Positive tensor products of maps
and $n$-tensor-stable positive qubit maps

Sergey N Filippov and Kamil Yu Magadov

Moscow Institute of Physics and Technology, Institutskii Per. 9, Dolgoprudny,
Moscow Region 141700, Russia

E-mail: sergey.filippov@phystech.edu

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Abstract
We analyze positivity of a tensor product of two linear qubit maps, $\Phi_1 \otimes \Phi_2$. Positivity of maps $\Phi_1$ and $\Phi_2$ is a necessary but not a sufficient condition for positivity of $\Phi_1 \otimes \Phi_2$. We find a non-trivial sufficient condition for positivity of the tensor product map beyond the cases when both $\Phi_1$ and $\Phi_2$ are completely positive or completely co-positive. We find necessary and (separately) sufficient conditions for $n$-tensor-stable positive qubit maps, i.e. such qubit maps $\Phi$ that $\Phi^\otimes n$ is positive. Particular cases of 2- and 3-tensor-stable positive qubit maps are fully characterized, and the decomposability of 2-tensor-stable positive qubit maps is discussed. The case of non-unital maps is reduced to the case of appropriate unital maps. Finally, $n$-tensor-stable positive maps are used in characterization of multipartite entanglement, namely, in the entanglement depth detection.

Keywords: positive map, tensor-stable positive map, qubit map, unital map, non-unital map, entanglement annihilation, entanglement depth

(Some figures may appear in colour only in the online journal)

1. Introduction

Tensor product structures play a vital role in quantum information theory: entanglement of quantum states is defined with respect to a particular bipartition [1] or multipartition (see, e.g. the reviews [2, 3]); communication via quantum channels involves multiple uses of the same channel, which results in the map of the form $\Phi^\otimes n$ (see, e.g. [4]); propagation of multipartite physical signals through separated communication lines $\Phi_1$ and $\Phi_2$ is described by a tensor product of corresponding maps $\Phi_1 \otimes \Phi_2$; local operations and measurements have the tensor product structure too. Properties of quantum channels may drastically change with tensoring as it takes place, for instance, in superactivation of zero-error capacities [5, 6].
Positive maps, in their turn, are an important auxiliary tool in quantum information theory and are widely used in the analysis of bipartite entanglement [7–16], multipartite entanglement [17, 18], entanglement distillation [19–22], distinguishability of bipartite states [23–25], description of open system dynamics [26–28], monotonicity of relative entropy [29], and evaluation of quantum channel capacities [30].

Positivity of linear maps under tensor powers was analyzed in the recent seminal paper [31], where the notions of $n$-tensor-stable positive and tensor-stable positive maps were introduced. Tensor-stable positive maps were found to provide new bounds on quantum channel capacities.

The aim of this paper is to study positivity of the tensor product maps $\Phi_1 \otimes \Phi_2$, where both $\Phi_1$ and $\Phi_2$ are qubit maps ($\mathcal{M}_2 \mapsto \mathcal{M}_2$). We focus special attention on 2-tensor-stable positive maps $\Phi$, i.e. such maps $\Phi$ that $\Phi^{\otimes 2}$ is positive, and then extend our results to 3- and $n$-tensor-stable positive maps.

The paper is organized as follows.

In section 2, we review notations and general properties of linear maps, and formulate some sufficient conditions for positivity of the tensor product map $\Phi_1 \otimes \Phi_2$. In section 3, an exact characterization of bipartite locally depolarizing positive maps is presented. In section 4, sufficient conditions for positivity of the tensor product unital map $\Phi_1 \otimes \Phi_2$ are derived. In section 5, we find the necessary and sufficient condition for 2-tensor positivity of unital qubit maps. In section 7, 2-tensor-stable positivity of non-unital qubit maps is studied by a reduction to the problem of 2-tensor-stable positivity of corresponding unital maps. In section 8, criteria for 3-tensor positivity of unital qubit maps are found and checked numerically. In section 9, we find necessary and (separately) sufficient conditions for $n$-tensor-stable positive maps. Section 10 is devoted to witnessing particular forms of multipartite entanglement via $n$-tensor-stable positive maps. In section 11, brief conclusions are given.

2. Notations and general properties

Consider a finite dimensional Hilbert space (unitary space) $\mathcal{H}_d$, $\dim \mathcal{H} = d$, and the set $B(\mathcal{H}_d)$ of operators acting on $\mathcal{H}_d$. The operator $R \in B(\mathcal{H}_d)$ is called positive semidefinite if $\langle \psi | R | \psi \rangle \geq 0$ for all vectors $| \psi \rangle \in \mathcal{H}_d$ (hereafter we use the Dirac notation). For positive semidefinite operators $R$ we write $R \geq 0$. We will denote the set of all positive semidefinite operators by $B(\mathcal{H}_d)^+$. The linear map $\Phi : B(\mathcal{H}_d)^+ \rightarrow B(\mathcal{H}_d)^+$ is called positive. By $\text{Id}_k$ denote the identity transformation on $B(\mathcal{H}_k)$. The linear map $\Phi$ is called $k$-positive if the map $\Phi \otimes \text{Id}_k$ is positive. 2-positive maps of the form $\Phi^{\otimes 2}$ are analyzed in the paper [32] and play an important role in the distillation problem [21]. A linear map $\Phi$ is called completely positive if it is $k$-positive for all $k \in \mathbb{N}$ (see, e.g. [33]). By $\mathcal{T}_d$ we denote the transposition map on $B(\mathcal{H}_d)$ associated with some orthonormal basis $\{|i\rangle \}_{i=1}^d$ in $\mathcal{H}_d$, $\mathcal{T}^d = \sum_{i,j} |i \rangle \langle j | \otimes |j \rangle \langle i |$. Maps of the form $\Phi \circ \mathcal{T}$, where $\Phi$ is completely positive, are called completely co-positive. Duality relations between cones of different maps are discussed, e.g. in [34].

A linear map $\Phi : B(\mathcal{H}_d) \rightarrow B(\mathcal{H}_d)$ is called $n$-tensor-stable positive if the map $\Phi^{\otimes n}$ is positive [31]. Obviously, if $m > n$, then the set of $n$-tensor-stable positive maps comprises the set of $m$-tensor-stable positive maps (nested structure). A linear map $\Phi : B(\mathcal{H}_d) \rightarrow B(\mathcal{H}_d)$ is called tensor-stable positive (or tensor product positive) if it is $n$-tensor-stable positive for all $n \in \mathbb{N}$ [31, 35]. Completely positive and completely co-positive maps $\Phi$ are trivial tensor-stable positive maps [31].

In subsequent sections, we exploit some properties of maps with regard to their action on entangled states. Quantum states are described by density operators, i.e. positive...
semidefinite operators \( \varrho \in \mathcal{B}(\mathcal{H}_d)^+ \) with unit trace, \( \text{tr}_\varrho = \sum_{i=1}^d \langle i | \varrho | i \rangle = 1 \). A positive semidefinite operator \( R \in (\mathcal{B}(\mathcal{H}_d) \otimes \mathcal{B}(\mathcal{H}_d))^+ \) is called separable [1] if it can be represented in the form 
\[
\varrho = \sum_k R_k^{(1)} \otimes R_k^{(2)}, \quad \text{where} \quad R_k^{(1)} \in \mathcal{B}(\mathcal{H}_d)^+ \quad \text{and} \quad R_k^{(2)} \in \mathcal{B}(\mathcal{H}_d)^+. 
\]
otherwise \( R \) is called entangled. Denote the cone of separable operators by \( \mathcal{S}(\mathcal{H}_d \otimes \mathcal{H}_d) \). We will refer to completely positive maps \( \Phi : \mathcal{B}(\mathcal{H}_d) \mapsto \mathcal{B}(\mathcal{H}_d) \) of the form \( \Phi = \sum \varrho \otimes \varrho \), where \( \varrho = \sum |i\rangle \otimes |i\rangle \) is a maximally entangled state, \( \varrho \) is a maximally entangled state, via the so-called Choi–Jamiołkowski isomorphism [46–48] reviewed in [49, 50]:

\[
\Omega_\varrho = (\varrho \otimes I_d) |\psi_\varrho \rangle \langle \psi_\varrho |, 
\]

\[
\Phi[X] = d \text{tr}[\Omega_\varrho (I \otimes X^T)], 
\]

where \( |\psi_\varrho \rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle \) is a maximally entangled state, \( I \) is the identity operator on \( \mathcal{H}_d \), \( \text{tr}_\varrho[Y] = \sum_{i=1}^d (I \otimes (i)) Y (I \otimes |i\rangle \langle i|) \) denotes the partial trace operation for operators \( Y \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \).

Let us remind the known properties of Choi operator:

1. \( \Phi \) is positive if and only if \( \Omega_\varrho \) is block-positive, i.e. \( \langle \varphi | \otimes \langle \chi | \Omega_\varrho |\varphi \rangle \otimes |\chi\rangle \geq 0 \) for all \( |\varphi\rangle, |\chi\rangle \in \mathcal{H}_d \) [47];

2. \( \Phi \) is completely positive (quantum operation) if and only if \( \Omega_\varrho \geq 0 \) [48];

3. \( \Phi \) is entanglement breaking if and only if \( \Omega_\varrho \) is separable (see, e.g. [40]);

4. \( \Phi \) is positive entanglement annihilating if and only if \( \text{tr}[\Omega_\varrho \xi_{1|1} \otimes R_{2|2}] \geq 0 \) for all \( R \in (\mathcal{B}(\mathcal{H}_d) \otimes \mathcal{B}(\mathcal{H}_d))^+ \) and all block-positive operators \( \xi_{1|2} \in \mathcal{B}(\mathcal{H}_d) \otimes \mathcal{B}(\mathcal{H}_d) \) [43].

The general problem addressed in this paper is to determine under which conditions a tensor product of two linear maps \( \Phi_1 \otimes \Phi_2 \) of the form \( \Phi_1 : \mathcal{B}(\mathcal{H}_d) \mapsto \mathcal{B}(\mathcal{H}_d) \) and \( \Phi_2 : \mathcal{B}(\mathcal{H}_d) \mapsto \mathcal{B}(\mathcal{H}_d) \) is a positive map. Acting on a factorized positive operator \( R_1 \otimes R_2 \geq 0 \), it is not hard to see that the positivity of maps \( \Phi_1 \) and \( \Phi_2 \) is a necessary condition. This condition, however, is not sufficient in general as \( (\mathcal{B}(\mathcal{H}_d))^+ \otimes (\mathcal{B}(\mathcal{H}_d))^+ \subsetneq (\mathcal{B}(\mathcal{H}_d) \otimes \mathcal{B}(\mathcal{H}_d))^+ \). Characterization of the cone \( (\mathcal{B}(\mathcal{H}_d))^+ \otimes (\mathcal{B}(\mathcal{H}_d))^+ \) is given in [51]. For instance, the maps \( \Phi_1 = I_d \) and \( \Phi_2 = T \) are both positive, but the map \( \Phi_1 \otimes \Phi_2 = I_d \otimes T \) is not positive. An apparent sufficient condition for positivity of the map \( \Phi_1 \otimes \Phi_2 \) is \( \{ \Phi_1 \otimes \Phi_2 \text{ is completely positive or completely co-positive} \} \), which takes place if \( \Phi_1 \) and \( \Phi_2 \) are both completely positive, or if \( \Phi_1 \) and \( \Phi_2 \) are both completely co-positive.

Generalization of the problem to a number of maps \( \Phi_1, \ldots, \Phi_n \) is to determine when the map \( \otimes_{k=1}^n \Phi_k \) is positive. Setting all the maps \( \Phi_k = 1 \) we get the problem of characterizing \( n \)-tensor-stable positive maps posed in [31].

We restrict our analysis to the case of linear qubit maps \( \Phi : \mathcal{B}(\mathcal{H}_2) \mapsto \mathcal{B}(\mathcal{H}_2) \). It was shown in [31] that all tensor-stable positive qubit maps are trivial (completely positive or completely co-positive). However, \( n \)-tensor-stable positive qubit maps for a fixed \( n \) are not necessarily trivially and their characterization is still missing, so we partially fill this gap in the present paper. Also, we provide a full characterization for the cases \( n = 2 \) and \( n = 3 \). First, we obtain results for unital maps, i.e. such linear maps \( \Phi \) that \( \Phi[I] = I \). Then, we extend these results to the case of non-unital maps.

Denote the concatenation of two maps \( \Phi \) and \( \Lambda \) by \( \Phi \circ \Lambda \), i.e. \( (\Phi \circ \Lambda)[X] = \Phi [\Lambda[X]] \).
Proposition 1. Suppose a map $\Phi_1 \otimes \Phi_2$ is positive entanglement annihilating and $\mathcal{P}$ is positive, then the maps $\Phi_1 \otimes (\mathcal{P} \circ \Phi_2)$ and $(\mathcal{P} \circ \Phi_1) \otimes \Phi_2$ are positive.

Proof. By definition of positive entanglement annihilating map, for any positive semidefinite operator $R$ we have: $(\Phi_1 \otimes \Phi_2)[R] = \sum_k R^{(1)}_k \otimes R^{(2)}_k \geq 0$, where $R^{(1)}_k \geq 0$ and $R^{(2)}_k \geq 0$. Since $\mathcal{P}[R^{(2)}_k] \geq 0$, the operator $(\Phi_1 \otimes (\mathcal{P} \circ \Phi_2))[R] = \sum_k R^{(1)}_k \otimes \mathcal{P}[R^{(2)}_k] \geq 0$ for all $R \geq 0$. Similarly, $((\mathcal{P} \circ \Phi_1) \otimes \Phi_2)[R] \geq 0$ for all $R \geq 0$. □

Proposition 1 enables one to use known criteria for entanglement-annihilating maps [43, 52] to find corresponding criteria for positive maps. Particular results of that kind are found in section 4.

Proposition 2. If $\Phi_1$ is entanglement breaking and $\Phi_2$ is positive, then the map $\Phi_1 \otimes \Phi_2$ is positive.

Proof. Since $\Phi_1$ is entanglement breaking, the operator $(\Phi_1 \otimes \text{Id})[R] = \sum_k R^{(1)}_k \otimes R^{(2)}_k$ is separable for any positive semidefinite $R$. Then, $(\Phi_1 \otimes \Phi_2)[R] = \sum_k R^{(1)}_k \otimes \Phi_2[R^{(2)}_k] \geq 0$ in view of positivity of $\Phi_2$. □

3. Depolarizing qubit maps

To illustrate the problem of positivity of tensor product maps, let us consider an exactly solvable case of depolarizing qubit maps. The action of a depolarizing qubit map $D_q$ is defined as follows:

$$D_q[X] = qX + (1 - q)\text{tr}[X]I/2,$$

The map $D_q$ is known to be positive if $q \in [-1, 1]$ and completely positive if $q \in [-1/3, 1]$ (see, e.g. [53, 54]). In what follows, we analyze when the two-qubit map $D_{q_1} \otimes D_{q_2}$ is positive. Entanglement-annihilating properties of the map $D_{q_1} \otimes D_{q_2}$ and their generalizations (acting in higher dimensions) are considered in papers [43, 55, 56].

Due to the convex structure of positive operators, if $(D_{q_1} \otimes D_{q_2})[\langle \psi \rangle \langle \psi \rangle] \geq 0$ for all $|\psi \rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$, then the map $D_{q_1} \otimes D_{q_2}$ is positive. Since the norm of a vector $|\psi \rangle$ is not relevant for the analysis of positivity, let us consider pure input states $\omega = |\psi \rangle \langle \psi |$ with $\langle \psi |\psi \rangle = 1$. We use the Schmidt decomposition $|\psi \rangle = \sqrt{p} |\phi \rangle \otimes |\chi \rangle + \sqrt{1 - p} |\phi_\perp \rangle \otimes |\chi_\perp \rangle$, where $\{ |\phi \rangle, |\phi_\perp \rangle \}$ and $\{ |\chi \rangle, |\chi_\perp \rangle \}$ are suitable orthonormal bases in Hilbert spaces of the first and second qubits, respectively, and $p$ and $p_\perp$ are real non-negative numbers such that $p + p_\perp = 1$.

Action of the two-qubit map $D_{q_1} \otimes D_{q_2}$ on $\omega$ yields

$$\omega_{\text{out}} = (D_{q_1} \otimes D_{q_2})[\omega] = q_1 q_2 \omega + \frac{1}{2} (1 - q_1) q_2 J \otimes \omega_2$$

$$+ \frac{1}{2} q_2 (1 - q_2) \omega_1 \otimes I + \frac{1}{4} (1 - q_1) (1 - q_2) I \otimes I,$$

with the reduced states $\omega_1 = p |\phi \rangle \langle \phi | + p_\perp |\phi_\perp \rangle \langle \phi_\perp |$ and $\omega_2 = p |\chi \rangle \langle \chi | + p_\perp |\chi_\perp \rangle \langle \chi_\perp |$. The condition $\omega_{\text{out}} \geq 0$ reduces to

$$\begin{pmatrix}
A_+ + B_+ & 0 & 0 & C \\
0 & A_- + B_- & 0 & 0 \\
0 & 0 & A_- - B_- & 0 \\
0 & 0 & 0 & A_+ - B_+
\end{pmatrix} \geq 0,$$

where

$$A_\pm = p |\phi \rangle \langle \phi | \pm p_\perp |\phi_\perp \rangle \langle \phi_\perp |$$

and

$$B_\pm = p |\chi \rangle \langle \chi | \pm p_\perp |\chi_\perp \rangle \langle \chi_\perp |.$$
where $A_k = 1 \pm q_4 q_2$, $B_k = (2p - 1)(q_1 \pm q_2)$, and $C = 4\sqrt{p(1-p)}$ $q_4 q_2$. After some algebra, we obtain that the condition (5) holds true for all $0 \leq p \leq 1$ if

$$q_4 q_2 \geq -\frac{1}{3}, \quad -1 \leq q_1 \leq 1, \quad -1 \leq q_2 \leq 1.$$  

(6)

Inequalities (6) define the conditions under which the two-qubit map $D_q \otimes D_q$, is positive. We depict the corresponding area of parameters $q_1$ and $q_2$ in figure 1. Note that $D_q \otimes D_q$ is completely positive if $-1 \leq q_1, q_2 \leq 1$. Analogously, $D_q \otimes D_q$ is completely co-positive if $-1 \leq q_1, q_2 \leq \frac{1}{2}$.

Let us demonstrate the use of proposition 1. Consider the reduction map $R[X] = \text{tr}[X]I - X$, which is known to be positive in qubit case [20]. The concatenation $R \circ D_q = D_q$, i.e. the depolarizing map with parameter $-q$. The map $D_q \otimes D_q$ is known to be positive entanglement annihilating if $q_j(-q_2) \leq \frac{1}{2}$ and $-1 \leq q_1, q_2 \leq 1$. According to proposition 1, these relations are sufficient for positivity of the map $D_q \otimes (R \circ D_q) = D_q \otimes D_q$. In this particular case, these relations turn out to be the same as the necessary and sufficient conditions (6).

4. Unital qubit maps

A unital qubit map $\Phi$ satisfies $\Phi[I] = I$ and can be expressed in the form [53, 54]

$$\Phi[X] = W(\Theta[VXV^\dagger])W^\dagger,$$

(7)

where $V$ and $W$ are appropriate unitary operators such that the map $\Theta$ has the Pauli form, i.e.

$$\Theta[X] = \frac{1}{2} \sum_{j=0}^3 \lambda_j \text{tr}[\sigma_j X] = \sum_{j=0}^3 q_j \sigma_j \sigma_j,$$

(8)

where $\sigma_0 = I$ and $\{\sigma_j\}_{j=1}^3$ is a conventional set of Pauli operators. Thus, up to a unitary pre-processing ($V \cdot V^\dagger$) and postprocessing ($W \cdot W^\dagger$) the unital map $\Phi$ reduces to the map $\Theta$. From equation (7) it is not hard to see that the two-qubit unital map $\Phi_1 \otimes \Phi_2$ is positive if and only if $\Theta_1 \otimes \Theta_2$ is positive. In this section, we will consider properties of maps $\Theta_1 \otimes \Theta_2$.

The relation between parameters $\{\lambda_j\}$ and $\{q_j\}$ in formula (8) is given by

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix},$$

(9)

i.e. $q = \frac{1}{2}H\lambda$, where $q = (q_0, q_1, q_2, q_3)^\top$, $H$ is the $4 \times 4$ Hadamard matrix, and $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)^\top$.

For hermiticity-preserving maps $\Theta$ the parameters $\{\lambda_j\}$ and $\{q_j\}$ are real. Positive maps correspond to parameters $\lambda_0 \geq 0$, $-\lambda_0 \leq \lambda_1, \lambda_2, \lambda_3 \leq \lambda_0$. Completely positive maps correspond to $q_j \geq 0$, $j = 0, \ldots, 3$. Trace preserving maps are those with $\lambda_0 = 1$. Completely positive trace preserving unital qubit maps are called unital qubit channels and are essentially the random unitary channels [57].

**Proposition 3.** A unital two-qubit map $\Theta_1 \otimes \Theta_2$ is positive if $\Theta_1^2$ and $\Theta_2^2$ are both entanglement breaking.
**Proof.** Consider a map \( \Upsilon = \Upsilon \circ R \), where \( R \) is the qubit reduction map. Then \( \Upsilon^2 = \Upsilon^2 \) is the entanglement breaking map. If \( \Upsilon_1^2 \) and \( \Upsilon_2^2 \) are both entanglement breaking, then the map \( \Upsilon_1 \otimes \Upsilon_2 \) is positive entanglement annihilating according to the proposition 1 of [52]. By proposition 1 of the present paper, the map \( \Upsilon_1 \otimes (\Upsilon \circ \Upsilon_2) = \Upsilon_1 \otimes \Upsilon_2 \) is positive. □

The 'power' of proposition 3 can be illustrated by the example of the local two-qubit depolarizing map \( D_{q_1} \otimes D_{q_2} \). Since \( D_{q_1}^2 = D_{q_1} \), the maps \( D_{q_1}^2 \) and \( D_{q_2}^2 \) are both entanglement breaking if \( q_1^2, q_2^2 \leq \frac{1}{3} \) [39], i.e. \( -\frac{1}{\sqrt{3}} \leq q_1, q_2 \leq \frac{1}{\sqrt{3}} \). Corresponding region of parameters is depicted in figure 1. Clearly, proposition 3 provides only a sufficient but not a necessary condition for positivity.

Being applied to the unital map \( \Upsilon \otimes \Upsilon \), proposition 3 guarantees that the map \( \Upsilon \otimes \Upsilon \) is positive if \( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq \lambda_0^2 \) (\( \Upsilon^2 \) is entanglement breaking). The corresponding ball (for \( \lambda_0 = 1 \)) is depicted in figure 2. Usual powers of linear maps (self-concatenations \( \Upsilon^n \)) also find applications in quantum information theory, for instance, in noise quantification [58, 59].

**Remark 1.** Comparison of formulas (7) and (8) clarifies that, in general, \( \Phi^2 = \Upsilon^2 \). This fact is analogous to filtering \( \Upsilon \circ U \circ \Upsilon \) [58–60], where the intermediate (unitary) map \( U \) is used to prevent \( \Upsilon^2 \) from becoming entanglement breaking.

**Remark 2.** In contrast to the entanglement annihilating property, which states that \( \Upsilon_1 \otimes \Upsilon_2 \) is positive entanglement annihilating if both \( \Upsilon_1^2 \) and \( \Upsilon_2^2 \) are entanglement breaking, complete positivity of the maps \( \Upsilon_1^2 \) and \( \Upsilon_2^2 \) does not imply that \( \Upsilon_1 \otimes \Upsilon_2 \) is positive. Counterexample is the case \( \Upsilon_1 = \text{Id} \) and \( \Upsilon_2 = \Upsilon \).

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**Figure 1.** Shaded area is the region of parameters \( q_1 \) and \( q_2 \), where the map \( D_{q_1} \otimes D_{q_2} \) is positive. Solid line regions correspond to completely positive (CP) and completely co-positive (CcP) maps \( D_{q_1} \otimes D_{q_2} \). Proposition 3 detects positivity of the map \( D_{q_1} \otimes D_{q_2} \) inside the dashed line region.
In this section, we analyze positivity of two-qubit unital maps $\Phi^\otimes_2$. By equation (7), $\Phi$ is 2-tensor-stable positive if and only if $\Upsilon$ is 2-tensor-stable positive. Without loss of generality one can impose the trace-preserving condition, $\lambda_0 = 1$, then the remaining three parameters $\{\lambda_j\}_{j=1}^3$ in formula (8) are scaling coefficients of the Bloch ball axes. Thus, the map $\Upsilon$ is given by a point in the Cartesian coordinate system $(\lambda_1, \lambda_2, \lambda_3)$ and can be readily visualized.

**Proposition 4.** $\Upsilon$ is 2-tensor-stable positive if and only if $\Upsilon^2$ is completely positive.

**Proof.** Necessity. Let us prove that if $\Upsilon \otimes \Upsilon$ is positive, then $\Upsilon^2$ is completely positive. Suppose $(\Upsilon \otimes \Upsilon)(R) \geq 0$ for all $R \geq 0$. Let $R$ be equal to $|\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It is not hard to see that

$$
(\Upsilon \otimes \Upsilon)[|\psi_+\rangle\langle\psi_+|] = \frac{1}{4} \begin{pmatrix}
1 + \lambda_3^2 & 0 & 0 & \lambda_1^2 + \lambda_2^2 \\
0 & 1 - \lambda_3^2 & \lambda_1^2 - \lambda_2^2 & 0 \\
0 & \lambda_1^2 - \lambda_2^2 & 1 - \lambda_3^2 & 0 \\
\lambda_1^2 + \lambda_2^2 & 0 & 0 & 1 + \lambda_3^2
\end{pmatrix} = \Omega R^2,
$$

where $R = |\psi_+\rangle\langle\psi_+|$ and $\Omega = |\psi_+\rangle\langle\psi_+|$. 

**Figure 2.** Regions of parameters $\lambda_1, \lambda_2, \lambda_3$ determining particular properties of the unital qubit map $\Upsilon$ defined by equation (8). Tetrahedrons represent trivial tensor-stable positive maps (yellow one corresponds to completely positive maps, blue one corresponds to completely co-positive maps). Sphere corresponds to positive entanglement annihilating maps $\Upsilon \otimes \Upsilon$ (see proposition 3).

5. **2-tensor-stable positive unital qubit maps**

In this section, we analyze positivity of two-qubit unital maps $\Phi^\otimes_2$. By equation (7), $\Phi$ is 2-tensor-stable positive if and only if $\Upsilon$ is 2-tensor-stable positive. Without loss of generality one can impose the trace-preserving condition, $\lambda_0 = 1$, then the remaining three parameters $\{\lambda_j\}_{j=1}^3$ in formula (8) are scaling coefficients of the Bloch ball axes. Thus, the map $\Upsilon$ is given by a point in the Cartesian coordinate system $(\lambda_1, \lambda_2, \lambda_3)$ and can be readily visualized.

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$$
(\Upsilon \otimes \Upsilon)[|\psi_+\rangle\langle\psi_+|] = \frac{1}{4} \begin{pmatrix}
1 + \lambda_3^2 & 0 & 0 & \lambda_1^2 + \lambda_2^2 \\
0 & 1 - \lambda_3^2 & \lambda_1^2 - \lambda_2^2 & 0 \\
0 & \lambda_1^2 - \lambda_2^2 & 1 - \lambda_3^2 & 0 \\
\lambda_1^2 + \lambda_2^2 & 0 & 0 & 1 + \lambda_3^2
\end{pmatrix} = \Omega R^2,
$$

where $R = |\psi_+\rangle\langle\psi_+|$ and $\Omega = |\psi_+\rangle\langle\psi_+|$. 

**Figure 2.** Regions of parameters $\lambda_1, \lambda_2, \lambda_3$ determining particular properties of the unital qubit map $\Upsilon$ defined by equation (8). Tetrahedrons represent trivial tensor-stable positive maps (yellow one corresponds to completely positive maps, blue one corresponds to completely co-positive maps). Sphere corresponds to positive entanglement annihilating maps $\Upsilon \otimes \Upsilon$ (see proposition 3).
i.e. $\Omega \geq 0$ and the map $\Upsilon^2$ is completely positive.

Taking into account the explicit form of the Choi matrix (10), we get

\[
\{ \Upsilon^2 \text{ is CP} \} \iff \begin{cases} 
1 + \lambda_1^2 \geq \lambda_2^2 + \lambda_3^2, \\
1 + \lambda_2^2 \geq \lambda_1^2 + \lambda_3^2, \\
1 + \lambda_3^2 \geq \lambda_1^2 + \lambda_2^2, 
\end{cases}
\]

(11)
or, concisely, $1 \pm \lambda_1^2 \geq |\lambda_2^2 \pm \lambda_3^2|$. Each of inequalities (11) defines an interior of the one-sheet hyperboloid in the space of parameters $(\lambda_1, \lambda_2, \lambda_3)$. Intersection of these three hyperboloids is depicted in figure 3. Gorges (throats) of those hyperboloids are exactly three mutually perpendicular great circles (orthodromes) of a unit sphere.

Sufficiency. Suppose $\Upsilon^2$ is completely positive and $\lambda_0 = 1$, then parameters $(\lambda_1, \lambda_2, \lambda_3)$ satisfy the inequalities (11).

Let us use the alternative description of the map $\Upsilon$, namely, $\Upsilon[X] = \sum_{i=0}^{3} q_{i}\sigma_i \otimes \sigma_i X \otimes \sigma_i$. Then $(\Upsilon \otimes \Upsilon)[X] = \sum_{i,j=0}^{3} q_{ij}\sigma_i \otimes \sigma_j X \otimes \sigma_j \otimes \sigma_j$. To demonstrate positivity of the map $\Upsilon \otimes \Upsilon$, it suffices to show that $\langle \varphi | (\Upsilon \otimes \Upsilon)[|\psi\rangle \langle \psi|] | \varphi \rangle \geq 0$ for all $|\psi\rangle, |\varphi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$. We have
\[ \langle \varphi | (\Upsilon \otimes \Upsilon) | \psi \rangle \langle \psi | (\psi \otimes \psi) | \varphi \rangle = \sum_{i,j=0}^{3} q_i q_j \left| \langle \varphi | \sigma_i \otimes \sigma_j | \psi \rangle \right|^2 \]
\[ = q^2 A q = \frac{1}{4} \lambda H^A H^A \lambda, \quad (12) \]

where the matrix elements
\[ A_{ij} = \frac{1}{2} \left( \left| \langle \varphi | \sigma_i \otimes \sigma_j | \psi \rangle \right|^2 + \left| \langle \varphi | \sigma_j \otimes \sigma_i | \psi \rangle \right|^2 \right) \geq 0. \quad (13) \]

Thus, the symmetric matrix \( A \) has non-negative entries only and, according to the Perron–Frobenius theorem, the absolute value of its minimal eigenvalue, \( |\lambda_-| \), cannot exceed its maximal eigenvalue, \( \lambda_+ > 0 \) (see, e.g. [61]). Since the Hadamard matrix \( H \) is unitary, the eigenvalues of matrices \( H^A H \) and \( A \) coincide. It means that the absolute value \( |\lambda_-| \) of any negative coefficient \( \lambda \) in the diagonal representation of the quadratic form \( \lambda H^A H \lambda \) is less or equal than the maximal positive coefficient. Consequently, the principal curvatures \( k_1, k_2 \) of a quadric surface \( \lambda H^A H \lambda = 0 \) satisfy \(-1 \leq k_1, k_2 \leq 1\). On the other hand, each of equalities \( (11) \) defines a surface with boundary principal curvatures \( \min k_1 = -1, \max k_2 = 1 \). Thus, no quadric \( \lambda H^A H \lambda = 0 \) can intersect the interior region of all inequalities \( (11) \) without intersecting the regions of completely positive maps or completely co-positive maps (two tetrahedrons in figure 2). Roughly speaking, all quadric surfaces \( \lambda H^A H \lambda = 0 \) are more ‘flat’ than those of equation \( (11) \). Therefore, the interior region of all inequalities \( (11) \) is the interior set of all figures \( \lambda H^A H \lambda \geq 0 \), which implies \{ \Upsilon^2 \text{ is CP} \} \Rightarrow \{ \Upsilon \otimes \Upsilon \text{ is positive} \}. \]

Inequalities \( (11) \) specify a non-convex geometrical figure in the space of parameters \( (\lambda_1, \lambda_2, \lambda_3) \). However, any interior point of that figure corresponds to a convex sum of some boundary map \( \Upsilon \) and the completely depolarizing map \( D_0 \) and corresponds to a 2-tensor-scalar positive map since the boundary map does so. Parameterizing the surface of hyperboloids, one can also find numerical evidence of proposition 4.

Any one-sheet hyperboloid is doubly ruled, i.e. it has two distinct generatrices that pass trough every point. Without loss of generality, let us consider a particular hyperboloid fragment with vertices \((1, 1, 1), (0, 0, 1), (1, -1, 1), \) and \((1, 0, 0)\) in the space \( (\lambda_1, \lambda_2, \lambda_3) \). These vertices correspond to the maps \( \text{Id}, Z, \top, \) and \( X \), respectively. The first family of generatrices is given by straight lines passing through points \((x, x, 1)\) and \((1, -f(x), f(x))\), where \( f(x) = (1 - x)^{1/2} (1 + x) \) and \( x \in (0, 1) \). The second family of generatrices is formed by straight lines passing through points \((y, -y, 1)\) and \((1, f(y), f(y))\), \( y \in (0, 1) \). Any point inside the involved hyperboloid fragment is defined by a pair of parameters \((x, y) \in (0, 1)^2 \) and corresponds to the following map:
\[ \Upsilon = \frac{x(1-y)\text{Id} + y(1-x)\top + 2xyX + (1-x)(1-y)Z}{1+xy}, \quad (14) \]
whose parameters read
\[ \lambda_1 = \frac{x + y}{1 + xy}, \quad \lambda_2 = \frac{x - y}{1 + xy}, \quad \lambda_3 = \frac{1 - xy}{1 + xy}. \quad (15) \]

Then one can check block-positivity of the Choi matrix of the map \( \Upsilon \otimes \Upsilon \), i.e. to validate inequality
\[ \langle \varphi^{\Lambda A^B} \otimes \langle \chi^{\Lambda A^B} | (\Omega Y^\Lambda Y^A \otimes \Omega Y^B Y^B^\top) | \varphi^{\Lambda A^B} \rangle \otimes | \chi^{A^B} \rangle \geq 0 \quad (16) \]
Numerically for all two qubit states $|\varphi^{AB}\rangle$ and $|\chi^{AB}\rangle$ and all $0 \leq x, y \leq 1$.

Alternatively, positivity of the operator $\varrho = (\Upsilon \otimes \Upsilon)[|\psi\rangle\langle\psi|]$ is guaranteed by the requirement that all coefficients $s_i, i = 1, \ldots, 4$ of the characteristic polynomial

$$\det(\lambda I - \varrho) = \lambda^N - s_1\lambda^{N-1} + s_2\lambda^{N-2} - \ldots + (-1)^N s_N$$

(17)

are non-negative, i.e. $\rho \geq 0 \Leftrightarrow s_k \geq 0$, $k = 1, \ldots, 4$ [62]. The coefficients $s_1 = \text{tr}[\varrho]$, $s_2 = \frac{1}{2}(s_1^2 - \text{tr}[\varrho]^2)$, and in general, iteratively, $s_k = \frac{1}{2}(s_{k-1}\text{tr}[\rho] - s_{k-2}\text{tr}[\rho^2] + \ldots + (-1)^{k-1}\text{tr}[\rho^{k-1}])$. Then one can numerically check positivity of matrices $(\Upsilon \otimes \Upsilon)[|\psi\rangle\langle\psi|]$ for all two-qubit states $|\psi\rangle$ and parameters $0 \leq x, y \leq 1$.

### 6. Decomposability

Following the results of [63, 64], we will refer to the map of the form $\Phi_1 + \Phi_2$, where $\Phi_1$ is completely positive and $\Phi_2$ is completely co-positive, as decomposable.

All positive qubit maps $\Phi : B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_2)$ are known to be decomposable [63]. Decomposability of extremal positive unital maps on $M_2$ is analyzed in [65]. However, even if $\Phi$ is decomposable, it does not imply that $\Phi \otimes \Phi$ is decomposable as it contains terms $\Phi_1 \otimes \Phi_2$ and $\Phi_2 \otimes \Phi_1$ which are not necessarily positive. Moreover, there exist examples of indecomposable maps $B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_2)$ [66]. This means that the decomposability of positive tensor powers $\Phi \otimes \Phi$ (as well as the more general property of $k$-decomposability [67]) is still an open problem even for qubit maps $\Phi$. In what follows, we make some steps toward understanding of this problem and consider examples of non-trivial 2-tensor-stable unital maps $\Upsilon$ such that $\Upsilon \otimes \Upsilon$ is decomposable.

**Example 1.** Positive map $\Upsilon \otimes \Upsilon$ with parameters $\lambda_1 = \frac{1}{\sqrt{2}}$, $\lambda_2 = 0$, $\lambda_3 = \frac{1}{\sqrt{2}}$ is decomposable.

In fact, it is not hard to see that

$$\Upsilon \otimes \Upsilon = \frac{1}{2} F \circ (\text{Id} \otimes \text{Id} + \Upsilon \otimes \Upsilon),$$

(18)

where $F$ is a two-qubit map of the form $F[X] = \frac{1}{2} \sum_{i,j=0}^3 \lambda_i \text{tr}[\sigma_i \otimes \sigma_j X] \sigma_i \otimes \sigma_j$ with $\lambda_{00} = 1$, $\lambda_{01} = \lambda_{02} = \lambda_{03} = \frac{1}{\sqrt{2}}$, $\lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{22} = \frac{1}{2}$, $\lambda_{21} = \lambda_{31} = \lambda_{33} = \frac{1}{3}$, and $\lambda_{22} = 0$. Eigenvalues of the Choi matrix $\Omega_F$ are all non-negative, consequently, $F$ is completely positive, $F \circ (\Upsilon \otimes \Upsilon)$ is completely co-positive, and $\Upsilon \otimes \Upsilon$ is decomposable.

**Example 2.** Consider the one-parametric family of maps $(\mu \Upsilon_1 + (1 - \mu) \Upsilon_2)^{\otimes 2}$, where $\Upsilon_1$ is given by parameters $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$, $\Upsilon_2$ is given by parameters $\lambda_0 = \lambda_3 = 1$, $\lambda_1 = -\lambda_2 = \frac{1}{3}$. Let us show that $(\mu \Upsilon_1 + (1 - \mu) \Upsilon_2)^{\otimes 2}$ is positive and decomposable for all $0 \leq \mu \leq 1$. Note that $\mu \Upsilon_1 + (1 - \mu) \Upsilon_2$ is non-trivial 2-tensor-stable positive for $0 < \mu < \frac{3}{11}$.

The map $\Upsilon_1^{\otimes 2}$ is completely positive, and the map $\Upsilon_2^{\otimes 2}$ is completely co-positive. Let us note that

$$\Upsilon_1 \otimes \Upsilon_2 = F \circ \left(\frac{3}{4} G_1 \otimes \text{Id} + \frac{1}{4} \Upsilon \otimes G_2\right).$$

(19)
where $G_1 = Z(\lambda_0 = \lambda_3 = 1, \lambda_1 = \lambda_2 = 0)$, $G_2 = D_{1/3}$ (depolarizing map with parameter 1/3), and $F$ is a two-qubit map $F[X] = \frac{1}{2} \sum_{j=0}^{3} \lambda_j \text{tr}[\sigma_j \otimes \sigma X] \sigma_j \otimes \sigma$ with $\lambda_0 = \lambda_3 = (4 + \delta_{00})/5$, $\lambda_2 = (2 - \delta_{00})/5$. Since the eigenvalues of the Choi matrix $\Omega$ are all non-negative, $F \circ (G_1 \otimes \text{Id})$ is completely positive as the concatenation of completely positive maps, and $F \circ (\mathbb{T} \otimes G_2)$ is completely co-positive as the concatenation of completely positive and completely co-positive maps. (Note that $G_2$ is entanglement breaking and $G_2 \otimes \mathbb{T}$ is completely co-positive.) Thus, $T_1 \otimes T_2$ is decomposable. Analogously, $T_2 \otimes T_1$ is decomposable. Finally, $(\mu \mathbb{T}_1 + (1 - \mu)\mathbb{T})^{\otimes 2}$ is decomposable as the convex sum of decomposable maps.

The examples above stimulate us to make a conjecture that all positive unital two-qubit maps of the form $\mathbb{T} \otimes \mathbb{T}$ are decomposable.

### 7. Non-unital qubit maps

Similarly to equation (7), an interior map of the cone of positive non-unital qubit maps $\Phi : (B(\mathcal{H}_2))^2 \rightarrow (B(\mathcal{H}_2))^2$ can be represented in the form of the following concatenation [68, 69]:

$$\Phi[X] = B(\mathcal{T}[AXA])B^t,$$

(20)

where $\mathcal{T}$ is given by equation (8) and $A, B \in B(\mathcal{H}_2)$ are positive-definite operators. As $A$ and $B$ are non-degenerate, the condition $\langle \varphi | (\Phi_1 \otimes \Phi_2) | \psi \rangle | \varphi \rangle \geq 0$ holds for all $| \psi \rangle, | \varphi \rangle \in \mathcal{H}_4$ if and only if $\langle \tilde{\varphi} | (\mathcal{T}_1 \otimes \mathcal{T}_2) | \tilde{\psi} \rangle | \tilde{\varphi} \rangle \geq 0$ holds for all $| \tilde{\psi} \rangle, | \tilde{\varphi} \rangle \in \mathcal{H}_4$, since $| \tilde{\psi} \rangle = A_1 \otimes A_2 | \psi \rangle$ and $| \tilde{\varphi} \rangle = B_1^\dagger \otimes B_2^\dagger | \varphi \rangle$. Thus, the positivity of a tensor product of non-unital maps $\Phi_1 \otimes \Phi_2$ is equivalent to the positivity of the tensor product of corresponding unital maps $\mathcal{T}_1 \otimes \mathcal{T}_2$.

A qubit map $\Phi$ can be expressed in an appropriate basis by its matrix form $E_{ij} = \frac{1}{2} \text{tr} [\sigma_j \Phi[\sigma_i]]$ as follows [53, 54]:

$$E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\lambda_1 & \lambda_1 & 0 & 0 \\
\lambda_2 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}.$$  

(21)

To demonstrate the idea of reducing the problem to the case of unital maps, let us consider a four-parametric family of maps $\Phi$ whose matrix representation reads

$$E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}.$$  

(22)

Such a family comprises the description of extremal completely positive qubit maps [54].

**Proposition 5.** Let $\Phi$ be a map defined by the matrix representation (22) with $1 - |t| - |\lambda_0| > 0$ and

$$A^{-1} = \begin{pmatrix}
a + b & 0 \\
0 & a + b
\end{pmatrix}, \quad B^{-1} = \frac{1}{2} \begin{pmatrix}
b & 0 \\
0 & b
\end{pmatrix},$$

(23)

$$a_{\pm} = \sqrt{1 + t - \lambda_3}, \quad b_{\pm} = \pm \frac{1}{2}(1 + t)^2 - \lambda_3^2,$$

(24)
then the map \(\tilde{Y}[Y] = B^{-1}\Phi[A^{-1}Y(A^{-1})']|B^{-1}y]\) is proportional to a unital map and the corresponding coefficients in equation (8) equal

\[
\bar{\lambda}_0 = \frac{1}{2} \left[ (1 - \lambda_0^2 - r^2) \left( 1 + \lambda_0^2 - r^2 + \sqrt{(1 - t)^2 - \lambda_0^2} \left( 1 + r^2 - \lambda_0^2 \right) \right) \right],
\]

(25)

\[
\bar{\lambda}_1 = \lambda_1 \sqrt{\left( (1 - \lambda_0^2 - r^2) \left( 1 - (1 - t)^2 - \lambda_0^2 \right) \left( 1 + r^2 - \lambda_0^2 \right) \right)},
\]

(26)

\[
\bar{\lambda}_2 = \lambda_2 \sqrt{\left( (1 - \lambda_0^2 - r^2) \left( 1 - (1 - t)^2 - \lambda_0^2 \right) \left( 1 + r^2 - \lambda_0^2 \right) \right)},
\]

(27)

\[
\bar{\lambda}_3 = \frac{1}{2} \left[ (1 - \lambda_0^2 - r^2) \left( 1 + \lambda_0^2 - r^2 - \sqrt{(1 - t)^2 - \lambda_0^2} \left( 1 + r^2 - \lambda_0^2 \right) \right) \right].
\]

(28)

**Proof.** Since the concatenation of maps corresponds to the product of their matrix representations, a straightforward calculation yields the map \(\tilde{Y}\) which has the form (8). \(\square\)

Thus, the map \(\Phi\) given by equation (22) is positive when (i) \(|t| > ||\lambda||\) and \(k = 1, 2, 3\) or (ii) \(|t| = ||\lambda|| = 0\) and \(\lambda_1^2, \lambda_2^2 \leq 1 - |t|\). The latter one can readily be obtained by considering Bloch ball transformations. Fixing parameter \(t\), one can visualize conditions (i)–(ii) in the reference frame \((\lambda_1, \lambda_2, \lambda_3)\) (see figure 4).

**Remark 3.** The result of [69] is that the matrices \(A^{-1}\) and \(B^{-1}\) can be chosen positive-definite for a map \(\Phi\) from the interior of the cone of positive maps. In proposition 5, we have considered non-degenerate matrices \(A^{-1}\) and \(B^{-1}\).

It is not hard to see that in contrast to the case of unital maps, the condition \((\Phi \otimes \Phi)[|\psi_+\rangle\langle\psi_+|] \geq 0\), where \(|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\), is not a sufficient condition for positivity of a non-unital map \(\Phi \otimes \Phi\). In fact, direct calculation of eigenvalues of \((\Phi \otimes \Phi)[|\psi_+\rangle\langle\psi_+|]\) results in the following conditions:

\[
\begin{aligned}
1 - t^2 + \lambda_1^2 - \lambda_2^2 - 2t\lambda_3 - \lambda_3^2 &\geq 0, \\
1 - t^2 - \lambda_1^2 + \lambda_2^2 - 2t\lambda_3 - \lambda_3^2 &\geq 0, \\
1 - t^2 - \sqrt{4t^2 + (\lambda_1^2 + \lambda_2^2)^2} + \lambda_3^2 &\geq 0, \\
1 - t^2 + \sqrt{4t^2 + (\lambda_1^2 + \lambda_2^2)^2} + \lambda_3^2 &\geq 0.
\end{aligned}
\]

(29)

The area of parameters \(\lambda_0, \lambda_2, \lambda_3\) satisfying inequalities (29) for a fixed value of parameter \(t\) is shown in figure 5.

However, from equation (20) and proposition 4 it follows that a map \(\Phi\) from the interior of cone of positive maps is 2-tensor-stable positive if and only if its action on the pure state \(A^{-1} \otimes A^{-1} |\psi_+\rangle\) results in the positive-semidefinite operator. Consequently, such a map \(\Phi\) is 2-tensor-stable positive if and only if

\[
\bar{\lambda}_0^2 \pm \bar{\lambda}_1^2 \geq \bar{\lambda}_2^2 \pm \bar{\lambda}_3^2.
\]

(30)

The region of parameters \(\lambda_0, \lambda_2, \lambda_3\), where the map \(\Phi \otimes \Phi\) is positive for a fixed \(t\), is shown in figure 6. Although figure 6 looks like an intersection of regions depicted in figures 4 and 5, it is not.
Finally, one can proceed analogously to find necessary and sufficient conditions for 2-tensor-stable positivity of maps defined by the matrix representation (21).

8. 3-tensor-stable positive qubit maps

Let us proceed to higher-order tensor-stable positive maps, namely, a unital subclass of 3-tensor-stable positive qubit maps $\Phi$. Similarly to the results of section 5, the map $\Phi^{\otimes 3}$ is positive if and only if $\Upsilon^{\otimes 3}$ is positive, with the diagonal map $\Upsilon$ being parameterized by equation (8).

First, we analytically find necessary conditions for positivity of the map $\Upsilon^{\otimes 3}$.

**Proposition 6.** If the unital qubit map $\Upsilon$ is 3-tensor-stable positive, then the following 12 inequalities are satisfied:

$$1 - \lambda_1^3 - 3 \lambda_1 \lambda_2^2 + 3 \lambda_3^2 \geq 0, \quad (31)$$

$$1 + \lambda_1^3 + 3 \lambda_1 \lambda_2^2 + 3 \lambda_3^2 \geq 0, \quad (32)$$

where $(i, j, k)$ is a permutation of indices $(1, 2, 3)$, i.e. $i, j, k = 1, 2, 3$ and $i \neq j \neq k \neq i$.

**Proof.** Consider the three-qubit Greenberger–Horne–Zeilinger state [70, 71]

$$|\text{GHZ} \rangle = \sqrt{2} (|000 \rangle + |111 \rangle)$$

(33)
written in the basis, in which the map $\Upsilon$ has the form (8). Let us define permutations generated by the following matrices:

\[
\begin{align*}
\sigma_1 + \sigma_2 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}, \\
\sigma_1 + \sigma_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
\sigma_2 + \sigma_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 - i \\ 1 + i & 0 \end{pmatrix}.
\end{align*}
\] (34)

The physical meaning of unitary transformation $u_i \cdot u^*_i$ is the rotation of the Bloch ball such that the $i$th direction becomes inverted and the other two perpendicular directions ($j$th and $k$th) become interchanged. Denote $U_0 = I \otimes I \otimes I$ and $U_i = u_i \otimes u_i \otimes u_i$, $i = 1, 2, 3$. We generate the following transformations of the GHZ state:

\[
\rho_{ij} = U_i U_j \langle \text{GHZ} | (U_i U_j)^\dagger \rangle, \quad i, j = 0, 1, 2, 3.
\] (36)

If $\Upsilon$ is 3-tensor-stable positive, then

\[
(\Upsilon \otimes \Upsilon \otimes \Upsilon) [\rho_{ij}] \geq 0, \quad i, j = 0, 1, 2, 3,
\] (37)

which is a number of constraints on parameters $\lambda_0, \lambda_2, \lambda_3$. Intersection of these constraints results in 12 inequalities (31)–(32).
In view of complexity of inequalities (31)–(32), we have used numerical methods to analyze the block-positivity of the Choi operator $\Omega \otimes \Omega$ with respect to the cut $|123|123$. It turns out that $\Omega \otimes \Omega$ is block positive whenever parameters $\lambda_1, \lambda_2, \lambda_3$ satisfy (31)–(32). Therefore, there is a numerical evidence that proposition 6 provides not only a necessary but also a sufficient condition for positivity of the map $\Upsilon \otimes \Upsilon$.

The region of parameters $\lambda_1, \lambda_2, \lambda_3$ satisfying (31)–(32) is depicted in figure 7. One can readily see that 3-tensor-stable positive maps occupy a subset of 2-tensor-stable positive maps and contain the set of tensor-stable positive qubit maps, which is known to consist of trivial maps only (completely positive and completely co-positive ones) [31].

Using the relation (20) and proposition 6, the full characterization of non-unital 3-tensor-stable positive qubit maps follows straightforwardly.

9. $n$-tensor-stable positive qubit maps

A general positive qubit map $\Phi$ is $n$-tensor-stable positive if and only if the corresponding Pauli map $\Upsilon$ is $n$-tensor stable positive (specified by formula (7) for unital maps $\Phi$ and by formula (20) for non-unital maps $\Phi$).

**Proposition 7.** Suppose the Pauli qubit map $\Upsilon$ is $n$-tensor-stable positive, then

$$| (1 + \lambda_2)^p (1 - \lambda_2)^q + (1 - \lambda_2)^p (1 + \lambda_2)^q |$$

$$\geq | (\lambda_j + \lambda_k)^p (\lambda_j - \lambda_k)^q + (\pm 1)^p (\lambda_j - \lambda_k)^p (\lambda_j + \lambda_k)^q | ,$$

(38)
for all permutations \((i, j, k)\) of indices \((1, 2, 3)\) and all \(p, q \in \mathbb{Z}_+\) such that \(p + q = n\).

**Proof.** Consider a generalized GHZ state of \(n\) qubits, \(|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^\otimes n + |1\rangle^\otimes n)\). Note that \((\text{GHZ}_n)|\text{GHZ}_m\rangle = \frac{1}{2}[(\sigma_1\sigma_3)^\otimes n + \sigma_1^\otimes n + \sigma_3^\otimes n + (\sigma_1\sigma_3)^\otimes n]\), where \(\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)\). Since \(\Upsilon[\sigma_1\sigma_3] = \sigma_0 \pm \lambda_3\sigma_3\) and \(\Upsilon[\sigma_1] = \lambda_0\sigma_0 \pm i\lambda_2\sigma_2\), the operator
\[
\Upsilon^\otimes n|\text{GHZ}_n\rangle|\text{GHZ}_m\rangle = 2^{-(n+1)}[(\sigma_0 + \lambda_3\sigma_3)^\otimes n \\
+ (\lambda_0\sigma_0 + i\lambda_2\sigma_2)^\otimes n + (\lambda_0\sigma_0 - i\lambda_2\sigma_2)^\otimes n + (\sigma_0 - \lambda_3\sigma_3)^\otimes n]
\]
has so-called X-form in the conventional basis. In accordance with Sylvester’s criterion, such an operator is positive semidefinite if and only if \((1 + \lambda_3)^p(1 - \lambda_3)^q + (1 - \lambda_3)^p(1 + \lambda_3)^q \geq \left| (\lambda_0 + \lambda_2)^p(\lambda_0 - \lambda_2)^q + (\lambda_0 - \lambda_2)^p(\lambda_0 + \lambda_2)^q \right|\) for all \(p, q \in \mathbb{Z}_+, p + q = n\). Continuing the same line of reasoning for states \(u_0^\otimes n|\text{GHZ}_n\rangle\) and \((u_iu_j)^\otimes n|\text{GHZ}_n\rangle\), where \(u_i\) and \(u_j\) are either identity operators or have the form \((34)-(35)\), we get permutations of \((\lambda_0, \lambda_2, \lambda_3)\) accompanied with the appropriate sign changes. All the obtained inequalities are summarized in equation \((38)\).

**Corollary 1.** The Pauli qubit map \(\Upsilon\) satisfies equation \((38)\) for \(n \geq 2\) if
\[
\sum_{i=1}^{3} \frac{\lambda_i^p}{p!} \leq 1.
\]

**Proof.** In the space of parameters \((\lambda_0, \lambda_2, \lambda_3)\), the geometrical figure \((38)\) comprises the figure \((39)\). The surfaces of two figures touch at points satisfying \(\lambda_i = \lambda_j = 0, |\lambda_k| = 1\), or \(|\lambda_i| = |\lambda_j| = 2^{-\frac{1}{n-1}}, \lambda_k = 0\). \(\square\)
The statement of Corollary 1 is valid for all $n = 2, 3, \ldots$ and stimulates the discussion of recurrence relation between $n$- and $(n+1)$-tensor-stable positive maps. In fact, suppose the map $\Phi$ is $n$-tensor-stable positive. Is it possible to modify $\Phi$ and construct a map $\tilde{\Phi}$, which is $(n+1)$-tensor-stable positive? The following proposition provides an affirmative answer to this question.

**Proposition 8.** Suppose $\Phi$ is $n$-tensor-stable positive and $\Phi_{EB}$ is entanglement breaking, then the map

$$\Phi = \mu \Phi + (1 - \mu) \Phi_{EB}$$

is $(n+1)$-tensor-stable positive whenever

$$\frac{\mu}{1 - \mu} \leq \frac{m_{\Phi}}{|m_{\Phi}|}, \quad m_{\Phi} = \min_{\rho \in \Theta} \Phi^{\otimes(n+1)}[\rho].$$

**Proof.** Expanding $\tilde{\Phi}^{\otimes(n+1)}$, we notice that the maps $\Phi_{EB} \otimes \Phi^{\otimes n}, \Phi_{EB}^{\otimes 2} \otimes \Phi^{\otimes (n-1)}, \ldots, \Phi_{EB}^{\otimes n} \otimes \Phi$ are all positive by proposition 2 as $\Phi^{\otimes n}$ is positive by the statement. Hence, if the map

$$\mu \Phi^{\otimes(n+1)} + (1 - \mu) \Phi_{EB}^{\otimes(n+1)}$$

is positive, then the map $\Phi^{\otimes(n+1)}$ is positive too. On the other hand, the map (42) is positive whenever the minimal output eigenvalue is non-negative, which results in formula (41) and concludes the proof.

Applying proposition 8 to the Pauli channels $\Upsilon$, we get a recurrent sufficient condition for $(n+1)$-tensor-stable positivity.

**Proposition 9.** Let the Pauli map $\Upsilon$ with parameters $\lambda_1, \lambda_2, \lambda_3$ be $n$-tensor-stable positive and $|\lambda_1| + |\lambda_2| + |\lambda_3| \geq 1$, then the map $\hat{\Upsilon}$ with parameters

$$\hat{\lambda}_i = \frac{(|\lambda_1| + |\lambda_2| + |\lambda_3|)^{-1} + x}{1 + x} \lambda_i,$$

$$0 \leq x \leq \frac{1}{2} \left(1 - \frac{\max_{1,2,3} \lambda_i}{|\lambda_1| + |\lambda_2| + |\lambda_3|} \right) \sqrt{\frac{2}{\max_{1,2,3} \lambda_i}},$$

is $(n+1)$-tensor-stable positive.

**Proof.** We use proposition 8, where the map $\Phi_{EB}$ has parameters $\lambda_{EB}^k = \lambda_k / (|\lambda_1| + |\lambda_2| + |\lambda_3|)$. Then $m_{\Phi_{EB}} \geq 2^{-k+1} |1 - \max_{1,2,3} \lambda_{EB}^k|^{n+1}$ and $m_{\Phi} \geq -\frac{1}{2} \max_{1,2,3} \lambda_k$ as $\Phi^{\otimes n}$ is positive and trace-preserving. Substituting the obtained values in equation (41) and using the explicit form of $\Phi_{EB}$, we get parameters (43). 

**Example 3.** Consider a family of the Pauli maps $\Upsilon$ with $\lambda_1 = \lambda_3 = t$ and $\lambda_2 = 0$ (see figure 7(b)).

$\Upsilon$ is positive $(n = 1)$ if $|t| \leq 1$. By proposition 9, $\hat{\Upsilon}$ is 2-tensor-stable positive if $|t| \leq 0.63$, which is in agreement with the exact result $|t| \leq \frac{1}{\sqrt{2}} \approx 0.71$ (proposition 4).

Let $\Upsilon$ be 2-tensor-stable positive, i.e., $|t| \leq \frac{1}{\sqrt{2}}$, then proposition 9 implies that $\hat{\Upsilon}$ is 3-tensor-stable positive if $|t| \leq 0.55$, which is in agreement with the result of section 8, $|t| \leq 2^{-2/3} \approx 0.63$. 

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If $\Upsilon$ is 3-tensor-stable positive, i.e. $|r| \leq 2^{-23}$, then proposition 9 implies that $\bar{\Upsilon}$ is 4-tensor-stable positive if $|r| \leq 0.532$.

10. Witnessing entanglement

Positive maps are often used to detect different types of entanglement [7–17, 18]. In this section, we find particular applications of $n$-tensor-stable positive maps in partial characterization of the entanglement structure.

A general density operator $\rho$ of $N$ qubits adopts the resolution

$$\rho = \sum_{p_j, j} p_j \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(p_j)}, \quad p_j \geq 0,$$

where $N$ qubits are divided into $k$ parts. For each fixed resolution of the state $\rho$ define the maximal number of qubits in the parts,

$$\max_{m=1, \ldots, k} \# \rho_j^{(m)},$$

and specifies the minimal physical resources needed to create such a state, namely, the minimal number of qubits to be entangled. The state $\rho$ is called fully separable if $R_{\text{ent}} = 0$ and genuinely entangled if $R_{\text{ent}} = N$.

The following result enables one to detect the entanglement depth via $n$-tensor-stable positive maps.

**Proposition 10.** Let $\rho$ be an $N$-qubit state. Suppose $\Phi$ is an $n$-tensor-stable positive qubit map and $\Phi \otimes \rho \not\geq 0$ (contains negative eigenvalues), then $R_{\text{ent}}[\rho] \geq n + 1$. 

**Proof.** Suppose $R_{\text{ent}}[\rho] \leq n$, then there exists a resolution (44) such that each state $\rho_j^{(m)}$ comprises at most $n$ qubits. Therefore, $\Phi \otimes \# \rho_j^{(m)} \rho_j^{(m)} \geq 0$ in view of the nested structure of $k$-tensor-stable positive maps. Thus, $\Phi \otimes \rho \not\geq 0$, which leads to a contradiction with the statement of proposition. Hence, $R_{\text{ent}}[\rho] \geq n + 1$. □

In what follows, we illustrate the use of proposition 10 for detecting particular forms of multipartite entanglement.

**Example 4.** A depolarized GHZ state of three qubits

$$\rho_q^{\text{GHZ}} = q|\text{GHZ}\rangle \langle \text{GHZ}| + (1 - q) \frac{1}{8} I, \quad 0 \leq q \leq 1,$$

is not fully separable if $q \geq 0.26$ as there exists a positive Pauli map $\Upsilon$ (with $|\lambda| \leq 1$) such that $\Upsilon \otimes 3 \rho_q^{\text{GHZ}} \not\geq 0$. Also, $\rho_q^{\text{GHZ}}$ is genuinely entangled if $q \geq 0.71$ as there exists a 2-tensor-stable positive Pauli map $\Upsilon$ (with parameters (11)) such that $\Upsilon \otimes 2 \rho_q^{\text{GHZ}} \not\geq 0$.

**Example 5.** Consider a depolarized W state of three qubits

$$\rho_q^{\text{W}} = q|\text{W}\rangle \langle \text{W}| + (1 - q) \frac{1}{8} I, \quad 0 \leq q \leq 1,$$

where $|\text{W}\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle)$. The state $\rho_q^{\text{W}}$ is not fully separable if $q \geq 0.31$ as there exists a positive Pauli map $\Upsilon$ (with $|\lambda| \leq 1$) such that $\Upsilon \otimes 3 \rho_q^{\text{W}} \not\geq 0$. Analogously, $\rho_q^{\text{W}}$ is...
genuinely entangled if $q \geq 0.86$ as there exists a 2-tensor-stable positive Pauli map $\Upsilon$ (with parameters (11)) such that $\Upsilon \otimes q^W \not\equiv 0$.

11. Conclusions

We have addressed the problem of positivity of tensor products $\bigotimes_{i=1}^n \Phi_i = \Phi_1 \otimes \Phi_2 \otimes \ldots \otimes \Phi_n$ of linear maps $\Phi_i$. In addition to the apparent implications $\{\text{All } \Phi_i \text{ are completely positive}$ $\} \lor \{\text{All } \Phi_i \text{ are completely co-positive} \} \Rightarrow \{\bigotimes_{i=1}^n \Phi_i \text{ is positive} \}$, we have managed to find non-trivial sufficient conditions for positivity of $\Phi_1 \otimes \Phi_2$, in particular, for unital qubit maps $\Phi_1$ and $\Phi_2$.

2- and 3-tensor-stable positive qubit maps are fully characterized. Namely, the explicit criteria for unital maps are found (equations (11), (31)–(32), respectively), and the analysis of non-unital maps is reduced to the case of unital ones.

Basing on the examples of decomposable positive maps $\Phi \otimes^2$, we have conjectured that all positive two-qubit maps $\Phi \otimes^2$ are decomposable.

For $n$-tensor-stable positive qubit maps we have found necessary and (separately) sufficient conditions. The first necessary condition involves algebraic inequalities on parameters $\lambda_1, \lambda_2, \lambda_3$ of degree $n$. Another condition has a concise form and clearly shows the nested structure of maps. The sufficient conditions have a recurrent form and enable one to find $(n+1)$-tensor-stable positive maps once $n$-tensor-stable positive maps are known. Entanglement breaking channels play a vital role in the derivation of those recurrent formulas. Due to the relation (20), the results obtained for unital maps can be readily transferred to non-unital maps.

Finally, we have discussed the application of positive maps with tensor structure to characterization of multipartite entanglement. A criterion for quantifying the entanglement depth (resource intensiveness, producibility) via $n$-tensor-stable positive maps is found and illustrated by a number of examples, which detect the genuine entanglement and the absence of full separability in depolarized GHZ and W states.

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