Constant Solutions of Reflection Equations 
and Quantum Groups

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ABSTRACT

To the Yang-Baxter equation an additional relation can be added. This is the reflection equation which appears in various places, with or without spectral parameter. For example, in factorizable scattering on a half-line, integrable lattice models with non-periodic boundary conditions, non-commutative differential geometry on quantum groups, etc. We study two forms of spectral parameter independent reflection equations, chosen by the requirement that their solutions be comodules with respect to the quantum group coaction leaving invariant the reflection equations. For a variety of known solutions of the Yang-Baxter equation we give the constant solutions of the reflection equations. Various quadratic algebras defined by the reflection equations are also given explicitly.

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1. Introduction

The cornerstone of the quantum inverse scattering method is the Yang-Baxter equation. It is also important in the theory of quantum groups [1, 2, 3]. The simplest physical interpretation of the Yang-Baxter equation (YBE) is connected with factorizable scattering on a line and with the braid group [4, 5]. These interpretations can naturally be extended to include an additional relation. Such an extension will be called in this paper for simplicity Reflection Equation (RE), for it was first obtained in the study of factorizable scattering on a half line. Together with the scattering matrix (called $R$-matrix in this paper), which is a solution of the YBE, there appears an additional object describing the reflection at the endpoint of the half line. It will be called the $K$-matrix. Originally, the reflection equation

$$
 R(\lambda - \nu)K_1(\lambda)R'(\lambda + \nu)K_2(\nu) = K_2(\nu)R''(\lambda + \nu)K_1(\lambda)R'''(\lambda - \nu)
$$

(1.1)

was written in the “spectral parameter” dependent form [6]. This is a matrix equation for matrices acting in the tensor product space $V_1 \otimes V_2$, and $K_1 = K \otimes I$, $K_2 = I \otimes K$ are standard notations* of the quantum inverse scattering method (QISM). The various $R$-matrices $R'$, $R''$ and $R'''$ are related to $R$ or among themselves by certain conjugations appropriate to the specific problem. The significance of constant solutions to the YBE for quantum group theory and braid group representations leads us naturally to the analysis of RE (1.1) without spectral parameter. One typical form of RE reads

$$
 RK_1\tilde{R}K_2 = K_2RK_1\tilde{R},
$$

(1.2)

where $\tilde{R} = PRP$, and $P$ is the permutation operator of the tensor product of the two spaces $V_1 \otimes V_2$. This form has a direct interpretation in terms of the generators of the braid group for a solid handlebody (e.g. for genus 1 there is only

* The suffices of the $R$-matrices like $R = R_{12}$ indicating the base space $V_1 \otimes V_2$ are suppressed in most cases.
one additional generator $\tau$ [7, 8], since they satisfy

$$\sigma_1 \tau \sigma_1 \tau = \tau \sigma_1 \tau \sigma_1.$$  \hfill (1.3)

This is just relation (1.2) if we identify $K_1 = \tau$, $PR = \sigma_1$, and use the fact $K_1 = PK_2P$.

The RE (1.2) is closely related to the quantum group $A(R)$ [3], which is generated by the elements $t_{ij}$ of the quantum group matrix $T$ having relations

$$RT_1 T_2 = T_2 T_1 R.$$ \hfill (1.4)

Any solution $K$ of (1.2) is a two-sided comodule of the quantum group $A(R)$, i.e. the transformed matrix

$$K_T = TKT^{-1},$$

$$(K_T)_{ij} = \sum_{m,n} t_{im} k_{mn} (T^{-1})_{nj},$$ \hfill (1.5)

is also a solution of (1.2), provided that the quantum group generators commute with the entries of the $K$-matrix

$$[t_{ij}, k_{mn}] = 0.$$ \hfill (1.6)

The inverse of the quantum group matrix $T$ in (1.5) should be understood as the antipode map $\gamma(T)$ of the quantum group as a Hopf algebra.

A different version of a reflection equation without spectral parameter is given by [9]

$$RK_1 R^{t_1} K_2 = K_2 R^{t_1} K_1 R,$$ \hfill (1.7)

where the superscript $t_1$ denotes transposition w.r.t. the first space. This RE is different as a two-sided comodule, because it requires the transformation property

$$K_T = TKT^t,$$

$$(K_T)_{ij} = \sum_{m,n} t_{im} k_{mn} t_{jn},$$ \hfill (1.8)

in order that $K_T$ is a solution of (1.7). In (1.8) we denoted the transposed quantum group matrix by $T^t$. 

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One can consider a more general form of RE

\[ R^{(1)}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R^{(4)}, \quad (1.9) \]

which is related with a generalization of the QISM to the case of non-ultralocal commutation relations and to a lattice regularized version of the Kac-Moody algebras [10 – 13].

In this paper, we restrict ourselves to two types of RE’s, the first is RE (1.7), which will be referred to as RE1 and the other reflection equation is (1.2), denoted as RE2. The reason for choosing these equations is the criterion that the \( K \)-matrix should be a two-sided comodule with respect to the corresponding quantum group coaction, which leaves the reflection equation invariant. The comodule property is closely related to the interpretations of reflection equations without spectral parameter. Namely, as a first interpretation we think of the RE as giving a set of quadratic relations for the entries \( k_{ij} \) of the \( K \)-matrix which define an associative algebra \( \mathcal{A} \). The second interpretation of the RE is that it gives an equation for a c-number \( K \)-matrix, whose solutions define one dimensional representations of the above mentioned quadratic algebra \( \mathcal{A} \) [9]. These solutions can be used to integrate non-ultralocal commutation relations [11, 13], also for braid group representations [8] and some integrable models [14]. These two interpretations of the RE are in fact inseparable from each other due to the above mentioned comodule property. Namely, starting from a c-number solution of an RE we can construct a new \( K \)-matrix \( K_T \) by means of (1.5) or (1.8), which is no longer a c-number due to the quantum group matrix \( T \) but it satisfies the corresponding RE, respectively. In other words \( K_T \) precisely obeys the quadratic algebra \( \mathcal{A} \) defined by the corresponding RE.

We would like to remark that different forms of RE’s (with or without spectral parameter) appeared recently in papers on various subjects: quantum current algebras and conformal field theory [15], modified Knizhnik-Zamolodchikov equations [16], integrable spin chains with non-periodic boundary conditions [17 – 20], non-commutative differential geometry on quantum groups [21, 22], twisted Yangians [23], and so on.
This paper is organized as follows. Some of the known results for the reflection equations without spectral parameter are briefly reviewed in section 2. Two types of reflection equations are selected based on the requirement of the two-sided co-module property under the coaction of the corresponding quantum groups. Section 2 also contains discussions and some new results on various general properties of the reflection equations, their constant solutions and the corresponding quadratic algebras for the $R$-matrix (spin 1/2 representation and the universal $R$-matrix) of the simplest quantum algebra $sl_q(2)$. Section 3 gives the new results on the constant solutions to the two types of reflection equations for a wide class of known constant solutions of the YBE. The final section is devoted for summary and discussion.

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2. Properties of Reflection Equations

2.1. Reflection equation 1

It was pointed out in [17] that the quadratic algebra defined by an appropriate reflection equation can be considered as a two-sided comodule

$$K_T = TKT^\sigma$$

(2.1)

for the corresponding quantum group and an anti-automorphism $\sigma$. It should be reminded that the entries of $K$ commute with those of $T$ as mentioned in (1.6). As the first example we specify $\sigma$ to be the transposition $\sigma : T \to T^t$, or $\sigma(t_{ij}) = t_{ji}$. In this case the reflection equation has the form RE1 as introduced in the previous section (1.7)

$$RE1 : \quad RK_1 R^{t_1} K_2 = K_2 R^{t_1} K_1 R,$$

(2.2)

where $t_1$ means transposition in the first space. The invariance of this equation w.r.t. the coaction (2.1) follows from the defining relations of the quantum group
generators

\[ RT_1 T_2 = T_2 T_1 R, \]  

(2.3)

which can be transformed also into the forms

\[
RT_1^t T_2^t = T_2^t T_1^t R,
T_2 R^t_1 T_1^t = T_1^t R^t_1 T_2,
T_1 R^t_1 T_2^t = T_2^t R^t_1 T_1,
\]

(2.4)

provided the \( R \)-matrix has the following properties \( (R^t = R^t_{1,01}) \)

\[
R = \mathcal{P} R^t \mathcal{P}, \quad R^t_1 = \mathcal{P} R^t_1 \mathcal{P}.
\]

(2.5)

They are certainly true for the \( sl_q(N) \) \( R \)-matrix which is the main subject of our research, but not for its multiparameter generalizations or in some other cases. In case \( R \)-matrices do not have the properties (2.5), the two-sided comodule property (1.8) or (2.1) is valid for a generalized equation

\[
RK_1 R^t_1 K_2 = K_2 \tilde{R}^t_1 K_1 \tilde{R}^t,
\]

(2.6)

instead of the original equation (2.2). In such situations we call (2.6) the RE1 belonging to the given \( R \)-matrix. The \( R \)-matrices on the right hand side are defined as \( \tilde{R}^t_1 = \mathcal{P} R^t_1 \mathcal{P} \) and \( \tilde{R}^t = \mathcal{P} R^t \mathcal{P} \). For each given \( R \)-matrix the RE1 defines an associative quadratic algebra which will be denoted by \( \mathcal{A}_1 \), and similarly by \( \mathcal{A}_2 \) for the RE2 to be discussed shortly.

In the remainder of this section we will study in some detail the basic example of \( sl_q(2) \) for both reflection equations RE1 (1.7), (2.2) and RE2 (1.2). The \( R \)-matrix is well known

\[
R = \begin{pmatrix}
q & 1 \\
1 & \omega \\
\omega & 1 \\
q & \end{pmatrix}, \quad R^t_1 = \begin{pmatrix}
qu & \omega \\
1 & \\
\omega & 1 \\
qu & \end{pmatrix}, \quad \omega = q - q^{-1}.
\]

(2.7)

We adopt the convention that all matrix elements not written explicitly are zeros. The notation for the \( R \)-matrices is such that the upper index pair \( (ij) \) of \( R^{ij}_{kl} \)
numbers the rows in the natural order and likewise the lower index pair \((kl)\) numbers the columns. In the basis (2.7) the permutation operator is given by

\[
P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\] (2.8)

For the \(R\)-matrix (2.7) the quadratic algebra \(A_1\) is generated by the four elements of the \(K\)-matrix

\[
K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\] (2.9)

which satisfy the relations [9]

\[
\begin{align*}
[\alpha, \beta] &= \omega \alpha \gamma, & [\alpha, \delta] &= \omega(q\beta \gamma + \gamma^2), & [\beta, \delta] &= \omega \gamma \delta, \\
\alpha \gamma &= q^2 \gamma \alpha, & [\beta, \gamma] &= 0, & \gamma \delta &= q^2 \delta \gamma.
\end{align*}
\] (2.10)

This algebra \(A_1\) has two central elements

\[
c_1 = \beta - q\gamma, \quad c_2 = \alpha \delta - q^2 \beta \gamma.
\] (2.11)

Under the coaction (2.1) (or (1.8)) these central elements transform homogeneously with respect to \(GL_q(2)\)

\[
c_1(K_T) = (det_q T)c_1(K), \quad c_2(K_T) = (det_q T)^2 c_2(K).
\] (2.12)

Here \(T\) is the matrix of the \(GL_q(2)\) quantum group generators satisfying the well known relations [2, 3] which follow from (2.3)

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad ab = qba, \quad bd = qdb, \quad ad - qbc = da - q^{-1}bc, \\
ac = qca, \quad cd = qdc, \quad bc = cb.
\] (2.13)

For the \(SL_q(2)\) case \(i.e.\ det_q T = ad - qbc = da - q^{-1}bc = 1\), the central elements are invariant. If we restrict the above quadratic algebra \(A_1\) by requiring that \(c_1(K) = 0\)
(i.e. $\beta = q\gamma$) then (2.10) reduces to the well known $GL_{q^2}(2)$ relations, namely the quantum group $GL_q(2)$ with the parameter $q$ replaced by $q^2$

$$\begin{align*}
\alpha\beta &= q^2\beta \alpha, \\
\beta\delta &= q^2\delta \beta, \\
\alpha\delta - q^2\beta\gamma &= \delta \alpha - q^{-2}\beta\gamma, \\
\alpha\gamma &= q^2\gamma \alpha, \\
\gamma\delta &= q^2\delta \gamma, \\
\beta\gamma &= \gamma \beta.
\end{align*}$$

(2.14)

Therefore the second condition of the “degeneracy”, $c_2 = 0$, simply corresponds to the vanishing quantum determinant of the above mentioned “$GL_{q^2}(2)$”. It is interesting to note that the above relations with $\beta = q\gamma$ are also satisfied by $K = TT^t$, which is obtained by (1.8) from a c-number solution $K = 1$ (the $2 \times 2$ unit matrix) to be discussed shortly. A similar phenomenon has been known for some time [24, 25]. Namely, if $T$ is a $GL_q(2)$ matrix then $T^n$ is a $GL_{q^n}(2)$ matrix.

Now we are interested in c-number solutions of the reflection equation. From the algebra (2.10) it is easy to conclude that there are only two types of non-trivial solutions [9]: (i) the $K$-matrix proportional to the quantum invariant metric $\varepsilon_q$ of $sl_q(2)$; (ii) an upper triangular $K$-matrix with three arbitrary parameters (excluding the irrelevant overall factor leaves essentially two arbitrary parameters)

$$K^{(1)} \sim \varepsilon_q = \begin{pmatrix} 1 \\ -q \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} \alpha & \beta \\ \delta \end{pmatrix}. \quad (2.15)$$

It is easy to see that the $A_1$ relations become trivial for $\gamma = 0$, which gives the c-number solution $K^{(2)}$. Similarly we get $K^{(1)}$ by putting $\alpha = \delta = 0$. The two-sided coaction (2.1) has $K^{(1)}$ as a fixed point due to the well known fact [24] that $T \varepsilon_q T^t = (\det_q T) \varepsilon_q$ and that for $SL_q(2)$ the quantum determinant is unity $\det_q T = 1$.

It is well known that the $R$-matrix (2.7) is the spin $(\frac{1}{2}, \frac{1}{2})$ representation of the universal $R$-matrix of the quasi-triangular Hopf algebra $sl_q(2)$ generated by three elements $J, X_+, X_-$ satisfying the relations

$$\begin{align*}
[J, X_{\pm}] &= \pm X_{\pm}, \quad (2.16) \\
[X_+, X_-] &= [2J]_q = (q^{2J} - q^{-2J})/(q - q^{-1}). \quad (2.17)
\end{align*}$$
The universal $R$-matrix is \[ R_U = q^{2J \otimes J} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n; q^{-2}!]} f^n \otimes e^n, \] (2.18)

where $[n; q] = (1-q^n)/(1-q)$ and

\[
e = q^J X_+, \quad f = q^{-J} X_-.
\] (2.19)

Hence it is possible to generalize RE in the tensor product $V_1 \otimes V_2$ of two irreducible representations of $q$-spin $s_1$ and $s_2$. In the analysis of the quantum Liouville equation on a strip \[14\] a quasi-group like element was constructed

\[ K_U = q^{J^2} \sum_{n=0}^{\infty} \frac{f_n}{[n]!} q^{-nJ} (X_+)^n, \] (2.20)

which just turned out to be a universal solution of RE1 for $sl_q(2)$ with universal $R$-matrix (2.18) and

\[ R_U^{t_1} = \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n; q^{-2}!]} e^n \otimes e^n q^{2J \otimes J}. \] (2.21)

In this case RE1 is understood to be defined in the tensor product of two copies of the quantum algebras $sl_q(2) \otimes sl_q(2)$. It is very remarkable that the universal $K$-matrix in fact depends on one parameter only, as the coefficients $f_n$ are defined by the recurrence relation related to that of the $q$-Hermite polynomials

\[ f_{n+1} = f_1 f_n + (q^{2n} - 1)f_{n-1}, \quad f_0 = 1, \] (2.22)

and the free parameter is $f_1$. The proof that $K_U$ indeed is a solution of RE1 follows from its property \[14\]

\[ \Delta(K_U) = (K_U)_1 R_U^{t_1}(K_U)_2 \] (2.23)

and the definition of the canonical element, i.e. the universal $R$-matrix, for the quasi-triangular Hopf algebra. The universal $R$-matrix intertwines the coproduct
and the permuted coproduct [1]

\[ R_U \Delta(\cdot) = \Delta'(\cdot)R_U, \quad (2.24) \]

where \( \Delta' \) is the permuted coproduct \( \Delta'(\cdot) = \mathcal{P}\Delta(\cdot)\mathcal{P} \). The relations (2.23) and (2.24) lead to

\[ R_U(K_U)_1 R_U^l_1(K_U)_2 = (K_U)_2 R_U^l_1(K_U)_1 R_U \quad (2.25) \]

when taking into account that (2.21) is invariant w.r.t. the similarity transformation by the permutation operator (namely the second equation of (2.5) holds for \( R_U \)). The constant solution \( K^{(2)} \) in (2.15) corresponds to the case of the \( q \)-spin \((\frac{1}{2}, \frac{1}{2})\) representation of the universal solution (2.20) for \( sl_q(2) \).

Let us point out the invariance of RE1 w.r.t. the transformation

\[ K' = e^{\kappa J} K e^{\kappa J}, \quad \kappa : \text{arbitrary constant} \quad (2.26) \]

which explains the difference in the number of independent parameters of \( K^{(2)} \) in (2.15) (two parameters) and in (2.20) (one parameter, \( f_1 \) ), as it can be used to transform away one parameter in (2.15) by properly adjusting the arbitrary constant \( \kappa \). This invariance follows from the relations

\[ [R, J_1 + J_2] = 0, \quad [R^h_1, J_1 - J_2] = 0, \quad (2.27) \]

which are valid for the universal \( R \)-matrix as well. The free parameter \( f_1 \) of the universal solution (2.20) is closely related to the central element \( c_1 \) of the quadratic algebra \( \mathcal{A}_1 \), namely \( c_1 \propto f_1 \).

The first solution \( K^{(1)} \) in (2.15) of RE1 can also be extended to higher representations. It is proportional to the element \( w \) of the quantum Weyl group [26], which can be defined in each finite dimensional representation \( V_j \), \( \dim V_j = 2j + 1 \),
as

\[ w \simeq q^{-J} \varepsilon_j, \quad (\varepsilon_j)_{m,m'} = (-1)^{j-m} \delta_{m,-m'}, \quad -j \leq m, m' \leq j, \quad (2.28) \]

or by its commutation properties with the generators of \( sl_q(2) \)

\[ wJ = -Jw, \quad wX_\pm = -q^{\pm 1} X_\mp w. \quad (2.29) \]

It can be shown that \( K_U = w \) indeed is a solution of RE1 due to the following properties of the universal \( R \)-matrix

\[ w_1 R_U^{t_1} w_1^{-1} = w_1^{-1} R_U^{t_1} w_1 = R_U^{-1}. \quad (2.30) \]

2.2. Reflection equation 2

In the lattice version of the current algebras or the Kac-Moody algebras [10, 11] another type of reflection equation appears which we denote as RE2

\[ RK_1 \tilde{R} K_2 = K_2 R K_1 \tilde{R}. \quad (2.31) \]

It defines the set of relations of an associative quadratic algebra \( A_2 \) generated by the entries of the \( K \)-matrix, and it is a two-sided comodule w.r.t. the quantum group coaction (1.5), namely the antiautomorphism \( \sigma \) is the inverse \( \sigma(T) = T^{-1} \).

Let us again consider the \( sl_q(2) \) \( R \)-matrix (2.7) and this time denote the entries of the \( K \)-matrix by

\[ K = \begin{pmatrix} u & x \\ y & z \end{pmatrix}, \quad (2.32) \]

then we find that the component form of the algebra (2.31) reads

\[ \begin{align*}
ux &= q^{-2} x u, \\
uz &= 0, \\
xz &= -q^{-1} \omega uz, \\
uy &= q^2 y u, \\
xy &= q^{-1} \omega (uz - u^2), \\
yz &= q^{-1} \omega y u.
\end{align*} \quad (2.33) \]

The central elements of the above algebra \( A_2 \) are also known [21] and they are
invariant under the $GL_q(2)$ coaction in contrast to the $SL_q(2)$ invariance for $A_1$,

\[ c_1 = u + q^2 z, \quad c_2 = uz - q^2 yx, \]
\[ c_1(K_T) = c_1(K), \quad c_2(K_T) = c_2(K). \]  

The quadratic algebras (2.10) and (2.33) are equivalent to each other by a simple substitution: \textit{i.e.} the following substitution in (2.10)

\[ \{\alpha, \beta, \gamma, \delta\} \rightarrow \{y, -z/q, u, -x/q\} \]  

produces (2.33) in which $q$ is replaced by $1/q$. Generally, however, this is not the case for the quadratic algebras defined by $sl_q(N)$ $R$-matrices, as for $N > 2$ there is no relation between $T^t$ and the inverse (antipode) $T^{-1} = \gamma(T)$ contrary to the $sl_q(2)$ case where we have

\[ T^t = \varepsilon_q T^{-1} \varepsilon_q^{-1}, \]  

and $\varepsilon_q$ is the $2 \times 2$ quantum metric in (2.15).

The c-number solution of RE2 for the $sl_q(2)$ $R$-matrix (2.7) can be easily obtained as a one-dimensional representation of the above $A_2$. Similarly to the case of $A_1$ there are two of them,

\[ K^{(1)} \simeq \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} x \\ y & z \end{pmatrix}. \]  

A glance at the $A_2$ relations shows that they become trivial for $u = 0$, which gives $K^{(2)}$. The above solutions are related to the solutions (2.15) of RE1 and they are mapped to each other by the substitution mentioned above. As in the case of RE1, $K^{(1)}$ is a fixed point under the coaction of the quantum group (1.5). Obviously RE2 has always a solution proportional to the unit matrix for any choice of the $R$-matrix. It is interesting to note that RE2 can be rewritten into the following equivalent form [10]

\[ RK_1 R^{-1} K_2 = K_2 \tilde{R}^{-1} K_1 \tilde{R}, \quad \tilde{R}^{-1} = PR^{-1}P. \]  

Next let us discuss certain invariance properties of the $sl_q(N)$ $R$-matrices, which are important for the understanding of their multiparametric generalizations. For
the $R$-matrices of $sl_q(N)$ the reflection equation (2.31) is invariant as is (2.2) w.r.t. the similarity transformation

$$K' = e^{\tau J} K e^{-\tau J},$$

(2.39)
due to

$$[R, J_1 + J_2] = 0, \quad [\tilde{R}, J_1 + J_2] = 0,$$

(2.40)
where $J$ belongs to the Cartan subalgebra and $\tau$ is some parameter. There is an additional transformation for the $sl_q(N)$ $R$-matrix [27]

$$R' = U_1^2 R U_2^{-2},$$

(2.41)

which transforms a solution of the YBE to the solution $R'$ if the $R$-matrix is symmetric, i.e.

$$[R, U_1 U_2] = 0, \quad U \in \text{Mat}(C^n).$$

(2.42)
The corresponding transformation of the $K$-matrix is

$$K' = U K U^{-1}$$

(2.43)
and the RE2 (2.31) is invariant under (2.41) and (2.43). This transformation gives relations among solutions and quadratic algebras of the reflection equations (2.31) for the one-parameter and some multiparameter $R$-matrices which we will discuss in the following section.

2.3. Comodule Properties

Before concluding this section we find it instructive to remark on the two-sided comodule properties (2.1) (or (1.5), (1.8)) of the reflection equations in general. For concreteness let us choose RE1 for a generic $R$-matrix but the situation for RE2 is essentially the same.
The two-sided comodule property simply states that given a solution $K$ of RE1 we can associate with it another solution $K_T$ (possibly in a different space) by (1.5) provided: (i) $T$ satisfies the FRT-relations (2.3) and (ii) each element of $K$ should commute with each element $t_{ij}$ of $T$. The general solutions of the FRT-relations describe the quantum group $A(R)$. We may say that $K$ is a comodule with respect to the quantum group $A(R)$ with the coaction $\delta$:

$$\delta : A \rightarrow A(R) \otimes A.$$ (2.44)

The quantum group $A(R)$ is known to have various representations. Suppose we choose one of them, $\rho(t_{ij})$ which acts on a linear space $V_\rho$, $\rho(t_{ij}) \in \text{Mat}(V_\rho)$, then $K_T$ is a new solution of RE1 in a new space

$$\text{Mat}(V_\rho) \otimes K.$$

Of course we can treat $T$ as a general quantum group matrix without specifying any representation as we have done in this section. Then the new solution $K_T$ lies in a space which is written symbolically as in (2.44).

Some concrete examples are in order. Suppose we have a solution of the YBE without spectral parameter acting in $V_1 \otimes V_2 \otimes V_3$ ($V_1 \simeq V_2$, $V_3$ : arbitrary)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ (2.45)

This provides a solution of the FRT-relations in which $V_3$ is considered as the “quantum space”, i.e.

$$\rho(T_1) = R_{13}.$$ (2.45)

The $(l, m)$ element of the representation matrix is given by $\rho(t_{ij})^l_m = (R_{13})^l_{jm}$. Thus we find a new solution $K_T$ (1.8) of RE1, whose components are matrices on $V_3$.

In this way we can interpret the symmetry properties of $K$ (2.26) which explained the equivalence of the two solutions (2.15) (two parameters) and (2.20) (one
parameter $f_1$). We can choose the trivial representation of $T$ in $GL_q(2)$, (2.13), $\rho_{\text{trivial}}(T) = \text{diag}(a, d)$, $a, d \in \mathbb{C}$. Then the two-sided comodule property simply means that if $K$ is a c-number solution of the RE1 then

$$\begin{pmatrix} a \\ d \end{pmatrix} K \begin{pmatrix} a \\ d \end{pmatrix}$$

is another c-number solution. This is the essence of the symmetry argument (2.26), (2.27).

Having clarified the meaning of the two-sided comodule properties (2.1) of RE1 and RE2 we are ready to discuss a more general form of the reflection equations (without spectral parameter) (1.9)

$$R^{(1)} K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R^{(4)}. \quad (2.47)$$

This equation defines the relations among the generators of an associative algebra. The algebra has the two-sided comodule property

$$K \rightarrow K_{T,S} = TKS; \quad (2.48)$$

provided that $T$ and $S$ satisfy the following two sets of equations

$$R^{(1)} T_1 T_2 = T_2 T_1 R^{(1)}, \quad R^{(4)} S_1 S_2 = S_2 S_1 R^{(4)}, \quad (2.49)$$

$$S_1 R^{(2)} T_2 = T_2 R^{(2)} S_1, \quad T_1 R^{(3)} S_2 = S_2 R^{(3)} T_1. \quad (2.50)$$

The first set (2.49) simply means that $T$ and $S$ belong to (possibly different) quantum groups specified by $R^{(1)}$ and $R^{(4)}$, respectively. The two equations in the second set (2.50) are compatible to each other e.g. when $R^{(3)} = \tilde{R}^{(2)}$. They require certain relations among $T$ and $S$. Namely $R^{(2)}$ intertwines $S_1$ and $T_2$ and $R^{(3)}$ should intertwine $T_1$ and $S_2$ which are necessary for the two-sided comodule property.
As before the elements of $T$ and $S$ should commute with the elements of $K$,

$$[t_{ij}, k_{lm}] = [s_{ij}, k_{lm}] = 0.$$  \hspace{1cm} (2.51)

It is conceptually straightforward to give concrete examples of the two-sided co-
module transformations for the general form of RE, which corresponds to (2.45) 
(cf.[12, 13]).

3. Constant Solutions of Reflection Equations

3.1. $sl_q(2)$ Spin 1 Representation

In the preceding section we have discussed the constant solutions to RE1 and 
RE2 for $sl_q(2)$ $R$-matrix. The quantum metric type and the upper-triangular sol-
lutions were found, and they already exhausted the list of solutions of RE1 for 
$sl_q(2)$. As we have seen, both of them can be generalized to universal solutions for 
RE1, therefore we expect this type of solutions in the higher spin representations 
of $sl_q(2)$. But, in principle, there can be other solutions in these cases in addition, 
which are not reductions of universal ones but tied to particular representations.

To examine this point we now discuss the constant solutions for the spin one 
case. The $R$-matrix can be derived from the universal $R$-matrix (2.18)

$$R = \begin{pmatrix}
q^2 & 1 & q^{-2} \\
1 & \Delta & 1 \\
q^{-1}\Delta & 1 & q^{-1}\Delta \\
q^{-1}\omega\Delta & q^{-1}\Delta & q^{-2} \\
\Delta & 1 & q^2
\end{pmatrix}, \quad \Delta = (q + q^{-1})\omega. \hspace{1cm} (3.1)$$

By inserting this $R$-matrix into RE1 we find four constant solutions. Two of them
are given by
\[ K^{(1)} = \begin{pmatrix} 1 & -q^{-1} \\ -q & 1 \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} a_{11} & a_{12} & h_{13} \\ a_{22} & h_{23} \\ h_{33} \end{pmatrix}, \]
where \( h_{13}, h_{23}, \) and \( h_{33} \) are functions of the free parameters \( a_{11}, a_{12}, \) and \( a_{22} \) given by \( h_{13} = \omega a_{22} + a_{12}^2/(q + q^{-1})a_{11}, \) \( h_{23} = qa_{12}a_{22}/a_{11}, \) \( h_{33} = q^2a_{22}^2/a_{11}. \) It is clear that \( K^{(1)} \) corresponds to the metric type universal solution and \( K^{(2)} \) to the universal triangular solution [14]. The other two solutions have two- and one-parameter dependence with vanishing determinants
\[ K^{(3)} = \begin{pmatrix} a_{23}^2/a_{33}Y \\ a_{23} \\ a_{33} \end{pmatrix}, \quad Y = q + 1/q, \quad K^{(4)} = \begin{pmatrix} a_{13} \end{pmatrix}. \]

At first sight they appear to be independent of the universal solution \( K^{(2)} \). But one can show that \( K^{(3)} \) can be obtained from \( K^{(2)} \) in an appropriate scaling limit and likewise \( K^{(4)} \) from \( K^{(3)} \). So in this particular case the two types of the universal solutions contain all the solutions and there are no additional solutions mentioned above. In the following we use non-vanishing determinants as a criterion for selecting ‘interesting’ solutions. We also omit the solutions which can be obtained from the others by specializing the parameters and/or by scaling limits.

3.2. sl_q(N) for \( N \geq 3 \)

Next we proceed to \( sl_q(3) \) and later to \( sl_q(N) \) for generic \( N \) in the fundamental representation. The \( R \)-matrix of \( sl_q(3) \) is given by \( R = \text{diag}(q, 1, 1, 1, q, 1, 1, 1, q) \) and three lower-diagonal elements \( R^{21}_{12} = R^{31}_{13} = R^{32}_{23} = \omega, \) from which we get eight solutions for RE1, only two of them have non-vanishing determinants
\[ K^{(1)} = \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ f_{22} & f_{23} \\ f_{33} \end{pmatrix}, \quad X = 1 + q^{-1} \]
where the functions \( f_{ij} \) depend on the parameters in the first row \( f_{ii} = a_{1i}^2/a_{11}X^2, \) \( f_{ij} = a_{1i}a_{1j}/a_{11}X. \) By comparing with (3.2) we see that these solutions are rather
different. The other six solutions can obviously be understood as an embedding of the \( sl_q(2) \) solutions into \( sl_q(3) \) ones

\[
K^{(3)} = \begin{pmatrix}
  a_{12} \\
  -qa_{12}
\end{pmatrix}, \quad K^{(6)} = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{22} & a_{22}
\end{pmatrix},
\]

\[
K^{(4)} = \begin{pmatrix}
  a_{13} \\
  -qa_{13}
\end{pmatrix}, \quad K^{(7)} = \begin{pmatrix}
  a_{11} & a_{13} \\
  a_{33} & a_{33}
\end{pmatrix},
\]

\[
K^{(5)} = \begin{pmatrix}
  a_{23} \\
  -qa_{23}
\end{pmatrix}, \quad K^{(8)} = \begin{pmatrix}
  a_{22} & a_{23} \\
  a_{33} & a_{33}
\end{pmatrix}.
\]

It is interesting to note the difference between \( K^{(2)} \) with \( a_{13} = 0 \) and \( K^{(6)} \). In the latter \( a_{22} \) is a free parameter whereas in the former \( f_{22} \) is not.

The embeddings of the lower dimensional solutions into the higher dimensional ones are a general feature which we encounter again and again for higher N. For example, if we consider \( sl_q(4) \) with \( R \)-matrix given by the following expression

\[
R = \text{diag}(q, 1, 1, 1, 1, q, 1, 1, 1, 1, q, 1, 1, 1, 1, q) \quad \text{plus six non-vanishing lower-diagonal elements} \quad R^{ij} = \omega, \quad 1 \leq j < i \leq 4,
\]

then we find 13 solutions among which only three have non-vanishing determinant. Two of them are given by

\[
K^{(1)} = \begin{pmatrix}
  a_{11} & a_{22} & a_{33} & a_{44}
\end{pmatrix}, \quad K^{(2)} = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  f_{22} & f_{23} & f_{24} & f_{24} \\
  f_{33} & f_{34} & f_{34} & f_{44}
\end{pmatrix},
\]

where the \( f_{ij} \) are the same functions as for \( sl_q(3) \). The third solution is a double embedding of the metric type \( sl_q(2) \) solution into \( sl_q(4) \)

\[
K^{(3)} = \begin{pmatrix}
  a_{12} \\
  -qa_{12} & -qa_{34} \\
  a_{34} & -qa_{34}
\end{pmatrix},
\]

and all others are degenerate embeddings of \( sl_q(2) \). Of course, there are a lot of
truncations of $K^{(1)}$ and $K^{(2)}$ obtained by putting to zero some of their elements, among them are also several $sl_q(3)$ embeddings.

Clearly, the solutions $K^{(1)}$, $K^{(2)}$ and $K^{(3)}$ suggest that this structure is a general feature of $sl_q(N)$ for $N > 2$. However, $sl_q(2)$ is special in that its solutions are less restricted.

To analyze $sl_q(N)$ we use the expression of the $R$-matrix in its fundamental representation in terms of matrix units ($e_{ij}^{ab} = \delta_{ia} \delta_{jb}$)

$$R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \omega \sum_{i > j} e_{ij} \otimes e_{ji}. \quad (3.7)$$

We find that RE1 is equivalent to the following set of equations for the entries $a_{ij}$ of $K$ which are c-numbers:

\[
\begin{align*}
\text{i > j :} & \quad a_{ij}(a_{ij} + qa_{ji}) = 0, \quad a_{ii}a_{ij} = 0, \quad a_{jj}a_{ij} = 0, \\
\text{i > j > k :} & \quad a_{jj}a_{ik} = 0, \quad a_{ik}a_{jk} = 0, \quad Xa_{jj}a_{ki} - a_{ji}a_{kj} = 0, \\
& \quad a_{ij}a_{ik} = 0, \quad a_{ik}a_{kj} = 0, \quad Xa_{ii}(a_{kj} - qa_{jk}) - a_{ji}a_{ki} = 0, \\
& \quad a_{ij}a_{ki} = 0, \quad a_{ki}a_{jk} = 0, \quad Xa_{kk}(a_{ji} - qa_{ij}) - a_{ki}a_{kj} = 0, \\
& \quad a_{ij}a_{jk} = 0, \quad a_{ji}a_{ik} = 0, \\
& \quad a_{ij}a_{kj} = 0, \quad a_{ji}a_{jk} = 0, \quad X = 1 + q^{-1}, \\
\text{i > j > k > l :} & \quad a_{ik}a_{jl} = 0, \quad a_{ik}a_{lj} = 0, \quad a_{ki}a_{jl} = 0, \\
& \quad a_{il}a_{jk} = 0, \quad a_{il}a_{kj} = 0, \quad a_{li}a_{jk} = 0, \\
& \quad a_{ij}a_{lk} - a_{ji}a_{kl} = 0, \\
& \quad a_{ij}a_{kl} + a_{li}a_{kj} - a_{ji}a_{lk} + \omega a_{ij}a_{lk} = 0. \quad (3.8)
\end{align*}
\]

Let us examine some consequences of these equations for the structure of possible solutions.

First, we note that there are no equations containing only $a_{ii}$, hence it follows that a diagonal $K$-matrix with arbitrary elements like $K^{(1)}$ is a solution. An explanation of this fact based on the two-sided comodule property is straightforward. From the
explicit form of the $sl_q(N)$ $R$-matrix in the fundamental representation (3.7) one can show that $R$ and $R^{t_1}$ commute,

$$[R, R^{t_1}] = 0,$$

which means that the $N$ dimensional unit matrix is a solution of RE1. By applying the symmetry transformation (2.26) (or the generalization of (2.46)) to the unit matrix solution we get $K^{(1)}$ in (3.3) and (3.5) and in general an arbitrary diagonal $N \times N$ matrix as a solution of RE1 for $sl_q(N)$ case. In other words all the free parameters in the diagonal solutions (e.g. $K^{(1)}$ in (3.3) and (3.5)) can be “gauged away”.

Second, for an upper-triangular $K$-matrix the above system of equations reduces drastically to

$$i > j > k: \quad X_{kk}a_{ji} - a_{ki}a_{kj} = 0,$$

$$X_{jj}a_{ki} - a_{ji}a_{kj} = 0,$$

$$X_{ii}a_{kj} - a_{ji}a_{ki} = 0,$$

$$i > j > k > l: \quad a_{li}a_{kj} - a_{ji}a_{lk} = 0.$$  \hspace{1cm} (3.10)

Choosing $a_{1k}, k = 1, \ldots, N$ as independent variables, it is a simple exercise to verify that the following generalization of $K^{(2)}$ to $sl_q(N)$

$$\{ a_{jj} = \frac{a_{jj}^2}{a_{11}X^2}, \quad a_{ji} = \frac{a_{jj}a_{1i}}{a_{11}X}, \quad i > j > 1 \}$$  \hspace{1cm} (3.11)

indeed satisfies these four equations. It is interesting to point out that in contrast to the $sl_q(2)$ case, all the free parameters $(a_{11}, \ldots, a_{1N})$ of the above upper-triangular solution for $sl_q(N)$ $N > 2$ can be made to unity (“gauged away”) by the transformation (2.26). For example, for $K^{(2)}$ in (3.3) we have

$$K^{(2)} \rightarrow UK^{(2)}U^{t_1}, \quad U = \text{diag}(1/\sqrt{a_{11}}, \sqrt{a_{11}/a_{12}}, \sqrt{a_{11}/a_{13}}).$$

Therefore the two types of generic solutions $K^{(1)}$ and $K^{(2)}$ of RE1 for $sl_q(N)$ $N > 2$ have essentially no free parameter.
Third, it is easy to see that there are no lower-triangular solutions, as in this case the first equation in (3.8) reduces to $a_{ij}^2 = 0, i > j$, hence all entries of a lower-triangular $K$-matrix vanish.

Finally, if we do not put any restrictions on $K$ we get non-triangular $sl_q(2)$ type embeddings, the structure of which is indicated by the very first equation in (3.8).

3.3. $sl_q(N|M)$ cases

Next we consider $R$-matrices related directly to supersymmetric algebras, which are, however, solutions of the non-graded Yang-Baxter equations by the well known transformation ($p(j) = 0, 1$ is the parity of the index $j$)

$$R^{ij}_{kl}(\text{YBE}) = (-1)^{p(i)p(j)} R^{ij}_{kl}(\text{graded YBE}).$$

We also consider the non-graded reflection equations only. The simplest example is $sl_q(1|1)$ connected with [27]

$$R = \begin{pmatrix} q & 1 \\ \omega & 1 \\ -q^{-1} \end{pmatrix},$$

which enforces nilpotency of two quantum group generators. In contrast to the $sl_q(2)$ case, the RE1 for $sl_q(1|1)$ has only one solution

$$K^{(1)} = \begin{pmatrix} a_{11} & a_{12} \\ f \end{pmatrix},$$

where $f = -a_{12}^2/a_{11}\omega$. Namely, the counterpart of the $sl_q(2)$ solution $\varepsilon_q$ is lacking. For completeness we include the quadratic algebra defined by RE1 and above $R$-matrix, given by

$$\alpha\beta = \beta\alpha + \omega\alpha\gamma, \quad \beta\delta = \delta\beta + \omega\gamma\delta,$$

$$\alpha\gamma = q^2\gamma\alpha, \quad \gamma\delta = q^{-2}\delta\gamma,$$

$$\alpha\delta = \delta\alpha + q\omega\gamma\beta, \quad \beta^2 = -\omega\alpha\delta,$$

$$\beta\gamma = -q^2\gamma/\beta, \quad \gamma^2 = 0.$$  \hfill (3.14)

This gives an interesting example of the deformation of two Grassmannian quanti-
ties $\beta$ and $\gamma$, which have $\beta^2 = \gamma^2 = \beta \gamma + \gamma \beta = 0$ in the classical limit, i.e. $q = 1$. For $q \neq 1$, $\beta^2$ and $\beta \gamma$ relations are deformed whereas $\gamma^2$ remains vanishing. No linear central element $c_1$ exists for this algebra. It is obvious that we get a trivial representation of the algebra by setting $\gamma = 0$, which is given by the above solution $K^{(1)}$ (3.13). It is clear that an equivalent formulation of $sl_q(1|1)$ can be obtained from the above $R$-matrix by the transformation $q = -1/q'$, which does not affect $\omega$.

The three dimensional case, i.e. $sl_q(N|M)$, $N + M = 3$ is described by the $R$-matrix $R = \text{diag}(\alpha, 1, 1, 1, \beta, 1, 1, 1, \gamma)$ plus three more non-vanishing elements $R^{21}_{12} = R^{31}_{13} = R^{32}_{23} = \omega$, and $\alpha, \beta, \gamma \in \{q, -q^{-1}\}$. There are eight different solutions of the non-graded Yang-Baxter-equation for different choices of $(\alpha, \beta, \gamma)$. However, due to the transformation $q = -1/q'$ there are only four independent ones. Clearly, one of them is the $sl_q(3)$ case discussed before (when $\alpha = \beta = \gamma$), the others describe $sl_q(2|1)$. We do not discuss all of them as they are very similar, but as an illustration choose $\alpha = \beta = -1/\gamma = q$. This gives three independent solutions of RE1, an upper-triangular one

$$K^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ f_{22} & f_{23} \\ f_{33} \end{pmatrix},$$

(3.15)

with $f_{22} = a_{12}^2/a_{11}X^2$, $f_{23} = a_{12}a_{13}/a_{11}X$, $f_{33} = -a_{13}^2/a_{11}\omega$, and two further solutions

$$K^{(2)} = \begin{pmatrix} a_{12} \\ -qa_{12} \\ a_{33} \end{pmatrix}, \quad K^{(3)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$ 

(3.16)

We obtain less solutions than for $sl_q(3)$, but still they are rather similar. Especially, comparing $K^{(1)}$ of $sl_q(1|1)$ and $sl_q(2|1)$ with their $sl_q(2)$ and $sl_q(3)$ counterparts we note that only the element in the lower right corner is changed. If we describe $sl_q(N-1|1)$ case by the $sl_q(N)$ $R$-matrix just replacing the element $R^{NN}_{NN} = q$ by $R^{NN'}_{NN} = -1/q$, then it seems that there is a general upper-triangular solution for $sl_q(N-1|1)$, given precisely by (3.11) except that $a_{NN} = a_{1N}^2/a_{11}X^2$ is changed to
\[ a_{NN} = -a_{1N}^2/a_{11} \omega. \] At least, we have checked this explicitly for \( sl_q(3|1) \). Obviously \( K^{(3)} \) above corresponds to \( K^{(6)} \) in (3.4) of \( sl_q(3) \). An interesting phenomenon is that \( K^{(2)} \) above has one more free parameter (\( a_{33} \)) than its \( sl_q(3) \) counterpart, \( K^{(3)} \) in (3.4).

### 3.4. Multiparameter \( R \)-matrices

We now proceed to a different class of models, namely multiparameter deformations. It will be interesting to see how previous results for \( K \) are affected by additional parameters. As already suggested by the supersymmetric examples we will note that the more complicated the \( R \)-matrix, the less structure the \( K \)-matrix usually can have.

We will discuss first \( gl_{p,q}(2) \) described by the \( R \)-matrix \([27 - 30]\)

\[
R_{p,q} = \begin{pmatrix}
q & p \\
\omega & p^{-1}
\end{pmatrix},
\]

which can be obtained from the 1-parameter \( R \)-matrix using the general transformation (2.41)

\[
R_{p,q} = U_1 R_q U_2^{-1}, \quad U = \begin{pmatrix} p^{1/2} \\ p^{-1/2} \end{pmatrix}.
\]

This \( R \)-matrix does not satisfy the conditions in (2.5) so we have to use the generalized reflection equation (2.6) here and in the multiparameter case in general, still we refer to that reflection equation also as \( \text{RE1} \) since it is related to the same comodule structure. The above transformation does not leave \( \text{RE1} \) invariant, in contrast to \( \text{RE2} \), hence we expect different solutions compared to the 1-parameter case. Indeed we find four solutions, one of them is of the quantum metric type as in (2.15)

\[
K^{(1)} = \begin{pmatrix}
a_{12} \\
-pqa_{12}
\end{pmatrix},
\]

\[ - 23 - \]
whereas the others all have vanishing determinant

$$K^{(2)} = \begin{pmatrix} a_{11} \\ \end{pmatrix}, \quad K^{(3)} = \begin{pmatrix} a_{12} \\ \end{pmatrix}, \quad K^{(4)} = \begin{pmatrix} a_{22} \\ \end{pmatrix}. \quad (3.20)$$

As in the 1-parameter case $K^{(1)}$ is stable under the comodule transformation up to the quantum determinant $det_{p,q}T = ad - pqbc = da - p^{-1}q^{-1}eb$ which is not central for the 2-parameter quantum group $GL_{p,q}(2)$, i.e. we have $TK^{(1)}T^t = (det_{p,q}T)K^{(1)}$. In particular the quadratic algebra defined by RE1 will not be isomorphic to the 1-parameter algebra. It is given by

$$\alpha\beta = p^{-2}\beta\alpha + p^{-1}\omega\alpha\gamma, \quad \beta\gamma = \gamma\beta,$$

$$\alpha\gamma = p^{-2}q^2\gamma\alpha, \quad \beta\delta = p^{-2}\delta\beta + p^{-1}\omega\gamma\delta, \quad (3.21)$$

$$\alpha\delta = p^{-4}\delta\alpha + p^{-2}\omega(q\beta\gamma + p^{-1}\gamma^2), \quad \gamma\delta = p^{-2}q^2\delta\gamma,$$

and for $p = 1$ we get the 1-parameter algebra (2.10). For this algebra there exists no central element which is linear in the elements of $K$. However, $c_1(K) = \beta - qp^{-1}\gamma$ has simple relations: $c_1\alpha = p^2\alpha c_1$, $c_1\delta = p^{-2}\delta c_1$, $[c_1,\beta] = [c_1,\gamma] = 0$ and transforms homogeneously $c_1(K_T) = (det_{p,q}T)c_1(K)$.

We now take a look at $gl(3)$ which has a 4-parameter deformation. We find that the structure of constant solutions is completely analogous to the $gl_{p,q}(2)$ case. The $R$-matrix is given by $R = \text{diag}(q, p_1, p_2, p_1^{-1}, q, p_3, p_2^{-1}, p_3^{-1}, q)$ plus the non-vanishing elements $R^{21}_{12} = R^{31}_{13} = R^{32}_{23} = \omega [28,29]$. It is worthwhile to point out that these multiparameter constant solutions of the YBE were already contained implicitly in [31] written some ten years ago (cf. formulae (25) and (26) therein). This was also noted recently in [32]. Unlike in the $N = 2$ case it cannot be obtained by such a simple transformation as (2.41), because in this way it is possible to add only $(N-1)$ independent parameters to the 1-parameter $R$-matrix of $sl_q(N)$. Anyway, in total we have nine solutions, and three of them are generalized $gl_{p,q}(2)$ embeddings

$$K^{(1)} = \begin{pmatrix} a_{12} \\ -p_1qa_{12} \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} a_{13} \\ -p_2qa_{13} \end{pmatrix},$$
\[ K^{(3)} = \begin{pmatrix} a_{23} \\ -p_3q a_{23} \end{pmatrix}. \] (3.22)

The other six solutions have one non-zero element only, located in the upper-triangle, i.e. contained in the set \( \{a_{ij}, i \leq j\} \). Looking at the \( N = 2, 3 \) solutions one gets an idea what solutions for \( N \geq 4 \) one might expect in multiparameter cases.

3.5. RE2

Finally we look for constant solutions of the second reflection equation (1.2) or (2.31). We shall not go through the complete analysis again, but just pick up a few basic examples. We have already discussed the \( sl_q(2) \) case in detail in section two. So let us proceed to \( sl_q(3) \) using the \( R \)-matrix given at the beginning of section 3.2.

RE2 is given by

\[ RK_1 \tilde{R} K_2 = K_2 R K_1 \tilde{R}, \]

and it has seven independent solutions for \( sl_q(3) \), two of them have non-vanishing determinants

\[ K^{(1)} = \begin{pmatrix} a_{11} \\ a_{11} \\ a_{11} \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} a_{13} \\ a_{22} \\ a_{33} \end{pmatrix}. \] (3.23)

where the function \( g_{31} \) is given by \( g_{31} = a_{22}(a_{22} - a_{33})/a_{13} \). Again, as for RE1 we get a triangular 3-parameter solution and a diagonal one, the latter is just the identity matrix which will always be a solution, as is evident from the structure of RE2. However, as for RE1 we cannot transform away all two independent parameters (disregarding an overall scaling parameter) by similarity transformation (2.39) but only one and the other one remains.

For \( sl_q(4) \) with \( R \)-matrix given below (3.4) we get three solutions with non-vanishing determinant. Besides the unity solution \( K^{(1)} \) there are the following
where the functions \( g_{14} \) and \( g_{44} \) are of quite different forms, namely \( g_{14} = a_{23}a_{32}/a_{14} \) and \( g_{44} = a_{22} - (a_{14}a_{41})/a_{22} \). They are also different from the corresponding function in the \( sl_q(3) \) case above. Besides them we found 14 independent solutions, many of them also had three parameters as in (3.24). The solutions \( K^{(2)} \) in (3.23) and (3.24) strongly suggest the generic solution of RE2 for the fundamental representation of \( sl_q(N) \) having the counter diagonal form,

\[
K \simeq \mathcal{D}, \quad (\mathcal{D})_{ij} = \delta_{i,N+1-j}. \tag{3.25}
\]

By using several properties of \( \mathcal{D} \) (\( e_{ij} \) is the matrix unit used in (3.7)),

\[
\mathcal{D}^2 = 1, \quad e_{\bar{m}m} = \mathcal{D}e_{\bar{m}m}, \quad \bar{m} = N + 1 - m,
\]

one can prove for the \( R \)-matrix in the fundamental representation of \( sl_q(N) \) the following commutation relation

\[
[R, R_\mathcal{D}] = 0, \quad R_\mathcal{D} = \mathcal{D}_2R\mathcal{D}_2 = \mathcal{D}_1\tilde{R}\mathcal{D}_1, \tag{3.26}
\]

which corresponds to (3.9) in the RE1. It is easy to show that \( \mathcal{D} \) satisfies RE2 by using (3.26). By applying the transformation (2.39), we get a counter diagonal solution with \( N/2 \) (or \( (N - 1)/2 \)) arbitrary parameters.

Next we treat again \( sl_q(1|1) \) with \( R \)-matrix (3.12), which gives the following quadratic algebra for RE2

\[
\begin{align*}
\alpha \beta &= q^{-2} \beta \alpha, \\
\alpha \gamma &= q^2 \gamma \alpha, \\
\alpha \delta &= \delta \alpha, \\
\beta \delta &= \delta \beta + q \omega \alpha \beta, \\
\beta \gamma &= -q^2 \gamma \beta + q \omega (\alpha^2 - \alpha \delta), \\
\gamma \delta &= \delta \gamma - q \omega \gamma \alpha, \\
\beta^2 &= 0, \\
\gamma^2 &= 0.
\end{align*} \tag{3.27}
\]

The central element linear in the entries of \( K \) is given by \( c_1 = \alpha - \delta \). It is obvious
that this algebra is not isomorphic to the algebra (3.14). This can be seen also by the forms of the constant solutions

\[ K^{(1)} \simeq \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} \alpha_{22} \end{pmatrix}, \quad (3.28) \]

where \( K^{(1)} \) clearly cannot be transformed to the corresponding solution \( K^{(1)} \) in (3.13) of RE1.

Finally some remarks on the multiparameter cases are in order. For RE2 the two-sided comodule property requires no conditions such as (2.5) on the \( R \)-matrices so that we can use the same equation RE2 for multi-parameter cases, too. As stated before, transformation (2.41) with an arbitrary diagonal matrix \( U \) leaves RE2 invariant for the \( R \)-matrices related to \( sl_q(N) \). Especially, the invariance under the transformation (3.18), which yields the 2-parameter \( R \)-matrix, means that the reflection equation for this \( R \)-matrix reduces to the reflection equation for the 1-parameter \( R \)-matrix. This is an important difference between RE1 and RE2. A similar statement holds for \( gl_{p,q}(3) \). Using transformation (2.41) it is possible to obtain a 3-parameter deformation of a special form which is completely equivalent to the 1-parameter case \( sl_q(3) \). However, the general 4-parameter deformation \([28, 29]\) cannot be obtained by a transformation (2.41) applied to the 1-parameter \( R \)-matrix. It has three rather uninteresting solutions with vanishing determinant plus the unity solution. This proves that the quadratic algebras defined by RE1 and RE2 for the \( gl_{p,q}(3) \) \( R \)-matrix in fact are not isomorphic.

4. Conclusions

In this paper we have studied various constant solutions of the reflection equations associated with a wide class of known \( R \)-matrices. Many interesting properties of the reflection equations have been uncovered through the examination of these explicit solutions, some of which are obtained by a formula manipulation program on a computer. The study of constant solutions of the reflection equations has also revealed rich algebraic structures which are directly related but not identical.
with the quantum groups. To name some of them, the quantum homogeneous spaces, generalization of the braid groups, non-commutative differential geometry on quantum groups \cite{21,22} and the fusion procedures for the $K$-matrices \cite{9,19}, etc. In particular, the relations (2.33) of the quadratic algebra $A_2$ coincide with the relations of the “coordinates of the $q$-deformed Minkowski space” \cite{33} and the central elements (2.34) correspond to the time and the invariant length.

A variety of constant solutions presented here are expected to give good starting points for constructing such objects and other possible applications. As another application of the constant solution $K$, let us mention the construction of local fields in terms of exchange algebra fields. A direct generalization into this direction with higher rank groups seems to require the non-standard $R$-matrices associated with non-affine Toda field theory \cite{34} for which the analysis turns out highly non-trivial.

One could go on exploring various aspects of reflection equations on many fronts. For example, understanding the reflection equations for the $R$-matrices related to the quantum orthogonal $SO_q(N)$ and symplectic $Sp_q(2N)$ groups would be a nice challenge. These groups are characterized, in addition to (2.3), by $TCT^t = C$ with the matrix $C$ of the invariant quadratic forms \cite{3}. These constant matrices $C$ satisfy the corresponding RE1. The simplest example would be the matrix $\varepsilon_q$ in (2.15) for the $sl_q(2)$ which may be considered as $sp_q(2)$. Now that a complete list of constant solutions to the YBE in two dimensions \textit{i.e.} $\dim V_1 = \dim V_2 = 2$ \cite{35} is available, one might be tempted to a systematic study of the quadratic algebras and their one-dimensional representations \textit{i.e.} constant solutions of the reflection equations related with lower dimensional quantum groups. It would be nice to have connections between the constant solutions and the spectral parameter dependent solutions of RE as in the cases of the YBE.
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