Lepton flavor violating decays of $\mu$ and $\tau$ leptons in a gauge group

$$SU(2)_L \times SU(2)_R \times SU(2)_Y$$

Fayyazuddin

National Centre for Physics,
Quaid-i-Azam University Campus,
Islamabad 45320, Pakistan

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The electroweak unification group $SU(2)_L \times SU(2)_R \times SU(2)_Y$ is proposed for the charged lepton flavor violating decays of the muon ($\mu$) and tau ($\tau$) leptons. The group $SU(2)_Y$ is in the lepton space. The left-handed leptons and anti-leptons are assigned to the fundamental representation $(2,2,\bar{2})$ of the semi-simple group. The gauge group $SU(2)_Y$ is spontaneously broken to $U(1)_Y$ where $Y = -L = \pm 1$ is the hypercharge, by introducing a scalar multiplet $\Sigma$ which belongs to the triplet representation $3$ of the $SU(2)_Y$ and is singlet under $SU(2)_L \times SU(2)_R$. At this stage charged vector bosons $Y^\pm$ of $SU(2)_Y$ which mediate the lepton flavor violating decays acquire masses and are decoupled with one Higgs scalar $H^0_\Sigma$. The residual group $SU(2)_L \times SU(2)_R \times U(1)_Y$ has all the features of the left-right electroweak unification group extensively studied in the literature. The probability for lepton flavor violating decays is $$(\sin^2 \theta_W - 2 \sin^2 \theta_W) \left( \frac{m_{Y^+}}{m_Y} \right)^4.$$ 

I. INTRODUCTION

In the standard model, lepton number and baryon number are conserved, i.e. $\Delta L = 0$ and $\Delta B = 0$. Bounds on the lifetimes of electron and proton are

$$\tau_e > 4.6 \times 10^{36}\text{ years}, \quad \tau_p > 10^{31}\text{ years}. \quad (1)$$

For the leptons, there is another conservation law, viz the lepton number for each generation is conserved. No process with $\Delta L_e \neq 0$, $\Delta L_\mu \neq 0$ and $\Delta L_\tau \neq 0$ is allowed. The left-handed current
for weak decays in the standard model is

\[
J^{\mu}_{\text{lepton}} = (\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau) \gamma^\mu (1 - \gamma^5) \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} = \left[ \bar{\nu}_e \gamma^\mu (1 - \gamma^5) e + \cdots \right]
\]

(2)

\[
J^{\mu}_{\text{quark}} = (\bar{u}, \bar{c}, \bar{t}) \gamma^\mu (1 - \gamma^5) V_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \left[ \bar{u} \gamma^\mu (1 - \gamma^5) (V_{ud}d + V_{us}s + V_{ub}b) + \cdots \right]
\]

(3)

where \( V_{\text{CKM}} \) is the Cabibbo–Kobayashi–Maskawa (CKM) matrix. Unlike the quark-sector, where the flavor changing weak decays are allowed, in the lepton-sector no lepton flavor changing decays are allowed.

In the standard model, the neutrinos are only left-handed and hence are massless. For massless neutrinos, no linkage between three generations. Mixing between neutrinos is possible if all the neutrinos are not massless. Mixing requires that mass eigenstates \( \nu_i, (i = 1, 2, 3) \) are different from flavor eigenstates. In this case the oscillations are possible. In the neutrino oscillations, \( \nu_e \rightarrow \nu_\mu \) and \( \nu_\mu \rightarrow \nu_\tau \) have been experimentally observed.

One has to go beyond the standard model to explore the charged lepton flavor violating decays. In this context, the electroweak unification gauge group \( SU(2)_L \times SU(2)_R \times SU(2)_Y \) is proposed, the group \( SU(2)_Y \) is in the leptonic space. The concept of isospin in the leptonic space was first introduced in 1975. The left-handed leptons and antileptons are assigned to the fundamental representation \( (2, 2, \bar{2}) \) of the gauge group:

\[
\Psi = \begin{pmatrix} SU(2)_Y \\ SU(2)_L + (\nu_n, e^c_m) \end{pmatrix}
\]

(4)

where \( n \) and \( m \) are the flavor indices and the superscript \( c \) denotes the charge-conjugation. The multiplets \( (\nu_n, e^c_m)^T_L \) and \( (e^c_m, -N^c_m)^T_L \) belong to the fundamental representations of \( SU(2)_L \) and \( SU(2)_R \), respectively, whereas two doublets \( (\nu_n, e^c_m)_L \) and \( (e_n, -N^c_m)_L \) belong to the representation \( \bar{2} \) of \( SU(2)_Y \). There are three sets of vector bosons \( (W^\pm_L, W^0_{L\mu}), (W^\pm_{R\mu}, W^0_{R\mu}) \) and \( (Y^\pm_\mu, Y^0_\mu) \) belonging to the adjoint representation of each \( SU(2) \) gauge group. Out of six charged vector bosons, the four \( W^\pm_{L\mu}, W^\pm_{R\mu} \) are coupled to the left-handed and right-handed weak currents \( J^{\pm\mu}_L \) and \( J^{\pm\mu}_R \) respectively. The remaining two charged vector bosons \( Y^\pm_\mu \) are coupled to the lepton flavor violating currents \( J^{\pm\mu}_Y \). The gauge bosons \( Y^\pm_\mu \) mediate the lepton number \( L_e, L_\mu \) and \( L_\tau \) violating processes. The linear combinations of three neutral vector bosons \( W^0_{L\mu}, W^0_{R\mu} \) and \( Y^0_\mu \) give three physical vector...
bosons $A_\mu$, $Z_\mu$, and $Z'_\mu$ coupled is the electromagnetic current $J^{em\mu}$, the weak neutral currents $J^{Z\mu}$ and the $J^{Z'\mu}$, respectively.

Note that for the gauge group $SU(2)$, the representations 2 and $\bar{2}$ are equivalent and it is anomaly free unlike $SU(N)$ for $N > 2$ gauge groups which are not anomaly free. Hence, the gauge group $SU(2)_L \times SU(2)_R \times SU(2)_Y$ is anomaly free.

II. INTERACTION LAGRANGIAN

The gauge invariant Lagrangian for the fundamental representation $(2, 2, \bar{2})$ is given by

$$\mathcal{L} = \text{Tr}[i\bar{\Psi}\gamma^\mu \nabla_\mu \Psi],$$

with $\Psi$ defined in Eq. (4) and

$$\nabla_\mu = \partial_\mu + \frac{i}{2} \tau^a \cdot W_{L\mu} + \frac{i}{2} \tau^a \cdot W_{R\mu} - \frac{i}{2} (\tau^a \cdot Y_\mu)^\dagger,$$

where $\tau^a$ are the Pauli matrices. Thus the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{i}{2} \left[ (\bar{\nu}_n, \tilde{e}_n)_{L} \gamma^\mu \left( \begin{array}{c} W^0_{L\mu} \sqrt{2} W^+_{L\mu} \\ \sqrt{2} W^-_{L\mu} \end{array} \right) \left( \begin{array}{c} \nu_n \\ \tilde{e}_n \end{array} \right)_{L} + (\tilde{\bar{e}}^c_n, -\tilde{N}^c_n)_{L} \gamma^\mu \left( \begin{array}{c} W^0_{B\mu} \sqrt{2} W^+_{R\mu} \\ \sqrt{2} W^-_{R\mu} \end{array} \right) \left( \begin{array}{c} e^c_n \\ -N^c_n \end{array} \right)_{L} \right]$$

From the above equation we can separate the charged and neutral parts of the interaction Lagrangian as

$$\mathcal{L}_{\text{int}}^{\text{charge}} = -\frac{1}{2\sqrt{2}} \left\{ g \left[ \bar{\nu}_n \gamma^\mu (1 - \gamma^5) e_n W^+_{L\mu} + \tilde{N}^c_n \gamma^\mu (1 + \gamma^5) e_n W^+_{R\mu} + \text{h.c.} \right] - g Y \left[ (\tilde{e}^c_n \gamma^\mu (1 - \gamma^5) \nu_n - \tilde{N}^c_n \gamma^\mu (1 + \gamma^5) \nu_n) Y^+_{\mu} + \text{h.c.} \right] \right\}$$

$$\mathcal{L}_{\text{int}}^{\text{neutral}} = -\frac{1}{4} \left\{ g \left[ (\bar{\nu}_n \gamma^\mu (1 - \gamma^5) \nu_n - \tilde{e}^c_n \gamma^\mu (1 - \gamma^5) e_n) W^0_{L\mu} + (\tilde{N}^c_n \gamma^\mu (1 + \gamma^5) N_n - \tilde{e}^c_n \gamma^\mu (1 + \gamma^5) e_n) W^0_{R\mu} \right] - g Y \left[ \bar{\nu}_n \gamma^\mu (1 - \gamma^5) \nu_n + \tilde{e}^c_n \gamma^\mu (1 + \gamma^5) e_n + \tilde{e}^c_n \gamma^\mu (1 - \gamma^5) e_n + \tilde{N}^c_n \gamma^\mu (1 + \gamma^5) N_n \right] Y^0_{\mu} \right\}$$

In order to express the $\mathcal{L}_{\text{int}}^{\text{neutral}}$ in terms of physical vector bosons $A_\mu$, $Z_\mu$ and $Z'_\mu$, we note that electric charge $Q$ is given by

$$Q = I_{3L} + I_{3R} + I_{3Y}$$

$$g \quad g \quad gY$$

$$W^0_{L\mu} \quad W^0_{R\mu} \quad Y^0_{\mu}$$
Below we define the gauge bosons and the couplings in the mass eigenbases,

\[
\begin{align*}
A_\mu^e &= \frac{W^0_{L\mu}}{g} + \frac{B_\mu}{g'}, \\
B_\mu &= \frac{W^0_{R\mu}}{g} + \frac{Y^0_\mu}{g'}, \\
Z'_\mu &= \frac{W^0_{R\mu}}{g} - \frac{Y_{0\mu}}{g},
\end{align*}
\]

\[
\begin{align*}
\frac{1}{e^2} &= \frac{1}{g^2} + \frac{1}{g'^2}, \\
\frac{e}{g} &= \sin \theta_W, \\
\frac{e}{g'} &= \cos \theta_W, \\
g_Y &= \frac{g \tan \theta_W}{\sqrt{1 - \tan^2 \theta_W}}.
\end{align*}
\]

From the above definitions, one can obtain \(W^0_{L\mu}, W^0_{R\mu}\) and \(Y^0_\mu\) in terms of the physical vector bosons \(A_\mu, Z_\mu\) and \(Z'_\mu\) as:

\[
\begin{align*}
g_W^0 &= e A_\mu + g \cos \theta_W Z_\mu, \\
g_W^0 - g_Y Y_{0\mu} &= \frac{g}{\cos \theta_W} Z_\mu + \frac{g \tan^2 \theta_W}{\sqrt{1 - \tan^2 \theta_W}} Z'_\mu, \\
g_W^0 &= e A_\mu - g \frac{\sin^2 \theta_W}{\cos \theta_W} Z_\mu + g \sqrt{1 - \tan^2 \theta_W} Z'_\mu, \\
g_Y^0 &= e A_\mu - g \frac{\sin^2 \theta_W}{\cos \theta_W} Z_\mu - g \frac{\tan^2 \theta_W}{\sqrt{1 - \tan^2 \theta_W}} Z'_\mu, \\
g_W^0 - g_Y^0 &= \frac{g}{\sqrt{1 - \tan^2 \theta_W}} Z'_\mu, \\
g(W^0_{L\mu} - W^0_{R\mu}) &= \frac{g}{\cos \theta_W} Z_\mu - g \sqrt{1 - \tan^2 \theta_W} Z'_\mu.
\end{align*}
\]

Using above relations, the neutral current interaction Lagrangian is given by

\[
\mathcal{L}^{\text{neutral}} = -\frac{1}{4} \left\{ e \left[ -4 \bar{e} \gamma^\mu e_n \right] A_\mu + g \left[ \bar{\nu}_n \gamma^\mu (1 - \gamma^5) - e_n \gamma^\mu (1 - \gamma^5) e_n \\
- 4 \sin^2 \theta_W (-e_n \bar{\gamma} e_n) \right] Z'_\mu + g \left[ (\bar{N}_n \gamma^\mu (1 + \gamma^5) N_n - e_n \gamma^\mu (1 + \gamma^5) e_n) \\
+ \tan^2 \theta_W (\bar{\nu}_n \gamma^\mu (1 - \gamma^5) \nu_n - e_n \gamma^\mu (1 - \gamma^5) e_n + 4 e_n \gamma^\mu e_n) \right] \frac{Z'_\mu}{\sqrt{1 - \tan^2 \theta_W}} \right\}
\]

We conclude from Eq. (8) and Eq. (13), that except for lepton number violating term coupled to \(Y_\mu\), we get exactly the same result for the lepton sector as those given by the left-right symmetric gauge group \(SU(2)_L \times SU(2)_R \times U(1)_{Y_1}\), see for instance [4].

III. SPONTANEOUS BREAKING OF THE GAUGE GROUP \(SU(2)_L \times SU(2)_R \times SU(2)_Y\)

In the first stage, the group \(SU(2)_Y\) is spontaneously broken to \(U(1)_{Y_1}\), where \(Y_1\) is the hypercharge, by a scalar multiplet \(\Sigma\) which belong to singlet representation of \(SU(2)_L, SU(2)_R\) and to
triplet representation of $SU(2)_Y$, i.e. $\Sigma = (1, 1, 3)$ and can be written in the following form

$$\Sigma = \begin{pmatrix} H_{\Sigma}^+ \\ v_{\Sigma} + H_{\Sigma}^0 \\ H_{\Sigma}^- \end{pmatrix}$$

(14)

where $\langle \Sigma \rangle \equiv (0, v_{\Sigma}, 0)^T$ is the vacuum expectation value (vev) of $\Sigma$. The mass term is given by

$$\mathcal{L}^{\Sigma}_{\text{mass}} = -\frac{1}{4} g_Y^2 \left[ (\langle \tilde{\Sigma} \rangle \cdot \langle \tilde{\Sigma} \rangle) (\tilde{Y}_\mu \cdot \tilde{Y}_\mu) - (\langle \tilde{\Sigma} \rangle \cdot \tilde{Y}_\mu)(\langle \tilde{\Sigma} \rangle \cdot \tilde{Y}_\mu) \right]$$

$$= -\frac{1}{4} g_Y^2 V^2 \left[ 2 Y_+^{\mu} Y_-^{\mu} \right],$$

(15)

where the mass of the gauge bosons $Y^{\pm}$ is given by

$$m_{Y^{\pm}}^2 = \frac{1}{4} g_Y^2 (2 v_{\Sigma}^2) = \frac{1}{4} g_Y^2 \frac{\tan^2 \theta_W}{1 - \tan^2 \theta_W} (2 v_{\Sigma}^2).$$

(16)

The would be Goldstone bosons $H_{\Sigma}^\pm$ have been absorbed in $Y_\mu^{\pm}$ to give them longitudinal components and masses. The vector bosons $Y_\mu^{\pm}$ are decoupled with one heavy Higgs scalar $H_{\Sigma}^0$ and the electroweak unification group $SU(2)_L \times SU(2)_R \times SU(2)_Y$ is broken to $SU(2)_L \times SU(2)_R \times U(1)_{Y_1}$. We are left with seven massless vector bosons $\left( W_{L\mu}^+, W_{L\mu}^0, W_{R\mu}^+, W_{R\mu}^0 \right)$ and a singlet $Y_\mu^0$ belonging to $SU(2)_L, SU(2)_R$ and $U(1)_{Y_1}$, respectively and the two doublets

\begin{align*}
\begin{pmatrix}
\nu_n \\
e_n
\end{pmatrix}_L, \\
\begin{pmatrix}
e^c_n \\
-N^c_n
\end{pmatrix}_L
\end{align*}

(17)

belonging to representation 2 of $SU(2)_L$ and $SU(2)_R$ with hypercharge $Y_1 = -L = \pm 1$. The singlet vector boson $Y_\mu^0 = B_{1\mu}, g_Y = g_1$.

In the second stage, the gauge group $SU(2)_L \times SU(2)_R \times U(1)_{Y_1}$ (left-right symmetric group) is spontaneously broken to $U(1)_{em}$ by three sets of scalars [4]:

$$\Delta_R : (1, 2, 2), \quad \langle \Delta_R \rangle = \begin{pmatrix} 0 \\ v_{R}/2 \\ 0 \end{pmatrix}$$

(18)

$$\Delta_L : (2, 1, 2), \quad \langle \Delta_L \rangle = \begin{pmatrix} 0 \\ v_{L}/\sqrt{2} \\ 0 \end{pmatrix} \approx 0$$

(19)

$$\phi : (2, 2, 0), \quad \langle \phi \rangle = \begin{pmatrix} \kappa \\ 0 \\ 0 \end{pmatrix}$$

(20)

The multiplet $\Delta_R$, generates the mass terms for the $W_{R}^{\pm}$ and $Z'$ gauge bosons by using Eq [12] as

$$\mathcal{L}^{W_{R}^{\pm}, Z'}_{\text{mass}} = -\frac{1}{8} g_{R}^2 \left[ 2 g_{R}^2 W_{R}^{\pm} W_{R}^{-} + 2 \left( g_{W_{R}^{0\mu}} - g_{B_{1\mu}} \right)(g_{W_{R}^{0\mu}} - g_{B_{1\mu}}) \right]$$

$$= -\frac{1}{8} g_{R}^2 v_{R}^2 \left[ 2 W_{R}^{\pm} W_{R}^{-} + \frac{2}{1 - \tan^2 \theta_W} Z'^{\mu} Z'^{\mu} \right].$$

(21)
Hence
\[ m_{W_R^\pm}^2 = \frac{1}{4} g^2 v_R^2, \quad m_{Z'}^2 = \frac{1}{4} \frac{2 g^2 v_R^2}{1 - \tan^2 \theta_W} \tag{22} \]

At this stage the left-right symmetric group is broken to \( SU(2)_L \times U(1)_Y \). The scalar multiplet \( \phi \) breaking this group to \( U(1)_{em} \). The multiplet \( \phi \) generate the mass term for the \( W_L^\pm \) and \( Z \) as,
\[
L_{mass}^{W_L^\pm, Z} = \frac{1}{4} g^2 \left( \kappa^2 + \kappa'^2 \right) \left[ \left( 2 W_L^{\mu+} W_L^{-\mu} + 2 W_R^{\mu+} W_R^{-\mu} \right) + \left( W_{L\mu}^0 - W_{R\mu}^0 \right) \left( W_{L\mu}^0 - W_{R\mu}^0 \right) \right]
- g^2 \kappa \kappa' \left( W_L^{+\mu} W_{R\mu}^- + W_R^{+\mu} W_{L\mu}^- \right)
= \frac{1}{4} g^2 \kappa^2 \left[ 2 W_L^{+\mu} W_L^{-\mu} + 2 W_R^{+\mu} W_R^{-\mu} + \frac{1}{\cos^2 \theta_W} Z^\mu Z^- \right] - \frac{\sqrt{1 - \tan^2 \theta_W}}{\cos \theta_W} \left( Z'^\mu Z^- + Z^- Z'^\mu \right)
+ \left( 1 - \tan^2 \theta_W \right) Z'^\mu Z^- \tag{23}
\]

where in the last step we used Eq. (12) and the fact that \( \kappa' \ll \kappa \) (which one can select). Hence with \( \kappa' \ll \kappa \ll v_R \), and \( \kappa = v_L/\sqrt{2} \), we get
\[
m_{W_L^\pm}^2 = \frac{1}{4} g^2 v_L^2, \quad m_{Z'}^2 = \frac{1}{4} \frac{g^2}{\cos^2 \theta_W} v_L^2 \tag{25}
\]

The scalar multiplet \( \Delta_R \) gives the Majorana mass term to the right-hand neutrino \( N_n \):
\[
L_{mass}^{\text{Majorana}} = - \left( \nu_n^T, -N_n^{cT} \right)_L C^{-1} i \tau_2 \langle \Delta_R \rangle \left( \begin{array}{c} e_n \\ -N_n^c \\ -N_n^c \end{array} \right)_L + \text{h.c.}
= - \left( \nu_n^T, -N_n^{cT} \right)_L C^{-1} \left( \begin{array}{cc} 0 & 0 \\ 0 & v_R/2 \end{array} \right) \left( \begin{array}{c} e_n \\ -N_n^c \end{array} \right)_L + \text{h.c.}
= - \frac{v_R}{2} \left[ N_{nL}^{cT} C^{-1} N_{nL} - N_{nL}^c C N_{nL}^{cT} \right]
= - \frac{v_R}{2} \left[ N_{nR}^c C^{-1} N_{nR} + \text{h.c.} \right]. \tag{26}
\]

The multiplet \( \phi \) generates Dirac masses for the leptons. The Dirac mass term for the leptons is
\[
L_{mass}^{\text{Dirac}} = - \left[ h_{11n} \left( \nu_n^T, e_n^r \right)_L C^{-1} \langle \phi \rangle \left( i \tau_2 \begin{array}{c} e_n^c \\ -N_n^c \end{array} \right)_L + h_{21n} \left( \nu_n^T, e_n^r \right)_L C^{-1} \left( \phi \right) \left( \begin{array}{c} e_n^c \\ -N_n^c \end{array} \right)_L \right] + \text{h.c.}
= - \left[ - h_{11n} \left( \kappa \nu_n^T C^{-1} N_{nL}^c + h_{11n} \kappa^T C^{-1} N_{nL}^c \right) - h_{21n} \left( \kappa \nu_n^T C^{-1} N_{nL}^c + \kappa^T C^{-1} N_{nL}^c \right) \right] + \text{h.c.}
= - \left[ \left( h_{11n} \kappa + h_{21n} \kappa' \right) \bar{N}_{nR} \nu_{nL} + \left( h_{11n} \kappa' + h_{21n} \kappa \right) \left( \bar{\nu}_{nR} \nu_{nL} \right) \right] + \text{h.c.}
= - \left[ h_{11n} \kappa \left( \bar{N}_{nR} \nu_{nL} + \text{h.c.} \right) + h_{21n} \kappa \left( \bar{\nu}_{nR} \nu_{nL} + \text{h.c.} \right) \right]
= - \frac{v_L}{\sqrt{2}} \left[ h_{11n} \left( \bar{v}_{nL} N_{nR} + h.c. \right) + h_{21n} \left( \bar{v}_{nL} \nu_{nR} + h.c. \right) \right]. \tag{27}
\]
above in the second-last line above we used the approximation \( \kappa' \ll \kappa \ll v_R \). On diagonalization, it gives Majorana mass terms \( m_{\nu_L} = \frac{m_{\nu}^2}{4M_{\nu_R}} \). Moreover, the multiplet \( \phi \) also generate the quark masses, the mass term for quarks:

\[
L_{\text{mass}}^{\text{quark}} = -\frac{v_L}{\sqrt{2}} \left[ h_{1q_n} (\bar{u}_nL u_{nR} + \text{h.c.}) + h_{2q_n} (\bar{d}_nL d_{nR} + \text{h.c.}) \right]
\]  

(28)

We end this section with the following remark. The content of the gauge vector bosons and breaking of the gauge group \( SU(2)_L \times SU(2)_R \times U(1)_{Y_1} \) by the Higgs scalar multiplets are characteristic of the gauge groups and are independent of the fermionic content of the model. In the left-right symmetric gauge group \( SU(2)_L \times SU(2)_R \times U(1)_{Y_1} \), with \( Y_1 = B - L \) considered in Ref. [4] the left-handed quarks and antiquarks doublets

\[
\begin{pmatrix}
u_n \\ d'_n \\
\end{pmatrix}_L, \quad \begin{pmatrix} u_n \\ d_n' \\
\end{pmatrix}_L
\]

have \( Y_1 = B, Y_1 = \pm \frac{1}{3} \) whereas lepton (antilepton) multiplets have \( Y_1 = -L, Y_1 = \mp 1 \) as given in Eq. (17). The vector bosons \( W^\pm_{L\mu} \) and \( W^\pm_{R\mu} \) are coupled to leptons and quarks with no difference, where as the vector boson \( B_{1\mu} \) associated with \( U(1)_{Y_1} \) is coupled to leptons with \( Y_1 = -L \) and to quarks with \( Y_1 = B \). The coupling of Higgs scalar with \( Y_1 = 0 \) and \( Y_1 = 2 \) are coupled to all the fermions of the group.

IV. EFFECTIVE LAGRANGIAN FOR CHARGED LEPTON NUMBER VIOLATING DECAYS

In the Standard Model (SM) charged lepton (\( \mu \) and \( \tau \)) decays are mediated by the vector bosons \( W^\pm_{L\mu} \). From the first term of Eq. (8), the effective Lagrangian for these decays is given by

\[
L_{\text{eff}}^{\text{SM}} = \frac{G_F}{\sqrt{2}} \left[ \bar{\nu}_{\mu} \gamma^\mu (1 - \gamma^5) e_n \right] \left[ \bar{e}_m \gamma^\mu (1 - \gamma^5) \nu_m \right],
\]

(29)

where \( G_F = \sqrt{2}g^2/8m_W^2 \), is the Fermi constant. After Fierz reordering

\[
L_{\text{eff}}^{\text{SM}} = \frac{G_F}{\sqrt{2}} \left[ \bar{e}_m \gamma^\mu (1 - \gamma^5) e_n \right] \left[ \bar{\nu}_{\mu} \gamma^\mu (1 - \gamma^5) \nu_m \right].
\]

(30)

In particular, for \( \mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e \):

\[
L_{\text{eff}}^{\text{SM}}(\mu\text{-decay}) = \frac{G_F}{\sqrt{2}} \left[ \bar{e}_m \gamma^\mu (1 - \gamma^5) \mu \right] \left[ \bar{\nu}_\mu \gamma^\mu (1 - \gamma^5) \nu_m \right].
\]

(31)

The effective Lagrangian for the charged lepton flavor violating (LFV) decays mediated by \( Y^\pm_{\mu} \):

\[
L_{\text{eff}}^{\text{LFV}} = \frac{G_Y}{\sqrt{2}} \left[ \bar{e}_m \gamma^\mu (1 - \gamma^5) \nu_n \right] \left[ \bar{\nu}_n \gamma^\mu (1 - \gamma^5) e_m' \right],
\]

(32)
with 

\[
\frac{G_Y}{\sqrt{2}} = \frac{g_Y^2}{8m_W^2} = \frac{g_Y^2}{8} \frac{\tan^2 \theta_W}{1 - \tan^2 \theta_W m_Y^2}.
\]

Fierz reordering gives

\[
L_{\text{eff}}^{\text{LFV}} = \frac{G_Y}{\sqrt{2}} \left[ \bar{e}_m \gamma^\mu (1 - \gamma^5) e_{m'} \right] [\bar{\nu}_{m'} \gamma^\mu (1 - \gamma^5) \nu_n] \\
= -\frac{G_Y}{\sqrt{2}} [\bar{e}_m C^{-1} \gamma^\mu (1 - \gamma^5) e_{m'}^T] [\bar{\nu}_{m'} \gamma^\mu (1 - \gamma^5) \nu_n] \\
= -\frac{G_Y}{\sqrt{2}} [\bar{e}_m \gamma^\mu (1 + \gamma^5) e_m] [\bar{\nu}_{m'} \gamma^\mu (1 - \gamma^5) \nu_n].
\]

There is a choice of the possible assignments of three generations of leptons. The natural assignment is as follows:

\[
(i) \quad D(e) : D(123) = \begin{pmatrix} \nu_e & e^+ \\ e^- & -N_e^c \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu & \mu^+ \\ \mu^- & -N_\mu^c \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau & \tau^+ \\ \tau^- & -N_\tau^c \end{pmatrix}_L
\]

(34)

For the assignment \( (i) \), the \( \mu \) and \( \tau \) decays are as follows

\[
\mu^- \rightarrow \bar{\nu}_\mu + e^- + \nu_e, \quad \Delta L_\mu = -2, \quad \Delta L_e = 2, \quad \nu_e \leftrightarrow \nu_\mu, \\
\tau^- \rightarrow \bar{\nu}_\tau + \mu^- + \nu_\mu, \quad \Delta L_\tau = -2, \quad \Delta L_\mu = 2, \quad \nu_\mu \leftrightarrow \nu_\tau, \\
\tau^- \rightarrow \bar{\nu}_\tau + e^- + \nu_e, \quad \Delta L_\tau = -2, \quad \Delta L_e = 2, \quad \nu_e \leftrightarrow \nu_\tau.
\]

(35, 36, 37)

These decays stimulate the neutrino oscillations \( \nu_e \rightarrow \nu_\mu \) in the \( \mu \)-decay and \( \nu_\mu \rightarrow \nu_\tau, \nu_\tau \rightarrow \nu_e \) in the \( \tau \)-decay.

The lepton flavor violating effective Lagrangian for \( \mu^- \rightarrow \bar{\nu}_\mu + e^- + \nu_e \) can be written as

\[
L_{\text{eff}}^{\text{LFV}}(\mu \text{-decay}) = -\frac{G_Y}{\sqrt{2}} \left[ \bar{e}_m \gamma^\mu (1 + \gamma^5) e_{m'} \right] [\bar{\nu}_{m'} \gamma^\mu (1 - \gamma^5) \nu_m]
\]

(38)

and similarly for the \( \tau \)-decays, replace \( \mu \rightarrow \tau, \nu_\mu \rightarrow \nu_\tau, \) and \( \nu_e \rightarrow \nu_\mu \), in the above expression.

Using the permutation \( D(123) \rightarrow D(231) \):

\[
(ii) \quad D(231) = \begin{pmatrix} \nu_e & \mu^+ \\ e^- & -N_\mu^c \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu & \tau^+ \\ \mu^- & -N_\tau^c \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau & e^+ \\ \tau^- & -N_e^c \end{pmatrix}_L
\]

(39)

For the assignment \( (ii) \), the lepton flavor violating decays are

\[
\mu^- \rightarrow \bar{\nu}_e + e^- + \nu_\tau, \quad \Delta L_\mu = -1, \quad \Delta L_\tau = 1, \quad \nu_\mu \rightarrow \nu_\tau, \\
\tau^- \rightarrow \bar{\nu}_\mu + \mu^- + \nu_e, \quad \Delta L_\tau = -1, \quad \Delta L_e = 1, \quad \nu_\tau \rightarrow \nu_e, \\
\tau^- \rightarrow \bar{\nu}_\mu + e^- + \nu_\tau, \quad \Delta L_\mu = -1, \quad \Delta L_e = 1, \quad \nu_e \rightarrow \nu_\mu.
\]

(40, 41, 42)
These decays also stimulate the neutrino oscillations $\nu_\mu \rightarrow \nu_\tau$, $\nu_\tau \rightarrow \nu_e$, $\nu_e \rightarrow \nu_\mu$ in $\mu$ and $\tau$ decays. In this assignment (ii), the charged LFV effective Lagrangian for the decay $\mu^- \rightarrow \bar{\nu}_e + e^- + \nu_\tau$ is

$$L_{\text{eff}}^{\text{LFV}}(\mu\text{-decay}) = -\frac{G_Y}{\sqrt{2}} [\bar{e} \gamma^\mu (1 + \gamma^5) \mu] [\bar{\nu}_e \gamma_\mu (1 - \gamma^5) \nu_\tau]. \quad (43)$$

The Feynman amplitude for the $\mu$ decay in the Standard Model is given by [c.f. Eq. (30)]

$$|M_{\mu\text{-decay}}^{\text{SM}}|^2 = \sum_{\text{spin}} |F_{\mu\text{-decay}}^{\text{SM}}|^2 \sim \frac{G_F^2}{2} 4 \ p_1 \cdot k_1 \ p_1 \cdot k_2, \quad (44)$$

and for the lepton flavor violating $\mu$ decay is given by [c.f. Eq. (35)]

$$|M_{\mu\text{-decay}}^{\text{LFV}}|^2 = \sum_{\text{spin}} |F_{\mu\text{-decay}}^{\text{LFV}}|^2 \sim \frac{G_Y^2}{2} 4 \ p_1 \cdot k_1 \ p_2 \cdot k_2, \quad (45)$$

where $p_1$, $p_2$, $k_1$ and $k_2$ are the 4-momenta of $\mu$, $e$, $\nu_\mu$ and $\nu_e$. From Eq. (44), the decay width $d\Gamma$, for $\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_\tau$ is given by [1]

$$d\Gamma_{\mu\text{-decay}}^{\text{SM}} = \frac{G_F^2}{12\pi^3} m_\mu p_\mu dE_e [3W E_e - 2E_e^2 - m_e^2], \quad \text{where} \quad W \equiv \frac{m_\mu^2 + m_e^2}{2m_\mu}. \quad (46)$$

After integration we get,

$$\Gamma_{\mu\text{-decay}}^{\text{SM}} = \frac{1}{\tau_\mu} = \frac{G_F^2}{192\pi^3} m_\mu^5 [1 - \frac{8m_e^2}{m_\mu^2}], \quad (47)$$

From Eq. (45), we get exactly the same expressions for the LFV case as those given in Eq. (46) and Eq. (47) with $G_F^2 \rightarrow G_Y^2$. For $\tau$ decays, replace $m_\mu \rightarrow m_\tau$, $m_e \rightarrow m_\mu$ for $\tau \rightarrow \mu$ and for $\tau \rightarrow e$, $m_\mu \rightarrow m_\tau$. Hence, for the assignment (i)

$$R_{\mu\text{-decay}}^{\text{LFV}} = \frac{\Gamma_{\mu\text{-decay}}^{\text{LFV}}(\mu^- \rightarrow \bar{\nu}_e + e^- + \nu_\tau)}{\Gamma_{\mu\text{-decay}}^{\text{SM}}(\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e)} = \frac{G_Y^2}{G_F^2} \frac{g_1^4}{g_1^4} = \left( \frac{\sin^2 \theta_W}{1 - 2\sin^2 \theta_W} \right)^2 \left( \frac{m_{W_L}}{m_Y} \right)^4,$$

Moreover, for the $\tau$-decay we get the same ratio, i.e. $R_{\tau\text{-decay}}^{\text{LFV}} = R_{\mu\text{-decay}}^{\text{LFV}}$. Similar expressions for the decays $\mu^- \rightarrow \bar{\nu}_e + e^- + \nu_\tau$, $\tau^- \rightarrow \bar{\nu}_\mu + \mu^- + \nu_\tau$ and $\tau^- \rightarrow \bar{\nu}_\mu + \mu^- + \nu_e$ for the assignment (ii).

The probability to observe lepton flavor violating $\mu$ and $\tau$ decays must be less than $10^{-6}$, since the SM decay rate for $\mu$ decay is in agreement with the experimental value up to six places of decimals.

For example, using $\sin^2 \theta_W = 0.23$, $m_{W_L} \approx 80.38$ GeV, the probability to observe lepton flavor violating decay is $2.9 \times 10^{-8}$ for $m_Y = 50m_{W_L} \approx 4$ TeV and $1.5 \times 10^{-9}$ for $m_Y = 100m_{W_L} \approx 8$ TeV and for $m_Y = 65$ TeV the probability is $5.0 \times 10^{-13}$.

The energy spectrum given in Eq. (46) is modified by $L_{\text{eff}}^{\text{LFV}}$ for flavor violating decays. The Feynman amplitude for $\mu$ decay into electron is given by

$$F_{\mu\text{-decay}}^{\text{SM}} = \frac{G_F}{\sqrt{2}} [\bar{u}(p_2)\gamma^\mu (1 - \gamma^5) u(p_1)] [\bar{u}(k_1)\gamma_\mu (1 - \gamma^5) v(k_2)],$$

$$F_{\mu\text{-decay}}^{\text{LFV}} = \frac{G_Y}{\sqrt{2}} [\bar{u}(p_2)\gamma^\mu (1 + \gamma^5) u(p_1)] [\bar{u}(k_2)\gamma_\mu (1 - \gamma^5) v(k_1)], \quad (48)$$
which leads to

\[ |M_{\mu\text{-decay}}|^2 \sim G_F^2 \, 4 \, p_2 \cdot k_1 \, p_1 \cdot k_2 - 2G_F G_Y (-4m_\mu m_e \, k_1 \cdot k_2) \]  

From Eq. (49), the modified energy spectrum is given by

\[
d\Gamma = \frac{G_F^2}{12\pi^3} m_\mu p_e dE_e \left[ 3W E_e - 2E_e^2 - m_e^2 - 12\left( \frac{\sin^2 \theta W}{1 - \sin^2 \theta W} \right) \frac{m_{W_L}^2}{m_Y^2} m_e \times (W - E_e) \right]. \]  

V. CONCLUSIONS

The electroweak unification group \( SU(2)_L \times SU(2)_R \times SU(2)_Y \) is broken to \( SU(2)_L \times SU(2)_R \times U(1)_{Y_1} \) by a scalar multiple \( \Sigma \) that belongs to the triplet representation of \( SU(2)_Y \) and singlet of \( SU(2)_L \times SU(2)_R \), the vector boson \( Y^\pm \) acquired mass \( m_{Y^\pm}^2 = \frac{1}{2\tan^2 \theta_W} g^2 v_\Sigma^2 \), where \( v_\Sigma^2 \gg v_R^2 \gg v_L^2 \). The vector bosons \( Y^\pm \) are decoupled, with one heavy Higgs scalar \( H_0^\Sigma \) and the residual group \( SU(2)_L \times SU(2)_R \times U(1)_{Y_1} \) where \( Y_1 \) is the hypercharge has all the features of the left-right symmetric electroweak unification group \( SU(2)_L \times SU(2)_R \times U(1)_{Y_1} \) with hypercharge \( Y_1 = B - L \). \( Y_1 = B = \pm \frac{1}{3} \) for the quark multiplets and \( Y_1 = -L = \mp 1 \) for the lepton multiplets. Addition of quark multiplet does not change any other feature of the group. The probability to observe charged lepton flavor violating decays is \( \leq 10^{-9} \).

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