Decoding the final state in binary black hole mergers

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Abstract

We demonstrate that in binary black hole (BBH) mergers there is a direct correlation between the frequency of the gravitational wave at peak amplitude and the quasi-normal mode frequency and damping time of the final black hole. Since the ringing frequency and damping time of a black hole are related to its mass and spin, the correlation discovered also provides a connection between the parameters of the final black hole and the frequency of the gravitational wave at peak amplitude. This correlation could potentially assist with the analysis of gravitational wave observations from BBH mergers.

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(Some figures may appear in colour only in the online journal)

Introduction

Because of their expected luminosity, mergers of binary black holes (BBHs) will be main targets of gravitational wave (GW) interferometric detectors like LIGO\cite{1}, Virgo\cite{2} and KAGRA\cite{3}. Detecting merging BBHs may not necessarily require exquisite knowledge of the GW signal. Characterizing the binary (i.e. eccentricity, masses, spins, sky location, orientation and distance) is a different story. Matched filtering is currently considered our best option, but not surprisingly the challenge in this case is constructing waveforms or templates that effectively cover the parameter space, a task very likely necessitating large amounts of computing resources.
Alternatives that reduce the computational cost in GW data analysis are very desirable, in particular options that exploit bulk features or partial information about the BBH. In this paper, we identify a feature in the GWs emitted during the merger of BBHs that could potentially help pinpoint the mass and spin of the final black hole (BH). The feature discovered is a correlation between the quasi-normal mode (QNM) frequency $\omega_{\text{qn}}$ and damping time $\tau_{\text{qn}}$ (or equivalently the quality factor $Q \equiv \omega_{\text{qn}} \tau_{\text{qn}}/2$) of the final BH with the frequency $\omega_{\text{mx}}$ of the GW at peak amplitude. Since $\omega_{\text{qn}}$ and $\tau_{\text{qn}}$ (or $Q$) are related to the mass $M_h$ and spin parameter $a$ of the final BH [4], the correlation we found also provides a connection of $\omega_{\text{mx}}$ with $M_h$ and $a$. Interestingly, not too long ago [5], a correlation with a similar character was found for neutron star binaries between the frequency peak of the post-merger GW emission and the physical properties of the nuclear equation of state. It is important to emphasize that our claim does not imply that the QNM ringing of the final BH depends exclusively on $\omega_{\text{mx}}$. Very likely, there are other factors such as the amplitude of the GWs that influence the final state.

In retrospect, it should not be too surprising that such a correlation exists. By the time the amplitude of the GW peaks, the horizons of the coalescing BHs have already merged, and most of the "hair" in the binary (masses, spins, eccentricity, etc) has been lost. Nonetheless, it is interesting that, although at peak amplitude the BH system is still dynamically nonlinear, the frequency of the GWs is already correlated with the QNM ringing of the final BH.

### Numerical relativity simulation bank

For over six years, our numerical relativity effort has produced an extensive archive of waveforms, with more than 512 high resolution simulations of inspiraling BBHs. The simulations were obtained with our Maya code [6–11]. Maya uses the Einstein Toolkit.
which is based on the CACTUS [13] infrastructure and CARPET [14] mesh refinement, with thorns generated by the Kranc [15] library.

For the present work, we considered 269 simulations, 97 with non-precessing and 172 with precessing binaries. Among the non-precessing binaries, 43 have non-spinning BHs with mass ratios ranging between 1:1 and 1:10; 39 have total BH spin aligned with the angular momentum and mass ratios ranging between 1:1 and 1:7; and the last 15 have total BH spin anti-aligned with orbital angular momentum and mass ratios between 1:1 and 1:4. Of the non-precessing binaries, 18 have a mass ratio of 1:1. Regarding the 172 precessing binary simulations, individual BH dimensionless spin parameters range between 0 and 0.7 and mass ratios primarily between 2:3 and 1:4, with a few equal mass and one each of 1:6 and 1:7. As mentioned before, we used a bank of high resolution simulations from previous studies. Nonetheless, we selected a small subset of the cases, and repeated the simulations at even

![Figure 2. The quasi-normal mode ringdown frequency $\tilde{\omega}_{\text{qu}}$ (top), the decay time $\tilde{t}_{\text{qu}}$ (middle), and the quality factor $Q$ (bottom) versus $\tilde{\omega}_{\text{mx}}$ from $\Psi_4$ for the (2,2) (blue squares), (3,3) (red triangles), (4,4) (green circles), and (5,5) (purple crosses) modes. The data only include simulations from non-precessing binaries.](image-url)
higher resolutions. The changes observed due to convergence were small enough to conclude that the correlation we found was not a numerical artifact.

For each simulation, we extract the Weyl Scalar $\Psi_4$. In data analysis, it is conventional to work with the strains polarizations $h_+ \text{ and } h_\times$, which are related to $\Psi_4$ by $\Psi_4 = h_+ - i h_\times \equiv \Psi^*$, with the star denoting complex conjugation and over-dots time derivatives. The correlation we have found shows in both $\Psi_4$ and the strains $h_+ \text{ and } h_\times$. To avoid the inaccuracies introduced while constructing the strains, we will center the discussion around $\Psi_4$.

As is customary, we decompose $\Psi_4$ into spin-weighted spherical harmonics, namely

$$r M \Psi_4(t; \theta, \phi) = \sum_{l,m} A_{lm}(t) e^{i\phi_{lm}(t)} Y_{lm}(\theta, \phi),$$

with both $A_{lm}$ and $\phi_{lm}$ real functions. In equation (1), $M$ denotes the sum of the BH masses and $r$ the distance to the binary. The frequency of the $(l, m)$ mode is given by $\omega_{lm} = \phi_{lm}$. To simplify notation, unless explicitly stated, we will drop the mode labels $(l, m)$.

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**Figure 3.** Black crosses are the same points as in the top panel of figure 2 for the (2,2) mode but with the vertical axis on the right. The points are replotted according to their mass ratio value $q$ (left axis) with a shape and color according to their total spin: non-spinning (green circles), aligned (blue squares), and anti-aligned (red diamonds) with the orbital angular momentum.

**Table 1.** Fitting coefficients for equations (2)–(4).

| Mode | 22 | 33 | 44 | 55 |
|------|----|----|----|----|
| $f_1$ | 0.136 ± 0.009 | 0.093 ± 0.006 | 0.141 ± 0.004 | 0.145 ± 0.004 |
| $f_2$ | 1.099 ± 0.012 | 0.996 ± 0.016 | 1.030 ± 0.032 | 1.016 ± 0.028 |
| $g_1$ | 3.556 ± 0.025 | 2.742 ± 0.008 | 2.485 ± 0.002 | 2.368 ± 0.003 |
| $g_2$ | 2.328 ± 0.069 | 1.225 ± 0.047 | 0.752 ± 0.022 | 0.283 ± 0.038 |
| $g_3$ | 1.204 ± 0.047 | 1.050 ± 0.067 | 1.177 ± 0.138 | 0.834 ± 0.122 |
| $h_1$ | 2.648 ± 0.066 | 2.129 ± 0.023 | 1.937 ± 0.006 | 1.813 ± 0.008 |
| $h_2$ | 3.002 ± 0.182 | 2.143 ± 0.136 | 1.749 ± 0.057 | 1.453 ± 0.099 |
| $h_3$ | 0.913 ± 0.124 | 0.937 ± 0.193 | 0.726 ± 0.354 | 0.284 ± 0.321 |
Typically, the angles $\theta$ and $\phi$ in equation (1) are relative to a coordinate system with the origin at the center-of-mass of the binary and with the $z$-axis aligned with the orbital angular momentum at the beginning of the simulation. However, given our interest in the period between coalescence and QNM ringing, for the decomposition in equation (1), we align the $z$-axis of the coordinate system with the spin vector of the final BH.

Figure 4. Same as in figure 2 but with the data from the $(3,3)$ (triangles, red), $(4,4)$ (circles, green), and $(5,5)$ (crosses, purple) modes shifted to lie on top of the $(2,2)$ (squares, blue) mode. The values of the shifts are reported in table 2.

Table 2. Values of the shifts used in figure 4 to bring the data from the $(3,3)$, $(4,4)$ and $(5,5)$ modes to lie on top of the $(2,2)$ mode.

| Mode | $\Delta \ln (\hat{\omega}_{33})$ | $\Delta \ln (\hat{\omega}_{44})$ | $\Delta \ln (\hat{\omega}_{55})$ | $\Delta \ln (\hat{\omega}_{22})$ | $\Delta \ln (\hat{\omega}_{55})$ |
|------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 33   | -0.4299                         | -0.4647                         | 0.0311                          | -0.4336                         |                                 |
| 44   | -0.6887                         | -0.7658                         | 0.0457                          | -0.7200                         |                                 |
| 55   | -0.9068                         | -0.9952                         | 0.0577                          | -0.9376                         |                                 |

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Figure 1 shows a typical evolution of the amplitude $A_{m\ell}$ (top panel) and frequency $\omega_{m\ell}$ (bottom panel) for different $\Psi_4$ modes. The amplitudes of the modes reach a maximum at $t \sim 2030 M$, with $t$ being the retarded time. The amplitudes have been rescaled, so each of them have the same maximum values; that is, $A_{22} = 1.48 A_{33} = 2.32 A_{44} = 3.21 A_{55}$. The case depicted is that of a binary with mass ratio 1:4 and non-spinning BHs. The vertical line at $t \approx 2004 M$ denotes the merger of the holes, i.e. when the common apparent horizon is first found. Notice that merger occurs $t \sim 25 M$ before the amplitudes reach their maximum value. Before the merger, the modes have the characteristic chirp-like behavior (i.e. a monotonic increase of amplitude and frequency). A few $M$s after peak amplitude, the waveforms become a sum of QNMs with the fundamental mode dominating, that is, $\Psi_4 \propto e^{-\tau^2/\tau} \sin(\omega_{m\ell} t)$.

Figure 5. Same as figure 2 but also including precessing binaries. Non-precessing are denoted with dark symbols and precessing with light symbols. Notice the increase in the spread.
Connecting $\omega_{mx}$ with QNM ringing

Figure 2 shows, from top to bottom, the dimensionless quantities $\omega_{mn} \equiv M_{h} \hat{q}_{n}$, $\tau_{mn} \equiv M_{h} \hat{a}_{n}$ and $Q$ as a function of $\omega_{mx} \equiv M_{h} \hat{a}_{mx}$ for non-precessing binaries. The QNM ringing quantities $\omega_{mn}$ and $\tau_{mn}$ were calculated from $M_{h}$ and $a$ using the tabulated values in [16], with $M_{h}$ obtained from the area of the apparent horizon (related to the irreducible mass) using the Christodoulou formula and $a$ using the isolated/dynamical horizon framework [17].

Each point in figure 2 represents one simulation, with points clustered according to their mode: blue squares for the (2,2) mode, red triangles for the (3,3) mode, green circles for the (4,4) mode and purple crosses for the (5,5) mode. These are the most dominant modes of GW emission. In order to identify the type of simulation in figure 3, the same points are replotted according to their mass ratio value $q$ (left axis) with a shape and color according to their total

Figure 6. The quasi-normal mode ringdown frequency $\hat{\omega}_{q}$ (top), the decay time $\hat{\tau}_{q}$ (middle), and the quality factor $Q$ (bottom) versus $\hat{\omega}(\Delta t)$, where $\hat{\omega}(\Delta t)$ denotes the frequency of the GW at time $\Delta t$ before $\Psi_{4}$ peaks for the (2,2) mode. From left to right $\Delta t = \{100, 50, 25, 10, 0\} M$. 

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spin: non-spinning as green circles, aligned as blue squares, and anti-aligned with the orbital angular momentum as red diamonds. The (2,2) mode data of the top panel of figure 2 are also included as black crosses with the vertical axis on the right.

For each mode, the correlation of $\omega_{mx}$ with $\tilde{\omega}_{qn}$ and $\tilde{\tau}_{qn}$ and $Q$ is evident in figure 2. Moreover, the data seem to imply that

$$\ln \left( \tilde{\omega}_{qn} \right) = f_1 + f_2 \ln \left( \tilde{\omega}_{mx} \right)$$

$$\ln \left( \tilde{\tau}_{qn} \right) = g_1 + g_2 \ln \left( \tilde{\omega}_{mx} \right) + g_3 \left[ \ln \left( \tilde{\omega}_{mx} \right) \right]^2$$

$$\ln \left( Q \right) = h_1 + h_2 \ln \left( \tilde{\omega}_{mx} \right) + h_3 \left[ \ln \left( \tilde{\omega}_{mx} \right) \right]^2.$$  

In table 1, we report the values of the fitting parameters for each mode. Notice that $f_2 \approx 1$, thus $\tilde{\omega}_{qn} \approx e^{f_1} \tilde{\omega}_{mx}$.

Furthermore, there also seems to be a self-similarity among the modes; that is, with the appropriate shifts, it is possible to cluster all the modes, while still preserving their original correlation characteristics. This is shown in figure 4, where the data from modes (3,3), (4,4) and (5,5) have been shifted to lie on top of the (2,2) mode. The shifts are calculated as the average of the shift for each simulation. For instance, for $\ln \left( \tilde{\omega}_{qn} \right)$, the shift for the (3,3) mode is obtained from

$$\Delta \ln \left( \tilde{\omega}_{qn} \right)_{(3,3)} = \left\langle \ln \left( \tilde{\omega}_{qn} \right) \right|_{(3,3)} - \ln \left( \tilde{\omega}_{qn} \right)_{(2,2)} \right\rangle,$$  

where the angle brackets denote the average over all the simulations. The values of the shifts are reported in table 2. Notice from equations (2) and (5) that

$$\left[ \Delta \ln \left( \tilde{\omega}_{qn} \right) - \Delta \ln \left( \tilde{\omega}_{mx} \right) \right]^{(3,3)} = f_1^{(3,3)} - f_1^{(2,2)},$$

where we have used that $f_2^{(3,3)} \approx f_2^{(2,2)} \approx 1$. The values reported in Tables 1 and 2 are consistent with equation (6). Namely, $\left[ \Delta \ln \left( \tilde{\omega}_{qn} \right) - \Delta \ln \left( \tilde{\omega}_{mx} \right) \right]^{(3,3)} = -0.035$ and $f_1^{(3,3)} - f_1^{(2,2)} = -0.0414$, with the difference related to the goodness of the fit given by equation (2).

Up until this point, we have only included non-precessing BBHs. The effect of precession is shown in figure 5, where we have included the remaining 172 simulations with precessing binaries. Non-precessing are denoted with dark symbols and precessing with light symbols. It is clear that, although the shapes are preserved, there is a noticeable increase in the spread of the points.

Lastly, it is interesting to investigate whether or not $\tilde{\omega}_{mx}$ is special. To do so, we have repeated the analysis, but instead of using $\tilde{\omega}_{mx}$, we use the frequency of the GW at a time $\Delta t$ before peak amplitude. The results for the (2,2) mode are depicted in figure 6. The clusters of points from left to right are for $\Delta t = \{100, 50, 25, 10, 0\} M$, respectively. Qualitatively, the data in figure 6 suggest that the correlation persists for $\Delta t \lesssim 10 M$, but very likely not around the merger, i.e. $\Delta t \approx 25 M$, or earlier.

**Discussion**

Our BBH simulations have unveiled a correlation between the frequency $\tilde{\omega}_{mx}$ of the GW around peak amplitude and the QNM ringing of the final BH. The correlation could prove helpful in the construction of templates and data analysis. Furthermore, the correlation of $\omega_{qn}$
and $\tau_\text{m}$ with $\omega_{\text{mx}}$ could in principle be used in both directions. Namely, if one is able to estimate $\omega_{\text{mx}}$ in the GW via for instance excess-power, then our correlation could tell us about the final BH. But one could also envision, given the mass and spin of the final BH of interest, using the correlation to constrain the frequency of the GW soon after merger, when the power of emission is the strongest. Specifically, for BBHs with large masses, e.g. $50 M_\odot \lesssim M \lesssim 500 M_\odot$, only the last few cycles, merger and ring-down lie within the sweet spot of the detector. The search basically reduces to that of a perturbed intermediate mass BH [18], for which sine-gaussians or chirplets are useful representations of the signal [19]. The correlations in our work could be used to fix the characteristic frequency of the chirplet, i.e. $\omega_{\text{mx}}$, given the range of masses and spins of the final BH.

Further, recent work has shown a connection between the ringdown properties and the initial BH parameters [20–22]. Our current work would extend this, allowing both the final state and the initial state of the system to be identified by just the burst region of the waveform.

The correlation between $\hat{\omega}_{\text{mx}}$ and the QNM could also aid in the tuning of phenomenological models to BBH mergers [23–25]. The relationship given in equations (2)–(4) provides a method for adding ringdown to the waveform using just the peak frequency of the waveform. Perhaps most importantly, the method applies to all the modes excited during the merger.

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References

[1] Harry G M and the LIGO Scientific Collaboration 2010 Class. Quantum Grav. 27 084006
[2] Accadia T et al 2011 Class. Quantum Grav. 28 114002
[3] Aso Y et al 2013 The KAGRA Collaboration Phys. Rev. D 88 043007
[4] Echeverria F 1989 Phys. Rev. D 40 3194–203
[5] Bauswein A and Janka H T 2012 Phys. Rev. Lett. 108 011101
[6] Haas R, Shcherbakov R V, Bode T and Laguna P 2012 Astrophys. J. 749 117
[7] Healy J, Bode T, Haas R, Pazos E, Laguna P, Shoemaker D M and Yunes N 2012 Class. Quantum Grav. 29 232002
[8] Bode T, Laguna P and Matzner R 2011 Phys. Rev. D 84 064044
[9] Bode T, Bogdanovic T, Haas R, Healy J, Laguna P et al 2012 Astrophys. J. 744 45
[10] Bode T, Haas R, Bogdanovic T, Laguna P and Shoemaker D 2010 Astrophys. J. 715 1117–31
[11] Healy J, Levin J and Shoemaker D 2009 Phys. Rev. Lett. 103 131101
[12] Einstein Toolkit: www.einsteinToolkit.org
[13] Cactus Computational Toolkit: www.cactuscode.org
[14] Schnetter E, Hawley S H and Hawke I 2004 Class. Quantum Grav. 21 1465–88
[15] Husa S, Hinder I and Lechner C 2006 Comput. Phys. Commun. 174 983–1004
[16] Berti E, Cardoso V and Will C M 2006 Phys. Rev. D 73 064030
[17] Dreyer O, Krishnan B, Shoemaker D and Schnetter E 2003 Phys. Rev. D 67 024018
[18] Aasi J et al 2014 Phys. Rev. D 89 102006
[19] Chassande Mottin É, Miele M, Mohapatra S and Cadonati L 2010 Class. Quantum Grav. 27 194017
[20] Kamaretsos I, Hannam M and Sathyaprakash B 2012 Phys. Rev. Lett. 109 141102
[21] London L, Healy J and Shoemaker D 2014 arXiv:1404.3197
[22] Meidam J, Agathos M, Van den Broeck C, Veitch J and Sathyaprakash B 2014 Phys. Rev. D 90 064009
[23] Buonanno A et al 2009 Phys. Rev. D 79 124028
[24] Ajith P et al 2011 Phys. Rev. Lett. 106 241101
[25] Hinder I et al 2014 Class. Quantum Grav. 31 025012