Tiling Iterated Function Systems

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ABSTRACT. This paper provides an approach to the study of self-similar tilings and substitution tilings, in the setting of graph-directed iterated function systems, where the tiles may be fractals and the tiled set maybe a complicated unbounded subset of $\mathbb{R}^M$.

1. Introduction

This paper describes an approach to the study of self-similar tilings and substitution tilings, in the setting of graph-directed iterated function systems, where the tiles may be fractals and the tiled set maybe a complicated unbounded subset of $\mathbb{R}^M$. See [27] for formal background on iterated function systems (IFS). We are concerned with graph directed IFS as defined here, but see also [5, 9, 20, 23, 24, 29, 41].

There are relationships between this work and Solomyak [37, 38], Anderson and Putnam [1], and many other works on tiling theory, including those in the References.

In this introduction we use plenty of inverted commas. Corresponding precise concepts are clarified later.

This paper is a development of [17]. It is a companion to [18], but the approach is very different. The paper [18] uses graph-theoretic language and “tiling hierarchies”, whereas here we think more in terms of “code space” and “canonical tilings”. This paper is also distinct from [18] because (i) it considers the case of purely fractal tilings, where tiles may have empty interiors, (ii) it considers continuity properties of the map from “code space” to “tiling space”, and (iii) it is concerned with the formal description of some patches of tiles in terms of their addresses.

This paper is a sequel to [16], which concerns generalized tilings associated with IFS attractors. The main results are Theorem 8 and Theorem 17.

A key notion in [16], [17], and [18], is that of “rigid tilings” and the associated tiling spaces. We show that “rigidity” always implies “aperiodicity”.

2. Foundations

2.1. Graph iterated function systems. Let $\mathcal{F}$ be a finite set of invertible contraction mappings $f : \mathbb{R}^M \rightarrow \mathbb{R}^M$ each with contraction factor $0 < \lambda < 1$, that is $\|f(x) - f(y)\| \leq \lambda \|x - y\|$ for all $x, y \in \mathbb{R}^M$. We suppose

$$\mathcal{F} = \{f_1, f_2, ..., f_N\}, \ N > 1$$
Let $G = (\mathcal{E}, \mathcal{V})$ be a strongly connected aperiodic directed graph with edges $\mathcal{E}$ and vertices $\mathcal{V}$ with
\[ \mathcal{E} = \{e_1, e_2, ..., e_N\}, \quad \mathcal{V} = \{v_1, v_2, ..., v_N\}, \quad 1 \leq V < N \]
$G$ is strongly connected means there is a path, a sequence of consecutive directed edges, from any vertex to any vertex. $G$ is aperiodic means that if $W$ is the $V \times V$ matrix whose $ij$th entry is the number of edges directed from vertex $j$ to vertex $i$, then there is some power of $W$ whose entries are all strictly positive.

We call $(\mathcal{F}, G)$ a graph IFS. The directed graph $G$ provides the orders in which functions of $\mathcal{F}$ may be composed. The sequence of successive directed edges $e_{\sigma_1}e_{\sigma_2} \cdots e_{\sigma_k}$ is associated with the composite function
\[ f_{\sigma_1}f_{\sigma_2} \cdots f_{\sigma_k} := f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k} \]
The edges may be referred to by their indices $\{1, 2, ..., N\}$ and the vertices by $\{1, 2, ..., V\}$.

2.2. Paths in $G$. Let $N$ be the strictly positive integers and $N_0 = N \cup \{0\}$. For $N \in N$, $[N] := \{1, 2, \ldots, N\}$.

$\Sigma$ is the set of directed paths in $G$, each with an initial vertex. A path $\sigma \in \Sigma$ is written $\sigma = \sigma_1\sigma_2 \cdots$ corresponding to the sequence of successive directed edges $e_{\sigma_1}e_{\sigma_2} \cdots$ in $G$. The length of $\sigma$ is $|\sigma| \in N_0 \cup \{\infty\}$. A metric $d_\Sigma$ on $\Sigma$ is
\[ d_\Sigma(\sigma, \omega) := 2^{-\min\{k \in N : \sigma_k \neq \omega_k\}} \quad \text{for} \quad \sigma \neq \omega, \]
where $\sigma_k = \sigma_k$ for all $k \leq |\sigma|$, $\sigma_k = 0$ for all $k > |\sigma|$. Then $(\Sigma, d_\Sigma)$ is a compact metric space.

The set $\Sigma_\ast \subset \Sigma$ is the directed paths of finite lengths, and $\Sigma_\infty \subset \Sigma$ is the directed paths of infinite length. For $\sigma \in \Sigma$, let $\sigma^- \in \mathcal{V}$ be the initial vertex and, if $\sigma \in \Sigma_\ast$, let $\sigma^+ \in \mathcal{V}$ be the terminal vertex; and for $v \in \mathcal{V}$ let
\[ \Sigma_v := \{\sigma \in \Sigma_\infty : \sigma^- = v\} \]
For $\sigma \in \Sigma$, $k \in N$,
\[ \sigma|k := \begin{cases} \sigma_1\sigma_2...\sigma_k & \text{if} \quad |\sigma| > k \\ \sigma^+_1 & \text{if} \quad |\sigma| \leq k \end{cases} \]
\[ f_{\sigma|k} := \begin{cases} f_{\sigma_1}f_{\sigma_2} \cdots f_{\sigma_k} & \text{if} \quad |\sigma| > k \\ f_{\sigma^+_1} & \text{if} \quad |\sigma| \leq k \end{cases} \quad f_v := \chi_{A^v} \]
where $\chi_{A^v}$ is the characteristic function of $A^v \subset \mathbb{R}^M$, see Definition $\mathbb{H}^{iii})$.

$G^\dagger = (\mathcal{E}^\dagger, \mathcal{V})$ is the graph $G$ modified so that the directions of all edges are reversed. The superscript $\dagger$ means that the superscripted object relates to $G^\dagger$. For example, $\Sigma_\ast^\dagger$ is the set of directed paths in $G^\dagger$ of finite length, $\Sigma_\infty^\dagger$ is the set of directed paths in $G^\dagger$, each of which starts at a vertex and is of infinite length, and $\Sigma^\dagger = \Sigma_\ast^\dagger \cup \Sigma_\infty^\dagger$. While $G$ is associated with compositions of functions in $\mathcal{F}$, in this paper $G^\dagger$ is associated with compositions of their inverses.

2.3. Addresses and Attractors. Let $H$ be the nonempty compact subsets of $\mathbb{R}^M$ and let $d_H$ be the Hausdorff metric.Singletons in $H$ are identified with points in $\mathbb{R}^M$. 
Definition 1. The attractor $A$ of the graph IFS $(\mathcal{F}, \mathcal{G})$, its components $A_{\sigma}$, and the address map $\pi : \Sigma \cup V \to \mathbb{H}$, are defined as follows.

(i) $\pi(\sigma) := \lim_{k \to \infty} f_{\sigma|k}(x)$ for $\sigma \in \Sigma_{\infty}$, fixed $x \in \mathbb{R}^M$

(ii) $A := \pi(\Sigma_{\infty})$

(iii) $\pi(\nu) := A_{\nu} := \pi(\Sigma_{\nu})$ for all $\nu \in \mathcal{V}$

(iv) $\pi(\sigma) := f_{\sigma}(A_{\sigma^+})$ for all $\sigma \in \Sigma_{\ast}$

Theorem 1. Let $(\mathcal{F}, \mathcal{G})$ be a graph IFS.

1. $\pi : \Sigma \cup \mathcal{V} \to \mathbb{H}$ is well-defined.
2. $\pi : \Sigma \cup \mathcal{V} \to \mathbb{H}$ is continuous.

3. $\pi(\sigma) = \bigcap_{k=1}^{\infty} \pi(\sigma|k)$ for all $\sigma \in \Sigma$

4. $f_{\sigma}(A_{\sigma^+}) \subseteq A_{\sigma^-}$ for all $\sigma \in \Sigma_{\ast}$.

Proof. (1) For all $\sigma \in \Sigma_{\infty}$, $\pi(\sigma)$ is well-defined by (i), independently of $x$, because $\mathcal{F}$ is strictly contractive [27]. It follows that $A$ is well-defined by (ii). Also it follows that $A_{\nu}$ and $\pi(\nu)$ are well-defined by (iii), for all $\nu \in \mathcal{V}$, in turn, $\pi(\sigma)$ is well-defined for all $\sigma \in \Sigma_{\ast}$ by Definition (iv).

(2) $\pi$ is continuous because for all $\sigma \in \Sigma_{\infty}$

$$d_{\mathbb{H}}(\pi(\sigma|k), \pi(\sigma|l)) \leq \lambda^{\min(k,l)} \max_{v,w} d_{\mathbb{H}}(A_{\nu}, A_{\nu'})$$

(3) and (4) follow from Definition (iv). □

Definition 2. Define $\sigma \in \Sigma_{\infty}$ to be disjunctive if, given any $k \in \mathbb{N}$ and $\theta \in \Sigma_{k}^1$, there is $p \in \mathbb{N}$ so that $\theta = \sigma_p \sigma_{p+1} \cdots \sigma_{p+k}$.

Likewise, $\theta \in \Sigma_{\infty}^1$ is disjunctive if, given any $k \in \mathbb{N}$ and $\sigma \in \Sigma_{k}$, there is $p \in \mathbb{N}$ so that $\sigma = \theta_p \theta_{p+1} \cdots \theta_{p+k}$.

Theorem 2. Let $(\mathcal{F}, \mathcal{G})$ be a graph IFS. Let $\theta \in \Sigma_{\infty}^1$, $x_0 \in \mathbb{R}^M$, and $x_n = f_{\theta_n}(x_{n-1})$ for all $n \in \mathbb{N}$. Then

$$\bigcap_{k \in \mathbb{N}} \left( \bigcup_{n=k}^{\infty} x_n \right) \subseteq A$$

with equality when $\theta \in \Sigma_{\infty}^1$ is disjunctive.

Proof. $\Omega(\{x_n : n \in \mathbb{N}\}) := \bigcap_{k \in \mathbb{N}} \left( \bigcup_{n=k}^{\infty} x_n \right)$ is an $\Omega$–limit set. Specifically it is the set of accumulation points of $\{x_n : n \in \mathbb{N}\}$ in $\mathbb{R}^M$. Since $\pi$ is continuous

$$\Omega(\{x_n : n \in \mathbb{N}\}) = \Omega \left( \{f_{\theta_n \theta_{n-1} \cdots \theta_1}(x_0) : n \in \mathbb{N}\} \right)$$

$$= \pi(\Omega(\{\theta_n \theta_{n-1} \cdots \theta_1 : n \in \mathbb{N}\}))$$

The $\Omega$–limit set of $\{\theta_n \theta_{n-1} \cdots \theta_1 : n \in \mathbb{N}\}$ is contained in or equal to $\Sigma_{\infty}$, with equality when $\theta \in \Sigma_{\infty}^1$ is disjunctive. □

The identity in Theorem 2 underlies the Chaos Game algorithm for calculating pictures of attractors, see [12].
2.4. Shift maps.

Definition 3. The shift map \( S : \Sigma \cup \mathcal{V} \rightarrow \Sigma \cup \mathcal{V} \) is defined by \( S(\sigma_1 \sigma_2 \cdots) = \sigma_2 \sigma_3 \cdots \) for all \( \sigma \in \Sigma \), \( Sv = v \) for all \( v \in \mathcal{V} \), with the conventions \( S^k \sigma = \sigma | k = \sigma_i^+ \) when \( k \geq |\sigma| \).

Theorem 3. Let \((\mathcal{F}, \mathcal{G})\) be a graph IFS.

1. \( S : \Sigma \cup \mathcal{V} \rightarrow \Sigma \cup \mathcal{V} \) is well-defined.
2. \( S(\Sigma \cup \mathcal{V}) = \Sigma \cup \mathcal{V} \).
3. \( S : \Sigma \cup \mathcal{V} \rightarrow \Sigma \cup \mathcal{V} \) continuous.
4. \( f_{\sigma|k} \circ \pi \circ S^k(\sigma) = \pi(\sigma) \) for all \( \sigma \in \Sigma \), for all \( k \in \mathbb{N}_0 \).

Proof. (1) and (2) can be checked.

3. \( S \) is continuous at every point in \( \Sigma_* \cup \mathcal{V} \) because this subset of \( \Sigma \cup \mathcal{V} \) is discrete and it is mapped onto itself by \( S \). A calculation using the metric \( d_{\Sigma} \) proves that \( S \) is continuous at every point in \( \Sigma_{\infty} \).

4. If \( \sigma = \sigma_1 \) and \( k = 0 \) then
\[
f_{\sigma|k} \circ \pi \circ S^k(\sigma) = \chi_{A_{\sigma_1^+}} \circ \pi(\sigma_i^+) = \chi_{A_{\sigma_1^+}}(A_{\sigma_1^+}) = \pi(\sigma_i^+)
\]
If \( \sigma = \sigma_1 \) and \( k = 1 \), then
\[
f_{\sigma|k} \circ \pi \circ S^k(\sigma) = f_{\sigma_1} \circ \pi(\sigma_i^+) = f_{\sigma_1}(A_{\sigma_i^+}) = \pi(\sigma_1)
\]
If \( \sigma \in \Sigma_{\infty} \) and \( k \in \mathbb{N} \), then
\[
f_{\sigma|k} \circ \pi \circ S^k(\sigma) = f_{\sigma_1 \sigma_2 \cdots}(\pi(\sigma_{k+1} \sigma_{k+2} \cdots)) = \lim_{m \rightarrow \infty} \pi(\sigma_{k+1} \sigma_{k+2} \cdots \sigma_m) = \pi(\sigma)
\]
The remaining cases follow similarly. \( \square \)

2.5. Disjunctive orbits, ergodicity, subshifts of finite type. Let \( T = S|_{\Sigma_{\infty}} \). The dynamical system \( T : \Sigma_{\infty} \rightarrow \Sigma_{\infty} \) is chaotic in the purely topological sense of Devaney [25]: it has a dense set of periodic points, it is sensitively dependent on initial conditions, and it is topologically transitive. Topologically transitive means that if \( Q \) and \( R \) are open subsets of \( \Sigma_{\infty} \), then there is \( K \in \mathbb{N} \) so that
\[
Q \cap T^K R \neq \emptyset
\]
This is true because the set of disjunctive points in \( \Sigma_{\infty} \) is dense in \( \Sigma_{\infty} \) and the orbit under \( T \) of a disjunctive point passes arbitrarily close to any given point in \( \Sigma_{\infty} \).

However, \( T : \Sigma_{\infty} \rightarrow \Sigma_{\infty} \) also possesses many invariant normalized Borel measures, each having support \( \Sigma_{\infty} \) and such that \( T \) is ergodic with respect to each. An example of such a measure \( \mu_T \) may be constructed by defining a Markov process on \( \Sigma_{\infty} \) using \( \mathcal{G} \) and probabilities \( \mathcal{P} = \{p_e > 0 : e \in \mathcal{E}\} \) where \( \sum_{d+ \in \mathcal{E}} p_d = 1 \) for all \( e \in \mathcal{E} \). Then \( \mu_T \) is the unique normalized measure on the Borel subsets \( \mathcal{B} \) of \( \Sigma_{\infty} \) such that
\[
\mu_T(b) = \sum_{e \in \mathcal{E}} p_e \mu_T(eb \cap \Sigma_{\infty}) \text{ for all } b \in \mathcal{B}
\]
where \( eb := \{ \sigma \in \Sigma_\infty : \sigma_1 = e, S\sigma \in b \} \). In particular, \( \mu_P \) is invariant under \( T \), that is
\[
\mu_P(b) = \mu_P(T^{-1}b) \text{ for all } b \in B
\]

The key point (1) in Theorem 4 is well known: \( T \) is ergodic with respect to \( \mu \). That is, if \( Tb = T^{-1}b \) for some \( b \in B \), then either \( \mu_P(b) = 0 \) or \( \mu_P(b) = 1 \). As a consequence, the set of disjunctive points has full measure, independent of \( P \).

**Theorem 4.** Let \((\mathcal{F}, \mathcal{G})\) be a graph IFS. Let \((\Sigma_\infty, B, T, \mu_P)\) be the dynamical system described above. Let \( D \) be the disjunctive points in \( \Sigma_\infty \). Then

1. Parry [31]: \((\Sigma_\infty, B, T, \mu_P)\) is ergodic
2. \( D = TD = T^{-1}D \in B \)
3. \( \mu_P(D) = 1 \), and \( \mu_P(\Sigma_\infty \setminus D) = 0 \)

**Proof.**
1. This is a standard result in ergodic theory, see for example [31].
2. It is readily checked that \( D \in B \) and that \( T^{-1}D = D = TD \).
3. Since \((\Sigma_\infty, B, T, \mu)\) is ergodic and \( D = T^{-1}D \), it follows that \( \mu(D) \in \{0, 1\} \).

Also we have
\[
1 = \mu(\Sigma_\infty) = \mu(D) + \mu(\Sigma_\infty \setminus D)
\]

So either \( \mu(D) = 1 \) and \( \mu(\Sigma_\infty \setminus D) = 0 \) or vice-versa. Now notice that
\[
\Sigma_\infty \setminus D \subseteq \bigcup_{x \in \Sigma_\infty} D_x
\]

where \( D_x = \{ \sigma \in \Sigma_\infty : S^n\sigma \notin c[x] \forall n \in \mathbb{N}_0 \} \) where \( c[x] \) is the cylinder set
\[
c[x] := \{ z \in \Sigma_\infty : z = xy, y \in \Sigma_\infty \}.
\]

In particular
\[
\mu(\Sigma_\infty \setminus D) \leq \sum_{x \in \Sigma_\infty} \mu(D_x)
\]

But \( \mu(D_x) = 0 \) as proved next, so \( \mu(\Sigma_\infty \setminus D) = 0 \).

Proof that \( \mu(D_x) = 0 \): Let \( f : \Sigma_\infty \to \mathbb{R} \) be defined by \( f(\sigma) = 0 \) if \( \sigma \in c[x] \) and \( f(\sigma) = 1 \) if \( \sigma \in \Sigma_\infty \setminus c[x] \). Since \( f \in L_1(\mu) \), by the ergodic theorem we have
\[
\int_{\Sigma_\infty} fd\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\sigma) \text{ for } \mu\text{-almost all } \sigma \in \Sigma_\infty.
\]

But \( \int fd\mu = 1 - \mu(c[x]) > 0 \) because the support of \( \mu \) is \( \Sigma_\infty \), and \( \Sigma_\infty \) contains a cylinder set disjoint from \( c[x] \) because \( |\mathcal{E}| \geq 2 \), and all cylinder sets have strictly positive measure. Also \( f(T^k\sigma) = 0 \) for all \( x \in D_x \) so
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\sigma) = 0 \text{ for all } x \in D_x
\]

so
\[
\int_{\Sigma_\infty} fd\mu \neq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\sigma) \text{ for all } x \in D_x \text{, so } \mu(D_x) = 0. \quad \square
\]
3. Tilings

3.1. Similitudes. A similitude is an affine transformation \( f : \mathbb{R}^M \rightarrow \mathbb{R}^M \) of the form \( f(x) = \lambda O(x) + q \), where \( O \) is an orthogonal transformation and \( q \in \mathbb{R}^M \) is the translational part of \( f(x) \). The real number \( \lambda > 0 \), a measure of the expansion or contraction of the similitude, is called its scaling ratio. An isometry is a similitude of unit scaling ratio and we say that two sets are isometric if they are related by an isometry.

3.2. Tiling iterated function systems.

Definition 4. The graph IFS \((\mathcal{F}, \mathcal{G})\) is said to obey the open set condition (OSC) if there are non-empty bounded open sets \( \{ \mathcal{O}_v : v \in \mathcal{V} \} \) such that for all \( d, e \in \mathcal{E} \) we have \( f_e(\mathcal{O}_e^+) \subset \mathcal{O}_e^- \) and \( f_e(\mathcal{O}_e^+) \cap f_d(\mathcal{O}_d^+) = \emptyset \) whenever \( e^- = d^- \).

The OSC for graph IFS is discussed in [10] and [23].

Definition 5. Let \( \mathcal{F} = \{ \mathbb{R}^M : f_1, f_2, \ldots, f_N \} \), with \( N \geq 2 \), be an IFS of contractive similitudes where the scaling factor of \( f_n \) is \( n = \frac{s}{a_n} \) where \( a_n \in \mathbb{N} \) and \( \text{gcd}\{a_1, a_2, \ldots, a_N\} = 1 \). Let the graph IFS \((\mathcal{F}, \mathcal{G})\) obeys the OSC. Let \( A_v \cap A_w = \emptyset \) for all \( v \neq w \), and let \( A_v \) span \( \mathbb{R}^M \). Then \((\mathcal{F}, \mathcal{G})\) is called a tiling iterated function system (tiling IFS).

The requirement \( A_v \cap A_w = \emptyset \) whenever \( v \neq w \) is without loss of generality in the following sense. By means of changes of coordinates applied to some of the maps of the IFS, we can move \( A_v \) to \( T_v A_v \), where \( T_v : \mathbb{R}^M \rightarrow \mathbb{R}^M \) is a translation, while holding \( A_w \) fixed for all \( w \neq v \). To do this, let

\[
\bar{f}_e = \begin{cases} 
T_v f_e T_v^{-1} & \text{if } e^+ = v \text{ and } e^- = v \\
T_v f_e & \text{if } e^+ \neq v \text{ and } e^- = v \\
f_e T_v^{-1} & \text{if } e^+ = v \text{ and } e^- \neq v \\
f_e & \text{if } e^+ \neq v \text{ and } e^- \neq v
\end{cases}
\]

and let \( \bar{\mathcal{F}} = \{ f_e : e \in \mathcal{E} \} \). Then the components of the attractor of \( \{ \bar{\mathcal{F}}, \mathcal{G} \} \) are \( \bar{A}_w = A_w \) for \( w \neq v \) and \( \bar{A}_v = T_v A_v \). By repeating this process for each vertex, we can modify the IFS so that different components of the attractor have empty intersections. Only the relative positions of the components are changed, while their geometries are unaltered, and (3.1) holds. This being the case, the OSC is simply “there are non-empty open sets \( \{ \mathcal{O}_v : v \in \mathcal{V} \} \) such that \( f_e(\mathcal{O}_e^+) \cap f_d(\mathcal{O}_d^+) = \emptyset \) for all \( d, e \in \mathcal{V} \) with \( d \neq e \).

Definition 6. The critical set of the tiling IFS \((\mathcal{F}, \mathcal{G})\) is

\[ \mathcal{C} := \bigcup_{\substack{d \neq e \in \mathcal{E} \atop d, e \in \mathcal{E}}} f_d(A_d^+) \cap f_e(A_e^+) \]

The following theorem tells us that the critical set of a tiling IFS is small, both topologically and measure theoretically, compared to the attractor.

Theorem 5. Let \((\mathcal{F}, \mathcal{G})\) be a tiling IFS, let \( \mathcal{C} \) be the critical set and let \( D \) be the disjunctive points in \( \Sigma_\infty \).
(1) Bedford [19] and Mauldin and Williams [29]: The Hausdorff dimension $D_H(A)$ of the attractor $A$ of $(\mathcal{F}, \mathcal{G})$ is the unique $t \in [0, M]$ such that the spectral radius of the matrix
\[ W_{w,v}(t) = \sum_{\{e \in E : e^+ = v, e^- = w\}} s^{|a_e|} \]
equals one. Also $0 < \mu_H(A) < \infty$ where $\mu_H$ is, up to a strictly positive constant factor, the Hausdorff measure on $A$.

(2) $C \subset \pi(\Sigma^\infty \setminus D)$

(3) $C \cap \mathring{A^o} = \emptyset$ where $A^o$ is the interior of $A$

(4) $\mu_P(\pi^{-1}(C)) = 0$ for all $P$.

(5) If $\sum_v W_{w,v}(t) = 1$ then $\mu_H = \mu_P \circ \pi^{-1}$ where $\mu_P$ is the stationary measure on $\Sigma^\infty$ obtained when $p_v = s^{D_H(A)a_v}$ in the Markov process described before Theorem 4. In this case $\mu_H(C) = 0$

**Proof.** (1) To apply [29] there must be exactly one edge of $G$ incoming to each vertex, but this can always be contrived, without changing either the dimension or the geometries of the components of the attractor, as we describe here. If $v \in V$ is such that $d = \left| \sum_{d^+ = v} 1 \right| > 1$, then introduce new vertices $v^{(1)}, v^{(2)}, \ldots, v^{(d)}$ and new components of the attractor $A_{v^{(1)}} = A_{v^{(2)}} = \ldots = A_{v^{(d)}} = A_v$, and replace the $d$ outgoing edges to $v$ by one outgoing edge to each of the new vertices and one incoming map from each of the new components of the attractor. Then translate the coincident attractors so that they have empty intersections and modify the maps accordingly, as above, and introducing additional maps, related to the original ones by changes of coordinates but with the same scaling ratios. This ensures that there is exactly one outward pointing edge at each vertex of $G$. This reduces the present situation to that in [29], who makes this assumption. Clearly the dimension of the attractor is unaltered.

We also have $0 < \mu_H(A) < \infty$ by [29] Theorem 3. Note that [29] Theorem 3 requires a different separation condition than the OSC, but both [10] Theorem 2.1 and [23] refer to [29] Theorem 3 as though the two conditions are equivalent, and we have assumed that this is true.

(2) This is the generalization to the graph-directed case of the definitions and argument in [7] Proposition 2.2]. We need the dynamical boundary $\partial A$ of the attractor $A$ of $(\mathcal{F}, \mathcal{G})$, namely
\[ \partial A := \bigcup_{n=1}^{\infty} \mathcal{F}|_{\mathcal{G}^{-n}(C)} \cap A \]
where $\mathcal{F}|_{\mathcal{G}^{-n}(C)} = \bigcup_{\theta \in \Sigma^+} f_{-\theta}(C \cap A_{\theta^+})$, see equation (3.2) for notation. We present the proof in parts (a) and (b) for the case $V = 1$. The proof is assumed to carry over to the tiling IFS case.

(a) The OSC implies, for similitudes, the open set $\mathcal{O} = \bigcup_{v \in V} \mathcal{O}_v$ can be chosen so that $\mathcal{O} \cap A \neq \emptyset$ [36], which implies $A \setminus \partial A \neq \emptyset$ because in this case $\mathcal{O} \cap \partial A = \emptyset$ by [30] Theorem 2.3 via (iii) implies (i) implies (ii)].
(b) $A \cap \partial A \neq \emptyset$ implies $\partial A \cap \pi(D) = \emptyset$ because if $x = \pi(\sigma) \in \mathcal{C}$ with $\sigma \in D$ then $\partial A = A$ as in [7] Proposition 2.2 Prop 2.2. It follows that $\mathcal{C} \subset \pi(\Sigma_{\infty} \setminus D)$.

(3) This is [7] Proposition 2.1] carried over to the tiling IFS case, using the non-overlappingness of $A$, namely $A \setminus \partial A \neq \emptyset$.

(4) We have
\[ \mu_{A}(\mathcal{C}) \leq \mu_{A}(\pi(\Sigma_{\infty} \setminus D)) \] by (2)
\[ = \mu_{H}(\pi^{-1}(\Sigma_{\infty} \setminus D)) \text{ since } \mu_{A} = \mu_{H} \circ \pi^{-1} \]
\[ = \mu_{H}(\Sigma_{\infty} \setminus D) = 0 \] by Theorem [19] (3)

(5) Using the thermodynamic formalism [19] **find exact reference or use MAGIC01| and the assumption that $\sum_{\omega} W_{\omega}(s) = 1$, we find that $\mu_{H} = \mu_{P} \circ \pi^{-1}$ is, up to a positive multiplicative constant, the Hausdorff measure obtained when
\[ p_{e} = s^{D_{H}(A)_{a_{e}}} / \sum_{d^{+} = e^{+}} s^{D_{H}(A)_{a_{d}}} \]

3.3. Tilings in this paper. According to Grunbaum and Sheppard [26] a tiling is a partition of $\mathbb{R}^{2}$ by closed sets. Here we consider tilings of subsets of $\mathbb{R}^{M}$ such as fractal blow-ups where tiles are components of attractors of IFSs. In this case tiles may have empty interiors and the question of what it means for tiles to be non-overlapping has to be answered. Here we simply say that two tiles $t_{1}$ and $t_{2}$ that belong to a tiling are non-overlapping if their intersection is small both topologically and measure theoretically, relative to the tiles themselves. This matches the customary situation: in a tiling of $\mathbb{R}^{2}$ such as a partition into triangles, tiles have positive two-dimensional Lebesgue measure, intersections of distinct tiles have zero two-dimensional Lebesgue measure and are subsets of their topological boundaries.

3.4. The tiling map. Define subsets of $\Sigma_{*}$ as follows:
\[ \Omega_{k} = \{ \sigma \in \Sigma_{*} : \xi^{-}(\sigma) \leq k < \xi(\sigma) \} \]
\[ \Omega_{0} = \{ N \} \]
\[ \Omega_{k}^{v} = \{ \sigma \in \Omega_{k} : \sigma^{-} = v \} \]
\[ \Omega_{0}^{v} = \{ \sigma_{1} \in \{ N : \sigma_{1}^{-} = v \} \}

for all $k \in \mathbb{N}$, $v \in \mathcal{V}$. Here $\xi : \Sigma_{*} \rightarrow \mathbb{N}_{0}$ is defined for all $\sigma \in \Sigma_{*}$ by
\[ \xi(\sigma) = \sum_{k=1}^{\mid \sigma \mid} a_{\sigma_{k}} \]
\[ \xi^{-}(\sigma) = \sum_{k=1}^{\mid \sigma \mid - 1} a_{\sigma_{k}} \]
\[ \xi(\emptyset) = \xi^{-}(\emptyset) = 0 \]

Tilings are associated with $\theta \in \Sigma^{\dagger}$. Edges in $\Sigma^{\dagger}$ are directed oppositely to those in $\Sigma$. For all $\theta \in \Sigma^{\dagger}$, $k \in \mathbb{N}$, $k \leq |\theta|$

(3.2)
\[ \theta|k := \theta_{1} \theta_{2} \cdots \theta_{k}, \theta|0 := \theta_{1}^{-} \]
\[ f^{-\theta}|k := f_{\theta_{1}}^{-1} f_{\theta_{2}}^{-1} \cdots f_{\theta_{k}}^{-1}, f^{-\theta}_{|0} := \chi_{\mathcal{A}_{\theta}|0} \]

Definition 7. The tiling map $\Pi$ from $\Sigma^{\dagger}$ to collections of subsets of $\mathcal{H}(\mathbb{R}^{M})$ defined as follows. For $\theta \in \Sigma^{\dagger}$,
\[ \Pi(\theta) = f_{-\theta} \pi \left( \Omega_{\xi(\theta)}^{\theta} \right), \Pi(\theta|0) = \pi \left( \Omega_{\theta|0}^{-} \right) \]
and for $\theta \in \Sigma_{\infty}^+$,
\[ \Pi(\theta) = \bigcup_{k \in \mathbb{N}} \Pi(\theta|k) \]

For $\sigma \in \Omega^+_{\xi(\theta)}$ and $\theta \in \Sigma^+$, the set $f_{-\pi}(\sigma)$ is called a tile and $\Pi(\theta)$ is called a tiling. The support of the tiling $\Pi(\theta)$ is the union of its tiles, and $\Pi(\theta)$ is said to tile its support.

Let $\mathcal{U}$ be a group of isometries on $\mathbb{R}^M$ that contains the group of isometries on $\mathbb{R}^M$ generated by the maps $\{f_{e} : e \in E\} \cup \{f_0\}$ where $f_0(x) := sx$ for all $x \in \mathbb{R}^M$.

**Theorem 6.** Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS.

1. For all $\theta \in \Sigma_{\infty}^+$, if $t_1, t_2 \in \Pi(\theta)$ with $t_1 \neq t_2$, then $t_1 \cap t_2$ is small both topologically and measure theoretically, compared to $t_1$. That is, $\mu_\mathcal{F}(t_1 \cap t_2) = 0$ and $t_1^c \cap t_2 = t_2^c \cap t_1 = \emptyset$ where $t^c$ is the interior of $t$. 

2. For all $\theta \in \Sigma_{\infty}^+$ the sequence of tilings $\{\Pi(\theta|k)\}_{k=1}^\infty$ obeys
\[ \Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2) \subset \cdots \]

3. $\Pi(\theta)$ is a tiling of a subset of $\mathbb{R}^M$ that is bounded when $\theta \in \Sigma_{\infty}^+$ and unbounded when $\theta \in \Sigma_{\infty}^+$.

4. For all $\theta \in \Sigma_{\infty}^+$
\[ \Pi(\theta) = \lim_{k \to \infty} f_{-\pi|k}(\{\pi(\sigma) : \sigma \in \Omega^+_{\xi(\theta|k)}, \sigma^- = \theta^+\}) \]

5. Any tile $t \in \Pi(\theta)$ can be written $t = s^iEA_v$ for some isometry $E \in \mathcal{U}$, $i \in \{1, 2, ..., \max a_v\}$ and $v \in \mathcal{V}$.

**Proof.** (1) $\Pi(\theta|0)$ is a tiling in the sense described in Section 3.3 $\Pi(\theta|0) = \pi\left(\Omega^+_{\theta^-}\right) = \pi\{e \in N : e^- = \theta^-\} = \{f_e(A_{e-}) : e^- = \theta^-\}$ has support $A_{e-}$ and its tiles are supposed to be $\{f_e(A_{e+}) : e^- = \theta^-\}$. We need to check (i) that they are components of attractors of tiling IFSs and (ii) that their intersections are relatively small. (i) is true because for each $e \in N$, the set $f_e(A_{e+})$ is a component of the attractor of the tiling IFS $(f_e F f^{-1}_e, \mathcal{G})$. (ii) This follows from Theorem 5 parts (3) and (4).

Similarly, $\Pi(\theta|k)$ and $\Pi(\theta)$ are tilings as in Section 3.3 the tiles are components of attractors of appropriately shifted versions of the original tiling IFS and their intersections are isometric to subsets of the critical set of the original tiling IFS.

(2) The proof is algebraic, independent of topology, essentially the same as for the case where $A_v$ has nonempty interior [15], and similar to the case where $V = 1$ [16]. Briefly,
\[ \Pi(\theta|k + 1) = \{f_{-\pi|k}^{-1}, \ldots, f_{\pi|k}^{-1}, f_{\sigma|\sigma}^{-1}(A_{\sigma|\sigma}^+) : \xi(\sigma_1, \ldots, \sigma_{|\sigma|-1}) \leq \xi(\theta_1, \ldots, \theta_{|\sigma|}) < \xi(\sigma_1, \ldots, \sigma_{|\sigma|}) \}
\]
\[ \supset \{f_{-\pi|k}^{-1}, \ldots, f_{\pi|k}^{-1}, f_{\sigma|\sigma}^{-1}(A_{\sigma|\sigma}^+) : \xi(\sigma_2, \ldots, \sigma_{|\sigma|-1}) \leq \xi(\theta_2, \ldots, \theta_{|\sigma|}) < \xi(\sigma_2, \ldots, \sigma_{|\sigma|}) \}
\]
\[ = \{f_{-\pi|k}^{-1}, \ldots, f_{\pi|k}^{-1}, f_{\sigma|\sigma}^{-1}(A_{\sigma|\sigma}^+) : \xi(\sigma_1, \ldots, \sigma_{|\sigma|-2}) \leq \xi(\theta_1, \ldots, \theta_{|\sigma|-1}) < \xi(\sigma_1, \ldots, \sigma_{|\sigma|-1}) \}
\]
\[ = \Pi(\theta|k) \]

(3) For $\theta \in \Sigma_{\infty}^+$, $\Pi(\theta) = f_{-\pi}(\Omega^+_{\xi(\theta)})$, so the support of $\Pi(\theta)$ is $f_{-\pi}(\Omega^+_{\xi(\theta)})$. Here $f_{-\pi}$ is a similitude of expansion factor $|s|^{-\xi(\theta)}$ which diverges with $|\theta|$, and $A_{\theta^+}$ spans $\mathbb{R}^M$. 

(4) This follows from (3).
(5) For \( t \in \Pi(\theta) \) we have \( t = f_{-(\theta|k)}f_\sigma(A_k) \) for some \( k, \theta, \sigma \) and \( v \), with \( \xi^- (\sigma) \leq \xi(\theta|k) < \xi(\sigma) \). Here \( f_{-(\theta|k)}f_\sigma = s^{-m}E \) where \( m = \xi(\theta|k) - \xi(\sigma) \) is an integer that lies between 1 and \( \max \{ \alpha_e \} \). \( E \in \mathcal{U} \) is an isometry of the form \( s^m f_{-(\theta|k)}f_\sigma \) for some \( m \).

\[ \square \]

4. Continuity properties of \( \Pi : \Sigma \rightarrow \Theta \).

4.1. A convenient compact tiling space. Let

\[ \Theta = \{ \Pi(\theta) : \theta \in \Sigma^+ \} \]

Let \( \rho : \mathbb{R}^M \rightarrow \mathbb{S}^M \) be the usual \( M \)-dimensional stereographic projection to the \( M \)-sphere, obtained by positioning \( \mathbb{S}^M \) tangent to \( \mathbb{R}^M \) at the origin. Let \( \mathbb{H}(\mathbb{S}^M) \) be the non-empty closed (w.r.t. the usual topology on \( \mathbb{S}^M \)) subsets of \( \mathbb{S}^M \). Let \( d_{\mathbb{H}(\mathbb{S}^M)} \) be the Hausdorff distance with respect to the round metric on \( \mathbb{S}^M \), so that \( (\mathbb{H}(\mathbb{S}^M), d_{\mathbb{H}(\mathbb{S}^M)}) \) is a compact metric space. Let \( \mathbb{H}(\mathbb{H}(\mathbb{S}^M)) \) be the nonempty compact subsets of \( (\mathbb{H}(\mathbb{S}^M), d_{\mathbb{H}(\mathbb{S}^M)}) \), and let \( d_{\mathbb{H}(\mathbb{H}(\mathbb{S}^M))} \) be the associated Hausdorff metric. Then \( (\mathbb{H}(\mathbb{H}(\mathbb{S}^M)), d_{\mathbb{H}(\mathbb{H}(\mathbb{S}^M))}) \) is a compact metric space. Finally, define a metric \( d_\Theta \) on \( \Theta \) by

\[ d_\Theta(T_1, T_2) = d_{\mathbb{H}(\mathbb{H}(\mathbb{S}^M))}(\rho(T_1), \rho(T_2)) \]

for all \( T_1, T_2 \in \Theta \).

**Theorem 7.** \((\Theta, d_\Theta)\) is a compact metric space.

**Proof.** Straightforward and omitted. \( \square \)

See also for example [1] [21] [35] [37] [40] where related metrics and topologies are defined.

4.2. Continuity. The following definition generalizes a related concept for the case where \( \Lambda \) is a topological disk and \( |V| = 1 \), see [14]. For \( \theta \in \Sigma^+_\infty \) define \( I(\theta) \subset \Sigma^\infty \) to be the set of limit points of \( \{ \theta_{l+m+1}...\theta_{m+1} : l, m \in \mathbb{N} \} \). Define

\[ H_v = \bigcup \{ f_{\omega}^{-1}f_\omega(A_{\omega}) : \sigma^+ = \omega, \sigma, \omega \in \Sigma, \sigma \neq \omega_1 \} \]

This is the union of all images of \( A_\nu \) under its neighbor maps and is a generalization of the same definition in the case \( V = 1 \). [2] [3] [4]. Define the central open sets to be

\[ O_v = \{ x \in \mathbb{R}^M : d(x, A_v) < d(x, H_v) \} \]

It appears that \( (\mathcal{F}, \mathcal{G}) \) obeys the OSC” if and only if “\( A_\nu \) is not contained in \( H_v \) for all \( v \in V \)” , see [2] [3] [4].

Call \( \theta \in \Sigma^+_\infty \) reversible if

\[ \Sigma^+_\text{rev} := I(\theta) \cap \{ \sigma \in \Sigma^\infty : \pi(\sigma) \subset \cup \nu \} \neq \emptyset \]

Equivalently, \( \theta \in \Sigma^+_\text{rev} \) if the following holds: there exists \( \sigma \in \Sigma^\infty \) with \( \pi(\sigma) \subset \cup \nu \) such that, for all \( L, M \in \mathcal{N} \) there is \( m \geq M \) so that

\[ \sigma_1\sigma_2...\sigma_L = \theta_{m+1}\theta_{m+L-1}...\theta_1 \]

Equivalently, in terms of the notion of “full” words, see [14], \( \theta \in \Sigma^+_\text{rev} \) if there is a nonempty compact set \( A' \subset \cup \nu \) such that for any positive integer \( M \) there exists \( n > m \geq M \) so that

\[ f_{\theta_n}f_{\theta_{n-1}}...f_{\theta_m}(A_{\theta_m^+}) \subset A' \]
THEOREM 8. Let \((\mathcal{F}, \mathcal{G})\) be a tiling IFS. Then
\[
\Pi|_{\Sigma_{rev}} : \Sigma_{rev} \subset \Sigma_\infty^{\dagger} \to T
\]
is continuous and
\[
\Pi : \Sigma_\infty^{\dagger} \to T
\]
is upper semi-continuous in this sense: if \(\Pi(\theta^{(n)})\) is a sequence of tilings that converges to a tiling \(T \in T\) as \(\theta^{(n)}\) converges to \(\theta \in \Sigma_\infty^{\dagger}\), then \(\Pi(\theta) \subset T\).

PROOF. Proof of upper semi-continuity: let \(\{\theta^{(n)}\}\) be a sequence of points in \(\Sigma_\infty^{\dagger}\) that converges to \(\theta\) and such that \(\lim \Pi(\theta^{(n)}) = T\) with respect to the tiling metric. Let \(m\) be given. Then there is \(l_m\) so that for all \(n \geq l_m\) we have \(\theta m = \theta^{(n)} m\) and hence \(\Pi(\theta m) = \Pi(\theta^{(n)} m) \subset \Pi(\theta^{(n)})\). Hence we have \(\Pi(\theta m) \subset \lim \Pi(\theta^{(n)})\) and hence, since this is true for all \(m\), \(\Pi(\theta) \subset \lim \Pi(\theta^{(n)})\).

Proof that \(\Pi|_{\Sigma_{rev}^{\dagger}} : \Sigma_{rev}^{\dagger} \to T\) is continuous involves blow-ups of central opens sets. Analogously to the definition of \(\Pi\), define a mapping \(\Xi\) from \(\Sigma^{\dagger}\) to subsets of \(\mathbb{H}(\mathbb{R}^M)\) as follows. For \(\theta \in \Sigma^{\dagger}\), \(\theta \neq \emptyset\),
\[
\Xi(\theta) := \{f^{\sigma}f_{\sigma}(O_{\sigma^{+}}) : \sigma \in \Omega_{\xi(\theta)}^{\theta^{+}}\},
\]
and for \(\theta \in \Sigma_\infty^{\dagger}\)
\[
\Xi(\theta) := \bigcup_{k \in \mathbb{N}} \Xi(\theta|k).
\]
As is the case for \(\Pi\), increasing families of sets are obtained: each collection \(\Xi(\theta)\) comprises a covering by compact sets of a subset of \(\mathbb{R}^M\), the subset being bounded when \(\theta \in \Sigma_\infty^{\dagger}\) and unbounded when \(\theta \in \Sigma_\infty^{\dagger}\). For all \(\theta \in \Sigma_\infty^{\dagger}\) the sequence of sets \(\{\Xi(\theta|k)\}_{k=1}^{\infty}\) is nested according to
\[
\Xi(\theta|1) \subset \Xi(\theta|2) \subset \Xi(\theta|3) \subset \cdots
\]
and we have \(\{\Xi(\theta|k)\}\) converges to \(\Xi(\theta)\) in the metric introduced in Section 4.1.
In particular, when reversible, the new tiles, those in \(\Xi(\theta|k+1)\) that are not in \(\Xi(\theta|k)\), are located further and further away from the origin as \(k\) increases. The result follows. \(\Box\)

5. Symbolic structure : canonical symbolic tilings and symbolic inflation and deflation

Write \(\Omega_k^{(v)}\) to mean any of \(\Omega_k^{(v)}\) or \(\Omega_k^{(v)}\). The following lemma tells us that \(\Omega_k^{(v)}\) can be obtained from \(\Omega_k^{(v)}\) by adding symbols to the right-hand end of some strings in \(\Omega_k^{(v)}\) and leaving the other strings unaltered.

LEMMA 1. (Symbolic Splitting) For all \(k \in \mathbb{N}\) and \(v \in \mathcal{V}\) the following relations hold:
\[
\Omega_{k+1}^{(v)} = \left\{ \sigma \in \Omega_k^{(v)} : k+1 < \xi(\sigma) \right\} \cup \left\{ \sigma j \in \Sigma_{(v)} : \sigma \in \Omega_k^{(v)}, k+1 = \xi(\sigma) \right\}.
\]

PROOF. Follows at once from definition of \(\Omega_k^{(v)}\). \(\Box\)

Define \(\alpha^{-1}_s : \Omega_k^{(v)} \to 2\Omega_{k+1}^{(v)}\) by
\[
\alpha^{-1}_s \sigma = \left\{ \begin{array}{ll}
\sigma & \text{if } k+1 < \xi(\sigma) \\
\sigma e : \sigma_{|\sigma^+} = e^-, e \in \mathcal{E} & \text{if } k+1 = \xi(\sigma)
\end{array} \right\}
\]
Then
\[ \{ \sigma \in \alpha_s^{-1}(\omega) : \omega \in \Omega_{k+1}^{(v)} \} = \Omega_k^{(v)} \]

This defines symbolic inflation or “splitting and expansion” of \( \Omega_k^{(v)} \), some words in \( \Omega_{k+1}^{(v)} \) being the same as in \( \Omega_k^{(v)} \) while all the other words in \( \Omega_k^{(v)} \), namely those \( \sigma \) for which \( k + 1 = \xi(\sigma) \), are split. The inverse operation is symbolic deflation or “amalgamation and shrinking”, described by the function
\[ \alpha_s : \Omega_{k+1}^{(v)} \rightarrow \Omega_k^{(v)}, \quad \alpha_s(\Omega_{k+1}^{(v)}) = \Omega_k^{(v)} \]

where \( \alpha_s(\sigma) \) is the unique \( \omega \in \Omega_k^{(v)} \) such that \( \sigma = \omega \beta \) for some \( \beta \in \Sigma_\ast \). Note that \( \beta \) may be the empty string.

We can use \( \Omega_k^{(v)} \) to define a partition of \( \Omega_{m}^{(v)} \) for \( m \geq k \). The partition of \( \Omega_{k+j}^{(v)} \) is \( \Omega_{k+j}^{(v)} \sim \sim \) where \( x \sim y \) if \( \alpha_j(x) = \alpha_j(y) \).

**Corollary 1.** (Symbolic Partitions) For all \( m \geq k \geq 0 \), the set \( \Omega_k^{(v)} \) defines a partition \( P_{m,k}^{(v)} \) of \( \Omega_m^{(v)} \) according to \( p \in P_{m,k}^{(v)} \) if and only if there is \( \omega \in \Sigma_\ast \) such that
\[ p = \{ \omega\beta \in \Omega_m^{(v)} : \beta \in \Omega_k^{(v)} \}. \]

**Proof.** This follows from Lemma \( \boxed{\text{II}} \) for any \( \theta \in \Omega_m^{(v)} \) there is a unique \( \omega \in \Omega_k^{(v)} \) such that \( \theta = \omega \beta \) for some \( \beta \in \Sigma_\ast \). Each word in \( \Omega_m^{(v)} \) is associated with a unique word in \( \Omega_k^{(v)} \). Each word in \( \Omega_k^{(v)} \) is associated with a set of words in \( \Omega_m^{(v)} \). \( \square \)

According to Lemma \( \boxed{\text{II}} \), \( \Omega_{k+1}^{(v)} \) may be calculated by tacking words (some of which may be empty) onto the right-hand end of the words in \( \Omega_k^{(v)} \). We can invert this description by expressing \( \Omega_k^{(v)} \) as a union of predecessors \( (\Omega_j^{(v)}) \)’s with \( j < k \) of \( \Omega_k^{(v)} \) with words tacked onto their other ends, that is, their left-hand ends.

**Corollary 2.** (Symbolic Predecessors) For all \( k \geq a_{\text{max}} + 1 \), for all \( v \in \mathcal{V} \), for all \( l \in \mathbb{N}_0 \),
\[ \Omega_k^{(v)} = \bigcup_{\omega \in \Omega_l^{(v)}} \omega \Omega_{k-l}^{(v)} \]

**Proof.** It is easy to check that the r.h.s. is contained in the l.h.s.

Conversely, if \( \sigma \in \Omega_k^{(v)} \) then there is unique \( \omega \in \Omega_l^{(v)} \) such that \( \sigma = \omega \beta \) for some \( \beta \in \Sigma_\ast \) by Corollary \( \boxed{\text{II}} \). Because \( \omega \beta \in \Sigma_\ast \) it follows that \( \beta_1 \) is an edge in \( \omega \) directed, namely the vertex \( \omega \sim \) Finally, since \( \xi(\omega \beta) = \xi(\omega) + \xi(\beta) \) it follows that \( \beta \in \Omega_k^{(v)} \). \( \square \)

6. Canonical tilings and their relationship to \( \Pi(\theta) \)

**Definition 8.** We define the canonical tilings of the tiling IFS \( (F, G) \) to be
\[ T_k := s^{-k} \pi(\Omega_k), \quad T_v := s^{-k} \pi(\Omega_k^v) \]

\( k \in \mathbb{N}, v \in \mathcal{V} \), also
\[ T_0 := \Pi(0) := \cup_{v \in \mathcal{V}} T_v^0, T_0 := \Pi(e|0) := \{ f_v(A^v) : e^- = v \}, \]
\[ T_{-1} := sA_v, T_{-1} := \cup_{v \in \mathcal{V}} sA_v \]
A canonical tiling may be written as a disjoint union of isometries applied of other canonical tilings as described in Lemma 2. We may say “$T_k$ can be written as an isometric combination of canonical tilings”.

**Lemma 2.** For all $k \geq a_{\text{max}} + 1$, for all $l \in \mathbb{N}_0$, for all $v \in V$

$$
T_k^v = \bigcup_{\omega \in \Omega^l_1} E_{k,\omega} T_k^+ \quad \text{and} \quad T_k = \bigcup_{\omega \in \Omega^l_1} E_{k,\omega} T_k^+ \quad \text{where} \quad E_{k,\omega} = s^{-k} f_\omega s^{k-\xi(\omega)} \in \mathcal{U} \quad \text{is an isometry.}
$$

**Proof.** Direct calculation using Corollary 2.

**Theorem 9.** For all $\theta \in \Sigma_1^+$,

$$
\Pi(\theta) = E_\theta T_{\xi(\theta)}^+ \quad \text{where} \quad E_\theta = f_{-\theta}s^{\xi(\theta)} \in \mathcal{U} \quad \text{Also if} \quad l \in \mathbb{N}_0, \quad \text{and} \quad \xi(\theta) \geq a_{\text{max}} + l, \quad \text{then}
$$

$$
\Pi(\theta) = \bigcup_{\omega \in \Omega^{l+}_1} E_{\theta,\omega} T_{\xi(\theta)-\xi(\omega)}^+ \quad \text{where} \quad E_{\theta,\omega} = f_{-\theta}s^{\xi(\theta)-\xi(\omega)} \in \mathcal{U} \quad \text{is an isometry.}
$$

**Proof.** Writing $\theta = \theta_1 \theta_2 \ldots \theta_k$ so that $|\theta| = k$, we have from the definitions

$$
\Pi(\theta_1 \theta_2 \ldots \theta_k) = f_{-\theta_1 \theta_2 \ldots \theta_k} \{ \pi(\sigma) : \sigma \in \Omega^+_{\xi(\theta_1 \theta_2 \ldots \theta_k)} \}
$$

$$
= f_{-\theta_1 \theta_2 \ldots \theta_k} s^{\xi(\theta_1 \theta_2 \ldots \theta_k)} s^{-\xi(\theta_1 \theta_2 \ldots \theta_k)} \{ \pi(\sigma) : \sigma \in \Omega^+_{\xi(\theta_1 \theta_2 \ldots \theta_k)} \}
$$

$$
= E_\theta T_{\xi(\theta)}^+ \quad \text{where} \quad E_\theta = f_{-\theta}s^{\xi(\theta)}
$$

The last statement of the theorem follows similarly from Lemma 2.

**7. All tilings in $T_\infty$ are quasiperiodic**

We recall from [16] the following definitions. A subset $P$ of a tiling $T$ is called a patch of $T$ if it is contained in a ball of finite radius. A tiling $T$ is quasiperiodic if, for any patch $P$, there is a number $R > 0$ such that any disk centered at a point in the support of $T$ and of radius $R$ contains an isometric copy of $P$. Two tilings are locally isomorphic if any patch in either tiling also appears in the other tiling. A tiling $T$ is self-similar if there is a similitude $\psi$ such that $\psi(t)$ is a union of tiles in $T$ for all $t \in T$. In this case $\psi$ is called a self-similarity for $T$. We say that the tiling IFS $(\mathcal{F}, \mathcal{G})$ is coprime when, for each pair $v, w \in V$, there are $\sigma, \omega \in \Sigma_1$ with $\sigma^+ = \omega^+ = v$ and $\sigma^- = \omega^- = w$ such that the greatest common factor of $\xi(\sigma)$ and $\xi(\omega)$ is 1.

**Theorem 10.** Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS.

1. Every tiling in $T_\infty$ is quasiperiodic.
2. If $(\mathcal{F}, \mathcal{G})$ is coprime, then each pair of tilings in $T_\infty$ are locally isomorphic.
3. If $\theta \in \Sigma_1^\text{e}$ is eventually periodic, then $\Pi(\theta)$ is self-similar: if $\theta = \alpha \beta$ for some $\alpha, \beta \in \Sigma_1^\text{e}$, then $f_{-\alpha} f_{-\beta} (f_{-\alpha})^{-1}$ is a self-similarity for $\Pi(\theta)$.

**Proof.** This uses Theorem 9 and follows similar lines to [16] proof of Theorem 2.
8. Addresses

Addresses, both relative and absolute, are described in [16] for the case \(|V| = 1\). See also [6]. Here we add information and generalize. The relationship between these two types of addresses is subtle.

8.1. Relative addresses. Write \(T_k^{(v)}\) to mean any of \(T_k^r\) or \(T_k^w\).

Definition 9. The relative address of \(t \in T_k^{(v)}\) is defined to be \(\varnothing.\pi^{-1}s^{k}(t) \in \varnothing.\Omega_k^{(v)}\). The relative address of a tile \(t \in T_k\) depends on its context, its location relative to \(T_k\), and depends in particular on \(k \in \mathbb{N}_0\). Relative addresses also apply to the tiles of \(\Pi(\theta)\) for each \(\theta \in \Sigma_1^+\) because \(\Pi(\theta) = E_\theta T^{\theta(\theta)}(\theta)\) where \(E_\theta = f_{-\theta}s^{\xi(\theta)}\) (by Theorem 3) is a known isometry applied to \(T_{\xi(\theta)}\). Thus, the relative address of \(t \in \Pi(\theta)\) relative to \(\Pi(\theta)\) is \(\varnothing.\pi^{-1}f_{-\theta}^{-1}(t)\), for \(\theta \in \Sigma_1^+\). When it is clear from context we may drop the symbols “\(\varnothing.\)”.

Lemma 3. The tiles of \(T_k\) are in bijective correspondence with the set of relative addresses \(\varnothing.\Omega_k\). The tiles of \(T_k^v\) are in bijective correspondence with the set of relative addresses \(\varnothing.\Omega_k^v\).

Proof. The correspondences are provided by the bijective map

\[H : \varnothing.\Omega_k \rightarrow T_k\]

defined by \(H(\varnothing.\sigma) = s^{-k}\pi(\sigma)\). We have \(T_k = s^{-k}\pi(\Omega_k)\) so \(H\) maps \(\varnothing.\Omega_k\) onto \(T_k\). Also \(H\) is one-to-one: if \(\beta \neq \gamma\), for \(\beta, \gamma \in \Sigma_1\), then \(f_\beta(A) \neq f_\gamma(A)\) because \(H(\varnothing.\beta) = H(\varnothing.\gamma)\) implies \(\pi(\beta) = \pi(\gamma)\) which implies \(\beta = \gamma\) because the tiling IFS obeys the open set condition and \(A_v \cap A_w = \emptyset\) for \(v \neq w\). If the requirement \(A_v \cap A_w = \emptyset\) does not hold, it may not be true that \(H : \varnothing.\Omega_k \rightarrow T_k\) is one-to-one; but it remains true that \(H|_{\varnothing.\Omega_k^v} : \varnothing.\Omega_k^v \rightarrow T_k^v\) is bijective.

For precision we should write “the relative address of \(t\) relative to \(T_k\)” however, when the context \(t \in T_k\) is clear, we may simply refer to “the relative address of \(t\)”.

Example 1. (Standard 1D binary tiling) For the IFS \(\mathcal{F}_0 = \{\mathbb{R}; f_1, f_2\}\) with \(f_1(x) = 0.5x, f_2(x) = 0.5x + 0.5\) we have \(\Pi(\theta)\) for \(\theta \in \Sigma_1^+\) is a tiling by copies of the tile \(t = [0, 0.5]\) whose union is an interval of length \(2^{\theta}\) and is isometric to \(T_0\) and represented by \(tttt...t\) with relative addresses in order from left to right

\(\varnothing.111...11, \varnothing.111...12, \varnothing.111...21, ..., \varnothing.222...22\),

the length of each string (address) being \(|\theta| + 1\). Notice that here \(T_k\) contains \(2^{\theta} - 1\) copies of \(T_0\) (namely \(tt\)) where a copy is \(ET_0\) where \(E \in \mathcal{T}_{\mathcal{F}_0}\), the group of isometries generated by the functions of \(\mathcal{F}_0\).

Example 2. (Fibonacci 1D tilings) \(\mathcal{F}_1 = \{ax, a^2x + 1 - a^2, a + a^2 = 1, a > 0\}\), \(\mathcal{T} = \mathcal{T}_{\mathcal{F}_1}\) is the largest group of isometries generated by \(\mathcal{F}_1\). The tiles of \(\Pi(\theta)\) for \(\theta \in \Sigma_1^+\) are isometries that belong to \(\mathcal{T}_{\mathcal{F}_1}\) (the group of isometries generated by the IFS) applied to the tiling of \([0, 1]\) provided by the IFS, writing the tiling \(T_0\) as \(ls\) where \(l\) is a copy of \([0, a]\) and (here) \(s\) is a copy of \([0, a^2]\) we have:

\(T_0 = ls\) has relative addresses \(\varnothing.1, \varnothing.2\) (i.e. the address of \(l\) is 1 and of \(s\) is 2)
\(T_1 = lsl\) has relative addresses \(\varnothing.11, \varnothing.12, \varnothing.2\)
$T_2 = \text{islls}$ has relative addresses $\emptyset, 1.11, 1.12, 1.21, 1.22$.

$T_3 = \text{isllslls}$ has relative addresses $\emptyset, 1.111, 1.112, 1.212, 1.212, \ldots$.

We remark that $T_k$ comprises $F_k+1$ distinct tiles and contains exactly $F_k$ copies (under maps of $T_{x_1}$) of $T_0$, where $\{F_k : k \in \mathbb{N}_0\}$ is a sequence of Fibonacci numbers $\{1, 2, 3, 5, 8, 13, 21, \ldots\}$.

Note that $T_4 = \text{isllsllslls}$ contains two “overlapping” copies of $T_2$.

The following theorem defines hierarchies of canonical tilings and point out that any relative address is naturally associated with a collection of hierarchies.

**Theorem 11.** Let $(\mathcal{F}, \mathcal{G})$ be a tiling IFS. The following hierarchy of canonical tilings is associated with any given relative address $\sigma \in \Sigma_k$:

\begin{equation}
F_0T_0^{\sigma_{|\theta|}} \subset F_1T_1^{\sigma_{|\theta|}} \subset \cdots \subset F_kT_k^{\sigma_{|\theta|}} \subset T_k^{\sigma_{|\theta|}} \subset T_k^{\sigma_{|\theta|}} \subset T_k^{\sigma_{|\theta|}} \subset \cdots
\end{equation}

where $F_k = s^{-\xi(\sigma)}(f_{-\sigma_1 \sigma_2 \ldots \sigma_k})^{-1}s^{-\xi(\sigma)} \in \mathcal{U}$ for $k = 0, 1, \ldots, \xi(\sigma).

**Proof.** The chain of inclusions

$\Pi(\sigma_{|\theta|}) \subset \Pi(\sigma_{|\theta|}) \subset \Pi(\sigma_{|\theta|}) \subset \cdots \subset \Pi(\sigma_{|\theta|})$

can be rewritten

$T_0^{\sigma_{|\theta|}} \subset f_{-\sigma_1 \sigma_2 \ldots \sigma_k}s^{-\xi(\sigma)}T_1^{\sigma_{|\theta|}} \subset \cdots
\end{equation}

Applying the isometry $E = s^{-\xi(\sigma)}f_{\sigma} \in \mathcal{U}$ on the left throughout to obtain

$s^{-\xi(\sigma)}f_{\sigma_1 \sigma_2 \ldots \sigma_k}T_0^{\sigma_{|\theta|}} \subset s^{-\xi(\sigma)}f_{\sigma_1 \sigma_2 \ldots \sigma_k}T_1^{\sigma_{|\theta|}} \subset \cdots$

which is equivalent to equation (8.1). \qed

**8.2. Absolute addresses.** The set of absolute addresses associated with $(\mathcal{F}, \mathcal{G})$ is

$\mathcal{A} := \{\theta, \sigma : \emptyset, 1.11, 1.12, 1.21, 1.22\}$.

Define $\hat{\Pi} : \mathcal{A} \to \{t \in T : T \in \mathbb{T}\}$ by

$\hat{\Pi}(\theta, \omega) = f_{\sigma_1 \sigma_2 \ldots \sigma_k}(A_{\sigma_1})$.

The condition $\theta_{|\sigma|} \neq \sigma_1$ is imposed. We say that $\theta, \sigma$ is an absolute address of the tile $f_{\sigma_1 \sigma_2 \ldots \sigma_k}(A)$. It follows from Definition 5 that the map $\hat{\Pi}$ is surjective: every tile of $\{t \in T : T \in \mathbb{T}\}$ possesses at least one absolute address.

Although tiles have unique relative addresses, relative to the $T_k^{\nu}$ to which they are being treated as belonging, they may have many different absolute addresses.
The tile $[1, 1.5]$ of Example 11 has the two absolute addresses 1.21 and 21.211, and many others.

**8.3. Relationship between relative and absolute addresses.**

**Theorem 12.** If $t \in \Pi(\theta)$ with $\theta \in \Sigma^1_+$ has relative address $\omega$ relative to $\Pi(\theta)$, then an absolute address of $t$ is $\theta_1 \theta_2 \ldots \theta_i, S^{[\theta]^{-1}} \omega$ where $l \in \mathbb{N}$ is the unique index such that

$$t \in \Pi(\theta_1 \theta_2 \ldots \theta_i) \quad \text{and} \quad t \notin \Pi(\theta_1 \theta_2 \ldots \theta_{i-1})$$

**Proof.** Recalling that

$$\Pi(\theta|0) \subset \Pi(\theta|1) \subset \Pi(\theta|2) \subset \ldots \subset \Pi(\theta|\theta|\theta|\theta|\ldots|1) \subset \Pi(\theta),$$

we have disjoint union

$$\Pi(\theta) = \Pi(\theta|0) \cup (\Pi(\theta|1)|\Pi(\theta|0))) \cup (\Pi(\theta|1)|\Pi(\theta|2)|\Pi(\theta|1))) \cup \ldots \cup (\Pi(\theta)|\Pi(\theta|\theta|\theta|\theta|\ldots|1)) .$$

So there is a unique $l$ such that Equation (8.2) is true. Since $t \in \Pi(\theta)$ has relative address $\emptyset.\sigma$ relative to $\Pi(\theta)$ we have

$$\emptyset.\sigma = \emptyset.\pi^{-1} f_{-\theta}^{-1}(t)$$

and so an absolute address of $t$ is

$$\theta.\sigma|_{\text{cancel}} = \theta.\pi^{-1} f_{-\theta}^{-1}(t)|_{\text{cancel}}$$

where $|_{\text{cancel}}$ means equal symbols on either side of “.” are removed until there is a different symbol on either side. Since $t \in \Pi(\theta)$ the terms $\theta_{i+1}\theta_{i+2}\ldots\theta_{|\theta|}$ must cancel yielding the absolute address

$$\theta.\sigma|_{\text{cancel}} = \theta_{1}\theta_{2}\ldots\theta_{l+1}\ldots\theta_{|\sigma|} \quad \square$$

**8.4. Inflation and deflation of canonical tilings.** We say that the canonical tiling $T^v_k$ is indexed when $k \in \mathbb{N}$ and $v \in \mathcal{V}$ are known. Here we define the deflation operator $\alpha$ and its inverse, the inflation operator $\alpha^{-1}$, both restricted to indexed canonical tilings, by

$$\alpha T^v_k = T^v_{k-1}, \quad \alpha^{-1} T^v_{k-1} = T^v_k$$

for all specified $k \in \mathbb{N}, v \in \mathcal{V}$. We extend the domains of $\alpha$ and $\alpha^{-1}$ to include any given isometry $E \in \mathcal{U}$ applied to $T^v_k$, by defining

$$\alpha ET^v_k = (sEs^{-1}) \alpha T^v_k = (sEs^{-1}) T^v_{k-1}$$

$$\alpha^{-1} ET^v_{k-1} = (s^{-1}Es) T^v_k$$

for all $k \in \mathbb{N}, v \in \mathcal{V}$.

Note that the tiling $\alpha^{-1} T^v_{k-1}$ may be calculated by replacing each tile $t \in T^v_{k-1}$ whose relative address (relative to $T^v_{k-1}$) $\emptyset.\sigma$ obeys $\xi(\sigma) = k-1$ by the set of tiles in $T^v_k$ whose relative addresses (relative to $T^v_k$) are $\emptyset.\sigma i$ where $i^- = \sigma^+$; and (ii) replacing each tile $t \in T^v_{k-1}$ whose relative address $\emptyset.\sigma$ obeys $\xi(\sigma) > k-1$ by $s^{-1}t$. Conversely, $\alpha T^v_k$ can be calculated by replacing each tile in $T^v_k$ whose relative addresses (relative to $T^v_k$) take the form $\emptyset.\sigma i$ where $i^- = \sigma^+$ for some fixed $\sigma$ with $\xi(\sigma) = k$, by the tile in $T^v_{k-1}$ whose relative address (relative to $T^v_{k-1}$) is $\emptyset.\sigma$. 

Remark 1. A wrong description, that gives the right idea in the case of rigid tilings, defined in Section 9.1. of how to calculate deflation and inflation of $T_k^v$ is the following. “$\alpha T_k^v$ can be calculated by replacing each set of tiles in $T_k^v$ that is equal to copy of $ET_0^v$ (for some $w$ and some isometry $E$) by $sEAS$, and replacing each and every other tile $t \in T_k^v$ by $st$. Similarly, $\alpha^{-1} T_{k-1}$ can be calculated by forming $s^{-1} T_{k-1}$ and then replacing each tile which is isometric to a copy of $A_w$ by a copy of $T_0^v$.” This description is wrong in general because there may occur spurious copies of $T_0^v$ in $T_k^v$. If the tiling is rigid then such spurious copies cannot occur.

In summary, inflation and deflation, represented by operators $\alpha$ and $\alpha^{-1}$ are well-defined on any tiling $T$ of the form $ET_k^v$ when $v, k$, and $E$ are known. In particular, we have the following result.

Theorem 13. Let $(F,G)$ be a tiling IFS. For all $\theta \in \Sigma^1$, $n \in [N]$, $k \in N_0$, with $E_{\theta|k} := f_{-\theta|k}s^{-\theta|k}$, we have the following identities

\[
\begin{align*}
\alpha^{a_1} \Pi(\theta) &= s^{a_1} f_{\theta|1}^{-1} \Pi(S \theta) \\
\alpha^{-a_2} \Pi(\theta) &= s^{-a_2} f_{\theta|0} \Pi(n \theta) \\
\Pi(S \theta) &= \alpha^{\xi(\theta|k)} E_{\theta|k}^{-1} \Pi(\theta)
\end{align*}
\]

Proof. This follows from $\Pi(\theta) = E_{\theta} T_{\theta|k}^{s^{\xi(\theta)}}$ where $E_{\theta} = f_{-\theta}s^{\xi(\theta)}$.

8.5. When is $ET_k^v$ recognizable? If, given the tiling $ET_k^v$, we can uniquely determine $E \in U$, $v \in V$, $k \in N_0$, then we can apply $\alpha$ without requiring that $E, v, k$ are known a priori.

Definition 10. We say that the canonical tilings $\{T_k^v\}$ are recognizable if the map $Z : V \times N_0 \times U \to \{ET_k^v : E \in U, v \in V, k \in N_0\}$ defined by $Z(v, k, E) = ET_k^v$ is bijective.

Lemma 4. Let $(F,G)$ be a tiling IFS.

(*) $A_v = Es^k A_v$ for some $E \in U, v \in V, k \in N_0, \Rightarrow E = Id, k = 0, \text{ and } v = w$, then the canonical tilings are recognizable.

Proof. If $ET_k^v = E'n'T_k^{v'}$, then $Es^{-k}A_v = E''s^{-k'}A_{v'}$, which implies $A_v = (s^k E^{-1} E'' s^{-k}) s^{-k'} A_{v'}$ which can be rewritten in the form $A_v = E' s^k A_{v'}$ where $v' = v'', k' = k - k'', E' = (s^k E^{-1} E'' s^{-k})$. So if (*) holds, then $(s^k E^{-1} E'' s^{-k}) = I, v' = v = v'', \text{ and } k' = 0$, which implies $E'' = E, v'' = v$, and $E'' = E$.

8.6. Isometric combinations of canonical tilings. Call a tiling $T$ an isometric combination of canonical tilings if it can be written in the form

\[T = \bigcup_{i \in I} E_i T_{k_i}^{v_i}\]

where $I$ is a countable index set, $v_i \in V, k_i \in N_0$ for all $i \in I$, and it is assumed that the $T_{k_i}^{v_i}$ are indexed. For example the tiling $\Pi(\theta)$ where $\theta$ is given is an isometric combination of canonical tilings for all $\theta \in \Sigma^1$. Inflation and deflation may not be well-defined when $T$ is represented as an isometric combination of canonical tilings. For example it can occur that $T = T_k^v = \bigcup_{i \in I} E_i T_{k_i}^{v_i}$ but $\alpha T \neq \bigcup_{i \in I} \alpha (E_i T_{k_i}^{v_i})$ as the following example shows.
Example 3.

\[ A_1 = f_1(A_1) \cup f_2(A_1) \cup f_3(A_3), \quad A_2 = f_4(A_1) \cup f_5(A_2) \]

\[ T_0^1 = \{ [0, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1] \}, \quad T_0^2 = \{ [2, \frac{5}{2}], [\frac{5}{2}, 3] \} \]

\[ T_1^1 = T_0^1 \cup ET_0^2, \quad Ex = x - 1 \]

\[ \alpha T_1^1 = T_0^1 \neq \alpha T_0^1 \cup \alpha ET_0^2 = T_0^1 \cup sE[0, 1] \]

Note that \( EsT_0^2 \subset T_0^1 \) where \( Es[2, 3] = [\frac{1}{2}, 1] \) and \( s = \frac{1}{2} \).

9. Rigid tiling IFSs

In this Section 9 we define the notion a rigid tiling IFS \((F, G)\) and of a rigid tiling \(T\). We extend the definitions of \(\alpha\) and \(\alpha^{-1}\) so that they act locally on individual tiles, regardless of their contexts as members of canonical tilings, in such a way that if \(T\) is a rigid tiling and \(T = \cup_{i \in I} E_i T_{k_i}^v\) with \(v_i \in V\) and \(k_i \in \mathbb{N}\) is an isometric combination, then

\[ \alpha T = \cup_{i \in I} \alpha (E_i T_{k_i}^v) = \cup_{i \in I} sE_i s^{-1} T_{k_i}^v \]

and similarly for \(\alpha^{-1}\) independently of the specific representation of \(T\) as an isometric combination.

9.1. Definitions of a rigid tiling and a rigid tiling IFS. Recall that \(U\) is a group of isometries on \(\mathbb{R}^M\) that contains the group of isometries on \(\mathbb{R}^M\) generated by the maps \(\{f_e : e \in E\} \cup \{f_0\}\) where \(f_0(x) := sx\) for all \(x \in \mathbb{R}^M\).

Definition 11. If \(P\) and \(Q\) are sets of subsets of \(\mathbb{R}^M\) we say “\(P\) meets \(Q\)”, to mean that \(P \cap Q \neq \emptyset\) and \((\cup P) \cap (\cup Q) = \cup (P \cap Q)\). We also say that “\(P\) is a copy of \(Q\)” to mean “there is \(E \in U\) such that \(P = \{p : p \in P\} = \{Eq : q \in Q\} = EQ\).”

For example, “\(T_k^v\) meets a copy of \(T_l^w\)” is shorthand for “there is \(E \in U\) such that \(T_k^v \cap ET_l^w \neq \emptyset\) and \(T_k^v \cap ET_l^w\) tiles \(s^{-k} A_v \cap s^{-l} A_w\)”.

Definition 12. The tilings \(T := \{\Pi(\theta) : \theta \in \Sigma^1\}\) and the tiling IFS \((F, G)\) are each said to be rigid (with respect to \(U\)) when the following three statements are true:

A(i) if \(E \in U\) and \(k \in \{0, 1, ..., a_{\text{max}} - 1\}\) are such that \(T_k^v\) meets \(Es^k T_{k}^v\) then \(E = Id, k = 0\), and \(v = w\);

A(ii) if \(E \in U, v, w \in V,\) and \(ET_0^v\) tiles \(A_w\) then \(E = Id\) and \(v = w\);

A(iii) \(A_w = Es^k A_v\) for some \(E \in U, v, w \in V, k \in \mathbb{N}_0, \Rightarrow E = Id, k = 0,\) and \(v = w\).

Definition 12 is a generalization of the definition of “strongly rigid” for the case \(|V| = 1\) in [16]. It is an extension and modification of the definition of rigid in [18].

For tiles with non-empty interiors, \(U\) is the group of translations on \(\mathbb{R}^M\), and \(a_{\text{max}} = 1\), rigidity is largely equivalent to recognizability [11] and to the unique composition property [38]. Rigidity extends these concepts to tilings involving more than one scaling factor, more general groups, and more general fractal tilings.

Lemma 5. Let the family of tilings \(T := \{\Pi(\theta) : \theta \in \Sigma^1\}\) and the tiling IFS \((F, G)\) be rigid. If \(s^k T_{k}^v\) meets \(ET_l^w\) for some \(k, l \in \mathbb{N}_0, v, w \in V, E \in U\), then \(k = 0\) and \(T_{k}^v \subset ET_l^w\).
PROOF. If \(s^k T_0^v\) meets \(ET_0^w\) then \(k = 0, E = Id, v = w\). In particular, if \(s^k T_0^v\) meets \(ET_0^w\) then \(k = 0, \text{and } T_0^v \subset ET_0^w\). Suppose that if \(s^k T_0^v\) meets \(ET_0^w\) then \(k = 0, \text{and } T_0^v \subset ET_0^w\), for all \(l = 0, 1, 2, \ldots L\).

If \(s^k T_0^v\) meets \(ET_{L+1}^w\), but does not meet any copy of \(T_0^v\) contained in \(ET_{L+1}^w\) we can apply \(T\) to \(ET_0^w\) and at the same time shrink \(s^k T_0^v\) without modification, yielding that \(s^{k+1} T_0^v\) meets \(T_0^w\) where \(E' = sE s^{-1} \in U\). This implies \(k = -1\) which is false. We conclude that \(s^k T_0^v\) meets a copy of \(T_0^v\) contained in \(ET_{L+1}^w\) which implies \(k = 0\) and \(T_0^v \subset ET_{L+1}^w\). \(\square\)

**Theorem 14.** If the family of tilings \(T := \{\Pi(\theta) : \theta \in \Sigma^1\}\) and the tiling IFS \((\mathcal{F}, \mathcal{G})\) are rigid then the following four statements are true.

- \(B(i)\) if \(E \in U, v, w \in V, \text{and } T_0^v \text{ meets } ET_0^w \text{ then } v = w \text{ and } E = Id\);
- \(B(ii)\) if \(E \in U, v, w \in V, \text{and } k, l \in N_0 \text{ are such that } T_k^v \text{ meets } ET_l^w \text{ then }\)
  
  \[
  \begin{align*}
  &\text{either } T_k^v \subset ET_l^w \text{ or } ET_l^w \subset T_k^v \\
  &B(iii) \text{ if } E \in U, v, w \in V, \text{and } ET_0^w \text{ tiles } A_w \text{ then } E = Id \text{ and } v = w; \\
  &B(iv) \text{ } A_w = E s^k A_v \text{ for some } E \in U, v \in V, k \in N_0, \Rightarrow E = Id, k = 0, \text{ and } v = w
  \end{align*}
  \]

- \(B(i)\) if \(|V| = 1 \text{ or } \text{if each } T_0^v \text{ possesses a tile isometric to } s^a A_w, \text{ then the two sets of conditions, } \{A(i), A(ii), A(iii)\} \text{ and } \{B(i), B(ii), B(iii), B(iv)\} \text{ are equivalent.}\)

- \(B(i)\) if \(|V| = 1 \text{ or } \text{if each } T_0^v \text{ possesses a tile isometric to } s^a A_w, \text{ then the two sets of conditions, } \{A(i), A(ii), A(iii)\} \text{ and } \{B(i), B(ii), B(iii), B(iv)\} \text{ are equivalent.}\)

**Proof.** Follows from Lemma 5. \(\square\)

**Corollary 3.** Let the family of tilings \(T := \{\Pi(\theta) : \theta \in \Sigma^1\}\) and the tiling IFS \((\mathcal{F}, \mathcal{G})\) be rigid. If \(T, \psi \in \Sigma^1, \Pi(\theta) \text{ meets } E \Pi(\varphi), \text{ then }\)

- \(\{A(i), A(ii), A(iii)\} \text{ and } \{B(i), B(ii), B(iii), B(iv)\} \text{ are equivalent.}\)

**9.2. Inflation and deflation of rigid tilings.** Let \(Q\) be the set of all tilings \(T\) that can be written in the form \(T = \bigcup_{i \in I} E_i T_{k_i}^v\) where \(i\) is a countable index set, \(E_i \in U, k_i \in N_0, \text{ and } v_i \in V \text{ for all } i \in I\). Let \(Q'\) be the set of all tilings \(T'\) that can be written in the form \(T = \bigcup_{i \in I} E_i T_{k_i}^v\) where \(i\) is a countable index set, \(E_i \in U, k_i \in N_0, \text{ and } v_i \in V \text{ for all } i \in I\).

The following definition extends the domains of \(\alpha\) and \(\alpha^{-1}\) to \(Q\) and \(Q'\) respectively, in the case of rigid tilings. It relies on the fact, assured by Lemma 5, that no “spurious copies” of any \(T_0^v\) can occur in any tiling in \(Q\).

**Definition 13.** Let \((\mathcal{F}, \mathcal{G})\) be a rigid tiling IFS. \(\textbf{Deflation } \alpha : Q \rightarrow Q'\) is defined by \(\alpha(T) = \{\alpha(t) : t \in T\} \text{ for all } T \in Q, \text{ where }\)

\[
\alpha(t) = \begin{cases} 
  sE A_v & \text{if } t \in ET_0^w \subset T \text{ for some } E \in U, v \in V, \\
  st & \text{otherwise}
\end{cases}
\]

\(ET_0^w\) is called the set of \(\textbf{partners}\) of \(t \in ET_0^w\). If \(t_1 \text{ and } t_2 \text{ are partners of } t\), then \(\alpha(t_1) = \alpha(t_2)\). \(\textbf{Inflation } \alpha^{-1} : Q' \rightarrow Q\) is defined by \(\alpha^{-1}(T) = \{\alpha^{-1}(t) : t \in T\} \text{ for all } T \in Q', \text{ where }\)

\[
\alpha^{-1}(t) = \begin{cases} 
  s^{-1} t & \text{if } t \neq Es A_v \text{ for any } E \in U, v \in V, \\
  ET_0^w & \text{if } t = Es A_v
\end{cases}
\]

for all \(T \in Q'\).

Conditions \(A(ii)\) and \(A(iii)\) ensures that inflation, represented by the operator \(\alpha^{-1}\), is well-defined on \(Q'\). Call a tile in any tiling in \(Q'\) which is isometric to \(sA_v\) for some \(v \in V\) a \(\textbf{large tile}\). To inflate a tiling \(T'\) in \(Q'\), first replace each large
tile in \(T'\) by the corresponding unique (by \(A(ii)\)) copy of \(sT^w_k\) (for all \(v\)), yielding a set of sets \(T'\), and then apply the simitude \(s^{-1}\) to \(T'\) to yield \(T \in \mathcal{Q}\). Similarly, deflation is well-defined, because by Lemma 4 no copies of \(s^kT^w_0\) with \(k > 0\) can occur in any \(T^w_0\).

Rigidity, in fact condition condition \(A(iii)\) which is the same as (*) in Lemma 4 ensures that, given the canonical tiling \(T^k_v\), we can infer the values of the indices \(v\) and \(k\). Rigidity ensures that canonical tilings are recognizable.

For rigid tilings \(\alpha : \mathcal{Q} \to \mathcal{Q}'\) and \(\alpha^{-1} : \mathcal{Q}' \to \mathcal{Q}\) are well-defined. Every copy of \(T^w_0\) in \(T^w_k\) is related via \(\alpha^{-1}\) to a large tile in \(T^w_{k-1}\). There is a one-to-one correspondence between the large tiles in \(T^w_{k-1}\) and copies of \(T^w_0\) in \(T^w_k\). In particular we find that \(\alpha\) and \(\alpha^{-1}\) in Definition 13 are consistent with the definition in Section 8.4. The following theorem says that inflation and deflation are well defined, in particular they behave well with respect to isometric combinations of tilings.

**Theorem 15.** If \((\mathcal{F}, \mathcal{G})\) is rigid, then the following statements are true for all \(E, E' \in \mathcal{U}, k, l \in \mathbb{N}, v, w \in \mathcal{V}\), and index sets \(\mathcal{I}\) and \(\mathcal{J}\),

\(\begin{align*}
(i) & \quad ET^w_i \subset T^w_k \iff sEs^{-1}A_v \in T^w_{k-1} \\
(ii) & \quad \alpha\text{ and }\alpha^{-1}\text{ in Definition 13 are consistent with the definition in Section 8.4} \\
(iii) & \quad \text{if } ET^w_i \subset E'T^w_i, \text{ then} \\
& \quad \alpha(ET^w_i) \subset \alpha(E'T^w_i) \quad \text{and} \quad \alpha^{-1}(ET^w_i) \subset \alpha^{-1}(E'T^w_i) \\
(iv) & \quad \text{if } \cup_{i \in \mathcal{I}} E_i \text{T}^v_{k_i} \in \mathcal{Q}, \text{ then} \\
& \quad \alpha(\cup_{i \in \mathcal{I}} E_i T^v_{k_i}) = \cup_{i \in \mathcal{I}} \alpha(E_i T^v_{k_i}) \quad \text{and} \quad \alpha^{-1}(\cup_{i \in \mathcal{I}} E_i T^v_{k_i}) = \cup_{i \in \mathcal{I}} \alpha^{-1}(E_i T^v_{k_i}) \\
(v) & \quad \text{if } \cup_{i \in \mathcal{I}} E_i \text{T}^v_{k_i} \in \mathcal{Q}, \text{ then} \\
& \quad \alpha(\cup_{i \in \mathcal{I}} E_i T^v_{k_i}) = \alpha(\cup_{j \in \mathcal{J}} E_j T^v_{k_j}) \quad \text{and} \quad \alpha^{-1}(\cup_{i \in \mathcal{I}} E_i T^v_{k_i}) = \alpha^{-1}(\cup_{j \in \mathcal{J}} E_j T^v_{k_j}) \\
(vi) & \quad \text{for all } \theta \in \Sigma^1, n \in [N], k \in \mathbb{N}_0, \text{with } E_{\theta|k} := f_{-\theta|k}s^\xi(\theta|k), \text{ the following identities hold} \\
& \quad \alpha^{a\theta}_\Pi(\theta) = s^{a\theta}f_{\theta^{-1}}\Pi(S\theta) \\
& \quad \alpha^{-a\theta}_\Pi(\theta) = s^{-a\theta}f_{\Pi(n\theta)} \\
& \quad \Pi(S^k\theta) = \alpha^{\xi(\theta|k)}E_{\theta|k}^{-1}\Pi(\theta)
\end{align*}\)

**Proof.** These statements can be checked with the aid of Theorem 13. \(\square\)

### 9.3. Characterization of isometric rigid tilings

Define for all \(k \in \mathbb{N}\) and \(v, w \in \mathcal{V}\)

\[\Lambda^v_k = \{ \sigma \in \Sigma^+ : \xi(\sigma) = k, \sigma^- = v, \sigma^+ = w \} \subset \Omega^v_k\]

**Theorem 16.** Let \((\mathcal{F}, \mathcal{G})\) be a rigid tiling IFS. For all \(k \in \mathbb{N}_0\) there is a bijection between \(\Lambda^v_k\) and the set of isometric copies of \(T^w_0\) contained in \(T^v_k\). The bijection is provided by the map \(H : \Lambda^v_k \to \mathcal{R}(H) \subset T^v_k\) defined by

\[H(\sigma) = s^k f_\sigma(T^w_0)\]

where \(\mathcal{R}(H)\) is the range of \(H\).
PROOF. (i) It is readily checked that $H(\Lambda_{k,w}^v) \subset T_k^w$.
(ii) Suppose $H(\sigma) = H(\omega)$ for $\sigma, \omega \in \Lambda_{k,w}^v$. Then $\xi(\sigma) = \xi(\omega) = k$, $\sigma^+ = w$, $\sigma^- = v$ and
\[ s^{-k}f_{\sigma}(T_0^w) = s^{-k}f_{\omega}(T_0^w) \Rightarrow f_{\sigma}(A_0) = f_{\omega}(A_0) \Rightarrow \sigma = \omega \]
(iii) Suppose that $ET_0^w \subset T_k^w$ is an isometric copy of $T_0^w$ that is contained in $T_k^w$. Then we need to show that $ET_0^w$ is in $R(H)$. We have
\[ \alpha ET_0^w \subset \alpha T_k^w \Rightarrow sE s^{-1}sA_w \in T_{k-1}^w \Rightarrow sE s^{-1}sA_w = s^{-k+1}f_{\sigma}(A_w) \]
for some $\sigma$ such that $\sigma^+ = w$, $\sigma^- = v$, $\xi(\sigma) = k$, because the r.h.s. must be a tile in $T_{k-1}^w$ congruent to $sA_w$. It follows that $E = s^{-k}f_{\sigma}$ where $\sigma \in \Lambda_{k,w}^v$ and so $H(\sigma) = ET_0^w \in R(H)$, because the any copy of $T_0^w$ in $ET_k^v$ must have arisen by application of $\alpha^{-1}$ to a copy of $sA_w$ in $T_{k-1}^w$. □

THEOREM 17. Let $(\mathcal{F}, \mathcal{G})$ be a rigid tiling IFS.
(i) If $\theta, \psi \in \Sigma_1^1$, $S^{p \theta} = S^q \psi$, $E = f_{-\theta|p}(f_{-\psi|q})^{-1}$, and $\xi(\theta|p) = (\psi|q)$, then $\Pi(\theta) = E \Pi(\psi)$ where $E$ is an isometry.
(ii) Let $(\mathcal{F}, \mathcal{G})$ be rigid, and let $\Pi(\theta) = E \Pi(\psi)$ where $E \in \mathcal{U}$ is an isometry, for some pair of disjunctive addresses $\theta, \psi \in \Sigma_1^1$. Then there are integers $p, q$ such that $S^{p \theta} = S^q \psi$, $E = f_{-\theta|p}(f_{-\psi|q})^{-1}$, $\theta|p = (\psi|q)$, and $\xi(\theta|p) = (\psi|q)$.

PROOF. Part (i) is readily checked.
Proof of (ii).
(A) Begin by choosing $L \in \mathbb{N}_0$ such that $\Pi(\theta|0) \cap E \Pi(\psi|0) \neq \emptyset$. Note that $\Pi(\theta|0) \subset E \Pi(\psi|0)$.
(B) Let $l \in \mathbb{N}_0$ with $l \geq L$. Using Corollary 3 we can choose $k = k_1$ so that
\[ (9.1) \]
$\Pi(\theta|k) \subset E \Pi(\psi|l) \subset \Pi(\theta|k+1)$
(C) Using Theorem 13 we can apply $\alpha^{\xi(\theta|k)}$ to $\Pi(\theta|k) \subset E \Pi(\psi|l)$ to obtain
\[ \alpha^{\xi(\theta|k)} \Pi(\theta|k) \subset \alpha^{\xi(\theta|k)} E \Pi(\psi|l) \]
Writing $w = (\theta|k)^+ = v = (\psi|l)^+$ and using the first part of Theorem 3 we now have
\[ s^{\xi(\theta|k)} f_{-\theta|k} T_w^w \subset s^{\xi(\theta|k)} E f_{-\theta|p}(f_{-\psi|q})^{-1} T_{\xi(\theta|l)-\xi(\theta|k)}^w \]

Now apply the Theorem 15 to conclude that there is $\sigma \in \Lambda_{\xi(\theta|k)-\xi(\theta|k)}^v$ with $\sigma^+ = v$ and $\sigma^- = w$ so that
\[ s^{-\xi(\psi|l)+\xi(\theta|k)} f_{-\theta|l}^{-1} E^{-1} f_{-\theta|k} f_{-\theta|k-1} T_w^w = s^{-\xi(\psi|l)+\xi(\theta|k-1)} f_{\sigma} T_0^w \]
This implies
\[ E = f_{-\theta|k} f_{\sigma}^{-1} \left( f_{-\theta|l}^{-1} \right)^{-1} \]
We also have $E \Pi(\psi|l) \subset \Pi(\theta|k+1)$ which, following the same steps, yields
\[ E = f_{-\theta|k+1} f_{\sigma} \left( f_{-\theta|l}^{-1} \right)^{-1} \]
for some $\tilde{\sigma} \in \Lambda_{\xi(\theta|k)-\xi(\theta|k)}^v$ where $x = \theta|k+1$, $y = \psi|l$ and $\tilde{\psi} = v$. Comparing the two expression for $E$ we conclude
\[ f_{-\theta|k} f_{\sigma} \left( f_{-\theta|l}^{-1} \right)^{-1} = f_{-\theta|k+1} f_{\sigma} \left( f_{-\theta|l}^{-1} \right)^{-1} \]
\[ \Rightarrow f_{-\theta|k} = f_{\sigma}^{-1} f_{\tilde{\sigma}}^{-1} \]
which implies either \( \tilde{\sigma} = \emptyset, \sigma = \theta_k, \) and \( v = w, \) or \( \sigma = \emptyset \) and \( \tilde{\sigma} = \theta_k \) and \( w = x. \) It follows that either \( E = f_{-\psi|k}(f_{-\psi|l})^{-1} \) or \( f_{-(\theta+1)|k}(f_{-\psi|l})^{-1}. \) That is, one or other of the two inclusion symbols in (9.1) can be replaced by an equality sign. It follows that either \( E = f_{-\psi|k}(f_{-\psi|l})^{-1} \) where \( \xi(\theta|k) = \xi(\psi|l) \) or \( f_{-(\theta+1)|k}(f_{-\psi|l})^{-1} \) where \( \xi(\theta + 1|k) = \xi(\psi|l). \)

(D) The rest of the proof follows from the arbitrarily large size of \( l. \) \( \square \)

See also [18], where a related result is proved for the case where the tiles have nonempty interiors.

**Corollary 4.** If \( (F,G) \) is rigid (with respect to \( \mathcal{U} \)) then \( \Pi(\theta) = \Pi(\psi) \) for some \( E \in \mathcal{U} \) and some disjunctive \( \theta \in \Sigma_{\infty} \) if and only if \( E = Id. \) In particular, if \( \mathcal{U} \) contains the group of Euclidean translations on \( \mathbb{R}^m, \) then \( \Pi(\theta) \) is aperiodic for all disjunctive \( \theta \in \Sigma_{\infty}. \)

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