Coexistence of Composite-Bosons and Composite-Fermions in $\nu = \frac{1}{2} + \frac{1}{2}$ Quantum Hall Bilayers

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In bilayer quantum Hall systems at filling fractions near $\nu = 1/2 + 1/2$, as the spacing $d$ between the layers is continuously decreased, intra-layer correlations must be replaced by inter-layer correlations, and the composite fermion (CF) Fermi seas at large $d$ must eventually be replaced by a composite boson (CB) condensate or “111 state” at small $d$. We propose a scenario where CBs and CFs coexist in two interpenetrating fluids in the transition. Trial wavefunctions describing these mixed mixed CB-CF states compare very favorably with exact diagonalization results. A Chern-Simons transport theory is constructed that is compatible with experiment.

Bilayer quantum Hall systems show a remarkable variety of phenomena [1]. Perhaps the most studied case is when the electron density in each of the two layers is such that $\nu = n\phi_0/B = \frac{1}{2}$, where $n$ is density, $\phi_0 = 2\pi\hbar c/e$ is the flux quantum and $B$ is the magnetic field perpendicular to the sample. At this filling fraction, it is known that at least two types of states can occur depending on the spacing $d$ between the layers. For large $d$ the two layers must be essentially independent $\nu = \frac{1}{2}$ states, which are thought to be well described as compressible composite fermion (CF) Fermi seas with strong intralayer correlations and no interlayer correlations [2]. For small enough values of $d$ one should have an interlayer coherent “111 state” which can be described as a composite boson (CB) condensate with strong interlayer correlations and intralayer correlations which are weaker than that of the CF Fermi sea [1]. The nature of the transition between CFs and CBs is the focus of this paper. (Throughout this paper we will assume zero interlayer tunnelling, assume the spins are fully polarized, and set the in-plane field to zero.)

Previous numerical work [4] suggested that the transition between the CF Fermi sea and the CB 111 state may be first order. (A number of more exotic mechanisms have also been proposed [5]). However, experiments clearly show that interlayer correlations and coherence turn on somewhat continuously as $d/\ell_B$ is reduced [6–8] ($\ell_B = (\hbar/eB)^{1/2}$ is the magnetic length). Based on a picture of a first order transition and percolating puddles of one phase within the other, Ref. [3] predicts the drag resistivity tensor $\rho^D$ should roughly obey the semicircle relation $(\rho^D_{xx})^2 + (\rho^D_{xy} + \pi\hbar/e^2)^2 = (\pi\hbar/e^2)^2$ which agrees reasonably well with experiment [8]. On the other hand, the first order transition model has trouble accounting for the strong interlayer correlations that appear to occur even deep into the putative Fermi liquid state [8]. We are thus motivated to look for a more continuous transition from the CF Fermi liquid to the CB 111 state. The picture we have in mind is a family of states interpolating between these endpoints where each state is specified by the number of CFs and the number of CBs with the total number of CBs plus CFs remaining fixed. A first order transition could also be described as a mixture of CBs and CFs but with phase separation of the two fluids. Here we instead consider states where the CB and CF fluids interpenetrate. Since the wavefunction must be fully antisymmetric in terms of the original electrons, one might think of the electron having some CF character and some CB character. This interpolation between CF and CB is advantageous since it allows us (by varying the ratio of CFs to CBs) to construct states with varying degrees of inter- versus intra-layer correlations. We find that such mixed CF-CB states agree well with exact diagonalizations. Further, we find that a Chern-Simons version of our mixed Bose-Fermi theory is consistent with experimental observation, and in particular also predicts the above mentioned semicircle relation.

**CF Fermi Sea and 111 State:** Near $\nu = \frac{1}{2}$ in a single layer, the Jain CF [2] picture is given by attaching 2 zeros of the wavefunction to each particle. Thus we write the electron wavefunction as $\Psi = \mathcal{P}[\Phi_f(z_1,\ldots,z_{2N},\bar{z}_N)\prod_{i<j}(z_i - z_j)^2]$ where $\mathcal{P}$ represents projection onto the lowest Landau level, where for convenience here and elsewhere the gaussian factors $\exp[-\sum_i|z_i|^2/(4\ell_B^2)]$ will not be written explicitly, and $z_i = x_i + iy_i$ is the complex representation of the position. In the above equation $\Phi_f$ represents the fermionic wavefunction of the CFs in the effective magnetic field $B = B - 2n\phi_0$ where $n$ is the density. Generally, we will assume the CFs are weakly interacting so that the wavefunction $\Phi_f$ can be written as a single Slater determinant appropriate for noninteracting fermions in the effective field $B$. At $\nu = 1/2$, $B$ is zero and $\Phi_f$ represents a filled Fermi sea wavefunction.

In Chern-Simons fermion theory [9,2], each electron is exactly transformed into a fermion bound to 2 flux quanta. At mean field level, the fermions see magnetic field $B = B - 2n\phi_0$ which is zero at $\nu = \frac{1}{2}$. The effective (mean) electric field seen by a fermion is given analo-
gously by $\mathcal{E} = \mathbf{E} - 2\epsilon \mathbf{J}$ where $\mathbf{E}$ is the actual electric field, $\mathbf{J}$ is the fermion current (which is equal to the electrical current) and $\epsilon = (2\pi\hbar/c^2)\tau$ where $\tau = i\sigma_y$ is the 2 by 2 antisymmetric unit tensor (here $\sigma_y$ is the Pauli matrix). Defining a transport equation for the weakly interacting fermions $\mathcal{E} = \rho_f \mathbf{J}$ (where $\rho_f$ is approximated as simple Drude or Boltzmann transport for fermions in zero magnetic field), we obtain the RPA expression for the electrical resistivity $\rho = \rho_f + 2\epsilon$

We now turn to the double layer systems and focus on filling fraction $\nu_1 = \nu_2 = \frac{1}{2}$. If the two layers are very far apart, then we should have two independent $\nu = \frac{1}{2}$ systems. Thus, we would have a simple CF liquid state in each layer with the total wavefunction being just a product of the wavefunctions for each of the two layers. It should be noted that such a state completely neglects the correlation effects of the interlayer interaction.

On the other hand, if the two layers are brought very close together, the intra- and inter-layer interactions will be roughly the same strength. In this case, it is known that the system will instead be described by the so-called 111 state. The wavefunction for this state is written as $\Psi = \prod_{i<j}(z_i - z_j) \prod_{i<j}(w_i - w_j) \prod_{i,j}(z_i - w_j)$ where the $z$’s represent the electron coordinates in the first layer and the $w$’s represent electron coordinates in the second layer. Here each electron is bound to a single zero of the wavefunction within its own layer as well as being bound to a single zero of the wavefunction in the opposite layer. Of course by Fermi statistics, each electron must be bound to at least one zero within its own layer so the only additional binding here is interlayer. Thus, when $d$ is small, the electron binds a zero in the opposite layer (to form a CB inter-layer dipole) whereas when $d$ is large the electron minimizes its energy by binding to a zero within its own layer (forming a CF dipole [2]).

One can also write a Chern-Simons boson theory for the 111 state. Here, each electron is exactly modeled as a boson bound to 1 flux quantum where the bosons see the Chern-Simons flux from both layers. Thus, the effective magnetic (mean) field seen by a boson in layer $\alpha$ is given by $\mathbb{B}^\alpha = B - \phi_0(n_1^\alpha + n_2^\alpha)$ with $n_1^\alpha$ being the density in layer $\alpha$. Here, at total filling fraction of $\nu_1 + \nu_2 = 1$, the bosons in each layer see an effective field of zero and can condense to form a superfluid (or quantum Hall) state. The effective electric field seen by the bosons in layer $\alpha$ is similarly given by $\mathbb{E}^\alpha = \mathbb{E}^\alpha - \epsilon(\mathbf{J}_1^\alpha + \mathbf{J}_2^\alpha)$ where $\mathbb{E}^\alpha$ is the actual electric field in layer $\alpha$. This can be supplemented by a transport equation for the bosons in zero magnetic field $\mathbb{E}^\alpha = \rho_b^\alpha \mathbf{J}_b^\alpha$. If the bosons are indeed condensed, then we can set $\rho_b^\alpha = 0$. This results in the perfect Hall drag indicative of the 111 state where $\mathbb{E}_1 = \mathbb{E}_2 = \epsilon(\mathbf{J}_1 + \mathbf{J}_2)$.

**Transition Wavefunctions:** At intermediate $d$ we need to ask what the energetic price is for binding within the layer (to form a CF) versus out of the layer (to form a CB). The important thing to note is that (as we might expect for fermions) each fermion put into the Fermi sea costs successively more energy (each having a higher wavevector than the last). Thus, it might be advantageous for some of the fermions at the top of the Fermi sea to become unbound from their zeros within the layer and bind to the other layer — falling into the boson condensate. We imagine having some number $N_\beta^\alpha$ of electrons in layer $\alpha$ that act like CFs (filling a Fermi sea) and $N_\beta^\alpha$ that act like CBs (which can condense). Of course we should have $N_\beta^\alpha + N_\beta^\alpha = N^\alpha$ the total number of electrons in layer $\alpha$. We can write down mixed Fermi-Bose wavefunctions for double layer systems as follows:

$$\Psi = \mathcal{A} \mathcal{P} \left[ \prod_{j<i;N_j^\beta<i} (z_i - z_j) \prod_{j<i;N_j^\beta<i} (w_i - w_j) \prod_{i,j;N_j^\beta<i} (z_i - w_j) \prod_{i,j;N_j^\beta<i} (w_i - z_j) \right]$$

We have chosen to order the particles so that particles $i = 1, \ldots, N_j^\beta$ are fermions and $i = N_j^\beta + 1, \ldots, N^\alpha$ are bosons. The antisymmetrization operator $\mathcal{A}$ antisymmetrizes only over particle coordinates within each layer, and $\mathcal{P}$ is the lowest Landau level projection. Here, $\Phi_f^\alpha$ is the CF wavefunction of $N_j^\beta$ fermions in layer $\alpha$. The first line of the wavefunction is thus the fermionic part, including the CF wavefunctions and also the Jastrow factors. Here the Jastrow factors bind two zeros to each fermion within a layer in the sense that the wavefunction vanishes as $z^2$ as two fermions in the same layer approach each other (before antisymmetrization and projection). The second line of the wavefunction binds zeros to each boson such that the wavefunction vanishes as $z$ as any particle (boson or fermion) approaches that boson from either layer. (The bosons are assumed condensed so the explicit boson wavefunction is unity).

It is perhaps easier to describe these Jastrow factors in terms of an effective Chern-Simons description. Within such a description, we write expressions for the effective magnetic field $\mathbb{B}^\alpha$ seen by either species in layer $\alpha$ as

$$\mathbb{B}_f^\alpha = B - 2\phi_0 n_\alpha^\alpha - \phi_0 (n_1^\alpha + n_2^\alpha) \quad (2)$$

$$\mathbb{B}_b^\alpha = B - \phi_0 (n_1^\alpha + n_2^\alpha + n_1^\alpha + n_2^\alpha). \quad (3)$$
We note, however, that unlike in the prior 111 case and Fermi liquid case, the transformation to a Chern-Simons theory here is not exact (even before making a mean field approximation) because of complications associated with arbitrarily choosing which particles are bosons and which are fermions. It seems plausible that as a low energy description, to the extent that zeros are bound tightly to electrons, such a mixed Bose-Fermi theory will be sensible. However, by making such an approximation we explicitly neglect processes by which a zero of the wavefunction is transferred from one electron to another (changing a boson into a fermion or vice versa).

Since at $\nu = \frac{1}{2}$ the effective magnetic field seen by bosons or fermions is zero, we will take $\Phi_f^\alpha$ to be a filled Fermi sea of $N_f^\alpha$ CFs in layer $\alpha$. Similarly the CBs in zero field condense into a $k = 0$ ground state (as assumed in the wavefunction). If we have two layers of matched density we must have $N_f^1 = N_f^2$ in the ground state. However, the overall number of CFs versus CBs is a matter of energetics and presumably varies continuously as $d$ changes. When $N_f = 0$ in both layers, this wavefunction is the 111 state, whereas $N_f = 0$ (or $N_f = N$) consists of 2 uncorrelated layers of CFs.

We generate wavefunctions with 0, 1, 2, 3, 4, 5 fermions per layer (and correspondingly 5, 4, 3, 2, 1, 0 bosons per layer). At low spacing $N_f = 0$ in both layers, this wavefunction is the 111 state, whereas $N_f = N$ is the 111 state, whereas $N_f = 0$ (or $N_f = N$) consists of 2 uncorrelated layers of CFs.

Numerical Calculations: We have considered a finite sized bilayer sphere with 5 electrons per layer and a monopole of flux $9q_0$ at its center. We first generate mixed CB-CF wavefunctions of the form of Eq. 1. The method of numerical generation is involved and will be discussed in a forthcoming paper. We note that in the Jastrow factors of Eq. 1 the fermions ($z_i, w_i \leq N_f$) experience one less flux quantum than the bosons ($z_i, w_i > N_f$). Thus, when the CBs are in zero effective magnetic field (so they can condense), the CFs experience a single flux quantum. Thus, $\Phi_f$ is modified to represent fermions on a sphere with a monopole of charge $q_0$ in the center.

We next perform an exact lowest Landau level diagonalization on the bilayer sphere using a pure coulomb interaction $v_{11}(r) = e^2/r$ within a layer and $v_{12}(r) = e^2/\sqrt{r^2 + d^2}$ between the two layers where $r$ is the chord distance between two points. We vary the spacing $d$ between the two layers and obtain exact ground states at each spacing $d$. In Figure 1, we show the overlap (squared) of each of our trial wavefunctions with the exact ground state at each spacing $d$. We also show the relative energy of the various ground states as a function of $d/l_B$. It is clear that the mixed CB-CF states have very high overlap with the exact ground state and very low energy in the transition region. This supports the picture of the transition from CF to CB occurring through a set of ground states with interpenetrating CF-CB mixtures.

In Figure 2 we show the inter- and intra-layer electron pair correlation functions ($g_{12}$ and $g_{11}$). We see that the mixed CB-CF states allow us to interpolate between the types of correlations that exist in the limiting 111 and Fermi liquid states.

Chern-Simons RPA: We can calculate the resistivities and drag resistivities of these mixed Bose-Fermi states by using the Chern-Simons RPA approach. Analogous to Eqs. 2 and 3 we can write effective electric (mean) fields seen by the bosons or fermions in layer $\alpha$:

$$\mathcal{E}_f^\alpha = E^\alpha - 2eJ_f^\alpha - \epsilon(J_b^\alpha + J_f^\alpha)$$

$$\mathcal{E}_b^\alpha = E^\alpha - \epsilon(J_f^\alpha + J_b^\alpha + J_f^\alpha + J_f^\alpha)$$

The CF current in layer $\alpha$ with the total current in layer $\alpha$ given by $J_f^\alpha = J_b^\alpha + J_f^\alpha$. We supplement Eqs. 4, 5 with transport equations for the fermions (bosons) in each layer $\mathcal{E}_f^{\alpha(b)} = \rho_{f(b)}(J_f^{\alpha(b)})$. Finally, for a drag experiment we fix $J^2 = 0$ and fix...
\( \mathbf{J}^1 \) finite. We then solve for \( \mathbf{E}^1 \) and \( \mathbf{E}^2 \) yielding the in-layer resistivity (\( \mathbf{E}^1 = \rho^{11} \mathbf{J}^1 \)) and the drag resistivity (\( \mathbf{E}^2 = \rho^{D} \mathbf{J}^1 \)). Assuming layer symmetry so \( \rho_{\beta(f)}^0 \) is not a function of \( \alpha \), the results of such a calculation yield \( \rho^{11} = (G + H)/2 \) and \( \rho^{D} = (G - H)/2 \) where

\[
G = (\rho_0^{-1} + \rho_f^{-1})^{-1} + 2\epsilon, \quad \text{and} \quad H^{-1} = \rho_0^{-1} + (\rho_0 + 2\epsilon)^{-1}.
\]

At filling fraction \( \nu_1 = \nu_2 = \frac{1}{2} \) both \( \rho_0 \) and \( \rho_f \) are diagonal (since CBs and CFs are in zero effective field). Furthermore, from the temperature or with disorder). For the 111 state \( \rho^{11} \approx \rho^{D} \), which is also reasonably consistent with published data \([8]\).

Our mixed CB-CF theory can be generalized for unequal densities as well as for filling fractions away from \( \nu_1 = \nu_2 = \frac{1}{2} \). Further, one may be able to treat the effects of an in-plane magnetic field. Indeed, such an approach was already used in Ref. \([12]\) to understand bilayer tunnelling experiments in tilted fields.

In summary, we have constructed a theory describing the crossover between a CF liquid at large \( d \) to the 111 CB state at small \( d \) which can be thought of as a two-fluid model. Comparisons of trial wavefunctions to exact diagonalization are very favorable, and the corresponding Chern-Simons transport theory appears to be in reasonable agreement with experimental data.

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**FIG. 2.** Interlayer(top) and Intralayer (bottom) electron pair correlation functions \( g_{12} \) and \( g_{11} \) as a function of (arc) distance for each of the trial wavefunctions (5 electrons per layer again). As we go from the Fermi liquid state(5 Fermions) to the 111 state (0 Fermions), replacing CFs with CBs, the short range intralayer correlations are reduced and the interlayer correlations build up. The deviation of \( g_{12} \) from unity in the Fermi liquid state is caused by the fact that two \( \rho_{0,XX} \) and four measurable quantities (\( \rho_{XX} \), \( \rho_{XY} \), \( \rho_{2XX} \), \( \rho_{2YY} \)) which enables us to derive experimental predictions, such as \( \rho_{2XX} + \rho_{2YY} = 4\pi / e^2 \) \([11]\). Other more complicated relationships can also be derived.

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