A CONNECTION WITH SKEW SYMMETRIC TORSION
AND KÄHLER CURVATURE TENSOR ON
QUASI-KÄHLER MANIFOLDS WITH NORDEN METRIC

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Abstract
There is considered a connection with skew symmetric torsion on a quasi-Kähler
manifold with Norden metric. Some necessary and sufficient conditions are derived
for the corresponding curvature tensor to be Kählerian. In the case when this ten-
sor is Kählerian, some relations are obtained between its scalar curvature and the
scalar curvature of other curvature tensors. Conditions are given for the considered
manifolds to be isotropic-Kähler.

Key words: almost complex manifold, Norden metric, nonintegrable structure,
skew symmetric torsion, quasi-Kähler manifolds

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1. Preliminaries

Let \((M, J, g)\) be a \(2n\)-dimensional almost complex manifold with Norden
metric, i.e. \(M\) is a differentiable manifold with an almost complex structure
\(J\) and a metric \(g\) such that

\[
J^2 x = -x, \quad g(Jx, Jy) = -g(x, y)
\]

for arbitrary \(x, y\) of the algebra \(\mathfrak{X}(M)\) on the smooth vector fields on \(M\).
The associated metric \(\tilde{g}\) of \(g\) on \(M\) is defined by \(\tilde{g}(x, y) = g(x, Jy)\). Both
metrics are necessarily of signature \((n, n)\). The manifold \((M, J, \tilde{g})\) is an
almost complex manifold with Norden metric, too.

Further, \(x, y, z, w\) will stand for arbitrary elements of \(\mathfrak{X}(M)\).

A classification of the almost complex manifolds with Norden metric is given
in [1]. This classification is made with respect to the tensor field \(F\) of type
\((0,3)\) defined by

\[
F(x, y, z) = g((\nabla x J) y, z),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). The tensor \(F\) has the following
properties

\[
F(x, y, z) = F(x, z, y) = F(x, Jy, Jz).
\]

Among the basic classes \(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3\) of this classification, the almost com-
plex structure is nonintegrable only in the class \(\mathcal{W}_3\). This is the class of the
so-called \textit{quasi-Kähler manifolds with Norden metric}, which we call briefly \(\mathcal{W}_3\)-manifolds. This class is characterized by the condition
\[ \mathfrak{S} F(x, y, z) = 0, \] (1.4)
where \(\mathfrak{S}\) is the cyclic sum by three arguments. The special class \(\mathcal{W}_0\) of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition \(F(x, y, z) = 0\).

Let \(R\) be the curvature tensor of \(\nabla\), i.e. \(R(x, y)z = \nabla_x (\nabla_y z) - \nabla_y (\nabla_x z) - \nabla_{[x,y]} z\). The corresponding tensor of type \((0, 4)\) is determined by \(R(x, y, z, w) = g(R(x, y)z, w)\).

The following Ricci identity for almost complex manifolds with Norden metric is known
\[ (\nabla_x F)(y, z, w) - (\nabla_y F)(x, z, w) = R(x, y, Jz, w) - R(x, y, z, Jw). \] (1.5)

The square norm of \(\nabla J\) is defined by
\[ ||\nabla J||^2 = g^{ij} g^{ks} (\nabla e_i J) e_k (\nabla e_j J) e_s. \] (1.6)

In \cite{2} the following equation is proved for a \(\mathcal{W}_3\)-manifold
\[ ||\nabla J||^2 = -2g^{ij} g^{ks} (\nabla e_i J) e_k (\nabla e_j J) e_s. \] (1.7)

An almost complex manifold with Norden metric \((M, J, g)\) is Kählerian iff \(\nabla J = 0\). It is clear that we have \(||\nabla J||^2 = 0\) for such a manifold, but the inverse one is not always true. An almost complex manifold with Norden metric with \(||\nabla J||^2 = 0\) is called an \textit{isotropic-Kählerian} in \cite{2}.

The Ricci tensor \(\rho\) for the curvature tensor \(R\) and the scalar curvature \(\tau\) for \(R\) are defined respectively by
\[ \rho(x, y) = g^{ij} R(e_i, x, y, e_j), \quad \tau = g^{ij} \rho(e_i, e_j), \] (1.8)
and their associated quantities \(\rho^*\) and \(\tau^*\) are determined respectively by
\[ \rho^*(x, y) = g^{ij} R(e_i, x, y, Je_j), \quad \tau^* = g^{ij} \rho(e_i, Je_j). \] (1.9)

Similarly, the Ricci tensor and the scalar curvature are determined for each \textit{curvature-like tensor} (curvature tensor) \(L\), i.e. for the tensor \(L\) with the following properties:
\[ L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \] (1.10)
\[ \mathfrak{S} L(x, y, z, w) = 0 \quad \text{(first Bianchi identity)}. \] (1.11)

A curvature-like tensor is called a \textit{Kähler tensor} if it has the property
\[ L(x, y, Jz, Jw) = -L(x, y, z, w). \] (1.12)
The characteristic condition (1.14) for $W_3$ is equivalent to each of the following conditions [2]:

\[(1.13) \quad S_{x,y,z} F(Jx, y, z) = 0,\]

\[(1.14) \quad (\nabla_x J) Jy + (\nabla_y J) Jx + (\nabla_{Jx} J) y + (\nabla_{Jy} J) x = 0.\]

The following identity for a $W_3$-manifold is known from [3]:

\[(1.15) \quad S_{x,y,z} \{ R(x, Jy, Jz, w) - R(x, Jy, z, Jw) + R(Jx, y, z, Jw) - R(Jx, y, Jz, w) \} = -S_{x,y,z} g( (\nabla_x J) y + (\nabla_y J) x, (\nabla_z J) w + (\nabla_w J) z ).\]

2. A CONNECTION WITH SKEW SYMMETRIC TORSION ON A $W_3$-MANIFOLD

A linear connection $\nabla'$ on an almost complex manifold with Norden metric $(M, J, g)$ preserving $J$ and $g$, i.e. $\nabla' J = \nabla' g = 0$, is called a natural connection [4]. If $T$ is a torsion tensor of $\nabla'$, i.e. $T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]$, then the corresponding tensor field of type $(0,3)$ is determined by $T(x, y, z) = g(T(x, y), z)$.

The connections with skew symmetric torsion are of particular interest in the string theory [5]. In mathematics this connection was used by Bismut [6] to prove the local index theorem for non-Kähler Hermitian manifolds.

In this paper we consider a natural connection $\nabla'$ with skew symmetric torsion on quasi-Kähler manifolds with Norden metric whose curvature tensor has the properties of the curvature tensor of a Kähler manifold with Norden metric. This connection is determined by

\[(2.1) \quad \nabla'_x y = \nabla_x y + Q(x, y),\]

where

\[(2.2) \quad Q(x, y) = \frac{1}{4} \left\{ (\nabla_x J) y - (\nabla_{Jx} J) y - 2(\nabla_{Jy} J) x \right\}.\]

For the torsion tensor $T$ of $\nabla'$ we have $T(x, y) = 2Q(x, y)$. We denote

\[(2.3) \quad Q(y, z, w) = g(Q(y, z), w)\]

and according to (1.2), (1.3), (2.2) and (2.3) we obtain

\[(2.4) \quad Q(y, z, w) = -\frac{1}{4} S_{y,z,w} F(y, z, Jw).\]
3. CONDITIONS FOR THE CURVATURE TENSOR OF THE CONNECTION $\nabla'$ ON $W_3$-MANIFOLDS TO BE KÄHLERIAN

Let $R'$ be the curvature tensor of the connection $\nabla'$ on a $W_3$-manifold $(M, J, g)$ determined by (2.1), i.e.

$$R'(x, y, z) = \nabla_x' (\nabla_y' z) - \nabla_y' (\nabla_x' z) - \nabla_{[x, y]}' z.$$  

The corresponding tensor of type $(0,4)$ is determined by $R'(x, y, z, w) = g(R'(x, y)z, w)$. According to (2.1), (2.2) and (2.3), we have

$$g(\nabla_x' y, z) = g(\nabla_x y, z) + Q(x, y, z).$$

Since $\nabla g = \nabla' g = 0$ then (3.1), (3.2) and (2.1) imply

$$R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w)$$

$$- g(Q(y, z), Q(x, w)) + g(Q(x, z), Q(y, w)).$$

The last equality implies the property (1.10) for $R'$ and since $\nabla' J = 0$ then (1.12) is valid, too. Therefore $R'$ becomes Kählerian if the condition (1.11) is fulfilled for this tensor. Because of (3.3) the equality (1.11) is valid for $R'$ iff

$$\mathcal{S}_{x, y, z} \{ (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w)$$

$$- g(Q(y, z), Q(x, w)) + g(Q(x, z), Q(y, w)) \} = 0.$$  

Since $Q$ is a totally skew symmetric tensor then (3.4) gets the form

$$\mathcal{S}_{x, y, z} \{ (\nabla_x Q)(y, z, w) \} = \mathcal{S}_{x, y, z} \{ g(Q(y, z), Q(x, w)) \}.$$  

The last equality implies immediately

$$(\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w)$$

$$= - (\nabla_x Q)(x, y, w) + \mathcal{S}_{x, y, z} \{ g(Q(x, y), Q(z, w)) \}$$

and then (3.3) gets the form

$$R'(x, y, z, w) = R(x, y, z, w)$$

$$- (\nabla_x Q)(x, y, w) + g(Q(x, y), Q(z, w)).$$

In (3.6) we substitute $y \leftrightarrow w$ and we add the obtained equality to (3.6). Then we receive

$$R'(x, y, z, w) + R'(z, y, x, w) = R(x, y, z, w) + R(z, y, x, w)$$

$$+ g(Q(x, y), Q(z, w)) + g(Q(z, y), Q(x, w)).$$

Now we substitute $z \leftrightarrow w$ in (3.7) and then we subtract the obtained equality from (3.7). Using the properties of $R$ and $R'$ in the last equality we finally obtain the following identity, which is equivalent to (3.4):

$$3R'(x, y, z, w) = 3R(x, y, z, w) + 2g(Q(x, y), Q(z, w))$$

$$+ g(Q(z, y), Q(x, w)) + g(Q(x, z), Q(y, w)).$$
In this way we proved the following

**Theorem 3.1.** Let \((M, J, g)\) be a \(\mathcal{W}_3\)-manifold and \(\nabla'\) be the connection determined by (2.1). Then the curvature tensor \(R'\) for \(\nabla'\) is Kählerian iff the condition (3.8) is valid.

Obviously the tensor \(P\) defined by

\[
P(x, y, z, w) = 2g(Q(x, y), Q(z, w)) + g(Q(z, y), Q(x, w)) + g(Q(x, z), Q(y, w))
\]  

satisfies the properties (1.10) and (1.11), i.e. \(P\) is a curvature-like tensor.

Then from Theorem 3.1 we obtain the following

**Corollary 3.2.** Let \((M, J, g)\) be a \(\mathcal{W}_3\)-manifold with a Kähler curvature tensor \(R'\) for the connection \(\nabla'\) determined by (2.1). Then the tensor \(P\) determined by (3.9) is Kählerian iff the curvature tensor \(R\) is Kählerian.

Using (1.3), (1.5), (2.3), (2.4), (3.1) and the first Bianchi identity for \(R\), we get the following identity, which is equivalent to (3.5):

\[
\mathcal{S}_{x,y,z} \{(\nabla_w F)(x, z, Jy)\} = A(x, y, z, w),
\]  

where

\[
A(x, y, z, w) = \mathcal{S}_{x,y,z} \{R(x, y, Jz, Jw) + R(Jx, Jy, z, w)
\]

\[
+ 4g(Q(x, y), Q(z, w)) - g(\nabla_x J, Jy, Jz) - (\nabla_y J, x, \nabla_z J) w\}
\]  

According to the properties of \(F\), from (3.10) and (3.11) we obtain

\[
A(Jx, y, z, w) + A(x, Jy, z, w) + A(x, y, Jz, w) - A(Jx, Jy, Jz, w) = 0.
\]  

Because of (3.11) the last equality implies

\[
\mathcal{S}_{x,y,z} \{g(\nabla_x J, Jy + (\nabla_J x) J y, (\nabla_w J) z + (\nabla_J z) J w - (\nabla_J J) w)\}
\]

\[
= 2 \mathcal{S}_{x,y,z} \{g(Q(x, y), Q(z, w)) + g(Q(Jx, y), Q(z, w))
\]

\[
+ g(Q(x, Jy), Q(z, w)) - g(Q(Jx, Jy), Q(Jz, w))\}.\]

Having in mind \(Q(x, Jy) = JQ(x, y) - (\nabla_x J) J y\) and (1.14), from (3.13) we get the following identity, equivalent to (3.5)

\[
\mathcal{S}_{x,y,z} \{g(\nabla_x J, Jy + (\nabla_J x) J y, (\nabla_J z) J w + (\nabla_J J) w)\} = 0.
\]  

Then the following theorem is satisfied.

**Theorem 3.3.** Let \((M, J, g)\) be a \(\mathcal{W}_3\)-manifold and \(\nabla'\) be the connection determined by (2.1). Then the curvature tensor \(R'\) for \(\nabla'\) is Kählerian iff the condition (3.14) is valid.
It is easy to verify that the properties (1.10), (1.11) and (1.12) are valid for the tensor $H$ defined by
\[(3.15)\quad H(x, y, z, w) = g((\nabla_x J) J y + (\nabla_{Jx} J) y, (\nabla_z J) J w + (\nabla_{Jz} J) w).\]

Then Theorem 3.3 implies the following

**Corollary 3.4.** Let $(M, J, g)$ be a $\mathcal{W}_3$-manifold and $\nabla'$ be the connection determined by (2.1). Then the curvature tensor $R'$ for $\nabla'$ is Kählerian iff the tensor $H$ determined by (3.15) is Kählerian.

4. **Scalar curvatures on a $\mathcal{W}_3$-manifold with Kähler curvature tensor of the connection $\nabla'$**

Let $(M, J, g)$ be a $\mathcal{W}_3$-manifold with Kähler curvature tensor of the connection and $\nabla'$ be determined by (2.1). Then the tensor $H$ determined by (3.15) is also Kählerian whereas the curvature tensor $R$ and the tensor $P$ determined by (3.9) are curvature-like. We denote the scalar curvatures of $R$, $R'$, $P$ and $H$ by $\tau$, $\tau'$, $\tau^*(P)$ and $\tau^*(H)$, respectively, and their associated scalar curvatures by $\tau^*$, $\tau'^*$, $\tau^*_{\ast}(P)$ and $\tau^*_{\ast}(H)$, respectively. We denote the associated square norm of $\nabla J$ with respect to $\tilde{g}$ by $\|\nabla J\|^2$.

The equalities (3.8) and (3.9) imply immediately
\[(4.1)\quad 3\tau' = 3\tau + \tau(P),\]
\[(4.2)\quad 3\tau'^* = 3\tau^* + \tau^*(P),\]
\[(4.3)\quad \tau(P) = 3g^{ij}g^{ks}g(Q(e_i, e_k), Q(e_s, e_j)).\]

We obtain $g^{ij}F(e_i, e_j, z) = g^{ij}F(e_i, Je_j, z) = 0$ from (1.4). The last equality and (2.4) imply $g^{ij}Q(e_i, e_j) = 0$. Then, having in mind (4.3), we get $\tau(P) = \frac{3}{8} \left(3\|\nabla J\|^2 + 2\|\nabla J\|^{*2}\right)$. Because of the antisymmetry of $Q$, (4.3) implies
\[(4.4)\quad \tau(P) = \frac{9}{8} \|\nabla J\|^2.\]

From (4.1) and (4.4) we have
\[(4.5)\quad \tau' = \tau + \frac{3}{8} \|\nabla J\|^2.\]

By virtue of (3.9) we get $\tau^*(P) = 3g^{ij}g^{ks}g(Q(e_i, e_k), Q(Je_s, e_j))$, from where
\[(4.6)\quad \tau^*(P) = -\frac{3}{8} \|\nabla J\|^2.\]

Then, according to (1.15) and (4.2) we have
\[(4.7)\quad \tau'^* = \tau^* - \frac{1}{8} \|\nabla J\|^2.\]
The equalities (4.5) and (4.7) imply
\[ (4.8) \quad \tau' + 3\tau'' = \tau + 3\tau^*. \]

Using (1.14) and (3.15) we obtain
\[ (4.9) \quad \tau(H) = \tau^*(H) = 2\|\nabla J\|^2. \]

Then, from (4.5), (4.7) and (4.9) the following equalities are valid
\[ (4.10) \quad \tau' = \tau + \frac{3}{16}\tau(H), \]
\[ (4.11) \quad \tau^* = \tau^* - \frac{1}{16}\tau(H). \]

By virtue of (4.4), (4.5), (4.6), (4.7) and (4.9), we get the following

**Theorem 4.1.** Let \((M, J, g)\) be a \(\mathcal{W}_3\)-manifold with Kähler curvature tensor \(R'\) of the connection \(\nabla'\) determined by (2.1). Then \((M, J, g)\) is an isotropic-Kähler manifold iff an arbitrary one of the quantities \(\tau - \tau', \tau^* - \tau^*, \tau(P), \tau^*(P), \tau(H), \tau^*(H)\) is zero.

Now, let \((M, J, g)\) be a 4-dimensional \(\mathcal{W}_3\)-manifold. Since \(R'\) is a Kähler tensor, according to [7] we have
\[ (4.12) \quad R' = \nu' (\pi_1 - \pi_2) + \nu^* \pi_3, \]
where \(\nu' = \frac{\tau'}{8}, \nu^* = \frac{\tau^*}{8}\) and
\[
\begin{align*}
\pi_1(x, y, z, w) &= g(y, z)g(x, w) - g(x, z)g(y, w), \\
\pi_2(x, y, z, w) &= g(y, Jz)g(x, Jw) - g(x, Jz)g(y, Jw), \\
\pi_3(x, y, z, w) &= -g(y, z)g(x, Jw) + g(x, z)g(y, Jw), \\
&\quad -g(y, Jz)g(x, w) + g(x, Jz)g(y, w).
\end{align*}
\]

According to (4.3), (4.7), (4.12) and (3.9), from (3.8) we obtain
\[ (4.13) \quad R = \frac{1}{8} \left\{ \left( \tau + \frac{3}{8}\|\nabla J\|^2 \right) (\pi_1 - \pi_2) + \left( \tau^* - \frac{1}{8}\|\nabla J\|^2 \right) \pi_3 \right\} - \frac{1}{3} P. \]

Then we have the following

**Theorem 4.2.** Let \((M, J, g)\) be a 4-dimensional \(\mathcal{W}_3\)-manifold with Kähler curvature tensor \(R'\) of the connection \(\nabla'\) determined by (2.1). Then \((M, J, g)\) is an isotropic-Kähler manifold iff
\[ R = \frac{1}{8} \{ \tau (\pi_1 - \pi_2) + \tau^* \pi_3 \} - \frac{1}{3} P. \]

Because of (4.9) and (4.13) the following theorem is valid.

**Theorem 4.3.** Let \((M, J, g)\) be a 4-dimensional \(\mathcal{W}_3\)-manifold with Kähler curvature tensor \(R'\) of the connection \(\nabla'\) determined by (2.1). Then we have
\[ R = \frac{1}{128} \{ (16\tau + \tau(H)) (\pi_1 - \pi_2) + (16\tau^* - \tau(H)) \pi_3 \} - \frac{1}{3} P. \]
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