Spin relaxation in a generic two-dimensional spin-orbit coupled system

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We study the relaxation of a spin density injected into a two-dimensional electron system with generic spin-orbit interactions. Our model includes the Rashba as well as linear and cubic Dresselhaus terms. We explicitly derive a general spin-charge coupled diffusion equation. Spin diffusion is characterized by just two independent dimensionless parameters: $\gamma_R$ and $\gamma_D$, which control the interplay between different spin-orbit couplings. The real-time representation of the diffusion matrix (Green’s function of the diffusion equation) is evaluated analytically. The diffusion describes space-time dynamics of the injected spin distribution. We explicitly study two regimes: The first regime corresponds to negligible spin-charge coupling and is characterized by standard charge diffusion decoupled from the spin dynamics. It is shown that there exist several qualitatively different dynamic behaviors of the spin density, which correspond to various domains in the $(\gamma_R, \gamma_D)$ parameter space. We discuss in detail a few interesting phenomena such as an enhancement of the spin relaxation times, real space oscillatory dynamics, and anisotropic transport. In the second regime, we include the effects of spin-charge coupling. It is shown that the spin-charge coupling leads to an enhancement of the effective charge diffusion coefficient. We also find that in the case of strong spin-charge coupling, the relaxation rates formally become complex and the spin/charge dynamics is characterized by real time oscillations. These effects are qualitatively similar to those observed in spin-grating experiments [Weber et al., Nature 437, 1330 (2005)].

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I. INTRODUCTION

Spin-orbit coupled systems host an amazing variety of interesting effects, which are currently the subject of intense experimental and theoretical investigation. The interest in this field stems from the fact that spin-orbit coupling opens the possibility of controlling the electron spin using purely electrical means, with potential applications in spintronics and quantum computing. The microscopic origin of the spin-orbit interaction in a low-dimensional system is the absence of inversion symmetry in the semiconductor crystal and confining potential. This leads to the appearance of the Rashba and Dresselhaus (linear and cubic) spin-orbit terms in the effective Hamiltonian. The spin-orbit interaction results in a variety of macroscopic effects such as antilocalization, electric-field induced spin accumulation, spin-galvanic effects, etc.

One particularly important aspect of spin transport in disordered systems is Dyakonov-Perel spin relaxation. The source of this effect is the precession of the electron spin around a momentum-dependent magnetic field. The electron momentum changes randomly due to electron scatterings off of impurities, which in turn randomizes spin precession. One should note here that a large spin relaxation rate is typically considered an undesirable effect from the experimental point of view. On the other hand, the Dyakonov-Perel spin relaxation rate is proportional to the strength of spin-orbit interactions. On the other hand, the existence of relatively strong spin orbit-interaction is needed for the appearance of non-trivial spin transport. It is therefore important to understand the interplay between different phenomena in spin transport and search for possibilities to reduce the negative effects of spin relaxation.

We should mention here that the issue of spin relaxation has been extensively studied in experiments. In particular, Weber et al. have used the transient spin grating method to study spin dynamics in two-dimensional semiconductor systems. This experimental technique is based on optical injection of spin polarized electrons in a two-dimensional quantum well, which generates a spin density wave with a wave-vector in the plane of the two-dimensional system and the spin polarization perpendicular to the plane. Particularly interesting results revealed by these experiments were an enhancement of the charge diffusion coefficient and an unusual time-dependence of the spin grating amplitude, which exhibited oscillatory features in addition to the conventional exponential decay.

Motivated by the aforementioned experimental results, we study theoretically spin transport and spin relaxation in a two-dimensional electron system with generic spin orbit interaction. We focus our attention on the interplay between different types of spin-orbit couplings and the effects of the spin-charge coupling on the spin relaxation dynamics.
arising from the lack of inversion symmetry of the semiconductor crystal, contains terms both linear and cubic in the change of the Fermi surface due to the presence of spin-orbit interaction. The Dresselhaus spin-orbit coupling, the two-dimensional momentum \( p \), the Fermi velocity, and \( \gamma \) are in the presence of a short-range impurity potential and we investigate the role of the spin-orbit interaction in the coupled spin and charge diffusion behavior, which are qualitatively similar to the oscillatory behavior observed in experiment.

We should note here that the real-time oscillations appear only beyond the formal regime of validity of the diffusion approximation. However, they give a tentative indication that the experimentally observed oscillations may originate from the spin-charge coupling.

Our paper is structured as follows:

In Sec. II we introduce our model Hamiltonian for the non-interacting disordered two-dimensional electron system with spin-orbit interaction and derive a general spin-charge coupled diffusion equation. We introduce natural dimensionless units of length and time and show that in these units, the spin part of the diffusion equation is characterized by two dimensionless parameters, \( \gamma_R \) and \( \gamma_D \). These parameters are constrained to lie within a circle of radius two in the \( (\gamma_R, \gamma_D) \) parameter space. We also introduce the matrix Green’s function of the diffusion equation (diffuson), whose real time representation describes the spin relaxation dynamics of interest.

In Sec. III we consider the spin sector of the diffusion equation (which is justified if the spin-charge coupling is relatively small). We focus our attention on the diagonal \( z \)-component of the diffuson, which describes the dynamics of injected out-of-plane spin density. We show that there exist different domains in the \( (\gamma_R, \gamma_D) \)-parameter space, which correspond to qualitatively different dynamic regimes. We explicitly derive the real space-time behavior of the diagonal component of the diffuson in various regimes. We also discuss the spin dynamics as a function of momentum and derive the spin relaxation spectrum as a function of spin-orbit coupling parameters. In particular, we study in detail the “symmetry point” in the parameter space \( (\gamma_R = \gamma_D) \), when the relaxation time becomes infinite in certain directions. We study the behavior of the spin relaxation spectrum in the vicinity of the symmetry point.

In Sec. IV we discuss the effects of spin charge coupling on the spin relaxation spectrum. We show that the spin charge coupling may formally lead to a complex spin relaxation rate, which physically translates into an oscillatory behavior of the spin density. We discuss the validity of the diffusion equation approximation and the implication of our results for understanding spin transport experiments. Our conclusions and a discussion of several open questions are presented in the last Section V.

II. THE DIFFUSION EQUATION

In this section we introduce the model and derive a general form of the spin-charge coupled diffusion equation. The main results of this section are equations (11) and (13) representing the most general form of the diffusion equation in an electron system with arbitrary spin-orbit interaction. In this and subsequent Sections, we use the units \( \hbar = k_B = 1 \).

We start with the following Hamiltonian, which describes the conduction band electrons in a III-V type semiconductor quantum well grown in the [001] direction (set as the \( z \) axis),

\[
H = \frac{p^2}{2m} + \mathbf{h}(\mathbf{p}) \cdot \mathbf{\hat{\sigma}},
\]

(1)

where \( m \) is the effective electron mass, \( \mathbf{\hat{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \) is the Pauli matrix, and \( \mathbf{h}(\mathbf{p}) = (h_x, h_y, h_z) \) are functions of the two-dimensional momentum \( \mathbf{p} \) describing the spin-orbit interaction. We assume that the noninteracting electrons are in the presence of a short-range impurity potential and we investigate the role of the spin-orbit interaction in the coupled spin and charge transport using the diffusion approximation. In general, the second term of the Hamiltonian, which describes the spin-orbit interaction, contains both Rashba and Dresselhaus contributions. The Rashba interaction\(^{11,12}\), which is due to the inversion asymmetry of the quantum well confining potential, has the form

\[
\mathbf{h}^R(\mathbf{p}) = \alpha v_F (p_y, -p_x),
\]

(2)

where \( v_F \) is the Fermi velocity, and \( \alpha \) is a dimensionless coupling constant. Throughout this work we will assume that the spin-orbit coupling energy is much smaller than the Fermi energy, i.e. \( |\alpha| \ll 1 \). Consequently, we will neglect the change of the Fermi surface due to the presence of spin-orbit interaction. The Dresselhaus spin-orbit coupling\(^{12}\), arising from the lack of inversion symmetry of the semiconductor crystal, contains terms both linear and cubic in \( \mathbf{p} \),

\[
\mathbf{h}^{D_1}(\mathbf{p}) = \begin{pmatrix} \beta_1 v_F (p_x, -p_y) \\ \beta_2 v_F (p_y, -p_x) \end{pmatrix},
\]

(3)

\[
\mathbf{h}^{D_3}(\mathbf{p}) = -4\beta_3 \frac{v_F}{p_F} (p_x p_y^2, -p_y p_x^2),
\]

(4)
where \( \beta_1 \) and \( \beta_3 \) are dimensionless coupling constants with \( |\beta_3| \ll 1 \), and \( p_F = mv_F \) is the Fermi momentum.

In the presence of disorder, the complete description of the coupled spin and charge diffusive transport at long lengthscales compared to the mean-free path is given by a set of linear differential equations that, in its most general form, can be written as

\[
\left( \frac{\partial}{\partial t} - D \nabla^2 \right) \rho_i = (-\Gamma^{ij} + P^{ijm} \nabla_m + C^{ij} \nabla) \rho_j
\]

(5)

where \( \partial/\partial t \) represents the time derivative, \( \rho_0 \) is the charge density, while \( \rho_1 = \rho_x \), \( \rho_2 = \rho_y \), and \( \rho_3 = \rho_z \) are the densities of the corresponding spin components. In equation (5) \( D \) represents the diffusion constant and can be expressed in terms of the mean scattering time \( \tau \) as \( D = \tau v_F^2 / 2 \). \( \Gamma^{ij} \) describes the Dyakonov-Perel spin relaxation, \( P^{ijm} \) is responsible for the precession of the inhomogeneous spin polarization, and the last term, having non-zero elements \( C^{ij} = C^{ji} \), describes the spin-orbit mixing of spin and charge degrees of freedom. For convenience, throughout this article we will express times in units of spin relaxation time, \( \tau_s \), and lengths in units spin relaxation length, \( L_s \), unless otherwise stated. The quantities \( \tau_s \) and \( L_s \) depend explicitly on the parameters of the 2D electron system,

\[
L_s = \frac{1}{2k_F \sqrt{\alpha^2 + (\beta_1 - \beta_3)^2 + \beta_3^2}}
\]

(6)

\[
\tau_s = \frac{2\tau}{g^2 \sqrt{\alpha^2 + (\beta_1 - \beta_3)^2 + \beta_3^2}}
\]

(7)

where \( g = 2v_F k_F \tau \) is a coefficient proportional to the dimensionless conductance. Using the units (6,7) we can write the coefficients from Eq. (5) as dimensionless coupling constants. For example, the diagonal contributions to the spin relaxation terms are \( \Gamma_{xx} = \Gamma_{yy} = 1 \) \( \Gamma_{zz} = 2 \). The final matter of convenience concerns the parametrization of the spin-orbit interaction. This interaction can be described in terms of the original coupling constants \( (\alpha, \beta_1, \beta_3) \) or, alternatively, in terms of a new set of parameters, \( (\Gamma, \gamma_R, \gamma_D) \), defined by

\[
\Gamma = \alpha^2 + \beta_1^2 + \beta_3^2
gamma_R = \frac{2\alpha}{\sqrt{\alpha^2 + (\beta_1 - \beta_3)^2 + \beta_3^2}}
\]

\[
gamma_D = \frac{2|\beta_1 - \beta_3|}{\sqrt{\alpha^2 + (\beta_1 - \beta_3)^2 + \beta_3^2}}
\]

(8)

The parameter \( \Gamma \) represents a measure of the overall strength of the spin-orbit interaction, while \( \gamma_R \) and \( \gamma_D \) characterize the interplay between the Rashba and Dresselhaus contributions. By convention the sign of \( \Gamma \) is the same as the sign of \( \beta_3 \). We derive the diffusion equations using the standard density matrix formalism\(^{26}\), used previously in Refs. 27,28,29,30. Eq. (5) can be re-written in the form

\[
[\delta_{ij} - \Pi_{ij}] \rho_j = 0,
\]

(9)

where, for a homogeneous system, \( \delta_{ij} - \Pi_{ij}(\omega, k) = (-i\omega + k^2)\delta_{ij} + \Gamma^{ij} + P^{ijm} k_m + C^{ij} k \) and summations over repeated indices are assumed. The matrix \( \Pi \) represents the kernel of the diffusion equation and can be expressed in terms of retarded and advanced Green functions as

\[
\hat{\Pi}_{ij}(\omega, k) = \frac{1}{4\pi \nu_F \tau} \int \frac{d^2 k'}{(2\pi)^2} Tr \left\{ \hat{\sigma}_i \hat{G}^R(\omega/2, k + k' / 2) \hat{\sigma}_j \hat{G}^A(\omega/2, k - k' / 2) \right\},
\]

(10)

where \( \hat{G}^{R(A)} \) is the \( 2 \times 2 \) retarded (advanced) Green function matrix averaged over disorder, \( \hat{\sigma}_i \) are the Pauli matrices, \( \nu_F \) is the density of states at the Fermi energy and \( Tr\{...\} \) involves the trace over spin indices. The diffusion approximation represents the low frequency, long wavelength limit of Eq. (10). This approximation is valid as long as the scattering time is much shorter than the spin precession time, i.e. in the weak spin-orbit coupling limit \( \Gamma v_F k_F \tau \ll 1 \). Taking the appropriate small frequency and small momentum limit, we obtain

\[
\hat{1} - \hat{\Pi}(\omega, k) = \begin{pmatrix}
  s & -i g(\mu_k k_x + \mu_0 k_y) & g(\mu_k k_x + \mu_0 k_y) & 0 \\
  -i g(\mu_0 k_x + \mu_1 k_y) & s & 0 & 0 \\
  g(\mu_k k_x + \mu_0 k_y) & 0 & s & 0 \\
  0 & 0 & 0 & s
\end{pmatrix},
\]

(11)

where \( \hat{1} \) is the \( 4 \times 4 \) unit matrix, \( s = -i\omega + k^2 + 1 \) and the spin-charge coupling is characterized by the parameters

\[
\mu_0 = \frac{(3\beta_3 - \beta_1)(\alpha^2 - \beta_1^2 + \beta_3^2) - \beta_1 \beta_3^3}{\sqrt{\alpha^2 + (\beta_1 - \beta_3)^2 + \beta_3^2}},
\]

\[
\mu_1 = \frac{\alpha(\alpha^2 - \beta_1^2 + 6\beta_3^2)}{\sqrt{\alpha^2 + (\beta_1 - \beta_3)^2 + \beta_3^2}}.
\]

(12)
In addition, it will be convenient to consider the diffusion problem within a coordinate system that is rotated in the x-y plane counterclockwise with $\pi/2$ relative to the initial coordinate system. The rotated coordinates are $r_+ = (y + x)/\sqrt{2}$ and $r_- = (y - x)/\sqrt{2}$, while the new components of the spin density will be $\rho_+ = (\rho_y + \rho_x)/\sqrt{2}$ and $\rho_- = (\rho_y - \rho_x)/\sqrt{2}$, in addition to $\rho_z$ and the charge density $\rho_0$ that remain unchanged. Relative to the rotated coordinate system, diffusion is described by the operator

$$\hat{1} - \hat{\Pi}(\omega, \mathbf{k}) = \begin{pmatrix} s - 1 & i\xi(\mu_0 - \mu_1)k_- & i\xi(\mu_0 + \mu_1)k_+ & 0 \\ i\xi(\mu_0 - \mu_1)k_- & s + \frac{1}{2}\gamma R\gamma D & 0 & -i(\gamma R + \gamma_D)k_+ \\ i\xi(\mu_0 + \mu_1)k_+ & 0 & s - \frac{1}{2}\gamma R\gamma D & -i(\gamma R - \gamma_D)k_- \\ i(\gamma R + \gamma_D)k_+ & i(\gamma R - \gamma_D)k_- & s + 1 & 0 \end{pmatrix}, \quad (13)$$

where the four columns correspond to $\rho_0$, $\rho_+$, $\rho_-$, and $\rho_z$, respectively. Equations (11) and (13) are the main results of this section.

The physical problem that we will be solving on the basis of Eqs. (11, 13) is to describe the transport of an injected spin density in the presence of a general type of spin-orbit coupling. Assuming that a spin (or charge) density with components $\rho_i(0, \mathbf{r})$ was injected into the system at $t = 0$, the density profile at any time $t > 0$ will be

$$\rho_i(t, \mathbf{r}) = \int d\mathbf{r}' D_{ij}(t, \mathbf{r}, \mathbf{r}')\rho_j(0, \mathbf{r}'), \quad (14)$$

where $\hat{D}$ is the diffuson, i.e., the Green’s function of the diffusion equation. The physical meaning of the diffuson can be understood if we assume that initially we inject a $\delta$-like density profile $\rho_j(0, \mathbf{r}) = \delta(\mathbf{r})$. The spin and charge densities at any later time are given by the matrix elements of the diffuson, $\rho_i(t, \mathbf{r}) = D_{ij}(t, \mathbf{r})$. For a homogeneous system the diffuson is the inverse of the diffusion kernel,

$$\hat{D} = (\hat{1} - \hat{\Pi})^{-1}. \quad (15)$$

Consequently, solving the transport problem for an injected spin density implies inverting the matrix in Eq. (11) or (13) and performing the appropriate Fourier transforms. We notice that the spin and charge dynamics is controlled by the poles of $\hat{D}(\omega, \mathbf{k})$. These poles determine four relaxation modes $i\omega_\alpha(\mathbf{k}) = 1/\tau_\alpha(\mathbf{k})$, which follow from the equation:

$$\Delta(\omega, \mathbf{k}) \equiv \det[\hat{1} - \hat{\Pi}(s, \mathbf{k})] = 0 \quad (16)$$

A time dependent matrix element of the diffuson can be written in general as

$$D_{ij}(t, \mathbf{k}) = \sum_{l=0}^{3} A_l(\mathbf{k})e^{-i\omega_l(\mathbf{k}) t}, \quad (17)$$

where the amplitudes $A_l(\mathbf{k})$ are functions of momentum. We should emphasize here that the long time relaxation of the injected spin density is uniquely determined by the lowest minimum of the relaxation rate modes. In a standard diffusion problem, the charge relaxation rate is $1/\tau_0(\mathbf{k}) = D\mathbf{k}^2$ (where $D$ is the diffusion constant, which is equal to one in our special units) and the minimum obviously corresponds to $\mathbf{k} = 0$. By contrast, in the presence of spin-orbit interaction, the $k$-dependence of the relaxation rate is more complicated, with possible minima at finite momenta. The detailed dependence of the relaxation rate on the spin-orbit interaction is analyzed in the following sections.

### III. THE SOLUTION OF THE TIME-DEPENDENT DIFFUSION EQUATION: THE SPIN SECTOR

In this section we discuss the spin relaxation spectrum and derive the diffuson in momentum space and real space. The main results of section III A are Eqs. (20), describing the spin relaxation spectrum and Fig. 1 which summarizes the properties of the spectrum. The behavior in the vicinity of the symmetry points characterized by an infinite spin relaxation time is illustrated in Fig. 2. Eqs. (30) and (33) describe spin diffusion in real space.

#### A. Momentum space picture

The diffusion approximation leading to equation (11) is rigourously valid only in the weak spin-orbit coupling regime characterized by $\Gamma v_F k_F \tau \ll 1$. In this limit, the spin-charge coupling is small, $\delta|\mu_{1,2}| \ll 1$, and can be neglected.
in the leading approximation. The goal of this section is to determine the role played by the interference between different spin-orbit interaction terms in spin diffusion. In particular, motivated by experiments that probe spin dynamics optically, we concentrate on the properties of the out-of-plane component of the spin density. To address this problem, we have to determine the element \( D_{zz} = [1 - \Pi^{-1}]_{zz} \) of the Green’s function for the diffusion equation (the diffusion). The general expression for the diagonal \( z \)-component of the diffusion in the absence of spin-charge coupling is

\[
D_{zz}(s, k) = \frac{s^2 - (\gamma_R \gamma_D / 2)^2}{\Delta(s, k)},
\]

where \( \Delta(s, k) \) is a third order polynomial in \( s \),

\[
\Delta(s, k) = s^2(s + 1) - s \left( (\gamma_R^2 + \gamma_D^2) k^2 + 4 \gamma_R \gamma_D k_x k_y + \left( \frac{\gamma_D^2}{2} \right)^2 \right) + \gamma_R^2 \gamma_D^2 k^2 + \gamma_R \gamma_D \left( \gamma_R + \gamma_D \right) k_x k_y - \left( \frac{\gamma_D^2}{2} \right)^2
\]

(18)

The spin dynamics is determined by the three spin relaxation rate modes \( i\omega_j(k) = -s_j(k) + k^2 + 1 \), obtained by solving the equation \( \Delta(s, k) = 0 \). For later reference, let us identify the modes that control the spin dynamics, together with the charge mode, according to their zero momentum values,

\[
i\omega_0(0) = 0,
\]

\[
i\omega_1(0) = 1 - \frac{\gamma_R \gamma_D}{2},
\]

\[
i\omega_2(0) = 1 + \frac{\gamma_R \gamma_D}{2},
\]

\[
i\omega_3(0) = 2,
\]

(19)

where the mode \( i\omega_0 \) is responsible for charge diffusion and has the standard form \( i\omega_0(k) = k^2 \) if we neglect spin-charge coupling. Let us first focus on the experimentally relevant special case of momentum parallel to the [110] or [110] directions corresponding to \( k_\perp \). Inverting the matrix from Eq. (13) we obtain,

\[
D_{zz}(s, k_\perp) = \frac{s \pm \frac{2 \gamma_D^2}{\gamma_R}}{s^2 + s \left( 1 \pm \frac{2 \gamma_D^2}{\gamma_R} \right) - k_\perp^2 \left( \gamma_R \pm \gamma_D \right)^2 \pm \frac{2 \gamma_D^2}{\gamma_R}}.
\]

(20)

Notice that, for k-vectors oriented along these special directions, one of the modes \( i\omega_1 \) or \( i\omega_2 \) does not contribute to the dynamics and, consequently, the problem simplifies significantly. The time dependence of \( D_{zz} \) can be obtained by Fourier transforming (20) with respect to the frequency. This time dependence is uniquely determined by the roots of the secular equation \( \Delta(s, k_\perp) = 0 \), i.e. by

\[
i\omega_1(k_-) = \frac{3}{2} + k_-^2 + \frac{\gamma_R \gamma_D}{4} - \frac{1}{2} \sqrt{\left( 1 + \frac{\gamma_R \gamma_D}{2} \right)^2 + 4 k_-^2 (\gamma_R - \gamma_D)^2},
\]

\[
i\omega_2(k_+) = \frac{3}{2} + k_+^2 + \frac{\gamma_R \gamma_D}{4} - \frac{1}{2} \sqrt{\left( 1 - \frac{\gamma_R \gamma_D}{2} \right)^2 + 4 k_+^2 (\gamma_R + \gamma_D)^2},
\]

\[
i\omega_3(k_\pm) = \frac{3}{2} + k_\pm^2 \pm \frac{\gamma_R \gamma_D}{4} + \frac{1}{2} \sqrt{\left( 1 \pm \frac{\gamma_R \gamma_D}{2} \right)^2 + 4 k_\pm^2 (\gamma_R \pm \gamma_D)^2}.
\]

(21)

As mentioned above, the modes \( i\omega_1(k_+) = 1 + k_+^2 - \gamma_R \gamma_D / 2 \) and \( i\omega_2(k_-) = 1 + k_-^2 + \gamma_R \gamma_D / 2 \) do not contribute to the dynamics of the out-of-plane component of the spin. The explicit time dependence of \( D_{zz} \) for k-vectors along the ‘+’ direction is

\[
D_{zz}(t, k_+) = \frac{1}{2} \left[ e^{-i\omega_2(k_+) t} + e^{-i\omega_3(k_+) t} \right] - \frac{(1 - \frac{2 \gamma_D}{\gamma_R})}{2 \sqrt{(1 - \frac{2 \gamma_D}{\gamma_R})^2 + 4 k_+^2 (\gamma_R + \gamma_D)^2}} \left[ e^{-i\omega_2(k_+) t} - e^{-i\omega_3(k_+) t} \right].
\]

(22)

A similar expression can be obtained for \( D_{zz}(t, k_-) \) by replacing \( k_+ \) with \( k_- \), \( i\omega_2 \) with \( i\omega_1 \) and changing the sign of \( \gamma_D \). The spin dynamics at large times, \( t \gg \tau_s \), is controlled by the minima of the relaxation rate modes \( i\omega_j(k_\perp) \). The mode \( i\omega_3 \) has always the minimum at \( k = 0 \), \( i\omega_2(0) = 2 \). On the other hand, the position and the values of the minima for the other two modes depend on the spin-orbit coupling parameters. The mode \( i\omega_1 \) has a minimum \( i\omega_1(0) = 1 - \gamma_R \gamma_D / 2 \) at zero wave-vector if \( \gamma_R^2 + \gamma_D^2 - 5 \gamma_R \gamma_D / 2 < 1 \). Otherwise, the dispersion curve has a local
maximum at $k = 0$ and the minimum occurs at a finite wave-vector $k_0^0$. Similarly, the mode $i\omega_2$ has either a zero wave-vector minimum, if $\gamma_R^2 + \gamma_D^2 + 5\gamma_R\gamma_D/2 < 1$, or otherwise a finite k-vector minimum at $k_0^0$. The finite wave-vectors that describe the positions of the minima are

$$|k_{\pm}^0| = \frac{\sqrt{(\gamma_R \pm \gamma_D)^4 - (1 + \frac{2\gamma_R\gamma_D}{2})^2}}{2|\gamma_R \pm \gamma_D|},$$

while the frequencies corresponding to the minima in the dispersion curves are

$$i\omega_1(k_{+}^0) = \frac{3}{2} - \frac{\gamma_R\gamma_D}{4} - \frac{(1 + \frac{2\gamma_R\gamma_D}{2})^2}{4(\gamma_R - \gamma_D)^2} - \frac{(\gamma_R - \gamma_D)^2}{4},$$

$$i\omega_2(k_{+}^0) = \frac{3}{2} + \frac{\gamma_R\gamma_D}{4} - \frac{(1 - \frac{2\gamma_R\gamma_D}{2})^2}{4(\gamma_R + \gamma_D)^2} - \frac{(\gamma_R + \gamma_D)^2}{4}.$$ (24)

A summary of the conditions necessary for the existence of these minima in terms of the spin-orbit coupling parameters $\gamma_R$ and $\gamma_D$ is presented in Fig. 1. In the red domain the condition $\gamma_R^2 + \gamma_D^2 + 5\gamma_R\gamma_D/2 < 1$ is satisfied and all the dispersion curves have minima at zero momentum. The blue regions are characterized by a finite wave-vector minimum of the $i\omega_2$ mode along the $k_+$ direction and a zero momentum minimum of the $i\omega_1$ mode, while the yellow sectors correspond to the $[i\omega_1(k_{+}^0), i\omega_1(0)]$ pair of minima. Finally, in the green regions both modes, $i\omega_1$ and $i\omega_2$ have finite k-vector minima corresponding to $k_{+}^0$ and $k_{+}^0$, respectively. Notice that spin diffusion is isotropic only if $\gamma_R\gamma_D = 0$, i.e. for pure Rashba spin-orbit interaction ($\beta_1 = \beta_3 = 0$ or $(\gamma_R, \gamma_D) = (\pm2,0)$, the points R and R’ in Fig. 1, pure Dresselhaus spin-orbit coupling ($\alpha = 0$ or $\gamma_R = 0$, the segment D’D in Fig. 1), or a combination of Rashba and Dresselhaus contributions with the linear and cubic Dresselhaus terms being equal ($\beta_1 = \beta_3$ or $\gamma_D = 0$ and $\alpha \neq 0$, the segment RR’ without the end points). In all the other cases the spin diffuses anisotropically as a result of the interference between the spin-orbit coupling terms.

To be more specific, let us consider now the case $\gamma_R > 0$, $\gamma_D > 0$, i.e. the quarter defined by the points R, O, and D in Fig. 1. We assume that a uniform out-of-plane spin density was injected into the system and ask about its
time evolution. The answer can be obtained using Eq. (22) for \( k_+ = 0 \). For a uniform system, the out-of-plane spin diffusion equation that takes the trivial form \( D_{zz}(t, 0) = \exp(-2t) \), which corresponds to \( i\omega_3(0) = 2 \) representing the uniform spin relaxation rate of the \( S_z \)-component. If we consider now the limit of small non-zero momenta an expand all the quantities up to second order terms in \( k \), we obtain

\[
D_{zz}(t, k) \approx \frac{(\gamma_R - \gamma_D)^2}{(1 + \frac{2\gamma_D}{\gamma})^2} k^2 e^{-i\omega_1(k)t} + \frac{(\gamma_R + \gamma_D)^2}{(1 - \frac{2\gamma_D}{\gamma})^2} k^2 e^{-i\omega_2(k)t} + e^{-i\omega_3(k)t},
\]

where the dispersion relations at small \( k \)-vectors take the form

\[
\begin{align*}
  i\omega_1(k) &\approx i\omega_1(0) + k^2 - \frac{(\gamma_R - \gamma_D)^2}{1 + \frac{2\gamma_D}{\gamma}} k^2, \\
  i\omega_2(k) &\approx i\omega_2(0) + k^2 - \frac{(\gamma_R + \gamma_D)^2}{1 - \frac{2\gamma_D}{\gamma}} k^2, \\
  i\omega_3(k) &\approx i\omega_3(0) + k^2 + \frac{(\gamma_R - \gamma_D)^2}{1 + \frac{2\gamma_D}{\gamma}} k^2 + \frac{(\gamma_R + \gamma_D)^2}{1 - \frac{2\gamma_D}{\gamma}} k^2,
\end{align*}
\]

where \( k^2 = k_x^2 + k_y^2 = k_x^2 + k_z^2 \). The initial decay of the spin density is dominated by the third term in Eq. (25) that has the form \( \exp(-2(2 + D_k^+ k_x^2 + D_- K_z^2) t) \). Here, \( D_{\pm} \) are effective diffusion constants along the \( k_\pm \) directions. This result shows explicitly that the diffusion of the out-of-plane component of the spin is, in general, anisotropic, and that the anisotropy increases rapidly as we approach the symmetry point. Another consequence of these relations is that the asymptotic behavior at \( t \gg \tau_s \) is controlled by the \( i\omega_1 \) mode, unlike the uniform case described solely by \( i\omega_3 \). In the isotropic case, the relaxation times corresponding to these modes differ by a factor of two, \( \tau_1/\tau_3 = i\omega_3/i\omega_1 = 2 \). As we approach the symmetry point, this ratio diverges, \( \tau_1/\tau_3 = 4/(2 - \gamma_R\gamma_D) \). At the same time, the pre-factor of the first term in Eq. (25) vanishes as we approach the \( OS_1 \) line (see Fig. 1) characterized by equal Rashba and Dresselhaus couplings, or if the wave-vector is oriented along the \( k_+ \) direction. We conclude that in general the asymptotic spin relaxation at small (but not-vanishing) \( k \)-vectors differs strongly from the spin relaxation at \( k = 0 \). In addition, the small wave-vector behavior is, in general, anisotropic as a result of the interference between different spin-orbit coupling terms.

Next, we focus on the finite momentum minima of the dispersion relations (23) and (24). The existence of these minima for the \( i\omega_1 \) and \( i\omega_2 \) modes implies an increase in the spin lifetime relative to \( k = 0 \) for the corresponding mode by a factor \( i\omega_{1(2)}(0)/i\omega_{1(2)}(k_\pm^0) \). For example, in the pure Rashba case we obtain \( k_+^0 = \sqrt{15}/4 \) in units of \( 1/L_x \), or \( k_+^0 = \alpha k_F \sqrt{15}/2 \) in standard units, and an increase in the lifetime by a factor of \( 16/7 \), in agreement with previous calculations based on the Rashba model.

Another important special case is represented by the symmetry points characterized by equal Rashba and Dresselhaus couplings, more precisely defined by \( |\gamma_R| = |\gamma_D| = \sqrt{2} \) (the \( S_z \) points in Fig. 1). At a symmetry point the minimum in the dispersion curves vanishes, \( i\omega_{1(2)}(\sqrt{2}) = 0 \), reflecting the absence of spin relaxation at the corresponding wave-vector, \( k_+^0 = \sqrt{2} \) (or \( k_0^0 = \sqrt{2} \) if \( \gamma_R \gamma_D < 0 \)). The existence of a finite \( k \)-vector minimum in the dispersion curves is illustrated in Fig. 2. The black curve with circles corresponds to the symmetry point \( \gamma_R = \gamma_D = \sqrt{2} \) and has a gapless minimum at \( k_+^0 = \sqrt{2}/L_x \). As we move away from the symmetry point, a gap develops (red and green lines). How is the gap behaving as we vary the coupling parameters in the vicinity of the symmetry point? To answer this question, we define \( d_\gamma \) as the distance in the parameter space \( (\gamma_R, \gamma_D) \) from the symmetry point

\[
d_\gamma = [(\gamma_R - \sqrt{2})^2 + (\gamma_D - \sqrt{2})^2]^{1/2}.
\]

If we move away from the symmetry point by changing the ratio between the Rashba and linear Dresselhaus interactions in the absence of a cubic Dresselhaus term, i.e. by varying the ratio \( \alpha/\beta_1 \) and keeping \( \beta_3 = 0 \) (see the inset of Fig. 2 red curve), the gap remains very small in comparison with the zero momentum frequency \( i\omega_2(0) \approx 2/\tau_s \). In contrast, the gap develops rapidly if we move along a direction in parameter space corresponding to \( \gamma_R = \gamma_D \) (green line). This observation suggests that, in order to observe experimentally the “absence” of spin relaxation at a certain finite momentum, the minimization of the cubic Dresselhaus interaction should be a major concern. In addition, we have to remember about the small momenta relaxation times given by equations (25) and (26), in particular the one associated with the \( i\omega_1 \) mode. To summarize the results concerning relaxation times, we refer to the diagram in Fig. 1 For a uniform system \((k=0)\), the only mode that contributes to the dynamics is \( i\omega_3 \), and the corresponding spin relaxation time of the out-of-plane component \( S_z \) is \( \tau_s/2 \), regardless of the spin-orbit coupling parameters. The system can have a slower relaxation at non-zero momenta that can be either arbitrarily small or of order \( 1/L_x \), depending on the values of the coupling parameters \( \gamma_R \) and \( \gamma_D \). In the red region, all three modes \( i\omega_j(k) \) have the minimum at
FIG. 2: Momentum dependence of the relaxation rate mode $\omega_{2}(k_{\gamma})$ for spin-orbit coupling parameters in the vicinity of the symmetry point $\gamma_{R} = \gamma_{D} = \sqrt{2}$ (point $S_{1}$ in Fig. 1). In this regime, the $\omega_{2}$ mode has a finite wave-vector minimum along the $k_{\gamma}$ direction. At the symmetry point, the dispersion curve becomes gapless (black curve and circles), meaning that the spin relaxation time at $k_{\gamma}^{0} = \sqrt{2}/L_{s}$ is infinite. As we depart from the symmetry point, a finite gap opens. The spin-orbit coupling parameters for the red curve correspond to a point on the $R_{S1}D$ arc in Fig. 1, while the green curve is for a point on the $S_{1}O$ segment, both points being in the vicinity of $S_{1}$. The gap size as a function of the distance $d_{\gamma}$ from the symmetry point [see Eq. (24) for the definition of $d_{\gamma}$] is shown in the inset. The red curve correspond to a spin orbit interaction with vanishing cubic Dresselhaus contribution $\beta_{3} = 0$ (the $R_{S1}D$ arc), while the green curve was obtained for $\gamma_{R} = \gamma_{D} < \sqrt{2}$ (the $S_{1}O$ segment). To observe a substantial enhancement of the spin relaxation time at finite momentum, i.e. to have a small gap in the relaxation rate curve, the cubic Dresselhaus spin-orbit interaction $\beta_{3}$ should be minimal.

$k = 0$ and the largest spin relaxation time, $\tau_{\text{max}}(0) = \tau_{s}/(1 - |\gamma_{R}\gamma_{D}|)/2$, is obtained at small k-vectors. In the blue (yellow) region, in addition to this small momentum minimum, we have the minima at $k_{\gamma}^{0}$ ($k_{\gamma}^{0}$) given by equations (23) and (24) and to determine the slowest spin relaxation we have to compare $\tau_{\text{max}}(0)$ with $\tau_{\text{max}}(k_{\gamma}^{0}) = \tau_{s}/i\omega_{2}(k_{\gamma}^{0})$. Finally, in the green regions, the largest spin relaxation time is associated with one of the minima at $k_{\gamma}^{0}$. The diffusion is isotropic if $\gamma_{R}\gamma_{D} = 0$ and anisotropic otherwise, especially in the vicinity of the symmetry points.

B. Real space picture

We turn now our attention to the real space behavior of $D_{zz}$, corresponding to the diffusion of a delta function-like spin density injected in the origin at $t = 0$. While the analytical treatment of the general case its rather complicated, we can easily solve special cases, like the isotropic case ($\gamma_{R}\gamma_{D} = 0$), or the symmetry point $\gamma_{R} = \pm \gamma_{D} = \sqrt{2}$. In particular, for the isotropic case we obtain from Eq. (11), after inverting the matrix and performing a Fourier transform with respect to the frequency,

$$D_{zz}(t, k) = \frac{1}{2} \left[ 1 + \frac{1}{\sqrt{1 + 4\gamma^{2}k^{2}}} \right] e^{-\left(\frac{k^{2}}{2} + \frac{1}{2}\sqrt{1 + 4\gamma^{2}k^{2}}\right)t} + \frac{1}{2} \left[ 1 - \frac{1}{\sqrt{1 + 4\gamma^{2}k^{2}}} \right] e^{-\left(\frac{k^{2}}{2} - \frac{1}{2}\sqrt{1 + 4\gamma^{2}k^{2}}\right)t},$$

(28)

where $\gamma = \gamma_{R}$ if $\gamma_{D} = 0$ and $\gamma = \gamma_{D}$ if we have a pure Dresselhaus spin-orbit coupling. The real space dependence is obtained after performing another Fourier transform, with respect to momentum, which reduces to the one dimensional integral

$$D_{zz}(t, r) = \frac{1}{4\pi} \int_{0}^{\infty} dk \ J_{0}(kr) D_{zz}(t, k),$$

(29)

where $J_{0}(kr)$ represents a Bessel function of the first kind. The time dependence of $D_{zz}(t, r)$ at distance $r$ form the origin is shown in Fig. 3 for the particular case of pure Rashba coupling $\gamma = 2$. The curves represent the time
evolution of the out-of-plane spin density at different positions in the plane after spin was injected in the origin at 
\( t = 0 \). Notice the oscillations in the direction of the spin polarization (see Fig. 1). The analytical expressions for these
oscillations can be derived in the large time limit using a saddle point approximation for the integral over momenta
[more precisely in the limit \( t/\tau_s \gg (r/L_s)^2 \)]. For the isotropic case this asymptotic behavior has the form
\[
D_{zz}(t, r) \approx \frac{\gamma^2 - 1}{8\gamma \sqrt{\pi t}} J_0 \left( \frac{\sqrt{\gamma^4 - 1}}{2\gamma} r \right) e^{\frac{-2\gamma^2 - \gamma^4 - 1}{4\gamma} t}, \quad \text{if } |\gamma| > 1, \\
D_{zz}(t, r) \approx \frac{\gamma^2}{4\pi(1 - \gamma^2)t^2} e^{-t}, \quad \text{if } |\gamma| < 1,
\]
where \( \gamma = \gamma_R \) or \( \gamma = \gamma_D \), as appropriate. The asymptotic behavior is qualitatively different for \( |\gamma_R| > 1 \) (or \( |\gamma_D| > 1 \)), the green regions in Fig. 1 and \( |\gamma_R| < 1 \) (or \( |\gamma_D| < 1 \), red zone. In the first case, \( D_{zz}(t, r) \) oscillates in space at large distances away from the origin, \( r/L_s \gg 1 \), with a period \( \lambda = 2\pi/(L_s k^0) \) and an amplitude that decays as \( \exp[-i\omega(k^0)t]/\sqrt{\pi t} \), where \( k^0 = k^0_+ = k^0_- = i\omega(k^0) = i\omega_1(k^0) = i\omega_2(k^0) \) are the parameters for the finite momentum minima of the dispersion relations described by equations (23) and (24). By contrast, in the red region, spin diffusion at large times, \( t/\tau_s \gg (r/L_s)^2 \), is independent of position. In addition, the spin relaxation time has the value \( \tau_s \), independent of the spin-orbit coupling, and the time dependence of the pre-factor is \( 1/t^2 \), instead of the \( 1/\sqrt{t} \) dependence for the oscillating case.

For \( \gamma_R \gamma_D \neq 0 \) the dynamics of the \( S_z \) spin component is anisotropic. The simplest analytical treatment of such a case obtains at the symmetry points \( |\gamma_R(D)| = \sqrt{2} \). Starting with Eq. (13), we get, after inverting the matrix and performing a Fourier transform with respect to the frequency,
\[
D_{zz}(t, \mathbf{k}) = \frac{1}{4\pi t} e^{-\frac{\gamma^2}{4\pi^2}[k_+^2 + k_-^2]t} \left[ 1 + \frac{1}{2} e^{-\frac{\gamma^2(\sqrt{2}k_+)^2}{4\pi^2}t} + \frac{1}{2} e^{-\frac{\gamma^2(\sqrt{2}k_-)^2}{4\pi^2}t} \right],
\]
where we assumed that \( \gamma_R \gamma_D = 2 \), which corresponds to the symmetry points \( S_1 \) and \( S_1' \) in Fig. 1. A similar equation can be obtained for \( \gamma_R \gamma_D = -2 \) by exchanging \( k_+ \) and \( k_- \). This equation does not acquire any spin-charge coupling corrections, as the coupling vanishes at the symmetry points, \( \mu_0 = \mu_1 = 0 \), regardless of the strength of the spin-orbit interaction. Next, after performing the Fourier transform with respect to momentum, we obtain the exact real space dependence at the symmetry point,
\[
D_{zz}(t, r) = \frac{1}{4\pi t} e^{-\frac{\gamma^2}{4\pi^2} \cos(\sqrt{2} r_+)}.
\]
FIG. 4: Real space dependence of the out-of-plane spin density in a system with isotropic spin-orbit interaction. The curves represent \( D_{zz}(t, r) \) multiplied by the exponential factor \( \exp(t/\tau_m) \), where \( t = 36 \) (in units of \( \tau_s \)) and \( \tau_m = (6\gamma_2 - \gamma_4 - 1)/(4\gamma_2) \) represents the maximum relaxation time corresponding to the finite \( k^0 \) minimum of the relaxation rate. At short distances, \( r < \sqrt{1} = 6 \), the solutions are described by Eq. (30). The period of the spatial oscillations increases as we decrease \( \gamma: \gamma = 2 \) (pure Rashba or pure linear Dresselhaus interaction) - red line and circles, \( \gamma = 1.2 \) - green line and triangles, \( \gamma = 1 \) - blue line and stars. Notice that, as we approach the non-oscillatory regime characterized by \( \gamma < 1 \), the spin density becomes independent of distance in the asymptotic limit \( t \gg r^2 \) (see Eq. (33) and the \( \gamma = 1 \) curve). The inset shows the spatial dependence of the spin density of a pure Rashba system (\( \gamma = 2 \)) at different times. Again, we multiplied the results by the exponential factors \( \exp(t/\tau_m) \). Notice that in the limit described by Eq. (30), the curves are proportional, with proportionality factors 1/3 (black curve and diamonds, \( t = 9 \)), 1/4 (cyan with squares, \( t = 16 \)) and 1/6 (red line and circles, \( t = 36 \)).

where \( r^2 = r_+^2 + r_-^2 \) and we assumed that \( \gamma_R \gamma_D = 2 \). Because of the gapless minimum of the \( i\omega_2(k_+) \) mode, the large time decay is proportional to \( 1/t \), rather than exponential. In addition, the spin density oscillates along the \( r_+ \) direction, due to the finite momentum minimum, while it becomes independent of \( r_- \) in the large time limit. For a general anisotropic case, the asymptotic behavior is determined by the minimum of the lowest relaxation rate. A characterization of the locations of the lowest minimum in the plane of the spin-orbit coupling parameters is provided in Fig. 5. We determine the asymptotic spin dynamics using a quadratic approximation for the dispersion curves in the vicinity of the minima. The approximate solutions of the secular equation \( \Delta(s, k) \) for wave-vectors with arbitrary \( k_+ \) and small \( k_- \) components are given in Appendix A. Assuming that \( \gamma_R \gamma_D > 0 \), the zero momentum minimum of the \( i\omega_1 \) mode represents the lowest energy contribution for a system with spin-orbit coupling parameters corresponding to the pink zone in Fig. 4. The asymptotic \( S_z \)-spin diffusion in the zone I of the parameter space (see Fig. 4) is described by

\[
D_{zz}^{I}(t, r) \approx \frac{2tC_0^{(1)} - r^2}{16\pi t^3} \left( \frac{\gamma_R - \gamma_D}{1 + \frac{2\gamma_D}{\gamma_R}} \right)^2 \left( C_0^{(1)} \right)^2 e^{-\frac{1}{4} \left( \frac{r^2}{r_+^2} + \frac{r^2}{r_-^2} \right) - (1 - \frac{2\gamma_D}{\gamma_R})t},
\]

where the coefficient \( C_0^{(1)} \) is given by Eq. (34). Eq. (33) is valid for times larger than the spin relaxation time, \( t \gg \tau_s \), when \( \gamma_R \neq \gamma_D \). When the couplings are equal, the \( i\omega_1 \) mode whose \( k = 0 \) minimum generates the asymptotic behavior described by Eq. (33) does not contribute to the spin dynamics. Instead, for \( \gamma_R = \gamma_D \) corresponding to the OP segment in Fig. 4, the lowest minimum is the zero momentum minimum of the \( i\omega_2 \) mode and it will generate an expression similar to Eq. (33), but with \( r_+ \rightarrow r_\pm \) and \( \gamma_D \rightarrow -\gamma_D \). Notice that in this regime spin diffusion is anisotropic only for times of order \( r^2 \) or smaller and becomes isotropic at longer times. Finally, for the anisotropic regime controlled by a finite momentum minimum in the dispersion relations, the blue and yellow regions and the segment \( PS_1 \) in Fig. 5, the asymptotic expression for \( D_{zz} \) can be obtained using the quadratic expansions given in Appendix A in particular equations (37) and (38). We obtain the asymptotic expression for the diffusion in the
regime described by zone II of the parameter space (see Fig. [5])

\[
D^{II}_{zz}(t, r) \approx \frac{1}{4\pi t} \frac{D^{(2)}_{k_0}}{B^{(2)}_{k_0^+} C^{(2)}_{k_0^+}} \exp \left[ -\frac{1}{4t} \left( \frac{r^2}{B^{(2)}_{k_0^+}} + \frac{r^2}{C^{(2)}_{k_0^+}} \right) - i\omega_2(k_0^+)t \right] \cos(k_0^0 r_+),
\]

where \(D^{(2)}_{k_0} = 1 - (1 - \gamma_R\gamma_D/2)/(\gamma_R + \gamma_D)^2\), \(B_{k_0^+}^{(2)}\) and \(C_{k_0^+}^{(2)}\) are given in Appendix A and \(i\omega_2(k_0^+)\) represents the \(k_0^+\) minimum of the \(i\omega_2\) mode given explicitly by equations (28) and (24). Equation (34) describes the large time dynamics of the out-of-plane spin component when the \(i\omega_2(k_0^+)\) represents lowest energy minimum, i.e., for the blue zone and the segment \(PS_1\) in Fig. [5]. A similar expression can be obtained when the dynamics is controlled by the \(i\omega_1(k^-)\) minimum (yellow zone III) using the correspondence rules \(i\omega_2 \leftrightarrow i\omega_1, r_+ \leftrightarrow r_-\) and \(\gamma_D \rightarrow -\gamma_D\):

\[
D^{III}_{zz}(t, r) \approx \frac{1}{4\pi t} \frac{D^{(2)}_{k_0}}{B^{(2)}_{k_0^-} C^{(2)}_{k_0^-}} \exp \left[ -\frac{1}{4t} \left( \frac{r^2}{B^{(2)}_{k_0^-}} + \frac{r^2}{C^{(2)}_{k_0^-}} \right) - i\omega_1(k_0^-)t \right] \cos(k_0^0 r_-),
\]

where \(D^{(2)}_{k_0} = 1 - (1 + \gamma_R\gamma_D/2)/(\gamma_R - \gamma_D)^2\) and the coefficients \(B_{k_0^-}^{(2)}\) and \(C_{k_0^-}^{(2)}\) are given in Appendix A. Notice that Eq. (34,35) reduce to Eq. (31) at the symmetry points \(\gamma_R = \gamma_D = \pm \sqrt{2}\) where it becomes exact. Within this regime, the solution of the diffusion equation is characterized by real space oscillations with a period \(\lambda = 2\pi L_s/k_0^0\) determined by the position of the minimum in k-space while, at times much larger than \(\tau_s r^2/L_s\), the solution becomes independent of \(r\). The relaxation time, \(\tau_s/(i\omega_2(k_0^+))\), can be much larger than the relaxation time in the regime controlled by the zero momentum minimum, \(\tau_s/(i\omega_1(0))\), and diverges as we approach the symmetry points. On the other hand, when we approach the \(\gamma_R = 0\) or \(\gamma_D = 0\) axes, the solution becomes isotropic and some of the
approximations used in deriving equations (33) and (34) are no longer valid. Instead, the asymptotic behavior in the isotropic regime is described by Eq. (30).

IV. THE ROLE OF THE SPIN-CHARGE COUPLING

In this section we study the effects of spin-charge coupling on spin and charge diffusion. The main results of this section are equations (37) and (42) describing the behavior of the relaxation rate modes in the presence of the spin-charge coupling.

In the absence of spin-charge coupling, the charge diffuses according to the standard diffusion equation, while a spin density characterized by a given k-vector decays, in general, as a sum of three exponentials. Recent spin grating experiments have shown that, in certain conditions, the time decay of an optically injected spin density may include an oscillatory component. In this section, we explore the possibility that the appearance of these oscillations is due to spin-charge coupled dynamics. To this end, we re-derive the Green’s function of the diffusion equation starting from the polarizability matrix Eq. (11), or from Eq. (13), but without neglecting the spin-charge coupling terms proportional to $\mu_0$ and $\mu_1$. Inverting the polarizability matrix produces, in general, a complicated expression for the diffusion. However, there are several special cases in which significant simplifications occur. These cases involve, on the one hand, special values of the spin-orbit coupling parameters, namely the pure Rashba interaction, $\gamma_D = 0$, $\gamma_R = \pm 2$, the pure Dresselhaus coupling, $\gamma_D = 0$, and the symmetry points $\gamma_R = \pm \sqrt{2}, \gamma_D = \pm \sqrt{2}$. On the other hand, the inversion of $\Pi(\omega, \mathbf{k})$ can be performed analytically for arbitrary values of the spin-orbit coupling parameters along the special directions in momentum space corresponding to $k_\perp$ and $k_\parallel$. Our strategy is to obtain explicit analytical expressions for these special cases, and use them, in conjunction with numerical results for arbitrary sets of parameters, to extract and characterize the main features of the general solution. We start with the isotropic case corresponding to a pure Rashba coupling, or a pure Dresselhaus spin-orbit interaction. The modes that control the coupled spin-charge dynamics are the solutions of the secular equation

$$\Delta(s, \mathbf{k}) \equiv [s(s + 1) - \gamma^2 k^2][s(s - 1) + g^2 \mu^2 k^2] = 0, \quad (36)$$

where $\Delta(s, \mathbf{k})$ is the determinant of the $\tilde{\Pi}(s, \mathbf{k})$ matrix, $s = -i\omega + 1 + k^2$ and $\gamma = \pm 2$ for the Rashba case, or $\gamma = \gamma_D \in [-2, 2]$ for a pure Dresselhaus spin-orbit interaction. The spin-charge coupling is measured by the parameter $\mu_1$ and we have $\mu_1 = \mu_1 = \text{sign}(\alpha)\alpha^2$ in the Rashba case and $\mu_1 = \mu_0 = [(3\beta_3 - \beta_1)(\beta_3^2 - \beta_3^2) - \beta_1\beta_3^2^2] / [(\beta_1 - \beta_3)^2 + \beta_3^2]^1/2$ for the Dresselhaus coupling. The full expression of the diffusion matrix is given in Appendix D. The real time behavior of the diffusion propagator can be obtained after performing a Fourier transform of $\tilde{\Pi}(\omega, \mathbf{k})$ with respect to the frequency. This time dependence is determined by the roots $s_j = -i\omega_j + 1 + k^2$ of equation (36), and has the general form (17). For the pure cases described by the equations (30) and (31), the four modes that control the coupled spin-charge dynamics are

\[
\begin{align*}
  i\omega_0(k) &= \frac{1}{2} + k^2 - \frac{1}{2} \sqrt{1 - 4g^2\mu^2 k^2} \\
  i\omega_1(k) &= \frac{1}{2} + k^2 + \frac{1}{2} \sqrt{1 - 4g^2\mu^2 k^2} \\
  i\omega_2(k) &= \frac{3}{2} + k^2 - \frac{1}{2} \sqrt{1 + 4\gamma^2 k^2} \\
  i\omega_3(k) &= \frac{3}{2} + k^2 + \frac{1}{2} \sqrt{1 + 4\gamma^2 k^2}
\end{align*}
\]

(37)

where $g = 2v_F k_F \nu$, $k^2 = k_x^2 + k_y^2$ and the k-vector is expressed, as usual, in units of $1/L_s$. Equation (37) has several key features. First, we notice that the mode responsible for the charge dynamics in the absence of spin-charge coupling, namely $i\omega_0$, couples with only one spin mode, $i\omega_1$, while the other spin modes, $i\omega_2$ and $i\omega_3$, remain unaffected by the spin-charge coupling. Consequently, the modes $i\omega_2$ and $i\omega_3$ are real and positive for any value of the momentum, and the corresponding terms in Eq. (17) will decay exponentially in time. On the other hand, the solutions $i\omega_0$ and $i\omega_1$ are real only if $2g0^2 k_F L_s = 2\tau v_F k_F \nu < 1$. This condition is always satisfied in the weak spin-orbit coupling regime, in which the diffusion equation formalism applies. We conclude that within the diffusive regime a system will never exhibit oscillatory dynamics. Below, we show that this conclusion holds for arbitrary spin-orbit coupling parameters. Nonetheless, we can formally extrapolate Eq. (37) outside its strict domain of validity and apply it to the strong spin-orbit interaction regime. In this limit, i.e., for $\tau v_F k_F \nu > 1$, the modes $i\omega_0$ and $i\omega_1$ acquire an imaginary component, which generates oscillatory terms in the solution (17). We interpret this second major feature of Eq. (37) as an indication that oscillatory dynamics is a signature of strong spin orbit interactions. While the diffusion equation formalism cannot offer a quantitative description of this regime, it should represent a good indicator for the range of
where the wave vector $k$ is expressed in units of $1/L$, the out-of-plane spin component does not couple to the charge, due to symmetry. For example, from Eq. (B1) we observe that the time evolution of the diagonal parameters where the oscillatory behavior is likely to appear and for the degree in which various components of the spin and charge sectors are affected. For example, from Eq. (B1) we observe that the time evolution of the diagonal parameters where the oscillatory behavior is likely to appear and for the degree in which various components of the spin-charge coupling, the charge relaxes on the same time scale as the spin degrees of freedom. We mentioned above that for a pure Rashba (or pure Dresselhaus) spin-orbit coupling, the charge does not couple with the out-of-plane spin component due to symmetry. In the example shown in Fig. 6 similar symmetry reasons apply to the $S_z$ spin component, due to the particular choice of momentum along the $x$-axis. However, in general all spin components will couple to the charge. For example, the general expression for the $D_{00}$ matrix element describing the coupling between the in-plane $S_x$ spin component (see Eq. 38).

The time dependence of the diagonal components of the diffusion matrix in the weak spin-orbit coupling (left panel) and strong spin-orbit coupling (right panel) regimes. The matrix elements are evaluated for the pure Rashba case ($\gamma_R = 2, \gamma_D = 0$) at a momentum $(k_x, k_y) = (0.5/L_x, 0)$. In the weak spin-orbit coupling limit all the quantities decay exponentially, while in the opposite limit the charge and the $S_y$ channels exhibit oscillatory behavior. In the pure Rashba (or pure Dresselhaus) case the out-of-plane spin component does not couple to the charge, due to symmetry. For $k$ parallel to the $x$-axis, this is also the case for the in-plane $S_x$ spin component (see Eq. 38).

Explicitly, the time dependence of the matrix element $D_{00}$ describing charge relaxation is given by

$$D_{00}(t, k) = \begin{cases} 
\cosh \left( \frac{t}{2} \sqrt{1 - 4g^2 \mu_k^2 k^2} \right) + \frac{\sinh \left( \frac{t}{2} \sqrt{1 - 4g^2 \mu_k^2 k^2} \right)}{\sqrt{1 - 4g^2 \mu_k^2 k^2}} e^{-\left(\frac{1}{2} + k^2\right)t}, & \text{if } 2g\mu k < 1, \\
\cos \left( \frac{t}{2} \sqrt{4g^2 \mu_k^2 k^2 - 1} \right) + \frac{\sin \left( \frac{t}{2} \sqrt{4g^2 \mu_k^2 k^2 - 1} \right)}{\sqrt{4g^2 \mu_k^2 k^2 - 1}} e^{-\left(\frac{1}{2} + k^2\right)t}, & \text{if } 2g\mu k > 1,
\end{cases}$$

where the wave vector $k$ is expressed in units of $1/L_x$ and time in units of spin relaxation time, $\tau_s$. Notice that in the absence of spin-charge coupling, $D_{00}$ reduces to the standard solution of the diffusion equation, $D_{00}(t, k) = \exp(-k^2t)$, while the time dependence of the matrix elements associated with the spin degrees of freedom is determined by $s_2$ and $s_3$ only. This behavior is illustrated in Fig. 6 which shows the time dependence of all diagonal matrix elements $D_{ii}(t, k)$ for $k = (0.5, 0)$. The curves in the left panel were calculated in the weak spin-orbit coupling regime corresponding to $\Gamma = 0.0005$ and $g = 1000$. Notice that in the charge channel we have the standard exponential decay, $\exp(-k^2t)$, while in the spin channels the effect of the finite relaxation time $\tau_x = \tau_y = \tau_z/2$ is evident. By contrast, in the strong-coupling limit, the charge and $S_y$ channels exhibit oscillatory behavior. Also notice that, as a result of the spin-charge coupling, the charge relaxes on the same time scale as the spin degrees of freedom. We mentioned above that for a pure Rashba (or pure Dresselhaus) spin-orbit coupling, the charge does not couple with the out-of-plane spin component due to symmetry. In the example shown in Fig. 6 similar symmetry reasons apply to the $S_z$ spin component, due to the particular choice of momentum along the $x$-axis. However, in general all spin components will couple to the charge. For example, the general expression for the $D_{00}$ matrix element describing the coupling between

![Figure 6: Time dependence of the diagonal components of the diffusion matrix in the weak spin-orbit coupling (left panel) and strong spin-orbit coupling (right panel) regimes. The matrix elements are evaluated for the pure Rashba case ($\gamma_R = 2, \gamma_D = 0$) at a momentum $(k_x, k_y) = (0.5/L_x, 0)$. In the weak spin-orbit coupling limit all the quantities decay exponentially, while in the opposite limit the charge and the $S_y$ channels exhibit oscillatory behavior. In the pure Rashba (or pure Dresselhaus) case the out-of-plane spin component does not couple to the charge, due to symmetry. For $k$ parallel to the $x$-axis, this is also the case for the in-plane $S_x$ spin component.](image)
\[
D_{0z}(s, k) = \frac{g}{2\Delta(s, k)} (k_x^2 - k_y^2) \left[ \gamma_R \gamma_D (\gamma_D \mu_0 - \gamma_R \mu_1) + 2s (\gamma_D \mu_1 - \gamma_R \mu_0) \right],
\]

where \( \Delta(s, k) \) is the determinant of \( \hat{D}^{-1} \). Let us consider a set of parameters for which \( D_{0z} \) is non-zero. We show in Fig. 7 the time dependence of the diagonal matrix elements in the presence of both Rashbe and Dresselhaus terms, so that \( \gamma_R = 1 \) and \( \gamma_D = -1 \). In terms of the original coupling constants, this case corresponds to \( \alpha \approx 0.47 \Gamma \), \( \beta_1 \approx 0.19 \Gamma \), and \( \beta_3 \approx 0.66 \Gamma \). Notice that in the strong-coupling limit (right panel in Fig. 7) all channels exhibit oscillatory behavior, although the relative amplitude for the \( S_z \) component is very small. Again, we observe that \( D_{00} \) decays on a time scale of order \( \tau_s \), instead of the standard exponential decay \( \exp(-k^2 t) \). We conclude that strong spin-orbit interaction can generate two effects: 1) an oscillatory structure in the decay curves, and 2) an increase in the decay rate of the charge channel. The first effect should be observable above a certain critical momentum of order \( k_c \approx g\mu_i/L_s \). The second effect can be understood as a re-normalization of the diffusion constant. In the limit of small momenta equation (38) reduces to

\[
D_{00}(t, k) \approx \exp[-(1 + 2g^2 \mu_i^2)k^2 t].
\]

Consequently, in this limit the charge diffuses at long lengthscales according to the standard diffusion equation but with a re-normalized diffusion constant. This effective diffusion constant is enhanced by a factor \( 1 + 2g^2 \mu_i^2 \) over its bare value:

\[
D^{\text{eff}} \approx (1 + 2g^2 \mu_i^2) v_F^2 \tau/2.
\]

Formally, the enhancement factor can be very large. However, we have to remember that these conclusions are based on an extrapolation of the diffusion equation formalism outside its domain of validity and, therefore, should be seen as having qualitative rather than quantitative relevance. Strictly speaking, in the strong spin-orbit coupling regime the diffusion equation formalism breaks down and the notion of a diffusion coefficient becomes meaningless. The coupled spin-orbit dynamics is no longer described by a set of differential equations of type (11) and (13).
Our calculation using the diffusion equation formalism indicates that oscillatory dynamics may appear in a system with strong spin-orbit interaction as a consequence of spin-charge coupling. The effective strength of this coupling depends not only on the overall strength of the spin-orbit interaction, i.e. on $\Gamma$, but also on the interplay between different types of spin-orbit couplings. The net result of these competing effects is contained in the expressions of the spin-charge coupling coefficients $\mu_0$ and $\mu_1$ given by Eq. (12). In the examples considered above (see Fig. [4] and [7]), we have $(\mu_0, \mu_1) = (0, \Gamma^2)$ for the pure Rashba case and $(\mu_0, \mu_1) = (1.09 \Gamma^2, 1.40 \Gamma^2)$ for the $\gamma_R - 1 = -\gamma_D$ case. For comparison, the maximum values of the spin-charge coupling coefficients are $|\mu_0| \approx 1.10 \Gamma^2$ and $|\mu_1| \approx 1.65 \Gamma^2$. However, for other values of the spin-orbit coupling parameters these coefficients, and implicitly the spin-charge coupling effects, may be much weaker. In particular the special case corresponding to the symmetric points defined by $\gamma_R = \gamma_D = \pm \sqrt{2}$ deserves special attention (similar considerations can be made for the case $\gamma_R = -\gamma_D$ by interchanging the '+' and '-' components). In this case $\mu_0 = \mu_1 = 0$ and, consequently, the spin-charge coupling vanishes. Using the rotated coordinate system and expression (13) for the polarizability matrix, we obtain for the diffusion

$$D^{(\sigma)}(\omega, k) = \begin{pmatrix}
\frac{i}{s-1} & 0 & 0 & \frac{2\sqrt{2g}k_i}{(s+1)^2-8k_i^2} \\
0 & \frac{s+1}{(s+1)^2-8k_i^2} & 0 & 0 \\
0 & 0 & \frac{s+1}{(s+1)^2-8k_i^2} & 0 \\
0 & 0 & 0 & \frac{2\sqrt{2g}k_i}{(s+1)^2-8k_i^2}
\end{pmatrix}$$

(41)

The effects of the spin-charge coupling are totally absent at the symmetry points, regardless of the strength of the spin-orbit interaction. In addition, the diffusion matrix is independent of the $k_+$ wave-vector component and has a gapless mode for $k_+ = \pm \sqrt{2}$, as discussed in the previous section.

To understand the effects of spin-orbit coupling in a system with arbitrary spin-orbit interaction, we calculate the diffusion matrix corresponding to wave-vectors along the $k_\pm$ directions. From Eq. (15) we obtain the general expression for the matrix $D^{(\sigma)}(s, k_\pm)$ given in Appendix [3]. The real time behavior can be obtained by Fourier transforming [12] with respect to the frequency. Each matrix element has a time dependence of the form given by Eq. (17), except that this time the k-vector is restricted to the $k_\pm$ directions. The four modes that control the coupled spin-charge dynamics are

\[
\begin{align*}
\dot{i}\omega_0(k_+) &= \frac{1}{2} + k_+^2 - \frac{2\gamma_0}{4} - \frac{1}{2} \sqrt{(1 - \frac{2\gamma_0}{2})^2 - 4g^2(\mu_0 + \mu_1)^2k_+^2} \\
\dot{i}\omega_1(k_+) &= \frac{1}{2} + k_+^2 - \frac{2\gamma_1}{4} + \frac{1}{2} \sqrt{(1 - \frac{2\gamma_1}{2})^2 - 4g^2(\mu_0 + \mu_1)^2k_+^2} \\
\dot{i}\omega_2(k_+) &= \frac{3}{2} + k_+^2 + \frac{2\gamma_0}{4} - \frac{1}{2} \sqrt{(1 - \frac{2\gamma_0}{2})^2 + 4(\gamma_R + \gamma_D)^2k_+^2} \\
\dot{i}\omega_3(k_+) &= \frac{3}{2} + k_+^2 + \frac{2\gamma_1}{4} + \frac{1}{2} \sqrt{(1 - \frac{2\gamma_1}{2})^2 + 4(\gamma_R + \gamma_D)^2k_+^2}
\end{align*}
\]

(42)

where $\mu_0$ and $\mu_1$ are expressed in terms of the spin-orbit coupling parameters in Eq. (12). Similar expressions can be written for the $k_-$ direction, using the correspondence rules '+' $\leftrightarrow$ '-', $\gamma_D$ $\rightarrow$ $-\gamma_D$, $\mu_1$ $\rightarrow$ $-\mu_1$ and $i\omega_1$ $\leftrightarrow$ $i\omega_2$. As in the isotropic case discussed above, the charge mode $i\omega_0$ couples with only one spin mode, $i\omega_1$ for the $k_+$ direction and $i\omega_2$ for the $k_-$ direction, while the other spin modes remain unaffected by the spin-charge coupling. Also, for these special k-space directions, the dynamics of the out-of-plane component of the spin is determined by the spin modes that do not couple to the charge, i.e. $D_{zz}$ in Eq. (12) depends on $i\omega_2(k_+)$ and $i\omega_3(k_+)$ only. We conclude that oscillatory dynamics, the re-normalization of the diffusion constants, or any other spin-charge coupling effect should not be observable in the relaxation of an injected out-of-plane spin density with a wave-vector along the $k_\pm$ directions. However, for an arbitrary direction, the effects of the spin-charge coupling are present, as we have seen in Fig. [7] as all the modes contribute to the spin dynamics. The set of four relaxation rate modes, has some main qualitative features that are independent of direction: i) there are always at least two purely real modes, one of them being $i\omega_3$, and ii) an imaginary component may be acquired by the charge mode $i\omega_0$ and one of the spin modes $i\omega_1$ or $i\omega_2$, exactly which one of them depending on the direction in k-space. The appearance of the imaginary component takes place above a certain minimum value of the wave-vector satisfying the condition $2g|\mu_0 + \mu_1|k_{\text{min}} = 1 + \gamma_R\gamma_D/2$. We mention that the existence of such an imaginary component does not necessarily generates an oscillatory relaxation of the spin and charge densities. The general solution in this case contains two exponentially decaying terms an addition to the oscillatory term with exponentially decaying amplitude and the existence of an oscillatory component in the relaxation curves is determined by the relative intensity of these contributions. Finally, at small values of the wave-vector, the coupled spin-charge relaxation is always monotonic. However, the relaxation is, in general, anisotropic and the diffusion coefficients get re-normalized.
V. SUMMARY

In this paper, we studied spin diffusion in a generic spin-orbit coupled system. We found that there are a number of different dynamic regimes and phenomena controlled by the relative values of the spin-orbit coupling parameters. These unusual phenomena include an enhancement of the spin polarization lifetime at finite wave-vectors, anisotropic spin diffusion, an oscillatory behavior of the spin relaxation in real space, an enhancement of the effective charge diffusion coefficient, and possible real-time oscillations in spin diffusion dynamics. The existence and manifestations of these phenomena depend in a non-trivial manner on the strength of the various terms contributing to the spin-orbit interaction.

One of the main results of the paper is a general analytic form of the spin-charge coupled diffusion equation in the presence of all possible types of spin-orbit couplings. The parameters of this equation can be conveniently parameterized by three dimensionless coupling constants: The overall strength of the spin-orbit coupling (Γ) and two Rashba-type (γ_R) and Dresselhaus-type (γ_D) contributions to spin precession. The last two parameters must satisfy the physical constraint γ_R^2 + γ_D^2 ≤ 4. We showed that spin diffusion can be completely characterized by the parameters of these dimensionless parameters (γ_R, γ_D). We have identified the domains in the parameter space characterized by either non-zero or zero k-vector maxima of the spin relaxation spectrum, which determine the asymptotic dynamics at large times. One particularly important example we considered is the vicinity of the symmetry point |γ_R| = |γ_D| = √2, where the lifetime becomes arbitrarily long for certain finite k-vectors. Our analysis showed that, in order to observe experimentally this slow spin relaxation, special attention should be paid to minimize the cubic Dresselhaus spin-orbit interaction.

We have also qualitatively considered the regime of strong spin-charge coupling and extrapolated the results of the diffusion equation formalism to the strong spin-orbit interaction regime. We found that the enhancement of the charge diffusion coefficient and the oscillatory spin relaxation observed in recent spin-grating experiments can be qualitatively understood as a spin-charge coupling effect. We showed that the coupling of spin and charge is in general strongly k-dependent and can be characterized by two parameters µ_0 and µ_1, which depend on the spin-orbit interaction. These coupling parameters depend not only on the overall interaction strength Γ, but also on the interplay between the Rashba and Dresselhaus contributions. We argued that a necessary condition to observe an oscillatory behavior is a strong spin-orbit interaction, corresponding to the crossover from diffusive transport to the ballistic regime. For a given direction in k-space, the effect should be observable for k larger than a certain minimum wave-vector. However, we predict that the effect is not observable along certain directions in k-space for which spin-charge coupling is prevented by symmetry. For example, the out-of-plane spin polarization does not couple with the charge sector in the pure Rashba or pure Dresselhaus case. The coupling is also absent in general for directions in k-space corresponding to k_+ and k_− (see text). In addition, we predict that the effect is absent if the competition between the Rashba and Dresselhaus terms generates small coupling constants µ_0 and µ_1, regardless of the overall strength of the spin-orbit interaction. This is the case, for example, in the vicinity of the special symmetry points where µ_0 and µ_1 vanish. We should reiterate here that the diffusion approximation may capture spin-charge coupled dynamics only qualitatively if the spin-charge coupling is strong. To obtain a quantitative theory in this case, one should study the dynamics of the matrix distribution functions in the Boltzmann equation.

APPENDIX A: SOLUTION OF THE SPIN DIFFUSION EQUATION IN THE LIMIT OF SMALL k_−

The secular equation Δ(s, k) = 0 has simple analytical solutions if one of the wave-vector components k_− or k_+ vanishes. The asymptotic real time and real space spin dynamics is controlled by the minima of the dispersion curves iω_p(k) that are either at zero momentum, or at a finite value along the k_+ or k_− directions. To determine this asymptotic behavior, we need the solution of the secular equation in the vicinity of the minima. Assuming that k_− is small, we obtain for an arbitrary value of k_+ the following approximate solutions of the secular equation

\[ s_1(k) = \frac{\gamma_R \gamma_D}{2} = -\xi_1(k_+) k_+^2 + O(k_+^4), \]

\[ s_{2,3}(k) = -\frac{1}{2} - \frac{\gamma_R \gamma_D}{4} \pm \frac{1}{2} \sqrt{\left(1 - \frac{\gamma_R \gamma_D}{2}\right)^2 + 4k_+^2 (\gamma_R + \gamma_D)^2 - \xi_2,3(k_+) k_+^2 + O(k_+^4)}, \]

where the coefficients of the quadratic contributions in k_− are

\[ \xi_1(k_+) = \frac{(\gamma_R - \gamma_D)^2}{\gamma_R \gamma_D} \frac{\left(\gamma_R + \gamma_D\right)^2}{k_+^2 - \left(1 + \frac{2k_+}{\gamma_D}\right)}, \]
\[ \xi_{2,3}(k_+) = -\frac{\xi_1(k_+)}{2} + \frac{(\gamma_R - \gamma_D)^2 + \xi_1(k_+) (\frac{2}{3} + \frac{3}{2} \gamma_R \gamma_D)}{\sqrt{(1 - \frac{2\gamma^2_D}{2}) + 4k_+^2(\gamma_R + \gamma_D)^2}}. \]

Similar expressions can be obtained for wave-vectors with small \(k_+\) and arbitrary \(k_-\) components using the correspondence rules \(s_1 \leftrightarrow s_2, k_+ \leftrightarrow k_-\) and \(\gamma_D \rightarrow -\gamma_D\). We notice that this quadratic approximation is valid away from level crossing points. For example, the denominator of \(\xi_1\) vanishes for a set of parameters corresponding to a degeneracy point for the \(i\omega_1\) and \(i\omega_2\) modes.

The quadratic expansion around the \(k = 0\) minimum of the \(i\omega_1\) mode has the form

\[ i\omega_1(k) = 1 - \frac{\gamma_R \gamma_D}{2} + k_+^2 + C_0^{(1)}(\gamma_R, \gamma_D) k_+^2 + \mathcal{O}(k^4), \quad (A3) \]

with

\[ C_0^{(1)}(\gamma_R, \gamma_D) = 1 - \frac{(\gamma_R - \gamma_D)^2}{1 - \frac{2\gamma^2_D}{2}}. \quad (A4) \]

This zero momentum minimum exists provided \(C_0^{(1)}(\gamma_R, \gamma_D) > 0\), i.e. for spin-orbit coupling parameters corresponding to the red and blue zones in Fig. 1 but it does not contribute to the dynamics along the line with equal Rashba and Dresselhaus couplings, \(\gamma_R = \gamma_D\). Similarly, we obtain for the zero momentum minimum of the \(i\omega_2\) mode that exists within the red and yellow regions in Fig. 1 the expression

\[ i\omega_2(k) = 1 + \frac{\gamma_R \gamma_D}{2} + k_-^2 + C_0^{(2)}(\gamma_R, \gamma_D) k_-^2 + \mathcal{O}(k^4), \quad (A5) \]

with

\[ C_0^{(2)}(\gamma_R, \gamma_D) = 1 - \frac{(\gamma_R + \gamma_D)^2}{1 - \frac{2\gamma^2_D}{2}}. \quad (A6) \]

For the finite momentum minimum of the \(i\omega_2\) mode at \(k_0^0\) that exists within the blue and green zones (see Fig. 1) we obtain from equations (A1) and (A2),

\[ i\omega_2(k) \approx \frac{3}{2} + \frac{\gamma_R \gamma_D}{4} - \frac{(1 - \frac{2\gamma^2_D}{2})^2}{4(\gamma_R + \gamma_D)^2} - \frac{(\gamma_R + \gamma_D)^2}{4} + B^{(2)}_{k_0^0}(k_+ - k_-^0)^2 + C^{(2)}_{k_0^0} k_-^2, \quad (A7) \]

where the coefficients of the quadratic terms are

\[ B^{(2)}_{k_0^0}(\gamma_R, \gamma_D) = 1 - \frac{(1 - \frac{2\gamma^2_D}{2})^2}{(\gamma_R + \gamma_D)^2}, \quad (A8) \]

\[ C^{(2)}_{k_0^0}(\gamma_R, \gamma_D) = 1 - \frac{(\gamma_R - \gamma_D)^2}{(\gamma_R + \gamma_D)^2} \frac{\gamma_R^2 + \gamma_D^2 + 2\gamma_R \gamma_D - 1}{\gamma_R^2 + \gamma_D^2 + \frac{2}{3}\gamma_R \gamma_D - 1}. \]

Similar expressions can be obtained for the finite momentum minimum of the \(i\omega_1\) mode, corresponding to the yellow and green regions in Fig. 1 using the correspondence rules: \(i\omega_2 \rightarrow i\omega_1\), \(k_+ \leftrightarrow k_-\), and \(\gamma_D \rightarrow -\gamma_D\).

**APPENDIX B: EXPLICIT EXPRESSIONS OF THE DIFFUSION MATRIX IN THE PRESENCE OF SPIN-CHARGE COUPLING**

The diffusion matrix propagator for pure Rashba or pure Dresselhaus spin-orbit interaction can be obtained by inverting the polarizability matrix from Eq. (11), after setting \(\gamma_D = 0\) and \(\mu_1 = 0\) (for the Rashba case) or \(\gamma_R = 0\) and \(\mu_1 = 0\) (for Dresselhaus spin-orbit interaction). We obtain

\[
D(\omega, k) = \begin{pmatrix}
\frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} \\
\frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} \\
\frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} & \frac{\mu_1 k_y}{(s-s_0)(s-s_1)} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
where $\Delta(s, k) = (s - s_0)(s - s_1)(s - s_2)(s - s_3)$ is the determinant of the polarizability matrix. The solutions of the secular equation $\Delta = 0$ are $s_j(k) = -i\omega_j(k) + 1 + k^2$ with the dispersion relations for the modes $\omega_j$ given by Eq. (13).

For a general set of spin-orbit interaction parameters, one can easily invert the matrix in Eq. (13) and obtain the diffusion matrix for wave vectors oriented along the ‘+’ and ‘−’ axes of the rotated coordinate system. Explicitly, for the $k_+$ direction we have

$$
\hat{D}(\omega, k_+) = \begin{pmatrix}
0 & \frac{-i(\gamma R + \gamma D)k_+}{(s-s_0)(s-s_1)} & \frac{-i(\gamma R - \gamma D)k_+}{(s-s_0)(s-s_1)} \\
\frac{-i(\mu + \mu_1)k_+}{(s-s_0)(s-s_1)} & 0 & \frac{-i(\mu_1 + \mu)k_+}{(s-s_0)(s-s_1)} \\
0 & \frac{(\gamma R - \gamma D)k_+}{(s-s_0)(s-s_1)} & \frac{\gamma R - \gamma D}{(s-s_0)(s-s_1)}
\end{pmatrix},
$$

(B2)

where $s_j = -i\omega_j + 1 + k^2$ represent the four modes described by equation (12). Notice that the charge couples only with the $S_-$ spin component, i.e. with the in-plane component perpendicular to the wave-vector. A similar expression can be obtained for the $k_-$ direction using the standard correspondence rules ‘+’ $\leftrightarrow$ ‘−’ and $\gamma_D \to -\gamma_D$.

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