The Poincaré coset models ISO(d-1,1)/IR^n and T-duality

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Abstract

We generalize a family of Lagrangians with values in the Poincaré group ISO(d − 1, 1), which contain the description of spinning strings in flat (d − 1) + 1 dimensions, by including symmetric terms in the world-sheet coordinates. Then, by promoting a subgroup $H \sim \mathbb{R}^n$, $n \leq d$, which acts invariantly from the left on the element of ISO(d − 1, 1), to a gauge symmetry of the action, we obtain a family of $\sigma$-models. They describe bosonic strings moving in (generally) curved, and in some cases degenerate, space-times with an axion field. Further, the space-times of the effective theory admit in general T-dual geometries. We give explicit results for two non degenerate cases.

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1 Introduction

Recently several string actions which naturally describe curved space-times with singularities have been obtained from WZWN models. Following the coset construction [1, 2, 3], in Ref. [4] we analyzed a particular WZWN action in the Poincaré ISO(2, 1) group.

In a recent paper [5] a very general family of Lagrangians in the Poincaré group $\text{ISO}(d-1, 1)$ was studied and shown to describe diverse closed, bosonized, spinning strings in $(d-1)+1$-dimensional Minkowski space-time depending on the values of the constants which parameterize the family. In this paper, we start from the family of actions cited above and add further contributions amounting to terms which are symmetric in the world-sheet indices. Then we apply a gauging procedure which generalizes the one used in Ref. [4]: we raise a subgroup $H$ to a local symmetry of the action, where $H$ turns out to be necessarily isomorphic to $\mathbb{R}^n$, $n \leq d$, because of the prescription that the gauge field belongs to the algebra of $H$ itself, and show that the gauged action generates a family of effective actions for $\sigma$-models with $N = d(d+1)/2 - \dim(H)$ degrees of freedom. The latter can be viewed as effective theories describing the dynamics of a bosonic string moving in a (generally) curved background which can also contain an axion field.

Further, since both the metric and axion fields are independent of at least $d - \dim(H)$ out of $N$ degrees of freedom, it is easy to prove that the usual T-duality considerations apply [6]. One can include a dilaton field at a higher order in the loop expansion and build dual spaces.

The main idea behind the procedure we use is actually quite simple, and it is worth displaying the way it works on a toy model to show its main features. Consider the following 2-dimensional action,

$$S(x^1, x^2) = \frac{1}{2} \int dt \left[ (\partial_t x^1)^2 + (\partial_t x^2)^2 \right] ,$$  \hfill (1.1)

and gauge one of the coordinates, e.g. $x^2$, by the minimal coupling prescription, $\partial_t x^2 \to \partial_t x^2 + A^2$, where $A^2$ is a gauge field,

$$S_g(x^1, x^2, A^2) = S + \frac{1}{2} \int dt A^2 \left( A^2 + 2 \partial_t x^2 \right) .$$ \hfill (1.2)

Since $S_g$ is quadratic in $A^2$, one can define an effective action by integrating out $A^2$ in the path integral,

$$\int [dx^1] [dx^2] [dA^2] e^{-S_g(x^1, x^2, A^2)} \equiv \int [dx^1] e^{-S_{eff}(x^1)} .$$ \hfill (1.3)

This is equivalent to solving the equation of motion for $A^2$, $\delta_{A^2} S_g = 0$, and substituting back the result into $S_g$. Thus one obtains

$$S_{eff}(x^1) = \frac{1}{2} \int dt (\partial_t x^1)^2 ,$$ \hfill (1.4)

which is a trivial result and equals the one which we get by assuming $A^2$ is a pure gauge, $A^2 = -\partial_t x^2$. But suppose we now perform the following (canonical) transformation,

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \equiv \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} ,$$ \hfill (1.5)

with $\theta_{11} \theta_{22} - \theta_{12} \theta_{21} \neq 0$, then we gauge e.g. $\tilde{x}$ introducing a gauge field $\tilde{A}$ and repeat the process above. This time we obtain

$$S_{eff}(x) = \frac{1}{2} \int dt \frac{(\theta_{11} \theta_{22} - \theta_{12} \theta_{21})^2}{\theta_{12}^2 + \theta_{22}^2} (\partial_t x)^2 ,$$ \hfill (1.6)
which is different from the result one would get by setting $\tilde{A} = -\partial_t \tilde{x}$, namely

$$S(x, \tilde{x} = \text{const}) = \frac{1}{2} \int dt (\theta_{11}^2 + \theta_{21}^2) (\partial_t x)^2.$$  

(1.7)

We can rephrase this conclusion by saying that the canonical transformation in Eq. (1.5) introduces cross terms of the form $\partial_t x \partial_t \tilde{x}$ in the action and these in turn generate the following mapping:

$$(\theta_{11}^2 + \theta_{21}^2) \rightarrow \frac{(\theta_{11} \theta_{22} - \theta_{12} \theta_{21})^2}{\theta_{12}^2 + \theta_{22}^2},$$  

(1.8)

which becomes trivial for $\theta_{11} \theta_{12} + \theta_{21} \theta_{22} = 0$ (this also rules out rotations).

Of course, the previous model applies to quadratic actions only. Whenever we encounter Lagrangians which are linear in the gauge field we will revert to the pure gauge sector, as we did in Ref. [4], and obtain degenerate metrics (apart from two exceptional cases).

This in brief describes both our coset construction and compactification to get T-dual solutions. However, due to the high degree of generality of the model we start with, we were not able to draw any explicit conclusions other than formal mappings like the one shown in Eq. (1.8) for the toy model above. In particular, we cannot say much about the reduced space-time in general, although we show that, with particular choices of the parameters involved and for $d = 2, 3$, it is actually possible to complete the analysis.

In the next Section we describe the ungauged model, with particular attention to the derivation of the equations of motion and their comparison with the models introduced in Ref. [5]. In Section 3 we give the general formal treatment of the action when one gauges a subgroup $H \sim \mathbb{R}^n$, identify the whole set of its symmetries and introduce an effective action in the form of a $\sigma$-model. We obtain expressions for actions which can be either quadratic or linear in the gauge field, but in the latter case we show there are only two cases with non-degenerate metrics. In Section 4 we describe the T-dual procedure as applied to our model and show the properties of our model which are related to its multiple isometries. We then study the properties of the effective action under T-duality transformations. In Section 5 we specialize to the simplest, 2-dimensional, non degenerate case and perform an explicit analysis to find one effective background and one of its T-duals. In Section 6 we prove that the (non degenerate) metrics for all the models with $d = 3$ which are linear in the gauge field reduce to the case already treated in Ref. [4], which we now revise.

2 The ungauged action

We recall here that the Poincaré group in $(d - 1) + 1$ space-time dimensions, $ISO(d - 1, 1)$, is the semidirect product of the Lorentz group $SO(d - 1, 1)$ with the space-time translation group $T(d - 1, 1) \sim \mathbb{R}^{(d-1,1)} \sim \mathbb{R}^d$. Therefore we write its elements $g$ using the notation $g = (\Lambda, x)$, where $\Lambda \in SO(d-1,1)$ and $x \in \mathbb{R}^d$.

Given the map $g : \mathcal{M} \mapsto ISO(d - 1, 1)$ from the 2-dimensional manifold $\mathcal{M}$, parametrized by the coordinates $\sigma^\alpha$ ($\sigma^0 \equiv \tau$, $\sigma^1 \equiv \sigma$), to $ISO(d - 1, 1)$, we consider the very general action given by

$$S(\Lambda, x; K) = S_1 + S_2 + S_3 ,$$  

(2.1)

where

$$S_1 = \frac{1}{2} \int_{\mathcal{M}} d^2 \sigma (g^{\alpha \beta} + \epsilon^{\alpha \beta}) K^{(1)}_{ij} V^i_{\alpha} V^j_{\beta}$$

2
\[ S_2 = \frac{1}{2} \int_M d^2 \sigma \left( g^{\alpha \beta} + \epsilon_{\alpha \beta} \right) K_{ij}^{(2)} V_{i} W_{j}^{\beta} \]
\[ S_3 = \frac{1}{8} \int_M d^2 \sigma \left( g^{\alpha \beta} + \epsilon_{\alpha \beta} \right) K_{ijkl}^{(3)} W_{ij} W_{kl}^{\beta} . \]

Summation is assumed among upper and lower repeated latin indices \( i, j, \ldots = 0, \ldots, d - 1 \) according to the usual Lorentzian scalar product rule \( A_i B^i \equiv A^i \eta_{ij} B^j \), where \( \eta_{ij} = (-, +, \ldots, +) \) is the Minkowski tensor in \((d - 1) + 1\)-dimensions.

The Lagrangians in Eqs. (2.2) contain two kinds of contribution: the first one is proportional to the area element on \( M \), \( d^2 \sigma \epsilon_{\alpha \beta} = d\sigma^\alpha \wedge d\sigma^\beta \), with \( \epsilon_{\alpha \beta} = -\epsilon^{\beta \alpha} \) (\( \epsilon^r \sigma^r = +1 \)) the Levi-Civita symbol in two dimensions; the second one is proportional to the constant symmetric 2-dimensional matrix \( g^{\alpha \beta} \) with \( |\det g^{\alpha \beta}| = 1 \). Since the constants \( K \) are assumed to satisfy

\[
K_{ij}^{(1)} = -K_{ji}^{(1)} \\
K_{ijk}^{(2)} = -K_{ikj}^{(2)} \\
K_{ijkl}^{(3)} = -K_{jikl}^{(3)} = -K_{ijlk}^{(3)} = -K_{klji}^{(3)},
\]

it turns out that the contributions proportional to \( g^{\alpha \beta} \) drop out both of \( S_1 \) (because of the skewsymmetry of \( K_{ij}^{(1)} \) in the indices \( i, j \)) and \( S_3 \) (because of the skewsymmetry of \( K_{ijkl}^{(3)} \) under the exchange of the pairs of indices \( (i, j), (k, l) \)) and thus only \( S_2 \) contains it. When \( g^{\alpha \beta} \equiv 0 \), the action \( S(K) \) coincides with the model introduced in Ref. [5] and describes a different kind of closed bosonized spinning string moving in \((d - 1) + 1\) Minkowski space-time with coordinates \( x^k, k = 0, \ldots, d - 1 \) depending on the values of the constants \( K \).

The 1-forms \( V^i, W^{ij} \), with components

\[ V^i_\alpha \equiv (\Lambda^{-1})^i_r \partial_r x^r \]
\[ W^{ij}_\alpha \equiv (\Lambda^{-1})^i_r \partial_r \Lambda^{rj} , \]

are obtained by projecting the (left invariant) Maurer-Cartan form \( g^{-1} dg \) on the basis of the Poincaré algebra \( iso(d - 1, 1) \). Thus it immediately follows that the action \( S \) in Eq. (2.1) is invariant under the left rigid action of the Poincaré group, \( g \rightarrow g' g, g' = (\theta, y) \in ISO(d - 1, 1), \)

\[ S(\theta \Lambda, \theta x + y; K) = S(\Lambda, x; K) . \]

(2.5)

It is also invariant under the right rigid action, \( g \rightarrow g g', g' = (\theta, y) \in ISO(d - 1, 1), \)

\[ S(\Lambda \theta, \Lambda y + x; K) = S(\Lambda, x; K') , \]

(2.6)

provided the constants \( K \equiv (K_{ij}^{(1)}, K_{ijk}^{(2)}, K_{ijkl}^{(3)}) \) map to new values \( K' \) according to an expression given in Ref. [5]. The action \( S_2 \) alone, depending on the choice of \( g^{\alpha \beta} \), may actually be invariant under a semi-local transformation, as we report in Section 6.

If we use the convention \( \Lambda_{ij} \equiv (\Lambda^{-1})^j_i \), the action \( S \) can be written more explicitly in terms of the elements \( \Lambda \) and \( x \) as

\[ S_1 = \frac{1}{2} \int_M d^2 \sigma \epsilon_{\alpha \beta} K_{ij}^{(1)} \Lambda_r^i \Lambda_s^j \partial_\alpha x^r \partial_\beta x^s \]
\[ S_2 = \int_M d^2 \sigma \left( g^{\alpha \beta} + \epsilon_{\alpha \beta} \right) K_{ijk}^{(2)} \Lambda_r^i \Lambda_s^j \partial_\alpha x^r \partial_\beta x^k \]
\[ S_3 = \frac{1}{8} \int_M d^2 \sigma \epsilon_{\alpha \beta} K_{ijkl}^{(3)} \Lambda_r^i \Lambda_s^j \partial_\alpha \Lambda^{rj} \partial_\beta \Lambda^{sl} . \]

(2.7)
The equations of motion \( \delta_x S = 0 \), which follow from the variation \( x \to x + \delta x \), with \( \delta x \) an infinitesimal \( (d-1)+1 \) vector, amount to linear momentum conservation, 

\[
\partial_{\alpha} P_i^{(\alpha)} = \partial_{\alpha} \left( \epsilon^{\alpha\beta} P_{\beta i}^{(1)} + (g^{\alpha\beta} + \epsilon^{\alpha\beta}) P_{\beta i}^{(2)} \right) = 0 ,
\]

where the only two linear momentum currents that are not identically zero follow from \( S_1 \) and \( S_2 \) and are respectively given by 

\[
P_i^{(1)\alpha} = \epsilon^{\alpha\beta} \Lambda_i^r K^{(1)}_{rs} V_{\beta s} \equiv V_{is}^{\alpha\beta} \partial_s x^s
\]

\[
P_i^{(2)\alpha} = (g^{\alpha\beta} + \epsilon^{\alpha\beta}) \Lambda_i^r K^{(2)}_{rst} W_{\beta st} \equiv W_{ist}^{\alpha\beta} \partial_s \Lambda^t .
\]

Upon integrating on a fixed \( \tau \) slice of the world-sheet, one finds that the conserved charges are given by 

\[
P_i^{(1)} + P_i^{(2)} = \int d\sigma \left[ P_{\sigma i}^{(1)} + (1 + g^{\tau\sigma}) P_{\sigma i}^{(2)} + g^{\tau\tau} P_{\tau i}^{(2)} \right] ,
\]

where \( \tau \) and \( \sigma \) are the world sheet coordinates and use has been made of the periodicity in \( \sigma \) to discard boundary terms. For the particular choice we will make in Section 6, \( g^{\alpha\beta} = \pm \eta^{\alpha\beta} = \pm \text{diag}(-1,1) \) (the Minkowski metric tensor on the world-sheet), one finds that the conserved linear momentum following from \( S_2 \) coincides with the spatial integral of \( P_{i}^{(2)} \), where 

\[
\sigma^\pm = \tau \pm \sigma
\]

are light-cone coordinates on the world-sheet.

Similarly, from the variation \( \Lambda \to \Lambda + \delta \Lambda \), \( \delta \Lambda = \Lambda \rho \) and \( \delta x = \rho x \), with \( \rho_{ij} = -\rho_{ji} \) an infinitesimal \( \text{so}(d-1,1) \) matrix, the equations \( \delta_{\Lambda} S = 0 \) lead to angular momentum conservation 

\[
\partial_{\alpha} J_{ij}^{(\alpha)} = \partial_{\alpha} \left( \epsilon^{\alpha\beta} J_{\beta ij}^{(1)} + (g^{\alpha\beta} + \epsilon^{\alpha\beta}) J_{\beta ij}^{(2)} + \epsilon^{\alpha\beta} J_{\beta ij}^{(3)} \right) = 0 ,
\]

where the three angular momentum currents following from \( S_1, S_2 \) and \( S_3 \) read 

\[
J_{ij}^{(1)\alpha} = L_{ij}^{(1)\alpha} = x_i \wedge P_j^{(1)\alpha}
\]

\[
J_{ij}^{(2)\alpha} = L_{ij}^{(2)\alpha} + S_{ij}^{(2)\alpha} ,
\]

\[
J_{ij}^{(3)\alpha} = S_{ij}^{(3)\alpha} = -\frac{1}{2} \epsilon^{\alpha\beta} \Lambda_i^r \Lambda_j^s K_{rstp}^{(3)} W_{\beta p} .
\]

It is thus clear that one obtains terms which can be interpreted as non zero intrinsic angular momentum (spin) \( S_{ij} \) without the use of Grassmann variables. If one expands Eq. (2.12) one finds that the conserved charges are given by 

\[
J_{ij}^{(1)} + J_{ij}^{(2)} + J_{ij}^{(3)} = \int d\sigma \left[ J_{\sigma ij}^{(1)} + (1 + g^{\tau\sigma}) J_{\sigma ij}^{(2)} + g^{\tau\tau} J_{\tau ij}^{(2)} + J_{\sigma ij}^{(3)} \right] ,
\]

and, for \( g^{\alpha\beta} = \pm \eta^{\alpha\beta} \), one obtains \( J^{(2)} = \int d\sigma J_{\pm}^{(2)} \). Again, this will be the case for the model studied in Section 6.

To summarize, the difference between the models in Ref. [5] and ours is given by the contribution to linear and angular momentum proportional to \( g^{\alpha\beta} \) (see Eqs. (2.9), (2.13)).
3 The gauged action

We can modify the action $S$ in Eq. (2.1) in such a way as to make it invariant under the local left action of a subgroup $H$ of the whole Poincaré group,

$$g \rightarrow hg = (\theta \Lambda, \theta x + y),$$

(3.1)

with $h(\tau, \sigma) = (\theta, y) \in H$. For this purpose we introduce a gauge connection $A_\alpha(\tau, \sigma) = (\omega_\alpha, \xi_\alpha)$ and the corresponding covariant derivative $D_\alpha g \equiv \partial_\alpha g + A_\alpha$.

We require that $A_\alpha$ belongs to the algebra $H$ of the group $H$ (so that it has as many components as the elements of $H$ have). Since for every element $g \in \text{ISO}(d-1, 1)$ one has

$$D_\alpha (hg) \simeq \partial_\alpha g + \partial_\alpha (\delta h g) + A_\alpha,$$

(3.2)

where $\delta h = (\delta \theta, \delta y) \in H$, it follows that $H$ must act invariantly from the left on the elements of $\text{ISO}(d-1, 1)$, that is

$$\delta L g = \delta h g = (\delta \theta \Lambda, \delta \theta x + \delta y) \in H, \quad \forall g = (\Lambda, x) \in \text{ISO}(d-1, 1).$$

(3.3)

The only possible non trivial choices for $H$ are then subgroups of the translation group $\mathbb{R}^d$, that is

$$\begin{cases}
\theta = 0 \\
y \in \mathbb{R}^n \oplus \mathbb{I}_{d-n} \sim \mathbb{R}^n,
\end{cases}$$

(3.4)

with $n \leq d$, $\mathbb{I}_{d-n}$ being the identity in $d-n$ dimensions, for which $\delta_L g = \delta h$, $\forall g \in \text{ISO}(d-1, 1)$. Thus one also has the following form for the gauge field

$$\begin{cases}
\omega_\alpha^k \equiv 0 & k = 0, \ldots, d-1 \\
\xi_\alpha^k \equiv 0 & k \notin H,
\end{cases}$$

(3.5)

where $k \notin H$ is a shorthand notation for $x^k \notin H$.

Further, the gauge field must change under an infinitesimal $\text{ISO}(d-1, 1)$ transformation according to

$$\begin{cases}
\xi_\alpha^i \rightarrow \xi_\alpha^i - \partial_\alpha (\delta x^i) & i \in H \\
\xi_\alpha^i \rightarrow \xi_\alpha^i = 0 & i \notin H,
\end{cases}$$

(3.6)

where $\delta x$ is any allowed infinitesimal variation of $x \in \mathbb{R}^d$, including $(d-1) + 1$ Lorentz transformations.

The gauged action then reads

$$S_g(\Lambda, x, \xi; K) = S_{1g} + S_{2g} + S_{3g},$$

(3.7)

with

$$\begin{align*}
S_{1g} &= \frac{1}{2} \int d^2 \sigma \, \mathcal{Y}_{rs}^{\alpha \beta} \left( \partial_\alpha x + \xi_\alpha \right)^r \left( \partial_\beta x + \xi_\beta \right)^s \equiv S_1 + \Delta S_1(\xi) \\
S_{2g} &= \int d^2 \sigma \, \mathcal{W}_{rsk}^{\alpha \beta} \left( \partial_\alpha x + \xi_\alpha \right)^r \partial_\beta \Lambda^{sk} \equiv S_2 + \Delta S_2(\xi) \\
S_{3g} &= S_3,
\end{align*}$$

(3.8)
with $\mathcal{V}$ and $\mathcal{W}$ defined in Eq. (2.9). The new term

$$\Delta S_1 = \int d^2 \sigma \sum_{s \in H} \left[ \mathcal{P}^{(1)\alpha}_s \xi^s_\alpha + \frac{1}{2} \sum_{r \in H} \mathcal{V}^{\alpha\beta}_{rs} \xi^r_\alpha \xi^s_\beta \right],$$

(3.9)
is bilinear in the gauge field $\xi$ while

$$\Delta S_2 = \int d^2 \sigma \sum_{s \in H} \mathcal{P}^{(2)\alpha}_s \xi^s_\alpha,$$

(3.10)
is linear in $\xi$ ($\sum_{r,s,\ldots} \in H$ means the indices $r, s, \ldots$ are summed only over $r, s, \ldots \in H$, while all other latin indices are not restricted).

### 3.1 Symmetries

We now describe the symmetries of the new action. To simplify the notation, we momentarily turn to the Euclidean case ISO($d$) which is the semidirect product of the rotation group SO($d$) with $\mathbb{R}^d$ (this can be achieved by complexifying the time-like variable $x^0 \mapsto ix^d$). We then consider the following four subgroups:

- the two groups of $n$- and $(d-n)$-dimensional translations, with $n \leq d$,
  
  $$H_n = \mathbb{R}^n \oplus \mathbb{I}_{d-n} \quad \text{and} \quad H_{d-n} = \mathbb{I}_n \oplus \mathbb{R}^{n-d},$$
  
  (3.11)
such that $\mathbb{R}^d \sim H_n \oplus H_{d-n}$,

and

- the following two rotation subgroups of the whole rotation group:
  
  $$R_n = SO(n) \oplus \mathbb{I}_{d-n} \quad \text{and} \quad R_{d-n} = \mathbb{I}_n \oplus SO(d-n).$$
  
  (3.12)

We also write the Euclidean connection $\bar{\xi}$ as a $d$-dimensional vector $\bar{\xi} \equiv (\xi^k, \phi^\mu)$, $k = 1, \ldots, n$, $\mu = n + 1, \ldots, d$, so that we separate its components into $\xi \in H_n$ and $\phi \in H_{d-n}$.

The ungauged action obtained by Euclideanizing $S$ in Eq. (2.1) is invariant under the left rigid action of ISO($d$). The Euclideanized action $S_g$ obtained by gauging the group $H = H_n$ becomes invariant under the left local action of $H_n$, and it is still invariant under the left rigid action of $H_{d-n}$. However it is no longer invariant under the left rigid action of the whole SO($d$) group because $\Delta S_1(\xi)$ in Eq. (3.9) and $\Delta S_2(\xi)$ in Eq. (3.10) are not. In fact, in order to preserve the invariance in $\Delta S_1$ and $\Delta S_2$, the gauge field $\xi$ must transform according to the (Euclidean version) of the first constraint in Eq. (3.6),

$$\bar{\xi}^i_\alpha \to \bar{\xi}^i_\alpha - \theta^i_\alpha \partial_\alpha x^j, \quad \forall i = 1, \ldots, d,$$

(3.13)

but this would mix $\xi$ components with $\phi$ components of the connection for a general rotation $\theta \in SO(d)$. At the same time, according to Eq. (3.5) or, equivalently, the second constraint in Eq. (3.6), it must be possible to set $\phi$ to zero (or $\bar{\xi}^i = 0, \forall i = n + 1, \ldots, d$), since these components correspond to the group $H_{d-n}$ that we are not gauging. This implies that the only
rigid rotations that leave $S_g$ left invariant are the ones which do not mix $\xi$ with $\phi$ and thus belong to $R_n$ or $R_{d-n}$. To summarize:

$$S_g(\theta \Lambda, \theta x + y, \xi; K) = S_g(\Lambda, x, \xi; K) \iff \begin{cases} y(\tau, \sigma) \in H_n \lor y \in H_{d-n} \\ \theta \in R_n \lor \theta \in R_{d-n} \end{cases}$$  \hspace{1cm} (3.14)$$

The same argument above applied to right rigid transformations would require

$$\bar{\xi}_\alpha \rightarrow \bar{\xi}_\alpha - (\partial_\alpha \Lambda) y , \quad \forall \Lambda \in SO(d) ,$$  \hspace{1cm} (3.15)

which necessarily mixes $\xi$ and $\phi$ components if $y \neq 0$. This singles out the whole $SO(d)$ subgroup, so that

$$S_g(\Lambda \theta, \Lambda x + y, \xi; K) = S_g(\Lambda, x, \xi; K') \iff \begin{cases} y \equiv 0 \\ \theta \in SO(d) \end{cases}$$  \hspace{1cm} (3.16)$$

It is now straightforward to translate these conclusions back to the Lorentzian framework.

As a simple corollary, one obtains that the action $S_g$ in Eq. (3.7) is invariant under the left rigid action of $ISO(d-1,1)$ iff $n = 0$ or $n = d$.

### 3.2 Equations of motion

In order to study the equations of motion following from the action $S_g$ it is more convenient to rewrite

$$S_g = S^{(g)} + S^{(H)} + S^{(I)} ,$$  \hspace{1cm} (3.17)

where

$$S^{(g)} = S_3 + \int d^2 \sigma \sum_{r \notin H} \left[ W_{r \beta}^{\alpha \beta} \partial_\alpha x^r \partial_\beta \Lambda_{pq} + \frac{1}{2} \sum_{s \notin H} V_{rs}^{\alpha \beta} \partial_\alpha x^r \Lambda_{pq} \right] ,$$  \hspace{1cm} (3.18)

does not contain elements $x \in H$,

$$S^{(H)} = \int d^2 \sigma \sum_{r \in H} \left[ W_{r \beta}^{\alpha \beta} \partial_\alpha x^r \partial_\beta \Lambda_{pq} + \frac{1}{2} \sum_{s \in H} V_{rs}^{\alpha \beta} \Lambda_{pq} \left( \partial_\beta x^s + \xi_\beta^s \right) \right] \left( \partial_\alpha x^r + \xi_\alpha^r \right) ,$$  \hspace{1cm} (3.19)

contains only terms proportional to elements of $H$, and

$$S^{(I)} = \int d^2 \sigma \sum_{r \in H, s \notin H} V_{rs}^{\alpha \beta} \left( \partial_\alpha x^r + \xi_\alpha^r \right) \partial_\beta x^s ,$$  \hspace{1cm} (3.20)

contains mixed terms.

The equations of motion $\delta x S_g = 0$ together with the transformation law for the gauge field in Eq. (3.6) split the theory into two sectors:

1. when $\delta x^i \notin H$ one requires $\delta x S^{(g)} = 0$ and obtains

$$\partial_\alpha P_i^{(g)\alpha} = \partial_\alpha \left( P_i^{(1g)\alpha} + P_i^{(2g)\alpha} \right) = 0 , \quad i \notin H ,$$  \hspace{1cm} (3.21)
so that the \( d - \dim(H) = d - n \) linear momentum currents \( P_i^{(g)\alpha} \) with \( i \notin H \) and
\[
P_i^{(1g)\alpha} = \sum_{j \notin H} V^{\alpha\beta}_{ij} \partial_\beta x^j,
\]
\[
P_i^{(2g)\alpha} = P_i^{(2)\alpha},
\]
are still conserved;

2. when \( \delta x^i \in H \) one gets \( \delta x S^{(H)} = \delta x S^{(I)} = 0 \) identically and thus the corresponding currents
\[
P_i^{(H)\alpha} \equiv P_i^\alpha - P_i^{(g)\alpha} = \sum_{s \in H} V_{is}^{\alpha\beta} \partial_\beta x^s, \quad i \in H
\]
are not conserved.

From the variation \( \delta_\Lambda S_g = 0 \) and Eq. (3.6) one obtains an analogous splitting into three sectors:

1. in the sector for which \( \delta_\Lambda k_j = \Lambda_k^i \rho_{ij}, \ i, j \notin H \) one has
\[
\partial_\alpha \left[ J_{ij}^\alpha + 2 \sum_{r \in H} W^{\alpha\beta}_{rij} \xi_{ij}^r + x_i \wedge \sum_{r \in H} V_{jr}^{\alpha\beta} \xi_{ij}^r \right] = 0 \quad i, j \notin H.
\]

Thus the angular momentum currents \( J_{ij}^\alpha \) with both indices \( i, j \notin H \) are no longer conserved but couple to the gauge field;

2. when \( \delta_\Lambda k_j = \Lambda_k^i \rho_{ij}, \ i, j \in H \) one obtains an analogous relation
\[
\partial_\alpha \left[ S_{ij}^\alpha + 2 \sum_{r \in H} W^{\alpha\beta}_{rij} \xi_{ij}^r \right] = \left( P_i^\alpha + \sum_{r \in H} V_{ir}^{\alpha\beta} \xi_{ij}^r \right) \wedge (\partial_\alpha x_i + \xi_{ij}) \quad i, j \in H;
\]
in which only the spin \( S_{ij} \) appears on the L.H.S. because now \( \partial_\alpha P_i^\alpha \neq 0, \ i \in H; \)

3. finally, in the sector in which \( \delta_\Lambda k_j = \Lambda_k^i \rho_{ij}, \ i \notin H, \ j \in H \) one obtains the following non trivial equation
\[
\partial_\alpha \left[ J_{ij}^\alpha + 2 \sum_{r \in H} W^{\alpha\beta}_{rij} \xi_{ij}^r + x_i \wedge \sum_{r \in H} V_{jr}^{\alpha\beta} \xi_{ij}^r \right] =
\]
\[
\left( P_i^\alpha + \sum_{r \in H} V_{ir}^{\alpha\beta} \xi_{ij}^r \right) (\partial_\alpha x_j + \xi_{ij}) - \partial_\alpha x_i \left( P_j^\alpha + \sum_{r \in H} V_{jr}^{\alpha\beta} \xi_{ij}^r \right), \quad i \notin H, \ j \in H
\]
which mixes terms from the two previous sectors.

To summarize, whenever one considers only quantities which do not contain elements of \( H \), the equations of motion for the linear momenta look the same as the ungauged ones and amount again to linear momentum conservation. The angular momentum, instead, is no longer conserved, even in that sector of the theory, because of the presence of \( S^{(H)} \) and \( S^{(I)} \). Further, due to these latter contributions to the action, the constraints displayed in Eqs. (3.25), (3.26) must also be satisfied, together with the equations of motion for the gauge field, \( \delta_\xi S_g = 0 \), which we will study in the following Subsection.
3.3 Eliminating $\xi$: the quadratic case

Now that we have introduced the gauge field $\xi$, we want to eliminate it from the action. One way to achieve this goal is to integrate out $\xi$ first in the path integral

$$
\int [d\Lambda] \left[ dx \not\in H \right] \left[ dx \in H \right] [d\xi] e^{-S(\Lambda,x)-\Delta S(\Lambda,x,\xi)} \equiv \int [d\Lambda] \left[ dx \not\in H \right] e^{-S_{\text{eff}}(\Lambda,x)},
$$

where, from Eqs. (3.9), (3.10) one has

$$
\Delta S \equiv \Delta S_1 + \Delta S_2 = \int d^2\sigma \sum_{s \in H} \left[ \mathcal{P}_s^\alpha \xi_s^\alpha + \frac{1}{2} \sum_{r \in H} \mathcal{V}_{rs}^{\alpha\beta} \xi_r^\alpha \xi_s^\beta \right],
$$

with $\mathcal{P}_s^\alpha$ defined in Eq. (2.8).

When $\mathcal{V}_{rs}^{\alpha\beta} \neq 0$, for some $r, s \in H$, $\Delta S$ is quadratic in $\xi$ and the above mentioned integration corresponds to solving the equations of motion for $\xi$, namely $\delta_\xi \Delta S = 0$, and substituting back the result into the action. We also notice that, since $\mathcal{V}^{\alpha\beta} = -\mathcal{V}^{\beta\alpha}$, the condition $\mathcal{V}_{rs}^{\alpha\beta} \neq 0$ implies that $\dim(H) = n \geq 2$ and thus excludes the ungauged case.

It is easy to find that $\delta_\xi \Delta S = 0$ implies

$$
\mathcal{P}_s^\alpha + \sum_{r \in H} \mathcal{V}_{sr}^{\alpha\beta} \xi_r^\gamma = 0, \quad s \in H.
$$

One can try to go further if we assume that $\mathcal{V}$ is invertible inside the $H$ sector (this will put constraints on the constants $K^{(1)}$ and could also exclude part of the subgroup $SO(d-1,1)$) and define $\mathcal{V}_{rs}^{\alpha\beta}$ such that

$$
\sum_{a \in H} \mathcal{V}_{ar}^{\alpha\gamma} \mathcal{V}_{as}^{\gamma\beta} = \delta_\alpha^\beta \delta_r^s, \quad \forall r, s \in H.
$$

Then Eq. (3.29) above can be inverted to give

$$
\xi_r^\gamma = -\sum_{s \in H} \mathcal{V}_{sr}^{\alpha\beta} \mathcal{P}_s^\beta, \quad r \in H,
$$

and the correction to the (ungauged) action becomes

$$
\Delta S = -\frac{1}{2} \int d^2\sigma \sum_{r, s \in H} \mathcal{P}_r^\alpha \mathcal{V}_{rs}^{\alpha\beta} \mathcal{P}_s^\beta.
$$

By simply expanding the linear momentum current $\mathcal{P}$ according to Eq. (3.23) one then easily proves that $\Delta S$ cancels out every dependence on $x \in H$ from the (ungauged) action, since

$$
S_1(x \not\in H, x \in H) - \frac{1}{2} \int d^2\sigma \sum_{r, s \in H} \mathcal{P}_r^{(1)\alpha} \mathcal{V}_{r}^{\alpha\beta} \left( \mathcal{P}_s^{(1)\beta} + 2 \mathcal{P}_s^{(1g)\beta} \right) = S_1(x \not\in H)
$$

$$
S_2(x \not\in H, x \in H) - \int d^2\sigma \sum_{r, s \in H} \mathcal{P}_r^{(1)\alpha} \mathcal{V}_{r}^{\alpha\beta} \mathcal{P}_s^{(2g)\beta} = S_2(x \not\in H)
$$

The remaining terms in Eq. (3.32) lead to corrections to the three (ungauged) contributions of the action according to the following scheme

$$
S_{1\text{eff}}^q(x \not\in H) = S_1(x \not\in H) - \frac{1}{2} \int d^2\sigma \sum_{r, s \in H} \mathcal{P}_r^{(1g)\alpha} \mathcal{V}_{r}^{\alpha\beta} \mathcal{P}_s^{(1g)\beta}
$$
the effective theory can be made invariant under the local action of the group space-time dimensions, $N$ subgroup of the Lorentz group $SO$.

This defines a total effective action

$$S_{\text{eff}}^q(\Lambda, x \notin H, K) = S_{\text{eff}}^q + S_{2\text{eff}}^q + S_{3\text{eff}}^q,$$

which differs from the one obtained by simply setting the terms containing $x \in H$ to zero. This is due to the presence of cross terms of the kind discussed in the Introduction.

We observe that the number of degrees of freedom in the effective action $S_{\text{eff}}^q$ is at most $N \equiv \dim(ISO(d-1,1)) - \dim(H) = d(d+1)/2 - n$, $n < d$, and that the only case (with $n \geq 2$) which is invariant under the left rigid action of $ISO(d-1,1)$, namely $n = d$, is given by $S_{\text{eff}}^q = S_{3\text{eff}}^q$ and has $N = d(d-1)/2$.

Further, although the gauged action $S_g$ is not well behaved under the action of any local subgroup of the Lorentz group $SO(d-1,1)$, as we have shown in Subsection 3.1, it turns out that the effective theory can be made invariant under the local action of the group $SO(N-D, D)$ in $N$ space-time dimensions, $0 \leq 2D \leq N$ (the precise signature must be computed for each case explicitly). This is not difficult to prove. Let us consider the $N$-dimensional vector $\bar{x} \equiv (x, t)$ whose components are the $d-n$ translations $x \notin H$ and the $d(d-1)/2$ independent parameters $t$ of the Lorentz group $SO(d-1,1)$. We notice that the effective action $S_{\text{eff}}^q$ written in terms of $\bar{x}$ is of the form

$$S_{\text{eff}} = \frac{1}{2} \int d^2 \sigma \bar{M}_{ab} \partial^a \bar{x}^b \partial^b \bar{x}^b,$$

where

$$\bar{M}_{ab} = \bar{M}_{ab}^{(1)} + \bar{M}_{ab}^{(2)} + \bar{M}_{ab}^{(3)}.$$

is an $N \times N$ matrix whose elements depend on (some of) the coordinates $t$ (and the constants $K$) only. It can be written in the following block form:

$$\bar{M}^{(1)} = \begin{bmatrix} \epsilon^{\alpha\beta} B_{xx}^{(1)} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{M}^{(3)} = \begin{bmatrix} 0 & 0 \\ 0 & g^{\alpha\beta} G_{tt}^{(3)} + \epsilon^{\alpha\beta} B_{tt}^{(3)} \end{bmatrix}.$$

where the symmetric matrix $G_{tt}$ is $d(d-1)/2 \times d(d-1)/2$ dimensional and comes from the corrections to $S_3$ sketched in the last of Eqs. (3.34), and the antisymmetric matrices $B_{xx}$ and $B_{tt}$ are respectively $(d-n) \times (d-n)$ and $d(d-1)/2 \times d(d-1)/2$ dimensional;

$$\bar{M}^{(2)} = \begin{bmatrix} 0 \\ (g^{\alpha\beta} - 2 \epsilon^{\alpha\beta}) U_{xt}^{(2)} T \\ (g^{\alpha\beta} + 2 \epsilon^{\alpha\beta}) U_{xt}^{(2)} \end{bmatrix}.$$

where the matrix $U_{xt}$ is $(d-n) \times d(d-1)/2$ dimensional. Thus $S_{\text{eff}}$ can also be written

$$S_{\text{eff}} = \frac{1}{2} \int d^2 \sigma \left[ g^{\alpha\beta} \bar{G}_{ab} \partial^a \bar{x}^a \partial^b \bar{x}^b + \epsilon^{\alpha\beta} \bar{B}_{ab} \partial^a \bar{x}^a \partial^b \bar{x}^b \right],$$

where $\bar{G}$ is the symmetric part of $\bar{M}$,

$$\bar{G} = \begin{bmatrix} 0 & U_{xt}^{(2)} T \\ U_{xt}^{(2)} & G_{tt}^{(3)} \end{bmatrix}.$$
and \( \bar{B} \) is the antisymmetric part,
\[
\bar{B} = \begin{bmatrix}
B_{xx}^{(1)} & 2U_{xt}^{(2)} \\
-2U_{xt}^{(2)} & B_{tt}^{(3)}
\end{bmatrix}.
\tag{3.42}
\]

Now \( S_{\text{eff}} \) in Eq. (3.40) is in the form of a \( \sigma \)-model which describes a bosonic string moving in a (generally curved) \((N - D) + D\)-dimensional background parameterized by the coordinates \( \bar{x} \). Such a model, with the addition of a possible dilaton field (see Section 4), is invariant under the local action of the group \( SO(N - D, D) \) as claimed, although the matter source which generates it might be unphysical.

The symmetric matrix \( \bar{G} \) is not zero iff \( S_2 \neq 0 \) and \( g^{\alpha \beta} \neq 0 \). It plays the role of the metric tensor in \( N \) dimensions, its signature thus determining the number of time-like coordinates \( D \).

The matrix \( \bar{B} \) is the antisymmetric potential of the axion field \( \bar{H}^{\alpha \beta} \equiv \partial_a \bar{B}^{bc} + \partial_c \bar{B}^{ab} + \partial_b \bar{B}^{ca} \).

### 3.4 Eliminating \( \xi \): the linear case

When \( V_{\alpha \beta}^{rs} = 0, \forall r, s \in H \), one has that \( \Delta S \) is linear in \( \xi \). This case includes both the ungauged action (for which \( n = 0 \)) and all the 1-dimensional subgroups \( H \sim \mathbb{R} \).

Upon taking \( \delta_\xi \Delta S = 0 \) one would get
\[
\mathcal{P}_{s}^{(g)\alpha} = 0, \quad s \in H,
\tag{3.43}
\]
but one now is not allowed to substitute this result back into the action. However, we observe that, since the action \( S_g \) does not contain a kinetic term for \( \xi \), one can force \( \xi \) to be a pure gauge,
\[
\xi_{\alpha}^r = -\partial_{\alpha} x^r, \quad r \in H, \tag{3.44}
\]
by adding a Lagrange multiplier term,
\[
\int d^2 \sigma \lambda_r \epsilon^{\alpha \beta} \partial_{\alpha} \xi_\beta^r, \tag{3.45}
\]
to the exponent in Eq. (3.27) and then integrating out \( \lambda \).

The relation in Eq. (3.44) reduces Eq. (3.24) to the conservation of the angular momentum current \( J_{ij}^{(g)} \), \( i, j \notin H \), defined by
\[
J_{ij}^{(g)} = J_{ij}^{(1g)} + J_{ij}^{(2g)} + J_{ij}^{(3g)}, \tag{3.46}
\]
where
\[
J_{ij}^{(1g)\alpha} = \mathcal{L}_{ij}^{(1g)\alpha} = x_i \wedge \mathcal{P}_j^{(1g)\alpha},
\]
\[
J_{ij}^{(2g)\alpha} = \mathcal{L}_{ij}^{(2g)\alpha} + S_{ij}^{(2g)\alpha}, \quad \begin{cases}
\mathcal{L}_{ij}^{(2g)\alpha} = \mathcal{L}_{ij}^{(2)\alpha} \\
S_{ij}^{(2g)\alpha} = S_{ij}^{(2)\alpha} - S_{ij}^{(2H)\alpha} = 2 \sum_{r \notin H} W_{\alpha \beta} \partial_\beta x^r
\end{cases}
\]
\[
J_{ij}^{(3g)\alpha} = J_{ij}^{(3)\alpha}. \tag{3.47}
\]
It thus follows that one is left with only the equations of motion for \((\Lambda, x \notin H)\), \(\delta x S^{pg}_{eff} = \delta \Lambda S^{pg}_{eff} = 0\) derived by varying the effective action obtained this time by setting to zero terms for which \(x \in H\) in the ungauged action,

\[
S^{pg}_{eff}(\Lambda, x \notin H; K) = S^{pg}_{1eff} + S^{pg}_{2eff} + S^{pg}_{3eff},
\]

where

\[
S^{pg}_{1eff} = \frac{1}{2} \int d^2 \sigma \sum_{r,s \notin H} V^{\alpha \beta}_{rs} \partial_\alpha x^r \partial_\beta x^s,
\]

\[
S^{pg}_{2eff} = \int d^2 \sigma \sum_{r \notin H} W^{\alpha \beta}_{rs} \partial_\alpha x^r \partial_\beta \Lambda^s,
\]

\[
S^{pg}_{3eff} = S_3,
\]

and the sums \(\sum_{r,s,\ldots \notin H}\) run only over the indices corresponding to the translations not included in \(H\).

The effective action \(S^{pg}_{eff}\) expressed in terms of the coordinates \(\bar{x}\) is again of the same form given in Eq. (3.40) but, since \(S^{pg}_{3eff} = S_3\), the matrix \(G^{(3)}_{tt} \equiv 0\) in Eq. (3.41) and one finds that the metric tensor is represented by an \(N \times N\) square matrix of dimension \(N = d(d + 1)/2 - n\),

\[
\bar{G} = \begin{bmatrix} 0 & U^{(2)}_{xt} \\ U^{(2)T}_{xt} & 0 \end{bmatrix},
\]

where the matrix \(U^{(2)}_{xt}\) is again \((d - n) \times d(d - 1)/2\) dimensional.

It is easy to prove that each matrix of the block form above is degenerate and admits

\[
N_d = \left| \frac{d(d-1)}{2} - (d-n) \right|,
\]

zero eigenvalues. This implies that the dimension of the non-degenerate subspace is only

\[
N - N_d = \begin{cases} 2(d-n), & \frac{d(d-1)}{2} - (d-n) \geq 0 \\ d(d-1), & \frac{d(d-1)}{2} - (d-n) < 0 \end{cases}
\]

so that, if one gauges the whole \(d\)-dimensional translation group \((n = d)\) the effective theory \(S^{pg}_{eff} = S_3\) has no metric structure and becomes spatially empty. Only internal (originally interpreted as spin) degrees of freedom (the ones contained in \(S_3\)) survive, and the present reduction scheme looks quite singular.

Contrary to the metric tensor, the antisymmetric matrix \(\bar{B}\) is, in general, non-singular due to the presence of \(B_{tt}\) in Eq. (3.42) and one obtains an axion field potential in a \(N\)-dimensional space. This makes the overall picture quite pathological, unless one regards the extra (degenerate) dimensions as pure parameters of the theory. In so doing, one ends up with a \(N_d\)-parameter family of \(N - N_d\) dimensional \(\sigma\)-models.

However, there are exceptional cases in which the metric \(\bar{G}\) is not singular. This happens when \(U^{(2)}_{xt}\) is square, or \(N_d = 0\), for which one obtains that the matrix \(\bar{G}\) is even dimensional and necessarily has nonzero eigenvalues \(\pm G_i\) with \(i = 1, \ldots, N/2\). There are only two such cases:
1. \( d = 2, \ n = 1 \) with \( N = 2 \); and
2. \( d = 3, \ n = 0 \) with \( N = 6 \),

which we will examine in Sections 5 and respectively 6, where we prove that this last case reduces to the one studied in Ref. [4].

4 T-duality

Since both \( \hat{G} \) and \( \hat{B} \) in Eq. (3.40) can depend at most on the \( d(d-1)/2 \) Lorentz parameters \( t \), the action \( S_{\text{eff}}^{p} \) clearly displays at least \( d-n \) target space isometries corresponding to the \( d-n \) (translational) coordinates \( x \) in Eq. (3.36). This fact can be used to introduce transformations that define T-dual spaces (see Ref. [6] and references therein for a general exposition).

The linear case, for which we have just shown that the metric is degenerate, is a special case because the effective dimension of the space-time is lower than \( N \). In this case one could eliminate \( N_d \) dimensions by simply diagonalizing \( \hat{G} \) and one would end up with \( (N-N_d) = d(d-1)/2 \) isometries. However this process would mix \( x \) and \( t \) coordinates, thus making less transparent the identification of the isometric directions. For this reason in the present Section we neglect the degeneracies of the metric structure in \( S_{\text{eff}}^{q} \) and work in the full \( N \)-dimensional space. This allows us to develop a formal treatment which is valid for both \( S_{\text{eff}}^{p} \) and \( S_{\text{eff}}^{q} \). One has only to remember that in the following sections \( G_{tt} = 0 \) for \( S_{\text{eff}} = S_{\text{eff}}^{p} \).

4.1 Dual Actions

To begin with, we want to show some of the general features of our model which are related to the isometry being (possibly) more than 1-dimensional in the case in which none of the Lorentz parameters \( t \) corresponds to an isometry of the action.

First one doubles the \( d - n \) coordinates \( x_1 \equiv x \) by adding an equal number of new coordinates \( x_2 \). Then one introduces a parent action in the new \( N + (d - n) = d(d + 3)/2 - 2n \) dimensional space with coordinates \( (x_1, x_2, t) \),

\[
S_{N+d-n}(x_1, x_2, t) = S_{xt} + S_t ,
\]

where

\[
S_{xt} = \frac{1}{2} \int d^2 \sigma \left[ D_{ab}^{\alpha \beta} \left( \partial_\alpha x_1^a \partial_\beta x_1^b + \partial_\alpha x_2^a \partial_\beta x_2^b \right) + 2 \Sigma_{ab}^{\alpha \beta} \partial_\alpha x_1^a \partial_\beta x_2^b \right.
\]

\[
\left. + 2 N_{ai}^{\alpha \beta} \left( \partial_\alpha x_1^a \partial_\beta t^i + \partial_\alpha x_2^a \partial_\beta t^i \right) \right] ,
\]

\[
S_t = \frac{1}{2} \int d^2 \sigma \left[ T_{ij}^{\alpha \beta} \partial_\alpha t^i \partial_\beta t^j + \Phi R^{(2)} \right] ,
\]

and we have also defined

\[
D_{ab}^{\alpha \beta} \equiv \epsilon^{\alpha \beta} B_{xx,ab} \\
N_{ai}^{\alpha \beta} \equiv \left( g^{\alpha \beta} + 2 \epsilon^{\alpha \beta} \right) U_{xt,ai} \\
\Sigma_{ab}^{\alpha \beta} \equiv g^{\alpha \beta} \Sigma_{ab}^S + \epsilon^{\alpha \beta} \Sigma_{ab}^A \\
T_{ij}^{\alpha \beta} \equiv g^{\alpha \beta} G_{tt,ij} + \epsilon^{\alpha \beta} B_{tt,ij} ,
\]

(4.3)
with \(a, b = 1, \ldots, d-n\) and \(i, j = 1, \ldots, d (d-1)/2\). Here the matrices \(\Sigma_{ab}^S(t)\) (symmetric), \(\Sigma_{ab}^A(t)\) (antisymmetric) and the dilaton field \(\Phi(t)\), which couples to the world-sheet scalar curvature \(R^{(2)}\), have been introduced so that the world-sheet theory described by \(S_{N+d-n}\) is conformal. It is well known that a dilaton \(\Phi\) must be included whenever the metric is a non-vacuum solution of the Einstein equations. Therefore it must satisfy [1, 7]

\[
\nabla_r \nabla_s \Phi = R_{rs} ,
\]

where \(\nabla_r\) is the covariant derivative in the target background space-time and \(R_{rs}\) is the Ricci tensor. When the dimension of space-time is not 26, one must also add the central term \((N + d - n - 26)/3\).

Suppose now we impose the condition that the coordinates \(x_2\) are periodic

\[
x_2 \equiv x_2 + 2\pi .
\]

In so doing, we are actually compactifying coordinates which correspond to one of the two copies of the (non-compact) translational parameters of the Poincaré group in the gauged action in Eq. (3.7). We are then left with only \((d - n) + d - 1\) non-compact \((x_1\) and boost parameters) and \(d (d-3)/2 + 1\) compact (rotation angles) coordinates. The action \(S_{N+d-n}\) is now manifestly invariant under the \(U(1)^{d-n}\) affine symmetries acting on \(x_2\) which are generated by the \(d - n\) currents

\[
J_{2\alpha} = D_{ab}^{\alpha\beta} \partial_\beta x_2^b + \Sigma_{ab}^{\alpha\beta} \partial_\beta x_1^b + N_{ai}^{\alpha\beta} \partial_\beta t^i .
\]

One can gauge this \((d - n)\)-dimensional symmetry by minimal coupling, introducing a gauge field \(A_2\) such that

\[
\partial_\alpha x_2 \rightarrow \partial_\alpha x_2 + A_{2\alpha} ,
\]

and one gets a gauged action given by

\[
S_{N+d-n}^g(x_1, x_2, t, A_2) = S_{N+d-n} + \int d^2 \sigma A_{2\alpha}^a \left[ J_{2\alpha}^a + \frac{1}{2} D_{ab}^{\alpha\beta} A_{2\beta}^b \right] .
\]

When \(A_2\) is a pure gauge, \(A_{2\alpha} = -\partial_\alpha x_2\), one is led back to the effective action in \(N\) dimensions \(S_{\text{eff}}\) displayed in Eq. (3.40) (plus the possible dilaton field) which does not contain \(x_2\).

The equations of motion \(\delta_{A_2} S_{N+d-n}^g = 0\) imply that

\[
A_{2\alpha}^a = -D_{ab}^{\alpha\beta} J_{2b}^\beta ,
\]

where we have assumed that \(D_{ab}^{\alpha\beta}\) is invertible and we have defined \(D_{ab}^{\alpha\beta}\) such that

\[
D_{\alpha\gamma}^a D_{cb}^{\gamma\beta} = \delta_{\alpha}^a \delta_{\beta}^b .
\]

On substituting the solution for \(A_2\) into Eq. (4.8) one obtains an effective action

\[
S_N \equiv S_{N+d-n} - \frac{1}{2} \int d^2 \sigma J_{2\alpha}^a D_{\alpha\beta}^{ab} J_{2b}^\beta ,
\]

where \(S_N\) does not depend on \(x_2\) and is given by

\[
S_N(x_1, t) = \frac{1}{2} \int d^2 \sigma \left[ (T_{ij}^{\alpha\beta} + \frac{1}{2} N_{ai}^{\alpha\gamma} D_{\gamma\lambda}^{ab} N_{bj}^{\lambda\beta}) \partial_\alpha t^i \partial_\beta t^j + \Phi R^{(2)} \right] + \frac{1}{2} \int d^2 \sigma \left[ N_{ai}^{\alpha\beta} + \frac{1}{2} N_{bi}^{\alpha\gamma} D_{\gamma\lambda}^{be} \Sigma^{\beta}_{ca} + \frac{1}{2} \Sigma^{\alpha\gamma}_{ac} D_{\gamma\lambda}^{cd} N_{di}^{\lambda\beta} \right] \partial_\alpha x_2^a \partial_\beta t^i .
\]
The conclusion is that $S_N$ can be obtained from $S_{N+d-n}$ by setting $\partial x_2 = 0$ and changing

\[
T_{ij}^{\alpha\beta} \rightarrow T_{ij}^{\alpha\beta} - \frac{1}{2} N_{ai}^{\alpha\gamma} D_{ij}^{\gamma\lambda} N_{bi}^{\lambda\beta},
\]
\[
N_{ai}^{\alpha\beta} \rightarrow N_{ai}^{\alpha\beta} - \frac{1}{2} N_{bi}^{\alpha\gamma} D_{ij}^{\beta\lambda} \Sigma_{\alpha\beta}^{\gamma\lambda} + \frac{1}{2} \Sigma_{ab}^{\alpha\gamma} D_{ij}^{bc} N_{ai}^{\lambda\beta},
\]

which are the T-duality transformations in the present context.

On compactifying and gauging $x_1$ instead of $x_2$, one would obtain the same action as the one given in Eq. (4.8) with only an interchange of the labels for the two sets of $d-n$ isometric coordinates $x_1$ and $x_2$.

But suppose we define new coordinates which are linear combinations of $x_1$ and $x_2$, thus rotating and scaling the directions in which we compactify in the $(x_1, x_2)$ subspace. This would change the quantities in Eq. (4.3).

One example is given by

\[
\begin{aligned}
x &\equiv (x_1 + x_2)/2 \\
\tilde{x} &\equiv (x_2 - x_1)/2.
\end{aligned}
\]

We now impose the periodicity requirement on the coordinates $\tilde{x}$,

\[
\tilde{x} = \tilde{x} + 2\pi.
\]

The contribution $S_{xt}$ in the action $S_{N+d-n}$ becomes

\[
S_{xt} = \frac{1}{2} \int d^2 \sigma \left[ D_{ab}^{\alpha\beta} \partial_\alpha x^a \partial_\beta \tilde{x}^b + \tilde{D}_{ab}^{\alpha\beta} \partial_\alpha \tilde{x}^a \partial_\beta \tilde{x}^b + 2 N_{ai}^{\alpha\beta} \partial_\alpha x^a \partial_\beta t^i - 2 \Sigma_{ab}^{\alpha\beta} \partial_\alpha x^a \partial_\beta \tilde{x}^b \right],
\]

where

\[
D_{ab}^{\alpha\beta} \equiv 2 g^{\alpha\beta} \Sigma_{ab}^S + 2 \epsilon^{\alpha\beta} \left( B_{xx,ab} + \Sigma_{ab}^A \right),
\]
\[
\tilde{D}_{ab}^{\alpha\beta} \equiv -2 g^{\alpha\beta} \Sigma_{ab}^S + 2 \epsilon^{\alpha\beta} \left( B_{xx,ab} - \Sigma_{ab}^A \right),
\]
\[
N_{ai}^{\alpha\beta} \equiv \left( g^{\alpha\beta} + 2 \epsilon^{\alpha\beta} \right) U_{xt,ai},
\]
\[
\Sigma_{ab}^{\alpha\beta} \equiv \epsilon^{\alpha\beta} \Sigma_{ab}^A.
\]

The whole action $S_{N+d-n}$ is now manifestly invariant under the $U(1)^{d-n}$ affine symmetries acting on $\tilde{x}^a$ which are generated by the currents

\[
\tilde{J}_a^\alpha = \tilde{D}_{ab}^{\alpha\beta} \partial_\beta \tilde{x}^b - 2 \Sigma_{ab}^{\alpha\beta} \partial_\beta x^b.
\]

As before, one can gauge this $(d-n)$-dimensional symmetry by minimal coupling introducing a gauge field $\tilde{A}$ such that

\[
\partial_\alpha \tilde{x} \rightarrow \partial_\alpha \tilde{x} + \tilde{A}_a,
\]

and one gets a gauged action given by

\[
S_{N+d-n}^g(x, \tilde{x}, \tilde{A}) = S_{N+d-n} + \frac{1}{2} \int d^2 \sigma \tilde{A}_a^{\alpha} \left[ \tilde{J}_a^\alpha + \tilde{D}_{ab}^{\alpha\beta} \tilde{A}_b^\beta \right],
\]

15
which, on using the equations of motion for the gauge field, \( \delta \tilde{A}_N^g = 0 \),

\[
\tilde{A}_N^g = -\tilde{D}_{\alpha \beta}^a \tilde{J}_b^\alpha ,
\]

becomes the new effective action in \( N \) dimensions,

\[
S_N(x, t) = S_t + \frac{1}{2} \int d^2 \sigma \left[ D^{\alpha \beta}_{ab} \partial_\alpha x^a \partial_\beta x^b + 2 N^{\alpha \beta}_{ai} \partial_\alpha x^a \partial_\beta t^i \right],
\]

where the T-duality transformation is

\[
D \to D' = D - 4 \Sigma \tilde{D}^{-1} \Sigma .
\]

Now \( S_N \) does not contain \( \tilde{x} \) and is again different from the action \( S_{\text{eff}} \) in Eq. (3.40).

If one instead compactifies and gauges \( x \), the corresponding currents are given by

\[
J_a^\alpha = D_{ab}^{\alpha \beta} \partial_\beta x^b + N_{ai}^{\alpha \beta} \partial_\beta t^i - 2 \Sigma_{ab}^{\alpha \beta} \partial_\beta \tilde{x}^b .
\]

One obtains an effective action which depends on \( \tilde{x} \) alone and can be obtained by setting \( \partial x = 0 \) everywhere in Eq. (4.16) above, with

\[
\begin{align*}
T & \to T - N D^{-1} N \\
\tilde{D} & \to \tilde{D} - 4 \Sigma D^{-1} \Sigma \\
N & \to N + 2 N D^{-1} \Sigma + 2 \Sigma D^{-1} N .
\end{align*}
\]

One could then conclude that a different linear combination of the form given in Eq. (1.5) for our toy model,

\[
\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\]

in place of Eq. (4.14) would lead to a different \( N \)-dimensional \( \sigma \)-model and all of them are T-duals of \( S_{\text{eff}} \) in Eq. (3.40). However, in order to prove that the space-times corresponding to different choices of the matrix \( \Theta \) are physically different, one should be able to compare the scalar curvatures and the other invariant quantities of General Relativity, including the axion field.

### 4.2 Dualizing one isometric coordinate

In the following Sections we will consider examples in which we want to compute the dual of the action with respect to one isometric coordinate. Thus we now specialize the results obtained so far to this simpler case.

From the definition in Eq. (4.3) it follows that, when the set of isometric coordinates \( x_1 \) we gauge is given by \( x_1^1 \) only, \( D_{11}^{\alpha \beta} = 0 \) and no quadratic term in \( A_1 \) nor \( A_2 \) will ever appear. This implies the gauge field must be a pure gauge and we obtain the action \( S_{\text{eff}} \) we started from before we doubled the coordinate \( x_1 \).

In order to get a non trivial answer, we have to perform a transformation of the type displayed in Eq. (4.26), \( e.g. \) the one in Eq. (4.14). Then we have

\[
\begin{align*}
D_{11}^{\alpha \beta} &= 2 g^{\alpha \beta} \Sigma_{11}^S \\
\tilde{D}_{11}^{\alpha \beta} &= -2 g^{\alpha \beta} \Sigma_{11}^S ,
\end{align*}
\]
which we can expect to be in general different from zero. If we gauge $\tilde{x}^1$, we obtain a generating current

$$\tilde{J}_1^\alpha = -2g^{\alpha\beta}\Sigma^S_{11} \left( \partial_\beta \tilde{x}^1 + \partial_\beta x^1 \right) + \sum_{b>1} \left[ \tilde{D}_{1b}^{\alpha\beta} \partial_\beta \tilde{x}^b - 2 \Sigma^\alpha_{1b} \partial_\beta x^b \right],$$

(4.28)

for the corresponding $U(1)$ symmetry, in which we singled out the first term in the sum.

If the terms with $b > 1$ vanish in the expression for the current above, the ungauged action (before one doubles $x^1$) reads

$$S_{\text{eff}} = S_t + \int d^2\sigma N^\alpha_{1i} \partial_\alpha x^1 \partial_\beta t^i,$$

(4.29)

while the gauged action becomes

$$S^g_{N+d-n}(x, \tilde{x}, \tilde{A}^1) = S_{N+d-n} - \int d^2\sigma \tilde{A}^1_\alpha g^{\alpha\beta}\Sigma^S_{11} \left[ \partial_\beta \tilde{x}^1 + \partial_\beta x^1 + \tilde{A}^1_\beta \right],$$

(4.30)

and one obtains

$$S_N(x, t) = S_t + \int d^2\sigma \left[ g^{\alpha\beta}\Sigma^S_{11} \partial_\alpha x^1 \partial_\beta x^1 + N^\alpha_{1i} \partial_\alpha x^1 \partial_\beta t^i \right],$$

(4.31)

where the metric tensor $\tilde{G}$ has now acquired the new component $G_{xx,11} = 2 \Sigma^S_{11}$.

5 The lowest dimensional case: ISO(1, 1)

We now consider the $(d = 2)$-dimensional case for which the algebra is simple enough to allow one to carry the computation to the end.

Every element $\Lambda \in SO(1, 1)$ can be written as function of the only boost parameter $t \in \mathbb{R}$,

$$\Lambda^i_j = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad \Lambda^i_j = \begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix}. \quad (5.1)$$

The relevant 1-forms in Eq. (2.4) become

$$V^i = \begin{bmatrix} \cosh t \, dx^1 - \sinh t \, dx^2 \\ \cosh t \, dx^2 - \sinh t \, dx^1 \end{bmatrix}, \quad W^{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} dt. \quad (5.2)$$

It is then easy to find that

$$S_1 = S_3 = 0$$

$$S_2 = \int d^2\sigma g^{\alpha\beta} \left[ \left( K_2 \sinh t - K_1 \cosh t \right) \partial_\alpha x^1 \partial_\beta t^i \\
+ \left( K_1 \sinh t - K_2 \cosh t \right) \partial_\alpha x^2 \partial_\beta t^i \right],$$

(5.3)

where total derivatives are discarded as usual and $(K_1, K_2)$ are the only relevant independent constants satisfying Eq. (2.3) in $d = 2$.

The (conserved) linear momentum currents related to $S_2$ are given by

$$\mathcal{P}_1^\alpha = g^{\alpha\beta} (K_1 \cosh t - K_2 \sinh t) \partial_\beta t$$

$$\mathcal{P}_2^\alpha = g^{\alpha\beta} (K_2 \cosh t - K_1 \sinh t) \partial_\beta t.$$
Upon varying the parameter $t$ one finds that the following quantity is also conserved

\[ J^\alpha = P_1^\alpha x^2 - P_2^\alpha x^1 + g^{\alpha\beta} (K_1 \cosh t - K_2 \sinh t) \partial_{\beta} x^1 \\
+ g^{\alpha\beta} (K_2 \cosh t - K_1 \sinh t) \partial_{\beta} x^2. \]  

(5.5)

It is quite obvious that, since only the term proportional to $g^{\alpha\beta}$ survives, no axion field will appear in the resulting $\sigma$-models. Further, one can gauge only 1-dimensional subgroups, since eliminating both $x^1$ and $x^2$ leads to $S_{\text{eff}}^{\text{pg}} = 0$.

### 5.1 Gauging a 1-dimensional subgroup

Since $S_2$ is linear in both $x^1$ and $x^2$, if we gauge a 1-dimensional subgroup, e.g. the one corresponding to $x^2$, we obtain that the gauge field $A^2$ is a pure gauge, $A^2 = -\partial x^2$. This is actually the first of the two exceptional cases listed in Section 3.4.

The conserved currents which survive are given by $P_1$ and $J(x^2 = 0)$ above. We then define

\[ \begin{cases} 
X &\equiv x^1 + t \\
T &\equiv x^1 - t,
\end{cases} \]  

(5.6)

and we obtain

\[ S_{\text{eff}}^{\text{pg}} = \frac{1}{2} \int d^2 \sigma g^{\alpha\beta} f(X - T) (\partial_\alpha X \partial_\beta X - \partial_\alpha T \partial_\beta T), \]  

(5.7)

where $f(t) \equiv K_2 \sinh t - K_1 \cosh t$. The diagonal form of the metric tensor is thus given by

\[ \tilde{G} = \begin{bmatrix} f & 0 \\
0 & -f \end{bmatrix}, \]  

(5.8)

which becomes singular (both components vanish) for $f(t_s) = 0 \Leftrightarrow \tanh t_s = K_1/K_2, K_2 \neq 0$.

The curvature of space-time and the Ricci tensor are zero everywhere, so $\tilde{G}$ above represents a vacuum solution. The singularity $t = t_s$ is a light-like volume singularity whose location depends on the ratio of the constants $K_1/K_2$. For example, when $K_1 = 0$ and $K_2 = 1$, one finds that $\tilde{G}$ becomes singular along the light-cone $X = T$.

### 5.2 T-dual form

The action $S_{\text{eff}}^{\text{pg}}$ is of the form given in Eq. (4.29) with $S_t = \int d^2 \sigma \Phi R^{(2)}$ and $N_{11}^{\alpha\beta} = 2 g^{\alpha\beta} f(t)$. If we introduce a coordinate $x_1^1$ and define $x$ and $\tilde{x}$ according to Eq. (4.14), we can then dualize with respect to the coordinate $\tilde{x}$ and obtain a new metric tensor whose diagonal form is given by

\[ \tilde{G} = \begin{bmatrix} \Sigma^S_{11} - \sqrt{(\Sigma^S_{11})^2 + f^2(t)} & 0 \\
0 & \Sigma^S_{11} + \sqrt{(\Sigma^S_{11})^2 + f^2(t)} \end{bmatrix}. \]  

(5.9)

Regardless of the explicit form of $\Sigma^S_{11} = \Sigma^S_{11}(t)$, $\det(\tilde{G}) = f^2$ and one obtains the same volume singularity for $f(t_s) = 0$. The scalar curvature is again zero everywhere.

The causal structure determined by $\tilde{G}$ in Eq. (5.9) is different from the one given by the metric tensor in Eq. (5.8), since in the latter case one has an overall change of sign when going through $f = 0$, however in the former this cannot happen.

Further, according to Eq. (4.4), since the Ricci tensor is still zero, the presence of a non-vanishing $\Sigma^S_{11}(t)$ does not affect the dilaton.
6 \quad S = S_2 \text{ in } d = 3 \text{ dimensions}

The second exceptional case listed in Section 3.4 has \( d = 3 \) and \( n = 0 \). Since we are mainly interested in the metric structure of the effective theory, we only consider \( S = S_2(K) \) according to the general form given in Eq. (3.50).

First we prove that, when \( g^{\alpha\beta} = \eta^{\alpha\beta} \) there is actually only one such action, namely the one with \( K_{ijk} = \epsilon_{ijk} \), and one recovers the model previously studied in Ref. [4]. In fact, every matrix \( K_{ijk}^{(2)} \) with the symmetry properties displayed in Eq. (2.3) can be written

\[
K_{ijk}^{(2)} = \sum_{l=0}^{2} A_i^{(l)} \Sigma_j^{(l)} K_{jk}^{(l)},
\]

where

\[
\Sigma^{(0)} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}, \quad \Sigma^{(1)} = \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0 
\end{bmatrix}, \quad \Sigma^{(2)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 
\end{bmatrix},
\]

and \( A_i^{(l)} \) are nine arbitrary constants such that \( \det(A_i^{(l)}) \neq 0 \). The action \( S_2 \) then can be written

\[
S_{(2,1)}(\Lambda, y, K) = \frac{1}{2} \int d^2 \sigma \ A_i^{(l)} \Sigma_j^{(l)} \partial_- y^{i} (\partial_+ \Lambda \Lambda^{-1})^{jk}
\]

\[
= \frac{1}{2} \int d^2 \sigma \ A_i^{(l)} \partial_- y^{i} \text{Tr} \left[ \Sigma^{(l)} (\partial_+ \Lambda \Lambda^{-1}) \right],
\]

where \( y \in \mathbb{R}^3 \) and \( \sigma^\pm \) have been defined in Eq. (2.11). The trace in the integrand above can now be evaluated assuming a specific parameterization of the Lorentz group \( SO(2,1) \). As in Ref. [4], we write any matrix \( \Lambda_{ij} \) as a product of two rotations (of angles \( \alpha \) and \( \gamma \)) and a boost (\( \beta \)),

\[
\Lambda = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha 
\end{bmatrix} \begin{bmatrix}
\cosh \beta & 0 & \sinh \beta \\
0 & 1 & 0 \\
\sinh \beta & 0 & \cosh \beta 
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \gamma & -\sin \gamma \\
0 & \sin \gamma & \cos \gamma 
\end{bmatrix},
\]

and we obtain

\[
\text{Tr} \left[ \Sigma^{(0)} (\partial_+ \Lambda \Lambda^{-1}) \right] = \mathcal{P}_+^0 = \partial_+ t^0 + \cosh t^1 \partial_+ t^2
\]

\[
\text{Tr} \left[ \Sigma^{(1)} (\partial_+ \Lambda \Lambda^{-1}) \right] = \mathcal{P}_+^1 = \cos t^0 \partial_+ t^1 + \sin t^0 \sinh t^1 \partial_+ t^2
\]

\[
\text{Tr} \left[ \Sigma^{(2)} (\partial_+ \Lambda \Lambda^{-1}) \right] = \mathcal{P}_+^2 = \sin t^0 \partial_+ t^1 - \cos t^0 \sinh t^1 \partial_+ t^2.
\]

Finally, on defining

\[
x^i \equiv \sum_{j=0}^{2} A_j^{(i)} y^j,
\]

one gets

\[
S_{(2,1)}(\Lambda, x; K) = \int d^2 \sigma \mathcal{P}_{+}^{k} \partial_+ x_k = S_{(2,1)}(\Lambda, x; \epsilon_{ijk}),
\]
as claimed.

This result is very peculiar and follows from the fact that there are \(d^2 (d-1)/2\) independent elements in \(K_{ijk}^{(2)}\). This number is equal to the square of the space-time dimension for \(d = 3\) and one can thus use all these constants to build the linear combination given in Eq. (6.8). In general (for \(d > 3\) one has
\[
d^2 (d-1)/2 > d^2,
\]
and one can not eliminate in this way \(d^2 (d-3)/2\) elements of \(K_{ijk}^{(2)}\).

The three linear momentum currents in Eq. (6.7) define a metric tensor \(\bar{G}\) in 6 dimensions of the form given in Eq. (3.50) with
\[
U_{xt} = \begin{bmatrix}
1 & 0 & \cosh t^1 \\
0 & \cos t^0 & \sin t^0 \sinh t^1 \\
0 & \sin t^0 & -\cos t^0 \sinh t^1
\end{bmatrix}.
\]

The Ricci tensor computed from the metric \(\bar{G}\) has the following non-zero components,
\[
\bar{R}_{tt} = -1 \\
\bar{R}_{t\ell t} = -\cosh(t^1) \\
\bar{R}_{t\ell t} = 1 \\
\bar{R}_{t\ell t} = -1,
\]
and its trace \(\bar{R} = 0\). The subspace \(t^1 = 0\) is a volume singularity, since
\[
\det(\bar{G}) = f^2(t^1),
\]
with \(f \equiv \sinh t^1\).

We now analyze two degenerate cases following from \(S_{(2,1)}\).

### 6.1 Gauging a 1-dimensional translation

We notice that \(S_{(2,1)}\) is already invariant under the following semi-local action of the Poincaré group:
\[
g \rightarrow h_L \sigma^+ g \ h_R^{-1} \sigma^-, \quad (\theta_L, y_L) \in ISO(2,1),
\]
where \(h_{L/R} = (\theta_{L/R}, y_{L/R}) \in ISO(2,1)\). However, it is not invariant under the fully local action of any subgroup \(H\) of \(ISO(2,1)\) given by \(g \rightarrow h_L \ g \ h_R^{-1} = (\theta_L \Lambda^{-1} \theta_R^{-1}, -\theta_L \Lambda^{-1} y_R + \theta_L x + y_L)\), where \(h_{L/R} = h_{L/R}(-\sigma^- \sigma^+) = (\theta_{L/R}, y_{L/R}) \in H\), due to the dependence of \(h_L\) on \(\sigma^-\) and of \(h_R\) on \(\sigma^+\). To promote \(H\) to a gauge symmetry of the action we introduce again the gauge field \(A_{\pm} = (\omega_{\pm}, \xi_{\pm}) \in iso(2,1)\), and the covariant derivatives \(D_\pm g = \partial_\pm g + A_\pm\). The requirement that \(H\) acts invariantly, \(\delta g = h_L \ g \ (0, y_L)\), leads to \(h_{L/R} = (0, y_{L/R} \in \mathbb{R}^n), n \leq 3\), so that
\[
\begin{cases}
\omega_+ = \xi_+ \equiv 0 \\
\xi_k \equiv 0 \quad k \not\in H
\end{cases}
\]

If we gauge the 1-dimensional subgroup corresponding to $x^0$, the gauged action $S_g(x, t, \xi^0)$ is linear in $\xi^0$ and after we eliminate $t^0$ as an irrelevant parameter, we obtain the effective action [4]

$$S_{\text{eff}}(x^1, x^2, t^1, t^2) = \int d^2 \sigma \left[ -\partial_+ t^1 \partial_- x^1 + \sinh t^1 \partial_+ t^2 \partial_- x^2 \right]$$

$$= \int d^2 \sigma \left( \eta^{\alpha\beta} + \epsilon^{\alpha\beta} \right) \left[ \partial_\alpha t^1 \partial_\beta x^1 - f \partial_\alpha t^2 \partial_\beta x^2 \right] \, ,$$

(6.16)

where again $f = \sinh t^1$. The metric tensor is \(N - N_d = 5 - 1 = 4\)-dimensional and is given by

$$\tilde{G} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -f \\ 1 & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \end{bmatrix} \, .$$

(6.17)

The axion field potential in this non-degenerate 4-dimensional subspace is given by

$$\tilde{B} = 2 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -f \\ -1 & 0 & 0 & 0 \\ 0 & f & 0 & 0 \end{bmatrix} \, ,$$

(6.18)

its field strength $\tilde{H}$ having $\tilde{H}_{124} = -2 \cosh t^1$ as the only non-zero component.

The Ricci tensor in this frame of reference has one non-vanishing component,

$$\tilde{R}_{t^1 t^1} = \frac{1}{2} \frac{\sinh^2 t^1 - 1}{\sinh^2 t^1} \, ,$$

(6.19)

and the scalar curvature $\tilde{R}$ is zero.

The signature of the metric is $2 + 2$ and never changes, as can be inferred by noting that the determinant of $\tilde{G}$ is again given by the expression in Eq. (6.13), and the eigenvalues of $\tilde{G}$ are given by ($\pm 1, \pm f$). This corrects an erroneous statement in the last part of Ref. [4].

### 6.2 T-dual form

Both the metric tensor $G$ and the axion field $B$ given above depend only on $t^1$, so that in this case we have a Lorentz parameter ($t^2$) which is irrelevant, as are the two translational parameters $x^1$ and $x^2$.

The dual effective theory obtained upon introducing $x$ and $\tilde{x}$ as given in Eq. (4.14) with $x_1^1 \equiv x^1$ contains the same axion field potential $\tilde{B}$ given in Eq. (6.18) above, but the metric tensor acquires a new component

$$\tilde{G}_{\tilde{x} x} = 2 \Sigma(t^1) \, .$$

(6.20)

This extra term generates new non-zero components for the Ricci tensor,

$$\tilde{R}_{\tilde{x} x} = 2 \Sigma \Sigma \frac{\sqrt{f^2 + 1}}{f} + \tilde{\Sigma}$$

$$\tilde{R}_{x t^1} = -\tilde{\Sigma} - \frac{f}{\sqrt{f^2 + 1}}$$

$$\tilde{R}_{x^2 t^2} = \Sigma f + \Sigma \sqrt{f^2 + 1} \, ,$$

(6.21)
where $\tilde{\Sigma} \equiv \partial_t \Sigma$, and a non-vanishing scalar curvature,

$$
\dot{R} = -3 \Sigma - 2 \frac{\tilde{\Sigma}}{f^2} - 4 \sqrt{f^2 + 1} \frac{\dot{\Sigma}}{f}.
$$

(6.22)

However the determinant of the metric is still given by Eq. (6.13), while the eigenvalues of $\bar{G}$ become $(\Sigma \pm \sqrt{\Sigma^2 + 1}, \pm f)$.

### 6.3 Gauging a 2-dimensional translation

Upon gauging a 2-dimensional subgroup one obtains an effective action in $N - N_d = 4 - 2 = 2$ dimensions of the same type as the one in Eq. (3.40), but with $\bar{G}$ a constant symmetric matrix with signature 1+1 and $\bar{B}$ a constant antisymmetric matrix.

On dualizing with respect to the unique translational coordinate which is left, one obtains a metric tensor whose diagonal form is

$$
\bar{G} = \begin{bmatrix}
\Sigma - \sqrt{\Sigma^2 + 1} & 0 \\
0 & \Sigma + \sqrt{\Sigma^2 + 1}
\end{bmatrix},
$$

(6.23)

in which there are no singularities regardless of the specific form of $\Sigma = \Sigma(t)$.

### 7 Conclusions

In this paper we have examined a new class of $\sigma$-models, which are generated by gauging a subgroup of the Poincaré group $ISO(d - 1, 1)$ which acts invariantly from the left. The fact that this group is a noncompact, semi-direct product group differentiates our coset model from all such models studied heretofore. There are several intriguing results in this investigation. Our starting point is a model with values in $ISO(d - 1, 1)$, which describes spinning strings in flat $(d - 1) + 1$ dimensions. After promoting a translation subgroup to a gauge symmetry, however, the resulting action describes spinless strings moving in curved space-times and interacting with an axion field. If the effective action is obtained from a pure gauge field, the resulting metric tensor is in general degenerate. The degeneracy is equal to the difference between the number of relevant (Lorentz) coordinates and the number of isometries. Finally, the effective actions inherently possess T-duals.

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