Matrix string states in pure 2d Yang Mills theories.

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Abstract
We quantize pure 2d Yang-Mills theory on a torus in the gauge where the field strength is diagonal. Because of the topological obstructions to a global smooth diagonalization, we find string-like states in the spectrum similar to the ones introduced by various authors in Matrix string theory. We write explicitly the partition function, which generalizes the one already known in the literature, and we discuss the role of these states in preserving modular invariance. Some speculations are presented about the interpretation of 2d Yang-Mills theory as a Matrix string theory.

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1 Introduction

In the last few years a lot of interesting results have been obtained on two dimensional gauge theories like QCD2 and pure Yang-Mills theory. Due to the invariance of 2d Yang-Mills theory under area preserving diffeomorphisms and its almost topological nature its partition functions and a number of observables have been calculated exactly [1, 2, 3, 4, 5, 6, 7] on arbitrary Riemann surfaces. In the large $N$ limit the existence of a deconfining phase transition on the sphere and on the cylinder has been recognized [8, 9, 10, 11, 12] as a result of a condensation of instanton contributions. Perhaps the most interesting development has been the recognition that in the large $N$ limit two dimensional YM theory is a string theory. In fact the partition function of $U(N)$ Yang Mills theory on a two dimensional Riemann surface $M_G$ of genus $G$ counts the number of homotopically distinct maps from a Riemannian world-sheet $W_g$ of genus $g$ to $M_G$ [14]. A new and seemingly unrelated connection between string theory and two dimensional gauge theories has been developed in [15, 16, 17]. By combining the conjecture of Banks et al. [18] with the compactification of an extra spatial dimension [19] it is argued that type IIA string theory can be identified with the large $N$ limit of two-dimensional $\mathcal{N} = 8$ supersymmetric Yang-Mills theory. In this context the eight non compact space dimensions are represented by the eight scalar fields $X_i$ of the $\mathcal{N} = 8$ supermultiplet belonging to the adjoint representation of $U(N)$. In the limit $g_{YM} \to \infty$ (which is the $g_s \to 0$ limit for the string coupling $g_s = 1/g_{YM}$) these eight matrix fields commute and can be simultaneously diagonalized. A smooth global diagonalization however is in general not possible because the $N$ eigenvalues can undergo a permutation $P$ as one goes round a non-contractable loop in the compactified dimension. As a result the spectrum contains states that are associated to the cycles of $P$ and can be identified with string states. Supersymmetry plays a crucial role in this scheme, as it ensures the cancellation of the Fadeev-Popov determinants (Vandermonde determinants of the eigenvalues).

In this paper we show that a similar spectrum of states arises, by the same mechanism, in pure Yang-Mills theory on a torus. We choose the gauge in which the field strength $F$, treated in a first order formalism as an independent auxiliary field, is diagonal, and analyze the sectors arising from non trivial permutations of its eigenvalues as one goes round the two independent cycles of the torus. A fermionic symmetry between the ghost-antighost sec-
tor and the non-diagonal part of the gauge fields leads to the cancellation of the Vandermonde determinants, provided the Riemann surface on which the theory is defined has zero curvature. This limits our analysis to the torus, and leaves the problem of its extension to general Riemann surfaces open to speculations. Consider now the theory on the torus as the theory of an infinite cylinder taken at finite temperature and denote by $P$ be the permutation of the eigenvalues of $F$ as we go round the compact space dimension. We shall find that the states that propagate along the cylinder are in correspondence with the decomposition of $P$ into cycles. More precisely they can be described as a gas of free fermions (or bosons), where each fermion is associated to a cycle of $P$ and is labeled by two quantum numbers: the discretized momentum $n$ and the length $k$ of the cycle. The resulting partition function is therefore different from the one so far produced in the literature, which corresponds to the truncation to the states with only cycles of order $k = 1$. The states associated to non trivial permutations $P$ are described by holonomies whose eigenvalues are not generic: the sets of eigenvalues on which $P$ acts as a cyclic permutation, say of order $k$, are spaced like the $k$-th roots of unity. The truncation to $k = 1$, that corresponds to the standard quantization, although consistent treats the compactified space and time dimensions on a different footing by allowing arbitrary permutations of the eigenvalues to occur only in the time direction, thus breaking modular invariance. Our generalization is characterized by arbitrary commuting permutations along the two generators of the torus, and hence preserves modular invariance.

The plan of the paper is as follows: in Section 2 we discuss the quantization of YM2 in the gauge where $F$ is diagonal (unitary gauge); in Section 3 we calculate the contribution of the new sectors, derive the partition function on the torus and discuss the role of modular invariance; in Section 4 we obtain the same results by calculating the functional integral on a cylinder and then sewing the two ends of the cylinder; in Section 5 we discuss our results, especially in connection with quantization in other gauges, and add a few concluding remarks.

## 2 YM2 in the Unitary gauge

We begin by reviewing the main steps involved in the calculation of the partition function of YM2 on an arbitrary Riemann surface using the so
called Unitary (or torus) gauge. The full details can be found in Ref. [6]. Let us consider the partition function

$$Z(\Sigma_g, t) = \int [dA][dF] \exp \left\{ -\frac{t}{2} \text{tr} \int_{\Sigma_g} d\mu F^2 + i \text{tr} \int_{\Sigma_g} f(A) F \right\} ,$$

(1)

where $d\mu$ is the volume form on $\Sigma_g$ and $f(A)$ is given by

$$f(A) = dA - i A \wedge A .$$

(2)

In Eq.s (1) and (2) $F$ is a $N \times N$ hermitian matrix and $A$ is a one form on $\Sigma_g$ with values on the space of hermitian matrices. The usual Yang-Mills action can be recovered from (1) by performing the Gaussian integral over $F$. The Unitary gauge consists in conjugating the $N \times N$ hermitian matrix $F$ into a diagonal form, namely into its Cartan sub-algebra. This can always be done, at least locally, by a gauge transformation $g$:

$$g^{-1}Fg = \text{diag}(\lambda) .$$

(3)

The gauge fixed action, including the appropriate Faddeev-Popov ghost term, can be written as the sum of two terms:

$$S_{\text{BRST}}(\Sigma_g, t) = S_{\text{Cartan}} + S_{\text{off-diag}} ,$$

(4)

where $S_{\text{Cartan}}$ involves the diagonal part of $A_\mu$ and exhibits a residual $U(1)^N$ gauge invariance:

$$S_{\text{Cartan}} = \int_{\Sigma_g} \sum_{i=1}^{N} \left( \frac{t}{2} \lambda_i^2 d\mu - i \lambda_i dA_i \right) ,$$

(5)

where $A_i$ is the $i$-th diagonal term of the matrix form $A$. The Faddeev-Popov ghost term and the off-diagonal part of $A$ are contained in $S_{\text{off-diag}}$ which can be cast into the following form:

$$S_{\text{off-diag}} = \int_{\Sigma_g} d\mu \sum_{i>j} (\lambda_i - \lambda_j) \left[ \hat{A}_{ij}^0 \hat{A}_{ji}^0 - \hat{A}_{ij}^1 \hat{A}_{ji}^1 + i (c^{ij} \bar{c}^{ji} + \bar{c}^{ij} c^{ji}) \right] ,$$

(6)

where $A_{ij}^a = E_a^\mu A_{ij}^\mu$ and $E_a^\mu$ denotes the inverse of the two dimensional vierbein. $c^{ij}$ and $\bar{c}^{ij}$ are respectively the ghost and anti-ghost corresponding to
the gauge condition $F^{ij} = 0$. The action (6) has some remarkable properties: it contains the same number of fermionic and bosonic degrees of freedom and it is symmetric, for each value of the composite index $[ij]$, with respect to a set of symmetry transformation with Grassmann-odd parameters, which with abuse of language we shall call supersymmetries. They are summarized by the following equations:

$$\begin{align*}
\delta \hat{A}_0 &= i(\eta c + \zeta \bar{c}) , \\
\delta \hat{A}_1 &= i(\xi c + \chi \bar{c}) , \\
\delta c &= -\chi \hat{A}_0 + \zeta \hat{A}_1 , \\
\delta \bar{c} &= -\xi \hat{A}_0 + \eta \hat{A}_1,
\end{align*}$$

(7)

where $\eta, \zeta, \xi$ and $\chi$ are the fermionic parameters and the index $[ij]$ has been omitted in all fields. One would expect as a result of the supersymmetry a complete cancellation of the bosonic and fermionic contributions in the partition function. This is not true in general because the supersymmetry is broken on a generic Riemann surface by the measure of the functional integral. This anomaly arises because the supersymmetric partners of the ghost anti-ghost fields are the zero forms $\hat{A}^{ij}_a$, which are the component of the one form $A$ in the base of the vierbein. The functional integral however is on the one form $A$, and on a curved surface the ‘number’ of zero forms and one forms does not coincide (as it is easily seen on a lattice like in Regge calculus). The mismatch of fermionic and bosonic degrees of freedom results into an anomaly that has been explicitly calculated in [6]:

$$\int \prod_{i>j}[d^c d^{\bar{c}}][d^A d^{\bar{A}}] e^{-S_{off-diag}} = \exp \left[ \frac{1}{8\pi} \int_{\Sigma_g} \sum_{i>j} \log(\lambda_i - \lambda_j) \right].$$

(8)

Two considerations are in order here: first that the anomaly vanishes for surfaces with zero curvature, such as the torus or the infinite cylinder, second that for constant eigenvalues $\lambda_i$, the r.h.s. of (8) reduces to $\prod (\lambda_i - \lambda_j)^{2-2g}$ and it becomes divergent for $g > 1$ when two eigenvalues coincide. We are not going to go through the whole calculation of the partition function, which can be found elsewhere [6]; the point is that the gauge fixing and the following calculation of the functional integral for the $U(1)^N$ gauge invariant action (4) leads to constant and integer values for the eigenvalues $\lambda_i$: $\lambda_i \rightarrow n_i$. The resulting partition function of YM2 on $\Sigma_g$ for the group $U(N)$ is then given by:

$$Z(\Sigma_g, t) = \sum_{\{n_i\}} \frac{1}{\prod_{i>j}(n_i - n_j)^{2g-2}} e^{-2\pi^2 t \sum_i n_i^2}.$$

(9)
There is nothing in the above derivation of (9) to stop two or more integers $n_i$ from being coincident. On the other hand such terms (which we shall call “non regular” following the terminology of Ref. [6]) are divergent for $g > 1$ and need to be regularized. The regularization suggested in [6] consists in adding small mass terms to $A^{ij}$. These terms preserve the residual $U(1)^N$ gauge invariance but they break explicitly the supersymmetry of $S_{\text{off-diag}}$. Correspondingly the contribution to the partition function coming from $S_{\text{off-diag}}$ is modified in the following way:

$$
\prod_{i>j} \frac{(n_i - n_j)^2}{(n_i - n_j)^{2g}} \to \prod_{i>j} \frac{(n_i - n_j)^2}{(n_i - n_j - m_{ij})^{2g}}.
$$

(10)

In (10) we have kept the ghost-antighost contribution, which is not divergent and is not affected by the regularization, separate from the one coming from the $A^{ij}$. Clearly after the regularization the terms with two or more coincident $n_i$’s vanish due to effect of the ghost contribution, while the would be divergent terms coming from $A^{ij}$ remain finite also for $g > 1$. As a result all “non regular” terms are altogether suppressed. Although not entirely satisfactory this procedure reproduces the well known partition function obtained both with other gauge choices and on the lattice, and it seems appropriate in YM2 on Riemann surfaces with non vanishing curvature. On flat surfaces however, like the torus and the infinite cylinder, the anomaly of the fermionic symmetry (7) vanishes and no regularization is required. Hence there is no reason to add to the action terms that would break that symmetry explicitly. On the other hand if the supersymmetry (7) is preserved nothing prevents non regular terms from appearing in the partition function. The integers $n_i$ have been interpreted on a torus (or on a cylinder) as the discretized momenta of a gas of free fermions (or bosons). Non regular terms would naturally be identified with fermions (or bosons) carrying the same integer momentum. However it will be shown in the following section that a non regular term with for instance two coincident $n_i$’s can arise either as two states with the same momentum, or as one state where the two eigenvalues are exchanged as we go round a non contractable loop. These states are a new feature in YM2 and they are the exact analogue of the stringy states described by Dijkgraaf, E. Verlinde and H. Verlinde (DVV) in the context of Matrix string theory

\footnote{The interpretation of the eigenvalues as bosons is associated to a quantization which is done on the algebra rather than on the group manifold [21].}
It is remarkable however that we do not need supersymmetric YM to obtain the DVV states as the cancellation of the Vandermonde determinants is ensured by the fermionic symmetry described above. This seems to be a peculiarity of YM2, possibly related to its interpretation as a string theory.

3 The partition function on the torus

We will now concentrate on the calculation of the partition function (1) on the torus, defined as a square with identified opposite sides. If we introduce a set of Euclidean coordinates \((\tau, x)\), then all fields will obey periodic boundary conditions in both directions:

\[
A_\mu(\tau + 2\pi, x) = A_\mu(\tau, x + 2\pi) = A_\mu(\tau, x), \\
F(\tau + 2\pi, x) = F(\tau, x + 2\pi) = F(\tau, x).
\] (11)

We have chosen for convenience to have periodicity \(2\pi\) in both directions; this is not restrictive as a rescaling of the coordinates can be absorbed in a redefinition of the coupling \(t\). We now proceed to fix the gauge according to Eq.(3). At any given point \((\tau, x)\) the group element that conjugate the matrix \(F\) into its Cartan sub-algebra is defined up to an element of the Weyl group, namely in our case up to an element \(P\) of the permutation group. So if \(g(\tau, x)\) is the U\((N)\) transformation that diagonalizes \(F(\tau, x)\), any transformation \(Pg(\tau, x)\) will also diagonalize \(F(\tau, x)\) to a form corresponding to a different permutation of the eigenvalues. If we require \(g(\tau, x)\) to be continuous with its first derivatives, then it is clear that \(g(\tau, x)\) will in general be multi-valued with boundary conditions of the type:

\[
g(\tau + 2\pi, x) = Pg(\tau, x), \\
g(\tau, x + 2\pi) = Qg(\tau, x).
\] (12)

This reflects the possibility that as we go around a closed loop the eigenvalues cross over, and undergo a permutation:

\[
\lambda_i(\tau + 2\pi, x) = \lambda_{P(i)}(\tau, x), \\
\lambda_i(\tau, x + 2\pi) = \lambda_{Q(i)}(\tau, x).
\] (13)

Consistency requires that the two permutations \(P\) and \(Q\) commute:

\[PQ = QP.\] (14)
After gauge fixing also the gauge field $A_\mu$ obeys generalized boundary conditions:

$$
A_\mu(\tau + 2\pi, x) = P^{-1}A_\mu(\tau, x)P ,
A_\mu(\tau, x + 2\pi) = Q^{-1}A_\mu(\tau, x)Q .
$$

(15)

We shall give here an explicit example of a configuration $F(\tau, x)$ that satisfies the periodic boundary conditions (11), but whose eigenvalues are permuted as $x \to x + 2\pi$. Consider for $N = 2$ the following periodic configuration:

$$
F(\tau, x) = \sin x \sigma_3 + (1 - \cos x) \sigma_2 ,
$$

(16)

where $\sigma_i$ are the Pauli matrices. The eigenvalues are given by the equation

$$
\lambda^2(\tau, x) = 2(1 - \cos x) = 4 \sin^2 \frac{x}{2}
$$

(17)

namely, if we require continuity of $\lambda(\tau, x)$, by

$$
\lambda_{\pm}(\tau, x) = \pm 2 \sin \frac{x}{2} .
$$

(18)

The eigenvalues are therefore exchanged as $x \to x + 2\pi$:

$$
\lambda_{\pm}(\tau + 2\pi, x) = \lambda_{\pm}(\tau, x) ,
\lambda_{\pm}(\tau, x + 2\pi) = \lambda_{\mp}(\tau, x) .
$$

(19)

Notice that in order to have an exchange of the eigenvalues in both the $x$ and the $\tau$ direction it would be enough to replace at the r.h.s. of (16) $x$ with $x + \tau$. In conclusion every configuration of the field $F(\tau, x)$ belongs to a topological sector labelled by an ordered pair of commuting permutations. In the previous example $F(\tau, x)$ belongs to the $(1, Q)$ sector with $Q = (1, 2)$. In Appendix A we give an explicit construction of all pairs of commuting permutations. We have already remarked that the transformation $g(\tau, x)$ that diagonalizes $F$ is defined only up to an element of the Weyl group, namely that if $g(\tau, x)$ diagonalizes $F$, then so does $Rg(\tau, x)$ with $R \in S_N$. There is therefore a residual ambiguity in the gauge fixing which could be removed by fixing for instance the order of the eigenvalues at a specific point. We

\footnote{Similar examples were given in Ref. [22]}

7
can avoid doing that by simply dividing the functional integral by $N!$ to account for the multiplicity of the gauge equivalent copies. Notice that fields obeying boundary conditions of the type (13) and (15) are gauge equivalent to fields obeying the same boundary conditions with $(P, Q)$ replaced by $(RPR^{-1}, RQR^{-1})$. Thus *gauge-inequivalent* topologically distinct sectors are in one-to-one correspondence with pairs of *conjugacy classes* of commuting permutations. The functional integral over each sector gives a *partition function* $Z(t, P, Q)$, that we shall evaluate shortly. The total partition function will be obtained by summing over all sectors with suitable relative weights $c(P, Q)$:

$$Z(\Sigma_1, t) = \frac{1}{N!} \sum_{P,Q} c(P, Q)Z(t, P, Q) , \quad (20)$$

where the factor $1/N!$ is inserted to account for the gauge ambiguity discussed above. The problem of determining the weights $c(P, Q)$ will be discussed later.

The fundamental feature, shared by all the different sectors, is the exact cancellation between the Faddeev-Popov determinant and the contribution of the non-diagonal part of the gauge field $A_{\mu}$. As discussed in the previous section this follows from the supersymmetry (7) which is unbroken in case of zero curvature surfaces.

As a result, we are left with the $U(1)^N$ invariant part of the action, which now reads

$$Z(P, Q, t) = \int \left( \prod_i [dA_{\mu}^{(i)}][d\lambda_i] \right) \exp \left\{ -\int_0^{2\pi} d\tau dx \sum_i \left[ \frac{t}{2} \lambda_i^2 - i\lambda_i \left( \partial_0 A_i^{(i)} - \partial_i A_0^{(i)} \right) \right] \right\} . \quad (21)$$

This would be just $N$ copies of QED on a torus, except for the fact that the $N$ copies are mixed by the boundary conditions, which are of the type described in Eq. (13) for all the fields involved:

Of course in the trivial sector $(P = Q = 1)$ the result is trivial and coincides with the $N$th power of the partition function of QED:

$$Z(1, 1, t) = (Z_{\text{QED}}(t))^N = \sum_{n_i} \exp \left( -2\pi^2 t \sum_{i=1}^{N} n_i^2 \right) , \quad (22)$$
where the sum over the integers $n_i$ is unrestricted; in particular, coincident values of different $n_i$’s are not excluded. Let us proceed to study non-trivial sectors, by considering first a special case, in which the permutation $P$ is given by $r_k$ cycles of length $k$, and $Q$ acts as a cyclic permutation of the $r_k$ cycles of $P$. An example, where $P$ consists of three cycles of length two, is illustrated in Fig. 1, where the cycles of $P$ are represented by continuous lines joining the different points. Different choices for $Q$ are given in Fig. 1(a,b) where the dotted lines represent the $Q$-cycles. The two cases correspond to $Q$ consisting of 2 cycles of length 3 or 1 cycle of length 6. It is easy to convince oneself that in the situation described above $k r_k$ eigenvalues obeying the boundary conditions (13) are equivalent to one eigenvalue satisfying the boundary conditions

$$\lambda(\tau + 2k\pi, x) = \lambda(\tau, x), \quad (23)$$

$$\lambda(\tau, x + 2r_k\pi) = \lambda(\tau + 2S\pi, x), \quad (24)$$

where $S$ is an integer shift, which in the notations of appendix A is given by $S = \sum_\alpha s(k, \alpha)$. In the example of Fig. 1, this is illustrated by Fig. 2, where the universal covering of the torus and the fundamental region are represented (by dotted lines). The opposite sides of the fundamental region

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**Fig. 1:** A permutation $P$ and a commuting permutation $Q$ (dashed lines) consisting of 2 cycles (a) or 1 cycle (b).
can be identified only modulo a permutation of the eigenvalues. Fig.s 2(a,b) show the fundamental regions of a torus of area $kr_k$ corresponding to the boundary conditions (24) in the cases of Fig.s 1(a,b).

In conclusion the partition function for a non-trivial sector with $P$ given by $r_k$ cycles of length $k$ and $Q$ acting as a permutation of such cycles coincides with the partition functions of QED defined on a torus of area $kr_k$ times the original torus (the QED partition function on a torus does not depend on the modular parameter of the torus but only on its area), namely

$$Z_{\text{QED}}(kr_k t) = \sum_n \exp \left( -2\pi^2 kr_k t n^2 \right). \quad (25)$$

A pair $(P, Q)$ of commuting permutations consists in general of several blocks of connected cycles, like the one discussed above and pictured as an example in Fig. 1(a,b). Correspondingly its partition function will consist of the product of QED partition function defined on tori of area proportional to the number of points in each block. For instance the sector corresponding to the pair of permutations illustrated in Fig. 3 has a partition function given by:

$$Z_{\text{Fig.3}}(t) = \sum_{n_1, n_2, n_3} \exp \left[ -2\pi^2 t (6n_1^2 + 5n_2^2 + n_3^2) \right]. \quad (26)$$

The general expression for the partition function of the $(P, Q)$ sector is

$$Z_{PQ}(t) = \prod_{k=1}^{N} \prod_{h=1}^{r_h} \left[ Z_{\text{QED}}(hkt) \right]^{s_h(k)}, \quad (27)$$
where the $r_k$ is the number of cycles of length $k$ in $P$. $Q$ acts on these as a permutation $\pi_k$ and the exponent $s_h(k)$ is the number of cycles of length $h$ in $\pi_k$.

The complete partition function is obtained as a sum over all different sectors with weights $c(P,Q)$ according to Eq. (20). So the problem is to determine to what extent the coefficients $c(P,Q)$ can be fixed from consistency requirements. In principle the different sectors correspond to disconnected parts of the functional integral, and could be added with arbitrary coefficients. It is shown in appendix B, that in the BRST invariant formulation they correspond to gauge fixing functions which are non connected to each other, so that BRST invariance does not tell us anything about their relative weights. On the other hand we may require that the partition function is unchanged if we perform a Dehn twist, or more generally a modular transformation, on the torus. The generators of the modular group $S$ and $T$ act on a given sector $(P,Q)$ in the following way:

\[
\begin{align*}
S : & \quad (P,Q) \to (Q,P), \\
T : & \quad (P,Q) \to (PQ,Q).
\end{align*}
\]

It is easy to check that the resulting pair of permutations still commute and that the dimension of the connected blocks of cycles, which determines the decomposition of the partition function in terms of QED partition functions, is left unchanged by modular transformations. For instance it is clear from Fig.s 2(a,b) that the blocks described in Fig.s 1(a,b) are obtained from each
other by a modular transformation of the torus. The invariance of \( c(P, Q) \) under the modular group implies then that \( c(P, Q) = c(P', Q') \) if \((P, Q)\) and \((P', Q')\) are related by a modular transformation. One would also expect to recover the standard partition function found in the literature by summing over a subset of sectors \((P, Q)\), namely the subset where one permutation, say \( P \), is the identity. This can be understood if one follows the standard derivation of the partition function (see for instance Ref. [23]), which is obtained from the kernel of the cylinder by identifying the holonomies at the borders and taking the trace. This automatically involves a sum over permutations \( Q \), in fact the eigenvalues of the holonomies are identified up to a permutation when the two edges are sewn together. There is no sign in this derivation of the sectors with a non trivial permutation \( P \) associated to the other cycle of the torus. This problem will be analyzed in Section 4, where the theory on a cylinder that includes all \((P, Q)\) sectors is developed. We just anticipate here that non trivial permutations \( P \) correspond to holonomies where the eigenvalues belonging to the same cycle of order \( k \) of \( P \) (namely on which \( P \) acts as a cyclic permutation) are proportional to the \( k \)-th roots of unity as shown in Eq. (83). So within each sector the trace is an integral over a number of invariant angles equal to the number of cycles of \( P \), and the standard group integration automatically projects over the trivial sector. Even in the \( P = 1 \) sector an ambiguity is present when the trace over the holonomies is taken. In the standard quantization this corresponds to an integration over the group manifold and the wave functions of the states at the edges of the cylinder are antisymmetric with respect to the exchange of the eigenvalues. Correspondingly a factor \((-1)^{|Q|}\) is obtained when the sum over the permutations is taken, and its effect is to cancel all non regular terms in the partition function (see for instance Ref. [23]). It is also possible however to quantize over the algebra rather than over the group. In this case the wave functions are symmetric and the result coincides with the one given in Ref. [21].

### 3.1 Modular invariant partition functions

Let us consider the partition function

\[
Z_N(t) = \frac{1}{N!} \sum_{PQ} c(P, Q) Z_{PQ}(t) ,
\]  

(30)
where the coefficients $c(P,Q)$ satisfy the requirements of modular invariance

\[
c(P, Q) = c(Q, P),
\]

\[
c(P, Q) = c(PQ, Q).
\] (31)

It is more convenient to work directly on the grand canonical partition function defined by

\[
Z(t, q) = \sum_N Z_N(t) q^N.
\] (32)

If one imposes on $c(P, Q)$ only the constraint (31) of modular invariance, the partition function (30) has as many free parameters as the number of commuting permutation not related by a modular transformation. In order to further restrict the possible choices we shall consider the case where $c(P, Q) = \pm 1$ for all pairs $(P, Q)$. As we shall see, this leads to partition functions that in the sub-sector where $Q = 1$ coincide with the standard partition function on the torus or with the one obtained by quantizing on the algebra rather than the group $[21]$. The simplest case is when $c(P, Q) = 1$. In order to calculate $Z(t, q)$ in this case, let us review some combinatorial formulas. The number of permutations $P$ with a given structure in cycles, namely with $r_k$ cycles of order $k$, is given by $N! / \prod_k (r_k! k^{r_k})$ and the number of permutations $Q$ commuting with $P$ are $\prod_k r_k! k^{r_k}$. As shown in Appendix A, $Q$ acts, for each $k$, as a permutation of the $r_k$ cycles of order $k$ in $P$. Let $s_h(k)$ be the number of cycles of order $h$ in such permutation. The set of numbers $s_h(k)$ characterizes completely the decomposition into connected blocks of the pair $(P, Q)$. The number of pairs $(P, Q)$ corresponding to a given choice of $s_h(k)$ can be easily calculated and is given by $N! / \prod_{h,k} [s_h(k)! h^{s_h(k)}]$. In conclusion the grand canonical partition function (32) can be written as

\[
Z_b(t, q) = \prod_{h,k} \sum_{s_h(k)} q^{h s_h(k)} \left( \sum_n e^{-2\pi^2 h k n^2} \right)^{s_h(k)} / s_h(k)! h^{s_h(k)}.
\] (33)

The sums over $s_h(k)$ and $h$ can be done explicitly, leading to the result

\[
Z_b(t, q) = \prod_{n=-\infty}^{\infty} \prod_{k=1}^{\infty} \frac{1}{1 - q^k e^{-2\pi^2 k t n^2}},
\] (34)

which can be interpreted as the grand canonical partition function of a collection of free bosons:

\[
Z_b(t, q) = \text{Tr} q^N e^{-tE},
\] (35)
where the trace is defined on an Hilbert space generated by harmonic oscillators $a_k(n)$ and $a_k(n)\dagger$ and the operators $N_c$ and $E$ are defined by

$$N_c = \sum_{k,n} ka_k(n)\dagger a_k(n),$$
$$E = \sum_{k,n} kn^2 a_k(n)\dagger a_k(n).$$

(36)

$Z_b(t, q)$ can be rewritten as an infinite product of Dedekind functions, with modular parameters $\tau_n$ which are functions of $n$:

$$Z_b(t, q) = e^{\alpha_0} \prod_{n=-\infty}^{\infty} [\eta(\tau_n)]^{-1}$$

(37)

with

$$\eta(\tau) = e^{\frac{2\pi i}{12} \tau} \prod_{k=1}^{\infty} (1 - e^{2\pi i \tau})$$

(38)

and

$$\tau_n = i(\mu + \pi tn^2),$$

(39)

where $\mu$ is the chemical potential, defined by $q \equiv e^{-\pi \mu}$ and

$$\alpha_0 = -\frac{\pi}{12} \sum_{n=-\infty}^{\infty} (\mu + \pi tn^2).$$

(40)

This sum is divergent and must be regularized. Remarkably the zeta function regularization gives just $\alpha_0 = 0$, and our partition function becomes exactly an infinite product of Dedekind functions.

If we restrict the permutation $P$ in the $x$ direction to be the identity, namely we restrict the product in (34) to $k = 1$, the expansion of the r.h.s. of (34) in powers of $q$ reproduces the partition function on a torus obtained by Hetrick in [21] by quantizing YM2 on the algebra rather than on the group.

Let us consider now the partition functions where $c(P, Q) = \pm 1$. These are obtained by inserting in the sum in (33) a sign $(-1)^f$, where $f$ is an integer and is a modular invariant function of $h$, $k$ and $s_h(k)$. There are two such modular invariant quantities one can construct: $\sum_{h,k} hks_h(k) = N$
and $\sum_{h,k} s_h(k)$. The latter is the number of the connected blocks of cycles in the given sector. Both quantities are preserved by modular transformations. The introduction in (33) of a factor $(-1)^{\sum_{h,k} s_h(k)} = (-1)^N$ just changes the overall sign of the partition functions with odd values of $N$; in the grand canonical partition function it is equivalent to the substitution $q \rightarrow -q$. The insertion of a factor $(-1)^{\sum_{h,k} s_h(k)}$ is more interesting as it turns the bosonic partition function (34) into a fermionic one:

$$Z_f(t,q) = \prod_{n=-\infty}^{\infty} \prod_{k=1}^{\infty} \left(1 - q^k e^{-2\pi^2 k t n^2}\right) = \prod_{n=-\infty}^{\infty} \left[\eta(\tau_n)\right]. \quad (41)$$

$Z_f(t,q)$ can be written as a trace on a Hilbert space generated by fermionic (anti-commuting) oscillators $b_k(n)$ and $b_k(n)^\dagger$.

$$Z_f = \text{Tr} (-1)^F q^N e^{-tE}, \quad (42)$$

where the operators $F$ (fermionic number), $N_c$ and $E$ are given by

$$N_c = \sum_{k,n} k b_k(n)^\dagger b_k(n),$$
$$E = \sum_{k,n} k n^2 b_k(n)^\dagger b_k(n),$$
$$F = \sum_{k,n} b_k(n)^\dagger b_k(n). \quad (43)$$

The restriction to $P = 1$ leads in this case to the standard partition function for YM2 on a torus. This case was already discussed in [23], and the equivalence of the standard approach with the present formulation restricted to $k = 1$ can be seen by comparing (41) with Eq. (41) of Ref. [23].

A particularly interesting limit, in both fermionic and bosonic partition function, is $t \rightarrow \infty$. This is the limit where the matrix string theory of Ref. [14] has an infrared fixed point described by a conformal field theory. In

\footnote{The slight discrepancy between the two expressions in the case of even $N$ is due to a different coupling of the U(1) factor within the U($N$) group. In fact the discrepancy disappears in the case of SU($N$) where the quantized momentum $n$ is shifted by the “center of mass momentum” $\beta$ and an integration over $\beta$ is included in the definition of the trace. This was discussed in Ref. [24] and it can be shown that the same prescription applies in the present generalization.}
this limit the free string is recovered as the string coupling $g_s$ is essentially $1/t$. For $t \to \infty$ only the $n = 0$ excitations survive in Eq.s (34) and (41). In the case of $Z_b(q,t)$ we recover (apart from an exponential prefactor) the partition function of the conformal field theory of a single boson living on a rectangle with Dirichlet boundary conditions and a ratio $\mu$ between the two sides. However this rectangle has nothing to do with the torus on which the original YM theory is defined.

Another interesting limit is $t \to 0$. In this limit Yang-Mills theory becomes a BF theory and $Z_b(q,t)$ becomes formally an infinite product of partition functions identical to the one discussed in the $t \to \infty$ limit. This clearly shows that the limit is singular. The singularity can be handled by using in (33) the Poisson summation formula and writing

$$
\sum_n e^{-\frac{2\pi^2 \hbar k n^2}{h t}} = \frac{1}{\sqrt{2\pi \hbar k t}} + O(e^{-\text{const} t}) \quad (t \to 0),
$$

which implies

$$
Z_{b,t}(t, q) = e^{\pm \frac{1}{\sqrt{2\pi \hbar k t}} \sum_{h,k} \frac{h^h k^k}{k^h + k^k} + O(e^{-\text{const} t})},
$$

where the $+$ and $-$ sign at the exponent refer to $Z_b$ and $Z_f$ respectively. It is apparent from (44) and (45) that a $1/\sqrt{t}$ singularity is associated to each connected block in the $(P,Q)$ into cycles. This means that at fixed $N$ the leading most singular term is of order $t^{-\frac{N}{2}}$ and comes from the $(P = 1, Q = 1)$ sector, namely from cycles of order 1. On the contrary in the $t \to \infty$ limit the mean value at fixed $N$ of the length of a cycle can be estimated [23] and found to be larger that $O(\sqrt{N \log \sqrt{N}})$. This might be a signal that in the large $N$ limit at some critical value $t_c$ a phase transition occurs from a short cycle to a long cycle regime.

From the physical point of view the situation can be described as follows: we have two types of degrees of freedom, the momentum excitations labelled by $n$ and the string degrees of freedom labelled by the length $k$. Correspondingly we have two free parameters: the YM coupling $t$ and the chemical potential $\mu$ that set the mass scale for the corresponding excitations. From the point of view of a Matrix string theory interpretation [16],

Note that while the dependence of the states’ energy from $k$ is fixed, the dependence from $n$ reflects the form of the $\text{tr} F^2$ term in the original action [1]. Replacing $\text{tr} F^2$ with the trace of an arbitrary potential $V(F)$ would amount to substitute $n^2$ with $V(n)$ in the partition functions.
\( t \simeq 1/g_s \) is the inverse of the string coupling constant. In the strong coupling limit of the string \( (t \to 0) \) the string breaks up into a gas of partonic constituents (the \( k = 1 \) states) while in the weak coupling regime the tension effects prevail and long strings are energetically favoured. Notice that the chemical potential \( \mu \) was not present in the original YM theory, but it was introduced because our results have a natural interpretation in terms of the grand canonical partition function. Its introduction from the very beginning would amount to writing the U(\( N \))gauge action as

\[
S_N(t, \mu) = \text{tr}_N \int dx \, d\tau \left( \frac{t}{2} F_{(N)}^2 - if(A_{(N)}) F_{(N)} \right) + 2\pi \mu N , \tag{46}
\]

where the labels \( N \) are to denote the dimension of the matrices. The partition function is then defined by

\[
Z(t, \mu) = \sum_N \int [dA_{(N)}] [dF_{(N)}] e^{-S_N(t, \mu)} . \tag{47}
\]

This establishes a close analogy with the IKKT matrix string theory for type IIB strings \[20\] where a similar sum over \( N \) is involved.

### 4 Path integral on the cylinder

Let us consider the path-integral (1) with \( \Sigma \) a cylinder, that we can represent as a square of area \( 4\pi^2 \), with periodic identification in the space-like \( x \) direction. In this section we shall perform the calculation of the functional integral and derive the kernel on the cylinder as a function of the degrees of freedom at the edges. Finally, by sewing the two edges of the cylinder together we shall reproduce the partition functions on a torus obtained in the previous section.

As discussed in Section 3, we fix the unitary gauge by performing the gauge transformation \( g(\tau, x) \) that diagonalizes \( F \). The continuity of \( g(\tau, x) \) leads one to consider the generalized boundary conditions

\[
g(\tau, x + 2\pi) = Qg(\tau, x) , \tag{48}
\]

where \( Q \) is a permutation. Thus in the unitary gauge the gauge fields \( A_{(u)}^\mu \) and the eigenvalues \( \lambda_i \) of \( F \) satisfy

\[
A_{(u)}^\mu(\tau, x + 2\pi) = Q^{-1} A_{(u)}^\mu(\tau, x)Q , \tag{49}
\]
\[ \lambda_i(\tau, x + 2\pi) = \lambda_{Q(i)}(\tau, x) . \]  

(50)

It is convenient to write the last condition using for the index \( i \) the multi-index notation introduced in Appendix A: \( i \rightarrow (k, \alpha, n) \) where the set of three indices label the \( n \)-th element of the \( \alpha \)-th cycle of length \( k \) in \( Q \). The range of the indices is then \( \alpha = 1, \ldots r_k \) with \( \sum k r_k = N \), and \( n = 1, \ldots k \). In this notation Eq. (50) reads

\[ \lambda_{k,\alpha,n}(\tau, x + 2\pi) = \lambda_{k,\alpha,n+1}(\tau, x) . \]

(51)

where here and in the following the index \( n \) is understood \( \text{mod} \ k \).

In order to understand the effect of the non trivial boundary conditions (51), let us first study the topologically non-trivial Wilson loop

\[ W(\tau) \equiv P \exp\{-i \int_0^{2\pi} dx A_1(\tau, x)\} \in U(N) \]

(52)

and denote by \( W^{(u)}(\tau) \) its expression in the unitary gauge. \( W(\tau) \) and \( W^{(u)}(\tau) \) are related by the gauge transformation \( g(\tau, x) \) taken at the end points \( x = 0 \) and \( x = 2\pi \):

\[ W(\tau) = g^{-1}(\tau, 0)W^{(u)}(\tau)g(\tau, 2\pi) = g^{-1}(\tau, 0)W^{(u)}(\tau)Q g(\tau, 0) , \]

(53)

According to Eq. (53), the eigenvalues \( e^{i\theta(\tau)} \) of \( W(\tau) \) coincide with the eigenvalues of \( W^{(u)}(\tau)Q \). In the unitary gauge, on the other hand, the non diagonal matrix elements of \( A_1^{(u)}(\tau, x) \) are forced to vanish as a result of the functional integral over \( A_0^{(u)}(\tau, x) \) with the action (3), and \( W^{(u)}(\tau) \) is therefore diagonal. It is easy to see that with \( W^{(u)}(\tau) \) diagonal the matrix \( W^{(u)}(\tau)Q \) has in the multi-index notation the form

\[ (W^{(u)}(\tau)Q)_{k,\alpha,n;k',\alpha',n'} = \delta_{k,k'}\delta_{\alpha,\alpha'}\delta_{n,n'} - 1 e^{i\phi_{k,\alpha,n}} , \]

(54)

where \( \phi_{k,\alpha,n} \) are the invariant angles of \( W^{(u)}(\tau) \). The eigenvalues of the matrix at the r.h.s. of (54) can be easily calculated to be

\[ \theta_{k,\alpha,n} = \theta_{k,\alpha} + 2\pi i \frac{n}{k} , \]

(55)

with

\[ \theta_{k,\alpha}(\tau) = \frac{1}{k} \sum_{n=1}^{k} \phi_{k,\alpha,n}(\tau) = -\frac{1}{k} \sum_{n=1}^{k} \int_0^{2\pi} dx A_1^{k,\alpha,n}(\tau, x) , \]

(56)
where \( A^{k,\alpha,n}_{\mu}(\tau, x) \) are the diagonal elements of the gauge fields in the unitary gauge. In conclusion, the eigenvalues of the original Wilson loop \( W(\tau, x) \) are not independent; rather, for each cycle of length \( k \) of \( Q \) they are distributed as the \( k \)-th roots of unity shifted by a common value \( \theta_{k,\alpha} \) which is defined modulo \( 2\pi/k \) instead of modulo \( 2\pi \).

Let us go back to the functional integral (1), and observe that due to the cancellation of the Vandermonde determinants as a result of the fermionic symmetry (7), we are left with a collection of QED-type actions as in Eq. (21). The fields in (21) whose \( U(N) \) index belong to the same cycle in the cycle decomposition of \( Q \), are related to each other by the boundary conditions (51) and they can be reduced to one field \( \tilde{\lambda}_{k,\alpha}(\tau, x) \) with \( x \) ranging in the interval \((0, 2\pi)\) instead of \((0, 2\pi)\):

\[
\tilde{\lambda}_{k,\alpha}(\tau, x) = \begin{cases} 
\lambda_{k,\alpha,1}(\tau, x) , & 0 \leq x < 2\pi , \\
\lambda_{k,\alpha,2}(\tau, x - 2\pi) , & 2\pi \leq x < 4\pi , \\
\vdots & \\
\lambda_{k,\alpha,k}(\tau, x - 2(k-1)\pi) , & 2(k-1)\pi \leq x < 2k\pi .
\end{cases}
\] (57)

Similarly a \( U(1) \) gauge field \( \tilde{A}^{k,\alpha}_{\mu}(\tau, x) \) with period \( 2\pi k \) in \( x \) can be defined from \( A^{k,\alpha,n}_{\mu}(\tau, x) \). In conclusion, the functional integral on the cylinder for the sector corresponding to a permutation \( Q \) decomposes into a product of functional integrals, one for each cycle of \( Q \), with the action being the one of a QED defined on a cylinder of length \( 2\pi k \) in the compactified direction:

\[
Z_{cyl}^{(t)} = \prod_k \prod_{\alpha=1}^{r_k} [d\tilde{A}^{k,\alpha}_{\mu}] [d\tilde{\lambda}_{k,\alpha}] 
\exp \left\{ -\sum_{\alpha=1}^{r_k} \int_0^{2\pi} d\tau \int_0^{2\pi k} dx \left[ \frac{t}{2} \tilde{\lambda}_{k,\alpha}^2 - i\tilde{\lambda}_{k,\alpha} \left( \partial_0 \tilde{A}^{k,\alpha}_1 - \partial_1 \tilde{A}^{k,\alpha}_0 \right) \right] \right\},
\] (58)

where \( k \) is the length of the cycle. As discussed in Section 3, after Eq. (19), the sum over the sectors involves a further gauge fixing related to the fact that the diagonal gauge is defined up to an arbitrary permutation of the eigenvalues: if \( g(\tau, x) \) is a gauge transformation that diagonalizes \( F \), so is \( Rg(\tau, x) \) with \( R \) an arbitrary permutation. It satisfies

\[
Rg(\tau, x + 2\pi) = RQR^{-1} Rg(\tau, x),
\] (59)

which show that sectors characterised by permutations \( Q \) and \( Q' = RQR^{-1} \), belonging to the same conjugacy class, are gauge equivalent. This implies
that the sum over all sectors involves a sum over the conjugacy classes rather than a sum over the permutations $Q$. Even so there is still a residual gauge transformation given by the permutations $R$ that commute with $Q$ (i.e. $R \in C(Q), C(Q)$ being called the centralizer of $Q$). In the following we shall denote with $P$ a generic permutation belonging to $C(Q))$. As described in Appendix A, such a permutation $P$ acts on the multi-index $(k, \alpha, n)$ by

$$(k, \alpha, n) \rightarrow (k, \pi_k(\alpha), n + s(k, \alpha)),$$  

(60)

where $\pi_k \in S_{r_k}$ is a permutations of $r_k$ elements and $s(k, \alpha)$ is an integer mod $k$. It follows from this equation and the definition (57) of $\hat{\lambda}_{k,\alpha}(\tau, x)$, that the gauge transformation $P$ acts on $\hat{\lambda}_{k,\alpha}(\tau, x)$ and $\hat{A}_{\mu}^{k,\alpha}(\tau, x)$ in the following way:

$$\tilde{\lambda}_{k,\alpha}(\tau, x) \rightarrow P \tilde{\lambda}_{k,\pi_k(\alpha)}(\tau, x - 2\pi s(k, \alpha)),$$

$$\tilde{A}_{\mu}^{k,\alpha}(\tau, x) \rightarrow P \tilde{A}_{\mu}^{k,\pi_k(\alpha)}(\tau, x - 2\pi s(k, \alpha)).$$  

(61)

Also the eigenvalues of the Wilson loop given in Eq.(55) and (56) can be expressed in terms of the redefined fields $\tilde{A}_{\mu}^{k,\alpha}$:

$$k \theta_{k,\alpha,n}(\tau) = k \theta_{k,\alpha}(\tau) + 2\pi n = -\int_0^{2\pi} d\tau \tilde{A}_{\mu}^{k,\alpha}(\tau) + 2\pi n.$$  

(62)

Let us proceed now to calculate the QED functional integrals in (58) by using a standard procedure (see [23]). We expand the fields appearing in (58) in their Fourier components in the compact $x$ direction: $\tilde{A}_{\mu}^{k,\alpha}(\tau, x) = \sum_m \tilde{A}_{\mu,m}^{k,\alpha}(\tau) \exp(imx/k)$, and similarly for $\tilde{\lambda}_{k,\alpha}(\tau, x)$. The U(1) gauge is fixed by choosing a Coulomb gauge $\partial_\tau \tilde{A}_{\mu}^{k,\alpha} = 0$, so that the only non-vanishing Fourier component of $\tilde{A}_{\mu}^{k,\alpha}$ is the zero mode which coincides, according to Eq. (62), with $-\theta_{k,\alpha}(\tau)/(2\pi)$. The functional integration over $\tilde{A}_{\mu}^{k,\alpha}$ and the Gaussian integration over the zero-mode of $\tilde{\lambda}_{k,\alpha}$ are straightforward and we remain with

$$Z^{\text{cyl}}(Q, t) = \prod_k \int \prod_{\alpha=1}^{r_k} [d\theta_{k,\alpha}] \exp \left\{ -\frac{k}{4\pi t} \sum_{\alpha=1}^{r_k} \int_0^{2\pi} d\tau (\partial_\tau \theta_{k,\alpha})^2 \right\}.$$  

(63)

For each length $k$ of the cycle $Z^{\text{cyl}}(Q, t)$ describes the quantum mechanics of $r_k$ free particles of mass $\mu = k/(2\pi t)$, that move on a circle of radius.
2\pi/k. In fact, according to Eq. (53) and following discussion the coordinates \( \theta_{k,\alpha}(\tau) \) are defined modulo \( 2\pi/k \). Given the boundary conditions at \( \tau = 0 \) and \( \tau = 2\pi \), namely \( \theta_{k,\alpha}(0) \) and \( \theta_{k,\alpha}(2\pi) \), the transition amplitude from the initial to the final configuration can be computed from (63) by using the methods described in [23]. We have:

\[
K_Q(\theta_{k,\alpha}(0), \theta_{k,\alpha}(2\pi)) = \frac{1}{(2\pi)^N} \prod_k \left( \frac{k}{2\pi} \right)^{r_k} \sum_{l_{k,\alpha}} \exp \left\{ -\frac{k}{8\pi^2 t} \sum_{\alpha=1}^{r_k} \left( \theta_{k,\alpha}(2\pi) - \theta_{k,\alpha}(0) - \frac{2\pi l_{k,\alpha}}{k} \right)^2 \right\} , \tag{64}
\]

where the sum over the winding numbers different \( k^{r_k/2} l_{k,\alpha} \) ensures the periodicity in the configuration space of the \( \theta_{k,\alpha} \)'s. The sums over \( l_{k,\alpha} \) can be performed by using the well known modular transformation for the function \( \theta_3 \) (see for instance Eq. (28) in [23]):

\[
K_Q(\theta_{k,\alpha}(0), \theta_{k,\alpha}(2\pi)) = \prod_k \left( \frac{k}{2\pi} \right)^{r_k} \sum_{n_{k,\alpha}} \exp \left\{ -\sum_{\alpha=1}^{r_k} -2\pi^2 k n_{k,\alpha}^2 - i \sum_{\alpha=1}^{r_k} k n_{k,\alpha} (\theta_{k,\alpha}(2\pi) - \theta_{k,\alpha}(0)) \right\} , \tag{65}
\]

where the integers \( n_{k,\alpha} \) can be interpreted as discretized momenta of the particles moving in the compactified configuration space. This is rather straightforward in the Hamiltonian formalism. In fact from (63) we find the Hamiltonian

\[
H_Q = -\sum_k \frac{\pi t}{k} \sum_{\alpha=1}^{r_k} (\partial/\partial \theta^{k,\alpha})^2 \tag{66}
\]

which, due to the periodicity on \( \theta^{k,\alpha} \), has discrete energy levels:

\[
E(n_{k,\alpha}) = \sum_k \pi k t \sum_{\alpha=1}^{r_k} n_{k,\alpha}^2 , \tag{67}
\]

in agreement with Eq. (63). Finally we observe that the residual gauge symmetry under permutations \( P \) that commute with \( Q \) given in Eq. (61) reduces to a permutation symmetry among the coordinates of the \( r_k \) indistinguishable particles:

\[
\theta_{k,\alpha}(\tau) \xrightarrow{P} \theta_{k,\pi_k(\alpha)}(\tau) . \tag{68}
\]
The configuration space is then an orbifold with respect to the permutation group \( S_{r_k} \):

\[
(S^1)^{r_k}/S_{r_k}.
\]

(69)

It is consistent to think of these particles both as bosons or as fermions, and we shall consider the two cases in the next subsection where we shall sew the cylinder to get the path-integral on the torus.

4.1 Sewing the cylinder

The partition function on the torus, studied in Section 3, can be reproduced from the results of the previous subsection by sewing the two ends of the cylinder, that is by imposing periodicity also in the \( \tau \) direction. This has to be done keeping in due account the residual gauge invariance generated by the permutations that commute with \( Q \). In the Hamiltonian language the partition function on the torus is given as a finite temperature trace:

\[
Z(t) = \sum_{\{Q\}} \text{Tr}(e^{-\beta H_Q P_Q}).
\]

(70)

where \( \beta \) is the inverse temperature and it is given in our case by \( \beta = \Delta \tau = 2\pi \). We shall consider both bosonic and fermionic partition functions. In the former case \( P_Q \) is just a projection operator onto the states that are invariant under the permutations \( P \) that commute with \( Q \). This corresponds to projecting over states whose wave functions are completely symmetric under \( (68) \). In the fermionic case two modifications are required: the wave functions are chosen to be antisymmetric and a factor \((-1)^F\) counting the number of fermions is included in the trace. In our case \( F \) is the number of antisymmetrized wave functions and so \((-1)^F = (-1)^{\sum r_k} \). Calculating the trace at the r.h.s. of (70) is the same as identifying in Eq. (65) \( \theta_{k,\alpha}(0) \) and \( \theta_{k,\alpha}(2\pi) \) up to an arbitrary permutation \( \pi_k(\alpha) \) of the index \( \alpha \) coming from the (anti)symmetrization of the wave functions and then integrating over the \( \theta_{k,\alpha} \)'s. In fact Eq. (70) can be rewritten in terms of the normalized wave functions

\[
\langle \theta_{k,\alpha}|n_{k,\alpha} \rangle = \prod_k \left( \frac{k}{2\pi} \right)^{r_k} e^{-i\sum_{\alpha=1}^{r_k} n_{k,\alpha} \theta_{k,\alpha}}
\]

(71)
as

\[ Z(t) = \sum_{\{Q\}} \Pi_{k,\alpha} \int_0^{2\pi} d\theta_{k,\alpha} \sum_{n_{k,\alpha}} \langle n_{k,\alpha} | e^{-2\pi H_Q} | n_{k,\alpha} \rangle \langle n_{k,\alpha} | P_Q | \theta_{k,\alpha} \rangle. \]  

(72)

The integrand at the r.h.s. of (72) is exactly the r.h.s. of (65) with the ends of the cylinder identified up to the effect of the projection operator \( P_Q \) which is in the bosonic case to symmetrize the wave function:

\[ \langle n_{k,\alpha} | P_Q | \theta_{k,\alpha} \rangle_s = \prod_k \frac{k^{r_k/2}}{(2\pi)^{r_k/2} r_k!} \sum_{\pi_k \in S_{r_k}} e^{-i \sum_{k,\alpha} k n_{k,\alpha} \theta_{k,\alpha}}. \]  

(73)

In order to obtain the fermionic partition function the wave function has to be antisymmetrized, namely\(^6\)

\[ \langle n_{k,\alpha} | \theta_{k,\alpha} \rangle_a = \prod_k \frac{k^{r_k/2}}{(2\pi)^{r_k/2} r_k!} \sum_{\pi_k \in S_{r_k}} (-1)^{N_{\pi_k}} e^{-i \sum_{k,\alpha} k n_{k,\alpha} \theta_{k,\alpha}}. \]  

(74)

The integration over the angles \( \theta_{k,\alpha} \) gives as a result a set of \( \delta \)-functions in the momenta \( n_{k,\alpha} \), whose structure in related to the cycles of \( \pi_k \), since it forces the momenta associated to the same cycle to coincide. In the end for each cycle of order \( h \) of \( \pi_k \) we have one integer momentum and the corresponding partition function is the one of QED on a torus of area \( h k t \), in complete agreement with the discussion of Section 3. The combinatorial factors are also easily checked. Let \( s_h(k) \) be the number of cycles of order \( h \) in \( \pi_k \), with \( \sum_{h=1}^{r_k} h s_h(k) = r_k \); then the total number of permutations in \( \pi_k \) with a given cycle decomposition is \( r_k!/(\prod_{h=1}^{r_k} s_h(k)!) \). By inserting this degeneracy into (73) and (72) we find for the bosonic case

\[ Z_N^b(t) = \sum_{\{r_k\}} \delta \left( \sum_{k=1}^{N} k r_k - N \right) \sum_{\{s_h(k)\}} \delta \left( \sum_{h=1}^{r_k} h s_h(k) - r_k \right) \]

\(^6\)We can replace the antisymmetrized wave function \( \langle n_{k,\alpha} | \theta_{k,\alpha} \rangle_a \) with the ratio \( \langle n_{k,\alpha} | \theta_{k,\alpha} \rangle_a / \prod_k \text{J}(k \theta_{k}) \), where \( \text{J}(k \theta_{k}) \) is the Vandermonde determinant for unitary matrices: \( \text{J}(k \theta_{k}) = \prod_{\alpha < \beta} 2 \sin((k \theta_{k,\alpha} - k \theta_{k,\beta})/2). \) The integration over \( \theta_{k,\alpha} \) should then be done with an integration volume \( \text{J}^2(k \theta_{k}) \prod_k k d \theta_{k,\alpha} \) for each \( k \), namely with the Haar measure of SU\((r_k)\) with invariant angles \( k \theta_{k,\alpha} \). This corresponds, in the trivial sector \( Q = 1 \), to fixing the holonomies at the edge of the cylinder and doing a group invariant integration when the two edges are sewn together (see for instance [23]).
In the fermionic case, besides a factor \((-1)^F\) with \(F = \sum_k r_k = \sum h_s_h(k)\) one has to introduce also a factor \((-1)^{\sum_k |\pi_k|} = (-1)^{\sum_{h,k}(h-1)s_h(k)}\), due to the antisymmetrization of the wave functions. Combining these two signs we have simply to insert \((-\sum_{r_k} s_h(k))\). We have then

\[
Z_N^f(t) = \sum_{\{r_k\}} \delta \left( \sum_{k=1}^N kr_k - N \right) \sum_{\{s_h(k)\}} \delta \left( \sum_{h=1}^N hs_h(k) - r_k \right) \times \prod_{k=1}^N \prod_{h=1}^r \frac{[-Z_{\text{QED}}(hkt)]^{s_h(k)}}{s_h(k)!h^{s_h(k)}}. \quad (76)
\]

The grand-canonical partition function is obtained by inserting (73) or (76) into

\[
Z^b,f(t, q) = \sum_{\{s_h(k)\}} Z_N^b(t) q^N. \quad (77)
\]

The sum over \(s_h(k)\), now unconstrained, can be performed and the result of the previous section is easily reproduced:

\[
Z^b,f(t, q) = \exp \left( \mp \sum_k \sum_n \log \left( 1 - q^k e^{-2\pi^2 ktn^2} \right) \right) = \prod_{kn} \left( 1 - q^k e^{-2\pi^2 ktn^2} \right)^{+1}. \quad (78)
\]

5 Concluding remarks

The analysis developed in the previous sections led us to a rather surprising conclusion: quantization of YM2 on a torus by using the unitary gauge and preserving all classical symmetries defines a theory that has a richer structure than the one obtained so far in the literature by using various gauges or lattice regularization. In the conventional formulation the partition function on the torus has been known for some time \([3, 4]\) and it has been given an interpretation in terms of \(N\) free fermions \([23, 23, 24]\) on a circle. The corresponding grand canonical ensemble coincides with the restriction to \(k = 1\) of the grand canonical partition function we obtain and that is given in
The new states with $k > 1$ introduced by our analysis are related to non-trivial permutations $P$ in the compact space direction and they are in one to one correspondence with the cycles of length $k$ in $P$ in complete analogy with the states introduced by [16] in the context of Matrix String Theory.

Naturally one would like to reproduce the same results in different gauges and understand, for instance, how the new $k > 1$ states appear in the gauge $\partial_1 A_1 = 0$ with $A_1$ diagonal. We do not have yet the full answer to this problem but we can point out some clues, and in one case a positive evidence, that the new set of states are required if one wants to preserve modular invariance, or in general invariance under discrete diffeomorphisms, in the quantization.

The first clue consists in the fact that the restriction to the $k = 1$ states corresponds, as we have seen, to a truncation of the full theory to one where the sum over all pairs of commuting permutations $(P, Q)$ is replaced by the sum over the subset of pairs of the form $(1, Q)$, which is clearly not a modular invariant subset.

On the other hand it would not be surprising if the gauge choice $\partial_1 A_1 = 0$, which unlike the unitary gauge is not manifestly modular invariant, turned out not to be the most convenient to reveal topological structures linked to non-trivial permutations on one of the cycles of the torus. In fact it is not even granted that $\partial_1 A_1 = 0$ is admissible, in the sense that it might project onto the trivial topological sector in one of the cycles of the torus\footnote{An example of this type is the gauge choice $A_1 = 0$, which is not admissible if the $x$ direction is compactified.}.

It is useful at this point to remember that the $(P, Q)$ sectors are related to the topological obstructions to a global smooth diagonalization of $F$ on the torus. There are other topological obstructions of the same type in the theory. Consider a Wilson loop that winds once around a cycle of the torus:

$$W_1(t) = P \exp \left\{ i \int_0^{2\pi} dx A_1(x, t) \right\} . \quad (79)$$

If the loop is moved once around the other cycle of the torus, its eigenvalues will in general undergo a permutation $Q$:

$$\text{diag } W_1(t + 2\pi) = Q^{-1} \text{diag } W_1(t) \, Q . \quad (80)$$

The same argument obviously applies to $W_0(x)$, when $x$ is increased of $2\pi$:

$$\text{diag } W_0(x + 2\pi) = P^{-1} \text{diag } W_0(x) \, P . \quad (81)$$
However $W_1(t)$ and $W_0(x)$ in general do not commute and they cannot be diagonalized simultaneously.

Consider now the theory on a cylinder with the dimension $x$ compactified and the edges in correspondence with $t = 0$ and $t = 2\pi$. The partition function of the torus is obtained by identifying $\text{diag} W_1(0)$ and $\text{diag} W_1(2\pi)$ up to a gauge transformation, namely, in the gauge $\partial_1 A_1 = 0$ with $A_1$ diagonal, by identifying $\text{diag} W_1(0)$ and $\text{diag} W_1(2\pi)$ up to an arbitrary permutation $Q$ of the eigenvalues. What about the sectors corresponding to the non trivial permutation $P$ of the eigenvalues of $W_0(x)$ (Eq.(81))? Have they been taken into account automatically by sewing the two ends of the cylinder with a group integration and a sum over all permutations $Q$? According to our discussion in the unitary gauge the answer to the last question is no. In fact it has been shown in the last section that in a sector where

$$\text{diag} F(x + 2\pi , t) = P^{-1} \text{diag} F(x , t) P$$

(82)

the independent eigenvalues of $W_1(0)$ and $W_1(2\pi)$ are in one to one correspondence with the cycles of $P$. More precisely the eigenvalues of $W_1$ corresponding to a cycle of $P$ of length $k$ are of the form

$$e^{i \frac{\phi}{r_k} + \frac{2\pi r}{k}} \ (r = 0, \cdots , k-1).$$

(83)

When sewing the ends of the cylinder, only eigenvalues corresponding to cycles of the same length can be identified, which is tantamount to restrict the permutation $Q$ to commute with $P$.

So in the unitary gauge the standard integration over the group manifold parametrized by the invariant angles of the holonomy projects onto the trivial sector $P = 1$ in the compactified direction of the cylinder. The sum over a complete set of states requires instead to consider a permutation $P$ and decompose it into cycles. Let $r_k$ be the number of cycles of length $k$ and $\phi^{(k)}_\alpha (\alpha = 1, \cdots , r_k)$ the invariant angles associated to each cycle. The identification of the eigenvalues when sewing the ends of the cylinder is done modulo permutations of the angles corresponding to the same length $k$ and the integration volume is not the one of $U(N)$ but rather of $U(r_1) \otimes U(r_2) \otimes U(r_3) \cdots$.

---

8This is the eigenvalue of a Wilson loop that winds $k$ times around the cylinder. All its eigenvalues associated to a cycle of length $k$ then coincide and the corresponding invariant angles are periodic of period $2\pi$.  

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It would be desirable to show that the same complete set of states is required in other gauges if one wants to include the configurations corresponding to a non-trivial $P$ in Eq. (81). This is technically not easy because $W_1$ and $W_0$ do not commute and the corresponding functional integrals are more involved. There is however one case in which $W_0$ and $W_1$ commute, namely the model with $t = 0$ (the BF theory) where the functional integral over $F$ leads to a $\delta[f(A)]$. This case is studied in Appendix C where it is shown that by a suitable gauge transformation

$$W_1 \rightarrow \text{diag} W_1(t) P , \quad W_0 \rightarrow \text{diag} W_0(x) Q ,$$

(84)

and that the eigenvalues of $\text{diag} W_1(t)$ (resp. $\text{diag} W_0(x)$) corresponding to the same cycle of $P$ (resp. $Q$) coincide. Moreover Eqs. (80) and (81) hold and $[P, Q] = 0$. The eigenvalues of $\text{diag} W_1(t) P$ and of $\text{diag} W_0(x) Q$ then follow the pattern of Eq. (83) and the sum over a complete set of states is done accordingly to the prescription discussed above.

Although $t = 0$ is a singular point, where the partition function becomes the volume of the moduli space of a flat connection, it is nevertheless important that the results obtained in the unitary gauge scheme are consistently reproduced in this case by diagonalizing the non contractable Wilson loops.

We remarked earlier about the close analogy between the spectrum obtained here and the states in Matrix string theory described in [16]. Moreover, just as in Matrix string theory, in the $t \rightarrow \infty$ limit the states with $n_i > 0$ decouple and we are left with the partition function of a conformal field theory.

In the present framework the states of the spectrum do not interact. The interaction may be implemented by allowing the gauge fixing matrix $M(\tau, x)$ introduced in appendix B to have branched points. Take for instance a square root branch point at $\tau = 0$ involving two eigenvalues $\lambda_i$ and $\lambda_j$. The two eigenvalues do not cross each other for $\tau < 0$, but they do for $\tau > 0$. So if $i$ and $j$ are contained in the same cycle (string) at $\tau < 0$ the cycle (string) will break into two for $\tau > 0$ (similar mechanisms are discussed in [16, 27, 28]). In general string interaction will be described by configurations where the eigenvalues live on higher genus Riemann surfaces which are branched coverings of the original torus. A different, although possibly related, problem is how to quantize YM2 on a surface with non vanishing curvature, namely how to consistently regularize the divergences appearing in Eq. (8), while
preserving the structure discovered in the case of the torus and invariance under discrete diffeomorphisms (which might amount to the same thing). This is an open problem whose solution, if it exists, might require a functional integration over all metrics, namely quantization of 2d gravity itself or alternatively a supersymmetric extension of the model.

Another open question concerns the relevance of the states with \( k > 1 \) to the large \( N \) limit, and in particular to the interpretation of YM2 as a string theory given by Gross and Taylor [14]. We remark that the large \( N \) limit originally introduced by ’t Hooft [13] and considered in [14] corresponds to scale \( t \) with \( N \) according to \( t = \tilde{t}/N \) with constant \( \tilde{t} \). In the large \( N \) limit \( t \) goes to zero and the partition function is dominated by the contribution from small cycles as discussed in Section 3. On the contrary the long string states, analogue to the Matrix string states of [16], are the leading contributions at large \( t \). Furthermore the scaling of \( t \) with \( N \) is not compatible with the grand canonical partition function formulation, which is the natural framework to describe the states of arbitrary length and requires summing over all \( N \) at fixed \( t \). All these considerations point to the fact that the string picture emerging from the analogy with the Matrix string theory is distinct from the one of Gross and Taylor, although it is possible that the two pictures are related by some strong-weak coupling duality\[9\]. The grand canonical formulation contains a new parameter \( \mu \), the chemical potential. This is reminiscent of the IKKT matrix model [20] where a sum over all matrix sizes is required to make contact with superstring theory. In conclusion the quantization of 2d Yang-Mills theory with \( U(N) \) gauge group seems naturally to lead to some more general underlying theory. It is possible then that the analogy with Matrix string theory of [16] is more than just a formal analogy, and that a deeper understanding of the stringy nature of 2d Yang-Mills theory may provide us with a deeper insight of Matrix theory as well.

Note added

Immediately after the first version of this paper was submitted to hep-th, the partition function of the DVV model in the IR limit was computed in [29] where the possibility that the computation might give the exact result

\[9\]For this duality to be apparent in our formulas it would be necessary to include string interaction to all orders.
was also pointed out. The result (eq. 2 in [29]) coincides exactly with the logarithm of our “bosonic” partition function $Z^b_N$ at fixed $N$ (see eq. (33)):

$$Z^{DVV}_N(t) = \log Z^b_N(t) = \sum_{h} \frac{1}{\hbar} \sum_{n} e^{-2\pi^2 h\hbar t n^2} = \sum_{h|N} e^{-2\pi^2 N\hbar t n^2}. \quad (a)$$

This nicely substantiates our concluding remark that the relation between $U(N)$ YM theory on a cylinder or torus and Matrix Strings is more than a formal analogy. The result (a) arises because the matter fields $X_i$ and $\psi_\alpha$ of the DVV model do not contribute, because of supersymmetry, to the partition function (see eq. (30) in [29]) which is then due entirely to the $U(N)$ gauge field. However, the structure of fermionic 0-modes is argued to effectively kill the “disconnected” contributions, i.e. those arising from configurations which in our language have more than one “connected block” of eigenvalues; this clearly accounts for the logarithm.

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A. Commuting permutations

The topological sectors described above are labelled by ordered pairs of commuting permutations. Therefore we need an explicit construction of all the permutations of \( N \) elements commuting with a given permutation \( P \).

Let \( r_k, (k = 1, \ldots, N), \) with \( \sum_k r_k = N \), be the number of cycles of length \( k \) in the permutation \( P \). Let us denote the elements of the set \( \{1, \ldots, N\} \) with a three–index notation based on how they transform under \( P \):

\[
a_{n}^{k,\alpha} \quad (k = 1, \ldots, N) \ (\alpha = 1, \ldots, r_k) \ (n = 1, \ldots, k)
\] (A.1)

is the \( n\)–th elements in the \( \alpha\)–th cycle of length \( k \). Therefore

\[
P(a_{n}^{k,\alpha}) = a_{n+1}^{k,\alpha},
\] (A.2)

where \( n + 1 \) is understood mod \( k \).

Let \( Q \) be a permutation commuting with \( P \) and consider its action on the cycle \( (a_{1}^{k,\alpha}, \ldots, a_{k}^{k,\alpha}) \) of \( P \). We have

\[
QP(a_{n}^{k,\alpha}) = Q(a_{n+1}^{k,\alpha})
\] (A.3)

and therefore, using \( PQ = QP \),

\[
P \left( Q(a_{n}^{k,\alpha}) \right) = Q(a_{n+1}^{k,\alpha}),
\] (A.4)

which means that \( (Q(a_{1}^{k,\alpha}), \ldots, Q(a_{k}^{k,\alpha})) \) is a cycle of length \( k \) in \( P \). Hence, there exists a permutation \( \pi = S_{r_{k}} \) of \( r_{k} \) elements such that the following equality between cycles holds

\[
(Q(a_{1}^{k,\alpha}), \ldots, Q(a_{k}^{k,\alpha})) = (a_{1}^{k,\pi_{k}(\alpha)}, \ldots, a_{n}^{k,\pi_{k}(\alpha)}).
\] (A.5)

This implies that there exist \( r_{k} \) integers

\[
s(k, \alpha) \ (\alpha = 1, \ldots, r_k) \ (1 \leq s(k, \alpha) \leq k)
\] (A.6)

such that

\[
Q(a_{n}^{k,\alpha}) = a_{n+s(k,\alpha)}^{k,\pi_{k}(\alpha)},
\] (A.7)

where \( n + s(k, \alpha) \) is understood mod \( k \).

We have thus shown that a permutation \( Q \), commuting with \( P \), is completely determined by assigning for each \( k = 1, \ldots, N \)
• a permutation $\pi_k \in S_{r_k}$, where $r_k$ is the number of cycles of length $k$
in $P$;
• a set of $r_k$ integers $1 \leq s(k, \alpha) \leq k$.

The permutation $Q$ is then defined by Eq. $(A.7)$. This shows in particular
that the number of permutations $Q$, commuting with $P$, is

$$|C(P)| = \prod_k r_k! k^{r_k}.$$  \hspace{1cm} (A.8)

### B. BRST formalism

We develop in this appendix the BRST formalism for YM2 on a torus
in the Unitary gauge and discuss how the non trivial sectors considered in
Section 3 arise in this context. The first order action introduced in $(\text{II})$ is
invariant under gauge transformations

$$\delta A = d\epsilon - i[A, \epsilon] ,$$
$$\delta F = -i[F, \epsilon] . \hspace{1cm} (B.1)$$

Correspondingly, the BRST and anti-BRST transformations are given by

$$sA = dc - i[A, c] ; \hspace{1cm} \bar{s}A = d\bar{c} - i[A, \bar{c}] ,$$
$$sF = -i[F, c] ; \hspace{1cm} \bar{s}F = -i[F, \bar{c}] ,$$
$$sc = icc ; \hspace{1cm} \bar{s}\bar{c} = i\bar{c}\bar{c} ,$$
$$s\bar{c} = icc + b ; \hspace{1cm} \bar{s}c = i\bar{c}c - b ,$$
$$sb = icb ; \hspace{1cm} \bar{s}b = i\bar{c}b , \hspace{1cm} (B.2)$$

where all fields are hermitian $N \times N$ matrices. In order to fix the gauge let
us introduce a matrix $M(\tau, x)$ and add to the action a BRST and anti-BRST
invariant term of the type

$$S_{g.f} = \int_0^{2\pi} d\tau dx \text{ tr } s\bar{s}(MF) \hspace{1cm} (B.3)$$

which, using the BRST transformations $(B.2)$, takes the form

$$S_{g.f} = \int_0^{2\pi} d\tau dx \text{ tr } (Mc\bar{c} - M\bar{c}Fc + M\bar{c}cF - MccF + ib[F, M]) . \hspace{1cm} (B.4)$$
The functional integration over the auxiliary field \( b \) leads to a \( \delta \)-function of argument \([F, M]\), which implies that in the base of the eigenvectors of \( M \) the matrix \( F \) is diagonal and the unitary gauge is implemented. It is convenient therefore to rewrite Eq. (B.4) in the base of the eigenvectors of the gauge-fixing matrix \( M \). As already discussed in Section 3, when the non trivial sectors were introduced, matrices on a torus can be divided into classes characterised by a pair of commuting permutations \((P, Q)\). They are the permutations of the eigenvalues obtained if we go round the non contractable loops \((a, b)\) that generate the fundamental group of the torus\(^{10}\). It easy to see that a gauge fixing matrix \( M \) belonging to the class \((P, Q)\) defines a functional integral over field configurations of the sector \((P, Q)\) defined in Section 3. In fact if we denote by \( \alpha_i(\tau, x) \) the eigenvalues of \( M \), we have

\[
\alpha_i(\tau + 2\pi, x) = \alpha_{P(i)}(\tau, x) , \quad \alpha_i(\tau, x + 2\pi) = \alpha_{Q(i)}(\tau, x) ,
\]  
(B.5)

and the same boundary conditions are obeyed by all the other field in the base of the eigenvectors of \( M \). In this base the BRST invariant action can be explicitly written as

\[
S_{\text{BRST}} = \int_0^{2\pi} d\tau dx \sum_{i,j} \left[ i b_{ij}(\alpha_i - \alpha_j) F_{ji} + (\lambda_i - \lambda_j)(\alpha_i - \alpha_j) c_{ij} \tilde{c}_{ji} + \right.
\]

\[
+ (\lambda_i - \lambda_j) A_{0,ij}^i A_{1,ji}^i \right]
\]

\[
+ \int_0^{2\pi} d\tau dx \sum_i \left[ \frac{1}{2} \lambda_i^2 - i \lambda_i (\partial_0 A_{1}^i(\tau) - \partial_1 A_{0}^i(\tau)) \right] ,
\]  
(B.6)

where the diagonal elements of \( F \) have been denoted \( \lambda_i \) and the diagonal elements of \( A_\mu \) by \( A_\mu^{(i)} \). Besides, as already mentioned, all fields appearing in (B.4) satisfy the same boundary condition (B.5) as \( \alpha_i \). It is clear that the unitary gauge condition \( F_{ji} = 0 \) for \( i \neq j \) is implemented by the functional integral over \( b_{ij} \) and BRST invariance ensures that the dependence from the eigenvalues \( \alpha_i \) of the gauge fixing matrix \( M \) cancel, as it can be seen by performing explicitly the functional integration over both \( b_{ij} \) and the ghost anti-ghost fields. The supersymmetry (7), suitably modified\(^{11}\), on the other

\(^{10}\) Clearly the assumption of continuity \( M \) and of its first derivatives must be made here.

\(^{11}\) The factor \((\alpha_i - \alpha_j)\) in the term containing the ghost fields should be absorbed by redefining \( c \) and \( \tilde{c} \) to reproduce (7).
hand, ensures that the Vandermonde determinants $\Delta(\lambda)$ coming from the integration over the ghost system are exactly canceled by the result of the integration over the non diagonal part of $A_\mu$.

Finally it should be noticed that the gauge has not been completely fixed in (B.6), the action being still invariant under a local U(1) symmetry for each eigenvalue $\lambda_i$. Correspondingly the diagonal part of $c, \bar{c}$ and $b$ do not appear in (B.6).

In conclusion the sectors described in Section 3 are generated by gauge fixing condition which are not connected by smooth variations of the the gauge fixing matrix $M$, hence BRST invariance does not fix the relative weight of the different sectors in the partition function.

C. Topological obstructions in the BF model

Let us consider the action (1) with $t = 0$. The functional integral over $F$ produces a $\delta(f(A))$ which has the solution:

$$A_\mu(x, \tau) = ig^{-1}(x, \tau)\partial_\mu g(x, \tau) . \quad (C.1)$$

Consider now the non contractable Wilson loops

$$W_0(x, \tau) = P \exp \left\{ i \int_\tau^{\tau+2\pi} A_0(x, t)dt \right\} = g^{-1}(x, \tau)g(x, \tau + 2\pi) , \quad (C.2)$$

$$W_1(x, \tau) = P \exp \left\{ i \int_x^{x+2\pi} A_1(y, \tau)dy \right\} = g^{-1}(x, \tau)g(x + 2\pi, \tau) \quad (C.2)$$

As $A_\mu(x, \tau)$ are defined globally on the torus, namely they are periodic in both variables, it follows from (C.2) that $W_0(x, \tau)$ and $W_1(x, \tau)$ are also periodic in both $x$ and $\tau$. By using the explicit form of $W_0(x, \tau)$ and $W_1(x, \tau)$ in terms of $g(x, \tau)$ one easily finds

$$[W_0(x, \tau), W_1(x, \tau)] = 0 . \quad (C.3)$$

Notice also that from the definition above we have

$$g(x, \tau + 2\pi) = g(x, \tau)W_0(x, \tau) ,$$

$$g(x + 2\pi, \tau) = g(x, \tau)W_1(x, \tau) . \quad (C.4)$$
Due to Eq. (C.3) it is always possible to find \textit{locally} a unitary transformation $U(x, \tau)$ that diagonalizes both $W_0(x, \tau)$ and $W_1(x, \tau)$:

$$W_0(x, \tau) = U^{-1}(x, \tau)w_0(x)U(x, \tau),$$
$$W_1(x, \tau) = U^{-1}(x, \tau)w_1(\tau)U(x, \tau),$$

(C.5)

where $w_0(x)$ and $w_1(\tau)$ are the diagonal matrices displaying the eigenvalues of $W_0(x, \tau)$ and $W_1(x, \tau)$. Notice that the eigenvalues of $W_0(x, \tau)$ are only function of $x$ and the ones of $W_1(x, \tau)$ only of $\tau$; in fact it follows from the definition (C.2) that for instance $W_0(x, \tau')$ is related to $W_0(x, \tau)$ by a unitary transformation. There are in general topological obstructions to a global and smooth diagonalization of $W_0(x, \tau)$ and $W_1(x, \tau)$; as a result $w_0(x + 2\pi)$ and $w_1(\tau + 2\pi)$ will coincide with $w_0(x)$ and $w_1(\tau)$ only up to an element of the Weyl group, namely, for $U(N)$, up to a permutation:

$$w_0(x + 2\pi) = Pw_0(x)P^{-1}, \quad w_1(\tau + 2\pi) = Qw_1(\tau)Q^{-1}.$$  

(C.6)

Due to the periodicity of $W_0(x, \tau)$ and $W_1(x, \tau)$ we must have also

$$U(x + 2\pi, \tau) = PU(x, \tau), \quad U(x, \tau + 2\pi) = QU(x, \tau),$$

(C.7)

which entails

$$[P, Q] = 0.$$  

(C.8)

It follows from (C.5) that the eigenvalues of $w_0(x)$ and $w_1(\tau)$ are not all independent. In fact if we shift $\tau$ (resp. $x$) by $2\pi$ in the first (resp. second) of Eqs (C.3) we obtain from Eq. (C.6) and the periodicity of $W_0(x, \tau)$ (resp $W_1(x, \tau)$) the following constraints:

$$w_1(\tau) = P^{-1}w_1(\tau)P; \quad w_0(x) = Q^{-1}w_0(x)Q.$$  

(C.9)

These conditions are satisfied if all the eigenvalues of $w_1(\tau)$ which are mapped into each other by $P$ coincide; in other words, the eigenvalues of $w_1(\tau)$ are associated to the cycles of $P$. The same applies to $w_0(x)$ and $Q$. We want to remark at this point that the unitary transformation (C.5) is \textit{not} a gauge transformation. In fact the correct gauge transformation of $W_0(x, \tau)$ and $W_1(x, \tau)$ with the unitary matrix $U(x, \tau)$ is given by

$$U(x, \tau)W_0(x, \tau)U^{-1}(x, \tau + 2\pi) = w_0(x)Q^{-1},$$
$$U(x, \tau)W_1(x, \tau)U^{-1}(x + 2\pi, \tau) = w_1(\tau)P^{-1}.$$  

(C.10)
The correct gauge transformed Wilson loops are then $w_0(x)Q^{-1}$ and $w_1(\tau)P^{-1}$ rather than $w_0(x)$ and $w_1(\tau)$. If we diagonalize one of them, say $w_0(x)Q^{-1}$, by a constant gauge transformation $^{12}$ then we may conclude that the eigenvalues of the gauge transformed Wilson loop have the form given in (8.3) for each cycle of $Q$. The implications of this can be better understood from the following example. Consider the theory on a cylinder with compactified dimensions $x$ and $\tau$ ranging between 0 and $2\pi$. Suppose we identify the states with the holonomies on the boundaries, namely with the configurations of $w_1(\tau)$ that satisfy the first of Eq.s (C.9) with $P = 1$. We can obtain the torus by sawing the two ends of the cylinder, that is by identifying the holonomies at the ends up to a permutation $Q$ of the eigenvalues according to the second of Eq.s (C.6). The sum over $Q$ implies that in the channel obtained by cutting the torus at constant $x$ we recover the whole spectrum of states satisfying the constraints (C.9) with arbitrary $Q$.

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