HALF-FLAT STRUCTURES INDUCING EINSTEIN METRICS ON HOMOGENEOUS SPACES

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ABSTRACT. In this paper we consider half-flat SU(3)-structures and the subclasses of coupled and double structures. In the general case we show that the intrinsic torsion form $w_1$ is constant in each of the two subclasses. We then consider the problem of finding half-flat structures inducing Einstein metrics on homogeneous spaces. We give an example of a left invariant half-flat (non coupled and non double) structure inducing an Einstein metric on $S^3 \times S^3$ and we show there does not exist any left invariant coupled structure inducing an $\text{Ad}(S^3)$-invariant Einstein metric on it. Finally, we show that there are no coupled structures inducing the Einstein metric on Einstein solvmanifolds and on homogeneous Einstein manifolds of nonpositive sectional curvature.

1. Introduction

An SU(3)-structure on a 6-dimensional smooth manifold $N$ is the data of a Riemannian metric $h$, an orthogonal almost complex structure $J$, a 2-form $\omega$ related with $h$ and $J$ via the identity $\omega(\cdot, \cdot) = h(J\cdot, \cdot)$ and a $(3,0)$-form $\Psi$ of nonzero constant length. Using the results on stable forms ([17], [23]) it can be shown that such a structure actually depends only on $\omega$ and $\psi_+ := \text{Re}(\Psi)$.

The intrinsic torsion of an SU(3)-structure lies in a 42 dimensional space and depends only on the exterior derivatives of $\omega, \psi_+$ and $\psi_- := \text{Im}(\Psi) = J^*\psi_+$ as showed in [8]. When the intrinsic torsion vanishes, i.e. when $\omega, \psi_+$ and $\psi_-$ are all closed, the manifold has holonomy contained in SU(3) and the metric $h$ is Ricci-flat.

When the forms $\psi_+$ and $\omega^2 = \omega \wedge \omega$ are both closed, the torsion lies in the 21 dimensional space $W_1^- \oplus W_2^- \oplus W_3$ and the SU(3)-structure is called half-flat. Half-flat structures are the initial values for the Hitchin flow equations and are used as starting point to construct 7 dimensional manifolds with holonomy contained in $G_2$ (see for example [4], [8], [10], [17]).

There are three classes of SU(3)-structures contained in the half-flat one which have been frequently considered in literature: the nearly Kähler ($W_1^-$), the double or co-coupled ($W_1^- \oplus W_3$) and the coupled ($W_1^- \oplus W_2^-$) (see [24]). It is well known that the metric $h$ induced by a nearly Kähler structure is always Einstein, i.e. the Ricci tensor of $h$ satisfies the identity

$$\text{Ric}(h) = \mu h$$

for some real number $\mu$. Moreover there exist some examples of half-flat structures with torsion class $W_1^- \oplus W_2^- \oplus W_3$ and $W_1^- \oplus W_3$ inducing an Einstein metric, but up to now it seems the coupled case has not been studied.
In \cite{13} invariant coupled structures on 6-dimensional nilpotent Lie groups have been classified and it was shown that there is only one case in which the coupled structure induces a Ricci soliton metric. Moreover, this coupled structure was used to construct a locally conformal calibrated $G_2$-structure inducing an Einstein metric on the rank one solvable extension of the nilpotent Lie algebra.

More in general, it is not difficult to show that on the cylinder and on the cone over a 6-manifold admitting a coupled structure there exists a locally conformal calibrated $G_2$-structure induced by the coupled one. On the other hand, it is possible to show that a parallel $G_2$-structure on a 7-dimensional manifold induces a coupled structure $(h, J, \omega, \Psi)$ on the oriented hypersurfaces having $J$-invariant second fundamental form (\cite{7}).

One of the simplest cases that can be considered when one is looking for examples of special geometric structures with (or without) torsion is the one of left invariant structures on homogeneous spaces, since in this case the starting analytic problem on the manifold (e.g. the problem of solving the PDEs deriving from the definition of half-flat structure) can often be reduced to an algebraic problem on the tangent space to a point. Following this idea, this paper focuses the attention to the case of left-invariant half-flat structures on 6 dimensional homogeneous manifolds.

In \cite{26}, Schulte-Hengesbach considers half-flat structures on Lie groups and, in particular, he describes left invariant half-flat structures on $S^3 \times S^3$ and gives two examples of half-flat structures inducing an Einstein metric on it. One of these examples consists of a double structure and the other is the unique (up to homotheties and sign) left-invariant nearly Kähler structure existing on this manifold, as showed by Butruille in \cite{6}. Moreover the Einstein metrics are the only two currently known examples of left invariant Einstein metrics on $S^3 \times S^3$ and can be characterized as the only $\text{Ad}(S^1)$-invariant Einstein metrics existing on it up to isometries and homotheties. It is then natural to ask whether there exist left invariant half-flat structures (neither coupled nor double) and left invariant coupled structures on $S^3 \times S^3$ inducing any of these metrics. We will show that in the first case the answer is positive by giving an explicit example, while for the coupled case we will prove that the answer is negative.

Left invariant half-flat structures on $S^3 \times S^3$ have also been studied by Madsen and Salamon in \cite{20}, where they describe them using the representation theory of $\text{SO}(4)$ and matrix algebra and show that the moduli space they define is essentially a finite dimensional symplectic quotient. In this paper they also consider the subclasses of coupled, co-coupled (double) and nearly Kähler structures, in particular they give some examples of double structures, give another proof of Butruille’s result in their setting and construct a 1-parameter family of double structures which is a solution for the Hitchin flow.

Conversely to the case of the compact manifold $S^3 \times S^3$, in the case of noncompact homogeneous manifolds it can be shown using the results of Heber (\cite{16}) and Lauret (\cite{19}) that every Einstein solvmanifold has a unique left invariant Einstein metric up to isometries and homotheties. Einstein solvmanifolds constitute the unique example of noncompact homogeneous Einstein manifolds known up to now and it has been conjectured by Alekseevskii that they might be the only case that can occur (\cite{3, 7.57}).

We then focus on 6 dimensional Einstein solvmanifolds, which have been classified by Nikitenko and Nikonorov in \cite{21}, and we look for left invariant half-flat structures defined
on them inducing the Einstein metric. Also in this case we will prove that there are no coupled structures satisfying the property we are looking for. Moreover, using another result contained in [21], we are able to conclude that this happens for 6 dimensional homogeneous Einstein manifolds of nonpositive sectional curvature too. This gives a constraint to the existence of coupled Einstein structures on 6 dimensional homogeneous manifolds and leads to ask whether the non positivity of the sectional curvature gives a constraint in a more general setting.

This paper is organized as follows: in section 2 we recall the definition of SU(3)-structures and we focus on the half-flat class and its subclasses, in section 3 we consider the case of left-invariant half-flat structures on $S^3 \times S^3$ and in section 4 after recalling some properties of 6 dimensional Einstein solvmanifolds, we study the case of left invariant half-flat structures inducing Einstein metrics on them.

All the algebraic computations in section 3 and 4 have been done with the aid of the software Maple.

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2. Preliminaries on SU(3)-structures

Let $N$ be a 6-dimensional smooth manifold. It admits an SU(3)-structure if the structure group of its frame bundle can be reduced from $\text{GL}(6, \mathbb{R})$ to $\text{SU}(3)$. The existence of an SU(3)-structure is equivalent to the existence of a Hermitian structure $(h, J, \omega)$ and a $(3, 0)$-form $\Psi$ of nonzero constant length such that the Riemannian metric $h$, the orthogonal almost complex structure $J$ and the 2-form $\omega$ are related via the identity

$$\omega(\cdot, \cdot) = h(J\cdot, \cdot),$$

the forms $\omega$, $\psi_+ := \Re(\Psi)$ and $\psi_- := \Im(\Psi) = J^*\psi_+$ are compatible in the sense that

$$\omega \wedge \psi_\pm = 0$$

and satisfy the normalization condition

$$\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3 = 4\text{Vol}(h),$$

where $\text{Vol}(h)$ is the Riemannian volume form of $h$.

Using the results on stable forms contained in the works [17], [23], it is possible to show that an SU(3)-structure actually depends only on $\omega$ and $\psi_+$ and thus it is possible to give another characterization for SU(3)-structures, which is the one we will use in this work. To describe this characterization let us consider a 6 dimensional real vector space $V$, we say that a differential $k$-form on $V$ is stable if its orbit under the natural action of $\text{GL}(V)$ on $\Lambda^k(V^*)$ is open. One can show that a 2-form $\sigma$ on $V$ is stable if and only if $\sigma^3 \neq 0$, that is if and only if it is non degenerate, moreover if we denote by $A : \Lambda^6(V^*) \to V \otimes \Lambda^6(V^*)$ the canonical isomorphism given by $A(\gamma) = v \otimes \Omega$, where $i_v \Omega = \gamma$, and define for a fixed $\rho \in \Lambda^3(V^*)$

$$K_\rho : V \to V \otimes \Lambda^6(V^*), \quad K_\rho(v) = A((i_v \rho) \wedge \rho)$$

this formula gives a criterion for the stability of $\sigma$.
and
\[ \lambda : \Lambda^3(V^*) \to (\Lambda^6(V^*))^{\otimes 2}, \quad \lambda(\rho) = \frac{1}{6} \text{tr} K_\rho^2, \]
we have that a 3-form \( \rho \) is stable if and only if \( \lambda(\rho) \neq 0 \) and whenever this happens it is possible to define a volume form by \( \sqrt{|\lambda(\rho)|} \in \Lambda^6(V^*) \) and an endomorphism
\[ J_\rho = -\frac{1}{\sqrt{|\lambda(\rho)|}} K_\rho, \]
which is an almost complex structure when \( \lambda(\rho) < 0 \).

The existence of an SU(3)-structure on \( N \) is equivalent to the existence of a pair of differential forms \( (\omega, \psi_+) \in \Lambda^2(N) \times \Lambda^3(N) \) such that for each \( p \in N \) the forms \( \omega(p) \) and \( \psi_+(p) \) on \( T_p N \) are stable with \( \lambda(\psi_+(p)) < 0 \), compatible, satisfy the normalization condition and define a Riemannian metric \( h_\rho(\cdot, \cdot) = \omega_\rho(\cdot, J \cdot) \), where \( J = J_\rho(p) \) is the almost complex structure induced by \( \psi_+(p) \). The Riemannian metric \( h \) can also be described in terms of \( \omega \) and \( \psi_+ \) as
\[ h(X, Y) \omega^3 = -3(i_X \omega) \wedge (i_Y \psi_+) \wedge \psi_+, \]
for any \( X, Y \in \mathfrak{X}(N) \).

The intrinsic torsion of an SU(3)-structure lies in a 42-dimensional space
\[ W_1^+ \oplus W_1^- \oplus W_2^+ \oplus W_2^- \oplus W_3 \oplus W_4 \oplus W_5 \]
given by the sum of irreducible SU(3)-modules and completely determined by \( d\omega, d\psi_+ \) and \( d\psi_- \) (see [8]).

In this paper we are mainly interested in SU(3)-structures having both \( \omega^2 \) and \( \psi_+ \) closed, known as half-flat SU(3)-structures in literature. In this case the intrinsic torsion lies in the space \( W_1^+ \oplus W_2^- \oplus W_3 \) and in terms of the exterior derivatives of \( \omega, \psi_+ \) and \( \psi_- \) this reads:
\begin{equation}
\begin{aligned}
d\omega &= -\frac{3}{2}w_1^- \psi_+ + w_3, \\
d\psi_+ &= 0, \\
d\psi_- &= w_1^- \omega^2 - w_2^- \wedge \omega,
\end{aligned}
\end{equation}
where \( w_1^- \in C^\infty(N) \cong W_1^- , w_2^- \in \Lambda_0^{1,1}(N) \cong W_2^- , w_3 \in \Lambda_0^{2,1}(N) \cong W_3 \) are the (non vanishing) intrinsic torsion forms.

There are three interesting families of SU(3)-structures with nonzero torsion contained in the family of half-flat ones, we recall here the definitions.

**Definition 2.1.** An SU(3)-structure is said **nearly Kähler** if \( \nabla J \) is skew-symmetric, that is \( (\nabla_X J)X = 0 \) for every \( X \in \mathfrak{X}(N) \), **coupled** if \( d\omega = c \psi_+ \) for some non vanishing \( c \in C^\infty(N) \) and **double** (or co-coupled) if \( d\psi_- = k \omega^2 \) for some non vanishing \( k \in C^\infty(N) \).

It can be shown that the intrinsic torsion of a nearly Kähler structure lies in \( W_1^- \) while it follows easily from the definition that the intrinsic torsion of a coupled lies in \( W_1^- \oplus W_2^- \) and the intrinsic torsion of a double lies in \( W_1^- \oplus W_3 \). In each case the expressions of \( d\omega, d\psi_+ \) and \( d\psi_- \) can be obtained from (1) having in mind the torsion class to which each family belongs. Looking at these, it is easy to see that coupled and double structures can be thought as a generalization of the nearly Kähler. For each class it is also possible to write the Ricci tensor and the scalar curvature of \( h \) in terms of the non vanishing intrinsic
torsion forms using the results of [5]. It is then evident that the metric induced by a nearly Kähler structure is Einstein, while in the general case it seems that there are no restrictions for the Ricci curvature of the metric induced by a coupled or a double.

It is well known that in the nearly Kähler case the only non vanishing torsion form $w_1^-$ is constant, using the fact that the coupled and the double structures are in particular half-flat, we can prove that the same is true in these two cases.

**Lemma 2.2.** Let $N$ be a 6-dimensional connected smooth manifold endowed with an SU(3)-structure $(\omega, \psi_+)$. If $(\omega, \psi_+)$ is coupled, then there exists a nonzero real constant $c$ such that

$$d\omega = c\psi_+,$$

if $(\omega, \psi_+)$ is double, then there is a nonzero real constant $k$ such that

$$d\psi_-=k\omega^2.$$  

**Proof.** If $(\omega, \psi_+)$ is coupled, using the notations of (1) we know that

$$d\omega = -\frac{3}{2}w_1^-\psi_+,$$

where $w_1^-$ is a smooth nonzero function. Taking the exterior derivative of $d\omega$ we obtain

$$0 = d(w_1^-\psi_+),$$

$$= d\omega_1^- \wedge \psi_+ + w_1^-d\psi_+,$$

so

$$d\psi_+ = -\frac{1}{w_1^-}d\omega_1^- \wedge \psi_+.$$  

Now, $d(\omega^2) = 2d\omega \wedge \omega = 0$ since $\omega$ and $\psi_+$ are compatible, thus the considered class of SU(3)-structures is contained in the half-flat one if and only if

$$d\omega_1^- = 0,$$

that is $w_1^- = c$ for some nonzero real constant $c$ on $N$ connected.

In the double case we can argue in a similar way: starting from

$$d\psi_- = w_1^-\omega^2,$$

we take the exterior derivative of both sides obtaining

$$0 = d\omega_1^- \wedge \omega^2 + w_1^-d\omega^2$$

and conclude observing that $d\omega^2 = 2d\omega \wedge \omega = 0$ since $\omega \wedge \psi_+ = 0 = \omega \wedge w_3$ and that wedging 1-forms by $\omega^2$ is an isomorphism.  

**Remark 2.3.** The defining conditions of an half-flat structure $d\psi_+ = 0$ and $d\omega^2 = 0$ are obviously satisfied by the subclasses of coupled, double and nearly Kähler, thus to make distinction between these classes is necessary to look at the non vanishing components of the intrinsic torsion (for example by computing $d\omega$ and $d\psi_-$). Moreover, we could sometimes emphasize the fact that we are considering half-flat structures with torsion in $W_1^- \oplus W_2^- \oplus W_3$ by saying that the structure is half-flat non coupled and non double.
3. Einstein half-flat structures on $S^3 \times S^3$

In this section we focus on the homogeneous manifold $S^3 \times S^3$ and we consider left invariant half-flat structures on it inducing Einstein metrics, looking for what happens in the subclasses.

As a Lie group the manifold is $SU(2) \times SU(2)$ and we can describe the left invariant tensors only working on the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In particular, a left invariant metric on $S^3 \times S^3$ can be identified with an inner product on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and a left invariant $SU(3)$-structure $(\omega, \psi_+)$ on $S^3 \times S^3$ can be identified with a 2-form $\omega$ and a 3-form $\psi_+$ defined on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ satisfying the defining properties of such a structure.

Let $(e_1, e_2, e_3)$ denote the standard basis for the first copy of $\mathfrak{su}(2)$, $(e_4, e_5, e_6)$ denote it for the second one and let $(e^1, e^2, e^3)$ and $(e^4, e^5, e^6)$ denote their dual bases. Then we have the following structure equations for the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

\[ de^1 = e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12}, \]
\[ de^4 = e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45}, \]

where we are using the shortening $e^{ijk\cdots}$ for the wedge product $e^i \wedge e^j \wedge e^k \wedge \cdots$.

The problem of classifying left invariant Einstein metrics on $S^3 \times S^3$ is still open and only two examples are known. These can be characterized as follows:

**Theorem 3.1 ([22]).** Let $h$ be a left invariant Einstein metric on the Lie group $SU(2) \times SU(2)$ which is $\text{Ad}(S^1)$-invariant for some embedding $S^1 \subset SU(2) \times SU(2)$, then $h$ is isometric up to homotheties either to the standard metric or to Jensen’s metric.

With respect to the basis $(e_1, e_2, e_3, e_4, e_5, e_6)$ and up to scalar multiples, the matrix associated to the standard metric is the identity matrix and the one associated to the Jensen metric is the following

\[
\begin{pmatrix}
\frac{2\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 \\
0 & \frac{2\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} & 0 \\
0 & 0 & \frac{2\sqrt{3}}{3} & 0 & 0 & -\frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & 0 & 0 & \frac{2\sqrt{3}}{3} & 0 & 0 \\
0 & -\frac{\sqrt{3}}{3} & 0 & 0 & \frac{2\sqrt{3}}{3} & 0 \\
0 & 0 & -\frac{\sqrt{3}}{3} & 0 & 0 & \frac{2\sqrt{3}}{3}
\end{pmatrix}
\]

In [26], Schulte-Hengesbach gives an example of a left invariant half-flat structure on $S^3 \times S^3$ inducing the standard metric and of one inducing the Jensen metric, we describe them in the following examples.

**Example 3.2 ([26]).** The pair of stable forms

\[
\omega = -e^{14} - e^{25} - e^{36}, \\
\psi_+ = \frac{1}{\sqrt{2}} (e^{123} - e^{156} + e^{246} - e^{345} + e^{126} - e^{135} + e^{234} - e^{456}),
\]
is compatible, normalized and induces the standard metric, thus it defines an SU(3)-
structure on $\text{su}(2) \oplus \text{su}(2)$. Moreover $d\psi_+ = 0$, $d\omega^2 = 0$, $d\psi_- = \frac{1}{\sqrt{2}} \omega^2$ and $d\omega$ is not
proportional to $\psi_+$, i.e. it is a double structure.

**Example 3.3 ([26]).** The following pair of stable, compatible, normalized forms

$$\omega = -\frac{\sqrt{3}}{18} (e^{14} + e^{25} + e^{36}) ,$$
$$\psi_+ = \frac{\sqrt{3}}{34} (-e^{234} + e^{156} + e^{135} - e^{246} - e^{126} + e^{345}) ,$$

induces the Jensen metric, therefore it defines an SU(3)-structure on $\text{su}(2) \oplus \text{su}(2)$ which is
nearly Kähler since $d\omega = 3\psi_+$ and $d\psi_- = -2\omega^2$.

**Remark 3.4.** The signs of the differential forms in the previous examples are different from
those in the original examples of [26], this is due to the fact that in this paper we are using
a convention in the definition of an SU(3)-structure which is slightly different from the one
used by the other author in his works.

We can also give a new example of a left invariant half-flat structure (non coupled and non double) inducing the Jensen metric:

**Example 3.5.** The pair

$$\omega = \frac{\sqrt{3}}{2} \left( -e^{14} + e^{25} + e^{36} \right) ,$$
$$\psi_+ = e^{123} + e^{135} - e^{246} - e^{126} + e^{345} - e^{456} ,$$

defines an SU(3)-structure on $\text{su}(2) \oplus \text{su}(2)$ and induces a metric which is (proportional to) the Jensen metric. Moreover, it can be checked that this SU(3)-structure is half-flat since both $\psi_+$ and $\omega^2$ are closed and it is neither coupled nor double since $d\omega$ is not proportional to $\psi_+$ and $d\psi_-$ is not proportional to $\omega^2$.

Summarizing, on $S^3 \times S^3$ there exist left invariant half-flat and nearly Kähler structures
inducing the Jensen metric and left invariant double structures inducing the standard metric.
We will prove now that it is not possible to find left invariant coupled structures on $S^3 \times S^3$ inducing either the standard or the Jensen metric. In the proof we will use some classical properties of algebraic varieties, the reader can find them for example in [11].

**Theorem 3.6.** $S^3 \times S^3$ does not admit left invariant coupled SU(3)-structures $(\omega, \psi_+)$
inducing an $\text{Ad}(S^1)$-invariant Einstein metric.

**Proof.** Let us consider a left invariant coupled structure $(\omega, \psi_+)$ on $S^3 \times S^3$ which we identify with a 2-form $\omega$ and a 3-form $\psi_+$ defined on $\text{su}(2) \oplus \text{su}(2)$ and such that $d\omega = \frac{1}{c} \psi_+$, $c \in \mathbb{R} - \{0\}$. Since $\omega^2$ is closed, it follows that $\omega \in \text{su}^*(2) \oplus \text{su}^*(2)$ (cf. [26], Lemma 1.1 p. 81), thus

$$\omega = a_{14} e^{14} + a_{15} e^{15} + a_{16} e^{16} + a_{24} e^{24} + a_{25} e^{25} + a_{26} e^{26} + a_{34} e^{34} + a_{35} e^{35} + a_{36} e^{36} ,$$
where $a_{ij}$ are real coefficients. Imposing the coupled condition $\psi_+ = cd\omega, c \neq 0$, we have that

$$\psi_+ = \frac{-2a^2}{\sqrt{-\lambda}}(a_{14}a_{25}a_{36} - a_{14}a_{26}a_{35} - a_{15}a_{24}a_{36} + a_{15}a_{26}a_{34} + a_{16}a_{24}a_{35} - a_{16}a_{25}a_{34}),$$

for $i = 1, \ldots, 6$

$$H_{1,4} = \frac{-2a^2}{\sqrt{-\lambda}}(a_{14}^2 + a_{14}a_{15}^2 + a_{14}a_{16}^2 + a_{14}a_{24}^2 - a_{14}a_{25}^2 - a_{14}a_{26}^2 + a_{14}a_{34}^2 - a_{14}a_{35}^2 - a_{14}a_{36}^2 + 2a_{15}a_{24}a_{25} + 2a_{15}a_{24}a_{35} + 2a_{15}a_{34}a_{35} + 2a_{16}a_{24}a_{26} + 2a_{16}a_{34}a_{36}),$$

$$H_{1,5} = \frac{-2a^2}{\sqrt{-\lambda}}(a_{14}^2 + a_{14}a_{15}a_{24} + a_{14}a_{15}a_{25} + a_{14}a_{15}a_{26} + a_{14}a_{15}a_{34} + a_{14}a_{15}a_{35} + a_{14}a_{15}a_{36} - a_{15}a_{24}a_{34} - a_{15}a_{24}a_{35} - a_{15}a_{24}a_{36} + 2a_{16}a_{25}a_{26} + 2a_{16}a_{34}a_{36},$$

$$H_{1,6} = \frac{-2a^2}{\sqrt{-\lambda}}(a_{14}^2 + a_{14}a_{24}a_{25} + a_{14}a_{24}a_{26} + a_{14}a_{24}a_{34} + a_{14}a_{24}a_{35} + a_{14}a_{24}a_{36} + a_{16}a_{24}a_{25} + a_{16}a_{24}a_{26} + a_{15}a_{25}a_{26} + a_{15}a_{25}a_{34} + a_{15}a_{25}a_{35} + a_{15}a_{25}a_{36} + a_{16}a_{25}a_{34} + a_{16}a_{25}a_{35} + a_{16}a_{25}a_{36} + 2a_{16}a_{34}a_{35} + 2a_{16}a_{34}a_{36},$$

and from direct computations we can see that the compatibility condition $\omega \wedge \psi_+$ holds. It is now possible to compute $\lambda = \lambda(\psi_+)$, which turns out to be a homogeneous polynomial of degree 4 in the coefficients $a_{ij}$, the almost complex structure $J = J_{\psi_+}$ and $h(\cdot, \cdot) = \omega(\cdot, J \cdot)$. With respect to the basis $(e_1, \ldots, e_6)$ the matrix $H$ associated to $h$ is symmetric and the nonzero entries are the following:

$$H_{i,j} = h(e_i, e_j).$$

Observe that up to multiplication by $\sqrt{-\lambda}$, the nonzero terms are all homogeneous polynomials of third degree in the $a_{ij}$.

We are looking for coupled structures inducing either the standard metric or the Jensen metric which with respect to the considered basis can be written as the identity matrix and as [2], respectively. Thus, since $\omega \wedge \psi_+ = 0, d\psi_+ = 0$ and $d\omega^2 = 0$, we first have to solve the system obtained by imposing that the matrix $H$ is proportional to the identity matrix or to the matrix [2] under the assumption $\lambda < 0$ and then, if we find solutions of this system, we need to impose that the normalization condition is satisfied in order to obtain what we want.
Case 1: the standard metric

Since rescaling a metric with a positive coefficient does not change the Ricci tensor, we are looking for solutions of the equation

\[ H = \alpha I, \]

where \( \alpha \) is a positive real number.

Since the entries in the diagonal of \( H \) are all equal, we only have to solve the system of equations

\[ H_{i,j} = 0, \quad i = 1, 2, 3, \quad j = 4, 5, 6 \]

under the assumptions \( H_{1,1} \neq 0 \) and \( \lambda < 0 \).

For \( i, j = 1, \ldots, 6 \), we let

\[ \tilde{H}_{i,j} := \sqrt{-\lambda} H_{i,j}. \]

Then, as already observed, the \( \tilde{H}_{i,j} \) are homogeneous polynomials of degree 3 in the \( a_{ij} \) and under our assumptions \( H_{i,j} = 0 \) if and only if \( \tilde{H}_{i,j} = 0 \) for \( i = 1, 2, 3, j = 4, 5, 6 \).

Since we have a system of equations involving homogeneous polynomials of the same degree and we are looking for solutions defined up to a multiplicative constant, let us consider the projective space \( \mathbb{CP}^8 \) with coordinate ring

\[ \mathbb{C}[a_{14}, a_{15}, a_{16}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36}] \]

and the homogeneous ideals

\[ P := \langle \tilde{H}_{1,1} \rangle, \]

\[ Q := \langle \tilde{H}_{1,4}, \tilde{H}_{2,4}, \tilde{H}_{3,4}, \tilde{H}_{1,5}, \tilde{H}_{2,5}, \tilde{H}_{3,5}, \tilde{H}_{1,6}, \tilde{H}_{2,6}, \tilde{H}_{3,6} \rangle. \]

What we are looking for is the set of points \([a_{14} : \ldots : a_{36}]\) lying in the projective variety \( V(Q) \) but not in \( V(P) \) and for which \( \lambda < 0 \). It is known that

\[ \overline{V(Q) - V(P)} \subseteq V(Q : P), \]

where \( Q : P \) is the ideal quotient of \( Q \) by \( P \). In our case it is possible to show that \( Q : P = (1) \), therefore \( V(Q : P) = V((1)) = \emptyset \).

We have then proved that on \( S^3 \times S^3 \) there are no invariant coupled structures inducing the standard metric.

Case 2: the Jensen metric

Following the same idea of the previous case and looking at the entries of the matrix \( [2] \), we have now to consider the ideals \( P \) and

\[ R := \langle \tilde{H}_{1,5}, \tilde{H}_{1,6}, \tilde{H}_{2,4}, \tilde{H}_{2,6}, \tilde{H}_{3,4}, \tilde{H}_{3,5}, \tilde{H}_{2,5} - \tilde{H}_{3,6}, \tilde{H}_{3,6} - \tilde{H}_{1,4} \rangle \]

and look for those points lying in the projective variety \( V(R) \) but not in \( V(P) \) and for which \( \lambda < 0 \) and \( \tilde{H}_{1,1} = -2 \tilde{H}_{1,4} \). Now

\[ R : P = \langle a_{15}, a_{16}, a_{24}, a_{26}, a_{34}, a_{35}, a_{14} - a_{36}, a_{25} - a_{36} \rangle, \]

then

\[ V(R : P) = \{ [\gamma : 0 : 0 : 0 : \gamma : 0 : 0 : 0 : \gamma] : \gamma \in \mathbb{C} - \{0\} \} \]

is a point in \( \mathbb{CP}^8 \) and since \( \mathbb{C} \) is algebraically closed and \( R \) is a radical ideal

\[ V(R : P) = \overline{V(R) - V(P)}. \]
Moreover the other requested conditions are satisfied, indeed:

\[ \lambda = -3c^4 \gamma^4 < 0, \]
\[ \tilde{H}_{1,1} = -2\gamma^3 c^2 = -2\tilde{H}_{1,4}. \]

The coupled structures we are interested in are obtained when \( \gamma \) is a negative real number, in this case we have:

\[ \omega = \gamma(e^{14} + e^{25} + e^{36}), \]
\[ \psi_+ = c\gamma(e^{234} - e^{156} + e^{246} + e^{126} - e^{345}), \]
\[ \psi_- = \frac{c\gamma}{\sqrt{3}}(2e^{123} - e^{126} + e^{135} - e^{156} - e^{234} + e^{246} - e^{345} + 2e^{456}). \]

The forms \( \omega \) and \( \psi_+ \) are stable, \( \omega \wedge \psi_+ = 0 \) and the normalization condition implies

\[ c = \pm \sqrt{-\frac{2\gamma}{\sqrt{3}}}, \]

in both cases we have a nearly Kähler structure. \( \Box \)

4. Einstein half-flat structures on 6-solvmanifolds

In section 3 we saw that on the compact manifold \( S^3 \times S^3 \) there are examples of left invariant half-flat, double and nearly Kähler structures inducing an Einstein metric and that there are no left invariant coupled structures inducing any of the Einstein metrics known up to now. In this section we turn our attention to the noncompact homogeneous case.

It is well known that noncompact homogeneous Einstein manifolds have non positive Ricci curvature. Moreover, if the Ricci curvature is zero, then the Riemannian metric is flat (\( \mathbb{H} \)) and the manifold is isometric to the product of a flat torus and the Euclidean space. Up to now the only known examples of noncompact homogeneous Einstein manifolds are Einstein solvmanifolds, that is simply connected solvable Lie groups endowed with a left invariant Einstein metric. It has been conjectured by Alekseevskii that any noncompact homogeneous Einstein manifold might be of this kind (see \([3]\) for the statement of the conjecture and refer to the recent works \([2]\) and \([18]\) and the references therein for more details and the most recent results on it).

Let \( (S,h) \) be a solvmanifold, we can identify the left invariant metric \( h \) on the simply connected solvable Lie group \( S \) with the inner product \( h_0 \) determined by it on the solvable Lie algebra \( \mathfrak{s} \) of \( S \), the pair \( (\mathfrak{s}, h_0) \) is said a metric solvable Lie algebra. Two metric Lie algebras \( (\mathfrak{s}, h_0) \) and \( (\mathfrak{s}', h_0') \) are isomorphic if there exists a Lie algebra isomorphism \( F : \mathfrak{s} \rightarrow \mathfrak{s}' \) which is also an isometry of Euclidean spaces.

In \([19]\), Lauret showed that any Einstein solvmanifold is standard, i.e. the orthogonal complement \( \mathfrak{a} \) to \( \mathfrak{n} = [\mathfrak{s}, \mathfrak{s}] \) is always an abelian subalgebra of \( \mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} \). The dimension of \( \mathfrak{a} \) is the algebraic rank of \( \mathfrak{s} \).

The properties of standard Einstein solvmanifolds have been studied by Heber in \([16]\). In particular, he showed that a standard Einstein metric is unique up to isometry and scaling among left invariant metrics and that, up to metric Lie algebra isomorphisms, a standard metric Lie algebra with Einstein inner product is an Iwasawa-type algebra.
It was proved by Dotti in \cite{12} that solvmanifolds \((S, h)\) with unimodular solvable Lie group \(S\) and Einstein metric \(h\) are flat.

The 6 dimensional Einstein solvmanifolds have been classified by Nikitenko and Nikonorov in \cite{21}. The result is given in the next Theorem. Instead of the Lie algebra structure equations given (in the original formulation of the Theorem) by the nontrivial Lie brackets of the basis vectors, we write here the structure equations in terms of the Chevalley-Eilenberg differential of the basis 1-forms, since we will use these in our next computations.

**Theorem 4.1** \cite{21}. Let \((\mathfrak{s}, h)\) be a 6 dimensional nonunimodular metric solvable Lie algebra with Einstein inner product \(h\) such that \(\text{Ric}(h) = -r^2h\), where \(r > 0\). Then \((\mathfrak{s}, h)\) is isomorphic to one of the metric Lie algebras contained in Table 1. For each algebra \((e_1, \ldots, e_6)\) is a \(h\)-orthonormal basis with dual basis \((e^1, \ldots, e^6)\).

**Table 1.** 6 dimensional nonunimodular metric solvable Lie algebras with Einstein inner product \(h = \sum_{i=1}^{6} (e^i)^2\). The Lie algebra \(\mathfrak{s}_{10}\) depends on a parameter \(0 \leq t \leq \frac{1}{\sqrt{22}}\) and \(\mathfrak{s}_{12}\) depends on two parameters \(0 \leq s \leq t \leq 1\).

| \(\mathfrak{s}_n\) | Structure equations \((de^1, de^2, de^3, de^4, de^5, de^6)\) |
|-----------------|----------------------------------------------------------|
| \(\mathfrak{s}_1\) | \((\frac{r}{\sqrt{2}})_{16}, \frac{1}{\sqrt{2}} e^{26}, \frac{1}{\sqrt{2}} e^{36}, \frac{r}{\sqrt{2}} e^{46}, -\frac{r}{\sqrt{2}} e^{34} - \frac{r}{\sqrt{2}} e^{56}, 0\) |
| \(\mathfrak{s}_2\) | \(2r \sqrt{\frac{2}{105}} e^{16}, r \sqrt{\frac{2}{105}} e^{26}, \frac{r}{\sqrt{2}} e^{36}, -\frac{r}{\sqrt{2}} e^{46} - \frac{r}{\sqrt{2}} e^{34} + \frac{r}{\sqrt{2}} e^{56}, 0\) |
| \(\mathfrak{s}_3\) | \((\frac{r}{\sqrt{19}})_{16}, \frac{r}{\sqrt{2}} e^{26}, -r \sqrt{6 \frac{e^{13}}{60} + 3 \frac{e^{36}}{60}}, -r \sqrt{6 \frac{e^{13}}{60} + 3 \frac{e^{36}}{60}}, -r \frac{e^{14} - 2r + 2r^{23} + \frac{r}{\sqrt{2}} e^{56}}{19} 0\) |
| \(\mathfrak{s}_4\) | \((\frac{r}{\sqrt{2}})_{16}, 3 \frac{r}{\sqrt{2}} e^{26}, -r \sqrt{6 \frac{e^{13}}{60} + 3 \frac{e^{36}}{60}}, -r \sqrt{6 \frac{e^{13}}{60} + 3 \frac{e^{36}}{60}}, -r \frac{e^{14} + 2r^{23} + \frac{r}{\sqrt{2}} e^{56}}{19} 0\) |
| \(\mathfrak{s}_5\) | \((\frac{r}{3})_{16}, \frac{r}{\sqrt{2}} e^{26}, -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_6\) | \((\frac{r}{3})_{16}, 3 \frac{r}{\sqrt{2}} e^{26}, -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_7\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_8\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_9\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_{10}\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_{11}\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_{12}\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
| \(\mathfrak{s}_{13}\) | \((\sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \sqrt{4 \frac{e^{13}}{60} + r^{23}}, -r \frac{4 \sqrt{4 \frac{e^{13}}{60} + r^{23},} -r \frac{23 + r^{23}}{19} e^{56}}{19} 0\) |
Remark 4.2. All the solvable metric Lie algebras appearing in Table 1 are of Iwasawa-type. It is worth emphasizing here that for each of the $s_i$, the inner product, with respect to which the basis $(e_1, \ldots, e_6)$ is orthonormal, is the only one having the property of being Einstein. This follows from the result of Heber and the fact that two solvable metric Lie algebras of Iwasawa-type are isomorphic if and only if the corresponding solvmanifolds are isometric as Riemannian manifolds.

As a consequence of the previous Theorem, for the 6 dimensional homogeneous Einstein manifolds of nonpositive sectional curvature they showed what follows.

**Theorem 4.3** ([21]). Let $(M, h)$ be a 6 dimensional, connected, simply connected, homogeneous Einstein manifold of nonpositive sectional curvature, then it is symmetric or isometric to one of the solvmanifolds of negative sectional curvature generated by the metric Lie algebras $s_5, s_8$. Moreover, in the symmetric case $(M, h)$ is obtained as the solvmanifold corresponding to the metric Lie algebras $s_1, s_9, s_{10}$ for $t = \frac{1}{\sqrt{22}}, s_{11}, s_{12}$ for $(s, t) = (0, 0)$ and $(s, t) = (1, 1)$ and $s_{13}$.

Now we will focus on the problem of finding left invariant half-flat structures on 6 dimensional Einstein solvmanifolds inducing the Einstein (non Ricci-flat) metric, these are in 1-1 correspondence with half-flat structures inducing the Einstein inner product on 6 dimensional nonunimodular solvable metric Lie algebras.

It is worth recalling here an obstruction to the existence of half-flat structures on 6-dimensional Lie algebras showed by Freibert and Schulte-Hengesbach in [14]. This result is a refinement of the one obtained by Conti in [9].

**Proposition 4.4** ([14]). Let $g$ be a 6-dimensional Lie algebra with volume form $\Omega \in \Lambda^6(g^*)$. If there exists a nonzero $\alpha \in g^*$ such that

$$\alpha \wedge J^*_\rho \alpha \wedge \sigma = 0$$

for all closed 3-forms $\rho \in \Lambda^3(g^*)$ and closed 4-forms $\sigma \in \Lambda^4(g^*)$, where for any $X \in g$

$$J^*_\rho \alpha(X) \Omega = \alpha \wedge (i_X \rho) \wedge \rho,$$

then $g$ does not admit any half-flat $SU(3)$-structure.

In [14] and [15] the authors completely classified the left invariant half-flat structures on 6-dimensional decomposable Lie groups (using also the classification contained in [25]) and on 6-dimensional indecomposable Lie groups with 5-dimensional nilradical. These classifications will be useful in the proof of the following

**Theorem 4.5.** There are no half-flat $SU(3)$-structures inducing the Einstein metric on the rank 1 solvable metric Lie algebras $s_i, i = 1 \ldots 9$, and on the rank 2 solvable metric Lie algebra $s_{12}$ and there are no coupled $SU(3)$-structures inducing the Einstein metric on the rank 2 metric Lie algebras $s_{10}, s_{11}$ and on the rank 3 metric Lie algebra $s_{13}$.

**Proof.** We will prove the theorem as follows: in the list of Einstein solvable metric Lie algebras we first exclude the ones that do not admit a half-flat structure using the results of [14] and [15], then we will show the result by direct computations in the remaining cases.

The rank 1 Lie algebra $s_9$ is indecomposable and has abelian nilradical, therefore it does not admit any half-flat structure by Proposition 4 of [15]. By Theorem 2 of [15] we have that
the Lie algebras $\mathfrak{s}_i$ with $i = 1, 2, 4, 5, 7, 8$ do not admit any half-flat structure since they are isomorphic to the Lie algebras $A_{6,8}^{1,0}, A_{6,9}^{3/3}, A_{6,9}^{9/2}, A_{6,9}^{2/5,1}, A_{6,6}^{9/2}, A_{6,6}^{2/3,3/1}$ respectively, whereas the Lie algebras $\mathfrak{s}_3, \mathfrak{s}_6$ admit a half-flat structure since the former is isomorphic to $A_{6,9}^{6}$ and the latter to $A_{6,9}^{7,6}$. By Theorem 1 of [14] we also have that on $\mathfrak{s}_{13}$ there exist half-flat structures since it is isomorphic to $\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$. Finally, applying Proposition 4.4 to $\mathfrak{s}_{12}$ with $\alpha = e^6$ we obtain that this Lie algebra does not admit any half-flat structure.

We can now start with the second part of the proof, since this involves some long computations, we will give here the detailed proof only for the Lie algebra $\mathfrak{s}_6$ and we will give an idea on how the result is obtained in the other cases.

Let us consider the Lie algebra $\mathfrak{s}_6$, whose structure equations are given in Table 1. We consider a generic 2-form

$$\omega = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{16} + b_6 e^{23} + b_7 e^{24} + b_8 e^{25} + b_9 e^{26} + b_{10} e^{34} + b_{11} e^{35} + b_{12} e^{36} + b_{13} e^{45} + b_{14} e^{46} + b_{15} e^{56}$$

and a generic 3-form

$$\psi_+ = a_1 e^{123} + a_2 e^{124} + a_3 e^{125} + a_4 e^{126} + a_5 e^{134} + a_6 e^{135} + a_7 e^{136} + a_8 e^{145} + a_9 e^{146} + a_{10} e^{156} + a_{11} e^{234} + a_{12} e^{235} + a_{13} e^{236} + a_{14} e^{245} + a_{15} e^{246} + a_{16} e^{256} + a_{17} e^{345} + a_{18} e^{346} + a_{19} e^{356} + a_{20} e^{456},$$

where $a_i$ and $b_j$ are real constants, and we impose the conditions they have to satisfy in order to be a half-flat SU(3)-structure inducing the Einstein metric. What we have to do is to solve the equations obtained by imposing

$$\begin{cases}
\omega \wedge \psi_+ = 0 \\
d\psi_+ = 0 \\
d\omega^2 = 0
\end{cases}$$

under the assumptions $\lambda(\psi_+) < 0, \omega^3 \neq 0$. Moreover, since we are considering a basis which is orthonormal with respect to the Einstein metric (see Theorem 4.4), we have also to impose that the entries of the matrix $H$ associated to $h(\cdot, \cdot) = \omega(\cdot, J\psi_+ \cdot)$ with respect to the basis $(e_1, \ldots, e_6)$ satisfy

$$H_{i,j} = 0, \text{ for } 1 \leq i, j \leq 6 \text{ and } i \neq j,$$

$$H_{i,i} - H_{i+1,i+1} = 0, \text{ for } 1 \leq i \leq 5,$$

where $H_{i,j} = h(e_i, e_j)$. In this case we have 6 polynomial equations of degree 2 in the unknowns $a_i$ and $b_j$ coming from the first condition in (6), 13 linear equations in the $a_i$ from the second condition, 5 equations of degree 2 in the $b_j$ from the third condition and 35 equations of degree 3 in the $a_i$ and $b_j$ from (6), i.e. 59 equations in 35 unknowns that have to be solved under the two constraints given by $\lambda(\psi_+) < 0$ and $\omega^3 \neq 0$. Looking at the expression of $\lambda(\psi_+)$ we deduce that $a_6 \neq 0$, otherwise we would have $\lambda(\psi_+) = 0$. Moreover we have also to assume that $b_3, b_8$ and $b_{12}$ are nonzero, otherwise we would obtain a contradiction after solving some equations. Using these constraints we can solve the equations deriving from (6) and $H_{i,j} = 0$ for $i \neq j$. After some computations we obtain a diagonal form for the matrix $H$, imposing that the elements on the diagonal are all equal implies that $\lambda(\psi_+) = 4a_6^3$, which contradicts our previous assumption.
For the Lie algebra $s_3$ we can argue in a similar way, but instead of working on it, we can show the result on the Lie algebra $A_{6,99}$ since the computations are less involved. The structure equations of $A_{6,99}$ are given for example in [15], with respect to a basis $(f_1, \ldots, f_6)$ with dual basis $(f^1, \ldots, f^6)$ they are

\[(5f^{16} + f^{25} + f^{34}, 4f^{26} + f^{35}, 3f^{36} + f^{45}, 2f^{46}, f^{56}, 0).\]

Observe that the matrix $H$ associated to the Einstein inner product with respect to the basis $(f_1, \ldots, f_6)$ is not proportional to the identity but it is still diagonal, thus we still have to solve the equations $H_{i,j} = 0$ for $i \neq j$. One can show that having $b_1 = 0$ or $a_6 = 0$ will give a contradiction after solving some equations, therefore we have to assume that they are not zero. What we obtain after some computations under these constraints is that an entry of the matrix $H$ is zero if and only if $\omega^3 = 0$, which can not be possible.

We can now turn our attention to the Lie algebras $s_{10}, s_{11}$ and $s_{13}$, we will show that none of these admits a coupled structure inducing the Einstein metric. The way in which we proceed is similar to the one used for $s_6$, but in this case we will consider a generic $\omega$ of the form (3) and $\psi_+ = c d\omega$ for $c \in \mathbb{R} - \{0\}$. Observe that the second condition of (5) is satisfied since $\psi_+$ is now an exact 3-form. For each Lie algebra we will consider the structure equations given in Table 1.

Consider $s_{10}$, this is a 1-parameter family of Lie algebras depending on $t \in \left[0, \frac{1}{\sqrt{22}}\right]$. Since the entries in the diagonal of $H$ can not be zero, we have that $b_{10} \neq 0$ and $b_2 \neq \pm b_6$. The way in which we solve the equations depends on whether $t \in \left\{0, \frac{7}{15}, \frac{1}{\sqrt{22}}\right\}$ or not; if $t = \frac{1}{\sqrt{22}}$ we have that $H_{3,3} = 0$ and then $h$ is not an inner product, in the remaining cases we always reach a contradiction given by the fact that an equation we have to solve is zero if and only if $\lambda(\psi_+) = 0$.

For $s_{11}$ we have that $b_{10} \neq 0$, otherwise $H_{4,4} = 0$, using this fact we can solve some equations but we reach again a contradiction involving the condition $\lambda(\psi_+) < 0$.

In the last case $s_{13}$ we can see that $b_1, b_2, b_6 \neq 0$ and $b_3 \neq b_5 \sqrt{2}$ from the fact that the entries in the diagonal of $H$ can not be zero. Solving the equations under these assumptions we obtain again that the constraint on $\lambda(\psi_+)$ can not be satisfied.

From the fact that the class of coupled structures is a subclass of the half-flat one, we can use the result of the previous theorem together with Theorem 4.1 to obtain:

**Corollary 4.6.** Let $(\mathfrak{s}, h)$ be a 6 dimensional nonunimodular solvable metric Lie algebra with $h$ Einstein. Then on $\mathfrak{s}$ there are no coupled $SU(3)$-structures inducing the Einstein inner product.

Moreover, from the previous theorem and the Theorem 4.3 we obtain a constraint for the existence of coupled structures inducing Einstein metrics on homogeneous spaces:

**Corollary 4.7.** Let $(M, h)$ be a 6 dimensional, connected, simply connected, homogeneous Einstein manifold of nonpositive sectional curvature. Then there are no left invariant coupled $SU(3)$-structures on $M$ inducing the Einstein metric.

**Remark 4.8.** In this case it is in principle possible to use the properties of algebraic varieties to find solutions as we did in the proof of Theorem 4.6. However the computations here
are more involved since we have more unknowns (35 or 15 instead of 9) and more equations arising from the fact that some defining conditions for an SU(3)-structure that were easily verified in the case of $S^3 \times S^3$ have to be imposed in this case.

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