GATEWAY-LIKE ABSURDLY BENIGN TRAVERSABLE WORMHOLE SOLUTIONS

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A class of wormhole solutions is constructed that has restricted polar degrees of freedom to achieve a gateway-like configuration. This compels the use of distribution-valued metrics and connections, which further compels the use of neutrix product of distributions, to define distribution-valued curvature, the Einstein tensor, and other relevant quantities. The solution requires a space–time with non-Riemannian effects like nonmetricity to be consistent and well defined, due to the nonassociativity of the neutrix product. Finally, the ideal gateway configuration where the negative energy requirement is zero is derived.

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1. Introduction

Wormholes are a class of solutions in gravity (Einstein’s or modified) that connect two distant points in space, allowing travel in a finite coordinate time of the order of the average human lifespan. Therefore, wormholes, if plausible (and traversable), are an important mode of travel over vast interstellar distances. Wormholes were first conceived as Einstein–Rosen bridges [1], but they are not traversable due to a singularity present at their throat [2]. The first class of traversable wormhole solutions was derived by Ellis [3], Bronnikov [4], and Clément [5]. A subclass of these solutions, called absurdly benign traversable wormholes (ABTW) [6], [7], was first studied by Morris and Thorne [8]. However, wormhole solutions of this subclass require a great amount of exotic matter or negative energy in their construction [8]. Here, “absurdly benign” means that the wormhole is safe for human travel without suffering damage due to the tidal forces. ABTWs also make a frequent presence in science fiction franchises as “gateways” to different parts of the universe. However, unlike the ABTWs in literature, which are always considered to be spherically symmetric, these fictional “gateways” are not spherically symmetric. They are mostly disk-like and behave literally like a “gateway.”

In this paper, taking an inspirational note of the above, we try to develop “gateway-like” wormhole solutions by restricting the polar degrees of freedom of the simplest Morris–Thorne wormhole (MTW).
The final version is not exactly disk-like but a bulging convex section of a sphere with the throat radius determining its convexity. This restriction, however, forces us to deal with discontinuous metrics, for which an appropriate formalism is developed to work with them consistently. The metric is treated like a distribution; using the neutrix product defined on the space of distributions, the distribution-valued connections and curvature are defined. Due to the nonassociativity of the product, it is more convenient to work with the Palatini formalism of gravity and treat the connection as a separate distribution. This allows us to use the product consistently. Eventually, we work out the ideal gateway-like solution that requires no negative energy. We see in what follows that the negative energy matter is replaced with hyperfluids.

2. Wormhole basics

The simplest example of a Morris–Thorne wormhole metric is given by

\[ ds^2 = -dt^2 + dl^2 + (l^2 + r_0^2)(d\theta^2 + \sin^2 \theta \, d\phi^2) \]

where \( l \in (-\infty, \infty) \). The entrance or the throat of the wormhole lies at \( l = 0 \); \( r_0 \) is called the throat radius. With this metric, we can compute the Einstein tensor to obtain

\[
\begin{align*}
G_{tt} &= -\frac{r_0^2}{(l^2 + r_0^2)^2}, \\
G_{ll} &= -\frac{r_0^2}{(l^2 + r_0^2)^2}, \\
G_{\theta\theta} &= \frac{r_0^2}{l^2 + r_0^2}, \\
G_{\phi\phi} &= \frac{r_0^2}{l^2 + r_0^2} \sin^2 \theta.
\end{align*}
\]

Assuming that the Einstein tensor is supported by a stress–energy tensor such that the Einstein equations hold, the energy density observed by a static observer \( u^\mu = \delta^\mu_t \) is given by

\[ 8\pi \rho = 8\pi T_{\mu\nu} u^\mu u^\nu = G_{tt}. \]

We note that the energy density is negative. For completeness, we compute the total negative energy required:

\[ E = \frac{1}{8\pi} \int d^3x \sqrt{-G} G_{tt} = -\frac{\pi r_0}{2}. \]

Because the wormhole is spherically symmetric, the throat of the wormhole is spherical. This is clearly wasteful because we may not have enough negative energy to build a full spherical wormhole of the desired radius at all times. Therefore, with the expectation of being able to reduce the amount of negative energy required, we restrict the polar degrees of freedom \( \theta \) such that the wormhole more closely resembles a gateway.\(^3\)

3. Stargate metric

The polar degrees of freedom are restricted by requiring

\[
\begin{align*}
dx^2_+ &= -dt^2 + dl^2 + (l^2 + r_0^2)(d\theta^2 + \sin^2 \theta \, d\phi^2), & \theta < \theta_0, \\
dx^2_- &= -dt^2 + dl^2 + l^2(d\theta^2 + \sin^2 \theta \, d\phi^2), & \theta > \theta_0.
\end{align*}
\]

where + denotes the part of space–time where \( \theta < \theta_0 \) and – denotes the part of space–time where \( \theta > \theta_0 \), i.e., \( \mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \cup \Sigma \). Let the metric on \( \mathcal{M}^\pm \) be \( g^\pm \), where \( g^+ \) is the wormhole metric and \( g^- \) is the

\(^3\)Henceforth, the term “stargate” is used to refer to our gateway-like traversable wormhole solutions.
flat metric and $\Sigma$ is the “discontinuity surface.”\footnote{Which we also call the interface.} Now, we write the full metric $g$ on $\mathcal{M}$ as a distribution using $\Theta$-functions,

$$g_{\alpha\beta} = \Theta(f)g^+_{\alpha\beta} + \Theta(-f)g^-_{\alpha\beta}$$

(6)

where $f = \theta_0 - \theta$ and $\Sigma$ is given by $f = 0$. Due to the conical discontinuity at $\theta = \theta_0$, one may naively expect a conical singularity at the origin. But a naive computation of the curvature tensor in metric gravity shows that the only singularities are of the form $\delta(\theta - \theta_0)$, arising due to the discontinuity (see the Appendix for additional details).

Therefore, instead of the full spherical throat of the MTW, only an angular section of the throat is accessible. It is now more appropriate to call it an entrance. Now, we try all possible ways to support this solution through some matter fields. If we consider the derivative of (6), we find that

$$\partial_\gamma g_{\alpha\beta} = \Theta(f)\partial_\gamma g^+_{\alpha\beta} + \Theta(-f)\partial_\gamma g^-_{\alpha\beta} + \delta(f)n_\gamma[g_{\alpha\beta}]$$

(7)

where $n_\gamma = \partial_\gamma f$ and $[g] = g^+ - g^-$. We immediately run into problems, however, because the curvature tensor $R \sim \partial g \cdot \partial g$ leads to a product of distributions. To deal with these products consistently, we have to introduce a well-defined algebra where product among distributions and product among distributions and ordinary\footnote{By ordinary, we mean at least $C^1$ differentiable.} functions are well defined and reduce to the usual product for ordinary functions. Such a product exists and is called the neutrix product developed by van der Corput [9], Fisher [10] and Mikusiński [11]. The product is denoted by $\circ$. Some $\circ$-products of distributions are

$$\Theta^2(x) = \Theta(x) \circ \Theta(x) = \Theta(x), \quad \Theta(x) \circ \Theta(-x) = 0,$$

$$\Theta(-x) \circ \delta(x) = \Theta(x) \circ \delta(x) = \frac{1}{2}\delta(x),$$

$$\delta^2(x) = \delta(x) \circ \delta(x) = 0.$$  

These are the only ones relevant to our computation. Throughout the paper, all products of distribution-valued quantities are of the neutrix kind. However, it must be noted that this product is not associative [10]:

$$\left(\Theta(x) \circ \delta(x)\right) \circ \Theta(-x) = \frac{1}{2}\delta(x) \circ \Theta(-x) = \frac{1}{4}\delta(x), \quad \left(\Theta(x) \circ \Theta(-x)\right) \circ \delta(x) = 0.$$  

(9)

In any computation, we therefore try to avoid having neutrix products of more than two factors. A review of the neutrix products of distributions is given in Appendix A.

### 3.1. Metric-compatible connection.

We consider the distribution-valued metric defined above (throughout the paper, objects with $+$ superscripts refer to the wormhole and those with $-$ superscripts refer to the flat space)

$$g_{\alpha\beta} = \Theta(f)g^+_{\alpha\beta} + \Theta(-f)g^-_{\alpha\beta}.$$  

(10)

We consider the distribution-valued connection

$$\Gamma^\gamma_{\alpha\beta} = \Theta(f)\Gamma^+_{\alpha\beta} + \Theta(-f)\Gamma^-_{\alpha\beta} + \delta(f)\left(\frac{\bar{g}^{-1}\gamma}{2}(n_\gamma[g_{\tau\beta}] + n_\beta[g_{\tau\alpha}] - n_\tau[g_{\alpha\beta}]\right)$$

(11)

where $\Gamma^+$ is the metric connection for the MTW, $\Gamma^-$ is the metric connection for flat space in radial coordinates, and

$$\bar{g}_{\alpha\beta} = \frac{g^+_{\alpha\beta} + g^-_{\alpha\beta}}{2}, \quad [g_{\alpha\beta}] = g^+_{\alpha\beta} - g^-_{\alpha\beta}.$$  

(12)
We then find that
\[ \nabla_{\mu} g_{\alpha \beta} = 0, \quad \nabla_{\mu} \delta^{\alpha \beta} = 0, \]
showing that the distribution-valued connection is compatible with the metric (see Appendix B). We note that the connection compatible with the metric is not the Levi-Civita or the metric connection. The derivative of the connection then becomes
\[ \partial_{\delta} \Gamma^{\gamma}_{\alpha \beta} = \Theta(f) \partial_{\delta} \Gamma^{+ \gamma}_{\alpha \beta} + \Theta(-f) \partial_{\delta} \Gamma^{- \gamma}_{\alpha \beta} + \delta(f) n_{\delta} [\Gamma^{\gamma}_{\alpha \beta}] + \partial_{\delta} (\delta(f) A^{\gamma}_{\alpha \beta}), \]
where
\[ A^{\gamma}_{\alpha \beta} = \frac{(g^{-1})^{\gamma \tau}}{2} \left( n_{\alpha} [g_{\tau \beta}] + n_{\beta} [g_{\tau \alpha}] - n_{\tau} [g_{\alpha \beta}] \right) \]
and \([\Gamma] = \Gamma^{+} - \Gamma^{-}\). For clarity, we write the nonzero components of \([\Gamma], [g]; \]
\[ [\Gamma^0_\theta] = [\Gamma^0_\phi] = \frac{l}{r^2 + r_0^2 - \frac{l}{L}}, \]
\[ [g_{\theta \theta}] = r_0^2, \quad [g_{\phi \phi}] = r_0^2 \sin^2 \theta. \]
Therefore, we have
\[ R^{\mu}_{\nu \rho \sigma} = \Theta(f) R^{+ \mu}_{\nu \rho \sigma} + \Theta(-f) R^{- \mu}_{\nu \rho \sigma} + \delta(f) n_{[\rho} [\Gamma^{\mu}_{\nu]_{\sigma}] + \]
\[ + \nabla_{\tau} (\delta^{\mu}_{\rho} \delta(f) A^{\rho}_{\nu \sigma} - \delta^{\mu}_{\sigma} \delta(f) A^{\rho}_{\nu \rho}), \]
where we implicitly assume a neutrix product and use (8). It is worth noting that in the limit \( \theta_0 = \pi \), the curvature tensor becomes
\[ R^{\mu}_{\nu \rho \sigma} |_{\theta_0 = \pi} = \Theta(\pi - \theta) R^{+ \mu}_{\nu \rho \sigma} - \delta(\pi - \theta) \left( -n_{[\rho} [\Gamma^{\mu}_{\nu]_{\sigma}] + \frac{4r_0^2 \theta}{(2l^2 + r_0^2)^2} (\delta^{\rho}_{\sigma} \delta^{\theta}_{\theta} - \delta^{\rho}_{\theta} \delta^{\theta}_{\sigma}) \right), \]
and hence the MTW behavior is reproduced up to a \( \delta \)-function term. However, if we consider the Riemann curvature density of weight 1 defined by \( \tilde{R}^{\mu}_{\nu \rho \sigma} = \sqrt{g} R^{\mu}_{\nu \rho \sigma}, \) then we find that \( \tilde{R}^{\mu}_{\nu \rho \sigma} |_{\theta_0 = \pi} = \tilde{R}^{+ \mu}_{\nu \rho \sigma}, \) reproducing the MTW behavior completely. Therefore, this \( \delta \)-function term is just an artifact of the distribution-valued curvature and is not a real singularity. Because we only deal with double products of distribution-valued connections, Riemann tensor (18) is well defined. However, to compute the Einstein tensor \( G_{\mu \nu}, \) we have to contract the above with \( g_{\mu \nu} \circ g^{\rho \sigma}, \) and the third term in (17) then leads to a product of distributions of the form
\[ \Theta(x) \circ \delta(x) \circ \Theta(-x), \]
which we have shown to be inconsistent. Hence, the Einstein tensor is ill defined due to the nonassociativity of the product. This means that the stargate metric cannot be supported by a metric-compatible connection.

3.2. Nonmetricity. We saw in the preceding section that due to the nonassociativity of the neutrix product, we were unable to consistently define the Einstein tensor. Due to this nonassociativity, it is convenient to define the metric and the connection as independent distributions, which is precisely the Palatini formalism of general relativity. Henceforth, we abandon the metricity condition altogether and consider the connection
\[ \Gamma^{\gamma}_{\alpha \beta} = \Theta(f) \Gamma^{+ \gamma}_{\alpha \beta} + \Theta(-f) \Gamma^{- \gamma}_{\alpha \beta}, \]
for which we find that
\[ \nabla_{\mu} g_{\alpha \beta} = \delta(f) n_{\mu} [g_{\alpha \beta}] \]
which is clearly not metric compatible. The Riemann tensor is then given by

$$R^\mu_{\nu\rho\sigma} = \Theta(f) R^+_{\nu\rho\sigma} + \Theta(-f) R^-_{\nu\rho\sigma} + \delta(f) n_{[\mu} [\Gamma^\mu_{\sigma]_{\nu}]}. \quad (22)$$

Here as well, in the limit \(\theta_0 = \pi\), the curvature tensor reproduces the MTW behavior up to a \(\delta\)-function term:

$$R^\mu_{\nu\rho\sigma}|_{\theta_0=\pi} = \Theta(\pi - \theta) R^+_{\nu\rho\sigma} + \delta(\pi - \theta) n_{[\mu} [\Gamma^\mu_{\sigma]_{\nu}]].$$

However, we can again see that \(\tilde{R}^\mu_{\nu\rho\sigma}|_{\theta_0=\pi} = \tilde{R}^+_{\nu\rho\sigma}\), hence reproducing the MTW behavior completely.

### 3.2.1. Energy requirement

We consider the distribution-valued Riemann curvature tensor (22), where we contract \(\mu\) and \(\rho\) indices to obtain a distribution-valued Ricci tensor

$$R_{\nu\sigma} = \Theta(f) R^+_{\nu\sigma} + \Theta(-f) R^-_{\nu\sigma} + \delta(f) (n_{[\mu} [\Gamma^\mu_{\sigma]_{\nu}] - n_{[\sigma} [\Gamma^\mu_{\mu]_{\nu}]}). \quad (23)$$

Now, to obtain the Ricci scalar, we have to evaluate the product of distributions \(g^{\nu\sigma} \circ R_{\nu\sigma}\). This product exists and is given by

$$R \equiv g^{\nu\sigma} \circ R_{\nu\sigma} = \Theta(f) R^+ + \Theta(-f) R^-.$$ \hspace{1cm} (24)

The \(\delta\)-function part vanishes (see Eq. (31)). To compute the Einstein tensor, we need the product \(g_{\nu\sigma} \circ R\), or more explicitly, \(g_{\nu\sigma} \circ g^{\alpha\beta} \circ R_{\alpha\beta}\). This is a triple neutrix product. Because the product is nonassociative, a triple product may not have a consistent value. However, Ricci scalar (24) only has the \(\Theta\) functions, over which the neutrix products are always associative. Therefore, in this case,

$$(g_{\nu\sigma} \circ g^{\alpha\beta}) \circ R_{\alpha\beta} = g_{\nu\sigma} \circ (g^{\alpha\beta} \circ R_{\alpha\beta}) = g^{\alpha\beta} \circ (g_{\nu\sigma} \circ R_{\alpha\beta}) = g_{\nu\sigma} \circ R \hspace{1cm} (25)$$

which means that \(g_{\nu\sigma} \circ R\) exists and hence the distribution-valued Einstein tensor exists as well,\(^6\)

$$G_{\nu\sigma} = \Theta(f) G^+_{\nu\sigma} + \Theta(-f) G^-_{\nu\sigma} + \delta(f) (n_{[\mu} [\Gamma^\mu_{\sigma]_{\nu}] - n_{[\sigma} [\Gamma^\mu_{\mu]_{\nu}]]) \hspace{1cm} (26)$$

where

$$G_{\mu\nu} = R_{(\mu,\nu)} - \frac{1}{2} g_{\mu\nu} R. \hspace{1cm} (27)$$

Assuming that the above is sourced by the stress–energy tensor \(T_{\mu\nu}\), the energy density observed by a static observer \(u^\mu = \delta^\mu_t\) is given by

$$8\pi \rho = 8\pi T_{\mu\nu} u^\mu u^\nu = G_{tt}. \hspace{1cm} (28)$$

Because \(G_{tt}\) only has \(\theta\)-function terms, any product with other \(\theta\)-function-based distribution is associative. It is easy to see that

$$\sqrt{h} = \Theta(f) \sqrt{h^+} + \Theta(-f) \sqrt{h^-} \hspace{1cm} (29)$$

where \(h_{\mu\nu}\) is the metric on a constant-time hypersurface. Therefore, the total energy is given by

$$E = \frac{1}{8\pi} \int d^3x \sqrt{h} \cdot G_{tt} = \frac{1}{8\pi} \int d^3x \Theta(f) \sqrt{h} \cdot G^t_t =$$

$$= \frac{r_0}{8} \int_{\theta < \theta_0} d\Omega^2_S = -\frac{\pi r_0}{2} \sin^2 \frac{\theta_0}{2} \hspace{1cm} (30)$$

because \(G^t_t = 0\). We note that the negative energy is more significant than that for the simplest Morris–Thorne wormhole. This seems to imply some significant and promising progress.

\(^6\)We note that the vanishing of the \(\delta\) function leads to an associative triple neutrix product allowing an unambiguous computation of the Einstein tensor. This is the reason that makes the nonmetricity setting preferable to the metric-compatible case.
3.2.2. Interface stress–energy tensor. However, if we explicitly compute the $\delta$-function component of Einstein tensor (26) using (16), we find that

$$G_{\nu\sigma}\big|_\delta = n_\mu [\Gamma^\mu_{\sigma\nu} - n_\nu [\Gamma^\mu_{\nu}]_{\mu} = 0.$$ \hfill (31)

The Palatini space–time is also endowed with a hypermomentum tensor density or a superpotential

$$-\nabla_\lambda (\sqrt{-g} g^\mu\nu) + \nabla_\sigma (\sqrt{-g} g^{\sigma\mu}) = -\Delta^\mu\nu.$$ \hfill (32)

This superpotential is a consequence of the variation due to the connection:

$$\Delta^\mu\nu = -\frac{\delta S}{\delta \Gamma^\lambda_{\mu\nu}}.$$ \hfill (33)

More explicitly, in our case,

$$\Delta^\mu\nu = \delta_0 (f) [\delta^\lambda_{\mu} (-\delta^\mu_\nu + \delta^\nu_\nu)] r^2 \sin \theta_0.$$ \hfill (34)

This hypermomentum can be sourced using some hyperfluid $\Delta^\lambda_{\mu\nu}$, which is a special kind of fluid that has intrinsic spin and other microstructures [12]–[14]. The hyperfluids obey a continuity equation in the absence of torsion [12],

$$\tilde{\nabla}^\mu T_{\mu\nu} + \nabla_\rho \nabla_\sigma \Delta^\rho\sigma - R^\lambda_{\rho\sigma\nu} \Delta^\rho\sigma = 0.$$ \hfill (35)

where $\tilde{\nabla}$ is the covariant derivative with respect to the metric connection and $T_{\mu\nu}$ is the stress–energy tensor density. However, it turns out that the continuity equation is not satisfied in this configuration (see E.2). Therefore, such a configuration, promising as it may be, is unphysical. Also, there are no geodesics that pass through the interface (see Sec. C). Hence, the interface in this case is actually nontraversable.

4. The ideal stargate

An ideal stargate configuration should have the minimum (ideally, zero) negative energy requirement and a traversable interface, and the matter supporting the configuration should be physical. Despite the lack of success in the preceding section, there is still hope in abandoning metric compatibility, because we have many suitable connection choices that can give rise to the desired stargate configuration. We find that this is most ideally achieved by the choice

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^{-\gamma}_{\alpha\beta} \quad \text{(flat-connection)}$$ \hfill (36)

with the metric still given by (10). If we look at the equations of motion due to the Palatini variation for a torsion-free connection

$$R_{(\mu,\nu)} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} - \nabla_\lambda (\sqrt{-g} g^{\mu\nu}) + \nabla_\sigma (\sqrt{-g} g^{\sigma (\mu}) = -\Delta^\mu\nu,$$ \hfill (37)

then, with connection (36) and metric (10), we have\hfill (38)

$$T_{\mu\nu} = 0,$$

$$\Delta^\mu\nu = \delta_0 (f) [\delta^\lambda_{\mu} (-\delta^\mu_\nu + \delta^\nu_\nu)] r^2 \sin \theta_0 + \Theta (f) \chi^\mu\nu,$$ \hfill (39)

$$\chi^\mu\nu = 2 \frac{r^2}{l} \sin \theta [\delta^\lambda_{\mu} (-\delta^\mu_\nu + \delta^\nu_\nu) - \delta^\nu_\nu \delta^\mu_\lambda],$$ \hfill (40)

\hfill (40)

In the case $\theta_0 = 0$, we obtain the usual flat space, while for $\theta_0 = \pi$, we obtain the full spherical wormhole completely supported by the hyperfluids (see Appendix D for additional details).
which can be made to work given the appropriate hyperfluid. And, in contrast to Sec. 3.2, continuity equation (35) is trivially satisfied by this configuration (see E.3). This choice means not only that there is no need for negative energy but also that the geodesic equation has consistent solutions through the interface. And most importantly, there are no tidal forces because there is no curvature, i.e., we have a “benign” stargate configuration (see Sec. D). If we allow torsion, then

$$\Gamma^\gamma_{\alpha\beta} = (\Lambda^{-1})^\gamma_\rho \partial_\beta \Lambda^\rho_\alpha$$  \hspace{1cm} (inertial-connection) \hspace{1cm} (41)$$

where $\Lambda \in GL(4, \mathbb{R})$ also gives the zero-energy configuration while still satisfying the continuity equation. However, the geodesic deviation equation acquires contributions due to the torsion, and therefore the stargate solutions may not be “benign.”

\textbf{Hyperfluid source.} Hyperfluids are usually thought to be ordinary fluids with some microstructure inherent to them. For instance, a fluid made of fermions has a spin microstructure. The macroscopic properties of the fluid source the stress–energy tensor $T_{\mu\nu}$, while the microstructures of the fluid source $\Delta^{\mu}_{\nu\lambda}$. We can also imagine different kinds of fluids with very exotic microstructures depending on their fundamental constituents, for instance, strings, quarks, etc. However, the hyperfluid required in our ideal stargate configuration must have a zero stress–energy tensor and a nonzero hypermomentum. This is only possible if somehow the hyperfluid of our interest does not couple to the metric and only couples to the connection. It is very difficult to propose an action that can allow that and even more difficult to motivate such an action. As an example, however, we consider an additional interaction of the form$^8$

$$S_{\text{int}} = \eta \int d^4x \sqrt{\det(m_{\mu\nu} + \alpha R_{\mu\nu})} + \beta \int d^4x \sqrt{m} + \gamma \int d^4x \sqrt{\det(R_{\mu\nu})},$$  \hspace{1cm} (42)$$

where $m_{\mu\nu}$ is an auxiliary tensor field and $\eta$, $\beta$, and $\gamma$ are coupling constants. The equation of motion for $m_{\mu\nu}$ and a torsionless $\Gamma^{\mu}_{\nu\lambda}$ is given by

$$\left(\eta \frac{\sqrt{\det(m_{\mu\nu} + \alpha R_{\mu\nu})}}{\sqrt{m}} + \beta\right) m_{\mu\nu} = -\alpha \beta R_{\mu\nu},$$

$$\eta \Pi^{\mu}_{\lambda}(m, \Gamma) + \gamma \Pi^{\mu\nu}(0, \Gamma) = -\Delta^{\mu\nu}_{\lambda},$$

$$\Pi^{\mu\nu}(m, \Gamma) = -\nabla_{[\lambda} \left[ \frac{\sqrt{\det(m_{\rho\sigma} + \alpha R_{\rho\sigma})}}{(m + \alpha R)^{-1}} \right]^{[\nu]}_{\mu] \delta_{\lambda}^\rho] +$$

$$+ \nabla_{\sigma} \left[ \frac{\sqrt{\det(m_{\rho\sigma} + \alpha R_{\rho\sigma})}}{(m + \alpha R)^{-1}} \right]^{[\nu]}_{\mu] \delta_{\lambda}^\rho],$$  \hspace{1cm} (43)$$

where $\Delta^{\mu\nu}_{\lambda}$ is given by (37). It is not very difficult to see that for $\beta = -\eta$ and $\gamma = 0$, $m_{\mu\nu} = g_{\mu\nu}$ solves the above equation of motion, i.e., $g_{\mu\nu}$ is a stargate solution and $\Gamma^{\mu}_{\nu\lambda}$ is a flat connection. What we see in (42) is an interaction that only involves the connection and it helps source a nonzero hypermomentum with zero stress–energy tensor. However, it seems quite difficult to motivate such interactions from known phenomena or theories of gravity. It might be possible, however, to eliminate or constrain such kind of interactions by considering no-ghost theorems and locality.

5. Discussion and conclusion

We started with a metric-compatible connection and tried to accommodate what is essentially a discontinuous metric. After a series of trials and errors, we finally settled with a nonmetric space–time with a flat connection to achieve a “benign” stargate configuration. The discontinuous metric and nonmetricity are

$^8$This is very reminiscent of Born–Infeld gravity [15].
non-Riemannian effects, and it is therefore unsurprising that the stargate solution is eventually supported
by hyperfluids that they have all the necessary microstructures to source such non-Riemannian degrees
of freedom [13], [14]. One can in principle develop a class of stargate configurations where the need for
hyperfluids may be further reduced to the vicinity of the “entrance.” However, that is a repetitive and
perhaps, not so illuminating exercise.

We also saw that it was a bad idea to support discontinuous metrics with a distribution-valued connection
(or more simply, discontinuous connections) because in presence of such connections, geodesics (metric
or affine) through the interface are absent. Trivial accessibility to the stargate is important for its utility
and ease of use. Therefore, it seems any stargate solution must necessarily have a continuous connection to
ensure that.

Also, the matter fields required must at least satisfy the continuity equation. The hyperfluid sources
supporting a zero-energy stargate are peculiar enough already because such sources must only couple to the
connection and not the metric. We made attempts to exemplify such a scenario using (42), but it lacked
any fundamental or phenomenological motivation. However, it can be checked whether such interactions
are free of ghosts or are local to ascertain their sensibility. Such examples, if sensible, can source hyperfluids
for ideal stargates or wormholes. It is possible to source such kinds of hyperfluids via the space–time itself.
The existence of quantum gravity necessitates an underlying space–time microstructure and vice versa.
Such microstructures would provide some corrections to the usual gravitational action, perhaps of the kind
we worked with in (42). Maybe this is the phenomenological motivation we have been missing. This line of
thought, however, is beyond the scope of this paper and will be seriously considered in our future studies.

Having alluded to a possibility of operating wormholes without negative energy, we now, finally, have
to confront the problem of wormhole construction. Even if we have all the exotic matter needed, it is
not clear how to use the exotic matter to assemble a wormhole. Also, every step of its assembly process
must be compatible with general relativity, and therefore, what we are finally looking at is a time-dependent
solution from its construction to deconstruction after its use. Also, such a wormhole assembly requires us to
manipulate topology, which is forbidden by classical general relativity, which is fundamentally Riemannian
in nature [8]. That is where the non-Riemannian nature of our solution and the sources that support it may
prove to be useful. This is because we can in principle consider the stargate solution as an intermediate
step in the full spherical wormhole assembly. How this step can be initiated from scratch is again a topic
beyond the scope of this paper and will therefore be considered seriously in our future studies.

Appendix A: Review of product of distributions

We review the product of distributions developed by van der Corput [9], Fisher [10]. Let $\rho$ be a fixed
infinitely differentiable function such that

$$
\rho(x) = 0 \text{ for all } |x| \geq 1, \quad \rho(x) \geq 0, \quad \rho(x) = \rho(-x), \quad \int_{-1}^{1} \rho(x) dx = 1. \quad (A.1)
$$

We define a function $\delta_n(x)$ by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$; $\{\delta_n\}$ is now a sequence of infinitely differentiable functions converging to the delta function $\delta(x)$ as $n \to \infty$. We consider an arbitrary distribution $f$ and define

$$
f_n(x) = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t) dt, \quad n = 1, 2, \ldots. \quad (A.2)
$$

Then $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution $f$.

**Definition 1.** Let $f$ and $g$ be arbitrary distributions and let $f_n = f * \delta_n, g_n = g * \delta_n$. Then the product of $f$ and $g$ exists and is equal to the distribution $h = f \cdot g$ on the open interval $(a, b)$, where $-\infty \leq a < b \leq \infty$, if and only if $\{f_n \cdot g_n\}$ is a regular sequence converging to $h$ on an open interval $(a, b)$.
**Definition 2.** [9]. Let \( f \) and \( g \) be arbitrary distributions and let \( f_n = f * \delta_n \), \( g_n = g * \delta_n \). Then the neutrix product of \( f \) and \( g \) exists and is equal to the distribution \( h = f \circ g \) on the open interval \((a, b)\), where \(-\infty \leq a < b \leq \infty\), if and only if

\[
\text{N-lim}_{n \to \infty} f_n g_n = h
\]  

where \( \text{N-lim} \) is called the neutrix limit, \( N \) is the neutrix, having the domain \( N' = \{1, 2, \ldots\} \) and the range \( N'' \) the real numbers, with negligible functions finite linear sums of the functions \( n^\lambda \ln^{r-1} n, \ln^r n \) for \( \lambda \geq 0 \) and \( r = 1, 2, \ldots \) and all functions that converge to zero in the normal sense as \( n \to \infty \).

We note that if \( \lim_{n \to \infty} f_n g_n = h \), then \( f \circ g \) reduces to \( f \cdot g \), and hence the neutrix product is in a sense a generalization of the product in Definition 1. Hence, we can state the following theorem due to Fisher–Lin–Zhi [16].

**Theorem 1.** Let \( f \) and \( g \) be arbitrary distributions and let \( f \cdot g \) exist and be equal to \( h \). Then the neutrix product \( f \circ g \) exists and is equal to \( h \).

Using the above definition of the product of distributions, we can prove several theorems involving products of distributions, such as a theorem by Fisher [10].

**Theorem 2.** Let \( x_\lambda^+ = x^{\lambda} \Theta(x) \) and \( x_\lambda^- = |x|^{\lambda} \Theta(-x) \). Then the product \( x_\lambda^+ \cdot x_\mu^- \) exists and is given by

\[
x_\lambda^+ \cdot x_\mu^- = 0 \quad \text{if} \quad \lambda, \mu, \lambda + \mu > -1.
\]

Also,

\[
x_\lambda^+ \cdot \delta^{(r)}(x) = \frac{(-1)^r r!}{2} \delta(x), \quad r = 0, 1, 2, \ldots,
\]

where \( \delta^{(r)} \) is the \( r \)th derivative of the delta function.

We also quote some useful theorems by Mikusiński [11] and Koh–Kuan [17].

**Theorem 3.** We have the equality \( \delta(x) \cdot x^{-1} = -\frac{1}{2} \delta'(x) \).

**Theorem 4.** Under a neutrix product, we have

\[
\delta^{2l+1}(x) = 0, \quad \delta^{2l+1}(x) = C_l \delta^{(2l)}(x), \quad l = 0, 1, 2, \ldots,
\]

where

\[
C_l = \frac{1}{2^l l! (2l + 1)^{l+1/2} \pi^l}.
\]

Using the above theorems, the following corollary is easy to show.

**Corollary 1.** Setting \( \lambda = \mu = 0 \) in Fisher’s theorem, we obtain \( \Theta(x) \cdot \Theta(-x) = 0 \), and setting \( r = 0 \), we obtain

\[
\Theta(x) \cdot \delta(x) = \frac{1}{2} \delta(x).
\]

Another product of distributions that we need is \( \Theta(x)^2 = \Theta(x) \cdot \Theta(x) = \Theta(x) [10] \), by definition of the Heaviside \( \Theta \) function. Now, we have all the distributions we need, with their well-defined products.
Appendix B: Metric-compatible connection for a distribution-valued metric

We consider the metric (10)
\[ g_{\alpha \beta} = \Theta(f)g^+_{\alpha \beta} + \Theta(-f)g^-_{\alpha \beta}. \] (B.1)

For the distribution-valued connection
\[ \Gamma^\gamma_{\alpha \beta} = \Theta(f)\Gamma^\gamma_{\alpha \beta}^+ + \Theta(-f)\Gamma^\gamma_{\alpha \beta}^- + \delta(f)\frac{(g^{-1})^{\gamma \tau}}{2}(n_\alpha [g_\tau \beta] + n_\beta [g_\tau \alpha] - n_\tau [g_{\alpha \beta}]), \] (B.2)
we have
\[ \nabla_\mu g_{\nu \lambda} = \delta(f)n_\mu [g_{\nu \lambda}] - g_{\tau \lambda} \circ \delta(f)\frac{(g^{-1})^{\tau \delta}}{2}(n_\nu [g_\delta \mu] + n_\nu [g_\delta \mu] - n_\delta [g_{\nu \mu}]) - \]
\[ - g_{\tau \nu} \circ \delta(f)\frac{(g^{-1})^{\tau \delta}}{2}(n_\tau [g_\delta \lambda] + n_\lambda [g_\delta \mu] - n_\delta [g_{\lambda \mu}]). \] (B.3)

Because the connections \( \Gamma^\pm \) are compatible with the metric \( g^\pm \), the \( \theta \) functions do not appear in the above. Now, it can be shown that the \( \delta \)-function part cancels. Using the product of distributions listed in (8), \( g_{\mu \nu} \circ \delta(f) = \bar{g}_{\mu \nu} \delta(f) \), we have \( \nabla_\mu g_{\nu \lambda} = 0 \), where
\[ \bar{g}_{\lambda \tau}(g^{-1})^{\tau \delta} = \delta_\lambda^\delta \] (B.4)
was also used. Now, it can be verified that
\[ g^{\alpha \beta} = \Theta(f)(g^+)^{\alpha \beta} + \Theta(-f)(g^-)^{\alpha \beta} \] (B.5)
is the inverse of (10). Similarly,
\[ \nabla_\mu g^{\nu \lambda} = \delta(f)n_\mu [g^{\nu \lambda}] + g^{\lambda \tau} \circ \delta(f)\frac{(g^{-1})^{\nu \delta}}{2}(n_\tau [g_\delta \mu] + n_\mu [g_\delta \tau] - n_\delta [g_{\tau \mu}]) + \]
\[ + g^{\nu \tau} \circ \delta(f)\frac{(g^{-1})^{\nu \delta}}{2}(n_\tau [g_\delta \mu] + n_\mu [g_\delta \tau] - n_\delta [g_{\tau \mu}]). \] (B.6)

Using the product of distributions, we obtain
\[ \nabla_\mu g^{\nu \lambda} = \delta(f)n_\mu [g^{\nu \lambda}] + g^{\lambda \tau} \delta(f)\frac{(g^{-1})^{\nu \delta}}{2}(n_\tau [g_\delta \mu] + n_\mu [g_\delta \tau] - n_\delta [g_{\tau \mu}]) + \]
\[ + g^{\nu \tau} \delta(f)\frac{(g^{-1})^{\nu \delta}}{2}(n_\tau [g_\delta \mu] + n_\mu [g_\delta \tau] - n_\delta [g_{\tau \mu}]). \] (B.7)

Noting that \( \bar{g}_{\lambda \tau}(g^{-1})^{\tau \delta} = -[g^{\lambda \tau}]^{-1} [g^{\nu \delta}]^{-1} \), where \([g]^{-1}\) is the inverse of \([g]\), we can cancel the \( \delta \)-function contribution and obtain \( \nabla_\mu g^{\nu \lambda} = 0 \), which shows that (B.2) is a metric-compatible connection.

Appendix C: Geodesics through the interface for distribution-valued connections

To access the stargate, the interface must be traversable. This means that there must exist geodesics through the interface. We consider the geodesic equation for a metric connection given by
\[ \frac{d^2 x^\mu}{d \tau^2} + \Gamma^\mu_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta = 0 \] (C.1)

Footnote: For any object \( A \) with a discontinuity, on the “discontinuity surface” \( \Sigma \) we define \( \bar{A} = (A^+ + A^-)/2 \), \([A] = A^+ - A^- \), where \( A^+ \) and \( A^- \). \([A]\) is the discontinuity in \( A \) across \( \Sigma \) and \( \bar{A} \) is the mean on \( \Sigma \).
where in our case the metric connection is given by
\[
\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Theta(f)\Gamma^{\gamma+}_{\alpha\beta} + \Theta(-f)\Gamma^{-\gamma}_{\alpha\beta} + \delta(f)\tilde{\gamma}^{\gamma\tau}_{\tau} \left( n_{\alpha}[g_{\tau\beta}] + n_{\beta}[g_{\tau\alpha}] - n_{\tau}[g_{\alpha\beta}] \right). \tag{C.2}
\]

Because we seek a path through the interface, we consider \( \dot{\phi} = 0 \), i.e., \( \phi = \Phi = \text{const.} \). Then
\[
\ddot{t} = 0, \quad \ddot{l} - l\dot{\theta}^2 = 0, \tag{C.3}
\]
\[
\ddot{\theta} + \left( \Theta(f)\frac{l}{l^2 + r_0^2} + \Theta(-f)\frac{1}{l} \right) \dot{l} \dot{\theta} + \frac{1}{2} \delta(f)\dot{\theta}^2 \left( \frac{r_0^2}{l^2 + r_0^2} + \frac{r_0^2}{l^2} \right) = 0. \tag{C.4}
\]

Due to the triple distribution product in the second term in (C.4), there are no consistent nontrivial solutions to the above. Therefore, there are no metric geodesics that cross the interface. Because all distribution-valued connections also face the same issue, there are no affine or metric geodesics through the interface for any distribution-valued connections.

**Appendix D: Tidal forces in the ideal stargate**

Because the ideal stargate configuration arises for a uniform connection, the affine geodesic equation has consistent solutions through the interface. However, because the metric is still distribution-valued, there are no consistent metric geodesics that go through the interface. We now consider the family of affine geodesics parameterized by \( \tau \). The equation for geodesic deviation then has the form
\[
\frac{D^2\xi^\mu}{D\tau^2} - R^\mu_{\nu\rho\sigma} u^\nu \xi^\sigma = 0, \tag{D.1}
\]
where \( D/D\tau = u^\mu \Delta \mu \) and \( \xi^\mu \) is the deviation vector. Because we are only interested in spatial deviations of the geodesics, we introduce the projector
\[
h^\mu_{\nu} = \delta^\mu_{\nu} - u^\mu u^\nu u^2 \tag{D.2}
\]
where the indices are raised and lowered using (10). We then define
\[
\eta^\nu \equiv \xi^\mu h^\mu_{\nu} = \xi^\nu - u^\nu \xi^\mu u^\mu u^2 \tag{D.3}
\]
where we have used \( Du^\mu/D\tau = 0 \) because \( u^\mu = dx^\mu/d\tau \) is the tangent to the affine geodesic. Using the above, we find that
\[
\frac{D\xi^\nu}{D\tau} = \frac{D\eta^\nu}{D\tau} + u^\nu \frac{D}{D\tau} \left( \frac{\xi^\mu u^\mu}{u^2} \right). \tag{D.4}
\]

Now, we introduce the tetrads such that
\[
e^0_\mu = \frac{u^\mu}{|u|}, \quad e^a_\mu e^b_\nu \delta_{ab} = -h^\mu_{\nu}, \tag{D.5}
\]
which allows writing (D.4) as
\[
e^0_\nu \frac{D\xi^\nu}{D\tau} = e^0_\nu \frac{D\eta^\nu}{D\tau} \tag{D.6}
\]
where we used \( e^a_\nu u^\nu = e^a_\nu e^0_\mu \delta_{\mu\nu} |u| = 0 \). Using the Liebniz rule, we find that
\[
\frac{D\xi^a}{D\tau} - \xi^\nu \frac{De^0_\nu}{D\tau} = \frac{D\eta^a}{D\tau} - \eta^\nu \frac{De^0_\nu}{D\tau}. \tag{D.7}
\]

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With (D.3), we obtain
\[
\frac{D\xi^a}{D\tau} = \frac{D\eta^a}{D\tau} + u^\nu \left( \frac{x^{\mu} u_\mu}{u^2} \right) \frac{D\nu}{D\tau}.
\] (D.8)

Using the identity
\[
0 = \frac{D}{D\tau} (e^a_\nu u^\nu) = u^\nu \frac{D e^a_\nu}{D\tau}
\] (D.9)
we finally have
\[
\frac{D\xi^a}{D\tau} = \frac{D\eta^a}{D\tau}
\] (D.10)
which may be used to rewrite the geodesic deviation equation as
\[
\frac{D^2\eta^a}{D\tau^2} + K^a_b \eta^b = 0, \quad K^a_b = -R^a_{\nu\rho\sigma} e^\nu_\mu v^\rho v^\sigma e^\mu_b.
\] (D.11)

For a flat-connection, as for the ideal stargate configuration, \( K^a_b = 0 \), and therefore no tidal forces are present,
\[
\frac{D^2\eta^a}{D\tau^2} = 0
\] (D.12)
showing that the ideal stargate configuration is also “absurdly benign.”

Appendix E: Continuity equation

E.1. Derivation. We present a much simpler version of the derivation given by Iosifidis [12]. We consider a matter action \( S_m \) that is general coordinate invariant. Given an arbitrary translation of coordinates \( x^\mu \rightarrow x^\mu + \xi^\mu \), we require
\[
\delta \xi S_m = \int d^4x \left[ \frac{\delta L}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta L}{\delta \Gamma^\lambda_{\mu\nu}} \delta \Gamma^\lambda_{\mu\nu} \right] = 0.
\] (E.1)
Assuming that the connection is torsion-free, it can be shown that
\[
\delta \xi g_{\mu\nu} = -2\tilde{\nabla}_{(\mu} \xi_{\nu)}, \quad \delta \xi \Gamma^\lambda_{\mu\nu} = \xi^\mu R^\lambda_{\nu(\mu\nu}\alpha} - \nabla_{(\nu} \nabla_{\mu)\xi}^\lambda,
\] (E.2)
where \( \tilde{\nabla} \) is the covariant derivative corresponding to the metric connection. Using this in (E.1), we obtain
\[
\delta \xi S_m = \int d^4x \left[ -2\mathcal{T}^{\mu\nu} \tilde{\nabla}_{\mu} \xi_{\nu} - \Delta^\mu_{\nu} \left( \xi^\alpha R^\lambda_{\mu\nu\alpha} - \nabla_{\nu} \nabla_{\mu} \xi^\lambda \right) \right],
\] (E.3)
where we used the definitions
\[
\mathcal{T}^{\mu\nu} := \frac{\delta L}{\delta g_{\mu\nu}}, \quad \Delta^\mu_{\nu} := -\frac{\delta L}{\delta \Gamma^\lambda_{\mu\nu}}.
\] (E.4)
where \( \mathcal{T} \) is the stress–energy density and \( \Delta^\lambda_{\mu\nu} \) is the hypermomentum density. After integration by parts, we obtain
\[
\tilde{\nabla}_{\mu} \mathcal{T}^{\mu\nu} + \nabla_{\rho} \nabla_{\sigma} \Delta^\rho_{\nu} - R^\lambda_{\rho\sigma\nu} \Delta^\rho_{\nu} = 0.
\] (E.5)
E.2. Unphysical stargate configuration: \( T_{\mu \nu} \neq 0, \Delta_{\nu}^{\nu} \neq 0 \). For (32), it can be shown that the identity \( \tilde{\nabla}_{\mu} \Delta_{\lambda}^{\mu
u} = -\sqrt{-g}g^{\nu \mu}(\tilde{R}_{\mu \lambda} + R_{\lambda \mu}) \) (E.6) holds, where \( \tilde{R}_{\mu} = R_{\mu \nu} g^{\nu \alpha}, \tilde{\nabla}_{\mu} = 2S_{\mu} - \nabla_{\mu} \). However, in our case, connection (20) leads to \( \nabla_{\mu} \Delta_{\lambda}^{\mu \nu} = 0 \). Hence, the continuity equation simply reduces to

\[ (E.7) \]

\[ \hat{\nabla}^{\mu} T_{\mu \nu} - R_{\rho \sigma \nu}^{\lambda} \Delta_{\rho \sigma}^{\lambda} = 0. \]

The metric connection corresponding to (10) is just

\[ (E.8) \]

\[ \tilde{\Gamma}_{\alpha \beta}^{\gamma} = \Theta ((f)\Gamma_{\alpha \beta}^{\gamma} + \Theta ((-f)\Gamma_{\alpha \beta}^{\gamma} + \delta(\bar{f}\rho_{\gamma}) (n_{\alpha} g_{\gamma \beta} + n_{\beta} g_{\gamma \alpha} - n_{\gamma} g_{\alpha \beta})). \]

Using this in (E.7), we obtain

\[ (E.9) \]

\[ \hat{\nabla}^{\mu} T_{\mu \nu} - R_{\rho \sigma \nu}^{\lambda} \Delta_{\rho \sigma}^{\lambda} = \delta(\bar{f}) r_{0}^{2} \sin \theta \left[ 8 l^{6} + 4 l^{4} r_{0}^{2} - r_{0}^{6} \right] \frac{8 l^{4} (l^{2} + r_{0}^{2})^{2}}{8 l^{4} (l^{2} + r_{0}^{2})^{2}} \neq 0. \]

Therefore, there are no physical configurations of fluid and hyperfluid that can give rise to this stargate configuration.

E.3. Ideal stargate configuration: \( T_{\mu \nu} = 0, \Delta_{\nu}^{\nu} \neq 0 \). The ideal stargate configuration is achieved when \( \Gamma_{\nu \lambda}^{\mu} = \Gamma_{\nu \lambda}^{-\mu} \) throughout the space–time. This leads to the vanishing of the Riemann tensor and commutativity of the covariant derivatives, which due to (E.6) implies \( \nabla_{\mu} \Delta_{\lambda}^{\mu \nu} = 0 \) and the continuity equation is therefore trivially satisfied.

Appendix F: Limit behavior of the physical stargate configuration

We discuss various limit cases of the stargate solution. In the limit case \( \theta_{0} = 0 \), Eqs. (38) and (39) give

\[ (F.1) \]

\[ T_{\mu \nu} |_{\theta_{0}=0} = 0, \quad \Delta_{\lambda}^{\mu \nu} |_{\theta_{0}=0} = 0. \]

This is not surprising because flat space is assumed. In the limit case \( \theta_{0} = \pi \),

\[ (F.2) \]

\[ T_{\mu \nu} |_{\theta_{0}=\pi} = 0, \quad \Delta_{\lambda}^{\mu \nu} |_{\theta_{0}=\pi} = 2 \Theta (f) \frac{r_{0}^{2}}{l} \sin \theta \left[ \delta_{\alpha}^{\beta} (\delta_{\lambda}^{\mu} \delta_{\gamma}^{\nu} + \delta_{\lambda}^{\nu} \delta_{\gamma}^{\mu}) - \delta_{\alpha}^{\lambda} \delta_{\gamma}^{\mu} \right] = 2 \frac{r_{0}^{2}}{l} \sin \theta \left[ \delta_{\alpha}^{\beta} (\delta_{\lambda}^{\mu} \delta_{\gamma}^{\nu} + \delta_{\lambda}^{\nu} \delta_{\gamma}^{\mu}) - \delta_{\alpha}^{\lambda} \delta_{\gamma}^{\mu} \right]. \]

This is the full spherical Morris–Thorne wormhole that is supported entirely by a hypermomentum and has zero negative energy. This hypermomentum may be sourced by some hyperfluid as discussed in Sec. 4.

Appendix G: Lack of conical singularities

We consider the metric

\[ (G.1) \]

\[ ds^{2} = -dt^{2} + dl^{2} + [l^{2} + r_{0}^{2} h(\theta)](d\theta^{2} + \sin^{2} \theta d\phi^{2}) \]
(where for \( h(\theta) = \Theta(f) \), (5) can be recovered; for now, \( h(\theta) \) is considered arbitrary). One of the Ricci tensor components in metric gravity is given by

\[
R_{\theta\theta} = \frac{l r_0^2 h'(\theta)}{[l^2 + r_0^2 h(\theta)]^2}. \tag{G.2}
\]

This component is suspicious. Indeed, using

\[
\lim_{z \to 0} z^{\Delta - d} \frac{z^\Delta}{(z^2 + x_\mu x^\mu)^\Delta} = \frac{\delta^{(d)}(x)}{C\Delta}, \quad C\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)}, \tag{G.3}
\]

where \( d \) is the dimension of the coordinate \( x^\mu \), we see that as \( l \to 0 \),

\[
R_{\theta\theta} \overset{l \to 0}{=} \frac{\pi r_0 h'(\theta)}{l^2 + r_0^2 h(\theta)} \delta(\sqrt{h(\theta)}) \tag{G.4}
\]

with a naive \( \delta \)-function singularity at the origin. Now, we use the identity

\[
\delta(f(x)) = \sum_{x_i: f(x_i) = 0} \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{G.5}
\]

whence we have

\[
R_{\theta\theta} \overset{l \to 0}{=} 2\pi r_0 h'(\theta) \sum_{\theta_i: h(\theta_i) = 0} \frac{\sqrt{h(\theta_i)}}{|h'(\theta_i)|^2} \delta(\theta - \theta_i). \tag{G.6}
\]

If we place the overall factor inside the sum, we have

\[
R_{\theta\theta} \overset{l \to 0}{=} 2\pi \sum_{\theta_i: h(\theta_i) = 0} \frac{h'(\theta_i)}{|h'(\theta_i)|^2} \frac{r_0 \sqrt{h(\theta_i)}}{l^2 + r_0^2 h(\theta_i)} \delta(\theta - \theta_i). \tag{G.7}
\]

We use (E.6) again, because \( \sqrt{h(\theta_i)} \to 0 \). We see that \( l^2 \) is a 3D norm, and therefore

\[
R_{\theta\theta} \overset{l \to 0}{=} -4\pi^3 r_0^2 \delta^{(3)}(l) \sum_{\theta_i: h(\theta_i) = 0} \frac{h'(\theta_i)}{|h'(\theta_i)| l^2} |h(\theta_i)| \delta(\theta - \theta_i) = 0. \tag{G.8}
\]

Because \( h(\theta) \) was considered arbitrary, this must hold even for \( h(\theta) = \Theta(f) \). The full Ricci tensor in the limit \( l \to 0 \) then becomes

\[
R_{\theta\theta} \overset{l \to 0}{=} -\frac{2}{r_0^2 h(\theta)}, \quad R_{\theta\theta} \overset{l \to 0}{=} \frac{h(\theta)/l^2 - h(\theta) [h'(\theta) \cot \theta + h''(\theta)]}{2h(\theta)^2}, \tag{G.9}
\]

\[
R_{\phi\phi} \overset{l \to 0}{=} R_{\theta\theta} \sin^2 \theta.
\]

It can be seen that a \( \delta \)-function singularity of the type \( \delta(\theta_0 - \theta) \) occurs in the above when \( h(\theta) = \Theta(f) \). There are no additional \( \delta \)-function singularities at the origin.

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