On Generalizations of the Spectrum Condition

RAINER VERCH

Institut für Theoretische Physik,
Universität Göttingen,
Bunsenstr. 9,
D-37073 Göttingen, Germany

E-mail: verch@theorie.physik.uni-goettingen.de

Dedicated to Sergio Doplicher and John E. Roberts
on the occasion of their 60th birthdays

Abstract. It is well known that the spectrum condition, i.e. the positivity of the energy in every inertial coordinate system, is one of the central conceptual ingredients in model-independent approaches to relativistic quantum field theory. When one attempts to formulate quantum field theory in a model-independent manner on a curved background spacetime, it is not immediately clear which concepts replace the spectrum condition. The present work is devoted to reviewing facets of this situation, thereby focussing on one particular approach that attempts to generalize the notion of energy-momentum spectrum by the notion of “wavefront set”, which may be seen as an asymptotic high-frequency part of the spectrum.

1 Introduction

The relativistic spectrum condition, stating that in relativistic quantum field theory the spectrum of the energy should be positive in all inertial Lorentz-frames, is one of the basic ingredients in all model-independent approaches to quantum field theory, and together with the principle of locality, it is responsible for remarkable results of the model-independent approach, such as the Reeh-Schlieder Theorem, the PCT-Theorem, and the connection between spin and statistics [22, 37, 18]. Moreover, more recent developments have clearly indicated that there is in quantum field theory a deep and subtle connection between the Tomita-Takesaki modular objects and the spectrum condition. This is a very active line of research, and promises to provide new and interesting insights concerning the operator algebraic structure of quantum field theory. The reader is referred to the recent review by Borchers [4] on these matters.

1 Contributed to the proceedings of “Mathematical Physics in Mathematics and Physics”, held in Siena, June 20-25, 2000, in honour of Sergio Doplicher and John E. Roberts
Stimulated by developments during, roughly, the past three decades, it has been realized that quantum field theory in curved spacetime (QFT in CST) is a subject that promises to have physical relevance (see [14, 43] as general references). The major impetus came from Hawking’s theoretical arguments for particle emission by black holes [19] derived in the framework of QFT in CST (cf. also [12]). There are other phenomena which also belong to the area of QFT in CST, like the Casimir effect [7, 24, 14], whose experimental verification has recently reached an astonishing degree of accuracy [26]. As regards relevance to cosmology, there are suggestions that by QFT in CST methods one may account for the recently observed accelerated expansion of the universe [29]. Furthermore, one may view QFT in CST as a preliminary, semiclassical approach to quantum effects in gravitation, hoping that the insights gained from QFT in CST may provide some guideline at least towards rudiments of that much sought for theory of quantum gravity.

At any rate, there is reason enough to consider the mathematical and conceptual foundations of QFT in CST a subject worthy of interest. When embarking on that subject, one notices right at the beginning that on a spacetime manifold, isometry groups are generically absent, and so the usual, flat space version of the spectrum condition can obviously not be formulated. In connection to this circumstance, there is no natural candidate for a vacuum state and, in turn, there is no natural choice of a set of physical states.

A guideline to finding a replacement of the spectrum condition for QFT in CST originated from the study of free fields. An important initial step was the approach by Wald [41] to defining the expectation value of the energy-momentum tensor for states of a free scalar field whose two-point functions are of Hadamard form. Following a number of investigations (see, e.g. [14, 43] and references therein), it was realized that those Hadamard states are a good choice of a set of physical states, comprising e.g. the set of so-called “adiabatic vacuum states” that had been proposed by Parker for cosmological model spacetimes (cf. [28, 27, 23]). A major advance in the understanding of why Hadamard states are in a sense “vacuum-like” as regards their spectral behaviour was reached in Radzikowski’s PhD thesis [31]. Radzikowski showed that for a free scalar field the Hadamard form of a two-point function can be characterized, in a one-to-one fashion, by a specific form of the wavefront set of that two-point function. This specific form can naturally be read as the generally covariant version of the form of the wavefront set of a vacuum two-point function in flat space; it is in a certain way asymmetric and this signifies a high-frequency, short-distance remnant of the spectrum condition (namely, the conicity of the energy-momentum spectrum).

The characterization of Hadamard form in terms of conditions on the form of the wavefront set has another advantage: The Hadamard form can only be prescribed for fields whose dynamics is governed by a free wave-equation, while conditions on the wavefront set of \(n\)-point functions generalizing the form of the wavefront set of \(n\)-point vacuum expectation values in flat spacetime can be formulated also for arbitrary, interacting field theories. This route has been taken in [4], where suggestions for conditions on the wavefront set of \(n\)-point functions of scalar fields on curved spacetime have been made which are to be viewed as generalizing the flat space spectrum condition. With the help of these conditions, which are now referred to as “microlocal
spectrum condition”, abbreviated µSC, it is possible to define Wick-powers, and their
time-ordered products, of free quantum field theories on curved spacetime, and more-
over, to modify the “causal perturbation theory” approach by Epstein and Glaser so
as to obtain a causal, local, perturbative construction of $P(\phi)_4$ scalar quantum field
theories on general (globally hyperbolic) curved spacetime. These very interesting
results have been obtained recently by Brunetti and Fredenhagen [3].

However, the notion of wavefront set, and therefore, concepts of a microlocal
spectrum condition, so far required the formulation of a quantum field theory in terms
of quantum fields, or “Wightman distributions”. From a purely operator-algebraic
point of view, it appears highly desirable to have a generalized notion of the wavefront
set concept which is directly applicable to algebraic quantum field theory. We have
made a first attempt in that direction in [10], where we defined the “asymptotic
correlation spectrum” of a state, which may be viewed as the generalization of the
wavefront set to the operator-algebraic framework of quantum field theory. Much of
the present work (Sections 4 and 5) is devoted to this topic.

This work is organized as follows. In Sec. 2 we recall a few basic facts about the
spectrum condition in flat spacetime. Sec. 3 is concerned with a summary of aspects
of quantum field theory in curved spacetime. It begins with a collection of some gen-
eral facts in Sec. 3.1. In Sec. 3.2, the free scalar field is considered. The microlocal
spectrum condition will be discussed in Sec. 3.3. In Sec. 4 we present the counterpart
of the wavefront set concept in algebraic quantum field theory, the “asymptotic cor-
relation spectrum” of a state. Most of the material in that section is taken from [10].
In Sec. 5 we discuss some generalizations of the asymptotic correlation spectrum to
quantum field theory in curved spacetime.

2 Spectrum Condition

Our discussion will be staged in the framework of the operator-algebraic approach
to quantum field theory, and so we begin by recalling the basic structures of that
framework. This will be done, to start with, for Minkowski-space $\mathbb{R}^4$ as underlying
spacetime (of dimension = 4, but the setting is easily generalized to any dimension
$\geq 2$).

In the operator-algebraic approach a quantum field theory is described by a col-
lection of objects $(\{A(\mathcal{O})\}_{\mathcal{O}\subset \mathbb{R}^4}, (\alpha_x)_{x\in \mathbb{R}^4}, \omega^0)$, where it is assumed that the following
properties hold:

(a) $\{A(\mathcal{O})\}_{\mathcal{O}\subset \mathbb{R}^4}$ is a local net of $C^*$-algebras indexed by the bounded open regions
$\mathcal{O}$ in $\mathbb{R}^4$, i.e. all $A(\mathcal{O})$ are $C^*$-algebras containing a common unit element, and
the conditions of

- isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow A(\mathcal{O}_1) \subset A(\mathcal{O}_2)$, and
- locality: $\mathcal{O}_1 \subset \mathcal{O}^\bot_2 \Rightarrow A_1A_2 = A_2A_1 \ \forall A_j \in A(\mathcal{O}_j)$

are fulfilled. Here $\mathcal{O}^\bot_2$ denotes the causal complement set of $\mathcal{O}_2$ in the underlying
spacetime (here, Minkowski-space).
(b) \((\alpha_x)_{x \in \mathbb{R}^4}\) is an automorphism group acting on the local net, i.e. the \(\alpha_x\) are automorphisms of \(\mathcal{A}\), the smallest \(C^\ast\)-algebra generated by all the \(\mathcal{A}(\mathcal{O})\), and there holds 
\[\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + x),\]
expressing that the \(\alpha_x\) act covariantly as translations.

(c) \(\omega^0\) is a vacuum state, i.e. \(\omega^0\) is a state on \(\mathcal{A}\) so that \(x \mapsto \omega^0(A\alpha_x(B)C)\) is continuously for all \(A, B, C \in \mathcal{A}\), and moreover, for all \(f \in S(\mathbb{R}^4)\) whose Fourier-transforms \(\hat{f}\) have support outside of the closed forward lightcone \(\overline{V}_+\), it holds that
\[
\int f(x) \omega^0(A^\ast \alpha_x(A)) \, d^4x = 0 \quad \forall \, A \in \mathcal{A}.
\]

One may then consider the GNS-representation \((\mathcal{H}^0, \pi^0, \Omega^0)\) of \(\mathcal{A}\) corresponding to the vacuum state \(\omega^0\). The von Neumann algebras \(\pi^0(\mathcal{A}(\mathcal{O}))\) will be denoted by \(\mathcal{R}(\mathcal{O})\). They contain all the observables of the underlying quantum field theory which can be measured at times and locations in the spacetime region \(\mathcal{O}\). Since \(\omega^0\) is a vacuum state, it follows that there is a continuous unitary group \((U(x))_{x \in \mathbb{R}^4}\) implementing \((\alpha_x)_{x \in \mathbb{R}^4}\) in the GNS-representation \(\pi^0\), and the GNS-vector \(\Omega^0\) is left invariant under the action of \(U(x)\), as can be deduced from (2.1). Moreover, (2.1) implies that the unitary group \((U(x))_{x \in \mathbb{R}^4}\) fulfills the spectrum condition. And this means that, if \((P_\mu) = (P_0, P_1, P_2, P_3)\) denote the generators of the unitary group, so that \(U(x) = e^{i P_\mu x^\mu}\), then \((P_0)^2 \geq (P_1)^2 + (P_2)^2 + (P_3)^2\). In other words, the joint spectrum of the \(P_\mu\) is contained in \(\overline{V}_+\). The vacuum state \(\omega^0\) is therefore a translation-invariant state of lowest energy; the existence of such a state may be interpreted as a stability property of the dynamics governing the quantum field theory described by \((\{\mathcal{A}(\mathcal{O})\})_{\mathcal{O} \in \mathbb{R}^4}, (\alpha_x)_{x \in \mathbb{R}^4}, \omega^0)\).

The above stated conditions (a,b,c) may be viewed as minimal conditions for the mathematical description of a quantum field theory in the operator-algebraic framework. They are usually supplemented by further conditions expressing additional properties of the quantum field theory to be described. (See [18] for ample discussion.) One such condition is, e.g., Poincaré-covariance. Another condition typically imposed is that \(\omega^0\) be a pure state on \(\mathcal{A}\), which can be shown to be equivalent to asymptotic spacelike clustering. A further condition is to strengthen weak continuity (with respect to the vacuum folium) of \((\alpha_x)_{x \in \mathbb{R}^4}\) to strong continuity, meaning that \(||\alpha_x(A) - A|| \to 0\) for \(x \to 0\) holds for all \(A \in \mathcal{A}\). (As has been pointed out in [8], given the vacuum representation \(\pi^0\) or any other representation of \(\mathcal{A}\) in which \((\alpha_x)_{x \in \mathbb{R}^4}\) acts weakly continuously, each \(\mathcal{R}(\mathcal{O})\) contains a weakly dense subalgebra on which the action of \((\alpha_x)_{x \in \mathbb{R}^4}\) is strongly continuous. Thus the assumption of strong continuity for the translations doesn’t appear to be too restrictive.)

Under the above stated assumptions (a) and (b) together with strong continuity of \((\alpha_x)_{x \in \mathbb{R}^4}\), Doplicher [3] proved that \(\mathcal{A}\) admits a vacuum state \(\omega^0\) if and only if the spectral ideal \(\mathcal{J} \subset \mathcal{A}\) is proper. Here, the spectral ideal \(\mathcal{J}\) is the left ideal in \(\mathcal{A}\) generated

\[V_+ = \{x \in \mathbb{R}^4 : (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 > 0, \, x^0 > 0\}.\]
by all $A$ having the property that $\int f(x) \alpha_x(A) \, d^4x = 0$ holds for all $f \in L^1(\mathbb{R}^4)$ whose Fourier-transforms are supported outside of $V_+$. This characterization shows that the existence of vacuum states may be seen as a property of the algebraic structure of the algebra of observables $\mathcal{A}$ relative to the action of the translations.

### 3 Quantum Field Theory on Curved Spacetime

#### 3.1 Generalities

The flat space spectrum condition clearly hinges upon the presence of the translation group. When considering quantum field theories on a curved spacetime, there is in general no counterpart of the translation group, and in general, there is not even any time-symmetry group. These circumstances make it difficult to formulate what should be a vacuum state for a quantum field theory on a curved spacetime. Even worse, it is not even clear what the set of physical states should be for quantum fields in a curved spacetime. Let us briefly recall how the set $S_{\text{phys}}$ of physical states may be determined from the vacuum state for a given quantum field theory $(\{\mathcal{A}(O)\}_{O \subset \mathbb{R}^4}, \{\alpha_x\}_{x \in \mathbb{R}^4}, \omega^0)$ on Minkowski spacetime: Here, one usually takes $S_{\text{phys}}$ to consist of all states $\omega$ on $\mathcal{A}$ which are locally normal to the vacuum state $\omega^0$. This means that there is for each bounded open region $O \subset \mathbb{R}^4$ a density matrix $\rho_{\omega,O}$ on $\mathcal{H}^0$ so that

$$\omega(A) = \text{Tr}(\rho_{\omega,O} \cdot \pi^0(A))$$

holds for all $A \in \mathcal{A}(O)$. That definition of “physical state” is on one hand broad enough and allows “charged” states (from which charge-carrying fields, originally not contained in $\mathcal{A}$, can be constructed [10]), on the other hand it avoids pathologies like states having infinite particle density which would be highly unphysical [18, 32].

However, one can outline a basic approach to the description of QFT in CST in the operator-algebraic setting. (We should like to mention that the formulation we are going to give here is patterned after several precursors, as e.g. given in [8, 16]. No claim of originality is made at this stage.) To this end, a curved spacetime will be modelled mathematically by a pair $(M, g)$ where $M$ is a 4-dimensional smooth manifold and $g$ a smooth metric on this manifold of Lorentzian signature. To avoid any causal pathologies, we shall assume that $(M, g)$ is globally hyperbolic. This means that the manifold $M$ can be smoothly foliated in Cauchy-surfaces, where a Cauchy-surface is a 3-dimensional sub-manifold which is intersected exactly once by each inextendible, $g$-causal curve in $M$. We refer to [20, 42] for further discussion and presentation of examples; it should nevertheless be mentioned that the class of globally hyperbolic spacetimes contains most of the spacetime models thought to describe physically relevant situations (like Robertson-Walker, de Sitter, Schwarzschild-Kruskal and, of course, Minkowski-spacetime). Also, it is worth mentioning that global hyperbolicity isn’t related to the existence of spacetime isometries.

Assuming now that a globally hyperbolic spacetime $(M, g)$ has been given to us, we formulate the basic mathematical structure of a quantum field theory on this “background spacetime” as being described by a collection of objects $(\{\mathcal{A}(O)\}_{O \subset M}, \{\alpha_\gamma\}_{\gamma \in G}, S^0_{\text{phys}})$ with the following properties:
(a') \( \{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset M} \) is a family assigning to each bounded (i.e., relatively compact) open region \( \mathcal{O} \) in \( M \) a \( \mathcal{C}^* \)-algebra \( \mathcal{A}(\mathcal{O}) \) in such a way that all these algebras have a common unit element and so that the conditions of isotony and locality (which can be taken over literally from (a) above) are fulfilled.

(b') \( G \) denotes the group of proper, orthochronous isometries of the spacetime \( (M, g) \), and \( (\alpha_\gamma)_{\gamma \in G} \) is a group of automorphisms of \( \mathcal{A} \), the \( \mathcal{C}^* \)-algebra generated by all \( \mathcal{A}(\mathcal{O}) \), with \( \alpha_{\gamma_1} \alpha_{\gamma_2} = \alpha_{\gamma_1 \gamma_2} \) and the covariance property

\[
\alpha_\gamma(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\gamma(\mathcal{O})).
\]

(There may frequently occur the case that \( G \) contains just the identical map; then the present condition is effectively void.)

(c') \( S^0_{\text{phys}} \) is a subset of the set of physical states, selected by a suitable generalization of the spectrum condition. The GNS-representations \( \pi_1 \) and \( \pi_2 \) of \( \mathcal{A} \) corresponding to any pair of states \( \omega_1, \omega_2 \in S_{\text{phys}}^0 \) are assumed to be locally quasi-equivalent (quasi-equivalent when restricted to \( \mathcal{A}(\mathcal{O}) \) for any bounded region \( \mathcal{O} \)), and we suppose that \( \gamma \mapsto \omega(\alpha_\gamma(B)C) \) is continuous (for \( \gamma \) ranging over continuous parts of \( G \)).

It is clear that (a’) and (b’) are natural generalizations of (a) and (b) above with similar meaning. Concerning (c’), what has essentially been changed in comparison to (c) is that the existence of one particular distinguished state has been replaced by a whole set of states which are supposed to be distinguished by a certain, generalized form of the spectrum condition. The present formulation of a mathematical framework is again to be viewed as, in a sense, consisting of “minimal” requirements; as we shall see, to make precise mathematical sense of “generalized form of the spectrum condition” in the present abstract operator-algebraic setting one needs additional structure, in particular a generalization of (b’) is needed for the case that \( G \) is very small. In the case that the operator-algebras \( \pi(\mathcal{A}(\mathcal{O})))'' \) for \( \pi \) a GNS-representation of an \( \omega \in S_{\text{phys}}^0 \) are generated by quantum fields, we view the microlocal spectrum condition of [4] as a candidate for that generalized form of the spectrum condition. But we shall follow the historical course of events and will first look at the example of the free scalar field in the next subsection. Before turning there, a word on the condition of local quasi-equivalence is in order. The members \( \omega \) in the set \( S_{\text{phys}}^0 \) should be locally normal to states \( \omega^0 \in S_{\text{phys}}^0 \). However, as the members within \( S_{\text{phys}}^0 \) aren’t further distinguished, consistency requires that each state \( \omega \) on \( \mathcal{A} \) which is locally normal to some \( \omega^0 \in S_{\text{phys}}^0 \) is also locally normal to any other \( \hat{\omega}^0 \in S_{\text{phys}}^0 \). And this is just equivalent to the condition of local quasi-equivalence formulated in (c’) above.

### 3.2 Free scalar field and Hadamard states

The following treatment of the free scalar field on a globally hyperbolic spacetime \( (M, g) \) is due to Dimock [8].

The classical free scalar field obeys the field equation

\[
(\nabla^a \nabla_a + m^2) \varphi = 0,
\]

(3.1)
where $\varphi$ is a real-valued smooth function on $M$, $\nabla_a$ denotes the covariant derivative of the spacetime metric $g$, and $m > 0$ is a constant. On a globally hyperbolic spacetime, the Cauchy-problem for the wave-equation (3.1) is well-posed, and there is a unique pair of advanced/retarded solutions of (3.1), i.e. continuous linear operators $E^\pm : C_0^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ so that $E^\pm(\nabla^a \nabla_a + m^2)f = f = (\nabla^a \nabla_a + m^2)E^\pm f$ holds for all $f \in C_0^\infty(M, \mathbb{R})$, and so that the support of $E^\pm f$ is contained in the causal future/past of $\text{supp } f$. The difference of the fundamental solutions takes test-functions to solutions of (3.1) and is called the propagator, or commonly also the commutator function. One can show that on the quotient space $K = C_0^\infty(M, \mathbb{R})/\ker(E)$ there is a symplectic form $\sigma(\cdot, \cdot)$ given by

$$\sigma([f], [h]) = \int_M f \cdot E_h \, d\mu$$

for all $[f], [h] \in K$, where we have denoted the quotient map $C_0^\infty(M, \mathbb{R}) \to K$ by $f \mapsto [f]$ and the metric-induced measure on $M$ by $d\mu$. Then it is standard to associate to the symplectic space $(K, \sigma)$ a $C^*$-algebra $\mathcal{A}(K, \sigma)$ generated by a family $\{W([f]) : [f] \in K\}$ of unitary elements satisfying the relations

$$W([f])^* = W([-f]), \quad W([f])W([h]) = e^{-i\sigma([f], [h])/2}W([f] + [h])$$

for all $[f], [h] \in K$; these are called the canonical commutation relations in Weyl-form. A net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset M}$ of local $C^*$-algebras fulfilling the conditions of isotony and locality (a') can then be obtained by setting

$$\mathcal{A}(\mathcal{O}) = C^*$-algebra generated by all $W([f])$, supp $f \subset \mathcal{O}.$

Moreover, one can show that this net of $C^*$-algebras fulfills also condition (b'). See [8, 43] for further discussion.

In a next step, one has to make a choice of $\mathcal{S}^0_{\text{phys}}$. First, one collects states that are suitably regular and have a simple structure. For the Weyl-algebra $\mathcal{A}(K, \sigma)$, a natural choice is to consider as candidates the so-called quasi-free states. They take the form

$$\omega(W([f])) = e^{-\omega_2(f, f)/2} \forall [f] \in K,$$

where $\omega_2(\cdot, \cdot)$ is the two-point function of $\omega$, defined by

$$\omega_2(f, h) = -\partial_s \partial_t|_{t=s=0}\omega(W(s[f])W(t[h]))$$

for all $f, h \in C_0^\infty(M, \mathbb{R})$. In other words, quasi-free states $\omega$ are determined by their two-point functions $\omega_2$ via (3.2). To restrict attention to quasi-free states when specifying conditions for the initial set $\mathcal{S}^0_{\text{phys}}$ of physical states thus constitutes a considerable simplification as now one needs only impose conditions on the two-point functions. The question is, then, what the two-point function $\omega_2$ of a physical state of the free scalar on a curved spacetime field should look like. As a technical condition it seems natural to assume that $\omega_2$ is a distribution. Above that, further input is
required. The suggestion by Wald [41] was that two-point functions of physical states should have Hadamard form. We won’t pause to discuss the motivations for that since this has been done in some depth in the literature [14, 43]. However, we give a sketch of the definition of Hadamard form. One says that
\[
\omega_2(f, h) = \lim_{\varepsilon \to 0} \int_{M \times M} \left( G_\varepsilon(x, y) + H_\omega(x, y) \right) f(x) h(y) \, d\mu(x) \, d\mu(y),
\]
where \( H_\omega \in C^\infty(M \times M) \) contains the dependence on the state while the singular part, represented by \( \lim_{\varepsilon \to 0} G_\varepsilon \) is the same for all Hadamard states and given – qualitatively – by
\[
G_\varepsilon(x, y) = \frac{U(x, y)}{s(x, y) + i\varepsilon(x, y)} + V(x, y) \ln(s(x, y) + i\varepsilon(x, y)).
\]
(3.3)
Here, \( s(x, y) \) denotes the square of the geodesic distance from \( x \) to \( y \), \( \varepsilon(x, y) \) is of order \( \varepsilon \) and has positive/negative sign according if \( x \) lies in the future/past of \( y \), and \( U \) and \( V \) are smooth functions which are determined by the wave-operator \( (\nabla^a \nabla_a + m^2) \) by means of the so-called Hadamard recursion relations (see [13, 17] for their modern formulation as well as references to the original works by Hadamard). This definition is only qualitative since \( s(x, y) \) (and likewise, \( U(x, y) \) and \( V(x, y) \)) need not be defined globally for all \( x, y \in M \), and in fact it took some time until a completely satisfactory definition of Hadamard form was first reached at in [25].

Since for all Hadamard forms their singular parts are identical, the difference of any pair of Hadamard forms \( \omega_2 \) and \( \hat{\omega}_2 \) is given by the smooth integral kernel \( H_\omega - H_{\hat{\omega}} \).

One can show that this fact is sufficient in order that the GNS-representations \( \pi \) and \( \hat{\pi} \) corresponding to any pair of quasifree states \( \omega \) and \( \hat{\omega} \) on \( \mathcal{A}[K, \sigma] \) having two-point functions of Hadamard form – such states will henceforth be called Hadamard states – are locally quasi-equivalent [33]. Thus, if one chooses for the free scalar field on a globally hyperbolic spacetime as initial collection of physical states \( S_0^{\text{phys}} \) the set of Hadamard states, then condition (c’) above is clearly fulfilled.

While the Hadamard condition thus appears as a reasonable selection criterion for physical states of the free scalar field (which may similarly be generalized to other fields obeying linear wave equations, cf. e.g. [34] and references cited therein), it is not immediately clear what the Hadamard form has to do with with a “suitable generalization of the spectrum condition”, which we had desired above to distinguish the set \( S_0^{\text{phys}} \). In particular, since Hadamard forms are only definable with respect to linear wave-equations, it is at this stage not at all evident how to generalize the Hadamard form criterion to more general quantum field theories. These points have been significantly clarified in the PhD thesis of Radzikowski [31] who noticed that the wavefront set of a Hadamard form assumes a distinguished shape.

### 3.3 Wavefront sets and microlocal spectrum condition

In order to present and discuss Radzikowski’s findings, we first have to introduce the notion of the wavefront set of a scalar distribution. There are several equivalent
definitions that one can give, but perhaps the simplest approach is the one we take here. See [21] for further discussion.

Let $n \in \mathbb{N}$ and $v \in \mathcal{D}'(\mathbb{R}^n)$. One calls $(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ a regular directed point for $v$ if there are $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi(x) \neq 0$, and a conical open neighbourhood $\Gamma$ of $k$ in $\mathbb{R}^n \setminus \{0\}$ (i.e. $\Gamma$ is an open neighbourhood of $k$, and $k \in \Gamma \iff \mu k \in \Gamma \forall \mu > 0$), such that

$$\sup_{\hat{k} \in \Gamma} (1 + |\hat{k}|)^N |\hat{\chi}(\hat{k})| \leq C_N < \infty$$

holds for all $N \in \mathbb{N}$, where $\hat{\chi}v$ denotes the Fourier transform of the distribution $\chi \cdot v$.

**Definition 3.1.** $WF(v)$, the wavefront set of $v \in \mathcal{D}'(\mathbb{R}^n)$, is defined as the complement in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ of the set of all regular directed points for $v$.

Thus, $WF(v)$ consists of pairs $(x, k)$ of points $x$ in configuration space, and $k$ in Fourier space, so that the Fourier transform of $\chi \cdot v$ isn’t rapidly decaying along the direction $k$ for large $|k|$, no matter how closely $\chi$ is concentrated around $x$.

If $\phi : U \to U'$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, and $v \in \mathcal{D}'(U)$, then it holds that $WF(\phi_*v) = (t^D\phi)^{-1}WF(v)$ where $t^D\phi$ denotes the transpose of the tangent map (or differential) of $\phi$, with $(t^D\phi)^{-1}(x, k) = (\phi(x), (t^D\phi)^{-1} \cdot k)$ for all $(x, k) \in WF(v)$ and $\phi_*(v(f)) = v(f \circ \phi)$, $f \in \mathcal{D}(U')$. This transformation behaviour of the wavefront set allows it to define the wavefront set $WF(v)$ of a scalar distribution $v \in \mathcal{D}'(X)$ on any $n$-dimensional manifold $X$ [as usual, we take manifolds to be Hausdorff, connected, 2nd countable, $C^\infty$ and without boundary] by using coordinates: Let $\kappa : U \to \mathbb{R}^n$ be a coordinate system around a point $q \in X$. Then the inverse dual tangent map is an isomorphism $(t^D\kappa)^{-1} : T^*_qX \to \mathbb{R}^n$. We will use the notational convention $(q, \xi) \in T^*X \iff \xi \in T^*_qX$. Then let $(q, \xi) \in T^*X \setminus \{0\}$ and $(x, k) := (t^D\kappa)^{-1}(q, \xi) = (\kappa(q), (t^D\kappa)^{-1} \cdot \xi)$, so that $(x, k)$ is in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

**Definition 3.2.** We define $WF(v)$ by saying that $(q, \xi) \in WF(v)$ iff $(x, k) \in WF(\kappa^*v)$ where $\kappa^*v$ is the chart expression of $v$.

Owing to the transformation properties of the wavefront set under local diffeomorphisms one can see that this definition is independent of the choice of the chart $\kappa$, and moreover, $WF(v)$ is a subset of $T^*X \setminus \{0\}$, the cotangent bundle without the zero section.

It is straightforward to deduce from the definition that

$$WF(Av) \subset WF(v), \quad v \in \mathcal{D}'(X),$$

for any partial differential operator $A$ with smooth coefficients. (This generalizes to pseudodifferential operators $A$.) It is also worth noting that $WF(v)$ is a closed conic subset of $T^*X \setminus \{0\}$ where conic means $(q, \xi) \in WF(v) \iff (q, \mu \xi) \in WF(v) \forall \mu > 0$. Another important property is the following: Denote by $p_{M^*}$ the base projection of $T^*X$, i.e. $p_{M^*} : (q, \xi) \mapsto q$. Then for all $v \in \mathcal{D}'(X)$ there holds

$$p_{X^*}WF(v) = \text{sing supp } v \quad (3.4)$$

where sing supp $v$ is the singular support of $v$. 

9
Definition 3.3. For $v \in \mathcal{D}'(X)$, sing supp $v$ is defined as the complement of all points $q \in X$ for which there is an open neighbourhood $U$ and a smooth $n$-form $\alpha_U$ on $U$ so that
\[ v(h) = \int_U h \cdot \alpha_U \quad \text{for all} \quad h \in \mathcal{D}(U). \]

In other words, $v$ is given by an integral over a smooth $n$-form exactly if \( \text{WF}(v) \) is empty.

Now let $(M, g)$ be a globally hyperbolic spacetime. Then define the set of “null-covectors”
\[ N = \{(q, \xi) \in T^*M : g^{\sigma\rho}(q)\xi_\sigma\xi_\rho = 0\}. \] (3.5)

The spacetime possesses a time-orientation, i.e. there is on $M$ a vector field $w$ which is timelike, hence everywhere non-zero, and, by definition, future pointing. With its help one can introduce the following two disjoint future/past-oriented parts of $N$,
\[ N_\pm = \{(q, \xi) \in N : \pm \xi(w) > 0\}. \]

On the set $N$ one can introduce an equivalence relation as follows:
Definition 3.4. One defines
\[ (q, \xi) \sim (q', \xi') \]
if there is an affinely parametrized lightlike geodesic $\gamma$ with $\gamma(t) = q$, $\gamma(t') = q'$ and
\[ g^{\sigma\rho}(q)\xi_\rho = \left( \frac{d}{ds} \right|_{s=t} \gamma(s))^\sigma, \quad g^{\sigma\rho}(q')\xi'_\rho = \left( \frac{d}{ds} \right|_{s=t'} \gamma(s))^\sigma. \]

In other words, $\xi$ and $\xi'$ are co-parallel to the lightlike geodesic $\gamma$ connecting the base points $q$ and $q'$, and therefore $\xi$ and $\xi'$ are parallel transports of each other along that geodesic.

Equipped with that notation, we can now formulate Radzikowski’s result, which rests to some extent on previous work by Duistermaat and Hörmander [11].

Theorem 3.5 (Radzikowski). Let $\omega_2 \in \mathcal{D}'(M \times M)$ be the two-point function of a state on the Weyl-algebra $A[K, \sigma]$ of the free scalar field on the globally hyperbolic spacetime $(M, g)$. Then $\omega_2$ is of Hadamard form if and only if
\[ \text{WF}(\omega_2) = \{(q, \xi; q', \xi') \in N_- \times N_+ : (q, \xi) \sim (q', -\xi')\}. \] (3.6)

What is so attractive about this characterization of Hadamard forms? First, (3.6) is just the generally covariant generalization of the form of the wavefront set for the two-point function of the Klein-Gordon field’s vacuum state in flat Minkowski spacetime. Secondly, it expresses an asymptotic high-frequency remnant of the spectrum condition, which in flat spacetime may be expressed as a restriction on the Fourier-space support of the translation group as in condition (c) of Section 2. We will make this somewhat more precise in the next section. Moreover, and quite importantly, a condition of the type (3.4) can be generalized to other than just free quantum fields.
A significant step in this direction has been taken by Brunetti, Fredenhagen and Köhler [4]. We briefly sketch their “microlocal spectrum condition” ($\mu$SC).

Assume that $(M, g)$ is, as before, a globally hyperbolic spacetime, and let $\mathcal{B}_M$ denote the Borchers-algebra over the manifold $M$. That is, $\mathcal{B}_M$ is the free tensor-algebra of scalar test-functions, in symbols $\mathcal{B}_M = \mathbb{C} \oplus_{n \in \mathbb{N}} (\otimes^n \mathcal{D}(M))$; an algebraic structure can be defined on $\mathcal{B}_M$ in a canonical way [4]. A state $\omega$ on $\mathcal{B}_M$ is a positive linear functional which is uniquely specified by a sequence $(\omega_m)_{m \geq 0}$ where $\omega_0 \in \mathbb{C}$ and the $m$-point functions (or $m$-point distributions) $\omega_m \in (\otimes^m \mathcal{D}(M))^\prime$ are the restrictions of $\omega$ to $\otimes^m \mathcal{D}(M)$. The approach by Brunetti, Fredenhagen and Köhler is to impose restrictions on $\text{WF}(\omega_m)$ which are to viewed as generalizations of the flat space spectrum condition. For the vacuum state $\omega$ in flat spacetime, the spectrum condition amounts to restricting the support of the Fourier-transform of $\omega_m$ for each $m$ in a specific way, encoding that the energy-momentum spectrum is “conic” and that $\omega$ is translation-invariant [37, 1]. The terminology “microlocal spectrum condition” refers to the fact that the wavefront set is the microlocal version of the Fourier-space support of a distribution, in the sense that the distribution is localized around a point and support properties of its Fourier-transform are replaced by rapid decay properties. This notion is, as the above stated transformation properties of the wavefront set show, independent of the chosen coordinate system, while the support of a distribution’s Fourier-transform is a manifestly coordinate-dependent concept.

This indicates once more the utility of the notion of wavefront set for generalizing the spectrum condition to quantum field theory in curved spacetime.

To eventually formulate the microlocal spectrum condition, it is necessary to introduce further notation. Let $G$ be a non-directed graph with $n$ vertices $\{v_1, \ldots, v_n\}$ and a collection of connecting edges $E_G = \{e_1, \ldots, e_N\}$. More precisely, a directed edge $\vec{e}_{ij} = (v_i, e, v_j)$ is an edge connecting the source-vertex $v_i$ to the range-vertex $v_j$, and to say that the graph $G$ is non-directed means that, if $\vec{e}_{ij}$ is contained in $E_G$, then also its opposite directed edge, $(\vec{e}_{ij})' = \vec{e}_{ji}$ is contained in $E_G$. Note that there may be several different edges in $E_G$ connecting the same source- and range-index pair, and it is also allowed that there are isolated vertices in $\{v_1, \ldots, v_n\}$ which aren’t source- or range-vertices of any directed edge in $E_G$. Now an immersion of a non-directed graph $G$ (with vertices $\{v_1, \ldots, v_n\}$) into the spacetime $(M, g)$ is defined as a map $\iota(.)$ with the following properties: (1) to each vertex $v_i$, it assigns a point $p_i = \iota(v_i)$ in $M$, (2) to each directed edge $\vec{e}_{ij} \in E_G$ it assigns a covector $(p_i, \xi) = \iota(\vec{e}_{ij}) \in T^*_p M$, with $p_i = \iota(v_i)$, together with a smooth curve $\gamma_{ij}$ connecting $p_i$ and $p_j = \iota(v_j)$, (3) for $(\vec{e}_{ij})' = \vec{e}_{ji}$, and $(p_j, \xi') = \iota(\vec{e}_{ji})$, it is required that $\gamma_{ji} = \gamma_{ij}$ and that $\xi'$ is the parallel transport of $-\xi$ along $\gamma_{ij}$, (4) if $i > j$, then (the dual of) the covector $(p_i, \xi) = \iota(\vec{e}_{ij})$ is causal and future-directed.

With this notation, the microlocal spectrum condition of [4] reads as follows.

**Definition 3.6.** A state $\omega$ on $\mathcal{B}_M$ with $m$-point functions $\omega_m$ is said to satisfy the microlocal spectrum condition ($\mu$SC) iff

$$\text{WF}(\omega_m) \subset \Gamma_m \quad \text{for all } m,$$

where $\Gamma_m$ is defined as the set of all $(p_1, \xi_1; \ldots; p_m, \xi_m) \in (T^* M^m)\setminus\{0\}$ so that there exists a non-directed graph $G$ with $m$ vertices $\{v_1, \ldots, v_m\}$ together with an
immersion $\iota(\cdot)$ into $(M, g)$ having the properties

$$p_i = \iota(v_i) \quad \text{and} \quad (p_i, \xi_i) = \sum_j \iota(\vec{e}_{ij}).$$

We remark that $\Gamma_m$ is a covariant generalization of the set bounding the Fourier-space support of the Wightman $m$-point functions in Minkowski-spacetime; the condition that (the dual of) $\xi_m$ be future-pointing and causal corresponds to the spectrum condition while the requirement that $(p_i, \xi_i) = \sum_j \iota(\vec{e}_{ij})$ is the microlocal remnant of translation invariance. We refer to [4] for more discussion on this point.

The set $\Gamma_m$ may be quite “large” as regards the relative position of the basepoints $p_1, \ldots, p_m$, since the connecting curves $\gamma_{ij}$ appearing in the definition of a graph-immersion are only required to be smooth. In this respect, it is at present not completely clear if the definition of a graph-immersion shouldn’t be more restrictive. In [3], a graph-immersion is defined in a more restrictive manner, where the $\gamma_{ij}$ are required to be lightlike geodesics with dual tangent $\xi_i$ at $p_i$. With this modified, more restrictive definition of graph-immersions and correspondingly, of $\Gamma_m$, it is shown in [3] that the $m$-point functions of quasifree Hadamard states for the free scalar field fulfill the bound $\text{WF}(\omega_m) \subset \Gamma_m$. This property is an important technical tool for the local perturbative construction of $P(\phi)_4$ theories in globally hyperbolic curved spacetimes developed in [3].

4 Asymptotic Correlation Spectrum

The previous considerations have shown that the wavefront set of the $m$-point correlation functions $\omega_m$ for states on the Borchers algebra is a very useful concept in order to formulate generalized versions of the spectrum condition for quantum field theory in curved spacetime. However, the description of a quantum field theory in terms of $m$-point correlation functions, or equivalently, in terms of quantum fields (operator-valued distributions) is not an intrinsic concept from the point of view of algebraic quantum field theory as outlined in Section 2. One would like to generalize the concept of wavefront set in such a way that it becomes an intrinsic notion within the framework of algebraic quantum field theory, say, in Minkowski spacetime to start with, using only the structural assumptions (a) and (b) of Sec. 2 which are also prerequisite to the spectrum condition. Such an algebraic variant of the wavefront set would then be an invariant of a state, or of a set of states, in a similar manner as the spectrum condition, and would be independent of the the various choices of different quantum fields that one may have to generate the same net of von Neumann algebras $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O} \subset \mathbb{R}^4}$.

We begin our discussion by collecting the relevant assumptions. Suppose that $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset \mathbb{R}^n}$ is an isotonous family of $*$-algebras indexed by the bounded open regions in $\mathbb{R}^n$. That is to say, to each bounded open region $\mathcal{O} \subset \mathbb{R}^n$ there is assigned a (not necessarily unital) $*$-algebra $\mathcal{A}(\mathcal{O})$, and the condition of isotony $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. Then one can form the algebra $\mathcal{A}^o = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$ generated by all local algebras $\mathcal{A}(\mathcal{O})$, and we suppose that $\mathcal{A}(\mathcal{O})^o$ is endowed with a locally convex topology in
such a way that it becomes a topological *-algebra. We denote by $S_{A^e}$ the set of all continuous semi-norms on $A^e$. Moreover, we suppose that there operates on $A^e$ an equi-continuous action of the translation group $(\alpha_x)_{x \in \mathbb{R}^n}$ fulfilling the condition of covariance, $\alpha_x(A(0)) = A(0 + x)$. (The condition of equi-continuity says that for each $\sigma \in S_{A^e}$ there is $\sigma' \in S_{A^e}$ and $r > 0$ with $\sigma(\alpha_x(A)) \leq \sigma'(A)$ for $|x| < r$, and $\sigma(\alpha_x(A) - A) \to 0$ as $x \to 0$ for each $a \in A^e$.) There is yet another condition concerning the structure of the $A(0)$ that we wish to impose here. Namely, we suppose that for each bounded open region $\mathcal{O}$, it holds that $A(\mathcal{O}) = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k(\mathcal{O})$ is the union of an ascending sequence of vectorspaces $\mathcal{V}_k(\mathcal{O}) \subseteq \mathcal{V}_{k+1}(\mathcal{O})$ with $\alpha_x(\mathcal{V}_k(\mathcal{O})) \subseteq \mathcal{V}_k(\mathcal{O} + x)$ for each $\mathcal{O}$ and $k$, with $\mathcal{V}_k(\mathcal{O}) \cdot \mathcal{V}_{k'}(\mathcal{O}) \subseteq \mathcal{V}_{k''}(\mathcal{O})$ for some $k''$ depending on $k$ and $k'$, and with the property that given $\sigma \in S_{A^e}$, $\mathcal{O} \subset \mathbb{R}^n$ open and bounded, and $N, k \in \mathbb{N}$, there is some $\sigma' \in S_{A^e}$ so that

$$\sigma(A_1 \cdots A_N) \leq \sigma'(A_1) \cdots \sigma'(A_N)$$

holds for all $A_1, \ldots, A_N \in \mathcal{V}_k(\mathcal{O})$.

The just listed structural properties are typical of the Borchers algebra, and will be made use of in order to incorporate the Borchers algebra formulation of quantum field theory in our approach to generalizing the wavefront set to the operator algebraic setting in quantum field theory. However, the partitioning of local algebras into subspaces $\mathcal{V}_k(\mathcal{O})$ would in general not be regarded as an intrinsic element in the algebraic framework of quantum field theory and is presently mainly to be seen as a technical device in order to treat the Borchers algebra at equal footing with $C^*$-algebras in our approach.

Supposing the validity of the just stated assumptions, we define, for each $0 \leq \mu \leq 1$, $p \in \mathbb{R}^n$ and for each bounded open region $\mathcal{O} \subset \mathbb{R}^n$, $A^{(\mu)}_p(\mathcal{O})$ as the set of all families $(A_\lambda)_{\lambda > 0}$ with the following properties:

(i) There is $k \in \mathbb{N}$ so that $A_\lambda \in \mathcal{V}_k(\lambda^\mu \mathcal{O} + p)$, $\lambda > 0$,

(ii) There is some $\lambda_0 > 0$ with $A_\lambda = 0$ for $\lambda > \lambda_0$,

(iii) For each $\sigma \in S_{A^e}$ there is $s \in \mathbb{R}$ so that

$$\sup_{\lambda} \lambda^s \sigma(A_\lambda) < \infty.$$

The elements in $A^{(\mu)}_p(\mathcal{O})$ are called testing families. Note that $A^{(\mu)}_p(\mathcal{O})$ inherits in a natural way a linear structure by defining algebraic operations on testing families pointwise for each $\lambda$. (If each $\mathcal{V}_k(\mathcal{O})$ is an algebra, then so is $A^{(\mu)}_p(\mathcal{O})$.) Another natural operation is to shift a testing family so that their localization properties change: We may define $\alpha_x(A_\lambda)_{\lambda > 0} = (\alpha_x(A_\lambda))_{\lambda > 0}$, then $\alpha_x(A^{(\mu)}_p(\mathcal{O})) = A^{(\mu)}_{p+x}(\mathcal{O})$. Equipped with that notation, we can now introduce the following definition.

**Definition 4.1.** Let $\varphi$ be a continuous linear functional on $A^e$, and let $N \in \mathbb{N}$. For $0 \leq \mu \leq 1$ and $N \in \mathbb{N}$, we call an element

$$(p_1, \ldots, p_N; \xi_1, \ldots, \xi_N) \in (\mathbb{R}^n)^N \times ((\mathbb{R}^n)^N \setminus \{0\})$$

13
a regular directed point of order $N$ and degree $\mu$ for $\varphi$ provided that the following holds: There exists an open, bounded neighbourhood $\mathcal{O}$ of the origin in $\mathbb{R}^n$, an open neighbourhood $V_{(N)}$ of $(\xi_1, \ldots, \xi_N)$ in $((\mathbb{R}^n)^N \setminus \{0\})$, and some $h \in \mathcal{D}((\mathbb{R}^n)^N)$ with $h(0) \neq 0$ so that for each $(A^{(j)}_\lambda)_{\lambda > 0} \in \mathcal{A}_{\mathcal{D}}^{(\mu)}(\mathcal{O})$ one has

$$\sup_{k \in V_{(N)}} \left| \int e^{-i\lambda^{-1}\mathcal{K} \mathcal{L} h(y) \varphi(\alpha_{y_1}(A^{(1)}_\lambda) \cdots \alpha_{y_N}(A^{(N)}_\lambda)) \, d^n y_1 \cdots d^n y_N} \right| = O(\lambda^\infty)$$

as $\lambda \to 0$. Here, $k = (k_1, \ldots, k_N)$ and $y = (y_1, \ldots, y_N)$ denote $N$-tuples of vectors in $\mathbb{R}^n$ and correspondingly, $k \cdot y$ denotes the sum of the scalar products $k_j \cdot y_j$, $j = 1, \ldots, N$.

The set of all regular directed points of order $N$ and degree $\mu$ of $\varphi$ is denoted by $\text{reg}^{(N,\mu)}(\varphi)$. The complement of that set in $(\mathbb{R}^n)^N \times ((\mathbb{R}^n)^N \setminus \{0\})$ is denoted by $\text{ACS}^{(N,\mu)}(\varphi)$ and will be called the asymptotic correlation spectrum of $\varphi$ of order $N$ and degree $\mu$.

Before adding a few remarks about the definition of the asymptotic correlation spectrum, we give an example which ought to illustrate why this notion may be viewed a generalization of the wavefront set. Take as local algebras the sets $\mathcal{A}(\mathcal{O}) = \mathcal{D}(\mathcal{O})$, with the pointwise multiplication of functions, and $\mathcal{V}(\mathcal{O}) = \mathcal{D}(\mathcal{O})$ for all $k \in \mathbb{N}$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. In this case one obtains:

**Lemma 4.2.**

$$\text{ACS}^{(1,\mu)}(u) = \text{WF}(u) \quad \text{for all } 0 \leq \mu \leq 1.$$  

The proof of this statement is easily obtained by a simple variation of Prop. 2.1 in [40].

In [40], we have introduced the algebras of testing families only for the case $\mu = 1$. This was strongly inspired by the “scaling algebra” approach to the analysis of short distance behaviour in quantum field theory introduced in [3]. The dependence on $\mu$ that has been added here is but one possible way of generalization. It is obvious that $\mathcal{A}_{\mathcal{D}}^{(\mu)}(\mathcal{O})$ becomes larger as $\mu$ decreases, and so one gets $\text{ACS}^{(N,\mu)}(\varphi) \subset \text{ACS}^{(N,\mu')}(\varphi)$ for $\mu > \mu'$. The case $\mu = 0$ is in some way distinguished from the cases $\mu > 0$. Let us consider this the case $\mu = 0$ under the assumption that the $\mathcal{A}(\mathcal{O})$ are $C^*$-algebras, and $\mathcal{V}(\mathcal{O}) = \mathcal{A}(\mathcal{O})$ for all $k$. Then the condition that $(p_1, \ldots, p_N; \xi_1, \ldots, \xi_N) \in \text{reg}^{(N,0)}(\varphi)$ can be formulated in the following way: There is a conic open neighbourhood $\Gamma$ of $(\xi_1, \ldots, \xi_N) \in ((\mathbb{R}^n)^N \setminus \{0\})$, an open neighbourhood $\mathcal{O}$ of the origin in $\mathbb{R}^n$ and some $h \in \mathcal{D}((\mathbb{R}^n)^N)$ with $h(0) \neq 0$, so that there is for each $R \in \mathbb{R}_+$ some $C_R > 0$ with

$$\sup_{A^{(j)}} \sup_{k \in \Gamma} (1 + |k|)^R \left| \int e^{-i\mathcal{K} h(y) \varphi(\alpha_{y_1}(A^{(1)}) \cdots \alpha_{y_N}(A^{(N)})) \, d^n y_1 \cdots d^n y_N} \right| < C_R$$

where supremum is formed over all $A^{(j)} \in \mathcal{A}(\mathcal{O} + x_j)$ with $\|A^{(j)}\| \leq 1$, $j = 1, \ldots, N$. (Such a definition of regular directed points has been suggested to the author by K. Fredenhagen.) In other words, elements in $\text{reg}^{(N,0)}(\varphi)$ are simultaneously and uniformly regular directed points (in the sense of not being contained in
the wavefront set) of all distributions \( \varphi_{A^{(1)}, ..., A^{(N)}} \) given by \( \varphi_{A^{(1)}, ..., A^{(N)}}(y_1, \ldots, y_N) = \varphi(a_{y_1}(A^{(1)}) \cdots a_{y_N}(A^{(N)})); \) therefore one obviously has

\[
\text{ACS}^{(N,0)}(\varphi) \supset \text{closure} \left[ \bigcup_{A^{(j)}} \text{WF}(\varphi_{A^{(1)}, ..., A^{(N)}}) \right],
\]

but there is no assertion if the reverse inclusion holds.

As may be expected, the basic properties of ACS\((N,\mu)(\varphi)\) are similar to those of the wavefront set. For instance, ACS\((N,\mu)(\varphi)\) is a closed subset of \((\mathbb{R}^n)^N \times ((\mathbb{R}^n)^N \setminus \{0\})\) and conic in the \(\xi_j\). For a proof and some more discussion, see Prop. 3.2 in [40].

We shall provide a few more examples. Let \(\mathcal{B}\) denote the Borchers algebra over \(n\)-dimensional Minkowski spacetime. For \(\mathcal{O}\) a bounded open region in \(\mathbb{R}^n\), we define the subspaces \(\mathcal{V}_k(\mathcal{O}) = \mathbb{C} \oplus_{m=1}^k (\otimes^m \mathcal{D}(\mathcal{O}))\) of \(\mathcal{B}\) and the local algebras \(A(\mathcal{O}) = \bigcup_{k=1}^\infty \mathcal{V}_k(\mathcal{O})\).

The action of the translations (\(\tau_x f\))(y) = \(f(y - x)\) on test-functions lifts to an equi-

continuous group action (\(\alpha_x\)) by automorphisms to \(A^0\) (note that \(A^0 = \mathcal{B}\)).

Define the sets of testing families \(A_\mu(\mathcal{O})\) with respect to these data, and let \(\omega_m, m \in \mathbb{N}_0\) denote the \(m\)-point functions of a state \(\omega\) on \(A^0\). Then it is not difficult to check that

\[
\text{WF}(\omega_m) \subset \text{ACS}^{(m,\mu)}(\omega).
\]

In general, one won’t expect equality to hold here, since with the just given definition the \(A_\mu(\mathcal{O})\) are algebras which are quite large. However, if we restrict the choice of \(\mathcal{V}_k(\mathcal{O})\) to \(\mathcal{V}_k(\mathcal{O}) = \mathcal{D}(\mathcal{O})\) for all \(k\), and if \(A_\mu(\mathcal{O})\) is defined accordingly, then equality holds in \((4.1)\).

Another example arises from quantum field theories on Minkowski spacetime in the \(C^*\)-algebraic framework described in Sec. 2. Suppose that we are given such an algebraic quantum field theory, \((\{A(\mathcal{O})\}_{\mathcal{O} \subseteq \mathbb{R}^4}, (\alpha_x)_{x \in \mathbb{R}^4}, \omega^0)\). Define \(A_\mu(\mathcal{O})\) with respect to \(\mathcal{V}_k(\mathcal{O}) = A(\mathcal{O})\) for all \(k\). Furthermore, let \(\omega\) be a state on \(A\) which is induced by a \(C^\infty\)-vector for the energy, \(\psi\), in the vacuum GNS-Hilbertspace \(\mathcal{H}^0\) (so that \(\omega(A) = \langle \psi, \pi^0(A)\psi \rangle\)). We recall that \(\psi\) is \(C^\infty\) for the energy if \(\psi \in \text{dom}((P_0)^s)\) for all \(s > 0\) where \(P_0\) is the energy-operator (generator of the time-translations) in \(\mathcal{H}^0\). In this situation, we obtain

**Theorem 4.3.** It holds that

\[
\text{ACS}^{(N,\mu)}(\omega) \subset \Gamma_N^\circ
\]

where \(\Gamma_N^\circ\) is the set of all \((p_1, \ldots, p_n; \xi_1, \ldots, \xi_N) \in ((\mathbb{R}^4)^N \setminus \{0\})\) with the following properties: There exists a non-directed graph \(G_N\) with \(N\) vertices, and with all pairs of distinct vertices connected by exactly one directed edge and its inverse, together with an immersion \(\iota(.)\) of \(G_N\) into Minkowski-spacetime, where each curve \(\gamma_{ij}\) is a straight geodesic line segment (which may degenerate to a point if \(\iota(v_i) = \iota(v_j)\)), at least one of which is causal, so that

\[
p_i = \iota(v_i), \quad \xi_i = \sum_j \iota(\bar{e}_{ij}).
\]
The proof of this theorem may be inferred by combining Thm. 4.6 in [4] with Thm. 5.1 in [40].

This shows that in flat spacetime, the spectrum condition places an upper bound on $\text{ACS}^{(N,\mu)}(\omega)$ which is of the form of a microlocal spectrum condition, and in turn shows that the asymptotic correlation spectrum may serve as a generalization of the wavefront set in the operator algebraic framework. There is a further result in support of this point of view: Suppose that one has, in the vacuum representation of the algebraic quantum field theory $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset \mathbb{R}^4}$, a scalar quantum field (Wightman field) $\mathcal{D}(\mathbb{R}^4) \ni f \mapsto \Phi(f)$ affiliated to the local von Neumann algebras $\mathcal{R}(\mathcal{O})$ and denote by

$$\omega_N(f_1 \otimes \cdots \otimes f_N) = \langle \psi, \Phi(f_1) \cdots \Phi(f_N) \psi \rangle$$

the $N$-point distributions corresponding to a state $\omega$ induced by a unit vector $\psi$ in the domain of the quantum field. In this case one obtains an analogue of (4.1), namely

$$\text{WF}(\omega_N) \subset \text{ACS}^{(N,\mu)}(\omega).$$

We refer to [40] for a proof and further discussion.

5 Quantum Field Theory on Curved Spacetime, Encore

We have seen that the asymptotic correlation spectrum appears as a viable generalization of the wavefront set in the operator algebraic approach to quantum field theory in Minkowski spacetime. The next step consists in generalizing the notion of asymptotic correlation spectrum to quantum field theory in curved spacetime. To this end, we are again faced with the difficulty that there is no counterpart of the translation group acting by isometries on a curved spacetime, since the translation group played a significant role in formulating conditions on the regular directed points. Nevertheless, it appears that the basic idea underlying the definition of regular directed points of a functional may be suitably generalized so as to cover also the situation where the spacetime manifold possesses no non-trivial isometries. We will consider that situation at the end of this section.

First, we will focus at the situation where some isometries are still present. While a more general investigation of the asymptotic correlation spectrum for general group actions is on the way [38], we will here restrict attention to the simplest case. We assume that there is a smooth, one-parametric group $\{\theta_t\}_{t \in \mathbb{R}}$ acting by isometries on the globally hyperbolic spacetime $(M, g)$. Moreover, we assume that its generating vector field $X = \frac{d}{dt} \theta_t^*$ is time-like and future-pointing. Thus $(M, g)$ is stationary. By $N_X = \{(p, \xi) \in T^*M : \xi(X_p) = 0\}$ we denote the co-normal bundle of $X$.

Then let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \subset M}$ be a net of topological *-algebras on $M$ satisfying the assumptions of the previous section with obvious changes. Furthermore, we suppose that there is an equi-continuous group action $(\alpha_t)_{t \in \mathbb{R}}$ by automorphisms on $\mathcal{A}^0$ which is covariant with respect to $(\theta_t)_{t \in \mathbb{R}}$, i.e. $\alpha_t(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\theta_t(\mathcal{O}))$. The definition of the sets
\( A_{\mu}(\emptyset) \) of testing families \( (A_{\lambda})_{\lambda > 0} \), for \( \emptyset \) an open neighbourhood of \( p \in M \), is similar as in the last section except that the localization condition is replaced by

\[(i') \text{ There is } k \in \mathbb{N} \text{ so that } A_{\lambda} \in \mathcal{V}_{k}(\exp(\lambda \mu \exp^{-1}(\emptyset))), \lambda > 0,
\]

where \( \exp \) is the exponential map at \( p \) (and it is understood that \( \emptyset \) is in the domain of \( \exp^{-1} \)).

Now let \( \varphi \) be a continuous linear functional on \( \mathcal{A}^{0} \). We define:

**Definition 5.1.** \( \text{reg}^{(N,\mu)}_{(\alpha_{i})}(\varphi) \) is defined as the set of all \( N \)-tuples of covectors \( (p_{1}, \xi_{1}; p_{2}, \xi_{2}; \ldots ; p_{N}, \xi_{N}) \in ((T^{*}M)^{N} \setminus (N_{X})^{N}) \) with the following property: There is an open neighbourhood \( W_{(N)} \) of \( (p_{1}, \xi_{1}; \ldots ; p_{N}, \xi_{N}) \in ((T^{*}M)^{N} \setminus (N_{X})^{N}) \), and there are open neighbourhoods \( O_{j} \) of \( p_{j} \) \( (j = 1, \ldots, N) \) and a function \( h \in \mathcal{D}(\mathbb{R}^{N}) \) with \( h(0) \neq 0 \), so that for all \( (A_{\lambda})_{\lambda > 0} \in \mathcal{A}_{\mu}(O_{j}) \) it holds that

\[
\sup_{(\xi', \xi) \in W_{(N)}} \left| \int e^{-i\lambda^{-1}\xi' \cdot \xi(X)} h(\xi')\varphi(\alpha_{t_{1}}(A_{\lambda}^{(1)}) \cdots \alpha_{t_{N}}(A_{\lambda}^{(N)})) \, dt_{1} \cdots dt_{N} \right| = O(\lambda^{\infty})
\]

as \( \lambda \to 0 \). We have written \( \xi = (t_{1}, \ldots, t_{N}) \) and \( (\xi', \xi) = (p_{1}', \xi_{1}', \ldots; p_{N}', \xi_{N}') \), and in the phase-factor \( \xi' \cdot \xi(X) = \sum_{j=1}^{N} t_{j} \cdot \xi_{j}'(X_{p_{j}}) \). The set \( \text{ACS}^{(N,\mu)}_{(\alpha_{i})}(\varphi) \) is now defined as the complement of \( \text{reg}^{(N,\mu)}_{(\alpha_{i})}(\varphi) \) in \( ((T^{*}M)^{N} \setminus (N_{X})^{N}) \).

**Theorem 5.2.** Let \( \omega \) be a continuous state on \( \mathcal{A}^{0} \) and assume that \( \omega \) is a grund state, or a KMS-state at inverse temperature \( \beta > 0 \) or the action of \( (\alpha_{t})_{t \in \mathbb{R}} \). Then it holds that \( \text{ACS}^{(2,\mu)}_{(\alpha_{t})}(\varphi) \) is either empty, or

\[
\text{ACS}^{(2,\mu)}_{(\alpha_{t})}(\varphi) = \{(p, \xi; p', \xi') \in ((T^{*}M)^{2} \setminus (N_{X})^{2}) : \xi'(X_{p'}) > 0, \xi(X_{p}) + \xi'(X_{p'}) = 0 \}.
\]

As an application of this last result, let \( \{\mathcal{A}(\emptyset)\}_{\emptyset \subset M} \) be the net of local \( C^{*} \)-algebras constructed for the free scalar field fulfilling the wave-equation \( f^{2} \) on \( (M, g) \). This net carries an automorphic action \( (\alpha_{t})_{t \in \mathbb{R}} \) which is covariant with respect to \( (\theta_{t})_{t \in \mathbb{R}} \) \( f \). Suppose that \( \omega \) is a quasifree state on \( \mathcal{A} = \mathcal{A}[K, \sigma] \) which is a grund state or a KMS-state at inverse temperature \( \beta > 0 \) for \( (\alpha_{t})_{t \in \mathbb{R}} \). Since the two-point function \( \omega_{2} \) of \( \omega \) is a distributional solution of the wave-equation in both entries, it holds by a general result that \( \text{WF}(\omega_{2}) \) must be contained in the set \( N \) of null-covectors defined in \( f^{2} \). Combining this with the statement of the last theorem yields that

\[
\text{WF}(\omega_{2}) = \{(q, \xi; q', \xi') \in N_{-} \times N_{+} : (q, \xi) \sim (q', -\xi') \},
\]

i.e. that \( \omega_{2} \) is of Hadamard form \( f^{2} \). This generalizes similar results which have been obtained, by other methods, for a special class of static spacetimes in the ground state case \( f^{2} \) and for static spacetimes with compact Cauchy-surfaces in the KMS case \( f^{2} \); the spacetimes covered by these previous works do not, however, include some
interesting situations like black holes while our result does. Moreover, the argument can be extended from the particular example of a free scalar field to the case of vector (-bundle) fields over \((M, g)\) satisfying a wave-equation and suitably generalized versions of the canonical commutation relations, and to Dirac-fields fulfilling canonical anti-commutation relations, in any spacetime-dimension \(\geq 2\) \cite{33, 34}. It should also be noted that ground states and KMS states as well as mixtures of such states are passive states for which the 2nd law of thermodynamics holds (i.e. one cannot extract energy from such states by cyclic processes), cf. \cite{30}. Thus, in the case of linear quantum fields obeying wave-equations on stationary spacetimes, one can see that the microlocal spectrum condition (or equivalently, the Hadamard condition) is implied by passivity. This is further support to the idea that the microlocal spectrum condition selects states which, in a suitable sense, are dynamically stable.

Finally we shall give a – tentative – outline how one may proceed in order to obtain a notion of asymptotic correlation spectrum in case that the underlying spacetime manifold \((M, g)\) admits no non-trivial isometries. In that case, there is no obvious definition of Fourier-integrals of the form

\[
\int e^{-i\lambda^{-1} \xi \cdot y} h(y) \varphi(\alpha(y)(A_\lambda)) \, d^ny
\]  

(5.1)

that we have used above in testing the regularity of directions \(\xi\) for a given functional \(\varphi\) upon letting \((A_\lambda)_{\lambda > 0}\) range through a collection of testing families suitably localized at the base point \(p\) to which the direction \(\xi\) is affixed. But we may think of the expression (5.1) as

\[
\varphi(A(\lambda^{-1} \xi, \lambda)),
\]  

(5.2)

i.e. the functional \(\varphi\) tested by “symbols” of the form

\[
A(\xi, \lambda) = \int e^{-i\xi \cdot y} h(y) \alpha(y)(A_\lambda) d^ny.
\]  

(5.3)

One may then be inclined to take the right hand side of (5.3) as a specific example of abstractly defined “testing symbols” \(A(\xi, \lambda)\) which are characterized by a suitable asymptotic high energy/short distance behaviour as would ensue for the right hand side of (5.3), if a spacetime-translation group action were present. In other words, one may generalize the approach of the last chapter by introducing suitable classes of “testing symbols” \(A(\xi, \lambda)\) and by defining regular directions of \(\varphi\) via the asymptotic \(\lambda \to 0\) behaviour of the quantities (5.2) for all such testing families.

Then the question arises which conditions on the testing symbols one should impose, and how to implement the just sketched idea. In the remainder of this work, we shall make some suggestions towards that question; however, we should warn the reader that these suggestions so far haven’t been tested in examples, and should be taken cum grano salis. Our starting point is a net \(\{A(O)\}_{O \subset M}\) of \(C^*-\)algebras indexed by the bounded open regions of some (globally hyperbolic) spacetime \((M, g)\) \((\text{dim } M = 4)\). This net is assumed to comply with the conditions of isotony and locality and moreover, it will be assumed that each \(A(O)\) is a \textit{von Neumann} algebra.
acting on some Hilbertspace $H$. That is to say, we assume that some Hilbertspace representation (or, equivalently, a suitable set of states) has been chosen, and the basic approach is to provide a definition of test-objects that allow it to decide if that representation fulfills, in a suitably generalized sense, a (microlocal) spectrum condition, in which case the representation may be regarded as physical.

Let a point $p \in M$ be given, and let $\mathcal{O}$ be an open, geodesically convex neighbourhood of $p$. We shall consider functions

$$T^*\mathcal{O} \times (0, 1) \ni (x, \xi; \lambda) \mapsto A(x, \xi; \lambda) \in \mathcal{A}(\mathcal{O})$$

with the following properties:

(I) In any coordinate system for $T^*\mathcal{O}$, and for all multi-indices $\alpha, \beta \in \mathbb{N}_0^4$, the weak partial derivatives

$$D^\alpha_x D^\beta_\xi A(x, \xi; \lambda)$$

exist, are jointly (weakly) continuous in $x, \xi, \lambda$, and are contained in $\mathcal{A}(\mathcal{O})$.

(II) In suitable coordinates,

$$\sup_{x \in K} \sup_{k \in V} ||D^\alpha_x D^\beta_\xi A(x, \xi; \lambda)||_{\xi = \lambda^{-1}k} \leq C_{K,V,\alpha,\beta}(1 + \lambda^{-1})^{m + |\alpha| - |\beta|}$$

hold for each compact subset $K \subset \mathcal{O}$ and each bounded subset $V \subset T^*K$ with suitable constants $m, C_{K,V,\alpha,\beta} > 0$. (This property is essentially what characterizes operator-valued symbols in microlocal analysis, see e.g. [36].)

We collect all functions $A(\cdot, \cdot; \cdot)$ with the just described properties in a set denoted by $\text{Sym}(p, \mathcal{O})$. We call it the set of testing symbols around $p$ localized in $\mathcal{O}$.

Let us give an example of such testing symbols in a concrete case: Take $p \in M$, and choose a coordinate system $(y^\nu)$ with $y(p) = 0$ around $p$. Let $f$ be a smooth test-function supported in a sufficiently small coordinate ball around $p$, and define, in coordinates,

$$f_{x,\lambda}(y') = f(y'/\lambda^s) - x,$$

where $s \geq 1$. Here, we have identified $x$ with its coordinate expression $y(x)$. Denote by $w_\lambda^{(\lambda)}$ the Weyl-operator $\pi(W([f_{x,\lambda}]))$ in the GNS-representation $\pi$ corresponding to a quasifree state on the CCR-algebra of the Klein-Gordon field on $(M, g)$. Then a testing symbol is obtained by setting

$$A(x, \xi; \lambda) = \int e^{-i\xi \cdot y'} h(y) w_\lambda^{(\lambda)} d^4 y$$

for $x$ in a sufficiently small neighbourhood of $p$ and $h \in \mathcal{D}(\mathbb{R}^4)$ having support sufficiently close to 0; the coordinates used for $\xi$ are those induced by the chosen coordinate system.

One can now introduce a notion of generalized asymptotic correlation spectrum of order 2 (the case of arbitrary order $N$ can be treated similarly, we consider only $N = 2$ for the sake of simplicity). Let $\omega$ be a state on $\mathcal{A}$, the quasi-local algebra generated by the local von Neumann algebras $\mathcal{A}(\mathcal{O})$, and let $p, p' \in M$. We say that $(p, \xi; p', \xi') \in (T^*M \times T^*M)\{0\}$ is a generalized regular directed point of order 2 for
\( \omega \) if there are open neighbourhoods \( \Theta \) of \( p \) and \( \Theta' \) of \( p' \), and an open neighbourhood \( W(2) \) of \( (p, p'; \xi, \xi') \in (T^* M \times T^* M) \setminus \{0\} \), so that

\[
\sup_{(x,k; x', k') \in W(2)} |\omega(A(x, \lambda^{-1} k; \lambda)A'(x', \lambda^{-1} k'; \lambda))| = O(\lambda^{\infty}) \quad \text{for} \quad \lambda \to 0
\]

holds for all testing symbols \( A(x, k; \lambda) \in \Sym(p, \Theta) \) and \( A'(x', k'; \lambda) \in \Sym(p', \Theta') \). Then \( \text{gACS}^{(2)}(\omega) \), the \textit{generalized asymptotic correlation spectrum of order 2} of \( \omega \), is defined as the complement in \( (T^* M \times T^* M) \setminus \{0\} \) of the set of all generalized regular directed points of order 2 for \( \omega \).

From the example for testing-symbols above it is fairly plausible that, in the case where the local von Neumann algebras \( \mathcal{A}(\Theta) \) are generated by a quantum field \( \mathcal{D}(M) \ni f \mapsto \Phi(f) \), and where \( \omega_2(f_1 \otimes f_2) = \omega(\Phi(f_1)\Phi(f_2)) \) denotes the corresponding two-point functions, one should find

\[
\WF(\omega_2) \subset \text{gACS}^{(2)}(\omega).
\]

However, in some sense the set of testing-symbols \( \Sym(p, \Theta) \) is too big: It is not related in any obvious way to a “dynamics” of the quantum field theory given by the net of von Neumann algebras \( \{\mathcal{A}(\Theta)\}_{\Theta \subseteq M} \). But then, on a general spacetime manifold it is not clear how a notion of a dynamics is to be formulated. The approach which we suggest is, therefore, that candidates for physical states should “select” their own (asymptotic) dynamics from the sets of testing symbols: Consider the case that for some state \( \omega \) on \( \mathcal{A} \), there are subspaces \( \Sym_\omega(p, \Theta) \subset \Sym(p, \Theta) \) such that the generalized asymptotic correlation spectra \( \text{gACS}_\omega^{(2)}(\omega') \), defined with respect to \( \Sym_\omega(p, \Theta) \) instead of \( \Sym(p, \Theta) \), have the property that, e.g.,

\[
\text{gACS}_\omega^{(2)}(\omega') \subset \{(p, \xi; p', \xi') \in (T^* M \times T^* M) \setminus \{0\} : g^{\mu\nu}\xi_{\mu}\xi_{\nu} \geq 0, \quad g^{\mu\nu}\xi'_{\mu}\xi'_{\nu} \geq 0, \quad \xi(X) < 0, \quad \xi'(X) > 0\}
\]

holds, for any timelike vector-field \( X \), for a dense set of normal states \( \omega' \) (including \( \omega \) itself). Then one would be inclined to call such a state dynamically stable once the symbol-spaces \( \Sym_\omega(p, \Theta) \) are sufficiently stable under algebraic operations like multiplication of symbols, or under the convolution

\[
A \star A'(x, \xi, \lambda) = \int A(x, \xi - \xi', \lambda)A'(x, \xi', \lambda) \, d^4 \xi'
\]

in suitable coordinates; moreover, \( \Sym_\omega(p, \Theta) \) would have to be sufficiently “big” (e.g. \( \Sym_\omega(p, \Theta)'' = \mathcal{A}(\Theta) \)). Further desiderata that one would like to impose on elements of \( \Sym_\omega(p, \Theta) \) are suitable (asymptotic) forms of geometric modular action. These matters remain to be explored; we just wished to point out that it appears well possible to extend the microlocal approach to generalizing the spectrum condition to quantum field theory in curved spacetime in the operator-algebraic setting. It seems also possible to further extend these ideas to reach at notions of “spectrum condition” for generally covariant theories, but this is still quite speculative.
References

[1] Borchers, H.J., On the structure of the algebra of field operators, Nuovo Cimento 24, 214 (1962);
   — Algebraic aspects of Wightman field theory in: Sen, R.N., Weil, C. (eds.), Statistical Mechanics and Field Theory, Israel Universities Press, Jerusalem, 1972

[2] Borchers, H.J., On revolutionizing quantum field theory with Tomita’s modular theory. J. Math. Phys. 41, 3604 (2000)

[3] Brunetti, R., Fredenhagen, K., Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds, Commun. Math. Phys. 208, 623 (2000)

[4] Brunetti, R., Fredenhagen, K., Köhler, M, The microlocal spectrum condition and Wick polynomials of free fields in curved spacetimes, Commun. Math. Phys. 180, 633 (1996)

[5] Buchholz, D., Verch, R., Scaling algebras and renormalization group in algebraic quantum field theory. Rev. Math. Phys. 7, 1195 (1995)

[6] Buchholz, D., Verch, R., Scaling algebras and renormalization group in algebraic quantum field theory. II. Instructive examples. Rev. Math. Phys. 10, 775 (1998)

[7] Casimir, H.B.G., On the attraction between two perfectly conducting plates, Konink. Nederl. Akad. Wetens., Proc. Sec. Sci. 51, 793 (1948)

[8] Dimock, J., Algebras of local observables on a manifold, Commun. Math. Phys. 77, 219 (1980)

[9] Doplicher, S., An algebraic spectrum condition. Comm. Math. Phys. 1, 1 (1965)

[10] Doplicher, S., Roberts, J.E., Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics, Commun. Math. Phys. 131, 51 (1990)

[11] Duistermaat, J.J., Hörmander, L., Fourier integral operators. II, Acta Mathematica 128, 183 (1972)

[12] Fredenhagen, K., Haag, R., On the derivation of Hawking radiation associated with the formation of a black hole, Commun. Math. Phys. 127, 273 (1990)

[13] Friedlander, F.G., The wave equation in a curved spacetime, Cambridge University Press, Cambridge, 1975

[14] Fulling, S.A., Aspects of quantum field theory in curved spacetime, Cambridge University Press, Cambridge, 1989

[15] Fulling, S.A., Narcowich, F.J., Wald, R.M., Singularity structure of the two-point function in quantum field theory in curved spacetime, II, Ann. Phys. (N.Y.) 136, 243 (1981)
[16] Guido, D., Longo, R., Roberts, J.E., Verch, R., Charged sectors, spin and statistics in quantum field theory on curved spacetimes, math-ph/9906019, to appear in Rev. Math. Phys.

[17] Günther, P., Huygens principle and hyperbolic equations, Academic Press, Boston, 1988

[18] Haag, R., Local quantum physics, 2nd edn., Springer-Verlag, Berlin 1996

[19] Hawking, S.W., Particle creation by black holes, Commun. Math. Phys. 43, 199 (1975)

[20] Hawking, S.W., Ellis, G.F.R., The large scale structure of space-time, Cambridge University Press, Cambridge, 1973

[21] Hörmander, L., The Analysis of Linear Partial Differential Operators I, Springer Verlag, Berlin, 1983

[22] Jost, R., The general theory of quantum fields, American Math. Soc., Providence, R.I., 1965

[23] Junker, W., Hadamard states, adiabatic vacua and the construction of physical states for scalar quantum fields on curved spacetime, Rev. Math. Phys. 8, 1091 (1996)

[24] Kay, B.S., Casimir effect in quantum field theory, Phys. Rev. D20, 3052 (1979)

[25] Kay, B.S., Wald, R.M., Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon, Phys. Rep. 207, 49 (1991)

[26] Lamoreaux, S.K., Demonstration of the Casimir force in the 0.6 to 6 micrometers range, Phys. Rev. Lett. 78, 5 (1997); Mohideen, U., Roy, A., Precision measurement of the Casimir force from 0.1 to 0.9 micrometers, Phys. Rev. Lett. 81, 4549 (1998)

[27] Lüders, C., Roberts, J.E., Local quasiequivalence and adiabatic vacuum states. Comm. Math. Phys. 134, 29 (1990)

[28] Parker, L., Quantized fields and particle creation in expanding universes, Phys. Rev. 183, 1057 (1969)

[29] Parker, L., Raval, A., Nonperturbative effects of vacuum energy on the recent expansion of the universe, Phys. Rev. D60, 063512 (1999)

[30] Pusz, W., Woronowicz, S.L., Passive states and KMS states for general quantum systems, Commun. Math. Phys. 58, 273 (1978)

[31] Radzikowski, M.J., Micro-local approach to the Hadamard condition in quantum field theory in curved spacetime, Commun. Math. Phys. 179, 529 (1996)

[32] Roberts, J.E., Lectures on algebraic quantum field theory. In: The algebraic theory of superselection sectors. Introduction and recent results, ed. D. Kastler, pp. 1-112. World Scientific, Singapore, New Jersey, London, Hang Kong 1990

[33] Sahlmann, H., Verch, R., Passivity and microlocal spectrum condition, math-ph/0002021, Commun. Math. Phys., in print
[34] Sahlmann, H., Verch, R., Microlocal spectrum condition and Hadamard form for vector-valued quantum fields in curved spacetime, math-ph/0008029

[35] Schmüdgen, K., *Unbounded operator algebras and representation theory*, Birkhäuser, Basel, 1990

[36] Schulze, B.-W., *Pseudo-differential operators on manifolds with singularities*. Studies in Mathematics and its Applications, 24. North-Holland, Amsterdam, 1991

[37] Streater, R.F., Wightman, A.S., *PCT, spin and statistics, and all that*, Benjamin, New York, 1964

[38] Strohmaier, A., Verch, R., Wollenberg, M., work in progress

[39] Verch, R., Local definiteness, primarity and quasiequivalence of quasifree Hadamard quantum states in curved spacetime, Commun. Math. Phys. **160**, 507 (1994)

[40] Verch, R., Wavefront sets in algebraic quantum field theory, Commun. Math. Phys. **205**, 337 (1999)

[41] Wald, R.M., The back-reaction effect in particle creation in curved spacetime, Commun. Math. Phys. **54**, 1 (1977)

[42] Wald, R.M., *General relativity*, University of Chicago Press, Chicago, 1984

[43] Wald, R.M., *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, Chicago, 1994