Induced vacuum bosonic current in a compactified cosmic string spacetime

E. A. F. Bragança¹∗, H. F. Santana Mota²† and E. R. Bezerra de Mello³‡

¹,³Departamento de Física, Universidade Federal da Paraíba
58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil

¹INFN, Laboratori Nazionali di Frascati,
Via Enrico Fermi 40, 00044 Frascati, Italy

²Department of Physics and Astronomy, University of Sussex
Falmer, Brighton BN1 9QH, U.K.

September 29, 2015

Abstract

We analyze the bosonic current densities induced by a magnetic flux running along an idealized cosmic string considering that the coordinate along its axis is compactified. We also consider the presence of a magnetic flux enclosed by the compactified axis. To develop this analysis, we calculate the complete set of normalized bosonic wave functions obeying a quasiperiodicity condition along the compactified dimension. We show that in this context only the azimuthal and axial currents take place.

PACS numbers: 98.80.Cq, 11.10.Gh, 11.27.+d

1 Introduction

Cosmic strings are linear topological defects which may have been created in the early universe as a consequence of phase transitions and are predicted in the context of the standard gauge field theory of elementary particle physics [1, 2, 3]. The formation of cosmic string can have astrophysical and cosmological consequences. For instance, emission of gravitational waves and high energy cosmic rays by strings such as neutrinos and gamma-rays, along observational data, can help to constraint the product of the Newton’s constant, \( G \), and the linear mass density of the string, \( \mu_0 \) [2].

The geometry associated with an infinity and straight cosmic string is locally flat but topologically conical, having a planar angle deficit given by \( \Delta \phi = 8\pi G \mu_0 \) on the two-surface orthogonal to the string. This conical structure alters the vacuum fluctuations associated with quantum fields and changes the vacuum expectation values (VEVs) of physical observables like

∗E-mail: deduardo@lnf.infn.it
†E-mail: hm288@sussex.ac.uk
‡E-mail: emello@fisica.ufpb.br
the energy-momentum tensor, $\langle T_{\mu\nu} \rangle$. Moreover, the presence of a magnetic flux running along the string’s axis provides additional contributions to these VEVs associated with charged fields [18, 13, 14, 15, 16, 17, 18, 19, 20]. Specifically an important quantity induced in this context is the current density, $\langle j^\mu \rangle$.

The presence of compact dimensions also induces topological quantum effects on matter field. An interesting application of field theoretical models that present compact dimensions can be found in nanophysics. The long-wavelength description of the electronic states in graphene can be formulated in terms of the Dirac-like theory in three-dimensional spacetime, with the Fermi velocity playing the role of the speed of light [21].

2 Wightman function

Here we consider a $(3+1)$-dimensional cosmic string spacetime. By using cylindrical coordinates $(x^1, x^2, x^3) = (r, \phi, z)$ with the string on the 1-dimensional hypersurface $r = 0$, the corresponding geometry is described by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dr^2 - r^2 d\phi^2 - dz^2 .$$

(2.1)

The coordinates take values in the following intervals: $r \geq 0, 0 \leq \phi \leq 2\pi/q$ and $-\infty < t < +\infty$. The parameter $q \geq 1$ codifies the presence of the cosmic string. Moreover, we assume that the direction along the $z$-axis is compactified to a circle with length $L$, so $0 \leq z \leq L$.

We are interested in determining the induced vacuum current density, $\langle j^\mu \rangle$, associated with a charged scalar quantum field, in the presence of a magnetic fluxes running along the core of the string and enclosed by it. To develop this analyze, we shall calculate the corresponding complete set of normalized bosonic wave function.

The equation that governs the quantum dynamics of this system is

$$\left[ \frac{1}{\sqrt{|g|}} D_\mu \left( \sqrt{|g|} g^{\mu\nu} D_\nu \right) + m^2 + \xi R \right] \varphi(x) = 0 ,$$

(2.2)

where $D_\mu = \partial_\mu + i e A_\mu$ and $g = \det(g_{\mu\nu})$. We considered the presence of an non-minimal coupling, $\xi$, between the field and the geometry represented by the Ricci scalar, $R$. However, for a thin and infinitely straight cosmic string, $R = 0$ for $r \neq 0$. To develop our analysis, will be assumed the $z$-axis is compactified to a circle with length $L$, $0 \leq z \leq L$. The compactification is achieved by imposing the quasiperidicity condition on the matter field

$$\varphi(t, r, \phi, z + L, x^4) = e^{2\pi i \beta} \varphi(t, r, \phi, z, x^4) ,$$

(2.3)

with a constant phase $\beta$, $0 \leq \beta \leq 1$. The special cases $\beta = 0$ and $\beta = 1/2$ correspond to the untwisted and twisted fields, respectively, along the $z$-direction. In addition, we shall consider the presence of the following constant vector potential

$$A_\mu = (0, 0, A_\phi, A_z) ,$$

(2.4)

with $A_\phi = -q \Phi_\phi/(2\pi)$ and $A_z = -\Phi_z/L$, being $\Phi_\phi$ and $\Phi_z$ the corresponding magnetic fluxes. In quantum field theory, the condition (2.3) alters the spectrum of the vacuum fluctuations compared with the case of uncompactified dimension and, as a consequence, the induced vacuum current density changes.

\textsuperscript{1} The calculation of the VEV of physical observables associated with scalar and fermionic fields in the cosmic string spacetime has been developed in Refs. [11, 5, 6, 7, 8, 9, 10, 11, 12].

\textsuperscript{2} In the standard cosmic string spacetime we have $q^{-1} = 1 - 4\mu_0$, being $\mu_0$ the linear mass density of the string.
Using Eqs. (2.1) and (2.2), and considering the general expression \( \varphi(x) = CR(x)e^{-i\omega t + iqn\phi + ikz} \), the normalized expression takes the form

\[
\varphi_\sigma(x) = \left[ \frac{q\lambda}{4\pi\omega qL} \right]^{\frac{1}{2}} J_{q|n+\alpha|}(\lambda r)e^{-i\omega t + iqn\phi + ikz},
\]

being specified by the set quantum numbers, \( \sigma = (\lambda, k_z, n) \). Also, we have

\[
\lambda = \sqrt{\omega^2 - \tilde{k}_z^2 - m^2}, \quad \alpha = \frac{eA_\phi}{q} = \frac{\Phi_\phi}{\Phi_0}, \quad \tilde{k}_z = k_z + eA_z,
\]

with \( \Phi_0 = 2\pi/e \) being the quantum flux. As a consequence of the condition (2.3), the quantum number \( k_z \) is discretized as follows:

\[
k_z = k_l = \frac{2\pi}{L}(l + \beta) \quad \text{with} \quad l = 0, \pm 1, \pm 2, \ldots.
\]

And under this circumstance, the energy takes the form

\[
\omega = \omega_l = \sqrt{m^2 + \lambda^2 + \tilde{k}_l^2},
\]

where

\[
\tilde{k}_z = \tilde{k}_l = \frac{2\pi}{L}(l + \tilde{\beta}), \quad \tilde{\beta} = \beta + \frac{eA_zL}{2\pi} = \beta - \frac{\Phi_z}{\Phi_0}.\]

The properties of the vacuum state are described by the corresponding positive frequency Wightman function, \( W(x, x') = \langle 0|\varphi(x)\varphi^*(x')|0 \rangle \), where \( |0 \rangle \) stands for the vacuum state. Having the complete set of normalized mode functions, \( \{ \varphi_\sigma(x), \varphi^*_\sigma(x') \} \), satisfying the periodicity condition (2.3), we can evaluate the corresponding Wightman function as:

\[
W(x, x') = \sum_\sigma \varphi_\sigma(x)\varphi^*_\sigma(x') \quad \text{with} \quad \sum_\sigma = \sum_{n=-\infty}^{+\infty} \int_0^\infty d\lambda \sum_{l=-\infty}^{+\infty} \ (2.10)
\]

The mode functions in Eq. (2.5) are specified by the set of quantum numbers \( \sigma = (n, \lambda, k_l) \), with the values in the ranges \( n = 0, \pm 1, \pm 2, \ldots, 0 < \lambda < \infty \) and \( k_l = 2\pi(l + \beta)/L \) with \( l = 0, \pm 1, \pm 2, \ldots \).

Substituting (2.5) into the sum (2.10) we obtain

\[
W(x, x') = \frac{q}{4\pi L} \sum_\sigma e^{iqn\Delta\phi} \lambda J_{q|n+\alpha|}(\lambda r)J_{q|n+\alpha|}(\lambda r') \frac{e^{-i\omega_l \Delta t + ikz\Delta z}}{\omega_l},
\]

where \( \Delta\phi = \phi - \phi', \Delta t = t - t' \) and \( \Delta z = z - z' \).

Now, we are in position to calculate the induced vacuum bosonic current density, \( \langle j_{\mu} \rangle \). This calculation will be developed in the next section.

### 3 Bosonic current

The bosonic current density operator is given by,

\[
\begin{align*}
    j_{\mu}(x) &= ie \left[ \varphi^*(x)D_{\mu}\varphi(x) - (D_{\mu}\varphi)^*\varphi(x) \right] \\
    &= ie \left[ \varphi^*(x)\partial_{\mu}\varphi(x) - \varphi(x)(\partial_{\mu}\varphi(x))^* - 2e^2A_{\mu}(x)|\varphi(x)|^2. \quad (3.1)
\end{align*}
\]
Its vacuum expectation value (VEV) can be evaluated in terms of the positive frequency Wightman function as shown below:

\[
\langle j_\mu(x) \rangle = ie \lim_{x' \to x} \{ (\partial_\mu - \partial'_\mu) W(x, x') + 2ieA_\mu W(x, x') \}. \tag{3.2}
\]

Writing the parameter \( \alpha \) in Eq. (2.6) in the following form

\[
\alpha = n_0 + \alpha_0 \quad \text{with} \quad |\alpha_0| < \frac{1}{2}, \tag{3.3}
\]

where \( n_0 \) is an integer number, we can see that the above VEV is a periodic function of the magnetic fluxes \( \Phi_\phi \) and \( \Phi_z \) with period equal to the quantum flux. In fact, as we will see, the VEV of the current density depends on \( \alpha_0 \) only.

In the next section we will calculate the current densities. It has been shown in Ref. [20], that the charge density and the radial current density vanish for the system in consideration. So, here we will focus only in the calculations of the azimuthal and axial current densities.

### 3.1 Azimuthal current

The VEV of the azimuthal current density is given by:

\[
\langle j_\phi(x) \rangle = ie \lim_{\phi' \to \phi} \{ (\partial_\phi - \partial_\phi') W(x, x') + 2ieA_\phi W(x, x') \}. \tag{3.4}
\]

Substituting (2.11) into the above equation, we get the form al expression for the azimuthal bosonic current density below,

\[
\langle j_\phi(x) \rangle = \frac{-eq}{4\pi L} \sum_{n=-\infty}^{\infty} q(n + \alpha) \int_0^{\infty} d\lambda \lambda J^2_{q|n+\alpha|}(\lambda r) \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{m^2 + \lambda^2 + k^2}}. \tag{3.5}
\]

To develop the summation over the quantum number \( l \) we shall apply the Abel-Plana summation formula in the form used in Ref. [22], which is given by

\[
\sum_{l=-\infty}^{\infty} g(l + \tilde{\beta}) f(|l + \tilde{\beta}|) = \int_0^{\infty} du \ [g(u) + g(-u)] f(u) + i \int_0^{\infty} du \ [f(iu) - f(-iu)] \sum_{\lambda=\pm 1} \frac{g(i\lambda u)}{e^{2\pi(u+i\lambda\tilde{\beta})} - 1}. \tag{3.6}
\]

Taking \( g(u) = 1 \) and

\[
f(u) = \frac{1}{\sqrt{(2\pi u/L)^2 + \lambda^2 + m^2}}, \tag{3.7}
\]

it is possible to decompose the expression to \( \langle j_\phi \rangle \) as the sum of the two contributions as shown below:

\[
\langle j_\phi \rangle = \langle j_\phi \rangle_{cs} + \langle j_\phi \rangle_c. \tag{3.8}
\]

The first contribution, \( \langle j_\phi \rangle_{cs} \) corresponds to the azimuthal current density in the geometry of the 3-dimensional cosmic string spacetime without compactification. It is given by the first integral of (3.6). The second term, \( \langle j_\phi \rangle_c \), is induced by the compactification of the string along its axis and is provided by the second integral.

For the first contribution we have:

\[
\langle j_\phi(x) \rangle_{cs} = -\frac{eq}{2\pi L} \sum_{n=-\infty}^{\infty} q(n + \alpha) \int_0^{\infty} d\lambda \lambda J^2_{q|n+\alpha|}(\lambda r) \int_0^{\infty} \frac{dy}{\sqrt{y^2 + \lambda^2 + m^2}}, \tag{3.9}
\]
where we have introduced a new variable $y = \frac{2\pi u}{L}$. Using the identity
\[
\frac{1}{\sqrt{m^2 + \lambda^2 + k^2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty ds \ e^{-(m^2+\lambda^2+k^2)s^2},
\] (3.10)
the integral over $\lambda$ can be evaluated by using Ref. [23]. Finally, writing $\alpha$ in the form (3.3) we obtain
\[
\langle j_\phi(x) \rangle_{\text{cs}} = -\frac{em^4r^2}{\pi^2} \left[ \sum_{k=1}^{[q/2]} \sin(2\pi/q) \sin(2\pi \alpha_0) f_2(2mr \sin(k\pi/q)) \right]
\]
\[
+ \frac{q}{\pi} \int_0^\infty dy \ g(q, \alpha_0, 2y) \frac{\sinh(2y)}{\cosh(qy) - \cos(q\pi)} f_2(2mr \cosh(y)),
\] (3.12)
where we have defined $w = \frac{r^2}{2s^2}$. Using the result for the summation over $n$ found in Ref. [20], we obtain
\[
\langle j_\phi(x) \rangle_{\text{cs}} = -\frac{em^4r^2}{\pi^2} \left[ \sum_{k=1}^{[q/2]} \sin(2\pi/q) \sin(2\pi \alpha_0) f_2(2mr \sin(k\pi/q)) \right]
\]
\[
+ \frac{q}{\pi} \int_0^\infty dy \ g(q, \alpha_0, 2y) \frac{\sinh(2y)}{\cosh(qy) - \cos(q\pi)} f_2(2mr \cosh(y)),
\] (3.12)
where we use the notation
\[
f_\nu(x) = \frac{K_\nu(x)}{x^\nu}.
\] (3.13)
In the above equation
\[
g(q, \alpha_0, 2y) = \sin(q\pi \alpha_0) \sinh[(1 - |\alpha_0|) 2qy] - \sinh(2qy \alpha_0) \sin[(1 - |\alpha_0|) \pi q].
\] (3.14)

We can see that where $\alpha_0 = 0$, $\langle j_\phi(x) \rangle_{\text{cs}}$ vanishes. From Eq. (3.12), we can see that $\langle j_\phi(x) \rangle_{\text{cs}}$ is an odd function of $\alpha_0$, with period equal to the quantum flux $\Phi_0$. We plot in Fig. 1 the behavior of the azimuthal current density as function of $\alpha_0$ for specific values of the parameter $q$, considering $mr = 0.5$. From the figure is possible to conclude that the main influence of this parameter is amplify the oscillatory nature of the azimuthal current with respect to $\alpha_0$.

![Figure 1: The azimuthal current density without compactification is plotted, in units of “$m^4e$”, in terms of $\alpha_0$ for $mr = 0.5$ and $q = 1.5, 2.5$ and 3.5.](image)

At large distances from the string, $mr \gg 1$, and considering $q > 2$, the azimuthal current due the cosmic string is dominated by the first term of (3.12), with $k = 1$ and is given by
\[
\langle j_\phi(x) \rangle_{\text{cs}} \approx -\frac{em^4r^2}{(2\pi)^2} \left( \frac{m}{\pi r \sin(\pi/q)} \right)^{1/2} \sin(2\pi \alpha_0) \cot(\pi/q) e^{-2mr \sin(\pi/q)},
\] (3.15)
where we see an exponential decay.

To obtain the contribution for the azimuthal current induced by the compactification we substitute the second term of (3.6) into Eq. (3.5). Also using the series expansion $(e^u - 1)^{-1} = \sum_{l=1}^{\infty} e^{-lu}$ and the following representation for the Macdonald function [23]

\[
K_\nu(x) = \frac{1}{2} \left( \frac{x}{\nu} \right) \int_0^\infty dt \frac{e^{-t - \frac{x^2}{4}}}{t^{\nu+1}},
\]

we obtain

\[
\langle j_\phi(x) \rangle_c = -\frac{e^{q^2}}{\pi r^2} \sum_{l=1}^{\infty} \cos(2\pi l\tilde{\beta}) \int_0^\infty dw \, e^{-w[1+\frac{l^2}{2r^2}]} \sum_{n=-\infty}^{\infty} \left( n + \alpha_0 \right) I_{q|n+\alpha_0|}(w),
\]

where we have used $\alpha$ in the form (3.3) and defined $w = \frac{2r^2}{L^2}$. Using the result for the summation on $n$ found in Ref. [20] in the above equation the final expression for the azimuthal current induced by the compactification is

\[
\langle j_\phi(x) \rangle_c = -\frac{2emr^2}{\pi^2} \sum_{l=1}^{\infty} \cos(2\pi l\tilde{\beta}) \left\{ \frac{q/2}{\sum_{k=1}^{q/2} \sin(2k\pi/q) \sin(2k\pi\alpha_0)}f_2\left[mL\sqrt{l^2 + \rho_k^2}\right] \right. \\
+ \frac{q}{\pi} \int_0^\infty dy \frac{g(q, \alpha_0, 2y) \sinh(2y)}{\cosh(2qy) - \cos(q\pi)}f_2\left[mL\sqrt{l^2 + \eta^2(y)}\right] \right\},
\]

where we have defined

\[
\rho_k = \frac{2r \sin(k\pi/q)}{L}, \quad \eta(y) = \frac{2r \cosh(y)}{L}.
\]

From Eq. (3.18) it is possible to see that the contribution induced by the compactification for the azimuthal current is an even function of the parameter $\tilde{\beta}$ and is an odd function of the magnetic flux along the core of the string, with period equal to the quantum flux. This contribution vanishes for the case where $\alpha_0 = 0$. In Fig. 2 we plot the behavior of the compactified azimuthal current density as a function of $\alpha_0$ for different values of the parameter $q$ and considering $mr = 0.5$ and $mL=1$. Is possible to see that effect of the parameter $q$ is increase the oscillatory nature of the azimuthal current while the parameter $\tilde{\beta}$ changes the direction of oscillation.

For large values of the length of the compact dimension $mL \gg 1$, assuming that $mr$ is fixed, the main contribution comes from the $l = 1$ term. It is given by:

\[
\langle j_\phi(x) \rangle_c \approx -\frac{\sqrt{2}emr^2}{\pi^2 L^2} \cos(2\pi\tilde{\beta})e^{-mL} \left[ \frac{q/2}{\sum_{k=1}^{q/2} \sin(2k\pi/q) \sin(2k\pi\alpha_0)} \right. \\
+ \frac{q}{\pi} \int_0^\infty dy \frac{g(q, \alpha_0, 2y) \sinh(2y)}{\cosh(2qy) - \cos(q\pi)} \right],
\]

where we can see an exponential decay. This means that in this limit, the contribution for the total azimuthal current density is dominated by $\langle j_\phi(x) \rangle_{cs}$.

### 3.2 Axial current

The VEV of the axial current is given by

\[
\langle j_z(x) \rangle = ie \lim_{z' \rightarrow z} \left\{ (\partial_z - \partial_{z'})W(x, x') + 2ieA_zW(x, x') \right\}.
\]
Using the fact that $A_z = -\Phi_z / L$, a formal expression for this current can be provided. It reads,

$$
\langle j_z(x) \rangle = -\frac{eq}{2\pi L} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \lambda J_{q|n+\alpha|}^2(\lambda r) \sum_{l=-\infty}^{\infty} \frac{\tilde{k}_l}{\sqrt{m^2 + \lambda^2 + \tilde{k}_l^2}},
$$

where $\tilde{k}_l$ is given by (2.6). To evaluate the summation over the quantum number $l$ we use again Eq. (3.6). For this case we have $g(u) = 2\pi u / L$ and $f(u)$ is given by Eq. (3.7). Due the fact that $g(u)$ is an odd function, the first term on the right-hand side of (3.6) vanishes. Thus, the only contribution for the axial current is due to the second term of (3.6), that means that the axial current is due only to the compactification. Adopting similar steps that we made to derive $\langle j_z(x) \rangle_{cs}$, for this case we obtain:

$$
\langle j_z(x) \rangle_{c} = \frac{geL}{2\pi^2} \sum_{l=1}^{\infty} l \sin(2\pi l \tilde{\beta}) \int_{0}^{\infty} dw \, we^{-w^2 + \frac{m^2 l^2}{2w^2}} \sum_{n=-\infty}^{\infty} I_{q|n+\alpha|}(w),
$$

where we have defined the variable $w$ as the same as in (3.17). The summation on $n$ in the above equation also was developed in Ref. [20]. Using that result, is possible to write the axial current as follows

$$
\langle j_z(x) \rangle_{c} = \frac{em^2}{\pi^2 L} \sum_{l=1}^{\infty} \frac{\sin(2\pi l \tilde{\beta})}{l} K_{2}(lmL) + \langle j_z(x) \rangle_{c}^{(q,0)}.
$$

The first contribution in the right-hand side of the above equation is independent of the $\alpha_0$ and the parameter $q$. It is a pure topological term, induced by the compactification only. The second contribution comes from the magnetic flux and planar angle deficit. It is:

$$
\langle j_z(x) \rangle_{c}^{(q,0)} = \frac{2em^4 L}{\pi^2} \sum_{l=1}^{\infty} \frac{l \sin(2\pi l \tilde{\beta})}{l} \left\{ \sum_{k=1}^{[q/2]} \cos(2k\pi\alpha_0) f_2 \left[ mL\sqrt{l^2 + \rho_k^2} \right] 
- \frac{q}{\pi} \int_{0}^{\infty} dy \frac{f(q, \alpha_0, 2y)}{\cosh(2qy) - \cos(q\pi)} f_2 \left[ mL\sqrt{l^2 + \eta^2(y)} \right] \right\}
$$

where $\rho_k$ and $\eta(y)$ are given by (3.19) and

$$
f(q, \alpha_0, 2y) = \sin((1 - |\alpha_0|)\pi q) \cosh(|\alpha_0|2qy) + \sin(|\alpha_0|\pi q) \cosh((1 - |\alpha_0|)2qy).
$$
The Eq. (3.25) is an odd function of the parameter $\tilde{\beta}$ and is an even function of $\alpha_0$, with period equal to the quantum flux. It is possible to see that this contribution vanishes for the case where $q = 1$ and $\alpha_0 = 0$. In Fig. 3, we plot the behavior of this expression as function of $\tilde{\beta}$ taking $mr = 0.4$, $mL = 1$ and $q = 1.5$, $2.5$, $3.5$. By these plots we can see that the amplitude of the current increases with $q$ and the effect of $\alpha_0$ is to change the orientation of the current.

Figure 3: $\langle j_z(x) \rangle_c^{(q, \alpha_0)}$ is plotted, in units of “$m^3 e$”, in terms of $\tilde{\beta}$ for the values $mr = 0.4$, $mL = 1$ and $q = 1.5$, $2.5$, $3.5$. The left plot is for $\alpha_0 = 0$ while the right plot is for $\alpha_0 = 0.25$.

Considering $mL \gg 1$ and $mr$ fixed, the main contribution to (3.25) comes from the $l = 1$ term. So, we have

$$
\langle j_z(x) \rangle_c^{(q, \alpha_0)} \approx \frac{8em^3}{(2\pi)^3} \sin(2\pi \tilde{\beta}) e^{-mL} \left\{ \sum_{k=1}^{[q/2]} \cos(2k\pi \alpha_0) - \frac{q}{\pi} \int_0^\infty dy f(q, \alpha_0, 2y) \frac{\cosh(2qy)}{\cos(q\pi)} \right\},
$$

(3.27)

4 Conclusion

Here, we have investigated the induced bosonic current density in a compactified cosmic string spacetime, considering the presence of the magnetic fluxes on the azimuthal and axial directions. In order to do that, we imposed a quasiperiodicity condition, with arbitrary phase $\tilde{\beta}$, on the solution of the Klein-Gordon equation. We constructed the positive frequency Wightman function (2.11), that is necessary to calculate the induced bosonic current. We have seen that the compactification induces the azimuthal current density to decompose in two parts. The first one corresponds to the expression in the geometry of a cosmic string without compactification, while the second one is due the compactification. The first contribution is an odd function on the parameter $\alpha_0$, with period the quantum flux, and the second contribution in an even function on the parameter $\tilde{\beta}$. Both contributions vanish for the case where the parameter $\alpha_0$ is equal to zero. In the limit of large values of the parameter of the compactification, the main contribution come from the first contribution.

We have also shown that the VEV of the axial current density has a purely topogical origin. This VEV can be expressed as the sum of two terms. One of them is given by the Eq. (3.22) and is independent of the radial distance $r$, the cosmic string parameter $q$ and the $\alpha_0$. The other contribution is given by the Eq. (3.25) and is due to the magnetic fluxes and the planar angle deficit. This contribution is an odd function of the parameter $\tilde{\beta}$ and is an even function of the parameter $\alpha_0$.
Acknowledgments

The author E. A. F. B thanks the Brazilian agency CAPES and the INFN for the financial support. H. F. S. M thanks the Brazilian agency CAPES for the financial support. E. R. B. M thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for the partial financial support.

References

[1] A. Vilenkin and E. P. S. Shellard, Cosmic strings and other topological defects. Cambridge monographs on mathematical physics. Cambridge Univ. Press, Cambridge, 1994.

[2] M. Hindmarsh and T. Kibble, Cosmic strings, Rept.Prog.Phys. 58 (1995) 477–562, [hep-ph/9411342].

[3] J. M. Hyde, A. J. Long, and T. Vachaspati, Dark Strings and their Couplings to the Standard Model, Phys.Rev. D89 (2014) 065031, [arXiv:1312.4573].

[4] B. Linet, Quantum Field Theory in the Space-time of a Cosmic String, Phys.Rev. D35 (1987) 536–539.

[5] B. Allen and E. P. S. Shellard, On the evolution of cosmic strings, in The formation and evolution of cosmic strings : proceedings of a workshop supported by the SERC and held in Cambridge, 3-7 July, 1989 (G. W. Gibbons, S. W. Hawking, and T. Vachaspati, eds.), (Cambridge), pp. 421–448, Cambridge University Press, 1990.

[6] M. Guimaraes and B. Linet, Selfinteraction and quantum effects near a point mass in three-dimensional gravitation, Class.Quant.Grav. 10 (1993) 1665–1680.

[7] P. Davies and V. Sahni, Quantum gravitational effects near cosmic strings, Class.Quant.Grav. 5 (1988) 1.

[8] T. Souradeep and V. Sahni, Quantum effects near a point mass in (2+1)-Dimensional gravity, Phys.Rev. D46 (1992) 1616–1633, [hep-ph/9208219].

[9] V. P. Frolov and E. Serebryanyi, Vacuum Polarization in the Gravitational Field of a Cosmic String, Phys.Rev. D35 (1987) 3779–3782.

[10] B. Linet, Euclidean spinor Green’s functions in the space-time of a straight cosmic string, J.Math.Phys. 36 (1995) 3694–3703, [gr-qc/9412050].

[11] J. Moreira, E.S., Massive quantum fields in a conical background, Nucl.Phys. B451 (1995) 365–378, [hep-th/9502016].

[12] V. B. Bezerra and N. R. Khusnutdinov, Vacuum expectation value of the spinor massive field in the cosmic string space-time, Class.Quant.Grav. 23 (2006) 3449–3462, [hep-th/0602048].

[13] J. Dowker, Vacuum Averages for Arbitrary Spin Around a Cosmic String, Phys.Rev. D36 (1987) 3742.

[14] M. Guimaraes and B. Linet, Scalar Green’s functions in an Euclidean space with a conical-type line singularity, Commun.Math.Phys. 165 (1994) 297–310.
[15] J. Spinelly and E. Bezerra de Mello, *Vacuum polarization of a charged massless scalar field on cosmic string space-time in the presence of a magnetic field*, *Class.Quant.Grav.* **20** (2003) 873–888, [hep-th/0301169].

[16] J. Spinelly and E. Bezerra de Mello, *Vacuum polarization by a magnetic field in the cosmic string space-time*, *Int.J.Mod.Phys.* **A17** (2002) 4375–4384.

[17] J. Spinelly and E. Bezerra de Mello, *Vacuum polarization of a charged massless fermionic field by a magnetic flux in the cosmic string space-time*, *Int.J.Mod.Phys.* **D13** (2004) 607–624, [hep-th/0306103].

[18] J. Spinelly and E. Bezerra de Mello, *Vacuum polarization by a magnetic flux in a cosmic string background*, *Nucl.Phys.Proc.Suppl.* **127** (2004) 77–83, [hep-th/0305142].

[19] J. Spinelly and E. Bezerra de Mello, *Spinor Green function in higher-dimensional cosmic string space-time in the presence of magnetic flux*, *JHEP* **0809** (2008) 005, [arXiv:0802.4401].

[20] E. A. F. Bragança, H. F. Santana Mota, and E. R. Bezerra de Mello, *Induced vacuum bosonic current by magnetic flux in a higher dimensional compactified cosmic string spacetime*, *Int. J. Mod. Phys.* **D24** (2015), 1550055, [arXiv:1410.1511].

[21] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, *The electronic properties of graphene*, *Reviews of Modern Physics* **81** (2009) 109–162, [arXiv:0709.1163].

[22] S. Bellucci, A. Saharian, and V. Bardeghyan, *Induced fermionic current in toroidally compactified spacetimes with applications to cylindrical and toroidal nanotubes*, *Phys.Rev. D82* (2010) 065011, [arXiv:1002.1391].

[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Academic Press, 1980.