CONVERGENCE RATE ANALYSIS OF THE
FORWARD-DOUGLAS-RACHFORD SPLITTING SCHEME∗

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Abstract. Operator splitting schemes are a class of powerful algorithms that solve complicated
monotone inclusion and convex optimization problems that are built from many simpler pieces. They
give rise to algorithms in which all simple pieces of the decomposition are processed individually.
This leads to easily implementable and highly parallelizable or distributed algorithms, which often
obtain nearly state-of-the-art performance.

In this paper, we analyze the convergence rate of the forward-Douglas-Rachford splitting (FDRS)
algorithm, which is a generalization of the forward-backward splitting (FBS) and Douglas-Rachford
splitting (DRS) algorithms. Under general convexity assumptions, we derive the ergodic and non-
ergodic convergence rates of the FDRS algorithm, and show that these rates are the best possible.
Under Lipschitz differentiability assumptions, we show that the best iterate of FDRS converges as
quickly as the last iterate of the FBS algorithm. Under strong convexity assumptions, we derive
convergence rates for a sequence that strongly converges to a minimizer. Under strong convexity
and Lipschitz differentiability assumptions, we show that FDRS converges linearly. We also provide
examples where the objective is strongly convex, yet FDRS converges arbitrarily slowly. Finally, we
relate the FDRS algorithm to a primal-dual forward-backward splitting scheme and clarify its place
among existing splitting methods. Our results show that the FDRS algorithm automatically adapts
to the regularity of the objective functions and achieves rates that improve upon the tight worst case
rates that hold in the absence of smoothness and strong convexity.

Key words. forward-Douglas-Rachford splitting, Douglas-Rachford splitting, forward-backward
splitting, generalized forward-backward splitting, nonexpansive operator, averaged operator, fixed-
point algorithm, primal-dual algorithm

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1. Introduction. Operator-splitting schemes are algorithms for splitting comp-
licated problems arising in PDE, monotone inclusions, optimization, and control into
many simpler subproblems. The achieved decomposition can give rise to inherently
parallel and, in some cases, distributed algorithms. These characteristics are particu-
larly desirable for large-scale problems that arise in machine learning, finance, control,
image processing, and PDE [8].

In optimization, the Douglas-Rachford splitting (DRS) algorithm [27] minimizes
sums of (possibly) nonsmooth functions \( f, g : H \to (-\infty, \infty) \) on a Hilbert space \( H \):

\[
\minimize_{x \in H} f(x) + g(x).
\]  

During each step of the algorithm, DRS applies the proximal operator, which is the
basic subproblem in nonsmooth minimization, to \( f \) and \( g \) individually rather than to
the sum \( f + g \). Thus, the key assumption in DRS is that \( f \) and \( g \) are easy to minimize
independently, but the sum \( f + g \) is difficult to minimize. We note that many complex
objectives arising in machine learning [8] and signal processing [15] are the sum of
nonsmooth terms with simple or closed-form proximal operators.

The forward-backward splitting (FBS) algorithm [29] is another technique for
solving (1.1) when \( g \) is known to be smooth. In this case, the proximal operator of \( g \)
is never evaluated. Instead, FBS combines gradient (forward) steps with respect to \( g \)

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1
and proximal (backward) steps with respect to \( f \). FBS is especially useful when the proximal operator of \( g \) is complex and its gradient is simple to compute.

Recently, the forward-Douglas-Rachford splitting (FDRS) algorithm [10] was proposed to combine DRS and FBS and extend their applicability (see Algorithm 1). More specifically, let \( V \subseteq \mathcal{H} \) be a closed vector space and suppose \( g \) is smooth. Then FDRS applies to the following constrained problem:

\[
\text{minimize } f(x) + g(x). \tag{1.2}
\]

Throughout the course of the algorithm, the proximal operator of \( f \), the gradient of \( g \), and the projection operator onto \( V \) are all employed separately.

The FDRS algorithm can also apply to affinely constrained problems. Indeed, if \( V = V_0 + b \) for a closed vector subspace \( V_0 \subseteq \mathcal{H} \) and a vector \( b \in \mathcal{H} \), then Problem (1.2) can be reformulated as

\[
\text{minimize } f(x + b) + g(x + b). \tag{1.3}
\]

For simplicity, we only consider linearly constrained problems.

The FDRS algorithm is a generalization of the generalized forward-backward splitting (GFBS) algorithm [31], which solves the following problem:

\[
\text{minimize } \sum_{i=1}^{n} f_i(x) + g(x) \tag{1.4}
\]

where \( f_i : \mathcal{H} \to (-\infty, \infty] \) are closed, proper, convex and (possibly) nonsmooth. In the GFBS algorithm, the proximal mapping of each function \( f_i \) is evaluated in parallel. We note that GFBS can be derived as an application of FDRS to the equivalent problem:

\[
\min_{x_1, \ldots, x_n \in \mathcal{H}^n} \sum_{i=1}^{n} f_i(x_i) + g \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right). \tag{1.5}
\]

In this case, the vector space \( V = \{(x, \ldots , x) \in \mathcal{H}^n \mid x \in \mathcal{H}\} \) is the diagonal set of \( \mathcal{H}^n \) and the function \( f \) is separable in the components of \((x_1, \ldots , x_n)\).

The FDRS algorithm is the only primal operator-splitting method capable of using all structure in Equation (1.2). In order to achieve good practical performance, the other primal splitting methods require stringent assumptions on \( f, g, \) and \( V \). Primal DRS cannot use the smooth structure of \( g \), so the proximal operator of \( g \) must be simple. On the other hand, primal FBS and forward-backward-forward splitting (FBFS) [32] cannot separate the coupled nonsmooth structure of \( f \) and \( V \), so minimizing \( f(x) \) subject to \( x \in V \) must be simple. In contrast, FDRS achieves good practical performance if it is simple to minimize \( f \), evaluate \( \nabla g \), and project onto \( V \).

Modern primal-dual splitting methods [30, 11, 23, 16, 18, 33, 9, 7, 6, 13, 25, 5, 14] can also completely decompose problem (1.2), but they introduce extra dual variables and are, thus, less memory efficient. It is unclear whether FDRS will perform better than existing primal-dual methods when memory is not a concern. However, it is easier to choose algorithm parameters for FDRS and, hence, it can be more convenient to use in practice.
application: constrained quadratic programming and support vector machines. Let \( d \) and \( m \) be natural numbers. Suppose that \( Q \in \mathbb{R}^{d \times d} \) is a symmetric positive semi-definite matrix, \( c \in \mathbb{R}^d \) is a vector, \( C \subseteq \mathbb{R}^d \) is a constraint set, \( A \in \mathbb{R}^{m \times d} \) is a linear map, and \( b \in \mathbb{R}^m \) is a vector. Consider the constrained quadratic programming problem:

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \langle Qx, x \rangle + \langle c, x \rangle \\
\text{subject to:} & \quad x \in C \\
& \quad Ax = b.
\end{align*}
\]

Problem (1.6) arises in the dual form soft-margin kernelized support vector machine classifier [19] in which \( C \) is a box constraint and \( b \) is 0. Note that by the argument in (1.3), we can always assume that \( b = 0 \).

Define the smooth function \( g(x) = (1/2)\langle Qx, x \rangle + \langle c, x \rangle \), the nonsmooth indicator function \( f(x) = \chi_C(x) \) (which is 0 on \( C \) and \( \infty \) elsewhere), and the vector space \( V = \{ x \in \mathbb{R}^d \mid Ax = 0 \} \). Evidently, this notation immediately casts the constrained quadratic programming problem in the form (1.2) and, thus, FDRS can be applied. This splitting is particularly nice because \( \nabla g(x) = Qx + c \) is simple whereas the proximal operator of \( g \) requires a matrix inversion

\[
\text{prox}_{\gamma g} = (I + \gamma Q)^{-1} \circ (I - \gamma c),
\]

which is quite expensive for large scale problems. In addition, the proximal operator of \( f \) is just the projection onto \( C \).

1.1. Goals, challenges, and approaches. This work seeks to characterize the convergence rate of the FDRS algorithm applied to Problem (1.2). Recently, [21] has shown that the optimal convergence rate of the fixed-point residual (FPR) (see Equation (1.27)) of the FDRS algorithm is \( o(1/(k+1)) \). To the best of our knowledge, nothing else is known about the convergence rate of FDRS. Furthermore, it is even unclear how the FDRS algorithm relates to other splitting algorithms. We seek to fill this gap.

The techniques used in this paper are based on [20, 21, 22]. These techniques are quite different from those used in classical objective error convergence rate analysis. The classical techniques do not apply because the FDRS algorithm is driven by the fixed-point iteration of a nonexpansive operator, not by the minimization of a model function. Thus, we must explicitly use the properties of nonexpansive operators in order to derive convergence rates for the objective error.

We summarize our contributions and techniques as follows:

(i) We analyze the objective error convergence rates (Theorems 3.1 and 3.5) of the FDRS algorithm under general convexity assumptions. We show that FDRS is, in the worst case, nearly as slow as the subgradient method yet nearly as fast as the proximal point algorithm (PPA) in the ergodic sense. Our nonergodic rates are shown by relating the objective error to the FPR through a fundamental inequality. We also show that the derived rates are tight by appealing to known counterexamples (Theorems 3.4 and 3.7).

(ii) We show that if \( f \) or \( g \) is strongly convex, then a natural sequence of points converges strongly to a minimizer. Furthermore, the best iterate converges with rate \( o(1/(k+1)) \), the ergodic iterate converges with rate \( O(1/(k+1)) \), and the nonergodic iterate converges with rate \( o(1/\sqrt{k+1}) \). The results follow by showing that a certain
sequence of squared norms is summable. We also show that some of the derived rates are optimal by constructing a novel counterexample (Theorem 6.6).

(iii) We show that if \( f \) is differentiable and \( \nabla f \) is Lipschitz, then the best iterate of the FDRS algorithm has objective error of order \( o(1/(k+1)) \) (Theorem 5.3). This rate is an improvement over the tight \( o(1/\sqrt{k+1}) \) convergence rate for nonsmooth \( f \). The result follows by showing that the objective error is summable.

(iv) We establish scenarios under which FDRS converges linearly (Theorem 6.1) and show that linear convergence should not be expected under other scenarios (Theorem 6.6).

(v) We show that even if \( f \) and \( g \) are strongly convex, the FDRS algorithm can converge arbitrarily slowly (Theorem 6.5).

(vi) We show that the FDRS algorithm is the limiting case of a recently developed primal-dual forward-backward splitting algorithm (Section 7) and, thus, clarify how FDRS relates to existing algorithms.

Our analysis builds on the techniques and results of [10, 21, 22]. The rest of this section contains a brief review of these results.

1.2. Notation and facts. Most of the definitions and notation that we use in this paper are standard and can be found in [3].

Throughout this paper, we use \( \mathcal{H} \) to denote (a possibly infinite dimensional) Hilbert space. In fixed-point iterations, \((\lambda_j)_{j \geq 0} \subset \mathbb{R}_+\) will denote a sequence of relaxation parameters, and

\[
\Lambda_k := \sum_{i=0}^{k} \lambda_i
\]  

(1.7)

is its kth partial sum. Given the sequence \((x^j)_{j \geq 0} \subset \mathcal{H}\), we let

\[
\overline{x}^k = \frac{1}{\Lambda_k} \sum_{i=0}^{k} \lambda_i x^i
\]

denote its kth average with respect to the sequence \((\lambda_j)_{j \geq 0}\). We call a convergence result \textit{ergodic} if it applies to the sequence \((\overline{x}^j)_{j \geq 0}\), and \textit{nonergodic} if it applies to the sequence \((x^j)_{j \geq 0}\).

For any subset \( C \subseteq \mathcal{H} \), we define the distance function:

\[
d_C(x) := \inf_{y \in C} \|x - y\|.
\]  

(1.8)

In addition, we define the indicator function \( \chi_C : \mathcal{H} \to \{0, \infty\} \) of \( C \): for all \( x \in C \) and \( y \in \mathcal{H}\setminus C \), we have \( \chi_C(x) = 0 \) and \( \chi_C(y) = \infty \).

Given a closed, proper, and convex function \( f : \mathcal{H} \to (-\infty, \infty] \), the set \( \partial f(x) = \{ p \in \mathcal{H} \mid \text{for all } y \in \mathcal{H}, f(y) \geq f(x) + \langle y - x, p \rangle \} \) denotes its subdifferential at \( x \) and \( \nabla f(x) \in \partial f(x) \) denotes a subgradient. (This notation was used in [4, Eq. (1.10)].) If \( f \) is Gâteaux differentiable at \( x \in \mathcal{H} \), we have \( \partial f(x) = \{ \nabla f(x) \} \) [3, Proposition 17.26].

The Legendre-Fenchel transform of a function \( f \) is

\[
f^*(y) := \sup_{x \in \mathcal{H}} \langle y, x \rangle - f(x).
\]
Let \( I_H : \mathcal{H} \to \mathcal{H} \) denote the identity map on \( \mathcal{H} \). For any point \( x \in \mathcal{H} \) and scalar \( \gamma \in \mathbb{R}_{++} \), we let
\[
\text{prox}_{\gamma f}(x) := \arg \min_{y \in \mathcal{H}} f(y) + \frac{1}{2\gamma} \| y - x \|^2 \quad \text{and} \quad \text{refl}_{\gamma f}(x) := 2\text{prox}_{\gamma f}(x) - I_H,
\]
which are known as the \textit{proximal} and \textit{reflection} operators, respectively.

The subdifferential of the indicator function \( \chi_V \) where \( V \subseteq \mathcal{H} \) is a closed vector subspace is defined as follows: for all \( x \in \mathcal{H} \),
\[
\partial \chi_V(x) = \begin{cases} V^\perp & \text{if } x \in \mathcal{H}; \\ \emptyset & \text{otherwise} \end{cases}
\]
(1.9)
where \( V^\perp \) is the orthogonal complement of \( V \). Evidently, if \( P_V(\cdot) = \arg \min_{y \in V} \| y - \cdot \|^2 \) is the projection onto \( V \), then
\[
\text{prox}_{\gamma \chi_V} = P_V \quad \text{and} \quad \text{refl}_{\gamma \chi_V} = 2P_V - I_H = P_V - P_{V^\perp},
\]
and these operators are independent of \( \gamma \).

Let \( \lambda > 0 \), let \( L \geq 0 \), and let \( T : \mathcal{H} \to \mathcal{H} \) be a map. The map \( T \) is called \textit{L-Lipschitz} continuous if \( \|Tx - Ty\| \leq L\|x - y\| \) for all \( x, y \in \mathcal{H} \). The map \( T \) is called \textit{nonexpansive} if it is 1-Lipschitz. We also use the notation:
\[
T_\lambda := (1 - \lambda)I_H + \lambda T.
\]
(1.10)
If \( \lambda \in (0, 1) \) and \( T \) is nonexpansive, then \( T_\lambda \) is called \textit{\( \lambda \)-averaged} [3, Definition 4.23].

We call the following identity the \textit{cosine rule}:
\[
\|y - z\|^2 + 2(y - x, z - x) = \|y - x\|^2 + \|z - x\|^2, \quad \forall x, y, z \in \mathcal{H}.
\]
(1.11)
We will use Young’s inequality for real numbers several times throughout this paper: for all \( a, b \geq 0 \) and \( \varepsilon > 0 \), we have
\[
ab \leq a^2/(2\varepsilon) + \varepsilon b^2/2.
\]
(1.12)

**1.3. Assumptions.**

**Assumption 1 (Convexity).** Every function we consider is closed, proper, and convex.

Unless otherwise stated, a function is not necessarily differentiable

We also assume the existence of a particular solution to 1.2

**Assumption 2 (Solution Existence).** We assume that \( \text{zer}(\partial f + \nabla g + \partial \chi_V) \neq \emptyset \)

Note that this assumption is slightly stronger than the existence of a minimizer, because \( \text{zer}(\partial f + \nabla g + \partial \chi_V) \subseteq \text{zer}(\partial (f + g + \chi_V)) \), in general [3, Remark 16.7]. Nevertheless, this assumption is standard.

Finally we assume that \( \nabla g \) is sufficiently nice.

**Assumption 3 (Differentiability).** The function \( g \) is differentiable, \( \nabla g \) is \( (1/\beta) \)-Lipschitz, and \( P_V \circ \nabla g \circ P_V \) is \( (1/\beta V) \)-Lipschitz.

Assumption 3 is also used in the original paper on FDRS [10].
1.4. The FDRS algorithm. The FDRS algorithm generates a sequence \((z^j)_{j \geq 0}\) via the following algorithm:

**Algorithm 1:** Relaxed Forward-Douglas-Rachford splitting (relaxed FDRS)

input : \(z^0 \in \mathcal{H}, \gamma \in (0, \infty), (\lambda_j)_{j \geq 0} \in (0, \infty)\)

for \(k = 0, 1, \ldots\) do

\[
z^{k+1} = (1 - \lambda_k)z^k + \lambda_k \left( \frac{1}{2}I_{\mathcal{H}} + \frac{1}{2}\text{refl}_{\gamma f} \circ \text{refl}_{\chi_V} \right) \circ (I - \gamma P_V \circ \nabla g \circ P_V)(z^k);
\]

For now, we do not further specify the stepsize parameters. See section 1.6 for choices that guarantee convergence and, see Lemma 2.1 and Fig. 1 for why the algorithm works.

Evidently, Algorithm 1 has the alternative expression: for all \(k \geq 0\),

\[
z^{k+1} = (T_{\text{FDRS}})_{\lambda_k}(z^k)
\]

where

\[
T_{\text{FDRS}} := \left( \frac{1}{2}I_{\mathcal{H}} + \frac{1}{2}\text{refl}_{\gamma f} \circ \text{refl}_{\chi_V} \right) \circ (I - \gamma P_V \circ \nabla g \circ P_V). \tag{1.13}
\]

After we note that \(T_{\text{FDRS}}\) is nonexpansive (Part 7 of Theorem 1.1), it follows that the FDRS algorithm is a special case of the Krasnosel’ski–Mann (KM) iteration [26, 28, 12].

When \(g = 0\), FDRS reduces to the relaxed DRS algorithm [27] applied to \(f + \chi_V\):

\[
z^{k+1} = (1 - \lambda_k)z^k + \lambda_k \left( \frac{1}{2}I_{\mathcal{H}} + \frac{1}{2}\text{refl}_{\gamma f} \circ \text{refl}_{\chi_V} \right) (z^k). \tag{1.14}
\]

When \(V = \mathcal{H}\), FDRS reduces to the relaxed FBS algorithm [29] applied to \(f + g\):

\[
z^{k+1} = (1 - \lambda_k)z^k + \lambda_k \text{prox}_{\gamma f} \circ (I - \gamma \nabla g)(z^k).
\]

When \(f = 0\), FDRS reduces to the relaxed FBS algorithm applied to \(\chi_V + g \circ P_V\):

\[
z^{k+1} = (1 - \lambda_k)z^k + \lambda_k P_V \circ (z - \gamma P_V \circ \nabla g \circ P_V)(z^k).
\]

For general \(f, g\) and \(V\), the primal DRS and FBS algorithms are not capable splitting Problem (1.2) in the same way as (1.13). Indeed, the DRS algorithm cannot use the smooth structure of \(g\), and the FBS algorithm requires the evaluation of

\[
\text{prox}_{\gamma(f + \chi_V)}(\cdot) = \arg \min_{x \in V} f(x) + \frac{1}{2\gamma} \| x - \cdot \|^2.
\]

The FDRS algorithm eliminates these difficult subproblems and replaces them with tractable ones.

In the following sections, we recall basic properties of \(T_{\text{FDRS}}\) that will be useful in our convergence analysis.

1.5. Proximal, averaged and FDRS operators. We briefly review some operator-theoretic properties.

**Proposition 1.1.** Let \(\lambda > 0\), let \(\gamma > 0\), let \(\alpha > 0\), and let \(f : \mathcal{H} \to (-\infty, \infty]\) be closed, proper, and convex.
1. Optimality conditions of prox: Let \( x \in \mathcal{H} \). Then \( x^+ = \text{prox}_{\gamma f}(x) \) if, and only if,
\[
\nabla f(x^+) := \frac{1}{\gamma} (x - x^+) \in \partial f(x^+).
\]

2. Optimality conditions of \( \text{prox}_{\gamma V} \): Let \( x \in \mathcal{H} \). Then \( x^+ = \text{prox}_{\gamma V}(x) = P_V x \) if, and only if,
\[
\nabla \chi_V(x^+) := \frac{1}{\gamma} (x - x^+) \in \partial \chi_V(x^+).
\]
In addition, \( \gamma \nabla \chi_V(x^+) = P_{V^\perp} x \in V^\perp \).

3. Averaged operator contraction property: A map \( T : \mathcal{H} \to \mathcal{H} \) is \( \alpha \)-averaged (see Equation (1.10)) if, and only if, for all \( x, y \in \mathcal{H} \),
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1 - \alpha}{\alpha} \|(I_{\mathcal{H}} - T)x - (I_{\mathcal{H}} - T)y\|^2. \tag{1.15}
\]

4. Composition of averaged operators: Let \( \alpha_1, \alpha_2 \in (0,1) \). Suppose that \( T_1 : \mathcal{H} \to \mathcal{H} \) and \( T_2 : \mathcal{H} \to \mathcal{H} \) are \( \alpha_1 \) and \( \alpha_2 \)-averaged operators, respectively. Then for all \( x, y \in \mathcal{H} \), the map \( T_1 \circ T_2 : \mathcal{H} \to \mathcal{H} \) is
\[
\alpha_{1,2} := \frac{\alpha_1 + \alpha_2 - 2\alpha_1 \alpha_2}{1 - \alpha_1 \alpha_2} \in (0,1) \tag{1.16}
\]
averaged.

5. Wider relaxations: A map \( T : \mathcal{H} \to \mathcal{H} \) is \( \alpha \)-averaged if, and only if, \( T_\alpha \) (Equation (1.10)) is \( \lambda \alpha \)-averaged for all \( \lambda \in (0,1/\alpha) \).

6. Proximal operators are \((1/2)\)-averaged: The operator \( \text{prox}_{\gamma f} : \mathcal{H} \to \mathcal{H} \) satisfies the following contraction property:
\[
\|\text{prox}_{\gamma f}(x) - \text{prox}_{\gamma f}(y)\|^2 \leq \|x - y\|^2 - \|\text{prox}_{\gamma f}(x) - (y - \text{prox}_{\gamma f}(y))\|^2. \tag{1.17}
\]

Therefore, the operator \( \text{refl}_{\gamma f} = 2\text{prox}_{\gamma f} - I_{\mathcal{H}} \) is nonexpansive.

7. Averaged property of the FDRS operator: Suppose that \( \gamma \in (0,2\beta) \). Then the operator \( T_{\text{FDRS}} \) (Equation (1.13)) is averaged with weight
\[
\alpha_{\text{FDRS}} := \frac{2\beta}{4\beta - \gamma}. \tag{1.18}
\]

Proof. Part 1 follows from [3, Proposition 16.34]; Part 2 follows from Part 1 and the identity: \( I_{\mathcal{H}} = P_V + P_{V^\perp} \); Part 3 follows from [3, Proposition 4.25]; Part 4 follows from [17]; Part 5 follows from [3, Proposition 4.28]; Part 6 follows from [3, Corollary 23.10]. Part 7 follows from two facts: The operator \(((1/2)I_{\mathcal{H}} + (1/2)\text{refl}_{\gamma f}) \circ \text{refl}_{\gamma V}\) is \((1/2)\)-averaged by Part 6, and \( I - \gamma P_{\mathcal{H}} \circ \nabla \gamma P_{\mathcal{V}} \) is \((\gamma/2\beta)\)-averaged by [10, Proposition 4.1 (ii)]. Thus, Part 4 proves Equation (1.18). \( \square \)

Remark 1. In general, we desire smaller \( \alpha_{\text{FDRS}} \) because the relaxation parameters \( (\lambda_j)_{j \geq 0} \) will be chosen less than \( 1/\alpha_{\text{FDRS}} \). Note that the averaged coefficient in Equation (1.16) is quite new [17] and improves on the previously known estimate in [12, Lemma 2.2]. Consequently, the expression for \( \alpha_{\text{FDRS}} \) is also new and improves upon the previous constant
\[
\max \left\{ \frac{2}{3}, \frac{2\gamma}{\gamma + 2\beta} \right\}.
\]
See the [17, Remark 2.5 (i)] for verification that Equation (1.18) is smaller than the above equation.

The following proof is essentially contained in [17, Theorem 2.4 (iii)]. We reproduce it in order to derive a bound.

**Proposition 1.2.** Let \( \alpha_1, \alpha_2 \in (0,1) \). Suppose that \( T_1 : \mathcal{H} \to \mathcal{H} \) and \( T_2 : \mathcal{H} \to \mathcal{H} \) are \( \alpha_1 \) and \( \alpha_2 \)-averaged operators, respectively, and that \( z^* \) is a fixed point of \( T_1 \circ T_2 \). Define \( \alpha_1, \alpha_2 \in (0,1) \) as in Equation (1.16). Let \( z^0 \in \mathcal{H} \), let \( \varepsilon \in (0,1) \), and consider a sequence \( (\lambda_j)_{j \geq 0} \subseteq (0, (1 - \varepsilon)(1 + \varepsilon \alpha_1, \alpha_2)/\alpha_1, \alpha_2) \). Let \( (z_j)_{j \geq 0} \) be generated by the following iteration: for all \( k \geq 0 \), let

\[
z^{k+1} = (T_1 \circ T_2)_{\lambda_k}(z^k).
\]

Then

\[
\sum_{i=0}^\infty \lambda_k \| T_2 z^k - T_2 z^* \|^2 \leq \frac{\alpha_2(1 + 1/\varepsilon) \| z^0 - z^* \|^2}{1 - \alpha_2}.
\]

**Proof.** First we comment on the size of \( \lambda_k \). Note that

\[
\lambda_k \leq \frac{(1 - \varepsilon)(1 + \varepsilon \alpha_1, \alpha_2)}{\alpha_1, \alpha_2} \implies \lambda_k \leq 1/\alpha_1, \alpha_2 - \varepsilon^2 \text{ and } \lambda_k - 1 \leq \frac{1 - \alpha_1, \alpha_2 \lambda_k}{\alpha_1, \alpha_2 \varepsilon}.
\]

See [17, Theorem 2.4 (iii)] for a proof. We will need these bounds throughout the rest of our proof.

For all \( k \geq 0 \), set

\[
p^k = \frac{1 - \alpha_1}{\alpha_1} \| (I_{\mathcal{H}} - T_1) \circ T_2(z^k) - (I_{\mathcal{H}} - T_1) \circ T_2(z^*) \|^2 + \frac{1 - \alpha_2}{\alpha_2} \| (I_{\mathcal{H}} - T_2)(z^k) - (I_{\mathcal{H}} - T_2)(z^*) \|^2.
\]

By iteratively applying Equation (1.15), we get \( \| T_1 \circ T_2(z^k) - T_1 \circ T_2(z^*) \|^2 \leq \| z^k - z^* \|^2 - p^k \).

Part 5 of Proposition 1.1 shows that \( (T_1 \circ T_2)_{\lambda_k} \) is \((\alpha_1, \alpha_2 \lambda_k)\)-averaged. Thus, by Equation (1.15), we have

\[
\| z^{k+1} - z^* \|^2 \leq \| z^k - z^* \|^2 - \frac{\lambda_k(1 - \alpha_1, \alpha_2)}{\alpha} \| T_1 \circ T_2(z^k) - z^k \|^2.
\]

Therefore,

\[
\sum_{i=0}^\infty \frac{\lambda_k(1 - \alpha_1, \alpha_2)}{\alpha_1, \alpha_2} \| T_1 \circ T_2(z^k) - z^k \|^2 \leq \| z^0 - z^* \|^2.
\]

Finally, we recall the following vectorial identity (see [3, Corollary 2.4]): for all \( x, y \in \mathcal{H} \) and all \( \lambda \in \mathbb{R} \), we have

\[
\| \lambda x + (1 - \lambda)y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \| x - y \|^2.
\]

Therefore, by Equation (1.20), we have

\[
\| z^{k+1} - z^* \|^2
\]

\[
= (1 - \lambda_k) \| z^k - z^* \|^2 + \lambda_k \| T_1 \circ T_2(z^k) - T_1 \circ T_2(z^*) \|^2 - \lambda_k(1 - \lambda_k) \| z^k - T_1 \circ T_2(z^k) \|^2
\]

\[
\leq \| z^k - z^* \|^2 - \lambda_k p^k + \lambda_k(\lambda_k - 1) \| z^k - T_1 \circ T_2(z^k) \|^2
\]

\[
\leq \| z^k - z^* \|^2 - \lambda_k p^k + \frac{\lambda_k(1 - \alpha_1, \alpha_2 \lambda_k)}{\alpha_1, \alpha_2 \varepsilon} \| z^k - T_1 \circ T_2(z^k) \|^2.
\]
Thus, $\sum_{i=0}^{\infty} \lambda_k p^k \leq (1 + 1/\varepsilon) \|z^0 - z^*\|^2$. The claimed bound follows by algebraic manipulation. \(\square\)

1.6. Convergence properties of FDRS. This section will summarize the main convergence properties, such as boundedness and summability, that we use to deduce convergence rates of the FDRS algorithm.

First we show that the parameter $\gamma$ can (possibly) be increased, which can result in faster convergence rates in practice. The proof follows by constructing a new Lipschitz differentiable function $h$ so that the triple $(f, h, V)$ generates the same FDRS operator, $T_{\text{FDRS}}$, as $(f, g, V)$. This result was not included in [10].

**Lemma 1.3.** Define a function $h := g \circ P_V$. (1.21)

Then the FDRS operator associated to $(f, g, V)$ is identical to the FDRS operator associated to $(f, h, V)$. Let $1/\beta$ be the Lipschitz constant of $\nabla h$. Then $\beta_V \geq \beta$. In addition, let $\gamma \in (0, 2\beta_V)$. Then $T_{\text{FDRS}}$ is $\alpha_{\text{FDRS}}^V$-averaged where

$$\alpha_{\text{FDRS}}^V := \frac{2\beta_V}{4\beta_V - \gamma}. \quad (1.22)$$

**Proof.** The bound $\beta_V \geq \beta$ follows because

$$\|\nabla h(x) - \nabla h(y)\| = \|P_V \circ g \circ P_V(x) - P_V \circ g \circ P_V(y)\|$$

$$\leq \|\nabla g \circ P_V(x) - \nabla g \circ P_V(y)\|$$

$$\leq (1/\beta)\|P_V(x) - P_V(y)\|$$

$$\leq (1/\beta)\|x - y\|$$

for all $x, y \in H$. The averaged property of $T_{\text{FDRS}}$ and the equivalence of FDRS operators follows from Part 7 of Proposition 1.1 and the identity $\nabla h = P_V \circ \nabla g \circ P_V$. \(\square\)

Throughout the rest of the paper, we will study the problem

$$\min_{x \in V} f(x) + h(x),$$

which is equivalent to Problem (1.2) because $g|_V = h|_V$.

Most of our results do not require that $(z^j)_{j \geq 0}$ converges. However, for completeness we include the following weak convergence result.

**Proposition 1.4.** Let $\gamma \in (0, 2\beta_V)$, let $(\lambda_j)_{j \geq 0} \subseteq (0, 1/\alpha_{\text{FDRS}}^V)$, and suppose that

$$\sum_{i=0}^{\infty} \frac{1 - \lambda_i \alpha_{\text{FDRS}}^V}{\lambda_i \alpha_{\text{FDRS}}^V} = \infty.$$ 

Then the FDRS algorithm weakly converges to a fixed point of $T_{\text{FDRS}}$.

**Proof.** Apply [10, Proposition 3.1] with the new averaged parameter $\alpha_{\text{FDRS}}^V$. \(\square\)

The following theorem recalls several results on convergence rates for the iteration of averaged operators [21]. In addition, we show that $(\lambda_j\|\nabla h(z^j) - \nabla h(z^*)\|^2)_{j \geq 0}$ is a summable sequence [10] whenever $(\lambda_j)_{j \geq 0}$ is chosen properly. The summability of this sequence is crucial to our analysis.

**Theorem 1.5.** Suppose that $(z^j)_{j \geq 0}$ is generated by Algorithm 1 with $\gamma \in (0, 2\beta_V)$ and $(\lambda_j)_{j \geq 0} \subseteq (0, 1/\alpha_{\text{FDRS}}^V)$. Let $z^*$ be a fixed point of $T_{\text{FDRS}}$. Then the following hold:
1. **Fejér monotonicity:** the sequence \((\|z^j - z^*\|^2)_{j \geq 0}\) is monotonically nonincreasing. In addition, for all \(z \in H\) and \(\lambda \in (0, 1/\alpha_{\text{FDRS}}^V)\), we have
\[
\|T_{\text{FDRS}}(\lambda z) - z^*\| \leq \|z - z^*\|.
\]

2. **Summable fixed-point residual:** The sum is finite:
\[
\sum_{i=0}^{\infty} \frac{1 - \lambda_k \alpha_{\text{FDRS}}^V}{\lambda_k \alpha_{\text{FDRS}}^V} \|z^{i+1} - z^i\|^2 \leq \|z^0 - z^*\|^2.
\]

3. **Convergence rates of fixed-point residual:** For all \(k \geq 0\), let \(\tau_k := \inf_{j \geq 0} \tau_j > 0\). Suppose that \(\tau_k := \inf_{j \geq 0} \tau_j > 0\). Then for all \(\lambda > 0\) and \(k \geq 0\),
\[
\|T_{\text{FDRS}}(\lambda z^k) - z^k\|^2 \leq \frac{\lambda^2 \|z^0 - z^*\|^2}{\tau(k+1)} \quad \text{and} \quad \|T_{\text{FDRS}}(\lambda z^k) - z^k\|^2 = O\left(\frac{1}{k+1}\right). \tag{1.24}
\]

4. **Gradient summability:** Let \(\varepsilon \in (0, 1)\) and suppose that
\[
(\lambda_j)_{j \geq 0} \subseteq \left(0, \frac{(1-\varepsilon)(1+\varepsilon \alpha_{\text{FDRS}}^V)}{\alpha_{\text{FDRS}}^V}\right). \tag{1.25}
\]
Then the following gradient sum is finite:
\[
\sum_{i=0}^{\infty} \lambda_i \|\nabla h(z^i) - \nabla h(z^*)\|^2 \leq \frac{\gamma(1+\varepsilon)}{\varepsilon(2\beta_V - \gamma)} \|z^0 - z^*\|^2. \tag{1.26}
\]

**Proof.** Parts 1, 2, and 3 are a direct consequence of [21, Theorem 1] applied to the \(\alpha_{\text{FDRS}}^V\)-averaged operator \(T_{\text{FDRS}}\).

Part 4 is a direct application of Proposition 1.2 applied to the \((1/2)\)-averaged operator \(T_1 = \frac{1}{2}I_H + \frac{1}{2}\text{refl}_f \circ \text{refl}_{\chi V}\) (see Part 6 of Theorem 1.1) and the \((\gamma/(2\beta_V))\)-averaged operator \(T_2 = I - \gamma \nabla h\) (from the Baillon-Haddad Theorem [1] and [3, Proposition 4.33]).

Throughout the rest of this paper, the term
\[
\|T_{\text{FDRS}}z^k - z^k\|^2 = \frac{1}{\lambda_k} \|z^{k+1} - z^k\|^2 \tag{1.27}
\]
will be called the fixed-point residual (FPR).

**Remark 2.** Note that the convergence rate proved for \(\|T_{\text{FDRS}}z^k - z^k\|^2\) in Equation (1.24) is optimal for the \(T_{\text{FDRS}}\) operator [21, Section 6.1.1].

### 2. Subgradients and fundamental inequalities
In this section, we identify several key algebraic identities that hold for the FDRS algorithm. In addition, we prove a key relationship between the FPR and the objective error at the \(k\)-th iteration (Propositions 2.3 and 2.4).

In first-order optimization algorithms, we only have access to (sub)gradients and function values. Consequently, the FPR is usually the squared norm of a linear combination of (sub)gradients of the objective functions. For example, the gradient descent algorithm for a differentiable function \(f\) generates a sequence of iterates by using forward gradient steps: given \(z^0 \in H\) and \(\gamma > 0\), for all \(k \geq 0\), define
\[
z^{k+1} = z^k - \gamma \nabla f(z^k).
\]
For this algorithm, the FPR is a multiple of the squared norm of the gradient:
\[
\|z^{k+1} - z^k\|^2 = \gamma^2 \|\nabla f(z^k)\|^2.
\]

In splitting algorithms, the FPR is usually more complicated because the subgradients are generated via forward-gradient or proximal (backward) steps (see Part 1 of Proposition 1.1) that are taken at different points. Thus, unlike the gradient descent algorithm where the objective error
\[
f(z^k) - f(x^*) \leq \langle z^k - x^*, \nabla f(x^k) \rangle
\]
can be bounded with the subgradient inequality, splitting algorithms for two or more functions can usually only bound the objective error when some or all of the functions are evaluated at separate points — unless a Lipschitz continuity assumption is imposed. In order to use this Lipschitz assumption, we need to bound the difference between the variables assigned to each function, i.e., to enforce consensus among the variables, which is where the FPR convergence rate becomes useful.

2.1. A subgradient representation of FDRS. In this section, we will write the FDRS algorithm in terms of the (sub)gradients that are implicitly and explicitly computed in each iteration of the algorithm. Our goal is to prove the identities in Fig. 1 and Lemma 2.1.

The way to read Fig. 1 is as follows: FDRS projects \( z \) onto \( V \) to get \( x_h = z - \gamma \nabla \chi_V(x_h) \). The reflection of \( z \) across \( V \) is \( x_h - \gamma \nabla \chi_V(x_h) = z - 2\gamma \nabla \chi_V(x_h) \). Then FDRS takes a forward-gradient with respect to \( \nabla h(x_h) \) from the reflected point \( x_h - \gamma \nabla \chi_V(x_h) \) and a proximal (backward) step with respect to \( f \) to get \( x_f \). Finally, we move from \( x_f \) to \( T_{\text{FDRS}}z \) by traveling along the positive subgradient \( \gamma \nabla \chi_V(x_h) \).

The following is an interesting consequence of these identities: If we apply \( P_V \) to all of the points in Fig. 1, then we collapse the square to a line. Thus, if we iteratively apply FDRS with \( \lambda_k \equiv 1 \), then the “collapsed” FDRS iteration would move from \( x_h^k \) to \( x_h^{k+1} \) and it would look very similar to a forward-backward algorithm.

**Lemma 2.1 (FDRS identities).** Let \( z \in H \). Define auxiliary points \( x_h \) and \( x_f \) by the formulas:
\[
x_h := \text{proj}_V z \quad \text{and} \quad x_f := \text{prox}_{\gamma f} \circ \text{refl}_{\chi_V} \circ (I_H - \gamma \nabla h)(z).
\]
Then the identities hold
\[ x_h := z - \gamma \nabla \chi_V(x_h) \quad \text{and} \quad x_f := x_h - \gamma \left( \nabla \chi_V(x_h) + \nabla h(x_h) + \nabla f(x_f) \right) \quad (2.1) \]

where \( \nabla \chi_V(x_h) = (1/\gamma)P_{V^\perp}(z) \) and \( \nabla f(x_f) \) is uniquely defined by Part 1 of Proposition 1.1. In addition, each FDRS step has the following form:
\[ (T_{\text{FDRS}})\lambda(z) - z = \lambda(x_f - x_h) = -\gamma \lambda \left( \nabla \chi_V(x_h) + \nabla h(x_h) + \nabla f(x_f) \right). \quad (2.2) \]

In particular, \( T_{\text{FDRS}}(z) = x_f + \gamma \nabla \chi_V(x_h) \).

**Proof.** The identity for \( x_h = z - \gamma \nabla \chi_V(x_h) \) follows from Part 1 of Proposition 1.1. Note that by the Moreau identity \( P_{V^\perp} = I - P_V \), we have \( \nabla \chi_V(x_h) = P_{V^\perp}z \). Note that by definition, \( \nabla h(z) = P_V \circ \nabla g \circ P_V(z) = P_V \circ \nabla g(x_h) = \nabla h(x_h) \) and \( \nabla h(z) \in V \). Thus, we get the identity for \( x_f \):
\[
\proxf \circ \refl_{\chi_V} \circ (I_{\mathcal{H}} - \gamma \nabla h)(z) = \proxf \circ (I_{\mathcal{H}} - \gamma \nabla h)(z) - \gamma \nabla f(x_f) = P_V(z - \gamma \nabla h(z)) + (P_V - I_{\mathcal{H}})(z - \gamma \nabla h(z)) - \gamma \nabla f(x_f) = x_h - \gamma \nabla h(z) - P_{V^\perp}z - \gamma \nabla f(x_f) = x_h - \gamma \left( \nabla \chi_V(x_h) + \nabla h(x_h) + \nabla f(x_f) \right). 
\]

Finally, given the identity \( (T_{\text{FDRS}})\lambda(z) - z = \lambda(T_{\text{FDRS}}(z) - z) \), Equation (2.2) will follow as soon as we show \( T_{\text{FDRS}}(z) = x_f + z - x_h = x_f + \gamma \nabla \chi_V(x_h) \):
\[
\left( \frac{1}{2}I_{\mathcal{H}} + \frac{1}{2} \proxf \circ \refl_{\chi_V} \right)(z - \gamma \nabla h(z)) = \left( \proxf \circ \refl_{\chi_V} + I_{\mathcal{H}} - P_V \right)(z - \gamma \nabla h(z)) = \proxf \circ \refl_{\chi_V}(z - \gamma \nabla h(z)) + (I_{\mathcal{H}} - P_V)(z - \gamma \nabla h(z)) = x_f + P_{V^\perp}(z - \gamma \nabla h(z)) = x_f + \gamma \nabla \chi_V(x_h)
\]

where the first equality follows by algebraic expansion. \( \square \)

**2.2. Optimality conditions of FDRS.** The following lemma characterizes the zeros of \( \partial f + \nabla h + \partial \chi_V \) in terms of the fixed points of the FDRS operator. The intuition is the following: If \( z^* \) is a fixed point of \( T_{\text{PRS}} \), then the base of the rectangle in Figure 1 has length zero. Thus, \( z^* := x^*_h = x^*_f \), and if we travel around the perimeter of the rectangle, we will start and begin at \( z^* \). This argument shows that \( \gamma \nabla f(x^*) + \gamma \nabla h(x^*) + \gamma \nabla \chi_V(x^*) = 0 \), i.e., \( z^* \in \zer(\partial f + \nabla h + \partial \chi_V) \).

**Lemma 2.2** (FDRS optimality conditions). The following set equality holds:
\[
\zer(\partial f + \nabla h + \partial \chi_V) = \{ P_V z \mid z \in \mathcal{H}, T_{\text{FDRS}} z = z \}
\]

That is, if \( z^* \) is a fixed point of \( T_{\text{FDRS}} \), then \( x^* := P_V z^* = x^*_h = x^*_f \) is a minimizer of Problem (1.2), and
\[
z^* - x^* = P_{V^\perp}(z^*) = \gamma \nabla \chi_V(x^*_h) \in \partial \chi_V(x^*). \]
Proof. Let \( x \in \text{zer}(\partial f + \nabla h + \partial \chi_V) \). Choose subgradients \( \tilde{\nabla} f(x) \in \partial f(x) \) and \( \tilde{\nabla} \chi_V(x) \in \partial \chi_V(x) = V^\perp \) (Equation (1.9)) such that \( \tilde{\nabla} f(x) + \nabla h(x) + \tilde{\nabla} \chi_V(x) = 0 \) and set \( z := x + \gamma \tilde{\nabla} \chi_V(x) \). We claim that \( z \) is a fixed point of \( T_{\text{FDRS}} \). From Lemma 2.1, we get the points:

\[
x_h := P_V(z) = x \quad \text{and} \quad x_f := \text{prox}_{\gamma f} \circ \text{refl}_{\chi_V} \circ (I_\mathcal{H} - \gamma \tilde{\nabla} h)(z).
\]

But \( \tilde{\nabla} \chi_V(x_h) + \nabla h(x_h) \in -\partial f(x) \), and

\[
\text{refl}_{\chi_V} \circ (I_\mathcal{H} - \gamma \tilde{\nabla} h)(z) = P_V(z - \gamma \tilde{\nabla} h(z)) + (P_V - I_\mathcal{H})(z - \gamma \tilde{\nabla} h(z))
= x - \gamma \tilde{\nabla} h(x) - P_{V^\perp} z
= x - \gamma \tilde{\nabla} h(x) - \gamma \tilde{\nabla} \chi_V(x)
= x + \gamma \tilde{\nabla} f(x).
\]

Therefore, \( x_f = \text{prox}_{\gamma f} \gamma f(x + \gamma \tilde{\nabla} f(x)) = x \) (see Part 1 of Proposition 1.1). Thus, by Lemma 2, \( T_{\text{FDRS}} z = z + x_f - x_h \). Therefore, we have proved the first inclusion.

On the other hand, suppose that \( z \in \mathcal{H} \) and \( T_{\text{FDRS}} z = z \). Then \( x := x_h = P_V z \), and \( 0 = T_{\text{FDRS}} z - z = x_f - x_h = \gamma \left( \tilde{\nabla} \chi_V(x_h) + \nabla h(x_h) + \tilde{\nabla} f(x_f) \right) \). Because \( x_f = x_h \), we get \( x \in \text{zer}(\partial f + \nabla h + \partial \chi_V) \). \( \square \)

2.3. Fundamental inequalities. In this section, we prove two fundamental inequalities that relate to the FPR (see Equation (1.27)) to the objective error. Without any Lipschitz continuity assumption, it seems impossible to bound the true objective error \( (f + h + \chi_V)(x) - (f + h + \chi_V)(x^*) \). Thus, we focus on bounding a modified objective error where \( h + \chi_V \) and \( f \) are not necessarily evaluated at the same point. This modified objective error is no longer positive. Therefore, we provide upper and lower bounds in Propositions 2.3 and 2.4.

Throughout the rest of the paper, we utilize the following notation: The functions \( f \) and \( g \) are \( \mu_f \) and \( \mu_g \)-strongly convex, respectively, where we allow the possibility that \( \mu_f = 0 \) (i.e., no strong convexity). In addition, we assume that \( f \) is \((1/\beta_f)\)-Lipschitz differentiable, where we allow the possibility that \( \beta_f = 0 \). If \( \beta_f > 0 \), then \( \tilde{\nabla} f = \nabla f \). With these assumptions, we get the following lower bounds \([3, \text{Theorem 18.15}]\): For all \( x, y \in \text{dom}(f) \), \( \tilde{\nabla} f(y) \in \partial f(y) \), and \( \tilde{\nabla} f(x) \in \partial f(x) \), we have

\[
f(x) \geq f(y) + \langle x - y, \tilde{\nabla} f(y) \rangle + \max \left\{ \frac{\mu_f}{2} \|x - y\|^2, \frac{\beta_f}{2} \|\tilde{\nabla} f(x) - \tilde{\nabla} f(y)\|^2 \right\}.
\]

Similarly, for all \( x, y \in \mathcal{H} \), we have

\[
h(x) \geq h(y) + \langle x - y, \nabla h(y) \rangle + \max \left\{ \frac{\mu_g}{2} \|P_V x - P_V y\|^2, \frac{\beta_V}{2} \|\nabla h(x) - \nabla h(y)\|^2 \right\}.
\]

These lower bounds motivate the following notation:

\[
S_f(x, y) := \max \left\{ \frac{\mu_f}{2} \|x - y\|^2, \frac{\beta_f}{2} \|\tilde{\nabla} f(x) - \tilde{\nabla} f(y)\|^2 \right\} \tag{2.3}
\]

\[
S_h(x, y) := \max \left\{ \frac{\mu_g}{2} \|P_V x - P_V y\|^2, \frac{\beta_V}{2} \|\nabla h(x) - \nabla h(y)\|^2 \right\} \tag{2.4}
\]
Choose subgradients $\tilde{S}$. Note that

$$\leq ||z - x||^2 - ||z^+ - x||^2 + \left(1 - \frac{2}{\lambda}\right) ||z^+ - z||^2 + 2\gamma\langle\nabla h(x_h), z - z^+\rangle$$ (2.5)

**Proof.** In the following derivation, we use the subgradient inequality, Lemma 2.1, the cosine rule, and the inclusion $\nabla \chi_V(x_h) \in V^\perp$:

$$2\gamma\lambda(f(x_f) + h(x_h) - f(x) - h(x) + S_f(x_f, x) + S_h(x_h, x))$$

$$\leq 2\gamma\lambda \left(\langle \nabla f(x_f), x_f - x \rangle + \langle \nabla h(x_h), x_h - x \rangle + \langle \nabla \chi_V(x_h), x_h - x \rangle\right)$$

$$= 2\gamma\lambda \left(\langle \nabla f(x_f) + \nabla h(x_h) + \nabla \chi_V(x_h), x_f - x \rangle + \langle \nabla h(x_h) + \nabla \chi_V(x_h), x_h - x_f \rangle\right)$$

$$= 2\langle z - z^+, x_f - x \rangle + 2\gamma\langle \nabla h(x_h) + \gamma\nabla \chi_V(x_h), z - z^+\rangle$$

$$= 2\langle z - z^+, x_f + \gamma\nabla \chi_V(x_h) - x \rangle + 2\gamma\langle \nabla h(x_h), z - z^+\rangle$$

$$= 2\langle z - z^+, T_{\text{FDRS}}z - x \rangle + 2\gamma\langle \nabla h(x_h), z - z^+\rangle$$

$$= 2\langle z - z^+, z - x \rangle + \frac{2}{\lambda}\langle z - z^+, z^+ - z \rangle + 2\gamma\langle \nabla h(x_h), z - z^+\rangle$$

(1.11) $\Rightarrow ||z - x||^2 - ||z^+ - x||^2 + \left(1 - \frac{2}{\lambda}\right) ||z^+ - z||^2 + 2\gamma\langle \nabla h(x_h), z - z^+\rangle$.

**Proposition 2.4 (Lower fundamental inequality).** Let $z^* \in \mathcal{H}$ be a fixed point of $T_{\text{FDRS}}$, and let $x^* := P_{\mathcal{H}}z^*$ be a minimizer of Problem (1.2) (see Lemma 2.2). Choose subgradients $\nabla f(x^*) \in \partial f(x^*)$ and $\nabla \chi_V(x^*) \in \partial \chi_V(x^*)$ such that $\nabla f(x^*) + \nabla h(x^*) + \nabla \chi_V(x^*) = 0$. Then for all $x_f \in \text{dom}(f)$ and $x_h \in \mathcal{V}$, we have

$$f(x_f) + h(x_h) - f(x^*) - g(x^*) \geq \langle x_f - x_h, \nabla f(x^*) \rangle + S_f(x_f, x^*) + S_h(x_h, x^*)$$ (2.6)

**Proof.** By the subgradient inequality and the inclusion $\nabla \chi_V(x_h) \in V^\perp$:

$$f(x_f) - f(x^*) \geq \langle x_f - x^*, \nabla f(x^*) \rangle + S_f(x_f, x^*)$$

$$h(x_h) - h(x^*) \geq \langle x_h - x^*, \nabla h(x^*) \rangle + S_h(x_h, x^*)$$

$$\langle x_h - x^*, \nabla \chi_V(x^*) \rangle = 0.$$ 

Therefore, we add these equations to get

$$f(x_f) + h(x_h) - f(x^*) - g(x^*)$$

$$\geq \langle x_f - x^*, \nabla f(x^*) + \nabla h(x^*) + \nabla \chi_V(x^*) \rangle + \langle x_f - x_h, \nabla f(x^*) \rangle$$

$$+ S_f(x_f, x^*) + S_h(x_h, x^*)$$

$$= \langle x_f - x_h, \nabla f(x^*) \rangle + S_f(x_f, x^*) + S_h(x_h, x^*).$$
3. Objective convergence rates. In this section, we analyze the ergodic and nonergodic convergence rates of the FDRS algorithm applied to Problem (1.2).

Throughout the rest of the paper, $z^*$ will denote an arbitrary fixed point of $T_{\text{FDRS}}$, and we define a minimizer of Problem (1.2) using Lemma 2.2:

$$x^* := P_V z^*.$$  

All of our bounds will be produced on objective errors of the form:

$$f(x_j^k) + h(x_h^k) - f(x^*) - g(x^*) \quad \text{and} \quad f(x_h^k) + h(x_h^k) - f(x^*) - g(x^*). \quad (3.1)$$

The objective error on the left hand size of Equation (3.1) can be negative. Thus, we bound the absolute value of the objective error. In addition, we bound $||x_j^k - x_h^k||$. If $f$ is evaluated at $x_h^k$, then we must have $x_h^k \in \text{dom}(f)$. Because $x_h^k \in V$, the objective error on the right hand size of Equation (3.1) is positive. Consequently, $x_h^k$ is the natural point at which to measure the convergence rate. To produce this type of bound, we must make a Lipschitz assumption on $f$. Note that in both cases, we have the identity $h(x_h^k) = (g \circ P_V)(x_h^k) = g(x_h^k)$.

We choose not to measure the absolute values of the objective errors

$$f(x_j^k) + h(x_h^k) - f(x^*) - g(x^*)$$

because, in general, $h(x_j^k) \neq g(x_j^k)$ and $x_j^k \notin V$. However, we note that it is easy to modify our analysis to do so. The main difference would be to apply the descent theorem [3, Theorem 18.15] on the term:

$$2\gamma \langle \nabla h(x_h^k), z^k - z^{k+1} \rangle = 2\gamma \lambda \langle \nabla h(x_h^k), x_h^k - x_j^k \rangle \leq 2\gamma \lambda (h(x_h^k) - h(x_j^k)) + \frac{\gamma \lambda}{\beta} ||x_j^k - x_h^k||^2.$$

and modify the upper fundamental inequality in Proposition 2.3.

Finally, all of our lower bounds will involve the subgradient norms $||\tilde{\nabla} f(x^*)||$, $||\nabla h(x^*)||$, and $||\tilde{\nabla} \chi_V(x^*)||$. We always assume that these norms to be minimal over all $\tilde{\nabla} f(x^*)$ satisfying $\tilde{\nabla} f(x^*) + \nabla h(x^*) + \tilde{\nabla} \chi_V(x^*) = 0$ (See Proposition 2.3). We make this assumption throughout the rest of the paper.

3.1. Ergodic convergence rates. In this section, we analyze the ergodic convergence rate of the FDRS algorithm. The key idea is to use the telescoping property of the upper and lower fundamental inequalities, together with the summability of the difference of gradients shown in Part 4 of Theorem 1.5. See section (1.2) for the distinction between ergodic and nonergodic convergence rates.

THEOREM 3.1 (Ergodic convergence of FDRS). Let $\gamma \in (0, 2\beta_V)$, let $\varepsilon \in (0, 1)$, and suppose that $(\lambda_j)_{j \geq 0}$ satisfies Equation (1.25). Then we have the following convergence rate:

$$2\varepsilon \leq \frac{||z^0 - z^*||}{\Lambda_k} \leq f(x_j^k) + h(x_h^k) - f(x^*) - h(x^*) \leq \left( ||z^0 - z^*|| + 2\gamma ||\nabla h(x^*)|| + \frac{1 + \varepsilon}{x_j^k(2\beta_V - \gamma)} ||z^0 - z^*|| \right) ||z^0 - z^*|| \leq \frac{2\gamma \lambda}{2\gamma \Lambda_k}.$$
In addition the following feasibility bound holds:

\[ \|\bar{x}_k - x_k\| \leq \frac{2}{\Lambda_k} \|z^0 - z^*\|. \quad (3.2) \]

**Proof.** Fix \( k \geq 0 \). The feasibility bound follows from Part 1 of Theorem 1:

\[ \|\bar{x}_k - x_k\| = \left\| \frac{1}{\Lambda_k} \sum_{i=0}^{k} \left( z^{i+1} - z^i \right) \right\| = \frac{1}{\Lambda_k} \|z^0 - z^{k+1}\| \leq \frac{1}{\Lambda_k} (\|z^0 - z^*\| + \|z^* - z^{k+1}\|) \]

\[ \leq \frac{2}{\Lambda_k} \|z^0 - z^*\|. \quad (3.3) \]

See [21, Theorem 5] for a general tool to derive such ergodic rates.

Now we prove the objective convergence rates. For all \( k \geq 0 \), let \( \eta_k := 2/\lambda_k - 1 \). Note that \( \eta_k > 0 \). Indeed, by expanding the upper bound in Equation (1.25), we have \( \lambda_k < 1/\alpha_{FDRS}^V - \varepsilon^2 \leq 2 - \varepsilon^2 \) (see Equation (1.19)). Furthermore, \( 1/\eta_k = \lambda_k/(2 - \lambda_k) \leq \lambda_k/(2 - 2 + \varepsilon^2) = \lambda_k/\varepsilon^2 \). Now, by the Cauchy-Schwarz inequality and Young’s inequality for real numbers (Equation (1.12)), we have

\[ 2\gamma(\nabla h(x_k^i), z^i - z^{i+1}) = 2\gamma(\nabla h(x_k^i), z^i - z^{i+1}) + 2\gamma(\nabla h(x_k^i) - \nabla h(x_k^*), z^i - z^{i+1}) \]

\[ \leq 2\gamma(\nabla h(x_k^*), z^i - z^{i+1}) + \frac{\varepsilon^2}{\eta_k} \|\nabla h(x_k^i) - \nabla h(x_k^*)\|^2 \]

\[ + \eta_k \|z^i - z^{i+1}\|^2. \quad (3.4) \]

Therefore, by Jensen’s inequality, the Cauchy-Schwarz inequality, the upper fundamental inequality in Proposition (2.3), and the bound \( \|z^0 - z^{k+1}\| \leq 2\|z^0 - z^*\| \) (see Equation (3.3)), we have

\[ f(\bar{x}_k^i) + h(\bar{x}_k^i) - f(x^*) - h(x^*) \]

\[ \leq \frac{1}{2\gamma\Lambda_k} \sum_{i=0}^{k} \lambda_i \left( f(\bar{x}_k^i) + h(\bar{x}_k^i) - f(x^*) - h(x^*) \right) \]

\[ \leq \frac{1}{2\gamma\Lambda_k} \sum_{i=0}^{k} \left( \|z^i - x^*\|^2 - \|z^{i+1} - x^*\|^2 - \eta_k \|z^{i+1} - z^i\|^2 + 2\gamma(\nabla h(x_k^i), z^i - z^{i+1}) \right) \]

\[ \leq \frac{1}{2\gamma\Lambda_k} \left( \|z^0 - x^*\|^2 + 2\gamma\|\nabla h(x^*)\|\|z^0 - z^*\| + (1 + \varepsilon\gamma^2)\|z^0 - z^*\|^2/(\varepsilon^2(2\gamma - \gamma)) \right) \]

\[ \leq \frac{2\gamma\Lambda_k}{2\gamma\Lambda_k} \left( \|z^0 - x^*\|^2 + 2\gamma\|\nabla h(x^*)\|\|z^0 - z^*\| + (1 + \varepsilon\gamma^2)\|z^0 - z^*\|^2/(\varepsilon^2(2\gamma - \gamma)) \right) \]

\[ \leq 2\|z^0 - z^*\|^2 \quad (3.5) \]

where the last inequality uses the Cauchy-Schwarz inequality and the bound \( \|z^0 - z^{k+1}\| \leq 2\|z^0 - z^*\| \) (see Equation (3.3)).

The lower bound in Proposition 2.4 and the Cauchy-Schwarz inequality show that

\[ f(\bar{x}_k^i) + h(\bar{x}_k^i) - f(x^*) - h(x^*) \geq \langle \bar{x}_k^i - x_k^i, \nabla f(x^*) \rangle \]

\[ \geq -\|\bar{x}_k^i - x_k^i\|\|\nabla f(x^*)\| \]

\[ \geq -\frac{2\|z^0 - z^*\|\|\nabla f(x^*)\|}{\Lambda_k}. \quad (3.2) \]
In general, \( x_k^s \notin \text{dom}(f) \). However, the conclusion of Theorem 3.1 can be improved if \( f \) is Lipschitz continuous. The following proposition gives a sufficient condition for Lipschitz continuity on a ball.

**Proposition 3.2 (Lipschitz continuity on a ball).** Suppose that \( f : \mathcal{H} \to (-\infty, \infty) \) is proper and convex. Let \( \rho > 0 \), and let \( x_0 \in \mathcal{H} \). Then, whenever \( \delta = \sup_{x,y \in B(x_0,2\rho)} |f(x) - f(y)| < \infty \), it follows that \( f \) is \((\delta/\rho)\)-Lipschitz on \( B(x_0, \rho) \).

**Proof.** See [3, Proposition 8.28]. \( \square \)

To use this fact, we need to show that the sequences \((x_j^s)_{j \geq 0}\), and \((x_k^s)_{j \geq 0}\) are bounded. Recall that \( x_k^s = P_V(z^s) \) and \( x_j^s = \text{prox}_{f_j}(\text{refl}_{\mathcal{H}}(I_{\mathcal{H}} - \gamma \nabla h)(z^s)) \) for \( s \in \{\ast, k\} \). Proximal, reflection, and forward-gradient maps are nonexpansive (see Proposition 1.1 for proximal and reflection operators and see the Baillon-Haddad Theorem [1] and [3, Proposition 4.33] for forward-gradient operators), so we have the following simple bound:

\[
\max\{\|x^k_j - x^s\|, \|x^s_j - x^s\|\} \leq \|z^k - z^s\| \leq \|z^0 - z^s\|.
\]

Thus, \((x_j^s)_{j \geq 0}, (x_k^s)_{j \geq 0} \subseteq B(x^s, \|z^0 - z^s\|)\). By the convexity of the ball, we also have \((x_j^s)_{j \geq 0}, (x_k^s)_{j \geq 0} \subseteq B(x^s, \|z^0 - z^s\|)\).

**Corollary 3.3 (Ergodic convergence of FDRS with Lipschitz \( f \)).** Let the notation be as in Theorem 3.1. Let \( L \geq 0 \) and suppose that \( f \) is \( L \)-Lipschitz on \( B(x^s, \|z^0 - z^s\|) \). Then

\[
0 \leq f(x^k_j) + h(x^k_j) - f(x^s) - h(x^s)
\leq \frac{\left(\|z^0 - z^s\| + 2\gamma \|
abla h(x^s)\| + \frac{(1+\gamma)\|z^0 - z^s\|}{L(1+\gamma)}\right) \|z^0 - z^s\|}{2k\lambda_k} + \frac{2L\|z^0 - z^s\|}{k\lambda_k}.
\]

**Proof.** The proof follows from by combining the upper bound in Theorem 3.1 with the following bound:

\[
f(x^k_j) + h(x^k_j) - f(x^s) - h(x^s) \leq f(x^k_j) + h(x^k_j) - f(x^s) - h(x^s) + L\|x^k_j - x^s\|
\leq f(x^k_j) + h(x^k_j) - f(x^s) - h(x^s) + \frac{2L\|z^0 - z^s\|}{k\lambda_k}.
\]

\( \square \)

When \( g = 0 \), the FDRS algorithm reduces to the DRS algorithm applied to \( f + \chi_V \) (see (1.14)). We use this fact to deduce the following ergodic lower complexity bound for FDRS.

**Theorem 3.4 (Ergodic lower complexity bound).** The convergence rate in Corollary 3.3 is tight up to constant factors.

**Proof.** The result follows by setting \( \mathcal{H} = \mathbb{R}, g = 0, f = |\cdot|, \) and \( V = \mathbb{R} \) and applying [21, Proposition 8]. \( \square \)

**3.2. Nonergodic convergence rates.** In this section, we analyze the nonergodic convergence rate of the FDRS algorithm in the case that \((\lambda_j)_{j \geq 0}\) is bounded away from 0 and \(1/\alpha_{\text{FDRS}}^V\). The proof uses Theorem 1.5 to bound the fundamental inequalities in Propositions 2.3 and 2.4.

**Theorem 3.5 (Nonergodic convergence of FDRS).** For all \( k \geq 0 \), let \( \lambda_k \in (0,1/\alpha_{\text{FDRS}}^V) \). Suppose that \( \tau := \inf_{j \geq 0}(1 - \alpha_{\text{FDRS}}^V\lambda_k)\lambda_k/\alpha_{\text{FDRS}}^V > 0 \). Then the feasi-
bility term satisfies
\[ \|x^*_j - x^*_k\| \leq \frac{\|z^0 - z^*\|}{\sqrt{z(k+1)}} \] and \[ \|x^*_j - x^*_k\| = o\left(\frac{1}{\sqrt{k+1}}\right). \]

In addition, the objective error satisfies:
\[ -\frac{\|z^0 - z^*\|\|\nabla f(x^*)\|}{\sqrt{z(k+1)}} \leq f(x^*_j) + h(x^*_k) - f(x^*) - g(x^*) \]
\[ \leq \frac{\left(\|z^0 - z^*\| + (1 + \gamma/\beta_V)\|z^0 - z^*\| + \gamma\|\nabla h(x^*)\|\right)\|z^0 - z^*\|}{\gamma\sqrt{z(k+1)}} \]

and \[ |f(x^*_j) + h(x^*_k) - f(x^*) - g(x^*)| = o(1/\sqrt{k+1}). \]

Proof. First we note that \( \left(\|\nabla h(x^*_k)\|\right)_{k \geq 0} \) is bounded: for all \( k \geq 0 \),
\[ \|\nabla h(x^*_k)\| \leq \|\nabla h(x^*_k) - \nabla h(x^*)\| + \|\nabla h(x^*)\| = \|\nabla h(z^k) - \nabla h(z^*)\| + \|\nabla h(x^*)\| \]
\[ \leq \frac{1}{\beta_V} \|z^k - z^*\| + \|\nabla h(x^*)\| \]
\[ \leq \frac{1}{\beta_V} \|z^0 - z^*\| + \|\nabla h(x^*)\| \quad (3.5) \]

because \( \left(\|z^j - z^*\|\right)_{j \geq 0} \) is decreasing (see Part 1 of Theorem 1.5).

Next, for any \( \lambda > 0 \), define \( z_\lambda = (T_{FDHS})_\lambda(z^k) \). Observe that \( x^*_j \) and \( x^*_k \) do not depend on the value of \( \lambda_k \). Therefore, by the fundamental inequality in Proposition 2.3 and the identities in Lemma 2.1, we have
\[ f(x^*_j) + h(x^*_k) - f(x^*) - g(x^*) \]
\[ \leq \inf_{\lambda \in [0,1/\alpha_{FDHS}]} \frac{1}{2\gamma \lambda} \left( \|z^k - x^*\|^2 - \|z_\lambda - x^*\|^2 + \left(1 - \frac{2}{\lambda}\right) \|z_\lambda - z^k\|^2 \right) \]
\[ + 2\gamma \|\nabla h(x^*_k), z^k - z_\lambda\| \]
\[ = \inf_{\lambda \in [0,1/\alpha_{FDHS}]} \frac{1}{2\gamma \lambda} \left( 2(z_\lambda - x^*, z^k - z_\lambda) + 2 \left(1 - \frac{1}{\lambda}\right) \|z_\lambda - z^k\|^2 \right) \]
\[ + 2\gamma \|\nabla h(x^*_k), z^k - z_\lambda\| \]
\[ \leq \frac{1}{2\gamma} \left( 2(z_1 - x^*, z^k - z_1) + 2\gamma \left(\frac{1}{\beta_V} \|z^0 - z^*\| + \|\nabla h(x^*)\|\right) \|z_1 - z^k\| \right) \quad (3.6) \]
\[ \leq \frac{1}{\gamma\sqrt{z(k+1)}} \left(\|z^0 - z^*\| + (1 + \gamma/\beta_V)\|z^0 - z^*\| + \gamma\|\nabla h(x^*)\|\right) \frac{\|z^0 - z^*\|}{\gamma\sqrt{z(k+1)}} \]
\[ \leq \frac{\|z^0 - z^*\| + (1 + \gamma/\beta_V)\|z^0 - z^*\| + \gamma\|\nabla h(x^*)\|\right) \frac{\|z^0 - z^*\|}{\gamma\sqrt{z(k+1)}} \]

where the last line follows from the bound: \( \|z_1 - x^*\| \leq \|z_1 - z^*\| + \|z^* - x^*\| \leq \|z^0 - z^*\| + \|z^* - x^*\| \) (see Part 1 of Theorem 1.5).
The lower bound follows from the lower fundamental inequality in Proposition (2.4) and the convergence rate in Part 3 of Theorem 1.5:

\[
    f(x^*_j) + h(x^*_j) - f(x^*) \geq \langle x^*_j - z^k, \nabla f(x^*) \rangle = \frac{1}{h_k} (z^{k+1} - z^k, \nabla f(x^*)) \tag{3.7}
\]

\[
\begin{align*}
    & \geq \frac{\|z^0 - z^*\| \|\nabla f(x^*)\|}{\sqrt{1 + k}} \tag{1.24}
\end{align*}
\]

The \(o(1/\sqrt{k + 1})\) rates follow from Equations (3.6) and (3.7), and the corresponding rates for the FPR in Equation (1.24). \(\square\)

If \(f\) is Lipschitz continuous, we can evaluate the entire objective function at \(z^k\).

The proof of the following corollary is analogous to Corollary 3.3. We ask the reader to recall from Section 3.1 that \((x^*_j)_{j \geq 0}, (x^j_k)_{j \geq 0} \subseteq B(x^*, \|z^0 - z^*\|)\).

**Corollary 3.6** (Nonergodic convergence of FDRS with Lipschitz \(f\)). Let the notation be as in Theorem 3.1. Let \(L \geq 0\) and suppose that \(f\) is \(L\)-Lipschitz on \(B(x^*, \|z^0 - z^*\|)\). Then

\[
0 \leq f(x^k_h) + h(x^k_h) - f(x^*) - h(x^*) \leq \frac{\|z^* - x^*\| + (1 + \gamma/\beta) \|z^0 - z^*\| + \gamma \|\nabla h(x^*)\|}{\sqrt{1 + k}} \|z^0 - z^*\| + L \|z^0 - z^*\| \tag{3.8}
\]

and \(f(x^k_h) + h(x^k_h) - f(x^*) - h(x^*) = o(1/\sqrt{k + 1})\).

**Proof.** The proof follows by combining the upper bound in Theorem 3.5 with the following bound:

\[
    f(x^k_h) + h(x^k_h) - f(x^*) - h(x^* \leq f(x^k_h) + h(x^k_h) - f(x^*) - h(x^*) + L \|x^k_h - x^k_h\| 
\]

\[
\leq f(x^k_h) + h(x^k_h) - f(x^*) - h(x^*) + \frac{L \|z^0 - z^*\|}{\sqrt{1 + k}}. 
\]

The \(o(1/\sqrt{k + 1})\) rate follows because \(\|x^k_h - x^k_h\| = \|T_{\text{FDRS}} z^k - z^k\| = o(1/\sqrt{k + 1})\) (see Equations (2.2) and (1.24)) and \(|f(x^k_j) + h(x^k_h) - f(x^*) - h(x^*)| = o(1/\sqrt{k + 1})\) (see Theorem 3.5). \(\square\)

The following theorem is the first nonergodic lower complexity bound for FDRS.

**Theorem 3.7** (Nonergodic lower complexity bound). Let \(g = 0\) and let \(\lambda_k = 1\) for all \(k \geq 0\). There exists a Hilbert space \(H\) and two closed vector subspaces \(U, V \subseteq H\) such that for every \(\alpha > 1/2\), there exists a point \(z^0 \in H\) and a parameter \(\gamma > 0\) such that if \(f = d_U\) (see (1.8)) and \((z^j)_{j \geq 0}\) is generated by FDRS, then

\[
    f(x^k_h) - f(x^*) = \Omega \left( \frac{1}{(k + 1)^{\alpha}} \right). 
\]

**Proof.** FDRS reduces to DRS when \(g = 0\) (see Equation (1.14)). Therefore, the result follows from [21, Theorem 11]. \(\square\)

**4. Strong convexity.** In this section, we show that \((x^j_k)_{j \geq 0}, (x^j_k)_{j \geq 0}, \) and their ergodic variants converge strongly whenever \(f\) or \(g\) is strongly convex. The techniques in this section are similar to those in Section 3, but we use a modified fundamental inequality.
Let $z \in \mathcal{H}$, let $\lambda > 0$, and let $z^+ = (T_{\text{FDRS}})_{\lambda}$. By Equation (1.11), we have
\[ \|z - x^*\|^2 - \|z^+ - x^*\|^2 = \|z - z^+\|^2 - \|z^+ - z^*\|^2 + 2(z - z^+, z^* - x^*). \]

Therefore, by Proposition 2.3, we have
\[
2\gamma \lambda (f(x) + h(x) - f(x^*) - h(x^*) + S_f(x, x^*) + S_h(x, x^*)) \\
\leq \|z - z^*\|^2 - \|z^+ - z^*\|^2 + 2(z - z^+, z^* - x^*) \\
+ \left(1 - \frac{2}{\lambda}\right)\|z^+ - z\|^2 + 2\gamma \langle \nabla h(x), z - z^+ \rangle. \tag{4.1}
\]

Before we prove the theorem, we quote a result that originally appeared in [21, Lemma 3].

**Lemma 4.1 (Summable sequence convergence rate).** Let $(a_j)_{j\geq0}$ be a sequence of positive numbers such that $\sum_{i=0}^{\infty} a_i < \infty$. Define the sequence of “best” indices: for all $k \geq 0$, let $k_{\text{best}} = \arg\min_{0 \leq j \leq k} a_k$. Then for all $k \geq 0$, we have
\[ a_{k_{\text{best}}} \leq \frac{\sum_{i=0}^{k} a_i}{(k + 1)}. \]

and $a_{k_{\text{best}}} = o(1/(k + 1))$.

We are now ready to prove the theorem:

**Theorem 4.2 (Auxiliary term bound).** Let $\gamma \in (0, 2\beta_V)$, let $(\lambda_j)_{j\geq0} \subseteq (0, 1/\alpha_{\text{FDRS}})$, let $z^0 \in \mathcal{H}$, and suppose that $(z^j)_{j\geq0}$ is generated by Algorithm 1. Then for all $k \geq 0$, we have
\[
4\gamma \lambda_k (S_f(x^k, x^*) + S_h(x^k, x^*)) \\
\leq \|z^k - z^*\|^2 - \|z^{k+1} - z^*\|^2 + \left(1 - \frac{2}{\lambda_k}\right)\|z^{k+1} - z^k\|^2 \\
+ 2\gamma \langle \nabla h(x^k) - \nabla h(x^*), z^k - z^{k+1} \rangle. \tag{4.2}
\]

Furthermore, we have the following convergence rates:

1. **Best iterate convergence:** Let $\varepsilon \in (0, 1)$ and suppose that $(\lambda_j)_{j\geq0}$ satisfies Equation (1.25). If $\underline{\Delta} := \inf_{j \geq 0} \lambda_j > 0$, then
\[ S_f(x^k_{\text{best}}, x^*) + S_h(x^k_{\text{best}}, x^*) \leq \frac{1 + \frac{(1+\varepsilon)\gamma^3}{\varepsilon^2(2\beta_V - \gamma)}}{4\gamma \lambda_k (k + 1)} \|z^0 - z^*\|^2.
\]

and, thus, $S_f(x^k_{\text{best}}, x^*) = o(1/(k + 1))$ and $S_h(x^k_{\text{best}}, x^*) = o(1/(k + 1))$.

2. **Ergodic convergence:** Let $\varepsilon \in (0, 1)$ and suppose that $(\lambda_j)_{j\geq0}$ satisfies Equation (1.25). We have
\[ \overline{S}_f^k + \overline{S}_h^k \leq \frac{1 + \frac{(1+\varepsilon)\gamma^3}{\varepsilon^2(2\beta_V - \gamma)}}{4\gamma \lambda_k} \|z^0 - z^*\|^2.
\]

where
\[ \overline{S}_f^k := \max \left\{ \frac{\mu f}{2} \|x_f^k - x^*\|^2, \frac{\beta f}{2} \left\| \frac{1}{\Lambda_k} \sum_{i=0}^{k} \nabla f(x_f^i) - \nabla f(x^*) \right\|^2 \right\}
\]

and $\overline{S}_h^k$ is similarly defined.
3. **Nonergodic convergence:** Suppose that \( \sum := \inf_{j \geq 0} (1 - \alpha_{\text{FDRS}}^V \lambda_k) \alpha_{\text{FDRS}}^V > 0 \). Then

\[
S_f(x^k_j, x^*) + S_h(x^k_h, x^*) \leq \frac{(1 + \gamma/\beta_V) \| z^0 - z^* \|^2}{\gamma \sqrt{2(k + 1)}},
\]

and \( S_f(x^k_j, x^*) + S_h(x^k_h, x^*) = o(1/\sqrt{k + 1}) \).

**Proof.** Inequality (4.2) follows by combining the upper inequality in Equation (4.1) and lower inequality in Proposition 2.4:

\[
4\gamma \lambda_k (S_f(x^k_j, x^*) + S_h(x^k_h, x^*)) \\
\leq (2) 2\gamma \lambda_k (f(x^k_j) + h(x^k_h) - f(x^*) - h(x^*) + S_f(x^k_j, x^*) + S_h(x^k_h, x^*)) \\
- 2\gamma \lambda_k (x^k_j - x^k_h, \tilde{\nabla} f(x^*)) \\
\leq \| z^k - z^* \|^2 - \| z^{k+1} - z^* \|^2 + 2 \| z^k - z^{k+1} + z^* - x^* \| - 2\gamma \lambda_k (x^k_j - x^k_h, \tilde{\nabla} f(x^*)) \\
+ \left(1 - \frac{2}{\lambda_k}\right) \| z^{k+1} - z^k \|^2 + 2\gamma \langle \nabla h(x^k_h), z^k - z^{k+1}\rangle \\
= \| z^k - z^* \|^2 - \| z^{k+1} - z^* \|^2 + \left(1 - \frac{2}{\lambda_k}\right) \| z^{k+1} - z^k \|^2 \\
+ 2\gamma \langle \nabla h(x^k_h) - \nabla h(x^*), z^k - z^{k+1}\rangle
\]

where the last equality follows from \( z^* - x^* + \tilde{\nabla} f(x^*) = \gamma \tilde{\nabla} \chi_V (x^*) + \tilde{\nabla} f(x^*) = -\nabla h(x^*) \) (see Lemma 2.1).

Now let \( \eta_k = 2/\lambda_k - 1 \). By the argument in Equation (3.4), we have

\[
2\gamma \langle \nabla h(x^k_h) - \nabla h(x^*), z^k - z^{k+1}\rangle \leq \frac{\gamma^2}{\eta_k} \| h(x^k_h) - h(x^*) \|^2 + \eta_k \| z^k - z^{k+1} \|^2. \tag{4.3}
\]

Hence, for all \( k \geq 0 \), we have (using \( 1/\eta_k \leq \lambda_k/\varepsilon^2 \) as in Equations (3.4) and (1.19))

\[
4\gamma \Delta \sum_{i=0}^k (S_i(x^k_j, x^*) + S_h(x^k_h, x^*)) \\
\leq \sum_{i=0}^k 4\gamma \lambda_k (S_f(x^k_j, x^*) + S_h(x^k_h, x^*)) \\
\leq \sum_{i=0}^k \left( \| z^i - z^* \|^2 - \| z^{i+1} - z^* \|^2 - \eta_k \| z^{i+1} - z^i \|^2 \\
+ 2\gamma \langle \nabla h(x^k_h) - \nabla h(x^*), z^i - z^{i+1}\rangle \right) \\
\leq \sum_{i=0}^k \left( \| z^i - z^* \|^2 - \| z^{i+1} - z^* \|^2 + (\gamma^2 \lambda_i/\varepsilon^2) \| \nabla h(x^k_h) - \nabla h(x^*) \|^2 \right) \\
\leq \| z^0 - z^* \|^2 - \| z^{k+1} - z^* \|^2 + \frac{(1 + \varepsilon) \gamma^3}{\varepsilon^3 (2\beta_V - \gamma)} \| z^0 - z^* \|^2.
\]

The “best” convergence rates now follow by taking \( k \to \infty \) and using Lemma 4.1. In
addition, apply we apply Jensen’s inequality to the convex function \( \| \cdot \|^2 \) to get
\[
S_f^k + S_h^k \leq \frac{1}{\lambda_k} \sum_{i=0}^{k} \lambda_i (S_f(x_f^i, x^*) + S_h(x_h^i, x^*)) \leq \frac{1}{\lambda_k} \left( \frac{1}{1 - \frac{\lambda_i}{\lambda_{i-1}}} \right) \| z^0 - z^* \|^2.
\]

The nonergodic convergence rates require an argument similar to Theorem 3.5: For all \( \lambda > 0 \), define \( z_\lambda = (T_{\text{FDRS}})_\lambda (z^k) \). Observe that \( S_f(x_f^*, x^*) \) and \( S_h(x_h^*, x^*) \) do not depend on the value of \( \lambda_k \). Therefore, we can use Equation (4.2) to get the following upper bounds:
\[
S_f(x_f^k, x^*) + S_h(x_h^k, x^*)
\leq \inf_{\lambda \in [0, 1/\omega_{\text{FDRS}}]} \left( \frac{1}{2 \gamma \lambda} \right) \left( 2 \| z^k - z^* \|^2 - \| z_\lambda - z^* \|^2 + \left( 1 - \frac{1}{\lambda} \right) \| z_\lambda - z^k \|^2 + 2 \gamma \langle \nabla h(x_h^k) - \nabla h(x^*), z^k - z_\lambda \rangle \right)
\leq \frac{1}{2 \gamma} \left( 2 \langle z_1 - z^*, z^k - z_1 \rangle + \frac{2 \gamma}{\beta_V} \| z^k - z^* \| \| z_1 - z^k \| \right) \leq \frac{1 + \gamma \beta_V}{\sqrt{1/(k+1)}} \| z^0 - z^* \|^2 \tag{4.4}
\]
where the Equation (4.4) uses the \((1/\beta_V)\)-Lipschitz continuity of \( \nabla h \) and the identity \( \nabla h(x_h^k) - \nabla h(x^*) = \nabla h(z^k) - \nabla h(z^*) \), and the last line uses the Fejér property \( \| z_1 - z^* \| \leq \| z^k - z^* \| \leq \| z^0 - z^* \| \) (see Part 1 of Theorem 1.5).

The \( o(1/\sqrt{k+1}) \) rates follow from Equations (4.4) and the corresponding rates for the FPR in Equation (1.24).

**Remark 3.** See Section 6.1 for a proof that the nonergodic “best” rates are optimal. It is not clear if we can improve the general nonergodic rates to \( o(1/(k+1)) \).

**5. Lipschitz differentiability.** In this section, we study the FDRS algorithm under the following assumption:

**Assumption 4.** The function \( f \) is differentiable and \( \nabla f \) is \((1/\beta_f)\)-Lipschitz where \( \beta_f > 0 \).

Under Assumption 4, we will show that the objective value
\[
f(x_h^k) + h(x_h^k) - f(x^*) - h(x^*) = f(x_h^k) + g(x_h^k) - f(x^*) - g(x^*)
\]
is summable. Therefore, by Lemma 4.1, the “best” objective error converges with rate \( o(1/(k+1)) \).

The following Theorem will be used several times throughout our analysis.

**Theorem 5.1 (Descent theorem).** Suppose that \( f : \mathcal{H} \to (-\infty, \infty] \) is closed, proper, convex, and differentiable. If \( \nabla f \) is \((1/\beta_f)\)-Lipschitz, then for all \( x, y \in \text{dom}(f) \), we have the upper bound
\[
f(x) \leq f(y) + \langle x - y, \nabla f(y) \rangle + \frac{1}{2 \beta} \| x - y \|^2 \tag{5.1}
\]
Proof. See [3, Theorem 18.15(iii)]. □

We are now ready to prove the upper bound.

**Proposition 5.2** (Fundamental inequality under the Lipschitz derivative assumption). Let \( \gamma \in (0, 2\beta_V) \), let \( \lambda > 0 \), let \( z \in \mathcal{H} \), let \( z^+ = (T_{FDRS})_\lambda(z) \), let \( z^* \) be a fixed point of \( T_{FDRS} \), and let \( x^* = P_V z^* \). If Assumption 4 holds, then

\[
2\gamma\lambda(f(x_h) + h(x_h) - f(x^*) - g(x^*)) \leq \begin{cases} 
\|z - z^*\|^2 - \|z^+ - z^*\|^2 + \left(1 + \frac{\gamma - \beta_f}{\beta_f}\right)\|z - z^+\|^2 & \text{if } \gamma \leq \beta_f \\
\left(1 + \frac{\gamma - \beta_f}{2\beta_f}\right)((z - z^*)^2 - \|z^+ - z^*\|^2 + \|z - z^+\|^2) & \text{if } \gamma > \beta_f.
\end{cases}
\]

(5.2)

**Proof.** Because \( \nabla f \) is \((1/\beta_f)\)-Lipschitz, we have

\[
f(x_h) \leq f(x_f) + \langle x_h - x_f, \nabla f(x_f) \rangle + \frac{1}{2\beta_f} \|x_h - x_f\|^2;
\]

(5.3)

\[
S_f(x_f, x^*) \geq \frac{\beta_f}{2} \|\nabla f(x_f) - \nabla f(x^*)\|^2.
\]

(5.4)

By applying the identity \( z^* - x^* = \gamma \bar{\nabla} \chi_V(x^*) = -\gamma \nabla f(x^*) - \gamma \nabla h(x^*) \), the cosine rule (1.11), and the identity \( z - z^+ = \lambda(x_h - x_f) \) (see Equation (2.2)) multiple times, we have

\[
2\langle z - z^+, z^* - x^* \rangle + 2\gamma\lambda\langle x_h - x_f, \nabla f(x_f) \rangle \\
= 2\lambda\langle x_h - x_f, \gamma \bar{\nabla} \chi_V(x^*) + \gamma \nabla f(x_f) \rangle \\
= 2\lambda(\gamma \bar{\nabla} \chi_V(x_h) + \gamma \nabla h(x_h) + \gamma \nabla f(x_f), \gamma \nabla f(x_f) - \gamma \nabla f(x^*) - (z - z^+, \gamma \nabla h(x^*)) \\
= \lambda \left(\|\gamma \nabla f(x_f) - \gamma \nabla f(x^*)\|^2 + \|x_h - x_f\|^2 \\
- \|\gamma \bar{\nabla} \chi_V(x_h) + \gamma \nabla h(x_h) - \gamma \bar{\nabla} \chi_V(x^*) - \gamma \nabla h(x^*)\|^2 \right) - \langle z - z^+, \gamma \nabla h(x^*) \rangle.
\]

(5.5)

By Equation (2.2) (i.e., \( z - z^+ = \lambda(x_h - x_f) \)), we have

\[
\left(1 - \frac{2}{\lambda}\right)\|z - z^+\|^2 + \lambda \left(1 + \frac{\gamma - \beta_f}{\beta_f}\right)\|x_h - x_f\|^2 = \left(1 + \frac{\gamma - \beta_f}{\beta_f}\right)\|z - z^+\|^2.
\]
Therefore,
\[
2\gamma \lambda (f(x_h) + h(x_h) - f(x^*) - h(x^*)) \\
\leq 2\gamma \lambda (f(x_f) + h(x_h) - f(x^*) - h(x^*)) + 2\gamma \lambda (x_h - x_f, \nabla f(x_f)) + \frac{\gamma \lambda}{\beta_f} \|x_h - x_f\|^2 \\
\leq \|z - z^*\|^2 - \|z^* - z\|^2 \leq \|z^* - z\|^2 + 2(\|z - z^*\|^2) + 2\gamma \lambda (x_h - x_f, \nabla f(x_f)) \\
+ \left(1 - \frac{2}{\lambda}\right) \|z^* - z\|^2 + 2\gamma \lambda (\nabla h(x_h), z - z^*) + \frac{\gamma \lambda}{\beta_f} \|x_h - x_f\|^2 - 2\gamma \lambda S_f(x_f, x^*) \\
\leq \|z - z^*\|^2 - \|z^* - z\|^2 + \left(1 - \frac{2}{\lambda}\right) \|z^* - z\|^2 + \lambda \left(\frac{\gamma}{\beta_f} + 1\right) \|x_h - x_f\|^2 \\
+ \lambda \gamma \lambda (\nabla f(x_f) - \nabla f(x^*))^2 + 2\gamma \lambda (\nabla h(x_h) - \nabla h(x^*), z - z^*) - 2\gamma \lambda S_f(x_f, x^*) \\
\leq \|z - z^*\|^2 - \|z^* - z\|^2 + \left(1 + \frac{\gamma - \beta_f}{\beta_f \lambda}\right) \|z - z^*\|^2 \\
+ 2\gamma \lambda (\nabla h(x_h) - \nabla h(x^*), z - z^*) + \gamma \lambda (\gamma - \beta_f) \|\nabla f(x_f) - \nabla f(x^*)\|^2. \quad (5.6)
\]

If \(\gamma \leq \beta_f\), then we can drop the last term. If \(\gamma > \beta_f\), then we apply the upper bound in Equation (4.2) to get:
\[
\gamma \lambda (\gamma - \beta_f) \|\nabla f(x_f) - \nabla f(x^*)\|^2 \\
\leq \frac{(\gamma - \beta_f)}{2\beta_f} \left(\|z - z^*\|^2 - \|z^* - z\|^2 + \left(1 - \frac{2}{\lambda}\right) \|z^* - z\|^2 \\
+ 2\gamma \lambda (\nabla h(x_h) - \nabla h(x^*), z - z^*)\right)\).
\]

The result follows by using the above inequality in Equation (5.6) together with the following identity:
\[
\left(1 + \frac{\gamma - \beta_f}{\beta_f \lambda}\right) \|z - z^*\|^2 + \frac{(\gamma - \beta_f)}{2\beta_f} \left(1 - \frac{2}{\lambda}\right) \|z^* - z\|^2 = \left(1 + \frac{\gamma - \beta_f}{2\beta_f}\right) \|z - z^*\|^2.
\]

The next theorem will show that the upper bound in Proposition 5.2 is summable and, as a consequence, we will have \(o(1/(k + 1))\) convergence.

**Theorem 5.3** (”Best” convergence rates under the Lipschitz derivative assumption). Let \(\gamma \in (0, 2\beta_V)\), let \(\varepsilon \in (0, 1)\), and suppose that \((\lambda_j)_{j \geq 0}\) satisfies Equation (1.25). Suppose that \(\varepsilon := \inf_{j \geq 0} (1 - \alpha^F_{\text{FDRS}} \lambda_j) / \alpha^F_{\text{FDRS}} > 0\) and let \(\lambda := \inf_{j \geq 0} \lambda_j > 0\). Let \(z^0 \in \mathcal{H}\), let \(z^*\) be a fixed point of \(T_{\text{FDRS}}\), and let \(x^* = P_V z^*\). If Assumption 4 holds, then
\[
f(x_h^{k_{\text{best}}}) + h(x_h^{k_{\text{best}}}) - f(x^*) - h(x^*) \\
\leq \left(1 + \frac{1}{2} + \frac{(1+\varepsilon)\gamma}{\varepsilon(2\beta_V - \gamma)} + \frac{1}{\beta_f}\right) \|z^0 - z^*\|^2 \\
\times \left\{\begin{array}{ll}
1 & \text{if } \gamma \leq \beta_f; \\
1 + \frac{2\beta_f}{2\beta_f} & \text{if } \gamma > \beta_f.
\end{array}\right.
\]
and \(f(x_h^{k_{\text{best}}}) + h(x_h^{k_{\text{best}}}) - f(x^*) - g(x^*) = o(1/(k + 1))\) where the \(k_{\text{best}}\) index sequence is defined in Lemma 4.1.
Proof. First recall that
\[
\sum_{i=0}^{\infty} \|z^{i+1} - z^i\|^2 \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{1 - \lambda_k \alpha_{\text{FDRS}}}{\lambda_k \alpha_{\text{FDRS}}} \|z^{i+1} - z^i\|^2 \leq \frac{1}{2} \|z^0 - z^\ast\|^2.
\]

Next, we use the Cauchy-Schwarz inequality and Young's inequality for real numbers (Equation (1.12)) to show that
\[
\sum_{i=0}^{\infty} 2\gamma \langle \nabla h(x_h^i) - \nabla h(x^\ast), z^i - z^{i+1}\rangle \leq \sum_{i=0}^{\infty} \left( \lambda_i \gamma^2 \|\nabla h(x_h^i) - \nabla h(x^\ast)\|^2 + \frac{1}{\lambda_i} \|z^i - z^{i+1}\|^2 \right)
\]
\[
\leq (\frac{1+\varepsilon}{\varepsilon(2\beta_V - \gamma)} + \frac{1}{\Delta \lambda}) \|z^0 - z^\ast\|^2
\]

If we combine the previous two sum bounds with Equation (5.2), we get
\[
\sum_{i=0}^{\infty} (f(x_h^i) + h(x_h^i) - f(x^\ast) - h(x^\ast)) \leq \frac{1}{2} \left( \frac{(1+\varepsilon)\gamma^3}{\Delta \lambda} + \frac{1}{\Delta \lambda} \right) \|z^0 - z^\ast\|^2 \times \begin{cases} 
1 & \text{if } \gamma \leq \beta_f; \\
1 + \frac{\beta_f}{2\beta_f} & \text{if } \gamma > \beta_f.
\end{cases}
\]

The convergence rate now follows from Lemma 4.1. \(\square\)

Remark 4. There appears to be no way to remove the “best” qualifier using our current techniques. Note that the best rate we can expect for Lipschitz differentiable \(f\) is \(o(1/(k + 1))\) \cite[Theorem 12]{21}.

6. Linear convergence. In this section, we prove that

FDRS converges linearly whenever \(\mu_g \beta_f > 0\) or \(\mu_f \beta_f > 0\).

In addition, in Section 6.1 we provide examples of \(f\) and \(g\) such that \((\mu_f + \mu_g)^2 > 0\) and \(\beta_f = 0\), but FDRS does not converge linearly. In fact, we show that FDRS can converge \(\text{arbitrarily slowly}\) under this assumption.

Theorem 6.1 (Linear convergence). Let \(\gamma \in (0, 2\beta_V)\), let \((\lambda_j)_{j \geq 0} \subseteq (0, 1/\alpha_{\text{FDRS}}),\) let \(z^0 \in \mathcal{H},\) let \(z^\ast\) be a fixed point of \(T_{\text{FDRS}},\) and let \(x^\ast = P_V z^\ast\). Let \(c > 1/2,\) let \(\gamma < 2\beta_V / c,\) and let \((\lambda_j)_{j \geq 0} \subseteq (0, (2c - 1)/c)\). For all \(\lambda \in (0, (2c - 1)/c),\) let

\[
C_1(\lambda) = \left(1 - \frac{\lambda}{3} \min \left\{ \frac{2\gamma \mu_g}{(1 + \gamma/\beta_V)^2}, \frac{2\beta_f}{\gamma}, \frac{2c - 1}{c} - \lambda \right\} \right)^{1/2},
\]

and let

\[
C_2(\lambda) = \left(1 - \frac{\lambda}{3} \min \left\{ \frac{2\gamma \mu_f}{(1 + \gamma/\beta_f)^2}, \frac{2\beta_V - c\gamma}{\gamma} \frac{1}{4} \left( \frac{2c - 1}{c} - \lambda \right) \right\} \right)^{1/2}.
\]

Then for all \(k \geq 0,\) we have
\[
\|z^{k+1} - z^\ast\| \leq \|z^k - z^\ast\| \times \begin{cases} 
C_1(\lambda_k) & \text{if } \mu_g \beta_f > 0; \\
C_2(\lambda_k) & \text{if } \mu_f \beta_f > 0;
\end{cases}
\]
\[(6.1)\]
and, consequently, we have the bound:

$$\|z^{k+1} - z^*\| \leq \|z^0 - z^*\| \times \begin{cases} \prod_{j=0}^{k} C_1(\lambda_j) & \text{if } \mu_g \beta_f > 0; \\ \prod_{j=0}^{k} C_2(\lambda_j) & \text{if } \mu_f \beta_f > 0. \end{cases}$$

Therefore, the sequence \((z^j)_{j \geq 0}\) converges linearly to \(z^*\) with rate \(C < 1\) if either of the following two conditions are met: \(\mu_g \beta_f > 0\) and \(C := \sup_{j \geq 0} C_1(\lambda_j) < 1\), or \(\mu_f \beta_f > 0\) and \(C := \sup_{j \geq 0} C_2(\lambda_j) < 1\).

Proof. Equation (4.2) shows that for all \(k \geq 0\), we have

$$2 \gamma \lambda_k \mu_f \|x_f^k - x^*\|^2 + 2 \gamma \lambda_k \beta_f \|\tilde{\nabla} f(x_f^k) - \tilde{\nabla} f(x^*)\|^2$$
$$+ 2 \gamma \lambda_k \mu_g \|x_h^k - x^*\|^2 + 2 \gamma \lambda_k \beta_V \|\nabla h(x_h^k) - \nabla h(x^*)\|^2$$
$$\leq \|z^k - z^*\|^2 - \|z^{k+1} - z^*\|^2 + \left(1 - \frac{2}{\lambda_k}\right) \|z^{k+1} - z^k\|^2$$
$$+ 2 \gamma \langle \nabla h(x_h^k) - \nabla h(x^*), z^k - z^{k+1}\rangle$$

In addition, by the Cauchy-Schwarz inequality and and Young’s inequality for real numbers (Equation (1.12)), we have

$$2 \gamma \langle \nabla h(x_h^k) - \nabla h(x^*), z^k - z^{k+1}\rangle \leq c \gamma^2 \lambda_k \|\nabla h(x_h^k) - \nabla h(x^*)\|^2 + \frac{1}{c \lambda_k} \|z^k - z^{k+1}\|^2.$$

Therefore, for all \(k \geq 0\),

$$2 \gamma \lambda_k \mu_f \|x_f^k - x^*\|^2 + 2 \gamma \lambda_k \beta_f \|\tilde{\nabla} f(x_f^k) - \tilde{\nabla} f(x^*)\|^2$$
$$+ 2 \gamma \lambda_k \mu_g \|x_h^k - x^*\|^2 + 2 \gamma \lambda_k \beta_V (c \gamma) \|\nabla h(x_h^k) - \nabla h(x^*)\|^2$$
$$\leq \|z^k - z^*\|^2 - \|z^{k+1} - z^*\|^2 + \left(1 - \frac{2c - 1}{c \lambda_k}\right) \|z^{k+1} - z^k\|^2.$$

Recall that we assume \(1 - (2c - 1)/(c \lambda_k) < 0\) and \(2 \beta_V - c \gamma > 0\).

Now suppose that \(\beta_f \mu_g > 0\). The following identity follows from from Lemma 2.1:

$$z^k = T_{FDRS}(z^k) + (z^k - T_{FDRS}(z^k)) = x_h^k - \gamma \nabla h(x_h^k) - \gamma f(x_h^k) + \frac{1}{\lambda_k}(z^k - z^{k+1}).$$

Intuitively, this identity results from tracing the perimeter of Fig. 1 from \(x_h \) to \(x_f \) to \(T_{FDRS}z^k\) to \(z^k\). Likewise, we have \(z^* = x^* - \gamma \nabla h(x^*) - \gamma \nabla f(x^*).\)

Note that \(I_h + \gamma \nabla h\) is not necessarily nonexpansive if \(c < 1\) and \(2 \beta_V \leq \gamma < 2 \beta_V/c\). However, we always have the bound:

$$\|(x_h^k - \gamma \nabla h(x_h^k)) - (x^* - \gamma \nabla h(x^*))\| \leq \|x_h^k - x^*\| + \gamma \|\nabla h(x_h^k) - \nabla h(x^*)\|$$
$$\leq (1 + \gamma/\beta_V) \|x_h^k - x^*\|. \quad (6.2)$$

Now, let

$$C_1' = 3 \max \left\{ \frac{(1 + \gamma/\beta_V)^2}{2 \gamma \lambda_k \mu_g}, \frac{\gamma^2}{2 \gamma \lambda_k \beta_f}, \frac{1}{\lambda_k^2 \left(\frac{2c - 1}{c \lambda_k} - 1\right)} \right\}. \quad (6.3)$$
By the convexity of $\| \cdot \|^2$, we have
\[
\| z^k - z^* \|^2 \\
\leq 3(1 + \gamma / \beta V)^2 \| x^k_h - x^* \|^2 + 3\gamma^2 \| \nabla f(x^k_f) - \nabla f(x^*) \|^2 + \frac{3}{\lambda_k} \| z^{k+1} - z^k \|^2 \\
\leq C'_1 \left( 2\gamma \lambda_k h \| x^k_h - x^* \|^2 + 2\gamma \lambda_k \beta_f \| \nabla f(x^k_f) - \nabla f(x^*) \|^2 \\
+ \left( \frac{2c - 1}{c\lambda_k} - 1 \right) \| z^{k+1} - z^k \|^2 \right) \\
\leq C'_1 \| z^k - z^* \|^2 - C'_1 \| z^{k+1} - z^* \|^2.
\]

Therefore,
\[
\| z^{k+1} - z^* \| \leq \left( 1 - \frac{1}{C'_1} \right)^{1/2} \| z^k - z^* \|.
\]

Now assume that $\beta_f \mu_f > 0$. Observe that:
\[
z^k = x^k_h - \gamma \nabla h(x^k_h) - \gamma \nabla f(x^k_f) + \frac{1}{\lambda_k} (z^k - z^{k+1}) \\
= x^k_f - \gamma \nabla h(x^k_h) - \gamma \nabla f(x^k_f) + \frac{2}{\lambda_k} (z^k - z^{k+1})
\]

where we use the identity $x^k_h - x^k_f = (1/\lambda_k)(z^k - z^{k+1})$ (see Equation (2.2)). The proof of this case is similar to the case $\beta_f \mu_h > 0$ except that we use the above identity for $z^k$, the bound $\| (x^k_f - \gamma \nabla f(x^k_f)) - (x^* - \gamma \nabla f(x^*)) \|^2 \leq (1 + \gamma / \beta_f)^2 \| x^k_f - x^* \|^2$, and the following constant $C'_2$ in place of $C'_1$:
\[
C'_2 = 3 \max \left\{ \left( \frac{1 + \gamma / \beta_f}{2}\right)^2, \frac{\gamma^2}{\gamma \lambda_k (2\beta V - c\gamma)}, \frac{4}{\lambda_k^2 \left( \frac{2c - 1}{c\lambda_k} - 1 \right)} \right\}. \tag{6.4}
\]

We can then derive a contraction of the form
\[
\| z^{k+1} - z^* \| \leq \left( 1 - \frac{1}{C'_2} \right)^{1/2} \| z^k - z^* \|.
\]

In both cases, the linear convergence rate for $(z^j)_{j \geq 0}$ follows by unfolding the derived contraction in Equation (6.1). \qed

**Remark 5.** The convergence rates for $(z^j)_{j \geq 0}$ immediately imply linear convergence rates for $\| z^{k+1} - z^* \|^2$ (see Inequality (1.15)) and, hence, for the objective error $f(x^k_h) + h(x^k_h) - f(x^*) - h(x^*)$ (see Inequality (5.2)). In addition, from the identity $x^k_h - x^* = P_{V_{z^*}}(z^k - z^*)$, it follows that $(x^k_h)_{j \geq 0}$ converges linearly. Finally, because $x^k_f - x^* = x^k_h - x^* + (x^k_f - x^k_h) = x^k_h - x^* + (1/\lambda_k)(z^{k+1} - z^k)$, it follows that $(x^k_f)_{j \geq 0}$ also converges linearly. We do not explicitly compute these rates because of limited space.

**Remark 6.** The constant $c$ in Theorem 6.1 is somewhat mysterious, but it seems unavoidable. Let us examine some choices: If we set $c = 2$, then we can choose $(\lambda_j)_{j \geq 0} \subseteq (0, 3/2)$ and $\gamma \in (0, \beta V)$. If $c = 1$, then we can choose $(\lambda_j)_{j \geq 0} \subseteq (0, 1)$ and
\( \gamma \in (0, 2\beta V) \). Thus, a smaller \( c \) leads to larger \( \gamma \) and smaller \( (\lambda_j)_{j \geq 0} \), while a larger \( c \) leads to smaller \( \gamma \) and larger \( (\lambda_j)_{j \geq 0} \).

**Remark 7.** The functions \( C_1 \) and \( C_2 \) in Theorem 6.1 are not necessarily the best possible. Indeed, when \( \gamma \in (0, 2\beta V) \), the map \( I - \gamma \nabla h \) is nonexpansive (from the Baillon-Haddad Theorem [1] and [3, Proposition 4.33]). Therefore, we can replace the bound in Equation (6.2) with \( \| (x^k_h - \gamma \nabla h(x^k_h)) - (x^* - \gamma \nabla h(x^*)) \| \leq \| x^k_h - x^* \| \). Several other minor improvements of this form are possible. They are not the main focus of this paper, so we omit them.

### 6.1. Arbitrarily slow convergence for strongly convex problems.

In general, we cannot expect linear convergence of FDRS when \( f \) is not differentiable—even if \( f \) and \( g \) are strongly convex. In this section, we construct an example to prove this claim. We also show that FDRS applied to this example converges in norm, but does so arbitrarily slowly. The following example is based on [2, Section 7] and [21, Example 1].

**The main example.** Let \( \mathcal{H} = \ell_2^2(N) = \mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \). Let \( R_\theta \) denote counter-clockwise rotation in \( \mathbb{R}^2 \) by \( \theta \) degrees. Let \( e_0 := (1, 0) \) denote the standard unit vector, and let \( e_{\theta} := R_\theta e_0 \). Suppose that \( (\theta_j)_{j \geq 0} \) is a sequence of angles in \((0, \pi/2)\) such that \( \theta_i \rightarrow 0 \) as \( i \rightarrow \infty \). For all \( i \geq 0 \), let \( c_i := \cos(\theta_i) \). We let

\[
V := \mathbb{R}^2 e_0 \times \mathbb{R}^2 e_0 \times \cdots \quad \text{and} \quad U := \mathbb{R}^2 e_{\theta_0} \times \mathbb{R}^2 e_{\theta_0} \times \cdots . \tag{6.5}
\]

Note that [2, Section 7] proves the identity

\[
(P_V)_i = \begin{bmatrix}
\cos^2(\theta_i) & \sin(\theta_i) \cos(\theta_i) \\
\sin(\theta_i) \cos(\theta_i) & \sin^2(\theta_i)
\end{bmatrix} \quad \text{and} \quad (P_U)_i = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.
\]

We now begin our extension of this example. Choose \( a \geq 0 \) and set \( f = \chi_U + (a/2) \| \cdot \|^2 \) and \( g = (1/2) \| \cdot \|^2 \). Note that \( \mu_g = 1 \) and \( \mu_f = a \). In addition, for \( h = g \circ P_V \), we have

\[
(\nabla h(x))_i = (P_V \circ I_\mathcal{H} \circ P_V)_i = (P_V)_i.
\]

Thus, \( \nabla h \) is 1-Lipschitz, and, hence, \( \beta V = 1 \) and we can choose \( \gamma = 1 < 2\beta V \) and

\[
\alpha_{FDRS}^V = \frac{2\beta V}{4\beta V - \gamma} = 2/3,
\]

so we can choose \( \lambda_k \equiv 1 < 1/\alpha_{FDRS}^V \). We also note that \( \text{prox}_{\gamma \chi_U} = (1/(1+a))P_U \).

Now, the DRS operator \( N = (1/2)I_\mathcal{H} + (1/2) \text{refl}_f \circ \text{refl}_U \) has the following form: for all \( i \geq 0 \),

\[
(N)_i := \left( \frac{1}{2}I_\mathcal{H} + \frac{1}{2} \text{refl}_f \circ \text{refl}_U \right)_i
= \frac{1}{a+1} (P_V)_i (2(P_V)_i - I_{\mathbb{R}^2}) + I_{\mathbb{R}^2} - (P_V)_i
= \frac{1}{a+1} (P_V)_i \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
= \frac{1}{a+1} \begin{bmatrix}
\cos^2(\theta_i) & -\sin(\theta_i) \cos(\theta_i)
\sin(\theta_i) \cos(\theta_i) & \cos^2(\theta_i) + a
\end{bmatrix}
\]

where the second equality follows by direct expansion. Therefore, the FDRS operator has the following form:

\[
T_{\text{FDRS}} := (N \circ (I - P_V)) = \bigoplus_{i \geq 0} \frac{1}{a+1} \begin{bmatrix}
0 & -\sin(\theta_i) \cos(\theta_i)
0 & \cos^2(\theta_i) + a
\end{bmatrix} . \tag{6.6}
\]
Note that for all $i \geq 0$, the operator $(T_{\text{FDRS}})_i$ has eigenvector
\[
z_i = -\frac{\cos(\theta_i) \sin(\theta_i)}{a + \cos^2(\theta_i)} + 1,
\]
with eigenvalue $b_i := (a + c_i^2)/(a + 1) < 1$. Each component also has the eigenvector $(1,0)$ with eigenvalue 0. Thus, the only fixed point of $T_{\text{FDRS}}$ is $0 \in \mathcal{H}$. Finally, we note that
\[
\|z_i\|^2 = \frac{c_i^2(1 - c_i^2)}{(a + c_i^2)^2} + 1 \quad \text{and} \quad \|(P_Y)z_i\|^2 = \frac{c_i^2(1 - c_i^2)}{(a + c_i^2)^2}.
\]

**Slow convergence proofs.** We know that $z^{k+1} - z^k \to 0$ from Equation (1.24). Therefore, because $T_{\text{FDRS}}$ is linear, [3, Proposition 5.27] proves the following lemma.

**Lemma 6.2** (Strong convergence for linear operators). Any sequence $(z^j)_{j \geq 0} \subseteq \mathcal{H}$ generated by the $T_{\text{FDRS}}$ operator in Equation (6.6) converges strongly to 0. Consequently, the sequences $(x^j_0)_{j \geq 0} = (P_Y z^j)_{j \geq 0}$ and $(x^j_f)_{j \geq 0}$ converges strongly to zero.

The following auxiliary Lemma explicitly appeared in [21, Lemma 6], but it can be traced back to the proof of [24, Theorem 4.2].

**Lemma 6.3** (Arbitrarily slow sequence convergence). Suppose that $F : \mathbb{R}_+ \to (0,1)$ is a function that is monotonically decreasing to zero. Then there exists a monotonic sequence $(b_j)_{j \geq 0} \subseteq (0,1)$ such that $b_k \to 1^-$ as $k \to \infty$ and an increasing sequence of integers $(n_j)_{j \geq 0} \subseteq \mathbb{N} \cup \{0\}$ such that for all $k \geq 0$,
\[
\frac{b_{k+1}}{n_{k+1}} > F(k + 1) e^{-1}.
\]

The following is a simple corollary of Lemma 6.3.

**Corollary 6.4.** Let the notation be as in Lemma 6.3. Then for all $\eta \in (0,1)$, we can find a sequence $(b_j)_{j \geq 0} \subseteq (\eta,1)$ that satisfies the conditions of the lemma.

**Proof.** Choose any sequence $(b_j') \subseteq (0,1)$ and $(n_j)_{j \geq 0} \subseteq \mathbb{N} \cup \{0\}$ which satisfies the Lemma. Then, choose a new sequence: for all $k \geq 0$, let $b_j = \max\{b_j', \eta\}$. Note that $(b_j)_{j \geq 0}$ is monotonic and converges to 1 from the left. In addition $b_k \geq b'_k$ for all $k \geq 0$, so Inequality (6.8) holds. \(\Box\)

We are now ready to show that FDRS can converge arbitrarily slowly.

**Theorem 6.5** (Arbitrarily slow FDRS). For every function $F : \mathbb{R}_+ \to (0,1)$ that strictly decreases to zero, there is a point $z^0 \in \ell_2^2(\mathbb{N})$ and two closed subspaces $U$ and $V$ with zero intersection, $U \cap V = \{0\}$, such that the FDRS sequence $(z^j)_{j \geq 0}$ generated with the functions $f = \chi_U + (a/2) \| \cdot \|^2$ and $g = (1/2) \| \cdot \|^2$, relaxation parameters $\lambda_k \equiv 1$, and stepsize $\gamma = 1$ satisfies the following bound:
\[
\|z^k - z^*\| \geq e^{-1} F(k),
\]
but $(\|z^j - z^*\|)_{j \geq 0}$ converges to 0.

**Proof.** For all $i \geq 0$, define $z_i^0 = (1/\|z_i\|(i+1))z_i$, then $\|z_i^0\| = 1/(i+1)$ and $z_i^0$ is an eigenvector of $(T_{\text{FDRS}})_i$ with eigenvalue $b_i = (a + c_i^2)/(a + 1)$. Define the concatenated vector $z_i^0 = (z_i^0)_{i \geq 0}$. Note that $z_i^0 \in \mathcal{H}$ because $\|z_i^0\|^2 = \sum_{i=0}^{\infty} 1/(i + 1)^2 < \infty$. Thus, for all $k \geq 0$, we let $z^{k+1} = T_{\text{FDRS}} z^k$.

Now, recall that $z^* = 0$. Thus, for all $n \geq 0$ and $k \geq 0$, we have
\[
\|z^k - z^*\|^2 = \|T_{\text{FDRS}} z^0\|^2 = \sum_{i=0}^{\infty} b_i^2(k+1) \|z_i^0\|^2 = \sum_{i=0}^{\infty} \frac{b_i^2(k+1)}{(i+1)^2} \geq \frac{b_0^2(k+1)}{(n+1)^2}.
\]
Thus, \( \|z^k - z^*\| \geq b^{(k+1)}_n/(n + 1) \). To get the lower bound, we choose \( b_n \) and the sequence \( (n_j)_{j \geq 0} \) using Corollary 6.4 with any \( \eta \in (a/(a + 1), 1) \). Then we solve for the coefficients: \( c_n = \sqrt{b_n(1 + a)} - a > 0 \). \( \square \)

**Remark 8.** Theorems 6.5 and 4.2 show that the sequence \( (z^j)_{j \geq 0} \) can converge arbitrarily slowly even if \( (x^j_f)_{j \geq 0} \) and \( (x^j_h)_{j \geq 0} \) converge with rate \( o(1/\sqrt{k + 1}) \), which is a strange phenomenon.

Theorem 4.2 shows that the sequences \( (x^j_h)_{j \geq 0} \) and \( (x^j_f)_{j \geq 0} \) cannot converge arbitrarily slowly. However, we can still show that this sequence does not converge linearly.

**Theorem 6.6.** There exists a sequence \( (c_i)_{i \geq 0} \) so that \( (x^i_h)_{i \geq 0} \) and \( (x^i_f)_{i \geq 0} \) converge strongly, but not linearly. In particular, for any \( \alpha > 1/2 \), there is an initial point \( z^0 \in \mathcal{H} \) so that for all \( k \geq 1 \),

\[
\|x^k_h - x^*\|^2 \geq \frac{1}{(k + 1)^{2\alpha}} \quad \text{and} \quad \|x^k_f - x^*\|^2 \geq \frac{(a + 1/2)^2}{(a + 1)^4(i + 1)^{2\alpha}}.
\]

In addition, \( \|\nabla h(x^k_h) - \nabla h(x^k_f)\|^2 = \|x^k_h - x^*\|^2 \). Thus, the nonergodic “best” convergence rates in Part 3 of Theorem 4.2 are tight.

**Proof.** For all \( i \geq 0 \), let \( c_i = (i/(i + 1))^{1/2} \). Let \( \kappa_a = (1/2) + 2(a + 1)^2 \), and let

\[
z^0 = \sqrt{2\alpha \kappa_a} e^{(1/(a + 1))} \times \left( \frac{z_i}{\|z_i\|(i + 1)^\alpha} \right)_{i \geq 0}.
\]

Then \( \|z^0\|^2 = 2\alpha \kappa_a e^{2/(a + 1)} \sum_{i=0}^{\infty} (1/(i + 1)^{2\alpha}) < \infty \) and, hence, \( z^0 \in \mathcal{H} \). Note that for all \( i \geq 1 \), we have

\[
\|z_i\|^2 (a + c_i^2)^2 \geq \frac{(1 - c_i^2)}{c_i^2} \quad \text{(6.7)}
\]

because \( c_i^2 \in [1/2, 1) \). In addition, for all \( i \geq 1 \), we have

\[
\|(P_V)z_i^0\|^2 = \frac{2\alpha \kappa_a e^{2/(a + 1)} \|z_i\|^2}{i/(i + 1)^{2\alpha}} \|(P_V)z_i^0\|^2 \geq \frac{2\alpha \kappa_a e^{2/(a + 1)} c_i^2 (1 - c_i^2)}{\|z_i\|^2 (a + c_i^2)^2 (i + 1)^{2\alpha}} \frac{1}{\|z_i\|^2 (a + c_i^2)^2 (i + 1)^{2\alpha}} \geq \frac{2\alpha \kappa_a e^{2/(a + 1)} (6.7)}{(i + 1)^{1+2\alpha}} \geq \frac{2\alpha \kappa_a e^{2/(a + 1)} (6.9)}{(i + 1)^{1+2\alpha}}
\]

where the third equality follows because \( 1 - c_i^2 = 1 - i/(i + 1) = 1/(i + 1) \).

Now, for all \( k \geq 0 \), let \( z^{k+1} = T_{FDRS} z^k \). Again, for all \( i \geq 0 \), let \( c_i = (a + c_i^2)/(a + 1) \) be the eigenvalue of \( (T_{FDRS})_i \) associated to \( z_i \). Note that \( b_i x^0 \geq e^{-2/(1+a)} \) whenever \( i \geq k \geq 0 \). Therefore, for all \( k \geq 1 \), we have

\[
\|x^k_h - x^*\|^2 = \|(P_V)T_{FDRS} z^0\|^2 = \sum_{i=k}^{\infty} b_i^{2(k+1)} \|(P_V)z_i^0\|^2 \geq \sum_{i=k}^{\infty} b_i^{2(k+1)} \frac{2\alpha \kappa_a e^{2/(a + 1)} (6.7)}{(i + 1)^{1+2\alpha}} \geq \frac{2\alpha}{(i + 1)^{1+2\alpha}} \geq \frac{1}{(k + 1)^{2\alpha}}. \quad (6.10)
\]
where we use $x^* = 0$ and the last inequality follows from the lower integral approximation of the sum. Finally, we show the gradient identity: $\|\nabla h(x_k^i) - \nabla h(x_k^*\|^2 = \|P_{\gamma}x_k^i - P_{\gamma}x_k^*\|^2 = \|x_k^i - x_k^*\|^2$.

Now we prove the bound for $(x_j^i)_{i \geq 0}$. For all $k \geq 0$, we have $x_k^i = T_{FDRS}z_k^i - \gamma \nabla h(x_k^i) = T_{FDRS}z_k^i - P_{\gamma}z_k^i = (T_{FDRS} - P_{\gamma})T_{FDRS}z_k^i$ (see Equation (2.1)). In addition, for all $i \geq 0$, we have

$$\left(\frac{1}{a+1}\begin{array}{cc}0 & -\cos(\theta)\sin(\theta) \\ \cos^2(\theta) + a - (a+1) & 0 \end{array}\right) = \left(\frac{1}{a+1}\begin{array}{cc}0 & \cos(\theta) \\ \sin(\theta) & 0 \end{array}\right).$$

Thus, for all $i \geq 0$, we have

$$\|\left(\frac{1}{a+1}\begin{array}{cc}0 & -\cos(\theta)\sin(\theta) \\ \cos^2(\theta) + a - (a+1) & 0 \end{array}\right)\| = \frac{2\alpha\kappa_\gamma e^{2/(a+1)}\sin^2(\theta)(\cos^2(\theta) + \sin^2(\theta))}{\|z_i\|^2(a+1)^2(i+1)^{2\alpha}} \leq \frac{2\alpha\kappa_\gamma e^{2/(a+1)}(1 - c_i^2)}{\|z_i\|^2(a+1)^2(i+1)^{2\alpha}} \geq \frac{(a+1)^2}{a+1} \frac{(a+1/2)^2}{(a+1)^2(i+1)^{2\alpha}}$$

where the last inequality follows because $1 - c_i^2 = 1 - i/(i+1) = 1/(i+1)$ and $\kappa_\gamma \|z_i\|^2 \geq (a+c_i^2)/c_i^2$? Note that for all $i \geq 1$, we have $(a+c_i^2)/c_i^2 \geq (a+1/2)^2$ because $c_i^2 \in [1/2, 1)$. Therefore, for all $k \geq 1$, we have

$$\|x_k^i - x_k^*\|^2 = \|T_{FDRS} - P_{\gamma}\|^2 \geq \sum_{i=k}^{\infty} b_i^{2(k+1)} \frac{2\alpha\kappa_\gamma e^{2/(a+1)}(a+c_i^2)^2}{c_i^2(a+1)^2(i+1)^{2\alpha}} \geq \frac{(a+1/2)^2}{(a+1)^4(i+1)^{2\alpha}}$$

where we use similar arguments to those used in Equation (6.10). \qed

**Remark 9.** The results of this section show that any proof of linear convergence when $(\mu_\gamma + \mu_f)^2 > 0$ and $\beta_f = 0$ must explicitly use finite-dimensional arguments.

### 7. Primal-dual splittings.

In this section, we reformulate FDRS as a primal-dual algorithm applied to the dual of Problem (1.23).

**Lemma 7.1 (FDRS is a primal-dual algorithm).** Let $\gamma = 1/\gamma$, and suppose that $(z^j)_{j \geq 0}$ is generated by the FDRS algorithm with $\lambda_k \equiv 1$. For all $k \geq 0$, let

$$y^k := -\nabla h(x_k^i).$$

Then for all $k \geq 0$, we have the recursive update rule:

$$\begin{cases} y^{k+1} = P_{\gamma}y^k - \tau x_k^i \\
 x_k^{i+1} = \text{prox}_{\gamma f}(x_k^i - y_k^k + \gamma(2y^{k+1} - y^k)) \end{cases} \quad (7.1)$$

**Proof.** Fix $k \geq 0$. By Lemma 2.1, $z^{k+1} = x_k^i - \gamma y_k^i$ and, hence, $(-1/\gamma)z^{k+1} = y_k^i - \tau x_k^i$. Thus, the formula for the sequence $(y^j)_{j \geq 0}$ follows from the definition $y^{k+1} = -\nabla h(x_k^i) = -(1/\gamma)P_{\gamma}z^{k+1}$. \qed
Now observe that
\[ x^k_f = P_V x_f^k + P_{V^\perp} x_f^k = P_V (z^{k+1} + \gamma y^k) + P_{V^\perp} (z^{k+1} + \gamma y^k) = x_h^{k+1} + \gamma (y^k - y^{k+1}). \]

Furthermore, \( \nabla h(x_f^k) = \nabla h(P_V x_f^k) = \nabla h(P_V (z^{k+1} + \gamma y^k)) = \nabla h(x_h^{k+1}). \) Thus, we have the following identity:
\[
\begin{align*}
x_f^{k+1} &\overset{(2.2)}{=} x_h^{k+1} - \gamma \left( \nabla h(x_h^{k+1}) + \nabla h(x_{f_h}^{k+1}) + \nabla f(x_f^{k+1}) \right) \\
&= \text{prox}_{\gamma f}(x_h^{k+1} - \gamma \nabla h(x_h^{k+1}) + \gamma y^{k+1}) \\
&= \text{prox}_{\gamma f}(x_h^{k+1} - \gamma \nabla h(x_f^{k+1}) + \gamma (2y^{k+1} - y^k)).
\end{align*}
\]

Therefore, the result follows.

The algorithm in Equation (7.1) is the primal-dual forward-backward algorithm of Vũ and Condat [33, 18] applied to the following dual problem:
\[
\min_{x \in V^\perp} (f + h)^*(x) \tag{7.2}
\]
where \((f + h)^*(\cdot) = \sup_{x \in \mathcal{H}} \langle x, \cdot \rangle - (f + h)(x)\) is the Legendre-Fenchel transform of \(f + h\) [3, Definition 13.1]. In order to guarantee convergence of this algorithm, [33, Theorem 3.1] requires the strict inequalities \(\gamma \tau < 1\) and
\[
2\beta \nu > \frac{1}{\min\{1/\gamma, 1/\tau\} (1 - \sqrt{\gamma \tau})}, \tag{7.3}
\]
whereas FDRS only requires \(\gamma < 2\beta \nu\), which is much weaker.

Thus, the FDRS algorithm is a limiting case of Vũ and Condat’s algorithm, much like the DRS algorithm [27] is a limiting case of Chambolle and Pock’s primal-dual algorithm [11]. In addition, the convergence rate analysis in Section 3 cannot be subsumed by the recent convergence rate analysis of the primal-dual gap of Vũ and Condat’s algorithm [20], which only applies when Equation (7.3) is satisfied and \(\gamma \tau < 1\). Note that the original paper on FDRS did not present this connection [10, Remark 6.3 (iii)].

8. Conclusion. In this paper, we provided a comprehensive convergence rate analysis of the FDRS algorithm under general convexity, strong convexity, and Lipschitz differentiability assumptions. In almost all cases, the derived convergence rates are shown to be optimal. In addition, we showed that the FDRS algorithm is the limiting case of a recently developed primal-dual forward-backward operator splitting algorithm and, thus, clarify how it relates to existing algorithms. All of the derived convergence rates follow from two fundamental inequalities (Propositions 2.3 and 2.4) and a simple diagram (Fig. 1). Future work on FDRS might focus on evaluating the practical performance of the algorithm on realistic problems. There is a large opportunity here because the power of this algorithm has yet to be fully explored.

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