Regular second order perturbations of extreme mass ratio black hole binaries

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Abstract

We report on the first results of self-consistent second order metric perturbations produced by a point particle moving in the Schwarzschild spacetime. The second order waveforms satisfy a wave equation with an effective source term build up from products of first order metric perturbations and its derivatives. We have explicitly regularized this source term at the particle location as well as at the horizon and at spatial infinity.

1 Introduction

The observation of gravitational waves opens a new window onto our universe and we also expect that the observation of gravitational waves will provides a direct experimental test of general relativity.

The space mission LISA will primarily detect gravitational waves from inspiraling solar-mass compact objects captured by a supermassive black hole residing in the core of active galaxies. For these Extreme Mass Ratio Inspirals (EMRI) we use the black hole perturbation approach, where the compact object is approximated by a point particle orbiting a massive black hole. There are two nontrivial problems to consider in this approach: The self-force problem and the second order gravitational perturbations problem. Due to the self-force the orbit of the particle deviates from the background geodesic, i.e. the spacetime is perturbed by the particle itself. It is essential to take this deviation into account in order to predict the orbital evolution to the required order. For the headon configuration studied in this paper this was achieved in \cite{1}. The gravitational self-force is, however, not easily obtainable for more general trajectories.

We require the second perturbative order calculations to derive the precise gravitational waveforms to be used as templates for gravitational wave data analysis. In general, this computation has to be done by numerical integration. Hence, it is important to derive a well-behaved second order effective source. In this paper, we will focus on this later problem.

2 Second order metric perturbations

We consider second order metric perturbations (MP), \( \tilde{g}_{\mu \nu} = g_{\mu \nu} + h^{(1)}_{\mu \nu} + h^{(2)}_{\mu \nu} \), with expansion parameter \( \mu/M \) corresponding to the mass ratio of the holes and where \( g_{\mu \nu} \) is the Schwarzschild metric. The Hilbert-Einstein tensor and the stress-energy tensor up to the second perturbative order is given by

\[ G_{\mu \nu}[\tilde{g}_{\mu \nu}] = G^{(1)}_{\mu \nu}[h^{(1)}] + G^{(1)}_{\mu \nu}[h^{(2)}] + G^{(2)}_{\mu \nu}[h^{(1)}], \quad T_{\mu \nu} = T^{(1)}_{\mu \nu} + T^{(2, SF)}_{\mu \nu} + T^{(2, h)}_{\mu \nu}, \]  

where

\[ G^{(1)}_{\mu \nu}[h] = -\frac{1}{2} h_{\mu \nu ; \alpha} h^{\alpha} + R_{\alpha \mu \beta \nu} h^{\alpha \beta} - \frac{1}{2} h_{\mu \nu} - \frac{1}{2} g_{\mu \nu} (h_{\lambda \alpha ; \alpha \lambda} - h_{\lambda \lambda ; \alpha}), \]

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the same differential behavior. Then, we note that

can see this as follows. First, from Eqs. (Z:C7d) and (Z:C7e), we find that

\[ \psi = \psi_{\text{RW}} \]

where the suffix RW stands for the RW gauge. This function

harmonics coefficients of the energy-momentum tensor are

\[ E \]

For the even part, which has the even parity behavior, (−1)\[ \ell \] (\ell = 0, 1, 2,...), we have

instance, denote the equation (1) in [3], [4] and [5], respectively.

summarized in the time domain in [4, 5]. In the following, equations numbers (Z:1), (L1:1) and (L2:1) for

Zerilli formalism [2, 3]. The basic formalism has been given in Zerilli’s paper [3], and it has been

Before considering the second order, it is necessary to discuss the first order MP, i.e., the Regge-Wheeler-

force

h

\[ \text{even} \]

z = \{T(\tau), R(\tau), \Theta(\tau), \Phi(\tau)\} for the particle orbit, the deviation from the geodesic by the self-force

\[ T_{\mu \nu}^{(2, SF)} \] (See Ref. [1]), and finally \[ T_{\mu \nu}^{(2, h)} \], which is purely affected by the first order MP,

\[ \frac{\partial}{\partial \tau} H^\text{RW}_{2 \ell m}(t, r) - \frac{r}{2} \frac{\partial}{\partial t} K^\text{RW}_{2 \ell m}(t, r) + \sum_{\ell = 0}^\infty \left[ 2 \ell + 1 \right] K^\text{RW}_{2 \ell m}(t, r) \]

where we have used the determinant \( \hat{g} = g(1 + h^{(1)}) \) up to the first perturbative order.

3 First order metric perturbations in the RW gauge

Before considering the second order, it is necessary to discuss the first order MP, i.e., the Regge-Wheeler-

Zerilli formalism [2, 3]. The basic formalism has been given in Zerilli’s paper [3], and it has been

summarized in the time domain in [4, 5]. In the following, equation numbers (Z:1), (L1:1) and (L2:1) for

\[ \psi^{\ell m}_{\text{even}}(t, r) = \frac{2 r}{\ell (\ell + 1)} \left[ K^\text{RW}_{\ell m}(t, r) + 2 \left( \frac{r - 2 M}{r^2 + r \ell - 2 r + 6 M} \right) \left( H^\text{RW}_{2 \ell m}(t, r) - r \frac{\partial}{\partial t} K^\text{RW}_{2 \ell m}(t, r) \right) \right] , \]

where the suffix RW stands for the RW gauge. This function \( \psi^{\ell m}_{\text{even}} \) obeys the Zerilli equation,

\[ \hat{Z}^{\text{even}}_{\ell} \psi^{\ell m}_{\text{even}}(t, r) = \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial r^2} - V^\text{even}(r) , \]

where \( r^* = r + 2 M \log(r/2M - 1) \), the potential \( V^\text{even} \) and the source \( S^\text{even}_{\ell m} \) are given in Eqs. (L1:1-2) and

(L1:A.3). The reconstruction of the MP under the RW gauge have been expressed in Eqs. (L2:B.9-12).

In the following, we consider a particle falling radially into a Schwarzschild black hole as the first

order source. The equation of motion of the test particle is given by

\[ \left( \frac{dR}{dt} \right)^2 = - \left( 1 - \frac{2 M}{R} \right) \frac{1}{E^2} + \left( 1 - \frac{2 M}{R} \right) \frac{dT(t)}{d\tau} , \]

where \( E \) and \( R \) are the energy and the location of the particle, respectively. The non-vanishing tensor

harmonics coefficients of the energy-momentum tensor are \( A_{\ell m}, A^0_{\ell m} \) and \( A^{(1)}_{\ell m} \). Because of the symmetry

of the problem we have only to consider even parity modes, i.e., described by the Zerilli equation.

In the head on collision case, the MP in the RW gauge are \( C^0 \) (continuous across the particle). One

can see this as follows. First, from Eqs. (Z.C7d) and (Z.C7e), we find that \( H^\text{RW}_{2 \ell m}(= H_{0 \ell m}) \) and \( K_{\ell m} \) have

the same differential behavior. Then, we note that \( \partial_t K_{\ell m} \sim \theta(r - R(t)) \), because the left hand side of
behaves as a step function near the particle. This means that \( K_{1m} \) (and also \( H_{21m} \)) is \( C^0 \). Using this, \( \partial_r H_{11m} \sim \theta(r - R(t)) \) is derived from Eq. (Z:C7e), i.e., \( H_{11m} \) is \( C^0 \). (See Ref. [6].)

Using the above fact, we can take up second derivatives of the function \( \psi_{1m}^{\text{even}} \) with respect to \( t \) and \( r \). These quantities allow us to calculate the coefficients of the \( \delta \)-terms in the second order source.

Next, we consider the \( \ell = 0 \) perturbations (\( \ell = 1 \) modes can be completely eliminated in the center of mass coordinate system). In the RW gauge there are four non-vanishing coefficients of the MP, \( H_{000} \), \( H_{100} \), \( H_{200} \) and \( K_{00} \). The gauge transformation has two extra degrees of freedom. We could choose the gauge so that \( H_{100} = K_{00} = 0 \), i.e., the Zerilli gauge, but it is difficult to treat the second order source, since the MP are not \( C^0 \).

We instead consider here a new (singular) gauge transformation, chosen to make the metric perturbations \( C^0 \) and to obtain an appropriate second order source behavior. To derive this gauge transformation, we have also considered a regularization of the second order source at \( r = \infty \) and the horizon at the same time\(^3\). Note that we succeeded in choosing the gauge such that all of the above MP behave as \( C^0 \) at the location of the particle and vanish at \( r = \infty \) and \( r = 2M \).

### 4 Second order Zerilli equation

Since the first order MP contains only even parity modes, we can discuss the second order MP for the even parity modes only, i.e., in terms of the Zerilli function,

\[
\chi_{20}^{Z}(t, r) = \frac{1}{2r + 3M} \left( r^2 \frac{\partial}{\partial t} \chi_{20}(t, r) - (r - 2M) \mathcal{H}_{120}(t, r) \right). \tag{7}
\]

Here, we have considered the contribution from the \( \ell = 0 \) and \( 2 \) modes of the first order to the \( \ell = 2 \) mode of the second order since this gives the leading contribution to gravitational radiation. We also choose the RW gauge to second order. This Zerilli function satisfies the equation, \( \hat{\mathcal{Z}}^{\text{even}}_{20}(t, r) = \mathcal{S}^{Z}_{20}(t, r) \) with

\[
\mathcal{S}^{Z}_{20}(t, r) = \frac{8 \pi \sqrt{3} (r - 2M)^2}{3(2r + 3M)} \frac{\partial}{\partial t} \mathcal{B}_{20}(t, r) + \frac{8 \pi (r - 2M)^2}{2r + 3M} \frac{\partial}{\partial t} \mathcal{A}_{20}(t, r) - \frac{8 \sqrt{3} \pi (r - 2M)}{3} \frac{\partial}{\partial t} \mathcal{F}_{20}(t, r)
- \frac{4 \sqrt{2} i \pi (r - 2M)^2}{2r + 3M} \frac{\partial}{\partial r} \mathcal{A}_{20}^{(1)}(t, r) - \frac{8 \sqrt{2} i \pi (r - 2M)(5r - 3M)M}{r(2r + 3M)^2} \mathcal{A}_{20}^{(1)}(t, r)
- \frac{8 \sqrt{3} i \pi (r - 2M)^2}{3(2r + 3M)} \frac{\partial}{\partial r} \mathcal{B}_{20}^{(0)}(t, r) + \frac{32 \sqrt{3} i \pi (3M^2 + r^2)(r - 2M)}{3r(2r + 3M)^2} \mathcal{B}_{20}^{(0)}(t, r). \tag{8}
\]

The functions \( \mathcal{B}_{20} \) etc. are derived from the second order quantities, \( G^{(2)}_{\mu\nu}(h^{(1)}, h^{(1)}), T_{\mu\nu}^{(2,h)} \) (and \( T_{\mu\nu}^{(2, SF)} \)) by the same tensor harmonics expansion as for the first order.

The delta function \( \delta^4(x - z(\tau)) \) in \( T_{\mu\nu}^{(2,h)} \) includes \( \delta^2(\Omega - \Omega(\tau)) = \sum_{\ell m} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega(\tau)) \). We have considered only the contribution from the \( \ell = 0 \) and \( 2 \) modes of the first order perturbations. Consistently, we use only the three components, \( h^{(1)}_{(\ell = 2)} Y_{2m}(\Omega) Y_{2m}^*(\Omega(\tau)) \), \( h^{(1)}_{(\ell = 2)} Y_{00}(\Omega) Y_{00}^*(\Omega(\tau)) \) and \( h^{(1)}_{(\ell = 0)} Y_{2m}(\Omega) Y_{2m}^*(\Omega(\tau)) \).

We may wonder if there is any \( \delta^2 \)-term in the second order source. The answer is “No”. This is because in the case of the head on collision, the MP under the RW gauge are \( C^0 \); \( R_{\mu\nu}^{(2,h)}(h^{(1)}, h^{(1)}) \) includes second derivatives and we need one more derivative to construct \( \mathcal{S}^{Z}_{20} \). \( (h^{(1)})^2 \) is \( C^0 \) and its third derivative yields \( C^0 \times \delta' \) and \( \theta \times \delta \) as the most singular terms. On the other hand, \( T_{\mu\nu}^{(2,h)} \) includes only \( C^0 \times \delta \) terms.

From Eq. (8), we obtain the second order source as

\[
\mathcal{S}^{Z}_{20}(t, r) = (^{2,2}) \mathcal{S}^{Z}_{20}(t, r) + (^{0,2}) \mathcal{S}^{Z}_{20}(t, r), \tag{9}
\]

where \( (^{2,2}) \mathcal{S}^{Z}_{20} \) and \( (^{0,2}) \mathcal{S}^{Z}_{20} \) are the contribution from \( (\ell = 2) \cdot (\ell = 2) \) and \( (\ell = 0) \cdot (\ell = 2) \), respectively.

Note that while the above source term is locally well behaved near the particle location, some terms diverge as \( r \to \infty \). So, we need to consider some regularization for the asymptotic behavior [7].

\(^3\) We could regularize and fix the gauge for the second order source. One could also proceed by first fixing the gauge.
In order to obtain a well behaved source for large values of $r$, we define a renormalized Zerilli function by
\[
\tilde{\chi}^{Z}_{20}(t, r) = \chi^{Z}_{20}(t, r) - \chi^{\text{reg.}(2,2)}_{20} - \chi^{\text{reg.}(0,2)}_{20} ;
\]
where
\[
\chi^{\text{reg.}(2,2)}_{20} = \frac{1}{\sqrt{5}} \frac{r^2}{\sqrt{2} \pi} \left( \frac{r - 2M}{r + 3M} \right) \ln \left( \frac{r + 3M}{2M} \right) H^{RZ}_{00}(t, r) \frac{\partial}{\partial t} H^{RW}_{220}(t, r) 
+ 6 \left( 1 - 5 \frac{M}{r} \right) H^{RZ}_{00}(t, r) \frac{\partial}{\partial r} H^{RW}_{220}(t, r) + 4 \left( 1 + 3 \frac{M}{r} \right) H^{RZ}_{220}(t, r) \frac{\partial}{\partial t} H^{RW}_{220}(t, r) 
+ 5 \left( 1 - 4 \frac{M}{r} \right) H^{RZ}_{00}(t, r) \frac{\partial}{\partial r} H^{RW}_{220}(t, r) - 18 \left( 1 - 5 \frac{M}{r} \right) H^{RZ}_{00}(t, r) H^{RW}_{220}(t, r) \ln \left( \frac{r}{2M} \right) H^{RZ}_{200}(t, r) \frac{\partial}{\partial t} H^{RW}_{220}(t, r) \right) ,
\]

where the suffix RZ means some gauge choice as discussed in Sec. 3. Finally, the best suited equation to solve numerically for $\tilde{\chi}^{Z}_{20}$ is then
\[
\tilde{S}^{Z, \text{even}}_{20}(t, r) = \frac{1}{\sqrt{5}} \frac{r^2}{\sqrt{2} \pi} \left( \frac{r - 2M}{r + 3M} \right) \ln \left( \frac{r + 3M}{2M} \right) H^{RZ}_{00}(t, r) \frac{\partial}{\partial t} H^{RW}_{220}(t, r) 
+ 6 \left( 1 - 5 \frac{M}{r} \right) H^{RZ}_{00}(t, r) \frac{\partial}{\partial r} H^{RW}_{220}(t, r) + 4 \left( 1 + 3 \frac{M}{r} \right) H^{RZ}_{220}(t, r) \frac{\partial}{\partial t} H^{RW}_{220}(t, r) 
+ 5 \left( 1 - 4 \frac{M}{r} \right) H^{RZ}_{00}(t, r) \frac{\partial}{\partial r} H^{RW}_{220}(t, r) - 18 \left( 1 - 5 \frac{M}{r} \right) H^{RZ}_{00}(t, r) H^{RW}_{220}(t, r) \ln \left( \frac{r}{2M} \right) H^{RZ}_{200}(t, r) \frac{\partial}{\partial t} H^{RW}_{220}(t, r) \right) ,
\]

5 Discussion

In this paper, we obtained the regularized second order effective source of the Zerilli equation in the case of a particle falling radially into a Schwarzschild black hole. Using this source, we are able to compute the second order contribution to gravitational radiation by numerical integration.

To be fully second order consistent we have to include the term $T_{\mu\nu}^{(2, SF)}$ which is derived from the self-force on a particle. The self-force for a headon collision has been calculated in [1], and in a circular orbit around a Schwarzschild black hole in [3], but have not been obtained in the general case yet.

To prove that there is no $\delta^2$ term in the second order source, we have used the fact that the first order MP in the RW gauge is $C^0$. In the general orbit cases (including circular orbits), the first order MP is not $C^0$ in the RW gauge, but it is $C^0$ in the Lorenz gauge [9]. This gauge choice favors the study of the second order perturbations for generic orbits.

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