A New Noncommutative Product on the Fuzzy Two-Sphere Corresponding to the Unitary Representation of $SU(2)$ and the Seiberg-Witten Map

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Abstract

We obtain a new explicit expression for the noncommutative (star) product on the fuzzy two-sphere which yields a unitary representation. This is done by constructing a star product, $\star_\lambda$, for an arbitrary representation of $SU(2)$ which depends on a continuous parameter $\lambda$ and searching for the values of $\lambda$ which give unitary representations. We will find two series of values: $\lambda = \lambda_j^{(A)} = 1/(2j)$ and $\lambda = \lambda_j^{(B)} = -1/(2j+2)$, where $j$ is the spin of the representation of $SU(2)$. At $\lambda = \lambda_j^{(A)}$ the new star product $\star_\lambda$ has poles. To avoid the singularity the functions on the sphere must be spherical harmonics of order $\ell \leq 2j$ and then $\star_\lambda$ reduces to the star product $\star$ obtained by Prešnajder [1]. The star product at $\lambda = \lambda_j^{(B)}$, to be denoted by $\star_\lambda$, is new. In this case the functions on the fuzzy sphere do not need to be spherical harmonics of order $\ell \leq 2j$. The star product $\star_\lambda$ has no singularity for negative values of $\lambda$ and we can move from one representation $\lambda = \lambda_j^{(B)}$ to another $\lambda = \lambda_j^{(B)}$ smoothly on the negative $\lambda$ line. Because in this case there is no cutoff on the order of spherical harmonics, the degrees of freedom of the gauge fields on the fuzzy sphere coincide with those on the commutative sphere. Therefore, although the field theory on the fuzzy sphere is a system with finite degrees of freedom, we can expect the existence of the Seiberg-Witten map between the noncommutative and commutative descriptions of the gauge theory on the sphere. We will derive the first few terms of the Seiberg-Witten map for the $U(1)$ gauge theory on the fuzzy sphere by using power expansion around the commutative point $\lambda = 0$.

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1 Introduction

The fuzzy sphere is a 2d surface on which the coordinates are represented by operators \( \hat{x}^a (a = 1, 2, 3) \) that satisfy a condition

\[
(\hat{x}^a)^2 = R^2 \mathbf{1}
\]

as well as a commutation relation

\[
[\hat{x}^a, \hat{x}^b] = 2i \lambda \epsilon_{abc} \hat{x}^c,
\]

where \( \lambda \) is a constant parameter, \( \mathbf{1} \) an identity operator and \( R \) the radius of the sphere. \[1\]

Because (2) is the \( su(2) \) algebra, for a unitary representation \( j \) \((2j = 0, 1, 2, \ldots)\), \( \hat{x}^a \) is represented by a hermitian \((2j + 1) \times (2j + 1)\) matrix. The \((2j + 1)\) components of this matrix can be decomposed into \( 2j + 1 \) irreducible representations of \( SU(2) \) with angular momentum \( \ell = 0, 1, \ldots, 2j \). These representations are called spherical harmonics on the fuzzy sphere \( \hat{Y}_{\ell m} \) and are \((2j + 1) \times (2j + 1)\) matrices. Therefore the algebra of the spherical harmonics on the fuzzy sphere is noncommutative. Furthermore in representation \( j \) the product of \( \hat{Y}_{\ell m} \) and \( \hat{Y}_{\ell' m'} \) \((\ell, \ell' = 0, 1, \ldots, 2j)\) are expanded in a linear combination of \( \hat{Y}_{\ell'' m''} \) with \( \ell'' = 0, 1, \ldots, \min(\ell + \ell', 2j) \).

\[
\hat{Y}_{\ell m} \hat{Y}_{\ell' m'} = \sum_{\ell'' m'' \leq \min(\ell + \ell', 2j)} C_{\ell_m, \ell_m'}^{\ell'' m''} \hat{Y}_{\ell'' m''}
\]

Therefore in contrast to the ordinary spherical harmonics \( Y_{\ell m} \) there is a truncation of the angular momentum.

Field theories on a fuzzy sphere have been studied by many people. \[2\] In a gauge theory was constructed in the context of a spherical D2-brane in \( SU(2) \) WZW model. The dynamical degrees of freedom are finite-size matrices. In the case of a flat space and infinite matrices the algebra of matrices can be realized by that of functions. The multiplication rule of the matrices are realized in the algebra of functions by the Moyal product.\[7\] For finite matrices on the fuzzy sphere the corresponding noncommutative (star) product was considered in \[2\] \[9\] \[10\] \[12\] \[14\] \[15\] \[16\]. In \[10\] an explicit formula for the star product on fuzzy sphere was constructed using the coherent state method.\[11\]

\[
f \ast g = f \cdot g + \sum_{m=1}^{2j} \frac{(2j - m)!}{m!(2j)!} J^{ab_1} J^{a_2 b_2} \ldots J^{a_m b_m} \partial_{a_1} \ldots \partial_{b_m} f \partial_{b_1} \ldots \partial_{b_m} g.
\]

Here \( J^{ab} = x_a x_b^2 \delta^{ab} - x_a x_b + i x_\epsilon \epsilon_{abc} x^c \) and \( x = \sqrt{(x^a)^2} \). The summation stops at \( m = 2j \) and \( f, g \) must be polynomials, i.e. spherical harmonics \( Y_{\ell m} \) of order \( \ell \leq 2j \). Extension to the fuzzy complex projective space \( CP^{N-1} \) was performed in \[17\]. There is also a star product...
in the integral form. In [12] by performing the stereographic projection of the sphere on
to the plane and using generalized coherent states on the complex plane [13] another star
product in the integral form was constructed. In [18] this product was also derived from the
matrix model of [5].

In this paper we will derive the following expression for the star produc
t on the fuzzy
sphere. (sec.2)

\[ f \star \lambda g = f g + \sum_{m=1}^{\infty} C_m(\lambda) J^{a_1 b_1} J^{a_2 b_2} \cdots J^{a_m b_m} \partial_{a_1} \cdots \partial_{a_m} f \partial_{b_1} \cdots \partial_{b_m} g. \] (5)

Here \( C_m(\lambda) = \frac{\lambda^m}{m!(1-\lambda)(1-2\lambda)\cdots(1-(m-1)\lambda)}. \) Note that the summation extends to \( m = \infty. \)
This product corresponds to an arbitrary representation of \( SU(2) \) including non-unitary
ones. \( \lambda \) is a parameter introduced in (2) and this product gives the realizatio
n of (2). We
will show the associativity of this product (5). For application to the field theories on the
fuzzy two-sphere the values of \( \lambda \) must be selected by the condition of unitary representation.
We will show that there exist two values of \( \lambda \) for a single unitary representation \( j \) of \( SU(2), \)
\( \lambda_j^A = 1/(2j) \) and \( \lambda_j^B = -1/(2j + 2). \) (sec.3) For \( \lambda = \lambda_j^A \) the product (2) reduces to
(4). The coefficient \( C_m(\lambda_j^A) = (2j - m)!/m!(2j)! \) is singular for \( m \geq 2j + 1. \) To truncate
the summation the functions on the fuzzy sphere must be polynomials of order \( 2j, \) i.e.
\[ f(x) = \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} a_{\ell m} x^\ell Y_{\ell m}. \] The algebra of the spherical harmonics \( Y_{\ell m} (\ell \leq 2j) \) with
the noncommutative product (3) coincides with that of the spherical harmonics on the fuzzy
sphere \( \hat{Y}_{\ell m}. \)

The product (3) for \( \lambda = \lambda_j^B, \) which we will denote by \( \bullet, \) is new and has interesting
properties. The coefficient \( C_m(\lambda_j^B) = (-1)^m(2j + 1)!/m!(2j + 1 + m)! \) is not singula
and there is no restriction on the angular momentum of the functions, i.e. \( f(x) = \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} a_{\ell m} x^\ell Y_{\ell m}. \) Furthermore, contrar
y to the case of the product (4) the spherical harmonics \( Y_{\ell m} \) with the product \( \bullet \) do not realize the algebra of \( \hat{Y}_{\ell m} \) explicitly! Especially, a
product of polynomials of orders \( \ell \) and \( \ell' \) yields a polynomial of order \( \ell + \ell'. \) It turns out,
however, that the integral of the star product of \( Y_{\ell m}, Y_{\ell' m'}^* \) corresponds to the trace of the
product of \( \hat{Y}_{\ell m}, \hat{Y}_{\ell' m'}^* \) and the integral vanishes for \( \ell \neq \ell' \) or \( \ell = \ell' > 2j. \) Therefore just the
combination of the star product and the integration gives a realization of the fuzzy sphere algebra.

For noncommutative theories in the flat space, the Seiberg-Witten map [19] between
noncommutative and commutative descriptions has been investigated [21] [22]. The Seiberg-
Witten map is a transformation from the gauge fields in the commutative description to those
in the noncommutative description in such a way that two field configurations in the gauge
equivalence class in one description are mapped on to the two field configurations in the gauge
equivalence class in the other description. A crucial problem in the study of the Seiberg-Witten map for theories on the fuzzy sphere is that while the angular momentum of the gauge fields in the noncommutative description is at most \( \ell = 2j \), that in the commutative description is arbitrary. Therefore it is difficult to establish a map from the commutative gauge fields to the noncommutative gauge fields. Furthermore the order of the polynomials will change when we move from \( \lambda = \lambda_j^{(A)} \) to \( \lambda = 0 \) along the positive \( \lambda \) line. However, if we use the star product with \( \lambda = \lambda_j^{(B)} \), then the order of spherical harmonics on the fuzzy sphere are not restricted and we can consider a map to commutative gauge fields, although the theory in the noncommutative description has only finite degrees of freedom. The star product (5) has poles on the positive \( \lambda \) line and interpolation between \( \lambda = \lambda_j^{(A)} \) and \( \lambda = 0 \) along the positive \( \lambda \) line is not possible. On the contrary there is no singularity on the negative line and the interpolation from \( \lambda = 0 \) to \( \lambda = \lambda_j^{(B)} \) will make sense. In this sense the second series of unitary representations \( \lambda_j^{(B)} \) seems preferable for the study of the Seiberg-Witten map. In sec.4 we will derive the first two nontrivial orders \( O(\lambda^n) \) \( (n = 1, 2) \) of the Seiberg-Witten map by using power expansion around \( \lambda = 0 \). We will give a brief summary in sec.5.

2 Star Product for Arbitrary Representations

In [10] the star product (4) on the fuzzy two-sphere was obtained. This product corresponds to the spin \( j \) representation of \( SU(2) \). The star product (4) can be formally derived by rewriting the factorials in (4) by gamma functions and replacing \( 2j \) inside the gamma functions by \( 1/\lambda \). However, additionally, the upper limit of summation, \( 2j \), must be replaced by \( \infty \). Therefore the associativity of (4) is not a priori guaranteed. We will examine this problem in this section. This product corresponds to arbitrary representations of \( SU(2) \), including non-unitary ones. In the next section we will select the values of \( \lambda \) for unitary representations and find a new star product \( \star \).

Instead of beginning with (4) we will assume the following form for the star product \( \star_\lambda \).

\[
\begin{align*}
f(x) \star_\lambda g(x) &= f(x)g(x) + \lambda J^{ab} \partial_a f(x) \partial_b g(x) \\
&\quad + \sum_{n=2}^{\infty} \lambda^n \sum_{m=2}^{n} \chi_{m,m}^{(n)} J^{a_1 b_1} J^{a_2 b_2} \cdots J^{a_n b_n} \partial_{b_1} \cdots \partial_{a_n} f \partial_{b_1} \cdots \partial_{b_m} g,
\end{align*}
\]

where \( \chi_{m,m}^{(n)} \) \( (2 \leq m \leq n) \) is a constant and will be determined by the requirement of the associativity. Here \( \lambda \) was introduced in (2) and works as an expansion parameter. We can

\[\text{Replacement of } N = 2j \text{ by } 1/\epsilon = 1/\lambda \text{ and power expansion of (4) around } \epsilon = 0 \text{ was suggested in (5). It was, however, concluded that only the product (4) gives a unitary representation. The upper limit of summation was not specified explicitly and there was no proof of the associativity for an arbitrary value of } \lambda.\]
easily show that this product realizes the algebra (3). $J^{ab}$ is a function defined by

$$J^{ab}(x) = x^2\delta_{ab} - x^ax^b + ix\epsilon_{abc}x^c,$$

where $\epsilon_{abc}$ is a completely anti-symmetric unit tensor ($\epsilon_{123} = +1$), and is a projector ($J^{ab}J^{bc} = 2x^2J^{ac}$).

First of all we note the following identities of $J^{ab}$.

1. $x^aJ^{ab} = x^bJ^{ab} = 0$  (8)

2. $J^{ab}J^{ac} = J^{ba}J^{ca} = 0$  (9)

3. $J^{ab}\partial_aJ^{cd} = J^{ca}\partial_aJ^{db}$  (10)

4. $J^{ba}J^{dc}\partial_a\partial_cJ^{ef} = -J^{bf}J^{de} - J^{be}J^{df}$  (11)

5. $J^{a_1b_1}J^{a_2b_2}\ldots J^{a_nb_n}\partial_{b_1}\partial_{b_2}\ldots\partial_{b_n}J^{cd} = 0$  ($n \geq 3$)  (12)

Identities 1..5 follow directly from the definition (7).

Identities 1,2 guarantee that $x$ is a constant with respect to the $\star_\lambda$ product: *i.e.* for any functions $f(x)$ of $x$ and $g(x^a)$ we obtain

$$f(x) \star_\lambda g(x^a) = g(x^a) \star_\lambda f(x) = f(x) g(x^a).$$

For example

$$J^{ab}\partial_a f(x)\partial_b g = J^{ab}\frac{df(x)}{dx}\partial_b g = 0,$$

$$J^{ab}\partial_a f(x)\partial_b g = J^{ab} \{\delta_{ac}f'(x)/x + (x^a x^c/x) (f'(x)/x)\} \partial_b g = 0,$$

where $f'(x) \equiv df(x)/dx$. This is a necessary condition because $\star_\lambda$ is the product on the sphere.

Let us check the associativity of (3). To the 1st order both $(f \star_\lambda g) \star_\lambda h$ and $f \star_\lambda (g \star_\lambda h)$ are given by

$$f g h + \lambda J^{ab}(\partial_a f \partial_b g h + \partial_a f g \partial_b h + f \partial_a g \partial_b h) + O(\lambda^2)$$

(16)
and $\star_\lambda$ is associative to this order. To the next order one can show that the difference
$(f \star_\lambda g) \star_\lambda h - f \star_\lambda (g \star_\lambda h)$ is given by
\[
\lambda^2 (J^{ab} \partial_a J^{cd} - J^{ca} \partial_a J^{db}) \partial_c f \partial_d g \partial_b h
+ \lambda^2 (1 - 2\chi_{2,2}^{(2)}) J^{ac} J^{bd} (\partial_a \partial_b \partial_c g \partial_d h - \partial_a f \partial_b g \partial_c \partial_d h).
\] (17)
The 1st term vanishes due to identity 3 and the other term vanishes if the condition
\[
2\chi_{2,2}^{(2)} - 1 = 0
\] is satisfied.

To the next order we can show by using identity 4 that the associativity imposes the conditions
\[
3\chi_{3,3}^{(3)} - \chi_{2,2}^{(2)} = 0, \quad 2\chi_{2,2}^{(2)} - 2\chi_{2,2}^{(2)} = 0.
\] (19)
Therefore we have
\[
\chi_{2,2}^{(2)} = \frac{1}{2}, \quad \chi_{3,3}^{(3)} = \frac{1}{6}, \quad \chi_{2,2}^{(3)} = \frac{1}{2}.
\] (20)

We can work out a similar analysis to higher orders. Study of a few more higher orders shows that the condition of the associativity leads to the recursion relation
\[
m \chi_{m,m}^{(n)} - m(m-1) \chi_{m,m}^{(n-1)} - \chi_{m-1,m-1}^{(n-1)} = 0, \quad \chi_{n,n}^{(n)} = \frac{1}{n!}.
\] (21)
Here identities 1-5 must be used. The solution to (21) is not difficult to obtain.
\[
\chi_{m,m}^{(n)} = \frac{1}{m!} \sum_{\{P\}} (m-1)^{P_1} (m-2)^{P_2} \ldots 2^{P_{n-2}} 1^{P_{n-1}},
\] (22)
where $P_i$ is the partition of $n - m$ into $m - 1$ nonnegative integers. First few examples of $\chi$ can be explicitly written as
\[
\chi_{2,2}^{(n)} = \frac{1}{2}, \quad \chi_{3,3}^{(n)} = \frac{1}{6}(2^{n-2} - 1), \quad \chi_{4,4}^{(n)} = \frac{1}{48}(3^{n-2} - 2^{n-1} + 1).
\] (23)
Now the summation over $n$ in (21) can be performed. For
\[
C_m(\lambda) = \sum_{n=m}^{\infty} \lambda^n \chi_{m,m}^{(n)}
\] (24)
we immediately obtain
\[
C_2(\lambda) = \frac{\lambda^2}{2!(1 - \lambda)}, \quad C_3(\lambda) = \frac{\lambda^3}{3!(1 - \lambda)(1 - 2\lambda)}, \quad C_4(\lambda) = \frac{\lambda^4}{4!(1 - \lambda)(1 - 2\lambda)(1 - 3\lambda)}.
\] (25)
and generally, we get a formula
\[
C_m(\lambda) = \frac{\lambda^m}{m!(1 - \lambda)(1 - 2\lambda) \ldots (1 - (m - 1)\lambda)}.
\] (26)
We also define $C_1(\lambda) \equiv \lambda$.

In summary we get the associative product $\star_\lambda$ on the fuzzy two-sphere for arbitrary representation including non-unitary ones.

$$f \star_\lambda g = f g + \sum_{m=1}^{\infty} C_m(\lambda) \partial_{a_1} \cdots \partial_{a_m} f \partial_{b_1} \cdots \partial_{b_m} g. \quad (27)$$

The coefficient $C_m(\lambda)$ has poles at $\lambda = 1, 1/2, \ldots, 1/(m-1)$ and we must restrict the functional space when we set $\lambda$ to these values.

It is possible to show that with the integration on the sphere this star product satisfies the typical property of the trace of matrices.

$$\int d\Omega \; f \star_\lambda g = \int d\Omega \; g \star_\lambda f \quad (28)$$

where $d\Omega$ is the standard measure on the unit sphere. In the polar coordinate system it is given by $d\Omega = \sin \theta d\theta d\phi$. Therefore $\int d\Omega$ plays the role of the trace.

3 Unitary representation of the Fuzzy Sphere Algebra and New Product

For application to field theory on the fuzzy two-sphere we must pick up unitary representations. We will determine the allowed values of $\lambda$. From the product (27) we can derive the fuzzy sphere algebra.

$$[x^a, x^b]_{\star_\lambda} \equiv x^a \star_\lambda x^b - x^b \star_\lambda x^a = 2i\lambda \epsilon_{abc} x^c \quad (29)$$

The normalized variable $y^a \equiv x^a/(2\lambda x)$ satisfies the standard $su(2)$ algebra

$$[y^a, y^b]_{\star_\lambda} = i\epsilon_{abc} y^c \quad (30)$$

and for unitary representations Casimir operator $y^a \star y^a$ must take discrete values $j(j + 1)$, $(2j = 0, 1, 2, 3, \ldots)$. By using (27) we get

$$R^2 = x^a \star_\lambda x^a = (x^a)^2 + \lambda(x^2 \delta_{aa} - (x^a)^2) = (1 + 2\lambda)x^2 \quad (31)$$

and then we obtain

$$y^a \star_\lambda y^a = (1 + 2\lambda)/(2\lambda)^2 = j(j + 1). \quad (32)$$

This equation has two series of solutions.

(A) $\lambda = \lambda_j^{(A)} = \frac{1}{2j} \quad (2j = 1, 2, \ldots), \quad (33)$
\[ \lambda = \lambda_j^{(B)} = -\frac{1}{2j + 2} \quad (2j = 1, 2, \ldots) \quad (34) \]

Let us first consider the case (A). The coefficient is given by

\[ C_m \left( \frac{1}{2j} \right) = \frac{(2j - m)!}{m!(2j)!} \quad (35) \]

When \( m \geq 2j \), this coefficient is infinite. Therefore the functions \( f(x) \), \( g(x) \) must be polynomials of order at most \( 2j \). This restriction corresponds to the size \((2j + 1) \times (2j + 1)\) of the matrices in the spin \( j \) representation. The summation must now be restricted to \( 0 \leq m \leq 2j \). The product (27) with this coefficient agrees with the product (4) obtained in [10]. If we do not restrict the order of \( f(x) \), \( g(x) \), the product (4) does not satisfy the associativity. The product reduces to the commutative one in the \( j \to \infty \) limit, where the matrix size becomes infinite.

Let us turn to the case (B). The coefficient

\[ C_m \left( -\frac{1}{2j + 2} \right) = (-1)^m \frac{1}{m!} \frac{(2j + 1)!}{(2j + 1 + m)!} \quad (36) \]

does not diverge for any \( m \) and there is no restriction on the functions \( f \), \( g \) on the fuzzy sphere. We will denote the corresponding product by \( \bullet \).

\[ f \bullet g = f + \sum_{m=1}^{\infty} (-1)^m \frac{1}{m!} \frac{(2j + 1)!}{(2j + 1 + m)!} \frac{1}{m} J^{a_{1}b_{1}} J^{a_{2}b_{2}} \ldots J^{a_{m}b_{m}} \partial_{a_{1}} \partial_{a_{2}} \ldots \partial_{a_{m}} f \partial_{b_{1}} \partial_{b_{2}} \ldots \partial_{b_{m}} g. \quad (37) \]

This also reduces to the ordinary commutative product for \( j \to \infty \).

Let us investigate the case \( j = 1/2 \) in detail. For the product (4) with \( \lambda = \lambda_{1/2}^{(A)} \), functions must be a linear combination of 1 and \( x^a \) and we get the multiplication rule

\[ x^a \star x^b = x^2 \delta_{ab} + i x \epsilon_{abc} x^c, \]

\[ 1 \star x^a = x^a \star 1 = x^a. \quad (38) \]

This is the algebra of the Pauli matrices \( \sigma_a \) and the 2 by 2 unit matrix \( I_2 \). \( x^a x^b \) does not appear on the RHS of \( x^a \star x^b \) and the angular momentum is truncated. For the product (37) we get

\[ x^a \bullet x^b = \frac{4}{3} (x^a x^b - \frac{1}{3} x^2 \delta_{ab}) + \frac{1}{9} x^2 \delta_{ab} - \frac{i}{3} x \epsilon_{abc} x^c, \]

\[ \frac{20}{9} x^a x^b x^c + \ldots - \frac{i}{27} x^3 \epsilon_{abc}. \quad (39) \]

Therefore the algebra of (Pauli) matrices is not realized. When we integrate these products over the sphere, however, we obtain

\[ \int d\Omega \ x^a \bullet x^b = \frac{4\pi}{9} x^2 \delta_{ab}, \quad \int d\Omega \ x^a \bullet x^b = -\frac{4\pi i}{27} x^3 \epsilon_{abc}. \quad (40) \]
These correspond to $\text{Tr} \sigma_a \sigma_b, \text{Tr} \sigma_a \sigma_b \sigma_c$. Therefore the combination of the star product and the integration realizes the matrix multiplication rule.

We also note that for the $\ell = 2$ spherical harmonics, $f(x) = \alpha_{ab} x^a x^b$, $g(x) = \beta_{ab} x^a x^b$ ($\alpha_{ab}$ and $\beta_{ab}$ are symmetric, traceless constant tensors), we obtain

$$
\int d\Omega (\alpha_{ab} x^a x^b) \star (\beta_{cd} x^c x^d) = \frac{8\pi}{15} \frac{(1 + 2\lambda)(1 + 3\lambda)}{1 - \lambda} x^4 \alpha_{ab} \beta_{ab}.
$$

(41)

For $j \geq 1$ and $-1/4 \leq \lambda < 0$ this does not vanish. However, when $j = 1/2$ and $\lambda = -1/3$, this integral vanishes. As this example shows the space of functions is actually finite dimensional.

In the case (B) the angular momentum is not cut off at some finite value and the relation of the algebra of functions to that of $\hat{Y}_{\ell m}$ is not clear. Only after integration the relation of the algebra of functions to that of matrices is manifest. However, this product does realize the fuzzy sphere algebra. It is known that a star product is not unique. Given a star product $\star$ and a differential operator $\hat{D}$, a new product defined by

$$
f \hat{\star} g = \hat{D}^{-1} \left( (\hat{D} f) \star (\hat{D} g) \right)
$$

(42)
gives another star product. Clearly, the relation between $\star$ and $\bullet$ is not of this type. We need more understanding of the relation between $\star$ and $\bullet$.

## 4 Seiberg-Witten Map on the Fuzzy Sphere

In this section we will construct the Seiberg-Witten map[13] for the gauge theory on the two-sphere by using perturbation theory in $\lambda$ up to order $O(\lambda^2)$. After expansion we will set $\lambda = \lambda_j^{(B)}$. So we will use $\bullet$ for the star product in what follows. We will consider only the $U(1)$ gauge theory for simplicity.

On the fuzzy sphere the gauge transformation is defined[13] by

$$
\delta \hat{A}_a = -i L_a \hat{\Lambda} - i [\hat{A}_a, \hat{\Lambda}] \bullet.
$$

(43)

Here $L_a$ is the angular momentum operator $L_a = -i \epsilon_{abc} x^b \partial_c$ and satisfies the relation $[L_a, L_b] = i \epsilon_{abc} L_c$. The strength of the gauge field is defined[13] by

$$
\hat{F}_{ab} = -i L_a \hat{A}_b + i L_b \hat{A}_a - i [\hat{A}_a, \hat{A}_b] \bullet - \epsilon_{abc} \hat{A}_c
$$

(44)

and transforms under[13] by the formula $\delta \hat{F}_{ab} = i [\hat{F}_{ab}, \hat{\Lambda}] \bullet$.

In commutative gauge theory the gauge field $A_a$ transforms as $\delta A_a = -i L_a \Lambda$ and a field strength

$$
F_{ab} = -i L_a A_b + i L_b A_a - \epsilon_{abc} A_c
$$

(45)
is invariant.

We will use the cohomological approach to the Seiberg-Witten map. We introduce the ghost field \( \hat{C} \) and \( C \). In the commutative description the BRST transformation is defined by

\[
sC = iC^2 = 0, \\
sA_a = -iL_a C,
\]

where \( s \) is the BRST operator. Analogously, in the noncommutative description we have

\[
s\hat{C} = i\hat{C} \cdot \hat{C}, \\
s\hat{A}_a = -iL_a \hat{C} + i[\hat{C}, \hat{A}_a].
\]

We will expand \( \hat{C} \) and \( \hat{A}_a \) in formal power series in \( \lambda \) whose coefficients are local polynomials in \( C \) and \( A_a \).

\[
\hat{C} = C + \lambda C^{(1)} + \lambda^2 C^{(2)} + \cdots, \\
\hat{A}_a = A_a + \lambda A_a^{(2)} + \lambda^2 A_a^{(3)} + \cdots
\]

The fields \( \hat{C}, \hat{A}_a \) reduce to \( C, A_a \) at \( \lambda = 0 \). We will substitute the expansion (48) into (47) and obtain eqs for \( C^{(i)}, A_a^{(i)} \). We get

\[
sC^{(1)} = -i\epsilon^a x^b x^c \partial_a C \partial_b C, \\
sA_a^{(2)} = -iL_a C^{(1)} - 2\epsilon_{bcd} x^d \partial_b C \partial_c A_a,
\]

\[
sC^{(2)} = -2\epsilon_{abc} x^c \partial_a C \partial_b C^{(1)} - \epsilon_{abc} x^c (x^2 \delta_{de} - x^d x^e) \partial_a \partial_b C \partial_c C, \\
sA_a^{(3)} = -iL_a C^{(2)} - 2\epsilon_{bcd} x^d (\partial_b C \partial_c A_a^{(2)} + \partial_b C^{(1)} \partial_c A_a) \\
+ 2i(x^2 \delta_{bc} - x^b x^c) \epsilon_{def} x^f \partial_a \partial_d A_e \partial_c C.
\]

The solution to the 1st eq of (49) is given by

\[
C^{(1)} = \frac{i}{x} \epsilon^{abc} x^a A_b L_a C.
\]

Similarly \( A_a^{(2)}, C^{(2)}, A_a^{(3)} \) can be obtained. Only the results are presented here.

\[
C^{(2)} = \frac{i}{x^2} \epsilon_{abc} x^b x^d A_c A_d L_a C - \frac{i}{x^2} \epsilon_{abc} \epsilon_{def} x^b x^d A_c F_{ae} L_f C \\
- \frac{1}{x^2} \epsilon_{abc} \epsilon_{def} x^b x^d A_c A_e L_f L_a C + \frac{i}{x} L_a A_d \epsilon_{abc} x^b L_c L_d C + \frac{i}{x} A_a \epsilon_{abc} x^b L_c C,
\]

\[
A_a^{(2)} = \frac{1}{x} \epsilon_{bcd} x^c A_d (iL_b A_a + F_{ab}),
\]

\[
A_a^{(3)} = L_a \left( \frac{1}{x} x^c A_c (x^d A_d L_b A_b + i A_b F_{bd} x^d) + \frac{1}{x} x^c A_c L_b A_b - \frac{1}{x} \epsilon_{bcd} F_{bc} x^e L_d A_e \right) \\
- 3F_{ab} F_{bc} A_e + A_e A_d L_c L_d A_a - A_c A_e (L_b L_b A_a - iL_b F_{ab}) \\
- 2i F_{ab} A_b L_c A_c + 2i F_{ab} A_c L_c A_b - i A_b L_b F_{ac} A_c - i L_b A_a F_{bc} A_c \\
+ \frac{i}{x} \epsilon_{bcd} x^b L_c L_c A_a L_c A_d - \frac{1}{x} L_a F_{ab} \epsilon_{bcd} x^e L_d A_e
\]

(51)
In this way one can compute the Seiberg-Witten map for the gauge theory on the two-sphere order by order in perturbation theory in powers of $\lambda\ (=\lambda_j^{(B)})$.

5 Summary

In this paper the star product $\star_\lambda$ for the field theories on the fuzzy sphere corresponding to an arbitrary representation of $SU(2)$ including non-unitary representation is constructed and the associativity of this product is proved. By imposing the condition of unitary representation we obtained a new noncommutative product $\bullet$ for $\lambda=\lambda_j^{(B)}$. This new product has a novel feature that the product of spherical harmonics with the product $\bullet$ does not realize the mulitplication rule of the matrices, i.e. spherical harmonics on the fuzzy sphere but after integration over the sphere the result agrees with the trace of the product of the corresponding matrices. The functions on the sphere are not restricted to be spherical harmonics of order $2j$ ($j$ is the spin of the representation) but the integration eliminates spherical harmonics of order larger than $2j$.

The product $\star_\lambda$ has singularities on the positive $\lambda$ line, while there is no singularity for negative $\lambda$. Therefore if we want to move from one unitary representation $j$ to another $j'$ passing through non-unitary representations, especially if we consider the Seiberg-Witten map, the negative value $\lambda=\lambda_j^{(B)}$ seems suitable. Along this move the gauge fields can be spherical harmonics of any order. We derived the first few terms of the Seiberg-Witten map by power expansions around $\lambda=0$. To derive an exact form of the map without the power expansions like that derived in the flat case a further investigation is necessary. An investigation such as that of the correspondence of the Dirac-Born-Infeld actions in the noncommutative and commutative descriptions will be reported elsewhere.

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Note Added

(1) We can prove the following formula for the inner product of ordinary spherical harmonics.

$$\int d\Omega \left(Y_{\ell m}\right)^* \star_\lambda Y_{\ell' m'} = \prod_{k=1}^\ell \left(\frac{1+(k+1)\lambda}{1-(k-1)\lambda}\right) \delta_{\ell\ell'} \delta_{mm'}$$

(52)
Especially, for $\lambda = \lambda_j^{(B)}$ we obtain

$$\int d\Omega \ (Y_{\ell m})^* \cdot Y_{\ell' m'} = \prod_{k=1}^{\ell} \left( \frac{2j-k+1}{2j+k+1} \right) \delta_{\ell \ell'} \delta_{mm'}.$$  \hspace{1cm} (53)

The inner product is positive definite for $\ell = 0, 1, \ldots, 2j$ and the norm vanishes for $\ell > 2j$. This shows that our new star product $\cdot$ with $\lambda = \lambda_j^{(B)}$ defines a unitary representation. This also completes the observation made around eq (41) concerning the finite dimensionality of the functional space.

The proof of eq (52) will be presented elsewhere.

\(\text{(2)}\) Generally, the solution to (49) is not unique and has ambiguity of $s$-exact terms. We also found a simpler solution.

$$C^{(2)} = -2 \epsilon_{abc} \epsilon_{def} \frac{x^c x^f}{x^2} A_a L_b A_d L_e C$$

$$-2i \epsilon_{abc} \frac{x^c x^d}{x^2} A_a A_d L_b C - i \epsilon_{abc} \frac{x^c}{x} L_d A_a L_d L_b C,$$

$$A_a^{(3)} = 2 \epsilon_{bcd} \epsilon_{efg} \frac{x^d x^g}{x^2} A_b (-L_c A_e L_f A_a + F_{ce} F_{fa} - i A_c L_c F_{fa})$$

$$+2 \epsilon_{bcd} \frac{x^d x^e}{x^2} A_b A_c (F_{ca} - i L_c A_a) + \epsilon_{bcd} \frac{x^d}{x} L_e A_b L_e (F_{ca} - i L_c A_a) \hspace{1cm} (54)$$

$C^{(1)}$ and $A_a^{(2)}$ are the same as in (50), (51).
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