DIRAC QUANTIZATION
OF FREE MOTION
ON CURVATURE SURFACES

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Abstract

We give an explicit operator realization of Dirac quantization of
free particle motion on a surface of codimension 1. It is shown that the
Dirac recipe is ambiguous and a natural way of fixing this problem is
proposed. We also introduce a modification of Dirac procedure which
yields zero quantum potential. Some problems of abelian conversion
quantization are pointed out.

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tential.
1. Introduction

We consider a problem of quantum motion in curved spaces. It is well-known that in the case of Euclidean spaces the correct Hamiltonian is $\hat{H} = -\frac{\hbar^2}{2}\Delta$. Podolsky [12] in 1928 proposed that for arbitrary space it should be replaced by $\hat{H} = -\frac{\hbar^2}{2}\Delta_{LB}$ with $\Delta_{LB}$ being the Laplace-Beltrami operator. This postulate is a direct and geometrically clear generalization of the dynamics in Euclidean spaces.

If one wants to get the theory by some canonical procedure, he or she encounters a severe problem. For any given classical theory there is an infinite number of quantum theories with a proper $\hbar \to 0$ limit. Quantization is not unique. In Euclidean spaces Dirac recipe in Cartesian coordinates yields experimentally correct result. In curved spaces we do not have a notion of Cartesian coordinates and can’t make a choice of the theory. A possible way out is to embed the space under consideration into some Euclidean space and to quantize the new theory as a theory with second-class constraints by Dirac brackets formalism [5] or by abelian conversion method [6, 11]. The results are different and depend on the embedding. In this article we restrict ourselves to codimension 1 surfaces.

In section 2 we describe the Dirac approach to motion on spheres [10, 9] and develop an explicit operator realization of it. If one demands the momenta to be differentials (instead of self-adjointness condition) the quantum potential would be zero. In section 3 we generalize our consideration to the case of arbitrary surface and show that the Dirac procedure is ambiguous. Dirac quantum potential depends on the choice of equation of surface; a way of fixing this freedom is proposed. At the same time the zero-potential quantization as in section 2 is well defined. The zero potential may be obtained for spheres by the abelian conversion method [10] too. In section 4 we point out some obstructions on the way of generalizing it to arbitrary surfaces and show that in general case one can’t get zero potential by abelian conversion.

2. Dirac quantization for spheres

We start with a free particle motion on $(n-1)$-dimensional sphere, $\sum_{i=1}^{n} x_i^2 = R^2$, in $n$-dimensional Euclidean space. It can be considered
as a system with two second-class constraints \([10]\)

\[
\phi_1 \equiv \sum_{i=1}^{n} x_i^2 - R^2 = 0, \quad (1)
\]

\[
\phi_2 \equiv \sum_{i=1}^{n} x_i p_i = 0 \quad (2)
\]

where \(p_i\) are canonical momenta. The Poisson bracket \(\{\phi_1, \phi_2\} = 2 \delta^2\) is not zero (it is the definition of second-class constraints), and hence in quantum theory the constraints \([11]\) and \([2]\) cannot be set equal zero simultaneously even for a physical sector \([7]\). This problem can be overcome by introducing the Dirac brackets:

\[
\{f, g\}_D = \{f, g\} - \sum_{a=1}^{2} \sum_{b=1}^{2} \{f, \phi_a\} \Delta_{ab} \{\phi_b, g\}, \quad (3)
\]

where \(\Delta_{ab}\) is the matrix inverse of \(\{\phi_a, \phi_b\}\). Now \(\{\phi_1, \phi_2\}_D = 0\) and for canonical variables we have \([10]\)

\[
\{x_i, x_j\}_D = 0, \quad (4)
\]

\[
\{x_i, p_j\}_D = \delta_{ij} - \frac{x_i x_j}{\delta^2}, \quad (5)
\]

\[
\{p_i, p_j\}_D = \frac{1}{\delta^2} (p_i x_j - p_j x_i). \quad (6)
\]

Dirac bracket is degenerate and does not define any symplectic manifold but it can be regarded as a Poisson structure \([11]\) obtained by factorization of original Poisson bracket algebra over motions in unphysical direction. One can get it by the following replacement:

\[
\vec{p} \to \vec{p} - \frac{\vec{x}}{\delta} \cdot \vec{p} \left( \frac{\vec{x}}{\delta} \right), \quad (7)
\]

so that all different values of radial momentum are identified. Another possible interpretation is made in \([2, 3]\) in terms of first-class functions algebra factorized over functions vanishing on the constraint surface.

Once we have the Dirac structure, the quantization can be performed in the usual way \([5]\). From \((1)-(6)\) we get

\[
[\hat{x}_i, \hat{x}_j] = 0, \quad (8)
\]
\[
[\hat{x}_i, \hat{p}_j] = i\hbar \left( \delta_{ij} \hat{I} - \frac{\hat{x}_i \hat{x}_j}{\sum_{l=1}^{n} \hat{x}_l^2} \right),
\]

(9)

\[
[\hat{p}_i, \hat{p}_j] = \frac{i\hbar}{\sum_{l=1}^{n} \hat{x}_l^2} (\hat{p}_i \hat{x}_j - \hat{p}_j \hat{x}_i).
\]

(10)

In (10) the operator ordering problem is solved; we show that this ordering is correct (satisfies the Jacobi identity) by providing an explicit operator realization of the algebra (8)-(10). We choose coordinate operators to be the usual ones \(\hat{x}_i = x_i \hat{I}\) and search for corresponding differential operators of momenta. One could solve the task by use of (7):

\[
-\frac{i\hbar}{\nabla} \rightarrow -\frac{i\hbar}{\nabla} - \frac{\hat{x}}{|\hat{x}|} \left( \frac{\hat{x}}{|\hat{x}|} \cdot \left(-\frac{i\hbar}{\nabla}\right) \right).
\]

This choice is in some sense unique. Indeed, we demand \(\hat{p}_i\) to be differentiations, i.e. to obey the Leibnitz rule. From (9) one gets \(\hat{p}_i(x_j) = -i\hbar \left( \delta_{ij} - \frac{x_i x_j}{\hat{x}^2} \right)\). By the Leibnitz rule we have for any polynomial

\[
\hat{p}_k \left( \sum_{\{\alpha\}} C_{\{\alpha\}} \prod_{i=1}^{\vert \alpha \vert} x_{\alpha_i} \right) = -i\hbar \sum_{\{\alpha\}} C_{\{\alpha\}} \cdot \left( \sum_{i=1}^{\vert \alpha \vert} \delta_{k \alpha_i} \prod_{j=1}^{i-1} x_{\alpha_j} \right) \left( \prod_{j=i+1}^{\vert \alpha \vert} x_{\alpha_j} \right) - \prod_{i=1}^{\vert \alpha \vert} x_{\alpha_i} \left( \frac{x_k}{\sum_{l=1}^{n} \hat{x}_l^2} \right)
\]

and extend this definition to analytic functions by continuity:

\[
\hat{p}_i = -i\hbar \left( \frac{\partial}{\partial x_i} - \frac{x_i}{|\hat{x}|} \sum_{j=1}^{n} \frac{x_j}{|\hat{x}|} \frac{\partial}{\partial x_j} \right);
\]

(11)

it’s the projection (7) of basis vectors \(-i\hbar \frac{\partial}{\partial x_i}\) onto the surface (11).
The calculation of commutators in (10) is straight-forward. It yields

\[ [\hat{p}_i, \hat{p}_j] = -\frac{\hbar^2}{x^2} \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right) = \frac{i\hbar}{x^2} (x_j \hat{p}_i - x_i \hat{p}_j) = \frac{i\hbar}{x^2} (\hat{p}_i x_j - \hat{p}_j x_i). \]

So, the second constraint is satisfied identically, \( \sum_{i=1}^{n} \hat{x}_i \hat{p}_i \equiv 0 \), and we fix the physical sector simply by \( \Psi_{\text{phys}} = \psi(x) \delta \left( \sum_{i=1}^{n} x_i^2 - R^2 \right) \).

The problem is that \( \hat{p}_i \) are not self-adjoint. At the sacrifice of Leibnitz rule we can introduce new self-adjoint momenta:

\[ \hat{\tilde{p}}_i = \frac{1}{2} (\hat{p}_i + \hat{p}_i^\dagger) = -i\hbar \left( \frac{\partial}{\partial x_i} - \frac{x_i}{|x|} \sum_{j=1}^{n} \frac{x_j}{|x|} \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \frac{x_i x_j}{|x|^2} \right) \right) = \hat{p}_i + i\hbar \frac{n-1}{2} \cdot \frac{x_i}{|x|^2} \hat{I}. \]

It is easy to check that the algebra (8)-(10) remains the same. The constraint (2) is now \( \hat{\phi}_2 = \sum_{i=1}^{n} (\hat{x}_i \hat{\tilde{p}}_i + (\hat{x}_i \hat{\tilde{p}}_i)^\dagger) \equiv 0 \).

**Theorem 1.** The Hamiltonian \( \hat{H}^{(D)} = \frac{1}{2} \sum_{i=1}^{n} \hat{\tilde{p}}_i^2 \) contains quantum potential \( V_q^{(D)} = \frac{\hbar^2 (n-1)^2}{8R^2} \).

**Proof.** By direct calculation we have

\[ \hat{\tilde{p}}_i \left( \psi(x) \cdot \delta \left( \sum_{i=1}^{n} x_i^2 - R^2 \right) \right) = \hat{p}_i (\psi(x)) \cdot \delta \left( \sum_{i=1}^{n} x_i^2 - R^2 \right), \]
$$\hat{H}^{(D)}(\psi(x)) =$$

$$= \left( \frac{1}{2} \sum_{i=1}^{n} \left( \hat{p}_i^2 + i\hbar(n-1) \frac{x_i}{|x|} \hat{p}_i \right) \right) \Psi(x) + \frac{\hbar^2(n-1)^2}{8R^2} \Psi(x) =$$

$$= -\frac{\hbar^2}{2} \left( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} \frac{x_i}{|x|} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \frac{x_j}{|x|} \frac{\partial}{\partial x_j} \right) \Psi(x) + \frac{\hbar^2(n-1)^2}{8R^2} \Psi(x) =$$

$$= -\frac{\hbar^2}{2} \Delta_{LB}(\psi(x)) + \frac{\hbar^2(n-1)^2}{8R^2} \psi(x),$$

where $\Delta_{LB}$ is the Laplace-Beltrami operator on sphere (for calculation of $\Delta_{LB}$ see section 3). So, the Dirac quantum potential is $V_q^{(D)} = \frac{\hbar^2(n-1)^2}{8R^2}$.

This result coincides with the conclusion of [10], but the approach of [10] is purely algebraic. What we presented here is an explicit operator realization of it, which clarifies the geometric properties.

The same procedure may lead to Podolsky theory if one takes the definition [11] and Hamiltonian $\hat{H}^{(P)} = \frac{1}{2} \sum_{i=1}^{n} \hat{p}_i^2 \hat{p}_i$ which equals $-\frac{\hbar^2}{2} \Delta_{LB}$ for the physical sector functions. The quantum potential is zero: $V_q^{(P)} = 0$. Thus one preserves an important property of momenta operators, the Leibnitz rule, so that they are differentials on the algebra of smooth functions. These operators are not self-adjoint and can’t represent observables. But they do not have any clear physical meaning being projections of generators of motions along the coordinate lines of n-dimensional flat space, which are somewhat esoteric for an observer living on the sphere. Natural observables on the sphere are generators of $SO(n)$ rotations, and they are self-adjoint (proportional to $i[\hat{p}_i, \hat{p}_j]$).

We should note that operators $\hat{p}_i$ are not self-adjoint with respect to Lebesgue measure in the other space $\mathbb{R}^n$. In order to get zero quantum potential with self-adjoint momenta one could try to find another measure for which these operators would be self-adjoint. But the potential $\frac{\hbar^2(n-1)^2}{8R^2}$ can be obtained algebraically [10], without use of any particular measure. Moreover, the desired measure does not exist. Indeed, for a measure $G(x)d^n x$ the operators $\hat{p}_i$ would be sym-
metric if and only if 
\[ \frac{\partial}{\partial x_i} - x_i \sum_{j=1}^{n} \frac{x_j}{R} \frac{\partial}{\partial x_i} \] 
\[ G(x) = \frac{(n-1)x_i}{R^2} G(x). \] After multiplication by \( x_i \) and summation over \( i \) one has \( G = 0 \). Due to the reasons mentioned in the Introduction we prefer to quantize in Cartesian coordinates with the standard Lebesgue measure.

3. Dirac quantization for arbitrary surfaces

We consider motions on a codimension 1 surface \( f(x_i) = 0 \). This theory has two constraints \[ \phi_1 \equiv f(x) = 0, \] \[ \phi_2 \equiv \sum_{i=1}^{n} (\partial_i f)p_i = 0. \] These constraints are of the second class because \( \{\phi_1, \phi_2\} = (\nabla f)^2 \neq 0 \). We introduce the Dirac brackets by \[ \{x_i, x_j\}_D = 0, \] \[ \{x_i, p_j\}_D = \delta_{ij} - \frac{(\partial_i f)(\partial_j f)}{(\nabla f)^2}, \] \[ \{p_i, p_j\}_D = \frac{1}{(\nabla f)^2} \sum_{k=1}^{n} ((\partial_j f)(\partial^2_{jk} f) - (\partial_i f)(\partial^2_{jk} f))p_k. \] We propose the following operators for the quantum description with non-selfadjoint momenta: \( \hat{x}_i = x_i \hat{I} \) and

\[ \hat{p}_i = -i\hbar \left( \frac{\partial}{\partial x_i} - \frac{(\partial_i f)}{|\nabla f|} \sum_{j=1}^{n} \frac{(\partial_j f)}{|\nabla f|} \frac{\partial}{\partial x_j} \right). \]

Here we used the factorization over unphysical motions again.

Lemma 2. The commutator algebra (corresponding to \[ 14 \], \[ 16 \]) is \[ [\hat{x}_i, \hat{x}_j] = 0, \] \[ [\hat{x}_i, \hat{p}_j] = \delta_{ij}. \]
\[ [\hat{x}_i, \hat{p}_j] = i\hbar \left( \delta_{ij} - \frac{(\partial_i f)(\partial_j f)}{\nabla f^2} \right) \hat{I}, \quad (18) \]

\[ [\hat{p}_i, \hat{p}_j] = \frac{i\hbar}{(\nabla f)^2} \sum_{k=1}^{n} \left( (\partial_j f)(\partial_{ik}^2 f) - (\partial_i f)(\partial_{jk}^2 f) \right) \hat{p}_k. \quad (19) \]

**Proof.** For operators under consideration (17) and (18) are obvious while (19) can be proved by a direct calculation:

\[
[\hat{p}_i, \hat{p}_j] = -\hbar^2 \left( \frac{\partial}{\partial x_i}, -\frac{(\partial_j f)}{|\nabla f|} \sum_{k=1}^{n} \frac{(\partial_k f)}{|\nabla f|} \frac{\partial}{\partial x_k} \right) + \\
+ \left( -\frac{(\partial_j f)}{|\nabla f|} \sum_{m=1}^{n} \frac{(\partial_m f)}{|\nabla f|} \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_j} \right) + \\
+ \left( \frac{(\partial_j f)}{|\nabla f|} \sum_{m=1}^{n} \frac{(\partial_m f)}{|\nabla f|} \frac{\partial}{\partial x_m}, \frac{(\partial_j f)}{|\nabla f|} \sum_{k=1}^{n} \frac{(\partial_k f)}{|\nabla f|} \frac{\partial}{\partial x_k} \right) = \\
= \hbar^2 \sum_{k=1}^{n} \frac{(\partial_j f)(\partial_{ik}^2 f) - (\partial_i f)(\partial_{jk}^2 f)}{(\nabla f)^2} \\
\cdot \left( \frac{\partial}{\partial x_k} - \frac{(\partial_k f)}{|\nabla f|} \sum_{m=1}^{n} \frac{(\partial_m f)}{|\nabla f|} \frac{\partial}{\partial x_m} \right). \]

One also has \( \sum_{i=1}^{n} (\partial_i f) \hat{p}_i \equiv 0 \). The physical sector is defined by
\[ \Psi_{phys} = \psi(x)\delta(f(x)) \] and the Hamiltonian \( \hat{H}(p) = \frac{1}{2} \sum_{i=1}^{n} \hat{p}_i^\dagger \hat{p}_i \) equals

\[ \hat{H}(p) = \frac{1}{2} \sum_{i=1}^{n} \left( \hat{p}_i + i\hbar \left( \sum_{j=1}^{n} \frac{\partial (\partial_i f)(\partial_j f)}{\left(\nabla f\right)^2} \right) i \right) \hat{p}_i = \]

\[ = -\frac{\hbar^2}{2} \left( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} \frac{\partial_i f}{|\nabla f|} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \frac{\partial_j f}{|\nabla f|} \frac{\partial}{\partial x_j} \right) - \left( \sum_{i=1}^{n} \frac{\partial (\partial_i f)}{\nabla f} \right) \left( \sum_{j=1}^{n} \frac{\partial_j f}{\nabla f} \frac{\partial}{\partial x_j} \right) = \]

\[ = -\frac{\hbar^2}{2} \left( \hat{\Delta} - \left( \frac{\partial}{\partial \nabla} \right)^2 - \text{div}(\nabla) \cdot \frac{\partial}{\partial \nabla} \right) \]

where \( \hat{\Delta} \) is the Laplace operator in the Euclidean space and \( \nabla = \frac{\nabla f}{|\nabla f|} \) is a unit vector normal to the surface (12).

Now we follow the standard Dirac procedure and replace our operators by self-adjoint ones:

\[ \hat{\tilde{p}}_i = \hat{p}_i + \frac{i\hbar}{2} \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} \frac{(\partial_i f)(\partial_j f)}{(\nabla f)^2} \right) \right) \]. (20)

It violates the relation (19), but one can overcome this problem by changing the operator ordering. Indeed, from (19) we have

\[ [\hat{p}_i^\dagger, \hat{p}_j^\dagger] = i\hbar \sum_{k=1}^{n} p_k^\dagger \frac{(\partial_j f)(\partial_k^2 f) - (\partial_i f)(\partial_j^2 f)}{(\nabla f)^2} \]

for \( \hat{p}_i^\dagger = \hat{p}_i + i\hbar \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} \frac{(\partial_i f)(\partial_j f)}{(\nabla f)^2} \right) \). It is not difficult to deduce the following commutational relation from it:

\[ [\hat{p}_i^\dagger, \hat{p}_j^\dagger] = i\hbar \sum_{k=1}^{n} \left( \frac{(\partial_j f)(\partial_k^2 f) - (\partial_i f)(\partial_j^2 f)}{(\nabla f)^2} \right) \hat{p}_k^\dagger + \]

\[ + \hat{p}_k \frac{(\partial_j f)(\partial_k^2 f) - (\partial_i f)(\partial_j^2 f)}{(\nabla f)^2} \].
which differs from \[10\] only by operator ordering. The Hamiltonian
\[
\hat{H}^{(\mathcal{D})} = \frac{1}{2} \sum_{i=1}^{n} \hat{p}_{i}^{2} = \hat{H}^{(\mathcal{P})} + V_{q}(x) \]
contains the quantum potential
\[
V_{q} = -\frac{\hbar^{2}}{8} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{\partial}{\partial x_{j}} \left( \frac{\partial f}{\nabla f} \right) \right)^{2} + 
+ \frac{\hbar^{2}}{4} \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_{i}} - \sum_{k=1}^{n} \frac{\partial_{f}(\partial_{f})(\partial_{f})(\partial_{f})}{(\nabla f)^{2}} \frac{\partial}{\partial x_{k}} \right) \left( \sum_{j=1}^{n} \frac{\partial}{\partial x_{j}} \left( \frac{\partial f}{\nabla f} \right) \right).
\]
Unfortunately both Hamiltonians, \(\hat{H}^{(\mathcal{D})}\) and \(\hat{H}^{(\mathcal{P})}\), are ambiguous; they take different values for those functions which represent one and the same surface. The problem exists even for spheres. We first prove it for \(\hat{H}^{(\mathcal{D})}\).

**Theorem 3.** Dirac quantization procedure is ambiguous.

**Proof.** Indeed, any surface can be represented by its tangent paraboloid at some point: \(f(y) = y_{n} - \frac{1}{2} \sum_{\alpha=1}^{n-1} k_{\alpha} y_{\alpha}^{2} + \mathcal{O}(y_{\alpha}^{3})\), \(y_{\alpha}\)'s are Cartesian coordinates. We have \(n_{n} = -\frac{\partial f}{|\nabla f|} = -(1 + \sum_{\alpha=1}^{n-1} k_{\alpha} y_{\alpha}^{2})^{-1/2} + \mathcal{O}(y_{\alpha}^{2}) = -1 + \mathcal{O}(y_{\alpha}^{2})\); \(n_{\alpha} = -\frac{\partial_{f}}{|\nabla f|} = k_{\alpha} y_{\alpha} + \mathcal{O}(y_{\alpha}^{2})\) and \(\partial_{f}^{2} f = 0\). One can neglect \(\mathcal{O}(y_{\alpha}^{3})\) terms in the calculation, because
\[
\sum_{j=1}^{n} \frac{\partial}{\partial y_{j}} \left( \frac{\partial f}{\nabla f} \right) \left( \frac{\partial f}{\nabla f} \right) = -\sum_{\alpha=1}^{n-1} k_{\alpha} + \mathcal{O}(y_{\alpha}) \sum_{j=1}^{n} \frac{\partial}{\partial y_{j}} \left( \frac{\partial f}{\nabla f} \right) = \sum_{\beta=1}^{n-1} k_{\alpha} k_{\beta} y_{\alpha} + k_{\alpha} y_{\alpha} + \mathcal{O}(y_{\alpha}) \quad \text{and}
\]
\[
\sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial_{f}(\partial_{f})(\partial_{f})(\partial_{f})}{(\nabla f)^{2}} = \mathcal{O}(y_{\alpha}) \quad \text{(if } k = n \text{ then we get zero identically, if } k \neq n \text{ then } \partial_{f} f = \mathcal{O}(y_{\alpha})).
\]
The obtained quantum potential in the vicinity of the point \(\vec{y} = 0\) equals
\[
V_{q} = \frac{\hbar^{2}}{8} \left( \sum_{\alpha=1}^{n-1} k_{\alpha} \right)^{2} + 2 \sum_{\alpha=1}^{n-1} k_{\alpha}^{2} + \mathcal{O}(y_{\alpha}).
\]
For a sphere the principal curvatures are \(k_{\alpha} = \frac{1}{R}\) and at the chosen point we have \(V_{q} = \frac{\hbar(\alpha^{2}-1)}{8R^{2}}\), which differs from the result of \[10\] and section 2. So, the Dirac recipe is ambiguous. \(\square\)
To fix the freedom, let’s consider a curvilinear coordinate system in a neighbourhood of (12). We suppose that $z_n$ is just a distance from the surface (with a proper sign, of course) and coordinate lines of $z_1, z_2, \ldots z_{n-1}$ are orthogonal to that of $z_n$. We propose the following choice of function $f(x)$: it should be equal $z_n$. Such smooth function exists in the whole vicinity of any orientable surface. After that we have $|\nabla f| = 1$ and $\partial_i n_k = \partial_k n_i$, $\sum_{k=1}^{n} n_k \partial_k n_i = 0$ where $n_k = \partial_k f$.

**Theorem 4.** The Dirac Hamiltonian for our choice of the function $f(x)$ is $\hat{H}^{(D)} = -\frac{\hbar^2}{2} \Delta_{LB} + \frac{\hbar^2}{8} \left( \sum_{\alpha=1}^{n-1} k_\alpha \right)^2$.

**Proof.** The quantum potential is

$$V_q^{(D)} = \frac{\hbar^2}{4} \sum_{i=1}^{n} \left( n_i \sum_{k=1}^{n} n_k \partial_k \right) \left( \sum_{j=1}^{n} \partial_j n_i n_j \right) - \frac{\hbar^2}{8} \left( \sum_{j=1}^{n} \partial_j n_i n_j \right)^2$$

$$= \frac{\hbar^2}{4} \left( (\text{div}(\vec{n}))^2 + \sum_{i=1}^{n} n_i \partial_i \cdot \text{div}(\vec{n}) - \sum_{k=1}^{n} n_k \partial_k \cdot \text{div}(\vec{n}) \right) -$$

$$- \frac{\hbar^2}{8} (\text{div}(\vec{n}))^2 = \frac{\hbar^2}{8} (\text{div}(\vec{n}))^2 = \frac{\hbar^2}{8} \left( \sum_{\alpha=1}^{n-1} k_\alpha \right)^2$$.

For spheres it yields the previous result $\frac{\hbar^2 (n-1)^2}{8 R^2}$.

The kinetic part of the of $\hat{H}^{(D)}$ is obtained by the following lemma:

**Lemma 5.** The Laplace-Beltrami operator on the surface $f(x) = 0$ is

$$\Delta_{LB} = \tilde{\Delta} - \left( \frac{\partial}{\partial \vec{n}} \right)^2 - \text{div}(\vec{n}) \cdot \frac{\partial}{\partial \vec{n}}$$.

**Proof.** In curvilinear coordinates $z_i$ the metric tensor is

$$\tilde{g}_{ik} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix}$$.

The definition of Laplace operator reads

$$\tilde{\Delta} = \sum_{i=1}^{n} \sum_{k=1}^{n} \tilde{g}^{-1/2} \partial_i \tilde{g}^{1/2} \tilde{g}^{ik} \partial_k = \partial_n^2 + \left( \tilde{g}^{-1/2} \partial_n \tilde{g}^{1/2} \right) \partial_n + \Delta_{LB}$$.
with $\Delta_{LB}$ being the Laplace-Beltrami operator on a surface $z_n = \text{const}$. The constraint (12) is $z_n = 0$. Let’s take another surface, $z_n = \epsilon$:

![Diagram of a surface with normal vectors and areas](image)

We have $\text{div} (\vec{\nu}) = \frac{dS'}{dV} + \mathcal{O}(\epsilon) = \frac{dS'}{dS} + \mathcal{O}(\epsilon)$, hence $dS' = dS(1 + \epsilon \text{div} (\vec{\nu}) + \mathcal{O}(\epsilon^2))$ and $\tilde{g}^{-1/2} \partial_n \tilde{g}^{1/2} = \text{div} (\vec{\nu})$.

It proves that $\tilde{\Delta} = \partial_n^2 + \text{div} (\vec{\nu}) \cdot \partial_n + \Delta_{LB}$ and $\tilde{H}^{(D)} = -\frac{k^2}{2} \Delta_{LB} + \frac{\hbar^2}{8} \left( \sum_{\alpha=1}^{n-1} k_{\alpha} \right)^2$.

while $\hat{H}^{(P)} = -\frac{k^2}{2} \Delta_{LB}$ exactly as in Podolsky theory with $V_q(P) = 0$.

In general the unit normal vector $\vec{\nu} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$ would not be orthogonal to the surfaces $z_n = \text{const} \neq 0$ and the result of the Lemma 5 would not be true. The second normal derivative $(\frac{\partial}{\partial \vec{\nu}})^2$ would yield an additional first order differential term to $\Delta_{LB}$. For a parabola $f(x) = x_2 - \frac{k^2}{2} x_1^2 = 0$ we have $n_1 = \frac{k x_1}{\sqrt{1+k^2 x_1^2}}$ and $n_2 = -\frac{1}{\sqrt{1+k^2 x_1^2}}$. One can easily see that at the surface $f(x) = 0$

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial \vec{\nu}^2} - \text{div} (\vec{\nu}) \frac{\partial}{\partial \vec{\nu}} = \Delta_{LB} - \frac{k^2 x_1}{(1+k^2 x_1^2)^2} \frac{\partial}{\partial x_1},$$

where $\Delta_{LB} = \frac{\partial}{\partial t}$ with $t_1 = \frac{1}{\sqrt{1+k^2 x_1^2}}$, $t_2 = \frac{k x_1}{\sqrt{1+k^2 x_1^2}}$. So, the kinetic part of the Hamiltonian in the Dirac recipe is also ambiguous.

4. Some remarks on abelian conversion

The abelian conversion method [[6]] consists of introducing new canonical pair of variables $Q$, $K$ and first class constraints $\sigma_1$, $\sigma_2$: $\{\sigma_1, \sigma_2\} = 0$; $\sigma_1 = \phi_1$, $\sigma_2 = \phi_2$ if $Q = 0$ and $K = 0$. In our case it would be $\sigma_1 = f(x) + K$ and $\sigma_2 = \vec{\nu} \cdot \vec{p} + Q$. The next step is to find a new Hamiltonian such that $H_S = H$ if $Q = 0$ and $K = 0$ and $\{H_S, \sigma_1\} = \{H_S, \sigma_2\} = 0$. The physical sector is obtained by setting $\sigma_1 = \sigma_2 = 0$. 

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For a sphere it yields zero quantum potential, see \cite{10,9}. In this section we point out some difficulties in the way of applying this method to arbitrary surfaces. For spheres the authors of \cite{10,9} had the result of the form

\[ H_S = H_S(\sigma_1, \sigma_2, \sum_{i<k} (x_i p_k - x_k p_i)^2) \]

and it is not difficult to see that

\[ \sum_{i<k} (x_i p_k - x_k p_i)^2 = (\sum_i x_i^2)(\sum_i p_i^2 - (\sum_i n_i p_i)^2). \]

It allows to get the correct answer because \( \sigma_2^2 = (\sum_i n_i p_i)^2 \) if \( Q = 0 \).

In general case let’s try to search for \( H_S \) in a form

\[ H_S = H_S(\sigma_1, \sigma_2, g(x)(\sum_i p_i^2 - (\sum_i n_i p_i)^2)). \]

The equation for \( g(x) \) can be obtained by use of relations

\[ \{H_S, \sigma_1\} = -\sum_i n_i \frac{\partial H_S}{\partial p_i} = 0, \quad (21) \]

\[ \{H_S, \sigma_2\} = \sum_i n_i \frac{\partial H_S}{\partial x_i} - \sum_{i,k} p_k(\partial_k n_i) \frac{\partial H_S}{\partial p_i} = 0. \quad (22) \]

From (22) we have

\[ \sum_{i,k} p_i p_k (n_j \partial_j g(x)(\delta_{ik} - n_i n_k) - 2g(x)(\partial_i n_k)) = 0. \]

For spheres it has a non-zero solution because \( \partial_i n_k \sim \delta_{ik} - n_i n_k \). But this is not true for arbitrary surfaces. Hence the result of \cite{10} can’t be generalized directly.

Moreover, we show that on this way quadratic in momenta \( p_i \) physical Hamiltonian is not possible in general. Let’s try to find it in a form

\[ H_S = H_S(\sigma_1, \sigma_2, \sum_{i,k} C_{ik}(x)p_i p_k + \sum_i D_i p_i + E(x)) \]

with symmetric matrix \( C_{ik} \). Equations (21) and (22) yield:

\[ \sum_i n_i C_{ik} = 0, \quad \forall \ k, \quad (23) \]

\[ \sum_i n_i \partial_i C_{ik} - \sum_i C_{ik} \partial_i n_k - \sum_i C_{ik} \partial_l n_l = 0, \quad \forall \ l, k, \quad (24) \]
\[
\sum_i n_i D_i = 0, \\
\sum_i n_i \partial_i D_k - \sum_i D_i \partial_i n_k = 0, \ \forall \ k, \\
\sum_i n_i \partial_i E = 0.
\]

For \(C_{ik}\) we have more equations than variables. This system does not have non-zero solution in general case. The problem appears even for a parabola, \(f(x) = x_2 - \frac{k}{2}x_1^2\). From (23) one has \(C_{11} = kx_1C_{12}\) and \(C_{22} = \frac{C_{12}}{kx_1}\). After that (24) turns to yield three different equations for one function \(C_{12}(x)\). This system is not solvable.

One could consider a coordinate system \(z_1, z_2\) from section 3 with \(n = 2\). Then we have \(C_{22} = C_{12} = 0\) and \(\partial_{z_2} C_{11} = 0\). This system is solvable, of course, but the coordinates \(z\) are not Cartesian.

Let’s consider Cartesian coordinates and function \(f(x) = z_n\). In this case the unit normal equals \(n_i = \partial_i f\) and has some additional properties: \(\partial_i n_k = \partial_k n_i\); \(\sum_i n_i \partial_i n_k = 0\). With these properties equations (23) - (24) are solvable. Indeed, we have

\[
C_{\alpha n} = -\frac{1}{n_n} \sum_{\beta=1}^{n-1} n_\beta C_{\alpha \beta}, \quad C_{nn} = \frac{1}{n_n^2} \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^{n-1} n_\alpha n_\beta C_{\alpha \beta}
\]

from (23) and analogous relations for the normal

\[
\partial_n n_\alpha = \partial_\alpha n_n = -\frac{1}{n_n} \sum_{\beta=1}^{n-1} n_\beta \partial_\beta n_\alpha, \quad \partial_n n_n = \frac{1}{n_n^2} \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^{n-1} n_\alpha n_\beta \partial_\alpha n_\beta.
\]

After that for \(C_{\alpha, \beta}\) equation (24) yields

\[
\sum_{i=1}^{n} n_i \partial_i C_{\alpha \beta} - \sum_{\zeta=1}^{n-1} \sum_{\gamma=1}^{n-1} C_{\alpha \zeta} \partial_\gamma n_\beta \left(\delta_{\zeta \gamma} + \frac{n_\zeta n_\gamma}{n_n^2}\right) - \\
- \sum_{\zeta=1}^{n-1} \sum_{\gamma=1}^{n-1} C_{\beta \zeta} \partial_\gamma n_\alpha \left(\delta_{\zeta \gamma} + \frac{n_\zeta n_\gamma}{n_n^2}\right) = 0 \quad (25)
\]

To this moment everything is solvable (provided that we made a good choice of direction of \(n\)-th axis). And it’s not difficult to see that
remaining equations in (24) take the form

\[-\frac{1}{n} \sum_{\alpha=1}^{n-1} n_{\alpha} \left[ \sum_{i=1}^{n} n_i \partial_i C_{\alpha\beta} - \sum_{\zeta=1}^{n-1} \sum_{\gamma=1}^{n-1} C_{\alpha\zeta} \partial_{\gamma} n_{\beta} \left( \delta_{\zeta\gamma} + \frac{n_{\zeta} n_{\gamma}}{n^2} \right) - \right. \]

\[\left. - \sum_{\zeta=1}^{n-1} \sum_{\gamma=1}^{n-1} C_{\beta\zeta} \partial_{\gamma} n_{\alpha} \left( \delta_{\zeta\gamma} + \frac{n_{\zeta} n_{\gamma}}{n^2} \right) \right] = 0,\]

and follow directly from (25).

So, quadratic in momenta Hamiltonian is possible in Cartesian coordinates if one admits the special definition of function $f(x)$. But even after that this method can’t yield the Podolsky theory for arbitrary surface, because it would mean that the quantum physical Hamiltonian is $-\hbar^2 \Delta_{LB}$ on the surface. And it follows from Lemma 5 that for it to take place with momenta $\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i}$, we should have in the classical limit a Hamiltonian with quadratic in momenta term proportional to $\sum_i p_i^2 - (\sum_i n_i p_i)^2$. But generally it’s not the case.

5. Conclusion

Quantum theory contains more information than classical one. That’s why it is not possible to find a unique quantization for a given classical theory. In the quantum world it is no longer enough to say that some particle moves in a certain curved space. One should know the nature of this motion. If it’s just some potential force which makes the particle to stay at the curved surface, one should use the thin layer quantization method [4, 5, 7]. But if there is no outer space (apart from our formalism) the result should not depend on any extrinsic properties of the physical space, so that Podolsky theory seems to be the preferable one. We reproduced this theory by our modification of Dirac quantization procedure. Still there may be some physical systems to which the original Dirac quantization should be applied. The
Dirac recipe turned out to be ambiguous, but we proposed a natural way to overcome this ambiguity.

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