A double-dimensional approach to formal category theory

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Abstract

Whereas formal category theory is classically considered within a 2-category, in this paper a double-dimensional approach is taken. More precisely we develop such theory within the setting of hypervirtual double categories, a notion extending that of virtual double category (also known as fc-multicategory) by adding cells with nullary target.

This paper starts by introducing the notion of hypervirtual double category, followed by describing its basic theory and its relation to other types of double category. After this the notion of ‘weak’ Kan extension within a hypervirtual double category is considered, together with three strengthenings. The first of these generalises Borceux-Kelly’s notion of Kan extension along enriched functors, the second one generalises Street’s notion of pointwise Kan extension in 2-categories, and the third is a combination of the other two; these stronger notions are compared. The notion of yoneda embedding is then considered in a hypervirtual double category, and compared to that of a good yoneda structure on a 2-category; the latter in the sense of Street-Walters and Weber. Conditions are given ensuring that a yoneda embedding \( y : A \to \hat{A} \) defines \( \hat{A} \) as the free small cocompletion of \( A \), in a suitable sense.

In the second half we consider formal category theory in the presence of algebraic structures. In detail: to a monad \( T \) on a hypervirtual double category \( K \) several hypervirtual double categories \( T\text{-Alg}_{(v,w)} \) of \( T \)-algebras are associated, one for each pair of types of weak coherence satisfied by the \( T \)-algebras and their morphisms respectively. This is followed by the study of the creation of, amongst others, left Kan extensions by the forgetful functors \( T\text{-Alg}_{(v,w)} \to K \). The main motivation of this paper is the description of conditions ensuring that yoneda embeddings in \( K \) lift along these forgetful functors, as well as ensuring that such lifted algebraic yoneda embeddings again define free small cocompletions, now in \( T\text{-Alg}_{(v,w)} \). As a first example we apply the previous to monoidal structures on categories, hence recovering Day convolution of presheaves and Im-Kelly’s result on free monoidal cocompletion, as well as obtaining a “monoidal Yoneda lemma”.

Motivation

Central to classical category theory is the Yoneda lemma which, for a locally small category \( A \), describes the position of the representable presheaves \( A(-,x) \) within...
the category \( \hat{A} \) of all presheaves \( A^{op} \to \text{Set} \). More precisely, in its “parametrised” form, the Yoneda lemma states that the yoneda embedding \( y : A \to \hat{A} : x \mapsto A(-, x) \) satisfies the following property: any profunctor \( J : A \to B \)—that is a functor \( J : A^{op} \times B \to \text{Set} \)—induces a functor \( J^\lambda : B \to \hat{A} \) equipped with bijections

\[
\hat{A}(y x, J^\lambda y) \cong J(x, y),
\]

that combine to form an isomorphism \( \hat{A}(y -, J^\lambda -) \cong J \) of profunctors. Indeed, \( J^\lambda \) can be defined as \( J y := J(-, y) \).

The main observation motivating this paper is that, given a monoidal structure \( \otimes \) on \( A \), the above “lifts” to a “monoidal Yoneda lemma” as follows. Recall that a monoidal structure on \( A \) induces such a structure \( \otimes \) on \( \hat{A} \), that is given by Day convolution \cite{Day70}

\[
(p \otimes q)(x) := \int_{u,v \in A} A(x, u \otimes v) \times pu \times qv,
\]

and with respect to which \( y : A \to \hat{A} \) forms a pseudomonoidal functor—that is, it comes equipped with coherent isomorphisms \( \hat{y} : y x \otimes y y \cong y(x \otimes y) \). Thus a pseudomonoidal functor, \( (y, \hat{y}) : (A, \otimes) \to (\hat{A}, \hat{\otimes}) \) satisfies the following monoidal variant of the property above: any lax monoidal profunctor \( J : A \to B \)—equipped with coherent morphisms \( J : J(x_1, y_1) \times J(x_2, y_2) \to J(x_1 \otimes x_2, y_1 \otimes y_2) \)—induces a lax monoidal functor \( J^\lambda : B \to \hat{A} \) equipped with an isomorphism of lax monoidal profunctors \( \hat{A}(y -, J^\lambda -) \cong J \). In detail: we can take \( J^\lambda \) to be as defined before, equipped with coherence morphisms \( J^\lambda : J^\lambda y_1 \hat{\otimes} J^\lambda y_2 \Rightarrow J^\lambda (y_1 \otimes y_2) \) that are induced by the composites

\[
A(x, u \otimes v) \times J(u, y_1) \times J(v, y_2) \xrightarrow{id \times J} A(x, u \otimes v) \times J(u \otimes v, y_1 \otimes y_2) \to J(x, y_1 \otimes y_2),
\]

where the unlabelled morphism is induced by the functoriality of \( J \) in \( A \).

The principal aim of this paper is to formalise the way in which the classical Yoneda lemma in the presence of a monoidal structure leads to the monoidal Yoneda lemma, as described above; thus allowing us to

(a) generalise the above to other types of algebra, such as double categories and representable topological spaces;

(b) characterise lax monoidal profunctors \( J \) whose induced lax monoidal functors \( J^\lambda \) are pseudomonoidal, and likewise for other types of algebraic morphism;

(c) recover classical results in the setting of monoidal categories, such as \( (y, \hat{y}) \) exhibiting \( (\hat{A}, \hat{\otimes}) \) as the “free monoidal cocompletion” of \( (A, \otimes) \) (see \cite{IK86}), and generalise these to other types of algebra.

**Hypervirtual double categories**

While the classical Yoneda lemma has been formalised in the setting of 2-categories, in the sense of the yoneda structures of Street and Walters \cite{SW78} (see also \cite{Web07}), it is unclear how the monoidal Yoneda lemma, as described above, fits into this framework. Indeed the problem is that it concerns two types of morphism between monoidal categories, both lax monoidal functors and lax monoidal profunctors, and it is not clear how to arrange both types into a 2-category equipped with a yoneda structure. In particular lax monoidal structures on representable profunctors \( C(f -, -) \), where \( f : A \to C \) is a functor, correspond to colax monoidal structures on \( f \) (see Theorem 6.15 below for the generalisation of this to other types of algebra).
The natural setting in which to consider two types of morphism is that of a (pseudo-)double category, where one type is regarded as being horizontal and the other as being vertical, while one considers square-shaped cells

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow^f & \swarrow & \downarrow^g \\
C & \xrightarrow{K} & D
\end{array}
\]

between them. Double categories however form a structure that is too strong for some of the types of algebra that we would like to consider, as it requires compositions for both types of morphism while, for example, the composite of “double profunctors” \( J: A \rightarrow B \) and \( H: B \rightarrow E \), between double categories \( A, B \) and \( E \), does in general not exist. While the latter is a consequence of, roughly speaking, the algebraic structures of \( J \) and \( H \) being incompatible in general, composites of profunctor-like morphisms may also fail to exist because of size issues. Indeed, recall from [FS95] that, for a locally small category \( A \), the category of presheaves \( \hat{A} \) need not be locally small. Consequently in describing the classical Yoneda lemma for locally small categories it is, on one hand, necessary to consider categories \( A, B, \ldots \) that might have large hom-sets while, on the other hand, the property satisfied by the yoneda embedding is stated in terms of ‘small’ profunctors—that is small-set-valued functors \( J: A^{op} \times B \rightarrow \text{Set} \)—between such categories; in general, such profunctors do not compose either. This classical situation is typical: for a description of most of the variations of the Yoneda lemma given in this paper, one considers a double-dimensional setting whose objects \( A, B, \ldots \) are “large in size” while the size of the horizontal morphisms \( J: A \rightarrow B \) between them is “small”.

In view of the previous we, instead of double categories, consider a weaker notion as the right notion for our double-dimensional approach to formal category theory. This weaker notion extends slightly that of ‘virtual double category’ [CS10] which, analogous to the generalisation of monoidal category to ‘multicategory’, does not require a horizontal composition but, instead of the square-shaped cells above, has cells of the form below, with a sequence of horizontal morphisms as horizontal source. Introduced by Burroni [Bur71] under the name ‘fc-catégorie’, where fc denotes the ‘free category’-monad, virtual double categories have also been called ‘fc-multicategories’ [Lei04].

\[
\begin{array}{ccc}
A_0 & \xrightarrow{J_1} & A_1 & \cdots & A_n & \xrightarrow{J_n} & A_n \\
\downarrow^f & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow^g \\
C & \xrightarrow{K} & D
\end{array}
\]

Both settings mentioned above can be considered as a virtual double category: there is a virtual double category with (possibly large) categories as objects, functors as vertical morphisms and small profunctors as horizontal morphisms, and likewise one with (possibly large) double categories, ‘double functors’ and ‘small’ double profunctors. These virtual double categories however miss an ingredient crucial to the theory of (double) categories: they do not contain transformations \( f \Rightarrow g \) between (double) functors \( f \) and \( g: A \rightarrow C \) into (double) categories \( C \) with large hom-sets. Indeed, for such transformations to be represented by cells of the form above we need the small (double) profunctor \( K \) to consist of the hom-sets of \( C \), which is not possible if some of them are large.

In order to remove the inadequacy described above, we extend to the notion of virtual double category to also contain cells with empty horizontal target, as shown
below, so that transformations $f \Rightarrow g$, as described above, are represented by cells with both empty horizontal source and target. We will call this extended notion ‘hypervirtual double category’; a detailed definition is given in the first section below.

\[
\begin{array}{c}
A_0 \xrightarrow{f} A_1 \xrightarrow{J_1} \cdots \xrightarrow{J_{n-1}} A_n \xrightarrow{g} C
\end{array}
\]

Overview

We start, in Section 1, by introducing the notion of hypervirtual double category. Every hypervirtual double category $\mathcal{K}$ contains both a virtual double category $U(\mathcal{K})$, consisting of the cells in $\mathcal{K}$ that have nonempty horizontal target, as well as a ‘vertical 2-category’ $V(\mathcal{K})$, consisting of cells with both horizontal target and source empty. We obtain examples of hypervirtual double categories by considering monoids and bimodules in virtual double categories, just like one does in monoidal categories, followed by restricting the size the bimodules. The archetypal example of a hypervirtual double category, that of small profunctors between large categories, is thus obtained by considering monoids and bimodules in the pseudo double category of spans of large sets, followed by restricting to small-set-valued profunctors. Given a ‘universe enlargement’ $V \subset V'$ of monoidal categories, we consider the enriched variant of the previous, resulting in the hypervirtual double category of $V$-enriched profunctors between $V'$-enriched categories. The description of the 2-category of hypervirtual double categories—in particular its equivalences—closes the first section.

In Section 2 and Section 3 the basic theory of hypervirtual double categories is introduced which, for a large part, consists of a straightforward generalisation of that for virtual double categories. Invaluable to both theories are the notions of restriction and composition of horizontal morphisms: these generalise respectively the restriction $K \circ (f^{op} \times g)$ of a profunctor $K$, along functors $f$ and $g$, as well as the composition of profunctors. As a special cases of restriction, the notions of companion and conjoint generalise that of the (op-)representable profunctors $C(f -, -)$ and $C(-, f -)$ induced by a functor $f: A \to C$, and at the end of Section 2 we characterise the full sub-hypervirtual double category of $\mathcal{K}$, obtained by restricting to representable horizontal morphisms, in terms of its vertical 2-category $V(\mathcal{K})$. As part of the theory of composition of horizontal morphisms in Section 3 horizontal units are considered: the small unit profunctor of a category $A$ exists, for example, precisely if $A$ has locally small hom-sets. Finishing the third section is a theorem proving, in the presence of all horizontal units, the equivalence of the notions of hypervirtual double category and virtual double category.

Having introduced the basic theory of hypervirtual double categories we begin studying ‘formal category theory’ within such double categories. We start Section 4 by rewriting the classical notion of left Kan extension in the vertical 2-category $V(\mathcal{K})$, in terms of the companions in $\mathcal{K}$, leading to a notion of ‘weak’ left Kan extension in $\mathcal{K}$. We then consider three strengthenings of this notion: the first of these can be thought of as generalising the notion of ‘Kan extension along enriched functors’, in the sense of e.g. [Kel82]; the second as generalising the notion of ‘pointwise Kan extension in a 2-category’, in the sense of Street [Str74]; while the third is a combination of the previous two. Besides studying the basic theory of these notions we will compare them among each other, as well as make precise their relation to the aforementioned classical notions.

Being one of the main goals of this paper, Section 5 introduces the notion
of yoneda embedding in a hypervirtual double category as a vertical morphism $y: A \rightarrow \hat{A}$ satisfying two axioms. As is the case for the classical yoneda embedding, the first of these asks $y$ to be dense, while the second is the ‘yoneda axiom’: this formally captures the fact that, in the classical setting, any small profunctor $J: A \rightarrow B$ induces a functor $J^\lambda: B \rightarrow \hat{A}$, as was described above as part of the motivation. These two conditions are closely related to the axioms satisfied by morphisms that make up a ‘good yoneda structure’ on a 2-category, in the sense of Weber [Web07]. Slightly strengthening the original notion of ‘yoneda structure’, introduced by Street and Walters in [SW78], a good yoneda structure consists of a collection of yoneda embeddings that satisfy a ‘yoneda axiom’ with respect to a specified collection of ‘admissible’ morphisms (informally these are to be thought of as ‘small in size’). In contrast, our position of regarding all horizontal morphisms as being small enables us to consider just a single yoneda embedding. We make precise the relation between the yoneda embeddings in a hypervirtual double category $K$ and the existence of a good yoneda structure on the vertical 2-category that it $V(K)$ contains. Given a yoneda embedding $y: A \rightarrow \hat{A}$, the main result of Section 5 gives conditions ensuring that it defines $\hat{A}$ as the ‘free small cocompletion’ of $A$, in a suitable sense.

In Section 6 we consider formal category theory within hypervirtual double categories in the presence of ‘algebraic structures’; the archetypal example being that of monoidal structures on categories. Like in 2-dimensional category theory, algebraic structures in a hypervirtual double category $K$ are defined by monads on $K$; in fact, any monad $T$ on $K$ induces a strict 2-monad $V(T)$ on the vertical 2-category $V(K)$. To each monad $T$ we will associate several hypervirtual double categories $T Alg_{(v, w)}$ of weak algebras of $T$, one for each pair of weaknesses $v, w \in \{\text{pseudo, lax, colax}\}$ specifying the type of coherence satisfied by the algebra structures and algebraic morphisms. The vertical parts $V(T Alg_{(v, w)})$ of these hypervirtual double categories are defined to coincide with the 2-categories $V(T) Alg_{(v, w)}$ of weak $V(T)$-algebras in the classical sense, while their notion of horizontal morphism generalises that of ‘horizontal $T$-morphism’ introduced by Grandis and Paré in the setting of pseudo double categories [GP04]. The theorem closing this section characterises (op-)representable horizontal $T$-morphisms.

The remaining sections, Section 7 and Section 8, are devoted to describing the creativity of the forgetful functors $U: T Alg_{(v, w)} \rightarrow K$. Generalising the main theorem of [Kou15a] to the setting of hypervirtual double categories, the main result of Section 7 describes the creation of algebraic left Kan extensions by $U$; it can be regarded as extending Kelly’s classical result on ‘doctrinal adjunctions’ [Kel74] to left Kan extensions. As the main result of this paper, in Section 8 we describe the lifting of algebraic yoneda embeddings along $U$, followed by making precise its consequences, as listed at the end of the motivation above. We treat in detail the lifting of a monoidal Yoneda embedding $(y, \bar{y}): (A, \otimes) \rightarrow (\hat{A}, \hat{\otimes})$, as described in the motivation, and recover Im and Kelly’s result [IK86] showing that $(y, \bar{y})$ defines $(\hat{A}, \hat{\otimes})$ as the ‘free monoidal cocompletion’ of $(A, \otimes)$.

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1 Hypervirtual double categories

1.1 Definition of hypervirtual double category

We start by introducing the notion of hypervirtual double category. In doing so we use the well-known notion of a directed graph, by which we mean a parallel pair of functions $A = (A_1 \xleftarrow{s} A_0)$ from a class $A_1$ of edges to a class $A_0$ of vertices. An edge $e$ with $(s, t)(e) = (x, y)$ is denoted $x \xrightarrow{e} y$; the vertices $x$ and $y$ are called its
source and target. Remember that any graph $A$ generates a free category $\mathbf{fc} A$, whose underlying graph has the same vertices as $A$ while its edges $x \to y$ are (possibly empty) paths $\mathbf{fc} (x = x_0 \xrightarrow{e_0} x_1 \xrightarrow{e_1} \cdots \xrightarrow{e_n} x_n = y)$ of edges in $A$; we write $|\mathbf{fc} A| := n$ for their lengths. Composition in $\mathbf{fc} A$ is given by concatenation $(\mathbf{fc} f, \mathbf{fc} g) \mapsto \mathbf{fc} f \circ \mathbf{fc} g$ of paths, while the empty path $(x)$ forms the identity at $x \in A_0$. We denote by $\mathbf{bl}_A \subseteq \mathbf{fc} A$ the subgraph consisting of all paths of length $\leq 1$, which we think of as obtained from $A$ by “freely adjoining base loops”.

Given an integer $n \geq 1$ we write $n' := n - 1$.

**Definition 1.1.** A hypervirtual double category $\mathcal{K}$ consists of

- a class $\mathcal{K}_0$ of objects $A, B, \ldots$

- a category $\mathcal{K}_v$ with $\mathcal{K}_{v0} = \mathcal{K}_0$, whose morphisms $f : A \to C, g : B \to D, \ldots$ are called vertical morphisms;

- a directed graph $\mathcal{K}_h$ with $\mathcal{K}_{h0} = \mathcal{K}_0$, whose edges are called horizontal morphisms and denoted by slashed arrows $J : A \Rightarrow B, K : C \Rightarrow D, \ldots$;

- a class of cells $\phi, \psi, \ldots$ that are of the form

\begin{equation}
\begin{aligned}
A_0 \xrightarrow{f} A_n \\
\downarrow \phi \downarrow g \\
C \xrightarrow{K} D
\end{aligned}
\end{equation}

where $f$ and $K$ are paths in $\mathcal{K}_h$ with $|K| \leq 1$;

- for any path of cells

\begin{equation}
\begin{aligned}
A_{10} \xrightarrow{J_1} A_{1m_1} \xrightarrow{K_1} A_{2m_2} \\
\quad \quad \quad \downarrow \phi_1 \downarrow f_1 \downarrow \phi_2 \downarrow f_2 \quad \cdots \\
C_0 \xrightarrow{K_0} C_1 \xrightarrow{K_1} C_2 \\
\quad \quad \quad \downarrow h_1 \downarrow f_1 \downarrow \phi_n \downarrow f_n \\
A_{n'm_n} \xrightarrow{J_n} A_{nm_n}
\end{aligned}
\end{equation}

of length $n \geq 1$ and a cell $\psi$ as on the left below, a vertical composite as on the right;

\begin{equation}
\begin{aligned}
C_0 \xrightarrow{K_0} C_1 \xrightarrow{K_1} C_2 \xrightarrow{K_n} C_n \\
\downarrow h \downarrow \psi \downarrow k \\
E \xrightarrow{L} F
\end{aligned} \quad \quad \quad \quad \quad \quad \quad
\begin{aligned}
A_{10} \xrightarrow{L_1} A_{1m_1} \xrightarrow{L_2} A_{2m_2} \xrightarrow{L_n} A_{nm_n} \\
\quad \downarrow h_0 \downarrow f_0 \downarrow \psi \circ (\phi_1, \ldots, \phi_n) \downarrow k \circ f_n \\
E \xrightarrow{K} F
\end{aligned}
\end{equation}

- horizontal identity cells as on the left below, one for each $J : A \Rightarrow B$;

\begin{equation}
\begin{aligned}
A \xrightarrow{(J)} B \\
\downarrow \text{id}_A \downarrow \text{id}_B \\
A \xrightarrow{(J)} B
\end{aligned}
\end{equation}

- vertical identity cells as on the right above, one for each $f : A \Rightarrow C$, that are preserved by vertical composition: $\text{id}_h \circ (\text{id}_f) = \text{id}_{h \circ f}$; we write $\text{id}_A := \text{id}_{\text{id}_A}$.
The composition above is required to satisfy the \textit{associativity axiom}
\[
\chi \circ (\psi_1 \circ (\phi_1, \ldots, \phi_{1m_1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nm_n})) = \chi \circ (\psi_1, \ldots, \psi_n) \circ (\phi_{11}, \ldots, \phi_{nm_n}),
\]
whenever the left-hand side makes sense, as well as the \textit{unit axioms}
\[
\begin{align*}
\id_C \circ (\phi) &= \phi,
\id_K \circ (\phi) &= \phi, \\
\phi \circ (\id_A) &= \phi, \\
\phi \circ (\id_{J_1}, \ldots, \id_{J_n}) &= \phi.
\end{align*}
\]
and
\[
\psi \circ (\phi_1, \ldots, \phi_i, \id_{J_1}, \ldots, \id_{J_{i+1}}, \ldots, \phi_n) = \psi \circ (\phi_1, \ldots, \phi_n)
\]
whenever these make sense and where, in the last axiom, \(0 \leq i \leq n\).

For a cell \(\phi\) as in (1) above we call the vertical morphisms \(f\) and \(g\) its \textit{vertical source} and \textit{target} respectively, and call the path of horizontal morphisms \(J = (J_1, \ldots, J_n)\) its \textit{horizontal source}, while we call \(K\) its \textit{horizontal target}. We write \(|\phi| := (|J|, |K|)\) for the \textit{arity} of \(\phi\). A \((n,1)\)-ary cell will be called \textit{unary}, \((n,0)\)-ary cells \textit{nullary} and \((0,0)\)-ary cells \textit{vertical}.

When writing down paths \((J_1, \ldots, J_n)\) of length \(n \leq 1\) we will often leave out parentheses and simply write \(A_0 := (A_0)\) or \(J_1 := (A_0 \stackrel{J_1}{\to} A_1)\). Likewise in the composition of cells: \(\psi \circ \phi_1 := \psi \circ (\phi_1)\). We will often denote unary cells simply by \(\phi: (J_0, \ldots, J_n) \Rightarrow K\), and nullary cells by \(\psi: (J_0, \ldots, J_n) \Rightarrow C\), leaving out their vertical source and target. When drawing compositions of cells it is often helpful to depict them in full detail and, in the case of nullary cells, draw their horizontal target as a single object, as shown below.

\[
\begin{array}{cccc}
\begin{array}{cccccc}
A_0 & \stackrel{J_1}{\to} & A_1 & \cdots & A_n & \stackrel{J_n}{\to} & A_n \\
\downarrow f & & & & & \downarrow g \\
C & \stackrel{\psi}{\to} & C & & & \downarrow \phi \\
\end{array} & \quad & \begin{array}{cccccc}
A_0 & \stackrel{J_1}{\to} & A_1 & \cdots & A_n & \stackrel{J_n}{\to} & A_n \\
\downarrow f & & & & & \downarrow g \\
C & \stackrel{\psi}{\to} & C & \leftrightsquigarrow & & \downarrow \phi \\
\end{array}
\end{array}
\]

A cell with identities as vertical source and target is called \textit{horizontal}. A horizontal cell \(\phi: J \Rightarrow K\) with unary horizontal source is called \textit{invertible} if there exists a horizontal cell \(\psi: K \Rightarrow J\) such that \(\phi \circ \psi = \id_K\) and \(\psi \circ \phi = \id_J\); in that case we write \(\phi^{-1} := \psi\). When drawing diagrams we shall often depict identity morphisms by the equal sign (=), while we leave identity cells empty. Also, because composition of cells is associative, we will usually leave out bracketings when writing down composites.

For convenience we use the ‘whisker’ notation from 2-category theory and define
\[
h \circ (\phi_1, \ldots, \phi_n) := \id_h \circ (\phi_1, \ldots, \phi_n) \quad \text{and} \quad \psi \circ f := \psi \circ \id_f,
\]
whenever the right-hand side makes sense. Moreover, for any path
\[
\begin{array}{cccccc}
A_0 & \stackrel{J_1}{\to} & A_1 & \cdots & A_n & \stackrel{J_n}{\to} & A_n \\
\downarrow f & & & & & \downarrow g \\
C & \stackrel{\psi}{\to} & C & \leftrightsquigarrow & & \downarrow \phi \\
\end{array}
\]
with \(|K| + |L| \leq 1\) we define the horizontal composite \(\phi \circ \psi: J \leftrightsquigarrow H \Rightarrow K \leftrightsquigarrow L\) by
\[
\phi \circ \psi := \id_{K \leftrightsquigarrow L} \circ (\phi, \psi),
\]
where \(\id_{K \leftrightsquigarrow L}\) is to be interpreted as the identity \(\id_C: C \to C\) in case \(K \leftrightsquigarrow L = (C)\). The following lemma follows easily from the associativity of the vertical composition in \(K\).
Lemma 1.2. Horizontal composition \((\phi, \psi) \mapsto \phi \odot \psi\), as defined above, satisfies the associativity and unit axioms

\[
(\phi \odot \psi) \odot \chi = \phi \odot (\psi \odot \chi), \quad (\text{id}_f \odot \phi) = \phi \quad \text{and} \quad (\phi \odot \text{id}_g) = \phi
\]

whenever these make sense. Moreover, horizontal and vertical composition satisfy the interchange axioms

\[
(\psi \circ (\phi_1, \ldots, \phi_n)) \odot (\chi \circ (\xi_1, \ldots, \xi_m)) = (\psi \circ \chi) \circ (\phi_1, \ldots, \phi_n, \xi_1, \ldots, \xi_m)
\]

and

\[
\psi \circ (\phi_1, \ldots, (\phi_{i'} \odot \phi_i), \ldots, \phi_n) = \psi \circ (\phi_1, \ldots, \phi_{i'}, \phi_i, \ldots, \phi_n)
\]

whenever they make sense.

Notice that by removing the nullary cells from Definition 1.1 (including the vertical identity cells) we recover the classical notion of virtual double category, in the sense of [CS10] or Section 5.1 of [Lei04], where it is called fc-multicategory. Virtual double categories where originally introduced by Burroni [Bur71], who called them ‘fc-catégories’. Likewise if instead we remove all horizontal morphisms, so that the only remaining cells are the vertical ones then, using both compositions \(\circ\) and \(\odot\), we recover the notion of 2-category. We conclude that every hypervirtual double category \(K\) contains a virtual double category \(U(K)\) consisting of its objects, vertical and horizontal morphisms, as well as unary cells. Likewise \(K\) contains a vertical 2-category \(V(K)\), consisting of its objects, vertical morphisms and vertical cells.

Every hypervirtual double category has a horizontal dual as follows.

Definition 1.3. Let \(K\) be a hypervirtual double category. The horizontal dual of \(K\) is the hypervirtual double category \(K^{co}\) that has the same objects and vertical morphisms, that has a horizontal morphism \(J^{co}: A \to B\) for each \(J: B \to A\) in \(K\), and a cell \(\phi^{co}\) as on the left below for each cell \(\phi\) in \(K\) as on the right.

\[
\begin{array}{cccc}
A_0 & \xrightarrow{J_0} & A_1 & \cdots & A_n & \xrightarrow{J_n} & A_n' & \xrightarrow{J_n'} & A_0 \\
 f & \downarrow & & & & \downarrow \phi^{co} & \downarrow g & \downarrow \psi & \downarrow \phi \\
 C & \xrightarrow{\phi^{co}} & D & & & & & & \xrightarrow{\phi^{co}} & K & \xrightarrow{J} & C
\end{array}
\]

Identities and compositions in \(K^{co}\) are induced by those of \(K\):

\[
\text{id}_{J^{co}} := (\text{id}_f)^{co}, \quad \text{id}_f := (\text{id}_f)^{co} \quad \text{and} \quad \psi^{co} \circ (\phi_1^{co}, \ldots, \phi_n^{co}) := (\psi \circ (\phi_n, \ldots, \phi_1))^{co}.
\]

1.2 Monoids and bimodules

Our main source of hypervirtual double categories is virtual double categories, as we will explain in this subsection. Briefly, given a virtual double category \(K\) we will consider the hypervirtual double category \(\text{Mod}(K)\) of ‘monoids’ and ‘bimodules’ in \(K\), the latter in the sense of Section 5.3 of [Lei04]; see also Section 2 of [CS10]. Often we will then consider a sub-hypervirtual double category of \(\text{Mod}(K)\) by “restricting the size of bimodules”. For instance, instead of considering ‘large profunctors’ between large categories, it is preferable to consider ‘small profunctors’ between large categories.

Definition 1.4. Let \(K\) be a virtual double category.
- A monoid $A$ in $\mathcal{K}$ is a quadruple $A = (A, \alpha, \bar{\alpha}, \tilde{\alpha})$ consisting of a horizontal morphism $\alpha: A \rightarrow A$ in $\mathcal{K}$ equipped with multiplication and unit cells

$$
\begin{array}{ccc}
A & \overset{\alpha}{\longrightarrow} & A \\
\downarrow \bar{\alpha} & & \downarrow \bar{\alpha} \\
A & \overset{\alpha}{\longrightarrow} & A
\end{array}
$$

and

$$
\begin{array}{ccc}
A & \overset{\alpha}{\longrightarrow} & A \\
\downarrow \tilde{\alpha} & & \downarrow \tilde{\alpha} \\
A & \overset{\alpha}{\longrightarrow} & A
\end{array}
$$

that satisfy the associativity and unit axioms $\bar{\alpha} \circ (\bar{\alpha}, \text{id}_A) = \bar{\alpha} \circ (\text{id}_A, \bar{\alpha})$ and $\tilde{\alpha} \circ (\tilde{\alpha}, \text{id}_A) = \text{id}_A = \tilde{\alpha} \circ (\text{id}_A, \tilde{\alpha})$.

- A morphism $A \rightarrow C$ of monoids is a vertical morphism $f: A \rightarrow C$ in $\mathcal{K}$ that is equipped with a cell

$$
\begin{array}{ccc}
A & \overset{\alpha}{\longrightarrow} & A \\
\downarrow f & & \downarrow f \\
C & \overset{\gamma}{\longrightarrow} & C
\end{array}
$$

that satisfies the associativity and unit axioms $\gamma \circ (f, f) = f \circ \bar{\alpha}$ and $\gamma \circ f = f \circ \bar{\alpha}$.

- A bimodule $A \rightarrow B$ between monoids is a horizontal morphism $J: A \rightarrow B$ in $\mathcal{K}$ that is equipped with left and right action cells

$$
\begin{array}{ccc}
A & \overset{\alpha}{\longrightarrow} & A \\
\downarrow \lambda & & \downarrow \lambda \\
A & \overset{J}{\longrightarrow} & B
\end{array}
$$

and

$$
\begin{array}{ccc}
A & \overset{J}{\longrightarrow} & B \\
\downarrow \rho & & \downarrow \rho \\
A & \overset{\beta}{\longrightarrow} & B
\end{array}
$$

that satisfy the usual associativity, unit and compatibility axioms for bimodules:

$$
\begin{align*}
\lambda \circ (\bar{\alpha}, \text{id}_J) &= \lambda \circ (\text{id}_A, \lambda); & \rho \circ (\text{id}_J, \bar{\beta}) &= \rho \circ (\rho, \text{id}_B); \\
\lambda \circ (\tilde{\alpha}, \text{id}_J) &= \text{id}_J = \rho \circ (\text{id}_J, \tilde{\beta}); & \rho \circ (\lambda, \text{id}_B) &= \lambda \circ (\text{id}_A, \rho).
\end{align*}
$$

- A cell

$$
\begin{array}{ccc}
A_0 & \overset{J_1}{\longrightarrow} & A_1 & \cdots & A_n & \overset{J_n}{\longrightarrow} & A_n \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
C & \overset{k}{\longrightarrow} & D
\end{array}
$$

of bimodules, where $n \geq 1$, is given by a cell $\phi$ in $\mathcal{K}$ between the underlying morphisms, satisfying the external equivariance axioms

$$
\begin{align*}
\phi \circ (\lambda, \text{id}_{J_2}, \ldots, \text{id}_{J_n}) &= \lambda \circ (f, \phi) \\
\phi \circ (\text{id}_{J_1}, \ldots, \text{id}_{J_{n-1}}, \rho) &= \rho \circ (\phi, \bar{g})
\end{align*}
$$

and the internal equivariance axioms

$$
\begin{align*}
\phi \circ (\text{id}_{J_1}, \ldots, \text{id}_{J_{n-1}}, \rho, \text{id}_{J_1}, \ldots, \text{id}_{J_{n-1}}, \ldots, \text{id}_{J_n}) \\
= \phi \circ (\text{id}_{J_1}, \ldots, \text{id}_{J_{n-1}}, \text{id}_{J_1}, \ldots, \text{id}_{J_{n-1}}, \ldots, \text{id}_{J_n})
\end{align*}
$$

for $2 \leq i \leq n$. 

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of bimodules is given by a cell \( \phi \) in \( K \) between the underlying morphisms, satisfying the external equivariance axiom 
\[ \lambda \circ (\bar{f}, \phi) = \rho \circ (\phi, \bar{g}) . \]

For the next proposition notice that any module \( C = (C, \gamma, \bar{\gamma}, \tilde{\gamma}) \) in a virtual double category \( K \) induces a bimodule \( \gamma : C \Rightarrow C \), both whose actions are given by multiplication \( \bar{\gamma} : (\gamma, \gamma) \Rightarrow \gamma \).

**Proposition 1.5.** Given a virtual double category \( K \) consider to each cell \( \phi \) of bimodules in \( K \), as on the left below, a new unary cell \( \bar{\phi} \) as on the right, that is of the same form, and to each cell \( \psi \) of bimodules as on the left, with horizontal target \( \gamma \) as described above, a new nullary cell \( \bar{\psi} \) as on the right.

Monoids in \( K \), the morphisms and bimodules between them, together with the cells \( \bar{\phi} \) and \( \bar{\psi} \) above, form a hypervirtual double category \( \text{Mod}(K) \). Composition of cells in \( \text{Mod}(K) \) is given by

\[ \bar{\psi} \circ (\bar{\phi}_1, \ldots, \bar{\phi}_n) := (\psi' \circ (\phi_1, \ldots, \phi_n)) \]

where \( \psi' \) is any cell of the right form in \( K \) that is obtained by composing \( \psi \) with actions, on its horizontal sources and/or target, of the horizontal targets \( \gamma_i \) of the cells \( \phi_i \) that are nullary. The identity cells in \( \text{Mod}(K) \), for bimodules \( J : A \Rightarrow B \) and morphisms \( f : A \to C \) of monoids, are given by

\[ \text{id}_J := (\text{id}_J) \quad \text{and} \quad \text{id}_f := (\bar{\gamma} \circ f) . \]

That the composition \( \bar{\psi} \circ (\bar{\phi}_1, \ldots, \bar{\phi}_n) \) above does not depend on the choice of \( \psi' \) follows from the equivariance axioms for \( \psi \) (Definition 1.4). It is straightforward to show that these axioms, together with associativity and unitality of composition in \( K \), imply the associativity and unit axioms for the composition in \( \text{Mod}(K) \). Later on we will see that \( K \to \text{Mod}(K) \) forms the object-function of the composition of a pair of 2-functors, the first described in Proposition 3.8 and the second in Theorem 3.25.

To give examples, we assume given a category \( \text{Set}' \) of large sets and functions between them, as well as a full subcategory \( \text{Set} \subset \text{Set}' \) of small sets and their functions, such that the morphisms in \( \text{Set} \) form a large set in \( \text{Set}' \).

**Example 1.6.** Let \( V = (V, \otimes, I) \) be a monoidal category. The virtual double category \( V\text{-Mat} \) of \( V \)-matrices has large sets and functions as objects and vertical morphisms, while a horizontal morphism \( J : A \Rightarrow B \) is a \( V \)-matrix, given by a family \( J(x, y) \) of...
$\mathcal{V}$-objects indexed by pairs $(x, y) \in A \times B$. A cell $\phi$ in $\mathcal{V}$-$\text{Mat}$, of the form as above, consists of a family of $\mathcal{V}$-maps

$$\phi_{(x_0, \ldots, x_n)} : J_1(x_0, x_1) \otimes \cdots \otimes J_n(x_n, x_n) \to K(f_{x_0}, g_{x_n})$$

indexed by sequences $(x_0, \ldots, x_n) \in A_0 \times \cdots \times A_n$, where the tensor product is to be interpreted as the unit $I$ in case $n = 0$.

The hypervirtual double category $\mathcal{V}$-$\text{Prof} := \text{Mod}(\mathcal{V}$-$\text{Mat}$) of monoids and bimodules in $\mathcal{V}$-$\text{Mat}$ is that of $\mathcal{V}$-enriched categories, $\mathcal{V}$-functors, $\mathcal{V}$-profunctors and $\mathcal{V}$-natural transformations. In some more detail, a horizontal morphism $J : A \to B$ in $\mathcal{V}$-$\text{Prof}$, between $\mathcal{V}$-categories $A$ and $B$, is a $\mathcal{V}$-profunctor in the sense of Section 7 of [DS97]: it consists of a family of $\mathcal{V}$-objects $J(x, y)$, indexed by pairs of objects $x \in A$ and $y \in B$, that is equipped with associative and unital actions

$$\lambda : A(x_1, x_2) \otimes J(x_2, y) \to J(x_1, y) \quad \text{and} \quad \rho : J(x, y_1) \otimes B(y_1, y_2) \to J(x, y_2)$$

satisfying the usual compatibility axiom for bimodules. If $\mathcal{V}$ is closed symmetric monoidal, so that it can be considered as enriched over itself, then $\mathcal{V}$-functors $J : A \Rightarrow B$ can be identified with $\mathcal{V}$-functors of the form $J : \text{Mod}(A_{op} \times B) \to \mathcal{V}$.

A vertical cell $\phi : f \Rightarrow g$ in $\mathcal{V}$-$\text{Prof}$, between $\mathcal{V}$-functors $f$ and $g : A \to C$, is a $\mathcal{V}$-natural transformation $f \Rightarrow g$ in the usual sense; see for instance Section 1.2 of [Kel82]. We conclude that the vertical 2-category $\mathcal{V}$-$\text{Prof}$, that is contained in $\mathcal{V}$-$\text{Prof}$, equals the 2-category $\mathcal{V}$-$\text{Cat}$ of $\mathcal{V}$-categories, $\mathcal{V}$-functors and the $\mathcal{V}$-natural transformations between them.

Taking $\mathcal{V} = \text{Set}$ we recover the archetypal virtual double category $\text{Set}$-$\text{Prof}$ of locally small categories, functors, (small) profunctors $J : A_{op} \times B \to \text{Set}$ and transformations. Analogously $\text{Set'}$-$\text{Prof}$ is the virtual double category of categories (with possibly large hom-sets), functors, large profunctors $J : A_{op} \times B \to \text{Set'}$ and transformations.

**Example 1.7.** Let $\mathcal{E}$ be a category with pullbacks. The virtual double category $\text{Span}(\mathcal{E})$ of spans in $\mathcal{E}$ has as objects and vertical morphisms those of $\mathcal{E}$, while its horizontal morphisms $J : A \Rightarrow B$ are spans $A \leftarrow J \rightarrow B$ in $\mathcal{E}$. A cell $\phi$ in $\text{Span}(\mathcal{E})$, of the form as above, is a morphism

$$\phi : J_1 \times_A \cdots \times_A J_n \to K$$

in $\mathcal{E}$ lying over $f$ and $g$. The virtual double category $\text{Prof}(\mathcal{E}) := \text{Mod}(\text{Span}(\mathcal{E}))$ of monoids and bimodules in $\text{Span}(\mathcal{E})$ is that of internal categories, functors, profunctors and transformations in $\mathcal{E}$. Analogous to previous example we have $\mathcal{V}$-$\text{Prof}(\mathcal{E}) = \text{Cat}(\mathcal{E})$, that is the 2-category of internal categories, functors and transformations in $\mathcal{E}$; the latter in the usual sense of e.g. Section 1 of [Str74].

The following example forms the main motivation for extending the notion of virtual double category to that of hypervirtual double category.

**Example 1.8.** Taking $\mathcal{V} = \text{Set'}$ in Example 1.6 we write $(\text{Set, Set'})$-$\text{Prof} \subset \text{Set'}$-$\text{Prof}$ for the sub-hypervirtual double category that is obtained by restricting to small profunctors. In detail: $(\text{Set, Set'})$-$\text{Prof}$ consists of all (possibly large) categories and functors, only those profunctors $J : A \Rightarrow B$ with small sets $J(x, y)$ for all $(x, y) \in A \times B$, and all transformations between such profunctors (including the nullary ones).

Thus we have an ascending chain of hypervirtual double categories

$$\text{Set}$-$\text{Prof} \subset (\text{Set, Set'})$-$\text{Prof} \subset \text{Set'}$-$\text{Prof},$$

and we take the view that the classical theory of locally small categories is best considered in $(\text{Set, Set'})$-$\text{Prof}$. As motivation for this view, on one hand remember from
that, for a locally small category $A$, the category $\text{Set}^{\text{A}^{\text{op}}}$ of small presheaves on $A$ is locally small too precisely if $A$ is essentially small, so that small presheaves on $A$ need not form an object in $\text{Set}-\text{Prof}$, whereas they do in $(\text{Set},\text{Set}')-\text{Prof}$. On the other hand, writing $y: A \to \text{Set}^{\text{A}^{\text{op}}}$ for the yoneda embedding, Yoneda’s lemma supplies, for each small profunctor $J: A \leftrightarrow B$ in $(\text{Set},\text{Set}')-\text{Prof}$, a functor $J^*: B \to \text{Set}^{\text{A}^{\text{op}}}$ equipped with a natural isomorphism of small profunctors $J \cong \text{Set}^{\text{A}^{\text{op}}}(y, J^*)$; of course such a $J^*$ does not exist for the properly large profunctors $J$ that exist in $\text{Set}'-\text{Prof}$.

Finally, for an advantage of working in the hypervirtual double category $(\text{Set},\text{Set}')-\text{Prof}$ rather than the virtual double category $U(\text{Set},\text{Set}')-\text{Prof}$ that it contains notice that, for any two functors $f$ and $g: A \to C$ into a large category $C$, the natural transformations $\phi: f \Rightarrow g$ can be considered in the former but not the latter. Indeed in $(\text{Set},\text{Set}')-\text{Prof}$ they form the vertical cells $\phi: f \Rightarrow g$, which are removed from $U(\text{Set},\text{Set}')-\text{Prof})$. On the other hand such natural transformations correspond to cells in $\text{Set}'-\text{Prof}$ of the form below, where $I_C$ is the ‘unit profunctor’ given by the large hom-sets $I_C(x,y) = C(x,y)$ (see Example 3.3), but these do not exist in either $(\text{Set},\text{Set}')-\text{Prof}$ or $U(\text{Set},\text{Set}')-\text{Prof})$.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi & & \downarrow \psi \\
C & \xrightarrow{g} & C
\end{array}
\]

As a variation on $(\text{Set},\text{Set}')-\text{Prof}$ the following example describes the hypervirtual double category $(\text{Set},\text{Set}')-\text{Prof}^S$ of small profunctors indexed by a category $S$. The case where $S$ is the category $G_1 = (1 \rightrightarrows 0)$ will be important to us, as the ‘free strict double category’-monad is defined on $(\text{Set},\text{Set}')-\text{Prof}^{G_1}$; see Example 6.3.

Example 1.9. Let $S$ be a small category. We first describe the hypervirtual double category $\text{Set}'-\text{Prof}^S := \text{Prof}(\text{Set}^S)$ of profunctors internal to the functor category $\text{Set}^S$ (Example 1.7). Its objects are large $S$-indexed categories, i.e. functors $A: S \to \text{Cat}(\text{Set}')$, while its vertical morphisms $f: A \to C$ are the $S$-indexed functors between them, that is natural transformations $A \Rightarrow C$. For $s \in S$ we write $A_s := A(s)$; likewise $A_u := A(u): A_s \to A_t$ for each map $u: s \to t \in S$. A horizontal morphism $J: A \Rightarrow B$ in $\text{Set}'-\text{Prof}^S$ is a large $S$-indexed profunctor between $A$ and $B$, consisting of a family of profunctors $J_s: A_s \Rightarrow B_s$, one for each $s \in S$, that is equipped with $(1,1)$-ary cells $J_{st}$ in $\text{Set}'-\text{Prof}$, as on the left below, one for each $u: s \to t \in S$. The assignment $u \mapsto J_{st}$ is required to be natural, that is $J_{suv} = J_{su} \circ J_{uv}$ and $J_{ins} = \text{id}_{J_{ns}}$.

An $S$-indexed cell $\phi$ in $\text{Set}'-\text{Prof}^S$, of the form as in (1), consists of a family of cells $\phi_s$ as on the right below, one for each $s \in S$, that are natural in $s$, in the sense that $\phi_t \circ (J_{1u}, \ldots, J_{nu}) = K_{1u} \circ \phi_s$ if $|K| = 1$, and $\phi_t \circ (J_{1u}, \ldots, J_{nu}) = C_u \circ \phi_s$ if $|K| = 0$. Finally, compositions and identities in $\text{Set}'-\text{Prof}^S$ are simply given indexwise: for instance $(\psi \circ (\phi_1, \ldots, \phi_n))_s = \psi_s \circ (\phi_{1s}, \ldots, \phi_{ns})$, and so on.

\[
\begin{array}{ccc}
A & \xrightarrow{J_s} & B \\
\downarrow J_s & & \downarrow J_s \\
A_t & \xrightarrow{J_t} & B_t
\end{array}
\quad
\begin{array}{ccc}
A_0 & \xrightarrow{J_{s0}} & A_1 & \cdots & A_{u/s} & \xrightarrow{J_{ns}} & A_n \\
\downarrow \phi_s & & \downarrow \phi_s & & \downarrow \phi_s & & \downarrow \phi_s \\
C & \xrightarrow{K} & D
\end{array}
\]

Analogous to the definition of $(\text{Set},\text{Set}')-\text{Prof}$ as a sub-hypervirtual double category of $\text{Set}'-\text{Prof}$, in the previous example, we define $(\text{Set},\text{Set}')-\text{Prof}^S$ as the sub-hypervirtual double category of $\text{Set}'-\text{Prof}^S$ that is obtained by restricting to small

\footnote{Indeed we take $J^S(y) := J(-, y)$.}
S-indexed profunctors $J: A \to B$, i.e. those with $J_s(x,y) \in \text{Set}$ for all $s \in S$, $x \in A_s$ and $y \in B_s$.

### 1.3 Universe enlargements

Notice that, analogous to Example 1.8, we can consider sub-hypervirtual double categories $(\text{Ab}, \text{Ab}')$-Prof $\subset \text{Ab}'$-Prof, $(\text{Cat}, \text{Cat}')$-Prof $\subset \text{Cat}'$-Prof, etc., where $\text{Ab} \subset \text{Ab}'$, $\text{Cat} \subset \text{Cat}'$, etc., are the embeddings obtained by considering abelian groups, categories, etc., in the categories of sets $\text{Set}$ and $\text{Set}'$ respectively. Again we prefer to work in e.g. $(\text{Ab}, \text{Ab}')$-Prof instead of $\text{Ab}$-Prof or $\text{Ab}'$-Prof, for reasons similar to the ones given in Example 1.8.

More generally, we will follow Kelly in Section 3.11 of [Kel82] and enrich both in a monoidal category $\mathcal{V}$ as well as in a ‘universe enlargement’ $\mathcal{V}'$ of $\mathcal{V}$, as follows. We call a category $\mathcal{C}$ small (resp. large) (co-)complete if the (co-)limit of any diagram $D: S \to \mathcal{C}$ exists as soon as the set of morphisms $S_1$ of $S$ is small (resp. large); that is $S_1 \in \text{Set}$ (resp. $S_1 \in \text{Set}'$). By a closed monoidal category $\mathcal{V}$ we mean a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I)$ that is equipped with, for every object $X \in \mathcal{V}$, a right adjoint $[X, -]$ to the endofunctor $X \otimes -$ on $\mathcal{V}$. These right adjoints combine to form an internal hom functor $[-, -]: \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

Remember that a normal monoidal functor $F: \mathcal{V} \to \mathcal{V}'$ between monoidal categories $\mathcal{V}$ and $\mathcal{V}' = (\mathcal{V}', \otimes', I')$ is equipped with coherent natural isomorphisms $\hat{F}: FX \otimes' FY \cong F(X \otimes Y)$, for any pair $X,Y \in \mathcal{V}$, while it preserves the unit $FI = I'$. If $\mathcal{V}$ and $\mathcal{V}'$ are symmetric monoidal, with symmetries denoted $s$ and $s'$, then a monoidal functor $F: \mathcal{V} \to \mathcal{V}'$ is called symmetric monoidal whenever its coherence isomorphisms $\hat{F}$ make the diagrams below commute.

$$
\begin{array}{ccc}
FX \otimes' FY & \xrightarrow{\hat{F}} & F(X \otimes Y) \\
\downarrow{s'} & & \downarrow{F_s} \\
FY \otimes' FX & \xrightarrow{\hat{F}} & F(Y \otimes X)
\end{array}
$$

Finally if $\mathcal{V}$ and $\mathcal{V}'$ are both closed, with internal hom functors $[-, -]$ and $[-, -]'$, then a (symmetric) monoidal functor $F: \mathcal{V} \to \mathcal{V}'$ is called closed as soon as the canonical maps $F[X,Y] \to [FX, FY]'$, that are adjunct to the composites below, are invertible.

$$
FX \otimes F[X,Y] \xrightarrow{\hat{F}} F(X \otimes [X,Y]) \xrightarrow{F_{\text{ev}}} FY
$$

**Definition 1.10.** Let $\mathcal{V}$ be a (closed) monoidal category whose set of morphisms is large. By a universe enlargement of $\mathcal{V}$ we mean a closed monoidal category $\mathcal{V}'$ that is equipped with a full and faithful (closed) normal monoidal functor $\mathcal{V} \to \mathcal{V}'$ satisfying the following axioms.

- (a) $\mathcal{V}'$ is locally large, that is $\mathcal{V}'(X,Y) \in \text{Set}'$ for all $X,Y \in \mathcal{V}'$;
- (b) $\mathcal{V}'$ is large cocomplete and large complete;
- (c) $\mathcal{V} \to \mathcal{V}'$ preserves all limits;
- (d) $\mathcal{V} \to \mathcal{V}'$ preserves large colimits.

A universe enlargement $\mathcal{V} \to \mathcal{V}'$ is called symmetric whenever both $\mathcal{V}$ and $\mathcal{V}'$ are symmetric monoidal categories, while $\mathcal{V} \to \mathcal{V}'$ is a symmetric monoidal functor.

We will see in Example 1.20 below that any universe enlargement $\mathcal{V} \to \mathcal{V}'$ induces an ‘inclusion’ $\mathcal{V}$-Prof $\to \mathcal{V}'$-Prof of hypervirtual double categories, so that
$\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-profunctors can be regarded as if enriched in $\mathcal{V}'$. We write $RV \subset \mathcal{V}'$ for the replete image of the enlargement, that is the full subcategory consisting of $\mathcal{V}'$-objects isomorphic to objects in the image of $\mathcal{V} \to \mathcal{V}'$. Since $\mathcal{V} \to \mathcal{V}'$ is full and faithful it factors as an equivalence $\mathcal{V} \cong RV \subset \mathcal{V}'$, which we shall use to regard $\mathcal{V}$ as a subcategory of $\mathcal{V}'$; consequently, by a $\mathcal{V}$-object we shall mean either an object in $\mathcal{V}$ or one in $RV$.

We shall also see that the factorisation $\mathcal{V} \to \mathcal{V}'$ through $RV$ induces a factorisation $\mathcal{V}-\text{Prof} \to \mathcal{V}'-\text{Prof}$ as an ‘equivalence’ of hypervirtual double categories, which we likewise use to identify $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-profunctors with their $RV$-enriched counterparts; using the term $\mathcal{V}$-category (resp. $\mathcal{V}$-functor and $\mathcal{V}$-profunctor) in both cases.

We shall also see that the embeddings $\text{Set} \subset \text{Set}'$, $\text{Ab} \subset \text{Ab}'$ and $\text{Cat} \subset \text{Cat}'$, of the categories of small sets, small abelian groups and small categories into the categories of their large counterparts, are symmetric universe enlargements in the above sense, as long as $\text{Set}$ has infinite sets. More generally we have the following result, which summarises Sections 3.11 and 3.12 of [Kel82].

**Theorem 1.11** (Kelly). Let $\mathcal{V}$ be a (closed) (symmetric) monoidal category, with tensor product $\otimes$, whose set of morphisms is large. The (symmetric) monoidal structure on $\mathcal{V}$ induces a closed (symmetric) monoidal structure on the category $\text{Set}_{\mathcal{V}}^{\mathcal{V}}$ of large presheaves on $\mathcal{V}$, whose tensor product $\otimes'$ is given by Day convolution (see [Day70] or Example 8.2 below):

$$p \otimes' q := \int^{x,y \in \mathcal{V}} \mathcal{V}(-, x \otimes y) \times px \times qy$$

for presheaves $p$ and $q$: $\mathcal{V}^{\mathcal{V}} \to \text{Set}'$. With respect to this structure the yoneda embedding $y: \mathcal{V} \to \text{Set}_{\mathcal{V}}^{\mathcal{V}}$ satisfies axioms (a) to (c) of the previous definition.

Next let $y: \mathcal{V} \to \mathcal{V}'$ denote the factorisation of $y$ through the full subcategory $\mathcal{V}' \subset \text{Set}_{\mathcal{V}^{\mathcal{V}}}$ consisting of presheaves that preserve all large limits in $\mathcal{V}^{\mathcal{V}}$. The induced closed (symmetric) monoidal structure on $\text{Set}_{\mathcal{V}}^{\mathcal{V}}$ again induces such a structure on $\mathcal{V}'$, with respect to which $y'$ forms a (symmetric) universe enlargement.

**Example 1.12.** Given a universe enlargement $\mathcal{V} \to \mathcal{V}'$ consider a $\mathcal{V}'$-profunctor $J: A \Rightarrow B$ in $\mathcal{V}'$-Prof (see Example 1.6). We will call $J$ a $\mathcal{V}$-profunctor whenever $J(x,y)$ is a $\mathcal{V}$-object for all pairs $x \in A$ and $y \in B$. Analogous to Example 1.8, we denote by $(\mathcal{V}, \mathcal{V}')$-Prof the sub-hypervirtual double category of $\mathcal{V}'$-Prof that consists of all $\mathcal{V}'$-categories and $\mathcal{V}'$-functors, as well as $\mathcal{V}$-profunctors and their transformations.

### 1.4 The 2-category of hypervirtual double categories

Having introduced the notion of hypervirtual double categories we next consider the functors between them, as well as their transformations.

**Definition 1.13.** A functor $F: \mathcal{K} \to \mathcal{L}$ between hypervirtual double categories consists of a functor $F_v: \mathcal{K}_v \to \mathcal{L}_v$ (which will be denoted $F$) as well as assignments mapping the horizontal morphisms and cells of $\mathcal{K}$ to those of $\mathcal{L}$, as shown below, in
a way that preserves compositions and identities strictly.

\[
\begin{array}{cccc}
J: A & \mapsto & B & \mapsto \quad FJ: FA & \mapsto & FB \\
A_0 \xrightarrow{J_1} A_1 & \cdots & A_n \xrightarrow{J_n} A_n & \mapsto \quad FA_0 \xrightarrow{FJ_1} FA_1 & \cdots & FA_n \xrightarrow{FJ_n} FA_n \\
\downarrow f & & \downarrow g & \mapsto \quad \downarrow ff & & \downarrow fg \\
C & \xrightarrow{\phi} & K & \mapsto \quad FC & \xrightarrow{F\phi} & FD \\
A_0 \xrightarrow{J_1} A_1 & \cdots & A_n' \xrightarrow{J_n} A_n & \mapsto \quad FA_0 \xrightarrow{FJ_1} FA_1 & \cdots & FA_n \xrightarrow{FJ_n} FA_n \\
\downarrow f & & \downarrow g & \mapsto \quad \downarrow ff & & \downarrow fg \\
C & \xrightarrow{\phi} & K & \mapsto \quad FC & \xrightarrow{F\phi} & FD
\end{array}
\]

**Definition 1.14.** A transformation \(\xi: F \Rightarrow G\) of functors \(F, G: K \to L\) between hypervirtual double categories consists of a natural transformation \(\xi: F_v \Rightarrow G_v\) as well as a family of \((1, 1)\)-ary cells

\[
\begin{array}{ccc}
FA & \xrightarrow{FJ} & FB \\
\xi_A & \downarrow & \xi_n \\
GA & \xrightarrow{GJ} & GB
\end{array}
\]

in \(L\), one for each \(J: A \mapsto B \in K\), that satisfies the naturality axiom

\(G\phi \circ \xi_J = \xi_K \circ F\phi\)

whenever this makes sense, where \(\xi_J := (\xi_{J_1}, \ldots, \xi_{J_n})\) if \(J = (J_1, \ldots, J_n)\) and \(\xi_A := \xi_A\) if \(J = (A)\).

Recall that the definition of hypervirtual double category reduces to that of virtual double category by restricting it to unary cells, together with removing the references to vertical identity cells; see the discussion following Lemma 1.2. Likewise the definitions above reduce to that of functor and transformation for virtual double categories as given in Section 3 of [CS10]. The latter combine into a 2-category of virtual double categories which we denote \(\text{VirtMultiCat}\). Remember that every hypervirtual double category \(K\) contains a 2-category \(V(K)\) and a virtual double category \(U(K)\). In the following, which is easily checked, \(2\text{-Cat}\) denotes the 2-category of 2-categories, strict 2-functors and 2-natural transformations.

**Proposition 1.15.** Hypervirtual double categories, their functors and the transformations between them form a 2-category \(\text{HypVirtMultiCat}\). The assignments \(K \mapsto V(K)\) and \(K \mapsto U(K)\) extend to strict 2-functors

\[
\begin{array}{c}
V: \text{HypVirtMultiCat} \to 2\text{-Cat} \\
U: \text{HypVirtMultiCat} \to \text{VirtMultiCat}
\end{array}
\]

**Example 1.16.** Every lax monoidal functor \(F: V \to W\) between monoidal categories induces a functor \(F\text{-Mat}: V\text{-Mat} \to W\text{-Mat}\) between the virtual double categories of matrices in \(V\) and \(W\) (Example 1.6) in the evident way. Likewise the components of any monoidal transformation \(\xi: F \Rightarrow G\) form the cell-components of an induced transformation \(\xi\text{-Mat}: F\text{-Mat} \Rightarrow G\text{-Mat}\). In fact the assignments \(F \mapsto F\text{-Mat}\) and \(\xi \mapsto \xi\text{-Mat}\) combine to form a strict 2-functor \((-)\text{-Mat}: \text{MonCat}_l \to \text{VirtMultiCat}\), where \(\text{MonCat}_l\) denotes the 2-category of monoidal categories, lax monoidal functors and monoidal transformations.
Example 1.17. Similarly any pullback-preserving functor \( F: \mathcal{D} \to \mathcal{E} \), between categories with pullbacks, induces a functor \( \text{Span}(F): \text{Span}(\mathcal{D}) \to \text{Span}(\mathcal{E}) \) between the virtual double categories of spans in \( \mathcal{D} \) and \( \mathcal{E} \) (see Example 1.7). This too extends to a strict 2-functor \( \text{Span}(\cdot): \text{Cat}_{\text{pb}} \to \text{VirtMultiCat} \), where \( \text{Cat}_{\text{pb}} \) denotes the 2-category of categories with pullbacks, pullback-preserving functors and all natural transformations between them.

Proposition 1.18. The assignment \( \mathcal{K} \to \text{Mod}(\mathcal{K}) \) of Proposition 1.5 extends to a strict 2-functor \( \text{Mod}: \text{VirtMultiCat} \to \text{HypVirtMultiCat} \).

Sketch of the proof. The image \( \text{Mod}F: \text{Mod}K \to \text{Mod}L \) of a functor \( F: \mathcal{K} \to \mathcal{L} \) between virtual double categories is simply given by applying \( F \) indexwise; e.g. it maps a monoid \( A = (A, \alpha, \beta, \varepsilon) \) in \( \mathcal{K} \) to the monoid \( (\text{Mod}F)(A) := (F\alpha, F\beta, F\varepsilon) \) in \( \mathcal{L} \). The image \( \text{Mod}\xi: \text{Mod}F \to \text{Mod}G \) of a transformation \( \xi: F \Rightarrow G \) has as components the monoid morphisms \( (\text{Mod}\xi)_A := (\xi_A, \xi_\alpha): FA \to GA \), one for each monoid \( A \) in \( \mathcal{K} \), as well as the cells of bimodules \( (\text{Mod}\xi)_J := \xi_J: FJ \Rightarrow GJ \), one for each bimodule \( J = (J, \lambda, \rho): A \Rightarrow B \) in \( \mathcal{K} \).

\[ \square \]

1.5 Equivalence of hypervirtual double categories

Now that we have a 2-category of hypervirtual double categories we can consider (adjoint) equivalences between such double categories.

Definition 1.19. An equivalence \( \mathcal{K} \simeq \mathcal{L} \) of hypervirtual double categories \( \mathcal{K} \) and \( \mathcal{L} \) is an equivalence in the 2-category \( \text{HypVirtMultiCat} \). That is, it is given by pair of functors \( F: \mathcal{K} \to \mathcal{L} \) and \( G: \mathcal{L} \to \mathcal{K} \) that is equipped with invertible transformations \( \eta: \text{id}_\mathcal{L} \cong GF \) and \( \varepsilon: FG \cong \text{id}_\mathcal{L} \).

If \( \eta \) and \( \varepsilon \) satisfy the triangle identities, that is they define \( F \) and \( G \) as adjoint morphisms in \( \text{HypVirtMultiCat} \), then the quadruple \( (F, G, \eta, \varepsilon) \) is called an adjoint equivalence \( \mathcal{K} \simeq \mathcal{L} \).

Remember that, in any 2-category, an equivalence \( \mathcal{K} \simeq \mathcal{L} \) of hypervirtual double categories as follows. Firstly applying the composite 2-functor \( (-) \text{-} \text{Prof} := \text{Mod} \circ (-) \text{-Mat} \), which extends the assignment \( \mathcal{V} \to \mathcal{V} \text{-Prof} \) of Example 1.6 to \( \mathcal{V} \to \mathcal{V}' \) gives a functor \( \mathcal{V} \text{-Prof} \to \mathcal{V}' \text{-Prof} \) of hypervirtual double categories, as in the top of the diagram below.

\[ \begin{array}{ccc}
\mathcal{V} \text{-Prof} & \xrightarrow{\mathcal{V}} & \mathcal{V}' \text{-Prof} \\
\cap & \mathcal{V} \text{-Prof} \subset & (\mathcal{V}, \mathcal{V}') \text{-Prof} \\
\mathcal{V} \text{-Prof} & \cap & (\mathcal{V}, \mathcal{V}') \text{-Prof} \\
\end{array} \]

Next consider the factorisation \( \mathcal{V} \cong \mathcal{R} \mathcal{V} \subset \mathcal{V}' \) through the replete image \( \mathcal{R} \mathcal{V} \). Because the enlargement, and hence its factorisation \( \mathcal{V} \cong \mathcal{R} \mathcal{V} \), is a monoidal functor, it follows from Kelly’s result on ‘doctrinal adjunctions’ \([\text{Kel74}]\) (also see Section 7.3 below) that \( \mathcal{V} \cong \mathcal{R} \mathcal{V} \) is in fact an equivalence of monoidal categories. Consequently its image under \( (-) \text{-Prof} \), as on the left in the diagram above, is an equivalence of hypervirtual double categories as well.

Finally notice that \( \mathcal{R} \mathcal{V} \text{-Prof} \subset \mathcal{V}' \text{-Prof} \) factors through the hypervirtual double category \( (\mathcal{V}, \mathcal{V}') \text{-Prof} \) of \( \mathcal{V} \)-profunctors between \( \mathcal{V}' \)-categories, that was described in Example 1.12. Since each of the functors \( \mathcal{V} \text{-Prof} \cong \mathcal{R} \mathcal{V} \text{-Prof} \subset (\mathcal{V}, \mathcal{V}') \text{-Prof} \subset \mathcal{V}' \text{-Prof} \) is full and faithful, in the sense below, it follows that both \( \mathcal{V} \text{-Prof} \to (\mathcal{V}, \mathcal{V}') \text{-Prof} \) and \( \mathcal{V} \text{-Prof} \to \mathcal{V}' \text{-Prof} \) are full and faithful too.
The goal of this section is to prove that, just like in classical category theory (see e.g. Section IV.4 of [ML98]), giving an equivalence $\mathcal{K} \simeq \mathcal{L}$ is the same as giving a functor $F: \mathcal{K} \to \mathcal{L}$ that is 'full, faithful and essentially surjective'. The following definitions generalise those for functors between double categories, as given in Section 7 of [Shu08].

We start with the notions full and faithful. Let $F: \mathcal{K} \to \mathcal{L}$ be a functor between hypervirtual double categories. Its restriction $J \mapsto FJ$ to horizontal morphisms preserves sources and targets, so that it extends to an assignment $J = (J_1, \ldots, J_n) \mapsto FJ := (FJ_1, \ldots, FJ_n)$ on paths. Next, for each pair of morphisms $f: A_0 \to C$ and $g: A_n \to D$ in $\mathcal{K}$, and paths $\overset{J}{\Rightarrow}: A_0 \to A_n$ and $\overset{K}{\Rightarrow}: C \Rightarrow D$, where $|K| \leq 1$, the functor $F$ restricts to the following assignment, between classes of cells of the forms as shown:

$$ A_0 \xrightarrow{\overset{J}{\Rightarrow}} A_n \quad \{ f \xrightarrow{\phi} g \} \quad \overset{F}{\Rightarrow} \quad \{ FA_0 \xrightarrow{\overset{FJ}{\Rightarrow}} FA_n \} \quad \overset{FC}{\Rightarrow} \quad \{ FC \xrightarrow{\overset{FK}{\Rightarrow}} FD \} $$

**Definition 1.21.** A functor $F: \mathcal{K} \to \mathcal{L}$ between hypervirtual double categories is called locally faithful (resp. locally full) if, for each $f: A_0 \to C$, $g: A_n \to D$, $\overset{J}{\Rightarrow}: A_0 \to A_n$ and $\overset{K}{\Rightarrow}: C \Rightarrow D$ in $\mathcal{K}$, the assignment above is injective (resp. surjective). If moreover the restriction $F: \mathcal{K}_v \to \mathcal{L}_v$, to the vertical categories, is faithful (resp. full), then $F$ is called faithful (resp. full).

The following is a direct translation of Definition 7.6 of [Shu08] to the setting of hypervirtual double categories.

**Definition 1.22.** A functor $F: \mathcal{K} \to \mathcal{L}$ of hypervirtual double categories is called essentially surjective if we can simultaneously make the following choices:

- an object $A_K \in \mathcal{K}$ for each object $A \in \mathcal{L}$, together with an isomorphism $\sigma_A: FA_K \cong A$ in $\mathcal{L}$;

- a horizontal morphism $J_K: A_K \Rightarrow B_K$ for each horizontal morphism $J: A \Rightarrow B$ in $\mathcal{L}$, together with an invertible cell

$$ \begin{array}{ccc} FA_K & \xrightarrow{FJ_K} & FB_K \\ \sigma_A & \downarrow & \sigma_B \\ A & \xrightarrow{J} & B. \end{array} $$

**Proposition 1.23.** A functor $F: \mathcal{K} \to \mathcal{L}$ between hypervirtual double categories is part of an adjoint equivalence $\mathcal{K} \simeq \mathcal{L}$ if and only if it is full, faithful and essentially surjective.

**Sketch of the proof.** The ‘only if’-part is straightforward; we will sketch the ‘if’-part. First, because $F$ is essentially surjective, we can choose objects $A_K \in \mathcal{K}$, for each $A \in \mathcal{L}$, and horizontal morphisms $J_K: A_K \Rightarrow B_K \in \mathcal{K}$, for each $J: A \Rightarrow B \in \mathcal{L}$, as in the definition above, together with isomorphisms $\sigma_A: FA_K \cong A$ and $\sigma_J: FJ_K \cong J$. Using the full and faithfulness of $F$ these choices can be extended to a functor $(-)_K: \mathcal{L} \to \mathcal{K}$ as follows: for each vertical morphism $f: A \Rightarrow C$ in $\mathcal{L}$ we define $f_K: A_K \Rightarrow C_K$ to be the unique map in $\mathcal{K}$ such that $Ff = \sigma_C^{-1} \circ f \circ \sigma_A$, and for each cell $\phi: \overset{J}{\Rightarrow} \overset{K}{\Rightarrow}$ in $\mathcal{L}$ we define $\phi_K: \overset{J}{\Rightarrow} \overset{K}{\Rightarrow}$ in $\mathcal{K}$ to be the unique cell in $\mathcal{K}$ such that $F\phi_K = \sigma_K^{-1} \circ \phi \circ \sigma_J$, where the notation $\sigma_J$ is as in Definition 1.14. Using that $F$
is faithful it is easily checked that these assignments preserve the composition and identities of $L$.

Now the isomorphisms $(\sigma_A)_{A \in L}$ and $(\sigma_J)_{J \in L}$ combine to form a transformation $\sigma : F \circ (-) \circ K \cong \text{id}_L$. Conversely, a transformation $\eta : \text{id}_K \cong (-) \circ K \circ F$ is obtained by defining $\eta_A$, where $A \in K$, to be unique with $F \eta_A = \sigma_{FA}$ and defining $\eta_J$, where $J : A \Rightarrow B \in K$, such that $F \eta_J = \sigma_{FJ}$. Checking that $\eta$ and $\sigma$ satisfy the triangle identities is easy.

\[ \square \]

## 2 Restriction of horizontal morphisms

Two notions that are invaluable to the theory of double categories and their generalisations are that of restricting horizontal morphisms and that of composing horizontal morphisms. In this section we consider a variation of the former notion, to one that is appropriate for hypervirtual double categories, following for a large part Section 7 of [CS10]; in the next section we consider the latter.

### 2.1 Cartesian cells

Restrictions of horizontal morphisms are defined by ‘cartesian cells’, as follows.

**Definition 2.1.** A cell $\psi : J \Rightarrow K$ with $|J| \leq 1$, as in the right-hand side below, is called cartesian if any cell $\chi$, as on the left-hand side, factors uniquely through $\psi$ as a cell $\phi$ as shown.

\[
\begin{array}{cccc}
X_0 \xrightarrow{H_0} X_1 & \cdots & X_n \xrightarrow{H_n} X_n \\
\downarrow \chi & & \downarrow \chi^\phi & \downarrow \chi^k \\
A & = & B & = \\
\downarrow f & & \downarrow g & \\
C & \xrightarrow{K} & D & \xrightarrow{K}
\end{array}
\]

Vertically dual, provided that the cell $\phi$ in the right-hand side above is unary, it is called weakly cocartesian if any cell $\chi$ factors uniquely through $\phi$ as shown.

If an $(1,n)$-ary cartesian cell $\psi$ like above exists then its horizontal source $J : A \Rightarrow B$ is called the restriction of $K$ along $f$ and $g$, and denoted $K(f,g) := J$. If $K = (C \xrightarrow{K} D)$ is of length $n = 1$ then we will call $K(f,g)$ unary; in the case that $K = (C)$ we call $C(f,g)$ nullary. By their universal property any two cartesian cells defining the same restriction factor through each other as invertible horizontal cells. We will often not name cartesian cells, but simply depict them as

\[
\begin{array}{cccc}
A & \xrightarrow{f} & B \\
\downarrow f & & \downarrow g & \\
C & \xrightarrow{K} & D.
\end{array}
\]

**Example 2.2.** In the hypervirtual double category $( '\mathcal{V}', \mathcal{V}' )\text{-Prof}$ of $\mathcal{V}'$-profunctors between $\mathcal{V}'$-categories (Example 1.12) all unary restrictions $K(f,g)$ exist: they are simply the $\mathcal{V}$-profunctors given by the family of $\mathcal{V}$-objects $K(fx,gy)$, for all $x \in A$ and $y \in B$, that is equipped with actions induced by those of $K$. The cartesian cell $K(f,g) \Rightarrow K$ simply consists of the identities on the $\mathcal{V}$-objects $K(fx,gy)$.

On the other hand, the nullary restriction $C(f,g)$ of two $\mathcal{V}'$-functors $f$ and $g$ does not exist in general, but it does whenever all of the hom-objects $C(fx,gy)$
are \( \mathcal{V} \)-objects: in that case the cartesian cell \( C(f,g) \Rightarrow C \) consists of the identities on these hom-objects. Specifically, in \((\text{Set}, \text{Set}')\)-Prof all nullary restrictions \( C(f,g) \) exist as soon as \( C \) is locally small. We will see in Example 2.13 below that, in the case that either \( f \) or \( g \) is an identity \( \mathcal{V}' \)-functor, the previous condition is necessary for the existence of \( C(f,g) \) as well.

Similarly, the hypervirtual double category \( \mathcal{V} \)-Prof of \( \mathcal{V} \)-profunctors (Example 1.6) admits all (both unary and nullary) restrictions.

**Example 2.3.** Analogous to the situation for \((\text{Set}, \text{Set}')\)-Prof, in the hypervirtual double category \((\text{Set}, \text{Set}')\)-Prof\(^{\text{null}}\) of \( \mathcal{V} \)-indexed profunctors (Example 1.9) all unary restrictions exist, while the nullary restriction \( C(f,g): A \Rightarrow B \) exists as soon as the hom-sets \( C_s(f_s x, g_s y) \) are small for all \( s \in S \), \( x \in A_s \) and \( y \in B_s \). In either case the restrictions can be defined indexwise, by setting \( K_s(f_s, g_s) := K(f_s, g_s) \) for all \( s \in S \).

**Example 2.4.** For any isomorphism \( f: A \to C \) the vertical identity cell \( \text{id}_f \) is cartesian.

Where it is often easy to give examples of restrictions, giving examples of extensions is usually harder. Fortunately, as we shall see in the next section (Corollary 3.24), the extension of \( J \) along \( f \) and \( g \) above coincides with the ‘horizontal composite’ \( (C, f) \circ J \circ D(g, \text{id}) \) whenever it exists, where \( C(f, \text{id}): C \Rightarrow A_0 \) and \( C(g, \text{id}): A_n \Rightarrow D \) are nullary restrictions in the above sense. Analogously, in Lemma 3.20 we will see that the restriction of \( K \) along \( f \) and \( g \) coincides with the composite \( (C(f, \text{id})) \circ K \circ D(\text{id}, g) \). Thus most results concerning weakly cocartesian cells are left to the next section, except for a characterisation of certain such cells in \((\mathcal{V}, \mathcal{V}')\)-Prof, which is given at the end of this subsection.

Cartesian cells satisfy the following pasting lemma. As a consequence, taking restrictions is ‘pseudofunctorial’ in the sense that \( K(f,g)(h,k) \cong K(f \circ h, g \circ k) \) and \( K(\text{id}, \text{id}) \cong K \).

**Lemma 2.5** (Pasting lemma). If the cell \( \phi \) in the composite below is cartesian then the full composite \( \phi \circ \psi \) is cartesian if and only if \( \psi \) is.

\[
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow \psi & \downarrow \psi \\
A \xrightarrow{f} B \\
\downarrow \phi & \downarrow \phi \\
C \xrightarrow{K} D
\end{array}
\]
of the former virtual double categories having all restrictions and all 'horizontal units' (which we consider in the next section) are called 'virtual equipments'. Virtual equipments that have all 'horizontal composites' as well can be regarded as 'bicategories equipped with proarrows', in the original sense of Wood [Woo82], by combining the results in Appendix C of [Shu08] with Proposition 3.9 below; whence the term 'equipment'. Because important hypervirtual double categories such as $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ do not have all units nor all nullary restrictions, but do have all unary restrictions, we have chosen the following generalisation of 'equipment' as appropriate for hypervirtual double categories.

**Definition 2.6.** An hypervirtual equipment is a hypervirtual double category that has all unary restrictions $K(f, g)$.

In the hypervirtual double category $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$, of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories, full and faithfulness of $\mathcal{V}'$-functors is related to cartesianness as follows.

**Proposition 2.7.** In $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ let $f : A \to C$ be a $\mathcal{V}'$-functor such that the hom-objects $C(f x, f y)$ are $\mathcal{V}$-objects for all $x, y \in A$. The identity cell $id_f$ is cartesian if and only if $f$ is full and faithful.

**Proof.** The 'if'-part is easy: if the actions $\bar{f} : (x, y) \mapsto (f(x), f(y))$ of $f$ on hom-objects are invertible then the unique factorisation of a cell $\chi : H_1, \ldots, H_n \Rightarrow C$ through $id_f$, as in Definition 2.1, is given by composing the components of $\chi$ with the inverses of $\bar{f}$.

For the converse, assume that $id_f$ is cartesian and remember that, by the assumption on the hom-objects $C(f x, f y)$, the restriction $C(f, f)$ exists; see Example 2.2.

\[
\begin{array}{ccc}
A & \xrightarrow{id_f} & A \\
\downarrow{id_f} & & \downarrow{id_f} \\
C & \xrightarrow{f} & C
\end{array}
\]

\[
\begin{array}{ccc}
A & \overset{\text{cart}}{\xrightarrow{f}} & A \\
\downarrow{\text{cart}} & & \downarrow{\text{cart}} \\
C & \xrightarrow{\chi} & C
\end{array}
\]

Consider the unique factorisations $\phi$ and $\psi$ in the identities above: the components of $\phi$ are simply the actions $f : A(x, y) \to C(f x, f y)$, and we claim that the components $C(f x, f y) \to A(x, y)$ of $\psi$ form their inverses. This claim is a straightforward consequence of the identities $\phi \circ \psi = id_{C(f, f)}$ and $\psi \circ \phi = id_A$, which themselves follow from the equations $\text{cart} \circ \phi \circ \psi = id_f \circ \psi = \text{cart}$ and $id_f \circ \psi \circ \phi = \text{cart} \circ \phi = id_f$, as well as the uniqueness of factorisations through $\text{cart}$ and $id_f$. We conclude that $f$ is full and faithful, completing the proof. 

In view of the above result we make the following definition.

**Definition 2.8.** A vertical morphism $f : A \to C$ is called full and faithful if both its identity cell $id_f$ is cartesian and the restriction $C(f, f)$ exists.

It is clear that a full and faithful morphism $f : A \to C$, in the above sense, is full and faithful in the vertical 2-category $\mathcal{V}(K)$, of objects, vertical morphisms and vertical cells in $K$, in the classical sense; that is, for each object $X \in K$ the functor $\mathcal{V}(K)(X, f) : \mathcal{V}(K)(X, A) \to \mathcal{V}(K)(X, C)$, given by postcomposition with $f$, is full and faithful (see e.g. Example 2.18 of [Web07]). The converse to this holds as soon as $K$ has ‘cocartesian tabulations’, as we shall see in Proposition 4.29.

Closing this subsection we characterise weakly cocartesian cells that are of the form as on the left below, in the hypervirtual double categories $\mathcal{V}$-$\text{Prof}$ and
$(\mathcal{V}, \mathcal{V}')$-Prof. Here $I$ denotes the unit $\mathcal{V}$-category consisting of a single object $*$ and hom-object $I(*,*) = I$, the unit of $\mathcal{V}$. Because the universe enlargement $\mathcal{V} \to \mathcal{V}'$ preserves the monoidal unit strictly, we can regard $I$ as the unit $\mathcal{V}'$-category as well. We identify $\mathcal{V}$-functors $I \to A$ with objects in $A$ and $\mathcal{V}$-profunctors $I \to I$ with $\mathcal{V}$-objects; cells between such profunctors are identified with $\mathcal{V}$-maps.

\[
\begin{array}{ccc}
I & \xrightarrow{J_0} & A_0 \\
\downarrow s & \searrow \phi & \downarrow t \\
C & \xrightarrow{J_1} & D
\end{array}
\]

First consider a general path $J = (A_0 \xrightarrow{J_0} A_1, \ldots, A_n \xrightarrow{J_n} A_n)$ of $\mathcal{V}$-profunctors. For any pair of objects $x \in A_0$ and $y \in A_n$ we denote by $J_t(x, y)$ the embedding into $\mathcal{V}$ of its subcategory consisting of the spans of the form below where, for each $i = 1, \ldots, n'$, both $v_i$ and $w_i$ range over all objects in $A_i$.

\[
\begin{array}{c}
J_0(x, v_1) \otimes A_1(v_1, w_1) \otimes J_2(w_1, v_2) \otimes A_2(v_2, w_2) \otimes \cdots \otimes A_n(v_n, w_n) \otimes J_n(w_n, y) \\
\text{id} \otimes \ldots \otimes \lambda \otimes \rho \otimes \ldots \otimes \rho \otimes \text{id}
\end{array}
\]

If $\mathcal{V}$ is closed symmetric monoidal, so that each $J_i: A_i \to A_i$ can be identified with a $\mathcal{V}$-functor $J_i: A_i^{op} \otimes A_i \to \mathcal{V}$, then colimit of $J_t(x, y)$, if it exists, is easily checked to coincide with the coend $\int^{u_1 \in A_1 \ldots, u_n \in A_n} J_1(x, u_1) \otimes \cdots \otimes J_n(u_n, y)$. Consequently we will use this coend notation for the colimit of $J_t(x, y)$, regardless of $\mathcal{V}$ being closed symmetric monoidal.

Returning to a path of $\mathcal{V}$-profunctors $J: I \to I$, as on the left of (4) above, notice that for any cocone $J_t(*, *) \Rightarrow \Delta X$, with $X \in \mathcal{V}$, its naturality with respect to the spans above coincides with the internal equivariance axioms satisfied by the cells in $\mathcal{V}$-Prof with source $J$. We conclude that giving a unary cell $\psi$ as on the right of (4), where $s \in C$ and $t \in D$, is the same as giving a cocone $\gamma: J_t \Rightarrow \Delta L(s, t)$. The following characterisation of weakly coCartesian cells in $\mathcal{V}$-Prof is now straightforward.

**Proposition 2.9.** A cell $\psi$ in $\mathcal{V}$-Prof, of the form as on the left of (4), is weakly coCartesian precisely if its corresponding cocone $J_t(*, *) \Rightarrow \Delta K$ is colimiting; that is, it defines the $\mathcal{V}$-object $K$ as the coend $\int^{u_1 \in A_1 \ldots, u_n \in A_n} J_1(*, u_1) \otimes \cdots \otimes J_n(u_n, *)$.

Furthermore, given a universe enlargement $\mathcal{V} \to \mathcal{V}'$, the inclusions

\[
\mathcal{V}$-Prof $\to (\mathcal{V}, \mathcal{V}')$-Prof $\to \mathcal{V}'$-Prof
\]

both preserve and reflect weakly coCartesian cells of the form as on the left in (4).

**Sketch of the proof.** Reflection along $(\mathcal{V}, \mathcal{V}')$-Prof $\to \mathcal{V}'$-Prof is clear; for preservation notice that, for any $\mathcal{V}'$-object $X$, cocones $J_t(*, *) \Rightarrow \Delta X$ correspond to nullary cells $\psi$ as on the right of (4), with $C$ the $\mathcal{V}'$-category consisting of objects $0$ and $1$, together with hom-objects $C(0, 0) = I' = C(1, 1)$ and $C(0, 1) = X$, while $\psi$ has vertical morphisms $s = 0$ and $t = 1$.

For reflection and preservation along $\mathcal{V}$-Prof $\to \mathcal{V}'$-Prof notice that the $\mathcal{V}'$-cocone $J_{t}(*, *) \Rightarrow \mathcal{V}'$ corresponding to a cell $\phi: J_{t} \Rightarrow K$ in $\mathcal{V}$-Prof is isomorphic to the composition of its corresponding $\mathcal{V}$-cocone $J_{t}(*, *) \Rightarrow \mathcal{V}$ with $\mathcal{V}$ $\to$ $\mathcal{V}'$. Use that universe enlargements preserve large colimits and are full and faithful. \[\square\]
2.2 Companions and conjoints

Here we consider nullary restrictions of the forms $C(f, \text{id})$ and $C(\text{id}, f)$, where $f : A \to C$ is a vertical morphism. These have been called respectively ‘horizontal companions’ and ‘horizontal adjoints’ in the setting of double categories [GP04]; we follow [CS10] in calling them ‘companions’ and ‘conjoints’. As foreshadowed in the discussion preceding Lemma 2.5 companions and conjoints can be regarded as building blocks out of which restrictions and extensions can be built, as will be explained in next section.

**Definition 2.10.** Consider a vertical morphism $f: A \to C$ in a hypervirtual double category. The nullary restriction $C(f, \text{id}) : A \dashv C$, if it exists, is called the companion of $f$ and denoted $f^\ast$, while $C(\text{id}, f) : C \dashv A$, if it exists, is called its conjoint and denoted $f^\ast$.

Notice that the notions of companion and conjoint are swapped when moving from $K$ to its horizontal dual $K^{co}$.

Although companions and conjoints are defined as nullary restrictions, the following lemma and its horizontal dual show that they can equivalently be defined as extensions along $f$. More precisely it gives a bijective correspondence between the cartesian cells $\phi$ defining a horizontal morphism $J : A \Rightarrow C$ as the companion of $f$ and the weakly cocartesian cells $\psi$ defining $J$ as the extension of $(A)$ along $\text{id}_A$ and $f$, in such a way that each corresponding pair $(\psi, \phi)$ satisfies the identities below, which are called the companion identities. Analogous identities are satisfied by corresponding pairs of a cartesian and weakly cocartesian cell defining a conjoint; these are called the conjoint identities.

**Lemma 2.11.** Consider a factorisation of the identity cell $\text{id}_f$ of $f$: $A \to C$ as on the left below. The following conditions are equivalent: $\psi$ is cartesian; the identity on the right holds; $\phi$ is weakly cocartesian.

![Diagram](image)

**Proof.** We will show that both $\psi$ being cartesian, as well as $\phi$ being weakly cocartesian, implies the identity on the right, while the latter implies the (co-)cartesianness of $\psi$ and $\phi$. For the first implication, notice that the identity on the left above implies that composing the left-hand side of the identity on the right either with $\psi$ or $\phi$ results in $\psi$ or $\phi$ again, respectively. By the uniqueness of factorisations through (co-)cartesian cells, it follows that the identity on the right holds as soon as $\psi$ is cartesian or $\phi$ is weakly cocartesian.

For the converse assume that both identities above hold; we will show that $\psi$ is cartesian and that $\phi$ is weakly cocartesian. For the first consider a cell $\chi$ as on the left below; we have to show that it factors uniquely through $\psi$. That it factors through $\psi$ follows from the identity on the left above as follows, where the last identity is one of the interchange axioms.

$$\chi = (\text{id}_f \circ \text{id}_h) \circ \chi = (\psi \circ \phi \circ h) \circ \chi = \psi \circ ((\phi \circ h) \circ \chi).$$

To show the uniqueness of this factorisation consider a second factorisation $\chi = \psi \circ \chi'$. Using the identity on the right above we have

$$(\phi \circ h) \circ \chi = (\phi \circ h) \circ (\psi \circ \chi') = (\phi \circ \psi) \circ \chi' = \chi',$$
showing that the factorisation obtained before coincides with $\chi'$. This concludes
the proof of $\psi$ being cartesian.

\[
\begin{array}{c}
X_0 \xrightarrow{h_1} X_1 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{h_n} X_n \\
\downarrow_{f \circ h} \swarrow \downarrow_{k} \searrow \downarrow_{k \circ f} \\
E \xrightarrow{h} A \\
\end{array}
\]

To show that $\phi$ is weakly cocartesian under the identities (6) consider any unary

cell $\xi$ as on the right above. That it factors through $\phi$ is shown by

$\xi = \xi \circ \text{id}_{k \circ f} = (\xi \circ (k \circ \psi) \circ \phi) = (\xi' \circ \phi \circ \psi) \circ \phi$.

To see that this factorisation is unique, suppose that $\xi = \xi' \circ \phi$ as well. Then

$\xi \circ (k \circ \psi) = (\xi' \circ \phi) (k \circ \psi) = \xi' (\phi \circ \psi) = \xi'$,

which concludes the proof that $\phi$ is weakly cocartesian.

As an immediate consequence we find that functors of hypervirtual double

categories preserve companions and conjoints.

**Corollary 2.12.** Any functor between hypervirtual double categories

preserves cartesian and weakly cocartesian cells that define companions and conjoints.

**Proof.** This follows immediately from the fact that functors preserve vertical
composition strictly, so that the companion and conjoint identities of (the horizontal
dual of) the previous lemma are preserved.

The lemma above can also be used to show that the sufficient conditions for the
existence of nullary restrictions $C(f, g)$ in the hypervirtual equipments $(\mathcal{V}, \mathcal{V}')\text{-Prof}$

and $(\text{Set}, \text{Set}')\text{-Prof}^S$, that were given in Example 2.2 and Example 2.3 above, are

necessary in the case of companions and conjoints, as follows.

**Example 2.13.** In Example 2.2 we saw that the companion $f_*$ of a $\mathcal{V}'$-functor

$f: A \to C$ exists in $(\mathcal{V}, \mathcal{V}')\text{-Prof}$ as soon as the hom-objects $C(f, g)$ are $\mathcal{V}$-objects,

for all $x \in A$ and $y \in C$. For the converse consider cells $\psi: J \Rightarrow C$ and $\phi: A \Rightarrow J$

as in the lemma above: it is straightforward to check that the companion identities for $\phi$ and $\psi$ imply that the composites below are inverses for the components

$J(x, y) \to C(f(x), y)$ of $\psi$, showing that $C(f, g) \cong J(x, y)$.

$C(f, g)$ $\phi \circ \text{id}$

$J(x, f(x)) \otimes' C(f, g) \otimes' J(x, y)$

Given an indexing category $S$, the previous argument can be applied at each

index $s \in S$ to show that the companion $f_s$ of an $S$-indexed functor $f: A \to C$

exists in $(\text{Set}, \text{Set}')\text{-Prof}^S$ precisely if the hom-sets $C_s(f, g)$ are small for all $s \in S$,

$x \in A_s$ and $y \in C_s$.

The remainder of this subsection records some useful properties of companions.

**Lemma 2.14.** In a hypervirtual equipment consider a composite $g \circ f$ of vertical
morphisms. If $g$ is an isomorphism then $(g \circ f)_*$ exists as soon as $f_*$ does.
Proof. Assuming that $f_*$ exists, we consider the composite on the left-hand side below, whose top cartesian cell defines the restriction $C(f, g^{-1})$ of $f_*$ along $g^{-1}$.

\[
\begin{array}{c}
A \xrightarrow{f} C \\
\overline{g^{-1}} \downarrow \\\nA \xrightarrow{f_*} C
\end{array}
\]

Since the vertical identity cell for $g^{-1}$ is cartesian (Example 2.4) the left-hand side factors uniquely as a cell $\phi$ as shown. Applying the pasting lemma (Lemma 2.5) we find that $\phi$ is cartesian, thus defining $C(f, g^{-1})$ as the companion of $g \circ f$.

Recall that the objects, vertical morphisms and vertical cells of any hypervirtual double category $K$ form a 2-category $V(K)$. The next lemma reformulates the notion of adjunction in $V(K)$ in terms of companions and conjoints in $K$.

**Lemma 2.15.** In a hypervirtual double category $K$ let $f: A \to C$ be a vertical morphism whose companion $f_*$ exists. Consider vertical cells $\eta$ and $\varepsilon$ below as well as their factorisations through $f_*$, as shown.

\[
\begin{array}{c}
A \xrightarrow{f} C \\
\overline{g} \downarrow \\\nA \xrightarrow{f_*} C
\end{array}
\]

The following are equivalent:

(a) $(\eta, \varepsilon)$ defines an adjunction $f \dashv g$ in $V(K)$;

(b) $(\eta', \varepsilon')$ forms a pair that defines $f_*$ as the conjoint of $g$ in $K$.

Proof. Notice that condition (b) is equivalent to $\eta'$ and $\varepsilon'$ satisfying the conjoint identities $\eta' \circ \varepsilon' = \text{id}_g$ and $\eta' \circ \varepsilon' = \text{id}_{f_*}$, by the horizontal dual of the previous lemma; we claim that these correspond to the two triangle identities for $\eta$ and $\varepsilon$. Indeed we have

\[(f \circ \eta) \circ (\varepsilon \circ f) = \text{id}_f \Leftrightarrow (f \circ \eta' \circ \text{cocart}) \circ (\text{cart} \circ \varepsilon' \circ f) = \text{id}_f \]

\[
\Leftrightarrow \text{cart} \circ (\eta' \circ \varepsilon') \circ \text{cocart} = \text{id}_f \Leftrightarrow \eta' \circ \varepsilon' = \text{id}_{f_*},
\]

where the second equivalence follows from one of the interchange axioms, and the third from the vertical companion identity $\text{cart} \circ \text{id}_{f_*} \circ \text{cocart} = \text{id}_f$ together with the fact that factorisations through (co-)cartesian cells are unique. Likewise

\[(\eta \circ g) \circ (\varepsilon \circ g) = \text{id}_g \Leftrightarrow (\eta' \circ \text{cocart} \circ g) \circ (g \circ \text{cart} \circ \varepsilon') = \text{id}_g \]

\[
\Leftrightarrow \eta' \circ (\text{cocart} \circ \text{cart}) \circ \varepsilon' = \text{id}_g \Leftrightarrow \eta' \circ \varepsilon' = \text{id}_g,
\]

where we used the horizontal companion identity.

For later use we record the following well-known elementary result for 2-categories; see for example Section 2.4 of [Web07], where the notion of (absolute) left Kan extension is recalled as well.

25
Lemma 2.16. For a cell

\[
\begin{array}{c}
A \\
\downarrow \eta \\
\downarrow f \\
\downarrow g \\
\downarrow \eta \\
A
\end{array}
\]

in a 2-category the following are equivalent:

(a) \( \eta \) is the unit of an adjunction \( f \dashv g \);

(b) \( \eta \) defines \( g \) as the left Kan extension of \( f \) along \( \text{id}_A \), which is preserved by \( f \);

(c) \( \eta \) defines \( g \) as the absolute left Kan extension of \( f \) along \( \text{id}_A \).

Furthermore, under these conditions, \( f \) is full and faithful if and only if \( \eta \) is invertible.

2.3 Representability

In this final subsection we consider the (op-)representability of horizontal morphisms, in the sense below. Our aim is to characterise the sub-hypervirtual double categories of representable and oprepresentable horizontal morphisms, that are contained in any hypervirtual double category \( K \), in terms of strict double category \((Q \circ V)(K)\) of ‘quintets’ in the vertical 2-category \( V(K) \).

Definition 2.17. A vertical morphism \( j : A \to B \) is said to represent the horizontal morphism \( J : A \rightarrowrightarrow B \) if there exists a cartesian cell as on the left below, that is \( J \) forms the companion of \( j \); in this case we say that \( J \) is representable. Horizontally dual, \( J \) is called oprepresentable whenever there exists a cartesian cell as on the right.

\[
\begin{array}{c}
A \xrightarrow{J} B \\
\downarrow \text{cart} \\
B
\end{array}
\quad \quad \quad
\begin{array}{c}
A \xrightarrow{J} B \\
\downarrow \text{cart} \quad \text{h} \\
A
\end{array}
\]

For a hypervirtual double category \( K \) we write \( \text{Rep}(K) \subseteq K \) for the sub-hypervirtual double category that consists of all objects, all vertical morphisms, the representable horizontal morphisms, and all cells between those. The subcategory \( \text{opRep}(K) \) generated by the oprepresentable horizontal morphisms is defined analogously; notice that \( \text{opRep}(K) = (\text{Rep}(K^{\text{co}}))^{\text{co}} \). Because functors of hypervirtual double categories preserve companions and conjoints (Corollary 2.12), they preserve (op-)representable horizontal morphisms as well; whence the following.

Proposition 2.18. The assignments \( K \mapsto \text{Rep}(K) \) and \( K \mapsto \text{opRep}(K) \) extend to strict 2-endofunctors \( \text{Rep} \) and \( \text{opRep} \) on \( \text{HypVirtMultiCat} \).

In \cite{Ehr63}, Ehresmann defined, inside any 2-category \( C \), a quintet to be a cell of
While the quintets of \( C \) most naturally arrange as a ‘strict double category’ (see Example 3.10 below), for now we will think of them as forming a hypervirtual double category \( Q(C) \) as follows.

**Definition 2.19.** Let \( C \) be a 2-category. The hypervirtual double category \( Q(C) \) of quintets in \( C \) has as objects those of \( C \), while both its vertical and horizontal morphisms are morphisms in \( C \). A unary cell \( \phi \) in \( Q(C) \), as in the middle above, is a cell \( \phi \) in \( C \) as on the right, while the nullary cells of \( Q(C) \) are cells in \( C \) as on the right but with \( k = \text{id}_C \). Composition in \( Q(C) \) is induced by that of \( C \) in the evident way.

We abbreviate \( Q^\circ(C) := (Q(C^\circ))^\circ \). Thus, to each morphism \( j : A \to B \) in \( C \) there is a horizontal morphism \( j^\circ : B \Rightarrow A \) in \( Q^\circ(C) \), and to each cell \( \phi \) as on the left below there is a unary cell \( \phi^\circ \) in \( Q^\circ(C) \) as on the right.

**Proposition 2.20.** The assignments \( \mathcal{C} \mapsto Q(C) \) and \( \mathcal{C} \mapsto Q^\circ(C) \) above extend to strict 2-functors \( Q : 2\text{-Cat} \to \text{HypVirtMultiCat} \) and \( Q^\circ : 2\text{-Cat} \to \text{HypVirtMultiCat} \).

**Proof.** The image \( QF : QC \to QD \) of a strict 2-functor \( F : C \to D \) is simply given by letting \( F \) act on objects, morphisms and cells. The image \( Q\xi : QF \Rightarrow QG \) of a 2-natural transformation \( \xi : F \Rightarrow G \) is given by \( (Q\xi)_A := \xi_A \) on objects, while the cell \( (Q\xi)_j : Fj \Rightarrow Gj \) in \( Q(D) \), where \( j : A \to B \), is the quintet given by the naturality square \( Gj \circ \xi_A = \xi_B \circ Fj \). Finally \( \mathcal{C} \mapsto Q^\circ(C) \) is extended by the composite of strict 2-functors \( Q^\circ := (-)^\circ \circ Q \circ (-)^\circ \).

Remember that any hypervirtual double category \( \mathcal{K} \) contains a 2-category \( V(\mathcal{K}) \) of vertical morphisms and vertical cells. We denote by \( (Q \circ V)_*(\mathcal{K}) \subseteq (Q \circ V)(\mathcal{K}) \)
the sub-hypervirtual double category generated by all vertical morphisms, the horizontal morphisms \( j: A \to B \) that admit companions in \( \mathcal{K} \), and all quintets between them. Notice that this extends to a sub-2-endofunctor \((Q \circ V)_* \subseteq Q \circ V\) on \( \text{HypVirtMultiCat} \), because functors between hypervirtual double categories preserve cartesian cells that define companions (Corollary 2.12). The sub-2-endofunctor \((Q^{\text{co}} \circ V)^* \) is defined likewise, by mapping each \( \mathcal{K} \) to the sub-hypervirtual double category \((Q^{\text{co}} \circ V)^*(\mathcal{K}) \subseteq (Q^{\text{co}} \circ V)(\mathcal{K})\) that is generated by horizontal morphisms \( j^{\text{co}}: B \to A \) that correspond to vertical morphism \( j: A \to B \) that admit conjoints in \( \mathcal{K} \).

**Theorem 2.21.** Let \( \mathcal{K} \) be a hypervirtual double category. Choosing, for each \( j: A \to B \) in \((Q \circ V)_*(\mathcal{K})\), a cartesian cell \( \varepsilon_j \) that defines the companion of \( j \) in \( \mathcal{K} \), induces a functor \((-)_*: (Q \circ V)_*(\mathcal{K}) \to \text{Rep}(\mathcal{K})\) that is part of an equivalence

\[
(Q \circ V)_*(\mathcal{K}) \simeq \text{Rep}(\mathcal{K}).
\]

Letting \( \mathcal{K} \) vary, these functors combine to form a pseudonatural transformation \((-)_*: (Q \circ V)_* \Rightarrow \text{Rep} \) of strict 2-endofunctors on \( \text{HypVirtMultiCat} \).

Analogously, choosing cartesian cells that define conjoints induces an equivalence \((Q^{\text{co}} \circ V)^*(\mathcal{K}) \simeq \text{opRep}(\mathcal{K})\). Their underlying functors too combine to form a pseudonatural transformation \((-)^*: (Q^{\text{co}} \circ V)^* \Rightarrow \text{opRep} \).

**Proof.** We will construct the functors \((-)_*: (Q \circ V)_*(\mathcal{K}) \to \text{Rep}(\mathcal{K})\); show that they are full, faithful and essentially surjective, so that they are part of equivalences by Proposition 1.23 and prove that they are pseudonatural in \( \mathcal{K} \). The functors \((-)^*: (Q^{\text{co}} \circ V)^*(\mathcal{K}) \to \text{opRep}(\mathcal{K})\) can then be defined as the composites \((-)^* := (-)^{\text{co}} \circ (-)_* \circ (-)^{\text{co}}\), where we use that companions in \( \mathcal{K}^{\text{co}} \) correspond to conjoints in \( \mathcal{K} \), so that \(((Q \circ V)_*(\mathcal{K}^{\text{co}}))^{\text{co}} \simeq (Q^{\text{co}} \circ V)^*(\mathcal{K})\) and \((\text{Rep}(\mathcal{K}))^{\text{co}} \simeq \text{opRep}(\mathcal{K})\).

We take \((-)_*\) to be the identity on objects and vertical morphisms, while letting it map each horizontal morphism \( j: A \to B \) in \((Q \circ V)_*(\mathcal{K})\) to its chosen companion \( j_*: A \to B \) in \( \mathcal{K} \). This leaves its action on cells: we define the image \( \phi_* \) of a unary quintet \( \phi \), as in the composite on the left-hand side below, to be the unique factorisation as shown. The image \( \phi_* \) of a nullary quintet \( \phi \), as in the composite on the left-hand side above but with \( k = \text{id}_{C'} \), is simply defined to be that composite.

\[
\begin{array}{ccc}
A_0 & \overset{j_{n*}}{\longrightarrow} & A_n \\
\downarrow{f} & \nearrow{\varepsilon_{j_{n*}}} & \nearrow{\varepsilon_{j_{n*}}} \\
A_1 & \ddots & A_n \\
\downarrow{k} & \swarrow{\phi_*} & \swarrow{\phi_*} \\
D & \overset{g}{\longrightarrow} & D \\
\end{array}
\quad \overset{=}{\Rightarrow} \quad
\begin{array}{ccc}
A_0 & \overset{j_{n*}}{\longrightarrow} & A_n \\
\downarrow{f} & \nearrow{\varepsilon_{j_{n*}}} & \nearrow{\varepsilon_{j_{n*}}} \\
A_1 & \ddots & A_n \\
\downarrow{k} & \swarrow{\phi_*} & \swarrow{\phi_*} \\
D & \overset{g}{\longrightarrow} & D \\
\end{array}
\]

\[ (7) \]

It is clear that \( \phi \to \phi_* \) preserves identities. To see that it preserves composites \( \psi \circ (\phi_1, \ldots, \phi_n) \) too, we first assume that \( \psi \) is unary. Consider the following equation, where we denoted all cartesian cells that define the chosen companions simplicy by ‘\( \varepsilon \)’. Its identities follow from the identity above for \( \psi \), as well as for \( \phi_1, \ldots, \phi_n \), and
the definition of composition in \((Q \circ V)(\mathcal{K})\).

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

\[
\begin{array}{c}
\phi_1 \circ \cdots \circ \phi_n \\
\downarrow \\
\psi
\end{array}
\]

We conclude that \(\psi \circ (\phi_1, \ldots, \phi_n)\) and \((\psi \circ (\phi_1, \ldots, \phi_n))_*\) coincide after composition with the cartesian cell that defines the horizontal target of \(\psi_*\). By uniqueness of factorisations through cartesian cells we conclude that \(\psi \circ (\phi_1, \ldots, \phi_n)_*\) is preserved by \((-)_*\). The case of \(\psi\) being nullary is similar: simply use the equation obtained by removing the cartesian cells \(\varepsilon\) in the left and right-hand side above. This completes the definition of the functor \((-)_*\).

To prove that \((-)_*\) is part of an equivalence we will show that it is full, faithfull and essentially surjective, and then apply Proposition 1.23. That it is essentially surjective and full and faithful on vertical morphisms is clear. It remains to show that it is locally full and faithful, that is full and faithful on cells. To see this we denote, for each \(j: A \to B\) in \((Q \circ V)_*(\mathcal{K})\), by \(\eta_j\) the weakly cocartesian cell that corresponds to \(\varepsilon_j\), such that the pair \((\varepsilon_j, \eta_j)\) satisfies the companion identities; see Lemma 2.11. To show faithfulness, consider cells \(\phi: j_1 \Rightarrow k\) in \((Q \circ V)_*(\mathcal{K})\) such that \(\phi_1 \circ \psi = \psi_1\). It follows that the left-hand sides of (7) coincide for \(\phi\) and \(\psi\) so that, by precomposing both with \((\eta_{j_1}, \ldots, \eta_{j_n})\), \(\phi = \psi\) follows from the vertical companion identities. To show fullness on unary cells, consider \(\psi: (j_1, \ldots, j_n) \Rightarrow k\) in \((Q \circ V)_*(\mathcal{K})\). We claim that the composite

\[
\phi := \varepsilon_k \circ \psi \circ (\eta'_{j_1}, \ldots, \eta'_{j_n})
\]

where \(\eta'_{j_i} := \eta_{j_i} \circ j_i \circ \cdots \circ j_1\) for each \(i = 1, \ldots, n\), is mapped to \(\psi\) by \((-)_*\). Indeed, plugging \(\phi\) into the left-hand side of (7) we find \(\varepsilon_k \circ \psi = \varepsilon_k \circ \phi\), by using the horizontal companion identities, so that \(\psi = \phi\) follows. The case of \(\psi\) nullary is similar; simply take \(\phi := \psi \circ (\eta'_{j_1}, \ldots, \eta'_{j_n})\) instead.

We now turn to proving that the functors \((-)_*\) combine to form a pseudonatural transformation \((Q \circ V)_* \Rightarrow \text{Rep}\) of strict 2-endofunctors on \text{HypVirtMultiCat}. This means that we have to give an invertible transformation \(\nu_F\) as on the left below, for each functor \(F: \mathcal{K} \to \mathcal{L}\) of hypervirtual double categories. We take \(\nu_F\) to consist of identities \((\nu_F)_A = \text{id}_{F(A)}\) on objects and, for each \(j: A \Rightarrow B\) in \((Q \circ V)_*(\mathcal{K})\), the unique factorisation \((\nu_F)_j: F(j) \Rightarrow (F j)_*\) as on the right below. The latter is
invertible since \( F \varepsilon_j \), on the left-hand side, is cartesian by Corollary 2.12.

\[
\begin{align*}
(Q \circ V)_*(K) & \xrightarrow{(\varepsilon)_*} \text{Rep}(K) \\
(Q \circ V)_*(F) & \xrightarrow{\varepsilon^\nu_{FP}} \text{Rep}(F) \\
(Q \circ V)_*(L) & \xrightarrow{(\varepsilon)_*} \text{Rep}(L)
\end{align*}
\]

We have to show that the components of \( \nu_F \) are natural with respect to the cells of \( (Q \circ V)_*(K) \), in the sense of Definition 1.14. We will do so in case of a unary cell \( \phi: (j_1, \ldots, j_n) \Rightarrow k \); the case of nullary cells is similar. Consider the following equation, where \( \varepsilon'_{Fj_i} := Fg \circ Fj_1 \circ \cdots \circ Fj_{i+1} \circ \varepsilon_{Fj_i} \) and \( \varepsilon'_{j_i} = g \circ j_n \circ \cdots \circ j_{i+1} \circ \varepsilon_{j_i} \) for each \( i = 1, \ldots, n \), as in the left-hand side of (7). The identities follow from (7) for \( F\phi \), the identity above, \( F \) preserves composition, the \( F \)-image of (7) for \( \phi \) and the identity above again. Since factorisations through \( \varepsilon_{Fk} \), in the left and right-hand side below, are unique, we conclude that the components of \( \nu_F \) are natural with respect to \( \phi \). This completes the definition of the transformation \( \nu_F \).

\[
\varepsilon_{Fk} \circ (F\phi)_* \circ ((\nu_F)_{j_1}, \ldots, (\nu_F)_{j_n}) = (F\phi \circ \varepsilon'_{Fj_1} \circ \cdots \circ \varepsilon'_{Fj_n}) \circ ((\nu_F)_{j_1}, \ldots, (\nu_F)_{j_n}) = F\phi \circ \varepsilon'_{j_1} \circ \cdots \circ \varepsilon'_{j_n} = F(\varepsilon_k \circ \phi_*) = \varepsilon_{Fk} \circ (\nu_F)_k \circ F(\phi_*)
\]

Finally we have to show that the transformations \( \nu_F \) are natural with respect to the transformations \( \xi: F \Rightarrow G \) in HypVirtMultiCat, and that they are compatible with compositions and identities, that is \( \nu_{Gd} = \text{id} \) and \( \nu_{GF} \circ G\nu_F = \nu_{GF} \). Since the latter is straightforward to prove, we will only prove the former. Thus, for each \( j: A \Rightarrow B \) in \( (Q \circ V)_*(K) \), we have to show that \( (\nu_G)_j \circ \xi_{(j)} = (\xi)_* \circ (\nu_F)_j \).

Consider the equation

\[
\varepsilon_{Gj} \circ (\nu_G)_j \circ \xi_{(j)} = G\varepsilon_j \circ \xi_{(j)} = \xi_B \circ F\varepsilon_j = \xi_B \circ \varepsilon_{Fj} \circ (\nu_F)_j = \varepsilon_{Gj} \circ (\xi)_* \circ (\nu_F)_j,
\]

where we have used the defining identities for \( (\nu_G)_j \) and \( (\nu_F)_j \), the naturality of \( \xi \), identity (7) for \( \xi_j \), and the fact that the latter is simply the quintet given by the naturality square \( Gj \circ \xi_A = \xi_B \circ Fj \); see the proof of Proposition 2.20. Using the cartesianess of \( \varepsilon_{Gj} \) we conclude that \( (\nu_G)_j \circ \xi_{(j)} = (\xi)_* \circ (\nu_F)_j \), proving the naturality of the transformations \( \nu_F \). This concludes the proof.

### 3 Composition of horizontal morphisms

We now turn to compositions of horizontal morphisms in hypervirtual double categories, as well as units for such compositions. Analogous to the situation for virtual double categories (see Section 2 of [DPP06] or Section 5 of [CS10]), these are defined by horizontal cells that satisfy a stronger variant of the universal property for weakly cocartesian cells.

#### 3.1 Cocartesian paths of cells

We start with both strengthening, as well as extending to paths, Definition 2.1 of weakly cocartesian cell.
**Definition 3.1.** A path of unary cells $(\phi_1, \ldots, \phi_n)$, as in the right-hand side below, is called *weakly cocartesian* if any cell $\psi$, as on the left-hand side, factors uniquely through $(\phi_1, \ldots, \phi_n)$ as shown.

If moreover the following conditions are satisfied then the path $(\phi_1, \ldots, \phi_n)$ is called *cocartesian*.

(a) The restrictions $J'(\text{id}, f_0)$ and $J''(f_n, \text{id})$ exist for all horizontal morphisms $J': A' \to C_0$ and $J'': C_n \to A''$;

(b) any path of the form below, where $p, q \geq 1$, is weakly cocartesian.

If a cocartesian horizontal cell of the form

$$A_0 \xrightarrow{J_1} A_1 \cdots A_n \xrightarrow{J_n} A_n$$

exists with $n \geq 1$ then we call $K$ the *(horizontal) composite* of $(J_1, \ldots, J_n)$ and write $(J_1 \circ \cdots \circ J_n) := K$; in the case that $n = 0$ we call $K$ the *(horizontal) unit* of $A_0$ and write $I_{A_0} := K$, while we call $A_0$ *unital*. By their universal property any two cocartesian horizontal cells defining the same composite or unit factor through each other as invertible horizontal cells. Like weakly cocartesian cells, we shall denote single cocartesian cells in our drawings simply by “cocart”.

**Example 3.2.** Let $\mathcal{V}'$ be a monoidal category whose tensor product $\otimes'$ preserves large colimits on both sides. Then the composite of a path $(J_1, \ldots, J_n): A_0 \to A_n$ of $\mathcal{V}'$-profunctors exists in $\mathcal{V}'$-$\text{Prof}$ as soon as for each $x \in A_0$ and $y \in A_n$ the coend, on the right-hand side below and in the sense of Proposition 2.9 exists.

$$(J_1 \circ \cdots \circ J_n)(x, y) := \int_{u_1 \in A_1, \ldots, u_n' \in A_n'} J_1(x, u_1) \otimes' \cdots \otimes' J_n(u_n', y)$$

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In that case, using the assumption on $\otimes'$, there is exactly one way of extending the assignment above into a $\mathcal{V}'$-profunctor $(J_1 \circ \cdots \circ J_n) : A_0 \Rightarrow A_n$ such that colimiting cocones combine into a horizontal cell \((J_1, \ldots, J_n) \Rightarrow (J_1 \circ \cdots \circ J_n)\).

In Proposition 3.17 below we will see that this cell is cocartesian; in fact it will be shown that it defines \((J_1 \circ \cdots \circ J_n)\) as a `pointwise' horizontal composite, in the sense of Section 3.3. The notion of pointwise horizontal composite will be important when we consider Kan extensions.

**Example 3.3.** Let $\mathcal{V} \to \mathcal{V}'$ be a universe enlargement (Definition 1.10) such that the tensor product $\otimes'$ of $\mathcal{V}'$ preserves large colimits on both sides. Remember that $\mathcal{V}'$ is assumed to be large cocomplete $\mathcal{V}'$-Prof so that it has all horizontal composites by the previous example. We will see in Proposition 3.17 below that the inclusion $\langle \mathcal{V}, \mathcal{V}' \rangle$-$\text{Prof} \to \mathcal{V}'$-$\text{Prof}$ reflects all cocartesian cells so that, for any path $(J_1, \ldots, J_n) : A_0 \Rightarrow A_n$ of $\mathcal{V}$-profunctors between $\mathcal{V}$-categories, the horizontal composite $(J_1 \circ \cdots \circ J_n)$ exists in $\langle \mathcal{V}, \mathcal{V}' \rangle$-$\text{Prof}$ as soon as the $\mathcal{V}'$-coends above are $\mathcal{V}$-objects, for each $x \in A_0$ and $y \in A_n$. This is the case, for instance, when the $\mathcal{V}'$-categories $A_1, \ldots, A_n$ are small $\mathcal{V}$-categories and $\mathcal{V}$ is small cocomplete.

In Lemma 3.12 below it is shown that, in general, the unit $I_A$ of an object $A$ coincides with the nullary restriction $A(\text{id}, \text{id})$ (see Definition 2.1). Specifically, as an easy consequence of that lemma, a $\mathcal{V}'$-category $A$ is unital in $\langle \mathcal{V}, \mathcal{V}' \rangle$-$\text{Prof}$ if and only if it is a $\mathcal{V}$-category; in that case the cocartesian cell $A \Rightarrow I_A$ consists of the unit $\mathcal{V}$-maps $A : I \to A(x,x)$ of $A$.

**Example 3.4.** If $\mathcal{E}$ has reflexive coequalisers preserved by pullback then the hypervirtual double category $\text{Prof}(\mathcal{E})$ of profunctors internal to $\mathcal{E}$ (Example 1.7) has all horizontal composites. The composite of internal profunctors is an `internal coend'.

**Example 3.5.** Given a path of small $\mathcal{S}$-indexed profunctors $A_0 \xrightarrow{J_0} A_1 \cdots A_n \xrightarrow{J_n} A_0$ in $\langle \text{Set}, \text{Set}' \rangle$-$\text{Prof}^\mathcal{S}$ (see Example 1.9), it is easily checked that their composite $(J_1 \circ \cdots \circ J_n)$ can be defined indexwise; that is it exists whenever, for each $s \in \mathcal{S}$, the composite $(J_{1s} \circ \cdots \circ J_{ns})$ exists in $\langle \text{Set}, \text{Set}' \rangle$-$\text{Prof}$. Combining Example 2.13 and Lemma 3.12 shows that an $\mathcal{S}$-indexed category $A$ is unital precisely if $A_s$ is locally small for each $s \in \mathcal{S}$.

Because the notion of weakly cocartesian cell in hypervirtual double categories restricts to the corresponding notion for virtual double categories, so do the notions of horizontal composite and horizontal unit restrict to the corresponding notions for virtual double categories.

**Example 3.6.** Let $\mathcal{K}$ be a virtual double category. It was shown in Proposition 5.5 of [CS10] that the virtual double category $\text{Mod}(\mathcal{K})$ of monoids and bimodules in $\mathcal{K}$ (see Definition 1.4) has all units. More precisely, the unit bimodule $I_A : A \Rightarrow A$ of a monoid $A = (A, \alpha, \bar{\alpha}, \bar{\alpha})$ is simply $I_A = \alpha$ equipped with left and right actions given by the multiplication $\bar{\alpha} : (\alpha, \alpha) \Rightarrow \alpha$ of $A$, while the unit cell $\bar{\alpha}$ forms the cocartesian cell $A \Rightarrow I_A$ in $\text{Mod}(\mathcal{K})$.

(Weakly) cocartesian paths, like cartesian cells, satisfy a pasting lemma as follows. For a proof of the cocartesian case use the pasting lemma for cartesian cells (Lemma 2.5).

**Lemma 3.7** (Pasting lemma). Consider a configuration of unary cells below, where the vertical source of $\psi_1$ is denoted by $h_0 : C_{10} \to E_0$ and the vertical target of $\psi_n$ by $h_n : C_{mn} \to E_n$.

\[
\begin{array}{cccc}
\phi_{11} & \cdots & \phi_{1m_1} & \phi_{21} & \cdots & \phi_{2m_2} & \cdots & \phi_{n1} & \cdots & \phi_{nm_n} \\
\psi_1 & \psi_2 & \cdots & \psi_n
\end{array}
\]
First assume that the path \((\phi_{11}, \ldots, \phi_{nm})\) is weakly cocartesian. Then the path 
\((\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m_1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nm}))\) is weakly cocartesian if and only if 
\((\psi_1, \ldots, \psi_n)\) is.

Secondly assume that the path \((\phi_{11}, \ldots, \phi_{nm})\) is cocartesian and that, for any 
horizontal morphisms \(J': A' \to E_0\) and \(J'' : E_n \to A''\), the restrictions \(J'(\text{id}, h)\) and \(J''(h, \text{id})\) exist. Then the path 
\((\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m_1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nm}))\) is cocartesian if and only if 
\((\psi_1, \ldots, \psi_n)\) is.

By applying the pasting lemma to compositions \(\psi \circ (\phi_1, \ldots, \phi_n)\) of horizontal cells we find that the collection of horizontal composites and horizontal units in any 
hypermultiplicative double category is coherent, in the sense that if all of the composites 
\((J_{11} \circ \cdots \circ J_{1m_1}), \ldots, (J_{n1} \circ \cdots \circ J_{nm})\) exist, then the composite \((J_{11} \circ \cdots \circ J_{nm})\) exists if and only if

\[
\left( (J_{11} \circ \cdots \circ J_{1m_1}) \circ \cdots \circ (J_{n1} \circ \cdots \circ J_{nm}) \right)
\]

does, in which case they are canonically isomorphic. Notice that this also includes 
isomorphisms of the form \((I_A \circ J) \cong J \cong (J \circ I_B)\), for any \(J : A \to B\), and similar.

In Section 3.5 below we will see that the notions of hypervirtual double category and virtual double category coincide as soon as all horizontal units exist. Consequently we do not distinguish between these notions and refer to either as a
unital virtual double category. Furthermore, in Corollary 3.13 below it is shown that any unital virtual double category that is also a hypervirtual equipment (Definition 2.6) admits all (both unary and nullary) restrictions; such a double category, e.g. \(\mathcal{V} Prof\) (see Example 2.2), we will call a unital virtual equipment. The latter notion coincides with that of ‘virtual equipment’, that was studied in [CS10].

We denote by \(\text{VirtMultiCat}_a\) the full sub-2-category of \(\text{HypVirtMultiCat}\) consisting of unital virtual double categories; in Corollary 3.14 below we will see that, unlike functors between unital virtual double categories, any functor between unital hypervirtual double categories preserves horizontal units. We follow [CS10] in calling a functor \(F : K \to L\) strong if it preserves horizontal composites too; that is, its image of any horizontal cocartesian cell is again cocartesian.

The following is Proposition 5.14 of [CS10].

**Proposition 3.8** (Cruttwell-Shulman). Let \(K\) be a virtual double category. Monoids in \(K\), the morphisms and bimodules between them, as well as the unary cells between those, in the sense of Definition 1.1, combine to form a unital virtual double category \(\text{Mod}(K)\). Moreover, the assignment \(K \mapsto \text{Mod}(K)\) extends to a strict 2-functor \(\text{Mod} : \text{VirtMultiCat} \to \text{VirtMultiCat}_a\) that is right pseudo-adjoint to the forgetful 2-
functor \(\text{VirtMultiCat}_a \to \text{VirtMultiCat}\).

To complete the picture, we will briefly describe the classical notion of ‘pseudo double category’, as introduced by Grandis and Paré in the Appendix to [GP99]; see also Section 2 of [Shu08]. In our terms, a pseudo double category is a virtual double category containing (1,1)-ary cells only, which is equipped with a horizontal composition

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow^f & & \downarrow^g \\
C & \xrightarrow{K} & D \\
\end{array}
\Rightarrow
\begin{array}{ccc}
A & \xrightarrow{J \circ H} & E \\
\downarrow^f & & \downarrow^h \\
C & \xrightarrow{K \circ L} & F \\
\end{array}
\]

as well as horizontal units \(I_A : A \to A\), that come with horizontal coherence cells of the forms \((J \circ H) \circ M \cong J \circ (H \circ M)\), \(I_A \circ J \cong J\) and \(J \circ I_B \cong J\). A pseudo double category with identity cells as coherence cells is called a strict double category.
Any pseudo double category gives rise to a virtual double category which has the same objects and morphisms, while its cells \((J_1, \ldots, J_n) \Rightarrow K\) are cells \(J_1 \odot \cdots \odot J_n \Rightarrow K\) in the double category. The following result, which is Proposition 2.8 of [DPP06] and Theorem 5.2 of [CS10] characterises the virtual double categories obtained in this way as unital virtual double categories with all horizontal composites.

**Proposition 3.9** (MacG. Dawson, Paré and Pronk). A virtual double category is induced by a pseudo double category if and only if it has all horizontal composites and units.

In view of the proposition above, by a double category we shall mean either a hypervirtual double category that has all horizontal composites and units or, equivalently, a pseudo double category in the classical sense. Finally, following [CS10] again, by an equipment we shall mean a double category that has all restrictions.

We close this subsection with two examples of strict double categories.

**Example 3.10.** The hypervirtual double category \(Q(C)\) of quintets in a 2-category \(C\) (Definition 2.19) is clearly a strict double category: the horizontal composite \((j \odot k)\) of two composable morphisms in \(C\) is simply their composite \(k \circ j\).

**Example 3.11.** Let \(2\) denote the ‘walking arrow’-category \((\bot \to \top)\), and notice that it admits all limits and colimits. Assuming the cartesian structure \((\land, \top)\) on \(2\), where \(\land\) denotes conjunction, the virtual double category \(2\text{-Prof}\) of 2-profunctors is a strict equipment as follows. Its objects are large preordered sets \((X, \leq)\), with reflexive and transitive ordering \(\leq\), while its vertical morphisms are monotone functions. A horizontal morphism \(J: A \Rightarrow B\) in \(2\text{-Prof}\) is a modular relation, that is \(J \subseteq A \times B\) such that \(x_1 \leq x_2, (x_2, y_1) \in J\) and \(y_1 \leq y_2\) implies \((x_1, y_2) \in J\); we sometimes shorten \((x, y) \in J\) to \(xJy\). Cells in \(2\text{-Prof}\) are uniquely determined by their sources and targets, with a cell

\[
\begin{array}{ccc}
A & \overset{J}{\longrightarrow} & B \\
\downarrow^{f} & \equiv & \downarrow^{g} \\
C & \longrightarrow & D
\end{array}
\]

existing precisely if \((fx, gy) \in K\) for all \((x, y) \in J\). Finally, the composite \(J \odot H\) of modular relations \(J: A \Rightarrow B\) and \(H: B \Rightarrow E\) is given by the usual composition of relations:

\[
x(J \odot H)z :\Leftrightarrow \bigvee_{y \in B} (xJy \land yHz),
\]

where \(x \in A, z \in E\) and \(\bigvee\) denotes disjunction.

### 3.2 Units in terms of restrictions

The following lemma implies that, for any object \(A\), unit \(I_A\) coincides with the restriction \(A(id, id)\).

**Lemma 3.12.** Consider cells \(\phi\) and \(\psi\) as in the identities below, and assume that either identity holds. The following conditions are equivalent: (a) \(\psi\) is cartesian; (b) both identities hold; (c) \(\phi\) is weakly cocartesian; (d) \(\phi\) is cocartesian; (e) \(\phi\) is
Consequently a cell \( \phi \), of the form as in the above, is cocartesian precisely if it is cartesian.

Proof. We claim that, under the assumption of the identity (A), the implications (a) \( \iff \) (b) \( \iff \) (c) \( \iff \) (d) \( \Rightarrow \) (e) hold. To see this first notice that the interchange axioms (Lemma 1.2) imply \( \phi \circ \psi = \phi \circ \psi' \), so that (a), (b) and (c) are equivalent by Lemma 2.11. Clearly (d) \( \Rightarrow \) (c), while (a) \( \Rightarrow \) (e) follows from applying the pasting lemma to (A). Thus it suffices to show that (b) \( \Rightarrow \) (d). To do so consider any cell \( \chi : J \Rightarrow K \), where \( J = (A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_i} A_i \xrightarrow{f_i'} A_i') \). We have to show that it factors uniquely through \( \phi \) as in Definition 3.1. Assuming (b), it is easily seen that this factorisation is given by \( \chi' := \chi \circ (\text{id}_{J_1}, \ldots, \text{id}_{J_i'}, \psi, \text{id}_{J_i}, \ldots, \text{id}_{J_n}) \); indeed, that \( \chi' \) composed with \( \phi \) gives back \( \chi \) follows from (A), while uniqueness of \( \chi' \) follows from (J).

Next we show that under (e) the identities (A) and (J) are equivalent. If (e) holds, that is \( \phi \) is cartesian, then there exists a unique cell \( \psi' \) such that \( \text{id}_A = \phi \circ \psi' \). Because \( \phi \circ \psi' \circ \phi = \phi \) and \( \phi \) is cartesian, \( \psi' \circ \phi = \text{id}_A \) follows. If (A) holds then \( \psi = \psi \circ \phi \circ \psi' = \psi' \) follows, so that \( \text{id}_A = \psi \circ \phi \circ \psi = \psi \). On the other hand if (J) then \( \psi = \psi' \circ \phi \circ \psi = \psi' \), so that \( \text{id}_A = \psi' \circ \phi = \psi \). We have shown that (A) \( \iff \) (J) under assumption of (e).

From the above we conclude that all five conditions are equivalent as soon as (A) holds. We complete the proof by showing that (J) \( \Rightarrow \) (A) under each condition. Assume (J). If (a) holds then \( \text{id}_A \) factors as \( \text{id}_A = \psi \circ \phi' \); hence \( \phi = \phi \circ \psi \circ \phi' = \phi' \) so that (A) follows. If (c) or (d) holds then \( \text{id}_A \) factors as \( \text{id}_A = \psi' \circ \phi ; \) hence \( \psi = \psi' \circ \phi \circ \psi = \psi' \) so that (A) follows. Of course (b) implies (A), and we have already shown that (A) \( \iff \) (J) under (e). This completes the proof of the main assertion.

For the final assertion notice that if \( \phi \) is cocartesian then we can obtain a factorisation of the form (A), while if it is cartesian then we can obtain one of the form (J), so that the equivalence follows from applying the main assertion.

The following corollary shows that a hypervirtual double category has all restrictions whenever it has all unary restrictions and horizontal units.

**Corollary 3.13.** Consider a unital object \( C \) and let \( \text{id}'_C \) denote the factorisation of \( \text{id}_C \) through the cocartesian cell defining its unit, as in the previous lemma. A nullary cell \( \phi \), as on the left-hand side below, is cartesian if and only if its factorisation \( \phi' \) through \( \text{id}'_C \) is cartesian.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi' / g & & \downarrow g \\
C & = & C
\end{array}
\]
Consequently, if the horizontal unit $I_C$ exists in a hypervirtual equipment then so do all nullary restrictions of the form $C(f,g)$. In particular, in that case the companion and conjoint of any morphism $f : A \to C$ exist.

**Proof.** Since the factorisation $\text{id}_C'$ is cartesian by the previous lemma, this follows immediately from the pasting lemma.

Unlike functors between virtual double categories, functors between hypervirtual double categories necessarily preserve horizontal units, as follows.

**Corollary 3.14.** Any functor between hypervirtual double categories preserves corec-tc cartesian cells that define horizontal units.

**Proof.** Consider a cocartesian cell $\psi$ defining the horizontal unit of an object $A$. By the lemma above the vertical identity cell $\text{id}_A$ factors uniquely through $\psi$ as a cartesian cell $\phi$ which, by Corollary 2.12, is preserved by any functor $F$. Since $F$ preserves the factorisation of $\text{id}_A$, mapping it to $\text{id}_{FA} = F\psi \circ F\phi$, where now $F\phi$ is cartesian, it follows from the lemma above again that $F\psi$ is cocartesian.

Recall that a vertical morphism $f : A \to C$ is called full and faithful whenever the restriction $C(f,f)$ exists and the vertical identity $\text{id}_f$ is cartesian.

**Corollary 3.15.** Consider a vertical morphism $f : A \to C$. If the restriction $C(f,f)$ exists then $f$ is full and faithful precisely if the factorisation of $\text{id}_f$, as shown below, is cocartesian.

$$
\begin{array}{c}
\begin{tikzcd}
A \\
\text{id}_f
\end{tikzcd}
\end{array}
\text{cart}
\begin{array}{c}
\begin{tikzcd}
A \\
\text{id}_f
\end{tikzcd}
\end{array}
$$

**Proof.** This follows from

$\text{id}_f$ is cartesian $\iff \text{id}_f'$ is cartesian $\iff \text{id}_f'$ is cocartesian

where the equivalences follow from the pasting lemma and Lemma 3.12.

### 3.3 Pointwise horizontal composites

Here we return to Example 3.2 to show that, under the condition considered there, the horizontal composite $(J_1 \circ \cdots \circ J_n)$ of a path $(J_1, \ldots, J_n) : A_0 \Rightarrow A_n$ in $\mathcal{V}'$-Prof, of $\mathcal{V}'$-profunctors between $\mathcal{V}'$-categories, exists as soon as the coends on the right below do, for each pair $x \in A_0$ and $y \in A_n$.

$$(J_1 \circ \cdots \circ J_n)(x,y) := \int_{u_1 \in A_1, \ldots, u_n' \in A_n'} J_1(x, u_1) \otimes' \cdots \otimes' J_n(u_n', y)$$

In fact we first capture, in the definition below, the sense in which the assignment above defines $(J_1 \circ \cdots \circ J_n)$ as a horizontal composite that is ‘pointwise’. This notion, which can be thought of informally as “any restriction of $(J_1 \circ \cdots \circ J_n)$ is again a composite”, will be used often in the next section, where we study Kan extensions. While the definition is stated in the general terms of a path $(\phi_1, \ldots, \phi_n)$ of unary cells, to treat the example above we will apply it to the just the single horizontal cell $(J_1, \ldots, J_n) \Rightarrow (J_1 \circ \cdots \circ J_n)$ that consists of the colimiting cocones defining the coends above.
Definition 3.16. Consider a path $\phi = (\phi_1, \ldots, \phi_n)$ of unary cells and of length $n \geq 1$, and assume that $\phi_n$ has non-nullary horizontal source as well as trivial vertical target, as shown in the composite in the left-hand side below. The path $\phi$ is called right pointwise cocartesian if for any morphism $f: B \to A_{nm_n}$ the following holds: the restriction $J_{nm_n}(id, f)$ exists if and only if $k_n(id, f)$ does, and in that case the path $(\phi_1, \ldots, \phi_n)$ is cocartesian, where $\phi_n'$ is the unique factorisation in

$$
\begin{array}{c}
A_{n0} \xrightarrow{J_{n1}} A_{n1} \xrightarrow{\cdots} A_{n(m_n)'} \xrightarrow{J_{n(m_n)'}} B \\
A_{n0} \xrightarrow{J_{n1}} A_{n1} \xrightarrow{\cdots} A_{n(m_n)'} \xrightarrow{\phi_n'} \xrightarrow{K_n(id, f)} B \\
\end{array}
$$

The notion of left pointwise cocartesian path is horizontally dual. A path of unary cells that is both left and right pointwise cocartesian is called pointwise cocartesian.

Notice that any right pointwise, or left pointwise, cocartesian path is cocartesian in particular, by taking $f = id_{A_{nm_n}}$ in the above. Of course a single horizontal cell $\phi: (J_1, \ldots, J_n) \Rightarrow K$, where $n \geq 1$, is called pointwise cocartesian whenever the singleton path $(\phi)$ is pointwise cocartesian. In that case we call $K$ the pointwise composite of $J_1, \ldots, J_n$.

Recall from (5) the construction of a diagram $\sum K(x, y)$ in $\mathcal{V}$, for any path $\mathbf{J}: A_0 \to A_n$ of $\mathcal{V}'$-profunctors, as well as objects $x \in A_0$ and $y \in A_n$.

Proposition 3.17. Let $\mathcal{V}'$ be a monoidal category whose tensor product $\otimes'$ preserves large coends on both sides; let $\mathbf{J}: A_0 \to A_n$ be a path of $\mathcal{V}'$-profunctors of length $n \geq 1$. A horizontal cell $\phi: \mathbf{J} \Rightarrow K$ in $\mathcal{V}'$-$\text{Prof}$ is pointwise cocartesian precisely if, for each $x \in A_0$ and $y \in A_n$, its restriction $(J_1(x, id), \ldots, J_n(id, y)) \Rightarrow K(x, y)$ defines $K(x, y)$ as the coend $\sum_{u_1 \in A_1, \ldots, u_n \in A_n} J_1(x, u_1) \otimes' \cdots \otimes' J_n(u_n, y)$, in the sense of Proposition 2.9

Moreover, given a universe enlargement $\mathcal{V} \to \mathcal{V}'$ with $\mathcal{V}'$ as above, the inclusion $(\mathcal{V}, \mathcal{V}')$-$\text{Prof} \to \mathcal{V}'$-$\text{Prof}$ reflects all (pointwise) cocartesian cells while preserving the horizontal pointwise ones. In particular a cell $\psi$ as above is cocartesian in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ if the induced $\mathcal{V}'$-cocones $(f^*, J_1, \ldots, J_n, g_*)^\mathcal{V}(x, y) \Rightarrow \Delta K(x, y)$ define the $\mathcal{V}$-objects $K(x, y)$ as the $\mathcal{V}'$-coends below, for any $x \in C$ and $y \in D$. In that case if $g = id_{A_n}$ then $\psi$ is right pointwise cocartesian.

Proof. The ‘precisely’-part follows immediately from the previous definition and Proposition 2.9, as the restrictions $(J_1(x, id), \ldots, J_n(id, y)) \Rightarrow K(x, y)$ are obtained from $\phi$ by composing it with the cartesian cells defining $J_1(x, id)$ and $J_n(id, y)$ and then factorising the result through the cartesian cell that defines $K(x, y)$.  

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For the ‘if’-part consider $\mathcal{V}'$-functors $f : A' \to A_0$ and $g : A'' \to A_n$; we have to show that any cell $\chi$ in $\mathcal{V}'$-$\text{Prof}$, of the form below, factors uniquely through the restriction $\phi' : (J_1(f, \text{id}), \ldots, J_n(\text{id}, g)) \Rightarrow K(f, g)$ of $\phi$.

\[
\begin{array}{ccc}
A'_1 \cdots A'_{p'} & \xrightarrow{J'_{q'}} A_1 \cdots A''_{p''} & \xrightarrow{J''_{q''}} A_1 \cdots A''_{q''} \\
\downarrow \cong & \Downarrow \chi & \Downarrow \Delta L \\
C & \xrightarrow{L} & D
\end{array}
\]

To do so notice that $\chi$ restricts, for any sequences of objects $x' \in \text{ob} A'_0 \times \cdots \times \text{ob} A'$ and $x'' \in \text{ob} A'' \times \cdots \times \text{ob} A''_n$, as a family of cocones below, where $\Delta L(f x', g x'')$ is the diagram described in (5).

\[
\begin{array}{c}
J'_1(x'_0, x'_1) \otimes \cdots \otimes J'_{p'}(x'_{p'}, x') \otimes J''_{q'}(f x', g x'') \\
\otimes J''_{q''}(x'', x'') \Rightarrow \Delta L(h x'_0, k x''_q)
\end{array}
\]

By the assumptions, this cocone factors uniquely through

\[
\begin{array}{c}
J'_1(x'_0, x'_1) \otimes \cdots \otimes J'_{p'}(x'_{p'}, x') \otimes J''_{q'}(x'', x'') \Rightarrow \Delta L(h x'_0, k x''_q)
\end{array}
\]

where $\phi'_{(x', x'')}$ denotes the restriction of $\phi'$ to $x' \in A'$ and $x'' \in A''$, as a $\mathcal{V}$-map

\[
\begin{array}{c}
\chi'_{x', x''} : J'_1(x'_0, x'_1) \otimes \cdots \otimes J'_{p'}(x'_{p'}, x') \otimes J''_{q'}(x'', x'') \Rightarrow L(h x'_0, k x''_q)
\end{array}
\]

It remains to check that these $\mathcal{V}$-maps combined satisfy the equivariance axioms of Definition 1.4, so that they form a cell $\chi' : (J'_1, \ldots, J'_{p'}, K(f, g), J''_{q'}, \ldots, J''_{q''}) \Rightarrow L$, which is the factorisation of $\chi$ through $\phi'$. This is straightforward: it follows from the uniqueness of factorisations through the restrictions $\phi'_{(x', x'')}$, together with the equivariance axioms for $\phi'$ and $\chi$.

Because $(\mathcal{V}, \mathcal{V}')$-$\text{Prof} \to \mathcal{V}'$-$\text{Prof}$ is full and faithful reflection of cocartesian cells follows, while its preservation of horizontal cocartesian cells follows from the first assertion combined with its preservation of horizontal weakly cocartesian cells (Proposition 2.9). We conclude that a cell $\psi$, as in the statement, is cocartesian in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ as soon as it is cocartesian in $\mathcal{V}'$-$\text{Prof}$. Using Corollary 3.24, the latter is implied by $\text{cart} \circ \psi \circ \text{cart}$, with the cocartesian cells defining $f^*$ and $g_*$, being cocartesian in $\mathcal{V}'$-$\text{Prof}$, from which the final assertion follows. That, in the case $g = \text{id}_{A_n}$, any restriction of $\psi$ along $J_n(\text{id}, h)$, where $h : B \to A_n$ and in the sense of Definition 3.16, is again cocartesian is clear, so that $\psi$ is right pointwise cocartesian in this case. This completes the proof.

Pointwise cocartesian paths are coherent in the following sense.

**Lemma 3.18.** If the path $\phi = (\phi_1, \ldots, \phi_n)$ is right pointwise cocartesian then any path of the form $(\phi_1, \ldots, \phi'_m)$, as in Definition 3.16, is again right pointwise cocartesian.

**Proof.** Consider $f : B \to A_{nm}$ as in Definition 3.16, and let $\phi'_m$ be the factorisation in (8). Then, for any $h : C \to B$ the following are equivalent: $J_{nm}(\text{id}, f)(\text{id}, h)$ exists; $J_{nm}(\text{id}, h \circ f)$ exists; $K_{n}(\text{id}, h \circ f)$ exists; $K_{n}(\text{id}, f)(\text{id}, h)$ exists, by using the pasting lemma for cartesian cells and the fact that $\phi$ is pointwise cocartesian. This shows that the first assertion of Definition 3.16 holds. Next consider the unique factorisation $\phi''_m$ in $\phi'_m \circ (\text{id}, \ldots, \text{id}, \text{cart}) = \text{cart} \circ \phi''_m$, where the cartesian cells
define \(J_{nmn}(\text{id}, f)(\text{id}, h)\) and \(K_n(\text{id}, f)(\text{id}, h)\) respectively; we have to show that \((\phi_1, \ldots, \phi''_n)\) is cocartesian. To see this consider the following equation, where the identities follow from the definitions of \(\phi'_n\) and \(\phi''_n\) respectively.

\[
\begin{array}{c}
\phi''_n \\
c \\
c \\
\end{array}
\begin{array}{c}
\phi'_n \\
c \\
c \\
\end{array} =
\begin{array}{c}
\cdots \\
\phi_n \\
\phi_n \\
\end{array}
\begin{array}{c}
\phi_n \\
c \\
c \\
\end{array}
\begin{array}{c}
\cdots \\
\phi_n \\
\phi_n \\
\end{array}
\]

Since the composites of cartesian cells in the left-hand and right-hand sides are again cartesian by the pasting lemma, we conclude that \((\phi_1, \ldots, \phi''_n)\) is cocartesian, as \((\phi_1, \ldots, \phi_n)\) is pointwise cocartesian. This concludes the proof. \(\square\)

The pasting lemma for cocartesian paths (Lemma 3.7) induces one for pointwise cocartesian paths, as follows.

**Lemma 3.19** (Pasting lemma). Consider a configuration of unary cells as below and assume that the path \((\phi_1, \ldots, \phi_{nmn})\) is right pointwise cocartesian. The path \((\psi_1 \circ (\phi_1, \ldots, \phi_{1m1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nmn})\) is right pointwise cocartesian if and only if \((\psi_1, \ldots, \psi_n)\) is.

\[
\begin{array}{cccccccc}
\phi_{11} & \cdots & \phi_{1m1} & \phi_{21} & \cdots & \phi_{2m2} & \cdots & \\
\psi_1 & & & \psi_2 & & & & \\
\cdots & & & \cdots & & & & \\
\phi_{n1} & \cdots & \phi_{nmn} & & & & & \\
\psi_n & & & & & & & \\
\end{array}
\]

**Proof.** That both the path \((\phi_{11}, \ldots, \phi_{n1})\) as well as one of the paths \((\psi_1, \ldots, \psi_n)\) or \((\psi_1 \circ (\phi_1, \ldots, \phi_{1m1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nmn})\) is right pointwise cocartesian means that the vertical targets of \(\phi_{nmn}\) and \(\psi_n\) are both the identity on a single object, say, \(A\). It also means that, for any \(f: B \to A\), that the existence of the following restrictions along \(f\) are equivalent: that along the horizontal target of \(\psi_n\); that along the horizontal target of \(\phi_{nmn}\); that along the last horizontal source of \(\phi_{nmn}\). In the case that these restrictions exist we can obtain factorisations \(\psi'_n\) and \(\phi''_{nmn}\), as in Definition 3.16, such that the equations

\[
\begin{array}{cccccccc}
\phi_{n1} & \cdots & \phi_{nmn} & & & & & \\
\psi_n & & & & & & & \\
\end{array}
\begin{array}{cccccccc}
\phi_{n1} & \cdots & \phi_{nmn} & & & & & \\
\psi_n & & & & & & & \\
\end{array} =
\begin{array}{cccccccc}
\phi_{n1} & \cdots & \phi_{nmn} & & & & & \\
\psi_n & & & & & & & \\
\end{array}
\begin{array}{cccccccc}
\phi_{n1} & \cdots & \phi_{nmn} & & & & & \\
\psi_n & & & & & & & \\
\end{array}
\]

hold, where ‘\(c\)’ denotes any of the three cartesian cells defining the restrictions along \(f\). From this we conclude that the unique factorisation that corresponds to \(\psi_n \circ (\phi_{n1}, \ldots, \phi_{nmn})\), as in Definition 3.16 and with respect to the restrictions along \(f\), equals \(\psi'_n \circ (\phi_{n1}, \ldots, \phi''_{nmn})\). The proof now follows from the fact that, assuming that \((\phi_1, \ldots, \phi''_{nmn})\) is cocartesian, the cocartesianness of \((\psi_1, \ldots, \psi'_n)\) is equivalent to that of \((\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nmn})\)\), by the pasting lemma (Lemma 3.7). \(\square\)

### 3.4 Restrictions and extensions in terms of companions and conjoints

Here we make precise the fact that restrictions and extensions can be defined by composing horizontal morphisms with companions and conjoints, as was described in the discussion following Example 2.2.

We start with restrictions. In the setting of virtual double categories the ‘only if’-part of the following lemma has been proved as Theorem 7.16 of [CS10]. In the next section we will see that, when considered in a hypervirtual equipment, the composite of \(f* \circ K^* \circ g*\) considered below is in fact a ‘pointwise’ composite.
Lemma 3.20. In a hypervirtual double category \( \mathcal{K} \) assume that the companion \( f^*: A \Rightarrow C \) and the conjoint \( g^*: D \Rightarrow B \) exist. Then, for each path \( \mathcal{K}(f, g) \) of length \( \leq 1 \), the restriction \( \mathcal{K}(f, g) \) exists if and only if the horizontal composite of the path \( f^* \mathcal{K} g^* \) does, and in that case they are isomorphic.

In detail, for a factorisation as above (where the empty cell is the vertical identity cell \( \text{id}_C \) if \( |\mathcal{K}| = 0 \) the following are equivalent: \( \psi \) is cartesian; \( \phi \) is cocartesian; the identity below holds.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{\text{cart}} & & \downarrow{\text{cart}} \\
C & \xrightarrow{k} & D
\end{array}
\quad \begin{array}{ccc}
D & \xrightarrow{g^*} & B \\
\downarrow{\phi} & & \downarrow{\psi} \\
A & \xleftarrow{f} & B
\end{array}
\]

(9)

Analogous assertions hold for one-sided restrictions: \( K(f, \text{id}) \) exists precisely if \( f^* \Rightarrow \mathcal{K} \) does, while \( K(\text{id}, g) \) exists if and only if \( \mathcal{K} \circ g^* \) does. Finally if \( \mathcal{K} \) is a hypervirtual equipment then the cocartesian cell \( \phi: f^* \mathcal{K} g^* \Rightarrow J \) above is pointwise cocartesian; the same holds for cocartesian cells of the forms \( (f^*, K) \Rightarrow K(f, \text{id}) \) and \( (K, g_* \Rightarrow K(\text{id}, g)) \).

Proof. Assuming that (9) holds, it follows from the companion identities and the conjoint identities (see Lemma 2.11 and its horizontal dual) that precomposing the composite on the left-hand side above with \( \phi \) results in \( \phi \), while postcomposing it with \( \psi \) gives back \( \psi \). Using the uniqueness of factorisations through (co-)cartesian cells, we conclude that either \( \psi \) or \( \phi \) being (co-)cartesian implies the identity above.

Conversely, assume that identities above holds; we will prove that \( \psi \) is cartesian and \( \phi \) cocartesian. For the first it suffices to show that the following assignment of cells is invertible. To see that it is notice that, as a consequence of the assumed identities, the inverse can be given as \( \chi \mapsto \phi \circ (\text{cocart} \circ h, \chi, \text{cocart} \circ k) \), where the weakly cocartesian cells define \( f^* \) and \( g^* \) respectively.

\[
\begin{array}{ccc}
X_0 & \xrightarrow{H_1} & X_1 & \ldots & X_{n-1} & \xrightarrow{H_n} & X_n \\
\downarrow{h} & & \downarrow{\chi'} & & \downarrow{k} & & \downarrow{\psi \circ} \\
A & \xrightarrow{f \circ h} & C & \xrightarrow{g \circ k} & B
\end{array}
\quad \begin{array}{ccc}
X_0 & \xrightarrow{H_1} & X_1 & \ldots & X_{n-1} & \xrightarrow{H_n} & X_n \\
\downarrow{f} & & \downarrow{h} & & \downarrow{\chi} & & \downarrow{\psi} \\
A & \xleftarrow{C} & \xrightarrow{K} & \xrightarrow{D}
\end{array}
\]

To prove that \( \phi \) is cocartesian is to show that, for any paths \( J': A'_0 \Rightarrow A'_1 = A \) and \( J'': B = B'_0 \Rightarrow B'_1 \) the assignment

\[
\begin{array}{ccc}
A'_0 & \xrightarrow{J'} & A & \xleftarrow{C} & D \\
\downarrow{\phi \circ} & & \downarrow{\text{id} \circ \phi} & & \downarrow{\psi \circ} \\
E & \xrightarrow{p} & F & \xleftarrow{L} & P
\end{array}
\quad \begin{array}{ccc}
A'_0 & \xrightarrow{J'} & A & \xleftarrow{C} & D \\
\downarrow{\phi \circ} & & \downarrow{\text{id} \circ \phi} & & \downarrow{\psi \circ} \\
E & \xrightarrow{p} & F & \xleftarrow{L} & P
\end{array}
\]

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is a bijection. That it is follows from the assumed identities: the inverse is given by \( \chi \mapsto \chi \circ (\text{id}, \text{cocart}, \psi, \text{cocart}, \text{id}) \). This completes the proof of the main assertion.

For the final assertion, assume that \( K \) is a hypervirtual equipment. We will prove that the cartesian cells of the forms \((f, K) \Rightarrow K\) are pointwise cartesian; the other two cases can then be easily derived. Thus we consider morphisms \( p: X \rightarrow A \) and \( q: Y \rightarrow D; \) we have to show that the composite of the top two rows in the left-hand side below factors as a cartesian cell through \( K(f \circ p, q) \). To see this consider the equation below, whose identities follow from applying \((9)\) to the bottom two cells of the left-hand side; applying \((9)\) to the top two cells of the third composite; factorising the composite of the two cartesian cells in the third composite through the bottom cartesian cell in the right-hand side which, by the pasting lemma, results in a cartesian cell as shown.

By the uniqueness of factorisations through the bottom cartesian cell in the left-hand side and right-hand side above, we conclude that the composite of the top two rows in the former factors as a cocartesian cell followed by a cartesian one, as required.

A horizontal dual argument applies to cocartesian cells \((K, g^*) \Rightarrow (K, g)\). The proof for a cocartesian cell of the general form \((f, K, g^*) \Rightarrow K(f, g)\) then follows by writing it as a composite of the cocartesian cells \((f, K) \Rightarrow (K, f, K)\) and \((K(f, id), g^*) \Rightarrow (K(f, g), g)\), followed by applying the pasting lemma (Lemma 3.19). This completes the proof.

As corollaries we find that functors of hypervirtual double categories behave well with respect to restrictions and full and faithful morphisms, as follows. The first of these is a variation on the corresponding result for functors between double categories; see Proposition 6.8 of [Shu08].

**Corollary 3.21.** Let \( F: K \rightarrow \mathcal{L} \) be a functor between hypervirtual double categories. Consider morphisms \( f: A \rightarrow C \) and \( g: B \rightarrow D \) in \( K \) and let \( K: C \rightarrow D \) be a path of length \( \leq 1 \). If the companion \( f_*: A \Rightarrow C \) and the conjoint \( g^*: D \Rightarrow B \) exist then \( F \) preserves both the cartesian cell defining the restriction \( K(f, g) \) as well as the cocartesian cell defining the composite of the path \( f_* \Rightarrow K \Rightarrow g^* \).

Under the same conditions the cartesian cells defining the restrictions of the form \( K(f, id) \) and \( K(id, g) \), as well as the cocartesian ones defining the composites of the form \((f, \circ K)\) and \((K \circ g^*)\), are preserved by \( F \).

**Proof.** This follows from the fact that \( F \) preserves the identities of the previous lemma, as well as the (weakly co-)cartesian cells that they contain; the latter by Corollary 2.12.

**Corollary 3.22.** A full and faithful morphism \( f: A \rightarrow C \) is preserved by any functor as soon as its companion \( f_* \) and conjoint \( f^* \) exist.

**Proof.** This follows from the fact that \( F \) preserves the factorisation of \( id_f \) that is considered in Corollary 3.15. Indeed by the previous corollary it preserves the
cartesian cell defining \( C(f, f) \) while, by Corollary 3.14, it preserves the cartesian cell that defines \( C(f, f) \) as the horizontal unit too.

Next we turn to describing (weakly) cocartesian cells in terms of companions and conjoints.

**Lemma 3.23.** Let \( f: A \to C \) be a vertical morphism. If the restriction \( J''(f, \text{id}) \) exists for every \( J''': C \to A'' \) then the weakly cocartesian cell defining the companion \( f_*: A \to C \) is cocartesian. A horizontal dual result holds for the weakly cocartesian cell defining the conjoint \( f^*: C \to A \).

**Proof.** Consider paths \( J': A_0 \to A \) and \( J'': C \to A''_q \); we have to show that the bottom path in

is weakly cocartesian. By the pasting lemma we may equivalently show that the full path above is weakly cocartesian, where the top weakly cocartesian cell corresponds to the cartesian cell as in the previous lemma; it defines \( J_1''(f, \text{id}) \) as the horizontal composite of \( f_* \) and \( J''_1 \). But this follows easily from the fact that the latter two cells compose as the cartesian cell defining \( f_* \) and the identity on \( J''_1 \) (see Lemma 3.20), and the horizontal companion identity for \( f_* \) (see Lemma 2.11).

Together with the pasting lemma for cocartesian paths (Lemma 3.7) the previous result implies the following corollary, whose first part is a variation of Theorem 7.20 of [CS10] for virtual double categories.

**Corollary 3.24.** If the conjoint of \( f: A_0 \to C \) and the composite \( (f^* \circ J_1 \circ \cdots \circ J_n) \) exist then the composite on the left below is weakly cocartesian.

If for any horizontal morphisms \( J': A' \to C \) and \( J''': D \to A'' \) the restrictions \( J'(\text{id}, f) \) and \( J'''(g, \text{id}) \) exist then it is cocartesian as well, while it is right pointwise cocartesian as soon as the composite \( (f^* \circ J_1 \circ \cdots \circ J_n) \) is; in that case any weakly cocartesian cell as on the right above is (right pointwise) cocartesian.

Analogous results hold for composites of the forms \( (J_1 \circ \cdots \circ J_n \circ g_*) \) and \( (f^* \circ J_1 \circ \cdots \circ J_n \circ g_*) \), where \( g: A_n \to D \).
3.5 The equivalence of unital hypervirtual double categories and unital virtual double categories

Closing this section, here we will show the equivalence of the notions of hypervirtual double category and virtual double category in the case that all horizontal units exist. We shall denote by \( \text{VirtMultiCat}_u \subset \text{VirtMultiCat} \) the locally full sub-2-category consisting of virtual double categories that have all horizontal units, and the \textit{normal} functors—that preserve the cocartesian cells defining horizontal units—between them, and all transformations between those. Likewise \( \text{HypVirtMultiCat}_u \subset \text{HypVirtMultiCat} \) denotes the full sub-2-category consisting of hypervirtual double categories that have all horizontal units; remember that any functor between hypervirtual double categories preserves horizontal units (Corollary 3.14).

Recall the strict 2-functor \( U : \text{HypVirtMultiCat}_u \to \text{VirtMultiCat}_u \) of Proposition 1.15, which maps any hypervirtual double category \( K \) to its underlying virtual double category \( U(K) \) consisting of its objects, morphisms and unary cells. Clearly the cocartesian cells in a hypervirtual double category \( K \) form again cocartesian cells in \( U(K) \), so that \( U \) restricts to a strict 2-functor \( U : \text{HypVirtMultiCat}_u \to \text{VirtMultiCat}_u \).

**Theorem 3.25.** Let \( K \) be a virtual double category that has all horizontal units and suppose that, for each object \( A \) in \( K \), a cocartesian cell \( \eta_A \) that defines its unit \( I_A \) has been chosen. Consider to each unary cell \( \phi \) of \( K \), as on the left below, a new unary cell \( \bar{\phi} \) as on the right, that is of the same form, and to each unary cell \( \psi \) as on the left a new nullary cell \( \bar{\psi} \) as on the right.

\[
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \cdots A_n \xrightarrow{J_n} A_n \\
\phi \\
C \xrightarrow{K} D
\end{array}
\quad
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \cdots A_n \xrightarrow{J_n} A_n \\
\bar{\phi} \\
C \xrightarrow{\bar{K}} D
\end{array}
\quad
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \cdots A_n \xrightarrow{J_n} A_n \\
\psi \\
C \xrightarrow{I_C} \bar{C}
\end{array}
\quad
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \cdots A_n \xrightarrow{J_n} A_n \\
\bar{\psi} \\
C \xrightarrow{\bar{I}_C} \bar{C}
\end{array}
\]

The objects and morphisms of \( K \), together with the unary cells \( \bar{\phi} \) and the nullary cells \( \bar{\psi} \), form a hypervirtual double category \( N(K) \) that has all horizontal units. Composition of cells in \( N(K) \) is given by

\[
\bar{\psi} \circ (\bar{\phi}_1, \ldots, \bar{\phi}_n) := \psi' \circ (\phi_1, \ldots, \phi_n)
\]

where \( \psi' \) is the unique factorisation of \( \psi \) through the cocartesian path of cells \((\eta_{\bar{\phi}_1}, \ldots, \eta_{\bar{\phi}_n})\), where \( \eta_{\bar{\phi}_i} := \eta_{C_{\phi_i}} \) if \( \bar{\phi}_i \) is nullary with horizontal target \( C_{\phi_i} \) and \( \eta_{\bar{\phi}_i} := \text{id}_{K_i} \) if \( \bar{\phi}_i \) is unary with horizontal target \( K_i : C_{\phi_i} \to C_i \). The identity cells in \( K \), for morphisms \( J : A \to B \) and \( f : A \to C \), are given by

\[
\text{id}_J := \bar{\text{id}}_J \quad \text{and} \quad \text{id}_f := \bar{\eta}_C \circ f.
\]

The strict 2-functor \( U : \text{HypVirtMultiCat}_u \to \text{VirtMultiCat}_u \) and the assignment \( K \mapsto N(K) \) extend to form a strict 2-equivalence \( \text{HypVirtMultiCat}_u \simeq \text{VirtMultiCat}_u \).

**Example 3.26.** Let \( K \) be any virtual double category. As we have seen in Example 3.6, the monoids and bimodules in \( K \) form a unital virtual double category \( \text{Mod}(K) \). The corresponding unital hypervirtual double category was described in Proposition 1.5.
Proof of Theorem 3.25. That the composition for \( N(\mathcal{K}) \) defined above satisfies the associativity and unit axioms is a straightforward consequence of the those axioms in \( \mathcal{K} \), together with the uniqueness of factorisations \( \psi' \) through cocartesian paths.

To show that \( N(\mathcal{K}) \) has all horizontal units let \( A \) be any object in \( N(\mathcal{K}) \); we claim that \( \eta_A: A \Rightarrow I_A \) defines \( I_A \) as the horizontal unit of \( A \). To see this, consider the identity of \( I_A \) as a nullary cell \( \text{id}_{I_A}: I_A \Rightarrow A \) in \( N(\mathcal{K}) \); we will show that \( \eta_A \) and \( \text{id}_{I_A} \) satisfy the identities of Lemma 3.12. Indeed, we have \( \text{id}_{I_A} \circ \eta_A = (\text{id}_{I_A} \circ \eta_A) = \eta_A = \text{id}_A \) (the identity cell for \( A \) in \( N(\mathcal{K}) \)). On the other hand we have

\[
\eta_A \circ \text{id}_{I_A} = \text{id}_{I_A} \circ (\eta_A \circ \text{id}_{I_A}) = (\text{id}_{I_A}) = \text{id}_{I_A},
\]

where the right-hand side denotes the identity cell for \( I_A \) in \( N(\mathcal{K}) \) and where \( \text{id}'_{I_A} \) is the unique factorisation \( \text{id}'_{I_A} \circ (\text{id}_{I_A} \circ \eta_A) = \text{id}_{I_A} \) in \( \mathcal{K} \), as before. The first identity above follows from the definition \( \circ \) (see the discussion preceding Lemma 1.2), while the second identity follows from the fact that

\[
\text{id}'_{I_A} \circ (\eta_A \circ \text{id}_{I_A}) \circ \eta_A = \text{id}'_{I_A} \circ \text{id}_{I_A} \circ \eta_A = \text{id}_{I_A} \circ \eta_A
\]

in \( \mathcal{K} \), so that \( \text{id}'_{I_A} \circ (\eta_A \circ \text{id}_{I_A}) = \text{id}_{I_A} \) by the uniqueness of factorisations through \( \eta_A \).

We conclude that \( N(\mathcal{K}) \) forms a well-defined hypervirtual double category that has all horizontal units. Next we extend the assignment \( \mathcal{K} \rightarrow N(\mathcal{K}) \) to a strict 2-functor \( N: \text{VirtMultiCat}_* \rightarrow \text{HypVirtMultiCat}_* \). For the action of \( N \) on morphisms consider a normal functor \( F: \mathcal{K} \rightarrow \mathcal{L} \) between unital virtual double categories. Since \( F \) preserves the cocartesian cells \( \eta_A \) of \( \mathcal{K} \) we can obtain, for each object \( A \in \mathcal{K} \), an invertible horizontal cell \( (F_\eta)_A: F\eta_A \Rightarrow F_A \) in \( \mathcal{L} \) that is the unique factorisation in

\[
\begin{align*}
\begin{array}{ccc}
FA & \xrightarrow{\eta_A} & FA \\
\downarrow{F\eta_A} & & \downarrow{F\eta_A} \\
FA & \xrightarrow{F\eta_A} & FA
\end{array}
\end{align*}
\]

We define \( NF: N(\mathcal{K}) \rightarrow N(\mathcal{L}) \) as follows. On objects and morphisms it simply acts like \( F \). To define its action on cells we define, for each \( \phi \) in \( N(\mathcal{K}) \), the cell \( \delta_\phi \in \mathcal{L} \) by \( \delta_\phi = (F_\phi)_C \) if \( \phi \) is nullary, and \( \delta_\phi = \text{id}_{FK} \) otherwise, and set \( (NF)\phi = (\delta_\phi \circ F\phi) \).

That this assignment preserves identity cells is easily checked; that it preserves any composition \( \psi \circ (\phi_1, \ldots, \phi_n) \) in \( N(\mathcal{K}) \), as in (10), is shown by

\[
(NF)(\psi) \circ ((NF)(\phi_1), \ldots, (NF)(\phi_n))
\]

which the third identity is shown as follows. The cells \( (F\psi)' \) and \( \psi' \), on either side, are the factorisations in \( F\psi = (F\psi)' \circ (\eta_{(NF)(\phi_1)}, \ldots, \eta_{(NF)(\phi_n)}) \) and \( \psi = \psi' \circ (\eta_{\phi_1}, \ldots, \eta_{\phi_n}) \) respectively. The identity follows from the fact that

\[
(NF)(\psi) = (F\psi)' \circ (\eta_{\phi_1}, \ldots, \eta_{\phi_n})
\]

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together with the uniqueness of factorisations through the path \((F \eta_{\partial_1}, \ldots, F \eta_{\partial_n})\), which is cocartesian because \(F\) is normal. This concludes the definition of \(N\) on morphisms.

Next consider a transformation \(\xi: F \Rightarrow G\) between normal functors \(F\) and \(G: K \rightarrow L\) of unital virtual double categories. We claim that the components of \(\xi\) again form a transformation \(NF \Rightarrow NG\), which we take to be the image \(N\xi\). For instance, to see that the components of \(\xi\) are natural with respect to a nullary cell \(\phi: J \Rightarrow C\) in \(N(K)\), notice that

\[
(NG)(\phi) \circ (\bar{\xi}_{J_n}, \ldots, \bar{\xi}_{J_1}) = (G_I)_C \circ G\phi \circ (\bar{\xi}_{J_n}, \ldots, \bar{\xi}_{J_1}) = (G_I)_C \circ \xi_{IC} \circ F\phi \\
= (\eta_{GC} \circ \xi_C)' \circ (F_I)_C \circ F\phi = \xi_C \circ (NF)(\phi)
\]

where the last identity follows from the definition of \(\text{id}_{\xi_C}\) in \(N(L)\), while the penultimate one follows from the fact that

\[
(G_I)_C \circ \xi_{IC} \circ F\eta_C = (G_I)_C \circ G\eta_C \circ \xi_C = \eta_{GC} \circ \xi_C \\
= (\eta_{GC} \circ \xi_C)' \circ \eta_{FC} = (\eta_{GC} \circ \xi_C)' \circ (F_I)_C \circ F\eta_C,
\]

by using that \(F\eta_C\) is cocartesian.

That the assignments \(K \mapsto N(K)\), \(F \mapsto NF\) and \(\xi \mapsto N\xi\) combine the form a strict 2-functor \(N: \text{VirtMultiCat}_u \rightarrow \text{HypVirtMultiCat}_u\) follows easily from the uniqueness of the factorisations \((11)\). It is also clear that the identity \((U \circ N)(K) = K\) extends to an identity of strict 2-functors \(U \circ N = \text{id}\). Thus, it remains to show the existence of an invertible 2-natural transformation \(\tau: \text{id} \cong N \circ U\). Given a unital hypervirtual double category \(K\) we define the functor \(\tau_K: K \rightarrow (N \circ U)(K)\) as follows. It is the identity on objects and morphisms, it is given by \(\phi \mapsto \phi\) on unary cells and by \(\psi \mapsto \eta_C \circ \psi\) on nullary cells \(\psi: J \Rightarrow C\). That these assignments preserve composites and identity cells is easily checked; that the family \(\tau = (\tau_K)_K\) is 2-natural is clear. Finally, an inverse functor \(\sigma: (N \circ U)(K) \rightarrow K\) can be given as the identity on objects and morphisms, as \(\phi \mapsto \phi\) on unary cells and as \(\psi \mapsto \varepsilon_C \circ \psi\) on nullary cells \(\psi: J \Rightarrow C\), where \(\varepsilon_C\) is the nullary cartesian cell \(I_C \Rightarrow C\) that corresponds to \(\eta_C: C \Rightarrow I_C\) in Lemma \(3.12\). This completes the proof.

\[\square\]

## 4 Kan extensions

Now that we have most of the necessary terminology of hypervirtual double categories in place, we can begin studying ‘formal category theory’ within such double categories. In this section we generalise the notion of Kan extension in double categories to the setting of a hypervirtual double category \(K\), and consider three strengthenings of this notion. The first of these generalises the notion of ‘Kan extension along enriched functors’, as introduced by Dubuc in \([\text{Dub}70]\) (see also Section 4 of \([\text{Kel}82]\)), while the second generalises the notion of ‘pointwise Kan extension in a 2-category’, that was introduced by Streit in \([\text{Str}74]\); the third then combines the latter two strengthenings. We will compare these strengthenings among each other, as well as compare them with the classical such notions for the vertical 2-category \(V(K)\) that is contained in \(K\). In the next section we consider the notion of yoneda embedding.

### 4.1 Weak Kan extensions

We start with the notion of ‘weak’ left Kan extension, which generalises the notion of Kan extension for double categories that was given in Section 2 of \([\text{GP}08]\).
Definition 4.1. The nullary cell $\eta$, in the composite on the right-hand side below, is said to define $l: A_n \to M$ as the weak left Kan extension of $d: A_0 \to M$ along $(J_1, \ldots, J_n)$ if any nullary cell $\phi$, as on the left-hand side, factors uniquely through $\eta$ as a vertical cell $\phi'$, as shown.

\[
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 & \cdots & A_n \xrightarrow{J_n} A_n \\
\downarrow d & & \downarrow k \\
M & & M
\end{array}
\quad = \quad \begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 & \cdots & A_n' \xrightarrow{J_n} A_n \\
\downarrow d & & \downarrow \eta \\
M & & M
\end{array}
\]

As usual any two nullary cells defining the same weak left Kan extension factor through each other as invertible vertical cells.

The definition above generalises the classical notion\(^1\) of left Kan extension in a 2-category in the following sense. Remember that any hypervirtual double category $\mathcal{K}$ contains a 2-category $V(\mathcal{K})$ consisting of its objects, vertical morphisms and vertical cells.

Proposition 4.2. In a hypervirtual double category $\mathcal{K}$ consider a vertical cell $\eta$, as on the left-hand side below, and assume that the companion $j_*$ of $j: A \to B$ exists.

\[
\begin{array}{ccc}
& A \xrightarrow{\eta} B \\
\downarrow l & \downarrow \quad \downarrow \quad \downarrow \eta' / l \\
M & & M
\end{array}
\quad = \quad \begin{array}{ccc}
& A. \xrightarrow{j} B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \eta' / l \\
M & & M
\end{array}
\]

The vertical cell $\eta$ defines $l$ as the left Kan extension of $d$ along $j$ in $V(\mathcal{K})$ precisely if its factorisation $\eta'$, as shown, defines $l$ as the weak left Kan extension of $d$ along $j_*$ in $\mathcal{K}$.

Proof. This is a simple consequence of the universal property of the weakly cocartesian cell defining $j_*$ as follows. Consider the diagram of assignments below, between collections of cells that are of the form as drawn. The identity above implies that the diagram commutes.

\[
\begin{array}{ccc}
& A \xrightarrow{j_*} B \\
\downarrow \quad \downarrow \quad \downarrow \eta' / k \\
M & & M
\end{array} \quad \begin{array}{ccc}
& B \xrightarrow{\eta \otimes -} \{ \begin{array}{c} \phi \vdash \psi \end{array} \} \\
\downarrow \quad \downarrow \eta \otimes (- \circ j) \\
M & & M
\end{array} \quad \begin{array}{ccc}
& A \xrightarrow{\eta / \psi} B \\
\downarrow \quad \downarrow \quad \downarrow \eta / \psi / k \\
M & & M
\end{array}
\]

Now the cell $\eta'$ defines $l$ as a weak left Kan extension in $\mathcal{K}$ precisely if the top assignment on the left is bijective, while $\eta$ defines $l$ as a left Kan extension in $V(\mathcal{K})$ precisely if the top assignment on the right is bijective. The proof follows from the fact that the bottom assignment is bijective, by the universal property of the weakly cocartesian cell that defines $j_*$.

\[\square\]

\(^1\)For a definition see for instance Section 2 of [Web07].
4.2 (Pointwise) Kan extensions

Definition 4.1 is strengthened to give a notion of left Kan extension in hypervirtual double categories as follows; this generalises the corresponding notion for double categories, that was given in Section 3 of [Kou14] under the name ‘pointwise left Kan extension’.

Definition 4.3. The nullary cell $\eta$, in the composite on the right-hand side below, is said to define $l : A_n \to M$ as the left Kan extension of $d : A_0 \to M$ along $(J_1, \ldots, J_n)$ if any nullary cell $\phi$, this time of the more general form as on the left-hand side below, factors uniquely through $\eta$ as a nullary cell $\phi'$, as shown.

Clearly every left Kan extension is a weak left Kan extension, by restricting the universal property above to cells $\phi$ with $m = 0$. The following result, which is an immediate consequence of Proposition 4.11 below, leads us to the definition of pointwise left Kan extensions. Recall that an object $A$ is called unital whenever its horizontal unit $I_A : A \Rightarrow A$ exists.

Corollary 4.4. Consider a nullary cell $\eta$ as in the composite below, where $n \geq 1$, and suppose that the object $A_n$ is unital. The cell $\eta$ defines $l$ as the left Kan extension of $d$ along $(J_1, \ldots, J_n)$ precisely if, for each vertical morphism $f : B \to A_n$ such that $J_n(id, f)$ exists, the full composite defines $l \circ f$ as the left Kan extension of $d$ along $(J_1, \ldots, J_n(id, f))$.

\[
\begin{array}{ccc}
A_0 & \xrightarrow{J_1} & A_1 \\
 & \vdots & \\
 & \xrightarrow{J_n} & A_n \\
\end{array}
\]

\[
\xrightarrow{\eta}
\]

\[
\begin{array}{ccc}
A_0 & \xrightarrow{J_1} & A_1 \\
 & \vdots & \\
 & \xrightarrow{J_n(id, f)} & B \\
\end{array}
\]

\[
\xrightarrow{\phi'}
\]

Proof. For the ‘if’-part take $f = id_{A_n}$ and use that $J_n(id, id) \cong J_n$. For the ‘only if’-part remember $A_n$ being unital ensures that all conjoints $f^* : A_n \Rightarrow B$ exist, by Corollary 3.13, and apply Proposition 4.11 below.

Definition 4.5. A nullary cell $\eta$ as in the composite below, where $n \geq 1$, is said to define $l$ as the pointwise (weak) left Kan extension of $d$ along $(J_1, \ldots, J_n)$ if, for any $f : B \to A_n$ such that $J_n(id, f)$ exists, the full composite defines $l \circ f$ as the (weak) left Kan extension of $d$ along $(J_1, \ldots, J_n(id, f))$.

\[
\begin{array}{ccc}
A_0 & \xrightarrow{J_1} & A_1 \\
 & \vdots & \\
 & \xrightarrow{J_n(id, f)} & B \\
\end{array}
\]

\[
\xrightarrow{\eta}
\]

\[
\begin{array}{ccc}
A_0 & \xrightarrow{J_1} & A_1 \\
 & \vdots & \\
 & \xrightarrow{J_n} & A_n \\
\end{array}
\]

\[
\xrightarrow{\phi'}
\]

Clearly every pointwise left Kan extension is a pointwise weak left Kan extension; that it is a left Kan extension too follows from the following lemma. As a
consequence of Corollary 4.4, all left Kan extensions in a unital virtual equipment, such as $V$-Prof of $V$-profunctors (Example 1.6), are pointwise left Kan extensions. In Section 4.6 we will see that all pointwise weak left Kan extensions in a hypervirtual double category $K$ are pointwise left Kan extensions as soon as $K$ has ‘cocartesian tabulations’; e.g. $K = (\text{Set}, \text{Set}')$-Prof of Example 1.12. It follows that in a unital virtual equipment admitting cocartesian tabulations, such as $2$-Prof (see Example 3.11 and Example 4.27), the notions of left Kan extension, pointwise weak left Kan extension, and pointwise left Kan extension coincide.

Lemma 4.6. Consider a nullary cell $\eta$ as in the composite above, that defines $l$ as the pointwise (weak) left Kan extension of $d$ along $(J_1, \ldots, J_n)$. The following hold:

(a) $\eta$ defines $l$ as the (weak) left Kan extension of $d$ along $(J_1, \ldots, J_n)$;

(b) for each $f: B \to A_n$ the composite above defines $l \circ f$ as the pointwise (weak) left Kan extension of $d$ along $(J_1, \ldots, J_n(id, f))$.

Proof. For (a) take $f = \text{id}_{A_n}$ in the previous definition and use that $J_n(id, id) \cong J_n$. For (b) remember that $J(id, f)(id, g) \cong J(id, g \circ f)$ for composable $g: C \to B$ and $f: B \to A_n$.

To close this subsection we give an important example of pointwise left Kan extension, that of the restriction along a vertical morphism, as follows. In fact, such Kan extension is ‘absolute’, in the following sense: a nullary cell $\eta$, as in Definition 4.1, is said to define $l$ as an absolute pointwise (weak) left Kan extension of $l$ along $j^*$ if, for any vertical morphism $g: M \to N$, the composite $g \circ \eta$ defines $g \circ l$ as the (pointwise) (weak) left Kan extension of $g \circ d$ along $(J_1, \ldots, J_n)$.

Proposition 4.7. Let $j: B \to A$ be a vertical morphism. Any nullary cartesian cell

\[
\begin{array}{ccc}
A & \overset{j^*}{\longrightarrow} & B \\
\downarrow & \nearrow \phi' & \\
A & \underset{j}{\longrightarrow} & A,
\end{array}
\]

that defines the conjoint of $j$, defines $j$ as the absolute pointwise left Kan extension of $\text{id}_A$ along $j^*$.

Proof. To prove that the cartesian cell above defines $j$ as an absolute left Kan extension we have to show that, for all $g: A \to N$, the assignment below, between collections of cells that are of the form as shown, is a bijection.

\[
\begin{array}{ccc}
\{ & \uparrow & \\
B & \overset{H_1}{\longrightarrow} & B_1, \ldots, B_m, \overset{H_m}{\longrightarrow} B_m \\
\Downarrow \phi' & \nearrow \phi & \\
N \times k & \underset{g \circ \phi' (\text{cart})}{\longrightarrow} & \\
\}
\end{array} = \begin{array}{ccc}
\{ & \uparrow & \\
A & \overset{\text{id}_A, \ldots, \text{id}_A, j^* \circ \text{cart}}{\longrightarrow} & B_1, \ldots, B_m, H_m \\
\Downarrow g & \nearrow \phi & \\
N \times k & \underset{g \circ \phi}{\longrightarrow} & \\
\}
\end{array}
\]

To see that it is, we claim that an inverse can be given by $\phi \mapsto \phi \circ (\text{cocart}, \text{id}, \ldots, \text{id})$, where cocart denotes the weakly cocartesian cell that corresponds to the cartesian cell defining $j^*$, as described before Lemma 2.11. The proof of the claim is a straightforward calculation using the conjoint identities satisfied by cart and cocart (horizontally dual to those in Lemma 2.11) as well as the interchange axioms (Lemma 1.2). We conclude that $j$ forms the absolute left Kan extension of $\text{id}_A$ along $j^*$. That it is in fact an absolute pointwise left Kan extension follows from the simple fact that $j^*(\text{id}, f) \cong (j \circ f)^*$ for any $f: C \to B$. 

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4.3 Pasting lemmas

In this subsection we describe two important pasting lemmas for Kan extension as well as their consequences, which will be used throughout. We begin with the horizontal pasting lemma for Kan extensions which, in view of Proposition 4.20 below, generalises a classical result on the iteration of Kan extensions of enriched functors; see page 42 of [Dub70] or Theorem 4.47 of [Kel82].

**Lemma 4.8** (Horizontal pasting lemma). Consider horizontally composable nullary cells

\[
\begin{array}{cccc}
A_0 & A_1 & \cdots & A_{n'} \\
j_1 & j_n & H_1 & B_1 & \cdots & B_{m'} & H_m & B_m \\
\downarrow d & \downarrow \eta & \downarrow i & \downarrow \xi & \downarrow k \\
M & \\
\end{array}
\]

in a hypervirtual double category, and suppose that \( \eta \) defines \( l \) as the left Kan extension of \( d \) along \( (J_1, \ldots, J_n) \). Then \( \eta \circ \xi \) defines \( k \) as the (weak) left Kan extension of \( d \) along \( (J_1, \ldots, J_n, H_1, \ldots, H_m) \) precisely if \( \xi \) defines \( k \) as the (weak) left Kan extension of \( l \) along \( (H_1, \ldots, H_m) \).

Analogously \( \eta \circ \xi \) defines \( k \) as the pointwise (weak) left Kan extension precisely if \( \xi \) does.

**Proof.** We will consider the case of left Kan extensions; the pointwise/weak cases are analogous. Given any path \( \phi : B_m \Rightarrow C_p \) and any morphism \( h : C_p \Rightarrow M \) consider the following commutative diagram of assignments between collections of cells, that are of the form as shown.

\[
\begin{array}{cccc}
B_m & \xrightarrow{K} & C_p \\
\downarrow \phi & \downarrow \eta \circ \xi & \downarrow \eta \circ \xi \\
M & \xrightarrow{h} & M \\
\end{array}
\]

The proof now follows from the fact that, by definition, the cell \( \eta \) (resp. \( \xi \) or \( \eta \circ \xi \)) defines \( l \) (resp. \( k \)) as a left Kan extension whenever the assignment given by horizontal composition with \( \eta \) (resp. \( \xi \) or \( \eta \circ \xi \)) above is a bijection. □

Next is the vertical pasting lemma.

**Lemma 4.9** (Vertical pasting lemma). Assume that, in the composite below, the path \( (\phi_1, \ldots, \phi_n) \) is cocartesian. The nullary cell \( \eta \) defines \( l \) as the (weak) left Kan extension of \( d \) along \( (K_1, \ldots, K_n) \) precisely if the full composite defines \( l \) as the (weak) left Kan extension of \( d \circ f_0 \) along \( (J_{11}, \ldots, J_{nm}) \).

\[
\begin{array}{cccc}
A_{10} & A_{11} & \cdots & A_{1m_1} \\
j_{11} & j_{1m_1} & H_{11} & A_{1m_1} \\
\downarrow f_0 & \downarrow f_{1m_1} & \downarrow \phi_1 & \downarrow \xi \\
C_0 & C_1 & \cdots & C_{nm} \\
\downarrow d & \downarrow \eta & \downarrow i & \downarrow l \\
K_1 & K_2 & \cdots & K_n \\
\end{array}
\]

Analogously, if \( (\phi_1, \ldots, \phi_n) \) is right pointwise cocartesian (Definition 3.16) then \( \eta \) defines \( l \) as a pointwise (weak) left Kan extension of \( d \) along \( (K_1, \ldots, K_n) \) precisely.
if the full composite defines \( l \) as the pointwise (weak) left Kan extension of \( d \circ f_0 \) along \((J_{11}, \ldots, J_{nm})\).

**Proof.** We will consider in the case of pointwise left Kan extensions; the other cases follow by, in the following, either taking \( f = \text{id}_{A_{nm}} \) or choosing the path \( H \) to be empty. Hence we assume that \((\phi_1, \ldots, \phi_n)\) is right pointwise cocartesian.

Let \( f : B \to A_{nm} \) be any morphism and remember from Definition 3.16 that the restriction \( J_{nm} \circ (\text{id}, f) \) exists if and only if \( K_n \circ (\text{id}, f) \) does. In that case \((\phi_1, \ldots, \phi_n')\) is cocartesian, where \( \phi_n' \) is obtained from \( \phi_n \) as in the factorsation (8). We consider the following assignments between collections of cells, that are of the form as shown, where \( \zeta := \eta \circ (\text{id}, \ldots, \text{id}, \text{cart}) \), with \( \text{cart} \) the cartesian cell defining \( K_n \circ (\text{id}, f) \), and \( \theta := \eta \circ (\phi_1, \ldots, \phi_n) \circ (\text{id}, \ldots, \text{id}, \text{cart}) \), with \( \text{cart} \) the cartesian cell defining \( J_{nm} \circ (\text{id}, f) \). It follows from (8) that the diagram commutes.

Now notice that \( \eta \) defines \( l \) as a pointwise left Kan extension precisely if, for every \( f : B \to A_{nm} \), the top assignment is a bijection, while \( \eta \circ (\phi_1, \ldots, \phi_n) \) does so precisely if the left assignment is a bijection, for every \( f : B \to A_{nm} \). Thus the proof follows from the fact that \((\phi_1, \ldots, \phi_n')\) is cocartesian, so that the assignment on the right is a bijection.

Applying the vertical pasting lemma to a single horizontal cocartesian cell we obtain the following corollary.

**Corollary 4.10.** In a hypervirtual double category \( \mathcal{K} \) consider a nullary cell \( \eta \) as on the left-hand side below.

If the composite \((J_1 \circ \cdots \circ J_n)\) exists then \( \eta \) defines \( l \) as the (weak) left Kan extension of \( d \) along \((J_1, \ldots, J_n)\) precisely if its factorisation \( \eta' \), as shown, defines \( l \) as the (weak) left Kan extension of \( d \) along \((J_1 \circ \cdots \circ J_n)\).

Moreover if the cocartesian cell above is right pointwise (Definition 3.16), then the analogous result for cells \( \eta \) defining \( l \) as a pointwise (weak) left Kan extension holds.

The following result was used in the proof of Corollary 4.4.
Proposition 4.11. Assume that the nullary cell \( \eta \) in the composites below defines \( l \) as the left Kan extension of \( d \) along \((J_1,\ldots,J_n)\) and let \( f : B \to A_0 \) be a morphism whose conjoint exists. The composite on the left defines \( l \circ f \) as the left Kan extension of \( d \) along \((J_1,\ldots,J_n,f^*)\).

If the restriction \( J_n(id,f) \) exists as well then the composite on the right defines \( l \circ f \) as the left Kan extension of \( d \) along \((J_1,\ldots,J_n(id,f))\).

Proof. The composite on the left above can be rewritten as \( \eta \circ (l \circ \text{cart}) \) by the axioms for horizontal composition (Lemma 1.2). Since both \( \eta \) and \( l \circ \text{cart} \) define left Kan extensions, the first by assumption and the latter by the previous proposition, so does their composite by the horizontal pasting lemma.

For the second assertion assume that the restriction \( J_n(id,f) \) exists and is defined by the cartesian cell in the composite above. By Lemma 3.20 we know that \( J_n(id,f) \) forms the horizontal composite of \( J_n \) and \( f^* \), whose defining cocartesian cell \( \varepsilon : (J_n,f^*) \Rightarrow J_n(id,f) \) is the unique factorisation of the composite \( id_{J_n} \circ \text{cart} \), in the composite on the left, through the cartesian cell defining \( J_n(id,f) \), in the composite on the right. We conclude that the composite on the left factors through that on the right as the path of cells \( (id_{J_1},\ldots,id_{J_n'},\varepsilon) \). Since \( \varepsilon \) is cocartesian it follows from the vertical pasting lemma that either composite defines \( l \circ f \) as a left Kan extension if the other does, from which the proof follows.

Similar to the previous result is the following.

Proposition 4.12. Let \( f : A \to C \) be a morphism whose conjoint exists. The nullary cell \( \eta \) in the composite below defines \( l \) as the (pointwise) (weak) left Kan extension of \( d \circ f \) along \((J_1,\ldots,J_n)\) precisely if the full composite defines \( l \) as the (pointwise) (weak) left Kan extension of \( d \) along \((f^*,J_1,\ldots,J_n)\).

Proof. The composite above can be rewritten as \( (d \circ \text{cart}) \circ \eta \) by the axioms for horizontal composition, so that the proof follows from Proposition 4.7 and the horizontal pasting lemma.

The following generalises a well known result for 2-categories; see Proposition 22 of [Str74]. Remember that a vertical morphism \( f : A \to C \) is called full and faithful (Definition 2.8) whenever the restriction \( C(f,f) \) exists and the identity cell \( id_f \) is cartesian.

Proposition 4.13. Let \( f : A_n \to B \) be a full and faithful morphism whose companion \( f_* \) exists and assume that the nullary cell \( \eta \) in the composite below defines
as the pointwise (weak) left Kan extension of \( d \) along \((J_1, \ldots, J_n, f_*)\). The full composite defines \( l \circ f \) as the (weak) left Kan extension of \( d \) along \((J_1, \ldots, J_n)\); in particular if \( n = 0 \) then the composite, in that case a vertical cell, is invertible.

**Proof.** By applying Corollary 3.15 to \( f : A_n \to B \), and by factorising the identity considered there through the cartesian cell that defines \( f_* \), we find that the weakly cocartesian cell in the composite above equals the composite below where \( \text{id}'_f \), which is the factorisation of \( \text{id}_f \) through \( B(f, f) \), is cocartesian. That \( \eta \) composed with the cartesian cell below defines \( l \circ f \) as the (weak) left Kan extension of \( d \) along \((J_1, \ldots, J_n, B(f, f))\) by definition (Definition 4.5). That the resulting composite \( \eta \circ (\text{id}_{J_1}, \ldots, \text{id}_{J_n}, \text{cart}) \), after being composed with \( \text{id}'_f \), again defines \( l \circ f \) as a (weak) left Kan extension follows from the vertical pasting lemma. Since the resulting composite coincides with the composite above, this proves the main assertion.

The final assertion, for the case \( n = 0 \), follows immediately from the easy lemma below.

**Lemma 4.14.** A vertical cell defines a (weak) left Kan extension if and only if it is invertible.

The following proposition is well known for 2-categories. We say that a vertical morphism \( f : M \to N \) is **cocontinuous** if, for any nullary cell

That defines \( l \) as the left Kan extension of \( d \) along \((J_1, \ldots, J_n)\), the composite \( f \circ \zeta \) defines \( f \circ l \) as the left Kan extension of \( f \circ d \) along \((J_1, \ldots, J_n)\). Notice that if \( l \) is a pointwise left Kan extension then \( f \circ l \) is again pointwise.

**Proposition 4.15.** Left adjoints are cocontinuous.

**Proof.** Let \( f : M \to N \) be left adjoint to \( g : N \to M \), with vertical cells \( \eta : \text{id}_M \Rightarrow g \circ f \) and \( \varepsilon : f \circ g \Rightarrow \text{id}_N \) forming the unit and counit. Consider any nullary cell \( \zeta \) as above, that defines \( l \) as a left Kan extension; we have to show that \( f \circ \zeta \) defines \( f \circ l \)
as a left Kan extension. To this end consider the commuting diagram of assignments between collections of cells below.

\[
\begin{array}{c}
A_n \Rightarrow B_1 \cdots \Rightarrow B_m \Rightarrow B_m' \\
\downarrow f \circ l \quad \Downarrow \phi' \quad \Downarrow k \\
N \quad \Downarrow \quad \Downarrow \quad \Downarrow \\
J_1 \Rightarrow A_1 \cdots \Rightarrow A_n' \Rightarrow A_n' \\
\downarrow f \circ d \\
N \quad \Downarrow \phi \quad \Downarrow k \\
\end{array}
\]

\[
(\eta \circ l) \circ (g \circ -) \quad \Downarrow \\
A_n \Rightarrow B_1 \cdots \Rightarrow B_m \Rightarrow B_m' \\
\downarrow l \quad \Downarrow \psi \quad \Downarrow g \circ k \\
\Downarrow \psi' \quad \Downarrow \quad \Downarrow \\
J_1 \Rightarrow A_1 \cdots \Rightarrow A_n' \Rightarrow A_n' \\
\downarrow d \\
N \quad \Downarrow \psi \\
\end{array}
\]

The vertically drawn assignments above are bijections. Indeed, the triangle identities for \(\eta\) and \(\varepsilon\) imply that their inverses are given by composition with \(\varepsilon\) on the right. The bottom assignment is a bijection as well, because \(\zeta\) defines \(l\) as a left Kan extension. We conclude that the top assignment is a bijection too, showing that \(f \circ \zeta\) defines \(f \circ l\) as left Kan extension as required.

The following result shows that the condition of Definition 4.3 can be simplified in a double category. Combined with Corollary 4.10 it follows that the latter, when considered in a double category, coincides with Definition 3.10 of [Kou14].

**Proposition 4.16.** When considered in a double category the condition of Definition 4.3 can be reduced to the existence of unique factorisations of nullary cells of the form \(\phi\): \((J_1, \ldots, J_n, H) \Rightarrow M\).

We shall prove a more general result instead, in the form of the following lemma; applying it to the cocartesian cells that define composites of paths \((H_1, \ldots, H_m)\), as considered in Definition 4.3, proves the proposition above.

**Lemma 4.17.** Consider a path of cells

\[
A_0 \Rightarrow A_1 \cdots \Rightarrow A_n' \Rightarrow A_n' \\
\Downarrow j_1 \quad \Downarrow \cdots \Downarrow j_n \quad \Downarrow j_n \\
A_0 \Rightarrow A_1 \cdots \Rightarrow A_n' \\
\Downarrow j_0 \quad \Downarrow \cdots \Downarrow j_n \\
\Downarrow j_n \\
K_1 \quad \Downarrow \phi_1 \quad \Downarrow C_1 \\
\Downarrow \cdots \quad \Downarrow \cdots \Downarrow \cdots \\
K_m \quad \Downarrow \phi_m \quad \Downarrow C_m \\
\Downarrow \cdots \quad \Downarrow \cdots \Downarrow \cdots \\
M \quad \Downarrow f_m \\
\Downarrow \cdots \quad \Downarrow \cdots \Downarrow \cdots \\
C_m
\]

with \(n, m \geq 1\) and assume that the subpath \((\phi_1, \ldots, \phi_m)\) is cocartesian. Remember that, for each object \(M\), precomposition with the full path gives a bijective correspondence between nullary cells of the form

\[
A_0 \Rightarrow A_1 \cdots \Rightarrow A_n' \Rightarrow A_n' \\
\Downarrow d \quad \Downarrow \nu \quad \Downarrow h \\
M \quad \Downarrow h \circ f_m \\
\Downarrow \cdots \quad \Downarrow \cdots \Downarrow \cdots \\
C_m
\]

and those of the form

\[
A_0 \Rightarrow A_1 \cdots \Rightarrow A_n' \Rightarrow A_n' \\
\Downarrow d \quad \Downarrow \nu \quad \Downarrow h \circ f_m \\
M \quad \Downarrow h \circ f_m \\
\Downarrow \cdots \quad \Downarrow \cdots \Downarrow \cdots \\
C_m
\]

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see Definition 3.1. Under this correspondence a unique factorisation of the cells ψ through the nullary cell η on the left below corresponds to a unique factorisation of the cells χ through the composite on the right.

Proof. Consider the following assignments between collections of nullary cells, that are of the form as shown. Notice that the diagram commutes.

Now the vertical assignments are bijective, because the path of those in Section 2.21 of [Kou15a]. Notice that the horizontal composite on the right.

4.4 Pointwise Kan extensions in \((\mathcal{V}, \mathcal{V}')\)-Prof

As an example, here we study pointwise left Kan extensions in the hypervirtual equipment \((\mathcal{V}, \mathcal{V}')\)-Prof, of \(\mathcal{V}\)-profunctors between \(\mathcal{V}'\)-categories (see Example 1.12), and show that they can be described in terms of (generalised) \('\mathcal{V}\)-weighted colimits', as one would expect. The results of this section are straightforward generalisations of those in Section 2.21 of [Kou15a].

In the case of a closed symmetric monoidal category \(\mathcal{V}\) the notion of Kan extension along \(\mathcal{V}\)-functors evolved as follows. Firstly Day and Kelly gave a definition for \(\mathcal{V}\)-functors into a cocomplete \(\mathcal{V}\)-category in [DK69], which was extended to \(\mathcal{V}\)-functors into a copowered \(\mathcal{V}\)-category by Dubuc [Dub70], and later to \(\mathcal{V}\)-functors into a general \(\mathcal{V}\)-category by Borceux and Kelly [BK75]. Following this, [Kel82] describes how to remove the assumption of a closed structure on \(\mathcal{V}\), by considering a universe enlargement \(\mathcal{V} \to \mathcal{V}'\) with \(\mathcal{V}'\) instead being closed symmetric monoidal.

Throughout this subsection let \(\mathcal{V} \to \mathcal{V}'\) be a universe enlargement (Definition 1.10); remember that we assume only a monoidal structure on \(\mathcal{V}\), and a closed monoidal structure on \(\mathcal{V}'\). Recall that we write \(I\) for the unit \(\mathcal{V}\)-category, with the unit \(I = I(\ast, \ast)\) of \(\mathcal{V}\) as its single hom-object, and that we regard \(I\) as the unit \(\mathcal{V}'\)-category as well. We identify \(\mathcal{V}'\)-functors \(I \to A\) with objects in \(A\) and \(\mathcal{V}\)-profunctors \(I \to I\) with \(\mathcal{V}\)-objects; cells between such profunctors are identified with \(\mathcal{V}\)-maps. Notice that the horizontal composite \((H_1 \otimes \cdots \otimes H_n)\) of \(\mathcal{V}\)-profunctors \(H_i: I \to I\) always exists; it corresponds to the tensor product of \(\mathcal{V}\)-objects \(H_1 \otimes' \cdots \otimes' H_n\).

By a \(\mathcal{V}\)-weight on a \(\mathcal{V}\)-category \(A\) we mean a path \((A \xrightarrow{A_0} A_1, \ldots, A_n \xrightarrow{A_n} I)\) of \(\mathcal{V}\)-profunctors between \(\mathcal{V}\)-categories. It is called small if each of the \(\mathcal{V}\)-categories
$A, A_1, \ldots, A_n$ is small. As usual we denote a singleton $\mathcal{V}$-weight $(J)$ on $A$ simply by $J$.

**Definition 4.18.** Let $(J_1, \ldots, J_n)$ a $\mathcal{V}$-weight on a $\mathcal{V}$-category $A$, $d: A \to M$ be a $\mathcal{V}^e$-functor, and $l$ an object of $M$. A cell $\eta$ in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$, as in the right-hand side below, is said to define $l$ as the $(J_1, \ldots, J_n)$-weighted colimit of $d$ if every cell $\phi$ below factors uniquely through $\eta$ as shown.

\[
\begin{array}{c}
  A \xrightarrow{J_i} A_1 \cdots A_n' \xrightarrow{J_n} I \xrightarrow{M} I \\
  d \xrightarrow{H} \eta \xrightarrow{k} I = A \xrightarrow{J_i} A_1 \cdots A_n' \xrightarrow{J_n} I \xrightarrow{M} I \\
\end{array}
\]

Since $\mathcal{V} \to \mathcal{V}'$ induces a full and faithful inclusion $\mathcal{V}$-$\text{Prof} \to (\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ (see Example 1.20) it follows that, in the case $\eta$ is contained in $\mathcal{V}$-$\text{Prof}$, the existence of the factorisations above reduces to the existence of such factorisations in $\mathcal{V}$-$\text{Prof}$.

When reduced to the case of $(1,0)$-ary cells $\eta: J_1 \Rightarrow M$ in $\mathcal{V}$-$\text{Prof}$, the definition above extends the classical definition of weighted colimit as follows. Note that if $\mathcal{V}$ is closed symmetric monoidal then singleton $\mathcal{V}$-weights $J$ on $A$ can be identified with $\mathcal{V}$-functors $J: A^{op} \to \mathcal{V}$, using the isomorphism $A^{op} \otimes I \cong A^{op}$ (see Example 1.6). This recovers the classical definition of $\mathcal{V}$-weight given in e.g. Section 3.1 of [Kel82], where such $\mathcal{V}$-functors are called ‘indexing types’. Using this identification the cell $\eta: J_1 \Rightarrow M$ can be regarded as a $\mathcal{V}$-natural transformation between the $\mathcal{V}$-functors $J_1: A^{op} \to \mathcal{V}$ and $M(d, l): A^{op} \to \mathcal{V}$.

**Proposition 4.19.** Let $d, J_1, l$ and $\eta$ be as above. If $\mathcal{V}$ is closed symmetric monoidal then the cell $\eta$ defines $l$ as the $J$-weighted colimit of $d$, in the above sense, precisely if the pair $(l, \eta)$, where $\eta$ is regarded as a $\mathcal{V}$-natural transformation $J_1 \Rightarrow M(d, l)$ of $\mathcal{V}$-functors $A^{op} \to \mathcal{V}$, forms the ‘colimit of $d$ indexed by $J_1$’ in the sense of Section 3.1 of [Kel82].

**Proof.** Since the factorisations of the definition above reduce to factorisations in $\mathcal{V}$-$\text{Prof}$, as discussed above, the horizontal dual of the proof of Proposition 2.23 of [Kou15a] applies verbatim; the only difference is that there $A$ is assumed to be small.

Next we describe Kan extensions in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ in terms of weighted colimits and, as a consequence, obtain an ‘enriched variant’ of Proposition 4.2. Remember that the vertical cells of $\mathcal{V}$-$\text{Prof}$ are $\mathcal{V}$-natural transformations between functors, in the classical sense.

**Proposition 4.20.** Consider a cell $\eta$ in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ as in the composite on the left below, where $A_0, \ldots, A_n$ are $\mathcal{V}$-categories. It defines $l$ as the pointwise left Kan extension of $d$ along $(J_1, \ldots, J_n)$ if and only if, for each $y \in A_n$, the full composite defines by as the $(J_1, \ldots, J_n(id, y))$-weighted colimit of $d$. In particular, pointwise left Kan extensions along a $\mathcal{V}$-weight $J: A \Rightarrow I$ coincide with $J$-weighted colimits.

\[\begin{array}{c}
  A_0 \xrightarrow{J_i} A_1 \cdots A_n' \xrightarrow{J_n(id, y)} I \\
  A_0 \xrightarrow{d} A_1 \cdots A_n' \xrightarrow{J_n} A_n \\
  \downarrow \text{cart} \downarrow y \downarrow \text{cart} \\
  A \xrightarrow{J_i} A_1 \cdots A_n' \xrightarrow{J_n} A_n \\
  \downarrow \text{cocart} \downarrow y \downarrow \text{cocart} \\
  B = A \xrightarrow{l} B \\
\end{array}\]
Finally assume that \( \mathcal{V} \) is closed symmetric monoidal, and consider a factorisation in \( \mathcal{V} \text{-}\text{Prof} \) as on the right above. The cell \( \zeta' \) defines \( l \) as a pointwise left Kan extension in \( (\mathcal{V}, \mathcal{V}')\text{-}\text{Prof} \), precisely if the \( \mathcal{V} \)-natural transformation \( \zeta : d \Rightarrow l \circ j \) ‘exhibits \( l \) as the left Kan extension of \( d \) along \( j' \)’, in the sense of Section 4.1 of \( [\text{Kel}82] \).

Proof. The ‘only if’-part of the first assertion follows immediately from Definition 4.5 by restricting the universal property of Definition 4.3, for the composite on the left above, to cells of the form \((J_1, \ldots, J_n (\text{id}, y), H) \Rightarrow M \), where \( H : I \Rightarrow I \).

For the ‘if’-part consider any \( \mathcal{V}' \)-functor \( f : B \rightarrow A_n \) as well as any cell \( \phi \) as in the composite on the left below; we have to show that it factors uniquely through the composition on the right, as a cell \( \phi' : (H_1, \ldots, H_m) \Rightarrow M \).

In order to obtain the \((x, x_1, \ldots, x_m)\)-component of \( \phi' \), where \( x \in B \) and each \( x_i \in B_i \), consider the full composite on the left, and notice that it can be regarded as a cell \((J_1, \ldots, J_n (\text{id}, f x), H_1 (x, x_1) \otimes \cdots \otimes H_m (x_m, x_m)) \Rightarrow M \). By assumption on \( \eta \), the latter cells factor uniquely through the composite on the left of (12), with \( y = f x \), as \( \mathcal{V}'\)-maps

\[ \phi'_{(x, x_1, \ldots, x_m)} : H_1 (x, x_1) \otimes \cdots \otimes H_m (x_m, x_m) \Rightarrow M (f x, k x) \]

which, we claim, combine to form a cell \( \phi' : (H_1, \ldots, H_m) \Rightarrow M \). The unique factorisations above then guarantee that \( \phi' \) is unique such that \( \phi = \eta \circ \phi' \). The proof of the claim consists of showing that the \( \mathcal{V}'\)-maps \( \phi'_{(x, x_1, \ldots, x_m)} \) are \( \mathcal{V}'\)-natural in each of the coordinates \( x, x_1, \ldots, x_m \). This is a straightforward consequence of the \( \mathcal{V}'\)-naturality of the cells \( \eta \) and \( \phi \), together with the uniqueness of the factorisations above, which we leave to the interested reader (see also the proof of Proposition 2.24 of \( [\text{Kou}15a] \), where this claim was proved in the horizontal dual case for cells \( \eta \) and \( \phi \) with \( n = m = 1 \)). This completes the proof of the first assertion.

To prove the second assertion notice that \( \zeta' \) is given by the \( \mathcal{V}\)-maps

\[ B (j x, y) \xrightarrow{\zeta} M ((j x, l y) \xrightarrow{M (\zeta x, l y)} M (d x, l y)) \]

for pairs \( x \in A \) and \( y \in B \). The proof follows from applying to \( \zeta \) the first assertion above, Proposition 4.19 and the horizontal dual of condition (ii) of Theorem 4.6 of \( [\text{Kel}82] \), which lists equivalent conditions defining a \( \mathcal{V} \)-natural transformation to ‘exhibit a right Kan extension’. \( \square \)

As is to be expected the pointwise left Kan extension along a \( \mathcal{V} \)-profunctor \( J : A \Rightarrow B \) can be constructed out of weighted colimits whenever \( B \) is a \( \mathcal{V} \)-category, as follows.

**Proposition 4.21.** In \( (\mathcal{V}, \mathcal{V}')\text{-}\text{Prof} \) consider a \( \mathcal{V} \)-profunctor \( J : A \Rightarrow B \) between \( \mathcal{V} \)-categories \( A \) and \( B \), as well as a \( \mathcal{V}' \)-functor \( d : A \Rightarrow M \). Assume given, for each \( y \in B \), a cell \( \eta_y \) as on the left below, that defines the \( J (\text{id}, y) \)-weighted colimit of \( d \).
There is exactly one way of extending the assignment \( l : y \mapsto ly \) to a \( \mathcal{V}' \)-functor \( l : B \to M \), such that the cells \( \eta_y \) combine to form a cell \( \eta \) as on the right. Moreover \( \eta \) defines \( l \) as the pointwise left Kan extension of \( d \) along \( J \).

**Proof.** For each \( y, z \in B \) we define the action of \( l \) on the hom-object \( B(y, z) \) to be the unique factorisation below, where the top cell in the composite on the left-hand side is given by the action \( \rho : J(x, y) \otimes B(y, z) \to J(x, z) \) of \( B \) on \( J \).

\[
\begin{array}{c}
A \overset{J(id, y)}{\rightarrow} B(y, z) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
A \overset{J(id, z)}{\rightarrow} I \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
M \quad \quad \quad \quad \quad \quad M
\end{array}
\]

\[
\begin{array}{c}
A \overset{J(id, y)}{\rightarrow} J(id, z) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
A \overset{J(id, y)}{\rightarrow} I \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
M \quad \quad \quad \quad \quad \quad M
\end{array}
\]

That these factorisations combine to form a \( \mathcal{V}' \)-functor \( l : B \to M \), that is they preserve composition and units, follows easily from the fact that the action of \( B \) on \( J \) is associative and unital, together with the uniqueness of the factorisations through the cells \( \eta_y \). Having constructed \( l : B \to M \), notice that the factorisations above imply that the cells \( \eta_y \) combine to form a nullary cell \( \eta : J \Rightarrow M \) as asserted; in fact, their uniqueness implies that \( l \) is unique with this property. That \( \eta \) defines \( l \) as a pointwise left Kan extension follows from the previous proposition and the fact that, for each \( y \in B \), it restricts along \( J(id, y) \) to a cell that defines a weighted colimit.

\( \square \)

We record the following simple lemma for later use. Remember that the 2-category \( \mathcal{V}'\text{-Cat} \) of \( \mathcal{V}' \)-categories admits a monoidal structure when \( \mathcal{V}' \) is symmetric monoidal; see for instance Section 1.4 of [Kel82]. It is straightforward to show that this structure extends to a ‘monoidal structure’ on the hypervirtual double category \( \mathcal{V}'\text{-Prof} \) of \( \mathcal{V}' \)-profunctors, in a sense that is the evident adaptation of that of ‘monoidal double category’, as given in Definition 9.1 of [Shu08]. While we shall not make this precise, we will use its underlying tensor product \( \otimes : \mathcal{V}'\text{-Prof} \times \mathcal{V}'\text{-Prof} \rightarrow \mathcal{V}'\text{-Prof} \). On \( \mathcal{V}' \)-categories and \( \mathcal{V}' \)-functors it is given as usual, while the tensor product \( J \otimes H : A \otimes C \rightarrow B \otimes D \) of \( \mathcal{V}' \)-profunctors is given by \((J \otimes H)((x, z)(y, w)) \coloneqq J(x, y) \otimes H(z, w)\). Its action on a pair of cells in \( \mathcal{V}'\text{-Prof} \) is likewise given by tensoring their components, as well as using the symmetric structure of \( \mathcal{V}' \). Given a symmetric universe enlargement \( \mathcal{V} \rightarrow \mathcal{V}' \), notice that the tensor product on \( \mathcal{V}'\text{-Prof} \) restricts to one on \( (\mathcal{V}, \mathcal{V}')\text{-Prof} \).

**Lemma 4.22.** Let \( \mathcal{V} \rightarrow \mathcal{V}' \) be a symmetric universe enlargement, \( A \) and \( B \) be \( \mathcal{V} \)-categories, and \( J \) a \( \mathcal{V} \)-weight on \( B \). A cell \( \eta \) in \((\mathcal{V}, \mathcal{V}')\text{-Prof}\), as in the composite below, defines \( l \) as a pointwise left Kan extension of \( d \) along \( I_A \otimes J \) precisely if, for each \( u \in A \), the full composite, whose top cell is induced by the unit \( A_u : I' \rightarrow A(u, u) \), defines \( lu \) as the \( J \)-weighted colimit of \( d \circ (u \otimes \text{id}) \).
Proof. By Proposition 4.20 the cell \( \eta \) defines \( \varepsilon \) as a pointwise left Kan extension precisely if, for all \( u \in A \), its restriction along \((I_A \otimes J)(\text{id}, u) \cong u^* \otimes J\) defines \( lu \) as a weighted colimit. Using Proposition 3.17 it is easy to see that \( A_u \otimes \text{id}_J \) factors through \( u^* \otimes J \) as a right pointwise cocartesian cell, so that the proof follows from the vertical pasting lemma (Lemma 4.9).

4.5 Tabulations

Here the notion of tabulation, that was introduced by Grandis and Paré in [GP99] in the setting of double categories, is generalised to the setting of hypervirtual double categories. In the next subsection we show that in hypervirtual double categories that have all ‘cocartesian tabulations’, all pointwise Kan weak extensions are pointwise Kan extensions (see Definition 4.5).

Definition 4.23. Given a horizontal morphism \( J : A \to B \) in a hypervirtual double category \( K \), the tabulation \( \langle J \rangle \) of \( J \) consists of an object \( \langle J \rangle \) equipped with a unary cell \( \pi \) as on the left below, satisfying the following 1-dimensional and 2-dimensional universal properties.

\[
\begin{array}{c}
\pi_A \\
\downarrow \pi \\
A \xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
\phi_A \\
\downarrow \phi \\
A \xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
X \\
\phi_X \\
\downarrow \phi \\
\downarrow \phi_B \\
A \xrightarrow{J} B
\end{array}
\]

Given another unary cell \( \phi \) as on the right above, the 1-dimensional property states that there exists a unique vertical morphism \( \phi' : X \to \langle J \rangle \) such that \( \pi \circ \text{id}_{\phi'} = \phi \).

The 2-dimensional property is the following. Suppose we are given a further unary cell \( \psi \) as in the identity below, which factors through \( \pi \) as \( \psi' : Y \to \langle J \rangle \), like \( \phi \) factors as \( \phi' \). Then for any pair of cells \( \xi_A \) and \( \xi_B \) as below, so that the identity holds, there exists a unique cell \( \xi' \) as on the right below such that \( \pi_A \circ \xi' = \xi_A \) and \( \pi_B \circ \xi' = \xi_B \).

\[
\begin{array}{c}
X_0 \\
\downarrow \phi_A \\
\downarrow \phi_X \\
A \xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
X_0 \\
\downarrow \phi_A \\
\downarrow \phi_X \\
A \xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
X \circ \xi' \\
\phi_X \\
\downarrow \phi_X \\
\downarrow \phi_B \\
A \xrightarrow{J} B
\end{array}
\]

We call \( \langle J \rangle \) cocartesian whenever its defining cell \( \pi \) is cocartesian.

We shall briefly use the vertical dual notion as well: the cotabulation \( [J] \) of \( J \) is defined by a nullary cell \( \sigma \) below, satisfying 1-dimensional and 2-dimensional universal properties that are vertical dual to those for \( \langle J \rangle \). We call \( [J] \) cartesian whenever \( \sigma \) is cartesian.

\[
\begin{array}{c}
A \xrightarrow{J} B \\
\downarrow \sigma_A \\
\downarrow \phi_A \\
[J]
\end{array}
\]

Example 4.24. In the hypervirtual equipment \((\text{Set}, \text{Set}')\)-Prof, of small profunctors between large categories, the tabulation \( \langle J \rangle \) is the graph of \( J : A^{\text{op}} \times B \to \text{Set} \) as follows. It has triples \((x, u, y)\) as objects, where \((x, y) \in A \times B \) and \( u : x \to y \) in \( J \), while a map \((x, u, y) \to (x', u', y')\) is a pair \((s, t) : (x, y) \to (x', y') \) in \( A \times B \) making

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the diagram

\[
x \begin{array}{c}
\xymatrix{
x \ar[r]^{u} 
& y \\
x' \ar[r]_{u'} 
& y'
}
\end{array}
\]

commute in \( J \). The functors \( \pi_A \) and \( \pi_B \) are the projections while the cell \( \pi : \langle J \rangle \Rightarrow J \) maps \( (x, u, y) \) to \( u \in J(x, y) \). It is straightforward to check that \( \pi \) satisfies the universal properties above, and that it is cocartesian.

Vertically dual, the cotabulation \( \langle J \rangle \) is \textit{cograph} of \( J \), as follows. Its set of objects is the disjoint union \( \text{ob}\langle J \rangle := \text{ob} A \sqcup \text{ob} B \) while its hom-sets are given by

\[
\langle J \rangle(x, y) := \begin{cases} 
A(x, y) & \text{if } x, y \in A; \\
J(x, y) & \text{if } x \in A \text{ and } y \in B; \\
B(x, y) & \text{if } x, y \in B; \\
\emptyset & \text{otherwise.}
\end{cases}
\]

The functors \( \sigma_A \) and \( \sigma_B \) are the embeddings of \( A \) and \( B \) into \( \langle J \rangle \), while the cell \( \sigma : J \Rightarrow \langle J \rangle \) consists of the identities on the sets \( J(x, y) \). Again it is straightforward to check that \( \sigma \) satisfies the universal properties and that it is cartesian.

\textbf{Example 4.25.} The hypervirtual equipment \( (\text{Set}, \text{Set'})\)-\text{Prof} of \( \text{Set}\)-indexed profunctors (see Example 1.9) has all cocartesian tabulations as well as cartesian cotabulations: in each index they can be constructed as in the previous example.

\textbf{Example 4.26.} In the strict equipment \( 2\text{-Prof} \) of modular relations, described in Example 3.11, the tabulation \( \langle J \rangle \) of \( J : A \Rightarrow B \) is the set \( J \subseteq A \times B \) itself, equipped with the product order

\[
(x_1, y_1) \leq (x_2, y_2) \iff (x_1 \leq x_2) \land (y_1 \leq y_2).
\]

Taking the monotone maps \( \pi_A \) and \( \pi_B \) to be the projections, it is straightforward to check that the evident cell \( \langle J \rangle \Rightarrow J \) satisfies the universal properties and is weakly cocartesian; applying Corollary 3.24 we conclude that it is cocartesian.

\textbf{Example 4.27.} In the hypervirtual equipment \( (\text{Cat}, \text{Cat'})\)-\text{Prof} the tabulation \( \langle J \rangle \) of a (small) 2-profunctor \( J : A^{op} \times B \Rightarrow \text{Cat} \), where \( A \) and \( B \) are (possibly large) 2-categories, has as underlying category the graph of the profunctor underlying \( J \), while its cells \((s, t) \Rightarrow (s', t')\) are pairs \((\delta, \varepsilon)\) of cells \( \delta : s \Rightarrow s' \) in \( A \) and \( \varepsilon : t \Rightarrow t' \) in \( B \) that make the diagram on the left below commute in \( J \).

\[
\xymatrix{
x \ar[r]^{u} 
& y \\
x' \ar[r]_{u'} 
& y'
}
\]

\[\langle x, * \rangle = \begin{cases} x \cdot \frac{u}{v} & \text{and } \langle * , y \rangle = \begin{cases} x' \cdot \frac{w}{v'} & \text{and } (x, v) \Rightarrow (x', v') \end{cases} \end{cases}\]

\[
K(*,*) = \begin{cases} x' \cdot \frac{w'}{v'} & \text{and } (x, v) \Rightarrow (x', v') \end{cases}
\]

Tabulations of 2-profunctors fail to be opcartesian in general. As an example consider two 2-profunctors \( J \) and \( K : 1 \Rightarrow 1 \), where \( 1 \) denotes the terminal 2-category with single object \(*\), whose single images \( J(*,*) \) and \( K(*,*) \) are the ‘free living’ cell and parallel pair of arrows respectively, as shown above. The tabulation \( \langle J \rangle \) is discrete with objects \((*, u, *) \) and \((*, v, *)\), so that the assignments \( (*, u, *) \mapsto u' \) and \((*, v, *) \mapsto v'\) define a cell \( \phi : \langle J \rangle \Rightarrow K \); it is easily checked that \( \phi \) does not factor through \( \pi : \langle J \rangle \Rightarrow J \).

Consider a hypervirtual double category \( K \) that has all tabulations. Choosing a tabulation \( \langle J \rangle \) for every \( J : A \Rightarrow B \) in \( K \) gives an assignment \( J \mapsto \langle J \rangle \) which
we extend to paths $\mathcal{J} = (A_0 \xrightarrow{J_1} A_1, \ldots, A_n \xrightarrow{J_n} A_n)$ by mapping $\mathcal{J}$ to the top $(J_1, \ldots, J_n)$ of the ‘piramid’ below, where each $\pi_i$ denotes the chosen tabulation of $J_i(\pi_{A_{n'i}}, \text{id})$: $(J_1, \ldots, J_{i-1}) \Rightarrow A_i$. For each $1 \leq i \leq n$ we denote the composite $(J_1, \ldots, J_n) \rightarrow (J_1, \ldots, J_{n'i}) \rightarrow \cdots \rightarrow (J_1, \ldots, J_{i}) \xrightarrow{\pi_{A_{i}}} A_i$ again by $\pi_{A_{i'}}$.

![Diagram](image_url)

importantly, if the tabulations of $\mathcal{K}$ are cocartesian then so is each of the rows above and hence, by the pasting lemma (Lemma 3.7), so is the path consisting of the columns that make up the piramid.

**Example 4.28.** For profunctors $J_1: A_0 \Rightarrow A_1, \ldots, J_n: A_{n'} \Rightarrow A_n$, the tabulation $(J_1, \ldots, J_n)$ in $(\text{Set}, \text{Set'})$-$\text{Prof}$ is (isomorphic to) the category whose objects are given by paths $(x_0 \xrightarrow{u_1} x_1, \ldots, x_{n'} \xrightarrow{u_n} x_n)$, where each $x_i \in A_i$ and each $u_i \in J_i$, while its morphisms are sequences of commutative squares

$$
\begin{array}{cccc}
& x_0 & \xrightarrow{u_1} & x_1 \\
& \downarrow s_0 & & \downarrow s_1 \\
& x'_0 & \xrightarrow{u'_1} & x'_1
\end{array}
\quad
\begin{array}{cccc}
& x_{n'} & \xrightarrow{u_n} & x_n \\
& \downarrow s_{n'} & & \downarrow s_n \\
& x'_{n'} & \xrightarrow{u'_n} & x'_n
\end{array}
$$

where each $s_i \in A_i$.

Besides horizontal composites, the tabulations $(J_1, \ldots, J_n)$ can also be used to reduce $(n, m)$-ary cells to $(1, m)$-ary ones, as follows. Similar to Corollary 4.10, this reduction preserves cells that define Kan extensions, as shown below.

**Proposition 4.29.** Consider a hypervirtual double category $\mathcal{K}$ that has cocartesian tabulations and assume that a tabulation $(J_1, \ldots, J_n)$ has been chosen for each path of horizontal morphisms $(J_1, \ldots, J_n)$, as in (13). The following hold.

(a) There exists a bijective correspondence between nullary cells $\phi$, of the form as on the left below, and nullary cells $\psi$, of the form as in the middle, which preserves cells that define $k$ as a (weak) left Kan extension.

(b) If $A_n$ is unital then the cells $\psi$ in turn are in bijective correspondence with nullary cells $\chi$, that are of the form as on the right below; this too preserves cells defining (weak) left Kan extensions.

(c) Given a vertical morphism $f: M \rightarrow N$ such that the restriction $N(f, f)$ exists, $f$ is full and faithful in $\mathcal{K}$ precisely if it is so in the vertical 2-category $V(\mathcal{K})$ (see Definition 2.8).
Proof. The correspondence of (a) is given by precomposing the nullary cells \( \phi \) with the cocartesian path consisting of all cells that make up the pyramid (13), except the top one. That this preserves cells defining (weak) left Kan extensions follows immediately from the vertical pasting lemma (Lemma 4.9).

For the correspondence of (b), between the nullary cells \( \psi \) and \( \chi \), assume that the horizontal unit \( I_{A_n} \) exists. The correspondence is given by the assignment on the left below, where \( \psi' \) is the unique factorisation of \( \psi \) through the cocartesian cell that defines \( I_{A_n} \). That this assignment is bijective follows from the fact that both the latter cell and the path \( (\pi_n, \text{cart}) \) are cocartesian. That it preserves cells defining (weak) left Kan extensions follows from the same fact, using the vertical pasting lemma.

The ‘only if’-part of (c) is clear. For the ‘if’-part, let \( f: M \to N \) be a vertical morphism and assume that it is full and faithful in \( V(K) \). Consider the commutative diagram of assignments between classes of nullary cells below, where the cells \( \phi \) are as in the statement of the proposition; the cells \( \xi \) are of the form as on the left above, and the unlabelled assignments are given by precompostion with the path of cells (13).

Now the assumption on \( f \) means that the assignment on the right is a bijection and, because the path (13) is cocartesian, so are the horizontal maps. We conclude that the assignment on the left is a bijection too, so that \( \text{id}_f \) is cartesian; that is \( f \) is full and faithful in \( K \). This completes the proof.

Closing this subsection, the following proposition describes the relation between representable nullary restrictions in a hypervirtual double category \( K \) and absolute left liftings (as introduced in Section 1 of [SW78]; or see Section 2.4 of [Web07]) in its vertical 2-category \( V(K) \).

**Proposition 4.30.** In a hypervirtual double category \( K \) consider the factorisation below. The vertical cell \( \psi \) defines \( j \) as an absolute left lifting of \( f \) along \( g \) in \( V(K) \) whenever its factorisation \( \psi' \) is cartesian in \( K \). The converse holds as soon as \( K \)
has all cocartesian tabulations.

\[
\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{\text{cocart}} & C
\end{array}
\]

**Proof.** For any vertical morphisms \( p: X \to A \) and \( q: X \to B \) consider the commuting diagram of assignments between collections of cells in \( K \) on the left below, where the assignment on the left is given by composing with the cartesian cell that defines \( j_\ast \). By definition, the vertical cell \( \psi \) defines \( j_\ast \) as the absolute left lifting of \( f \) along \( g \) in \( V(K) \) when the bottom assignment is a bijection, so that the proof of the first assertion follows from the fact that, assuming that \( \psi' \) is cocartesian, both the assignment on the left and that at the top are bijections.

For the converse assume that \( K \) has cocartesian tabulations and that \( \psi \) defines \( j_\ast \) as an absolute left lifting. It follows that, for any vertical morphisms \( p \) an \( q \), the top assignment in the diagram on the left above is a bijection; that is, any vertical cell \( \chi \) as in the collection on the right factors uniquely through \( \psi' \). To prove that \( \psi' \) is cartesian in \( K \) we have to show that any nullary cell \( \chi' \) as in the collection on the right above factors uniquely through \( \psi' \) as well. But this follows the fact that any such \( \chi' \) corresponds to a cell of the form \( \chi \), by composing it with the piramid-shaped cocartesian path that defines the tabulation \( \langle H_1, \ldots, H_m \rangle \), as in (13); under this correspondence the factorisation of \( \chi' \) through \( \psi' \) corresponds to that of \( \chi \). This completes the proof.

**4.6 Pointwise Kan extensions in terms of pointwise weak Kan extensions**

In the theorem below we prove that the notions of pointwise weak Kan extension and pointwise Kan extension coincide in hypervirtual double categories that have cocartesian tabulations. This is analogous to the situation for double categories, as described in Section 5 of [Kou14], and reminiscent of Street’s definition of pointwise Kan extension in 2-categories (see [Str74]), which is given in terms of ordinary Kan extensions.

**Theorem 4.31.** In a hypervirtual double category that has cocartesian tabulations, all pointwise weak left Kan extensions are pointwise left Kan extensions.
Proof. Consider a nullary cell $\eta$ as on the left below and assume that it defines $l$ as the pointwise weak left Kan extension of $d$ along $(J_1, \ldots, J_n)$. Below we prove that $\eta$ defines $l$ as a left Kan extension; thus, from applying the proof to composites of $\eta$ with a cartesian cell defining restrictions $J_n(id, f)$, which again define weak left Kan extensions by Lemma 4.6(b), we conclude that $\eta$ defines $l$ as a pointwise left Kan extension.

Hence consider a cell $\phi$ as on the right above; we have to show that it factors uniquely through $\eta$. To do so we consider the tabulation $\langle H_1, \ldots, H_m \rangle$ analogous to (13); remember that, since the tabulations used in constructing $\langle H_1, \ldots, H_m \rangle$ are cocartesian, they form a pyramid-shaped cocartesian path, as was discussed following (13). It follows that, under precomposition with this path, cells of the form $\phi$ on the right above correspond bijectively to cells $\psi$ as on the left below. Using Lemma 4.17 we conclude that, under this correspondence, a unique factorisation of the cells $\phi$ through the composite on the right above corresponds to a unique factorisation of the cells $\psi$ through the composite on the left below, as vertical cells of the form $l \circ \pi_{A_n} \Rightarrow k \circ \pi_{B_m}$.

That the cells $\psi$ do factorise uniquely through the composite on the right follows, by definition, from the assumption that $\eta$ defines $l$ as a pointwise weak left Kan extension. This completes the proof.

In closing this section we show how Theorem 4.31 can be used to extend Proposition 4.2 to the pointwise case, in the setting of hypervirtual equipments (Definition 2.6) that have cocartesian tabulations. This generalises the corresponding result for double categories given in [Kou14].

Proposition 4.32. Consider the following factorisation in a hypervirtual equipment that has cocartesian tabulations.

The vertical cell $\eta$ defines $l$ as the pointwise left Kan extension of $d$ along $j$ in the 2-category $V(K)$, in the sense of Section 4 of [Str74] (or see Section 2.4 of [Web07]), precisely if its factorisation $\eta'$ defines $l$ as the pointwise left Kan extension of $d$ along $j_*$ in $K$. 

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Street’s notion of pointwise Kan extension in a 2-category uses the well known notion of comma object; see [Str74], or [Web07] where it is called lax pullback. Instead of recalling the definition of comma object we record the following lemma, which relates it to that of tabulation.

**Lemma 4.33.** Consider vertical morphisms \( f: A \to C \) and \( g: B \to C \) in a hyper-virtual double category \( \mathcal{K} \). If both the nullary cartesian cell and the tabulation below exist then their composite defines \( \langle C(f,g) \rangle \) as the comma object \( f/g \) of \( f \) and \( g \) in \( V(\mathcal{K}) \).

\[
\begin{array}{c}
\pi_A \\
\downarrow \phi \\
\downarrow \gamma \\
A \\
\rightarrow \\
C(\mathcal{K},g) \\
\rightarrow \\
B \\
\rightarrow \\
\end{array}
\]

**Proof.** This follows immediately from the fact that the universal properties defining the comma object of \( f \) and \( g \) in \( V(\mathcal{K}) \) correspond precisely to universal properties satisfied by the tabulation of \( C(f,g) \) in \( \mathcal{K} \), by factorising them through the nullary cartesian cell that defines \( C(f,g) \).

**Proof of Proposition 4.32.** Given any \( f: C \to B \) we consider the composite on the left below, where \( \eta = \eta' \circ \text{cocart} \) and where \( \text{cart} \circ \pi \) defines the comma object \( j/f \), by the previous lemma. Hence \( \eta \) defines \( l \) as the pointwise left Kan extension in the 2-category \( V(\mathcal{K}) \) if, for any morphism \( f \), the composite on the left defines \( l \circ f \) as a left Kan extension. The latter in turn means that the top assignment in the commutative diagram on the right, of assignments between collections of cells in \( \mathcal{K} \) as shown, is a bijection.

\[
\begin{array}{c}
j/f \\
\pi_A \\
\downarrow \downarrow \\
A \\
\rightarrow \\
C \\
\rightarrow \\
B \circ \text{cocart} \\
\rightarrow \\
M \\
\end{array}
\]

\[
\begin{array}{c}
\{ A \downarrow \psi \} \\
\{ B \downarrow \phi \} \\
\{ C \downarrow \k \} \\
\end{array}
\]

The bottom assignment on the other hand is a bijection, for every \( f \), precisely when \( \eta' \) defines \( l \) as the pointwise left Kan extension of \( d \) along \( j_* \): this is a consequence of Theorem 4.31 and the fact that the composite \( \text{cocart} \circ \text{cart}: B(j,f) \to j_* \), in the composite on the left, is cartesian. The proof now follows from the fact that the right assignment above is a bijection, as \( \pi \) is cocartesian.

The following is Example 2.24 of [Kou13].

**Example 4.34.** Recall from Example 4.27 that tabulations of 2-profunctors are in general not opcartesian. As a consequence the equivalence described in Proposition 4.32 above fails to hold in the hypervirtual equipment \((\text{Cat}, \text{Cat}'\)-Prof). For a
counter example consider the 2-natural transformation of 2-functors below, where 1 is the terminal 2-category and the collapsing 2-functor on the left has the ‘free living’ cell as source and the free living parallel pair of arrows as target. It is straightforward to check that this transformation defines the collapsing 2-functor as the enriched left Kan extension of \( x' \) along \( x \), in the sense of e.g. Section 4.1 of [Kel82], while the pointwise left Kan extension of \( x' \) along \( x \) in the 2-category 2-Cat, of 2-categories, 2-functors and 2-natural transformations, and in the sense of Section 4 of [Str74], does not exist.

\[
\begin{array}{c}
1 \\
\downarrow \text{(identity at } x') \\
\downarrow \text{(collapse onto } x') \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
\downarrow \text{(identity at } x') \\
\downarrow \text{(collapse onto } x') \\
\end{array}
\]

5 Yoneda embeddings

We are now ready to introduce the main notion of this paper, that of yoneda embedding in hypervirtual double categories. Informally a yoneda embedding is a vertical morphism that is ‘dense’ (as defined below) and that satifies an axiom that “formally captures the Yoneda’s lemma”. These two conditions are closely related to the axioms satisfied by morphisms that make up a ‘yoneda structure’ on a 2-category, as introduced by Street and Walters in [SW78]; see also [Web07]. Our position of regarding horizontal morphisms as “being small in size” enables us to give a relatively simple definition of a single yoneda embedding. This in contrast to the definition of yoneda structure on a 2-category, which consists of a collection of yoneda embeddings that satisfy a “formal Yoneda’s lemma” with respect to a specified collection of ‘admissible’ morphisms (informally these are to be thought of as “small in size”).

5.1 Definition of yoneda embedding

We start with the notion of density. Recall from Section 5.1 of [Kel82] that one way of defining a \( \mathcal{V} \)-functor \( f: A \to M \) to be dense is to require that its identity \( \mathcal{V} \)-natural transformation \( \text{id}_f: f \Rightarrow f \) defines \( \text{id}_M \) as the (enriched) left Kan extension of \( f \) along itself. As we have seen in Proposition 4.20, the latter is equivalent to asking that the cartesian cell defining the companion \( f_*: A \Rightarrow M \), in \((\mathcal{V}, \mathcal{V}')\)-Prof, defines \( \text{id}_M \) as the pointwise left Kan extension of \( f \) along \( f_* \). In general, for a vertical morphism \( f: A \to M \) in any hypervirtual double category, further equivalent conditions are given by the following lemma. We call \( f \) dense if these equivalent conditions are satisfied.

Lemma 5.1. For a vertical morphism \( f: A \to M \) in a hypervirtual double category the following conditions are equivalent:

(a) if a cell \( \eta \), as on the left below, is cartesian then it defines \( I \) as the left Kan extension of \( f \) along \( J \);

(b) if a cell \( \eta \) as below is cartesian then it defines \( I \) as the pointwise left Kan extension of \( f \) along \( J \).

If the companion \( f_*: A \Rightarrow M \) exists then the following condition is equivalent too:

(c) the nullary cartesian cell on the right below, that defines the companion \( f_* \), defines \( \text{id}_M \) as the pointwise left Kan extension of \( f \) along \( f_* \).
Proof. (b) ⇒ (a) is clear. For the converse, assume that (a) holds and consider any cartesian cell as in the composite on the left hand side below. Since \( \eta \) is cartesian the full composite is cartesian too by the pasting lemma. Hence, applying (a) we find that the composite defines \( l \circ g \) as a left Kan extension, showing that (b) holds.

(b) ⇒ (c) is clear. For the converse consider a cartesian cell \( \eta \) as on the right above and let \( \eta' \) be its factorisation as shown; it is cartesian because \( \eta \) is, by the pasting lemma. Assuming (c) we may apply Lemma 4.6(b) to find that \( \eta \) defines \( l \) as a pointwise left Kan extension; we conclude that (c) ⇒ (b).

Following the definition of dense morphism we define yoneda embeddings.

**Definition 5.2.** A dense vertical morphism \( y : A \to \hat{A} \) in a hypervirtual double category is called a **yoneda embedding** if it satisfies the yoneda axiom: for every horizontal morphism \( J : A \nrightarrow B \) there exists a cartesian cell

\[
\begin{array}{ccc}
A & \xrightarrow{J} & C \\
\downarrow \text{cart} & & \downarrow g \\
A & \xrightarrow{J} & B \\
\downarrow \eta & & \downarrow \eta' \\
M & & M
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{f} & M \\
\downarrow \text{cart} & & \downarrow \text{cart} \\
A & \xrightarrow{f} & M
\end{array}
\]

(b) ⇒ (c) is clear. For the converse consider a cartesian cell \( \eta \) as on the right above and let \( \eta' \) be its factorisation as shown; it is cartesian because \( \eta \) is, by the pasting lemma. Assuming (c) we may apply Lemma 4.6(b) to find that \( \eta \) defines \( l \) as a pointwise left Kan extension; we conclude that (c) ⇒ (b).

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\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \text{cart} & & \downarrow l \\
\hat{A}
\end{array}
\]

Notice that the density of \( y : A \to \hat{A} \) implies that the vertical morphism \( J^\circ \) above is unique up to vertical isomorphism. If the yoneda embedding above exists then we call \( \hat{A} \) the **object of presheaves on** \( A \), or **presheaf object** for short. Yoneda embeddings \( y : A \to \hat{A} \) such that all restrictions \( \hat{A}(y, g) \) exist, for any \( g : B \to \hat{A} \), are especially pleasant to work with; we call them **good** yoneda embeddings. Notice that, when considered in a hypervirtual equipment, the latter condition reduces to the existence of the companion \( y_\ast : A \to \hat{A} \), since \( \hat{A}(y, g) \cong y_\ast(\text{id}, g) \).

At the end of this subsection we will show that a \( V' \)-category \( A \) admits a good yoneda embedding in \( (V, V')\)-Prof (Example 1.12) precisely if \( A \) is a \( V \)-category. First we record some basic properties of yoneda embeddings; the following are variations of Lemma 3.2 and Corollary 3.5 of [Web07].

**Lemma 5.3.** Let \( y : A \to \hat{A} \) be a yoneda embedding. If the nullary cell below defines \( l \) as the (weak) left Kan extension of \( J \) along \( y \) then it is cartesian and, moreover, it defines \( l \) as a pointwise left Kan extension.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow y \text{cart} & & \downarrow l \\
\hat{A}
\end{array}
\]
Proof. By the yoneda axiom there exists a morphism $J^\lambda : B \to \hat{A}$ such that $J$ is the nullary restriction of $\hat{A}$ along $y$ and $g$, and the cartesian cell $\varepsilon$ defining $J$ as this restriction defines $J^\lambda$ as a pointwise left Kan extension of $y_A$ along $J$ by Lemma 5.1(b). Since $\eta$ defines the same left Kan extension we conclude that it factors through $\varepsilon$ as an invertible vertical cell $g \cong l$; it follows that, like $\varepsilon$, $\eta$ is cartesian and defines $l$ as a pointwise left Kan extension.

Lemma 5.4. For a yoneda embedding $y : A \to \hat{A}$ the following hold:

(a) if $y$ is a good yoneda embedding then it is full and faithful, while $A$ is unital;

(b) any $f : A \to C$, such that the companion $f_*$ and the restriction $C(f, f)$ exist, is full and faithful precisely if the composite

$$
\begin{array}{c}
A \\
\cocart \searrow f \\
A \overrightarrow{f_*} C \\
y \downarrow \cart \searrow t^\lambda_A
\end{array}
$$

is invertible, where the cartesian cell exists by the yoneda axiom.

Proof. (a). That $y$ is full and faithful follows from applying part (b) to the vertical companion identity of $y$ (see Lemma 2.11); here we use that $y$ is a good yoneda embedding, so that the restriction $A(y, y)$ exists. That $A$ is unital then follows from Corollary 3.15.

(b). First notice that the composite above can be rewritten as the composite below, where the cell $\id'_f$ is the factorisation of $\id_f$ through the cartesian cell defining $C(f, f)$, as in Corollary 3.15. Indeed this follows from the fact that, when composed with the cartesian cell defining $f_*$, both the weakly cocartesian cell above as well as the composite of the two top cells below equal $\id_f$.

$$
\begin{array}{c}
A \\
\cocart \searrow \id'_f \\
A \overrightarrow{C(f, f)} \overrightarrow{f_*} C \\
y \downarrow \cart \searrow t^\lambda_A
\end{array}
$$

The proof follows by noticing that the following conditions are equivalent: $f$ is full and faithful; $\id'_f$ is cocartesian (by Corollary 3.15); $\id'_f$ is cartesian (by Lemma 3.12); the composite above is cartesian (by the pasting lemma); the composite defines a left Kan extension (because $y$ is dense and the previous lemma); the composite is invertible (by Lemma 4.14).

As an example we now consider yoneda embeddings in the hypervirtual double category $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ of $\mathcal{V}$-profunctors between $\mathcal{V}'$-categories, that is associated to a universe enlargement $\mathcal{V} \to \mathcal{V}'$ as described in Example 1.12. In the case that $\mathcal{V}'$ is closed symmetric monoidal the construction of such yoneda embeddings is of course classical, see for instance Section 2 of [Kel82]. The nuts and bolts description
below, which leaves many easy details to check, shows that the classical construction extends to the case in which \( \mathcal{V}' \) is merely closed monoidal.

Let \( A \) be a (possibly large) \( \mathcal{V}' \)-category; by a \( \mathcal{V} \)-presheaf \( p \) on \( A \) we mean a \( \mathcal{V} \)-profunctor \( p : A \Rightarrow I \). Thus \( p \) consists of \( \mathcal{V} \)-objects \( px \), one for each \( x \in A \), equipped with an action of \( A \) given by \( \mathcal{V}' \)-maps

\[
\lambda : A(x, y) \otimes' py \to px,
\]

that is associative and unital. If both \( \mathcal{V} \) and \( \mathcal{V}' \) are closed symmetric monoidal then \( \mathcal{V} \)-presheaves on a \( \mathcal{V} \)-category \( A \) can be identified with \( \mathcal{V} \)-functors \( A^{\text{op}} \to \mathcal{V} \).

The \( \mathcal{V} \)-presheaves on \( A \) arrange into a \( \mathcal{V}' \)-category \( \hat{\mathcal{A}} \) as follows. To define the hom-object \( \hat{\mathcal{A}}(p, q) \) of \( \mathcal{V} \)-presheaves \( p \) and \( q \), we write \([p, q]'\) for the inclusion into \( \mathcal{V}' \) of its subcategory consisting of the cospans

\[
[px, qx]' \quad \quad \quad \quad [py, qy]',
\]

under the hom-tensor adjunction \( X \otimes' \dashv [X, -]' \) of \( \mathcal{V}' \). Since \( A \) is large the limit of \([p, q]'\) exists in \( \mathcal{V}' \), which we take as the hom-object \( \hat{\mathcal{A}}(p, q) \). In the case that \( \mathcal{V}' \) is closed symmetric monoidal it easy to check that \( \hat{\mathcal{A}}(p, q) \) forms the end \( \int_{x \in A}[px, qx]' \).

Composition \( \hat{\mathcal{A}}(p, q) \otimes' \hat{\mathcal{A}}(q, r) \to \hat{\mathcal{A}}(p, r) \) is the factorisation of the cone \( \Delta(\hat{\mathcal{A}}(p, q) \otimes' \hat{\mathcal{A}}(q, r)) \Rightarrow [p, q]' \) that uniquely extends the family of \( \mathcal{V}' \)-maps adjunto to

\[
px \otimes' \hat{\mathcal{A}}(p, q) \otimes' \hat{\mathcal{A}}(q, r) \Rightarrow px \otimes' [px, qx]' \otimes' [qx, rx]' \otimes' rx \quad \text{ev}_{\hat{\mathcal{A}}(p, q) \otimes' \hat{\mathcal{A}}(q, r)} \to px \otimes' [px, qx]' \otimes' [qx, rx]' \otimes' rx
\]

where the first map is given by projections. Similarly the units \( I' \to \hat{\mathcal{A}}(p, p) \) are induced by the adjuncts to the unitsors \( px \otimes' I' \xrightarrow{\omega} px \) of \( \mathcal{V}' \). This completes the definition of the \( \mathcal{V}' \)-category \( \hat{\mathcal{A}} \) of \( \mathcal{V} \)-presheaves on \( A \); notice that, because \( A \) and \( \mathcal{V} \) are large while \( \mathcal{V}' \) is locally large, \( \hat{\mathcal{A}} \) is again a large \( \mathcal{V}' \)-category.

**Proposition 5.5.** Let \( \mathcal{V} \to \mathcal{V}' \) be a universe enlargement. A large \( \mathcal{V}' \)-category \( A \) admits a good yoneda embedding in \( (\mathcal{V}, \mathcal{V}') \)-Prof if and only if it is a \( \mathcal{V} \)-category. More precisely, in that case \( y : A \to \hat{\mathcal{A}} \) can be given with the \( \mathcal{V}' \)-category \( \hat{\mathcal{A}} \) as defined above and \( y x = A(-, x) \) while, for each \( \mathcal{V} \)-profunctor \( J : A \Rightarrow B \), the cartesian cell

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow y & & \downarrow y \\
\hat{\mathcal{A}} & \xrightarrow{A} & \hat{\mathcal{A}}
\end{array}
\]

has \( J^! y = J(-, y) \) as vertical target.

Finally also assume that \( \mathcal{V} \) is closed monoidal, and that \( \mathcal{V} \to \mathcal{V}' \) is a closed monoidal functor. For any \( \mathcal{V} \)-presheaves \( p \) and \( q \) on \( A \) the diagram \([p, q]'\) in \( \mathcal{V}' \), as described above, factors (up to natural isomorphism) through \( \mathcal{V} \to \mathcal{V}' \) as a diagram \([p, q]\) in \( \mathcal{V} \), that is obtained by replacing the inner-homs \([-, -]'\) of \( \mathcal{V}' \) in \((14)\) by those \([-, -]\) of \( \mathcal{V} \). In this case \( \hat{\mathcal{A}} \) is a \( \mathcal{V} \)-category if and only if, for any pair of \( \mathcal{V} \)-presheaves \( p \) and \( q \) on \( A \), the factorisation diagram \([p, q]\) in \( \mathcal{V} \) has a limit.
Proof. For the ‘only if’-part notice that the existence of a good Yoneda embedding implies that $A$ is unital by Lemma 5.4(a) and thus a $\mathcal{V}$-category as we saw in Example 3.3.

For the ‘if’-part assume that $A$ is a $\mathcal{V}$-category, so that $y$ can be defined on objects as described above. In detail: $yx$ is the $\mathcal{V}$-presheaf given by the hom-objects $(y \times)(s) = A(s, x)$, equipped with actions $A(s, t) \otimes (yx)(t) \to (yx)(s)$ that are given by the composition of $A$. On hom-objects $y$ is given by the unique factorisations $\hat{y}: A(x, y) \to \hat{A}(y, x, y)$ of the cones $\Delta A(x, y) \Rightarrow [y, y, y]$ that are unique extensions of the adjuncts to $(yx)(s) \otimes' A(x, y) \to (yx)(s)$, which in turn are again given by the composition of $A$. A straightforward calculation shows that $\hat{y}$ preserves composition and units.

It remains to show that $y$ is dense, admits a companion $y_+: A \to \hat{A}$, and satisfies the Yoneda axiom. For the latter consider a $\mathcal{V}$-profunctor $J: A \to B$: we have to supply a cartesian cell as above. As mentioned in the statement, the $\mathcal{V}$-functor $J^*: B \to \hat{A}$ maps $y \in B$ to the $\mathcal{V}$-presheaf given by $(J^*y)(s) = J(s, y)$ on objects, while its action $A(s, t) \otimes (J^*y)(t) \to (J^*y)(s)$ is simply that of $A$ on $J$. Its action on hom-objects $J^*: B(y, z) \to \hat{A}(y, z)$ is induced by the adjuncts to the actions $J(x, y) \otimes B(y, z) \to J(x, z)$ of $B$ on $J$. Checking that $J^*$ preserves composition and units is again straightforward while, in order to show that the cartesian cell above exists, it suffices to show that $J(x, y) \cong \hat{A}(y, x, J^*)$, natural in $x$ and $y$. First notice that natural $\mathcal{V}'$-maps $J(x, y) \to \hat{A}(y, x, J^*)$ can be obtained from the adjuncts of the maps $(yx)(s) \otimes' J(x, y) \to (J(x, y))(s)$ that form the action of $A$ on $J$; it is then not hard to show that the composites

$$
\hat{A}(y, x, J^*) \xrightarrow{\exists} I' \otimes' \hat{A}(y, x, J^*) \xrightarrow{\exists} A(x, x) \otimes' [A(x, x), J(x, y)]' \xrightarrow{\exists} J(x, y)
$$

form inverses, where the middle map is the tensor product of the unit $I' \to A(x, x)$ and the projection $\hat{A}(y, x, J^*) \to [A(x, x), J(x, y)]'$.

Next, by applying the above to a $\mathcal{V}$-presheaf $p: A \to I$ we obtain an isomorphism $px \cong \hat{A}(y, p)$. We conclude that $\hat{A}(y, p)$ is a $\mathcal{V}$-object for each $x \in A$ and $p \in \hat{A}$ so that, by Example 2.2 the companion $y_*: A \to \hat{A}$ exists. Finally to prove that $y$ is dense we will show that, for any $\mathcal{V}$-profunctor $J: A \to B$, the cartesian cell above defines $J^*: B \to \hat{A}$ as a pointwise left Kan extension. Remember that, by Proposition 4.20 we may equivalently show that its restriction along $J(id, y): A \to I$ defines $J^*y$ as the $J(id, y)$-weighted colimit of $y$. Hence it suffices to prove that, for every $\mathcal{V}$-presheaf $p$ on $A$, the cartesian cell in the composite on the right below, that consists of the isomorphisms $px \cong \hat{A}(y, x, p)$, defines $p: I \to \hat{A}$ as the $\hat{A}(y, p)$-weighted colimit of $y$.

![Diagram](https://example.com/diagram.png)

To see this consider any cell $\phi$ in $(\mathcal{V}, \mathcal{V}')$-Prof that is of the form as on left above; we have to prove that it factors as shown. To see this remember that $\phi$ is given by a family $\mathcal{V}'$-maps $\phi_x: px \otimes' H \to \hat{A}(y, q)$ that are natural in $x \in A$. After composition with $\hat{A}(y, q) \cong qz$ it is easy to check that this family corresponds to a cone $\phi': \Delta H \Rightarrow \hat{A}(p, q)'$, under the adjunctions $px \otimes' \dashv [px, -]'$. The latter in turn factors uniquely through the limiting cone $\Delta \hat{A}(p, q) \Rightarrow [p, q]'$ as a $\mathcal{V}'$-map $\phi': H \to \hat{A}(p, q)$. A straightforward calculation will show that $\phi'$, when regarded as a cell as in the composite on the right above, forms the unique factorisation of $\phi$ as required. This completes the construction of the Yoneda embedding $y: A \to \hat{A}$ for a $\mathcal{V}$-category $A$. 

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Briefly, on the final assertion: having constructed the diagram \([p, q]\) in \(V\) as described, an isomorphism between its composite with \(V \to V'\) and \([p, q]'\) can be obtained from the assumption that \(V \to V'\) is closed monoidal (see Definition 1.10). Now use the fact that \(V \to V'\) is full and faithful, and that it preserves limits.

Closing this subsection the following proposition compares, in the case of a suitable hypervirtual equipment \(K\), the collection of good yoneda embeddings in \(K\) to the notion of good yoneda structure on the vertical 2-category \(V(K)\); the latter, as introduced by Weber in [Web07], is a strengthening of the original notion of yoneda structure given by Street and Walters in [SW78]. In the statement and proof of the proposition below the terminology and notation of [Web07] are used; for the notion of (co-)tabulation see Definition 4.23.

**Proposition 5.6.** Consider the following types of structure on a hypervirtual equipment \(K\):

1. (E) a collection of good yoneda embeddings \(y_A : A \to \hat{A}\), one for each unital object \(A\) of \(K\);
2. (S) a good yoneda structure on \(V(K)\), in the sense of Section 3 of [Web07], in which a vertical morphism \(f : A \to C\) is admissible precisely if its companion \(f_* : A \Rightarrow C\) exists in \(K\).

If \(K\) has all opcartesian tabulations then a structure of type (E) induces one of type (S). Conversely a structure of type (S) induces one of type (E) whenever \(K\) has cartesian cotabulations \([J]\) for all horizontal morphisms \(J : A \Rightarrow B\) with \(A\) unital as well, in a way such that the insertion \(\sigma_A : A \Rightarrow [J]\), as is part of the defining cell \(\sigma\) below, admits a companion \(\sigma_{A*}\).

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\sigma_A \downarrow & & \downarrow \sigma_{f*} \\
& \Rightarrow [J] & \\
\end{array}
\]

That the hypothesis of \(\sigma_A\) having a companion does not depend on the choice of \(\sigma\) and \([J]\) is a consequence of Lemma 2.14.

**Proof.** Assume that \(K\) has opcartesian tabulations, and that a good yoneda embedding \(y_A : A \to \hat{A}\) is given for each unital object \(A\) in \(K\); we define a good yoneda structure on \(V(K)\) as follows. Firstly we define a vertical morphism \(f : A \to C\) to be admissible if its companion \(f_*\) exists, as described above; that these form a right ideal follows from the fact that \((g \circ f)_* \cong g_*(f, \text{id})\) (a consequence of the pasting lemma) and the fact that \(K\) has all unary restrictions. Recall that an object \(A\) is called admissible whenever its identity \(\text{id}_A\) is admissible which, by Lemma 3.12, is equivalent to \(A\) being unital. Thus, secondly, for every admissible \(A\) we can take the yoneda embedding \(y_A : A \to \hat{A}\) the chosen one; that it is admissible follows from the fact that it is a good yoneda embedding, see Definition 5.2.

Lastly we have to supply, for any vertical morphism \(f : A \to C\) with both \(A\) and \(f\) admissible, a morphism \(C(f, 1) : C \to \hat{A}\) that is equipped with a cell \(\chi_f\): we take \(C(f, 1) := (f_*)^\lambda\), that is given by the yoneda axiom and equipped with the cartesian
cell in the composite on the left below, while we set $\chi^f$ to be the full composite.

$$\chi^f := \begin{array}{c}
A \\
\circ \text{cart} \Downarrow f
\end{array} \xrightarrow{\text{cocart}} \begin{array}{c}
A \\
\circ \text{cart} \Downarrow B(f,1)
\end{array}$$

$$\begin{array}{c}
\hat{\phi} \\
\circ \text{cart} \Downarrow C
\end{array} \xrightarrow{\text{cocart}} \begin{array}{c}
\hat{\phi} \\
\circ \text{cart} \Downarrow C
\end{array}$$

Having completed giving the data for the yoneda structure on $V(\mathcal{K})$; it remains to check that it satisfies two axioms. The first of these states that the cells $\chi^f$ define $f$ as an absolute left lifting of $y_A$ along $B(f,1)$, which follows from Proposition 4.30. The second axiom states that, for any cell $\phi$ as on the right above with $A$ and $f$ admissible, defines $g$ as a pointwise left Kan extension of $y_A$ along $f$ as soon as it defines $f$ as an absolute left lifting of $y_A$ along $g$. This is a consequence of combining Proposition 4.30, the density of $y_A$ (Lemma 5.1(b)) and Proposition 4.32.

For the converse assume that $\mathcal{K}$ has all cartesian coatabulations $[J]$ for $J: A \Rightarrow B$ with $A$ unital, in the sense described in the statement. Consider a good yoneda structure on $V(\mathcal{K})$ in which $f: A \Rightarrow C$ is admissible precisely if $f_*$ exists in $\mathcal{K}$. Hence, for every unital $A$, an admissible morphism $y_A: A \Rightarrow \hat{A}$ is given, which we claim to be a good yoneda embedding in our sense. Indeed, to see that it satisfies the yoneda axiom consider any $J: A \Rightarrow B$ and let $\sigma$, as in the statement, be the cell defining its cotabulation $[J]$; we have to provide the corresponding $J^\chi: B \Rightarrow \hat{A}$ as well as its defining cartesian cell (see Definition 5.2). We set $J^\chi := \left[ [J](\sigma_A,1) \circ \sigma_B, [J](\sigma_A,1) \right]$ where $[J](\sigma_A,1)$ is supplied by the good yoneda structure; it is equipped with a vertical cell $\chi^f: y_A \Rightarrow [J](\sigma_A,1) \circ \sigma_A$ that defines $\sigma_A$ as an absolute left lifting of $y_A$ through $[J](\sigma_A,1)$. Hence, by Lemma 4.30, the factorisation $\chi^f$ through $\sigma_{A*}$, as in the composite below, is cartesian. Since the defining cell $\sigma$ of $[J]$ is assumed to be cartesian it factors through $\sigma_{A*}$ as a cartesian cell $\sigma'$ as well, forming the top cell below. Applying the pasting lemma we find that the full composite is cartesian, so that we can take it to be the cartesian cell defining $J^\chi = [J](\sigma_A,1) \circ \sigma_B$.

$$
\begin{array}{c}
A \\
\circ \text{cart} \Downarrow \sigma_B
\end{array} \xrightarrow{\text{cocart}} \begin{array}{c}
A \\
\circ \text{cart} \Downarrow [J]
\end{array}$$

$$\begin{array}{c}
\hat{\phi} \\
\circ \text{cart} \Downarrow \chi^f
\end{array} \xrightarrow{\text{cocart}} \begin{array}{c}
\hat{\phi} \\
\circ \text{cart} \Downarrow [J](\sigma_A,1)
\end{array}$$

It remains to prove that $y_A$ is dense. This follows easily from Proposition 3.4(1) of [Web07], showing that the identity cell $id_{\hat{A}}$ defines $id_{\hat{A}}$ as the pointwise left Kan extension of $y_A$ along $y_A$ in $V(\mathcal{K})$, together with Proposition 4.32 and Lemma 5.1(c) above. This completes the proof.

5.2 Restriction of presheaves

Vertical morphisms $f: A \Rightarrow C$ induce morphisms $\hat{f}: \hat{C} \Rightarrow \hat{A}$ between presheaf objects given by “restriction along $f$” as follows. Given yoneda embeddings $y_A: A \Rightarrow \hat{A}$ and $y_C: C \Rightarrow \hat{C}$, as well as any morphism $f: A \Rightarrow C$ such that the companion $(y_C \circ f)_*: A \Rightarrow \hat{C}$ exists, we set $\hat{f} := (y_C \circ f)_*$ as defined by the cartesian cell below that exists by the yoneda axiom; notice that it defines $\hat{f}$ uniquely up to isomorphism,
because $y_A$ is dense.

\[
\begin{array}{c}
A \\
y_A \downarrow \quad \text{cart} \quad \hat{f} \\
\hat{A}
\end{array}
\]

(15)

Example 5.7. For any $\mathcal{V}$-functor $f: A \rightarrow C$ in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ the $\mathcal{V}'$-functor $\hat{f}: \hat{C} \rightarrow \hat{A}$ exists. It is given by precomposition with $f$; more precisely, we have $\hat{f}(p) \cong p(f, \text{id})$ for every $\mathcal{V}$-presheaf $p \in \hat{C}$.

Following Section 3 of [Web07], we pause to investigate the existence of left and right adjoints to $\hat{f}$; in particular Proposition 5.8 and Corollary 5.10 below are adaptations of Proposition 3.7 and Theorem 3.20(2) given there, to the setting of hypervirtual double categories. We start with the description of the right adjoint.

Proposition 5.8. In a hypervirtual equipment consider yoneda embeddings $y_A: A \rightarrow \hat{A}$ and $y_C: C \rightarrow \hat{C}$ as well as a morphism $f: A \rightarrow C$ such that $(y_C \circ f)_*$ exists, so that $\hat{f}: \hat{C} \rightarrow \hat{A}$ can be defined as above. If the companions of $y_C$, $f$, $\hat{f}$ and $\hat{f} \circ y_C$ exist then $\hat{f}$ has a right adjoint $f^\flat := (\hat{f} \circ y_C)_*$ that is defined by the cartesian cell

\[
\begin{array}{c}
C \\
y_C \downarrow \quad \text{cart} \quad \hat{f} \\
\hat{C}
\end{array}
\]

Proof. First recall that $(\hat{f} \circ y_C)_*$ forms the restriction of $\hat{f}_*$ along $y_C$ so that, by Lemma 3.20 it forms the pointwise horizontal composite of $(y_C, \hat{f}_*)$. Now consider the identity below; notice that the cartesian cells in either side define pointwise left Kan extensions because $y_C$ is dense (see Lemma 5.1). By Corollary 4.10, the full composite on the left again defines a pointwise left Kan extension so that, by the horizontal pasting lemma, it factors through the cartesian cell on the right-hand side as a cell $\eta$, as shown, which too defines a pointwise left Kan extension. Using Lemma 2.16 and Proposition 4.2, it follows that we can complete the proof by showing that $\hat{f} \circ \eta$ defines $f \circ f^\flat$ as a left Kan extension.

To see this notice that the identity above induces that below, where the cell $\text{cart}'$ in the left-hand side denotes the factorisation of the cartesian cell defining $f^\flat$. 

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through $y_{C*}$; it is again cartesian by the pasting lemma.

\[
\begin{array}{c}
A \xrightarrow{f} C \xrightarrow{C*} \hat{A} \\
\downarrow \quad \downarrow \\
A \xrightarrow{f*} C' \xrightarrow{\hat{C}} \tilde{A} \\
\end{array}
\]

\[
\begin{array}{c}
A \xrightarrow{f} C \xrightarrow{C*} \hat{A} \\
\downarrow \quad \downarrow \\
A \xrightarrow{f*} C' \xrightarrow{\hat{C}} \tilde{A} \\
\end{array}
\]

We have seen that the composite of the bottom two cells of the left-hand side above defines a pointwise left Kan extension, so that the full composite defines a pointwise left Kan extension by the vertical pasting lemma (Lemma 4.9). Since the same applies to the first column of the right-hand side, we conclude from the horizontal pasting lemma that $\hat{f} \circ \eta$ defines $\hat{f} \circ f^\circ$ as a left Kan extension, as required.

The following proposition, which is a variation on Lemma 3.18 of [Web07], allows us to describe the left adjoint to $\hat{f}: \hat{A} \to \hat{C}$, in the corollary that follows.

**Proposition 5.9.** Let $y: A \to \hat{A}$ be a good yoneda embedding. Any pointwise weak left Kan extension $l: \hat{A} \to M$ of $d: A \to M$ along $y*$, such that the companion $d*$ exists, has a right adjoint $d^\lambda$ defined by the cartesian cell

\[
\begin{array}{c}
A \xrightarrow{d*} M \\
\downarrow y^\ast \text{cart} \quad \downarrow d^\lambda \\
\hat{A} \\
\end{array}
\]

**Corollary 5.10.** Let $y_A: A \to \hat{A}$ and $y_C: C \to \hat{C}$ be good yoneda embeddings. Consider a morphism $f: A \to C$ such that $(y_C \circ f)^*$ exists, so that $\hat{f}: \hat{C} \to \hat{A}$ can be defined as in (15). The pointwise left Kan extension $f^\sharp: \hat{C} \to \hat{A}$ of $y_C \circ f$ along $y_{A*}$, if it exists, is left adjoint to $\hat{f}$. The existence of $f^\sharp$ is implied by that of the right pointwise cocartesian cell

\[
\begin{array}{c}
A \xrightarrow{y_A*} \hat{A} \\
\downarrow f \text{cocart} \quad \downarrow K^\gamma \\
C \xrightarrow{k} \hat{A} \\
\end{array}
\]

**Proof.** Since the companion $(y_C \circ f)^*$ exists, it follows from the previous proposition that $f^\sharp$ has a right adjoint that is given by $(y_C \circ f)^\gamma$. But the latter equals $\hat{f}$ by definition: see (15). For the second assertion take $f^\sharp := K^\gamma$ as given by the yoneda axiom. That it forms the pointwise weak left Kan extension of $y_C \circ f$ along $y_{A*}$ follows from applying the vertical pasting lemma (Lemma 4.9) to the cartesian cell defining $K^\gamma$ composed with the cocartesian cell above.

In the proof of Proposition 5.9 we will use the following lemma.
Lemma 5.11. Let \( y : A \rightarrow \hat{A} \) be a good yoneda embedding in a hypervirtual double category \( K \). A vertical cell

\[
\begin{array}{c}
\hat{A} \\
\downarrow^j \\
B \\
\downarrow_\eta \\
\hat{A}
\end{array}
\]

defines \( l \) as the left Kan extension of \( \text{id}_\hat{A} \) along \( j \) in \( V(K) \) precisely if \( \eta \circ y \) defines \( l \) as the left Kan extension of \( y \) along \( j \circ y \).

Proof. Consider the commuting diagram of assignments below, between collections of cells that are of the form as shown. The two bijections here are given by composition with the cartesian and weakly cocartesian cell that define \( y^* \); that they are bijections follows from the fact that the cartesian cell defines \( \text{id}_\hat{A} \) as a left Kan extension, and the universal property of weakly cocartesian cells.

\[
\begin{array}{c}
B \\
\downarrow_{\hat{A}}
\end{array}
\quad \eta \circ (\cdot j) \quad \begin{array}{c}
\{ \{ \} \} \\
\{ \{ \} \}
\end{array}
\quad \begin{array}{c}
\hat{A} \\
\downarrow^j \\
B
\end{array}
\quad \begin{array}{c}
y \\
\downarrow_\eta
\end{array}
\quad \begin{array}{c}
\hat{A} \\
\downarrow^k
\end{array}
\quad \{ \{ \} \}
\quad \begin{array}{c}
\hat{A} \\
\downarrow^j
\end{array}
\quad \begin{array}{c}
\hat{A} \\
\downarrow^j
\end{array}
\end{array}
\]

The proof now follows from the fact that, by definition, \( \eta \) defines \( l \) as a left Kan extension if the top left assignment is a bijection, while \( \eta \circ y \) does so whenever the bottom assignment is a bijection.

Proof of Proposition 5.9. Let us write \( \eta \) for the cell that defines \( l \) as the pointwise weak left Kan extension of \( d \) along \( y_* \), as on the left below. Notice that, because \( y \) is full and faithful (Lemma 5.4), composing \( \eta \) with the weakly cocartesian cell defining \( y_* \) results in an invertible vertical cell by Proposition 4.13. Writing \( \eta' \) for the factorisation of \( \eta \) through \( d_* \), it follows that the composite of the top two cells in the left-hand side of the equation below is weakly cocartesian, so that the full composite defines \( d_* \) as the left Kan extension of \( y \) along \( l \circ y \) in \( V(K) \), by Proposition 4.2.

\[
\begin{array}{c}
A \\
\downarrow_{y_*}
\end{array}
\quad \begin{array}{c}
\hat{A} \\
\downarrow^j
\end{array}
\quad \begin{array}{c}
A \\
\downarrow_{y_*}
\end{array}
\quad \begin{array}{c}
\hat{A} \\
\downarrow^j
\end{array}
\]

Now the composite of the bottom two cells in the left-hand side above factorises uniquely through the cartesian cell defining \( y_* \) as a vertical cell \( \zeta \), as shown in the
first identity. Using the companion identity for $y_\ast$, the second identity follows. We conclude that $\zeta \circ y$ defines $d_\ast^\lambda$ as the left Kan extension of $y$ along $l \circ y$ in $V(K)$ which, by the lemma above, implies that $\zeta$ defines $d_\ast^\lambda$ as the left Kan extension of $\text{id}_{\hat{A}}$ along $l$. We will shown that it is preserved by $l$, so that the proof follows from Lemma 2.16. To this end consider the identity below, where $\text{cart}'$ denotes the factorisation of the cartesian cell defining $d_\ast^\lambda$ through $y_\ast$; the identity itself follows from the first identity above.

\[
\begin{array}{c}
\begin{array}{c}
A \\
\langle \text{cocart}, y \rangle \\
A \\
\downarrow q' \quad l \\
A \\
d_\ast \\
\downarrow q \quad \eta \\
\hat{A} \\
p \\
M
\end{array}
\end{array}
\]

By the pasting lemma $\text{cart}'$ is again cartesian; hence, because $\eta$ defines a pointwise weak left Kan extension, so does the composite of the bottom two cells in the left-hand side above. Remembering that $\eta' \circ \text{cocart}$ is weakly cocartesian, it follows that the full left-hand side defines $l \circ d_\ast^\lambda$ as a left Kan extension in $V(K)$ by Proposition 4.2. Since $\eta \circ \text{cocart}$ in the right-hand side is invertible we conclude that $l \circ \zeta \circ y$ defines $l \circ d_\ast^\lambda$ as a left Kan extension which, by the previous lemma, implies that $l \circ \zeta$ does too, as required.

5.3 Exact paths of cells

In the definition below the classical notion of 'carré exact', as studied by Guitart [Gui80], is generalised to the setting of hypervirtual double categories. This notion will be used throughout the remainder.

Definition 5.12. Consider a path $\phi = (\phi_1, \ldots, \phi_n)$ of unary cells as in the composite below, where $n \geq 1$, and let $d: C_0 \to M$ be any vertical morphism. The path $\phi$ is called (weak) left $d$-exact if for any nullary cell $\eta$ as below, that defines $l$ as the (weak) left Kan extension of $d$ along $(K_1, \ldots, K_n)$, the full composite defines $l \circ f_n$ as the (weak) left Kan extension of $d \circ f_0$ along $(J_{11}, \ldots, J_{nmn})$.

\[
\begin{array}{c}
\begin{array}{c}
A_{10} \\
\downarrow f_0 \quad f_1 \\
C_0 \\
\downarrow \phantom{\eta} & d \\
\hat{C}_1 \\
\downarrow \phantom{\eta} & M
\end{array}
\end{array}
\]

If $\phi$ is (weak) left $d$-exact for any $d: C_0 \to M$, where $M$ varies, then it is called (weak) left exact.

Analogously, the path $\phi$ above is called pointwise (weak) left $d$-exact if for any cell $\eta$ above, that defines $l$ a pointwise (weak) left Kan extension of $d$ along

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\((K_1, \ldots, K_n)\), the full composite defines \(l \circ f_n\) as the pointwise (weak) left Kan extension of \(d \circ f_0\) along \((J_{11}, \ldots, J_{nm})\). A path that is pointwise (weak) left \(d\)-exact for all \(d; C_0 \to M\) is called pointwise (weak) left exact.

Moreover, we say that \(\phi\) satisfies the left Beck-Chevalley condition if the restriction \(K_n(id, f_n)\) exists and the path \((\phi_1, \ldots, \phi_n, \phi'_n)\) is right pointwise cocartesian (Definition 3.16), where \(\phi'_n\) is the unique factorisation in

\[
\begin{array}{c}
A_{10} \xrightarrow{J_{11}} A_{11} \cdots A_{nm}\phi_n' \xrightarrow{J_{nm}} A_{nm} \\
\downarrow f_{1n} \hspace{5cm} \downarrow f_n \\
C_{1n} \xrightarrow{K_n(id, f_n) \text{cart}} A_{nm} \xrightarrow{f_n} C_{1n}.
\end{array}
\]

A single cell \(\phi\) is said to satisfy the left Beck-Chevalley condition if the single path \((\phi)\) does.

We shall also use the horizontally dual condition: we say that the path \(\phi\) above satisfies the right Beck-Chevalley condition if the restriction \(K_1(f_0, id)\) exists and \((\phi'_1, \phi_2, \ldots, \phi_n)\) is left pointwise cocartesian, where \(\phi'_1\) is the unique factorisation of \(\phi_1\) through the cartesian cell defining \(K_1(f_0, id)\).

**Example 5.13.** Consider a cell

\[
A \xrightarrow{f} B \xrightarrow{g} D
\]

in the hypervirtual equipment \((\mathbf{Set}, \mathbf{Set'})\)-Prof of (unenriched) profunctors, and assume that \(C\) is small. It follows that both the conjoint \(f^*\) and the pointwise composite \((f^* \circ J)\) exist; see Example 2.2 and Example 3.3. The latter forms the extension of \(J\) along \(f\) by Corollary 3.24 whose defining cocartesian cell is right pointwise, so that \(\phi\) satisfies the left Beck-Chevalley condition precisely if its factorisation \(\phi''(f^* \circ J) \Rightarrow K(id, g)\), that is induced by the cell

\[
(f^*, J) \Rightarrow K(id, g) : (x \xrightarrow{z} fy, y \xrightarrow{u} z) \mapsto (x \xrightarrow{z} fy \xrightarrow{\phi u} gz),
\]

is invertible. If \(J = j, \) and \(K = k,\) for functors \(j : A \to B\) and \(k : C \to D,\) so that \(\phi\) corresponds to a transformation \(\phi : k \circ f = g \circ j,\) this recovers the definition of exactness given in [Gui80].

The Beck-Chevalley condition is preserved under concatenation, as follows. In particular, a path \((\phi_1, \ldots, \phi_n)\) satisfies the left Beck-Chevalley condition whenever each of the cells \(\phi_i\) do.

**Lemma 5.14.** Consider composite paths of cells \(\phi = (\phi_1, \ldots, \phi_n)\) and \(\psi = (\psi_1, \ldots, \psi_m)\). If they both satisfy the left Beck-Chevalley condition then so does their concatenation \(\phi \circ \psi = (\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m)\).

**Proof.** That \(\psi\) satisfies the left Beck-Chevalley condition means that \((\psi_1, \ldots, \psi'_m)\) is right pointwise cocartesian, where \(\psi'_m\) is the unique factorisation of \(\psi_m\) through the restriction of its horizontal target along its vertical target, as in the definition above. We have to show that the concatenation \((\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi'_m)\) is again right pointwise cocartesian. To see this we substitute \(\phi_n = \text{cart} \circ \phi'_n\), again as in the
definition above, thus obtaining the composite of paths that is drawn schematically below.

\[
\begin{array}{cccc}
\phi_1 & \cdots & \phi_n' \\
\text{cart} & \cdots & \psi_1 & \cdots & \psi_m'
\end{array}
\]

It is clear from the definition of cocartesian paths (Definition 3.1) that the bottom row here is right pointwise cocartesian, because \((\psi_1, \ldots, \psi_m')\) is so. Clearly also the top row is right pointwise cocartesian, so that the full path is right-cocartesian by the pasting lemma Lemma 3.19, as required.

The following generalises Theorem 1.7 of [Gui80].

**Proposition 5.15.** For the path \(\phi = (\phi_1, \ldots, \phi_n)\) of Definition 5.12 the implication (a) \(\Rightarrow\) (b) holds for the following conditions:

(a) \(\phi\) satisfies the left Beck-Chevalley condition;

(b) \(\phi\) is pointwise weak left exact as well as pointwise left exact.

Moreover, in the case that \(n = 1\) while both the restriction \(K_1(\text{id}, f_1)\) and the yoneda embedding \(y: C_0 \to \hat{C}_0\) exist, consider the further condition

(c) \(\phi_1\) is pointwise left \(y\)-exact.

Then the conditions (a) and (c) are equivalent whenever the right pointwise cocartesian cell below exists, while (b) and (c) are equivalent as soon as both the conjoint \(f_0^*: C_0 \to A_{10}\) and the right pointwise composite \((f_0^* \circ J_{11} \circ \cdots \circ J_{1m_1})\) exist.

\[
\begin{array}{cccc}
A_{10} & A_{11} & \cdots & A_{1m_1} \\
\text{cocart} & \cdots & \text{cart} & \cdots \\
C_0 & H & A_{1m_1}
\end{array}
\]

**Proof.** To show (a) \(\Rightarrow\) (b), consider a nullary cell \(\eta: (K_1, \ldots, K_n) \Rightarrow M\) that defines the pointwise (weak) left Kan extension of a vertical morphism \(d: C_0 \to M\), as in Definition 5.12 above. Writing \(\phi_n = \text{cart} \circ \phi_n'\) as before, it follows that \(\eta \circ (\text{id}, \ldots, \text{cart})\) defines a pointwise (weak) left Kan extension by Lemma 4.6(b), so that the full composite \(\eta \circ (\text{id}, \ldots, \text{cart}) \circ (\phi_1, \ldots, \phi_n') = \eta \circ (\phi_1, \ldots, \phi_n)\) defines a pointwise (weak) left Kan extension by the pointwise extension for Kan extensions (Lemma 4.9), showing that (b) holds.

Restricting to \(n = 1\), it is clear that (b) \(\Rightarrow\) (c), so that proving (c) \(\Rightarrow\) (a) and (c) \(\Rightarrow\) (b), under the respective assumptions, suffices. To prove (a) \(\Rightarrow\) (c) consider the factorisation \(\phi_1'\) of \(\phi_1\) as given in Definition 5.12; we have to show that it is right pointwise cocartesian. We write \(\phi_1''\) for its further factorisation through the right pointwise cocartesian cell above, so that the identity on the left below follows. Here, on both sides, the bottom nullary cartesian cell is that given by the yoneda axiom (Definition 5.2); remember it defines \(g\) as a pointwise weak left Kan extension because \(y\) is dense (Lemma 5.1). Assuming condition (c), it follows that the full
composite on the left-hand side defines $l \circ f_1$ as a pointwise left Kan extension.

Since the right-hand side of the identity above defines a pointwise left Kan extension, so does the composite of its bottom three cells, as follows from the vertical pasting lemma (Lemma 4.9) applied to its right pointwise cocartesian top cell. From Lemma 5.3 we conclude that the latter composite is cartesian so that, by the pasting lemma, the cell $\phi''_1$ is cartesian. Being a horizontal cell, it follows that $\phi''_1$ is invertible; hence $\phi'_1 = \phi''_1 \circ \text{cocart}$ is right pointwise cocartesian, as required.

Finally, to prove (c) ⇒ (b), consider the composite on the right above. Assuming that the right pointwise composite $(f_0 \circ J_1 \circ \cdots \circ J_{m_1})$ exists, we may apply (c) ⇒ (a) ⇒ (b) to find that it is pointwise left exact. Using Proposition 4.12 we conclude that $\phi_1$ is itself pointwise left exact. This concludes the proof.

### 5.4 Presheaf objects as free cocompletions

We now turn to proving that, under mild conditions, a presheaf object $\hat{M}$ forms the ‘free cocompletion of $M$’, in a sense that will be made precise. This generalises the situation of $\mathcal{V}$-enriched category theory, where the $\mathcal{V}$-category of presheaves $\hat{M}$ on a small $\mathcal{V}$-category $M$ forms the free cocompletion of $M$ under small weighted colimits; see for instance Theorem 4.51 of [Kel82]. On the other hand, for any yoneda embedding $y : M \to \hat{M}$ in a 2-category in the sense of [SW78] and [Web07], the conditions ensuring that $\hat{M}$ is cocomplete seem to be unclear.

In the result below we start by characterising the cocompleteness of presheaf objects, that are defined by good yoneda embeddings, in terms of left exact cells. In particular it follows that presheaf objects are cocomplete in a strong sense, that is they admit many pointwise left Kan extensions. Remember that the yoneda embedding $y$ is called good if all restrictions $\hat{M}(y, d)$ exist, where $d : A \to \hat{M}$; for the notion of pointwise composition see Definition 3.16.

**Proposition 5.16.** Let $y : M \to \hat{M}$ be a good yoneda embedding. The pointwise left Kan extension of $d : A \to \hat{M}$ along $J : A \to B$ exists precisely when a pointwise left $y$-exact cell of the form below does. This is the case, for instance, if $\phi$ defines $K$ as the right pointwise composite $(\hat{M}(y, d) \circ J)$. 

$$
\begin{array}{ccc}
M & \xrightarrow{\phi} & B \\
\downarrow & & \downarrow \\
\hat{M}(y, d) & \xrightarrow{J} & \hat{M}
\end{array}
$$

**Proof.** For the ‘when’-part consider $l := K^\lambda$ given by the yoneda axiom and defined by the cartesian cell in the left-hand side below. This cell defines $l$ as a pointwise left
Kan extension because \( y \) is dense (Lemma 5.1) so that, by the assumption on \( \phi \), the full composite on the left-hand side again defines \( l \) as a pointwise left Kan extension. Next consider the cartesian cell in the right-hand side, which exists because \( y \) is a good yoneda embedding. This too defines a left Kan extension, again by density of \( y \), so that the left-hand side factors through it as a cell \( \eta \), as shown. Applying the horizontal pasting lemma (Lemma 4.8) we conclude that \( \eta \) defines \( l \) as the pointwise left Kan extension of \( d \) along \( J \).

\[
\begin{array}{c}
\xymatrix{
M \ar[r]^{\tilde{M}(y,d)} & A \ar@{.>}[d]^J \\
M \ar[u]^\phi \ar[r]_y & B
}
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
M \ar[r]^{\tilde{M}(y,d)} & A \ar@{.>}[d]^J \\
M \ar[r]_{\text{cart}} & B
}
\end{array}
\]

For the ‘precisely’-part assume that the cell \( \eta \) in the right-hand side above defines \( l \) as the pointwise left Kan extension of \( d \) along \( J \). We compose it with the cartesian cell defining the restriction \( \tilde{M}(y,d) \), which exists because \( y \) is a good yoneda embedding, and factor the result through the cartesian cell defining the restriction \( K := \tilde{M}(y,l) \), obtaining a horizontal cell \( \phi \) as shown. Now the cartesian cells above define pointwise left Kan extensions because \( y \) is dense; it follows that the full right-hand side does as well, by the horizontal pasting lemma. We conclude that composing \( \phi \) with a cell that defines the pointwise left Kan extension of \( y \) along \( K \) results in a composite that again defines a pointwise left Kan extension, showing that \( \phi \) is pointwise left \( y \)-exact as required. Completing the proof, the final assertion follows directly from Corollary 4.10.

Notice in particular that presheaf objects \( \hat{N} \) in hypervirtual equipments, that are defined by good yoneda embeddings, admit all pointwise left Kan extensions of the identity \( \text{id}_{\hat{N}} \), along any horizontal morphisms \( J : \hat{N} \Rightarrow B \): indeed in this case the pointwise composite \( (\hat{N}(y,\text{id}) \circ J) \) coincides with the restriction \( J(y,\text{id}) \); see Lemma 3.20. Thus the object \( M = \hat{N} \) satisfies the equivalent conditions of the lemma below; we will call such objects \( M \) total. This term is motivated by [SW78], where an ‘admissible’ object \( C \) in a 2-category is called total when its yoneda embedding \( y : C \Rightarrow \hat{C} \) admits a left adjoint. Moreover in [DS86] a \( \mathcal{V} \)-category \( M \) is called total when it admits, for all \( \mathcal{V} \)-presheaves \( J : M^{\mathcal{V}^\text{op}} \Rightarrow \mathcal{V} \), the \( J \)-weighted colimit of the identity \( \text{id}_{\hat{M}} \); that this is equivalent to the condition (a) below, considered in \( \mathcal{V} \)-\text{Prof}, follows easily from the results of Section 4.4. The equivalences (c) \( \Leftrightarrow \) (a) \( \Leftrightarrow \) (b) below are generalisations of Theorems 5.2 and 5.3 of [Kel86], where they are proved in the case of \( (\mathcal{V},\mathcal{V}')\)-\text{Prof}.

**Lemma 5.17.** For an object \( M \) the following conditions are equivalent:

(a) for any \( J : M \Rightarrow B \) the pointwise left Kan extension of \( \text{id}_{\hat{M}} \) along \( J \) exists;

(b) for any right pointwise cocartesian cell

\[
\begin{array}{c}
A \xrightarrow{J} B \\
\xymatrix{d \ar[d]_{\text{cocart}} & \\
M \ar[r]_K & B}
\end{array}
\]

the pointwise left Kan extension of \( d \) along \( J \) exists.
Given a good yoneda embedding \( y: M \rightarrow \hat{M} \) the following condition is equivalent too:

(c) \( y \) admits a left adjoint \( c: \hat{M} \rightarrow M \).

**Proof.** (a) \( \Rightarrow \) (b) follows from applying the vertical pasting lemma (Lemma 4.9) to the composites as on the left below, where the cell \( \eta \) defines \( l: B \rightarrow M \) as the pointwise left Kan extension of \( \text{id}_M \) along \( K \).

(b) \( \Rightarrow \) (a) simply follows from taking the identity on \( J \) as cocartesian cell.

(c) \( \Rightarrow \) (a) follows from considering composites as on the second left below, where the cartesian cell is given by the yoneda axiom (Definition 5.2) and \( \varepsilon: c \circ y \Rightarrow \text{id}_M \) denotes the counit of \( c \dashv y \); the latter is invertible because \( y \) is full and faithful (see Lemma 5.4 and the horizontal dual of Lemma 2.16). Use the fact that the left adjoint \( c \) is cocontinuous; see Proposition 4.15.

To prove the second triangle identity \((c \circ \zeta) \circ (\delta \circ c) = \text{id}_C\), notice that, after composition with the cartesian cell defining \( y_* \), it coincides with the left-hand and right-hand sides of the equation above. The first identity here follows from the definition of \( \zeta \), while the second one follows by precomposition with the weakly cocartesian cell cocart that defines \( y_* \), the uniqueness of factorisations through the latter, the identity \( \gamma \circ \text{cocart} = \delta^{-1} \) and the interchange axiom (Lemma 1.2). The second triangle identity now follows from the fact that the left-hand side above defines \( c \) as a pointwise left Kan extension: indeed, its composition with the invertible cell \( \delta^{-1} = \gamma \circ \text{cocart} \) equals \( \gamma \) by the horizontal companion identity (see Lemma 2.11), and \( \gamma \) defines \( c \) as a pointwise left Kan extension. This completes the proof. \( \square \)
Having seen that presheaf objects are cocomplete in a strong sense, we now describe the way in which they form 'small cocompletions'. As there is no clear notion of 'smallness' for objects in a general hypervirtual double category, we regard the notion of 'smallness' as variable, as follows.

**Definition 5.18.** Let $K$ be a hypervirtual double category. By a left extension diagram in $K$ we mean a span of the form $M \xleftarrow{d} A \xrightarrow{\phi} B$. A collection $S$ of left extension diagrams is called an ideal if $(f \circ d, J) \in S$ for all $(d, J) \in S$ and $f \in K$ composable with $d$; given such an ideal and an object $M$ we write $S(M) \subset S$ for the subcollection of spans of the form $M \xleftarrow{d} A \xrightarrow{J} B$. Moreover:

- $M$ is called $S$-cocomplete if, for any $(d, J) \in S(M)$, the pointwise left Kan extension of $d$ along $J$ exists;
- $f: M \to N$ is called $S$-cocontinuous if, for any $(d, J) \in S(M)$ and any nullary cell $\eta$ that defines a morphism $l: B \to M$ as the pointwise left Kan extension of $d$ along $J$, the composite $f \circ \eta$ defines $f \circ l$ as a pointwise left Kan extension;
- $w: M \to \hat{M}$ is said to define $\hat{M}$ as the free $S$-cocompletion of $M$ if $\hat{M}$ is $S$-cocomplete and, for any $S$-cocomplete $N$, the composite

$$V_{S\text{-cocts}}(K)(\hat{M}, N) \subseteq V(K)(\hat{M}, N) \xrightarrow{V(K)(w, N)} V(K)(M, N)$$

is an equivalence, where $V_{S\text{-cocts}}(K)$ denotes the sub-2-category of $S$-cocontinuous morphisms in $V(K)$.

We can now state the main theorem of this subsection, using the notion of left exactness (Definition 5.12). Afterwards some examples are given. Recall that, for condition (e) below, it suffices that the cell $\phi$ defines $K$ as the right pointwise composite $(\hat{M}(y, d) \circ J)$ (Definition 3.16).

**Theorem 5.19.** Let $y: M \to \hat{M}$ be a good yoneda embedding in a hypervirtual double category $K$ and let $S$ be an ideal of left extension diagrams in $K$. If

(c) a pointwise left $y$-exact cell

$$
\begin{array}{ccc}
M & \xrightarrow{\hat{M}(y, d)} & A \\
\downarrow \phi & & \downarrow J \\
M & \xrightarrow{\phi} & B
\end{array}
$$

exists for every $(d, J) \in S(\hat{M});$

(y) $(f, y_\ast) \in S$ for all $f: M \to N$,

then $y$ defines $\hat{M}$ as the free $S$-cocompletion of $M$.

**Proof.** Condition (e) ensures that $\hat{M}$ is $S$-cocomplete by Proposition 5.16. That precomposition with $y$ induces an equivalence $V_{S\text{-cocts}}(K)(\hat{M}, N) \simeq V(K)(M, N)$, for any $S$-cocomplete $N$, is shown in Lemma 5.23 below.

**Example 5.20.** Taking $\mathcal{K} = (\mathcal{V}, \mathcal{V}')$-Prof (Example 1.12), consider the ideal of left extension diagrams

$$S = \{ (d, J) \mid J: A \Rightarrow B \text{ is a } \mathcal{V}\text{-profunctor between } \mathcal{V}\text{-categories, with } A \text{ small} \}.$$
If \( \mathcal{V} \) is closed monoidal, small cocomplete and small complete, and with \( \otimes \) preserving small colimits on both sides then, for any small \( \mathcal{V} \)-category \( M \), the yoneda embedding \( y: M \to \hat{M} \) satisfies the conditions of the theorem (as follows from Example 3.3 and Proposition 5.5), so that the \( \mathcal{V} \)-category \( \hat{M} \) of presheaves forms the free \( \mathcal{S} \)-cocompletion of \( M \).

If \( \mathcal{V} \) is closed symmetric monoidal then, in view of the results of Section 4.4, \( \mathcal{S} \)-cocompleteness coincides with the classical notion of (small) cocompleteness for \( \mathcal{V} \)-categories, in the sense of e.g. of Section 3.2 of [Kel82]. The theorem in this case recovers the fact that, for a small \( \mathcal{V} \)-category \( M \), the \( \mathcal{V} \)-category of presheaves \( \hat{M} \) forms the free small cocompletion of \( M \); see Theorem 4.51 of [Kel82].

**Example 5.21.** To give a useful ideal \( \mathcal{C} \) of left extension diagrams in a general hypervirtual double category \( K \), let us call a horizontal morphism \( J: A \Rightarrow B \) left composable when the right pointwise composite \( (H \circ J) \) exists for any \( H: C \Rightarrow A \); we set

\[
\mathcal{C} = \{ (d, J) \mid J \text{ is left composable} \}.
\]

Clearly a good yoneda embedding \( y: M \to \hat{M} \) in \( K \) satisfies the conditions of the theorem as soon as its companion \( y_* \) is left composable so that, in that case, \( \hat{M} \) forms the free \( \mathcal{C} \)-cocompletion of \( M \).

Notice that if \( K = (\mathcal{V}, \mathcal{V}')\)-Prof, with \( \mathcal{V} \) as in the previous example, then the ideal \( \mathcal{S} \) considered there is contained in \( \mathcal{C} \), so that any \( \mathcal{C} \)-cocomplete \( \mathcal{V} \)-category is \( \mathcal{S} \)-cocomplete. Under what conditions the converse holds I do not know.

In the proof of Lemma 5.23 below the following proposition is used, which is a weak variant of Proposition 5.9.

**Lemma 5.22.** Let \( y: M \to \hat{M} \) and \( \mathcal{S} \) be as in Theorem 5.19. Any pointwise left Kan extension along \( y_* \) is \( \mathcal{S} \)-cocontinuous.

**Proof.** Suppose that the cell \( \zeta \) in the composite below defines \( l \) as the pointwise left Kan extension of \( e \) along \( y_* \). We have to show that for any left extension diagram \( (d, J) \in \mathcal{S}(\hat{M}) \) and any cell \( \eta \), as below, that defines a morphism \( l: B \Rightarrow \hat{M} \) as the pointwise left Kan extension of \( d \) along \( J \), the composite \( k \circ \eta \) defines \( k \circ l \) as a pointwise left Kan extension as well.

\[
\begin{array}{ccc}
M & \xrightarrow{\hat{M}(y, d)} & A \\
\downarrow \text{cart'} & & \downarrow J \\
M & \xrightarrow{\hat{M}} & \hat{M} \\
\downarrow \varepsilon & & \downarrow k \\
N & & 
\end{array}
\]

To see this first notice that, by the uniqueness of Kan extensions, we may without loss of generality assume that \( \eta \) is the cell obtained in the ‘when’-part of Proposition 5.16, as the unique factorisation in the left-hand side of (16); here we use that the pointwise left \( y \)-exact cell \( \phi: \hat{M}(y, d), J \Rightarrow K \) as considered there, exists by condition (e) of Theorem 5.19. Now the full left-hand side of (16), after factorising it through the cartesian cell defining \( y_* \), coincides with the top row in the composite above, while its right-hand side factors through \( y_* \) as a cartesian cell (by the pasting lemma for cartesian cells), precomposed with \( \phi \). Using Lemma 4.6(b) and the assumption that \( \phi \) is pointwise left \( y \)-exact, we conclude that the full composite above defines \( k \circ l \) as a pointwise left Kan extension. Finally, because the factorisation cart' too is cartesian by the pasting lemma, we find that the first column
\( \zeta \circ \text{cart}' \) above defines a left Kan extension as well. Using the horizontal pasting lemma we conclude that second column, that is \( l \circ \zeta \), also defines a pointwise left Kan extension, as required.

Lemma 5.23. Let \( y : M \to \hat{M} \) and \( S \) be as in Theorem 5.19. For any object \( N \) the top leg of the diagram

\[
\begin{align*}
V(K)(\hat{M}, N) \xrightarrow{V(K)(y, N)} & \ V(K)(M, N) \\
\cup \ \ \cup & \ \ \ \cup \\
V_{S, \text{cocts}}(K)(\hat{M}, N) & \to V(K)(M, N)'
\end{align*}
\]

factors through the full subcategory \( V(K)(M, N)' \) of \( V(K)(M, N) \), that is generated by all \( g : M \to N \) whose pointwise left Kan extension along \( y_* \) exists in \( K \), as an equivalence as shown.

In particular, if \( N \) is \( S \)-cocomplete, then the factorisation above reduces to an equivalence \( V_{S, \text{cocts}}(K)(\hat{M}, N) \simeq V(K)(M, N) \).

Proof. Firstly, for the final assertion, simply notice that condition \((y)\) of Theorem 5.19 ensures that \( V(K)(M, N)' = V(K)(M, N) \) for all \( S \)-cocomplete \( N \). Next to see that the top leg of the diagram above factors consider any \( S \)-cocontinuous \( f : \hat{M} \to N \); we have to show that the pointwise left Kan extension of \( f \circ y \) along \( y_* \) exists. Since \((y, y_*) \in S \) by the same condition it clearly does: it is \( f \) itself, defined by the composite on the left below.

To prove that the factorisation is an equivalence we will show that it is essentially surjective and full and faithful; for the former consider any \( g \in V(K)(M, N)' \). By definition of \( V(K)(M, N)' \) the pointwise left Kan extension \( l : \hat{M} \to N \) of \( g \) along \( y_* \) exists; we denote its defining cell by \( \eta \), as in the middle below. By Lemma 5.22 \( l \) is \( S \)-cocontinuous while, by precomposing \( \eta \) with the weakly cocartesian cell defining \( y_*, \) we obtain a vertical isomorphism \( g \cong f \circ y \), as follows from the fact that \( y \) is full and faithful (Lemma 5.4) and Proposition 4.13. This shows essential surjectivity.

Finally, to prove full and faithfulness, consider any vertical cell \( \phi : f \Rightarrow g \). We have to show that there exists a unique vertical cell \( \phi' : f \Rightarrow g \) such that \( \phi = \phi' \circ y \).

\[
\begin{align*}
M & \xrightarrow{y_*} \hat{M} \\
\text{cart} \quad & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \\
\downarrow & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \\
\hat{M} & \xrightarrow{y} M \\
\downarrow f & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \\
N & \xrightarrow{y_*} \hat{N}
\end{align*}
\]

Since \( y \) is dense, the cartesian cell in the right-hand side of the identity above defines a left Kan extension, by Lemma 5.1. It follows that its composition with \( f \) does too, by \( S \)-cocompleteness of \( f \) and condition \((y)\) of Theorem 5.19 again, so that the composite on the left-hand side factors uniquely as a cell \( \phi' \) as shown. Composing both sides with the weakly cocartesian cell that defines \( y_* \), we conclude that \( \phi' \) is unique such that \( \phi = \phi' \circ y \), as required. This completes the proof.

6 Algebras of monads

With our study of ‘ordinary’ category theory within hypervirtual double categories complete, we now turn to such theory in the presence of ‘algebraic structures’; the archetypal example being that of monoidal structures on categories.
Like in 2-dimensional category theory, algebraic structures in a hypervirtual double category \( \mathcal{K} \) are defined by monads on \( \mathcal{K} \). In fact, any monad \( T \) on \( \mathcal{K} \) induces a strict 2-monad \( V(T) \) on the vertical 2-category \( V(\mathcal{K}) \), under the 2-functor \( V: \text{HypVirtMultiCat} \to \text{2-Cat} \). Given a monad \( T \) we will consider a hypervirtual double category \( T-\text{Alg} \) of (weak) algebras of \( T \), whose vertical part \( V(T-\text{Alg}) \) coincides with the 2-category \( V(T)-\text{Alg} \) of (weak) \( V(T) \)-algebras in the classical sense. The notion of horizontal morphism in \( T-\text{Alg} \) generalises that of ‘horizontal \( T \)-morphism’ introduced by Grandis and Paré in the setting of pseudo double categories [GP04]; see also [Kou15a].

In closing this section we will use the characterisation of (op-)representable horizontal morphisms given in Theorem 2.21 to characterise (op-)representable horizontal \( T \)-morphisms.

### 6.1 Monads on a hypervirtual double category

We start with monads on hypervirtual double categories. They are simply monads in the 2-category \( \text{HypVirtMultiCat} \) as follows.

**Definition 6.1.** By a monad \( T \) on a hypervirtual double category \( \mathcal{K} \) we mean a monad \( T = (T, \mu, \iota) \) on \( \mathcal{K} \) in the 2-category \( \text{HypVirtMultiCat} \). As usual, \( T \) consists of an endofunctor \( T: \mathcal{K} \to \mathcal{K} \), equipped with multiplication and unit transformations \( \mu: T^2 \Rightarrow T \) and \( \iota: \text{id}_{\mathcal{K}} \Rightarrow T \) which satisfy the associativity and unit identities \( T \circ T \mu = T \circ \mu T \) and \( \mu \circ T \iota = \text{id}_T = \mu \circ \iota T \).

We call \( T \) strong whenever its underlying endofunctor is strong, that is preserves \( \otimes \) on both sides. The ‘free monoid’-monad \( TA \) is given by \( TA := \prod_{n \geq 0} A^{	imes n} \); likewise, on a function \( f: A \to C \) it is given by \( T f := \prod_{n \geq 0} f^{	imes n} \), while its image of the \( \mathcal{V} \)-matrix \( J: A \Rightarrow B \) is given by

\[
(TJ)(\underline{x}, \underline{y}) = \begin{cases} \bigotimes_{i=1}^{n} J(x_i, y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\
\emptyset & \text{otherwise}, \end{cases}
\]

where \( |\underline{x}| \) and \( |\underline{y}| \) denote the lengths of \( \underline{x} \) and \( \underline{y} \). The components of the image \( T\phi: (TJ_1, \ldots, TJ_n) \Rightarrow TK \) of a cell \( \phi \) are given by tensor products of the components of \( \phi \), after using the while the symmetry of \( \mathcal{V} \) to put the tensor factors in the right order. The multiplication of \( T \), finally, is given by concatenation of sequences on sets, while on \( \mathcal{V} \)-matrices it is induced by the associator of \( \mathcal{V} \).
Applying $\text{Mod}$ to $T$ we obtain the ‘free strict monoidal $V$-category’-monad on $V$-$\text{Prof}$, which we again denote $T$. It maps a $V$-category $A$ to the freely generated strict monoidal $V$-category $TA := \coprod_{n \geq 0} A^{\otimes n}$, while its image of a $V$-profunctor $J : A \Rightarrow B$ consists of the $V$-objects $(TJ)(x, y)$ above, on which $TA$ and $TB$ act indexwise. For further details we refer to Example 3.11 of [Kou15a] where, in the case of a cocomplete $V$ whose tensor product preserves colimits on both sides, the restriction of $T$ to the double category of small $V$-categories is described in detail.

For a closed symmetric monoidal $V'$ that is large cocomplete, so that $V'$-$\text{Prof}$ is a double category (Example 3.3), showing that $T$ is a strong monad on $V'$-$\text{Prof}$ is straightforward. In proving this the “Fubini theorem” for coends is useful; see Section 2.1 of [Kel82] for the dual theorem for ends. Now, given a symmetric universe enlargement $V \to V'$ with $V$ as in the above, notice that $T$ restricts to a strong monad on $(V, V')$-$\text{Prof}$, whose strongness is a consequence of the reflection of cocartesian cells along the inclusion $(V, V')$-$\text{Prof} \to V'$-$\text{Prof}$ (Proposition 3.17).

**Example 6.3.** Let $G_1$ be the indexing category $G_1 = (1 \iff 0)$, so that the functor category $\text{Set}^{G_1}$ is that of large directed graphs. Consider the ‘free category’-monad $\text{Set}^{G_1} : \text{Set}^{G_1}$ to its free category $TA$, with the same vertices as $A$ and, as edges, (possibly empty) paths $x = (x_0 e_1 x_1 e_2 \cdots e_n x_n)$ of edges in $A$; formally

$$(TA)_0 = A_0 \quad \text{and} \quad (TA)_1 = A_0 \amalg \coprod_{n \geq 1} A_1 \times_{A_0} A_1 \times_{A_0} \cdots \times_{A_0} A_1.$$  

Multiplication of $T$ is given by concatenation of paths, while its unit inserts edges as singleton paths.

Checking that $T$ preserves the indexwise pullbacks of $\text{Set}^{G_1}$ is straightforward; see for instance Proposition 2.3 of [DPP06]. Hence we may apply $\text{Prof}(\_ ) = \text{Mod} \circ \text{Span}(\_ )$ (see Example 1.17 and Proposition 1.18) to obtain a monad on the equipment $\text{Set}'$-$\text{Prof}^{G_1} := \text{Prof}(\text{Set}^{G_1})$ of large $G_1$-indexed profunctors: this is the ‘free strict double category’-monad, which we again denote $T$. We shall briefly describe $\text{Set}'$-$\text{Prof}^{G_1}$ and the action of $T$.

A $G_1$-indexed category $A = (A_1 \xrightarrow{\sigma} A_0)$ will be regarded as being a virtual double category $A$ that has $(1, 1)$-ary cells only; its objects and vertical morphisms form the category $A_0$ while its horizontal morphisms and cells form $A_1$. The free strict double category $TA$ has the same objects and vertical morphisms as $A$, while its horizontal morphisms and cells are (possible empty) paths of horizontal morphisms and cells in $A$. A large $G_1$-indexed profunctor $J : A \Rightarrow B$ consists of large profunctors $J_0 : A_0 \Rightarrow B_0$ and $J_1 : A_1 \Rightarrow B_1$, equipped with natural transformations $J_{\sigma} : J_1 \Rightarrow J_0(\sigma_A, \sigma_B)$ and $J_\tau : J_1 \Rightarrow J_0(\tau_A, \tau_B)$. We think of the elements $u \in J_0(A, C)$ as vertical morphisms $u : A \to C$, and of those $\theta \in J_1(J, K)$ as cells as below, where $u = J_{\theta}(\theta)$ and $v = J_{\theta}(\theta)$.

$$
\begin{array}{c}
A \xrightarrow{J} B \\
\downarrow u \quad \downarrow v \\
C \xrightarrow{K} D
\end{array}
$$

Analogous to its action on $G_1$-indexed categories, $T$ maps $J : A \Rightarrow B$ to the $G_1$-indexed profunctor $TJ : TA \Rightarrow TB$ that has the same vertical morphisms as $J$ while its cells are paths of cells of $J$.

Finally, notice that $T$ restricts to the hypervirtual equipment $(\text{Set}, \text{Set}')$-$\text{Prof}^{G_1}$ of small $G_1$-indexed profunctors between large $G_1$-indexed categories, that is those
\( \mathcal{J} : \mathcal{A} \to \mathcal{B} \) with small sets \( \mathcal{J}_0(A,C) \) and \( \mathcal{J}_1(J,K) \) of vertical morphisms and cells. We remark that \( T \) is not strong; for an example of a composite \( \mathcal{G}_1 \)-indexed pro-functor that is not preserved by \( T \) see Proposition 3.25 of [Kou15a].

### 6.2 Algebras of 2-monads

Here we recall from [Str74] several notions of weak algebra of a 2-monad, and apply them to the vertical parts \( V(T) \) of monads \( T \) on hypervirtual double categories.

**Definition 6.4.** Let \( T = (T, \mu, \iota) \) be a strict 2-monad on a 2-category \( C \).

- A lax \( T \)-algebra \( A \) is a quadruple \( A = (A, a, \bar{a}, \bar{\alpha}) \) consisting of an object \( A \) in \( C \) equipped with a morphism \( a : TA \to A \), its structure morphism, and cells \( \bar{a} \) and \( \bar{\alpha} \) as on the left below, its associator and unitor, that satisfy the usual coherence axioms; see for instance Section 2 of [Str74] or Section 3 of [Kou15a].

A lax \( T \)-algebra \( A \) with an invertible unitor \( \bar{a} \) is called normal. If both the associator and the unitor are invertible then \( A \) is called a pseudo \( T \)-algebra; if they are identity cells then \( A \) is called strict.

\[
\begin{array}{ccc}
T^2A & \xrightarrow{\mu_A} & TA \\
\downarrow{T} & & \downarrow{T} \\
TA & \xrightarrow{a} & A
\end{array}
\]

- Given lax \( T \)-algebras \( A = (A, a, \bar{a}, \bar{\alpha}) \) and \( C = (C, c, \bar{c}, \bar{\beta}) \), a lax \( T \)-morphism \( A \to C \) is a morphism \( f : A \to C \) of \( C \) that is equipped with a structure cell \( \bar{c} \) as on the right above, which is required to satisfy an associativity and unit axiom; see [Str74] or [Kou15a].

Dually, in the notion of an colax \( T \)-morphism \( h : A \to C \) the direction of the structure cell \( h : h \circ a \Rightarrow \bar{c} \circ T \iota h \) is reversed. A (co-)lax morphism is called a pseudo \( T \)-morphism if its structure cell is invertible.

- Given lax \( T \)-morphisms \( f \) and \( g : A \to C \), a \( T \)-cell \( f \Rightarrow g \) is a cell \( \phi : f \Rightarrow g \) in \( C \) satisfying

\[
\begin{array}{ccc}
TA & \xrightarrow{\phi} & TA \\
\downarrow{T_g} & & \downarrow{T_g} \\
TC & \xrightarrow{\phi} & TC
\end{array}
\]

Likewise a \( T \)-cell between colax \( T \)-morphisms \( h \) and \( k : A \to C \) is a cell \( \phi : h \Rightarrow k \) satisfying \( h \circ (c \circ T \phi) = (\phi \circ a) \circ \bar{k} \).

Lax \( T \)-algebras, lax \( T \)-morphisms and the \( T \)-cells between them form a 2-category denoted \( T\text{-Alg}_{(l,1)} \). We denote by \( T\text{-Alg}_{(l,1)} \) and \( T\text{-Alg}_{(p,1)} \) the sub-2-categories consisting of normal lax \( T \)-algebras and pseudo \( T \)-algebras. Likewise (normal) lax \( T \)-algebras, or pseudo \( T \)-algebras, together with colax \( T \)-morphisms and their \( T \)-cells form a hierarchy of 2-categories \( T\text{-Alg}_{(p,1)} \subset T\text{-Alg}_{(l,1)} \subset T\text{-Alg}_{(l,1)} \).

Writing \( T^\circ \) for the induced strict 2-monad on \( C^\circ \), by an colax \( T \)-algebra we mean a lax \( T^\circ \)-algebra; that is the notion of colax \( T \)-algebra is obtained by reversing
the direction of the associator and unitor cells of lax $T$-algebras. The notions of lax $T$-morphism and colax $T$-morphism for colax $T$-algebras are defined analogously to those for lax $T$-algebras: in fact the 2-category of colax $T$-algebras, lax $T$-morphisms and $T$-cells is defined as $T\text{-}\text{Alg}_{(c,1)} := (T^\text{co}\text{-}\text{Alg}_{(1,1)})^\text{op}$, while that of colax $T$-algebras, colax $T$-morphism and $T$-cells is defined as $T\text{-}\text{Alg}_{(c,2)} := (T^\text{co}\text{-}\text{Alg}_{(2,1)})^\text{op}$. As before we denote their sub-2-categories consisting of normal colax $T$-algebras by $T\text{-}\text{Alg}_{(nc,1)}$ and $T\text{-}\text{Alg}_{(nc,2)}$. Notice that we need not distinguish between “pseudo lax algebras” and “pseudo colax algebras”, since inverting associator and unitor cells induces an isomorphism $T\text{-}\text{Alg}_{(ps,1)} \cong (T^\text{co}\text{-}\text{Alg}_{(ps,1)})^\text{op}$, and similar for colax morphisms.

Example 6.5. Consider the ‘free strict monoidal $\mathcal{V}$-category’-monad $T$ on $\mathcal{V}\text{-}\text{Prof}$ (Example 6.2), and its vertical part $V(T)$ on $\mathcal{V}\text{-}\text{Cat} = V(\mathcal{V}\text{-}\text{Prof})$. A lax $V(T)$-algebra is a lax monoidal $\mathcal{V}$-category, that is a large $\mathcal{V}$-category $A$ equipped with tensor product $\mathcal{V}$-functors

$$\odot: TA \to A: (x_1, \ldots, x_n) \mapsto (x_1 \odot \cdots \odot x_n),$$

where $n \geq 0$, together with $\mathcal{V}$-natural unitors$^1$: $x \to (x)$ and associators

$$a: \left((x_{11} \odot \cdots \odot x_{1 m_1}) \odot \cdots \odot (x_{n1} \odot \cdots \odot x_{nm_n})\right) \mapsto (x_{11} \odot \cdots \odot x_{nm_n})$$

satisfying the usual coherence axioms; see for instance Definition 3.1.1 of [Lei04]. In a monoidal $\mathcal{V}$-category unitors and associators are invertible; in a colax monoidal $\mathcal{V}$-category their direction is reversed.

A lax $V(T)$-morphism $f: A \to C$ between lax monoidal $\mathcal{V}$-categories is a lax monoidal $\mathcal{V}$-functor, that is a $\mathcal{V}$-functor $f: A \to C$ equipped with $\mathcal{V}$-natural compositors $f_\odot: (fx_1 \odot \cdots \odot fx_n) \to f(x_1 \odot \cdots \odot x_n)$, that are compatible with the coherence maps of $A$ and $C$; see Definition 3.1.3 of [Lei04]. In the notion of colax monoidal $\mathcal{V}$-functor the direction of the compositors is reversed. The $V(T)$-cells finally are monoidal $\mathcal{V}$-natural transformations $\xi: f \Rightarrow g$, whose components satisfy the coherence axiom $f_\odot \circ (\xi_{x_1} \odot \cdots \odot \xi_{x_n}) = (\xi_{x_1} \odot \cdots \odot \xi_{x_n}) \circ g_\odot$. We remark that, in relation to the classical definitions of monoidal category/functor/translation, the preceding (co-)lax definitions are often called unbiased because, unlike the classical notions, they are not “biased” towards binary tensor products.

Example 6.6. Consider the ‘free strict double category’-monad $T$ on $\text{Set'}\text{-}\text{Prof}^{G_1}$, (Example 6.3), and its vertical part $V(T)$ on the 2-category $\text{Cat}^{G_1}$ of large $G_1$-indexed categories. A lax $V(T)$-algebra $A$ is a lax double category as follows. It consists of a large $G_1$-indexed category $A$ equipped with compatible horizontal compositions for its horizontal morphisms and cells:

$$\begin{array}{c}
(A_0 \xrightarrow{J_0} A_1, \ldots, A_{n'} \xrightarrow{J_{n'}} A_n) \mapsto \alpha (J_1 \odot \cdots \odot J_n) \\
\begin{pmatrix}
A_0 \\
C_0
\end{pmatrix} \\
\begin{pmatrix}
A_1 \\
C_1
\end{pmatrix} \\
\begin{pmatrix}
A_{n'} \\
C_{n'}
\end{pmatrix} \\
\begin{pmatrix}
A_n \\
C_n
\end{pmatrix}
\end{array}$$

The second of these is required to be functorial with respect to the vertical composition in $A$; this condition is known as the interchange axiom. Furthermore, $A$ is equipped with horizontal unitor cells $i: J \Rightarrow (J)$, one for each horizontal morphism $J: A \Rightarrow B$, and horizontal associator cells

$$\alpha: ((J_{11} \odot \cdots \odot J_{1 m_1}) \odot \cdots \odot (J_{n1} \odot \cdots \odot J_{n m_n})) \Rightarrow (J_{11} \odot \cdots \odot J_{nm_n}),$$

$^1$As is the custom, by a map $i: x \to (x)$ in the $\mathcal{V}$-category $A$ we mean a $\mathcal{V}$-map $i: I \to A(x,(x))$. 87
one for each double path $J$ of horizontal morphisms. These coherence cells are required to be natural with respect to the vertical composition of cells in $A$ and to satisfy the usual coherence axioms, analogous to those of a lax monoidal category. For each $A \in A$ we call the horizontal composite $(A) : A \rightarrow A$ of the empty path $(A)$ the (horizontal) unit of $A$. By requiring that the associator and unitor cells be invertible we obtain the notion of ‘weak double category’, considered in Section 5.2 of [Lei04]; notice that this an “unbiased” variant of the ‘pseudo double categories’ that were considered after Proposition 3.8.

Lax double functors $F : A \rightarrow C$ are defined analogously to lax monoidal functors: they are $G_1$-indexed functors equipped with natural horizontal compositor cells

$$F_\otimes : (F j_1 \circ \cdots \circ F j_n) \Rightarrow F(j_1 \circ \cdots \circ j_n)$$

satisfying the usual coherence axioms. Likewise double transformations $\xi : F \Rightarrow G$ are $G_1$-indexed natural transformations satisfying the coherence axiom

$$G_\otimes \circ (\xi j_1 \circ \cdots \circ \xi j_n) = \xi(j_1 \circ \cdots \circ j_n) \circ F_\otimes.$$  

### 6.3 Hypervirtual double categories of $T$-algebras

We turn to the hypervirtual double categories $T$-$\text{Alg}_{(v, w)}$ associated to a monad $T = (T, \mu, i)$ on a hypervirtual double category $K$. Remember that $T$ induces a strict 2-monad on the vertical 2-category $V(K)$ that is contained in $K$. To start, by an (co-)lax $T$-algebra we shall simply mean an (co-)lax $V(T)$-algebra, and by an (co-)lax vertical $T$-morphism between such algebras we shall mean an (co-)lax $V(T)$-morphism. The following definition generalises to hypervirtual double categories the notion of ‘horizontal $T$-morphism’, given in [Kou15a], which is itself a slight generalisation of the ‘algebraic arrows’ that were introduced in Section 7 of [GP04].

**Definition 6.7.** Let $T = (T, \mu, i)$ be a monad on a hypervirtual double category $K$. Given lax $T$-algebras $A = (A, a, \bar{a}, \tilde{a})$ and $B = (B, b, \bar{b}, \tilde{b})$, a horizontal $T$-morphism $A \Rightarrow B$ is a horizontal morphism $J : A \rightarrow B$ equipped with a structure cell

$$\begin{array}{c}
TA \xrightarrow{TJ} TB \\
A \xrightarrow{J} B
\end{array}$$

satisfying the associativity and unit axioms below.

A horizontal $T$-morphism $J : A \rightarrow B$ is called (op-)representable whenever its underlying morphism $J : A \rightarrow B$ is (op-)representable in $K$. It is said to satisfy the left Beck-Chevalley condition whenever its structure cell $J$ satisfies the left Beck-
Chevalley condition (Definition 5.12).

\[
\begin{array}{c}
T^2A \xrightarrow{T^2J} T^2B \\
\downarrow \quad \quad \downarrow \\
TA \xrightarrow{J} TB 
\end{array}
\]

\[
\begin{array}{c}
TA \xrightarrow{\mu_J} TB \\
\downarrow \quad \quad \downarrow \\
A \xrightarrow{J} B 
\end{array}
\]

Equation 6.8: Let \( T \) be the ‘free strict monoidal \( V \)-category’-monad on \((V, V')\)-Prof, described in Example 6.2. Given lax monoidal \( V' \)-categories \( A \) and \( B \), a horizontal \( T \)-morphism \( A \rightarrow B \) is a monoidal \( V \)-profunctor, that is a \( V \)-profunctor \( J: A \rightarrow B \) equipped with a monoidal structure given by \( V' \)-maps

\[
J_\otimes: J(x_1, y_1) \otimes' \cdots \otimes' J(x_n, y_n) \rightarrow J((x_1 \otimes \cdots \otimes x_n), (y_1 \otimes \cdots \otimes y_n)),
\]

where \( x_1, \ldots, x_n \in A \), \( y_1, \ldots, y_n \in B \) and, in the right-hand side, the tensor products of \( A \) and \( B \) are denoted by \( \otimes \). These \( V' \)-maps are required to be compatible with the actions of \( TA \) and \( TB \), and to satisfy associativity and unit axioms; see Example 3.23 of [Kou15a].

We remark that if \( A \) is a monoidal \( V' \)-category, that is with invertible associator and unitor, while \( V' \) is closed symmetric monoidal, so that \( V \) can be regarded as a monoidal \( V' \)-category itself, then monoidal \( V' \)-profunctors \( J: A \rightarrow B \) in the sense above can be identified with lax monoidal \( V' \)-functors \( A^{op} \otimes' B \rightarrow V' \) whose images are \( V \)-objects.

A monoidal \( V \)-profunctor \( J: A \rightarrow B \) satisfies the left Beck-Chevalley condition whenever, for each \( x \in A \) and \( y_1, \ldots, y_n \in B \), the family of \( V' \)-maps

\[
\begin{array}{c}
A(x, (u_1 \otimes \cdots \otimes u_n)) \otimes' J(u_1, y_1) \otimes' \cdots \otimes' J(u_n, y_n) \\
\downarrow \quad \quad \downarrow \\
A(x, (u_1 \otimes \cdots \otimes u_n)) \otimes' J((u_1 \otimes \cdots \otimes u_n), (y_1 \otimes \cdots \otimes y_n)) \\
\downarrow \\
J(x, (y_1 \otimes \cdots \otimes y_n))
\end{array}
\]

defines the \( V \)-object \( J(x, (y_1 \otimes \cdots \otimes y_n)) \) as the coend

\[
\int_{u_1, \ldots, u_n \in A} A(x, (u_1 \otimes \cdots \otimes u_n)) \otimes' J(u_1, y_1) \otimes' \cdots \otimes' J(u_n, y_n)
\]

in \( V' \), as follows from Definition 5.12 and Proposition 3.17. In the case \((V, V') = (\text{Set, Set'})\) this means that any morphism \( p: x \rightarrow (y_1 \otimes \cdots \otimes y_n) \) in \( J \) factors as

\[
\begin{array}{c}
x \xrightarrow{p} (y_1 \otimes \cdots \otimes y_n), \\
\downarrow s \\
(u_1 \otimes \cdots \otimes u_n)
\end{array}
\]

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where $s \in A$ and each $q_i \in J$, while these factorisations are required to be unique up to the equivalence relation defined by the coend above.

**Example 6.9.** Let $T$ be the ‘free strict double category’-monad on $(\text{Set}, \text{Set})$-$\text{Prof}^{G_1}$; we will call a horizontal $T$-morphism $A \Rightarrow B$, between large lax double categories $A$ and $B$, a (small) double profunctor. It is given by a small $G_1$-indexed profunctor $\mathcal{J} : A \Rightarrow B$ equipped with ‘horizontal compositions’

$$
\begin{pmatrix}
A_0 \overset{J_1}{\to} A_1 & \cdots & A_{n-1} \overset{J_n}{\to} A_n \\
\downarrow u_0 & & \downarrow u_n
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
A_0^{(J_1 \circ \cdots \circ J_n)} \overset{(J_1 \circ \cdots \circ J_n)}{\to} A_n \\
\downarrow u_0 & \cdots & \downarrow u_n
\end{pmatrix}
$$

one for each path $\theta : J \Rightarrow K$ of cells in $\mathcal{J}$. These compositions are required to be natural with respect to the actions of $A$ and $B$, and to satisfy the following coherence axioms. The associativity axioms states that the diagram below commutes in $\mathcal{J}$, for every double path $\theta : J \Rightarrow K$ of cells, while the unit axiom states that $i_{\theta} \circ \theta = (\theta) \circ i_A$, for every cell $\theta : J \Rightarrow K$ in $\mathcal{J}$.

\[
\begin{array}{c}
((J_{11} \circ \cdots \circ J_{1m_1}) \circ \cdots \circ (J_{n1} \circ \cdots \circ J_{nm_n}))^{\bar{a}_A} \Rightarrow (J_{11} \circ \cdots \circ J_{nm_n}) \\
((\theta_{11} \circ \cdots \circ \theta_{1m_1}) \circ \cdots \circ (\theta_{n1} \circ \cdots \circ \theta_{nm_n}))^{\bar{a}_A} \Rightarrow (\theta_{11} \circ \cdots \circ \theta_{nm_n})
\end{array}
\]

Next is the definition of $T$-cell. Notice that, when restricted to vertical cells, it coincides with that of $V(T)$-cell' in the sense of Definition 6.4. In Proposition 6.12 below we will see that lax $T$-algebras, ‘weak’ $T$-morphisms, horizontal $T$-morphisms and the $T$-cells between them form a hypervirtual double category.

**Definition 6.10.** Let $T$ a monad on a hypervirtual double category and consider a cell

$$
\begin{array}{c}
A_0 \overset{J_1}{\to} A_1 \cdots \overset{J_n}{\to} A_n \\
\downarrow f & \downarrow \phi & \downarrow g \\
C \overset{K}{\to} D
\end{array}
$$

where $f$ and $g$ are lax vertical $T$-morphisms while $J$ and $K$ are paths of horizontal $T$-morphisms. It is called a $T$-cell whenever the identity

$$
\begin{array}{c}
TA_0 \overset{T_{J_1}}{\to} TA_1 \cdots \overset{T_{J_n}}{\to} TA_n \\
\downarrow \phi & \downarrow \psi & \downarrow \gamma
\end{array}
\Rightarrow
\begin{array}{c}
TA_0 \overset{T_{J_1}}{\to} TA_1 \cdots \overset{T_{J_n}}{\to} TA_n \\
\downarrow \phi & \downarrow \psi & \downarrow \gamma
\end{array}
$$

holds, where in the left-hand side $K := \bar{K}$ if $K = (C \Rightarrow D)$ and $\bar{K} := \text{id}_C$ if $K = (C)$; analogously the path $(J_1, \ldots, J_n)$ of structure cells in the right-hand side is to be interpreted as the identity cell $\text{id}_{A_n}$ if $J$ is empty.

Likewise a cell $\phi$ as above, but with colax vertical $T$-morphisms $f$ and $g$, is called a $T$-cell whenever $f \circ (\bar{K} \circ \phi) = (\phi \circ (J_1, \ldots, J_n)) \circ g$.

In the case of the ‘free strict monoidal $V$-category’-monad, $(1, 1)$-ary $T$-cells were described in Example 3.25 of [Kou15a].
Example 6.11. For a $G_1$-indexed cell $\xi$ in $(\text{Set}, \text{Set}')$-$\text{Prof}^{G_1}$, as in the top right below, where $F$ and $G$ are lax double functors and the paths $J$ and $K$ are small double profunctors, the $T$-cell axiom states the following. Consider matrices of cells as on the left, where each row $(\theta_1, \ldots, \theta_m)$ forms a composable path of cells in $J_i$; remember that $\xi$ consists of cells $\xi((\theta_1, \ldots, \theta_m))$: $FJ_{0k} \Rightarrow GJ_{nk}$ in $K$, one for each column $(\theta_1, \ldots, \theta_n)$. The $T$-axiom for $\xi$ means that the diagram on the bottom right commutes, for each matrix $(\theta_{ik})$; in that case we call $\xi$ a double transformation.

\[
\begin{array}{ccc}
A_0 & \overset{J_0}{\rightarrow} & A_1 \\
\downarrow \theta_1 & & \downarrow \theta_1 \\
\vdots & & \vdots \\
A_{m0} & \overset{J_{m0}}{\rightarrow} & A_{m1} \\
\downarrow \theta_m & & \downarrow \theta_m \\
A_{nm} & \overset{J_{nm}}{\rightarrow} & A_{nm1}
\end{array}
\]

\[
\begin{array}{ccc}
A_0 & \overset{J_0}{\rightarrow} & A_1 & \cdots & A_n & \overset{J_n}{\rightarrow} & A_n \\
\downarrow F & & \downarrow \xi & & \downarrow G \\
C & \rightarrow & D
\end{array}
\]

(\xi((\theta_1, \ldots, \theta_1) \odot \cdots \odot (\theta_1, \ldots, \theta_m)))

\[
\begin{array}{ccc}
A_{01} & \overset{J_{01}}{\rightarrow} & A_{11} \\
\downarrow \phi_1 & & \downarrow \phi_1 \\
\vdots & & \vdots \\
A_{nmj} & \overset{J_{nmj}}{\rightarrow} & A_{nmj1} \\
\downarrow \phi_{mj} & & \downarrow \phi_{mj} \\
A_{nm} & \overset{J_{nm}}{\rightarrow} & A_{nm1}
\end{array}
\]

\[
\begin{array}{ccc}
A_{01} & \overset{J_{01}}{\rightarrow} & A_{11} \\
\downarrow \phi_1 & & \downarrow \phi_1 \\
\vdots & & \vdots \\
A_{nmj} & \overset{J_{nmj}}{\rightarrow} & A_{nmj1} \\
\downarrow \phi_{mj} & & \downarrow \phi_{mj} \\
A_{nm} & \overset{J_{nm}}{\rightarrow} & A_{nm1}
\end{array}
\]

\[
\begin{array}{ccc}
FJ_{01} & \odot \cdots & FJ_{0m} \\
\downarrow F \odot \xi & & \downarrow F \odot \xi \\
GJ_{11} & \cdots & GJ_{nm}
\end{array}
\]

Proposition 6.12. Let $T$ be a monad on a hypervirtual double category $K$ and let ‘weak’ mean either ‘lax’ or ‘colax’. The structure on $K$ lifts to make lax $T$-algebras, weak vertical $T$-morphisms, horizontal $T$-morphisms and $T$-cells into a hypervirtual double category $T\text{-Alg}_{(l,w)}$, such that $V(T\text{-Alg}_{(l,w)}) = V(T\text{-Alg}_{(l,w)})$.

Analogous to the notation for 2-monads we set $T\text{-Alg}_{(c,l)} := (T^{co}\text{-Alg}_{(l,c)})^{co}$ and $T\text{-Alg}_{(c,c)} := (T^{co}\text{-Alg}_{(l,l)})^{co}$. For $v \in \{c, l\}$ we write $T\text{-Alg}_{(v, ps)}$ for the sub-hypervirtual double category of $T\text{-Alg}_{(v, l)}$ (or, equivalently, $T\text{-Alg}_{(v, c)}$) obtained by restricting to pseudo vertical $T$-morphisms. By $T\text{-Alg}_{(v, ps, lbc)} \subset T\text{-Alg}_{(v, ps)}$ we denote the sub-hypervirtual double category obtained by further restricting to horizontal $T$-morphisms that satisfy the left Beck-Chevalley condition (see Definition 6.7).

Proof. We treat the case of lax morphisms; that of colax morphisms is similar. The structure on $T\text{-Alg}_{(l,l)}$ is completely determined by the requirement that it is lifted from $K$, and that it reduces to $V(T\text{-Alg}_{(l,l)})$. In particular the structure cell of a vertical composite $h \circ f$ is given by $h \circ f := (h \circ T f) \odot (h \circ f)$, as usual. Checking that this structure on $T\text{-Alg}_{(l,l)}$ is well-defined is straightforward. Indeed that the composite $\psi \circ (\phi_1, \ldots, \phi_n)$ of $T$-cells, as in (3), is again a $T$-cell is shown by the equality (drawn schematically, leaving out all details except the shape of cells)

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
T(\psi \circ (\phi_1, \ldots, \phi_n)) & \overset{f_0}{\rightarrow} & K \\
\downarrow L & & \downarrow \xi \\
T\phi_1 & \cdots & T\phi_n \\
\downarrow h & & \downarrow \psi \\
J_{\phi_1} & \cdots & J_{\phi_n} \\
\downarrow f_0 & & \downarrow \xi \\
\cdots & \cdots & \cdots
\end{array}
\]
where the first identity follows from the fact that $T$ preserves composites and the $T$-cell axiom for $\psi$, and the second from the $T$-cell axioms for $\phi_1,\ldots,\phi_n$ and the interchange axioms (Lemma 1.2). This leaves checking that the identity cells of $K$ form $T$-cells, which clearly is the case.

The following example recovers, for a strict 2-monad $T$, the strict double category of $T$-algebras, lax and colax $T$-morphisms, and the appropriate notion of cell between those, that is considered in Example 4.8 of [Shu11].

**Example 6.13.** Let $T = (T, \mu, \iota)$ be a strict 2-monad on a 2-category $C$. Its image under the strict 2-functor $Q: 2\text{-Cat} \to \text{HypVirtMultiCat}$ of Proposition 2.20 forms a monad on the strict double category $Q(C)$ of quintets in $C$ (see Definition 2.19 and Example 3.10). In this case $Q(T)\text{-Alg}_{\{1,1\}}$ has lax $T$-algebras in $C$ as objects, lax $T$-morphisms as vertical morphisms and colax $T$-morphisms as horizontal morphisms. Composing the latter in the usual way, $Q(T)\text{-Alg}_{\{1,1\}}$ is again a strict double category. Its cells are quintets $\phi$ in $C$ as on the left below, where $f: A \to C$ and $g: B \to D$ are lax $T$-morphisms and $j: A \to B$ and $k: C \to D$ are colax $T$-morphisms, such that the identity on the right is satisfied. Quintets like these are called ‘generalised $T$-transformations’ in [Shu11]; we will call them $T$-quintets.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C & \xrightarrow{j} & B \\
C & \xrightarrow{k} & D & \xrightarrow{g}
\end{array}
\]

\[
\begin{array}{ccc}
Tf & \xleftarrow{T\phi} & TA & \xleftarrow{Tj} & TB \\
TC & \xleftarrow{Tk} & TD & \xleftarrow{Tg} & D
\end{array}
\]

Example 6.14. If in the previous example $T$ is the restriction of ‘free strict double category’-monad, as in Example 6.6, to the 2-category of $G_1$-indexed categories, functors and transformations, then a $T$-quintet $\xi: K \circ F \Rightarrow G \circ H$, where $F$ and $G$ are lax double functors and $H$ and $K$ are colax ones, is given by a $G_1$-indexed transformation $\xi: K \circ F \Rightarrow G \circ J$ that makes the diagrams

\[
\begin{align*}
K(FJ_1 \odot \cdots \odot FJ_n) & \xrightarrow{K\phi} (KFJ_1 \odot \cdots \odot KFJ_n) \\
KFJ_1 \odot \cdots \odot J_n & \xrightarrow{\xi(J_1 \odot \cdots \odot J_n)} (GHJ_1 \odot \cdots \odot GHJ_n) \\
GHJ_1 \odot \cdots \odot J_n & \xrightarrow{G\phi} G(HJ_1 \odot \cdots \odot HJ_n)
\end{align*}
\]

commute, for each path $\downarrow_{J}$ of horizontal morphisms in the source of $F$ and $H$. We will call these quintets **double quintets**.

The (biased variant of the) notion of double quintet was introduced by Grandis and Paré in Section 2.2 of [GP04], without name. In fact, the sub-strict double category of $Q(T)\text{-Alg}_{\{1,1\}}$ consisting of all pseudo double categories coincides with (the biased variant of) the strict double category $\text{Dbl}$ considered there, which forms an important object of study in the subsequent [GP08] and [GP07]. We remark that, unfortunately, when applied to $\text{Dbl}$ our notion of pointwise right Kan extension (horizontally dual to Definition 4.5) is not the right one. Indeed the right notion, which generalises those of e.g. restriction and tabulation in a pseudo double category.
(see [GP07]), defines the pointwise right Kan extension of a lax double functor $D: A \to M$ along an colax double functor $J: A \to B$ as a normal lax double functor $R: B \to M$ (i.e. one that preserves horizontal units strictly), equipped with a universal double quintet $\varepsilon: R \circ J \Rightarrow D$. I do not know how to (smoothly) reconcile these notions.

6.4 Representable horizontal $T$-morphisms

Let $T$ be a monad on a hypervirtual double category $\mathcal{K}$. In this section we will characterise representable horizontal $T$-morphisms (Definition 6.7) in terms of colax $T$-morphisms. This characterisation is a consequence of that of Theorem 2.21, which characterises representable horizontal morphisms in $\mathcal{K}$.

In detail, we will characterise the full sub-hypervirtual double categories

$$\text{Rep}(T)-\text{Alg}_{(1,1)} \subseteq T-\text{Alg}_{(1,1)} \supseteq \text{opRep}(T)-\text{Alg}_{(1,1)}$$

generated by (op-)representable horizontal $T$-morphisms, where $\text{Rep}$ and $\text{opRep}$ denote the strict 2-endofunctors on $\text{HypVirtMultiCat}$ of Proposition 2.18, that restrict to (op-)representable horizontal morphisms. To do so recall from Theorem 2.21 the sub-2-endofunctor $(Q \circ V)_* \subseteq Q \circ V$ on $\text{HypVirtMultiCat}$, that maps $\mathcal{K}$ to the sub-hypervirtual double category $(Q \circ V)_*(\mathcal{K}) \subseteq (Q \circ V)(\mathcal{K})$ consisting of all objects, all vertical morphisms, those horizontal morphisms $j: A \Rightarrow B$ that admit companions in $\mathcal{K}$, and all quintets between them. Horizontally dual, the sub-2-endofunctor $(Q^\circ \circ V)^* \subseteq (Q^\circ \circ V)$ restricts to horizontal morphisms $\bar{j}: A \Rightarrow B$ that admit conjoints. We will show that the sub-hypervirtual double categories of $T-\text{Alg}_{(1,1)}$ above are equivalent to hypervirtual double categories $(Q \circ V)_*(T)-\text{Alg}_{(1,1)}$ and $(Q^\circ \circ \circ V)^*(T)-\text{Alg}_{(1,1)}$ of $T$-quintets respectively; see Example 6.13.

**Theorem 6.15.** Let $T$ be a monad on a hypervirtual double category $\mathcal{K}$. Choosing, for each $j: A \Rightarrow B$ in $(Q \circ V)_*(\mathcal{K})$, a cartesian cell $\varepsilon_j$ in $\mathcal{K}$ that defines the companion $j^\varepsilon$, induces a functor

$$(-)^T: (Q \circ V)_*(T)-\text{Alg}_{(1,1)} \to \text{Rep}(T)-\text{Alg}_{(1,1)}$$

that is part of an equivalence $(Q \circ V)_*(T)-\text{Alg}_{(1,1)} \simeq \text{Rep}(T)-\text{Alg}_{(1,1)}$ of hypervirtual double categories. While it restricts to the identity on lax $T$-algebras and lax $T$-morphisms, $(-)^T$ maps each colax $T$-morphism $(j, j)$: $A \Rightarrow B$ to the horizontal $T$-morphism $(j^\varepsilon, j^\varepsilon): A \Rightarrow B$, where $j^\varepsilon$ is the unique factorisation in

$$\begin{array}{ccc}
TA & \xrightarrow{j^\varepsilon} & TB \\
\downarrow \alpha & = & \downarrow \alpha \\
A & \xrightarrow{j} & B
\end{array} \quad \begin{array}{ccc}
TA & \xrightarrow{j^\varepsilon} & TB \\
\downarrow \beta & = & \downarrow \beta \\
B & \xrightarrow{j^\varepsilon} & B
\end{array}$$

Horizontally dual, choosing cartesian cells that define conjoints induces an equivalence $(Q^\circ \circ \circ V)^*(T)-\text{Alg}_{(1,1)} \simeq \text{opRep}(T)-\text{Alg}_{(1,1)}$.

**Proof.** A formal proof can be given as follows. First show that the assignment $T \mapsto T-\text{Alg}_{(1,1)}$ extends to a strict 2-functor $(-)-\text{Alg}_{(1,1)}: \text{Mnd} \to \text{HypVirtMultiCat}$, where $\text{Mnd}$ denotes the 2-category of monads on hypervirtual double categories, the lax morphisms between them and the cells between those, in the sense of [Str72] or Section 6.1 of [Lei04]. The pseudonaturality of $(-)_*: (Q \circ V)_*(\mathcal{K}) \simeq \text{Rep}(\mathcal{K})$
The following “hands on”-approach is quicker. By Theorem 2.21 a choice of companions $j_*$, for each horizontal morphism $j: A \to B$ in $(Q \circ V)_+(K)$, induces an equivalence $(-)_*: (Q \circ V)_+(K) \simeq \text{Rep}(K)$ that restricts to the identity on objects and vertical morphisms, while it maps $j: A \to B$ to its companion $j_*: A \to B$. First we define $(-)_T^*: (Q \circ V)_+(T)-\text{Alg}_{(1,1)} \to \text{Rep}(T)-\text{Alg}_{(1,1)}$ on objects and morphisms as in the statement. To check that, for each colax $T$-morphism $(j, \tilde{j}): A \to B$, its image $(j_*, \tilde{j}_*)$ satisfies the coherence axioms of Definition 6.7, consider the invertible horizontal cell $\gamma_j: T\tilde{j}_* \cong (Tj)_*$ that is the factorisation of $T\varepsilon_j$ through $\varepsilon_T\tilde{j}$; here we use that $T$ preserves $\varepsilon_j$ by Corollary 2.12. Comparing the factorisation above with the action (7) of $(-)_*$ on quintets in $K$, it follows that $\tilde{j}_* = (\tilde{j})_* \circ \gamma_j$. Similarly, by postcomposing the coherence axioms for $\tilde{j}_*$ with $\varepsilon_j$, we find that they coincide with the $(-)_*$ images of the coherence axioms for $j$ which, in the case of associativity, are precomposed with $T\gamma_j$. We conclude that the coherence axioms for $\tilde{j}_*$ are induced by those for $j$. Even better, because $(-)_*$ is full and faithful, it follows that the assignment $j \mapsto \tilde{j}_*$ gives a bijection between the colax $T$-morphism structures on $j$ and the horizontal $T$-morphism structures on $j_*$; this we will use later.

Next, on $T$-quintets we let $(-)_T^*$ act simply by $\phi \mapsto \phi_*$, as in (7). Similar to the argument above, postcomposing the $T$-cell axiom for $\phi_*$ with $\varepsilon_k$, shows that it coincides with the $(-)_*$-image of the $T$-quintet axiom for $\phi$ after precomposing it with the invertible horizontal cells $\gamma_1, \ldots, \gamma_n$. We conclude that the $T$-quintet axiom for $\phi$ and the $T$-cell axiom for $\phi_*$ are equivalent, showing that the assignment $\phi \mapsto \phi_*$ is well-defined. Even better, combined with the fact that $(-)_*$ is full and faithful, we conclude that $(-)_T^*: (Q \circ V)_+(T)-\text{Alg}_{(1,1)} \cong \text{Rep}(T)-\text{Alg}_{(1,1)}$, whose definition is now complete, is again full and faithful.

To prove that $(-)_T^*$ is an equivalence it remains to show that it is essentially surjective, by Proposition 1.23. To see this consider any horizontal $T$-morphism $(J, \tilde{J})$: $A \to B$, with $J$ represented by $j: A \to B$. By factoring $\varepsilon_j$ through the cartesian cell that defines $J$ as a companion of $j$ we obtain a horizontal isomorphism $\delta: j_* \cong J$. Clearly the composite $\tilde{j}_*: [T\tilde{j}_* \xrightarrow{T\varepsilon} TJ \xrightarrow{\tilde{J}} J \xrightarrow{\delta^{-1}} j_*$] forms a horizontal $T$-morphism structure on $j_*$ so that, as we have already seen, it is the image of $(j, \tilde{j})$ under $(-)_T^*$ for some colax $T$-morphism structure cell $j$. Finally, by definition of $j_*$, the cell $\delta$ forms a horizontal $T$-cell $(j_*, \tilde{j}_*) \cong (J, \tilde{J})$, showing that $(-)_T^*$ is essentially surjective. This completes the proof.

7 Creativity of the forgetful functors $U: T-\text{Alg} \to K$

The classical notion of the creation of limits by ordinary functors can be generalised to the creation of “universal constructions” by functors between hypervirtual double categories, where by universal constructions we mean restrictions, horizontal composites, Kan extensions, etc. In this section, after making this notion precise, we describe the creation of universal constructions by the forgetful functors $U: T-\text{Alg}_{(v, w)} \to K$, where $T$ is a monad on a hypervirtual double category $K$ and $v, w \in \{c, 1, ps\}$.

In any hypervirtual double category $K$ consider a “formal universal construction” $U$, whether it exists or not, that is defined by a universal cell. For example $U$ can be one of

- “the restriction of $K$: $C \to D$ along $f: A \to C$ and $g: B \to D$;
- “the horizontal composite of $A_0 \xrightarrow{j_0} A_1, \ldots, A_n \xrightarrow{j_n} A_n$;”

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- “the left Kan extension of \(d: A_0 \to M\) along \(A_0 \xrightarrow{d} A_1, \ldots, A_n \xrightarrow{d} A_n\).”

Hence, for any universal construction \(U\) in \(\mathcal{K}\) as above, as well as a cell \(\phi\) in \(\mathcal{K}\), we have a proposition “\(\phi\) defines \(U\) in \(\mathcal{K}\)” that either holds or fails. Furthermore given any functor \(F: \mathcal{K} \to \mathcal{L}\), we can assign to each formal universal construction \(U\) in \(\mathcal{K}\) its “image” \(F[U]\), by replacing each reference to a \(\mathcal{K}\)-morphism in \(U\) by its image under \(F\). For example, the image of the first universal property above is “the restriction of \(FK: FC \to FC\) along \(Ff: FA \to FC\) and \(Fg: FB \to FD\)”. The following definition is a direct translation of the classical notion of limit-creating functor, see e.g. Section V.1 of [ML98].

**Definition 7.1.** Let \(F: \mathcal{K} \to \mathcal{L}\) be a functor of hypervirtual double categories, and consider a formal universal construction \(U\) in \(\mathcal{K}\). We say that \(F\) *creates* \(U\) if for any cell \(\psi\) that defines \(F[U]\) in \(\mathcal{L}\) the following holds:

- there exists a unique cell \(\phi\) in \(\mathcal{K}\) such that \(F\phi = \psi\);

- \(\phi\) defines \(U\) in \(\mathcal{K}\).

We remark that creation of \(U\) by \(F\) implies that \(F\) *reflects* \(U\), that is for all cells \(\phi\) in \(\mathcal{K}\) we have: if \(F\phi\) defines \(F[U]\) in \(\mathcal{L}\) then \(\phi\) defines \(U\) in \(\mathcal{K}\). Notice that creation of \(U\) by \(F\), together with the existence of \(F[U]\) in \(\mathcal{L}\), implies *preservation* of \(U\) by \(F\): if a cell \(\phi\) defines \(U\) in \(\mathcal{K}\) then \(F\phi\) defines \(F[U]\) in \(\mathcal{L}\).

### 7.1 Restrictions and horizontal composites

We start with the creation of restrictions and horizontal composites in \(T\text{-}\text{Alg}_{(c,\text{ps})}\). Recall that \(T\text{-}\text{Alg}_{(c,\text{ps})}\) has as horizontal morphisms the horizontal \(T\)-morphisms whose structure cells satisfy the left Beck-Chevalley condition (Definition 5.12). The following result generalises Proposition 4.1 of [Kou15a].

**Proposition 7.2.** For a monad \(T\) on a hypervirtual double category \(\mathcal{K}\) the following hold for each \(v \in \{c, l, \text{ps}\}\):

(a) \(U: T\text{-}\text{Alg}_{(c, v)} \to \mathcal{K}\) creates restrictions \(\mathcal{K}(f, g)\) where \(g\) is a pseudo \(T\)-morphism;

(b) \(U: T\text{-}\text{Alg}_{(l, v)} \to \mathcal{K}\) creates restrictions \(\mathcal{K}(f, g)\) where \(f\) is a pseudo \(T\)-morphism;

(c) \(U: T\text{-}\text{Alg}_{(\text{ps}, v)} \to \mathcal{K}\) creates all restrictions;

(d) \(U: T\text{-}\text{Alg}_{(v, \text{ps})} \to \mathcal{K}\) creates unary restrictions \(\mathcal{K}(\text{id}, g)\) that are preserved by \(T\).

**Sketch of the proof.** We will sketch the proof of part (b) in the case \(v = l\). Consider a pseudo \(T\)-morphism \(f: A \to C\), a lax vertical \(T\)-morphism \(g: B \to D\) and a path \(\mathcal{K}\) of horizontal \(T\)-morphisms of length \(\leq 1\), and assume given a cartesian cell \(\phi\) in \(\mathcal{K}\) as in the composite on the left-hand side below. We obtain a structure cell \(J\) on the horizontal source \(J: A \Rightarrow B\) of \(\phi\) by taking the unique factorisation in

\[
\begin{array}{ccl}
TA & \xrightarrow{TJ} & TB \\
A & \xrightarrow{f} & TC \\
& \searrow c & \swarrow d \\
C & \xrightarrow{\mathcal{K}} & D
\end{array}
\]

\[
\begin{array}{ccl}
TA & \xrightarrow{\mathcal{K}} & TB \\
A & \xrightarrow{f} & B \\
& \searrow g & \swarrow h \\
C & \xrightarrow{\mathcal{K}} & D
\end{array}
\]
where, in the left-hand side, \( f^{-1} \) denotes the inverse of the structure cell of \( f \), while 

\[ \overline{K} := \overline{K} \text{ if } \overline{K} = (\overline{C} \overset{K}{\to} \overline{D}) \text{ and } \overline{K} := \text{id}_c \text{ if } \overline{K} = (\overline{C}). \]

We claim that \( J \) forms a well-defined \( T \)-structure on \( J \), that this structure is unique in making \( \phi \) into a \( T \) and that, thus a \( T \)-cell, \( \phi \) is cartesian in \( T\text{-Alg}(\mathbb{L}, \mathbb{I}) \). In checking these claims the proof of Proposition 4.1 of [Kou15a] applies almost verbatim, except that here \( \overline{K} \) might be empty and that the factorisations \( \overline{H} \Rightarrow J \) through \( \phi \) have paths \( \overline{H} \) as horizontal source, rather than a single horizontal \( T \)-morphism \( H \).

For the proof of part (d) notice that the inclusion \( T\text{-Alg}(c, ps, lbc) \to \) \( T\text{-Alg}(c, ps) \) reflects restrictions. Since \( U \colon T\text{-Alg}(c, ps) \to \mathcal{K} \) creates unary restrictions, as is sketched above, it suffices to show that, when taking \( f = \text{id}_C \) in the above, the structure cell \( J \) satisfies the left Beck-Chevalley condition whenever \( K \) does, provided that \( T\phi \) is again cartesian. To see this consider the following equation: in the first identity we factor \( K \) as a right pointwise cocartesian cell \( K' \) through \( K(\text{id}, d) \), as is possible by the Beck-Chevalley condition, while in the second identity the composite \( K' \circ T\phi \) (where \( T\phi \) is again cartesian) is factored through the restriction \( H := K(\text{id}, d \circ Tg) \); that the resulting cell \( K'' \) is again right pointwise cocartesian follows from Lemma 3.18.

\[
\begin{array}{ccc}
\text{TC} & \overset{TJ}{\to} & \text{TB} \\
\downarrow T\phi & \Downarrow Tg & \downarrow b \\
\text{TC} & \overset{TK}{\to} & \text{TD} \\
\downarrow T\phi & \Downarrow Tg & \downarrow b \\
C & \overset{K}{\to} & D \\
\end{array}
\]

Finally notice that the composite of the bottom two rows in the right-hand side above is cartesian, by the pasting lemma and the fact that \( g \) is invertible. Since, by definition, each of the composites above factors through \( \phi \) as the structure cell \( J \) we conclude, by factoring the right-hand side and using the pasting lemma again, that \( J \) coincides with \( K'' \) composed with the cartesian cell that defines \( H \) as the restriction of \( K(\text{id}, d) \) along \( Tg \). Since \( K'' \) is right pointwise cocartesian, this shows that \( J \) satisfies the left Beck-Chevalley condition.

Example 7.3. In \( \langle V, V' \rangle\)-Prof the canonical monoidal structure on the restriction \( K(f, g) \) of a monoidal \( V \)-profunctor \( K \colon C \to D \) (Definition 6.8) along a monoidal \( V' \)-functor \( f \colon A \to C \) and a lax monoidal \( V' \)-functor \( g \colon B \to D \) is given by the \( V \)-maps

\[
K(fx_1, gy_1) \otimes \cdots \otimes K(fx_n, gy_n) \xrightarrow{K\otimes} K((fx_1 \otimes \cdots \otimes fx_n), (gy_1 \otimes \cdots \otimes gy_n))
\]

where the second map is given by the actions of the structure transformations \( f^{-1} \) and \( g \).

The following proposition describes the creation of horizontal composites by the forgetful functors \( U \colon T\text{-Alg}(v, w) \to \mathcal{K} \). Recall that a horizontal \( T \)-morphism \( (J, \overline{J}) \colon A \to B \) is said to satisfy the left Beck-Chevalley condition as soon as its structure cell \( J \) does, in the sense of Definition 5.12.

Proposition 7.4. Consider a monad \( T \) on a hypertychic double category \( \mathcal{K} \) and let \( v, w \in \{c, l, ps\} \). A horizontal composite \( (J_1 \otimes \cdots \otimes J_n) \) of horizontal \( T \)-morphisms is created by \( U \colon T\text{-Alg}(v, w) \to \mathcal{K} \) whenever it is preserved by both \( T \) and \( T^2 \).
In the case \( v = 1 \) or \( ps \) an analogous result holds for right pointwise horizontal composites \((J_1 \circ \cdots \circ J_n)\) (Definition 3.16), provided that the restrictions \( J_n(id, f) \) and \((J_1 \circ \cdots \circ J_n)(id, f)\) exist in \( \mathcal{K} \), for all \( f: B \to A_n \), and that they are preserved by \( T \).

Finally, if the horizontal \( T \)-morphisms \( J_1, \ldots, J_n \) satisfy the left Beck-Chevalley condition then, under the assumptions above, \( U: T-\text{Alg}_{(v, ps, lbc)} \to \mathcal{K} \) creates the horizontal composite \((J_1 \circ \cdots \circ J_n)\) as well.

**Proof.** We consider the case \((v, w) = (1, 1)\); the other cases are similar. Let \( \phi \) be a cocartesian cell in \( \mathcal{K} \), as in the left-hand side below, and assume that \( T\phi \) is again cocartesian. It follows that the left-hand side factors uniquely through \( T\phi \) as a cell \( K \), as shown:

\[
\begin{array}{ccc}
T A_0 & \xrightarrow{TJ_1} & T A_1 \\
\downarrow a_0 & & \downarrow a_1 \\
A_0 & \xrightarrow{J_1} & A_1
\end{array}
\quad = \quad
\begin{array}{ccc}
T A_0 & \xrightarrow{TJ_n} & T A_n \\
\downarrow a_n & & \downarrow a_n \\
A_0 & \xrightarrow{J_n} & A_n
\end{array}
\quad = \quad
\begin{array}{ccc}
T A_0 & \xrightarrow{TJ_1} & T A_1 & \cdots & T A_n & \xrightarrow{TJ_n} & T A_n \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\
K & & K & & K & & K
\end{array}
\quad = \quad
\begin{array}{ccc}
T A_0 & \xrightarrow{T\phi} & T A_n \\
\downarrow \phi & & \downarrow \phi \\
A_0 & \xrightarrow{\phi} & A_n
\end{array}
\]

We claim that \( K \) makes \( K \) into a horizontal \( T \)-morphism, that is it satisfies the associativity and unit axioms of Definition 6.7. To prove the associativity axiom consider the equality below, where the identities follow from the factorisation above and its \( T \)-image; the associativity axioms for \( J_1, \ldots, J_n \) and the interchange axiom; the factorisation above; the naturality of \( \mu \) with respect to \( \phi \). Notice the left-hand and right-hand sides below equal the corresponding sides of the associativity axiom for \( K \) after composition with \( T^2 \phi \). Since the latter is cocartesian the associativity axiom itself follows.

\[
\begin{array}{ccc}
\phi & = & \phi
\end{array}
\]

Using the fact that \( \phi \) is cocartesian, the unit axiom for \( K \) can be deduced from that for \( J_1, \ldots, J_n \) in a similar way. The uniqueness of \( K \) follows from the fact that its defining unique factorisation above forms the \( T \)-cell axiom for \( \phi \).

It remains to show that \( \phi \) is cocartesian as a \( T \)-cell. Hence consider paths \( J^* : A_0 \Rightarrow A_0 \) and \( J'' : A_n \Rightarrow A_n \) of horizontal \( T \)-morphisms; we have to prove that any \( T \)-cell \( \psi : J^* \sim J \sim J'' \Rightarrow K \) factors uniquely through \( \phi \) in \( T-\text{Alg}_{(1,1)} \), in the sense of Definition 3.1. Since \( \phi \) is cocartesian in \( \mathcal{K} \) such a unique factorisation \( \psi = \psi' \circ (id_{J^*}, \ldots, id_{J''}, \phi, id_{J''}, \ldots, id_{J'}) \) certainly exists in \( \mathcal{K} \), and it suffices to show that \( \psi' \) satisfies the \( T \)-cell axiom. To see this consider the equation below, where the identities follow from the \( T \)-image of the factorisation of \( \psi \); the \( T \)-cell
axiom for \( \psi \); the factorisation of \( \psi \); the \( T \)-cell axiom for \( \phi \).

\[
\begin{array}{l}
\begin{array}{c}
\cdots \\
\begin{array}{l}
J_0 \\
\cdots \\
J_n \\
\cdots \\
J_n \\
\cdots \\
J_0 \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c}
\cdots \\
\begin{array}{l}
\phi \\
\cdots \\
\phi \\
\cdots \\
\phi \\
\cdots \\
\phi \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c}
\cdots \\
\begin{array}{l}
K \\
\cdots \\
K \\
\cdots \\
K \\
\cdots \\
K \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

Since \( T\phi \) is assumed to be cocartesian, the top rows in both the left-hand and right-hand sides above are cocartesian. Hence, because their bottom two rows equal the sides of the \( T \)-cell axiom for \( \psi' \), the axiom follows. This completes the proof of the first assertion.

For the second assertion, we now assume that both \( \phi \) and \( T\phi \), in the above, are right pointwise cocartesian in \( K \); we have to show that \( \phi \), as a \( T \)-cell, is right pointwise cocartesian in \( T\text{-Alg}_{(1,1)} \). To this end consider any lax \( T \)-morphism \( f : B \to A_n \).

By assumption the restriction \( J_n(\text{id}, f) \) exists in \( K \) so that, because \( \phi \) is right pointwise cocartesian, the restriction \( K(\text{id}, f) \) does too; it follows from the previous proposition that these restrictions created in \( T\text{-Alg}_{(1,1)} \). Thus we obtain the following factorisation in \( T\text{-Alg}_{(1,1)} \); we have to show that the \( T \)-cell \( \phi' \) is cocartesian.

\[
\begin{array}{l}
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \xrightarrow{J_n(\text{id}, f)} B \\
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \xrightarrow{J_n(\text{id}, f)} B \\
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \xrightarrow{J_n(\text{id}, f)} B \\
\end{array}
\end{array}
\]

To see this, use the assumption that both \( \phi \) and \( T\phi \) are right pointwise cocartesian, and that \( T \) preserves the cartesian cells above: it follows that both \( \phi' \) and \( T\phi' \) are cocartesian. Applying the second part of the proof of the non-pointwise case above to the \( T \)-cell \( \phi' \), we find that it is cocartesian.

Finally assume that each of the \( J_1, \ldots, J_n \) satisfies the Beck-Chevalley condition. Since the restrictions \( J_n(\text{id}, f) \) and \( K(\text{id}, f) \), where \( f : B \to A_n \), are assumed to exist and be preserved by \( T \), it follows from Proposition 7.2 that such restrictions are created by both \( U : T\text{-Alg}_{(v, ps, lbc)} \to K \) and \( U : T\text{-Alg}_{(v, ps)} \to K \), and hence preserved by the inclusion \( T\text{-Alg}_{(v, ps, lbc)} \to T\text{-Alg}_{(v, ps)} \). Since this inclusion (being locally full and faithful) clearly reflects cocartesian cells, it follows that it reflects right pointwise cocartesian cells as well. Therefore, to prove creation of \( (J_1 \circ \cdots \circ J_n) \) along \( U : T\text{-Alg}_{(v, ps, lbc)} \to K \), it suffices to prove that the structure cell \( K \), as obtained above, satisfies the left Beck-Chevalley condition; that is we have to show its factorisation \( K' \) through \( K(\text{id}, A_n) \) to be right pointwise cocartesian. By the pasting lemma (Lemma 3.19) we may equivalently show that \( K' \circ T\phi \) is right pointwise cocartesian; to do so consider the following equation whose identities follow from the definition of \( K \) and the factorisation of \( J_n \) through \( J(\text{id}, A_n) \) and that
of \( \phi \circ (\id, \cdots, \cart) \), in the third composite, through \( K(\id, a_n) \).

\[
\begin{array}{ccc}
  T\phi & = & J_1 \cdots J_n \\
  \phi & = & J_1 \cdots J_n \\
  \phi' & = & J_1 \cdots J_n
\end{array}
\]

Now observe that \((J_1, \ldots, J_n)\) satisfies the left Beck-Chevalley condition, because all of \(J_1, \ldots, J_n\) do and Lemma 5.14. By definition it follows that \((J_1, \ldots, J_n)\) is right pointwise cocartesian and, by Lemma 3.18, so is \(\phi\). Using the pasting lemma again we conclude that \(\phi' \circ (J_1, \ldots, J_n) = K' \circ T\phi\) is right pointwise cocartesian, as required. This completes the proof. \(\square\)

### 7.2 Kan extensions

The main results of this section, Theorem 7.7 and Theorem 7.8 below, describe the creation of algebraic Kan extensions by the forgetful functors \(U: \Alg_{\id, w} \to \mathcal{K}\) and \(U: \Alg_{\c, w} \to \mathcal{K}\) respectively. The second of these, concerning colax \(T\)-algebras, generalises the main result of [Kou15a], which describes creation of algebraic Kan extensions in the case of a double category \(\mathcal{K}\). The latter in turn is a generalisation of a result by Getzler, on the lifting of pointwise left Kan extensions along symmetric monoidal enriched functors, that was given in [Get09], where it was used to obtain a coherent way of freely generating many types of generalised operad. In [Web15] a variant of Theorem 7.7, considered in the setting of 2-categories, is used to obtain algebraic Kan extensions along ‘morphisms of internal algebra classifiers’. Theorem 7.8 can also be regarded as an extension of Kelly’s result on ‘doctrinal adjunction’, given in [Kel74], as we shall see in the next section.

**Remark 7.5.** Related to the results of this section, in the forthcoming [Kou15b] the ‘lifting’ of left Kan extensions that preserve algebraic structures defined by ‘colax-idempotent’ 2-monads \(T\), in the sense of e.g. [KL97], is considered. The ‘free category with finite products’-monad as a prime example, such monads are special in that their pseudo \(T\)-algebra structures are ‘essentially unique’, like the ‘structure of finite products’ on a category is essentially unique. Moreover, any morphism \(l: B \to M\) between pseudo \(T\)-algebras is uniquely a colax \(T\)-morphism. In [Kou15b] we consider the situation where such an colax \(T\)-morphism \((l, l): B \to M\) is the left Kan extension of morphisms \(d: A \to M\) and \(j: A \to B\), where \(A\) is not necessarily a \(T\)-algebra. Generalising a classical result on finite-product-preserving left Kan extensions, by Adámek and Rosický [AR01], the main results of [Kou15b] relate the invertibility of \(l\) to conditions involving the morphisms \(d\) and \(j\).

The following definition is used in stating the theorems.

**Definition 7.6.** Let \(T\) be a monad on a hypervirtual double category \(\mathcal{K}\). Consider a (co-) lax \(T\)-algebra \(M = (M, m, \bar{m}, \bar{m})\) as well as a vertical morphism \(d: A_0 \to M\) and a path \(\underline{j} = (J_1, \ldots, J_n): A_0 \to A_n\) of horizontal morphisms in \(\mathcal{K}\). We say that the algebraic structure of \(M\) preserves the (pointwise) (weak) left Kan extension of \(d\) along \((J_1, \ldots, J_n)\) if, for any cell

\[
A_0 \xrightarrow{J_1} A_1 \xrightarrow{\cdots} A_n' \xrightarrow{J_n} A_n
\]

that defines \(l\) as the (pointwise) (weak) left Kan extension of \(d\) along \((J_1, \ldots, J_n)\), the composite \(m \circ T\eta\) defines \(m \circ Tl\) as the (pointwise) (weak) left Kan extension of \(m \circ Td\) along \((TJ_1, \ldots, TJ_n)\).
In Example 7.13 we will see that the monoidal structure of any (op-)lax monoidal \( \mathcal{V} \)-category \( M \), whose tensor product preserves small \( \mathcal{V} \)-weighted colimits in each variable, preserves pointwise left Kan extensions along \( \mathcal{V} \)-profunctors \( J: A \Rightarrow B \) with \( A \) a small \( \mathcal{V} \)-category.

We start with the creation of algebraic left Kan extensions between lax algebras.

For the notion of left exactness see Definition 5.12.

**Theorem 7.7.** Let \( T = (T, \mu, \iota) \) be a monad on a hypervirtual double category \( \mathcal{K} \) and let ‘weak’ mean either ‘colax’, ‘lax’ or ‘pseudo’. Given lax \( T \)-algebras \( A_0, \ldots, A_n \) and \( M \), consider the following conditions on a path of horizontal \( T \)-morphisms \((A_0 \xrightarrow{d} A_1, \ldots, A_n \xrightarrow{d} A_n)\) and a weak vertical \( T \)-morphism \( d: A_0 \Rightarrow M \), where \( m = \ldots \)

\( \text{The following hold:} \)

\( \text{(p) the algebraic structure of } M \text{ preserves both the (weak) left Kan extension of } d \text{ along } (J_1, \ldots, J_n) \text{ and that of } m \circ Td \text{ along } (TJ_1, \ldots, TJ_n); \)

\( \text{(e) the path of structure cells } (\bar{J}_1, \ldots, \bar{J}_n) \text{ is (weak) left } d \text{-exact, while its } T \text{-image } (T\bar{J}_1, \ldots, T\bar{J}_n) \text{ is (weak) left } (d \circ a_0) \text{-exact; } \)

\( \text{(l) the forgetful functor } U: T \text{-Alg}_{(1, w)} \Rightarrow \mathcal{K} \text{ creates the (weak) left Kan extension of } d \text{ along } (J_1, \ldots, J_n). \)

\( \text{The following hold:} \)

\( \text{(a) if ‘weak’ means ‘lax’ then (p) implies (l); } \)

\( \text{(b) if ‘weak’ means ‘colax’ then (e) implies (l); } \)

\( \text{(c) if ‘weak’ means ‘pseudo’ then any two of (p), (e) and (l) imply the third. } \)

Moreover if the restrictions \( J_\circ(\text{id}, f) \) exist in \( \mathcal{K} \) and are preserved by \( T \), for all \( f: B \Rightarrow A_n \), then the result analogous to (a) for pointwise (weak) left Kan extensions holds as well. In this case the forgetful functor \( U: T \text{-Alg}_{(1, \text{ps, lbc})} \Rightarrow \mathcal{K} \) too creates the pointwise left Kan extension of \( d \) along \( (J_1, \ldots, J_n) \) (which now satisfy the left Beck-Chevalley condition), under the assumption of (p).

We remark that in the proof below of the pointwise case of (a) above we use the fact that, under the above assumptions, all restrictions in \( T \text{-Alg}_{(1, l)} \) of the form \( J_\circ(\text{id}, f) \) are created by \( U: T \text{-Alg}_{(1, l)} \Rightarrow \mathcal{K} \), as we have seen in Proposition 7.2(b).

That the same need not be true in the colax case \( w = c \), prevents us from being able to prove a pointwise variant of (b) above. It is possible however to show that \( U: T \text{-Alg}_{(1, c)} \Rightarrow \mathcal{K} \) creates left Kan extensions satisfying a variation of Definition 4.5 by restricting it to restrictions \( J_\circ(\text{id}, f) \) in \( T \text{-Alg}_{(1, c)} \) with \( f: B \Rightarrow A_n \) a pseudo \( T \)-morphism, for such restrictions are created by \( U: T \text{-Alg}_{(1, c)} \Rightarrow \mathcal{K} \); see Proposition 7.2(a).

**Proof.** Part (a): ‘weak’ means ‘lax’. In this case \( d: A_0 \Rightarrow M \) is equipped with a lax \( T \)-morphism structure cell \( d: m \circ Td \Rightarrow d \circ a_0 \). We shall treat the creation of pointwise left Kan extensions; a proof for the other cases can be obtained by either choosing \( f = \text{id}_{A_n} \) in (18) below, taking the path \( H = (H_1, \ldots, H_m) \) to be empty, or both. Consider a nullary cell \( \eta \) in \( \mathcal{K} \), as in the composite on the left-hand side below, that defines \( l: A_n \Rightarrow M \) as the pointwise left Kan extension of \( d \) along \( (J_1, \ldots, J_n) \). We will equip \( l \) with the structure of a lax \( T \)-morphism, with respect to which \( \eta \) forms a \( T \)-cell. Thus a \( T \)-cell, we will show that \( \eta \) defines \( l \) as a pointwise left Kan extension in \( T \text{-Alg}_{(1, l)} \).

Under the assumption of (p) the composite \( m \circ T\eta \) in the right-hand side below defines \( m \circ Tl \) as a left Kan extension so that the right-hand side, where \( \bar{d} \) is the
We claim that $\bar{l}$ makes $l$ into a lax $T$-morphism, that is it satisfies the associativity and unit axioms. To see that it satisfies the former consider the equation of composites below, whose identities are explained below. Notice that its left-hand and right-hand sides are given by the corresponding sides of the associativity axiom for $\bar{l}$, composed with $m \circ Tm \circ T^2\eta$ on the left. By condition (p) the latter defines $m \circ Tm \circ T^2\eta$ as the left Kan extension of $m \circ Tm \circ T^2d$ along $(T^2J_1, \ldots, T^2J_n)$, so that the axiom itself follows from uniqueness of factorisations through $m \circ Tm \circ T^2\eta$.

The identities above follow from the $T$-image of (17); the identity (17) itself; the associativity axioms for $J_1, \ldots, J_n$ (see Definition 6.7); the associativity axiom for $\bar{d}$; the identity (17) again; the naturality of $\mu$.

The unit axiom for $\bar{l}$ follows similarly from the equation below, whose sides form those of the unit axiom composed with $\eta$, and the fact that the latter defines $\bar{l}$ as a left Kan extension. The identities here follow from the naturality of $\iota$; the identity (17); the unit axiom for $\bar{d}$; the unit axioms for $J_1, \ldots, J_n$.

Now notice that the factorisation (17) forms the $T$-cell axiom for $\eta$ (Definition 6.10). In fact, since the factorisation is unique, the lax $T$-structure cell $\bar{l}$ is
unique in making \( \eta \) into a \( T \)-cell. Thus a \( T \)-cell, it remains to prove that \( \eta \) defines \((l, l)\) as the pointwise left Kan extension of \( d \) along \((J_1, \ldots, J_n)\) in \( T \text{-Alg}_{(1,1)} \). In order to do so, consider any nullary \( T \)-cell \( \phi \) as on the left-hand side below, where \( J_n(\text{id}, f) \) denotes any \( T \)-restriction of \( J_n \) along any lax \( T \)-morphism \( f: B \to A_n \).

Since we assume the underlying restriction \( J_n(\text{id}, f) \) to exist in \( K \) the cartesian \( T \)-cell defining \( J_n(\text{id}, f) \) is created, and thus preserved, by \( U: T \text{-Alg}_{(1,1)} \to K \).

Because \( \eta \) defines \( l \) as a pointwise left Kan extension in \( K \), the cell \( \phi \) factors uniquely in \( K \) as shown, and it remains to prove that the factorisation \( \phi' \) is a \( T \)-cell. To see this consider the following equation of composites in \( K \), where ‘\( c \)’ denotes the cartesian cell that defines \( J_n(\text{id}, f) \). Its identities follow from the \( T \)-cell axiom for \( \eta \); the definition of \( J_n(\text{id}, f) \) (see the proof of Proposition 7.2); the identity above; the \( T \)-cell axiom for \( \phi \); the \( T \)-image of the identity above. Now notice that the left-hand and right-hand sides of the equation below are the sides of the \( T \)-cell axiom for \( \phi' \), composed with the composite \( m \circ T \eta \circ \text{id}, \ldots, T \text{cart} \) on the left. Since \( m \circ T \eta \) here defines a pointwise left Kan extension by the assumption of (p), the full composite does too, because we assume the cartesian cell to be preserved by \( T \). From this we conclude that the \( T \)-cell axiom for \( \phi' \) holds, which completes the proof of part (a).

**Part (b): ‘weak’ means ‘colax’**. In this case \( d: A_0 \to M \) is equipped with an colax \( T \)-morphism structure cell \( d: d \circ a_0 \Rightarrow m \circ T d \). We shall treat the creation of left Kan extensions; a proof for weak Kan extensions is obtained by taking the path \( \overline{H} = (H_1, \ldots, H_m) \) in (18) to be empty. Consider a nullary cell \( \eta \) in \( K \), as in the composite on the right-hand side below, that defines \( l: A_n \to M \) as the pointwise left Kan extension of \( d \) along \((J_1, \ldots, J_n)\). Under the assumption of (e),
the composite \( \eta \circ (\tilde{J}_1, \ldots, \tilde{J}_n) \) in the left-hand side below defines \( l \circ a_n \) as a left Kan extension, so that the right-hand side factors as a vertical cell \( l \circ a_n \Rightarrow m \circ Tl \) as shown.

\[
\begin{align*}
T \cdot A_0 & \xrightarrow{TJ_1} T \cdot A_1 \xrightarrow{TJ_n} T \cdot A_n \\
A_0 & \xrightarrow{a_n} Td \\
& \xrightarrow{T\eta} Tl \\
& \xrightarrow{d} m \\
& \xrightarrow{M} \\
\end{align*}
\]

Analogous to the proof of part (a) above, we can show that the cell \( l \) satisfies the associativity and unit axioms, so that it makes \( l : A_n \rightarrow M \) into an colax \( T \)-morphism.

Indeed, the associativity axiom follows from the identity below, which itself follows from the factorisation above, the associativity axiom for \( d \), the naturality of \( \mu \) and the associativity axioms for \( \tilde{J}_1, \ldots, \tilde{J}_n \). As before notice that the identity below is the associativity axiom for \( l \) composed on the right with the composite \( \eta \circ (\tilde{J}_1 \circ T \tilde{J}_1, \ldots, \tilde{J}_n \circ T \tilde{J}_n) \). As a consequence of condition (e), this composite defines a left Kan extension, so that the associativity axiom itself follows.

\[
\begin{align*}
T \cdot J_1 & \xrightarrow{T \cdot \eta} T \cdot l \\
& \xrightarrow{\eta} l \\
& \xrightarrow{m} \\
\end{align*}
\]

The unit axiom for \( \tilde{l} \) follows in the same way from the identity on the left below, which itself follows from the unit axioms for \( \tilde{J}_1, \ldots, \tilde{J}_n \) and \( d \), the identity (19) and the naturality of \( \iota \).

\[
\begin{align*}
\cdots & \xrightarrow{\eta} \tilde{a}_n \\
& \xrightarrow{\iota} \tilde{m} \\
\end{align*}
\]

Analogous to the reasoning in the proof of part (a), the cell \( \tilde{l} \) is unique in making \( \eta \) into a \( T \)-cell and, to show that \( \eta \) defines \( (l, \tilde{l}) \) as a left Kan extension in \( T \text{-Alg}_{(1, c)} \), we have to prove that any unique factorisation \( \phi' \) of a \( T \)-cell \( \phi \), as in (18), where now \( d \) and \( k \) are colax \( T \)-morphisms, and where we take \( f = \text{id}_{A_n} \), is again a \( T \)-cell. That its \( T \)-cell axiom holds after composition with \( \eta \circ (\tilde{J}_1, \ldots, \tilde{J}_n) \), as shown below, follows from the \( T \)-cell axiom for \( \eta \), the factorisation (18) and the \( T \)-cell axiom for \( \phi \). By assumption (e) the composite \( \eta \circ (\tilde{J}_1, \ldots, \tilde{J}_n) \) defines a left Kan extension, so that the \( T \)-cell axiom follows. This completes the proof of part (b).

\[
\begin{align*}
\cdots & \xrightarrow{\eta} \tilde{a}_n \\
& \xrightarrow{\iota} \tilde{m} \\
\end{align*}
\]

**Part (c):** ‘weak’ means ‘pseudo’. In this case \( d : A_0 \rightarrow M \) is equipped with an invertible \( T \)-morphism structure cell \( d : m \circ Td \Rightarrow d \circ a_0 \). To see that any two of (e), (p) and (l) imply the third, first notice that the locally full inclusions \( T \text{-Alg}_{(1, 1)} \supset T \text{-Alg}_{(1, p)} \subset T \text{-Alg}_{(1, c)} \) reflect cells defining (weak) left Kan extensions.
Assuming \((e)\) and \((p)\), we may apply \((a)\) to find that the \((\text{weak})\) left Kan extension \(l: A_0 \to M\) of \(d\) along \((J_1, \ldots, J_n)\) is created by \(U: T\Alg_{(l, \text{ps})} \to \mathcal{K}\), with structure cell \(\bar{l}\) defined by the factorisation \((17)\). Using the fact that \((J_1, \ldots, J_n)\) is \((\text{weak})\) left \(d\)-exact, by \((e)\), and that \(d\) is invertible, it follows that the left-hand side of this factorisation defines a left-hand Kan extension, from which we conclude that \(\bar{l}\) is invertible. We conclude that its defining \(T\)-cell \(\eta\) defines \((l, \bar{l})\) as a \((\text{weak})\) left Kan extension in \(T\Alg_{(l, \text{ps})}\), that is condition \((l)\) holds.

\[
\begin{array}{c}
\begin{array}{c}
T\bar{d} \\
\downarrow \quad \cdot \quad \cdot \quad \cdot \\
TJ_1 \quad J_1 \quad \cdot \quad \cdot \\
\downarrow \quad \cdot \quad \cdot \quad \cdot \\
J \quad \cdot \quad \cdot \\
\eta \quad \cdot \quad \cdot \\
\downarrow \quad \cdot \quad \cdot \\
\bar{l} \quad \cdot \quad \cdot \\
\end{array}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{array}{c}
T^2\eta \\
\downarrow \quad \cdot \quad \cdot \quad \cdot \\
T^2J_1 \quad J_1 \quad \cdot \quad \cdot \\
\downarrow \quad \cdot \quad \cdot \quad \cdot \\
J \quad \cdot \quad \cdot \\
\eta \quad \cdot \quad \cdot \\
\downarrow \quad \cdot \quad \cdot \\
\bar{l} \quad \cdot \quad \cdot \\
\end{array}
\end{array}
\]

Next assume that both conditions \((p)\) and \((l)\) hold. From \((l)\) it follows that the \((\text{weak})\) left Kan extension \(l: A_0 \to M\) of \(d\) along \((J_1, \ldots, J_n)\) in \(\mathcal{K}\) admits an invertible \(T\)-morphism structure cell that is unique in making its defining cell \(\eta\) into a \(T\)-cell in \(T\Alg_{(l, \text{ps})}\). Of course, the same structure cell makes \(\eta\) into a \(T\)-cell in \(T\Alg_{(l, \bar{l})}\), so that it must coincide with the structure cell \(\bar{l}\) that we obtain by applying part \((a)\), as the unique factorisation in \((17)\). Thus, \(\bar{l}\) in the right-hand side of \((17)\) is invertible, and it follows that the full right-hand side defines a \((\text{weak})\) left Kan extension. Composing both sides on the left with \(d^{-1}\) we find that \(\eta \circ (J_1, \ldots, J_n)\), in the left-hand side, defines a \((\text{weak})\) left Kan extension too, showing that \((J_1, \ldots, J_n)\) is \((\text{weak})\) left \(d\)-exact. This proves the first half of \((e)\); for the second half, that is \((TJ_1, \ldots, TJ_n)\) is \((\text{weak})\) left \((d \circ a_0)\)-exact, a similar argument can applied to the identity above, which is itself a consequence of \((17)\). This shows that \((p)\) and \((l)\) together imply \((e)\).

The argument showing that \((e)\) and \((l)\) together imply \((p)\) is horizontally dual to the previous one: in this case the unique factorisation \(\bar{l}\) in \((19)\) is invertible; it follows that \(m \circ T\eta\) in its left-hand side defines a \((\text{weak})\) left Kan extension. Hence the algebraic structure of \(M\) preserves the \((\text{weak})\) left Kan extension of \(d\) along \((J_1, \ldots, J_n)\); that is the first half of \((p)\) holds. That the \((\text{weak})\) left Kan extension of \(m \circ Td\) along \((TJ_1, \ldots, TJ_n)\) is preserved as well follows from applying a similar argument to the identity above, which itself is a consequence of \((19)\).

Finally, we consider the creation of pointwise left Kan extensions by the forgetful functor \(U: T\Alg_{(l, \text{ps}, \text{lbc})} \to \mathcal{K}\). Notice that, in the ‘pseudo’-case above, the proof that \((p)\) and \((e)\) implies \((l)\) only uses the first part of \((e)\), namely the \(d\)-exactness of \((J_1, \ldots, J_n)\). Analogously, if \((J_1, \ldots, J_n)\) is a path in \(T\Alg_{(l, \text{ps}, \text{lbc})}\), so that \((J_1, \ldots, J_n)\) satisfies the left Beck-Chevalley condition by Lemma 5.14 and thus is pointwise left \(d\)-exact by Proposition 5.15, it follows that the left Kan extension \((l, \bar{l})\) created by \(U: T\Alg_{(l, \text{ps})} \to \mathcal{K}\), of \(d\) along \((J_1, \ldots, J_n)\), is a pseudo \(T\)-morphism provided that \(\bar{l}\) is a pointwise Kan extension in \(\mathcal{K}\). Now notice that, because the restrictions \(J_n (\text{id, } f)\) are assumed to exist in \(\mathcal{K}\) and be preserved by \(T\), it follows from Proposition 7.2 that the inclusion \(T\Alg_{(l, \text{ps}, \text{lbc})} \to T\Alg_{(l, \text{ps})}\) preserves such restrictions. Together with the fullness of this inclusion, from this we conclude that \((l, \bar{l})\) forms a pointwise left Kan extension in \(T\Alg_{(l, \text{ps}, \text{lbc})}\) too, which completes the proof.
Next is the colax case. We will only sketch its proof in the lax case \( w = 1 \), since it is similar to parts of that above and parts of that of the main result of [Kou15a].

**Theorem 7.8.** Let \( T = (T, \mu, \iota) \) be a monad on a hypervirtual double category \( \mathcal{K} \) and let ‘weak’ mean either ‘colax’, ‘lax’ or ‘pseudo’. Given colax \( T \)-algebras \( A_0, \ldots, A_n \) and \( M \), consider the following conditions on a path of horizontal \( T \)-morphisms \( (A_0 \xrightarrow{J_1} A_1, \ldots, A_n \xrightarrow{J_n} A_n) \) and a weak vertical \( T \)-morphism \( d: A_0 \to M \), where \( m \) and \( a_0 \) denote the structure maps of \( M \) and \( A_0 \):

\( (p) \) the algebraic structure of \( M \) preserves the (weak) left Kan extension of \( d \) along \( (J_1, \ldots, J_n) \), while both paths \( (\mu_{J_1}, \ldots, \mu_{J_n}) \) and \( (\iota_{J_1}, \ldots, \iota_{J_n}) \) are (weak) left \((m \circ Td)\)-exact; 

\( (e) \) the path of structure cells \( (\bar{1}, \ldots, \bar{J}_n) \) is (weak) left \( d \)-exact, while both paths \( (\mu_{J_1}, \ldots, \mu_{J_n}) \) and \( (\iota_{J_1}, \ldots, \iota_{J_n}) \) are (weak) left \((d \circ a_0)\)-exact; 

\( (l) \) the forgetful functor \( U: T-\text{Alg}_{(c, w)} \to \mathcal{K} \) creates the (weak) left Kan extension of \( d \) along \( (J_1, \ldots, J_n) \).

The following hold:

\( (a) \) if ‘weak’ means ‘lax’ then \((p)\) implies \((l)\); 

\( (b) \) if ‘weak’ means ‘colax’ then \((e)\) implies \((l)\); 

\( (c) \) if ‘weak’ means ‘pseudo’ then any two of \((p), (e)\) and \((l)\) imply the third.

Moreover if the restrictions \( J_n, (\text{id}, f) \) exist in \( \mathcal{K} \) and are preserved by \( T \), for all \( f: B \to A_n \), then the result analogous to \((a)\) for pointwise (weak) left Kan extensions holds as well. In this case the forgetful functor \( U: T-\text{Alg}_{(c, \text{ps}, \text{lbc})} \to \mathcal{K} \) too creates the pointwise left Kan extension of \( d \) along \( (J_1, \ldots, J_n) \) (which now satisfy the left Beck-Chevalley condition), under the assumption of \((p)\).

**Sketch of the proof of part \((a)\): ‘weak’ means ‘lax’.** In this case \( d: A_0 \to M \) comes with a lax \( T \)-morphism structure cell \( d: m \circ Td \Rightarrow d \circ a_0 \). Using the assumption \((p)\) we obtain, just like in the proof of the previous theorem, a cell \( l: m \circ Tl \Rightarrow l \circ a_0 \) as the unique factorisation in \((17)\). The proof that \( l \) makes \( l \) into a lax \( T \)-morphism, that is it satisfies the associativity and unit axioms, is essentially the same as the horizontal dual of that given in proof of Theorem 5.7(a) of [Kou15a], except that the setting there is that of double categories. Briefly, the proof follows from \((17)\), the associativity and unit axioms for \( d \), those for \( \bar{J}_1, \ldots, \bar{J}_n \) (see Definition 6.7), the naturality of \( \mu \) and \( \iota \), and the fact that the composites

\[ m \circ \mu_M \circ T^2 \eta = m \circ T \eta \circ (\mu_{J_1}, \ldots, \mu_{J_n}) \quad \text{and} \quad m \circ \iota_M \circ \eta = m \circ T \eta \circ (\iota_{J_1}, \ldots, \iota_{J_n}) \]

define left Kan extensions, where the latter is a consequence of the assumption \((p)\) and the left exactness assumption on the paths \( (\mu_{J_1}, \ldots, \mu_{J_n}) \) and \( (\iota_{J_1}, \ldots, \iota_{J_n}) \). As before the unique factorisation \((17)\) shows that \( l \) is unique in making \( \eta \) into a \( T \)-cell in \( T-\text{Alg}_{(c, 1)} \). Since the type of algebraic structure on objects does not figure in proving that \( \eta \), as a \( T \)-cell, defines \((l, l)\) as a (pointwise) (weak) left Kan extension in \( T-\text{Alg}_{(c, 1)} \), the argument given in the proof of Theorem 7.7(a) applies verbatim.

For completeness we show that, in the case of pseudo \( T \)-algebras, the conditions of Theorem 7.7 and Theorem 7.8 are equivalent, as follows.
Lemma 7.9. Let $T$ be a monad on a hypervirtual double category. Consider pseudo $T$-algebras $A_0, \ldots, A_n$ and $M$, a path $(A_0 \xrightarrow{J} A_1, \ldots, A_n \xrightarrow{J_n} A_n)$ of horizontal $T$-morphisms, and a vertical morphism $d \colon A_0 \to M$. If the (pointwise) (weak) left Kan extension of $d$ along $(J_1, \ldots, J_n)$ exists then the conditions (p), of Theorem 7.7 and Theorem 7.8 respectively, are equivalent, and so are their conditions (e).

Proof. We treat the case of left Kan extensions; the proof for the others is the same. Writing $\eta$ for the cell defining the left Kan extension of $d$ along $(J_1, \ldots, J_n)$, notice that the first halves of the conditions (p) coincide: they say that $m \circ T \eta$ defines the left Kan extension of $m \circ T d$ along $(TJ_1, \ldots, TJ_n)$. Showing that their other halves are equivalent too means showing that $m \circ TM \circ T^2 \eta$ defines a left Kan extension if and only if both $m \circ T \eta \circ (\mu_{J_1}, \ldots, \mu_{J_n})$ and $m \circ T \eta \circ (\iota_{J_1}, \ldots, \iota_{J_n})$ do. Using the fact that the coherence cells $\tilde{m} \colon m \circ T m \Rightarrow m \circ \mu_M$ and $\tilde{m} \colon \id_M \Rightarrow m \circ \iota_M$ are invertible, the latter follows from the identities below, which themselves are consequences of the interchange axiom (Lemma 1.12) and the naturality of $\mu$ and $\iota$ (see Definition 1.13).

$$
\tilde{m} \circ T^2 \eta = (\tilde{m} \circ T^2 d) \circ (m \circ \mu_M \circ T^2 \eta) = (\tilde{m} \circ T^2 d) \circ (m \circ T \eta \circ (\mu_{J_1}, \ldots, \mu_{J_n}))
$$

$$
\tilde{m} \circ \eta = (\tilde{m} \circ d) \circ (m \circ \mu_M \circ \eta) = (\tilde{m} \circ d) \circ (m \circ T \eta \circ (\iota_{J_1}, \ldots, \iota_{J_n}))
$$

The first halves of (e) coincide as well: they say that $\eta \circ (J_1, \ldots, J_n)$ defines the left Kan extension of $d \circ a_0$ along $(TJ_1, \ldots, TJ_n)$. Showing that their other halves are equivalent means showing that $\eta \circ (J_1 \circ T \eta \circ (\mu_{J_1}, \ldots, \mu_{J_n})$ and $\eta \circ (J_1 \circ \mu_{J_1}, \ldots, J_n \circ \mu_{J_1})$ do. Using the fact that the coherence cells $\tilde{a}_0, \tilde{a}_0, \tilde{a}_n$ and $\tilde{a}_n$ are invertible, the latter follows from the identities

$$
\eta \circ (J_1 \circ T \eta \circ (\mu_{J_1}, \ldots, \mu_{J_n})) \circ (l \circ \tilde{a}_n) = (d \circ \tilde{a}_0) \circ (\eta \circ (J_1 \circ \mu_{J_1}, \ldots, J_n \circ \mu_{J_1}));
$$

$$
\eta \circ (l \circ \tilde{a}_n) = (d \circ \tilde{a}_0) \circ (\eta \circ (J_1 \circ \iota_{J_1}, \ldots, J_n \circ \iota_{J_1})),
$$

which themselves follow from the coherence axioms for the structure cells $J_1, \ldots, J_n$; see Definition 6.7. This completes the proof. \qed

7.3 Adjunctions

Here we show how Theorem 7.7 and Theorem 7.8 above can be used to recover and extend Kelly’s result on ‘doctrinal adjunction’, when applied to the vertical part $V(T)$ of a monad on a unital virtual equipment $K$. More precisely, in that case part (a) of the theorem below implies the ‘sufficient’-part of Theorem 1.5 of [Kel74]. In Remark 5.25 of [Shu11] it is also observed that Kelly’s result is naturally described in the language of double categories, by using the so-called “mate correspondence” that is induced by companions.

Consider a morphism $f \colon A \to C$ in a hypervirtual double category $K$. If the companion $f_* : A \to C$ exists then we can regard the right adjoint of $f$ as a universal construction, in the sense of Definition 7.1: it is the absolute left Kan extension of $\id_A$ along $f_*$, as follows from combining Lemma 2.16 and Proposition 4.2.

Theorem 7.10. Let $T$ be a monad on a hypervirtual double category $K$ and let ‘weak’ mean either ‘colax’ or ‘lax’. Consider a pseudo $T$-morphism $f : A \to C$ between weak $T$-algebras, whose companion $f_*$ exists in $K$. The following hold:

(a) $U : T \Alg_{(e,1)} \to K$ creates the right adjoint of $f$;

(b) $U : T \Alg_{(e,p)} \to K$ creates the right adjoint of $f$ precisely if the unique factorisation $f_*$, in the right-hand side below, is left $f$-exact.
Proof. First notice that, \( f : A \to C \) being a pseudo \( T \)-morphism, the companion \( f_* \) is created by the forgetful functors \( U : T\text{-Alg}(v, w) \to K \), in either case \( w = 1 \) or \( ps \), by Proposition 7.2; its \( T \)-structure cell \( \bar{f}_* \) is given by the factorisation above. It follows that the right adjoint of \( f \) in \( T\text{-Alg}(v, w) \) can be regarded as a universal construction as described above.

To start consider a nullary cell

\[
\begin{array}{ccc}
A & \xrightarrow{f_*} & C \\
\downarrow{\eta/g} & & \downarrow{\eta/g} \\
A & & A
\end{array}
\]

in \( K \) and assume that it defines \( g \) as the absolute left Kan extension of \( \text{id}_A \) along \( f_* \) or, equivalently, that the vertical cell \( \eta \circ \text{cocart} : \text{id}_A \Rightarrow g \circ f \), where \( \text{cocart} \) is the cocartesian cell that defines \( f_* \), defines an adjunction \( f \dashv g \) in \( V(K) \); this follows from combining Proposition 4.2 and Lemma 2.16. Because \( T \) preserves weakly cocartesian cells defining companions (Corollary 2.12), it follows that the images \( T^n\eta \), where \( n \geq 0 \), again define \( T^n g \) as an absolute left Kan extensions, since \( T^n \eta \circ T^n \text{cocart} = T^n (\eta \circ \text{cocart}) \) defines \( T^n f \dashv T^n g \) in \( V(K) \). From this we conclude that each of the composites

\[
\begin{align*}
& c \circ T \eta, \quad c \circ T \circ T^2 \eta, \quad a \circ T \eta \circ \mu_{f_*}, \quad a \circ T \eta \circ \iota_{f_*}, \\
& c \circ T f \circ T \eta, \quad c \circ T c \circ T^2 f \circ T^2 \eta, \quad c \circ T f \circ T \eta \circ \mu_{f_*} \quad \text{and} \quad c \circ T f \circ T \eta \circ \iota_{f_*}
\end{align*}
\]

defines a left Kan extension in \( K \); here the naturality of \( \mu \) and \( \iota \) is used, e.g. \( a \circ T \eta \circ \mu_{f_*} = a \circ \mu_A \circ T^2 \eta \) for the third composite.

The first row above being left Kan extensions ensures that we may apply either Theorem 7.7(a) or Theorem 7.8(a) to obtain a lax \( T \)-structure \( \bar{g} \) on \( g \) that is unique in making \( \eta \) into a \( T \)-cell, such that it defines \( (g, \bar{g}) \) as the left Kan extension of \( \text{id}_A \) along \( (f_*, \bar{f}_*) \) in \( T\text{-Alg}_{(v, 1)} \). In more detail, \( \bar{g} \) is obtained as the unique factorisation \( \eta \circ \bar{f}_* = (a \circ \eta) \circ \bar{g} \) (compare (17)); it follows that \( \bar{g} \) is invertible if and only if \( \bar{f}_* \) is left \( \text{id}_A \)-exact so that, in that case, \( \eta \) defines \( (g, \bar{g}) \) as a left Kan extension in \( T\text{-Alg}_{(v, ps)} \) as well.

It remains to show that \( \eta \circ \text{cocart} \), as a \( T \)-cell, defines \( (f, \bar{f}) \dashv (g, \bar{g}) \) in \( T\text{-Alg}_{(v, 1)} \). Again using Proposition 4.2 and Lemma 2.16, for this it suffices to show that \( f \circ \eta \) defines \( (f, \bar{f}) \circ (g, \bar{g}) \) as a left Kan extension. That it does follows from applying either Theorem 7.7(a) or Theorem 7.8(a) again, which is possible because the second row of composites above define left Kan extensions. This concludes the proof. \( \square \)

### 7.4 Cocomplete \( T \)-algebras

In this final subsection we show how the creation of left Kan extensions can be used to “lift” cocomplete \( T \)-algebras along the forgetful functors \( U : T\text{-Alg}_{(v, 1)} \to K \) and \( U : T\text{-Alg}_{(v, ps, lbc)} \to K \), where \( v \in \{c, l, ps\} \). Given an ideal \( S \) of left extension...
diagrams $M \xleftarrow{d} A \xrightarrow{f} B$ in $\mathcal{K}$ (see Definition 5.18), we denote for each $v \in \{c, l, ps\}$ by

$$\mathcal{S}(v, i) := \{(d, J) \mid (Ud, UJ) \in \mathcal{S}\}$$

the ideal of left extension diagrams in $T\text{-Alg}_{(v, i)}$ that is the preimage of $\mathcal{S}$ under the forgetful functor $U: \text{T-Alg}_{(v, i)} \to \mathcal{K}$. Similarly we write

$$\mathcal{S}(v, ps, lbc) := \{(d, J) \in \mathcal{S}(v, ps) \mid J \text{ satisfies the left Beck-Chevalley condition}\}$$

for the restriction of $\mathcal{S}(v, ps)$ to horizontal $T$-morphisms that satisfy the left Beck-Chevalley condition (see Definition 6.7).

**Proposition 7.11.** Let $T = (T, \mu, i)$ be a monad on a hypervirtual equipment $\mathcal{K}$ that preserves unary restrictions. Let $\mathcal{S}$ be an ideal of left extension diagrams in $\mathcal{K}$, and let ‘weak’ mean either ‘colax’ or ‘lax’. Consider a weak $T$-algebra $M = (M, m, \bar{m}, \bar{m})$ whose underlying object $M$ is $\mathcal{S}$-cocomplete in $\mathcal{K}$ and whose algebraic structure preserves, for each $(d, J) \in \mathcal{S}(M)$, the pointwise left Kan extension of $d$ along $J$. The following hold:

- (a) if ‘weak’ means ‘colax’ then $M$ is both $\mathcal{S}(c, i)$-cocomplete in $\text{T-Alg}_{(c, i)}$ and $\mathcal{S}(c, ps, lbc)$-cocomplete in $\text{T-Alg}_{(c, ps, lbc)}$ provided that, for each $(d, J) \in \mathcal{S}(M)$, the cells $\mu_J$ and $\iota_J$ of are pointwise left $(m \circ Td)$-exact;

- (b) if ‘weak’ means ‘lax’ then $M$ is both $\mathcal{S}(l, i)$-cocomplete in $\text{T-Alg}_{(l, i)}$ as well as $\mathcal{S}(l, ps, lbc)$-cocomplete in $\text{T-Alg}_{(l, ps, lbc)}$ provided that its algebraic structure preserves, for each $(d, J) \in \mathcal{S}(M)$, the pointwise left Kan extension of $m \circ Td$ along $TJ$.

In either case, for any $(d, J) \in \mathcal{S}(v, i)(M)$ (resp. $\mathcal{S}(v, ps, lbc)(M)$), the pointwise left Kan extension of $d$ along $J$ is created by the forgetful functor $U: \text{T-Alg}_{(v, i)} \to \mathcal{K}$ (resp. $U: \text{T-Alg}_{(v, ps, lbc)} \to \mathcal{K}$).

Finally, any lax (resp. pseudo) $T$-morphism $(f, \phi): M \to N$, between weak $T$-algebras that both satisfy the appropriate condition above, is $\mathcal{S}(v, i)$-cocontinuous (resp. $\mathcal{S}(v, ps, lbc)$-cocontinuous) as soon as its underlying morphism $f$ is $\mathcal{S}$-continuous.

Notice that, as a consequence of $\mathcal{K}$ being a hypervirtual equipment and Proposition 7.2, the inclusion $\text{T-Alg}_{(ps, 1)} \to \text{T-Alg}_{(l, 1)}$ preserves restrictions of the form $J(id, f)$ and hence reflects pointwise left Kan extensions. It follows that any pseudo $T$-algebra $M$ is $\mathcal{S}(ps, l)$-cocomplete in $\text{T-Alg}_{(ps, l)}$ whenever it is $\mathcal{S}(l, l)$-cocomplete in $\text{T-Alg}_{(l, l)}$. An analogous implication holds for $\mathcal{S}(v, ps, lbc)$-cocompleteness.

**Proof.** In either case consider $(d, J) \in \mathcal{S}(v, i)$ (resp. $(d, J) \in \mathcal{S}(v, ps, lbc)$); we have to show that the pointwise left Kan extension of $d$ along $J$ exists in $\text{T-Alg}_{(v, i)}$ (resp. $\text{T-Alg}_{(v, ps, lbc)}$). The $\mathcal{S}$-cocompleteness of $M$ in $\mathcal{K}$ ensures that the pointwise left Kan extension of the morphisms underlying $d$ and $J$ exists in $\mathcal{K}$. Since the remaining assumptions on $M$ allow us to apply the final part of either Theorem 7.7 or Theorem 7.8, it follows that the pointwise left Kan extension of $d$ along $J$ is created by the forgetful functor $U: \text{T-Alg}_{(v, i)} \to \mathcal{K}$ (resp. $U: \text{T-Alg}_{(v, ps, lbc)} \to \mathcal{K}$).

For the final assertion consider a lax $T$-morphism $(f, \phi): M \to N$ between weak $T$-algebras, whose $\mathcal{S}(v, i)$-cocompleteness is lifted along $U: \text{T-Alg}_{(v, i)} \to \mathcal{K}$ in the above sense, and assume that $f$ is $\mathcal{S}$-continuous. Let $((d, d), (J, J)) \in \mathcal{S}(v, i)(M)$ and let $(l, l): B \to M$ be the lax $T$-morphism that is the pointwise left Kan extension of $(d, d)$ along $(J, J)$ created by $U$; we have to show that $(f \circ l, f \circ l)$ forms a pointwise left Kan extension as well. But this is clear: $((f \circ d, f \circ d), (J, J)) \in \mathcal{S}(v, i)$ because the latter is an ideal, so that the pointwise left Kan extension of $(f \circ d, f \circ d)$ along $(J, J)$ is created, and hence reflected, along the forgetful functor $U$. By the $\mathcal{S}$-cocontinuity of $f$ the composite $f \circ l$ forms the pointwise left Kan extension of $f \circ d$.
along $J$ in $\mathcal{K}$, from which we conclude that $(f \circ l, T \circ d)$ forms that of $(f \circ l, T \circ d)$ along $(J, J)$. Clearly the previous argument applies to the case of $S_{(\psi, \text{ps}), \text{hbc}}$-cocontinuity as well; this completes the proof.

**Example 7.12.** Let $\mathcal{V} \to \mathcal{V}'$ be a symmetric universe enlargement (Definition 1.10) and let $S$ be the ideal of left extension diagrams in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ described in Example 5.20, consisting of pairs $(d, J)$ where $J: A \to B$ is a $\mathcal{V}$-profunctor between $\mathcal{V}$-categories with $A$ small. Consider the ‘free strict monoidal $\mathcal{V}'$-category’-monad $T = (T, \mu, i)$ on $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$, that was described in Example 6.2 whose (co-)lax algebras are (op-)lax monoidal $\mathcal{V}'$-categories $M = (\mathcal{M}, \otimes, a, i)$ (Example 6.5).

Using Proposition 3.17 it is straightforward to show that, for any $\mathcal{V}$-profunctor $J: A \to B$, the cells $\mu_J$ and $\iota_J$ satisfy the left Beck-Chevalley condition (Definition 5.12), so that by Proposition 5.15 they are pointwise left exact. Moreover, given an (op-)lax monoidal category $M$ it follows from the next example that, for each $(d, J) \in S$, its monoidal structure preserves both the pointwise left Kan extension of $d$ along $J$ as well as that of $m \circ T d$ along $J$, provided that its tensor product $\otimes$ preserves small $\mathcal{V}$-weighted colimits in each variable. Thus, if moreover $M$ is small $\mathcal{V}$-cocomplete as a $\mathcal{V}$-category, by the proposition above it admits all monoidal pointwise left Kan extensions of (op-)lax monoidal $\mathcal{V}'$-functors $d: A \to M$ with $A$ small. Moreover, if $d$ is a monoidal $\mathcal{V}'$-functor (with invertible compositors) then a pointwise left Kan extension of $d$ along a monoidal $\mathcal{V}$-profunctor $J: A \to B$ is again monoidal whenever $J$ satisfies the left Beck-Chevalley condition; see Example 6.8.

In [IK86] a monoidal $\mathcal{V}$-category $M$ is called ‘monoidally cocomplete’ if it is small cocomplete as a $\mathcal{V}$-category and its tensor product $- \otimes -$ preserves small $\mathcal{V}$-weighted colimits on both sides. Thus, in terms of the previous and in view of the proposition above, monoidal cocompleteness implies $S_{(\psi, \text{ps}), \text{hbc}}$-cocontinuity in $T^-\text{Alg}(\mathcal{V}, \mathcal{V}')$.

The following example is adapted from Example 5.3 in [Kou15a]. Recall the tensor product on $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$, as described before Lemma 4.22.

**Example 7.13.** Given an (op-)lax monoidal $\mathcal{V}'$-category $M = (\mathcal{M}, \otimes, a, i)$, assume that each of its $n$-ary tensor products $\otimes_n := [M^{\otimes_n} \to TM \otimes_n M]$ preserves small $\mathcal{V}$-weighted colimits in each variable; of course if $M$ is a monoidal category, with invertible $a$ and $i$, then it suffices that its binary tensor product $\otimes_2: M \otimes' M \to M$ does so.

Given any $\mathcal{V}$-profunctor $J: A \to B$ with $A$ a small $\mathcal{V}$-category, we claim that the algebraic structure of $M$ preserves all pointwise left Kan extensions along $J$, in the sense of Definition 7.6. To see this let $\eta$ be any cell in $(\mathcal{V}, \mathcal{V}')$-$\text{Prof}$ that defines a pointwise left Kan extension along $J$ into $M$; we have to show that $m \circ T \eta$, as in the bottom of the left-hand side below, again defines a pointwise left Kan extension. Equivalently by Proposition 4.20 we may show that, for each $y = (y_1, \ldots, y_n) \in TB$, the composite of the bottom three cells below defines $(l y_1 \otimes \cdots \otimes l y_n)$ as a $T J(id, y)$-
weighted colimit of $m \circ Td$.

Next we will describe a cell $\chi$, as in the left hand side above, that is right pointwise cocartesian, so that by the vertical pasting lemma (Lemma 4.9), we may equivalently show that the whole composite on the left defines $(ly_1 \otimes \cdots \otimes ly_n)$ as a weighted colimit. At the sequences of objects $\underline{x}_0 \in A^\otimes n$, $\underline{x}_1 \in A^\otimes n', \ldots, x_n \in A$ the component of $\chi$ is given by

$$n' \prod_{i=1}^{n'} A(x_{0i}, x_{1i}) \otimes' J(x_{0i}, y_{ni}) \otimes' \prod_{i=1}^{n''} A(x_{1i}, x_{2i}) \otimes' J(x_{1i}, y_{ni}) \otimes' \cdots \otimes' J(x_{n'}, y_{1i})$$

$$\cong \prod_{i=1}^{n'} A(x_{0i}, x_{1i}) \otimes' \cdots \otimes' A(x_{(n-1)i}, x_{(n-1)i}) \otimes' J(x_{(n-1)i}, y_{1i})$$

where the isomorphism reorders the factors and the unlabelled map is given by the action of $A$ on $J$. Using Proposition 3.17 checking that $\chi$ is right pointwise cocartesian is straightforward.

Finally consider the right-hand side above, where the nullary cells $\delta: I_A \Rightarrow M$ are simply given by the action of $d$ on the hom-sets of $A$, and where each cell $\eta_i$ denotes the restriction of $\eta$ along $J(id, y_i)$. That the two sides coincide follows from the equivariance of the cells $\eta_i$, with respect to the actions of $A$. Hence, using the horizontal pasting lemma (Lemma 4.8), we conclude that the claim follows if each of the cells in the right-hand side defines a pointwise left Kan extension. That that is the case, finally, follows easily from Lemma 4.22 and the assumption that the tensor products of $M$ preserve $J(id, y_i)$-weighted colimits in each variable.

8 Lifting algebraic yoneda embeddings

Given a monad $T$ on a hypervirtual double category $\mathcal{K}$, the following theorem describes the lifting of yoneda embeddings along the forgetful functor $U: T{\text{-Alg}}[c,1] \to \mathcal{K}$. Stating and proving this result forms the main motivation of this paper. Recall that an (op-)lax $T$-algebra $A = (A, a, \tilde{a})$ is called normal if its unitor $\tilde{a}$ is invertible.

**Theorem 8.1.** Let $T = (T, \mu, i)$ be a monad on a hypervirtual double category $\mathcal{K}$. Consider an colax $T$-algebra $A = (A, a, \tilde{a})$ and a good yoneda embedding $y: A \to \hat{A}$ in $\mathcal{K}$. Assume that

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- the conjoint $a^*$ exists;
- the right pointwise composite $(a^* \odot T y_*)$ exists (Definition 3.16);
- the cell $\mu_J$ is pointwise left $(y \circ a)$-exact for each $J : A \rightarrow B$ (Definition 5.12);
- the cell $\mu_T y_*$ is pointwise left $(y \circ a \circ \mu_A)$-exact, while $\iota_{y_*}$ is pointwise left $(y \circ a)$-exact;
- $T$ preserves any unary cartesian cell with $y_*$ as horizontal target.

The morphism $v := (a^* \odot T y_*) : T \hat{A} \rightarrow \hat{A}$, that is given by the yoneda axiom (Definition 5.2), and that comes equipped with a cartesian cell

![Diagram](https://i.imgur.com/3Q5zQ.png)

extends to an colax $T$-algebra structure $(v, \bar{v}, \hat{v})$ on $\hat{A}$. With respect to this structure $y : A \rightarrow \hat{A}$ admits a pseudo $T$-morphism structure that makes it into a good yoneda embedding in $TAlg_{(c, 1)}$. The following hold:

(a) $\hat{A}$ is normal precisely if $A$ is;

(b) assuming that $\bar{a}$ is invertible, the associator $\hat{v}$ is invertible if and only if the $T$-image of the cocartesian cell defining $(a^* \odot T y_*)$ is left $(y \circ a)$-exact (e.g. if the cocartesian cell is preserved by $T$).

If the conditions above are satisfied, so that the lift of $y$ forms a pseudo $T$-morphism between pseudo $T$-algebras, then $y$ forms a good yoneda embedding in $TAlg_{(ps, lbc)}$ as well.

Finally, if the restrictions $H(id, a)$ exist in $K$, for all $H : C \rightarrow A$, then the lift of $y$ forms a good yoneda embedding in $TAlg_{(c, ps, lbc)}$ (resp. $TAlg_{(ps, ps, lbc)}$) too.

**Example 8.2.** Let $\mathcal{V} \rightarrow \mathcal{V}'$ be a symmetric universe enlargement, such that $\mathcal{V}$ has an initial object that is preserved by its tensor product $\otimes$ on both sides, and let $T = (T, \mu, i)$ be the ‘free strict monoidal $V'$-category’-monad on $(\mathcal{V}, \mathcal{V}')$-Prof, that was described in Example 6.2. Remember that $T$ is strong and preserves cocartesian cells, while it is easily checked to preserve all cartesian cells. That the cells $\mu_J$ and $\iota_J$, for each $\mathcal{V}$-profunctor $J : A \rightarrow B$, are pointwise left exact was briefly described in Example 7.12.

Thus given a monoidal $\mathcal{V}$-category $A = (A, \odot, a, i)$ (Example 6.5), whose yoneda embedding $y : A \rightarrow \hat{A}$ exists by Proposition 5.5, we may apply the theorem above as soon as the $\mathcal{V}'$-coends on the right below are $\mathcal{V}$-objects; e.g. in the case that $A$ is small and $\mathcal{V}$ is small cocomplete.

$$(p_1 \odot \cdots \odot p_n)(x) \cong \int_{u_1, \ldots, u_n \in A} A(x, (u_1 \odot \cdots \odot u_n)) \otimes' p_1 u_1 \otimes' \cdots \otimes' p_n u_n$$

In that case the monoidal structure on $A$ lifts to one on $\hat{A}$, which we will again denote by $\otimes : TA \rightarrow A$. Its defining cartesian cell, in the theorem above, ensures that its action on $\mathcal{V}$-presheaves $p_1, \ldots, p_n$ on $A$ is given by the coends above, as shown; that is $(p_1 \odot \cdots \odot p_n)$ is the *Day convolution* of the presheaves $p_1, \ldots, p_n$, as introduced (in a biased form) in Section 4 of [Day70].
With respect to the monoidal structure on $\hat{A}$ above, the yoneda embedding $y : A \to \hat{A}$ lifts to form a pseudomonoidal functor between $A$ and $\hat{A}$ such that it becomes a good yoneda embedding in the hypervirtual double categories $T$-$\text{Alg}(ps,1)$ and $T$-$\text{Alg}(ps,ps,inc)$, of monoidal $V$-profunctors and monoidal $V$-profunctors that satisfy the left Beck-Chevalley condition (see Example 6.8). In particular it satisfies the following monoidal Yoneda’s lemma: for any monoidal $V$-profunctor $J : A \Rightarrow B$ there exists a lax monoidal functor $J^\lambda : B \to \hat{A}$ such that $J \cong \hat{A}(y, J^\lambda)$ as monoidal $V$-profunctors. On objects $J^\lambda$ is given by $J^\lambda y := J(-, y)$, while is lax monoidal structure $J^\lambda y(z) \Rightarrow J^\lambda y(z')$ is given by $\eta_{y} : J^\lambda y(z) \Rightarrow J^\lambda y(z')$.

The proof consists of the following steps: complete the definition of the colax $T$-algebra structure on $\hat{A}$, define the pseudo $T$-morphism structure on $y : A \to \hat{A}$ and show that $(y, \hat{y})$ forms a good yoneda embedding in $T$-$\text{Alg}(c,1)$. It is useful to abbreviate by $\eta$ the composite on the left below, and by $\eta'$ its factorisation through the cartesian cell defining $y_*$, as shown.

$$
\eta := \begin{array}{ccc}
TA & \xrightarrow{\theta} & T\hat{A} \\
\downarrow a^* & & \downarrow \alpha \\
A & \xrightarrow{y} & \hat{A}
\end{array}
= \begin{array}{ccc}
TA & \xrightarrow{T\theta} & T\hat{A} \\
\downarrow a^* & & \downarrow \alpha \\
A & \xrightarrow{y_*} & \hat{A}
\end{array}
$$

We claim that $\eta$ defines $\nu$ as a pointwise left Kan extension. To see this first notice that, by Proposition 4.12, we may equivalently prove that its composition with the cartesian cell defining $a^*$ does which, by the conjoint identities (horizontally dual to those in Lemma 2.11), coincides with the composite of the bottom two rows on the left above. To see that the latter composite defines a pointwise left Kan extension remember that $y$ is dense (Lemma 5.1), so that the bottom cartesian cell defines a pointwise left Kan extension, followed by applying Corollary 4.10 to the cocartesian cell defining $(a^* \otimes T\hat{y}_*)$, which is right pointwise by assumption.

Step 1: the colax $T$-algebra structure on $\hat{A}$. Having obtained a structure morphism $\nu : T\hat{A} \to \hat{A}$, it remains to define associator and unitor cells $\hat{v} : v \circ \mu_{\hat{A}} \Rightarrow v \circ T\nu$ and $\tilde{v} : v \circ i_{\hat{A}} \Rightarrow \text{id}_{\hat{A}}$, and show that $(\hat{A}, v, \tilde{v})$ satisfies the usual coherence axioms for colax algebras; see (23) below. In order to define the associator $\hat{v}$ notice that the composite $\eta \circ \mu_{\hat{y}_*}$ in the right-hand side below defines $v \circ \mu_{\hat{A}}$ as a left Kan extension, since $\mu_{\hat{y}_*}$ is assumed pointwise left $(y \circ a)$-exact; we take $\hat{v}$ to be the unique

$$
\begin{array}{c}
\eta \circ \mu_{\hat{y}_*} : \eta \circ \mu_{\hat{y}_*} \\
\eta \circ \mu_{\hat{y}_*} \circ \mu_{\hat{y}_*} : \eta \circ \mu_{\hat{y}_*} \circ \mu_{\hat{y}_*} \\
\eta \circ \mu_{\hat{y}_*} \circ \mu_{\hat{y}_*} \circ \mu_{\hat{y}_*} : \eta \circ \mu_{\hat{y}_*} \circ \mu_{\hat{y}_*} \circ \mu_{\hat{y}_*} \\
\end{array}
$$

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factorisation of the left-hand side through this composite, as shown.

$$\begin{array}{c}
\begin{array}{ccc}
T^2A & T^2A & T^2\hat{A} \\
\mu_A & \mu_A & \mu_{\hat{A}} \\
T^2A & T^2A & T^2\hat{A} \\
\eta_y & \eta_y & \eta_{\hat{y}} \\
T^2\hat{A} & T^2\hat{A} & T^2\hat{A} \\
\end{array}
\end{array}
\right\} = 
\begin{array}{ccc}
T^2A & T^2A & T^2\hat{A} \\
\mu_A & \mu_A & \mu_{\hat{A}} \\
T^2A & T^2A & T^2\hat{A} \\
\eta_y & \eta_y & \eta_{\hat{y}} \\
\eta_{\hat{y}} & \eta_{\hat{y}} & \eta_{\hat{y}} \\
T^2\hat{A} & T^2\hat{A} & T^2\hat{A} \\
\end{array}
\right\}
$$

(21)

Analogously the composite $\eta \circ \iota_{y_\ast}$ in the right-hand side below defines $v \circ \iota_{\hat{y}}$ as a left-Kan extension, by the assumption that $\iota_{y_\ast}$ is pointwise left $(y \circ a)$-exact; we take the unitor $\bar{v}$ to be the factorisation as shown.

$$\begin{array}{c}
\begin{array}{ccc}
A & A & \hat{A} \\
\iota_A & \iota_{\hat{A}} & \iota_{\hat{A}} \\
A & A & \hat{A} \\
\eta_{y_\ast} & \eta_{y_\ast} & \eta_{\hat{y}} \\
A & A & \hat{A} \\
\end{array}
\end{array}
\right\} = 
\begin{array}{ccc}
A & A & \hat{A} \\
\iota_A & \iota_{\hat{A}} & \iota_{\hat{A}} \\
A & A & \hat{A} \\
\eta_{y_\ast} & \eta_{y_\ast} & \eta_{\hat{y}} \\
A & A & \hat{A} \\
\end{array}
\right\}
$$

(22)

Before proving the coherence axioms for $(v, \bar{v}, \bar{v})$ we consider the assertions (a) and (b) of the statement. To prove (a) first notice that $\bar{v}$ is invertible if and only if $y \circ \bar{a}$ is, where we use that $y$ is full and faithful (Lemma 5.4) so that $\text{id}_y$ is cartesian. Both the first column in the right-hand side above as well as the second column in the right-hand side define left Kan extensions; the latter because of the density of $y$ (Lemma 5.1). That $\bar{v}$ being invertible is implied by $y \circ \bar{a}$ being so now follows from uniqueness of left Kan extensions. Conversely, if $\bar{v}$ is invertible then both sides above define left Kan extensions so that, composing the left-hand side with the weakly cocartesian cell defining $y_\ast$, we find that $y \circ \bar{a}$ is invertible by using Proposition 4.13.

To prove (b) assume that $\tilde{a}$ is invertible. Composing both sides of (21) with $\tilde{a}^{-1}$ we see that $\bar{v}$ is invertible if and only if the composite $\eta \circ T\eta'$ in the left-hand side defines a left Kan extension. The defining identity (20) of $\eta'$ implies that we can rewrite $\eta' = \text{cart}' \circ \text{cocart} \circ (\text{cocart}, \text{id}_{T\eta_\ast})$, where $\text{cart}'$ is the factorisation $(a^* \circ T\eta_\ast) \Rightarrow y_\ast$, of the cartesian cell defining $v$ through $y_\ast$, which is again cartesian by the pasting lemma (Lemma 2.5), and where the cocartesian cells are those already in the composite $y$. Composing $\eta \circ T\eta'$ on the left with the $T$-image of the weakly cocartesian cell that defines $\iota_{y_\ast}$, we conclude that it defines a left Kan extension if and only if $\eta \circ T\text{cart}' \circ T\text{cocart}$ does; here we use Proposition 4.12 and the horizontal conjoint identity for $a^*$ (horizontally dual to that in Lemma 2.11). In the latter composite $\eta \circ T\text{cart}'$ defines the left Kan extension of $y \circ a$ along $(a^* \circ T\eta_\ast)$, because $\eta$ defines $v$ as a pointwise left Kan extension while $T$ preserves the cartesian cell $\text{cart}'$ by assumption. We conclude that $\eta \circ T\text{cart}' \circ T\text{cocart}$ defining a left Kan extension is the same as the $T$-image of the cocartesian cell, which defines $(a^* \circ T\eta_\ast)$, being left $(y \circ a)$-exact; since we have shown that the former is equivalent to $\bar{v}$ being invertible, the proof of (b) follows.

We return to proving the associativity and unit axioms for $(v, \bar{v}, \bar{v})$. Below they
are drawn schematically where, as always, empty cells denote identity cells.

\[
\begin{array}{c}
\mu_{y, s} \\
\eta
\end{array}
\begin{array}{c}
v \\
\bar{v}
\end{array}
\begin{array}{c}
Tv \\
\bar{v}
\end{array} =
\begin{array}{c}
v \\
\bar{v}
\end{array}
\begin{array}{c}
v \\
\bar{v}
\end{array}
\begin{array}{c}
Tv \\
\bar{v}
\end{array} = \text{id}_v \quad (23)
\]

To prove the associativity axiom, on the left above, consider the equation below, whose identities follow the associativity axiom for \( T \); the identity (21); the \( T \)-image of (21) after both of its sides have been factorised through the cartesian cell defining \( y_\ast \); the associativity axiom for \( A \); the identity (21) again; the naturality of \( \mu \); the identity (21) once more. Now notice that the left and right-hand sides below coincide with both sides of the associativity axiom for \( \hat{A} \) after composition on the left with \( \eta \circ \mu_y \circ \mu_{T y} \). By the exactness assumptions on \( \mu_y \) and \( \mu_{T y} \), the latter composite defines \( v \circ \mu_{\hat{A}} \circ \mu_{T \hat{A}} \) as a left Kan extension, so that the associativity axiom follows.

\[
\begin{array}{c}
\mu_{y, s} \\
\eta
\end{array}
\begin{array}{c}
v \\
\bar{v}
\end{array}
\begin{array}{c}
Tv \\
\bar{v}
\end{array} =
\begin{array}{c}
Tv \eta' \\
\eta
\end{array}
\begin{array}{c}
Tv' \\
\eta
\end{array} =
\begin{array}{c}
Tv \eta' \\
\eta
\end{array}
\begin{array}{c}
Tv' \\
\eta
\end{array} = \text{id}_v \quad (23)
\]

To prove the first unit axiom, in the middle of (23), consider the following equation, where ‘c’ denotes the cartesian cell defining \( y_\ast \), and whose identities follow from the identity (21); the naturality of \( \iota \); the first unit axiom for \( A \); the identity (20) and the unit axiom for \( T \). Notice that the left and right-hand sides below coincide with the first unit axiom for \( \hat{A} \) composed on the left with \( \eta \circ \mu_y \circ \iota_{Ty} = \eta \). Since the latter defines a left Kan extension the unit axiom follows.

\[
\begin{array}{c}
\iota_{y, s} \\
\eta
\end{array}
\begin{array}{c}
v \\
\bar{v}
\end{array}
\begin{array}{c}
Tv \\
\bar{v}
\end{array} =
\begin{array}{c}
Tv \eta' \\
\eta
\end{array}
\begin{array}{c}
Tv' \\
\eta
\end{array} =
\begin{array}{c}
Tv \eta' \\
\eta
\end{array}
\begin{array}{c}
Tv' \\
\eta
\end{array} = \text{id}_v \quad (23)
\]
This leaves the second unit axiom, on the right of (23). Analogous to the arguments above, it follows from the equation below, whose identities follow from the unit axiom for \(T\); the identity (21); the \(T\)-image of (22) after factorising it through the cartesian cell defining \(y_*\); the second unit axiom for \(A\). This completes the definition of the colax \(T\)-algebra \(\hat{A} = (\hat{A}, v, \overline{v}, \tilde{v})\).

\[
\begin{array}{c}
\eta \\
\downarrow \\
TA \\
\downarrow \\
y \\
\downarrow \\
A
\end{array}
\]

\[
\begin{array}{c}
\tilde{v} \\
\downarrow \\
\tilde{v}_y \\
\downarrow \\
v \\
\downarrow \\
y \\
\downarrow \\
A
\end{array}
\]

\[
\begin{array}{c}
\eta \\
\downarrow \\
A \\
\downarrow \\
\hat{A}
\end{array}
\]

\[
\begin{array}{c}
\eta \\
\downarrow \\
A \\
\downarrow \\
\hat{A}
\end{array}
\]

Step 2: the pseudo \(T\)-structure on \(y: A \to \hat{A}\). Consider the composite \(\gamma\) on the left below; we claim that it makes \(y: A \to \hat{A}\) into an colax \(T\)-morphism, that is it satisfies the associativity and unit axioms. Indeed: these follow directly from composing both sides of the identities (21) and (22) with the \((T^2\text{-image of})\) the cocartesian cell that defines \(y_*\), by using the naturality of \(\mu\) and \(\iota\) together with the fact that \(\eta' = (\text{cocart} \circ a) \odot \eta\) (for the latter notice that composing either side with the cartesian cell defining \(y_*\) results in \(\eta\)).

\[
\begin{array}{c}
\gamma := \\
\tilde{v} \\
\downarrow \\
\tilde{v}_y \\
\downarrow \\
v \\
\downarrow \\
y \\
\downarrow \\
A
\end{array}
\]

\[
\begin{array}{c}
\eta \\
\downarrow \\
A \\
\downarrow \\
\hat{A}
\end{array}
\]

It remains to show that \(\gamma\) is invertible. Since the cell \(\eta\) in the composite \(\gamma\), defines \(v\) as a pointwise left Kan extension it suffices to prove that \(Ty\) is full and faithful, by Proposition 4.13. To this end consider the the factorisation of \(id_T\) on the right above and notice that both cartesian cells here are preserved by \(T\): the bottom one by Corollary 2.12 and the one in the middle row by assumption. Because \(y\) is full and faithful (Lemma 5.4) the top cell \(id_y'\) is cocartesian by Corollary 3.15; it is preserved by \(T\) as well by Corollary 3.14. We conclude that \(id_{Ty}\) factors as a cocartesian cell through the restriction \(TA(y, y)\), so that \(Ty\) is full and faithful by applying Corollary 3.15 once more. We write \(\tilde{y} := \gamma^{-1}\), completing the definition of \((y, \tilde{y}): A \to \hat{A}\) as a pseudo \(T\)-morphism.

Step 3: \((y, \tilde{y}): A \to \hat{A}\) forms a good yoneda embedding in \(T\)-Alg\(_{(c, l)}\). To complete the proof we show that, with the \(T\)-algebra structures \((v, \tilde{v}, \tilde{v})\) and \(\tilde{y}\) on \(\hat{A}\) and \(y\) above, \((y, \tilde{y})\) defines a good yoneda embedding in \(T\)-Alg\(_{(c, l)}\). First notice that, because \((y, \tilde{y})\) is a pseudo \(T\)-morphism, for any lax \(T\)-morphism \(g: B \to \hat{A}\) the restriction \(\hat{A}(y, g)\) in \(T\)-Alg\(_{(c, l)}\) is created by the forgetful functor \(U: T\)-Alg\(_{(c, l)} \to \mathcal{K}\),

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by Proposition 7.2(b). Thus, once we have shown that \((y, \bar{y})\) is a yoneda embedding, it is in fact a good yoneda embedding, in the sense described after Definition 5.2.  
Now to show that \((y, \bar{y})\) satisfies the yoneda axiom (Definition 5.2), consider a horizontal \(T\)-morphism \(J : A \to B\). By the yoneda axiom for \(y\) in \(K\) there exists a cartesian cell as on the left below; we have make it into a cartesian cell in \(T-\text{Alg}_{c,l}\), by supplying a lax \(T\)-morphism structure for \(\mu_J : \hat{A} \to B\).

\[
\begin{array}{c}
A \xrightarrow{J} B \\
y \xrightarrow{\text{cart}} \bar{J} = \\
A 
\end{array}
\]

Writing \(\text{cart}'\) for the factorisation as shown above, remember that it is again cartesian by the pasting lemma, and that it is preserved by \(T\) by assumption. Because \(\eta\) defines a pointwise left Kan extension, it follows that the composite \(\eta \circ \text{cart}'\) in the right-hand side below defines \(v \circ T \bar{J}\) as a left Kan extension; consequently the left-hand side, where \(J\) denotes the structure cell of \(J\), factors uniquely as a vertical cell \(\bar{J}\) as shown.

\[
\begin{array}{c}
TA \xrightarrow{TJ} TB \\
A \xrightarrow{J} B \\
y \xrightarrow{\text{cart}/\bar{J}} A \\
\end{array}
\]

\[
\begin{array}{c}
TA \xrightarrow{TJ} TB \\
A \xrightarrow{T\gamma} \hat{A} \\
y \xrightarrow{\text{cart}/\bar{J}} A \\
\end{array}
\]

We claim that \(\bar{J}\) makes \(J\) into a lax \(T\)-morphism and that, with this structure, the cartesian cell defining \(J\) lifts to form a cartesian \(T\)-cell in \(T-\text{Alg}_{c,l}\). In view of the proof of Proposition 7.2 the latter follows immediately from the fact that the first column of the right-hand side above coincides with \(\gamma \circ (v \circ T \text{cart})\), which in turn follows from the definition of \(\gamma\), substituting (25), and the horizontal companion identity for \(y\). This leaves proving that \((J, \bar{J})\) forms a lax \(T\)-morphism; that is it satisfies the associativity and unit axioms.

That the two sides of the associativity axiom coincide after composition on the left with the composite \(\eta \circ \mu_J \circ T^2 \text{cart}'\) is shown below, where the identities follow from the identity (21); the \(T\)-image of the identity above factorised through the cartesian cell defining \(y\); the identity above; the associativity axiom for \(J\) (Definition 6.7); the identity above; the naturality of \(\mu\). The associativity axiom itself now follows from the fact that \(\eta \circ \mu_J \circ T^2 \text{cart}' = \eta \circ T \text{cart}' \circ \mu_J\) defines a left
Kan extension, as $\mu_J$ is assumed to be left $(y \circ a)$-exact.

The unit axiom for $\bar{J}$ follows likewise from the following equation, where $\eta \circ \iota_y \circ \text{cart}'$ defines a left Kan extension by the assumption on $\iota_y$, and whose identities follow from the naturality $\iota$; the identities (26); the unit axiom for $\bar{J}$; the identity (22).

To finish the proof we will show that $(y, \bar{y}) : A \rightarrow \hat{A}$ is dense in $T\text{-Alg}(c, l)$, see Lemma 5.1. Hence consider a cartesian $T$-cell as on the left below; we have to show that it defines the lax $T$-morphism $l : B \rightarrow \hat{A}$ as a left Kan extension of $y$ along $J$. As before it will be useful to consider the factorisation $\zeta'$ of $\zeta$ through the cartesian cell defining $y_*$, as shown.

To start we notice that $\zeta$ is preserved by the forgetful functor $U : T\text{-Alg}_{(c, 1)} \rightarrow K$, so that it is cartesian in $K$. This follows immediately from the fact that $U$ creates the restriction $\hat{A}(y, l)$ in $T\text{-Alg}_{(c, 1)}$, as we have already seen, so that the existence of an invertible horizontal $T$-cell $J \cong \hat{A}(y, l)$ follows. Of course this isomorphism is preserved by the forgetful functor $U$, and we conclude that $\zeta$ is cartesian in $K$.

Thus any $T$-cell $\phi$ as on the left below factors uniquely through $\zeta$ in $K$ as shown, and we have to prove that the factorisation $\phi'$ forms a $T$-cell.

That it does follows from the equation below, whose identities follow from the identity $\gamma \circ (\psi \circ T\zeta) = \eta \circ T\zeta'$ (a consequence of the definitions of $\gamma$ and $\zeta'$, as well
as the horizontal companion identity for \( \gamma \); the \( T \)-cell axiom for \( \zeta \); the factorisation above; the \( T \)-cell axiom for \( \phi \); the \( T \)-image of the factorisation above; the identity \( \gamma \circ (v \circ T \zeta) = \eta \circ T \zeta' \) again.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T \zeta' \\
\eta
\end{array}
\begin{array}{c}
\begin{array}{c}
R_1 \\
\phi'
\end{array}
\begin{array}{c}
\begin{array}{c}
R_n
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T \zeta \\
\phi
\end{array}
\begin{array}{c}
\begin{array}{c}
R_1 \\
\phi'
\end{array}
\begin{array}{c}
\begin{array}{c}
R_n
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
T \phi' \\
\eta
\end{array}
\begin{array}{c}
\begin{array}{c}
T \phi \\
k
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Finally, notice that \( \zeta' \) is cartesian in \( K \) because \( \zeta \) is, by the pasting lemma; hence \( T \) preserves \( \zeta' \) by assumption. Because \( \eta \) defines \( v \) as a pointwise left Kan extension it follows that the first column \( \eta \circ T \zeta' \), in the left-hand and right-hand sides above, defines \( v \circ Tl \) as a pointwise left Kan extension. Since these sides coincide with \( T \)-cell axiom for \( \phi' \) after it has been composed with \( \eta \circ T \zeta' \) on the left, we conclude that the \( T \)-cell axiom itself holds.

This completes the proof of the main assertion of the theorem. Next we remark on the case in which \( A \) and \( \hat{A} \) are pseudo \( T \)-algebras. Since the inclusion \( T-\text{Alg}(\text{ps},1) \rightarrow T-\text{Alg}(c,1) \) is locally full (see Definition 1.21) it creates cartesian cells; it follows immediately that \( y \), as a pseudo \( T \)-morphism, also satisfies the yoneda axiom in \( T-\text{Alg}(\text{ps},1) \). Since the inclusion reflects cells defining left Kan extensions as well the argument above, that shows that \( y \) is dense in \( T-\text{Alg}(c,1) \), can be applied to show that \( y \) is dense in \( T-\text{Alg}(\text{ps},1) \) as well; we conclude that therein it forms a yoneda embedding too.

Finally assume that the restrictions \( H(id,a) \) exist in \( K \), for any \( H : C \Rightarrow A \); we will show that the pseudo \( T \)-morphism \( (y,\hat{y}) : A \rightarrow \hat{A} \) forms a good yoneda embedding in \( T-\text{Alg}(c,1) \). To see that it satisfies the yoneda axiom in \( T-\text{Alg}(c,1) \) consider a horizontal \( T \)-morphism that satisfies the left Beck-Chevalley condition.

By Proposition 5.15 it follows that the structure cell \( J \) is pointwise left exact so that the lax \( T \)-structure cell \( J^\wedge \) on \( J^\wedge : B \rightarrow \hat{A} \), that is obtained in (26), is invertible. Since the inclusion \( T-\text{Alg}(c,ps,\text{lbc}) \rightarrow T-\text{Alg}(c,1) \) is locally full it follows that the cartesian \( T \)-cell in \( T-\text{Alg}(c,1) \), that comes with \( (J^\wedge, J^\wedge) \), is cartesian in \( T-\text{Alg}(c,ps,\text{lbc}) \) as well.

Next, to prove that \( (y,\hat{y}) \) is dense in \( T-\text{Alg}(c,ps,\text{lbc}) \), again by the locally fullness of the latter in \( T-\text{Alg}(c,1) \) it suffices to prove that any cartesian cell \( \zeta \) in \( T-\text{Alg}(c,ps,\text{lbc}) \), of the form as on the left-hand side in (27), is cartesian in \( T-\text{Alg}(c,1) \). The latter in turn follows from showing that, for any pseudo \( T \)-morphism \( g : B \rightarrow \hat{A} \), the restriction \( \hat{A}(y,g) \) in \( T-\text{Alg}(c,1) \), that is created by \( U : T-\text{Alg}(c,1) \rightarrow K \), satisfies the left Beck-Chevalley condition, so that it is cartesian in \( T-\text{Alg}(c,ps,\text{lbc}) \). To see this remember from the proof of Proposition 7.2 that the created \( T \)-structure \( \hat{A}(y,g) \) on \( \hat{A}(y,g) \) is the unique factorisation of the composite on the left-hand side below.
through the cartesian cell that defines $\hat{A}(y, g)$, where $\gamma = \bar{y}^{-1}$ as before.

Now notice that, by using Corollary 3.24, all assumptions for the equivalence (a) $\Leftrightarrow$ (c) of Proposition 5.15 are satisfied. Thus, it suffices to prove that $\hat{A}(y, g)$ is pointwise left $y$-exact. Now remember that the cartesian cell defining $\hat{A}(y, g)$ also defines $g$ as the pointwise left Kan extension of $y$ along $\hat{A}(y, g)$, by density of $y$ (Lemma 5.1); it follows that we have to show that the composite on the left-above defines $g \circ b$ as a pointwise left Kan extension. To see this, finally, consider the full equation above, whose first identity follows from the definition of $\gamma$, as on the left of (24), while the second one follows from composing the cocartesian and cartesian cell in the middle composite to form the cartesian cell $\text{cart}'$ in the right-hand side, which is the factorisation through $y_*$ of the cartesian cell that defines $\hat{A}(y, g)$; it is again cartesian by the pasting lemma. At the beginning of this proof we saw that $\eta$ defines $v$ as a pointwise left Kan extension; together with the assumption that $T$ preserves the cartesian cell $\text{cart}'$, and the fact that $\bar{g}$ is invertible, we conclude that the right-hand side above, and thus all composites above, define $g \circ b$ as a pointwise left Kan extension. This completes the proof. $\square$

In the following proposition we consider the uniqueness of the colax structures $(v, \bar{v}, \tilde{v})$ on presheaf objects $\hat{A}$, as obtained in the previous theorem.

**Proposition 8.3.** When regarded as an colax $T$-morphism the lifted yoneda embedding $(y, \gamma): (A, a, \bar{a}, \tilde{a}) \to (\hat{A}, v, \bar{v}, \tilde{v})$, obtained in Theorem 8.1, is unique as follows. Any colax $T$-morphism $(y, \phi): (A, a, \bar{a}, \tilde{a}) \to (\hat{A}, w, \bar{w}, \tilde{w})$ between colax $T$-algebras factors uniquely through $(y, \gamma)$ as an colax $T$-morphism $(\text{id}, \phi'): \hat{A} \to \hat{A}$. If $(y, \phi)$ is a pseudo $T$-morphism then $(\text{id}, \phi')$ is too precisely if the composite below defines $w$ as a pointwise left Kan extension of $w \circ T y$ along $T y_*$.

$$
\begin{array}{ccc}
TA & \xrightarrow{T y} & T \hat{A} \\
\downarrow^{T y} & \text{cart} & \downarrow^{T y} \\
\hat{A} & \xrightarrow{w} & A
\end{array}
$$

Proof. The invertible colax structure cell $\gamma: y \circ a \Rightarrow v \circ T y$ for $y$ was defined in (24) in terms of the composite cell $\eta$, as in the right-hand side below, which in turn was defined in (20). In the discussion that followed $\eta$ was shown to define a pointwise left Kan extension; we take the colax structure cell $\phi'$ on $\text{id}_{\hat{A}}$ to be the
unique factorisation through $\eta$ as shown below.

\[
\begin{array}{ccc}
TA & T\hat{A} \\
\downarrow{\gamma} & \downarrow{\hat{\gamma}} \\
\hat{A} & A
\end{array}
\]

We claim that $\phi'$ makes $\text{id}_{\hat{A}}$ into an colax $T$-morphism $(\hat{A}, v, \bar{v}, \bar{w}) \to (A, w, \bar{w}, \bar{w})$; notice that, by precomposing both sides above with the $T$-image of the weakly cocartesian cell defining $y_*$, it follows from the definition (24) of $\gamma$ that $\phi'$ is unique such that $(\text{id}, \phi') \circ (y, \gamma) = (y, \phi)$. Also notice that the final assertion can be read off: if $\phi$ is invertible then, as follows from the identity obtained by composing both sides above with $\phi^{-1}$ on the left together with uniqueness of Kan extensions, $\phi'$ is invertible precisely if $w \circ T\text{cart}$ defines $w$ as a pointwise left Kan extension.

We return to the claim that $\phi'$ forms an colax $T$-structure cell for $\text{id}_{\hat{A}}$; that is it satisfies the associativity and unit axioms. The former follows from the equation below, where $\eta \circ \mu_y^* \eta$ defines a left Kan extension because $\eta$ does and the left exactness assumption on $\mu_y$, in Theorem 8.1, and where the identities follow from the definition (21) of $\bar{v}$; the identity above and the definition (20) of $\eta'$; the $T$-image of the identity above; the associativity axiom for $\phi'$; the naturality of $\mu$; the identity above.

\[
\begin{array}{ccc}
\mu_y & T\phi' \\
\eta & \phi' \\
\end{array} =
\begin{array}{ccc}
T\phi' & T\phi' \\
\eta & \phi' \\
\end{array} =
\begin{array}{ccc}
T\eta & T\phi' \\
\phi & \phi \\
\end{array} =
\begin{array}{ccc}
T\phi & T\phi \\
\phi & \phi \\
\end{array}
\]

Similarly the unit axiom follows from the following equation, where $\eta \circ \iota_y^*$ defines a left Kan extension by the exactness assumption on $\iota_y$, and where the identities follow from the identity defining $\phi'$ above; the naturality of $\iota$; the unit axiom for $\phi'$; the definition (22) of $\bar{v}$.

\[
\begin{array}{ccc}
\iota_y & \bar{w} \\
\eta & \bar{w} \\
\end{array} =
\begin{array}{ccc}
\iota_y & \bar{w} \\
\phi & \bar{w} \\
\end{array} =
\begin{array}{ccc}
\iota_y & \bar{w} \\
\phi & \bar{w} \\
\end{array} =
\begin{array}{ccc}
\iota_y & \bar{v} \\
\eta & \bar{v} \\
\end{array}
\]

This completes the proof.

By combining Theorem 5.19, Proposition 7.11 and the previous proposition, the theorem below describes the sense in which a lifted algebraic yoneda embedding, as obtained in Theorem 8.1, defines a free cocompletion.

**Theorem 8.4.** Let $T = (T, \mu, \iota)$ be a monad on a hypervirtual equipment $K$ such that $T$ preserves unary restrictions. Given an colax $T$-algebra $M = (M, m, \bar{m}, \tilde{m})$ and a good yoneda embedding $y : M \to \hat{M}$ in $K$, assume that the hypotheses of Theorem 8.1 are satisfied so that $y$ lifts to a yoneda embedding $(y, \hat{y}) : (M, m, \bar{m}, \tilde{m}) \to (\hat{M}, v, \bar{v}, \bar{v})$ in $T\text{-Alg}_c$. Let $S$ be an ideal of left extension diagrams in $K$ such that
(c) for each \((d, J) \in \mathcal{S}(\overline{M})\) there exists a pointwise left \(\gamma\)-exact cell

\[
\begin{array}{ccc}
M \xrightarrow{\phi} A & \xrightarrow{J} & B \\
\downarrow & & \\
\overline{M} & \overset{\gamma}{\rightarrow} & B
\end{array}
\]

whose \(T\)-image is pointwise left \((\gamma \circ \text{om})\)-exact;

(m) for each \((d, J) \in \mathcal{S}(\overline{M})\) both cells \(\mu_J\) and \(\nu_J\) are pointwise left \((m \circ Td)\)-exact;

(y) \((f, \gamma) \in \mathcal{S}\) for all \(f: M \to N\).

The yoneda embedding \((y, \hat{\mathcal{S}})\) defines \(\overline{M}\) as the free \(\mathcal{S}_{(c,1)}\)-cocompletion of \(M\) in \(T\)-\(\mathcal{Alg}_{(c,1)}\) while, if it lifts to a yoneda embedding in \(T\)-\(\mathcal{Alg}_{(c, \text{ps}, \text{lbc})}\) as well, then it defines \(\overline{M}\) as the free \(\mathcal{S}_{(c, \text{ps}, \text{lbc})}\)-cocompletion of \(M\) there too. Moreover \(\overline{M}\) is \(\mathcal{S}\)-cocomplete in \(\mathcal{K}\) and, for each \((d, J) \in \mathcal{S}_{(c,1)}(\overline{M})\) (resp. \(\mathcal{S}_{(c, \text{ps}, \text{lbc})}(\overline{M})\)) the pointwise left Kan extension of \(d\) along \(J\) is created by \(U: T\)-\(\mathcal{Alg}_{(c,1)} \to \mathcal{K}\) (resp. \(U: T\)-\(\mathcal{Alg}_{(c, \text{ps}, \text{lbc})} \to \mathcal{K}\)).

Finally let \(N\) be an colax \(T\)-algebra that satisfies the hypotheses of Proposition \(7.11\), so that its \(\mathcal{S}_{(c,1)}\)-cocompleteness is lifted along \(U: T\)-\(\mathcal{Alg}_{(c,1)} \to \mathcal{K}\). A lax \(T\)-morphism \((f, \hat{f})\): \(\overline{M} \to N\) is \(\mathcal{S}_{(c,1)}\)-cocontinuous if and only if its underlying morphism \(f\) is \(\mathcal{S}\)-cocontinuous. The analogous result for \(\mathcal{S}_{(c, \text{ps}, \text{lbc})}\)-cocontinuity of pseudo \(T\)-morphisms \(\overline{M} \to N\) in \(T\)-\(\mathcal{Alg}_{(c, \text{ps}, \text{lbc})}\) holds as well.

**Proof.** We first prove the final assertion. The assumptions (e) and (y) above ensure that we may apply Theorem \(5.19\) to find that \(\overline{M}\) is \(\mathcal{S}\)-cocomplete in \(\mathcal{K}\). To prove that the forgetful functors \(U: T\)-\(\mathcal{Alg}_{(c,1)} \to \mathcal{K}\) and \(U: T\)-\(\mathcal{Alg}_{(c, \text{ps}, \text{lbc})} \to \mathcal{K}\) create all algebraic pointwise left Kan extensions of \(T\)-morphisms \(d\) along a horizontal \(T\)-morphisms \(J\), with \((d, J)\) ranging over \(\mathcal{S}_{(c,1)}(\overline{M})\) (resp. \(\mathcal{S}_{(c, \text{ps}, \text{lbc})}(\overline{M})\)), it suffices to show that \(\overline{M} = (\overline{M}, \nu, \hat{\nu}, \hat{\nu})\) satisfies the hypotheses of Proposition \(7.11\)(a). Besides \(\mathcal{S}\)-cocompleteness of \(\overline{M}\) and assumption (m) above, the hypothesis that remains to be checked asks that the algebraic structure of \(\overline{M}\) preserves the pointwise left Kan extension of \(d\) along \(J\), for each \((d, J) \in \mathcal{S}(\overline{M})\). From the proof of Proposition \(5.16\) we know how to construct this Kan extension: it is defined by the cell \(\gamma\) that is the factorisation defined in \(16\), and we have to show that \(\nu \circ T\gamma\) again defines a pointwise left Kan extension.

\[
\begin{array}{ccc}
M \xrightarrow{\overline{M}(y, d)} A & \xrightarrow{J} & B \\
\downarrow & & \\
\overline{M} & \overset{\gamma}{\rightarrow} & B
\end{array}
\]

\[
\begin{array}{ccc}
TM \xrightarrow{T\overline{M}(y, d)} TA & \xrightarrow{TJ} & TB \\
\downarrow T\gamma & & \\
TM & \overset{T\gamma}{\rightarrow} & TB
\end{array}
\]

(28)

To see this we first factor either side of \(16\) through \(y_\ast\), as shown on the left above. Here \(\phi\) is the pointwise left \(\gamma\)-exact cell whose existence is assumed by condition (e) above, while \(\text{cart}'\) and \(\text{cart}''\) are the factorisations of the cartesian cells in \(16\) through \(y_\ast\); they are again cartesian by the pasting lemma. Notice that

\[
\begin{array}{ccc}
M \xrightarrow{y} \overline{M} \\
\downarrow & & \\
\overline{M} & \overset{\gamma}{\rightarrow} & B
\end{array}
\]
$T$ cart’ is again cartesian by the hypotheses of Theorem 8.1: so that it is pointwise left $(y \circ m)$-exact by Lemma 4.6(b). It follows that the $T$-image of the composite $T \circ \phi$ in the left-hand side is pointwise left $(v \circ T \gamma)$-exact, by assumption (c) and the fact that $y \circ m \cong v \circ T \gamma$: a consequence of $(y, \bar{y})$ being a pseudo $T$-morphism.

Now consider the composite on the right above, whose top row is the $T$-image of the right-hand side of the identity on the left. Applying the previous proposition to the identity on $(\hat{M}, v, \bar{v}, \bar{\bar{v}})$ we find that the composite $v \circ T$ cart here defines $v$ as a pointwise left Kan extension. As we have seen the $T$-image of both factorisations through $y_*$, on either side of the identity on the left, is pointwise left $(v \circ T \gamma)$-exact, so that it follows that the full composite on the right defines $v \circ T I$ as a pointwise left Kan extension. Finally, since $T$ preserves the cartesian cell cart’, its first column defines $v \circ T d$ as a pointwise left Kan extension, so that by the horizontal pasting lemma (Lemma 4.8) we may conclude that its second column, that is $v \circ T \eta$, defines a $v \circ T I$ as a pointwise left Kan extension, as required. This completes the proof of the final assertion of the theorem.

To prove the first assertion, that is $(y, \bar{y})$ defines $\bar{M} = (\hat{M}, v, \bar{v}, \bar{\bar{v}})$ as the free $S_{(c, 1), c}$-cocompletion of $M$ in $T$-$\text{Alg}_{(c, 1)}$, it suffices to show that we can apply Theorem 5.19 to the lifted Yoneda embedding $(y, \bar{y})$ in $T$-$\text{Alg}_{(c, 1)}$. That the second hypothesis of the latter theorem is satisfied, that is $((f, f), (y, \bar{y})) \in S_{(c, 1)}$ for each lax $T$-morphism $(f, f) : M \to N$, follows immediately from the assumption $(y)$ above. It remains to prove its first hypothesis, which asks for a pointwise left $(y, \bar{y})$-exact horizontal $T$-cell $(\bar{M} \gamma, d, J) \Rightarrow K$ to exist for each $((d, d), (J, J)) \in S_{(c, 1)}(\bar{M})$. We will show that the horizontal cell $\phi$, that is supplied by the assumption (c) above, lifts to form such a $T$-cell.

To see this first notice that $((d, d), (J, J)) \in S_{(c, 1)}(\bar{M})$ implies $(d, d) \in S(\bar{M})$, so that the pointwise left Kan extension $l : B \to \bar{M}$ of $d$ along $J$ exists in $K$: indeed, it is defined by the cell $\eta$ in the left-hand side of the identity on the left above. We already know that the pointwise left Kan extension of $(d, d)$ along $(J, J)$ is created by $U : T$-$\text{Alg}_{(c, 1)} \to K$, that is there exists a lax $T$-structure $\bar{l}$ on $l$ that is unique in making $\eta$ a $T$-cell that defines $l$ as this Kan extension. Likewise the nullary restriction $K$ of $\gamma$ along $l$ in $T$-$\text{Alg}_{(c, 1)}$, as in the left-hand side of the identity above, is created by $U$; see Proposition 7.2. We claim that the thus created horizontal $T$-structure cell $K$ on $K$ makes $\phi$ into a $T$-cell. Indeed the equation below shows that its $T$-cell axiom holds after composition with the cartesian cell that defines $K$, where ‘$c$’ has been used to denote both the cartesian cell defining $K$ and that defining $\bar{M} \gamma, d$, and where $\gamma := y^{-1}$. The identities follow from the definition of $K$ (see the proof of Proposition 7.2); the $T$-image of the identity on the left of (28); the $T$-cell axiom for $\eta$; the definition of $\bar{M}(y, d)$ (again see Proposition 7.2); the identity on the left of (28) itself.

\[
\begin{array}{ccc}
T\phi \\
\bar{K} \\
c \\
\gamma & T\phi &=& T\gamma \\
Tc & i &=& \bar{M}(y, d) \\
J & \eta &=& \gamma \\
\end{array}
\]

Thus a $T$-cell, it remains to prove that $\phi$ is pointwise left $(y, \bar{y})$-exact in $T$-$\text{Alg}_{(c, 1)}$. To see this we return to the identity on the left of (28) once more, now regarding both its sides as compositions of $T$-cells. By the density of the lifted Yoneda embed-
ding \((y, \bar{y})\) (Lemma 5.1) both composites of cartesian \(T\)-cells, in either side, define pointwise left Kan extensions. Because \(\eta\) was reflected as a \(T\)-cell that defines \((l, \bar{l})\) as a pointwise left Kan extension, it follows from the horizontal pasting lemma that the full left-hand side does so too. From this we conclude that \(\phi\), now as a \(T\)-cell, is pointwise left \((y, \bar{y})\)-exact. This concludes the proof of the first assertion.

Next, if the lift of \(y\) also forms a good yoneda embedding in \(T\-\text{Alg}_{(c, ps, lbc)}\), then the \(T\)-cell \(\phi: (\hat{M}(y, d), J) \Rightarrow K\) considered above is in fact a \(T\)-cell between horizontal \(T\)-morphisms satisfying the left Beck-Chevalley condition, provided that \(((d, d), (J, J)) \in S_{(c, ps, lbc)}\). Indeed, in that case both \(\hat{M}(y, d)\) and \(K\) are nullary restrictions of \(\hat{M}\) along \((y, \bar{y})\) and a pseudo \(T\)-morphism \((d \text{ and } l)\) respectively, and such restrictions satisfy the the left Beck-Chevalley condition in \(T\-\text{Alg}_{(c, ps, lbc)}\); indeed \(\eta\) does because \((y, \bar{y})\) forms a good yoneda embedding, so that \(\hat{M}(y, d)\) and \(K\) do by Proposition 7.2(d). We conclude that in this case too the hypotheses of Theorem 5.19 are satisfied, so that the lift of \(y\) defines \(\hat{M}\) as the free \(S_{(c, ps, lbc)}\)-cocompletion of \(M\) in \(T\-\text{Alg}_{(c, ps, lbc)}\) as well.

To complete the proof consider a lax \(T\)-morphism \((f, \bar{f}): \hat{M} \rightarrow N\) where \(N\) is an colax \(T\)-algebra satisfying the hypotheses of Proposition 7.11 so that its \(S_{(c, l)}\)-cocompleteness lifts along \(U: T\-\text{Alg}_{(c, l)} \rightarrow K\), while \((f, \bar{f})\) is \(S_{(c, l)}\)-cocontinuous whenever \(f\) is \(S\)-cocontinuous. Conversely assume that \((f, \bar{f})\) is \(S_{(c, l)}\)-cocontinuous. Using the density of \(y\) in \(T\-\text{Alg}_{(c, l)}\) (Lemma 5.1) it follows that the composite of \(f\) and the cartesian \(T\)-cell defining \(y\) defines \(f\) as the pointwise left Kan extension of \(f \circ y\) along \(y\). Because \(N\) is assumed to be \(S\)-cocomplete in \(K\), the latter pointwise left Kan extension is created, and hence preserved, by \(U: T\-\text{Alg}_{(c, l)} \rightarrow K\); it follows that \(f\) forms a pointwise left Kan extension along \(y\) in \(K\) too, so that it is \(S\)-cocontinuous by Proposition 5.22. That the same argument applies in the case of \(S_{(c, ps, lbc)}\)-cocontinuity too is clear. This concludes the proof.

**Example 8.5.** Let \(\mathcal{V} \rightarrow \mathcal{V}'\) be a symmetric universe enlargement such that \(\mathcal{V}\) has an initial object preserved by \(\otimes\) on both sides, let \(T\) be the ‘free strict monoidal \(\mathcal{V}'\)-category’-monad on \((\mathcal{V}, \mathcal{V}')\)-Prof (Example 6.2), and let \(S\) be the ideal of left extension diagrams in \((\mathcal{V}, \mathcal{V}')\)-Prof described in Example 5.20, consisting of pairs \((d, J)\) where \(J: A \Rightarrow B\) is a \(\mathcal{V}\)-profunctor between \(\mathcal{V}\)-categories with \(A\) small. It follows from the discussion at the beginning of Example 8.2 that the hypotheses of the previous theorem are satisfied so that, for any monoidal \(\mathcal{V}\)-category \(M = (M, \otimes, a, i)\), the presheaf \(\mathcal{V}\)-category \(\hat{M}\) equipped with Day convolution (Example 8.2) forms both the free \(S_{(c, l)}\)-cocompletion of \(M\) in \(T\-\text{Alg}_{(c, l)}\) as well as the free \(S_{(c, ps, lbc)}\)-cocompletion of \(M\) in \(T\-\text{Alg}_{(c, ps, lbc)}\). In particular for any small \(\mathcal{V}\)-cocomplete monoidal \(\mathcal{V}\)-category \(N\), whose tensor product \(\otimes\) preserves small \(\mathcal{V}\)-weighted colimits in each variable (see Example 7.13), we obtain an equivalence of categories of \((\mathcal{V}\)-small cocomplete) lax monoidal functors

\[
\text{MonCat}_{(l, \text{cocts})}(\hat{M}, N) \simeq \text{MonCat}_{l}(M, N)
\]

that is given by precomposition with \(y: M \rightarrow \hat{M}\) and that restricts to an equivalence of categories of monoidal functors. This recovers Theorem 5.1 of [IK86].

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