Simple functional equations for generalized Selberg zeta functions with Tate motives

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Abstract

We prove that for a compact locally symmetric Riemannian space $M$ of rank 1 there exist infinitely many automorphic Tate motives $f$ such that the generalized Selberg zeta function $Z_{M(f)}(s)$ satisfies a simple functional equation in the sense that it has no gamma factors.

Key Words: Selberg zeta functions, functional equations, Tate motives

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Introduction

Let

$$Z_M(s) = \prod_{P \in \text{Prim}(M)} \prod_{\lambda} (1 - N(P)^{-s - \lambda}),$$

be the Selberg zeta function of a compact locally symmetric Riemannian space $M$ of rank 1, where $\text{Prim}(M)$ denotes the set of primitive closed geodesics on $M$ with $N(P) = \exp(\text{length}(P))$. It is known that $Z_M(s)$ has analytic continuation to all $s \in \mathbb{C}$ with a functional equation

$$Z_M(2\rho_M - s) = Z_M(s)S_M(s),$$

where

$$S_M(s) = \exp\left(\text{vol}(M) \int_0^{s-\rho_M} \mu_M(it)dt\right).$$

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Here $\mu_M(t)$ is the Plancherel measure and $2\rho_M, \text{ vol}(M) \in \mathbb{Z}_{>0}$.

Actually, the “gamma factor” $S_M(s)$ can be described explicitly in terms of the multiple sine function (or the multiple gamma function) as in [1]; see §1 of the text. In this paper we tacitly assume that the dimension $\text{dim}(M)$ is even since otherwise (that is the case of $M = \Gamma \backslash SO(1, 2n + 1)/K$) the factor $S_M(s)$ is a simple exponential function.

Now, we introduce the action of $f(x) \in \mathbb{Z}[x, x^{-1}]$ on $Z_M(s)$ as the “(absolute) Tate motif” (see Manin [5]). More concretely, for a Laurent polynomial

\[ f(x) = \sum_{k \in \mathbb{Z}} a(k)x^k \in \mathbb{Z}[x, x^{-1}], \]

we define

\[ Z_{M(f)}(s) = \prod_k Z_M(s-k)^{a(k)}, \]

which is considered as a generalized Selberg zeta function.

The purpose of this paper is to prove the following result.

**Theorem** Let $M$ be a compact locally symmetric Riemannian space of rank 1 as above. Then $Z_{M(f)}(s)$ has simple functional equations

\[ Z_{M(f)}(D + 2\rho_M - s)^C = Z_{M(f)}(s) \]

for infinitely many automorphic $f$. Here the automorphy of $f$ means that

\[ f(x^{-1}) = Cx^{-D}f(x) \]

with $C = \pm 1$ and $D \in \mathbb{Z}$.

Our proof shows that the functional equation of $Z_{M(f)}(s)$ for an automorphic

\[ f(x) = \sum_{k \in \mathbb{Z}} a(k)x^k \in \mathbb{Z}[x, x^{-1}], \]

is given as

\[ Z_{M(f)}(D + 2\rho_M - s)^C = Z_{M(f)}(s)S_{M(f)}(s) \]

where

\[ S_{M(f)}(s) = \prod_{k \in \mathbb{Z}} S_M(s-k)^{a(k)} \]

in general: see Theorem 2 in §2. Hence the crucial point is to show that

\[ S_{M(f)}(s) = 1 \]

for infinitely many $f$. Thus we do not need any “gamma factors” in such cases. For more explicit construction of such $f$ see Theorem 3 in §3.
We notice that the original Selberg zeta function $Z_M(s)$ was studied by Selberg [6, 7] for a compact Riemann surface $M$ of genus $g \geq 2$. It is written as

$$Z_M(s) = \prod_{P \in \text{Prim}(M)} \prod_{n=0}^{\infty} \left( 1 - N(P)^{-s-n} \right).$$

The functional equation is

$$Z_M(1-s) = Z_M(s) S_M(s)$$

with

$$S_M(s) = \exp \left( (4 - 4g) \int_0^{s-\frac{1}{2}} \pi t \tan(\pi t) dt \right) = (S_2(s)S_2(s+1))^{2-2g},$$

where

$$S_r(s) = S_r(s, (1, \cdots, 1))$$

is the multiple sine function in [1]. We refer to the paper [2] concerning this case.

\section{The gamma factor}

We recall the explicit calculation of $S_M(s)$ by multiple sine functions and multiple gamma functions in [1] according to the classification of $M = \Gamma \backslash G/K$.

\textbf{Theorem 1 ([1]).} The factors $S_M(s)$ are written explicitly as follows.

1. $S_M(s) = \begin{cases} (S_{2n}(s)S_{2n}(s+1))^{\text{vol}(M)/2} & G = SO(1, 2n) \\ \left( \prod_{k=0}^n S_{2n}(s+k) \right)^{\text{vol}(M)/2} & G = SU(1, n) \\ \left( \prod_{k=0}^{2n-1} S_{4n}(s+k) \right)^{\text{vol}(M)} & G = Sp(1, n) \\ (S_{16}(s)S_{16}(s+1)^{10}S_{16}(s+2)^{28} \times S_{16}(s+3)^{28}S_{16}(s+4)^{28}S_{16}(s+5)^{28})^{\text{vol}(M)} & G = F_4. \end{cases}$

2. $S_M(s) = \frac{\Gamma_M(s)}{\Gamma_M(2\rho_M - s)}$
with
\[
\Gamma_M(s) = \begin{cases} 
\frac{(\Gamma_{2n}(s)\Gamma_{2n}(s+1))^{\text{vol}(M)(-1)^{\dim(M) - 1}}}{\prod_{k=0}^{2n-1} \Gamma_{4n}(s+k)\Gamma_{4n}(s+k+1)} & G = SO(1, 2n) \\
\frac{1}{\prod_{k=0}^{n-1} \Gamma_{2n}(s+k)\Gamma_{2n}(s+k+1)} & G = SU(1, n) \\
\frac{\Gamma_{16}(s+1)\Gamma_{16}(s+2)^{28} \times \Gamma_{16}(s+3)\Gamma_{16}(s+4)^{10} \Gamma_{16}(s+5))^{\text{vol}(M)}}{\prod_{k=0}^{n-1} \Gamma_{2n}(s+k)\Gamma_{2n}(s+k+1)} & G = F_4.
\end{cases}
\]

We refer to [1] for the proofs. In the case of compact Riemann surfaces $M$ of genus $g \geq 2$, we have
\[
M = \Gamma \setminus SO(1, 2)/K
\]
and $\text{vol}(M) = 2g - 2$ with $\dim(M) = 2$.

## 2 Generalized functional equations

We show the functional equation for $Z_{M(f)}(s)$ with automorphic Tate motif $f(x) \in \mathbb{Z}[x, x^{-1}]$ satisfying
\[
f(x^{-1}) = Cx^{-D}f(x).
\]

**Theorem 2.** For each automorphic $f$ we have the functional equation
\[
Z_{M(f)}(D + 2\rho_M - s)^C = Z_{M(f)}(s)S_{M(f)}(s).
\]

**Proof.** First we remark that the following equivalence is easily seen:
\[
f(x^{-1}) = Cx^{-D}f(x) \iff a(D - k) = Ca(k) \quad (\forall k \in \mathbb{Z}).
\]

Then we see that
\[
Z_{M(f)}(D + 2\rho_M - s)^C = \prod_k Z_{M}(D + 2\rho_M - s - k)^{Ca(k)}
\]
\[
= \prod_k Z_{M}(2\rho_M - s + (D - k))^{a(D-k)}
\]
\[
= \prod_k Z_{M}(2\rho_M - s)(s - k))^{a(k)},
\]

where we replaced $k$ by $D - k$. Thus the functional equation for $Z_{M}(s)$ gives
\[
Z_{M(f)}(D + 2\rho_M - s)^C = \prod_k (Z_{M}(s - k)S_{M}(s - k))^{a(k)}
\]
\[
= Z_{M(f)}(s)S_{M(f)}(s).
\]

\[\square\]
3 Vanishment of gamma factors

Now we prove the main result.

**Theorem 3.** Let $f(x) \in (x - 1)^{\dim(M)} \mathbb{Z}[x, x^{-1}]$ be automorphic. Then it holds that

$$S_{M(f)}(s) = 1.$$  

Namely, we have

$$Z_{M(f)}(D + 2\rho_M - s)^C = Z_{M(f)}(s)$$

for such $f$.

**Proof.** We denote by $S^{f}_r(s)$ the action of $f(x) \in \mathbb{Z}[x, x^{-1}]$ on $S_r(x)$. We remark that

$$S^{fg}_r(s) = (S^{f}_r)^g.$$  

In fact for

$$f(x) = \sum_k a(k)x^k,$$

$$g(x) = \sum_l b(l)x^l$$

it holds that

$$f(x)g(x) = \sum_{k,l} a(k)b(l)x^{k+l}$$

and

$$S^{fg}_r(s) = \prod_{k,l} S_r(s - (k + l))^{a(k)b(l)}.$$  

On the other hand it follows that

$$S^{f}_r(s) = \prod_k S_r(s - k)^{a(k)}$$

and that

$$(S^{f}_r)^g(s) = \prod_l \left( \prod_k S_r((s - l) - k)^{a(k)} \right)^{b(l)}$$

$$= \prod_{k,l} S_r(s - (k + l))^{a(k)b(l)}.$$  

Thus $S^{fg}_r = (S^{f}_r)^g$.

Let $r = \dim(M)$ and put

$$f_r(x) = (1 - x^{-1})^r.$$
We show that \( S_f^r(s) = -1 \).

The case \( r = 1 \) is easy:

\[
S_f^1(s) = \frac{S_1(s)}{S_1(s + 1)} = \frac{2\sin(\pi s)}{2\sin(\pi(s + 1))} = -1.
\]

In general, the relation (see [1])

\[
S_f(1-x^{-1})(s) = \frac{S_r(s)}{S_r(s + 1)} = S_{r-1}(s)
\]

gives

\[
S_f^r(s) = S_{r-1}^r(s) = \cdots = S_1^r(s) = -1
\]

Let

\[
f(x) \in (x-1)^r\mathbb{Z}[x, x^{-1}]
\]

and

\[
f(x) = f_r(x)g(x).
\]

Then we have

\[
S_f^r(s) = \pm 1.
\]

Actually,

\[
S_f^r(s) = S_r^fg(s) = (S_r^f)^g(s) = (-1)^g(1) = \pm 1.
\]

Thus by the explicit formula for \( S_M(s) \) in §1, we see that

\[
S_M(f)(s) = 1.
\]

\[\square\]

**Remark.** The above arguments give the implication

\[
f(1) = f'(1) = \cdots = f^{(r-1)}(1) = 0 \implies S_f^r(s) = \pm 1.
\]

We refer to [3][4] concerning the converse and generalizations.

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