Subclasses of $\lambda$-bi-pseudo-starlike functions with respect to symmetric points based on shell-like curves

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ABSTRACT

In this paper we define the subclass $\mathcal{PSL}_\lambda^{\Sigma}(\alpha, \tilde{p}(z))$ of the class $\Sigma$ of bi-univalent functions defined in the unit disk, called $\lambda$-bi-pseudo-starlike, with respect to symmetric points, related to shell-like curves connected with Fibonacci numbers. We determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions $f \in \mathcal{PSL}_\lambda^{\Sigma}(\alpha, \tilde{p}(z))$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}_\lambda^{\Sigma}(\alpha, \tilde{p}(z))$ and for the special cases $\alpha = 0$, $\alpha = 1$ and $\tau = -0.618$ we state corollaries improving the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

RESUMEN

En este artículo definimos la subclase $\mathcal{PSL}_\lambda^{\Sigma}(\alpha, \tilde{p}(z))$ de la clase $\Sigma$ de funciones bi-univalentes definidas en el disco unitario, llamadas $\lambda$-bi-pseudo-estrelladas, con respecto a puntos simétricos, relacionadas a curvas espirales en conexión con números de Fibonacci. Determinamos los coeficientes iniciales de Taylor-Maclaurin $|a_2|$ y $|a_3|$ para funciones $f \in \mathcal{PSL}_\lambda^{\Sigma}(\alpha, \tilde{p}(z))$. Más aún determinamos el resultado de Fekete-Szegö para la clase de funciones $\mathcal{PSL}_\lambda^{\Sigma}(\alpha, \tilde{p}(z))$ y para los casos especiales $\alpha = 0$, $\alpha = 1$ y $\tau = -0.618$ enunciamos corolarios mejorando los coeficientes iniciales de Taylor-Maclaurin $|a_2|$ y $|a_3|$.

Keywords and Phrases: Analytic functions, bi-univalent, shell-like curve, Fibonacci numbers, starlike functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f$ which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $U$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

The Koebe one quarter theorem [4] ensures that the image of $U$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z, \ (z \in U) \text{ and } f(f^{-1}(w)) = w \quad (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions defined in the unit disk $U$. Since $f \in \Sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots. \quad (1.2)$$

We notice that the class $\Sigma$ is not empty. For example, the functions $z$, $z_1 - z$, $-\log(1 - z)$ and $\frac{1}{2}\log\left(\frac{1 + z}{1 - z}\right)$ are members of $\Sigma$. However, the Koebe function is not a member of $\Sigma$. In fact, Srivastava et al. [15] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by (see [2, 3, 9, 15, 16, 17]).

An analytic function $f$ is subordinate to an analytic function $F$ in $U$, written as $f \prec F$ ($z \in U$), provided there is an analytic function $\omega$ defined on $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(U) \subset F(U), \ z \in U$$

(for details see [4, 8]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that $f$ is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Recently, in [14], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition:

**Definition 1.1.** The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$. 

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^\ast$.

The function $\tilde{p}$ is not univalent in $U$, but it is univalent in the disc $|z| < (3 - \sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\pi i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|},$$

which shows that the number $|\tau|$ divides $[0, 1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since $\tau$ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers $\tau^n$ as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of $\tau$ and 1. The resulting recurrence relationships yield Fibonacci numbers $u_n$:

$$\tau^n = u_n\tau + u_{n-1}.$$

In [11] Raina and Sokół showed that

$$\tilde{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau z - \tau^2z^2} = \left( t + \frac{1 + \tau}{2} \right) \frac{t}{1 - t - t^2},$$

$$= \frac{1}{\sqrt{5}} \left( t + \frac{1 + \tau}{2} \right) \left( \frac{1}{1 - (1 - \tau)t} - \frac{1}{1 - \tau t} \right)$$

$$= \left( t + \frac{1 + \tau}{2} \right) \sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1})\tau^n z^n,$$

where

$$u_n = \frac{(1 - \tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1 - \sqrt{5}}{2}, \quad t = \tau z \quad (n = 1, 2, \ldots).$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers $u_n$, such that $u_0 = 0, u_1 = 1, u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \ldots$. And they got

$$\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n$$

$$= 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n z^n$$

$$= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots.$$

Let $\mathcal{P}(\beta), 0 \leq \beta < 1$, denote the class of analytic functions $p$ in $U$ with $p(0) = 1$ and $\text{Re}\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.
Theorem 1.2. [6] The function \( \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2} \) belongs to the class \( \mathcal{P}(\beta) \) with \( \beta = \sqrt{5}/10 \approx 0.2236 \).

Now we give the following lemma which will use in proving.

Lemma 1.3. [10] Let \( p \in \mathcal{P} \) with \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \), then

\[
|c_n| \leq 2, \quad \text{for} \quad n \geq 1. \tag{1.6}
\]

2 Bi-Univalent function class \( \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z)) \)

In this section, we introduce a new subclass of \( \Sigma \) associated with shell-like functions connected with Fibonacci numbers and obtain the initial Taylor coefficients \( |a_2| \) and \( |a_3| \) for the function class by subordination.

Firstly, let \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \), and \( p \prec \tilde{p} \). Then there exists an analytic function \( u \) such that \( |u(z)| < 1 \) in \( U \) and \( p(z) = \tilde{p}(u(z)) \). Therefore, the function

\[
h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.1}
\]

is in the class \( \mathcal{P} \). It follows that

\[
u(z) = \frac{c_1 z}{2} + \left( c_2 - \frac{c_1^2}{2} \right) \frac{z^2}{2} + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) \frac{z^3}{2} + \cdots \tag{2.2}
\]

and

\[
\tilde{p}(u(z)) = 1 + \tilde{p}_1 c_1 z + \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) \tilde{p}_1 + \frac{c_1^3}{4} \tilde{p}_2 \right\} z^2 + \cdots \tag{2.3}
\]

And similarly, there exists an analytic function \( v \) such that \( |v(w)| < 1 \) in \( U \) and \( p(w) = \tilde{p}(v(w)) \). Therefore, the function

\[
k(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + d_1 w + d_2 w^2 + \cdots \tag{2.4}
\]

is in the class \( \mathcal{P}(0) \). It follows that

\[
v(w) = \frac{d_1 w}{2} + \left( d_2 - \frac{d_1^2}{2} \right) \frac{w^2}{2} + \left( d_3 - d_1 d_2 + \frac{d_1^3}{4} \right) \frac{w^3}{2} + \cdots \tag{2.5}
\]

and

\[
\tilde{p}(v(w)) = 1 + \tilde{p}_1 d_1 w + \left\{ \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) \tilde{p}_1 + \frac{d_1^3}{4} \tilde{p}_2 \right\} w^2 + \cdots \tag{2.6}
\]
The class $L_\alpha(\alpha)$ of $\lambda$-pseudo-starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) were introduced and investigated by Babalola [1] whose geometric conditions satisfy
\[
\Re \left( \frac{z(f'(z))^\lambda}{f(z)} \right) > \alpha, \quad \lambda > 0.
\]
He showed that all pseudo-starlike functions are Bazilević of type $(1 - \frac{1}{\lambda})$ order $\alpha^\lambda$ and univalent in open unit disk $U$. If $\lambda = 1$, we have the class of starlike functions of order $\alpha$, which in this context, are $1$-pseudo-starlike functions of order $\alpha$. A function $f \in A$ is starlike with respect to symmetric points in $U$ if for every $r$ close to 1, $r < 1$ and every $z_0$ on $|z| = r$ the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as $z$ traverses the circle $|z| = r$ in the positive direction.

This class was introduced and studied by Sakaguchi [13] presented the class $S_\ast^\ast$ of functions starlike with respect to symmetric points. This class consists of functions $f(z) \in S$ satisfying the condition
\[
\Re \left( \frac{2zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in U.
\]
Motivated by $S_\ast^\ast$, Wang et al. [18] introduced the class $K_\ast$ of functions convex with respect to symmetric points, which consists of functions $f(z) \in S$ satisfying the condition
\[
\Re \left( \frac{2zf'(z)}{(f(z) - f(-z))'} \right) > 0, \quad z \in U.
\]
It is clear that, if $f(z) \in K_\ast$, then $zf'(z) \in S_\ast^\ast$. For such a function $\phi$, Ravichandran [12] presented the following subclasses: A function $f \in A$ is in the class $S_\ast^\ast(\phi)$ if
\[
\frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z), \quad z \in U,
\]
and in the class $K_\ast(\phi)$ if
\[
\frac{2zf'(z)}{(f(z) - f(-z))'} \prec \phi(z), \quad z \in U.
\]
Motivated by aforementioned works [1, 13, 12, 18] and recent study of Sokól [14] (also see [11]), in this paper we define the following new subclass $f \in PSL_{\alpha,\Sigma}^\lambda(\hat{p}(z))$ of $\Sigma$ named as $\lambda$-bi-pseudo-starlike functions with respect to symmetric points, related to shell-like curves connected with Fibonacci numbers, and determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Further we determine the Fekete-Szegő result for the function class $PSL_{\alpha,\Sigma}^\lambda(\hat{p}(z))$ and the special cases are stated as corollaries which are new and have not been studied so far.

**Definition 2.1.** For $0 \leq \alpha \leq 1; \lambda > 0; \lambda \neq \frac{1}{2}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $PSL_{\alpha,\Sigma}^\lambda(\alpha, \hat{p}(z))$ if the following subordination hold:
\[
\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\alpha \left( \frac{2zf'(z)}{(f(z) - f(-z))'} \right)^{1-\alpha} \hat{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau z - \tau^2z^2} (2.7)
\]
and
\[
\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^\alpha \left( \frac{2wg'(w)}{(g(w) - g(-w))'} \right)^{1-\alpha} \hat{p}(w) = \frac{1 + \tau^2w^2}{1 - \tau w - \tau^2w^2} (2.8)
\]
where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2).
Specializing the parameter $\lambda = 1$ we have the following definitions, respectively:

**Definition 2.2.** For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}_{s,\Sigma}^1(\alpha, \tilde{p}(z)) \equiv \mathcal{ML}_{s,\Sigma}(\alpha, \tilde{p}(z))$ if the following subordination hold:

$$\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\alpha \left( \frac{2(zf'(z))^\prime}{[f(z) - f(-z)]^2} \right)^{1-\alpha} < \tilde{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau^2z^2}$$

and

$$\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^\alpha \left( \frac{2(wg'(w))'^\prime}{[g(w) - g(-w)]^2} \right)^{1-\alpha} < \tilde{p}(w) = \frac{1 + \tau^2w^2}{1 - \tau^2w^2}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2).

Further by specializing the parameter $\alpha = 1$ and $\alpha = 0$ we state the following new classes $\mathcal{SL}_{s,\Sigma}^1(\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}^1(\tilde{p}(z))$ respectively.

**Definition 2.3.** A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}_{s,\Sigma}^1(1, \tilde{p}(z)) \equiv \mathcal{SL}_{s,\Sigma}^1(\tilde{p}(z))$ if the following subordination hold:

$$\frac{2zf'(z)}{f(z) - f(-z)} < \tilde{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau^2z^2}$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} < \tilde{p}(w) = \frac{1 + \tau^2w^2}{1 - \tau^2w^2}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2).

**Definition 2.4.** A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}_{s,\Sigma}^1(0, \tilde{p}(z)) \equiv \mathcal{KL}_{s,\Sigma}^1(\tilde{p}(z))$ if the following subordination hold:

$$\frac{2(zf'(z))^\prime}{[f(z) - f(-z)]^2} < \tilde{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau^2z^2}$$

and

$$\frac{2(wg'(w))^\prime}{[g(w) - g(-w)]^2} < \tilde{p}(w) = \frac{1 + \tau^2w^2}{1 - \tau^2w^2}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2).

**Definition 2.5.** For $\lambda > 0; \lambda \neq \frac{1}{5}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}_{s,\Sigma}^1(\tilde{p}(z))$ if the following subordination hold:

$$\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\lambda < \tilde{p}(z) = \frac{1 + \tau^2z^2}{1 - \tau^2z^2}$$

and

$$\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^\lambda < \tilde{p}(w) = \frac{1 + \tau^2w^2}{1 - \tau^2w^2}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2).
Definition 2.6. For $\lambda > 0; \lambda \neq \frac{1}{3}$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{GSL}_{\alpha, \Sigma}^{\lambda}(\tilde{p}(z))$ if the following subordination hold:

$$
\left( \frac{2[(z(f'(z)))']^\lambda}{[f(z) - f(-z)]^\eta} \right) < \tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}
$$

(2.17)

and

$$
\left( \frac{2[(w(g'(w)))']^\lambda}{[g(w) - g(-w)]^\eta} \right) < \tilde{p}(w) = \frac{1 + \tau^2 w^2}{1 - \tau w - \tau^2 w^2}
$$

(2.18)

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{PSL}_{\alpha, \Sigma}^{\lambda}(\alpha, \tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

Theorem 2.7. Let $f$ be given by (1.1) be in the class $\mathcal{PSL}_{\alpha, \Sigma}^{\lambda}(\alpha, \tilde{p}(z))$. Then

$$
|a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}^2 - 2}}
$$

(2.19)

and

$$
|a_3| \leq \frac{2\lambda|\tau| [2\lambda(\alpha - 2)^2 - \{5\lambda(\alpha - 2)^2 + 4 - 3\alpha\}^2]}{(3\lambda - 1)(3 - 2\alpha) \{4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}^2 - 2\}^2}
$$

(2.20)

where $0 \leq \alpha \leq 1; \lambda > 0$ and $\lambda \neq \frac{1}{3}$.

Proof. Let $f \in \mathcal{PSL}_{\alpha, \Sigma}^{\lambda}(\alpha, \tilde{p}(z))$ and $g = f^{-1}$. Considering (2.7) and (2.8), we have

$$
\left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \right)^\alpha \left( \frac{2[(z(f'(z)))']^\lambda}{[f(z) - f(-z)]^\eta} \right)^{1-\alpha} = \tilde{p}(u(z))
$$

(2.21)

and

$$
\left( \frac{2w(g'(w))^\lambda}{g(w) - g(-w)} \right)^\alpha \left( \frac{2[(w(g'(w)))']^\lambda}{[g(w) - g(-w)]^\eta} \right)^{1-\alpha} = \tilde{p}(v(w))
$$

(2.22)

for some Schwarz functions $u$ and $v$ where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in U$ and $g$ is given by (1.2). Since

$$
\left( \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} \right)^\alpha \left( \frac{2[(z(f'(z)))']^\lambda}{[f(z) - f(-z)]^\eta} \right)^{1-\alpha}
$$

$$
= 1 - 2\lambda(\alpha - 2)a_2z + \{2\lambda^2(\alpha - 2)^2 + 2\lambda(3\alpha - 4)a_2^2 + (3\lambda - 1)(3 - 2\alpha)a_3\}z^2 + \cdots
$$

and

$$
\left( \frac{2w(g'(w))^\lambda}{g(w) - g(-w)} \right)^\alpha \left( \frac{2[(w(g'(w)))']^\lambda}{[g(w) - g(-w)]^\eta} \right)^{1-\alpha}
$$

$$
= 1 + 2\lambda(\alpha - 2)a_2w + \{2\lambda^2(\alpha - 2)^2 + 2\lambda(5 - 3\alpha) + 2\lambda(2\alpha - 3)a_2^2 + (3\lambda - 1)(2\alpha - 3)a_3\}w^2 + \cdots
$$
Thus we have

\[
1 - 2\lambda(\alpha - 2)a_2z + \{2\lambda^2(\alpha - 2)^2 + 2\lambda(3\alpha - 4)|a_2^2 + (3\lambda - 1)(3 - 2\alpha)a_3\}z^2 + \cdots \\
= 1 + \frac{\tilde{p}_1c_1z}{2} + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_1 + \frac{c_1^3}{4}\tilde{p}_2\right]z^2 \\
+ \left[\frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\tilde{p}_1 + \frac{1}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_2 + \frac{c_1^3}{8}\tilde{p}_3\right]z^3 + \cdots. \tag{2.23}
\]

and

\[
1 + 2\lambda(\alpha - 2)a_2w + \{2\lambda^2(\alpha - 2)^2 + 2\lambda(5 - 3\alpha) + 2(2\alpha - 3)|a_2^2 + (3\lambda - 1)(2\alpha - 3)a_3\}w^2 \\
= 1 + \frac{\tilde{p}_1d_1w}{2} + \left[\frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_1 + \frac{d_1^3}{4}\tilde{p}_2\right]w^2 \\
+ \left[\frac{1}{2}(d_3 - d_1d_2 + \frac{d_1^3}{4})\tilde{p}_1 + \frac{1}{2}d_1\left(d_2 - \frac{d_1^2}{2}\right)\tilde{p}_2 + \frac{d_1^3}{8}\tilde{p}_3\right]w^3 + \cdots. \tag{2.24}
\]

It follows from (1.5), (2.23) and (2.24) that

\[-2\lambda(\alpha - 2)a_2 = \frac{c_1\tau}{2}, \tag{2.25}\]

\[2\lambda^2(\alpha - 2)^2 + 2\lambda(3\alpha - 4)|a_2^2 + (3\lambda - 1)(3 - 2\alpha)a_3 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{3}{4}c_1^2\tau^2, \tag{2.26}\]

and

\[2\lambda(\alpha - 2)a_2 = \frac{d_1\tau}{2}. \tag{2.27}\]

\[2\lambda^2(\alpha - 2)^2 + 2\lambda(5 - 3\alpha) + 2(2\alpha - 3)|a_2^2 + (3\lambda - 1)(2\alpha - 3)a_3 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tau + \frac{3}{4}d_1^2\tau^2. \tag{2.28}\]

From (2.25) and (2.27), we have

\[c_1 = -d_1, \tag{2.29}\]

and

\[a_2^2 = \frac{(c_1^2 + d_1^2)}{32\lambda^2(\alpha - 2)^2}\tau^2. \tag{2.30}\]

Now, by summing (2.26) and (2.28), we obtain

\[
[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)]a_2^2 = \frac{1}{2}(c_2 + d_2)\tau - \frac{1}{4}(c_1^2 + d_1^2)\tau + \frac{3}{4}(c_1^2 + d_1^2)\tau^2. \tag{2.31}
\]

By putting (2.30) in (2.31), we have

\[2\left[8\lambda^2(\alpha - 2)^2 - (20\lambda^2(\alpha - 2)^2 - 2(\lambda + 2\alpha - 3))\tau\right]a_2^2 = (c_2 + d_2)\tau^2. \tag{2.32}\]

Therefore, using Lemma 1.3 we obtain

\[|a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2(\alpha - 2)^2 + (10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3)\tau}}. \tag{2.33}\]

Now, so as to find the bound on $|a_3|$, let’s subtract from (2.26) and (2.28). So, we find

\[2(3\lambda - 1)(3 - 2\alpha)a_3 - 2(3\lambda - 1)(3 - 2\alpha)a_3^2 = \frac{1}{2}(c_2 - d_2)\tau. \tag{2.34}\]
Hence, we get
\[ 2(3\lambda - 1)(3 - 2\alpha)|a_3| \leq 2|\tau| + 2(3\lambda - 1)(3 - 2\alpha)|a_2|^2. \quad (2.35) \]

Then, in view of (2.33), we obtain
\[ |a_4| \leq \frac{2\lambda|\tau|\left[2\lambda(\alpha - 2)^2 - \{5\lambda(\alpha - 2)^2 + 4 - 3\alpha\}\right]}{(3\lambda - 1)(3 - 2\alpha)(4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\}\tau)}. \quad (2.36) \]

If we can take the parameter \( \lambda = 1 \) in the above theorem, we have the following the initial Taylor coefficients \( |a_2| \) and \( |a_3| \) for the function classes \( \mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z)) \).

**Corollary 2.8.** Let \( f \) given by (1.1) be in the class \( \mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z)) \). Then
\[ |a_2| \leq \frac{|\tau|}{\sqrt{4(\alpha - 2)^2 - 2(5\alpha^2 - 21\alpha + 21)\tau}} \quad (2.37) \]
\[ |a_3| \leq \frac{|\tau|\left[2(\alpha - 2)^2 - \{5\alpha^2 - 23\alpha + 24\}\right]}{(3 - 2\alpha)(4(\alpha - 2)^2 - \{10\alpha^2 - 42\alpha + 42\}\tau)}. \quad (2.38) \]

Further by taking \( \alpha = 1 \) and \( \alpha = 0 \) and \( \tau = -0.618 \) in Corollary 2.8, we have the following improved initial Taylor coefficients \( |a_2| \) and \( |a_3| \) for the function classes \( \mathcal{SL}_{s,\Sigma}^*(\tilde{p}(z)) \) and \( \mathcal{KL}_{s,\Sigma}(\tilde{p}(z)) \) respectively.

**Corollary 2.9.** Let \( f \) given by (1.1) be in the class \( \mathcal{SL}_{s,\Sigma}^*(\tilde{p}(z)) \). Then
\[ |a_2| \leq \frac{|\tau|}{\sqrt{4 - 10\tau}} \simeq 0.19369 \quad (2.39) \]
\[ |a_3| \leq \frac{|\tau|(1 - 3\tau)}{2 - 5\tau} \simeq 0.3465. \quad (2.40) \]

**Corollary 2.10.** Let \( f \) given by (1.1) be in the class \( \mathcal{KL}_{s,\Sigma}(\tilde{p}(z)) \). Then
\[ |a_2| \leq \frac{|\tau|}{\sqrt{16 - 42\tau}} \simeq 0.0954 \quad (2.41) \]
\[ |a_3| \leq \frac{4|\tau|(1 - 3\tau)}{3(8 - 21\tau)} \simeq 0.17647. \quad (2.42) \]

**Corollary 2.11.** Let \( f \) given by (1.1) be in the class \( \mathcal{PSL}_{s,\Sigma}^*(\tilde{p}(z)) \). Then
\[ |a_2| \leq \frac{|\tau|}{\sqrt{4\lambda^2 - \{10\lambda^2 - \lambda + 1\}\tau}} \quad (2.43) \]
\[ |a_3| \leq \frac{2\lambda|\tau|\left[2\lambda - \{5\lambda + 1\}\right]}{(3\lambda - 1)(4\lambda^2 - \{10\lambda^2 - \lambda + 1\}\tau)} \quad (2.44) \]

where \( \lambda > 0 \) and \( \lambda \neq \frac{1}{3} \).
Corollary 2.12. Let $f$ given by (1.1) be in the class $\mathcal{GS}L^A_{s,\Sigma}(\tilde{p}(z))$. Then

$$|a_2| \leq \frac{|\tau|}{\sqrt{16\lambda^2 - \{40\lambda^2 - \lambda + 3\} \tau}}$$

(2.45)

and

$$|a_3| \leq \frac{2\lambda|\tau| [8\lambda - \{20\lambda + 4\} \tau]}{3(3\lambda - 1) [16\lambda^2 - \{40\lambda^2 - \lambda + 3\} \tau]}$$

(2.46)

where $\lambda > 0$ and $\lambda \neq \frac{1}{3}$.

3 Fekete-Szegö inequality for the function class

$\mathcal{PS}L^A_{s,\Sigma}(\alpha, \tilde{p}(z))$

Fekete and Szegö [7] introduced the generalized functional $|a_3 - \mu a_2^2|$, where $\mu$ is some real number. Due to Zaprawa [19], in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{PS}L^A_{s,\Sigma}(\alpha, \tilde{p}(z))$.

Theorem 3.1. Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let $f$ given by (1.1) be in the class $\mathcal{PS}L^A_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)} & 0 \leq |h(\mu)| \leq \frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)} \\
4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)}
\end{cases}$$

where

$$h(\mu) = \frac{(1 - \mu)\tau^2}{4[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\} \tau]}.$$  

(3.1)

Proof. From (2.32) and (2.34) we obtain

$$a_3 - \mu a_2^2 = \frac{(1 - \mu)(c_2 + d_2)\tau^2}{4[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\} \tau]} + \frac{\tau(c_2 - d_2)}{4(3\lambda - 1)(3 - 2\alpha)}$$

$$= \left( \frac{(1 - \mu)\tau^2}{4[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\} \tau]} + \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)} \right) c_2$$

$$+ \left( \frac{(1 - \mu)\tau^2}{4[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\} \tau]} - \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)} \right) d_2.$$  

So we have

$$a_3 - \mu a_2^2 = \left( h(\mu) + \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)} \right) c_2 + \left( h(\mu) - \frac{\tau}{4(3\lambda - 1)(3 - 2\alpha)} \right) d_2$$

(3.2)

where

$$h(\mu) = \frac{(1 - \mu)\tau^2}{4[4\lambda^2(\alpha - 2)^2 - \{10\lambda^2(\alpha - 2)^2 - \lambda - 2\alpha + 3\} \tau]}.$$  

Then, by taking modulus of (3.2), we conclude that
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)}. \end{cases} \]

Taking \( \mu = 1 \), we have the following corollary.

**Corollary 3.2.** If \( f \in \mathcal{P}\mathcal{S}\mathcal{L}_{s, \Sigma}^{\lambda}(\alpha, \tilde{p}(z)) \), then

\[ |a_3 - a_2^2| \leq \frac{|\tau|}{4(3\lambda - 1)(3 - 2\alpha)}. \tag{3.3} \]

If we can take the parameter \( \lambda = 1 \) in Theorem 3.1, we can state the following:

**Corollary 3.3.** Let \( f \) given by (1.1) be in the class \( \mathcal{M}\mathcal{S}\mathcal{L}_{s, \Sigma}^{\ast}(\alpha, \tilde{p}(z)) \) and \( \mu \in \mathbb{R} \). Then we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{8(3 - 2\alpha)}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{8(3 - 2\alpha)} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{8(3 - 2\alpha)} \end{cases} \]

where

\[ h(\mu) = \frac{(1 - \mu)\tau^2}{4[4(\alpha - 2)^2 - \{10(\alpha - 2)^2 - 2\alpha + 2\}\tau]}. \]

Further by fixing \( \lambda = 1 \) taking \( \alpha = 1 \) and \( \alpha = 0 \) in the above corollary, we have the following the Fekete-Szegö inequalities for the function classes \( \mathcal{S}\mathcal{L}_{s, \Sigma}^{\ast}(\tilde{p}(z)) \) and \( \mathcal{K}\mathcal{L}_{s, \Sigma}(\tilde{p}(z)) \), respectively.

**Corollary 3.4.** Let \( f \) given by (1.1) be in the class \( \mathcal{S}\mathcal{L}_{s, \Sigma}^{\ast}(\tilde{p}(z)) \) and \( \mu \in \mathbb{R} \). Then we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{24}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{24} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{24} \end{cases} \]

where \( h(\mu) = \frac{(1 - \mu)\tau^2}{8[2 - 5\tau]} \).

**Corollary 3.5.** Let \( f \) given by (1.1) be in the class \( \mathcal{K}\mathcal{L}_{s, \Sigma}(\tilde{p}(z)) \) and \( \mu \in \mathbb{R} \). Then we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{8}, & 0 \leq |h(\mu)| \leq \frac{|\tau|}{8} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{8} \end{cases} \]

where \( h(\mu) = \frac{(1 - \mu)\tau^2}{8[8 - 21\tau]} \).

By assuming \( \lambda \in \mathbb{R}; \lambda > \frac{1}{3} \) and taking \( \alpha = 1 \) and \( \alpha = 0 \) we have the following the Fekete-Szegö inequalities for the function classes \( \mathcal{P}\mathcal{S}\mathcal{L}_{s, \Sigma}^{\lambda}(\tilde{p}(z)) \) and \( \mathcal{G}\mathcal{S}\mathcal{L}_{s, \Sigma}^{\lambda}(\tilde{p}(z)) \), respectively.
Corollary 3.6. Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let $f$ given by (1.1) be in the class $\mathcal{PSL}^\lambda_{s, \Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\tau}{4(3\lambda - 1)}, & 0 \leq |h(\mu)| \leq \frac{\tau}{4(3\lambda - 1)} \\
4|h(\mu)|, & |h(\mu)| \geq \frac{\tau}{4(3\lambda - 1)}
\end{cases}
\]
where
\[
h(\mu) = \frac{(1 - \mu)\tau^2}{4[4\lambda^2 - \{10\lambda^2 - \lambda + 1\}\tau]}.
\]

Corollary 3.7. Let $\lambda \in \mathbb{R}$ with $\lambda > \frac{1}{3}$ and let $f$ given by (1.1) be in the class $\mathcal{GSL}^\lambda_{s, \Sigma}(\tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\tau}{12(3\lambda - 1)}, & 0 \leq |h(\mu)| \leq \frac{\tau}{12(3\lambda - 1)} \\
4|h(\mu)|, & |h(\mu)| \geq \frac{\tau}{12(3\lambda - 1)}
\end{cases}
\]
where
\[
h(\mu) = \frac{(1 - \mu)\tau^2}{4[16\lambda^2 - \{40\lambda^2 - \lambda + 3\}\tau]}.
\]

Conclusions

Our motivation is to get many interesting and fruitful usages of a wide variety of Fibonacci numbers in Geometric Function Theory. By defining a subclass $\lambda$-bi-pseudo-starlike functions with respect to symmetric points of $\Sigma$ related to shell-like curves connected with Fibonacci numbers we were able to unify and extend the various classes of analytic bi-univalent function, and new extensions were discussed in detail. Further, by specializing $\alpha = 0$ and $\alpha = 1$ and $\tau = -0.618$ we have attempted at the discretization of some of the new and well-known results. Our main results are new and better improvement to initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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