Default Bayesian Model Selection of Constrained Multivariate Normal Linear Models

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Abstract

A default Bayes factor is proposed for evaluating multivariate normal linear models with competing sets of equality and order constraints on the parameters of interest. The default Bayes factor is based on generalized fractional Bayes methodology where different fractions are used for different observations and where the default prior is centered on the boundary of the constrained space under investigation. First, the method is fully automatic and therefore can be applied when prior information is weak or completely unavailable. Second, using group specific fractions, the same amount of information is used from each group resulting in a minimally informative default prior having a matrix Cauchy distribution. This results in a consistent default Bayes factor. Third, numerical computation can be done using parallelization which makes it computationally cheap. Fourth, the evidence can be updated in a relatively simple manner when observing new data. Fifth, the selection criterion can be applied relatively straightforwardly in the presence of missing data that are missing at random. Finally the methodology can be used for default model selection and hypothesis testing of commonly used models such as (M)AN(C)OVA, (multivariate) multiple regression, or repeated measures.

Keywords: Default Bayesian statistics, model selection, Bayesian updating, missing data.
1 Introduction

In this paper a default Bayes factor is presented for the selection problem of $T$ multivariate normal linear models with competing sets of equality and order constraints on the parameters of interest,

$$M_t : \mathbf{R}_{t,E}\theta = \mathbf{r}_{t,E} \& \mathbf{R}_{t,O}\theta > \mathbf{r}_{t,O},$$

(1)

where $\theta$ is a vector of (adjusted) means and regression coefficients of interest, and $[\mathbf{R}_{t,E}|\mathbf{r}_{t,E}]$ and $[\mathbf{R}_{t,O}|\mathbf{r}_{t,O}]$ are augmented matrices containing the coefficients of the $r_{t,E}$ equality constraints and $r_{t,O}$ order constraints under $M_t$, respectively, for $t = 1, \ldots, T$. Models containing such equality and order constraints are becoming increasingly used in social research as scientific expectations are often formulated using equality as well as order constraints on the parameters of interest. This can be seen from the increasing literature in applied scientific fields in the last decade where these methods are used (Well et al., 2008; Van de Schoot et al., 2006; Kluytmans et al., 2012; Braeken et al., 2015; Vrinten et al., 2016; van Schie et al., 2016; de Jong et al., 2017; Flore et al., 2019; Dogge et al., 2019; Zondervan-Zwijnenburg et al., 2019), as well as the introductory and tutorial papers that have been written (Wagenmakers, 2007; Masson, 2011; van de Schoot et al., 2011; Hoijtink et al., 2019).

A useful and increasingly popular method for such a model selection problem is the Bayes factor, a Bayesian criterion originally proposed by Jeffreys (1961). A well-known property of the Bayes factor is that it can be very sensitive to the chosen prior distribution of the parameters that are tested, and therefore arbitrary prior specification should be avoided. Prior specification for the parameters under all the $T$ separate models based on prior beliefs can be a challenging and time-consuming endeavour however (Berger, 2006). Furthermore noninformative improper priors can also not be used as the Bayes factor would depend on undefined constants. For this reason there has been an extensive development of so-called default or automatic Bayes factors where a small subset of the data is used for prior specification (e.g. Spiegelhalter & Smith, 1982; O’Hagan, 1995; Berger & Pericchi, 1996; Berger & Mortera, 1999; Moreno et al., 1998; Berger & Pericchi, 2004; Klugkist et al., 2005; Mulder et al., 2009; Rouder et al., 2009; Klugkist et al., 2010; Hoijtink, 2011; Gu et al., 2014, 2017; Consolandi & Paroli, 2017; Böing-Messing et al., 2017; Mulder & Fox, 2018; Mulder & Olsson-Collentine, 2019, and the references therein).
The Bayes factor that is considered here builds upon the work of O’Hagan (1995, 1997) who proposed the fractional Bayes factor for default Bayesian model selection and hypothesis testing in the case of weak prior information. In the fractional Bayes factor default prior specification is implicitly tackled by splitting the data in a fraction, $b$, that is used for specifying a default (fractional) prior while the remaining fraction, $1 - b$, is used for marginal likelihood computation (Gilks, 1995). By setting $b$ to be minimal, maximal information from the data is used for model selection (Berger & Mortera, 1995).

The original fractional Bayes factor however is not designed for evaluating models with order constraints on the parameters of interest; it was particularly developed for precise hypothesis testing. Social science researchers however often formulate their expectations using order constraints (e.g., ‘group 1 is expected to score higher on average than group 2, and group 2 is expected to score higher on average than group 3’ (Klugkist et al., 2005), or ‘the effect of the first predictor on the outcome variable is expected to be stronger than the effects of the other predictors’ (Bracken et al., 2013)). The problem of the original fractional Bayes factor for order-constrained model selection is that it does not always properly incorporate the complexity of an order-constrained model. To allow for order-constrained model selection using fractional Bayes methodology, the underlying default prior it is centered on the boundary of the constrained space. Given the sets of constraints in (1) this implies that the prior location $\theta_0$ satisfies, $R_t \theta_0 = r_t$, where $R'_t = [R'_{t,E} R'_{t,O}]$ and $r'_t = [r'_{t,E} r'_{t,O}]$. Under this adjusted fractional prior, the prior probability of the order constraints, a key quantity in the Bayes factor which quantifies the size of relative complexity of an order-constrained model, would be equal for models with opposite constraints. For example, for the simple test of $H_1 : \beta \leq 0$ versus $H_2 : \beta > 0$, the prior probabilities of $\beta < 0$ or $\beta > 0$ will be equal to $\frac{1}{2}$ because the adjusted prior will be centered at 0 under the encompassing prior (while still containing minimal information due to the fraction of the data), thereby appropriately capturing the relative size of the order constrained subspaces. This is not achieved under the default prior in the original fractional Bayes factor which is centered around the likelihood. For this simple test, the proposed default Bayes factor only depends on the posterior probabilities that the constraints hold under a larger unconstrained model (see also Lindley, 1995, for a related discussion on fractional Bayes factors for one-sided testing).

The original fractional Bayes factor based on a single fraction ‘$b$’ also is not
suitable for Bayesian model comparison in case of unbalanced data from different populations (De Santis & Spezzaferri, 2001; Hoijtink, Gu, & Mulder, 2018). For this reason a default Bayes factor is proposed that uses group (or population) specific fractions, \( b_j \) for group \( j \), for \( j = 1, \ldots, J \), in order to control the amount of information that is used from the data of the different populations for default prior specification. This will allow us to tune the amount of information in the default prior such that it is minimal (regardless of the sample sizes of the different groups), resulting in maximal information for model selection. It will be shown that the proposed default Bayes factor is consistent for the model selection problem in (1) based on this minimally informative default prior.

Another key property of the proposed Bayes factor is that it is relatively fast to compute for the model selection problem in (1). This is due to the analytic expression of the default Bayes factor for specific cases, and due to the fast Monte Carlo estimate for the general case. The posterior and prior probabilities that the order constraints hold can be computed using available cdf’s functions of the multivariate normal and multivariate Student t distribution, and therefore estimates based on the proportions of posterior and prior draws satisfying the constraints, which can be inefficient (Mulder et al., 2012), can be avoided. Bayesian updating of the evidence in light of new data can also be done relatively fast without needing to store the complete data matrices.

Finally the proposed selection criterion can be computed relatively straightforwardly in the presence of missing data that are missing at random. Instead of list-wise deletion, which results in a loss of information and possible bias, a multiple imputation method can be used (Rubin, 1987, 1996; Hoijtink, Gu, Mulder, & Rosseel, 2018) for which only imputed data sets are needed under the larger unconstrained model. Therefore we would not need to generate imputed data sets under all the separate constrained models under investigation, which would be the case when using the BIC for instance. Note that although missing data is ubiquitous in statistical practice, it has received surprisingly little attention in model selection problems however.

The proposed methods has several advantages in comparison to other methods that are available for this selection problem for the multivariate normal linear model. First the BIEMS program (Mulder et al., 2012) can be relatively slow because the default prior is based on an empirical expected posterior prior which requires a lot of MCMC sampling (Pérez & Berger, 2002; Mulder et al., 2009, 2010). Furthermore, in contrast to other fast de-
fault Bayes factors, such as the BIC (Raftery, 1995) or Bain (Gu et al., 2017), the proposed default Bayes factor does not make use of (large sample) Gaussian approximations of the likelihood or (unit information) default prior, and is therefore more suitable for small to moderate samples. In contrast to the multivariate Gaussian priors under the BIC and Bain, the underlying default prior in the proposed Bayes factor has a matrix Student $t$ distribution with one degree of freedom (or matrix Cauchy distribution) (which follows naturally from the employed fractional Bayes methodology). Finally the Bayes factors based on mixtures of $g$ priors (Liang et al., 2008, who considered a univariate regression model) require the specification of a prior scale for the anticipated effects before observing the data (Rouder et al., 2012). If prior information is absent or researchers want to refrain from making such a choice, the proposed default Bayes factor would be recommendable.

This paper is organized as follows. In Section 2 a general formulation of the model is given. Section 3 presents the default Bayes factor for a constrained model selection problem of the form \((1)\). It is explained how to compute the default Bayes factor, analytic expressions are presented for special cases, consistency is proven as the sample size grows, and the computation is explained in the presence of missing observations. In Section 4 the method is applied to an empirical model selection problem. We end the paper with a short discussion.

2 Multivariate normal linear model

The multivariate normal linear model can be written as

\[
Y = X\Theta + E
\]

with $N \times P$ matrix $Y = [y_1 \cdots y_N]'$ where $y_i$ is the $i$-th observation of the $P$ dependent observations, $N \times P$ matrix $E = [e_1 \cdots e_N]'$ where $e_i \sim N(0, \Sigma)$ is the $i$-th $P$ dimensional error vector with unstructured $P \times P$ covariance matrix $\Sigma$, $N \times K$ matrix $X = [x_1 \cdots x_N]'$ where $x'_i = (d'_i, w'_i)$ is the $i$-th observation of the independent variables out of which the first $J$ are dummy variables that indicate group membership, i.e., $d_{ij} = 1$ if the $i$-th observation belongs to group $j$ and zero elsewhere, and the remaining $L$ elements are the predictor variables, and matrix of (adjusted) group means and regression
Thus, the $i$-th dependent observation is distributed as follows
\[ y_i \sim N(\Theta'x_i, \Sigma). \] (2)

Subsequently, the interest is in a set of $T$ models with competing linear equality and order constraints in (1) on the (adjusted) means and regression coefficients in $\theta = \text{vec}(\Theta)$. Many common testing problems are special cases of this, such as univariate ($P = 1$) or multivariate ($P > 1$) linear regression when testing the regression coefficients or interaction effects, univariate or multivariate analysis of variance ($L = 0$) or covariance ($L > 0$) when testing the (adjusted) group means, or repeated measures models with an unrestricted covariance matrix ($P > 2$) when testing the repeated measures means.

### 3 A Bayes factor for default model selection

Following O’Hagan (1995), the marginal likelihood under the original fractional Bayes factor of a constrained model $M_t$ of the form (1) under (2) is defined by
\begin{align}
p_t(Y, b) &= \frac{\int_{\theta} \int_{\Sigma} p_t(Y|X, \Theta, \Sigma) \pi_t^N(\Theta, \Sigma) d\Theta d\Sigma}{\int_{\theta} \int_{\Sigma} p_t(Y|X, \Theta, \Sigma) \pi_t^N(\Theta, \Sigma) d\Theta d\Sigma} \\
&= \frac{\int_{\theta} \int_{\Sigma} p(Y|X, \Theta, \Sigma)|\Sigma|^{-\frac{P+1}{2}} d\Theta d\Sigma}{\int_{\theta} \int_{\Sigma} p(Y|X, \Theta, \Sigma)|\Sigma|^{-\frac{P+1}{2}} d\Theta d\Sigma} \\
&(3)
\end{align}

where $M_t = \{\theta | R_{t,E} \theta = r_{t,E}, \ R_{t,O} \theta > r_{t,O}\}$ denotes the parameter subspace that satisfies the constraints under model $M_t$, $p_t(Y|X, \Theta, \Sigma) = p(Y|X, \Theta, \Sigma) I(\Theta \in M_t)$ denotes the likelihood of the data under model $M_t$, where $p(Y|X, \Theta, \Sigma)$ denotes the likelihood under an unconstrained model, the improper noninformative independence Jeffreys prior is used, i.e., $\pi_t^N(\Theta, \Sigma) \propto |\Sigma|^{-\frac{P+1}{2}}$, and
where the fraction, $0 < b < 1$, which controls the amount of information in the data that is used for prior specification, while the remaining fraction, $1 - b$, is used for model selection. Note that the likelihood under $M_t$ can be replaced by the unconstrained likelihood in (1) because the integral is computed over the constrained subspace $M_t$.

The original fractional Bayes factor is not suitable for order-constrained model selection (Mulder, 2014). This can be seen as follows. When model $M_t$ only contains order constraints and we consider data that (strongly) supports these constraints, the mass underneath the likelihood and the fraction of the likelihood in (4) are almost completely located in the order-constrained subspace. Therefore the marginal likelihood will be virtually the same as an unconstrained model without order constraints. The Bayes factor will there for not function as an “Occam’s razor” where the simpler order-constrained model is preferred over the more complex unconstrained model in the case of an almost equal fit. To correct for this, the integrand in the numerator in (4) will be integrated over an adjusted parameter space, $M^*_t = \{ \theta | R_{t,E}(\theta - \hat{\theta}) = 0, R_{t,O}(\theta - \hat{\theta}) > 0 \}$. This (implicitly) results in a default prior that is centered on the boundary of the constrained space having prior uncertainty based on a fraction of the information in the prior.

Furthermore, to properly control the amount of information that is used for default prior specification and to avoid inconsistent behavior, different fractions are needed for observations coming different groups (De Santis & Spezzaferri, 2001; Hoijtink, Gu, Mulder, & Rosseel, 2018). For this reason, group specific fractions will be used for the likelihood that is raised to $b$ in the numerator in (3). The generalized fraction of the likelihood will be defined by

$$p(Y|X, \Theta, \Sigma)^b = \prod_{i=1}^{N} p(y_i|x_i, \Theta, \Sigma)^{h_i}.$$  

We come back to the choice of the fractions later in this paper.

Consequently, the proposed default marginal likelihood can therefore be expressed as

$$p_t^*(Y, b) = \frac{\int_\Sigma \int_{\Theta \in M_t} p(Y|X, \Theta, \Sigma)|\Sigma|^{-\frac{P+4}{2}} d\Theta d\Sigma}{\int_\Sigma \int_{\Theta \in M^*_t} p(Y|X, \Theta, \Sigma)^b|\Sigma|^{-\frac{P+4}{2}} d\Theta d\Sigma}. \tag{5}$$

1 The adjusted parameter space can be seen as a movement of the original constrained parameter space $M^*_t = \{ \theta | R_{t,E}(\theta - \hat{\theta}) = r_{t,E}, R_{t,O}(\theta + \hat{\theta} - \theta_0) > r_{t,O} \}$, where $\theta_0$ satisfies $[R'_{t,E} R'_{t,O}]^{'}\theta_0 = [r'_{t,E} r'_{t,O}]^{'}$.  

7
The default Bayes factor of a constrained model of the form $M_t$ against an unconstrained model $M_u$ can then be expressed as a multivariate Savage-Dickey density ratio [Dickey, 1971] multiplied with a ratio of posterior and prior probabilities that the order constraints hold conditional on that the equality constraints hold.

**Lemma 1** The default Bayes factor for a constrained model $M_t$ of the form \[ f \] against an unconstrained alternative model $M_u$ based on the marginal likelihood in (3) can be expressed as

\[
B_{tu} = \frac{f_t^E(Y, X)}{c_t^E(Y, X)} \times \frac{f_t^O(Y, X)}{c_t^O(Y, X)},
\]

where

\[
f_t^E(Y, X) = \pi(t, E \theta = r_{t,E})Y, X
\]

\[
c_t^E(Y, X, b) = \pi(t, E \theta = r_{t,E})Y, X, b
\]

\[
f_t^O(Y, X) = Pr_t(R_{t,O} \theta > r_{t,O})R_{t,E} \theta = r_{t,E}, Y, X
\]

\[
c_t^O(Y, X, b) = Pr_t(R_{t,O} \theta > r_{t,O})R_{t,E} \theta = r_{t,E}, Y, X, b
\]

where the marginal unconstrained posterior and default prior for $\Theta$ under $M_u$ follow matrix-t distributions given by

\[
\pi_u(\Theta|Y, X) = T_{K \times P}((\Theta_0 - X0)^{-1}S, N - K - P + 1)
\]

\[
\pi_u^*(\Theta|Y, X, b) = T_{K \times P}(\Theta_0, (X_0^bX_0^-1S_b, \sum_{i=1}^N b_i - K - P + 1),
\]

from which the conditional and marginal distributions used for computing $f_t^E, f_t^O, c_t^E$, and $c_t^O$ naturally follow, and where $\Theta_0 = vec(\Theta_0)$ satisfies $[R_{t,E}^r R_{t,O}^r]^{*}\theta_0 = [r_{t,E}^r r_{t,O}^r]^*$, the LS estimate equals $\hat{\Theta} = (X^T X)^{-1}X^T Y$, the sum of square matrix in the posterior equals $S = (Y - X\hat{\Theta})^T(Y - X\hat{\Theta})$, and the sums of square matrix in the default prior equals $S_b = (Y_b - X_b\Theta_b)^T(Y_b - X_b\Theta_b)$, with $\Theta_b = (X_b^bX_b)^{-1}X_bY_b$, where $Y_b$ and $X_b$ are the stacked matrices of $y_{i,b}$ and $x_{i,b}^r$, with $y_{i,b} = \sqrt{b_i}y_i$ and $x_{i,b}^r = \sqrt{b_i}x_i$.

A proof is given in Appendix A. The posterior quantities in the numerators in (6) are denoted with a “$f$” because they can be interpreted as measures of relative fit of the constrained model $M_t$ relative to the unconstrained model.
The prior quantities in the denominators in (6) are measures of relative complexity of $M_t$ and therefore denoted with a “$c$”. Furthermore the superscript of these symbols denote which part of the constraints of model $M_t$ it evaluates (either the equality constraints “$E$” or the order constraints conditional that the equality constraints hold “$O$”). Because the location hyperparameter in the unconstrained default prior $\pi_u$ satisfies $[r'_{t,E} R'_{t,O}]\theta_0 = [r'_{t,E} r'_{t,O}]$, we say that the prior is centered on the boundary of the constrained space of $M_t$.

### 3.1 Bayes factor computation

Unlike the matrix normal distribution, the equivalent marginal posterior of the vectorization $\theta$ does note follow a multivariate Student $t$ distribution; only the marginal distributions of the separate columns or rows of $\Theta$ have multivariate $t$ distributions ([Box & Tiao, 1973, p. 443]. Consequently, a linear combination of the elements in $\theta$, say, $\zeta_t = (\zeta'_{t,E}, \zeta'_{t,O})' = H_t\theta$, does not have a multivariate student $t$ distribution or other known distributional form for a coefficient matrix $R_{t,E}$ in general. Therefore, the posterior density in the numerator in the first term in (6) does not have an analytic form. A Monte Carlo estimate can be obtained relatively easy however. First we define the one-to-one transformation, $\zeta_t = (\zeta'_{t,E}, \zeta'_{t,O})' = H_t\theta$, where $H_t = [R'_{t,E} D_t]'$, with $D_t$ being a $(PK - r_{t,E}) \times PK$ matrix such that $\zeta_{t,O}$ contains the parameters in $\theta$ that not constrained with equalities. Conditionally on $\Sigma$, the transformed parameters, $\zeta_t$, have a multivariate normal conditional posterior, $N(\mu_{\zeta_t}, \Psi_{\zeta_t})$, with $\mu_{\zeta_t} = H_t\hat{\theta}$ and $\Psi_{\zeta_t} = H_t[\Sigma \otimes (X'X)^{-1}]H'_t$. Then,

$$
\pi_u(R_{t,E}\theta = r_{t,E}|Y, X) = \int \pi_u(\zeta_{t,E} = r_{t,E}, \zeta_{t,O}|Y, X)d\zeta_{t,O} 
= \int \int \pi_u(\zeta_{t,E} = r_{t,E}, \zeta_{t,O}|Y, X, \Sigma)\pi_u(\Sigma|Y, X)d\Sigma d\zeta_{t,O} 
\approx S^{-1}\sum_{s=1}^{S} \mathcal{N}_{PK}(r_{t,E}; R_{t,E}\hat{\theta}, R_{t,E}[\Sigma^{(s)} \otimes (X'X)^{-1}]R'_{t,E}),
$$

where $\Sigma^{(s)} \sim \mathcal{I}W(N - K, S)$, for $s = 1, \ldots, S$, and $\mathcal{N}_{PK}(r; \mu, \Psi)$ denotes a $PK$-variate normal density with mean vector $\mu$ and covariance matrix $\Psi$ evaluated at $r$, which has an analytic expression (in R, for instance, it can be computed using the dmvnorm function from the mvtnorm package). This
Monte Carlo estimate can be obtained via parallelized computation, and it is therefore computationally cheap.

The conditional posterior probability in the numerator in the second term can be obtained in a similar manner. First note that the order constraints under $M_t$ for the transformed parameter vector are equivalent to $\tilde{R}_{t,O} \zeta_{t,O} > r_{t,O}$, where $\tilde{R}_{t,O}$ consists of the columns corresponding to the parameters of $\theta$ that are not constrained with equalities. Furthermore, a property of the multivariate normal distribution is that the conditional posterior of $\zeta_{t,O}$ given $\zeta_{t,E} = r_{t,E}$ has a multivariate normal distribution with mean $\mu_{\zeta_{t,O}|E} = \mu_{\zeta_{t,O}} + \Psi_{\zeta_{t,O}|E}(r_{t,E} - \mu_{\zeta_{t,E}})$ and covariance matrix $\Psi_{\zeta_{t,O}|E} = \Psi_{\zeta_{t,O}} - \Psi_{\zeta_{t,O}|E}\Psi_{\zeta_{t,E},E}^{-1}\Psi_{\zeta_{t,E},O}$, where the indices $E$ and $O$ refer to the appropriate parts of the mean vector and covariance matrix of $\zeta_t$. Now when $\tilde{R}_{t,O}$ is of full row rank, which is generally the case, a transformed parameter can be defined, $\eta_{t,O} = \tilde{R}_{t,O} \zeta_{t,O}$, which has a multivariate normal distribution with mean vector $\mu_{\eta_{t,O}} = \tilde{R}_{t,O} \mu_{\zeta_{t,O}|E}$ and covariance matrix $\Psi_{\eta_{t,O}} = \tilde{R}_{t,O} \Psi_{\zeta_{t,O}|E}\tilde{R}_{t,O}'$. The posterior probability in the numerator in the second term in (6) can then be computed via a Monte Carlo estimate, similar as in (9),

$$\Pr_u(\theta_{t,O} > r_{t,O}|\theta_{t,E}, Y, X) \approx S^{-1} \sum_{s=1}^S \Phi_N(-r_{t,O}; -\mu_{\eta_{t,O}}, \Psi_{\eta_{t,O}}^{(s)}),$$

where $\Psi_{\eta_{t,O}}^{(s)}$ is computed using $\Psi_{\zeta_{t,O}}^{(s)} = H_s[\Sigma^{(s)} \otimes (X'X)^{-1}]H_s'$, with $\Sigma^{(s)} \sim IW(N - K, S)$, for $s = 1, \ldots, S$, and $\Phi_N$ denotes the multivariate normal cdf. Note that the cdf can be computed using standard statistical software (e.g., using the pmvnorm function from the mvtnorm package in R). When $\tilde{R}_{t,O}$ is not of full row rank, the above method cannot be used. To get the conditional posterior probability, a numerical sampling estimate can be used via the R-function bain in the bain package (Gü et al., 2019).

Note that if the sample size is sufficiently large, the matrix $t$ can be well approximated using a matrix normal distribution, $N_{K \times P}(\Theta, (X'X)^{-1}, (N - K - P + 1)^{-1}S)$ (Box & Tiao, 1973, p. 447). In this case no Monte Carlo estimate would be needed as the posterior density and the posterior probability could directly be computed using the approximated matrix (or multivariate) normal distribution of $\Theta$ (or $\theta$).

\footnote{An example of a set of order constraints that corresponds to a matrix $[\tilde{R}_{t,O}|r_{t,O}]$ that is not of full row rank is when a subset of parameters is expected to be larger/smaller than another set of parameters, e.g., $(\beta_1, \beta_2) > (\beta_3, \beta_4)$.}
3.2 Minimally informative default prior

The group specific fractions will be chosen such that the information in the default prior is minimal, so that maximal information in the data is used for model selection. Following Berger & Mortera (1995), alternative choices for the fractions, such as \( b = \log n \) or \( b = \sqrt{n} \) as suggested by O’Hagan (1995), would not be recommendable as the amount of information in the prior would then diverge as the sample size grows.

The group dependent fractions are chosen such that from each group the same amount of information, say, the information in \( m \) independent observations, is taken. This can be achieved by setting \( b_i = \frac{m}{n_j} \) if the \( i \)-th observation belongs to the \( j \)-th group, i.e., \( d_{ij} = 1 \), where \( n_j \) is the sample size of group \( j \). Note that in the original fractional Bayes factor with a single fraction for one group, the minimal fraction would be \( b = \frac{m}{n} \) where \( m \) is the minimal sample size to get a finite marginal likelihood. Given the matrix \( t \) distribution of the default prior for \( \Theta \) in (8), a minimally informative prior is obtained when the prior degrees of freedom of this distribution is equal to one, i.e., \( \sum_{i=1}^{n} b_i - K - P + 1 = 1 \). This is achieved by setting \( m = \frac{P+K}{J} \). To keep the notation simple, \( b \) will denote the minimal fraction throughout the remainder of the paper. The marginal fractional prior for \( \Theta \) in (8) then has a matrix Cauchy distribution,

\[
\pi^*_u(\Theta|Y, X, b) = C_{K \times P}(\Theta_0, S_b, (X_b^t X_b)^{-1}). \tag{10}
\]

The matrix variate Cauchy distribution has not been reported a lot in the literature. Some of its properties have been presented by Bandekar & Nagar (2003), for example. This paper presents another important application of this distribution for Bayesian model selection problems under the multivariate normal linear model.

The prior density in the denominator in the first term in (6) and the prior probability in the denominator in the second term in (6) based on the above matrix Cauchy prior can be computed using the same Monte Carlo estimate as was shown for their respective posterior counterparts. Note that if the order constraints under \( M_t \) are solely specified between parameters in the same column or same row of \( \Theta \), the prior probability can also be computed using a multivariate normal distribution with the same location and covariance structure. This is because the probability of a set of order constraints is invariant to the exact distributional form as long as the mean lies on the boundary, and an elliptical distribution is used with the same covari-
ance structure. Finally note that because the amount of prior information is kept minimal regardless of the sample size, the default prior cannot be approximated with a matrix normal distribution for larger samples, unlike the posterior.

### 3.3 Analytic expression for special cases

The marginal distribution of a column (or row\textsuperscript{3}) of a matrix random variable with a matrix Student $t$ distribution has a multivariate Student $t$ distribution (Box & Tiao, 1973, p. 442-443). This implies that the unconstrained marginal prior and posterior of the $p$-th column of $\Theta$, denoted by $\theta_p$, are distributed as

$$
\pi_u(\theta_p | Y, X) = T_{K, p}((N - K - P + 1)^{-1}s_{pp}(X'X)^{-1}, N - K - P + 1)
$$

$$
\pi_u(\theta_p | Y, X, b) = C_{K}(\theta_{0,p}, s_{b,pp}(X'_bX_b)^{-1}),
$$

where $s_{b,pp}$ and $s_{pp}$ denote the $(p,p)$-th element of $S_b$ and $S$, respectively. Thus, using standard calculus, it can be shown that for a constrained model with only constraints on the elements in column $p$, i.e., $M_t : R_{t,E} \theta_p = r_{t,E} \& R_{t,O} \theta_p > r_{t,O}$, the posterior and prior quantities in (6) are equal to

$$
f_{t}^{E}(Y, X) = T(r_{t,E} ; R_{t,E} \hat{\theta}_p, (N - K - P + 1)^{-1}s_{pp}R_{t,E}(X'X)^{-1}R'_{t,E},
N - K - P + 1)
$$

$$
c_{t}^{E}(Y, X, b) = C(r_{t,E} ; R_{t,E} \theta_{0,p}, s_{b,pp}R_{t,E}(X'_bX_b)^{-1}R'_{t,E})
$$

$$
f_{t}^{O}(Y, X) = \Psi_{T}(r_{t,O}; -\mu_{f,0}; \Psi_{f}, N - K - P + 1)
$$

$$
c_{t}^{O}(Y, X, b) = \Psi_{C}(0; 0, \Psi_{c})
$$

where $\Psi_{T}(x; \mu, \Psi, \nu)$ is the cdf of a multivariate Student $t$ distribution at $x$ with location $\mu$, scale matrix $\Psi$, and $\nu$ degrees of freedom, $\Psi_{C}(x; \mu, \Psi)$ is the cdf of a multivariate Cauchy distribution at $x$ with location $\mu$ and scale

\textsuperscript{3}Note that when $\Theta \sim T_{K,P}(\nu, M, \Psi, \Phi)$, then $\Theta' \sim T_{P,K}(\nu, M', \Phi, \Psi)$ (Box & Tiao, 1973, p. 442)
matrix $\Psi$, and

$$
\mu_{\eta,0}^f = R_{t,O}\hat{\theta}_p + R_{t,O}(X'X)^{-1}R'_t(E)(R_{t,E}(X'X)^{-1}R'E)^{-1}(r_{t,E} - R_{t,E}\hat{\theta}_p)
$$

$$
\Psi_{\eta,0}^f = s_{pp} + \frac{(r_{t,E} - R_{t,E}\hat{\theta}_p)'(R_{t,E}(X'X)^{-1}R'_t,E)^{-1}(r_{t,E} - R_{t,E}\hat{\theta}_p)}{N - K - P + 1 + r_{t,E}}
$$

$$
\Psi_{\eta,0}^c = \frac{s_{b,pp}}{1+r_{t,E}}R_{t,O}(X_b'X_b)^{-1}\left(I_{K-r_{t,E}} - R_{t,E}(X_b'X_b)^{-1}R'_t,E(R_{t,E}(X_b'X_b)^{-1}X_b'X_b)^{-1}R'_t,E\right)^{-1}
$$

where $r_{t,E}$ is the number of rows of $R_{t,E}$. Note that the cdf’s can be computed using standard functions in statistical software (e.g., using pmvt in the mvtnorm-package [Genz et al., 2016]).

Hence, the proposed default Bayes factor has an analytic expression for univariate testing problems (e.g., AN(C)OVA or linear regression), for multivariate/univariate $t$ tests, or in other testing problems where the constraints are formulated solely on the elements of one specific column or row of $\Theta$.

### 3.4 Consistency

A Bayes factor is called consistent if the evidence goes to infinity for the true constrained model against the alternative models as the sample goes to infinity. Consistency is therefore a fundamental property that a model selection criterion should have because it ensures that the true model will always be selected as long as the sample is large enough.

**Lemma 2** Given a set of competing nonnested multivariate normal linear models with equality and order constraints on the location parameters $\theta$ of the form (1), the default Bayes factor defined in (6) with group specific minimal fractions is consistent.

Here a sketch of the proof is given. First note that the unconstrained posterior density in the numerator in the first term in (6) goes to infinity if the equality constraints hold, and to zero if they do not hold. Second, the conditional posterior probability in the numerator in the second term goes to 1 if the constraints hold and to 0 if they do not hold. The quantities in the denominators depend on the unconstrained matrix Cauchy prior in (10) with scale matrices $(X_b'X_b)^{-1}$ and $S_b$. As the sample sizes, $n_j$ for all groups go to
infinity, these scale matrices converge to finite scale matrices which depend on the population distributions of the observed variables. For example, the scale matrix $\sum_{j=1}^{J} n_j^{-1} X_j' X_j$ in (11) converges to
\[
\begin{bmatrix}
1 & 0 & 0 & E\{w'_1\} \\
0 & \ddots & 0 & \vdots \\
0 & 0 & 1 & E\{w'_J\} \\
E\{w_1\} & \cdots & E\{w_J\} & \sum_j E\{w_j w'_j\}
\end{bmatrix}.
\]

Similar results hold for the matrices $\sum_{j=1}^{J} n_j^{-1} X_j' Y_j$ and $\sum_{j=1}^{J} n_j^{-1} Y_j' Y_j$ in (12) in the limit. The unconstrained default prior distribution therefore converges to some fixed matrix Cauchy distribution. This implies that the value of the prior density in the denominator in the first term in (5) converges to some positive constant as well as the conditional prior probability in the numerator in the second term for all constrained models under consideration. Consequently the Bayes factor $B_{tu}$ for a true constrained model $M_t$ goes to infinity, and $B_{tu}$ converges to zero for an incorrect model. The proposed default Bayes factor is therefore consistent.

### 3.5 Sequential Bayesian updating

Similar as the original fractional Bayes factor, the proposed default Bayes factor is not coherent when sequential updating of the evidence when observing new data $(Y_{new}, X_{new})$. This implies that the evidence has to be recomputed when new data are observed because the fractional prior will (slightly) change. This can be done very efficiently as discussed below.

The default fractional prior in (8) depends on the data via the sufficient statistics,
\[
X'_b X_b = \sum_{i=1}^{n} b_i x'_i = \frac{P+K}{J} \sum_{j=1}^{J} n_j^{-1} X_j' X_j,
\]
where (with a slight abuse of notation) $X_j$ denotes the stacked matrix of the covariates $x'_i$ for group $j$, and similarly for the fractional sums of squares matrix,
\[
S_b = Y'_b Y_b - Y'_b X_b (X'_b X_b)^{-1} X_b Y_b
\]
\[
= \frac{P+K}{J} \left( \sum_{j=1}^{J} n_j^{-1} X'_j Y_j - (\sum_{j=1}^{J} n_j^{-1} Y'_j Y_j) (\sum_{j=1}^{J} n_j^{-1} X'_j X_j)^{-1} (\sum_{j=1}^{J} n_j^{-1} X'_j Y_j) \right),
\]
14
where (with a slight abuse of notation) $\mathbf{Y}_j$ denotes the stacked matrix of the outcome variables $\mathbf{y}_i'$ for group $j$. Thus, when observing new data matrices, $\mathbf{X}_{new,j}$ and $\mathbf{Y}_{new,j}$, we only need to update the group specific sufficient statistics, i.e.,

\[
\begin{align*}
\mathbf{X}'_j \mathbf{X}_j & \rightarrow \mathbf{X}'_j \mathbf{X}_j + \mathbf{X}'_{new,j} \mathbf{X}_{new,j} \\
\mathbf{X}'_j \mathbf{Y}_j & \rightarrow \mathbf{X}'_j \mathbf{Y}_j + \mathbf{X}'_{new,j} \mathbf{Y}_{new,j} \\
\mathbf{Y}'_j \mathbf{Y}_j & \rightarrow \mathbf{Y}'_j \mathbf{Y}_j + \mathbf{Y}'_{new,j} \mathbf{Y}_{new,j},
\end{align*}
\]

to obtain the updated fractional prior.

The unconstrained posterior depends on the same sufficient statistics for the complete data set, i.e., $\mathbf{X}' \mathbf{X}$, $\mathbf{X}' \mathbf{Y}$, and $\mathbf{Y}' \mathbf{Y}$. These can be updated in a similar manner as above when observing new data.

### 3.6 Missing data

If the data matrices $\mathbf{Y}$ and $\mathbf{X}$ contain missing observations that are missing are random (MAR), the default Bayes factors can still be computed relatively straightforwardly using the multiple imputation method of Hoijtink, Gu, Mulder, & Rosseel (2018). Note that there is a huge body of literature about the superiority of multiple imputation over list-wise deletion (Rubin, 1987, 1996).

To compute the default Bayes factor in the case of missing data, it is important to note that the posterior and prior quantities in (6) are computed under the unconstrained model. Therefore we only need to obtain unbiased estimates of the unconstrained prior and posterior using multiple imputation under the unconstrained model. Subsequently the posterior and prior quantities based on the partly missing data matrices can be computed by taking the arithmetic averages of the respective quantities based on the imputed data sets. Let us denote the observed data matrices by $\mathbf{Y}^o$ and $\mathbf{X}^o$, and data matrices with the missing observations by $\mathbf{Y}^m$ and $\mathbf{Y}^m$. For example, the posterior unconstrained density evaluated at the equality constraints of $M_t$
can be computed as
\[
f_t^E(Y^o, X^o) = \pi_u(R_E \theta = r_E | Y^o, X^o)
\]
\[
\approx M^{-1} \sum_{m=1}^{M} \pi_u(R^E \theta = r^E | Y^o, Y^m, X^o, X^m) \pi_u(Y^m, X^m | Y^o, X^o) dX^m dY^m
\]
\[
\approx M^{-1} \sum_{m=1}^{M} N(r_E; R_E \theta^{o,(m)}, R_E[\Sigma^{(m)} \otimes (X^{o,(m)^r} X^{o,(m)})^{-1}] R'_E)
\]
where \(\Sigma^{(m)} \sim \mathcal{IW}(N-K, S^{o,(m)})\), with \(S^{o,(m)} = (Y^{o,(m)} - \Theta^{o,(m)} X^{o,(m)}) (Y^{o,(m)} - \Theta^{o,(m)} X^{o,(m)})'\), and \(\Theta^{o,(m)} = (X^{o,(m)^r} X^{o,(m)})^{-1} X^{o,(m)} Y^{o,(m)}\), \(Y^{o,m}\) and \(X^{o,m}\) are the \(m\)-th draws of the data matrices of the dependent variables and predictor variables with missing observations, respectively, sampled from the unconstrained posterior \(\pi_u(Y^m, X^m | Y^o, X^o)\), and \(Y^{o,m}\) and \(X^{o,m}\) denote the complete data matrix that combines the observed and missing data matrices. Effectively, the posterior density based on the data with missing observations is computed as the arithmetic average of posterior densities obtained from different imputed data sets.

Similarly, the prior density can be obtained via
\[
c_t^E(Y^o, X^o, b) = \pi_u^*(R_E \theta = r_E | Y^o, X^o, b)
\]
\[
\approx M^{-1} \sum_{m=1}^{M} \pi_u^*(R^E \theta = r^E | Y^o, Y^m, X^o, X^m, b) \pi_u(Y^m, X^m | Y^o, X^o) dX^m dY^m
\]
\[
\approx M^{-1} \sum_{m=1}^{M} N(r_E; R_E \theta_0, R_E[\Sigma^{(m)} \otimes (X^{o,(m)^r} X^{o,(m)})^{-1}] R'_E),
\]
where \(\Sigma^{(m)} \sim \mathcal{IW}(P, S^{o,(m)}_b)\). Note that the sampling distribution of the missing observations, \(\pi_u(Y^m, X^m | Y^o, X^o)\), is the same as used for the posterior.
4 Empirical applications

4.1 One way ANOVA

Informative hypotheses evaluation in the context of a one way analysis of variance is illustrated using one of the studies from the OSF reproducibility project psychology (Open Science Collaboration, 2015). Monin et al. (2008) investigate the attraction to “moral rebels”, that is, persons that take an unpopular but morally laudable stand. There are three groups in their experiment: in Group 1 participants rate their attraction to “a person that is obedient and selects an African American person from a police line up of three”; in Group 2 participants execute a self-affirmation task intended to boost their self-confidence after which they rate “a moral rebel who does not select the African American person”; and, in Group 3 participants execute a bogus writing task after which they rate “a moral rebel”. The authors expect that the attraction to moral rebels is higher in the group executing the self-affirmation task (that boosts the confidence of the participants in that group) than in the group executing the bogus writing task, possibly even higher than in the group that rates the attraction of the obedient person. Their data will henceforth be referred to as the Monin data. Corresponding to their study are the following competing constrained models:

\[ M_1: \mu_1 = \mu_2 = \mu_3 \]
\[ M_2: \mu_2 > \mu_1 > \mu_3 \]
\[ M_3: \text{neither } M_1, \text{ nor } M_2, \]

where, \( \mu_1, \mu_2, \) and \( \mu_3 \) denote the mean attractiveness scores in Groups 1, 2, and 3, respectively. Note that model \( M_3 \) denotes the complement model encompasses the subspace of \( \mathbb{R}^3 \) for \( \mathbf{\mu} \) that does satisfy the constraints of \( M_1 \) and \( M_2 \).

In Table 1 and 2 the main results are presented. For model with no equality constraints or order constraints, the measures of relative fit and complexity omitted. The posterior model probabilities were computed using equal prior model probabilities (i.e., \( P(M_t|Y) = B_{tu}/\sum_{t'} B_{t'u} \)). Note that the sufficient statistics correspond to the data reported by Monin, Sawyer, and Marques (2006). As can be seen, for the three models under consideration, the order-constrained model \( M_2 \) receives is the best with a posterior probability of .963, followed by the complement model \( M_3 \), with a posterior
Table 1: Unconstrained estimates for the “Monin” application.

| parameter | estimate | standard error | N  |
|-----------|----------|----------------|----|
| $\mu_1$   | 1.88     | 0.464          | 19 |
| $\mu_2$   | 2.54     | 0.464          | 19 |
| $\mu_3$   | 0.02     | 0.375          | 29 |

Table 2: Bayesian model selection for the “Monin” application.

| Model | $f_t^E$ | $c_t^E$ | $f_t^O$ | $c_t^O$ | $B_{tu}$ | $P(M_t|y)$ |
|-------|---------|---------|---------|---------|----------|------------|
| $M_1$ | 5.42e−5 | 8.45e−3 |         |         | 0.006    | 0.001      |
| $M_2$ |         |         | 0.842   | 0.167   | 5.05     | 0.963      |
| $M_3$ |         |         | 0.158   | 0.833   | 0.189    | 0.036      |

probability of .036, and finally the equality-constrained null model received least evidence with a posterior probability of .001. This can be interpreted as very strong evidence for the order-constrained model. Note that the estimates and standard errors presented in Table 1 also indicate evidence for the order-constrained model. Further note that the default prior probability that the order constraints of $M_2$ and $M_3$ hold under the unconstrained model equal $c^O_2 = \frac{1}{6}$ and $c^O_3 = \frac{5}{6}$, which are exactly equal to the probabilities that the order constraints hold under the unconstrained model when assuming that each ordering is equally likely a priori. This is a direct consequence of centering the unconstrained prior on the boundary of the order-constrained space. The presented default Bayes factors and posterior model probabilities confirm this suspicion by providing strong evidence in favor of the order-constrained model against the competing models.

4.2 Multivariate multiple regression

Stevens (1996) (Appendix A) presented data concerning the effect of the first year of the Sesame street series on the knowledge of 240 children in the age range 34 to 69 months. To illustrate informative hypothesis evaluation in the context of a multivariate multiple regression, the outcome variables $y_1$ and $y_2$, which are the knowledge of numbers and the knowledge of letters of children after watching Sesame Street, respectively, are regressed on $x_1$ and $x_2$, which are the knowledge of numbers and the knowledge of letters of children before watching Sesame Street for a year. In this application all data
are standardized. The following multivariate multivariate multiple regression model will be used for $i = 1, \ldots, N$, where $N = 240$ denotes the sample size:

$$
y_{i1} = \mu_{11} + \beta_{11}x_{i1} + \beta_{21}x_{i2} + e_{i1}
$$

$$
y_{i2} = \mu_{12} + \beta_{12}x_{i2} + \beta_{22}x_{i2} + e_{i2}
$$

$$
\begin{bmatrix}
e_{i1} \\
e_{i2}
\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_1 & \sigma_{12} \\ \sigma_{12} & \sigma^2_2 \end{bmatrix}\right).
$$

In this context expectations were formulated on the effects \textit{within} each dimension (letter knowledge and number knowledge), and on the effects \textit{between} the two dimensions. Within the knowledge dimensions, two competing expectations were formulated. First, it was expected that letter knowledge after watching Sesame Street can better be predicted by letter knowledge before watching Sesame Street than by number knowledge before watching Sesame Street. A similar expectation can be formulated for the number knowledge dimension. Furthermore it was expected that all effects were positive, i.e., $\beta_{11} > \beta_{21} > 0$ & $\beta_{22} > \beta_{12} > 0$. Second, it was expected that there was no effect of number knowledge before watching Sesame Street on letter knowledge after watching Sesame Street, and no effect of letter knowledge before watching Sesame Street on number knowledge after watching Sesame Street. The other effects were assumed positive, i.e., $\beta_{11} > \beta_{21} = 0$ & $\beta_{22} > \beta_{12} = 0$. Between the knowledge dimensions, it was expected that the effect of number knowledge of the pre-measurement on the post-measurement was equal, smaller, or larger than the effect of letter knowledge of the pre-measurement on the post-measurement, i.e., $\beta_{11} = \beta_{22}$ or $\beta_{11} < \beta_{22}$ or $\beta_{11} > \beta_{22}$.

Combining these different expectations we can formulate 7 competing constrained models:

$M_1 : \beta_{11} > \beta_{21} > 0$ & $\beta_{22} > \beta_{12} > 0$ & $\beta_{11} = \beta_{22}$

$M_2 : \beta_{11} > \beta_{21} > 0$ & $\beta_{22} > \beta_{12} > 0$ & $\beta_{11} < \beta_{22}$

$M_3 : \beta_{11} > \beta_{21} > 0$ & $\beta_{22} > \beta_{12} > 0$ & $\beta_{11} > \beta_{22}$

$M_4 : \beta_{11} > \beta_{21} = 0$ & $\beta_{22} > \beta_{12} = 0$ & $\beta_{11} = \beta_{22}$

$M_5 : \beta_{11} > \beta_{21} = 0$ & $\beta_{22} > \beta_{12} = 0$ & $\beta_{11} < \beta_{22}$

$M_6 : \beta_{11} > \beta_{21} = 0$ & $\beta_{22} > \beta_{12} = 0$ & $\beta_{11} > \beta_{22}$

$M_7 : \text{not } M_1, \ldots, \text{ or } M_6$. 

19
Table 3: Unconstrained estimates for the Sesame Street application.

| parameter | estimate | standard error | correlation matrix |
|-----------|----------|----------------|-------------------|
| \(\beta_{11}\) | .647     | .082           | 1.00              |
| \(\beta_{21}\) | .040     | .103           | -.717 1.00        |
| \(\beta_{12}\) | .428     | .090           | .708 -.508 1.00   |
| \(\beta_{22}\) | .242     | .113           | -.508 .708 -.717 1.00 |

Table 4: Bayesian Hypothesis Evaluation for the Sesame Street application.

| Model | \(f^E_t\) | \(c^E_t\) | \(f^O_t\) | \(c^O_t\) | \(B_{tu}\) | \(P(M_t|y)\) |
|-------|-----------|-----------|-----------|-----------|-----------|--------------|
| \(M_1\) | 1.50      | 1.51      | 1.00      | .129      | 7.67      | .110         |
| \(M_2\) |           |           |           |           |           |              |
| \(M_3\) |           |           |           |           |           |              |
| \(M_4\) | .000      | 94.9      | 1.00      | .831      | .000      | .000         |
| \(M_5\) | .000      | 11.9      | .884      | .348      | .000      | .000         |
| \(M_6\) | .000      | 11.9      | .117      | .244      | .000      | .000         |
| \(M_7\) |           |           |           |           |           |              |

The MLEs with standard errors and correlations, and the Bayes factors and posterior model probabilities (assuming equal prior model probabilities) together with the measures of relative fit and complexity (if available) are presented in Table 3 and 4 respectively. The Bayes factors and posterior model probabilities show that the data provides most evidence for the order-constrained model \(M_3\) with a posterior probability of .855, while constrained models \(M_1\), \(M_2\), and the complement model \(M_7\) also receive some mild evidence with posterior probabilities of .110, .023, and .012, respectively. Interestingly the estimates are not in agreement with one of the five order constraints under model \(M_3\), namely \(\beta_{22} > \beta_{12}\). This seemingly conflicting result can be explained from the fact that the posterior probability that the constraints of \(M_3\) hold under \(M_u\), i.e., \(f^O_{3} \approx .193\), is 59.5 times larger than the default prior probability that the constraints hold, i.e., \(c^O_{3} \approx .003\), as the Bayes factor quantifies the change in support prior to posterior (Lavine & Chervish, 1999) while balancing between model complexity and model fit (Berger & Mortera, 1999; Mulder et al., 2010).
5 Concluding remarks

A default Bayes factor was proposed for evaluating multivariate normal linear models with competing sets of equality and order constraints on the parameters of interest. The methodology has the following attractive features. First the method can be used for evaluating statistics models with equality as well as order constraints on the parameters of interest. The possibility of order constrained testing is particularly useful in the applied sciences where researchers often formulate their scientific expectations using order constraints. Second, the method is fully automatic and therefore can be applied when prior information is weak or completely unavailable. The default prior is based on a minimal fraction of the information in the observed data of every group so that maximal information is used for model selection. Third, the Bayes factor is relatively simple to compute via Monte Carlo estimation that can be done in parallel. The Bayes factor has analytic expressions for special cases. Fourth the criterion is consistent which implies that the true constrained model will always be selected it the sample is large enough. Fifth, in the presence of missing data that are missing at random, the Bayes factor can be computed relatively easily using a multiple imputation method only under the unconstrained model. In sum, the method gives substantive researchers a simple tool for quantifying the evidence between competing scientific expectations, updating the evidence as new data emerge, while also correcting for missing data that are missing at random for many popular models including (multivariate) linear regression, (M)AN(C)OVA, repeated measures. The methodology will be implemented in the R-package ‘BFpack’ that is scheduled for later this year.

In this paper the Bayes factor was used as a confirmatory tool for model selection among a specific set of models with equality and/or order constraints. Equal model prior model probabilities were considered because all models were (approximately) equally plausible based on substantive justifications. In a more exploratory setting other choices may be preferable, (e.g., see Scott & Berger, 2006, who considered a model selection problem of many competing equality constrained models). It will be interesting to investigate how prior model probabilities should be specified in such exploratory settings when models may contain equality as well as order constraints on the parameters of interest.
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A Proof of Lemma 1

The constrained model in (1) can equivalently be written in the parameterization \( \zeta_t = (\zeta_{t,E}, \zeta_{t,O})' = H_t \theta, \) where \( H_t = [R_{t,E} \ D_t]' \) and \( D_t \) is a \((PK - r_{r,t}) \times PK\) matrix with independent rows of the form \((1, 0, \ldots, 0)\) with a 1 in the \( PK - r_{r,t} \) columns that correspond to parameters that are not constrained with an equality constraint, such the transformation is one-to-one, as \( M_t : \zeta_{t,E} = r_{t,E} \) & \( \tilde{R}_{t,O} \zeta_{t,O} > r_{t,O}, \) with \( \tilde{R}_{t,O} \) are the columns of \( R_{t,O} \) of the parameters that are not equality constrained. Furthermore, the adjusted parameter space in the denominator in (5) becomes

\[
M_t^* = \{ \zeta | \zeta_{t,E} - \hat{\zeta}_{t,E} = 0, \ \tilde{R}_{t,O}(\zeta_{t,O} - \hat{\zeta}_{t,O}) > 0 \},
\]

where \( \hat{\zeta}_{t,E} = R_{t,E} \hat{\theta}, \) and \( \hat{\zeta}_{t,O} = D_t \hat{\theta}. \) The marginal likelihood under \( M_t \) in
and the marginal likelihood under an unconstrained alternative model, \( M_u \), equals

\[
p_u(Y, b) = \frac{\int_{\mathcal{M}_t} \int_{\mathcal{M}_l} p(Y|X, \zeta_t, \zeta_l, \Sigma) |\Sigma|^{-\frac{p+1}{2}} d\zeta_l d\Sigma}{\int_{\mathcal{M}_t} \int_{\mathcal{M}_l} p(Y|X, \zeta_t, \zeta_l, \Sigma) |\Sigma|^{-\frac{p+1}{2}} d\zeta_l d\Sigma}.
\]

Thus, the Bayes factor can be written as

\[
B_{1u,b} = \frac{p_t(Y, b)}{p_u(Y, b)} = \frac{\int_{\mathcal{M}_t} \int_{\mathcal{M}_l} p(Y|X, \zeta_t, \zeta_l, \Sigma) |\Sigma|^{-\frac{p+1}{2}} d\zeta_l d\Sigma}{\int_{\mathcal{M}_t} \int_{\mathcal{M}_l} p(Y|X, \zeta_t, \zeta_l, \Sigma) |\Sigma|^{-\frac{p+1}{2}} d\zeta_l d\Sigma}.
\]

where

\[
p_t^*(\zeta_t, \zeta_l | Y, X, b) = \pi_t^*(\zeta_t, \zeta_l | Y, X, b) = \pi_t(\zeta_t + \hat{\zeta}_t - r_t, \zeta_l + \hat{\zeta}_l - \zeta_{l,0}|Y, X, b).
\]
The unconstrained marginal and conditional posteriors follow naturally from Bayes’ theorem,

$$
\pi_u(\theta, \Sigma|Y, X) \propto |\Sigma|^{-\frac{N+1}{2}} p(Y|X, \Theta, \Sigma) \\
\propto |\Sigma|^{-\frac{N+1}{2}} \exp\{-\frac{1}{2} \text{tr} \Sigma^{-1}(Y - X\Theta)'(Y - X\Theta)\} \\
\propto \pi_u(\Theta|Y, X, \Sigma) \pi_u(\Sigma|Y, X),
$$

with \(\pi(\Theta|Y, X, \Sigma) = \mathcal{N}(\hat{\Theta}, (X'X)^{-1}, \Sigma)\) \hspace{1cm} (15)

\(\pi(\Sigma|Y, X) = \mathcal{IW}(N - K, S)\) \hspace{1cm} (16)

where the least squares estimate is given by \(\hat{\Theta} = (X'X)^{-1}X'Y\) and the sums of square matrix equals \(S = (Y - X\hat{\Theta})'(Y - X\hat{\Theta})\). Furthermore, \(\mathcal{N}_{K,P}\) and \(\mathcal{IW}\) denote a matrix normal distribution for a \(K \times P\) matrix and an inverse Wishart distribution, respectively. Note that the conditional posterior distribution for \(\Theta\) is equivalent to a multivariate normal on the vectorization, \(\pi(\theta|Y, X, \Sigma) = \mathcal{N}(\hat{\theta}, \Sigma \otimes (X'X)^{-1})\). Integrating the covariance matrix out results in a marginal posterior for \(\Theta\) having a \(K \times P\) matrix Student \(t\) distribution,

\(\pi(\Theta|Y, X, \Sigma) = \mathcal{T}_{K,P}(\hat{\Theta}, (X'X)^{-1}, \Sigma, N - K - P + 1)\).

The unconstrained default prior is obtained by first raising the likelihood of the \(i\)-th observation to a fraction \(b_i\), i.e.,

\(p(y_i|x_i, \Theta, \Sigma)^{b_i} \propto |\Sigma|^{-\frac{b_i}{2}} \exp\{-\frac{b_i}{2}(y_i - \Theta'x_i)'\Sigma^{-1}(y_i - \Theta'x_i)\} = |\Sigma|^{-\frac{b_i}{2}} \exp\{-\frac{1}{2}(y_{i,b_i} - \Theta'x_{i,b_i})'\Sigma^{-1}(y_{i,b_i} - \Theta'x_{i,b_i})\},\)

where \(y_{i,b_i} = \sqrt{b_i}y_i\) and \(x_{i,b_i} = \sqrt{b_i}x_i\). The likelihood raised to observation specific fractions is then defined as

\(p(Y|X, \Theta, \Sigma)^b = \prod_{i=1}^{n} p(y_i|x_i, \Theta, \Sigma)^{b_i} \propto |\Sigma|^{-\frac{1}{2} \sum_{i=1}^{n} b_i} \exp\{-\frac{1}{2} \text{tr} \Sigma^{-1}(Y_b - X_b\Theta)'(Y_b - X_b\Theta)\} \propto |\Sigma|^{-\frac{1}{2} \sum_{i=1}^{n} b_i} \exp\{-\frac{1}{2} \text{tr} \Sigma^{-1}S_b\} \exp\{-\frac{1}{2} \text{tr} \Sigma^{-1}(\Theta - \hat{\Theta}_b)'X'_bX_b(\Theta - \hat{\Theta}_b)\},\)

where the least squares estimate is given by \(\hat{\Theta}_b = (X'_bX_b)^{-1}X'_bY_b\) and the sums of square matrix equals \(S_b = (Y_b - X_b\hat{\Theta}_b)'(Y_b - X_b\hat{\Theta}_b)\), and \(Y_b\) and
\( \mathbf{X}_b \) are the stacked matrices of \( \mathbf{y}_{i,b}' \) and \( \mathbf{x}_{i,b}' \), respectively. In combination with the improper noninformative independence Jeffreys’ prior, the fractional default prior based on generalized fractional Bayes methodology can then be written as

\[
\pi_u(\Theta, \Sigma | \mathbf{Y}, \mathbf{X}, b) \propto |\Sigma|^{-\frac{P+1}{2}} p(Y|\mathbf{X}, \Theta, \Sigma)^b
\]

with \( \pi_u(\Theta|\Sigma, \mathbf{Y}, \mathbf{X}, b) = \mathcal{N}_{K,P}(\hat{\Theta}_b, (\mathbf{X}_b' \mathbf{X}_b)^{-1}, \Sigma) \),

\[
\pi_u(\Sigma|\mathbf{Y}, \mathbf{X}, b) = \mathcal{W}(\sum_{i=1}^{N} b_i - K, S_b),
\]

so that

\[
\pi_u(\Theta|\mathbf{Y}, \mathbf{X}, b) = \mathcal{T}_{K,P}(\hat{\Theta}_b, S_b, (\mathbf{X}_b' \mathbf{X}_b)^{-1}, \sum_{i=1}^{N} b_i - K - P + 1).
\]

Finally, integrating the unconstrained prior \( \pi_u \) over the adjusted subspace \( \mathcal{M}_t' \) in step 4 of (14) is equivalent to integrating adjusted unconstrained priors \( \pi^* \) over \( \mathcal{M}_t \),

\[
\pi^*_u(\Theta|\Sigma, \mathbf{Y}, \mathbf{X}, b) = \mathcal{N}_{K,P}(\Theta_0, (\mathbf{X}_b' \mathbf{X}_b)^{-1}, \Sigma)
\]

\[
\pi^*_u(\Sigma|\mathbf{Y}, \mathbf{X}, b) = \mathcal{W}(\sum_{i=1}^{N} b_i - K, S_b),
\]

\[
\pi^*_u(\Theta|\mathbf{Y}, \mathbf{X}, b) = \mathcal{T}_{K,P}(\sum_{i=1}^{N} b_i - K - P + 1, \Theta_0, S_b, (\mathbf{X}_b' \mathbf{X}_b)^{-1}).
\]