LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH GENERALIZED SYMMETRIC METRIC CONNECTION

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Abstract. In this paper, we defined new metric connection for Lorentzian para-Sasakian manifolds which is called generalized symmetric metric connection of type $(\alpha, \beta)$. Quarter-symmetric and semi-symmetric connections are two samples of this connection such that $(\alpha, \beta) = (0, 1)$ and $(\alpha, \beta) = (1, 0)$, respectively. We get some basic concept with respect to generalized symmetric metric connection in a LP-Sasakian manifolds. Finally we consider CR-submanifolds with respect to generalized metric connection.

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1. Introduction

In [5], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors ([1], [2], [4], [7], [20], [18]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S.Golab [4]. A linear connection is said
to be a quarter-symmetric connection if its torsion tensor $\mathcal{T}$ is of the form

$$\mathcal{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y,$$

for any vector fields $X, Y$ on a manifold, where $u$ is a 1-form and $\varphi$ is a tensor of type $(1,1)$. If $\varphi = I$, then the quarter-symmetric connection is reduced to a semi-symmetric connection. Hence quarter-symmetric connection can be viewed as a generalization of semi-symmetric connection.

A linear connection $\nabla$ is said to be generalized symmetric connection if its torsion tensor $T$ is of the form

$$T(X,Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\varphi X - u(X)\varphi Y\},$$

for any vector fields $X, Y$ on a manifold, where $\alpha$ and $\beta$ are smooth functions. $\varphi$ is a tensor of type $(1,1)$ and $u$ is a 1-form. Moreover, the connection $\nabla$ is said to be a generalized symmetric metric connection if there is a Riemannian metric $g$ in $M$ such that $\nabla g = 0$, otherwise it is non-metric.

In the equation (2), if $\alpha = 0$ ($\beta = 0$), then the generalized symmetric connection is called $\beta$-quarter-symmetric connection ($\alpha$-semi-symmetric connection), respectively. Moreover, if we choose $(\alpha, \beta) = (1,0)$ and $(\alpha, \beta) = (0,1)$, then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Hence a generalized symmetric connection can be viewed as a generalization of semi-symmetric connection and quarter-symmetric connection. This two connection are important for both the geometry study and applications to physics.

On the other hand, in 1989, K. Matsumoto [8] introduced the notion of Lorentzian para-Sasakian manifolds. I. Mihai and R. Rosca [10] studied the same manifolds independently and they obtained several results on such manifolds. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [9], I. Mihai, A.A. Shaikh and U. C. De [11]. S. K. Srivastava and R. P. Kushwaha studied Lorentzian para-Sasakian manifolds admitting a special semi symmetric recurrent metric connection [17]. In [12] S.Y. Perktas, E. Kilic and M, M, Tripathi investigated curvature tensors with respect to semi-symmetric connection in a Lorentzian para-Sasakian manifold. It is shown that a Lorentzian para-Sasakian manifold with semi-symmetric non-metric connection is an $\eta$-Einstein manifold [13]. In [19] O. Bahadir
get some results with Lorentzian para-Sasakian manifold with quarter-symmetric non-metric connection.

In the present paper, we defined new connection for Lorentzian para-Sasakian manifold, generalized symmetric metric connection. This connection is the generalized form of semi-symmetric metric connection and quarter-symmetric metric connection. Section 2 is devoted to preliminaries. In section 3, we get generalized symmetric metric connection for a Lorentzian para-Sasakian manifold. In section 4, we calculate curvature tensor and ricci tensor of Lorentzian para-Sasakian with respect to generalized symmetric metric connection. Moreover it is shown that if a Lorentzian para-Sasakian manifold is Ricci semi-symmetric with respect to generalized symmetric metric connection, then the manifold is a generalized $\eta$– Einstein manifold with respect to generalized symmetric metric connection. section 5, we study CR-submanifolds a Lorentzian para-Sasakian with respect to generalized symmetric metric connection. Furthermore, we get integrability conditions of the distributions on CR-submanifolds.

2. Preliminaries

A differentiable manifold of dimension $n$ is called Lorentzian para-Sasakian (briefly, LP-Sasakian) [8, 10], if it admit a $(1, 1)$ tensor field $\phi$, a contravariant vector field $\xi$, a $1-$ form $\eta$ and Lorentzian metric $g$ which satify

$$\eta(\xi) = -1, \quad (3)$$
$$\phi^2(X) = X + \eta(X)\xi, \quad (4)$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (5)$$
$$g(X, \xi) = \eta(X), \quad (6)$$
$$\nabla_X \xi = \phi X, \quad (7)$$
$$\nabla_X \phi (Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (8)$$

where $\nabla$ is Levi-Civita connection with respect to the Lorentzian metric $g$. It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad rank \phi = n - 1, \quad (9)$$
If we write
\[ \Phi(X, Y) = g(\phi X, Y), \]  
for any vector field \( X \) and \( Y \), then the tensor field \( \Phi(X, Y) \) is a symmetric (0,2) tensor field \[8\]. Also, since the vector \( \eta \) is closed in an LP-Sasakian \[8, 11\] manifold, we have
\[ (\nabla_X \eta)Y = \Phi(X, Y), \quad \Phi(X, \xi) = 0, \]  
for any vector field \( X \) and \( Y \).

Let \( M \) be an \( n \)-dimensional LP-Sasakian manifold. Then the following relations hold \[11, 9\]:
\[ g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \]  
\[ R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \]  
\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \]  
\[ R(\xi, X)\xi = X + \eta(X)\xi, \]  
\[ S(X, \xi) = (n - 1)\eta(X), \]  
\[ S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \]  
for any vector fields \( X, Y \) and \( Z \), where \( R \) and \( S \) are the curvature and Ricci tensors of \( M \), respectively.

A LP-Sasakian manifold \( M \) is said to be generalized \( \eta \)-Einstein if its Ricci tensor \( S \) is of the form
\[ S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + cg(\phi X, Y), \]  
for any \( X, Y \in \Gamma(TM) \), where \( a, b \) and \( c \) are scalar functions such that \( b \neq 0, c \neq 0 \). If \( c = 0 \) then \( M \) is called \( \eta \)-Einstein manifold.

Let \( M \) be a submanifold of a LP-Sasakian manifold \( M' \). The Gauss and Weingarten formulas are given
\[ \nabla_X Y = \nabla'_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM'), \]  
\[ \nabla_X N = -A_N X + \nabla'_X N, \quad \forall N \in \Gamma(T^\perp M'), \]  
where \( \{\nabla_X Y, A_N X\} \) and \( \{h(X, Y), \nabla'_X N\} \) belong to \( \Gamma(TM') \) and \( \Gamma(T^\perp M') \), respectively.
3. Generalized Symmetric Metric Connection in a LP- Sasakian manifold

Let $\nabla$ be a linear connection and $\nabla$ be a Levi-Civita connection of an LP-Sasakian manifold $M$ such that

$$\nabla_X Y = \nabla_X Y + H(X, Y),$$

for any vector field $X$ and $Y$. Where $H$ is a tensor of type $(1, 2)$. For $\nabla$ to be a generalized symmetric metric connection of $\nabla$, we have

$$H(X, Y) = \frac{1}{2} [T(X, Y) + T'(X, Y) + T'(Y, X)],$$

where $T$ is the torsion tensor of $\nabla$ and

$$g(T'(X, Y), Z) = g(T(Z, X), Y).$$

From (2) and (23) we get

$$T'(X, Y) = \alpha \{\eta(X)Y - g(X, Y)\xi\} + \beta \{\eta(Y)\phi X - g(\phi X, Y)\xi\}.\quad (24)$$

Using (2), (22) and (24) we obtain

$$H(X, Y) = \alpha \{\eta(Y)X - g(X, Y)\xi\} + \beta \{\eta(Y)\phi X - g(\phi X, Y)\xi\}.\quad (25)$$

**Corollary 1.** For an LP-Sasakian manifold, generalized symmetric metric connection $\nabla$ of type $(\alpha, \beta)$ is given by

$$\nabla_X Y = \nabla_X Y + \alpha \{\eta(Y)X - g(X, Y)\xi\} + \beta \{\eta(Y)\phi X - g(\phi X, Y)\xi\}.\quad (26)$$

If we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, generalized symmetric metric connection is reduced a semi-symmetric metric connection and quarter-symmetric metric connection as follows:

$$\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi,\quad (27)$$

$$\nabla_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.\quad (28)$$

From (27), (28), (11) and (23) we have the following proposition
Proposition 2. Let $M$ be an LP- Sasakian manifold with generalized symmetric metric connection. We have the following relations:

\[
(\nabla_X \phi)Y = [(1 - \beta)g(X,Y) + (2 - 2\beta)\eta(X)\eta(Y) - \alpha\Phi(X,Y)]\xi + (1 - \beta)\eta(Y)X - \alpha\eta(Y)\phi X,
\]

\[
\nabla_X \xi = (1 - \beta)\phi X - \alpha X - \alpha\eta(X)\xi,
\]

\[
(\nabla_X \eta)Y = (1 - \beta)\Phi(X,Y) - \alpha g(\phi X, \phi Y),
\]

for any $X, Y, Z \in \Gamma(TM)$.

4. Curvature Tensor

Let $M$ be an $n$- dimensional LP-Sasakian manifold. The curvature tensor $\mathcal{R}$ of the generalized symmetric metric connection $\nabla$ on $M$ is defined by

\[
\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

Using (26), (29) and Proposition 2 we have

\[
\mathcal{R}(X, Y)Z = \mathcal{R}(X, Y)Z + K_1(Y, Z)X - K_1(X, Z)Y + K_2(Y, Z)\phi X - K_2(X, Z)\phi Y
\]

\[
+ \{K_3(X, Y)Z - K_3(Y, X)Z\} \xi,
\]

where

\[
K_1(Y, Z) = (\alpha \beta - \alpha)\Phi(Y, Z) + \alpha^2 g(Y, Z) + (\alpha^2 + \beta - \beta^2)\eta(Y)\eta(Z),
\]

\[
K_2(Y, Z) = (\beta^2 - 2\beta)\Phi(Y, Z) - \alpha(1 - \beta)g(Y, Z),
\]

\[
K_3(X, Y)Z = \{((\alpha^2 + \beta)g(Y, Z) + \alpha\beta\Phi(Y, Z)\} \eta(X),
\]

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

From (31) we have the following lemma

Lemma 3. Let $M$ be an $n$- dimensional LP- Sasakian manifold with generalized symmetric metric connection. Then we have the following equations:

\[
\mathcal{R}(X, Y)\xi = (1 - \beta + \beta^2)(\eta(Y)X - \eta(X)Y) + \alpha(1 - \beta)(\eta(X)\phi Y - \eta(Y)\phi X),
\]

\[
\mathcal{R}(\xi, Y)Z = \{-\alpha\Phi(Y, Z) + (1 - \beta)g(Y, Z) - \beta^2\eta(Y)\eta(Z)\} \xi + (-1 + \beta - \beta^2)\eta(Z)Y + \alpha(1 - \beta)\eta(Z)\phi Y
\]

\[
\mathcal{R}(\xi, Y)\xi = (1 - \beta + \beta^2)(\eta(Y)\xi + Y) + \alpha(\beta - 1)\phi Y,
\]

for any $X, Y, Z \in \Gamma(TM)$. 
The Ricci tensor $\overline{S}$ of an LP-Sasakian manifold $M$ with respect to generalized symmetric metric connection $\nabla$ is given by

$$\overline{S}(Y, Z) = \sum_{i=1}^{n} \varepsilon_{i} g(R(e_{i}, Y)Z, e_{i}).$$

where $X, Y \in \Gamma(TM)$, $\{e_{1}, e_{2}, ..., e_{n}\}$ is an orthonormal frame and $\varepsilon_{i} = g(e_{i}, e_{i})$.

From (30) we have

$$\overline{S}(Y, Z) = \sum_{i=1}^{n} \varepsilon_{i} \{g(R(e_{i}, Y)Z, e_{i}) + K_{1}(Y, Z)e_{i} - K_{1}(e_{i}, Z)g(Y, e_{i}) + K_{2}(Y, e_{i})g(\phi e_{i}, e_{i}) - K_{2}(e_{i}, Z)g(\phi Y, e_{i}) + \{K_{3}(e_{i}, Y)Z - K_{3}(Y, e_{i})Z\} \eta(e_{i})\}.$$  \hspace{1cm} (35)

Since the Ricci tensor of the Levi-Civita connection $\nabla$ is given by

$$S(X, Y) = \sum_{i=1}^{n} \varepsilon_{i} g(R(e_{i}, X)Y, e_{i}),$$

then by using (31), (32), (33) and (35) we get

$$\overline{S}(Y, Z) = S(Y, Z) + \{-(n-1)(1 - \beta + \beta^{2}) + \alpha(\beta - 1)trace\Phi\} \eta(Y)\eta(Z).$$  \hspace{1cm} (36)

$\Phi$ and Ricci tensor of the Levi-Civita connection are symmetric, thus from (36) we have the following corollary:

**Corollary 4.** Let $M$ be an $n$–dimensional LP–Sasakian manifold. Then the Ricci tensor $\overline{S}$ of generalized symmetric metric connection $\nabla$ is symmetric.

**Lemma 5.** Let $M$ be an $n$–dimensional LP–Sasakian manifold. Then we have.

$$\overline{S}(Y, \xi) = \{ (n-1)(1 - \beta + \beta^{2}) + \alpha(\beta - 1)trace\Phi \} \eta(Y),$$  \hspace{1cm} (37)

$$\overline{S}(\phi Y, \phi Z) = \overline{S}(Y, Z) + \{ (n-1)(1 - \beta + \beta^{2}) + \alpha(\beta - 1)trace\Phi \} \eta(Y)\eta(Z),$$  \hspace{1cm} (38)

for any $X, Y \in \Gamma(TM)$.

**Proof.** Using (3), (9) and (16) in the equation (36), we get (37)

By using (5), (9) and (17) in the equation (36), we have (38). \hfill $\square$

**Theorem 6.** Let $M$ be an $n$–dimensional LP–Sasakian manifold. If $M$ is Ricci semi-symmetric with respect to generalized symmetric metric connection , then we have the following statements:
(i) \( M \) is generalized \( \eta \) Einstein manifold with respect to generalized symmetric metric connection of type \((\alpha, \beta)\),

(ii) \( M \) is \( \eta \) Einstein manifold with respect to generalized symmetric metric connection of type \((0, \beta)\),

(iii) \( M \) is Einstein manifold with respect to generalized symmetric metric connection of type \((\alpha, 0) \) \((\alpha \neq 1)\).

**Proof.** Let \( \overline{R}(X,Y)\mathcal{S} = 0 \) be on \( M \) for any \( X, Y, Z, U \in \Gamma(TM) \), then we have

\[
\mathcal{S}(\overline{R}(X,Y)Z,U) + \mathcal{S}(Z,\overline{R}(X,Y)U) = 0. \tag{39}
\]

If we choose \( Z = \xi \) and \( X = \xi \) in (39), we get

\[
\mathcal{S}(\overline{R}(\xi,Y)\xi,U) + \mathcal{S}(\xi,\overline{R}(\xi,Y)U) = 0. \tag{40}
\]

Using Lemma 3 and Lemma 5 in (40), we obtain

\[
(1 - \beta + \beta^2)\mathcal{S}(Y,U) + \alpha(\beta - 1)\mathcal{S}(\Phi Y,U) \tag{41}
\]

\[
= \{(n - 1)(1 - \beta + \beta^2) + \alpha(\beta - 1) \} \{-\alpha\Phi(Y,U) + (1 - \beta)g(Y,U) - \beta^2 \eta(Y)\eta(U)\}.
\]

If one substitutes \( Y = \phi Y \) in the equation (41) and using (37), we get

\[
(1 - \beta + \beta^2)\mathcal{S}(\phi Y,U) + \alpha(\beta - 1)\mathcal{S}(Y,U) \tag{42}
\]

\[
= \{(n - 1)(1 - \beta + \beta^2) + \alpha(\beta - 1) \} \{(1 - \beta)\Phi(Y,U) - \alpha g(Y,U) - \alpha \beta \eta(Y)\eta(U)\}.
\]

From the (41) and (42), we obtain

\[
(1 - \beta + \beta^2)\mathcal{S}(Y,U) \tag{43}
\]

\[
= \{(n - 1)(1 - \beta + \beta^2) + \alpha(\beta - 1) \} \{\alpha \beta \Phi(Y,U) - (1 - \beta)(1 - \beta + \beta^2 - \alpha^2)g(Y,U)

+ (-\beta^4 + \beta^3 - \beta^2 + \alpha^2 \beta^2 - \beta \alpha^2)\eta(Y)\eta(U)\}.
\]

Thus, for \( \alpha = 0 \) and \( \beta = 0 \) we get the following equations:

\[
\mathcal{S}(Y,U) = (n - 1)(1 - \beta)g(Y,V) - (n - 1)\beta^2 \eta(Y)\eta(U), \tag{44}
\]

\[
(1 - \alpha^2)\mathcal{S}(Y,U) = (1 - \alpha^2)(n - 1 - \alpha \text{ trace } \Phi)g(Y,U). \tag{45}
\]

Equations (43), (44) and (45) tell us proof is completed. \( \square \)
5. CR-SUBMANIFOLDS OF AN LP-SASAKIAN MANIFOLD WITH GENERALIZED
SYMMETRIC METRIC CONNECTION

Definition 7. [20] An \( n \)-dimensional Riemannian manifold \( M \) of an LP-Sasakian
manifold \( M' \) is called a CR-submanifold if \( \xi \) is tangent to \( M \) and there exists on \( M \)
a differentiable distribution \( D: x \to D_x \subset T_x(M) \) such that
(i) \( D \) is invariant under \( \phi \), i.e. \( \phi D \subset D \).
(ii) The orthogonal complement distribution \( D^\perp: x \to D^\perp_x \subset T^\perp_xM \) of the distribu-
tion \( D \) on \( M \) is totally real \( \), i.e. \( \phi D^\perp \subset T^\perp M \).

Definition 8. [20] The distribution \( D \) (resp., \( D^\perp \)) is called horizontal (resp., ver-
tical) distribution. The pair \( (D, D^\perp) \) is called \( \xi \)-horizontal (resp., \( \xi \)-vertical) if \( \xi \in \Gamma(D) \) (resp., \( \xi \in \Gamma(D^\perp) \)). The CR-submanifold is also called \( \xi \)-horizontal
(resp., \( \xi \)-vertical) if \( \xi \in \Gamma(D) \) (resp., \( \xi \in \Gamma(D^\perp) \)).

The orthogonal complement \( \phi D^\perp \) in \( T^\perp M \) is given by
\[
TM = D \oplus D^\perp, \quad T^\perp M = \phi D^\perp \oplus \mu,
\]
where \( \phi \mu = \mu \).

Let \( M \) be a CR-submanifold of an LP-Sasakian manifold \( M' \) with generalized
symmetric metric connection \( \nabla \). For any \( X \in \Gamma(TM') \) and \( N \in \Gamma(T^\perp M') \) we can write
\[
X = PX + QX, \quad PX \in \Gamma(D), \quad QX \in \Gamma(D^\perp), \quad (46)
\]
\[
\phi N = BN + CN, \quad BN \in \Gamma(D^\perp), \quad CN \in \Gamma(\mu). \quad (47)
\]
The Gauss and weingarten formulas with respect to \( \nabla \) are given, respectively,
\[
\nabla_X Y = \nabla'_{X} Y + \overline{h}(X,Y) \quad (48)
\]
\[
\nabla_X N = -\overline{A}_N X + \nabla^\perp_X N \quad (49)
\]
for any \( X, Y \in \Gamma(TM') \), where \( \nabla'_{X}, \overline{A}_N X \in \Gamma(TM') \). Here, \( \nabla', \overline{h} \) and \( \overline{A}_N \)
are called the induced connection on \( M \), the second fundamental form and the
Weingarten mapping with respect to \( \nabla \). From \( [19] \), \( [20] \) and \( [18] \) we have
\[
\nabla'_{X} Y + \overline{h}(X,Y) = \nabla_{X} Y + h(X,Y) + \alpha \{ \eta(Y)X - g(X,Y)\xi \} + \beta \{ \eta(Y)\phi X - g(\phi X,Y)\xi \}
\]
Using (46) and (47) in above the equation and comparing the tangential and normal components on both sides we obtain

\[
P\nabla Y = P\nabla Y + \alpha \eta(Y)PX - \alpha g(X,Y)P\xi + \beta \eta(Y)\phi PX - \beta g(\xi, Y)P\xi \tag{50}
\]

\[
\tilde{h}(X,Y) = h(X,Y) + \beta \eta(Y)\phi QX \tag{51}
\]

\[
Q\nabla Y = Q\nabla Y + \alpha \eta(Y)QX - \alpha g(X,Y)Q\xi - \beta \phi QX. \tag{52}
\]

for any \(X, Y \in \Gamma(TM').\)

**Theorem 9.** Let \(M\) be a CR-submanifold of an LP-Sasakian manifold \(M'\) with generalized symmetric metric connection \(\tilde{\nabla}\). Then we have the following expression:

(i) If \(M\) \(\xi\)− horizontal, \(X, Y \in \Gamma(D)\) and \(D\) is parallel with respect to \(\tilde{\nabla}\), then the induced connection \(\nabla\) is a generalized symmetric metric connection.

(ii) If \(M\) is \(\xi\)− vertical, \(X, Y \in \Gamma(D^\perp)\) and \(D^\perp\) is parallel with respect to \(\tilde{\nabla}\) then the induced connection \(\nabla\) is a generalized symmetric non-metric connection.

(iii) The Gauss formula with respect to generalized symmetric metric connection is of the form

\[
\nabla X Y = \nabla X Y + \beta \eta(Y)\phi QX. \tag{53}
\]

(iv) The weingarten formula with respect to generalized symmetric metric connection is of the form

\[
\nabla X N = -A X + \nabla X N + \alpha \eta(N)X + \beta \eta(N)\phi X - \beta g(\phi X, N)\xi. \tag{54}
\]

**Proof.** Using (48) and (51) we have (iii). Moreover, from (26) and Weingarten formula, we get (iv).

In view of (50), if \(M\) \(\xi\)− horizontal, \(X, Y \in \Gamma(D)\) and \(D\) is parallel with respect to \(\tilde{\nabla}\), we obtain

\[
\nabla X Y = \nabla X Y + \alpha \eta(Y)X - \alpha g(X,Y)\xi + \beta \eta(Y)\phi X - \beta g(\phi X, Y)\xi. \tag{55}
\]

This equation is verifying (i).

On the other hand, In view of (52) if \(M\) is \(\xi\)− vertical, \(X, Y \in \Gamma(D^\perp)\) and \(D^\perp\) is parallel with respect to \(\tilde{\nabla}\), we have

\[
\nabla X Y = \nabla X Y + \alpha \eta(Y)X - \alpha g(X,Y)\xi - \beta g(\phi X, Y)\xi. \tag{56}
\]

Using (56) we get

\[
(\nabla X g)(Y, Z) = \beta \{\eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y)\}. \tag{57}
\]
Thus, we have (ii).

□

**Lemma 10.** Let $M$ be a CR-submanifold of an LP-Sasakian manifold $M'$ with generalized symmetric metric connection. Then

\[ h(X, \phi PY) + \beta \eta(Y)\phi QX + \nabla_X^\perp QY = Ch(X, Y) - \alpha \eta(Y)\phi QX + \phi Q
\]

\[ P^\perp X \phi PY - PA_{\phi QY}X - \beta g(\phi X, \phi QY)P\xi = K(X, Y)P\xi + (1 - \beta)\eta(Y)PX \]

\[ -\alpha \eta(Y)\phi PX + \phi P\nabla_X Y + \beta \eta(Y)\eta(QX)P\xi, \]

\[ Q^\perp X \phi PY - QA_{\phi QY}X - \beta g(\phi X, \phi QY)Q\xi = K(X, Y)Q\xi + (1 - \beta)\eta(Y)QX \]

\[ + B\eta(Y) + \beta \eta(Y)QX + \beta \eta(Y)\eta(QX)Q\xi, \]

for any $X, Y \in \Gamma(TM)$, where $K(X, Y) = (1 - \beta)g(X, Y) + (2 - 2\beta)\eta(Y)\eta(Y) - \alpha g(X, \phi Y)$.

**Proof.** We know that $\nabla_X \phi Y = (\nabla_X \phi)Y + \phi(\nabla_X Y)$.

By virtue of Proposition(2, 33) and (34), we get

\[ \nabla_X \phi PY + h(X, \phi PY) + \beta \eta(Y)\phi QX - A_{\phi QY}X + \nabla_X^\perp QY - \beta g(\phi X, \phi QY)\xi = (1 - \beta)\eta(Y)X - \alpha \eta(Y)\phi X \]

\[ + \{(1 - \beta)g(X, Y) + (2 - 2\beta)\eta(Y)\eta(Y) - \alpha g(X, \phi Y))\xi + \phi \nabla_X Y + \phi h(X, Y) + \beta \eta(Y)(QX + \eta(QX))\xi. \]

Using (10) and (17) and the above equation, comparing the normal, horizontal and vertical components, we have (58)-(60). □

**Lemma 11.** Let $M$ be a $\xi$-vertical CR-submanifold of an LP-Sasakian manifold $M'$ with generalized symmetric metric connection. Then

\[ \phi P[Y, Z] = A_{\phi Y}Z - A_{\phi Z}Y + (\beta - 1)\{\eta(Z)Y - \eta(Y)Z \}
\]

for any $Y, Z \in \Gamma(D^\perp)$.

**Proof.** For any $Y, Z \in \Gamma(D^\perp)$, We know that $\nabla_X \phi Y = (\nabla_X \phi)Y + \phi(\nabla_X Y)$. Using Proposition(2, 33) and (34), we get

\[ -A_{\phi Z}Y + \nabla_Y^\perp \phi Z - \beta g(\phi Y, \phi Z)\xi = \{(1 - \beta)g(Y, Z) + (2 - 2\beta)\eta(Y)\eta(Z) - \alpha g(Y, \phi Z))\xi \]

\[ + (1 - \beta)\eta(Z)Y - \alpha \eta(Z)\phi Y + \phi \nabla_Y Z + \phi h(Y, Z) + \beta \eta(Z)\phi^2 QY \]

By using (58), we obtain

\[ \phi \nabla_Y Z = -A_{\phi Z}Y + \{-g(Y, Z) + (1 - \beta)\eta(Y)\eta(Z) + \alpha g(Z, \phi Y)\}Z \]

\[ - B\eta(Y, Z) + (\beta - 1)\eta(Z)Y - \beta \eta(Z)(\phi QY + \phi^2 QY). \]
Interchanging $Y$ and $Z$, we have
\[
\phi\nabla_Z Y = -A_{\phi Y} Z + \{-g(Y, Z) + (-2 + \beta)\eta(Y)\eta(Z) + \alpha g(Z, \phi Y)\}\xi \\
- Bh(Y, Z) + (\beta - 1)\eta(Y)Z - \beta\eta(Y)(\phi QZ + \phi^2 QZ).
\]

By subtracting, we completed the proof. \hfill \Box

This lemma is verifying the following theorem.

**Theorem 12.** Let $M$ be a $\xi-$ vertical CR-submanifold of an LP-Sasakian manifold $M'$ with generalized symmetric metric connection. Then the distribution $D^\perp$ is integrable if and only if
\[
A_{\phi Y} Z - A_{\phi Z} Y = (\beta - 1)\{\eta(Y)Z - \eta(Z)Y\},
\]
for any $Y, Z \in \Gamma(D^\perp)$.

**Corollary 13.** Let $M$ be a $\xi-$ vertical CR-submanifold of an LP-Sasakian manifold $M'$ with generalized symmetric metric connection of type $(\alpha, 1)$. Then the distribution $D^\perp$ is integrable if and only if
\[
A_{\phi Y} Z = A_{\phi Z} Y,
\]
for any $Y, Z \in \Gamma(D^\perp)$.

**Corollary 14.** Let $M$ be a $\xi-$ vertical CR-submanifold of an LP-Sasakian manifold $M'$ with semi-symmetric metric connection. Then the distribution $D^\perp$ is integrable if and only if
\[
A_{\phi Y} Z - A_{\phi Z} Y = \eta(Z)Y - \eta(Y)Z,
\]
for any $Y, Z \in \Gamma(D^\perp)$.

**Proposition 15.** Let $M$ be a $\xi-$ vertical CR-submanifold of an LP-Sasakian manifold $M'$ with generalized symmetric metric connection. Then
\[
\phi Ch(X, Y) = Ch(\phi X, Y) = Ch(X, \phi Y)
\]
for any $X, Y \in \Gamma(D)$.

**Proof.** From (60) we get
\[
Q\nabla_X \phi Y = \{(1 - \beta)g(X, Y) - \alpha g(X, \phi Y)\}Q\xi + Bh(X, Y)
\]
(65)
and
\[
Q\nabla_{\phi X}\phi Y = \{(1 - \beta)g(\phi X, Y) - \alpha g(X, Y)\}Q\xi + Bh(\phi X, Y).  \tag{66}
\]
Interchanging \(X\) and \(Y\) in (65) we have
\[
Q\nabla_{Y}^\prime\phi X = \{(1 - \beta)g(Y, X) - \alpha g(\phi Y, \phi X)\}Q\xi + Bh(\phi X, Y).  \tag{67}
\]
Replacing \(X\) by \(\phi X\) we obtain
\[
Q\nabla_{Y}^\prime\phi X = \{(1 - \beta)g(\phi X, Y) - \alpha g(X, Y)\}Q\xi + Bh(\phi X, Y).  \tag{68}
\]
Subtracting (66) from (68)
\[
Q(\nabla_{\phi X}^\prime\phi Y - \nabla_{Y}^\prime X) = 0.  \tag{69}
\]
Thus, we get
\[
\nabla_{\phi X}^\prime\phi Y - \nabla_{Y}^\prime X \in D.  \tag{70}
\]
Moreover, from (58), we find
\[
h(X, \phi Y) = Ch(X, Y) + \phi Q\nabla_{X}^\prime Y  \tag{71}
\]
Replacing \(X\) by \(\phi X\) and \(Y\) by \(\phi Y\) in (71) we obtain
\[
h(\phi X, Y) = Ch(\phi X, \phi Y) + \phi Q\nabla_{\phi X}^\prime \phi Y.  \tag{72}
\]
Interchanging \(X\) and \(Y\) in (71), we get
\[
h(\phi X, Y) = Ch(X, Y) + \phi Q\nabla_{Y}^\prime X.  \tag{73}
\]
Subtracting (72) from (73) and using (70), we have
\[
Ch(\phi X, \phi Y) = Ch(X, Y),
\]
Replacing \(X\) by \(\phi X\) in the last equation we find
\[
Ch(\phi^2 X, \phi Y) = Ch(\phi X, Y),
\]
Thus, we obtain
\[
Ch(X, \phi Y) = Ch(\phi X, Y).
\]
Using (65) we obtain
\[
Q\nabla_{X}^\prime \phi^2 Y = \{(1 - \beta)g(X, \phi Y) - \alpha g(X, \phi^2 Y)\}Q\xi + Bh(X, \phi Y).  \tag{74}
\]
Thus,

\[ Q\nabla'_X Y = \{(1 - \beta)g(X, \phi Y) - \alpha g(X, Y)\}Q\xi + Bh(X, \phi Y). \]  

(75)

Using (75) in (71), we have

\[ h(X, \phi Y) = Ch(X, Y) + \phi Bh(X, \phi Y). \]  

(76)

Applying \(\phi\) on both sides, we obtain

\[ \phi h(X, \phi Y) = \phi Ch(X, Y) + \phi Bh(X, \phi Y). \]  

(77)

Using (47) in (77), proof is completed.

**Theorem 16.** Let \( M \) be a \(\xi\)-horizontal CR-submanifold of an LP-Sasakian manifold \( M' \) with generalized symmetric metric connection. Then the distribution \( D \) is integrable if and only if

\[ h(\phi X, Y) = h(\phi Y, X), \]

for any \( X, Y \in \Gamma(D) \).

**Proof.** From (60) we get

\[ Q\nabla'_X \phi Y = B h(X, Y). \]

Replacing \( X \) by \( \phi X \) we have

\[ Q\nabla'_\phi X \phi Y = B \phi h(\phi X, Y). \]

Interchanging \( X \) and \( Y \) we obtain

\[ Q\nabla'_\phi X \phi Y = B h(\phi X, \phi Y). \]

Subtracting last two equations, we find

\[ Q[\phi X, \phi Y] = B\{h(\phi X, Y) - h(\phi Y, X)\}. \]

Proof is completed.

\(\square\)
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