ab} M_{ab} + e_\mu^a P_a, \tag{1.3} \]
as well as the curvature
\[ R_{\mu\nu} \equiv 2 \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu] = \frac{1}{2} R_{\mu\nu}^{ab} M_{ab} + R_{\mu\nu}^a P_a. \tag{1.4} \]
The components \( R_{\mu\nu}^{ab} \) are the Lorentz curvature and \( R_{\mu\nu}^a \) to the torsion. The action is the first-order Einstein-Hilbert action
\[ S \sim \int d^4x \, e R(e, \omega), \tag{1.5} \]
and the equations of motion that follow from (1.5) are fully equivalent to Einstein’s vacuum equations
\[ R_{\mu\nu}^a = 0, \quad G_{\mu\nu} = 0, \tag{1.6} \]
although this formulation allows for the coupling of fermions to gravity (the Cartan-Sciama-Kibble (CSK) theory, see, e.g. [1, 4]).

Similarly, SUGRA can be seen as the gauge theory of the Poincaré superalgebra. The gauge potential has one more component \( \psi_\mu^{\alpha} \)
\[ \{M_{ab}, P_a, Q^\alpha\} \longrightarrow A_\mu \equiv \frac{1}{2} \omega_\mu^{ab} M_{ab} + e_\mu^a P_a + \bar{\psi}_\mu^{\alpha} Q^\alpha. \tag{1.7} \]

\(^4\)Strictly speaking it is not a gauge theory of the Poincaré group, but it can be constructed following the same steps, up to certain point.
that compensates for local SUSY transformations and is known as the Rarita-Schwinger field, that describes a massless spin-3/2 particle: the gravitino. Its two possible helicity states \((\pm \frac{3}{2})\) are the superpartners of the two helicity states of the graviton \((\pm 2)\). The QFT will have the same number of bosonic and fermionic states at each mass level, a property of linearly realized SUSY.

The curvature is

\[
R_{\mu \nu} = 2 \partial_{[\mu} A_{\nu]} + [A_{\mu}, A_{\nu}] = \frac{1}{2} R_{\mu \nu}{}^{a b} M_{a b} + R_{\mu \nu}{}^{a} P_{a} + \bar{R} \mu \nu \alpha Q^{\alpha}, \tag{1.8}
\]

and the action \((N = 1, d = 4 \text{ SUGRA})\) is just the CSK theory for a Rarita-Schwinger field couplet to gravity

\[
S \sim \int d^4 x \ e \left\{ R(e, \omega) + e^{\mu \nu \rho \sigma} \bar{\psi}^{\mu} \gamma_{5} \gamma_{\mu} D_{\rho} \omega(\omega) \psi_{\sigma} \right\}, \tag{1.9}
\]

but turns out to be invariant under local SUSY transformations

\[
\delta \epsilon^{a} \psi_{\mu} = -i \bar{\epsilon} \gamma^{a} \psi_{\mu}, \quad \delta \psi_{\mu} = D_{\mu} \epsilon. \tag{1.10}
\]

Now there is non-vanishing torsion, proportional to the fermions

\[
T_{\mu \nu}{}^{a} \sim \bar{\psi} \gamma^{a} \psi_{\nu}. \tag{1.11}
\]

Setting all the fermions to zero is always a consistent truncation and any purely bosonic GR solution will also be a solution of \(N = 1, d = 4 \text{ SUGRA}\).

Generalizing the superalgebra we gauge we can generalize the SUGRA theory\footnote{Of course, we can always add supersymmetric matter, but here we are not interested in this possibility.} There are three main ways (that can be combined) to generalize the Poincaré superalgebra:

1 Adding more supercharges

Adding more supercharges \(Q^{i \alpha}, i = 1 \ldots N\), one is left with \(N\)-extended \(d = 4\) Poincaré superalgebras. These turn out to admit central charges

\[
Q^{i j} = -Q^{j i}, \quad P^{i j} = -P^{j i}, \tag{1.12}
\]

that appear in the anticommutator of two supercharges

\[
\{Q^{i \alpha}, Q^{j \beta}\} = i \delta^{ij} (\gamma^{a} C^{-1})^{\alpha \beta} P_{a} - i (C^{-1})^{\alpha \beta} Q^{i j} - (\gamma_{5} C^{-1})^{\alpha \beta} P^{i j}, \tag{1.13}
\]

and commute with all generators. Gauging them we obtain \(N\)-extended Poincaré SUGRAS. The gauge potential contains \(N\) gravitini \(\psi_{\mu}{}^{i \alpha}\) and also \(N(N - 1)/2\) Abelian vector fields \(A_{i j}^{\mu}\)

\[
A_{\mu} \equiv \frac{1}{2} \bar{\omega}_{\mu}{}^{a b} M_{a b} + e_{\mu}{}^{a} P_{a} + \frac{1}{2} A_{i j}^{\mu} Q^{i j} + \bar{\psi}^{i \alpha} Q^{i \alpha}. \tag{1.14}
\]
The action now contains the kinetic terms of $N$ gravitini and of $N(N-1)/2$ Abelian vector fields $A^{ij \mu}$ with field strengths $F^{ij \mu \nu} = 2 \partial_{[\mu} A^{ij \nu]}$, but this is not the whole story, as the counting of bosonic and fermionic states immediately shows: there are additional scalar fields and fermionic fields in the theory that cannot be accounted for with our heuristic formulation\footnote{A more rigorous formulation can be found in \cite{6}.}. The scalars appear always in a non-linear $\sigma$-model, couple in a non-trivial fashion to the vector fields and have no potential.

$N = 8$ (in $d = 4$) is the maximum \footnote{These can be understood as deformations of the Poincaré algebra in which the generators of translations $P_a$ do not commute. There are other possible deformations, like the Heisenberg algebras, that but, so far, no SUGRA has been constructed gauging them. Further, we could think of taking the product of some spacetime group and other compact group that would play the role of internal symmetries, but the corresponding SUGRAs arise naturally from the $N$-extended $AdS$ SUGRAs we are going to consider.} if we do not want to deal with the problem of higher spin fields in interaction or more than one graviton.

2 Using a different spacetime bosonic superalgebra

The natural candidates are those superalgebras with a meaningful bosonic subalgebra\footnote{A more rigorous formulation can be found in \cite{6}.}: $dS$ or $AdS$. $dS$ leads to inconsistent field theories and thus, we are left with $N$-extended $d = 4$, $AdS$ superalgebras. The supercharges $Q^{i \alpha}$ transform as spinors under the bosonic ($AdS$) subalgebra generated by the $\hat{M}_{\hat{a}}$’s. On top of these, we are forced to introduce bosonic $SO(N)$ generators $T^{ij} = -T^{ij}$ that rotate the supercharges and also appear in their anticommutator:

\[
\{ Q^{i \alpha} , Q^{j \beta} \} = i \delta^{ij} m^{\alpha \beta} \hat{M}_{\hat{a}} - i (C^{-1})^{\alpha \beta} T^{ij} ,
\]

\[
[T^{ij} , T^{kl}] = \delta^{ik} T^{jl} + \delta^{jl} T^{ik} - \delta^{il} T^{jk} - \delta^{jk} T^{il} ,
\]

\[
[Q^{i \alpha} , T^{jk} ] = 2 \delta^{ij} Q^{k | \alpha} ,
\]

(1.15)

Accordingly we have to introduce $SO(N)$ an vector field with $N(N-1)/2$ components $A^{ij \mu}$ and $N$ $SO(N)$-charged gravitini $\psi^{\mu \alpha}$

\[
A_\mu \equiv \frac{1}{2} \hat{\omega}_{\mu}^{\hat{a}} \hat{M}_{\hat{a}} + \frac{1}{2} A^{ij \mu} T^{ij} + \bar{\psi}^{i \mu \alpha} Q^{i \alpha} .
\]

(1.16)

The resulting theory is known as a “gauged SUGRA” and for $N = 1, 2$ the Lagrangian has a negative cosmological constant whose value is related to the gauge coupling constant. For $N > 2$ there also scalars present and there is a potential which has an extremum at a negative value and acts as a negative cosmological constant.
3 Using $d > 4$ spacetime bosonic superalgebras

In the Poincaré case this is straightforward but only up to $d = 11$, which is allowed only for $N = 1$. Beyond $d = 11$ we run into the same problems we found in going beyond $N = 8$ in $d = 4$ \[7\]. These limits are related since $N = 8, d = 4$ SUGRAs can be derived from $N = 1, d = 11$ SUGRA \[3\] by dimensional reduction. AdS superalgebras only exist up to $d = 7$ \[7\].

The main feature of higher-dimensional superalgebras is that they admit quasi-central charges $Z_{[a_1...a_p]}$ that commute with the supercharges but transform as $p$-forms under Lorentz transformations. An important example is the $N = 1, d = 11$ Poincaré superalgebra (a.k.a. “M-superalgebra”, see \[10\] for a discussion of the great amount of information that it contains), which admits two quasi-central charges $p = 2, 5$

\[
\{ Q^\alpha, Q^\beta \} = i (\gamma^a C^{-1})^{\alpha\beta} P_a + \frac{1}{2} (\gamma^{ab} C^{-1})^{\alpha\beta} Z_{ab} + \frac{i}{5!} (\gamma^{a_1...a_5} C^{-1})^{\alpha\beta} Z_{a_1...a_5}.
\]

(1.17)

The gauge potential must include now a potential $C_{\mu}^{ab}$

\[
A_\mu \equiv \frac{1}{2} \omega_\mu^{ab} M_{ab} + e_\mu^a P_a + \frac{1}{2} C_\mu^{ab} Z_{ab} + \bar{\psi}_\mu Q^\alpha,
\]

(1.18)

which actually appears in the SUGRA action as a 3-form potential $C_{\mu\nu\rho}$ with field strength $G_{\mu\nu\rho\sigma} = 4 \partial_\mu C_{\nu\rho\sigma}$.

$(p + 1)$-form potentials naturally couple to the worldvolume of extended objects of $p$ spatial dimensions ("$p$-branes"). Higher-dimensional SUGRAs, are, therefore, theories associated to $p$-branes and, indeed, one finds classical solutions that include the $(p + 1)$-dimensional Poincaré group in their isometry group and represent the long-range fields sourced by a flat $p$-brane. $N = 1, d = 11$ SUGRA can couple to a 2-brane and, through the dual 6-form potential, to a 5-brane. The quasi-central charges that appear in the superalgebra correspond to these objects and there are classical solutions associated to them. One of the main properties of these solutions is that they are supersymmetric.

2 From SUGRA Back to SUSY

In the previous section we have seen heuristically how to construct SUGRA theories based on a superalgebra. Most SUGRAs (actions and transformation laws), though, have not been constructed in this way and the question arises as to what superalgebra can be associated to a given SUGRA.

\[8\]A very complete guide to the literature on SUGRAs in diverse dimensions is \[9\].

\[9\]for a review on $p$-branes see, for example \[11\].
Actually, it is the vacuum of the theory what can be connected to a superalgebra and it so happens that many SUGRAs admit several vacua. Here we are going to show how to find the superalgebra associated to a given SUGRA vacuum. We need first a definition of vacuum: in GR, the vacuum is the maximally symmetric solution (i.e. Minkowski, $AdS$ or $dS$). In SUGRA we can define it as a maximally supersymmetric solution.

2.1 Supersymmetric Solutions

GR is invariant under any diffeomorphism but any given solution is only invariant under a finite isometry group generated by its Killing vectors $k_{(I)}$, that satisfy a Lie algebra

$$[k_{(I)}, k_{(J)}] = f_{IJ}^K k_{(K)}.$$  \hfill (2.1)

We can call symmetric any solution with a non-trivial isometry group.

A SUGRA theory is invariant under any local supersymmetry transformation, but a given solution will only be invariant under some. By analogy, we call a SUGRA solution supersymmetric if there exists a (Killing) spinor $\kappa$ such that

$$\delta_{\kappa} = 0 \implies \begin{cases} \delta_{\kappa} B = \bar{\kappa} F = 0, \\ \delta_{\kappa} F = \partial \kappa + B \kappa = 0. \end{cases} \hfill (2.2)$$

A supersymmetric solution will always be symmetric (see later). Its symmetries and supersymmetries must form a supergroup, and, the infinitesimal generators must form a superalgebra.

How can this superalgebra be found? The answer to this question was given in [12]:

\textbf{RECIPE}

1. First we associate each Killing spinor (vector) to an odd (even) element of the superalgebra

$$\kappa_{(A)}^a \rightarrow Q_{(A)} \quad \text{(Odd)}$$

$$k_{(I)}^\mu \rightarrow P_{(I)} \quad \text{(Even)}$$

\begin{equation}
\begin{cases} 
\text{SUPERALGEBRA} \\
\text{GENERATORS}
\end{cases}
\end{equation} \hfill (2.3)

The superalgebra is determined by the three sets of structure constants $f_{IJ}^K, f_{AB}^I, f_{AB}^B$

2. The structure constants $f_{IJ}^K$ of the even subalgebra are those of the isometry Lie algebra

$$[k_{(I)}, k_{(J)}] = f_{IJ}^K k_{(K)}.$$ \hfill (2.4)
3. The structure constants $f_{AB}^I$ are given by the decomposition of the bilinears

$$-i\bar{\kappa}_A\gamma^\mu\kappa_B\epsilon_a \equiv f_{AB}^Ik_{(I)}.$$

(2.5)

4. The structure constants $f_{AI}B$ are given by the spinorial Lie derivatives

$$\mathbb{L}_{k_{(I)}}\kappa_A \equiv f_{AI}B\kappa_{(B)}.$$

(2.6)

Some of these rules deserve a comment:

- **Rule 3**: If $\kappa_1$ and $\kappa_2$ are Killing spinors then one can show that the bilinear $-i\bar{\kappa}_1\gamma^\mu\kappa_2$ is a Killing vector, and, therefore, a linear combination of the $k_{(I)}^\mu$'s. This ensures that the $f_{AB}^I$'s are well-defined constants.

- **Rule 4**: The spinorial Lie derivative (see [13, 14]) is a particular case of a generalized $G$-reductive Lie derivative (see, e.g. [15]). In pedestrian/physicist terms it is just a gauge-covariant Lie derivative

$$\mathbb{L}_v = \mathcal{L}_v + W(v),$$

(2.7)

where $\mathcal{L}_v$ is the standard Lie derivative and $W(v)$ is a gauge compensator. Spinors are defined up to local (gauge) Lorentz transformations and

$$\mathcal{L}_v\psi \equiv v^\mu\nabla_\mu\psi + \frac{1}{4}\nabla_{[\mu}v_{\nu]}\gamma^{\mu\nu}\psi.$$

(2.8)

Charged spinors are also defined up to local phases and more terms have to be added to their Lie derivative. One of its main properties is that, taken w.r.t. Killing vectors, it transforms Killing spinors into Killing spinors. This ensures that the $f_{AI}B$'s are also well-defined constants.

### 2.2 Vacua and Their Superalgebras

Let us now review the known maximally supersymmetric solutions of the different SUGRAs. Applying the above recipe we can derive their symmetry superalgebras.

#### 2.2.1 Poincaré SUGRAs

Minkowski spacetime is always a solution, maximally symmetric and supersymmetric. Its symmetry superalgebra is (not surprisingly) the Poincaré superalgebra:

1. Minkowski’s Killing vectors in Cartesian coordinates are

\[ k_{(a)} = \partial_a \rightarrow P_a , \]
\[ k_{ab} = 2 x_{[a} \partial_{b]} \rightarrow M_{ab}. \]  

(2.9)

The Killing spinor equation in Cartesian coordinates with \( e^\mu_a = \delta^\mu_a \) is

\[ \partial_a \kappa = 0 , \]  

(2.10)

and has four independent solutions

\[ \kappa_{(a)}^\beta = \delta_{(a)}^\beta \rightarrow Q_{(a)} . \]  

(2.11)

2. \( \{ P_a , M_{ab} \} \) satisfy the Poincaré algebra.

3. The Killing spinor bilinears give, trivially

\[ -i \bar{\kappa}_{(a)}^\gamma \kappa_{(\beta)} \partial_a = -i (C \gamma^a)_{\alpha\beta} k_{(a)} \Rightarrow \{ Q_{(a)} , Q_{(\beta)} \} = -i (C \gamma^a)_{\alpha\beta} P_{(a)} . \]  

(2.12)

4. The translational Killing vectors are covariantly constant and they annihilate the Killing spinors

\[ \nabla_{\mu} k_{(a)}^\nu = 0 \Rightarrow \mathbb{L}_{k_{(a)}} \kappa_{(a)}^\beta = 0 \Rightarrow [Q_{(a)} , P_{(a)}] = 0 . \]  

(2.13)

The rotational Killing vectors have non-trivial covariant derivatives and act as Lorentz rotations on the Killing spinors

\[ \nabla_{[\mu} k_{(ab)]^\nu} = 2 e_{[a|\mu} e_{b|\nu]} \Rightarrow \mathbb{L}_{k_{(ab)}} \kappa_{(a)}^\beta = \frac{1}{2} (\gamma_{ab})^\beta_\gamma \kappa_{(a)}^\gamma , \]
\[ \Rightarrow [Q_{(a)} , M_{ab}] = Q_{(\beta)} \frac{1}{2} (\gamma_{ab})^\beta_\alpha . \]  

(2.14)

We could raise now the spinorial indices using the inverse charge conjugation matrix to have the algebra in the form Eq. (1.2).

### 2.2.2 Anti-de Sitter Supergravities

\( AdS \) is always a maximally symmetric and supersymmetric solution of all gauged supergravities. Its symmetry superalgebra can be derived most easily using a construction that exploits the fact that it is a homogeneous space that we are going to review in Section 2.3.
2.2.3 Extended Poincaré Supergravities

Minkowski is always a maximally supersymmetric solution of all extended Poincaré SUGRAS, but there are additional maximally supersymmetric (but not maximally symmetric) solutions of essentially two types: products of $AdS$ spaces and spheres that appear as near-horizon limits and homogenous $pp$-wave spacetimes [10, 17] of the type found by Kowalski-Glikman [18, 19] (KG solutions) and which are related by a Penrose limit [20, 21, 22, 23]. A complete table follows:

\[
N = 1, \ d = 11 : \quad \{ \begin{array}{l}
AdS_7 \times S^4 \\
AdS_4 \times S^7
\end{array} \} \xrightarrow{\text{Penrose limit}} \quad KG11
\]

\[
N = 2B, \ d = 10 : \quad \{ AdS_5 \times S^5 \} \xrightarrow{\text{Penrose limit}} \quad KG10
\]

\[
N = \left( \begin{array}{c}
2, 0 \\
4, 0
\end{array} \right), \ d = 6 : \quad \{ AdS_3 \times S^3 \} \xrightarrow{\text{Penrose limit}} \quad KG6
\]

\[
N = 2, \ d = 5 : \quad \{ \begin{array}{l}
AdS_3 \times S^2 \\
AdS_2 \times S^3 \\
AdS_2 \ast S^2 \quad \text{Gödel}
\end{array} \} \xrightarrow{\text{Penrose limit}} \quad KG5
\]

\[
N = 2, \ d = 4 : \quad \{ AdS_2 \times S^2 \} \xrightarrow{\text{Penrose limit}} \quad KG4
\]

The only two exceptional cases occur in $d = 5$: the $AdS_2 \ast S^2$ solution which is the near-horizon limit of the rotating extreme $d = 5$ black hole [31, 32, 33] and a Gödel-like solution [28].

2.3 The Symmetry Superalgebras of Homogeneous Spacetimes

All known maximally supersymmetric SUGRA vacua are homogeneous spaces. The less well known cases are

\[
AdS_2 \ast S^2 \sim \frac{[SO(2,1) \times SO(3)]}{SO(2)} \quad \text{[34],}
\]

\[
H_{pp} \sim \frac{H(d-2)}{l_{d-2}} \quad \text{[10]},
\]

\[
\text{Gödel}_5 \sim \frac{H(2n+2)}{U(1)}
\]

(2.15)
where $H(2n + 2)$ stands for the Heisenberg algebra. This makes easy to find the Killing spinors and supersymmetry algebras. It can be shown \[35\] that

1. The Killing spinors are (in an appropriate basis!)

$$\kappa_\alpha^\beta = u_\beta^\alpha, \quad u = e^{x^a \Gamma_s(P(a))},$$

where $\Gamma_s(P(a))$ stands for the generators in the spinorial representation.

2. In most cases, the bilinears can be decomposed easily

$$-i\bar{\kappa}_\alpha^\gamma \kappa_\beta^\beta \epsilon_a = -i(\bar{\epsilon} \Gamma_s(T(I)))_{\alpha \beta} \kappa_\alpha^\gamma , \quad \Rightarrow f_{\alpha \beta}^I = -i(\bar{\epsilon} \Gamma_s(T(I)))_{\alpha \beta} .$$

3. The spinorial Lie derivatives w.r.t. the Killing vectors are given by

$$\mathbb{L}_{k(I)} \kappa_\alpha^\beta = \kappa_\gamma^\gamma \Gamma_s(T(I))^\gamma_{\alpha} , \quad \Rightarrow f_{\alpha I}^\gamma = \Gamma_s(T(I))^\gamma_{\alpha} .$$

3 Some General Results on Supersymmetric Solutions

To end, let us review the few general results that exist on (not necessarily maximally) supersymmetric solutions.

3.1 $N = 1, d = 4$ Poincaré Supergravity

The Killing spinor equation is

$$\nabla_\mu \kappa = 0 \Rightarrow -\frac{1}{4} R_{\mu \nu}^{ab} \gamma_{ab} \kappa = 0 .$$

Only two kinds of solutions known: Minkowski spacetime (maximally supersymmetric) and $pp$-waves spacetimes admitting a covariantly constant null Killing vector $\ell^\mu$ , whose Killing spinors satisfy

$$\ell_\mu \gamma^\mu \kappa = 0 ,$$

that has a 2-dimensional space of solutions (1/2 out of the 4 possible).

In Euclidean signature also the Gibbons-Hawking metrics \[36\] used to construct self-dual gravitational instantons preserve 1/2 of the supersymmetries

$$\left\{ \begin{array}{l}
    ds^2 = H^{-1}(d\tau + A_i dx^i)^2 + H dx^i dx^i , \\
    \epsilon_{ijk} \partial_j A_k = \partial_i H \Rightarrow \partial_i \partial_i H = 0 .
\end{array} \right.$$
3.2 \( N = 2, d = 4 \) Poincaré Supergravity

This theory is just Einstein-Maxwell coupled to \( \psi^i_\mu \) and all solutions of Einstein-Maxwell are solutions of \( N = 2, d = 4 \) Poincaré supergravity as well. The Killing spinor equation is

\[
[\delta^i_j \nabla_\mu + \frac{1}{4} F(\sigma^2)^{ij}] \kappa^j = 0 \Rightarrow \left\{ C^{ab}_{\mu\nu} \gamma_{ab} + 2i \gamma (F_{\mu\nu} + i^* F_{\mu\nu} \gamma_5) i \sigma^2 \right\} \kappa = 0.
\]  

(3.4)

These integrability conditions were fully solved by Tod [37], who found two kind of solutions that preserve generically one half of the supersymmetries:

- **Israel-Wilson-Perjés solutions**
  \[
  \begin{cases}
    ds^2 = |H|^{-2} (dt + \omega)^2 - |H|^2 dx^i dx^i, \\
    A_t = 2 \text{Re} \ H, \quad A_\bar{t} = -2 \text{Re} (i H), \\
    \omega = \omega_i dx^i, \quad \epsilon_{ijk} \partial_j \omega_k = \pm \text{Im} (\bar{H} \partial_k H), \Rightarrow \partial_i \partial_i H = 0.
  \end{cases}
  \]  

(3.5)

These include the extreme RN \( (M^2 = Q^2) \) black hole (whose near-horizon geometry is the maximally supersymmetric Bertotti-Robinson spacetime, \( AdS_2 \times S^2 \)), the extreme Taub-NUT \( (M^2 + N^2 = Q^2) \), KN with \( M^2 = Q^2 \) and multicenter solutions [38, 39].

- **Gravito-electromagnetic pp-waves**
  \[
  \begin{cases}
    ds^2 = 2du (dv + K du) - 2d\xi d\bar{\xi}, \\
    F_{\xi u} = \partial_\xi C, \quad K = \text{Re} f + \frac{1}{4} |C|^2, \\
    \partial_\bar{\xi} f = \partial_\xi C = 0.
  \end{cases}
  \]  

(3.6)

The \( Hpp \)-waves are a particular (maximally supersymmetric) case with \( K = A_{ij} x^i x^j \). \( KG4 \) is a particular case with \( A_{ij} \sim \delta_{ij} \).

Kowalski-Glikman proved that Minkowski, Bertotti-Robinson and the \( KG4 \) solutions are the only vacua of \( N = 2 \) supergravity [18].

3.3 \( N = 4, d = 4 \) Poincaré Supergravity

The theory consists of the metric \( g_{\mu\nu} \), six vectors \( A^{\mu i} \), a complex scalar field \( \tau \) (also known as axidilaton), four gravitini \( \psi^i_\mu \) and four dilatini \( \chi^i \).

The most general families of supersymmetric solutions were obtained by Tod [40], who identified several families:

- **SUPER-Israel-Wilson-Perjés solutions** [11] Include all the IWP spacetimes of \( N = 2 d = 4 \), but now none of them seems to be maximally supersymmetric.
• Waves (pp and more) Again, none of them seems to be maximally
supersymmetric.

3.4 \( N = 2, d = 5 \) Poincaré SUGRA

This theory is an interesting modification of the 5-dimensional Einstein-
Maxwell theory that leads to very interesting new solutions like the rotating
black string of \[42\]. The action is

\[
S \sim \int d^4x \ e \left\{ R(e, \omega) - \frac{1}{4} F^2 - \frac{1}{\sqrt{|g|}} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} A_8 \right\}, \quad (3.7)
\]

where the new Chern-Simons term changes the Maxwell equation.Recently
it was shown in \[28\] how to construct all the supersymmetric solutions of
this theory, although not all of them can be written explicitly. This method
exploits the identities satisfied by the Killing spinor bilinears \(-i \bar{\kappa} \gamma^a \kappa\). The
most interesting of the new results is the presence of a G\ödel-like maximally
supersymmetric solution.

these are all the general results known. In higher dimensions and higher
\(N\) many supersymmetric solutions are known but no general classification
scheme exists. This seems a very promising direction of research, worth
pursuing.

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