DISTRIBUTION OF THE MINIMAL DISTANCE OF RANDOM LINEAR CODES

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Abstract. In this paper, we study the distribution of the minimal distance (in the Hamming metric) of a random linear code of dimension $k$ in $\mathbb{F}_q^n$. We provide quantitative estimates showing that the distribution function of the minimal distance is close (super-polynomially in $n$) to the cumulative distribution function of the minimum of $(q^k-1)/(q-1)$ independent binomial random variables with parameters $\frac{1}{q}$ and $n$. The latter, in turn, converges to a Gumbel distribution at integer points when $\frac{k}{n}$ converges to a fixed number in $(0, 1)$. In a sense, our result shows that apart from identification of the weights of parallel codewords, the probabilistic dependencies introduced by the linear structure of the random code, produce a negligible effect on the minimal code weight. As a corollary of the main result, we obtain an asymptotic improvement of the Gilbert–Varshamov bound for $2 < q < 49$.

1. Introduction

Let $\mathbb{F}_q$ be a finite field. A linear code $C$ is a subspace of $\mathbb{F}_q^n$ where $n$ is the length of the code. The parameter $q$ of the field is referred to as the alphabet size. The size of $C$ is the number of elements in $C$. For a (not necessarily linear) code with size $M$, alphabet size $q$, and length $n$, the information rate $R$ is defined to be $\log_q(M)/n$. For a linear code this number is equal to $k/n$, where $k$ is the dimension of the code as a vector space.

Another fundamental parameter is the relative minimal distance. Let the Hamming distance between any two codewords $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$ in $\mathbb{F}_q^n$ be given by

$$d(u, v) := |\{1 \leq i \leq n, u_i \neq v_i\}|,$$

and the Hamming weight of a codeword $u$ be defined as $\text{wt}(u) := d(u, 0)$. For linear codes, the minimal distance between two distinct codewords in a code is equal to the minimal weight over all nonzero codewords. It is well-known that a code with a minimal distance $d$ can correct up to $\frac{d-1}{2}$ errors. The relative minimal distance $\delta$ is defined as the ratio $\frac{d}{n}$.

In coding theory, the trade-off between the code rate $R$ and error-correcting ability $\delta$ is a central topic of study. Let $q$ be fixed. For linear codes, Manin [9] has proved that there exists a function $\alpha_q(\cdot)$ with the following property: for any $\delta_0 \in (0, 1 - 1/q)$ and any $R_0 \leq \alpha_q(\delta_0)$, there is an infinite sequence of linear codes with the relative minimal distance converging to $\delta_0$ and the rate converging to $R_0$; on the other hand, for every $R_0 > \alpha_q(\delta_0)$, such a sequence does not exist. An explicit description of $\alpha_q(\cdot)$ remains a major open problem (see [5] [13] [7], as well as [10] for an upper bound for $\alpha_q$). Considerable work has been done to
obtain explicit constructions for linear codes with good rate and relative minimal distance
(we refer, in particular, to [14]).

Rather than considering special codes, one may be interested in studying the statistical
properties on the space of all linear codes, using probabilistic methods. A classical result
in this direction is the Gilbert–Varshamov argument. Gilbert [4] and Varshamov [15] inde-
pendently gave upper bound for the size of a (not necessarily linear) code given \( n \) and \( d \).
Let \( A_q(n, d) \) be the maximal size of a code of length \( n \) over \( \mathbb{F}_q \) and with minimum distance \( d \). Then

\[
A_q(n, d) \geq \frac{q^n}{\sum_{j=0}^{d-1} \binom{n}{j} (q-1)^j},
\]

and, moreover, there are linear codes that can achieve this bound i.e. there exists a linear
code over \( \mathbb{F}_q \) with dimension at least \( n - \lfloor \log_q \sum_{j=0}^{d-1} (q-1)^j \rfloor \). The proof of the result can
be obtained by a union bound argument.

Recall that the \( q \)-ary entropy function is defined by

\[
H_q(x) := x \log_q(q-1) - x \log_q(x) - (1-x) \log_q(1-x).
\]

In [3], it was shown that for \( q = 2 \) and given a rate \( R_0 \) and \( \varepsilon > 0 \), the probability that
a random linear code of length \( n \) and rate \( R_0 \), uniformly distributed on the set of linear
codes of the given length and rate, has the minimal distance \( d < n(\delta_0 - \varepsilon) \), is exponentially
small in \( n \). Here \( 0 < \delta_0 < \frac{1}{2} \) is the solution of the equation \( R_0 = 1 - H_q(\delta_0) \). On the
other hand, if we fix any \( \delta_0 \) satisfying \( 0 \leq \delta_0 < 1 - \frac{1}{q} \) and \( 0 < \varepsilon \leq 1 - H_q(\delta_0) \), then the
Gilbert–Varshamov argument implies that there exist infinitely many linear codes with a
rate \( R \geq 1 - H_q(\delta_0) - \varepsilon \). By taking \( \varepsilon \to 0 \), one would obtain a lower bound for the function
\( \alpha_q(\delta) \) mentioned above:

\[
\alpha_q(\delta) \geq 1 - H_q(\delta).
\]

In fact, as was proved in [1], the following law of large numbers holds for the minimal
distance of a sequence of random linear codes: if \( n \to \infty \) and the rate \( k/n \) converges to
a number \( R_0 \in (0, 1) \) then the relative minimal distance converges (almost surely) to the
number \( \delta_0 \) given by the equation \( R_0 = 1 - H_q(\delta_0) \). Moreover, the probability that a random
linear code of length \( n \) has the relative minimal distance outside of the interval \( [\delta_0 - \varepsilon, \delta_0 + \varepsilon] \),
is exponentially small in \( n \) (we remark here that in the same paper it was shown that the
minimal distance of random non-linear codes is asymptotically worse than in the linear
setting).

Our goal in this paper is to obtain a more precise description of the distribution of the
minimal distance of random linear codes. The main statement is the following

**Theorem 1.1.** For any prime power \( q \) and any real numbers \( R_1 < R_2 \) in \( (0, 1) \) there is
\( c(R_1, R_2, q) > 0 \) with the following property. Let positive integers \( k, n \) satisfy \( R_1 \leq k/n \leq R_2 \), and let \( C \) be the random linear code uniformly distributed on the set of all linear codes
in \( \mathbb{F}_q^n \) of dimension \( k \). Denote by \( F_{\text{dmin}} \) the cumulative distribution function of the minimal
distance of \( C \). Further, let \( w_{\text{min}} \) be the minimal weight of \( \frac{q^k - 1}{q-1} \) i.i.d. uniform random vectors
in \( \mathbb{F}_q \), and \( F_{\text{wmin}} \) be its cumulative distribution function. Then

\[
\sup_{x \in \mathbb{R}} |F_{\text{dmin}}(x) - F_{\text{wmin}}(x)| = O\left( \exp(-c(R_1, R_2, q) \sqrt{n}) \right).
\]

A surprising feature of this result is that the distribution of the minimal distance can be approximated by a c.d.f. of the minimum of i.i.d. binomial variables with precision superpolynomial in \( n \). In a sense, this result asserts that dependencies between codeword weights introduced by the linear structure of the code, produce a negligible effect on the distribution of the minimal weight.

The proof of the result is based on analysis of moments of certain functionals associated with the code. We remark that in a recent work by Linial and Mosheiff [8], the authors calculated centered moments for number of codewords of a random linear code with a given weight. The approach used in that paper influenced our work.

As an immediate corollary of our result, we obtain the following statement which gives an \( \Theta(n^{1/2}) \) improvement over the classical Gilbert–Varshamov bound:

**Corollary 1.1.** For any prime power \( q \), any \( \alpha \in (0, 1) \), any integer \( n \), and \( d \in [\alpha n, (1 - \alpha)(n - n/q)] \) there is a linear code of size at least

\[
cn^{1/2} \sum_{j=0}^{d-1} \frac{q^n}{\binom{n}{j}(q-1)^j},
\]

where \( c > 0 \) may only depend on \( \alpha \) and \( q \).

We note that existence of non-linear codes of size at least \( cn \sum_{j=0}^{d-1} \frac{q^n}{\binom{n}{j}(q-1)^j} \) has been previously established in [13, 16]. Linear double-circulant binary codes beating the Gilbert–Varshamov bound were considered in [2]. To our best knowledge, the above improvement for \( 2 < q < 49 \) is new.

Further, we obtain an explicit limit theorem for the distribution of the minimal distance. Due to the discrete nature of our random variable, the convergence to a Gumbel distribution can only be established on the points along certain arithmetic progressions:

**Theorem 1.2** (The limit theorem for the minimal distance). Let \( q \) be a prime power, and let \( R_1 < R_2 \) be numbers in \( (0, 1) \). Let \( (k_n) \) be a sequence of positive integers such that \( R_1 \leq k_n/n \leq R_2 \) for all large \( n \). For any \( n \) let \( d_{\text{min}}(n) \) be the minimal distance of the random linear code uniformly distributed on the set of linear codes of length \( n \) and dimension \( k_n \). Further, for any \( n \) let \( d_0(n) \) be the largest integer satisfying

\[
u(n) := \frac{q^{k_n} - 1}{q - 1} \sum_{i=0}^{d_0(n)} \binom{n}{i} \left( 1 - \frac{1}{q} \right)^i q^{-i-n} \leq 1.
\]

Denote by \( \xi_n \) the random variable

\[
\xi_n := (d_0(n) - d_{\text{min}}(n)) \log \frac{(q - 1)(n - d_0(n))}{d_0(n)} - \log u(n).
\]
Then, as $n \to \infty$, we have

$$\sup \left\{ |\mathbb{P}\{\xi_n < t\} - G(t)\| : t \in \log \frac{(q-1)(n-d_0(n))}{d_0(n)} \mathbb{Z} - \log u(n) \right\} \to 0,$$

where $G$ is the Gumbel law given by $G(t) = e^{-e^{-t}}$.

The paper is organized as follows. In Section 2, we consider some auxiliary results for the binomial distribution, including a limiting result for the minimum of i.i.d. binomial random variables. At the end of the section, we show how the main result of the paper implies Theorem 1.2.

In Section 3, we consider the set of random vectors $\{Y_a : a \in F_q^n \setminus \{0\}\}$ uniformly distributed in $F_q^n$ that are mutually independent up to the constraint that $Y_a = Y_b$ whenever $a$ and $b$ are parallel. We study moments of the random variable that counts number of codewords with weights less than or equal to $d$ in this configuration as well as that of random linear code ensemble and give a quantitative comparison between them.

Finally, in Section 4 we give the comparison of the c.d.f. of minimum distance between these two ensembles. Due to the discrete nature of this problem, either c.d.f. can be obtained by solving a set of linear equations involving quantities we computed in previous sections. Then we give a quantitative comparison by estimating the truncation errors and moment differences.

2. Auxiliary results for the binomial distribution

Our goal in this section is to obtain quantitative estimates for the distribution of the minimum of i.i.d. binomial random variables (with specially chosen parameters). Although the material of this section is rather standard, we prefer to include it in the exposition for the reader’s convenience.

Let $1 \leq m \leq (q-1)^n$ and let $X_1, \ldots, X_m$ be i.i.d. vectors uniformly distributed in $F_q^n$. Here, we are interested in estimates of the quantities

$$\mathbb{P}\{\min_{i \leq m} \text{wt}(X_i) \leq d\}, \quad d \geq 0,$$

where $\text{wt}(X_i)$ is the number of non-zero components of $X_i$. Denote

$$\rho_d := \mathbb{P}\{\text{wt}(X_1) \leq d\} = \sum_{i=0}^d \binom{n}{i} \left(1 - \frac{1}{q}\right)^i q^{i-n}.$$

We start by recording the following approximations to $\rho_d$:

**Proposition 2.1.** For any $\alpha \in (0, 1)$ there is $C_\alpha > 0$ with the following property. Assume that $n \geq 1$ and $C_\alpha \log(n) \leq d \leq (1 - \alpha)(1 - 1/q)n$. Then we have

$$\frac{\rho_d}{\binom{n}{d} q^{-n} (q-1)^d} = (1 + O_\alpha(\log n/n))\frac{n - d + 1}{n - \left(\frac{q}{q-1}\right) d + 1}. \quad (1)$$
Furthermore, for any positive integer \( t \leq \sqrt{d} \), we have

\[
\frac{\rho_{d+t}}{\rho_d} = \left(1 + O_\alpha \left( \frac{\log n}{n} + \frac{t^2}{d} \right) \right) \left( \frac{(q-1)(n-d)}{d} \right)^t.
\]

Proof. We have

\[
\frac{\rho_d}{\binom{n}{d} q^{-n(q-1)d}} = 1 + \frac{1}{q-1} \frac{d}{n-d+1} + \left( \frac{1}{q-1} \right)^2 \frac{d}{n-d+1} \frac{d-1}{n-d+2} + \ldots \n + \left( \frac{1}{q-1} \right)^d \frac{d(d-1) \cdots 1}{(n-d+1)(n-d+2) \cdots n} \leq \frac{1}{1 - \frac{d}{(n-d+1)(q-1)}} \frac{n-d+1}{n - \left( \frac{q}{q-1} \right) d + 1}.
\]

On the other hand, for any positive integer \( t \leq d \) we have

\[
\frac{\rho_d}{\binom{n}{d} q^{-n(q-1)d}} \geq 1 + \frac{1}{q-1} \frac{d}{n-d+1} + \left( \frac{1}{q-1} \right)^2 \frac{d}{n-d+1} \frac{d-1}{n-d+2} + \ldots \n + \left( \frac{1}{q-1} \right)^t \frac{d(d-1) \cdots (d-t+1)}{(n-d+1)(n-d+2) \cdots (n-d+t)} \geq \frac{1 - \left( \frac{d-t+1}{(n-d+t)(q-1)} \right)^t}{1 - \frac{d-t+1}{(n-d+t)(q-1)}} \frac{n-d+1 + (t-1)}{n - \left( \frac{q}{q-1} \right) (d-t+1) + 1}.
\]

Observe that

\[
\frac{n-d+1 + (t-1)}{n - \left( \frac{q}{q-1} \right) (d-t+1) + 1} = \frac{n-d+1 + (t-1)}{n-d+1} \frac{n - \left( \frac{q}{q-1} \right) d + 1}{n - \left( \frac{q}{q-1} \right) (d-t+1) + 1} \frac{n-d+1}{n - \left( \frac{q}{q-1} \right) d + 1}
\]

\[
= \left( 1 + \frac{t-1}{n-d+1} \right) \left( 1 + \frac{\frac{q}{q-1} (t-1)}{n - \left( \frac{q}{q-1} \right) d + 1} \right)^{-1} \frac{n-d+1}{n - \left( \frac{q}{q-1} \right) d + 1}.
\]

With \( d \leq (1 - \alpha) \left( 1 - \frac{1}{q} \right) n \), we have

\[
n-d+1 > n - \left( \frac{q}{q-1} \right) d + 1 \geq \alpha n.
\]

Thus, if \( t \leq \alpha n \), we obtain
\[
\frac{n - d + 1 + (t - 1)}{n + \frac{q}{q - 1}(d - t + 1) + 1} = \left(1 + O\left(\frac{t}{\alpha n}\right)\right) \frac{n - d + 1}{n - \left(\frac{q}{q - 1}\right) d + 1}. 
\]

Taking \( t = -\frac{\log n}{\log(1 - \frac{1}{2}\alpha)} \), and using the assumption on \( d \), we get for all large enough \( n \)
\[
\left(\frac{d - t + 1}{(n - d + t)(q - 1)}\right)^t \leq \left(\frac{(1 - \alpha)(1 - \frac{1}{q})n + 1}{(n - (1 - \frac{1}{q})n)(q - 1)}\right)^t \leq \left(1 - \frac{1}{2\alpha}\right)^t = \frac{1}{n}.
\]

Combining the above, we get (1).

Next, observe that for \( t \leq n - d \) we have
\[
\left(\frac{n - d - t + 1}{d + t}\right)^t \left(\frac{n}{d}\right) \leq \left(\frac{n - d}{d + 1}\right)^t \left(\frac{n}{d}\right)
\]
and
\[
\frac{n - d - t + 1}{d + t} = \frac{n - d - t + 1}{d + t} \frac{d + 1}{n - d} = \left(1 - \frac{t - 1}{n - d}\right) \left(1 - \frac{t - 1}{d + t}\right)n - d = \left(1 + O_\alpha\left(\frac{t}{d + t}\right)\right)\frac{n - d}{d + 1}.
\]

Hence, when \( t \leq \sqrt{d} \), we have
\[
\frac{n - d - t}{d + t} = \left(1 + O_\alpha\left(\frac{t^2}{d}\right)\right)\left(\frac{n - d}{d + 1}\right)^t.
\]

Combining this with (1) and (3), we get
\[
\frac{\rho_{d+t}}{\rho_d} = \left(\frac{q - 1}{d}\right)^t \left(1 + O_\alpha\left(\frac{\log n}{n} + \frac{t^2}{d}\right)\right).
\]

\(\square\)

The next proposition provides an approximation of the minimum of independent binomial variables in terms of the Gumbel distribution. Although the computations seem to be rather standard, we prefer to include them for reader's convenience.

**Proposition 2.2.** Fix \( q \geq 2 \) and \( \alpha \in (0, 1) \). Let \( q^{\alpha n} \leq m \leq q^{(1-\alpha)n} \) and let \( d_0 \) be the largest integer such that \( \rho_{d_0} m \leq 1 \). Let \( X_1, \ldots, X_m \) be i.i.d. binomial random variables with parameters \( n \) and \( \frac{1}{q} \), i.e.
\[
P\{X_j = a\} = \binom{n}{a} \left(\frac{1 - \frac{1}{q}}{6}\right)^a q^{a-n}, \quad a = 0, 1, \ldots, n,
\]
and set \( Y := \min_{j=1,\ldots,m} X_j \). Then
\[
\mathbb{P}\left\{ Y - d_0 > \frac{-t}{\log \frac{(q-1)(n-d_0)}{d_0}} - \frac{\log(\rho_{d_0} m)}{\log \frac{(q-1)(n-d_0)}{d_0}} \right\} = o_{\alpha,q}(1) + \exp \left( - e^{-t} \right),
\]
for all \( t \in \log \frac{(q-1)(n-d_0)}{d_0} \mathbb{Z} - \log(\rho_{d_0} m) \).

**Proof.** Without loss of generality, we can assume that \( n \geq C_{\alpha,q} \) for a large constant \( C_{\alpha,q} \) depending on \( \alpha, q \). Observe that there is \( c \in (0,1) \) depending only on \( \alpha \) and \( q \) such that the condition on \( m \) implies \( cn \leq d_0 \leq (1-c)(n-n/q) \). For any integer \( d \) we have
\[
\mathbb{P}\{ Y \leq d \} = 1 - (1 - \mathbb{P}\{ X_1 \leq d \})^m,
\]
where in our notation,
\[
\mathbb{P}\{ X_1 \leq d \} = \rho_d.
\]
Thus,
\[
1 - \mathbb{P}\{ Y \leq d \} = \exp \left( - (1 + o_{\alpha,q}(1)) \rho_d m \right) = \exp \left( - (1 + o_{\alpha,q}(1)) \rho_{d_0} m \frac{\rho_d}{\rho_{d_0}} \right),
\]
whenever \( d \) is an integer less equal to \( (1-c/2)(n-n/q) \), so that \( \rho_d = o_{\alpha,q}(1) \). Further, applying the second assertion of Proposition 2.1 we get that for any integer \( d \) with \( |d-d_0| = o(\sqrt{d_0}) \),
\[
\frac{\rho_d}{\rho_{d_0}} = (1 + o_{\alpha,q}(1)) \left( \frac{(q-1)(n-d_0)}{d_0} \right)^{d-d_0}.
\]
Hence,
\[
1 - \mathbb{P}\{ Y \leq d \} = \exp \left( - (1 + o_{\alpha,q}(1)) \rho_{d_0} m \left( \frac{(q-1)(n-d_0)}{d_0} \right)^{d-d_0} \right), \quad |d-d_0| = o(\sqrt{d_0}).
\]
Further, note that by our conditions on \( m \) we have \( \rho_{d_0} m \geq \tilde{c} \) for some \( \tilde{c} > 0 \) depending only on \( \alpha \) and \( q \). Moreover, \( \frac{(q-1)(n-d_0)}{d_0} > \frac{1}{1-c} \), by the above observation for \( d_0 \). Thus, we can write
\[
1 - \mathbb{P}\{ Y \leq d \} = o_{\alpha,q}(1) + \exp \left( - \rho_{d_0} m \left( \frac{(q-1)(n-d_0)}{d_0} \right)^{d-d_0} \right)
\]
for all integers \( d \), or, in other form,
\[
1 - \mathbb{P}\left\{ Y - d_0 \leq \frac{t}{\log \frac{(q-1)(n-d_0)}{d_0}} - \frac{\log(\rho_{d_0} m)}{\log \frac{(q-1)(n-d_0)}{d_0}} \right\} = o_{\alpha,q}(1) + \exp \left( - e^t \right),
\]
for all \( t \in \log \frac{(q-1)(n-d_0)}{d_0} \mathbb{Z} + \log(\rho_{d_0} m) \). The result follows. \( \square \)

It is not difficult to see that the above proposition and the main theorem of the paper imply Theorem 1.2.
3. Moments comparison for parallel codes

Fix \( a \in \mathbb{R}^k \) and \( d \geq 0 \). Given the independent random vectors \( X_1, \ldots, X_k \) uniform on \( \mathbb{F}_q^n \), we define

\[
Z_d := \sum_{a \in \mathbb{F}_q^k \setminus \{0\}} W_a(d), \quad d \geq 0,
\]

where \( W_a(d) \) is the indicator of the event

\[
\left\{ \text{wt} \left( \sum_{i=1}^k a_i X_i \right) \leq d \right\}.
\]

For any \( a, b \in \mathbb{F}_q^k \setminus \{0\} \), we say \( a \) and \( b \) are parallel if there exists \( f \in \mathbb{F}_q \setminus \{0\} \) such that \( a = fb \) (here the multiplication is in the field \( \mathbb{F}_q \)). Notice that if \( a \) and \( b \) are parallel, then, \( \sum_{i=1}^k a_i X_i \) and \( \sum_{i=1}^k b_i X_i \) are parallel. In particular, the supports of the linear combinations are the same, and thus \( W_a(d) = W_b(d) \) whenever \( a \) and \( b \) are parallel.

Let \( \{Y_a\}_{a \in \mathbb{F}_q^k \setminus \{0\}} \) be random vectors uniformly distributed on \( \mathbb{F}_q^k \) and mutually independent up to the constraint that \( Y_a = Y_b \) whenever \( a \) and \( b \) are parallel. Define

\[
\tilde{Z}_d := \sum_{a \in \mathbb{F}_q^k \setminus \{0\}} \tilde{W}_a(d)
\]

where \( \tilde{W}_a(d) \) is the indicator function of the event \( \{\text{wt}(Y_a) \leq d\} \).

The goal of this section is to compare the moments of \( \tilde{Z}_d \) and \( Z_d \) assuming certain constraints on the parameters \( n, k \) and \( d \). The main statement of the section is

**Proposition 3.1.** For any \( \lambda_0 \in (0, 1) \), there are \( \tilde{c}_{\mathbb{F}_q}(\lambda_0, q) > 0 \) and \( C_{\mathbb{F}_q}(\lambda_0, q) > 0 \) with the following property. Suppose \( d, n \in \mathbb{N} \) satisfy \( \frac{2}{n} \leq \lambda_0 (1 - \frac{1}{q}) \), and \( d^2/n^{3/2} \geq C_{\mathbb{F}_q}(\lambda_0, q) \).

Then for any positive integer \( m \leq \tilde{c}_{\mathbb{F}_q}(\lambda_0, q) d^2/n^3 \) such that \( q^k \rho_d \geq \exp \left( -\frac{(q-1)\lambda_0 d^2}{n^2 m} \right) \), we have

\[
\mathbb{E}Z_d^m = (1 + O(\exp(-(q-1)\lambda_0 d^2/n^3))) + O(2^{-k/2}) \mathbb{E}\tilde{Z}_d^m.
\]

Before proving the proposition we need to make some preparatory work.

Let \( \ell \leq m \leq k \) be positive integers. Suppose \( I_1, \ldots, I_\ell \) is a partitioning of \([m]\) into non-empty set. Denote by \( \Omega(I_1, \ldots, I_\ell) \) the collection of all \( m \)-tuples \((a^1, \ldots, a^m) \in (\mathbb{F}_q^k \setminus \{0\})^m \) such that \( a^i \) is parallel to \( a^j \) if and only if \( i, j \in I_t \) for some \( t \in [\ell] \). Further, define

\[
(5) \quad \Omega_\ell := \{ (v^1, \ldots, v^\ell) \in (\mathbb{F}_q^k \setminus \{0\})^\ell : v^1, \ldots, v^\ell \text{ are pairwise non-parallel} \}.
\]

Note that there is a natural \((q-1)^{m-\ell}\)–to–one mapping from \( \Omega(I_1, \ldots, I_\ell) \) onto \( \Omega_\ell \) which assigns \((a_{\min(j \in I_t)})_{t=1}^\ell \) to each \((a^1, \ldots, a^m)\).

Now, in view of the above remarks,

\[
Z_d^m = \sum_{a^1, \ldots, a^m \in \mathbb{F}_q^k \setminus \{0\}} \prod_{i=1}^m W_{a^i}(d) = \sum_{\ell=1}^m \sum_{I_1, \ldots, I_\ell} \sum_{v^1, \ldots, v^\ell \in \Omega_\ell} (q-1)^{m-\ell} \prod_{i=1}^\ell W_{v^i[d]}(d),
\]

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where the second summation is taken over all partitions \( I_1, \ldots, I_\ell \) of \([m]\) into non-empty sets. Notice that \( W_{\varphi}^{[1]}(d) = W_{\varphi}(d) \), so we can simplify the above representation to

\[
Z_d^m = \sum_{\ell=1}^{m} S(m, \ell) (q - 1)^{m-\ell} \left( \sum_{v^1, \ldots, v^\ell \in \Omega_\ell} \prod_{i=1}^{\ell} W_{\varphi}(d) \right),
\]

where \( S(m, \ell) \) is the number of ways to partition \([m]\) into \( \ell \) non-empty sets (a Stirling number of the second kind). The above formula works for \( \hat{Z}_d^m \) as well, up to replacing \( W_{\varphi}(d) \) with \( \hat{W}_{\varphi}(d) \).

The central technical statement of the section is the following

**Proposition 3.2.** For any \( \lambda_0 \in (0, 1) \) there are \( c_{\lambda_0, 2}(\lambda_0, q) > 0 \) and \( c_{\lambda_0, 3}(\lambda_0, q) > 0 \) with the following property. Suppose \( d, n \in \mathbb{N} \) satisfy \( d/n \leq \lambda_0(1 - 1/q) \) and \( d \geq c_{\lambda_0, 2}(\lambda_0, q) \). Suppose further that \( s \leq k \), and \((v^1, v^2, \ldots, v^s)\) are linearly independent vectors in \( \mathbb{F}_q \), and that \( v^{s+1} = \sum_{i=1}^{s} c_i v^i \) for some \( c_i \in \mathbb{F}_q \setminus \{0\} \). Then

\[
\mathbb{E} \prod_{i=1}^{s+1} W_{\varphi}(d) \leq C \rho_d^s \exp(-c_{\lambda_0, 3}(\lambda_0, q)d^4/n^3)
\]

where \( C > 0 \) is a universal constant.

Let us postpone the proof, and proceed with the argument. As a corollary of the above statement, we have

**Corollary 3.3.** Suppose \( d, n \in \mathbb{N} \) are as in Proposition 3.2. Suppose further that \( \ell \leq k \), and \( v^1, v^2, \ldots, v^\ell \) are non-zero vectors in \( \mathbb{F}_q \) such that the rank of \((v^1, v^2, \ldots, v^\ell)\) is \( r < \ell \). Then

\[
\mathbb{E} \prod_{i=1}^{\ell} W_{\varphi}(d) \leq C \rho_d^r \exp(-c_{\lambda_0, 3}(\lambda_0, q)d^4/n^3).
\]

**Proof.** By rearranging the indices, we may assume there exists \( s \) such that the vectors \( v^1, \ldots, v^s \), \( v^{s+2}, \ldots, v^{r+1} \) form a linearly independent set, and \( v^{s+1} = \sum_{i=1}^{s} c_i v^i \) where \( c_i \in \mathbb{F}_q \setminus \{0\} \). Next,

\[
\mathbb{E} \prod_{i=1}^{\ell} W_{\varphi}(d) \leq \mathbb{E} \prod_{i=1}^{r+1} W_{\varphi}(d) = \mathbb{E} \prod_{i=1}^{s+1} W_{\varphi}(d) \prod_{j=s+2}^{r+1} W_{\varphi}(d) = \mathbb{E} \prod_{i=1}^{s+1} W_{\varphi}(d) \rho_d^{r-s},
\]

and the result follows by applying Proposition 3.2. \( \square \)

Next, we need to estimate cardinality of the set of \( \ell \)-tuples of vectors \((v^1, v^2, \ldots, v^\ell)\) in \( \Omega_\ell \) with a given rank \( r \).

**Lemma 3.4.** For \( r \leq \ell \leq k \), denote

\[
\Omega_{r, \ell} := \left\{ (v^1, v^2, \ldots, v^\ell) \in \Omega_\ell : \dim \left( \text{span} \left( v^1, \ldots, v^\ell \right) \right) = r \right\}.
\]
Then
\[ |\Omega_{r,\ell}| \leq \binom{\ell}{r} q^{r}\prod_{i=0}^{r-1} (q^k - q^i). \]

When \( r = \ell \), equality holds, implying
\[ \frac{|\Omega_{r,\ell}|}{|\Omega_{\ell,\ell}|} \leq \binom{\ell}{r} \frac{q^{r-\ell}}{\prod_{i=0}^{\ell-1} (q^k - q^i)}. \]

Proof. Let \( I \subset [\ell] \) with \( |I| = r \). We will consider vectors \((v^1, \ldots, v^\ell)\) such that \( \{v^i\}_{i \in I} \) forms a linearly independent set, and \( v^j \) lies in the span of \( \{v^i\}_{i \notin I} \) for each \( j \notin I \). Without loss of generality, we will assume \( I = [r] \). The cardinality of the set of all \( r \)-tuples \((v^1, \ldots, v^r)\) of linearly independent vectors is
\[ \left( q^k - 1 \right) \left( q^k - q \right) \left( q^k - q^2 \right) \cdots \left( q^k - q^{r-1} \right) \]
where \( q^k - q^{i-1} \) represents the number of choices of \( v^i \) which is not a linear combination of \( v^1, \ldots, v^{i-1} \) with the assumption that \( v^1, \ldots, v^{i-1} \) are linearly independent. Having chosen \( v^1, \ldots, v^r \), the vectors \( v^{r+1}, \ldots, v^{\ell} \) are vectors in the span of \( v^1, \ldots, v^\ell \), which has cardinality \( q^\ell \). Therefore, there are at most \( (q^r)^{\ell-r} \prod_{i=0}^{r-1} (q^k - q^i) \) choices for \((v^1, \ldots, v^\ell)\) satisfying the above conditions. We obtain the desired bound because there are \( \binom{\ell}{r} \) subsets of \([\ell]\) with cardinality \( r \). \( \square \)

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. We set
\[ c_{3.1}(\lambda_0, q) := \frac{c_{3.2}(\lambda_0, q)}{4 + 2 \log q} \quad \text{and} \quad C_{3.1}(\lambda_0, q) := \max(C_{3.2}(\lambda_0, q), 1 / c_{3.1}(\lambda_0, q)). \]

Assume that \( d, n, k \) satisfy the assumptions of the proposition.

Fix any \( \ell \leq m \). For each \( r \leq \ell \), let \( \Omega_{r,\ell} \) be defined as in Lemma 3.4. For \((v^1, \ldots, v^\ell) \in \Omega_{\ell}\), we have \( \mathbb{E} \prod_{i=1}^{\ell} W_{v^i}(d) = \rho_d^\ell \) whenever \( v^1, \ldots, v^\ell \) form a linearly independent set. Thus,
\[ \mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{\ell}} \prod_{i=1}^{\ell} W_{v^i}(d) = \mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{\ell}} \prod_{i=1}^{\ell} \widetilde{W}_{v^i}(d) = \rho_d^\ell |\Omega_{\ell,\ell}|. \]

Further, take any \( r < \ell \), and observe that, in view of Corollary 3.3, we have
\[ \mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{r,\ell}} \prod_{i=1}^{\ell} W_{v^i}(d) \leq \rho_d^r \exp(-\alpha_{3.2}(\lambda_0, q)d^4/n^3) |\Omega_{r,\ell}|. \]
Applying the estimate from Lemma 3.4 to the last expression, we obtain
\[
\mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{\ell, \ell}} \prod_{i=1}^\ell W_{v^i}(d) \leq C \rho_d^\ell |\Omega_{\ell, \ell}| \exp\left(-c_{3.2}(\lambda_0, q) d^4/n^3\right) \left(\frac{\ell}{\ell - r}\right) \frac{\rho_d^{r-\ell} (q^r)^{\ell-r}}{\prod_{i=r}^{\ell-1} (q^k - q^i)} \leq C \rho_d^\ell |\Omega_{\ell, \ell}| \exp\left(-c_{3.2}(\lambda_0, q) d^4/n^3\right)
\]

Further, using our assumptions on the parameters, we get
\[
\frac{2^\ell q^{\ell^2}}{(\rho_d q^k)^{\ell-r}} \leq \exp\left(q_{3.1}(\lambda_0, q) (1 + \log q) d^4/n^3\right) \exp\left(\frac{q_{3.1}(\lambda_0, q) d^4(\ell - r)}{n^3 \ell}\right) \leq \exp\left(-r + \frac{1}{2} q_{3.2}(\lambda_0, q) d^4/n^3\right).
\]

Thus,
\[
\mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{\ell, \ell}} \prod_{i=1}^\ell W_{v^i}(d) = (1 + O(\exp(-q_{3.1}(\lambda_0, q) d^4/n^3))) \rho_d^\ell |\Omega_{\ell, \ell}|.
\]

On the other hand, applying Lemma 3.4, we get
\[
|\Omega_{\ell, \ell}| = \prod_{i=0}^{\ell-1} (q^k - q^i) \geq (1 - O(q^{\ell-k})) |\Omega_{\ell}|.
\]

Hence,
\[
\mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{\ell, \ell}} \prod_{i=1}^\ell W_{v^i}(d) = (1 + O(\exp(-q_{3.1}(\lambda_0, q) d^4/n^3))) + O(2^{-\ell}) \mathbb{E} \sum_{v^1, \ldots, v^\ell \in \Omega_{\ell, \ell}} \prod_{i=1}^\ell \tilde{W}_{v^i}(d).
\]

The result follows by applying formula (6).

□

The rest of the section is devoted to proving Proposition 3.2.

Lemma 3.5. Suppose \(M \in GL_k(\mathbb{F}_q)\) is a fixed invertible \(k \times k\) matrix over the field \(\mathbb{F}_q\). Let \(Y\) be a random vector uniformly distributed in \(\mathbb{F}_q^k\). Then, the image \(MY\) is uniformly distributed in \(\mathbb{F}_q^k\).

As a corollary, we obtain

Corollary 3.6. Suppose \(a^1, \ldots, a^\ell \in \mathbb{F}_q^k \setminus \{0\}\) are linearly independent fixed vectors, and let, as before, \(X_1, \ldots, X_k\) be i.i.d. random vectors uniformly distributed in \(\mathbb{F}_q^n\). Then the random vectors
\[
\sum_{i=1}^k a_i^{\ell'} X_i, \quad \ell' = 1, \ldots, \ell,
\]
are mutually independent and uniformly distributed over \(\mathbb{F}_q^n\).
Proof. It is sufficient to show that for each $t \in [n]$, the random variables $\sum_{i=1}^{k} a_{i}^{\ell} X_{i}(t)$, $\ell = 1, \ldots, \ell$, are mutually independent and uniform over $\mathbb{F}_{q}$, where by $X_{i}(t)$ we denote the $t$-th component of $X_{i}$.

If $\ell < k$ then we can find vectors $a^{\ell+1}, a^{\ell+2}, \ldots, a^{k}$ such that $a^{1}, \ldots, a^{k}$ are linearly independent. Let $M$ be the $k \times k$ matrix with rows $a^{1}, \ldots, a^{k}$ and let $Y$ be the random vector in $\mathbb{F}_{q}^{k}$ with components $X_{i}(t), \ldots X_{k}(t)$. Applying the above lemma, we get that the components of the vector $MY$ are mutually independent and uniform over $\mathbb{F}_{q}$. The result follows.

Lemma 3.7. For any $\lambda_{0} \in (0, 1)$ there is $c(\lambda_{0}, q) > 0$ with the following property. Suppose $d, n \in \mathbb{N}$ satisfy $\frac{d}{n} \leq \lambda_{0}(1 - \frac{1}{d})$. Let $\varepsilon = \min \left\{ \frac{1}{4}, \frac{1 - \lambda_{0}}{2} \right\}$ and take $\kappa, \gamma > 0$ such that $\gamma d$ and $\kappa d$ are integers, $1 - \varepsilon \leq \kappa \leq \frac{n}{d}$, and $(1 - \varepsilon) \leq \gamma \leq 1$. Let $V = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{F}_{q}^{n}$, and let $Y$ be a random vector uniformly distributed in $\mathbb{F}_{q}^{n}$. Then,

$$\mathbb{P} \{ \text{wt} (V + Y) \leq d \mid \text{wt} (Y) = \gamma d \} \leq 3 \exp(-c(\lambda_{0}, q)d^{4}/n^{3}).$$

Proof. Let $J$ be a random subset of size $\gamma d$ uniformly distributed in $[n]$. Let $u_{1}, \ldots, u_{n}$ be i.i.d random variables uniformly distributed in $\mathbb{F}_{q} \setminus \{0\}$, independent from $J$. Then the conditional distribution of $Y$ given $\text{wt} (Y) = \gamma d$ coincides with the distribution of

$$U = (u_{1}1_{J}(1), \ldots, u_{n}1_{J}(n))^{\top},$$

where $1_{J}()$ is the indicator function of $J$. Thus,

$$\mathbb{P} \{ \text{wt} (V + Y) \leq d \mid \text{wt} (Y) = \gamma d \} = \mathbb{P} \{ \text{wt} (V + U) \leq d \}.$$

Notice that

$$\text{wt} (V + U) = \kappa d + \gamma d - ||[\kappa d] \cap J|| - |\{i \in [\kappa d] \cap J : u_{i} = -1\}|. \quad (7)$$

The random variable $||[\kappa d] \cap J||$ has the hypergeometric distribution given by

$$\mathbb{P} ||[\kappa d] \cap J|| = \ell = \binom{n}{\kappa d}^{-1} \binom{n - \kappa d}{\gamma d - \ell} \binom{\kappa d}{\ell}, \quad \ell \geq 0.$$ 

The expected size of $[\kappa d] \cap J$ is $\frac{\kappa d}{n} \gamma d \leq \kappa \gamma \lambda_{0} (\frac{q-1}{q}) d$. Applying the large deviation inequality for the hypergeometric distribution (see, e.g. Talagrand [12]), we then get

$$\mathbb{P} \left\{ \left| ||[\kappa d] \cap J|| - \frac{\kappa d}{n} \gamma d \right| \geq t \gamma d \right\} \leq 2 \exp(-2t^{2} \gamma d), \quad t > 0. \quad (8)$$

Let $\mathcal{E}_{1}(s)$ be the event that $||[\kappa d] \cap J|| - \frac{\kappa d}{n} \gamma d \leq \kappa s \left( \frac{q-1}{q} \right) \gamma d$, where $s > 0$ will be determined later. Further, let $\mathcal{E}_{2}(s)$ be the event that

$$|\{i \in [\kappa d] \cap J : u_{i} = -1\}| \leq \left( \frac{1}{q-1} + s \right) ||[\kappa d] \cap J||.$$

By Hoeffding’s inequality, we have

$$\mathbb{P} (\mathcal{E}_{2}(s) \mid J) \leq \exp \left( -2s^{2} ||[\kappa d] \cap J|| \right). \quad (9)$$
Whenever both $\mathcal{E}_1(s)$ and $\mathcal{E}_2(s)$ hold, we have, in view of (7),

$$
\text{wt}(V + U) \geq \kappa d + \gamma d - \left(1 + \frac{1}{q - 1} + s\right) ||\kappa d\cap J||
$$

$$
\geq \kappa d + \gamma d - \left(\frac{q}{q - 1} + s\right)\kappa\gamma d\left(\frac{d}{n} + s\frac{q - 1}{q}\right).
$$

Now, we set $s := \min\left(\frac{qd}{2n(q-1)}, \frac{1 - \lambda_0}{6}\right)$, so that the above relation implies

$$
\text{wt}(V + U) \geq d\left(\kappa + \gamma - \frac{\lambda_0 + 1}{2}\gamma\right) \text{ everywhere on } \mathcal{E}_1(s) \cap \mathcal{E}_2(s).
$$

If $\kappa\frac{\lambda_0 + 1}{2} \geq 1$ then

$$
\kappa + \gamma - \frac{\lambda_0 + 1}{2}\gamma \geq \kappa + 1 - \kappa\frac{\lambda_0 + 1}{2} = 1 + \kappa\frac{1 - \lambda_0}{2},
$$

which implies that $\text{wt}(U + V) > d$ everywhere on $\mathcal{E}_1(s) \cap \mathcal{E}_2(s)$.

Next, if $\kappa\frac{\lambda_0 + 1}{2} < 1$ then we have

$$
\kappa + \gamma - \frac{\lambda_0 + 1}{2}\gamma \geq (1 - \varepsilon) + (1 - \varepsilon)(1 - 3\varepsilon)\frac{\lambda_0 + 1}{2}
$$

$$
= 1 + \frac{1 - \lambda_0}{2} - 2\varepsilon - \frac{1 - \lambda_0}{2} - \varepsilon\frac{\lambda_0 + 1}{2}.
$$

Using the definition $\varepsilon = \min\left(\frac{1}{4}, \frac{1 - \lambda_0}{6}\right)$, we get

$$
\kappa + \gamma - \frac{\lambda_0 + 1}{2}\gamma \geq 1 + 1 - 3\varepsilon\frac{1 - \lambda_0}{2} > 1
$$

which, again, implies that $\text{wt}(U + V) > d$ everywhere on $\mathcal{E}_1(s) \cap \mathcal{E}_2(s)$.

It remains to estimate the probability of $\mathcal{E}_1(s) \cap \mathcal{E}_2(s)$. In view of (9), the definition of $\mathcal{E}_1(s)$, and the condition $s \leq \frac{qd}{2n(q-1)}$, we have

$$
\mathbb{P}(\mathcal{E}_2^c(s) \mid \mathcal{E}_1(s)) \leq \exp(-c(\lambda_0, q)d^4/n^3),
$$

whereas (8) yields

$$
\mathbb{P}(\mathcal{E}_1^c(s)) \leq 2\exp(-c(\lambda_0, q)d^3/n^2)
$$

for some $c(\lambda_0, q) > 0$ depending only on $\lambda_0$ and $p$. Hence,

$$
\mathbb{P}\{\text{wt}(V + Y) \leq d \mid \text{wt}(Y) = \gamma d\} \leq \mathbb{P}\{(\mathcal{E}_1(s) \cup \mathcal{E}_2(s))^c\} \leq 3\exp(-c(\lambda_0, q)d^4/n^3),
$$

and the result follows. \hfill \Box

**Proof of Proposition 3.2** Notice that

$$
W_{v'}(d) = \mathbf{1}\{\text{wt}(\sum_{i=1}^{k} c_i v_i' X_i) \leq d\} = \mathbf{1}\{\text{wt}(\sum_{i=1}^{k} c_i v_i' X_i) \leq d\}.
$$

Further, since the vectors $c_1 v_1, \ldots, c_s v^s$ are linearly independent, by Corollary 3.6 the joint distribution of the vectors

$$
\sum_{i=1}^{k} c_1 v_i^1 X_i, \sum_{i=1}^{k} c_2 v_i^2 X_i, \ldots, \sum_{i=1}^{k} c_s v_i^s X_i, \sum_{i=1}^{k} \left(\sum_{t=1}^{s} c_t v_i^t\right) X_i
$$
is the same as that of $Y_1, \ldots, Y_s, \sum_{t=1}^s Y_t$, where $Y_1, \ldots, Y_s$ are i.i.d. copies of $X_1$. Thus,

$$
\mathbb{E} \left( \prod_{i=1}^s W_{v_i}(d) \right) W_{\sum_{i=1}^s c_i v_i}(d)
= \mathbb{P} \{ \forall i \in [s] \text{ wt}(Y_i) \leq d, \text{ and wt}(S + Y_s) \leq d \}
= \mathbb{E}(1_{\{\text{wt}(Y_1) \leq d, \forall i \leq s-2\}}) \mathbb{P} \{ \text{wt}(Y_{s-1}) \leq d, \text{ wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d \mid Y_1, \ldots, Y_{s-2} \},
$$

where $S := \sum_{i=1}^{s-1} Y_i$. Set $P_1 := \mathbb{P} \{ \text{wt}(Y_{s-1}) \leq d, \text{ wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d \mid Y_1, \ldots, Y_{s-2} \}$. We shall break $P_1$ into two components:

$$
P_1 = \mathbb{P} \{ \text{wt}(Y_{s-1}) \leq d, \text{ wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d, \text{ wt}(S) < (1 - \varepsilon)d \mid Y_1, \ldots, Y_{s-2} \}
+ \mathbb{P} \{ \text{wt}(Y_{s-1}) \leq d, \text{ wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d, \text{ wt}(S) \geq (1 - \varepsilon)d \mid Y_1, \ldots, Y_{s-2} \}
\leq \mathbb{P} \{ \text{wt}(Y_s) \leq d, \text{ wt}(S) < (1 - \varepsilon)d \mid Y_1, \ldots, Y_{s-2} \}
+ \mathbb{P} \{ \text{wt}(Y_{s-1}) \leq d, \text{ wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d, \text{ wt}(S) \geq (1 - \varepsilon)d \mid Y_1, \ldots, Y_{s-2} \}.
$$

Notice that $Y_1, \ldots, Y_{s-2}, S$ and $Y_s$ are mutually independent (and uniform on $\mathbb{F}_q^n$), so we have

$$
A = \mathbb{P} \{ \text{wt}(Y_1) \leq d, \text{ wt}(Y_2) < (1 - \varepsilon)d \} \leq \rho_d^2 \exp(-c(\lambda_0, q)d)
$$
for some $c(\lambda_0, q) > 0$, where we have used the second assertion of Proposition 2.1 For the second summand,

$$
B = \mathbb{E}(1_{\{\text{wt}(Y_{s-1}) \leq d\}} 1_{\{\text{wt}(Y_s) \leq d\}} 1_{\{\text{wt}(S + Y_s) \leq d\}} 1_{\{\text{wt}(S) \geq (1 - \varepsilon)d\}} \mid Y_1, \ldots, Y_{s-2})
= \mathbb{E}\left( \mathbb{E}(1_{\{\text{wt}(Y_{s-1}) \leq d\}} 1_{\{\text{wt}(S + Y_s) \leq d\}} \mid Y_1, \ldots, Y_{s-2}, 1_{\{\text{wt}(Y_{s-1}) \leq d\}} 1_{\{\text{wt}(S) \geq (1 - \varepsilon)d\}} \mid Y_1, \ldots, Y_{s-2}) \right)
= \mathbb{E}(1_{\{\text{wt}(Y_{s-1}) \leq d\}} 1_{\{\text{wt}(S + Y_s) \leq d\}} \mid \text{ wt}(S)) \cdot 1_{\{\text{wt}(Y_{s-1}) \leq d\}} 1_{\{\text{wt}(S) \geq (1 - \varepsilon)d\}} \mid Y_1, \ldots, Y_{s-2}),
$$

where we used the fact that the conditional expectation

$$
\mathbb{E}(1_{\{\text{wt}(Y_s) \leq d\}} 1_{\{\text{wt}(S + Y_s) \leq d\}} \mid Y_1, \ldots, Y_{s-2}, 1_{\{\text{wt}(Y_{s-1}) \leq d\}} 1_{\{\text{wt}(S) \geq (1 - \varepsilon)d\}}, \text{ wt}(S))
$$

is actually measurable with respect to $\text{wt}(S)$. On the event $\{\text{wt}(Y_{s-1}) \leq d, \text{ wt}(S) \geq (1 - \varepsilon)d\}$ we have

$$
\mathbb{E}(1_{\{\text{wt}(Y_s) \leq d\}} 1_{\{\text{wt}(S + Y_s) \leq d\}} \mid \text{ wt}(S))
= \mathbb{P} \{ \text{wt}(Y_s) \leq (1 - \varepsilon)d, \text{ wt}(S + Y_s) \leq d \mid \text{ wt}(S) \}
+ \mathbb{P} \{(1 - \varepsilon)d < \text{wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d \mid \text{ wt}(S) \}
\leq \mathbb{P} \{ \text{wt}(Y_s) \leq (1 - \varepsilon)d \} + \mathbb{P} \{(1 - \varepsilon)d < \text{wt}(Y_s) \leq d, \text{ wt}(S + Y_s) \leq d \mid \text{ wt}(S) \}
\leq \rho_d \exp(-c(\lambda_0, q)d) + \mathbb{E}(1_{\{(1 - \varepsilon)d < \text{wt}(Y_s) \leq d\}} \mathbb{P} \{ \text{wt}(S + Y_s) \leq d \mid \text{ wt}(S), \text{ wt}(Y_s) \} \mid \text{ wt}(S))
$$

where we again used the second assertion of Proposition 2.1. To estimate the second summand in the last expression, we apply Lemma 3.7 First, let us make the following observation.
For any fixed $n \times n$ diagonal matrix $D$ whose diagonal entries are non-zero elements in $F$ and any $v \in \mathbb{F}^n$, we have $w(t) = w(Dv)$, and, furthermore, $DX_1$ and $X_1$ have the same distribution. Similarly, for any fixed $n \times n$ permutation matrix $P$, $w(t) = w(Pv)$ for any vector $v \in \mathbb{F}^n$, and $PX_1$ and $X_1$ have the same distribution. Now, note that we can construct a random permutation matrix $P$ and a random diagonal matrix $D$ with non-zero diagonal elements, both measurable with respect to $S$, such that everywhere on the probability space $PDS = V$, where $V = \left( \begin{array}{c} 1, \ldots, 1, 0, \ldots, 0 \end{array} \right)^\top$. Then we have

$$P \{ w(S + Y_s) \leq d \mid w(S), w(Y_s) \} = P \{ w(V + PDY_s) \leq d \mid w(S), w(Y_s) \} = P \{ w(V + Y_s) \leq d \mid w(S), w(Y_s) \},$$

where we used that $Y_s$ is independent from $P, D, S$.

The above observation allows us to use Lemma 3.7: conditioning on any value $k_1 \in [(1 - \varepsilon)d, n]$ of $w(S)$ and any value $k_2 \in [(1 - \varepsilon)d, d]$ of $w(Y_s)$, we have

$$P \{ w(S + Y_s) \leq d \mid w(S) = k_1, w(Y_s) = k_2 \} \leq 3 \exp(-c'(\lambda_0, q)d^4/n^3)$$

for some $c'(\lambda_0, q) > 0$ depending only on $\lambda_0$ and $q$. Thus, everywhere on $\{ w(Y_{s-1}) \leq d, w(S) \geq (1 - \varepsilon)d \}$ we have

$$\mathbb{E} \{ 1_{\{ w(Y_s) \leq d \}} I_{\{ w(S + Y_s) \leq d \}} \mid w(S) \} \leq 4 \rho_d \exp(-c''(\lambda_0, q)d^4/n^3),$$

where $c''(\lambda_0, q) := \min(c(\lambda_0, q), c'(\lambda_0, q))$. That, in turn, implies

$$B \leq 4 \rho_d^2 \exp(-c''(\lambda_0, q)d^4/n^3).$$

Then, we have

$$P_1 \leq 5 \rho_d^2 \exp(-c''(\lambda_0, q)d^4/n^3)$$

and the result follows since $\mathbb{P} \left\{ w(Y_i) \leq d, \forall i \leq s - 2 \right\} = \rho_d^{s-2}$. \hfill \Box

4. Analysis of the distribution of the minimal distance

The goal of this section is to prove our main result comparing the distributions of the minimal distance of the random linear code, with the minimum $w_{\min}$ of the weights of the random vectors $Y_a$, $a \in \mathbb{F}_q^k \setminus \{0\}$ (defined earlier in the paper).

First, we state the “technical” version of the result:

**Theorem 4.1.** For any $\lambda_0 \in (0, 1)$ there are $c_{d, l}(\lambda_0, q) > 0$ and $C_{d, l}(\lambda_0, q) > 0$ with the following property. Let $n \geq 1$, and take any $L \geq e$. Assume further that $k$ satisfies $C_{d, l}(\lambda_0, q)L \log L \leq k \leq n$, and take any $d$ such that

$$C_{d, l}(\lambda_0, q)\sqrt{Ln^{3/4}} \leq d \leq \lambda_0 \left( 1 - \frac{1}{q} \right)n,$$
and \( q \cdot (\lambda_0, q) L \geq q^k \rho_d \geq \exp \left( - \frac{4\gamma L (\lambda_0, q) d^2}{n^{1/2}} \right) \). Let, as before, \( X_1, \ldots, X_k \) be i.i.d. random vectors uniformly distributed in \( \mathbb{F}_q^n \), and denote

\[
d_{\min} := \min \left\{ \text{wt} \left( \sum_{i=1}^k a_i X_i \right), \ a \in \mathbb{F}_q^k \setminus \{0\} \right\}.
\]

Then

\[
\left| \mathbb{P}\{d_{\min} \leq d\} - \mathbb{P}\{w_{\min} \leq d\} \right| = O(\exp(-L)).
\]

The theorem provides some freedom of the choice of the parameters, and includes a regime when the ratio \( k/n \) converges to one when \( n \to \infty \). At the same time, we would like to provide a cleaner statement for the most important regime when \( k/n \) is “separated” from both 0 and 1. We will obtain Theorem 1.1 as a corollary of Theorem 4.1.

For each \( r \geq 0 \), we let

\[
M_d(r) := \mathbb{P}\{Z_d = r\}, \quad \tilde{M}_d(r) := \mathbb{P}\{\tilde{Z}_d = r\},
\]

so that

\[
\mathbb{P}\{d_{\min} \leq d\} = \mathbb{P}\{Z_d > 0\} = \sum_{r=1}^{\infty} M_d(r);
\]

\[
\mathbb{P}\{w_{\min} \leq d\} = \mathbb{P}\{Z_d > 0\} = \sum_{r=1}^{\infty} \tilde{M}_d(r).
\]

Observe further that the numbers \( M_d(r) \) and \( \tilde{M}_d(r) \) satisfy the relations

\[
\sum_{r=1}^{\infty} M_d(r) r^m = \mathbb{E}Z_d^m, \quad \sum_{r=1}^{\infty} \tilde{M}_d(r) r^m = \mathbb{E}\tilde{Z}_d^m, \quad m \geq 1.
\]

These identities, together with the relations between \( \mathbb{E}Z_d^m \) and \( \mathbb{E}\tilde{Z}_d^m \) obtained in the previous section, will allow us to compare \( M_d(r) \) with \( \tilde{M}_d(r) \), hence bound the distance between the distributions of \( d_{\min} \) and \( w_{\min} \). Let us start by recording a moment growth estimate for \( \tilde{Z}_d \):

**Lemma 4.2.** We have

\[
\left( \mathbb{E}\tilde{Z}_d^\ell \right)^{1/\ell} \leq C_{4.2} \begin{cases} \frac{q^k \rho_d}{q-1}, & \text{if } \ell \leq \frac{q^k \rho_d}{q-1}, \\ \frac{\ell}{\log(e \ell (q-1)/(q^k \rho_d))}, & \text{if } \ell > \frac{q^k \rho_d}{q-1}. \end{cases}
\]

*Here, \( C_{4.2} > 0 \) is a universal constant.*

*Proof.* Notice that

\[
\mathbb{E}\tilde{Z}_d^\ell = \sum_{m=1}^{\ell} S(\ell, m) (q-1)^{\ell-m} |\Omega_m| \rho_d^m,
\]
where we use the notation from the previous section. The cardinality of $\Omega_m$ is

\[ |\Omega_m| = (q^k - 1) \left(q^k - (q - 1) - 1\right) \left(q^k - 2(q - 1) - 1\right) \cdots \left(q^k - (m - 1)(q - 1) - 1\right) \leq q^{km}. \]

Thus,

\[ \ell \sum_{m=1}^{\ell} S(\ell, m) (q - 1)^{\ell - m} |\Omega_m| \rho_d^m \leq (q - 1)^{\ell} \sum_{m=1}^{\ell} S(\ell, m) \left(\frac{q^k \rho_d}{q - 1}\right)^m. \]

Let $\lambda := \frac{q^k \rho_d}{q - 1}$. We will use an upper estimate for $S(\ell, m)$ from [11, Theorem 3]:

\[ S(\ell, m) \leq \frac{1}{2} \binom{\ell}{m} m^{\ell - m} \leq 2^{\ell+1} m^{\ell - m}. \]

With this bound and the substitution $m = t \ell$ we have

\[ \sum_{m=1}^{\ell} S(\ell, m) \lambda^m \leq \ell 2^{\ell+1} \left(\max_{t \in [0,1]} \left((t \ell)^{(1-t)} \lambda^t\right)\right)^\ell. \]

When $\ell \leq \lambda$, we get

\[ \max_{t \in [0,1]} \left((t \ell)^{(1-t)} \lambda^t\right) \leq \lambda \]

which finishes the proof of the first inequality. Now we assume $\ell \geq \lambda$. We will use a standard argument from calculus. Consider the derivative

\[ \frac{d}{dt} \left((t \ell)^{(1-t)} \lambda^t\right) = \frac{d}{dt} \exp(\log(t \ell)(1 - t) + \log(\lambda)t) \]

\[ = \left(\frac{1}{\ell} (1 - t) - \log(t \ell) + \log(\lambda)\right) (t \ell)^{(1-t)} \lambda^t \]

\[ = \left(\frac{1}{\ell} - \log(t) - \left(1 + \log \left(\frac{\ell}{\lambda}\right)\right)\right) (t \ell)^{(1-t)} \lambda^t. \]

Notice that $\frac{1}{t} - \log(t) - \left(1 + \log \left(\frac{\ell}{\lambda}\right)\right)$ is a monotone decreasing function which takes value $\infty$ at $t = 0$ and $-\log \left(\frac{\ell}{\lambda}\right) < 0$ at $t = 1$. Thus, the maximum of $(t \ell)^{(1-t)} \lambda^t$ is achieved when

\[ \frac{1}{t} - \log(t) = \left(1 + \log \left(\frac{\ell}{\lambda}\right)\right). \]

Now we fix $t \in (0, 1)$ to be the constant satisfying the above equation. Since $\frac{1}{t} \geq -\log(t) \geq 0$ on $t \in [0, 1]$, we have

\[ \frac{1}{1 + \log \left(\frac{\ell}{\lambda}\right)} \leq t \leq \frac{2}{1 + \log \left(\frac{\ell}{\lambda}\right)}. \]

Furthermore, we have $\lambda \leq \frac{\ell}{1 + \log \left(\frac{\ell}{\lambda}\right)}$ since $\frac{x}{1 + \log(x)} \geq 1$ for $x \geq 1$. We conclude that

\[ \max_{t \in [0,1]} (t \ell)^{(1-t)} \lambda^t \leq \frac{2\ell}{1 + \log \left(\frac{\ell}{\lambda}\right)}. \]
and the statement of the lemma follows.

Next, fix an integer parameter \( h \geq 1 \) (its value will be defined later), and define the \( h \times h \) square matrix \( B = (b_{ij}) \) as
\[
b_{ij} = j^i, \quad i, j = 1, \ldots, h.
\]
The next lemma can be easily checked by a straightforward computation.

**Lemma 4.3.** Let \( B = (b_{ij}) \) be as above. Then \( B \) is invertible, and the entries of the inverse matrix \( B^{-1} = (b'_{ij}) \) are given by
\[
b'_{ij} = \begin{cases} 
(-1)^{j-1} \frac{\sum_{1 \leq m_1 < \cdots < m_{h-j} \leq h, \ m_1 \ldots m_{h-j} \neq i} m_1 \ldots m_{h-j}}{\prod_{1 \leq m \leq h, m \neq i} (m-i) i} & \text{if } j < h; \\
1 & \text{if } j = h.
\end{cases}
\]

In what follows, we will not need a precise formula for the entries of the inverse; just a crude upper bound will be sufficient:

**Corollary 4.4.** With the above notation, we have
\[
|b'_{ij}| \leq \frac{h^{h-j}}{((\lfloor h/2 \rfloor - 1)!)^2} \leq C_{4.4}^{-j},
\]
where \( C_{4.4} > 0 \) is a universal constant.

Denote the vector \( (M_d(1), \ldots, M_d(h))^\top \) by \( V \), and the vector \( (\tilde{M}_d(1), \ldots, \tilde{M}_d(h))^\top \) by \( \tilde{V} \). Further, let \( U := (\mathbb{E}Z_d, \ldots, \mathbb{E}Z_d^h)^\top \), and \( \tilde{U} := (\mathbb{E}\tilde{Z}_d, \ldots, \mathbb{E}\tilde{Z}_d^h)^\top \), and, finally, define the “error vectors”
\[
E := \left( \sum_{r=h+1}^\infty r^i M_d(r) \right)_{i=1}^h, \quad \tilde{E} := \left( \sum_{r=h+1}^\infty r^i \tilde{M}_d(r) \right)_{i=1}^h.
\]
In view of the above,
\[
BV + E = U, \quad B\tilde{V} + \tilde{E} = \tilde{U},
\]
whence the difference \( V - \tilde{V} \) can be expressed as
\[
V - \tilde{V} = B^{-1}(U - \tilde{U}) - B^{-1}(E - \tilde{E}).
\]
Let us first estimate \( B^{-1}(U - \tilde{U})\):

**Lemma 4.5.** Suppose \( d, n \in \mathbb{N} \) satisfy \( \frac{d}{n} \leq \lambda_0(1 - \frac{1}{q}) \), and \( d^2/n^{3/2} \geq C_{4.4}(\lambda_0, q) \). Assume additionally that \( h \geq q^k \rho_d \geq \exp \left( -\frac{\mathbb{E}(\lambda_0, q) d^4}{n^2 h} \right) \), \( h \log_2 C_{4.4}^2 + h \log_2 C_{4.3}^2 + h + h \log(hp-h) \leq k/4 \) and \( h \leq \frac{\log_2 C_{4.4}^2 + \log_2 C_{4.3}^2 + 2 + \log(q-1)}{n^{3/2}} \). Then the absolute value of each component of the vector \( B^{-1}(U - \tilde{U}) \) is bounded above by
\[
O \left( \exp \left( -\frac{1}{2} C_{4.3}(\lambda_0, q) d^4 / n^{3/2} \right) + 2^{-k/4} \right).
\]
Proof. Fix any \(i, j \leq h\). Applying Proposition 3.1 in combination with Lemma 4.2 we get

\[
|E Z_d^j - \mathbb{E} Z_d^j| \leq O\left(\exp\left(-\frac{\lambda_0}{2}(\lambda_0, q)d^4/n^3\right) + 2^{-k/2}\right) C_{3,1}^j \begin{cases}
(q^k \rho_d)^j, & \text{if } j \leq \frac{e^{q^k \rho_d}}{q-1}, \\
\left(\frac{j(q-1)}{\log(j(q-1)/(q^k \rho_d))}\right)^j, & \text{if } j > \frac{e^{q^k \rho_d}}{q-1}.
\end{cases}
\]

On the other hand, according to Corollary 4.4, we have

\[
|b'_{ij}| \leq C_{1,3}^h.
\]

Hence, the \(i\)-th component of \(B^{-1}(U - \tilde{U})\) can be bounded from above by

\[
O\left(C_{3,1}^h \left(\exp\left(-\frac{\lambda_0}{2}(\lambda_0, q)d^4/n^3\right) + 2^{-k/2}\right) C_{4,2}^h (h(q-1)^h)\right).
\]

Using the assumptions on parameters, we get the result. \(\square\)

By a slightly more careful argument, we get an estimate on the term \(B^{-1}(E - \tilde{E})\):

**Lemma 4.6.** Suppose \(d, n \in \mathbb{N}\) satisfy \(\frac{d}{n} \leq \lambda_0(1 - \frac{1}{q})\), and \(d^2/n^{3/2} \geq C_{3,1}^h(\lambda_0, q)\). Assume additionally that

\[
e^{-8C_{3,2}^h \lambda_0(d^4/q)h} \geq q^k \rho_d \geq \exp\left(\frac{3.1(\lambda_0, q)d^4}{4n^3h}\right),
\]

and \(h \leq \frac{3.1(\lambda_0, q)}{4} \frac{d^2}{n^{3/2}}\). Then

\[
\sum_{r=h+1}^{\infty} M_d(r), \sum_{r=h+1}^{\infty} \tilde{M}_d(r) = O(2^{-h}),
\]

and the absolute value of each component of the vector \(B^{-1}(E - \tilde{E})\) is bounded above by \(O(2^{-h})\).

**Proof.** Set \(m := 4h\), and observe that \(m(q-1)/e \geq q^k \rho_d \geq \exp\left(\frac{3.1(\lambda_0, q)d^4}{n^3m}\right)\) and, furthermore, \(m \leq \frac{3.1(\lambda_0, q)d^2}{n^{3/2}}\). By applying the comparison Proposition 3.1, the bound given by Lemma 4.2 and Markov’s inequality, we get for any \(r > h\):

\[
M_d(r), \tilde{M}_d(r) = O\left(C_{3,2}^h \left(\frac{m(q-1)}{\log(m(q-1)/(q^k \rho_d))}\right)^m\right).
\]

Further, our conditions on \(h\) and \(q^k \rho_d\) imply that

\[
\log(m(q-1)/(q^k \rho_d)) \geq 8C_{3,2}^h C_{4,2}^h (q-1) = 2C_{4,2}^h C_{4,1}^h n(q-1)/h.
\]

Thus,

\[
M_d(r), \tilde{M}_d(r) = O\left(h^m r^{-m} (2C_{4,1}^h)^{-m}\right),
\]

and we get the first assertion of the lemma after summing up. Further, taking into account the definition of the vectors \(E\) and \(\tilde{E}\), we obtain for every \(j \leq h\):

\[
E_j, \tilde{E}_j = O\left(\sum_{r=h+1}^{\infty} h^m r^{-m} (2C_{4,1}^h)^{-m}\right) = O(h^j (2C_{4,1}^h)^{-m}).
\]
At the same time, by Corollary 4.4 we have $|b_{ij}^h| \leq C_{4.4}^h h^{-j}$, so that the $i$-th component of the vector $B^{-1}E$ can be bounded above by $O(2^{-h})$. The same is true for $B^{-1}E$, and the result follows.

**Proof of Theorem 4.1.** We start by defining the constants. Let

$$G_{4.4}(\lambda_0, q) := \max \left( 64 (2 + \log(16q - 16) + \log_2 G_{3.1} + \log_2 G_{4.2}), \sqrt{C_{4.3}(\lambda_0, q)} \right),$$

$$4 \left( \frac{\sqrt{C_{4.3}(\lambda_0, q)}}{\log_2 G_{4.2} + \log_2 G_{2.2} + 4 + \log(q - 1)} \right)^{-1/2},$$

and

$$C_{4.4}(\lambda_0, q) := \min \left( e^{-8C_{4.3}(\lambda_0, q)}, q^{\frac{3}{4}C_{4.3}(\lambda_0, q)/4} \right),$$

and fix any $L, d, k$ satisfying the conditions of the theorem. We set $h := \lfloor 16L \rfloor$. Observe that

$$h \log_2(G_{4.3}G_{4.2}) + h + h \log(hq - h) \leq k/4, \quad h \leq \frac{C_{3.1}(\lambda_0, q)}{\log_2 G_{4.2} + \log_2 G_{2.2} + 4 + \log(q - 1)} \frac{d^2}{n^{3/2}},$$

and that the product $q^k \rho_d$ satisfies

$$e^{-8C_{4.3}(\lambda_0, q)} h \geq q^k \rho_d \geq \exp \left( - \frac{C_{3.1}(\lambda_0, q)d^4}{4n^3h} \right).$$

A combination of Lemmas 4.5 and 4.6 then gives

$$|\mathbb{P}\{Z_d > 0\} - \mathbb{P}\{\tilde{Z}_d > 0\}| \leq \|V - \tilde{V}\|_1 + \sum_{r=h+1}^{\infty} M_d(r) + \sum_{r=h+1}^{\infty} \tilde{M}_d(r)$$

$$= O(2^{-h}) + O(h 2^{-h}) + O \left( h \exp \left( - \frac{1}{2} \frac{C_{3.1}(\lambda_0, q)d^4}{n^3} \right) + h 2^{-h/4} \right).$$

Our definition of $h$ then implies that

$$|\mathbb{P}\{Z_d > 0\} - \mathbb{P}\{\tilde{Z}_d > 0\}| = O(2^{-h/8}),$$

and the result follows.

**Lemma 4.7.** Let $H_1, H_2$ be two $k$-dimensional subspaces of $\mathbb{F}_q^n$. Then,

$$\mathbb{P}\{H_1 = \text{span}(X_1, \ldots, X_k)\} = \mathbb{P}\{H_2 = \text{span}(X_1, \ldots, X_k)\}$$

where $X_1, \ldots, X_k$ are i.i.d. random vectors uniformly distributed in $\mathbb{F}_q^n$. As a consequence, conditioned on the event that $X_1, \ldots, X_k$ are linearly independent, span$(X_1, \ldots, X_k)$ is a random $k$-dimensional subspace uniformly distributed over all $k$-dimensional subspaces of $\mathbb{F}_q^n$.

**Proof.** Let $M$ be the $n \times k$ matrix with columns $X_1, \ldots, X_k$. Then, $M$ is uniformly chosen among all $n \times k$ matrices with coefficients in $\mathbb{F}_q$.

Let $H$ be a $k$-dimensional subspace. Then $\mathbb{P}\{H = \text{span}(\text{column vectors of } M)\}$ is equal to the ratio of the number of $n \times k$ matrices whose column vectors span $H$, and the number
of all $n \times k$ matrices over $\mathbb{F}_q$. The number of $n \times k$ matrices whose column vectors span $H$ is
\[
(q^k - 1) (q^k - q) (q^k - q^2) \cdots (q^k - q^{k-1})
\]
where $q^k - q^{i-1}$ represents the number of choices of the $i$-th column from $H$ which is not a linear combination of first $i-1$ columns, with the assumption that the first $i-1$ columns are independent. Since it does not depend on $H$, it is the same for all $k$-dimensional subspaces. Thus, the first statement follows.

Further, the collection of $n \times k$ matrices whose column vectors span $H$ is a subset of the collection of $n \times k$ matrices whose column vectors are linearly independent. Thus,
\[
\mathbb{P} \{ H = \text{span (column vectors of } M) \mid \text{column vectors of } M \text{ are linearly independent} \}
\]

Fix any $R_1 < R_2$ in the interval $(0, 1)$. Without loss of generality, we can assume that $n$ is large. First, we observe that there exist numbers $\lambda_0 = \lambda_0(R_1, R_2, q) \in (0, 1)$, $\tilde{c} = \tilde{c}(R_1, R_2, q) > 0$, and $c' = c'(R_1, R_2, q) > 0$ such that
\[
q^{R_1 n} \rho_{[\lambda_0(n-n/q)]} \geq \exp(c'n),
\]
and
\[
q^{R_2 n} \rho_{[\tilde{c}n]} \leq \exp(-c'n).
\]
These estimates can be obtained, in particular, with help of Proposition 2.1.

Set $L := \tilde{c}^2 \sqrt{n}/C^2_{\text{ex}}(\lambda_0, q)$. Since $k \geq R_1 n$ and $n$ is large, we can assume
\[
k \geq C_{\text{ex}}(\lambda_0, q)L \log L.
\]
Further, let $d_1$ be the smallest integer in the interval
\[
\left[\text{C}_{\text{ex}}(\lambda_0, q)\sqrt{L}n^{3/4}, \lambda_0 \left(1 - \frac{1}{q}\right)n\right]
\]
such that $q^k \rho_{d_1} \geq \exp\left(-\frac{\text{C}_{\text{ex}}(\lambda_0, q) d_2}{n^{3/2}}\right)$, and let $d_2$ be the largest integer in the same interval, such that
\[
q^k \rho_{d_2} \leq q^{\text{C}_{\text{ex}}(\lambda_0, q)L}.
\]
Note that since $C_{\text{ex}}(\lambda_0, q)\sqrt{L}n^{3/4} = \tilde{c}n$, the numbers $d_1$ and $d_2$ are well defined (for large enough $n$), and, moreover,
\[
q^k \rho_{d_1} \leq \exp(-c'' \sqrt{n}), \quad q^k \rho_{d_2} \geq c'' \sqrt{n}
\]
for some $c''(R_1, R_2, q) > 0$.

Let $X_1, \ldots, X_k$ be i.i.d. random vectors uniformly distributed in $\mathbb{F}_q^n$. Denote by $\mathcal{E}$ the event that the vectors are linearly independent. By Lemma 4.7, conditioned on $\mathcal{E}$, the linear span of $X_1, \ldots, X_k$ is equidistributed with $\mathcal{C}$. For any $d \in [d_1, d_2]$, applying Theorem 4.1, we get
\[
|\mathbb{P}\{d_{\text{min}} \leq d\} - \mathbb{P}\{w_{\text{min}} \leq d\}| = O(\exp(-L)),
\]
whence
\[|\mathbb{P}\{d_{\text{min}} \leq d \mid \mathcal{E}\} - \mathbb{P}\{w_{\text{min}} \leq d\}| = O(\exp(-L)) + \mathbb{P}(E^c).\]

On the other hand, it is not hard to check that \(\mathbb{P}(E^c) = O(\exp(-\hat{c}n))\) for some \(\hat{c} = \hat{c}(R_2) > 0\). Thus, we obtain the required estimate for the difference \(|F_{d_{\text{min}}}(x) - F_{w_{\text{min}}}(x)|\) within the interval \(x \in [d_1, d_2]\). To complete the proof, it remains to notice that, in view of (10), we have
\[F_{w_{\text{min}}}(d_1) \leq q^k \rho_{d_1} \leq \exp(-c'' \sqrt{n}),\]
and
\[1 - F_{w_{\text{min}}}(d_2) \leq (1 - \rho_{d_2})(q^k - 1)/(q - 1) \leq \exp(-c'' \sqrt{n}/q).\]

Finally, we consider the improvement of the Gilbert–Varshamov bound implied by our argument. We shall state the result in a probabilistic form:

**Corollary 4.8.** Let \(q\) be a prime power and \(\alpha \in (0, \frac{1}{2})\). There exists constants \(c, C > 0\) depending on \(q\) and \(\alpha\) such that, for a sufficiently large integer \(n\) and \(\alpha n \leq d \leq (1 - \alpha)(1 - \frac{1}{q})n\), with probability greater than \(\exp(-c \sqrt{n})\), a uniform random \([k + \frac{1}{2} \log_q(n) - C]\)-dimensional linear code has the minimal distance at least \(d\) where \(k\) is the largest integer such that
\[
\frac{1}{q} \sum_{j=0}^{d-1} \binom{n}{j} (q - 1)^j < q^k \leq \frac{q^n}{\sum_{j=0}^{d-1} \binom{n}{j} (q - 1)^j}.
\]

(i.e. the dimension in Gilbert–Varshamov’s bound)

**Proof.** Notice that \(k\) is the largest integer satisfying \(q^k \rho_{d-1} \leq 1\). The Gilbert–Varshamov result states that there exists a \(k\)-dimensional linear code with the minimal distance at least \(d\).

Let \(t \geq 0\) be a positive integer which we will determine later. Further, let \(w_{\text{min}}\) be the minimal weight of \(\frac{q^{k+t-1}}{q-1}\) i.i.d. random vectors uniformly distributed over \(\mathbb{F}_q^n\), and let \(d_{\text{min}}\) be the minimal distance of the uniform random \((k + t)\)-dimensional linear code in \(\mathbb{F}_q^n\). We have
\[
\mathbb{P}\{w_{\text{min}} \geq d\} = (1 - \rho_{d-1})^{q^{k+t-1}/q-1} \geq \exp\left(- \frac{2q^{k+t-1} - 1}{q - 1}\right) \geq \exp(-2q^{k+t}) \geq \exp(-2q^t).
\]

Recall the \(q\)-ary entropy function
\[
H_q(x) = x \log_q(q - 1) - x \log_q(x) - (1 - x) \log_q(1 - x)
\]
which appears in the Gilbert–Varshamov bound. It is a monotone increasing function on 
\((0, 1 - \frac{1}{q})\) with \(H_q(0) = 0\) and \(H_q(1) = 1\). Furthermore, for \(x \in (1, 1 - \frac{1}{q})\),

\[
H_q(x) = \frac{1}{n} \log_q \left( \sum_{i=0}^{xn} \binom{n}{i} (q-1)^i \right) + o(1) = \frac{1}{n} \log_q (\rho_x n^q) + o(1)
\]

whenever \(xn\) is an integer. (See [6, Proposition 3.3.1])

With \(q^k \rho_d \leq 1 < q^{k+1} \rho_d\), we have

\[
H_q \left( \frac{d}{n} \right) = 1 - \frac{k}{n} + o(1).
\]

Therefore, there exist \(0 < R_1 < R_2 < 1\) depending only on \(q, \alpha\) such that

\[
R_1 \leq \frac{k}{n} \leq R_2.
\]

Now we apply Theorem 1.1 to get

\[
P \{ d_{\min} \geq d + t \} \geq P \{ w_{\min} \geq d + t \} - |P \{ d_{\min} \geq d + t \} - P \{ w_{\min} \geq d + t \}| \\
\geq \exp(-2q^t) - \exp(-c_{\alpha,q}\sqrt{n})
\]

where \(c_{\alpha,q} = c(R_1, R_2, q)\). Choosing

\[
t = \frac{1}{2} \log_q n + \log_q \left( \frac{c_{\alpha,q}}{4} \right)
\]

we obtain the desired bound.

\[
\square
\]

It is not difficult to check that the above corollary implies Corollary 1.1 from the introduction.

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