BOUNDED COMPLEMENTS FOR WEAK FANO PAIRS 
WITH ALPHA-INVARIANTS AND VOLUMES BOUNDED 
BELOW 

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Abstract. We show that fixed dimensional klt weak Fano pairs with 
alpha-invariants and volumes bounded away from 0 and the coefficients 
of the boundaries belonging to a fixed DCC set \( S \) form a bounded 
family. Moreover, such pairs admit a strong \( \epsilon \)-lc \( R \)-complement for some 
fixed \( \epsilon > 0 \). This is an improvement of \[5\].

1. Introduction 

Throughout this paper, we work over an uncountable algebraically closed 
field of characteristic 0, for instance, the complex number field \( \mathbb{C} \).

In the birational geometry, as the first step of moduli theory, it is inter-
esting to consider whether a certain kind of family of varieties satisfy certain 
finiteness. For varieties of Fano type with bounded log discrepancies, Birkar 
shows in \[3\] Theorem 1.1 that

Theorem 1.1. Fix a positive integer \( d \) and a positive real number \( \epsilon \). The 
projective varieties \( X \) satisfying

(1) \( \dim X = d \),
(2) there exists a boundary \( B \) such that \( (X,B) \) is \( \epsilon \)-lc, and
(3) \( -(K_X + B) \) is nef and big,

form a bounded family.

Theorem 1.1 was known as the Borisov-Alexeev-Borisov (BAB) Conjec-
ture for decades before Birkar proved it. Equivalently, we can state Theorem 
1.1 in the following form of boundedness of varieties of Calabi–Yau type.

Theorem 1.2. Fix a positive integer \( d \) and a positive real number \( \epsilon \). The 
projective varieties \( X \) satisfying

(1) \( \dim X = d \),
(2) there exists a boundary \( B \) such that \( (X,B) \) is \( \epsilon \)-lc, and
(3) \( K_X + B \sim_R 0 \) and \( B \) is big,

form a bounded family.

In Theorem 1.1 it is necessary to take \( \epsilon > 0 \). In fact, klt Fano threefolds 
do not even form a birational family (see \[19\]). Nevertheless, Jiang shows 
in \[12\] that if we bound the alpha-invariants and the volumes from below, we have

Date: February 25, 2020.
Theorem 1.3 ([12, Theorem1.6]). Fix a positive integer $d$ and a positive real number $\theta$. The normal projective klt Fano (i.e. $\mathbb{Q}$-Fano in [12]) varieties $X$ satisfying

1. $\dim X = d$, and
2. $\alpha(X)^d (-K_X)^d > \theta$,

form a bounded family.

Inspired by Theorem 1.3, it is natural to ask if certain boundedness holds for varieties of Fano type or under more general setting. Thanks to boundedness of complements by Birkar (Theorem 3.9), if the coefficients of the boundaries are well controlled, then the boundedness, which is one of our main theorems, holds as follows.

Theorem 1.4. Fix a positive integer $d$, positive real numbers $\theta$ and $\delta$ and a finite set $\mathcal{R}$ of rational numbers in $[0,1]$. The set of all klt Fano pairs $(X,B)$ satisfying

1. $\dim X = d$,
2. the coefficients of $B \in \Phi(\mathcal{R})$,
3. $\alpha(X,B) > \theta$, and
4. $(-(K_X + B))^d > \theta$

is log bounded. Moreover, there exists a natural number $k$, depending only on $d$, $\theta$, $\delta$ and $\mathcal{S}$ such that there exists a strong klt $k$-complement $K_X + \Theta$ of $K_X + B$.

Combining Theorem 1.4 with boundedness of complements for DCC coefficients of boundaries by Han, Liu and Shokurov, we can weaken the assumption of the coefficients of boundaries as in the following theorem.

Theorem 1.5. Fix a positive integer $d$, positive real numbers $\theta$ and $\delta$ and a DCC set $\mathcal{S}$ of real numbers in $[0,1]$. The set of all klt Fano pairs $(X,B)$ satisfying

1. $\dim X = d$,
2. the coefficients of $B \in \mathcal{S}$,
3. $\alpha(X,B) > \theta$, and
4. $(-(K_X + B))^d > \theta$

is log bounded. Moreover, there exists a natural number $k$, finite sets $\Gamma_1 = \{a_i\}_i \in [0,1]$ with $\sum_i a_i = 1$ and $\Gamma_2 \in [0,1] \cap \mathbb{Q}$ depending only on $d$, $\theta$, $\delta$ and $\mathcal{S}$ such that there exists a strong klt $(k,\Gamma_1,\Gamma_2)$-complement $K_X + \Theta$ of $K_X + B$.

Remark 1.6. Chi Li, Yuchen Liu and Chenyang Xu show the following boundedness theorem using normalized volume.

Theorem 1.7 ([17, Corollary 6.14]). Fix a positive integer $d$ and a positive real number $\theta$. Then the set of all projective varieties $X$ satisfying

1. $\dim X = d$,
2. $(X,B)$ is a klt weak Fano pair for some effective $\mathbb{Q}$-divisor $B$ on $X$, and
3. $\alpha(X,B)^d(-(K_X + B))^d > \theta$,

is bounded.
2. Preliminaries

We adopt the standard notation and definitions in [13] and [16], and will freely use them.

2.1. Pairs and singularities. A sub-pair \((X, B)\) consists of a normal projective variety \(X\) and an \(\mathbb{R}\)-divisor \(B\) on \(X\) such that \(K_X + B\) is \(\mathbb{R}\)-Cartier. \(B\) is called the sub-boundary of this pair.

A log pair \((X, B)\) is a sub-pair with \(B \geq 0\). We call \(B\) a boundary in this case.

Let \(f : Y \to X\) be a log resolution of the log pair \((X, B)\), write

\[ K_Y = f^*(K_X + B) + \sum a_i F_i, \]

where \(F_i\) are distinct prime divisors. For a non-negative real number \(\epsilon\), the log pair \((X, B)\) is called

(a) \(\epsilon\)-kawamata log terminal (\(\epsilon\)-klt, for short) if \(a_i > -1 + \epsilon\) for all \(i\);
(b) \(\epsilon\)-log canonical (\(\epsilon\)-lc, for short) if \(a_i \geq -1 + \epsilon\) for all \(i\);

Usually we write \(X\) instead of \((X, 0)\) in the case when \(B = 0\). Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense. Also note that \(\epsilon\)-lc singularities only make sense if \(\epsilon \in [0, 1]\), and \(\epsilon\)-klt singularities only make sense if \(\epsilon \in [0, 1]\).

Similarly, sub-\(\epsilon\)-klt and sub-\(\epsilon\)-lc sub-pairs can be defined.

The log discrepancy of the divisor \(F_i\) is defined to be \(a(F_i, X, B) = 1 + a_i\). It does not depend on the choice of the log resolution \(f\).

\(F_i\) is called a non-klt place of \((X, B)\) if \(a_i \leq -1\). A subvariety \(V \subset X\) is called a non-klt center of \((X, B)\) if it is the image of a non-klt place. The non-klt locus \(\text{Nklt}(X, B)\) is the union of all non-klt centers of \((X, B)\). We recall the Kollár-Shokurov connectedness lemma.

Lemma 2.1 (cf. [20], [21] and [14, Theorem 17.4]). Let \((X, B)\) be a log pair, and let \(\pi : X \to S\) be a proper morphism with connected fibers. Suppose \(- (K_X + B)\) is \(\pi\)-nef and \(\pi\)-big. Then \(\text{Nklt}(X, B) \cap X_s\) is connected for any fiber \(X_s\) of \(\pi\).

2.2. Fano pairs and Calabi–Yau pairs. A projective pair \((X, B)\) is a Fano (resp. weak Fano, resp. Calabi–Yau) pair if it is lc and \(-(K_X + B)\) is ample (resp. \(- (K_X + B)\) is nef and big, resp. \(K_X + B \equiv 0\)). A projective variety \(X\) is called Fano, (resp. Calabi–Yau) if \((X, 0)\) is Fano (resp. Calabi–Yau). It is called \(\mathbb{Q}\)-Fano if it is klt and Fano. It is called of Fano type if \((X, B)\) is klt weak Fano for some boundary \(B\).

2.3. Bounded pairs. A collection of varieties \(\mathcal{D}\) is said to be bounded (resp. birationally bounded) if there exists \(h : Z \to S\) a projective morphism of schemes of finite type such that each \(X \in \mathcal{D}\) is isomorphic (resp. birational) to \(Z_s\) for some closed point \(s \in S\).

A couple \((X, D)\) consists of a normal projective variety \(X\) and a reduced divisor \(D\) on \(X\). Note that we do not require \(K_X + D\) to be \(\mathbb{Q}\)-Cartier here.

We say that a collection of couples \(\mathcal{D}\) is log birationally bounded (resp. log bounded) if there is a quasi-projective scheme \(Z\), a reduced divisor \(E\) on \(Z\), and a projective morphism \(h : Z \to S\), where \(S\) is of finite type and \(E\) does not contain any fiber, such that for every \((X, D) \in \mathcal{D}\), there is a closed
point \( s \in S \) and a birational (resp. isomorphic) map \( f : Z_s \rightarrow X \) such that \( E_s \) contains the support of \( f_s^{-1}D \) and any \( f \)-exceptional divisor.

A set of log pairs \( P \) is \emph{log birationally bounded} (resp. \emph{log bounded}) if the set of the corresponding couples \( \{(X, \text{Supp}B)|(X, B) \in P\} \) is log birationally bounded (resp. log bounded).

2.4. Volumes. Let \( X \) be a \( d \)-dimensional normal projective variety and \( D \) a Cartier divisor on \( X \). The \emph{volume} of \( D \) is the real number

\[
\text{vol}(X, D) = \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.
\]

For more backgrounds on the volume, see [18, 2.2.C]. By the homogeneous property and continuity of the volume, we can extend the definition to \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisors. Moreover, if \( D \) is a nef \( \mathbb{R} \)-divisor, then \( \text{vol}(X, D) = D^d \).

2.5. Complements.

\textbf{Definition 2.2.} Let \((X, B)\) be a pair and \( n \) a positive integer. We write \( B = \lfloor B \rfloor + \{B\} \). An \emph{n-complement} of \( K_X + B \) is a divisor of the form \( K_X + B^+ \) such that

1. \((X, B^+)\) is lc,
2. \( n(K_X + B^+) \sim 0 \), and
3. \( nB^+ \geq n\lfloor B \rfloor + [(n+1)\{B\}] \).

If moreover, \( B^+ \geq B \) (resp. \((X, B^+)\) is klt, resp. \((X, B^+)\) is \( \epsilon \)-lc, where \( \epsilon > 0 \)), we say that the complement is \emph{strong} (resp. klt, resp. \( \epsilon \)-lc).

We say that \((X, B)\) has an \( \mathbb{R} \)-complement or is \emph{complementary} if there exists \( B \geq B \) such that \((X, B)\) is lc. In this case, we call \( K_X + B \) an \( \mathbb{R} \)-complement of \( K_X + B \).

\textbf{Definition 2.3.} For a subset \( \mathcal{R} \) of \([0, 1]\), we define the set of \emph{hyperstandard multiplicities} associated to \( \mathcal{R} \) to be

\[
\Phi(\mathcal{R}) = \left\{ 1 - \frac{r}{m} | r \in \mathcal{R}, \ m \in \mathbb{N} \right\}.
\]

Note that the only possible accumulating point of \( \Phi(\mathcal{R}) \) is 1 if \( \mathcal{R} \) is finite. Birkar shows the following boundedness of complements.

\textbf{Theorem 2.4 ([2, Theorem 1.7]).} Fix a positive integer \( d \) and a finite set \( \mathcal{R} \) of rational numbers in \([0, 1]\). Then there exists a positive integer \( n \) depending only on \( d \) and \( \mathcal{R} \), such that if \((X, B)\) is a projective pair with

1. \((X, B)\) is lc dimension \( d \),
2. the coefficients of \( B \in \Phi(\mathcal{R}) \),
3. \( X \) is of Fano type, and
4. \(- (K_X + B) \) is nef,

then there is an \( n \)-complement \( K_X + B^+ \) of \( K_X + B \) such that \( B^+ \geq B \).

Based on Theorem 3.9 and ACC for log canonical threshold [9 Theorem 1.1]], Filipazzi and Moraga showed the following existence of bounded complements for DCC coefficients of the boundaries.

\textbf{Theorem 2.5 ([6, Theorem 1.2]).} Fix a positive integer \( d \) and a closed DCC set \( \mathcal{I} \) of rational numbers in \([0, 1]\). Then there exists a positive integer \( n \) depending only on \( d \) and \( \mathcal{I} \), such that if \((X, B)\) is a projective pair with
conditions are satisfied.

\[ \sum \] is a normal variety and \( X \) is of Fano type, and

\[ -(K_X + B) \] is nef.

then there is an \( n \)-complement \( K_X + B^+ \) of \( K_X + B \) such that \( B^+ \geq B \).

**Theorem 2.6** ([7 Theorem 5.22]). Let \( d \) be a natural number and \( \Gamma \subset [0, 1] \) be a closed DCC set. Then there exists a finite subset \( \Gamma_1 \) of \( \Gamma \) and a projection \( g : \Gamma \to \Gamma_1 \) such that \( g(g(x)) = g(x) \), \( g(x) \geq x \) and \( g(y) \geq g(x) \) for every \( x \leq y \in \Gamma \), depending only on \( d \) and \( \Gamma \) satisfying the following. Suppose \( X, B := \sum b_i B_i \) is lc of dimension \( d \) where \( b_i \in \Gamma \), \( B_i \geq 0 \) is \( \mathbb{Q} \)-Cartier Weil divisor for any \( i \), \( (X, B) \) has an \( \mathbb{R} \)-complement and \( X \) is of Fano type, then

\begin{enumerate}
  \item \( (X, \sum g(b_i) B_i) \) is lc, and
  \item \( -(K_X + \sum g(b_i) B_i) \) is pseudo-effective.
\end{enumerate}

We recall the construction of \((n, \Gamma_1, \Gamma_2)\)-complement by Han, Liu and Shokurov.

**Theorem 2.7** ([7 Theorem 1.13]). Let \( d \) be a natural number \( \delta > 0 \) be a positive real number and \( \Gamma \subset [0, 1] \) be a closed DCC set. Then there exist integers \( n > 0 \) and \( r > 0 \), finite sets \( \Gamma_1 \subset [0, 1] \), \( \{a_i\}_{i=1}^r \in (0, 1] \), \( \Gamma_2 = \bigcup_{i=1}^r \Gamma_i' \subset Q \cap [0, 1] \), a projection \( g : \Gamma \to \Gamma_1 \) and bijections \( g_i : \Gamma_1 \to \Gamma_i' \) for every \( 1 \leq i \leq r \) which only depend only on \( d \) and \( \Gamma \) satisfying the following.

\( \Gamma_1 \) and \( g \) are given by Theorem 2.6. \( g(x) = \sum_{i=1}^r a_i (g_i \circ g(x)) \) and \( \sum_{i=1}^r a_i = 1 \). \( |g_i(x) - x| < \delta \) for every \( 1 \leq i \leq r \) and \( x \in \Gamma_1 \). Assume \( X \) is a normal variety and \( B \) is an effective divisor on \( X \) with the following conditions:

\begin{enumerate}
  \item \( X \) is of Fano type,
  \item \( \dim X \leq d \),
  \item the coefficients of \( B \in \Gamma \),
  \item \( (X, B) \) has an \( \mathbb{R} \)-complement, and
  \item we can write \( B := \sum b_j B_j \) where \( b_j \in \Gamma \) and \( B_j \geq 0 \) is a \( \mathbb{Q} \)-Cartier Weil divisor for any \( j \),
\end{enumerate}

then \( (X, \sum g(b_j) B_j) \) and \( (X, \sum g_i \circ g(b_j) B_j) \) are lc for every \( K_X + \sum_j g_i \circ g(b_j) B_j \) has a strong \( n \)-complement for every \( 1 \leq i \leq r \). In particular, \( (X, \sum g(b_j) B_j) \) has an \( \mathbb{R} \)-complement.

**Definition 2.8** ([7 Definition 1.12]). Assume \( (X, B) \) is a normal pair, \( n \) is a positive integer, and \( \Gamma_1 \subset [0, 1] \) and \( \Gamma_2 \subset Q \cap [0, 1] \) are two finite sets. We say that \( K_X + B^+ \) is an \((n, \Gamma_1, \Gamma_2)\)-complement of \((X, B)\) if the following conditions are satisfied.

\begin{enumerate}
  \item \( B^+ \geq B \),
  \item there is a finite set \( \{a_i\}_{i=1}^r \subset \Gamma_1 \) with \( \sum a_i = 1 \),
  \item there are divisors \( B_i \geq 0 \) on \( X \) with coefficients of \( B_i \in \Gamma_2 \) for every \( 1 \leq i \leq r \),
  \item \( \sum a_i B_i = B^+ \), and
  \item \( K_X + B_i \) is an \( n \)-complement of itself for each \( 1 \leq i \leq r \).
\end{enumerate}
Definition 2.9. Under the notation of Definition 2.8, if \((X, B_i)\) are ckt, then we say that the \((n, \Gamma_1, \Gamma_2)\)-complement \(K_X + B^+\) is ckt. Note that this implies that \((X, B_i)\) and \((X, B^+)\) are \(\frac{1}{n}\)-lc.

Theorem 2.10. Under the notation of Theorem 2.4, for each \(1 \leq i \leq r\) there exist a closed DCC set of rational numbers \(\mathcal{S}_i\) and a function \(g'_i : \Gamma \to \mathcal{S}_i\) such that \(B \leq \sum_i (a_i \sum_j g'_i(b_j)B_j)\) and \(\sum_j g'_i(b_j)B_j \leq \frac{B + \sum_i g(b)B_i}{n}\) for every \(1 \leq i \leq r\).

Proof. By shrinking \(\Gamma_1\), we may assume that it is the image of \(g\) and so \(g\) is identity on \(\Gamma_1\). We may also assume that 0 is not in \(\Gamma_1\). We define a finite rational set \(\Gamma^-\) and a function \(g^- : \Gamma \to \Gamma^-\) as following. For \(\gamma \in \Gamma_1\), let \(\gamma^-\) be any rational number in the interval

\[
(\max\{0, x|x \in \Gamma, g(x) < \gamma\}, \min\{x|x \in \Gamma, g(x) \geq \gamma\}).
\]

Note that here the maximum and minimum exist and \(\max\{0, x|x \in \Gamma, g(x) < \gamma\} < \min\{x|x \in \Gamma, g(x) \geq \gamma\}\). Let \(g^-(x) = g(x)^-\) for every \(x \in \Gamma\). It follows by construction that \(g^-(x) \leq x < g(x)\) for every \(x \in \Gamma\). By abuse of notations, we denote the functions \(g_i \circ g\) as \(g_i\). Since \(\Gamma_1\) and \(\Gamma^-\) are finite, by possibly replacing \(\delta\), we may assume that \(g(x) - g_i(x) \leq g(x) - g^-(x)\) for every \(x \in \Gamma\). Therefore we have \(2(g_i(x) - g^-(x)) \geq (g(x) - g^- (x))\) for every \(x \in \Gamma\). Let \(f : \mathbb{Q} \to \mathbb{N}\) be any bijection. We now define the functions \(g'_i\) as following. Assume that \(x \in \Gamma\), let \(x' = g^-(x) + \frac{(g(x) - g^-(x))(x - g^-(x))}{g(x) - g^-(x)}\). Then

\[
g_i(x) - x' = g_i(x) - g^-(x) - \frac{(g_i(x) - g^-(x))(x - g^-(x))}{g(x) - g^-(x)}
\]

\[
= (g_i(x) - g^-(x)) \frac{g(x) - g^-(x) - (x - g^-(x))}{g(x) - g^-(x)}
\]

\[
= (g_i(x) - g^-(x)) \frac{g(x) - x}{g(x) - g^-(x)}
\]

\[
\geq 0,
\]

where the equality holds if and only if \(g(x) = x\). Let \(g'_i(x)\) be the rational number in \([x', \frac{x' + g(x)}{2}]\) with the smallest \(f\) value. Note that here, if \(x' = \frac{x' + g(x)}{2}\), then it is a rational number in \(\Gamma_1\). Let \(\mathcal{S}_i\) be the closure or the image of \(g'_i\).

We now argue that \(\mathcal{S}_i\) is a rational DCC set. It is enough to show that the image of \(g'_i\) is a DCC set with rational accumulation points.

Fix \(i\) for each \(1 \leq i \leq r\). Let \(\{x_i\}_i\) be a sequence of elements in \(\Gamma\) such that \(\{g'_i(x_i)\}_i\) is and strictly converges to some point \(x_0 \in \mathcal{S}_i\). Suppose either \(\{g'_i(x_i)\}_i\) is strictly decreasing or \(x_0\) is irrational. By construction, \(g'_i(x_i)\) is the element with the smallest \(f\) value in \(\mathbb{Q} \cap [x'_i, \frac{x'_i + g_i(x_i)}{2}]\), where \(x'_i = g^-(x_i) + \frac{(x_i - g^-(x_i))(x_i - g^-(x_i))}{(g(x_i) - g^-(x_i))}\). Because the image of \(g^-\), \(g_i\) and \(g^+\) are finite, \(\frac{(g_i(x) - g^-(x))}{(g(x) - g^-(x))} > 0\) for every \(x \in \Gamma\), and \(\Gamma\) is DCC, it follows that \(\{x'_i\}_i\) and \(\{\frac{x'_i + g_i(x_i)}{2}\}_i\) are both DCC. Passing through a subsequence, we may assume that \(\{x'_i\}_i\) and \(\{\frac{x'_i + g_i(x_i)}{2}\}_i\) are non-decreasing. Passing to a subsequence, we may assume the sequence \(\{g_i(x_i)\}_i\) is a constant sequence.
Further, if \( \{x_i^r\}_l \) is a constant sequence, \( \{ \frac{x_i^r + g(x_i)}{2} \}_l \) is also a constant sequence, and this implies \( \{ g_i'(x_i) \}_l \) is also constant, which is a contradiction.

Now we may assume that \( \{x_i^r\}_l \) and \( \{ \frac{x_i^r + g(x_i)}{2} \}_l \) are strictly increasing and \( \{x_i^r\}_l \) converges to \( x_0^r \leq g_i(x_i) \in \Gamma_i' \). If \( x_0^r = g_i(x_i) \), then neither \( \{ g_i'(x_i) \}_l \) is strictly decreasing nor \( x_0 = g_i(x_i) \) is irrational, which is a contradiction. So we have \( x_0^r < g_i(x_i) \). Let \( y \) be a rational number in \( [x_0^r, \frac{2}{x_i^r + g(x_i)}] \). Passing through a subsequence again, we may assume that \( \frac{x_i^r + g(x_i)}{2} > y \) for every \( l \).

This implies that \( x_i^r < x_0^r < y < \frac{x_i^r + g(x_i)}{2} \) for every \( l \). Then \( f(g_i'(x_i)) \leq f(y) \) for every \( l \) because \( g_i'(x_i) \) is the rational number in \( [x_i^r, \frac{2}{x_i^r + g(x_i)}] \) with the smallest \( f \) value. But the set \( \{ f(g_i'(x_i)) \} \) is infinite because \( \{ g_i'(x_i) \}_l \) is strictly converging. This contradicts our assumption that \( f \) is a bijection.

It then remains to show that \( B \leq \sum_i (a_i \sum_j g_i(b_j)B_j) \) and \( \sum_j g_i'(b_j)B_j \leq \frac{B + 3 \sum_j g(b_j)B_j}{4} \) for every \( 1 \leq i \leq r \). We simply compute that

\[
\sum_i \left( a_i \sum_j g_i'(b_j)B_j \right) = \sum_j \left( a_i \sum_j g_i(b_j)B_j \right) + \left( g_i(b_j) - g_i'(b_j) \right) \frac{\sum_i \left( a_i g_i(b_j) \right) - \sum_i \left( a_i g_i'(b_j) \right) \sum_j g_i(b_j)B_j}{2} = \sum_j \left( g_i(b_j) \right) \frac{\sum_i \left( a_i g_i(b_j) \right) - \sum_i \left( a_i g_i'(b_j) \right) \sum_j g_i(b_j)B_j}{2} = \sum_j b_j B_j = B.
\]

On the other hand, we have

\[
\sum_j g(b_j)B_j - \sum_j g_i'(b_j)B_j = \sum_i \left( a_i \sum_j g_i(b_j)B_j \right) - \sum_i \left( a_i \sum_j g_i'(b_j)B_j \right) = \sum_j \sum_i \left( a_i g_i(b_j) - g_i'(b_j) \right) B_j.
\]

We compute that for each \( i \),

\[
g_i(b_j) - g_i'(b_j) \geq g_i(b_j) - g_i(b_j) + \frac{\left( g_i(b_j) - g_i'(b_j) \right) \left( b_j - g_i'(b_j) \right)}{\left( g_i(b_j) - g_i'(b_j) \right)} = \frac{\left( g_i(b_j) - g_i'(b_j) \right) \left( b_j - g_i'(b_j) \right)}{\left( g_i(b_j) - g_i'(b_j) \right)} = \frac{\left( g_i(b_j) - g_i'(b_j) \right) \left( g_i(b_j) - g_i'(b_j) \right) - \left( b_j - g_i'(b_j) \right)}{\left( g_i(b_j) - g_i'(b_j) \right)} = \frac{1}{2} \left( g_i(b_j) - g_i'(b_j) \right) \frac{\left( g_i(b_j) - g_i'(b_j) \right) - \left( b_j - g_i'(b_j) \right)}{\left( g_i(b_j) - g_i'(b_j) \right)} = \frac{1}{2} \left( g_i(b_j) - g_i'(b_j) \right) \frac{\left( g_i(b_j) - b_j \right)}{\left( g_i(b_j) - g_i'(b_j) \right)} \geq \frac{g(b_j) - b_j}{4}.
\]
It follows that \( \sum_j g(b_j)B_j - \sum_j g'(b_j)B_j \geq \frac{\sum_j g(b_j)B_j - B}{4} \). Therefore, we have
\[
\sum_j g'(b_j)B_j \leq \sum_j g(b_j)B_j - \frac{\sum_j g(b_j)B_j - B}{4} = \frac{B + 3\sum_j g(b_j)B_j}{4}.
\]

2.6. \( \alpha \)-invariants and log canonical thresholds.

**Definition 2.11.** Let \((X, B)\) be a projective lc pair and let \(D\) be an effective \(\mathbb{R}\)-Cartier divisor, we define the log canonical threshold of \(D\) with respect to \((X, B)\) to be
\[
\text{lct}((X, B), D) = \sup \{ t \in \mathbb{R} \mid (X, B + tD) \text{ is lc} \}.
\]
The log canonical threshold of \(|D|_\mathbb{R}\) with respect of \((X, B)\) is defined to be
\[
\text{lct}((X, B), |D|_\mathbb{R}) = \inf \{ \text{lct}((X, B), M) \mid M \in |D|_\mathbb{R} \}.
\]

**Definition 2.12.** Let \((X, B)\) be a projective lc pair such that \(|-(K_X + B)|_\mathbb{R}\) is non-empty, we define the \(\alpha\)-invariant of \((X, B)\) to be
\[
\alpha(X, B) = \text{lct}((X, B), |-(K_X + B)|_\mathbb{R}).
\]
In the case when \(B = 0\), we usually write \(\alpha(X) := \alpha(X, 0)\) for convenience.

Now we consider \(\alpha(X, B)^d(-(K_X + B))^d\) as an invariant for \(d\)-dimensional klt Fano pairs \((X, B)\). It is well known that this invariant has an upper bound, which can be given by the following lemma.

**Lemma 2.13** ([15, Theorem 6.7.1]). Let \((X, B)\) be a klt pair of dimension \(d\). Then we have
\[
\text{lct}((X, B), |H|_\mathbb{R})^dH^d \leq d^d
\]
for any nef and big \(\mathbb{Q}\)-Cartier divisor \(H\) on \(X\).

2.7. Potentially birational divisors.

**Definition 2.14** (cf. [9, Definition 3.5.3]). Let \(X\) be a projective normal variety, and \(D\) a big \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(X\). Then, we say that \(D\) is potentially birational if for any two general points \(x\) and \(y\) of \(X\), there is an effective \(\mathbb{Q}\)-divisor \(\Delta \sim_{\mathbb{Q}} (1 - \epsilon)D\) for some \(0 < \epsilon < 1\), such that, after possibly switching \(x\) and \(y\), \((X, \Delta)\) is not klt at \(y\), lc at \(x\) and \(x\) is a non-klt center.

2.8. Descending chain condition.

**Definition 2.15.** A set of real numbers \(\mathcal{S}\) is said to satisfy descending chain condition (DCC for short) if for every non-empty subset \(S\) of \(\mathcal{S}\), there is a minimum element in \(S\). \(\mathcal{S}\) is called a DCC set if it satisfies DCC.
3. Proofs of Theorems

Now we restate and prove the theorems in Section 1. We recall the following key Lemma of [5].

**Theorem 3.1** ([5, Theorem 3.1]). Fix a positive integer \(d\) and a positive real number \(\theta\). Then there is a number \(m\) depending only on \(d\) and \(\theta\) such that if \(X\) is a projective normal variety satisfying

1. \(\dim X = d\),
2. there exists a boundary \(B\) such that \((X, B)\) is klt,
3. there is a nef \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(H\) on \(X\) with \(\text{lct}((X, B), |H|) > \theta\), and
4. \(H^d > \theta\),

then \(|K_X + [mH]|\) defines a birational map and \(mH\) is potentially birational.

**Corollary 3.2** ([5, Corollary 3.2]). Fix positive integers \(d, n\), and a positive real number \(\theta\). Then there is a number \(m\) depending only on \(d, n\) and \(\theta\) such that if \((X, B)\) is a weak Fano pair satisfying

1. \(\dim X = d\),
2. \(\alpha(X, B) > \theta\),
3. \(K_X + B\) is a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor,
4. \((-K_X - B)^d > \theta\), and
5. there is an \(n\)-complement \(K_X + B^+\) of \(K_X + B\) with \(B^+ \geq B\),

then \(|[m(B^+ - B)]|\) defines a birational map.

We recall the following theorem by Hacon and Xu.

**Theorem 3.3** ([10, Theorem 1.3]). Fix a positive integer \(d\) and a DCC set \(\mathcal{I}\) of rational numbers in \([0,1]\). The set of all projective pairs \((X, B)\) satisfying

1. \((X, B)\) is klt log Calabi–Yau of dimension \(d\),
2. \(B\) is big, and
3. the coefficients of \(B \in \mathcal{I}\),

forms a bounded family.

Then we recall the following log-version of [2, Lemma 2.26].

**Lemma 3.4.** Fix positive integers \(d, k\) and a non-negative real number \(\epsilon\). Let \(\mathcal{P}\) be a set of klt weak Fano pairs of dimension \(d\). Assume that for every element \((Y, B_Y) \in \mathcal{P}\), there is a \(k\)-complement (resp. \(\mathbb{R}\)-complement) \(K_Y + B_Y^+\) of \(K_Y + B_Y\) such that \((Y, B_Y^+)\) is \(\epsilon\)-lc and \(B_Y^+ \geq B_Y\). Let \(\mathcal{Q}\) be the set of projective pairs \((X, B)\) such that

1. there is \((Y, B_Y) \in \mathcal{P}\) and a birational map \(X \to Y\),
2. there is a common resolution \(\phi: W \to Y\) and \(\psi: W \to X\), and
3. \(\phi^*(K_Y + B_Y) \geq \psi^*(K_X + B)\).

Then for every element \((X, B) \in \mathcal{Q}\), there is a \(k\)-complement \(K_X + B^+\) of \(K_X + B\) such that \((X, B^+)\) is \(\epsilon\)-lc and \(B^+ \geq B\).

**Proof.** Let \(K_X + B^+\) be the crepant pullback of \(K_Y + B_Y^+\) to \(X\). Then \((X, B^+)\) is \(\epsilon\)-lc. Since \(k(K_Y + B_Y^+) \sim 0\) (resp. \(K_Y + B_Y^+ \sim_{\mathbb{R}} 0\)) and \(\phi^*(K_Y + B_Y) \geq \psi^*(K_X + B), B^+ - B \geq \psi_* \phi^*(B_Y^+ - B_Y) \geq 0\) and \(K_X + B^+\) is a strong \(k\)-complement (resp. \(\mathbb{R}\)-complement) of \(K_X + B\). \(\square\)
Lemma 3.5. Let $(X, B)$ be a projective lc weak Fano pair with $\alpha(X, B) \leq 1$. Suppose $\phi : X \to Y$ is a partial MMP (of any divisor). Let $B_Y = \phi_*(B)$. Then $(Y, B_Y)$ is lc and $\alpha(Y, B_Y) \geq \alpha(X, B)$. If moreover $(X, B)$ is klt, then $(Y, B_Y)$ is also klt.

Proof. Since $(X, B)$ is weak Fano, there is an $\mathbb{R}$-complement $K_X + \overline{F}$ of $K_X + B$. Let $\overline{F}_Y = \phi_*(\overline{F})$, then $K_Y + \overline{F}_Y$ is an $\mathbb{R}$-complement of $K_Y + B_Y$. In particular, $(Y, B_Y)$ is lc and $|-(K_Y + B_Y)|_\mathbb{R}$ is non-empty. Let $t = \alpha(X, B)$. Now we consider the following argument for any $M_Y \in |-(K_Y + B_Y)|_\mathbb{R}$. Let $K_X + B + M$ be the crepant pullback of $K_Y + B_Y + M_Y$ to $X$. As in the proof of Lemma 3.4, we have $M \in |-(K_X + B)|_\mathbb{R}$ and $\phi_*(M) = M_Y$. By assumption, $(X, B + tM)$ is lc and $-(K_X + B + tM) \sim_{\mathbb{R}} -(1-t)(K_X + B)$ is nef and big if $t \neq 1$. So there is an effective divisor $G$ on $X$ such that $(X, B + tM + G)$ is lc Calabi–Yau. Let $G_Y = \phi_*(G)$. Then $(Y, B_Y + tM_Y + G_Y)$ is an $\mathbb{R}$-complement of $K_Y + B_Y + tM_Y$. In particular, $(Y, B_Y + tM_Y)$ is lc. Therefore, we have $\alpha(Y, B_Y) \geq t$. If moreover $(X, B)$ is klt, then we may assume $(X, B)$ is klt and therefore $(Y, B_Y)$ is klt.

Next, we recall the following proposition of Birkar with a small observation.

Proposition 3.6 ([2, Proposition 4.4]). Fix positive integers $d, v$ and a positive real number $\epsilon$. Then there exists a bounded set of couples $\mathcal{P}$ and a positive real number $c$ depending only on $d, v$ and $\epsilon$ satisfies the following. Assume

- $X$ is a normal projective variety of dimension $d$,
- $B$ is an effective $\mathbb{R}$-divisor with coefficient at least $\epsilon$,
- $M$ is a $\mathbb{Q}$-divisor with $|M| := |\lfloor M\rfloor|$ defining a birational map,
- $M - (K_X + B)$ is pseudo-effective,
- $\operatorname{vol}(M) < v$, and
- $\mu_D(B + M) \geq 1$ for every component $D$ of $M$.

Then there is a projective log smooth couple $(\overline{W}, \Sigma_{\overline{W}}) \in \mathcal{P}$, a birational map $\overline{W} \to X$ and a common resolution $X'$ of this map such that

1. $\operatorname{Supp}(\Sigma_{\overline{W}})$ contains the exceptional divisor of $\overline{W} \to X$ and the birational transform of $\operatorname{Supp}(B + M)$, and
2. there is a resolution $\phi : W \to X$ such that $M_W := \phi^* M \sim A_W + R_W$ where $A_W$ is the movable part of $|M_W|$, $|A_W|$ is base point free, $\psi : X' \to X$ factors through $W$ and $A_{X'} \sim 0/\overline{W}$, where $A_{X'}$ is the pushdown of $A_W$ on $X'$.

Moreover, if $M$ is nef and $M_{\overline{W}}$ is the pushdown of $M_{X'} := \psi^* M$. Then each coefficient of $M_{\overline{W}}$ is at most $c$.

Note that in the original statement of [2, Proposition 4.4], $M$ is assumed to be nef. We observe from Birkar’s proof that the nefness of $M$ is used only when showing the existence of $c$ and is not necessary when showing (1) and (2) of proposition 3.6.

Now we are ready to show the main theorem of this paper. The idea is to follow the strategy of [2, Proposition 7.13], which is to construct a klt complement with coefficients in a finite set depending only on $d, \theta$ and $\mathcal{P}$, and then apply Theorem 3.3.
Theorem 3.7. Fix a positive integer $d$, a positive real number $\theta$ and a closed DCC set $\mathcal{I}$ of rational numbers in $[0, 1]$. The set $\mathcal{D}$ of all klt weak Fano pairs $(X, B)$ satisfying

1. $\dim X = d$,
2. the coefficients of $B \in \mathcal{I}$,
3. $\alpha(X, B) > \theta$, and
4. $(-(K_X + B))^d > \theta$

forms a log bounded family. Moreover, there is a rational number $k$ depending only on $d$, $\theta$, $\mathcal{I}$, such that every element $(X, B) \in \mathcal{D}$, there is a strong klt $k$-complement $K_X + \Theta$ of $K_X + B$.

Proof. By Theorem 3.3 it is enough to show the existence of $k$.

By Lemma 3.4, replacing by a small $\mathbb{Q}$-factorialisation of $X$, we may assume $X$ is $\mathbb{Q}$-factorial.

By Theorem 2.5, there is a $n$-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$.

Let $m$ be given by Corollary 3.2 such that $|\lfloor m(B^+ - B) \rfloor|$ defines a birational map. Replacing $n$ and $m$ by $2mn$, we may assume $n = m > 1$. On the other hand, by Lemma 2.13, $\vol(|\lfloor m(B^+ - B) \rfloor|) \leq \vol(m(B^+ - B)) < v$ for some $v$ depending only on $d$, $m$ and $\theta$.

Let $M$ be a general element of $|\lfloor m(B^+ - B) \rfloor|$. By Proposition 3.6, there is a bounded set of couples $P$ depending only on $d$, $m$ and $\theta$, such that there is a projective log smooth couple $(\overline{W}, \Sigma_{\overline{W}}) \in P$, a birational map $\overline{W} \dashrightarrow X$ and a common resolution $X'$ of this map such that

1. $\text{Supp} \Sigma_{\overline{W}}$ contains the exceptional divisor of $\overline{W} \dashrightarrow X$ and the birational transform of $\text{Supp}(B^+ + M)$, and
2. there is a resolution $\phi : W \rightarrow X$ such that $M_W := \phi^* M \sim A_W + R_W$ where $A_W$ is the movable part of $|M_W|$, $|A_W|$ is base point free, $X' \rightarrow X$ factors through $W$ and $A_{X'} \sim 0/\overline{W}$, where $A_{X'}$ is the pullback of $A_W$ on $X'$.

Since $M$ is a general element of $|\lfloor m(B^+ - B) \rfloor|$, we may assume $M_W = A_W + R_W$ and $A_W$ is general in $|A_W|$. In particular, if $A_{\overline{W}}$ is the pushdown of $A_W|_{X'}$ to $\overline{W}$, then $A_{\overline{W}} \leq \Sigma_{\overline{W}}$. Let $M, A, R$ be the pushdowns of $M_W$, $A_W$, $R_W$ to $X$. Since $|A_{\overline{W}}|$ defines a birational contraction and $A_{\overline{W}} \leq \Sigma_{\overline{W}}$, there exists $l \in \mathbb{N}$ depending only on $P$ such that $lA_{\overline{W}} \sim G_{\overline{W}}$ for some $G_{\overline{W}} \geq 0$ whose support contains $\Sigma_{\overline{W}}$. Let $K_{\overline{W}} + B^+_{\overline{W}}$ be the crepant pullback of $K_X + B^+$ to $\overline{W}$. Then $(\overline{W}, B^+_{\overline{W}})$ is sub-lc and

$$\text{Supp} B^+_{\overline{W}} \subseteq \text{Supp} \Sigma_{\overline{W}} \subseteq \text{Supp} G_{\overline{W}}.$$

Let $G$ be the pushdown of $G_X := G_{\overline{W}}|_{X'}$ to $X$. Since $A_{X'}$ is the pullback of $A_{\overline{W}}$, we obtain $tA_{X'} \sim G_{X'}$, and $tA \sim G$. Therefore, $G + lR + t\{m(B^+ - B)\} \sim_{\mathbb{Q}} m(B^+ - B)$.

Take a positive rational number $t \leq (lm)^{-\delta} \theta$, then

$$(X, B + t(G + lR + t\{m(B^+ - B)\}))$$

is klt. Moreover, we have

$$-K_X - B - t(G + lR + t\{m(B^+ - B)\}) \sim_{\mathbb{Q}} B^+ - B - t(lm(B^+ - B)).$$
By replacing $t$, we may assume $t < \frac{1}{lm}$. Since

$$B^+ - B - t(lm(B^+ - B)) = (1 - tl)(B^+ - B) \geq 0,$$

$B^+ - B - t(lm(B^+ - B))$ is nef and big. Therefore

$$(X, B + t(G + lR + l\{m(B^+ - B)\}))$$

is klt weak Fano.

Now we argue that the coefficients of $B + t(G + lR + l\{m(B^+ - B)\})$ are in a closed DCC set of rational numbers in $[0, 1]$. It is clear that the coefficients of $B + t(G + lR + l\{m(B^+ - B)\})$ are in a set of rational numbers in $[0, 1]$, so it is enough to show that they are in a set with rational accumulation points. Write

$$B + tl\{m(B^+ - B)\} = (1 - tm)B + tlB^+ - tl\{m(B^+ - B)\}.$$

Since $m, l$ and $t$ are determined by the fixed terms $d, \theta$ and $\mathcal{R}$, the coefficients of $tlB^+ - tl\{m(B^+ - B)\} + t(G + lR)$ are in a fixed finite rational set. By assumption, the coefficients of $B$ are in a DCC set with rational accumulation points $\mathcal{R}$. Therefore the coefficients of $(1 - tm)B + tlB^+ - tl\{m(B^+ - B)\} + t(G + lR)$ are in a DCC set with rational accumulation points.

By Theorem 2.3, there is a positive integer $n'$ depending only on $d, \mathcal{R}, m, l$ and $t$ such that there is an $n'$-complement $K_X + \Omega$ of $K_X + B + t(G + lR + l\{m(B^+ - B)\})$, such that

$$\Omega \geq B + t(G + lR + l\{m(B^+ - B)\}).$$

On the other hand, let

$$\Delta_W := B^+_W + \frac{t}{m}A_W - \frac{t}{lm}G_W.$$

Then $(\overline{W}, \Delta_W)$ is sub-$2\epsilon$-klt for some $\epsilon > 0$ depending only on $\mathcal{P}, t, l$ and $m$ since $\text{Supp}B^+_W \subseteq \text{Supp}X \subseteq \text{Supp}G_W$, $(W, \Sigma_W)$ is log smooth, $K_W + \Delta_W$ is sub-$\epsilon$-lc, and $A_W$ is not a component of $[B^+_W]$. Moreover, $K_W + \Delta_W \sim_{Q} 0$.

Let

$$\Delta := B^+ + \frac{t}{m}A - \frac{t}{lm}G.$$

Then $K_X + \Delta \sim_{Q} 0$. $(X, \Delta)$ is sub-klt since $K_X + \Delta$ is the crepant pullback of $K_W + \Delta_W$.

Let $\Theta = \frac{1}{2}\Delta + \frac{1}{2}\Omega$. Then

$$\Theta = \frac{1}{2}B^+ + \frac{t}{2m}A - \frac{t}{2lm}G + \frac{1}{2}\Omega \geq \frac{1}{2}B^+ + \frac{t}{2m}A - \frac{t}{2lm}G + \frac{1}{2}B + \frac{t}{2}(G + lR) \geq \frac{t}{2lm}G + \frac{t}{2}G \geq 0.$$

Since $(X, \Delta)$ is sub-$\epsilon$-klt, $K_X + \Delta \sim_{Q} 0$ and $(X, \Omega)$ is lc log Calabi–Yau, $(X, \Theta)$ is klt log Calabi–Yau. The coefficients of $\Theta$ belong to a fixed finite set depending only on $t, l, m$ and $n'$. Moreover, $\Theta \geq \frac{1}{2}B^+ + \frac{1}{2}\Omega \geq \frac{1}{2}B + \frac{1}{2}B = B$.

Recall that $n(K_X + B^+) \sim 0$, $lA \sim G$ and $n'(K_X + \Omega) \sim 0$. Let $k$ be an integer such that $k(\frac{1}{2m}, \frac{1}{2lm}, \frac{1}{2}) \subseteq \mathbb{N}$, then $K_X + \Theta$ is a $k$-complement of $K_X + B$, with $(X, \Theta)$ $\epsilon$-lc and $\Theta \geq B$. It follows that $(X, \Theta)$ is $\frac{1}{k}$-lc. \quad \Box
Theorem 3.8. Fix a positive integer $d$, a positive real number $\theta$ and a DCC set $\mathcal{S}$ of real numbers in $[0,1]$. The set $\mathcal{D}'$ of all klt weak Fano pairs $(X, B)$ satisfying

1. $\dim X = d$,
2. the coefficients of $B \in \mathcal{S}'$,
3. $\alpha(X, B) > \theta$, and
4. $(-K_X + B)^d > \theta$

forms a log bounded family. Moreover, there is a rational number $k$, finite sets $S_1 \subset [0,1]$ and $S_2 \subset \mathbb{Q} \cap [0,1]$, depending only on $d, \theta$ and $\mathcal{S}'$, such that for every element $(X, B) \in \mathcal{D}'$, there is a strong klt $(S_1, S_2, k)$-complement $K_X + \Theta$ of $K_X + B$.

Proof. As in the proof of Theorem 3.7, by Theorem 3.3 and by Lemma 3.4, we may assume $X$ is $\mathbb{Q}$-factorial and it is enough to show the existence of $k$. Replacing $\mathcal{S}'$ by its closure, we may assume it is closed.

Let $\Gamma_1, \Gamma_2, r, a_i, g, \Gamma_i, g_i, \mathcal{S}$ and $g_i'$ be as given in Theorem 2.10. Let $(X, B) \in \mathcal{D}'$ and write $B = \sum_j b_j B_j$, where $B_j$ are distinct Weil $\mathbb{Q}$-Cartier divisors. Then we have $B \leq \sum_i (a_i \sum_j g'_i(b_j)B_j)$ and $\sum_j g'_i(b_j)B_j \leq \frac{B + 3\sum_j g_i(b_j)B_j}{4}$ for every $1 \leq i \leq r$. Furthermore, $(X, \sum_j g_i(b_j)B_j)$ is lc and $-(K_X + \sum_j g_i(b_j)B_j)$ is pseudo-effective by Theorem 2.6. So $(X, \sum_j g_i(b_j)B_j)$ is klt for every $1 \leq i \leq r$.

For each fixed $i$, we consider the following argument. Since $X$ is of Fano type and $-(K_X + \sum_j g'_i(b_j)B_j)$ is pseudo-effective, we may run a $(K_X + \sum_j g'_i(b_j)B_j)$-MMP and reach a model $Y_i$ such that $-(K_{Y_i} + \sum_j g'_i(b_j)B_{Y_i,j})$ is nef, where $B_{Y_i,j}$ is the strict transform of $B_j$ on $Y_i$. Therefore, by Lemma 3.3 (i) $\text{vol}(Y_i, -(K_{Y_i} + \sum_j b_j B_{Y_i,j})) \geq \text{vol}(X, -(K_X + B)) > \theta$, and (ii) $\alpha(Y_i, \sum_j b_j B_{Y_i,j}) > \theta$.

By Lemma 3.3, we may replace $X$ by $Y$ and replace $B_j$ by $B_{Y_i,j}$ except that $(X, \sum_j g'_i(b_j)B_j)$ is klt weak Fano but $(X, B)$ might not be weak Fano any more.

We estimate that

$$\text{vol}(X, -(K_X + \sum_j g'_i(b_j)B_j))$$

$$\geq \text{vol}(-(K_X + \sum_j g_i(b_j)B_j) + \sum_j g_i(b_j)B_j - \sum_j g'_i(b_j)B_j)$$

$$\geq \text{vol}(-(K_X + \sum_j g_i(b_j)B_j) + \sum_j g_i(b_j)B_j - \frac{B + 3\sum_j g_i(b_j)B_j}{4})$$

$$\geq \text{vol}(\frac{1}{4}(K_X + \sum_j g_i(b_j)B_j) + \frac{g_i(b_j)B_j - \sum_j g_i(b_j)B_j}{4})$$

$$= (\frac{1}{4})^d \text{vol}(-(K_X + B)).$$
On the other hand,  
\[
\alpha(X, \sum_j g'_j(b_j)B_j) = \text{lct}((X, \sum_j g'_j(b_j)B_j), | - (K_X + \sum_j g'_j(b_j)B_j)|_\mathbb{R}) \\
\geq \frac{1}{4} \text{lct}((X, \frac{B + 3 \sum_j g(b_j)B_j}{4}), | - (K_X + B)|_\mathbb{R}) \\
= \frac{1}{4} \alpha(X, B),
\]
where that last inequality follows from the following argument. If \(0 < t \leq \frac{1}{4} \text{lct}((X, B), | - (K_X + B)|_\mathbb{R})\), then for any \(M \in | - (K_X + B)|_\mathbb{R}\), \((X, B + 4tM)\) is lc. It follows that \((X, \frac{B + 4tM + 3 \sum_j g(b_j)B_j}{4})\) is lc by the linearity of log discrepancies because \((X, \sum_j g(b_j)B_j)\) is lc. Therefore we have  
\[
t \leq \text{lct}((X, \frac{B + 3 \sum_j g(b_j)B_j}{4}), | - (K_X + B)|_\mathbb{R}).
\]

As a conclusion,  
\[\text{vol}(X, -(K_X + \sum_j g'_j(b_j)B_j)) \text{ and } \alpha(X, \sum_j g'_j(b_j)B_j)\]
are both bounded below away from \(0\) and \((X, \sum_j g'_j(b_j)B_j)\) is a klt weak Fano pair.

By Theorem 3.7 for each \(i\), there is a rational number \(k_i\) depending only on \(d\) and \(\mathcal{F}\) such that there is a strong klt \(k_i\)-complement \(K_X + \Theta_i\) of \(K_X + \sum_j g_i(b_j)B_j\). Replacing \(k_i\) by \(k := \prod_{h=1}^i k_h\) for each \(i\), we may assume that \(k_i = k\). Let \(S_1 = \{a_i\}_i\) and \(S_2 = \frac{1}{2} \mathbb{N} \cap [0,1]\). Let \(\Theta := \sum_{i=1}^r a_i \Theta_i\). Then \(K_X + \Theta\) is the complement we want since \(K_X + \Theta \geq K_X + \sum_j (a_i \sum_j g'_j(b_j)B_j) \geq K_X + B\).

\[\square\]

Theorem 3.9. Fix a positive integer \(d\), and a positive real number \(\theta\). Let \(\mathcal{P}\) be a log bounded set of all klt Fano pairs \((X, B)\) satisfying

1. \(\dim X = d\),
2. \(\alpha(X, B) > \theta\), and
3. \((- (K_X + B))^d > \theta\).

Then there is a natural number \(k\), depending only on \(d, \theta\) and \(\mathcal{P}\), such that for every \((X, B) \in \mathcal{P}\), there is a strong klt \(k\)-complement \(K_X + \Theta\) of \(K_X + B\).

Proof. By Lemma 3.3 we may assume \(X\) is \(\mathbb{Q}\)-factorial. Since \(\mathcal{P}\) is log bounded and \((- (K_X + B))^d > \theta\), there is a positive real number \(a\) such that for every \((X, B) \in \mathcal{P}\), \(- (K_X + B + a\text{Supp}B)\) is effective. Since \(\alpha(X, B) > \theta\), by replacing \(a\), we may assume that \((X, B + a\text{Supp}B)\) is klt. Therefore, there is a natural number \(k\) such that for every \((X, B) \in \mathcal{P}\), there is a divisor \(\overline{B} \geq B\) on \(X\) such that \(\overline{B} \leq B + \frac{1}{k} a\text{Supp}B\) and \(kB\) is an integral divisor. Therefore, \((X, 2\overline{B} - B)\) is klt and \(- K_X - 2\overline{B} + B\) is pseudo-effective. Replacing a further, we may assume that \((X, 2\overline{B} - B)\) has an \(\mathbb{R}\)-complement. We may run a \(-(K_X + \overline{B})\)-MMP and reach a model \(Y\) such that \(-(K_Y + \overline{B}_Y)\) is nef, where \(\overline{B}_Y\) and \(B_Y\) are the strict transforms of \(\overline{B}\) and \(B\) on \(Y\) respectively. Therefore, by Lemma 3.5 \((Y, \overline{B}_Y)\) is a klt weak Fano pair, and \((Y, 2\overline{B}_Y - B_Y)\) has an \(\mathbb{R}\)-complement. By Lemma 3.3 and the fact
that volume is invariant under small birational maps and is increased by pushing-forward through morphisms, we have

\( (i) \ \text{vol}(Y, -(K_Y + B_Y)) \geq \text{vol}(X, -(K_X + B)) > \theta, \) and \\
\( (ii) \ \alpha(Y, B_Y) > \theta. \)

By Lemma 3.4, we may replace \( X, B \) and \( B \) by \( Y, B_Y \) and \( B_Y \) respectively except that \((X, B)\) is klt weak Fano but \((X, B)\) might not be weak Fano any more. By the same computation as in Theorem 3.8, we can bound \( \alpha(X, B) \) and \((−(K_X + B))^d\) below away from 0. Now we apply Theorem 3.7 and we are done. □

**Corollary 3.10.** Fix a positive integer \( d \) and positive real numbers \( \theta \) and \( \delta \). Let \( \mathcal{P} \) be the set of all projective pairs \( (X, B) \) satisfying

1. \((X, B)\) is a klt weak Fano pair of dimension \( d \),
2. \( B \) is an effective \( \mathbb{Q} \)-divisor on \( X \),
3. \( \alpha(X, B)^d(−(K_X + B))^d > \theta \), and
4. coefficients of \( B > \delta \).

Then there is a natural number \( k \), depending only on \( d, \theta \) and \( \delta \), such that for every \((X, B) \in \mathcal{P}, \) there is a strong klt \( k \)-complement \( K_X + \Theta \) of \( K_X + B \).

**Proof.** This follows from Theorem 1.7 and Theorem 3.9. □

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