Towards Differential Privacy for Symbolic Systems

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Abstract

In this paper, we develop a privacy implementation for symbolic control systems. Such systems generate sequences of non-numerical data, and these sequences can be represented by words or strings over a finite alphabet. This work uses the framework of differential privacy, which is a statistical notion of privacy that makes it unlikely that privatized data will reveal anything meaningful about underlying sensitive data. To bring differential privacy to symbolic control systems, we develop an exponential mechanism that approximates a sensitive word using a randomly chosen word that is likely to be near it. The notion of “near” is given by the Levenshtein distance, which counts the number of operations required to change one string into another. We then develop a Levenshtein automaton implementation of our exponential mechanism that efficiently generates privatized output words. This automaton has letters as its states, and this work develops transition probabilities among these states that give overall output words obeying the distribution required by the exponential mechanism. Numerical results are provided to demonstrate this technique for both strings of English words and runs of a deterministic transition system, demonstrating in both cases that privacy can be provided in this setting while maintaining a reasonable degree of accuracy.

I. INTRODUCTION

Control systems appear in a wide range of applications and are used in a wide range of problem formulations. As control applications have become increasingly reliant upon user data, there has arisen interest in protecting individuals’ privacy in some applications, e.g., in smart power grids [1], [2]. In response, there has been some work on privacy in control, and sensitive user data have been made private in multi-agent control systems [3], [4], convex optimization [5], [6], [7], linear-quadratic control [8], and a range of filtering and estimation problems [9], [10]. All of these problems protect sensitive numerical data by adding carefully calibrated noise to such data before they are shared.

However, methods based on additive noise do not readily extend to non-numerical data, nor to sequences of them. Symbolic control systems generate sequences of non-numerical data, which are analogous to trajectories for ordinary control systems, and these are typically represented as words or strings over a finite alphabet. A symbolic
A symbolic trajectory can therefore reveal one’s actions or positions over time, and this setting incurs privacy concerns similar to those with trajectories of numerical data. Simultaneously, agents may need to share these trajectories with other agents to jointly coordinate their activities, though a network may contain untrusted agents, or communications may be subject to eavesdropping. Thus there is a need to share symbolic trajectories in a way that preserves their accuracy while providing strong, provable privacy guarantees to users. Because existing approaches do not readily extend to the symbolic setting, fundamentally new approaches are required to ensure that sensitive symbolic data of this kind can safely be shared.

Accordingly, in this paper we develop a general-purpose method for providing privacy to sensitive words generated by symbolic control systems. To do so, we adopt the framework of differential privacy. Differential privacy is a statistical notion of privacy that makes it unlikely for an eavesdropper or adversary to learn anything meaningful about sensitive data from its differentially private form [13]. Its key features include immunity to post-processing, in that transformations of privatized data to not weaken privacy guarantees, and robustness to side information, in that learning additional information about data-producing entities does not weaken differential privacy by much.

Differential privacy originates in the database literature in computer science, and it has been applied to protect individual database entries in response to queries of the database as a whole [14]. It was later extended to trajectory-valued data and applied in the control setting in [9], and has seen applications in both its database and trajectory forms in a range of control settings [3], [4], [8], [9]. Differential privacy is most commonly implemented using the Laplace and Gaussian Mechanisms, which add Laplacian and Gaussian noise, respectively, to sensitive data before sharing them.

Differential privacy has also been applied to non-numerical data using the exponential mechanism, which randomly generates responses to non-numerical queries based on how well those responses approximate the non-private response [15]. The exponential mechanism has been applied, for example, to data aggregation [16] and data release [17] problems, as well as pricing and auction problems [13], [15]. To bring privacy to the symbolic control setting, we will develop an exponential mechanism for words over a finite alphabet.

Doing so first requires defining differential privacy in a manner that captures the privacy needs of symbolic control systems, and the first contribution of this paper is formally defining differential privacy for this setting. Next, actually implementing differential privacy requires defining “quality” in a meaningful way. The notion of “quality” we use is based on the Levenshtein distance from a word. The Levenshtein distance counts how many insertions, substitutions, and deletions are required to change one word into another. Given a sensitive word (representing a sensitive symbolic trajectory), our differential privacy implementation therefore outputs nearby words (in the Levenshtein sense) with high probability and, conversely, outputs distant words with low probability.

The second contribution of this paper then comes from the exponential mechanism itself, and, for a given sensitive word, we provide the distribution over possible output words required to implement differential privacy. A naïve approach to generating samples from this distribution would iterate over all possible output words, compute their
quality scores, and then select one as a private output. However, actually executing these operations can be very computationally demanding, and this naïve implementation would require computing all pair-wise Levenshtein distances among all possible output words, which can incur substantial computational expense.

Instead, to preclude the need for large scale computation, we specify a more efficient means of generating output words, and this implementation constitutes this paper’s third contribution. In particular, we construct a Levenshtein automaton that generates output words one letter at a time in a manner that obeys the probability distribution required by differential privacy. The state of the Levenshtein automaton is defined to be the letter most recently added to the output word, and transition probabilities are constructed between letters to determine which letter should be added to the output word next. Levenshtein automata can be constructed efficiently, and our implementation provides a substantial computational improvement over the naïve approach. In this preliminary study, we consider a restricted form of the Levenshtein distance, namely, we allow substitutions but not deletions or insertions, and we defer the use of the full Levenshtein distance to a future publication. These modifications require only removing certain transitions from the automaton that we construct, and this can likewise be done efficiently, as will be shown. This implementation will be demonstrated on both English words and a deterministic transition system to show its applicability in symbolic control systems and beyond.

We note that a Levenshtein automaton can be represented as a graph, with each letter a node in the graph and the transition probabilities acting as edge weights. Differential privacy has been applied to graphs in various ways, including to protect topological characteristics [18], [19] and the edge weights within a graph [20]. This paper differs from those works because we use graphs merely for the implementation of the exponential mechanism over words, and we are not applying privacy to any graph structure.

The rest of the paper is organized as follows. Section II presents the relevant background on Levenshtein automata and differential privacy. Then, Section III formally states the differential privacy problem that is the focus of the paper. Section IV next defines the exponential mechanism for words that we use, and Section V uses Levenshtein automata to provide an efficient means of generating samples from this distribution. These results are demonstrated for transition systems in Section VI where we apply our theoretical results to the problem of generating private runs of such systems. Section VII then provides numerical results for both strings in a general setting and the transition system setting. Finally, Section VIII provides concluding remarks and directions for future research.

II. Preliminaries

In this section, we define our notation and establish some mathematical preliminaries for the developments below.

A. Languages

An alphabet is a collection of symbols Σ. A word over Σ is a concatenation of symbols \( w = \sigma_0 \sigma_1 \ldots \) such that \( \sigma_i \in \Sigma \) for all \( i \). We use the notation \( \Sigma^* \) to denote the set of all finite words over \( \Sigma \). Any subset \( L \subseteq \Sigma^* \) (equivalently \( L \in 2^{\Sigma^*} \)) is called a finite language.
B. Finite State Automata

Definition 1: A (nondeterministic) finite state automaton (NFA) is a tuple \( A = (Q, \Sigma, q_0, \delta, F) \), where \( Q \) is a set of states, \( \Sigma \) is an input alphabet, \( q_0 \in Q \) is the initial state, \( \delta \subseteq Q \times \Sigma \times Q \) is the transition relation between states, and \( F \subseteq Q \) is the set of accepting states.

Definition 2: A run on an NFA \( A = (Q, \Sigma, q_0, \delta, F) \) induced by word \( w = \sigma_0 \sigma_1 \ldots \in \Sigma^* \) is a word \( q = q_0 q_1 \ldots \in Q^* \) such that \( q_0 = q_0 \) and \( (q_i, w_i, q_{i+1}) \in \delta \). Automaton \( A \) accepts a word \( w \) if the final state of the induced run is an accepting state \( q_f \in F \). We call the set of all words accepted by the automaton its language, denoted by \( L(A) \).

Definition 3: A deterministic finite state automaton (DFA) is an NFA with deterministic transition function \( \delta : Q \times \Sigma \rightarrow Q \).

C. Transition Systems

Definition 4: A deterministic transition system (DTS) is given as a tuple \( TS = (S, s_0, Act, T) \), where
- \( S \) is a finite state space,
- \( s_0 \in S \) is an initial state,
- \( Act \) is an input set
- \( T : S \times Act \rightarrow S \) is a deterministic transition function such that applying input \( a_1 \) in state \( s_1 \) will move the system to state \( T(s_1, a_1) \).

Definition 5: A plan for a DTS is a sequence of actions \( a = a_0 a_1 a_2 \ldots \in Act^* \). A plan \( a \) is in the input language of a DTS, denoted \( a \in L_i(TS) \), if it induces a run \( r(a) = s_0 s_1 \ldots \in S^* \) such that \( s_0 = s_0 \) and \( s_{i+1} = T(s_i, a^i) \) \( \forall i \).

D. Levenshtein Distances and Automata

To compare two strings, we introduce the notion of Levenshtein distance [21] and Levenshtein automata [22]. These tools can be used to measure the difference or edit distance between two strings.

Definition 6 (Levenshtein Distance [21]): The Levenshtein distance between words \( w_1, w_2 \), denoted \( d_L(w_1, w_2) \), is the minimum number of changes—insertions, substitutions, or deletions—that can be applied to \( w_1 \) to convert it to \( w_2 \).

For example, the Levenshtein distance between “sample” and “examples” is 3, since the “s” at the beginning of “sample” must be substituted for an “e” or an “x,” the remaining letter (“e” or “x”, whichever was not substituted for the “s”) must be added, and an “s” must be added to the end of the word. We can identify whether a string is within a specific Levenshtein distance of another string using a Levenshtein automaton.

Definition 7 (Levenshtein Automaton [22]): For a string \( x \) and a distance \( k \in \mathbb{N} \), the Levenshtein automaton is an NFA \( A_{x,k} = (Q, \Sigma, q_0, \delta, F) \) such that \( L(A_{x,k}) \) is the set of all words with Levenshtein distance less than or equal to \( k \) from \( x \).
For a given string $x$ and a distance $k$, we can construct the corresponding Levenshtein automaton $A_{x,k}$ as follows, using Algorithm 1.

**Algorithm 1 Levenshtein Automaton Construction**

1. procedure $\text{MAKELEVENSHTEIN}(\Sigma, x \in \Sigma^*, k \in \mathbb{N})$
2. $Q \leftarrow \{q_{i,e} \mid i \in \{0, 1, \ldots, |x|\}, e \in \{0, 1, \ldots, k\}\}$
3. $F \leftarrow \{q_{i,e} \in Q \mid i = |x|\}$
4. $\delta \leftarrow \emptyset$
5. $q_0 \leftarrow q_{0,0}$
6. for $i \in \{0, 1, \ldots, |x|\}$ do
7. for $e \in \{0, 1, \ldots, k\}$ do
8. $\delta \leftarrow \delta \cup \{(q_{i,e}, x_i, q_{i+1,e})\}$ ▷ Correct transition
9. if $l < k$ then
10. $\delta \leftarrow \delta \cup \{(q_{i,e}, *, q_{i+1,e})\}$ ▷ Deletion
11. $\delta \leftarrow \delta \cup \{(q_{i,e}, e, q_{i,e+1})\}$ ▷ Insertion
12. $\delta \leftarrow \delta \cup \{(q_{i,e}, *, q_{i+1,e+1})\}$ ▷ Substitution
13. return $A_{x,k} = (Q, \Sigma, q_0, \delta, F)$

Fig. 1: (1a) Levenshtein automaton $A_{x,k}$ for all words of distance less than or equal to 2 from the word $x = abc$ from alphabet $\{a, b, c\}$. Accepting states are noted with double circles. (1b) $A_{|x|,x}$ for all words of length exactly 3 and distance less than or equal to 2 from the word $abc$ from alphabet $\{a, b, c\}$.

**Remark 1:** In this paper, for simplicity, we only consider substitutions and ignore insertions and deletions. That is, we ignore lines 10 and 11 of Algorithm 1 when constructing the automaton. We call this distance the substitution Levenshtein distance $d_{\text{sL}}$, which is equal to the Hamming distance [23]. Full considerations of insertions and deletions is a topic of future research.

**Remark 2:** The automaton $A_{x,k}$ generated by Algorithm 1 can be pruned to a DFA that accepts only those words
whose length is the same as the length of the input word, $|x|$, denoted $A^{ix}_{x,k}$. An example of this is shown in Figure [1]. The details of the DFA construction are beyond the scope of this paper and are therefore omitted.

E. Differential Privacy

We provide here only a high-level discussion of differential privacy as background, and further details will be provided in developing our privacy implementation below. The underlying goal of differential privacy is to make similar pieces of sensitive data produce outputs with approximately equal probability distributions. The definition of “similar” for sensitive data is specified by an adjacency relation. Adjacency is frequently specified in terms of a metric, e.g., the $\ell_p$-metric on a space of trajectories [9] or the counting metric on the space of databases [13], and two pieces of data are adjacent if the distance between them is bounded above by a pre-specified constant. Differential privacy then requires that adjacent sensitive data produce approximately indistinguishable outputs.

The notion of approximate indistinguishability is made precise by specifying a relationship between the probability distributions of outputs corresponding to adjacent inputs. For adjacent sensitive data $D_1$ and $D_2$, a randomized map $M$ provides $\epsilon$-differential privacy if

$$\Pr[M(D_1) \in S] \leq e^\epsilon \Pr[M(D_2) \in S]$$

for all $S \subseteq \text{range}(M)$. The parameter $\epsilon$ controls the degree of indistinguishability between the distributions of $M(D_1)$ and $M(D_2)$, and thus the degree of privacy afforded to users. Smaller values of $\epsilon$ provide stronger privacy guarantees, and typical values range from $0.1$ to $\ln 3$. As noted in the introduction, differential privacy is immune to post-processing, so that any transformation of $M(D_1)$ or $M(D_2)$ is also $\epsilon$-differentially private, and robust to side information, so that learning additional information about a data-producing entity does not weaken this privacy by much.

Given a privacy parameter $\epsilon$, an adjacency relation, and some form of sensitive data, the principal challenge in implementing differential privacy is finding the randomized map $M$ that satisfies the above definition. Maps of this kind are called mechanisms for differential privacy, and we formally define the problem of finding such a mechanism for words in the next section. This problem will then be solved in Section IV which provides the details of our privacy implementation.

III. PROBLEM FORMULATION

In this section, we provide the essential privacy definitions that underlie this work, and then we formally state the problems that are the focus of the remainder of the paper.

A. Word Differential Privacy

Here, we define a novel concept of differential privacy, called word differential privacy, that is appropriate for describing privacy for sequences of states in symbolic systems.
**Definition 8 (Word adjacency):** The adjacency relation between words \( w_1, w_2 \in \Sigma^* \) is defined as
\[
\text{Adj}_{w,k} = \{(w_1, w_2) \mid d_L(w_1, w_2) \leq k\}.
\] (2)

**Definition 9 (Substitution Word adjacency):** The substitution adjacency relation between words \( w_1, w_2 \in \Sigma^* \) is defined as
\[
\text{Adj}_{s,w,k} = \{(w_1, w_2) \mid d_s L(w_1, w_2) \leq k\}.
\] (3)

**Definition 10 (Word Differential Privacy):** Fix a probability space \((\Omega, \mathcal{F}, \Pr)\). A mechanism \(M_w : \Sigma^* \times \Omega \rightarrow \Sigma^*\) is word \(\epsilon\)-differentially private if
\[
\Pr_{\Omega}[M_w(w_1) \in L] \leq e^{\epsilon} \Pr_{\Omega}[M_w(w_2) \in L]
\]
\[
\forall (w_1, w_2) \in \text{Adj}_{w,k} \forall L \in 2^{\Sigma^*}.
\] (4)

**Definition 11 (Substitution Word Differential Privacy):** Fix a probability space \((\Omega, \mathcal{F}, \Pr)\). A mechanism \(M_w : \Sigma^* \times \Omega \rightarrow \Sigma^*\) is substitution word \(\epsilon\)-differentially private if
\[
\Pr_{\Omega}[M_w(w_1) \in L] \leq e^{\epsilon} \Pr_{\Omega}[M_w(w_2) \in L]
\]
\[
\forall (w_1, w_2) \in \text{Adj}_{s,w,k} \forall L \in 2^{\Sigma^*}.
\] (5)

Essentially, a word differential privacy mechanism approximates sensitive sequences of symbols with randomized versions of them. These randomizations must have similar distributions for two sequences that are nearby (in sense of Definitions 8 and 9), and this is captured by the relationships between probability distributions in Definitions 10 and 11. Nearby symbolic trajectories are therefore made approximately indistinguishable to any recipient of their privatized forms, as well as any eavesdroppers who gain access to them, and these recipients are therefore unlikely to determine the exact underlying sensitive word. This approximate indistinguishability criterion is the basic idea behind differential privacy, and it is this idea that we apply to symbolic trajectories in this work. In this paper, we restrict ourselves to substitution word differential privacy. The extension to word differential privacy is a topic of future research.

**B. Problems**

Here, we consider two problems involving substitution word differential privacy. First, we consider the problem of synthesizing a differentially private mechanism for an arbitrary sequence of characters from a given alphabet

**Problem 1:** Fix a probability space \((\Omega, \mathcal{F}, \Pr)\). Given an alphabet \(\Sigma\) and a word \(w\), find a mechanism \(M_w : \Sigma^* \times \Omega \rightarrow \Sigma^*\) that is substitution word \(\epsilon\)-differentially private.

Next, as a first step towards differential privacy for symbolic systems, we consider how to synthesize a mechanism for privatizing runs of a deterministic transition system.

**Problem 2:** Fix a probability space \((\Omega, \mathcal{F}, \Pr)\). Given a transition system \(TS\) and a run \(x\), find a mechanism \(M_w,TS : \Sigma^* \times \Omega \rightarrow L(TS)\) that is substitution word \(\epsilon\)-differentially private.

A solution to Problem 2 would enable a designer to privatize a desired run of a deterministic transition system such that an agent that observes repeated executions can determine the desired trajectory only up to bounded precision. The price that must be paid for this is, of course, deviating from the desired run. This relationship
between performance and privacy is handled by tuning the single parameter $\epsilon$, and this will be shown in more detail in Section VII.

IV. THE EXPONENTIAL MECHANISM FOR WORDS

In this section we define the exponential mechanism for words over a finite alphabet. We first define the notion of utility we use for our privacy implementation, and then we bound the sensitivity of this utility. With this sensitivity bound established, we then formally define the distribution from which words should be drawn in order to preserve differential privacy. Section V below then provides the means of efficiently generating samples from this distribution.

A. Utility for Words in a Language

An exponential mechanism is defined with respect to a utility function, which quantifies the quality of each possible output. The choice of utility function here should therefore reflect the quality of outputting a certain word in response to a given sensitive input word. In this work, we seek to privatize the input word by randomly outputting a word which is close to it. Formalizing this idea, we now define the Levenshtein utility, which simply captures the idea that a private output is of higher quality when it is closer to the input.

**Definition 12: (Substitution Levenshtein Utility)** Fix a constant $\alpha > 0$, an alphabet $\Sigma$, and a language $L \in 2^\Sigma^*$. Then, for an input word $w_i \in L$, outputting the word $w_o \in L$ provides Substitution Levenshtein utility equal to

$$u_\alpha(w_i, w_o) = \frac{1}{d^*_L(w_i, w_o) + \alpha}. \quad (6)$$

Here, the inclusion of $\alpha$ ensures that $u_\alpha$ is always defined, and smaller values of $\alpha$ will give higher values of $u_\alpha$ when $w_o$ is close to $w_i$. This choice of utility has the benefit of decreasing rapidly as output words disagree more with the input word, which will more strongly bias the output of the exponential mechanism toward better output words while maintaining privacy.

B. Sensitivity Bounds

The next step in defining the exponential mechanism is to calculate the sensitivity of the utility function $u_\alpha$. In particular, for a fixed output word $w_o$, we must provide a bound on how much $u_\alpha(\cdot, w_o)$ can differ across two adjacent input words, and this bound will be used in defining the distribution over possible output words below. Mathematically, we must bound the quantity

$$\Delta u_\alpha := \max_{v \in L} \max_{w_1, w_2 \in L, (w_1, w_2) \in \text{Adj}_{w, k}^*} |u_\alpha(w_1, v) - u_\alpha(w_2, v)|, \quad (7)$$

and we have the following lemma that does so.

**Lemma 1:** Fix $\alpha > 0$ and $k \in \mathbb{N}$. Then the sensitivity of $u_\alpha$ is bounded via

$$\Delta u_\alpha \leq \frac{k}{\alpha(k + \alpha)}. \quad (8)$$

**Proof:** Without loss of generality we may remove the absolute value signs and set

$$\Delta u_\alpha := \max_{v \in L} \max_{w_1, w_2 \in L, (w_1, w_2) \in \text{Adj}_{w, k}^*} u_\alpha(w_1, v) - u_\alpha(w_2, v), \quad (9)$$
because we can relabel \( w_1 \) and \( w_2 \) to make the right-hand side non-negative. Expanding the right-hand side, we find

\[
\Delta u_\alpha := \max_{v \in L} \max_{w_1, w_2 \in L} \frac{1}{d_L^c(w_1, v) + \alpha} - \frac{1}{d_L^c(w_2, v) + \alpha},
\]

and its non-negativity requires that \( d_L^c(w_1, v) \leq d_L^c(w_2, v) \). To reduce the number of variables in the maximization, we set

\[
d_L^c(w_2, v) = d_L^c(w_1, v) + c
\]

where \( c \geq 0 \). Using the triangle inequality we have

\[
d_L^c(w_2, v) \leq d_L^c(w_2, w_1) + d_L^c(w_1, v) \leq d_L^c(w_1, v) + k,
\]

which follows from the adjacency of \( w_1 \) and \( w_2 \). Combining this with Equation (11) gives

\[
d_L^c(w_1, v) + c \leq d_L^c(w_1, v) + k,
\]

which gives \( c \in \{0, \ldots, k\} \).

Returning to Equation (10) we find that

\[
\Delta u_\alpha := \max_{v \in L} \max_{c \in \{0, \ldots, k\}} \frac{c}{\alpha(c + \alpha)},
\]

For every \( v \in L \), maximizing over \( w_1 \) is easily done by setting \( w_1 = v \), which now gives

\[
\Delta u_\alpha := \max_{c \in \{0, \ldots, k\}} \frac{c}{\alpha(c + \alpha)},
\]

where we have removed the maximization over \( v \) because all dependence upon \( v \) is now eliminated. The right-hand side in Equation (15) is maximized by maximizing \( c \), which completes the proof.

Although we frequently expect this bound to be attained, we write it as an inequality to account for the case that \( L \) does not contain any words exactly distance \( k \) apart.

C. Distribution Over Output Words

Given the above bound on sensitivity, the final step needed to define the exponential mechanism is determining the required distribution over output words. The standard definition of the exponential mechanism [13] says that, for a given sensitive input word \( x \), a candidate word \( w \) should be output with probability \( p(w; x) \) satisfying the proportionality relation

\[
p(w; x) \sim \exp \left( \frac{\epsilon u_\alpha(x, w)}{2\Delta u_\alpha} \right).
\]

For the case of word adjacency (cf. Definition 5), one would need to determine a proportionality constant \( K_x \), and this would require computing the distance to every possible output word from every possible sensitive input word \( x \). Computing the Levenshtein distance has time complexity \( \mathcal{O}(n^2) \) [24], though the implied constants can be very large for longer strings and large alphabets. Computing all possible pairwise distances can therefore easily become intractable. However, for substitution word adjacency (cf. Definition 9), which is the focus of this paper, explicitly computing \( K_x \) can be avoided, and this is shown in the next section.

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V. GENERATING DIFFERENTIALLY PRIVATE WORDS FROM A FIXED ALPHABET

In the previous section, we defined an exponential mechanism for substitution word $\epsilon$-differential privacy. In this section, we propose an efficient method for generating samples $w' \sim p(\cdot; x)$. We propose to do this by synthesizing appropriate randomized policies $\mu_{\epsilon,x} : Q \times \Sigma \rightarrow [0, 1]$ over the Levenshtein automaton associated with word $x$, and these policies will randomly select each letter in an output word to implement the exponential mechanism for words. Formally, we have the following formulation.

**Problem 3:** Given a Levenshtein automaton $A_{k,x}$ as constructed by Algorithm 1, synthesize a policy $\mu_{\epsilon,x}$ such that

$$\prod_{e=0}^{|[x]|-1} \mu_{\epsilon,x}(q^e, \sigma^e) = p(\sigma^0:k; x).$$

(17)

Note that in this approach, we are restricting ourselves to generating privatized output words $w$ that are the same length as $x$ and are allowing arbitrary symbols from an alphabet $\Sigma$ to be selected. That is, we are privatizing words in the language $L = \Sigma^{|x|}$. This means that $\Delta u_\alpha$ achieves its maximum value $\frac{k}{\alpha(k+\alpha)}$ as long as $k \leq |x|$. Given these assumptions, we propose the following procedure for indirectly synthesizing $\mu_{\epsilon,x}$ by sampling a Levenshtein distance $\ell$ and computing the policy $\mu_{\epsilon,x,\ell}$ for only those words that are distance $\ell$ from $x$.

1) For a given input word $x$ and an adjacency relation $\text{Adj}^e_{w,k}$, fix a desired substitution Levenshtein distance $\ell$ by drawing from the distribution

$$\rho(\ell; |x|, k) = \frac{\exp\left(\frac{\epsilon\alpha(k+\alpha)}{2(\ell+\alpha)}\right)}{\sum_{\lambda=1}^{m} \exp\left(\frac{\epsilon\alpha(k+\alpha)}{2(\lambda+\alpha)}\right)}$$

(18)

Note that $\rho(\ell; |x|, k) = \Pr_{p(w;x)}[d_L(w, x) = \ell]$, the probability of selecting an output word $w$ distance $\ell$ from $x$.

2) Construct the subset of $A_{k,x,\ell}^{|x|} \subseteq A_{k,x}^{|x|}$ that is backwards reachable from $q_{|x|,\ell}$ (the accepting state for words of length $|x|$ and distance exactly $\ell$) and denote the states of $A_{k,x,\ell}^{|x|}$ as $Q_{k,x,\ell}$

3) Synthesize $\mu_{\epsilon,x,\ell} : Q_{k,x,\ell} \times \Sigma \rightarrow [0, 1]$

A. **Synthesizing distance-restricted policies**

In this section, we describe the procedure used to construct $\mu_{\epsilon,x,\ell}$. First, we note that $p(w;x)$ is a function of $d_L(w, x)$, and thus all strings of the same Levenshtein distance to $x$ should be equiprobable. In other words,

$$\prod_{e=0}^{|[x]|-1} \mu_{\epsilon,x,\ell}(q^e, \sigma^e) = \pi_{|x|,\ell,\Sigma} \forall q^0 \cdots q^{|x|} \in L(A_{k,x,\ell})$$

(19)

The procedure outlined in Algorithm 2 uses this principle to construct $\mu_{\epsilon,x,\ell}$. We assign a function $V : Q_{k,x,\ell} \rightarrow \mathbb{N}$ such that $V(q)$ is the number of unique paths from $q$ that end in $q_{|x|,\ell}$. Then, the weighting of the policy at each point is equal to the proportion of unique paths that can be reached by applying the symbol compared to the total number of unique paths reachable from the current state.
Algorithm 2 Distance-Restricted Policy Construction

1: **procedure** SYNTHESIZEPOLICY($A_{x,k,\ell}^{|x|}$)

2: \[ V(q_{x,\ell}) \leftarrow 1 \]

3: \[ CurrQ \leftarrow \{q_{x,\ell}\} \]

4: \[ counter = 0 \]

5: **while** \( counter < \ell \) **do**

6: \[ \text{for } q \text{ s.t. } (q, \sigma, q') \in \delta_{x,\ell}, q' \in CurrQ \text{ do} \]

7: \[ V(q) \leftarrow \sum_{\sigma \in \Sigma((q, \sigma, q'') \in \delta_{x,\ell})} V(q'') \]

8: \[ q \in ActiveQ \]

9: \[ \text{for } q \in ActiveQ, (q, \sigma, q') \in \delta_{x,\ell} \text{ do} \]

10: \[ \mu_{x,\ell}(q, \sigma) \leftarrow \frac{V(q)}{\sum_{(q'', \sigma) \in \delta_{x,\ell}} V(q'')} \]

11: \[ CurrQ \leftarrow ActiveQ \]

12: \[ counter \leftarrow counter + 1 \]

**return** \( \mu_{x,\ell} \)

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Fig. 2: Levenshtein automaton for all words of length 3 and distance equal to 2 from the word \textit{abc} from alphabet \{a, b, c\}. The value of \( V(q) \) is shown in blue. The value of \( \mu_{e,x,\ell}(q, \sigma) \) is shown in red.

Fig. 3: Example transition system. For simplicity, the actions that enable a transition to state \( s_i \) is simply labeled \( s_i \). Thus, plans and runs are equivalent for this system.
VI. GENERATING DIFFERENTIALLY PRIVATE RUNS FOR A TRANSITION SYSTEM

Here, we extend the principles used to generate differentially private words presented in Section V to generate differentially private runs of a system. We do this via the introduction of the product Levenshtein automaton.

Definition 13: Let $A^{[x]}_{x,k} = (Q, \Sigma, q_0, \delta, F)$ be a Levenshtein automaton and let $TS = (S, s_0, \Sigma, T)$ be a deterministic transition system. The Product Levenshtein Automaton, $A^{[x]}_{x,k,TS} = (Q_S, \Sigma, q_0 \times s_0, \delta_{TS}, F_{TS})$ where

- $Q_S \subseteq Q \times S$
- $\delta_{TS} = Q \times S \times \Sigma \times Q \times S$ such that $(q, s, \sigma, q', s') \in \delta_{TS} \iff (q, \sigma, q') \in \delta \land T(s, \sigma) = s'$
- $F_{TS} = \{(q_f, s) \in Q_S | q_f \in F\}$

In other words, $A^{[x]}_{x,k,TS}$ is the synchronous product of $A^{[x]}_{x,k}$ and $TS$. Further, $w \in L(A^{[x]}_{x,k,TS}) \iff w \in L(A^{[x]}_{x,k}) \land w \in L(TS)$. That is, every accepting word in the product corresponds to a sequence of inputs that when applied to $TS$ will result in a run that is within substitution Levenshtein distance $k$ of $w$. An example of a product Levenshtein automaton is shown in Figure 4(a).

Because the product Levenshtein automaton is a Levenshtein automaton, we can use the exact same procedure as in Section V with using $A^{[x]}_{x,k,TS}$ instead of $A^{[x]}_{x,k}$ and ensuring that the maximum distance used to compute $\rho$ is the minimum of $k$ and $\max_{v \in L(TS)} d_{L}(v, x)$. An example of applying Algorithm 2 to $A^{[x]}_{x,k,TS}$ is shown in Figure 4. (a) Product Levenshtein automaton for all traces of length 3 and Levenshtein distance less than or equal to three from $s_1 s_2 s_3$ from the transition system shown in Figure 3. (b) Restricted product automaton for traces of Levenshtein distance exactly 2 from $s_1 s_2 s_3$. The value of $V(q)$ is shown in blue. The value of $\mu_{e,x}(q, \sigma)$ is shown in red.
VII. EXPERIMENTS

In this section, we present results of some computational experiments that demonstrate the procedures in Sections V and VI. We developed a package in the Julia programming language called LevenshteinPrivacy.jl that implements the procedures in Sections V and VI. The code uses the LightGraphs.jl framework. All experiments were performed on a Windows Desktop PC with a 1.90 GHz processor and 16.0GB of RAM.

A. Case Study 1: Strings

For this set of experiments, we demonstrate the procedure from Section V by generating differentially private versions of the string “american control conference 2019”. The alphabet is comprised of all the unique characters in the input string. The Levenshtein automaton contains 561 states and 1056 edges. The automaton was constructed in 6.6s and generating 40 privatized strings required 1.53s of computation time. Figure 5a shows outputs from these experiments with different values of the privacy parameter $\epsilon$. As we can see, as $\epsilon$ decreases (strength of privacy increases), the outputs become less recognizable until they become almost entirely gibberish.

B. Case Study 2: Transition System

In this case study, we demonstrate the procedure outlined in Section VI. We constructed a transition system with 225 states and edges that corresponds to a 15 by 15 “grid world”. The path we wish to privatize is shown in Figure 5b(i). The resulting product automaton constructed by Algorithm has 6194 states and 18762 edges. The computation time for constructing the product automaton was 193s and the time required to generate 100 samples with $\epsilon = 0.01$ was 6.4s.

VIII. CONCLUSIONS AND FUTURE WORK

In this work we presented a method for providing differential privacy to words over a finite alphabet. An exponential mechanism was devised to generate possible output words, and the theory of Levenshtein automata was applied to efficiently generate samples from this distribution. Numerical results validated these theoretical developments and demonstrated their efficiency.

The first natural extension of this work is to the full Levenshtein distance, which will allow for not only substitutions as in this work, but also deletions and insertions. A key challenge in doing so is efficiently generating samples from the distribution over possible outputs. This work did so by considering strings of a fixed output length, which corresponds to using the substitution Levenshtein distance rather than the full Levenshtein distance. Making this extension will require fundamental innovations beyond this work, though successfully making this extension will significantly broaden the scope of this work.

1https://github.com/JuliaGraphs/LightGraphs.jl
(a) Samples of differentially private versions of the string “american control conference 2019” generated with different values of the privacy parameter $\epsilon$.

(b) (i) Input trajectory (ii)-(vi) Example trajectories generated by differentially private mechanism with $\epsilon = 5$.

Fig. 5: Numerical results for generate private strings over a finite alphabet and private runs of a transition system.
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