1-form symmetry
versus
large N QCD

Aleksey Cherman  Theo Jacobson
University of Minnesota

Collaborators:

Aleksey Cherman  Maria Neuzil
SU(N) pure Yang-Mills

It’s believed that pure YM theory confines test quarks in e.g. the fundamental representation.

- Can be stated as selection rules for line operators:

\[
\left\langle \frac{1}{N} \text{tr}_F P(\vec{x}) \right\rangle = 0
\]

in any finite volume

Quark free energy diverges

\[
\lim_{A(C) \to \infty} \left\langle \frac{1}{N} \text{tr}_F W(C) \right\rangle = 0
\]

in any scheme

Linear quark-antiquark potential
$\mathbb{Z}_N$ 1-form symmetry in YM theory

Gaiotto, Kapustin, Seiberg, Willett 2014

YM has $N$ codimension-2 topological symmetry generators $U_k$:

$$U_k(M_{d-2})U_n(M_{d-2}) = U_{k+n \mod N}(M_{d-2})$$

$$\langle U_k(M_{d-2})W_F(C) \rangle = e^{2\pi i \frac{k}{N} \text{Link}(C,M_{d-2})} \langle W_F(C) \rangle$$

- Action of $U_k(M_{d-2})$'s on Wilson loops is local: phase can be determined from a computation in a tiny region near $W_R(C)$
$\mathbb{Z}_N$ 1-form symmetry in YM theory

These selection rules are consequences of the $\mathbb{Z}_N$ 1-form symmetry.

- The confining phase is thus identified as a phase where the $\mathbb{Z}_N$ 1-form symmetry is not spontaneously broken.

$$\left\langle \frac{1}{N} \text{tr}_F P(\vec{x}) \right\rangle = 0$$

in any finite volume

$$\lim_{A(C) \to \infty} \left\langle \frac{1}{N} \text{tr}_F W(C) \right\rangle = 0$$

in any scheme
SU(N) QCD at large $N$

With $N_F \sim \mathcal{O}(1)$ fundamental-rep quark fields, quark loops are suppressed at large $N$.

- Large $N$ QCD obeys the same selection rules as pure YM:
  
  \[
  \langle \frac{1}{N} \text{tr}_F P(\vec{x}) \rangle = 0 \quad \text{in any finite volume}
  \]
  
  \[
  \lim_{A(C) \to \infty} \langle \frac{1}{N} \text{tr}_F W(C) \rangle = 0 \quad \text{in any scheme}
  \]

- Confinement is well-defined in large $N$ QCD!
Confinement in large N QCD

We normally expect that selection rules are consequences of symmetries.

So are the Wilson loop selection rules consequences of a symmetry in large N QCD?

• Natural guess: at large N, there’s a $\mathbb{Z}_N$ 1-form symmetry which explains the selection rules, just as in YM theory.

• Curiously, this guess is not right.

• Rest of the talk will explain why.
Obstructions to 1-form symmetry

Two basic issues:

- Existence of open Wilson lines in large N QCD.
- Large N quark loop suppression isn’t quite universal.

I’ll explain these issues, then show how things work explicitly in a calculable example, 2d scalar QCD on the lattice using the hopping expansion.

Upshot: there are no non-trivial topological codimension-2 operators in large N QCD with an action on Wilson loops.
Large N QCD has open Wilson lines: $\left\langle \frac{1}{N} \text{tr} \bar{Q}(x)e^{i \int_{x'}^{x} a Q(x')} \right\rangle = \mathcal{O}(1)$

- Suppose it has topological $U_k(M_{d-2})$ operators. Then

- This is inconsistent, so $U_k(M_{d-2})$ cannot be topological operators in large N QCD.
Closed versus open Wilson lines

• Given the assumption that $U_k(M_{d-2})$ is topological, its action on a Wilson line on a curve $C$ can be calculated by “shrinking”:

• Data can be obtained from an infinitesimal neighborhood of $C$ - no info on whether $C$ is open or closed!

• So failure of topological property on open Wilson lines implies failure in general.
Quark loop suppression?

- If quark loop contributions are universally suppressed at large N, how could correlation functions of $U_k(M_{d-2})$ be different in QCD versus pure YM theory?

- We’ll see that quark loops aren’t universally suppressed!

- At large N, the interesting $U_k(M_{d-2})$ operators have $k \sim N$.
  
  - $\left\langle \frac{1}{d_R} W_R(C) \right\rangle = 0$ for $n_R \sim N$, so $n_R \sim 1$ is most interesting.
  
  - $U_1(M_{d-2})$ acts trivially on $W_R(C)$ with N-ality $n_R \sim 1$

\[
U_1(\Sigma_{d-2}) W_R(C) = e^{\frac{2\pi i}{N} n_R} W_R(C) = W_R(C) + \mathcal{O}\left(\frac{1}{N}\right)
\]
Quark loop non-suppression

• In QFTs with exact $\mathbb{Z}_N$ 1-form symmetry, the fusion rules force

$$\langle U_k(M_{d-2}) \rangle = e^{\frac{2\pi i k}{N} \ell}, \; \ell \in \mathbb{Z}$$

• In pure YM on $\mathbb{R}^d$, $\ell = 0$.

• Let’s suppose $U_k$’s are all topological up to $1/N$ corrections in e.g. 2d QCD on $\mathbb{R}^2$, and factorize:

$$\langle U_1(x) \rangle = 1 + \mathcal{O}(1/N)$$

$$\langle U_1(x)U_1(x) \rangle = \langle U_1(x) \rangle \langle U_1(x) \rangle + \mathcal{O}(1/N)$$

• NB: in $d = 2$ the $U_k$’s are local operators.
Quark loop non-suppression

• We’re supposing that

\[ \langle U_1(x) \rangle = 1 + \mathcal{O}(1/N) \]
\[ \langle U_1(x)U_1(x) \rangle = 1 + \mathcal{O}(1/N) \]

• There are corrections to the fusion rule \( \langle U_k(x) \rangle = \langle U_1(x) \rangle^k \)

from the two-point functions: \( \sim k^2 \) pairs from \( k U_1(x) \)'s.

• So correction to \( \langle U_k(x) \rangle = \langle U_1(x) \rangle^k \) is \( \sim k^2/N \).

• Suggests trouble sets in already for \( k \sim \sqrt{N} \)!

  • correlation function is inconsistent with \( Z_N \) symmetry

• We’ll see that, indeed, corrections are \( \mathcal{O}(1) \) for \( k \sim \sqrt{N} \).
2d scalar QCD

• Let’s see how it all works in the simplest possible example.
  • 2d scalar SU(N) QCD, on the lattice, treated using the hopping ( = large mass) expansion.

\[ Z = \prod_\ell \int d\phi_\ell \prod_x \int d\phi_x d\phi_\dagger_x \prod_p e^{-s_{\text{YM}}(u_p)} \prod_\ell e^{-s_{\text{H}}(\phi_\dagger_\ell u_\ell \phi_\ell')} \prod_x e^{-m^2 \phi_\dagger_x \phi_x} \]

\( p = \text{plaquettes} \quad \ell = \text{links} \quad x = \text{sites} \)

• Rich enough to share the key features of real QCD, but simple enough to study explicitly.
2d scalar QCD

\[ Z = \prod_\ell \int d u_\ell \prod_x \int d \phi_x d \phi_x^\dagger \prod_p e^{-s_{\text{YM}}(u_p)} \prod_\ell e^{-s_{\text{H}}(\phi_x^\dagger u_\ell \phi_{x'})} \prod_x e^{-m^2 \phi_x^\dagger \phi_x} \]

- In 2d there is a maximally-nice gauge action called the ‘heat kernel’ action:

\[ e^{-s_{\text{YM}}(u_p)} = \sum_\alpha d_\alpha \chi_\alpha (u_p) e^{-g^2 c_\alpha A_p} \]

  Migdal 1975,
  Menotti+Onofri, 1981

- For pure YM get continuum results even on coarse lattices

- The hopping term (scalar kinetic term) is

\[ s_{\text{H}}(\phi_x^\dagger u_\ell \phi_{x'}) = -\kappa \phi_x^\dagger u_\ell \phi_{x'} + \text{h.c.} \]
Hopping expansion

• For small $\kappa (ma \gg 1)$, matter field integral gives a sum over all possible Wilson loop insertions $W_F(\Gamma)$, representing quark world-lines. Schematically:

$$\langle O_{\text{glue}} \rangle = \langle O_{\text{glue}} \rangle_0 + \sum_{\text{loops } \Gamma} \left( \frac{\kappa}{m^2} \right)^{L_\Gamma} \langle O_{\text{glue}} W_F[\Gamma] \rangle_0 + \cdots$$

• Physically, $\kappa/m^2 \sim 1/(m^2a^2)$. Large mass expansion!

• For us the two interesting $O_{\text{glue}}$ operators are Wilson loops $W_F(C)$ and $U_k(x)$’s, the generators of the $\mathbb{Z}_N$ 1-form symmetry of the 2d YM theory.
Wilson loop in pure YM

- At 0th order in the hopping expansion, Wilson loop expectation value is, of course, same as in pure YM:

\[
\frac{1}{N} \langle W_F(C) \rangle = e^{-\sigma A[C]} \left[ 1 + \mathcal{O}(\kappa^4) \right], \quad \sigma = \frac{\lambda}{2} + \mathcal{O}(1/N)
\]

- Area law behavior ⇔ 2d pure YM confines.
Wilson loop in QCD

- Smallest possible loop on the lattice is built from 4 links, so first corrections comes at $\kappa^4$. Calculation gives:

$$\frac{1}{N}\langle W_F(C) \rangle = \frac{1}{N}\langle W_F(C) \rangle_0 \left\{ 1 + A[C] \left( \frac{\kappa}{m^2} \right)^4 \frac{2}{N} \sinh \left( \frac{\lambda}{2} \right) + O(\kappa^6) \right\}$$

- The $\kappa^4$ term is coming from a quark loop, and it’s $1/N$ suppressed as expected. 

✓
Perimeter law behavior

- Working to higher order, we find a perimeter-law piece:

\[
\frac{1}{N} \langle W_F(C) \rangle = e^{-\sigma A[C]} + \frac{1}{N} e^{-\mu L[C]} + \cdots, \quad \mu = \log \frac{m^2}{\kappa}
\]

- Perimeter term also comes from a 1/N suppressed quark loop
  - If \( N \to \infty \) with loop geometry fixed, confinement!

- 2d QCD contains all the necessary physics to explore our puzzles.
1-form symmetry generators $U_k(x)$

- Several equivalent definitions.
  - As a Gukov-Witten operator.
    - Delete $x$ from spacetime, and pick the (conjugacy class of) the SU(N) holonomy $g$ around $x$.
    - The choice $g = e^{2\pi i/N} \mathbb{1} = \omega \mathbb{1}$ defines the $U_k(x)$’s.
  - On the lattice, ‘thin center-vortex’ definition is more convenient:
    \[
    U_k(\tilde{x} = \star p) = \exp \left[ s_{\text{YM}}(u_p) - s_{\text{YM}}(\omega^{-k} u_p) \right]
    \]
  - This is the definition we use.
Expectation value of $U_k(\tilde{x})$

- Let's write $\langle \mathcal{O} \rangle = \frac{\langle \langle \mathcal{O} \rangle \rangle}{Z}$. Then

$$\langle U_k(\tilde{x}) \rangle = \langle U_k(\tilde{x}) \rangle_0 + \left( \frac{\kappa}{m^2} \right)^4 \sum_p \langle U_k(\tilde{x}) \text{tr}(u_p + u_p^\dagger) \rangle_0 + \mathcal{O}(\kappa^6)$$

Smallest possible \textquotedblleft hopping\textquotedblright Wilson loop
Expectation value of $U_k(\tilde{x})$

- Let's write $\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} \rangle}{Z}$. Then

$$
\langle U_k(\tilde{x}) \rangle = \langle U_k(\tilde{x}) \rangle_0 + \left( \frac{\kappa}{m^2} \right)^4 \sum_p \langle U_k(\tilde{x}) \text{tr}(u_p + u_p^\dagger) \rangle_0 + \mathcal{O}(\kappa^6)
$$

Smallest possible "hopping" Wilson loop

$$
\langle U_k(\tilde{x}) \rangle = 1 + \left( \frac{\kappa}{m^2} \right)^4 \left[ 2(A - 1) N e^{-g^2 c_F} + N(\omega^k + \omega^{-k}) e^{-g^2 c_F} \right] + \mathcal{O}(\kappa^6)
$$

$$
Z = 1 + \left( \frac{\kappa}{m^2} \right)^4 2A N e^{-g^2 c_F} + \ldots
$$

$$
\langle W_F(c) \rangle_0 = N e^{-g^2 c_F A(c)}
$$

$$
c_F = \frac{N^2 - 1}{2N}
$$
Expectation value of $U_k(\tilde{x})$

- Putting things together and working through $\mathcal{O}(\kappa^8)$, we get

$$\langle U_k(\tilde{x}) \rangle = 1 - \left( \frac{\kappa}{m^2} \right)^4 2N \ e^{-g^2 c_F} \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right) +$$

$$\frac{1}{2} \left[ \left( \frac{\kappa}{m^2} \right)^4 2N \ e^{-g^2 c_F} \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right) \right]^2 + \cdots$$

- As is natural to guess from the result above, the result exponentiates:

$$\langle U_k(\tilde{x}) \rangle = \exp \left[ - \left( \frac{\kappa}{m^2} \right)^4 2N \ e^{-g^2 c_F} \left( 1 - \cos \left( \frac{2\pi k}{N} \right) \right) + O(\kappa^6) \right]$$
Expectation value of $U_k(\tilde{x})$

- We’ve learned that $\langle U_k(\tilde{x}) \rangle \sim e^{-k^2/N} \sim e^{-N}$ for $k \sim N$, within the hopping expansion.
- This means that for $k \sim N$, at large $N$ we can write

$$\langle U_k(\tilde{x}) \rangle = 0$$

- Meanwhile, in pure YM, $\langle U_k(\tilde{x}) \rangle = 1$, and $1 - 0 \sim \mathcal{O}(1)$. "Quark loops" give $\mathcal{O}(1)$ correction!
- Finally, for $k \sim \sqrt{N}$, $\langle U_k(\tilde{x}) \rangle \in (0,1)$.
- Not consistent with $U_k(\tilde{x})$ generating a $\mathbb{Z}_N$ symmetry.
Fate of $\mathbb{Z}_N$ 1-form symmetry in large N QCD

- Large N QCD doesn’t have a $\mathbb{Z}_N$ 1-form symmetry.
- Open Wilson line considerations are inconsistent with the existence of topological codim-2 operators.
- One might have hoped that codim-2 operators would be topological thanks to quark loop suppression. But there’s no reason to expect that in the cases of interest.
- We verified that the would-be 1-form symmetry generators fail to be topological in a calculable setting.
- Fundamental matter contributions are not suppressed!
Outlook

• Wilson loops in large N QCD obey the same selection rules as pure YM.
  • They are not explained by a $\mathbb{Z}_N$ 1-form symmetry.
• Is there some symmetry principle that could explain them?
  • If yes, seems to require some appropriate generalization of “generalized global symmetry.” Is there one?
  • If no, we’d have to accept selection rules without symmetries. Seems very strange!
• Are there examples apart from large N QCD?
  • Same obstructions apply to non-invertible symmetry in large N YM
Thank you for listening!
Backup: exponentiation of $\langle U_k(\tilde{x}) \rangle$

- The heat kernel action on each plaquette can be written as

$$\sum_{\alpha} d_{\alpha} \chi_{\alpha}(u) e^{-g^2 c_{\alpha}} = \exp \left[ \sum_{\alpha} \text{Re} \frac{1}{g_{\alpha}^2} \chi_{\alpha}(u) \right]$$

- Inserting $U_k(\tilde{x}) = \text{changing weight of one plaquette } p = \star \tilde{x}$:

$$g_{\alpha}^2 \rightarrow e^{\frac{2\pi i k}{N} n_{\alpha}} g_{\alpha}^2 \quad n_{\alpha} = \text{N-ality}$$

- Clustering arguments then imply

$$\tilde{Z}(g^2, k) = e^{-A\tilde{F}(k)} = e^{-A(F + \frac{1}{A} f(k))} \Rightarrow \langle U_k(\tilde{x}) \rangle = e^{-f(k)}$$

where $f(k)$ has a nice $1/N$ expansion, akin to free energy $F$. 
Backup: rescaling

• We could try defining $V_k(x) \equiv \frac{U_k(x)}{|\langle U_k(x) \rangle|}$, which is forced to have a unit VEV both in YM and in large N QCD.

• Immediate conflict with $\mathbb{Z}_N$ fusion rule is avoided.

• But at large N these operators are quite singular. Correlation functions do not satisfy cluster decomposition!

• At least for $\kappa \ll 1$, $\langle V_k(x)^\dagger V_k(0) \rangle$ with $k \sim N$ diverges for any separation $r \sim N^0$, only decays once $r \gtrsim \sqrt{N}$.

• Can’t interpret $V_k(x)$ as generators of a $\mathbb{Z}_N$ 1-form symmetry.