GLOBAL SMALL SOLUTIONS TO A SPECIAL $2_{\frac{1}{2}}$-D COMPRESSIBLE VISCOUS NON-RESISTIVE MHD SYSTEM

BOQING DONG, JIAHONG WU, AND XIAOPING ZHAI

Abstract. This paper solves the global well-posedness and stability problem on a special $2_{\frac{1}{2}}$-D compressible viscous non-resistive MHD system near a steady-state solution. The steady-state here consists of a positive constant density and a background magnetic field. The global solution is constructed in $L^p$-based homogeneous Besov spaces, which allow general and highly oscillating initial velocity. The well-posedness problem studied here is extremely challenging due to the lack of the magnetic diffusion, and remains open for the corresponding 3D MHD equations. Our approach exploits the enhanced dissipation and stabilizing effect resulting from the background magnetic field, a phenomenon observed in physical experiments. In addition, we obtain the solution’s optimal decay rate when the initial data is further assumed to be in a Besov space of negative index.

MSC 2020: 35Q35, 35A01, 35A02, 76W05

Key Words: Global solutions; Non-resistive compressible MHD; Decay rates

1. Introduction and the main results

The small data global well-posedness problem on the three-dimensional (3D) compressible viscous non-resistive magnetohydrodynamic (MHD) equations remains an challenging open problem. Mathematically the concerned MHD equations are given by

\begin{align*}
\partial_t \rho + \text{div} (\rho \mathbf{v}) &= 0, \\
\rho (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \mu \Delta \mathbf{v} - (\lambda + \mu) \nabla \text{div} \mathbf{v} + \nabla P &= (\nabla \times \mathbf{B}) \times \mathbf{B}, \\
\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) &= 0, \\
\text{div} \mathbf{B} &= 0, \\
\end{align*}

where $\rho$ denotes the density of the fluid, $\mathbf{v}$ the velocity field, and $\mathbf{B}$ the magnetic field. The parameters $\mu$ and $\lambda$ are shear viscosity and volume viscosity coefficients, respectively, which satisfy the standard strong parabolicity assumption,

$$\mu > 0 \quad \text{and} \quad \nu := \lambda + 2\mu > 0.$$ 

The pressure $P = P(\rho)$ is in $C^1$ such that $P' > 0$ and that $P(\bar{\rho}) = 0$ for some positive constant reference density $\bar{\rho}$. The compressible MHD equations model the motion of electrically conducting fluids in the presence of a magnetic field. The compressible MHD equations can be derived from the isentropic Navier-Stokes-Maxwell system by taking the zero dielectric constant limit [31]. When the effect of the magnetic field can be neglected or $\mathbf{B} = 0$, (1.1) reduces to the compressible Navier-Stokes equations.

The goal of this paper is to solve the small data global well-posedness problem on a very special two-and-half-dimensional ($2_{\frac{1}{2}}$-D) compressible viscous non-resistive MHD equations (to
be specified later). In addition, we are also interested in the precise large-time behavior of the solutions.

Due to its wide physical applications and mathematical challenges, the compressible MHD equations have attracted the interests of many physicists and mathematicians (see, e.g., [2, 7–11, 13, 16, 39, 45, 46] and the references therein). We briefly recall some results concerning the multi-dimensional barotropic compressible MHD equations, which are closely related to our investigation here. Ducomet and Feireisl [9] considered the heat-conducting fluids together with the influence of radiation, and obtained the global existence of weak solutions with finite energy initial data. Hu and Wang [20] proved the global existence of weak solutions to the 3D isentropic compressible MHD system via the Lions-Feireisl theory. We remark that there are essential differences between the vacuum case and the non-vacuum case. The global weak solution in the case of vacuum was obtained in the work of Li, Xu and Zhang [33]. The local well-posedness in the framework of critical Besov spaces was shown by Bian and Yuan [2] when there is full dissipation and no vacuum. In the case of vacuum and no magnetic diffusion, Li, Su and Wang [35] proved the local existence and uniqueness of strong solutions. The small data global well-posedness problem is extremely difficult when there is no magnetic diffusion. Wu and Wu [39] presented a systematic approach to the small data global well-posedness and stability problem on the 2D compressible non-resistive MHD equations. Initial- and boundary-value problems under some additional compatibility conditions for the 3D compressible MHD equations were examined by Fan and Yu [10] and local solutions were obtained even when there is a vacuum. Zhu [46] extended the result obtained in [35] to the case of allowing non-negativity of the initial density. We mention that there are many interesting results on the zero Mach limit results on the incompressible MHD equations (see, e.g., [8, 11, 19, 23, 29, 36]).

If we neglect the effect of the magnetic field, the system (1.1) reduces to the compressible Navier-Stokes equations, which have also been studied by many researchers, see [3], [4], [5], [6], [21], [26], [28], [34], [41], [42], [44] and the references therein.

Although the small data global well-posedness on the 2D compressible MHD equations without magnetic diffusion has been successfully settled, this same problem on the 3D counterpart appears to be inaccessible at this moment. This paper focuses on a very special $2\frac{1}{2}$-D compressible MHD system. The motion of fluids takes place in the plane $\mathbb{R}^2$ while the magnetic field acts on fluids only in the vertical direction, namely

$$v = (v^1(t, x_1, x_2), v^2(t, x_1, x_2), 0) \overset{\text{def}}{=} (u, 0),$$

$$\rho \overset{\text{def}}{=} \rho(t, x_1, x_2), \quad B \overset{\text{def}}{=} (0, 0, m(t, x_1, x_2)).$$

Then (1.1) is reduced to

$$\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\rho \partial_t u + u \cdot \nabla u - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla P + \frac{1}{2} \nabla m^2 &= 0, \\
\partial_t m + \text{div} (mu) &= 0.
\end{aligned}$$

(1.2)

Clearly $(\rho^{(0)}, u^{(0)}, m^{(0)})$ with

$$\rho^{(0)} = 1, \quad u^{(0)} = 0, \quad m^{(0)} = 1.$$
solves (1.2). We intend to understand the well-posedness and stability problem on the system governing the perturbation \((a, u, b)\), where \(a = \rho - 1, \ b = m - 1\).

It is easy to check that \((a, u, b)\) satisfies
\[
\begin{aligned}
\partial_t a + \text{div } u + u \cdot \nabla a + a \text{ div } u &= 0, \\
\partial_t u + u \cdot \nabla u - \Delta u - \nabla \text{ div } u + 2\nabla a + \nabla b &= F(a, u, b), \\
\partial_t b + \text{div } u + u \cdot \nabla b + b \text{ div } u &= 0,
\end{aligned}
\tag{1.3}
\]
where
\[
F(a, u, b) \overset{\text{def}}{=} I(a) \nabla b + k(a) \nabla a + (I(a) - 1)b(\nabla b) - I(a)(\Delta u + \nabla \text{div } u)
\tag{1.4}
\]
with
\[
I(a) \overset{\text{def}}{=} \frac{a}{1 + a}, \quad k(a) \overset{\text{def}}{=} -\frac{P'(1 + a)}{1 + a} + P'(1) \quad \text{and} \quad P'(1) = 2. \tag{1.5}
\]

As the first step of our main results, we provide a local well-posedness result in the Besov space.

**Proposition 1.1.** (Local well-posedness) Let \(1 < p < 4\). Assume \(u_0 \in \dot{B}^{\frac{2}{p} - 1}_{p, 1}(\mathbb{R}^2)\), \((a_0, b_0) \in \dot{B}^{\frac{2}{p}}_{p, 1}(\mathbb{R}^2)\) with \(1 + a_0\) bounded away from zero. Then there exists a positive time \(T\) such that the system (1.3) has a unique solution \((a, u, b)\) satisfying
\[
(a, b) \in C([0, T]; \dot{B}^{\frac{2}{p}}_{p, 1}(\mathbb{R}^2)), \quad u \in C([0, T]; \dot{B}^{\frac{2}{p}}_{p, 1}(\mathbb{R}^2)) \cap L^1([0, T]; \dot{B}^{\frac{2}{p} - 1}_{p, 1}(\mathbb{R}^2)).
\]

Before stating our main results, we introduce some notation. Let \(S(\mathbb{R}^2)\) be the Schwartz space on \(\mathbb{R}^2\) and \(S'(\mathbb{R}^2)\) be its dual space. For any \(z \in S'(\mathbb{R}^2)\), the lower and higher frequency parts are expressed as
\[
z^\ell \overset{\text{def}}{=} \sum_{j \leq j_0} \hat{\Delta}_j z \quad \text{and} \quad z^h \overset{\text{def}}{=} \sum_{j > j_0} \hat{\Delta}_j z
\]
for some fixed integer \(j_0 \geq 1\) (the value of \(j_0\) is fixed in the proofs of the main theorems). The corresponding truncated semi-norms are defined as follows:
\[
\|z\|_{\dot{B}^{\frac{2}{p}}_{2, 1}} \overset{\text{def}}{=} \|z^\ell\|_{\dot{B}^{\frac{2}{p}}_{2, 1}} \quad \text{and} \quad \|z\|_{\dot{B}^{\frac{2}{p} - 1}_{p, 1}} \overset{\text{def}}{=} \|z^h\|_{\dot{B}^{\frac{2}{p} - 1}_{2, 1}}.
\]

Let \(P = I - \nabla \Delta^{-1} \nabla\cdot\) be the projection onto the divergence-free vector fields and \(Q = I - P = \nabla \Delta^{-1} \nabla\cdot\).

The small data global well-posedness and stability result on (1.3) is stated in the following theorem.

**Theorem 1.2.** (Global well-posedness) Let \(2 < p < 4\). For any \((a_0^\ell, Qu_0^\ell, b_0^\ell) \in \dot{B}^{\frac{2}{p} - 1}_{2, 1}(\mathbb{R}^2)\), \((a_0^h, b_0^h) \in \dot{B}^{\frac{2}{p}}_{p, 1}(\mathbb{R}^2)\) and \((Pu_0, Qu_0^h) \in \dot{B}^{\frac{2}{p} - 1}_{p, 1}(\mathbb{R}^2)\), there exists a positive constant \(c_0\) such that, if
\[
\|(a_0^\ell, Qu_0^\ell, b_0^\ell)\|_{\dot{B}^{\frac{2}{p} - 1}_{2, 1}} + \|(a_0^h, b_0^h)\|_{\dot{B}^{\frac{2}{p}}_{p, 1}} + \|(Pu_0, Qu_0^h)\|_{\dot{B}^{\frac{2}{p} - 1}_{p, 1}} \leq c_0,
\tag{1.6}
\]
then the system (1.3) has a unique global solution \((a, u, b)\) so that
\[
(a^\ell, b^\ell) \in C_b(\mathbb{R}^+; \dot{B}^0_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^2_{2,1}), \quad (a^h, b^h) \in C_b(\mathbb{R}^+; \dot{B}^2_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{2-s}_{p,1}),
\]
\[
\mathcal{Q}u^\ell \in C_b(\mathbb{R}^+; \dot{B}^0_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^2_{2,1}), \quad (\mathcal{P}u, \mathcal{Q}u^h) \in C_b(\mathbb{R}^+; \dot{B}^{2-s-1}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{2+s+1}_{p,1}).
\]
Moreover, there exists some constant \(C\) such that
\[
\mathcal{X}(t) \leq Cc_0,
\]
where
\[
\mathcal{X}(t) \overset{\text{def}}{=} \|(a^\ell, \mathcal{Q}u^\ell, b^\ell)\|_{L^\infty_t(B^0_{2,1})} + \|(a^h, b^h)\|_{L^\infty_t(B^2_{p,1})} + \|(P_u, \mathcal{Q}u^h)\|_{L^\infty_t(B^{2-s}_{p,1})} + \|(B_u, \mathcal{Q}u^h)\|_{L^1_t(B^{2+s+1}_{p,1})}.
\]

It is natural and physically important to study the large-time behavior of the global solution obtained in (1.2). The large-time behavior has always been a prominent topic on the fluid equations. Important results have been established for the compressible Navier-Stokes equations (see, e.g., \([6, 40, 44]\)) and the compressible MHD equations (see, e.g., \([20, 31]\)).

What is special here is that the system concerned here is partially dissipated with no damping or dissipation in the equations of \(\rho\) and \(b\). We show that, when the low modes of the initial data are in a Besov space with suitable negative index, then the Sobolev norm of the solution is shown to decay at an optimal rate. The proof relies on the enhanced dissipation resulting from the interaction between the velocity and the magnetic field.

**Theorem 1.3.** (Optimal decay) Let \((a, u, b)\) be the global small solutions addressed by Theorem 1.2. For any \(0 < \sigma \leq \frac{4}{p} - 1\), and \((a^\ell_0, u^\ell_0, b^\ell_0) \in \dot{B}^{-\sigma}_{2,\infty}(\mathbb{R}^2)\), we have the following time-decay rate
\[
\|\Lambda^{\gamma_1}(a, u, b)\|_{L^p} \leq C(1 + t)^{-\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{2\gamma_1 + \sigma}{2}}, \quad \forall \gamma_1 \in \left(\frac{2}{p} - 1 - \sigma, \frac{2}{p} - 1\right].
\]

**Remark 1.4.** Let \(p = 2\), one can deduce from the decay estimates that
\[
\|\Lambda^{\gamma_1}(a, u, b)\|_{L^2} \leq C(1 + t)^{-\frac{4}{p} - \frac{s}{2}},
\]
which coincides with the heat flows, thus our decay rate is optimal in some sense.

Finally, we mention the small data global well-posedness result for a closely related system of inhomogeneous incompressible MHD equations. The general inhomogeneous incompressible MHD equations are of the form
\[
\begin{aligned}
\partial_t \rho + \text{div} (\rho v) &= 0, \\
\rho (\partial_t v + v \cdot \nabla v) - \mu \Delta v + \nabla P &= (\nabla \times B) \times B, \\
\partial_t B - \nabla \times (v \times B) &= 0, \\
\text{div } v &= \text{div } B = 0, \\
(\rho, v, B)|_{t=0} &= (\rho_0, v_0, B_0).
\end{aligned}
\]
If we set
\[
v = (v^1(t, x_1, x_2), v^2(t, x_1, x_2), 0) \overset{\text{def}}{=} (u, 0),
\]
\[
\rho \equiv \rho(t, x_1, x_2), \quad B \equiv (0, 0, b(t, x_1, x_2)),
\]

then (1.8) is reduced to
\[
\begin{cases}
\partial_t \rho + \text{div} (\rho \mathbf{u}) = 0, \\
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla P + \frac{1}{2} \nabla b^2 = 0, \\
\partial_t b + \text{div} (b \mathbf{u}) = 0, \\
\text{div} \mathbf{u} = 0,
\end{cases}
\]
(1.9)

Different from the compressible MHD equations, the combination \(\Pi := P + \frac{1}{2} b^2\) can be regarded as new pressure and the new system (1.9) is decoupled into equations of \((\rho, \mathbf{u})\) and the equation of \(b\). We can solve the equations of \((\rho, \mathbf{u})\) first and then get the solution of \(b\) through the third equation of (1.9). Now we write\(\rho = 1 + a\), inspired by [43] and the previous well-posedness result on the compressible MHD equations, we obtain the following global well-posedness result on (1.9). We shall not provide a detailed proof for this result.

**Theorem 1.5.** Let \(p \in (1, 4), (a_0, \mathbf{u}_0, b_0) \in \dot{B}^2_{p, 1}(\mathbb{R}^2)\) with \(\text{div} \mathbf{u}_0 = 0\) and \(1 + a_0\) bounded away from zero. Then (1.9) has a unique local solution \((a, \mathbf{u}, \nabla \Pi, b)\) on \([0, T]\) such that
\[
(a, b) \in C([0, T]; \dot{B}^2_{p, 1}(\mathbb{R}^2)) \cap \dot{L}^\infty_t (\dot{B}^2_{p, 1}(\mathbb{R}^2)), \quad \nabla \Pi \in L^1_t (\dot{B}^{2-1}_{p, 1}(\mathbb{R}^2)),
\]
\[
\mathbf{u} \in C([0, T]; \dot{B}^{2-1}_{p, 1}(\mathbb{R}^2)) \cap \dot{L}^\infty_t (\dot{B}^{2-1}_{p, 1}(\mathbb{R}^2)) \cap L^1_t (\dot{B}^{2+1}_{p, 1}(\mathbb{R}^2)).
\]
Moreover, if \(a_0 \in L^p(\mathbb{R}^2)\), then (1.9) has a unique global solution \((a, \mathbf{u}, \nabla \Pi, b)\) such that for any \(t > 0\),
\[
\|(a, b)\|_{\dot{L}^\infty_t (\dot{B}^2_{p, 1})} + \|\mathbf{u}\|_{\dot{L}^\infty_t (\dot{B}^{2-1}_{p, 1})} + \|\mathbf{u}\|_{L^1_t (\dot{B}^{2+1}_{p, 1})} + \|\nabla \Pi\|_{L^1_t (\dot{B}^{2-1}_{p, 1})} \leq C \exp \left( C \exp \left( Ct^{\frac{3}{2}} \right) \right)
\]
for some time-independent constant \(C\).

## 2. Preliminaries

This section reviews Besov spaces and related facts to be used in the subsequent sections. We start with the Littlewood-Paley decomposition. To define it, we fix a smooth radial non-increasing function \(\chi\) supported in the ball \(B(0, \frac{4}{3})\) of \(\mathbb{R}^2\), and with value 1 on \(B(0, \frac{2}{3})\) such that, for \(\varphi(\xi) = \chi(\frac{2}{3}) - \chi(\xi),\)
\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1 \text{ in } \mathbb{R}^2 \setminus \{0\} \quad \text{and} \quad \text{Supp } \varphi \subset \left\{ \xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\}.
\]
The homogeneous dyadic blocks \(\check{\Delta}_j\) are defined on tempered distributions by
\[
\check{\Delta}_j u \overset{\text{def}}{=} \varphi(2^{-j} D) u \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u).
\]
For any homogeneous function \(A\) of order 0 and smooth outside 0, we have
\[
\forall p \in [1, \infty], \quad \|\check{\Delta}_j (A(D) u)\|_{L^p} \leq C \|\check{\Delta}_j u\|_{L^p}.
\]
Definition 2.1. Let $p, r$ be in $[1, +\infty]$ and $s$ in $\mathbb{R}$, $u \in \mathcal{S}'(\mathbb{R}^2)$. We define the Besov norm by
\[
\|u\|_{B^s_{p,r}} \overset{\text{def}}{=} \left\| (2^{js}\|\dot{\Delta}_j u\|_{L^p})_j \right\|_{\ell^r(\mathbb{Z})}.
\]
We then define the homogeneous Besov spaces by $\dot{B}^s_{p,r} = \left\{ u \in \mathcal{S}'(\mathbb{R}^2), \|u\|_{B^s_{p,r}} < \infty \right\}$, where $u \in \mathcal{S}'(\mathbb{R}^2)$ means that $u \in \mathcal{S}'(\mathbb{R}^2)$ and $\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$ (see Definition 1.26 of [1]).

When employing parabolic estimates in Besov spaces, it is somehow natural to take the time-Lebesgue norm before performing the summation for computing the Besov norm. So we next introduce the following Besov-Chemin-Lerner space $\dot{L}^q_T(\dot{B}^s_{p,r})$ (see [1]):
\[
\dot{L}^q_T(\dot{B}^s_{p,r}) = \left\{ u \in (0, +\infty) \times \mathcal{S}_h'(\mathbb{R}^2) : \|u\|_{\dot{L}^q_T(\dot{B}^s_{p,r})} < +\infty \right\},
\]
where
\[
\|u\|_{\dot{L}^q_T(\dot{B}^s_{p,r})} \overset{\text{def}}{=} \left\| 2^{ks}\|\dot{\Delta}_k u(t)\|_{L^q(0,T;L^p)} \right\|_{\ell^r}.\]
The index $T$ will be omitted if $T = +\infty$ and we shall denote by $\tilde{C}_b([0,T];\dot{B}^s_{p,r})$ the subset of functions of $\tilde{L}^\infty_T(\dot{B}^s_{p,r})$ which are also continuous from $[0,T]$ to $\dot{B}^s_{p,r}$.

By the Minkowski inequality, we have the following inclusions between the Chemin-Lerner space $\tilde{L}^\lambda_T(\dot{B}^s_{p,r})$ and the Bochner space $L^\lambda_T(\dot{B}^s_{p,r})$:
\[
\|u\|_{\tilde{L}^\lambda_T(\dot{B}^s_{p,r})} \leq \|u\|_{L^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda \leq r, \quad \|u\|_{\tilde{L}^\lambda_T(\dot{B}^s_{p,r})} \geq \|u\|_{L^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if } \lambda \geq r.
\]
The following Bernstein’s lemma will be repeatedly used throughout this paper.

Lemma 2.2. Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^2$. A constant $C$ exists so that for any positive real number $\lambda$, any non-negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$, and any couple of real numbers $(p, q)$ with $1 \leq p \leq q \leq \infty$, there hold
\[
\text{Supp} \, \dot{u} \subset \lambda \mathcal{B} \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1}\lambda^{k+2\left(\frac{1}{p} - \frac{1}{q}\right)}\|u\|_{L^p},
\]
\[
\text{Supp} \, \dot{u} \subset \lambda \mathcal{C} \Rightarrow C^{k-1}\lambda^k\|u\|_{L^p} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p},
\]
\[
\text{Supp} \, \dot{u} \subset \lambda \mathcal{C} \Rightarrow \|\sigma(D)u\|_{L^q} \leq C_{\sigma,m}\lambda^{m+2\left(\frac{1}{p} - \frac{1}{q}\right)}\|u\|_{L^p}.
\]

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Here, we recall the decomposition in the homogeneous context:
\[
uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v) = \dot{T}_u v + \dot{T}_v u,
\]
where
\[
\dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v,
\]
and
\[
\tilde{\Delta}_j v = \sum_{|j-j'| \leq 1} \dot{\Delta}_j v, \quad \dot{T}_v u = \sum_{j \in \mathbb{Z}} \dot{S}_{j+2} u \dot{\Delta}_j u.
\]
The paraproduct $\dot{T}$ and the remainder $\dot{R}$ operators satisfy the following continuous properties.
Lemma 2.3. Suppose that $s \in \mathbb{R}, \delta > 0$, and $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$. Then we have

1. The paraproduct $\hat{T}$ is a bilinear, continuous operator from $L^{\infty} \times \dot{B}^{s}_{p_1,r_1}$ to $\dot{B}^{s}_{p,r}$, and from $\dot{B}^{-\delta}_{\infty,r_1} \times \dot{B}^{s}_{p_2,r_2}$ to $\dot{B}^{s-\delta}_{p,r}$ with $\frac{1}{r} = \min \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\}$.

2. The remainder $\hat{R}$ is bilinear continuous from $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ to $\dot{B}^{s_1 + s_2}_{p,r}$ with $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $\frac{1}{s} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$.

From Lemma 2.3, we may deduce the following corollary concerning the product estimates.

Corollary 2.4. For any $2 \leq p < 4$, $1 - \frac{1}{p} \leq s < 0$, there hold

$$\|fg\|_{\dot{B}^{s}_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s}_{p,1}} \|g\|_{\dot{B}^{s}_{p,\infty}}$$

and

$$\|fg\|_{\dot{B}^{s}_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s}_{p,1}} \|g\|_{\dot{B}^{s}_{p,\infty}}$$

Lemma 2.5. ( [40, Proposition A.1]) Let $1 \leq p, q \leq \infty$, $s_1 \leq \frac{2}{q}$, $s_2 \leq 2 \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$ and $s_1 + s_2 > 2 \max \left\{ 0, \frac{1}{p} + \frac{1}{q} - 1 \right\}$. For any $(u, v) \in \dot{B}^{s_1}_{q,1}(\mathbb{R}^2) \times \dot{B}^{s_2}_{p,1}(\mathbb{R}^2)$, we have

$$\|uv\|_{\dot{B}^{s_1 + s_2}_{p,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{p,1}} \|v\|_{\dot{B}^{s_2}_{p,1}}$$

Lemma 2.6. Let $2 \leq p < 4$. For any $u \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2), v^f \in \dot{B}^{0}_{2,1}(\mathbb{R}^2)$ and $v^h \in \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$, we have

$$\|(uv)^f\|_{\dot{B}^{2}_{2,1}} \lesssim \left( \|v^f\|_{\dot{B}^{0}_{2,1}} + \|v^h\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \right) \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}$$

Proof. We first use Bony’s decomposition to write

$$\dot{S}_{j_0+1}(uv) = \dot{T}_u \dot{S}_{j_0+1}v + \dot{S}_{j_0+1}(\dot{T}_v u + \dot{R}(v, u)) + [\dot{S}_{j_0+1}, \dot{T}_u]v.$$  

(2.5)

Applying Lemma 2.3, we have

$$\|\dot{T}_u \dot{S}_{j_0+1}v\|_{\dot{B}^{2}_{2,1}} \lesssim \|u\|_{L^{\infty}} \|v^f\|_{\dot{B}^{2}_{2,1}} \lesssim \|v^f\|_{\dot{B}^{2}_{2,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}},$$

and, for $\frac{1}{p} = \frac{1}{2} - \frac{1}{p^*},$

$$\|\dot{S}_{j_0+1}(\dot{T}_v u + \dot{R}(v, u))\|_{\dot{B}^{0}_{2,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p^*}}_{p^*,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}.$$  

(2.6)

By Lemma 6.1 in [5], the term with the commutator can be bounded

$$\|[\dot{S}_{j_0+1}, \dot{T}_u]v\|_{\dot{B}^{0}_{2,1}} \lesssim \|\nabla u\|_{\dot{B}^{\frac{2}{p^*}}_{p^*,1}} \|v\|_{\dot{B}^{\frac{2}{p^*}}_{p^*,1}} \lesssim \|v\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \|u\|_{\dot{B}^{\frac{2}{p}}_{p,1}}.$$  

(2.7)

Thus, the combination of (2.5)–(2.7) shows the validity of (2.4).

We also need the following classical commutator’s estimate.

Lemma 2.7. ( [1, Lemma 2.100]) Let $1 \leq p \leq \infty$, $-2 \min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\} < s \leq \frac{2}{p}$. For any $v \in \dot{B}^{s}_{p,1}(\mathbb{R}^2)$ and $\nabla u \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, there holds

$$\|[\hat{\Lambda}, u \cdot \nabla]v\|_{L^p} \lesssim d_2 2^{-js} \|\nabla u\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \|v\|_{\dot{B}^{s}_{p,1}}.$$
Finally, we recall a composition result and the parabolic regularity estimate for the heat equation to end this section.

Lemma 2.8. ([1]) Let \( G \) with \( G(0) = 0 \) be a smooth function defined on an open interval \( I \) of \( \mathbb{R} \) containing \( 0 \). Then the following estimates
\[
\| G(a) \|_{\dot{B}^s_{p,1}} \lesssim \| a \|_{\dot{B}^s_{p,1}} \quad \text{and} \quad \| G(a) \|_{\dot{L}^q_T(\dot{B}^s_{p,1})} \lesssim \| a \|_{\dot{L}^q_T(\dot{B}^s_{p,1})}
\]
hold true for \( s > 0 \), \( 1 \leq p, q \leq \infty \) and \( a \) valued in a bounded interval \( J \subset I \).

Lemma 2.9 ([1]). Let \( \sigma \in \mathbb{R} \), \( T > 0 \), \( 1 \leq p, r \leq \infty \) and \( 1 \leq q_2 \leq q_1 \leq \infty \). Let \( u \) satisfy the heat equation
\[
\partial_t u - \Delta u = f, \quad u|_{t=0} = u_0.
\]
Then there holds the following a priori estimate
\[
\| u \|_{\dot{L}^q_T(\dot{B}^\sigma_{p,r})} \lesssim \| u_0 \|_{\dot{B}^\sigma_{p,r}} + \| f \|_{\dot{L}^{q_2}_T(\dot{B}^{\sigma-2+\frac{2}{q_2}}_{p,r})}.
\]

3. The proof of Proposition 1.1

We prove Proposition 1.1 by a fixed point theorem under the Lagrangian coordinates. Firstly, we shall convert (1.2) into its Lagrangian formulation. Assume temporarily that \( u = u(t, x) \) is a \( C^1 \) vector field, namely,
\[
u \in L^1_{loc}(\mathbb{R}^+; C^1_b(\mathbb{R}^2; \mathbb{R}^2)).
\]
By virtue of Cauchy-Lipschitz theorem, the unique trajectory \( X(t, \cdot) \) of \( u \), defined by the ODE
\[
\begin{align*}
\frac{d}{dt}X(t, y) &= u(t, X(t, y)), \\
X(0, y) &= y,
\end{align*}
(3.1)
\]
is a \( C^1 \)-diffeomorphism over \( \mathbb{R}^2 \) for every \( t \geq 0 \). Let us introduce \( A(t, y) = (D_y X(t, y))^{-1} \), \( J(t, y) = \det DX(t, y) \), and \( \mathcal{A}(t, y) = \text{adj} DX(t, y) \) (the adjugate of \( DX \), i.e., \( \mathcal{A} = JA \)). For any scalar function \( \phi = \phi(x) \) and any vector field \( w = w(x) \), it is easy to see that
\[
(\nabla \phi) \circ X = A^T \nabla (\phi \circ X),
(3.2)
\]
and
\[
(\text{div} w) \circ X = \text{Tr}[AD(w \circ X)],
(3.3)
\]
where \( \text{Tr} \) denotes the trace of a square matrix. On the other hand, using an integration by part argument as in the appendix of [4], we also have
\[
(\text{div} w) \circ X = J^{-1} \text{div} (\mathcal{A}(w \circ X)).
(3.4)
\]
Applying (3.2) and (3.3), we see that
\[
(\nabla \text{div} w) \circ X = A^T \nabla \text{Tr}(AD(w \circ X)).
(3.5)
\]
By writing \( \Delta = \text{div} \nabla \), we get from (3.2) and (3.4) that
\[
(\Delta w) \circ X = J^{-1} \text{div} (\mathcal{A} A^T \nabla (w \circ X)).
(3.6)
\]
Now we introduce new unknowns in Lagrangian coordinates. Denote
\[
\bar{\rho}(t, y) \overset{\text{def}}{=} \rho(t, X(t, y)), \quad \bar{m}(t, y) \overset{\text{def}}{=} m(t, X(t, y)), \quad \text{and} \quad \bar{u}(t, y) \overset{\text{def}}{=} u(t, X(t, y)). \tag{3.7}
\]

The continuity equation in (1.2) has a unique weak solution \( \rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2) \) such that \( J\bar{\rho} \equiv \rho_0 \). Using (3.5), (3.6) and the chain rule, one can formally convert the system (1.2) into its Lagrangian formulation that reads
\[
\begin{cases}
\partial_t (J\bar{\rho}) = 0, \\
\partial_t (J\bar{m}) = 0, \\
\rho_0 \partial_t \bar{u} - \mu \text{div} (\mathcal{A}_u \bar{u} \nabla \bar{u}) - (\mu + \lambda) \nabla \text{Tr} (A_u D\bar{u}) + A^T \nabla P(\bar{\rho}, \bar{m}) = 0, \\
(\bar{\rho}, \bar{m}, \bar{u})|_{t=0} = (\rho_0, m_0, u_0),
\end{cases}
\tag{3.8}
\]

where we associate \( \mathcal{A}_u \) and \( A_u \) with the new velocity \( \bar{u} \), namely,
\[
\mathcal{A}_u = \text{adj} DX_u, \quad \text{and} \quad A_u = (DX_u(t, y))^{-1}
\]
with
\[
X_u(t, y) = y + \int_0^t \bar{u}(\tau, y) \, d\tau. \tag{3.9}
\]

We shall prove the local well-posedness of the nonlinear system (3.8) using the contraction mapping theorem. As the process is similar to the compressible Navier-Stokes equations discussed in [4], here we omit the details for convenience.

4. The proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2 in the following three subsections.

4.1. Low-frequency estimates. To study the coupling among \( a, b \) and \( \mathcal{Q}u \), it is convenient to set
\[
\varphi = \Lambda^{-1} \text{div} u,
\]
where \( \Lambda^{s}z \overset{\text{def}}{=} \mathcal{F}^{-1}(|\xi|^s \mathcal{F}z)(s \in \mathbb{R}) \). Since \( \varphi \) and \( \mathcal{Q}u = \nabla \Delta^{-1} \text{div} u \) can be converted into each other by a zeroth-order homogeneous Fourier multiplier, it suffices to bound \( \varphi \) in order to control \( \mathcal{Q}u \). Now one can infer from (1.3) that
\[
\begin{cases}
\partial_t a + \Lambda \varphi = f_1, \\
\partial_t \varphi - 2\Delta \varphi - 2\Lambda a - \Lambda b = f_2, \\
\partial_t b + \Lambda \varphi = f_3,
\end{cases}
\tag{4.1}
\]

where

\[
f_1 \overset{\text{def}}{=} - \text{div} (au), \quad f_2 \overset{\text{def}}{=} \Lambda^{-1} \text{div} \left( - (u \cdot \nabla u) + \mathcal{F}(a, u, b) \right), \quad f_3 \overset{\text{def}}{=} - \text{div} (bu).
\]

In this subsection, we prove the following crucial lemma.

Lemma 4.1. Let \( k_0 \) be some integer, and \( z^\ell \overset{\text{def}}{=} \mathcal{S}_{k_0}z \). For any \( t \geq 0 \), there holds that
\[
\| (a^\ell, \varphi^\ell, b^\ell) \|_{L^\infty(\mathcal{B}^0_{2,1})} + \| (a^\ell, \varphi^\ell, b^\ell) \|_{L^1(\mathcal{B}^2_{2,1})} \\
\lesssim \| (a_0^\ell, \varphi_0^\ell, b_0^\ell) \|_{\mathcal{B}^0_{2,1}} + \| ((f_1)^\ell, (f_2)^\ell, (f_3)^\ell) \|_{L^1(\mathcal{B}^0_{2,1})}. \tag{4.2}
\]
Proof. Setting $f_k = \hat{\Delta}_k f$, applying the operator $\hat{\Delta}_k S_{k_0}$ to the equations in (4.1), then multiplying (4.1) by $a_k^\ell$, (4.1) by $\varphi_k^\ell$, (4.1) by $b_k^\ell$, respectively, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|a_k^\ell\|^2_{L^2} + \|\varphi_k^\ell\|^2_{L^2} + \|b_k^\ell\|^2_{L^2} \right) + 2 \|\Lambda \varphi_k^\ell\|^2_{L^2} = \langle (f_1)^\ell_k, a_k^\ell \rangle + \langle (f_2)^\ell_k, \varphi_k^\ell \rangle + \langle (f_3)^\ell_k, b_k^\ell \rangle \quad (4.3)$$

where we have used the following cancellations

$$\langle 2\Lambda \varphi_k^\ell, a_k^\ell \rangle - \langle 2\Lambda a_k^\ell, \varphi_k^\ell \rangle - \langle \Lambda b_k^\ell, \varphi_k^\ell \rangle + \langle \Lambda \varphi_k^\ell, b_k^\ell \rangle = 0. \quad (4.4)$$

To capture the dissipation of $a$ and $b$, we need to consider the time derivative of the mixed terms involved in $\langle \varphi_k^\ell, \Lambda a_k^\ell \rangle$ and $\langle \varphi_k^\ell, \Lambda b_k^\ell \rangle$

$$- \frac{d}{dt} \langle \varphi_k^\ell, \Lambda a_k^\ell \rangle + 2 \|\Lambda a_k^\ell\|^2_{L^2} - \|\Lambda \varphi_k^\ell\|^2_{L^2} + \langle \Lambda b_k^\ell, \Lambda a_k^\ell \rangle = -2 \langle \Delta \varphi_k^\ell, \Lambda a_k^\ell \rangle - \langle (f_2)^\ell_k, \Lambda a_k^\ell \rangle - \langle \Lambda (f_1)^\ell_k, \varphi_k^\ell \rangle, \quad (4.5)$$

$$- \frac{d}{dt} \langle \varphi_k^\ell, \Lambda b_k^\ell \rangle + 2 \|\Lambda b_k^\ell\|^2_{L^2} - \|\Lambda \varphi_k^\ell\|^2_{L^2} + \langle \Lambda a_k^\ell, \Lambda b_k^\ell \rangle = -2 \langle \Delta \varphi_k^\ell, \Lambda b_k^\ell \rangle - \langle (f_2)^\ell_k, \Lambda b_k^\ell \rangle - \langle \Lambda (f_3)^\ell_k, \varphi_k^\ell \rangle. \quad (4.6)$$

To eliminate the highest order terms on the right-hand sides of (4.5) and (4.6), we next estimate $\|\Lambda a_k^\ell\|^2_{L^2}$ and $\|\Lambda b_k^\ell\|^2_{L^2}$. From (4.1), we have

$$\partial_t \Lambda a_k^\ell + \Lambda^2 \varphi_k^\ell = \Lambda (f_1)^\ell_k. \quad (4.7)$$

Testing (4.7) by $\Lambda a_k^\ell$ yields

$$\frac{1}{2} \frac{d}{dt} \|\Lambda a_k^\ell\|^2_{L^2} = \langle \Delta \varphi_k^\ell, \Lambda a_k^\ell \rangle + \langle (f_1)^\ell_k, \Lambda^2 a_k^\ell \rangle. \quad (4.8)$$

Similarly, we have

$$\frac{1}{2} \frac{d}{dt} \|\Lambda b_k^\ell\|^2_{L^2} = \langle \Delta \varphi_k^\ell, \Lambda b_k^\ell \rangle + \langle (f_3)^\ell_k, \Lambda^2 b_k^\ell \rangle. \quad (4.9)$$

Denote

$$L_k^{2, \text{def}} = \frac{1}{2} \left[ 4 \|a_k^\ell\|^2_{L^2} + 2 \|\varphi_k^\ell\|^2_{L^2} + 2 \|b_k^\ell\|^2_{L^2} - \langle \varphi_k^\ell, \Lambda a_k^\ell \rangle - \langle \varphi_k^\ell, \Lambda b_k^\ell \rangle + 2 \|\Lambda a_k^\ell\|^2_{L^2} + 2 \|\Lambda b_k^\ell\|^2_{L^2} \right].$$

Summing up (4.3) $\times 2$, (4.5), (4.6), (4.8) $\times 2$ and (4.9) $\times 2$, we obtain

$$\frac{1}{2} \frac{d}{dt} L_k^{2} + 2 \|\Lambda \varphi_k^\ell\|^2_{L^2} + 2 \|\Lambda a_k^\ell\|^2_{L^2} + 2 \|\Lambda b_k^\ell\|^2_{L^2} + 2 \langle \Lambda a_k^\ell, \Lambda b_k^\ell \rangle$$

$$= 4 \langle (f_1)^\ell_k, a_k^\ell \rangle + 2 \langle (f_2)^\ell_k, \varphi_k^\ell \rangle + 2 \langle (f_3)^\ell_k, b_k^\ell \rangle - \langle (f_2)^\ell_k, \Lambda a_k^\ell \rangle - \langle (f_3)^\ell_k, \Lambda b_k^\ell \rangle$$

$$- \langle (f_2)^\ell_k, \Lambda b_k^\ell \rangle - \langle \Lambda (f_3)^\ell_k, \varphi_k^\ell \rangle + 2 \langle (f_1)^\ell_k, \Lambda^2 a_k^\ell \rangle + 2 \langle (f_3)^\ell_k, \Lambda^2 b_k^\ell \rangle. \quad (4.10)$$

It’s straightforward to deduce from the low-frequency cut-off and Young’s inequality that

$$L_k^{2} \approx \|a_k^\ell, \Lambda a_k^\ell, \varphi_k^\ell, b_k^\ell, \Lambda b_k^\ell\|^2_{L^2},$$

which leads to

$$\frac{1}{2} \frac{d}{dt} L_k^{2} + 2^{2k} L_k^{2} \lesssim \|(f_1)^\ell_k, (f_2)^\ell_k, (f_3)^\ell_k\|^2_{L^2} L_k. \quad (4.11)$$
Dividing by $\mathcal{L}_k$ formally on both hand sides of (4.11), and then integrating from 0 to $t$, we finally get desired estimate (4.2) by summing up over $j \leq j_0$. This proves the lemma.

From Lemma 4.1 and the definitions of $\varphi$, $f_1$, $f_2$, and $f_3$, the low frequency part of $(a, Q\mathbf{u}, b)$ can be bounded by

$$
\|(a^\ell, Q\mathbf{u}^\ell, b^\ell)\|_{L^\infty(B_{p,1}^0)} + \|(a^\ell, Q\mathbf{u}^\ell, b^\ell)\|_{L^1(B_{p,1}^0)} \\
\lesssim \|(a_0^\ell, Q\mathbf{u}_0^\ell, b_0^\ell)\|_{B_{p,1}^2} + \|(\text{div } (a\mathbf{u}))^\ell\|_{L^1(B_{p,1}^0)} \\
+ \|(\text{div } (b\mathbf{u}))^\ell\|_{L^1(B_{p,1}^0)} + \|(\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{L^1(B_{p,1}^0)} + \|(F(a, \mathbf{u}, b))^\ell\|_{L^1(B_{p,1}^0)}.
$$

(4.12)

In the following, we estimate successively each of terms on the right hand side of (4.12). To simplify the writing, we introduce the following notation:

$$
\mathcal{E}_\infty(t) \overset{\text{def}}{=} \|(a, Q\mathbf{u}, b)^\ell\|_{B_{p,1}^0} + \|(a, b)^h\|_{B_{p,1}^0} + \|(P\mathbf{u}, Q\mathbf{u})^\ell\|_{B_{p,1}^2} + \|(P\mathbf{u}, Q\mathbf{u})^h\|_{B_{p,1}^2}.
$$

$$
\mathcal{E}_1(t) \overset{\text{def}}{=} \|(a, Q\mathbf{u}, b)^\ell\|_{B_{p,1}^2} + \|(a, b)^h\|_{B_{p,1}^2} + \|(P\mathbf{u}, Q\mathbf{u})^\ell\|_{B_{p,1}^2} + \|(P\mathbf{u}, Q\mathbf{u})^h\|_{B_{p,1}^2}.
$$

In view of Lemma 2.6, there holds

$$
\|(\text{div } (a\mathbf{u}))^\ell\|_{B_{p,1}^0} + \|(\text{div } (b\mathbf{u}))^\ell\|_{B_{p,1}^0} \\
\lesssim \|(a\mathbf{u})^\ell\|_{B_{p,1}^2} (\|a^\ell\|_{B_{p,1}^2} + \|a^h\|_{B_{p,1}^2} + \|b^\ell\|_{B_{p,1}^2} + \|b^h\|_{B_{p,1}^2}) \\
+ (\|a\|_{B_{p,1}^2} + \|b\|_{B_{p,1}^2}) (\|Q\mathbf{u}^\ell\|_{B_{p,1}^2} + \|Q\mathbf{u}^h\|_{B_{p,1}^2}) \\
\lesssim \|(P\mathbf{u})^\ell\|_{B_{p,1}^2}^2 + \|Q\mathbf{u}^\ell\|_{B_{p,1}^2}^2 + \|Q\mathbf{u}^h\|_{B_{p,1}^2}^2 + \|a^\ell\|_{B_{p,1}^2}^2 + \|a^h\|_{B_{p,1}^2}^2 + \|b^\ell\|_{B_{p,1}^2}^2 + \|b^h\|_{B_{p,1}^2}^2 \\
\lesssim \|(P\mathbf{u})^\ell\|_{B_{p,1}^2}^\frac{2}{p} + \|Q\mathbf{u}^\ell\|_{B_{p,1}^2}^\frac{2}{p} + \|Q\mathbf{u}^h\|_{B_{p,1}^2}^\frac{2}{p} + (\|a^\ell\|_{B_{p,1}^2}^2 + \|a^h\|_{B_{p,1}^2}^2 + \|b^\ell\|_{B_{p,1}^2}^2 + \|b^h\|_{B_{p,1}^2}^2) \\
\lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t).
$$

(4.13)

To bound $\|\mathbf{u} \cdot \nabla \mathbf{u}^\ell\|_{B_{p,1}^2}$, we obtain from $\mathbf{u} = P\mathbf{u} + Q\mathbf{u}$ and Lemma 2.6 that

$$
\|\mathbf{u} \cdot \nabla \mathbf{u}^\ell\|_{B_{p,1}^2} \lesssim \|(P\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{B_{p,1}^2} + \|(Q\mathbf{u} \cdot \nabla \mathbf{u})^\ell\|_{B_{p,1}^2} \\
\lesssim \|(P\mathbf{u})^\ell\|_{B_{p,1}^2} \|\nabla \mathbf{u}\|_{B_{p,1}^2} + (\|Q\mathbf{u}^\ell\|_{B_{p,1}^2} + \|Q\mathbf{u}^h\|_{B_{p,1}^2}) \|\nabla \mathbf{u}\|_{B_{p,1}^2}.
$$

(4.14)

Due to

$$
\|\nabla \mathbf{u}\|_{B_{p,1}^2} \lesssim \|(P\mathbf{u})^\ell\|_{B_{p,1}^2} + \|Q\mathbf{u}^\ell\|_{B_{p,1}^2} + \|Q\mathbf{u}^h\|_{B_{p,1}^2} \lesssim \mathcal{E}_1(t),
$$

we infer from (4.14) that

$$
\|\mathbf{u} \cdot \nabla \mathbf{u}^\ell\|_{B_{p,1}^2} \lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t).
$$

(4.15)
We now turn to bound the terms contained in $F(a, u, b)$. Thanks to Lemma 2.6 and Lemma 2.8, we get
\[
\|(I(a)(\nabla b))\|_{B_{2,1}^{0}} \lesssim \|I(a)\|_{B_{p}^{p}} (\|\nabla b\|_{B_{2,1}^{0}} + \|\nabla b\|_{B_{p}^{p}}) \\
\lesssim \|a\|_{B_{p}^{p}} (\|\nabla b\|_{B_{2,1}^{0}} + \|\nabla b\|_{B_{p}^{p}}) \\
\lesssim (\|a\|_{B_{2,1}^{0}} + \|a\|_{B_{p}^{p}}) (\|b\|_{B_{2,1}^{1}} + \|b\|_{B_{p}^{p}}) \\
\lesssim \mathcal{E}_{\infty}(t) \mathcal{E}_{1}(t). \tag{4.16}
\]
Similarly, we can bound $\|(k(a)(\nabla a))\|_{B_{2,1}^{0}}$ as
\[
\|k(a)(\nabla a)\|_{B_{2,1}^{0}} \lesssim \mathcal{E}_{\infty}(t) \mathcal{E}_{1}(t). \tag{4.17}
\]
Thanks to Lemma 2.6 again and the interpolation inequality, there holds
\[
\|(b(\nabla a))\|_{B_{2,1}^{0}} \lesssim \|b\|_{B_{p}^{p}} (\|\nabla b\|_{B_{2,1}^{0}} + \|\nabla b\|_{B_{p}^{p}}) \\
\lesssim (\|b\|_{B_{2,1}^{0}} + \|b\|_{B_{p}^{p}}) (\|b\|_{B_{2,1}^{0}} + \|b\|_{B_{p}^{p}}) \\
\lesssim \|b\|_{B_{2,1}^{0}} \|b\|_{B_{2,1}^{0}} + \|b\|_{B_{p}^{p}}^2 \\
\lesssim \mathcal{E}_{\infty}(t) \mathcal{E}_{1}(t). \tag{4.18}
\]
Similarly,
\[
\|(I(a)(b(\nabla a)))\|_{B_{2,1}^{0}} \lesssim \|I(a)\|_{B_{p}^{p}} (\|\nabla b\|_{B_{2,1}^{0}} + \|\nabla b\|_{B_{p}^{p}}) \\
\lesssim (1 + \|b\|_{B_{p}^{p}}) (\|b\|_{B_{2,1}^{0}} + \|b\|_{B_{p}^{p}}) (\|b\|_{B_{2,1}^{0}} + \|b\|_{B_{p}^{p}}) \\
\lesssim (1 + \mathcal{E}_{\infty}(t)) \mathcal{E}_{\infty}(t) \mathcal{E}_{1}(t). \tag{4.19}
\]
As we set $F\ u$ in the $L^p$ type spaces, we cannot use Lemma 2.6 directly to bound the term $I(a)\Delta u$. For an integer $j_0 \geq 0$, we use Bony’s decomposition to rewrite this term into
\[
\dot{S}_{j_{0}+1}Q(I(a)\Delta u) = \dot{S}_{j_{0}+1}Q(\hat{T}_{\Delta u}I(a) + \hat{R}(\Delta u, I(a))) \\
+ \hat{T}_{I(a)}\Delta \dot{S}_{j_{0}+1}Q u + [\dot{S}_{j_{0}+1}Q, \hat{T}_{I(a)}] \Delta u. \tag{4.20}
\]
The first term can be bounded by Lemmas 2.3 and 2.8,
\[
\left\|(\dot{S}_{j_{0}+1}Q(\hat{T}_{\Delta u}I(a) + \hat{R}(\Delta u, I(a))))\right\|_{B_{2,1}^{0}} \\
\lesssim \|I(a)\|_{B_{p}^{p}} (\|\Delta u\|_{B_{2,1}^{0}} + \|\Delta u\|_{B_{p}^{p}}) \\
\lesssim (\|a\|_{B_{p}^{p}} + \|a\|_{B_{p}^{p}}) (\|Q u\|_{B_{2,1}^{0}} + \|Q u\|_{B_{p}^{p}}). \tag{4.21}
\]
Similarly, we have
\[
\left\|(\hat{T}_{I(a)}\Delta \dot{S}_{j_{0}+1}Q u)\right\|_{B_{2,1}^{0}} \lesssim \|I(a)\|_{L^\infty} \|\Delta \dot{S}_{j_{0}+1}Q u\|_{B_{2,1}^{0}}
\]
The commutator term is estimated by using Lemma 6.1 in [5] that

$$\lesssim \|a\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} \|Qu^f\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}$$

$$\lesssim (\|a^f\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} + \|a^h\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}) \|Qu^f\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}. \quad (4.22)$$

The commutator term is estimated by using Lemma 6.1 in [5] that

$$\left\|\left[\hat{S}_{j_0+1}Q, T_{I(\alpha)}\right] \Delta u\right\|_{\tilde{B}^{\frac{2}{p}}_{p,1}} \lesssim \|\nabla I(a)\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} \|\nabla^2 u\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}, \quad \left(\frac{1}{p^*} + \frac{1}{p} = \frac{1}{2}\right),$$

$$\lesssim \|\nabla I(a)\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} \|u\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}$$

$$\lesssim \|a\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} \|u\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}$$

$$\lesssim (\|a^f\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} + \|a^h\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}) \|Qu^f\|_{\tilde{B}^{\frac{2}{p}}_{p,1}} + \|(P_u, Qu^h)\|_{\tilde{B}^{\frac{2}{p}}_{p,1}}, \quad (4.23)$$

where we have used the embedding $\tilde{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2) \hookrightarrow \tilde{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, $2 < p < 4$.

The term $I(a)\nabla \text{div} u$ can be estimated in a similar manner. As a result, we have

$$\|\left(I(a)(\Delta u + \nabla \text{div} u)\right)^f\|_{\tilde{B}^{\frac{2}{p}}_{p,1}} \lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t). \quad (4.24)$$

Plugging (4.13), (4.14)-(4.19), and (4.24) into (4.12) gives

$$\|(a^f, Qu^f, b^f)\|_{L^{\infty}_x(B^{\frac{2}{p}-1}_{p,1})} + \|(a^f, Qu^f, b^f)\|_{L^1_t(B^{\frac{2}{p}}_{p,1})}$$

$$\lesssim \|(a^f, Qu^f_0, b^f_0)\|_{\tilde{B}^{\frac{2}{p}}_{p,1}} + \int_0^t (1 + \mathcal{E}_\infty(s)) \mathcal{E}_\infty(s) \mathcal{E}_1(s) \, ds. \quad (4.25)$$

4.2. High-frequency estimates. In this subsection, we shall introduce two effective velocity to capture the damping effect of $a$ and $b$ respectively. First, denoting

$$d = a + \frac{1}{2}b, \quad (4.26)$$

we infer from (1.3) that $(d, Qu)$ satisfies

$$\begin{cases}
\partial_t d + \frac{3}{2} \text{div} u = -\text{div} (du), \\
\partial_t Qu - 2\Delta Qu + 2\text{div} d = -Qu \cdot \nabla u + Qu \cdot \text{div} (a, b).
\end{cases} \quad (4.27)$$

Now, we define the first effective velocity $G$ as follows

$$G \overset{\text{def}}{=} Qu - \Delta^{-1} \nabla d. \quad (4.28)$$

Then $G$ satisfies

$$\partial_t G - 2\Delta G = \frac{3}{2} G + \frac{3}{2} \Delta^{-1} \nabla d + Qu (du) - Qu \cdot \nabla u + Qu F(a, u, b). \quad (4.29)$$

Applying the heat estimate (2.9) for the high frequencies of $G$ only, we get

$$\|G^h\|_{L^{\infty}_x(B^{\frac{2}{p}-1}_{p,1})} + \|G^h\|_{L^1_t(B^{\frac{2}{p}}_{p,1})}$$

$$\lesssim \|G^h_0\|_{\tilde{B}^{\frac{2}{p}-1}_{p,1}} + \|G^h\|_{L^1_t(B^{\frac{2}{p}}_{p,1})} + \|(\Delta)^{-1} \nabla d^h\|_{L^1_t(B^{\frac{2}{p}}_{p,1})}$$

$$+ \|Qu (du)^h\|_{L^1_t(B^{\frac{2}{p}}_{p,1})} + \|Qu (u, \nabla u)^h\|_{L^1_t(B^{\frac{2}{p}}_{p,1})} + \|Qu F(a, u, b)^h\|_{L^1_t(B^{\frac{2}{p}}_{p,1})}. \quad (4.30)$$
The important point is that, owing to the high frequency cut-off at $|\xi| \sim 2^{j_0}$,
\[
\|G^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} \lesssim 2^{-2j_0} \|G^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} \quad \text{and} \quad \|d^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} \lesssim 2^{-2j_0} \|d^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})}.
\]
Hence, if $j_0$ is large enough then the term $\|G^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})}$ may be absorbed by the right hand side.

In view of (4.28), we have
d\frac{3}{2}d + u \cdot \nabla d = -\frac{3}{2} \text{div } G - d \text{ div } u. \quad (4.31)

Applying $\dot{\Delta}_j$ to (4.31) and using a commutator argument give rise to
\[
\partial_t \dot{\Delta}_j d + \frac{3}{2} \dot{\Delta}_j d + u \cdot \nabla \dot{\Delta}_j d = -[\dot{\Delta}_j, u \cdot \nabla]d - \frac{3}{2} \dot{\Delta}_j \text{div } G - \dot{\Delta}_j (d \text{ div } u). \quad (4.32)
\]
A standard energy argument leads to
\[
\|\dot{\Delta}_j d(t)\|_{L^p} + \int_0^t \|\dot{\Delta}_j d\|_{L^p} \, ds \\
\lesssim \|\dot{\Delta}_j d_0\|_{L^p} + \frac{1}{p} \int_0^t \|\text{div } u\|_{L^\infty} \|\dot{\Delta}_j d\|_{L^p} \, ds \\
+ \int_0^t \|\dot{\Delta}_j, u \cdot \nabla\|_{L^p} \, ds + \int_0^t \|\text{div } G\|_{L^p} \, ds + \int_0^t \|\dot{\Delta}_j (d \text{ div } u)\|_{L^p} \, ds \quad (4.33)
\]
from which we can further get
\[
\|d^h\|_{L^\infty_t(B^\frac{2}{p-1}_{p,1})} + \frac{3}{2} \|d^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} \\
\lesssim \|d_0^h\|_{B^\frac{2}{p-1}_{p,1}} + \frac{3}{2} \|G^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} + \int_0^t \|\nabla u\|_{B^\frac{2}{p-1}_{p,1}} \|d\|_{B^\frac{2}{p-1}_{p,1}} \, ds. \quad (4.34)
\]
Multiplying (4.30) by a suitable large constant and adding to (4.34), we obtain
\[
\|G^h\|_{L^\infty_t(B^\frac{2}{p-1}_{p,1})} + \|d^h\|_{L^\infty_t(B^\frac{2}{p-1}_{p,1})} + \|G^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} + \|d^h\|_{L^1_t(B^\frac{2}{p-1}_{p,1})} \\
\lesssim \|G_0^h\|_{B^\frac{2}{p-1}_{p,1}} + \|d_0^h\|_{B^\frac{2}{p-1}_{p,1}} + \int_0^t \|\nabla u\|_{B^\frac{2}{p-1}_{p,1}} \|d\|_{B^\frac{2}{p-1}_{p,1}} \, ds \\
+ \int_0^t \|u \cdot \nabla u\|_{B^\frac{2}{p-1}_{p,1}} + \|(d^u)^h\|_{B^\frac{2}{p-1}_{p,1}} ds + \int_0^t \|F(a, u, b)^h\|_{B^\frac{2}{p-1}_{p,1}} \, ds. \quad (4.35)
\]
To obtain their respective high frequency estimates for $a$ and $b$, we need to introduce another new effective velocity $\Gamma$,
\[
\Gamma \overset{\text{def}}{=} Q u - \frac{1}{2} \Delta^{-1} \nabla a. \quad (4.36)
\]
Then $\Gamma$ and $a$ satisfy
\[
\partial_t \Gamma - 2 \Delta \Gamma = \frac{1}{2} \Gamma + \frac{3}{4} \Delta^{-1} \nabla a + \frac{1}{2} Q(a u) - Q(a, \nabla u) - \nabla d + Q F(a, u, b), \quad (4.37)
\]
\[
\partial_t a + \frac{1}{2} a + u \cdot \nabla a = -\text{div } \Gamma - a \text{ div } u. \quad (4.38)
\]
As in the derivation of (4.30) and (4.34), we have
\[
\| \mathbf{\Gamma}^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} + \| \mathbf{\Gamma}^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} \\
\lesssim \| \mathbf{\Gamma}_0^h \|_{B_{p,1}^{\frac{2}{p}-1}} + \int_0^t \left( \| \mathbf{\Gamma}^h \|_{B_{p,1}^{\frac{2}{p}-1}} + \| a^h \|_{B_{p,1}^{\frac{2}{p}-2}} \right) ds + \int_0^t \| d^h \|_{B_{p,1}^{\frac{2}{p}}} ds \\
+ \int_0^t \| u \cdot \nabla u \|_{B_{p,1}^{\frac{2}{p}-1}} + \| (au)^h \|_{B_{p,1}^{\frac{2}{p}-1}} ds + \int_0^t \left\| (F(a, u, b))^h \right\|_{B_{p,1}^{\frac{2}{p}-1}} ds,
\]
(4.39)
and
\[
\| a^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}+1})} + \| a^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})} \\
\lesssim \| a_0^h \|_{B_{p,1}^{\frac{2}{p}}} + \int_0^t \| a^h \|_{B_{p,1}^{\frac{2}{p}} ds} + \int_0^t \| \nabla u \|_{B_{p,1}^{\frac{2}{p}}} \| a \|_{B_{p,1}^{\frac{2}{p}}} ds.
\]
(4.40)
Combining with (4.35), (4.39) and (4.40), we finally get
\[
\| G^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} + \| a^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}})} + \| G^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \| d^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})} \\
+ \| \mathbf{\Gamma}^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} + \| a^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}})} + \| \mathbf{\Gamma}^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \| a^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})} \\
\lesssim \| G_0^h \|_{B_{p,1}^{\frac{2}{p}}} + \| d_0^h \|_{B_{p,1}^{\frac{2}{p}}} + \| \mathbf{\Gamma}_0^h \|_{B_{p,1}^{\frac{2}{p}}} + \| a_0^h \|_{B_{p,1}^{\frac{2}{p}}} + \int_0^t \| u \|_{B_{p,1}^{\frac{2}{p}}} \| d \|_{B_{p,1}^{\frac{2}{p}}} ds \\
+ \int_0^t \left( \| u \cdot \nabla u \|_{B_{p,1}^{\frac{2}{p}-1}} + \| (du)^h \|_{B_{p,1}^{\frac{2}{p}-1}} \right) ds + \int_0^t \left\| (F(a, u, b))^h \right\|_{B_{p,1}^{\frac{2}{p}-1}} ds \\
+ \int_0^t \left( \| (a \text{ div } u)^h \|_{B_{p,1}^{\frac{2}{p}}} + \| (au)^h \|_{B_{p,1}^{\frac{2}{p}-1}} + \| \mathbf{Q}_u \|_{B_{p,1}^{\frac{2}{p}+1}} \| ah \|_{B_{p,1}^{\frac{2}{p}}} \right) ds.
\]
(4.41)
On the one hand, in view of $\mathbf{\Gamma} \overset{\text{def}}{=} \mathbf{Q} u - \frac{1}{2} \Delta^{-1} \nabla a$ and the embedding relation in the high frequency, there holds
\[
\| \mathbf{Q} u^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} \lesssim \| \mathbf{\Gamma}^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} + \| \Delta^{-1} \nabla a^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} \\
\lesssim \| \mathbf{\Gamma}^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}-1})} + \| a^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}})},
\]
(4.42)
\[
\| \mathbf{Q} u^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} \lesssim \| \mathbf{\Gamma}^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \| \Delta^{-1} \nabla a^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} \\
\lesssim \| \mathbf{\Gamma}^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}+1})} + \| a^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})}.
\]
(4.43)
On the other hand, from $d = a + \frac{1}{2} b$, we have
\[
\| b^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}})} \lesssim \| d^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}})} + \| a^h \|_{L^\infty_t(B_{p,1}^{\frac{2}{p}})},
\]
\[
\| b^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})} \lesssim \| d^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})} + \| a^h \|_{L^1_t(B_{p,1}^{\frac{2}{p}})}.
\]
(4.44)
As a result, we can rewrite (4.41) into
\[
\begin{align*}
\|(a^h, b^h)\|_{L^\infty_t(B^2_{p,1})} &+ \|\mathbb{Q}u^h\|_{L^\infty_t(B^{2-1}_{p,1})} + \|(a^h, b^h)\|_{L^1_t(B^{2}_{p,1})} + \|\mathbb{Q}u^h\|_{L^1_t(B^{2+1}_{p,1})} \\
&\lesssim \|a_0^h\|_{B^{2}_{p,1}} + \|b_0^h\|_{B^{2+1}_{p,1}} + \|\mathbb{Q}u_0^h\|_{B^{2-1}_{p,1}} + \\
&\quad + \int_0^t \|u\|_{B^{2+1}_{p,1}} \|(a, b)\|_{B^{2}_{p,1}} ds + \int_0^t \|u \cdot \nabla u\|_{B^{2-1}_{p,1}} + \|\mathbb{Q}u^h\|_{B^{2+1}_{p,1}} ds \\
&\quad + \int_0^t \|(a \div u)^h\|_{B^{2}_{p,1}} + \|\mathbb{Q}u^h\|_{B^{2-1}_{p,1}} ds + \int_0^t \|\mathbb{F}(a, u)\|_{B^{2+1}_{p,1}} ds. \tag{4.45}
\end{align*}
\]
Finally, we estimate the incompressible part of the velocity field. Applying the operator \(\mathbb{P}\) to the second equation of (1.3), we find that \(\mathbb{P}u\) satisfies the heat equation
\[
\partial_t \mathbb{P}u - \Delta \mathbb{P}u = -\mathbb{P}(u \cdot \nabla u) + \mathbb{P}\mathbb{F}(a, u, b). \tag{4.46}
\]
By Lemma 2.9, we can get
\[
\begin{align*}
\|\mathbb{P}u\|_{L^\infty_t(B^{2+1}_{p,1})} &+ \|\mathbb{P}u\|_{L^1_t(B^{2-1}_{p,1})} \\
&\lesssim \|\mathbb{P}u_0\|_{B^{2-1}_{p,1}} + \|\mathbb{Q}u_0^h\|_{B^{2+1}_{p,1}} + \|(a, b)\|_{L^1_t(B^{2}_{p,1})} + \|\mathbb{Q}u^h\|_{L^1_t(B^{2+1}_{p,1})}. \tag{4.47}
\end{align*}
\]
Combining (4.45) with (4.47) gives
\[
\begin{align*}
\|(a^h, b^h)\|_{L^\infty_t(B^{2}_{p,1})} &+ \|(a^h, b^h)\|_{L^1_t(B^{2+1}_{p,1})} \\
&\lesssim \|a^h_0, b^h_0\|_{B^{2}_{p,1}} + \|(a, b)\|_{L^1_t(B^{2}_{p,1})} + \|\mathbb{P}u\|_{B^{2+1}_{p,1}} + \|\mathbb{Q}u^h\|_{B^{2-1}_{p,1}} + \\
&\quad + \int_0^t \|u\|_{B^{2+1}_{p,1}} \|(a, b)\|_{B^{2}_{p,1}} ds + \int_0^t \|u \cdot \nabla u\|_{B^{2-1}_{p,1}} + \|\mathbb{Q}u^h\|_{B^{2+1}_{p,1}} ds \\
&\quad + \int_0^t \|(a \div u)^h\|_{B^{2}_{p,1}} + \|\mathbb{Q}u^h\|_{B^{2-1}_{p,1}} ds + \int_0^t \|\mathbb{F}(a, u)\|_{B^{2+1}_{p,1}} ds. \tag{4.48}
\end{align*}
\]
We now bound the terms on the right hand side of (4.48). First, according to Lemma 2.5,
\[
\|u \cdot \nabla u\|_{B^{2+1}_{p,1}} \lesssim \|u\|_{B^{2}_{p,1}}^2 \\
\lesssim \|\mathbb{P}u\|_{B^{2}_{p,1}}^2 + \|\mathbb{Q}u^f\|_{B^2_{p,1}}^2 + \|\mathbb{Q}u^h\|_{B^{2}_{p,1}}^2 \\
\lesssim \|\mathbb{P}u\|_{B^{2+1}_{p,1}} \|\mathbb{P}u\|_{B^{2}_{p,1}} + \|\mathbb{Q}u^f\|_{B^{2+1}_{p,1}} \|\mathbb{Q}u^f\|_{B^2_{p,1}} + \|\mathbb{Q}u^h\|_{B^{2-1}_{p,1}} \|\mathbb{Q}u^h\|_{B^{2+1}_{p,1}} \\
\lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t). \tag{4.49}
\]
By the embedding relation in high frequency and the Young inequality, we get
\[
\|(a^h)^h\|_{B^{2+1}_{p,1}} \lesssim \||(a^h)^h\|_{B^{2}_{p,1}} \lesssim \|a\|_{B^{2}_{p,1}}^2 + \|u\|_{B^{2}_{p,1}}^2 \lesssim \mathcal{E}_\infty(t) \mathcal{E}_1(t). \tag{4.50}
\]
Similarly,
\[ \left\| (b u)^h \right\|_{B^{\frac{3}{p}}_{p,1}} \lesssim E_{\infty}(t) E_1(t), \]
\[ \left\| (a \text{div } u)^h \right\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \left\| a \right\|_{B^{\frac{3}{p}}_{p,1}} ^a \left\| \text{div } u \right\|_{B^{\frac{3}{p}}_{p,1}} ^a \]
\[ \lesssim (\left\| a^f \right\|_{B^{\frac{2}{p}}_{1,1}} ^a + \left\| a^h \right\|_{B^{\frac{3}{p}}_{p,1}} ^a) (\left\| Q u^f \right\|_{B^{\frac{2}{p}}_{2,1}} ^a + \left\| Q u^h \right\|_{B^{\frac{3}{p}}_{p,1}} ^a) \]
\[ \lesssim E_{\infty}(t) E_1(t). \quad (4.51) \]

At last, we deal with each term in \( F(a, u, b) \). In view of Lemmas 2.5, 2.8 and the Young inequality, there holds
\[ \left\| (I(a) \nabla b)^h \right\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \left\| I(a) \right\|_{B^{\frac{3}{p}}_{p,1}} \left\| \nabla b \right\|_{B^{\frac{3}{p}}_{p,1}} \]
\[ \lesssim \left\| a \right\|_{B^{\frac{2}{p}}_{p,1}} ^a \left\| b \right\|_{B^{\frac{2}{p}}_{p,1}} ^b \]
\[ \lesssim \left\| a \right\|_{B^{\frac{2}{p}}_{p,1}} ^a \left\| b \right\|_{B^{\frac{2}{p}}_{p,1}} ^b \]
\[ \lesssim \left\| a^f \right\|_{B^{\frac{2}{p}}_{1,1}} ^a + \left\| a^h \right\|_{B^{\frac{3}{p}}_{p,1}} ^a + \left\| b^f \right\|_{B^{\frac{2}{p}}_{1,1}} ^b + \left\| b^h \right\|_{B^{\frac{3}{p}}_{p,1}} ^b \]
\[ \lesssim E_{\infty}(t) E_1(t). \quad (4.52) \]

The term \( \left\| (k(a) \nabla a)^h \right\|_{B^{\frac{3}{p}}_{p,1}} \) can be bounded in the same manner. The last two terms in \( F(a, u, b) \) can be estimated directly via Lemmas 2.5 and 2.8,
\[ \left\| ((I(a) - 1)b(\nabla b))^h \right\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \left\| (I(a)) \right\|_{B^{\frac{3}{p}}_{p,1}} ^{\frac{2}{p}} + 1 \left\| b \right\|_{B^{\frac{3}{p}}_{p,1}} ^b \left\| \nabla b \right\|_{B^{\frac{3}{p}}_{p,1}} ^b \]
\[ \lesssim \left\| a \right\|_{B^{\frac{2}{p}}_{p,1}} ^a \left\| 1 + E_{\infty}(t) \right\|_{B^{\frac{2}{p}}_{p,1}} ^{\frac{2}{p}} \]
\[ \lesssim (1 + E_{\infty}(t)) E_{\infty}(t) E_1(t), \quad (4.53) \]

and
\[ \left\| (I(a)(\Delta u + \nabla \text{div } u))^h \right\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \left\| I(a) \right\|_{B^{\frac{3}{p}}_{p,1}} \left\| u \right\|_{B^{\frac{3}{p}}_{p,1}} \]
\[ \lesssim \left\| a \right\|_{B^{\frac{2}{p}}_{p,1}} ^a \left( \left\| \text{div } u \right\|_{B^{\frac{3}{p}}_{p,1}} + \left\| Q u^f \right\|_{B^{\frac{2}{p}}_{2,1}} + \left\| Q u^h \right\|_{B^{\frac{3}{p}}_{p,1}} \right) \]
\[ \lesssim E_{\infty}(t) E_1(t). \quad (4.54) \]

Collecting the estimates above, we finally get from (4.48) that
\[ \left\| (a^h, b^h) \right\|_{L^2_e(\tilde{B}^{\frac{3}{p}}_{p,1})} + \left\| (\text{div } u, Q u^h) \right\|_{L^\infty(\tilde{B}^{\frac{3}{p}}_{p,1})} \]
\[ + \left\| (a^h, b^h) \right\|_{L^1(\tilde{B}^{\frac{3}{p}}_{p,1})} + \left\| (\text{div } u, Q u^h) \right\|_{L^1(\tilde{B}^{\frac{3}{p}}_{p,1})} \]
\[ \lesssim \left\| (a^0, b^0) \right\|_{\tilde{B}^{\frac{3}{p}}_{p,1}} + \left\| (\text{div } u_0, Q u^0) \right\|_{\tilde{B}^{\frac{3}{p}}_{p,1}} + \int_0^t (1 + E_{\infty}(s)) E_{\infty}(s) E_1(s) ds. \quad (4.55) \]
4.3. **Proof of Theorem 1.2.** In this subsection, we shall give the proof of Theorem 1.2 by the local existence result and the continuation argument. Denote

$$\mathcal{X}(t) \overset{\text{def}}{=} \| (a^\ell, Q u^\ell, b^\ell) \|_{L^\infty(\mathcal{B}^0_{2,1})} + \| (a^h, b^h) \|_{L^\infty(B^0_{p,1})} + \| (\mathcal{P} u, Q u^h) \|_{L^\infty(B^{2-\ell}_{p,1})}$$

$$+ \| (a^\ell, Q u^\ell, b^\ell) \|_{L^1(\mathcal{B}^0_{2,1})} + \| (a^h, b^h) \|_{L^1(B^0_{p,1})} + \| (\mathcal{P} u, Q u^h) \|_{L^1(B^{2-\ell}_{p,1})},$$

$$\mathcal{X}_0 \overset{\text{def}}{=} \| (a_0^\ell, Q u_0^\ell, b_0^\ell) \|_{B^0_{2,1}} + \| (a_0^h, b_0^h) \|_{B^{2-\ell}_{p,1}} + \| (\mathcal{P} u_0, Q u_0^h) \|_{B^{2-\ell}_{p,1}}.$$  (4.56)

It follows from (4.25) and (4.45) that

$$\mathcal{X}(t) \leq \mathcal{X}_0 + C(\mathcal{X}(t))^2 (1 + C\mathcal{X}(t)).$$  (4.57)

Under the setting of initial data in Theorem 1.2, there exists a positive constant $C_0$ such that $\mathcal{X}_0 \leq C_0 \epsilon$. Due to the local existence result which has been achieved by Proposition 1.1, there exists a positive time $T$ such that

$$\mathcal{X}(t) \leq 2C_0 \epsilon, \quad \forall t \in [0, T].$$  (4.58)

Let $T^*$ be the largest possible time of $T$ for what (4.58) holds. Now, we only need to show $T^* = \infty$. By the estimate of (4.57), we can use a standard continuation argument to prove that $T^* = \infty$ provided that $\epsilon$ is small enough. We omit the details here. This finishes the proof of Theorem 1.2.

\[ \Box \]

5. **The proof of Theorem 1.3**

In this section, we shall follow the method (independent of the spectral analysis) used in [12] and [40] to get the decay rate of the solutions constructed in the previous section. From the proof of Theorem 1.2, we can get the following inequality (see the derivation of (4.25) and (4.55) for more details):

$$\frac{d}{dt} (\| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}} + \| (a, b) \|^h_{B^2_{2,1}} + \| (a, b) \|^h_{B^{2-\ell}_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}})$$

$$\lesssim (1 + \| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}})$$

$$\times (\| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}}) (\| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}})).$$  (5.1)

By Theorem 1.2,

$$\| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}} \leq C_0.$$  (5.2)

Inserting (5.2) in (5.1) yields

$$\frac{d}{dt} (\| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}})$$

$$+ \bar{c} (\| (a, b) \|^h_{B^0_{2,1}} + \| (a, b) \|^h_{B^h_{p,1}} + \| u \|^h_{B^{2-\ell}_{p,1}}) \leq 0.$$  (5.3)
In order to derive the decay estimate of the solutions given in Theorem 1.2, we need to get a Lyapunov-type differential inequality from (5.3). According to (5.2) and the embedding relation in the high frequency, it’s obvious for any $\beta > 0$ that

$$\|(a, b)\|_{B^{\frac{2}{p}, 1}_p}^h \geq C(\|(a, b)\|_{B^{\frac{2}{p}, 1}_p}^h)^{1+\beta},$$  \hspace{1cm} (5.4)

and

$$\|u\|_{B^{\frac{2}{p}+1, 1}_p}^h \geq C(\|u\|_{B^{\frac{2}{p}+1, 1}_p}^h)^{1+\beta}.$$  \hspace{1cm} (5.5)

Thus, to get the Lyapunov-type inequality of the solutions, we only need to control the norm of $\|(a, u, b)\|_{B^{2, 1}_p}^\ell$. This process can be obtained from the fact that the solutions constructed in Theorem 1.2 can propagate the regularity of the initial data in Besov space with low regularity, see the following Proposition 5.1. This will ensure that one can use interpolation to get the desired Lyapunov-type inequality.

**Proposition 5.1.** Let $(a, u, b)$ be the solutions constructed in Theorem 1.2. Assume further that $(a_0^\ell, u_0^\ell, b_0^\ell) \in \dot{B}_{2, \infty}^{-\sigma}(\mathbb{R}^2)$ for some $0 < \sigma \leq \frac{4}{p} - 1$. Then there exists a constant $C_0 > 0$ depending on the norm of the initial data such that for all $t \geq 0$,

$$\|(a, u, b)(t, \cdot)\|_{\dot{B}_{2, \infty}^{-\sigma}} \leq C_0.$$  \hspace{1cm} (5.6)

**Proof.** It follows from (4.3) that

$$\frac{1}{2} \frac{d}{dt}(2 \|a_k\|^2_{L^2} + \|u_k\|^2_{L^2} + \|b_k\|^2_{L^2}) + 2 \|\Lambda u_k\|^2_{L^2} = \langle 2(f_1)_k, a_k \rangle + \langle (f_2)_k, u_k \rangle + \langle (f_3)_k, b_k \rangle.$$  \hspace{1cm} (5.7)

By performing a routine procedure, one obtains

$$\|(a, u, b)\|_{\dot{B}_{2, \infty}^{-\sigma}}^\ell \lesssim \|(a_0, u_0, b_0)\|_{\dot{B}_{2, \infty}^{-\sigma}} + \int_0^t \|\langle f_1, f_2, f_3 \rangle\|_{\dot{B}_{2, \infty}^{-\sigma}}^\ell ds.$$  \hspace{1cm} (5.8)

To bound the nonlinear terms in $f_1, f_2, f_3, f_4$, we first establish the following two important estimates.

**Lemma 5.2.** Let $2 \leq p < 4$, there hold

$$\|fg\|_{\dot{B}_{2, \infty}^{-\sigma}} \lesssim \|f\|_{\dot{B}_{2, \infty}^{-\sigma}} \|g\|_{B^{\frac{2}{p}, 1}_p}, \quad -\frac{2}{p} < \sigma \leq \frac{2}{p},$$  \hspace{1cm} (5.9)

$$\|fg\|_{\dot{B}_{2, \infty}^{-\sigma}} \lesssim \|f\|_{\dot{B}_{2, \infty}^{-\sigma}} \|g\|_{\dot{B}_{p, \infty}^{\frac{2}{p}}}, \quad 0 < \sigma \leq \frac{4}{p} - 1.$$  \hspace{1cm} (5.10)

**Proof.** To obtain (5.9), we use Bony’s decomposition to rewrite

$$fg = \hat{T}f g + \hat{T}g f + \hat{R}(f, g).$$  \hspace{1cm} (5.11)

Due to Hölder’s inequality, Bernstein’s inequality and the embedding $\dot{B}_{p, 1}^{\frac{2}{p}}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we get

$$\|\Delta_j(\hat{T}f g)\|_{L^2} \lesssim \sum_{|k-j| \leq 1} \|\Delta_j(\hat{S}_{k-1}f \Delta_k g)\|_{L^2}$$
Similarly, for any $\sigma > -\frac{2}{p}$, the second term in (5.11) can be estimated as follows,

$$\|\hat{\Delta}_j(T_g f)\|_{L^2} \lesssim \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} \|\hat{\Delta}_k g\|_{L^{2p/(p-2)}} \|\hat{\Delta}_k f\|_{L^p}$$

$$\lesssim \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} 2^{k'\left(\frac{2}{p} + \sigma\right)} 2^{-\sigma k'} \|\hat{\Delta}_k' g\|_{L^2} \|\hat{\Delta}_k f\|_{L^p}$$

$$\lesssim 2^{\sigma j} \|f\|_{B_{2,p}^s} \|g\|_{B_{2,\infty}^{-\sigma}}. \quad (5.13)$$

In view of the fact that $\sigma \leq \frac{2}{p}$, we can deal with the last term in (5.11)

$$\|\hat{\Delta}_j \hat{R}(f, g)\|_{L^2} \lesssim \sum_{k \geq j-3} \sum_{|k-k'| \leq 1} \|\hat{\Delta}_j (\hat{\Delta}_k f \hat{\Delta}_k' g)\|_{L^2}$$

$$\lesssim 2^{\frac{2}{p} j} \sum_{k \geq j-3} \sum_{|k-k'| \leq 1} \|\hat{\Delta}_k f \hat{\Delta}_k' g\|_{L^{2p/(p-2)}}$$

$$\lesssim 2^{\frac{2}{p} j} \sum_{k \geq j-3} \sum_{|k-k'| \leq 1} 2^{-k^{\frac{2}{p}} - k_2^{\frac{2}{p}}} \|\hat{\Delta}_k f\|_{L^2} 2^{-k'\sigma} 2^{k'\sigma} \|\hat{\Delta}_k' g\|_{L^2}$$

$$\lesssim 2^{\frac{2}{p} j} \sum_{k \geq j-3} 2^{-k\left(-\sigma + \frac{2}{p}\right)} c(k) \|f\|_{B_{2,p}^s} \|g\|_{B_{2,\infty}^{-\sigma}}$$

$$\lesssim 2^{\sigma j} \|f\|_{B_{2,p}^s} \|g\|_{B_{2,\infty}^{-\sigma}}. \quad (5.14)$$

where $\|c(k)\|_{l_1} = 1$. The combination of (5.12)–(5.14) gives (5.9).

Next we prove (5.10). We also use Bony’s decomposition to write $f g = \hat{T}_f g + \hat{T}_g f + \hat{R}(f, g)$. It follows from the Hölder inequality and the Bernstein inequality that

$$\|\hat{\Delta}_j(\hat{T}_f g)\|_{L^2} \leq \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} \|\hat{\Delta}_j (\hat{\Delta}_k' f \hat{\Delta}_k g)\|_{L^2}$$

$$\lesssim \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} \|\hat{\Delta}_k' f\|_{L^{2p/(p-2)}} \|\hat{\Delta}_k g\|_{L^p}$$

$$\lesssim \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} 2^{k'\left(\frac{2}{p} - 1\right)} \|\hat{\Delta}_k' f\|_{L^p} \|\hat{\Delta}_k g\|_{L^p}$$

$$\lesssim 2^{\sigma j} \|f\|_{B_{2,p}^s} \|g\|_{B_{2,\infty}^{-\sigma + \frac{2}{p}}}. \quad (5.15)$$
Due to $0 < \sigma \leq \frac{4}{p} - 1$,
\[
\|\dot{\Delta}_j \dot{R}(f, g)\|_{L^2} \leq \sum_{k \geq j-3} \sum_{|k-k'| \leq 1} \|\dot{\Delta}_j (\dot{\Delta}_k f \dot{\Delta}_k g)\|_{L^2} \\
\lesssim 2^{j(\frac{2}{p} - 1)} \sum_{k \geq j-3} \sum_{|k-k'| \leq 1} \|\dot{\Delta}_k f\|_{L^p} \|\dot{\Delta}_k g\|_{L^p} \\
\lesssim 2^{j(\frac{2}{p} - 1)} \sum_{k \geq j-3} 2^{k(\sigma - \frac{2}{p} + 1)} c(k) \|f\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \|g\|_{\dot{B}^{-\sigma + \frac{2}{p}}_{p,\infty}} \\
\lesssim 2^{j\sigma} \|f\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \|g\|_{\dot{B}^{-\sigma + \frac{2}{p} - 1}_{p,\infty}},
\]
where $\|c(k)\|_n = 1$. For the term $\dot{T}_g f$, we obtain
\[
\|\dot{\Delta}_j (\dot{T}_g f)\|_{L^2} \leq \sum_{|k-j| \leq 4k' \leq k-2} \|\dot{\Delta}_j (\dot{\Delta}_k g \dot{\Delta}_k f)\|_{L^2} \leq \sum_{|k-j| \leq 4k' \leq k-2} \|\dot{\Delta}_k g\|_{L^\infty} \|\dot{\Delta}_k f\|_{L^p} \\
\lesssim \sum_{|k-j| \leq 4k' \leq k-2} 2^{k(\frac{2}{p} - 1)} \|\dot{\Delta}_k g\|_{L^p} \|\dot{\Delta}_k f\|_{L^p} \\
\lesssim 2^{j(\sigma + 1 - \frac{2}{p})} \|f\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \|g\|_{\dot{B}^{-\sigma + \frac{2}{p} - 1}_{p,\infty}}.
\]
from which we can get
\[
\|\dot{T}_g f\|_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim \|\dot{T}_g f\|_{\dot{B}^{-\sigma + \frac{2}{p} - 1}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \|g\|_{\dot{B}^{-\sigma + \frac{2}{p} - 1}_{p,\infty}}.
\]
The combination of (5.15), (5.16) and (5.18) gives the desired estimate (5.10). This proves the lemma.

**Remark 5.3.** As a consequence of (5.10), there hold the following three estimates
\[
\|fg\|_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \|g\|_{\dot{B}^{-\sigma + 1}_{2,\infty}}, \quad 0 < \sigma \leq \frac{4}{p} - 1. \tag{5.19}
\]
\[
\|fg\|_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim \|f\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \|g\|_{\dot{B}^\frac{2}{p} + 1}, \quad 0 < \sigma \leq \frac{4}{p} - 1. \tag{5.20}
\]
\[
\|f^h g^h\|_{\dot{B}^{-\sigma}_{2,\infty}} \lesssim \|f^h\|_{\dot{B}^{\frac{2}{p} - 1}_{p,1}} \left(\|g^h\|_{\dot{B}^\frac{2}{p} + 1} + \|g^f\|_{\dot{B}^{-\sigma + 1}_{2,\infty}}\right), \quad 0 < \sigma \leq \frac{4}{p} - 1. \tag{5.21}
\]
To simplify the notation, we set
\[
\mathcal{D}_\infty(t) \overset{\text{def}}{=} \|(a, u, b)\|_{\dot{B}^0_{2,1}} + \|(a, b)\|_{\dot{B}^\frac{2}{p} - 1_{p,1}} + \|u\|_{\dot{B}^\frac{2}{p} + 1_{p,1}}.
\]
\[
\mathcal{D}_1(t) \overset{\text{def}}{=} \|(a, u, b)\|_{\dot{B}^2_{2,1}} + \|(a, b)\|_{\dot{B}^\frac{2}{p} - 1_{p,1}} + \|u\|_{\dot{B}^\frac{2}{p} + 1_{p,1}}.
\]
From (5.9), one has
\[
\|u \cdot \nabla a^\ell\|_{\dot{B}^{-\sigma}_{2,\infty}} \|a \text{ div } u^\ell\|_{\dot{B}^{-\sigma}_{2,\infty}}
\]
Using the second estimate in (5.19), we have

\[
\| \mathbf{u} \cdot \nabla h \|_{B_{2,\infty}^\sigma} + \| \text{div} \mathbf{u}^h \|_{B_{2,\infty}^\sigma} \\
\lesssim \| \mathbf{u} \|_{B_{2,\infty}^{\sigma+1}} \| \nabla a^h \|_{B_{2,\infty}^\sigma} + \| \nabla \mathbf{u} \|_{B_{2,\infty}^{\sigma+1}} + \| \nabla \mathbf{u} \|_{B_{2,\infty}^{\sigma+1}} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma} \\
\lesssim \| \mathbf{u} \|_{B_{2,\infty}^{\sigma+1}} + \| \nabla \mathbf{u} \|_{B_{2,\infty}^{\sigma+1}} + \| \mathbf{u} \|_{B_{2,\infty}^{\sigma+1}} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma} \\
\lesssim \mathcal{D}_1(t) \| (a^\ell, \mathbf{u}^\ell) \|_{B_{2,\infty}^\sigma},
\]

(5.23)

which, together with (5.22), gives

\[
\| f_1 \|_{B_{2,\infty}^\sigma} \lesssim \mathcal{D}_1(t) \| (a^\ell, \mathbf{u}^\ell) \|_{B_{2,\infty}^\sigma} + \mathcal{D}_\infty(t) \mathcal{D}_1(t).
\]

(5.24)

Along the same lines, we have

\[
\| f_2 \|_{B_{2,\infty}^\sigma} \lesssim \mathcal{D}_1(t) \| (b^\ell, \mathbf{u}^\ell) \|_{B_{2,\infty}^\sigma} + \mathcal{D}_\infty(t) \mathcal{D}_1(t).
\]

(5.25)

Next, we bound the terms in \( f_2 \). The estimates of \( \mathbf{u} \cdot \nabla \mathbf{u}, I(a)(\Delta \mathbf{u} + \nabla \text{div} \mathbf{u}) \) follow from essentially the same procedures as \( \text{div}(\mathbf{a} \mathbf{u}) \) so that

\[
\| \mathbf{u} \cdot \nabla \mathbf{u} \|_{B_{2,\infty}^\sigma} \lesssim \| \mathbf{u} \cdot \nabla \mathbf{u} \|_{B_{2,\infty}^\sigma} + \| \mathbf{u} \cdot \nabla \mathbf{u}^h \|_{B_{2,\infty}^\sigma} \\
\lesssim (\| \mathbf{u} \|_{B_{2,\infty}^\sigma} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma}) \| \mathbf{u} \|_{B_{2,\infty}^\sigma} + (\| \mathbf{u} \|_{B_{2,\infty}^\sigma} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma}) \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \\
\lesssim \mathcal{D}_1(t) \| \mathbf{u} \|_{B_{2,\infty}^\sigma} + \mathcal{D}_\infty(t) \mathcal{D}_1(t)
\]

(5.26)

and

\[
\| I(a)(\Delta \mathbf{u} + \nabla \text{div} \mathbf{u}) \|_{B_{2,\infty}^\sigma} \lesssim (\| a^0 \|_{B_{2,\infty}^\sigma} + \| a^0 \|_{B_{2,\infty}^\sigma}) \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \\
+ (\| \mathbf{u} \|_{B_{2,\infty}^\sigma} + \| \mathbf{u} \|_{B_{2,\infty}^\sigma}) \| \mathbf{u} \|_{B_{2,\infty}^\sigma} \\
\lesssim (1 + \mathcal{D}_\infty(t)) \mathcal{D}_\infty(t) \| (a^\ell, \mathbf{u}^\ell) \|_{B_{2,\infty}^\sigma} + \mathcal{D}_\infty(t) \mathcal{D}_1(t).
\]

(5.27)

According to the definition of \( I(a) \), it’s not hard to check that

\[
I(a) = a - aI(a).
\]

Now, using (5.19), (5.20) and Lemma 2.8, we get

\[
\| a \nabla b \|_{B_{2,\infty}^\sigma} \lesssim \| a^\ell \nabla b^\ell \|_{B_{2,\infty}^\sigma} + \| a^\ell \nabla b^h \|_{B_{2,\infty}^\sigma} + \| a^h \nabla b^\ell \|_{B_{2,\infty}^\sigma} + \| a^h \nabla b^h \|_{B_{2,\infty}^\sigma} \\
\lesssim \| \nabla b^\ell \|_{B_{2,\infty}^\sigma} \| a^\ell \|_{B_{2,\infty}^\sigma} + \| \nabla b^h \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma} \\
+ \| a^h \|_{B_{2,\infty}^\sigma} \| \nabla b^\ell \|_{B_{2,\infty}^\sigma} + \| \nabla b^h \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma} \\
+ \| a^h \|_{B_{2,\infty}^\sigma} \| \nabla b^\ell \|_{B_{2,\infty}^\sigma} + \| \nabla b^h \|_{B_{2,\infty}^\sigma} \| a^h \|_{B_{2,\infty}^\sigma}.
\]
Thanks to (5.9) and Lemma 2.8, we obtain

\[
\int_{\mathbb{T}^d} b^\ell \cdot \nabla b^\ell \, dx + \int_{\mathbb{T}^d} a^\ell \cdot \nabla a^\ell \, dx = 0.
\]

Combining (5.28), (5.29) and (5.30), we have

\[
\int_{\mathbb{T}^d} b^\ell \cdot \nabla b^\ell \, dx + \int_{\mathbb{T}^d} a^\ell \cdot \nabla a^\ell \, dx \leq \mathcal{D}_\infty(t) \| (a^\ell, b^\ell) \|_{B_{2,\infty}} > \mathcal{D}_\infty(t) \mathcal{D}_1(t). \tag{5.28}
\]

Next, we write

\[
aI(a) \nabla b = aI(a) \nabla b^\ell + aI(a) \nabla b^h.
\]

Thanks to (5.9) and Lemma 2.8, we obtain

\[
\| aI(a) \nabla b^\ell \|_{B_{2,\infty}^{-\sigma}} \lesssim \| aI(a) \|_{B_{p,1}^\sigma} \| \nabla b^\ell \|_{B_{2,\infty}^{-\sigma}} \lesssim \| a \|_{B_{p,1}^\sigma}^2 \| b^\ell \|_{B_{2,\infty}^{-\sigma}} \lesssim (\| a^\ell \|_{B_{2,1}^\sigma}^2 + \| a^h \|_{B_{p,1}^\sigma}^2) \| b^\ell \|_{B_{2,\infty}^{-\sigma}} \lesssim \mathcal{D}_\infty(t) \mathcal{D}_1(t) \| b^\ell \|_{B_{2,\infty}^{-\sigma}}. \tag{5.29}
\]

Thanks to (5.21), Lemma 2.8 and Corollary 2.4, we can get

\[
\| aI(a) \nabla b^h \|_{B_{2,\infty}^{-\sigma}} \lesssim \| \nabla b^h \|_{B_{p,1}^{\sigma+1}} (\| aI(a) \|_{B_{p,\infty}^{-\sigma+\frac{d}{2}}} + \| a \|_{B_{p,\infty}^{-\sigma+\frac{d}{2}-1}}) \lesssim \| \nabla b^h \|_{B_{p,1}^{\sigma}} \| a \|_{B_{p,1}^\sigma} (\| a^\ell \|_{B_{2,\infty}^{-\sigma+1}} + \| a^h \|_{B_{p,1}^\sigma}) \lesssim \| b^h \|_{B_{p,1}^{\sigma}} (\| a^\ell \|_{B_{2,1}^\sigma} + \| a^h \|_{B_{p,1}^\sigma}) (\| a^\ell \|_{B_{2,\infty}^{\sigma}} + \| a^h \|_{B_{p,1}^\sigma}) \lesssim (1 + \mathcal{D}_\infty(t)) \mathcal{D}_1(t) \| a \|_{B_{2,\infty}^{-\sigma}} + (1 + \mathcal{D}_\infty(t)) \mathcal{D}_\infty(t) \mathcal{D}_1(t). \tag{5.30}
\]

Combining (5.28), (5.29) and (5.30), we have

\[
\| I(a) \nabla b \|_{B_{2,\infty}^{-\sigma}} \lesssim (1 + \mathcal{D}_\infty(t)) \mathcal{D}_1(t) \| (a, b) \|_{B_{2,\infty}^{-\sigma}} + (1 + \mathcal{D}_\infty(t)) \mathcal{D}_\infty(t) \mathcal{D}_1(t). \tag{5.31}
\]

By using the same method, we can deal with the rest terms \( k(a) \nabla a, b(\nabla b), \) and \( I(a) b(\nabla b). \) Collecting the estimates above, we obtain

\[
\| f_2 \|_{B_{2,\infty}^{-\sigma}} \lesssim (1 + \mathcal{D}_\infty(t)) \mathcal{D}_1(t) \| (a, u, b) \|_{B_{2,\infty}^{-\sigma}} + (1 + \mathcal{D}_\infty(t)) \mathcal{D}_\infty(t) \mathcal{D}_1(t). \tag{5.32}
\]

Inserting (5.24), (5.25) and (5.32) in (5.8) gives

\[
\| (a, u, b) \|_{B_{2,\infty}^{-\sigma}} \lesssim \| (a_0, u_0, b_0) \|_{B_{2,\infty}^{-\sigma}} + \int_0^t (1 + \mathcal{D}_\infty(s)) \mathcal{D}_\infty(s) \mathcal{D}_1(s) \, ds + \int_0^t (1 + \mathcal{D}_\infty(s)) \mathcal{D}_1(s) \| (a, u, b) \|_{B_{2,\infty}^{-\sigma}} \, ds. \tag{5.33}
\]

Gronwall’s inequality then implies

\[
\| (a, u, b)(t, \cdot) \|_{B_{2,\infty}^{-\sigma}} \leq C_0 \tag{5.34}
\]

for all \( t \geq 0, \) where \( C_0 > 0 \) depends on the norm of the initial data. This completes the proof of Proposition 5.1.
Now, we prove the Lyapunov-type inequality from (5.3). For any $0 < \sigma \leq \frac{4}{p} - 1$, it follows from an interpolation inequality that
\[
\| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \leq C \left( \| (a, u, b) \|_{\dot{B}_{2,\infty}^{\frac{\sigma}{2}}} \right)^{\frac{\theta_1}{2}} \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \right)^{1 - \theta_1} \leq C \left( \| (a, u, b) \|_{\dot{B}_{2,\infty}^{\frac{\sigma}{2}}} \right)^{\frac{\theta_1}{2}} \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \right)^{1 - \theta_1}, \quad \theta_1 = \frac{2}{2 + \sigma} \in (0, 1),
\]
which, together with Proposition 5.1, implies
\[
\| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \geq c_0 \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \right)^{-\frac{1}{\theta_1}}. \tag{5.35}
\]
Taking $\beta = 1 + \theta_1 > 0$ in (5.4) and (5.5) and combining with (5.35), we deduce from (5.3) that
\[
\frac{d}{dt} \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} + \| (a, b) \|_{\dot{B}_{2,p,1}^{\frac{\beta}{2}}} \right) + \tilde{c}_0 \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} + \| (a, b) \|_{\dot{B}_{2,p,1}^{\frac{\beta}{2}}} \right)^{1 + \frac{\beta}{2}} \leq 0. \tag{5.36}
\]
Solving this differential inequality directly, we obtain
\[
\| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} + \| (a, b) \|_{\dot{B}_{2,p,1}^{\frac{\beta}{2}}} \leq C(1 + t)^{-\frac{\beta}{2}}. \tag{5.37}
\]
For any $\frac{2}{p} - 1 - \sigma < \gamma_1 < \frac{2}{p} - 1$, by the interpolation inequality, we have
\[
\| (a, u, b) \|_{\dot{B}_{p,1}^{\gamma_1}} \leq C \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \right)^{\frac{\theta_2}{\gamma_1 + 1 - \frac{\sigma}{2}}} \leq C \left( \| (a, u, b) \|_{\dot{B}_{2,1}^{\frac{\sigma}{2}}} \right)^{\frac{\theta_2}{\gamma_1 + 1 - \frac{\sigma}{2}}}, \quad \theta_2 = \frac{\frac{2}{p} - 1 - \gamma_1}{\sigma} \in (0, 1),
\]
which, together with Proposition 5.1, gives
\[
\| (a, u, b) \|_{\dot{B}_{p,1}^{\gamma_1}} \leq C(1 + t)^{-\frac{\sigma(1 - \theta_2)}{2}} = C(1 + t)^{-\frac{\sigma(1 - \theta_2)}{2}}. \tag{5.38}
\]
In the light of $\frac{2}{p} - 1 - \sigma < \gamma_1 < \frac{2}{p} - 1$, we see that
\[
\| (a^h, u^h, b^h) \|_{\dot{B}_{p,1}^{\gamma_1}} \leq C(\| (a, b) \|_{\dot{B}_{p,1}^{\frac{\sigma}{2}}} + \| u \|_{\dot{B}_{p,1}^{\frac{\beta}{2}}}) \leq C(1 + t)^{-\frac{\beta}{2}},
\]
which, together with (5.38), yields
\[
\| (a, u, b) \|_{\dot{B}_{p,1}^{\gamma_1}} \leq C(\| (a, u, b) \|_{\dot{B}_{p,1}^{\frac{\sigma}{2}}} + \| (a, u, b) \|_{\dot{B}_{p,1}^{\frac{\beta}{2}}}) \leq C(1 + t)^{-\frac{\beta}{2} - \frac{\sigma}{2}} + C(1 + t)^{-\frac{\sigma}{2}} \leq C(1 + t)^{-\frac{\beta}{2} - \frac{\sigma}{2}}.
\]
Hence, thanks to the embedding relation $\dot{B}_{p,1}^{0}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$, one infers that
\[
\| \Lambda^{\gamma_1}(a, u, b) \|_{L^p} \leq C(1 + t)^{-\frac{\beta}{2} - \frac{\sigma}{2}}.
\]
This completes the proof of Theorem 1.3. \qed
Dong was partially supported by the National Natural Science Foundation of China under grant 11871346, the NSF of Guangdong Province under grant 2020A1515010530, NSF of Shenzhen City (Nos.JCYJ20180305125554234, 20200805101524001). Wu was partially supported by the National Science Foundation of the USA under grant DMS 2104682, the Simons Foundation grant (Award number 708968) and the AT&T Foundation at Oklahoma State University. Zhai was partially supported by the National Natural Science Foundation of China under grant11601533, and the Science and Technology Program of Shenzhen under grant 20200806104726001.

REFERENCES

[1] H. Bahouri, J.Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren Math. Wiss., vol. 343, Springer-Verlag, Berlin, Heidelberg, 2011.
[2] D. Bian, B. Guo, Local well-posedness in critical spaces for the compressible MHD equations, *Appl. Anal.*, 95 (2016), 239–269.
[3] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, 141 (2000), 579–614.
[4] R. Danchin, A Lagrangian approach for the compressible Navier-Stokes Equations, *Ann. Inst. Fourier, Grenoble*, 64 (2014), 753–791.
[5] R. Danchin, L. He, The incompressible limit in $L^p$ type critical spaces, *Math. Ann.*, 366 (2016), 1365–1402.
[6] R. Danchin, J. Xu, Optimal time-decay estimates for the compressible Navier-Stokes equations in the critical $L^p$ framework, *Arch. Ration. Mech. Anal.*, 224 (2017), 53–90.
[7] P. A. Davidson, *Introduction to Magnetohydrodynamics*, 2nd ed. Cambridge University Press, Cambridge, 2017.
[8] C. Dou, S. Jiang, Q. Ju, Global existence and the low Mach number limit for the compressible magnetohydrodynamic equations in a bounded domain with perfectly conducting boundary, *Z. Angew. Math. Phys.*, 64 (2013), 1661–1678.
[9] B. Ducomet, E. Feireisl, The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars, *Commun. Math. Phys.*, 266 (2006), 595–629.
[10] J. Fan, W. Yu, Strong solution to the compressible magnetohydrodynamic equations with vacuum, *Nonlinear Anal. Real World Appl.*, 10 (2009), 392–409.
[11] E. Feireisl, A. Novotny, Y. Sun, Dissipative solutions and the incompressible inviscid limits of the compressible magnetohydrodynamic system in unbounded domains, *Discrete Contin. Dyn. Syst.*, 34 (2014), 121–143.
[12] Y. Guo, Y. Wang, Decay of dissipative equations and negative sobolev spaces, *Comm. Part. Differ. Equ.*, 37 (2012), 2165–2208.
[13] C. Hao, Well-posedness to the compressible viscous magnetohydrodynamic system, *Nonlinear Anal. Real World Appl.*, 12 (2011), 2962–2972.
[14] L. He, J. Huang, C. Wang, Global stability of large solutions to the 3D compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, 234 (2019), 1167–1222.
[15] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations* 120 (1995), 215–254.
[16] D. Hoff, E. Tsyganov, Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics, *Z. Angew. Math. Phys.*, 56 (2005), 791–804.
[17] G. Hong, X. Hou, H. Peng, C. Zhu, Global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum, *SIAM J. Math. Anal.*, 49 (2017), 2409–2441.
[18] X. Hu, D. Wang, Global solutions to the three-dimensional full compressible magnetohydrodynamic flows, *Commun. Math. Phys.*, 283 (2008), 255–284.
[19] X. Hu, D. Wang, Low Mach number limit of viscous compressible magnetohydrodynamic flows, *SIAM J. Math. Anal.*, 41 (2009), 1272–1294.
[20] X. Hu, D. Wang, Global existence and large-time behavior of solutions to the threedimensional equations of compressible magnetohydrodynamic flows, *Arch. Ration. Mech. Anal.*, 197 (2010), 203–238.

[21] X. Huang, J. Li, Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, *Comm. Pure Appl. Math.*, 65 (2012), 549–585.

[22] J. Jia, J. Peng, J. Gao, Well-posedness for compressible MHD systems with highly oscillating initial data, *J. Math. Phys.*, 57 (2016), 081514.

[23] S. Jiang, Q. Ju, F. Li, Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients, *SIAM J. Math. Anal.*, 42 (2010), 2539–2553.

[24] S. Jiang, Q. Ju, F. Li, Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions, *Commun. Math. Phys.*, 297 (2010), 371–400.

[25] S. Jiang, F. Li, Rigorous derivation of the compressible magnetohydrodynamic equations from the electromagnetic fluid system, *Nonlinearity*, 25 (2012), 1735–1752.

[26] Q. Ju, Y. Wang, Z. Xin, Global classical solution to two-dimensional compressible Navier-Stokes equations with large data in $\mathbb{R}^2$, *Phys. D*, 376/377 (2018), 180–194.

[27] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, Ph.D. Thesis, Kyoto University, 1983.

[28] Z. Lei, Z. Xin, On scaling invariance and type-I singularities for the compressible Navier-Stokes equations, *Sci. China Math.*, 57 (2014), 2271–2286.

[29] F. Li, Y. Mu, D. Wang, Local well-posedness and low Mach number limit of the compressible magnetohydrodynamic equations in critical spaces, *Kinetic Related Models*, 10 (2017), 741–784.

[30] F. Li, H. Yu, Optimal decay rate of classical solutions to the compressible magnetohydrodynamic equations, *Proc. Roy. Soc. Edinburgh Sect. A*, 141 (2011), 109–126.

[31] F. Li, H. Yu, Optimal decay rate of classical solutions to the compressible magnetohydrodynamic equations, *Proc. R. Soc. Edinb. A*, 141 (2011), 109–126.

[32] H. Li, Y. Wang, Z. Xin, Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, 232 (2019), 557–590.

[33] H. Li, X. Xu, J. Zhang, Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum, *SIAM J. Math. Anal.*, 45 (2013), 1356–1387.

[34] J. Li, Z. Xin, Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum, *Ann. PDE.*, 5 (2019), Paper No. 7, 37 pp.

[35] X. Li, N. Su, and D. Wang, Local strong solution to the compressible magnetohydrodynamic flow with large data, *J. Hyperbolic Differ. Equ.*, 08 (2011), 415–436.

[36] Y. Li, Convergence of the compressible magnetohydrodynamic equations to incompressible magnetohydrodynamic equations, *J. Differential Equations*, 252 (2012), 2725–2738.

[37] Y. Li, Y. Sun, Global weak solutions to a two-dimensional compressible MHD equations of viscous non-resistive fluids, *J. Differential Equations*, 267 (2019), 3827–3851.

[38] B. Lv and B. Huang, On strong solutions to the Cauchy problem of the two-dimensional compressible magnetohydrodynamic equations with vacuum, *Nonlinearity*, 28 (2015), 509–530.

[39] J. Wu, Y. Wu, Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion, *Adv. Math.*, 310 (2017), 759–888.

[40] Z. Xin, J. Xu, Optimal decay for the compressible Navier-Stokes equations without additional smallness assumptions, *J. Differential Equations*, 274 (2021), 543–575.

[41] Z. Xin, W. Yan, On blowup of classical solutions to the compressible Navier-Stokes equations, *Comm. Math. Phys.*, 321 (2013), 529–541.

[42] Z. Xin, S. Zhu, Well-posedness of the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and far field vacuum, *J. Math. Pures Appl.*, 152 (2021), 94–144.

[43] H. Xu, Y. Li, X. Zhai, On the well-posedness of 2D incompressible Navier-Stokes equations with variable viscosity in critical spaces, *J. Differential Equations*, 260 (2016), 6604–6637.

[44] X. Zhai, Z. Chen, Long-time behavior for three dimensional compressible viscous and heat-conductive gases, *J. Math. Fluid Mech.*, 22 (2020), 38.

[45] X. Zhong, On local strong solutions to the 2D Cauchy problem of the compressible non-resistive magnetohydrodynamic equations with vacuum, *J. Dynam. Differential Equations*, 32 (2020), 505–526.
[46] S. Zhu, On classical solutions of the compressible magnetohydrodynamic equations with vacuum, *SIAM J. Math. Anal.*, 47 (2015), 2722–2753.

(B. Dong) School of Mathematics and Statistics, Shenzhen University, Shenzhen, 518060, China.  
Email address: bqdong@szu.edu.cn

(J. Wu) Department of Mathematics, Oklahoma State University, 401 Mathematical Sciences, Stillwater, OK 74078, USA.  
Email address: jiahong.wu@okstate.edu

(X. Zhai) School of Mathematics and Statistics, Shenzhen University, Shenzhen, 518060, China.  
Email address: zhaixp@szu.edu.cn (Corresponding author)