PIECEWISE AFFINE APPROXIMATIONS FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. BV functions cannot be approximated well by piecewise constant functions, but we will show in this work that a good approximation is still possible with (countably) piecewise affine functions. In particular, this approximation is area-strictly close to the original function and the $L^1$-difference between the traces of the original and approximation functions on the mesh can be made arbitrarily small. Necessarily, the mesh needs to be adapted to the singularities of the BV function to be approximated, and consequently, the proof is based on a blow-up argument together with explicit constructions of the mesh. We also establish a $W^{1,1}$-error estimate for approximations by piecewise affine functions, which is used in the proof of BV-approximations and is also of independent interest.

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1. INTRODUCTION

Functions of bounded variation have important applications in many branches of mathematical physics, among them optimization [4] free-discontinuity problems [3] (this and the previous reference also contains good introductions to the theory of BV functions) and hyperbolic systems of conservation laws [10]. However, the theory of numerical approximations of such functions is not very well developed and indeed, because of the nature of singularities of BV functions, deeper analytical insight is required. In this work we aim to contribute to such insight by considering the basic question whether BV function can be approximated well by (countably) piecewise affine functions. Before we start our investigation, though, we remark that any good approximation by piecewise-constant functions must fail, even though such functions of course lie in BV, this will be shown in Proposition 3 below.

For $d > 1$ an integer, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. It is well-known that the space of piecewise affine functions with countably many pieces is (norm-)dense in $W^{1,p}(\Omega;\mathbb{R}^m)$, where $1 \leq p < \infty$. Such a result is interesting for example in proving lower semicontinuity theorems, see for example [9, 13, 14]. Such an approximation result is also easily seen to be true in the space $\text{BV}$ of functions of bounded variation, if we switch from norm convergence to the so-called strict convergence (see below).

For some applications, however, one needs a better approximation result, in which also the trace differences over the boundaries of the mesh are controlled. This need becomes immediately obvious for example in the analysis of energy functionals in free-discontinuity problems, which typically involve a term measuring jumps over the discontinuity surface (see for example [8]), and where, to produce good approximants, one needs to control the "jump error".
Consequently, the purpose of this note is to prove the following approximation theorem, related to a weaker result in [17] and numerical and analytical studies [2,5,6,21,25].

**Theorem 1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $u \in BV(\Omega; \mathbb{R}^m)$. Then, for every $\varepsilon > 0$ there exist a countable family $\mathcal{R}$ of (rotated) rectangles and simplices $R \subset \Omega$ and $v \in W^{1,1}(\Omega; \mathbb{R}^m)$ with the following properties:

(i) The sets in $\mathcal{R}$ are non-overlapping and $(L^d + |Du|)(\Omega \setminus \bigcup \mathcal{R}) = 0$.
(ii) For each $R \in \mathcal{R}$ the restriction $v|_R$ is affine.
(iii) $\|u - v\|_{L^1(\Omega; \mathbb{R}^m)} + |(Du)(\Omega) - (Dv)(\Omega)| < \varepsilon$.
(iv) $\sum_{R \in \mathcal{R}} \int_{\partial R} |u - v| \, d\mathcal{H}^{d-1} < \varepsilon$.
(v) $v|_{\partial \Omega} = u|_{\partial \Omega}$.

Here, if $\mu = \frac{du}{dx} \, L^d + \mu^s$ is the Lebesgue–Radon–Nikodým decomposition of the measure $\mu$, we define the measure (related to the area functional)

$$\langle \mu \rangle(A) := |(\mu, L^d)|(A) = \int_A \sqrt{1 + \left| \frac{d\mu}{dL^d} \right|^2} \, dx + |\mu^s|(A)$$

for every Borel set $A \subset \Omega$.

A sequence $(u_j) \subset BV(\Omega; \mathbb{R}^m)$ is said to converge **strictly** to $u \in BV(\Omega; \mathbb{R}^m)$ if $\|u_j - u\|_{L^1} \to 0$ and $|Du_j|(\Omega) \to |Du|(\Omega)$ as $j \to \infty$; it is said to converge **(\cdot)-strictly** if even $(Du_j)(\Omega) \to (Du)(\Omega)$. The latter convergence is the “natural” replacement for the strong convergence in $BV(\Omega; \mathbb{R}^m)$ since smooth functions are (\cdot)-strictly but not strongly dense in $BV(\Omega; \mathbb{R}^m)$. In fact, we even have the following stronger result (see Lemma 1 of [15] or Lemma B.1 in [7] for a proof under additional assumptions):

**Lemma 2.** Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded and open set (no regularity assumption is imposed on $\partial \Omega$) and let $u \in BV(\Omega; \mathbb{R}^m)$. There exists a sequence $(v_j) \subset (W^{1,1} \cap C^\infty)(\Omega; \mathbb{R}^m)$ such that

$$v_j \to u \quad \text{\((\cdot)-strictly\)} \quad \text{and} \quad v_j - u \in W^{1,1}_0(\Omega; \mathbb{R}^m) \quad \text{for all } j \in \mathbb{N}.$$ 

If $u \in W^{1,1}(\Omega; \mathbb{R}^m)$, then we can in addition arrange that $v_j \to u$ strongly in $W^{1,1}(\Omega; \mathbb{R}^m)$.

Many reasonable integral functionals with linear growth are continuous with respect to the (\cdot)-strict convergence by Reshetnyak’s continuity theorem [22] and extensions, see for example Theorem 4 and the appendix of [16]; lower semicontinuity in this situation is discussed in [11]. For some special examples of positively 1-homogeneous integrands see [20].

Thus, the main result of this paper establishes density of (countably) piecewise affine functions in BV in the strict sense and such that additionally conditions (iv), (v) are satisfied. We remark that it is easy to satisfy conditions (i)–(iii) of Theorem 1 by simply mollifying $u$, which gives a strictly close smooth function (by Lemma 2) and then strongly approximating this smooth function by a piecewise affine function. Unfortunately, on this route one loses control of the sum of integrals in (iv), and it is precisely this condition, which is important for instance in applications to free-discontinuity problems.

In the course of the proof of the above theorem, we will rely on a result about approximations in $W^{1,1}$, which we factor out for its independent interest, see Theorem 7. It tells us that in $W^{1,1}$, we can prove actual approximation error bounds for any mesh, with the error
bound depending only the $L^1$-modulus of continuity of the weak gradient of the approximated function and regularity/uniformity constants of the mesh. Hence, in norm-compact subsets of $W^{1,1}$ we get a uniform rate of convergence.

We finally point out that the corresponding result for approximation by pure jump functions, that is, (countably) piecewise constant functions is not true—this also gives another reason why condition (iv) in Theorem 1 is of interest:

**Proposition 3.** Let $m \in \mathbb{N}$ be a natural number and assume that $u \in C_c^\infty(B(0,1),\mathbb{R}^m)$ is not identically zero. Then there can be no sequence of piecewise constant maps $u_j \in BV(B(0,1),\mathbb{R}^m)$ such that $u_j \to u$ $(\cdot)$-strictly in BV.

**Proof.** Let $F : \mathbb{R}^{m \times d} \to \mathbb{R}$ be any continuous integrand satisfying $F \equiv 0$ on $\nabla u(B(0,1))$, which is a bounded set by assumption, and $F(z) = |z|$ for large values of $|z|$, hence $F^\infty \equiv 1$. If $u_j$ were piecewise constant BV functions such that $u_j \to u$ $(\cdot)$-strictly in BV, then by Reshetnyak’s continuity theorem [22] (also see [24] and the appendix of [16]), we would have

$$|Du_j| \to |Du|$$

and, since $Du_j = D^s u_j$,

$$|Du_j| = F^\infty \left( \frac{dD^s u_j}{d|D^s u_j|} \right) |D^s u_j| \to F \left( \frac{dDu}{d|Du|} \right) E^d = 0,$$

where both convergences are weak$^*$ in $C^0_0(B(0,1))^*$. But this is clearly impossible because $Du \neq 0$. 

For the usual strict convergence, consider the continuous and positively 1-homogeneous integrand

$$F : \mathbb{R}^{2 \times 2} \to \mathbb{R}, \quad F(A) := |\det A|^{1/2}.$$  

Then, for any $u_j \to u$ strictly in $BV(\mathbb{R}^2;\mathbb{R}^2)$, Reshetnyak’s (original) continuity theorem implies

$$F \left( \frac{dDu_j}{d|Du_j|} \right) |Du_j| \to F \left( \frac{dDu}{d|Du|} \right) |Du| =: F(Du).$$

However, if the $u_j$’s are pure jump functions, then rank $\frac{dDu_j}{d|Du_j|} \leq 1$ $|Du_j|$-almost everywhere, but $F$ is zero on the rank-one cone, and hence $F(Du) = 0$ as a measure would follow, which clearly is not generally true. Thus it is not possible to approximate $u$ with $F(Du) \neq 0$ by piecewise constant functions.

This observation should be contrasted with the case of real-valued BV-functions (corresponding to $m = 1$) and the usual strict convergence, where a discretisation of the coarea formula allows approximation by piecewise constant functions, see [26, Theorem 3].

Finally, we note the following connection to quasiconvexity theory: Assume for a moment that we could find a piecewise constant sequence $u_j \to u \in BV_0(\Omega;\mathbb{R}^m)$ strictly ($m \geq 2$ and zero boundary values) and that there exists a positively 1-homogeneous rank-one convex, but not quasiconvex function $F : \mathbb{R}^{m \times d} \to \mathbb{R}$. Then, by Reshetnyak’s theorem and the fact that $|D^s u_j|$-a.e. it holds that rank $\frac{dD^s u_j}{d|D^s u_j|} = 1$,

$$\int_\Omega F \left( \frac{dDu}{d|Du|} \right) d|Du| = \lim_{j \to \infty} \int_\Omega F \left( \frac{dD^s u_j}{d|D^s u_j|} \right) d|D^s u_j| \geq \lim_{j \to \infty} |D^s u_j|(\Omega) \cdot F(0).$$

But then $F$ must be quasiconvex as well, a contradiction!
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2. Approximation on a triangulation

Let \( \tau \) be a \( d \)-dimensional simplex in \( \mathbb{R}^d \), that is, \( \tau \) is the convex hull of \( d + 1 \) affinely independent points \( v_0, \ldots, v_d \) of \( \mathbb{R}^d \) (the vertices of \( \tau \)). Each point \( x \in \tau \) admits a unique representation (the barycentric representation) as a convex combination of the vertices of \( \tau \):

\[
x = \sum_{j=0}^{d} \lambda_j(x)v_j,
\]

where \( \lambda_j : \tau \to [0, 1], \) \( j = 1, \ldots, d \), are affine functions with \( \sum_{j=0}^{d} \lambda_j \equiv 1 \). If \( f : \tau \to \mathbb{R}^m \) is a continuous mapping, then the Lagrange interpolation of \( f \) on \( \tau \) is the (unique) affine mapping \( a : \mathbb{R}^d \to \mathbb{R}^m \) that agrees with \( f \) at the vertices of \( \tau \). Using the \( \lambda_j \)'s, it can be written as

\[
a(x) = \sum_{j=0}^{\infty} \lambda_j(x)f(v_j).
\]

Lemma 4. There exists a constant \( C = C(d,m) \), only depending on the dimensions, with the following property: Assume that \( f : \tau \to \mathbb{R}^m \) is a \( C^2 \) mapping with Lagrange interpolant \( a \) on \( \tau \). Then,

\[
\int_{\tau} |\nabla f - \nabla a| \, dx \leq C(\text{diam} \tau)^d \sum_{j=0}^{d} \int_{\tau} \frac{|\nabla^2 f(x)|}{|x - v_j|^{d-1}} \, dx.
\]

Proof. We start from the formula

\[
\sum_{j=0}^{d} (x - v_j) \otimes \nabla \lambda_j(x) = -I \in \mathbb{R}^{d \times d},
\]

which is a consequence of \( \sum_j \nabla \lambda_j \equiv 0 \). Thus,

\[
\nabla f(x) - \nabla a(x) = -\nabla f(x) \left( \sum_{j=0}^{d} (x - v_j) \otimes \nabla \lambda_j \right) + \sum_{j=0}^{d} (f(x) - f(v_j)) \otimes \nabla \lambda_j
\]

\[
= \sum_{j=0}^{d} \left( \int_{0}^{1} (\nabla f(v_j + t(x - v_j)) - \nabla f(x)) \cdot (x - v_j) \, dt \right) \otimes \nabla \lambda_j
\]

\[
= \sum_{j=0}^{d} \left( \int_{0}^{1} \int_{0}^{1} (x - v_j)^T \nabla^2 f(v_j + s(x - v_j)) (x - v_j) \, ds \, dt \right) \otimes \nabla \lambda_j
\]

Consequently, we find

\[
\int_{\tau} |\nabla f(x) - \nabla a(x)| \, dx
\]

\[
\leq \sum_{j=0}^{d} \int_{\tau} |\nabla \lambda_j| \int_{0}^{1} |\nabla^2 f(v_j + s(x - v_j))| \cdot |x - v_j|^2 \, ds \, dx
\]

\[
= \sum_{j=0}^{d} \int_{0}^{1} \int_{0}^{1} |\nabla^2 f(v_j + s(x - v_j))| \cdot |x - v_j|^2 \, dH^{d-1}(x) \, ds \, dh, \quad (2.1)
\]
where we used (an elementary version of) the coarea formula for the last equality. Now, for fixed \( h \in (0, 1) \) we change variables via 
\[
y = \Phi(x, s) := v_j + s(x - v_j),
\]
where \( (x, s) \in \{ \lambda_j = h \} \times (0, 1) \) and \( y \in \tau_h := \tau \cap \{ \lambda_j \geq h \} \), for which we can estimate 
\[
d\mathcal{H}^{d-1}(x) \, ds = \frac{dy}{|\det D\Phi|} \sim \frac{dy}{s^{d-1}h(diam \tau)} \sim \frac{(diam \tau)^{d-2}}{|y - v_j|^{d-1}h} \, dy,
\]
since \( (diam \tau)s \sim |y - v_j| \). Furthermore, \( |x - v_j| \sim (diam \tau)h \) and hence 
\[
\int_0^1 \int_{\{\lambda_j = h\}} |\nabla^2 f(v_j + s(x - v_j))| \cdot |x - v_j|^2 \, d\mathcal{H}^{d-1}(x) \, ds \\
\leq C(diam \tau)^2 \int_0^1 \int_{\{\lambda_j = h\}} |\nabla^2 f(v_j + s(x - v_j))| \cdot h^2 \, d\mathcal{H}^{d-1}(x) \, ds \\
\leq C(diam \tau)^d \int_{\tau_h} \frac{|\nabla^2 f(y)|}{|y - v_j|^{d-1}} \, h \, dy.
\]
Plugging this in (2.1), we conclude 
\[
\int_{\tau} |\nabla f(x) - \nabla a(x)| \, dx \leq C(diam \tau)^d \sum_{j=0}^d \int_{\tau_h} \frac{|\nabla^2 f(y)|}{|y - v_j|^{d-1}} \, h \, dy \, dh \\
\leq C(diam \tau)^d \sum_{j=0}^d \int_{\tau} \frac{|\nabla^2 f(y)|}{|y - v_j|^{d-1}} \, dy.
\]
This is the claim. \( \square \)

Next we consider triangulations (we here always understand triangulations to be possibly infinite).

**Definition 5.** A triangulation of \( \Omega \subset \mathbb{R}^d \) is a family \( \mathcal{T} = \bigcup_{j \in \mathbb{N}} \tau_j \) with 
(i) \( \tau_j = a_j + M_j \tau_0 \) for a fixed reference simplex \( \tau_0 \subset \mathbb{R}^d \) and some \( a_j \in \mathbb{R}^d \), \( M_j \in \text{GL}(d) \).
(ii) \( \Pi = \bigcup_{j \in \mathbb{N}} \tau_j \).
(iii) \( \tau_j \cap \tau_k = \emptyset \) for all \( j \neq k \).

The triangulation \( \mathcal{T} \) is called regular if there exists a constant \( \alpha > 0 \) with 
\[
\alpha^{-1} \leq |\det M_j| \leq \alpha \quad \text{for all} \ j \in \mathbb{N}.
\]

We now have the following lemma:

**Lemma 6.** Let \( u \in BV(D; \mathbb{R}^m) \), where \( D \subset \mathbb{R}^d \) is a bounded polygonal domain. Furthermore assume:

(i) \( \mathcal{T} = \bigcup_{j \in \mathbb{N}} \tau_j \) is a (possibly infinite) regular triangulation of \( D \).
(ii) \( \mathcal{T} \) satisfies a bounded overlap property, that is, there exists a constant \( K \in \mathbb{N} \) such that every \( x \in D \) lies in at most \( K \) different sets \( \tau + B(0, diam \tau) \), where \( \tau \in \mathcal{T} \).

Then, there exists a piecewise affine function \( a : D \rightarrow \mathbb{R}^m \) with respect to the triangulation \( \mathcal{T} \) and a constant \( C = C(d, m) \) such that 
\[
\int_{D} |a| \, dx \leq C \int_{D} |u| \, dx \quad \text{and} \quad \int_{D} |\nabla a| \, dx \leq C|Du|(D). \tag{2.2}
\]

**Proof.** We only show the second inequality, the first is similar. We also implicitly consider \( u \) to be extended by a constant to \( \tilde{u} \in BV(D + B(0, R) ; \mathbb{R}^m) \) such that \( |Du|(D + B(0, R)) \leq C|Du|(D) \), where \( R = \max_j diam \tau_j \); this is possible for example by Lemma 9.5.5 in [18].
Apply Lemma 4 to the regularized mapping \( f = \varphi_{\varepsilon} \ast u \), where \( \varphi \in C_c^\infty(\mathbb{R}^d) \) with \( \text{supp} \varphi \subset B(0,1) \) and \( (\varphi)_\varepsilon(x) := \varepsilon^{-d} \varphi(x/\varepsilon) \) for \( \varepsilon > 0 \) to be chosen later. For integers \( 1 \leq k, l \leq d \) we have
\[
\partial_k f = \frac{1}{\varepsilon} (\partial_k \varphi)_\varepsilon \ast (\partial_k u - z) \quad \text{for any } z \in \mathbb{R}^m.
\]

Let \( \tau \) be a fixed simplex in the triangulation \( \mathcal{T} \). Consequently, we may estimate using Lemma 4,
\[
\int_\tau |\nabla f(x) - \nabla a(x)| \, dx \\
\leq \frac{C}{\varepsilon} \sum_{j=0}^d (\text{diam } \tau)^d \int_\tau \int_{B(0,\varepsilon)} \frac{|(\nabla \varphi)_\varepsilon(y) \cdot |\nabla u(x-y) - z|}{|x-v_j|^{d-1}} \, dy \, dx \\
\leq \frac{C}{\varepsilon} \sum_{j=0}^d (\text{diam } \tau)^d \varepsilon^{-d} \|
abla \varphi\|_{L^\infty} \int_\tau \int_{B(0,\varepsilon)} \frac{|\nabla u(x-y) - z|}{|x-v_j|^{d-1}} \, dy \, dx \\
\leq \frac{C}{\varepsilon^{d+1}} \|
abla \varphi\|_{L^\infty} (\text{diam } \tau)^d \sum_{j=0}^d \int_{\tau + B(0,\varepsilon)} |\nabla u(w) - z| \, \int_{B(0,\varepsilon)} \frac{dy}{|w+y-v_j|^{d-1}} \, dw.
\]

Here we remark that all constants can be chosen independently of \( \tau \), because the triangulation is regular. Taking \( \varepsilon = \text{diam } \tau \), we note
\[
\int_{B(0,\varepsilon)} \frac{dy}{|w+y-v_j|^{d-1}} \leq \int_{B(v_j,3\varepsilon)} \frac{dy}{|w+y-v_j|^{d-1}} = \int_{B(0,3\varepsilon)} \frac{dx}{|x|^{d-1}} = 3\omega_d \varepsilon,
\]
and so,
\[
\int_\tau |\nabla f - \nabla a| \, dx \leq C \inf_z \int_{\tau + B(0,\text{diam } \tau)} |\nabla u - z| \, dx,
\]
hence, setting \( z = 0 \) and summing these integrals over all simplices \( \tau \) using the bounded overlap property of the triangulation,
\[
\int_D |\nabla u| \, dx \leq C \int_{D+B(0,\text{diam } \tau)} |\nabla u| + |\nabla f| \, dx,
\]
which immediately implies the claim. \( \square \)

Finally, we remark that for \( W^{1,1} \)-functions \( u \), we also have convergence to the original function on uniform regular meshes and we can even quantify the speed of convergence in terms of the \( L^1 \)-modulus of continuity for \( \nabla u \), defined as
\[
\omega(h) := \sup_{|w|\leq h} \int_D |u(x+w) - u(x)| \, dx.
\]
The result then is:

**Theorem 7.** In the situation of Lemma 4 assume additionally that \( u \in W^{1,1}(\Omega; \mathbb{R}^m) \) and that the triangulation \( \mathcal{T} \) is uniform, that is, there exists a constant \( \gamma \in (0,1) \) such that
\[
\gamma k \leq \text{diam } \tau \leq k \quad \text{for all } \tau \in \mathcal{T},
\]
where
\[
k := \sup_{\tau \in \mathcal{T}} \text{diam } \tau.
\]
Then, there exists a constant $C$, only depending on the dimensions and the parameter of regularity of the mesh $\mathcal{T}$, such that

$$
\int_D |\nabla u - \nabla a| \, dx \leq \frac{C}{\gamma^d} \omega(3k).
$$

It is well-known that $\omega(h) \to 0$ as $h \to 0$ (proof by approximation with smooth functions), but of course the speed of convergence is highly dependent on $u$, so in general there is no “rate” of convergence. However, by the Fréchet–Kolmogorov characterization of compact sets in $W^{1,1}$, see for example [12], there is a uniform rate of convergence for all functions $u$ in a fixed compact subset of $W^{1,1}(\Omega; \mathbb{R}^m)$.

**Proof.** We abbreviate $\tau^+ := \tau + B(0, \text{diam } \tau)$ and plug $z := (\nabla u)_{\tau^+} := \int_{\tau^+}$ into (2.3). Then we estimate

$$
\int_\tau |\nabla f - \nabla a| \, dx \leq C \int_{\tau^+} |\nabla u - (\nabla u)_{\tau^+}| \, dx
\leq C \int_{\tau^+} \int_{\tau^+} |\nabla u(x) - \nabla u(y)| \, dy \, dx
\leq C \int_{\tau^+} \int_{B(0, \text{diam } \tau^+)} |\nabla u(x) - \nabla u(x + w)| \, dw \, dx
\leq C \int_{B(0, \text{diam } \tau^+)} \int_{\tau^+} |\nabla u(x) - \nabla u(x + w)| \, dx \, dw.
$$

Thus, summing over all $\tau \in \mathcal{T}$ and using the bounded overlap property of the system $\{\tau^+\}_{\tau \in \mathcal{T}}$,

$$
\int_D |\nabla f - \nabla a| \, dx \leq \frac{C}{\gamma^d k^d} \int_{B(0, 3k)} \int_D |\nabla u(x) - \nabla u(x + w)| \, dx \, dw \leq \frac{C}{\gamma^d} \omega(3k).
$$

Using the fact that $\|\nabla u - \nabla f\|_{L^1(D; \mathbb{R}^m)}$ can be made arbitrarily small, this finishes the proof. \(\square\)

From the preceding theorem we immediately get convergence with a rate if $\nabla u$ is more regular: For example, if $u \in W^{2,1}(D; \mathbb{R}^m)$, then we have convergence with a linear rate (on regular uniform meshes).

3. **Proof of the main result**

We first establish the following local lemma:

**Lemma 8.** Let $Q$ be an open (possibly rotated) cube in $\mathbb{R}^d$, let $Q_0 \subset\subset Q$ be a concentric similar open subcube, and $u \in \text{BV}(Q; \mathbb{R}^m)$. We further assume:

(i) The inner cube $Q_0$ is the disjoint union of a finite number of similar (closed) rectangles $S_1, \ldots, S_m$ that are translations of one another along a single axis parallel to one of the sides of $Q_0$, and with the length $\eta$ of the short side of the $S_j$, it holds that $\text{dist}(\partial Q, Q_0) = \eta$ (like in Figure 7).
\( \sum_{k=1}^{n} \int_{\partial S_k} |u - w| \, d\mathcal{H}^{d-1} < \varepsilon \| Du \|_1(Q). \)

Then, on \( A := Q \setminus Q_0 \) there exists a countably piecewise affine mapping \( v \in W^{1,1}(A; \mathbb{R}^m) \) satisfying \( v = u \) on \( \partial Q \), \( v = w \) on \( \partial Q_0 \), and

\[
\| v \|_{L^1(A; \mathbb{R}^m)} < C \| u \|_{L^1(A; \mathbb{R}^m)} \quad \text{and} \quad (Dv)(A) < C(L^d + \| Du \|_1(A)). \tag{3.1}
\]

Moreover, if \( T \) is the triangulation of \( A \) corresponding to \( v \), then

\[
\sum_{\tau \in T} \int_{\partial \tau} |u - v| \, d\mathcal{H}^{d-1} \leq C \| Du \|_1(A). \tag{3.2}
\]

Here, \( C = C(d,m) \) is a dimensional constant.

**Proof of Lemma 8.** We start by constructing a triangulation \( T = \bigcup_{j \in \mathbb{N}} \tau_j \) of \( A \) as in Figure 1. This triangulation is of Whitney-type towards the outer boundary \( \partial Q \),

\[
\text{diam } \tau \sim \text{dist}(\tau, \partial Q).
\]

Our \( T \) also has the property that the simplices match the elements \( S_1, \ldots, S_n \) at the inner boundary \( \partial Q_0 \), meaning that for each simplex \( \tau \in T \) we have \( \tau \cap \partial Q_0 \subset \partial S_k \) for some \( k \).

At this stage we invoke Lemma 6. This allows us to find \( v \in W^{1,1}(A; \mathbb{R}^m) \) with \( v = u \) on \( \partial Q \), \( v = w \) on \( \partial Q_0 \) and such that (3.1) holds. For the latter note that \( |1 - |A||^2 - |A| \leq 1 \), hence the second statement in (3.1) follows from (2.2).
Now for \( r > 0 \) and \( \tau \in \mathcal{T} \) we let \( \tau^{(r)} \) denote the simplex obtained by dilating \( \tau \) by \( r \) about its centre of mass. In view of our construction of \( \mathcal{T} \), we can find an absolute \( r = r(c, d) > 1 \), such that for each \( \tau \in \mathcal{T} \), a substantial fraction of \( \tau^{(r)} \) touches the outside of \( Q \); more precisely, we require that \( \mathcal{L}^d(\tau^{(r)} \setminus Q) > 2\delta \mathcal{L}^d(\tau) \) for all \( \tau \in \mathcal{T} \) and for some constant \( \delta > 0 \). We define for each \( \tau \in \mathcal{T} \), \( \tau^* := \tau^{(r)} \setminus Q_0 \). Then, 
\[
\mathcal{L}^d(\tau^* \setminus Q) = \mathcal{L}^d(\tau^{(r)} \setminus Q) > 2\delta \mathcal{L}^d(\tau^*) \quad \text{for all } \tau \in \mathcal{T}.
\]

For checking (3.2) we note that we can extend \( u - v \) to \( \mathbb{R}^d \) by zero, and that \( D(u - v) = Dv - Du \) for this extension since \( v = u \) on \( \partial Q \). Since \( u - v \equiv 0 \) on \( \tau^* \setminus Q \) we have by standard Sobolev and Poincaré inequalities (see for instance Theorem 4.4.2 in [27], the specific form of the constant can be derived by a simple scaling argument as in Remark 3.50 in [3])
\[
\int_{\partial \tau} |u - v| \, d\mathcal{H}^{d-1} \leq C \left( (\text{diam } \tau)^{-1} \int_{\tau} |u - v| \, dx + |D(u - v)||(\tau) \right)
\]
\[
\leq C \left( (\text{diam } \tau)^{-1} \int_{\tau} |u - v| \, dx + (|Du| + |Dv|)(\tau^* \cap Q) \right)
\]
\[
\leq C(\text{diam } \tau^*, 1)(|Du| + |Dv|)(\tau^* \cap Q)
\]
\[
\leq C(|Du| + |Dv|)(\tau^* \cap Q).
\]

Consequently, we get using (3.1), the bounded overlap property of the system \( \tau^*, \tau \in \mathcal{T} \), and the result of Lemma 6 (in particular the fact that its constant is universal) that
\[
\sum_{\tau \in \mathcal{T}} \int_{\partial \tau} |u - v| \, d\mathcal{H}^{d-1} \leq C \sum_{\tau \in \mathcal{T}} (|Du| + |Dv|)(\tau^* \cap Q)
\]
\[
\leq C(|Du| + |Dv|)(A)
\]
\[
\leq C|Du|(A).
\]

Of course, the constant \( C \) depends on the \( r \) we chose, but since the geometry of the triangulation is fixed, this can be chosen absolutely. \( \square \)

We can now show our main result.

**Proof of Theorem 7** We introduce the following notation: Let \( C = (-1, 1)^d \) and for \( x_0 \in \mathbb{R}^d, r > 0 \) set \( Q(x_0, r) := x_0 + rC \). For every unit vector \( n \in \mathbb{S}^{d-1} \) select a rotation \( P \in \text{SO}(d) \) with \( P_e_1 = n \), where \( e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^d \). Let \( Q_n(x_0, r) := x_0 + rP(C) \), so that \( Q_n(x_0, r) \) is an open cube with center \( x_0 \), sidelength \( 2r \), and two faces orthogonal to \( n \).

**Step 1.** For \( L^d \)-almost every \( x_0 \in \Omega \) and \( 0 < r < \text{dist}(x_0, \partial \Omega)/\sqrt{d} \) put
\[
u_r(y) := \frac{u(x_0 + ry) - \bar{u}(x_0)}{r}, \quad y \in C,
\]
where \( \bar{u}(x_0) \) is the value of the precise representative of \( u \) at \( x_0 \) (for this and other properties of BV-functions see [3]). We recall that for \( L^d \)-almost all \( x_0 \in \Omega \), \( u \) is approximately differentiable at \( x_0 \) and so, as \( r \downarrow 0 \),
\[
u_r \to u_0 \quad (\cdot)\text{-strictly in } BV(C; \mathbb{R}^m) \quad \text{with } u_0(x) := \nabla u(x_0)x, \quad x \in C,
\]
with the approximate gradient \( \nabla u(x_0) \) of \( u \) at \( x_0 \). The fact that one can indeed choose a sequence \( r_n \downarrow 0 \) such that \( u_r \to u_0 \quad (\cdot)\text{-strictly} \) follows for example from Lemma 3.1 in [23] about the construction of strictly converging blow-ups, applied to the measure \( (Du, L^d) \). Since the trace operator is strictly continuous, we also have that
\[
\int_{\partial C} |u_r - u_0| \, d\mathcal{H}^{d-1} \to 0 \quad \text{as } r \downarrow 0;
\]

(3.4)
see for instance [16], pp. 53–54, and the references given there. Let us denote the set of such points \( x_0 \in \Omega \) by \( G_1 \).

Now, for

\[
 w(x) = \tilde{u}(x_0) + \nabla u(x_0) \cdot (x - x_0), \quad x \in Q(x_0, r),
\]

from (3.3), (3.4) we get by a change of variables that there exists \( 0 < r(x_0) < 1 \) such that for \( r < r(x_0) \) we have

\[
\int_{Q(x_0, r)} |u - \tilde{u}| \, dx < \varepsilon r \mathcal{L}^d(Q(x_0, r)) \\
|\langle Du \rangle(Q(x_0, r)) - \langle Dw \rangle(Q(x_0, r))| < \varepsilon \mathcal{L}^d(Q(x_0, r)),
\]

\[
\int_{\partial Q(x_0, r)} |u - \tilde{u}| \, d\mathcal{H}^{d-1} < \varepsilon \mathcal{L}^d(Q(x_0, r)).
\]

Step 2. Next, for \( |D^s u| \)-almost all \( x_0 \in \Omega \), henceforth fixed, we have \( \alpha_r := \frac{|Du(Q_n(x_0, r))|}{r^d} \to \infty \) as \( r \downarrow 0 \), and, defining

\[
u_r(y) := \frac{u(x_0 + ry) - u_{x_0,r}}{\alpha_r}, \quad y \in Q_n(0, 1),
\]

where \( u_{x_0,r} = \int_{Q_n(x_0, r)} u \, dx \), by Lemma 3.1 of [23] we can find a sequence of \( r \)'s going to 0 (not specifically labelled) such that

\[
u_r \to \nu_0 \text{ strictly in } \text{BV}(Q_n(0, 2); \mathbb{R}^n)
\]

and

\[
u_0(y) = b \psi(y \cdot n), \quad y \in Q_n(0, 2),
\]

where \( \psi: (-1, 1) \to \mathbb{R} \) is increasing and bounded. The fact that the blow-up limit \( u_0 \) can indeed be chosen one-directional can be proved using the rigidity result in Lemma 3.2 of [23] (also see Remark 5.8 in loc. cit.), or using Alberti’s rank one theorem [1], see Theorem 3.95 in [3].

It is not hard to see that we additionally may assume that the sequence of \( r \)'s is chosen such that \( |Du|((\partial Q_n(x_0, r) \cap \Omega) \setminus (G_1 \cup G_2)) = 0 \); we again refer to [16], pp. 54–55, for these assertions. Denote the set of such points \( x_0 \in \Omega \) by \( G_2 \), and observe that \( (\mathcal{L}^d + |Du|)(\Omega \setminus (G_1 \cup G_2)) = 0 \).

Let \( N \) satisfy

\[
N > \frac{\max\{2^{d+1}, (d - 1)2^d\} |b|}{\varepsilon} (\psi(2) - \psi(-2)).
\]

For the function \( \psi \) appearing in (3.8) we claim that we can require that the equidistant partition

\[-1 = t_0 < t_1 < \ldots < t_N = 1 \quad \text{with} \quad t_{j+1} - t_j = \frac{2}{N},
\]

consists only of continuity points of \( \psi \). This can be achieved as follows: Select \( 0 < \theta < 1 \) such that for the modified function \( \psi_\theta(t) := \psi(t + \theta) \) all the \( t_j \) are continuity points. Since \( \psi \) has only countably many discontinuity points, such \( \theta \) always exists. This corresponds to a blow-up sequence of the form

\[
u_r(y) := \frac{u(x_0 + r\theta n + ry) - u_{x_0,r}}{\alpha_r}, \quad y \in Q_n(0, 1),
\]

and in the following we need to replace \( Q_n(x_0, r) \) by \( Q_n(x_0 + r\theta n, r) \). We note that \( N \) still satisfies

\[
N > \frac{\max\{2^{d+1}, (d - 1)2^d\} |b|}{\varepsilon} (\psi(1) - \psi(-1)).
\]
In the following we will however suppress such a possible shift for ease of notation.

Define $\varphi$ as the piecewise affine function satisfying $\varphi(t_j) = \psi(t_j)$ for each $j = 0, \ldots, N$, and note that with (the rotation $P$ as above),

$$S_j := P[[t_j, t_{j+1}) \times (-1,1)^{d-1}], \quad j = 0, \ldots, N-1,$$

we have

$$\int_{Q_n} |u_0(y) - b \varphi(y \cdot n)| \, dx = 2^{d-1}|b| \int_{t_{j-1}}^{t_j} |\psi - \varphi| \, ds \leq \frac{2^d|b|}{N} \sum_{j=0}^{N-1} \psi(t_{j+1}) - \psi(t_j) < \frac{\varepsilon}{2},$$

(3.9)

and

$$\sum_{j=0}^{N-1} \int_{\partial S_j} |u_0(y) - b \varphi(y \cdot n)| \, d\mathcal{H}^{d-1}(y) = |b| \sum_{j=0}^{N-1} \int_{\partial S_j} |\psi(y \cdot n) - \varphi(y \cdot n)| \, d\mathcal{H}^{d-1}(y) \leq \frac{(d-1)2^{d-1}|b|}{N} \sum_{j=0}^{N-1} \psi(t_{j+1}) - \psi(t_j) < \frac{\varepsilon}{2}.$$  (3.10)

Next, in view of (3.7) and our choice of partition points $t_j$, we infer from the trace theorem that

$$\sum_{j=0}^{N-1} \int_{\partial S_j} |u_r - u_0| \, d\mathcal{H}^{d-1} \to 0 \text{ as } r \downarrow 0.$$  (3.11)

For a point $x_0 \in G_2$, the mapping $x \mapsto b \varphi(x \cdot n)$ defined above is piecewise affine. Split the the rotated cube $Q_n(x_0, r)$ into $k$ rectangles $S_j(x_0, r) := x_0 + rS_j$ and define the corresponding piecewise affine map

$$w(x) := u_{x_0, r} + r x_0 \varphi\left(\frac{x - x_0}{r}, n\right), \quad x \in Q_n(x_0, r).$$

(3.12)

Hence, changing variables in (3.7), (3.11) and using (3.9), (3.10) as well as the estimate $|\sqrt{1 - |A|^2} - |A|| \leq 1$ it follows that there exists $r(x_0) > 0$ such that for $r < r(x_0)$,

$$\int_{Q_n(x_0, r)} |u - w| \, dx < \varepsilon r |Du|(Q_n(x_0, r)),

|\langle Du(Q_n(x_0, r)) - (Du)(Q_n(x_0, r))\rangle| < \varepsilon \langle Du(Q_n(x_0, r)) + 2\mathcal{L}(Q_n(x_0, r)),

\sum_{j=0}^{N-1} \int_{\partial S_j(x_0, r)} |u - w| \, d\mathcal{H}^{d-1} \leq \varepsilon |Du|(Q_n(x_0, r)).$$

(3.13)

**Step 3.** For every $x_0 \in G := G_1 \cup G_2$ we have so far constructed a (rotated) cube $Q_0 = Q_n(x_0, r)$ with $n$ and $r$ depending on $x_0$, and for all $x_0 \in G_2$ this cube is further subdivided into rectangles $S_j(x_0, r) \ (j = 0, \ldots, N-1).$ Now, for every such $Q_0$ we choose a slightly larger concentric similar cube $Q = Q(x_0, r) \supset Q_0$ with the properties

$$\|u\|_{L^1(\overline{Q} \setminus Q_0, \mathbb{R}^n)} < \varepsilon \|u\|_{L^1(Q_0, \mathbb{R}^n)}, \quad (\mathcal{L}^d + (Du))(\overline{Q} \setminus Q_0) < \varepsilon (Du)(Q_0)$$  (3.14)

and in the case $x_0 \in G_2$, we further require

$$\text{dist}(\partial Q, Q_0) = \eta = \frac{2r}{N},$$
where $\eta$ is the length of the short side of the $S_j(x_0,r)$’s. This can be achieved by subdividing the rectangles and correspondingly shrinking $Q$.

Now we invoke Lemma 8 with $Q_0$ and $Q$ and with $w$ as in (3.5) or (3.12) for $x_0 \in G_1$ or $x_0 \in G_2$, respectively. In particular, this yields a (countably) piecewise affine function $v_Q$ with the properties stated in that lemma.

**Step 4.** We next apply the Morse Covering Theorem 19 (or see Theorem 5.51 in [3]) to cover $(\mathcal{L}^d + |Du|)$-almost all of $\Omega$ with (rotated) cubes $Q$ from the above family. After subdividing the cubes $Q_0 = Q_0(x_0,r)$ corresponding to points $x_0 \in G_2$ into the rectangles $S_j(x_0,r)$ and the set $Q(x_0,r) \setminus Q_0(x_0,r)$ into the simplices constructed in Lemma 8, we thereby find a family $\mathcal{R}$ satisfying (i) in the statement of Theorem 1.

For the remaining properties (ii)–(v), we write

$$v = \sum_{R \in \mathcal{R}} a_R \mathbb{1}_R,$$

where $a_R$ denotes the affine map corresponding to the rectangle or simplex $R \in \mathcal{R}$ (in particular $v_Q = \sum_{R \subseteq Q} a_R$ for any $v_Q$ from the previous step). Because the rectangles/simplices are disjoint, the map $v$ is well-defined and it is clear that (ii) is satisfied. For the remaining assertions we consider for $j \in \mathbb{N}$ the mapping

$$v_j = u \mathbb{1}_{H_j} + \sum_{\mathcal{L}^d(R) > \frac{1}{j}} a_R \mathbb{1}_R, \quad \text{with} \quad H_j = \Omega \setminus \left( \bigcup_{\mathcal{L}^d(R) > \frac{1}{j}} R \right).$$

Since the above sum is finite, we infer that $v_j \in \text{BV}(\Omega; \mathbb{R}^m)$ and

$$Du_j = Du \mathbb{1}_{H_j} + u \otimes \nu_{H_j} \mathcal{H}^{d-1} \mathbb{1}_\partial H_j + \sum_{\mathcal{L}^d(R) > \frac{1}{j}} (\nabla a_R \mathcal{L}^d \mathbb{1}_R + a_R \otimes \nu_R \mathcal{H}^{d-1} \mathbb{1}_\partial R).$$

Here, $\nu_{H_j}$ and $\nu_R$ are the unit inner normals to $H_j$ and $R$, respectively. Employing (3.6), (3.13),

$$|Du_j|(\Omega) = |Du|(H_j) + \sum_{\mathcal{L}^d(R) > \frac{1}{j}} \left( |\nabla a_R| \mathcal{L}^d(R) + \int_{\partial R} |a_R - u| \, d\mathcal{H}^{d-1} \right)$$

$$= |Du|(H_j) + \sum_{\mathcal{L}^d(R) > \frac{1}{j}} |\nabla a_R| \mathcal{L}^d(R) + O(\varepsilon)(\mathcal{L}^d + |Du|(\Omega)).$$

Since $|Du|(H_j) \to 0$ as $j \to \infty$, we see that $v \in \text{BV}(\Omega; \mathbb{R}^m)$.

Concerning (iii), we estimate using (3.6), (3.13), (3.1), (3.14),

$$|\langle Du \rangle(\Omega) - \langle Du \rangle(\Omega)| \leq \sum_Q \left[ |\langle Du \rangle(Q_0) - \langle Du \rangle(Q_0) + |\langle Du \rangle + \langle Du \rangle|(Q \setminus Q_0) \right]$$

$$\leq \sum_Q \left[ \varepsilon(\mathcal{L}^d + |Du|)(Q_0) + C(\mathcal{L}^d + |Du|)(Q \setminus Q_0) \right] + \mathcal{L}^d(Z)$$

$$\leq C \sum_Q (\mathcal{L}^d + |Du|)(Q_0) + \mathcal{L}^d(Z)$$

$$\leq \varepsilon C(\mathcal{L}^d + |Du|)(\Omega) + \mathcal{L}^d(Z),$$
where the summation is over all cubes used to cover Ω; inside the sum \( Q_0 = Q_n(x_0, r) \) refers to the inner cube. The term \( \mathcal{L}^d(Z) \) originates from the additional Lebesgue measure for the cubes corresponding to singular points \( x \in G_Q \). Because they only occur on a \( \mathcal{L}^d \)-negligible set, we can make this term disappear in the limit \( \varepsilon \to 0 \).

By a similar calculation, also (iv) holds (this time we use (3.2) instead of (3.1)):

\[
\sum_{R \in \mathcal{R}} \int_{\partial R} |u - v| \, d\mathcal{H}^{d-1}
\]

\[
= \sum_{Q} \left[ \sum_{j=0}^{N-1} \int_{\partial S_j(x_0, r)} |u - v| \, d\mathcal{H}^{d-1} + \sum_{\tau \in \mathcal{T}} \int_{\partial \tau} |u - v| \, d\mathcal{H}^{d-1} \right]
\]

\[
\leq \sum_{Q} \left[ \varepsilon (\mathcal{L}^d + |Du|)(Q) + C|Du|(Q_n(x_0, r) \setminus Q_0(x_0, r)) \right]
\]

\[
< \varepsilon (1 + C)(\mathcal{L}^d + |Du|)(\Omega).
\]

For (v) we only need to observe that \( v_j = u \) on \( \partial \Omega \) for all \( j \in \mathbb{N} \) and \( v_j \to u \) strictly, hence the trace is preserved.

The only remaining part to check is whether the constructed mapping \( v \) is of class \( W^{1,1}(\Omega; \mathbb{R}^m) \). The features we shall use here are that for every cube \( Q \) as before, \( v|_Q \in W^{1,1}(Q; \mathbb{R}^m) \), \( v = u \) on \( \partial Q \), and that we may assume that \( |Du|(\partial Q) = 0 \). We have

\[
Du = \sum_{Q} (Dv \mathbb{1}_{\Omega} + v|_{\partial Q} \otimes \nu_Q) \mathcal{H}^{d-1}(\partial Q).
\]

By assumption \( v|_{\partial Q} = u|_{\partial Q} \) and the latter coincides also with the outer trace of \( u \) on \( \partial Q \) since \( |Du|(\partial Q) = 0 \). Keeping in mind that \( \sum_{Q \in \mathcal{Q}} 1_Q = 1 \mathbb{1}_{\Omega} \mathcal{L}^d \)-almost everywhere, and hence in the sense of \( L^1(\mathbb{R}^d) \), we find

\[
\sum_{Q \in \mathcal{Q}} v|_{\partial Q} \otimes \nu_Q \mathcal{H}^{d-1}(\partial Q) = \sum_{Q \in \mathcal{Q}} u|_{\partial Q} \otimes D(1_Q) = u \otimes \nu_{\Omega} \mathcal{H}^{d-1}(\partial \Omega).
\]

This concludes the proof. \( \square \)

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