A GENERATING FUNCTION FOR NON–STANDARD
ORTHOGONAL POLYNOMIALS INVOLVING DIFFERENCES:
THE MEIXNER CASE

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Abstract. In this paper we deal with a family of non–standard polynomials orthogonal with respect to an inner product involving differences. This type of inner product is the so–called ∆–Sobolev inner product. Concretely, we consider the case in which both measures appearing in the inner product correspond to the Pascal distribution (the orthogonal polynomials associated to this distribution are known as Meixner polynomials). The aim of this work is to obtain a generating function for the ∆–Meixner–Sobolev orthogonal polynomials and, by using a limit process, recover a generating function for Laguerre–Sobolev orthogonal polynomials.

1. Introduction

Classical Meixner polynomials $m_n(x; \beta, c)$, or Meixner polynomials of the first kind according to the terminology used in [4], are those polynomials associated to the Pascal distribution

$$
\mu(k) = \frac{c^k(\beta)_k}{k!}, \quad k = 0, 1, 2, \ldots, \quad 0 < c < 1, \quad \beta > 0,
$$

that is, they are orthogonal with respect to the discrete standard inner product

$$
(f, g) = \int f g d\mu(k) = \sum_{k=0}^{+\infty} f(k)g(k)\frac{c^k(\beta)_k}{k!},
$$

and they satisfy a second order difference equation (see, for instance, [9]). Some authors in the literature use the distribution $(1 – c)^\beta \mu(k)$ to have a probability measure. In this paper, following the classical texts [4] and [9] or, more recently [5], we will consider the discrete measure $\mu(k)$.

In [1], a generalization of the above inner product is introduced. The authors consider the ∆–Sobolev inner product

$$
(f, g)_S = \sum_{k=0}^{+\infty} f(k)g(k)\frac{c^k(\beta)_k}{k!} + \lambda \sum_{k=0}^{+\infty} \Delta f(k)\Delta g(k)\frac{c^k(\beta)_k}{k!},
$$

with $\beta > 0, 0 < c < 1, \lambda > 0$, and where $\Delta$ is the usual forward difference operator defined by $\Delta f(k) = f(k + 1) – f(k)$.

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As we can observe, \( (1) \) is a non-standard inner product, that is, \((xf, g)_S \neq (f, xg)_S\). Thus, the corresponding sequence of orthogonal polynomials does not satisfy a three-term recurrence relation, and in general, the nice algebraic and differential properties of standard orthogonal polynomials do not hold any more.

We denote by \( \{S_n\} \) the sequence of polynomials orthogonal with respect to \( (1) \), normalized by the condition that \( S_n(x) \) and the Meixner polynomial \( m_n(x; \beta, c) \) have the same leading coefficient \((n = 0, 1, 2, \ldots)\). The polynomials \( S_n(x) \) are the so-called \( \Delta \)-Meixner–Sobolev orthogonal polynomials. As we have already mentioned, the polynomials \( S_n(x) \) were introduced in [1] where several algebraic and difference relations between the families of polynomials \( S_n(x) \) and \( m_n(x; \beta, c) \) were established. Asymptotic results for \( S_n(x) \) when \( n \to +\infty \), have been obtained in [2].

The main goal of this paper is to obtain a generating function for the polynomials \( S_n(x) \). Furthermore, we will be able to recover the results obtained in [8] for Laguerre–Sobolev orthogonal polynomials, that is, using a limit process we obtain the generating function for Laguerre–Sobolev orthogonal polynomials from the generating function for \( \Delta \)-Meixner–Sobolev orthogonal polynomials. Thus, we are in some sense working in one of the direction pointed out in the recent survey about Sobolev orthogonal polynomials on unbounded supports [7] (second item of Section 4).

The structure of the paper is as follows: in Section 2 we state some well-known results on classical Meixner polynomials which will be used along the paper. Section 3 gives the basic relations on \( \Delta \)-Meixner–Sobolev polynomials. In particular, it is shown that a generating function for the \( \Delta \)-Meixner–Sobolev polynomials can be reduced to a generating function involving the classical Meixner polynomials (Proposition 3.5). In Section 4 a generating function for \( \Delta \)-Meixner–Sobolev polynomials is derived. The main results are stated in Theorem 4.1 and 4.3. Finally, in Section 5 we recover the generating function for Laguerre–Sobolev orthogonal polynomials obtained in [8].

### 2. Classical Meixner Polynomials

Let \( \beta, \) and \( c \) be real numbers such that \( c \neq 0, 1, \) and \( \beta \neq 0, -1, -2, \ldots \). It is well known that classical Meixner polynomials \( m_n(x; \beta, c) \) can be defined by their explicit representation in terms of the hypergeometric function \( _2F_1 \) (see, for instance, [4] p. 175–177] where a different normalization is used),

\[
(2) \quad m_n(x; \beta, c) = \frac{(\beta)_n}{n!} _2F_1(-n, -x; \beta; 1 - c^{-1}) = \frac{(\beta)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} (\frac{(-x)_k}{(\beta)_k}) \left(\frac{1}{c} - 1\right)^k,
\]

where \((a)_n\) denotes the usual Pochhammer symbol,

\[
(a)_0 = 1, \quad (a)_n = a(a + 1) \cdots (a + n - 1), \quad n \geq 1.
\]

Observe that [2] provides \( m_n(x; \beta, c) \) as a polynomial of exact degree \( n \) with leading coefficient

\[
(3) \quad \frac{1}{n!} \left(1 - \frac{1}{c}\right)^n.
\]
If $\beta > 0$ and $0 < c < 1$, classical Meixner polynomials are orthogonal with respect to the inner product,

$$ (f, g) = \sum_{k=0}^{+\infty} f(k)g(k) \frac{c^k(\beta)_k}{k!}, $$

and then,

$$ \sum_{k=0}^{+\infty} (m_n(k; \beta, c))^2 \frac{c^k(\beta)_k}{k!} = \frac{(\beta)_n}{n! c^n (1-c)\beta}, \quad n = 0, 1, 2, \ldots $$

Simplifying expression (2), we get

$$ m_n(x; \beta, c) = \sum_{k=0}^{n} (\beta + k)n-k \frac{(-x)_k}{k!} \frac{1}{c} - 1 \right)^k. $$

Observe that, for every value of the parameter $\beta$, expression (6) defines a polynomial of exact degree $n$, and leading coefficient (3). In this way, we can define Meixner polynomials for all $\beta \in \mathbb{R}$.

Very simple manipulations of the explicit representation (6) show that the main algebraic properties of the classical Meixner polynomials still hold for the general case $\beta \in \mathbb{R}$, and $c \in \mathbb{R} \setminus \{0, 1\}$, although the orthogonality given in (4) holds only for $\beta > 0$ and $0 < c < 1$. In particular, for $n \geq 1$, Meixner polynomials satisfy a three-term recurrence relation

$$ c (n+1) m_{n+1}(x; \beta, c) = [x (c-1) + \beta c + n (c+1)] m_n(x; \beta, c) - (n + \beta - 1) m_{n-1}(x; \beta, c), $$

with the initial conditions $m_{-1}(x; \beta, c) = 0$, and $m_0(x; \beta, c) = 1$.

Moreover, the following relations are satisfied:

$$ m_n(x; \beta, c) - m_{n-1}(x; \beta, c) = m_n(x; \beta - 1, c), $$

$$ \Delta [m_n(x; \beta, c) - m_{n-1}(x; \beta, c)] = \frac{c - 1}{c} m_{n-1}(x; \beta, c). $$

The generating function for classical Meixner polynomials plays an important role in this work. This generating function can be found, for instance, in [4, p. 176] or [5, p. 175]. Here, we give an elementary proof of this result for general values of the parameter $\beta$.

**Lemma 2.1.** For $|\omega| < c < 1$ and $\beta \in \mathbb{R}$, we have

$$ \sum_{n=0}^{+\infty} m_n(x; \beta, c) \omega^n = \left(1 - \frac{\omega}{c}\right)^x (1-\omega)^{-x-\beta} $$

**Proof.** From (6), we get

$$ \sum_{n=0}^{+\infty} m_n(x; \beta, c) \omega^n = \sum_{k=0}^{+\infty} \frac{(-x)_k}{k!} \left(\frac{1}{c} - 1\right)^k \omega^k \sum_{n=k}^{+\infty} \frac{(\beta + k)n-k}{(n-k)!} \omega^{n-k} $$

$$ = \sum_{k=0}^{+\infty} \frac{(-x)_k}{k!} \left(\frac{1}{c} - 1\right)^k \sum_{j=0}^{+\infty} \frac{\beta + k}{j!} \omega^j. $$
Finally, using the well–known formula
\begin{equation}
\sum_{j=0}^{+\infty} \frac{(\alpha)_j}{j!} \omega^j = (1 - \omega)^{-\alpha}, \quad |\omega| < 1,
\end{equation}
we obtain
\begin{align*}
\sum_{n=0}^{+\infty} m_n(x; \beta, c) \omega^n &= \sum_{k=0}^{+\infty} \frac{(-x)_k}{k!} \left( \frac{1}{c} - 1 \right) (1 - \omega)^{-\beta - k} \\
&= (1 - \omega)^{-\beta} \sum_{k=0}^{+\infty} \frac{(-x)_k k! (\frac{1}{c} - 1 \omega)^k}{1 - \omega} \\
&= (1 - \omega)^{-\beta} \left( 1 - \frac{\frac{1}{c} - 1 \omega}{1 - \omega} \right)^x = (1 - \omega)^x (1 - \omega)^{-\beta - \omega},
\end{align*}
for $|\omega| < c < 1$.

We want to remark that, in this paper, we will use the previous Lemma for $\beta > -1$.

3. $\Delta$–Meixner–Sobolev orthogonal polynomials

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the $\Delta$–Sobolev inner product
\begin{equation}
(f, g)_\Delta = \sum_{k=0}^{+\infty} f(k) g(k) \frac{c^k (\beta)_k}{k!} + \lambda \sum_{k=0}^{+\infty} \Delta f(k) \Delta g(k) \frac{c^k (\beta)_k}{k!},
\end{equation}
with $\beta > 0, 0 < c < 1, \lambda > 0$. The polynomials $\{S_n\}$ are the so–called $\Delta$–Meixner–Sobolev orthogonal polynomials, and they are normalized by the condition that the leading coefficient of $S_n(x)$ equals the leading coefficient of $m_n(x; \beta, c), n \geq 0$. Observe that $S_0(x) = m_0(x; \beta, c)$, and $S_1(x) = m_1(x; \beta, c)$.

The following result is obtained in [1].

**Lemma 3.1.** There exist positive constants $a_n$ depending on $\beta, c$ and $\lambda$, such that
\begin{equation}
m_n(x; \beta, c) - m_{n-1}(x; \beta, c) = S_n(x) - a_{n-1} S_{n-1}(x), \quad n \geq 1.
\end{equation}

**Proof.** Put
\begin{equation*}
m_n(x; \beta, c) - m_{n-1}(x; \beta, c) = m_n(x; \beta - 1, c) = S_n(x) + \sum_{i=0}^{n-1} c_i^{(n)} S_i(x).
\end{equation*}
Then
\begin{equation*}
c_i^{(n)} (S_i, S_i)_\Delta = (m_n - m_{n-1}, S_i)_\Delta.
\end{equation*}
Applying (7), (8), and (11) to the right–hand side, we obtain
\begin{equation*}
c_i^{(n)} = 0, \quad 0 \leq i \leq n - 2,
\end{equation*}
and
\[ c_{n-1}^{(n)}(S_{n-1}, S_{n-1})_S = -\sum_{k=0}^{+\infty} m_{n-1}(k; \beta, c) S_{n-1}(k) \frac{(\beta) e^k}{k!} \]
\[ = -\sum_{k=0}^{+\infty} (m_{n-1}(k; \beta, c))^2 \frac{(\beta) e^k}{k!}. \]

The following recurrence relation for the coefficients \( \{a_n\} \) in \((12)\) is also obtained in \([1]\). Here, we write this recurrence relation in an analogous form useful for our purposes.

**Lemma 3.2.** The sequence \( \{a_n\} \) in \((12)\) satisfies
\[ a_n = \frac{n + \beta - 1}{n + \beta - 1 + \left(1 + \lambda \left(1 - \frac{1}{c}\right)^2\right) c n - c n a_{n-1}}, \quad n \geq 1, \]
with
\[ a_0 = 1. \]

**Proof.** Write
\[ R_0(x) = S_0(x), \]
\[ R_n(x) = S_n(x) - a_{n-1} S_{n-1}(x), \quad n \geq 1, \]
then for \( n \geq 1, \)
\[ (R_{n+1}, R_n)_S + a_n(R_n, R_n)_S + a_n a_{n-1} (R_n, R_{n-1})_S = 0. \]

After computing the \( \Delta \)–Sobolev inner products with \((3), (8), (11), \) and \((12)\), we obtain \((12)\), for \( n \geq 1. \)

Finally, since \( S_0(x) = m_0(x; \beta, c), \) and \( S_1(x) = m_1(x; \beta, c), \) relation \((12)\) implies
\[ a_0 = 1. \]

In order to simplify the notations, from now on, we will denote by
\[ \eta := 1 + \lambda \left(1 - \frac{1}{c}\right)^2 > 1. \]

Then, relation \((12)\) reads
\[ a_n = \frac{n + \beta - 1}{n + \beta - 1 + \eta c n - c n a_{n-1}}, \quad n \geq 1, \]
with \( a_0 = 1. \)

To derive a generating function for \( \Delta \)–Meixner–Sobolev orthogonal polynomials, we need more information about the sequence \( \{a_n\}. \) The asymptotic behavior of this sequence was established in \([2\text{ Prop. 5}]\). Again, we introduce this result in an adequate form useful for our objectives, and we also give an alternative and elemental proof.
Lemma 3.3. The sequence \( \{ a_n \} \) is convergent, and
\[
a = \lim_{n \to \infty} a_n = \frac{1 + \eta c - \sqrt{(1 + \eta c)^2 - 4c}}{2c},
\]
is the smallest root of the equation
\[
c z^2 - (1 + \eta c) z + 1 = 0.
\]

Proof. First, we observe that a simple induction argument applied on Lemma 3.2
gives \( 0 < a_n \leq 1 \), for all \( n \geq 0 \).

Suppose that \( a = \lim_{n \to +\infty} a_n \) exists, then (13) implies
\[
a = \frac{1}{1 + \eta c - ca},
\]
that is, \( a \) is a solution of the equation
\[
c z^2 - (1 + \eta c) z + 1 = 0.
\]
Since \( a_n \leq 1 \) for all \( n \geq 0 \), we have \( a \leq 1 \). Hence
\[
a = \frac{1 + \eta c - \sqrt{(1 + \eta c)^2 - 4c}}{2c} < 1.
\]
Now, we prove that \( \{ a_n \} \) is indeed convergent to \( a \). With (13) and (14), we have
\[
\left| \frac{1}{a_n} - a \right| = \eta c \left( \frac{n}{n + \beta - 1} - 1 \right) - c \left( \frac{n}{n + \beta - 1} a_{n-1} - a \right).
\]
Then, using \( 0 < a_{n-1} \leq 1 \), and \( 0 < a \leq 1 \), we get
\[
\left| a_n - a \right| = |a_n| |a| \left| \frac{1}{a_n} - 1 \right| <
\]
\[
\eta c \left| \frac{n}{n + \beta - 1} - 1 \right| + c \left| \frac{n}{n + \beta - 1} \right| \left| a_{n-1} - a \right| + a \left| \frac{\beta - 1}{n + \beta - 1} \right|.
\]
Hence
\[
\limsup |a_n - a| \leq c \limsup |a_{n-1} - a|.
\]
Since \( c < 1 \), the lemma follows.

From the sequence \( \{ a_n \} \) we construct a sequence \( \{ q_n(\eta) \} \) of polynomials in \( \eta \).

Lemma 3.4. Define the sequence \( \{ q_n(\eta) \} \) by
\[
q_0(\eta) = 1, \quad q_{n+1}(\eta) = \frac{q_n(\eta)}{a_n}, \quad n \geq 0.
\]
Then \( q_n(\eta) \), for \( n \geq 1 \), is a polynomial in \( \eta \) (and therefore in \( \lambda \)) such that \( \deg q_n = n - 1 \), satisfying the three–term recurrence relation
\[
(n + \beta - 1)q_{n+1}(\eta) = (n + \beta - 1 + \eta c n) q_n(\eta) - c n q_{n-1}(\eta), \quad n \geq 1,
\]
with initial conditions \( q_0(\eta) = q_1(\eta) = 1 \).

Proof. The recurrence relation (15) is just relation (13) rewritten in terms of \( q_n(\eta) \).
Since \( a_0 = 1 \), then \( q_1 = 1 \), and thus (15) implies that, for \( n \geq 1 \), \( q_n \) is a polynomial
in \( \eta \) of degree \( n - 1 \).
Note that in the limit case \( \lambda = 0 \), we have \( S_n(x) = m_n(x; \beta, c) \) for all \( n = 0, 1, 2, \ldots \). Therefore, \( \eta = 1 \), \( a_n = q_n(\eta) = 1 \), for all \( n = 0, 1, 2, \ldots \), and \( a = 1 \).

Next result shows that the formal power series, (i.e. generating function) for \( \Delta \)-Meixner–Sobolev orthogonal polynomials can be reduced to a formal power series involving Meixner polynomials.

**Proposition 3.5.** We have

\[
\sum_{n=0}^{+\infty} q_n(\eta) S_n(x) \omega^n = \frac{1}{1-\omega} \sum_{n=0}^{+\infty} q_n(\eta) m_n(x; \beta - 1, c) \omega^n.
\]

**Proof.** Equation (12) gives

\[
q_n(\eta) m_n(x; \beta - 1, c) = q_n(\eta) S_n(x) - q_{n-1}(\eta) S_{n-1}(x),
\]

and therefore

\[
q_n(\eta) S_n(x) = \sum_{k=0}^{n} q_k(\eta) m_k(x; \beta - 1, c).
\]

Thus, we have

\[
\sum_{n=0}^{+\infty} q_n(\eta) S_n(x) \omega^n = \sum_{n=0}^{+\infty} \left[ \sum_{k=0}^{n} q_k(\eta) m_k(x; \beta - 1, c) \right] \omega^n
\]

\[
= \sum_{k=0}^{+\infty} q_k(\eta) m_k(x; \beta - 1, c) \sum_{n=k}^{+\infty} \omega^{n-k} = \sum_{n=0}^{+\infty} \omega^n \sum_{k=0}^{+\infty} q_k(\eta) m_k(x; \beta - 1, c) \omega^k
\]

\[
= \frac{1}{1-\omega} \sum_{k=0}^{+\infty} q_k(\eta) m_k(x; \beta - 1, c) \omega^k.
\]

\[\square\]

4. Generating function for \( \Delta \)-Meixner–Sobolev polynomials

In this section, we will obtain a generating function for \( \Delta \)-Meixner–Sobolev orthogonal polynomials with \( \beta > 0 \) by means of Proposition 3.5 where Meixner polynomials \( m_n(x; \beta - 1, c) \) are considered. The general approach uses the explicit expression for Meixner polynomials \( [2] \), where \( \beta \in \mathbb{R} \), \( \beta \neq 0, -1, -2, \ldots \). Note that, in the case \( \beta = 1 \), Meixner polynomials \( m_n(x; 0, c) \) defined by \( [2] \) appear in Proposition 3.5. Therefore, we have to distinguish \( \beta = 1 \) and \( \beta \neq 1 \). We begin with the particular case \( \beta = 1 \) due to their simplicity. From now on, we will denote

\[
G_M(x, \omega, \lambda) := \sum_{n=0}^{+\infty} q_n(\eta) S_n(x) \omega^n.
\]

4.1. Case \( \beta = 1 \). In this case the generating function is stated in the following theorem.

**Theorem 4.1.** Let \( \{S_n\} \) be the sequence of orthogonal polynomials associated with the \( \Delta \)-Sobolev inner product \( [14] \), with \( \beta = 1 \), and normalized by the condition that the leading coefficient of \( S_n \) equals the leading coefficient of \( m_n(x; 1, c) \). Let \( \{q_n(\eta)\} \) be defined by the recurrence relation

\[
q_{n+1}(\eta) = (1 + \eta c) q_n(\eta) - c q_{n-1}(\eta), \quad q_0(\eta) = q_1(\eta) = 1.
\]
defined by the recurrence relation

\[ h_n(x) = \frac{1 + \eta c - \sqrt{(1 + \eta c)^2 - 4c}}{2c} \frac{\gamma}{a - a^2 c} (1 - \omega a + \delta (1 - \omega a)^2), \]

where

\[ a = \frac{1 + \eta c - \sqrt{(1 + \eta c)^2 - 4c}}{2c}, \quad \gamma = \frac{a - a^2 c}{1 - a^2 c}, \quad \delta = \frac{1 - a}{1 - a^2 c}. \]

Proof. If \( \beta = 1 \) the second order difference equation \((19)\) is reduced to \((17)\) and, therefore we have

\[ q_n(\eta) = \frac{1}{1 - a^2 c} \left( (a - a^2 c) \frac{1}{a^n} + (1 - a)(ac)^n \right). \]

Thus, the theorem follows from Proposition 3.3 and Lemma 2.1.

**Remark.** It is important to note that, in the limit case \( \lambda = 0 \), we recover the generating function for classical Meixner polynomials \((13)\) from \((18)\), since in this situation \( q_n(\eta) = 1 \), for all \( n = 0, 1, 2, \ldots \), \( a = 1 \), \( \gamma = 1 \), and \( \delta = 0 \).

4.2. Case \( \beta \neq 1 \). Now, we suppose \( \beta > 0 \) and \( \beta \neq 1 \). We will deduce a generating function for the polynomials \( S_n(x) \) starting from relation \((16)\). First, we need a generating function for the polynomials \( q_n(\eta) \).

**Lemma 4.2.** Let \( \beta > 0, \beta \neq 1, \) and let \( \{q_n(\eta)\} \) be the sequence of polynomials defined by the recurrence relation \((15)\). Put

\[ F(\omega) = \sum_{n=0}^{+\infty} q_n(\eta) (\beta - 1) a \frac{\omega^{n \lambda}}{n!}, \]

with \( |\omega| < a < 1 \). Then,

\[ F(\omega) = \left( 1 - \frac{\omega}{a} \right)^{-(\beta - 1)\gamma} (1 - \omega a + \delta (1 - \omega a)^2)^{-(\beta - 1)\delta}, \]

where \( a, \gamma \) and \( \delta \) are defined in \((19)\).

Proof. Observe that the ratio test shows that the series in the right-hand side of \((20)\) is convergent if \( |\omega| < a < 1 \). To simplify, if we write

\[ h_n(\eta) = q_n(\eta) (\beta - 1) a \frac{\omega^{n \lambda}}{n!}, \quad n \geq 0, \]

then

\[ F(\omega) = \sum_{n=0}^{+\infty} h_n(\eta) \omega^n. \]

From \((15)\), we obtain the recurrence relation for \( \{h_n(\eta)\} \) as follows

\[ (n + 1)h_{n+1}(\eta) = [n(1 + \eta c) + \beta - 1] h_n(\eta) - c(n + \beta - 2) h_{n-1}(\eta), \quad n \geq 1, \]

with \( h_0(\eta) = 1 \), \( h_1(\eta) = \beta - 1 \).

Multiplying \((23)\) times \( \omega^n \), and summing over \( n = 1, 2, \ldots \), we obtain

\[ F'(\omega) - h_1(\eta) = (1 + \eta c)c F'(\omega) + (\beta - 1)(F(\omega) - h_0(\eta)) - c \omega^2 F'(\omega) - c(\beta - 1)\omega F'(\omega), \]

hence

\[ F'(\omega) [1 - (1 + \eta c) \omega + c \omega^2] = (\beta - 1) F(\omega)(1 - c \omega), \]
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with $1 + \eta c = 1/a + ca$. Then, we get

$$F'(\omega) \left(1 - \frac{\omega}{a}\right) (1 - \omega ca) = (\beta - 1)F(\omega)(1 - c \omega),$$

and, therefore, we have

$$\begin{aligned}
F'(\omega) &= (\beta - 1) \left( \frac{\gamma/a}{1 - \omega} + \frac{\delta ca}{1 - \omega} \right), \\
F(0) &= h_0(\eta) = 1,
\end{aligned}$$

where $\gamma$ and $\delta$ are defined in (19). Solving this initial value problem, we obtain (21). □

Remark. Note that in the limit case $\lambda = 0$, we have $a = 1$ and, therefore $\gamma = 1$, and $\delta = 0$. Thus, we deduce $F(\omega) = (1 - \omega)^{-\beta+1}$.

Now, we have the necessary tools to obtain a generating function for $\Delta$–Meixner–Sobolev orthogonal polynomials with $\beta \neq 1$.

**Theorem 4.3.** Let $\{S_n\}$ be the sequence of polynomials orthogonal with respect to the $\Delta$–Sobolev inner product (11) with $\beta \neq 1$, and normalized by the condition that the leading coefficient of $S_n(x)$ equals the leading coefficient of $m_n(x; \beta, c)$. Let $\{q_n(\eta)\}$ be defined by the recurrence relation (15). Then, for $|\omega| < ac < 1$,

$$G_M(x, \omega, \lambda) = \frac{1}{1 - \omega} (1 - ca \omega)^{-(\beta - 1)\delta} \left(1 - \frac{\omega}{a}\right)^{-(\beta - 1)\gamma} \left(1 - \frac{\omega}{ac}\right)^x \left(1 - \frac{\omega}{1 - \omega}\right)^{-x}$$

$$\times 2F_1 \left(-x, (\beta - 1)\delta; \beta - 1, \frac{\omega(c - 1)(1 - a^2)c}{(1 - ca \omega)(ac - \omega)}\right),$$

where $a$, $\gamma$ and $\delta$ are defined in (19).

Proof. We start giving two expressions for $k$–th derivative of $F(\omega)$ defined in (20). First, taking into account (22), we have

$$F^{(k)}(\omega) = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} h_n(\eta) \omega^{n-k}.$$ (25)

On the other hand, from (21), we get

$$F^{(k)}(\omega) = \sum_{s=0}^{k} \binom{k}{s} (1 - ca \omega)^{-(\beta - 1)\delta} \left(1 - \frac{\omega}{a}\right)^{-(\beta - 1)\gamma} \left(1 - \frac{\omega}{ac}\right)^x \left(1 - \frac{\omega}{1 - \omega}\right)^{-x}$$

$$\times \sum_{s=0}^{k} (-1)^k k! \binom{-(\beta - 1)\delta}{s} \binom{-(\beta - 1)\gamma}{k} \left(\frac{ca}{1 - ca \omega}\right)^s \left(\frac{1}{a - \omega}\right)^{k-s}.$$ (26)
Now, with (25) and the explicit representation of Meixner polynomials (2), we get

\[
\sum_{n=0}^{+\infty} q_n(\eta) m_n(x; \beta - 1, c) \omega^n =
\]

\[
\sum_{n=0}^{+\infty} q_n(\eta) \left[ \frac{(\beta - 1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)_k}{(\beta - 1)_k} \left( \frac{1}{c} - 1 \right)^k \right] \omega^n
\]

\[
= \sum_{k=0}^{+\infty} \frac{(-x)_k}{k! (\beta - 1)_k} \left( \frac{1}{c} - 1 \right)^k \omega^k \sum_{n=0}^{+\infty} \frac{n!}{(n-k)!} h_n(\eta) \omega^{n-k}
\]

\[
= \sum_{k=0}^{+\infty} \frac{(-x)_k}{k! (\beta - 1)_k} \left( \frac{1}{c} - 1 \right)^k \omega^k F^{(k)}(\omega).
\]

Thus, from (26), we obtain

\[
\sum_{n=0}^{+\infty} q_n(\eta) m_n(x; \beta - 1, c) \omega^n = (1 - ca \omega)^{-(\beta - 1)\delta} \left( 1 - \frac{\omega}{a} \right)^{-(\beta - 1)\gamma}
\]

\[
\times \sum_{s=0}^{+\infty} \left\{ \frac{(-x)_s}{s} \left( \frac{1}{c} - 1 \right)^s \omega^s \right\}
\]

\[
= (1 - ca \omega)^{-(\beta - 1)\delta} \left( 1 - \frac{\omega}{a} \right)^{-(\beta - 1)\gamma}
\]

\[
\times \sum_{s=0}^{+\infty} \left\{ \frac{(-x)_s}{s} \left( \frac{1}{c} - 1 \right)^s \frac{ca}{1 - ca \omega} \omega^s \right\}
\]

\[
= \left( 1 - ca \omega \right)^{-(\beta - 1)\delta} \left( 1 - \frac{\omega}{a} \right)^{-(\beta - 1)\gamma}
\]

\[
\times \sum_{s=0}^{+\infty} \left\{ \frac{(-x)_s}{s} \left( \frac{1}{c} - 1 \right)^s \frac{ca}{1 - ca \omega} \omega^s \right\}
\]

\[
= (1 - ca \omega)^{-(\beta - 1)\delta} \left( 1 - \frac{\omega}{a} \right)^{-(\beta - 1)\gamma}
\]

\[
\times \sum_{s=0}^{+\infty} \left\{ \frac{(-x)_s}{s} \left( \frac{1}{c} - 1 \right)^s \frac{ca}{1 - ca \omega} \omega^s \right\}
\]

\[
= \left( 1 - ca \omega \right)^{-(\beta - 1)\delta} \left( 1 - \frac{\omega}{a} \right)^{-(\beta - 1)\gamma}
\]

where in last equality we use \((-x)_s + m = (-x)_s (-x + s)_m\). If we denote

\[
\omega_1 := \left( 1 - \frac{1}{c} \right) \frac{ca \omega}{1 - ca \omega} = \frac{(c - 1)a \omega}{1 - ca \omega}, \quad \omega_2 := \frac{\omega}{a - \omega} \left( 1 - \frac{1}{c} \right) = \frac{(c - 1)\omega}{c(a - \omega)},
\]
the above expression yields
\[
\sum_{n=0}^{+\infty} q_n(\eta) m_n(x; \beta - 1, c) \omega^n = (1 - ca\omega)^{-(\beta-1)\delta} \left(1 - \frac{\omega}{a}\right)^{-(\beta-1)\gamma}
\]
\[
\times \sum_{s=0}^{+\infty} \left(\frac{\beta - 1}{s!}\right)_{s} \frac{(-\omega_1)^s}{s!} \sum_{m=0}^{+\infty} \left(\frac{\beta - 1}{m!}\right)_{m} (-x + s) \frac{(-\omega_2)^m}{m!}
\]
\[
= (1 - ca\omega)^{-(\beta-1)\delta} \left(1 - \frac{\omega}{a}\right)^{-(\beta-1)\gamma}
\]
\[
\times \sum_{s=0}^{+\infty} \left(\frac{\beta - 1}{s!}\right)_{s} \frac{(-\omega_1)^s}{s!} \frac{2F_1(s - x, (\beta - 1)\gamma; s + \beta - 1; -\omega_2)}{2F_1(s - x, (\beta - 1)\gamma; s + \beta - 1; -\omega_2)}
\]
where in the last equality we have been able to apply the Pfaff–Kummer transformation (see, for instance, \[5\] f. (1.4.9) or \[6\] p. 425)
\[
2F_1(a, b; c; z) = (1 - z)^{-b} 2F_1(c - a, b; c; \frac{z}{z-1}), \quad |z| < 1,
\]

since \(|\omega_2| < 1\) for \(|\omega| < ac\).

In order to simplify the above expression, we can observe that we are in situation to apply formula (65.2.2) in \[6\], i.e.,
\[
\sum_{k=0}^{+\infty} \frac{(a)_k(b)_k y^k}{(c)_k k!} 2F_1(c - a, c - b; c + k; z) = (1 - z)^{a+b-c} 2F_1(a, b; c; z + y - zy),
\]
since \((\beta - 1)(1 - \delta) = (\beta - 1)\gamma\). Therefore, after some simplifications, we get
\[
\sum_{n=0}^{+\infty} q_n(\eta) m_n(x; \beta - 1, c) \omega^n = (1 - ca\omega)^{-(\beta-1)\delta} \left(1 - \frac{\omega}{a}\right)^{-(\beta-1)\gamma} (1 + \omega_2)^x
\]
\[
\times 2F_1(-x, (\beta - 1)\delta; \beta - 1; \frac{\omega_2 - \omega_1}{\omega_2 + 1}).
\]
Finally, using Proposition 3.5 and the explicit expressions for \(\omega_1\), and \(\omega_2\) given in \[2\], we obtain \[3\].

**Remark.** In the limit case \(\lambda = 0\), we have
\[
\sum_{n=0}^{+\infty} m_n(x; \beta, c) \omega^n
\]
\[
= \frac{1}{1 - \omega} \left(1 - \omega\right)^{-(\beta-1)} \left(\frac{c - \omega}{c(1 - \omega)}\right)^x 2F_1\left(-x, 0; \beta - 1; \frac{-\omega(1 + c^2)}{(1 - c\omega)(c - \omega)}\right)
\]
\[
= (1 - \omega)^{-x-\beta} \left(1 - \frac{\omega}{c}\right)^x,
\]
and we obtain again the generating function for Meixner polynomials.
Remark. Of course, the case $\beta = 1$ in Theorem 4.1 can be deduced from (24), since as we can easily check

$$
\lim_{\beta \to 1} 2F_1\left(-x, (\beta - 1)\delta; \beta - 1; \frac{\omega_2 - \omega_1}{\omega_2 + 1}\right) = \gamma + \delta \sum_{k=0}^{+\infty} \frac{(-x)^k}{k!} \left(\frac{\omega_2 - \omega_1}{1 + \omega_2}\right)^k
$$

and the result follows from the explicit expressions for $\omega_1$ and $\omega_2$.

5. Generating function for Laguerre–Sobolev orthogonal polynomials

In this section, by using a limit process we will recover the generating function for the Laguerre–Sobolev orthogonal polynomials obtained in [8]. As it is well-known (see, for instance, [4, p. 177]) there exists a limit relation between Meixner and Laguerre orthogonal polynomials, namely

$$
\lim_{c \uparrow 1} c^n m_n^{(a+1,c)}\left(\frac{x}{1-c}\right) = L_n^{(a)}(x), \quad \alpha > -1,
$$

where $L_n^{(a)}(x)$ denotes the Laguerre polynomials with leading coefficient $(-1)^n/n!$ orthogonal with respect to the inner product

$$(f, g)_L = \int_0^{+\infty} f(x) g(x) x^\alpha e^{-x} dx .$$

In [3, Prop. 4.4] the authors give a formula which extends the limit relation (28) to the $\Delta$–Sobolev case in the framework of $\Delta$–coherence. It is important to note that in this paper Meixner polynomials are considered orthogonal with respect to the inner product $(1-c)^\beta(f, g)$ where $(f, g)$ is given in (4). Anyway, taking

$$
\beta = \alpha + 1 \quad \text{and} \quad \lambda = \frac{\tilde{\lambda}}{(1-c)^2}, \quad \tilde{\lambda} > 0,
$$

and using the same arguments as in [3], we can prove

$$
\lim_{c \uparrow 1} c^n S_n\left(\frac{x}{1-c}\right) = S_n^L(x),
$$

where $\{S_n^L\}$ are the so-called Laguerre–Sobolev polynomials with leading coefficient $(-1)^n/n!$ orthogonal with respect to the inner product

$$(f, g)_S = \int_0^{+\infty} f(x) g(x) x^\alpha e^{-x} dx + \tilde{\lambda} \int_0^{+\infty} f'(x) g'(x) x^\alpha e^{-x} dx .$$

Note that the values for $\beta$ and $\lambda$ given in (29) imply

$$
\eta = 1 + \frac{\tilde{\lambda}}{c^2}.
$$

In [8], a generating function for polynomials $S_n^L(x)$ was obtained. In fact, if we denote by $\{q_n^L(\tilde{\lambda})\}$ the sequence of polynomials defined by the recurrence relation

$$
(n + \alpha)q_{n+1}^L(\tilde{\lambda}) = \left[ n(\tilde{\lambda} + 2) + \alpha \right] q_n^L(\tilde{\lambda}) - nq_{n-1}^L(\tilde{\lambda}),
$$

then we have

$$
\sum_{n=0}^{+\infty} q_n^L(\tilde{\lambda}) x^n = \frac{1}{(1-x)(1-\tilde{\lambda}x)}.
$$
with \( q_n^L(\tilde{\lambda}) = q_n^L(\lambda) = 1 \), and
\[
G_L(x, \omega, \tilde{\lambda}) := \sum_{n=0}^{\infty} q_n^L(\tilde{\lambda}) S_n^L(x) \omega^n.
\]
Then, for \( |\omega| < \tilde{a} < 1 \), we get (see Theorems 2.1 and 3.1 in [8])

- For \( \alpha = 0 \)
  \[
  G_L(x, \omega, \tilde{\lambda}) = \frac{1}{(1 - \omega)(1 + \tilde{a})} \left[ \exp \left( \frac{-x\omega\tilde{a}}{1 - \omega\tilde{a}} \right) + \tilde{a} \exp \left( \frac{-x\omega/\tilde{a}}{1 - \omega/\tilde{a}} \right) \right]
  \]

- For \( \alpha \neq 0 \)
  \[
  G_L(x, \omega, \tilde{\lambda}) = \frac{1}{1 - \omega} \left( 1 - \tilde{a}\omega \right)^{\frac{\tilde{a}}{\alpha}} \left( 1 - \frac{\omega}{\tilde{a}} \right)^{\frac{\tilde{a}}{\alpha}} \exp \left( \frac{-x\omega/\tilde{a}}{1 - \omega/\tilde{a}} \right)
  \times _1F_1 \left( \frac{\alpha}{1 + \tilde{a}}; \alpha; \frac{x\omega(1 - \tilde{a}^2)}{(\tilde{a} - \omega)(1 - \omega\tilde{a})} \right),
  \]
where, in both cases,
\[
\tilde{a} = \frac{\tilde{\lambda} + 2 - \sqrt{\tilde{\lambda}^2 + 4\tilde{\lambda}}}{2}.
\]

Using again the values for \( \beta \) and \( \tilde{\lambda} \) given in (29) and taking limits when \( c \uparrow 1 \) in (15) we recover \( 31 \) with the same initial conditions. Therefore, we get
\[
\lim_{c \uparrow 1} q_n(\eta) = q_n^L(\tilde{\lambda}).
\]

Thus, using (30) and (32) we obtain
\[
\lim_{c \uparrow 1} G_M \left( \frac{x}{1 - c}, c\omega, \frac{\tilde{\lambda}}{(1 - c)^2} \right) = \lim_{c \uparrow 1} \sum_{n=0}^{+\infty} q_n(\eta) S_n \left( \frac{x}{1 - c} \right) c^n \omega^n
= \sum_{n=0}^{+\infty} q_n^L(\tilde{\lambda}) S_n^L(x) \omega^n = G_L(x, \omega, \tilde{\lambda}),
\]
for \( |w| < \tilde{a} \) (note that \( \lim_{c \uparrow 1} a = \tilde{a} \) with \( \beta \) and \( \lambda \) given in (29)).

Therefore, we claim that we have recovered the generating functions for Laguerre–Sobolev orthogonal polynomials from the generating functions for \( \Delta \)-Meixner–Sobolev orthogonal polynomials.

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