Calculus on fractal subsets of real line – I: formulation

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Abstract. A new calculus based on fractal subsets of the real line is formulated. In this calculus, an integral of order $\alpha$, $0 < \alpha \leq 1$, called $F^\alpha$-integral, is defined, which is suitable to integrate functions with fractal support $F$ of dimension $\alpha$. Further, a derivative of order $\alpha$, $0 < \alpha \leq 1$, called $F^\alpha$-derivative, is defined, which enables us to differentiate functions, like the Cantor staircase, “changing” only on a fractal set. The $F^\alpha$-derivative is local unlike the classical fractional derivative. The $F^\alpha$-calculus retains much of the simplicity of ordinary calculus. Several results including analogues of fundamental theorems of calculus are proved.

The integral staircase function, which is a generalisation of the functions like the Cantor staircase function, plays a key role in this formulation. Further, it gives rise to a new definition of dimension, the $\gamma$-dimension.

$F^\alpha$-differential equations are equations involving $F^\alpha$-derivatives. They can be used to model sublinear dynamical systems and fractal time processes, since sublinear behaviours are associated with staircase-like functions which occur naturally as their solutions. As examples, we discuss a fractal-time diffusion equation, and one dimensional motion of a particle undergoing friction in a fractal medium.

1. Introduction

It is now a well established fact that fractals can model many structures found in nature \cite{1,2}. The geometry of fractals is also a well explored subject \cite{1,3,4,5,6}.

Fractals are often too irregular to have any smooth differentiable structure defined on them, and render the methods and techniques of ordinary calculus powerless or inapplicable. For example the derivative of a Lebesgue-Cantor staircase function is zero almost everywhere and therefore this function is not a solution of an ordinary differential equation. Consequently, ordinary calculus does not equip us to handle problems such as fractal time random walks, anomalous diffusion, dynamics on fractals, fields of fractally distributed sources etc., by setting up and solving ordinary differential equations.

During recent times, a few approaches have been developed to deal with various aspects of the problems mentioned above.

Several authors have recognized the need to use fractional derivatives and integrals to explore the characteristic features of fractal walks, anomalous diffusion, transport, etc. by setting up fractional kinetic equations, master equations and so on \cite{7,8,9,10,11,12,13}. Fractional derivatives are nonlocal operators and often are suitable for modelling processes with memory but not always suitable to handle the local scaling behaviour e. g. the behaviour of fractal functions. In \cite{14,15,16,17} this problem was circumvented by renormalising fractional derivatives and constructing
local fractional operators. This was further pursued in [18, 19]. A particular success of this approach was the demonstration of the striking fact that fractal and multifractal functions can be differentiated up to an order (fractional) determined by the Hölder exponent of the function (or dimension of its graph). In particular, Weierstrass’ nowhere differentiable function was shown [14, 15] to be differentiable up to order \((1 - \gamma)\), if \((1 + \gamma)\) is the box dimension of its graph.

Another remarkable development is analysis on fractals. Many important ideas and applications are developed in the realm of analysis on fractals. This approach has been extensively used for the treatment of diffusion, heat conduction, waves, etc. on fractals—see [20, 21, 22, 23] and several references therein.

There is a further beautiful development using a measure-theoretical approach [24, 25]. It consists of defining derivative as the inverse of the integral with respect to a measure and defining other operators using the derivative. This avoids the dependence on the structure of the underlying fractal.

While all these themes have increased our understanding and brought out many beautiful connections, a direct and simple approach involving fractional order operators on fractal sets is only moderately explored. Even though measure theoretical approach is elegant, Riemann integration like procedures have their own place. They are more transparent, constructive, and advantageous from an algorithmic point of view.

It indeed seems possible to develop such an appropriate calculus, tuned to these requirements. In the present paper, the first of a series devoted to these ideas, we undertake a systematic development of calculus on fractal subsets of real line, involving integrals and derivatives of appropriate orders \(0 < \alpha \leq 1\) based on a (fractal) set \(F \subset \mathbb{R}\), indicated in [16]. We call them \(F_{\alpha}\)-integral and \(F_{\alpha}\)-derivative respectively.

The organisation of the paper is as follows: We begin in section 2 by defining a mass function \(\gamma_{\alpha}(F, a, b)\) and integral staircase function \(S_{\alpha}^F(x)\) of an order \(\alpha\) for a set \(F\). The mass function \(\gamma_{\alpha}(F, a, b)\) gives us the content of a set \(F\) in the interval \([a, b] \subset \mathbb{R}\). Its definition is based on Riemann-like sums. The construction can be compared to the definition of Hausdorff measure, except that the covers are more restrictive: they are in the form of finite subdivisions of \([a, b]\). Though \(\gamma_{\alpha}\) is not a measure due to this simplification, it turns out to be proportional to Hausdorff measure for compact sets.

The integral staircase function \(S_{\alpha}^F(x) = \gamma_{\alpha}(F, a, x)\), obtained from the mass function by fixing \(a\), is a generalization of the well known functions such as the Lebesgue-Cantor staircase (or the Devil’s staircase) functions. The definitions of \(F_{\alpha}\)-integral and \(F_{\alpha}\)-derivative use the quantity \((S_{\alpha}^F(y) - S_{\alpha}^F(x))\) in place of the length \((y - x)\) of the interval \([x, y]\). In this respect, the definition of \(F_{\alpha}\)-integral is similar to that of Riemann-Stieltjes integral [26, 27].
In section 3 it is shown that the mass function leads to a new definition of dimension called $\gamma$-dimension, which is finer than the box dimension, though not as fine as the Hausdorff dimension. In later sections it is seen that the $F^\alpha$-integral or the $F^\alpha$-derivative give meaningful results when the $\gamma$-dimension of the underlying fractal $F$ is the same as $\alpha$.

In section 4 it is shown that the Hausdorff measure and the mass function agree for compact sets up to a proportionality constant. Using this property, the staircase function is calculated for the middle $\frac{1}{3}$ Cantor set.

Several sets can give rise to the same staircase function. A representative set from such an equivalence class of sets, with nice properties, needs to be chosen for defining the $F^\alpha$-derivative and proving the analogues of fundamental theorems. Section 5 assures the existence and uniqueness of such a set, called an $\alpha$-perfect set, associated with a staircase function.

We develop the rest of the theory in a way analogous to the standard calculus. In section 6 we introduce notations for limit and continuity using the topology of $F$ with the metric inherited from $\mathbb{R}$. This is done in order to distinguish them from limit and continuity on $\mathbb{R}$.

The ordinary integral of functions with fractal support $F \subset \mathbb{R}$ is zero or undefined depending on the definition of integral (Lebesgue or Riemann) and the nature of the support. The $F^\alpha$-integral defined in section 7 suits the needs of integration of such functions. It is further shown that the $F^\alpha$-integral of the characteristic function of $F$ is the staircase function associated with $F$ as indicated in 16.

Functions representing intermittent phenomena or fractal time evolution typically “change” only on a fractal. The Cantor staircase function is an example. The $F^\alpha$-derivative defined in section 8 is best suited to quantify the “rate of change” of such functions. This derivative is local unlike fractional derivatives. It is also not any kind of average derivative as in 33, 51, 52. It is more like the first order derivative in ordinary calculus, which makes its dynamical interpretation possible. Further in the same section, it is shown that the $F^\alpha$-derivative of the staircase function of $F$ is the characteristic function of $F$. Analogues of Rolle’s theorem, the law of the mean and Leibniz rule are discussed. In section 9 we prove the analogues of fundamental theorems of calculus. The formula for $F^\alpha$-integration by parts follows thereby.

The definitions of $F^\alpha$-integral and the $F^\alpha$-derivative reduce to those of usual Riemann integral and first order derivative respectively when $\alpha = 1$ and $F = \mathbb{R}$.

In section 10 we discuss examples including subdiffusion and motion of a particle undergoing friction in a fractal medium, and demonstrate the use of $F^\alpha$-differential equations as their models.

As an example, the $F^\alpha$-integral of $f(x) = x \chi_C(x)$ for the middle $\frac{1}{3}$ Cantor set $C$ is calculated in Appendix A. Repeated $F^\alpha$-derivatives and $F^\alpha$-integrals are discussed in Appendix B where we also calculate $F^\alpha$-derivatives and $F^\alpha$-integrals of powers $(S^\alpha_C(x))^n$. A few analogies between classical calculus and $F^\alpha$-calculus are tabulated in Appendix C.

We begin by defining the integral staircase function.

2. The mass function and the integral staircase

Let $F$ be a subset of the real line. In most of the interesting cases discussed below, $F$ would be a fractal. In this section we formulate the notion of the content or $\alpha$-mass
of $F$ in an interval $[a,b]$, i.e., mass of $F \cap [a,b]$, of order $\alpha$, $0 < \alpha \leq 1$.

In all the following discussion, $0 < \alpha \leq 1$ unless stated otherwise.

**Definition 1** The flag function $\theta(F,I)$ for a set $F$ and a closed interval $I$ is given by

$$\theta(F,I) = \begin{cases} 1 & \text{if } F \cap I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$  \hfill (1)

**Definition 2** A subdivision $P_{[a,b]}$ (or just $P$) of the interval $[a,b]$, $a < b$, is a finite set of points $\{a = x_0, x_1, \ldots, x_n = b\}$, $x_i < x_{i+1}$. Any interval of the form $[x_i, x_{i+1}]$ is called a component interval or just a component of the subdivision $P$. If $Q$ is any subdivision of $[a,b]$ and $P \subset Q$, then we say that $Q$ is a refinement of $P$. If $a = b$, then the set $\{a\}$ is the only subdivision of $[a,b]$.

**Definition 3** For a set $F$ and a subdivision $P_{[a,b]}$, $a < b$,

$$\sigma^\alpha[F,P] = \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} \theta(F,[x_i,x_{i+1}]).$$  \hfill (2)

If $a = b$, we define $\sigma^\alpha[F,P]$ to be zero.

We note that the sum in (2) contains a contribution from a component interval if and only if that component contains at least one point of $F$. Further, $\sigma^\alpha[F,P] \geq 0$ for any set $F$ and subdivision $P$ of $[a,b]$.

We remark that this definition, and in particular the factor $1/\Gamma(\alpha + 1)$ and the use of finite subdivisions, has been motivated by local fractional calculus \[16, 17\].

Now we introduce the coarse-grained mass:

**Definition 4** Given $\delta > 0$ and $a \leq b$, the coarse-grained mass $\gamma^\alpha_\delta(F,a,b)$ of $F \cap [a,b]$ is given by

$$\gamma^\alpha_\delta(F,a,b) = \inf_{P_{[a,b]}:|P| \leq \delta} \sigma^\alpha[F,P]$$  \hfill (3)

where

$$|P| = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i)$$  \hfill (4)

for a subdivision $P$, and the infimum in (3) is taken over all subdivisions $P$ of $[a,b]$ satisfying $|P| \leq \delta$.

Eventually, a limit of $\gamma^\alpha_\delta(F,a,b)$ as $\delta \to 0$ is taken in definition 5 below. But before that we examine some important properties of $\gamma^\alpha_\delta(F,a,b)$.

Let $a \leq b$ and $\delta_1 < \delta_2$. Then $\gamma^\alpha_{\delta_1}(F,a,b)$ is the infimum of $\sigma^\alpha[F,P]$ over a smaller class of subdivisions than $\gamma^\alpha_{\delta_2}(F,a,b)$. Thus:

**Lemma 5** Let $a \leq b$ and $\delta_1 < \delta_2$. Then $\gamma^\alpha_{\delta_1}(F,a,b) \geq \gamma^\alpha_{\delta_2}(F,a,b)$.

The following lemma shows that $\gamma^\alpha_\delta(F,a,b)$ is a monotonic increasing function of $b$ and a monotonic decreasing function of $a$. (Throughout the paper we distinguish between monotonic and strictly monotonic.)

**Lemma 6** Let $\delta > 0$ and $a < b < c$. Then, $\gamma^\alpha_\delta(F,a,b) \leq \gamma^\alpha_\delta(F,a,c)$ and $\gamma^\alpha_\delta(F,b,c) \leq \gamma^\alpha_\delta(F,a,c)$. 


Theorem 7 \( \gamma^\alpha(F,a,b) \) is continuous in \( b \) and \( a \).

Proof: We prove the continuity of \( \gamma^\alpha(F,a,b) \) in \( b \) (with \( \delta, \alpha \) and \( a \) fixed). Continuity in \( a \) can be proved in a similar manner.

Given \( \epsilon > 0 \), let

\[
\Delta' = (\epsilon \Gamma(\alpha + 1))^{\frac{1}{\alpha}} \quad \text{and} \quad \Delta = \min(\Delta', \delta).
\]

For \( \epsilon_1 > 0 \), there exists a subdivision \( P \), such that \( |P| \leq \delta \) and

\[
\sigma^\alpha[F,P] < \gamma^\alpha(F,a,b) + \epsilon_1.
\]

Now \( Q = P \cup \{ b + \Delta \} \) is a subdivision of \([a, b + \Delta]\). Therefore,

\[
\gamma^\alpha(F,a,b + \Delta) \leq \sigma^\alpha[F,Q]
\]

\[
= \sigma^\alpha[F,P] + \theta(F,[b,b + \Delta])\frac{\Delta^\alpha}{\Gamma(\alpha + 1)}
\]

\[
\leq \sigma^\alpha[F,P] + \epsilon
\]

\[
< \gamma^\alpha(F,a,b) + \epsilon_1 + \epsilon.
\]

As \( \epsilon_1 \) is arbitrary, we get \( \gamma^\alpha(F,a,b + \Delta) < \gamma^\alpha(F,a,b) + \epsilon \). As \( \gamma^\alpha(F,a,b) \) is a nondecreasing function of \( b \),

\[
\gamma^\alpha(F,a,b + t) < \gamma^\alpha(F,a,b) + \epsilon
\]

for \( 0 < t < \Delta \).

Summarizing, given \( \epsilon > 0 \), there exists a \( \Delta > 0 \) such that

\[
c - b < \Delta \implies \gamma^\alpha(F,a,c) - \gamma^\alpha(F,a,b) < \epsilon
\]

which implies that \( \gamma^\alpha(F,a,b) \) is continuous in \( b \) from right. The continuity from left follows on the replacement of \( b \) by \( b - \Delta \) and of \( b + \Delta \) by \( b \) in the above proof. •

As mentioned earlier, the mass function is the limit of the coarse-grained mass as \( \delta \to 0 \):

Definition 8 The mass function \( \gamma^\alpha(F,a,b) \) is given by

\[
\gamma^\alpha(F,a,b) = \lim_{\delta \to 0} \gamma^\alpha_{\delta}(F,a,b).
\]
We note that since $\gamma_\alpha^\delta(F, a, b)$ increases as $\delta$ decreases, $\gamma_\alpha^\delta(F, a, b)$ always exists and is a non-negative number, which may possibly be $+\infty$.

Another simple observation is that if $F \cap [a, b] = \emptyset$, then $\gamma_\alpha^\delta(F, a, b) = 0$ for any $\delta > 0$, and consequently $\gamma_\alpha^\delta(F, a, b) = 0$. This result can be extended so that it also applies to an open interval $(a, b)$:

**Lemma 9** If $F \cap (a, b) = \emptyset$, then $\gamma_\alpha^\delta(F, a, b) = 0$.

**Proof:** If $F \cap [a, b] = \emptyset$, then the result is obvious. If not, then $F \cap [a, b]$ contains one or both of $a$ and $b$. In that case, given $\epsilon > 0$, we can choose a subdivision $P$ of $[a, b]$ such that

$$(x_1 - x_0) \text{ and } (x_n - x_{n-1}) < \left(\frac{\epsilon \Gamma(\alpha + 1)}{2}\right)^\frac{1}{\delta}$$

where $\{x_0, \ldots, x_n\}$ are points of $P$. Then, $\sigma_\alpha^\delta[F, P] < \epsilon$ since $[x_1, x_{n-1}] \cap F = \emptyset$. But as $\epsilon$ is arbitrary, it follows that $\gamma_\alpha^\delta(F, a, b) = 0$ for any $\delta > 0$, so that $\gamma_\alpha^\delta(F, a, b) = 0$.

A property, desired of a mass function, is additivity. The following theorem asserts this.

**Theorem 10** Let $a < b < c$ and $\gamma_\alpha^\delta(F, a, c) < \infty$. Then

$$\gamma_\alpha^\delta(F, a, c) = \gamma_\alpha^\delta(F, a, b) + \gamma_\alpha^\delta(F, b, c).$$

(5)

**Proof:** Given $\delta > 0$, let $P_1$ be any subdivision of $[a, b]$ and $P_2$ be any subdivision of $[b, c]$ such that $|P_1| \leq \delta$ and $|P_2| \leq \delta$. Then, $P_1 \cup P_2$ is a subdivision of $[a, c]$, $|P_1 \cup P_2| \leq \delta$, and

$$\sigma_\alpha^\delta[F, P_1 \cup P_2] = \sigma_\alpha^\delta[F, P_1] + \sigma_\alpha^\delta[F, P_2].$$

Taking infimum over all subdivisions $P_1$ and $P_2$ such that $|P_1| \leq \delta$ and $|P_2| \leq \delta$, and noting that not all the subdivisions of $[a, c]$ can be written in the form $P_1 \cup P_2$, where $P_1$ is a subdivision of $[a, b]$ and $P_2$ is that of $[b, c]$, we get

$$\gamma_\alpha^\delta(F, a, c) \leq \inf_{|P_1| \leq \delta, |P_2| \leq \delta} \sigma_\alpha^\delta[F, P_1 \cup P_2]
= \gamma_\alpha^\delta(F, a, b) + \gamma_\alpha^\delta(F, b, c).$$

(6)

Now for every subdivision $P_{[a, c]}$, $|P| \leq \delta$, we can construct a subdivision $P' = P \cup \{b\}$. Obviously $|P'| \leq \delta$, and $P' = P_1 \cup P_2$ where $P_1$ is a subdivision of $[a, b]$ and $P_2$ is a subdivision of $[b, c]$.

Let $P = \{x_0, x_1, \ldots, x_n\}$. If $b \in P$, then $P = P'$ and $\sigma_\alpha^\delta[F, P] = \sigma_\alpha^\delta[F, P']$. Otherwise, let $[x_k, x_{k+1}]$ be the interval which contains $b$. Thus,

$$\sigma_\alpha^\delta[F, P \cup \{b\}] - \sigma_\alpha^\delta[F, P] = \theta(F, [x_k, b]) \left(\frac{b - x_k}{\Gamma(\alpha + 1)} + \theta(F, [b, x_{k+1}]) \left(\frac{x_{k+1} - b}{\Gamma(\alpha + 1)}\right) - \theta(F, [x_k, x_{k+1}]) \left(\frac{x_{k+1} - x_k}{\Gamma(\alpha + 1)}\right)\right).$$

Hence,

$$\sigma_\alpha^\delta[F, P \cup \{b\}] - \sigma_\alpha^\delta[F, P] \leq \frac{3 \delta^\delta}{\Gamma(\alpha + 1)}.$$
This implies that
\[
\sigma_\alpha[F,P] + \frac{2\delta^\alpha}{\Gamma(\alpha + 1)} \geq \sigma_\alpha[F, P \cup \{b\}]
\]
\[
= \sigma_\alpha[F, P_1] + \sigma_\alpha[F, P_2]
\]
\[
\geq \gamma^\alpha_\delta(F, a, b) + \gamma^\alpha_\delta(F, b, c)
\]
for all \(P\). Thus if we take infimum over all subdivisions \(P\) such that \(|P| \leq \delta\), we get
\[
\gamma^\alpha_\delta(F, a, c) + \frac{3\delta^\alpha}{\Gamma(\alpha + 1)} \geq \gamma^\alpha_\delta(F, a, b) + \gamma^\alpha_\delta(F, b, c) \tag{7}
\]

From (6) and (7) and taking the limit as \(\delta \to 0\), we get the result. \(
\)

Since each term in (3) is nonnegative for \(a \leq b \leq c\), an immediate consequence is

**Corollary 11** \(\gamma^\alpha(F, a, b)\) is increasing in \(b\) and decreasing in \(a\).

The next theorem states that \(\gamma^\alpha(F, a, x)\) takes all values in the range \((0, \gamma^\alpha(F, a, b))\) for \(x \in (a, b)\).

**Theorem 12** Let \(a < b\) and let \(\gamma^\alpha(F, a, b) \neq 0\) be finite. Let \(y\) be such that \(0 < y < \gamma^\alpha(F, a, b)\). Then there exists \(c, a < c < b\), such that \(\gamma^\alpha(F, a, c) = y\).

**Proof:** Let \(z = \gamma^\alpha(F, a, b) - y\).

Given a \(\delta > 0\), consider the set of all points \(x\) of \([a, b]\) such that \(\gamma^\alpha_\delta(F, x, b) \leq z\). This set is an interval of the form \([s_\delta, b]\) for some \(s_\delta\), \(a \leq s_\delta < b\), because \(\gamma^\alpha_\delta(F, x, b)\) is continuous (theorem [6]) and decreasing in \(x\) (corollary [11]). Since \(\gamma^\alpha_\delta(F, x, b)\) increases as \(\delta\) decreases (lemma [5]), \(s_\delta\) increases as \(\delta\) decreases.

Similarly the set of all points \(x\) of \([a, b]\) such that \(\gamma^\alpha_\delta(F, a, x) \leq y\) is an interval of the form \([a, t_\delta]\), \(a < t_\delta \leq b\), and \(t_\delta\) decreases as \(\delta\) decreases.

Let \(x \in (a, b)\). Then by theorem [10]
\[
\gamma^\alpha(F, a, b) = \gamma^\alpha(F, a, x) + \gamma^\alpha(F, x, b) \geq \gamma^\alpha_\delta(F, a, x) + \gamma^\alpha_\delta(F, x, b) \tag{8}
\]
\[
As \ y, z < \gamma^\alpha(F, a, b), \ there \ exists \ a \ \delta_0 > 0 \ such \ that \ \delta < \delta_0 \ implies \ that \ \gamma^\alpha_\delta(F, a, b) > y, z. \ In \ the \ rest \ of \ this \ proof, \ we \ only \ consider \ \delta < \delta_0 \ without \ mentioning. \]
\[
Since \ \gamma^\alpha_\delta(F, a, b) \geq y \ and \ \gamma^\alpha_\delta(F, a, u) \ is \ continuous \ and \ increasing \ in \ u, \ there \ exists \ an \ \ x \in (a, b) \ such \ that \ \gamma^\alpha_\delta(F, a, x) = y. \ This \ implies \ that \ x \in [a, t_\delta]. \ Further, \ from \ [8], \ it \ follows \ that \]
\[
z = \gamma^\alpha(F, a, b) - y = \gamma^\alpha(F, a, b) - \gamma^\alpha_\delta(F, a, x) \geq \gamma^\alpha_\delta(F, x, b)
\]
implying that \(x\) also belongs to \([s_\delta, b]\). This can happen only when \(s_\delta \leq t_\delta\).

Thus for each \(\delta\) there exists an interval \([s_\delta, t_\delta]\) such that
\[
x \in [s_\delta, t_\delta] \implies \gamma^\alpha_\delta(F, x, b) \leq z \ and \ \gamma^\alpha_\delta(F, a, x) \leq y.
\]

Let \(s = \sup_{0 < \delta < \delta_0} s_\delta\) and let \(t = \inf_{0 < \delta < \delta_0} t_\delta\). Now \(s_\delta\) increases and \(t_\delta\) decreases as \(\delta\) goes to zero, but as \(s_\delta \leq t_\delta\) for any \(\delta\). Thus \(s \leq t\) and
\[
[s, t] = \bigcap_{0 < \delta < \delta_0} [s_\delta, t_\delta].
\]

Consequently \(x \in [s, t]\) implies \(\gamma^\alpha_\delta(F, x, b) \leq z\) and \(\gamma^\alpha_\delta(F, a, x) \leq y\) for any \(\delta\). Hence
\[
x \in [s, t] \implies \gamma^\alpha(F, x, b) \leq z \ and \ \gamma^\alpha(F, a, x) \leq y \tag{9}
\]
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But as \( \gamma^{\alpha}(F, a, x) + \gamma^{\alpha}(F, x, b) = \gamma^{\alpha}(F, a, b) = y + z \), the inequalities in \( \text{(9)} \) must be equalities. Thus for a given \( y, 0 < y < \gamma^{\alpha}(F, a, b) \), there exists a set \([s, t] \subset [a, b]\) such that \( x \in [s, t] \implies \gamma^{\alpha}(F, a, x) = y \) which completes the proof. 

**Corollary 13** If \( \gamma^{\alpha}(F, a, b) \) is finite, \( \gamma^{\alpha}(F, a, x) \) is continuous for \( x \in (a, b) \).

This can be proved using the monotonicity of \( \gamma^{\alpha}(F, a, b) \) in \( a \) and \( b \).

**Remark:** The implication of this result is that no single point has a nonzero mass, or in other words, the mass function is atomless.

The scaling and translation properties of the mass function are similar to those of Hausdorff measure:

**Theorem 14** For \( F \subset \mathbb{R} \) and \( \lambda \in \mathbb{R} \), let \( F + \lambda \) denote the set

\[
F + \lambda = \{x + \lambda : x \in F\}
\]

and let \( \lambda F \) denote the set

\[
\lambda F = \{\lambda x : x \in F\}.
\]

Then,

(i) Translation:

\[
\gamma^{\alpha}(F + \lambda, a + \lambda, b + \lambda) = \gamma^{\alpha}(F, a, b)
\]

(ii) Scaling (\( \lambda \geq 0 \)):

\[
\gamma^{\alpha}(\lambda F, \lambda a, \lambda b) = \lambda^{\alpha} \gamma^{\alpha}(F, a, b)
\]

**Remark:** If the set \( F \) is self-similar so that \( \lambda_0 F \cap [\lambda_0 a, \lambda_0 b] = F \cap [\lambda_0 a, \lambda_0 b] \) for a particular \( \lambda_0 \), then the scaling property can be rewritten as

\[
\gamma^{\alpha}(F, \lambda_0 a, \lambda_0 b) = \lambda_0^{\alpha} \gamma^{\alpha}(F, a, b).
\]

An example is the middle \( \frac{1}{3} \) Cantor set \( C \), with \( a = 0, b = 1 \) and \( \lambda_0 = \frac{1}{3^{1/\alpha}} \).

Now we introduce one of the central notions of this paper, viz. the integral staircase function for a set \( F \) of the order \( \alpha \). This function, which is a generalization of functions like the Lebesgue-Cantor staircase function, describes how the mass of \( F \cap [a, b] \) increases as \( b \) increases.

**Definition 15** Let \( a_0 \) be an arbitrary but fixed real number. The integral staircase function \( S^{\alpha}_{F}(x) \) of order \( \alpha \) for a set \( F \) is given by

\[
S^{\alpha}_{F}(x) = \begin{cases} 
\gamma^{\alpha}(F, a_0, x) & \text{if } x \geq a_0 \\
-\gamma^{\alpha}(F, x, a_0) & \text{otherwise.}
\end{cases}
\]

The number \( a_0 \) can be chosen according to convenience. A few properties of \( S^{\alpha}_{F}(x) \) which are restatements of the corresponding properties of the mass function \( \gamma^{\alpha}(F, a, b) \) are as follows.

**Theorem 16** Let \( F \) be a subset of \( \mathbb{R} \), and let \( 0 < \alpha \leq 1 \). If \( \gamma^{\alpha}(F, a, b) \) is finite, then for all \( x, y \in (a, b) \) such that \( x < y \), the following statements hold:

(i) \( S^{\alpha}_{F}(x) \) is increasing in \( x \).

(ii) If \( F \cap (x, y) = \emptyset \), then \( S^{\alpha}_{F} \) is a constant in \( [x, y] \).

(iii) \( S^{\alpha}_{F}(y) - S^{\alpha}_{F}(x) = \gamma^{\alpha}(F, x, y) \).

(iv) \( S^{\alpha}_{F} \) is continuous on \( (a, b) \).

As an example, we calculate and show the graph of \( S^{\alpha}_{C} \) for the middle \( \frac{1}{3} \) Cantor set \( C \) in the section \( \text{[4]} \) after discussing some results required to calculate it.
3. The \( \gamma \)-dimension

We now consider the sets \( F \) for which the mass function \( \gamma^\alpha(F, a, b) \) gives the most useful information. Due to the similarity of the definitions of the mass function and the Hausdorff outer measure \([1,3,4,5,6]\), one might expect that the mass function can be used to define a fractal dimension. It is indeed the case. If \( 0 < \alpha < \beta \leq 1 \),

\[
\sigma^\beta[F, P] \leq |P|^{\beta - \alpha} \sigma^\alpha[F, P] \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)}
\]

so that

\[
\gamma^\beta(F, a, b) \leq \delta^{\beta - \alpha} \gamma^\alpha(F, a, b) \frac{\Gamma(\alpha + 1)}{\Gamma(\beta + 1)}
\]

Thus in the limit as \( \delta \to 0 \), we get

\[
\gamma^\beta(F, a, b) = 0 \quad \text{provided } \gamma^\alpha(F, a, b) < \infty \text{ and } \alpha < \beta.
\]

It follows that \( \gamma^\alpha(F, a, b) \) is infinite up to certain value of \( \alpha \), say \( \alpha_0 \) (if \( \alpha_0 < 1 \)). We call this number the \( \gamma \)-dimension of \( F \). \( \gamma^{\alpha_0}(F, a, b) \) itself may be zero, nonzero finite, or infinite. To make the notion of dimension precise,

**Definition 17** The \( \gamma \)-dimension of \( F \cap [a, b] \), denoted by \( \dim_\gamma(F \cap [a, b]) \), is

\[
\dim_\gamma(F \cap [a, b]) = \inf \{ \alpha : \gamma^\alpha(F, a, b) = 0 \} = \sup \{ \alpha : \gamma^\alpha(F, a, b) = \infty \}
\]

Now we compare the \( \gamma \)-dimension with the Hausdorff dimension and the box dimension. As the definition of Hausdorff measure involves arbitrary countable covers, it is expected that the Hausdorff dimension be finer than \( \gamma \)-dimension. This is shown to be the case below:

Let \( H^\delta_E(E) \) denote the coarse grained Hausdorff measure of a subset \( E \) of \( \mathbb{R} \), and \( H^\alpha(E) \) denote the Hausdorff measure. Let \( P \) be a subdivision with \( |P| \leq \delta \). Those components \([x_i, x_{i+1}]\) of \( P \) for which \( \theta(F, [x_i, x_{i+1}]) \) is nonzero, form a \( \delta \)-cover of \( F \cap [a, b] \). Thus,

\[
\sigma^\alpha[F, P] \geq \frac{1}{\Gamma(\alpha + 1)} H^\delta(F \cap [a, b]).
\]

Since this is true for any \( P \) such that \( |P| \leq \delta \), it follows that

\[
\gamma^\delta(F, a, b) \geq \frac{1}{\Gamma(\alpha + 1)} H^\delta(F \cap [a, b])
\]

for each \( \delta > 0 \). So taking limit as \( \delta \to 0 \),

\[
\gamma^\alpha(F, a, b) \geq \frac{1}{\Gamma(\alpha + 1)} H^\alpha(F \cap [a, b]). \quad (11)
\]

which also implies

\[
\dim_H(F \cap [a, b]) \leq \dim_\gamma(F \cap [a, b]).
\]

There exist sets for which the two definitions give different results. For example, if \( \mathbb{Q} \) denotes the set of rational numbers, then \( \dim_H(\mathbb{Q} \cap [0,1]) = 0 \), while \( \dim_\gamma(\mathbb{Q} \cap [0,1]) = 1 \). However, it will be shown in the next section that the two dimensions are equal for compact sets.
Next we compare the $\gamma$-dimension with the box dimension. Let $\dim_\gamma(F \cap [a, b]) = \alpha$. Then $\gamma^\beta(F, a, b)$ diverges for any $\beta < \alpha$. Thus for any $k > 0$, there exists $\delta_0 > 0$ such that $\delta < \delta_0 \implies \gamma^\beta(F, a, b) > k$.

Let $P$ be any subdivision such that $|P| \leq \delta$, and let $N_\delta(F \cap [a, b])$ be the number of nonzero terms in the sum $\sigma^\alpha[F, P]$. Then, for arbitrary but fixed $k > 0$, there exists $\delta_0 > 0$ such that $\delta < \delta_0 \implies \gamma^\beta(F, a, b) > k$.

Let $P$ be any subdivision such that $|P| \leq \delta$, and let $N_\delta(F \cap [a, b])$ be the number of nonzero terms in the sum $\sigma^\alpha[F, P]$. Then, for arbitrary but fixed $k > 0$ and $\delta < \delta_0$,

$$k < \gamma^\beta(F, a, b) \leq \frac{N_\delta(F \cap [a, b])\delta^\beta}{\Gamma(\beta + 1)}$$

where $0 < \beta < \alpha \leq 1$. Thus,

$$\ln(k) \leq \ln N_\delta(F \cap [a, b]) + \beta \ln(\delta) - \ln(\Gamma(\beta + 1))$$

which implies

$$-\beta \ln(\delta) \leq \ln N_\delta(F \cap [a, b]) - \ln(k) - \ln(\Gamma(\beta + 1)).$$

Dividing by $-\ln(\delta)$ (which is positive for $\delta < 1$),

$$\beta \leq \frac{\ln(N_\delta(F \cap [a, b]))}{-\ln(\delta)} - \frac{\ln(k) + \ln(\Gamma(\beta + 1))}{-\ln(\delta)}$$

Taking limit as $\delta \to 0$ and noting that the first term is the definition of the box dimension $\dim_B(F \cap [a, b])$ in the limit and the denominator of the second diverges, we get

$$\beta \leq \dim_B(F \cap [a, b]) = \lim_{\delta \to 0} \frac{\ln(N_\delta(F \cap [a, b]))}{-\ln(\delta)}.$$
Theorem 18 For a compact set \( F \subset \mathbb{R} \),
\[
\gamma^\alpha(F, a, b) = \frac{1}{\Gamma(\alpha + 1)} \mathcal{H}^\alpha(F \cap [a, b]).
\]

Proof: For \( \delta > 0 \), let \( \{A_i, i = 1, 2, \ldots\} \) be any countable cover of \( F \cap [a, b] \) such that \( \text{diam}A_i \leq \frac{\delta}{2} \) for all \( i \). The sets \( A_i \) need not be open or closed. Then
\[
\mathcal{H}^\alpha_{\delta/2}(F \cap [a, b]) \leq \sum_i (\text{diam}A_i)^\alpha.
\]

Consider closed intervals \( B_i = [u_i, v_i] \) where \( u_i = \inf A_i \) and \( v_i = \sup A_i \). Then \( A_i \subset B_i \) and \( \text{diam}B_i = \text{diam}A_i \). Thus \( \{B_i\} \) forms a cover of \( F \cap [a, b] \) and
\[
\sum_i (\text{diam}B_i)^\alpha = \sum_i (\text{diam}A_i)^\alpha \geq \mathcal{H}^\alpha_{\delta/2}(F \cap [a, b]).
\]

Given \( \epsilon \in (0, (\delta/2)^\alpha) \), let \( \{C_i\}_{i=1}^\infty \) be the open intervals
\[
C_i = \left( u_i - \frac{1}{2} \left( \frac{\epsilon}{2^i} \right)^{\frac{1}{\alpha}}, v_i + \frac{1}{2} \left( \frac{\epsilon}{2^i} \right)^{\frac{1}{\alpha}} \right).
\]
The class \( \{C_i\}_{i=1}^\infty \) thus forms an open cover of \( F \cap [a, b] \) and
\[
\text{diam}C_i = \text{diam}A_i + \left( \frac{\epsilon}{2^i} \right)^{\frac{1}{\alpha}} < \delta
\]
so that
\[
\sum_i (\text{diam}C_i)^\alpha = \sum_i \left( \text{diam}A_i + \left( \frac{\epsilon}{2^i} \right)^{\frac{1}{\alpha}} \right)^\alpha.
\]

A simple consequence of Jensen’s inequality \[38\], which for the case of two variables assures that \((s_1 + s_2)^t \leq s_1^t + s_2^t \) for \( s_1, s_2 > 0 \) and \( 0 < t < 1 \), is that
\[
\sum (\text{diam}C_i)^\alpha \leq \sum (\text{diam}A_i)^\alpha + \epsilon \sum \frac{1}{2^i} = \sum (\text{diam}A_i)^\alpha + \epsilon. \tag{12}\]

We now show that a finite cover consisting of closed intervals can be constructed. As \( F \) is compact, so is \( F \cap [a, b] \). Thus a finite subset of \( \{C_i\} \) covers \( F \cap [a, b] \). We denote this finite subcover by \( \{D_i, i = 1, \ldots, n\} \). The \( D_i \) are open intervals of the form \( (a_i, b_i) \). Without loss of generality we can choose this finite subcover \( \{D_i\} \) such that \( D_i \not\subset D_j \) whenever \( i \neq j \). Further, the sets are labeled such that \( a_i \leq a_{i+1} \). But as \( D_i \not\subset D_{i+1} \) and \( D_{i+1} \not\subset D_i \), it implies that \( a_i < a_{i+1} \) and \( b_i < b_{i+1} \).

Now we consider the closures \( \overline{D}_i \) of \( D_i \). As \( \{D_i\} \) is a finite subcover out of \( \{C_i\} \) and \( \{\overline{D}_i\} \) have the same diameters as \( D_i \), it follows from \[12\] that
\[
\sum (\text{diam}\overline{D}_i)^\alpha \leq \sum (\text{diam}A_i)^\alpha + \epsilon.
\]

Let \( I_1 = \overline{D}_1 \) and \( I_i = \overline{D}_i \setminus \overline{D}_{i-1} \) for \( 2 \leq i \leq n \). The collection \( \{I_i\} \) forms a finite cover of \( F \cap [a, b] \) by closed intervals, and
\[
\sum (\text{diam}I_i)^\alpha \leq \sum (\text{diam}A_i)^\alpha + \epsilon.
\]

The closed intervals \( I_i \) share at the most endpoints. The set of all the endpoints of \( I_i \), \( 1 \leq i \leq n \) forms a subdivision \( P \) of \( [a, b] \) which can be refined to a subdivision \( Q \) such that \( |Q| \leq \delta \) and
\[
\Gamma(\alpha + 1)s^\alpha[F, Q] = \sum (\text{diam}I_i)^\alpha \leq \sum (\text{diam}A_i)^\alpha + \epsilon.
\]
Therefore,
\[
\Gamma(\alpha + 1) \gamma_0^\alpha(F, a, b) \leq \sum (\text{diam} A_i)^\alpha + \epsilon.
\]
Since this relation holds for any countable cover \( \{ A_i \} \) of \( F \cup [a, b] \) such that \( \text{diam} A_i \leq \delta/2 \) and for arbitrary \( \epsilon > 0 \), it follows that
\[
\Gamma(\alpha + 1) \gamma_0^\alpha(F, a, b) \leq \mathcal{H}_2^\alpha(F \cap [a, b]).
\]
Consequently in the limit as \( \delta \to 0 \),
\[
\Gamma(\alpha + 1) \gamma_0^\alpha(F, a, b) \leq \mathcal{H}_2^\alpha(F \cap [a, b]).
\]
Equations (11) and (13) together imply the required equality. •

**Corollary 19** If \( F \subset \mathbb{R} \) is compact, then \( \dim_\gamma F = \dim_\mathcal{H} F \).

*Example:* We now discuss an important prototype example of \( S_0^\alpha F(x) \). Consider the middle 1/3 Cantor set \( C \) (hereafter referred to as the Cantor set). This set is compact and has a Hausdorff dimension \( \alpha = \log(2)/\log(3) \). Thus by theorem 18, \( \mathcal{H}_2^\alpha(C \cap [a, b]) = \Gamma(\alpha + 1) \gamma_0^\alpha(C, a, b) \) and \( \dim_\gamma C = \dim_\mathcal{H} C = \alpha = \log(2)/\log(3) \). Using the self-similarity of \( C \) and the monotonicity as well as scaling and translation properties of the mass function (theorem 14), we can calculate \( S_0^\alpha F \) at each point. A graph of \( \Gamma(\alpha + 1) S_0^\alpha C(x) \) i.e. \( \Gamma(\alpha + 1) \gamma_0^\alpha(C, 0, x) \) against \( x \) is shown in figure 1. This is the Lebesgue-Cantor Staircase function.

**Figure 1.** \( \Gamma(\alpha + 1) S_0^\alpha C(x) \): The integral staircase function for the Cantor set

### 5. \( \alpha \)-Perfect sets

The correspondence between sets \( F \) and their staircase functions \( S_0^\alpha F \) is many to one. For example, \( S_0^\alpha C' = S_0^\alpha C \) where \( C \) is the middle 1/3 Cantor set and \( C' = C \setminus \{ \frac{2}{3} \} \). Intuitively, it can be said that as the mass function is atomless, removing a single point from \( C \) does not change its value.

Another example is a set \( D = C \cup E \) where \( E \subset (\frac{1}{3}, \frac{2}{3}) \) satisfying \( \dim_\gamma E < \alpha \), \( \alpha = \ln(2)/\ln(3) \). Then it can be seen that \( S_0^\alpha D(x) = S_0^\alpha C(x) \) for all \( x \). Thus adding a lower dimensional set need not change the value of the staircase function either. We call the sets giving rise to the same staircase function as staircawise congruent:
Definition 20 Let $F \subset \mathbb{R}$ and $G \subset \mathbb{R}$ be such that $\dim F = \dim G = \alpha$, $\alpha \in (0,1]$. Then $F$ and $G$ are said to be staircasewise congruent if $S_F^\alpha(x)$ and $S_G^\alpha(x)$ are finite and equal for all $x \in \mathbb{R}$.

This congruence being an equivalence relation, we denote the equivalence class of sets containing $F$ by $\mathcal{E}_F^\alpha$. Thus if $G$ is in $\mathcal{E}_F^\alpha$, then $S_G^\alpha = S_F^\alpha$ and $E_G^\alpha = E_F^\alpha$.

The above examples intuitively suggest that not all points or subsets contribute to the staircase function. Now we proceed to select a representative set out of the equivalence class which, intuitively speaking, has exactly those points at which $S_F^\alpha$ “changes”. To choose only the points where a function “changes”, we need the following definition.

Definition 21 We say that a point $x$ is a point of change of a function $f$, if $f$ is not constant over any open interval $(c,d)$ containing $x$. The set of all points of change of $f$ is called the set of change of $f$ and is denoted by $\text{Sch}(f)$.

Thus $\text{Sch}(f_1) = \emptyset$ if $f_1 = \text{constant}$, while $\text{Sch}(f_2) = \mathbb{R}$ if $f_2(x) = x$. More importantly we note that if $G \in \mathcal{E}_F^\alpha$, then $S_G^\alpha(x) = S_F^\alpha(x)$ and therefore $\text{Sch}(S_G^\alpha) = \text{Sch}(S_F^\alpha)$. In other words, $\text{Sch}(S_F^\alpha)$ is determined by the equivalence class $\mathcal{E}_F^\alpha$.

The following theorem states that $\text{Sch}(S_F^\alpha)$ itself belongs to $\mathcal{E}_F^\alpha$.

Theorem 22 Let $F \subset \mathbb{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbb{R}$ for $\alpha = \dim F$ and $H = \text{Sch}(S_F^\alpha)$. Then $H$ belongs to $\mathcal{E}_F^\alpha$, i.e. $S_H^\alpha = S_F^\alpha$.

Proof: It will be sufficient, in view of definition 15, to prove that $\gamma^\alpha(H,a,b) = \gamma^\alpha(F,a,b)$ for any $a,b \in \mathbb{R}$.

We begin by noting that for $u < v$, if $F \cap [u,v] = \emptyset$, then $\gamma^\alpha(F,u,v) = 0$. Consequently $S_F^\alpha$ is constant on $(u,v)$ implying $(u,v) \cap H = \emptyset$. Then for any $\epsilon > 0$ such that $u + \epsilon \leq v - \epsilon'$,

$$\theta(H,[u + \epsilon',v - \epsilon']) = 0. \quad (14)$$

Next, let $\delta > 0$. Then given $\epsilon > 0$, there is a subdivision $P_{a,b} = \{y_0,y_1,\ldots,y_n\}$ such that $|P| \leq \delta$ and

$$\sigma^\alpha[F,P] \leq \gamma^\alpha_\delta(F,a,b) + \frac{\epsilon}{2}. \quad (15)$$

If $\theta(F,I) = 1$ for all components $I$ of $P$, then certainly

$$\sigma^\alpha[H,P] \leq \sigma^\alpha[F,P]. \quad (16)$$

Otherwise, let $K$ be the set of all points of the form

$$c' = c + \left(\frac{\epsilon \Gamma(\alpha + 1)}{2n}\right)^\frac{1}{\alpha}, \quad d' = d - \left(\frac{\epsilon \Gamma(\alpha + 1)}{2n}\right)^\frac{1}{\alpha} \quad (17)$$

where $c$ and $d$ are the endpoints of those components $I$ of $P$ such that $\theta(F,I) = 0$. Then $Q_{a,b} = P \cup K$ is a refined subdivision and $|Q| \leq |P|$. If $I = [c,d]$ is a component of $P$ such that $\theta(F,I) = 0$, then it contains three components of $Q$, viz. $[c,c']$, $[c',d']$ and $[d',d]$ where $c'$ and $d'$ are given by (17). The term in $\sigma^\alpha[H,Q]$ corresponding to $[c',d']$ is zero according to (14), and the remaining two contribute at the most $\epsilon/2n$ each. If $\theta(F,I) \neq 0$, then $I$ is also a component of $Q$ and the term corresponding to $I$ in $\sigma^\alpha[H,Q]$ is either zero or is exactly the same as the corresponding term in $\sigma^\alpha[F,P]$. Therefore,

$$\sigma^\alpha[H,Q] \leq \sigma^\alpha[F,P] + \frac{\epsilon}{2}. \quad (18)$$
Thus from (15), (16) and (18), we see that there exists a subdivision \( Q \), such that \( |Q| \leq \delta \) and
\[
\sigma^\alpha[H, Q] \leq \gamma_{\delta}^\alpha(F, a, b) + \epsilon.
\]
As \( \epsilon \) is arbitrary, we see that
\[
\gamma_{\delta}^\alpha(H, a, b) \leq \gamma_{\delta}^\alpha(F, a, b)
\]
(19)

Now we wish to rule out the possibility that
\[
\gamma_{\delta}^\alpha(H, a, b) < \gamma_{\delta}^\alpha(F, a, b).
\]
(20)

Suppose that (20) is true. Then there exists a subdivision \( P_1 = \{x_0, \ldots, x_n\} \) such that \( |P_1| \leq \delta \) and
\[
\sigma^\alpha[H, P_1] < \gamma_{\delta}^\alpha(F, a, b).
\]
(21)

From (6) we know that
\[
\gamma_{\delta}^\alpha(F, a, b) \leq \sum_{i=0}^{n-1} \gamma_{\delta}^\alpha(F, x_i, x_{i+1})
\]
so that
\[
\sigma^\alpha[H, P_1] < \sum_{i=0}^{n-1} \gamma_{\delta}^\alpha(F, x_i, x_{i+1}).
\]

For this equation to hold, there must be at least one \( k, 0 \leq k \leq n-1 \), such that
\[
\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha + 1)} \theta(H, [x_k, x_{k+1}]) < \gamma_{\delta}^\alpha(F, x_k, x_{k+1}) \leq \gamma_{\delta}^\alpha(F, x_k, x_{k+1}).
\]

As every quantity is non-negative, it follows that \( \gamma_{\delta}^\alpha(F, x_k, x_{k+1}) > 0 \), and \( S_F^\alpha \) is not constant in \([x_k, x_{k+1}]\). Therefore \( \text{Sch}(S_F^\alpha) \cap [x_k, x_{k+1}] \neq \emptyset \) i.e. \( H \cap [x_k, x_{k+1}] \neq \emptyset \). So \( \theta(H, [x_k, x_{k+1}]) = 1 \) and
\[
\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha + 1)} < \gamma_{\delta}^\alpha(F, x_k, x_{k+1}).
\]

As \( Q = \{x_k, x_{k+1}\} \) is a subdivision of \([x_k, x_{k+1}]\) such that \( |Q| \leq \delta \),
\[
\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha + 1)} = \sigma^\alpha[F, Q] < \gamma_{\delta}^\alpha(F, x_k, x_{k+1})
\]
which is a contradiction by the definition of \( \gamma_{\delta}^\alpha(F, x_k, x_{k+1}) \) implying that our assumption (20) is wrong. Thus, (16) is an equality for any \( \delta > 0 \), and therefore
\[
\gamma_{\delta}^\alpha(H, a, b) = \gamma_{b}^\alpha(F, a, b).
\]

\[\text{Lemma 23}\] Let \( F \subset \mathbb{R} \) be such that \( S_F^\alpha(x) \) is finite for all \( x \in \mathbb{R} \) for \( \alpha = \dim F \). Then the set \( H = \text{Sch}(S_F^\alpha) \) is perfect i.e. \( H \) is closed and every point of \( H \) is its limit point.

\[\text{Proof}\] Let \( y \) be a limit point of \( H \). Then any open interval \((c, d)\) containing \( y \) contains a point \( z \) of \( H = \text{Sch}(S_F^\alpha) \). Therefore \( S_F^\alpha \) is not constant on \((c, d)\). Hence \( y \in H \) implying that \( F \) is closed.

If \( x \in H \) is not a limit point of \( H \), then there exists an open interval \((c, d)\) containing \( x \) but no other point of \( H \) so that \( F \cap (c, x) = \emptyset \) and \( F \cap (x, d) = \emptyset \). This implies that \( S_F^\alpha(x) - S_F^\alpha(c) = (S_F^\alpha(x) - S_F^\alpha(c)) + (S_F^\alpha(c) - S_F^\alpha(c)) = 0 \) due to theorem (10 ii). Therefore \( x \) is not in \( H = \text{Sch}(S_F^\alpha) \) which is a contradiction.

Now we choose \( \text{Sch}(S_F^\alpha) \) as the “canonical” representative of \( \mathcal{E}_F^\alpha \):
Definition 24 Let $F \subset \mathbb{R}$ be such that $S^\alpha_F(x)$ is finite for all $x \in \mathbb{R}$ for $\alpha = \dim_F F$. Then the set $\text{Sch}(S^\alpha_F)$ is said to be $\alpha$-perfect, and is said to be the $\alpha$-perfect representative of $E^\alpha_F$.

Thus, every $E^\alpha_F$ contains a unique $\alpha$-perfect set. The next theorem states that it is the minimal closed set in $E^\alpha_F$.

Theorem 25 An $\alpha$-perfect set $F$ is the intersection of all the closed sets $G$ in $E^\alpha_F$. In other words, it is the minimal closed set in $E^\alpha_F$.

Proof: Let $G$ be the class of all closed sets in $E^\alpha_F$.

As $F$ is perfect, it is closed. Therefore,

$$F \supset \bigcap_{G \in G} G.$$  \hspace{1cm} (22)

Let $G_0 \in G$ and $x \notin G_0$. Then there is an open interval $(c, d)$ containing $x$ but no point of $G_0$, as $G_0$ is closed. This implies that $S^\alpha_{G_0}(c) = S^\alpha_{G_0}(d)$ by theorem 14(ii), and further $S^\alpha_F(c) = S^\alpha_F(d)$ as $G_0 \in E^\alpha_F$. Since $F$ is $\alpha$-perfect, we have $F \cap (c, d) = \emptyset$ implying that $x \notin F$. Therefore, $x \in F \implies x \in G_0$ for all $G_0 \in G$, so that

$$F \subset \bigcap_{G \in G} G.$$  \hspace{1cm} (23)

The proof is completed in view of (22) and (24). $\blacksquare$

The following lemma is a useful restatement of the definition of an $\alpha$-perfect set.

Lemma 26 Let $F \subset \mathbb{R}$ be $\alpha$-perfect and $x \in F$. If $y < x < z$, then either $S^\alpha_F(y) < S^\alpha_F(x)$ or $S^\alpha_F(x) < S^\alpha_F(z)$ (or both).

Thus for an $\alpha$-perfect set $F$, the lemma assures that if $x \in F$, then the values of $S^\alpha_F(y)$ must be different from $S^\alpha_F(x)$ at all points $y$ on at least one side of $x$.

As an example, we now show that the middle $\frac{1}{3}$ Cantor set $C$ is $\alpha$-perfect, for $\alpha = \log(2)/\log(3)$. Let $x \notin C$. As $C$ is closed, there is at least one open interval $(c, d)$ containing $x$, such that $C \cap (c, d) = \emptyset$. Therefore, $S^\alpha_C$ is constant on $(c, d)$ implying that $x \notin \text{Sch}(S^\alpha_C)$. Hence,

$$\text{Sch}(S^\alpha_C) \subset C.$$  \hspace{1cm} (24)

Let $x$ be a point of $C$. Then $x$ can be represented by $0.x_1x_2x_3\ldots$ where $x_i$ is the $i$th digit in the ternary representation of $x$. As $x \in C$, $x_i = 0$ or 2. Let $(c, d)$ be any open interval containing $x$. Then there is an integer $n > 0$ such that $(x - 3^{-n}, x + 3^{-n}) \subset (c, d)$. Let $D$ be the set of numbers $y = 0.y_1y_2y_3\ldots$ satisfying

$$y_i = \begin{cases} x_i & 1 \leq i \leq n + 1 \\ 0 \text{ or } 2 & i > n + 1 \end{cases}$$

where $y_i$ is the $i$th digit in the ternary representation of $y$. Then $D \subset C$ is a scaled down copy of $C$ by a factor $3^{-n-1}$ and $D \subset (x - 3^{-n}, x + 3^{-n}) \subset (c, d)$. Thus using the scaling property of $\gamma^n$ (theorem 13),

$$S^\alpha_C(d) - S^\alpha_C(c) \geq \frac{1}{\Gamma(\alpha + 1)}(3^{-n-1})^\alpha = \frac{1}{\Gamma(\alpha + 1)}2^{-n-1} > 0$$

where $\alpha = \ln(2)/\ln(3)$. This implies that $S^\alpha_C$ is not constant on $(c, d)$. Thus, $x \in C \implies x \in \text{Sch}(S^\alpha_C)$ i.e.

$$C \subset \text{Sch}(S^\alpha_C).$$  \hspace{1cm} (25)

From (24) and (26) we see that $C = \text{Sch}(S^\alpha_C)$ implying that $C$ is $\alpha$-perfect for $\alpha = \log(2)/\log(3)$. 
6. $F$-continuity

In this section we introduce the notation for limit and continuity using topology of $F \subset \mathbb{R}$ with the metric inherited from $\mathbb{R}$. Our purpose in doing so is to distinguish between these notions and ones on $\mathbb{R}$ when they both appear together.

**Definition 27** Let $F \subset \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $x \in F$. A number $\ell$ is said to be the limit of $f$ through the points of $F$, or simply $F$-limit, as $y \to x$, if given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$y \in F \text{ and } |y - x| < \delta \implies |f(y) - \ell| < \epsilon.$$  

If such a number exists, then it is denoted by

$$\ell = F\lim_{y \to x} f(y)$$

This definition does not involve values of the function at $y$ if $y \notin F$. Also, $F$-limit is not defined at points $x \notin F$.

We now introduce the notion of $F$-continuity which is continuity as far as the values of the function only on the set $F$ are concerned.

**Definition 28** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be $F$-continuous at $x \in F$ if

$$f(x) = F\lim_{y \to x} f(y)$$

We note that the notion of $F$-continuity is not defined at $x \notin F$.

It is clear that continuity of $f : \mathbb{R} \to \mathbb{R}$ at $x \in F$ implies $F$-continuity at $x$. But the converse is not true. We consider a few examples: Let $C$ be the middle $\frac{1}{3}$ Cantor set. Then the functions $f_1(x) = 1$ and $f_2(x) = x$ are continuous on $[0, 1]$; they are also $C$-continuous on $C \cap [0, 1]$. In contrast, consider $f_3(x) = \chi_C(x)$ and $f_4(x) = x \cdot \chi_C(x)$ where $\chi_C(x)$ is the characteristic function of $C$. These functions are $C$-continuous, but not continuous.

Now we define an analogue of uniform continuity:

**Definition 29** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be uniformly $F$-continuous on $E \subset F$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$x \in F, y \in E \text{ and } |y - x| < \delta \implies |f(y) - f(x)| < \epsilon.$$  

It is clear that uniform $F$-continuity on $E$ implies $F$-continuity on $E$. The converse is true only in certain cases:

**Theorem 30** If a function $f : \mathbb{R} \to \mathbb{R}$ is $F$-continuous on a compact set $E \subset F$, then it is uniformly $F$-continuous on $E$.

7. $F^\alpha$-Integration

In the definition of $F^\alpha$-integral below, values of the function only at the points of $F$ are considered. Further, instead of the lengths of subintervals, we consider the difference between the values of the staircase function $S^\alpha_F$ at the endpoints. In this respect, $F^\alpha$-integral is similar to Riemann-Stiltjes integral [26, 27].

**Definition 31** The class of functions $f : \mathbb{R} \to \mathbb{R}$ which are bounded on $F$ is denoted by $B(F)$. In other words,

$$f \in B(F) \iff -\infty < \inf_{x \in F} f(x) \leq \sup_{x \in F} f(x) < +\infty$$
As the first step, we now define upper and lower sums which approximate the value of the $F^\alpha$-integral.

**Definition 32** Let $f \in B(F)$. Let $I$ be a closed interval. Then,
\[
M[f, F, I] = \begin{cases} 
\sup_{x \in F \cap I} f(x) & \text{if } F \cap I \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]
and similarly
\[
m[f, F, I] = \begin{cases} 
\inf_{x \in F \cap I} f(x) & \text{if } F \cap I \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

**Definition 33** Let $S_F^\alpha(x)$ be finite for $x \in [a, b]$. Let $P$ be a subdivision of $[a, b]$ with points $x_0, \ldots, x_n$. The upper $F^\alpha$-sum and the lower $F^\alpha$-sum for the function $f$ over the subdivision $P$ are given respectively by
\[
U^\alpha[f, F, P] = \sum_{i=0}^{n-1} M[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) \tag{26}
\]
and
\[
L^\alpha[f, F, P] = \sum_{i=0}^{n-1} m[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) \tag{27}
\]

We emphasize the appearance of intersection $F \cap I$ in the definition of $M$ and $m$, and also the use of $(S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i))$ as in a Riemann-Stieltjes sum instead of $(x_{i+1} - x_i)$.

From the definition it is clear that
\[
U^\alpha[f, F, P] \geq L^\alpha[f, F, P]. \tag{28}
\]

The following lemma asserts that with refinements, the upper $F^\alpha$-sum decreases and the lower $F^\alpha$-sum increases, both monotonically.

**Lemma 34** Let $F \subset \mathbb{R}$ and $f \in B(F)$. If $Q$ is a refinement of a subdivision $P$, then $U^\alpha[f, F, Q] \leq U^\alpha[f, F, P]$ and $L^\alpha[f, F, Q] \geq L^\alpha[f, F, P]$.

**Proof:** To start with, let $P = \{x_0, x_1, \ldots, x_n\}$ and $Q = P \cup \{x'\}$ where $x' \in (x_i, x_{i+1})$. Let $I = [x_i, x_{i+1}]$, $I' = [x_i, x']$, and $I'' = [x', x_{i+1}]$. If there are no points of $F$ in $I$, then $M[f, F, I] = M[f, F, I'] = M[f, F, I''] = 0$. Otherwise there are two possibilities: either both $I'$ and $I''$ have points of $F$, or only one of them, say $I'$ without loss of generality, has points of $F$.

In the first case,
\[
M[f, F, I'] \leq M[f, F, I] \quad \text{and} \quad M[f, F, I''] \leq M[f, F, I].
\]

Thus we have,
\[
M[f, F, I'](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) = M[f, F, I'][(S_F^\alpha(x') - S_F^\alpha(x_i)) + (S_F^\alpha(x_{i+1}) - S_F^\alpha(x'))]
\[
\geq M[f, F, I'](S_F^\alpha(x') - S_F^\alpha(x_i)) + M[f, F, I''](S_F^\alpha(x_{i+1}) - S_F^\alpha(x')). \tag{29}
\]

In the second case, only $I'$ has the points of $F$. Consequently $S_F^\alpha(x_{i+1}) = S_F^\alpha(x')$ and $M[f, F, I'] = M[f, F, I]$. Thus
\[
M[f, F, I'](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) = M[f, F, I'](S_F^\alpha(x') - S_F^\alpha(x_i)) \tag{30}
\]

since $M[f,F,I^n] = 0$. Combining (29) and (30), we have
\[
M[f,F,I](S_F^r(x_{i+1}) - S_F^r(x_i)) \geq M[f,F,I'](S_F^r(x') - S_F^r(x_i)) + M[f,F,I''](S_F^r(x_{i+1}) - S_F^r(x')).
\]
Thus
\[
U^\alpha[f,F,Q] \leq U^\alpha[f,F,P].
\]
This conclusion can easily be extended for any refinement of $P$.

By a similar argument, we can prove that
\[
L^\alpha[f,F,Q] \geq L^\alpha[f,F,P]
\]
which completes the proof.

**Lemma 35** If $P$ and $Q$ are any two subdivisions of $[a, b]$, then
\[
U^\alpha[f,F,P] \geq L^\alpha[f,F,Q]
\]

**Proof:** As $P \cup Q$ is a refinement of both $P$ and $Q$, it follows from the above lemma and (28) that
\[
U^\alpha[f,F,P] \geq U^\alpha[f,F,P \cup Q] \geq L^\alpha[f,F,P \cup Q] \geq L^\alpha[f,F,Q].
\]

We are now ready to define the $F^\alpha$-integral.

**Definition 36** Let $F$ be such that $S_F^r$ is finite on $[a, b]$. For $f \in B(F)$, the lower $F^\alpha$-integral is given by
\[
\int_a^b f(x) \, d_F^r x = \sup_{P[a,b]} L^\alpha[f,F,P] \tag{31}
\]
and the upper $F^\alpha$-integral is given by
\[
\int_a^b f(x) \, d_F^r x = \inf_{P[a,b]} U^\alpha[f,F,P] \tag{32}
\]
Both the supremum and infimum are taken over all the subdivisions $P$ of $[a, b]$.

The $d_F^r x$ appearing in (31) and (32) has no separate meaning; it is just the notation. It is obvious that
\[
\int_a^b f(x) \, d_F^r x \leq \int_a^b f(x) \, d_F^r x \tag{33}
\]

**Definition 37** If $f \in B(F)$, we say that $f$ is $F^\alpha$-integrable on $[a, b]$ if
\[
\int_a^b f(x) \, d_F^r x = \int_a^b f(x) \, d_F^r x
\]
In that case the $F^\alpha$-integral of $f$ on $[a, b]$, denoted by $\int_a^b f(x) \, d_F^r x$, is given by the common value.

For future use we note the following obvious and useful criterion for proving $F^\alpha$-integrability:
Lemma 38 Let \( f \in B(F) \). Then \( f \) is \( \mathcal{I}_\alpha \)-integrable on \([a, b]\) if and only if, for any \( \epsilon > 0 \), there exists a subdivision \( P \) of \([a, b]\) such that
\[
U^\alpha[f, F, P] < L^\alpha[f, F, P] + \epsilon.
\]

Now we state a sufficient condition for \( \mathcal{I}_\alpha \)-integrability. The sufficient and necessary conditions will be discussed in a companion paper.

Theorem 39 Let \( F \) be such that \( F \cap [a, b] \) is compact and \( S^\alpha_F \) is finite on \([a, b]\). Let \( f \in B(F) \), and \( a < b \). If \( f \) is \( \mathcal{I}_\alpha \)-continuous on \( F \cap [a, b] \), then \( f \) is \( \mathcal{I}_\alpha \)-integrable on \([a, b]\).

Proof: Let \( S^\alpha_F(a) = S^\alpha_F(b) \), then the \( \mathcal{I}_\alpha \)-integral is zero, and the result is obvious.

Now consider the case where \( S^\alpha_F(a) \neq S^\alpha_F(b) \). The function \( f \) is uniformly \( \mathcal{I}_\alpha \)-continuous on \( F \cap [a, b] \) as \( F \cap [a, b] \) is compact. Thus, given \( \epsilon > 0 \), there is a \( \delta > 0 \) such that
\[
| f(x) - f(y) | < \epsilon \quad \Rightarrow \quad | f(y) - f(x) | < \frac{\epsilon}{S^\alpha_F(b) - S^\alpha_F(a)}.
\]

Let \( P \) be a subdivision such that \( |P| < \delta \). Then it can be seen that \( U^\alpha[f, F, P] < L^\alpha[f, F, P] + \epsilon \) which completes the proof in view of lemma 38.

The following property of \( \mathcal{I}_\alpha \)-integral is expected from any fair definition of an integral:

Theorem 40 Let \( a < b \) and \( f \) be an \( \mathcal{I}_\alpha \)-integrable function on \([a, b]\). Let \( c \in (a, b) \). Then, \( f \) is \( \mathcal{I}_\alpha \)-integrable on \([a, c]\) and \([c, b]\). Further,
\[
\int_a^b f(x) \, d\mathcal{I}_\alpha x = \int_a^c f(x) \, d\mathcal{I}_\alpha x + \int_c^b f(x) \, d\mathcal{I}_\alpha x
\]

This can be proved in a manner analogous to Riemann integral.

The linearity of \( \mathcal{I}_\alpha \)-integral follows from the definition:

Theorem 41 (i) If \( f \) is \( \mathcal{I}_\alpha \)-integrable on \([a, b]\), and \( \lambda \) is any real number, then
\[
\int_a^b \lambda f(x) \, d\mathcal{I}_\alpha x = \lambda \int_a^b f(x) \, d\mathcal{I}_\alpha x.
\]

(ii) If \( f \) and \( g \) are \( \mathcal{I}_\alpha \)-integrable functions on \([a, b]\), then
\[
\int_a^b (f(x) + g(x)) \, d\mathcal{I}_\alpha x = \int_a^b f(x) \, d\mathcal{I}_\alpha x + \int_a^b g(x) \, d\mathcal{I}_\alpha x.
\]

The following lemma states an obvious property:

Lemma 42 If \( f \) and \( g \) are \( \mathcal{I}_\alpha \)-integrable over \([a, b]\), and \( f(x) \geq g(x) \) for all \( x \in F \cap [a, b] \), then
\[
\int_a^b f(x) \, d\mathcal{I}_\alpha x \geq \int_a^b g(x) \, d\mathcal{I}_\alpha x
\]

Definition 43 If \( f \) is \( \mathcal{I}_\alpha \)-integrable on \([a, b]\), \( a < b \), then
\[
\int_a^b f(x) \, d\mathcal{I}_\alpha x = - \int_b^a f(x) \, d\mathcal{I}_\alpha x.
\]

A particularly simple but important example, as realized in [10], is the \( \mathcal{I}_\alpha \)-integral of the characteristic function \( \chi_F \) of the set \( F \):
Lemma 44 If $\chi_F(x)$ is the characteristic function of $F \subset \mathbb{R}$, then
\[
\int_a^b \chi_F(x) \, d^\alpha F = S^\alpha_F(b) - S^\alpha_F(a)
\]

Proof: For a closed interval $I \subset [a, b]$,
\[
M[\chi_F, F, I] = m[\chi_F, F, I] = \begin{cases} 
1 & \text{if } F \cap I \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]
so that $M[\chi_F, F, I]$ is zero for a closed interval $I = [c, d]$ only when $S^\alpha_F(d) - S^\alpha_F(c) = 0$. Thus
\[
U^\alpha[\chi_F, F, P] = L^\alpha[\chi_F, F, P] = S^\alpha_F(b) - S^\alpha_F(a)
\]
for any subdivision $P$ of $[a, b]$.

As a further example, in Appendix A we calculate the $C^\alpha$-integral of the function $f(x) = x \chi_C(x)$ where $C$ is the middle $\frac{1}{3}$ Cantor set, and $\alpha = \ln(2)/\ln(3)$ is its $\gamma$-dimension. The integral is given by equations (A.12) and (A.13).

8. $F^\alpha$-Differentiation

Like the first order derivative, the $F^\alpha$-derivative is a limit of a quotient. But here the limit is $F$-limit, and the denominator is the difference in the values of the staircase function $S^\alpha_F$ at two points. Moreover, intuitively speaking, $F$ is typically the set of change of the function, and $\alpha$ is typically the $\gamma$-dimension of $F$.

Definition 45 If $F$ is an $\alpha$-perfect set then the $F^\alpha$-derivative of $f$ at $x$ is
\[
D^\alpha_F(f(x)) = \begin{cases} 
F-\lim_{y \to x} \frac{f(y) - f(x)}{S^\alpha_F(y) - S^\alpha_F(x)} & \text{if } x \in F \\
0 & \text{otherwise}
\end{cases}
\]

if the limit exists.

Note that lemma 20 tells us that if $x \in F$, then we would find such points $y$ which are arbitrarily close to $x$ at least on one side of $x$ so that the denominator in the definition is not zero and the RHS in (36) makes sense.

We now state a necessary condition for the above limit to exist.

Theorem 46 If $D^\alpha_F(f(x))$ exists for all $x$ in $(a, b)$, then $f(x)$ is $F$-continuous in $(a, b)$.

The proof is straightforward.

The linearity of the $F^\alpha$-derivative is an immediate consequence of the definition 45. Thus:

Theorem 47 (i) Let $f$ be a function on $[a, b]$. If $D^\alpha_F(f(x))$ exists for all $x \in [a, b]$, then $D^\alpha_F(\lambda f(x))$ exists and
\[
D^\alpha_F(\lambda f(x)) = \lambda D^\alpha_F(f(x)).
\]
(ii) Let $f$ and $g$ be functions on $[a, b]$. If $D^\alpha_F(f(x))$ and $D^\alpha_F(g(x))$ exist for all $x \in [a, b]$, then $D^\alpha_F(f(x) + g(x))$ exists and
\[
D^\alpha_F(f(x) + g(x)) = D^\alpha_F(f(x)) + D^\alpha_F(g(x)).
\]
Now we calculate the derivative for two elementary functions. The first does not need a proof:

**Lemma 48** The $F^\alpha$-derivative of a constant function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = k \in \mathbb{R}$ is zero, i.e.

$$D_F^\alpha(f) = 0.$$  

This result is to be contrasted with the classical fractional derivative (Riemann-Liouville, and others) of a constant which is not zero in general [29, 30, 31, 32].

**Lemma 49** The derivative of the integral staircase itself is the characteristic function $\chi_F$ of $F$:

$$D_F^\alpha(S_F^\alpha(x)) = \chi_F(x).$$

**Proof:** If $x \notin F$, $D_F^\alpha(S_F^\alpha(x)) = 0$.

If $x \in F$, then

$$D_F^\alpha(S_F^\alpha(x)) = \lim_{y \to x} S_F^\alpha(y) - S_F^\alpha(x) = 1$$

This lemma together with lemma [44] can be viewed as the special cases of the fundamental theorems of calculus (section [4]) involving $S_F^\alpha$ and its derivative $\chi_F$.

An analogue of Rolle’s theorem is:

**Theorem 50** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\text{Sch} f \subset F$ where $F$ is $\alpha$-perfect, $D_F^\alpha(f(x))$ is defined for all $x \in [a, b]$, and $f(a) = f(b) = 0$. Then there is a point $c \in F \cap [a, b]$ such that $D_F^\alpha(f(c)) \geq 0$ and a point $d \in F \cap [a, b]$ such that $D_F^\alpha(f(d)) \leq 0$.

**Proof:** If $f$ is zero throughout $[a, b]$, then $D_F^\alpha(f(x)) = 0$ for all $x \in [a, b]$ and the result follows in this case.

If $f(y) > 0$ for some $y \in (a, b)$, then as $f$ is continuous, there exists an open interval $(c, d) \subset (a, b)$ containing $y$ such that $f(z) > 0$ for any $z \in (c, d)$. Let $(c_0, d_0)$ be largest such interval. Then $f(c_0) = f(d_0) = 0$. The point $c_0 \in \text{Sch} f \subset F$ as $f$ is positive on the right of $c_0$. So,

$$D_F^\alpha(c_0) = F \lim_{z \to c_0} \frac{f(z) - f(c_0)}{S_F^\alpha(z) - S_F^\alpha(c_0)} \geq 0.$$  

Similarly $d_0 \in F$ and

$$D_F^\alpha(d_0) = F \lim_{z \to d_0} \frac{f(z) - f(d_0)}{S_F^\alpha(z) - S_F^\alpha(d_0)} \leq 0,$$

with the same considerations. The points $c_0$ and $d_0$ can be identified as points $c$ and $d$ in the statement of the theorem.

If there are no points $y$ such that $f(y) > 0$ and neither is the function zero throughout, then we can choose a point $y$ such that $f(y) < 0$ and proceed in a similar manner. •

**Remark:** The following example shows that the analogue of Rolle’s theorem can not be made more strict which would have implied existence of a point $c \in F$ such that $D_F^\alpha(f(c)) = 0$. Let $C$ be the middle $\frac{1}{3}$ Cantor set. Define

$$f(x) = S_F^\alpha(x)$$

$$= 1 - S_F^\alpha(x)$$

for $0 \leq x \leq 0.5$ and $0.5 < x \leq 1$. 

•
This function satisfies \( f(0) = f(1) = 0 \). Further, it is continuous in the interval \([0,1]\).
Its set of change is \( C \). The \( C^\alpha \)-derivative is given by
\[
\mathcal{D}_C \alpha^\alpha(f(x)) = \chi_C(x) \quad 0 \leq x \leq 0.5 \\
\quad = -\chi_C(x) \quad 0.5 < x \leq 1.
\]
Thus, \( x \in C \implies \mathcal{D}_C \alpha^\alpha(f(x)) = \pm 1 \neq 0 \).

In general, it can be said that the “fragmented nature” of the fractal \( F \) does not allow us to make the analogue of Rolle’s theorem as strict as its original version.

Now we state the analogue of the law of the mean.

**Corollary 51** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that its set of change is contained in an \( \alpha \)-perfect set \( F \subset \mathbb{R} \), \( \mathcal{D}_F^\alpha(f(x)) \) exists at all points \( x \in [a,b] \) and \( S_F^\alpha(b) \neq S_F^\alpha(a) \). Then there exists a point \( c \in F \) such that
\[
\mathcal{D}_F^\alpha(f(c)) \geq \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)}
\]
and a point \( d \in F \) such that
\[
\mathcal{D}_F^\alpha(f(d)) \leq \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)}
\]

**Proof:** Let
\[
g(x) = (f(x) - f(a)) - \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)}(S_F^\alpha(x) - S_F^\alpha(a))
\]
so that the difference between \( f \) and \( g \) is a constant plus a multiple of \( S_F^\alpha(x) - S_F^\alpha(a) \).
Now \( g(a) = g(b) = 0 \) so that we can use the last theorem to say that there exists a point \( c \in F \) such that \( \mathcal{D}_F^\alpha(g(c)) \geq 0 \) and a point \( d \in F \) such that \( \mathcal{D}_F^\alpha(g(d)) \leq 0 \). This implies that
\[
\mathcal{D}_F^\alpha(f(c)) - \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)} \geq 0
\]
for some \( c \in F \), and
\[
\mathcal{D}_F^\alpha(f(d)) - \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)} \leq 0
\]
for some \( d \in F \), which lead to the required relations. •

We had seen earlier that the \( F^\alpha \)-derivative of a constant \( f(x) = k \) is zero. Now we see that these are the only functions whose \( F^\alpha \)-derivatives are zero:

**Corollary 52** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( \text{Sch}(f) \subset F \) and \( \mathcal{D}_F^\alpha(f(x)) = 0 \) for all \( x \in [a,b] \). Then \( f(x) = k \) where \( k \) is a constant on \([a,b]\).

**Proof:** Suppose, if possible, that the function is not a constant. Then there exist \( y \) and \( z, y < z, \) such that \( f(y) \neq f(z) \). This implies either \( f(y) < f(z) \) or \( f(y) > f(z) \).

**Case 1.** \( f(y) < f(z) \). Then there exists a \( c \in F \cap (y,z) \) such that
\[
\mathcal{D}_F^\alpha(f(c)) \geq \frac{f(z) - f(y)}{S_F(z) - S_F(y)} > 0.
\]

**Case 2.** \( f(y) > f(z) \). Then there exists a \( d \in F \cap (y,z) \) such that
\[
\mathcal{D}_F^\alpha(f(d)) \leq \frac{f(z) - f(y)}{S_F(z) - S_F(y)} < 0.
\]
In both the cases we have found a point where the derivative is not zero which contradicts our assumption.

Remark: Again due to the “fragmented nature” of the fractal $F$, the $F^\alpha$-differentiability of $f$ is not sufficient to guarantee the result. Further, The additional conditions that $f$ be a continuous function and $\text{Sch}(f) \subset F$ are necessary also in the second fundamental theorem which relies on the last corollary, and the integration by parts rule (theorem which depends on theorem).

The $F^\alpha$-derivative satisfies the analogue of Leibniz rule:

**Theorem 53** If the functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ are $F^\alpha$-differentiable, then $f(x) = u(x)v(x)$ is $F^\alpha$-differentiable, and

$$D^\alpha_F(u(x)v(x)) = D^\alpha_F(u(x))v(x) + u(x)D^\alpha_F(v(x)).$$

(37)

The proof is straightforward.

9. **Fundamental theorems of $F^\alpha$-calculus**

This section relates the $F^\alpha$-integration and $F^\alpha$-differentiation as “inverse processes” of each other.

The first fundamental theorem says that the $F^\alpha$-derivative is the inverse of indefinite $F^\alpha$-integral.

**Theorem 54** Let $F \subset \mathbb{R}$ be an $\alpha$-perfect set. If $f \in B(F)$ is an $F$-continuous function on $F \cap [a, b]$ and

$$g(x) = \int_a^x f(y) \, d^\alpha_F y$$

for all $x \in [a, b]$, then

$$D^\alpha_F(g(x)) = f(x)\chi_F(x).$$

**Proof:** If $x \notin F$, then $D^\alpha_F(g(x)) = 0$ by definition.

For $x \in F$, if there are points in $F$ arbitrarily close to $x$ on both sides of $x$, then we have to consider both the following cases:

(i) The set $F \cap (x, z)$ is never empty for $z > x$ and

$$g(z) - g(x) = \int_x^z f(y) \, d^\alpha_F y.$$

(ii) The set $F \cap (z, x)$ is never empty for $z < x$ and

$$g(x) - g(z) = \int_z^x f(y) \, d^\alpha_F y.$$

Otherwise, we have to consider only one of the cases which is applicable. We consider the first one; the second can be treated similarly.

In the first case, $F \cap (x, z)$ is not empty for any $z > x$. Taking the $F$-limit as $z \to x$, we get

$$D^\alpha_F(g(x)) = F_\lim_{z \to x} \int_x^z f(y) \, d^\alpha_F y.$$ 

(38)

Now,

$$m[f, F, [x, z]] \int_x^z \chi_F(y) \, d^\alpha_F y \leq \int_x^z f(y) \, d^\alpha_F y \leq M[f, F, [x, z]] \int_x^z \chi_F(y) \, d^\alpha_F y,$$
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and

\[ \int_x^z \chi_F(y) \, d_\alpha^a y = (S_\alpha^a(z) - S_\alpha^a(x)) \]

so that

\[ m[f, F, [x, z]] \leq \frac{\int_x^z f(y) \, d_\alpha^a y}{S_\alpha^a(z) - S_\alpha^a(x)} \leq M[f, F, [x, z]]. \tag{39} \]

As \( f \) is \( F \)-continuous,

\[ F\text{-}\lim_{z \to x} m[f, F, [x, z]] = F\text{-}\lim_{z \to x} M[f, F, [x, z]] = f(x) \tag{40} \]

From (38), (39) and (40), we get the required result.

The second fundamental theorem says that the \( F^\alpha \)-integral as a function of upper limit is the inverse of \( F^\alpha \)-derivative except for an additive constant.

**Theorem 55** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous, \( F^\alpha \)-differentiable function such that \( \text{Sch}(f) \) is contained in an \( \alpha \)-perfect set \( F \) and \( h : \mathbb{R} \to \mathbb{R} \) be \( F \)-continuous, such that

\[ h(x)\chi_F(x) = D_\alpha^a(f(x)). \]

Then

\[ \int_a^b h(x) \, d_\alpha^a x = f(b) - f(a). \]

**Proof:** If

\[ g(x) = \int_a^x h(x) \, d_\alpha^a x \]

then \( D_\alpha^a(g(x)) = h(x)\chi_F(x) \) by the last theorem. Therefore \( D_\alpha^a(g(x) - f(x)) = 0 \) for all \( x \in [a, b] \). Now corollary 52 implies that \( g(x) - f(x) = k \), a constant, or \( g(x) = f(x) + k \). Thus,

\[ f(b) - f(a) = g(b) - g(a) = g(b) = \int_a^b h(x) \, d_\alpha^a x \]

which proves the theorem. ●

The following theorem states that the \( F^\alpha \)-integration can be performed by parts, and can be proved by using fundamental theorem 55 and Leibniz rule (theorem 53):

**Theorem 56** Let the functions \( u : \mathbb{R} \to \mathbb{R}, v : \mathbb{R} \to \mathbb{R} \) be such that

(i) \( u \) is continuous on \( [a, b] \) and \( \text{Sch}(u) \subset F \);

(ii) \( D_\alpha^a(u) \) exists and is \( F \)-continuous on \( [a, b] \),

(iii) \( v \) is \( F \)-continuous on \( [a, b] \).

Then,

\[ \int_a^b u(x)v(x) \, d_\alpha^a x = \left[ u(x) \int_a^x v(x') \, d_\alpha^a x' \right]_a^b - \int_a^b D_\alpha^a(u(x)) \int_a^x v(x') \, d_\alpha^a x' \, d_\alpha^a x. \tag{41} \]

The proof is straightforward and omitted.

In Appendix B, we discuss examples of repeated \( F^\alpha \)-derivatives and \( F^\alpha \)-integrals. There we also calculate \( F^\alpha \)-derivatives and \( F^\alpha \)-integrals of powers \((S_\alpha^a(x))^n \).

Now that the analogies between \( F^\alpha \)-calculus and ordinary calculus have become clear, we summarise some of them in section Appendix C for a quick reference.
10. Examples and applications of $F^\alpha$-Differential equations

In this section we briefly touch a couple of examples of $F^\alpha$-differential equations. The $F^\alpha$-differential equations is the main topic of a subsequent work [39].

Firstly we revisit the local fractional diffusion equation proposed in [16] and also discussed partly in [17]. This equation is of the form

$$D^\alpha_{F,t}(W(x,t)) = \frac{\chi_F(t)}{2} \frac{\partial^2}{\partial x^2} W(x,t).$$

(42)

where the density $W$ is defined as a function of two arguments $(x,t) \in \mathbb{R} \times \mathbb{R}$ and with a slight change of notation $D^\alpha_{F,t}$ denotes the partial $F^\alpha$-derivative with respect to time $t$, $\chi_F$ being the characteristic function of $F$. (This equation may be compared with ordinary diffusion equation $\frac{\partial W}{\partial t} = D \frac{\partial^2}{\partial x^2} W(x,t)$.) The Riemann integral like prescription given in [16] had enabled one to construct a new exact solution. This solution is

$$W(x,t) = \frac{1}{(2\pi S^\alpha_F(t))^{\frac{1}{2}}} \exp \left( -\frac{-x^2}{2S^\alpha_F(t)} \right), \quad W(x,0) = \delta(x).$$

(43)

This can be recognized as a subdiffusive solution, since $S^\alpha_F$ is known to be bounded by $kt^\alpha$, $k$ constant, in simple cases including Cantor sets.

An important observation at this stage is that: equations like (42) are examples of fractal-time evolution processes.

**Motion in a fractally distributed medium**

As a second example, we consider one dimensional motion of a particle undergoing friction. First we recall the equation of motion in a continuous (i.e. nonfractal) medium. If the frictional force is proportional to the velocity, the equation of motion can be written as

$$\frac{dv}{dt} = -k(x)v$$

(44)

where $k(x)$, the coefficient of friction, may be dependent on the particle position $x$.

Equation (44) can be reexpressed by considering velocity $v$ as a function of position $x$. The equation can be written as

$$\frac{dv}{dx} \frac{dx}{dt} = -k(x)v.$$

Identifying $dx/dt = v$ and assuming $v \neq 0$, the equation becomes

$$\frac{dv}{dx} = -k(x)$$

(45)

which is readily solved by integrating $k(x)$ if $k(x)$, which models the frictional medium, is smooth.

If the underlying medium is a fractal, then (45) is inadequate to model the motion. Instead we propose the $F^\alpha$-differential equation of the form

$$D^\alpha_F(v(x)) = -k(x)$$

(46)

for this scenario. Here, the set $F$ is the support of $k(x)$ which describes the underlying fractal medium, and $\alpha$ is the $\gamma$-dimension of $F$. (If $F$ is not $\alpha$-perfect, then the set $\text{Sch}(S^\alpha_F)$ can be chosen instead.) The function $k(x)$ may be called fractional coefficient of friction due to its physical dimensions.
The solution of (46) is easily seen to be
\[ v(x) = v_0 - \int_{x_0}^{x} k(x') \, dx' \]
where \( v_0 \) and \( x_0 \) are the initial velocity and position respectively. In a simple case where \( k(x) \) is uniform on the fractal i.e. \( k(x) = \kappa \chi_F(x) \) where \( \kappa \) is a constant, (47) reduces to
\[ v(x) = v_0 - \kappa (S^\alpha_F(x) - S^\alpha_F(x_0)). \]
In the extreme cases we obtain back the classical behaviour: (i) If \( F \) is empty (frictionless case), then \( v(x) = v_0 \); (ii) If \( F = \mathbb{R} \) (uniform medium) then \( v(x) = v_0 - \kappa (x - x_0) \).

The time dependence of \( x \) is given by
\[ t(x) = \int_{x_0}^{x} \frac{1}{v(x')} \, dx' \]
where \( t(x) \) is the time required to reach the position \( x \).

11. Concluding remarks

In this paper we have developed a calculus on fractal subsets of the real line. This development involved the identification of the special role played by staircase functions associated with fractal sets, which may be compared with the role of independent variable itself in ordinary calculus. In particular, \( F^\alpha \)-integrals and \( F^\alpha \)-derivatives (of order \( \alpha \), \( 0 < \alpha \leq 1 \)) are defined using staircase functions for sets \( F \) of dimension \( \alpha \). In contrast with the classical fractional calculus, the notions of \( F^\alpha \)-derivatives and \( F^\alpha \)-integrals are specifically tailored for fractals of dimension \( \alpha \) and thus provide suitable operators on fractals. Further, they reduce to ordinary derivative and Riemann integral respectively, when \( F = \mathbb{R} \) and \( \alpha = 1 \).

Much of the development of the \( F^\alpha \)-calculus is carried in analogy with the ordinary calculus. Several results and techniques of ordinary calculus, including the Leibniz rule, the fundamental theorems of calculus, the technique of integration by parts etc. have analogues in this calculus. Specifically we have adopted Riemann approach for \( F^\alpha \)-integrations. This approach can possibly be generalised using Kurzweil-Henstock integration schemes \[40, 41\]. Work is in progress in this direction.

In the process of the development of the \( F^\alpha \)-integrals we have introduced \( \alpha \)-mass or mass function associated with a fractal subset \( F \) of the real line. This lead us to introduce the \( \gamma \)-dimension in section \[9\]. This dimension is finer than the box-dimension. Though it is not as fine as the Hausdorff dimension, it is specific to the development of calculus here and we expect it to be associated naturally with algorithms and numerical schemes based on the present calculus.

We have also discussed simple models based on \( F^\alpha \)-differential equations. The solutions of \( F^\alpha \)-differential equations naturally involve staircase-like functions. Staircase functions such as the Lebesgue Cantor staircase function are known to be bounded by sublinear power laws. Also, they “change” or “evolve” only on a fractal set. Thus, this framework may be useful in modelling many cases of sublinear behaviour, fractal time evolution, fields due to fractal charge distributions, etc. The \( F^\alpha \)-differentiability may be used to classify singular probability distribution functions.

Continuous-time dynamical systems are associated with ordinary differential equations, and discrete-time dynamical systems are associated with maps/
diffeomorphisms. But as realised in [16], the dynamical systems associated with \( F^\alpha \)-differential equations would be those evolving on fractal subsets of time-axis. It would also be of great interest to investigate correspondences between ordinary differential equations and \( F^\alpha \)-differential equations. These are explored in a companion paper [39].

There are many obvious directions in which considerations of this paper should be extended. Some of them are mentioned above. Other important directions would be extensions to multivariable case, development of differential equations and variational principles to mention a few. Work is in progress in these directions.

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Appendix A. \( C^\alpha \)-Integrating \( f(x) = x \chi_C(x) \)

As an example of \( F^\alpha \)-integration, here we calculate

\[
g(y) = \int_0^y x \chi_C(x) \, dx = \int_0^y x \, d_x^\alpha
\]

where \( C \) is the middle \( \frac{1}{3} \) Cantor set, and \( \alpha = \ln(2)/\ln(3) \) is its \( \gamma \)-dimension. The function \( f(x) = x \chi_C(x) \) is \( C \)-continuous on \([0,1]\), hence it is \( C^\alpha \)-integrable.

The set \( P_n = \{ x_i = i/n : 0 \leq i \leq n \} \) is a subdivision of \([0,1]\). For any component \([x_i, x_{i+1}]\) of \( P_n \), \( x_i \leq M[f,F,[x_i,x_{i+1}]] \) and \( x_{i+1} \geq M[f,F,[x_i,x_{i+1}]] \) if \( F \cap [x_i,x_{i+1}] \neq \emptyset \). Therefore,

\[
g(1) = \lim_{n \to \infty} \sum_{i=0}^{n} \left\{ \frac{i}{n} \left[ S_C^\alpha \left( \frac{i+1}{n} \right) - S_C^\alpha \left( \frac{i}{n} \right) \right] \right\} \leq L^\alpha[f,F,P_n]
\]

and

\[
\overline{g(1)} = \lim_{n \to \infty} \sum_{i=0}^{n} \left\{ \frac{i+1}{n} \left[ S_C^\alpha \left( \frac{i+1}{n} \right) - S_C^\alpha \left( \frac{i}{n} \right) \right] \right\} \leq U^\alpha[f,F,P_n].
\]

Further, it can be seen that

\[
\lim_{n \to \infty} (\overline{g(1)} - \underline{g(1)}) = 0.
\]

Thus, \( g(1) \) can be calculated using the limit

\[
g(1) = \lim_{n \to \infty} \sum_{i=0}^{n} \left\{ \frac{i}{n} \left[ S_C^\alpha \left( \frac{i+1}{n} \right) - S_C^\alpha \left( \frac{i}{n} \right) \right] \right\}.
\]

Similarly for integers \( m > 0 \),

\[
g \left( \frac{1}{3^m} \right) = \lim_{n \to \infty} \sum_{i=0}^{n} \left\{ \frac{i}{3^m n} \left[ S_C^\alpha \left( \frac{i+1}{3^m n} \right) - S_C^\alpha \left( \frac{i}{3^m n} \right) \right] \right\}.
\]

Using the self-similarity of \( C \) and scaling of \( S_C^\alpha \), we see from (A.2) and (A.3) that

\[
g \left( \frac{1}{3^m} \right) = \frac{1}{3^{m(1+\alpha)}} g(1) = \frac{1}{6^m} g(1).
\]
We make use of the ternary representation of numbers which simplifies many calculations involving the Cantor set. Any number \( y \in [0, 1] \) can be represented by the series

\[
y = \sum_{i=1}^{\infty} \frac{t_i(y)}{3^i}
\]

where \( t_i(y) = 0, 1 \) or 2 is the \( i \)th ternary digit of \( y \) after ternary point. The number \( y \) belongs to \( C \) if and only if \( y \) has a representation of the form (A.5) where \( t_i(y) = 0 \) or 2 for all \( i \).

An approximation of \( y \in [0, 1] \) by a finite number of digits is denoted by

\[
T_0(y) = 0 \quad \text{and} \quad T_n(y) = \sum_{i=1}^{n} \frac{t_i(y)}{3^i}.
\]

The sequence \( \{T_n(y)\}_{n=0}^{\infty} \) is a monotonically (but not strictly) increasing sequence whose limit is \( y \). Hence we can write

\[
g(y) = \sum_{i=1}^{\infty} \int_{T_{i-1}(y)}^{T_i(y)} x \chi(x) \, d\alpha^n x = \sum_{i=1}^{\infty} I_i(y)
\]

where

\[
I_i(y) = \int_{T_{i-1}(y)}^{T_i(y)} x \chi(x) \, d\alpha^n x.
\]

The quantities \( I_i(y) \) can be calculated using the self-similarity of \( C \), the scaling and translation properties of \( S^C \) (theorem \( \text{[13]} \), and \( \text{(A.4)} \)). Let \( y \in [0, 1] \) and let \( n \) be any integer such that \( i < n \Rightarrow t_i(y) = 0 \) or 2. Then \( i < n \Rightarrow T_i(y) \in C \). For calculating \( I_n(y) \), there are three cases corresponding to three possible values of \( t_n(y) \):

**Case** \( t_n(y) = 0 \): Here, \( T_{n-1}(y) = T_n(y) \) and \( I_n(y) = 0 \).

**Case** \( t_n(y) = 1 \): In this case,

\[
T_n(y) - T_{n-1}(y) = \frac{1}{3^n} = \sum_{i=n+1}^{\infty} \frac{2}{3^i}
\]

so that there is another sequence \( \{t_i(T_n(y))\} \) which does not contain the digit 1, hence \( T_n(y) \in C \). The set \( [T_{n-1}(y), T_n(y)] \cap C \) can be written as

\[
\{ z : i < n \Rightarrow t_i(z) = t_i(y); \ t_n(z) = 0; \ i > n \Rightarrow t_i(z) = 0 \text{ or } 2 \}
\]

Therefore it is a scaled down version of \( C \) by a factor \( 1/3^n \) and translated by \( T_{n-1}(y) \). Hence writing \( x = T_{n-1}(y) + (x - T_{n-1}(y)) \), we get

\[
I_n(y) = T_{n-1}(y) \int_{T_{n-1}(y)}^{T_n(y)} \chi_C(x) \, d\alpha^n x + \int_{T_{n-1}(y)}^{T_n(y)} (x - T_{n-1}(y)) \chi_C(x) \, d\alpha^n x
\]

\[
= T_{n-1}(y) \int_{0}^{1/3^n} \chi_C(x) \, d\alpha^n x + \int_{0}^{1/3^n} x \chi_C(x) \, d\alpha^n x
\]

\[
= T_{n-1}(y) \frac{1}{\Gamma(\alpha+1)} + \frac{1}{3^\alpha(1+\alpha)} g(1)
\]

\[
= \frac{T_{n-1}(y)}{\Gamma 2^n} + \frac{g(1)}{6^n}
\]

where \( \Gamma \) denotes \( \Gamma(a+1) \) for convenience. If \( y = T_n(y) \) then \( y \in C \). But if \( y > T_n(y) \), then as \( t_i(y) = 1 \) and \( t_i \neq 0 \) for some \( i > n \), therefore \( y \notin C \). Thus the half open interval \( (T_n(y), y) \) does not intersect \( C \) implying that \( I_k(y) = 0 \) for all \( k > n \).
Case $t_n(y) = 2$: Here, $T_n(y)$ clearly belongs to $C$. If $D$ is the set

$$D = \{ z : i < n \implies t_i(z) = t_i(y); \quad t_n(z) = 0; \quad i > n \implies t_i(z) = 0 \text{ or } 2 \}$$

then $D$ is a scaled down version of $C$ by a factor $1/3^n$, $D \subset [T_{n-1}(y), T_n(y)]$, and more specifically, $[T_{n-1}(y), T_n(y)] \cap C = D \cup \{ T_n(y) \}$. Therefore by arguments similar to the case $t_n(y) = 1$,

$$I_n(y) = \frac{T_{n-1}(y)}{1^{2n}} + \frac{g(1)}{6^n} \quad \text{(A.9)}$$

But unlike the case $t_n(y) = 1$, there is a possibility that $C \cap (T_n(y), y]$ is nonempty so that $I_k(y)$ need not be zero for all $k > n$.

Summarizing,

$$I_n(y) = \begin{cases} 
0 & \text{if } t_n(y) = 0 \text{ or } t_i(y) = 1 \text{ for some } i < n \\
\frac{T_{n-1}(y)}{1^{2n}} + \frac{g(1)}{6^n} & \text{otherwise.} 
\end{cases} \quad \text{(A.10)}$$

This description requires the value of $g(1)$. It can be found out by putting $y = 1$ in (A.10). If $y = 1$, then $t_i(y) = 2$ for all $i$. Also,

$$T_n(1) = \sum_{i=1}^{n} \frac{2}{3^i} = 1 - 3^{-n}.$$

Therefore,

$$I_n(1) = \frac{1 - 3^{-(n+1)}}{1^{2n}} + \frac{g(1)}{6^n}.$$

Substituting this in (A.10) and solving (A.7) for $g(1)$, we get

$$g(1) = \frac{1}{2\Gamma} \quad \text{(A.11)}$$

Thus,

$$g(y) = \int_0^y x\chi_C(x) d_F^\alpha x = \sum_{n=1}^{\infty} I_n(y) \quad \text{(A.12)}$$

where

$$I_n(y) = \begin{cases} 
0 & \text{if } t_n(y) = 0 \text{ or } t_i(y) = 1 \text{ for some } i < n \\
\frac{1}{\Gamma(\alpha + 1)} \left[ \frac{T_{n-1}(y)}{2^n} + \frac{1}{2 \cdot 6^n} \right] & \text{otherwise} 
\end{cases} \quad \text{(A.13)}$$

and $T_n(y)$ are given by equations (A.5) and (A.6).

Appendix B. Regarding repeated $F^\alpha$-integration and $F^\alpha$-derivative

$F^\alpha$-derivative

Many dynamical systems are modelled by differential equations involving second and higher order derivatives i.e. derivative operator applied repeatedly. The successive operation of the $D_F^\alpha$ operator is also possible and gives meaningful results. As an
example, let us $F^\alpha$-differentiate the function $g(x) = (S_\alpha^F(x))^2$ twice, where $F \subset \mathbb{R}$ is an $\alpha$-perfect set. By definition of the derivative,

$$x \notin F \implies D^\alpha_F g(x) = 0. \quad \text{(B.1)}$$

If $x \in F$, then

$$D^\alpha_F g(x) = F \lim_{y \to x} \frac{(S_\alpha^F(x))^2 - (S_\alpha^F(y))^2}{S_\alpha^F(y) - S_\alpha^F(x)}$$

$$= 2 S_\alpha^F(x). \quad \text{(B.2)}$$

Equations (B.1) and (B.2) can be combined to give

$$D^\alpha_F (S_\alpha^F(x))^2 = 2 S_\alpha^F(x) \chi_F(x) \quad \text{(B.3)}$$

where $\chi_F$ is the characteristic function of $F$. As a side remark, it is easy to generalize this to

$$D^\alpha_F ((S_\alpha^F(x))^n) = n(S_\alpha^F(x))^{n-1} \chi_F(x) \quad \text{(B.4)}$$

for any integer $n > 0$.

Now we take the second $F^\alpha$-derivative of $g$. As far as the operator $D^\alpha_F$ is concerned, the values of the function outside $F$ make no difference because of the $F$-limit in its definition. Thus,

$$(D^\alpha_F)^2(S_\alpha^F(x))^2 = D^\alpha_F(2 S_\alpha^F(x) \chi_F(x))$$

$$= 2 D^\alpha_F S_\alpha^F(x)$$

$$= 2 \chi_F(x) \quad \text{(B.5)}$$

Where the last step follows from lemma 49 and linearity (theorem 47).

Apart from the $\gamma$-dimension of $F$, the order $\alpha$ also has another significance. This will be clear from the following example. If $C$ is the Cantor set, then it is known [42] that $S_\alpha^C(x)$ is bounded by the power $\alpha$ of $x$ from below and above:

$$ax^\alpha \leq S_\alpha^C(x) \leq bx^\alpha \quad \text{(B.6)}$$

where $\alpha = \ln(2)/\ln(3)$ is the $\gamma$-dimension of $C$. Hence the function $g$ defined above is bounded by power $2\alpha$ of $x$:

$$ax^{2\alpha} \leq g(x) = (S_\alpha^C(x))^2 \leq bx^{2\alpha} \quad \text{(B.7)}$$

so that

$$x \in F \implies 2ax^\alpha \leq D^\alpha_C(g(x)) \leq 2bx^\alpha \quad \text{(B.8)}$$

and

$$x \in F \implies (D^\alpha_C)^2(g(x)) = 2. \quad \text{(B.9)}$$

This example demonstrates that $F^\alpha$-differentiation reduces the power of bounds by $\alpha$. 
### Appendix C. A few analogies between $F^\alpha$-calculus and ordinary calculus

The $F^\alpha$-calculus can be thought of as a generalization of ordinary calculus with Riemann approach. Table C1 shows a few analogies between various quantities.

#### References

1. Mandelbrot B B 1977 *The fractal geometry of nature* (Freeman and company)
2. Bunde A and Havlin S (Eds) 1995 *Fractals in Science* (Springer)
3. Falconer K 1985 *The geometry of fractal sets* (Cambridge university press)
4. Falconer K 1990 *Fractal geometry: Mathematical foundations and applications* (John Wiley and Sons)
5. Falconer K 1997 *Techniques in fractal geometry* (John Wiley and Sons)
6. Edgar G A 1998 *Integral, probability and fractal measures* (Springer-Verlag, New York)
7. Metzler R, Glöckle W G and Nonnenmacher T F 1994 Fractional model equation for anomalous diffusion *Physica A* 211 13–24
8. Metzler R, Barkai E and Klafter J 1999 Anomalous diffusion and relaxation close to thermal equilibrium: A fractional Fokker-Planck equation approach *Phys. Rev. Lett.* 82 3563
9. Hilfer R and Anton L 1995 Fractional master equations and fractal time random walks *Phys. Rev. E* 51 R848
Calculus on fractal subsets of real line – I: formulation

[10] Compte A 1996 Stochastic foundations of fractional dynamics Phys. Rev. E 53(4) 4191
[11] Zaslavsky G M 1994 Fractional kinetic equation for Hamiltonian chaos Physica D 76 110–122
[12] Metzler R, Barkai E and Klafter J 1999 Anomalous transport in disordered systems under the influence of external fields Physica A 266 343–350
[13] Hilfer R 2000 Fractional diffusion based on Riemann-Liouville fractional derivatives Jnl. Phys. Chem. B 104 3914
[14] Kolwankar K M and Gangal A D 1996 Fractional differentiability of nowhere differentiable functions and dimensions Chaos 6, 505
[15] Kolwankar K M and Gangal A D 1997 Holder exponents of irregular signals and local fractional derivatives Pramana 48, 49–68
[16] Kolwankar K M and Gangal A D 1998 Local fractional Fokker-Planck equation Phys. Rev. Lett. 80 214
[17] Kolwankar K M and Gangal A D 1999 Local Fractional Calculus: A Calculus for Fractal Space-Time Fractals: Theory and Applications in Engineering ed Dekking M, Levy Vehel J et al (Springer)
[18] Adda F B and Cresson J 2001 About Non-differentiable Functions J. Math. Anal. Appl. 263 721–737
[19] Babakhani A and Daftardar-Gejji V 2002 On calculus of local fractional derivatives J. Math. Anal. Appl. 270 66–79
[20] Barlow M T 1998 Diffusion on fractals, Lecture notes (Math. Vol. 1690, Springer)
[21] Kigami J 2000 Analysis on Fractals (Cambridge Univ. Press)
[22] Strichartz R S and Vinson J P 1999 Fractal differential equations on the Sierpinski gasket Jnl. Fourier Anal. and Appl. 5(2) 205–287
[23] Strichartz R S 2000 Taylor approximations on Sierpinski type fractals J. Funct. Anal. 174(1) 76–127
[24] Freiberg U and Zähle M 2002 Harmonic calculus on fractals—A measure geometric approach I Potential Anal. 16 265–277
[25] Freiberg U and Zähle M 2000 Harmonic calculus on fractals—A measure geometric approach II Preprint
[26] Widder D V 1974 Advanced Calculus, Second Ed. (Prentice-Hall India Pvt Ltd)
[27] Shilov G E and Gurevich B L 1966 Integral, measure and derivative: a unified approach (Prentice Hall, Inc.)
[28] Goldberg R R 1970 Methods of real analysis (Oxford and IBH Publishing Co. Pvt. Ltd.)
[29] Samko S G, Kilbas A A and Marichev O I 1993 Fractional Integrals and Derivatives—Theory and Applications (Gordon and Breach Science Publishers)
[30] Hilfer R 2000 Applications of Fractional Calculus in Physics (World Scientific Publ. Co., Singapore)
[31] Miller K S and Ross B 1993 An introduction to the fractional calculus and fractional differential equations (John Wiley, New York)
[32] Oldham K B and Spanier J 1974 The fractional calculus (Academic Press, New York)
[33] Bedford T and Fisher A M 1992 Analogues of the Lebesgue density theorem for fractal sets of reals and integers Proc. London Math. Soc. 64(3) 95–124
[34] Patzschke N and Zähle M 1992 Fractional Differentiation in the Self-Affine Case I - Random Functions Stochastic Proc. Appl. 43 165–175
[35] Patzschke N and Zähle M 1993 Fractional Differentiation in the Self-Affine Case II - Extremal Processes Stochastic Proc. Appl. 45 61–72
[36] Patzschke N and Zähle M 1993 Fractional Differentiation in the Self-Affine Case III - The Density of the Cantor Set Proc. Amer. Math. Soc. 117(1) 137–144
[37] Zähle M 1997 Fractional Differentiation in the Self-Affine Case V - The Local Degree of Differentiability Math. Nachr. 185 279–306
[38] Beckenbach E F and Bellman R 1961 Inequalities (Springer-Verlag) page 18
[39] Parvate A and Gangal A D In progress
[40] Gordon R A 1994 The integrals of Lebesgue, Denjoy, Perron and Henstock (American Mathematical Society)
[41] Bartle R G 2001 A modern theory of Integration (American Mathematical Society)
[42] Hille E and Tamarkin J D 1929 Remarks on a known example of a monotone continuous function American Mathematics Monthly 36 255–264