ON IDENTIFYING MAGNETIZED ANOMALIES USING GEOMAGNETIC MONITORING

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Abstract. We propose and investigate the inverse problem of identifying magnetized anomalies beneath the Earth using the geomagnetic monitoring. Suppose a collection of magnetized anomalies presented in the shell of the Earth. The presence of the anomalies interrupts the magnetic field of the Earth, monitored above the Earth. Using the difference of the magnetic fields before and after the presence of the magnetized anomalies, we show that one can uniquely recover the locations as well as their material parameters of the anomalies. Our study provides a rigorous mathematical theory to the geomagnetic detection technology that has been used in practice.

Keywords: Maxwell system, geomagnetism, magnetized anomalies, inverse problem, uniqueness

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1. Introduction

It is widely known that the Earth as well as most of the planets in the solar system all generate magnetic fields through the motion of electrically conducting fluids [10,16]. Earth’s magnetic field, also known as the geomagnetic field, is the magnetic field that extends from the Earth’s interior out into the space. Following the general discussion in [5], the geomagnetic field is described by the Maxwell system as follows.

Let the Earth be of a core-shell structure with \( \Sigma_c \) and \( \Sigma \), respectively, signifying the core and the Earth. It is assumed that both \( \Sigma_c \) and \( \Sigma \) are bounded simply-connected \( C^2 \) domains in \( \mathbb{R}^3 \) and \( \Sigma_c \subset \Sigma \). \( \Sigma_s := \Sigma \setminus \Sigma_c \) signifies the shell of the Earth. We note that in the literature, it is usually assumed that the Earth and its core are concentric balls of radii \( R_1 \) and \( R_0 \) with \( R_0 \ll R_1 \). However, we shall not impose such a restrictive assumption in our study. Let \( \varepsilon, \mu \) and \( \sigma \) be all real-valued \( L^\infty \) functions, such that \( \varepsilon \) and \( \mu \) are positive and \( \sigma \) is nonnegative. The functions \( \varepsilon, \mu \) and \( \sigma \) signify the electromagnetic (EM) medium parameters in \( \mathbb{R}^3 \), and are referred to as the electric permittivity, the magnetic permeability and the electric conductivity, respectively. Let \( \varepsilon_0 \) and \( \mu_0 \) denote, respectively, the permittivity and the permeability of the uniformly homogeneous free space \( \mathbb{R}^3 \setminus \Sigma \). The material distribution is described by

\[
\sigma(x) = \sigma_c(x)\chi(\Sigma_c), \quad \mu(x) = (\mu_c(x) - \mu_0)\chi(\Sigma_c) + \mu_0, \\
\varepsilon(x) = (\varepsilon_c(x) - \varepsilon_0)\chi(\Sigma_c) + (\varepsilon_s(x) - \varepsilon_0)\chi(\Sigma_s) + \varepsilon_0, \tag{1.1}
\]

where and also in what follows, \( \chi \) denotes the characteristic function. By (1.1), we know that the mediums in the core and shell of the Earth are respectively characterized by \( (\Sigma_c; \varepsilon_c, \mu_c, \sigma_c) \) and \( (\Sigma_s; \varepsilon_s, \mu_0) \). Let \( \mathcal{E}(x,t) \) and \( \mathcal{H}(x,t) \), \( (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ \), respectively, denote the electric and magnetic fields of the Earth. They satisfy the following Maxwell

\[
\n\]
Figure 1. Schematic illustration of identifying magnetized anomalies using the geomagnetic monitoring.

System for \((x, t) \in \mathbb{R}^3 \times \mathbb{R}_+\) (cf. [5])

\[
\begin{align*}
\nabla \times \mathcal{H}(x, t) &= \varepsilon(x) \partial_t \mathcal{E}(x, t) + \sigma(x) (\mathcal{E}(x, t) + \mu(x) \mathbf{v} \times \mathcal{H}(x, t)), \\
\nabla \times \mathcal{E}(x, t) &= -\mu(x) \partial_t \mathcal{H}(x, t), \\
\nabla \cdot (\mu(x) \mathcal{H}(x, t)) &= 0, \quad \nabla \cdot (\varepsilon(x) \mathcal{E}(x, t)) = \rho(x, t), \\
\mathcal{E}(x, 0) &= \mathcal{H}(x, 0) = 0,
\end{align*}
\]

(1.2)

where \(\rho(x, t) = \rho_c(x, t) \chi(\Sigma_c) \in H^1(\mathbb{R}_+, L^2(\Sigma_c))\) stands for the charge density of the Earth core, and \(\mathbf{v} \in L^\infty(\Sigma)\) is the fluid velocity of the Earth. In (1.2), \(\mathbf{v} \times \mu \mathcal{H}\) is the so-called motional electromotive force generated by the rotation of the Earth.

Next we suppose that a collection of magnetized anomalies presented in the shell of the Earth. Let \(D_l, l = 1, 2, \ldots, l_0\), denote the magnetized anomalies, where \(D_l, 1 \leq l \leq l_0\) is a simply-connected Lipschitz domain such that the corresponding material parameters are given by \(\varepsilon_l, \mu_l\) and \(\sigma_l\). It is assumed that \(\varepsilon_l, \mu_l\) and \(\sigma_l\) are all positive constants with \(\mu_l \neq \mu_0, 1 \leq l \leq l_0\). With the presence of the magnetized anomalies \((D_l; \varepsilon_l, \mu_l, \sigma_l), l = 1, 2, \ldots, l_0,\) in the shell of the Earth, the EM medium configuration in the space \(\mathbb{R}^3\) is then described by

\[
\begin{align*}
\sigma(x) &= \sigma_c(x) \chi(\Sigma_c) + \sum_{l=1}^{l_0} \sigma_l \chi(D_l), \\
\mu(x) &= (\mu_c(x) - \mu_0) \chi(\Sigma_c) + \sum_{l=1}^{l_0} (\mu_l - \mu_0) \chi(D_l) + \mu_0, \\
\varepsilon(x) &= (\varepsilon_c(x) - \varepsilon_0) \chi(\Sigma_c) + (\varepsilon_s(x) - \varepsilon_0) \chi(\Sigma_s \setminus \bigcup_{l=1}^{l_0} D_l) + \sum_{l=1}^{l_0} (\varepsilon_l - \varepsilon_0) \chi(D_l) + \varepsilon_0.
\end{align*}
\]

(1.3)

In the sequel, we let \((\mathcal{E}_0, \mathcal{H}_0)\) be the solution to the Maxwell system (1.2) associated with the medium configuration in (1.1), and \((\mathcal{E}, \mathcal{H})\) be the solution to (1.2) associated with (1.3). Let \(\Gamma\) be an open surface located away from \(\Sigma\). In the current article, we are mainly concerned with the following inverse problem,

\[
(\mathcal{H}(x, t) - \mathcal{H}_0(x, t)) \bigg|_{(x, t) \in \Gamma \times \mathbb{R}_+} \rightarrow \bigcup_{l=1}^{l_0} (D_l; \varepsilon_l, \mu_l, \sigma_l).
\]

(1.4)

That is, one intends to recover the magnetized anomalies by monitoring the change of the geomagnetic field away from the Earth. It is emphasized that in (1.4), we do not assume
that the medium configuration of the Earth, and the charge density and fluid velocity of the Earth core are known a priori. From a practical point view, the only thing known before the presence of the magnetized anomalies is the monitored geomagnetic field \( H_0 \) on \( \Gamma \). We would also like to emphasize that in practice, \( \mathbb{R}_+ \) in (1.4) can actually be replaced by a finite time interval, and we shall remark this point in Section 5. The magnetic anomaly detecting (MAD) technique has been used in various practical applications. The magnetometer that can measure minute variations in the Earth’s magnetic field has been used by military forces to detect submarines. The military MAD equipment is a descendent of geomagnetic survey or aeromagnetic survey instruments used to search for minerals by detecting their disturbance of the normal earth-field. The aim of this study is to provide a rigorous mathematical theory for this important applied technology. Indeed, we establish global uniqueness results for the nonlinear inverse problem (1.4) in certain practically important and generic scenarios.

The rest of the section is devoted to recasting the time-dependent inverse problem (1.4) to its counterpart in the frequency domain via the Fourier transform approach. We refer to [11,12] for the well-posedness of the forward Maxwell system (1.2), and in particular the unique existence of a pair of solutions \((E, \mathcal{H}) \in H^1(\mathbb{R}^d_+, H_{loc}(\text{curl}, \mathbb{R}^3))^2\). In the sequel, we shall make use of the following temporal Fourier transform for \((x, \omega) \in \mathbb{R}^3 \times \mathbb{R}_+\),

\[
J(x, \omega) = F_t(\mathcal{J}) := \frac{1}{2\pi} \int_{0}^{\infty} \mathcal{J}(x, t)e^{i\omega t} \, dt, \quad \mathcal{J} = E \text{ or } \mathcal{H},
\]

such that \(J(x, \omega) \in H_{loc}(\text{curl}, \mathbb{R}^3)\). Set

\[
E = F_t(E), \quad H = F_t(\mathcal{H}), \quad E_0 = F_t(E_0), \quad H_0 = F_t(H_0), \quad \rho = F_t(\rho). \tag{1.6}
\]

Throughout the paper, under a certain generic causality condition on the geomagnetic configuration of the Earth, we assume that the Fourier transforms in (1.6) all exist. In order to appeal for a general inverse problem study, we shall not explore this point in the current article; see also our remark concerning the geomagnetic configuration of the Earth after (1.4). With the above assumption, the time-dependent Maxwell system (1.2) is then reduced to the following time-harmonic system in the frequency domain,

\[
\begin{aligned}
\nabla \times H &= -i\omega \varepsilon E + \sigma(E + \mu \nabla \times H) \quad \text{in } \mathbb{R}^3, \\
\nabla \times E &= i\omega \mu H \quad \text{in } \mathbb{R}^3, \\
\nabla \cdot (\mu H) &= 0, \quad \nabla \cdot (\varepsilon E) = \rho \quad \text{in } \mathbb{R}^3, \\
\lim_{\|x\| \to \infty} \|x\|(\sqrt{\mu_0} H \times \hat{x} - \sqrt{\varepsilon_0} E) &= 0,
\end{aligned}
\tag{1.7}
\]

where the last limit is the Silver-Müller radiation condition and it holds uniformly in all directions \(\hat{x} := x/\|x\| \in S^2\). The Silver-Müller radiation condition characterizes the outward radiating waves (cf. [12]). That is, in order to establish the recovery results for the inverse problem (1.4), from a practical point of view, we shall only make use of the measurement data from the outward radiating EM waves. We refer to [13] for the related study on the unique existence of \((E, H) \in H_{loc}(\text{curl}, \mathbb{R}^3)^2\) to (1.7). In particular, we know that there holds the following asymptotic expansion as \(\|x\| \to +\infty\) (cf. [6,14]),

\[
H(x) = e^{ik_0 ||x||} H^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{\|x\|^2}\right), \quad k_0 := \omega \sqrt{\varepsilon_0 \mu_0}.
\tag{1.8}
\]

In a similar manner, one can derive the Maxwell system for the EM fields \(E_0\) and \(H_0\), as well as the corresponding magnetic far-field pattern \(H_0^\infty\). The inverse problem (1.4) can...
then be recast as
\[ H^\infty(\hat{x}; \omega) - H_0^\infty(\hat{x}; \omega) \big|_{(\hat{x}, \omega) \in \Gamma \times \Bbb{R}_+} \rightarrow \bigcup_{l=1}^{l_0} (D_l; \varepsilon_l, \mu_l, \sigma_l), \ \ \hat{x} := \{ \hat{x} \in S^2; \hat{x} = \frac{x}{\|x\|}, \ x \in \Gamma \} \subset S^2. \] (1.9)

On the other hand, by the real-analyticity of \( H^\infty \) and \( H_0^\infty \), together with the Rellich theorem (cf. [6]), we know that the inverse problem (1.9) is equivalent to the following one,
\[ H(x; \omega) - H_0(x; \omega) \big|_{(x, \omega) \in \partial \Sigma \times \Bbb{R}_+} \rightarrow \bigcup_{l=1}^{l_0} (D_l; \varepsilon_l, \mu_l, \sigma_l). \] (1.10)

We are mainly concerned with the theoretical unique recovery results for the aforementioned inverse problem (1.9), or equivalently (1.10). It is remarked that in our subsequent study of (1.9) or (1.10), we actually make use of the low-frequency asymptotics of the geomagnetic fields. That is, in (1.9) and (1.10), it is sufficient for us to know the geomagnetic fields with frequencies from a neighbourhood of the zero frequency.

The rest of the paper is organized as follows. In Section 2, we derive the asymptotic low-frequency approximation of the background magnetic field \( H_0 \) and show that the leading-order term is conservative with an explicit form of the corresponding potential function. Section 3 is devoted to the two-level asymptotic approximations of the perturbed magnetic field \( H \). One approximation is derived in terms of the frequency and the other one is derived in terms of the size of anomalies. Finally, in Section 4, we establish the unique recovery results on identifying the positions as well as the material properties of the magnetized anomalies.

2. Integral representation and asymptotics of \( H_0 \)

In this section, we present the integral representation of the magnetic field generated by the Earth core. We are mainly concerned with the magnetic field distribution outside the Earth core, namely, \( H_0 \) in \( \Bbb{R}^3 \setminus \Sigma_c \). Before proceeding, we present some preliminary knowledge on layer potential techniques (cf. [4,14]).

2.1. Layer potentials. Let \( \Gamma_k \) be the fundamental solution to the PDO \((\Delta + k^2)\), that is given by
\[ \Gamma_k(x) = -\frac{e^{ik\|x\|}}{4\pi\|x\|}, \ x \in \Bbb{R}^3 \text{ and } x \neq 0. \] (2.1)
For any bounded domain \( B \subset \Bbb{R}^3 \), we denote by \( S^k_B : H^{-1/2}(\partial B) \rightarrow H^1(\Bbb{R}^3 \setminus \partial B) \) the single layer potential operator given by
\[ S^k_B[\phi](x) := \int_{\partial B} \Gamma_k(x - y) \phi(y) ds_y, \] (2.2)
and \( K^k_B : H^{1/2}(\partial B) \rightarrow H^{1/2}(\partial B) \) the Neumann-Poincaré operator
\[ K^k_B[\phi](x) := \text{p.v.} \int_{\partial B} \frac{\partial \Gamma_k(x - y)}{\partial \nu_y} \phi(y) ds_y, \] (2.3)
where p.v. stands for the Cauchy principal value. In (2.3) and also in what follows, unless otherwise specified, \( \nu \) signifies the exterior unit normal vector to the boundary of the concerned domain. It is known that the single layer potential operator \( S^k_B \) satisfies the following trace formula
\[ \frac{\partial}{\partial \nu} S^k_B[\phi] \big|_\pm = (\pm \frac{1}{2} I + (K^k_B)^*)[\phi] \text{ on } \partial B, \] (2.4)
where \((\mathcal{K}_B^k)^*\) is the adjoint operator of \(\mathcal{K}_B^k\). In addition, for a density \(\Phi \in \text{TH}(\text{div}, \partial B)\), we define the vectorial single layer potential by
\[
A_B^k[\Phi](x) := \int_{\partial B} \Gamma_k(x - y)\Phi(y)dy, \quad x \in \mathbb{R}^3 \setminus \partial B.
\]
(2.5)

It is known that \(\nabla \times A_B^k\) satisfies the following jump formula
\[
\nu \times \nabla \times A_B^k[\Phi]|_\pm = \mp \frac{\Phi}{2} + M_B^k[\Phi] \quad \text{on} \ \partial B,
\]
(2.6)

where
\[
\forall x \in \partial B, \quad \nu \times \nabla \times A_B^k[\Phi]|_\pm (x) = \lim_{t \to 0^+} \nu \times \nabla \times A_B^k[\Phi](x \pm t\nu)
\]
and
\[
M_B^k[\Phi](x) = \text{p.v.} \nu \times \nabla \nabla \cdot A_B^k[\Phi](x).
\]
(2.7)

2.2. Integral representation and approximation. Let \((E_0, H_0)\) be the solution to (1.1) and (1.7). In this section, we consider the steady fields generated by the Earth’s core for \(\omega \in [0, \omega_0)\), with \(\omega_0 > 0\) a fixed and sufficiently small real number. For the subsequent use of the inverse problem study, we shall derive the integral representation and the low-frequency asymptotics of the fields.
By using the transmission conditions across $\partial \Sigma_c$, $(\mathbf{E}_0, \mathbf{H}_0)$ is the solution to the following transmission problem

$$
\begin{align*}
\n \nabla \times \mathbf{H}_0 &= -i \omega \varepsilon_c \mathbf{E}_0 + \sigma_c (\mathbf{E} + \mu_c \nabla \times \mathbf{H}_0), \quad \text{in } \Sigma_c \\
\n \nabla \times \mathbf{E}_0 &= i \omega \mu_c \mathbf{H}_0, \quad \text{in } \Sigma_c, \\
\n \nabla \cdot (\mu_c \mathbf{H}_0) &= 0, \quad \nabla \cdot (\varepsilon_c \mathbf{E}_0) = \hat{\rho}, \quad \text{in } \Sigma_c, \\
\n\left[ [\nu \times \mathbf{E}_0] \right] &= [ [\nu \times \mathbf{H}_0] ] = 0, \quad \text{on } \partial \Sigma_a, \\
\n \nabla \times \mathbf{H}_0 &= -i \omega \varepsilon_c \mathbf{E}_0, \quad \nabla \times \mathbf{E}_0 = i \omega \mu_0 \mathbf{H}_0, \quad \text{in } \Sigma_s, \\
\n \nabla \times \mathbf{H}_0 &= -i \omega \sigma_0 \mathbf{E}_0, \quad \nabla \times \mathbf{E}_0 = i \omega \mu_0 \mathbf{H}_0, \quad \text{in } \mathbb{R}^3 \setminus \Sigma, \\
\n\lim_{||x|| \to \infty} |x| |(\sqrt{\mu_0} \mathbf{H}_0 \times \hat{x} - \sqrt{\varepsilon_0} \mathbf{E}_0)| &= 0,
\end{align*}
$$

(2.9)

where $[ [\nu \times \mathbf{E}_0] ]$ and $[ [\nu \times \mathbf{H}_0] ]$ denote the jumps of $\nu \times \mathbf{E}_0$ and $\nu \times \mathbf{H}_0$ along $\partial \Sigma_a$, namely,

$$
[ [\nu \times \mathbf{E}_0] ] = (\nu \times \mathbf{E}_0)|_+ - (\nu \times \mathbf{E}_0)|_-, \quad [ [\nu \times \mathbf{H}_0] ] = (\nu \times \mathbf{H}_0)|_+ - (\nu \times \mathbf{H}_0)|_-.
$$

Using the potential theory, the solution $(\mathbf{E}_0, \mathbf{H}_0)$ to (2.9), outside $\Sigma_c$, can be represented by

$$
\mathbf{E}_0 = \begin{cases} 
\mu_0 \nabla \times A^{k_0}_{\Sigma_c} \Phi_C + \mu_0 \nabla \times A^{k_0}_{\Sigma} \Phi_S + \nabla \times \nabla \times A^{k_0}_{\Sigma} \Psi_S & \text{in } \mathbb{R}^3 \setminus \Sigma, \\
\mu_0 \nabla \times A^{k_0}_{\Sigma_c} \Phi_C + \mu_0 \nabla \times A^{k_0}_{\Sigma} \Phi_S + \nabla \times \nabla \times A^{k_0}_{\Sigma} \Psi_S & \text{in } \Sigma_s,
\end{cases}
$$

(2.10)

and

$$
\mathbf{H}_0 = \begin{cases} 
-i \omega^{-1} \left( \nabla \times \nabla \times A^{k_0}_{\Sigma_c} \Phi_C + \nabla \times \nabla \times A^{k_0}_{\Sigma} \Phi_S + \right. \\
\omega^2 \varepsilon_0 \nabla \times A^{k_0}_{\Sigma} \Psi_S & \text{in } \mathbb{R}^3 \setminus \Sigma, \\
-i \omega^{-1} \left( \nabla \times \nabla \times A^{k_0}_{\Sigma_c} \Phi_C + \nabla \times \nabla \times A^{k_0}_{\Sigma} \Phi_S + \right. \\
\omega^2 \varepsilon_0 \nabla \times A^{k_0}_{\Sigma} \Psi_S & \text{in } \Sigma_s,
\end{cases}
$$

(2.11)

where by the transmission conditions across $\partial \Sigma_c, (\Phi_S, \Psi_S, \Phi_C) \in \text{TH}(\text{div}, \partial \Sigma) \times \text{TH}(\text{div}, \partial \Sigma) \times \text{TH}(\text{div}, \partial \Sigma_c)$ satisfy

$$
\begin{align*}
\mu_0 \left( -I + M_{\Sigma_c}^{k_0} - M_{\Sigma}^{k_0} \right) \Phi_C + \mu_0 \left( L_{\Sigma}^{k_0} - L_{\Sigma_c}^{k_0} \right) \Psi_S &= 0 \\
= \mu_0 (M_{\Sigma_c, \Sigma_c}^{k_0} - M_{\Sigma, \Sigma}^{k_0}) \Phi_C & \text{on } \partial \Sigma, \\
\omega^2 \left( -\frac{\varepsilon_0 + \varepsilon_\infty}{2} I + \varepsilon_0 M_{\Sigma_c}^{k_0} - \varepsilon_\infty M_{\Sigma}^{k_0} \right) \Phi_C + \left( L_{\Sigma}^{k_0} - L_{\Sigma_c}^{k_0} \right) \Psi_S &= 0 \\
= (L_{\Sigma_c, \Sigma_c}^{k_0} - L_{\Sigma, \Sigma}^{k_0}) \Phi_C & \text{on } \partial \Sigma_c, \\
\mu_0 \left( -I + M_{\Sigma_c}^{k_0} \right) \Phi_C + \mu_0 M_{\Sigma_c, \Sigma_c}^{k_0} \Phi_C + \left( L_{\Sigma_c, \Sigma_c}^{k_0} \right) \Psi_S &= \nu \times \mathbf{E}_0 & \text{on } \partial \Sigma_c,
\end{align*}
$$

(2.12)

with $M_{\Sigma_c, \Sigma_c}^{k_0}, L_{\Sigma_c, \Sigma_c}^{k_0}, M_{\Sigma, \Sigma, \Sigma_c}^{k_0}$ and $L_{\Sigma, \Sigma, \Sigma_c}^{k}$ ($k = k_0, k_s$) defined by

$$
\begin{align*}
M_{\Sigma_c, \Sigma_c}^{k_0} &= \nu \times \nabla \times A_{\Sigma_c}^{k_0} \bigg|_{\partial \Sigma}, \\
L_{\Sigma_c, \Sigma_c}^{k_0} &= \nu \times \nabla \times A_{\Sigma_c}^{k_0} \bigg|_{\partial \Sigma}, \\
M_{\Sigma, \Sigma, \Sigma_c}^{k_0} &= \nu \times \nabla \times A_{\Sigma}^{k_0} \bigg|_{\partial \Sigma}, \\
L_{\Sigma, \Sigma, \Sigma_c}^{k} &= \nu \times \nabla \times A_{\Sigma}^{k} \bigg|_{\partial \Sigma_c}.
\end{align*}
$$

By direct asymptotic analysis one can obtain that (see also [3,8])

$$
M_{\Sigma}^{k_0} = M_{\Sigma}^{k_0} + \mathcal{O}(\omega^2), \quad \nabla \times A_{\Sigma}^{k_0} = \nabla \times A_{\Sigma}^{k_0} + \mathcal{O}(\omega^2).
$$

(2.13)
One can also verify that
\[ \mathcal{L}^k_{\Sigma, \Sigma_c} = \nu \times D^2 A^0_{\Sigma_c} \bigg|_{\partial \Sigma} + k^2 \nu \times \left( A^0_{\Sigma_c} + D^2 B_{\Sigma_c} \right) \bigg|_{\partial \Sigma} + O(\omega^3) \] (2.14)
and similar results hold for \( \mathcal{L}^k_{\Sigma, \Sigma_c}, \mathcal{M}^k_{\Sigma, \Sigma_c} \) and \( \mathcal{M}^{k_0}_{\Sigma, \Sigma_c} \). Here \( D^2 \) denotes \( \nabla \nabla \cdot \) and \( B_{\Sigma_c} : \text{TH}(\text{div}, \partial \Sigma_c) \to H^2(\Sigma \setminus \Sigma_c)^3 \) is defined by
\[ B_{\Sigma_c}[\Phi](x) := \frac{1}{4\pi} \int_{\partial \Sigma_c} \| x - y \| \Phi(y) ds_y, \quad \Phi \in \text{TH}(\text{div}, \partial \Sigma_c). \] (2.15)

We next show that (2.12) is uniquely solvable, and to that end we first prove a useful lemma.

**Lemma 2.1.** There holds the following asympototic expansion
\[ \mathcal{L}^k_{\Sigma, \Sigma_c}[\Psi_S] = \nu \times \nabla S^0_{\Sigma_c}[\nabla \partial B \cdot \Psi_S] \bigg|_{\partial \Sigma_c} + O(\omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)}). \] (2.16)

**Proof.** By using (2.14) and integration by parts, one has by straightforward asymptotic analysis that
\[ \mathcal{L}^k_{\Sigma, \Sigma_c}[\Psi_S] = \nu \times D^2 A^0_{\Sigma_c}[\Psi_S] \bigg|_{\partial \Sigma_c} + O(\omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)}) \]
\[ = \nu \times \nabla \nabla \cdot A^0_{\Sigma_c}[\Psi_S] \bigg|_{\partial \Sigma_c} + O(\omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)}) \]
\[ = \nu \times \nabla \int_{\partial \Sigma} \nabla \Gamma_0(\cdot - y) \cdot \Psi_S(y) ds_y + O(\omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)}) \]
\[ = - \nu \times \nabla \int_{\partial \Sigma} \nabla \partial B \cdot \Psi_S(y) ds_y + O(\omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)}) \]
\[ = \nu \times \nabla \int_{\partial \Sigma} \Gamma_0(\cdot - y) \nabla \partial B \cdot \Psi_S(y) ds_y + O(\omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)}). \] (2.17)

The proof is complete. \( \square \)

**Lemma 2.2.** (\( \Phi_S, \Psi_S, \Phi_C \)) \( \in \text{TH}(\text{div}, \partial \Sigma) \times \text{TH}(\text{div}, \partial \Sigma) \times \text{TH}(\text{div}, \partial \Sigma_c) \) is uniquely solvable in (2.12) for all sufficiently small \( \omega \in \mathbb{R}_+ \).

**Proof.** Denote by \( \nu \) the exterior unit normal vector on \( \partial \Sigma_c \) and \( \tilde{\nu} \) the exterior unit normal vector on \( \partial \Sigma \). First, we recall that \( \mathcal{M}^k_{\Sigma, \Sigma} \) are bounded on \( \text{TH}(\text{div}, \partial \Sigma) \), for \( D = \Sigma, \Sigma_c \) and \( k = k_0, k_s \) (see, e.g., [3, 15]). From (2.13), (2.14) and the first equation in (2.12), one can find that
\[ \| \Phi_S \|_{\text{TH}(\text{div}, \partial \Sigma_c)} = O\left( \omega^2\| \Psi_S \|_{\text{TH}(\text{div}, \partial \Sigma)} + \| \Phi_C \|_{\text{TH}(\text{div}, \partial \Sigma_c)} \right). \] (2.18)

By substituting (2.18) into the second equation of (2.12) and using asymptotic analysis, one can obtain that
\[ \left( - \frac{\varepsilon_s + \varepsilon_0}{2} I + (\varepsilon_0 - \varepsilon_s) M^0_{\Sigma} + O(\omega^2) \right) [\Psi_S] \]
\[ = \mu_0(\varepsilon_s - \varepsilon_0) \left( \tilde{\nu} \times (A^0_{\Sigma_c} + D^2 B_{\Sigma_c}) + \omega \tilde{\mathcal{B}}_{\Sigma_c} \right) \left[ \Phi_C \right], \] (2.19)

where \( \tilde{\mathcal{B}}_{\Sigma_c} \) is a bounded operator from \( \text{TH}(\text{div}, \partial \Sigma_c) \) to \( \text{TH}(\text{div}, \partial \Sigma) \). By taking the surface divergence on \( \partial \Sigma \) of both sides of (2.19) and using the formula \( \nabla_{\partial \Sigma} \cdot M^0_{\Sigma} = -(K_{\Sigma})^* \nabla_{\partial \Sigma} \cdot (A^0_{\Sigma_c} + D^2 B_{\Sigma_c}) \) (see [3, 7]), one can further obtain that
\[ \left( \frac{\varepsilon_s + \varepsilon_0}{2} I + (\varepsilon_0 - \varepsilon_s)(K_{\Sigma})^* + O(\omega^2) \right) [\nabla_{\partial \Sigma} \cdot \Psi_S] = \mu_0(\varepsilon_s - \varepsilon_0) \left( \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}) - \omega \nabla_{\partial \Sigma} \cdot \tilde{\mathcal{B}}_{\Sigma_c} \right) \left[ \Phi_C \right]. \] (2.20)
Note that \((\varepsilon_s + \varepsilon_0)/2I + (\varepsilon_0 - \varepsilon_s)(K^0_\Sigma)^*\) is invertible on \(L^2(\partial\Sigma)\), one thus has
\[
\nabla_{\partial\Sigma} \cdot \Psi_S = \mu_0 \left( \lambda_c I - (K^0_\Sigma)^* \right)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c})[\Phi_c] + O(\omega [\Phi_c]\|_{\text{TH}(\text{div}, \partial \Sigma_c)}),
\]
where \(\lambda_c\) is defined by
\[
\lambda_c := \frac{\varepsilon_s + \varepsilon_0}{2(\varepsilon_s - \varepsilon_0)}.
\]
Finally, by using (2.16) and substituting (2.18) and (2.21) into the last equation of (2.12), along with the help of (2.19), one can show by direct asymptotic analysis that
\[
\left( -\frac{I}{2} + M^0_{\Sigma_c} + \nu \times \nabla S^0_{\Sigma} (\lambda_c I - (K^0_\Sigma)^*)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}) + O(\omega) \right)[\Phi_c] = \mu_0^{-1} \nu \times E_0.
\]
Next, we prove the unique solvability of (2.22) when \(\omega \in \mathbb{R}_+\) is sufficiently small, which is equivalent to proving the invertibility of the operator
\[
\left( -\frac{I}{2} + M^0_{\Sigma_c} + \nu \times \nabla S^0_{\Sigma} (\lambda_c I - (K^0_\Sigma)^*)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}) \right)
\]
on \(\text{TH}(\text{div}, \partial \Sigma_c)\). Note that \(\nu \times \nabla S^0_{\Sigma} (\lambda_c I - (K^0_\Sigma)^*)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c})\) and \(M^0_{\Sigma_c}\) are compact operators on \(\text{TH}(\text{div}, \partial \Sigma_c)\). By using the Fredholm theory, it is sufficient to prove that the following homogeneous equation possesses only a trivial solution,
\[
\left( -\frac{I}{2} + M^0_{\Sigma_c} + \nu \times \nabla S^0_{\Sigma} (\lambda_c I - (K^0_\Sigma)^*)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}) \right)[\Phi] = 0.
\]
By taking the surface divergence of (2.23) one then has
\[
\left( -\frac{I}{2} - (K^0_\Sigma)^* \right)[\nabla_{\partial\Sigma_c} \cdot \Phi] = 0.
\]
By using the invertibility of \(\frac{I}{2} + (K^0_\Sigma)^*\) on \(L^2(\partial \Sigma_c)\), one thus has \(\nabla_{\partial\Sigma_c} \cdot \Phi = 0\). It can be verified that there exists only a trial solution to the following system (see Appendix A)
\[
\begin{align*}
\nabla \times E &= 0, \quad \nabla \cdot E = 0, \quad \text{in } (\mathbb{R}^3 \setminus \Sigma) \cup \Sigma_s, \\
\tilde{\nu} \times E|_+ &= \tilde{\nu} \times E|_-, \quad \text{on } \partial \Sigma, \\
\varepsilon_0 \tilde{\nu} \cdot E|_+ &= \varepsilon_s \tilde{\nu} \cdot E|_-, \quad \text{on } \partial \Sigma_c, \\
\nu \times E|_+ &= 0, \quad \int_{\partial \Sigma_c} \nu \cdot E|_+ = 0, \quad \text{on } \partial \Sigma_c, \\
E(x) &= \mathcal{O}(\|x\|^{-2}), \quad \|x\| \to \infty.
\end{align*}
\]
Furthermore, one can verify that
\[
E = \left( \nabla \times A^0_{\Sigma_c} + \nabla S^0_{\Sigma} (\lambda_c I - (K^0_\Sigma)^*)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}) \right)[\Phi]
\]
is also the solution to (2.25). Hence,
\[
\left( \nabla \times A^0_{\Sigma_c} + \nabla S^0_{\Sigma} (\lambda_c I - (K^0_\Sigma)^*)^{-1} \tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}) \right)[\Phi] = 0 \quad \text{in } \mathbb{R}^3 \setminus \Sigma_c.
\]
Then one has \(\tilde{\nu} \cdot E = 0\) on \(\partial \Sigma\), which together with the jump formula (2.4) further implies that
\[
\tilde{\nu} \cdot (\nabla \times A^0_{\Sigma_c}[\Phi]) = 0 \quad \text{on } \partial \Sigma.
\]
Therefore by (2.26) one has
\[
\nabla \times A^0_{\Sigma_c}[\Phi] = 0 \quad \text{in } \mathbb{R}^3 \setminus \Sigma_c.
\]
Finally, one has \(\Phi = 0\) by using
\[
\nu \times \nabla \times A^0_{\Sigma_c}[\Phi] = \left( -\frac{I}{2} + M^0_{\Sigma_c} \right)[\Phi] = 0 \quad \text{on } \partial \Sigma_c,
\]
and hence the invertibility of $-I/2 + \mathcal{M}_{\Sigma_c}^0$ on $\text{TH}(\text{div}, \partial \Sigma_c)$.

We have proved the unique solvability of (2.22). By using (2.18) and (2.19) one can thus find a unique solution to (2.12). This completes the proof. □

By virtue of Lemma 2.2, we can derive the following result

**Lemma 2.3.** Let $(E_0, H_0)$ be the solution to (2.9). Then for $\omega \in \mathbb{R}_+$ sufficiently small, one has

$$H_0 = \nabla S^0_{\Sigma_c} \left( \frac{I}{2} + (k_{\Sigma_c}^0)^* \right)^{-1} [\nu \cdot H_0]_{\partial \Sigma_c}^+ + O(\omega) \quad \text{in} \quad \mathbb{R}^3 \setminus \Sigma_c. \quad (2.28)$$

**Proof.** First, by using Lemma 2.2, one has

$$\Phi_C = \left( \mu_0^{-1} \left( - \frac{I}{2} + \mathcal{M}_{\Sigma_c}^0 + \nu \times \nabla S^0_{\Sigma_c} \left( \lambda_\varepsilon I - (k_{\Sigma_c}^0)^* \right)^{-1} \nu \cdot (\nabla \times A^0_{\Sigma_c}) \right)^{-1} + O(\omega) \right) [\nu \times E_0]. \quad (2.29)$$

Note that (2.19) and (2.18) imply

$$\|\Phi_S\|_{\text{TH}(\text{div}, \partial \Sigma_c)} = O(\|\Phi_C\|_{\text{TH}(\text{div}, \partial \Sigma_c)}), \quad \|\Phi_S\|_{\text{TH}(\text{div}, \partial \Sigma_c)} = O(\omega^2\|\Phi_C\|_{\text{TH}(\text{div}, \partial \Sigma_c)}). \quad (2.30)$$

Hence by using (2.11) and straightforward asymptotic analysis, there holds

$$H_0 = -i\omega^{-1} \nabla \times \nabla \times A^0_{\Sigma_c} [\Phi_C] + O(\omega) = -i\omega^{-1} \nabla S^0_{\Sigma_c} [\nabla_{\partial \Sigma_c} \cdot \Phi_C] + O(\omega) \quad \text{in} \quad \mathbb{R}^3 \setminus \Sigma_c. \quad (2.31)$$

Moreover, by using (2.29) one can show that

$$\nabla_{\partial \Sigma_c} \cdot \Phi_C = i\omega \left( \frac{I}{2} + (k_{\Sigma_c}^0)^* \right)^{-1} [\nu \cdot H_0]_{\partial \Sigma_c}^+ + O(\omega^2) \quad \text{on} \quad \partial \Sigma_c. \quad (2.32)$$

By substituting (2.32) into (2.31), one thus has (2.28).

The proof is complete. □

Lemma 2.3 shows that the leading-order term in the low-frequency asymptotic expansion of the magnetic field generated by the Earth’s core is a gradient filed, namely it is conservative. We would like to point out that the leading-order term of the low-frequency asymptotic expansion of the electric field can also be exactly calculated by following a similar argument. However, since our main concern is to use the monitoring of the magnetic field for detecting the anomalies, we choose not to give the details on that aspect.

### 3. Integral Representation and Asymptotics of $H$

In this section, we consider the case that the Earth’s magnetic field is perturbed by the anomalous magnetized objects, that is, $\sigma$, $\varepsilon$ and $\mu$ are replaced by (1.3), respectively. Henceforth, we denote by $E$ and $H$, respectively, the associated electric and magnetic fields. In the following, we define the wave numbers $q_l$, $l = 1, 2, \ldots, l_0$, by $q_l^2 := \omega^2 \mu_l \gamma_l$, $\gamma_l := \varepsilon_l + i\sigma_l/\omega$, where the sign of $q_l$ is chosen such that $\Re q_l \geq 0$ (see [6]). For the sake of simplicity, we denote by $S^2$ the unit sphere and define $D := \Sigma_s \setminus \bigcup_{l=1}^{l_0} D_l$. We also let $\nu_l$ be the exterior unit normal vector defined on $\partial D_l$, $l = 1, 2, \ldots, l_0$. By using the integral ansatz, one can have the following representation formula, whose proof is postponed to be given in Appendix B.
Lemma 3.1. Let \((E, H)\) be the solution to (1.3) and (1.7). Then there hold the following results,

\[
\begin{align*}
\mathbf{E} &= \begin{cases}
\dot{E}_0 + \nabla \times (\mu_0 A_{k0}^c [\Phi_0] + \nabla \times A_{k0}^f [\Psi_0]) \\
+ \nabla \times \sum_{l' = 1}^{l_0} (\mu_0 A_{k0}^{l'} [\Phi_{l'}] + \nabla \times A_{k0}^{l'} [\Psi_{l'}]) & \text{in } \mathbb{R}^3 \setminus \Sigma,
\end{cases} \\
\dot{H}_0 - i \omega^{-1} \nabla \times \left( \left( \omega^2 \varepsilon_0 A_{k0}^c [\Psi_0] + \nabla \times A_{k0}^f [\Phi_0] \right) \\
+ \sum_{l' = 1}^{l_0} \left( \omega^2 \varepsilon_0 A_{k0}^{l'} [\Psi_{l'}] + \nabla \times A_{k0}^{l'} [\Phi_{l'}] \right) \right) & \text{in } \mathbb{R}^3 \setminus \Sigma,
\end{align*}
\]

(3.1)

and

\[
\begin{align*}
\mathbf{H} &= \begin{cases}
\dot{H}_0 - i \omega^{-1} \nabla \times \left( \left( \omega^2 \varepsilon_0 A_{k0}^c [\Psi_0] + \nabla \times A_{k0}^f [\Phi_0] \right) \\
+ \sum_{l' = 1}^{l_0} \left( \omega^2 \varepsilon_0 A_{k0}^{l'} [\Psi_{l'}] + \nabla \times A_{k0}^{l'} [\Phi_{l'}] \right) \right) & \text{in } \mathbb{R}^3 \setminus \Sigma,
\end{cases} \\
- i \omega^{-1} \nabla \times \left( \left( \omega^2 \gamma l A_{k1}^c [\Psi_0] + \nabla \times A_{k1}^f [\Phi_0] \right) \\
+ \sum_{l' = 1}^{l_0} \left( \omega^2 \gamma l A_{k1}^{l'} [\Psi_{l'}] + \nabla \times A_{k1}^{l'} [\Phi_{l'}] \right) \right) & \text{in } D_l,
\end{align*}
\]

(3.2)

where \((\Phi_0, \Psi_0) \in \text{TH} (\text{div}, \partial \Sigma) \times \text{TH} (\text{div}, \partial \Sigma)\) and \((\Phi_l, \Psi_l) \in \text{TH} (\text{div}, \partial D_l) \times \text{TH} (\text{div}, \partial D_l)\), \(l = 1, 2, \ldots, l_0\) satisfy (B.1). The fields \((E_0, H_0)\) satisfy (1.1) and (1.7) in \((\mathbb{R}^3 \setminus \Sigma) \cup D\) with \(H_0\) given in (3.17) in the following, and they depend on the background fields \((E_0, H_0)\) and the boundary condition on \(\partial \Sigma_c\).

Based on Lemma 3.1, we next derive two critical asymptotic expansions of the electromagnetic fields \(E\) and \(H\). The first one is the low-frequency asymptotics of the aforementioned fields, and the leading-order terms are referred to as the steady fields. The second one is the asymptotic expansion of the steady fields in terms of the size of the anomalies.

3.1. First level approximation. In this section, we derive the steady parts of the perturbed magnetic field in (3.2). We have the following asymptotic expansion results.
Theorem 3.1. Let \((E,H)\) be the solution to (1.3) and (1.7). Then for \(\omega \in \mathbb{R}_+\) sufficiently small, there hold the following asymptotic expansions,

\[
\begin{aligned}
\hat{H}_0 &= \varepsilon_0 \nabla \times A^0_{\Sigma}[\Xi] + \nabla \mathcal{S}^0_{\Sigma}[\Theta] \\
&+ \sum_{l'=1}^{l_0} \left( \varepsilon_0 \nabla \times A^0_{D_{l'}}[\Psi_{l'}^{(0)}] - \mu_0 \nabla \mathcal{S}^0_{D_{l'}}[\Pi_{l'}] \right) + \mathcal{O}(\omega) \quad \text{in } \mathbb{R}^3 \setminus \Sigma, \\
&+ \sum_{l'=1}^{l_0} \left( \varepsilon_0 \nabla \times A^0_{D_{l'}}[\Psi_{l'}^{(0)}] - \mu_0 \nabla \mathcal{S}^0_{D_{l'}}[\Pi_{l'}] \right) + \mathcal{O}(\omega) \quad \text{in } \hat{D}, \\
&- \gamma_l \nabla \times A^0_{\Sigma}[\Xi] - \nabla \mathcal{S}^0_{\Sigma}[\Theta] + \sum_{l'=1}^{l_0} \left( \gamma_l \nabla \times A^0_{D_{l'}}[\Psi_{l'}^{(0)}] - \mu_0 \nabla \mathcal{S}^0_{D_{l'}}[\Pi_{l'}] \right) + \mathcal{O}(\omega) \quad \text{in } D_l,
\end{aligned}
\]

where \(\Xi, \Theta \in \text{TH}(\text{div}, \partial \Sigma)\) satisfy

\[
\Xi = \sum_{l'=1}^{l_0} \left( \chi_{l} I + \mathcal{M}^0_{\Sigma} \right)^{-1} \mathcal{M}^0_{\Sigma, D_{l'}}[\Psi_{l'}^{(0)}],
\]

\[
\Theta = (\varepsilon_0 - \varepsilon_0) \sum_{l'=1}^{l_0} \nu \cdot \left( \nabla \times A^0_{D_{l'}}[\Psi_{l'}^{(0)}] \right) - (\varepsilon_0 - \varepsilon_0) \nu \cdot \left( \nabla \times A^0_{\Sigma}[\Xi] \right),
\]

and \(\Psi_{l'}^{(0)} \in \text{TH}(\text{div}, \partial D_l), l = 1, 2, \ldots, l_0\) are defined in (B.21). \(\Pi_{l} \in L^2(\partial D_l), l = 1, 2, \ldots, l_0\) are defined by

\[
\Pi_{l} = \left( \mathcal{J}^\mu_D \right)^{-1} \left( \begin{array}{c} \nu_1 \cdot \hat{H}_0 - \mu_0 \cdot \nu_1 - \mu_0 \cdot \nu_2 - \mu_0 \cdot \nu_0 - \mu_0 \cdot \hat{H}_0 \\ \mu_1 - \mu_0 \\ \mu_2 - \mu_0 \\ \mu_0 - \mu_0 \end{array} \right) l
\]

\[
- \left( \mathcal{J}^\mu_D \right)^{-1} \left( \begin{array}{c} \omega \gamma_1 \mu_1 \mu_2 \mu_3 \\ \omega \gamma_2 \mu_2 \mu_3 \\ \omega \gamma_3 \mu_3 \mu_3 \\ \omega \gamma_4 \mu_4 \mu_3 \\ \omega \gamma_5 \mu_5 \mu_3 \\ \omega \gamma_6 \mu_6 \mu_3 \\ \omega \gamma_7 \mu_7 \mu_3 \\ \omega \gamma_8 \mu_8 \mu_3 \\ \omega \gamma_9 \mu_9 \mu_3 \\ \omega \gamma_{10} \mu_{10} \mu_3 \end{array} \right) l
\]

where \(\mathcal{J}^\mu_D\) is defined in (B.10) and \(C\) is defined by

\[
C := \nabla \times A^0_{\Sigma} \left( \chi_{l} I + \mathcal{M}^0_{\Sigma} \right)^{-1} \sum_{l'=1}^{l_0} \mathcal{M}^0_{\Sigma, D_{l'}}[\Psi_{l'}^{(0)}] - \nabla \times \sum_{l'=1}^{l_0} A^0_{D_{l'}}[\Psi_{l'}^{(0)}].
\]

The parameters \(\lambda_{\mu_l}\) and \(\gamma_{\mu_l}\) are defined by

\[
\lambda_{\mu_l} := \frac{\mu_1 + \mu_0}{2(\mu_1 - \mu_0)}, \quad \gamma_{\mu_l} := \frac{\gamma_l + \varepsilon_0}{2(\gamma_l - \varepsilon_0)}, \quad l = 1, 2, \ldots, l_0.
\]
Proof. By using direct asymptotic expansion with respect to $\omega$ in (3.2) and combing (B.4), (B.5), (B.11) and (B.20) one obtains that

\[
\mathbf{H} = \begin{cases}
\mathbf{H}_0 - i(\varepsilon_0 \nabla \times A^0_{\Sigma}[\omega \Phi_0] + D^2 A^0_{\Sigma}[\omega^{-1} \Phi_0]) \\
- i \sum_{l'=1}^{l_0} \left( \varepsilon_0 \nabla \times A^0_{D_{l'}}[\omega \Psi_{l'}] + D^2 A^0_{D_{l'}}[\omega^{-1} \Phi_{l'}] \right) + O(\omega) \quad \text{in } \mathbb{R}^3 \setminus \Sigma,
\end{cases}
\]

Substituting (3.10) into (3.8) and using (B.13) and (B.20) one can show (3.3). Combining the first equation in (B.1), (B.4), (B.5), (B.11) and using integration by parts one has

\[
\omega \Phi_0 = - \left( \lambda_s I + \mathcal{M}^0_{\Sigma} \right)^{-1} \sum_{l'=1}^{l_0} \mathcal{M}^0_{\Sigma,D_{l'}}[\omega \Psi_{l'}] + O(\omega),
\]

and

\[
D^2 A^0_{\Sigma}[\omega^{-1} \Phi_0] = (\varepsilon_s - \varepsilon_0)\nabla S^0_{\Sigma} \left[ \nu \cdot \left( \nabla \times \sum_{l'=1}^{l_0} A^0_{D_{l'}}[\omega \Psi_{l'}] \right) + \nu \cdot \left( \nabla \times A^0_{\Sigma}[\omega \Phi_0] \right) \right] + O(\omega).
\]

Substituting (3.10) into (3.8) and using (B.13) and (B.20) one can show (3.3). The proof is complete. \qed

Now we present the explicit forms of the fields $\mathbf{H}_0$ and $\mathbf{E}_0$ in $(\mathbb{R}^3 \setminus \Sigma) \cup \hat{D}$ that have been used in our earlier discussion. We remark that if there are no magnetized objects presented, namely $\mu_l = \mu_0$ and $\gamma_l = \varepsilon_s$, $l = 1, 2, \ldots, l_0$, then (3.3) is degenerated to $\mathbf{H} = \mathbf{H}_0 + O(\omega)$ in $\mathbb{R}^3 \setminus \Sigma_c$. In this case, (B.21) yields

\[
\Psi^{(0)}_l = \varepsilon_s^{-1} \nu_l \times \mathbf{H}_0 \quad \text{on } \partial D_l, \quad l = 1, 2, \ldots, l_0,
\]

and (3.5) yields

\[
\Pi_l = \mu_0^{-1} \nu_l \cdot \mathbf{H}_0 + O(\omega) \quad \text{on } \partial D_l, \quad l = 1, 2, \ldots, l_0.
\]

Substituting (3.11) and (3.12) into (3.3), and using the assumption that $\mathbf{H} = \mathbf{H}_0 + O(\omega)$, one obtains

\[
\mathbf{H}_0 = \mathbf{H}_0 - \varepsilon_s^{-1} \sum_{l'=1}^{l_0} \left( \varepsilon_0 \nabla \times A^0_{\Sigma} \left( \lambda_s I + \mathcal{M}^0_{\Sigma} \right)^{-1} \mathcal{M}^0_{\Sigma,D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) \\
- (\varepsilon_s - \varepsilon_0)\nabla S^0_{\Sigma} \left[ \nu \cdot \left( \nabla \times A^0_{D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) \right] \\
+ (\varepsilon_s - \varepsilon_0)\nabla S^0_{\Sigma} \nu \cdot \nabla A^0_{\Sigma} \left( \lambda_s I + \mathcal{M}^0_{\Sigma} \right)^{-1} \mathcal{M}^0_{\Sigma,D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) \\
+ \sum_{l'=1}^{l_0} \left( \varepsilon_0 \varepsilon_s^{-1} \nabla \times A^0_{D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) - \nabla S^0_{D_{l'}} \left[ \nu_{l'} \cdot \mathbf{H}_0 \right] \right) + O(\omega) \quad \text{in } \mathbb{R}^3 \setminus \Sigma,
\]

in ($\mathbb{R}^3 \setminus \Sigma$) $\cup \hat{D}$ that have been used in our earlier discussion. We remark that if there are no magnetized objects presented, namely $\mu_l = \mu_0$ and $\gamma_l = \varepsilon_s$, $l = 1, 2, \ldots, l_0$, then (3.3) is degenerated to $\mathbf{H} = \mathbf{H}_0 + O(\omega)$ in $\mathbb{R}^3 \setminus \Sigma_c$. In this case, (B.21) yields

\[
\Psi^{(0)}_l = \varepsilon_s^{-1} \nu_l \times \mathbf{H}_0 \quad \text{on } \partial D_l, \quad l = 1, 2, \ldots, l_0,
\]

and (3.5) yields

\[
\Pi_l = \mu_0^{-1} \nu_l \cdot \mathbf{H}_0 + O(\omega) \quad \text{on } \partial D_l, \quad l = 1, 2, \ldots, l_0.
\]

Substituting (3.11) and (3.12) into (3.3), and using the assumption that $\mathbf{H} = \mathbf{H}_0 + O(\omega)$, one obtains

\[
\mathbf{H}_0 = \mathbf{H}_0 - \varepsilon_s^{-1} \sum_{l'=1}^{l_0} \left( \varepsilon_0 \nabla \times A^0_{\Sigma} \left( \lambda_s I + \mathcal{M}^0_{\Sigma} \right)^{-1} \mathcal{M}^0_{\Sigma,D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) \\
- (\varepsilon_s - \varepsilon_0)\nabla S^0_{\Sigma} \left[ \nu \cdot \left( \nabla \times A^0_{D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) \right] \\
+ (\varepsilon_s - \varepsilon_0)\nabla S^0_{\Sigma} \nu \cdot \nabla A^0_{\Sigma} \left( \lambda_s I + \mathcal{M}^0_{\Sigma} \right)^{-1} \mathcal{M}^0_{\Sigma,D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) \\
+ \sum_{l'=1}^{l_0} \left( \varepsilon_0 \varepsilon_s^{-1} \nabla \times A^0_{D_{l'}} \nu_{l'} \times \mathbf{H}_0 \right) - \nabla S^0_{D_{l'}} \left[ \nu_{l'} \cdot \mathbf{H}_0 \right] \right) + O(\omega) \quad \text{in } \mathbb{R}^3 \setminus \Sigma.
\]
Lemma 3.2. There hold the following relations

\[
\nabla \times A_{D_i}^0 (\nu_\nu \times \hat{H}_0) = \nabla S_{D_i}^0 (\nu \cdot \hat{H}_0) + O(\omega) \quad \text{in } (\mathbb{R}^3 \setminus \Sigma) \bigcup \tilde{D},
\]

and

\[
\nabla \times A_{\Sigma}^0 (\lambda_\nu I + \mathcal{M}_\nu^0)^{-1} \mathcal{M}_{\Sigma, D_i}^0 [\nu_\nu \times \hat{H}_0]
\]

\[
= \begin{cases} 
\nabla S_{\Sigma}^0 (\lambda_\nu I + (\kappa_\nu^0)\nu^0)^{-1} \nu \cdot \nabla S_{D_i}^0 (\nu_\nu \cdot \hat{H}_0) + O(\omega) \quad \text{in } \mathbb{R}^3 \setminus \Sigma, \\
(\varepsilon_x - \varepsilon_y) \nabla S_{D_i}^0 (\nu_\nu \cdot \hat{H}_0) + \frac{\varepsilon_x}{\varepsilon_y} \nabla S_{\Sigma}^0 (\lambda_\nu I + (\kappa_\nu^0)\nu^0)^{-1} \nu \cdot \nabla S_{D_i}^0 (\nu_\nu \cdot \hat{H}_0) + O(\omega) \quad \text{in } \tilde{D}.
\end{cases}
\]

Proof. Note that the lower order term of \(\hat{H}_0\) is the gradient of a harmonic function (similar to (2.28)). The proof for (3.15) follows from a similar argument to that for the proof of Lemma 3.6. Using (3.15) and also a similar argument to that in the proof of Lemma 3.6 one can then prove (3.16). \(\square\)

By Lemma 3.2, (3.14), and (3.13), one can readily show that

Lemma 3.3. \(\hat{H}_0\) introduced in (3.2) satisfies

\[
\hat{H}_0 = \hat{H}_0 + (\varepsilon_x - \varepsilon_y) \varepsilon_x^{-1} \sum_{l'=1}^{l_0} \nabla S_{D_i}^0 (\nu_\nu \cdot \hat{H}_0) + O(\omega) \quad \text{in } (\mathbb{R}^3 \setminus \Sigma) \bigcup \tilde{D}.
\]

By using the jump formula on \(\partial D_i\), \(l = 1, 2, \ldots l_0\) and Lemma 3.3 one can further obtain

Lemma 3.4. \(\hat{H}_0\) introduced in (3.2) can be written as

\[
\hat{H}_0 = H_0 + \sum_{l'=1}^{l_0} \nabla S_{D_i}^0 (\varphi_{l'}) + O(\omega) \quad \text{in } (\mathbb{R}^3 \setminus \Sigma) \bigcup \tilde{D},
\]

where \(\varphi_{l'}\), \(l = 1, 2, \ldots, l_0\) satisfy

\[
\varphi_{l'} = \left(\left[J_D^{-1}\right]^{-1} [(\nu_1 \cdot H_0, \nu_2 \cdot H_0, \ldots, \nu_{l_0} \cdot H_0)^T]\right)_{l'},
\]

with the operator \(J_D\) defined in (B.15) with \(\lambda_{l'}\) replaced by \(\lambda_x\), \(l = 1, 2, \ldots, l_0\).
3.2. Second level approximation. In the subsequent analysis, we shall make use of the steady part of the magnetic field in the representation formula (3.3), namely the leading-order term in the asymptotic low-frequency expansion. By Lemma 2.3, we see that $H_0$ is a gradient field in $\mathbb{R}^3 \setminus \mathcal{B}$. In what follows, we let $H^{(0)}$ be the leading-order term of $H$ in (3.3), $H_0^{(0)}$ be the leading-term of $H_0$ and $H_0^{(0)}$ be the leading-order term of $H_0$ (and $\varphi_l^{(0)}$ is the leading term of $\varphi_l$, $l = 1, 2, \ldots, l_0$). We would like to point out that if $\sigma_l$, $l = 1, 2, \ldots, l_0$ are not identically zero, then the leading-order term in (3.3) may still depend on $\omega$. In such a case, one needs to perform further asymptotic analysis in terms of the frequency and we shall discuss this point at the suitable place in what follows.

In this section, we consider further asymptotic expansion of the steady fields in terms of the size of the magnetized anomalies. Indeed, from a practical point of view, the size of the magnetized anomalies ($D_l; \varepsilon_l, \mu_l, \sigma_l$), $l = 1, 2, \ldots, l_0$, introduced in (1.3), is much smaller than the size of the Earth. Hence, we can assume that

$$D_l = \delta \Omega + z_l, \quad l = 1, 2, \ldots, l_0,$$

(3.20)

where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$ and $\Omega \subset \subset \Sigma$, and $\delta \in \mathbb{R}_+$ is sufficiently small. Furthermore, we assume that $D_l$, $l = 1, 2, \ldots, l_0$ are sparsely distributed and far away from each other and $z_l$, $l = 1, 2, \ldots, l_0$, are far away from $\partial \Omega$ such that $x - z_l \gg \delta$, for any $x \in \partial \Omega$. With the above preparations, we are in a position to derive the asymptotic expansion of the steady geomagnetic field in terms of the size of the magnetized anomalies.

We first have the following lemma

**Lemma 3.5.** Suppose $D_l$, $l = 1, 2, \ldots, l_0$ are defined in (3.20) with $\delta \in \mathbb{R}_+$ sufficiently small. Let $K_D^*$ and $M_D$ be defined in (B.16) and (B.9), respectively. Then we have

$$K_D^* = K_\Omega^* + \mathcal{O}(\delta^2), \quad M_D = M_\Omega + \mathcal{O}(\delta^2),$$

(3.21)

where $K_\Omega^*$, $M_\Omega$ are $l_0 \times l_0$ and $3l_0 \times 3l_0$ matrix-valued operators defined by

$$K_\Omega^* := \text{diag}((K_\Omega^0)^*, (K_{\Omega_1}^0)^*, \ldots, (K_{\Omega_{l_0}}^0)^*), \quad M_\Omega := \text{diag}(M_{\Omega_1}, M_{\Omega_2}, \ldots, M_{\Omega_{l_0}}),$$

(3.22)

respectively.

**Proof.** We only prove the second assertion in (3.21), and the first one can be proved in a similar manner. For any $x, y \in \partial D_l$, we let $\bar{x} = \delta^{-1}(x - z_l)$, $\bar{y} = \delta^{-1}(y - z_l) \in \partial \Omega$, $l = 1, 2, \ldots, l_0$. Define $\tilde{\Phi}(\bar{y}) = \Phi(\bar{y})$. By using change of variables, one can show that there holds

$$M_{D_l}^0[\tilde{\Phi}](\bar{x}) = \nu_x \times \nabla \tilde{x} \times \int_{\partial D_l} \Gamma_0(x - y) \Phi(y)ds_y$$

$$\quad = - \frac{1}{4\pi} \nu_x \times \nabla \tilde{x} \times \int_{\partial \Omega} \frac{1}{\|x - y\|} \tilde{\Phi}(\bar{y})ds_y$$

$$\quad = - \frac{1}{4\pi} \nu_x \times \nabla \tilde{x} \times \int_{\partial \Omega} \frac{1}{\|\tilde{x} - \bar{y}\|} \tilde{\Phi}(\bar{y})ds_{\bar{y}} = M_{\Omega_1}^0[\tilde{\Phi}](\bar{x}).$$

(3.23)
On the other hand, letting \( x \in \partial D_l \) and \( y \in \partial D_m \) and \( \bar{x} = \delta^{-1}(x - z_l) \), \( \bar{y} = \delta^{-1}(y - z_m) \), and the following asymptotic expansion result

**Theorem 3.2.**

**Proof.** Note that

\[
\gamma_l \in \{ 1, 2, \ldots, \ell_0 \}, \quad \nu_l \in \{ 1, 2, \ldots, \ell_0 \}
\]

\[
\gamma_l = \begin{cases} 
\gamma_l \in \{ 1, 2, \ldots, \ell_0 \}, \nu_l \in \{ 1, 2, \ldots, \ell_0 \} & \text{in } \mathbb{R}^3 \setminus \overline{D_l}, \quad (3.25) \\
\gamma_l \in \{ 1, 2, \ldots, \ell_0 \}, \nu_l \in \{ 1, 2, \ldots, \ell_0 \} & \text{in } D_l.
\end{cases}
\]

**Lemma 3.6.** For any simply connected domain \( D_l \) and the gradient filed \( H^{(0)}_0 \) in \( \mathbb{R}^3 \), which is divergence free, there holds the following relation

\[
\frac{1}{\gamma_l - \varepsilon_s} \nabla \times A_{D_l}^0(\lambda_l I + M_{D_l}^0)^{-1}[
u_l \times H^{(0)}_0] = \begin{cases} 
\frac{1}{\gamma_l - \varepsilon_s} \nabla S_{D_l}^0(\lambda_l I + (K_{D_l}^0)^*)^{-1}[
u_l \cdot H^{(0)}_0] & \text{in } \mathbb{R}^3 \setminus \overline{D_l}, \quad (3.25) \\
\frac{1}{\gamma_l \nu_l} H^{(0)}_0 + \frac{\varepsilon_s}{\gamma_l \nu_l} \nabla S_{D_l}^0(\lambda_l I + (K_{D_l}^0)^*)^{-1}[
u_l \cdot H^{(0)}_0] & \text{in } D_l.
\end{cases}
\]

**Proof.** Note that \( H^{(0)}_0 \) is the gradient of a harmonic function. The proof of (3.25) follows from a similar argument to that in the proof of Lemma 5.5 in [3].

**Theorem 3.2.** Suppose \( D_l, l = 1, 2, \ldots, \ell_0 \) are defined in (3.20) with \( \delta \in \mathbb{R}_+ \) sufficiently small. Let \( (E, H) \) be the solution to (1.3) and (2.9). Then for \( x \in \mathbb{R}^3 \setminus \Sigma \), there holds the following asymptotic expansion result

\[
H^{(0)}(x) = H^{(0)}_0(x) - \delta^3 \sum_{l=1}^{\ell_0} \nabla(\nabla \Gamma_0(x - z_l)^T P_0 H^{(0)}_0(z_l))
- \delta^3 \sum_{l=1}^{\ell_0} \left( \varepsilon_s \nabla(\nabla \Gamma_0(x - z_l)^T D_l H^{(0)}_0(z_l)) - \mu_0 \nabla(\nabla \Gamma_0(x - z_l)^T M_l H^{(0)}_0(z_l)) \right) + \mathcal{O}(\delta^4),
\]

(3.26)

where \( P_0 \) is defined by

\[
P_0 := \int_{\partial \Omega} \tilde{y}(\lambda_c I - (K_{\Omega}^0)^*)^{-1}[
u_l]d\tilde{s},
\]

(3.27)

The polarization tensors \( D_l \) and \( M_l \) are \( 3 \times 3 \) matrices defined by

\[
D_l = \frac{1}{\gamma_l - \varepsilon_s} \int_{\partial \Omega} \tilde{y}(\lambda_l I + (K_{\Omega}^0)^*)^{-1}[(\lambda_c I - (K_{\Omega}^0)^*)^{-1}[
u_l]]d\tilde{s},
\]

(3.28)

and

\[
M_l = \frac{1}{\mu_l - \mu_0 \varepsilon_s - \varepsilon_0} \int_{\partial \Omega} \tilde{y}(\lambda_l I + (K_{\Omega}^0)^*)^{-1}[(\lambda_c I - (K_{\Omega}^0)^*)^{-1}[
u_l]]d\tilde{s},
\]

(3.29)

respectively, \( l = 1, 2, \ldots, \ell_0 \). More specifically, let \( D_l = ((D_l)_{mn}) \), \( m, n = 1, 2, 3 \), \( \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)^T \) and \( \nu_l = (\nu_{l1}^{(1)}, \nu_{l2}^{(2)}, \nu_{l3}^{(3)})^T \), we have

\[
(D_l)_{mn} = \frac{1}{\gamma_l - \varepsilon_s} \int_{\partial \Omega} \tilde{y}_m(\lambda_l I + (K_{\Omega}^0)^*)^{-1}[(\lambda_c I - (K_{\Omega}^0)^*)^{-1}[
u_{l1}^{(m)}]]d\tilde{s},
\]

where

\[
\tilde{y}_m = \lambda_l \tilde{y}_m(\lambda_l I + (K_{\Omega}^0)^*)^{-1}[(\lambda_c I - (K_{\Omega}^0)^*)^{-1}[
u_{l1}^{(m)}]]d\tilde{s}.
\]
where $\epsilon_{mn} = 1$ for $m = n$ and $\epsilon_{mn} = 0$ for $m \neq n$. $P_0$ and $M_l$ have similar forms.

**Proof.** First, we note that either $\omega \gamma_l$ or $1/(\gamma_l - \epsilon_l)$ is of order $\omega$, $l = 1, 2, \ldots, l_0$, no matter $\sigma$ is zero or nonzero. One can immediately find that the second term in (3.3) is of order $\omega$. By (3.3), it then can be seen that the leading-order term $H^{(0)}$ has the following form

$$H^{(0)} = \tilde{H}_0^{(0)} - \epsilon_0 \sum_{l=1}^{l_0} \nabla \times A_0^l \left( \lambda_l I + M_0^l \right)^{-1} \nu \times \nabla A_0^l [\Theta_i^l]$$

$$+ (\epsilon_s - \epsilon_0) \sum_{l=1}^{l_0} \nabla S_0^l \left[ \nu \cdot \left( \nabla \times A_0^l [\Theta_i^l] \right) \right]$$

$$- (\epsilon_s - \epsilon_0) \sum_{l=1}^{l_0} \nabla S_0^l \nu \cdot \nabla A_0^l \left( \lambda_l I + M_0^l \right)^{-1} \nu \times \nabla \times A_0^l [\Theta_i^l]$$

$$+ \sum_{l=1}^{l_0} \left( \epsilon_0 \nabla \times A_0^l [\Theta_i^l] - \mu_0 \nabla S_0^l [\Pi_i^l] \right) \text{ in } \mathbb{R}^3 \setminus \Sigma,$$

where $\Theta_i^l$ and $\Pi_i^l$ are defined by

$$\Theta_i^l = \left( (L_D)^{-1} \left[ \left( \frac{(l_1 \times \tilde{H}_0^{(0)})^T}{\gamma_1 - \epsilon_s}, \frac{(l_2 \times \tilde{H}_0^{(0)})^T}{\gamma_2 - \epsilon_s}, \ldots, \frac{(l_{l_0} \times \tilde{H}_0^{(0)})^T}{\gamma_{l_0} - \epsilon_s} \right) \cdot (e_l \otimes (1, 1, 1)^T), \right. \right.$$

$$\Pi_i^l = \left. \left( (K_D)^{-1} \left[ \left( \frac{l_1 \cdot \tilde{H}_0^{(0)}}{\mu_1 - \mu_0}, \frac{l_2 \cdot \tilde{H}_0^{(0)}}{\mu_2 - \mu_0}, \ldots, \frac{l_{l_0} \cdot \tilde{H}_0^{(0)}}{\mu_{l_0} - \mu_0} \right) \right] \right), \right.$$

respectively, $l = 1, 2, \ldots, l_0$. It can be verified that

$$\nabla \partial_D \cdot \Theta_i^l = 0, \quad l = 1, 2, \ldots, l_0.$$

Thus $\nabla \times A_0^l [\Theta_i^l]$ is a gradient field of harmonic function in $\mathbb{R}^3 \setminus \partial_D$. By using Lemma 3.2, one can derive that

$$H^{(0)} = \tilde{H}_0^{(0)} + \sum_{l=1}^{l_0} \left( \epsilon_0 \nabla \times A_0^l [\Theta_i^l] - \mu_0 \nabla S_0^l [\Pi_i^l] \right) \text{ in } \mathbb{R}^3 \setminus \Sigma,$$  (3.31)

As before, for $y \in \partial_D$, we let $\tilde{y} = \delta^{-1}(y - z_l) \in \partial \Omega$, and define $\tilde{\Theta}_i^l(\tilde{y}) := \Theta_i^l(\tilde{y})$, $\tilde{\Pi}_i^l(\tilde{y}) := \Pi_i^l(y)$, $l = 1, 2, \ldots, l_0$ and $\tilde{H}_0^{(0)}(\tilde{y}) := H_0^{(0)}(y)$. Then by Lemma 3.5, one has

$$\tilde{\Theta}_i^l(\tilde{y}) = \frac{1}{\gamma_l - \epsilon_s} (\lambda_l I + M_0^l)^{-1} [\nu \times \tilde{H}_0^{(0)}](\tilde{y}) + O(\delta^2),$$  (3.32)

and

$$\tilde{\Pi}_i^l(\tilde{y}) = \frac{1}{\mu_l - \mu_0} (\lambda_l I - (K_0^l)^*)^{-1} [\nu \cdot \tilde{H}_0^{(0)}](\tilde{y}) + O(\delta^2),$$  (3.33)

$l = 1, 2, \ldots, l_0$. Hence by using (3.25), there holds

$$\nabla \times A_0^l [\Theta_i^l] = \frac{1}{\gamma_l - \epsilon_s} \nabla \times A_0^l (\lambda_l I + M_0^l)^{-1} [\nu \times \tilde{H}_0^{(0)}](\tilde{y}) + O(\delta^4)$$

$$= \frac{1}{\gamma_l - \epsilon_s} \nabla S_0^l (\lambda_l I + (K_0^l)^*)^{-1} [\nu \cdot \tilde{H}_0^{(0)}](\tilde{y}) + O(\delta^4)$$  (3.34)

$$:= \nabla S_0^l [Q_l] + O(\delta^4) \text{ in } \mathbb{R}^3 \setminus \Sigma.$$

On the other hand, by the Taylor expansion, there holds

$$H_0^{(0)}(y) = H_0^{(0)}(z_l) + \delta \nabla H_0^{(0)}(z_l) \tilde{y} + O(\delta^2),$$  (3.35)
and so by using (3.18) one has
\[ \nu \cdot \hat{H}_0^{(0)}(\bar{y}) = \nu_1 \cdot \hat{H}_0^{(0)}(y) = \nu_1 \cdot H_0^{(0)}(z_l) + \left( \frac{I}{2} + (K_\Omega^0)^* \right) [\varphi_l^{(0)}](\bar{y}) + O(\delta), \] (3.36)
where \( \varphi_l^{(0)}(\bar{y}) := \varphi_l(y) = \varphi_l(y) + O(\omega) \) and by (3.19) and (3.35) one has
\[ \tilde{\varphi}_l^{(0)}(\bar{y}) = \left( \lambda_\gamma I - (K_\Omega^0)^* \right)^{-1} [\nu_1 \cdot H_0^{(0)}(z_l)] + O(\delta^2). \] (3.37)
For \( x - z_l \gg \delta \) there also holds
\[ \Gamma_0(x - y) = \Gamma_0(z_l - \delta \nabla \Gamma_0(x - z_l) + O(\delta^2). \] (3.38)
Define \( \tilde{Q}_l(\bar{y}) := Q_l(y) \), where \( Q_l \) is given in (3.34). By using change of variables and substituting (3.32)-(3.38) into (3.31) and using (3.18), one thus has
\[ H^{(0)}(x) = H_0^{(0)}(x) - \delta^3 \sum_{l=1}^{l_0} \nabla^2 \Gamma_0(x - z_l)P_0^*H_0^{(0)}(z_l) - \delta^3 \sum_{l=1}^{l_0} \varepsilon_0 \int_{\partial \Omega} \nabla^2 \Gamma_0(x - z_l)^T \tilde{y} \tilde{Q}_l \\
+ \delta^3 \sum_{l=1}^{l_0} \left( \mu_0 \frac{1}{\mu_l - \mu_0} \int_{\partial \Omega} \nabla^2 \Gamma_0(x - z_l) \tilde{y} (\lambda_\mu I - (K_\Omega^0)^*)^{-1} [\nu_1 \cdot \tilde{H}_0^{(0)}(\bar{y})] + O(\delta^4), \right. \]
\[ = H_0^{(0)}(x) - \delta^3 \sum_{l=1}^{l_0} \nabla^2 \Gamma_0(x - z_l)P_0^*H_0^{(0)}(z_l) - \delta^3 \sum_{l=1}^{l_0} \nabla^2 \Gamma_0(x - z_l)M_l^*H_0^{(0)}(z_l) \\
+ \delta^3 \sum_{l=1}^{l_0} \nabla^2 \Gamma_0(x - z_l)M_lH_0^{(0)}(z_l) + O(\delta^4). \] (3.39)
The first equality of (3.39) is obtained by using the following fact
\[ \int_{\partial \Omega} \tilde{Q}_l = O(\delta^2), \int_{\partial \Omega} \tilde{H}_0^{(0)} = O(\delta^2). \] (3.40)
Indeed, in order to show (3.40), we set 
\[ \phi_l(\bar{y}) := (\lambda_\gamma I + (K_\Omega^0)^*)^{-1} [\nu_1 \cdot \tilde{H}_0^{(0)}(\bar{y})]. \]
By using the jump formula (2.4) and integration by parts, one can show that there holds
\[ 0 = \int_{\partial \Omega} \nu_1 \cdot \tilde{H}_0^{(0)} = \int_{\partial \Omega} (\lambda_\gamma + 1/2) \phi_l - \int_{\partial \Omega} (1/2I - (K_\Omega^0)^*)[\phi_l] \\
= (\lambda_\gamma + 1/2) \int_{\partial \Omega} \phi_l - \int_{\partial \Omega} \nu \cdot \nabla S_{\Omega}^0[\phi_l]_+ = (\lambda_\gamma + 1/2) \int_{\partial \Omega} \phi_l, \]
which readily proves the first assertion in (3.40). The second assertion in (3.40) can be proven in a similar manner.

The proof is complete. \( \Box \)

For notational convenience, in the sequel, we introduce the matrix \( P_l \) by
\[ P_l := \mu_0 M_l - \varepsilon_0 D_l - P_0, \] (3.41)
where \( M_l \) and \( D_l \) are defined in (3.28) and (3.29), respectively. We have the following axillary results

**Lemma 3.7.** If \( \sigma_l \neq 0, l = 1, 2, \ldots, l_0 \) and \( \varepsilon_s = \varepsilon_0 \), then \( P_l = \mu_0 M_l + O(\omega) \) is nonsingular.
Proof. Since $\varepsilon_s = \varepsilon_0$, one immediately has $P_0 = 0$ from (3.27). Recall that $\gamma_l = \varepsilon_l + i\sigma l\omega^{-1}$. Since $\varepsilon_s = \varepsilon_0$ and $\sigma \neq 0$, it is straightforward to see from the definition of $D_l$ in (3.28) that

$$D_l = -i\omega \sigma^{-1} l \int_{\partial \Omega} \hat{y} \left( \frac{I}{2} + (K^{(0)}_l)^* \right)^{-1} [\nu] d\hat{y} + O(\omega^2).$$

Then one can obtain that

$$P_l = \mu_0 M_l + O(\omega) = \frac{\mu_0}{\mu_l - \mu_0} \int_{\partial \Omega} \hat{y} (\lambda_{\mu l} - 1/6)^{-1} (\lambda_{\varepsilon} - 1/6)^{-1} [\nu] d\hat{y} + O(\omega).$$

(3.42)

It is known that the polarization tensor $M_l$ in (3.42) is a positive definite matrix (see, e.g., [2, 4]).

The proof is complete. □

Lemma 3.8. Suppose $\Omega$ is a ball. Let $P_l$ be defined in (3.42). If there holds

$$\mu_l \varepsilon_s^2 + 2(\mu_0 - \mu_l)\varepsilon_s \gamma_l + 2(\mu_l + 2\mu_0)\varepsilon_0 \gamma_l \neq \mu_0 \varepsilon_s^2$$

(3.43)

then $P_l$ is nonsingular.

Proof. Since $\Omega$ is a ball, one has the following result (see, e.g., [1, 9])

$$(K^{(0)}_l)^* [\nu] = \frac{1}{6} \nu.$$  

(3.44)

Then one can calculate explicitly that

$$P_l = \frac{\mu_0}{\mu_l - \mu_0} \frac{\varepsilon_s}{\varepsilon_0 - \varepsilon_0} \int_{\partial \Omega} \hat{y} (\lambda_{\mu l} - 1/6)^{-1} (\lambda_{\varepsilon} - 1/6)^{-1} [\nu] d\hat{y}$$

$$= \frac{\varepsilon_0}{\gamma_l - \varepsilon_s \gamma - \varepsilon_0} \int_{\partial \Omega} \hat{y} (\lambda_{\varepsilon} + 1/6)^{-1} (\lambda_{\varepsilon} - 1/6)^{-1} [\nu] d\hat{y} - \int_{\partial \Omega} \hat{y} (\lambda_{\varepsilon} - 1/6)^{-1} [\nu] d\hat{y}$$

$$= \frac{3((\mu_l - \mu_0)\varepsilon_s^2 + 2(\mu_0 - \mu_l)\varepsilon_s \gamma_l + 2(\mu_l + 2\mu_0)\varepsilon_0 \gamma_l)}{(\varepsilon_s + 2\varepsilon_0)(\mu_l + 2\mu_0)(2\gamma_l + \varepsilon_s)} [\nu],$$

(3.45)

which proves that $P_k$ is nonsingular. □

We remark that $\Omega$ is not necessary to be a ball to ensure the nonsingularity of the matrix $P_l$. Indeed, one can also explicitly calculate $P_l$ if $\Omega$ is an ellipsoid and show that $P_l$ is nonsingular if the parameters $\mu_l$ and $\gamma_l$ are not quite special. One can find from (3.45) that if $\varepsilon_s = \varepsilon_0$ and $\sigma \neq 0$ then $P_l$ is nonsingular, which is indicated in Lemma 3.7. Starting from now on and throughout the rest of the paper, we always assume that $P_l$, $l = 1, 2, \ldots l_0$ are nonsingular.

3.3. Spherical harmonics expansion. In Theorem 3.2 we derived the necessary asymptotic expansion for our subsequent inverse problem study. Furthermore, at a certain point, we shall need the the expansion of the steady geomagnetic field on the surface of the Earth with respect to the spherical harmonic functions. To that end, we present the following lemma

Lemma 3.9. Let $z \in B_{R_0}$ be fixed, where $B_{R_0}$ stands for a ball of radius $R_0 \in \mathbb{R}_+$. Let $x \in \partial B_{R_1}$ and suppose $R_0 < R_1$. There holds the following asymptotic expansion

$$\nabla \Gamma (x - z) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n + 1) Y^m_n(\hat{x}) \hat{x} - \nabla_S Y^m_n(\hat{x})}{(2n + 1) R_1^{n+2}} Y^m_n(z) ||z||^n,$$

(3.46)

where $z = z/||z||$ and $\hat{x} = x/||x||$. $Y^m_n$ is the spherical harmonics of order $m$ and degree $n$. 

Proof. Suppose that \(\|\mathbf{x}\| > \|\mathbf{y}\|\), then there holds the following addition formula (cf. [6][14])

\[
\frac{1}{4\pi \|\mathbf{x} - \mathbf{y}\|} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n + 1} Y_n^m(\hat{x}) Y_n^m(\hat{y}) \|\mathbf{y}\|^{n+1}. \quad (3.47)
\]

Since \(\|\mathbf{x}\| = R_1\), by using the definition of surface gradient, one has

\[
\nabla Y_n^m(\mathbf{x}) \|\mathbf{x}\|^{n+1} = -((n + 1)Y_n^m(\hat{x})\mathbf{x} - \nabla_S Y_n^m(\hat{x}))R_1^{-(n+2)}. \quad (3.48)
\]

By substituting (3.48) into the gradient of (3.47), one can obtain (3.46).

The proof is complete. \(\square\)

By substituting (3.46) into (3.26), one can obtain the spherical harmonic expansion of the magnetic field.

4. Unique recovery results for magnetized anomalies

We are in a position to present the main unique recovery results in identifying the magnetized anomalies. In what follows, we let \(D_l^{(1)}\) and \(D_l^{(2)}\), \(l = 1, 2, \ldots, l_0\), be two sets of magnetic anomalies, which satisfy (3.20) with \(\mathbf{z}_l\) replaced by \(\mathbf{z}_l^{(1)}\) and \(\mathbf{z}_l^{(2)}\), respectively. Correspondingly, the material parameters \(\varepsilon_l, \sigma_l, \gamma_l\) and \(\mu_l\) are replaced by \(\varepsilon_l^{(1)}, \sigma_l^{(1)}, \gamma_l^{(1)}, \mu_l^{(1)}\) and \(\varepsilon_l^{(2)}, \sigma_l^{(2)}, \gamma_l^{(2)}, \mu_l^{(2)}\), respectively, for \(D_l^{(1)}\) and \(D_l^{(2)}\), \(l = 1, 2, \ldots, l_0\). Let \((\mathbf{E}_l, \mathbf{H}_l)\), \(j = 1, 2\), be the solutions to (1.3) and (2.9) with \(D_l\) replaced by \(D_l^{(1)}\) and \(D_l^{(2)}\), respectively. Denote by \(\mathbf{D}_l^{(1)}, \mathbf{D}_l^{(2)}, \mathbf{P}_l^{(1)}\) and \(\mathbf{P}_l^{(2)}\) the polarization tensors for \(D_l^{(1)}\) and \(D_l^{(2)}\), respectively, \(l = 1, 2, \ldots, l_0\).

Let \(\mathbf{H}_l^{(0)}\) and \(\mathbf{H}_l^{(2)}\) be the leading terms of \(\mathbf{H}_l\) and \(\mathbf{H}_l\), respectively. Then from (3.20), there holds the following for \(\mathbf{x} \in \mathbb{R}^3 \setminus \Sigma\),

\[
\mathbf{H}_l^{(0)}(\mathbf{x}) = \mathbf{H}_l^{(2)}(\mathbf{x}) + \delta^3 \sum_{l=1}^{l_0} \left( \nabla \left( \nabla \Gamma_0(\mathbf{x} - \mathbf{z}_l^{(j)})^T \mathbf{P}_l^{(j)} \mathbf{H}_l^{(0)}(\mathbf{z}_l^{(j)}) \right) \right) + O(\delta^4), \quad j = 1, 2, \quad (4.1)
\]

Lemma 4.1. If there holds

\[
\nu \cdot \mathbf{H}_1 = \nu \cdot \mathbf{H}_2 \neq 0 \quad \text{on} \quad \Gamma, \quad (4.2)
\]

then one has

\[
\sum_{m=-n}^{n} N_{n+1}(\hat{x})^T d_{l}^{n,m} = \sum_{m=-n}^{n} N_{n+1}(\hat{x})^T d_{l}^{n,m}, \quad \hat{x} \in \mathbb{S}^2 \quad (4.3)
\]

for any \(n \in \mathbb{N} \cup \{0\}\), where

\[
d_{l}^{n,m} = \sum_{l=1}^{l_0} \frac{Y_n^m(\mathbf{z}_l^{(j)})\|\mathbf{z}_l^{(j)}\|^n(\mathbf{P}_l^{(j)})^T \mathbf{H}_l^{(0)}(\mathbf{z}_l^{(j)})}{(n + 1)Y_n^m(\hat{x})\mathbf{x} - \nabla_S Y_n^m(\hat{x})}, \quad j = 1, 2, \quad (4.4)
\]

and

\[
N_{n+1}(\hat{x}) = (n + 1)Y_n^m(\hat{x})\mathbf{x} - \nabla_S Y_n^m(\hat{x}). \quad (4.5)
\]

Proof. First, by using (4.2) and unique continuation, one sees that

\[
\mathbf{H}_1 = \mathbf{H}_2 \quad \text{in} \quad \mathbb{R}^3 \setminus \Sigma,
\]

Then from (4.1) one has

\[
\sum_{l=1}^{l_0} \left( \nabla^2 \Gamma_0(\mathbf{x} - \mathbf{z}_l^{(1)}) \mathbf{D}_l^{(1)} \mathbf{H}_l^{(0)}(\mathbf{z}_l^{(1)}) \right) = \sum_{l=1}^{l_0} \left( \nabla^2 \Gamma_0(\mathbf{x} - \mathbf{z}_l^{(2)}) \mathbf{D}_l^{(2)} \mathbf{H}_l^{(0)}(\mathbf{z}_l^{(2)}) \right) \quad \text{in} \quad \mathbb{R}^3 \setminus \Sigma. \quad (4.6)
\]
Suppose \( R_1 \in \mathbb{R}_+ \) is sufficiently large such that \( \Sigma \subset B_{R_1} \). By (4.6) one readily has
\[
\nu \cdot \nabla \sum_{l=1}^{l_0} \nabla \Gamma_0(x - z_l^{(1)})^T P_l^{(1)} H_0^{(0)}(z_l^{(1)}) = \nu \cdot \nabla \sum_{l=1}^{l_0} \nabla \Gamma_0(x - z_l^{(2)})^T P_l^{(2)} H_0^{(0)}(z_l^{(2)}), \quad \text{on } \partial B_{R_1}.
\]
(4.7)

On the other hand, it can be verified that
\[
u_j(x) := \sum_{l=1}^{l_0} \nabla \Gamma_0(x - z_l^{(j)})^T P_l^{(j)} H_0^{(0)}(z_l^{(j)}), \quad j = 1, 2
\]
are harmonic functions in \( \mathbb{R}^3 \setminus \overline{B_{R_1}} \), which decay at infinity. Using this together with (4.7), and the maximum principle of harmonic functions, one can obtain that
\[
u_1(x) = \nu_2(x), \quad x \in \mathbb{R}^3 \setminus \overline{B_{R_1}},
\]
and therefore
\[
u_1(x) = \nu_2(x), \quad x \in \partial B_{R_1}.
\]
(4.8)

By substituting (3.46) into (4.8) one has
\[
\sum_{n=0}^{\infty} \sum_{m=-n}^{n} N_{n+1}^m(\hat{x})^T d_{n}^{m} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} N_{n+1}^m(\hat{x})^T d_{n}^{m},
\]
(4.9)

By taking \( R_1 \) sufficiently large and comparing the orders of \( R_1^{-1} \) one has (4.3).

The proof is complete. \( \square \)

4.1. **Uniqueness in recovering a single anomaly.** We present the uniqueness result in recovering a single anomaly.

**Theorem 4.1.** Suppose \( l_0 = 1 \) and \( \sigma_1^{(1)}, \sigma_2^{(2)} \neq 0 \). If there holds (4.2) then \( z_1^{(1)} = z_2^{(2)} \) and \( \mu_1^{(1)} = \mu_1^{(2)} \).

**Proof.** By the unique continuation principle, one has from (4.2) that \( H_1 = H_2 \) in \( \mathbb{R}^3 \setminus D_1^{(1)} \cup D_1^{(2)} \). First, we note that \( Y_0 = \frac{1}{2\sqrt{\pi}} \), and then by letting \( n = 0 \) and using (4.4) and (4.5), \( l_0 = 1 \), we have
\[
d_{0}^{(j)} = \frac{1}{2\sqrt{\pi}} M_{1}^{(j)} H_0^{(0)}(z_l^{(j)}) \quad \text{and} \quad N_{1}^{(j)}(\hat{x}) = \frac{1}{2\sqrt{\pi}} \hat{x}, \quad j = 1, 2.
\]
(4.10)

Hence by using (4.3) there holds
\[
\hat{x}^T P_1^{(1)} H_0^{(0)}(z_l^{(1)}) = \hat{x}^T P_1^{(2)} H_0^{(0)}(z_l^{(2)}), \quad \forall \hat{x} \in S^2,
\]
(4.11)

which readily implies that
\[
P_1^{(1)} H_0^{(0)}(z_l^{(1)}) = P_1^{(2)} H_0^{(0)}(z_l^{(2)}).
\]
(4.12)

In the following, we set \( c := P_1^{(1)} H_0^{(0)}(z_l^{(1)}) \). We claim that
\[
H_0^{(0)}(z_l^{(1)}) \neq 0.
\]

From (2.28) one has that \( H_0^{(0)} = \nabla u_0 \), where \( u_0 \) is a non-constant harmonic function. Hence, by the maximum principle of harmonic functions, \( \nabla u_0(z_l^{(1)}) \neq 0 \). This together with the assumption that \( P_1^{(1)} \) is a nonsingular matrix, one has \( c \neq 0 \). Now we set \( n = 1 \), and define
\[
Y_1(\hat{x}) := (Y_1^{-1}(\hat{x}), Y_0^1(\hat{x}), Y_1^1(\hat{x}^T)).
\]
(4.13)
Let \((\hat{x}, e_\phi, e_\theta)\) be the triplet of the local orthogonal unit vectors, where \(\phi\) and \(\theta\) depend on \(\hat{x}\). Note that

\[
\nabla s Y_n^m = \frac{1}{\sin \theta} \left( \frac{\partial Y_n^m(\hat{x})}{\partial \phi} e_\phi + \frac{\partial Y_n^m(\hat{x})}{\partial \theta} e_\theta \right).
\]

By using (4.3) again there holds

\[
\|z_1^{(1)}\| Y_1(z_1^{(1)})^T Q(\hat{x})(\hat{x}, e_\phi, e_\theta)^T c = \|z_1^{(2)}\| Y_1(z_1^{(2)})^T Q(\hat{x})(\hat{x}, e_\phi, e_\theta)^T c, \quad \hat{x} \in S^2,
\]

where \(Q(\hat{x})\) is a matrix defined by

\[
Q(\hat{x}) := \left[ 2Y_1(\hat{x}), -\frac{1}{\sin \theta} \frac{\partial Y_1(\hat{x})}{\partial \phi}, -\frac{\partial Y_1(\hat{x})}{\partial \theta} \right].
\]

Straightforward calculations show that \(Q(\hat{x})\) is nonsingular for any \(\hat{x} \in S^2\). Since \(\hat{x} \in S^2\) is arbitrarily given, (4.14) implies that

\[
\|z_1\| Y_1(z_1) = \|z_2\| Y_1(z_2).
\]

Then by direct calculations, one has \(\|z_1\| = \|z_2\|\) and \(z_1 = z_2\). We thus have \(D^{(1)}_1 = D^{(2)}_1\).

Hence, in the sequel, we let \(D_1 := D^{(1)}_1 = D^{(2)}_1\). Clearly, \(H_1 = H_2 \in \mathbb{R}^3 \setminus D_1\). Since \(\sigma^{(1)}_1, \sigma^{(2)}_1 \neq 0\), from (3.3), one can find that

\[
H_j = \hat{H}_0^{(0)} - \frac{\mu_0}{\mu^{(1)}_1 - \mu_0} \nabla S_D (\lambda^{(j)}_{\mu^{(1)}_1} I - (K^{(0)}_{D_1})^*)^{-1} [\nu_1 \cdot \hat{H}_0^{(0)}] + O(\omega) \quad \text{in} \quad \hat{D},
\]

By using the jump formula one further has that

\[
\frac{1}{\mu^{(1)}_1 - \mu_0} \left( \frac{I}{2} + (K^{(0)}_{D_1})^* \right) (\lambda^{(j)}_{\mu^{(1)}_1} I - (K^{(0)}_{D_1})^*)^{-1} [\nu_1 \cdot \hat{H}_0^{(0)}] = \frac{1}{\mu^{(2)}_1 - \mu_0} \left( \frac{I}{2} + (K^{(0)}_{D_1})^* \right) (\lambda^{(j)}_{\mu^{(2)}_1} I - (K^{(0)}_{D_1})^*)^{-1} [\nu_1 \cdot \hat{H}_0^{(0)}] \quad \text{on} \quad \partial D_1.
\]

Using the fact that \(\frac{I}{2} + (K^{(0)}_{D_1})^*\) is invertible on \(L^2(\partial D_1)\) and some elementary calculations, one has from (4.18) that

\[
(\mu^{(1)}_1 - \mu^{(2)}_1) \left( \frac{I}{2} + (K^{(0)}_{D_1})^* \right) (\lambda^{(j)}_{\mu^{(1)}_1} I - (K^{(0)}_{D_1})^*)^{-1} [\nu_1 \cdot \hat{H}_0^{(0)}] = 0 \quad \text{on} \quad \partial D_1.
\]

Note that \(\nu \cdot \hat{H}_0^{(0)} \in L^2(\partial D_1)\), where \(L^2(\partial D_1)\) is a subset of \(L^2(\partial D_1)\) with zero average on \(\partial D_1\). Since \(\nu_1 \cdot \hat{H}_0^{(0)} \neq 0\) (otherwise by the unique continuation of harmonic functions one has \(H_0^{(0)} = 0\) in \(\mathbb{R}^3 \setminus S\), and this cannot be true), one has from (3.18) that \(\nu_1 \cdot \hat{H}_0^{(0)} \neq 0\) on \(\partial D_1\). Using this and the fact that \(\frac{I}{2} + (K^{(0)}_{D_1})^*\) is invertible on \(L^2_0(\partial D_1)\), we finally have from (4.19) that \(\mu^{(1)}_1 = \mu^{(2)}_1\).

The proof is complete. \(\square\)

4.2 Uniqueness in recovering multiple anomalies.

**Theorem 4.2.** If there holds (4.3) then \(z_l^{(1)} = z_l^{(2)}\), \(l = 1, 2, \ldots, l_0\).
that the geomagnetic field should be monitored for all the time, can be relaxed to a finite part in the difference of the geomagnetic fields monitored before and after the presence of the wave frequency and the size of the anomalies. We mainly make use of the steady mathematical arguments rely on the asymptotic analysis of the geomagnetic fields with respect to the locations of multiple magnetized anomalies. For the case with a single anomaly, modelling as a type of nonlinear inverse problem and establish the global uniqueness in recovering magnetized anomalies using geomagnetic monitoring. We provide the mathematical developed in this work can also be used to show the identification of the magnetic permeability of the anomalies as well, similar to the single anomaly case (cf. Theorem 1.1). However, it would involve much more complicated analysis and we leave it for our future study.

Remark 4.1. We remark that for the recovery of multiple anomalies, we can only prove the uniqueness in identifying the positions of the anomalies. In principle, our arguments developed in this work can also be used to show the identification of the magnetic permeability of the anomalies as well, similar to the single anomaly case (cf. Theorem 1.1). However, it would involve much more complicated analysis and we leave it for our future study.

5. Concluding Remark

In this paper, we develop a mathematical theory for the applied technology of identifying magnetized anomalies using geomagnetic monitoring. We provide the mathematical modelling as a type of nonlinear inverse problem and establish the global uniqueness in recovering the locations of multiple magnetized anomalies. For the case with a single anomaly, we show that one can also identify the magnetic permeability of the anomaly. Our mathematical arguments rely on the asymptotic analysis of the geomagnetic fields with respect to the wave frequency and the size of the anomalies. We mainly make use of the steady part in the difference of the geomagnetic fields monitored before and after the presence of the magnetized anomalies. One can expect that the technical condition in [1,4], requiring that the geomagnetic field should be monitored for all the time, can be relaxed to a finite

Proof. With our earlier preparations, the proof follows from a similar argument to that of Theorem 7.8 in [4]. In the following, we only sketch it. Using the formula (4.7) and similar analysis in the proof of Lemma 4.1, one can show that

\[
\sum_{l=1}^{l_0} \left( \nabla \Gamma_0 (x - z_l^{(1)})^T P_l^{(1)} H_0^{(0)} (z_l^{(1)}) - \nabla \Gamma_0 (x - z_l^{(2)})^T P_l^{(2)} H_0^{(0)} (z_l^{(2)}) \right) = 0, \tag{4.20}
\]

holds in \(\mathbb{R}^3 \setminus \Sigma\). By straightforward calculations, one can further show that

\[
F(x) := \sum_{l=1}^{l_0} \left( (\nabla \Gamma_0 (x - z_l^{(1)}) - \nabla \Gamma_0 (x - z_l^{(2)}))^T P_l^{(1)} H_0^{(0)} (z_l^{(1)}) \\
- \nabla \Gamma_0 (x - z_l^{(2)})^T (P_l^{(2)} H_0^{(0)} (z_l^{(2)}) - P_l^{(1)} H_0^{(0)} (z_l^{(1)})) \right)
\]

\[
= \sum_{l=1}^{l_0} \left( (\nabla^2 \Gamma_0 (x - z_l^{(1)}) (z_l^{(1)} - z_l^{(2)}))^T P_l H_0^{(0)} (z_l^{(1)}) \\
- \nabla \Gamma_0 (x - z_l^{(2)})^T (P_l^{(2)} H_0^{(0)} (z_l^{(2)}) - P_l^{(1)} H_0^{(0)} (z_l^{(1)})) \right) = 0
\]

holds in \(\mathbb{R}^3 \setminus \Sigma\), where \(z_l' = z_l^{(1)} + t' z_l^{(2)}\) with \(t' \in (0, 1)\). Note that \(F(x)\) defined in (4.21) is also harmonic in \(\mathbb{R}^3 \setminus \bigcup_{l=1}^{l_0} (z_l^{(1)} \cup z_l^{(2)})\). By using the analytic continuation of harmonic functions, one thus has that \(F(x) \equiv 0\) in \(\mathbb{R}^3\). Define \(F := F_1 + F_2\), where

\[
F_1(x) := \sum_{l=1}^{l_0} (\nabla^2 \Gamma_0 (x - z_l') (z_l^{(1)} - z_l^{(2)}))^T P_l^{(1)} H_0^{(0)} (z_l^{(1)}),
\]

and

\[
F_2(x) := -\nabla \Gamma_0 (x - z_l^{(2)})^T (P_l^{(2)} H_0^{(0)} (z_l^{(2)}) - P_l^{(1)} H_0^{(0)} (z_l^{(1)})).
\]

Then by comparing the types of poles of \(F_1\) and \(F_2\), one immediately finds that \(F_1 = 0\) and \(F_2 = 0\) in \(\mathbb{R}^3\). Since \(H_0^{(0)}(x)\) does not vanish for \(x \in \mathbb{R}^3 \setminus \Sigma\), then one has \(P_l^{(1)} H_0^{(0)} (z_l^{(1)}) \neq 0\). Hence we have

\[
z_l^{(1)} - z_l^{(2)} = 0, \quad l = 1, 2, \ldots, l_0.
\]

The proof is complete. \(\square\)
time interval. In fact, in a forthcoming article, we not only develop an efficient numerical reconstruction scheme for the geomagnetic monitoring problem based on the theory in the current article, but also numerically verify that the monitoring can indeed be conducted within a finite time interval. Our study also opens up intriguing mathematical topics for further developments, including the identification of moving anomalies using the geomagnetic monitoring and the investigation of geomagnetic monitoring for different planets other than the Earth, such as the Sun.

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Appendix A. Uniqueness of Solution

We prove the uniqueness of a trivial solution to (2.25). Let $\mathbf{E}$ be the solution to (2.25). Since
\begin{equation}
\nabla \times \mathbf{E} = 0, \quad (\mathbb{R}^3 \setminus \Sigma) \cup \Sigma_s,
\end{equation}
and noting that $\mathbb{R}^3 \setminus \Sigma$ and $\Sigma_s$ are simply connected domains, one can find $u_1 \in H^1_{\operatorname{loc}}(\mathbb{R}^3 \setminus \Sigma)$ and $u_2 \in H^1(\Sigma_s)$, such that
\begin{equation}
\mathbf{E} = \begin{cases} 
\nabla u_1 & \text{in} \quad \mathbb{R}^3 \setminus \Sigma, \\
\nabla u_2 & \text{in} \quad \Sigma_s.
\end{cases}
\end{equation}
Furthermore, $\nabla \cdot \mathbf{E} = 0$ implies that $\Delta u_1 = 0$ and $\Delta u_2 = 0$. This together with the fact that $\mathbf{E} = O(\|\mathbf{x}\|^{-2})$ as $\mathbf{x} \to \infty$, and the Helmholtz decomposition, readily implies that $u_1 = O(\|\mathbf{x}\|^{-1})$. Then by using the transmission and boundary conditions in (2.25), and integration by parts, we have
\begin{equation}
\int_{\mathbb{R}^3 \setminus \Sigma} \epsilon_0 |\nabla u_1|^2 + \int_{\Sigma_s} \epsilon_s |\nabla u_2|^2 = -\int_{\partial \Sigma} \epsilon_0 \frac{\partial u_1}{\partial \nu} u_1 + \int_{\partial \Sigma} \epsilon_s \frac{\partial u_2}{\partial \nu} u_2 - \int_{\partial \Sigma_s} \epsilon_s \frac{\partial u_2}{\partial \nu} u_2 + \int_{\partial \Sigma_s} \epsilon_s \frac{\partial u_2}{\partial \nu} u_2,
\end{equation}
where $C$ is a constant. By (A.2) one thus has $u_1 = C_1$ and $u_2 = C_2$, where $C_1$ and $C_2$ are constants. Hence, $\mathbf{E} = 0.$

Appendix B. Integral Representations in Lemma 3.1

In this appendix, we present the proof of Lemma 3.1, namely the solution to (1.3) and (1.7) admits the integral representations (3.1) and (3.2). Let $\nu_l$ be the exterior unit normal vector to $\partial D_l$, $l = 1, 2, \ldots, l_0$. By using the transmission conditions on $\partial \Sigma$ and $\partial D_l$,
In the sequel, we prove that (B.1) is uniquely solvable when \( \omega \in \mathbb{R}_+ \) is sufficiently small. By asymptotic analysis (see (2.13) and (2.14)), one can find that

\[
\begin{align*}
\Phi_0 \in \mathcal{H}^1(\Omega) \quad &\quad \Rightarrow \quad \Phi_0 \in \mathcal{H}^1(\Omega) \\
\end{align*}
\]

where \( k, k' \in \{ k_0, k_s, s_1, s_2, \ldots, s_{l_0} \} \), \( D, D' \in \{ \Sigma, D_1, D_2, \ldots, D_{l_0} \} \) and \( \mu^{(k)}, \gamma^{(k)} \) are respectively the parameters \( \mu \) and \( \gamma \), which are related to \( k \). In other words,

\[
\begin{align*}
\mu^{(k_0)} &= \mu^{(k_s)}, \quad \mu^{(s_i)} = \mu, \\
\gamma^{(k_0)} &= \gamma^{(k_s)} = \gamma^{(s_i)} = \gamma. \\
\end{align*}
\]

In the sequel, we prove that (B.1) is uniquely solvable when \( \omega \in \mathbb{R}_+ \) is sufficiently small. By asymptotic analysis (see (2.13) and (2.14)), one can find that

\[
\begin{align*}
\mathcal{F}_{k_0, k_s} &= (\varepsilon_0 - \varepsilon_s) \mathcal{M}^{(0)}_{D_1, D_1} + \mathcal{O}(\omega^2), \\
\mathcal{F}_{k_s, k_0} &= (\varepsilon_0 - \varepsilon_s) \mathcal{M}^{(0)}_{D_1, D_1} + \mathcal{O}(\omega^2), \\
\mathcal{F}_{k_s, k_0} &= (k_s^2 - k_0^2) \mu \times (\mathcal{A}^{(0)}_{D_1, D_1} + D^2 \mathcal{B}_{D_1, D_1}) + \mathcal{O}(\omega^2), \\
\mathcal{F}_{k_0, k_s} &= (k_s^2 - k_0^2) \mu \times (\mathcal{A}^{(0)}_{D_1, D_1} + D^2 \mathcal{B}_{D_1, D_1}) + \mathcal{O}(\omega^2). \\
\end{align*}
\]

In (B.3), we only present the asymptotic expansion of some of the operators involved in (B.1), while the asymptotic behavior of the other operators can be derived in a similar manner. Then the first equation in (B.1) implies

\[
\| \Phi_0 \|_{\text{TH(div,}\partial \Sigma)} = \mathcal{O} \left( \omega^2 \left( \| \Phi_0 \|_{\text{TH(div,}\partial \Sigma)} + \sum_{l' = 1}^{l_0} \left( \| \Phi_{l'} \|_{\text{TH(div,}\partial D_{l'}}} + \| \Phi_{l'} \|_{\text{TH(div,}\partial D_{l'}}} \right) \right) \right). 
\]
By substituting (B.4) and (B.5) into the second equation of (B.1), one further has

$$\Psi_0 = - \left( \lambda_e I + M_{\Sigma}^0 \right)^{-1} \sum_{l'=1}^{l_0} \left( M_{\Sigma, D_{l'}}^0 [\Psi_{l'}] + \mu_0 \nu \times (A_{D_{l'}}^0 + D^2 B_{D_{l'}}) [\Phi_{l'}] \right) + O \left( \omega^2 \left( \sum_{l'=1}^{l_0} \|\Phi_{l'}\|_{TH(div,D_{l'})} + \|\Psi_{l'}\|_{TH(div,D_{l'})} \right) \right) \quad \text{on } \partial \Sigma.$$  

(B.5)

By substituting (B.4) and (B.5) into the third and fourth equations in (B.1), one obtains

$$\left( \lambda_{\mu_l} I + M_{D_{l_l}}^0 \right) [\Phi_{l}] + \sum_{l' \neq l}^{l_0} M_{D_{l_l}, D_{l'}}^0 [\Phi_{l'}]$$

$$+ \frac{\varsigma_l}{\mu_l - \mu_0} \nu_l \times \left( (A_{D_{l_l}}^0 + D^2 B_{\Sigma}) [\Psi_0] + \sum_{l'=1}^{l_0} (A_{D_{l'}}^0 + D^2 B_{D_{l'}}) [\Psi_{l'}] \right)$$

$$= \frac{1}{\mu_l - \mu_0} \nu_l \times \hat{E}_0 + O \left( \sum_{l'=1}^{l_0} \left( \varsigma_l^2 + \omega^2 \right) \|\Phi_{l'}\|_{TH(div,D_{l'})} + \omega^2 \|\Psi_{l'}\|_{TH(div,D_{l'})} \right) \quad \text{on } \partial D_l,$$

and

$$\left( \lambda_{\gamma_l} I + M_{D_{l_l}}^0 \right) [\omega \Psi_{l}] + \sum_{l' \neq l}^{l_0} M_{D_{l_l}, D_{l'}}^0 [\omega \Psi_{l'}]$$

$$+ \sum_{l'=1}^{l_0} \omega^{-1} \varsigma_l^2 - k_l^2 \nu_l \times (A_{D_{l_l}}^0 + D^2 B_{D_{l'}}) [\Phi_{l'}] + \omega M_{D_{l_l}, \Sigma}^0 [\Psi_0]$$

$$= i \frac{1}{\gamma_l - \varepsilon_s} \nu_l \times \hat{H}_0 + O \left( \sum_{l'=1}^{l_0} \left( \omega \varsigma_l^2 \|\Psi_{l'}\|_{TH(div,D_{l'})} + \omega^{-1} \varsigma_l^2 \|\Phi_{l'}\|_{TH(div,D_{l'})} \right) \right) \quad \text{on } \partial D_l,$$

(B.6)

for \( l = 1, 2, \ldots, l_0 \), where \( \lambda_{\mu_l} \) and \( \lambda_{\gamma_l} \) are defined in (3.7). We claim that the following equations are uniquely solvable

$$\left( \lambda_{\mu_l} I + M_{D_{l_l}}^0 \right) [\Phi_{l}] + \sum_{l'=1}^{l_0} M_{D_{l_l}, D_{l'}}^0 [\Phi_{l'}] = \frac{1}{\mu_l - \mu_0} \nu_l \times \hat{E}_0, \quad \text{on } \partial D_l, \quad l = 1, 2, \ldots, l_0.$$  

(B.7)

when \( D_l, l = 1, 2, \ldots, l_0 \) are far away from each other. For relevant details, we refer to Section 3.2. Denote by \( M_D \) the \( l_0 \)-by-\( l_0 \) matrix type operator defined on \( \text{TH}(\text{div}, \partial D_1) \times \text{TH}(\text{div}, \partial D_2) \times \cdots \times \text{TH}(\text{div}, \partial D_{l_0}) \)

$$M_D := \begin{pmatrix}
M_{D_{l_1}, D_{l_1}}^0 & M_{D_{l_1}, D_{l_2}}^0 & \cdots & M_{D_{l_1}, D_{l_0}}^0 \\
M_{D_{l_2}, D_{l_1}}^0 & M_{D_{l_2}, D_{l_2}}^0 & \cdots & M_{D_{l_2}, D_{l_0}}^0 \\
\vdots & \vdots & \ddots & \vdots \\
M_{D_{l_0}, D_{l_1}}^0 & M_{D_{l_0}, D_{l_2}}^0 & \cdots & M_{D_{l_0}, D_{l_0}}^0 
\end{pmatrix}.$$  

(B.9)

Then the operator

$$\mathbb{L}_D^0 := \begin{pmatrix}
\lambda_{\mu_1} I & 0 & \cdots & 0 \\
0 & \lambda_{\mu_2} I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{\mu_{l_0}} I 
\end{pmatrix} + M_D \quad \text{(B.10)}$$
is invertible on $\text{TH} (\text{div}, \partial D_1) \times \text{TH} (\text{div}, \partial D_2) \times \cdots \times \text{TH} (\text{div}, \partial D_{l_0})$. From \((B.6)\) one obtains that

$$
\Phi_l = \left( (l_D^0)^{-1} \left[ \left( \frac{\nu_1}{\mu_1 - \mu_0} \cdot (\mathbf{E}_0 + \mathbf{A}) \right)^T, \left( \frac{\nu_2}{\mu_2 - \mu_0} \cdot (\mathbf{E}_0 + \mathbf{A}) \right)^T, \ldots, \left( \frac{\nu_{l_0}}{\mu_{l_0} - \mu_0} \cdot (\mathbf{E}_0 + \mathbf{A}) \right)^T \right] \right) \cdot (e_l \otimes (1, 1, 1)^T)$$

$$+ \mathcal{O} \left( \sum_{l' = 1}^{l_0} \left( \frac{\nu_{l'}}{\mu_{l'}} \cdot \mathbf{E}_0 \right) \right), \quad l = 1, 2, \ldots, l_0,$$

where

$$\mathbf{A} := \frac{\mathbf{S}_l^2}{\mu_l - \mu_0} \left( (l_D^0 + \mathbf{D}^2\mathbf{A}_c + \mathbf{M}^0_{\nu})^{-1} \left[ \sum_{l' = 1}^{l_0} \left( \mathbf{M}^0_{\nu} \right) \left[ \mathbf{B}_l^* \right] - \sum_{l' = 1}^{l_0} \left( \mathbf{A}_{D,l'}^0 + \mathbf{D}^2\mathbf{B}_{D,l'} \right) \left[ \mathbf{B}_l^* \right] \right) \right).$$

Here, the notation $\otimes$ stands for the Kronecker product and $e_l$ is the $l_0$-dimensional Euclidean unit vector. By taking the surface divergence of both sides of \((B.11)\), one can further obtain


$$
\nabla_{\partial D_l} \cdot \Phi_l = -i\omega \mu_0 \left( (l_D^0)^{-1} \left[ \left( \frac{\nu_1}{\mu_1 - \mu_0} \cdot \mathbf{H}_0, \frac{\nu_2}{\mu_2 - \mu_0} \cdot \mathbf{H}_0, \ldots, \frac{\nu_{l_0}}{\mu_{l_0} - \mu_0} \cdot \mathbf{H}_0 \right) \right) \right) l$$

$$- \left( (l_D^0)^{-1} \left[ \left( \frac{\mathbf{B}}{\mu_l - \mu_0}, \frac{\mathbf{B}}{\mu_2 - \mu_0}, \ldots, \frac{\mathbf{B}}{\mu_{l_0} - \mu_0} \right) \right] \right) l$$

$$+ \mathcal{O} \left( \sum_{l' = 1}^{l_0} \left( \frac{1}{\sqrt{l}} + \omega^2 \right) \left[ \mathbf{B}_l^* \right] \right), \quad l = 1, 2, \ldots, l_0,$$

with

$$\mathbf{B} := \nabla \times \mathbf{A}_c \left( \lambda_\nu I + \mathbf{M}^0_{\nu} \right)^{-1} \sum_{l' = 1}^{l_0} \left( \mathbf{M}^0_{\nu} \right) \left[ \mathbf{B}_l^* \right] - \nabla \times \sum_{l' = 1}^{l_0} \left( \mathbf{A}_{D,l'}^0 \right) \left[ \mathbf{B}_l^* \right],$$

and the operator $\mathbf{J}_D^0$ defined on $L^2 (\partial D_1) \times L^2 (\partial D_2) \times \cdots \times L^2 (\partial D_{l_0})$ given by

$$\mathbf{J}_D^0 := \left( \begin{array}{cccc}
\lambda_{\mu_1} & 0 & \cdots & 0 \\
0 & \lambda_{\mu_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{\mu_{l_0}}
\end{array} \right) - \mathcal{K}^*_D,$$

where $\mathcal{K}^*_D$ is an $l_0$-by-$l_0$ matrix type operator defined on $L^2 (\partial D_1) \times L^2 (\partial D_2) \times \cdots \times L^2 (\partial D_{l_0})$

$$\mathcal{K}^*_D := \left( \begin{array}{cccc}
(\mathbf{K}_{D_1}^0)^* & \nu_1 \cdot \nabla \mathbf{S}^0_{D_2} & \cdots & \nu_1 \cdot \nabla \mathbf{S}^0_{D_{l_0}} \\
\nu_2 \cdot \nabla \mathbf{S}^0_{D_1} & (\mathbf{K}_{D_2}^0)^* & \cdots & \nu_2 \cdot \nabla \mathbf{S}^0_{D_{l_0}} \\
\vdots & \vdots & \ddots & \vdots \\
\nu_{l_0} \cdot \nabla \mathbf{S}^0_{D_1} & \nu_{l_0} \cdot \nabla \mathbf{S}^0_{D_2} & \cdots & (\mathbf{K}_{D_{l_0}}^0)^*
\end{array} \right).$$

Finally, by substituting \((B.5), (B.11)\) and \((B.13)\) into \((B.7)\), one can have

$$\left( \lambda_\nu I + \mathbf{M}^0_{D_1} \right) \left[ \omega \Psi_l \right] + \sum_{l' \neq l}^{l_0} \left( \mathbf{M}_{D_1,l'}^0 \right) \left[ \omega \Psi_l \right] - \mathbf{M}_{D_1}^0 \left( \lambda_\nu I + \mathbf{M}^0_{D_1} \right)^{-1} \sum_{l' = 1}^{l_0} \left( \mathbf{M}_{D_1,l'}^0 \right) \left[ \omega \Psi_l \right]$$

$$= \frac{1}{\gamma_l - \varepsilon_s} \mathbf{H}_0^* + \mathcal{O} \left( \sum_{l' = 1}^{l_0} \left[ \omega \mathbf{S}_l^2 \right] \left[ \mathbf{B}_l^* \right] \right), \quad l = 1, 2, \ldots, l_0,$$

with

$$\mathbf{S}_l^2 := \left[ \left( \omega \mathbf{S}_l^2 \right) \right] \left( \omega \mathbf{S}_l^2 \right),$$

$$\mathbf{B}_l^* := \left( \omega \mathbf{S}_l^2 \right) \left( \omega \mathbf{S}_l^2 \right),$$

and

$$\mathbf{M}_{D_1,l'}^0 := \left( \omega \mathbf{S}_l^2 \right) \left( \omega \mathbf{S}_l^2 \right).$$

Finally, by substituting \((B.5), (B.11)\) and \((B.13)\) into \((B.7)\), one can have

$$\left( \lambda_\nu I + \mathbf{M}^0_{D_1} \right) \left[ \omega \Psi_l \right] + \sum_{l' \neq l}^{l_0} \left( \mathbf{M}_{D_1,l'}^0 \right) \left[ \omega \Psi_l \right] - \mathbf{M}_{D_1}^0 \left( \lambda_\nu I + \mathbf{M}^0_{D_1} \right)^{-1} \sum_{l' = 1}^{l_0} \left( \mathbf{M}_{D_1,l'}^0 \right) \left[ \omega \Psi_l \right]$$

$$= \frac{1}{\gamma_l - \varepsilon_s} \mathbf{H}_0^* + \mathcal{O} \left( \sum_{l' = 1}^{l_0} \left[ \omega \mathbf{S}_l^2 \right] \left[ \mathbf{B}_l^* \right] \right), \quad l = 1, 2, \ldots, l_0,$$
Similarly, one can prove that (B.17) is uniquely solvable (see Section 3.2). Define
\[ N_D := \text{diag} \left( M_{D_1}, M_{D_2}, \ldots, M_{D_0} \right) \left( \lambda_s I + M^{(0)}_s \right)^{-1} \]
\[
\begin{pmatrix}
M^0_{\Sigma, D_1} & M^0_{\Sigma, D_2} & \cdots & M^0_{\Sigma, D_0} \\
M^0_{\Sigma, D_1} & M^0_{\Sigma, D_2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
M^0_{\Sigma, D_1} & M^0_{\Sigma, D_2} & \cdots & M^0_{\Sigma, D_0}
\end{pmatrix},
\]
and
\[ L^\gamma_D := \begin{pmatrix}
\lambda_{\gamma_1} I & 0 & \cdots & 0 \\
0 & \lambda_{\gamma_2} I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{\gamma_l} I
\end{pmatrix} + M_D - N_D. \tag{B.19} \]

One can show that
\[ \omega \Psi_l = i \Psi_l^{(0)} + O(\omega), \quad l = 1, 2, \ldots, l_0, \tag{B.20} \]
where \( \Psi_l^{(0)} \) satisfies
\[ \Psi_l^{(0)} = \left( (L^\gamma_D)^{-1} \left[ \frac{(\nu_1 \times \tilde{H}_0)^T}{\gamma_1 - \varepsilon_s}, \frac{(\nu_2 \times \tilde{H}_0)^T}{\gamma_2 - \varepsilon_s}, \ldots, \frac{(\nu_{l_0} \times \tilde{H}_0)^T}{\gamma_{l_0} - \varepsilon_s} \right] \right) \cdot (e_l \otimes (1, 1, 1)^T). \tag{B.21} \]

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