Multiserver Queueing Model subject to Single Exponential Vacation

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Abstract: A multi-server queueing model subject to single exponential vacation is considered. The arrivals are allowed to join the queue according to a Poisson distribution and services takes place according to an exponential distribution. Whenever the system becomes empty, all the servers goes for a vacation and returns back after a fixed interval of time. The servers then starts providing service if there are waiting customers otherwise they will wait to complete the busy period. The vacation times are also assumed to be exponentially distributed. In this paper, the stationary and transient probabilities for the number of customers during ideal and functional state of the server are obtained explicitly. Also, numerical illustrations are added to visualize the effect of various parameters.

1. INTRODUCTION

The single server queueing model with vacations have been studied by many researchers and applied to a variety of real time scenario like telecommunication and computer networks, manufacturing and production systems and many more. Multi server queueing models with vacation are more complex to analyze in comparison to single server vacation models. So, there are only few studies available in the literature on multiserver queueing model with vacation. The concept of server vacation in multi server queueing model is quite appropriate in many practical situations than the single server counterparts.
Parthasarathy and Sharafali [1] provided explicit analytical expressions for the time dependent probabilities of the number in the system at arbitrary time in terms of modified Bessel function of first kind using generating function methodology for a multi server queueing model. Al-seedy et al [2] extended the results by introducing the concept of balking and reneging in the multi server queueing model. Using the similar technique as above, Sheriff Ammar [3] obtained explicit expressions for the transient system size probabilities for the queue with heterogeneous servers and impatient behaviour. More recently, Dharmaraja and Rakesh Kumar [4] obtained the explicit time dependent probabilities of Markovian queueing model with heterogeneous servers and catastrophes.

The steady state analysis of multi server queueing model subject to single and multiple exponential vacation has been extensively studied in the literature. The $M/M/c$ queue with exponentially distributed vacation time was first analyzed by Levy and Yechiali[5] and Vinod[6]. Chao and Zhao[7] discussed the $GI/M/c$ queue subject to two classes of vacation, namely, station vacation (group vacation of all the servers together when the system becomes empty) and server vacation (wherein each server takes independent vacation when no customers wait in the queue) and obtained the steady state probabilities using matrix geometric approach. Tian et al[8] presented a more detailed analysis of an $M/M/c$ queueing model subject to vacation and proved several conditional stochastic decomposition results for the queue length and customer waiting time. These existing results on multi-server models require all the servers to take vacations simultaneously. However, in many practical situations, it is appropriate to assume some working server at any instant of time. Zhang and Tian [9] have studied the vacation model with a “partial server multiple vacation policy” in which some servers (not all) take multiple vacations. Further, Zhang and Tian [10] analyzed the $M/M/c$ queueing model with a single vacation policy for some idle server and obtained the stationary distribution of the system using matrix analytic method. Altman and Yechiali [11] gave a comprehensive analysis of both single and multi server queueing models subject to single and multiple vacation under stationary regime and presented many closed form results. Lin and Ke[12] presented the multi server queueing model where the servers work at different rate during the vacation period and provided explicit formula for the probability distributions of queue length and other system characteristics using matrix analytic method. More recently, Yue et al[13] discussed an $M/M/c$ queueing model subject to synchronous vacation and customer impatience and obtained explicit expressions for certain interesting performance measures in steady state.

This paper is the first of its kind to analyze the multi server queueing model subject to single exponential vacation in the time dependent regime and obtain a closed form expressions for the state probabilities of the system. Explicit analytical expressions aid in the better understanding of the various performance measures associated with the system. Further, single vacation policy is more
appropriate than a multiple vacation policy in the situation where the secondary job is only performed once by idle servers. In this model, it is assumed that all the servers leave the system for a vacation whenever the system is empty and return to the system simultaneously when the vacation is completed. This occurs in situation where a system consists of several interconnected machines that are inseparable, or when all the machines are run by a single operator. In this case, if the system (or the operator who runs the system) is used for a secondary task when it becomes empty (or available), all the servers (the operator) will then be utilized to perform a secondary task. During this period of time the system is unavailable to further arrivals to the system.

The paper is organized as follows: Section 2 describes the mathematical model; Section 3 obtains explicit expressions for the steady state probabilities of the model; Section 4 presents the transient analysis of the $M/M/c$ queueing model subject to single exponential vacation and hence obtains the closed form expressions for the time dependent probabilities of the system; Section 5 illustrates the behaviour of the state probabilities numerically for the appropriate choice of the parameters.

2. MODEL DESCRIPTION

Consider an $M/M/c$ queueing model subject to single exponential vacation. Arrivals are assumed to follow Poisson distribution with parameter $\lambda$. The $c$ servers provide service according to an exponential distribution each with parameter $\mu$. When the system is empty, all the $c$ servers go on a vacation wherein the vacation times follow exponential distribution with parameter $\theta$. After some time, all the $c$ server will return to the system and if they find customers they will start providing service, otherwise they remain idle.

Let $X(t)$ denote the number of customers in the system at time $t$. Define $J(t) = 1$, when the server is functional and $J(t) = 0$, when the server is ideal at time $t$. It is well known that $\{J(t), X(t), t \geq 0\}$ is a Markov process with state space $\Omega = \{(0,0), (1,0), (1,1), \ldots\}$. The state transistion diagram for the model is given in Figure 1.

![Figure 1. State Transition Diagram of an $M/M/c$ Queueing Model Subject to Single Exponential Vacation](image-url)
3. STATIONARY ANALYSIS

This section presents explicit expressions for the steady state system size probabilities of the above described model. Let \( \pi_k \) denote the stationary probability for the system to be in state ‘1’ with \( k \) customers and \( Q \) denote the stationary probability for the system to be in state ‘0’ with no customers. The system of equations governing the state probabilities under steady state are given by

\[
\theta Q - \lambda \pi_0 = 0, \quad (3.1)
\]

\[
\mu \pi_1 - \theta Q = 0, \quad (3.2)
\]

\[
\lambda \pi_{k-1} - (\lambda + k\mu)\pi_k + (k + 1)\mu \pi_{k+1} = 0, k = 1, 2 \ldots c - 1 \quad (3.3)
\]

and

\[
\lambda \pi_{k-1} - (\lambda + c\mu)\pi_k + c\mu \pi_{k+1} = 0, k = c, c + 1 \ldots \quad (3.4)
\]

Observe that from equation (3.1) and equation (3.2),

\[
\mu \pi_1 - \lambda \pi_0 = 0, \quad (3.5)
\]

and hence

\[
\pi_1 = \rho \pi_0, \quad (3.6)
\]

where \( \rho = \frac{\lambda}{\mu} \). Also, equation (3.1) gives

\[
Q = \frac{\lambda}{\theta} \pi_0. \quad (3.7)
\]

Equation (3.3), can be rewritten as

\[
(k + 1)\mu \pi_{k+1} - \lambda \pi_k = k\mu \pi_k - \lambda \pi_{k-1}
\]

\[
= (k - 1)\mu \pi_{k-1} - \lambda \pi_{k-2}
\]

\[
\vdots
\]

\[
= \mu \pi_1 - \lambda \pi_0 = 0.
\]

Hence

\[
\pi_{k+1} = \left( \frac{\lambda}{\mu} \right) \left( \frac{\pi_k}{k+1} \right).
\]

Rewriting the above equation, we get

\[
\pi_k = \frac{\rho^k}{k!} \pi_0 , k = 1, 2 \ldots c - 1. \quad (3.8)
\]

Similarly, equation (3.4) recursively yields

\[
\pi_k = \frac{1}{c^{k-c}(c)!} \rho^k \pi_0 , k = c, c + 1 \ldots. \quad (3.9)
\]

From the normalization condition, given by

\[
\sum_{k=0}^{c-1} \pi_k + \sum_{k=c}^{\infty} \pi_k = 1,
\]

we get

\[
\pi_0 = \left( \frac{\lambda}{\theta} + \sum_{k=0}^{c-1} \frac{\rho^k}{k!} + \frac{\rho^c}{(c-1)! (c-\rho)} \right)^{-1},
\]
subject to the stability condition given by $\frac{\rho}{c} < 1$. Thus, all the steady state system size probabilities are explicitly determined.

The mean number of customers in the system under steady state is given by

$$E(N) = \sum_{k=1}^{\infty} k\pi_k = \sum_{k=0}^{c-1} k\pi_k + \sum_{k=c}^{\infty} k\pi_k,$$

which on simplification yields

$$E(N) = \pi_0\rho \left( 1 + \rho + \frac{\rho^2}{2!} + \ldots + \frac{\rho^{c-1}}{(c-1)!} + \frac{\rho^c}{c(c!)} \left( 1 + \frac{\rho}{c} + \frac{\rho^2}{c(c!)} + \ldots \right) \right),$$

subject to the conditions $Q(0) = 1$ and $P_{\rho}(0) = 0$ for $k = 1, 2, 3, \ldots$. The solution to the above system of differential difference equations represents the transient state probabilities of the model under consideration and is given by Theorem 1.

**Theorem 1.**

The time dependent system size probabilities of the $M/M/c$ queueing model subject to single exponential vacation are given by

$$Q(t) = \theta Q(t) - \lambda P_0(t),$$

$$(4.1)$$

$$P_0(t) = \mu P_1(t) - \theta Q(t),$$

$$(4.2)$$

$$P_k(t) = \lambda P_{k-1}(t) - (\lambda + k\mu) P_k(t) + (k + 1)\mu P_{k+1}(t), k = 1, 2, \ldots, c - 1,$$

$$(4.3)$$

and

$$P_k(t) = \lambda P_{k-1}(t) - (\lambda + c\mu) P_k(t) + c\mu P_{k+1}(t), k = c, c + 1, \ldots,$$

$$(4.4)$$

subject to the conditions $Q(0) = 1$ and $P_{\rho}(0) = 0$ for $k = 1, 2, 3, \ldots$. The solution to the above system of differential difference equations represents the transient state probabilities of the model under consideration and is given by Theorem 1.

**Theorem 1.**

The time dependent system size probabilities of the $M/M/c$ queueing model subject to single exponential vacation are given by

$$Q(t) = \sum_{i=0}^{\infty} (\mu\theta)^i e^{-\theta t} \frac{t^i}{i!} * \sum_{i=0}^{\infty} (\lambda t)^i e^{-\lambda t} \frac{t^i}{i!} * (r_1(t))^{*i},$$

$$P_0(t) = \theta e^{-\lambda t} * Q(t),$$

$$P_k(t) = P_0(t) * r_1(t) * r_2(t) * \ldots * r_k(t), k = 1, 2, \ldots, c - 1,$$

and

$$P_k(t) = P_{\rho}(t) * r_1(t) * r_2(t) * \ldots * r_k(t), k = c, c + 1, \ldots,$$

where * denotes convolution,

$$r_k(t) = \sum_{i=0}^{\infty} \lambda ((k + 1)\mu)^i e^{-(k + 1)\mu t} \frac{t^i}{i!} * (r_{k+1}(t))^{*i}, k = 1, 2, \ldots, c - 1,$$
\[ r_c(t) = r(t) = \frac{1}{2c\mu} \frac{aI_1(\alpha(t))}{t} e^{-\lambda c t} \]

and

\[ \psi_k(t) = \frac{1}{(2c\mu)^k} \frac{kI_k(\alpha(t))}{t} e^{-\lambda c t}, k = 1, 2, 3 \ldots \]

Note that \( \alpha = 2\sqrt{\lambda \mu} \) and \( I_k(\alpha(t)) \) represents the modified Bessel function of the first kind.

**Proof**

Let \( \tilde{Q}(s) \) and \( \tilde{P}_k(s) \) denote the Laplace transform of \( Q(t) \) and \( P_k(t) (k = 1, 2 \ldots) \). Then, taking Laplace transform for the equations (4.1) to (4.4) leads to

\[
(s + \lambda) \tilde{P}_0(s) - \theta \tilde{Q}(s) = 0, \quad (s + \theta) \tilde{Q}(s) - \mu \tilde{P}_1(s) = 1, \tag{4.5}
\]

\[
-\lambda \tilde{P}_{k-1}(s) + (s + \lambda + k\mu) \tilde{P}_k(s) - (k + 1)\mu \tilde{P}_{k+1}(s) = 0, \quad k = 1, 2, \ldots, c - 1, \tag{4.6}
\]

and

\[
-\lambda \tilde{P}_{c-1}(s) + (s + \lambda + c\mu) \tilde{P}_c(s) - c\mu \tilde{P}_{c+1}(s) = 0, \quad k = c, c + 1, \ldots \tag{4.7}
\]

Assume,

\[
\tilde{P}_k(s) = \tilde{P}_{k-1}(s)\tilde{r}_c(s) = \tilde{P}_0(s)\tilde{r}_1(s)\tilde{r}_2(s) \ldots \tilde{r}_k(s), \quad k = 1, 2, \ldots, c - 1 \tag{4.9}
\]

and

\[
\tilde{P}_c(s) = \tilde{P}_{c-1}(s)r^{k-c+1}(s), \quad k = c, c + 1, \ldots \tag{4.10}
\]

where \( \tilde{r}_k(s) \) satisfies the recursive equation given by

\[
\tilde{r}_k(s) = \frac{s + \lambda + k\mu - (k + 1)\mu \tilde{r}_{k+1}(s)}{s + \lambda + c\mu - \sqrt{(s + \lambda + c\mu)^2 - 4\lambda c\mu}}.
\]

for \( k = 1, 2, \ldots, c - 1 \), such that \( \tilde{r}_c(s) = r(s) \) and

\[
r(s) = \frac{(s + \lambda + c\mu) - \sqrt{(s + \lambda + c\mu)^2 - 4\lambda c\mu}}{2c\mu}.
\]

We now prove that equation (4.9) and equation (4.10) satisfies the governing system of difference equation in the Laplace domain represented by equation (4.7) and equation (4.8). Towards this end, when \( k = c \), we get from equation (4.8),

\[
-\lambda \tilde{P}_{c-1}(s) + (s + \lambda + c\mu) \tilde{P}_c(s) - c\mu \tilde{P}_{c+1}(s) = 0
\]

From equation (4.10), \( \tilde{P}_c(s) = \tilde{P}_{c-1}(s)r(s) \) and \( \tilde{P}_{c+1}(s) = \tilde{P}_{c-1}(s)r^2(s) \).

Substituting the above expressions in equation (4.13) yields

\[
\tilde{P}_{c-1}(s)[-\lambda + (s + \lambda + c\mu)r(s) - c\mu r^2(s)] = 0.
\]

Since \( \tilde{P}_{c-1}(s) \) is not identically zero, we get

\[
-\lambda + (s + \lambda + c\mu)r(s) - c\mu r^2(s) = 0,
\]

and hence

\[
r(s) = \frac{(s + \lambda + c\mu) - \sqrt{(s + \lambda + c\mu)^2 - 4\lambda c\mu}}{2c\mu}.
\]

which coincides with equation (4.12). In general, for \( k = c, c + 1, c + 2 \ldots \), equation (4.10) is seen to satisfy equation (4.8) as

\[
-\lambda \tilde{P}_{k-1}(s) + (s + \lambda + c\mu) \tilde{P}_k(s) - c\mu \tilde{P}_{k+1}(s) = 0
\]

leads to

\[
\tilde{P}_{c-1}(s)r^{k-c}(s)[-\lambda + (s + \lambda + c\mu)r(s) - c\mu r^2(s)] = 0
\]

which is true from equation (4.14). Similarly, for \( k = 1, 2, \ldots, c - 1 \), substituting equation (4.9) in equation (4.7) yields
\[ \hat{P}_{k-1}(s) \left( -\lambda + (s + \lambda + k\mu) r_k(s) - (k + 1)\mu r_k(s) r_{k+1}(s) \right) = 0. \]

Since \( \hat{P}_{k-1}(s) \) is not identically zero, we get
\[ -\lambda + (s + \lambda + k\mu) r_k(s) - (k + 1)\mu r_k(s) r_{k+1}(s) = 0 \]
and hence
\[ r_k(s) = \frac{\lambda}{s + \lambda + k\mu - (k + 1)\mu r_{k+1}(s)}. \]
which is same as equation (4.11). In particular, when \( k = c - 1 \), we get from equation (4.7)
\[ -\lambda \hat{P}_{c-2}(s) + (s + \lambda + (c - 1)\mu) \hat{P}_{c-1}(s) - c\mu \hat{P}_c(s) = 0 \quad (4.15) \]
From equation (4.10), it is seen that
\[ \hat{P}_c(s) = \hat{P}_{c-1}(s) r(s), \]
and from equation (4.9),
\[ \hat{P}_{c-1}(s) = \hat{P}_{c-2}(s) r_{c-1}(s). \]
Therefore, equation (4.15) reduces to
\[ \hat{P}_{c-2}(s) \{ -\lambda + (s + \lambda + (c - 1)\mu) r_{c-1}(s) - \mu r_{c-1}(s) r(s) \} = 0. \]
As before, \( \hat{P}_{c-2}(s) \) cannot be identically zero and hence \( r_{c-1}(s) \) satisfies
\[ r_{c-1}(s) = \frac{\lambda}{s + \lambda + (c - 1)\mu - c\mu r(s)}, \]
where \( r(s) \) is explicitly given in equation (4.12). Similarly, when \( k = c - 2 \), we get
\[ r_{c-2}(s) = \frac{\lambda}{s + \lambda + (c - 2)\mu - (c - 1)\mu r_{c-1}(s)} \]
In general, for \( k = 1, 2, \ldots, c - 1 \)
\[ r_k(s) = \frac{\lambda}{s + \lambda + k\mu - (k + 1)\mu r_{k+1}(s)}, \]
Observe that with \( r_c(s) = r(s) \), \( r_{c-1}(s) \) is explicitly known in terms of \( r(s) \) and hence \( r_k(s) \) for \( k = c - 1, \ldots, 1 \) can be recursively determined in terms of \( r(s) \).
Therefore, it is verified that \( \hat{P}_k(s) \) expressed by equation (4.9) and equation (4.10) satisfies the governing system of differential equations in the Laplace domain as represented by equations (4.7) and (4.8). Also, we have expressed all the probabilities \( \hat{P}_k(s) \), in terms of \( \hat{P}_0(s) \) for \( k = 1, 2, \ldots \).

Further, from equation (4.6),
\[ \hat{Q}(s) = \frac{1}{s + \theta} + \frac{\mu}{s + \theta} \hat{P}_0(s) r_1(s) \quad (4.16) \]
\]
Now, from equation (4.5)
\[ \hat{P}_0(s) = \left( \frac{\theta}{s + \lambda} \right) \hat{Q}(s). \quad (4.17) \]
Hence all the probabilities are expressed in terms of \( \hat{Q}(s) \). Substituting equation (4.17) in equation (4.16) leads to
\[ \hat{Q}(s) = \sum_{i=0}^{\infty} \frac{(\mu\theta)^i (r_i(s))^i}{(s + \theta)^{i+1}(s + \lambda)^i} \quad (4.18) \]
where \( r_i(s) \) can be recursively obtained. Now, taking inverse Laplace transform of equation (4.17) and equation (4.18), yields
\[ P_0(t) = \theta e^{-\lambda t} * Q(t) \quad (4.19) \]
\[
Q(t) = \sum_{i=0}^{\infty} \left( \mu \theta \right)^i e^{-\theta t} \frac{t^i}{i!} * e^{-\lambda t} \frac{t^{i-1}}{(i-1)!} * (r_i(t))^i
\] (4.20)

where \(*\) denotes the \(i\) fold convolution of \(r_i(t)\) with itself and
\[
r_i(t) = \sum_{i=0}^{\infty} \lambda(2\mu)^i e^{-(\lambda + \mu)t} \frac{t^i}{i!} \ast (r_2(t))^i.
\]

In general,
\[
r_k(t) = \sum_{i=0}^{\infty} \lambda((k + 1)\mu)^i e^{-(\lambda + k\mu)t} \frac{t^i}{i!} \ast (r_{k+1}(t))^i
\]

Note that when \(k = c\), \(r_c(t) = r(t)\). Laplace inversion of equation (4.12) yields
\[
r_c(t) = r(t) = \frac{\alpha(\alpha(t))}{2\mu} e^{-(\lambda + c\mu)t}
\]

Therefore, \(r_{c-1}(t)\) ... \(r_2(t)\), \(r_1(t)\) are all explicitly known in terms of \(r(t)\).

Also, inverse Laplace transform of equation (4.9) yields
\[
P_k(t) = P_0(t) * r_1(t) * r_2(t) * ... * r_k(t), k = 1, 2 ... c - 1
\] (4.21)

Similarly taking inverse Laplace transform for equation (4.10) yields
\[
P_k(t) = P_{c-1}(t) * \psi_{k-c+1}(t), k = c, c + 1...
\] (4.22)

where
\[
\psi_k(t) = L^{-1}[r^k(s)] = \frac{1}{(2\mu)^k} \left( \frac{kl_k(\alpha(t))\alpha^k}{t} \right) e^{-(\lambda + c\mu)t}, k = 1, 2, ...
\]

Therefore, explicit analytical expressions are obtained for all the transient state probabilities of an \(M/M/c\) queuing model subject to single exponential vacation.

**Remark.**

Here, the steady state probabilities are deduced from the time dependent state probabilities by applying the final value theorem of Laplace transforms which states i.e., \(\lim_{t\to\infty} P_k(t) = \lim_{s\to0} sP_k(s) = \pi_k\).

Consider, equation (4.10)
\[
P_k(s) = \hat{\beta}_{c-1}(s) t^{k-c+1} (s)
\]

Taking \(\lim\) on either side, yields
\[
\lim_{s\to0} P_k(s) = \lim_{s\to0} \hat{\beta}_{c-1}(s) t^{k-c+1} (s)
\]

and hence
\[
\pi_k = \pi_{c-1} t^{k-c+1}
\] (4.23)

where \(r = \lim_{s\to0} r(s)\) and from equation (4.12), we get
\[
r = \left( \frac{\lambda + c\mu - \sqrt{(\lambda + c\mu)^2 - 4\lambda c\mu}}{2\mu} \right).
\]

which on simplification leads to \(r = \left( \frac{\lambda}{c\mu} \right)\). Therefore, equation (4.23) gives
\[
\pi_k = \pi_{c-1} \left( \frac{\lambda}{c\mu} \right)^{k-c+1}
\] (4.24)

for \(k = c, c + 1...\)To obtain the other steady state probabilities, consider equation (4.9)
\[
\hat{P}_k(s) = \hat{P}_{k-1}(s) r_k(s),
\]
Taking \( \lim_{s \to 0} \) on either side yields
\[
\lim_{s \to 0} s \hat{P}_k(s) = \lim_{s \to 0} s \hat{P}_{k-1}(s) \lim_{s \to 0} r_k(s),
\]
and hence
\[
\pi_k = \pi_{k-1} r_k \text{ for } k = 1, 2 \ldots c - 1.
\]
where
\[
r_k = \lim_{s \to 0} r_k(s) = \frac{\lambda}{\lambda + k \mu - (k + 1)\mu r_{k+1}}.
\]
Observe that, when \( k = c - 1 \),
\[
r_{c-1} = \frac{\lambda}{\lambda + (c - 1)\mu - c\mu c}.
\]
where
\[
r_c = \lim_{s \to 0} r_c(s) = \lim_{s \to 0} r(s),
\]
\[
= \frac{(\lambda + c\mu) - \sqrt{(\lambda + c\mu)^2 - 4\lambda c\mu}}{2c\mu},
\]
\[
= \frac{\lambda}{c^2}.
\]
Similarly
\[
r_{c-1} = \frac{(\lambda + (c - 1)\mu) - \sqrt{(\lambda + (c - 1)\mu)^2 - 4\lambda(c - 1)\mu}}{2(c - 1)\mu}
\]
\[
= \frac{\lambda}{(c - 1)\mu}.
\]
In general, for \( k = 1, 2, \ldots c - 1 \)
\[
r_k = \frac{\lambda}{k\mu}.
\]
Therefore
\[
\pi_k = \pi_{k-1} r_k = \pi_0, \quad k = 1, 2 \ldots c - 1 \quad (4.25)
\]
Substituting the equation (4.25) for \( k = c - 1 \) in equation (4.24) yields
\[
\pi_k = \frac{1}{c^{k-c}(c!)} \rho^k \pi_0, \quad k = c, c + 1 \ldots \quad (4.26)
\]
The expressions in equation (4.25) and equation (4.26) coincides with the equation (3.8) and (3.9) respectively.

5. NUMERICAL ILLUSTRATIONS

This section illustrates the behaviour of the time dependent state probabilities of the system during the functional state and vacation state of the server against time for appropriate choice of the parameter values. Though the system is of infinite capacity, the value of \( k \) is restricted to 25 and \( c \) to 5 for the purpose of numerical study.

Figure 2 depicts the variation of the system size probability, \( Q(t) \) against time for \( \lambda = 0.4 \) and \( \theta = 0.5 \) and varying values of \( \mu \) (namely 0.4, 0.6, 0.9 and 1.5). By our assumption, the system is initially assumed to be empty and the server is in vacation state. Therefore, the graph of \( Q(t) \) starts at 1 and converges to the corresponding steady state probabilities as time progresses. It is seen that for a fixed time \( t \), the probability, \( Q(t) \) increases with increases in \( \mu \).
Figure 2: Variations of $Q(t)$ against $t$ for varying values of $\mu$

![Figure 2](image.png)

Figure 3 depicts the variation of the system size probability, $Q(t)$ against time for $\lambda = 0.4$ and $\mu = 1.5$ and varying values of $\theta$ (namely $0.1$, $0.2$, $0.4$ and $0.8$). By our assumption, the system is initially assumed to be empty and the server is in vacation. Therefore, the graph of $Q(t)$ starts at 1 and converges to the corresponding steady state probabilities as time progresses. It is seen that for a fixed time $t$, the probability, $Q(t)$ decreases with increases in $\theta$ (which represents the parameter of the exponentially distributed vacation times).

Figure 3: Variations of $Q(t)$ against $t$ for varying values of $\theta$

![Figure 3](image.png)

Figure 4 and Figure 5 illustrates the variation of the system size probability during the functional state of the server against time for $\mu = 0.6$ and varying values of $\lambda$ ($0.1,0.2,0.3$ and $0.4$) and $\theta$ ($0.1,0.2,0.4$ and $0.8$) respectively. The probability, $P_0(t)$ decreases with
increases in the value of ‘λ’ and the probability, $P_0(t)$ increases with increases in the value of ‘θ’. However, $P_0(t)$ increases initially with respect to $t$, reaches a peak value and then decreases to converge to the corresponding steady state probabilities. Note that the expression obtained for $P_0(t)$ as given by equation (4.14) suggests its behaviour similar to that of $t^n e^{-t}$. For the same reason, $t^n$ dominates for the initial values of ‘$t$’ and $e^{-t}$ dominates as the value of ‘$t$’ increases as depicted in Figure 4 and Figure 5.

Figure 4: Variations of $P_0(t)$ against $t$ for varying values of λ

![Figure 4](image)

Figure 5: Variations of $P_0(t)$ against $t$ for varying values of θ

![Figure 5](image)

Figure 6 and Figure 7 illustrates the variation of the system size probability during the functional state of the server against time for $\theta = 0.5$ and varying values of $\lambda (0.1, 0.2, 0.3$ and $0.4)$ and $\mu (0.4, 0.6, 0.9$ and $1.5)$ respectively. The probability, $P_1(t)$ increases with increases in the value of ‘$\lambda$’ and the probability, $P_1(t)$ decreases with increases in the value of ‘$\mu$’.

Figure 6: Variations of $P_1(t)$ against $t$ for varying values of $\lambda$

![Figure 6](image)
Figure 7: Variations of $P_1(t)$ against $t$ for varying values of $\mu$

![Graph of $P_1(t)$ against $t$ for varying values of $\mu$.](image)

Figure 8 illustrates the variations of $Q(t)$ and $P_k(t)$ for varying values of $k$ against time for $c = 8, \lambda = 1.5, \mu = 3$ and $\theta = 1$. The probability, $P_k(t)$ decreases with increases in the value of ‘$k$’. Also, the system is initially assumed to be empty and the server is in vacation state. Therefore, the graph of $Q(t)$ starts at 1 and converges to the corresponding steady state probabilities as time progresses.

Figure 8: Variations of $Q(t)$ and $P_k(t)$ for varying values of $k$

![Graph of $Q(t)$ and $P_k(t)$ for varying values of $k$.](image)

6. CONCLUSION

This paper analyses a multi server queueing model subject to single exponential vacation in both stationary and transient regime. Explicit analytical expressions for the time dependent state probabilities during both functional and vacation states of the server are presented. Further, the steady state probabilities are deduced from the time dependent state probabilities by applying the final value theorem of Laplace transforms. For certain specific values of the parameters, numerical illustrations are added to depict the variations of the state probabilities against time. The convergence of the probabilities to the corresponding steady state values are also illustrated in the respective figures. The model can be further extended to multi server model with partial server vacation, $N$-Policy, finite capacity etc.
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