Kac-Moody Lie Algebras
Graded by Kac-Moody Root Systems

Hechmi Ben Messaoud and Guy Rousseau

Communicated by A. Valette

Abstract. We look to gradations of Kac-Moody Lie algebras by Kac-Moody root systems with finite dimensional weight spaces. We extend, to general Kac-Moody Lie algebras, the notion of $C-$admissible pair as introduced by H. Rubenthaler and J. Nervi for semi-simple and affine Lie algebras. If $\mathfrak{g}$ is a Kac-Moody Lie algebra (with Dynkin diagram indexed by $I$) and $(I, J)$ is such a $C-$admissible pair, we construct a $C-$admissible subalgebra $\mathfrak{g}^J$, which is a Kac-Moody Lie algebra of the same type as $\mathfrak{g}$, and whose root system $\Sigma$ grades finitely the Lie algebra $\mathfrak{g}$. For an admissible quotient $\rho : I \to \hat{I}$ we build also a Kac-Moody subalgebra $\mathfrak{g}^\rho$ which grades finitely the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is affine or hyperbolic, we prove that the classification of the gradations of $\mathfrak{g}$ is equivalent to those of the $C-$admissible pairs and of the admissible quotients. For general Kac-Moody Lie algebras of indefinite type, the situation may be more complicated; it is (less precisely) described by the concept of generalized $C-$admissible pairs.

Mathematics Subject Classification 2000: 17B67.
Key Words and Phrases: Kac-Moody algebra, $C-$admissible pair, gradation.

Introduction

The notion of gradation of a Lie algebra $\mathfrak{g}$ by a finite root system $\Sigma$ was introduced by S. Berman and R. Moody [8] and further studied by G. Benkart and E. Zelmanov [5], E. Neher [15], B. Allison, G. Benkart and Y. Gao [1] and J. Nervi [16]. This notion was extended by J. Nervi [17] to the case where $\mathfrak{g}$ is an affine Kac-Moody algebra and $\Sigma$ the (infinite) root system of an affine Kac-Moody algebra; in her two articles she uses the notion of $C-$admissible subalgebra associated to a $C-$admissible pair for the Dynkin diagram, as introduced by H. Rubenthaler [21].

We consider here a general Kac-Moody algebra $\mathfrak{g}$ (indecomposable and symmetrizable) and the root system $\Sigma$ of a Kac-Moody algebra. We say that $\mathfrak{g}$ is finitely $\Sigma-$graded if $\mathfrak{g}$ contains a Kac-Moody subalgebra $\mathfrak{m}$ (the grading subalgebra) whose root system relatively to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\Sigma$ and moreover the action of $ad(\mathfrak{a})$ on $\mathfrak{g}$ is diagonalizable with weights in $\Sigma \cup \{0\}$ and
finite dimensional weight spaces, see Definition 1.4. The finite dimensionality of weight spaces is a new condition, it was fulfilled by the non-trivial examples of J. Nervi [17] but it excludes the gradations of infinite dimensional Kac-Moody algebras by finite root systems as in [5]. Many examples of these gradations are provided by the almost split real forms of $g$, cf. 1.6. We are interested in describing the possible gradations of a given Kac-Moody algebra (as in [16], [17]), not in determining all the Lie algebras graded by a given root system $\Sigma$ (as e.g. in [1] for $\Sigma$ finite). We carry out completely this project when $g$ is affine or hyperbolic.

Let $I$ be the index set of the Dynkin diagram of $g$, we generalize the notion of $C$–admissible pair $(I,J)$ as introduced by H. Rubenthaler [21] and J. Nervi [16], [17], cf. Definition 2.1. For each Dynkin diagram $I$ the classification of the $C$–admissible pairs $(I,J)$ is easy to deduce from the list of irreducible $C$–admissible pairs due to these authors. We are able then to generalize in section 2 their construction of a $C$–admissible subalgebra (associated to a $C$–admissible pair) which grades finitely $g$:

**Theorem 1.** (cf. 2.6, 2.11, 2.14) Let $g$ be an indecomposable and symmetrizable Kac-Moody algebra, associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. Let $J \subseteq I$ be a subset of finite type such that the pair $(I,J)$ is $C$–admissible. There is a generalized Cartan matrix $A^J = (a_{k,l}^{J})_{k,l \in I'}$ with index set $I' = I \setminus J$ and a Kac-Moody subalgebra $g^J$ of $g$ associated to $A^J$, with root system $\Delta^J$. Then $g$ is finitely $\Delta^J$–graded with grading subalgebra $g^J$.

For a general finite gradation of $g$ with grading subalgebra $m$, we prove (in section 3) that $m$ also is indecomposable, symmetrizable and the restriction to $m$ of the invariant bilinear form of $g$ is non-degenerate (3.11 and 3.17). The Kac-Moody algebras $g$ and $m$ have the same type: finite, affine or indefinite; the first two types correspond to the cases already studied e.g. by J. Nervi. Moreover if $g$ is indefinite Lorentzian or hyperbolic, then so is $m$ (Propositions 3.6 and 3.27). We get also the following precise structure result for this general situation:

**Theorem 2.** Let $g$ be an indecomposable and symmetrizable Kac-Moody algebra, finitely graded by a root system $\Sigma$ of Kac-Moody type with grading subalgebra $m$.

1) We may choose the Cartan subalgebras $a$ of $m$, $h$ of $g$ such that $a \subset h$.

Then there is a surjective map $\rho_a : \Delta \cup \{0\} \rightarrow \Sigma \cup \{0\}$ between the corresponding root systems. We may choose the bases $\Pi_a = \{\gamma_s \mid s \in I\} \subset \Sigma$ and $\Pi = \{\alpha_i \mid i \in I\} \subset \Delta$ of these root systems such that $\rho_a(\Delta^+) \subset \Sigma^+ \cup \{0\}$ and $\{\alpha \in \Delta \mid \rho_a(\alpha) = 0\} = \Delta_J := \Delta \cap (\sum_{j \in J} \mathbb{Z}\alpha_j)$ for some subset $J \subset I$ of finite type.

2) Let $I_{re} = \{i \in I \mid \rho_a(\alpha_i) \in \Pi_a\}$, $I_{im} = \{i \in I \mid \rho_a(\alpha_i) \not\in \Pi_a \cup \{0\}\}$. Then $J = \{i \in I \mid \rho_a(\alpha_i) = 0\}$. We note $I_{re}$ (resp. $J^o$) the union of the connected components of $I \setminus I'_{im} = I'_{re} \cup J$ meeting $I'_{re}$ (resp. contained in $J$), and $J_{re} = J \cap I_{re}$. Then the pair $(I_{re}, J_{re})$ is $C$–admissible (eventually decomposable).

3) There is a Kac-Moody subalgebra $g(I_{re})$ of $g$, associated to $I_{re}$, which contains $m$. This Kac-Moody Lie algebra is finitely $\Delta(I_{re})^{J_{re}}$–graded, with grading subalgebra $g(I_{re})^{J_{re}}$. Both algebras $g(I_{re})$ and $g(I_{re})^{J_{re}}$ are finitely $\Sigma$–graded with grading subalgebra $m$.

It may happen that $I_{im}$ is non-empty, we then say that $(I,J)$ is a gener-
alized $C-$admissible pair and the gradation is imaginary. We give and explain precisely an example in section 5.

When $I_{im}$ is empty (i.e. when the gradation is real : 3.16), $I_{re} = I$, $J_{re} = J$, $g(I_{re}) = g$, $(I, J) = (I_{re}, J_{re})$ is a $C-$admissible pair and the situation looks much like the one described by J. Nervi in the finite [16] or affine [17] cases. Actually we prove that this is always true when $g$ is of finite type, affine or hyperbolic (Proposition 3.26). In this real case we get the gradation of $g$ with two levels: $g$ is finitely $\Delta^J-$graded with grading subalgebra $g^J$ as in Theorem 1 and $g^J$ is finitely $\Sigma-$graded with grading subalgebra $m$. But the gradation of $g^J$ by $\Sigma$ and $m$ is such that the corresponding set "J" described as in Theorem 2 is empty; we say (following [16], [17]) that it is a maximal gradation, cf. Definition 3.16 and Proposition 3.21.

To get a complete description of the real gradations, it remains to describe the maximal gradations; this is done in section 4. We prove in Proposition 4.1 that a maximal gradation $(g, \Sigma, m)$ is entirely described by a quotient map $\rho : I \rightarrow \overline{I}$ which is admissible i.e. satisfies two simple conditions (MG1) and (MG2) with respect to the generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. Conversely for any admissible quotient map $\rho$, it is possible to build a maximal gradation of $g$ associated to this map, cf. Proposition 4.5 and Remark 4.7.

1. Preliminaries

We recall the basic results on the structure of Kac-Moody Lie algebras and we set the notations. More details can be found in the book of Kac [12]. We end by the definition of finitely graded Kac-Moody algebras.

**Generalized Cartan matrices**

Let $I$ be a finite index set. A matrix $A = (a_{i,j})_{i,j \in I}$ is called a generalized Cartan matrix if it satisfies:

1. $a_{i,i} = 2$ ($i \in I$)
2. $a_{i,j} \in \mathbb{Z}^-$ ($i \neq j$)
3. $a_{i,j} = 0$ implies $a_{j,i} = 0$.

The matrix $A$ is called decomposable if for a suitable permutation of $I$ it takes the form $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where $B$ and $C$ are square matrices. If $A$ is not decomposable, it is called indecomposable.

The matrix $A$ is called symmetrizable if there exists an invertible diagonal matrix $D = \text{diag}(d_i, i \in I)$ such that $DA$ is symmetric. The entries $d_i, i \in I$, can be chosen to be positive rationals and if moreover the matrix $A$ is indecomposable, then these entries are unique up to a constant factor.

Any indecomposable generalized Cartan matrix is of one of three mutually exclusive types: finite, affine and indefinite ([12, Chap. 4]). A generalized Cartan matrix is said of finite type if each of its indecomposable factors is of finite type. An indecomposable and symmetrizable generalized Cartan matrix $A$ is called Lorentzian if it is non-singular and the corresponding symmetric matrix has signature $(++\ldots-)$; it is then of indefinite type.
An indecomposable generalized Cartan matrix $A$ is called strictly hyperbolic (resp. hyperbolic) if the deletion of any one vertex, and the edges connected to it, of the corresponding Dynkin diagram yields a disjoint union of Dynkin diagrams of finite (resp. finite or affine) type.

Note that a symmetrizable hyperbolic generalized Cartan matrix is non-singular and Lorentzian (cf. [14]).

**Kac-Moody algebras and groups (See [12] and [18]).**

Let $A = (a_{i,j})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix. Let $(\mathfrak{h}_\mathbb{R}, \Pi = \{\alpha_i, i \in I\}, \Pi' = \{\alpha'_i, i \in I\})$ be a realization of $A$ over the real field $\mathbb{R}$: thus $\mathfrak{h}_\mathbb{R}$ is a real vector space such that $\dim(\mathfrak{h}_\mathbb{R}) = |I| + \text{corank}(A)$, $\Pi$ and $\Pi'$ are linearly independent in $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}$ respectively such that $\langle \alpha_j, \alpha_i \rangle = a_{i,j}$. Let $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes \mathbb{C}$, then $(\mathfrak{h}, \Pi, \Pi')$ is a realization of $A$ over the complex field $\mathbb{C}$. It follows that, if $A$ is non-singular, then $\Pi'$ (resp. $\Pi$) is a basis of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$); moreover $\mathfrak{h}_\mathbb{R} = \{h \in \mathfrak{h} | \alpha_i(h) \in \mathbb{R}, \forall i \in I\}$ is well defined by the realization $(\mathfrak{h}, \Pi, \Pi')$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the complex Kac-Moody Lie algebra associated to $A$ : it is generated by $\{\mathfrak{h}, e_i, f_i, i \in I\}$ with the following relations

$$[\mathfrak{h}, \mathfrak{h}] = 0, \quad [e_i, f_j] = \delta_{i,j} \alpha_i^\vee \quad (i, j \in I);$$

$$[h, e_i] = (\alpha_i, h)e_i, \quad [h, f_i] = -\langle \alpha_i, h \rangle f_i \quad (h \in \mathfrak{h});$$

$$\text{(ad}e_i)^{1-a_{i,j}}(e_j) = 0, \quad \text{(ad}f_i)^{1-a_{i,j}}(f_j) = 0 \quad (i \neq j). \quad (1.1)$$

The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ decomposes as a direct sum of factors $\mathfrak{g}(A_i)$, where $A_1, \ldots, A_r$ are the indecomposable factors of $A$. It is said indecomposable if the corresponding generalized Cartan matrix $A$ is indecomposable and of finite, affine or indefinite type if $A$ is.

The derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$ is generated by the Chevalley generators $e_i, f_i$, $i \in I$, and the center $\mathfrak{c}$ of $\mathfrak{g}$ lies in $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i \in I} \mathbb{C} \alpha_i$. If the generalized Cartan matrix $A$ is indecomposable and non-singular, then $\mathfrak{g} = \mathfrak{g}'$ is a (finite or infinite)-dimensional simple Lie algebra, and the center $\mathfrak{c}$ is trivial.

The subalgebra $\mathfrak{h}$ is a maximal $\text{ad}(\mathfrak{g})$—diagonalizable subalgebra of $\mathfrak{g}$, it is called the standard Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system; then $\Pi$ is a root basis of $\Delta$ and $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^\pm = \Delta \cap \mathbb{Z}^\pm \Pi$ is the set of positive (or negative) roots relative to the basis $\Pi$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ corresponding to the root $\alpha$; then $\mathfrak{g} = \mathfrak{h} \oplus (\alpha \in \Delta \mathfrak{g}_\alpha)$.

The Weyl group $W$ of $(\mathfrak{g}, \mathfrak{h})$ is generated by the fundamental reflections $r_i$ ($i \in I$) such that $r_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee$ for $h \in \mathfrak{h}$; it is a Coxeter group on $\{r_i, i \in I\}$ with length function $w \mapsto l(w)$, $w \in W$. The Weyl group $W$ acts on $\mathfrak{h}'$ and $\Delta$, we set $\Delta^re = W(\Pi)$ (the real roots) and $\Delta^{im} = \Delta \setminus \Delta^re$ (the imaginary roots). If the generalized Cartan matrix $A$ is indecomposable, then any root basis of $\Delta$ is $W$—conjugate to $\Pi$ or $-\Pi$.

A Borel subalgebra of $\mathfrak{g}$ is a maximal completely solvable subalgebra. A parabolic subalgebra of $\mathfrak{g}$ is a (proper) subalgebra containing a Borel subalgebra. The standard positive (or negative) Borel subalgebra is $\mathfrak{b}^\pm := \mathfrak{h} \oplus (\alpha \in \Delta^\pm \mathfrak{g}_\alpha)$. A parabolic subalgebra $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) containing $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is called positive (resp. negative) standard parabolic subalgebra of $\mathfrak{g}$; then there exists a subset $J$
of $I$ (called the type of $p^\pm$) such that $p^\pm = p^\pm(J) := (\bigoplus_{\alpha \in \Delta_J} g_\alpha) + b^\pm$, where $\Delta_J = \Delta \cap (\oplus_{j \in J} \mathbb{Z}\alpha_j)$ (cf. [13]).

In [18], D.H. Peterson and V.G. Kac construct a group $G$, which is the connected and simply connected complex algebraic group associated to $g$ when $g$ is of finite type, depending only on the derived Lie algebra $g'$ and acting on $g$ via the adjoint representation $\text{Ad} : G \to \text{Aut}(g)$. It is generated by the one-parameter subgroups $U_\alpha = \exp(g_\alpha)$, $\alpha \in \Delta^+, \text{ and } \text{Ad}(U_\alpha) = \exp(\text{ad}(g_\alpha))$. In the definitions of J. Tits [22] $G$ is the group of complex points of $G_D$ where $D$ is the datum associated to $A$ and the $\mathbb{Z}$--dual $\Lambda$ of $\bigoplus_{i \in I} \mathbb{Z}\alpha_i$.

The Cartan subalgebras of $g$ are $G$--conjugate. If $g$ is indecomposable and not of finite type, there are exactly two conjugate classes (under the adjoint action of $G$) of Borel subalgebras : $G.b^+$ and $G.b^-$. A Borel subalgebra $b$ of $g$ which is $G$--conjugate to $b^+$ (resp. $b^-$) is called positive (resp. negative). It follows that any parabolic subalgebra $p$ of $g$ is $G$--conjugate to a standard positive (or negative) parabolic subalgebra, in which case, we say that $p$ is positive (or negative).

**Standard Kac-Moody subalgebras and subgroups**

Let $J$ be a non-empty subset of $I$. Consider the generalized Cartan matrix $A_J = (a_{ij})_{i,j \in J}$.

**Definition 1.1.** The subset $J$ is called of finite type if the corresponding generalized Cartan matrix $A_J$ is. We say also that $J$ is connected, if the Dynkin subdiagram, with vertices indexed by $J$, is connected or, equivalently, the corresponding generalized Cartan submatrix $A_J$ is indecomposable.

**Proposition 1.2.** Let $\Pi_J = \{\alpha_j, j \in J\}$ and $\Pi_J^\perp = \{\alpha_j^\perp, j \in J\}$. Let $h'_J$ be the subspace of $h$ generated by $\Pi_J^\perp$, and $h^J = \Pi_J^\perp = \{h \in h, \langle \alpha_j, h \rangle = 0, \forall j \in J\}$. Let $h''_J$ be a supplementary subspace of $h'_J + h^J$ in $h$ and let $h_J = h'_J \oplus h''_J$.

then, we have :
1) $(h_J, \Pi_J, \Pi_J^\perp)$ is a realization of the generalized Cartan matrix $A_J$. Hence $h''_J = \{0\}$, $h_J = h'_J$ when $A_J$ is regular (e.g. when $J$ is of finite type).
2) The subalgebra $g(J)$ of $g$, generated by $h_J$ and the $e_j, f_j, j \in J$, is the Kac-Moody Lie algebra associated to the realization $(h_J, \Pi_J, \Pi_J^\perp)$ of $A_J$.
3) The corresponding root system $\Delta(J) = \Delta(g(J), h_J)$ can be identified with $\Delta_J := \Delta \cap (\oplus_{j \in J} \mathbb{Z}\alpha_j)$.

**N.B.** The derived algebra $g'(J)$ of $g(J)$ is generated by the $e_j, f_j$ for $j \in J$; it does not depend of the choice of $h''_J$.

**Proof.** We may assume $g$ indecomposable.

1) Note that $\dim(h''_J) = \dim(h'_J \cap h^J) = \text{corank}(A_J)$. In particular, $\dim(h_J) - |J| = \text{corank}(A_J)$. If $\alpha \in \text{Vect}(\alpha_j, j \in J)$, then $\alpha$ is entirely determined by its restriction to $h_J$ and hence $\Pi_J$ defines, by restriction, a linearly independent
set in \( h_i^* \). As \( \Pi_J \) is linearly independent, assertion 1) holds. Assertions 2) and 3) are straightforward.

In the same way, the subgroup \( G_J \) of \( G \) generated by \( U_{\pm \alpha_j}, j \in J \), is equal to the Kac-Moody group associated to the generalized Cartan matrix \( A_J \); it is clearly a quotient; the well known equality is proven explicitly in [20, 5.15.2], it may be deduced from [22, th. 1], see also [19, 8.4.2].

The invariant bilinear form (See [12]).
We recall that the generalized Cartan matrix \( A \) is supposed symmetrizable. There exists a non-degenerate \( \text{ad}(g) \) invariant symmetric \( \mathbb{C} \)-bilinear form \((.,.)\) on \( g \), which is entirely determined by its restriction to \( h \), such that

\[
(\alpha_i, h) = \frac{(\alpha_i, \alpha_i)}{2} (\alpha_i, h), \quad i \in I, \ h \in h,
\]

and we may thus assume that

\[
(\alpha_i, \alpha_i) \text{ is a positive rational for all } i. \tag{1.2}
\]

The non-degenerate invariant bilinear form \((.,.)\) induces an isomorphism \( \nu : h \to h^* \) such that \( \alpha_i = \frac{2\nu(\alpha_i)}{(\alpha_i, \alpha_i)} \) and \( \alpha_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)} \) for all \( i \).

There exists a totally isotropic subspace \( h'' \) of \( h \) (relative to the invariant bilinear form \((.,.)\)) which is in duality with the center \( c \) of \( g \). In particular, \( h'' \) defines a supplementary subspace of \( h' \) in \( h \).

Note that any invariant symmetric bilinear form \( b \) on \( g \) satisfying \( b(\alpha_i, \alpha_i) > 0, \) \( \forall i \in I \), is non-degenerate and \( b(\alpha_i, h) = b(\alpha_i, \alpha_i) (\alpha_i, h), \forall i \in I, \forall h \in h \). It follows that, if \( g \) is indecomposable, the restriction of \( b \) to \( g' \) is proportional to that of \((.,.)\). In particular, if moreover \( A \) is non-singular, then the invariant bilinear form \((.,.)\) satisfying the condition 1.2 is unique up to a positive rational factor.

The Tits cone (See [12, Chap. 3 and 5]).
Let \( C := \{ h \in h_{\mathbb{R}}; \langle \alpha, h \rangle \geq 0, \forall i \in I \} \) be the fundamental chamber (relative to the root basis \( \Pi \)) and let \( X := \bigcup_{w \in W} w(C) \) be the Tits cone. We have the following description of the Tits cone:

1. \( X = \{ h \in h_{\mathbb{R}}; \langle \alpha, h \rangle < 0 \text{ only for a finite number of } \alpha \in \Delta^+ \} \).
2. \( X = h_{\mathbb{R}} \) if and only if the generalized Cartan matrix \( A \) is of finite type.
3. If \( A \) is indecomposable of affine type, then \( X = \{ h \in h_{\mathbb{R}}; \langle \delta, h \rangle > 0 \} \cup \mathbb{R} \nu^{-1}(\delta) \), where \( \delta \) is the lowest imaginary positive root of \( \Delta^+ \).
4. If \( A \) is indecomposable of indefinite type, then the closure of the Tits cone, for the metric topology on \( h_{\mathbb{R}} \), is \( \bar{X} = \{ h \in h_{\mathbb{R}}; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta^+_m \} \).
5. If \( h \in X \), then \( h \) lies in the interior \( \overset{o}{X} \) of \( X \) if and only if the fixer \( W_h \) of \( h \), in the Weyl group \( W \), is finite. Thus \( \overset{o}{X} \) is the union of finite type facets of \( X \).
6. If \( A \) is hyperbolic, then \( \overset{o}{X} \cup (-\overset{o}{X}) = \{ h \in h_{\mathbb{R}}; \langle h, h \rangle \leq 0 \} \) and the set of imaginary roots is \( \Delta^i_m = \{ \alpha \in Q \setminus \{0\}; (\alpha, \alpha) \leq 0 \} \), where \( Q = \mathbb{Z} \Pi \) is the root lattice.
Remark 1.3. Combining (3) and (4) one obtains that if $A$ is not of finite type then $\bar{X} = \{ h \in \mathfrak{h}_\mathbb{R} ; \langle \alpha, h \rangle \geq 0, \ \forall \alpha \in \Delta^+_m \}.$

Graded Kac-Moody Lie algebras

Definition 1.4. Let $\Sigma$ be a root system of Kac-Moody type. The Kac-Moody Lie algebra $\mathfrak{g}$ is said to be finitely $\Sigma-$graded if:

(i) $\mathfrak{g}$ contains, as a subalgebra, a Kac-Moody algebra $\mathfrak{m}$ whose root system relative to a Cartan subalgebra $\mathfrak{a}$ is equal to $\Sigma$.

(ii) $\mathfrak{g} = \sum_{\alpha \in \Sigma \cup \{0\}} V_\alpha$, with $V_\alpha = \{ x \in \mathfrak{g} ; [a, x] = \langle \alpha, a \rangle x, \ \forall a \in \mathfrak{a} \}$.

(iii) $V_\alpha$ is finite dimensional for all $\alpha \in \Sigma \cup \{0\}$.

We say that $\mathfrak{m}$ (as in (i) above) is a grading subalgebra, and $(\mathfrak{g}, \Sigma, \mathfrak{m})$ a gradation with finite multiplicities (or, to be short, a finite gradation).

Note that from (ii) the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is ad$(\mathfrak{g})-$diagonalizable, and we may assume that $\mathfrak{a}$ is contained in the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Lemma 1.5. Let $\mathfrak{g}$ be a Kac-Moody algebra finitely $\Sigma-$graded, with grading subalgebra $\mathfrak{m}$. If $\mathfrak{m}$ itself is finitely $\Sigma'-$graded (for some root system $\Sigma'$ of Kac-Moody type), then $\mathfrak{g}$ is finitely $\Sigma'-$graded.

Proof. If $\mathfrak{m}'$ is the grading subalgebra of $\mathfrak{m}$, we may suppose the Cartan subalgebras such that $\mathfrak{a}' \subset \mathfrak{a} \subset \mathfrak{h}$, with obvious notations. Conditions (i) and (ii) are clearly satisfied for $\mathfrak{g}$, $\mathfrak{m}'$ and $\mathfrak{a}'$. Condition (iii) for $\mathfrak{m}$ and $\Sigma'$ tells that, for all $\alpha' \in \Sigma'$, the set $\{ \alpha \in \Sigma \mid \alpha|_{\mathfrak{a}'} = \alpha' \}$ is finite. But $V_{\alpha'} = \oplus_{\alpha|_{\mathfrak{a}'} = \alpha'} V_\alpha$, so each $V_{\alpha'}$ is finite dimensional if this is true for each $V_\alpha$.

Examples 1.6.

1) Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, then $\mathfrak{g}$ is finitely $\Delta-$graded: this is the trivial gradation of $\mathfrak{g}$ by its own root system.

2) Let $\mathfrak{g}_\mathbb{R}$ be an almost split real form of $\mathfrak{g}$ (see [2]) and let $\mathfrak{t}_\mathbb{R}$ be a maximal split toral subalgebra of $\mathfrak{g}_\mathbb{R}$. Suppose that the restricted root system $\Delta' = \Delta(\mathfrak{g}_\mathbb{R}, \mathfrak{t}_\mathbb{R})$ is reduced of Kac-Moody type. In [4, §9], N. Bardy constructed a split real Kac-Moody subalgebra $\mathfrak{l}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$ such that $\Delta' = \Delta(\mathfrak{l}_\mathbb{R}, \mathfrak{t}_\mathbb{R})$, then $\mathfrak{g}$ is obviously finitely $\Delta'-$graded.

We get thus many examples coming from known tables for almost split real forms: see [2] in the affine case and [6] in the hyperbolic case.

3) When $\mathfrak{g}_\mathbb{R}$ is an almost compact real form of $\mathfrak{g}$, the same constructions should lead to gradations by finite root systems, as in [5] e.g.

2. Gradations associated to $C-$admissible pairs

In this section, we suppose the Kac-Moody Lie algebra $\mathfrak{g}$ indecomposable and symmetrizable, see however Remark 2.15. We shall build a finite gradation of $\mathfrak{g}$ associated to some good subset of $I$. 

We recall some definitions introduced by H. Rubenthaler ([21]) and J. Nervi ([16], [17]). Let $J$ be a subset of $I$ of finite type. For $k \in I \setminus J$, we denote by $I_k$ the connected component, containing $k$, of the Dynkin subdiagram corresponding to $J \cup \{k\}$, and let $J_k := I_k \setminus \{k\}$.

We are interested in the case where $I_k$ is of finite type for all $k \in I \setminus J$ : that is always true if $\mathfrak{g}$ is of affine type and $|I \setminus J| \geq 2$ or if $\mathfrak{g}$ is of hyperbolic type and $|I \setminus J| \geq 3$.

For $k \in I \setminus J$, let $\mathfrak{g}(I_k)$ be the simple subalgebra generated by $\mathfrak{g}_{\pm\alpha_i}$, $i \in I_k$, then $\mathfrak{h}_{I_k} = \mathfrak{h} \cap \mathfrak{g}(I_k) = \sum_{i \in I_k} \mathbb{C}\alpha_i$ is a Cartan subalgebra of $\mathfrak{g}(I_k)$. Let $H_k$ be the unique element of $\mathfrak{h}_{I_k}$ such that $\langle \alpha_i, H_k \rangle = 2\delta_{i,k}$, $\forall i \in I_k$.

**Definition 2.1.** We suppose the Dynkin diagram indexed by $I$ connected and consider a subset $J$ of finite type. We preserve the notations introduced above.

1) Let $k \in I \setminus J$.

(i) The pair $(I_k, J_k)$ is called admissible if $I_k$ is of finite type and there exist $E_k, F_k \in \mathfrak{g}(I_k)$ such that $(E_k, H_k, F_k)$ is an $\mathfrak{sl}_2$-triple.

(ii) The pair $(I_k, J_k)$ is called $C-$admissible if it is admissible and the simple Lie algebra $\mathfrak{g}(I_k)$ is $A_1-$graded by the root system, of type $A_1$, associated to the $\mathfrak{sl}_2$-triple $(E_k, H_k, F_k)$.

2) The pair $(I, J)$ is called $C-$admissible if the pairs $(I_k, J_k)$ are $C-$admissible for all $k \in I \setminus J$. It is said irreducible if, moreover, $|I \setminus J| = 1$.

Schematically, any $C-$admissible pair $(I, J)$ is represented by the Dynkin diagram, corresponding to $A$, on which the vertices indexed by $J$ are denoted by white circles $\circ$ and those of $I \setminus J$ are denoted by black circles $\bullet$.

**Remark 2.2.**

1) The admissibility of each $(I_k, J_k)$ is essential to build (in 2.6, 2.11) the grading subalgebra $\mathfrak{g}^J$ and its grading root system $\Delta^J$.

2) As $\mathfrak{g}(J)$ will be in the eigenspace $V_0$ of weight 0 for the grading by $\Delta^J$, it is necessary to assume $J$ of finite type to get a finite gradation.

3) $I_k$ is of finite type if, and only if, $\mathfrak{g}(I_k)$ is finite dimensional, and this is equivalent to the alternative assumption in (ii) that the $A_1-$gradation has finite multiplicities. It is clearly necessary to get, in Theorem 2.14, a finite gradation of $\mathfrak{g}$ by the root system $\Delta^J$. Moreover, even in a more general situation, the condition $I_k$ of finite type will naturally appear (3.14).

4) Note that the definition presented here, for $C-$admissible pairs, is equivalent to that introduced by Rubenthaler and Nervi (see [21], [16]) in terms of prehomogeneous spaces of parabolic type : if $(I_k, J_k)$ is $C-$admissible, define for $p \in \mathbb{Z}$, the subspace $d_{k,p} := \{X \in \mathfrak{g}(I_k) ; [H_k, X] = 2pX\}$; then $(d_{k,0}, d_{k,1})$ is an irreducible regular and commutative prehomogeneous space of parabolic type, and $d_{k,p} = \{0\}$ for $|p| \geq 2$. Then $(I_k, J_k)$ is an irreducible $C-$admissible pair. According to Rubenthaler and Nervi ([21, Table 1] or [16, Table 2]) the irreducible $C-$admissible pair $(I_k, J_k)$ should be among the list in Table 1 below.

5) Along our study of general finite gradations in section 3, we shall meet a situation of "generalized $C-$admissible pair" $(I, J)$ (3.16) where $J \subset I$ is of finite type and $I_k$ (for $k \in I' = I \setminus J$) is defined as above but perhaps not of
finite type. When \( k \) is in some subset \( I'_{\text{re}} \) of \( I' \), \((I_k, J_k)\) is \( C - \text{admissible} \) and the \( k \in I'_{\text{im}} = I' \setminus I'_{\text{re}} \) do not contribute to the root system \( \Sigma \) grading \( \mathfrak{g} \). But we do not know the good assumptions on these \((I_k, J_k)\) for \( k \in I'_{\text{im}} \) to get, conversely, a finite gradation of \( \mathfrak{g} \) by some root system. So we give no precise definition; it is expected in the work in preparation [7].

Table 1

List of irreducible \( C - \text{admissible pairs} \)

| \( A_{2n-1} \), \( n \geq 1 \) | \( 1 \quad 2 \ldots \quad n \quad \ldots \quad 2n-1 \) |
| \( B_n \), \( n \geq 3 \) | \( 1 \quad 2 \quad 3 \ldots \quad n \) |
| \( C_n \), \( n \geq 2 \) | \( 1 \quad 2 \quad 3 \ldots \quad n \) |
| \( D_{n,1} \), \( n \geq 4 \) | \( 1 \quad 2 \ldots \quad 2n-1 \) |
| \( D_{2n,2} \), \( n \geq 2 \) | \( 1 \quad 2 \ldots \quad 2n-1 \) |
| \( E_7 \) | \( 1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \) |

**Definition 2.3.** Let \( J \) be a subset of \( I \) and let \( i, k \in I \setminus J \). We say that \( i \) and \( k \) are \( J - \text{connected} \) relative to \( A \) if there exist \( j_0, j_1, \ldots, j_{p+1} \in I \) such that \( j_0 = i, j_{p+1} = k, j_s \in J, \forall s = 1, 2, \ldots, p \), and \( a_{j_s, j_{s+1}} \neq 0, \forall s = 0, 1, \ldots, p \).

**Remark 2.4.** Note that the relation “ to be \( J - \text{connected} \)” is symmetric on \( i \) and \( k \). As the generalized Cartan matrix \( A \) is assumed to be indecomposable, for any vertices \( i, k \in I \setminus J \) there exist \( i_0, i_1, \ldots, i_{p+1} \in I \setminus J \) such that \( i_0 = i, i_{p+1} = k \) and \( i_s \) and \( i_{s+1} \) are \( J - \text{connected} \) for all \( s = 0, 1, \ldots, p \).

Let us assume from now on that \((I, J)\) is a \( C - \text{admissible pair} \) and let \( I' := I \setminus J \). For \( k \in I' \), let \((E_k, H_k, F_k)\) be an \( \mathfrak{sl}_2 - \text{triple} \) associated to the irreducible \( C - \text{admissible pair} \) \((I_k, J_k)\).
Lemma 2.5. Let $k \neq l \in I'$, then:
1) $\langle \alpha_i, H_k \rangle \in \mathbb{Z}^-$.
2) the following assertions are equivalent:
   i) $k,l$ are $J-$connected
   ii) $\langle \alpha_i, H_k \rangle$ is a negative integer
   iii) $\langle \alpha_k, H_l \rangle$ is a negative integer

Proof. 1) One can write $H_k = \sum_{i \in I_k} n_i k_i \alpha_i$, where $n_i k_i$ are positive integers (see [21] or [17, 1.4.1.2]). As $l \notin I_k$, we have that $\langle \alpha_i, H_k \rangle = \sum_{i \in I_k} n_i k_i \langle \alpha_i, \alpha_i \rangle \in \mathbb{Z}^-$. 2) In view of Remark 2.4, it suffices to prove the equivalence between i) and ii). Since $I_k$ is the connected component of $J \cup \{k\}$ containing $k$, the assertion i) is equivalent to say that the vertex $l$ is connected to $I_k$, so there exists $i_k \in I_k$ such that $\langle \alpha_i, \alpha_i \rangle < 0$ and hence $\langle \alpha_i, H_k \rangle < 0$. ■

Proposition 2.6. Let $\mathfrak{h}' = \Pi^J = \{h \in \mathfrak{h}, \langle \alpha_j, h \rangle = 0, \forall j \in J\}$. For $k \in I'$, denote by $\alpha'_k = \alpha_k/\mathfrak{h}'$ the restriction of $\alpha_k$ to the subspace $\mathfrak{h}'$ of $\mathfrak{h}$, and $\Pi^J = \{\alpha'_k; k \in I'\}, \Pi^{J'} = \{H_k; k \in I'\}$. For $k,l \in I'$, put $a'_{k,l} = \langle \alpha_i, H_k \rangle$ and $A^J = (a'_{k,l})_{k,l \in I'}$. Then $A^J$ is an indecomposable and symmetrizable generalized Cartan matrix, $(\mathfrak{h}', \Pi^J, \Pi^{J'})$ is a realization of $A^J$ and $\text{corank}(A^J) = \text{corank}(A)$. 

Proof. The fact that $a'_{k,l} = 2$ follows from the definition of $H_k$ for $k \in I'$. If $k \neq l \in I'$, then by lemma 2.5, $a'_{k,l} \in \mathbb{Z}^-$ and $a'_{k,l} \neq 0$ if and only if $a'_{k,l} \neq 0$. Hence $A^J$ is a generalized Cartan matrix. As the matrix $A$ is indecomposable, $A_J$ is also indecomposable (see Remark 2.4). Clearly $\Pi^J = \{\alpha'_k; k \in I'\}$ is a linearly independent subset of the dual space $\mathfrak{h}'^*$ of $\mathfrak{h}'$, $\Pi^{J'} = \{H_k; k \in I'\}$ is a linearly independent subset of $\mathfrak{h}'$ and by construction $\langle \alpha_i, H_k \rangle = a'_{k,l}, \forall k,l \in I'$. We have to prove that $\dim(\mathfrak{h}') - |I'| = \text{corank}(A^J)$. As $J$ is of finite type, the restriction of the invariant bilinear form $(.,.)$ to $\mathfrak{h}_J$ is non-degenerate and $\mathfrak{h}_J$ is contained in $\mathfrak{h}' = \bigoplus_{i \in I} \mathfrak{g}_\alpha$. Therefore

$$\mathfrak{h}_J = \mathfrak{h}' \oplus \mathfrak{h}_J$$

and

$$\mathfrak{h}' = (\mathfrak{h}' \cap \mathfrak{h}_J) \oplus \mathfrak{h}_J.$$

It follows that $\dim(\mathfrak{h}' \cap \mathfrak{h}_J) = |I'| = \dim(\bigoplus_{k \in I'} \mathfrak{g}_H_k)$. As the subspace $\bigoplus_{k \in I'} \mathfrak{g}_H_k$ is contained in $\mathfrak{h}' \cap \mathfrak{h}_J$, we deduce that $\mathfrak{h}' \cap \mathfrak{h}_J = \bigoplus_{k \in I'} \mathfrak{g}_H_k$. Note that any supplementary subspace $\mathfrak{h}'^{J''}$ of $\mathfrak{h}' \cap \mathfrak{h}_J$ in $\mathfrak{h}_J$ is also a supplementary of $\mathfrak{h}'$ in $\mathfrak{h}$; hence, we have that $\text{corank}(A) = \dim(\mathfrak{h}'^{J''}) = \dim(\mathfrak{h}') - |I'|$. Let $c := \bigcap_{i \in I} \ker(\alpha_i)$ be the center of $\mathfrak{g}$ and let $c' = \bigcap_{k \in I'} \ker(\alpha'_k)$. Recall that $\text{corank}(A) = \dim(c)$ and $\text{corank}(A^J) = \dim(c'')$. It’s clear that $c'' = c$; hence $\text{corank}(A^J) = \dim(c'') = \text{corank}(A) = \dim(\mathfrak{h}'') - |I'|$.

It remains to prove that $A^J$ is symmetrizable. For $k \in I'$, let $R^J_k$ be the fundamental reflection of $\mathfrak{h}'$ such that $R^J_k(h) = h - \langle \alpha'_k, h \rangle H_k, \forall h \in \mathfrak{h}'$. Let $W^J$ be the Weyl group of $A^J$ generated by $R^J_k, k \in I'$. Let $(.,.)^J$ be the restriction
to \( \mathfrak{h}^J \) of the invariant bilinear form \( (.,.) \) on \( \mathfrak{h} \). Then \( (.,.)^J \) is a non-degenerate symmetric bilinear form on \( \mathfrak{h}^J \) which is \( W^J \)-invariant (see the lemma hereafter).

From the relation \( (R^J_k(H_k), R^J_l(H_l)) = (H_k, H_l)^J \) one can deduce that

\[
(H_k, H_l)^J = \frac{(H_k, H_l)}{2} a^J_{i,k}, \quad \forall k, l \in I'.
\]

Since \( (H_k, H_k)^J > 0, \forall k \in I' \), the generalized Cartan matrix \( ^tA^J \) (and so \( A^J \)) is symmetrizable.

**Lemma 2.7.** For \( k \in I' := I \setminus J \), let \( w^J_k \) be the longest element of the Weyl group \( W(I_k) \) generated by the fundamental reflections \( r_i, i \in I_k \). Then \( w^J_k \) stabilizes \( \mathfrak{h}^J \) and induces the fundamental reflection \( R^J_k \) of \( \mathfrak{h}^J \) associated to \( H_k \).

**Proof.** If one looks at the list above of the irreducible \( C \)-admissible pairs, one can see that \( w^J_k(\alpha_j) = -\alpha_j \) and that \( -w^J_k \) permutes the \( \alpha_j, j \in J_k \). Clearly \( w^J_k(\alpha_j) = \alpha_j, \forall j \in J \setminus J_k \). Hence \( w^J_k \) stabilizes \( \mathfrak{h}^J \) and its orthogonal subspace \( \mathfrak{h}^J = \mathfrak{h}^J \). Note that \( -w^J_k(H_k) \in \mathfrak{h}^J_k \) and it satisfies the same equations defining \( H_k \). Hence \( -w^J_k(H_k) = H_k = -R^J_k(H_k) \). Recall that \( \ker(\alpha^J_k) = \ker(\alpha_k) \cap (\bigcap_{j \in J} \ker(\alpha_j)) \); thus it is fixed by \( R^J_k \) and \( W^J_k \). Since \( \mathfrak{h}^J = \ker(\alpha^J_k) \oplus \mathbb{C}H_k \), the reflection \( R^J_k \) coincides with \( W^J_k \) on \( \mathfrak{h}^J \).

**Remark 2.8.** Actually we can now rediscover the list of irreducible \( C \)-admissible pairs given in Table 1. The black vertex \( k \) should be invariant under \( -w^J_k \) and the corresponding coefficient of the highest root of \( I_k \) should be 1 (an easy consequence of the definition 2.1 1) (ii) ).

**Example 2.9.** Consider the hyperbolic generalized Cartan matrix \( A \) of type \( HE_8^{(1)} = E_{10} \) indexed by \( I = \{-1, 0, 1, \ldots, 8\} \).

The following two choices for \( J \) define \( C \)-admissible pairs :

1) \( J = \{2,3,4,5\} \).

\[
\begin{array}{cccccccccc}
\bullet & 3 & \bullet & 4 & \bullet & 5 & \bullet & 6 & \bullet & 0 & \bullet
\end{array}
\]

The corresponding generalized Cartan matrix \( A^J \) is hyperbolic of type \( HF_4^{(1)} \):

\[
\begin{array}{cccccccc}
\bullet & 1 & \bullet & 2 & \bullet & 3 & \bullet & 4 & \bullet & 0 & \bullet & \bullet
\end{array}
\]

2) \( J = \{1,2,3,4,5,6\} \).

\[
\begin{array}{cccccccccc}
\bullet & 3 & \bullet & 4 & \bullet & 5 & \bullet & 6 & \bullet & 0 & \bullet & \bullet
\end{array}
\]

The corresponding generalized Cartan matrix \( A^J \) is hyperbolic of type \( HG_2^{(1)} \):

Note that the first example corresponds to an almost split real form of the Kac-Moody Lie algebra \( \mathfrak{g}(A) \) and \( A^J \) is the generalized Cartan matrix associated to the corresponding (reduced) restricted root system (see [6]) whereas the second example does not correspond to an almost split real form of \( \mathfrak{g}(A) \).

**Lemma 2.10.** For \( k \in I' \), set \( \mathfrak{s}(k) = \mathbb{C}E_{k} \oplus \mathbb{C}H_{k} \oplus \mathbb{C}F_{k} \). Then, the Lie algebra \( \mathfrak{g} \) is an integrable \( \mathfrak{s}(k) \)-module via the adjoint representation of \( \mathfrak{s}(k) \) on \( \mathfrak{g} \).
Note that $s(k)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ with standard basis $(E_k, H_k, F_k)$. It is clear that $\text{ad}(H_k)$ is diagonalizable on $\mathfrak{g}$ and $E_k = \sum_\alpha e_\alpha \in d_{k,1}$, where $\alpha$ runs over the set $\Delta_{k,1} = \{ \alpha \in \Delta(I_k); \langle \alpha, H_k \rangle = 2 \}$, $e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta(I_k)$, and $d_{k,1} := \{ X \in \mathfrak{g}(I_k); [H_k, X] = 2X \}$. Since $\Delta_{k,1} \subset \Delta^{re}$, $\text{ad}(e_\alpha)$ is locally nilpotent for $\alpha \in \Delta_{k,1}$. As $d_{k,1}$ is commutative (see Remark 2.2) we deduce that $\text{ad}(E_k)$ is locally nilpotent on $\mathfrak{g}$. The same argument shows that $\text{ad}(F_k)$ is also locally nilpotent. Hence, the Kac-Moody algebra $\mathfrak{g}$ is an integrable $s(k)$–module.

**Proposition 2.11.** Let $\mathfrak{g}^J$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}^J$ and $E_k, F_k$, $k \in I'$. Then $\mathfrak{g}^J$ is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}^J, \Pi^J, \Pi^{J(\vee)})$ of the generalized Cartan matrix $A^J$.

**Proof.** It is not difficult to check that the following relations hold in the Lie subalgebra $\mathfrak{g}^J$:

\[
\begin{align*}
[\mathfrak{h}^J, \mathfrak{h}^J] &= 0, \\
[E_k, F_l] &= \delta_{k,l}H_k \quad (k, l \in I'); \\
[h, E_k] &= \langle \alpha_k', h \rangle E_k, \\
[h, F_k] &= -\langle \alpha_k', h \rangle F_k \quad (h \in \mathfrak{h}^J, k \in I').
\end{align*}
\]

We have to prove the Serre’s relations:

\[
(\text{ad}E_k)^{1-s_{k,l}(E_l)}(E_l) = 0, \quad (\text{ad}F_k)^{1-s'_{k,l}(F_l)}(F_l) = 0 \quad (k \neq l \in I').
\]

For $k \in I'$, let $s(k) = C F_k \oplus CH_k \oplus CE_k$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let $k \neq l \in I'$; note that $[H_k, F_l] = -a_{k,l} F_l$ and $[E_k, F_l] = 0$, which means that $F_l$ is a primitive weight vector for $s(k)$. As $\mathfrak{g}$ is an integrable $s(k)$–module (see Lemma 2.10) the primitive weight vector $F_l$ is contained in a finite dimensional $s(k)$–submodule (see [12, 3.6]). The relation $(\text{ad}F_k)^{1-a_{k,l}(F_l)}(F_l) = 0$ follows from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (see[12, 3.2]). By similar arguments we prove that $(\text{ad}E_k)^{1-a'_{k,l}(E_l)}(E_l) = 0$.

Now $\mathfrak{g}^J$ is a quotient of the Kac-Moody algebra associated to $A^J$ and $(\mathfrak{h}^J, \Pi^J, \Pi^{J(\vee)})$. By [12, 1.7] it is equal to it.

**Definition 2.12.** The Kac-Moody Lie algebra $\mathfrak{g}^J$ is called the $C$–admissible algebra associated to the $C$–admissible pair $(I, J)$.

**Proposition 2.13.** The Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{g}^J$–module with finite multiplicities.

**Proof.** The $\mathfrak{g}^J$–module $\mathfrak{g}$ is clearly $\text{ad}(\mathfrak{h}^J)$–diagonalizable and $\text{ad}(E_k), \text{ad}(F_k)$ are locally nilpotent on $\mathfrak{g}$ for $k \in I'$ (see Lemma 2.10). Hence, $\mathfrak{g}$ is an integrable $\mathfrak{g}^J$–module. For $\alpha \in \Delta$, let $\alpha' = \alpha|_{\mathfrak{h}^J}$ be the restriction of $\alpha$ to $\mathfrak{h}^J$. Set $\Delta' = \{ \alpha' \in \Delta \setminus \{0\} \}$. Then the set of weights, for the $\mathfrak{g}^J$–module $\mathfrak{g}$, is exactly $\Delta' \cup \{0\}$. Note that for $\alpha \in \Delta$, $\alpha' = 0$ if and only if $\alpha \in \Delta(J)$. In particular, the weight space $V_0 = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta(J)} \mathfrak{g}_\alpha)$ corresponding to the null weight is finite dimensional. Let $\alpha = \sum_{i \in J} n_i \alpha_i \in \Delta$ such that $\alpha' \neq 0$. We will see that the corresponding weight space $V_{\alpha'}$ is finite dimensional. Note that $V_{\alpha'} = \bigoplus_{\beta - \alpha' \in \Delta} \mathfrak{g}_\beta$. Let $\beta = \sum_{i \in I} m_i \alpha_i \in \Delta$ such that $\beta' = \alpha' = \sum_{k \in I'} m_k \alpha'_k$, then $m_k = n_k$, $\forall k \in I'$, since $\Pi' = \{ \alpha'_k, k \in I' \}$ is free in $(\mathfrak{h}^J)^*$. In particular, $\beta$ and $\alpha$ are of the same
sign, and we may assume $\alpha \in \Delta^+$. Let $ht_J(\beta) = \sum_{j \in J} m_j$ be the height of $\beta$ relative to $J$, and let $W_J$ be the finite subgroup of $W$ generated by $r_j$, $j \in J$. Since $W_J$ fixes pointwise $\mathfrak{h}^J$, we deduce that $\gamma' = \beta'$, $\forall \gamma \in W_J \beta$, and so we may assume that $ht_J(\beta)$ is minimal among the roots in $W_J \beta$. From the inequality $ht_J(\beta) \leq ht_J(r_j(\beta))$, $\forall j \in J$, we get $\langle \beta, \alpha_j \rangle \leq 0$, $\forall j \in J$. Let $\rho_J$ be the half sum of positive coroots of $\Delta(J)$. It is known that $\langle \alpha_j, \rho_J \rangle = 1$, $\forall j \in J$. Note that $\langle \beta, \rho_J \rangle = \sum_{j \in J} m_j + \sum_{k \in P^+} n_k \langle \alpha_k, \rho_J \rangle = ht_J(\beta) + \sum_{k \in P^+} n_k \langle \alpha_k, \rho_J \rangle$. Hence, the condition $(\langle \beta, \rho_J \rangle \leq 0)$ implies $(ht_J(\beta) \leq \sum_{k \in P^+} -n_k \langle \alpha_k, \rho_J \rangle)$. Thus there is just a finite number of possibilities for $\beta$. It follows that $\alpha'$ is of finite multiplicity.

**Theorem 2.14.** Let $\Delta^J$ be the root system of the pair $(g^J, \mathfrak{h}^J)$, then the Kac-Moody Lie algebra $g$ is finitely $\Delta^J$-graded, with grading subalgebra $g^J$.

**Proof.** Let $\Delta' = \{\alpha', \alpha \in \Delta \} \setminus \{0\}$ be the set of non-null weights of the $g^J$-module $g$ relative to $\mathfrak{h}^J$. Let $\Delta'_+ = \{\alpha' \in \Delta', \alpha \in \Delta^+\}$ and $\Delta^J_+$ the set of positive roots of $\Delta^J$ relative to the root basis $\Pi^J$. We have to prove that $\Delta^J = \Delta^J_+$ or equivalently $\Delta'_+ = \Delta^J_+$. Let $Q^J = \mathbb{Z}\Pi^J$ be the root lattice of $\Delta^J$ and $Q^J_+ = \mathbb{Z}^+\Pi^J$. It is known that the positive root system $\Delta^J_+$ is uniquely defined by the following properties (see [12, Ex. 5.4]):

(i) $\Pi^J \subset \Delta^J_+ \subset Q^J_+$, $2\alpha'_i \notin \Delta^J_+$, $\forall i \in I'$;
(ii) if $\alpha' \in \Delta^J_+$, $\alpha' \neq \alpha'_i$, then the set $\{\alpha' + k\alpha'_i; k \in \mathbb{Z}\} \cap \Delta^J_+$ is a string $\{\alpha' - p\alpha'_i, ..., \alpha' + q\alpha'_i\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \alpha', H_i \rangle$;
(iii) if $\alpha' \in \Delta^J_+$, then $supp(\alpha')$ is connected.

We will see that $\Delta'_+$ satisfies these three properties and hence $\Delta'_+ = \Delta^J_+$. Clearly $\Pi^J \subset \Delta'_+ \subset Q^J_+$. For $\alpha \in \Delta$ and $k \in I'$, the condition $\alpha \in N\alpha_k$ implies $\alpha \in \Delta(I_k)^+$. As $(I_k, J_k)$ is $C$-admissible for $k \in I'$, the highest root of $\Delta(I_k)^+$ has coefficient 1 on the root $\alpha_k$ (cf. Remark 2.8). It follows that $2\alpha'_k \notin \Delta'_+$ and (i) is satisfied. By Proposition 2.13, $g$ is an integrable $g^J$-module with finite multiplicities. Hence, the propriety (ii) follows from [12, 3.6]. Let $\alpha \in \Delta_+$, then $supp(\alpha)$ is connected and $supp(\alpha') \subset supp(\alpha)$. Let $k, l \in supp(\alpha')$; if $k, l$ are $J$-connected in $supp(\alpha)$ relative to the generalized Cartan matrix $A$ (cf. 2.3), then by lemma 2.5, $k, l$ are linked in $I'$ relative to the generalized Cartan matrix $A'$. Hence, the connectedness of $supp(\alpha')$, relative to $A'$, follows from that of $supp(\alpha)$ relative to $A$ (see Remark 2.4) and (iii) is satisfied.

**Remark 2.15.** Note that the definition of $C$-admissible pair can be extended to decomposable Kac-Moody Lie algebras: thus if $I^1, I^2, ..., I^m$ are the connected components of $I$ and $J^k = J \cap I^k$, $k = 1, 2, ..., m$, then $(I, J)$ is $C$-admissible if and only if $(I^k, J^k)$ is for all $k = 1, 2, ..., m$. In particular, the corresponding $C$-admissible algebra is $g^J = \bigoplus_{k=1}^m g(I^k)^{J^k}$, where $g(I^k)^{J^k}$ is the $C$-admissible subalgebra of $g(I^k)$ corresponding to the $C$-admissible pair $(I^k, J^k)$, $k = 1, 2, ..., m$. 

Ben Messaoud and Rousseau

333
3. Real gradations

From now on we suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is symmetrizable and, starting from 3.5, indecomposable.

Let $\mathfrak{m}$ be a Kac-Moody subalgebra of $\mathfrak{g}$ and let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{m}$. Put $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ the corresponding root system. We assume that $\mathfrak{a} \subset \mathfrak{h}$ and that $\mathfrak{g}$ is finitely $\Sigma$-graded with $\mathfrak{m}$ as grading subalgebra. Thus $\mathfrak{g} = \sum_{\gamma \in \Sigma \cup \{0\}} V_{\gamma}$, with $V_{\gamma} = \{ x \in \mathfrak{g}; [a, x] = (\gamma, a)x, \forall a \in \mathfrak{a}\}$ is finite dimensional for all $\gamma \in \Sigma \cup \{0\}$. For $\alpha \in \Delta$, denote by $\rho_a(\alpha)$ the restriction of $\alpha$ to $\mathfrak{a}$. As $\mathfrak{g}$ is $\Sigma$-graded, one has $\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}$.

**Lemma 3.1.**

1) Let $\mathfrak{c}$ be the center of $\mathfrak{g}$ and denote by $\mathfrak{c}_a$ the center of $\mathfrak{m}$. Then $\mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}$. In particular, if $\mathfrak{g}$ is perfect, then the grading subalgebra $\mathfrak{m}$ is also perfect.

2) Suppose that $\Delta^{im} \neq \emptyset$, then $\rho_a(\Delta^{im}) \subset \Sigma^{im}$.

**Proof.**

1) It is clear that $\mathfrak{c} \cap \mathfrak{a} \subset \mathfrak{c}_a$. Since $\mathfrak{g}$ is $\Sigma$-graded, we deduce that $\mathfrak{c}_a$ is contained in the center $\mathfrak{c}$ of $\mathfrak{g}$, hence $\mathfrak{c}_a \subset \mathfrak{c} \cap \mathfrak{a}$. If $\mathfrak{g}$ is perfect, then $\mathfrak{g} = \mathfrak{g}'$, $\mathfrak{h} = \mathfrak{h}'$, $\mathfrak{c} = \{0\}$; so $\mathfrak{c}_a = \{0\}$, $\mathfrak{a} = \mathfrak{a}'$ and $\mathfrak{m} = \mathfrak{m}'$.

2) If $\alpha \in \Delta^{im}$, then $\mathbb{N}\alpha \subset \Delta$. Since $V_0$ is finite dimensional, $\rho_a(\alpha) \neq 0$ and $\mathbb{N}\rho_a(\alpha) \subset \Sigma$, hence $\rho_a(\alpha) \in \Sigma^{im}$. $

**Definition 3.2.** ([3, 5.2.6]) Suppose that $\Delta^{im} \neq \emptyset$. Let $\alpha, \beta \in \Delta^{im}$.

(i) The imaginary roots $\alpha$ and $\beta$ are said to be linked if $\mathbb{N}\alpha + \mathbb{N}\beta \subset \Delta$ or $\beta \in \mathbb{Q}^+ \alpha$.

(ii) The imaginary roots $\alpha$ and $\beta$ are said to be linkable if there exists a finite family of imaginary roots $(\beta_i)_{0 \leq i \leq n+1}$ such that $\beta_0 = \alpha$, $\beta_{n+1} = \beta$ and $\beta_i$ and $\beta_{i+1}$ are linked for all $i = 0, 1, ..., n$.

**Proposition 3.3.** ([3, 5.2.7]) Suppose that $\Delta^{im} \neq \emptyset$. Let $\Delta = \bigcup_{j=1}^{m} \Delta_j$ be the decomposition of $\Delta$ in indecomposable root systems. Suppose that $\Delta_1, \Delta_2, ..., \Delta_r$ ($r \leq m$) are the indecomposable root subsystems of $\Delta$ which are not of finite type. Then to be linkable is an equivalence relation on $\Delta^{im}$ and the equivalence classes are the $2r$ sets $\Delta^{im}_{\pm} \cap \Delta_j$, $j = 1, 2, ..., r$.

**Lemma 3.4.** Suppose that $\Delta^{im} \neq \emptyset$, then there exist root bases in $\Sigma$ and $\Delta$ such that $\rho_a(\Delta^{im}_{\pm}) \subset \Sigma^{im}_{\pm}$.

**Proof.** Fix a root basis $\Pi_a$ for the grading root system $\Sigma$. Let $\Delta = \bigcup_{j=1}^{m} \Delta_j$ be, as above, the decomposition of $\Delta$ in indecomposable root systems. Denote by $\Pi_j := \Pi \cap \Delta_j$ the root basis of $\Delta_j$, $j = 1, 2, ..., m$. If $\alpha, \beta$ are two imaginary linkable roots of $\Delta^{im}_{\pm}$, then $\rho_a(\alpha)$ and $\rho_a(\beta)$ are also linkable in $\Sigma^{im}$. By Proposition 3.3, $\rho_a(\alpha)$ and $\rho_a(\beta)$ are of the same sign. Since $\alpha$ and $\beta$ are of the same sign in $\Delta^{im}_{\pm}$ relative to the root basis $\Pi_j$, one can, if necessary, change the sign of $\Pi_j$ so that $\rho_a(\alpha)$ and $\rho_a(\beta)$ are positive imaginary roots of $\Sigma^{+}$ relative to the fixed root basis.
\(\Pi_a\). Hence we get a root basis of \(\Delta = \bigcup_{j=1}^m \Delta_j\) satisfying \(\rho_a(\Delta^m_+) \subset \Sigma^m_+\).

In the following, we will show that the indecomposable Kac-Moody Lie algebra \(\mathfrak{g}\) and the grading subalgebra \(\mathfrak{m}\) are of the same type.

**Lemma 3.5.** The Kac-Moody Lie algebra \(\mathfrak{g}\) is of indefinite type if and only if \(\Delta^m\) generates the dual space \((\mathfrak{h}/\mathfrak{c})^*\) of \(\mathfrak{h}/\mathfrak{c}\).

**Proof.** Note that the root basis \(\Pi = \{\alpha_i, i \in I\}\) induces a basis for the quotient vector space \((\mathfrak{h}/\mathfrak{c})^*\). It follows that the condition \((\Delta^m \neq \emptyset)\) implies \((\dim(\mathfrak{h}/\mathfrak{c})^* \geq 2)\). Suppose now that \(\mathfrak{g}\) is of indefinite type. Let \(\alpha \in \Delta^m_+\) be a positive strictly imaginary root satisfying \(\langle \alpha, \alpha_i \rangle < 0, \forall i \in I\); then, \(r_i(\alpha) = \alpha - \langle \alpha, \alpha_i \rangle \alpha_i \in \Delta^m_+\) for all \(i \in I\). In particular, the vector subspace \((\Delta^m_+)\) spanned by \(\Delta^m_+\) contains \(\Pi\) and hence is equal to \((\mathfrak{h}/\mathfrak{c})^*\). Conversely, if \(\Delta^m_+\) generates \((\mathfrak{h}/\mathfrak{c})^*\), then \(\Delta^m_+\) is non-empty and contains at least two linearly independent imaginary roots; hence \(\Delta\) can not be of finite or affine type.

**Proposition 3.6.**
1) If \(\Delta^m_+\) is not empty, then \(\mathfrak{m}\) is indecomposable.
2) The Kac-Moody algebra \(\mathfrak{g}\) and the grading subalgebra \(\mathfrak{m}\) are of the same type.
3) Suppose \(\mathfrak{g}\) Lorentzian, then \(\mathfrak{m}\) is also Lorentzian.

**N.B.** We will see below that \(\mathfrak{m}\) is always indecomposable (3.11) and symmetrizable (3.17).

**Proof.**
1) We saw in Lemma 3.4 that \(\rho_a(\Delta^m_+)\) is in a unique linkable equivalence class of \(\Sigma^m_+\). So, if \(\Sigma = \Sigma_1 \cup \Sigma_2\) is decomposable, we may assume \(\rho_a(\Delta^m_+) \subset \Sigma^m_+\). But there is \(\delta \in \Delta^m_+\) such that \(\alpha + n\delta \in \Delta_+\) for all \(\alpha \in \Delta_+\) and \(n \in \mathbb{N}\) [12, 4.3, 5.6 and 6.3]. So \(\rho_a(\alpha) + n\rho_a(\delta) \in \Sigma\) for \(n \gg 0\) and \(\rho_a(\alpha) \in \Sigma_1 \cup \{0\}\). As \(\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}\), we have \(\Sigma_2 = \emptyset\).
2) If \(\mathfrak{g}\) is of finite type, then \(\Delta\) is finite and hence \(\Sigma = \rho_a(\Delta) \setminus \{0\}\) is finite. If \(\mathfrak{g}\) is affine, let \(\delta\) be the lowest positive imaginary root. One can choose a root basis \(\Pi_a = \{\gamma_i, i \in I\}\) of \(\Sigma\) so that \(\delta := \rho_a(\delta)\) is a positive imaginary root. Note that \(\mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}' \subset \mathfrak{h}'\); in particular \(\delta(\mathfrak{a}') = \{0\}\) and \(\langle \delta, \gamma_i \rangle = 0, \forall i \in I\). It follows that \(\mathfrak{m}\) is affine (see [12, 4.3]).

Suppose now that \(\mathfrak{g}\) is of indefinite type. Thanks to Lemma 3.5, it suffices to prove that \(\Sigma^m\) generates \((\mathfrak{a}/\mathfrak{c}_a)^*\), where \(\mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}\) is the center of \(\mathfrak{m}\). The natural homomorphism of vector spaces \(\pi : \mathfrak{a} \to \mathfrak{h}/\mathfrak{c}\) induces a monomorphism \(\bar{\pi} : \mathfrak{a}/\mathfrak{c}_a \to \mathfrak{h}/\mathfrak{c}\). By duality, the homomorphism \(\bar{\pi}^* : (\mathfrak{h}/\mathfrak{c})^* \to (\mathfrak{a}/\mathfrak{c}_a)^*\) is surjective and \(\bar{\pi}^*(\Delta^m) \subset \Sigma^m\) generates \((\mathfrak{a}/\mathfrak{c}_a)^*\).
3) Suppose that \(\mathfrak{g}\) is Lorentzian (hence of indefinite type) and let \((\ldots, \cdot)\) be an invariant non-degenerate bilinear form on \(\mathfrak{g}\). Then, the restriction of \((\ldots, \cdot)\) to \(\mathfrak{h}_\mathbb{R}\) has signature \( (+, +, +, -)\) and any maximal totally isotropic subspace of \(\mathfrak{h}_\mathbb{R}\) relatively to \((\ldots, \cdot)\) is one dimensional. Let \(\mathfrak{a}_\mathbb{R} := \mathfrak{a} \cap \mathfrak{h}_\mathbb{R}\) and let \((\ldots, \cdot)_\mathbb{R}\) be the restriction of \((\ldots, \cdot)\) to \(\mathfrak{m}\). As \(\mathfrak{m}\) is of indefinite type, \(\dim(\mathfrak{a}) \geq 2\) and the restriction of \((\ldots, \cdot)_\mathbb{R}\) to \(\mathfrak{a}_\mathbb{R}\) is non-null. It follows that the orthogonal subspace \(\mathfrak{m}^\perp\)
of \( m \) relatively to \((\ldots)_a\) is a proper ideal of \( m \). Since \( m \) is perfect (because \( g \) is) we deduce that \( m^+ = \{0\} \) (cf. [12, 1.7]) and the invariant bilinear form \((\ldots)_a\) is non-degenerate. It follows that \( m \) is symmetrizable and the bilinear form \((\ldots)_a\) when restricted to \( a_R \) is non-degenerate; since \( m \) is of indefinite type, it can not be positive definite. Hence, the bilinear form \((\ldots)_a\) has signature \( (+\ldots,+,-) \) on \( a_R \) and then the grading subalgebra \( m \) is Lorentzian.

**Definition 3.7.** Let \( \Pi_a \) be a root basis of \( \Sigma \) and let \( \Sigma^+ \) be the corresponding set of positive roots. The root basis is said to be adapted to the root basis \( \Pi \) of \( \Delta \) if \( \rho_a(\Delta^+) \subset \Sigma^+ \cup \{0\} \).

We will see (3.10) that adapted root bases always exist.

**Lemma 3.8.** Let \( \Pi_a \) be a root basis of \( \Sigma \) such that \( \rho_a(\Delta^+_im) \subset \Sigma^+_im \) and let \( X_a \) be the corresponding positive Tits cone. Then we have \( X_a \subset \bar{X} \cap a \).

**Proof.** As \( \Delta^+_im \neq \emptyset \), one has \( \bar{X} = \{ h \in h_R; \langle \alpha, h \rangle \geq 0, \forall \alpha \in \Delta^+_im \} \) (see Remark 1.3). The lemma follows from Lemma 3.4.

**Lemma 3.9.** Suppose that \( \Delta^+_im \neq \emptyset \). Let \( p \in \bar{X} \) such that \( \langle \alpha, p \rangle \in \mathbb{Z}, \forall \alpha \in \Delta, \) and \( \langle \beta, p \rangle > 0, \forall \beta \in \Delta^+_im \). Then \( p \in \bar{X} \).

**Proof.** The result is clear when \( \Delta \) is of affine type since
\[
\bar{X} = \bar{X} = \{ h \in h_R; \langle \delta, h \rangle > 0 \}.
\]
Suppose now that \( \Delta \) is of indefinite type. If one looks to the proof of Proposition 5.8.c) in [12], one can show that an element \( p \in \bar{X} \) satisfying the conditions of the lemma lies in \( X \). As \( \Delta^+_im \) is \( W \)-invariant, we may assume that \( p \) lies in the fundamental chamber \( C \). Hence there exists a subset \( J \) of \( I \) such that \( \{ \alpha \in \Delta; \langle \alpha, p \rangle = 0 \} = \Delta_J = \Delta \cap \sum_{j \in J} \mathbb{Z} \alpha_j \). Since \( \Delta_J \cap \Delta^+_im = \emptyset \), the root subsystem \( \Delta_J \) is of finite type and \( p \) lies in the finite type facet of type \( J \). Thus \( p \in \bar{X} \) (see section 1).

**Theorem 3.10.** There exists a root basis \( \Pi_a \) of \( \Sigma \) which is adapted to the root basis \( \Pi \) of \( \Delta \). Moreover, there exists a finite type subset \( J \) of \( I \) such that \( \Delta_J = \{ \alpha \in \Delta; \rho_a(\alpha) = 0 \} \).

**N.B.** This is part 1) of Theorem 2.

**Proof.** Let \( \Pi_a = \{ \gamma_i, i \in \bar{I} \} \) be a root basis of \( \Sigma \) such that \( \rho_a(\Delta^+_im) \subset \Sigma^+_im \), where \( \bar{I} \) is just a set indexing the basis elements. Let \( p \in a \) such that \( \langle \gamma_i, p \rangle = 1, \forall i \in \bar{I} \) and let \( P = \{ \alpha \in \Delta; \langle \alpha, p \rangle \geq 0 \} \). If \( \Delta \) is finite, then \( P \) is clearly a parabolic subsystem of \( \Delta \) and the result is trivial. Suppose now that \( \Delta^+_im \neq \emptyset \); then \( p \) satisfies the conditions of the Lemma 3.9 and we may assume that \( p \) lies in the facet of type \( J \) for some subset \( J \) of finite type in \( \bar{I} \). In which case \( P = \Delta_J \cup \Delta^+ \) is the standard parabolic subsystem of finite type \( J \). Note that, for
$\gamma \in \Sigma^+$, one has $\langle \gamma, p \rangle = \text{ht}_a(\gamma)$ the height of $\gamma$ with respect to $\Pi_a$. It follows that $\{a \in \Delta; \rho_a(\alpha) = 0\} = \Delta_J$, in particular, $\rho_a(\Delta^+) = \rho_a(P) \subset \Sigma^+ \cup \{0\}$. Hence, the root basis $\Pi_a$ is adapted to $\Pi$.

Corollary 3.11. $\Sigma$ is indecomposable.

Proof. For $\gamma_1, \gamma_2 \in \Pi_a$, there are $\alpha_1, \alpha_2 \in \Delta_+$ such that $\gamma_i = \rho_a(\alpha_i)$. But $\gamma_i$ is not a sum in $\Sigma_s$, so, up to $\Delta_J$, $\alpha_i$ is not a sum: we may assume $\alpha_i \in \Pi$. As $\Delta$ is indecomposable, there is a root $\alpha \in \Delta \cap (\alpha_1 + \alpha_2 + \sum_{\alpha \in \Pi} \mathbb{Z}^+ \alpha)$. Now $\rho_a(\alpha) \in (\Sigma \cup \{0\}) \cap (\gamma_1 + \gamma_2 + \sum_{\gamma \in \Pi_a} \mathbb{Z}^+ \gamma) \subset \Sigma$ and $\gamma_1, \gamma_2$ have to be in the same connected component of $\Pi_a$.

From now on, we fix a root basis $\Pi_a = \{\gamma_s, s \in \bar{I}\}$, for the grading root system $\Sigma$, which is adapted to the root basis $\Pi = \{\alpha_i, i \in I\}$ of $\Delta$ (see Theorem 3.10). As before, let $J := \{j \in I; \rho_a(\alpha_j) = 0\}$ and $I' := I \setminus J$. For $k \in I'$, we denote, as above, by $I_k$ the connected component of $J \cup \{k\}$ containing $k$, and $J_k := J \cap I_k$.

Proposition 3.12.  
1) Let $s \in \bar{I}$, then there exists $k_s \in I'$ such that $\rho_a(\alpha_{k_s}) = \gamma_s$ and any preimage $\alpha \in \Delta$ of $\gamma_s$ is equal to $\alpha_k$ modulo $\sum_{j \in I_k} \mathbb{Z} \alpha_j$ for some $k \in I'$ satisfying $\rho_a(\alpha_k) = \gamma_s$.
2) Let $k \in I'$ such that $\rho_a(\alpha_k)$ is a real root of $\Sigma$. Then $\rho_a(\alpha_k) \in \Pi_a$ is a simple root.

Proof. This result was proved by J. Nervi for affine algebras (see [17, 2.3.10] and the proof of Prop. 2.3.12). The arguments used there are available for general Kac-Moody algebras.

We introduce the following notations:

$$I'_{re} := \{i \in I'; \rho_a(\alpha_i) \in \Pi_a\}; \quad I'_{im} := I' \setminus I'_{re},$$

$$I_{re} = \bigcup_{k \in I'_{re}} I_k; \quad J_{re} = I_{re} \cap J = \bigcup_{k \in I'_{re}} J_k; \quad J^o = J \setminus J_{re}$$

$$\Gamma_s := \{i \in I'; \rho_a(\alpha_i) = \gamma_s\}, \forall s \in \bar{I}.$$

Note that $J^o$ is not connected to $I_{re}$.

Remark 3.13.  
1) In view of Proposition 3.12, assertion 2), one has $\rho_a(\alpha_k) \in \Sigma^+_{im}, \forall k \in I'_{im}$.
2) $I = I_{re} \cup I'_{im} \cup J^o$ is a disjoint union.
3) If $I'_{im} = \emptyset$, then $I = I_{re} \cup J^o$. Since $I$ is connected (and $I_{re}$ is not connected to $J^o$) we deduce that $J^o = \emptyset, I = I_{re}$ and $I'_{re} = I' = I \setminus J$.
4) If $I'_{im} \neq \emptyset$, then $I_{re}$ may be non-connected (see the example in §5 below).

Proposition 3.14.  
1) Let $k \in I'_{re}$, then $I_k$ is of finite type.
2) Let $s \in \bar{I}$. If $|\Gamma_s| \geq 2$ and $k \neq l \in \Gamma_s$, then $I_k \cup I_l$ is not connected: $g(I_k)$ and $g(I_l)$ commute and are orthogonal.

3) For all $k \in I'_{re}$, $(I_k, J_k)$ is an irreducible $C-$admissible pair.

4) The derived subalgebra $m'$ of the grading algebra $m$ is contained in $g'(I_{re})$ (as defined in proposition 1.2).

**Proof.**

1) Suppose that there exists $k \in I'_{re}$ such that $I_k$ is not of finite type; then there exists an imaginary root $\beta_k$ whose support is the whole $I_k$. Hence, there exists a positive integer $m_k \in \mathbb{N}$ such that $\rho_{\alpha}(\beta_k) = m_k \rho(\alpha_k)$ is an imaginary root of $\Sigma$. It follows that $\rho_k(\alpha_k)$ is an imaginary root and this contradicts the fact that $k \in I'_{re}$.

2) Let $s \in \bar{I}$ such that $|\Gamma_s| \geq 2$ and let $k \neq l \in \Gamma_s$. Since $V_{n_{\gamma_s}} = \{0\}$ for all integer $n \geq 2$, the same argument used in 1) shows that $I_k \cup I_l$ is not connected, and $I_k$ and $I_l$ are its two connected components. In particular, $[g(I_k), g(I_l)] = \{0\}$ and $(g(I_k), g(I_l)) = \{0\}$.

3) Let $k \in I'_{re}$ and let $s \in \bar{I}$ such that $\rho_k(\alpha_k) = \gamma_s$. Let $(\bar{X}_s, \bar{H}_s, \bar{Y}_s)$ be an $\mathfrak{sl}_2-$triple in $m$ corresponding to the simple root $\gamma_s$. Let $V_{\gamma_s}$ be the weight space of $g$ corresponding to $\gamma_s$. In view of Proposition 3.12, assertion 1), one has:

$$V_{\gamma_s} = \bigoplus_{l \in \Gamma_s} V_{\gamma_s} \cap g(I_l).$$

Hence, one can write:

$$\bar{X}_s = \sum_{l \in \Gamma_s} E_l; \quad \bar{Y}_s = \sum_{l \in \Gamma_s} F_l,$$

with $E_l \in V_{\gamma_s} \cap g(I_l)$ and $F_l \in V_{-\gamma_s} \cap g(I_l)$. It follows from assertion 2) that

$$\bar{H}_s = \gamma_s = [\bar{X}_s, \bar{Y}_s] = \sum_{l \in \Gamma_s} [E_l, F_l] = \sum_{l \in \Gamma_s} H_l,$$

where $H_l := [E_l, F_l] \in \mathfrak{h}_l$, $\forall l \in \Gamma_s$. Then one has, for $k \in \Gamma_s$,

$$2 = \langle \gamma_s, \gamma_s \rangle = \langle \alpha_k, \gamma_s \rangle = \sum_{l \in \Gamma_s} \langle \alpha_k, H_l \rangle = \langle \alpha_k, H_k \rangle,$$

and for $j \in J_k$,

$$0 = \langle \alpha_j, \gamma_s \rangle = \sum_{l \in \Gamma_s} \langle \alpha_j, H_l \rangle = \langle \alpha_j, H_k \rangle.$$

In particular, $H_k$ is the unique semi-simple element of $\mathfrak{h}_{I_k}$ satisfying:

$$\langle \alpha_i, H_k \rangle = 2\delta_{i,k}, \forall i \in I_k.$$  \hspace{1cm} (3.4)

Hence, $(E_k, H_k, F_k)$ is an $\mathfrak{sl}_2-$triple in the simple Lie algebra $g(I_k)$ and since $V_{2\gamma_s} = \{0\}$, $(I_k, J_k)$ is an irreducible $C-$admissible pair for all $k \in \Gamma_s$. The statement 4) follows from the relation (3.2).
Corollary 3.15. The pair \((I_{re}, J_{re})\) is \(C\)-admissible (in the eventually decomposable sense of Remark 2.15). If \(I'_{im} = \emptyset\), then \(I_{re} = I\), \(J_{re} = J\) and \(\mathfrak{g}\) is finitely \(\Delta^J\)-graded, with grading subalgebra \(\mathfrak{g}^J\).

N.B. We have got part 2) of Theorem 2.

Proof. The first assertion is a consequence of Proposition 3.14. By Remark 3.13, when \(I'_{im} = \emptyset\), we have \(I = I_{re}\); hence, by Theorem 2.14, \(\mathfrak{g}\) is finitely \(\Delta^J\)-graded.

Definition 3.16. If \(I'_{im} \neq \emptyset\), then \((I, J)\) is called a generalized \(C\)-admissible pair and the gradation of \(\mathfrak{g}\) by \(\Sigma\) and \(m\) is said imaginary.
On the contrary if \(I'_{im} = \emptyset\), the gradation is said real.
If \(I'_{im} = J = \emptyset\), the Kac-Moody algebra \(\mathfrak{g}\) is said to be maximally finitely \(\Sigma\)-graded.

Corollary 3.17. The grading subalgebra \(m\) of \(\mathfrak{g}\) is symmetrizable and the restriction to \(m\) of the invariant bilinear form of \(\mathfrak{g}\) is non-degenerate.

Proof. Let \((\ldots)_a\) be the restriction to \(m\) of the invariant bilinear form \((\ldots)\) of \(\mathfrak{g}\). Recall from the proof of Proposition 3.14 that \(\gamma_s = \sum_{k \in \Gamma_s} H_k\), \(\forall s \in \tilde{I}\).
In particular \((\gamma_s, \gamma_s)_a = \sum_{k \in \Gamma_s} (H_k, H_k) > 0\). It follows that \((\ldots)_a\) is a non-degenerate invariant bilinear form on \(m\) (see §1) and that \(m\) is symmetrizable.

Corollary 3.18. Let \(\mathfrak{h}^J\) be the orthogonal of \(\mathfrak{h}_J\) in \(\mathfrak{h}\). For \(k \in I'_{im}\), write

\[ \rho_a(\alpha_k) = \sum_{s \in \tilde{I}} n_{s,k}\gamma_s. \]

For \(s \in \tilde{I}\), choose \(l_s\) a representative element of \(\Gamma_s\). Then \(a/c_a\) can be viewed as the subspace of \(\mathfrak{h}^J/c\) defined by the following relations :

\[ \langle \alpha_k, h \rangle = \langle \alpha_{l_s}, h \rangle, \forall k \in \Gamma_s, \forall s \in \tilde{I} \]

\[ \langle \alpha_k, h \rangle = \sum_{s \in \tilde{I}} n_{s,k}\langle \alpha_{l_s}, h \rangle, \forall k \in I'_{im}. \]

Proof. The subspace of \(\mathfrak{h}^J/c\) defined by the above relations has dimension \(|\tilde{I}|\) and contains \(a/c_a\) and hence it is equal to \(a/c_a\).

Proposition 3.19. Let \((\ldots)_a\) be the restriction to \(m\) of the invariant bilinear form \((\ldots)\) of \(\mathfrak{g}\).
1) Let \(a' = a \cap m'\) and let \(a''\) be a supplementary subspace of \(a'\) in \(a\) which is totally isotropic relatively to \((\ldots)_a\). Then \(a'' \cap \mathfrak{h} = \{0\}\).
2) Let \(A_{I_{re}}\) be the submatrix of \(A\) indexed by \(I_{re}\). Then there exists a subspace \(\mathfrak{h}_{I_{re}}\) of \(\mathfrak{h}\) containing \(a\) such that \((\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}})\) is a realization of \(A_{I_{re}}\). In particular, the Kac-Moody subalgebra \(\mathfrak{g}(I_{re})\) associated to this realization (in 1.2) contains the grading subalgebra \(m\).
3) The Kac-Moody algebra \( \mathfrak{g}(I_{\text{re}}) \) is finitely \( \Delta(I_{\text{re}})^{J_{\text{re}}} \)-graded and its grading subalgebra is the subalgebra \( \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} \) associated to the \( C \)-admissible pair \( (I_{\text{re}}, J_{\text{re}}) \) as in Proposition 2.11.

4) The Kac-Moody algebra \( \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} \) contains \( \mathfrak{m} \).

**Proof.**

1) Recall that the center \( \mathfrak{c}_a \) of \( \mathfrak{m} \) is contained in the center \( \mathfrak{c} \) of \( \mathfrak{g} \). Since \( \mathfrak{h}' = \mathfrak{c}^\perp \) and \( \mathfrak{c}_a \) is in duality with \( \mathfrak{a}'' \) relatively to \( (\ldots, \alpha) \), we deduce that \( \mathfrak{a}'' \cap \mathfrak{h}' = \{0\} \).

2) From the proofs of 3.17 and 3.14 we get \( \gamma'_{\mathfrak{h}_a} = \sum_{k \in \Gamma_s} H_k \in \sum_{k \in \Gamma_s} \mathfrak{h}_k = \mathfrak{h}'_{I_{\text{re}}} \). So \( \mathfrak{c}_a \subset \mathfrak{a}' \subset \mathfrak{h}'_{I_{\text{re}}} \subset \mathfrak{h}' \). It follows that \( (\mathfrak{h}'_{I_{\text{re}}} + \mathfrak{h}'_{\text{re}}) \) is contained in \( \mathfrak{c}_a^\perp \) the orthogonal subspace of \( \mathfrak{c}_a \) in \( \mathfrak{h} \). Since \( \mathfrak{a}'' \cap \mathfrak{c}_a^\perp = \{0\} \), one can choose a supplementary subspace \( \mathfrak{h}'_{\text{re}} \) of \( (\mathfrak{h}'_{I_{\text{re}}} + \mathfrak{h}'_{\text{re}}) \) containing \( \mathfrak{a}'' \). Let \( \mathfrak{h}_{I_{\text{re}}} = \mathfrak{h}'_{I_{\text{re}}} \oplus \mathfrak{h}'_{\text{re}} \), then, by Proposition 1.2, \( (\mathfrak{h}_{I_{\text{re}}}, \Pi_{I_{\text{re}}}, \Pi_{\text{re}}) \) is a realization of \( A_{I_{\text{re}}} \).

3) As in Corollary 3.15, assertion 3) is a simple consequence of Theorem 2.14.

4) The algebra \( \mathfrak{a} \) is in \( \mathfrak{h}_{I_{\text{re}}} \cap \Pi_{I_{\text{re}}} = (\mathfrak{h}_{I_{\text{re}}})^{J_{\text{re}}} \). By the proof of Proposition 3.14, for \( s \in \mathcal{T}, \mathcal{X}_s \) and \( \mathcal{Y}_s \) are linear combinations of the elements in \( \{E_k, F_k \mid k \in \Gamma_s\} \subset \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} \).

Hence \( \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} \) contains all generators of \( \mathfrak{m} \).

**Lemma 3.20.** Let \( \mathfrak{l} \) be a Kac-Moody subalgebra of \( \mathfrak{g} \) containing \( \mathfrak{m} \). Then \( \mathfrak{l} \) is finitely \( \Sigma \)-graded. In particular, the Kac-Moody subalgebra \( \mathfrak{g}(I_{\text{re}}) \) or \( \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} \) is finitely \( \Sigma \)-graded.

**N.B.** Proposition 3.19 and Lemma 3.20 finish the proof of Theorem 2.

**Proof.** Recall that the Cartan subalgebra \( \mathfrak{a} \) of \( \mathfrak{m} \) is \( \text{ad}_\mathfrak{g} \)-diagonalizable. Since \( \mathfrak{l} \) is \( \text{ad}(\mathfrak{a}) \)-invariant, one has \( \mathfrak{l} = \bigoplus_{\gamma \in \Sigma, \mathfrak{L} \cap \mathfrak{l}} V_{\mathfrak{L}} \cap \mathfrak{l} \). By assumption \( \{0\} \neq \mathfrak{m}_\gamma \subset V_{\mathfrak{L}} \cap \mathfrak{l} \) for all \( \gamma \in \Sigma \). Thus, \( \mathfrak{l} \) is finitely \( \Sigma \)-graded.

**Proposition 3.21.** If \( I_{\text{im}} = \emptyset \), then \( \mathfrak{g}(I_{\text{re}}) = \mathfrak{g} \) and the \( C \)-admissible subalgebra \( \mathfrak{g}^\mathfrak{l} \) is maximally finitely \( \Sigma \)-graded, with grading subalgebra \( \mathfrak{m} \).

**Proof.** This result follows from the facts that \( V_0 \cap \mathfrak{g}^\mathfrak{l} = \mathfrak{h}^\mathfrak{l} \) and \( \mathfrak{m} \subset \mathfrak{g}^\mathfrak{l} \) (see Prop. 3.19).

We now want a precise description of the gradation of \( \mathfrak{g}(I_{\text{re}}) \) by \( \Sigma \) and \( \mathfrak{m} \); particularly in the case (already mentioned in Remark 3.13) where \( \mathfrak{g}(I_{\text{re}}) \) (and so \( \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} \)) is decomposable.

Let \( I_{\text{re}}^1, I_{\text{re}}^2, \ldots, I_{\text{re}}^q \) be the connected components of \( I_{\text{re}} \) and \( J_{\text{re}} := J_{\text{re}} \cap I_{\text{re}}^i \), \( i = 1, 2, \ldots, q \). Then \( \mathfrak{g}(I_{\text{re}}) = \bigoplus_{i=1}^q \mathfrak{g}(I^i_{\text{re}}) \) and hence \( \mathfrak{g}(I_{\text{re}})^{J_{\text{re}}} = \bigoplus_{i=1}^q \mathfrak{g}(I^i_{\text{re}})^{J_{\text{re}}} \) (see Remark 2.15).

Retain the notations introduced just before Proposition 3.14 and those introduced in its proof. For \( s \in \mathfrak{I} \) and \( i = 1, 2, \ldots, q \), let \( \Gamma_s^i := \Gamma_s \cap I_{\text{re}}^i \). If \( \Gamma_s^i \) is non-empty, put \( E^i_s := \sum_{l \in \Gamma_s^i} E_l; F^i_s := \sum_{l \in \Gamma_s^i} F_l \) and \( H^i_s := \sum_{l \in \Gamma_s^i} H_l \). We take \( E^i_s = F^i_s = H^i_s = 0 \) if \( \Gamma_s^i \) is empty.
Note that \( \Gamma_s = \bigcup_{i=1}^{q} \Gamma_i \) (disjoint union) and from the proof of the Proposition 3.14 we get the following relations:

\[
\bar{X}_s = \sum_{i=1}^{q} E^i_s; \quad \bar{Y}_s = \sum_{i=1}^{q} F^i_s, \quad \forall s \in \bar{I}, \tag{3.5}
\]

\[
\bar{H}_s = \gamma_s = [\bar{X}_s, \bar{Y}_s] = \sum_{i=1}^{q} [E^i_s, F^i_s] = \sum_{i=1}^{q} H^i_s, \quad \forall s \in \bar{I}. \tag{3.6}
\]

**Lemma 3.22.** Let \( s \in \bar{I} \) and \( i \in \{1, 2, \ldots, q\} \) such that \( \Gamma_i \neq \emptyset \). Then we have

1) \( \Gamma^i_t \neq \emptyset \) for all \( t \in \bar{I} \) satisfying \( \langle \gamma_t, \gamma_i \rangle < 0 \).

2) \( \Gamma_i^\prime \neq \emptyset \), \( \forall t \in \bar{I} \).

**Proof.** To prove 1), suppose \( \Gamma_i^t = \emptyset \) for any \( t \) satisfying \( \langle \gamma_t, \gamma_i \rangle < 0 \). Let \( k \in \Gamma_s \), then \( \langle \gamma_s, \gamma_i \rangle = \sum_{j=1}^{q} \langle \alpha_k, H^j_t \rangle = 0 \), a contradiction since \( \langle \gamma_s, \gamma_i \rangle \) must be negative. Thus \( \Gamma_i^\prime \neq \emptyset \) if \( \Gamma_i^t \neq \emptyset \). The second statement follows from the connectedness of \( \bar{I} \) : For \( t \in \bar{I} \), there exists a sequence \( s_0 = s, s_1, \ldots, s_n = t \) in \( \bar{I} \) such that \( s_j \) is linked to \( s_{j+1} \) for all \( j = 0, 1, \ldots, n - 1 \). By 1) \( \Gamma_{s_j} \), is as \( \Gamma_i^\prime \), non-empty for all \( j = 0, 1, \ldots, n \). In particular \( \Gamma_i^t \neq \emptyset \). \( \blacksquare \)

**Lemma 3.23.** \( \Gamma_i^\prime \neq \emptyset \), \( \forall s \in \bar{I} \), and \( (H^i_s)_{s \in \bar{I}} \) is free for all \( i = 1, 2, \ldots, q \).

**Proof.** Recall that \( I_{re} = \bigcup_{k \in \mathcal{I}_{re}} I_k \), with all the \( I_k \) connected. Let \( i \in \{1, 2, \ldots, q\} \) and let \( k \in I_{re}^i \) such that \( I_k \subseteq I_{re}^i \). Let \( s \in \bar{I} \) such that \( \rho_\alpha(\alpha_k) = \gamma_s \), then \( k \in \Gamma_s^i \) and \( \Gamma^s_i \neq \emptyset \). By the Lemma 3.22, \( \Gamma^s_i \neq \emptyset \) for all \( t \in I \). Thus \( H^s_i \neq 0 \), \( \forall s \in I \); \( \forall i = 1, 2, \ldots, q \), and the freeness of \( (H^i_s)_{s \in \bar{I}} \) follows from that of \( (H_k)^{\bar{I}_{re}} \). \( \blacksquare \)

**Proposition 3.24.** For \( i = 1, 2, \ldots, q \), let \( p_i \) be the projection of \( g(I_{re}) \) on \( g(I_{re}^i) \) with kernel \( \bigoplus_{j \neq i} g(I_{re}^j) \) and let \( m_i := p_i(m) \). Then we have:

1) \( m_i \) is a Kac-Moody subalgebra of \( g(I_{re}^i) \) isomorphic to \( m \).

2) The Kac-Moody subalgebra \( g(I_{re}^i) \) is maximally finitely \( \Sigma_i \)-graded, where \( \Sigma_i \) is the root system of \( m_i \), relative to the Cartan subalgebra \( a_i := p_i(a) \).

**N.B.** Note that \( m \) is contained in \( \bigoplus_{i=1}^{q} m_i \). In particular, \( \bigoplus_{i=1}^{q} m_i \) is finitely \( \Sigma \)-graded.

If we identify \( \bigoplus_{i=1}^{q} m_i \) with \( m^q \), then the grading subalgebra \( m \) can be viewed as the diagonal subalgebra \( \Delta(m^q) \) of \( m^q \) : \( \Delta(m^q) := \{(X, X, \ldots, X) : X \in m \} \).

**Proof.** For \( i \in \{1, 2, \ldots, q\} \), \( p_i \) is a morphism of Lie algebras and \( m_i := p_i(m) \) is contained in \( g(I_{re}^i) \). For \( s \in I \), one has \( p_i(\gamma_s) = H^s_i \). Thus the restriction of \( p_i \) to \( a' := [a, a] = \bigoplus_{s \in \bar{I}} \mathbb{C} \gamma_s \) is injective by Lemma 3.23. Since \( m \) is indecomposable, \( p_i \) when restricted to \( m \) is still injective (see [12, 1.7]). Thus \( m_i = p_i(m) \) is isomorphic to \( m \) and we have the following commutative diagram:
For the second assertion, Let $a_i := p_i(a)$ and $\Sigma_i = \Delta(m_i, a_i)$. When restricted to $\mathfrak{m}$, $p_i$ induces an isomorphism of root systems $\psi_i : \Sigma_i \to \Sigma$ such that

$$\langle \alpha, a \rangle = \langle \psi_i^{-1}(\alpha), p_i(a) \rangle, \quad \forall \alpha \in \Sigma, \forall a \in \mathfrak{a}.$$ 

Note that for $\alpha \in \Sigma$ and $X \in \mathfrak{g}(I_{re})$ satisfying $[a, X] = \langle \alpha, a \rangle X$, $\forall a \in \mathfrak{a}$, one has $[a_i, p_i(X)] = \langle \psi_i^{-1}(\alpha), a_i \rangle p_i(X)$, $\forall a_i \in \mathfrak{a}_i$. Since $p_i$ is surjective and $\mathfrak{g}(I_{re})$ (resp. $\mathfrak{g}(I_{re})^{J_{re}}$) is finitely $\Sigma_i$-graded, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ (resp. $\mathfrak{g}(I_{re})^{J_{re}}$) is also finitely $\Sigma_i$-graded. For $k \in \mathcal{J}_{re}$, Let $\rho_k(\alpha_k)$ be the restriction of $\alpha_k$ to $\mathfrak{a}_k$. Then $(\rho_i(\alpha_k) = 0) \iff (\rho_a(\alpha_k) = 0) \iff (k \in \mathcal{J}_{re})$. By Proposition 3.21, $\mathfrak{g}(I_{re})^{J_{re}}$ is maximally finitely $\Sigma_i$-graded.

**Corollary 3.25.** If $\mathfrak{g}$ is Lorentzian then $I_{re}$ is connected.

**Proof.** If $\mathfrak{g}$ is Lorentzian, then by Proposition 3.6, the grading subalgebra $\mathfrak{m}$ and hence all the $\mathfrak{m}_i$ $(i = 1, 2, \ldots, q)$ are also Lorentzian. When restricted to $\bigoplus_{i=1}^{q} \mathfrak{a}_i$, the invariant bilinear form $(\cdot, \cdot)$ is still non-degenerate and has signature $(q(r - 1), q)$, where $r$ is the common rank of the $\mathfrak{m}_i$, $i = 1, 2, \ldots, q$. Hence $q = r$ and $I_{re}$ is connected.

**Proposition 3.26.** If $\mathfrak{g}$ is of finite, affine or hyperbolic type, then any finite gradation is real: $I_{im}^i = \emptyset$ and $(I, J)$ is a $C$–admissible pair.

**Proof.** The result is trivial if $\mathfrak{g}$ is of finite type. Suppose $I_{im}^i \neq \emptyset$ for one of the other cases. If $\mathfrak{g}$ is of affine type, then $I_{re}$ is of finite type and by Lemma 3.19, $\mathfrak{m}$ is contained in the finite dimensional semi-simple Lie algebra $\mathfrak{g}(I_{re})$. This contradicts the fact that $\mathfrak{m}$ is, as $\mathfrak{g}$, of affine type (see Proposition 3.6). If $\mathfrak{g}$ is hyperbolic, then it is Lorentzian and perfect (cf. section 1). By Lemma 3.20 and Corollary 3.25, $\mathfrak{g}(I_{re})$ is an indecomposable finitely $\Sigma_i$-graded Kac-Moody subalgebra of $\mathfrak{g}$. As $I_{re}$ is assumed to be a proper connected subset of $I$, $\mathfrak{g}(I_{re})$ is of finite or affine type, a contradiction since, by Proposition 3.6, $\mathfrak{m}$ must be Lorentzian. Hence $I_{im}^i = \emptyset$ in the two last cases.

**Proposition 3.27.** If $\mathfrak{g}$ is hyperbolic, then the grading subalgebra $\mathfrak{m}$ is also hyperbolic.

**Proof.** Recall that in this case $I_{re} = I$ (see Proposition 3.26 and Corollary 3.15). Let $\tilde{I}$ be a proper subset of $I$ and suppose that $\tilde{I}$ is connected.
Let \( I^1 = \bigcup_{s \in I^1} (\bigcup_{k \in \Gamma_s} I_k) \). Then, \( I^1 \) is a proper subset of \( I \). We may assume that the subalgebra \( m(I^1) \) of \( m \) is contained in \( g(I^1) \). Let \( \Sigma^1 := \Sigma(I^1) \) be the root system of \( m(I^1) \). Then, it is not difficult to check that \( g(I^1) \) is finitely \( \Sigma^1 \)-graded. The argument used in Proposition 3.24 shows that the indecomposable components of \( g(I^1) \) (which all are of finite or affine type) are finitely \( \Sigma^1 \)-graded. By Proposition 3.6, \( m(I^1) \) is of finite or affine type. Hence \( m \) is hyperbolic. \( \blacksquare \)

**Corollary 3.28.** The problem of classification of finite real gradations of \( g \) comes down first to classify the \( C \)-admissible pairs \((I,J)\) of \( g \) and then the maximal finite gradations of the corresponding admissible algebra \( g^J \). When \( g \) is of finite, affine or hyperbolic type, we get thus all finite gradations.

**Proof.** This follows from Proposition 3.26, Proposition 3.21 and Lemma 1.5. \( \blacksquare \)

### 4. Maximal gradations

We assume now moreover that \( g \) is maximally finitely \( \Sigma \)-graded. We keep the notations in section 3 but we have \( J = I^\prime_{\text{im}} = \emptyset \). So \( \overline{T} \) is a quotient of \( I \), with quotient map \( \rho \) defined by \( \rho_s(\alpha_k) = \gamma_{\rho(k)} \). For \( s \in \overline{T}, \Gamma_s = \rho^{-1}(\{s\}) \).

**Proposition 4.1.**

1) If \( k \neq l \in I \) and \( \rho(k) = \rho(l) \), then there is no link between \( k \) and \( l \) in the Dynkin diagram of \( A : \alpha_k(\alpha_l^\vee) = \alpha_l(\alpha_k^\vee) = 0 \) and \( (\alpha_k, \alpha_l) = 0 \).
2) \( \mathfrak{a} \subset \{ h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h) \} \) whenever \( \rho(k) = \rho(l) \).
3) For good choices of the simple coroots and Chevalley generators \( (\alpha_k^\vee, e_k, f_k)_{k \in I} \) in \( g \) and \( (\gamma_s^\vee, X_s, Y_s)_{s \in I^1} \) in \( m \), we have \( \gamma_s^\vee = \sum_{k \in \Gamma_s} \alpha_k^\vee \), \( X_s = \sum_{k \in \Gamma_s} e_k \) and \( Y_s = \sum_{k \in \Gamma_s} f_k \).
4) In particular, for \( s, t \in \overline{T}, \) we have \( \gamma_s(\gamma_t^\vee) = \sum_{k \in \Gamma_t} \alpha_i(\alpha_k^\vee) \) for any \( i \in \Gamma_s \).

**Proof.** Assertions 1) and 2) are proved in 3.14 and 3.18. For \( i \in \Gamma_s, \gamma_s = \rho_s(\alpha_i) \) is the restriction of \( \alpha_i \) to \( \mathfrak{a} \); so 4) is a consequence of 3).

For 3) recall the proof of Proposition 3.14. The \( sl_2 \)-triple \( (X_s, \gamma_s^\vee, Y_s) \) may be written \( \gamma_s^\vee = \sum_{k \in \Gamma_s} H_k, X_s = \sum_{k \in \Gamma_s} E_k \) and \( Y_s = \sum_{k \in \Gamma_s} F_k \) where \( (E_k, H_k, F_k) \) is an \( sl_2 \)-triple in \( g(I^1) \), with \( \alpha_k(H_k) = 2 \). But now \( J = I^\prime_{\text{im}} = \emptyset \), so \( I^1_k = \{k\} \) and \( g(I^1_k) = \mathbb{C} e_k \oplus \mathbb{C} \alpha_k^\vee \oplus \mathbb{C} f_k \), hence the result. \( \blacksquare \)

So the grading subalgebra \( m \) may be entirely described by the quotient map \( \rho \). We look now to the reciprocal construction.

So \( g \) is an indecomposable and symmetrizable Kac-Moody algebra associated to a generalized Cartan matrix \( A = (a_{i,j})_{i,j \in I} \). We consider a quotient \( \overline{T} \) of \( I \) with quotient map \( \rho : I \to \overline{T} \) and fibers \( \Gamma_s = \rho^{-1}(\{s\}) \) for \( s \in \overline{T} \). We suppose that \( \rho \) is an admissible quotient i.e. that it satisfies the following two conditions:

(MG1) If \( k \neq l \in I \) and \( \rho(k) = \rho(l) \), then \( a_{k,l} = \alpha_l(\alpha_k^\vee) = 0 \).

(MG2) If \( s \neq t \in \overline{T} \), then \( \overline{a}_{s,t} := \sum_{i \in \Gamma_s} a_{i,j} = \sum_{i \in \Gamma_s} \alpha_j(\alpha_i^\vee) \) is independent of the choice of \( j \in \Gamma_t \).
The matrix $\overline{A} = (\overline{a}_{s,t})_{s,t \in \mathcal{T}}$ is an indecomposable generalized Cartan matrix.

**Proof.** Let $s \neq t \in \overline{I}$ and let $j \in \Gamma_t$. By (MG1) one has $\overline{a}_{t,t} = \sum_{i \in \Gamma_t} a_{i,j} = a_{j,j} = 2$, and by (MG2) $\overline{a}_{s,t} := \sum_{i \in \Gamma_t} a_{i,j} \in \mathbb{Z}^- \ (\forall j \in \Gamma_t)$. Moreover, $\overline{a}_{s,t} = 0$ if and only if $a_{i,j} = 0 (= a_{j,i}), \ \forall (i,j) \in \Gamma_s \times \Gamma_t$. It follows that $\overline{a}_{s,t} = 0$ if and only if $\overline{a}_{t,s} = 0$, and $\overline{A}$ is a generalized Cartan matrix. Since $\overline{A}$ is indecomposable, $\overline{A}$ is also indecomposable. 

Let $\mathfrak{h}^\Gamma = \{ h \in \mathfrak{h} \ | \ \alpha_k(h) = \alpha_l(h) \ \text{whenever} \ \rho(k) = \rho(l) \}$, $\gamma^\vee_s = \sum_{k \in \Gamma_s} \alpha^\vee_k$ and $\mathfrak{a}' = \oplus_{s \in \mathcal{T}} \mathbb{C} \gamma^\vee_s \subset \mathfrak{h}^\Gamma$. We may choose a subspace $\mathfrak{a}''$ in $\mathfrak{h}^\Gamma$ such that $\mathfrak{a}'' \cap \mathfrak{a}' = \{0\}$, the restrictions $\overline{a}_i =: \gamma^\vee_{\rho(i)}$ to $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$ of the simple roots $\alpha_i$ (corresponding to different $\rho(i) \in \overline{I}$) are linearly independent and $\mathfrak{a}''$ is minimal for these two properties.

Proposition 4.3. $(\mathfrak{a}, \{ \gamma_s \ | \ s \in \mathcal{T} \}, \{ \gamma_s^\vee \ | \ s \in \mathcal{T} \})$ is a realization of $\overline{A}$.

**Proof.** Let $\ell$ be the rank of $\overline{A}$. Note that $\mathfrak{a}$ contains $\mathfrak{a}' = \oplus_{s \in \mathcal{T}} \mathbb{C} \gamma^\vee_s$; the family $(\gamma_s)_{s \in \mathcal{T}}$ is free in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and satisfies $\langle \gamma^\vee_t, \gamma^\vee_t \rangle = \overline{a}_{s,t}$, $\forall s,t \in \overline{I}$. It follows that dim($\mathfrak{a}$) $\geq 2|\overline{I}| - \ell$ (see [11, 14.1] or [12, Ex. 1.3]). As $\mathfrak{a}$ is minimal, we have dim($\mathfrak{a}$) $= 2|\overline{I}| - \ell$ (see [11, 14.2] for minimal realization). Hence $(\mathfrak{a}, \{ \gamma_s \ | \ s \in \mathcal{T} \}, \{ \gamma_s^\vee \ | \ s \in \mathcal{T} \})$ is a (minimal) realization of $\overline{A}$.

We note $\Delta^\rho = \sum_{s \in \mathcal{T}} \mathbb{Z} \gamma_s$ the root system associated to this realization.

We define now $\overline{X}_s = \sum_{k \in \Gamma_s} e_k$ and $\overline{Y}_s = \sum_{k \in \Gamma_s} f_k$. Let $\mathfrak{m} = \mathfrak{g}^\rho$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}$ and the elements $\overline{X}_s, \overline{Y}_s$ for $s \in \mathcal{T}$.

Proposition 4.4. The Lie subalgebra $\mathfrak{m} = \mathfrak{g}^\rho$ is the Kac-Moody algebra associated to the realization $(\mathfrak{a}, \{ \gamma_s \ | \ s \in \mathcal{T} \}, \{ \gamma_s^\vee \ | \ s \in \mathcal{T} \})$ of $\overline{A}$. Moreover, $\mathfrak{g}$ is an integrable $\mathfrak{g}^\rho$–module with finite multiplicities.

**Proof.** Clearly, the following relations hold in the Lie subalgebra $\mathfrak{g}^\rho$:

\[
\begin{align*}
[a, a] &= 0, \\
[a, \overline{X}_s] &= \langle \gamma_s, a \rangle \overline{X}_s, \\
[a, \overline{Y}_t] &= \delta_{s,t} \gamma^\vee_s (s, t \in \overline{I}); \\
[a, \overline{Y}_s] &= -\langle \gamma_s, a \rangle \overline{Y}_s (a \in \mathfrak{a}, s \in \overline{I}).
\end{align*}
\]

For the Serre’s relations, one has:

\[1 - \overline{a}_{s,t} \geq 1 - a_{i,j}, \ \forall (i,j) \in \Gamma_s \times \Gamma_t.\]

In particular, one can see, by induction on $|\Gamma_s|$, that:

\[(\mathrm{ad} \overline{X}_s)^{1 - \overline{a}_{s,t}}(e_j) = \sum_{i \in \Gamma_s} \mathrm{ad} e_i^{1 - \overline{a}_{s,t}}(e_j) = 0, \ \forall j \in \Gamma_t.\]

Hence:

\[(\mathrm{ad} \overline{X}_s)^{1 - \overline{a}_{s,t}}(\overline{X}_t) = 0, \ \forall s, t \in \overline{I},\]
and in the same way we obtain that:

$$(\text{ad}Y_s)^{1-n_s-t}(Y_t) = 0, \forall s, t \in I.$$ 

It follows that $\mathfrak{g}^\rho$ is a quotient of the Kac-Moody algebra $\mathfrak{g}(\mathcal{A})$ associated to $\mathcal{A}$ and $(\alpha, \{\gamma_s | s \in I\}, \{\gamma_s^\gamma | s \in I\})$ in which the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}(\mathcal{A})$ is embedded. By [12, 1.7] $\mathfrak{g}^\rho$ is equal to $\mathfrak{g}(\mathcal{A})$.

It’s clear that $\mathfrak{g}$ is an integrable $\mathfrak{g}^\rho$-module with finite dimensional weight spaces relative to the adjoint action of $\mathfrak{a}$, since for $\alpha = \sum_{i \in I} n_i \alpha_i \in \Delta^+$, its restriction $\rho_\alpha(\alpha)$ to $\mathfrak{a}$ is given by

$$\rho_\alpha(\alpha) = \sum_{s \in I} \left( \sum_{i \in \Gamma_s} n_i \right) \gamma_s$$

(4.1)

**Proposition 4.5.** The Kac-Moody algebra $\mathfrak{g}$ is maximally finitely $\Delta^\rho$-graded with grading subalgebra $\mathfrak{g}^\rho$.

**Proof.** As in Theorem 2.14, we will see that $\rho_\alpha(\Delta^+) \subset Q^\Gamma_+: = \bigoplus_{s \in I} \mathbb{Z}^+ \gamma_s$ satisfies, as $\Sigma^+ = \Delta^\rho_+$, the following conditions:

(i) $\gamma_s \in \rho_\alpha(\Delta^+) \subset Q^\Gamma_+, 2\gamma_s \notin \rho_\alpha(\Delta^+), \forall s \in I$.

(ii) if $\gamma \in \rho_\alpha(\Delta^+), \gamma \neq \gamma_s$, then the set $\{\gamma + k \gamma_s; k \in \mathbb{Z}\} \cap \rho_\alpha(\Delta^+)$ is a string $\{\gamma - p \gamma_s, ..., \gamma + q \gamma_s\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = (\gamma, \gamma_s^\gamma)$;

(iii) if $\gamma \in \rho_\alpha(\Delta^+)$, then $\text{supp}(\gamma)$ is connected.

Clearly $\{\gamma_s | s \in I\} \subset \rho_\alpha(\Delta^+) \subset Q^\Gamma_+$. For $\alpha \in \Delta$ and $s \in I$, the condition $\rho_\alpha(\alpha) \in \mathbb{N} \gamma_s$ implies $\alpha \in \Delta(\Gamma_s)^+ = \{\alpha_i; i \in \Gamma_s\}$ [see (4.1)]. It follows that $2\gamma_s \notin \rho_\alpha(\Delta^+)$ and (i) is satisfied. By Proposition 4.4, $\mathfrak{g}$ is an integrable $\mathfrak{g}^\rho$-module with finite multiplicities. Hence, the propriety (ii) follows from [12, 3.6]. Let $\alpha \in \Delta_+$ and let $s, t \in \text{supp}(\rho_\alpha(\alpha))$. By (4.1) there exists $(k, l) \in \Gamma_s \times \Gamma_t$ such that $k, l \in \text{supp}(\alpha)$, which is connected. Hence there exist $i_0 = k, i_1, ..., i_{n+1} = l$ such that $a_{ij} \in \text{supp}(\alpha), j = 0, 1, ..., n + 1$, and for $j = 0, 1, ..., n, i_j$ and $i_{j+1}$ are linked relative to the generalized Cartan matrix $\mathcal{A}$. In particular, $\rho(i_j) \neq \rho(i_{j+1}) \in \text{supp}(\rho_\alpha(\alpha))$ and they are linked relative to the generalized Cartan matrix $\bar{\mathcal{A}}, j = 0, 1, ..., n$, with $\rho(i_0) = s$ and $\rho(i_{n+1}) = t$. Hence the connectedness of $\text{supp}(\rho_\alpha(\alpha))$ relative to $\bar{\mathcal{A}}$. It follows that $\rho_\alpha(\Delta^+) = \Delta^\rho_+$ and hence $\rho_\alpha(\Delta) = \Delta^\rho$ (see [12, Ex. 5.4]). In particular, $\mathfrak{g}$ is finitely $\Delta^\rho$-graded with $J = \emptyset = J^\prime_{\text{im}}$.

**Corollary 4.6.** The restriction to $\mathfrak{m} = \mathfrak{g}^\rho$ of the invariant bilinear form $(., .)$ of $\mathfrak{g}$ is non-degenerate. In particular, the generalized Cartan matrix $\bar{\mathcal{A}}$ is symmetrizable of the same type as $\mathcal{A}$.

**Proof.** The first part of the corollary follows from Proposition 4.5 and Corollary 3.17. The second part follows from Proposition 3.6.

**Remark 4.7.** The map $\rho$ coincides with the map (also denoted $\rho$) defined at the beginning of this section using the maximal gradation of Proposition 4.5. Conversely, Proposition 4.1 tells that, for a general maximal finite gradation, $\rho$ is
admissible and $m = g^\rho$ for good choices of the Chevalley generators. So we get a good correspondence between maximal gradations and admissible quotient maps.

By Corollary 3.28 the real finite gradations of a Kac-Moody algebra $g$ are bijectively associated to pairs of a $C-$admissible pair $(I, J)$ and an admissible quotient map $\rho : I = I \setminus J \rightarrow \bar{I}$.

5. An example

The following example shows that imaginary gradations do exist. It shows in particular that, for a generalized $C-$admissible pair $(I, J)$, $J^\circ$ may be non-empty and $I_{re}$ may be non-connected. Moreover, the Kac-Moody algebra $g$ may be not graded by the root system of $g(I_{re})$.

The imaginary gradations will be studied in a forthcoming paper [7].

Example 5.1. Consider the Kac Moody algebra $g$ corresponding to the indecomposable and symmetric generalized Cartan matrix $A$:

$$A = \begin{pmatrix}
2 & -3 & -1 & 0 & 0 & 0 \\
-3 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & -1 & -1 & -1 \\
0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & -3 \\
0 & 0 & -1 & 0 & -3 & 2 \\
\end{pmatrix}$$

with the corresponding Dynkin diagram:

```
3 1
3 2
\(\bullet\)
3 4
3 5
3 6
```

Note that det($A$) = 275 and the symmetric submatrix of $A$ indexed by \{1, 2, 4, 5, 6\} has signature $(+++, --)$. Since det($A$) > 0, the matrix $A$ should have signature $(+++, --)$. Let $\Sigma$ be the root system associated to the strictly hyperbolic generalized Cartan matrix \(\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}\), the corresponding Dynkin diagram is the following:

```
H_{3,3}
\(\bullet\)
\(\bullet\)
\(\bullet\)
```

We will see that $g$ is finitely $\Sigma-$graded and describe the corresponding generalized $C-$admissible pair.

1) Let $\tau$ be the involutive diagram automorphism of $g$ such that $\tau(1) = 5$, $\tau(2) = 6$ and $\tau$ fixes the other vertices. Let $\sigma_n^\tau$ be the normal semi-involution of $g$ corresponding to the split real form of $g$. Consider the quasi-split real form $g_{\mathbb{R}}^1$ associated to the semi-involution $\tau \sigma_n^\tau$ (see [2] or [6]). Then $t_{\mathbb{R}} := h_{\mathbb{R}}$ is a maximal split toral subalgebra of $g_{\mathbb{R}}^1$. The corresponding restricted root system $\Delta' := \Delta(g_{\mathbb{R}}, t_{\mathbb{R}})$ is reduced and the corresponding generalized Cartan matrix $A'$ is...
with the corresponding Dynkin diagram:

Following N. Bardy [4, 9], there exists a split real Kac-Moody subalgebra \( m_R^1 \) of \( g_R^1 \) containing \( t_R \) such that \( \Delta' = \Delta(m_R^1, t_R) \). It follows that \( g \) is finitely \( \Delta' \)-graded.

2) Let \( m^1 := m_R^1 \otimes \mathbb{C} \) and \( t := t_R \otimes \mathbb{C} \). Denote by \( \alpha'_i := \alpha_i/t, i = 1, 2, 3, 4 \). Put \( \alpha'_1 = \alpha_1 + \alpha_5, \alpha'_2 = \alpha_2 + \alpha_6, \alpha'_3 = \alpha_3 \) and \( \alpha'_4 = \alpha_4 \). Let \( I^1 := \{1, 2, 3, 4\} \), then \( (t, \Pi' = \{\alpha'_i, i \in I^1\}, \Pi'^\vee = \{\alpha'_i, i \in I^1\}) \) is a realization of \( A' \) which is symmetrizable and Lorentzian.

Let \( m \) be the Kac-Moody subalgebra of \( m^1 \) corresponding to the submatrix \( \tilde{A} \) of \( A' \) indexed by \( \{1, 2\} \). Thus \( \tilde{A} = \begin{pmatrix} 2 & -3 & -1 & 0 \\ -3 & 2 & -1 & 0 \\ -1 & -1 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{pmatrix} \) is strictly hyperbolic. Let \( a := \mathbb{C}\alpha'_1 + \mathbb{C}\alpha'_2 \) be the standard Cartan subalgebra of \( m \) and let \( \Sigma = \Delta(m, a) \). For \( \alpha' \in \mathfrak{t}^* \), denote by \( \rho_1(\alpha') \) the restriction of \( \alpha' \) to \( a \). Put \( \gamma_s = \rho_1(\alpha'_s), \gamma_s' = \alpha'_s \), \( s = 1, 2 \). Then \( \Pi_a = \{\gamma_1, \gamma_2\} \) is the standard root basis of \( \Sigma \). One can see easily that \( \rho_1(\alpha'_1) = 0 \) and \( \rho_1(\alpha'_3) = 2(\gamma_1 + \gamma_2) \) is a strictly positive imaginary root of \( \Sigma \). Now we will show that \( m^1 \) is finitely \( \Sigma \)-graded.

Let \( \langle ., . \rangle_1 \) be the normalized invariant bilinear form on \( m^1 \) such that short real roots have length 1 and long real roots have square length 2. Then there exists a positive rational \( q \) such that the restriction of \( \langle ., . \rangle_1 \) to \( t \) has the matrix \( B_1 \) in the basis \( \Pi'^\vee \), where:

\[
B_1 = q \begin{pmatrix} 2 & -3 & -1 & 0 \\ -3 & 2 & -1 & 0 \\ -1 & -1 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}
\]

By duality, the restriction of \( \langle ., . \rangle_1 \) to \( t \) induces a non-degenerate symmetric bilinear form on \( \mathfrak{t}^* \) (see [12, 2.1]) such that its matrix \( B'_1 \) in the basis \( \Pi' \), is the following:

\[
B'_1 = q^{-1} \begin{pmatrix} 2 & -3 & -2 & 0 \\ -3 & 2 & -2 & 0 \\ -2 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}
\]

Hence, \( q \) equals 2.

Note that for \( \alpha' = \sum_{i=1}^4 n_i \alpha'_i \in \Delta'^+ \), we have that

\[
\langle \alpha', \alpha' \rangle_1 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2 - 3n_1n_2 - 2n_1n_3 - 2n_2n_3 - 2n_3n_4.
\]

We will show that \( \rho_1(\Delta'^+) = \Sigma^+ \cup \{0\} \). Note that \( \Sigma \) can be identified with \( \Delta' \cap (\mathbb{Z}\alpha'_1 + \mathbb{Z}\alpha'_2) \); hence \( \rho_1 \) is injective on \( \Sigma \) and \( \Sigma^+ \subset \rho_1(\Delta'^+) \).
Let $(\ldots)_a$ be the normalized invariant bilinear form on $\mathfrak{m}$ such that all real roots have length 2. Then the restriction of $(\ldots)_a$ to $\mathfrak{a}$ has the matrix $B_a$ in the basis $\Pi_a = \{\gamma_1, \gamma_2\}$, where:

$$B_a = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

Since $\bar{A}$ is symmetric, the non-degenerate symmetric bilinear form, on $\mathfrak{a}^*$, induced by the restriction of $(\ldots)_a$ to $\mathfrak{a}$, has the same matrix $B_a$ in the basis $\Pi_a$. In particular, we have that:

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a = 2[(n_1 + 2n_3)^2 + (n_2 + 2n_3)^2 - 3(n_1 + 2n_3)(n_2 + 2n_3)],$$

since $\rho_1(\alpha') = (n_1 + 2n_3)\gamma_1 + (n_1 + 2n_3)\gamma_2$.

Using (5.1), it is not difficult to check that

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a = 2[(\alpha', \alpha')_1 - (n_3 - n_4)^2 - 5n_3^2 - n_4^2]$$

(5.2)

Suppose $n_3 = 0$, then, since $\text{supp}(\alpha')$ is connected, we have that $\alpha' = n_1\alpha'_1 + n_2\alpha'_2$ or $\alpha' = \alpha'_1$. Hence $\rho_1(\alpha') = n_1\gamma_1 + n_2\gamma_2 \in \Sigma$ or $\rho_1(\alpha') = 0$.

Suppose $n_3 \neq 0$, then, since $(\alpha', \alpha')_1 \leq 2$, one can see, using (5.2), that

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a < 0.$$ 

As $\Sigma$ is hyperbolic and $\rho_1(\alpha') \in \mathbb{N}\gamma_1 + \mathbb{N}\gamma_2$, we deduce that $\rho_1(\alpha')$ is a positive imaginary root of $\Sigma$ (see [12, 5.10]). It follows that $\rho_1(\Delta'^+) = \Sigma^+ \cup \{0\}$.

To see that $\mathfrak{m}^1$ is finitely $\Sigma-$graded, it suffices to prove that, for $\gamma = m_1\gamma_1 + m_2\gamma_2 \in \Sigma^+ \cup \{0\}$, the set $\{\alpha' \in \Delta'^+, \rho_1(\alpha') = \gamma\}$ is finite. Note that if $\alpha' = \sum_{i=1}^4 n_i\alpha'_i \in \Delta'^+$ satisfying $\rho_1(\alpha') = \gamma$, then $n_1 + 2n_3 = m_i$, $i = 1, 2$. In particular, there are only finitely many possibilities for $n_i$, $i = 1, 2, 3$. The same argument as the one used in the proof of Proposition 2.13 shows also that there are only finitely many possibilities for $n_4$.

3) Recall that $\mathfrak{m} \subset \mathfrak{m}^1 \subset \mathfrak{g}$. The fact that $\mathfrak{g}$ is finitely $\Delta'-$graded with grading subalgebra $\mathfrak{m}^1$ and $\mathfrak{m}^1$ is finitely $\Sigma-$graded implies that $\mathfrak{g}$ is finitely $\Sigma-$graded (cf. Lemma 1.5). Let $I = \{1, 2, 3, 4, 5, 6\}$, then the root basis $\Pi_{\mathfrak{g}}$ of $\Sigma$ is adapted to the root basis $\Pi$ of $\Delta$ and we have $I_{re} = \{1, 2, 5, 6\}$ (not connected), $\Gamma_1 = \{1, 5\}$, $\Gamma_2 = \{2, 6\}$, $J = \{4\}$, $J_{re} = \emptyset$, $I'_{im} = \{3\}$ and $J^o = J = \{4\}$.

Note that, for this example, $\mathfrak{g}(I_{re})$, which is $\Sigma-$graded, is isomorphic to $\mathfrak{m} \times \mathfrak{m}$. This gradation corresponds to that of the pseudo-complex real form of $\mathfrak{m} \times \mathfrak{m}$ (i.e. the complex Kac-Moody algebra $\mathfrak{m}$ viewed as real Lie algebra) by its restricted reduced root system. Since the pair $(I_3, J_3) = (\{3, 4\}, \{4\})$ is not admissible, it is not possible to build a Kac-Moody algebra $\mathfrak{g}^1$ grading finitely $\mathfrak{g}$ and maximally finitely $\Sigma-$graded.

Acknowledgments. We thank the anonymous referee for his/her valuable comments and suggestions.
References

[1] Allison, B., G. Benkart and Y. Gao, *Central extensions of Lie algebras graded by finite root systems*, Math. Ann. 316 (2000), 499–527.

[2] Back-Valente, V., N. Bardy-Panse, H. Ben Messaoud and G. Rousseau, *Formes presque déployées d’algèbres de Kac-Moody, Classification et racines relatives*, J. of Algebra 171 (1995), 43–96.

[3] Bardy-Panse, N., «Systèmes de racine infinis», Mémoire de la S.M.F 65, 1996.

[4] Bardy, N., *Sous-algèbres birégulières d’une algèbre de Kac-Moody-Borcherds*, Nagoya Math. J. 156 (1999), 1–83.

[5] Benkart, G., and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. 126 (1996), 1–45.

[6] Ben Messaoud, H., *Almost split real forms for hyperbolic Kac-Moody Lie algebras*, J. Phys A. Math. Gen 39 (2006), 13659–13690.

[7] Ben Messaoud, H., and N. Fradi, *Imaginary gradations of Kac-Moody Lie algebras*, In preparation.

[8] Berman, S. and R. Moody, *Lie algebras graded by finite root systems*, Invent. Math. 108 (1992), 323–347.

[9] Borcherds, R. E., *Generalized Kac-Moody algebras*, J. of Algebra 115 (1988), 501–512.

[10] Bourbaki, N., «Groupes et algèbres de Lie», Chap 4, 5 et 6», Masson, Paris, 1981.

[11] Carter, R., “Lie algebras of finite and affine type,” Cambridge University Press, 2005.

[12] Kac, V. G., “Infinite dimensional Lie algebras,” Third Edition, Cambridge University Press, 1990.

[13] Kac, V. G., and S. P. Wang, *On automorphisms of Kac-Moody algebras and groups*, Advances in Math. 92 (1992), 129–195.

[14] Moody, R. V., *Root systems of hyperbolic type*, Advances in Mathematics 33 (1979), 144–160.

[15] Neher, E., *Lie algebras graded by J—graded root systems and Jordan pairs covered by grids*, Amer. J. Math. 118 (1996), 439–491.

[16] Nervi, J., *Algèbres de Lie simples graduées par un système de racines et sous-algèbres C—admissibles*, J. of Algebra 223 (2000), 307–343.

[17] —, *Affine Kac-Moody algebras graded by affine root systems*, J. of Algebra 253 (2002), 50–99.
[18] Peterson, D. H., and V. G. Kac, \textit{Infinite flag varieties and conjugacy theorems}, Proc. Natl. Acad. Sc. USA \textbf{80} (1983), 1778–1782.

[19] Rémy, B., «Groupes de Kac-Moody déployés et presque déployés», Astérisque \textbf{277}, 2002.

[20] Rousseau, G., \textit{Groupes de Kac-Moody déployés sur un corps local, II Masures ordonnées}, ArXiv 1009.0135v2.

[21] Rubenthaler, H., \textit{Construction de certaines sous-algèbres remarquables dans les algèbres de Lie semi-simples}, J. of Algebra \textbf{81} (1983), 268–278.

[22] Tits, J., \textit{Uniqueness and presentation of Kac-Moody groups over fields}, J. of Algebra \textbf{105} (1987), 542–573.