Finite-temperature dynamical correlations in massive integrable quantum field theories

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Abstract. We consider the finite-temperature frequency and momentum-dependent two-point functions of local operators in integrable quantum field theories. We focus on the case where the zero-temperature correlation function is dominated by a delta-function line arising from the coherent propagation of single-particle modes. Our specific examples are the two-point function of spin fields in the disordered phase of the quantum Ising and the O(3) nonlinear sigma models. We employ a Lehmann representation in terms of the known exact zero-temperature form factors to carry out a low-temperature expansion of two-point functions. We present two different but equivalent methods of regularizing the divergences present in the Lehmann expansion: one directly regulates the integral expressions of the squares of matrix elements in the infinite volume whereas the other operates through subtracting divergences in a large, finite volume. Our central results are that the temperature broadening of the lineshape exhibits a pronounced asymmetry and a shift of the maximum upwards in energy (‘temperature-dependent gap’). The field theory results presented here describe the scaling limits of the dynamical structure factor in the quantum Ising and integer spin Heisenberg chains. We discuss the relevance of our results for the analysis of inelastic neutron scattering experiments on gapped spin chain systems such as CsNiCl$_3$ and YBaNiO$_5$.

Keywords: correlation functions, form factors, integrable quantum field theory, spin chains, ladders and planes (theory)

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1. Introduction

Progress over the last two decades [1]–[10] has made it possible to determine zero-temperature dynamical response functions in massive integrable models of quantum field theory (QFT) by means of the ‘form factor bootstrap approach’. More precisely, response functions can be calculated exactly at low frequencies (several times the mass gap) and to high accuracy at intermediate frequencies. The $T = 0$ dynamics described by integrable QFTs exhibits a number of interesting phenomena such as dynamical mass generation, spin-charge separation and other kinds of quantum number fractionalization. The results obtained by the form factor bootstrap approach have important applications in condensed matter systems [11] such as quantum magnets [12], Mott insulators [13], doped ladder materials [14], carbon nanotubes [15] and ultra-cold atomic gases [16].

At the heart of the form factor bootstrap approach is the notion that in an integrable model the scattering of elementary excitations is purely elastic by virtue of the existence of an infinite number of local conservation laws. There is no particle production and the individual particle momenta are conserved in scattering events. Correspondingly, the scattering of $n$ particles in an integrable model can always be reduced to a sum of two-body scattering events. The resulting simplified nature of the exact Hamiltonian eigenstates in a massive integrable field theory permits the computation of zero-temperature correlation functions as follows. One first employs a Lehmann representation in terms of $n$-particle Hamiltonian eigenstates $|n; \{s_n\}\rangle$ with energy $E[\{s_n\}]$, where $\{s_n\}$ labels the corresponding sets of good quantum numbers:

$$\chi_O(\tau,x)|_{T=0} = \langle 0|T\tau O(\tau,x)O^\dagger(0,0)|0\rangle$$

$$= \sum_{n=0}^{\infty} e^{-\tau E[\{s_n\}]} \langle 0|O(x,0)|n; \{s_n\}\rangle \langle n; \{s_n\}|O^\dagger(0,0)|0\rangle \quad (\tau > 0).$$

As the energies are simply given as sums over the single-particle energies of the elementary excitations, we have reduced the computation of the response function to computing a set of matrix elements or ‘form factors’, $\langle 0|O(\tau,x)|n; \{s_n\}\rangle$.

To perform this computation we again exploit integrability. The ability to express $|n; \{s_n\}\rangle$ as a collection of $n$ distinct particles allows one to write down a set of algebraic constraints that the matrix elements must satisfy [1]. These constraints encode both the simplified form the scattering of $n$ particles takes in an integrable model together with analytic constraints coming from crossing symmetries present in a relativistic quantum field theory. While these constraints can be written down for eigenstates involving an arbitrary number, $n$, of particles, they become increasingly cumbersome to solve as $n$ increases. Fortunately, if we are interested merely in the behavior of the low energy spectral function, it is only necessary to compute matrix elements involving a few particles. The spectral function is obtained by Fourier transforming (1) with respect to space and imaginary time and then analytically continuing to real frequencies ($\omega_n \rightarrow \omega + i0$):

$$-\frac{1}{\pi} \text{Im} \chi_O(\omega,q) = 2\pi \sum_{n=0}^{\infty} \sum_{s_n} |\langle 0|O(0,0)|n; \{s_n\}\rangle|^2 \delta(\omega - E[\{s_n\}]) \delta(q - P[\{s_n\}])$$

$$- \epsilon \langle n; \{s_n\}|O^\dagger(0,0)|0\rangle^2 \delta(\omega + E_{s_n}) \delta(q + P[\{s_n\}]},$$

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where \( P[\{s_n\}] \) denotes the momentum of \( |n, \{s_n\}\rangle \) and \( \epsilon = \pm 1 \) depending on whether the field \( \mathcal{O} \) is bosonic or fermionic. The presence of the delta function in the above expression for the spectral function guarantees that at an energy \( \omega \) only eigenfunctions with this exact energy contribute. As the theory is massive, eigenstates with \( n \) particles will have a minimum energy, \( n\Delta \) (supposing the particles all have mass \( \Delta \)). Thus for energies \( \omega < n\Delta \), such eigenstates will not contribute to the spectral function. For low energies, only a small finite number of matrix elements need to be computed in order to obtain exact results for the spectral function. Even at higher energies, it has been typically found that the sum of matrix elements is strongly convergent, and that matrix elements with higher particle number make only an extremely small contribution to the spectral function [3], [17]–[19], [11].

While the above-described approach has been successful at computing zero-temperature correlation functions, it is not a settled question whether the form factor bootstrap approach can be used universally to gain information about \( T > 0 \) dynamical correlations. This question has been investigated in a number of specific instances [20]–[24]. On the basis of this past work, there appear, in various forms, two main difficulties in doing so. These difficulties become particularly acute in theories which are interacting. To illustrate these problems, we write out the corresponding form factor expansion for a Green’s function at finite temperature:

\[
-\frac{1}{\pi} \text{Im} \chi_{\mathcal{O}}(\omega, q) = \frac{2\pi}{Z} \sum_{n=m} \delta(\omega - E[\{s_n\}] + E[\{s_m\}])\delta(q - P[\{s_n\}] + P[\{s_m\}])
\times \left[ e^{-\beta E[\{s_m\}]} - \epsilon e^{-\beta E[\{s_n\}]} \right] \langle m; \{s_m\}|\mathcal{O}(0,0)|n; \{s_n\}\rangle^2.
\]

(3)

The first difficulty can be seen in that the form factor expansion now involves two sums over eigenstates, the first sum arising from the insertion of the resolution of the identity as before and the second sum coming from the Boltzmann trace associated with working at finite temperature. Concomitantly, working at a particular energy, \( \omega \), no longer guarantees that only a finite number of matrix elements will contribute to the spectral function. This problem was partially resolved in [23]. There it was advocated that the Boltzmann factor provides a natural small parameter, that is, terms involving eigenstates \( |m; \{s_m\}\rangle \) with many particles (and so \( E[\{s_m\}] \) large) play only a specific, limited role, at least at low temperatures. It was shown there that, for a certain class of correlation functions, it was possible to develop a low-temperature expansion of the response function. However, this work left unresolved how to perform the low-temperature expansion in general. In particular, it did not address how to develop the low-temperature expansion for response functions with singular features at zero temperature.

Such response functions form a class of great physical interest. As an important physical example, the spin response, \( S(\omega, q) \), of a gapped quantum spin chain as represented by either the quantum Ising model or the O(3) NLSM is given by \( S(\omega, q) = Z(q)\delta(\omega - \epsilon(q)) + \cdots \) at zero temperature, where the dots indicate very weak multiparticle scattering continua. At finite temperature this \( \delta \)-function response broadens out to a lineshape with finite width. But it has been unclear how to capture this broadening in the context of the form factor expansion as given in (3). It is one of the achievements of this paper to demonstrate how this can be done.
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The second difficulty is that, unlike the zero-temperature case, we are now faced with computing matrix elements between states, $|n; \{s_n\}\rangle$ and $|m; \{s_m\}\rangle$ both with finite particle number (as opposed to amplitudes $\langle 0|O(0,0)|n; \{s_n\}\rangle$ governing the transition between the vacuum and some eigenstate). In an infinite volume, matrix elements of the form $\langle m; \{s_m\}|O(0,0)|n; \{s_n\}\rangle$ possess singularities. Moreover these singularities are 'squared' as it is the quantity $|\langle m; \{s_m\}|O(0,0)|n; \{s_n\}\rangle|^2$ that determines the correlation function. Making sense of these squared singularities amounts to giving a sensible interpretation to terms of the form $\delta(0)'$.

It is perhaps the main achievement of this paper that we have demonstrated a set of regularization procedures for such singularities. These ultimately arise as a consequence of working in an infinite volume, where the momenta of particles found in the $n$-particle and $m$-particle states, $|n; \{s_n\}\rangle$ and $|m; \{s_m\}\rangle$, can be identical. In a matrix element of the form, $\langle m; \{s_m\}|O(0,0)|n; \{s_n\}\rangle$, identical momenta lead to divergences. One approach is thus to work in a large but finite volume. In such cases the momenta (at least in the relevant examples) of the particles composing the states $|n; \{s_n\}\rangle$ and $|m; \{s_m\}\rangle$ are never equal, leaving the matrix element $\langle m; \{s_m\}|O(0,0)|n; \{s_n\}\rangle$ finite. Such an approach is feasible as $\langle m; \{s_m\}|O(0,0)|n; \{s_n\}\rangle$ retains the same momentum dependences as in an infinite volume. Provided we work in a large, finite volume, the sole difference in evaluating the form factor $\langle m; \{s_m\}|O(0,0)|n; \{s_n\}\rangle$ for general momenta between the finite and infinite volume cases lies in taking into account that in finite volume the momenta of the states $|n; \{s_n\}\rangle$ and $|m; \{s_m\}\rangle$ are quantized [25]. Thus we can still use in a large, finite volume the infinite volume constraints that govern these matrix elements.

The use of finite volume regularization has precedent. For the special case of the quantum Ising model this particular problem has been solved by calculating the form factors on the cylinder [26,27]—that is, the form factors are computed in an arbitrary, not merely asymptotically large, finite volume. These results have then been used in [24] to recover the semiclassical expression obtained previously by Sachdev and Young [28] and in [29,30] to carry out low- and high-temperature expansions for the spin–spin correlation function. Form factors as computed on the cylinder have also been exploited very recently [33] to analyze time-dependent zero-temperature correlators in the Ising model. However, like in [31], the underlying theory always has had trivial scattering, that is the scattering matrix is momentum-independent and diagonal. It is presently not known how to generalize the results of [26,27] to integrable QFTs with non-trivial $S$ matrices.

In addition to our use of a finite-volume regularization scheme, we demonstrate in this paper a new regularization technique that operates in an infinite volume. In this regularization scheme, ambiguous terms such as $\delta(0)$ are absent by construction. We demonstrate explicitly that our two regularization schemes lead to the same answers. We do so in two examples, the quantum Ising model and the $O(3)$ nonlinear sigma model (NLSM). That the latter model is interacting and so highly non-trivial provides a strong indication that our infinite-volume regularization scheme is robust and so provides a candidate for a regulator that works generally. In particular, it would be useful to
check the scheme for cases involving form factors between two two-particle states or a two-particle and a three-particle state.

Our work, both in how we develop a low-temperature expansion, and in our particular choice of integrable models, the quantum Ising model and the O(3) NLSM, is motivated in large degree by recent inelastic neutron scattering experiments on several quantum magnets. A key objective of this recent experimental work has been to investigate how the spin dynamics crosses over from the strongly correlated zero-temperature quantum regime to the classical high-temperature regime [34]–[36]. In a system such as the spin-1 Heisenberg chain that supports a coherent, gapped, magnetic single-particle excitation at $T = 0$, the question arises of how the dominant feature in the dynamical structure factor, a delta function at zero temperature, broadens at finite temperatures. As field theories, the quantum Ising model and the O(3) NLSM describe the scaling limits of the (non-integrable) spin chain Hamiltonians used to model these various experiments.

Applying the methodology detailed in this paper, we analyze this finite-temperature lineshape. As our central finding in this regard, we demonstrate that the lineshape is always asymmetric in energy, a feature that becomes more pronounced as the temperature increases. For the Ising model we further demonstrate the emergence of a ‘temperature-dependent gap’. A subset of our results on the lineshape has been previously reported in [37].

At temperatures $T$ far below the spin gap $\Delta$ our technique complements previous semiclassical approaches to the study of the lineshape of quantum spin chains [28], [38]–[41]. Both in our methodology and in semiclassical approaches, the lineshape has been shown to be essentially Lorentzian for $T \ll \Delta$ and for energies, $\omega$, in the vicinity of the gap, $\Delta$. However, in our approach we can both study temperatures where the semiclassical is inaccurate as well as the entire lineshape, not merely $\omega \sim \Delta$.

The outline of this paper is as follows. In section 2 we set out our framework for deriving low-temperature expansions of dynamical correlation functions in massive integrable quantum field theories. In particular, we summarize how the space of Hamiltonian eigenstates in an integrable model is handled as well as how we develop the low-temperature expansion appropriate for computing the finite-temperature lineshape. In section 3 we apply the method to the quantum Ising chain. We introduce how we regularize the squares of form factors both by working in finite volume as well as in our new infinite-volume scheme. We show that in the context of the Ising model, these schemes are equivalent. In the following section, section 4, we present a detailed discussion of the results obtained for the quantum Ising model as well as a comparison to the semiclassical results of Sachdev and Young. In the next two sections we move on to the O(3) NLSM showing that our methodology also works for interacting theories with non-trivial (even non-diagonal) scattering matrices. In section 5 we consider the case of the retarded Green’s function of the vector field in the O(3) nonlinear sigma model, the quantity that corresponds to the spin response of a gapped Heisenberg spin chain near wavevector $\pi$. We again show that the two regularization schemes yield the same result for this correlation function. In section 6 we detail the results that so arise for the low-temperature dynamics for the O(3) NLSM and compare them to the semiclassical results of [39]. The final section, section 7, presents a summary and discussion of our results. Computational details on our new method for regularizing form factor squares directly in the infinite volume are presented in several appendices.

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2. General formalism

A defining feature of integrable quantum field theories is a basis of scattering states of 'elementary' excitations, which are eigenstates of the Hamiltonian. It is customary to construct these states from the so-called Faddeev–Zamolodchikov algebra:

\[
Z_a(\theta_1)Z_b(\theta_2) = S_{ab}^{c'b'}(\theta_1 - \theta_2)Z_{b'}(\theta_2)Z_{c'}(\theta_1),
\]
\[
Z_a^\dagger(\theta_1)Z_b^\dagger(\theta_2) = S_{ab}^{c'b'}(\theta_1 - \theta_2)Z_{b'}(\theta_2)Z_{c'}(\theta_1),
\]
\[
Z_a(\theta_1)Z_b^\dagger(\theta_2) = 2\pi \delta_{ab}\delta(\theta_1 - \theta_2) + S_{ba}^{b'a}(\theta_1 - \theta_2)Z_{b}(\theta_2)Z_{a}(\theta_1).
\]

(4)

Here \(\theta_{1,2}\) are rapidity variables, \(a, b\) are quantum numbers and \(S\) is the exact two-particle scattering matrix describing the purely elastic scattering of the elementary excitation. The \(S\) matrix is a solution to the Yang–Baxter equation, which can be thought of as a consistency condition for factorizable three-particle scattering. Using the Faddeev–Zamolodchikov operators, a Fock space of states can be constructed as follows. The vacuum is defined by

\[
Z_a(\theta)|0\rangle = 0.
\]

(5)

Multiparticle states are then obtained by acting with strings of creation operators \(Z_b^\dagger(\theta)\) on the vacuum

\[
|\theta_n \ldots \theta_1\rangle_{a_n \ldots a_1} = Z_{a_n}^\dagger(\theta_n) \ldots Z_{a_1}^\dagger(\theta_1)|0\rangle.
\]

(6)

Energy and momentum of the states (6) are by construction additive:

\[
E_s(\theta_1, \ldots, \theta_s) = \sum_{j=1}^{s} \epsilon(\theta_j), \quad \epsilon(\theta) = \Delta \cosh \theta,
\]

\[
P_s(\theta_1, \ldots, \theta_s) = \sum_{j=1}^{s} \frac{\Delta}{\theta} \sinh \theta_j.
\]

(7)

In terms of this basis the resolution of the identity is given by

\[
\mathbb{I} = |0\rangle\langle 0| + \sum_{n=1}^{\infty} \sum_{(a_i)} \int_{-\infty}^{\infty} d\theta_1 \ldots d\theta_n \frac{1}{(2\pi)^n n!} |\theta_n \ldots \theta_1\rangle_{a_n \ldots a_1} \langle \theta_1 \ldots \theta_n|.
\]

(8)

In the basis of scattering states introduced above, the following formal spectral representation for the retarded finite-temperature two-point function of the local operator \(\mathcal{O}\) holds:

\[
\chi_\mathcal{O}(\omega, q) = \frac{1}{2} \sum_{r,s=0}^{\infty} C_{r,s}^\mathcal{O}(\omega, q).
\]

(9)

Here \(C_{r,s}\) denotes the contribution with \(r\) particles in the thermal trace and \(s\) in the intermediate state:

\[
C_{r,s}^\mathcal{O}(\omega, q) = \int_0^\beta d\tau \int_{-\infty}^{\infty} dx e^{i\omega_n \tau - iqx} C_{r,s}(\tau, x) \bigg|_{\omega_n = -\omega +i0},
\]

\[
C_{r,s}(\tau, x) = - \sum_{\{a_j\}, \{a_j'\}} \int \frac{d\theta_1 \ldots d\theta_r}{(2\pi)^r r!} \int \frac{d\theta_1' \ldots d\theta_s'}{(2\pi)^s s!} e^{-\beta(E_r - E_s)} e^{-\tau(E_s - E_r)}
\]

\[
\times e^{-i(P_r - P_s)x} |\theta_1 \ldots \theta_r\rangle_{a_1 \ldots a_r} \langle \theta_1' \ldots \theta_s' | \mathcal{O}(0, 0) |\theta_1' \ldots \theta_s'|_{a_1' \ldots a_s'}|^2.
\]

(10)
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The partition function can formally be expressed as
\[ Z = \langle 0|0 \rangle + \sum_b \int \frac{d\theta}{2\pi} e^{-\beta c(\theta)} b(\theta) c_b \]
\[ + \sum_{b_1, b_2} \int \frac{d\theta_1 d\theta_2}{2(2\pi)^2} e^{-\beta c(\theta_1) + c(\theta_2)} b_1 b_2 (\theta_1 \theta_2 | \theta_2, \theta_1) b_2 b_1 + \cdots = \sum_{n=0}^{\infty} Z_n. \] (11)

Both the partition function and the Lehmann representations of correlation functions are ill-defined in the infinite-volume limit as the normalization condition for scattering states is (for \( \theta_1 > \theta_2 > \cdots > \theta_n \) and \( \theta'_1 > \theta'_2 > \cdots > \theta'_n \))
\[ a_1 \cdots a_n \langle \theta_1, \ldots, \theta_n | \theta_{n}' \ldots \theta_{n}' \rangle_{a_n \cdots a'_1} = \prod_{j=1}^{n} 2\pi \delta(\theta_j - \theta'_{j}) \delta_{a_j, a'_j}. \] (12)

The idea of a low-temperature expansion is to subtract these divergences in some way [21,23,24]. Here we proceed as follows. We separate the contributions \( C_{r,s} \) in the Lehmann representation of the two-point function (9) according to their different formal temperature dependences into
\[ C_{r,s}^{\mathcal{O}}(\omega, q) = E_{r,s}^{\mathcal{O}}(\omega, q) + F_{r,s}^{\mathcal{O}}(\omega, q), \] (13)
where
\[ E_{r,s}^{\mathcal{O}}(\omega, q) = \sum_{\{a_j, a'_j\}} \int \frac{d\theta_1 \cdots d\theta_r}{(2\pi)^r r!} \int \frac{d\theta'_1 \cdots d\theta'_s}{(2\pi)^s s!} 2\pi \delta(q + P_r - P_s) \]
\[ \times \frac{e^{-\beta E_r}}{\omega + i\delta - E_s + E_r} |a_1 \cdots a_r \langle \theta_1 \cdots \theta_r | \mathcal{O}(0,0) | \theta_{r}' \cdots \theta_{r}' \rangle_{a_r \cdots a'_1}|^2. \] (14)

The functions \( E_{r,s}, F_{r,s} \) are related by
\[ E_{r,s}^{\mathcal{O}}(\omega, q) = [F_{r,s}^{\mathcal{O}}(-\omega, -q)]^*. \] (15)

The matrix elements \( a_1 \cdots a_r \langle \theta_1 \cdots \theta_r | \mathcal{O}(0,0) | \theta_{r}' \cdots \theta_{r}' \rangle_{a_r \cdots a'_1} \) can be decomposed into a connected and a disconnected contribution. The latter is characterized by containing factors of \( \delta(\theta_j - \theta'_k) \), signaling that some of the particles do not encounter the operator \( \mathcal{O} \) in the process described by the matrix element. A fundamental assumption of our approach is that the disconnected contributions act to cancel the partition function in the denominator of (9). More precisely, we define quantities
\[ E_{j,k}^{\mathcal{O}} = E_{j,k}^{\mathcal{O}} - \sum_{m=1}^{n} Z_m E_{j-m,k-m}^{\mathcal{O}}, \]
\[ F_{j,k}^{\mathcal{O}} = F_{j,k}^{\mathcal{O}} - \sum_{m=1}^{n} Z_m F_{j-m,k-m}^{\mathcal{O}}, \quad n = 0, 1, 2, \ldots, \]
\[ \mathcal{E}_n^\mathcal{O} = \sum_{k=0}^{n-1} E_{k,n}^{\mathcal{O}} + \sum_{m=n}^{\infty} E_{n,m}^{\mathcal{O}}, \quad n = 0, 1, 2, \ldots, \]
\[ \mathcal{F}_n^\mathcal{O} = \sum_{k=0}^{n-1} F_{k,n}^{\mathcal{O}} + \sum_{m=n}^{\infty} F_{n,m}^{\mathcal{O}}, \quad n = 0, 1, 2, \ldots. \] (17) (18)

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The key assertion of our low-temperature expansion is that the quantities defined in this way are finite in the thermodynamic limit. Upon re-ordering of the infinite sums the two-point function (9) is expressed in terms of the $E_s^O$ and $F_s^O$ as

$$\chi^O(\omega, q) = \sum_{s=0}^{\infty} E_s^O(\omega, q) + F_s^O(\omega, q).$$

(19)

By construction $E_s^O$ and $F_s^O$ formally have a temperature dependence

$$E_s^O, F_s^O \sim \mathcal{O}(e^{-s\beta\Delta}),$$

(20)

and (19) hence constitutes a low-temperature expansion of $\chi^O(\omega, q)$. The two-point functions we analyze in detail below have the symmetry

$$\chi^O(\omega, q) = \chi^O(-\omega, -q),$$

(21)

which relates the positive and negative frequency regions. It is useful to combine $E_i$ and $F_i$ into quantities that exhibit the same symmetry

$$C_i^O(\omega, q) = E_i^O(\omega, q) + F_i^O(\omega, q).$$

(22)

In terms of these quantities the spectral representation takes the form

$$\chi^O(\omega, q) = \sum_{s=0}^{\infty} C_s^O(\omega, q).$$

(23)

A key property of the $C_i^O$ is that they are finite in the thermodynamic limit. We demonstrate this explicitly for the first nontrivial terms $C_1^O$ and $C_2^O$ below and postulate that it is true in general.

### 2.1. Resummation

Following the procedure set out above, the finite-temperature Lehman representation of the particular two-point functions analyzed below can be re-expressed in the form (23), where the quantities $C_r^O(\omega, q)$ are finite in the infinite-volume limit. However, in the cases we are interested in, the functions $C_r^O(\omega, q)$ are not uniformly small. In order to make this statement more precise let us denote the single-particle dispersion relation by

$$\varepsilon(q) = \sqrt{\Delta^2 + v^2 q^2}.$$ 

(24)

We observe that, as long as both $\omega \pm \varepsilon(q) \sim \mathcal{O}(1)$, the $C_r^O$ are of order $\mathcal{O}(e^{-3r\Delta})$ and hence (23) provides a good low-temperature expansion of the two-point functions we are interested in far away from the mass shell. On the other hand, when we approach the mass shell we have

$$C_r^O(\omega, q) \sim (\omega^2 - \varepsilon^2(q))^{r+1}.$$ 

(25)

In order to obtain an expression for the susceptibility close to the mass shell we therefore need to sum up an infinite number of terms in (23). For the cases considered below, the
zero-temperature two-point function is of the form
\[ C^0_0(\omega, q) = \frac{Z}{(\omega + i\delta)^2 - \epsilon^2(q)} + \cdots, \quad (26) \]
where the corrections are negligible in the regime of temperatures and frequencies we consider. We then introduce a quantity \( \Sigma^0(\omega, q) \) by defining
\[ \chi^0(\omega, q) = \frac{C^0_0(\omega, q)}{1 - C^0_0(\omega, q)\Sigma^0(\omega, q)}. \quad (27) \]
The low-temperature expansion (23) for \( \chi^0(\omega, q) \) then provides a way of determining low temperature approximations to \( \Sigma^0(\omega, q) \) in the following way. Assuming that a low-temperature expansion of the form
\[ \Sigma^0(\omega, q) = \sum_{n=1}^{\infty} \Sigma^n_0(\omega, q) \quad (28) \]
exists, we may determine the leading term at low temperatures by expanding
\[ \chi^0(\omega, q) = C^0_0(\omega, q) + [C^0_0(\omega, q)]^2 \Sigma^n_0(\omega, q) + [C^0_0(\omega, q)]^3 [\Sigma^n_0(\omega, q)]^2 + \cdots = C^0_0(\omega, q) + [C^0_0(\omega, q)]^2 \Sigma^1_0(\omega, q) + \cdots \quad (29) \]
and then comparing this expansion to (23). This gives
\[ \Sigma^1_0(\omega, q) = C^1_0(\omega, q)[C^0_0(\omega, q)]^{-2}, \]
\[ \Sigma^2_0(\omega, q) = -C^2_0(\omega, q)[\Sigma^1_0(\omega, q)]^2 + C^2_0(\omega, q)[C^0_0(\omega, q)]^{-2}, \]
\[ \Sigma^3_0(\omega, q) = \cdots. \quad (30) \]
We now turn to the implementation of the program set out above to the case of the quantum Ising model.

3. Quantum Ising model

The Hamiltonian of the transverse field Ising ferromagnet is given by
\[ H = \sum_n -J\sigma_n^z\sigma_{n+1}^z + h\sigma_n^x, \quad (31) \]
where we take \( J, h > 0 \). The phase diagram of the model (31) is shown in figure 1. At zero temperature the quantum Ising model (31) exhibits two phases:

(i) **Ordered phase:** This phase occurs for \( h < J \) and is characterized by spontaneously broken \( \mathbb{Z}_2 \) symmetry associated with long-range magnetic order along the \( z \) direction \( \langle \sigma_n^z \rangle \neq 0 \). \quad (32)

(ii) **Disordered phase:** This phase occurs for \( h > J \). There is no spontaneous symmetry breaking and the only ordered moment is along the \( x \) direction.

The two phases are related by the Kramers–Wannier duality transformation [42], which provides a map of operators and therefore correlation functions [43, 44]. By virtue of this map it is sufficient to consider the regime \( h > J \) only. For the remainder of this paper we will concentrate on this parameter regime.
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Figure 1. Phase diagram of the quantum Ising chain. At zero temperature ferromagnetic long range order occurs for $h < J$.

3.1. Scaling limit in the disordered phase

The quantum Ising model (31) can be solved exactly [45, 46]. However, rather than analyzing the lattice model (31) directly we concentrate on the simpler scaling limit defined by

$$J, h \to \infty, \quad a_0 \to 0, \quad |J - h| \text{ fixed,} \quad v = a_0 \sqrt{Jh} \text{ fixed.} \quad (33)$$

Here $a_0$ is the lattice spacing. In the scaling limit the lattice spin operators turn into continuum fields $\sigma_n^z \to C a_0^{1/8} \sigma(x)$, while the Hamiltonian is expressed in terms of left and right-moving Majorana fermions:

$$H = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \left[ i v (\bar{\psi} \partial_x \psi - \psi \partial_x \bar{\psi}) - i \Delta \psi \bar{\psi} \right]. \quad (34)$$

The excitation spectrum of (34) follows from a mode expansion of the Majorana field and is given by

$$\varepsilon(q) = \sqrt{\Delta^2 + v^2 q^2}. \quad (35)$$

In the disordered phase excitations can be thought of in terms of simple spin flips, as can be seen by considering the limit $h \gg J$. Let us denote the corresponding annihilation and creation operators by $Z(\theta)$ and $Z^\dagger(\theta)$, respectively. They fulfill the simple Faddeev–Zamolodchikov algebra:

$$A(\theta_1) A(\theta_2) = SA(\theta_2) A(\theta_1),$$

$$A^\dagger(\theta_1) A^\dagger(\theta_2) = SA^\dagger(\theta_2) A^\dagger(\theta_1),$$

$$A(\theta_1) A^\dagger(\theta_2) = 2\pi \delta(\theta_1 - \theta_2) + SA^\dagger(\theta_2) A(\theta_1), \quad (36)$$

where the scattering matrix is $S = -1$. The ground state is then defined as

$$A(\theta)|0\rangle = 0, \quad (37)$$

and a basis of scattering states is given by

$$|\theta_1, \ldots, \theta_n\rangle = A^\dagger(\theta_1) \cdots A^\dagger(\theta_n)|0\rangle. \quad (38)$$

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Energy and momentum of the scattering states are by construction additive:

\[ E_s(\theta_1, \ldots, \theta_s) = \sum_{j=1}^{s} \Delta \cosh \theta_j, \]
\[ P_s(\theta_1, \ldots, \theta_s) = \sum_{j=1}^{s} \frac{\Delta}{\nu} \sinh \theta_j. \]

(39)

In terms of the states (38) the resolution of the identity is

\[ \text{id} = |0\rangle \langle 0| + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} |\theta_1, \ldots, \theta_n\rangle \langle \theta_n, \ldots, \theta_1|. \]

(40)

For the calculation of correlation functions the knowledge of the matrix elements or form factors of local operators is necessary. In the disordered phase the non-vanishing form factors of \( \sigma^z \) contain an odd number of particles and are given by [46]–[49], [5]

\[ \langle 0| \sigma(0, 0)|\theta_1, \ldots, \theta_{2n+1}\rangle = i^n \bar{\sigma} \prod_{i<j} \tanh \frac{\theta_i - \theta_j}{2}. \]

(41)

It is customary to choose the normalization of the field \( \sigma(x) \) such that

\[ \lim_{x \to 0} \langle 0| \sigma(x) \sigma(0)|0\rangle = \frac{1}{|x|^{1/4}}, \]

(42)

which implies that

\[ \bar{\sigma} = 2^{1/2} e^{-1/8} A^{3/2} \left[ \frac{\Delta}{\nu} \right]^{1/8}, \quad A = 1.28242712910062 \ldots \]

(43)

We note that in this normalization the continuum field \( \sigma(x) \) is related to the lattice spin operator by \( \sigma^z_j \to 2^{1/24} e^{1/8} A^{-3/2} \sigma(x) \). In what follows we need more general matrix elements of the form

\[ \langle \theta'_1, \ldots, \theta'_k| \sigma(0, 0)|\theta_1, \ldots, \theta_n\rangle. \]

(44)

These can be calculated using crossing relations following [1]. The necessary identities are summarized in appendix A.

3.2. Spectral representation of the dynamical susceptibility

Our main interest is in calculating the retarded dynamical susceptibility at finite temperature, which is obtained by analytically continuing the Matsubara two-point function:

\[ \chi_\sigma(\omega, q) = \int_0^\beta d\tau dx \ e^{i\omega \tau - i qx} \chi_\sigma(\tau, x) \bigg|_{\omega_n \to \delta - i\omega}, \]
\[ \chi_\sigma(\tau, x) = -\langle T_\tau \sigma(\tau, x) \sigma(0, 0) \rangle. \]

(45)

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In the basis of scattering states introduced above, the following formal spectral representation for the finite-temperature dynamical susceptibility holds:

$$\chi_\sigma(\omega, q) = \frac{1}{Z} \sum_{r,s=0}^{\infty} C_{r,s}^\sigma(\omega, q).$$  \hspace{1cm} (46)

Here $C_{r,s}$ denotes the contribution with $r$ particles in the thermal trace and $s$ in the intermediate state and is of the form (10) without isotopic quantum numbers $a_j$ and $a'_k$.

As we have already stated, both the partition function and the Lehmann representations of correlation functions are ill-defined in the infinite-volume limit by virtue of the normalization condition of states (12). Setting this issue aside for a moment, we may cast $C_{r,s}^\sigma$ in the form

$$C_{r,s}^\sigma(\omega, q) = \int \frac{d\theta_1 \ldots d\theta_r}{(2\pi)^r r!} \int \frac{d\theta'_1 \ldots d\theta'_s}{(2\pi)^s s!} 2\pi \delta(q + P_r - P_s)$$

$$\times \frac{e^{-\beta E_r} - e^{-\beta E_s}}{\omega + i\delta - E_s + E_r} |\langle \theta_1 \ldots \theta_r | \sigma(0,0) | \theta'_s \ldots \theta'_1 \rangle|^2. \hspace{1cm} (47)$$

In order to implement the low-temperature expansion we separate the $C_{r,s}^\sigma$ according to their (formal) temperature dependences into $E_{r,s}^\sigma$ and $F_{r,s}^\sigma$ following (13), (14). At zero temperature the spectral sum simplifies dramatically as only terms with $r = 0$ or $s = 0$ remain.

### 3.3. Zero-temperature dynamical susceptibility

At $T = 0$ the leading contributions to the dynamical susceptibility at low frequencies are

$$\chi_\sigma(\omega, q) \approx [E_{0,1}^\sigma(\omega, q) + F_{1,0}^\sigma(\omega, q) + E_{0,3}^\sigma(\omega, q) + F_{3,0}^\sigma(\omega, q)]. \hspace{1cm} (48)$$

Here the one-particle contributions are

$$E_{0,1}^\sigma(\omega, q) = \frac{v\bar{\sigma}^2}{\varepsilon(q)} \frac{1}{\omega - \varepsilon(q) + i0},$$

$$F_{1,0}^\sigma(\omega, q) = - \frac{v\bar{\sigma}^2}{\varepsilon(q)} \frac{1}{\omega + \varepsilon(q) + i0}. \hspace{1cm} (49)$$

The three-particle terms can be cast in the form

$$E_{0,3}^\sigma(\omega, q) = \frac{v\bar{\sigma}^2}{\Delta} \int \frac{d\theta_1 \, d\theta_2 \, d\theta_3}{(2\pi)^3} \frac{1}{\cosh \theta_3}$$

$$\times \frac{\tanh^2((\theta_1 - \theta_2)/2) \, \tanh^2((\theta_1 - \theta_3)/2) \, \tanh^2((\theta_2 - \theta_3)/2)}{\omega - \Delta \sum_{j=1}^{3} \cosh \theta_j + i0}, \hspace{1cm} (50)$$

where

$$\theta_3 = \arcsinh \left( \frac{vq}{\Delta} - \sinh \theta_1 - \sinh \theta_2 \right). \hspace{1cm} (51)$$

We plot the real and imaginary parts of $E_{0,3}(\omega, q = 0)$ in figure 2. In order to plot these functions it is useful to separate off a dimensionful normalization factor

$$N_0 = \frac{v\bar{\sigma}^2}{\Delta^2}. \hspace{1cm} (52)$$

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Figure 2. Real and imaginary parts of $E_{03}^\sigma(\omega, q = 0)$.

We see that by virtue of the smallness of $E_{03}^\sigma$ the dynamical response at low energies is dominated by the coherent single-particle contributions $E_{0,1}^\sigma + F_{1,0}^\sigma$. This yields the following result for the temperature-independent part of the spectral representation for the dynamical susceptibility

$$C_0^\sigma(\omega, q) \approx E_{0,1}^\sigma(\omega, q) + F_{1,0}^\sigma(\omega, q) = \frac{2u\bar{\sigma}^2}{(\omega + i0)^2 - \varepsilon^2(q)},$$  \hspace{1cm} (53)

where $\varepsilon(q) = \sqrt{\Delta^2 + v^2q^2}$.

3.4. Infinite volume regularization

At low temperatures the next most important contributions arise from $C_{1,2}^\sigma$ and $C_{2,1}^\sigma$, which are formally given by

$$C_{1,2}^\sigma(\omega, q) = v \int \frac{d\theta_1 d\theta_2}{2(2\pi)^2} \frac{e^{-\Delta\beta c(\theta)} - e^{-\Delta\beta[c(\theta_1) + c(\theta_2)]}}{\omega + i0 - \Delta[c(\theta_1) + c(\theta_2) - c(\theta)]} \times |\langle\theta|\sigma(0)|\theta_2, \theta_1\rangle|^2 \delta(vq - \Delta[s(\theta_1) + s(\theta_2) - s(\theta)]),$$  \hspace{1cm} (54)

$$C_{2,1}^\sigma(\omega, q) = v \int \frac{d\theta_1 d\theta_2}{2(2\pi)^2} \frac{e^{-\Delta\beta[c(\theta_1) + c(\theta_2)]} - e^{-\Delta\beta c(\theta)}}{\omega + i0 + \Delta[c(\theta_1) + c(\theta_2) - c(\theta)]} \times |\langle\theta|\sigma(0)|\theta_2, \theta_1\rangle|^2 \delta(vq + \Delta[s(\theta_1) + s(\theta_2) - s(\theta)]),$$  \hspace{1cm} (55)

where $c(\theta) = \cosh \theta$ and $s(\theta) = \sinh \theta$. We note that $C_{2,1}$ can be obtained from $C_{1,2}$ as

$$C_{2,1}^\sigma(\omega, q) = [C_{1,2}^\sigma(-\omega, -q)]^*.$$  \hspace{1cm} (56)

In order to proceed further we need to evaluate the products of form factors. Following Smirnov [1] we have the following crossing relations for three-particle form factors of the spin field in the disordered phase:

$$\langle\theta_1, \theta_2|\sigma(0, 0)|\theta_3\rangle = \langle\theta_1 - i0, \theta_2 - i0|\sigma(0, 0)|\theta_3\rangle + 2\pi\bar{\sigma}[\delta(\theta_{32}) - \delta(\theta_{31})] = \langle\theta_1 + i0, \theta_2 + i0|\sigma(0, 0)|\theta_3\rangle - 2\pi\bar{\sigma}[\delta(\theta_{32}) - \delta(\theta_{31})] = \langle\theta_1 + i0, \theta_2 - i0|\sigma(0, 0)|\theta_3\rangle + 2\pi\bar{\sigma}[\delta(\theta_{32}) + \delta(\theta_{31})] = \langle\theta_1 - i0, \theta_2 + i0|\sigma(0, 0)|\theta_3\rangle - 2\pi\bar{\sigma}[\delta(\theta_{32}) + \delta(\theta_{31})].$$  \hspace{1cm} (57)

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where we have defined $\theta_{jk} = \theta_j - \theta_k$. The poles occurring in the form factors appearing on the RHS have been shifted away from the real rapidity axis. The delta-function contributions correspond to disconnected pieces of the form factor. It is clear from expression (57) that the absolute value squared of a form factor is ill-defined as a consequence of working in an infinite volume. As we will show below, the absolute value squared of a form factor contains divergent pieces that get canceled by a corresponding divergence in the partition function. In order to exhibit these cancellations we need to exhibit the divergences in the form factor squares explicitly. For the three-particle form factor of the spin field we do this as follows:

$$|\langle \theta_3 | \sigma(0, 0) | \theta_2, \theta_1 \rangle|^2 \equiv \lim_{\kappa \to 0} |\langle \theta_3 | \sigma(0, 0) | \theta_2, \theta_1 \rangle \langle \theta_1, \theta_2 | \sigma(0, 0) | \theta_3 + \kappa \rangle|. \quad (58)$$

The product of form factors on the RHS is now well defined, so that we can use (57) to extract the divergent pieces explicitly. For form factors involving more than three particles one needs to introduce several parameters $\kappa_j$, e.g.

$$|\langle \theta_4, \theta_5 | \sigma(0, 0) | \theta_3, \theta_2, \theta_1 \rangle|^2 \equiv \lim_{\kappa_{1,2} \to 0} |\langle \theta_4, \theta_5 | \sigma(0, 0) | \theta_3, \theta_2, \theta_1 \rangle \times \langle \theta_1, \theta_2, \theta_3 | \sigma(0, 0) | \theta_5 + \kappa_2, \theta_4 + \kappa_1 \rangle|. \quad (59)$$

In order to calculate the contributions $C_{1,2}^\sigma$ and $C_{2,1}^\sigma$ by means of contour integral techniques the following choice of analytically continuing $\theta_{1,2}$ into the complex plane is the most convenient:

$$|\langle \theta_3 | \sigma(0, 0) | \theta_2 \theta_1 \rangle|^2 \equiv \lim_{\kappa \to 0} (|\langle \theta_1 + i 0, \theta_2 - i 0 | \sigma(0, 0) | \theta_3 + \kappa \rangle + 2\pi i \bar{\sigma} [\delta(\theta_{32} + \kappa) + \delta(\theta_{31} + \kappa)]) 
\times (\langle \theta | \sigma(0, 0) | \theta_2 - i 0, \theta_1 + i 0 \rangle - 2\pi i \bar{\sigma} [\delta(\theta_{31} + \delta(\theta_{32})]). \quad (60)$$

The product of form factors on the RHS of equation (60) is evaluated in appendix B. Using (B.14) we can express the contribution $C_{1,2}^\sigma$ in the form

$$C_{1,2}^\sigma(\omega, q) = C_{1,2}^{\text{conn}}(\omega, q) + C_{1,2}^{\text{dis}}(\omega, q), \quad (61)$$

where the ‘connected’ $C_{1,2}^{\text{conn}}(\omega, q)$ and ‘disconnected’ $C_{1,2}^{\text{dis}}(\omega, q)$ parts correspond to the contributions of the first and second terms on the rhs of (B.14). The disconnected part is given by

$$C_{1,2}^{\text{dis}}(\omega, q) = \nu \bar{\sigma}^2 \lim_{\kappa \to 0} \int d\theta_1 \, d\theta_2 \, d\theta_3 \frac{e^{-\Delta \bar{\delta}(\theta_3)} - e^{-\Delta \bar{\delta}[c(\theta_1) + c(\theta_2)]}}{\omega + i 0 - \Delta [c(\theta_1) + c(\theta_2) - c(\theta_3)]} 
\times \delta(vq - \Delta [s(\theta_1) + s(\theta_2) - s(\theta_3)])(\delta(\kappa) \bar{\delta}(\theta_{31}) + \delta(\theta_{32}) \bar{\delta}(\theta_{31})) 
\times (1 - e^{-\beta\epsilon(q)} \int \frac{d\theta}{\omega + i 0 - \epsilon(q)} e^{-\beta \epsilon(q)} + \delta(\kappa) \int d\theta e^{-\beta \epsilon(c(\theta))} \right] 
\times \int \frac{d\theta}{\omega + i 0 - \epsilon(q)} e^{-\beta \epsilon(q)} + \delta(\kappa) \int d\theta e^{-\beta \epsilon(c(\theta))} \right]. \quad (62)$$

We note that most of the ‘cross-terms’ derived in appendix B cancel in the integral as they are antisymmetric in $\theta_1$ and $\theta_2$, while the remaining part of the integrand is symmetric.
The connected part of $C_{1,2}$ is given by

$$C_{1,2}^{\text{conn}}(\omega, q) = v\sigma^2 \int \frac{d\theta d\theta_+ d\theta_-}{(2\pi)^2} \left[ e^{-\Delta \beta c(\theta)} - e^{-2\Delta \beta c(\theta_-)} \right]$$

$$\times K(\theta_-, \theta_+, \theta) \frac{\delta(vq - \Delta [2s(\theta_+)c(\theta_-) - s(\theta)])}{\omega - \Delta [2c(\theta_+)c(\theta_-) - c(\theta)]},$$

where we have changed variables to

$$\theta_{\pm} = \frac{\theta_2 \pm \theta_1}{2},$$

and defined a function

$$K(\theta_-, \theta_+, \theta) = \tanh^2(\theta_-) \frac{\tanh^2((\theta_- - \theta + \theta_+ - i0)/2)}{\tanh^2((\theta_- + \theta - \theta_+ - i0)/2)}.$$

We can carry out the integration over $\theta_+$ using the momentum conservation delta function. The result is

$$C_{1,2}^{\text{conn}}(\omega, q) = v\sigma^2 \int \frac{d\theta d\theta_-}{(2\pi)^2} \frac{e^{-\Delta \beta c(\theta)} - e^{-\beta u(q, \theta, \theta_-)}}{\Omega(\theta, \omega) - u(q, \theta, \theta_-)} K(\theta_-, \theta_+^0(q, \theta, \theta_-), \theta).$$

Here we have introduced the notations

$$\Omega(\theta, \omega) = \omega + i\delta + \Delta \cosh(\theta),$$

$$Q(\theta, q) = q + \frac{\Delta}{v} \sinh(\theta),$$

$$\theta_+^0(q, \theta, \theta_-) = \text{arcsinh} \left( \frac{vQ(\theta, q)}{2\Delta c(\theta_-)} \right),$$

$$u(q, \theta, \theta_-) = \sqrt{(vQ(\theta, q))^2 + (2\Delta c(\theta_-))^2}.$$

In order to proceed it is useful to consider the integrand as a function of the complex variable $\theta_-$ (the analytic properties as a function of $\theta$ are not as nice). We want to move the integration contour away from the real axis in order to avoid the vicinities of the double poles. The branch point of the square roots and inverse hyperbolic functions occur only at $|\text{Im}(\theta_-)| = \pi/2$, which allows us to move the contour to a line parallel to the real axis in the lower half-plane. When doing this we may encounter a simple pole when

$$\Omega(\theta, \omega) - u(q, \theta, \theta_-) = 0. (68)$$

Two kinds of solutions to this equation exist in the strip $-(\pi/2) < \gamma \leq \text{Im}(\theta_-) \leq 0$ (for simplicity we assume $\omega > 0$ in the following)

(i) If $\Omega^2 - (vQ)^2$ is larger than $4\Delta^2$, we have a simple pole at

$$\alpha(\omega, q, \theta) = -\text{arccosh} \left( \frac{\bar{s}(\omega, q, \theta)}{2\Delta} \right) - i0,$$

$$\bar{s}(\omega, q, \theta) = [(\omega + \Delta \cosh \theta)^2 - (vq + \Delta \sinh \theta)^2]^{1/2}. (70)$$
(ii) If \(4\Delta^2 \cos^2 \gamma < \Omega^2 - (vQ)^2 < 4\Delta^2\), we have a pole at
\[
\mp(\omega, q, \theta) = -i \arccos \left( \frac{s(\omega, q, \theta)}{2\Delta} \right) - 0. \tag{71}
\]

Defining
\[
\theta_0(\omega, q, \theta) = \arcsinh \left( \frac{vq + \Delta \sinh \theta}{s(\omega, q, \theta)} \right), \tag{72}
\]
we may cast \(C^{\text{conn}}_{12}\) in the form
\[
C^{\text{conn}}_{12}(\omega, q) = -iv\sigma^2(1 - e^{-\beta\omega}) \int_{s_+} \frac{d\theta}{2\pi} e^{-\beta\Delta c(\theta)} K(\alpha, \theta_0, \theta) - v\sigma^2(1 - e^{-\beta\omega}) \int_{r_+} \frac{d\theta}{2\pi} e^{-\Delta c(\theta)} K(\alpha, \theta_0, \theta)
\]
\[
+ v\sigma^2 \int \frac{d\theta}{(2\pi)^2} \int \frac{d\theta}{\Omega(\omega) - u(q, \theta, \theta_-)} K(\theta, \theta_0, \theta_-). \tag{73}
\]

Here \(S\) is a straight line in the lower half-plane parallel to the real axis with imaginary part \(-i\gamma\). \(S_+\) are the segments of the real axis such that \(s^2 > 4\Delta^2\) and \(T^+_\gamma\) are the segments of the real axis such that \(4\Delta^2 \cos^2 \gamma \leq s^2 \leq 4\Delta^2\). The segments \(S_+, T^+_\gamma\) for positive frequencies \(\omega > 0\) can be characterized as follows:

(i) \(S_+\):
\[
\theta \in \begin{cases} 
(-\infty, \infty) & \text{if } \omega > |vq| \text{ and } s^2 > \Delta^2, \\
(-\infty, a_-) \cup [a_+, \infty) & \text{if } \omega > |vq| \text{ and } s^2 < \Delta^2, \\
(-\infty, a_-) & \text{if } 0 < \omega < vq, \\
[a_+, \infty) & \text{if } 0 < \omega < -vq,
\end{cases} \tag{74}
\]

where
\[
a_+ = \ln \left[ \frac{3\Delta^2 - s^2 \pm s - 10\Delta^2 s + 9\Delta^2}{2\Delta(\omega - vq)} \right]. \tag{75}
\]

(ii) \(T^+_\gamma\):
\[
\theta \in \begin{cases} 
[a_, a_+] & \text{if } \omega > |vq| \text{ and } \Delta^2 \gamma^2_+ < s^2 < \Delta^2, \\
[a_, a'_+] \cup [a'_+, a_+] & \text{if } \omega > |vq| \text{ and } s^2 < \Delta^2 \gamma^2_-, \\
[a_, a'_+] & \text{if } 0 < \omega < vq, \\
[a'_+, a_+] & \text{if } 0 < \omega < -vq,
\end{cases} \tag{76}
\]

where \(\gamma_\pm = 1 \pm 2 \cos \gamma\) (we have assumed that \(0 < \gamma < \pi/3\)) and
\[
a'_\pm = \ln \left[ \frac{-s^2 - \Delta^2 \gamma_+ \gamma_- \pm \sqrt{(s^2 + \Delta^2 \gamma_+ \gamma_-)^2 - 4\Delta^2 s^2}}{2\Delta(\omega - vq)} \right]. \tag{77}
\]

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The real part of $C_{1,2}^{\text{conn}}$ is given by the last two terms in (73). For fixed $\gamma$ there will always be a value of $\theta$ such that the $\theta_-$ integral in the third line of (73) is very close to a singularity at $\theta_-=0$. The problem occurs at $\theta$ such that

$$\Omega(\theta, \omega) - \sqrt{(vQ(\theta, q))^2 + 4\Delta^2 \cos^2(\gamma)} = 0.$$  \hspace{1cm} (78)$$

For $\omega > |vq|$ there are singularities at $\theta = a'_+ \approx -vq$ if $s^2 < E^2_\omega, \omega \approx \Delta^2$. We further observe that $F_{2,1} - Z_1 F_{1,0}$ is negligibly small in the parameter regime we are interested in ($\omega \approx \Delta$ and low temperatures). We therefore drop it in the following.

For $\gamma = \pi/3$ we have $\gamma_- = 0$, so that we basically can always stay away from this problem. If we want to know the answer for $s^2 < 0$ we can always calculate $f(\theta, \omega, q)$ for different values of $\gamma$ and in this way always stay away from having to deal with a singularity. This concludes our evaluation of $C_{2,1}(\omega, q)$ in the infinite-volume regularization. The contribution $C_{2,1}(\omega, q)$ can be obtained in the same way.

We are now in a position to calculate the leading contributions to the quantity $C_1^{\sigma}(\omega, q)$:

$$C_1^{\sigma} \approx E_{1,0}^{\sigma} + F_{0,1}^{\sigma} + (E_{1,2}^{\sigma} - Z_1 E_{0,1}^{\sigma}) + (F_{2,1}^{\sigma} - Z_1 F_{1,0}^{\sigma}).$$  \hspace{1cm} (80)$$

The leading corrections to $C_1^{\sigma}$ are $(E_{1,4}^{\sigma} - Z_1 E_{0,3}^{\sigma})$ and $(F_{4,1}^{\sigma} - Z_1 F_{3,0}^{\sigma})$ and these are expected to be small for the same reasons that $E_{0,3}^{\sigma}$ is negligible for $\omega \approx \Delta$. We further observe that $F_{2,1}^{\sigma} - Z_1 F_{1,0}^{\sigma}$ is negligibly small in the parameter regime we are interested in ($\omega \approx \Delta$ and low temperatures). We therefore drop it in the following.

As $Z_1$ is ill-defined in the infinite-volume limit we regulate it by ‘shifting the trace’ in the same way as (60):

$$Z_1 = \lim_{\kappa \to 0} \left[ \frac{d\theta}{2\pi} e^{-\beta \Delta c(\theta)} (\theta | \theta + \kappa) \right] = \lim_{\kappa \to 0} \delta(\kappa) \int d\theta \ e^{-\beta \Delta c(\theta)}.$$  \hspace{1cm} (81)$$

Combining (22), (61), (62), (73) and (81) we arrive at the following expression for the first subleading part of the dynamical susceptibility (23):

$$C_1^{\sigma}(\omega, q) \approx -iv\sigma^2 \int_{S_+} d\theta \ e^{-\beta \Delta c(\theta)} K(\alpha(\omega, q, \theta), \theta_0(\omega, q, \theta), \theta) \sqrt{s^2(\omega, q, \theta) - 4\Delta^2}$$

$$- iv\sigma^2 \int_{T_{\perp}^+} d\theta \ e^{-\Delta c(\theta)} K(\alpha(\omega, q, \theta), \theta_0(\omega, q, \theta), \theta) \sqrt{4\Delta^2 - s^2(\omega, q, \theta)}$$

$$+ iv\sigma^2 \int_{S_+} d\theta \ e^{-\beta \Delta c(\theta)} K(\theta_-, \theta_0(\omega, q, \theta), \theta_0(\omega, q, \theta), \theta) \sqrt{\Omega(\theta, \omega) - u(q, \theta, \theta_-)} u(q, \theta, \theta_-).$$  \hspace{1cm} (82)$$

Here $S_+$ and $T_{\perp}^+$ are the segments of the real axis characterized by $s^2(\omega, q, \theta) > 4\Delta^2$ and $4\Delta^2 \cos^2 \gamma \leq s^2(\omega, q, \theta) \leq 4\Delta^2$, respectively, and $S$ is the contour from $-\infty - i\gamma$ to $\infty - i\gamma$ parallel to the real axis.
3.5. Finite-volume regularization

Another way to regulate infinities in matrix elements is to work in a large, finite volume $R$. The Hamiltonian on a finite, periodic line of length $R$ is

$$H = \int_0^R \frac{dx}{2\pi} \left[ i\psi \bar{\partial}_x \bar{\psi} - \psi \partial_x \bar{\psi} - i\Delta \psi \bar{\psi} \right]. \quad (83)$$

The Hilbert space of the theory divides itself into two sectors: Neveu–Schwarz (NS) and Ramond (R). The NS sector consists of a Fock space built with even numbers of half-integer fermionic modes, i.e. states of the form

$$|p_1 \cdots p_{2N})_{NS} \equiv a_{p_1}^\dagger \cdots a_{p_{2N}}^\dagger |0\rangle_{NS}, \quad (84)$$

where a mode’s momentum satisfies

$$p_i = \frac{2\pi}{R} \left( n_i + \frac{1}{2} \right), \quad n_i \in \mathbb{Z}. \quad (85)$$

On the other hand, the R-sector consists of a Fock space composed of odd numbers of even integer fermionic modes:

$$|k_1 \cdots k_{2M+1})_R \equiv a_{k_1}^\dagger \cdots a_{k_{2M+1}}^\dagger |0\rangle_R, \quad k_i = \frac{2\pi}{R} n_i. \quad (86)$$

Energy $E(p_i)$ and momentum $P(p_i)$ of a NS state $|p_1 \cdots p_{2N})_{NS}$ are given simply by

$$E(\{p_i\}) = \sum_{i=1}^{2N} \varepsilon(p_i), \quad P(\{p_i\}) = \sum_{i=1}^{2N} p_i, \quad (87)$$

where as before $\varepsilon(p) = \sqrt{\Delta^2 + v^2 p^2}$. An identical relation holds for states in the R-sector. It is useful to parameterize the momenta in terms of a rapidity variable as

$$p_i = \Delta \sinh(\theta_{p_i}). \quad (88)$$

The finite volume form factors of the spin field have been determined in [26,50]. The non-vanishing form factors in the disordered phase are

$$R \langle k_1 \cdots k_{2M+1}|\sigma(0)|p_1 \cdots p_{2N}\rangle_{NS} = i^{M+N}C_R \prod_{i,j} g(\theta_{k_i})g(\theta_{p_j}) \times \prod_{i<j} \tanh\left(\frac{\theta_{k_i} - \theta_{k_j}}{2}\right) \prod_{i<j} \tanh\left(\frac{\theta_{p_i} - \theta_{p_j}}{2}\right) \prod_{i,j} \coth\left(\frac{\theta_{k_i} - \theta_{p_j}}{2}\right), \quad (89)$$

where for large $\Delta R$ we have

$$C_R = \bar{\sigma} + \mathcal{O}(e^{-\Delta R}),$$

$$g(\theta) = \frac{1 + \mathcal{O}(e^{-\Delta R})}{\sqrt{\Delta R} v^{-1} \cosh(\theta)}. \quad (90)$$

We note that the factors $g(\theta)$ disappear when the states are normalized in terms of rapidity variables. Importantly, up to exponentially small corrections, the matrix elements (89) have the same functional form as at $R = \infty$. The essential difference between large but
finite $R$ and the infinite-volume limit is that, at finite $R$, the momenta are quantized. This is a pattern that repeats itself for general integrable models, as emphasized in [25], and that we will exploit for our analysis of spin-1 chains.

Crucially, for large but finite $R$ the matrix elements (89) are finite. In the infinite-volume limit divergences develop in the factor

$$
\prod_{i,j} \coth \left( \frac{\theta_{k_i} - \theta_{p_j}}{2} \right),
$$

and occur when two momenta, $k_i$ and $p_j$, approach one another. However, a finite $R$ regulates these divergences by virtue of $k_i$ lying in the R sector with integer quantization and $p_j$ in the NS sector with half-integer quantization, respectively. Hence the two are never exactly equal. The finite-temperature Lehmann representation for the two-point function of the spin field on a ring of length $R$ takes the form

$$
\chi_R^R(\omega, q) = \frac{1}{Z^R} \sum_{r,s=0}^{\infty} C^R_{r,s}(\omega, q),
$$

where, for example,

$$
C^R_{2M+1,2N} \left( \omega, q = \frac{2\pi n q}{R} \right) = \int_0^R dx \int_0^\beta d\tau \ e^{i\omega n \tau - iqx} C^R_{2M+1,2N}(\tau, x) |_{\omega_n \to -i\omega}
$$

$$
= \sum_{\{k_j\},\{p_i\}} |_{R} \langle k_1 \cdots k_{2M+1} | \sigma(0) | p_1 \cdots p_{2N} \rangle_{NS}^2 \times \frac{e^{-\beta E(\{k_j\})} - e^{-\beta E(\{p_i\})}}{\omega + i\eta - E(\{p_i\}) + E(\{k_j\})} R \delta_{P(\{p_i\}) - P(\{k_j\})}.\label{eq:chir}
$$

$$
Z^R = 1 + \sum_{p \in R} e^{-\beta \epsilon(p)} + \sum_{p_1, p_2 \in NS} e^{-\beta \epsilon(p_1) + \epsilon(p_2)} + \cdots \equiv \sum_{n=0}^{\infty} Z_n^R.\label{eq:zr}
$$

Here $\eta$ is a positive infinitesimal. All terms in the expansion (93) are finite. As before, we re-order the spectral sums according to (22), which gives

$$
\chi^R_\sigma(\omega, q) = \sum_{r=0}^{\infty} C^R_r(\omega, q),
$$

where the $C^R_r$ are defined as the finite-volume analogs of (22).

### 3.6. Comparison between infinite- and finite-volume regularizations

In this section we establish the equivalence between the infinite-volume regularization used in section 3.2 to evaluate the leading order terms of $\chi^R_\sigma(\omega, q)$ and the finite-volume regularization (as $R \to \infty$) introduced in the preceding section. In particular we establish that $C^R_0 + C^R_1$, the first two terms in the temperature expansion of $\chi^R(\omega, q)$, and given by

$$
C^R_0 \approx E^R_{0,1} + F^R_{1,0},
$$

$$
C^R_1 \approx E^R_{1,0} + F^R_{0,1} + (E^R_{1,2} - Z^R_1 E^R_{0,1}) + (F^R_{2,1} - Z^R_1 F^R_{1,0}),
$$

are

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are equal to their counterparts in infinite volume once we take volume $R$ to infinity. The equivalence of $C_0$ in both schemes is straightforward. We have for $C_{0,1}^R$ and $C_{1,0}^R$ the following:

$$C_{0,1}^R(\tau, x) = E_{0,1}^R(\tau, x) + F_{0,1}^R(\tau, x) = -\sigma^2 v \sum_{p_i \in R} \frac{e^{-\varepsilon(p_i) + i}px}{R\varepsilon(p_i)},$$

$$C_{1,0}^R(\tau, x) = E_{1,0}^R(\tau, x) + F_{1,0}^R(\tau, x) = -\sigma^2 v \sum_{p_i \in R} \frac{e^{-\varepsilon(p_i) + \beta - \tau - i}px}{R\varepsilon(p_i)}. \quad (97)$$

Taking the Fourier transform we find

$$E_{0,1}^R(\omega, q) = \frac{\sigma^2 v}{\varepsilon(q)} \frac{1}{\omega + i0 - \varepsilon(q)}, \quad F_{0,1}^R(\omega, q) = \frac{\sigma^2 v}{\varepsilon(q)} \frac{-e^{-\beta\varepsilon(q)}}{\omega + i0 - \varepsilon(q)},$$

$$E_{1,0}^R(\omega, q) = \frac{\sigma^2 v}{\varepsilon(q)} \frac{e^{-\beta\varepsilon(q)}}{\omega + i0 + \varepsilon(q)}, \quad F_{1,0}^R(\omega, q) = \frac{\sigma^2 v}{\varepsilon(q)} \frac{-1}{\omega + i0 + \varepsilon(q)}. \quad (98)$$

In comparing this to (49) and using $F_{r,s}^R(\omega, q) = [E_{r,s}^R(-\omega, -q)]^*$ we see that the finite-and infinite-volume regularizations for $C_0^\sigma$ lead to the same result.

The above also establishes that the first two terms of $C_1^\sigma$, $E_{1,0}^\sigma + F_{0,1}^\sigma$, in the two schemes are the same. It leaves then to verify that

$$\lim_{R \to \infty} (E_{1,2}^R - Z_1^R F_{0,1}^R) = E_{1,2}^\sigma - Z_1 E_{0,1}^\sigma,$$

$$\lim_{R \to \infty} (F_{1,2}^R - Z_1^R E_{0,1}^R) = F_{1,2}^\sigma - Z_1 F_{0,1}^\sigma. \quad (99)$$

We first consider $(E_{1,2}^R - Z_1^R E_{0,1}^R)$. In a finite volume this term is equal to

$$E_{1,2}^R - Z_1^R E_{0,1}^R = \frac{\sigma^2 v^3}{2R^2} \sum_{p_2 \in R} \sum_{p_3 \in NS} e^{-\beta\varepsilon(p_2)} \sum_{p_1 \in R} \frac{\delta_{q,p_2+p_3-p_1}}{\varepsilon(p_1)\varepsilon(p_2)\varepsilon(p_3)}$$

$$\times \left\{ \frac{e^{-\beta\varepsilon(p_1)} \tanh^2(\theta_{23}/2) \coth^2(\theta_{12}/2) \coth^2(\theta_{13}/2)}{\varepsilon(p_1)\varepsilon(p_2)\varepsilon(p_3)} \right\}$$

$$- \frac{\sigma^2 v}{2R^2} \sum_{p_3 \in R} \frac{e^{-\beta\varepsilon(p_1)}}{\varepsilon(p_3)} \frac{1}{\omega + i0 - \varepsilon(p_3)} \sum_{p_1 \in R} e^{-\beta\varepsilon(p_1)}. \quad (100)$$

We rewrite $Z_1^R$ using the identity

$$Z_1^R = \sum_{p_1 \in R} e^{-\beta\varepsilon(p_1)} = \sum_{p_1 \in R} e^{-\beta\varepsilon(p_1)} \frac{1}{\pi^2} \sum_n \frac{1}{(n + 1/2)^2}$$

$$= \sum_{p_1 \in R} \sum_{p_2 \in NS} \frac{4e^{-\beta\varepsilon(p_1)}}{R^2(p_1 - p_2)^2}. \quad (101)$$

Inserting this into equation (100) we obtain

$$E_{1,2}^R - Z_1^R E_{0,1}^R = \frac{\sigma^2 v^3}{2R^2} \sum_{p_2 \in R} \sum_{p_3 \in NS} \left\{ \frac{\delta_{q,p_2+p_3-p_1}}{\varepsilon(p_1)\varepsilon(p_2)\varepsilon(p_3)} \right\}$$

$$\times \left\{ \frac{e^{-\beta\varepsilon(p_1)} \tanh^2(\theta_{23}/2) \coth^2(\theta_{12}/2) \coth^2(\theta_{13}/2)}{\varepsilon(p_1)\varepsilon(p_2)\varepsilon(p_3)} \right\}$$

$$- \frac{\sigma^2 v}{2R^2} \sum_{p_3 \in R} \frac{4e^{-\beta\varepsilon(p_1)}}{R^2(p_1 - p_2)^2}.$$
To do so we employ the identities
\[ E \left( p_1 + p_2 \right) \left( p_3 + p_4 \right) = \frac{1}{\varepsilon(p_3 + \pi/R)} \left( \omega + i0 - \varepsilon(p_3 + \pi/R) \right) v^2(p_1 - p_2)^2 \]
and
\[ E \left( p_2 + p_4 \right) \left( p_3 + p_4 \right) = \frac{1}{\varepsilon(p_2 + \pi/R)} \left( \omega + i0 - \varepsilon(p_2 + \pi/R) \right) v^2(p_1 - p_3)^2 \]
respectively, we can rewrite
\[ E_{1,2} - Z_{1}^{R} E_{0,1}^{R} = \frac{1}{8\pi^2} \int \frac{d\theta_1}{\varepsilon(s(\theta_1) - s(\theta_2))^2} \frac{f(\theta_1, \theta_2)}{(s(\theta_1) - s(\theta_2))^2} \]
\[ \times \left[ \frac{\delta(vq - \Delta(s(\theta_2) + s(\theta_3) - s(\theta_1)))}{\omega + i0 - \varepsilon(\theta_2) - \varepsilon(\theta_3) + \varepsilon(\theta_1)} \right] \tanh^2\left( \frac{\theta_{23}}{2} \right) \coth^2\left( \frac{\theta_{12}}{2} \right) \coth^2\left( \frac{\theta_{13}}{2} \right) \]
\[ - \frac{1}{\varepsilon(p_2 + \pi/R)} \left( \omega + i0 - \varepsilon(p_2 + \pi/R) \right) v^2(p_1 - p_3)^2 \]}
(102)

We now see that the above expression has no singular pieces of the form \((p_1 - p_2)^{-2}\) or \((p_1 - p_3)^{-2}\) as \(p_1\) approaches \(p_2\) or \(p_3\). There are singular terms of the form \((p_1 - p_2)^{-1}\) and \((p_1 - p_3)^{-1}\), but these can be seen to vanish upon summation and so can be ignored. Given the absence of singular terms we can take the limit \(R \to \infty\) and convert the summations to principal value integrals, i.e.
\[ E_{1,2} - Z_{1}^{R} E_{0,1}^{R} = \frac{1}{8\pi^2} \int \frac{d\theta_1}{\varepsilon(s(\theta_1) - s(\theta_2))^2} \frac{f(\theta_1, \theta_2)}{(s(\theta_1) - s(\theta_2))^2} \]
\[ \times \left[ \frac{\delta(vq - \Delta(s(\theta_2) + s(\theta_3) - s(\theta_1)))}{\omega + i0 - \varepsilon(\theta_2) - \varepsilon(\theta_3) + \varepsilon(\theta_1)} \right] \tanh^2\left( \frac{\theta_{23}}{2} \right) \coth^2\left( \frac{\theta_{12}}{2} \right) \coth^2\left( \frac{\theta_{13}}{2} \right) \]
\[ - \frac{1}{\varepsilon(p_2 + \pi/R)} \left( \omega + i0 - \varepsilon(p_2 + \pi/R) \right) v^2(p_1 - p_3)^2 \]}
(103)

The final step before being able to compare to the results stemming from our infinite-volume regularization scheme is to convert the principal value integrals into integrals along contours deformed by \(\pm i\eta\)'s about the singularities found at \(\theta_1 = \theta_2\) and \(\theta_1 = \theta_3\). To do so we employ the identities
\[ \int_{R_{\eta}(\theta_2)} d\theta_1 f(\theta_1, \theta_2) = \int_{C_{\eta}(\theta_2)} d\theta_1 f(\theta_1, \theta_2) \]
\[ + \int d\theta_1 \left[ \frac{\delta'(\theta_{21})}{c(\theta_2)c(\theta_1)} + \frac{2\delta(\theta_{12})}{\eta c^2(\theta_2)} \right] f(\theta_1, \theta_2) + \mathcal{O}(\eta), \]}
(104)

and
\[ \int_{R_{\eta}(\theta_1)} d\theta_2 f(\theta_1, \theta_2) \tanh^2(\theta_{12}/2) = \int_{C_{\eta}(\theta_1)} d\theta_2 f(\theta_1, \theta_2) \tanh^2(\theta_{12}/2) \]
\[ + \int d\theta_2 f(\theta_1, \theta_2) \left[ 4\pi i \delta'(\theta_{21}) + \frac{8}{\eta} \delta(\theta_{12}) \right] + \mathcal{O}(\eta), \]}
(105)

where \(f(\theta_1, \theta_2)\) is a test function and the integration contours are defined in figure 3. Deforming the contours into the lower half-plane instead changes the signs of the \(\delta'\) terms. Using (104) and (105) to deform the \(\theta_2\) and \(\theta_3\) integrals into the upper and lower half-plane respectively, we can rewrite \(E_{1,2}^{R} - Z_{1}^{R} E_{0,1}^{R}\) in the following way:
\[ E_{1,2}^{R} - Z_{1}^{R} E_{0,1}^{R} = \frac{1}{8\pi^2} \int \frac{d\theta_1 d\theta_2 d\theta_3 e^{-\beta\varepsilon(\theta_1)}}{\varepsilon(p_2 + \pi/R)} \left( \omega + i0 - \varepsilon(p_2 + \pi/R) \right) v^2(p_1 - p_3)^2 \]
\[ \times \tanh^2\left( \frac{\theta_{23}}{2} \right) \left[ \coth^2\left( \frac{\theta_{13} - i\eta}{2} \right) \coth^2\left( \frac{\theta_{12} + i\eta}{2} \right) + 16\pi^2 \delta'(\theta_{12}) \delta'(\theta_{13}) \right] \]
\[ + 4\pi i \coth^2\left( \frac{\theta_{12} + i\eta}{2} \right) \delta'(\theta_{13}) - 4\pi i \coth^2\left( \frac{\theta_{13} - i\eta}{2} \right) \delta'(\theta_{12}) \]}

\[ \text{doi:10.1088/1742-5468/2009/09/P09018} \]
Figure 3. Integration contours $R_\eta(\theta)$ and $C_\eta(\theta)$.

\[
E_{1,2}^R - Z_{1}^R F_{0,1}^R = \frac{v\tilde{\sigma}^2}{8\pi^2} \int d\theta_1 d\theta_2 d\theta_3 e^{-\beta \epsilon(\theta_1)} \frac{\delta(vq - \Delta s(\theta_2))}{\omega + i0} \frac{\epsilon(\theta_1)\epsilon(\theta_3)}{\Delta^2(s(\theta_1 - i\eta) - s(\theta_3))^2} \\
- \frac{v\tilde{\sigma}^2}{8\pi^2} \int d\theta_1 d\theta_2 d\theta_3 e^{-\beta \epsilon(\theta_1)} \frac{\delta(vq - \Delta s(\theta_3))}{\omega + i0} \frac{\epsilon(\theta_1)\epsilon(\theta_2)}{\Delta^2(s(\theta_1 + i\eta) - s(\theta_2))^2}.
\]

We note that all terms singular in $\eta$ that come about from deforming the contour of integration of $\theta_1$ about the singularities at $\theta_2$ and $\theta_3$ vanish (as they must as the principal value integral is well defined and finite). The last two integrals in (106) vanish as may be seen by deforming $\theta_3 \rightarrow \theta_3 + i\pi/2$ and $\theta_2 \rightarrow \theta_2 - i\pi/2$, respectively. After carrying out the integrals over the derivatives of delta functions we arrive at

\[
E_{1,2}^R - Z_{1}^R F_{0,1}^R = \frac{v\tilde{\sigma}^2}{8\pi^2} \int d\theta_1 d\theta_2 d\theta_3 e^{-\beta \epsilon(\theta_1)} \frac{\delta(vq - \Delta (s(\theta_2) + s(\theta_3) - s(\theta_1)))}{\omega + i0 - \epsilon(\theta_2) - \epsilon(\theta_3) + \epsilon(\theta_1)} \\
\times \left[ \tanh^2 \left( \frac{\theta_{13}}{2} \right) \coth^2 \left( \frac{\theta_{13} - i\eta}{2} \right) \coth^2 \left( \frac{\theta_{13} + i\eta}{2} \right) + 8\pi^2 \delta(\theta_{12})\delta(\theta_{13}) \right].
\]

We are now in a position to show that $E_{1,2}^R - Z_{1}^R F_{0,1}^R$ equals its value in our infinite volume regularization scheme. Indeed, combining (63), (62) and (81) and then keeping only the part of order $O(e^{-\beta \Delta})$ we obtain (107).

By an analogous consideration we can show that $\lim_{R \rightarrow \infty} F_{2,1}^R - Z_{1}^R F_{1,0}^R$ recovers the result of the infinite-volume regularization scheme.

3.7. Resummation

Above we have argued that the finite-temperature Lehmann representation can be re-expressed as

\[
\chi_\sigma(\omega, q) = \sum_{s=0}^{\infty} C^\sigma_s(\omega, q),
\]

where the quantities $C^\sigma_s(\omega, q)$ are finite in the infinite-volume limit. In particular

\[
C^\sigma_0(\omega, q) = \frac{2v\tilde{\sigma}^2}{(\omega + i\delta)^2 - \epsilon^2(q)} + \cdots
\]

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is the zero-temperature dynamical structure factor. We emphasize that we only consider frequencies close to $\Delta$ and low temperatures, so that we can neglect $n$-particle contributions to $C_0^\sigma$ with $n \geq 3$ as their contributions are vanishingly small. For $\omega - \varepsilon(q) \sim \mathcal{O}(1)$ $C_\sigma^\tau$ is of order $\mathcal{O}(e^{-\beta \Delta})$ and hence (108) provides a good low-temperature expansion of the dynamical susceptibility far away from the mass shell. On the other hand, when we approach the mass shell we have

$$C_\sigma^\sigma(\omega, q) \propto (\omega^2 - \varepsilon^2(q))^{s+1}.$$  \hspace{1cm} (110)

In order to obtain an expression for the susceptibility close to the mass shell we therefore have to carry out a resummation of terms. This is achieved by introducing a quantity $\Sigma(\omega, q)$ by (27) and using our results for $C_1^\sigma$ and $C_2^\sigma$ to carry out a low-temperature expansion of $\Sigma(\omega, q)$ using (30).

4. Results for the low-temperature dynamical susceptibility of the quantum Ising model

In order to present results obtained from the resummation (27) it is useful to define quantities

$$\chi^{(m)}_\sigma(\omega, q) = \frac{C_0^\sigma(\omega, q)}{1 - C_0^\sigma(\omega, q) \sum_{j=1}^{m} \Sigma^{(j)}(\omega, q)},$$  \hspace{1cm} (111)

where $\Sigma^{(j)}(\omega, q)$ are given by (30). Loosely speaking $\chi^{(m)}_\sigma(\omega, q)$ is the dynamical susceptibility obtained by calculating the $\Sigma(\omega, q)$ up to order $\mathcal{O}(e^{-\beta(n+1)\Delta})$. From the point of view of applications to experiment the relevant quantity is the dynamical structure factor, which is related to the retarded susceptibility by

$$S(\omega, q) = \frac{1}{\pi} \frac{1}{1 - e^{-\omega/T}} \text{Im} \chi_\sigma(\omega, q).$$  \hspace{1cm} (112)

In analogy with the definition of the $m$th order low-temperature approximation $\chi^{(m)}$ for the dynamical susceptibility we define

$$S^{(m)}(\omega, q) = \frac{1}{\pi} \frac{1}{1 - e^{-\omega/T}} \text{Im} \chi^{(m)}_\sigma(\omega, q).$$  \hspace{1cm} (113)

The leading order result $S^{(1)}(\omega, q)$ is most easily calculated using the infinite-volume regularization scheme and carrying out the integrals in (82) numerically. In order to determine $S^{(2)}(\omega, q)$ we employ the finite-volume regularization scheme instead. We calculate $S^{(2)}(\omega + i\eta, q)$ for system sizes up to $R = 800$. The non-zero imaginary part is introduced in order to suppress finite-size effects. If $\eta$ is sufficiently large the numerically accessible values of $R$ will coincide (within numerical accuracy) with the thermodynamic limit results. In order to obtain results for real frequencies we compute $S^{(2)}(\omega + i\eta, q)$ for a sequence of different $\eta$’s and then extrapolate to $\eta \to 0$. We have tested this method...
for $S^{(1)}(\omega, q)$, which we can calculate directly in the thermodynamic limit, and found it to work well.

At very low temperatures it is sufficient to calculate the leading approximation $S^{(1)}(\omega, q)$ as the corrections are exponentially small in $e^{-\Delta/T}$. We find that the difference between $S^{(1)}(\omega, q)$ and $S^{(2)}(\omega, q)$ is negligible for $T < 0.3\Delta$ and we discuss this regime first. Evaluation of $S^{(1)}(\omega, q)$ for low temperatures shows that, as expected, the $T = 0$ delta function at $\omega = \varepsilon(q)$ broadens with temperature. We find that the resulting peak scales as

\begin{align}
\text{peak height} & \propto \frac{\Delta}{T} \exp\left(\frac{\Delta}{T}\right), \\
\text{peak width} & \propto \frac{T}{\Delta} \exp\left(-\frac{\Delta}{T}\right).
\end{align}  \quad (114)

In order to exhibit the evolution of the structure factor as a function of frequency for fixed momentum $q$ with temperature it is therefore useful to rescale both the frequency axis and the structure factor. The result is shown in figure 4 for a temperature range $0.15\Delta \leq T < 0.3\Delta$, which corresponds to approximately a factor of 50 difference in peak height (and width). We see that the lineshape is asymmetric in frequency, with more spectral weight appearing at higher frequencies. The asymmetry increases with temperature. This effect is most easily quantified by comparison with a Lorentzian lineshape, which is done below.

If we increase the temperature beyond $T \approx 0.3\Delta$ the correction $\Sigma^{(2)}(\omega, q)$ (30) in the expression for the dynamical susceptibility (27), (30) is no longer negligible compared to the leading term $\Sigma^{(1)}(\omega, q)$. In figure 5 we show the leading $S^{(1)}(\omega, q)$ and improved $S^{(2)}(\omega, q)$ low-temperature approximation to the dynamical structure factor for $T = 0.45\Delta$ and $T = 0.5\Delta$. As expected the difference between the two is small sufficiently far away from the mass shell. The most important effect is the shift of the maximum to higher frequencies. This effect is known as the ‘temperature-dependent gap’.

Figure 4. Leading low-$T$ approximation $S^{(1)}(\omega, q = 0)$ of the dynamical structure factor for several temperatures. Both axes have been rescaled as discussed in the text.
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Figure 5. Leading $S^{(1)}(\omega, q = 0)$ and improved $S^{(2)}(\omega, q = 0)$ low-$T$ approximation of the dynamical structure factor for temperatures $T = 0.45 \Delta$ and $T = 0.5 \Delta$ respectively. The most important effect of the next to leading order approximation is the shift of the maximum upwards in frequency (‘temperature-dependent gap’).

4.1. Comparison to semiclassical results of Sachdev and Young

In [28, 38] Sachdev and Young have developed a semiclassical approach to determine the dynamical structure factor. Their result$^3$, valid for $T \ll \Delta$, is

$$S_{sc}(\omega, q) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} dx e^{i \omega t - i q x} K(x, t) R(x, t),$$

(115)

where

$$K(x, t) = \frac{\bar{\sigma}^2}{\pi} K_0(|\Delta| \sqrt{(x/c)^2 - t^2}),$$

$$R(x, t) = \exp \left( - \int_{-\infty}^{\infty} \frac{dk}{\pi} e^{-\varepsilon(k)/T} |x - v(k)t| \right),$$

(116)

$$v(k) \equiv \frac{d\varepsilon(k)}{dk} = \frac{v^2 k}{\varepsilon(k)}.$$

Here $\bar{\sigma}$ is defined in (43). At sufficiently low temperatures and when $|vq| \ll \sqrt{T \Delta}$ the semiclassical result is well approximated by a Lorentzian [38]

$$S_{Lor}(\omega, q) = \frac{\bar{\sigma}^2 \nu}{\pi \varepsilon(q)} \frac{1/\tau\phi}{(\omega - \varepsilon(q))^2 + 1/\tau\phi^2},$$

(117)

where

$$\tau\phi = \frac{\pi}{2T} e^{\Delta/T}.$$

(118)

We expect our low-temperature expansion to reproduce the semiclassical results at sufficiently low temperatures $T \ll \Delta$. The comparison of the improved result $S^{(2)}(\omega, q)$ to the leading order $S^{(1)}(\omega, q)$ suggests that the latter is a good approximation up to

$^3$ Our definition of the structure factor differs from [38] by a factor of $2\pi$.

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Figure 6. Dynamical structure factor for $T = 0.3\Delta$, $q = 0$. The result of the present work (red line) is still in good agreement with the semiclassical result of Sachdev and Young as well as the Lorentzian approximation to the latter. The asymmetry of the lineshape is more pronounced than the one predicted by the semiclassical result.

Figure 7. (a) Ratios of the dynamical structure factors calculated from equation (27) (this work) and (115) (Sachdev and Young) to the Lorentzian approximation (117) for $T = 0.2\Delta$ and $q = 0$. The non-monotonic behavior very close to the mass shell indicates a small shift of the maximum of $S^{(1)}(\omega, q = 0)$ away from the $T = 0$ gap. (b) Ratio of $S^{(1)}(\omega, q = 0)$ to the Lorentzian approximation (117) for $T = 0.1\Delta$ and $q = 0$. We see that the low-temperature approximation recovers the semiclassical result at sufficiently low temperatures.

temperatures of $T \approx 0.3\Delta$. In figure 6 we show a comparison of our low-temperature approximation to the semiclassical result (115) and the Lorentzian approximation (117) for $T = 0.3\Delta$. We see that, while the gross structure of the lineshapes is quite similar, there are considerable differences away from the mass shell. This implies that the semiclassical approximation is no longer quantitatively accurate for $T = 0.3\Delta$.

In order to exhibit the differences between the various approximations more clearly, we plot the ratios $S^{(1)}(\omega, q)/S_{\text{Lor}}(\omega, q)$ and $S_{\text{sc}}(\omega, q)/S_{\text{Lor}}(\omega, q)$ in figure 7(a) for $T = 0.2\Delta$. We see that the semiclassical approximation underestimates the asymmetry of the lineshape.
On the other hand, for sufficiently small temperatures the low-temperature expansion indeed recovers the semiclassical result as can be seen from figure 7(b).

The ratio $S^{(1)}(\omega, q)/S_{\text{cl}}(\omega, q)$ displays non-monotonic behavior close to the mass shell. This suggests that the maximum of $S^{(1)}(\omega, q)$ is, in fact, very slightly shifted away from the $T = 0$ gap. This is indeed the case.

### 4.2. Comparison to exact finite-temperature form factors

In [24, 27] the following form factor expansion for the two-point function of the spin field in the quantum Ising chain was derived:

$$
\chi_{\sigma}(\tau, x) = \sum_{r,s} D^\sigma_{r,s}(\tau, x),
$$

(119)

$$
D^\sigma_{r,s}(\tau, x) = -\frac{C^2(\beta)}{r!s!} e^{-\eta(\sigma)\nu|x|} \int \prod_{j=1}^r \left[ \frac{d\theta_j}{2\pi} f(\theta_j) \exp \left( \tau \Delta c(\theta_j) - i \frac{\Delta}{\nu} s(\theta_j) x - \eta_+(\theta_j) \right) \right]
$$

$$
\times \int \prod_{l=1}^s \frac{d\eta_l}{2\pi} f(\eta_l) \exp \left( (\beta - \tau) \Delta c(\eta_l) + i \frac{\Delta}{\nu} s(\eta_l) x - \eta_-(\eta_l) \right)
$$

$$
\times \prod_{n>m=1}^r \tanh^2 \left( \frac{\theta_n}{2} \right) \prod_{p>q=1}^s \tanh^2 \left( \frac{\eta_p}{2} \right) \prod_p \cot^2 \left( \frac{\theta_n - \eta_p + i0}{2} \right),
$$

(120)

where $c(\theta) = \cosh \theta$, $s(\theta) = \sinh \theta$, $\theta_{jk} = \theta_j - \theta_k$ and

$$
C(\beta) = \tilde{C} \exp \left[ \frac{(\Delta)^2}{2} \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} s(\theta_1) s(\theta_2) \ln \left| \coth \frac{\theta_{12}}{2} \right| \right],
$$

$$
n(\beta) = \frac{2\Delta}{\pi \nu} \sum_{k=1}^{\infty} \frac{1}{2k-1} K_1((2k-1)\beta \Delta) \approx e^{-\beta \Delta} \sqrt{\frac{2\Delta}{\pi \beta \nu^2}}.
$$

(121)

$$
\eta_{\pm}(\theta) = \pm 2 \int_{-\infty+i0}^{\infty+i0} \frac{d\theta'}{2\pi i \sinh(\theta - \theta')} \ln \left[ \frac{1 + e^{-\beta \Delta c(\theta')}}{1 - e^{-\beta \Delta c(\theta')}} \right].
$$

At low temperatures we have

$$
C(\beta) \approx \sigma,
$$

$$
e^{\eta_{\pm}(\theta)} \approx \left[ 1 - 2e^{-\beta \Delta c(\theta)} \right] \exp \left[ \pm \frac{2i}{\pi} \int_0^\infty \frac{dx}{\sinh x} \left( e^{-\beta \Delta c(\theta-x)} - e^{-\beta \Delta c(\theta+x)} \right) \right],
$$

(122)

$$
n(\beta) \approx e^{-\beta \Delta} \sqrt{\frac{2\Delta}{\pi \beta \nu^2}}.
$$

An important question is in which way this expansion is related to the low-temperature expansion based on the zero-temperature form factors. Fourier transforming and analytically continuing to real frequencies we find

$$
D^\sigma_{0,1}(\omega, q) = C^2(\beta) \int \frac{d\theta}{2\pi} \frac{e^{-\eta_(\theta)}}{\omega + i0 - \Delta c(\theta)} \frac{2n(\beta)}{n^2(\beta) + (q - (\Delta/\nu) s(\theta))^2},
$$

$$
D^\sigma_{1,0}(\omega, q) = -C^2(\beta) \int \frac{d\theta}{2\pi} \frac{e^{-\eta_(\theta)}}{\omega + i0 + \Delta c(\theta)} \frac{2n(\beta)}{n^2(\beta) + (q + (\Delta/\nu) s(\theta))^2}.
$$

(123)

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The main difference between $D_{r,s}$ and the corresponding zero-temperature quantities is the replacement of the momentum conservation delta function by a Lorentzian of width $n(\beta)$. We note that the imaginary part of $D_{0,1}$ has a square root divergence for $\omega \to \Delta$ and it is necessary to sum an infinite number of terms in (119) to get a meaningful answer [24]. In order to recover our low-temperature expansion from the Lehmann representation in terms of exact finite-temperature form factors we should expand the Lorentzian expressing approximate momentum conservation under the integral, e.g.

$$
\frac{2n(\beta)}{n^2(\beta) + q^2(\theta)} = 2\pi \delta(q(\theta)) + 4n(\beta) \frac{q^2(\theta) - \epsilon^2}{(q^2(\theta) + \epsilon^2)^2} - 4\pi n^2(\beta) \delta''(q(\theta)) + \mathcal{O}(n^3(\beta)),
$$

(124)

where $q(\theta) = q - (\Delta/v)s(\theta)$.

5. O(3) nonlinear $\sigma$-model

We now apply the methods outlined above to the O(3) nonlinear sigma model. Unlike the quantum Ising model the sigma model describes a strongly interacting theory featuring dynamical mass generation. The Lagrangian of the sigma model is given by

$$
\mathcal{L} = \frac{1}{2g} \int dx \left[ \frac{1}{v} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - v \partial_x \mathbf{n} \cdot \partial_x \mathbf{n} \right].
$$

(125)

The O(3) nonlinear sigma model describes the scaling limit of integer spin-$S$ Heisenberg models [51]

$$
H = J \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}, \quad \mathbf{S}_j^2 = S(S+1).
$$

(126)

The velocity and coupling constant of the sigma model are related to the lattice model parameters by

$$
g = \frac{2}{S}, \quad v = 2JSa_0,
$$

(127)

where $a_0$ is the lattice spacing [52]. The lattice spin operators, $\mathbf{S}_j$, are related to the continuum fields by

$$
\mathbf{S}_j \approx S(-1)^j \mathbf{n}(x) + \mathbf{l}(x), \quad x = ja_0,
$$

(128)

where $\mathbf{l}(x) \propto (a_0/vg) \mathbf{n}(x) \times \partial \mathbf{n}(x)/\partial t$. The dynamical susceptibilities of the lattice model are given by

$$
\chi_{ab}^{\text{lat}}(\omega, Q) = - \int_0^\beta d\tau \sum_l e^{i\omega_n \tau - iQla_0} \langle T_{\tau} S^a_l(\tau) S^b_{l+1} \rangle \bigg|_{\omega_n - \delta - i\omega}.
$$

(129)

Substituting (128) into (129) we see that in the vicinity of the antiferromagnetic wavenumber ($Q = \pi/a_0 + q$ with $|q| \ll \pi/a_0$) the lattice susceptibility at low energies $\omega \ll v/a_0$ can be expressed in terms of the two-point function of the $\mathbf{n}$ field:

$$
\chi_{zz}^{\text{lat}}(\omega, Q) \propto - \int_0^\beta d\tau \int dx \left[ e^{i\omega_n \tau - iqx} \langle T_{\tau} n^z(\tau, x) n^z(0) \rangle \right] \bigg|_{\omega_n - \delta - i\omega}.
$$

(130)
The field needs to be renormalized and in an appropriate scheme is related to the spin field $Φ$ by

$$n^a = ζΦ^a.$$  \hspace{1cm} (131)

In what follows we analyze the two-point function of the sigma model spin field:

$$χ_φ(ω, q) = -\int_0^\beta dτ d x e^{iω nτ−iqx}⟨T_τ Φ^a(τ, x)Φ^a(0, 0)⟩\bigg|_{ω_n−δ−iω},$$  \hspace{1cm} (132)

keeping in mind that at low energies we have

$$χ_\text{lat}^zz(ω, πa_0+q)∝χ_φ(ω, q).$$  \hspace{1cm} (133)

The O(3) nonlinear sigma model is integrable [53]–[55] and the exact spectrum and scattering matrix have been known for a long time. The elementary excitations of the sigma model are a triplet of massive particles with spins $S^z = ±1, 0$. It is useful to parameterize their energy and momentum in terms of a rapidity variable $θ$:

$$ε(θ) = Δ\cosh θ, \quad p(θ) = \frac{Δ}{v}\sinh θ.$$  \hspace{1cm} (134)

It is convenient to choose a basis such that the spin operators $S^a$ act on single-particle states as

$$S^a|θ⟩_b = iε_{abc}|θ⟩_c, \quad a = 1, 2, 3.$$  \hspace{1cm} (135)

A convenient basis for the Hilbert space is formed by scattering states of these elementary excitations. In order to describe it one introduces creation and annihilation operators $Z^a(θ)$ and $Z^a_\dagger(θ)$, respectively, fulfilling the Faddeev–Zamolodchikov algebra (4).

Here $a, b$ are O(3) quantum numbers and $S$ is the exact two-particle scattering matrix [54]:

$$S_{cd}^{ab}(θ) = \sigma_1(θ)δ_{ab}δ_{cd} + \sigma_2(θ)δ_{ac}δ_{bd} + \sigma_3(θ)δ_{ad}δ_{bc},$$  \hspace{1cm} (136)

$$σ_1(θ) = \frac{2πiθ}{(θ+iπ)(θ−2πi)}, \quad σ_2(θ) = \frac{θ(θ−iπ)}{(θ+iπ)(θ−2πi)} - \frac{2πi(θ−iπ)}{(θ+iπ)(θ−2πi)}.$$  \hspace{1cm} (137)

We note that the $S$ matrix is a solution to the Yang–Baxter equation and fulfills

$$[S_{cd}^{ab}(θ)]^* = S_{ab}^{cd}(−θ).$$  \hspace{1cm} (137)

Using the Faddeev–Zamolodchikov operators, a Fock space of states can be constructed as in (5) and (6). Energy and momentum are by construction additive and given by (7). In the basis of scattering states introduced above, formally the following spectral representation for the finite-temperature dynamical susceptibility holds:

$$χ_φ(ω, q) = \sum_{r,s=0}^∞ C^Φ_{r,s}(ω, q).$$  \hspace{1cm} (138)

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Here $C_{r,s}$ are given by (10)

$$C_{r,s}^\Phi (\omega, q) = \int_0^\beta d\tau \int dx \ e^{i\omega_n \tau - iqx} C_{r,s}^\Phi (\tau, x) \bigg|_{\omega_n = -i\omega}.$$  

$$C_{r,s}^\Phi (\tau, x) = -\int_0^{\beta} d\theta_1 \cdots d\theta_r \int_0^{\beta} d\theta'_1 \cdots d\theta'_s e^{-\beta E_r} e^{-\tau(E_r-E_s)-i(P_r-P_s)x}$$  

$$\times \sum_{b_1, \ldots, b_r} \sum_{b'_1, \ldots, b'_s} |b_1 \cdots b_r \theta_r \Phi^a(0, 0) \theta'_1 \cdots \theta'_s \Phi^b(0, 0)|^2.$$  

The partition function can formally be expressed as in (11). The low-temperature expansion is constructed by following the steps set out in section 2, e.g. the functions $E_{r,s}$ used in the decomposition (13) of $C_{r,s}^\Phi$ are

$$E_{r,s}^\Phi (\omega, q) = \sum_{b_1, \ldots, b_r} \sum_{b'_1, \ldots, b'_s} \int \frac{d\theta_1 \cdots d\theta_r}{(2\pi)^r r!} \int \frac{d\theta'_1 \cdots d\theta'_s}{(2\pi)^s s!} 2\pi \delta(q + P_r - P_s)$$  

$$\times e^{-\beta E_r} \frac{\omega + i\delta - E_r + E_s}{E_r + E_s} \langle b_1 \cdots b_r \theta_r | \Phi^a(0, 0) | b'_1 \cdots b'_s \theta'_s \Phi^b(0, 0) \rangle^2,$$

where $E_r$ and $P_r$ are defined in (7). Our aim is to determine the functions $E_{r,s}^\Phi$, $F_{r,s}^\Phi$ and $C_{r,s}^\Phi$ defined by (17), (18) and (22), respectively, and to use them to obtain a low-temperature expansion (23) of the susceptibility.

**5.1. Zero-temperature dynamical response**

The low energy dynamical response in the O(3) nonlinear sigma model has been worked out in [56, 57]. The zero-temperature form factors in the infinite volume have been determined by several groups [6]–[8]. The one- and three-particle form factors of the spin field are [7]

$$\langle 0 | \Phi^a(0) | \theta \rangle_b = \delta_{ab},$$

$$\langle 0 | \Phi^a(0) | \theta_3, \theta_2, \theta_1 \rangle_{a_3a_2a_1} = \frac{\pi^3}{2} \psi(\theta_{12}) \psi(\theta_{31}) \psi(\theta_{21})$$  

$$\times \delta_{a_1a_3} \delta_{a_2a_3} (\theta_{31} - 2\pi i) + \delta_{a_3a_1} \delta_{a_1a_2} \theta_{12},$$

where

$$\psi(\theta) = \frac{\theta - i\pi}{\theta(2\pi i - \theta)} \tanh^2 \left( \frac{\theta}{2} \right).$$  

Note that these differ slightly from [7] because we use a different normalization condition for the scattering states

$$a \langle \theta | \theta' \rangle_b = 2\pi \delta_{ab} \delta(\theta - \theta').$$
Some useful identities involving the function $\psi(\theta)$ are

$$
\psi(\theta + i\pi) = -\psi(i\pi - \theta) = -\frac{\theta \coth^2(\theta/2)}{\theta^2 + \pi^2},
\psi(\theta)\psi(\theta + i\pi) = \frac{1}{(\theta + i\pi)(\theta - 2\pi i)},
\psi(\theta + i\pi^+) = \psi(\theta + i\pi^-) + \frac{8i}{\pi} \delta(\theta).
$$

Here $\pi^\pm = \pi \pm 0$.

At $T = 0$ the leading contributions to the dynamical susceptibility at low frequencies are

$$
\chi_\Phi(\omega, q)|_{T=0} \approx [E_{0,1}^\Phi(\omega, q) + F_{1,0}^\Phi(\omega, q) + E_{0,3}^\Phi(\omega, q) + F_{3,0}^\Phi(\omega, q)].
$$

Here the one-particle contributions are

$$
E_{0,1}^\Phi(\omega, q) = |F_{1,0}^\Phi(-\omega, -q)|^* = \frac{v}{\varepsilon(q)} \frac{1}{\omega - \varepsilon(q) + i0}.
$$

The three-particle terms can be cast in the form [57]

$$
E_{0,3}^\Phi(\omega, q) + F_{3,0}^\Phi(\omega, q)
= \frac{2v\pi^6}{3} \int_{-\infty}^{\infty} \frac{dy}{(2\pi)^2} \frac{dz}{s^2 - \Delta^2 - 4\Delta^2 \cosh(z)[\cosh(z) + \cosh(y)]},
$$

where

$$
f(z) = \frac{z^2 + \pi^2}{z^2(\pi^2 + 4\pi^2)}[\tanh(z/2)]^4,
$$

$$
s^2 = (\omega + i0)^2 - v^2 q^2.
$$

We plot the real and imaginary parts of $E_{0,3}^\Phi(\omega, q = 0) + F_{0,3}^\Phi(\omega, q = 0)$ in figure 8. We see that, by virtue of the smallness of $E_{0,3}^\Phi + F_{3,0}^\Phi$, the dynamical response at low energies is dominated by the coherent single-particle contributions $E_{0,1}^\Phi + F_{1,0}^\Phi$. This yields the following result for the temperature-independent part of the spectral representation for the dynamical susceptibility:

$$
C_{0}^\Phi(\omega, q) \approx E_{0,1}^\Phi(\omega, q) + F_{1,0}^\Phi(\omega, q) = \frac{2v}{(\omega + i0)^2 - \varepsilon^2(q)},
$$

where $\varepsilon(q) = \sqrt{\Delta^2 + v^2 q^2}$.
5.2. Infinite-volume regularization

At low temperatures the next most important contributions arise from $E_{1,2}^\Phi$ and $F_{2,1}^\Phi$, which are formally given by

$$E_{1,2}^\Phi(\omega, q) = v \sum_{b_1, b_2} \int \frac{d\theta_1 d\theta_2}{2(2\pi)^2} \frac{e^{-\beta \Delta c(\theta)}}{\omega + i0 - \Delta [c(\theta_1) + c(\theta_2) - c(\theta)]}$$

$$\times |b_1 b_2 \langle \theta_1, \theta_2 | \Phi^a(0) | \theta_3 \rangle_b|^2 (\omega - \Delta [s(\theta_1) + s(\theta_2) - s(\theta)])$$

$$F_{2,1}^\Phi(\omega, q) = -v \sum_{b_1, b_2} \int \frac{d\theta_1 d\theta_2}{2(2\pi)^2} \frac{e^{-\beta \Delta c(\theta)}}{\omega + i0 + \Delta [c(\theta_1) + c(\theta_2) - c(\theta)]}$$

$$\times |b_1 b_2 \langle \theta_1, \theta_2 | \Phi^a(0) | \theta_3 \rangle_b|^2 (\omega + \Delta [s(\theta_1) + s(\theta_2) - s(\theta)])$$

where $c(\theta) = \cosh \theta$ and $s(\theta) = \sinh \theta$. In order to proceed further we need to evaluate the absolute value squares of form factors. The individual form factors can be analytically continued following Smirnov [1], see appendix A for a summary. The various ways of analytically continuing the three-particle form factors are

$$|b_1 b_2 \langle \theta_1, \theta_2 | \Phi^a(0) | \theta_3 \rangle_b|^2 \equiv \lim_{\kappa \to 0} |b_1 \langle \theta_3 | \Phi^a(0) | \theta_2, \theta_1 \rangle_{b_1 b_2} b_1 b_2 \langle \theta_1, \theta_2 | \Phi^a(0) | \theta_3 + \kappa \rangle_b$$

$$= \lim_{\kappa \to 0} \left| \{b_1 b_2 \langle \theta_1 - i0, \theta_2 + i0 | \Phi^a(0) | \theta_3 \rangle_b^* + 2\pi \delta(\theta_3) S_{b_1 b_2}^{ab} \langle \theta_2 | \right|_{\theta_1}$$

Figure 8. Real and imaginary parts of $E_{03}^\Phi(\omega, q = 0)$. At low frequencies both are small compared to $E_{01}^\Phi(\omega, q = 0)$.
\[ + 2\pi \delta(\theta_{31}) S_{b_1 b_2}^{a_b}(\theta_{21}) \times [b_{1 b_2}(\theta_1 + i0, \theta_2 - i0)\Phi^a(0)|\theta_3 + \kappa)_b + 2\pi \delta(\theta_{32} + \kappa) \delta_{ab} \delta_{ab_1} + 2\pi \delta(\theta_{31} + \kappa) \delta_{ab_2} \delta_{bb_1}], \]  

where in the second line we have used (153). Using the results of appendix C we can separate (154) into a connected contribution \( \Gamma_{\text{conn}} \) given by (C.3) and (C.7) and disconnected contributions \( \Gamma_{\text{dis},1} \) and \( \Gamma_{\text{dis},2} \) given by (C.4) and (C.5), respectively. Concomitantly \( E_{1,2} \) can be split into a connected and a disconnected part:

\[ E_{1,2}^\Phi(\omega, q) = E_{1,2}^{\text{conn}}(\omega, q) + E_{1,2}^{\text{dis}}(\omega, q). \]

The connected part is

\[ E_{1,2}^{\text{conn}}(\omega, q) = v \int \frac{d\theta_1 d\theta_2 d\theta_3}{2(2\pi)^2} \frac{e^{-\beta \Delta c(\theta_3)}}{\omega + i0 - \Delta[c(\theta_1) + c(\theta_2) - c(\theta_3)]} \times \delta(\omega q - \Delta[s(\theta_1) + s(\theta_2) - s(\theta_3)])L(\theta_1, \theta_2, \theta_3), \]

where

\[ L(\theta_1, \theta_2, \theta_3) = \frac{\pi^6}{2} \left[ \theta_{12}^2 + \theta_{13}^2 + \theta_{23}^2 + 4\pi^2 \right] \frac{\theta_{12}^2 + \pi^2}{\theta_{12}^2 (\theta_{12}^2 + 4\pi^2)} \tanh^4 \left( \frac{\theta_{12}}{2} \right) \times \frac{(\theta_{13} + i0)^2}{(\theta_{13} + \pi^2)^2} \coth^4 \left( \frac{\theta_{13} + i0}{2} \right) \frac{(\theta_{23} - i0)^2}{(\theta_{23} + \pi^2)^2} \coth^4 \left( \frac{\theta_{23} - i0}{2} \right). \]

The disconnected term, \( E_{1,2}^{\text{dis}} \), is equal to

\[ E_{1,2}^{\text{dis}}(\omega, q) = v \int \frac{d\theta_1 d\theta_2 d\theta_3}{2(2\pi)^2} \frac{e^{-\beta \Delta c(\theta_3)}}{\omega + i0 - \Delta[c(\theta_1) + c(\theta_2) - c(\theta_3)]} \times \delta(\omega q - \Delta[s(\theta_1) + s(\theta_2) - s(\theta_3)]) \sum_{j=1}^{2} \Gamma_{\text{dis}}^{\text{1j}}(\theta_1, \theta_2, \theta_3), \]

where \( \Gamma_{\text{dis}}^{\text{1j}} \) are given by (C.9) and (C.11), respectively. Upon evaluation, this term simplifies to

\[ E_{1,2}^{\text{dis}}(\omega, q) = v \int \frac{d\theta_1 d\theta_2 d\theta_3}{2(2\pi)^2} \frac{\delta(\omega q - \Delta[s(\theta_1) + s(\theta_2) - s(\theta_3)])e^{-\beta \Delta c(\theta_3)}}{\omega + i0 - \Delta[c(\theta_1) + c(\theta_2) - c(\theta_3)]} \times \frac{4}{(\theta_{12}^2 + \pi^2)(\theta_{12}^2 + 4\pi^2)} \left[ \pi^4 \delta(\theta_{31}) \delta(\theta_{32}) + \pi^2 \theta_{21} \delta'(\theta_{32}) + \delta(\theta_{32}) \left( 4\pi^2 \theta_{21} \sinh \theta_{21} + \frac{\theta_{21}^2 (\theta_{21}^2 + 5\pi^2)}{\theta_{21}^2 + \pi^2} \right) \right] + E_{0,1}(\omega, q) \mathcal{Z}_1, \]

where \( \mathcal{Z}_1 \) is defined below in (165). Note that, unlike the Ising model, the finite disconnected terms here involve derivatives of \( \delta \) functions.

To evaluate the connected term, \( E_{1,2}^{\text{conn}} \), we now proceed in complete analogy with the Ising case. We change variables to \( \theta_\pm = (\theta_2 \pm \theta_1)/2 \), carry out the \( \theta_+ \) integral using the momentum conservation delta function and then shift the \( \theta_- \) integration contour down in
the complex plane. This results in

$$E_{1,2}^{\text{conn}}(\omega, q) = -iv \int_{S_{+}} \frac{d\theta}{2\pi} e^{-\beta \Delta c(\theta)} K_{\text{SM}}(\alpha(\omega, q, \theta), \theta_{0}(\omega, q, \theta), \theta)$$

$$- v \int_{T_{+}} \frac{d\theta}{2\pi} e^{-\beta \Delta c(\theta)} K_{\text{SM}}(\alpha(\omega, q, \theta), \theta_{0}(\omega, q, \theta), \theta)$$

$$+ v \int \frac{d\theta}{(2\pi)^{2}} \int_{S} \frac{d\theta}{2\pi} e^{-\beta \Delta c(\theta)} K_{\text{SM}}(\theta_{-}, \theta_{0}(q, \theta_{-}), \theta)$$

where $\Omega(\theta, \omega), \theta_{0}(q, \theta, \omega)$ and $u(q, \theta, \omega)$ are given by (67), $\alpha$ by (69), $\tilde{s}(\omega, \theta)$ by (70), $\alpha$ by (71) and

$$K_{\text{SM}}(\theta_{+}, \theta_{-}, \theta_{3}) = L(\theta_{+} - \theta_{-}, \theta_{+} + \theta_{-}, \theta_{3}).$$

The remaining integrals in (160) are easily evaluated numerically.

We can further simplify the disconnected terms by performing some of the integrals. Doing so we obtain

$$E_{1,2}^{\text{dis}}(\omega, q) = \frac{v}{\varepsilon(q) \omega + i0 - \varepsilon(q)} \int_{-\infty}^{\infty} \frac{d\theta}{\varepsilon(q) \omega + i0 - \varepsilon(q)} e^{-\beta \Delta c(\theta)}$$

$$+ \frac{v}{\varepsilon(q) \omega + i0 - \varepsilon(q)} \left[ e^{-\beta \varepsilon(q)} + \int_{-\infty}^{\infty} \frac{d\theta}{\varepsilon(q)} e^{-\beta \Delta c(\theta) + \varepsilon(q)} g_{1}(\theta) \right]$$

$$+ \frac{v}{\varepsilon(q) \omega + i0 - \varepsilon(q)} \int_{-\infty}^{\infty} \frac{d\theta}{\varepsilon(q)} e^{-\beta \Delta c(\theta) + \varepsilon(q)} g_{2}(\theta),$$

$$g_{1}(\theta) = \frac{4}{(\theta^{2} + \pi^{2})(\theta^{2} + 4\pi^{2})} \left[ \theta^{2} + \frac{4\pi^{2} \theta}{\sinh \theta} + \frac{4\pi^{2} \theta^{2}}{\theta^{2} + \pi^{2}} \right]$$

$$+ 4\pi^{2} \left[ \Delta \beta s(\theta + \theta_{q}) + \frac{\Delta \varepsilon(q)}{\varepsilon(q)} c(\theta + \theta_{q}) \right] f(\theta) + \frac{\Delta c(\theta + \theta_{q})}{\varepsilon(q)} f(\theta),$$

$$g_{2}(\theta) = 4\pi^{2} f(\theta) \left[ s(\theta + \theta_{q}) - \frac{\varepsilon(q)}{\varepsilon(q)} c(\theta + \theta_{q}) \right].$$

Here

$$f(\theta) = \frac{1}{(\theta^{2} + \pi^{2})(\theta^{2} + 4\pi^{2})},$$

$$\theta_{q} = \arcsinh \left( \frac{\varepsilon(q)}{\Delta} \right).$$

The next step in the low-temperature expansion is to subtract the contributions due to the partition function following (17) and (11). The relevant contribution to the partition function is

$$Z_{1} \equiv \lim_{\kappa \to 0} \sum_{b} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\beta \Delta c(\theta)} b_{b}(\theta) \theta + \kappa)_{b}$$

$$= \lim_{\kappa \to 0} 3 \delta(\kappa) \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\beta \Delta c(\theta)}.$$
Using that $\mathcal{E}_1^\Phi \simeq E_{1,0}^\Phi + E_{1,2}^\Phi - Z_1 E_{0,1}^\Phi$ we obtain

$$
\mathcal{E}_1^\Phi = \frac{v}{\varepsilon(q)} \omega + i 0 - \varepsilon(q) \left[ e^{-\beta \varepsilon(q)} + \int_{-\infty}^{\infty} d\theta e^{-\beta \Delta c(\theta + \theta_s)} g_1(\theta) \right] 
+ \frac{v}{\varepsilon(q)} \left( \omega + i 0 - \varepsilon(q) \right)^2 \int_{-\infty}^{\infty} d\theta e^{-\beta \Delta c(\theta + \theta_s)} g_2(\theta) 
+ \frac{v}{\varepsilon(q)} \omega + i 0 + \varepsilon(q) + E_{1,2}^{\text{conn}}(\omega, q).
$$

(166)

Starting from $C_{2,1}^\Phi(\omega, q)$ we can determine the contribution $\mathcal{F}_1^\Phi$ in an analogous way. This then leads to the following result for the leading finite-temperature contribution to the expansion (23) for frequencies $\omega \approx \Delta$:

$$
C_1^\Phi(\omega, q) \approx \frac{2v}{(\omega + i 0)^2 - \varepsilon^2(q)} \int_{-\infty}^{\infty} d\theta e^{-\beta \Delta c(\theta + \theta_s)} g_1(\theta) 
+ \frac{2v\Delta((\omega + i 0)^2 + \varepsilon^2(q))}{\varepsilon(q)(\omega + i 0)^2 - \varepsilon^2(q)^2} \int_{-\infty}^{\infty} d\theta e^{-\beta \Delta c(\theta + \theta_s)} g_2(\theta) 
- iv \int_{S_+} d\theta e^{-\beta c(\theta)} \frac{K_{SM}(\alpha(\omega, q, \theta), 0(\omega, q, \theta), \theta)}{\hat{s}(\omega, q, \theta) \sqrt{\hat{s}^2(\omega, q, \theta) - 4\Delta^2}} 
- v \int_{T_+} d\theta e^{-\beta c(\theta)} \frac{K_{SM}(\pi(\omega, q, \theta), 0(\omega, q, \theta), \theta)}{\hat{s}(\omega, q, \theta) \sqrt{4\Delta^2 - \hat{s}^2(\omega, q, \theta)}} 
+ v \int \frac{d\theta}{(2\pi)^2} \int_s \frac{e^{-\beta \Delta c(\theta)} K_{SM}(\theta, \theta_0(\omega, q, \theta, \theta))}{\Delta c(\theta)} u(q, \theta, \theta_0) u(q, \theta, \theta).
$$

(167)

Here $S_+$ and $T_+$ are the segments of the real axis characterized by $\hat{s}^2(\omega, q, \theta) > 4\Delta^2$ and $4\Delta^2 \cos^2 \gamma \leq \hat{s}^2(\omega, q, \theta) \leq 4\Delta^2$, respectively, and $S$ is the contour from $-\infty - i\gamma$ to $\infty - i\gamma$ parallel to the real axis.

### 5.3. Finite-volume regularization

A second way of regularizing infinities in matrix elements is to work in a large, finite volume $R$. As was pointed out in [25], up to corrections that are exponentially small in system size the functional form of matrix elements remains the same as in the thermodynamic limit. The main effect of the finite volume is to quantize to momenta, or equivalently the rapidities, that parameterize the basis states. The quantization conditions in the finite volume are

$$
e^{iR\hat{\varphi}_0} \sinh \theta |\theta_1 \cdots \theta_n\rangle_{b_1 \ldots b_n} = S_{b_n \ldots b_1}^{a_n \ldots a_1} |\theta_{n1}\rangle \cdots S_{b_{2\ell-1}}^{a_2 \ldots a_1}(\theta_{21}) |\theta_1 \cdots \theta_n\rangle_{a_1 \ldots a_n},
$$

(168)

where $\theta_{kl} = \theta_k - \theta_l$. In the one-particle sector we simply have

$$
e^{iR\hat{\varphi}_0} \sinh \theta = 1.
$$

(169)

This is readily solved:

$$
\theta_j = \arcsinh \left( \frac{2\pi v j}{R\Delta} \right), \quad j \in \mathbb{Z}.
$$

(170)
The density of such one-particle states is
\[ \rho_1(\theta) = \frac{\Delta R}{v} \cosh \theta. \] (171)

In the two-particle sector the finite-volume quantization conditions are obtained from the $S$-matrix eigenvalues $S_a(\theta)$ corresponding to spin singlet, triplet and quintet representations:
\[ e^{iR(\Delta/v) \sinh \theta_1} = e^{-iR(\Delta/v) \sinh \theta_2} = S^{(a)}(\theta_2 - \theta_1), \quad a = 0, 1, 2. \] (172)

We have
\[ S^{(0)}(\theta) = \frac{\theta + 2\pi i}{\theta - 2\pi i}, \] (173)
\[ S^{(1)}(\theta) = \frac{\theta - \pi i \theta + 2\pi i}{\theta + \pi i \theta - 2\pi i}, \] (174)
\[ S^{(2)}(\theta) = \frac{\theta - \pi i}{\theta + \pi i}. \] (175)

In practice it is useful to consider the logarithmic form of the quantization conditions:
\[ Y_j^{(S)}(\theta_1, \theta_2) = \frac{R\Delta}{v} \sinh \theta_j - \sum_{k \neq j} \delta^{(S)}(\theta_j - \theta_k) = 2\pi I_j^{(S)}, \] (176)

where $S = 0, 1, 2$, $I_j^{(S)} \in \mathbb{Z} + (S + 1)/2$ and
\[ \delta^{(0)}(\theta) = 2 \arctan \left( \frac{\theta}{2\pi} \right), \]
\[ \delta^{(1)}(\theta) = 2 \arctan \left( \frac{\theta}{2\pi} \right) - 2 \arctan \left( \frac{\theta}{\pi} \right), \] (177)
\[ \delta^{(2)}(\theta) = -2 \arctan \left( \frac{\theta}{\pi} \right). \]

For later use we define the density of two-particle Bethe ansatz states with total spin $S$:
\[ \rho_2^{(S)}(\theta_1, \theta_2) = \det \frac{\partial Y_j^{(S)}(\theta_1, \theta_2)}{\partial \theta_k}. \] (178)

In Bethe ansatz solvable models each solution of the quantization conditions (Bethe ansatz equations) gives rise to the highest weight state of an entire multiplet of the global symmetry algebra [58]. For the sigma model this means that each solution of (176) gives the highest weight state of an O(3) multiplet. The entire multiplet is constructed from the highest weight state by acting with the spin lowering operator. This leaves the spatial part of the wavefunction unchanged, and all states in the multiplet are characterized by the same set of quantized rapidities.
Recalling the action of spin operators on single-particle states (135), we may construct the following basis of two-particle eigenstates with definite values of total spin:

\[
\begin{align*}
|\theta_1, \theta_2; 2, \pm 2\rangle &= \frac{1}{2} \{ |\theta_1 \theta_2\rangle_{11} - |\theta_1 \theta_2\rangle_{22} \pm i |\theta_1 \theta_2\rangle_{12} \pm i |\theta_1 \theta_2\rangle_{21} \}, \\
|\theta_1, \theta_2; 2, \pm 1\rangle &= \frac{1}{2} \{ |\theta_1 \theta_2\rangle_{13} + |\theta_1 \theta_2\rangle_{31} \pm i |\theta_1 \theta_2\rangle_{32} \}, \\
|\theta_1, \theta_2; 2, 0\rangle &= \frac{1}{\sqrt{6}} \{ 2 |\theta_1 \theta_2\rangle_{33} - |\theta_1 \theta_2\rangle_{11} - |\theta_1 \theta_2\rangle_{22} \}, \\
|\theta_1, \theta_2; 1, \pm 1\rangle &= \frac{1}{2} \{ |\theta_1 \theta_2\rangle_{13} - |\theta_1 \theta_2\rangle_{31} \pm i |\theta_1 \theta_2\rangle_{32} \}, \\
|\theta_1, \theta_2; 1, 0\rangle &= \frac{1}{\sqrt{2}} \{ |\theta_1 \theta_2\rangle_{12} - |\theta_1 \theta_2\rangle_{21} \}, \\
|\theta_1, \theta_2; 0, 0\rangle &= \frac{1}{\sqrt{3}} \{ |\theta_1 \theta_2\rangle_{33} + |\theta_1 \theta_2\rangle_{11} + |\theta_1 \theta_2\rangle_{22} \}.
\end{align*}
\] (179)

The form factors in the basis (179)–(181) are readily obtained. Using the crossing relations for different rapidities \(\theta_{1,2} \neq \theta_3\)

\[
b_{b,b} |\{ \theta_1, \theta_2 \}| \Phi^a(0, 0)|\theta_3\rangle_b = \langle 0| \Phi^a(0, 0) \theta_1 + i \pi, \theta_2 + i \pi, \theta_3 \rangle_{b,b,b},
\] (182)

we obtain the following result for the form factor squares involving the two-particle singlet, triplet and quintet states:

\[
\begin{align*}
\sum_b |\langle \theta_1, \theta_2; 0, 0 \Phi^a(0, 0)|\theta_3\rangle_b|^2 &= -\frac{\pi^6}{3} \psi(\theta_{12}) \psi(-\theta_{12})(\theta_{12}^2 + \pi^2) \\
&\times \psi(\theta_{13} + i \pi)^2 \psi(\theta_{23} + i \pi)^2, \\
\sum_b \sum_{\sigma=-1} |\langle \theta_1, \theta_2; 1, \sigma \Phi^a(0, 0)|\theta_3\rangle_b|^2 &= -\frac{\pi^6}{4} \psi(\theta_{12}) \psi(\theta_{21})(\theta_{13} + \theta_{23})^2 \\
&\times \psi(\theta_{13} + i \pi)^2 \psi(\theta_{23} + i \pi)^2, \\
\sum_b \sum_{\sigma=-2} |\langle \theta_1, \theta_2; 2, \sigma \Phi^a(0, 0)|\theta_3\rangle_b|^2 &= -\frac{5\pi^6}{12} \psi(\theta_{12}) \psi(\theta_{21})(\theta_{12}^2 + 4\pi^2) \\
&\times \psi(\theta_{13} + i \pi)^2 \psi(\theta_{23} + i \pi)^2.
\end{align*}
\] (183, 184, 185)

In order to proceed we now assume that the form factors in a large finite volume have the same functional form as in the infinite volume up to exponentially small corrections.

This is true for the quantum Ising model, for which the exact finite-volume form factors are known, and support in favor of this hypothesis for general massive integrable QFTs has been provided in [25]. We are now in a position to evaluate the leading contributions \(C_0, C_1\) (22) to the low-temperature expansion (23) of the two-point function (138) of the nonlinear sigma model in the finite-volume regularization scheme. The leading contribution is

\[
C_0^R(\omega, q) \approx E_{0,0}^R(\omega, q) + F_{1,0}^R(\omega, q) = \frac{\nu}{\varepsilon(q)} \left[ \frac{1}{\omega + i \varepsilon(q)} - \frac{1}{\omega + i0 + \varepsilon(q)} \right] \\ = \frac{2\nu}{(\omega + i0)^2 - \varepsilon^2(q)},
\] (186)

\[\text{doi:}10.1088/1742-5468/2009/09/P09018\]
where the momentum is quantized \( q = 2\pi j/R \) with \( j \) an integer. The first subleading contribution is

\[
C^R_1(\omega, q) = E^R_1(\omega, q) + F^R_1(\omega, q).
\]

Here \( E^R_1(\omega, q) \approx E^R_{1,0} + E^R_{1,2}(\omega, q) - Z^R_1E_{0,1}(\omega, q), \) where

\[
E^R_{1,2}(\omega, q) = \sum_{S=0}^{2} E^{RS}_{1,2}(\omega, q) = \sum_{S=0}^{2} \int_0^R dx e^{-iqx} E^{RS}_{1,2}(\omega, x),
\]

\[
E^{RS}_{1,2}(\omega, q) = \frac{1}{2} \sum_{b} \sum_{\sigma = -S}^{S} \sum_{\theta_1, \theta_2, \theta_3} \frac{W(\omega, x, \theta_1, \theta_2, \theta_3)}{\rho_1(\theta_3)\rho_2^{(S)}(\theta_1, \theta_2)} |\langle \theta_1, \theta_2, \theta_3 | \sigma | \Phi^a(0, 0) | \rangle|^2,
\]

\[
W(\omega, x, \theta_1, \theta_2, \theta_3) = \frac{e^{-\beta\Delta c(\theta_3)} e^{-\beta\Delta s(\theta_3) - s(\theta_1) - s(\theta_2)}}{\omega + i\delta - \Delta [c(\theta_1) + c(\theta_2) - c(\theta_3)]},
\]

\[
Z^R_1 = 3 \sum_{j} \exp \left(-\frac{2\beta j}{R}\right).
\]

Here the sums are over solutions to the Bethe ansatz equations (170) and (176). The contribution \( F^R_1(\omega, q) \) is obtained from \( E^R_1(\omega, q) \) using the symmetry

\[
F^R_1(\omega, q) = [E^R_1(\omega, -q)]^*.
\]

### 5.4. Evaluating \( E^R_{1,2} \) as \( R \to \infty \): comparing the finite and infinite volume regularization scheme

In this section we compute \( E^R_{1,2}(\omega, q) \) as \( R \to \infty \). We will show that \( \lim_{R \to \infty} E^R_{1,2}(\omega, q) - E^R_{0,1}(\omega, q) \) is the same as \( E_{1,2}(\omega, q) - E_{0,1}(\omega, q) \) as computed in the infinite-volume scheme. This then establishes again (as with the quantum Ising model) that the two regularization schemes are equivalent. However, in this case we will have shown this to be true in a non-trivial setting: that of an interacting theory with non-diagonal scattering.

To evaluate \( E^R_{1,2}(\omega, q) \) we first examine the contributions of each spin sector, i.e. the terms in the sum \( E^R_{1,2}(\omega, q) = \sum_S E^{RS}_{1,2}(\omega, q) \) (see equation (188)). We will take the Fourier transform in \( q \) in the end and consider \( E^{RS}_{1,2}(\omega, x) \) for the time being. The latter takes the form

\[
E^{RS}_{1,2}(\omega, x) = \frac{1}{2} \sum_{\theta_1, \theta_2, \theta_3} \frac{G^{(S)}(\omega, x, \{\theta_j\})}{\rho_1(\theta_3)\rho_2^{(S)}(\theta_1, \theta_2)} \tanh^2 \left( \frac{\theta_{12}}{2} \right) \coth^2 \left( \frac{\theta_{13}}{2} \right) \coth^2 \left( \frac{\theta_{23}}{2} \right),
\]

\[
G^{(S)}(\omega, x, \{\theta_j\}) \equiv -\psi(\theta_{12})\psi(\theta_{23}) - \psi(\theta_{13} + i\pi)^2 \psi(\theta_{23} + i\pi)^2 \times \coth^2 \left( \frac{\theta_{12}}{2} \right) \tanh^2 \left( \frac{\theta_{13}}{2} \right) \tanh^2 \left( \frac{\theta_{23}}{2} \right) U^{(S)}(\omega, q, \{\theta_j\}),
\]

where

\[
U^{(0)}(\omega, x, \{\theta_j\}) = \frac{\pi^6}{3} (\theta_{12}^2 + \pi^2) W(\omega, x, \{\theta_j\}),
\]

\[
U^{(1)}(\omega, x, \{\theta_j\}) = \frac{\pi^6}{4} (\theta_{13} + \theta_{23})^2 W(\omega, x, \{\theta_j\}),
\]

\[
U^{(2)}(\omega, x, \{\theta_j\}) = \frac{5\pi^6}{12} (\theta_{12}^2 + 4\pi^2) W(\omega, x, \{\theta_j\}).
\]
Like with the Ising model in section 3.6, our aim is in the $R \to \infty$ limit to convert these sums into integrals. To do so we must first isolate and subtract out the singular terms in $E_{1,2}^{RS}(\omega, x)$ in order to define finite integrals. Expanding $E_{1,2}^{RS}(\omega, x)$ in $\theta_3$ about $\theta_1$ and $\theta_2$, we obtain its singular pieces

$$E_{\text{sing}1,2}^{RS}(\omega, x) = \frac{1}{2} \sum_{\theta_3} \sum_{\theta_1 \neq \theta_2} \frac{1}{\rho_1(\theta_3) \rho_2^{(S)}(\theta_1, \theta_2)} \sum_{a=1}^{2} \Gamma_a^{(S)}(\omega, x, \{\theta_j\})$$

$$\equiv \sum_{a=1}^{2} E_{\text{sing}}^{(a)}(\omega, x), \quad (195)$$

The singular pieces, as indicated by the introduction of the quantities $E_{\text{sing}}^{(a)}(\omega, x)$, come in the form of both single ($a = 1$) and double poles ($a = 2$) in $\theta_3$ about $\theta_1$ and $\theta_2$. To regularize the sum we add and subtract the singular pieces from $E_{1,2}^{RS}(\omega, x)$:

$$E_{1,2}^{RS}(\omega, x) = \frac{1}{2} \sum_{\theta_3} \sum_{\theta_1 \neq \theta_2} \left[ G^{(S)}(\omega, x, \{\theta_j\}) \tanh^2(\theta_1/2) \coth^2(\theta_2/2) \coth^2(\theta_3/2) \rho_1(\theta_3) \rho_2^{(S)}(\theta_1, \theta_2) \right. $$

$$- \left. \sum_{a=1}^{2} \frac{\Gamma_a^{(S)}(\omega, x, \{\theta_j\})}{\rho_1(\theta_3) \rho_2^{(S)}(\theta_1, \theta_2)} \right] + E_{\text{finite}1,2}^{RS}(\omega, x)$$

$$\equiv E_{\text{finite}1,2}^{RS}(\omega, x) + E_{\text{sing}1,2}^{RS}(\omega, x). \quad (198)$$

The first term in the above equation is singularity free (i.e. the summand is finite as $\theta_3$ approaches either $\theta_1$ or $\theta_2$). We can then take $R \to \infty$, turning the sum into a principal value integral after which the integration contours can be modified so that they deform about the singularities in the same fashion as was done for the Ising model (see section 3.6). The result of doing so is

$$E_{\text{finite}1,2}^{RS}(\omega, x) = \int \frac{d\theta_1 d\theta_2 d\theta_3}{2(2\pi)^3} G^{(S)}(\omega, x, \{\theta_j\})$$

$$\times \tanh^2(\frac{\theta_1}{2}) \coth^2(\frac{\theta_{31} - i\eta}{2}) \coth^2(\frac{\theta_{32} + i\eta}{2})$$

$$+ \int \frac{d\theta_1 d\theta_2 d\theta_3}{2\pi} \delta(\theta_{32}) \delta(\theta_{31}) G^{(S)}(\omega, x, \{\theta_j\}). \quad (199)$$

doi:10.1088/1742-5468/2009/09/P09018
We note that the principal part integrals over the single pole terms vanish. If we now sum over the spin sectors, \(S\), then take the limit \(R \to \infty\) and finally Fourier transform with respect to \(x\) we find

\[
\sum_{S} E_{\text{finite}1,2}^{RS}(\omega, q) = \frac{1}{2} \int \frac{d\theta_1}{(2\pi)^3} d\theta_2 d\theta_3 \tilde{W}(\omega, q, \theta_1, \theta_2, \theta_3) L(\theta_1, \theta_2, \theta_3)
\]

\[
+ 8\pi^2 \int \frac{d\theta_1}{(2\pi)^3} d\theta_2 d\theta_3 \tilde{W}(\omega, q, \theta_1, \theta_2, \theta_3) \delta(\theta_12) \delta(\theta_13),
\]

where \(L(\theta_1, \theta_2, \theta_3)\) is defined in (157) and \(\tilde{W}(\omega, q, \theta_1, \theta_2, \theta_3)\) is

\[
\tilde{W}(\omega, q, \theta_1, \theta_2, \theta_3) = \frac{e^{-\beta\Delta c(\theta_3)} 2\pi \delta(vq + \Delta(s(\theta_3) - s(\theta_1) - s(\theta_2)))}{\omega + i\delta - \Delta[c(\theta_1) + c(\theta_2) - c(\theta_3)]}.
\]

We see that we have reproduced \(E_{\text{conn}}^{1,2}(\omega, q)\) as given in equation (156) plus a disconnected term. The remaining disconnected terms together with a term proportional to the partition function are found: \(E_{\text{sing}1,2}^{RS}(\omega, q)\), which we now turn to evaluate. We are able to carry out the sum over \(\theta_3\) courtesy of the identities

\[
\sum_{\theta_3} \frac{\cosh(\theta_3)}{\rho_1(\theta_3)(\sinh(\theta_3) - \sinh(\theta_2))} \bigg|_{\theta_3 \neq \theta_2 \in S} = \frac{i + S^{(S)}(\theta_12)}{2} - \frac{S^{(S)}(\theta_12)}{2},
\]

\[
\sum_{\theta_3} \frac{\cosh(\theta_2) \cosh(\theta_3)}{\rho_1(\theta_3)(\sinh(\theta_3) - \sinh(\theta_2))^2} \bigg|_{\theta_3 \neq \theta_2 \in S} = \frac{\Delta R}{2v} \frac{\cosh(\theta_2)}{1 - \text{Re} S^{(S)}(\theta_12)}.
\]

Here \(\theta_1 \neq \theta_2 \in S\) indicates that \(\theta_{1,2}\) are solutions of the Bethe ansatz equations (176) in the spin-\(S\) sector and \(S^{(S)}(\theta)\) are the \(S\) matrices (173). The first identity is established by replacing \(\theta_3\) by its corresponding integer through the quantization conditions (170) and carrying out the resulting sum using

\[
\cot(\pi z) = \frac{1}{\pi z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2}.
\]

Finally, the Bethe ansatz equations (176) in the two-particle sector are used to rewrite the result. The second identity in (202) can be established by using the derivative of the first.

The second identity in (202) allows us to evaluate the double pole term in \(E_{\text{sing}1,2}^{RS}(\omega, x)\) with the result

\[
E_{\text{sing}}^{(2)}(\omega, x) = \frac{2S + 1}{3v} \sum_{\theta_1, \theta_2 \in S} \Delta R \frac{\cosh(\theta_2) W(\omega, x, \theta_1, \theta_2, \theta_2) \rho_2^{(S)}(\theta_1, \theta_2)}{\rho_2^{(S)}(\theta_1, \theta_2)}
\]

\[
= \frac{2S + 1}{3v} \sum_{\theta_1, \theta_2 \in S} \Delta R \frac{\cosh(\theta_2) W(\omega, x, \theta_1, \theta_2, \theta_2) \rho_2^{(S)}(\theta_1, \theta_2)}{\rho_2^{(S)}(\theta_1, \theta_2)}
\]

\[
- \delta_{S,1} \frac{2S + 1}{3} \sum_{\theta_1} W(\omega, x, \theta_1, \theta_1, \theta_1) \rho_1(\theta_1)
\]

\[
= \frac{2S + 1}{3v} \Delta R \frac{d\theta_2}{(2\pi)^2} \cosh(\theta_2) W(\omega, x, \theta_1, \theta_2, \theta_2)
\]

\[
- \delta_{S,1} \frac{2S + 1}{3} \sum_{\theta_1} W(\omega, x, \theta_1, \theta_1, \theta_1) \rho_1(\theta_1)
\]

\[
= \frac{2S + 1}{3v} \Delta R \frac{d\theta_2}{(2\pi)^2} \cosh(\theta_2) W(\omega, x, \theta_1, \theta_2, \theta_2)
\]

\[
- \delta_{S,1} \frac{2S + 1}{3} \sum_{\theta_1} W(\omega, x, \theta_1, \theta_1, \theta_1) \rho_1(\theta_1).
\]
Here we need to subtract the term $\theta_1 = \theta_2$ only in the triplet sector because in the $S = 0, 2$ sectors solutions of the Bethe ansatz equations (172) and (173) with coinciding rapidities do not occur. In the triplet sector the spatial part of the wavefunction has to be antisymmetric (as the spin part is and we are dealing with bosons), which forbids solutions with coinciding rapidities. Turning the above integral back into a sum over rapidities subject to free quantization conditions (170) we arrive at

$$E^{(2)}_{\text{sing}}(\omega, x) = \frac{2S + 1}{3} \sum_{\theta_1} \exp(i(\Delta/v)s(\theta_1)) \sum_{\theta_2} e^{-\beta\Delta c(\theta_2)}$$

$$- \delta_{S,1} \int \frac{d\theta_1}{2\pi} W(\omega, x, \theta_1, \theta_1, \theta_1).$$

Carrying out the Fourier transform in $x$ and summing over spin sectors we obtain

$$E^{(2)}_{\text{sing}}(\omega, q) = E^{RS}_{0,1}(\omega, q) Z^R_{1} - \int \frac{d\theta_1}{2\pi} \tilde{W}(\omega, q, \theta_1, \theta_1, \theta_1),$$

where $\tilde{W}$ is the spatial Fourier transform of $W$. We thus obtain a term proportional to $R$ as $R \to \infty$ but which will be canceled out by a corresponding term, $E^{RS}_{0,1}(\omega, q) Z^R_{1}$, arising from the expansion of the partition function.

The first identity in (202) allows us to evaluate the single pole term in $E^{RS}_{\text{sing},1,2}(\omega, q)$. Unlike the double pole term, this leads to an expression entirely finite in the $R \to \infty$ limit:

$$\sum_{S} E^{(1)}_{\text{sing}}(\omega, x) = 16\pi^2 \int \frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3} \frac{W(\omega, x, \theta_1, \theta_2, \theta_3)}{(\theta^2_{12} + \pi^2)^2(\theta^2_{12} + 4\pi^2)}$$

$$\times \left[ \delta(\theta_{32}) \left\{ 2\pi^2 \left( \frac{2\theta_{21}(\theta^2_{12} + \pi^2)}{\sinh(\theta_{21})} + (\theta^2_{12} - \pi^2) \right) + (\theta^2_{12} + \pi^2)(2\pi^2 + \theta^2_{12}) \right\} \right]$$

$$+ \pi^2 \theta_{21}(\theta^2_{12} + \pi^2)\delta(\theta_{32}).$$

(207)

This then allows us to write down a complete expression for $E^{R}_{1,2}(\omega, q)$ as $R \to \infty$:

$$E^{R}_{1,2}(\omega, q) = \frac{1}{2} \int \frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3} \tilde{W}(\omega, q, \theta_1, \theta_2, \theta_3)L(\theta_1, \theta_2, \theta_3)$$

$$+ 4\pi^2 \int \frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3} \tilde{W}(\omega, q, \theta_1, \theta_2, \theta_3)\delta(\theta_{12})\delta(\theta_{13})$$

$$+ \sum_{S} E^{(1)}_{\text{sing}}(\omega, q) + E^{R}_{0,1}(\omega, q) Z^R_{1}.$$
The calculation of \( C_\Phi^0 \) is not possible in practice because it involves the five-particle form factor, which is tremendously complicated for the nonlinear sigma model [7]. We have shown that \( C_\Phi^0(\omega, q) \) exhibits a quadratic divergence when the frequency approaches the mass shell \( \omega \rightarrow \varepsilon(q) \), whereas \( C_\Phi^0(\omega, q) \) is only linearly divergent. We have argued that the higher-order \((\exp(-\Delta/T))\) terms in (209) exhibit stronger divergences and hence a resummation is required to get meaningful results for \( \omega \approx \varepsilon(q) \). Following the resummation procedure set out in section 2.1 we determine the quantity

\[
\Sigma^{(1)}(\omega, q) = \frac{C_\Phi^1(\omega, q)}{(C_\Phi^0(\omega, q))^2},
\]

and then use it to obtain a resummed low-temperature approximation (see (27)):

\[
\chi^{(1)}(\omega, q) = \frac{C_\Phi^0(\omega, q)}{1 - C_\Phi^0(\omega, q)\Sigma^{(1)}(\omega, q)},
\]

\[
S^{(1)}(\omega, q) = -\frac{1}{\pi} \frac{1}{1 - e^{-(\omega/T)}} \Im \chi^{(1)}(\omega, q).
\]

6. Results for the low-temperature dynamical susceptibility of the O(3) nonlinear sigma model

The leading-order result \( S^{(1)}(\omega, q) \) is most easily calculated using the infinite-volume regularization scheme and carrying out the integrals in (167) numerically. Evaluation of \( S^{(1)}(\omega, q) \) for low temperatures shows that, as expected, the \( T = 0 \) delta function at \( \omega = \varepsilon(q) \) broadens with temperature. We find that the resulting peak scales as

\[
\text{peak height} \propto \frac{\Delta}{T} \exp\left(\frac{\Delta}{T}\right),
\]

\[
\text{peak width} \propto \frac{T}{\Delta} \exp\left(-\frac{\Delta}{T}\right).
\]

In order to exhibit the evolution of the structure factor as a function of frequency for fixed momentum \( q \) with temperature it is therefore useful to rescale both the frequency axis and the structure factor. The result is shown in figure 9 for a temperature range \( 0.1\Delta \leq T < 0.3\Delta \), which corresponds to approximately a factor of 2000 difference in peak height (and width). We see that the lineshape is asymmetric in frequency, with more spectral weight appearing at higher frequencies. The asymmetry increases with temperature. This effect is most easily quantified by comparison with a Lorentzian lineshape, which is done below.

At very low temperatures we find that the lineshape at \( q = 0 \) is well approximated by a Lorentzian

\[
S^{(1)}(\omega, 0) \approx \frac{v}{\pi \Delta} \frac{\Gamma(T)}{(\omega - \Delta(T))^2 + \Gamma^2(T)}.
\]

A comparison of \( S^{(1)}(\omega, 0) \) to (214) is shown in figure 10 for two temperatures. We see that the agreement improves with decreasing temperature.
Finite-temperature dynamical correlations in massive integrable QFTs

Figure 9. Rescaled structure factor for the O(3) nonlinear sigma model for three different temperatures.

Figure 10. Comparison of the structure factor to the Lorentzian approximation (214).

6.1. Comparison to semiclassical results

In [39] Damle and Sachdev carried out a semiclassical analysis of the dynamical structure factor of the O(3) nonlinear sigma model. In contrast to the case of the quantum Ising model, the scattering matrix is not taken into account fully but approximated by its zero rapidity limit:

\[ S_{cd}^{ab}(\theta) \rightarrow -\delta_{ad}\delta_{bc}. \]  

(215)

We note that, compared to the Ising case, this additional approximation may impose tighter restrictions on the window of applicability of the semiclassical result. Damle and
Sachdev find the following form for the dynamical structure factor:

\[ S_{sc}(\omega, q) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} dx e^{i\omega t - iqx} F_0(x, t), \quad (216) \]

where the relaxation function \( R(x, t) \) is determined numerically and

\[ K(x, t) = \frac{Z}{\pi} \frac{\Delta}{\sqrt{(x/v)^2 - t^2}}. \quad (217) \]

Here \( Z \) is a normalization factor. At sufficiently low temperatures and when \( |vq| < \sqrt{T\Delta} \) the semiclassical result is approximately Lorentzian in form \cite{39,38}:

\[ S_{sc}(\omega, q) \approx S_{Lor}(\omega, q) = \frac{vZ}{\pi \varepsilon(q)} \frac{\alpha/\tau_0}{(\omega - \varepsilon(q))^2 + (\alpha/\tau_0)^2}, \quad (218) \]

where \( \alpha \approx 0.72 \) \cite{38} and

\[ \tau_0 = \frac{\sqrt{\pi}}{3T} e^{\Delta/T}. \quad (219) \]

We have shown above that our result is well approximated by a Lorentzian at low temperatures. However, unlike \cite{218}, the best fit to our result involves a temperature-dependent gap, see \cite{214}. The width of the Lorentzian is quite close to Damle and Sachdev’s result, e.g. we find that \( \Gamma(0.1T) = 0.736/\tau_\phi \). However, our value of \( \alpha \) is found to increase as \( T \to 0 \). In \cite{41} an analytic expression for the relaxation function \( R(x, t) \) was given:

\[ R(x, t) = C \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{(1 - \gamma^2)}{\gamma^2 + 2\gamma \cos \phi + 1} \exp \left[ -(1 - \cos \phi)\frac{1}{\tau_0} \left( \frac{e^{-u^2}}{\sqrt{\pi}} + \text{erf}(u) \right) \right], \quad (220) \]

where \( C \) is a normalization constant, \( \gamma = \frac{1}{2}, x/\xi_0, \bar{t}/\tau_0, u = \bar{x}/\bar{t} \) and

\[ \xi_0 = \frac{1}{3} \sqrt{\frac{2\pi v^2}{\Delta T} e^{\Delta/T}}. \quad (221) \]

At low temperatures and for \( \omega \) sufficiently close to \( \varepsilon(q) \) the Fourier transform \cite{216} is well approximated by \cite{39}

\[ S_{sc}(\omega, q) \approx \int_{-\infty}^{\infty} \frac{dt}{2\pi} dx e^{i\omega t - iqx} K(x, t) R(0, t). \quad (222) \]

Carrying out the integrals we obtain

\[ S_{sc}(\omega, q) \approx \frac{ZCV}{\pi \varepsilon(q)} \text{Re} \left[ \frac{\Gamma(T) + i(\omega - \varepsilon(q))}{(\omega - \varepsilon(q))^2 + \Gamma^2(T)} \left( 1 + \frac{1 - \gamma^2}{2\gamma \sqrt{\alpha^2 - 1}} \right) \right], \quad (223) \]

where

\[ \Gamma(T) = \frac{\gamma + (1/\gamma) + 2}{2\sqrt{\pi \tau_0}}, \quad \alpha = 1 - i\sqrt{\pi \tau_0}(\omega - \varepsilon(q)). \quad (224) \]

While this expression is very close to being a Lorentzian sufficiently far away from the mass shell, it exhibits a square root divergence as \( \omega \to \varepsilon(q) \). This precludes a comparison with the results of our low-temperature expansion.
7. Summary and discussion

In this work we have proposed a general method for determining frequency- and momentum-dependent two-point functions of local operators in massive integrable quantum field theories at low temperatures. We have applied this method to the calculation of response functions in the disordered phase of the quantum Ising model and the O(3) nonlinear sigma model. The methodology in this particular application possesses two crucial ingredients. First we have shown that there exists a systematic expansion of the spin–spin response function in terms of the small parameter $\exp(-\Delta/T)$, where $\Delta$ is the spectral gap and $T$ is the temperature. While such expansions have been known to exist previously [23], they were limited to certain correlation functions. Here we have shown that the spin–spin response function can be described sensibly by performing a low-temperature expansion of the spin–spin response’s ‘self-energy’ (or a quantity analogous to such). The second ingredient was a procedure to make sense of infinities that appear in the squares of matrix elements that arise for a Lehmann expansion of the correlators. In this paper we have developed a new regulator for the infinities that appear when working in an infinite volume and shown that this regulator reproduces results found by working in a large, finite volume and taking the thermodynamic limit only at the end of the calculation. We have accomplished this both in the quantum Ising model, a free fermionic theory, and the O(3) NLSM where the elementary excitations are strongly interacting with a non-diagonal, momentum-dependent scattering matrix.

A number of open problems remain on the technical level. In the present paper we have not analyzed the situation where we need more than one $\kappa$ parameter in the infinite-volume regularization scheme (see, e.g., (59)). It is important to establish the equivalence of infinite- and finite-volume regularization schemes in this more general case as well. We also have not presented a general proof that all terms (17) and (18) in the low-temperature expansion (19) are finite. We hope that these questions will be addressed in future work.

At zero temperature the response functions in both the Ising model and the O(3) NLSM are dominated by a delta function peak at the position of the single-particle dispersion. We have determined how this peak broadens at finite temperatures smaller than the gap. Our main result is that the lineshape at $T > 0$ exhibits a pronounced asymmetry in energy that increases with temperature. At very low temperatures $T < 0.1\Delta$ our results essentially reduce to those of previous semiclassical analyses [28, 39, 38], which concluded that the lineshape is, to a good approximation, Lorentzian. This shows that, while the semiclassical approximation gives a good account of the width and height of the lineshape only at very low temperatures.

A second important feature seen in our low-temperature expansion is what is known as the temperature-dependent gap. For sufficiently high temperatures the maximum of the lineshape is seen to shift upwards in energy when compared to the $T = 0$ gap. We found that this phenomenon emerges in the quantum Ising model once subleading terms in our low-temperature expansion are taken into account. The calculation of these terms involves five-particle form factors. While these are known for the O(3) NLSM as well, their

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complexity puts the calculation of subleading terms in the low-temperature expansion beyond the scope of this work.

The O(3) NLSM describes the scaling limit of integer-spin Heisenberg chains. While the agreement between the field theory and the Heisenberg lattice model is best for large spins (and low energies), the sigma model has been found to provide a reasonable approximation of the structure factor even in the extreme $S = 1$ case [59]. A number of inelastic neutron scattering experiments have measured the temperature dependence of the dynamical structure factor of quasi-one-dimensional spin-1 Heisenberg magnets [34, 35]. Our finding of an asymmetric lineshape are relevant to these experiments. In particular, the excess of spectral weight at high energies reported in [34] for CsNiCl$_3$ should at least be partly accounted for by a lineshape asymmetric in energy. However, a quantitative comparison of our theory to inelastic neutron scattering data on CsNiCl$_3$ is precluded by the presence of a Néel transition, driven by non-negligible interchain coupling, at a temperature comparable to the gap. It is likely that the interchain coupling will affect the precise lineshape in a significant way [60]. Similarly, in YBaNiO$_5$ a quantitative comparison to the finite-temperature structure factor is not straightforward because of the presence of an exchange anisotropy.

The presence of an asymmetric lineshape even at low temperatures is expected to be a general feature in quantum magnets that support coherent single-particle excitations at zero temperature. Theoretical studies of the alternating Heisenberg chain [61] suggest that asymmetric lineshapes occur in dimer systems. This has been confirmed by recent experiments on the quasi-one-dimensional alternating Heisenberg chain copper nitrate [62]. It would be interesting to investigate to what extent the same holds for the two- and three-dimensional cases [63].

The methods we have developed have a wider scope for applications. With regards to low-temperature dynamics in gapped integrable models, it would be interesting to investigate the case of the spin-1/2 Heisenberg–Ising chain. Here the dynamical structure factor has been measured for a number of materials [64]. In the limit of large gaps the transverse component of the structure factor has been determined by diagrammatic methods [65]. The excitation spectrum is quite different compared to the O(3) NLSM and the disordered phase of the quantum Ising model in that the lowest excitations are two-parametric. Concomitantly the dynamical structure factor is dominated by an incoherent two-particle continuum at zero temperature. At finite temperature a low-frequency resonance, known as the ‘Villain mode’, develops [66]. It should be possible to adapt our methods to analyze this case.

Our method of regularizing general matrix elements could be useful for studying non-integrable perturbations of integrable models, which is an area of considerable importance [67]. In fact, shortly after our work a preprint by Takacs appeared (arXiv:0907.2109), which addresses this problem for the double sine–Gordon model and obtains an independent derivation of the finite-volume regularization scheme.

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Appendix A. Crossing relations

In this appendix we summarize identities first given by Smirnov in [1] that allow us to analytically continue form factors. Let $A = \{\theta_1, \ldots, \theta_n\}$ with $\theta_1 < \theta_2 < \cdots < \theta_n$ and $B = \{\beta_1, \ldots, \beta_m\}$ with $\beta_1 < \beta_2 < \cdots < \beta_m$ and introduce notations

$$Z[A]_{a_1 \ldots a_n} = Z_{a_1}(\theta_1)Z_{a_2}(\theta_2) \ldots Z_{a_n}(\theta_n),$$

$$Z[A]_{a_n \ldots a_1} = Z_{a_n}(\theta_n)Z_{a_{n-1}}(\theta_{n-1}) \ldots Z_{a_1}(\theta_1).$$

(A.1)

Now let $A_1$ and $A_2$ be a partition of $A$. As a consequence of the periodicity axiom we have

$$Z[A]_{a_1 \ldots a_n} = Z[A_2]_{c_1 \ldots c_r} Z[A_1]_{c_{r+1} \ldots c_n} S(A_1 | A_2)_{c_1 \ldots c_n},$$

(A.2)

where $S(A_1 | A_2)$ is the product of two-particle scattering matrices needed to rearrange the order of Faddeev–Zamolodchikov operators in $Z[A]$ to arrive at $Z[A_2]Z[A_1]$. Similarly we have

$$Z[B]_{b_m \ldots b_1} = Z[B_1]_{d_m \ldots d_{s+1}} Z[B_2]_{d_s \ldots d_1} S(B_1 | B_2)^{d_m \ldots d_1}.$$

(A.3)

Finally we define

$$\delta[A, B]_{b_1 \ldots b_m} = \delta_{n,m} \prod_{j=1}^{n} 2\pi \delta(\theta_j - \beta_j) \delta_{\theta_j, \beta_j}.$$

(A.4)

For a local operator $\mathcal{O}$ we then can analytically continue form factors as

$$\langle 0 | Z[A]_{a_1 \ldots a_n} \mathcal{O}(0, 0) Z[B]_{b_m \ldots b_1} | 0 \rangle = \sum_{A = A_1 \cup A_2} S(A | A_1)_{a_1 \ldots a_n} S(B | B_2)_{b_m \ldots b_1} \delta[A_2, B_2]_{d_m \ldots d_1} \delta[A_1, B_1]_{d_m \ldots d_1}$$

$$\times \langle 0 | Z[A_1 + i0]_{c_{r+1} \ldots c_n} \mathcal{O}(0, 0) Z[B_1]_{d_m \ldots d_{s+1}} | 0 \rangle.$$  

(A.5)

Similarly, we could choose to analytically continue to the lower half-plane:

$$\langle 0 | Z[A]_{a_1 \ldots a_n} \mathcal{O}(0, 0) Z[B]_{b_m \ldots b_1} | 0 \rangle = \sum_{A = A_1 \cup A_2} S(A | A_2)_{a_1 \ldots a_n} S(B | B_2)_{b_m \ldots b_1} \delta[A_2, B_2]_{d_m \ldots d_1} \delta[A_1, B_1]_{d_m \ldots d_1}$$

$$\times \langle 0 | Z[A_1 - i0]_{c_{r+1} \ldots c_n} \mathcal{O}(0, 0) Z[B_1]_{d_m \ldots d_{s+1}} | 0 \rangle.$$  

(A.6)

For semi-local operators the above identities need to be modified as discussed in [1].

Appendix B. Products of form factors for the Ising model

Let us consider the product of form factors:

$$|\langle \theta_3 | \sigma(0,0) | \theta_2 \theta_1 \rangle|^2 = \lim_{\kappa \to 0} \langle \theta_3 | \sigma(0,0) | \theta_2 \theta_1 \rangle \langle \theta_2 \theta_1 | \sigma(0,0) | \theta_3 + \kappa \rangle.$$

(B.1)
Using the crossing relations (57) this can be rewritten as

\[
|\langle \theta_3 | \sigma(0, 0) | \theta_2 \rangle |^2 = \lim_{\kappa \to 0} \left\{ \langle \langle \theta_3 | \sigma(0, 0) | \theta_2 - i0, \theta_1 + i0 \rangle \rangle \right. \\
\times \left( \langle \theta_1 + i0, \theta_2 - i0 | \sigma(0, 0) | \theta_3 + \kappa \rangle + 2\pi \delta[\delta(\theta_{32} + \kappa) + \delta(\theta_{31} + \kappa)] \right) \right\}.
\] (B.2)

Multiplying out the various terms then gives

\[
|\langle \theta_3 | \sigma(0, 0) | \theta_2 \rangle |^2 = \lim_{\kappa \to 0} \left\{ \langle \theta_1 + i0, \theta_2 - i0 | \sigma(0, 0) | \theta_3 + \kappa \rangle \langle \theta_1 - i0, \theta_2 + i0 | \sigma(0, 0) | \theta_3 \rangle^* \\
+ 2\pi \delta[\delta(\theta_{32} + \kappa) + \delta(\theta_{31} + \kappa)] \langle \theta_1 - i0, \theta_2 + i0 | \sigma(0, 0) | \theta_3 \rangle^* \\
- 2\pi \delta[\delta(\theta_{32}) + \delta(\theta_{31})] \langle \theta_1 + i0, \theta_2 - i0 | \sigma(0, 0) | \theta_3 + \kappa \rangle \\
- (2\pi \delta)^2 [\delta(\theta_{32} + \kappa) + \delta(\theta_{31} + \kappa)] [\delta(\theta_{32}) + \delta(\theta_{31})] \right\}
= \lim_{\kappa \to 0} \left[ \Gamma^{\text{conn}} + \Gamma^{\text{dis,1}} + \Gamma^{\text{dis,2}} \right].
\] (B.3)

The connected part

\[
\Gamma^{\text{conn}} = \langle \theta_1 + i0, \theta_2 - i0 | \sigma(0, 0) | \theta_3 + \kappa \rangle \langle \theta_1 - i0, \theta_2 + i0 | \sigma(0, 0) | \theta_3 \rangle^*
\] (B.4)
does not contain any divergent pieces and the limit \( \kappa \to 0 \) can be taken straightforwardly. The product of delta functions gives

\[
\Gamma^{\text{dis,1}} = -(2\pi \delta)^2 \{ \delta(\kappa) [\delta(\theta_{32}) + \delta(\theta_{31})] + \delta(\theta_{31}) \delta(\theta_{32} + \kappa) + \delta(\theta_{32}) \delta(\theta_{31} + \kappa) \},
\] (B.5)

where \( \theta_{jk} = \theta_j - \theta_k \). The cross-terms are

\[
\Gamma^{\text{dis,2}} = 2\pi \delta \Gamma_{\text{cross,1}} + 2\pi \delta \Gamma_{\text{cross,2}},
\] (B.6)

where

\[
\Gamma_{\text{cross,1}} = \delta(\theta_{32} + \kappa) \langle \theta_1 - i\eta_1, \theta_2 + i\eta_2 | \sigma | \theta_3 \rangle^* - \delta(\theta_{32}) \langle \theta_1 + i\eta_1', \theta_2 - i\eta_2 | \sigma | \theta_3 + \kappa \rangle,
\]

\[
\Gamma_{\text{cross,2}} = \delta(\theta_{31} + \kappa) \langle \theta_1 - i\eta_1, \theta_2 + i\eta_2 | \sigma | \theta_3 \rangle^* - \delta(\theta_{31}) \langle \theta_1 + i\eta_1', \theta_2 - i\eta_2 | \sigma | \theta_3 + \kappa \rangle.
\] (B.7)

Here \( \eta_{1,2} \) are positive infinitesimals and we are interested in the limit \( \eta_{1,2} \to 0 \) at fixed \( \kappa \). Using the explicit form of the form factors we obtain

\[
\Gamma_{\text{cross,1}} = -i\delta(\theta_{32} + \kappa) \tanh \left( \frac{\theta_{12} + i\eta_1 + i\eta_2}{2} \right) \coth \left( \frac{\theta_{12} + \kappa + i\eta_1}{2} \right) \coth \left( \frac{\kappa - i\eta_2}{2} \right) \\
+ i\delta(\theta_{32}) \tanh \left( \frac{\theta_{12} + i\eta_1' + i\eta'_2}{2} \right) \coth \left( \frac{\theta_{12} - \kappa + i\eta_1'}{2} \right) \coth \left( \frac{\kappa + i\eta'_2}{2} \right).
\] (B.8)

Using that

\[
\tanh \left( \frac{\theta_{12} + i\eta_1 + i\eta_2}{2} \right) \coth \left( \frac{\theta_{12} + \kappa + i\eta_1}{2} \right) \\
\simeq 1 - (\kappa - i\eta_2) \left[ \frac{1}{\sinh \theta_{12}} - \frac{1}{\theta_{12}} + \frac{1}{\theta_{12} + \kappa + i\eta_1} \right],
\] (B.9)
this can be simplified to
\[
\Gamma_{\text{cross},1} = -2i\bar{\sigma}\delta(\theta_{32} + \kappa) \left[ \frac{1}{\kappa - i\eta_2} - \frac{1}{\sinh\theta_{12}} + \frac{1}{\theta_{12}} - \frac{1}{\theta_{12} + \kappa + i\eta_1} \right] \\
+ 2i\bar{\sigma}\delta(\theta_{32}) \left[ \frac{1}{\kappa + i\eta_2} + \frac{1}{\sinh\theta_{12}} - \frac{1}{\theta_{12}} + \frac{1}{\theta_{12} - \kappa + i\eta_1} \right] \\
= 2\pi\bar{\sigma}\delta(\kappa) \left[ \delta(\theta_{32} + \kappa) + \delta(\theta_{32}) \right] + i\bar{\sigma} \frac{2\kappa}{\kappa^2 + \eta_2^2} [\delta(\theta_{32}) - \delta(\theta_{32} + \kappa)] \\
+ 2i\bar{\sigma} \left[ \delta(\theta_{32} + \kappa) + \delta(\theta_{32}) \right] \left[ \frac{1}{\sinh\theta_{12}} - \frac{1}{\theta_{12}} \right] \\
+ 2i\bar{\sigma} \left[ \frac{\delta(\theta_{32})}{\theta_{12} - \kappa + i\eta_1} + \frac{\delta(\theta_{32} + \kappa)}{\theta_{12} + \kappa + i\eta_1} \right]. 
\] (B.10)

In the limit \( \kappa \to 0 \) this becomes
\[
\Gamma_{\text{cross},1} \to 4\pi\bar{\sigma}\delta(\theta_{32})\delta(\kappa) + 2i\bar{\sigma} \left[ \frac{\delta(\theta_{32})}{\theta_{12} - \kappa + i0} + \frac{\delta(\theta_{32} + \kappa)}{\theta_{12} + \kappa + i0} \right] \\
- 2i\bar{\sigma}\delta'(\theta_{32}) + 4i\bar{\sigma}\delta(\theta_{32}) \left[ \frac{1}{\sinh\theta_{12}} - \frac{1}{\theta_{12}} \right]. 
\] (B.11)

Similarly we find
\[
\Gamma_{\text{cross},2} \to 4\pi\bar{\sigma}\delta(\theta_{31})\delta(\kappa) + 2i\bar{\sigma} \left[ \frac{\delta(\theta_{31})}{\theta_{12} + \kappa + i0} + \frac{\delta(\theta_{31} + \kappa)}{\theta_{12} - \kappa + i0} \right] \\
+ 2i\bar{\sigma}\delta'(\theta_{31}) + 4i\bar{\sigma}\delta(\theta_{31}) \left[ \frac{1}{\sinh\theta_{12}} - \frac{1}{\theta_{12}} \right]. 
\] (B.12)

We note the emergence of a term involving the derivative of the delta function. Our final result for \( \Gamma^{\text{dis},2} \) is then
\[
\lim_{\kappa \to 0} \Gamma_{\text{dis},2} = \lim_{\kappa \to 0} \left\{ 8\pi^2\bar{\sigma}^2 [\delta(\theta_{31}) + \delta(\theta_{32})] \delta(\kappa) \\
+ 4\pi i\bar{\sigma}^2 [\delta'(\theta_{31}) - \delta'(\theta_{32})] \right\} \\
+ 8\pi i\bar{\sigma}^2 [\delta(\theta_{31}) + \delta(\theta_{32})] \left[ \frac{1}{\sinh\theta_{12}} - \frac{1}{\theta_{12}} \right] \\
+ 4\pi i\bar{\sigma}^2 \left[ \frac{\delta(\theta_{31}) + \delta(\theta_{32} + \kappa)}{\theta_{12} + \kappa + i0} + \frac{\delta(\theta_{31} + \kappa) + \delta(\theta_{32})}{\theta_{12} - \kappa + i0} \right]. 
\] (B.13)

The infinite-volume regularization for the form factor squared is then
\[
|\langle \theta_3 | \sigma(0, 0) | \theta_2 \theta_1 \rangle |^2 = \bar{\sigma}^2 \tanh^2 \left( \frac{\theta_{12}}{2} \right) \coth^2 \left( \frac{\theta_{13} + i0}{2} \right) \coth^2 \left( \frac{\theta_{23} - i0}{2} \right) \\
+ \lim_{\kappa \to 0} \left[ \Gamma^{\text{dis},1} + \Gamma^{\text{dis},2} \right], 
\] (B.14)

where \( \Gamma^{\text{dis},1} \) and \( \Gamma^{\text{dis},2} \) are given by (B.5) and (B.13), respectively. For our purposes we

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only need the part of $\Gamma^\text{dis} = \Gamma^\text{dis,1} + \Gamma^\text{dis,2}$ symmetric under $\theta_1 \leftrightarrow \theta_2$ as the remainder of the integrands we consider all have this symmetry. The symmetric part is

$$\lim_{\kappa \to 0} \frac{\Gamma^\text{dis}(\theta_1, \theta_2, \theta_3) + \Gamma^\text{dis}(\theta_2, \theta_1, \theta_3)}{2} = 4\pi^2 \sigma^2 \delta(\kappa) [\delta(\theta_{31}) + \delta(\theta_{32})] + 8\pi^2 \sigma^2 \delta(\theta_{31}) \delta(\theta_{32}).$$

(B.15)

**Appendix C. Products of form factors for the $O(3)$ nonlinear $\sigma$ model**

In this appendix we determine the form factor squared:

$$\Gamma(\theta_1, \theta_2, \theta_3) = \sum_{b,b_1,b_2} |b_{b_1,b_2} \langle \theta_1, \theta_2 | \Phi^a(0) | \theta_3 \rangle|^2. \quad \text{(C.1)}$$

Using (154), which is a consequence of the crossing relations discussed in appendix A, we obtain the decomposition

$$\Gamma(\theta_1, \theta_2, \theta_3) = \lim_{\kappa, \eta_1, \eta_2 \to 0} \left[ \Gamma^\text{conn}(\theta_1, \theta_2, \theta_3) + \Gamma^\text{dis,1}(\theta_1, \theta_2, \theta_3) + \Gamma^\text{dis,2}(\theta_1, \theta_2, \theta_3) \right]. \quad \text{(C.2)}$$

Here

$$\Gamma^\text{conn} = \sum_{b,b_1,b_2} b_{b_1,b_2} \langle \theta_1 - i\eta_1, \theta_2 + i\eta_2 | \Phi^a(0) | \theta_3 \rangle^* \times b_{b_1,b_2} \langle \theta_1 + i\eta_1, \theta_2 - i\eta_2 | \Phi^a(0) | \theta_3 + \kappa \rangle, \quad \text{(C.3)}$$

$$\Gamma^\text{dis,1} = \sum_b (2\pi)^2 S_{ab}^{(21)} [\delta(\theta_{32}) \delta(\theta_{32} + \kappa) + \delta(\theta_{31}) \delta(\theta_{31} + \kappa)]$$

$$+ \sum_b (2\pi)^2 S_{ab}^{(21)} [\delta(\theta_{31}) \delta(\theta_{32} + \kappa) + \delta(\theta_{32}) \delta(\theta_{31} + \kappa)], \quad \text{(C.4)}$$

$$\Gamma^\text{dis,2} = \sum_b 2\pi \delta(\theta_{32} + \kappa)_{ab} \langle \theta_1 - i\eta_1, \theta_2 + i\eta_2 | \Phi^a(0) | \theta_2 - \kappa \rangle^*$$

$$+ \sum_b 2\pi \delta(\theta_{31} + \kappa)_{ba} \langle \theta_1 - i\eta_1, \theta_2 + i\eta_2 | \Phi^a(0) | \theta_1 - \kappa \rangle^*$$

$$+ \sum_b 2\pi \delta(\theta_{32})_{ba} \langle \theta_2 - i\eta_2, \theta_1 + i\eta_1 | \Phi^a(0) | \theta_2 + \kappa \rangle$$

$$+ \sum_b 2\pi \delta(\theta_{31})_{ab} \langle \theta_2 - i\eta_2, \theta_1 + i\eta_1 | \Phi^a(0) | \theta_1 + \kappa \rangle. \quad \text{(C.5)}$$

Using crossing, the connected contribution can be written as

$$\Gamma^\text{conn} = \sum_{b,b_1,b_2} (0 | \Phi^a(0) | \theta_1 + i\pi^-, \theta_2 + i\pi^+, \theta_3)_{b_1,b_2}^*$$

$$\times \langle 0 | \Phi^a(0) | \theta_1 + i\pi^+, \theta_2 + i\pi^-, \theta_3 + \kappa \rangle_{b_1,b_2}. \quad \text{(C.6)}$$
Using the explicit expression (142) for the three-particle form factor this can be brought in the form

\[
\lim_{\kappa,\eta_1,\eta_2 \to 0} \Gamma_{\text{conn}} = \frac{\pi^6}{2} \left[ \theta_{12}^2 + \theta_{13}^2 + \theta_{24}^2 + 4\pi^2 \right] \frac{\theta_{12}^2 + \pi^2}{\theta_{12}^2(\theta_{12}^2 + 4\pi^2)} \tanh^4 \left( \frac{\theta_{12}}{2} \right) \times \left( \frac{\theta_{13} + i0}{(\theta_{13}^2 + \pi^2)^2} \coth^4 \left( \frac{\theta_{13} + i0}{2} \right) \right).
\]

(C.7)

The connected \(\Gamma_{\text{conn}}\) contribution is now in a form suitable for further analysis. For our purposes it is sufficient to determine the parts of \(\Gamma_{\text{dis},1}, \Gamma_{\text{dis},2}\) symmetric in \(\theta_1\) and \(\theta_2\):

\[
\Gamma_{\text{dis},j}(\theta_1, \theta_2, \theta_3) = \lim_{\kappa,\eta_1,\eta_2 \to 0} \frac{\Gamma_{\text{dis},j}(\theta_1, \theta_2, \theta_3) + \Gamma_{\text{dis},j}(\theta_2, \theta_1, \theta_3)}{2}, \quad j = 1, 2.
\]

(C.8)

We therefore concentrate on these parts only in the following. We have

\[
\lim_{\kappa \to 0} \Gamma_{\text{dis},1} = \lim_{\kappa \to 0} \sum_{b} 4\pi^2 \delta(\kappa) \Re S_{ab}^{\theta_1}(\theta_{21}) \left[ \delta(\theta_{32}) + \delta(\theta_{31}) \right] - 24\pi^2 \delta(\theta_{31})\delta(\theta_{32}).
\]

(C.9)

Using the explicit forms of the three-particle form factors and proceeding along the same lines as for the Ising model, we obtain after some lengthy calculations

\[
2\Gamma_{\text{dis},2}(\theta_1, \theta_2, \theta_3) = \sum_{b} (2\pi)^2 \delta(\theta_{32}) \delta(\kappa) \Re \left[ 1 - S_{ab}^{\theta_1}(\theta_{21}) \right]
+ \sum_{b} (2\pi)^2 \delta(\theta_{31}) \delta(\kappa) \Re \left[ 1 - S_{ab}^{\theta_1}(\theta_{21}) \right]
+ \sum_{b} 4\pi \delta(\theta_{32}) \Im S_{ab}^{\theta_1}(\theta_{12})
+ \sum_{b} (4\pi)^2 \delta(\theta_{32})(1 - \delta_{ab}) \Re \left[ \frac{1}{\theta_{21} + \pi i} \frac{1}{\theta_{21} - 2\pi i} \left[ \frac{2}{\sinh \theta_{21} - \frac{2\theta_{21}}{\theta_{21}^2 + \pi^2}} \right] \right]
+ \sum_{b} 8\pi \delta(\theta_{32}) \Im S_{ab}^{\theta_1}(\theta_{12}) \left[ \frac{2}{\sinh \theta_{21} - \theta_{21}^2 + \pi^2} + \frac{2\theta_{21}}{\theta_{21}^2 + \pi^2} \right]
+ \sum_{b} (2\pi)^2 \delta(\theta_{32} + \kappa) \delta(\theta_{31}) \Re \left[ 1 - S_{ab}^{\theta_1}(\theta_{21}) \right]
+ \sum_{b} (2\pi)^2 \delta(\theta_{31} + \kappa) \delta(\theta_{32}) \Re \left[ 1 - S_{ab}^{\theta_1}(\theta_{21}) \right]
+ \sum_{b} 4\pi \delta(\theta_{32} + \kappa) \frac{\theta_{31}}{\theta_{31}^2 + \eta^2} \Im S_{ab}^{\theta_1}(\theta_{12})
+ \sum_{b} 4\pi \delta(\theta_{32}) \frac{\theta_{31} + \kappa}{(\theta_{31} + \kappa)^2 + \eta^2} \Im S_{ab}^{\theta_1}(\theta_{12})
+ \theta_1 \leftrightarrow \theta_2.
\]

(C.10)
This is simplified further to
\[ \lim_{\kappa \to 0} r^{\text{dis}, 2} = 4\pi^2 [\delta(\theta_{32}) + \delta(\theta_{31})] \delta(\kappa) \left[ 3 - \sum_b \text{Re} S_{ab}^b(\theta_{21}) \right] - 16\pi^4 [\delta'(\theta_{32}) - \delta'(\theta_{31})] \left(\frac{\theta_{12}}{\theta_{12}^2 + \pi^2}(\theta_{12}^2 + 4\pi^2) + 32\pi^2 \delta(\theta_{32}) \delta(\theta_{31}) \right) + 16\pi^2 \delta(\theta_{32}) + \delta(\theta_{31}) \left[ \frac{\theta_{12}^2 + 4\pi^2 \theta_{12}}{\sinh \theta_{12}} + \frac{4\pi^2 \theta_{12}^2}{\theta_{12}^2 + \pi^2} \right]. \]

(C.11)

Our final result for the symmetrized (in \( \theta_1 \) and \( \theta_2 \)) disconnected parts of the form factor squared in the infinite volume scheme is
\[ \lim_{\kappa \to 0} \left[ \Gamma_+^{\text{dis}, 1} + r^{\text{dis}, 2} \right] = 12\pi^2 [\delta(\theta_{32}) + \delta(\theta_{31})] \delta(\kappa) + 8\pi^2 \delta(\theta_{32}) \delta(\theta_{31}) \left(\frac{\theta_{12}}{\theta_{12}^2 + \pi^2}(\theta_{12}^2 + 4\pi^2) \right) + 16\pi^2 \delta(\theta_{32}) + \delta(\theta_{31}) \left[ \frac{\theta_{12}^2 + 4\pi^2 \theta_{12}}{\sinh \theta_{12}} + \frac{4\pi^2 \theta_{12}^2}{\theta_{12}^2 + \pi^2} \right]. \]

(C.12)

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