Massive gravitons in arbitrary spacetimes

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We present two different versions of the consistent theory of massive gravitons in arbitrary spacetimes which are simple enough for practical applications. The theory is described by a non-symmetric rank-2 tensor whose equations of motion imply six algebraic and five differential constraints reducing the number of independent components to five. The theory reproduces the standard description of massive gravitons in Einstein spaces. In generic spacetimes it does not show the massless limit and always propagates five degrees of freedom, even for the vanishing mass parameter. We illustrate these features by an explicit calculation for a homogeneous and isotropic cosmological background. We find that the gravitons are stable if they are sufficiently massive, hence they may be a part of Dark Matter at present. We discuss also other possible applications.

I. INTRODUCTION

Equations of massive fields of spin 0, 1/2, 1, 3/2 in Minkowski space (the Klein-Gordon, Dirac, Proca, Rarita-Schwinger) directly generalize to curved space, but for the massive spin 2 field this does not work. The Fierz-Pauli (FP) theory of free massive gravitons [1] generalizes to curved space only for special spacetimes – Einstein spaces, whose Ricci tensor is proportional to the metric, $R_{\mu\nu} = \Lambda g_{\mu\nu}$ [2, 3]. In an arbitrary spacetime the theory shows six instead of five dynamical graviton polarizations, the extra polarization state being ghost-type. This feature was for a long time thought to be inevitable [4], hence all applications of the consistent massive spin-2 theory have been limited only to Einstein spaces. Quite recently, by applying the methods of the ghost-free massive gravity theory [5], it was shown that a consistent theory of massive gravitons can nevertheless be formulated for arbitrary backgrounds [6]. However, the graviton mass term obtained in [6] is very complicated and even the very demonstration of the existence of the constraint removing the sixth polarization requires tedious calculations.

In what follows we present two different versions of the consistent theory of massive gravitons in arbitrary spacetimes which are simple enough for practical applications. The essential property of this theory is that it propagates only five and not six degrees of freedom (DoF) and hence does not show the extra polarization state. In this sense the theory is ghost-free, which property is exceptional. As explained above, the existence of such a theory for arbitrary backgrounds was for a long time thought to be impossible. Our theory is probably equivalent to that considered in [6] since it is constructed in a similar way, but the equivalence is not immediately seen because our parametrization is quite different. Contrary to what is usually done, we describe the massive spin-2 field by a non-symmetric rank-2 tensor. Although this may seem odd, our parameterization gives simple results, which opens up the possibility to efficiently study massive gravitons in curved space.

We deliberately do not present details of our derivation – they are rather technical and will be given in a separate publication. Instead, we pass directly to the result: the theory described by equations (1)–(4) below. It is easy to check that these equations propagate only 5 DoF in an arbitrary spacetime, which is the key property. This property will be demonstrated in what follows by explicit calculations – by counting the constraints for a generic background and also by explicitly solving the equations for a cosmological background and counting the number of independent solutions. In the latter case we show that the system is free of ghosts and tachyons, hence massive gravitons are stable and could contribute to the Dark Matter. We therefore have a valid new theory in our disposal and we shall indicate some of its possible applications.

II. FIELD EQUATIONS

Our theory of massive spin-2 field is parameterized by a second rank tensor $X_{\mu\nu}$ which is apriori non-symmetric and has 16 independent components (one denotes $X = X^\sigma_{\sigma\nu}$). This field propagates in a spacetime with the standard (symmetric) Lorentzian metric $g_{\mu\nu}$ whose Christoffel connection determines the covariant derivative $\nabla_{\mu}$.
The field equations are
\[ E_{\mu\nu} \equiv \Delta_{\mu\nu} + M_{\mu\nu} = 0 \] (1)
with
\[
\Delta_{\mu\nu} = \frac{1}{2} \nabla^\sigma \nabla_\mu (X_{\sigma\nu} + X_{\nu\sigma}) + \frac{1}{2} \nabla^\sigma \nabla_\nu (X_{\sigma\mu} + X_{\mu\sigma}) \\
- \frac{1}{2} \square (X_{\mu\nu} + X_{\nu\mu}) - \nabla_\mu \nabla_\nu X \\
+ g_{\mu\nu} \left( \frac{1}{2} \mathcal{L}_{\gamma} X - \nabla^\alpha \nabla^\beta X_{\alpha\beta} + R^{\alpha\beta} X_{\alpha\beta} \right) \\
- R_{\sigma\nu} X_{\sigma\mu} - R_{\sigma\mu} X_{\sigma\nu}.
\] (2)
and we shall consider two different options for the mass term:

model I: \[ M_{\mu\nu} = \gamma_{\mu\alpha} X_{\nu}^{\alpha} - g_{\mu\nu} \gamma_{\alpha\beta} X^{\alpha\beta}, \]
\[ \gamma_{\mu\nu} = R_{\mu\nu} + \left( M^2 - \frac{R}{6} \right) g_{\mu\nu}; \] (3)

model II: \[ M_{\mu\nu} = -X_{\mu}^{\alpha} \gamma_{\nu\alpha} + X_{\gamma_{\mu\nu}}, \]
\[ \gamma_{\mu\nu} = R_{\mu\nu} - \left( M^2 + \frac{R}{2} \right) g_{\mu\nu}. \] (4)

Here \( M \) is the FP mass of gravitons. Since \( \Delta_{\mu\nu} \) is symmetric with respect to \( \mu \leftrightarrow \nu \), the asymmetric part of the field equations, \( E_{\mu\nu} \neq 0 \), yields algebraic conditions

I: \( \gamma_{\mu\alpha} X_{\nu}^{\alpha} = \gamma_{\nu\alpha} X_{\mu}^{\alpha} \); II: \( X_{\mu}^{\alpha} \gamma_{\nu\alpha} = X_{\nu}^{\alpha} \gamma_{\mu\alpha} \). (5)

The equations (1) can be obtained by varying the action
\[ I = \frac{1}{2} \int X^{\mu\nu} E_{\mu\nu} \sqrt{-g} d^4x \equiv \int L \sqrt{-g} d^4x \] (6)
(notice the order of indices) where, after integrating by parts, the Lagrangian \( L \) contains only the first derivatives of \( X_{\mu\nu} \) (see the Appendix). One can directly check that \( \delta I = \int E_{\mu\nu} \delta X^{\mu\nu} \sqrt{-g} d^4x \).

We shall only briefly indicate how these theories were obtained (details will be given separately). The strategy was to linearise equations of the ghost-free massive gravity [5]. When applied within the metric parameterization, this gives symmetric expressions for both the graviton kinetic and graviton mass terms, but the latter turns out to be extremely complicated [6]. We used instead the tetrad parameterization. Perturbations of the background tetrad are described in this approach by the non-symmetric \( X_{\mu\nu} \) while perturbations of the background potential comprise the mass term \( M_{\mu\nu} \), which is apriori non-symmetric. However, the kinetic term \( \Delta_{\mu\nu} \) is symmetric since it is obtained by linearising the background Einstein tensor. This procedure actually gives many different theories, but the above two models are the only ones for which the mass term \( M_{\mu\nu} \) depends on the background Ricci tensor linearly.

We shall now check that the number of propagating DoF in our models is indeed 5.

III. EINSTEIN SPACES

Let us consider first the special case where \( R_{\mu\nu} = \Lambda g_{\mu\nu} \). Then, in both models, the tensor \( \gamma_{\mu\nu} \) becomes proportional to the metric and the conditions (5) yield \( X_{\mu\nu} = X_{\nu\mu} \). The field equations reduce to
\[
\Delta_{\mu\nu} + M^2_{H(X_{\mu\nu} - X g_{\mu\nu})} = 0
\] (7)
where the Higuchi mass [3] is \( M^2_{H} = \Lambda/3 + M^2 \) in model I and \( M^2_{H} = \Lambda + M^2 \) in model II. The operator \( \Delta_{\mu\nu} \) is divergence free in this case (see the Appendix) and is invariant under \( X_{\mu\nu} \rightarrow X_{\mu\nu} + \nabla_\phi \gamma_{\mu\nu} \). Therefore, for \( M_H = 0 \) the theory (7) describes massless gravitons with two polarizations. For \( M_H \neq 0 \), taking the divergence of (7), one obtains four constraints \( \nabla^\mu X_{\mu\nu} = \nabla_{\nu} X \) which reduce the number of independent components of \( X_{\mu\nu} \) to 6 and bring equations (7) to the form
\[
- \square X_{\mu\nu} + \nabla_\mu \nabla_{\nu} X - 2R_{\mu\alpha\beta\gamma} X_{\alpha\beta} \gamma_{\gamma\nu} + \Lambda X g_{\mu\nu} + M^2_{H(X_{\mu\nu} - X g_{\mu\nu})} = 0.
\] (8)

Taking the trace of these yields \((2\Lambda - 3M^2_{H})X = 0\) hence, unless for \( M^2_{H} = 2\Lambda/3 \), one has \( X = 0 \). This is the fifth constraint reducing the number of DoF to five. For \( M^2_{H} = 2\Lambda/3 \equiv M^2_{PM} \) equations (8) admit the gauge symmetry \( X_{\mu\nu} \rightarrow X_{\mu\nu} + \nabla_\nu \gamma_{\mu\phi} + (\Lambda/3)g_{\mu\phi} \gamma_{\nu\phi} \) and there remain only DoF (the partially massless (PM) case [3]). As a result, our theory successfully reproduces the properties of massive gravitons in Einstein spaces.

IV. GENERIC SPACETIME

In an arbitrary spacetime geometry \( X_{\mu\nu} \) has no symmetries, but the six algebraic conditions (5) reduce the number of its independent components to ten. Next, one can see that there are in addition five differential constraints.

Consider first model I. Taking the divergence of \( E_{\mu\nu} \), the third and second derivatives of \( X_{\mu\nu} \) contained in \( \nabla^\mu E_{\mu\nu} \) cancel, while the first derivatives turn out to be all proportional to \( \gamma_{\mu\nu} \) (see the Appendix). Multiplying by the inverse \( \tilde{\gamma}_{\mu\nu} \) (one has \( \tilde{\gamma}_{\mu\nu} \gamma_{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta \) while the indices are moved by the metric, so that \( \gamma_{\mu\nu} = g_{\mu\alpha} \gamma^\alpha_{\nu} \) and \( \tilde{\gamma}_{\mu\nu} = g^{\mu\alpha} \gamma^\alpha_{\nu} \)) one obtains the four vector constraints

I: \[ C^\rho \equiv \tilde{\gamma}_{\mu\nu} \nabla_\nu E_{\mu\rho} = \nabla_\alpha \rho^{\alpha} - \nabla^{\rho} \rho + \mathcal{T}^\rho = 0 \] (9)
with \( \mathcal{T}^\rho = \tilde{\gamma}_{\mu\nu} \{ X^{\alpha\beta} (\nabla_\alpha G_{\beta\nu} - \nabla_\nu G_{\alpha\beta}) + \nabla^\mu \gamma_{\mu\alpha} X^{\alpha\nu} \} \).

Taking the divergence of this and combining with the trace of the equations, the second derivatives cancel yielding the fifth constraint

I: \[ C_5 = \nabla_\rho C^\rho + \frac{1}{2} E^\rho_{\mu} \]
\[ = -\frac{3}{2} M^2 X - \frac{1}{2} G^\mu\nu X_{\mu\nu} + \nabla_\rho \mathcal{T}^\rho = 0. \] (10)

Therefore, the number of propagating DoF is \( 10 - 5 = 5 \). If \( R_{\mu\nu} = \Lambda g_{\mu\nu} \) then \( \mathcal{T}^\rho = 0 \) and the constraints reduce
to the same as before: \( \nabla_\sigma X^{\sigma \rho} - \nabla^\rho X = 0 \) and \( (\Lambda/3 - M^2)X = (M^2 \rho_M - M^2 \rho_H)X = 0 \).

Consider now model II. Taking the divergence of \( E_{\mu \nu} \) and multiplying by \( \gamma^{\rho \nu} \) (and not by \( \tilde{\gamma}^{\rho \nu} \)) yields

\[
\text{II} : \quad C^\rho \equiv \gamma^{\rho \nu} \nabla_\nu E_{\mu \nu} = \Sigma^{\rho \alpha \beta} \nabla_\nu X_{\alpha \beta} = 0
\]

with \( \Sigma^{\rho \alpha \beta} = \gamma^{\rho \nu} \gamma^{\alpha \beta} - \gamma^{\rho \beta} \gamma^{\alpha \nu} \). Taking the divergence again and combining with the field equations yields (see the Appendix)

\[
\text{II} : \quad C_5 \equiv \nabla_\rho C^\rho + \frac{1}{2g^{00}} \Sigma^{0\alpha \beta} \left( 2E^\beta_\alpha - \delta^\beta_\alpha (E^\sigma_\sigma - \frac{1}{g^{00}} E^{00}) \right) = 0.
\]

A remark is in order here. This expression does in general contain second derivatives of \( X_{\mu \nu} \), but not the second time derivatives. The expression is not generally covariant and depends on the choice of time, but for any such a choice the second derivatives with respect to the corresponding time coordinate drop out from \( C_5 \). Therefore, \( C_5 = 0 \) is a constraint restricting the initial data.

Summarizing, the six algebraic conditions (5) and five differential constraints \( C^\rho = 0 \) and \( C_5 = 0 \) reduce the number of independent components of \( X_{\mu \nu} \) from 16 to 5, which matches the number of polarizations of massive particles of spin 2. When restricted to Einstein spaces, our theory reproduces the standard description of massive gravitons. However, its unusual feature is that, unless in Einstein spaces, the theory does not show the massless limit, since the mass term \( M_{\mu \nu} \) never vanishes for generic backgrounds, whatever the value of the mass parameter \( M \) is. Therefore, unless in Einstein spaces, the theory always propagated five DoF. We shall confirm this below also by explicitly solving the equations and counting the independent modes in the general solution.

V. COSMOLOGICAL BACKGROUND

As an application and in order to illustrate the above features, we explicitly construct the general solution of \( E_{\mu \nu} = 0 \) on a cosmological background with \( g_{\mu \nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 dx^2 \) where \( a(t) \) fulfills the Einstein equations

\[
3 \frac{\ddot{a}}{a} = \frac{\rho}{M_{\text{Pl}}} \equiv \rho, \quad 2 \frac{\dot{a}}{a} + \frac{\ddot{a}}{a} = -\frac{p}{M_{\text{Pl}}} \equiv -p.
\]

Here \( M_{\text{Pl}} \) is the Planck mass and \( \rho, p \) are the energy density and pressure of the background matter. The general solution can be represented as \( X_{\mu \nu}(t, x) = a^2(t) \sum_k X_{\mu \nu}(t, k) e^{ikx} \) where the Fourier amplitude splits into the sum of the tensor, vector, and scalar harmonics: \( X_{\mu \nu}(t, k) = X_{\mu \nu}^{(2)}(t) + X_{\mu \nu}^{(1)}(t) + X_{\mu \nu}^{(0)}(t) \). Since the spatial part of the background Ricci tensor is proportional to the unit matrix, \( R_{ik} \sim \delta_{ik} \), the algebraic constraints (5) imply that \( X_{ik} = X_{ki} \) hence \( X_{\mu \nu} \) has in this case only 13 independent components. Assuming the spatial momentum \( k \) to be directed along the third axis, \( k = (0, 0, k) \), the harmonics can be parameterized as

\[
X_{\mu \nu}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & D_+ & D_- & 0 \\ 0 & D_- & -D_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_{\mu \nu}^{(1)} = \begin{bmatrix} 0 & W_+^T & W_- & 0 \\ W_+^- & 0 & 0 & ikV_+ \\ W_- & 0 & 0 & ikV_- \\ 0 & ikV_+ & ikV_- & 0 \end{bmatrix}, \quad X_{\mu \nu}^{(0)} = \begin{bmatrix} S_+ & 0 & 0 & ikS_+ \\ 0 & S_- & 0 & 0 \\ 0 & 0 & S_- & 0 \\ ikS_+ & 0 & 0 & S_- - k^2 S \end{bmatrix},
\]

where \( D_\pm, V_\pm, S, W_\pm, S_\pm \) are functions of time. Injecting everything to \( E_{\mu \nu} = 0 \), the equations split into three independent groups – one for the tensor modes \( X_{\mu \nu}^{(2)} \), one for vector modes \( X_{\mu \nu}^{(1)} \), and one for scalar modes \( X_{\mu \nu}^{(0)} \).

In the tensor sector everything reduces to two decoupled second order equations for \( D_+ \) and for \( D_- \) describing the two tensor polarizations. In the vector sector the four amplitudes \( W_\pm \) can be expressed (see the Appendix) in terms of \( V_+ \) and \( V_- \) which fulfill two independent second order equations describing the two vector polarizations.

Most importantly, one finds that in the scalar sector the four \( S_\pm \) can be expressed in terms of one single amplitude \( S \) that fulfills a second order master equation (see the Appendix). Therefore, there is only one scalar polarization, hence the total number of polarizations is 5.

Injecting everything to the action (6), it splits into the sum of five independent terms of the form

\[
\int (K Y^2 - U Y^2) a^3 dt.
\]

Varying this yields the master equation in each sector. For the tensor modes one has \( U = D_+ \) or \( U = D_- \) and \( K = 1 \) while \( U = M_{\text{eff}}^2 + k^2/a^2 \). Here and in what follows we denote, depending on the model,

\[
\text{I} : \quad M_{\text{eff}}^2 = M^2 + \frac{1}{3} \rho, \quad m_H^2 = M_{\text{eff}}^2, \\
\text{II} : \quad M_{\text{eff}}^2 = M^2 - \rho, \quad m_H^2 = M^2 + \rho,
\]

where \( M_{\text{eff}} \) is the effective graviton mass (notice that \( M_{\text{eff}}^2 \) may be negative) while \( m_H \) reduces to the Higgs mass in the Einstein space limit.

For the vector modes one has \( U = V_+ \) or \( U = V_- \) and

\[
K \equiv K_{(1)} = \frac{k^2 m_H^4}{m_H^4 + (k^2/a^2)(m_H^2 - \epsilon/2)}
\]
where $\epsilon = \rho + p$ while the potential is $U = M_{\text{eff}}^2 k^2$.

$$K \equiv K_{(0)} = \frac{3k^4m_H^4(m_H^2 - 2H^2)}{(m_H^2 - 2H^2)[9m_H^4 + 6(k^2/a^2)(2m_H^2 - \epsilon)] + 4(k^4/a^4)(m_H^2 - \epsilon)}. \quad (18)$$

The potential in the scalar sector is more complicated (see the Appendix) but its asymptotic behaviour is simple. In all sectors one has $U/K \rightarrow M_{\text{eff}}^2$ for $k \rightarrow 0$ while for $k \rightarrow \infty$ one has $U/K \rightarrow c^2(k^2/a^2)$ where $c$ is the sound speed. One has for the vector and scalar modes, respectively,

$$c_{(1)}^2 = \frac{M_{\text{eff}}^2}{m_H^2}(m_H^2 - \epsilon/2), \quad (19)$$

$$c_{(0)}^2 = \frac{(m_H^2 - \epsilon)[m_H^4 + (2H^2 - 4M_{\text{eff}}^2 - \epsilon)m_H^2 + 4H^2M_{\text{eff}}^2]}{3m_H^4(2H^2 - m_H^2)},$$

while for the tensor modes $c_{(2)}^2 = 1$. For the vectors and scalars one has $c^2 < 1$ but $c^2 \rightarrow 1$ if $\rho \rightarrow 0$.

If $R_{\mu\nu} = \Lambda g_{\mu\nu}$ then $\rho = -p = \Lambda$ while $m_H^2 = M_{\text{H}}^2$ and $2H^2 = M_{\text{PM}}^2 = 2\Lambda/3$. The above formulas then imply that if $M_H > 0$ then $K_{(0)} = K_{(1)} = 0$, hence only the tensor modes propagate. The massless theory is recovered in this way. If $0 < M_H < M_{\text{PM}}$ then $K_{(0)} < 0$ (for $k \rightarrow \infty$) and the scalar polarization becomes (Higuchi) ghost [3]. If $M_H = M_{\text{PM}}$ then $K_{(0)} = 0$ and the scalar polarization is non-dynamical (the PM case).

All these features are well known for massive gravitons in Einstein spaces. However, for generic backgrounds, where $\rho, p$ are not constant, $m_H$ and $H$ become functions of time, and it is not possible to have $K_{(1)} = 0$ or $K_{(0)} = 0$ for all time moments, whatever the value of the FP mass $M$ is. Therefore, neither the massless nor PM cases are contained in the theory on generic backgrounds and it always propagates five polarizations. At most, there could be special backgrounds where gravitons become massless or PM for some values of $M$ [7].

A direct inspection of Eqs. (17)–(19) shows that if $\rho$ is small, $\rho \leq M^2$ ($\rho \leq M^2 M_{\text{Pl}}^2$), then $K > 0$ (for $k \rightarrow \infty$) and $c^2 > 0$, hence the system is free of ghosts and tachyons. The situation is more complex for large $\rho$. In model I the kinetic term $K_{(0)}$ changes sign for $\rho > 3M^2$ since $m_H^2 < 2H^2$ in this case, which corresponds to the Higuchi ghost. However, $c_{(0)}^2$ also changes sign at the same time, unless $p/\rho = -1$, so that the ghost and tachyon “compensate each other” only changing the overall sign of the action. In model II one always has $m_H^2 > 2H^2$ and the Higuchi ghost is absent, but since $M_{\text{eff}}^2$ may be negative, there could be tachyons in the vector sector. However, one finds in this case that $K > 0$ (always for $k \rightarrow \infty$) and that $c^2 > 0$ for any $\rho$ provided that $p/\rho < -2/5$. Therefore, model II is stable during the inflationary stage, whereas model I is stable if the graviton mass is large enough, $M \geq H$. Estimating that $\rho \approx (10^{18}\text{GeV})^4$ at the beginning of the radiation dominated stage [8], it follows that for $M \geq 10^{13}$ GeV one would have $\rho \leq M^2$ and hence both models I and II would be stable at all times after the inflation.

A much milder bound $M \geq 10^{-3}$ eV is needed to insure that both models are stable at present when $\rho$ is small. Assuming that the tensor $X_{\mu\nu}$ couples only to the gravity and hence massive gravitons do not have other decay channels, it follows that they could be a part of Dark Matter (DM) at present. Massive gravitons as DM candidates have actually been considered before [9], but only our description is consistent for arbitrary backgrounds.

One may wonder if the massive gravitons could also mimic the dark energy as in the other massive gravity models [10]. To calculate their backreaction on the space-time geometry, one adds the Einstein-Hilbert term to the action (6), which yields

$$I = \frac{1}{2} \int (M_{\text{Pl}}^2 R + X^{\mu\nu}E_{\mu\nu}) \sqrt{-g} d^4x. \quad (20)$$

Varying this with respect to the metric yields Einstein equations $M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu}$ to be solved together with $E_{\mu\nu} = 0$. The energy-momentum tensor $T_{\mu\nu}$ has a somewhat complicated structure, partly due to the non-minimal terms like $X^{\mu\nu}R_{\sigma\beta}X_{\sigma\beta}$ in the action (see the Appendix). We solved the equations in the homogeneous and isotropic sector, with $X_{\mu\nu} = X_{\mu\nu}(0)$ given by (14) for $k = 0$, and we found the solution only in model II and only for $M^2 < 0$: this is the de Sitter space with $\Lambda = -3M^2 > 0$. This would be a legitimate solution if we flipped the sign in front of $M^2$ in (5) from the very beginning, but such a modified theory would be very unstable since $K$ and $c^2$ in (17)–(19) would then be negative. We therefore conclude that our theory cannot mimic a positive $\Lambda$-term.

Since the potential $U$ can be negative, one might wonder if the gravitons could exhibit features like superluminality [11]. We have seen that for the cosmological background this problem does not arise, since the system is free form ghosts and tachyons. Other backgrounds should be studied separately.

VI. OTHER POSSIBLE APPLICATIONS

Apart from cosmology, the theory of massive spin 2 field can have other applications. For example, massive
gravitons in curved space can be used for the holographic description of superconductors [12] or electron-phonon interactions [13]. Up to now all applications have always been restricted to the Einstein spaces, but in our theory this is no longer necessary.

One can also study solutions with back-reacting massive gravitons described by (20), as for example static stars or black holes. Yet more interesting applications could be found by extending the theory to complex values of $X_{\mu\nu}$ via replacing in (20) $X^{\mu\nu}E_{\mu\nu} \rightarrow X^{\mu\nu}E_{\mu\nu} + X^{\mu\nu}E_{\mu\nu}$ with the bar denoting complex conjugation. Very recently it was shown [14] that the superradiance [15] of complex massive fields in the vicinity of spinning black holes leads to a spontaneous formation of massive clouds evolving towards stationary hairy black holes [16]. So far this phenomenon has been observed only for the Kazan Federal University.

VII. SUMMARY

We have constructed a new theory of a free massive spin-2 field in curved space. This theory is exceptional because it propagates not more than 5 DoF for any background. So far only one theory with this property has been discovered [6], to which theory our theory is probably equivalent, but we use a completely different parametrization in terms of a non-symmetric tensor $X_{\mu\nu}$, which yield simpler results. Our main goal was to show that our theory is self-consistent and that the number of independent DoF is indeed 5. We have shown this by counting the constraints and also by counting the independent modes in the general solution. It turns out that massive gravitons in the expanding universe are free of ghosts and tachyons.

We notice finally that the recent LIGO data [17] imply that the graviton mass should be less than $10^{-22}$ eV [18]. However, this bound applies rather to the mass of quanta of the background metric $g_{\mu\nu}$, and not to that of $X_{\mu\nu}$. It is consistent to assume that $X_{\mu\nu}$ does not directly interact with the ordinary matter and hence is not seen by the LIGO detector. Therefore the bound does not apply to the FP mass $M$.

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IX. APPENDIX

Below we sketch some technical details of the calculations presented in the main text above.

A. A. Lagrangian

The Lagrangian $L$ in the action $I = \int L\sqrt{-g}d^4x$ defined by Eq.(6) in the main text is $L = L(2) + L(0)$ where

$$
L(2) = -\frac{1}{4} \nabla^\alpha X^{\mu\nu}\nabla_\alpha X_{\mu\nu} + \frac{1}{8} \nabla^\alpha X^{\mu\nu}\nabla_\alpha X_{\mu\nu} + \nabla^\alpha X^{\mu\nu}\nabla_\alpha X_{\mu\nu} + \frac{1}{8} \nabla^\alpha X^{\mu\nu}\nabla_\alpha X_{\mu\nu}
$$

(A.1)

with $X^\mu_{\mu} = X_{\mu\nu} + X_{\nu\mu}$ and $X = X^\alpha_{\alpha}$. One has in model I

$$
L(0) = -\frac{1}{2} X^{\mu\nu}R_{\mu\nu}X_{\sigma\nu} + \frac{1}{2} (M^2 - \frac{R}{6})(X_{\mu\nu}X^{\mu\nu} - X^2)
$$

(A.2)

and in model II

$$
L(0) = -\frac{1}{2} X^{\mu\nu}R_{\mu\nu}X_{\sigma\nu} - \frac{1}{2} X^{\mu\nu}R^{\sigma\nu}X_{\sigma\mu}
- \frac{1}{2} X^{\nu\alpha}X^\rho_{\alpha\nu}R^\rho_{\mu\nu} + XR_{\mu\nu}X^{\mu\nu}
+ \frac{1}{2} (M^2 + \frac{R}{2})(X_{\mu\nu}X^{\mu\nu} - X^2); \quad (A.3)
$$

the order of indices being important. One can directly check that varying the action with respect to $X_{\mu\nu}$ yields the field equations,

$$
\delta I = \int E_{\nu\mu} \delta X^{\mu\nu} \sqrt{-g} d^4x.
$$

(A.4)

Varying with respect to the metric gives the energy-momentum tensor, $\delta I = -2 \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d^4x$.

A. B. Constraints

Here we sketch the derivation of the five constraints expressed by Eqs.(9)–(12) in the main text. Using

$$
(\nabla_{\nu} \nabla_{\mu} - \nabla_{\mu} \nabla_{\nu})X^{\beta}_{\beta} = R^\sigma_{\beta\mu\nu}X^\alpha_{\sigma} - R^\alpha_{\sigma\mu\nu}X^{\beta}_{\sigma}
$$

a direct calculation yields the following result for the divergence of $\Delta_{\mu\nu}$ defined by Eq.(2) in the main text:

$$
\nabla^{\mu} \Delta_{\mu\nu} = \gamma_{\nu\beta}(\nabla_\alpha X^{\alpha\beta} - \nabla^{\beta} X) + \gamma_{\nu\beta}(\nabla_\nu X^{\alpha\beta} - \nabla^{\alpha} X^{\beta}_{\nu}) + X^{\alpha\beta}\nabla_\alpha G_{\beta\mu}.
$$

(B.1)

Here we introduced the bold-faced $\gamma_{\alpha\beta} \equiv R_{\alpha\beta} + \phi g_{\alpha\beta}$ where $\phi$ can be set to any value, because the part of $\gamma_{\alpha\beta}$ proportional to $g_{\mu\nu}$ cancels in (B.1). If $R_{\mu\nu} = \Lambda g_{\mu\nu}$ then $\gamma_{\alpha\beta} \sim g_{\mu\nu}$ and (B.1) yields $\nabla^{\nu} \Delta_{\mu\nu} = 0$. 
The divergence of $M_{\mu\nu}$ in model I given by Eq. (3) in the main text is

$$\nabla^\mu M_{\mu\nu} = \gamma_{\alpha\beta}(\nabla^\alpha X^\beta - \nabla_\nu X^{\alpha\beta}) + X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta} - X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta} \quad (B.2)$$

with $\gamma_{\alpha\beta}$ defined by Eq. (3). Setting in (B.1) $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}$ and adding up with (B.2), the second line on the right in (B.1) cancels against the first line in (B.2), yielding

$$\nabla^\mu(\Delta_{\mu\nu} + M_{\mu\nu}) = \gamma_{\nu\beta}(\nabla_\alpha X^{\alpha\beta} - \nabla_\nu X^{\alpha\beta}) + X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta} - X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta}. \quad (B.3)$$

Multiplying this by the inverse $\tilde{\gamma}^{\rho\nu}$ of $\gamma_{\alpha\beta}$ one obtains

$$C^\rho \equiv \tilde{\gamma}^{\rho\nu}\nabla^\mu E_{\mu\nu} = \nabla_\alpha X^{\alpha\beta} - \nabla_\nu X^{\alpha\beta} + \tilde{\gamma}^{\rho\nu}(X^{\alpha\beta}\nabla_\alpha G_{\rho\nu} + X^{\alpha\nu}\nabla^\mu \gamma_{\mu\alpha} - X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta}), \quad (B.4)$$

which yields Eq. (10) in the main text. Acting on this with $\nabla_\rho$ and combining with the trace $E^\mu_\mu$ reproduces Eq. (10) in the main text.

The divergence of $M_{\mu\nu}$ in model II given by Eq. (4) in the main text is

$$\nabla^\mu M_{\mu\nu} = \gamma_{\nu\beta}(\nabla^\alpha X^\beta - \nabla_\nu X^{\alpha\beta}) + X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta} - X^{\alpha\beta}\nabla_\nu \gamma_{\alpha\beta} \quad (B.5)$$

with $\gamma_{\alpha\beta} = G_{\mu\nu} - M^2 g_{\mu\nu}$. Setting in (B.1) $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}$ and adding up with (B.5), the first and third lines on the right in (B.1) cancel against (B.5), hence

$$\nabla^\mu(\Delta_{\mu\nu} + M_{\mu\nu}) = \gamma_{\alpha\beta}(\nabla_\nu X^{\alpha\beta} - \nabla^\alpha X^\beta). \quad (B.6)$$

Multiplying this by $\gamma^{\rho\nu}$ yields

$$C^\rho \equiv \gamma^{\rho\nu}\nabla^\mu E_{\mu\nu} = \gamma^{\rho\nu}(\nabla_\nu X^{\alpha\beta} - \nabla^\alpha X^\beta) + \gamma^{\rho\nu}(\nabla_\nu X^{\alpha\beta} - \nabla^\alpha X^\beta), \quad (B.7)$$

which reproduces Eq. (11) in the main text. Acting with $\nabla_\rho$ gives

$$\nabla_\rho C^\rho = (\gamma^{\rho\nu}\gamma^{\nu\alpha} - \gamma^{\rho\beta}\gamma^{\nu\alpha})\nabla_\rho \nabla_\nu X^{\alpha\beta} + \nabla_\rho(\gamma^{\rho\nu}\gamma^{\nu\alpha} - \gamma^{\rho\beta}\gamma^{\nu\alpha})\nabla_\nu X^{\alpha\beta} = (\gamma^{\rho\nu}\gamma^{\nu\alpha} - \gamma^{\rho\beta}\gamma^{\nu\alpha})\nabla_\rho \nabla_\nu X^{\alpha\beta}, \quad (B.8)$$

where the dots denote terms not containing second time derivatives of $X_{\alpha\beta}$. The definition of $\Delta_{\mu\nu}$ in (2) in the main text implies that (here $i, k, m, n = 1, 2, 3$)

$$\Delta_{ik} = -g^{00}\tilde{X}_{(ik)} + g_{ik} g^{00} h^{nm} \tilde{X}_{nm} + \ldots \quad (B.9)$$

with $h^{ik} = g^{ik} - g^{0i} g^{0k}/g^{00}$ hence

$$\tilde{X}_{(ik)} = \frac{1}{g^{00}} \left( \frac{1}{2} g_{ik} h^{nm} \Delta_{nm} - \Delta_{ik} \right) + \ldots$$

$$= \frac{1}{g^{00}} \left( \frac{1}{2} g_{ik} h^{nm} E_{nm} - E_{ik} \right) + \ldots \quad (B.10)$$

It follows that the combination

$$\frac{1}{g^{00}}(\gamma^{00} \gamma_{ik} - \gamma^{0i} \gamma^{0k}) \left( \frac{1}{2} g_{ik} h^{nm} E_{nm} - E_{ik} \right) \quad (B.11)$$

has precisely the same second time derivatives as in $\nabla_\rho C^\rho$. Noting finally that

$$h^{nm} E_{nm} = \frac{1}{g^{00}}(g^{00} g^{mn} - g^{0m} g^{0n}) E_{nm} \quad (B.12)$$

$$= \frac{1}{g^{00}}(g^{00} g^{\nu\nu} - g^{0\nu} g^{0\nu}) E_{\nu\nu} = E^\alpha - \frac{1}{g^{00}} E^{00}$$

yields Eq. (12) in the main text.

### A. C. Solution in the expanding universe

Injecting Eqs. (13), (14) from the main text to the equations $E_{\mu\nu} = 0$ expressed by Eq. (1), the direct inspection reveals that the vector amplitudes $W_+^\pm$ in Eq. (14) can be expressed in terms of two independent amplitudes $V_\pm$ as

$$W_+^+ = \frac{P^2 m_H^2 V_+}{m_H^4 + P^2 (m_H^2 - \epsilon/2)},$$

$$W_+^- = \frac{P^2 [m_H^2 - \epsilon] V_-}{m_H^4 + P^2 (m_H^2 - \epsilon/2)} \quad (C.1)$$

where $m_H$ and $\epsilon$ are defined after Eq. (16) in the main text and $P = k/a$ is the physical momentum. The equations for $V_\pm$ reduce to those obtained by varying the effective action expressed by Eq. (15) in the main text.

Similarly, the four scalar amplitudes $S_\pm$ in Eq. (14) are expressed in terms of one single amplitude $S$ by the following relations:

$$S_+ = \frac{m_H^2 - \epsilon}{m_H^4} S, \quad (C.2)$$

$$S_- = \frac{2}{m_H^4} \left( S_- + a^2 H S_+^* \right),$$

$$S_+^* = \frac{1}{\tilde{H} a^2} S_-, \quad \frac{2H m_H^4 P^2 \tilde{S} + m_H^6 P^2 S - m_H^4 (2P^2 + 3m_H^2) S_-/a^2}{2H^2 [3m_H^2 + 2P^2 (2m_H^2 - \epsilon)]},$$

while
\[ S^- = a^2P^2 - 4Hm_H^2P^2 \frac{\dot{S}}{4P^4(m_H^2 - \epsilon)} + 6P^2(2m_H^2 - \epsilon)(m_H^2 - 2H^2) \] (C.3)

It is crucial that all four \( S^{\pm}_i \) are expressed in terms of one single amplitude \( S \) that fulfills the master equation obtainable by varying the effective action given by Eq. (15) in the main text. This shows that there is only one dynamical DoF in the scalar sector. Therefore, together with the tensor and vector modes, the theory propagates five DoF.

The effective action for the scalars contains the kinetic term \( K \) expressed by Eq. (18) in the main text. One has

\[ \frac{U}{K} = \frac{b_0 + b_2P^2 + b_4P^4 + b_6P^6}{C(c_0 + c_2P^2 + c_4P^4)} \] (C.4)

where

\[ C = 3m_H^4(m_H^2 - 2H^2), \]
\[ c_0 = 9m_H^4(m_H^2 - 2H^2), \]
\[ c_2 = 6(m_H^2 - 2H^2)(2m_H^2 - \epsilon), \]
\[ c_4 = 4(m_H^2 - \epsilon), \] (C.5)

and also

\[ b_0 = 27m_H^8M_{\text{eff}}^2(m_H^2 - 2H^2)^2, \]
\[ b_2 = 9m_H^4(m_H^2 - 2H^2)^2[4M_{\text{eff}}^2(2m_H^2 - \epsilon) - m_H^4], \]
\[ b_4 = 6m_H^4[8m_H^2 - (20H^2 + 9\epsilon)m_H^4 + (8H^4 + 20H^2\epsilon + 2H\dot{\rho} + \epsilon^2)m_H^4 - 4H^2(H\dot{\rho} + \epsilon^2)] + 12(M_{\text{eff}}^2 - m_H^4)[5m_H^4 - 6(2H^2 + \epsilon)m_H^6

+ (8H^4 + 14H^2\epsilon + \epsilon^2)m_H^4 - 4H^2(2H^2 + \epsilon)m_H^4 + 4H^4\epsilon^2], \]
\[ b_6 = 4(m_H^2 - \epsilon)^2[4M_{\text{eff}}^2(m_H^2 - H^2) + m_H^2(\epsilon - 2H^2 - m_H^2)], \]

Notice that these expressions contain \( \dot{\rho} \) and hence the third derivative of the background scale factor \( a(t) \). The ratio \( c_2^2 = b_6/(Cc_4) \) is the speed of sound expressed by Eq. (19) in the main text.

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