Deviations from the Area Law for Supersymmetric Black Holes

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ABSTRACT

We review modifications of the Bekenstein-Hawking area law for black hole entropy in the presence of higher-derivative interactions. In four-dimensional \(N = 2\) compactifications of string theory or M-theory these modifications are crucial for finding agreement between the macroscopic entropy obtained from supergravity and the microscopic entropy obtained by counting states in string or M-theory. Our discussion is based on the effective Wilsonian action, which in the context of \(N = 2\) supersymmetric theories is defined in terms of holomorphic quantities. At the end we briefly indicate how to incorporate non-holomorphic corrections.

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1 Introduction

It is one of the most intriguing properties of black holes in general relativity that one can derive a set of laws, called the laws of black hole mechanics, which are formally equivalent to the laws of thermodynamics [1]. For instance, the first law of thermodynamics, which relates variations of the internal energy and the work done to the variation of the entropy, has its counterpart in black hole mechanics. The first law for black holes in general relativity relates the variation of the mass and angular momentum of the black hole to the change in the area of its event horizon. This then leads to the celebrated Bekenstein-Hawking area law [2], which expresses the black hole entropy in terms of the horizon area.

The first law is quite a remarkable result as it relates global quantities of the black hole, such as its energy or mass and its angular momentum, which can be entirely determined from the behaviour of the fields at spatial infinity, to the horizon area which is defined at the inner boundary of the black hole solution. Also, the connection with thermodynamics suggests a possible interpretation of the entropy in terms of microstates. Such an interpretation has recently been provided in the context of string theory [3].

From the point of view of the field theory, be it fundamental or effective, it is rather surprising that variations near the outer boundary at spatial infinity are related to variations near the inner boundary at the horizon. Moreover, an effective field theory action will contain more than just the standard Einstein-Hilbert term and will depend on higher derivatives of the fields. So one may wonder whether there could be a natural principle that explains the behaviour implied by the first law of black hole mechanics for more generic field theories. Such a principle can be provided by making use of the concept of a surface charge, and a specific proposal for this charge was put forward by Wald [4]. The surface charge is related to the conventional Noether current. In a certain sense there is no Noether current associated with a local gauge symmetry, but there exists a current associated with the residual invariance of a background configuration. When evaluating this current subject to the field equations, current conservation becomes trivial and the current takes the form of an improvement term, i.e., the derivative of an antisymmetric tensor. This antisymmetric tensor, sometimes called the Noether potential, can be written down for arbitrary gauge parameters. Integration of this potential over the boundary of some (spatial) hypersurface leads to a surface charge, which, when restricting the gauge transformation parameters to those that leave a certain background invariant, is equal to the Noether charge in the usual sense. Variations of this surface charge that continuously connect solutions of the equations of motion thus relate the various surface contributions and this is how eventually the first law of black hole mechanics follows (this is, for example, reviewed in [4, 5, 6, 7]).

In this approach the entropy is related to the integral of the Noether potential over the horizon. The entropy defined in this way is a local geometric quantity and equals the Bekenstein-Hawking entropy (i.e., one fourth of the horizon area) for Einstein gravity. But for more general actions containing terms of higher order in derivatives the entropy formula contains additional terms. Since the low-energy effective action constitutes the macroscopic or ‘thermodynamic’ level of description of quantum gravity, the laws of black hole mechanics have to hold for such effective actions if they are more than accidental analogies.
Naturally, since string theory is a candidate for a theory of quantum gravity, one expects to be able to describe black holes also at the microscopic or ‘statistical mechanics’ level. Progress into this direction has been made after nonperturbative dualities, D-branes and the M-theory description of string theory were discovered (see, for example, [8, 9] for a review). In particular, quantitative agreement has been found between macroscopic black hole entropies extracted from supergravity solutions and microscopic entropies calculated by counting excitations of D-branes and M-branes. A crucial ingredient in establishing agreement is extended supersymmetry, because one needs to interpolate between different regimes of the theory, such as the low-energy regime and the regime of small string coupling. This interpolation is possible for extremal black holes which are BPS solitons of extended supergravity.

The resulting expressions for the entropy concern extremal charged black holes and depend exclusively on the electric and magnetic charges. These results are obtained in cases where (some of) these charges are large, but sometimes it is also possible to evaluate some of the subleading corrections. The comparisons between macroscopic and microscopic entropy have mostly been made in situations where the distinction between the Bekenstein-Hawking area law and more general definitions of macroscopic black hole entropy are immaterial. Only recently has it become apparent that microscopic entropy formulae contain subleading corrections which on the macroscopic level are due to higher-derivative interactions in the field theory [10, 11, 12]. In this case the correct macroscopic definition of entropy is crucial for finding agreement, as we will review below.

The structure of this paper is as follows. First we will review the derivation of the Noether potential for Yang-Mills theories and for gravity in the context of field theories with higher derivatives. Then we will discuss the laws of black hole mechanics and certain key notions of black hole physics in the framework of general effective Lagrangians. Particular emphasis will be put on Wald’s derivation of the first law and the related definition of black hole entropy as a Noether charge. Then we recall the special features of extremal black holes and their relation to supersymmetry. We briefly describe how special geometry encodes the couplings of vector multiplets to \( \mathcal{N} = 2 \) supergravity in the presence of a certain class of higher-order derivative terms proportional to the square of the Weyl tensor. This part of the discussion is based on the effective \( \mathcal{N} = 2 \) Wilsonian action, which is defined in terms of a holomorphic quantity. Then we review our work on the entropy of extremal \( \mathcal{N} = 2 \) black holes in the presence of these higher-derivative curvature terms, which uses a definition of the macroscopic entropy that deviates from the area law. We conclude with a few examples of black hole solutions arising in string theory compactifications. By recalling results of microscopic entropy calculations in the context of string and M-theory compactifications, we show that they perfectly agree with macroscopic entropy calculations based on Wald’s definition of macroscopic entropy. We also briefly discuss how to incorporate non-holomorphic corrections to the effective Lagrangian into macroscopic entropy formulae.

2 Noether potential and charge

For any local symmetry there exists a globally-defined Noether potential, denoted here by an antisymmetric tensor \( Q^{\mu\nu}(\phi, \xi) \), which is a local function of the fields and of the gauge transfor-
mation parameters, here generically denoted by $\phi$ and $\xi$ respectively. To derive this potential, one starts from a gauge-invariant action and one derives the Noether current as if one is dealing with a (rigid) residual symmetry associated with a certain background field configuration. To illustrate this we will first present the case of a Yang-Mills theory as a pedagogical example, with a gauge-invariant Lagrangian $L(F_{\mu\nu}, \nabla_{\rho}F_{\mu\nu}, \psi, \nabla_{\mu}\psi)$ depending on the field strengths $F_{\mu\nu}$, matter fields $\psi$ and first derivatives thereof. We begin by multiplying the gauge transformations with a test function $\epsilon(x)$. This test function as well as its derivatives will satisfy certain boundary conditions, such that the variation of the action is proportional to the field equations, in accord with a modified (in view of the higher-derivative interactions) version of Hamilton’s variational principle. Hence we have

$$
\begin{align*}
\delta A_\mu &= \epsilon(x) \nabla_\mu \xi(x), \\
\delta \psi &= \epsilon \psi, \\
\delta \nabla_\mu \psi &= \partial_\mu \epsilon \psi + \epsilon \nabla_\mu \psi.
\end{align*}
$$

Here the gauge field, the field strength and the transformation parameters $\xi$ are written as Lie-algebra valued quantities in the representation relevant for $\psi$. The explicit variation of the action now leads directly to the current (using that $\epsilon$ and its first derivative vanish at the boundary) and one finds (we suppress the gauge-invariant inner product notation),

$$
J^\mu = 2 L^{\mu\nu} \nabla_\nu \xi - 2 \nabla_\rho L^{(\rho,\mu)\nu} \nabla_\nu \xi + 2 L^{(\rho,\mu)\nu} \nabla_\nu \nabla_\rho \xi - L^{\mu,\rho\sigma}[F_{\rho\sigma}, \xi] + L^{\mu} \xi \psi,
$$

where

$$
L^{\mu\nu} = \frac{\partial L}{\partial F_{\mu\nu}}, \quad L^{(\rho,\mu)\nu} = \frac{\partial L}{\partial \nabla_\rho F_{\mu\nu}}, \quad L^\mu = \frac{\partial L}{\partial \nabla_\mu \psi}.
$$

Observe that the Bianchi identity implies $L^{[\rho,\mu\nu]} = 0$. The equations of motion for the gauge fields take the form

$$
-2 \nabla_\mu L^{\mu\nu} + 2 \nabla_\mu \nabla_\rho L^{\rho,\mu\nu} + L^{\mu,\rho\sigma}[F_{\rho\sigma}, \cdot] - L^{\nu} \psi = 0.
$$

We refrain from giving the matter field equation for $\psi$. By virtue of the combined field equations one can verify that this current is conserved.

However, one can also use the field equation (4) directly on the current so as to obtain

$$
J^\mu = \partial_\nu Q^{\mu\nu},
$$

where $Q^{\mu\nu}$ is the Noether potential, which in the case at hand takes the form

$$
Q^{\mu\nu} = 2 L^{\mu\nu} \xi - 2 \nabla_\rho L^{(\rho,\mu)\nu} \xi + L^{(\rho,\mu)\nu} \nabla_\rho \xi,
$$

and is thus a local function of the fields and of the transformation parameter. Obviously the fact that the Noether current is conserved is now trivial. Nevertheless the Noether potential is still tied directly to the invariant action, up to terms that are exact, i.e., that can be written as $\partial_\rho X^{\mu\nu\rho}$, with $X^{\mu\nu\rho}$ a totally antisymmetric tensor, and up to improvement terms that vanish in the symmetric background (see below). From the Noether potential one can determine the charge by integrating over a closed hypersurface of co-dimension two, in which case the ambiguity

\[d\text{Note that the variations are defined such that we do not differentiate with respect to independent field components. Specifically, the variation of the Lagrangian takes the form } \delta L = L^{\mu\nu} \delta F_{\mu\nu} + \cdots.\]
drops out. For a spacelike hypersurface we then determine the Noether charge in the usual sense, but written as a surface integral. This charge is associated with the residual gauge symmetry corresponding to parameters that satisfy $\nabla_\mu \xi = 0$ in the background. Here we observe that we have used a somewhat different algorithm for computing (2) than the more conventional one used, for instance, in [4, 5, 6]. Compared with the latter, this reflects itself in the presence of an additional improvement term in (2) equal to $\partial_\nu (L^\rho{}_{\mu\nu} \nabla_\rho \xi)$, and hence in the presence of the last term in (6). We thus see that both approaches yield the same Noether potential up to a term which vanishes in the symmetric background.

Here we have assumed that the Lagrangian is gauge invariant, which implies that the Noether current is proportional to the symmetry variations of the fields (as one can for instance verify directly for (2)). Consider now the situation where we are dealing with a continuous variety of solutions of the field equations which are left invariant under a corresponding continuous variety of residual gauge transformations. The Noether current associated with these residual transformations is then vanishing for every one of these solutions, from which it follows that, under changes of the solution (and the corresponding symmetry parameters $\xi$), the Noether potential must vary into a term that is exact, i.e.,

$$\delta Q^{\mu\nu} = \partial_\mu \omega^{\mu\nu\rho},$$

where $\omega$ is a totally antisymmetric tensor. This implies that the value of the integral of the Noether potential over the boundary of a hypersurface of co-dimension one must be constant and equal to zero for all these solutions, provided that the fields are regular on the hypersurface. While this example is therefore not so interesting, the situation changes when the Lagrangian is only invariant modulo a total derivative, because then the current does in general not vanish in symmetric background solutions. For Yang-Mills theory this happens, for example, when the Lagrangian contains Chern-Simons terms.

For the case of gravity with general coordinate invariance, the Lagrangian is never invariant under general coordinate transformations, so here the current will not necessarily vanish in a symmetric background. So let us follow the same procedure for gravity and construct the Noether potential, starting from an invariant action depending on the Riemann curvature, and on a matter field $\psi_{\mu\nu}$ (with no particular symmetry) and its first derivative. After multiplying the diffeomorphisms with a test function $\epsilon(x)$ with appropriate boundary conditions, the transformation rules read

$$\delta g_{\mu\nu} = -\epsilon(x) \left( \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) \right),$$
$$\delta \psi_{\mu\nu} = -\epsilon(x) \left( \nabla_\mu \xi^\sigma \psi_{\sigma\nu} + \nabla_\nu \xi^\sigma \psi_{\mu\sigma} + \xi^\sigma \nabla_\sigma \psi_{\mu\nu} \right).$$

We note the following useful equations for variations of connections and curvatures,

$$\delta \Gamma^\sigma_{\nu\rho} = \frac{1}{2} g^\sigma{}_{\lambda} \left[ \nabla_\nu \delta g_{\lambda\rho} + \nabla_\rho \delta g_{\nu\lambda} - \nabla_\lambda \delta g_{\nu\rho} \right],$$
$$\delta R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + 2 \nabla_{[\mu} \nabla_{\nu]} \delta g_{\rho\sigma]}.\]$$

The variation of the action, assuming the boundary conditions for $\epsilon$, now yields the current

$$J^\mu = \xi^\mu L - 2 L^{\mu\nu\rho\sigma} \left[ R_{\lambda\nu\rho\sigma} \xi^\lambda + \nabla_\nu \nabla_\rho \xi_\sigma \right] + 4 \nabla_\mu L^{\mu\nu\rho\sigma} \nabla_\nu \xi_\sigma$$
\[-L^{\mu,\rho\sigma}_{\psi} \left[ \nabla_{\mu} \xi^\lambda \psi_{\lambda\sigma} + \nabla_{\sigma} \xi^\lambda \psi_{\mu\lambda} + \xi^\lambda \nabla_{\lambda} \psi_{\rho\sigma} \right] + \frac{1}{2} (\nabla_{\lambda} \xi_{\rho} + \nabla_{\rho} \xi_{\lambda}) \left[ L^{\mu,\rho\sigma}_{\psi} \psi_{\lambda\sigma} + L^{\mu,\sigma\rho}_{\psi} \psi_{\lambda\sigma} + L^{\rho,\sigma\mu}_{\psi} \psi_{\lambda\sigma} \right. \\
+ \left. L^{\rho,\sigma\mu}_{\psi} \psi_{\lambda\sigma} - L^{\rho,\lambda\sigma}_{\psi} \psi_{\mu\sigma} - L^{\rho,\sigma\mu}_{\psi} \lambda_{\sigma\mu} \right], \] (10)

where
\[ L^{\mu\nu} = \frac{\partial L}{\partial g_{\mu\nu}}, \quad L^{\mu\nu\rho\sigma} = \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}, \quad L^{\rho,\mu\nu}_{\psi} = \frac{\partial L}{\partial \nabla_{\rho} \psi_{\mu\nu}}. \] (11)

Observe that \( L^{\mu\nu\rho\sigma} \) is antisymmetric in \([\mu\nu]\) and in \([\rho\sigma]\); furthermore it is symmetric under pair exchange, \( L^{\mu\nu\rho\sigma} = L^{\rho\sigma\mu\nu} \), and satisfies the cyclicity property \( L^{\mu\nu\rho\sigma} = 0 \).

The current (10) is conserved by virtue of the equations of motion for the metric,
\[
\frac{1}{2} g^{\mu\nu} L + L^{\mu\nu} + L^{\rho\sigma\lambda\mu} R_{\rho\sigma\lambda\mu} - 2 \nabla_{(\rho} \nabla_{\sigma)} L^{\mu\nu\rho\sigma} \\
+ \frac{1}{4} \nabla_{\lambda} \left[ L^{\lambda,\mu\rho}_{\psi} \psi_{\lambda\rho} + L^{\lambda,\rho\mu}_{\psi} \psi_{\lambda\rho} + L^{\mu,\rho\lambda}_{\psi} \psi_{\lambda\rho} \right. \\
+ \left. L^{\mu,\lambda\rho}_{\psi} \psi_{\lambda\rho} - L^{\mu,\rho\lambda}_{\psi} \psi_{\lambda\rho} - L^{\mu,\rho\lambda}_{\psi} \psi_{\lambda\rho} + (\mu \leftrightarrow \nu) \right] = 0, \] (12)

and of the equation of motion for \( \psi_{\mu\nu} \),
\[
\nabla_{\rho} L^{\rho,\mu\nu}_{\psi} - L^{\mu\nu}_{\psi} = 0, \] (13)

where
\[ L^{\mu\nu}_{\psi} = \frac{\partial L}{\partial \psi_{\mu\nu}}. \] (14)

as well as of the general covariance of the Lagrangian. The latter implies
\[
2 L^{\mu\nu} \nabla_{\mu} \xi_{\nu} + 4 L^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \nabla_{\lambda} \xi_{\lambda} + L^{\mu\nu}_{\psi} \left( \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} \right) + L^{\rho,\mu\nu}_{\psi} \left( \nabla_{\rho} \psi_{\mu\nu} + \nabla_{\mu} \psi_{\rho\nu} + \nabla_{\nu} \psi_{\rho\mu} \right) - \partial_{\rho} \psi_{\nu \rho} = 0, \] (15)

which in turn gives rise to the identity
\[
2 L^{\mu\nu}_{\xi_{\nu}} - 4 L^{\mu\nu\rho\sigma} R_{\rho\sigma\nu} \xi_{\lambda} + L^{\mu\nu}_{\psi} \xi_{\sigma} \nabla_{\rho} \psi_{\sigma\nu} + L^{\mu\nu}_{\psi} \xi_{\sigma} \nabla_{\nu} \psi_{\sigma\rho} \\
+ L^{\rho,\mu\nu}_{\psi} \xi_{\sigma} \nabla_{\rho} \psi_{\sigma\nu} + L^{\rho,\mu\nu}_{\psi} \xi_{\sigma} \nabla_{\nu} \psi_{\rho\sigma} + L^{\rho,\mu\nu}_{\psi} \xi_{\sigma} \nabla_{\rho} \psi_{\sigma\nu} = 0. \] (16)

Imposing the equations of motion (12) and (13) and the relation (16), the current takes the form \( J^{\mu} = \nabla_{\nu} Q^{\mu\nu} \) with the Noether potential equal to
\[
Q^{\mu\nu} = -2 L^{\mu\nu\rho\sigma} \nabla_{\rho} \xi_{\sigma} + 4 \nabla_{\rho} L^{\mu\nu\rho\sigma} \xi_{\sigma} \\
+ \frac{1}{2} \left[ - L^{\mu,\nu\rho}_{\psi} \psi_{\sigma \rho} - L^{\mu,\rho\nu}_{\psi} \psi_{\nu \rho} - L^{\mu,\sigma\rho}_{\psi} \psi_{\nu \rho} \\
+ L^{\sigma,\mu\rho}_{\psi} \psi_{\nu \rho} + L^{\sigma,\rho\mu}_{\psi} \psi_{\nu \rho} + L^{\sigma,\rho\mu}_{\psi} \psi_{\nu \rho} - (\mu \leftrightarrow \nu) \right] \xi_{\sigma}. \] (17)

When spacetime exhibits an isometry with a corresponding Killing vector \( \xi^\mu \), so that
\[
\nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} = 0, \] (18)
the corresponding Noether potential is only proportional to \( \xi^\mu \) and to its curl \( \nabla_{[\mu} \xi_{\nu]} \) in view of the identity
\[
\nabla_{\mu} \xi_{\nu} = R_{\nu \rho \mu}^{\sigma} \xi_{\sigma}. \] (19)

The Noether charge associated with the isometry is given as the integral of the Noether potential over the boundary of a spatial hypersurface. The variation of the charge under infinitesimal variations of the background solution will be discussed in the following section.
As was shown by Wald one can employ the Noether charge in order to find generalized definitions of the black hole entropy that are consistent with the first law of black hole mechanics. Although most of the following applies to black holes in any number of spacetime dimensions we will restrict ourselves to four dimensions. As we discussed already in the introduction, the description in terms of a surface charge can in principle explain how variations of the fields subject to the equations of motion at the horizon can be expressed in terms of the variations of quantities that are defined at spatial infinity. The crucial observation is that there exists a Hamiltonian, whose change under a variation of the fields can be expressed in terms of the corresponding change of the Noether charge and takes the following form

\[ \delta H = \delta \left( \int_{\mathcal{C}} d\Omega_{\mu} J^\mu \right) - \int_{\mathcal{C}} d\Omega_{\mu} \nabla_\mu (\xi^\mu \theta^\nu - \xi^\nu \theta^\mu), \]

(20)

where we integrate over a Cauchy surface \( \mathcal{C} \) with volume element \( d\Omega_{\mu} \), \( J^\mu \) is the Noether current associated with a particular Killing vector \( \xi^\mu \) (to be discussed below) and \( \theta^\mu \) is defined by the surface integral that one obtains when considering the change of the action under the field variation,

\[ \delta S = \int d^4x \partial_\mu (\sqrt{-g} \theta^\mu), \]

(21)

after subsequently imposing the field equations. We assume that the Killing vector field is timelike so that \( H \) can be associated with a Hamiltonian that governs the evolution along the integral timelike lines of \( \xi^\mu \).

For black holes, \( \xi^\mu \) is a horizon-generating Killing field, meaning that it is the normal vector of a null hypersurface, called the Killing horizon. The Cauchy surface in \( \mathcal{C} \) is chosen to extend from spatial infinity down to the Killing horizon where the Killing field turns lightlike. In Einstein gravity it can be shown that under certain assumptions, such as that the dominant-energy condition holds and that the matter field equations of motion have a well defined Cauchy problem, all event horizons are Killing horizons, but in more general theories this is not obvious. For static black holes, the event horizon is always a Killing horizon, irrespective of the precise form of the action. In that case the relevant Killing field is just the static Killing vector field. In the following we shall always assume that we are dealing with a Killing horizon.

When imposing the equations of motion, \( \mathcal{C} \) takes the form of surface integrals over the boundary \( \partial \mathcal{C} \) with surface element \( d\Sigma_{\mu\nu} \),

\[ \delta H = \int_{\partial \mathcal{C}} d\Sigma_{\mu\nu} \left( \delta Q^{\mu\nu} - \xi^\mu \theta^\nu + \xi^\nu \theta^\mu \right). \]

(22)

Here it is important that the last two terms are proportional to the Killing vector and not to its curl. Furthermore one assumes that these terms can be rewritten as variations of some (not necessarily globally defined) quantity. Because one can prove that \( \delta H = 0 \) (but not that \( H \) itself vanishes) whenever \( \xi^\mu \) is a Killing vector characterizing an invariance of the full background, it follows that the sum of the surface integrals in \( \mathcal{C} \) has to vanish! If one identifies, up to a certain proportionality factor, the resulting variations of the surface integrals at infinity with the mass and angular momentum variations that one has in the first law, then the surface integral
at the horizon defines the variation of the entropy. Hence the mass, the angular momentum and the entropy are all surface charges derived from the same current, and the entropy takes the form of an integral over the Noether potential \[4\],

\[
S = -\pi \int_{\Sigma_{\text{hor}}} Q^{\mu \nu} \epsilon_{\mu \nu} \left. \right|_{\xi^\mu = 0, \nabla_{[\mu} \xi_{\nu]} = \epsilon_{\mu \nu}}.
\]

Here \(\Sigma_{\text{hor}}\) denotes a spacelike cross section of the Killing horizon (which usually has the topology of \(S^2\)) and we have used \(d\Sigma_{\mu \nu} = \epsilon_{\mu \nu} \sqrt{h} \, d^2 x\). Here \(\epsilon_{\mu \nu}\) denotes the binormal, whose definition we will review below and which is normalized according to \(\epsilon_{\mu \nu} \epsilon^{\mu \nu} = -2\); \(\sqrt{h} \, d^2 x\) gives the surface element induced on \(\Sigma_{\text{hor}}\). We already mentioned that the Noether potential \(Q^{\mu \nu}\) can be decomposed according to \(Q^{\mu \nu} = Y^{\mu \nu \rho \sigma} \nabla_{[\rho} \xi_{\sigma]} + N^{\mu \nu \rho} \xi_{\rho}\). We shall discuss in due course how one is led to impose the conditions \(\xi^\mu = 0\) and \(\nabla_{[\mu} \xi_{\nu]} = \epsilon_{\mu \nu}\) on the Killing vector field at \(\Sigma_{\text{hor}}\) associated with the Noether potential in (23). Let us already point out that the contributions we are suppressing in (23) by imposing these conditions have actually been shown to vanish for non-extremal black holes [5]; this is more subtle for extremal black holes, where the surface gravity is vanishing. Nevertheless we simply adopt (23) as the definition for the entropy in both the extremal and non-extremal case. Note that in both cases (23) leads in principle to a well-defined result.

Finally, the normalization in (23) has been chosen such that we reproduce the Bekenstein-Hawking area law for static black holes in general relativity. To see this, we note that the Lagrangian associated to Einstein gravity, with the conventions of [12], reads \(8\pi L = -\frac{1}{2} R\), so that \(8\pi L^{\mu \nu \rho \sigma} = -\frac{1}{2} g^{\mu [\rho} g^{\sigma] \nu}\). Consequently we have, subject to the conditions in (23), that \(Q^{\mu \nu} = -2 L^{\mu \nu \rho \sigma} \epsilon_{\rho \sigma}\) which equals \((8\pi)^{-1} \epsilon^{\mu \nu}\) for Einstein gravity. Substitution of this result into (23) immediately yields one-fourth of the area, expressed in Planck units.

In the presence of higher-derivative curvature terms, the entropy of a static black hole solution will in general not any longer be simply given by the Bekenstein-Hawking area law. As shown in the previous section (c.f. (17)), when the Lagrangian depends on the Riemann curvature tensor but not on derivatives thereof, as well as on matter fields with at most second derivatives, then \(Q^{\mu \nu} = -2 L^{\mu \nu \rho \sigma} \epsilon_{\rho \sigma}\) when we impose the conditions on the Killing vector at the horizon, specified in (23). The entropy of the static black hole is then given by [15, 5, 6]

\[
S = 2\pi \int_{\Sigma_{\text{hor}}} L^{\mu \nu \rho \sigma} \epsilon_{\mu \nu} \epsilon_{\rho \sigma}.
\]

This is the result that we will need in the next section.

We will now review some of the concepts involved in the definition (23) of the black hole entropy [4, 5, 6, 16] in somewhat more detail. Wald’s construction of the entropy as a surface charge applies to stationary and to some extent also to non-stationary black holes in arbitrary spacetime dimensions, but in the following we will for concreteness restrict the discussion to four-dimensional static black hole solutions. A spacetime containing a black hole consists of an asymptotically flat region, which is separated by a (future) event horizon from an interior region, such that after crossing the horizon one is trapped in the interior region. Usually, the term ‘horizon’ designates either the ‘horizon at a given instant of time’, i.e. a spacelike two-surface \(\Sigma_{\text{hor}}\), or the corresponding worldvolume \(\Delta\) swept out in time by \(\Sigma_{\text{hor}}\), which is a hypersurface in spacetime. In order to avoid any confusion, we will consistently distinguish between these two cases and use the symbols \(\Sigma_{\text{hor}}\) and \(\Delta\) throughout.
A hypersurface can be defined by an equation $f(x^\mu) = 0$. Since the gradient $\nabla_\mu f$ is automatically normal to the surface (i.e. $t^\mu \nabla_\mu f = 0$ for all tangent vectors $t^\mu$) one can equally well specify this surface in terms of the normal vector field $n_\mu = \nabla_\mu f$. According to the Frobenius theorem, which formulates necessary and sufficient conditions for vector fields to define smoothly embedded submanifolds, a vector field $n^\mu$ is hypersurface orthogonal in the above sense if and only if

$$n_\mu \nabla_\nu n_\rho = 0 .$$  \hspace{1cm} (25)

If the normal vector field of a hypersurface is a null vector field, $n_\mu n^\mu = 0$, then the hypersurface is called a null hypersurface. Note that the normal vector field of a null hypersurface is also tangential to it. The (future) event horizon $\Delta$ is such a null hypersurface. Since we assumed that it is a Killing horizon we know that its normal vector is in fact a Killing vector field.

By taking a spacelike cross section of $\Delta$ one obtains a spacelike surface $\Sigma_{\text{hor}}$. In the two-dimensional space normal to $\Sigma_{\text{hor}}$ there exists one linearly independent antisymmetric tensor of second rank, $\epsilon_{\mu\nu}$, which is called the normal bivector or simply the binormal. We will normalize it according to $\epsilon_{\mu\nu} \epsilon^{\mu\nu} = -2$. The binormal can be explicitly written as a bivector, as follows. First we note that the space normal to $\Sigma_{\text{hor}}$ has signature $(-+)$, and therefore it can be spanned by two null vectors. Since $\Sigma_{\text{hor}}$ is contained in $\Delta$, one of the null vectors can be taken to be the normal $n^\mu$ of $\Delta$, that is, one of the null vectors is proportional to the Killing vector $\xi^\mu$. The other null vector, which we denote by $N^\mu$, is chosen such that $N_\mu n^\mu = -1$. Then, the bivector

$$\epsilon_{\mu\nu} = N_\mu n_\nu - N_\nu n_\mu$$ \hspace{1cm} (26)

is non-vanishing in the normal directions and has the required normalization.

The spacetime metric describing a static and spherically symmetric black hole solution can be written as

$$ds^2 = -e^{2g(r)} dt^2 + e^{2f(r)} (dr^2 + r^2 d\Omega^2)$$ \hspace{1cm} (27)

in isotropic coordinates $(t, r, \phi, \theta)$. In such an adapted coordinate system, the time coordinate $t$ and the radius $r$ denote the directions normal to the two-sphere $\Sigma_{\text{hor}}$, whose coordinates are $\phi$ and $\theta$. Thus, the associated binormal $\epsilon_{\mu\nu}$ has the non-vanishing components (up to an overall sign) $\epsilon_t r = -\epsilon_r t = \exp[g(r) + f(r)]$.

In order to formulate the laws of black hole mechanics, we need to define the notion of surface gravity $\kappa_S$ associated with a Killing horizon. Since the Killing vector field is null on the horizon (but in general $\xi^\mu \neq 0$), we can take $f = \xi^\mu \xi_\mu = 0$ as the defining equation of the horizon $\Delta$. Since $\nabla_\mu f$ is normal to $\Delta$, it must be proportional to $\xi_\mu$ itself. The coefficient of proportionality defines the surface gravity $\kappa_S$ of the black hole:

$$\nabla_\mu (\xi^\nu \xi_\mu) = -2\kappa_S \xi_\mu .$$ \hspace{1cm} (28)

Observe that this presupposes a certain intrinsic normalization for the Killing vector field, which is usually specified at spatial infinity. Subsequently one shows that

$$\kappa_S^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu) = \frac{1}{2} R_{\mu\nu\rho\sigma} \xi^\mu \xi^\nu \quad \text{and} \quad \xi_\mu \partial_\mu \kappa_S = 0 .$$ \hspace{1cm} (29)

The first equation follows by multiplying (28) with $\nabla^\mu \xi^\rho$ and by making use of the Killing equation (18) together with the Frobenius theorem (25) with $n^\mu = \xi^\mu$. On the other hand, it
follows from (19) that \( \xi^\rho \nabla_\rho \nabla_\mu \xi_\nu = 0 \), so that \( \kappa_S \) is constant along the integral curves of \( \xi^\mu \) on \( \Delta \). This is the last equation. The second equality is then obtained by applying a covariant derivative \( \nabla^\mu \) on (28) and by using (19). For a static and spherically symmetric black hole, the surface gravity \( \kappa_S \) is thus constant over all of \( \Delta \). The constancy of \( \kappa_S \) over \( \Delta \) is known as the zero-th law of black hole mechanics. We should point out here that the above considerations do not quite apply to extremal black holes, because they have zero surface gravity. We return to this point shortly.

The comparison with the laws of thermodynamics suggests to identify the surface gravity with the temperature of the black hole, up to a multiplicative constant. Then, the zero-th law is reinterpretated by stating that the Killing horizon is in thermodynamical equilibrium and that it radiates like a black body. This interpretation is confirmed by the phenomenon of Hawking radiation which is found when quantizing matter fields in a classical black hole background [2]. In this way the proportionality constant is fixed according to \( T = \kappa_S/2\pi \), where \( T \) denotes the Hawking temperature.

It is instructive to calculate the surface gravity for a Reissner-Nordström black hole in general relativity, which provides an example of a static and charged black hole. In a coordinate system where the curvature singularity is located at \( r = 0 \), the associated spherically symmetric spacetime line element is given by

\[
ds^2 = -e^{2h(r)} dt^2 + e^{-2h(r)} dr^2 + r^2 d\Omega^2 , \quad e^{2h(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} ,
\]

where \( M \) and \( Q \) denote the mass and the charge of the black hole, respectively. The cases \( Q = 0 \), \( M > |Q| \) and \( M = |Q| \) yield the Schwarzschild, the non-extremal and the extremal Reissner-Nordström black hole, respectively. The case \( M < |Q| \) is excluded by the generalized positivity theorem for the ADM mass when assuming the so-called dominant-energy condition, i.e., a non-negative energy density and a non-spacelike energy flow for every observer [17]. The surface gravity is computed to be [16]

\[
\kappa_S = \frac{\sqrt{M^2 - Q^2}}{2M(M + \sqrt{M^2 - Q^2}) - Q^2} ,
\]

which shows that charged and neutral black holes behave very differently when loosing energy by Hawking radiation. The Schwarzschild black hole (\( Q = 0 \)) will heat up in the process, because \( \kappa_S = (4M)^{-1} \), which suggests that it will completely evaporate into radiation. On the other hand a charged black hole will cool down when approaching the extremal limit, namely \( \kappa_S \to 0 \) for \( M \to Q \). The final state, an extremal black hole with \( \kappa_S = 0 \), has vanishing Hawking temperature and could therefore be a stable object. In the following, we will always call black hole solutions extremal if they have \( \kappa_S = 0 \). Note that this does not necessarily imply that they are extremal in the sense of being on the edge of developing a naked singularity.

We now turn to the first law of black hole mechanics which relates changes of the black hole mass to changes in the entropy as well as to changes of other quantities which characterize the black hole, such as its charges. In the following we will omit these latter changes for the sake of clarity. For a static black hole, the first law follows from (22) with \( \delta H = 0 \) by taking the Cauchy surface \( C \) to have a boundary \( \Sigma_{\text{hor}} \cup \Sigma_\infty \), where \( \Sigma_{\text{hor}} \) denotes a spacelike cross section of the event horizon, and where \( \Sigma_\infty \) denotes a two-sphere at spatial infinity. Inspection of (22) and
(23) (where in (23) we have adopted a different normalization for the Killing vector, see below) then suggests to identify the change in the mass and in the entropy of the black hole with

\[ \delta M = -\frac{1}{2} \int_{\Sigma} \left( \delta Q^{\mu\nu} - \xi^\mu \theta^\nu + \xi^\nu \theta^\mu \right), \]

\[ \frac{\kappa_S}{2\pi} \delta S = -\frac{1}{2} \int_{\Sigma_{\text{hor}}} \left( \delta Q^{\mu\nu} - \xi^\mu \theta^\nu + \xi^\nu \theta^\mu \right), \]  

in accordance with the first law of thermodynamics, \( \delta E = T \delta S + \cdots \), with \( E = M \) and \( T = \kappa_S/2\pi \). We note that the first law applies to non-extremal black holes, which have a non-vanishing surface gravity. We now proceed to express the entropy as a surface integral of a local geometrical quantity.

As already mentioned, the Noether potential has the generic form \( Q^{\mu\nu} = Y^{\mu\nu\rho\sigma} \nabla_{[\rho} \xi_{\sigma]} + N^{\mu\rho\nu} \xi_\rho \), where the tensors \( N^{\mu\rho\nu} \) and \( Y^{\mu\nu\rho\sigma} \) are local quantities constructed out of the Riemann tensor and its derivatives. The antisymmetric tensor \( \nabla_\mu \xi_\nu \) can be decomposed as follows,

\[ \nabla_\mu \xi_\nu = \kappa_S \epsilon_{\mu\nu} + t_{[\mu} \xi_{\nu]}, \]

where \( t_\mu \) is tangential to \( \Sigma \). This decomposition expresses the fact that according to the Frobenius theorem the non-vanishing components of \( \nabla_\mu \xi_\nu \) are those where at least one of the indices is not tangential to \( \Sigma \). The coefficient of \( \epsilon_{\mu\nu} \) is determined by contracting (33) with \( \nabla_\mu \xi_\nu \) and by comparing with (29), where one also uses the explicit realization (26) of \( \epsilon_{\mu\nu} \) as a bivector. By substituting the decomposition (33) into \( Q^{\mu\nu} \) we obtain \( Q^{\mu\nu} = \kappa_S Y^{\mu\nu\rho\sigma} \epsilon_{\rho\sigma} + [N^{\mu\nu\lambda} + Y^{\mu\nu\rho\sigma} t_{[\mu} \delta^\lambda_{\sigma]}] \xi_\lambda \). Observe that the last two terms of the integrand in (32) are proportional to the Killing vector and can thus be absorbed into \( N^{\mu\nu\rho} \), so they don’t need to be discussed separately.

For non-extremal black holes there is a theorem [18] which states that the horizon hypersurface \( \Delta \) contains, or can be analytically extended to contain, a spacelike cross section, called the bifurcation surface \( \Sigma_0 \), where the timelike Killing vector field has a zero, \( \xi^\mu = 0 \), so that \( \nabla_\mu \xi_\nu = \kappa_S \epsilon_{\mu\nu} \). In theories with matter one also has to assume that the matter fields can likewise be analytically continued and that all the fields are regular at the bifurcation surface. Thus, by evaluating the Noether potential on \( \Sigma_0 \) one can get rid of the terms in \( Q^{\mu\nu} \) proportional to the Killing vector field, so that we are left with \( \kappa_S/2\pi \) times the variation of the entropy defined in (23) but with \( \Sigma_{\text{hor}} \) replaced by \( \Sigma_0 \). As already mentioned, it can be shown [5] that, when replacing \( \Sigma_0 \) with any other spacelike cross section \( \Sigma_{\text{hor}} \) of the horizon, the resulting expression for the entropy is given by a similar expression, namely by (23), which indeed expresses the entropy as a surface integral of a local geometrical quantity over an arbitrary cross section of the horizon.

As stressed above, the above procedure for deriving the first law of black hole mechanics applies to non-extremal black holes. Nevertheless, the resulting expression for the entropy (23) remains well behaved in the extremal limit \( \kappa_S \to 0 \), as it is independent of \( \kappa_S \) and of the Killing field \( \xi^\mu \). Since extremal black holes do not possess a bifurcation surface, it is important that the entropy can be evaluated on any spacelike cross section \( \Sigma_{\text{hor}} \), as is the case with (23). Thus, we expect that the entropy of an extremal black hole, if computed from (23), will be non-vanishing in general. It should be pointed out though that the question whether or not an extremal black hole has a non-vanishing entropy, is a somewhat subtle issue that depends on the approach used.
to compute its entropy. For instance, in the context of semiclassical quantization of matter coupled to Einstein gravity, one finds that the entropy of an extremal Reissner-Nordstrøm black hole is non-vanishing and given by the Bekenstein-Hawking area law, provided the extremal limit is taken after quantization. If, on the other hand, the quantization is performed after extremalization (that is, if the quantization is applied to the part of phase space that only contains extremal configurations), then the resulting entropy is vanishing. This has been shown both in the Euclidean path integral framework [19] and in the Minkowskian canonical framework [20]. In the context of string theory, various microscopic state countings yield a non-vanishing entropy in all cases where extremal black holes have a non-vanishing horizon area. Thus, string theory favours to treat extremal black holes as limiting cases of non-extremal ones.

A comparison of Wald’s Noether charge approach with various other approaches can be, for instance, found in [15, 21] and references therein. Within their domain of applicability, all of these other approaches yield results in agreement with the Noether charge approach.

4 Supersymmetric black holes

The electrically charged static extremal Reissner-Nordstrøm solution of Einstein-Maxwell theory, which we briefly described in (30), enjoys several remarkable properties, as follows. It interpolates between two maximally symmetric spaces, namely Minkowski flat spacetime at spatial infinity and Bertotti-Robinson spacetime at the horizon. There exists a dyonic version of it, whose mass $M$ and entropy $S$ (the latter computed from the area law) are determined in terms of its electric and magnetic charges $q$ and $p$ as $M = |q + ip|$ and $S = \pi |q + ip|^2$. Since its temperature vanishes, this suggests that such a dyonic black hole is quantum mechanically stable. Moreover, its mass saturates the Bogomolnyi bound which follows from the generalized positivity theorem for the ADM mass [17]. There also exists a static multi-center version of it, which is described by the Majumdar-Papapetrou metric [22] and which resembles the static multi-monopole solutions of Yang-Mills-Higgs theories in the Prasad-Sommerfield limit.

All these properties can be explained in terms of a symmetry principle, namely supersymmetry, by embedding Einstein-Maxwell theory into $N = 2$ supergravity [17]. Then the extremal Reissner-Nordstrom black hole solution can be interpreted as a supersymmetric soliton which interpolates between two maximally supersymmetric vacua of $N = 2$ supergravity. Globally the solution is invariant under 4 of the 8 supersymmetries. Its mass formula follows from the $N = 2$ supersymmetry algebra and takes the form $M = |z|$, where $z$ denotes the central charge of the supersymmetry algebra. It is determined in terms of the electric and magnetic charges $q$ and $p$ associated with the gauge field, namely by $z = q + ip$.

More recently, following the work of [23], this has been extended to the study of static supersymmetric black hole solutions in four-dimensional theories describing the coupling of $n$ abelian vector multiplets to $N = 2$ supergravity in the presence of a certain class of $R^2$-terms [24, 12]. Such theories arise as low-energy effective field theories of string and M-theory compactifications on suitable compact manifolds. These effective Lagrangians contain, in general, various other terms describing the couplings of additional sectors, such as matter associated with hypermultiplets, to supergravity. These other sectors, however, play only a limited role in the following and we will therefore omit them.
Let us first review how the couplings of vector multiplets to $N = 2$ supergravity can be described in terms of special geometry. Since we will allow for the presence of certain $R^2$-terms, we will utilize the so-called superconformal framework, which provides a systematic and powerful approach for constructing these couplings \[24\]. It makes use of the fact that a conformal theory of $N = 2$ supergravity with suitable couplings to compensating multiplets is gauge equivalent to $N = 2$ Poincaré supergravity. In the superconformal framework, there is a multiplet, the so-called Weyl multiplet, which comprises the gravitational degrees of freedom, namely the graviton, two gravitini as well as various other superconformal gauge fields and also some auxiliary fields. One of these auxiliary fields is an anti-selfdual Lorentz tensor field $T^{abij}$, where $i, j = 1, 2$ denote chiral $SU(2)$ indices (we recall that the associated $SU(2)$ algebra is part of the $N = 2$ superconformal algebra). The field strengths corresponding to the various gauge fields in the Weyl multiplet reside in a so-called reduced chiral multiplet, denoted by $W^{abij}$, from which one then constructs the unreduced chiral multiplet $W^2 = (W^{abij} \hat{\varepsilon}_{ij})^2$ [26]. The lowest component field of $W^2$ is $\hat{A} = (T^{abij} \hat{\varepsilon}_{ij})^2$.

In addition, there are $n + 1$ abelian vector multiplets labelled by an index $I = 0, \ldots, n$. Each vector multiplet contains a complex scalar field $X^I$, a vector gauge field $W^I_\mu$ with field strength $F^I_{\mu \nu}$, as well as two gaugini and a set of auxiliary scalar fields. The couplings of these $n + 1$ vector multiplets to the Weyl multiplet are encoded in a holomorphic function $F(X^I, \hat{A})$, which is homogenous of degree two and consequently satisfies $X^IF_I + 2\hat{A}F_{\hat{A}} = 2F$, where $F_I = \partial F/\partial X^I$, $F_{\hat{A}} = \partial F/\partial \hat{A}$.

The field equations of the vector multiplets are subject to equivalence transformations corresponding to electric-magnetic duality, which do not involve the fields of the Weyl multiplet. These equivalence transformations are symplectic $\text{SP}(2n + 2; \mathbb{Z})$ transformations. Two complex $(2n + 2)$-component vectors can now be defined which transform linearly under $\text{SP}(2n + 2; \mathbb{Z})$ transformations, namely

$$V = \begin{pmatrix} X^I \\ F_I(X, \hat{A}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F^\pm_I \\ G^\pm_{\mu \nu I} \end{pmatrix},$$

(34)

where $(F^\pm_{\mu \nu}, G^\pm_{\mu \nu I})$ denotes the (anti-)selfdual part of $(F^I_{\mu \nu}, G_{\mu \nu I})$. The field strength $G^\pm_{\mu \nu I}$ is defined by the variation of the action with respect to $F^\pm_{\mu \nu}$. The precise definition reads $G^\pm_{\mu \nu I} = -4i (-g)^{-1/2} \partial S/\partial F^\pm_{\mu \nu}$. By integrating the gauge fields over two-dimensional surfaces enclosing their sources, it is possible to associate to $(F^\pm_{\mu \nu}, G^\pm_{\mu \nu I})$ a symplectic vector $(p^I, q_I)$ comprising the magnetic and electric charges. It is then possible to construct a complex quantity $Z$ out of the charges and out of $V$ which is invariant under symplectic transformations, as follows,

$$Z = e^{K/2} (p^I F_I(X, \hat{A}) - q_I X^I),$$

(35)

where $e^{-K} = i[\bar{X}^IF_I(X, \hat{A}) - \bar{F}_I(\bar{X}, \bar{A})X^I]$. This quantity will play a role in the following.

The associated (Wilsonian) Lagrangian describing the coupling of these vector multiplets to supergravity is quite complicated [27]. Here we only display those terms which will be relevant for the computation of the entropy of a static supersymmetric black hole,

$$8\pi \mathcal{L} = -\frac{1}{2} e^{-K} R + \frac{1}{2} (iF_{\hat{A}} \hat{C} + \text{h.c.}) + \cdots,$$

(36)
where \( \hat{C} = 64 C^{-\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + 16 \varepsilon_{ij} T^{\mu\nuij} f_{\mu}^\rho T_{\rho k l} \varepsilon^{kl} + \cdots \). Here \( C_{\mu\nu\rho\sigma} \) denotes the anti-selfdual part of the Weyl tensor \( C_{\mu\nu\rho\sigma} \), and \( f_{\mu}^\nu = \frac{1}{2} R_{\mu}^\nu - \frac{1}{12} R \delta_{\mu}^\nu + \cdots \). Eventually we will set \( e^{-\kappa} \) to unity in order to obtain a properly normalized Einstein-Hilbert term. In this way we fix the local scale invariance which is present in a superconformal formulation of the theory. We note that the Lagrangian contains \( C^2_{\mu\nu\rho\sigma} \)-terms, but no terms involving derivatives of the Riemann curvature tensor. By expanding the holomorphic function \( F(X, \hat{A}) \) in powers of \( \hat{A} \), \( F(X, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X) \hat{A}^g \), we see that the Lagrangian \( \mathcal{L} \) contains an infinite set of higher-derivative curvature terms of the type \( C^2_{\mu\nu\rho\sigma}(T^{abij}\varepsilon_{ij})^{2g-2} \) \( (g \geq 1) \) with field-dependent coupling functions \( F^{(g)}(X) \).

As alluded to above, we will view a static supersymmetric black hole solution of the Lagrangian \( \mathcal{L} \) as a solitonic interpolation between two \( N = 2 \) supersymmetric groundstates, namely flat spacetime at spatial infinity and the horizon, whose geometry we now proceed to determine. The spacetime line element associated with a static spherically symmetric solution is of the form \( ds^2 = -K_{ij} \varepsilon_{ij} + 2A \varepsilon_{ij} \) in isotropic coordinates. The near-horizon solution can be specified by imposing full \( N = 2 \) supersymmetry on the solution. This is achieved by requiring that the supersymmetry variations of all the fermions present in the theory vanish in the bosonic black hole background. We stress here that we do not analyze the equations of motion, as their validity is implied by full \( N = 2 \) supersymmetry. A careful analysis \( \mathcal{L} \) of the resulting restrictions on the bosonic background then shows that the \( X^I \) and \( \hat{A} \) are constant at the horizon. Since the black hole is charged and can carry both magnetic and electric charges \( (p^I, q_I) \), the associated quantity \( Z \) is therefore generically non-vanishing and constant at the horizon. Moreover, we also find that \( T^{01ij} = -i T^{23ij} = 2 \varepsilon_{ij} e^{-\kappa/2} \hat{Z}^{-1} \), while all other components of \( T^{abij} \) vanish. Therefore we have \( \hat{A} = -64 e^{-\kappa} Z^{-2} \). And finally, the near-horizon spacetime geometry is determined to be of the Bertotti-Robinson type, that is of the form \( \mathcal{L} \) with \( e^{2g(r)} = e^{-2j(r)} = e^{-\kappa} |Z|^{-2} r^2 \).

The requirement of \( N = 2 \) supersymmetry at the horizon does not by itself fix the actual values of the constants \( X^I \). To do so, we have to invoke the so-called fixed-point behaviour \( \mathcal{L} \) for the scalar fields \( X^I \) at the horizon (for a recent reference on the fixed-point behaviour, see \( \mathcal{L} \)). The fixed-point behaviour implies that regardless of what the values of the scalar fields are at spatial infinity, they always take the same values at the horizon. In the absence of higher-derivative terms it has been shown \( \mathcal{L} \) that the supersymmetric black hole solutions do indeed always exhibit fixed-point behaviour as a result of their residual \( N = 1 \) supersymmetry. In the presence of higher-derivative terms this has not yet been shown to be the case. In the following we will assume that such a fixed-point behaviour holds for black hole solutions of the Lagrangian \( \mathcal{L} \). From electric-magnetic duality one may then deduce that the values of the scalar fields \( X^I \) at the horizon are determined from a set of equations, called the stabilization equations, which take the following form \( \mathcal{L} \) \( \mathcal{L} \)

\[
Z \begin{pmatrix} X^I \\
F_I(X, \hat{A}) \end{pmatrix} - Z \begin{pmatrix} \bar{X}^I \\
\bar{F}_I(\bar{X}, \hat{A}) \end{pmatrix} = ie^{-\kappa/2} \begin{pmatrix} p^I \\
q_I \end{pmatrix} .
\]

(37)

At this point it is convenient to introduce rescaled variables \( Y^I = e^{\kappa/2} \hat{Z} X^I \) and \( \Upsilon = e^{\kappa} \hat{Z}^2 \hat{A} = -64 \). Using the homogeneity property of \( F \) mentioned earlier, it follows that the stabilization equations \( \mathcal{L} \) now simply read \( Y^I - \bar{Y}^I = ip_I \) and \( F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \Upsilon) = iq_I \). These equations then determine the value of the rescaled fields \( Y^I \) in terms of the charges carried by the black hole. On the other hand, it follows from \( \mathcal{L} \) that \( |Z|^2 = p^I F_I(Y, \Upsilon) - q_I Y^I \), which determines
the value of $|Z|$ in terms of $(p', q_I)$.

The entropy of the static black hole solution described above can now be computed from (24), using (36). The result takes the remarkably concise form

$$S = \pi \left[ |Z|^2 - 256 \text{Im} F_A \right].$$

(38)

The first term denotes the Bekenstein-Hawking entropy contribution, whereas the second term is due to Wald’s modification of the definition of the entropy in the presence of higher-derivative terms. Here we point out that this contribution does not actually originate from the $C_{\mu\nu\rho\sigma}^2$ terms in the action, because the Weyl tensor vanishes at the horizon, but from the term in $\hat{C}$ (see below (36)) proportional to the product of the Ricci tensor with the tensor field $T^{ab}ijT_{cdkl}$. Note that when switching on higher-derivative interactions the value of $|Z|$ changes and hence also the horizon area changes. There are thus two ways in which the presence of higher-derivative interactions modifies the black hole entropy, namely by a change of the near-horizon geometry and by an explicit deviation from the Bekenstein-Hawking area law. Also note that the entropy (38) is entirely determined in terms of the charges carried by the black hole, $S = S(q, p)$.

Let us now exhibit the generic dependence of the macroscopic entropy (38) on the charges. Let $Q$ denote a generic electric or magnetic charge carried by the black hole. For large charges $Q$, the stabilization equations (37) and the homogeneity property of $F$ imply that the generic dependence of the entropy (38) on the charges is given by

$$S = \pi \sum_{g=0}^{\infty} a_g Q^{2-2g}$$

(39)

with constant coefficients $a_g$. We recall that (38) encodes the contributions from a particular set of $R^2$-terms, namely terms proportional to $C_{\mu\nu\rho\sigma}^2$. In general, however, the effective Lagrangian will not only contain these particular terms, but also many other (even higher-derivative) curvature terms which, in principle, will lead to further contributions to the entropy. One might then worry that the inclusion of such contributions could wash out the contributions appearing in (39). In the context of string theory this appears to be unlikely, given that the coefficients multiplying the various powers of the charges in (39) encode topological information about the $N = 2$ compactification.

5 Examples

Let us now briefly discuss various classes of black hole solutions arising in string theory compactifications. We will use the rescaled variables $Y^I = e^{K/2} \bar{Z} X^I$, $\Upsilon = e^K \bar{Z}^2 \hat{A} = -64$ throughout.

Let us first consider type-IIA string theory compactified on a Calabi-Yau threefold, in the limit where the volume of the Calabi-Yau threefold is taken to be large. For the associated homogenous function $F(Y, \Upsilon)$ we take (with $I = 0, \ldots, n$ and $A = 1, \ldots, n$)

$$F(Y, \Upsilon) = \frac{D_{ABC} Y_A Y_B Y_C}{Y_0} + d_A \frac{Y_A}{Y_0} \Upsilon, \quad D_{ABC} = -\frac{1}{6} C_{ABC}, \quad d_A = -\frac{1}{24} \frac{1}{c_2 A},$$

(40)
where the coefficients $C_{ABC}$ denote the intersection numbers of the four-cycles of the Calabi-Yau threefold, whereas the coefficients $c_{2A}$ denote its second Chern-class numbers [24]. The Lagrangian (36) associated with this homogenous function thus contains a term proportional to $c_{2A} \text{Im} z^A C_{\mu \nu \rho \sigma}^2$, where $z^A = Y^A / Y^0$. The associated stabilization equations can be solved [12] for black holes with $p^0 = 0$, yielding $Y^I = Y^I(q,p)$. The result for the macroscopic entropy, which is computed from (38), reads [12]

$$ S = 2\pi \sqrt{\frac{1}{6} |q_0| (C_{ABC} p^A p^B p^C + c_{2A} p^A)} , $$

(41)

where $q_0 = q_0 + \frac{1}{12} D^{AB} q_A q_B$, $D_{AB} = D_{ABC} p^C$, $D_{AB} D^{BC} = \delta^C_A$. The expression (41) for the macroscopic entropy is in exact agreement with the microscopic entropy formula computed in [10, 11] via state counting.

Next, let us consider black hole solutions arising in heterotic string compactifications on $K_3 \times T_2$. The associated tree-level function is given by

$$ F(Y, \Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + c \frac{Y^1 \Upsilon}{Y^0} , \quad a = 2, \ldots, n \, , $$

(42)

where the real constants $\eta_{ab}$ and $c$ are related to the intersection numbers of two-cycles and to the second Chern-class number of $K_3$, respectively. For a function $F(Y, \Upsilon)$ of the form (42), the stabilization equations can be solved in full generality and the resulting expression for the entropy reads [30]

$$ S = \pi \sqrt{\langle M, M \rangle \langle N, N \rangle - (M \cdot N)^2} \sqrt{1 - \frac{512 c}{\langle N, N \rangle}} , $$

(43)

where the $M_I = (q_0, -p^1, q_2, q_3, \ldots, q_n)$ and the $N^I = (p^0, q_1, p^2, q_3, \ldots, p^n)$ now denote the electric and magnetic charges carried by the heterotic black hole. The bilinears $\langle M, M \rangle$ and $\langle N, N \rangle$ are given by

$$ \langle M, M \rangle = 2 \left( M_0 M_1 + \frac{1}{4} M_a \eta^{ab} M_b \right) \, , \quad \langle N, N \rangle = 2 \left( N^0 N^1 + N^a \eta_{ab} N^b \right) \, , $$

(44)

whereas $M \cdot N = M_I N^I$. These bilinears are invariant under tree-level target-space duality transformations of the charges [31]. We thus see that turning on a term proportional to $\text{Im} z^1 C_{\mu \nu \rho \sigma}^2 (z^1 = Y^1 / Y^0)$ in the Lagrangian leads again to a modification of the entropy.

There will be various corrections to (41) and to (43) from terms proportional to $C_{\mu \nu \rho \sigma}^2 \hat{A}^{g-1} (g \geq 2)$ in the effective Lagrangian of heterotic string theory compactified on a six-torus, where $S$ here denotes the heterotic dilaton field. The entropy of a black hole solution in the presence of such an $C_{\mu \nu \rho \sigma}^2$-term is given by an expression analogous to (43). As can be seen from (43), the analogous expression is not invariant under the exchange of the electric and the magnetic charges. The heterotic $N = 4$ theory is, however,
expected to be invariant under strong-weak coupling duality. Here we recall that it is essential to distinguish between the Wilsonian couplings of an effective theory and the physical (in general momentum-dependent) effective couplings. In theories with massless fields, the effective couplings are different from the Wilsonian couplings and do not share their analytic properties. In the context of heterotic \( N = 4 \) compactifications, the effective coupling function multiplying \( C^2_{\mu \nu \rho \sigma} \) is non-holomorphic and invariant under strong-weak coupling duality transformations \[32\], whereas the Wilsonian coupling function, which at tree-level is proportional to \( S \), is holomorphic although not invariant under strong-weak coupling duality. Thus, in order to arrive at a manifestly strong-weak coupling duality invariant expression for the entropy of a heterotic \( N = 4 \) black hole, its computation should be based on the effective rather than the Wilsonian coupling function of \( C^2_{\mu \nu \rho \sigma} \).

We will now restrict ourselves to black hole solutions in an \( N = 2 \) subsector of the heterotic \( N = 4 \) theory. Both the stabilization equations \[37\] and the entropy formula \[38\] were derived in the Wilsonian context and as such they are defined in terms of a holomorphic function \( F \). In the effective coupling approach, on the other hand, we expect that both \[37\] and \[38\] will receive non-holomorphic corrections so as to be consistent with strong-weak coupling duality. The function \( F \), in particular, will not any longer be holomorphic:

\[
F(Y, \bar{Y}, \Upsilon) = -\frac{Y^IY^a\eta_{ab}Y^b}{Y^0} + F^{(1)}(z^1, \bar{z}^1) \Upsilon ,
\]

where we require that \[45\] turns into the tree-level function \[32\] at weak coupling, that is \( F^{(1)}(z^1, \bar{z}^1) \rightarrow cz^1 \) as \( S + \bar{S} \rightarrow \infty \), where \( S = -iz^1 = -iY^1/Y^0 \). The associated stabilization equations now read

\[
Y^I - \bar{Y}^I = ip_I , \quad F_I(Y, \bar{Y}, \Upsilon) - \bar{F}_I(Y, \bar{Y}, \bar{\Upsilon}) = iq_I , \quad \Upsilon = -64
\]

The form of \( F^{(1)}(z^1, \bar{z}^1) \) can then be determined by requiring \[45\] to transform in a consistent way under strong-weak coupling duality transformations and also by requiring \( F^{(1)}(z^1, \bar{z}^1) \) to have the weak-coupling behaviour specified above \[30\]:

\[
F^{(1)}(S, \bar{S}) = -6i \frac{c}{\pi} \left( \log \eta^2(S) + \log(S + \bar{S}) \right),
\]

where \( \eta(S) \) denotes the Dedekind function. The associated non-holomorphic quantity \( |Z|^2 = p^IF_I(Y, \bar{Y}, \Upsilon) - q_IY^I \) is then invariant under strong-weak coupling duality. In analogy to \[38\] we thus propose the following strong-weak coupling duality invariant expression for the entropy of a black hole in an \( N = 2 \) subsector of the heterotic \( N = 4 \) theory \[30\]:

\[
S = \pi \left[ |Z|^2 - 256 \Im \left( F^{(1)}(S, \bar{S}) + 3i \frac{c}{\pi} \log(S + \bar{S}) \right) \right].
\]

The additional non-holomorphic piece in \[48\] is there to render the expression invariant under strong-weak coupling duality. We note that the value of the dilaton \( S \) at the horizon is, in principle, determined in terms of the charges \( M_I \) and \( N^I \) carried by the black hole through the stabilization equations \[46\], although in practice they are hard to solve. We refer to \[30\] for a more detailed discussion of these and related issues.

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