The logarithmic Minkowski inequality for cylinders

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Abstract In this paper, we prove that if $K$ is an $o$-symmetric cylinder and $L$ is an $o$-symmetric convex body in $\mathbb{R}^3$, then the logarithmic Minkowski inequality

$$\frac{1}{V(K)} \int_{S^2} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{3} \log \frac{V(L)}{V(K)}$$

holds, with equality if and only if $K$ and $L$ are relative cylinders.

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1. Introduction

The classical Brunn-Minkowski inequality is one of the core results within the Brunn-Minkowski theory (also called the mixed volume theory), which reads as follows: If $K$ and $L$ are convex bodies (compact convex sets with nonempty interiors) in Euclidean $n$-space $\mathbb{R}^n$ and $\lambda \in (0, 1)$, then

(1.1) \quad V((1 - \lambda)K + \lambda L) \geq (1 - \lambda) V(K)^{\frac{1}{n}} + \lambda V(L)^{\frac{1}{n}},

with equality if and only if $K$ and $L$ are homothetic (i.e., they coincide up to a translation and a dilatate). Here, $K + L = \{x + y : x \in K, y \in L\}$ is the Minkowski sum of convex bodies $K$ and $L$; and $V$ denotes the volume, i.e., $n$-dimensional Lebesgue measure. Because of the homogeneity of the Lebesgue measure, (1.1) is equivalent to say that if $\lambda \in (0, 1)$, then

(1.2) \quad V((1 - \lambda)K + \lambda L) \geq V(K)^{1-\lambda} V(L)^{\lambda},

with equality if and only if $K$ and $L$ are translates.

The Brunn-Minkowski inequality was actually inspired by issues around the isoperimetric problem and was for a long time considered to belong to geometry, where its significance was widely recognized. For example, it implies the clear fact that the function which gives the volumes of parallel hyperplane sections of a convex body is unimodal.
The fundamental geometric content of the Brunn-Minkowski inequality makes it a cornerstone of the Brunn-Minkowski theory, a beautiful and powerful apparatus for conquering all sorts of problems involving metric quantities such as volume and surface area.

If $h_K$ and $h_L$ are the support functions of convex bodies $K$ and $L$ (see their definitions in Section 2), the Minkowski combination $(1 - \lambda)K + \lambda L$ also can be expressed as an intersection of half-spaces,

$$(1 - \lambda)K + \lambda L = \bigcap_{u \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n : x \cdot u \leq (1 - \lambda)h_K(u) + \lambda h_L(u) \},$$

where $x \cdot u$ denotes the standard inner product of $x$ and $u$ in $\mathbb{R}^n$.

In the early 1960s, Firey [8] (see also Schneider [20, Section 9.1]) generalized the Minkowski combination of convex bodies to the $L_p$ Minkowski combination for each $p \geq 1$. In the 1990s, Lutwak [14, 15] showed that many classical results can be extended to the $L_p$ Brunn-Minkowski-Firey theory. If $K$ and $L$ are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, $p \in (1, \infty)$ and $\lambda \in (0, 1)$, then

$$(1.3) \quad V \big( (1 - \lambda) \cdot_p K +_p \lambda \cdot_p L \big)^{\frac{2}{p}} \geq (1 - \lambda)V(K)^{\frac{2}{p}} + \lambda V(L)^{\frac{2}{p}},$$

with equality if and only if $K$ and $L$ are dilatates. Here $(1 - \lambda) \cdot_p K = (1 - \lambda)\frac{1}{p} K$, $\lambda \cdot_p L = \lambda \frac{1}{p} L$. The $L_p$ combination $(1 - \lambda) \cdot_p K +_p \lambda \cdot_p L$ is defined by

$$(1.4) \quad (1 - \lambda) \cdot_p K +_p \lambda \cdot_p L = \bigcap_{u \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n : x \cdot u \leq ((1 - \lambda)h_K(u)^p + \lambda h_L(u)^p)^{\frac{1}{p}} \}.$$

The $L_p$ Brunn-Minkowski inequality (1.3) has an equivalent form: If $p > 1$, then

$$V \big( (1 - \lambda) \cdot_p K +_p \lambda \cdot_p L \big) \geq V(K)^{1-\lambda}V(L)^{\lambda},$$

with equality if and only if $K = L$. A unified approach used to generalize classical Brunn-Minkowski type inequalities to $L_p$ Brunn-Minkowski type inequalities, called the $L_p$ transference principle, is refined in the paper [23].

The definition (1.4) actually makes sense for all $p > 0$. The case where $p = 0$ is the limiting case given by (1.4) and is represented as

$$(1.5) \quad (1 - \lambda) \cdot K +_0 \lambda \cdot L = \bigcap_{u \in \mathbb{S}^{n-1}} \{ x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda}h_L(u)^{\lambda} \},$$

which is called the logarithmic Minkowski combination of convex bodies $K$ and $L$. The most significance conjecture on the logarithmic Minkowski combination is the logarithmic Brunn-Minkowski inequality.

Böröczky, Lutwak, Yang and Zhang [1] initially posed the logarithmic Brunn-Minkowski conjecture: If $K$ and $L$ are $o$-symmetric convex bodies in $\mathbb{R}^n$, then the inequality

$$V \big( (1 - \lambda) \cdot K +_0 \lambda \cdot L \big) \geq V(K)^{1-\lambda}V(L)^{\lambda}$$
holds. The logarithmic Brunn-Minkowski inequality is stronger than the classical Brunn-Minkowski inequality (1.1) and has an equivalent form, which is called the logarithmic Minkowski inequality (see [2])

\[
\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}.
\]

Here, \(dV_K = \frac{1}{n} h_K dS_K\) denotes the cone-volume measure of \(K\), where \(S_K\) is the surface area measure of \(K\). For the definitions of the cone-volume measure and the surface area measure, see Section 2.

In 2012, Böröczky, Lutwak, Yang and Zhang [1] showed the equivalence and established the planar logarithmic Brunn-Minkowski inequality when \(K\) and \(L\) are \(o\)-symmetric convex bodies in the plane.

Turning to higher dimensions, besides the cases of unconditional convex bodies by Saroglou [19] and complex bodies by Rotem [18], the conjecture was proved by Kolesnikov and Milman [13], when \(K\) is close to be an ellipsoid in the sense of Hausdorff metric by a combination of the local estimates. By using the continuity method, Chen, Huang, Li, and Liu [6, Corollary 1.1] proved the conjecture when \(K\) and \(L\) are \(o\)-symmetric convex bodies, and \(K\) is in a small \(C^0\) neighborhood of the unit ball. In [17], Putterman gave a proof of the equivalence of the inequality (1.6) to the local version of the inequality studied by Colesanti, Livshyts, and Marsiglietti [5] and by Kolesnikov and Milman [13]. The local form of the logarithmic Brunn-Minkowski conjecture for zonoids was established by van Handel [21], where a variant of the Bochner method is used in the proof. For more progress, see [3, 4, 7, 12, 16, 22].

Write \(K_{os}^n\) for the set of \(o\)-symmetric convex bodies in \(\mathbb{R}^n\). \(K\) is called a cylinder in \(\mathbb{R}^n\), if there exist convex sets \(K_i, i = 1, \ldots, m, 1 < m \leq n\), with \(\dim K_i \geq 1\) and \(\sum_{i=1}^m \dim K_i = n\), such that \(K = \sum_{i=1}^m K_i\). We call convex bodies \(K\) and \(L\) in \(\mathbb{R}^n\) relative cylinders, if \(K\) and \(L\) are cylinders with \(K = \sum_{i=1}^m K_i\) and \(L = \sum_{i=1}^m L_i\), such that \(K_i\) and \(L_i\) are dilatates, \(i = 1, \ldots, m\).

We prove the following results in this article.

**Theorem 1.1.** Suppose that \(K, L \in K_{os}^3\) and \(K\) is a cylinder. Then

\[
\frac{1}{V(K)} \int_{S^2} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{3} \log \frac{V(L)}{V(K)},
\]

with equality if and only if \(K\) and \(L\) are relative cylinders.

**Theorem 1.2.** For any \(n \geq 1\). The following assertions are equivalent.

(1) If \(K, L \in K_{os}^n\), then

\[
\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)},
\]
with equality if and only if $K$ and $L$ are dilatates, or $K$ and $L$ are relative cylinders.

(2) If $K, L \in \mathcal{K}_{os}^n$ and $V_K = V_L$, then $K = L$, or $K$ and $L$ are relative cylinders.

This article is organized as follows. Notations and necessary facts from the Brunn-Minkowski theory are listed in Section 2. To prove Theorem 1.2, some critical lemmas are provided in Section 3. The main results are shown in Section 4.

2. Preliminaries

For quick reference, we collect some basic facts on convex bodies. Good references are the books by Gardner [10], Gruber [11] and Schneider [20].

Let $S^{n-1}$ be the unit sphere of $\mathbb{R}^n$. Write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$. Let $\mathcal{K}_o^n \subseteq \mathcal{K}^n$ be the set of convex bodies with the origin $o$ in their interiors, and $\mathcal{K}_{os}^n \subseteq \mathcal{K}_o^n$ be the set of $o$-symmetric convex bodies.

Write $\text{int} K$ and $\text{bd} K$ for the interior and boundary of a set $K$, respectively. Write $\text{relint} K$ and $\text{relbd} K$ for the relative interior and relative boundary of $K$, that is, the interior and boundary of $K$ relative to its affine hull, respectively.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of convex set $K$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$  

From the definition, it follows immediately that, for $T \in \text{GL}(n)$, the support function of $TK = \{Tx : x \in K\}$ is given by

$$h_{TK}(x) = h_K(T^t x).$$  

Denote by $C(S^{n-1})$ the set of continuous functions defined on $S^{n-1}$, which is equipped with the metric induced by the maximal norm. Write $C^+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$. Write $C_e(S^{n-1})$ for the set of even functions in $C(S^{n-1})$. Write $C^+_e(S^{n-1})$ for the set of strictly positive even functions in $C(S^{n-1})$.

For nonnegative $f \in C(S^{n-1})$, define

$$[f] = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq f(u)\}.$$  

The set is called the Aleksandrov body (also known as the Wulff shape) of $f$. Obviously, $[f]$ is a compact convex set containing the origin. For a compact convex set containing the origin, say $K$, we have $K = [h_K]$. If $f \in C^+(S^{n-1})$, then $[f] \in \mathcal{K}_o^n$.

The Aleksandrov convergence lemma reads: If the sequence $\{f_j\}_j \subseteq C^+(S^{n-1})$ converges uniformly to $f \in C^+(S^{n-1})$, then $\lim_{j \to \infty} [f_j] = [f]$.

Denote by $V(K)$ the volume of convex body $K$ in $\mathbb{R}^n$. If $K$ is a lower dimensional convex set in $\mathbb{R}^n$, we write $|K|$ for the Hausdorff measure of $K$.

Write $K|_\xi$ for the image of orthogonal projection of $K$ onto the subspace $\xi$ of $\mathbb{R}^n$.  

Let $K \in \mathcal{K}^n$. The surface area measure $S_K$ of $K$ is a Borel measure on $\mathbb{S}^{n-1}$ defined for a Borel set $\omega \subseteq \mathbb{S}^{n-1}$ by
\[ S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)), \]
where $\nu_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map of $K$, defined on $\partial' K$, the set of points of $\partial K$ that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Let $K \in \mathcal{K}_0^n$. Its cone-volume measure $V_K$ is a Borel measure on $\mathbb{S}^{n-1}$ defined for a Borel set $\omega \subseteq \mathbb{S}^{n-1}$ by
\[ V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x). \]
Thus, $V_K(\cdot) = \frac{1}{n} h_K S_K(\cdot)$.

For $f \in C(\mathbb{S}^{n-1})$ and the general linear transformation $T \in \text{GL}(n)$, it follows that
\[
\int_{\mathbb{S}^{n-1}} f(u) dS_{TK}(u) = |\det T| \int_{\mathbb{S}^{n-1}} f(\langle T^{-t}u \rangle) |T^{-t}u| dS_K(u).
\]
Here, $\langle T^{-t}u \rangle = T^{-t}u/|T^{-t}u|$ denotes the unit vector of $T^{-t}u$.

The following form of Aleksandrov’s lemma ([20, Theorem 7.5.3]) and the logarithmic Minkowski inequality in the plane ([1, Theorem 1.4]) will be needed.

**Lemma 2.1.** Suppose $I \subseteq \mathbb{R}$ is an open interval containing 0 and that the function $h_t = h(t, u) : I \times \mathbb{S}^{n-1} \to (0, \infty)$ is continuous. If, as $t \to 0$, the convergence in
\[ \frac{h_t - h_0}{t} \to f = \frac{\partial h_t}{\partial t} \bigg|_{t=0}, \]
is uniform on $\mathbb{S}^{n-1}$, and if $K_t$ denotes the Wulff shape of $h_t$, then
\[ \lim_{t \to 0} \frac{V(K_t) - V(K_0)}{t} = \int_{\mathbb{S}^{n-1}} f \, dS_{K_0}. \]

**Lemma 2.2.** If $K$ and $L$ are $o$-symmetric convex bodies in the plane, then
\[
\int_{\mathbb{S}^1} \log \frac{h_L}{h_K} \, dV_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)},
\]
with equality if and only if, either $K$ and $L$ are dilatates or $K$ and $L$ are parallelograms with parallel sides.

### 3. Several Lemmas

In this section, we give some critical lemmas on the logarithmic Minkowski inequality and the cone-volume measure.

**Lemma 3.1.** If $K, L$ are convex bodies in $\mathbb{R}^n$ containing the origin in their interiors, then for $T \in \text{GL}(n)$,
\[
\frac{1}{V(TK)} \int_{\mathbb{S}^{n-1}} \log \frac{h_{TL}}{h_{TK}} \, dV_{TK} = \frac{1}{V(K)} \int_{\mathbb{S}^{n-1}} \log \frac{h_L}{h_K} \, dV_K.
\]
Proof. **Step 1.** We show that \((1 - \lambda) \cdot TK +_0 \lambda \cdot TL = T((1 - \lambda) \cdot K +_0 \lambda \cdot L)\) for any \(T \in \text{GL}(n)\). By (2.1), it follows that

\[
(1 - \lambda) \cdot TK +_0 \lambda \cdot TL = \bigcap_{u \in S^{n-1}} \{ x \in \mathbb{R}^n : x \cdot u \leq h_{TK}^{1-\lambda}(u)h_{TL}^\lambda(u) \}
\]

\[
= \bigcap_{u \in S^{n-1}} \{ x \in \mathbb{R}^n : T^{-1}x \cdot T^u \leq h_K^{1-\lambda}(T^u)h_L^\lambda(T^u) \}
\]

\[
= T \bigcap_{u \in S^{n-1}} \{ T^{-1}x \in \mathbb{R}^n : -x \cdot T^u \leq h_K^{1-\lambda}(T^u)h_L^\lambda(T^u) \}
\]

\[
= T((1 - \lambda) \cdot K +_0 \lambda \cdot L).
\]

**Step 2.** We show the relationship between the integral and the differential of \(\log V((1 - \lambda) \cdot K +_0 \lambda \cdot L)\). By Lemma 2.1, it follows that

\[
\frac{d}{d\lambda}(\log V((1 - \lambda) \cdot K +_0 \lambda \cdot L)) \bigg|_{\lambda=0} = \frac{1}{V(K)} \frac{d}{d\lambda}(V((1 - \lambda) \cdot K +_0 \lambda \cdot L)) \bigg|_{\lambda=0}
\]

\[
= \frac{1}{V(K)} \int_{S^{n-1}} h_K \log \frac{h_L}{h_K} dS_K
\]

\[
= \frac{n}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K.
\]

Combining these with that \(V(T((1 - \lambda) \cdot K +_0 \lambda \cdot L)) = |\det T|V((1 - \lambda) \cdot K +_0 \lambda \cdot L)\) for \(T \in \text{GL}(n)\), the desired equation is proved. 

The following lemma is essentially contained in Lemma 4.1 of [2]. For the completeness of this article, we present its proof in the following.

**Lemma 3.2.** Suppose that \(K, L \in \mathcal{K}_{os}^n\). If \(L\) solves the extremum problem

\[
\inf \left\{ \frac{1}{V(K)} \int_{S^{n-1}} \log h_Q dV_K - \frac{1}{n} \log V(Q) : Q \in \mathcal{K}_{os}^n \right\},
\]

then \(\frac{V_k(\cdot)}{V(L)} = \frac{V_K(\cdot)}{V(K)}\).

**Proof. Step 1.** Consider the minimization problem on \(C^+_e(S^{n-1})\),

\[
\inf \{ \Phi(f) : f \in C^+_e(S^{n-1}) \},
\]

where the functional \(\Phi : C^+_e(S^{n-1}) \to (0, \infty)\) is defined by

\[
\Phi(f) = \frac{1}{V(K)} \int_{S^{n-1}} \log f dV_K - \frac{1}{n} \log V([f]),
\]

for \(f \in C^+_e(S^{n-1})\). Since the functional \(V : C^+_e(S^{n-1}) \to (0, \infty)\) is continuous, it follows that the functional \(\Phi : C^+_e(S^{n-1}) \to (0, \infty)\) is continuous as well.
Step 2. We show that \( h_L \) solves \( \inf \{ \Phi(f) : f \in C^+_e(S^{n-1}) \} \).

In fact, let \( f \in C^+_e(S^{n-1}) \) and \( Q \) be the Wulff shape of \( f \). Then \( V([f]) = V(Q) = V([h_Q]) \) but \( h_Q \leq f \). Therefore, \( \Phi(h_Q) \leq \Phi(f) \). Hence,

\[
\inf \{ \Phi(h_Q) : Q \in K_{os}^n \} = \inf \{ \Phi(f) : f \in C^+_e(S^{n-1}) \}.
\]

Since \( \Phi(h_Q) = \frac{1}{V(K)} \int_{S^{n-1}} \log h_Q dV_K - \frac{1}{n} \log V(Q) \) and the hypothesis of this lemma is that \( L \) solves the left infimum, it yields that \( h_L \) solves \( \inf \{ \Phi(f) : f \in C^+_e(S^{n-1}) \} \).

Step 3. Suppose that \( g \in C_e(S^{n-1}) \) is arbitrary but fixed. Consider the family \( h_t \in C^+_e(S^{n-1}) \), where the function \( h_t = h(t, \cdot) : \mathbb{R} \times S^{n-1} \rightarrow (0, \infty) \) is defined by

\[
h_t = h(t, \cdot) = h_L e^{tg},
\]

and let \( L_t \) denote the Wulff shape of \( h_t \). Note that \( h_0 = h_L \).

Since \( g \) is bounded on \( S^{n-1} \), for \( h_t = h_L e^{tg} \), the convergence

\[
\frac{h_t - h_0}{t} \rightarrow gh_L \text{, as } t \rightarrow 0
\]

is uniformly on \( S^{n-1} \), via Lemma 2.1, it follows that

\[
\frac{d}{dt} V(L_t) \bigg|_{t=0} = \int_{S^{n-1}} gh_L dS_L.
\]

So the function \( t \rightarrow \Phi(h_t) \) is differentiable at \( t = 0 \). Combining this, with the fact that \( h_0 = h_L \) attains the minimum, it yields that

\[
0 = \frac{d}{dt} \Phi(h_t) \bigg|_{t=0} = \frac{1}{V(K)} \int_{S^{n-1}} g dV_K - \frac{1}{n} \frac{1}{V(L)} \frac{d}{dt} V(L_t) \bigg|_{t=0} = \frac{1}{V(K)} \int_{S^{n-1}} g dV_K - \frac{1}{n} \frac{1}{V(L)} \int_{S^{n-1}} gh_L dS_L.
\]

Since the above equation holds for arbitrary \( g \in C_e(S^{n-1}) \), it follows that

\[
\frac{V_L(\cdot)}{V(L)} = \frac{1}{nV(L)} h_L S_L(\cdot) = \frac{V_K(\cdot)}{V(K)},
\]

as desired. \( \square \)

Lemma 3.3. Suppose that \( \xi_1 \) and \( \xi_2 \) are orthogonal complementary subspaces in \( \mathbb{R}^n \) with \( 0 < \text{dim } \xi_i = k_i < n, i = 1, 2 \). If \( K_1 \) and \( K_2 \) are convex bodies in \( \xi_1 \) and \( \xi_2 \) containing the origin in their interiors, respectively, then the cone-volume measure of \( K_1 + K_2 \) in \( \mathbb{R}^n \) is concentrated on \( (\xi_1 \cup \xi_2) \) and

\[
(3.1) \quad V_{K_1+K_2}(\cdot) = \frac{k_1}{n} |K_2| V_{K_1}(\cdot) + \frac{k_2}{n} |K_1| V_{K_2}(\cdot).
\]
Proof. Observe that

\( (3.2) \) bd \((K_1 + K_2) = (\text{relbd } K_1 + \text{relint } K_2) \cup (\text{relbd } K_2 + \text{relint } K_1) \cup (\text{relbd } K_1 + \text{relbd } K_2). \)

Consider \( \mathbb{R}^n \) as the orthogonal sum of \( \xi_1 \) and \( \xi_2 \). Write \( y = (y_1, y_2) \in \mathbb{R}^n \) and identify \( y_1 \) with \((y_1, 0)\) and \( y_2 \) with \((0, y_2)\).

Assume that \( y_1 + y_2 \in \text{relbd } K_1 + \text{relint } K_2 \) with a unique unit normal. In the following, we show that \( \nu_{K_1+K_2}(y_1 + y_2) = \nu_{K_1+K_2}(y_1) = \nu_{K_1}(y_1) \). In fact, since \( y_2, o \in \text{relint } K_2 \), it follows that \( y_1 + y_2, y_1 \in y_1 + \text{relint } K_2 = \text{relint } (y_1 + K_2) \). Note that \( \text{relint } (y_1 + K_2) \) is relative open. So \( \nu_{K_1+K_2}(y_1 + y_2) = \nu_{K_1+K_2}(y_1) \) by Theorem 2.1.2 of [20] and the definition of the normal cone of convex bodies. Since \( (K_1 + K_2)|_{\xi_1} = K_1 \) and \( \nu_{K_1+K_2}(y_1) \in \xi_1 \), it follows that \( h_{K_1+K_2}(\nu_{K_1+K_2}(y_1)) = h_{K_1}(\nu_{K_1+K_2}(y_1)) \). Hence, \( y_1 \cdot \nu_{K_1+K_2}(y_1) = h_{K_1}(\nu_{K_1+K_2}(y_1)). \)

From the fact that \( y_1 \in \text{relbd } K_1 \) and the definition of the support function of convex bodies, it follows that \( \nu_{K_1+K_2}(y_1) = \nu_{K_1}(y_1) \) as desired.

Suppose that \( \omega \subseteq \mathbb{S}^{n-1} \cap \xi_1 \). Then \( \nu_{K_1+K_2}^{-1}(\omega) \subseteq \text{relbd } K_1 + \text{relint } K_2 \). Combining this, \( (K_1 + K_2)|_{\xi_1} = K_1 \), it follows that

\[
V_{K_1+K_2}(\omega) = \frac{1}{n} \int_{\omega} h_{K_1+K_2}(u) dS_{K_1+K_2}(u) = \frac{1}{n} \int_{\omega} h_{K_1}(u) dS_{K_1+K_2}(u) = \frac{1}{n} \int_{\omega} \nu_{K_1}^{-1}(\nu_{K_1}(y_1))d\mathcal{H}^{n-1}(y_1 + y_2) = \frac{1}{n} \int_{\omega} \nu_{K_1}^{-1}(\nu_{K_1}(y_1))K_2|d\mathcal{H}^{k_1-1}(y_1) = \frac{1}{n} \int_{\omega} h_{K_1}(u)K_2|dS_{K_1}(u) = \frac{k_1}{n} |K_2| \frac{1}{k_1} \int_{\omega} h_{K_1}(u) dS_{K_1}(u) = \frac{k_1}{n} |K_2| V_{K_1}(\omega).
\]

Similarly, we obtain \( V_{K_1+K_2}(\omega) \) as desired.

Since the \((n-1)\)-dimensional Hausdorff measure of \( \text{relbd } K_1 + \text{relbd } K_2 \) is zero, together with \((3.2)\), it follows that the surface area measure \( S_{K_1+K_2} \) is concentrated on \((\xi_1 \cup \xi_2)\), and thus the cone-volume measure \( V_{K_1+K_2} \) is as well. Hence,

\[
V_{K_1+K_2}(\cdot) = \frac{k_1}{n} |K_2| V_{K_1}(\cdot) + \frac{k_2}{n} |K_1| V_{K_2}(\cdot),
\]

as desired. \qed
4. The Logarithmic Minkowski Inequality

In this section, we present the proof of main results in this paper.

**Theorem 4.1.** Suppose that $K, L \in \mathcal{K}_{os}^3$ and $K$ is a cylinder. Then
\[
\frac{1}{V(K)} \int_{S^2} \log \frac{h_L}{h_K} \, dV_K \geq \frac{1}{3} \log \frac{V(L)}{V(K)},
\]
with equality if and only if $K$ and $L$ are relative cylinders.

**Proof.** Without loss of generality, by Lemma 3.1, assume that $K = \bar{K} + a[-u_0, u_0]$, where $\bar{K} \subseteq u_0^\perp$, $a > 0$, $u_0 \in S^2$. Then the cone-volume measure of $K$ is concentrated on $(u_0^\perp \cup \{\pm u_0\})$ and
\[
V_K(\cdot) = \frac{1}{3} a|\bar{K}|(\delta_{u_0}(\cdot) + \delta_{-u_0}(\cdot)) + \frac{4}{3} a V_{\bar{K}}(\cdot).
\]

For any $L \in \mathcal{K}_{os}^3$, it follows that $L \subseteq L|_{u_0^\perp} + L|_{lu_0}$. Here, $lu_0$ denotes the 1-dimensional subspace spanned by $u_0$. Combining this, that $h_L(u_0) = h_L(-u_0)$ and $h_{L|_{u_0^\perp}} = h_L$ on $u_0^\perp$, the inequality (1.6), that $K|_{u_0^\perp} = \bar{K}$ and $V(K) = 2h_K(u_0)|\bar{K}| = 2a|\bar{K}|$, it follows that
\[
\int_{S^2} \log \frac{h_L}{h_K} \, dV_K = \frac{1}{3} a|\bar{K}| \left( \log \frac{h_L(u_0)}{h_K(u_0)} + \log \frac{h_L(-u_0)}{h_K(-u_0)} \right) + \frac{4}{3} a \int_{S^2 \cap u_0^\perp} \log \frac{h_L}{h_K} \, dV_{\bar{K}}
\]
\[
= \frac{2}{3} a|\bar{K}| \log \frac{h_L(u_0)}{h_K(u_0)} + \frac{4}{3} a \int_{S^2 \cap u_0^\perp} \log \frac{h_{L|_{u_0^\perp}}}{h_K} \, dV_{\bar{K}}
\]
\[
\geq \frac{2}{3} a|\bar{K}| \log \frac{h_L(u_0)}{h_K(u_0)} + \frac{4}{3} a \frac{|\bar{K}|}{2} \log \frac{|L|_{u_0^\perp}|}{|K|}
\]
\[
= \frac{1}{3} V(K) \log \frac{h_L(u_0)}{h_K(u_0)} + \frac{1}{3} V(K) \log \frac{|L|_{u_0^\perp}|}{|K|}
\]
\[
= \frac{1}{3} V(K) \log \frac{h_L(u_0)|L|_{u_0^\perp}|}{h_K(u_0)|K|}
\]
\[
\geq \frac{1}{3} V(K) \log \frac{V(L)}{V(K)}.
\]

Assume that the equality holds. Then $V(L) = 2h_L(u_0)|L|_{u_0^\perp}|$. Thus, the inclusion $L \subseteq L|_{u_0^\perp} + L|_{lu_0}$ implies that $L = L|_{u_0^\perp} + L|_{lu_0}$ is a cylinder. Meanwhile, the equality of the logarithmic Minkowski inequality for $\bar{K}$ and $L|_{u_0^\perp}$ holds, which implies that $\bar{K}$ and $L|_{u_0^\perp}$ are dilatates, or $\bar{K}$ and $L|_{u_0^\perp}$ are parallelograms with parallel sides, i.e., relative cylinders. So $K$ and $L$ are relative cylinders. \(\square\)

The final theorem gives the relationship between the logarithmic Minkowski inequality and the uniqueness of the cone-volume measure.
Theorem 4.2. Let $n \geq 1$. The following assertions are equivalent.

(1) If $K, L \in \mathcal{K}^n_{os}$, then

$$\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)},$$

with equality if and only if $K$ and $L$ are dilatates, or $K$ and $L$ are relative cylinders.

(2) If $K, L \in \mathcal{K}^n_{os}$ and $V_K = V_L$, then $K = L$, or $K$ and $L$ are relative cylinders.

Proof. Assume that the assertion (1) holds. That is, for $K, L \in \mathcal{K}^n_{os}$,

$$\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)},$$

and

$$\frac{1}{V(L)} \int_{S^{n-1}} \log \frac{h_K}{h_L} dV_L \geq \frac{1}{n} \log \frac{V(K)}{V(L)},$$

when exchanging $K$ and $L$. Then

$$\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{V(L)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_L.$$

The equality holds if and only if $K$ and $L$ are dilatates, or $K$ and $L$ are relative cylinders. From that $V_K = V_L$, it follows that

$$\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{V(L)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_L = \frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K.$$

So the equality holds, which implies that $K = L$, or $K$ and $L$ are relative cylinders. Thus, the assertion (1) implies the assertion (2).

In the following, we show that the assertion (2) implies (1) by induction of dimension $n$. The proof is basically in the same spirit as Theorem 7.1 in [1].

Let $n = 1$. For $K = [-a, a]$ and $L = [-b, b]$ where $a, b > 0$, since the cone-volume measure $V_K, V_L$ are both concentrated on two directions $\{-1, 1\}$, $h_K(\pm 1) = a, h_L(\pm 1) = b$ and $V(K) = 2a, V(L) = 2b$, it follows that

$$\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K = \frac{1}{2a} \log \frac{b}{a} = \log \frac{2b}{2a} = \log \frac{V(L)}{V(K)}.$$

In this case, the equality always holds. So the implication for dimension 1 naturally holds.

Assume the implication (2) $\Rightarrow$ (1) is proved, when the space dimension is not greater than $n - 1$. In the following, it suffices to prove the assertion (1) for dimension $n$ under the assumption that the assertion (2) for dimension $n$ holds. We divide it into two cases.

Case 1. Assume that $K$ is a cylinder. Let $K = K_1 + K_2$, $K_i \subseteq \xi_i \in G_{n,k_i}, i = 1, 2$ and $k_1 + k_2 = n$. Without loss of generality, by Lemma 3.1, assume that $\xi_2 = \xi_1^\perp$. In fact, if $\xi_2 \neq \xi_1^\perp$, then there exists a linear transform $T \in \text{GL}(n)$ such that $T\xi_1 = \xi_1$ and
\[ T \xi_2 = \xi_1^\perp. \] So \( TK = TK_1 + TK_2 \) with \( TK_1 \subseteq \xi_1 \) and \( TK_2 \subseteq \xi_1^\perp \). Then the cone-volume measure \( V_K \) is concentrated on \( (\xi_1 \cup \xi_2) \), and
\[
dV_K = \frac{k_1}{n} |K_2| dV_{K_1} + \frac{k_2}{n} |K_1| dV_{K_2}.
\]
Here, recall that \(|K_i|\) denotes the \( k_i \)-dimensional volume of \( K_i \subseteq \xi_i \) and \( V_{K_i} \) denotes the \( k_i \)-dimensional cone-volume measure of \( K_i \), \( i = 1, 2 \).

In the following, we first show that the assertion (2) for dimension \( k_i \) in \( \xi_i \) holds. In fact, for convex bodies \( M_i, N_i \subseteq \xi_i \subseteq G_{n,k_i} \) with \( V_{M_i} = V_{N_i}, \ i = 1, 2 \), it suffices to prove that \( M_i = N_i \), or \( M_i \) and \( N_i \) are relative cylinders. Since \( M_1 + M_2 \) is a convex body in \( \mathbb{R}^n \), whose cone-volume measure is concentrated on \( S_{n-1} \cap (\xi_1 \cup \xi_2) \), it follows that
\[
dV_{M_1+M_2} = \frac{k_1}{n} |M_2| dV_{M_1} + \frac{k_2}{n} |M_1| dV_{M_2}.
\]
Similarly,
\[
dV_{N_1+N_2} = \frac{k_1}{n} |N_2| dV_{N_1} + \frac{k_2}{n} |N_1| dV_{N_2}.
\]
Hence, \( V_{M_1+M_2} = V_{N_1+N_2} \). By the assumption that the assertion (2) for dimension \( n \) holds, it follows that \( M_1 + M_2 = N_1 + N_2 \), or \( M_1 + M_2 \) and \( N_1 + N_2 \) are relative cylinders. Since \( (M_1 + M_2)|_{\xi_i} = M_i \), \( (N_1 + N_2)|_{\xi_i} = N_i \) and \( V_{M_i} = V_{N_i} \), it follows that \( M_i = N_i \), or \( M_i \) and \( N_i \) are relative cylinders by the construction of \( M_1 + M_2 \) and \( N_1 + N_2 \). So the assertion (2) for dimension \( k_i \) in \( \xi_i \) holds. Therefore the logarithmic Minkowski inequality for dimension \( k_i \) in \( \xi_i \) is proved by the induction.

Next, we show the assertion (1) for dimension \( n \) holds when \( K \) is a cylinder. For any convex body \( L \in \mathcal{K}^n_{os} \), it follows that \( L \subseteq L|_{\xi_1} + L|_{\xi_2} \). Combining this, (3.1), that \( h_{L|_{\xi_1}} = h_L \) on \( \xi_i \) and \( K|_{\xi_i} = K_i \), the logarithmic Minkowski inequality for dimension \( k_i \) in \( \xi_i \), that \( V(K) = |K_1||K_2| \) and \( V(L) \leq |L|_{\xi_1}|L|_{\xi_2} \), it follows that
\[
\int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K = \frac{k_1}{n} |K_2| \int_{S^{n-1} \cap \xi_1} \log \frac{h_L}{h_K} dV_{K_1} + \frac{k_2}{n} |K_1| \int_{S^{n-1} \cap \xi_2} \log \frac{h_L}{h_K} dV_{K_2}
\]
\[
= \frac{k_1}{n} |K_2| \int_{S^{n-1} \cap \xi_1} \log \frac{h_{L|_{\xi_1}}}{h_{K|_{\xi_1}}} dV_{K_1} + \frac{k_2}{n} |K_1| \int_{S^{n-1} \cap \xi_2} \log \frac{h_{L|_{\xi_2}}}{h_{K|_{\xi_2}}} dV_{K_2}
\]
\[
= \frac{k_1}{n} |K_2| \int_{S^{n-1} \cap \xi_1} \log \frac{|L|_{\xi_1}}{|K_1|} dV_{K_1} + \frac{k_2}{n} |K_1| \int_{S^{n-1} \cap \xi_2} \log \frac{|L|_{\xi_2}}{|K_2|} dV_{K_2}
\]
\[
\geq \frac{k_1}{n} |K_2| |K_1| \frac{1}{k_1} \log \left| \frac{|L|_{\xi_1}}{|K_1|} \right| + \frac{k_2}{n} |K_1| |K_2| \frac{1}{k_2} \log \left| \frac{|L|_{\xi_2}}{|K_2|} \right|
\]
\[
= \frac{1}{n} |K_1| |K_2| \log \left| \frac{|L|_{\xi_1}||L|_{\xi_2}}{|K_1||K_2|} \right|
\]
\[
\geq \frac{V(K)}{n} \log \frac{V(L)}{V(K)}.
\]
Thus, the logarithmic Minkowski inequality for dimension $n$ is proved. Assume that the equality holds. Then $V(L) = |L|_{\xi_1} |L|_{\xi_2}$. Thus, the inclusion $L \subseteq L|_{\xi_1} + L|_{\xi_2}$ implies that $L = L|_{\xi_1} + L|_{\xi_2}$ is a cylinder. Meanwhile, the equality of the $k_i$-dimensional logarithmic Minkowski inequality for $K_i$ and $L|_{\xi_i}$ holds, which implies that $K_i$ and $L|_{\xi_i}$ are dilatates, or $K_i$ and $L|_{\xi_i}$ are relative cylinders for $i = 1, 2$. So $K$ and $L$ are relative cylinders.

Case 2. Assume that $K$ is not a cylinder. Then $V_K$ satisfies the strict subspace concentration inequality. By Theorem 6.3 in Page 845 of [2], there exists a convex body $L_0 \in K_{\text{os}}^n$ such that $L_0$ is the solution to the extremum problem

$$\inf \left\{ \frac{1}{V(K)} \int_{S^{n-1}} \log h_Q dV_K - \frac{1}{n} \log V(Q) : Q \in K_{\text{os}}^n \right\}.$$

Moreover, the normalized cone-volume measure $\frac{V_{L_0}(\cdot)}{V(L_0)} = \frac{V_K(\cdot)}{V(K)}$ by Lemma 3.2. Together with the assertion (2), it follows that $L_0$ and $K$ are dilatates. Let $L_0 = \lambda K$, $\lambda > 0$. Then for any $L \in K_{\text{os}}^n$,

$$\frac{1}{V(K)} \int_{S^{n-1}} \log h_L dV_K - \frac{1}{n} \log V(L) \geq \frac{1}{V(K)} \int_{S^{n-1}} \log h_{L_0} dV_K - \frac{1}{n} \log V(L_0) = \frac{1}{V(K)} \int_{S^{n-1}} (\log \lambda + \log h_K) dV_K - \frac{1}{n} \log V(K) - \log \lambda$$

$$= \frac{1}{V(K)} \int_{S^{n-1}} \log h_K dV_K - \frac{1}{n} \log V(K).$$

That is,

$$\frac{1}{V(K)} \int_{S^{n-1}} \log \frac{h_L}{h_K} dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}.$$

The equality holds if and only if $L$ is a solution to the extremum problem, which implies that $L$ and $K$ are dilatates.

Combining the two cases, the assertion (1) for dimension $n$ holds. Therefore, the assertion (2) implies the assertion (1) as desired. □

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