Explicit Codes for the Wiretap Channel: A Unified Design Framework

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Abstract

A construction of explicit codes for the wiretap channel is proposed. Appropriate choices of the construction parameters yield, within a unified design framework, the achievability of the secrecy capacity of (i) Wyner's wiretap channel, (ii) the wiretap channel of type II, (iii) the wiretap channel of type II with noisy main channel, (iv) the hybrid Wyner's/type II wiretap channel, and the best known single-letter achievable secrecy rates for (v) the wiretap channel when uncertainties hold on the eavesdropper's channel statistics (compound model), (vi) the wiretap channel when the eavesdropper's channel statistics are arbitrarily varying. Results are obtained for strong secrecy, do not require any symmetry or degradation assumptions on the channel, and do not require a pre-shared secret between the legitimate users. The underlying construction idea is an efficient emulation of random binning via polar codes to obtain reliability, coupled with universal hashing implemented via invertible extractors to ensure strong secrecy.

I. INTRODUCTION

Wiretap channel models [2] represent a fundamental primitive to model eavesdropping at the physical layer [3], [4]. While many extensions and refinements of Wyner's seminal work [2] have been successfully developed, most results are concerned about existence results. Such results enable the understanding of secure communication limits in the presence of an eavesdropper but need to be completed by explicit coding scheme designs to bridge a gap between theory and practice.

Next, we describe three classes of wiretap channel models related to Wyner’s wiretap channel [2] as well as known associated results regarding non-constructive and explicit coding schemes. Then, we state our contribution in terms of explicit coding scheme designs for these three classes of models.

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In Wyner’s wiretap channel [2], a sender wishes to encode a message $M$ and send the result $X^{1:N}$ to a legitimate receiver by the $N$-time use of a degraded discrete memoryless channel described by the distribution $p_{Y^{1:N}Z^{1:N}|X^{1:N}} = \prod_{i=1}^{N} p_{Y|X}p_{Z|Y}$ such that (i) $M$ can be reconstructed from the legitimate receiver channel output $Y^{1:N}$, and (ii) $M$ is almost independent of the channel output of the eavesdropper $Z^{1:N}$, in the sense $\lim_{N \to \infty} \frac{1}{N} I(M; Z^{1:N}) = 0$. A refinement obtained in [5] is the characterization of the secrecy capacity for an arbitrary discrete memoryless wiretap channel described by the distribution $p_{Y^{1:N}Z^{1:N}|X^{1:N}} = \prod_{i=1}^{N} p_{YZ|X}$. Another refinement of [2] is a strengthening, called strong secrecy, of the security guarantee into $\lim_{N \to \infty} I(M; Z^{1:N}) = 0$, e.g., [6].

Explicit coding schemes based on low-density parity-check codes [7]–[9], polar codes [10]–[13], and invertible extractors [14], [15] have been successfully developed for special cases of Wyner’s model [2], in which the channels are at least assumed to be symmetric. Explicit and non-explicit codes for arbitrary channels based on polar codes have also been proposed in [12], [16]–[18].

The wiretap channel of type II, introduced in [19], is a model related to [2], where a sender encodes a message $M$ into a sequence of symbols $X^{1:N}$, which is sent to a legitimate receiver over a noiseless channel in the presence of an eavesdropper able to obtain $\mu$ symbols of its choice among the $N$-symbol sequence $X^{1:N}$. The secrecy capacity of the wiretap channel of type II is obtained in [19] and subsequently obtained in [20] when the communication channel between the legitimate users is noisy. The wiretap channel of type II and the original wiretap channel [2] can be unified in a hybrid model [21], where the eavesdropper has now access to $\mu$ symbols of its choice among $X^{1:N}$, as in [19], and a noisy version of the remaining symbols of the encoded message $X^{1:N}$, as in [2]. The secrecy capacity for this model is also derived in [21].

To the best of our knowledge, explicit coding schemes have only been proposed for the original wiretap channel of type II model [19], e.g., [6], [22].

An assumption made by all the aforementioned models is that the eavesdropper’s channel statistics are perfectly known by the legitimate users. While this greatly simplifies the analysis of the models, this might not always be a relevant assumption if there exists uncertainty, from the point of view of the legitimate users, about the physical location of the eavesdropper. To model such an uncertainty, several models assume that the eavesdropper’s channel statistics are known to belong to a given set of channel statistics without knowing to which specific element of the set they correspond. Such
models include (i) the compound wiretap channel [23], where the eavesdropper’s channel statistics is known to be fixed for all the channel uses, and (ii) the arbitrarily varying wiretap channel [24], [25], where the eavesdropper’s channel statistics can change at each channel use. For general channels, lower and upper bounds on the secrecy capacity have been derived in [23]–[25] but no capacity result is known for these two models.

To the best of our knowledge, only the explicit coding schemes in [14], [15] can be used for these models. These coding schemes, however, do not achieve the best known achievable rates [3], [23]–[25] (obtained non-constructively), when the channels are asymmetric or non-degraded.

Our contribution is an explicit coding scheme that achieves, by appropriate choices of its parameters, the secrecy capacity of (i) Wyner’s wiretap channel [2], (ii) the wiretap channel of type II [19], (iii) the wiretap channel of type II with noisy main channel [20], (iv) the hybrid Wyner’s/type II wiretap channel [21], and the best known single-letter achievable secrecy rates for (v) the wiretap channel when uncertainties hold on the eavesdropper’s channel statistics (compound model) [3], (vi) the wiretap channel when the eavesdropper’s channel statistics are arbitrarily varying [24], [25]. These achievability results are obtained for strong secrecy, do not require any symmetry or degradation assumptions on the channel, and do not require a pre-shared secret between the legitimate users.

The underlying idea of our construction is an efficient emulation of random binning via a block-Markov encoding with polar codes to obtain reliability, coupled with universal hashing implemented via invertible extractors to ensure secrecy. Our proposed construction improves upon known constructions with polar codes, which do not support type II wiretap channel or uncertainties on the eavesdropper’s channel, and improves upon known explicit codes relying on invertible extractors, which are not optimal for asymmetric or non-degraded channels. Note that Block-Markov encoding in polar coding schemes has first been used in [26], [27], for problems involving reliability constraints. Unlike problems that only involve reliability constraints, an additional difficulty of block-Markov encoding in our setting is to ensure a security constraint over all coding blocks jointly, despite potential inter-block dependencies.

The remainder of the paper is organized as follows. The problem statement, in Section III, provides a unified model from which the six models (i)-(vi) described above can be recover as special cases. Our proposed coding scheme can be found in Section IV, and a statement of our main results in Section V. Proofs are presented in Sections VI, VII, VIII. Section IX provides concluding remarks.
II. Notation

For $a, b \in \mathbb{R}_+$, define $[a, b] \triangleq \lfloor a \rfloor, \lceil b \rceil \cap \mathbb{N}$. The components of a vector $X^{1:N}$ of size $N$ are denoted with superscripts, i.e., $X^{1:N} \triangleq (X_1, X_2, \ldots, X_N)$. For any set $A \subset [1, N]$, let $X^{1:N}[A]$ be the components of $X^{1:N}$ whose indices are in $A$. For two distributions $p$ and $q$ defined over a finite alphabet $\mathcal{X}$, define the variational distance $\mathcal{V}(p, q) \triangleq \sum_{x \in \mathcal{X}}|p(x) - q(x)|$, and denote the Kullback-Leibler divergence between $p$ and $q$ by $\mathbb{D}(p\| q)$, with the convention $\mathbb{D}(p\| q) = +\infty$ if there exists $x \in \mathcal{X}$ such that $q(x) = 0$ and $p(x) > 0$. For joint probability distributions $p_{X|Y}$ and $q_{X|Y}$ defined over $\mathcal{X} \times \mathcal{Y}$, the conditional Kullback-Leibler divergence is written as $\mathbb{E}_{p_X}[\mathbb{D}(p_{Y|X}\| q_{Y|X})] \triangleq \sum_{x \in \mathcal{X}} p_X(x) \mathbb{D}(p_{Y|X=x}\| q_{Y|X=x})$. Unless otherwise specified, capital letters denote random variables, whereas lowercase letters designate realizations of associated random variables, e.g., $x$ is a realization of the random variable $X$. Let $\mathbb{I}\{\omega\}$ be the indicator function, which is equal to 1 if the predicate $\omega$ is true and 0 otherwise. For any $x \in \mathbb{R}$, define $[x]^+ \triangleq \max(0, x)$. Finally, GF($2^N$) denotes a finite field of order $2^N$.

III. Model and review of known results

Consider two finite alphabets $\mathcal{X} \triangleq \{0, 1\}$ and $\mathcal{Y}$, and $|\mathcal{S}|$ finite alphabets $(\mathcal{Z}_s)_{s \in \mathcal{S}}$, where $\mathcal{S}$ is a finite set. Consider also $|\mathcal{S}|$ transitions probability $(p_{YZ(s)|X})_{s \in \mathcal{S}}$. A wiretap channel is defined as a discrete memoryless channel with transition probability for one channel use $p_{YZ(s)|X}(y, z(s)|x)$ where $x \in \mathcal{X}$ is the channel input from the transmitter, $y \in \mathcal{Y}$ is the channel output observed by the legitimate receiver, $z(s) \in \mathcal{Z}_s$ is the channel output observed by the eavesdropper, $s \in \mathcal{S}$ is arbitrary, unknown to the legitimate users, and can potentially change for each channel use. In the following, we omit the index $s \in \mathcal{S}$ whenever $|\mathcal{S}| = 1$. Moreover, when the channel input is a sequence of $N$ symbols $X^{1:N}$, then, in addition to the channel output $Z^{1:N}(s)$, the eavesdropper, has access to $X^{1:N}[S] \triangleq (X^i)_{i \in S}$, where $S \subseteq [1, N]$ is chosen by the eavesdropper and such that $|S| = \alpha N$ for some $\alpha \in [0, 1]$.

**Definition 1.** For $B \in \mathbb{N}$, define $B \triangleq [1, B]$. A $(2^{NR}, N, B)$ code operates over $B$ encoding blocks and consists for each encoding Block $b \in B$ of

- A message set $\mathcal{M}_b \triangleq [1, 2^{NR_b}]$;
- A stochastic encoding function $f_b : \mathcal{M}_b \to \mathcal{X}^{1:N}$, used by the transmitter to encode a message $M_b$, uniformly distributed over $\mathcal{M}_b$ into $X^{1:N}_b \triangleq f_b(M_b)$;
- A deterministic decoding function used by the legitimate receiver $g_b : \mathcal{Y}^{1:N} \to \mathcal{M}_b$, to form $\hat{M}_b$ an estimate of $M_b$ given the channel outputs $Y^{1:N}_b$.  

The messages $M_{1:B} \triangleq (M_b)_{b \in B}$ are assumed mutually independent. Moreover, define the rate of the code as $R \triangleq \sum_{b \in B} R_b / B$, $M_{1:B} \triangleq (M_b)_{b \in B}$, and $\hat{M}_{1:B} \triangleq (\hat{M}_b)_{b \in B}$.

**Definition 2.** A rate $R$ is achievable if there exists a sequence of $(2^{NR}, N, B)$ codes such that

$$\lim_{N \to \infty} \mathbb{P}[\hat{M}_{1:B} \neq M_{1:B}] = 0,$$

(Reliability)

$$\lim_{N \to \infty} \max_{s \in \mathcal{S}^N} \max_{A \in A} I_{A} = 0,$$

(Strong secrecy)

where $A \triangleq \{(A_b)_{b \in B} : A_b \subseteq [1, N] \text{ and } |A_b| = \alpha N, \forall b \in B\}$, $(M_b, \hat{M}_b, Z_1^{1:N}(s_b), X_1^{1:N}[A])$ corresponds to the random variables in Block $b \in B$ for $A = (A_b)_{b \in B} \in A$ and $s_b \in \mathcal{S}^N$, $X_1^{1:N}[A] \triangleq (X_b^{1:N}[A_b])_{b \in B}$, and $Z_1^{1:N}(s) \triangleq (Z_b^{1:N}(s_b))_{b \in B}$ for $s = (s_b)_{b \in B} \in \mathcal{S}^{NB}$.

The supremum of such achievable rates is called secrecy capacity and denoted by $C_s$.

We now review known results for six special cases of the model.

**Theorem 1** ([2], [5]). Consider Wyner’s wiretap channel, i.e., $|\mathcal{S}| = 1$ and $\alpha = 0$. Then, the secrecy capacity is

$$C_s = \max_{U \subseteq \mathcal{X} \setminus \{Y, Z\}} \left[ I(U; Y) - I(U; Z) \right]^+.$$

(1)

**Theorem 2** ([19]). Consider the wiretap channel of type II introduced in [19], i.e., $|\mathcal{S}| = 1$, $p_{Z|X} = p_Z$, and for any $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $p_{Y|X}(y|x) = 1\{y = x\}$. Then, the secrecy capacity is

$$C_s = 1 - \alpha.$$  

(2)

**Theorem 3** ([20]). Consider the wiretap channel of type II with noisy main channel, i.e., $|\mathcal{S}| = 1$ and $p_{Z|X} = p_Z$. Then, the secrecy capacity is

$$C_s = \max_{U \subseteq \mathcal{X} \setminus Y} \left[ I(U; Y) - \alpha I(U; X) \right]^+.$$  

(3)

**Theorem 4** ([21]). Consider the hybrid Wyner’s/type II wiretap channel, i.e., $|\mathcal{S}| = 1$. Then, the secrecy capacity is

$$C_s = \max_{U \subseteq \mathcal{X} \setminus (Y, Z)} \left[ I(U; Y) - \alpha I(U; X) - (1 - \alpha) I(U; Z) \right]^+.$$  

(4)

**Theorem 5** ([23]). Consider the wiretap channel with compound eavesdropper channel, i.e., assume that $s = (s_b)_{b \in B} \in \mathcal{S}^{NB}$ is unknown to the legitimate users but all the components of $s_b$, $b \in B$, are identical. Assume also that $\alpha = 0$. Then, the secrecy capacity is lower-bounded as
\[
C_s \geq \max_{\forall \bar{s} \in \mathcal{S}, U \sim U - \{Y, Z(s)\}} \min_{s \in \mathcal{S}} \left[ I(U; Y) - I(U; Z(s)) \right]^+.
\]

Moreover, for a degraded wiretap channel, i.e., when for all \(s \in \mathcal{S}\), \(X - Y - Z(s)\), we have

\[
C_s = \max_{p_X} \min_{s \in \mathcal{S}} I(X; Y|Z(s)).
\]

**Theorem 6** ([24], [25]). Consider the wiretap channel with arbitrarily varying eavesdropper channel, i.e., assume that \(s \in \mathcal{S}^{NB}\) is unknown to the legitimate users. Assume also that \(\alpha = 0\). We define \(\mathcal{S}\) as the set of all the convex combinations of elements of \(\mathcal{S}\). If there exists a best channel for the eavesdropper, i.e., \(\exists s^* \in \mathcal{S}, \forall s \in \mathcal{S}, X - Z_{s^*} - Z_s\), then the secrecy capacity is lower-bounded as

\[
C_s \geq \max_{\forall \bar{s} \in \mathcal{S}, U \sim U - \{Y, Z(\bar{s})\}} \min_{s \in \mathcal{S}} \left[ I(U; Y) - I(U; Z(\bar{s})) \right]^+.
\]

Moreover, if there exists a best channel for the eavesdropper and for all \(\bar{s} \in \mathcal{S}\), \(X - Y - Z(\bar{s})\), then

\[
C_s = \max_{p_X} \min_{\bar{s} \in \mathcal{S}} I(X; Y|Z(\bar{s})).
\]

Our main result is a unified construction that yields, by appropriate choices of its parameters, explicit coding schemes that achieve the rates in Theorems 1-6. To the best of our knowledge, explicit coding schemes were previously known only for Theorem 1 provided that the legitimate users have access to a pre-shared secret-key with negligible rate, and for Theorem 2.

**IV. PROPOSED CODING SCHEME**

In this section, we describe an explicit coding scheme that will be shown to achieve the rates described in Theorems 1-6. The coding scheme consists of two parts, an initialization phase presented in IV-B, and the actual secure communication in Section IV-C. The initialization phase allows the legitimate users to share a secret key which is used in the second part of the coding scheme.

**A. Notation**

For \(s \in \mathcal{S}\), we consider an arbitrary joint distribution \(q_{U_0X_0Y_0Z_0(s)} \triangleq q_{U_0X_0Y_0Z_0|X_0}\) with \(|U| = |X| = 2\) and such that \(U - X - (Y, Z(s))\). Let \(K\) be a power of two, let \((U^{1:K}, X^{1:K})\) be distributed according to \(q_{U^{1:K}X^{1:K}} \triangleq \prod_{i=1}^{K} q_{U_iX_i}\), and define \(A^{1:K} \triangleq G_KU^{1:K}, V^{1:K} \triangleq G_KX^{1:K}\), where \(G_K \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \log K\).
is the matrix defined in [28]. We also define for \( \delta_K \triangleq 2^{-K^\beta}, \beta \in [0,1/2] \), the sets

\[ \mathcal{V}_U \triangleq \{ i \in [1,K] : H(A_i^i|A_1^{i-1}) > 1 - \delta_K \}, \]

\[ \mathcal{H}_U \triangleq \{ i \in [1,K] : H(A_i^i|A_1^{i-1}) > \delta_K \}, \]

\[ \mathcal{V}_{U|Y} \triangleq \{ i \in [1,K] : H(A_i^i|A_1^{i-1}Y^{1:N}) > 1 - \delta_K \}, \]

\[ \mathcal{H}_{U|Y} \triangleq \{ i \in [1,K] : H(A_i^i|A_1^{i-1}Y^{1:N}) > \delta_K \}, \]

\[ \mathcal{V}_X \triangleq \{ i \in [1,K] : H(V_i^i|V_1^{i-1}) > \delta_K \}, \]

\[ \mathcal{V}_{X|U} \triangleq \{ i \in [1,K] : H(V_i^i|V_1^{i-1}U^{1:N}) > \delta_K \}. \]

Observe that \( \mathcal{V}_{U|Y} \subset \mathcal{V}_U \subset \mathcal{H}_U \) and \( \mathcal{V}_{X|U} \subset \mathcal{V}_X \). We will use the following lemma, which also provides an interpretation for the sets \( \mathcal{H}_{U|Y} \) and \( \mathcal{H}_U \). For an interpretation of the sets \( \mathcal{V}_U, \mathcal{V}_{U|Y}, \mathcal{V}_X, \mathcal{V}_{X|U} \) in terms of randomness extraction or privacy amplification, we refer to [18], [29]–[31].

**Lemma 1** (Source coding with side information [28]). Consider a probability distribution \( p_{XY} \) over \( X \times Y \) with \( |X| = 2 \) and \( Y \) a finite alphabet. Let \( N \) be a power of two and consider \( (X_1^N, Y_1^N) \) distributed according to \( \prod_{i=1}^N p_{XY} \). Define \( A_1^N \triangleq X_1^NG_N \), and for \( \delta_N \triangleq 2^{-N^\beta} \) with \( \beta \in [0,1/2] \), the set \( \mathcal{H}_{X|Y} \triangleq \{ i \in [1,N] : H(A_i^i|A_1^{i-1}Y_1^{1:N}) > \delta_N \} \). Given \( A_1^N[\mathcal{H}_{X|Y}] \) and \( Y_1^N \), one can form \( \hat{A}_1^N \) by the successive cancellation decoder of [28] such that \( P[\hat{A}_1^N \neq A_1^N] \leq N\delta_N \). Moreover, \( \lim_{N \to \infty} |\mathcal{H}_{X|Y}|/N = H(X|Y) \).

**B. Coding scheme: Part I - Initialization**

The legitimate users perform the initialization phase described in Algorithms 1 and 2 to generate a secret key with length \( l_{key} \), which will be be specified later in Section VII-B.

The initialization phase operates over \( B_0 \) blocks of length \( N \triangleq KL \), where \( L, K \in \mathbb{N} \) are both power of two. We define \( B_0 \triangleq [1,B_0] \) and \( L \triangleq [1,L] \). Encoding at the transmitter and decoding at the receiver are described in Algorithms 1 and 2, respectively.

In each block \( b \in B_0 \), the encoder forms the key \( Key_b \) with length \( l'_{key} \triangleq l_{key}/B_0 \), as described in Algorithm 1. The encoder uses the following randomization sequences: \( R_b^{init} \triangleq (R_b^{init}, l \in L) \), where \( R_b^{init}, l \in L \), \( l \in L \), is a sequence of uniformly distributed bits over \( \{0,1\}^{R_{U|Y}|\mathcal{V}_{U|Y}|} \), and \( R_b^{init} \), a sequence of uniformly distributed bits over \( R^{init} \triangleq \{0,1\}^{N} \setminus \{0\} \).
Algorithm 1 Initialization at the transmitter

Require: Randomization sequences \((R_{b0}^{\text{init}})_{b \in B_0}\) and \((R_{b0}^{\text{init}'})_{b \in B_0}\)
1: for Block \(b \in B_0\) do
2: for Sub-block \(l \in L\) do
3: Draw \(A_{b,l}^{1:K}\) according to
\[
\tilde{P}_{A_{b,l}^{j:1}, A_{b,l}^{j:1}}(a_{b,l}^j|a_{b,l}^{j-1}) \triangleq \begin{cases} 
1/2 & \text{if } j \in V_U \\
q_{A_{b,l}^{j:1}, A_{b,l}^{j:1}}(a_{b,l}^j|a_{b,l}^{j-1}) & \text{if } j \in V_U^c
\end{cases}
\] (9)
4: Define \(\tilde{U}_{b,l}^{1:K} \triangleq \tilde{A}_{b,l}^{1:K}G_K\)
5: Perform channel prefixing by forming \(\tilde{V}_{b,l}^{1:K}\) as follows
\[
\tilde{P}_{V_{b,l}^{j:1}, V_{b,l}^{j:1}, U_{b,l}^{j:1}}(v_{b,l}^j|v_{b,l}^{j-1}, u_{b,l}^{j-1}) \triangleq \begin{cases} 
1/2 & \text{if } j \in V_X|U \\
q_{V_{b,l}^{j:1}, V_{b,l}^{j:1}, U_{b,l}^{j:1}}(v_{b,l}^j|v_{b,l}^{j-1}, u_{b,l}^{j-1}) & \text{if } j \in V_X|U^c
\end{cases}
\] (10)
6: Define \(\tilde{X}_{b,l}^{1:K} \triangleq \tilde{V}_{b,l}^{1:K}G_K\)
7: end for
8: Transmit \(\tilde{X}_{b}^{1:N} \triangleq \parallel_{l \in L} \tilde{X}_{b,l}^{1:K}\) over the channel. Denote the outputs by \(\tilde{Y}_{b}^{1:N} \triangleq \parallel_{l \in L} \tilde{V}_{b,l}^{1:K}\), \(\tilde{Z}_{b}^{1:N}(s_b) \triangleq \parallel_{l \in L} \tilde{Z}_{b,l}^{1:K}\) \((s_b,l)\)
9: Transmit with a channel code [32] \(D_b \triangleq \parallel_{l \in L} \left(\tilde{A}_{b,l}^{1:K}[\mathcal{H}_U|Y] \parallel \tilde{R}_{b,l}^{\text{init'}}\right) \parallel \tilde{A}_{b,l}^{1:K}[\mathcal{V}_U|Y]\)
10: Form \(\text{Key}_b \triangleq (R_{b}^{\text{init}} \circ \tilde{U}_{b}^{1:N})|_{1:l_{\text{key}}'}\), where \(\circ\) denotes multiplication in \(\text{GF}(2^N)\) and \(|_{1:l_{\text{key}}'}\) indicates that only the leftmost \(l_{\text{key}}'\) bits are kept
11: end for

Algorithm 2 Initialization phase at the receiver

Require: Randomization sequences \((R_{b}^{\text{init}})_{b \in B_0}\) and \((R_{b}^{\text{init'}})_{b \in B_0}\)
1: for Block \(b \in B_0\) do
2: Form an estimate \(\tilde{D}_b\) of \(D_b\)
3: for Sub-block \(l \in L\) do
4: From \((\tilde{D}_b, R_b)\) form an estimate \(\tilde{A}_{b,l}^{1:K}[\mathcal{H}_U|Y]\) of \(\tilde{A}_{b,l}^{1:K}[\mathcal{H}_U|Y]\)
5: From \(\tilde{Y}_{b,l}^{1:K}\) and \(\tilde{A}_{b,l}^{1:K}[\mathcal{H}_U|Y]\), form \(\tilde{A}_{b,l}^{1:K}\) an estimate of \(\tilde{A}_{b,l}^{1:K}\) with the successive cancellation decoder for lossless source coding with side information [28]
6: From \(\tilde{A}_{b,l}^{1:K}\) form \(\tilde{U}_{b,l}^{1:K} \triangleq \tilde{A}_{b,l}^{1:K}G_K\) an estimate of \(\tilde{U}_{b,l}^{1:K}\)
7: end for
8: Form \(\tilde{U}_{b}^{1:N} \triangleq \parallel_{l \in L} \tilde{U}_{b,l}^{1:K}\) an estimate of \(\tilde{U}_{b}^{1:N}\)
9: Form \(\tilde{\text{Key}}_b = (R_{b}^{\text{init}} \circ \tilde{U}_{b}^{1:N})|_{1:l_{\text{key}}'}\) an estimate of \(\text{Key}_b\)
10: end for
Remark 1. In line 7 of Algorithm 2, note that the channel code [32] requires a uniformly distributed message. While \( \|_{l \in \mathcal{L}} \overline{A}_{b,l}^{1:K}[H_{U|Y}] \) is not a sequence of uniformly distributed bits, \( \overline{D}_b \) is a sequence of uniformly distributed bits over \( [1, 2^{|H_{U|Y}^2}|] \).

C. Coding scheme: Part II - Secure communication

The encoding scheme operates over \( B \) blocks of length \( N \triangleq KL \), where \( L, K \in \mathbb{N} \) are both power of two. We define \( B \triangleq [1, B] \) and \( \mathcal{L} \triangleq [1, L] \). Encoding at the transmitter and decoding at the receiver are described in Algorithms 3 and 4, respectively.

In each block \( b \in B \), the transmitter encodes, as described in Algorithm 3, a message \( M_b \) uniformly distributed over \( [1, 2^{|M_b|}] \) and represented by a binary sequence with length

\[
|M_b| \triangleq \begin{cases} |M_1| & \text{if } b = 1 \\ |M_1| - L|V_U| & \text{otherwise} \end{cases}.
\]

Algorithms 3 and 4 depend on the parameter

\[
r \triangleq |M_1|,
\]

which will be specified later with the constraint \( L|V_U| < r < L|V_U| \).

In each block \( b \in B \), as described in Algorithm 3, the encoder uses \( R'_b \), a binary randomization sequence only known at the encoder, uniformly distributed over \( \{0, 1\}^{L|V_U|} \). The sequences \( R'_{1:B} \triangleq (R'_b)_{b \in B} \) are mutually independent. The length of the sequences \( (R'_b)_{b \in B} \) is defined for \( b \in B \) as \( |R'_b| \triangleq L|V_U| - r \).

In each block \( b \in B \), the encoder also uses, as described in Algorithm 3, \( R_b \), a binary randomization sequence with length \( L|V_U| \), uniformly distributed over \( \mathcal{R} \triangleq \{0, 1\}^{L|V_U|} \setminus \{0\} \). The sequences \( R_{1:B} \triangleq (R_b)_{b \in B} \) are mutually independent. Moreover, it is assumed that \( M_{1:B}, R_{1:B}, \text{and } R'_{1:B} \) are mutually independent.

We depict in Figure 1 the dependencies between two consecutive encoding blocks. In a given block, we also depict in Figure 2 a summary of the different phases in Algorithm 3 through which the encoder output is obtained.

Remark 2. In Algorithm 3, observe that \( T_{b}^{1:|V_U|L}, b \in B \), is uniformly distributed over \( \{0, 1\}^{L|V_U|} \) because \( (M_b)[M'_b|R'_b] \) is uniformly distributed over \( \{0, 1\}^{L|V_U|} \) and independent of \( R_b \). Hence, the \( L \) random variables \( (T_{b,l}^{1:|V_U|})_{l \in \mathcal{L}} \) are uniformly distributed over \( \{0, 1\}^{L|V_U|} \) and independent. When the
elements of $s_b$ are all equal to $s$, then, by construction, the conditional probability $\bar{p}_{Z_b^1;K}(s) | T_b^1;Y_b^1)$ is the same for all $l \in \mathcal{L}$, and the $L$ pairs $\left((T_b^1;Y_b^1), (Z_b^1;K(s)) \right)_{l \in \mathcal{L}}$ are independently and identically distributed according to the joint distribution $\bar{p}_{T_b^1;Y_b^1} Z_b^1;K(s)$.

**Remark 3.** In Algorithm 3, consider $\bar{X}_b^1;K \{ A_{b,l} \}, b \in \mathcal{B}$, $l \in \mathcal{L}$, where for all $l \in \mathcal{L}$, $A_{b,l} \subset [1, K]$ and $\sum_{l \in \mathcal{L}} |A_{b,l}| = \alpha N$ such that $\bar{X}_b^1;N \{ A_b \} \triangleq \{ \bar{X}_{b,l}^1;K \} \{ A_{b,l} \}$ corresponds to the $\alpha N$ symbols of the codewords emitted at the transmitter that the eavesdropper has chosen to have access to. Similar to Remark 2, the $L$ triplets $\left( (T_b^1;Y_b^1), \bar{X}_b^1;K \{ A_{b,l} \}, \bar{Z}_b^1;K(s_b) \right)_{l \in \mathcal{L}}$ are independent, however, they are not necessarily identically distributed because the components of $s_b$ are arbitrary, and because the sets $(A_{b,l})_{l \in \mathcal{L}}$ are chosen by the eavesdropper.

**Remark 4.** Similar to [33], Equation (14) could be replaced by

$$\bar{p}_{A_{b,l}^1;A_{b,l}^{-1}}(a_{b,l}^1|a_{b,l}^{-1}) \equiv \begin{cases} q_{A_{b}^1|A_{b}^{-1}}(a_{b,l}^1|a_{b,l}^{-1}) & \text{if } j \in \mathcal{V}_b \setminus \mathcal{H}_b^\ell, \\ \mathbb{I}\{a_{b,l}^1 = \arg \max_a q_{A_{b}^1|A_{b}^{-1}}(a|a_{b,l}^{-1})\} & \text{if } j \in \mathcal{H}_b^\ell. \end{cases} \quad (12)$$

![Fig. 1](image1.png)

**Fig. 1.** The construction of $M'_b$ in Algorithm 3 creates a dependency between Block $b \in [2, B]$ and Block $b - 1$.

![Fig. 2](image2.png)

**Fig. 2.** Summary of the steps in Algorithm 3 to obtain from $T_b^1;Y_b^1$ the encoder output $\bar{X}_b^1;N$ in Block $b \in \mathcal{B}$. 
Algorithm 3 Encoding

Require: Randomization sequences \((R_b)_{b \in B}\), \((R'_b)_{b \in B}\), and messages \((M_b)_{b \in B}\)
1: \((A_{b,l}^{1:K})_{l \in L} \leftarrow \emptyset\)
2: for Block \(b \in B\) do
3: \(M'_b \leftarrow \| A_{b-l,1}^{1:K} [V_U | Y] \) (with the convention \(M'_b = \emptyset\))
4: \(T_b^{1:|V_U| L} \leftarrow R_b^{-1} \circ (M_b \| M'_b \| R'_b)\), where \(\circ\) denotes multiplication in GF(2\(^{|V_U| L}\))
5: for Sub-block \(l \in L\) do
6: Define \(A_{b,l}^{1:K}\) as follows
7: and the bits \(A_{b,l}^{1:K} [Y_U] \) are drawn according to
8: Perform channel prefixing by forming \(\tilde{V}_{b,l}^{1:K}\) as follows
9: Define \(\tilde{X}_{b,l}^{1:K} \triangleq \tilde{V}_{b,l}^{1:K} G_K\)
10: Transmit \(\tilde{X}_{b,l}^{1:N} \triangleq \| \tilde{X}_{b,l}^{1:K}\) over the channel. Denote the outputs by \(\tilde{Y}_{b,l}^{1:N} \triangleq \| \tilde{Y}_{b,l}^{1:K}, \tilde{Z}_{b,l}^{1:N} (s_b) \triangleq \| \tilde{Z}_{b,l}^{1:K} (s_{b,l})\)
11: Transmit secretly \((\tilde{A}_{b,l}^{1:K} [H_U|Y \setminus V_U|Y])_{l \in L,s \in B}\), and \((\tilde{A}_{b,l}^{1:K} [V_U|Y])_{l \in L}\) by means of a one-time pad and a pre-shared secret.

Algorithm 4 Decoding

Require: Randomization sequences \((R_b)_{b \in B}\), \((\tilde{A}_{b,l}^{1:K} [H_U|Y \setminus V_U|Y])_{l \in L,s \in B}\), and \((\tilde{A}_{b,l}^{1:K} [V_U|Y])_{l \in L}\)
1: for Block \(b \in B\) from \(b = B\) to \(b = 1\) do
2: for \(l \in L\) do
3: From \(\tilde{Y}_{b,l}^{1:K}, \tilde{A}_{b,l}^{1:K} [V_U|Y]\) (an estimate of \(\tilde{A}_{b,l}^{1:K} [V_U|Y]\), and \(\tilde{A}_{b,l}^{1:K} [H_U|Y \setminus V_U|Y]\), form \(\tilde{A}_{b,l}^{1:K}\) an estimate of \(\tilde{A}_{b,l}^{1:K}\) with the successive cancellation decoder for lossless source coding with side information [28]
4: end for
5: From \(\| \tilde{A}_{b,l}^{1:K}\) and (13), determine an estimate \(\tilde{T}_{b,l}^{1:|V_U| L} \triangleq \| \tilde{A}_{b,l}^{1:K} [V_U]\) of \(T_b^{1:|V_U| L}\)
6: From \(R_b \circ \tilde{T}_{b}^{1:|V_U| L}\) and Line 4 in Algorithm 3, determine an estimate \(\tilde{M}_b'\) of \(M_b'\) and then, using Line 3 in Algorithm 3, an estimate \(\tilde{A}_{b-l,1}^{1:K} [V_U|Y]\) of \(\tilde{A}_{b-l,1}^{1:K} [V_U|Y]\) for all \(l \in L\)
7: From \(R_b \circ \tilde{T}_{b}^{1:|V_U| L}\) and Line 4 in Algorithm 3, determine \(\tilde{M}_b\) an estimate of \(M_b\)
8: end for
V. STATEMENT OF MAIN RESULTS

Our main result is the following theorem.

**Theorem 7.** Consider the coding scheme described in Algorithms 3, 4 combined with the initialization phase described in Algorithms 1, 2.

1) If all the components of \( s_b, b \in \mathcal{B} \), are identical, then the following secrecy rate is achieved

\[
I(U;Y) - \alpha I(U;X) - (1 - \alpha) \min_{s \in \mathcal{S}} I(U;Z(s)) + 1.
\]

2) Assume that the components of \( s_b, b \in \mathcal{B} \), are arbitrary. If there exists a best channel for the eavesdropper, then the following secrecy rate is achieved

\[
I(U;Y) - \alpha I(U;X) - (1 - \alpha) \min_{\bar{s} \in \bar{\mathcal{S}}} I(U;Z(\bar{s})) + 1.
\]

The proof of Theorem 7.1 is presented in two parts. First, in Section VI, the initialization phase, i.e., Algorithms 1, 2, is ignored and Theorem 7.1 is proved under the assumption that the legitimate users have a pre-shared key whose rate is negligible. Next, in Section VII, Theorem 7.1 is proved without this assumption by considering the initialization phase combined with Algorithms 3, 4. The proof of Theorem 7.2 largely relies on the proof of Theorem 7.1 and is presented in Section VIII. Finally, from Theorem 7, we immediately obtain the following result.

**Corollary 1.** The coding scheme described in Algorithms 3, 4 combined with the initialization phase described in Algorithms 1, 2, achieves the secrecy rates described in Theorems 1-6 with an appropriate choice of \( q_{UX} \).

VI. PROOF OF THEOREM 7.1 WHEN A PRE-SHARED SECRET KEY IS AVAILABLE

In this section, we prove Theorem 7.1 when the legitimate users have access to a pre-shared secret key whose rate is negligible. Hence, we ignore in this section the initialization phase, i.e., Algorithms 1, 2. We also assume in this section that all the components of \( s_b, b \in \mathcal{B} \), are identical and equal to \( s \). To simplify notation, we write \( s \) instead of \( s_b, b \in \mathcal{B} \).
A. Characterization of the distribution induced by the encoder

Let \( \tilde{p}_{U_{1:N} X_{1:N} Y_{1:N} Z_{1:N}}(s) \) denote the distribution induced by the encoding scheme described in Algorithm 3. Lemma 2 gives an approximation of \( \tilde{p}_{U_{1:N} X_{1:N} Y_{1:N} Z_{1:N}}(s) \) in terms of the distribution \( q_{UXYZ}(s) \) defined in Section IV-A. This result will be useful in our subsequent analysis.

**Lemma 2.** For \( b \in B \), we have

\[
\mathbb{D}(q_{U_{1:N} X_{1:N} Y_{1:N} Z_{1:N}}(s) \| \tilde{p}_{U_{1:N} X_{1:N} Y_{1:N} Z_{1:N}}(s)) \leq 2LK\delta_K ,
\]

where we have defined \( q_{U_{1:N} X_{1:N} Y_{1:N} Z_{1:N}}(s) \equiv \prod_{i=1}^N q_{UXYZ}(s) \).

**Proof.** See Appendix A. □

B. Reliability

We now show that the receiver is able to recover the original message with a vanishing error probability.

Define \( \hat{M}_{1:B} \equiv (\hat{M}_b)_{b \in B} \). Define for \( b \in B \), \( \hat{A}_{1:N} \equiv \| \hat{A}_{1:N}^{1:L} \), \( \hat{A}_{b}^{1:N} \equiv \| \hat{A}_{b}^{1:L} \), \( A_{1:N} \equiv A_{1:LK} \), \( E_{b-1} \equiv \{ A_{b}^{1:N} \neq \hat{A}_{b}^{1:N} \} \), and \( E_{A_b} \equiv \{ (Y_{b}^{1:N}, \hat{A}_{b}^{1:N}) \neq (Y_{b}^{1:N}, A_{b}^{1:N}) \} \). For \( b \in B \), consider a coupling [34, Lemma 3.6] between \( \tilde{p}_{Y_{1:N} A_{b}^{1:N}} \) and \( q_{Y_{1:N} A_{b}^{1:N}} \) such that \( \mathbb{P}[E_{A_b}] = \mathbb{V}(\tilde{p}_{Y_{1:N} A_{b}^{1:N}}, q_{Y_{1:N} A_{b}^{1:N}}) \). For \( b \in B \), consider \( (A_{b}^{1:N}, Y_{b}^{1:N}, \hat{A}_{b}^{1:N}, \hat{Y}_{b}^{1:N}) \) distributed according to this coupling, then

\[
\begin{align*}
\mathbb{P} \left[ \hat{M}_{1:B} \neq M_{1:B} \right] & \leq \sum_{b \in B} \mathbb{P} \left[ \hat{M}_b \neq M_b \right] \\
& \leq \sum_{b \in B} \mathbb{P} \left[ T_b^{1:N} \neq T_b^{1:N} \right] \\
& \leq \sum_{b \in B} \mathbb{P} \left[ \hat{A}_{b}^{1:N} \neq A_{b}^{1:N} \right] \\
& = \sum_{b \in B} \left[ \mathbb{P} \left[ \hat{A}_{b}^{1:N} \neq A_{b}^{1:N} \mid E_{A_b} \right] \mathbb{P}[E_{A_b}] + \mathbb{P} \left[ \hat{A}_{b}^{1:N} \neq A_{b}^{1:N} \mid E_{A_b} \right] \mathbb{P}[E_{A_b} + E_b] \right] \\
& \leq \sum_{b \in B} \left[ \mathbb{P} \left[ \hat{A}_{b}^{1:N} \neq A_{b}^{1:N} \mid E_{A_b} \right] + \mathbb{P}[E_{A_b} + E_b] \right] \\
& \leq \sum_{b \in B} \left[ \mathbb{P} \left[ \hat{A}_{b}^{1:N} \neq A_{b}^{1:N} \mid E_{A_b} \right] + \mathbb{P}[E_{A_b} + E_b] \right] \\
& \leq KL\delta_K + \sqrt{2\ln2\sqrt{2LK\delta_K}} + \mathbb{P} \left[ \hat{A}_{b+1}^{1:N} \neq A_{b+1}^{1:N} \right].
\end{align*}
\]
\[
(e) \leq \sum_{b \in B} \left[ (KL\delta_K + \sqrt{2\ln 2 \sqrt{\frac{2L}{K}\delta_K}})(B - b + 1) \right] \\
= (KL\delta_K + \sqrt{2\ln 2 \sqrt{\frac{2L}{K}\delta_K}})B(B + 1)/2,
\]

where (a) holds by Line 7 in Algorithm 4, (b) holds by (13), (c) holds by the union bound, (d) holds because \(P[\hat{A}_{b,l} \neq \tilde{A}_{b,l}[\mathcal{H}_U|Y,\mathcal{V}_U|Y]]_{l \in \mathcal{L}, b \in B}\) \(\leq K\delta_K\) by the error probability for distributed source coding [28] and because \(P[\mathcal{E}_{A_b}] = \mathcal{V}(\tilde{p}_{Y_{b}^{1:N}A_{b}^{1:N}}, q_{Y_{b}^{1:N}A_{b}^{1:N}}) \leq \sqrt{2\ln 2 \sqrt{\frac{2L}{K}\delta_K}}\) by Lemma 2 and Pinsker’s inequality, (e) holds by induction.

**C. Pre-shared key rate**

The coding scheme described in Algorithms 3 and 4 involves a one-time pad to securely transmit \((\tilde{A}_{b,l}^{1:b}|\mathcal{H}_U|Y,\mathcal{V}_U|Y)]_{l \in \mathcal{L}, b \in B}\) and \((\hat{A}_{b,l}^{1:b}|\mathcal{V}_U|Y)]_{l \in \mathcal{L}}\). This operation requires a pre-shared key with length \(l_{\text{OTP}} \triangleq LB|\mathcal{H}_U|Y,\mathcal{V}_U|Y| + L|\mathcal{V}_U|Y|\) and rate

\[
\frac{l_{\text{OTP}}}{NB} = \frac{|\mathcal{H}_U|Y| - |\mathcal{V}_U|Y|}{K} + \frac{|\mathcal{V}_U|Y|}{KB} \\
\leq \frac{|\mathcal{H}_U|Y| - |\mathcal{V}_U|Y|}{K} + \frac{1}{B} \\
= \delta(K) + 1/B,
\]

where \(\delta(K)\) is such that \(\lim_{K \to \infty} \delta(K) = 0\) since \(\lim_{K \to \infty} |\mathcal{H}_U|Y|/K = H(U|Y)\) [28], and \(\lim_{K \to \infty} |\mathcal{V}_U|Y|/K = H(U|Y)\) [29], [35].

**D. Blockwise Security Analysis**

We prove in this section that security holds in each block \(b \in B\) individually. We use a series of lemmas to obtain this result and determine acceptable values for the parameter \(r\) defined in (11). For \((X,Z)\) distributed according to \(p_{XZ}\), defined over the finite alphabet \(\mathcal{X} \times \mathcal{Z}\), recall that the \(\epsilon\)-smooth min-entropy of \(X\) given \(Z\) is defined as [36]

\[
H_{\infty}^\epsilon(p_{XZ}|p_Z) \triangleq \max_{r_{XZ} \in \mathcal{B}_r(p_{XZ})} \min_{z \in \text{Supp}(p_{Z})} \min_{x \in \mathcal{X}} \log \left( \frac{p_{Z}(z)}{r_{XZ}(x,z)} \right),
\]

where \(\text{Supp}(p_Z) \triangleq \{z \in \mathcal{Z} : p_Z(z) > 0\}\) and \(\mathcal{B}_r(p_{XZ}) \triangleq \{(r_{XZ} : \mathcal{X} \times \mathcal{Z} \to [0,1]) : \mathcal{V}(p_{XZ},r_{XZ}) \leq \epsilon\}\). We will also need the following version of the leftover hash lemma.
Lemma 3 ([36]). Let \( T \) and \( Z \) be random variables distributed according to \( p_{TZ} \) over \( T \times Z \). Let \( F : \mathcal{R} \times \{0,1\}^k \to \{0,1\}^r \) be a two-universal hash function. Let \( R \) be uniformly distributed over \( \mathcal{R} \). Then, we have for any \( z \in \mathcal{Z} \), for any \( \epsilon \in [0,1] \)
\[
\forall(p_{F(R,T),R,Z},p_{U_x}p_{U_y}p_{Z}) \leq 2\epsilon + \sqrt{2^r - H^*_\infty(p_{TZ}|p_Z)},
\]
where \( p_{U_x} \) and \( p_{U_y} \) are the uniform distribution over \( \{0,1\}^r \) and \( \mathcal{R} \), respectively.

Define the function \( F : \mathcal{R} \times \{0,1\}^{|Y_u|L} \to \{0,1\}^r \), \((R_b,T_b^{1:|Y_u|L}) \mapsto (R_b \odot T_b^{1:|Y_u|L})|_{1:r}\), where \( r \) is defined in (11) and \( |_{1:r} \) indicates that only the left-most \( r \) bits are kept. \( F \) is known to be a two-universal hash function [15]. We now would like to use Lemma 3 with the goal of making \((M_b\|M_b')\) almost independent from the eavesdropper channel observations. However, in the encoding scheme described in Algorithm 3, \((M_b\|M_b')\) is not defined as the output of a two-universal hash function as required in Lemma 3. To overcome this difficulty, we show in the following lemma that the distribution \( \bar{p} \) induced by the encoder in Algorithm 3 also describes a process for which \((M_b\|M_b')\) would be defined as \((M_b\|M_b') \triangleq F(R_b,T_b^{1:|Y_u|L})\).

Lemma 4. Fix \( b \in \mathcal{B} \). To simplify notation we write \( T_b \) instead of \( T_b^{1:|Y_u|L} \), \( \bar{X}_b(s) \) instead of \( \bar{Z}_b^{1:N}(s) \), \( \bar{X}_b \) instead of \( \bar{X}_b^{1:N} \), and \( Z_b(s) \) instead of \( Z_b^{1:N}(s) \). We also define \( \bar{M}_b \triangleq (M_b\|M_b') \) such that \( T_b \triangleq R_b^{-1} \odot (\bar{M}_b|R_b') \). Next, define
\[
\bar{q}_{M_bT_bX_aZ_a(s)R_a} \triangleq \bar{p}_{X_aZ_a(s)|T_b}\bar{q}_{T_b}\bar{q}_{R_a}\bar{q}_{\bar{M}_b|T_bR_a},
\]
with \( \bar{q}_{T_b} \) the uniform distribution over \( \{0,1\}^{|Y_u|L} \), \( \bar{q}_{R_a} \) the uniform distribution over \( \mathcal{R} \), and \( \forall \bar{m}_b, \forall t_b, \forall r_b, \nabla \bar{q}_{M_b|T_bR_a}(\bar{m}_b|t_b,r_b) \equiv 1 \{\bar{m}_b = F(t_b, t_b)\} \). We have
\[
\bar{p}_{\bar{M}_bT_bX_aZ_a(s)R_a} = \bar{q}_{M_bT_bX_aZ_a(s)R_a}.
\]

Proof. See Appendix B. \( \blacksquare \)

Let \( \mathcal{A}_b \subset [1,N] \) such that \( |\mathcal{A}_b| = \alpha N \) and consider \( \bar{X}_b^{1:N}[\mathcal{A}_b] \), the \( \alpha N \) symbols that the eavesdropper has chosen to have access to in Block \( b \in \mathcal{B} \). We study, by combining Lemmas 3, 4, the independence between \((R_b,\bar{Z}_b^{1:N}(s),\bar{X}_b^{1:N}[\mathcal{A}_b])\), i.e., all the knowledge at the eavesdropper in Block \( b \in \mathcal{B} \), and \((M_b\|M_b')\) as follows.
Lemma 5. Fix $b \in B$. We adopt the same notation as in Lemma 4 and also write $\tilde{X}_b[A_b]$ instead of $\tilde{X}_b^{1:N}[A_b]$ for convenience. We have for any $\gamma \in ]0, 1[$

$$\mathbb{V}(\tilde{p}_{M_b R_b Z_b(s)} X_b[A_b], \tilde{p}_{M_b R_b Z_b(s)} X_b[A_b]) \leq 2^{1-L\gamma} + \sqrt{2^{r-H(T_b|\tilde{Z}_b(s)X_b[A_b])} + N\delta(1)(K, L)},$$

(20)

where $\delta(1)(K, L) \triangleq (K^{-1} + 1)\sqrt{2L\gamma - 1}$.

Proof. See Appendix C.

Next, using Lemma 2, we lower bound the conditional entropy in (20) in the following lemma.

Lemma 6. Fix $b \in B$. We adopt the same notation as in Lemmas 4, 5. We have

$$H(T_b|\tilde{Z}_b(s)\tilde{X}_b[A_b]) \geq N[(1 - \alpha)H(U|Z(s)) + \alpha H(U|X) - \delta(2)(K, L)],$$

where we have defined

$$\delta(2)(K, L) \triangleq (LK)^{-1}H_b(LK\delta_K) + LK\delta_K + 2\sqrt{2\ln 2\sqrt{2LK\delta_K}} \log \frac{|X|\max_{s \in S} |Z_s|}{\sqrt{2}\ln 2\sqrt{2LK\delta_K}} + o(1),$$

with $H_b(\cdot)$ the binary entropy.

Proof. See Appendix D.

By combining Lemma 5 and Lemma 6 we obtain the following result.

Lemma 7. Fix $b \in B$. We adopt the same notation as in Lemma 6. We have for any $\gamma \in ]0, 1[$

$$\mathbb{V}(\tilde{p}_{M_b R_b Z_b(s)} X_b[A_b], \tilde{p}_{M_b R_b Z_b(s)} X_b[A_b]) \leq 2^{1-L\gamma} + \sqrt{2^{r-N[(1-\alpha)H(U|Z(s)) + \alpha H(U|X) - \delta(3)(K, L)]}},$$

where $\delta(3)(K, L) \triangleq \delta(1)(K, L) + \delta(2)(K, L)$, with $\delta(1)(K, L)$ defined in Lemma 5 and $\delta(2)(L, K)$ defined in Lemma 6.

Finally, we obtain security in a given block as follows.

Lemma 8. Fix $b \in B$. Choose

$$r \triangleq N \left( (1 - \alpha) \min_{s \in S} H(U|Z(s)) + \alpha H(U|X) - \delta(3)(K, L) - \xi \right),$$
with \( \xi > 0 \) and \( \delta^{(3)}(K, L) \) defined in Lemma 7. We have for \( L \) large enough
\[
I \left( M_b M'_b; \tilde{Z}_b(s) X_b[A_b] R_b \right) \leq \delta^{(4)}(K, L, \xi),
\]
where \( \delta^{(4)}(K, L, \xi) \triangleq (2^{1-L} + \sqrt{2^{-N \xi}}) \log \frac{2^N}{2^{1-L} + \sqrt{2^{-N \xi}}} \).

Proof. We adopt the same notation as in the previous lemmas. By definition of \( r \) and by Lemma 7, we have
\[
\forall (\tilde{Z}_b(s) X_b[A_b] R_b) \leq 2^{1-L} + \sqrt{2^{-N \xi}},
\]
(21)
We thus have
\[
I(M_b M'_b; \tilde{Z}_b(s) X_b[A_b] R_b) = I(M_b; \tilde{Z}_b(s) X_b[A_b] R_b)
\]
\[
\leq (f(\forall (\tilde{Z}_b(s) X_b[A_b] R_b))
\]
\[
\leq f(2^{1-L} + \sqrt{2^{-N \xi}}),
\]
(22)
where (a) holds by [37] with \( f : x \mapsto x \log(2^N/x) \), (b) holds for \( L \) large enough because \( f \) is increasing for small enough values.

E. Analysis of security over all blocks jointly

We obtain security over all blocks jointly from Lemma 8 as follows.

Lemma 9. For convenience, we define for \( i, j \in B \), \( \tilde{Z}_{1,i}(s) \triangleq (\tilde{Z}_b(s))_{b \in [1,i]} \), \( \tilde{X}_{1,i}[A] \triangleq (\tilde{X}_b^{1:N}[A_b])_{b \in [1,i]} \), \( R_{i,j} \triangleq (R_b)_{b \in [i,j]} \), and \( M_{i,j} \triangleq (M_b)_{b \in [i,j]} \). We have
\[
\max_{s \in \Theta} \max_{A \in A} I(M_{1:B}; \tilde{Z}_{1:B}(s) \tilde{X}_{1:B}[A] R_{1:B}) \leq 2B \delta^{(4)}(L, K, \xi),
\]
where \( \delta^{(4)}(L, K, \xi) \) is defined in Lemma 8.

Proof. For convenience, define for \( i \in B \), \( L_i \triangleq (\tilde{Z}_i(s), X_i[A_i], R_i) \) and \( L_{1:i} \triangleq (\tilde{Z}_{1:i}(s), \tilde{X}_{1:i}[A], R_{1:i}) \). We have
\[
I(M_{1:B}; L_{1:B}) \overset{(a)}{=} \sum_{i=0}^{B-1} I(M_{1:B}; L_{i+1}\mid L_{1:i})
\]

\[
\begin{align*}
(b) & \sum_{i=0}^{B-1} I(M_{1:i+1}; L_{i+1}| L_1) \\
& \leq \sum_{i=0}^{B-1} I(M_{1:i+1} L_{i+1}; L_{i+1}) \\
& = \sum_{i=0}^{B-1} I(M_{i+1}; L_{i+1}) + I(M_{1:i} L_{1:i}; L_{i+1}| M_{i+1}) \\
(c) & \leq B\delta^{(4)}(K, L, \xi) + \sum_{i=0}^{B-1} I(M_{1:i}, M'_{i+1} L_{1:i}; L_{i+1}| M_{i+1}) \\
(d) & = B\delta^{(4)}(K, L, \xi) + \sum_{i=0}^{B-1} I(M'_{i+1}; L_{i+1}| M_{i+1}) \\
(e) & = B\delta^{(4)}(K, L, \xi) + \sum_{i=0}^{B-1} I(M_{i+1} M'_{i+1}; L_{i+1}) \\
(f) & \leq 2B\delta^{(4)}(K, L, \xi),
\end{align*}
\]

where (a) holds by the chain rule, (b) holds by the chain rule and because \(I(M_{1:i+2:B}; L_{i+1}| L_1; M_{1:i+1}) \leq I(M_{1:i+2:B}; L_{1:i+1} M_{1:i+1}) = 0\) (c) holds by Lemma 8, (d) holds by the chain rule and because \((M_{1:i}, L_{1:i}) - M'_{i+1} - (L_{i+1}, M_{i+1})\) forms a Markov chain, (e) holds by independence between \(M'_{i+1}\) and \(M_{i+1}\), (f) holds by Lemma 8. Since (23) holds for any \(s \in S\) and any \(A \in \mathcal{A}\), we obtain the lemma.

\[\square\]

**F. Secrecy Rate**

The rate of the transmitted messages is

\[
\frac{\sum_{b \in B} |M_b|}{BN} \stackrel{(a)}{=} \frac{r + (B - 1)(r - L|V_{U|Y}|)}{BN} \\
\geq \frac{r}{N} - \frac{|V_{U|Y}|}{K} \\
\stackrel{(b)}{=} (1 - \alpha) \min_{s \in \mathcal{S}} H(U|Z(s)) + \alpha H(U|X) - \delta^{(3)}(K, L) - \xi - H(U|Y) - o(1) \\
= I(U; Y) - \alpha I(U; X) - (1 - \alpha) \min_{s \in \mathcal{S}} I(U; Z(s)) - \delta^{(3)}(K, L) - \xi - o(1),
\]

where (a) holds by (11), (b) holds by the choice of \(r\) in Lemma 8 and because \(\lim_{K \to \infty} |V_{U|Y}|/K = H(U|Y)\) by [29], [35], (c) holds by [29, Lemma 1], [35].
G. Randomness amortization

The randomness \((R_b)_{1:B}\) in the coding scheme of Section IV-C needs to be shared between the legitimate users. This can be done with negligible impact on the overall communication rate similar to [15] using an hybrid argument by repeating the coding scheme of Section IV-C with the same randomness \((R_b)_{1:B}\).

VII. PROOF OF THEOREM 7.1 (WITHOUT PRE-SHARED KEY)

The coding scheme of Section IV-C requires a pre-shared secret key between the legitimate users. We now consider the initialization phase, described in Algorithms 1, 2, to generate such a key with negligible impact on the overall communication rate. We study the reliability and the secrecy of the generated key in Sections VII-A and VII-B, respectively, the impact of the initialization phase on the overall communication rate in Section VII-C, and the joint secrecy of the initialization phase and the coding scheme of Section IV-C in Section VII-D. We adopt the same notation as in Section VI.

A. Key reliability

We have following lemma, whose proof is similar to the one of Lemma 2, and is thus omitted.

**Lemma 10.** For \(b \in \mathcal{B}_0\), the distribution induced by the encoder of Algorithm 1 is approximated as follows.

\[
\mathbb{D}(q_{U_b^{1:N}X_b^{1:N}Y_b^{1:N}Z_b^{1:N}(s)}\|\tilde{p}_{U_b^{1:N}X_b^{1:N}Y_b^{1:N}Z_b^{1:N}(s)}) \leq 2LK\delta_K.
\]

Then, we have

\[
\mathbb{P}\left[\text{Key}_b \neq \text{Key}_b\right] \leq \mathbb{P}\left[\tilde{U}_b^{1:N} \neq \tilde{U}_b^{1:N}\right] \leq B_0 L(\sqrt{2 \ln 2} + 2K\delta_K).
\]

where the last inequality holds similar to (17).

B. Key secrecy

We first show secrecy in a given Block \(b \in \mathcal{B}_0\). Let \(A_b \subset [1, N]\) such that \(|A_b| = \alpha N\) and consider \(\tilde{X}_b^{1:N}[A_b]\), the \(\alpha N\) symbols that the eavesdropper has chosen to have access to in Block \(b \in \mathcal{B}_0\). Define
$p_{\mathcal{K}}$ the uniform distribution over $\{0,1\}^{I_{\text{key}}}$. We have

$$\mathbb{V}(\tilde{p}_{\mathcal{K},R_b^{\text{init}}} Z_b(s) X_b[A_b] D_b R_b^{\text{init}}; p_{\mathcal{K}} R_b^{\text{init}} Z_b(s) X_b[A_b] D_b R_b^{\text{init}}) \leq 2\epsilon + \sqrt{2 l_{\text{key}} H^{\mathbb{P}_{\tilde{p}_{\mathcal{K},R_b^{\text{init}}} Z_b(s) X_b[A_b] D_b R_b^{\text{init}}}}(\tilde{p}_{\mathcal{K},R_b^{\text{init}}} Z_b(s) X_b[A_b] D_b R_b^{\text{init}})},$$

where (a) holds by Lemma 3, (b) holds by Lemma 15 with $\gamma \in [0,1]$ as in the proof of Lemma 5 with $\delta^{(1)}(K, L)$ defined in Lemma 5.

**Lemma 11.** For $b \in \mathcal{B}_0$, we have

$$H(\tilde{U}_b|\tilde{B}_b(s) \tilde{X}_b[A_b] D_b R_b^{\text{init}}) \geq N[I(U; Y) - \alpha I(U; X) - (1 - \alpha) \min_{s \in \mathcal{S}} I(U; Z(s)) - \delta^{(5)}(K, L)],$$

where

$$\delta^{(5)}(K, L) \triangleq 2\sqrt{2\ln 2} \sqrt{2LK\delta_K} \log \frac{|X|^{2\max_{s \in \mathcal{S}} |Z_s|}}{2\ln 2 \sqrt{2LK\delta_K}} + o(1).$$

**Proof.** We have

$$H(\tilde{U}_b|\tilde{Z}_b(s) \tilde{X}_b[A_b] D_b R_b^{\text{init}}) = H(\tilde{U}_b|\tilde{Z}_b(s) \tilde{X}_b[A_b]) - I(D_b R_b^{\text{init}}; \tilde{U}_b|\tilde{Z}_b(s) \tilde{X}_b[A_b])$$

$$\geq H(\tilde{U}_b|\tilde{Z}_b(s) \tilde{X}_b[A_b]) - L(|\mathcal{H}_U| + |\mathcal{H}_U \setminus \mathcal{V}_U|)$$

$$\geq H(\tilde{U}_b|\tilde{B}_b(s) \tilde{X}_b[A_b]) - NH(U|Y) - o(KL)$$

$$\geq N(1 - \alpha)H(U|Z(s)) + N\alpha H(U|X) - NH(U|Y)$$

$$- 2LK\sqrt{2\ln 2} \sqrt{2LK\delta_K} \log \frac{|X|^{2\max_{s \in \mathcal{S}} |Z_s|}}{2\ln 2 \sqrt{2LK\delta_K}} - o(KL),$$

where (a) holds because $\lim_{K \to \infty} |\mathcal{H}_U|/K = H(U|Y)$ [28], and $\lim_{K \to \infty} |\mathcal{V}_U|/K = H(U|Y)$ [29], [35], (b) holds similar to the proof of Lemma 6.

Choose

$$l_{\text{key}}^b \triangleq N[I(U; Y) - \alpha I(U; X) - (1 - \alpha) \min_{s \in \mathcal{S}} I(U; Z(s)) - \delta^{(1)}(K, L) - \delta^{(5)}(K, L) - \xi],$$

with $\xi > 0$. By combining (24) and Lemma 11, we obtain for $b \in \mathcal{B}_0$,

$$\mathbb{V}(\tilde{p}_{\mathcal{K},R_b^{\text{init}}} Z_b(s) X_b[A_b] D_b R_b^{\text{init}}; p_{\mathcal{K}} R_b^{\text{init}} Z_b(s) X_b[A_b] D_b R_b^{\text{init}}) \leq 2 \cdot 2^{-L^7} + \sqrt{2 - N\xi}.$$  

(25)
Lemma 12. We have for $L$ large enough

$$I \left( \text{Key}_b; \tilde{Z}_b(s) \tilde{X}_b[A_b]D_bR_b^{\text{init}}R_b^{\text{init}'} \right) \leq 2 \delta^{(4)}(K, L, \xi),$$

$$\log |\mathcal{K}_b| - H(\text{Key}_b) \leq \delta^{(4)}(K, L, \xi),$$

where $\mathcal{K}_b \triangleq \{0, 1\}^{l_{\text{key}}}$, and $\delta^{(4)}(K, L, \xi)$ is defined in Lemma 8.

Proof. The first inequality holds similar to the proof of Lemma 8 by using (25) in place of (21). The second inequality holds by [37, Lemma 2.7] and (25). ■

By mutual independence of all the $B_0$ blocks of the initialization phase, we obtain from Lemma 12 the following result.

Lemma 13. Define $\text{Key} \triangleq (\text{Key}_b)_{b \in B_0}$ and $\mathcal{K} \triangleq \mathcal{K}_b^{B_0}$. Let $\tilde{Z}^{\text{init}}(s)$ denote all the knowledge of the eavesdropper related to the initialization phase, i.e., $\tilde{Z}^{\text{init}}(s) \triangleq (\tilde{Z}_b(s), \tilde{X}_b[A_b], D_b, R_b^{\text{init}}, R_b^{\text{init}'} )_{b \in B_0}$. We have for $K$ large enough

$$\max_{s \in S} \max_{A \in A} I \left( \text{Key}; \tilde{Z}^{\text{init}}(s) \right) \leq 2 B_0 \delta^{(4)}(K, L, \xi),$$

$$\log |\mathcal{K}| - H(\text{Key}) \leq B_0 \delta^{(4)}(K, L, \xi).$$

C. Impact of the initialization phase on the overall communication rate

The initialization phase requires $\rho NB_0$ channel uses, for some fixed $\rho \in \mathbb{N}$, to generate the secret key and transmit $(D_b, R_b^{\text{init}}, R_b^{\text{init}'} )_{b \in B_0}$. We choose $B_0$ such that

$$B_0 = \left\lceil \frac{l_{\text{OTP}}}{l_{\text{key}}} \right\rceil,$$

where $l_{\text{OTP}} = o(NB)$ represents the key length necessary to perform the one-time pad that appears in Algorithms 1, 2. Hence, the impact of the initialization phase on the overall communication is

$$\rho NB_0 < \rho N \left( 1 + \frac{l_{\text{OTP}}}{l_{\text{key}}} \right) = \rho \frac{o(NB)}{l_{\text{key}}/N} = o(NB). \tag{26}$$

We deduce from (26) that the communication rate of the coding scheme of Section IV-C and the initialization phase (considered jointly) is the same as the communication rate of the coding scheme of Section IV-C alone.
D. Security of Algorithms 3, 4 and the initialization phase when considered jointly

Let $M_{\text{OTP}}$ be the sequence that needs to be secretly transmitted with a one-time pad in Algorithm 3. Let $C \triangleq M_{\text{OTP}} \oplus \text{Key}$ be the encrypted version of $M_{\text{OTP}}$ using Key, obtained in the initialization phase. Let $\tilde{Z}_B(s) \triangleq (\tilde{Z}_{1:B}(s), \tilde{X}_{1:B}[A], R_{1:B})$ denote all the observations of the eavesdropper related to the coding scheme of Section IV-C, excluding $C$. Let $Z_{\text{init}}(s)$, defined as in Lemma 13, denote all the observations of the eavesdropper related to the initialization phase. The following lemma shows that strong secrecy holds for the coding scheme of Section IV-C and the initialization phase considered jointly.

Lemma 14. We have

$$\max_{s \in \mathcal{S}} \max_{A \in \mathcal{A}} I(M_{1:B}; C\tilde{Z}_B(s)\tilde{Z}_{\text{init}}(s)) \leq (2B + 3B_0)\delta^{(4)}(K, L, \xi),$$

where $\delta^{(4)}(K, L, \xi)$ is defined in Lemma 8.

Proof. We have

$$I(M_{1:B}; C\tilde{Z}_B(s)\tilde{Z}_{\text{init}}(s)) = I(M_{1:B}; \tilde{Z}_B(s)\tilde{Z}_{\text{init}}(s)) + I(M_{1:B}; C|\tilde{Z}_B(s)\tilde{Z}_{\text{init}}(s))$$

$$\overset{(a)}{=} I(M_{1:B}; \tilde{Z}_B(s)) + I(M_{1:B}; C|\tilde{Z}_B(s)\tilde{Z}_{\text{init}}(s))$$

$$\leq I(M_{1:B}; \tilde{Z}_B(s)) + I(M_{1:B}\tilde{Z}_B(s)\tilde{Z}_{\text{init}}(s); C)$$

$$= I(M_{1:B}; \tilde{Z}_B(s)) + I(C; M_{1:B}\tilde{Z}_B(s)) + I(C; \tilde{Z}_{\text{init}}(s)|M_{1:B}\tilde{Z}_B(s)), \quad (27)$$

where $(a)$ holds by the chain rule and because $I(M_{1:B}; \tilde{Z}_{\text{init}}(s)|\tilde{Z}_B(s)) \leq I(M_{1:B}\tilde{Z}_B(s); \tilde{Z}_{\text{init}}(s)) = 0$.

Next, we have

$$I(C; M_{1:B}\tilde{Z}_B(s)) \leq \log|\mathcal{K}| - H(C|M_{1:B}\tilde{Z}_B(s))$$

$$\leq \log|\mathcal{K}| - H(\text{Key} \oplus M_{\text{OTP}}|M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s))$$

$$= \log|\mathcal{K}| - H(\text{Key}|M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s))$$

$$= \log|\mathcal{K}| - H(\text{Key}). \quad (28)$$

We also have

$$I(C; \tilde{Z}_{\text{init}}(s)|M_{1:B}\tilde{Z}_B(s)) \leq I(CM_{\text{OTP}}; \tilde{Z}_{\text{init}}(s)|M_{1:B}\tilde{Z}_B(s))$$

$$= I(\text{Key}M_{\text{OTP}}; \tilde{Z}_{\text{init}}(s)|M_{1:B}\tilde{Z}_B(s))$$
\begin{align*}
  \text{(b)} & \quad I(\text{Key}; \tilde{Z}^{\text{init}}(s)|M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s)) \\
  & \leq I(\text{Key}M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s); \tilde{Z}^{\text{init}}(s)) \\
  & \text{(c)} \quad I(\text{Key}; \tilde{Z}^{\text{init}}(s)),
\end{align*}

where \text{(b)} holds by the chain rule and because \(I(M_{\text{OTP}}; \tilde{Z}^{\text{init}}(s)|M_{1:B}\tilde{Z}_B(s)) \leq I(M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s); \tilde{Z}^{\text{init}}(s)) = 0\), \text{(c)} holds by the chain rule and because \(I(M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s); \tilde{Z}^{\text{init}}(s)|\text{Key}) \leq I(M_{\text{OTP}}M_{1:B}\tilde{Z}_B(s); \tilde{Z}^{\text{init}}(s)\text{Key}) = 0\).

By combining (27), (28), and (29), we obtain

\[
I(M_{1:B}; C\tilde{Z}_B(s)\tilde{Z}^{\text{init}}(s)) \leq I(M_{1:B}; \tilde{Z}_B(s)) + I(\text{Key}; \tilde{Z}^{\text{init}}(s)) + \log|{\mathcal K}| - H(\text{Key}).
\]

Finally, we obtain the lemma with Lemmas 9 and 13.

\section*{VIII. PROOF OF THEOREM 7.2}

We assume in the following that there exists a best channel for the eavesdropper [24], i.e., \(\exists s^* \in S, \forall s \in S, X - Z_{s^*} - Z_s\). Similar to the proof of Theorem 7.1, we proceed in two steps. We first ignore the initialization phase and assume that the legitimate users have access to a secret key to perform the one-time pad in Algorithms 3, 4. We only show blockwise security as the remainder of the proof is similar to the proof in Section VI. We also omit the second step that consists in analyzing the initialization phase jointly with Algorithms 3, 4, as it is similar to the analysis in Section VII.

\subsection*{A. Blockwise security analysis}

We adopt the same notation as in Section VI. We have the following inequality, whose proof is identical to the proof of Lemma 2. For \(b \in {\mathcal B}\), we have

\[
\mathbb{D}(q_{U^1:NX^1:NY^1:NZ^1:N(s_b)} \| \tilde{p}_{U^1:NX^1:NY^1:NZ^1:N(s_b)}) \leq 2LK\delta_K,
\]

where we have defined \(q_{U^1:NX^1:NY^1:NZ^1:N(s_b)} \triangleq \prod_{i=1}^N q_{UXYZ(s,b)}\). Next, similar to Lemma 5 using (30) in place of Lemma 2, we have for any \(\gamma \in [0,1]\)

\[
\mathbb{V}(\tilde{p}_{M_bR_bZ_b(s_b)}X_b[A_b]; \tilde{p}_{M_b\tilde{R}_bZ_b(s_b)}X_b[A_b]) \leq 2^{1-L} + 2\sqrt{2^{r-H(T_b|\tilde{Z}_b(s_b))}X_b[A_b]} + N\delta^{(5)}(K,L),
\]

Finally, we obtain the lemma with Lemmas 9 and 13.
where $\delta^{(1)}(K, L)$ is defined in Lemma 5. We then have

$$H \left( T_b | \tilde{Z}_b(s_b) \right) \geq H (U_b | Z_b(s_b) X_b[A_b]) - N \delta^{(2)}(K, L)$$

$$\geq H (U_b | Z_b(s^*) Z_b(s_b) X_b[A_b]) - N \delta^{(2)}(K, L)$$

$$(b) \geq H (U_b | Z_b(s^*) X_b[A_b]) - N \delta^{(2)}(K, L)$$

$$(c) = N(1 - \alpha) H(U | Z(s^*)) + N\alpha H(U | X) - N\delta^{(2)}(K, L),$$

where $(a)$ holds as in the proof of Lemma 6 with $\delta^{(2)}(K, L)$ defined in Lemma 6, $(b)$ holds because $(U_b, X_b - Z_b(s^*) - Z_b(s_b))$ forms a Markov chain, $(c)$ holds as in the proof of Lemma 6. Finally, from (31) and (32), we can conclude as in Section VI-D.

IX. CONCLUDING REMARKS

We developed an explicit coding scheme for the wiretap channel. We proved that within a unified coding scheme and with appropriate parameter choices, our coding scheme achieves the secrecy capacity of (i) Wyner’s wiretap channel, (ii) the wiretap channel of type II, (iii) the wiretap channel of type II with noisy main channel, (iv) the hybrid Wyner’s/type II wiretap channel, and the best known single-letter achievable secrecy rates for (v) the wiretap channel when uncertainties hold on the eavesdropper’s channel statistics (compound model), (vi) the wiretap channel when the eavesdropper’s channel statistics are arbitrarily varying.

Our coding scheme can also be applied to the problem of secret sharing, first introduced in [38] for channel models, and then extended in [39] to source models. Specifically, our construction can be applied to the case of a single dealer when the access structure is the set of all participants, i.e., when all the participants need to pool their share together to recover the secret.

While much remains to be done to reduce the overall blocklength of the coding scheme, our result provides the first explicit coding scheme for some wiretap channel models and improves previous constructions by achieving larger secrecy rates or relaxing assumptions such as symmetry/degradation of the channels and the necessity of a pre-shared secret key at the legitimate users.
APPENDIX A

PROOF OF LEMMA 2

Let \( b \in B \) and \( l \in L \). We have

\[
\mathbb{D}(q^{A^1 \mid K} \parallel \bar{p}_{A^1 \mid K}) = \sum_{j=1}^{K} \mathbb{E}_{q^{A^1 \mid j-1}} \mathbb{D}(q^{A^1 \mid j-1} \parallel \bar{p}_{A^1 \mid j-1})
\]

\[
= \sum_{j=1}^{K} \mathbb{E}_{q^{A^1 \mid j-1}} \mathbb{D}(q^{A^1 \mid j-1} \parallel \bar{p}_{A^1 \mid j-1})
\]

\[
= \sum_{j \in \mathcal{V}_U} (1 - H(A^1 \mid A^{1:j-1}))
\]

\[
\leq \sum_{j \in \mathcal{V}_U} \delta_K
\]

\[
\leq K \delta_K,
\]

where \((a)\) holds by the chain rule for relative entropy \([40]\), \((b)\) holds by \((14)\), \((c)\) holds because the bits \(\bar{A}^{1,K}_{b,l}[\mathcal{V}_U]\) are uniformly distributed, which is a consequence of the definition of \(\bar{A}^{1,K}_{b,l}[\mathcal{V}_U]\) in \((13)\) and the fact that the bits \(T_b^{1:|\mathcal{V}_U|} = R_b^{-1} \circ (M_b \| M'_b \| R'_b)\) are uniformly distributed since the bits \((M_b \| M'_b \| R'_b)\) are uniformly distributed, \((d)\) holds by definition of \(\mathcal{V}_U\). Next, we have

\[
\mathbb{D}(q^{U^1 \mid K \mid V^1 \mid K} \parallel \bar{p}_{U^1 \mid K \mid V^1 \mid K}) = \mathbb{E}_{q^{U^1 \mid K}} \mathbb{D}(q^{V^1 \mid K \mid U^1 \mid K} \parallel \bar{p}_{V^1 \mid K \mid U^1 \mid K}) + \mathbb{D}(q^{U^1 \mid K} \parallel \bar{p}_{U^1 \mid K})
\]

\[
\leq \mathbb{E}_{q^{U^1 \mid K}} \mathbb{D}(q^{V^1 \mid K \mid U^1 \mid K} \parallel \bar{p}_{V^1 \mid K \mid U^1 \mid K}) + K \delta_K
\]

\[
= \sum_{j=1}^{K} \mathbb{E}_{q^{U^1 \mid K \mid V^1 \mid j-1}} \mathbb{D}(q^{V^1 \mid j-1 \mid U^1 \mid K} \parallel \bar{p}_{V^1 \mid j-1 \mid U^1 \mid K}) + K \delta_K
\]

\[
= \sum_{j \in \mathcal{V}_X \mid U} \mathbb{D}(q^{V^1 \mid j-1 \mid U^1 \mid K} \parallel \bar{p}_{V^1 \mid j-1 \mid U^1 \mid K}) + K \delta_K
\]

\[
\leq \sum_{j \in \mathcal{V}_X \mid U} (1 - H(V^1 \mid j-1 \mid U^1 \mid K)) + K \delta_K
\]

\[
\leq \sum_{j \in \mathcal{V}_X \mid U} \delta_K + K \delta_K
\]

\[
\leq 2K \delta_K,
\]

where \((a)\) and \((c)\) hold by the chain rule for relative entropy \([40]\), \((b)\) holds by \((33)\) and because \(\mathbb{D}(q^{U^1 \mid K} \parallel \bar{p}_{U^1 \mid K}) = \mathbb{D}(q^{A^1 \mid K} \parallel \bar{p}_{A^1 \mid K})\) by invertibility of \(G_K\), \((d)\) holds by \((15)\), \((e)\) holds by uniformity of
the bits $\overline{V}_{b,l}^{1,K} [V_{X|U}]$ by (15), (f) holds by definition of $V_{X|U}$. Finally, we obtain

$$\mathbb{D}(q_{U^{1:N}X^{1:N}Y^{1:N}Z_{1:N}^{1:N}(s)} \| P_{U^{1:N}X^{1:N}Y^{1:N}Z_{1:N}^{1:N}(s)})$$

$$= \sum_{l \in \mathcal{L}} \mathbb{D}(q_{U^{1,N}X^{1,K}Y^{1,K}Z_{1,K}^{1,K}(s)} \| P_{U^{1,N}X^{1,K}Y^{1,K}Z_{1,K}^{1,K}(s)})$$

$$= \sum_{l \in \mathcal{L}} \left[ \mathbb{D}(q_{U^{1,K}X^{1,K}} \| P_{U^{1,K}X^{1,K}}) + \mathbb{E}_{q_{U^{1,K}X^{1,K}}} \mathbb{D}(q_{Y^{1,K}Z_{1,K}(s)} \| P_{Y^{1,K}Z_{1,K}(s)}|U^{1,K}X^{1,K}) \right]$$

$$= \sum_{l \in \mathcal{L}} 2K \delta_K$$

$$= 2LK\delta_K, \quad (35)$$

where (a) holds because the random variables $(\overline{U}_{b,l}^{1,K}, \overline{X}_{b,l}^{1,K}, \overline{Y}_{b,l}^{1,K}, \overline{Z}_{b,l}^{1,K}(s))$ across the different sub-blocks $l \in \mathcal{L}$ are independent by construction (see Algorithm 3 and Remark 2), (b) holds by the chain rule for relative entropy [40], (c) holds because $P_{Y^{1,K}Z_{1,K}(s)}|U^{1,K}X^{1,K} = P_{Y^{1,K}Z_{1,K}(s)}|U^{1,K}X^{1,K} = q_{Y^{1,K}Z^{1,K}(s)}|X^{1,K} = q_{Y^{1,K}Z^{1,K}(s)}|U^{1,K}X^{1,K}$, (d) holds by (34) because $\mathbb{D}(q_{U^{1,K}X^{1,K}} \| P_{U^{1,K}X^{1,K}}) = \mathbb{D}(q_{U^{1,K}Y^{1,K}} \| P_{U^{1,K}Y^{1,K}})$ by invertibility of $G_K$.

**APPENDIX B**

**PROOF OF LEMMA 4**

For any $(\bar{m}_b, t_b, x_b, z_b(s), r_b)$, we have

$$\bar{p}_{M_t,T_b,X_b,Z_b(s)R_b}(\bar{m}_b, t_b, x_b, z_b(s), r_b)$$

$$\overset{(a)}{=} \bar{p}_{X_b,Z_b(s)T_b}(x_b, z_b(s)|t_b)\bar{p}_{M_b}(\bar{m}_b)\bar{p}_{R_b}(r_b) \sum_{r_b'} \bar{p}_{R_b'}(r_b')\bar{p}_{T_b|R_b,M_b,R_b}(t_b|r_b', \bar{m}_b, r_b)$$

$$\overset{(b)}{=} \bar{p}_{X_b,Z_b(s)T_b}(x_b, z_b(s)|t_b)2^{-r_b} |\mathcal{R}|^{-1} \sum_{r_b'} 1 \{ t_b = r_b^{-1} \circ (\bar{m}_b||r_b') \}$$

$$= \bar{p}_{X_b,Z_b(s)T_b}(x_b, z_b(s)|t_b)2^{-|\mathcal{V}_U|L}|\mathcal{R}|^{-1} \sum_{r_b'} 1 \{ r_b \circ t_b = (\bar{m}_b||r_b') \}$$

$$\overset{(c)}{=} \bar{p}_{X_b,Z_b(s)T_b}(x_b, z_b(s)|t_b)2^{-|\mathcal{V}_U|L}|\mathcal{R}|^{-1} \{ F(r_b, t_b) = \bar{m}_b \}$$

$$= \bar{p}_{X_b,Z_b(s)T_b}(x_b, z_b(s)|t_b)\bar{q}_{T_b}(t_b)\bar{q}_{R_b}(r_b)\bar{q}_{X_b,T_b|R_b}(\bar{m}_b|t_b, r_b)$$

$$\overset{(d)}{=} \bar{q}_{M_t,T_b,X_b,Z_b(s)R_b}(\bar{m}_b, t_b, x_b, z_b(s), r_b),$$
where \( (a) \) holds because \( \tilde{p}_{\bar{M}_b R_b} = \tilde{p}_{X_b Z_b | T_b} \tilde{p}_{\bar{M}_b R_b} \tilde{p}_{T_b} | \bar{M}_b R_b \) and \( R_b' \) is independent of \( (\bar{M}_b, R_b) \), \( (b) \) holds by uniformity of \( \bar{M}_b, R_b, R_b' \), and by definition of \( T_b \), \( (c) \) holds because \( F(r_b, t_b) = \bar{m}_b \) \( \implies \) \( (\sum_{r_b} \mathbb{1} \{ r_b \odot t_b = (\bar{m}_b, r_b') \} = 1) \) (because \( \exists r_b' \in \{0, 1\}^{|Y_b|} \) such that \( r_b \odot t_b = (\bar{m}_b, r_b') \)) and \( F(r_b, t_b) \neq \bar{m}_b \) \( \implies \) \( (\sum_{r_b} \mathbb{1} \{ r_b \odot t_b = (\bar{m}_b, r_b') \} = 0) \), \( (d) \) holds by definition of \( \bar{q} \).

**APPENDIX C**

**PROOF OF LEMMA 5**

We will use the following lemma.

**Lemma 15** ( [41] ). *Let \( p_{X^L Z^L} \triangleq \prod_{i=1}^L p_{X_i Z_i} \) be a probability distribution over \( X^L \times Z^L \). For any \( \delta > 0 \), \( H^p_{\infty} (p_{X^L Z^L} | p_{Z^L}) \geq H(X^L | Z^L) - L \delta \), where \( e \triangleq 2^{\frac{-L \delta^2}{-L \log(1+|X^L|)}} \).

We have

\[
\mathbb{V}(\tilde{p}_{\bar{M}_b R_b Z_b(s) X_b[A_b]} \tilde{p}_{\bar{M}_b R_b Z_b(s) X_b[A_b]}^a) \leq 2^e + \sqrt{2^{r-H^p_{\infty}(\tilde{p}_{\bar{M}_b R_b Z_b(s) X_b[A_b]}))} - \sqrt{2^{r-H^p_{\infty}(\tilde{p}_{\bar{M}_b R_b Z_b(s) X_b[A_b]})) + L \delta^2 (K, L)}
\]

where \( (a) \) holds by Lemma 4 and the definition of \( \tilde{q} \), \( (b) \) holds by Lemma 3, \( (c) \) holds by Lemma 15, which can indeed be applied by Remark 3, with \( e \triangleq 2^{-L \gamma} \), \( \delta^0 (K, L) \triangleq \sqrt{2 L \gamma \log(2 |Y_b|) + 3} \), \( (d) \) holds by choosing \( \delta^1 (K, L) \triangleq (K^{-1} + 1) \sqrt{2L \gamma - 1} \geq \delta^0 (K, L) / K \).

**Remark 5.** An argument similar to the one in [6] to lower bound the min-entropy does not seem easily applicable in our case and would complexify the coding scheme with an extra round of reconciliation as in [42]. Lemma 15 appears to be a simpler alternative here.

**APPENDIX D**

**PROOF OF LEMMA 6**

We first introduce some notation for convenience. Define for any \( \mathcal{I} \subseteq [1, K] \), \( \tilde{A}_b[Z] \triangleq (\tilde{A}_b^{1:K} [Z])_{l \in \mathcal{I}} \) and \( A_b \triangleq (\tilde{A}_b^{1:K})_{l \in \mathcal{L}} \). For \( b \in \mathcal{B} \), consider \( (U_{b,l}^{1:K}, X_{b,l}^{1:K}, Z_{b,l}^{1:K}(s))_{l \in \mathcal{L}} \) distributed according to \( q_{U^{1:N} X^{1:N} Z^{1:N}(s)} \triangleq \prod_{i=1}^N q_{UXZ(s)} \) and define for \( l \in \mathcal{L} \), \( A_{b,l}^{1:K} \triangleq G_K U_{b,l}^{1:K} \). Next, define for any
\( \mathcal{I} \subseteq [1, K], A_b[\mathcal{I}] \triangleq (A_{b,i}^{1:K}[\mathcal{I}])_{i \in \mathcal{L}} \) and \( A_b \triangleq (A_{b,i}^{1:K})_{i \in \mathcal{L}} \). Define \( U_b[A_b] \triangleq (U_{b,i}^{1:K}[A_{b,i}])_{i \in \mathcal{L}}, U_b[A_b^c] \triangleq (U_{b,i}^{1:K}[A_{b,i}^c])_{i \in \mathcal{L}}, X_b[A_b] \triangleq (X_{b,i}^{1:K}[A_{b,i}])_{i \in \mathcal{L}}, X_b[A_b^c] \triangleq (X_{b,i}^{1:K}[A_{b,i}^c])_{i \in \mathcal{L}}, Z_b[A_b] \triangleq (Z_{b,i}^{1:K}(s)[A_{b,i}])_{i \in \mathcal{L}}, Z_b[A_b^c] \triangleq (Z_{b,i}^{1:K}(s)[A_{b,i}^c])_{i \in \mathcal{L}}. \)

We have

\[
H \left( \tilde{A}_b[\mathcal{V}_U] | \tilde{Z}_b(s) \tilde{X}_b[A_b] \right) - H (A_b[\mathcal{V}_U] | Z_b(s) X_b[A_b]) \\
= H \left( \tilde{A}_b[\mathcal{V}_U] | \tilde{Z}_b(s) \tilde{X}_b[A_b] \right) - H (A_b[\mathcal{V}_U] | Z_b(s) X_b[A_b]) + H (Z_b(s) X_b[A_b]) - H \left( \tilde{Z}_b(s) \tilde{X}_b[A_b] \right) \\
\geq -2LK \sqrt{2 \ln 2} / \sqrt{2LK\delta_K} \log \frac{|\mathcal{X}|^2 |Z_b|}{|\mathcal{X}|^2 \max_{s \in \mathcal{S}} |Z_s|} \\
\geq -2LK \sqrt{2 \ln 2} / \sqrt{2LK\delta_K} \log \frac{|\mathcal{X}|^2 \max_{s \in \mathcal{S}} |Z_s|}{|\mathcal{X}|^2 \sqrt{2LK\delta_K}} \\
= -\delta^*, \quad (36)
\]

where the first inequality holds by [37, Lemma 2.7] applied twice because for \( N \) large enough, \( \mathbb{V}(q_{A_b[\mathcal{V}_U] Z_b(s) X_b[A_b], \tilde{A}_b[\mathcal{V}_U] Z_b(s) X_b[A_b]}) \leq 2^{\ln 2} \sqrt{D(q_{A_b[\mathcal{V}_U] Z_b(s) X_b[A_b], \tilde{A}_b[\mathcal{V}_U] Z_b(s) X_b[A_b])} \leq \sqrt{2 \ln 2} \sqrt{2LK\delta_K} \leq 2 \ln 2 \sqrt{2LK\delta_K} \) where we have used Pinsker’s inequality, the chain rule for divergence, positivity of the divergence, and Lemma 2.

Then, we have

\[
H \left( T_b | \tilde{Z}_b(s) \tilde{X}_b[A_b] \right) \overset{(a)}{=} H \left( \tilde{A}_b[\mathcal{V}_U] | \tilde{Z}_b(s) \tilde{X}_b[A_b] \right) \\
\overset{(b)}{=} H (A_b[\mathcal{V}_U] | Z_b(s) X_b[A_b]) - \delta^* \\
= H (A_b[\mathcal{H}_U] | Z_b(s) X_b[A_b]) - H (A_b[\mathcal{H}_U \setminus \mathcal{V}_U] | A_b[\mathcal{V}_U] Z_b(s) X_b[A_b]) - \delta^* \\
\overset{(c)}{=} H (A_b[\mathcal{H}_U] | Z_b(s) X_b[A_b]) - L |\mathcal{H}_U \setminus \mathcal{V}_U| - \delta^* \\
\overset{(d)}{=} H (A_b[\mathcal{H}_U] | Z_b(s) X_b[A_b]) - H_b (LK\delta_K) - (LK)^2 \delta_K - o(LK) - \delta^* \\
\overset{(e)}{=} H (U_b | Z_b(s) X_b[A_b]) - H_b (LK\delta_K) - (LK)^2 \delta_K - o(LK) - \delta^*, \quad (37)
\]

where (a) holds by definition of \( \tilde{A}_b[\mathcal{V}_U] \), (b) holds by (36), (c) holds because \( \lim_{K \to \infty} |\mathcal{H}_U| / K = H(U) \).
by [28], and \( \lim_{K \to \infty} |\mathcal{V}_U|/K = H(U) \) by [29], [35], (d) holds by Fano’s inequality since the error probability in the reconstruction of \( U_b \) from \( A_b[H_U] \) is upper-bounded by \( LK\delta_K \) by Lemma 1 and the union bound, (e) holds because \( U_b - (Z_b(s)[A_b^c], X_b[A_b]) - Z_b(s)[A_b] \) forms a Markov chain.

Next, we have

\[
H\left(U_b|Z_b(s)[A_b^c]X_b[A_b]\right)
= H\left(U_b[A_b^c]|Z_b(s)[A_b^c]X_b[A_b]\right) + H\left(U_b[A_b]|U_b[A_b^c]Z_b(s)[A_b^c]X_b[A_b]\right)
\]
\[
= H\left(U_b[A_b^c]|Z_b(s)[A_b^c]\right) + H\left(U_b[A_b]|U_b[A_b^c]Z_b(s)[A_b^c]X_b[A_b]\right) - H\left(X_b[A_b]|U_b[A_b^c]Z_b(s)[A_b^c]\right)
\]
\[
= H\left(U_b[A_b^c]|Z_b(s)[A_b^c]\right) + H\left(U_b[A_b]|X_b[A_b]\right)
\]
\[
= N(1 - \alpha)H(U|Z(s)) + N\alpha H(U|X),
\]

(38)

where (a) holds because \( X_b[A_b] \) is independent of \( (U_b[A_b^c], Z_b(s)[A_b^c]) \), (b) holds because \( (U_b[A_b], X_b[A_b]) \) is independent of \( (U_b[A_b^c], Z_b(s)[A_b^c]) \) and \( X_b[A_b] \) is independent of \( (U_b[A_b^c], Z_b(s)[A_b^c]) \), (c) holds because \( q_{U^{1:N}X^{1:N}Z^{1:N}}(s) = \prod_{i=1}^{N} q_{U_iX_iZ_i}(s) \). We obtain the lemma by combining (37) and (38).

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