Boson stars, neutron stars and black holes in five dimensions

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July 27, 2016

Abstract

Different types of gravitating compact objects occurring in d=5 space-time are considered: boson stars, hairy black holes and perfect fluid solutions. All these solutions of the Einstein equations coupled to matter have well established counterparts in d=4; in particular neutron stars can be modelled more or less realistically by a perfect fluid. A special emphasis is set on the possibility -and/or the necessity- for these solutions to have an intrinsic angular momentum or spin. The influence of a cosmological constant on their pattern is also studied. Several physical properties are presented from which common features to boson and neutron stars clearly emerge. We finally point out qualitative differences of the gravitational interaction supporting these classical lumps between four and five dimensions.

PACS Numbers: 04.70.-s, 04.50.Gh, 11.25.Tq

1 Introduction

The theory of gravity proposed by Einstein a century ago is among the most successful models in theoretical physics. In particular, the 100 years old prediction of gravitational wave emission by compact binary systems has finally been observed in a direct way - two times - in 2015 by the gravitational wave observatories of the LIGO project. These historical events, GW150914 and the Christmas present GW151226 [1, 2] constitute the first real strong field test of Einstein’s gravity. Yet this successful theory has frustrating features and is not able to fully explain all of (cosmological) observations, or even galactic dynamics without introducing some unknown type of dark matter and/or an unexplained cosmological constant. On the other hand, from a theoretical perspective, the theory of gravity sits aside of the other 3 fundamental interactions, and is not compatible with a quantum description.

Compact objects such as black holes and neutron stars are studied in order to test gravity or what is beyond theoretically. Black holes are clean objects, in the sense that they are solutions of the vacuum equations, but they pudically hide a singular point where general relativity fails. On the contrary, neutron stars display no regularity problem, but are constituted by matter that is not well understood [3] because of the very high density regime inside these objects. Many models describing the microphysics inside a neutron star are available on the market, some of them being incompatible with some current observations, and still many of them being plausible candidates.

Such compact objects can be used as theoretical laboratories to test alternative models of gravity. There have been many studies in the last years constructing neutron star models in, say scalar-tensor theories, Galileon theories Eddington-inspired Born-Infeld theories, stringy gravity, and so on (see for instance [4, 5, 6], or [7] for a review).

Perhaps the simplest extension of pure gravity to a gravity-matter system consist to couple (minimally or not) a scalar field to the Einstein-Hilbert action. In this system, boson stars solutions can be constructed. These
are solitonic, compact regular equations characterized by a conserved quantity. For a review of boson stars, see e.g. [10]. Recently a new class of hairy black holes was constructed [11]. Interestingly these black holes need to spin sufficiently fast and their domain of existence is limited by a family of boson stars; this constitutes a new motivations for studying boson stars.

At the other side of the gravity community, higher dimensional models of gravity has been intensively studied. The main motivation being provided by string theory and the famous gauge/gravity duality. In this context, spacetimes with a (negative) cosmological constant is of particular relevance; stability of the AdS spacetime has been debated [12], and more recently, dynamical process has started being investigated more seriously [13] [14]. The prototype vacuum higher dimensional solution is the Myers-Perry black hole [15]. It played an important role in understanding the dependence of the gravitational interaction to the number of dimensions of space-time. Besides these solutions which present an event horizon and an essential singularity at the origin, boson star solutions can be constructed as well by supplementing gravity with an appropriate matter field. The first construction of this type was reported in [16, 17]. Enforcing the d-dimensional boson star to spin requires an appropriate choice of the boson fields as shown in [18]. The counterpart of the black holes of [11] to five dimensions was achieved in [19] in the case of equal angular momenta and in [21] for arbitrary angular momenta. The patterns of hairy black holes in d=4 and in d=5 present similitudes but the main difference is that the underlying boson stars and hairy black holes display a mass gap in d=5.

In this paper, we focus on higher dimensional (5-dimensional) self-gravitating perfect fluid. Contrary to the case of 4 spacetime dimensions, a non-perturbative analysis of rotations is technically much simpler, in the case where the two independent angular momenta are equal. In this case, the spatial isometry is enhanced (in 5 dimensions) from \( U(1) \times U(1) \) to \( U(1) \times SU(2) \). As a consequence, it is possible to describe the nonperturbative rotation in the form of a co-homogeneity 1 problem, i.e. using ordinary differential equations (ODEs), whereas the less symmetric case or the case of 4 spacetime dimensions is described by partial differential equations (PDEs). In many papers (see namely [8, 9]) the effect of rotation on neutron stars is emphasized by considering slow rotations, allowing to still describe the problem in a simpler mathematical form (ODEs) than the nonperturbative case (PDEs).

To our knowledge, the equations describing a gravitating perfect fluid in higher dimensions has been studied only marginally [22] [23]. Recently in Ref. [25], static neutron stars were emphasized in d dimensional space-time with the emphasis set on a positive cosmological constant. One natural extension of this work is to include the effect of rotation of the fluid. In this paper we attempted to address the effect of rotation both pertubatively (independent angular momenta) and non-pertubatively (equal angular momenta).

The paper is organised as follow: In Sect. 2 we present the general framework, the ansatz for the metric and recall the form of the Myers-Perry solutions. Sect. 3 is devoted to the spinning, compact objects supported by bosonic matter (boson stars and black holes). The solutions composed of perfect fluid-type matter are emphasized in Sect. 4. The effects of a cosmological constant on these latter is pointed out in Sect. 5. Some extensions of these results are mentioned in Sect. 6.

## 2 The model and equations

We will consider the Einstein equations supplemented by matter:

\[
G^\nu_\mu + \Lambda g^\nu_\mu = \frac{8\pi G_d}{c^4} T^\nu_\mu, \quad \text{"Matter equations"}
\]

for two types of matter field \( T^\nu_\mu \) corresponding to boson star and to a perfect fluid respectively. The form of the energy momentum tensors will be specified later as well as the "Matter equations". In this formula, \( \Lambda \) represents a cosmological constant and \( G_d \) the d-dimensional Newton constant.

For both cases, we will parametrize the metric according to

\[
ds^2 = -b(r)dt^2 + \frac{1}{f(r)}dr^2 + g(r)d\theta^2 + h(r)\sin^2 \theta (d\varphi_1 - w(r)dt)^2 + h(r)\cos^2 \theta (d\varphi_2 - w(r)dt)^2 + (g(r) - h(r))\sin^2 \theta \cos^2 \theta (d\varphi_1 - d\varphi_2)^2,
\]
where $\theta$ runs from 0 to $\pi/2$, while $\varphi_1$ and $\varphi_2$ are in the range $[0, 2\pi]$. The corresponding space-times possess two rotation planes at $\theta = 0$ and $\theta = \pi/2$ and the isometry group is $\mathbb{R} \times U(2)$. The metric above still leaves the diffeomorphisms related to the definitions of the radial variable $R$ unfixed; for the numerical construction, we will fix this freedom by choosing $g(r) = r^2$.

In the following, the matter fields will be choosen and parametrized in such a way that the general equations (1) reduces consistently to a system of differential equations in the functions $f, b, h, w$ and the functions parametrized the matter.

## 2.1 Vacuum solutions: Myers-Perry black holes

Before setting the matter fields, we find it useful -for completeness- to recall the form of the asymptotically flat, vacuum solutions i.e. for $\Lambda = 0$ and $T_{\mu\nu} = 0$ in (1). These are the Myers-Perry solutions [15]: they are spinning black holes with event horizon $R_H$ and horizon angular velocity $\Omega_H$. They have the form

\[
\begin{align*}
    f(r) &= 1 - \frac{1}{1 - r_H^2 \Omega_H^2 \left(\frac{r_H}{r}\right)^2} + \frac{r_H^2 \Omega_H^2}{1 - r_H^2 \Omega_H^2} \left(\frac{r_H}{r}\right)^4 \\
    b(r) &= 1 - \left(\frac{r_H}{r}\right)^2 \frac{1}{1 - (1 - (\frac{r_H}{r})^4)^2 \Omega_H^2} \\
    h(r) &= r^2 \left(1 + \frac{r_H^2 \Omega_H^2}{1 - r_H^2 \Omega_H^2} \left(\frac{r_H}{r}\right)^4\right) \\
    w(r) &= \frac{\Omega_H}{1 - (1 - (\frac{r_H}{r})^4)^2 \Omega_H^2} \left(\frac{r_H}{r}\right)^4 \\
\end{align*}
\]

It is well known that generic Myers-Perry solutions with an outside (event horizon) $r_H$ presents a second horizon at $r_H'$ with $0 < r_H' < r_H$. The solutions exist for $r \in [0, \infty]$; the functions $b(r), w(r)$ remain finite in the limit $r \to 0$ while $f(r), h(r)$ diverge in this limit. The solution present a singularity at the origin.

## 3 Boson stars

Boson stars are obtained by coupling scalar fields to the Einstein-Hilbert gravity. That is to say by using the action

\[ S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R - 2\Lambda - (16\pi G_5)(\partial_\mu \Pi^\mu \partial^\nu \Pi^{\nu} + M_0^2 \Pi^\mu \Pi^\mu)) \]

Here $R$ represents the Ricci scalar, $\Lambda = -6/\ell^2$ is the cosmological constant. The matter sector consists of a doublet of complex scalar fields with the same mass $M_0$ and denoted by $\Pi$ in (1). The corresponding form of the energy momentum tensor is obtained in the standard way :

\[ T_{\mu\nu} = \Pi^\mu_{\alpha} \Pi^\nu_{\beta} + \Pi^\mu_{\beta} \Pi^\nu_{\alpha} - g_{\mu\nu} \left[ \frac{1}{2} \eta^{\alpha\beta} (\Pi^\mu_{\alpha} \Pi^\beta_{\mu} + \Pi^\mu_{\beta} \Pi^\alpha_{\mu}) + M_0^2 \Pi^\mu \Pi^\nu \right] \]

The key point for the construction of classical solutions is to choose the scalar doublet in the form

\[ \Pi(x) = \phi(r) e^{i\omega t} \vec{\Pi} \]

where $\vec{\Pi}$ is a doublet of unit length that depends on the angular coordinates. The standard non spinning solutions are recovered by means of the particular form

\[ \vec{\Pi} = (1, 0)^t \]

Only one component of the scalar doublet is non-zero. This leads to a system of coupled equations for the fields $f(r), b(r), \phi(r)$ while $h(r) = r^2, w(r) = 0$. In contrast, spinning solutions can be obtained with the parametrization [18]

\[ \vec{\Pi} = (\sin \theta e^{i\varphi_1}, \cos \theta e^{i\varphi_2})^t \]

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All fields $f(r), b(r), h(r), w(r)$ and $\phi(r)$ corresponding Einstein-Klein-Gordon equations are then non-trivial and the full system admit regular, localized solutions: the spinning boson stars. They can be characterized by several physical parameters, some of them are associated with the globally conserved quantities. The charge $Q$ associated to the U(1) symmetry possesses a special relevance; in terms of the ansatz (2) it takes respectively the forms

$$Q = -\int \sqrt{-g} j^0 dr = 4\pi^2 \int_0^\infty \sqrt{b h r^2 (\omega + w) \phi^2 dr}, \quad j^\mu = -i(\Pi^\mu \Pi - \partial^\mu \Pi \phi).$$

(9)

The mass of the solution $M$ and the angular momentum $J$ can be obtained from the asymptotic decay of some components of the metric:

$$g_{tt} = -1 + \frac{8G_5 M}{3\pi r^2} + o\left(\frac{1}{r^3}\right), \quad g_{\varphi t} = -\frac{4G_5 J}{\pi r^2} \sin^2 \theta + o\left(\frac{1}{r^3}\right), \quad g_{\varphi \varphi} = -\frac{4G_5 J}{\pi r^2} \cos^2 \theta + o\left(\frac{1}{r^3}\right)$$

(10)

In the case of spinning solutions, it was shown [18] that the charge $Q$ is related to the sum of the two (equal) angular momentum of the solution: $Q = 2|J|$.

Note that the 5-dimensional Newton constant has units $[G] = \text{length}^2 \text{mass}^{-1}$. We will adopt the dimensions such that $8\pi G_5 = 1$ and $c = 1$. As a consequence, all physical dimensions can be expressed in the unit of a length; in particular $[M_0] = \text{length}^{-1}$. The following dimensionless quantities will be used on the plots to characterize the solutions:

$$x = r M_0, \quad \omega M_0^{-1}, \quad Q M_0, \quad M M_0^2$$

(11)

For both cases -spinning and non-spinning- the system of differential equations has to be supplemented by boundary conditions. The regularity of the equations at the origin requires

$$f(0) = 1, \quad b'(0) = 0, \quad h'(0) = 0, \quad w(0) = 0$$

(12)

and $\phi'(0) = 0$ (resp. $\phi(0) = 0$) in the spinning (resp. non-spinning) case. Then, assuming for the moment $\Lambda = 0$ and the metric to be asymptotically flat, localized solutions should obey the following conditions:

$$f(r \to \infty) = 1, \quad b(r \to \infty) = 1, \quad h(r \to \infty) = 1, \quad w(r \to \infty) = 0, \quad \phi(r \to \infty) = 0.$$ 

(13)

The construction of exact solutions obeying the equations-plus-boundary conditions can be achieved numerically; for this purpose, we used the routine COLSYS [26]. In principle, the frequency $\omega$ is fixed by hand and the value $\phi(0)$ (or $\phi'(0)$) is fine tuned up to an $\omega$-depending value such that all boundary conditions are obeyed. Alternatively, it is also convenient to use $\phi(0)$ (or $\phi'(0)$) as control parameter and to reconstruct the corresponding frequency by an appropriate reformulation of the equations.

The results show that spinning and non-spinning solutions exist for a finite interval of $\omega$: the plot of the mass as function of $\omega$ is shown on Fig. 1. The solid and dashed RED curves represent respectively the mass of the spinning and non-spinning boson stars. The limit $\omega \to 1$ correspond to $\phi(0) \to 0$ (non-spinning case) and $\phi'(0) = 0$ (spinning case). In fact in this limit the scalar fields tends uniformly to zero; interestingly the mass of the solution remain finite in this limit, demonstrating the occurrence of a mass gap [19]. This seems to be a peculiarity of the gravitational interaction in $d = 5$.

While increasing the control parameter $\phi(0)$ (or $\phi'(0)$) the frequency $\omega$ decreasing up to a minimal value, say $\omega_m$ (for instance we find respectively $\omega_m \approx 0.925$ and $\omega_m \approx 0.947$ for the spinning and non-spinning cases), increasing again the control parameter a succession of branches are produced which exist on smaller, nested intervals of $\omega$. The plot of the mass versus frequency then presents the form of a spiral. For simplicity we do not report the value of the charge $Q$ but the shape of the curves are qualitatively similar.

In the next section we will consider matter under the form of a perfect fluid for which specific relations are assumed for the different components of the energy-momentum tensor. In a purpose of comparison, we show the different components of the energy momentum tensor of a boson star on Fig. 2. This solution is generic, corresponding to $\phi(0) = 0.5$ and $\omega = 0.948$. The pressure-density relation is also provided in the lower figure.
Figure 1: Mass of the non spinning and spinning bosons stars as function of $\omega$ (solid and dashed red lines). The mass of spinning black holes corresponding to horizon $x_h = 0.3$ and $x_h = 0.4$ is shown by the black lines. The blue line represent the limit where black holes become extremal.

### 3.1 Black holes

In \cite{19}, it was demonstrated that the spinning boson stars can be deformed into spinning hairy black holes. Within the ansatz (2) used for the metric the black holes are characterized by an event horizon at $x = x_H$ and by the conditions $f(x_H) = 0, b(x_H) = 0$. We refrain to discuss the different conditions of regularity that the different functions should fullfill at the horizon, they are discussed at length in \cite{19} and \cite{20}. For brevity, we just mention the one of them -essential for the discussion- namely:

$$w(x_H) = \omega.$$  \hspace{1cm} (14)

Accordingly, the hairy black holes can be labelled by two parameters: the horizon $x_H$ and the angular velocity on the horizon $w(x_H)$. A sketch of the pattern of hairy black holes is given by Fig. 1, the masses of these black holes are set between the red lines (corresponding to bosons stars) and the blue lines (corresponding to extremal black holes). The mass-frequency relation for the black holes with $x_H = 0.3$ and $x_H = 0.4$ are represented by the black lines.
Figure 2: Up: The components of the energy momentum tensor of the non spinning boson star with $\phi(0) = 0.5$ as function of $x$. Bottom: The pressure as function of the density.
4 Perfect Fluid

In this section, we study perfect fluid-like compact objects in five dimensional gravity and compare their features with the boson star-solutions constructed in the previous section. The idea is to solve the Einstein Equations by taking the energy momentum of a perfect fluid, i.e.

\[ T^{\mu \nu} = (\rho + P) U^\mu U^\nu + P g^{\mu \nu} \, , \quad U^\mu U_\mu = -1. \] (15)

Here \( \rho, P \) represent the density and the pressure which we both assume to depend on \( r \) only and \( U^a \) is the four velocity of the fluid. Assuming \( U^1 = U^2 = 0 \) and \( U^4 = U^3 \), the normalization condition implies a relation between \( U^0 \) and \( U_4 \). Referring to the case \( d = 4 \), we further pose \( U^4(r) = \lambda \omega(r)/\sqrt{b(r)} \) where \( \lambda \) is a constant. Then the normalization condition leads to the relation

\[ U^0 = \frac{\lambda \omega^2 \pm \sqrt{b^2 + (\lambda^2 - 1)b \omega^2}}{\sqrt{b(b - \hbar \omega^2)}} \] (16)

So it exists a continuum of possible values for \( U \) (this contrasts to the case \( d = 4 \) where the normalisation leads to a unique solution). In the following, we will only study the two cases corresponding to \( \lambda^2 = 1 \). The relation between \( U^0 \) and \( U^4 \) then simplifies:

\[ U^0 = \epsilon \frac{1}{\sqrt{b}} \, , \quad U^3 = U^4 = \frac{\epsilon \omega}{\sqrt{b}} \, , \quad \epsilon = \pm 1 \quad : \text{Type I} \] (17)

or

\[ U^0 = -\frac{\epsilon}{\sqrt{b}} \frac{b + h \omega^2}{b - h \omega^2} \, , \quad U^3 = U^4 = \frac{\epsilon \omega}{\sqrt{b}} \quad : \text{Type II} \] (18)

4.1 Static fluid

Assume a static fluid with \( w = 0 \), the equations corresponding to \( w \) and \( h \) are trivially solved by \( h(r) = r^2 \), \( w(r) = 0 \) and the relevant Einstein equations are those of \( f \) and \( b \) and are completed by the conservation equation for the perfect fluid:

\[ P' + \frac{b'}{2b} (P + \rho) = 0 \] . (19)

We further impose an equation of state between \( \rho \) and \( P \) under the form of a polytrope: \( P = K \rho^{1+1/n} \), where \( K \) and \( n \) are constants. For definiteness we set \( n = 2 \). For later use, we introduce the function \( Z(r) \) such that \( \rho = Z^n \, , \quad P = K Z^{n+1} \).

Using an appropriate rescaling of the density function, it turns out that the system depends on the coupling parameters \( G_5 \) and \( K \) through the combination \( \kappa \equiv 8\pi G_5 K^{-n} \). The effective parameter \( \kappa \) can then be absorbed in a rescaling of the radial variable \( r \). 'Physical' quantities can be recovered by this dimensionless system by noticing that the radius \( R \) and the mass \( M \) of the star scale respectively according to \( K^{-n/2} R \) and \( K^n M \).

The Einstein-perfect-fluid equations lead to a system of three coupled equations for \( f, b, P \). The equations for \( f(r) \) and \( P(r) \) are of the first order while the equation for \( b(r) \) is of the second order. Four boundary conditions are then necessary to specify a solution. To obtain the solution with a given value of the radius, say \( R \), the system is first solved on the interval \( r \in [0, R] \). The requirements for the metric to be regular at the origin and for the matter to vanish for \( r \geq R \) (then ensuing a Schwarschild-Myers-Perry metric in this region) lead to

\[ f(0) = 1 \, , \quad b'(0) = 0 \, , \quad f(R) = b(R) \, , \quad P(R) = 0 \, . \] (20)

In the outside region \( r \in [R, \infty] \), the solution is matched by continuity with the appropriate Schwarschild-Myers-Perry solution.

In principle the radius \( R \) is a natural parameter to control the solutions but it turns out that varying \( \rho(0) \), the density at the center, is slightly more convenient for the construction: the radius \( R \) can then be identified as the first zero of \( P(r) \). Profiles of interior solutions corresponding to \( \rho(0) = 1 \) and \( \rho(0) = 0.25 \) are reported in Fig. 4.1: the corresponding radius are given in the caption.
Several parameters characterizing the $d=5$ perfect fluids are presented on Fig. 4 (left panels) and compared with the corresponding data available for the more known $d=4$ case (right panels).

The mass $M$ as function of the radius $R$ is reported on the upper part of the figure. The parameters $b(0)$ and $Z(0)$ (with $Z(r) \equiv P^2(r)$) are also given on the lower panel of the figure. We see that solutions exist for large enough values of the radius, say for $R > R_{\text{min}}$. Our numerical results reveal the existence...
of two branches of solutions. The main branch exists for $R \in [R_{\text{min}}, \infty]$ while the second branch exists for $R \in [R_{\text{min}}, R_c] \sim [26.8, 36.6]$. For the values of $R$ such that two solution exist, the solution with the lowest mass belong to the second branch. In the limit $R \to R_{\text{min}}$, the two masses coincide. The solutions the main branch exist for large values of $R$. The limit $R \to \infty$ in fact corresponds to the density and pressure tending to zero and the solution smoothly approaching Minkowski space-time. Along with the case of boson stars, the mass of the star does not approach zero in the limit of vanishing matter: both, boson stars and perfect fluid solutions present a mass gap.

We now comment on the critical phenomenon limiting the solutions of the second branch of solutions for $R \to R_c$. It turns out that this limit is approached for $\rho(0) \gg 1$, at the same time the metric parameter $b(0)$ tends to zero. This suggests that, taking the limit $\rho(0) \to \infty$, the metric becomes singular at the origin and that the mass of the limiting configuration remains finite. In fact, this phenomenon is qualitatively the same for the boson stars, see Fig. 4.1.

Owing these peculiar properties of the $d=5$ perfect fluid, it is natural to compare these solutions with the more conventional $d=4$ case. The corresponding data are reported respectively on the left and right panels of Fig. 4. On the mass-radius plot, the main difference is that, for large $R$ the mass presents a local maximum for $d=4$ while it keeps monotonically increasing with $R$ for $d=5$. In both cases, large values of $R$ corresponds to a vanishing of the perfect fluid energy momentum tensor and to a metric approaching $d$-dimensional Minkowski space-time. However only for $d=5$ the solutions present a mass gap.

For the small values of $R$ the $d=4$-pattern is simpler, presenting only one branch. The solution exist for $R \geq R_{\text{min}}$ with $R_{\text{min}} \approx 9.5$. In the limit $R \to R_{\text{min}}$ the central density diverges and the metric parameter $b(0)$ approaches zero.

### 4.2 Type I spinning fluid

We now investigate the possibility of a spinning perfect fluid by solving the full equations with a spinning perfect fluid as constructed at the top of this section. For type I the conservation law reads as above

$$P' + \frac{b'}{2b}(P + \rho) = 0 \quad ,$$

and is supplemented by the polytropic equation of state. The problem consists in integrating the full Einstein equations as an initial value problem from the origin up to some finite value of $r$ with the conditions

$$f(0) = 1, b(0) = 1, b'(0) = 0, h(0) = 0, h'(0) = 0, w(0) = \Omega, w'(0) = 0 \quad .$$

We keep in mind that the metric field $b$ can be renormalized at the end in order to obey $b(\infty) = 1$. These conditions are completed by

$$\rho(0) = \rho_0, \quad \rho'(0) = 0 \quad .$$

Integrating the system up to a large enough value of $R$, the goal was to look for the condition $\rho(R_S) = 0$ we could determine the radius of the star $R_S$ as a function of $\Omega, \rho_0$ and the constant $K$.

Unfortunately, the Einstein equations for the fields $\omega$ and $h$ are trivially fulfilled with $\omega(r) = \Omega$ and $h(r) = r^2$. In particular, the equation for $w(r)$ has the form

$$w'' = w' \left( \frac{5H - 12f - 8}{3xf} + \kappa \frac{x\rho}{f} + \text{"lower order"} \right) \quad , \quad H(x) \equiv \frac{h}{x^2} \quad .$$

We see that the most singular term implies $w(r) = \omega_0 + \frac{C}{r^2}$, excluding the possibility of non-trivial, regular solutions at the origin. This contrasts with the corresponding equations for $d=4$. In the equation above, we have checked that the term due to matter does not regularize the singular term. It is instructive to compare the above equation with its "boson-star" counterpart:

$$w'' = w' \left( \frac{5H - 12f - 8}{3xf} + \text{"lower order"} \right) + 2\alpha \frac{\phi^2}{x^2 fH}(\omega + w) \quad , \quad \alpha \equiv 8\pi G_5 M_0^2 \quad .$$

Here the contribution of matter can be used to suppress the singular term, leaving the possibility of existence of spinning solutions which were indeed found numerically.
4.3 Independent slow rotations

In light of the results discussed in the previous subsections, we have checked whether it was possible to excite a single rotation, or non equal independent rotations. The underlying hypothesis to test being that the fluid cannot be made rotating due to the symmetry enhancement of the equal angular momentum spacetime.

The anzatz for the metric is proposed in [21] and leads to a system of partial differential equations with boundary conditions. Due to the complexity of these equations, we limited the analysis to small rotation and linearized the equations in the two independent rotating functions. For this purpose, we have followed the standard procedure for computing the slow rotation equations to order 1 [24] namely, we considered a spherically symmetric background and added the rotations as first order perturbations around the spherically symmetric case.

For completeness, we present here the form of the metric truncated to first order in rotation:

\[ ds^2 = -b(r)dt^2 + \frac{dr^2}{f(r)} + g(r) \left( d\theta^2 + \sin^2 \theta (d\varphi - \epsilon \omega_1(r) dt)^2 + \cos^2 \theta (d\psi - \epsilon \omega_2(r) dt)^2 \right), \]  

where \( \omega_1, \omega_2 \) are the independent angular velocity functions, \( b, f \) are the spherically symmetric background functions, \( g \) is an arbitrary function (as above we set \( g = r^2 \)) and \( \epsilon \) is the small parameter controlling the slow rotation approximation.

The matter fields are still modelled by a perfect fluid of the form \( T_{\mu \nu} = (\rho + P) u_\mu u_\nu + P \delta_{\mu \nu} \), where \( u \) is such that \( u^2 = -1 \), at least to first order in \( \epsilon \), leading to \( u^a = \frac{1}{\sqrt{b}} (1, 0, \epsilon \omega_1, \epsilon \omega_2) \).

We find two additional equations for the functions \( \omega_1, \omega_2 \), given by

\[ \omega_i'' - \left( \frac{\kappa r P(r)}{3f(r)} + \frac{\kappa r \rho(r)}{3f(r)} - \frac{5}{r} \right) \omega_i' = 0, \]  

where \( i = 1, 2 \). Note that the equations are the same for \( \omega_1 \) and \( \omega_2 \); only the initial conditions makes the difference.

The vacuum solution is given by

\[ \omega_i = \Omega_i \left( 1 - \frac{I}{r^4} \right), \]  

where \( \Omega_i, I \) are constants of integration related to the rotation frequency of the configuration, measured by a distant observer, and to the moment of inertia of the configuration.

For a solution to be regular at the origin, the boundary conditions for \( \omega_i \) are

\[ \omega_i(0) = \Omega_i, \quad \omega_i'(0) = 0, \]  

for constant values \( \Omega_i \).

Due to the fact that there are no terms linear in \( \omega_i \), and no source term in equation (27), the only regular solutions are \( \omega_i = \Omega_i \in \mathbb{R} \), as in the equal angular momenta case studied in the previous section.

Here again, five dimensional spacetime cannot support perfect fluids in rotation, even in the case of unequal angular momenta.

4.4 Type II fluid

For type II the conservation equation is more involved and not worth to be written explicitly. From the beginning it is clear the condition \( b - hw^2 > 0 \) will limitate the radius of the star. We were able to construct numerically solutions with \( dw/dr \neq 0 \) by using a shooting method from the origin. However these solutions obtained for \( w(0) > 0 \) are hard to be interpreted for several reasons, namely : (i) after decreasing the density function reaches a local minimum and then increases; (ii) the functions \( w(r), b(r), \rho(r) \) have a tendency to diverge for a finite radius while \( f(r) \) approaches zero. These results are illustrated by mean of Fig. 4; the features of this graphic seem to be generic.

In summary, from the numerous cases studied, the result strongly suggest that the d=5 perfect fluid cannot be made spinning.
5 Effect of a cosmological constant

In the above section, we obtained several characteristics of the asymptotically flat boson stars and perfect fluid solutions in 5-dimensions. In this section we address the effect of a cosmological constant on these families of solutions. Asymptotically AdS spinning boson stars (and black holes) have been studied in details in [27]. A crucial difference is that the presence of the negative cosmological constant suppress the mass gap in the pattern of solutions. On the other hand, the results of [20] seem to exclude the existence of asymptotically deSitter spinning boson stars and black holes.

We therefore put the emphasis on the deformation of the static perfect fluid solutions by a cosmological constant. We checked that the following properties hold for several values of \( \lambda \), although -for clarity- the data on Fig[6] is reported for \( \Lambda = \pm 10^{-4} \).

\( \Lambda > 0 \) We solved the equations in the interior in the case of a positive cosmological constant by integrating from the origin and using \( \rho(0) \) as a shooting parameter. We observe that the function \( Z(r) \) related to the density \( \rho(r) \) crosses zero only for a finite range of the value \( \rho(0) \) of the central density, say \( \rho(0) \in [\rho_1, \rho_2] \). The maximal mass of the star is reached for \( \rho(0) = \rho_2 \); it coincides with the maximal value, say \( R_{max} \) of the radius.
This appears by means of the red curve of Fig. 6. The solutions exist for $R \in [R_{\text{min}}, R_{\text{max}}]$. Along the case $\Lambda = 0$ two branches of solutions exist for small values of the radius.

$\Lambda < 0$ In the case $\Lambda < 0$, the situation is quite different. The following features are worth pointing out:

- The solutions exist in the limit $\rho(0) \to 0$ and the mass naturally approaches zero in this limit (see blue line on Fig. 6). In other words, the presence negative cosmological constant suppresses the mass gap. It is remarkable that this feature also holds in the case of AdS boson stars (see [27]). The mass gap seems to be a characteristic of asymptotically flat d=5 compact objects.

- As a consequence of the first property, the family of gravitating perfect-fluid presents a configuration with a maximal mass. This is reached for a specific value of the radius say $R(\Lambda)$. The value $R(\Lambda)$ tends to infinity while $\Lambda \to 0$.

- The solutions exist only for a finite interval of values of the radius, say $R \in [R_{\text{min}}, R_{\text{max}}]$.

- In the neighbourhood of both $R_{\text{min}}$ and $R_{\text{max}}$ there are two branches of solutions with two different masses corresponding to one value of $R$.

Some of these features, but not all, agree with the results obtained in [25]. The differences might be set on account on a different choice of the equation of state and/or on the accuracy of the numerical method.

Figure 6: Several properties of d=5 solutions for $\Lambda = 0, \pm 10^{-4}$. 

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6 Outlook

Myers-Perry black holes provide a laboratory to test how the dimensionality of space-time affects the gravitational interaction on black holes and then to appreciate how special is the d=4 Kerr Black holes. Besides black holes several type of solutions have been constructed, namely by supplementing different sort of matter. One of the most recent beeing the hairy black holes of \[11\] which has to be supported by rotation. The recent interest for gravity and field theory in higher-dimensional space-times motivates naturally the construction of boson stars, black holes as well as perfect fluid solutions in higher dimensions. This paper constitutes an attempt in this direction, although focalizing to d=4 and d=5 only. The generalisations and extentions are multiple but we mention only two : (i) are spinning perfect fluid possible for \(d > 5\); (ii) can perfect fluid surrounded by a cloud of boson field exist as stationary solutions?.

Note added: While completing the manuscript, Ref. [28] appeared were five-dimensional compact objects are emphasized as well.

References

[1] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 6, 061102 (2016) [arXiv:1602.03837 [gr-qc]].
[2] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. 116, no. 24, 241103 (2016) [arXiv:1606.04855 [gr-qc]].
[3] J. M. Lattimer and M. Prakash, Astrophys. J. 550, 426 (2001) [astro-ph/0002232].
[4] B. Kleihaus, J. Kunz, S. Mojica and M. Zagermann, Phys. Rev. D 93, no. 6, 064077 (2016) [arXiv:1601.05583 [gr-qc]].
[5] A. Cisterna, T. Delsate and M. Rinaldi, Phys. Rev. D 92 (2015) no.4, 044050 [arXiv:1504.05189 [gr-qc]].
[6] P. Pani, V. Cardoso and T. Delsate, Phys. Rev. Lett. 107, 031101 (2011) [arXiv:1106.3569 [gr-qc]].
[7] E. Berti et al., Class. Quant. Grav. 32, 243001 (2015) [arXiv:1501.07274 [gr-qc]].
[8] S. S. Yazadjiev, D. D. Doneva and D. Popchev, arXiv:1602.04766 [gr-qc].
[9] A. Cisterna, T. Delsate, L. Ducobu and M. Rinaldi, Phys. Rev. D 93, no. 8, 084046 (2016)
[10] E. W. Mielke and F. E. Schunck, “Boson stars: Early history and recent prospects,” [gr-qc/9801063]
[11] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. 112 (2014) 221101
[12] P. Bizon and A. Rostworowski, Phys. Rev. Lett. 107, 031102 (2011) doi:10.1103/PhysRevLett.107.031102 [arXiv:1104.3702 [gr-qc]].
[13] J. V. Rocha, R. Santarelli and T. Delsate, Phys. Rev. D 89, no. 10, 104006 (2014) doi:10.1103/PhysRevD.89.104006 [arXiv:1402.4161 [gr-qc]].
[14] T. Delsate, J. V. Rocha and R. Santarelli, Phys. Rev. D 89, 121501 (2014) doi:10.1103/PhysRevD.89.121501 [arXiv:1405.1433 [gr-qc]].
[15] R. C. Myers and M. J. Perry, Ann. Phys. 172, 304 (1986).
[16] D. Astefanesei and E. Radu, Nucl. Phys. B 665, 594 (2003) [gr-qc/0309131].
[17] A. Prikas, Phys. Rev. D 69 (2004) 125008 [hep-th/0404037].
[18] B. Hartmann, B. Kleihaus, J. Kunz and M. List, Phys. Rev. D 82 (2010) 084022
[19] Y. Brihaye, C. Herdeiro and E. Radu, Phys. Lett. B 739 (2014) 1
[20] Y. Brihaye, C. Herdeiro and E. Radu, arXiv:1605.08901 [gr-qc].
[21] C. Herdeiro, J. Kunz, E. Radu and B. Subagyo, Phys. Lett. B 748 (2015) 30
[22] J. de Boer, K. Papadodimas and E. Verlinde, JHEP 1010, 020 (2010) arXiv:0907.2695 [hep-th].
[23] X. Arsiwalla, J. de Boer, K. Papadodimas and E. Verlinde, JHEP 1101, 144 (2011) arXiv:1010.5784 [hep-th].
[24] J. B. Hartle, Astrophys.J., 150 (1967), 1005-1029
[25] G. H. Bordbar, S. H. Hendi and B. E. Panah, arXiv:1502.02929 [gr-qc].
[26] U. Ascher, J. Christiansen and R. D. Russell, Math. Comput. 33 (1979), 659; ACM Trans. Math. Softw. 7 (1981), 209.
[27] O. J. C. Dias, G. T. Horowitz and J. E. Santos, JHEP 1107, 115 (2011) arXiv:1105.4167 [hep-th].
[28] P. Bhar, M. Govender and R. Sharma, “A comparative study between EGB gravity and GTR by modelling compact stars,” arXiv:1607.06664 [gr-qc].