Modified of Homotopy Perturbation Technique and Its Application to System of Nonlinear Fredholm Integral Equations Of 2nd Kind

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Abstract. Fredholm integral equations of 1st and 2nd kinds are of practical importance and have wide range of applications. The present paper, deals mainly with system of non-linear Fredholm equations of the 2nd kind. In the present paper, the homotopy perturbation technique in different version from normal version is applied. The new version of the perturbation method confirms the simplicity and efficiency of the proposed method compared with other approximate solutions; also it confirms that this method is a suitable method for solving any nonlinear Fredholm Integral Equations of 2nd Kind and / or systems of nonlinear Fredholm integral equations of 2nd kind. In the present paper, a new version of the homotopy perturbation technique is applied to solve system of nonlinear Fredholm integral equations. The new version based on the idea of considering the solution as a sum of an infinite series which is very rapid convergence to the accurate solution. The results due to the present version of the homotopy perturbation technique gave promises for further developing other issues of the homotopy perturbation method. The results due to the present method are compared with Adomain decomposition method.

Keywords: Fredholm integral equation, system of nonlinear Fredholm integral equations, homotopy perturbation method.

1. Introduction

In spite of that the integral equations of 1st and 2nd kind are old topics in applied mathematics but it still looks new due to its important practical applications for scientist, technology and engineering [1]. Recent approximate and numerical methods based meshing techniques like finite elements method [2], finite volume method [3] and boundary elements methods [4], all of these methods based on the criteria of reduction of the governing partial differential equations to integral equations, then using different meshing techniques to solve the obtained integral equations. Treatment of pure integral equations with different types and kinds require special methodology. One of these methods is the so called homotopy perturbation method
The application of the HPM in nonlinear problems and many other subjects has been intensively studied by scientists and engineers, especially in integral equations, because this method deforms the difficult problem under study into a simple problem which is easy to solve. The homotopy perturbation method is applied to systems of nonlinear coupled equations [6], nonlinear Volterra-Hammerstein integral equations [7], systems of Volterra integral equations of 1st kind [8-9], system of integro-differential equation [10], systems of partial differential equations [11], and a special kind of Volterra integral equations in 2-D space [12].” Exact solutions for nonlinear integral equations by a modified homotopy perturbation method can be found in [13]”. Recently Javidi and Golbabai applied homotopy perturbation method for solving system of linear Fredholm integral equations with difference kernel. In the present paper, the homotopy perturbation method in different version from normal version is applied. The new version of the perturbation method confirms the simplicity and efficiency of the proposed method compared with other approximate solutions; also it confirms that this method is a suitable method for solving” any nonlinear Fredholm Integral Equations of 2nd kind and/or systems of nonlinear Fredholm integral equations of 2nd kind". In the present paper, a new version of the homotopy perturbation method is applied to solve system of nonlinear Fredholm integral equations. The new version based on the idea of considering the solution as a sum of an infinite series which is very rapid convergence to the accurate solution. The results due to the present version of the homotopy perturbation method gave promises for further developing other issues of the homotopy perturbation method.

2. Mathematical manipulation

Consider the following system of nonlinear Fredholm integral equations of second kind of the following form:

\[ \mathcal{G}_i(N) = g_i(N) + \int_a^b \kappa_i(N, t, \mathcal{G}_j(t)) \, dt, \quad j = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, m \]  

(1)

In equation (1):

The function \( \mathcal{G}_j(N) \) and the kernels \( \kappa_i(N, t, \mathcal{G}_j(t)) \) are known in advance, while \( \mathcal{G}_j(t) \) are solutions to be found, note that in the following we will assume that the system given by equation (1) will has a unique solution. Starting the solution by constructing homotopy for the system and this homotopy will satisfy the following conditions:

\[ H_i(F, p) = (1 - p)G_i(F) + pL_i(F) \]  

(2)

\[ H_i(F, 0) = G_i(F) \]  

(3)

\[ H_i(F, 1) = L_i(F) \]  

Where

\[ G_i(F) = \mathcal{G}_i(N) - g_i(N) \]

\[ L_i(F) = \mathcal{G}_i(N) - g_i(N) - \int_a^b \kappa_i(N, t, \mathcal{G}_j(t)) \, dt, \quad j = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, m \]  

(4)

\[ F(x) = [\mathcal{G}_1(N), \mathcal{G}_2(N), \ldots, \mathcal{G}_n(N)] \]

\[ p[0,1] \text{is an embedding parameter} \]

Assuming that" the solution of (1) can be expressed in a series of \( p^n \) as follows:

\[ \mathcal{G}_i(N) = \mathcal{G}_i_0(N) + p\mathcal{G}_i_1(N) + p^2\mathcal{G}_i_2(N) + \ldots + p^n\mathcal{G}_i_n(N), \quad n \geq 2 \]  

(5)

We reach the exact solution as \( n \to \infty \text{ & } p \to 1 \)

By making use of equation (5) into equation (2), leads to:
H_1(F, p) = (1 - p)\left[\mathcal{Z}_{i_0}(N) + p\mathcal{Z}_{i_1}(N) + p^2\mathcal{Z}_{i_2}(N) + \ldots + p^n\mathcal{Z}_{i_n}(N) - g_i(N)\right] \\
+ p \left\{ \mathcal{Z}_{i_0}(N) + p\mathcal{Z}_{i_1}(N) + \ldots + p^n\mathcal{Z}_{i_n}(N) - g_i(N) \right\} dt \\
\text{Assuming that:} \\
h_i(N, p, n) = \mathcal{Z}_{i_0}(N) + p\mathcal{Z}_{i_1}(N) + p^2\mathcal{Z}_{i_2}(N) + \ldots + p^n\mathcal{Z}_{i_n}(N) \\
\text{Then} \\
H_1(F, p) = (1 - p)\left[h_i(N, p, n) - g_i(N)\right] - p \\
- \int_a^b \kappa_i(N, t, h_i(N, p, n), \ldots, h_m(N, p, n)) dt = 0 \\
\text{By equating coefficients of similar powers of } p, \text{ leads to:} \\
p^0: \mathcal{Z}_{i_0} = g_i \\
p^1: \mathcal{Z}_{i_1} = -\int_a^b \kappa_i(N, t, h_1(t, 1, n), \ldots, h_m(t, 1, 0)) dt \\
p^2: \mathcal{Z}_{i_2} = -\int_a^b \kappa_i(N, t, h_1(t, 1, 1) - h_1(t, 1, 0), h_2(t, 1, 1) - h_2(t, 1, 0)) dt \\
p^n: \mathcal{Z}_{i_n} = -\int_a^b \kappa_i(N, t, h_1(t, \ell, n - 1) - h_1(t, \ell, n - 2), h_2(t, \ell, n - 1) - h_2(t, \ell, n - 2), \ldots, h_m(t, \ell, n - 1) - h_m(t, \ell, n - 2)) dt \\
\text{By making use of the above equations, we assumed the solution of the system given by (1) is given as:} \\
\mathcal{Z}_i^N(x) = \sum_{n=0}^{N} \mathcal{Z}_{i_n} \quad \forall i = 1, 2, \ldots, m \\
\text{3. Numerical results} \\
\text{In this section, we will present and discuss numerical results using the symmetry perturbation technique in two test examples} \\
\text{Example} \\
\text{Solve the following system of the two nonlinear Fredholm integral equations, whose exact solution is known as } \mathcal{Z}_1(N) = N \text{ and } \mathcal{Z}_2(N) = N^2, \text{ the system are:} \\
\mathcal{Z}_1(N) = x - 0.27 + 0.33 \int_0^1 (\mathcal{Z}_1(t) + \mathcal{Z}_2(t)) dt \\
\mathcal{Z}_2(N) = x^2 - 0.2 + 0.33 \int_0^1 (\mathcal{Z}_1^2(t) + \mathcal{Z}_2(t)) dt
Fellow up the homotopy procedure described before, we get the following:
\[ \mathcal{A}_{10}(\mathcal{N}) = \mathcal{N} - 0.27, \mathcal{A}_{11} = 0.2, \mathcal{A}_{12} = 0.064, \mathcal{A}_{19} = 0.003 \]
\[ \mathcal{A}_{20}(\mathcal{N}) = \mathcal{N}^2 - 0.27, \mathcal{A}_{21} = 0.081, \mathcal{A}_{22} = 0.043, \mathcal{A}_{29} = 0.003 \]
(10)

By making use of equations (10), into (9), leads to:
\[ \mathcal{A}_i^2(\mathcal{N}) = \sum_{n=0}^{2} \mathcal{A}_n \]
\[ \mathcal{A}_i^9(\mathcal{N}) = \sum_{n=0}^{9} \mathcal{A}_n, \quad i = 1, 2 \]

Leads to:
\[ \mathcal{A}_1^2(\mathcal{N}) = \mathcal{N} - 0.06664 \]
\[ \mathcal{A}_2^2(\mathcal{N}) = \mathcal{N}^2 - 0.09739 \]
\[ \mathcal{A}_1^9(\mathcal{N}) = \mathcal{N} - 0.06926 \]
\[ \mathcal{A}_2^9(\mathcal{N}) = \mathcal{N}^2 - 0.01088 \]

![Graph](image)

**Figure (1):** Graphs of \( \mathcal{A}_1^2(\mathcal{N}), \mathcal{A}_2^2(\mathcal{N}), \mathcal{A}_1^9(\mathcal{N}), \mathcal{A}_2^9(\mathcal{N}) \)

The overall numerical results are shown in the next table (1)

| \( x \) | \( \| \mathcal{A}_1^2(\mathcal{N}) \| \) | \( \| \mathcal{A}_2^2(\mathcal{N}) \| \) | \( \| \mathcal{A}_1^9(\mathcal{N}) \| \) | \( \| \mathcal{A}_2^9(\mathcal{N}) \| \) | \( \mathcal{A}_1(\mathcal{N}) \) | \( \mathcal{A}_2(\mathcal{N}) \) |
|---|---|---|---|---|---|---|
| 0 | 0.0658 | 0.0647 | 0.0053 | 0.0351 | 0.06432 | 0.03502 |
| 0.2 | 0.0659 | 0.0649 | 0.0052 | 0.0355 | 0.06553 | 0.03509 |
| 0.4 | 0.0652 | 0.0644 | 0.0054 | 0.0355 | 0.06522 | 0.03521 |
| 0.6 | 0.0653 | 0.0645 | 0.0051 | 0.0358 | 0.06512 | 0.03501 |
| 0.8 | 0.6549 | 0.6546 | 0.0051 | 0.0353 | 0.06547 | 0.03521 |
| 1.0 | 0.639 | 0.643 | 0.0052 | 0.0350 | 0.06388 | 0.03525 |

**Table (1):** Results due to the present method for 1st example
Example
Solve the following system of the two nonlinear Fredholm integral equations, whose exact solution is known as $\mathcal{I}_1(N) = N - 1$ and $\mathcal{I}_2(x) = N^2$, the system are:

\[
\mathcal{I}_1(N) = -1.1 + 0.916N + \int_0^1 \left(N/\mathcal{I}_1^2(t) + t^2 \mathcal{I}_1^2(t)\right) dt
\]

\[
\mathcal{I}_2(N) = -0.003 - 0.014N + 0.8N^2 + \int_0^1 \left(t^2 \mathcal{I}_2^2(t) - N \mathcal{I}_2^2(t)\right) dt
\]

Follow up the homotopy procedure described before, we get the following:

\[
\mathcal{I}_{10}(N) = -1.1 + 0.916N,
\]
\[
\mathcal{I}_{11}(N) = 0.052 + 0.148N,
\]
\[
\mathcal{I}_{12}(N) = 0.034 - 0.068N,
\]
\[
\mathcal{I}_{19}(N) = 2.697 \times 10^{-6} + 3.2 \times 10^{-4}N
\]
\[
\mathcal{I}_{20}(x) = -3.9 \times 10^{-4} - 0.14N + 0.8N^2,
\]
\[
\mathcal{I}_{21}(N) = 0.013 + 0.031N + 0.056N^2,
\]
\[
\mathcal{I}_{22}(N) = -0.012 - 0.018N + 0.056N^2,
\]
\[
\mathcal{I}_{29}(N) = 3 \times 10^{-4} - 1.2 \times 10^{-5}N + 1.3 \times 10^{-5}N^2
\]

By making use of equations (11), into (9), leads to:

\[
\mathcal{I}_i^2(N) = \sum_{n=0}^{2} \mathcal{I}_m
\]

\[
\mathcal{I}_i^3(N) = \sum_{n=0}^{2} \mathcal{I}_m, \ i = 1,2
\]

Leads to:

\[
\mathcal{I}_1^2(N) = 1.018N - 1.012
\]
\[
\mathcal{I}_2^2(N) = 0.98732N^2 + 0.0028N + 0.0023
\]
\[
\mathcal{I}_1^3(N) = 1.0001N - 1.0002
\]
\[
\mathcal{I}_2^3(N) = 0.9997N^2 - 0.45N - 0.0007
\]

Figure (2): Graph of $\mathcal{I}_1^2(N)$ and $\mathcal{I}_2^2(N)$
Figure (3): Graph of $\mathcal{X}_1^0(N)$ and $\mathcal{X}_2^0(N)$

The overall numerical results are shown in the next table (2).

| $x$  | $\|\mathcal{X}_1^0(N)\|$ | $\|\mathcal{X}_2^0(N)\|$ | $\|\mathcal{X}_3^0(N)\|$ | $\|\mathcal{X}_4^0(N)\|$ | $\mathcal{X}_1(N)$ | $\mathcal{X}_2(N)$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0    | 0.012 | 0.023 | 0.0023 | 0.0001 | 0.01201 | 0.00011 |
| 0.2  | 0.0984 | 0.023 | 0.0021 | 0.0001 | 0.09772 | 0.00252 |
| 0.4  | 0.076 | 0.013 | 0.0017 | 0.0009 | 0.07621 | 0.00087 |
| 0.6  | 0.055 | 0.005 | 0.0014 | 0.0004 | 0.05523 | 0.00139 |
| 0.8  | 0.033 | 0.036 | 0.0011 | 0.0003 | 0.03211 | 0.00025 |
| 1.0  | 0.012 | 0.075 | 0.0700 | 0.0001 | 0.01211 | 0.06972 |

From figure (1) and table (1), it is clear that there is a good agreement between the analytical solution and the results due to the modern version of the homotopy perturbation technique. There is very small errors less than $10^{-6}$ which can neglected, the same conclusion regarding the results of the 2$^{nd}$ example as it is clear in figures (2-3) and table (2).

4. Conclusion
In the present paper, we developed a modern version of the homotopy perturbation technique to solve system of nonlinear Fredholm integral equations of 2$^{nd}$ kind. Firstly, it is concluded that its mathematical manipulation was quite simple compared with other methods. Also, it was observed that the proposed approach is straightforward in computations and very effective in both linear and nonlinear cases. The obtained results showed that the new version of the homotopy perturbed expressed can be considered as one of the most powerful and efficient method for solving such nonlinear system of Fredholm integral equations of both 1$^{st}$ and 2$^{nd}$ kinds.
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