The flag-dipole spinors hierarchy within the singular sector of the Lounesto’s classification

R. J. Bueno Rogerio

1Institute of Physics and Chemistry, Federal University of Itajubá, Itajubá, Minas Gerais, 37500-903, Brazil.

Abstract. The present essay show the possibility to originate singular spinors, e.g., flag-pole and Weyl spinors, from flag-dipole spinors. Such approach reveal that flag-dipole spinors can be understood as a fundamental structure within singular sector of the Lounesto’s classification.

I. INTRODUCTION

The well known Lounesto’s spinor classification is a comprehensive categorization based on the bilinear covariants (physical information) that discloses the possibility of a large variety of spinors. Comprising regular and singular spinors which includes the cases of Dirac, flag-dipole, Majorana (flag-pole) and Weyl spinors [1].

The Lounesto’s classification is a geometrical classification and usually classify spinors according to the spinor’s physical information, the criterion lies on the bi-spinorial densities [2, 3], which is given by the following set \( \Gamma = \{ \sigma, \omega, J, K, S \} \). The relativistic description of the electron, in the light of Lounesto’s interpretation, stands for: the invariant length \( \sigma = \psi \gamma_0 \psi \), the pseudo-scalar amount \( \omega = \psi \gamma_0 \gamma^5 \psi \), the conserved current density, defined as \( J = \psi \gamma_0 \gamma^\mu \psi \gamma_\mu \), the spin projection in the momentum direction read \( K = \psi \gamma_0 \gamma^\mu \gamma^5 \psi \gamma_\mu \), and, finally, the electromagnetic momentum density \( S = \psi \gamma_0 \gamma^\mu \gamma^\nu \psi \gamma_\mu \wedge \gamma_\nu \), in which \( \gamma \) stands for the Dirac gamma matrices [2, 4].

Thus, such physical amounts allow to define the following classification

1. \( \sigma \neq 0, \quad \omega \neq 0; \)
2. \( \sigma \neq 0, \quad \omega = 0; \)
3. \( \sigma = 0, \quad \omega \neq 0; \)
4. \( \sigma = 0 = \omega, \quad K \neq 0, \quad S \neq 0; \)
5. \( \sigma = 0 = \omega, \quad K = 0, \quad S \neq 0; \)
6. \( \sigma = 0, \quad \omega = 0, \quad K \neq 0, \quad S = 0, \)

for the first three classes the \( K \) and \( S \) bilinear forms are always non-null quantities, moreover, the bilinear \( J \) is non-null for all cases. The first three classes stands for the Dirac spinors, fourth and fifth classes engenders flag-dipole and flag-pole (Majorana) spinors, respectively, and the sixth class stands for the Weyl spinors. Quiet recently, a more involved and comprehensive description for the bilinear forms were developed in [5], in such approach the authors extend the interpretation for bilinear forms and also takes into account a more general dual structure to define the bilinear amounts. Besides, in [6] is shown the possibility to enlarge the Lounesto’s classification by assuming \( J = 0 \).

Most of the text-books show that the bi-spinorial densities (or bilinear forms) are restricted to a geometrical link - the Fierz-Pauli-Kofink identities. As it can be seen in [7–9], Lounesto’s classification can be divided into two distinct sets: a set described by regular spinors (namely Dirac spinors) and another set describing singular spinors (flag-dipole, flag-pole and Weyl spinors). The physical description of all the spinors that belong to the Lounesto’s classification is quite successful, except, until then, for the flag-dipole spinors. The flag-dipole spinors belong to a rare set of spinors in the literature. Its application in physics were listed in (very) specific frameworks [10–14, and references therein]. Our proposal in this manuscript is to show that these spinors that were discovered by chance by Lounesto, carry very interesting characteristics from the mathematical point of view.

Remarkably enough, as we can see in the current literature, spinors endowed with dual-helicity carry a new physical content and also peculiar characteristics, which gained much prominence after the theoretical discovery of the pioneering mass dimension one fermions, the so-called Elko spinors [15]. All the new contents carried throughout mass dimension one theory, had opened new windows to the a new theory known as Beyond the Standard Model Theory. The mass dimension one theory is still being constructed [12, 15], thus, we then believe that quite interesting content still hidden beyond the mass dimension one fermions. By that time, we have only two examples of mass dimension one fermions: the Elko and the flag-dipole spinors. Due to what was previously mentioned, we highlight the importance of understanding, in a well-posed mathematical level, the main features of the dual-helicity spinors present in the singular sector of the Lounesto’s classification.

We start our analysis based on the mathematical argumentation of flag-dipole spinors in [16, 17] and later verified in [7]. The flag-dipole and flag-pole spinors are objects endowed with dual helicity [7, 16]. Thus, when exploring the singular sector of the Lounesto’s classification, and more specifically the flag-dipole spinors, we can verify that such objects play a very important role within the classification; flag-dipole spinors have a hierarchical character, where they assume a main position, and, under certain circumstances, may originate flag-pole spinors and also Weyl spinors (also, through them, it is...

*Electronic address: rodolforogerio@gmail.com
possible to connect the singular sector to the regular sector of the Lounesto’s classification \([8]\). We would like to emphasize that the protocol that will be shown here show some restrictions, thus, for some cases it does not admit a reverse path. In such a way, we may then say that flag-dipole spinors can be understood as generating spinors.

The manuscript is organized as follows: In the next section we weave some discussions and observations concerning the dual-helicity spinors present in the singular sector of the Lounesto’s classification. Such approach open up the possibility to transmute from a flag-dipole spinor to a flag-pole and also to a Weyl spinor. Due to its mathematical structure and a peculiarity in the arbitrariness of the phase factors, we claim that the flag-dipole spinors are in a hierarchical level within the singular sector of the Lounesto’s classification. Finally, in Sect. III we conclude.

II. DISCUSSION

Suppose a given set of dual-helicity spinors \([18]\), belonging to the singular sector of the Lounesto’s classification, e.g.,

\[
\rho = \begin{bmatrix} \alpha \Theta \phi^*_1 \\ \beta \phi_1 \end{bmatrix} \quad \text{and} \quad \varrho = \begin{bmatrix} \varsigma \phi_r \\ \partial \Theta \phi^*_r \end{bmatrix}, \tag{1}
\]

where \(\alpha, \beta, \varsigma\) and \(\vartheta\) stands for arbitrary phase factors, \(\Theta\) is the Wigner time-reversal operator, which in the spin-1/2 representation reads

\[
\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2}
\]

A quick inspection of the right-hand and left-hand components in the rest-frame referential, \(\phi_r\) and \(\phi_l\) respectively, under action of the helicity operator directly provide

\[
\hat{\sigma} \cdot \hat{\rho} \phi^\pm = \pm \phi^\pm, \tag{3}
\]

where \(\hat{\rho}\) stands for parametrization of the unit vector \(\hat{\rho} = (\sin(\theta) \cos(\phi), \sin(\phi) \sin(\theta), \cos(\theta))\) and \(\hat{\sigma}\) stands for the Pauli matrix. Thus, the positive helicity component is given by

\[
\phi^+ = \sqrt{m} \begin{bmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{bmatrix}, \tag{4}
\]

and the negative helicity reads

\[
\phi^- = \sqrt{m} \begin{bmatrix} -\sin(\theta/2) e^{-i\phi/2} \\ \cos(\theta/2) e^{i\phi/2} \end{bmatrix}. \tag{5}
\]

Previous observations in \([18]\) allow one to define the following relations

\[
\Theta \phi^+ = \phi^-, \quad \Theta \phi^- = -\phi^+. \tag{6}
\]

Such a quick derivation allow one to claim that \(\Theta \phi^*_r\) and \(\Theta \phi^*_l\) transforms like a right-hand and left-hand component under Lorentz transformations \([18]\). Since the boost operator acted on each spinors component in (1), providing a constant factor, we hasten to advise that in Eq. (1) we omitted the Lorentz boost factors and the components helicity indexes, for the purpose of this work, this will not result in any physical or mathematical implications. Both spinors in (1) carry the same physical information, therefore, for the sake of simplicity, all the calculations and analysis will be performed taking into account only the \(\rho\) spinor.

It is important to emphasize, if the phases factors are constrained to \(|\alpha|^2 \neq |\beta|^2\), then, we are handling the most general dual-helicity spinor — the flag-dipole spinor \([8, 12, 19]\). Now, note if one impose to \(\rho\) to hold conjugacy under charge conjugation operator, \(C = \gamma_2 K\), in which \(K\) stands for the algebraic complex conjugation. Thus, it leads to

\[
C \rho^\pm = \pm \rho^\pm, \tag{7}
\]

where the upper indexes “+” and “-” are related with the charge conjugation operator eigenvalues. Such constraint force the \(\rho\) spinor to be written as it follows

\[
\rho^+ = \begin{bmatrix} i \beta^* \Theta \phi^*_1 \\ \beta \phi_1 \end{bmatrix} \quad \text{and} \quad \rho^- = \begin{bmatrix} -i \beta^* \Theta \phi^*_1 \\ \beta \phi_1 \end{bmatrix}. \tag{8}
\]

Note that if one set \(\beta = 1\), it matches with the Elko spinors described in \([15]\). The dual-helicity spinors in (8) now stands for flag-pole spinors, in agreement with previous calculations in \([8]\). As discussed in \([12, 20]\), the main difference among flag-dipole and flag-pole spinors, lies in the bilinear form \(K\), being non-null for the first case and null for the second, the aforementioned spinors are constrained under the following sets of bilinear forms \(\Gamma_{f,d} = \{0, 0, J, K, S\}\) and \(\Gamma_{f,p} = \{0, 0, J, 0, S\}\), respectively. Nonetheless, note that the simply fact to impose to a flag-dipole spinor hold conjugacy under charge conjugation operator, it leads to a flag-pole spinor. Restricting the degree of freedom of the phase factors, such a restriction causes the phase factors present in both components to have equal moduli and, thus, the \(K\) bilinear form is identically null.

Now, another interesting fact that we bring to scene is: if one set \(\alpha = 0\) or \(\beta = 0\) in (1) — never both null, because, in this scenario, we would have a non-physical case (null spinor) — it is easy to show that we will then obtain a Weyl spinor (class-6 spinor), leading, thus, to one representation of such spinor, which can be displayed
in the following fashion
\[
\rho_{W_1} = \begin{bmatrix} \alpha \Theta \phi_0^\dagger \\ 0 \end{bmatrix} \quad \text{and} \quad \rho_{W_2} = \begin{bmatrix} 0 \\ \beta \Theta \phi_0^* \end{bmatrix}.
\] (9)

Moreover, note that such task cannot be accomplished by the flag-pole spinor (8), due to the fact that flag-pole spinor carry only one phase factor (\(\beta\)), and imposing \(\beta = 0\) in eq.(8), we obtain a null spinor, in other words, leading us to conclude that Weyl spinors can not be directly experienced from flag-pole spinors.

We emphasize that the protocol that we are exploring seems to be unique. We are able to start with a flag-dipole spinor and, under certain requirements, transmute to a flag-pole spinor or to transmute from a flag-dipole spinor to a Weyl spinor, however, this programme do not admit be done in the opposite direction to what is being shown here. It is impossible to construct any other spinor from only one Weyl spinor [9].

The Dirac spinors (classes 1, 2 and 3 from the regular sector) have already been extensively explored and have much of their physics revealed and well understood. Our focus on the present manuscript is restricted to dual-helicity spinors, precisely because of the advances that such spinors have recently shown in the literature [12, 15, 19, 20]. Remarkably enough, we highlight that a very same procedure can be extended for the regular sector, \textit{mutatis mutandis}, leading also to some interesting observations, however, under very strict mathematical restrictions. If one define a single-helicity spinor, with two arbitrary phases factors, standing for \(\mathbb{C}\)-numbers [8], belonging, then, to Dirac class-1. Thus, if one imposes that only one of the phase factors to be equal to zero, then, we obtain a Weyl spinor, nonetheless, now, if one impose to such single-helicity spinor to satisfy the Dirac’s dynamic, automatically one reach that both phase factors must be equal each other, evincing that only the spinors belonging to the Louesto’s class-2 satisfy the Dirac’s dynamic [9].

III. CONCLUDING REMARKS

In this report we show the possibility of, under certain physical and mathematical requirements, to transmute from a flag-dipole spinor to a flag-pole spinor or to transmute to a Weyl spinor. Therefore, it is impossible to obtain a Weyl spinor from a flag-pole spinor and it seems to be impossible to obtains any other spinor from only one Weyl spinor. Thus, we show that flag-dipole spinors are important structures within the Louesto’s classification, playing a central hierarchical role within the singular sector within Louesto’s classification. The focus is to elucidate that flag-dipole spinors carry a more general structure among singular spinors.

We shall finalize evincing that a very similar analysis can be performed in the regular sector within Louesto’s classification, however, it does not bring a result as comprehensive as that presented for the singular sector.

IV. ACKNOWLEDGEMENTS

The author thanks Rodrigo de Castro Lima and Laura Duarte for discussions in the initial stages of this manuscript.

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