Note on Tests of the Factorization Hypothesis and the Determination of Meson Decay Constants

Volker Rieckert

Institut für Theoretische Physik der Universität Heidelberg
Philosophenweg 16, D-6900 Heidelberg, Germany
e-mail: K46 @ DHDURZ1

We discuss various tests of the factorization hypothesis making use of the close relationship between semi-leptonic and factorized nonleptonic decay amplitudes. It is pointed out that factorization leads to truly model-independent predictions for the ratio of nonleptonic to semi-leptonic decay rates, if in the nonleptonic decay a spin one meson of arbitrary mass or a pion take the place of the lepton pair. Where the decay constants of those mesons are known, these predictions represent ideal tests of the factorization hypothesis. In other cases they may be used to extract the decay constants. Currently available data on the decays $\bar{B}^0 \to D^+\pi^-, D^{*+}\pi^-, D^+\eta-, D^{*+}\eta^-$ are shown to be in excellent agreement with the factorization results. A weighted average of the four independent values for the QCD coefficient $a_1$ extracted from the data gives $a_1 = 1.15 \pm 0.06$ suggesting that it may be equal to the Wilson coefficient $c_1(\mu)$ evaluated at the scale $\mu = m_b$.

The dynamics of nonleptonic weak decays is strongly influenced by the confining color forces among the quarks. In contrast to semi-leptonic transitions, where the lepton current naturally factorizes and one is left with the hadronic matrix element of a color-singlet quark current, nonleptonic processes are complicated by the phenomenon of quark rearrangement due to the exchange of soft and hard gluons. The theoretical description involves matrix elements of local four-quark operators, which are much harder to deal with than current operators.

A great simplification can be accomplished if one is willing to adopt the factorization hypothesis, which relates the complicated nonleptonic decay amplitudes to products of meson decay constants and hadronic matrix elements of current operators similar to the ones encountered in semi-leptonic decays. Despite its remarkable success in the description of 2-body decays of $B$- and $D$-mesons, precise tests of the factorization hypothesis are of utmost importance in order to find out its realm of applicability as well as its limitations. While many tests have been suggested or already carried out [1–8], most of them do not simply test the factorization hypothesis, but rather factorization together with some phenomeno-
logical model or, alternatively, together with heavy-quark symmetry for dealing with the hadronic current matrix elements. It is the main objective of this short note to concentrate on such tests that do not suffer from additional uncertainties due to our unsatisfactory ways of dealing with non-perturbative QCD.

As there exist several versions of factorization in the literature, let us begin by giving an unambiguous prescription of how to calculate the rate of some exclusive nonleptonic $B$-decay in the factorization approximation. We will concentrate on $b \to c$ transitions, which are induced by the effective Hamiltonian

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cb} \left[ c_1(\mu) Q_1^{cb} + c_2(\mu) Q_2^{cb} \right] + \text{penguin operators.}$$

(1)

It consists of products of local four-quark operators with scale-dependent Wilson coefficients $c_i(\mu)$. The operators $Q_1$ and $Q_2$, written as products of color-singlet currents, are given by

$$Q_1^{cb} = \left[ (\bar{d} u)_{V-A} + (\bar{s} c)_{V-A} \right] (\bar{c} b)_{V-A} ,$$

$$Q_2^{cb} = (\bar{c} u)_{V-A} (\bar{d} b)_{V-A} + (\bar{c} c)_{V-A} (\bar{s} b)_{V-A} ,$$

(2)

where $d'$ and $s'$ denote weak eigenstates of the down and strange quarks, respectively, and $(\bar{c} b)_{V-A} = \bar{c} \gamma_\mu (1 - \gamma_5) b$ etc. The Wilson coefficients of so-called penguin operators $[9]$ in Eq. (1) are very small. Their contribution to the dominant decay amplitudes may be neglected.

If QCD was turned off, the Wilson coefficients of the operators $Q_1^{cb}$ and $Q_2^{cb}$ would be $c_1 = 1$ and $c_2 = 0$. These values are modified by hard gluon exchange. Evaluated at the scale $\mu = m_b \simeq 5.0$ GeV one finds in leading logarithmic approximation $[10]$: $c_1(m_b) = 1.12$ and $c_2(m_b) = -0.26$.

According to the factorization hypothesis one may now write the hadronic matrix elements of $Q_1^{cb}$ and $Q_2^{cb}$ as products of two current matrix elements $[11]$. As an example, we consider the decay amplitude of the transition $\bar{B}^0 \to D^+ \pi^-$, which in the factorization approximation is given by

$$A_{\text{fact}} = \frac{G_F}{\sqrt{2}} V_{cb} V_{ud}^* a_1 \langle \pi^- | (\bar{d} u)_A | 0 \rangle \langle D^+ | (\bar{c} b)_V | \bar{B}^0 \rangle .$$

(3)

Class I transitions like the one considered above, in which only a charged meson can be generated directly from a current, are proportional to the QCD coefficient $a_1$. Its relation to the Wilson coefficients will be discussed below. Correspondingly, those decays in which the meson generated directly from the current is neutral, like the $J/\Psi$-particle in the decay $\bar{B} \to \bar{K} J/\Psi$, are called class II, and their decay amplitudes are proportional to the QCD-coefficient $a_2$. Factorized amplitudes in which there is interference between $a_1$- and $a_2$-terms are categorized as class III.

Usually, form factor suppressed weak annihilation topologies ($W$-exchange and quark-annihilation diagrams) are neglected in the calculation of factorization amplitudes. This is not an inherent property of the factorization approximation. Rather it is necessary from a practical point of view, since little more is known about form factors at such large time-like
momentum transfer than that they should be strongly suppressed. What really is an inherent
property of the factorization approximation is the neglect of final state interactions (FSI).
However, unlike $D$-decays, the decays of $B$-mesons do not take place in a resonance region.
Thus one has good reason to believe that ignoring the effects of FSI is a good approximation
in $B$-decays.

Let us now turn to the relation between the Wilson coefficients and the QCD coefficients
$a_1$ and $a_2$. Naively, one would expect $a_1 = c_1(\mu_f) + \xi c_2(\mu_f)$ and $a_2 = c_2(\mu_f) + \xi c_1(\mu_f)$, with
$\xi = 1/N_c$, and $\mu_f$ denoting the factorization point, in $B$-decays usually identified with $m_b$.
However, experience in $D$-decays has shown that setting $\xi = 0$ allows for a better description
of the data, and it has been suggested to treat $\xi$ or even $a_1$ and $a_2$ independently as a free
parameters \[1\]. Thus one can test the factorization hypothesis by checking whether or
not the values for the QCD coefficient $a_1 (a_2)$ as extracted from different class I (class II)
transitions agree with each other. For $a_1$ also an absolute prediction becomes possible, by
observing that varying the parameter $\xi$ in the range $0 < \xi < 1/3$ induces no more than a
10% change in $a_1$. One would therefore expect $a_1 = 1.1 \pm 0.1$ which has been confirmed
in a recent extraction of $a_1$ from all available nonleptonic 2-body decays of $B$-mesons \[3\].
Hence, we have good reason to believe that ignoring the effects of FSI is a good approximation
in $B$-decays.

As first pointed out by Bjorken, the close relationship between factorized amplitudes and
semi-leptonic decay amplitudes provides the most direct test of the factorization assump-
tion \[4\]. To this end, a nonleptonic decay width is related to the corresponding differential
semi-leptonic decay width evaluated at the same $q^2$. Let us consider the ratios

$$
R_P^{(*)} = \frac{\Gamma(\overline{B}^0 \rightarrow D^{(*)+} \ell^- \overline{\nu}_\ell)}{d \{\Gamma(\overline{B}^0 \rightarrow D^{(*)+}) \}/dq^2} \big|_{q^2=m_{P^*}^2} = 6\pi^2 f_P^2 |A_{ij}|^2 |V_{ij}|^2 X_P^{(*)},
$$

(4)

where $f_P$ is the decay constant of the pseudoscalar meson $P$, $V_{ij}$ is the appropriate KM-
matrix element (associated with $P$) and (in the limit of vanishing lepton mass)

$$
X_P = \frac{(m_B^2 - m_P^2)^2}{|m_B^2 - (m_D + m_P)^2| |m_B^2 - (m_D - m_P)^2|} \left| \frac{H_0(m_P^2)}{F_0(m_P^2)} \right|^2,
$$

$$
X_P^* = \left[ m_B^2 - (m_D + m_P)^2 \right] \left[ m_B^2 - (m_D - m_P)^2 \right] \frac{|A_0(m_P^2)|^2}{m_P^2 \sum_{i=0 \pm} |H_i(m_P^2)|^2}.
$$

(5)

The helicity amplitudes $H_0(q^2)$ and $H_\pm(q^2)$ are defined in Ref. \[5\].

Bjorken has suggested this test with $P = \pi$, in which case $X_\pi \simeq X_\pi^* \simeq 1$ to within
less than 0.5% as can be shown by expanding those quantities in powers of $m_\pi^2/m_B^2$ \[6\].
For heavier pseudoscalar mesons, $X_P$ and $X_P^*$ become model dependent and may quite
substantially deviate from 1. In the infinite quark mass limit, one finds, for example, $X_{D_s} \simeq 1.36$ and $X_{D_s}^* \simeq 0.37$ \[7\].

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1Here and in the following, “$D^{(*)}$” stands for “$D$ or $D^{*}$"
We can get rid of this model dependence altogether by replacing the pseudoscalar meson \( P \) in Eq. (4) by a vector or pseudovector meson. In the factorization approximation one then finds

\[
R_V^{(*)} = \frac{\Gamma(\bar{B}^0 \to D^{(*)+} V^-)}{d \{\Gamma(\bar{B}^0 \to D^{(*)+} \ell^- \bar{\nu}_\ell)\}} / dq^2 |_{q^2=m_{V}^2} = 6\pi^2 f_V^2 |a_1|^2 |V_{ij}|^2 ,
\]

where now, of course, the KM-matrix element is associated with the (pseudo-)vector meson and \( f_V \) denotes its decay constant. The reason for all form factors and additional kinematical factors to cancel in the ratio can be easily understood. For zero lepton masses, the lepton pair that in the semi-leptonic decay is generated by the \((V - A)\) current carries spin one in its c.m. frame. Integrated over the lepton angles keeping \( q^\mu = (p_\ell + p_{\bar{\nu}})^\mu \) fixed, the production of the lepton pair is therefore kinematically equivalent to the production of a (pseudo-)vector particle with four-momentum \( q^\mu \) (summed over all polarizations of the (pseudo-)vector particle). Corrections to Eq. (6) due to finite lepton masses are of order \( m_{\ell}^2/m_V^2 \). With the \( \rho \)-meson being the lightest spin-one meson these corrections may safely be neglected for electrons and muons.

Setting \( V = \rho \), we can use Eq. (6) to obtain two independent values for \( a_1 \), since the decay constant \( f_\rho \) is known\(^2\). These values should be compared with those obtained from Eq. (4) with \( P = \pi \). However, as long as the differential \( q^2 \) spectrum of the semi-leptonic decay \( \bar{B}^0 \to D^{(*)+} \ell^- \bar{\nu}_\ell \) has not yet been measured, we must again resort to some form factor model in decays with a \( D \)-meson in the final state. In Table I, we have used the predictions of Ref. \( [6] \) for those two decays. They are based on an Isgur-Wise function extracted from data on the decay \( \bar{B}^0 \to D^{*+} \ell^- \bar{\nu}_\ell \) with perturbative QCD-corrections and (model-dependent) \( 1/m_Q \)-corrections added on. Nonleptonic decay data used in Table I has been taken from CLEO \( [14] \) and ARGUS \( [15] \). The ARGUS data as well as the predictions of Ref. \( [6] \) have been rescaled using the new CLEO measurement \( BR(D^{*+} \to D^0 \pi^+) = (68 \pm 2)\% \) \( [16] \).

The experimental number for the ratio \( R_\rho^{(*)}/R_\pi^{(*)} \), where some of the systematic uncertainties drop out, has been taken from CLEO alone. We observe that all four values for the QCD coefficient \( a_1 \) presented in Table I are in excellent agreement with each other and with the expectation from perturbative QCD, thus providing strong support for the factorization hypothesis in \( B \)-decays with large recoil. Taking the weighted average of all four values gives \( a_1 = 1.15 \pm 0.06 \), which suggests that, just like in \( D \)-decays, we may have \( \xi = 0 \), i.e. the QCD coefficient \( a_1 \) may be equal to the Wilson coefficient \( c_1(\mu) \) evaluated at the scale of the decaying quark.

As better statistics becomes available the decays \( \bar{B}^0 \to D^{(*)+} K^{(*)-} \) should be included in the above analysis, since the decay constants \( f_K \) and \( f_K^- \) are also known (the latter can be extracted from exclusive \( \tau \)-decay data). On the other hand, Eq. (6) may be used together with the experimentally determined value of the QCD coefficient \( a_1 \) to extract yet unknown decay constants of spin-one mesons like the \( a_1 \)-meson or the \( D_s^{(*)} \)-meson without resorting to some particular form factor model or to heavy-quark symmetry.

From our kinematical argument about the equivalence of the lepton pair in the semi-leptonic decay and the spin-one particle in the nonleptonic decay it is clear that Eq. (6)

\(^2\)We use \( f_\rho = 205 \text{ MeV} \)
must be valid, separately, for the different polarizations of the $D^*$-meson in the final state. This amounts to the factorization prediction that the polarization of the $D^*$-meson in the nonleptonic decay $\bar{B}^0 \rightarrow D^{*+} V^-$ should be equal to the polarization in the corresponding semi-leptonic decay at the same $q^2$. This prediction is currently being tested by the CLEO collaboration [14]. However, in interpreting the results of such a test, one has to bear in mind that in the semi-leptonic as well as in the nonleptonic case the $D^*$-polarization at the points $q^2 = 0$ and $q^2 = q^2_{\text{max}}$ is unambiguously determined by kinematics alone to be 100% longitudinal and 1/3 longitudinal, respectively. At zero recoil, there is no preferred direction and thus the value 1/3 just expresses the fact that there are two transverse, but only one longitudinal polarization. At $q^2 = 0$, corresponding to maximum recoil in the semi-leptonic decay, the left-handed electron or muon and the right-handed anti-neutrino go off parallel to each other, thereby forcing the $D^*$ into longitudinal polarization. In the corresponding nonleptonic decay $\bar{B}^0 \rightarrow D^{*} V^-$ we know (even without the factorization approximation!) that the decay amplitude must be proportional to the polarization vector of the (pseudo-)vector meson $V$. Now, for small $q^2 = m_V^2 \ll m_B^2/4$ the (pseudo-)vector meson $V$ is highly relativistic (in the $B$-meson rest frame) so that for longitudinal polarization of $V$ (and consequently $D^*$) the components of the polarization vector acquire very large values, causing longitudinal polarization to dominate.

The above discussion shows that in comparing polarizations in semi-leptonic and nonleptonic decays, one needs polarization data of quite high precision in order to make a statement about the validity of factorization. Thus, at low $q^2$, it is the amount of transverse polarization that has to be measured with a small relative uncertainty. Especially for the semi-leptonic decay such a high precision measurement of the $q^2$-dependence of the $D^*$-polarization seems hardly possible at present. Fortunately, this seems to be a case where heavy-quark symmetry predictions receive only minor corrections. In the infinite quark mass limit one finds for the ratio of transverse to longitudinal polarization at some fixed $q^2$

$$\frac{d\Gamma_T}{d\Gamma_L} = \frac{4q^2(m_B^2 + m_{D^*}^2 - q^2)}{(m_B - m_{D^*})^2 [(m_B + m_{D^*})^2 - q^2]}$$

(7)

which is subject to QCD- as well as $1/m_Q$-corrections. The general structure of the corrections can be found in Ref. [17]. While the QCD-corrections can be reliably calculated using perturbation theory (see e.g. Ref. [18]), the $1/m_Q$-corrections are model-dependent. We have calculated the corrections to Eq. (7) using the QCD-corrections of Ref. [18] and the $1/m_Q$-corrections resulting from a) an analysis of the wave function model of Bauer, Stech and Wirbel [19,20] and b) a sum rule calculation [21]. Although corrections to individual form factors in both models are as large as 30% at maximum recoil and furthermore vary strongly between both models, the correction factor to Eq. (7) in neither one of the two models deviates from one by more than 5% (though the deviations in the two models go in opposite directions). In Table II, we present the prediction for the $D^*$-polarization in semi-leptonic $B$-decays as a function of $q^2$ obtained from Eq. (7). The quoted errors result from the conservative estimate of a 10% relative uncertainty for $\Gamma_T/\Gamma_L$ at maximum recoil (i.e. at $q^2 = 0$), decreasing linearly to the point of zero recoil, where the polarization is fixed model-independently.

In nonleptonic $B$-decays, the only polarization measurement presently available is that of the $D^*$-polarization in the decay $\bar{B}^0 \rightarrow D^{*+} \rho^-$. CLEO finds $\Gamma_T/\Gamma_{\text{tot.}} = (10 \pm 9)\%$ [14],

5
which has to be compared with the 12% transverse polarization predicted for the semi-leptonic decay at $q^2 = m_\ell^2$ (see Table II). In order for this test to be sensitive to deviations from factorization, the experimental uncertainty will have to be reduced.

The situation may be more favorable in the decay $B^0 \to D^{*+} D_s^*$ with predicted 48% of transverse polarization, hopefully allowing for a measurement with smaller relative uncertainties. Also, it will be particularly interesting to see whether this decay obeys the factorization prediction, as this would indicate that the factorization assumption may be justified even in decays with only medium energy release. However, one should keep in mind that the QCD coefficient $a_1$ drops out of the ratio $\Gamma_T/\Gamma_L$, so that from polarization tests alone it will not be possible to decide whether the short range QCD corrections represented by the values of $a_1$ and $a_2$ are really independent of the energy release. To this end, one would like to test the validity of Eq. (6) with $V = D_s^*$ using a value for the decay constant of the $D_s^*$ as determined independently from a measurement of the rate for the decay $D_s \to \mu \bar{\nu}$ (employing $f_{D_s} \simeq f_{D_s^*}$, predicted by heavy-quark symmetry). In the absence of such a measurement $f_{D_s^*}$ may be taken from sum rule or lattice calculations.

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TABLES

TABLE I. Determination of the QCD coefficient $a_1$ from several nonleptonic $B$ decay modes as a test of the factorization assumption. The data are taken from CLEO and ARGUS. The theoretical predictions for the branching ratios in the last two rows are those of Ref. \[6\].

| Quantity | Experiment | Theory       | $a_1$       |
|----------|------------|--------------|-------------|
| $R_\pi^*$ [see Eq. (4)] | 1.29±0.22 | 0.97$a_1^2$ | 1.15±0.10   |
| $R_\rho^*$ [see Eq. (6)] | 3.0±0.7   | 2.37$a_1^2$ | 1.13±0.13   |
| $R_\rho^*/R_\pi^*$ | 2.5±0.6   | $f_\rho^2/f_\pi^2 = 2.4$ | —           |
| BR($B^0 \to D^+\pi^-$) | 0.28±0.05 | 0.214$a_1^2$ | 1.15±0.10   |
| BR($B^0 \to D^+\rho^-$) | 0.74±0.22 | 0.502$a_1^2$ | 1.21±0.18   |

TABLE II. Amount of transverse polarization (in %) of the $D^*$ in semi-leptonic $B$-decay.

| $q^2$ | 0 | $m_{\rho}^2$ | $m_{a_1}^2$ | $m_{D^*}^2$ | $q_{\text{max}}^2$ |
|-------|---|-------------|-------------|-------------|-------------------|
| d$\Gamma_T/d\Gamma_{\text{tot.}}$ | 0 | 12±1        | 26±2        | 48±1        | $\frac{2}{3}$    |