Novel Transition to fully absorbing state without long-range spatial order in Directed Percolation class

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Abstract

We study coupled Gauss maps in one dimension and observe a transition to band periodic state with 2 bands. This is a periodic state with period-2 in a coarse-grained sense. This state does not show any long-range order in space. We compute two different order parameters to quantify the transition a) Flipping rate $F(t)$ which measures departures from period-2 and b) Persistence $P(t)$ which quantifies the loss of memory of initial conditions. At the critical point, $F(t)$ shows a power-law decay with exponent 0.158 which is close to 1-D directed percolation (DP) transition. The persistence exponent at the critical point is found to be 1.51 which matches with several models in 1-D DP class. We also study the finite-size scaling and off-critical scaling to estimate other exponents $z$ and $\nu_\parallel$. We observe excellent scaling for both $F(t)$ as well as $P(t)$ and the exponents obtained are clearly in DP class. We believe that DP transition could be observed in systems where activity goes to zero even if the spatial profile could be inhomogeneous and lacking any long-range order.

Keywords: Non-equilibrium phase transition, directed percolation, coupled map lattice, persistence

1. Introduction

Nonequilibrium phase transitions are found in a variety of situations. In nature, we observe dynamic phases ranging from the synchronized flashing
of fireflies, spirals or rings in a chemical reaction, synchronized chirping of crickets, Turing patterns etc. The studies in this fascinating field have only started. The most studied transition from the viewpoint of phase transitions is the transition to absorbing state. They are further classified in classes such as directed percolation, compact directed percolation, parity conserving class, voter class, Manna universality class, etc [1]. Directed Ising universality class has been observed in systems such as Grassberger’s model A and B [2, 3], branching and annihilating random walks with two offsprings [4, 5, 6, 7], interacting monomer-dimer model [8] and nonequilibrium kinetic Ising model [9, 10]. In one dimension, voter universality class in voter model is equivalent to compact directed percolation. It can be mapped to an equilibrium model [1]. There has been a long-standing debate if pair contact process with diffusion (PCPD) is a new universality class [11, 12]. Directed percolation remains most studied and most observed universality class in this context. Even for PCPD, increasing evidence points to the possibility that for long enough simulations on large enough systems, it will be in directed percolation universality class [13, 14].

If we consider models with continuous variable values such as coupled map lattices, there have been fewer studies. Transition in logistic map with delay (when mapped on a pseudo spatiotemporal system) is found to be in directed Ising class [15, 16]. There are systems which share the transition to equilibrium systems. The transition to an antiferromagnetic state in coupled logistic maps is found to be in Glauber-Ising class [17]. These systems are studied in higher dimension as well. In two dimensions, the possibility of transitions in equilibrium, as well as nonequilibrium class has been studied. Work by Miller and Huse demonstrating the possibility of transition in Ising class in coupled map lattice in two dimensions attracted a lot of attention [18]. It was found that the nature of update matters and it can change universality class. Coupled map lattices with some specific maps been shown to be in $q = 3$ Potts class [19]. Chaté and Mannevile studied the transition to a laminar phase in coupled piecewise linear discontinuous maps and showed that the transition is in directed percolation universality class [20]. Transition to spatiotemporal intermittency in coupled circle maps is also found to be in directed percolation universality class [16, 21]. Chaté and Mannevile studied the transition to spatiotemporal intermittency in 2-D coupled maps and showed that the continuous transition is in directed percolation universality class [22]. In many of these cases, change in nature of update can change universality class. Transitions in DP class are often transitions to a
synchronized state and the state has long-range order. Even for transition is to a chaotic synchronized state, where infinite absorbing states are possible, there is an obvious long-range order [23]. The above transitions are marked by clean order parameters such as the number of active sites or number of domain walls.

Janssen and Grassberger conjectured that the transition generically belongs to directed percolation universality class if a) transition is to a unique absorbing state from a fluctuating active phase b) characterized by non-negative one-component order parameter c) couplings are short-range and d) the system has no special attributes such as unconventional symmetries, conservation laws or quenched randomness [24]. In particle systems, directed percolation was observed in Domany-Kinzel automata [25], threshold transfer processes and Ziff-Gulari-Barshad model [26].

We would demonstrate a possibility of DP transition in a system which does not follow Janssen-Grassberger conjecture not only in the sense that it does not have a unique absorbing state. The state is not synchronized or periodic in space and does not have a long-range order. There is no long-range spatial order even if we coarse-grain the variables, i.e. if the variable values are divided into classes depending upon their values. Furthermore, this transition does not belong to the damage-spreading class. This is a transition to a frozen state in coarse-grained period-2. This should open up the possibility of observing DP transition experimentally in a system which approaches a state which is periodic in time in a coarse-grained sense. Most studies in nonequilibrium phase transitions are in particle systems or systems in which variables take discrete values. The model studied in this paper has variables which take continuous values. Usually, the order parameter is obtained from the spatial profile at a given time instance such as the variance of the profile. For a particle system, it could be the number of isolated particles or active particles at a given time. In our case, the appropriate order parameter is obtained by observing the difference between the spatial profile at a given time step and previous time steps.

2. Model

The system consists of diffusively coupled Gauss maps. Gauss map is given as,

\[ f(x) = \exp(-\nu x^2) + \beta, \quad x \in \mathbb{R} \]
Figure 1: Gauss Map for various values of $\beta$.

Where $\nu$ and $\beta$ are the parameters. The above function is a Gaussian with variance proportional to $1/\nu$ shifted by $\beta$. The value of the function tends to $\beta$ as $x \to \infty$. We fixed the value of $\nu = 7.5$ while $\beta$ is our control parameter. Unlike logistic or tent map, this is not a map on the interval. The nature of the function for a few different values of $\beta$ is shown in Fig. 1. The number of fixed points changes from 3 to 1 as we increase $\beta$. For large values of $\beta$, the largest fixed point is stable. The bifurcation diagram for a single Gauss map is shown in Fig. 2.

We couple Gauss maps diffusively as follows,

$$x_i(t + 1) = (1 - \epsilon)f(x_i(t)) + \frac{\epsilon}{2}[f(x_{i+1}(t)) + f(x_{i-1}(t))]$$

where $x_i(t)$ is the variable value associated with the site $i$ at time $t$. We assume the periodic boundary conditions. We fix the coupling, $\epsilon = 0.4$ and vary $\beta$. The bifurcation diagram for $N = 100$ is shown in Fig. 3.

We have also shown the largest fixed point of the map as a reference. As mentioned above, this map $f(x) = \exp(-7.5x^2) + \beta$ has a stable fixed point for large values of $\beta$. Simple stability analysis indicates that the coupled map lattice also has a stable fixed point for large values of $\beta$ [27]. The transition to synchronization has been studied extensively in many works. We will investigate another transition in detail in this work. A clear two-band structure is seen for smaller values of $\beta$. Not only there is a two-band structure but the system is also frozen in this state for smaller values of $\beta$. 
Figure 2: Bifurcation Diagram of Single Gauss Map. The fixed point is indicated by a red dashed line.

Figure 3: Bifurcation diagram of Coupled Gauss Maps. The fixed point is indicated by a red dashed line.
Figure 4: Non-Frozen spatial pattern for $\epsilon = 0.4$ and $\beta = -0.65$. The plot is done after an even number of time steps. The fixed point is also shown as a reference.

There is a coarse-grained 2-periodicity. We have shown the spatial profile at two different time steps for $\beta = -0.65$ and $\beta = -0.69$ in Fig. 4 and 5. We have also shown the largest fixed point for respective values of $\beta$ as reference.

It is clear that even though the exact variable value at $i$'th site is not repeated, the sites with the variable value greater than the largest fixed point $x^*$, continue to have a value greater than $x^*$ and vice versa for $\beta = -0.69$. This behavior is obtained for $\beta < \beta_c = -0.6773$.

We also plot a space-time diagram at the point $\beta = \beta_c = -0.6773$ shown in Fig. 6. As mentioned above, the system approaches a state with no spatial period but has a coarse-grained period-2 in time. There is no periodicity in space even in coarse-grained variables. We associate $s_i(t) = 1$ for $x_i(t) > x^*$ and $s_i(t) = -1$ for $x_i(t) < x^*$. The snapshot of the final spatial pattern in these variables is shown in Fig. 7. We observe a predominantly 3-up, 4-down or 4-up 3-down or 3-up, 3-down pattern. However, there is no exact periodicity even in coarse-grained sense. There are infinite such states possible. Clearly, Janssen-Grassberger conjecture does not apply here. The largest fixed point $x^*$ of the map can be found using the bisection method or other root-finding algorithms. The system enters a 2-band attractor eventually. So we expect sites to have the same spin value at all even times and different value at all odd times. We quantify the transition to a 2-band-attractor state using two quantifiers, namely a) Flip rate $F(t)$: The fraction of sites $i$ such that $s_i(2t - 2) \neq s_i(2t)$. b) Persistence $P(t)$: Fraction of sites $i$ such that
Figure 5: Frozen spatial pattern for $\epsilon = 0.4$ and $\beta = -0.69$. The plot is done after an even number of time steps. The fixed point is also shown as a reference.

Figure 6: Space-time diagram at critical point $\beta_c = -0.6773$
\( s_i(2t') = s_i(0) \) for all \( t' \leq t \).

The flip rate \( F(t) \) is an indicator of activity in the lattice at a given time. This is similar to the density of active sites which is a standard order parameter for absorbing state in DP class. We also study persistence in this system and show that the results are consistent. We note that if the site \( i \) is persistent till time \( T \), it implies that \( s_i(2t) = s_i(0) \) for all \( t \leq T \). By definition, \( P(t) \) decreases monotonically in time. It is also possible that it saturates to a finite value i.e. some sites do not deviate even once from their initial state during the entire course of evolution. Thus nonzero persistence may be due to frozen states in which sites do not flip any longer or at least a fraction of sites does not flip any longer. The transition could be to a fully absorbing state or a partially absorbing state. The order parameter \( F(t) \) helps us to distinguish between these possibilities. In this study, both \( P(t) \) and \( F(t) \) tend to zero asymptotically at the critical point and a fully absorbing state is reached. Above the critical point, we have a fluctuating state where \( F(t) > 0 \) asymptotically and \( P(\infty) \) goes to zero since every site flips from its initial state sooner or later. Below the critical point, \( F(\infty) \sim 0 \) and some sites are stuck forever in their initial conditions leading to a nonzero asymptotic value of persistence \( P(\infty) > 0 \).

We carry out simulations for \( N = 2 \times 10^5 \) and average over at least \( 10^3 \) configurations. For \( \epsilon = 0.4 \), we observe a clear power-law decay of \( F(t) \) as a function of time \( t \) at \( \beta = \beta_c = -0.6773 \). The flip rate \( F(\infty) \) saturates for \( \beta > \beta_c \) and \( F(\infty) = 0 \) for \( \beta < \beta_c \). At the critical point, \( \beta = \beta_c \), the asymptotic behavior is expected to be \( F(t) \sim t^{-\delta} \). We indeed observe this behavior in Fig. 8. The exponent \( \delta = 0.158 \) which is very close to directed percolation value of 0.159 \[1\].

We also find \( P(t) \) as a function of time \( t \) for \( \beta = \beta_c \). We find \( P(t) \sim t^{-\theta} \) with \( \theta = 1.51 \) asymptotically (see Fig. 9). The exponent \( \theta \) is also known as persistence exponent. This value of the exponent is consistent with persistence exponents obtained in several other systems showing DP transition except for a couple of models \[13\]. Even though the persistence exponent is not universal, significant sub-class of models showing DP transition in 1-D have persistence exponent close to \( 3/2 \) \[21, 25, 26, 28, 29\]. This is another indicator that the transition is in DP universality class.

We also obtain other exponents such as dynamic exponent \( z \) as well as parallel (temporal) correlation length exponent \( \nu_\parallel \) using finite-size scaling and off-critical scaling. We conduct this exercise for both persistence \( P(t) \) as well as for flip rate \( F(t) \).
Figure 7: Coarse-grained spatial pattern at $\beta = -0.6973$ in the persistent region.

Figure 8: Order Parameter $F(t)$ as a function of time $t$ at the critical point, $\beta_c = -0.6773$. We carry out simulation for $2 \times 10^5$ sites and averaging is carried over $10^3$ configurations. Asymptotically, we obtain a clear power-law decay with exponent, $\delta = 0.158$. In the inset, $F(t)t^\delta$ is plotted against $t$. Power-law decay is indicated from the fact that $F(t)t^\delta$ tends to a constant.
Figure 9: Persistence $P(t)$ as a function of time $t$ at a critical point, $\beta_c = -0.6773$. We carry out simulation for $2 \times 10^5$ sites and average over $5 \times 10^4$ configurations. Asymptotically, we obtain a clear power-law decay with persistence exponent, $\theta = 1.51$. In the inset, $P(t)t^\theta$ is plotted against $t$. Power-law decay is indicated from the fact that $P(t)t^\theta$ tends to a constant.

For our order parameter which is flip rate, we expect the following scaling law to hold,

$$F_N(t) = t^{-\delta} F(t/N^z, t^z)$$

where $\Delta = |\beta - \beta_c|$ is a departure from the critical point. For $N \to \infty$, $t \to \infty$, and $\Delta = 0$, $F(t) \sim t^{-\delta}$. This fit with $\delta = 0.158$ is shown in Fig. 8.

For persistence we expect that the following asymptotic law to hold:

$$P_N(t) = t^{-\theta} G(t/N^z, t^z)$$

where $G$ is the scaling function. Again in the thermodynamic limit, at the critical point, we have $P(t) \sim t^{-\theta}$ and we observe an excellent fit with $\theta = 1.51$ which matches with estimates of local persistence exponent in several other one-dimensional models of DP [14, 25, 29] (See Fig. 9). For persistence, there is a significant departure at early times. We fit this departure by incorporating nonlinear correction. A standard nonlinear correction is given by $P(t) \sim Ct^{-\theta}(1 + c_1t^{-\gamma} \ldots)$ [30]. We ignore higher-order terms. The value of $\gamma$ can be found by plotting $P(t)t^\theta$ as a function of $t^{-\gamma}$ for various values of $\gamma$. We find good linear fit for $\gamma = \frac{1}{3}$, $C = 13$ and $c_1 = -1.7692$. The fit is shown in Fig. 10. We retain this correction for finding values of $z$ and $\nu_\parallel$. 
Figure 10: $P(t\theta)$ is plotted against $t^{-0.33}$ which gives a good linear fit for the nonlinear correction.

For finite-size scaling, we simulate the system for various lattice sizes and compute the order parameter $F(t)$ as well as $P(t)$ at $\beta = \beta_c$. The order parameter $F(t)$ is averaged over $3 \times 10^3$ configurations, while persistence is averaged over $10^5$ configurations. The absorbing state is expected to be reached for $t_c = N^z$. The flip rate scales as $F(t_c) = N^{-\delta z}$. We plot $F(t)/F(t_c)$ as a function of $t/t_c$ in Fig. 11 and obtain very good scaling collapse.

Similarly, we plot $P(t)/P(t_c)$ as a function of $t/t_c$ for various values of $N$ in Fig. 12 and obtain excellent scaling collapse. Thus, we observe an excellent scaling collapse for $z = 1.58$ for the flip rate as well as persistence as shown in Fig. 11 and 12. These scalings are consistent with the expected value of $z$ for 1-D DP class.

We also study off-critical scaling behavior for both order parameter as well as persistence to obtain $\nu_\parallel$. The size of the lattice is large and fixed at $N = 2 \times 10^5$ sites. Thus finite-size corrections can be neglected. We carry out extensive averaging over at least $10^3$ configurations. For $\Delta > 0$, we average over $10^5$ configurations for persistence. The critical time scales as $t_c \sim \Delta^{-\nu_\parallel}$. Thus $F(t_c) \sim t_c^{-\delta} = \Delta^{-\nu_\parallel \delta}$. Plotting $F(t)/F(t_c)$ as a function of $t/t_c$ gives an excellent scaling collapse for $\nu_\parallel = 1.73$ (see Fig. 13). This is an expected value for DP transition [3].

We plot $P(t)/P(t_c)$ as a function of $t/t_c$ and obtain an excellent scaling collapse for $\nu_\parallel = 1.73$. The fit is shown in Fig. 14. This value of $\nu_\parallel$ is
Figure 11: Order Parameter $F(t)/F(t_c)$ as a function of $t/t_c$ where $t_c = N^z$ for various values of $N$ at critical point $\beta_c = -0.6773$. $N$ ranges from 40 to 1280 (from top to bottom).

Figure 12: Persistence $P(t)/P(t_c)$ is plotted against $t/t_c$ where $t_c = N^z$ for various values of $N$ at critical point $\beta_c = -0.6773$. A clean finite-size scaling is obtained. $N$ ranges from 40 to 1280 (from bottom to top).
Figure 13: Order Parameter $F(t)/F(t_c)$ is plotted as a function of $t/t_c$ where $t_c = \Delta^{-\nu_1}$ for various values of $\Delta = |\beta - \beta_c|$. We carry out off-critical simulation for $2 \times 10^5$ sites.

consistent with the value 1.73 observed for 1-D DP transitions.

3. Damage-spreading

It has been argued that damage-spreading transitions are generically in DP universality class [31]. Essentially, we make identical $k$ copies of the spatially extended system and perturb the central site. We measure the difference between all the $N_P = k(k-1)/2$ copies at each instant and sum over those. We define two quantities, $d(t) = \frac{1}{N_P} \sum_{l=1}^{k-1} \sum_{m=l+1}^k \sum_{i=1}^n |x_i^l(t) - x_i^m(t)|$ and $D(t) = \frac{1}{N_P} \sum_{l=1}^{k-1} \sum_{m=l+1}^k \sum_{i=1}^n |s(x_i^l(t)) - s(x_i^m(t))|$. We observe that none of these quantities go to zero at our critical point. Thus the transition is not in the damage-spreading class. If we change fraction $p$ of sites in replica, results do not change.

To graphically demonstrate that the transition is not in the damage-spreading class, we made two identical copies of the system and perturbed the central site in one of those. Let us denote the variable value at site $i$ at time $t$ by $x_i(t)$ and the value in its replica by $y_i(t)$. Since this is a deterministic system, procedural differences on whether or not the same set of random numbers is used in simulating both systems is not relevant. We plot color-coded difference between these two values, i.e. $|x_i(t) - y_i(t)|$ as a function of $t$ and $i$ for different values of $\beta$ (shown in Fig. 15). The damage spreads almost linearly and spreads to all sites in the non-persistent region.
Figure 14: Persistence $P(t)/P(t_c)$ is plotted against $t/t_c$ where $t_c = \Delta^{-\nu_1}$ for various values of $\Delta = |\beta - \beta_c|$. We carry out simulations for $2 \times 10^5$ sites and average over $10^5$ configurations for $\Delta > 0$ and $10^3$ for $\Delta < 0$.

Figure 15: Damage spreading at (a) $\beta = -0.63$ (b) $\beta = -0.6773$ (c) $\beta = -0.73$ (d) $\beta = -0.7771$
Even in the persistent region, damage does not heal completely. But it does not spread to all sites and tends to remain localized in the persistent region. Thus the transition at $\beta_c$ is not in the damage-spreading class if we follow the standard definition of damage spreading as in [32]. It can be noted that the damaged sites remain damaged and undamaged sites remain undamaged below $\beta_c$. Thus the change in the state of damage can be quantifier which can quantify transition at $\beta_c$. However, the same information is obtained by flip rate $F(t)$ and no new significant information is obtained by defining damage in a different manner. The damage vanishes completely only for $\beta < -0.777$ which is far from the critical point. Both fine-grained damage $d(t)$, as well as coarse-grained damage $D(t)$, vanish at values of $\beta$ which is much smaller than the critical point.

We also studied lyapunov exponent. There is no significant change in largest lyapunov exponent at $\beta_c = -0.6773$. However, the point at which the damage-spreading transition occurs ($\beta \sim -0.78$) is the one at which the largest lyapunov exponent becomes negative. Thus the change in sign of largest lyapunov exponent correlates well with the damage-spreading transition (see Fig. 16).

4. Discussion

We study coupled Gauss maps in one dimension. The system has a transition to an absorbing state which is a period-2 state in a coarse-grained sense.
We found no evidence of long-range order in the system. (In fact, for coupled logistic maps in one dimension, a transition to coarse-grained period-2 state in space and time is observed. There is a transition to a state which shows long-range antiferromagnetic order in space and period-2 in time. There are two such absorbing states. This transition does not belong to DP class but to a Glauber-Ising class \[33\].) Usually, a vacuum state is an absorbing state for DP transitions and the density of active sites is an obvious order parameter. Here, we propose the quantity \( F(t) \). This is a fraction of sites which do not return to the same band after two time-steps. It shows power-law behavior at the critical point with exponent \( \delta \) which matches with systems showing DP transition.

Recently, another quantifier known as persistence has been extremely popular for spotting transitions to a fully or partially absorbing state. If the flipping rate eventually becomes zero, there can be a fraction of sites which did not flip even once during evolution. These are known as persistent sites. Persistence reaches a finite value asymptotically in absorbing state and goes to zero in the active state. At the critical point, persistence may show power-law decay with exponent known as local persistence exponent. In several systems in DP class, this exponent is found to be \( 3/2 \) or very close to it \[ 25 \] \[ 26 \] \[ 28 \] \[ 29 \] \[ 21 \]. We also find local persistence exponent \( \theta_l = 1.51 \) which matches with these models in DP class.

We carry out finite-size scaling as well as off critical scaling to find other exponents \( z \) and \( \nu_{||} \). These exponents are found to be 1.58 and 1.73 respectively which put the transition firmly in DP universality class. The exponents obtained for persistence are consistent with those obtained using order parameter. We find that the transition is not in the damage-spreading class.

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