Homotopy quantum field theory and the index gerbe

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November 1, 2018

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1 Introduction

In the recent paper [8] Turner and Willerton study the relation between gerbes with connection and thin-invariant rank-one field theories on a space $B$. Their main result is that if $H_1(B, \mathbb{Z})$ is torsion-free, then there is a one-to-one correspondence of gerbes with connection on $B$ and isomorphism classes of thin-invariant rank-one field theories on $B$. The gerbe corresponding to a thin-invariant rank-one field theory is given by the holonomy of the field theory.

In [6] Lott associates to an odd dimensional geometric family on $B$ (see [2] and Section 2 for this notion) an index gerbe.

The goal of the present paper is to analyze the construction of Turner and Willerton [8] in the case of the index gerbe. Starting from the geometric family we will construct a thin-invariant rank-one field theory on $B$ such that its holonomy is that of the index gerbe. Our construction works without any assumption on the manifold $B$.

If $H_1(B, \mathbb{Z})$ is torsion-free, then in view of [8], Thm. 3.5, the field theory which we construct in the present paper coincides with the one obtained by applying the construction of Turner and Willerton [8] to the index gerbe.

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For general $B$ the thin-invariant field theory associated to a geometric family seems to encode slightly more structure than the index gerbe of that family. The construction of the field theory given in the present paper induces a construction of the index gerbe which is independent of the constructions in [3] and [5].

2 The index gerbe

Let $\text{Gerbe}(B)$ denote the group of gerbes with connection on $B$. We refer to Hitchin [5] for a nice introduction to the geometric picture of gerbes. Lott’s index gerbe is constructed in [6] using Hitchin’s geometric picture. In the present paper we prefer to represent gerbes by Deligne cohomology classes or Cheeger-Simons differential characters.

Let $\mathcal{A}_B^q$ denote the sheaf of real smooth $q$-forms on $B$, $q \in \mathbb{N}_0$. We consider the complex of sheaves on $B$

$$\mathcal{K}(2,\mathbb{Z})_B : 0 \rightarrow \mathbb{Z}_B \rightarrow \mathcal{A}_B^0 \rightarrow \mathcal{A}_B^1 \rightarrow \mathcal{A}_B^2 \rightarrow 0,$$

where the constant sheaf $\mathbb{Z}_B$ sits in degree $-1$. The third Deligne cohomology of $B$ is by definition the second Čech hyper cohomology group of this complex

$$H^3_{\text{Del}}(B) := \check{H}^2(B, \mathcal{K}(2,\mathbb{Z})_B).$$

There is a natural isomorphism $\text{Gerbe}(B) \cong H^3_{\text{Del}}(B)$ (see Brylinski [1] and [5]).

Holonomy provides a natural isomorphism $H : H^3_{\text{Del}}(B) \cong \check{H}^2(B, U(1))$, where $\check{H}^2(B, U(1))$ is the group of Cheeger-Simons differential characters with values in $U(1)$ (see [5]). An element $\phi \in \check{H}^2(B, U(1))$ is a homomorphism from the group of smooth cycles in $B$ to $U(1)$ such that there exists a closed form $R^\phi \in \mathcal{A}_B^2(B, d = 0)$ with the property that

$$\phi(\partial C) = \exp \left(2\pi i \int_C R^\phi \right)$$

for all 3-chains $C$. The form $R^\phi$ is called the curvature of $\phi$. A construction of $H : H^3_{\text{Del}}(B) \cong \check{H}^2(B, U(1))$ was given in [3], Sec. 6.1. Thus we can represent gerbes in a third way, namely by Cheeger-Simons differential characters.

There is a natural homomorphism $R : H^3_{\text{Del}}(B) \rightarrow \mathcal{A}_B^3(B, d = 0)$, $x \mapsto R^x$, such that $R^x$ is the curvature of the Cheeger-Simons differential character $H(x)$. We call $R^x$ the curvature of $x$.

An odd dimensional geometric family $\mathcal{E}_{\text{geom}}$ over $B$ is given by (see [5], Sec.1.1)

- a fibre bundle $\pi : E \rightarrow B$ with closed odd dimensional fibres,
- a vertical orientation, a vertical spin structure, and a complex vector bundle $V$ over $E$
- a vertical Riemannian metric and a horizontal distribution for $\pi : E \rightarrow B$, a hermitian metric $h^V$ and a metric connection $\nabla^V$ on $V$ (the geometric structures).

By $\text{gerbe}(\mathcal{E}_{\text{geom}}) \in \text{Gerbe}(B)$ we denote the index gerbe constructed by Lott [5]. Let $\text{index}(\mathcal{E}) \in K^1(B)$ be the index of the family of selfadjoint twisted Dirac operators defined by $\mathcal{E}_{\text{geom}}$. We omit the subscript $\text{geom}$ since this index is independent of the choice of the geometric structures. Let $\text{ch}_1(\text{index}(\mathcal{E})) \in H^2_{\text{del}}(B)$ be the degree-one component of the Chern character in the de Rham cohomology of $B$. Under the assumption that $\text{ch}_1(\text{index}(\mathcal{E})) = 0$ in [5] we
constructed a natural class \( \text{index}^3_{\text{Del}}(\Omega_{\text{geom}}) \in H^3_{\text{Del}}(B) \) such that \( \text{index}^3_{\text{Del}}(\Omega_{\text{geom}}) = \Omega^3(\Omega_{\text{geom}}) \), where \( \Omega(\Omega_{\text{geom}}) = \int_{E/B} \tilde{A}(\nabla^T \pi) \text{ch}(\nabla) \) is the local index form (see [3], Sec. 3.2 for definitions). The class \( \text{index}^3_{\text{Del}}(\Omega_{\text{geom}}) \) coincides with \( \text{gerbe}(\Omega_{\text{geom}}) \) under the natural isomorphism \( \text{Gerbe}(B) \cong \hat{H}^2(B, U(1)) \).

The construction of the thin-invariant field theory given below will provide an independent construction of the index gerbe as an element in \( \hat{H}^2(B, U(1)) \).

### 3 Thin invariant field theories

The thin homotopy category \( \mathcal{T}_B \) of \( B \) is defined e.g. in [8]. Let \( S^1 \subset \mathbb{C} \) be the unit circle with a fixed orientation. By \( \bar{S}^1 \) we denote the circle with the opposite orientation. Let \( S_{m,n}, n, m \in \mathbb{N}_0 \), we denote the disjoint union of \( m \) copies of \( S^1 \) and \( n \) copies of \( \bar{S}^1 \). An object of \( \mathcal{T}_B \) is a smooth map \( \gamma : S_{m,0} \to B \).

Let \( I := [0,1] \) denote the unit interval with the canonical orientation. We consider the oriented surface \( C_{m,n} := I \times S_{m,n} \) which comes with a natural projection \( \text{pr} : C_{m,n} \to S_{m,n} \). By a surface \( \Sigma \) we mean a compact smooth oriented surface with boundary \( \partial \Sigma \) together with an orientation preserving collar \( \iota_2 : C_{m,n} \hookrightarrow \Sigma \) such that \( \iota_2(\{0\} \times S_{m,n}) = \partial \Sigma \). Here \( m, n \in \mathbb{N}_0 \). The image of \( S_{m,0} \) (resp. \( S_{0,n} \)) is called the ingoing (resp. outgoing) boundary of \( \Sigma \) and will be denoted by \( \partial^\text{in} \Sigma \) (resp. \( \partial^\text{out} \Sigma \)). By \( \iota_1^\text{in} (\Sigma) \) and \( \iota_1^\text{out} (\Sigma) \) we denote the set of connected components of \( \partial^\text{in} \Sigma \) and \( \partial^\text{out} \Sigma \).

A \( B \)-surface is a smooth map \( g : \Sigma \to B \) such that \( g \circ \iota_2 : C_{m,n} \to B \) factors over the projection \( \text{pr} \). In particular, a \( B \)-surface determines an ingoing object \( \gamma^\text{in} : S_{m,0} \to B \) and an outgoing object \( \gamma^\text{out} : S_{n,0} \to B \) such that \( g \circ \iota_2 = (\gamma \cup \gamma') \circ \text{pr} : C_{m,n} \to B \).

By a three manifold \( X \) we mean a compact three manifold with corners of codimension at most two and fixed collars. Let \( I_1(X) \) be the set of boundary faces \( \partial_i X, i \in I_1(X) \). Then \( \partial_i X, i \in I_1(X) \), should be a surface with collared boundary as above. The collars \( i_i X : I \times \partial_i X \to X \) are part of the data. Let \( I_2(X) \) denote the set of faces of codimension two. Let \( k \in I_2(X) \) and \( \partial_i X \) and \( \partial_j X \), \( i, j \in I_1(X) \), be the two boundary faces which meet at \( \partial_k X \). Then we require that \( i_{i,j} X \circ (\text{id} \times i_{k} X) = i_{j,k} X \circ (\text{id} \times i_{i} X) \circ (\text{id} \times i_{k} X) \) as maps from \( I^2 \times \partial k X \to X \), where \( \sigma \) interchanges the components in \( I^2 \). A \( B \)-three manifold is a smooth map \( h : X \to B \) such that the compositions \( h \circ i_{i,j} X : I \times \partial_j X \to B \) factor over the projections \( I \times \partial_j X \to \partial_i X \) and \( B \)-surfaces \( g_i : \partial_i X \to B \) for all \( i \in I_1(X) \).

Let \( \gamma^\text{in} : S_{m,0} \to B \) and \( \gamma^\text{out} : S_{n,0} \to B \) be two objects of \( \mathcal{T}_B \). A morphism \( \gamma^\text{in} \to \gamma^\text{out} \) is represented by a \( B \)-surface with ingoing object \( \gamma^\text{in} \) and outgoing object \( \gamma^\text{out} \). Two such \( B \)-surfaces \( g \) and \( g' \) define the same morphism, if they are thin homotopic, i.e. there is a homotopy \( h : I \times X \to B \) from \( g \) to \( g' \) such that \( \text{rank}(dh) \leq 2 \) everywhere. Here \( I \times X \) is a three manifold in the natural way, and \( h : I \times X \to B \) is required to be a \( B \)-three manifold.

Composition of morphisms is induced by glueing. The category \( \mathcal{T}_B \) has a product induced by disjoint union.

Let \( \text{vect}_1 \) denote the monoidal category of one dimensional complex vector spaces. A rank-one thin-invariant field theory (compare [8], Def. 3.2) is a monoidal functor \( E : \mathcal{T}_B \to \text{vect}_1 \) satisfying the following condition: There is a closed form \( R^E \in \mathcal{A}^3_B(B, d = 0) \) (the curvature of \( E \)) such that if \( h : C \to B \) is a \( B \)-three manifold with boundary \( g : \Sigma \to B \) such that \( \Sigma \) is closed, then

\[
E([g]) = \exp \left( 2\pi i \int_C h^* R^E \right).
\]

Here \([g]\) denotes the class of \( g \) in the endomorphisms of the empty object \( \Theta \) of \( \mathcal{T}_B \), and \( E([g]) \in \mathbb{C} \) under the natural identification of the linear endomorphisms of \( E(\Theta) \cong \mathbb{C} \) with \( \mathbb{C} \).
To any thin-invariant rank-one field theory $E$ with curvature $R^E$ on $B$ we can associate a gerbe on $B$ with the same curvature. If $g : \Sigma \to B$ is a closed $B$-surface, then let $[g, \Sigma]$ denote the corresponding cycle on $B$. The Cheeger-Simons differential character $\phi \in H^2(B, U(1))$ corresponding to the gerbe satisfies $\phi([g, \Sigma]) := E([g])$.

4 Taming

Let $\mathcal{E}_{\text{geom}}$ be an odd dimensional geometric family over $B$. By $D(\mathcal{E}_{\text{geom}}) := (D_b)_{b \in B}$ we denote the family of twisted Dirac operators given by $\mathcal{E}_{\text{geom}}$.

Let $g : M \to B$ be any smooth map from some manifold $M$ to $B$. A taming of the family $g^* \mathcal{E}_{\text{geom}}$ is given by a smooth family of selfadjoint smoothing operators $(Q_m)_{m \in M}$ such that $D_m + Q_m$ is invertible for all $m \in M$. It was shown in [7], that $g^* \mathcal{E}_{\text{geom}}$ admits a taming exactly if $\text{index}(g^* \mathcal{E}) = 0$ holds in $K^1(M)$.

Let $N \subset M$ and $\gamma : N \to B$ be the restriction of $g$ to $N$. Assume that we already have a taming $(\gamma^* \mathcal{E}_{\text{geom}})$. Then the obstruction against extending the taming to $M$ is the index element $\text{index}(g^* \mathcal{E}, (\gamma^* \mathcal{E}_{\text{geom}})) \in K^1(M, N)$. In order to construct this element we let $\mathcal{C}_N(M)$ be the algebra of continuous functions on $M$ which vanish on $N$. Let $U$ be some neighborhood of $N$ to which we can extend the family $(Q_n)_{n \in N}$ giving the taming $(\gamma^* \mathcal{E}_{\text{geom}})$. Let $\chi \in \mathcal{C}_N(M)$ be such that $\chi|_N = 1$ and $\chi|_{M \setminus U} = 0$. Then the family of operators $(D_m + \chi(m)Q_m)_{m \in M}$ defines the element of $KK_1(C, \mathcal{C}_N(M)) \cong K^1(M, N)$.

5 A thin-invariant rank-one field theory associated to an odd dimensional geometric family

**Theorem 5.1.** If $\mathcal{E}_{\text{geom}}$ is an odd dimensional geometric family on a connected manifold $B$ such that $\text{ch}_1(\mathcal{E}_{\text{geom}}) = 0$, then there is a natural rank-one thin invariant field theory $E$ with curvature $R^E = \Omega^3(\mathcal{E}_{\text{geom}})$ such that the associated gerbe is the index gerbe $\text{gerbe}(\mathcal{E}_{\text{geom}})$.

**Proof.** We fix a base point $b_0$ and a taming $\mathcal{E}_{b_0,t}$ of the fibre of $\mathcal{E}_{\text{geom}}$ over $b_0$.

Let $\gamma : S^1 \to B$ be a smooth map. Since $S^1$ is one dimensional we have an inclusion $\text{ch}_1 : K^1(S^1) \to H^1_{dR}(S^1)$. Since $\text{ch}_1(\text{index}(\gamma^* \mathcal{E})) = \gamma^* \text{ch}_1(\text{index}(\mathcal{E})) = 0$ the family $\gamma^* \mathcal{E}_{\text{geom}}$ admits a taming $(\gamma^* \mathcal{E}_{\text{geom}})_t$.

The set of homotopy classes of tamings of $\gamma^* \mathcal{E}_{\text{geom}}$ is a torsor over $K^1(S^1 \times I, \partial(S^1 \times I)) \cong \mathbb{Z}$. We employ the choice $\mathcal{E}_{b_0,t}$, in order to distinguish one of these classes. Let $M := S^1 \cup_I I$, where we identify $1 \in S^1$ and $1 \in I$. We extend $\gamma$ to $\tilde{\gamma} : M \to B$ such that $\tilde{\gamma}$ maps $0 \in I$ to $b_0$. Note that $K^1(S^1 \cup_I I, S^1 \cup \{0\}) \cong \mathbb{Z}$. The distinguished homotopy class of tamings on $\gamma^* \mathcal{E}_{\text{geom}}$ is now characterized by the property that it can be connected with $\mathcal{E}_{b_0,t}$ along the path $\tilde{\gamma}|_I$. Because of our assumption $\text{ch}_1(\text{index}(\mathcal{E})) = 0$ this class is independent of the choice of the path $\tilde{\gamma}|_I$.

We now construct the space $E(\gamma)$. Let $\gamma : S^1 \to B$ be a smooth map. We consider the set $E(\gamma)$ of pairs $(\lambda, (\gamma^* \mathcal{E}_{\text{geom}})_t)$, where $\lambda \in \mathbb{C}$ and $(\gamma^* \mathcal{E}_{\text{geom}})_t$ is a taming of $\gamma^* \mathcal{E}_{\text{geom}}$ in the distinguished homotopy class.

Let $(\gamma^* \mathcal{E}_{\text{geom}})_t$ and $(\gamma^* \mathcal{E}_{\text{geom}})'_t$ be two tamings in the distinguished class. We consider $\tilde{\gamma} : I \times S^1 \overset{p_1}{\to} S^1 \overset{p_1}{\to} B$. Then we have a taming of $\gamma^* \mathcal{E}_{\text{geom}}$ over $\{0\} \times S^1$ given by $(\gamma^* \mathcal{E}_{\text{geom}})_t$, and over $\{1\} \times S^1$ given by $(\gamma^* \mathcal{E}_{\text{geom}})'_t$. Since these belong to the same homotopy class we can extend this taming to a taming $(\gamma^* \mathcal{E}_{\text{geom}})_t$. Any two of these extensions are homotopic since $K^1(I \times I \times S^1, \partial(I \times I \times S^1)) = 0$. 

4
By \( \eta^2((\gamma^t E_{\text{geom}})_t) \in A^2_{I \times S^1}(I \times S^1) \) we denote the 2-form component of the \( \eta \) form as defined in [3], Sec. 3.3. We claim that \( \int_{I \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t) \) is independent of the choice of the taming. We consider a homotopy between two choices \((\gamma^t E_{\text{geom}})_t, (\gamma^t E_{\text{geom}})_t')\), i.e. \( \gamma : I \times I \times S^1 \xrightarrow{\pr} S^1 \xrightarrow{\gamma} B \) and a taming \((\gamma^t E_{\text{geom}})_t\). Then we have by [3], Prop. 3.2, that \( d\eta^2((\gamma^t E_{\text{geom}})_t) = \Omega^3(\gamma^t E_{\text{geom}}) \). Moreover \( \Omega^3(\gamma^t E_{\text{geom}}) = 0 \) since \( \gamma \) factors over a one dimensional manifold. Therefore by Stoke’s Lemma

\[
0 = \int_{I \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t) - \int_{\{0\} \times I \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t) - \int_{I \times \{1\} \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t) + \int_{I \times \{0\} \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t). \tag{1}
\]

Here (1) is equal to

\[
\int_{I \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t) - \int_{I \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t'). \tag{2}
\]

Moreover, \((\gamma^t E_{\text{geom}})_t|_{\{0\} \times I \times S^1} = \pr^* (\gamma^t E_{\text{geom}})_t\), where \( \pr : I \times \{0\} \times S^1 \to S^1 \). Thus for dimensional reasons \( \eta^2((\gamma^t E_{\text{geom}})_t)|_{\{0\} \times I \times S^1} = \pr^* \eta^2((\gamma^t E_{\text{geom}})_t) = 0 \). In a similar manner we have \( \eta^2((\gamma^t E_{\text{geom}})_t)|_{I \times \{1\} \times S^1} = 0 \). Thus the terms in (2) vanish. This finishes the proof of the claim.

We now define an equivalence relation on \( E(\gamma) \) such that \((\lambda, (\gamma^t E_{\text{geom}})_t) \sim (\lambda', (\gamma^t E_{\text{geom}})_t')\) if

\[
\lambda' = \lambda \exp \left( -2\pi i \int_{I \times S^1} \eta^2((\gamma^t E_{\text{geom}})_t) \right),
\]

where \((\gamma^t E_{\text{geom}})_t\) is any homotopy between \((\gamma^t E_{\text{geom}})_t\) and \((\gamma^t E_{\text{geom}})_t'\).

We set \( E(\gamma) := E(\gamma)/\sim \). Then \( E(\gamma) \) is a one dimensional complex vector space. If \( \gamma = \cup_{i=1, \ldots, m} \gamma_i : S_{m,0} \to B \), then we set \( E(\gamma) := \oplus_{i=1}^m E(\gamma_i) \).

We now define \( E([g]) \) for a morphism \([g]\) in \( \mathcal{T}_B \) represented by a \( B \)-surface \( g : \Sigma \to B \).

First assume that \( \Sigma \) is closed. We again have an inclusion \( \text{ch}_1 : K^1(\Sigma) \to H^1_{\text{dR}}(\Sigma) \). Thus \( \text{index}(g^* E) = 0 \) because of our assumption \( \text{ch}_1(\text{index}(E)) = 0 \), and there exists a taming \((g^* E_{\text{geom}})_t\). The set of homotopy classes of tamings is parameterized by \( K^1(I \times \Sigma, \partial I \times \Sigma) \cong K^0(\Sigma) \cong \mathbb{Z} \). We restrict the choices of tamings \((g^* E_{\text{geom}})_t\) by the following condition. Let \( s \in \Sigma \) be any point. We consider \( M := \Sigma \cup_{s=1} I \). We choose some extension \( \tilde{g} : M \to B \) of \( g \) by choosing a path from \( b_0 \) to \( g(s) \). We will only consider tamings \((g^* E_{\text{geom}})\) which can be connected with \( \tilde{g}|_{\{0\} \times B_{b_0,t}} \) along \( M \). This condition is independent of the choice of \( s \) and the path. The remaining choices correspond to the reduced \( K \)-theory \( \tilde{K}^0(\Sigma) \cong \mathbb{Z} \).

We define

\[
E(g) := \exp \left( 2\pi i \int_{\Sigma} \eta^2((g^* E_{\text{geom}})_t) \right) \in \mathbb{C} \cong \text{End}(E(\emptyset)).
\]

If \( \Sigma \) has a boundary, then for each boundary component \( i \in I_1(\Sigma) \) we obtain an object \( \gamma_i : S^1 \to B \). We choose tamings \((\gamma_i^t E_{\text{geom}})_t\), \( i \in I_1(\Sigma) \), in the distinguished components. The obstruction against extending these tamings to \( g^* E_{\text{geom}} \) belongs to \( K^1(\Sigma, \partial \Sigma) \). Since this group is trivial we can extend the given taming over the boundaries to a taming \((g^* E_{\text{geom}})\). The set of homotopy classes of these extensions is parameterized by \( K^1(I \times \Sigma, \partial(I \times \Sigma)) \cong K^0(\Sigma, \partial \Sigma) \cong \mathbb{Z} \).

Let \( z \in E(\gamma_{\text{in}}) \) be represented by

\[
\otimes_{i \in I_1(\Sigma)} (\lambda_i, (\gamma_i^t E_{\text{geom}})_t). \tag{3}
\]

Then we define

\[
E(g)z := \exp \left( 2\pi i \int_{\Sigma} \eta^2((g^* E_{\text{geom}})_t) \right) z' \in E(\gamma_{\text{out}}),
\]

where \( z' \in E(\gamma_{\text{out}}) \).
where \( z' \in E(\gamma_{out}) \) is represented by

\[
\otimes_{i \in \mathbb{F}^*} (\lambda_i, (\gamma_i^t E_{geom}))
\]

We must show that \( E(g) \) only depends on the class \( [g] \in \mathcal{I}_B \).

Let \( g \) and \( g' \) be thin homotopic by a homotopy \( h : I \times \Sigma \to B \). Furthermore, let \( (h^* E_{geom})_t \) be a taming which restricts to the tamings \((g^* E_{geom})_t \) on \( \{0\} \times \Sigma \) and \((g'^* E_{geom})_t \) on \( \{1\} \times \Sigma \). Then we have by \([3] \), Prop. 3.2., that \( d\eta^2((h^* E_{geom})_t) = \Omega^3(h^* E_{geom}) \). Since the homotopy \( h \) is thin we have \( \Omega^3(h^* E_{geom}) = h^* \Omega^3(E_{geom}) = 0 \). It follows from Stoke’s Lemma that

\[
\int_{\Sigma} \eta^2((g^* E_{geom})_t) - \int_{\partial \Sigma} \eta^2((h^* E_{geom})_t) + \int_{\partial \Sigma} \eta^2((h^* E_{geom})_t) = \int_{\Sigma} \eta^2((g'^* E_{geom})_t) \]  

This implies that \( E(g) = E(g') \).

Assume that we are given two tamings \((g^* E_{geom})_t \) and \((g'^* E_{geom})_t \) which coincide over the boundary of \( \Sigma \), and which are not homotopic. Let \( \Sigma = \Sigma_1 \cup_{\Sigma_2} \Sigma_3 \) be the surface obtained by doubling \( \Sigma \) along the boundary, and let \( \tilde{g} : \Sigma \to B \) be the map induced by \( g \). We consider the family of cylinders \( I \times \tilde{g}^* E \to \Sigma \) which is boundary tamed by \((g^* E_{geom})_t \) on \( \{0\} \times \tilde{g}^* E \), and by \((g'^* E_{geom})_t \) on \( \{1\} \times \tilde{g}^* E \) over the copy \( \Sigma \), and similarly by \((g^* E_{geom})_t \) and again \((g'^* E_{geom})_t \) over the other copy \( \Sigma \). This works since the two tamings \((g^* E_{geom})_t \) and \((g'^* E_{geom})_t \) coincide near \( \partial \Sigma \). The index \( \text{index}((I \times \tilde{g}^* E_{geom})_{int}) \in K^0(\Sigma) \) is the obstruction against extending the boundary taming to a taming (we refer to \([3] \), Sec. 2.4 and 2.5 for definitions). It follows from our restrictions on the choice of tamings \((g^* E_{geom})_t \) and \((g'^* E_{geom})_t \) that \( \dim \text{index}((I \times \tilde{g}^* E_{geom})_{int}) = 0 \).

Let \( \mathcal{F}_{geom} \) be any geometric family over \( \tilde{\Sigma} \) with closed even dimensional fibres such that

\[
\text{index}(\mathcal{F}) = -\text{index}((I \times \tilde{g}^* E_{geom})_{int})
\]

In fact we can realize \( \mathcal{F}_{geom} \) with zero dimensional fibres (see \([3] \), Sec. 3.1). Then \( \text{index}((I \times \tilde{g}^* E_{geom} + \mathcal{F}_{geom})_{int}) = 0 \), and the boundary taming \((I \times \tilde{g}^* E_{geom} + \mathcal{F}_{geom})_{int} \) can be extended to a taming. Using \([3] \), Prop. 3.2, (at *), we obtain

\[
\int_{\Sigma} \eta^2((g^* E_{geom})_t) - \int_{\Sigma} \eta^2((g^* E_{geom})_t) = \int_{\Sigma} \eta^2(\partial(I \times \tilde{g}^* E_{geom} + \mathcal{F}_{geom})) = -\int_{\Sigma} d\eta^1((I \times \tilde{g}^* E_{geom} + \mathcal{F}_{geom})_t) + \int_{\Sigma} \Omega^2(\mathcal{F}_{geom}) = \chi_2(\text{index}(\mathcal{F}), [\tilde{\Sigma}]) \in \mathbb{Z}.
\]

We conclude that

\[
\exp \left( 2\pi i \int_{\Sigma} \eta^2((g^* E_{geom})_t) \right) = \exp \left( 2\pi i \int_{\Sigma} \eta^2((g'^* E_{geom})_t) \right).
\]

We now have seen that \( E(g) \) only depends on the class \([g] \in \mathcal{I}_B \).

We have constructed for each choice of a taming \( \mathcal{E}_{\theta_0,t} \) a thin-invariant rank-one field theory. We must now show that this theory is independent of this choice.

Let \( \mathcal{E}_{\theta_0,t'} \) be another choice, and let \( E' \) be the corresponding field theory. We must define an equivalence \( Q : E \to E' \). Let \( \gamma : S^1 \to B \) be an object of \( \mathcal{I}_B \). Let \( (\lambda_x, (\gamma_x^t E_{geom})_t) \) represent \( x \in E(\gamma) \), and let \( (\gamma'^t E_{geom})_t \) be a taming in the distinguished class for the choice \( \mathcal{E}_{\theta_0,t'} \). Then we consider the geometric family \( I \times \gamma'^t E_{geom} \) over \( S^1 \) with a boundary taming by \((\gamma'^t E_{geom})_t \) and \((\gamma'^t E_{geom})_t \) at \( \{0\} \times E_{geom} \) and \( \{1\} \times E_{geom} \). We consider \( \text{index}((I \times \gamma'^t E_{geom})_{int}) \).
\(\gamma^r E_{geom}\) in \(K_0(S^1) \cong \mathbb{Z}\). Let \(\mathcal{F}\) be a geometric family over \(S^1\) with underlying bundle \(S^1 \to S^1\) and vector bundle \(\mathbb{C}^n \times S^1 \to S^1\), where we adjust \(n\) and the fibre wise orientation such that \(\text{index}(\mathcal{F}) = -\text{index}((I \times \gamma^r E_{geom})_{bt})\).

Since \(\text{index}((I \times \gamma^r E_{geom} \cup \mathcal{F}_{geom})_{bt}) = 0\) this boundary taming can be extended to a taming \((I \times \gamma^r E_{geom} \cup \mathcal{F}_{geom})_t\).

We define
\[
Q(\gamma)(x) = \left(\lambda \exp \left(-2\pi i \int_{\Sigma} \eta^1((I \times \gamma^r E_{geom} \cup \mathcal{F}_{geom})_t)\right), (\gamma^r E_{geom})_t\right).
\]

We claim that \(Q\) is well-defined. Let \((\gamma^r E_{geom})_t\) and \((\gamma^r E_{geom})_{t'}\) other choices of the tamings in the distinguished classes. Let \((\gamma^r E_{geom})_{t'}\) and \((\gamma^r E_{geom})_{t''}\) be corresponding homotopies of tamings from \((\gamma^r E_{geom})_{t'}\) to \((\gamma^r E_{geom})_t\) and from \((\gamma^r E_{geom})_{t'}\) to \((\gamma^r E_{geom})_t\), where \(\tilde{\gamma} : I \times S^1 \rightarrow I \times S^1\) is closed. By \([3\), Prop. 3.2,\] we have

\[
d\eta^1((I \times \gamma^r E_{geom} \cup \mathcal{F}_{geom})_{t'}) = -\eta^2((\gamma^r E_{geom})_{t'}) + \eta^2((\gamma^r E_{geom})_{t})\]

By Stoke’s Lemma

\[
-\int_{\Sigma} \eta^1((I \times \gamma^r E_{geom} \cup \mathcal{F}_{geom})_t) + \int_{\Sigma} \eta^1((I \times \gamma^r E_{geom} \cup \mathcal{F}_{geom})_{t'}) = -\int_{S^1} \eta^2((\gamma^r E_{geom})_t) + \int_{S^1} \eta^2((\gamma^r E_{geom})_{t'}) + \int_{S^1} \eta^2((\gamma^r E_{geom})_{t})\]

An inspection of the definition of \(E(\gamma)\) shows that this relation implies that \(Q(\gamma)\) is well-defined independent of the choice of representatives of \(E(\gamma), E'(\gamma), \) and of the additional tamings. We use the monoidal structures of \(T_B\) and vect\(_1\) in order to extend \(Q\) to general objects of \(T_B\).

It remains to show that \(Q\) is natural. Let \(g : \Sigma \rightarrow B\) be an \(X\)-surface representing a morphism \([g]\) from \(\gamma_m\) to \(\gamma_{Out}\). Let \((g^* E_{geom})_t\) and \((g^* E_{geom})_{t'}\) be tamings in the distinguished classes with respect to \(E_{bt}, E_{bt'}\). Then we consider the geometric family \(I \times g^* E_{geom}\) over \(\Sigma\) which is boundary tamed by \((g^* E_{geom})\), and \((g^* E_{geom})_{t'}\) at \([0] \times g^* E_{geom}\) and \([1] \times g^* E_{geom}\). We now want to kill \(\text{index}((I \times g^* E_{geom})_{bt}) \in K^0(\Sigma)\).

Let \(L_0 := C([-\text{dim} \text{index}((I \times g^* E_{geom})_{bt}))]) \times \Sigma \rightarrow \Sigma\) be a trivial vector bundle. If \(\Sigma\) is closed, then furthermore let \(L_1 \rightarrow \Sigma\) be a line bundle with \(c_1(L_1) = -c_1(\text{index}((I \times g^* E_{geom})_{bt}))\). We equip \(L_1\) with a hermitian metric and a metric connection. We let \(\mathcal{F}_{0,geom}\) be the geometric family with underlying fibre bundle \(\Sigma \rightarrow \Sigma\) and vector bundle \(L_0\), where we flip the orientation of the fibres if \(\text{dim} \text{index}((I \times g^* E_{geom})_{bt})) > 0\). If \(\Sigma\) is closed, then we let \(\mathcal{F}_{1,geom}\) be the union of geometric families \(\mathcal{F}_{1a,geom}\) and \(\mathcal{F}_{1b,geom}\). Here \(\mathcal{F}_{1a,geom}\) has the underlying fibre bundle \(\Sigma \rightarrow \Sigma\) and the vector bundle is \(L_1\), and \(\mathcal{F}_{1b,geom}\) has underlying fibre bundle \(\Sigma \rightarrow \Sigma\) with flipped orientation and the bundle is \(\Sigma \times \Sigma \rightarrow \Sigma\). Finally we set \(\mathcal{F}_{geom} := \mathcal{F}_{0,geom}\) if \(\Sigma\) has a non-trivial boundary, and \(\mathcal{F}_{geom} := \mathcal{F}_{0,geom} \cup \mathcal{F}_{1,geom}\) if \(\Sigma\) is closed. Then \(\text{index}((I \times g^* E_{geom} \cup \mathcal{F}_{geom})_{bt}) = 0\) and we can extend the boundary taming to a taming \((I \times g^* E_{geom} \cup \mathcal{F}_{geom})_t\).

Note that \(\Omega^2(I \times g^* E_{geom} \cup \mathcal{F}_{geom}) = \Omega^2(\mathcal{F}_{geom})\) and \(f_\Sigma \Omega^2(\mathcal{F}_{geom}) = c_1(\text{index}(\mathcal{F}))\), \([\Sigma] > \in \mathbb{Z}\), if \(\Sigma\) is closed, and \(f_\Sigma \Omega^2(\mathcal{F}_{geom}) = 0\), if \(\Sigma\) is not closed. By \([3\), Prop. 3.2,\] we have

\[
d\eta^1((I \times g^* E_{geom} \cup \mathcal{F}_{geom})_t) = -\eta^2((g^* E_{geom})_t) + \eta^2((g^* E_{geom})_t) + \Omega^2(\mathcal{F}_{geom})\]

By Stoke’s Lemma

\[
\int_{\Omega \Sigma} \eta^1((I \times g^* E_{geom} \cup \mathcal{F}_{geom})_t) = -\int_{\Sigma} \eta^2((g^* E_{geom})_t) + \int_{\Sigma} \eta^2((g^* E_{geom})_t) + \mathbb{Z}.
\]

This relation implies

\[
Q(\gamma)E(g) = E'(g)Q(\gamma).
\]
Next we show that the curvature of $E$ is $\Omega^3(E_{\text{geom}})$. Let $h : C \to B$ be a $B$-three manifold with boundary $g : \Sigma \to B$ such that $\Sigma$ is closed. If $\partial C = \emptyset$, then
\[
\int_C \Omega^3(E_{\text{geom}}) = \langle \text{ch}_3(\text{index}(E)), [C] \rangle \in \mathbb{Z}
\]
(see Lott \[6\], Prop. 8) so that
\[
\exp \left( 2\pi i \int_C \Omega^3(E_{\text{geom}}) \right) = 1 = E([g]) .
\]
If $\partial C \neq \emptyset$, then $C$ is homotopy equivalent to a two dimensional space. In this case vanishing of $\text{ch}_1(\text{index}(h^*E)) = h^*\text{ch}_1(\text{index}(E))$ implies that there exists a taming $(h^*E_{\text{geom}})_t$. We have by \[3\], Prop. 3.2,
\[
E([g]) = \exp \left( 2\pi i \int_{\partial C} \eta^2((h^*E_{\text{geom}})_t) \right) = \exp \left( 2\pi i \int_C d\eta^2((h^*E_{\text{geom}})_t) \right) = \exp \left( 2\pi i \int_C \Omega^3(E_{\text{geom}}) \right) .
\]
Finally we show that the gerbe associated to the rank-one thin-invariant field theory $E$ is the index gerbe gerbe $(E_{\text{geom}})_t$.

Let $g : \Sigma \to B$ be a closed $B$-surface defining the cycle $[g, \Sigma]$. Let $U$ be a tubular neighborhood of $g(\Sigma)$. We choose a taming $E_{U,t}$ which can be connected with $E_{b_0,t}$ along some path from $U$ to $b_0$. Then we obtain a taming $(g^*E_{\text{geom}})_t$. Then we have
\[
H(\text{gerbe}(E_{\text{geom}}))( [g, \Sigma] ) = \exp \left( 2\pi i \int_{\Sigma} g^*\eta^2(E_{U,t}) \right) = \exp \left( 2\pi i \int_{\Sigma} \eta^2((g^*E_{\text{geom}})_t) \right) = E([g]) .
\]

Remark: Let $E_{\text{geom}}$ be an odd-dimensional geometric family over $B$. Let $f : B \to S^1$ by any classifying map of $-c_1(\text{index}(E)) \in H^1(B, \mathbb{Z})$. Furthermore, let $F_{\text{geom}}$ be some odd dimensional geometric family over $S^1$ such that $c_1(\text{index}(F)) = 1 \in H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$. Then we have $\text{ch}_1(\text{index}(E + f^*F)) = 0$. One can show that the rank-one thin-invariant field theory associated to $E_{\text{geom}} + f^*F_{\text{geom}}$ is independent of the choice of $f$ and $F_{\text{geom}}$.

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