THE INDUCED REPRESENTATIONS
OF THE $\kappa$-POINCARE GROUP.
THE MASSIVE CASE

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Abstract. The induced representations of the $\kappa$-Poincare group for the massive case are described. It is shown that it extends many of the features of the classical case.

1. Introduction

Recently an interesting deformation of Poincare algebra - the so-called $\kappa$-Poincare algebra - has been constructed [1] (see also [2]). Its characteristic property is that the deformation parameter $\kappa$ is dimensionful. Some physical consequences of the deformed space-time symmetry have been discussed by H. Bacry [3] (see also [4]) who stressed its attractive features. The global counterpart of the $\kappa$-Poincare algebra was constructed by S. Zakrzewski [5]. The resulting quantum Poincare group can be described as follows. The group element is written in the following form

\begin{equation}
\begin{bmatrix}
A & v \\
0 & 1
\end{bmatrix}
\end{equation}

where $A = [A^\mu_\nu]_{\mu,\nu=0}^3$ and $v = [v^\mu]_{\mu=0}^3$ are selfadjoint elements subject to the

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following commutation rules

\[
\begin{align*}
[A^0_\mu, A^0_\nu] &= 0, \\
[v^r, v^k] &= 0, \\
[v^0, v^r] &= -\frac{i}{\kappa}v^r, \\
[A^\mu_\nu, v^0] &= \frac{i}{\kappa} (A^\mu_\nu A^0_\nu - \delta^\mu_\nu \delta^0_0), \\
[A^m_n, v^r] &= \frac{i}{\kappa} (A^m_n A^r_m - A^r_m), \\
[A^m_n, v^r] &= \frac{i}{\kappa} (A^m_n A^r_m - \delta_{mr} A^0_0), \\
[A^0_0, v^r] &= \frac{i}{\kappa} (A^0_0 A^r_0 - A^r_0).
\end{align*}
\]

(2)

Equipped with the standard matrix comultiplication the above structure defines a Hopf $*$-algebra.

Once the quantum Poincare group is defined we can try to explore its physical consequences. The necessary step in this direction is to construct its unitary representations, to define its action on quantum Minkowski space, to find covariant fields, etc. In this paper we concentrate on the first problem: to find the unitary representations. To this end we shall use the method of induced representations for quantum groups as formulated by A. Gonzales-Ruiz and L.A. Ibort [6].

II. Some subgroups

To start with let us analyse some subgroups of quantum $\kappa$-Poincare group defined above. Let us recall that, by definition, $A(K)$ is a quantum subgroup of a quantum group $A(G)$ if there exists an epimorphism of Hopf algebras $\Pi : A(G) \to A(K)$; we have

\[
(\Pi \otimes \Pi) \circ \Delta_G = \Delta_K \circ \Pi.
\]

(3)

Let $A(G)$ be our quantum Poincare group. We are interested in those subgroups $A(K)$ for which $\Pi(v^\mu)$ are independent generators. First let us note that, due to the first commutation rule (2) and the form of the coproduct, $\{\Pi(A^\mu_\nu)\}_\mu,\nu=0$ generate a subgroup (in the classical sense) of the classical Lorentz group. There are few choices of this subgroup which lead to the quantum subgroups of quantum Poincare group.

(i) Let us first take $\{\Pi(A^\mu_0)\}$ to be a trivial subgroup of the Lorentz group. Then $A(N)$ is generated by four elements $\bar{v}^\mu$ such that

\[
\begin{align*}
[\bar{v}^0, \bar{v}^k] &= -\frac{i}{\kappa} \bar{v}^k, \\
[\bar{v}^i, \bar{v}^k] &= 0, \\
\Delta(\bar{v}^\mu) &= \bar{v}^\mu \otimes I + I \otimes \bar{v}^\mu.
\end{align*}
\]

(4)
The epimorphism \( \Pi \) is given by
\[
\Pi(v^\mu) = \tilde{v}^\mu, \quad \Pi(A_{0}^\mu) = \delta^\mu_0 I.
\]

(ii) Let \( \{\Pi(A_{\mu}^\nu)\} \) be the rotation subgroup. Define \( A(K) \) as generated by \( \tilde{v}^\mu \) and \( M_j^i \) obeying:
\[
\begin{align*}
[\tilde{v}^0, \tilde{v}^k] &= -\frac{i}{\kappa} \tilde{v}^k, \quad [\tilde{v}^i, \tilde{v}^k] = 0, \\
[M_j^i, M_j^k] &= 0, \quad [M_j^i, \tilde{v}^\mu] = 0, \\
M_j^i M_j^k &= \delta_{ik} I, \quad M_j^i M_j^k = \delta_{ik} I, \\
\Delta(M_j^i) &= M_j^i \otimes M_j^i, \\
\Delta(\tilde{v}^i) &= M_j^i \otimes \tilde{v}^k + \tilde{v}^i \otimes I, \\
\Delta(\tilde{v}^0) &= I \otimes \tilde{v}^0 + \tilde{v}^0 \otimes I.
\end{align*}
\]

Epimorphism \( \Pi \) is defined as follows
\[
\begin{align*}
\Pi(v^\mu) &= \tilde{v}^\mu, \quad \Pi(A_0^\mu) = I, \\
\Pi(A_0^i) &= \Pi(A_0^i) = 0, \quad \Pi(A_j^i) = M_j^i.
\end{align*}
\]

It is easy to check that \( A(K) \) is a subgroup of quantum Poincare group.

We shall not discuss here which subgroups of (classical) Lorentz group generate (in the sense explained above) quantum subgroups of \( \kappa \)-Poincare group. Let us only note two interesting features. First, the whole Lorentz group does not form a subgroup of quantum Poincare group. Moreover, the same continues to hold for those subgroups of Lorentz group which are the stability groups of light-like or space-like fourvectors. To see this let us first note that space rotations are the automorphisms of our Poincare group. Indeed, if \( R \) is any \( c \)-number \( 3 \times 3 \) orthogonal matrix and
\[
\bar{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}
\]
then
\[
v^0 \to v^0, \quad v^i \to (Rv)^i, \quad A \to \bar{R}A\bar{R}^{-1}
\]
is an automorphism of \( \kappa \)-Poincare group. Therefore we can put the light-like (resp. space-like) fourvector in canonical position \( k^\mu = (k, 0, 0, k) \) (resp. \( k^\mu = (0, 0, 0, m) \)). Now assume that \( \bar{A}_\mu^\nu = \Pi(A_\mu^\nu) \) is a stability group of, say, light-like fourvector \( k^\mu \). Then \( \bar{A}_\mu^\nu k^\nu = \bar{k}^\mu \) implies
\[
\bar{A}_0^0 + \bar{A}_3^0 = I, \quad \bar{A}_0^3 + \bar{A}_3^3 = I, \quad \bar{A}_0^{1,2} + \bar{A}_3^{1,2} = 0;
\]
these are the standard conditions. However, further constraints follow from the commutation rules (2) and the fact that \( \Pi \) is a homomorphism. Taking the commutators of both sides of eqs. (8) with \( \tilde{v}^\mu = \Pi(v^\mu) \) we arrive finally at the following form of \( \bar{A}_\mu^\nu \):
\[
\bar{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
We see that, contrary to the classical case, the stability subgroup of light-like fourvector does not give rise to the subgroup of $\kappa$-Poincare group (in the sense defined above). However, the $U(1)$-subgroup of the stability group does the job. The same holds true for the space-like case: again only rotations around the third axis survive in the quantum case.

III. Representations

Let us now construct the induced representations for the massive case, i.e. we choose the quantum subgroup described in (ii). In order to describe explicitly these representations we could follow [6]. In this paper the quantum counterpart of the classical construction of induced representations is given. As it is well known [7] in order to construct the representations of a group $G$ induced from the representations of its subgroup $H \subset G$ one starts from the Hilbert space of square integrable functions on $G$ taking their values in the vector space carrying some representation of $H$. The group action is defined to be, say, right action: $f(g) \rightarrow f(gg_0)$. The essence of the method is the selection of invariant subspace by imposing the so-called coequivariance condition; in many cases the invariant subspace obtained in this way carries an irreducible representation of $G$. The whole construction can be generalized in a rather straightforward way to the quantum case [6]. However, in specific cases it is profitable to have more explicit characterization of the representation. This is achieved by solving explicitly the coequivariance condition which gives rise to the description of representation in terms of Hilbert space of functions defined on the right coset space $H \backslash G$. This is especially effective for the case of semidirect products in which one factor $N$ is abelian. Then the irreducible representations are induced from the stability subgroups of characters of $N$ and the relevant coset spaces get with relevant orbits in the dual group $\hat{N}$. In particular, in the case of Poincare group we obtain exactly Wigner’s construction.

Let us now apply the same idea to the case of $\kappa$-Poincare group. We start with abelian subgroup of translations (our counterpart of $N$) described in (ii). It is abelian in the sense that it is commutative. We define its dual in a similar way as in the classical case. Recall that a representation $\rho$ of a quantum group $A(N)$ is a map $\rho : \mathcal{H} \rightarrow A(N) \otimes \mathcal{H}$ satisfying $(\Delta_N \otimes I) \circ \rho = (I \otimes \rho) \circ \rho$ (or, for the right (co)representation: $\rho : \mathcal{H} \rightarrow \mathcal{H} \otimes A(N)$, $(I \otimes \Delta_N) \circ \rho = (\rho \otimes I) \circ \rho$). If $\mathcal{H}$ is one-dimensional, an unitary representation $\rho$ can be written as:

$$\rho : z \rightarrow a \otimes z, \quad z \in \mathbb{C},$$

where $a$ is unitary element of $A(N)$ and

$$\Delta_N(a) = a \otimes a. \quad (9)$$

The unitary elements of our algebra $A(N)$ obeying (9) are called characters. It is trivial to verify that a product of characters is again a character. It is not difficult to check that in our case the solution to eq. (9) reads

$$a = e^{i q_0 \tilde{v}_0} e^{i q_k \tilde{v}_k} \quad (10)$$

where $q_0, q_k$ are real numbers. Now, if $a'$ is another character, a small calculation gives

$$aa' = e^{i q_0' \tilde{v}_0} e^{i q_k' \tilde{v}_k} \quad (11a)$$
where

\[(11b) \quad q''_0 = q_0 + q'_0, \quad q''_k = q_k e^{-\frac{q'_0}{\kappa}} + q'_k.\]

To get slightly more symmetric form we redefine

\[(12) \quad p_0 \equiv q_0, \quad p_k = q_k e^{\frac{q'_0}{2\kappa}}.\]

Then we conclude that the dual group \(\hat{N}\) is the classical group with group manifold being \(\mathbb{R}^4\) and the composition law

\[(13) \quad \{p_0, p_k\} * \{p'_0, p'_k\} = \left\{ p_0 + p'_0, \frac{p_k e^{-\frac{i q'_0}{\kappa}} + p'_k e^{\frac{i q'_0}{\kappa}}}{2} \right\}.\]

The only difference with the classical case is that the dual group is no longer abelian but only solvable. However, we shall see that our space of states can still be viewed (in some sense) as consisting of functions concentrated on some orbit in \(\hat{N}\) and taking values in the space carrying a representation of \(A(K)\).

Let us now construct the representations of \(A(K)\). We shall follow closely the classical case. There the representation of \(A(K)\) from which an irreducible representation of Poincare group is induced is constructed out of representation of rotation subgroup and a character of translation subgroup invariant under the action of the former (cf. [7]). It is easy to see that, in quantum case, both rotations and translations also form the subgroups of \(A(K)\); by direct analysis of the classical case we infer also that the counterpart of an invariant character is here provided by a unitary element \(a\) constructed out of \(v^\mu\) and obeying

\[(14) \quad \Delta_K (a) = a \otimes a.\]

Taking the above into account we construct the representation of \(A(K)\) as follows. Let \(\mathcal{H}\) be a Hilbert space carrying unitary representation of classical rotation group

\[(15) \quad \mathcal{H} \ni \psi = a_i e_i \rightarrow D_{ij} (M) a_i e_j \in \mathcal{H}.\]

Then \(\rho : \mathcal{H} \rightarrow \mathcal{H} \otimes A(K)\) is defined as follows:

\[(16) \quad \rho (\psi) \equiv \rho (a_i e_i) = a_i e_j \otimes D_{ij} (M) e^{i m v^0}\]

where \(m\) is a numerical parameter, \(m \in \mathbb{R}^+,\) and \(\{e_i\}\) is the basis in \(\mathcal{H}\).

It is easy to check that \(\rho\) is really a representation. We have

\[(17a) \quad [(\rho \otimes I) \circ \rho] (\psi) = a_i e_k \otimes D_{kj} (M) e^{i m v^0} \otimes D_{ij} (M) e^{i m v^0}\]

and, on the other hand,

\[(17b) \quad [(I \otimes \Delta_K) \circ \rho] (\psi) = a_i e_j \otimes D_{ij} (M \otimes M) e^{i m (v^0 \otimes I + I \otimes v^0)}\]

\[= a_i e_j \otimes (D_{kj} (M) \otimes D_{ki} (M)) (e^{i m v^0} \otimes e^{i m v^0}).\]

*) note that rotations form a classical group
Note that the property (14) of \( a \equiv e^{imv_0} \) has played a crucial role above.

Now, we are ready to solve the coequivariance condition explicitly. Let us recall that the Hilbert space of the induced representation of quantum group \((A(G))\) is a subspace of \( \mathcal{H} \otimes A(G) \) consisting of the elements subject to the following coequivariance condition:

\[
\mathcal{H} \uparrow A(G) = \{ F \in \mathcal{H} \otimes A(G) : (I \otimes L)F = (\rho \otimes I)F \}.
\]

Here \( L \) is a left coaction of \( A(K) \) in \( A(G) \) [6]:

\[
L = (\Pi \otimes I) \circ \Delta_G.
\]

To solve the equation \((I \otimes L)F = (\rho \otimes I)F\) let us first make what can be called a counterpart of Mackey decomposition

\[
\left[ \begin{array}{cc} A & v \\ 0 & 1 \end{array} \right] = \left[ \begin{array}{cc} \hat{M} & v \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} \tilde{A} & 0 \\ 0 & 1 \end{array} \right]
\]

where the product on the right-hand side is an algebra product (and not the tensor one). The matrices \( \hat{M} \) and \( \tilde{A} \) are defined as follows

\[
\tilde{A} = \begin{bmatrix} A_0 & A_0^0 \\ A_i & \delta_{ij}I + \frac{A_i^0 A_j^0}{1 + A_0^2} \end{bmatrix}
\]

\[
\hat{M} = \begin{bmatrix} 1 & 0 \\ 0 & M \end{bmatrix}
\]

where

\[
M = [M^i_j]_{i,j=1}^3 = \left[ A^i_j - \frac{A^0_i A^0_j}{1 + A_0^2} \right]_{i,j=1}^3.
\]

Note that the matrix \( \hat{M} \), being formally orthogonal, is not an element of the subalgebra \( A(K) \), in particular, their elements do not commute with \( v^k \)'s. If we define the momenta \( p_\mu \) by

\[
p_\mu = m A_\mu^0
\]

we can write \( \tilde{A} \) as

\[
\tilde{A} = \begin{bmatrix} \frac{p_0}{m} & \frac{p_0}{m} \\ \frac{m}{m} & \delta_{ij}I + \frac{p_0}{m}(p_0 + m) \end{bmatrix}
\]

Now, we propose the following solution to the coequivariance condition

\[
F = e_i \otimes D_{ij}(M)e^{imv^0}f_j(\tilde{A}) \equiv e_i \otimes D_{ij}(M)e^{imv^0}f_j(p).
\]

The matrices \( D_{ij}(M) \) are constructed out of \( \mathcal{M} \)'s in the same way as the matrices of the representation of classical orthogonal group. There is no ambiguity here because all elements of \( \mathcal{M} \) commute among themselves. Note that, in principle,
we should take only integer spin representations because we have quantized the Poincare group and not $\text{ISL}(2, \mathbb{C})$.

Now, we can check that our Ansatz solves the coequivariance condition. One has

\begin{equation}
    (\rho \otimes I)(F) = e_k \otimes D_{ki}(M)e^{imv^0} \otimes D_{ij}(\mathcal{M})e^{imv^0} f_j(\tilde{A}).
\end{equation}

On the other hand,

\begin{equation}
    (I \otimes L)(F) = e_k \otimes \{(\Pi \otimes I)(D_{kj}(\Delta(\mathcal{M}))e^{im\Delta(v^0)} f_j(\Delta(\tilde{A}))\}.
\end{equation}

But

\begin{equation}
    (\Pi \otimes I)\Delta(\tilde{A}) = I \otimes \tilde{A},
\end{equation}

\begin{equation}
    (\Pi \otimes I)\Delta(\mathcal{M}) = M \otimes \mathcal{M},
\end{equation}

\begin{equation}
    D_{kj}(M \otimes M) = D_{ki}(M) \otimes D_{ij}(\mathcal{M}),
\end{equation}

\begin{equation}
    (\Pi \otimes I)(e^{imv^0}) = e^{imv^0} \otimes e^{imv^0}.
\end{equation}

Collecting (25)–(27) we see that (24) solves the coequivariance condition.

The induced representation

\begin{equation}
    \rho_R : \mathcal{H} \uparrow A(G) \to \mathcal{H} \uparrow A(G) \otimes A(G)
\end{equation}

is now defined in the same way as in the classical case [6]:

\begin{equation}
    \rho_R = I \otimes \Delta.
\end{equation}

For $F$ given by eq. (24) we have

\begin{equation}
    \rho_R(F) = (I \otimes \Delta)(F) = e_i \otimes \{(D_{ij}(\Delta(\mathcal{M}))e^{im(\lambda_\mu^0 \otimes v_\mu + v_\mu \otimes I)} f_j(\Delta(\tilde{A}))\}.
\end{equation}

Now, according to eqs. (22)–(24) we can identify our space of representation with linear space of functions of $p_\mu$. The $p_\mu$’s commute among themselves (although they do not commute with $v^\mu$); therefore, the above functions can be viewed as classical functions defined on hyperboloid $p^2 = m^2, p_0 > 0$, exactly as in the classical case.

In order to simplify eq. (30) let us note first that

\begin{equation}
    \Delta(p_\mu) = \Delta(mA_\mu^0) = mA_\mu^0 \otimes A_\mu^0 = p_\mu \otimes A_\mu^0,
\end{equation}

so the action on the support space is standard. Next, let us write eq. (30) in the form

\begin{equation}
    \rho_R(F) = e_i \otimes \{(D_{ik}(\mathcal{M})e^{imv^0} \otimes I)(D_{kl}^{-1}(\mathcal{M}) \otimes I)\}
    \cdot D_{ij}(\Delta(\mathcal{M}))(e^{imv^0} \otimes I)(e^{imv^0} \otimes I)e^{i(p_\mu \otimes v^\mu + m^2 v^0 \otimes I)} f_j(p_\mu \otimes A_\mu^0).
\end{equation}

Consider the expression $(D_{kl}^{-1}(\mathcal{M}) \otimes I)D_{ij}(\Delta(\mathcal{M}))$; using eqs. (21)–(23) we can at once identify this expression. It is simply equal to $D_{kj}(R(p, \Lambda))$, where $R(p, \Lambda)$ is a classical Wigner rotation corresponding to the momentum $p$ and transformation $\Lambda$; of course, $D_{kj}(R(p, \Lambda))$ is to be understood here as an element of the tensor
product of the algebra of functions on the hyperboloid $p^2 = m^2$ and the algebra $A(G)$.

As a next step we calculate

$$(e^{-imv_0^I} \otimes I) D_{kj}(R(p, A))(e^{imv_0^I} \otimes I).$$

Using the commutation rules among $v_0^I$ and $p_\mu \sim A_0^0$, we easily obtain that:

$$(e^{-imv_0^I} \otimes I) D_{kj}(R(p, A))(e^{imv_0^I} \otimes I) = D_{kj}(R(\tilde{p}, A))$$

where

$$\tilde{p}_0 = \frac{p_0 \cos \frac{m}{\kappa} + m \sin \frac{m}{\kappa}}{m \sin \frac{m}{\kappa}} + p_0 \sin \frac{m}{\kappa},$$

$$\tilde{p}_k = \frac{p_k}{m \cos \frac{m}{\kappa}} + p_0 \sin \frac{m}{\kappa}.$$ 

Finally,

$$e^{-imv_0^I} I_{\kappa}(p_\mu \otimes v^I + m v_0^I)$$

$$= \exp \left\{ i \kappa \ln \left( \cos \frac{m}{\kappa} + \frac{p_0}{m} \sin \frac{m}{\kappa} \right) \otimes v_0^I \right\}$$

$$\times \exp \left\{ i \kappa \frac{\sin \frac{m}{\kappa}}{m \cos \frac{m}{\kappa}} p_k \otimes v_0^I \right\}.$$ 

So, collecting all terms in eq. (32) we finally arrive at the following form of our representation:

$$\rho_R : f_i(p) \rightarrow D_{ij}(R(\tilde{p}, A)) \exp \left\{ i \kappa \ln \left( \cos \frac{m}{\kappa} + \frac{p_0}{m} \sin \frac{m}{\kappa} \right) \otimes v_0^I \right\}$$

$$\times \exp \left\{ i \kappa \frac{\sin \frac{m}{\kappa}}{m \cos \frac{m}{\kappa}} p_k \otimes v_0^I \right\} f_j(p_\mu \otimes A^0).$$

IV. Conclusions

Let us conclude with few remarks:

(i) the carrier space of the representation is, as in the classical case, the Hilbert space of square integrable (with respect to the standard measure $d^3x$) functions on hyperboloid $p^2 = m^2$;

(ii) in the limit $\kappa \rightarrow \infty$ one recovers the classical representation;

(iii) one can pose the problem of the infinitesimal representations, i.e. the representations of the ‘Lie algebra’; the Lie algebra of a quantum group is defined via a notion of duality of Hopf algebras. For the case of quantum $E(2)$ group this duality is fully understood [8], in this case, we have checked that, using the standard definition of infinitesimal transformations [9], one recovers the representation of Lie algebra from the induced representation of $E(2)$;
(iv) there are many interesting questions to be studied. First of all, as we have noted above, there are no quantum subgroups corresponding to light-like or space-like momenta. In the classical case we have to use all orbits in the group dual to the abelian subgroup of translations in order to exhaust all irreducible representations. So question arises what are the quantum counterparts of mass zero and imaginary mass representations;

(v) one can define a noncommutative Minkowski space for which the $\kappa$-Poincare group is the symmetry group. One can ask what is a quantum counterpart of classical Fourier transform relating Poincare-covariant states with covariant space-time functions. For example, whether there exists a generalization of Weinberg’s theory [10].

All the above questions will be addressed to in subsequent publications.

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