Strong convergence rate of the Euler scheme for SDEs driven by additive rough fractional noises

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Abstract. The strong convergence rate of the Euler scheme for SDEs driven by additive fractional Brownian motions is studied, where the fractional Brownian motion has Hurst parameter \( H \in \left( \frac{1}{3}, \frac{1}{2} \right) \) and the drift coefficient is not required to be bounded. The Malliavin calculus, the rough path theory and the 2D Young integral are utilized to overcome the difficulties caused by the low regularity of the fractional Brownian motion and the unboundedness of the drift coefficient. The Euler scheme is proved to have strong order \( 2H \) for the case that the drift coefficient has bounded derivatives up to order three and have strong order \( H + \frac{1}{2} \) for linear cases. Numerical simulations are presented to support the theoretical results.

1. Introduction

Stochastic differential equations driven by fractional Brownian motions with Hurst parameter \( H \in (0, 1) \) are basic models to characterize the randomness phenomena and have various applications in the fields of hydrology \([17]\), porous media \([3]\), oscillators \([10]\), explorations \([6]\), finance \([9]\) and so on. If \( H > \frac{1}{2} \), the fractional Brownian motion (fBm) exhibits a long-range dependence property. If \( H = \frac{1}{2} \), the fBm is equivalent to the standard Brownian motion so that the increments are independent. If \( H < \frac{1}{2} \), the fBm exhibits a short-range dependence property and the regularity of the sample paths is relatively low, in which case we call it rough fractional noise. In this article, we investigate the numerical approximation for the stochastic differential equation (SDE) driven by an additive rough fractional noise

\[
dX_t = a(X_t)dt + \sigma dB_t, \quad t \in (0, T],
\]

where \( X_0 \in \mathbb{R} \) is a deterministic initial value, the drift coefficient \( a \) is unbounded, and \( B = \{B_t\}_{t \in [0, T]} \) is the fBm with Hurst parameter \( H \in (\frac{1}{3}, \frac{1}{2}) \).

One of the main obstacles in the convergence analysis on numerical schemes for SDEs in the rough case is the low regularity of the noise, which leads to the lack of an explicit formulation for the covariance kernel of the noise. Meanwhile, the unboundedness of the drift coefficient and the correlation of the increments of the fBm make the interaction of the local errors between the numerical solution and the exact solution more complicated. These difficulties result in that the numerical

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analysis in this case is far from well-developed. To deal with the problems men-

tioned above, we apply the Malliavin calculus, the rough path theory and the 2D 

Young integral to establish the strong convergence rate of the Euler scheme for (1). 

More precisely, for \( n \in \mathbb{N}_+ \), denoting \( h = \frac{T}{n} \) and \( t_k = kh \), we focus on the following 

continuous interpolation of the Euler scheme 

\[
Y_t = Y_{t_k} + a(Y_{t_k})(t - t_k) + \sigma(B_t - B_{t_k}), \quad t \in (t_k, t_{k+1}], \ k = 0, \ldots, n - 1.
\]

(2) 

Our main result is stated in the following.

**Theorem 1.1.** Let \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \). Assume that \( a : \mathbb{R} \to \mathbb{R} \) has bounded derivatives 

to order three. Then it holds that 

\[
\left( \sup_{t \in [0,T]} \mathbb{E} \left| X_t - Y_t \right|^2 \right)^{1/2} \leq Ch^{2H},
\]

where \( X \) solves (1) and \( Y \) is given by the Euler scheme (2).

As \( H \) tends to \( \frac{1}{2} \), the strong convergence rate of the Euler scheme above goes 
to 1, which is consistent with the classical result that the Euler–Maruyama scheme 
for SDEs driven by additive standard Brownian motions has strong order 1 [18 
Chapter 1]. Moreover, comparing with [1, 13, 15], Theorem 1.1 reveals that the 
strong convergence rate of the Euler scheme in the above additive noise case is 
half order higher than those of the Euler-type schemes in the multiplicative noise 
case. In particular, if \( a \) is linear, we have that the strong convergence rate of the 
Euler scheme is improved to \( H + \frac{1}{2} \); see Corollary 3.3. We remark that the results 
of Theorem 1.1 and Corollary 3.3 can be extended directly to multi-dimensional 
cases. If the drift coefficient is bounded but less regular, we refer to [2] for the 
optimal strong convergence rate of the Euler scheme in Hölder spaces.

The rest of the article is arranged as follows. In Section 2 some preliminaries 
for the 2D Young integral and the Malliavin calculus are introduced. In Section 3 
we prove the strong convergence rate of the Euler scheme, i.e., Theorem 1.1 and 
Corollary 3.3. In Section 4 numerical simulations are given to verify our theoretical 
analysis.

2. 2D Young integral and Malliavin calculus

This section reviews basic concepts and results about the 2D Young integral 
and the Malliavin calculus associated to the fBm. We utilize \( C \) as a generic constant 
and \( G \) as a generic finite random variable, which may be different from line to line. 
We will make use of subscripts to emphasize the parameters that they depend on.

2.1. 2D Young integral. Let \( U, W \) be Banach spaces with norms \( \| \cdot \|_U \) and 
\( \| \cdot \|_W \), respectively. We denote by \( \mathcal{L}(U, W) \) the set of linear operators from \( U \) to 
\( W \).

**Definition 2.1.** For fixed \( p \geq 1 \) and \( T > 0 \), the \( p \)-variation of \( f : [0, T] \to U \) 
on \([s, t] \subseteq [0, T]\) is defined as 

\[
\| f \|_{p, \text{var};[s,t]} := \sup_{P \in \mathcal{D}([s,t])} \left( \sum_{k=0}^{N-1} \left\| f_{t_{k+1}} - f_{t_k} \right\|_U^p \right)^{1/p},
\]

where
where $\mathcal{P} = \{t_k : k = 0, \ldots, N, \ s = t_0 < t_1 < \cdots < t_N = t\}$ denotes a partition of $[s, t]$ and $\mathcal{D}([s, t])$ is the set of all such partitions. In addition, we define

$$C^p\text{-}\text{var}(U; [0, T]) := \{f : \|f\|_{C^p\text{-}\text{var}; [0, T]} < +\infty\}.$$

**Definition 2.2.** Fix $p \geq 1$ and $T > 0$. For $g : [0, T]^2 \to U$, let

$$g([u_i, u_{i+1}] \times [v_j, v_{j+1}]) := g_{u_{i+1}, v_{j+1}} - g_{u_i, v_{j+1}} - g_{u_i, v_j} + g_{u_{i+1}, v_j}.$$ 

The $p$-variation of $g$ on $[s, t] \times [u, v] \subseteq [0, T]^2$ is defined as

$$\|g\|_{p; [s, t] \times [u, v]} := \sup_{\pi \in \mathcal{D}([s, t] \times [u, v])} \left( \sum_{i,j} \|g([u_i, u_{i+1}] \times [v_j, v_{j+1}])\|_U^p \right)^{1/p},$$

where $\pi = \{(u_i, v_j)\}$ is a partition of $[s, t] \times [u, v]$ and $\mathcal{D}([s, t] \times [u, v])$ denotes the set of grid-like partitions of $[s, t] \times [u, v]$. Moreover, we define

$$C^p\text{-}\text{var}(U; [0, T]^2) := \{g : \|g\|_{p; [0, T]^2} < +\infty\}.$$

**Remark 2.1.** For $f : [0, T] \to U$, the $\beta$-Hölder semi-norm of $f$ on $[s, t] \subseteq [0, T]$ is denoted by

$$\|f\|_{\beta; [s, t]} := \sup_{s \leq u < v \leq t} \frac{\|f_v - f_u\|_U}{|v - u|^\beta}.$$ 

If $\|f\|_{\beta; [0, T]} < +\infty$, then we have $f \in C^{1/\beta\text{-}\text{var}}(U; [0, T])$. Moreover, if $g$ also satisfies $\|g\|_{\beta; [0, T]} < +\infty$, then the $1/\beta$-variation of the function $fg : (r_1, r_2) \mapsto f_{r_1}g_{r_2}$ defined on $[0, T]^2$ is finite.

**Definition 2.3.** Assume $f \in C^p\text{-}\text{var}(U; [0, T]^2)$ and $g \in C^q\text{-}\text{var}(W; [0, T]^2)$. If

$$\frac{1}{p} + \frac{1}{q} > 1,$$

then we say that $f$ and $g$ have complementary regularity.

**Lemma 2.2.** $(\text{[8, 21]})$ Given $f : [0, T]^2 \to \mathcal{L}(U, W)$ and $g : [0, T]^2 \to U$. Then the following 2D Young integral is defined as

$$\int_{[0,T]^2} \! f_{r_1, r_2} dgr_{r_1, r_2} := \lim_{|\pi| \to 0} \sum_{i,j} f_{u_i, v_j} g([u_i, u_{i+1}] \times [v_j, v_{j+1}]).$$

if the limit above exists. Moreover, if $f$ and $g$ have complementary regularity, then $\int_{[0,T]^2} \! f_{r_1, r_2} dgr_{r_1, r_2}$ exists, and it holds that

$$\left\| \int_{[0,T]^2} \! f_{r_1, r_2} dgr_{r_1, r_2} \right\|_W \leq C_{p,q}\|f\|_{p; [0, T]^2}\|g\|_{q; [0, T]^2},$$

where

$$\|f\|_{p; [0, T]^2} := \|f_0.0\|_{\mathcal{L}(U, W)} + \|f_0\|_{p; \text{var}; [0, T]} + \|f.0\|_{p; \text{var}; [0, T]} + \|f\|_{p; \text{var}; [0, T]^2}.$$ 

In particular, the result can also be restricted to $[s, t] \times [u, v] \subseteq [0, T]^2$. 

2.2. Malliavin calculus. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

**Definition 2.4.** The scalar-valued fractional Brownian motion \(B = \{B_t\}_{t \in [0,T]}\) is a continuous centered Gaussian process with \(B_0 = 0\) almost surely and the covariance

\[
R_{s,t} := \mathbb{E}[B_s B_t] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad s, t \in [0,T].
\]

Here, \(H \in (0,1)\) is called the Hurst parameter of \(B\).

Based on Definition 2.4, the regularity of the fBm, as well as the regularity of its covariance, is obtained.

**Lemma 2.3.** \([19]\) Chapter 5] For \(H \in (0,1)\) and \(p \geq 1\), there exists a constant \(C = C_p\) such that

\[
\sup_{0 \leq s < t \leq T} \frac{\|B_t - B_s\|_{L^p(\Omega)}}{|t-s|^H} \leq C.
\]

Meanwhile, for any \(\beta \in (0,H)\), there exists a nonnegative random variable \(G = G_{\beta,T} \in L^p(\Omega)\) for all \(p \geq 1\), such that \(\|B\|_{\beta;[0,T]} \leq G\) almost surely.

**Lemma 2.4.** \([8]\) Example 1] For \(H \in (0, \frac{1}{2})\), we have

\[
R \in C^{1/2H\text{-var}}(\mathbb{R}; [0,T]^2).
\]

More precisely, it holds that \(\|R\|_{V^{1/2H};[s,t]^2} \leq C_H |t-s|^{2H}\).

Combining Lemmas 2.2\&2.3, for a function \(f : [0, T]^2 \to \mathbb{R}\) sharing a similar regularity of \(B\), i.e., \(f \in C^{1/\beta\text{-var}}(\mathbb{R}; [0,T]^2)\) with \(\beta = H^{-}\), we obtain that

\[
\int_{[0,T]^2} f_{r_1, r_2} \, dR_{r_1, r_2}
\]

is well-defined as long as \(H \in (\frac{1}{3}, \frac{1}{2})\). In the following, based on the 2D Young integral, we introduce the Malliavin calculus associated to the fBm with Hurst parameter \(H \in (\frac{1}{3}, \frac{1}{2})\).

Noticing

\[
R_{s,t} = \int_{[0,s] \times [0,t]} dR_{r_1, r_2} = \int_{[0,T]^2} 1_{[0,s]}(r_1) 1_{[0,t]}(r_2) \, dR_{r_1, r_2}
\]

with \(1_{[0,t]}(\cdot)\) being the indicator function, we consider the inner product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} := R_{s,t},
\]

which yields a Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})\) being the closure of the space of all step functions on \([0,T]\) with respect to \(\langle \cdot, \cdot \rangle_{\mathcal{H}}\).

**Definition 2.5.** Given a random variable

\[
F = f(B_{t_1}, \ldots, B_{t_N}),
\]

where \(t_1, \ldots, t_N \in [0,T]\), and \(f : \mathbb{R}^N \to \mathbb{R}\) is a bounded smooth function with derivatives bounded up to any order, the Malliavin derivative of \(F\) is defined by

\[
DF := \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(B_{t_1}, \ldots, B_{t_N}) 1_{[0,t_i]}(\cdot).
\]
Furthermore, for $p \geq 1$, the space $\mathbb{D}^{1,p}$ is the closure of the set of random variables in terms of the norm

$$
\|F\|_{\mathbb{D}^{1,p}} := \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{L^p}^p]\right)^{\frac{1}{p}}.
$$

**Definition 2.6.** Given an $\mathcal{H}$-valued random variable $\varphi \in L^2(\Omega; \mathcal{H})$ satisfying

$$
\mathbb{E}\left[\|\varphi, DF\|_{\mathcal{H}}\right] \leq C_{\varphi}\|F\|_{L^2(\Omega)}, \quad F \in \mathbb{D}^{1,2},
$$

the adjoint operator $\delta$ of the derivative operator $D$ acts on $\varphi$ is $\delta(\varphi) \in L^2(\Omega; \mathbb{R})$ such that

$$
\mathbb{E}\left[\langle \varphi, DF \rangle_{\mathcal{H}}\right] = \mathbb{E}[F\delta(\varphi)]
$$

for all $F \in \mathbb{D}^{1,2}$. In this case, we say $\varphi \in \text{Dom}(\delta)$. Furthermore, the Skorohod integral of $\varphi$ with respect to $B$ is defined by

$$
\int_0^T \varphi_t \delta B_t := \delta(\varphi).
$$

In particular, for $t \in [0, T]$, $\int_0^t \varphi_u \delta B_u := \delta(\varphi_{[0,t]})$.

On the other hand, the fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$ can be naturally lifted to the rough path almost surely, which leads to the rough integral $\int_0^T \varphi_t dB_t$ and the solutions to SDEs in the sense of rough paths \cite{17, 16}. In the sequel, we introduce the transformation formula for the Skorohod integral and the rough integral, which is essential for us in numerical analysis.

**Lemma 2.5.** \cite{4, 20} Let $H \in (\frac{1}{2}, 1)$. Assume that the first order derivative of the function $\Phi$ is bounded and that $\varphi$ solves

$$
d\varphi_t = \Phi(\varphi_t) dB_t
$$
in the sense of rough path. Then it holds almost surely that

$$
\int_0^T \varphi_t dB_t = \int_0^T \varphi_t \delta B_t + H \int_0^T \Phi(\varphi_s) s^{H-1} ds + \int_{[0,T]^2} \mathbb{1}_{[0,r_2]}(r_1) \left[D_{r_1} \varphi_{r_2} - \Phi(\varphi_{r_2})\right] dR_{r_1,r_2}.
$$

### 3. Convergence analysis on the Euler scheme

In this section, we set $h = \frac{T}{n}$. and $t_k = kh, \ k = 0, \cdots, n$. For $t \in (t_k, t_{k+1}]$, define $|t| := t_k$ and $\lfloor t \rfloor := t_{k+1}$. Before proving the main results, we give lemmas for the solution of (1) and the covariance of the fBm.

**Lemma 3.1.** Assume that the derivative of $a$ is bounded. Then (1) admits a unique solution satisfying

$$
\mathbb{E}\left[\sup_{\tau \in [0,T]} |X_\tau|^p\right] + \mathbb{E}\left[\|X\|_{\beta, [0,T]}^p\right] \leq C, \quad p \geq 1, \ \beta < H.
$$

**Proof.** Since $a$ has bounded derivative, the existence and uniqueness of the solution to (1) is deduced from a standard argument by the contractive mapping principle. Moreover, based on

$$
\sup_{\tau \in [0,T]} |X_\tau| \leq |X_0| + \int_0^T \sup_{\tau \in [0,s]} |a(X_\tau)| ds + \sup_{\tau \in [0,T]} |B_\tau|
$$
\[ \leq |X_0| + C \int_0^t \left( 1 + \sup_{\tau \in [0,s]} |X_\tau| \right) ds + \sigma \sup_{\tau \in [0,T]} |B_\tau|, \]

Gronwall’s inequality gives

\[ \sup_{\tau \in [0,t]} |X_\tau| \leq C \left( 1 + \sup_{\tau \in [0,T]} |B_\tau| \right). \]

Then Lemma 2.3 yields

\[ \mathbb{E} \left[ \sup_{\tau \in [0,T]} |X_\tau|^p \right] \leq C, \quad p \geq 1. \]

On the other hand, we have

\[ |X_t - X_s| \leq \int_s^t |a(X_\tau)| d\tau + \sigma |B_t - B_s| \]

\[ \leq C \int_s^t (1 + |X_\tau|) d\tau + \sigma |B_t - B_s| \]

\[ \leq C \left( 1 + \sup_{\tau \in [0,T]} |B_\tau| \right) |t - s| + \sigma |B_t - B_s|, \]

which implies

\[ \mathbb{E} \left[ \|X\|_p^{\beta_{[0,T]}} \right] \leq C, \]

for any \( p \geq 1 \) and \( \beta < H \).

**Lemma 3.2.** Let \( R \) be the covariance of the fractional Brownian motion \( B \) with Hurst parameter \( H \in (0, \frac{1}{2}) \). Then it holds that

(3) \[ \int_0^T \int_0^T \| R \|_{V^{1/2H};[[t],t] \times [[s],s]} ds dt \leq Ch^{2H+1}, \]

and

(4) \[ \int_0^T \| R \|_{V^{1/2H};[[t],t]}^{2} dt + \int_0^T \| R \|_{V^{1/2H};[0,[[t]]]}^{2} d\tau dt \leq Ch^{2H}. \]

**Proof.** We decompose

\[ \int_0^T \int_0^T \| R \|_{V^{1/2H};[[t],t] \times [[s],s]} ds dt \]

into

\[ \int_0^T \int_0^T \| R \|_{V^{1/2H};[[t],t] \times [[s],s]} I_{[[t],t]}(s) ds dt \]

\[ + \int_0^T \int_0^T \| R \|_{V^{1/2H};[[t],t] \times [[s],s]} I_{[0,t] \setminus [[t],t]}(s) ds dt =: I_1 + I_2. \]

By means of Lemma 2.3, we get

\[ I_1 \leq \int_0^T \int_0^T h^{2H} I_{[[t],t]}(s) ds dt \leq Ch^{2H+1}. \]

For the part \( I_2 \), notice that if \( s \notin [[t], [t]] \), then the sets \([t] [[t], [t]] \) and \([s] [[s], [s]]\) are essentially disjoint. We claim that for any two essentially disjoint sets, \([a, b] \) and
\[ [c, d] \text{ with } a < b \leq c < d, \text{ the covariance of the increments of the fBm is negative. Indeed, due to } H < 1/2, \text{ it holds that} \]

\[
\mathbb{E} \left[ (B_b - B_a)(B_d - B_c) \right] \\
= \frac{1}{2} \left( (d - a)^{2H} - (d - b)^{2H} + (c - b)^{2H} - (c - a)^{2H} \right) \\
= H \left( \int_a^b (d - u)^{2H-1}du - \int_a^b (c - u)^{2H-1}du \right) \\
= H(2H - 1) \left( \int_a^b \int_c^d (v - u)^{2H-2}dvdu \right) < 0. 
\]

It then leads to

\[
\| R \|_{V^{1/2H}:[(t],|t|) \times [|s],|s|}^{1/2H} = \sup_{\pi} \sum_{i,j} \mathbb{E} \left[ R([u_i, u_{i+1}] \times [v_j, v_{j+1}]) \right]^{1/2H} \\
\leq \sup_{\pi} \left| \sum_{i,j} R([u_i, u_{i+1}] \times [v_j, v_{j+1}]) \right|^{1/2H} \\
= \mathbb{E} \left[ (B_s - B_{|s|})(B_t - B_{|t|}) \right]^{1/2H} 
\]

with \( \pi = \{ (u_i, v_j) \} \) being a partition of \( [|t|, t] \times [|s|, s] \), which yields

\[
\| R \|_{V^{1/2H}:[(t],|t|) \times [|s],|s|} \leq \mathbb{E} \left[ (B_s - B_{|s|})(B_t - B_{|t|}) \right] \\
= H(1 - 2H) \left( \int_{|t|}^t \int_{|s|}^s |v - u|^{2H-2}dvdu \right). 
\]

Then we obtain

\[
I_2 \leq C \int_0^T \int_0^T \int_{|t|}^t \int_{|s|}^s |v - u|^{2H-2}dvdu \mathbb{1}_{[0,T] \setminus [|t|,|t|]}(s)dsdt \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( \int_{0}^{t_i} + \int_{t_{i+1}}^{T} \right) \int_{|s|}^s |v - u|^{2H-2}dvdu dsdt \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_{|s|}^s |v - u|^{2H-2}dvdu dsdt \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{i+1} - u) \left( \int_{0}^{t_i} + \int_{t_{i+1}}^{T} \right) \int_{|s|}^{|v|} |v - u|^{2H-2}dvdsdu \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (t_{i+1} - u) \left( \int_{0}^{t_i} + \int_{t_{i+1}}^{T} \right) \int_{v}^{[v]} |v - u|^{2H-2}dvdsdu \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( \int_{0}^{t_i} + \int_{t_{i+1}}^{T} \right) (t_{i+1} - u)(|[v] - v)|v - u|^{2H-2}dvdu \\
\leq C\h^{2} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( \int_{0}^{t_i} + \int_{t_{i+1}}^{T} \right) |v - u|^{2H-2}dvdu. 
\]
By direct calculations, we derive
\[
\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( \int_0^{t_i} + \int_{t_i+1}^T \right) |v-u|^{2H-2} dv du \\
= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( \int_0^{t_i} (u-v)^{2H-2} dv + \int_{t_i+1}^T (v-u)^{2H-2} dv \right) du \\
= \frac{1}{1-2H} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left( (u-t_i)^{2H-1} - u^{2H-1} + (t_{i+1} - u)^{2H-1} - (T-u)^{2H-1} \right) du \\
- \frac{1}{1-2H} \int_0^T \left( u^{2H-1} + (T-u)^{2H-1} \right) du \\
= \frac{1}{2H(1-2H)} \left( \sum_{i=0}^{n-1} 2h^{2H} \right) - 2T^{2H} \leq Ch^{2H-1},
\]
which completes the proof of (3).

Similarly, we have
\[
\int_0^T \|R\|_{V^{1/2H},[0,T]} dt = \int_0^T \left| \mathbb{E} \left[ (B_t - B_{t_i}) (B_{t} - B_{t_i}) \right] \right| dt \\
= C \int_0^T \int_{[t]} \int_0^t |v-u|^{2H-2} dv dt du \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^t \int_0^{t_i} |v-u|^{2H-2} dv du dt \\
= C \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \int_0^{t_i} |v-u|^{2H-2} dv du dt \\
\leq Ch \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^{t_i} (u-v)^{2H-2} dv du \\
= Ch \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{1-2H} (u-t_i)^{2H-1} - u^{2H-1} \right) du \leq Ch^{2H}.
\]

Combining with
\[
\int_0^T \|R\|_{V^{1/2H},[t_i,t]} dt \leq Ch^{2H}
\]
implied by Lemma 2, the inequality (4) is obtained. □

Now we are in position to prove Theorem 1.3

**Proof.** By (1)-(2), we have
\[
X_t - Y_t = \int_0^t a(X_s) ds - \int_0^t a(Y_s) ds
\]
= \int_0^t (a(X_s) - a(Y_s))ds + \int_0^t (a(X_s) - a(X_s))ds,

which satisfies

\[ \mathbb{E}|X_t - Y_t|^2 \leq C \int_0^t \mathbb{E}|a(X_s) - a(Y_s)|^2 ds + C\mathbb{E}\left| \int_0^t (a(X_s) - a(X_s))ds \right|^2. \]

Taking the supremum with respect to the time variable and using the Lipschitz continuity of \(a\), we have

\[ \sup_{t \in [0,T]} \mathbb{E}|X_t - Y_t|^2 \leq C \int_0^t \mathbb{E}|X_t - Y_t|^2 ds + C \sup_{t \in [0,T]} \mathbb{E}\left| \int_0^t (a(X_s) - a(X_s))ds \right|^2. \]

Together with Gronwall’s inequality, in order to prove

\[ \left( \sup_{t \in [0,T]} \mathbb{E}|X_t - Y_t|^2 \right)^{1/2} \leq C h^{2H}, \]

it suffices to show

\[ \sup_{t \in [0,T]} \mathbb{E}\left| \int_0^t (a(X_s) - a(X_s))ds \right|^2 \leq C h^{4H}. \]

In the sequel, we focus on proving (5). The chain rule applied to \(a(X)\) implies that \(a(X)\) solves the rough differential equation

\[ da(X_t) = a'(X_t)a(X_t)dt + \sigma a'(X_t)dB_t. \]

Exploiting Lemma 2.5, we obtain

\[ a(X_s) - a(X_{[s]}) \]

\[ = \int_s^{[s]} a'(X_r)a(X_r)dr + \sigma \int_{[s]}^{s} a'(X_r)dB_r \]

\[ = \int_s^{[s]} a'(X_r)a(X_r)dr + \sigma \int_{[s]}^{s} a'(X_r)\delta B_r + \sigma^2 H \int_{[s]}^{s} a''(X_r)^{2H-1}dr \]

\[ + \sigma \int_{[0,T]^2} 1_{[s],r_2} 1_{[s],r_1} [D_{r_1} [a'(X_{s})]] - \sigma a''(X_{s})]dR_{r_1,r_2} \]

\[ =: J_1(s) + J_2(s) + J_3(s) + J_4(s). \]

It follows that

\[ \mathbb{E}\left| \int_0^u (a(X_s) - a(X_{[s]}))ds \right|^2 \]

\[ = \mathbb{E}\left[ \left( \int_0^u (J_1(t) + J_2(t) + J_3(t) + J_4(t))dt \right) \left( \int_0^u (J_1(s) + J_2(s) + J_3(s) + J_4(s))ds \right) \right] \]

\[ = \sum_{i,j=1}^4 \mathbb{E}\left[ \left( \int_0^u J_i(t)dt \right) \left( \int_0^u J_j(s)ds \right) \right] \]

\[ \leq \sum_{i,j=1}^4 \left( \mathbb{E}\left[ \left( \int_0^u J_i(t)dt \right)^2 \right] \right)^{1/2} \left( \mathbb{E}\left[ \left( \int_0^u J_j(s)ds \right)^2 \right] \right)^{1/2}. \]

It then remains to estimate \( \mathbb{E}\left[ \left( \int_0^u J_i(t)dt \right)^2 \right] \) for each \(i \in \{1, 2, 3, 4\}\).
For $J_1$, Lemma 3.1 leads to
\[
\mathbb{E} \left[ \left( \int_0^u J_1(t) dt \right)^2 \right]
= \mathbb{E} \left[ \int_0^u \int_t^u a'(X_r) a(X_s) dr dt + \int_0^u \int_s^u a'(X_r) a(X_s) ds dv \right]
\leq h^2 \int_0^u \int_t^u \mathbb{E} \left[ |a'(X_r) a(X_s)| dv dr \right] \leq Ch^2.
\]

For $J_2$, based on [19] Chapter 1, we have
\[
\mathbb{E} \left[ \left( \int_0^u J_2(t) dt \right)^2 \right]
= \sigma^2 \mathbb{E} \left[ \int_0^u \int_t^u a'(X_r) \delta B_r dr + \int_0^u \int_s^u a'(X_r) \delta B_s ds \right]
\leq \sigma^2 \int_0^u \int_t^u \mathbb{E} \left[ \left( \int_{[t,t]} X_r (r_1) X_{[s,s]} (r_2) a'(X_r) a'(X_s) dR_{r_1, r_2} \right)^2 \right] dt ds
+ \sigma^2 \int_0^u \int_t^u \mathbb{E} \left[ \left( \int_{[t,t]} X_r (r_1) X_{[s,s]} (r_2) a(X_r) a(X_s) dR_{u_1, u_2} \right)^2 \right] dt ds
\leq \sigma^2 A_1 + \sigma^2 A_2.
\]

According to the regularity of $R$ and $X$ given in Lemma 2.4 and Lemma 3.1, we get from the fact $H > 1/3$ that the functions
\[
f : [0, T]^2 \rightarrow \mathbb{R},
(r_1, r_2) \mapsto f_{r_1, r_2} := a'(X_{r_1}) a'(X_{r_2})
\]
and $R$ have complementary regularity almost surely. Moreover, for any $p \geq 1$ and $\beta < H$, it holds that
\[
\mathbb{E} \left[ \left\| f \right\|^p_{V^{1/\beta, [0, T]^2}} \right] \leq C.
\]
Then Lemma 2.2 and Lemma 3.2 produce
\[
|A_1| \leq C \int_0^u \int_0^u \left\| R \right\|_{V^{1/2, [t,t] \times [s,s]}}^2 dt ds \leq C h^{2H+1}.
\]

Meanwhile, the Malliavin derivative satisfies
\[
D_{u_1} [a'(X_{r_1})] = a''(X_{r_1}) D_{u_1} X_{r_1} = \sigma J_{r_1} \mathcal{F}_{u_1}^{-1} a''(X_{u_1}),
\]
where $\mathcal{J}$ and $\mathcal{J}^{-1}$ solve the linear system

$$
\begin{align*}
\mathcal{J}_t &= 1 + \int_0^t a'(X_s)\mathcal{J}_s ds, \\
\mathcal{J}_t^{-1} &= 1 + \int_0^t \mathcal{J}_s^{-1} a'(X_s) ds.
\end{align*}
$$

Since the second order derivative of $a$ is bounded, it implies that the functions

$$
\tilde{f} : [0, T]^2 \to \mathbb{R},
$$

$$(r_1, r_2) \mapsto \tilde{f}_{r_1, r_2} := \int_{[0, T]^2} \mathbb{I}_{[0, r_1]}(u_1) \mathbb{I}_{[0, r_2]}(u_2) \mathbb{E} \left[ D_{u_1} \left[ a'(X_{r_1}) \right] D_{u_2} \left[ a'(X_{r_2}) \right] \right] dR_{u_1, u_2}
$$

and $R$ have complementary regularity almost surely, and $\mathbb{E} \left[ \| \tilde{f} \|_{V^{1/2};[0, T]^2}^p \right] \leq C$. Then we deduce

$$
|A_2| \leq C \int_0^u \int_0^u \| R \|_{V^{1/2};[t, s] \times [t, s]} dtds \leq C^2 H^{2H+1}.
$$

The above estimates for $A_1$ and $A_2$ yield

$$
\mathbb{E} \left[ \left( \int_0^u J_2(t) dt \right)^2 \right] \leq C^2 H^{2H+1}.
$$

For $J_3$, due to $H > 1/3$ and Lemma 5.1 it holds that

$$
\mathbb{E} \left[ \left( \int_0^u J_3(t) dt \right)^2 \right] = \sigma^4 H^2 \mathbb{E} \left[ \int_0^u \int_0^u a''(X_r) r^{2H-1} dt dr \int_0^u \int_0^s a''(X_v) v^{2H-1} dv ds \right]
$$

$$
= \sigma^4 H^2 \mathbb{E} \left[ \int_0^u \int_0^{[r]} a''(X_r) r^{2H-1} dt dr \int_0^u \int_0^{[v]} a''(X_v) v^{2H-1} dv ds \right]
$$

$$
\leq C H^2 \int_0^u \int_0^u \mathbb{E} \left[ a''(X_r) a''(X_v) \right] r^{2H-1} v^{2H-1} dt dr \leq C^2 H^2.
$$

For $J_4$, recall that

$$
\mathbb{E} \left[ \left( \int_0^u J_4(t) dt \right)^2 \right] = \sigma^2 \mathbb{E} \left[ \left( \int_0^u \int_0^t \mathbb{I}_{[t, s]}(r_2) \mathbb{I}_{[0, r_1]}(r_1) \left[ D_{r_1} \left[ a'(X_{r_2}) \right] - \sigma a''(X_{r_2}) \right] dR_{r_1, r_2} dt \right) \right]
$$

$$
\times \left( \int_0^u \int_0^t \mathbb{I}_{[s, t]}(r_4) \mathbb{I}_{[0, r_1]}(r_1) \left[ D_{r_3} \left[ a'(X_{r_4}) \right] - \sigma a''(X_{r_4}) \right] dR_{r_2, r_4} ds \right).
$$

Based on the boundedness of the third order derivative of $a$, Section 6 implies that the continuous functions

$$
g : [0, T]^2 \to \mathbb{R},
$$

$$(r_1, r_2) \mapsto g_{r_1, r_2} := \mathbb{I}_{[0, r_2]}(r_1) \left[ D_{r_1} \left[ a'(X_{r_2}) \right] - \sigma a''(X_{r_2}) \right]$$
and $R$ have complementary regularity almost surely, and $E\left[\|g\|_{H_{1/2;[0,T]}^2}^{1/2}\right] \leq C$.

Using Lemma 3.2 and the formulation

$$
\int_0^u \int_{[0,T]^2} \mathbf{1}_{[[t],t]}(r_2) \mathbf{1}_{[0,r_2]}(r_1) \left[ D_{r_1} \left[ a' \left( X_{r_2} \right) \right] - \sigma a'' \left( X_{r_2} \right) \right] dR_{r_1,r_2} dt
$$

we have

$$
\int_0^u \int_{[0,T]^2} \mathbf{1}_{[[t],t]}(r_2) \mathbf{1}_{[0,r_2]}(r_1) g_{r_1,r_2} dR_{r_1,r_2} dt + \int_0^u \int_{[0,T]^2} \mathbf{1}_{[[t],t]}(r_2) \mathbf{1}_{[0,r_2]}(r_1) g_{r_1,r_2} dR_{r_1,r_2} dt,
$$

we have

$$
E \left[ \left( \int_0^u J_4(t) dt \right)^2 \right] \leq Ch^{4H},
$$

which completes the proof.

\[ \square \]

**Corollary 3.3.** Let $H \in \left( \frac{1}{4}, \frac{1}{2} \right)$. Assume that $a(x) = Ax$ with a constant $A$. Then it holds that

$$
\left( \sup_{t \in [0,T]} E \left[ X_t - Y_t \right]^2 \right)^{1/2} \leq C h^{H + \frac{1}{2}},
$$

where $X$ solves (1) and $Y$ is given by the Euler scheme (2).

**Proof.** Since $a(x) = Ax$, the second derivative of $a$ vanishes. Repeating the proof of Theorem 1.1 we have

$$
E \left[ \left( \int_0^u J_3(t) dt \right)^2 + \left( \int_0^u J_4(t) dt \right)^2 \right] \leq Ch^{2H + 1}
$$

and $J_3 = J_4 = 0$. Then the result is obtained.

\[ \square \]

**Remark 3.4.** In the case of $H > \frac{1}{2}$, the framework for Malliavin calculus holds with $R$ being more regular and we refer to [5, 9, 11, 12, 14] and references therein for the analysis on numerical schemes. In the case of $H \leq \frac{1}{2}$, more efforts should be paid to establish a conversion formula between the Skorohod integral and the rough integral. If $H \leq \frac{1}{4}$, the well-posedness of SDEs with multiplicative noises in multi-dimensional cases is still an open problem.

4. Numerical simulations

In this section, we give numerical simulations for the SDE

$$
dX_t = a(X_t) dt + \sigma dB_t, \quad t \in (0, 1]
$$

with $X_0 = 1$ and $B$ being the fBm with Hurst parameter $H \in \left( \frac{1}{4}, \frac{1}{2} \right)$.

**Example 4.1.** Let the diffusion coefficient $\sigma = 1$ and the drift coefficient

$$
a(x) = \begin{cases} 
\ln |x|, & |x| \geq 1, \\
\frac{1}{6}x^6 - \frac{3}{4}x^4 + \frac{3}{2}x^2 - \frac{11}{12}, & |x| < 1.
\end{cases}
$$

Since $a$ has bounded derivatives up to order three, Theorem 1.1 leads to that the mean square convergence rate of the Euler scheme (2) is $2H$.

**Example 4.2.** Let the diffusion coefficient $\sigma = 9$ and the drift coefficient $a(x) = 2x$. Due to the linearity of $a$, it follows from Corollary 3.3 that the mean square convergence rate of the Euler scheme (2) is $H + \frac{1}{2}$. 
In Figure 1 and Figure 2 the error
\[
\left( \sup_{t \in [0,T]} \mathbb{E} |X_t - Y_t|^2 \right)^{1/2}
\]
is presented for Example 4.1 and Example 4.2 respectively. We take the Hurst parameter of the fBm as $H = 0.35, 0.4, 0.45$. The exact solution is simulated by the numerical solution with a fine time step size $h = \frac{1}{T}$ and the expectation is approximated by 1000 sample paths. The numerical results support our theoretical analysis.

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