Toward quantized Picard-Vessiot theory, 
Further observations on our previous example

Katsunori Saito and Hiroshi Umemura
Graduate School of Mathematics
Nagoya University

Email m07026e@math.nagoya-u.ac.jp and umemura@math.nagoya-u.ac.jp

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Abstract

It is quite natural to wonder whether there is a difference-differential equations, the Galois group of which is a quantum group that is neither commutative nor co-commutative. Believing that there was no such linear equations, we explored non-linear equations and discovered a such equation [6], [5]. We show that the example is related with a linear equation. We treat only one charming example. It is not, however, an isolated example. We open a window that allows us to have a look at a quantized Picard-Vessiot theory.

1 Introduction

We believed for a long time that it was impossible to quantize Picard-Vessiot theory, Galois theory for linear difference or differential equations. Namely, there was no Galois theory for linear difference-differential equations, the Galois group of which is a quantum group that is, in general, neither commutative nor co-commutative. Our mistake came from a misunderstanding of preceding works Hardouin [2] and Masuoka and Yanagawa [4]. They studied linear \( q \)-SI \( \sigma \)-differential equations, \( qsi \) equations for short, under two assumptions on \( qsi \) base field \( K \) and \( qsi \) module \( M \):

1. The base field \( K \) contains \( \mathbb{C}(t) \).

2. On the \( K[\sigma, \theta^*] \)-module \( M \) the equality

\[
\theta^{(1)} = \frac{1}{(q-1)t}(\sigma - \text{Id}_M).
\]

holds. Under these conditions, their Picard-Vessiot extension is realized in the category of commutative algebras. The second assumption seems too restrictive as clearly explained
in [4]. If we drop one of these conditions, there are many linear non-
linear equations whose Picard-Vessiot ring is not commutative and Galois group is a quantum group that is neither commutative nor co-commutative.

We analyze only one favorite example (5) over the base field $C$, which is equivalent to the non-linear equation in [5]. The reader’s imagination would go far away. In the example, we have a Picard-Vessiot ring $R$ that is non-commutative, simple $qsi$ ring (Observation 6 and Lemma 7). The Picard-Vessiot ring $R$ is a torsor of a quantum group (Observation 9). We have the Galois correspondence (Observation 11) and non-commutative Tannaka theory (Observation 10).

We are grateful to A. Masuoka and K. Amano for teaching us their Picard-Vessiot theory and clarifying our idea.

2 Field extension $C(t)/C$ from classical and quantum view points

In §8 of the previous paper [5], we studied a non-linear $q$-SI $\sigma$-differential equation, which we call $qsi$ equation for short,

$$\theta^{(1)}(y) = 1, \quad \sigma(y) = qy,$$  \hspace{1cm} (1)

where $q \neq 0, 1$ is a complex number. Let $t$ be a variable over the complex number field $C$. We assume to simplify the situation that $q$ is not a root of unity. We denote by $\sigma: C(t) \to C(t)$ the $C$-automorphism of the field $C(t)$ of rational functions sending $t$ to $qt$. We introduce the $C$-linear operator $\theta^{(1)}: C(t) \to C(t)$ by

$$\theta^{(1)}(f(t)) := \frac{f(qt) - f(t)}{(q - 1)t} \quad \text{for every } f(t) \in C(t).$$

We set

$$\theta^{(m)} := \begin{cases} \text{Id}_{C(t)} & \text{for } m = 0, \\ \frac{1}{[m]_q} (\theta^{(1)})^m & \text{for } m = 1, 2, \ldots. \end{cases}$$

As we assume that $q$ is not a root of unity, the number $[m]_q$ in the formula is not equal to 0 and hence the formula determines the family $\theta^* = \{\theta^{(i)} | i \in \mathbb{N}\}$ of operators. So $(C(t), \sigma, \theta^*)$ is a $qsi$ field. See §7, [5] and $y = t$ is a solution for system (1).

The system (1) is non-linear in the sense that for two solutions $y_1, y_2$ of (1), a $C$-linear combination $c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in C$) is not a solution of the system.

However, the system is very close to a linear system. To illustrate this, let us look at the differential field extension $(C(t), \partial_t)/(C, \partial_t)$, where we denote the derivation $d/dt$ by $\partial_t$. The variable $t \in C(t)$ satisfies a non-linear differential equation

$$\partial_t t - 1 = 0.$$  \hspace{1cm} (2)

The differential field extension $(C(t), \partial_t)/(C, \partial_t)$ is, however, the Picard-Vessiot extension for the linear differential equation

$$\partial^2_t t = 0.$$  \hspace{1cm} (3)
To understand the relation between (2) and (3), we introduce the 2-dimensional \( \mathbb{C} \)-vector space
\[
E := \mathbb{C}t \oplus \mathbb{C} \subset \mathbb{C}[t].
\]
The vector space \( E \) is close under the action of the derivation \( \partial_t \) so that \( E \) is a \( \mathbb{C}[\partial_t] \)-module. Solving the differential equation associated with the \( \mathbb{C}[\partial_t] \)-module \( E \) is to find a differential algebra \( (L, \partial_t)/\mathbb{C} \) and a \( \mathbb{C}[\partial_t] \)-module morphism
\[
\varphi: E \rightarrow L.
\]
Writing \( \varphi(t) = f_1, \varphi(1) = f_2 \) that are elements of \( L \), we have
\[
\begin{bmatrix}
\partial_t f_1 \\
\partial_t f_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}.
\]
Since \( \partial_t t = 1, \partial_t 1 = 0 \), in the differential field \( (\mathbb{C}(t), \partial_t)/\mathbb{C} \), we find two solutions \( t(t, 1) \) and \( t(1, 0) \) that are two column vectors in \( \mathbb{C}(t)^2 \) satisfying
\[
\partial_t \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix} \tag{4}
\]
and
\[
\begin{vmatrix} t & 1 \\ 1 & 0 \end{vmatrix} \neq 0.
\]
Namely, \( \mathbb{C}(t)/\mathbb{C} \) is the Picard-Vessiot extension for linear differential equation (4).

We can argue similarly for the \( qsi \) field extension \( (\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C} \). You will find a subtle difference between the differential case and the \( qsi \) case. Quantization of Galois group arises from here.

Let us set
\[
M = \mathbb{C}t \oplus \mathbb{C} \subset \mathbb{C}[t]
\]
that is a \( \mathbb{C}[\sigma, \theta^*] \)-module. Maybe to avoid the confusion that you might have in Remark 3 below, writing \( m_1 = t \) and \( m_2 = 1 \), we had better define formally
\[
M = \mathbb{C}m_1 \oplus \mathbb{C}m_2
\]
as a \( \mathbb{C} \)-vector space on which \( \sigma \) and \( \theta^{(1)} \) operate by
\[
\begin{bmatrix}
\sigma(m_1) \\
\sigma(m_2)
\end{bmatrix} =
\begin{bmatrix}
q & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix}, \quad
\begin{bmatrix}
\theta^{(1)}(m_1) \\
\theta^{(1)}(m_2)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
m_1 \\
m_2
\end{bmatrix}. \tag{5}
\]
Solving \( \mathbb{C}[\sigma, \theta^*] \)-module \( M \) is equivalent to find elements \( f_1, f_2 \) in a \( qsi \) algebra \( (A, \sigma, \theta^*) \) satisfying the system of linear difference-differential equation
\[
\begin{bmatrix}
\sigma(f_1) \\
\sigma(f_2)
\end{bmatrix} =
\begin{bmatrix}
q & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}, \quad
\begin{bmatrix}
\theta^{(1)}(f_1) \\
\theta^{(1)}(f_2)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}. \tag{6}
\]
in the \( qsi \) algebra \( A \).
Lemma 1. Let $(L, \sigma, \theta^*)/\mathbb{C}$ be a qsi field extension. If a $2 \times 2$ matrix $Y = (y_{ij}) \in M_2(L)$ satisfies a system of difference-differential equations
\[
\sigma Y = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y \quad \text{and} \quad \theta^{(1)} Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y,
\]
then $\det Y = 0$.

Proof. It follows from (7)
\[
\sigma(y_{11}) = qy_{11}, \quad \sigma(y_{12}) = qy_{12}, \quad \sigma(y_{21}) = y_{21}, \quad \sigma(y_{22}) = y_{22}
\]
and
\[
\theta^{(1)}(y_{11}) = y_{21}, \quad \theta^{(1)}(y_{12}) = y_{22}, \quad \theta^{(1)}(y_{21}) = 0, \quad \theta^{(1)}(y_{22}) = 0.
\]
It follows from (8) and (9)
\[
\theta^{(1)}(y_{11}y_{12}) = \theta^{(1)}(y_{11})y_{12} + \sigma(y_{11})\theta^{(1)}(y_{12}) = y_{21}y_{12} + qy_{11}y_{22}
\]
and similarly
\[
\theta^{(1)}(y_{12}y_{11}) = y_{22}y_{11} + qy_{12}y_{21}.
\]
As $y_{11}y_{12} = y_{12}y_{11}$, equating (10) and (11), we get
\[
(q - 1)(y_{11}y_{22} - y_{12}y_{21}) = 0
\]
so that $\det Y = 0$. \hfill \Box

Corollary 2. Let $(K, \sigma, \theta)$ be a qsi field over $\mathbb{C}$. Then the qsi linear equation
\[
\sigma Y = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y \quad \text{and} \quad \theta^{(1)} Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y
\]
has no Picard-Vessiot extension $L/K$ in the following sense. There exists a solution $Y \in GL_2(L)$ to (12) such that the abstract field $L$ is generated by the entries of the matrix $Y$ over $K$. The field of constants of the qsi over-field $L$ coincides with the field of constants of the base field $K$.

Proof. This is a consequence of Lemma 1. \hfill \Box

Remark 3. Masuoka pointed out that Corollary 2 is compatible with Remark 4.4 and Theorem 4.7 of Hardouin [2]. See also Masuoka and Yanagawa [4]. They assure the existence of Picard-Vessiot extension for a $K[\sigma, \theta^*]$-module $N$ if the following two conditions are satisfied;

1. The qsi base field $K$ contains $(\mathbb{C}(t), \sigma, \theta^*)$,

2. The operation of $\sigma$ and $\theta^{(1)}$ on the module $N$ as well as on the base field $Y$, satisfy the relation
\[
\theta^{(1)} = \frac{1}{(q - 1)t}(\sigma - \text{Id}_N).
\]

In fact, even if the base field $K$ contains $(\mathbb{C}(t), \sigma, \theta^*)$, in $K \otimes_{\mathbb{C}} M$, we have by definition of the $\mathbb{C}[\sigma, \theta^*]$-module $M$,
\[
\theta^{(1)}(m_1) = m_2 \neq \frac{1}{t}m_1 = \frac{1}{(q - 1)t}(\sigma(m_1) - m_1).
\]
So $K \otimes_{\mathbb{C}} M$ does not satisfy the second condition above.
3 Quantum normalization of \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\)

We started from the qsi field extension \(\mathbb{C}(t)/\mathbb{C}\). The column vector \(\begin{bmatrix} t(1) \\ 1 \end{bmatrix} \in \mathbb{C}(t)^2\) is a solution to the system of equations (5), i.e. we have

\[
\begin{bmatrix} \sigma(1) \\ \theta(1)(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\begin{bmatrix} \sigma(t) \\ \theta(1)(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

By applying to the qsi field extension \((\mathbb{C}(t), \sigma, \theta^*)/\mathbb{C}\), the general procedure of [7], [3] that is believed to lead us to the normalization, we arrived at the Galois hull \(L = \mathbb{C}(t)[Q, Q^{-1}, X]\). This suggests an appropriate normalization of the non-commutative qsi ring extension \(\mathbb{C}(t)[Q, Q^{-1}]/\mathbb{C}\) is a (maybe the), qsi Picard-Vessiot extension of the system of equations (5). More precisely, \(Q\) is a variable over \(\mathbb{C}(t)\) satisfying the commutation relation

\[Qt = qtQ.\]

We understand \(R = \mathbb{C}[t, Q, Q^{-1}]\) as a sub-ring of \(S = \mathbb{C}[[t, Q]][t^{-1}, Q^{-1}]\). The ring \(S\) is a non-commutative qsi algebra by setting

\[
\sigma(Q) = qQ, \; \theta^{(1)}(Q) = 0 \quad \text{and} \quad \sigma(t) = qt, \; \theta^{(1)}(t) = 1
\]

and \(R = \mathbb{C}[t, Q, Q^{-1}]\) is a qsi sub-algebra. Thus we get a qsi ring extension \((R, \sigma, \theta^*)/\mathbb{C} = (\mathbb{C}[t, Q, Q^{-1}], \sigma, \theta^*)/\mathbb{C}\). We examine that \((R, \sigma, \theta^*)/\mathbb{C}\) is a non-commutative Picard-Vessiot extension for the systems of equations (5).

Observation 4. The \(\mathbb{C}[\sigma, \theta^*]\)-module \(M\) has two solutions linearly independent over \(\mathbb{C}\). In fact, setting

\[Y := \begin{bmatrix} Q & t \\ 0 & 1 \end{bmatrix} \in M_2(R), \quad (13)\]

we have

\[
\sigma Y = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} Y \quad \text{and} \quad \theta^{(1)} Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y. \quad (14)
\]

So the column vectors \(\begin{bmatrix} t(Q, 0) \\ 1 \end{bmatrix}, \begin{bmatrix} t(1) \\ 1 \end{bmatrix} \in R^2\) are \(\mathbb{C}\)-linearly independent solution of the system of equations (5).

Observation 5. The ring \(\mathbb{C}[t, Q, Q^{-1}]\) has no zero-divisors. We can consider the ring \(K\) of total fractions of \(\mathbb{C}[t, Q, Q^{-1}]\).

Proof. In fact, we have \(R \subset \mathbb{C}[[t, Q]][t^{-1}, Q^{-1}]\). In the latter ring every non-zero element is invertible.

Observation 6. The ring of qsi constants \(C_K\) coincide with \(\mathbb{C}\). The ring of \(\theta^*\) constants of \(\mathbb{C}[[t, Q]][t^{-1}, Q^{-1}]\) is \(\mathbb{C}(Q)\). Moreover as we assume that \(q\) is not a root of unity, the ring of \(\sigma\)-constants of \(\mathbb{C}(Q)\) is equal to \(\mathbb{C}\).

Lemma 7. The non-commutative qsi algebra \(R\) is simple. There is no qsi bilateral ideal of \(R\) except for the zero-ideal and \(R\).
Proof. Let $I$ be a non-zero qsi bilateral ideal of $R$. We take an element

$$0 \neq f := a_0 + ta_1 + \ldots + t^n a_n \in I,$$

where $a_i \in \mathbb{C}[Q, Q^{-1}]$ for $0 \leq i \leq n$. We may assume $a_n \neq 0$. Applying $\theta^{(n)}$ to the element $f$, we conclude that $0 \neq a_n \in \mathbb{C}[Q, Q^{-1}]$ is in the ideal $I$. Multiplying a monomial $bQ^i$ with $b \in \mathbb{C}$, we find a polynomial $h = 1 + b_1 Q + \ldots + b_s Q^s \in \mathbb{C}[Q]$ with $b_s \neq 0$ is in the ideal $I$. We show that 1 is in $I$ by induction on $s$. If $s = 0$, then there is nothing to prove. Assume that the assertion is proved for $s \leq m$. We have to show the assertion for $s = m + 1$. Then, since $Q^i$ is an eigenvector of the operator $\sigma$ with eigenvalue $q^i$ for $i \in \mathbb{N}$,

$$\frac{1}{q^{m+1} - q} (q^{m+1} h - \sigma(h)) = 1 + c_1 Q + \ldots + c_m Q^m \in \mathbb{C}[Q]$$

is an element of $I$ and by induction hypothesis 1 is in the ideal $I$. \qed

**Observation 8.** The extension $R/\mathbb{C}$ trivializes the $\mathbb{C}[\sigma, \theta^*]$-module $M$. Namely, there exist constants $c_1, c_2 \in R \otimes \mathbb{C} M$ such that

$$R \otimes \mathbb{C} M \simeq Rc_1 \oplus Rc_2.$$

**Proof.** In fact, it is sufficient to set

$$c_1 := Q^{-1} m_1 - Q^{-1} tm_2, \quad c_2 := m_2.$$

Then

$$\sigma(c_1) = c_1, \quad \sigma(c_2) = c_2, \quad \theta^{(1)}(c_2) = 0$$

and

$$\theta^{(1)}(c_1) = q^{-1} Q^{-1} \theta^{(1)}(m_1) - q^{-1} Q^{-1} m_2 = q^{-1} Q^{-1} m_2 - q^{-1} Q^{-1} m_2 = 0.$$

So we have an $(R, \sigma, \theta^*)$-module isomorphism $R \otimes \mathbb{C} M \simeq Rc_1 \oplus Rc_2$. \qed

**Observation 9.** The Hopf algebra $h_q = \mathbb{C}(u, u^{-1}, v)$ with $uv = quv$ co-acts on the non-commutative algebra $R$. Namely, we have an algebra morphism

$$R \to R \otimes \mathbb{C} h_q$$

sending

$$t \mapsto t \otimes 1 + Q \otimes v, \quad Q \mapsto Q \otimes u, \quad Q^{-1} \mapsto Q^{-1} \otimes u^{-1}.$$

This gives us an $1 \otimes R$-algebra isomorphism

$$R \otimes \mathbb{C} R \to R \otimes \mathbb{C} h_q$$

Moreover isomorphism (15) is $\mathbb{C}[\sigma, \theta^*]$-isomorphism, where $\mathbb{C}[\sigma, \theta^*]$ operates on the Hopf algebra $h_q$ trivially.
We study category $\mathcal{C}(\mathbb{C}[\sigma, \theta^*])$ of left $\mathbb{C}[\sigma, \theta^*]$-modules that are finite dimensional as $\mathbb{C}$-vector spaces. We notice first the internal homomorphism

$$\text{Hom}_{\mathbb{C}}((M_1, \sigma_1, \theta_1^*), (M_2, \sigma_2, \theta_2^*)) \in \text{ob}(\mathcal{C}(\mathbb{C}[\sigma, \theta]))$$

exists for two objects $(M_1, \sigma_1, \theta_1^*), (M_2, \sigma_2, \theta_2^*) \in \text{ob}(\mathcal{C}(\mathbb{C}[\sigma, \theta]))$. In fact, let $N := \text{Hom}(M_1, M_2)$ be the set of $\mathbb{C}$-linear maps from $M_1$ to $M_2$. It sufficient to consider two $\mathbb{C}$-linear maps

$$\sigma_h : N \rightarrow N$$

given by

$$\sigma_h(f) := \sigma_2 \circ f \circ \sigma_1^{-1}$$

and

$$\theta^{(1)}_h(f) := -\sigma_h f \circ \theta^{(1)}_1 + \theta^{(1)}_2 \circ f.$$ 

So we have $q\sigma_h \circ \theta^{(1)}_h = \theta^{(1)}_h \circ \sigma$. Since $q$ is not a root of unity, we define $\theta^{(m)}_h$ in an evident manner

$$\theta^{(m)}_h = \begin{cases} 
\text{Id}_N, & \text{if } m = 0, \\
\frac{1}{[m]^q} (\theta^{(1)}_h)^m, & \text{if } m \geq 1.
\end{cases}$$

Since $\mathbb{C}[\sigma, \theta^*]$ is a Hopf algebra, for two objects $M_1, M_2 \in \text{ob}(\mathcal{C}(\mathbb{C}[\sigma, \theta]))$ the tensor product $M_1 \otimes_{\mathbb{C}} M_2$ is defined as an object of $\mathcal{C}(\mathbb{C}[\sigma, \theta^*])$. However, as $\mathbb{C}[\sigma \theta^*]$ is not co-commutative, we do not have, in general, $M_1 \otimes_{\mathbb{C}} M_2 \simeq M_2 \otimes_{\mathbb{C}} M_1$. Taking the forgetful functor

$$\omega : \mathcal{C}(\mathbb{C}[\sigma, \theta^*]) \rightarrow \text{category of } \mathbb{C}\text{-vector spaces},$$

we get a non-commutative Tannakian category.

**Observation 10.** The non-commutative Tannakian category $\{\{M\}\}$ generates by the $\mathbb{C}[\sigma, \theta^*]$-module $M$ is equivalent to the category $\mathcal{C}(h_q)$ of right $h_q$-co-modules that are finite dimensional as $\mathbb{C}$-vector space.

**Proof.** We owe this proof to Masuoka and Amano. Our Picard-Vessiot ring $R$ is not commutative. However, by Observations 8 and Lemma 7, we can apply the arguments of the classical differential Picard-Vessiot theory according to Amano, Masuoka and Takeuchi [1], [7]. We first show that every $\mathbb{C}[\sigma, \theta^*]$-module $N \in \{\{M\}\}$ is trivialized over $R$. Then, the functor

$$\phi : \{\{M\}\} \rightarrow \mathcal{C}(h_q)$$

is given by

$$\phi(N) = \text{Constants of } \mathbb{C}[\sigma, \theta^*]\text{-module } R \otimes_{\mathbb{C}} N \text{ for } N \in \text{ob}(\{\{M\}\}).$$

In fact, the Hopf algebra $h_q$ co-acts on $R$ and so on the trivial $R$-module $R \otimes_{\mathbb{C}} N$ and consequently on the vector space of constants of $R \otimes_{\mathbb{C}} N$. 

**Observation 11.** We have the Galois correspondence between the elements of the two sets.

1. The set of intermediate $q$-division rings of $K/\mathbb{C}$:

$$\mathbb{C}, \mathbb{C}(t), \mathbb{C}(Q), K$$

with inclusions

$$\mathbb{C} \subset \mathbb{C}(t) \subset K \text{ and } \mathbb{C} \subset \mathbb{C}(Q) \subset K.$$ (16)
2. The set of quotient $\mathbb{C}$-Hopf algebras of $\mathfrak{h}_q$:

$$\mathfrak{h}_q, \mathfrak{h}_q/I, \mathfrak{h}_q/J, \mathbb{C}$$

with the sequences of the quotient morphisms corresponding to inclusions \((16)\)

$$\mathfrak{h}_q \to \mathfrak{h}_q/I \to \mathbb{C} \text{ and } \mathfrak{h}_q \to \mathfrak{h}_q/J \to \mathbb{C}, \tag{17}$$

where $I$ is the ideal of the Hopf algebra $\mathfrak{h}_q$ generated by $u - 1$ and similarly $J$ is the ideal of the Hopf algebra generated by $v$.

The extensions

$$K/\mathbb{C}, K/\mathbb{C}(Q), K/\mathbb{C}(t) \text{ and } \mathbb{C}(Q)/\mathbb{C}$$

are psi Picard-Vessiot extensions with Galois groups

$$\text{Gal} (K/\mathbb{C}) \simeq \mathfrak{h}_q, \quad \text{Gal} (K/\mathbb{C}(Q)) \simeq \mathbb{C}[G_{a\mathbb{C}}],$$
$$\text{Gal} (K/\mathbb{C}(t)) \simeq \mathbb{C}[G_{a\mathbb{C}}], \quad \text{and} \quad \text{Gal} (\mathbb{C}(Q)/\mathbb{C}) \simeq \mathbb{C}[G_{m\mathbb{C}}].$$

Here we denote by $\mathbb{C}[G]$ the Hopf algebra of the coordinate ring of an affine group scheme $G$ over $\mathbb{C}$.

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