COMONADIC BASE CHANGE FOR ENRICHED CATEGORIES

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Abstract. For our concepts of change of base and comonadicity, we work in the general context of the tricategory Caten whose objects are bicategories \( \mathcal{V} \) and whose morphisms are categories enriched on two sides. For example, for any monoidal comonad \( G \) on a cocomplete closed monoidal category \( \mathcal{C} \), the forgetful functor \( U : \mathcal{C}^G \to \mathcal{C} \) is comonadic when regarded as a morphism in Caten between one-object bicategories. Other examples are provided including that obtained from any comonoidal \( \mathcal{C} \)-enriched category.

We show that the forgetful pseudofunctor \( \mathcal{U} : \mathcal{V}^G \to \mathcal{V} \) from the bicategory of Eilenberg-Moore coalgebras for a comonad \( \mathcal{G} \) on \( \mathcal{V} \) in Caten induces a change of base pseudofunctor \( \mathcal{W} : \mathcal{V}^\mathcal{G}-\text{Mod} \to \mathcal{V}-\text{Mod} \) which is comonadic in a bigger version of Caten. We should emphasise that the right adjoints to \( \mathcal{U} \) and \( \mathcal{W} \) generally do not have right adjoint lax functors, confirming our need to work with two-sided enrichments.

We define Hopfness for such a comonad \( \mathcal{G} \) and prove that having that property implies \( \mathcal{U} \) creates left (Kan) extensions in the bicategory \( \mathcal{V}^\mathcal{G} \). We provide conditions under which Hopfness carries over from \( \mathcal{G} \) to the comonad \( \mathcal{G} = \mathcal{U} \circ \mathcal{R} \) generated by the adjunction \( \mathcal{W} \to \mathcal{R} \). This has implications for characterizing the absolute colimit completion of \( \mathcal{V}^\mathcal{G} \)-categories. A motivating example was the monoidal category of differential graded abelian groups obtained as the category of coalgebras for a Hopf monoid in the category of abelian groups. Examples include some involving base bicategories \( \mathcal{V} = \text{Spn}(\mathcal{E}) \) of spans in an ordinary category \( \mathcal{E} \) with pullbacks.

2010 Mathematics Subject Classification: 16E45; 16D90; 18A40
Key words and phrases: Hopf comonad; extension creation; 2-sided enrichment; differential graded category; bicategory of spans; monoidal comonad.

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Date: December 10, 2021.

The first author gratefully acknowledges the support of an International Macquarie University Research Scholarship while the second gratefully acknowledges the support of Australian Research Council Discovery Grants DP160101519 and DP190102432.
Monoidal comonads \( G \) on monoidal categories \( \mathcal{C} \) arise in many parts of mathematics. In the case where the monoidal category \( \mathcal{C} \) is braided, such a \( G \) can be obtained from a bimonoid \( B \) in \( \mathcal{C} \): the endofunctor of the comonad is \( B \otimes - \), the comultiplication and unit come from the comonoid structure on \( B \), while the monoid structure on \( B \) makes the comonad \( B \otimes - \) monoidal.

For a monoidal comonad \( G \) on a monoidal category \( \mathcal{C} \), the category \( \mathcal{C}^G \) of Eilenberg-Moore \( G \)-coalgebras is monoidal and the underlying functor \( U : \mathcal{C}^G \to \mathcal{C} \) is strong monoidal; see Moerdijk [29] and McCrudden [28]. The present paper studies such \( U \) for changing base in enriched category theory: each category \( \mathcal{A} \) with homs enriched in the base \( \mathcal{C}^G \) yields a category \( U_{\mathcal{A}} \mathcal{A} \) with homs enriched in the base \( \mathcal{C} \).

The setting for the present paper is hom enriched category theory, as developed for example by Betti et al.[6], where the base for enrichment is a bicategory \( \mathcal{V} \). In order to treat both \( U \) and the base change operation in the same context, we deal with change of base bicategory of the kind developed in [22] using two-sided enrichment. In Section 1 we recall the tricategory \( \text{Caten} \) (bold face was originally used for this) whose objects are bicategories, whose morphisms \( \mathcal{A} : \mathcal{W} \to \mathcal{V} \) are two-sided enriched categories, and whose 2-cells \( T : \mathcal{A} \Rightarrow \mathcal{B} \) are enriched functors. The homs of the tricategory are actually 2-categories, composition is 2-functorial (see (3.15) of [22]), and associativities and identities are up to isomorphism. It is the morphisms of Caten, more general than lax functors, that we use to change base.

A simple base change is obtained by whiskering in Caten with such a morphism; this defines a 2-functor

\[
\text{Caten}(\mathcal{X}, \mathcal{A}) = \mathcal{A} \circ - : \text{Caten}(\mathcal{X}, \mathcal{W}) \to \text{Caten}(\mathcal{X}, \mathcal{V}). \tag{0.1}
\]

It is routine then that, if \( \mathcal{A} \) has a right adjoint in Caten, then this 2-functor has a 2-adjoint. We also have the extension of the 2-categories Caten\((\mathcal{W}, \mathcal{V})\) to the bicategories Moden\((\mathcal{W}, \mathcal{V})\) where the morphisms are modules instead of functors. Recall that Caten\((1, \mathcal{V})\) is the usual 2-category \( \mathcal{V} \)-Cat of \( \mathcal{V} \)-categories, \( \mathcal{V} \)-functors and \( \mathcal{V} \)-natural transformations, while Moden\((1, \mathcal{V})\) is \( \mathcal{V} \)-Mod where the morphisms are \( \mathcal{V} \)-modules (also called profunctors and distributors). The base change of more interest here is the normal lax functor

\[
\mathcal{A} \circ - : \text{Moden}(\mathcal{X}, \mathcal{W}) \to \text{Moden}(\mathcal{X}, \mathcal{V}) \tag{0.2}
\]

of the kind occurring in Proposition 7.5 of [22].

Our paper has four goals. The first is to show that the lax functor (0.2) has a right adjoint in a bigger version CATEN of Caten if \( \mathcal{A} \) has a right adjoint in Caten. This is a point that we consider to be a useful addendum to [22].

In Section 3, we study pseudocomonads \( \mathcal{G} \) in the tricategory Caten (recalled in Section 1) of two-sided enriched categories and identify their Eilenberg-Moore construction. For a braided monoidal base category \( \mathcal{C} \), comonoidal \( \mathcal{C} \)-enriched categories provide examples in the case where the base bicategory \( \mathcal{V} \) has only one object with endohom category \( \mathcal{C} \).

To obtain examples at the bicategory level, in Section 4 we take any pullback-preserving comonad \( G \) on a category \( \mathcal{E} \) and construct a comonad \( \mathcal{I} \) on the bicategory \( \text{Spn}(\mathcal{E}) \) of spans in \( \mathcal{E} \) whose Eilenberg-Moore construction in Caten is the bicategory \( \text{Spn}(\mathcal{E}^G) \) of spans in \( \mathcal{E}^G \). Recall from [37] that \( \text{Spn}(\mathcal{E}) \)-enriched categories relate to indexed (or parametrized) categories.

Section 5 introduces differential systems which lead, in a simple way, to comonads in Caten. The Eilenberg-Moore construction for the comonad can be identified in terms of the differential system. This process abstracts that for obtaining chain complexes from graded objects.
Our second goal is Theorem 6.1 which provides mild conditions in order for a comonadic morphism \( \mathcal{W} \) in Caten with comonad \( \mathcal{G} \) to induce a comonadic base change morphism \( \mathcal{W} \) in CATEN with comonad \( \mathcal{G} \). In the comonoidal \( C \)-enriched category example, the Eilenberg-Moore construction generally leads to a bicategory which has more than one object.

Section 7 makes explicit when a comonad in Caten is Hopf and interprets it in some examples including the comonoidal \( C \)-enriched category case obtained from a Hopf \( C \)-algebroid in the sense Definition 21 of [13]. This leads to our third goal which is to show that Hopf comonadic morphisms in Caten create (preserve and reflect) left Kan extensions. Kan extensions and liftings envelope various categorical notions including adjunctions (hence duals) and (weighted) (co)limits. Theorem 8.1 generalises the works of [8, 10] which show that the forgetful functor from the category of algebras for a Hopf opmonoidal monad is strong closed (and a dual). Note in passing that Weber [43] describes conditions under which the forgetful functor from the bicategory of pseudo-algebras (for a 2-monad on a bicategory) reflects left Kan extensions.

Our fourth goal is reached in Section 9: the \( \mathcal{G} \) of Theorem 6.1 is Hopf if \( \mathcal{G} \) is Hopf and locally cocontinuous. We then apply the theorem to monoidal comonads whose coalgebras are graded, or differential graded, abelian groups; earlier versions appeared as [31, 30].

1. Enrichment on two sides

We begin by quickly reviewing the tricategory Caten, originally introduced and justified in [22] (where the tricategory was denoted by Caten while Caten denoted the bicategory obtained on ignoring the 3-cells). The general definition of tricategory as in [15] is rather involved; however, Caten is mildly more complicated than a bicategory: 1-cell composition is associative up to coherent isomorphisms rather than equivalences.

To shorten notation a little, we sometimes denote the hom category \( \mathcal{V}(V, V') \) of a bicategory \( \mathcal{V} \) by \( \mathcal{V}^\mathcal{V} \). Horizontal composition in \( \mathcal{V} \) will be denoted by tensor product \( \otimes \).

Objects of Caten are (small) bicategories for which we use symbols such as \( \mathcal{V}, \mathcal{W} \), and so on. An arrows \( \mathcal{A} : \mathcal{W} \rightarrow \mathcal{V} \), called a 2-sided enriched category, consists of

(i) a set \( \text{Ob}\mathcal{A} \) of objects together with a span

\[
\begin{array}{c}
\text{Ob}\mathcal{W} \\
\downarrow \\
\text{Ob}\mathcal{V}
\end{array}
\xymatrix{
\text{Ob}\mathcal{A} \ar@<1ex>[rr]^-(0,1) & & \text{Ob}\mathcal{A} \\
\downarrow \\
\text{Ob}\mathcal{A}
}
\]

assigning to each object \( A \) an object \( A_- \) in \( \mathcal{W} \) and \( A_+ \) in \( \mathcal{V} \);

(ii) homes \( \mathcal{A}(A, A') \), also denoted \( \mathcal{A}_A^{A'} \), defined to be functors

\[
\mathcal{A}_A^{A'} : \mathcal{W}^{A_+} \rightarrow \mathcal{V}^{A_+},
\]

(iii) unit and composition natural transformations

\[
\begin{array}{c}
1 \\
\downarrow \\
1
\end{array}
\xymatrix{
\mathcal{W}^{A_+} \\
\otimes \rightarrow \mathcal{W}^{A_+} \\
\mathcal{V}^{A_+} \\
\otimes \rightarrow \mathcal{V}^{A_+}
\}
\]

satisfying unit and associativity laws.

Composition of 2-sided enriched categories is given by composition of the spans (which uses pullback [4]), composition of the functors defining homes, and pasting of the unit and multiplication natural transformations.
\textbf{Example 1.1.} When $\mathbb{W} = 1$, $\mathcal{A}$ is precisely a category enriched in the bicategory $\mathcal{V}$.  

\textbf{Example 1.2.} An enriched category $\mathcal{A} : \mathbb{W} \to \mathcal{V}$ with precisely one object might be called a \textit{monad} from $\mathbb{W}$ to $\mathcal{V}$. If furthermore the bicategories $\mathbb{W}$ and $\mathcal{V}$ also each have precisely one object then such an $\mathcal{A}$ amounts to a monoidal functor $T : \mathcal{D} \to \mathcal{C}$ between monoidal categories (in the sense of [14]) where $\mathcal{C}$ and $\mathcal{D}$ are the endohom monoidal categories of the single objects of $\mathcal{V}$ and $\mathbb{W}$. When $\mathcal{D} = 1$ then $T$ is precisely a monoid in $\mathcal{C}$.

\textbf{Example 1.3.} Let $\mathcal{K}$ be a bicategory with two full subbicategories $\mathcal{M}$ and $\mathcal{P}$. Suppose $\Lambda$ is a set of morphisms $a \in \mathcal{K}(M, P)$ with $M \in \mathcal{M}$ and $P \in \mathcal{P}$ such that each functor $\mathcal{K}(a, P') : \mathcal{K}(P, P') \to \mathcal{K}(M, P')$ with $P' \in \mathcal{P}$ has a right adjoint ran$(a, -)$. We can define an enriched category $\mathcal{A} : \mathcal{M} \to \mathcal{P}$ having ob$\mathcal{A} = \Lambda$ with $a_- = M \in \mathcal{M}$ and $a_+ = P \in \mathcal{P}$.

The functor $\mathcal{A}^a$ is the composite $\mathcal{K}(M, M') \xrightarrow{\mathcal{K}(M, a')} \mathcal{K}(M, P') \xrightarrow{\text{ran}(a, -)} \mathcal{K}(P, P')$. The unit $\eta_a$ and composition $\mu_{a', a''}$ are obtained by using the universal property of the right adjoints ran$(a, -)$ and ran$(a', -)$. Example 2.3 (f) of [22] about two-sided monoidal actions is a special case where $\mathcal{M}$ and $\mathcal{P}$ each have one object and $\mathcal{K}$ has two objects.

A 2-cell $F : \mathcal{A} \to \mathcal{B}$, called an \textit{(enriched) functor}, consists of:

(iv) a map of spans $F : \text{Ob}\mathcal{A} \to \text{Ob}\mathcal{B}$

which means a function between the object sets such that

$$(FA)_- = A_- \quad \text{and} \quad (FA)_+ = A_+ \quad (1.3)$$

(v) natural transformations $F^A_B : \mathcal{A}^B \Rightarrow \mathcal{B}^A$ which are compatible with the unit and multiplication of $\mathcal{A}$ and $\mathcal{B}$.

A 3-cell $\psi : F \to E$, called an \textit{(enriched) natural transformation} is a family with components 2-cells $\psi_A : 1_{A_+} \Rightarrow \mathcal{B}^E_F(1_{A_-})$ in $\mathcal{V}$ satisfying an enriched naturality condition (the coherence 2-cells we omit are set out in [22])

$$\begin{array}{ccc}
\mathcal{A}^B_A(w) & \xrightarrow{\psi_A \otimes (F_A^B)_w} & \mathcal{B}^E_F(1_{A_-}) \otimes \mathcal{B}^F_A(w) \\
(1_{E_A^B})_w \otimes \psi_A & \downarrow & \\
\mathcal{B}^E_A(w) \otimes \mathcal{B}^F_A(1_{A_-}) & \xrightarrow{\mu} & \mathcal{B}^F_A(w). 
\end{array} \quad (1.4)$$

All axioms, compositions, whiskerings, and the fact that Caten is a tricategory are explained in detail in [22]. This construction in a larger universe, which contains our ambient category of sets, is denoted CATEN. Theorems below hold when Caten is substituted with CATEN, see the discussion on page 56 of [22].

As mentioned, it is important to realise that Caten is actually a fairly special kind of tricategory. Without the 3-cells it is a bicategory, while the homs Caten($\mathbb{W}$, $\mathcal{V}$) are 2-categories.

\textbf{Example 1.4.} When Ob$\mathcal{A}$ = Ob$\mathbb{W}$ and ($-)_-$ is the identity function, $\mathcal{A}$ is precisely a lax functor from $\mathbb{W}$ to $\mathcal{V}$, and 2-cells are ‘icons’ as so named in [25].

\textbf{Remark 1.5.} Recall that Proposition 2.7 of [22] characterizes left adjoints in Caten as those enriched categories $\mathcal{A} : \mathbb{W} \to \mathcal{V}$ which are pseudofunctors and are local left adjoints (that is, each functor $\mathcal{A}^B_A : \mathcal{W}_{A_-} \to \mathcal{V}_{A_+}$ has a right adjoint in Cat).
Example 1.6. Let $K$ be an object of a bicategory $\mathcal{K}$ and let $\mathcal{M}$ be a full subbicategory of $\mathcal{K}$. Suppose $\mathcal{M}$ is a sub-pseudofunctor of the restricted representable pseudofunctor $\mathcal{M} \to \mathcal{K} \to \mathcal{K}(K,-) \to \text{Cat}$ such that the right extension $\text{ran}(x,x') : A \to B$ of $x'$ along $x$ exists (see Example 1.3 for the notation) for all $A,B \in \mathcal{M}$, $x \in \mathcal{K}(A)$ and $x' \in \mathcal{K}(B)$. If each functor $\mathcal{K}(A)^{op} \times \mathcal{K}(A) \to \mathcal{M}(A,B)$ admits an end then $\mathcal{M} : \mathcal{M} \to \text{Cat}$ admits a right adjoint in Caten. Indeed, the right adjoint to the functor $\mathcal{K}(A) \to \mathcal{M}(A,B)$ takes the functor $h : \mathcal{K}(A) \to \mathcal{K}(B)$ to the end $\int_x \mathcal{K}(A)(x,h(x))$. In particular, $\mathcal{K}$ could be any bicategory whose homs are complete lattices and which admits all right extensions, and where $K$ is any object, $\mathcal{M} = \mathcal{K}$ and $\mathcal{M} = \mathcal{K}(K,-)$. Even more particularly, $\mathcal{K}$ could be the bicategory $\text{Rel}(\mathcal{E})$ of relations in any Grothendieck topos $\mathcal{E}$.

Instead of (enriched) functors we could have chosen enriched modules $M : \mathcal{A} \to \mathcal{B}$ as 2-cells. Such a module consists of

(vi) functors

$$M_A^A : \mathcal{A}_A \to \mathcal{B}_A$$

(vii) action natural transformations

$$M_A^A \times M_B^B \downarrow \lambda_A^B \Rightarrow M_A^B, \quad M_A^A \times B_B^B \downarrow \rho_B^B \Rightarrow M_B^A \quad (1.5)$$

compatible with each other, and with the units and compositions in $\mathcal{A}$ and $\mathcal{B}$.

A module morphism $\sigma : M = \Rightarrow N$ consists of natural transformations

$$\sigma_B^A : N_B^A \Rightarrow M_B^A$$

compatible with the actions (1.5).

Module morphisms compose and we obtain a category $\text{Moden}(\mathcal{A},\mathcal{B})$ of modules between $\mathcal{A}$ and $\mathcal{B}$.

When $\mathcal{B}$ is locally cocomplete (including having whiskering preserve the local colimits), $\text{Moden}(\mathcal{A},\mathcal{B})$ becomes a bicategory. Composition of modules is defined by the reflexive coequalizer

$$\sum_{B,B'} M_B^A \otimes B_B^B \otimes N_{B'}^B \xrightarrow{\sum_{B} \otimes 1} \sum_{B,B'} M_B^A \otimes N_{B'}^B \xrightarrow{\text{coeq}} (N \circ_{\mathcal{B}} M)_B^A . \quad (1.6)$$

In particular, $\text{Moden}(1,\mathcal{B})$ is the usual bicategory $\mathcal{B}$-$\text{Mod}$ of $\mathcal{B}$-categories and $\mathcal{B}$-modules between them. In general, $\text{Moden}(\mathcal{A},\mathcal{B})$ is equivalent to the bicategory $\text{Conv}(\mathcal{A},\mathcal{B})$, where $\text{Conv}$ denotes the internal hom in Caten for the cartesian product of bicategories; see [22]. The bicategory $\text{Conv}(\mathcal{A},\mathcal{B})$ is constructed by Day local convolution [12] as follows: $\text{obConv}(\mathcal{A},\mathcal{B}) = \text{ob}\mathcal{A} \times \text{ob}\mathcal{B}$ and $\text{Conv}(\mathcal{A},\mathcal{B})(((X,V),(X',V')) = [\mathcal{A}(X,X'),\mathcal{B}(V,V')]$ with composition functors

$$[\mathcal{A}(X',X''),\mathcal{B}(V',V'')] \times [\mathcal{A}(X,X'),\mathcal{B}(V,V')] \xrightarrow{\otimes} [\mathcal{A}(X,X''),\mathcal{B}(V,V'')]$$

defined by the coend formula

$$P \otimes Q = \int_{h,k} [\mathcal{A}(X,X'')(h \otimes k, -)] \bullet (P(h) \otimes Q(k)) . \quad (1.7)$$
Remark 1.7. If in (1.6) the enriched category $\mathcal{B}$ has only one object then we only need the existence of local reflexive coequalizers in $\mathcal{V}_{C,+}^A$.

Remark 1.8. As noted in Example 1.2, monoidal functors can be regarded as two-sided enriched categories. If $T, S : \mathcal{D} \to \mathcal{C}$ are monoidal functors between monoidal categories, a module $M : T \to S$ amounts to a functor $M : \mathcal{D} \to \mathcal{C}$ equipped with natural families $\lambda_{D,D'} : TD \otimes MD' \to M(D \otimes D')$ and $\rho_{D,D''} : MD' \otimes SD'' \to M(D' \otimes D'')$ providing left and right compatible actions. In this situation, Remark 1.7 applies to composition $T \xrightarrow{M} S \xrightarrow{N} R$: we only require the existence of reflexive coequalizers in $\mathcal{C}$, which are preserved by tensoring by an object on either side, rather than requiring all colimits. Let us denote this bicategory of monoidal functors and modules by $\text{Mod}_1(\mathcal{D}, \mathcal{C})$. When $\mathcal{D} = 1$, we obtain the usual bicategory $\text{Mod}(\mathcal{C})$ whose objects are monoids in $\mathcal{C}$ (called "rings" or "algebras" in the additive case) and bimodules as morphisms.

In [22] Example 7.4(b), a module $\mathcal{B}(T, S)$ was constructed from a cospan $\mathcal{A} \xrightarrow{S} \mathcal{X} \xleftarrow{T} \mathcal{B}$ of functors. We will discuss the case $F_* = \mathcal{B}(1, \mathcal{F})$ in a bit more detail. Each functor $F : \mathcal{A} \to \mathcal{B}$ defines a module $F_* : \mathcal{A} \to \mathcal{B}$ by taking

\[
(F_*)_B^A = \mathcal{B}_B^{FA} : \mathcal{B}_{B_-}^A \to \mathcal{V}_{C,+}^A
\]

\[
\lambda_{A, A'}^B = \otimes (\mathcal{A}_A^{A'} \times \mathcal{B}_B^{FA}) \xrightarrow{1(F_*^A \times 1_1)} \otimes (\mathcal{B}_F^{FA} \times \mathcal{B}^{FA}_B) \xrightarrow{\mu_{FA, F'F''_B}} \mathcal{B}_B^{FA'} \otimes (1.8)
\]

\[
\rho_{B, B'}^A = \otimes (\mathcal{B}_B^{FA} \times \mathcal{B}^{FA'}_B) \xrightarrow{\mu_B^{FA'}} \mathcal{B}_B^{FA \otimes FA'} (1.9)
\]

which is properly typed because of (1.3). The unit and associativity axioms for the $\mu$ and $\eta$ of $\mathcal{B}$ give their compatibility with $\rho$. Compatibility of $\lambda$ with the $\mu$ and $\eta$ of $\mathcal{A}$ follows by applying their compatibility with the functor $F$ followed by their unit and associativity laws.

Similarly, each natural transformation $\psi : F \to E$ has an induced module morphism

\[
\psi_* : F_* \to E_*
\]

\[
(\psi_*)_B^A := \mathcal{B}_B^{FA} \Rightarrow \mathcal{B}_B^{EA}
\]

\[
((\psi_*)_B^A)_w := \mathcal{B}_B^{FA}(w) \xrightarrow{\psi_A \otimes 1} \mathcal{B}_F^{EA}(1_{A_-}) \otimes \mathcal{B}_B^{FA}(w) \xrightarrow{\mu_{FA, FA}_B} \mathcal{B}_B^{EA}(w).
\]

To see that $\psi_*$ is compatible with $\lambda$, tensor diagram (1.4) by $\mathcal{B}^{FA}(w')$, whisker the resulting square on the right with the $\mu$ of $\mathcal{B}$, and add obvious commutative squares. Compatibility with $\rho$ follows from associativity of the $\mu$ for $\mathcal{B}$. Also, every module morphism between modules induced by functors gives rise to a natural transformation: given

\[
\sigma_B^A : \mathcal{B}_B^{FA} \Rightarrow \mathcal{B}_B^{EA}
\]

we can define a natural transformation $\sigma$ with components

\[
\sigma_A : 1_{A_-} \xrightarrow{\eta_{FA}} \mathcal{B}_F^{EA}(1_{A_-}) \xrightarrow{\mu_{FA, FA}^{-1}} \mathcal{B}_F^{FA}(1_{A_-}) \quad (1.11)
\]
and the natural transformation axiom (1.4) is proved by commutativity of

\[
\begin{array}{c}
\alpha_A^I(w) \\
\downarrow \psi_A \\
\mathcal{B}_F^A(1_{A_+}) \\
\downarrow \psi_A \\
\mathcal{B}_F^A(1_{A_-}) \\
\downarrow 1 \\
\mathcal{B}_F^A(1_{A_-}) \\
\downarrow \mu \\
\mathcal{B}_F^A(1_{A_-}) \\
\downarrow \sigma_B^\nu \\
\mathcal{B}_E^A(1_{A_-}) \\
\downarrow \sigma_B^\nu \\
\mathcal{B}_E^A(1_{A_-}) \\
\downarrow 1 \\
\mathcal{B}_E^A(1_{A_-}) \\
\downarrow \eta_B^\nu \\
\mathcal{B}_E^A(1_{A_-}) \\
\downarrow 1 \\
\mathcal{B}_E^A(1_{A_-}) \\
\downarrow 1 \\
\mathcal{B}_E^A(1_{A_-}) \\
\downarrow 1 \\
\mathcal{B}_E^A(1_{A_-}) \\
\end{array}
\]

where the hexagon and the bottom right square are compatibility conditions between module morphism \(\sigma\) and actions (1.8) and (1.9) respectively.

**Proposition 1.9.** The functor

\[(-)_* : \text{Caten}(\mathcal{W}, \mathcal{V})(\mathcal{A}, \mathcal{B}) \to \text{Moden}(\mathcal{W}, \mathcal{V})(\mathcal{A}, \mathcal{B})\]

is full and faithful.

**Proof.** The process (1.10) of turning a natural transformation into a module morphism has inverse defined by (1.11), as shown by the commuting diagrams (1.12).

We end this section by recalling the **duality** for two-sided enriched categories defined in Section 2.9 of [22]. There is an involutory trihomomorphism (in the sense of [15])

\[(-)^\circ : \text{Caten} \to \text{Caten}\]

which is a pseudofunctor between bicategories when 3-cells are ignored. The involution is defined on objects \(\mathcal{V}\) by \(\mathcal{V} \to \mathcal{V}^{\text{op}}\).

**2. Base change under a left adjoint in Caten**

As mentioned in the Introduction, a base change 2-functor of the form (0.1) will have a right adjoint 2-functor if the base change morphism has a right adjoint in Caten. The simple reason no longer applies when we ask for a Moden version. Indeed, we do prove that there is a right adjoint to the induced pseudofunctor although it is an adjoint enriched category, not an adjoint pseudofunctor.

Recall the base change result in [22] which appears there as Proposition 7.5 and states that right whiskering modules by a two-sided enriched category provides a normal lax functor.
Theorem 2.1. Let \( \mathcal{U} : \mathcal{X} \to \mathcal{Y} \) be a morphism in Caten between locally cocomplete bicategories and suppose \( \mathcal{U} \dashv \mathcal{R} \) is an adjunction in Caten with unit \( g : 1_Y \to \mathcal{R}\mathcal{U} \). Then the lax functor
\[
\mathcal{N} := \text{Moden}(\mathcal{X}, \mathcal{W}) \xrightarrow{\mathcal{W}:=\text{Moden}(\mathcal{X}, \mathcal{W})} \mathcal{M} := \text{Moden}(\mathcal{X}, \mathcal{Y}) ,
\]
given by right whiskering with \( \mathcal{U} \), a category and suppose \( \mathcal{U} \) is the terminal bicategory; the general case then follows from \( \text{Moden}(\mathcal{W}, \mathcal{Y}) \cong \text{Conv}(\mathcal{W}, \mathcal{Y})\)-Mod and Proposition 3.6.

Proof. We will consider the case when \( \mathcal{X} \) is the terminal bicategory; the general case then follows directly from the compatibility of \( \mathcal{W} \).

First we show that \( \mathcal{W} \) has local right adjoints \( \mathcal{R}\mathcal{A} : \mathcal{M}(\mathcal{U} \circ \mathcal{A}, \mathcal{U} \circ \mathcal{B}) \to \mathcal{N}(\mathcal{A}, \mathcal{B}) \) given by
\[
(\mathcal{U} \circ \mathcal{A} \xrightarrow{M} \mathcal{U} \circ \mathcal{B}, \alpha) \mapsto (\mathcal{A} \mathcal{R} M \mathcal{B}, \mathcal{R} \alpha)
\]
with the assignments defined by
\[
(\mathcal{R} M)^A_B := \mathcal{R} M^A_B \\
(\mathcal{R} A)^{B', A'}_{B, A} := (\mathcal{A} A'^{-1} \otimes \mathcal{R} M^A_B \otimes \mathcal{R} B' \otimes \mathcal{U} A' \otimes \mathcal{R} M^A_B \otimes \mathcal{U} B' \otimes \mathcal{U} A')
\]
\[
\mathcal{R} \sigma := \mathcal{R} \sigma^B_B = \sigma^B_B
\]
where \( \alpha \) denotes a 2-sided action (the analogous 1-sided actions are denoted by \( \lambda \) and \( \rho \)). Actions \( \mathcal{R} \alpha \) (or separately \( \mathcal{R} \lambda \) and \( \mathcal{R} \rho \)) are compatible with unit and composition in \( \mathcal{A} \) and \( \mathcal{B} \). For example, compatibility of \( \rho \) with composition is shown by commutativity of the diagram (2.13).

In the above diagram, the top pentagon commutes since \( g \) is a functor; there are naturality squares for \( \mu ; \) and the bottom right square is obtained by applying \( \mathcal{R} \) to compatibility of \( \rho \) with \( \mu(\mathcal{U} \circ \mathcal{B}) \). Similarly, functoriality of \( g \) with respect to identities leads to compatibility of \( \mathcal{R} \rho \) with \( \eta(\mathcal{B}) \). Compatibility of \( \mathcal{R} \sigma \) with \( \mathcal{R} \rho \) (and \( \mathcal{R} \lambda \)) follows directly from the compatibility of \( \sigma \) with \( \rho \) (and \( \lambda \)).

\( ^1 \)Note that \( \mathcal{N} \) and \( \mathcal{M} \) may have a large set (proper class) of objects.
The components of the unit and counit of the local adjunctions are given by components of $\eta$ and $\varepsilon$:

$$\tilde{\eta}_A : 1 \leadsto \mathcal{F} \circ (-), \quad (\tilde{\eta}_A)^A : N_A \leadsto \mathcal{F} N_A$$

$$\tilde{\varepsilon}_A : \mathcal{F} \circ \mathcal{F} (-) \leadsto 1, \quad (\tilde{\varepsilon}_A)^A : \mathcal{F} N_A \leadsto M_A.$$ 

They form module morphisms, as proved by diagrams:

$$\mathcal{F} \mathcal{M}_A \otimes \mathcal{F} \mathcal{M}_B,$$

and they satisfy the adjunction axioms because $g$ and $\varepsilon$ do. Since $\mathcal{F}$ has local right adjoints, it preserves local colimits, which, together with pseudofunctoriality of $\mathcal{F}$, gives sufficient conditions for pseudofunctoriality of $\mathcal{F}$:

$$\sum \mathcal{F} M_A \otimes \mathcal{F} B \otimes \mathcal{F} C \xrightarrow{\mathcal{F}(\rho \otimes 1)} \sum \mathcal{F} M_A \otimes \mathcal{F} C \xrightarrow{\text{coeq}} (\mathcal{F} N \circ \mathcal{F} \mathcal{M}^A),$$

Since $\mathcal{F}$ is a pseudofunctor and has local right adjoints, by Proposition 2.7 of [22] (see Remark 1.5), $\mathcal{F}$ extends to a 2-sided enriched category which is a right adjoint to $\mathcal{F}$. □
Explicitly, the unit for \( \mathcal{A} \) is a module morphism defined using the enriched functor
\[
\eta_{\mathcal{A}} := (g \circ \mathcal{A})^*,
\]
and the multiplication components
\[
\mathcal{A}(N) \circ_{\mathcal{A}} \mathcal{A}(M) \xrightarrow{\mu(\mathcal{A})} \mathcal{A}(N \circ_{\mathcal{A}} M)
\]
are given by the right column of
\[
\sum \mathcal{B} M^A_B \otimes \mathcal{B} B^B_B \otimes \mathcal{B} B^B_C \xrightarrow{\mathcal{B} \rho \otimes 1} \sum \mathcal{B} M^A_B \otimes \mathcal{B} N^B_C \xrightarrow{\coeq} (\mathcal{B}(N) \circ_{\mathcal{A}} \mathcal{B}(M))^A_C,
\]
where the top line is the coequalizer defining composition of those modules in \( \mathcal{A} \), the bottom line is the result of applying \( \mathcal{U} \) to the defining coequalizer for composition of those modules in \( \mathcal{B} \), morphisms \( i_B \) and \( i_{B^B} \) are the coproduct inclusions, and \( (-)_B \) denotes the induced map for mapping out of a coproduct.

**Remark 2.2.** Following Remark 1.8, we point out that Theorem 2.1 restricts to the one-object case. In particular, if \( \mathcal{C} \), \( \mathcal{D} \) and \( \mathcal{E} \) are monoidal categories and both \( \mathcal{C} \) and \( \mathcal{D} \) admit reflexive coequalizers preserved by tensoring with an object then each adjunction \( \mathcal{U} \dashv \mathcal{B} \) with \( \mathcal{U} : \mathcal{D} \rightarrow \mathcal{C} \) a strong monoidal functor induces an adjunction \( \mathcal{U} \dashv \mathcal{B} \) in Caten where \( \mathcal{U} \) is the pseudofunctor
\[
\text{Mod1}(\mathcal{E}, \mathcal{U}) : \text{Mod1}(\mathcal{E}, \mathcal{D}) \rightarrow \text{Mod1}(\mathcal{E}, \mathcal{C}).
\]
In particular, when \( \mathcal{E} = 1 \), we obtain a right adjoint in Caten for \( \mathcal{U} : \text{Mod}(\mathcal{D}) \rightarrow \text{Mod}(\mathcal{C}) \).

3. **Pseudocomonads in Caten and their Coalgebras**

Pseudocomonads in a tricategory are dual to pseudomonads; see [13, 24]. However, we will be more explicit in the case of interest here.

Let \( \mathcal{G} : \mathcal{V} \rightarrow \mathcal{V} \) be a pseudocomonad in Caten; that is, a 2-sided enriched category with enriched functors
\[
\mathcal{G} \xleftarrow{1_\mathcal{V}} \mathcal{G} \xrightarrow{d} \mathcal{G}^2
\]
and three enriched natural isomorphisms
\[
\mathcal{G} \xrightarrow{d} \mathcal{G}^2 \xleftarrow{\alpha} \mathcal{G}^d \quad \mathcal{G}^2 \xrightarrow{\lambda} \mathcal{G} \quad \mathcal{G}^2 \xrightarrow{\rho} \mathcal{G}
\]
satisfying two axioms. The existence of the span morphism \( \text{Ob}\mathcal{G} \xrightarrow{\text{Ob}} \text{Ob}\mathcal{V} \) forces
\[
G_+ = G_- = (\text{Ob})_+(G) =: G_0,
\]
for all objects \( G \). To avoid extra brackets, we are taking the objects of \( \mathcal{G}^3 \) to be triples \( (G, H, K) \) of objects of \( \mathcal{G} \) such that \( G_0 = H_0 = K_0 \). The two counit constraints \( \lambda \) and \( \rho \) give isomorphisms
\[
(\text{Obd})(G) \xrightarrow{(\lambda G_0 \cdot \mu_0)^{-1}} (G, G)
\]
which allow \( d \) to be replaced, up to isomorphism, by a comultiplication given by the diagonal function on objects, and whereby \( \alpha, \lambda \) and \( \rho \) become identities. Then \( \mathcal{G} \) is a comonad on \( \mathcal{V} \) in the bicategory obtained from Caten by ignoring 3-cells. We will henceforth assume we have made this replacement and simply refer to \( \mathcal{G} \) as a “comonad” in Caten. Then the remaining data for \( \mathcal{G} \) are given by endofunctors

\[
\mathcal{G}_G' : \mathcal{V}_{G_0}^G \to \mathcal{V}_{G_0}^G
\]

and natural transformations with components

\[
\mu_{\gamma, v} = (\mu_{\mathcal{G}_G'}^v)_\mathcal{V}_G : \mathcal{V}_G^G(v') \otimes \mathcal{V}_G^G(v) \to \mathcal{V}_G^G(v' \otimes v)
\]

\[
\eta = \eta_G : 1_{G_0} \to \mathcal{V}_G^G(1_{G_0})
\]

\[
d_v = (d_{\mathcal{G}_G'}^v)_\mathcal{V}_G : \mathcal{V}_G^G(v) \to (\mathcal{V}_G^G)^2(v)
\]

\[
e_v = (e_{\mathcal{G}_G'}^v)_\mathcal{V}_G : \mathcal{V}_G^G(v) \to v
\]

satisfying enriched functor compatibility axioms, which together with the comonad axioms, are the monoidal comonad axioms dual to the opmonoidal monad axioms appearing in [8]. In particular, each \((\mathcal{G}_G', e_{\mathcal{G}_G'}, d_{\mathcal{G}_G'})\) is a comonad on \( \mathcal{V}_{G_0}^G \) in the usual sense (in \( \text{Cat} \)).

We will proceed to define a bicategory \( \mathcal{V}^G \) with the same objects as \( \mathcal{G} \) and with homs the categories of Eilenberg-Moore-coalgebras

\[
\mathcal{V}^G(G, G') := \mathcal{V}(G_0, G_0')^{\mathcal{G}(G, G')}
\]

The identity coalgebra is \((1_{G_0}, \eta_G)\) and composition is given on coalgebras by

\[
\mathcal{V}^G(G', G'') \times \mathcal{V}^G(G, G') \to \mathcal{V}^G(G, G'')
\]

\[
(v', \gamma_v), (v, \gamma_v) \mapsto (v' \otimes v, (\mu_{\mathcal{G}_G'}^v)_\mathcal{V}_G \circ (v' \otimes v))(v' \otimes v).
\]

The assigned pair is indeed a coalgebra: compatibility with \( d \) is proved by

\[
\begin{array}{ccc}
\mathcal{G}^G(v' \otimes v) & \xrightarrow{\gamma \otimes \gamma} & \mathcal{G}^G(v' \otimes v) \\
\gamma \otimes \gamma & \downarrow & \mu \\
\mathcal{G}^G(v' \otimes v) & \xrightarrow{\mu} & \mathcal{G}^G(\mathcal{G}^G(v' \otimes v)) \\
\mu & \downarrow & \mu
\end{array}
\]

where the upper left square is a componentwise compatibility of local coalgebras \( \gamma \) with comultiplication, the bottom left square is naturality of \( \mu \), the triangle is the definition of composition for the composite category, and the remaining square is compatibility of the enriched functor \( d \) with compositions in its source and target (a typical bimonoid axiom). Similarly, compatibility of \((3.14)\) with \( e \) follows from its being an enriched functor. The assignment extends to coalgebra morphisms since \( \mu \) is natural. The unit and associativity isomorphisms are inherited from \( \mathcal{V} \); they are coalgebra morphisms and satisfy the monoidal axioms because they do in \( \mathcal{V} \).

There is an underlying (strict) functor \( \mathcal{U} : \mathcal{V}^G \to \mathcal{V} \) which sends \( G \) to the underlying object \( G_0 \) in \( \mathcal{V} \) and disregards the coalgebra structure on 1-cells. By construction, each \( \mathcal{U}_G \) has a right adjoint \( \mathcal{U}_G^{\mathcal{V}'} \), and by Theorem 2.7 of [22] the right adjoints are part of a

\[\text{When indices are omitted they can be deduced from the context. For example, } \mathcal{G}_G' (v) \text{ is the full notation for } \mathcal{G} v.\]
2-sided enriched category \( \mathcal{R} : \mathcal{V} \to \mathcal{V}^{\mathcal{G}} \) with the same objects as \( \mathcal{G} \), with span legs given by \( G_\pm = G_0 \) and \( G_\pm = G \), with multiplication having components
\[
\mathcal{R}_G^G(v') \otimes \mathcal{R}_G^G(v) \xrightarrow{(\mathcal{G}_G^G(v') \otimes \mathcal{G}_G^G(v))} \mathcal{R}_G^G(v' \otimes v)
\]
and unit having components
\[
1_G \xrightarrow{\eta_G} \mathcal{R}_G^G(1_{G_0}) := (1_{G_0}, \eta_G) \xrightarrow{\eta_G} (\mathcal{G}1_{G_0}, d_{1_{G_0}}).
\]
Now we have an adjunction in \( \text{Caten} \).

The counit and the unit of the adjunction are given by the enriched functors
\[
\mathcal{U} \circ \mathcal{R} = \mathcal{G} \xrightarrow{\eta} 1_\mathcal{V} \quad \text{and} \quad 1_{\mathcal{V}^{\mathcal{G}}} \xrightarrow{\eta} \mathcal{R} \circ \mathcal{U}
\]
where \((\text{obg})(G) = (G, G)\) and \(g_{(v, \gamma v)} = \gamma_v : (v, \gamma_v) \to (\mathcal{G}_G^G v, d_v)\).

**Example 3.1.** Let \( \mathcal{C} \) be a braided monoidal category. Then the 2-category \( \mathcal{C} \text{-Cat} \) of \( \mathcal{C} \)-enriched categories, enriched functors and natural transformations is monoidal; the symmetric case was already in \([14]\) and the braided in \([20]\). A comonoidal \( \mathcal{C} \)-category \( \mathcal{A} \) is a monoidale (= pseudomonoid) in \( \mathcal{C} \text{-Cat}^{\text{op}} \). As pointed out in Section 8 of \([13]\), the cotensor \( \mathcal{C} \)-functor \( D : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \) can be taken on objects to be the diagonal function \( DA = (A, A) \); then, on homs, \( D \) and the counit \( E \) provide morphisms
\[
D_A^A : \mathcal{A}(A, A') \to \mathcal{A}(A, A') \otimes \mathcal{A}(A, A') \quad \text{and} \quad E_A^A : \mathcal{A}(A, A') \to I
\]
in \( \mathcal{A} \). These are equipped with coassociativity and counital constraints satisfying the two axioms for a comonoidal. Consequently, we obtain, as follows, a pseudo-comonad \( \mathcal{G} \) on the bicategory \( \mathcal{V} = \Sigma \mathcal{C} \) with one object \( o \) and hom category \( \mathcal{V}(o, o) = \mathcal{C} \). The objects of \( \mathcal{G} \) are those of \( \mathcal{A} \). The endofunctor \( \mathcal{G}^A_A : \mathcal{V}^o \to \mathcal{V}^o \) is \( \mathcal{A}(A, A') \otimes - : \mathcal{C} \to \mathcal{C} \). Then, using the braiding on \( \mathcal{C} \) and composition in \( \mathcal{A} \), we have the composite
\[
\mathcal{A}(A', A'') \otimes X' \otimes \mathcal{A}(A', A') \otimes X \cong \mathcal{A}(A', A'') \otimes \mathcal{A}(A, A') \otimes X' \otimes X \to \mathcal{A}(A, A'') \otimes X' \otimes X
\]
defining \( \mu_X^{A''} \), while \( \eta : I \to \mathcal{A}(A, A) \cong \mathcal{A}(A, A) \otimes I \) uses the identity structure of \( \mathcal{A} \) and the right unit constraint of \( \mathcal{G} \). The morphisms \( d_X : \mathcal{A}(A, A') \otimes X \to \mathcal{A}(A, A') \otimes \mathcal{A}(A, A') \otimes X \) and \( \varepsilon_X : \mathcal{A}(A, A') \otimes X \to X \) are defined by tensoring with \( D_A^A \) and \( E_A^A \).

The bicategory \( \mathcal{V}^{\mathcal{G}} \) has the same objects as \( \mathcal{A} \); so it is not \( \Sigma \) of a monoidal category unless \( \mathcal{A} \) is a bimonoid in \( \mathcal{C} \). The hom category \( \mathcal{V}^{\mathcal{G}}(A, A') = \mathcal{C}^{\mathcal{A}(A, A') \otimes -} \) is the category of coalgebras for the comonad \( \mathcal{A}(A, A') \otimes - \) on \( \mathcal{C} \). For composition in the bicategory, we refer the reader back to \((3.14)\).

Now we present a version of Beck’s comonadicity theorem (for example, see \([3, 2]\) where “triple” means “monad”) for use in the rest of the chapter.

**Proposition 3.2.** Any 2-sided enriched category \( \mathcal{L} : \mathcal{W} \to \mathcal{V} \) such that
(a) \( \mathcal{L} \) has a right adjoint \( \mathcal{R} \) in \( \text{Caten} \),
(b) \( \mathcal{L} \) is locally conservative, and
(c) \( \mathcal{W} \) has, and \( \mathcal{L} \) preserves, local \( \mathcal{L} \)-split equalizers,
determines an identity-on-objects biequivalence $\mathcal{W} \sim \mathcal{V}_G$ over $\mathcal{V}$, where $G = L \circ R$ is the generated comonad.

Proof. By Proposition 2.7 of [22], $L$ has a right adjoint if and only if it is a pseudofunctor and each functor $L(L, L')$ has a right adjoint which we denote by $R(L, L')$. Then, the right adjoint $R$ has the same objects as $L$ (and $\mathcal{V}$, since $L$ is a pseudofunctor), and homs are precisely the $R(L, L')$. From the usual Beck comonadicity theorem it follows that $\mathcal{W}$ and $\mathcal{V}_G$ have compatibly equivalent homs; since they have the same objects, we obtain the desired biequivalence. \hfill \Box

Return now to our comonad $G$ on $\mathcal{V}$ in Caten. The category Caten($\mathcal{X}$, $\mathcal{V}$) has an induced comonad Caten($\mathcal{X}$, $G$) on it. In particular, when $\mathcal{X} = \mathcal{V}_G$ there is a natural coalgebra structure on $\mathcal{W}$ given by the enriched functor

$$\mathcal{W} \xrightarrow{\psi} G \circ \mathcal{W},$$

whose effect on homs has components the coactions $\gamma_v : v \rightarrow G v$.

Lemma 3.3. Let $\mathcal{X}$ be a bicategory. The functor

$$\text{Moden}(\mathcal{X}, \mathcal{V}_G)(\mathcal{A}, \mathcal{B}) \xrightarrow{\psi} \text{Moden}(\mathcal{X}, \mathcal{V})(\mathcal{W} \circ \mathcal{A}, \mathcal{W} \circ \mathcal{B})$$

is conservative. Moreover the source has, and the functor preserves, $(\mathcal{W} \circ -)$-split equalizers.

Proof. Take modules $M, N : \mathcal{A} \rightarrow \mathcal{B}$. A module morphism $\sigma : M \Rightarrow N$ has components

$$(\sigma_A^B)_x : M^A_B(x) \rightarrow N^A_B(x)$$

which are 2-cells in $\mathcal{V}_G$, natural in $x \in \mathcal{X}_{AB}$. To prove $\mathcal{W} \circ -$ is conservative, denote by $\psi : \mathcal{W} \circ M \Rightarrow \mathcal{W} \circ N$ the inverse of $\mathcal{W} \circ \sigma$. This precisely means that the component

$$(\psi_A^B)_x : \mathcal{W} N^A_B(x) \rightarrow \mathcal{W} M^A_B(x)$$

is an inverse of the component 2-cell $(\sigma_A^B)_x$ in $\mathcal{V}$. Since $\mathcal{W}$ is locally conservative, $(\psi_A^B)_x$ is also a coalgebra morphism. Hence, naturality squares for $\psi_B^A$ consist of the same arrows regardless of whether it is seen as a morphism from $\mathcal{W} N^A_B$ to $\mathcal{W} M^A_B$, or from $N^A_B$ to $M^A_B$. Compatibility of $\psi$ with actions for $M$ and $N$ follows from the same compatibility conditions for $\sigma$ and the fact that they are inverse to each other.

Consider a pair $\sigma, \chi : M \Rightarrow N$ with a split equalizer

$$E \xrightarrow{\xi} \mathcal{W} \circ M \xrightarrow{\psi} \mathcal{W} \circ N$$

meaning that we have the following componentwise formulas:

$$(\sigma_A^B)_x (\xi_A^B)_x = 1_{E^A_B(x)} , \quad (\psi_A^B)_x (\sigma_A^B)_x = 1_{M^A_B(x)} , \quad (\psi_A^B)_x (\chi_A^B)_x = (\xi_A^B)_x (\sigma_A^B)_x.$$}

This in particular means that the pair $(\sigma_A^B)_x, (\chi_A^B)_x : M^A_B(x) \rightarrow N^A_B(x)$ has a $\mathcal{W}^+_B$-split equalizer in $\mathcal{Y}^{A+0}_{B+0}$. Since $\mathcal{W}^+_B$ is comonadic, $(\xi_A^B)_x$ is an equalizer of $(\sigma_A^B)_x$ and $(\chi_A^B)_x$ in $\mathcal{Y}_{B+0}^{A+0}$, with an algebra structure $\gamma_{E^A_B(x)} :=

$$E^A_B(x) \xrightarrow{(\xi_A^B)_x} \mathcal{W} M^A_B(x) \xrightarrow{\gamma_{M^A_B(x)}} \mathcal{W}^+_B \mathcal{W} M^A_B(x) \xrightarrow{\mathcal{W}^+_B (\sigma_A^B)_x} \mathcal{W}^+_B E^A_B(x)$$

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on its source. The action components for the module \( E \) are coalgebra morphisms: the proof for left action (dually for right) comes from the diagram (3.17) (all indices can be deduced from the top left term).

\[
\begin{array}{ccc}
\mathcal{A}^A(x') \otimes E^B_B(x) & \xrightarrow{\lambda} & E \\
g \otimes \xi & \downarrow & \xi \\
\mathcal{A} \otimes M & \xrightarrow{\lambda} & M \\
1 \otimes g & \downarrow & g \\
\mathcal{A} \otimes \mathcal{G} M & \xrightarrow{\mu} & \mathcal{G}(\mathcal{A} \otimes M) & \xrightarrow{\mathcal{G} \lambda} & \mathcal{G} M \\
1 \otimes \mathcal{G} \varphi & \downarrow & \mathcal{G}(1 \otimes \varphi) & \downarrow & \mathcal{G} \varphi \\
\mathcal{A} \otimes \mathcal{G} E & \xrightarrow{\mu} & G(\mathcal{A} \otimes E) & \xrightarrow{\mathcal{G} \lambda} & \mathcal{G} E \\
\end{array}
\]

Diagrams for compatibility of actions of \( E \) with units and multiplications in \( \mathcal{A} \) and \( \mathcal{B} \) are the same as the ones for \( \mathcal{U} \circ \mathcal{A} \) and \( \mathcal{U} \circ \mathcal{B} \). This proves that \( E : \mathcal{A} \rightarrow \mathcal{B} \) is a module.

Components \((\xi_B)_x\) are natural in \( x \), and compatible with actions of \( E \) as coalgebra morphisms because they are natural and compatible as usual arrows. This proves that \( \xi \) is a module morphism between \( E \) (with coalgebra structure) and \( M \).

It remains to show that \( \xi \) is an equalizer of \( \sigma \) and \( \chi \), so assume \( L \stackrel{\omega}{\rightarrow} M \) is another \( \mathcal{V}/G \)-module morphism satisfying \( \sigma \omega = \chi \omega \). The components of the composite \( \varphi \omega \) are coalgebra maps since \( \mathcal{U} \) is locally comonadic, and naturality in \( x \) and compatibility with actions follows for \( \xi \).

\[\Box\]

Corollary 3.4. The functor

\[\text{Caten}(\mathcal{X}, \mathcal{V}/G) : \mathcal{A}, \mathcal{B}) \xrightarrow{\mathcal{U} \circ -} \text{Caten}(\mathcal{X}, \mathcal{V})(\mathcal{A} \circ \mathcal{A}, \mathcal{B} \circ \mathcal{B})\]

is conservative. Moreover the source has, and the functor preserves, \((\mathcal{U} \circ -)\)-split equalizers.

Proof. This is a direct consequence of Proposition 1.9, Lemma 3.3, and commutativity of \((-)_s\) with \( \mathcal{U} \circ - \).

\[\Box\]

Proposition 3.5. The bicategory \( \mathcal{V}/G \) is the object of Eilenberg-Moore \( \mathcal{G} \)-coalgebras, in the sense of [35], for the comonad \( \mathcal{G} \) in \( \text{Caten} \).

Proof. Mapping out of \( \mathcal{X} \) is a pseudofunctor \( \text{Caten}(\mathcal{X}, -) : \text{Caten} \rightarrow 2\text{-CAT} \) and therefore preserves adjunctions. In particular, applying it to (3.15) gives

\[\mathcal{X}' := \text{Caten}(\mathcal{X}, \mathcal{U})\]

\[\text{Caten}(\mathcal{X}, \mathcal{V}/G) \perp \text{Caten}(\mathcal{X}, \mathcal{V}) \]

\[\mathcal{R}' := \text{Caten}(\mathcal{X}, \mathcal{G})\]

The composite is isomorphic to \( \text{Caten}(\mathcal{X}, \mathcal{G}) \), and what remains to show is that \( \mathcal{X}' \) is comonadic in the sense of Proposition 3.2. It has a right adjoint \( \mathcal{R}' \), and the rest follows from Corollary 3.4.

\[\Box\]

Proposition 3.6.

\[\text{Conv}(\mathcal{X}, \mathcal{V}/G) \cong \text{Conv}(\mathcal{X}', \mathcal{V})^{\text{Conv}(\mathcal{X}, \mathcal{G})}\]
Proof. We have
\[
\text{obConv}(\mathcal{X}^e, \mathcal{Y}^e) = \text{ob}\mathcal{X} \times \text{ob}\mathcal{Y}^e \\
= \text{ob}\mathcal{X} \times \text{ob}\mathcal{Y}
\]
Furthermore,
\[
\text{Conv}(\mathcal{X}^e, \mathcal{Y}^e)((X, G), (X', G')) = [\mathcal{X}^e(X, X'), \mathcal{Y}^e(G, G')]
\]
\[
\cong [\mathcal{X}^e(X, X'), \mathcal{Y}^e(G_0, G'_0)_{\mathcal{G}(G,G')}]
\]
Finally recall that the underlying functor for a category of Eilenberg-Moore coalgebras creates colimits, so horizontal composition using the convolution formula (1.7) respects the above identifications.

4. Spans in categories of coalgebras

We prove an adjunction result for bicategories Spn(ℰ) of spans on a categories ℰ with pullbacks; see [4] much like Theorem 2.1 for the bicategories ℰ-Mod.

Then we provide a class of examples of comonads on bicategories Spn(ℰ) of the kind in Section 3. This begins with any pullback preserving comonad G on ℰ. In particular, ℰ could be a topos, in which case the category ℰ^G of G-coalgebras is also a topos (for example, see [18] Section A, Remark 4.2.3). Another motivation and application lies in the connection between categories enriched in bicategories of the form Spn(ℰ) and locally small categories parametrized (or indexed) over ℰ; see [37].

The objects of the bicategory Spn(ℰ) are those of ℰ. The morphisms (u, S, v) : A → A' are spans in ℰ; that is, diagrams of the form A \xrightarrow{u} S \xrightarrow{v} A'. The 2-cells f : (u, S, v) ⇒ (r, T, s) : A → A' are morphisms S \xrightarrow{f} T such that rf = u and sf = v. Span composition is defined using pullback:

\[(u', S', v') \otimes (u, S, v) = (A \xrightarrow{(u, S, v)} A' \xrightarrow{(u', S', v')} A'') := (A \xrightarrow{(up, P, v')q} A'')\]

where the square in the diagram (4.18) is a pullback. The identity morphism of A is the span A \xrightarrow{(1A, A, 1A)} A.

For any morphism h : A → B in ℰ, we write h_* and h^* for the morphisms A \xrightarrow{(1A, A, h)} B and B \xrightarrow{(h, A, 1A)} A, respectively, in Spn(ℰ) where we have an adjunction h_* ≃ h^*.

Remark 4.1. If ℰ is cocomplete and each morphism is exponentiable (= powerful) then the bicategory Spn(ℰ) admits colimits in the hom categories, all right extensions, and all right liftings. So Spn(ℰ) satisfies the local cocompleteness hypothesis on the ℰ in Theorem 6.1, for example.
Each functor \( U : \mathcal{F} \to \mathcal{E} \) between categories with pullbacks determines an oplax functor \( \mathcal{Y} : \text{Spn}(\mathcal{F}) \to \text{Spn}(\mathcal{E}) \) agreeing with \( U \) on objects and defined on hom categories by the functors \( \mathcal{Y}_X^\prime : \text{Spn}(\mathcal{F}(X,X')) \to \text{Spn}(\mathcal{E}(UX,UX')) \) which take the span \((r,T,s) : X \to X'\) in \( \mathcal{F} \) to the span \((Ur,UT,Ur) : UX \to UX' \) in \( \mathcal{E} \).

The following is routine.

**Lemma 4.2.** Suppose \( U : \mathcal{F} \to \mathcal{E} \) is a functor between categories with pullbacks. If the functor \( \mathcal{U} \) has a right adjoint \( R \) with unit \( 1 \mathcal{F} \to RU \) then the functor \( \mathcal{Y}_X^\prime \) defined by \( \mathcal{Y}_X^\prime(u, S, v) = (\rho_X)^* \otimes (Ru, RS, Rv) \otimes (\rho_X)_* \).

Our span analogue of Theorem 2.1 follows from Proposition 2.7 of [22] and our Lemma 4.2.

**Proposition 4.3.** Suppose \( U : \mathcal{F} \to \mathcal{E} \) is a pullback preserving functor between categories with pullbacks. Then \( \mathcal{Y} : \text{Spn}(\mathcal{F}) \to \text{Spn}(\mathcal{E}) \) is a pseudofunctor. If the functor \( \mathcal{U} \) has a right adjoint \( R \) then \( \mathcal{Y} \) has a right adjoint \( \mathcal{R} \) in Caten given on hom categories by the functors \( \mathcal{R}_X \) of Lemma 4.2.

Let \( G \) be a pullback-preserving comonad on a category \( \mathcal{E} \) which admits pullbacks. We shall define a two-sided enriched category \( \mathcal{G} : \text{Spn}(\mathcal{E}) \to \text{Spn}(\mathcal{E}) \). The objects are the \( G \)-coalgebras \((A, A \xrightarrow{\rho} GA) \) in \( \mathcal{E} \) with \((A, \gamma)_- = (A, \gamma)_+ = (A, \gamma)_0 : A \). The functor \( \mathcal{G}^{(A, \gamma)} : \text{Spn}(\mathcal{E})(A, A') \to \text{Spn}(\mathcal{E})(A, A') \) takes the span \((u, S, v) : A \to A'\) to the composite

\[
\gamma'^* \otimes (Gu, GS, Gv) \otimes \gamma_* : A \to A'.
\]

Note that coactions are monomorphisms so the identity span \( 1_A \) of \( A \) is a pullback of the cospan \( A \xrightarrow{\rho} GA \xleftarrow{\gamma} A \); so there is a unique invertible 2-cell \( \eta_{(A, \gamma)} : 1_A \Rightarrow \mathcal{G}^{(A, \gamma)}1_A \) in \( \text{Spn}(\mathcal{E}) \).

The natural transformation \( \mu_{(A', \gamma')}^{(A', \gamma')} \) has components defined by the composites

\[
\mathcal{G}^{(A', \gamma')}_{(A', \gamma')}(u', S', v') \otimes \mathcal{G}^{(A, \gamma)}(u, S, v) = \gamma'^* \otimes (Gu', GS', Gv') \otimes \gamma'_* \otimes (Gu, GS, Gv) \otimes \gamma_* \]

\[
\Rightarrow \gamma'^* \otimes (Gu', GS', Gv') \otimes (Gu, GS, Gv) \otimes \gamma_* \]

\[
\cong \gamma'^* \otimes (G(up), GP, G(v'q)) \otimes \gamma_* \]

\[
= \mathcal{G}^{(A', \gamma')}((u', S', v') \otimes (u, S, v))
\]

where the 2-cell is a whiskered counit of the adjunction \( \gamma'_* \dashv \gamma'^* \) and the isomorphism comes from the fact that \( G \) preserves the pullback (4.18).

Indeed, \( \mathcal{G} \) becomes a comonad on \( \text{Spn}(\mathcal{E}) \) as follows. The natural transformations

\[
\delta_{(A, \gamma)} \mathcal{G}^{(A', \gamma')} : \mathcal{G}^{(A', \gamma')} \Rightarrow (\mathcal{G}^{(A, \gamma)})^2 \]

\[
\epsilon_{(A, \gamma)} : \mathcal{G}^{(A, \gamma)} \Rightarrow 1_{\text{Spn}(\mathcal{E})}
\]

have component at \((u, S, v)\) induced on the limits of the rows of the diagrams (4.19) and (4.20) by those commuting diagrams. The limits exist in \( \mathcal{E} \) since they can be constructed by iterated pullbacks.
Proposition 4.4. For any category \( \mathcal{E} \) with pullbacks and any comonad \( G \) on \( \mathcal{E} \) which preserves pullbacks, the comonad \( \mathcal{G} \) just constructed satisfies an equivalence

\[
\operatorname{Spn}(\mathcal{E})^G \cong \operatorname{Spn}(\mathcal{E}^G)
\]

in \( \text{Caten} \).

Proof. Note that the category \( \mathcal{E}^G \) of \( G \)-coalgebras has pullbacks since \( \mathcal{E} \) does and \( G \) preserves them. The displayed bicategories do have the same objects, namely, the \( G \)-coalgebras. By definition, \( \operatorname{Spn}(\mathcal{E})^G((A, \gamma), (A', \gamma')) = \operatorname{Spn}(\mathcal{E})(A, A')^{g(A, \gamma'; A', \gamma')} \). An object of this category is a \( g(A, \gamma'; A', \gamma') \)-coalgebra; that is, a span \((u, S, v) : A \to A'\) together with a span morphism \((u, S, v) \to \gamma'^o \otimes (Gu, GS, Gv) \otimes \gamma_s \) satisfying two axioms for a coaction. Such span morphisms are in bijection with morphisms \( \sigma : S \to GS \) in \( \mathcal{E} \) such that the diagram

\[
\begin{array}{ccc}
A & \xleftarrow{u} & S \xrightarrow{v} & A' \\
\gamma & \downarrow & \sigma & \downarrow \gamma' \\
GA & \xleftarrow{Gu} & GS & \xrightarrow{Gv} GA'
\end{array}
\]

(4.21)

commutes while the two coaction conditions translate to the two conditions for \((S, \sigma)\) to be a \( G \)-coalgebra. So (4.21) tells us that \( u \) and \( v \) are \( G \)-coalgebra morphisms and we have a span in \( \mathcal{E}^G \). This extends to an isomorphism of categories

\[
\operatorname{Spn}(\mathcal{E})(A, A')^{g(A, \gamma'; A', \gamma')} \cong \operatorname{Spn}(\mathcal{E}^G)((A, \gamma), (A', \gamma')).
\]

It is easy to see that composition and identities in the bicategories correspond under these isomorphisms. \( \square \)

Corollary 4.5. If the functor \( U : \mathcal{F} \to \mathcal{E} \) of Proposition 4.3 is comonadic in \( \text{Cat} \) then \( \mathcal{U} : \operatorname{Spn}(\mathcal{F}) \to \operatorname{Spn}(\mathcal{E}) \) is comonadic in \( \text{Caten} \).

Example 4.6. Take any set \( X \) and let \( \mathcal{E} = \mathcal{P}X \) be the ordered set of subsets of \( X \). This \( \mathcal{E} \) is complete; pullbacks are intersections. We have the locally ordered bicategory \( \mathcal{F}X = \operatorname{Spn}(\mathcal{E}) \). Now let \( X \) be a topological space. Write \( \mathcal{O}X \) for the ordered set of open subsets of \( X \). For our example, take \( G \) to be the interior operator for the topology. This \( G \) is a finite-limit-preserving idempotent comonad on the complete cartesian-monoidal category \( \mathcal{E} = \mathcal{P}X \). We have \( \mathcal{E}^G = \mathcal{O}X \). By Proposition 4.4, the bicategory \( \operatorname{Spn}(\mathcal{O}X) \) is comonadic over \( \mathcal{F}X \) in \( \text{Caten} \).

5. Differential comonads

Let \( \mathcal{V} \) be a locally additive bicategory; that is, a bicategory whose homs are additive categories such that composition with a morphism, on either side, is an additive functor.

Definition 5.1. A differential system \( \mathcal{D} \) on \( \mathcal{V} \) consists of a set \( \operatorname{ob}\mathcal{D} \) (whose elements are called objects of \( \mathcal{D} \)), a function \((-)_0 : \operatorname{ob}\mathcal{D} \to \operatorname{ob}\mathcal{V} \), additive functors \( \mathcal{D}^D_D : \mathcal{V}^D_D \to \mathcal{V}^D_D \) for all objects \( D, D' \) of \( \mathcal{D} \), and families of 2-cells

\[
\tau_{D, v} : \mathcal{D}^D_D(v') \otimes v \to \mathcal{D}^D_D(v' \otimes v) \quad \text{and} \quad \tau_{D, v} : v' \otimes \mathcal{D}^D_D(v) \to \mathcal{D}^D_D(v' \otimes v) ,
\]

for all objects \( D, D' \) of \( \mathcal{D} \), and families of 2-cells
natural in \( \mathcal{V}^{D'}, v' \in \mathcal{V}^{D''} \), satisfying conditions (5.22)-(5.26) (in which brackets, associativity constraints, and unit constraints are omitted for easier reading).

\[
\begin{align*}
\mathcal{D}^{D''} (v'') \otimes v' \otimes v & \xrightarrow{\eta_{v'' \otimes v', \otimes v}} \mathcal{D}^{D''} (v'' \otimes v') \otimes v \\
\mathcal{D}^{D''} (v'') \otimes v' \otimes v & \xrightarrow{1_{v''} \otimes 1_{v'} \otimes \eta_{v'}} \mathcal{D}^{D''} (v'' \otimes v') \otimes v
\end{align*}
\]  

(5.22)

\[
\begin{align*}
v'' \otimes v' \otimes \mathcal{D}^{D'} (v) & \xrightarrow{1_{v''} \otimes 1_{v'} \otimes \eta_{v'}} \mathcal{D}^{D''} (v'') \otimes v' \otimes v \\
\mathcal{D}^{D''} (v'') \otimes v' \otimes \mathcal{D}^{D'} (v) & \xrightarrow{1_{v''} \otimes 1_{v'} \otimes \eta_{v'}} \mathcal{D}^{D''} (v'') \otimes v' \otimes \mathcal{D}^{D'} (v)
\end{align*}
\]  

(5.23)

\[
\begin{align*}
\mathcal{D}^{D'} (v) & \xrightarrow{1_{v''} \otimes 1_{D_0}} \mathcal{D}^{D'} (v) \otimes 1_{D_0} \xrightarrow{r_{v'', v} \otimes 1_{D_0}} \mathcal{D}^{D'} (v \otimes 1_{D_0}) \\
& = 1_{D'_0} \otimes \mathcal{D}^{D'} (v) \xrightarrow{1_{D'_0} \otimes \eta_{v'}} \mathcal{D}^{D'} (1_{D'_0} \otimes v)
\end{align*}
\]  

(5.25)

\[
\mathcal{D}^{D''} (v'' \otimes v') \circ r_{v'' \otimes v'} + \mathcal{D}^{D''} (r_{v'' \otimes v'}) \circ l_{\mathcal{D}^{D''} (v'' \otimes v')} = 0
\]  

(5.26)

**Proposition 5.2.** Suppose \( \mathcal{D} \) is a differential system on the locally additive bicategory \( \mathcal{V} \) whose hom categories have finite direct sums. The following defines a comonad \( \mathcal{G} \) on \( \mathcal{V} \) in Caten. The object span of \( \mathcal{G} \) is that of \( \mathcal{D} \). The functor \( \mathcal{G}^{D'} : \mathcal{V}^{D'_0} \rightarrow \mathcal{V}^{D'_0} \) is defined by

\[
\mathcal{G}^{D'} = 1_{\mathcal{V}^{D'_0}} \otimes \mathcal{D}^{D'}.
\]

The 2-cell \( \mu_{v', v} \) is defined by the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \ell_{v', v} & r_{v', v} & 0
\end{bmatrix}:
\]

\[
(v' \otimes v) \oplus (v' \otimes \mathcal{D}^{D'} (v)) \oplus (\mathcal{D}^{D''} (v') \otimes v) \oplus (\mathcal{D}^{D''} (v') \otimes \mathcal{D}^{D'} (v)) \Rightarrow (v' \otimes v) \oplus \mathcal{D}^{D''} (v' \otimes v).
\]

The 2-cell \( \eta : 1_{D_0} \Rightarrow 1_{D_0} \otimes \mathcal{D}^{D'} (1_{D_0}) \) is the first injection

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

The 2-cell \( \alpha_v \) is the matrix

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1 \\
0 & 0
\end{bmatrix}:
\]

\[
v \oplus \mathcal{D}^{D'} (v) \Rightarrow v \oplus \mathcal{D}^{D'} (v) \oplus \mathcal{D}^{D'} (v) \oplus (\mathcal{D}^{D'})^2 (v).
\]

The 2-cell \( \epsilon_v \) is the first projection

\[
\begin{bmatrix}
1 & 0
\end{bmatrix}:
\]

\[
v \oplus \mathcal{D}^{D'} (v) \Rightarrow v.
\]
Proof. Associativity of composition \( \mu \) for \( \mathcal{D} \) is proved by calculating two matrix products:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \ell & r & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 \otimes \ell & 1 \otimes r & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \otimes \ell & 1 \otimes r & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \ell & r & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ell \otimes 1 & r \otimes 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ell \otimes 1 & r \otimes 1 & 0 & 0
\end{bmatrix}
\]

which are equal by (5.22), (5.23) and (5.24). That \( \eta \) is the unit for \( \mu \) is proved by calculating two matrix products of the form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \ell & r & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \ell & r & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]

which are both identity \( 2 \times 2 \)-matrices by (5.25).

The proof that \( d \) and \( e \) are enriched functors and that they satisfy coassociativity and counital conditions for a comonad proceeds similarly using matrix multiplication. Only the condition relating \( d \) and \( \mu \) (see [22] diagram (2.16)) requires any of the differential system axioms, namely, condition (5.26); for a little more detail on this, see [30]. \( \square \)

A comonad arising as in Proposition 5.2 is called a differential comonad.

Example 5.3. Let \( \mathcal{C} \) be a braided monoidal additive category containing an object \( E \) with

\[
\sigma_{E,E} = -1_{E \otimes E} : E \otimes E \to E \otimes E,
\]

where \( \sigma_{X,Y} : X \otimes Y \to Y \otimes X \) is the braiding for \( \mathcal{C} \). Take \( \mathcal{Y} \) to be the one-object bicategory with endomorph \( \mathcal{C} \). The following defines a differential system \( \mathcal{D} \) on \( \mathcal{Y} \) (where we are writing as if \( \mathcal{C} \) were strict monoidal):

a. \( \text{ob} \mathcal{D} = \{ \{D\} \} \);

b. \( \mathcal{D} = \mathcal{D}_D^\mathcal{Y} := E \otimes - : \mathcal{C} \to \mathcal{C} \);

c. \( r_{Y,X} = 1_{E \otimes Y \otimes X} : E \otimes Y \otimes X \to E \otimes Y \otimes X \);

d. \( r_{Y,X} = \sigma_{Y,E} \otimes 1_X : Y \otimes E \otimes X \to E \otimes Y \otimes X \).

All the differential system conditions are obvious except that (5.26) translates to the equation \( \sigma_{E \otimes Y,E} + \sigma_{Y,E} = 0 \) which holds by the property \( \sigma_{E \otimes Y,E} = (\sigma_{E,E} \otimes 1_Y) \circ (1_E \otimes \sigma_{Y,E}) \) of the braiding and the condition \( \sigma_{E,E} = -1_{E \otimes E} \).

We have the following easy identification of Eilenberg-Moore coalgebra construction for a differential comonad in terms of the differential system.

Proposition 5.4. The bicategory \( \mathcal{Y}^\mathcal{D} \) for the differential comonad \( \mathcal{D} \) on \( \mathcal{Y} \) as in Proposition 5.2 has the same objects as the differential system \( \mathcal{D} \). The objects of the hom category \( \mathcal{Y}^\mathcal{D}(D,D') \) are pairs \((x,\delta)\) where \( x : D_0 \to D'_0 \) is a morphism and \( \delta : x \Rightarrow \mathcal{D}_D^\mathcal{Y}(x) \) is a 2-cell of \( \mathcal{Y} \) satisfying \( \mathcal{D}_D^\mathcal{Y}(x) \delta = 0 \). Morphisms \( \theta : (x,\delta) \Rightarrow (x_1,\delta_1) \) are 2-cells \( \theta : x \Rightarrow x_1 \) in \( \mathcal{Y} \) such that \( \theta \delta = \delta_1 \theta \). The composite \( D \xrightarrow{(x,\delta)} D' \xrightarrow{(x',\delta')} D'' \xrightarrow{\delta''} \) is \( D \xrightarrow{(x' \otimes x,\delta' \otimes \delta) + \ell_{x',x}(1_{x'} \otimes \delta)} D'' \).

Example 5.5. Consider the case of Example 5.3 where \( \mathcal{C} = \text{GAb} \) is the symmetric monoidal category of \( \mathbb{Z} \)-graded abelian groups. The objects are families \( A = (A_n)_{n \in \mathbb{Z}} \) of abelian groups with the morphisms families of abelian group morphisms. Elements of \( A_n \) are said to be of degree \( n \). We regard abelian groups as objects by putting them in degree 0 and putting
the abelian group 0 in all other degrees. The tensor product (as per [14]) is defined by the convolution formula
\[(A \otimes B)_n = \sum_{p+q=n} A_p \otimes B_q .\]
The unit object for this tensor product is the group \(\mathbb{Z}\) of integers under addition. The symmetry \(\sigma = \sigma_{A,B} : A \otimes B \to B \otimes A\) is defined by
\[\sigma(a \otimes b) = (-1)^{pq} b \otimes a\]
for \(a \in A_p\) and \(b \in B_q\). The suspension of \(A \in \text{GAb}\) is the graded abelian group \(SA\) defined by \(SA_n = A_{n-1}\). Then \(E = \text{SZ}\) has the property that \(\sigma_{E,E} = -1_{E \otimes E}\). Proposition 5.2 provides a monoidal comonad on \(\text{GAb}\) arising from this \(E\) (in fact, since \(E \otimes A \cong SA\), we have \(\mathcal{D} \cong S\) and Proposition 5.4 tells us that the monoidal category of Eilenberg-Moore coalgebras is the category \(\text{DGAb}\) of chain complexes of abelian groups with the usual symmetric monoidal structure (see [14]). This example can be generalised in the obvious way to replace \(\text{Ab}\) by any monoidal additive category with countable coproducts.

6. Comonadicity of comonadic base change

Recall the base change result in [22] which appears there as Proposition 7.5 and states that right whiskering by modules over a two-sided enriched category provides a normal lax functor.

**Theorem 6.1.** Let \(\mathcal{D}\) be a comonad in \(\text{Cat}\) on the locally cocomplete bicategory \(\mathcal{V}\) and let \(\mathcal{W} \rightarrow \mathcal{R}\) be as in diagram (3.15). Then the lax functor
\[\mathcal{N} := \text{Moden}(\mathcal{D}, \mathcal{V}) \xrightarrow{\mathcal{W} := \text{Moden}(-, \mathcal{W})} \mathcal{M} := \text{Moden}(\mathcal{D}, \mathcal{V}) ,\]
given by right whiskering with \(\mathcal{W}\), is comonadic in \(^3\text{CATEN}^3\).

**Proof.** As in the proof of Theorem 2.1, we only look at \(\mathcal{N} = \mathcal{V}^{\mathcal{D}}\)-\text{Mod}, and \(\mathcal{M} = \mathcal{V}\text{-Mod}\). By Theorem 2.1, we have a right adjoint \(\mathcal{R}\) to \(\mathcal{W}\). Moreover, \(\mathcal{W}\) satisfies the other two conditions of Proposition 3.2 as stated in Lemma 3.3. This is all we need to conclude that \(\mathcal{W}\) is comonadic.

**Remark 6.2.** Theorem 6.1 restricts to the one-object case. In the situation of Remark 2.2, if \(\mathcal{W} : \mathcal{D} \rightarrow \mathcal{C}\) is comonadic in \(\text{Cat}\) then \(\mathcal{W}\) is comonadic in \(\text{Caten}\).

**Example 6.3.** Recall from Example 2.1 of [37] that each Grothendieck fibration \(P : \mathcal{F} \rightarrow \mathcal{E}\) (in the sense of [16]), over the category \(\mathcal{E}\) with pullbacks, gives rise to a \(\text{Spn}(\mathcal{E})\)-enriched category \(\mathcal{A}\) when the fibration is locally small (or “has small homs”) in the two-sided sense; see [5, 34]. The objects of \(\mathcal{A}\) over \(X\) are the objects \(A\) of \(\mathcal{F}\) with \(PA = X\). The span \(\mathcal{A}(A,B) : X \rightarrow Y\), where \(PA = X\), \(PB = Y\), is defined universally (using local smallness) by a natural bijection
\[\text{Spn}(\mathcal{E})(X,Y)((u,S,v), \mathcal{A}(A,B)) \cong \mathcal{F}_S(u^*A, v^*B)\]
where \(\mathcal{F}_S\) is the fibre of \(P\) over \(S\) and \(u^*A \rightarrow A\), \(v^*B \rightarrow B\) are cartesian morphisms over \(u : S \rightarrow X\), \(v : S \rightarrow Y\), respectively. It follows from Theorem 6.1 that the \(\text{Spn}(\mathcal{E}^G)\)-enriched category arising from a locally small fibration over \(\mathcal{E}^G\) is a coalgebra for a base-change comonad on \(\text{Spn}(\mathcal{E})\)-\text{Mod}.

**Example 6.4.** Let \(\Delta\) denote the topologist’s simplex category: objects are finite non-empty linearly ordered sets \([n] = \{0,1,\ldots,n\}\), for \(n \geq 0\), and morphisms are order-preserving functions. Let \(\mathbb{N}\) be the discrete category of natural numbers. Restriction along the bijective-on-objects functor \([-] : \mathbb{N} \rightarrow \Delta^{\text{op}}\) taking \(n\) to \([n]\) is a comonadic functor \(U : [\Delta^{\text{op}}, \text{Set}] \rightarrow\)
that the bicategory of simplicial-set-enriched categories and modules between them is comonadic in \( \text{CATEN} \) over the bicategory of \( \mathbb{N} \)-graded-set-enriched categories and modules between them.

**Example 6.5.** Let \( \mathcal{V} \) be a locally presentable symmetric monoidal category and let \( \text{Comon}\mathcal{V} \) denote the category of comonoids and comonoid morphisms in \( \mathcal{V} \). Then \( \text{Comon}\mathcal{V} \) is locally-presentable symmetric monoidal with the forgetful functor \( U : \text{Comon}\mathcal{V} \to \mathcal{V} \) comonadic and symmetric strong monoidal; see [33]. We deduce from Theorem 6.1 that the bicategory of \( \mathcal{V} \)-enriched categories and modules between them is comonadic in \( \text{CATEN} \) over the bicategory of \( \mathcal{V} \)-enriched categories and modules between them. As shown in [41], an example of a \( \text{Comon}\mathcal{V} \)-enriched category is the category \( \text{Mon}\mathcal{V} \) of monoids and monoid morphisms in \( \mathcal{V} \); the homs are given by the measuring comonoid construction; see [1, 17] for consequential applications and developments.

**Example 6.6.** Consider the diagram (6.27) of monoidal additive categories and additive functors.

\[
\begin{array}{ccc}
\text{DGAb} & & \text{GAb} \\
\downarrow L & & \downarrow \Sigma \\
\text{GAb} & \xleftarrow{U} & \text{Ab}
\end{array}
\]  

(6.27)

Some of the notation was explained in Example 5.5. Of course, \( \text{Ab} \) is the monoidal category of abelian groups so that \( \text{Ab} \)-enriched categories are additive categories (no direct sums required). Categories enriched in \( \text{GAb} \) are graded categories or \( \mathcal{G} \)-categories while categories enriched in \( \text{DGAb} \) are differential graded categories or DG-categories. These were motivating examples of enriched categories for Eilenberg-Kelly [14]. The functor \( U \) forgets the differentials in the chain complexes and has adjoints \( L \dashv U \dashv R \) given by

\[
L(C)_n = C_{n+1} \oplus C_n , \quad R(C)_n = C_n \oplus C_{n-1} , \quad d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .
\]  

(6.28)

The functor \( \Sigma \) takes \( A = (A_n)_{n \in \mathbb{Z}} \) to the coproduct \( \Sigma A = \sum_{n \in \mathbb{Z}} A_n \) while its right adjoint \( C \) takes each abelian group \( X \) to the graded abelian group \( CX \) with \( X \) in all degrees. We already mentioned in Example 5.5 that \( U \) is comonadic but so too are \( \Sigma \) and \( \Sigma \circ U \). The comonad \( \Sigma \circ C \) generated by \( \Sigma \dashv C \) is given by tensoring with the Hopf ring \( \mathbb{Z}[x, x^{-1}] \) of Laurent polynomials with integer coefficients. The comonad \( \Sigma \circ U \circ R \circ C \) generated by \( \Sigma \circ U \dashv R \circ \Sigma \) is given by tensoring with the Hopf ring which is the object of study by Pareigis in [32]. So we obtain three applications of Theorem 6.1. To be explicit, the pseudofunctors

\[
\text{GAb-Mod} \xrightarrow{\Sigma} \text{Ab-Mod} , \quad \text{DGAb-Mod} \xrightarrow{U} \text{GAb-Mod} , \quad \text{DGAb-Mod} \xrightarrow{U \circ \Sigma} \text{Ab-Mod}
\]

are all comonadic in \( \text{CATEN} \).

**Example 6.7.** Refer back to the situation of Example 4.6. For \( X \) just a set, we will give an interpretation of the bicategory \( \mathscr{J}X \text{-Mod} \) in terms of families of ordered sets over the...
discretely ordered $X$. Let $\text{Idl}/X$ denote the 2-category of ordered objects and two-sided order ideals (in the sense of [9]) in the topos $\text{Set}/X$ of sets over $X$. The objects are ordered sets $A$ equipped with a function $p : A \to X$ such that $a \leq a'$ implies $p(a) = p(a')$. The morphisms $M : (A, p) \to (B, q)$ are relations $M \subseteq B \times A$ such that

(i) $bMa$ implies $p(a) = q(b)$ and

(ii) $bMa, b' \leq b, a \leq a'$ imply $b'Ma'$

where $bMa$ means $(b, a) \in M$. The $2$-cells are inclusions. Morphisms are composed as relations.

An invertible $2$-functor

$$\Gamma : \text{Idl}/X \longrightarrow \mathcal{S} X\text{-Mod}$$

is defined as follows. The $\mathcal{S} X$-category $\Gamma(A, p)$ has objects $(U, s)$ where $s$ is a section of $p$ over $U \subseteq X$; that is, $s : U \to A$ is a function such that $p(s(x)) = x$ for all $x \in U$. Put $\Gamma(A, p)((U, s), (U', s')) = \{ x \in X : s(x) = s'(x) \} \subseteq U \cap U'$ as a span from $U$ to $U'$ in $\mathcal{P} X$. The $\mathcal{S} X$-category composition and identity inclusions are obvious. For an ideal $M : (A, p) \to (B, q)$, the module $\Gamma M : \Gamma(A, p) \to \Gamma(B, q)$ is defined by $\Gamma M((V, t), (U, s)) = \{ x \in X : t(x)Ms(x) \}$. The inverse for $\Gamma$ takes the $\mathcal{S} X$-category $A$ to the set of objects $a$ over singleton subsets $a_+$ of $X$ ordered by $a \leq a'$ if and only if $a$ and $a'$ are over the same $\{ x \}$ and $x \in A(a, a')$.

Again let $X$ be a topological space with $G$ to be the interior operator for the topology. By Theorem 6.1, the bicategory $\text{Spn}(\mathcal{O} X)\text{-Mod}$ is comonadic over $\text{Idl}/X$ in Caten. Recall that Walters [42] showed that sheaves on the space $X$ are equivalent to symmetric Cauchy complete categories enriched in this bicategory $\text{Spn}(\mathcal{O} X)$. Hence, the bicategory of left adjoint morphisms in $\text{Spn}(\mathcal{O} X)\text{-Mod}$ is of interest.

7. Fusion Operators and Hopfness for Comonads in Caten

Fusion operators in monoidal categories were studied in [39] along with the example of those coming from bimonoids. A proof that a bimonoid has an antipode if and only if the fusion operator is invertible can be found in [7] Appendix. In [8], an opmonoidal monad on a monoidal category was called Hopf when a suitable fusion morphism was invertible. The dual notion of Hopf monoidal comonad was studied in a general context in [10]. Here we identify Hopf comonads in Caten.

**Definition 7.1.** For a comonad $\mathcal{G}$ on $\mathcal{V}$ in Caten, the left fusion operator at $v' \in \mathcal{V}(G_0', G_0''')$ and $(v, \gamma_v) \in \mathcal{V}^{\mathcal{G}}(G, G')$ is the composite $2$-cell

$$v^l_{v', v} : G^{G'''}_G v' \otimes v \xrightarrow{1 \otimes \gamma_v} G^{G'''}_G v' \otimes G^G_G v (\mu^G_{G, G'''} v') \otimes v \xrightarrow{G^{G'''}_G (v' \otimes v)} G^{G'''}_G (v' \otimes v).$$

Dually, in the sense of [22] Section 2.9, the right fusion operator at $v' \in \mathcal{V}(G_0', G_0''')$ and $(v, \gamma_v) \in \mathcal{V}^{\mathcal{G}}(G, G')$ is the composite $2$-cell

$$v^r_{v', v} : v' \otimes G^G_G v \xrightarrow{\gamma_v \otimes 1} G^{G'''}_G v' \otimes G^G_G v (\mu^G_{G, G'''} v') \otimes v \xrightarrow{G^{G'''}_G (v' \otimes v)} G^{G'''}_G (v' \otimes v).$$

The comonad $\mathcal{G}$ is called left (right) Hopf when the left (right) fusion operators are all invertible.

Since we will use left fusion operators mostly, we simply write $v_{v', v}$ for $v^l_{v', v}$. 
Proposition 7.2. The inverse fusion maps are $\mathcal{C}$-compatible in the first variable as well as compatible with any coalgebra structure existing on $v'$, in the sense that

\[
\begin{align*}
\mathcal{G}(v' \otimes v) & \xrightarrow{\gamma_{v' \otimes v}} \mathcal{G} v' \otimes v \\
\gamma_{v' \otimes v} & \xrightarrow{\gamma_{v' \otimes v}} \mathcal{G} v' \otimes v.
\end{align*}
\]

Proof. Refer to the commuting diagrams (7.29), (7.31), (7.30). □

Example 7.3. Comonads obtained by tensoring with a Hopf bimonoid in a braided monoidal category are all Hopf. In particular, the three comonads in Example 6.6 are Hopf.

Example 7.4. The comonad $\mathcal{G}$ on $\text{Spn}(\mathcal{P}X)$ coming from an interior operator (as in Example 4.6) is Hopf.

Remark 7.5. We refer back to Example 3.1. The notion of Hopf algebroid defined in Section 8 of [13] is designed for modules for a comonoidal $\mathcal{C}$-category $\mathcal{A}$; those modules are $\mathcal{C}$-functors $M : \mathcal{A} \to \mathcal{C}$. The monoidal structure on modules defined by the comonoidal structure on $\mathcal{A}$ is pointwise and the Hopfness implies that the closed structure is particularly simple. However, the Hopf property in the present paper is designed for comodules for a comonoidal $\mathcal{C}$-category $\mathcal{A}$. The two notions are different.
8. Extension creation

In this section we show that Hopf comonadic pseudofunctors in Caten create left extensions. This generalises results in \([8, 10]\).

Recall (see \([35]\) for example) that a diagram

\[
\begin{array}{ccc}
U & \xrightarrow{v} & V \\
\downarrow{u} & & \downarrow{h} \\
C & \xrightarrow{\kappa} & \end{array}
\]

in a bicategory \(\mathcal{W}\) is said to exhibit \(h\) as a left (Kan) extension of \(u\) along \(v\) when for each 2-cell \(\sigma : u \Rightarrow w \otimes v\) there exists a unique 2-cell \(\tau : h \Rightarrow w\) whose pasted composite with \(\kappa\) is \(\sigma\). Should it exist, we write \(\text{lan}_p v,u \kappa\) for such an \(h\).

A pseudofunctor \(H : W \to V\) between bicategories is said to create left extensions when, given a span \(C \xleftarrow{u} U \xrightarrow{v} V\) in \(W\) for which a left extension of \(H u\) along \(H v\) exists in \(V\), there exists a left extension of \(u\) along \(v\) in \(W\) which is taken by \(H\) to a left extension of \(H u\) along \(H v\).

**Theorem 8.1.** If the comonad \(G\) in Caten is left Hopf, then the forgetful pseudofunctor \(U : V^G \to V\) creates left extensions.

**Proof.** Consider a span \(G'' \xleftarrow{(u,\gamma_u)} G \xrightarrow{(v,\gamma_v)} G'\) of coalgebras for which the underlying span \(G'' \xleftarrow{u} G \xrightarrow{v} G'\) has a left extension \(k = \text{lan}(v,u)\) in \(V\) exhibited by a 2-cell \(\kappa : u \Rightarrow k \otimes v\). The universal property of the left extension is that the function taking 2-cells \(\phi : u \Rightarrow l \otimes v\) to 2-cells \(\phi : k \Rightarrow l\) is \(\phi = (\phi \otimes 1)\kappa\), is a bijection. In particular, there is a map \(\gamma_k : k \to Gk\) corresponding to \(\gamma_k = G\gamma u, G\gamma_v, G(k \otimes v) \xrightarrow{\gamma_{k,v}} G(k \otimes v)\) such that the diagram below commutes.

\[
\begin{array}{ccc}
u & \xrightarrow{\gamma_u} & G\gamma u & \xrightarrow{G\gamma_v} & G(k \otimes v) \\
\kappa \downarrow & & \kappa \downarrow & & \kappa \downarrow \\
k \otimes v & \xrightarrow{\gamma_k \otimes 1} & Gk \otimes v & \xrightarrow{\gamma_{k,v}} & Gk \otimes v
\end{array}
\]

(8.33)

The obtained arrow \(\gamma_k\) defines a coalgebra structure on \(k\), where the compatibility with \(d\) and \(e\) follows from the following two commutative diagrams.
The 2-cell \( \kappa \) is a coalgebra morphism as seen by substituting \( v^{-1} \) in (8.33).

To see that \( \kappa \) exhibits \( p_{k,\gamma}k \) as a left extension of \( (u, \gamma_u) \) along \( (v, \gamma_v) \), consider a coalgebra \( p_{l,\gamma}l : G_1 \to G_2 \), and a coalgebra morphism \( \varphi : u \Rightarrow l \otimes v \). In \( \mathcal{V} \), the left extension universal property gives \( \overline{\varphi} : k \Rightarrow l \). Using the commuting diagram

\begin{align*}
& u \\
\downarrow^\kappa \quad \downarrow^\gamma \quad \downarrow^e \quad \downarrow^\kappa \\
G u & \xrightarrow{\kappa} G (k \otimes v) \\
\downarrow^\varphi \quad \downarrow^\gamma \quad \downarrow^\kappa \quad \downarrow^\gamma \\
G k \otimes v & \xrightarrow{\gamma \otimes 1} G l \otimes v
\end{align*}

we indeed see that \( \overline{\varphi} \) is a coalgebra morphism. \( \square \)

**Remark 8.2.** Examination of the above proof allows us to obtain a more localised result.

To obtain that the left extension of \( (u, \gamma_u) \) along \( (v, \gamma_v) \) is created we do not need the global invertibility of the fusion operators. We only need that the function

\[ \mathcal{V}(u, v' \otimes v) : \mathcal{V}(u, G(v' \otimes v)) \]

should be injective for all \( v' \in \mathcal{V}(G_0', G_0') \) and surjective for \( v' = k \), while the function

\[ \mathcal{V}(u, G(k \otimes v)) : \mathcal{V}(u, G(k \otimes v)) \to \mathcal{V}(u, G(k \otimes v)) \]

should be injective.

**Remark 8.3.** A dual of Theorem 8.1 is that right Hopfness of \( \mathcal{G} \) in Caten implies \( \mathcal{U} : \mathcal{V} \to \mathcal{V} \) creates left liftings.

Recall that a morphism \( m \) having a right adjoint in a bicategory is equivalent to the existence of a left extension \( \text{lan}(m, 1_M) \) which is respected by \( m \); that is, \( m \circ \text{lan}(m, 1_M) \equiv \text{lan}(m, m) \).

**Corollary 8.4.** With a Hopf-comonadic \( \mathcal{U} : \mathcal{N} \to \mathcal{M} \), a morphism \( n \in \mathcal{N}(N, N') \) has a right adjoint if and only if \( \mathcal{U} n \) does.

**Proof.** Being a pseudofunctor, \( \mathcal{U} \) preserves adjoints.

The other way around, assume \( \mathcal{U} n \) has a right adjoint, so both \( \text{lan}(\mathcal{U} n, 1_{\mathcal{M}}) \) and \( \text{lan}(\mathcal{U} n, \mathcal{U} n) \) exist. From the previous theorem, \( \text{lan}(n, 1_N) \) exists and \( n \circ \text{lan}(n, 1_N) \) is taken to \( \mathcal{U}(n \circ \text{lan}(n, 1_N)) \equiv \mathcal{U} n \circ \text{lan}(\mathcal{U} n, 1_{\mathcal{M}}) \equiv \text{lan}(\mathcal{U} n, \mathcal{U} n) \) which creates \( \text{lan}(n, n) \). \( \square \)
9. Hopfness of comonadic base change

We now show that Hopfness of the pseudofunctor inducing the base change passes to the base change itself.

**Theorem 9.1.** In the situation of Theorem 6.1, let \( \mathcal{R} \) be the right adjoint of \( \mathcal{U} \). If \( \mathcal{R} \) preserves local colimits, and \( \mathcal{J} \) is left Hopf, then the induced comonad \( \mathcal{G} := \mathcal{U} \circ \mathcal{R} \) is right Hopf.

**Proof.** Again we consider the case when \( \mathcal{E} \) is the terminal bicategory. Recall that we are looking at \( \mathcal{N} = \mathcal{V}/\mathcal{M} \)-Mod, and \( \mathcal{M} = \mathcal{V}/\mathcal{M} \)-Mod.

At modules \( M \in \mathcal{M}(\mathcal{U} \circ \mathcal{A}, \mathcal{U} \circ \mathcal{B}) \) and \( N \in \mathcal{N}(\mathcal{B}, \mathcal{C}) \), the right fusion operator \( v_{N,M}^r \) for \( \mathcal{G} \) is given by the right column of the following diagram.

\[
\begin{array}{c}
\sum R M_B^A \otimes R B_B^C \otimes N_B^C & \xrightarrow{\mathcal{R} (\rho \otimes 1)} & \sum R M_B^A \otimes N_C^B \\
1 \otimes 1 \otimes g & \xrightarrow{1 \otimes g} & (g \circ 1) \lambda \\
\sum \mu^{(\mathcal{S})} (1 \otimes g \otimes 1) & \xrightarrow{\mathcal{R} (\rho \otimes 1)} & \sum \mu^{(\mathcal{S})} \ (\mathcal{R}(\text{coeq})) \\
& \xrightarrow{\mathcal{R}(\text{coeq})} & \mathcal{R}(\text{coeq})
\end{array}
\]

The middle column is a coproduct of left fusion operators \( v_{N,M}^r \) for \( \mathcal{G} \) and so is invertible. The left column can be rewritten as a sum of composites of the form \( v_{N,M}^r (v_{M,N}^l \otimes 1_{N_B^C}) \) and so is invertible. So \( v_{N,M}^r \) is invertible. \( \square \)

**Remark 9.2.** That \( \mathcal{J} \) is left Hopf while \( \mathcal{G} \) is right Hopf in Theorem 6.1 is expected since our notational conventions lead to a morphism reversing inclusion \( \mathcal{I} : \mathcal{W}^{\text{op}} \to \mathcal{W} \)-Mod; see Section 7.6 of [22].

**Example 9.3.** In the diagram (6.27) functors \( U \) and \( \Sigma \) forget differential and take a sum of all components of the graded abelian group. They are both part of Hopf-comonadic adjunctions, and by Theorem 8.1 create duals and cohoms. An abelian group \( A \) has a dual if and only if it is finitely generated and projective (\([40] \) has a proof). As a consequence of \( \Sigma \) being Hopf-comonadic, a graded abelian group \( A \) has a dual if and only if it has finitely many non-zero components each of which is finitely generated and projective. As a consequence of \( U \) being Hopf-comonadic, a chain complex \( A \) has a dual if and only if its underlying graded abelian group does.

**Example 9.4.** Since the \( U \) of diagram (6.27) is a left adjoint it preserves colimits, so by Theorem 6.1 the change of base functor \( \mathcal{U} \) creates Cauchy modules.

**Example 9.5.** As in Remark 4.1, suppose we have a cocomplete category \( \mathcal{E} \) with each slice category \( \mathcal{E}/E \) cartesian closed and we have a comonad \( G \) on \( \mathcal{E} \) which preserves pullbacks. If furthermore \( G \) preserves colimits then \( \mathcal{R} \) preserves local colimits as required by Theorem 9.1 with \( \mathcal{V} = \text{Spn}(\mathcal{E}) \).

Let us look, in particular, at the case where \( G = E \times - \), with the comonad structure defined by the diagonal comonoid structure on \( E \in \mathcal{E} \), so that \( \mathcal{E}^G = \mathcal{E}/E \). The comonad \( \mathcal{J} \)
on $\text{Spn}(\mathcal{E})$ in Caten has objects those of $\mathcal{E}/E$; that is, pairs $(X, p)$ where $X \xrightarrow{p} E$ in $\mathcal{E}$. The functor

$$\mathcal{G}_{(X,p)}^{(X',p')} : \text{Spn}(\mathcal{E})(X, X') \longrightarrow \text{Spn}(\mathcal{E})(X, X')$$

takes a span $(u, S, v) : X \rightarrow X'$ to the span $(uk, S, v): X \xrightarrow{} X'$ where $S \xrightarrow{k} S$ is the equalizer of $pu$ and $p'v$. Let us use the method of generalised elements in $\mathcal{E}$ to investigate the fusion operators: a $Z$-element of $A$ is a morphism $a : Z \rightarrow A$ and we write $a \in Z A$. The $Z$-elements of $\tilde{S}$ are the $x \in Z S$ such that $pu x = p'v x$; so $(u, S, v) : (X, p) \rightarrow (X', p')$ in $\text{Spn}(\mathcal{E}/E)$ if and only if $k : \tilde{S} \rightarrow S$ is invertible. We also see that

$$\mathcal{G}_{(X,p)}^{(X',p')}(u', S', v') \otimes \mathcal{G}_{(X,p)}^{(X',p')}(u, S, v) = (u \text{pr}_1, P, v' \text{pr}_2)$$

where a $Z$-element of $P$ is a pair $(x, x') \in Z S \times S'$ such that

$$vx = u' x', \quad pu x = p'v x, \quad p' u' x' = p'' v' x',$$

while

$$\mathcal{G}_{(X,p)}^{(X',p')}(u', S', v') \otimes (u, S, v) = (u \text{pr}_1, Q, v' \text{pr}_2)$$

where a $Z$-element of $Q$ is a pair $(x, x') \in Z S \times S'$ such that

$$vx = u' x', \quad pu x = p'' v' x'.$$

We see that $P$ is a subobject of $Q$ and the component of $\mu_{(X,p)}^{(X',p')}$ is the inclusion $P \hookrightarrow Q$ as a span morphism. It is now an easy calculation to see that the fusion operators are invertible: for, when $(u, S, v)$ is a span over $E$, we have the equation $pu = p'v$ which, together with equations (9.35), implies equations (9.34). So $\mathcal{G}$ is Hopf. It follows from Theorems 9.1 and 8.1 that the pseudofunctor

$$\widetilde{\mathcal{G}} : \text{Spn}(\mathcal{E}/E)\text{-Mod} \longrightarrow \text{Spn}(\mathcal{E})\text{-Mod}$$

creates left extensions and left liftings.

**Remark 9.6.** Theorem 9.1 restricts to the one-object case. In the situation of Remark 6.2, if the monoidal comonad $\mathcal{J} = \mathcal{R} \circ \mathcal{U}$ on $\mathcal{C}$ is Hopf and preserves reflexive coequalizers then the comonad $\widetilde{\mathcal{J}} = \widetilde{\mathcal{R}} \circ \widetilde{\mathcal{U}}$ in Caten is Hopf.

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