Formulæ for the Number of Parts
itions of $n$ into at most $m$ parts
(Using the Quasi-Polynomial Ansatz)

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Abstract
The purpose of this short article is to announce, and briefly describe, a Maple package, \textsc{partitions}, that (inter alia) completely automatically discovers, and then proves, explicit expressions (as sums of quasi-polynomials) for $p_m(n)$ for any desired $m$. We do this to demonstrate the power of “rigorous guessing” as facilitated by the quasi-polynomial ansatz.

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1. Introduction
Recall that a partition of a non-negative integer $n$ is a non-increasing sequence of positive integers $\lambda_1 \ldots \lambda_m$ that sum to $n$. For example the integer 5 has the following seven partitions: \{5, 41, 32, 311, 221, 2111, 11111\}. The bible on partitions is George Andrews’ \textit{magnum opus} [1].

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We denote by \( p_m(n) \) the number of partitions of \( n \) into at most \( m \) parts. By a classic theorem \([1, p. 8, Thm. 1.4]\), \( p_m(n) \) also equals the number of partitions of \( n \) into parts that are at most \( m \). There is an extensive literature concerning formulae for \( p_m(n) \), including contributions by Cayley, Sylvester, Glaisher, and Gupta. For additional references and historical notes, see George Andrews’ fascinating article \([2, \S 3]\) and Gupta’s Tables \([8, pp. i–xxxix]\). For an exhaustive history through 1920, see Dickson \([4, Ch. 3]\).

More recently, George Andrews’ student, Augustine O. Munagi, developed a beautiful theory of so-called \( q \)-partial fractions \([11]\), where the denominators in the decomposition are always expressions of the form \((1 - q^r)^s\), rather than powers of cyclotomic polynomials as is the case with the ordinary partial fraction decomposition. Accordingly, formulae for \( p_m(n) \) derived from the \( q \)-partial fraction decomposition of the generating function are most naturally expressed in terms of binomial coefficients.

It is well-known and easy to see that for any \( m \), \( p_m(n) \) is a sum of quasi-polynomials of periods 1, 2, 3, \ldots, \( m \). A quasi-polynomial of period \( r \) is a function \( f(n) \) on the integers such that there exist \( r \) polynomials \( P_1(n), P_2(n), \ldots, P_r(n) \) such that \( f(n) = P_i(n) \) if \( n \equiv i \pmod{r} \). We represent such a quasi-polynomial as a list \([P_1(n), \ldots, P_r(n)]\).

Thus, e.g., we have, for \( n \geq 0 \),

\[
p_1(n) = 1, \tag{1}
\]

\[
p_2(n) = \left\lfloor \frac{n}{2} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{-1}{4}, \frac{1}{4} \right\rfloor, \tag{2}
\]

\[
p_3(n) = \left\lfloor \frac{n^2}{12} + \frac{n}{2} + \frac{47}{72} \right\rfloor + \left\lfloor \frac{-1}{8}, \frac{1}{8} \right\rfloor + \left\lfloor \frac{-1}{9}, \frac{1}{9}, \frac{2}{9} \right\rfloor \tag{3}
\]

\[
p_4(n) = \left\lfloor \frac{n^3}{144} + \frac{5n^2}{48} + \frac{15n}{32} + \frac{175}{288} \right\rfloor + \left\lfloor \frac{-n + 5}{32}, \frac{n + 5}{32} \right\rfloor + \left\lfloor 0, \frac{-1}{9}, \frac{1}{9} \right\rfloor + \left\lfloor 0, -\frac{1}{8}, \frac{1}{8} \right\rfloor \tag{4}
\]

\[
p_5(n) = \left\lfloor \frac{n^4}{2880} + \frac{n^3}{96} + \frac{31n^2}{288} + \frac{85}{192}n + \frac{50651}{86400} \right\rfloor + \left\lfloor \frac{-n}{64} - \frac{15}{128}, \frac{n}{64} + \frac{15}{128} \right\rfloor + \left\lfloor \frac{-1}{27}, \frac{1}{27}, \frac{2}{27} \right\rfloor + \left\lfloor \frac{1}{16}, \frac{-1}{16}, \frac{-1}{16}, \frac{1}{16} \right\rfloor
\]
Eqs. (1)–(5) were given in 1856 by Cayley [3, p. 132] in a somewhat different form. In 1909, Glaisher [6] presented formulæ for \( p_m(n) \) for \( m = 1, 2, \ldots, 10 \). In 1958, Gupta [8] extended Glaisher’s results to the cases \( m = 11, 12 \). In his 2005 Ph.D. thesis [10], Munagi gave formulæ for the cases \( m = 1, 2, \ldots, 15 \). Munagi’s formulæ were derived with the aid of a Maple package he developed, and are of a somewhat different character than earlier contributions, as they follow from his theory of \( q \)-partial fractions [11].

2. The PARTITIONS Maple package

2.1. Overview

The purpose of this short article is to announce and briefly describe a Maple package, PARTITIONS, that completely automatically discovers and proves explicit expressions (as sums of quasi-polynomials) for \( p_m(n) \) for any desired \( m \). So far we only bothered to derive the formulæ for \( 1 \leq m \leq 70 \), but one can easily go far beyond.

Not only that, we can, more generally, derive (and prove!), completely automatically, expressions, as sums of quasi-polynomials, for the number of ways of making change for \( n \) cents in a country whose coins have denominations of any given set of positive integers.

Not only that, we can derive (and prove!), completely automatically, expressions (as sums of quasi-polynomials) for \( D_k(n) \), the number of partitions of \( n \) whose Durfee square has size \( k \), for any desired, (numeric) positive integer \( k \). (Recall that the size of the Durfee square of a partition \( \lambda_1 \ldots \lambda_m \) is the largest \( k \) such that \( \lambda_k \geq k \).

Not only that, we (or rather our computers (and of course yours, if it has Maple and is loaded with our package)) can derive asymptotic expressions, to any desired order, for both \( p_m(n) \) and \( D_k(n) \). As far we we know the formula for \( D_k(n) \) is brand-new, and the previous attempts for the asymptotic formula for \( p_m(n) \) by humans G.J. Rieger [14] and E.M. Wright [16] (of Hardy-and-Wright fame) only went as far as \( O(n^{-2}) \) and \( O(n^{-4}) \) respectively. We go all the way to \( O(n^{-100}) \)! (and of course can easily go far beyond).

Not only that, we implement George Andrews’ ingenious way [2, sec. 3] to convert any quasi-polynomial to a polynomial expression where one is also
allowed to use the integer-part function \( \lfloor n \rfloor \). This enabled our computers to
find Andrews-style expressions for \( p_m(n) \) for \( 1 \leq m \leq 70 \).

All these feats (and more!) are achieved by the Maple package \texttt{PARTITIONS}.

2.2. Using the \texttt{PARTITIONS} package

In order to use \texttt{PARTITIONS}, you must have \texttt{Maple}\textsuperscript{TM} installed on your
computer. Then download the file:
\[ \text{http://www.math.rutgers.edu/~zeilberg/tokhniot/PARTITIONS} \]
save it as \texttt{PARTITIONS}. Then launch Maple, and at the prompt, enter:
\[ \text{read PARTITIONS;} \]
and follow the on-line instructions. Let’s just highlight the most important
procedures.

\texttt{AS100(m,n)}: shows the \emph{pre-computed} first 100 terms of the asymptotic
expression, in \( n \), of \( p_m(n) \) for \emph{symbolic} \( m \).

\texttt{ASD80(k,n)}: shows the \emph{pre-computed} first 80 terms of the asymptotic
expression, in \( n \), of \( D_k(n) \) for \emph{symbolic} \( k \).

\texttt{BuildDBpmn(n,M)}: inputs a symbol \( n \) and a positive integer \( M \) and out-
puts a list of size \( M \) whose \( i \)-th entry is an expression for \( p_i(n) \) as a sum of
\( i \) quasi-polynomials

\texttt{DiscoverAS(m,n,k)}: discovers the asymptotic expansion to order \( k \) of
\( p_m(n) \) (the number of partitions of \( n \) into at most \( m \) parts) for large \( n \) and
fixed, but \emph{symbolic}, \( m \).

\texttt{DiscoverDAS(k,n,r)}: discovers the asymptotic expansion to order \( r \) of
\( D_k(n) \) (the number of partitions of \( n \) whose Durfee square has size \( k \)) for
large \( n \) and fixed, but \emph{symbolic} \( k \).

\texttt{Durfee(k,n)}: discovers (rigorously!) the quasi-polynomial expression, in
\( n \), for \( D_k(n) \), for any desired positive integer \( k \). It is extremely fast for small
\( k \), but of course gets slower as \( k \) gets larger.

\texttt{DurfeePC(k,n)}: does the same thing (much faster, of course!) using the
pre-computed expressions of \texttt{Durfee(k,n)}; for \( k \leq 40 \).

\texttt{EvalQPS(L,n,n0)}: evaluates the sum of the quasi-polynomials in the
variable \( n \) given in the list \( L \) at \( n = n_0 \).

\texttt{HRR(n,T)}: evaluates in floating point the sum of the first \( T \) terms of the
Hardy-Ramanujan-Rademacher formula for \( p(n) \), the number of unrestricted
partitions of $n$: 

$$p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k \geq 1} \sqrt{k} \sum_{0 \leq h < k \text{ gcd}(h,k)=1} e^{\pi i (s(h,k)-2nh/k)} \frac{d}{dn} \left( \sinh \left( \frac{\pi}{4} \sqrt{\frac{3}{3} \left( n - \frac{1}{24} \right)} \right) \right),$$

where $s(h,k) = \sum_{j=1}^{k-1} \left( \frac{j}{k} - \left\lfloor \frac{j}{k} \right\rfloor - \frac{1}{2} \right) \left( \frac{hj}{k} - \left\lfloor \frac{hj}{k} \right\rfloor - \frac{1}{2} \right)$ is the Dedekind sum.

Please be warned that for larger $n$ you need to increase $\text{Digits}$. In order to get reliable results you may want to use procedure $\text{HRRr}(n,T,k)$.

$\text{pmn}(m,n)$: discovers (rigorously!) the quasi-polynomial expression, in $n$, for $p_m(n)$, for any desired positive integer $m$. It is extremely fast for small $m$, but of course gets slower as $m$ gets larger.

$\text{pmnPC}(m,n)$: does the same thing (much faster, of course!) using the pre-computed expressions of $\text{pmn}(m,n)$; for $m \leq 70$.

$\text{pmnAndrews}(m,n)$: discovers (rigorously!) the Andrews-style expression, in $n$, for $p_m(n)$ for any desired positive integer $m$. Instead of using quasi-polynomials explicitly (that some humans find awkward), it uses the integer-part function $\lfloor n \rfloor$, denoted by $\text{trunc}(n)$ in Maple.

$\text{pn}(n)$: the number of partitions of $n$, $p(n)$, using Euler’s recurrence. It is useful for checking, since $p_n(n) = p(n)$.

$\text{pnSeq}(N)$: the list of the first $N$ values of $p(n)$. The output of $\text{pnSeq}(50000)$: can be gotten from

http://www.math.rutgers.edu/~zeilberg/tokhniot/oPARTITIONS9 where this list of 50000 terms is called $\text{pnTable}$.

$\text{pSn}(S,n,K)$: the more general problem where the parts are drawn from the list $S$ of positive integers. It outputs an explicit expression, as a sum of quasi-polynomials, for $p_S(n)$, the number of integer partitions of $n$ whose parts are drawn from the finite list of positive integers $S$. $K$ is a guessing parameter, that should be made higher if the procedure returns $\text{FAIL}$.

$\text{pmnNum}(m,n0)$: like $\text{pmn}(m,n)$; but for both numeric $m$ and $n0$. The output is a number. For $m \leq 70$ it is extremely fast, since it uses the pre-computed values of $p_m(n)$ gotten from $\text{pmnPC}(m,n)$; For example to get the number of integer partitions of a googol ($10^{100}$) into at most 60 parts, you would get, in 0.02 seconds, the 5783-digit integer, by simply typing

$\text{pmnNum}(60,10^{100})$.

One of us (DZ) posed this is a 100-dollar challenge to the users of the very useful Mathoverflow forum. This was taken-up, successfully, by user
joro\cite{5}, whose computer did it correctly in about 2 hours, using PARI. User joro generously suggested that instead of sending him a check, DZ should donate it in joro’s honor, to a charity of DZ’s choice, and the latter decided on the Wikipedia Foundation.

Sample input and output can be gotten from the “front” of this article:
http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/pmn.html.

3. Methodology: Rigorous Guessing

The idea of deriving formulæ for $p_m(n)$ and $p_S(n)$ with the aid of the partial fraction decomposition of the generating function dates back at least to Cayley \cite{3}. We ask Maple to convert the generating function

$$
\sum_{n \geq 0} p_m(n)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}
$$

or in the case of $p_S(n)$, where $S = \{s_1, s_2, \ldots, s_j\}$,

$$
\sum_{n \geq 0} p_S(n)q^n = \frac{1}{(1-q^{s_1})(1-q^{s_2})\cdots(1-q^{s_j})}
$$

into partial fractions. Then for each piece, Maple finds the first few terms of the Maclaurin expansion, and then fits the data with an appropriate quasi-polynomial using undetermined coefficients. The output is the list of these quasi-polynomials whose sum is the desired expression for $p_m(n)$ or $p_S(n)$. See the source-code for more details.

Example. Consider the case $m = 4$. We have Maple calculate that

$$
\sum_{n \geq 0} p_4(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)}
$$

$$
= \frac{17/72}{1-q} + \frac{59/288}{(1-q)^2} + \frac{1/8}{(1-q)^3} + \frac{1/24}{(1-q)^4} + \frac{1/8}{1+q} + \frac{1/32}{(1+q)^2} + \frac{(1+q)/9}{1+q+q^2}.
$$

(6)

At this point we could, as Cayley did, expand each term as a series in $q$, collect like terms, and then the coefficient of $q^n$ will be a formula for $p_4(n)$, but why bother? From Sylvester \cite{15} and Glaisher \cite{7}, we know that

$$
p_4(n) = \sum_{j=1}^{4} W_j(n),
$$
where each \( W_j(n) \) is a quasi-polynomial \([P_{j1}(n), P_{j2}(n), \ldots, P_{jj}(n)]\) of period \( j \). Further, \( W_j(n) \) is of degree \( \lfloor \frac{2m-j}{j} \rfloor \), and arises from those terms of (6) with denominator a power of the \( j \)-th cyclotomic polynomial. Instead, let us allow Maple to guess the correct quasi-polynomials: We know \textit{a priori} that \( W_1(n) \) is of the form \([a_0 + a_1 n + a_2 n^2 + a_3 n^3]\) and let Maple calculate the (beginning of the) Maclaurin series for the terms of (6) that contribute to \( W_1(n) \):

\[
\frac{17}{72} + \frac{59}{288} + \frac{1}{8} + \frac{1}{24} = \frac{175}{288} + \frac{19}{16} q + \frac{581}{288} q^2 + \frac{113}{36} q^3 + O(q^4).
\]

Thus,

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= \begin{bmatrix}
175/288 \\
19/16 \\
581/288 \\
113/36
\end{bmatrix},
\]

which immediately implies that

\[
W_1(n) = \left[ \frac{1}{144} n^3 + \frac{5}{48} n^2 + \frac{15}{32} n + \frac{175}{288} \right].
\]

Similarly, for \( W_2(n) \), which must be of the form

\[ [a_1 + a_3 n, a_0 + a_2 n], \]

we find

\[
\frac{1}{8} + \frac{1}{32} = \frac{5}{32} - \frac{3}{16} q + \frac{7}{32} q^2 - \frac{1}{4} q^3 + O(q^4),
\]

so that

\[
\begin{bmatrix}
1 & 0 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_2
\end{bmatrix}
= \begin{bmatrix}
5/32 \\
7/32
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
1 & 1 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_3
\end{bmatrix}
= \begin{bmatrix}
-3/16 \\
-1/4
\end{bmatrix},
\]

and thus

\[
W_2(n) = \left[ -\frac{5}{32} - \frac{n}{32}, \frac{5}{32} + \frac{n}{32} \right].
\]

Analogous reasoning yields \( W_3(n) = [0, -\frac{1}{5}, \frac{1}{5}] \) and \( W_4(n) = [0, -\frac{1}{8}, 0, \frac{1}{8}] \).
4. Conclusion

The present approach uses very naïve guessing to discover, and prove (rigorously!), formulas (or as Cayley and Sylvester would say, formulæ) for the number of partitions of the integer \( n \) into at most parts \( m \) parts for \( m \leq 70 \), and of course, one can easily go far beyond. The core of the idea goes back to Arthur Cayley, and is familiar to any second-semester calculus student: partial fractions! But dear Arthur could only go so far, so his good buddy, James Joseph Sylvester, designated a sophisticated theory of “waves” [15] that facilitated hand calculations, which were later dutifully carried out by J. W. L. Glaisher in [7]. But, with modern computer algebra systems (Maple in our case), one can go much further just using Cayley’s original ideas.

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References

[1] G. E. Andrews, The Theory of Partitions, Addison-Wesley, 1976. Reprinted by Cambridge University Press, 1984. First paperback edition, 1998.

[2] G. E. Andrews, Partitions: at the interface of \( q \)-series and modular forms, Ramanujan J. 7 (2003) 384-400.

[3] A. Cayley, Researches on the partition of numbers, Phil. Trans. Royal Soc. of London, 146 (1856) 127–140.

[4] L. E. Dickson, History of the Theory of Numbers, vol. 2: Diophantine Analysis, Carnegie Institute, Washington, 1920. Reissued by AMS Chelsea and Dover.

[5] “joro”, answer to Doron Zeilberger’s 100-dollar challenge, http://mathoverflow.net/questions/71092/how-many-integer-partitions-of-a-googol-10100-into-at-most-60-parts

[6] J. W. L. Glaisher, On the number of partitions of a number into a given number of parts, Quart. J. Pure Appl. Math. 40 (1909) 57–143.
[7] J. W. L. Glaisher, Formulæ for partitions into given elements, derived from Sylvester’s theorem, Quart. J. Pure Appl. Math. 40 (1909) 275–348.

[8] H. Gupta, Tables of Partitions, Royal Society Mathematical Tables, vol. 4, Cambridge University Press, 1958.

[9] G. H. Hardy and S. Ramanujan, Asymptotic formulæ in combinatory analysis, Proc. London Math. Soc. (2) 17 (1918), 75–115.

[10] A. O. Munagi, Restricted Partition Formulas, q-Partial Fractions, and the Rademacher Conjecture, Ph.D. thesis, University of Lagos, Nigeria, 2005.

[11] A. O. Munagi, Computation of $q$-partial fractions, INTEGERS 7 (2007) #A25.

[12] H. Rademacher, On the partition function $p(n)$, Proc. London Math. Soc. (2) 43 (1938) 241–254.

[13] H. Rademacher, On the expansion of the partition function in a series, Ann. Math. (2) 44 (1943) 416–422.

[14] G. J. Rieger, Über Partitionen, Math. Annalen 138 (1959), 356-362.

[15] J. J. Sylvester, On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order. With an excursus on rational fractions and partitions, Quart. J. Math. (Oxford Series (2)) 2 (1882) 85–108.

[16] E.M. Wright, Partitions into $k$ parts, Math. Annalen 142 (1961), 311-316.