Vortex Solution in 2+1 Dimensional Pure Yang–Mills Theory at High Temperatures

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Abstract
At high temperatures the $A_0$ component of the Yang–Mills field plays the role of the Higgs field, and the 1-loop potential $V(A_0)$ plays the role of the Higgs potential. We find a new stable vortex solution of the Abrikosov–Nielsen–Olesen type, and discuss its properties and possible implications.

Recently there has been renewed interest in the quantized $Z(N_c)$ vortices as possible candidates for the confinement mechanism [1, 2, 3]. If vortices are physical objects (and not lattice or gauge artifacts) playing a role in the dynamics of the vacuum fluctuations they have to be found as stable solutions of the effective action obtained from integrating out high frequencies of the field. In the zero-temperature case the one-loop effective action for vortices has been introduced in our previous work [4]. The zero-derivative term of the effective action, i.e. the 'potential energy' of the vortex has been found in that paper: it does not lead to any stable solution. This is probably not surprising since all derivatives of the effective action should be summed up before one gets to a definite conclusion on the existence of vortex solutions.

In this letter we address a related but simpler question on whether there are stable vortex solutions in the case of nonzero temperatures. Moreover, we restrict ourselves to
a simple and more ‘pure’ case of 2 + 1 dimensions. If the temperature $T$ is high enough vortices can be viewed as cylinders pointing in the ‘time’ direction, with a nontrivial profile in the transverse ‘spatial’ plane. We shall show that the Abrikosov–Nielsen–Olesen-type vortex solution indeed exists in this case, and we shall present its profile and energy per unit length, as functions of temperature.

The 2 + 1-dimensional YM theory has a coupling constant $g_3^2$ of the dimension of mass; high temperatures mean $T \gg g_3^2$. One can always choose a gauge with the ‘time’ component of the YM field $A_0^a(x)$ independent of ‘time’. In this gauge, one can integrate out the high-momenta components of the fields to obtain the low-momenta effective action for time-independent fields $A_\mu^a(x)$, $\mu = 0, 1, 2$. The separation scale between high and low momenta is given, naturally, by the temperature: high momenta means $p > 2\pi T$. The effective action is obtained by integrating over nonzero Matsubara frequencies, $\omega_n = 2\pi nT, n > 0$, and over zero ($n = 0$) Matsubara frequency but large spatial momenta. It can be expanded in powers of the spatial derivatives of the $A_\mu^a(x)$ fields, divided by appropriate powers of $T$. The zero-derivative term is the ‘potential energy’ $V(A_0)$. Since the spatial size of the vortex solution is expected to be larger than $1/T$ one can neglect all terms of the derivative expansion except the first, zero-derivative term $V(A_0)$. The latter can be written as

$$V(A_0) = \frac{2T^3}{\pi} \sin^2 \pi \nu \int_0^\infty dp \frac{p^2 \cosh p}{\sinh p (\sinh^2 p + \sin^2 \pi \nu)} = \pi T^3 \nu^2 \left( \ln \frac{1}{\nu^2} + 2.67575 \right) + O(\nu^4)$$

$$= \frac{T}{4\pi} (A_0^a)^2 \ln \frac{\text{const} \cdot T^2}{(A_0^a)^2} + O(A_0^4), \quad \nu = \frac{\sqrt{A_0^a A_0^a}}{2\pi T}. \quad (1)$$

It is a periodic function of the dimensionless variable $\nu$ with unit period, depicted in Fig.1. Notice that the potential energy (1) is nonanalytic at the minima $\nu = \text{integer}$. The fact that the second derivative of $V(A_0)$ has a logarithmic singularity at $\nu = \text{integer}$ is related to the infrared divergency of the Debye mass in 2 + 1 dimensions \[5, 4\], and will have important consequences for the vortex solution.

Adding $V(A_0)$ to the classical YM action, $F_{\mu\nu}^a F_{\mu\nu}^a/4g_3^2$, we get the energy functional in the spatial transverse plane:

$$E_\perp = \int d^2x \left\{ \frac{1}{2g_3^2} \left[ \epsilon_{ij} \left( \partial_i A_j^a + \frac{1}{2} f^{abc} A_i^b A_j^c \right) \right]^2 + \frac{1}{2g_3^2} \left( D_i^a A_i^a \right)^2 + V(A_0^a) \right\}$$

where $i, j = 1, 2$ denote spatial components, $D_i$ is the covariant derivative, $D_i^{ab} = \partial_i \delta^{ab} + f^{abc} A_i^c$.

Eq. (2) presents a 2-dimensional YM + Higgs system where $A_0^a$ plays the role of the adjoint Higgs field, with a peculiar form of the potential energy (1).

We choose the following vortex-type Ansatz for the fields, restricting ourselves to the $SU(2)$ colour group for simplicity:

$$A_i^a = \delta^{a3} \epsilon_{ij} n_j \frac{\mu(\rho)}{\rho}, \quad \rho = \sqrt{x_1^2 + x_2^2}, \quad n_j = \frac{x_j}{\rho}, \quad \mu(\rho) = \frac{x_3}{\rho} \quad (3)$$

$$A_0^a = \delta^{a1} \epsilon_{ij} n_j \nu(\rho) 2\pi T \quad (4)$$
where \( \mu(\rho) \) and \( \nu(\rho) \) are trial profile functions of the distance \( \rho \) from the vortex center. The Ansatz (3) corresponds to the radial component \( A^a_\rho = 0 \) and the azimuthal component \( A^a_\phi(\rho) = \delta^{a3} \mu(\rho) / \rho \). The closed Wilson loop in the representation labelled by spin \( J \), circling around the center of the vortex in the transverse plane at distance \( \rho \), is

\[
W_J(\rho) = \frac{1}{2J + 1} \, \text{Tr} \, \text{P} \exp i \int d\phi \rho A^a_\phi T^a = \frac{1}{2J + 1} \frac{\sin[(2J + 1)\pi \mu(\rho)]}{\sin[\pi \mu(\rho)]}, \quad \mu(\rho) = \sqrt{A^a_\rho A^a_\rho \rho},
\]

(5)

where \( \mu(\rho) \) is the magnetic field flux along the vortex inside a tube of radius \( \rho \). If \( \mu(\rho) \rightarrow 1 \) at large \( \rho \) the Wilson loop \( W_J(\rho) \rightarrow (-1)^{2J} \). In particular, in the fundamental representation one has \( W_{1/2} \rightarrow -1 \). This is the definition of the quantized \( Z(2) \) vortex. Our solution will be precisely of this type.

Let us introduce the dimensionless radius

\[
x = \rho g_3^{1/2} T^{3/4}
\]

(6)

and the dimensionless parameter

\[
\alpha = \frac{g_3}{\sqrt{T(2\pi)^2}}.
\]

(7)

At high temperatures \( \alpha \ll 1 \). In terms of these quantities the action functional for the vortex of length \( 1/T \) becomes
\[
S_{\text{vortex}} = \frac{E_{\perp}}{T} = \frac{1}{2\pi\alpha'} \int_0^\infty dx \left[ \frac{1}{2} \nu'^2 + \frac{1}{2} \nu^2 (1 - \mu)^2 + \alpha \frac{1}{2\alpha^2} \nu'^2 + \alpha v(\nu) \right],
\]
where we have introduced the dimensionless ‘Higgs potential’

\[
v(\nu) = \frac{2}{\pi} \sin^2 \pi \nu \int_0^\infty dp \frac{p^2 \cosh p}{\sinh p (\sinh^2 p + \sin^2 \pi \nu)} = \begin{cases} 
\pi \nu^2 \ln \frac{1}{\nu^2} + \ldots & \text{at } \nu \approx 0, \\
\pi (1 - \nu)^2 \ln \frac{1}{(1 - \nu)^2} + \ldots & \text{at } \nu \approx 1.
\end{cases}
\]

The Euler–Lagrange equations of motion for the profile functions \(\mu, \nu(\rho)\) are:

\[
\alpha \frac{d}{dx} \left( \frac{1}{x} \frac{d\mu}{dx} \right) = -\frac{1}{x} \nu^2 (1 - \mu),
\]
\[
\frac{d}{dx} \left( \frac{d\nu}{dx} \right) = \frac{1}{x} \nu (1 - \mu)^2 + \alpha x \frac{d\nu}{d\nu},
\]
\[
\frac{dv}{d\nu} = 2 \sin 2\pi \nu \int_0^\infty dp \frac{p \sinh p}{\sin^2 p + \sin^2 \pi \nu} = \begin{cases} 
2\pi \nu \left( \ln \frac{1}{\nu^2} + 1.67575 \right) & \text{at } \nu \approx 0, \\
-2\pi (1 - \nu) \left( \ln \frac{1}{(1 - \nu)^2} + 1.67575 \right) & \text{at } \nu \approx 1.
\end{cases}
\]

We look for the solution with the following boundary conditions:

\[
\mu(0) = \nu(0) = 0, \quad \mu(\infty) = \nu(\infty) = 1,
\]
corresponding to the quantized \(Z(2)\) vortex, with the ‘Higgs field’ \(A_0\) going from the trivial minimum at \(\rho = 0\) to a non-trivial one at \(\rho \to \infty\).

The behaviour of the profile functions near the origin and at infinity can be found analytically. At small \(x\) we get:

\[
\mu(x) = c_1 x^2 + c_2 x^4 + \ldots, \\
\nu(x) = d_1 x + d_2 x^3 \ln x + d_3 x^3 + \ldots.
\]

The coefficients \(c_1, d_1\) are arbitrary but the higher coefficients are determined from eqs. \([10, 11]\):

\[
c_2 = -\frac{d_1^2}{8\alpha}, \quad d_2 = -\frac{\alpha \pi d_1}{2}, \quad d_3 = \frac{d_1}{8} \left[ -2c_1 + \alpha \pi (3 + 2h - 4 \ln d_1) \right], \quad h = 1.67575\ldots
\]

At large values of \(x\) the analytical solution is:

\[
\mu(x) \approx 1 - e_1 \sqrt{x} \exp \left( -\frac{x}{\sqrt{\alpha}} \right),
\]
\[
\nu(x) \approx 1 - f_1 \exp \left( -\pi \alpha x^2 \right),
\]
Figure 2: Profile functions of the vortex for two values of temperature, corresponding to $\alpha = 0.15$ and 0.5.

where the constants $e_1, f_1$ are not determined by the equations. Notice that the ‘Higgs field’ $\nu(x)$ approaches its asymptotic value at infinity not as an exponent but as a gaussian. This is a consequence of the logarithmic divergence of the Debye mass in 2+1 dimensions. Being rewritten in original notations eq. (17) reads

$$|A_0^0(\rho)| \to 2\pi T \left[ 1 - f_1 \exp \left( -\frac{g_3^2 T}{4\pi \rho^2} \right) \right], \quad \text{at } \rho \to \infty.$$ 

We notice that the characteristic scale for the variation of $A_0(\rho)$, namely $1/g_3\sqrt{T}$, is, at high temperatures, much larger than $1/T$. It justifies neglecting derivative terms of $A_0$ in the effective action and leaving only the zero-derivative term $V(A_0)$, as in eq. (2).

The coefficients $c_1, d_1, e_1, f_1$ are determined by solving eqs. (10, 11) starting from the expansion (14) towards larger values of $x$ and starting from the asymptotic form (16, 17) towards smaller values of $x$, and matching the functions and their derivatives at some intermediate point $x \sim 1$; this is done numerically. The resulting profile functions $\mu(x)$ and $\nu(x)$ are shown in Fig.2, for two values of $\alpha = 0.15$ and 0.5.

Integrating the profile functions over the whole range of $x$ we get the action of the vortex

$$S_{\text{vortex}} = \frac{E_1}{T}$$

plotted in Fig.3, as function of temperature $T$. At $T \gg g_3^2$ the action is large, so that the vortices are exponentially suppressed. This is a theoretically ‘clean’ case: as explained above, the use of the energy functional (2) for finding the vortex solution is justified. The vortices are in fact short and broad cylinders oriented in the ‘time’ direction.
Figure 3: The vortex action as function of dimensionless temperature $T/g_3^2$.

Indeed, the length of the cylinders is $1/T$ while the characteristic radius where the ‘Higgs field’ $A_0$ reaches its asymptotic value $2\pi T$ is of the order of $1/g_3\sqrt{T} \gg 1/T$, see eq. (17). It should be mentioned, however, that the magnetic flux $\mu(\rho)$ reaches its asymptotic value of 1 at a smaller radius $\rho \sim 1/T$, see eq. (16).

To build the $2 + 1$-dimensional vacuum at high temperatures out of the vortices, one needs first of all to gauge rotate the ‘Higgs field’ $A_0^a$ (4) to, say, the third direction in colour space: one cannot add up vortices with different colour orientations of $A_0$ at infinity. This is achieved with a help of a $\phi$-dependent gauge transformation

$$A'_\mu = U^\dagger(\phi)A_\mu U(\phi) + i\delta_{\mu\phi} \frac{1}{\rho} U^\dagger(\phi) \frac{d}{d\phi} U(\phi), \quad U(\phi) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} e^{i\phi/2} & ie^{i\phi/2} \\ e^{-i\phi/2} & -ie^{-i\phi/2} \end{array} \right).$$ (19)

Under this gauge transformation the $A_0$ component of the YM field becomes proportional to $\tau_3$, $A_0^a = \delta^a_3 2\pi T \nu(\rho)$. The gauge transformation (19) is, however, discontinuous at $\phi = 2\pi$, therefore the azimuthal component $A'_\phi$ will now have a ‘Dirac-surface’ singularity at $\phi = 0$:

$$A'_\phi = \delta^a_3 \frac{1}{\rho} [1 - \mu(\rho) - 2\pi \delta(\phi)].$$ (20)

The Wilson loop (5) is, naturally, preserved by this gauge transformation: it remains $(-1)^{2J}$ at $\rho \to \infty$ (for the representation labelled by spin $J$).
Having oriented the ‘Higgs expectation value’ \( A_0(\infty) \) in one colour direction it is now possible to add up many vortices, each of them necessarily carrying a singular Dirac surface. Apart from the factor \( \exp(-S_{\text{vortex}}) \) the statistical weight of a vortex is determined by the fluctuation determinant, in particular by the zero modes of the solution. We expect three zero modes here, two of which are associated with the spatial position of the vortex center \( z_1 \), and one pure gauge cyclic mode associated with shifts in the ‘time’ direction. (The latter can be revealed if one uses a gauge with \( A_0 \) explicitly dependent on time.)

Denoting the (uncalculated) prefactor arising from the fluctuation determinant and from zero modes by \( \kappa(T, g_3) \) we can write the vortex partition function as that of a gas,

\[
Z_{\text{vortex}} = \sum_N \frac{1}{N!} \left( \int d^2 z \, \kappa \exp(-S_{\text{vortex}}) \right)^N,
\]  

implying that at large \( T \) vortices are dilute and thus neglecting their interactions. It follows immediately from eq. (21) that the spatial density of vortices (i.e. number per unit area) is

\[
n = \kappa \exp(-S_{\text{vortex}}).
\]

At large \( T \) the numbers of noninteracting vortices inside any large area are Poisson-distributed. Therefore, the average of large Wilson loops can be calculated as

\[
\langle W_J \rangle = \sum_N \left[ (-1)^{2J} \right]^N \frac{(n \cdot \text{Area})^N}{N!} e^{-n \cdot \text{Area}} = \begin{cases} 
e^{-2n \cdot \text{Area}} \quad \text{at } J = \text{half–integer}, \\ 1 \quad \text{at } J = \text{integer}, \end{cases}
\]

so that the string tension for half-integer representations is twice the vortex density [1], and zero for integer representations.

It should be recalled, however, that at \( T \to \infty \) the \( 2 + 1 \) dimensional system reduces to two dimensions with a coupling constant \( g_2^2 = g_3^2 T \), and in two dimensions there is a trivial perturbative confinement with a nonzero string tension in any representation,

\[
\sigma_2 = g_2^2 J(J + 1).
\]

The above vortex-induced string tension should be thus regarded as an exponentially small addition to eq. (24): it leads to a deviation from the ‘Casimir scaling’ of eq. (24). It would be instructive to study the high-temperature \( 2 + 1 \) dimensional YM theory in lattice simulations, if only because one can learn to identify physical vortices by comparing objects found from maximal center gauge fixing with the continuum profiles presented here. Such experience might be useful in higher dimensions.

As one lowers \( T \) the vortex action decreases; at \( T \approx g_3^2 \) the action becomes of the order of unity, therefore, the vortices are not suppressed anymore. Unfortunately, the quantitative theory fails at this point: first, because the effective action (2) is not accurate anymore, second, because the vortices become long and start to bend, third, because one can hardly neglect interactions between dense vortices. Much more serious efforts are needed to describe vortices in this case (if they exist). Nevertheless, to our mind it is useful to know that at least in the limit of high temperatures vortices do exist in a pure Yang–Mills theory, and their basic properties are established.
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