Optimal harvesting and stability of predator prey model with Monod-Haldane predation response function and stage structure for predator

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Abstract. This article deals with growth rate predator prey population model with stage structure for predator. The predator population is divided into two stages, mature and immature predator. Predation functional response in the model follows the Monod-Haldane type. Under consideration that the population is a valuable stock, the mature predator and prey population are then harvested with constant effort. This model is then analysed by found the equilibrium point and stabilities. There exists four equilibrium points and stability analyses is focused only for interior point. The stable interior point is related to the maximum profit problem. From the analyses we found that there exists a condition for the effort of harvesting so that the interior point is still stable and also obtained maximum profit.

Keywords: Predator prey model, Monod-Haldane function, stage structure, optimal harvesting

1. Introduction

In study of population dynamics, the growth rate of predator prey is one model that widely studied by some researchers. When there are two or more populations living in the same environment, there will be interaction between the populations and the interaction that occur most often is predation, that is, one population as a predator and the other as a prey. In the case of frequencies of interaction between prey and predator very often, the prey population will be extinct. One strategy to protect the prey from extinction due to predation is by controlling the growth rate of prey and its predator, [1].

The dynamics of predator prey models have been studied and developed by many researchers. There are researchers who consider more than two populations in the model, some researchers consider economic aspects and other aspects in the model. Prey as well as predator can be considered into two stages, immature and mature populations. Regarding to the stage structured for predator prey model. Research on predator prey models with stage structure was also studied in [2] with stage structure in prey population. Study the predator prey model with stage structure in [3] was found that the coexistence of both populations are strongly influenced by the population on the stage structure. Study of
population dynamics of predator prey with stage structure and some extensions can be found, for example in [4, 5, 6].

In ecological modeling, the population as a useful stock is then exploited. The predator prey model with harvesting also become a special topic. The predator prey model with harvesting has been studied in [7] and model with stage structure and harvesting has been studied in [8]. In this article, we develop a model proposed in [3] by incorporating selective harvesting factor to the prey and mature predator population. We analyze the existence of the interior equilibrium point and its stability. The stable equilibrium point is then linked to how maximizing the profit from exploitation of the populations.

2. Predator prey model with harvesting and stage structure

A predator prey model with stage structure for predator in [3] has been studied. The model follows predation functions with Monod-Haldane type. In this model, mature predator population is assumed has interaction with prey population. Whereas for immature predator has not interaction directly with the prey. The immature predator has not reproductive abilities and it interacts on the interactions of mature predator and prey population. The model is

\[
\begin{align*}
\frac{dx}{dt} &= \rho_1 x \left(1 - \frac{x}{k_1}\right) - \frac{\beta x z}{\phi + \tau x^2} \\
\frac{dy}{dt} &= \frac{c \beta x z}{\phi + \tau x^2} - (\alpha + \delta_1)y \\
\frac{dz}{dt} &= \alpha y - \delta_2 z.
\end{align*}
\]

(1)

The symbols \(\frac{dx}{dt}, \frac{dy}{dt}, \) and \(\frac{dz}{dt}\) define the growth rate of prey, immature predator, and mature predator respectively. Constant \(\rho_1\) is the intrinsic rate of prey population and \(k_1\) is carrying capacity for the prey when there is no predator population. A simplification functional response of Monod Haldane is \(\frac{\beta x z}{\phi + \tau x^2}\), [9]. The symbols \(\beta\) and \(c\) state the rate of predation and efficiency of predation for growth rate of immature predator. The symbol \(\alpha\) is the rate from immature predator becomes mature predator. The symbols \(\delta_1\) and \(\delta_2\) denote the rate mortality for immature and mature predator respectively.

Under consideration that the populations in the model are beneficial, then harvesting factor is incorporated in the growth rate of the population. We improve model (1) by considering selective harvesting in prey and mature predator populations. The model becomes

\[
\begin{align*}
\frac{dx}{dt} &= \rho_1 x \left(1 - \frac{x}{k_1}\right) - \frac{\beta x z}{\phi + \tau x^2} - q_1 E_1 x \\
\frac{dy}{dt} &= \frac{c \beta x z}{\phi + \tau x^2} - (\alpha + \delta_1)y \\
\frac{dz}{dt} &= \alpha y - \delta_2 z - q_2 E_2 z.
\end{align*}
\]

(2)

The symbols \(q_1\) and \(q_2\) denote catchability coefficient for prey and mature predator population respectively. The symbols \(E_1\) and \(E_2\) denote constant efforts of harvesting. There are four suitable equilibrium points to be analysed, namely alalah \(T_0 = (0, 0, 0)\), \(T_1 = (K, 0, 0)\), \(T_2 = (x_*, y_*, z_*)\), and \(T_3 = (x^*, y^*, z^*)\),

where \(K = \frac{k_1(\rho_1 - q_1 E_1)}{\rho_1}, \frac{(ac\beta) + \sqrt{A}}{(\alpha + \delta_1)}\),

\[
x_* = \frac{2 (\alpha + \delta_1)(q_2 E_2 + \delta_2)\zeta'}{(\alpha + \delta_1)}
\]

\[
y_* = \frac{c(\rho_1 - q_1 E_1)}{(\alpha + \delta_1)} \left(\frac{(ac\beta) + \sqrt{A}}{2 (\alpha + \delta_1)(q_2 E_2 + \delta_2)\zeta'}\right)
\]
\[
\rho_1 c \alpha \frac{(\alpha + \delta_1)(q_2 E_2 + \delta_2)}{((\alpha + \delta_1)(q_2 E_2 + \delta_2)) k_1 \left(2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\right)} \frac{(ac\beta) + \sqrt{A}}{2(\alpha + \delta_1)(q_2 E_2 + \delta_2) \tau},
\]
\[
z_3 = -\frac{\rho_1 c \alpha}{((\alpha + \delta_1)(q_2 E_2 + \delta_2)) k_1 \left(2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\right)} \frac{(ac\beta) + \sqrt{A}}{2(\alpha + \delta_1)(q_2 E_2 + \delta_2) \tau}^2,
\]
\[
x^* = \frac{2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\tau}{c(\rho_1 - q_1 E_1) \left(\frac{2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\tau}{\frac{\alpha \beta}{\phi + \tau x^*} + 2 \frac{\beta}{\phi + \tau x^*}}\right) - \frac{\rho_1 c \alpha}{((\alpha + \delta_1)(q_2 E_2 + \delta_2)) k_1 \left(2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\right)} \frac{(ac\beta) + \sqrt{A}}{2(\alpha + \delta_1)(q_2 E_2 + \delta_2) \tau}^2},
\]
\[
y^* = -\frac{\rho_1 c \alpha}{((\alpha + \delta_1)(q_2 E_2 + \delta_2)) k_1 \left(2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\right)} \frac{(ac\beta) + \sqrt{A}}{2(\alpha + \delta_1)(q_2 E_2 + \delta_2) \tau}^2,
\]
\[
z^* = -\frac{\rho_1 c \alpha}{((\alpha + \delta_1)(q_2 E_2 + \delta_2)) k_1 \left(2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\right)} \frac{(ac\beta) - \sqrt{A}}{2(\alpha + \delta_1)(q_2 E_2 + \delta_2) \tau}^2,
\]
and
\[
A = (ac\beta)^2 - 4(\alpha + \delta_1)(q_2 E_2 + \delta_2)\tau \phi (\alpha + \delta_1)(q_2 E_2 + \delta_2).
\]

There is only one possible positive equilibrium point for the model (2), as well as for model (1). Here, only the positive equilibrium point will be analysed. The possible positive equilibrium point and also stable for model (2) is written as \( T_3 = (x^*, y^*, z^*) \). The Jacobian matrix for model evaluated at equilibrium point \( T_3 \) is given by

\[
J = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix},
\]
where
\[
J_{11} = \rho_1 - 2 \frac{\rho_1 x^*}{k_1} - \frac{\beta x^*}{\phi + \tau x^*} + 2 \frac{\beta x^*}{\phi + \tau x^*}, \quad J_{12} = -\frac{\rho_1 c \alpha}{((\alpha + \delta_1)(q_2 E_2 + \delta_2)) k_1 \left(2(\alpha + \delta_1)(q_2 E_2 + \delta_2)\right)} \frac{(ac\beta) + \sqrt{A}}{2(\alpha + \delta_1)(q_2 E_2 + \delta_2) \tau}^2,
\]
\[
J_{13} = -\frac{\beta x^*}{\phi + \tau x^*}, \quad J_{21} = c \frac{\beta x^*}{\phi + \tau x^*} - 2 \frac{\beta x^*}{\phi + \tau x^*}, \quad J_{22} = -\alpha - \delta, \quad J_{23} = \frac{\beta x^*}{\phi + \tau x^*}, \quad J_{32} = \alpha, \quad J_{33} = -q_2 E_2 - \delta._n
\]

The characteristic equation associated with the Jacobian matrix \( J \) is given by
\[
f(\lambda) = \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3,
\]
where
\[
A_1 = -(J_{11} + J_{22} + J_{33}), \quad A_2 = J_{11} J_{22} + J_{11} J_{33} + J_{22} J_{33} - J_{23} J_{32}, \quad A_3 = J_{11} J_{23} J_{32} - J_{11} J_{22} J_{33} - J_{13} J_{23} J_{32} J_{21}.
\]
According to Hurwitz stability test [10], the equilibrium point \( T_3 \) is asymptotically stable when the conditions \( A_1 > 0, A_2 > 0, A_3 > 0, \) and \( A_1 A_2 > A_3 \) are satisfied.

3. Maximum profit in population harvesting

The stable equilibrium point of the model is then related to maximizing the profit for exploitation of the populations. We suppose that the prey and mature predator population are harvested with constant effort. From the exploitation we consider revenue and cost function. We define total revenue as \( TR = p_1 x^* E_1 + p_2 z^* E_2 \) and total cost \( TC = c_1 E_1 + c_2 E_2 \). The Constants, while the constants \( c_1 \) and \( c_2 \) denote unit cost of harvesting for prey and mature predator respectively.
The equilibrium point $T_3$ becomes a positive equilibrium point when $(E_1, E_2) \in D$, where
\[ D = \{(E_1, E_2) | 0 \leq E_1 \leq E_{1\text{max}}, 0 \leq E_2 \leq E_{2\text{max}}\} \] for some value of $E_{1\text{max}}$ and $E_{2\text{max}}$.

The profit function for harvesting of prey and mature predator at equilibrium point from the exploitation at the equilibrium point $T_3$ is given by
\[
\pi(E_1, E_2) = p_1 x^* E_1 + p_2 z^* E_2 - c_1 E_1 - c_2 E_2.
\]

We need to maximize the profit function (3) in the feasible region of $D$ for some values of $E_{1\text{max}}$ and $E_{2\text{max}}$. The critical values of the efforts $E_1$ and $E_2$ are determined by considering the first partial derivatives and the critical values of the efforts at domain $D$.

The profit function the equilibrium point $T_3$ is
\[
\pi(E_1, E_2) = p_1 x^* E_1 + p_2 z^* E_2 - c_1 E_1 - c_2 E_2
\]
where $e_1 = a t^2 k_1 + t^2 \delta_1 k_1$, $e_2 = 2 a s t k_1 + \alpha t^2 \delta_2 k_1 + 2 s t \delta_1 k_1 + t^2 \delta_1 \delta_2 k_1$, $e_3 = 2 a^2 c^2 \beta t k_1$, $a_2 = 2 a^2 c^2 \delta s k_1$, $a_3 = 2 a^2 c^2 \beta s k_1$, $a_4 = 2 a^2 c^2 \beta c s k_1$, $a_5 = 2 a^2 c^2 \beta \delta s k_1$, $a_6 = 2 a^2 c^2 \beta s k_1$, $a_7 = 2 a^2 c^2 \beta s k_1$, $a_8 = \alpha a^2 c^2 \beta c s k_1$, $a_9 = \alpha a^2 c^2 \beta c s k_1$, $a_{10} = \alpha a^2 c^2 \beta c s k_1$, $a_{11} = 2 a^2 c^2 \beta c s k_1$, $a_{12} = \alpha a^2 c^2 \beta c s k_1$.

4. Numerical simulations
For simulation on model (1) without harvesting, we take the parameters $\rho_1 = 1.5, k_1 = 100, \phi = 8, \beta = 0.16, c = 0.4, \alpha = 0.02, \tau = 0.001, \delta_1 = 0.01, \delta_2 = 0.2, E_1 = 0$ dan $E_2 = 0$ in appropriate units. We have positive equilibrium point $T_3 = (48.5480, 499.5783, 499.5978)$. The characteristic equation associated with Jacobian matrix evaluated at the equilibrium point is $f(\lambda) = \lambda^3 + 0.60696 \lambda^2 + 0.86699 \lambda + 0.25231$. From the characteristic equation, it is easy to check that the Routh-Hurwitz criterion follows and the eigen values are $\lambda_1 = -0.41089$, $\lambda_2 = -0.39129$, and $\lambda_3 = -0.15692$. All of eigen values are real and negative, then the equilibrium point $T_3$ is locally asymptotically stable. This means the prey and mature predator will sustain for a long period of time.

For simulation on model (2), we use the values of parameter $\rho_1 = 1.5, k_1 = 100, \phi = 8, \beta = 0.16, c = 0.4, \alpha = 0.02, \tau = 0.001, \delta_1 = 0.01, \delta_2 = 0.2, q_1 = 1, q_2 = 1$ in appropriate units. We get an interior equilibrium point $T_3 = (x^*, y^*, z^*)$, where
\[
\begin{align*}
x^* & = \frac{0.000128 - \sqrt{A_1}}{2} \\
y^* & = \frac{6.896551725 (E_1 + 1.5)(0.00128 - \sqrt{A_1})}{0.000029 E_2 + 0.0000058}
\end{align*}
\]
\begin{align*}
z^* &= \frac{0.001034482758 \left(0.00128 - \sqrt{A_1}\right)^2}{(E_2 + 0.2)(0.000029 E_2 + 0.0000058)} \\
&\quad - \frac{0.1379310345 (-E_1 + 1.5) \left(0.00128 - \sqrt{A_1}\right)^2}{(E_2 + 0.2)(0.000029 E_2 + 0.0000058)} \\
&\quad - \frac{0.001034482758 \left(0.00128 - \sqrt{A_1}\right)^2}{(E_2 + 0.2)(0.000029 E_2 + 0.0000058)},
\end{align*}

where \(A_1 = -0.000026912 E_2^2 - 0.0000107648 E_2 + 0.00000056192\).

From equation (3) we get

\[\pi(E_1, E_2) = \frac{50 E_1 \left(0.00128 - \sqrt{A_2}\right)}{0.00003 E_2 + 0.000006} - \frac{25 E_2}{0.0000000003 E_1^2 + 0.00000000016 E_2^2 + 0.000000000032 E_2 + 0.000000000022}
\]

\[\cdot (0.0000614 E_1 E_2 + 0.0000012288 E_1 - 0.00000023 E_2 + 0.0000000706)
\]
\[- 0.000048 E_1 E_2 \sqrt{A_2}
\]
\[- 0.00002112 \sqrt{A_2}
\]
\[- 0.0000096 E_1 \sqrt{A_2}
\]
\[+ 0.000072 E_2 \sqrt{A_2} + 0.0000003456 E_2^2\]
\[- 50 (E_1 + E_2),\]

where \(A_2 = -0.0000288 E_2^2 - 0.00001152 E_2 + 0.0000004864\).

We consider the range of efforts as \(D = \{(E_1, E_2)|0 \leq E_1 \leq 0.038, 0 \leq E_2 \leq 0.038\}\). The critical point of the efforts \(E_1\) and \(E_2\) are determined by using the first partial derivatives and the boundary of \(D\). In domain \(D\) we get four critical points for \(\pi(E_1, E_2)\), namely

\[P_1 = (E_1, E_2) = (0.006485, 0.00372)\]

as stationary point (saddle point),

\[P_2 = (E_1, E_2) = (0.002957),\]

\[P_3 = (E_1, E_2) = (0.03800, 0),\]

\[P_4 = (E_1, E_2) = (0.038, 0.03522),\]

\[P_5 = (E_1, E_2) = (0.03800, 0.03800).\]

Profit function which is evaluated at the critical point are given as follow:

1) \((E_1, E_2) = (0.00649, 0.00372)\) with profit \(\pi = 31.06066\).
2) \((E_1, E_2) = (0.002956)\) with profit \(\pi = 148.87029\).
3) \((E_1, E_2) = (0.03800, 0)\) with profit \(\pi = 182.58245\).
4) \((E_1, E_2) = (0.03800, 0.03522)\) with profit \(\pi = 408.16992\), and
5) \((E_1, E_2) = (0.03800, 0.03800)\) with profit \(\pi = 398.33448\).

From these critical points we get profit maximum with profit \(\pi = 408.16992\). The equilibrium point associated with the critical point is \(T_3 = (75.6780, 329.7842, 28.0403)\). The characteristic equation from Jacobian matrix evaluated at this equilibrium point is \(f(\lambda) = \lambda^3 + 1.12768\lambda^2 + 0.22874 \lambda + 0.000382\) which gives eigenvalues as \(\lambda_1 = -0.86319, \lambda_2 = -0.26279, \) and \(\lambda_3 = -0.00168\). This means that the equilibrium point \(T_3\) is locally asymptotically stable. In this situation, the prey and mature predator populations will sustain for a long period of time.

For initial population around the equilibrium point \(T_3\), we take \(z(0) = 45, y(0) = 90,\) and \(z(0) = 20\) we get trajectories for prey, immature, and mature predator with and without harvesting are given the following figures.
Figure 1. Trajectories for prey with and without harvesting

Figure 2. Trajectories for immature predator with and without harvesting

Figure 3. Trajectories for mature predator with and without harvesting

From Figure (1), (2), and (3) we know that all populations will sustain for a long time. However, there is a change in the value of the equilibrium point without harvesting $T_3 = (48.5480, 499.5783, 49.9578)$ and equilibrium point without harvesting $T_3 = (75.6780, 329.7842, 28.0403)$. The value of equilibrium point for prey and mature predator are decreasing but for immature predator is increasing in value.
5. Conclusions
One model of predator and prey populations with stage structure for predator population and Monod-Haldane functional response has been developed by adding harvesting factor to the prey and the mature predator populations. We found four equilibrium points, but only one is possible to become an interior equilibrium point. With suitable value of parameters, the interior equilibrium point without harvesting becomes locally asymptotically stable.

For the model with constant effort of harvesting, the stable equilibrium point is then linked to the problem of maximizing the profit function. We found a condition for the value of harvesting efforts that provides maximum profit and the interior equilibrium point remains stable. This means that the value of harvesting efforts keeps the prey and predator population sustainable and provides maximum profit for a long period of time.

Acknowledgements
This research is supported by Ministry of Research and Higher Education of Republic Indonesia via LP2M Hasanuddin University grant competition of Post Graduate Program Research Scheme with contract number: 1660/UN4.21/PL.00.00/2018.

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