The Free Energy in Scalar Electrodynamics at High Temperature

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Abstract

Massless scalar electrodynamics is studied at high temperature and zero chemical potential. I derive the free energy to order $\lambda^2$, $\lambda e^2$ and $e^4$ by effective field theory methods. The first step consists of the construction of an effective three-dimensional field theory that is valid on distance scales $R \gg 1/T$ and whose parameters can be written as power series in $\lambda$ and $e^2$. These coupling constants encode the contribution to physical quantities from the scale $T$. The second step is the use of the effective Lagrangian and perturbative calculations yield the contributions to the free energy from the scale $e T \sim \sqrt{\lambda T}$.

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1 Introduction

Quantum field theory at finite temperature have received considerably attention in recent years for a number of reasons. Two major applications are phase transitions in the early Universe and QCD at high temperature.

It was first observed by Kirzhnitz and Linde that symmetries which are spontaneously broken at zero temperature are normally restored at high temperature [1]. The phase transition which has been studied most extensively is the electroweak phase transition, which takes place when the Universe has a temperature of approximately 200 GeV, and where the Higgs field acquires a nonzero expectation value. One reason for investigating the electroweak phase transition is that it may be responsible for the present baryon asymmetry. In order to explain the excess of baryons over antibaryons, the electroweak phase transition must be sufficiently strongly first order [4]. For realistic values of the Higgs mass, this is not the case in the minimal standard model [2], and so one is led to investigate extensions, such as two Higgs doublet models [3] and the minimal supersymmetric standard model [4].

The hadrons in QCD also undergoes a phase transition when the temperature is raised to approximately 200 MeV. The hadrons turn into a plasma of free quarks and gluons. This quark-gluon plasma may be produced in heavy-ion collisions in the next generation of colliders. Hence, the study of QCD in the deconfined phase is very important.

It is now a well-known fact that bare or naive perturbation theory at high temperature breaks down for soft external momenta $k$ ($k \sim gT$, where $g$ is some generic coupling constant) due to infrared divergences [5]. Leading order results for physical quantities such as screening masses and damping rates receive contributions from all orders of perturbation theory. In order to do consistent perturbative calculations, it is necessary to apply the resummation program of Braaten and Pisarski [6] or extenstions thereof. In this approach one uses effective propagators, and in the case of non-Abelian gauge theories one uses effective vertices as well. This effective expansion resums the infinite subset of diagrams which contributed to physical quantities al leading order, and hence it is truly perturbative.

Resummation methods have been used by several authors to study the symmetry restoration at finite temperature of spontaneously broken gauge theories. Hebecker has studied the Abelian Higgs model to two-loop order in resummed perturbation theory [1], while Fodor and Hebecker have analyzed the electroweak phase transition, also in the two-loop approximation [7]. The Abelian Higgs model has been used a testing ground for different resummation methods in hot gauge theories [8]. However, it is also of interest in its own right. since it provides a model for a relativistic superconductor. The phase transition is then a transition from a superconducting phase (Higgs phase) to a normal
concluding phase (symmetric phase).

In some cases, as in the study of static quantities such as the effective potential, Arnold and Espinosa have invented a simplified resummation scheme \cite{9}. For static quantities, it is sufficient to dress the bosonic zero-frequency modes by thermal masses. For all other bosonic modes as well the fermionic modes, the Matsubara frequencies provide the necessary infrared cutoff. Arnold and Espinosa \cite{9} have used this resummation approach to study phase transitions in various gauge theories. It has also been used to compute the free energy in $g^2\Phi^4$-theory to order $g^5$ by Parwani and Singh \cite{10} (the result to order $g^4$ was first obtained by Frenkel, Saa and Taylor \cite{11}). Corresponding calculations in QCD have been carried out by Arnold and Zhai to order $g^4$ \cite{12}, and extended to $g^5$ by Kastening and Zhai \cite{13}. Finally, Blaizot et al. have used the method to compute the electric screening mass in scalar and spinor electrodynamics to order $e^4$ and $e^5$, respectively \cite{14}.

In recent years effective field theory methods have invaded the area of high temperature field theory. This was initiated by the papers of Ginsparg and Landsman \cite{15,16}. The basic idea is as follows. In the imaginary time formalism there is a summation of Matsubara frequencies in loop diagrams. These frequencies can be viewed as masses proportional equal to $2\pi n T$ for bosons and $(2n + 1)\pi T$ for fermions, where $n$ is positive integer. Hence, there is a mass hierarchy, and the fermionic modes as well as the nonzero bosonic modes ought to decouple at distance scales $R \gg 1/T$, analogous to the decoupling of heavy particles at zero temperature and low energy according to the Appelquist-Carrazone theorem \cite{17}.

There are several ways of constructing effective field theories at finite temperature. At the one-loop level, one would integrate out the heavy modes. In this context this means that the coupling constants are defined by computing the effects of the heavy modes on static correlators. Hence, only heavy modes circulate in loop diagrams. Beyond one loop, one must be more careful. It has been demonstrated by Jakovác that the above procedure generates non-local terms which cannot be expanded in local operators, and so it is difficult to obtain $L_{\text{eff}}$ \cite{18}. The reason is that there can be light and heavy modes simultaneously in multi-loop diagrams, and that one must take this into account in order to obtain the effective Lagrangian.

Kajantie et al. \cite{19} have put forward an effective field theory approach, which resolves the above problems. The coupling constants are now determined by a matching procedure. One calculates static correlators in the full and in the effective theory, and demand they be the same. At the one-loop level, this method coincides with the original approach to dimensional reduction \cite{16}. Beyond the one-loop approximation, one omits the contributions to a correlator where all the modes circulating are light, but contributions involving both heavy and light particles are retained. The effects from a loop diagram involving only static (light) modes will be taken care of by the effective Lagrangian. In many cases,
one needs the correlators at zero external momentum. One can then directly apply the effective potential, since this is the generator of one-particle irreducible Green’s functions at zero external momentum. One advantage of using the effective potential is that this often is easier to calculate than performing a direct calculation of loop corrections to the correlators.

There is an alternative effective field theory approach due to Braaten and Nieto [20]. Instead of explicitly constructing an effective theory for the zero mode one proceeds as follows. One writes down the most general Lagrangian in three dimensions, which respects the symmetries of the theory and includes all relevant degrees of freedom (i.e. no fermionic fields are present in $\mathcal{L}_{\text{eff}}$). The coupling constants are again determined by computing static correlators in the two theories and require that they match. One advantage of this approach is that we do not explicitly divide loop corrections to a static correlator into contributions from light and heavy modes. This method has been applied by Braaten and Nieto both to $g^2\Phi^4$-theory [20] and QCD [21]. They have confirmed the results of calculations of the free energy to order $g^5$ in these theories, first obtained by resummation methods by Parwani and Singh [10], and by Zhai and Kastening [13], respectively. They also computed the screening mass squared in $g^2\Phi^4$-theory to order $g^4$, which is a new result. The present author has applied the methods to scalar [22] and spinor QED [23], and reproduced the results by Blaizot et al. [14] for the electric screening mass in the two theories to order $e^4$ and $\lambda^3/2$, respectively. Similarly, the free energy result in QED to order $e^5$ found by Zhai and Kastening agrees with that obtained by effective field theory methods [23]. This effective field theory approach has also been used to calculate the screening mass squared in $g^2\Phi^4$ theory to order $g^5$ [24].

In contrast with other theories, the literature on SQED at very high temperature ($T \gg T_c$) is somewhat sparse. There are no papers specifically devoted to the calculation of the free energy at high temperature, although the result to order $e^3$ and $\lambda^{3/2}$ may be easily extracted from existing literature [3]. In this work, I apply the effective field theory methods of Braaten and Nieto to compute the free energy in SQED to order $\lambda^2$, $\lambda e^2$, and $e^4$. The result then represents the present calculational frontier.

The plan of the present paper is as follows. In section II, we discuss dimensional reduction and how effective field theory methods can be applied high temperature thermodynamics. In section III, the coefficients in the effective Lagrangian are determined. In section IV we apply the effective three-dimensional field theory to compute the free energy in scalar electrodynamics to order $\lambda^2$, $\lambda e^2$ and $e^4$. In section V we summarize and conclude. In two appendices, the notation and conventions are given. We also list the sum-integrals used in the full theory as well as the three-dimensional integrals needed in the effective theory.
Effective Field Theory and Dimensional Reduction

In the imaginary time formalism (ITF) of quantum field theory, there exists a path integral representation of the partition function \[^\{15\}\]:

\[
Z = \int \mathcal{D}\phi \exp \left[ -\int_0^\beta d\tau \int d^3x \mathcal{L}_E \right]. \tag{1}
\]

Here, \(\phi\) is a generic field, and \(\mathcal{L}_E\) is the Euclidean Lagrangian which is obtained from the usual Lagrangian after Wick rotation, \(t \rightarrow -i\tau\). Moreover, the boundary conditions in imaginary time are that bosonic fields are periodic in the time direction with period \(\beta\) and that fermionic fields are antiperiodic with the same period. If \(\Phi\) denotes a bosonic field and \(\Psi\) denotes a fermionic field we have

\[
\Phi(x, 0) = \Phi(x, \tau), \quad \Psi(x, 0) = -\Psi(x, \tau). \tag{2}
\]

The (anti)periodicity implies that we can decompose the fields into Fourier components characterized by their Matsubara frequencies:

\[
\Phi(x, \tau) = \beta^{-\frac{1}{2}} \sum_n \phi_n(x) e^{i\omega_n \tau}, \tag{3}
\]

\[
\Psi(x, \tau) = \beta^{-\frac{1}{2}} \sum_n \psi_n(x) e^{i\omega_n \tau}. \tag{4}
\]

Here

\[
\omega_n = \begin{cases} 
2n\pi T, & \text{bosons} \\
(2n + 1)\pi T, & \text{fermions}, \end{cases} \tag{5}
\]

where \(n\) is an integer. In four dimensions, bosonic fields are assigned a naive scaling dimension one, while fermionics field have a naive scaling dimension of 3/2. This implies that the bosonic fields in the three-dimensional effective theory have a naive scaling dimension of 1/2.

In the ITF, we can associate a free propagator \(\Delta_n(k)\) with the \(n\)th Fourier mode:

\[
\Delta_n(k) = \frac{1}{k^2 + \omega_n^2}. \tag{7}
\]

Here, \(k = |k|\). Hence, a quantum field theory at finite temperature may be viewed as an infinite tower of fields in three dimensions, where the Matsubara frequencies act as tree-level masses. The \(n = 0\) bosonic mode is called a light or static mode, while the \(n \neq 0\) modes as well as the fermionic modes are termed heavy or nonstatic.

At tree level the light modes are actually massless. If \(\Phi\) denotes the scalar field in \(g^2\Phi^4\)-theory, the scalar field acquires a thermal mass of order \(gT\). The thermal mass
reflects that static scalar fields are screened in the plasma, in complete analogy to the screening of static electric fields in QED. For the heavy modes this represents a perturbative correction which is down by a power of the coupling, and these modes are still characterized by a mass of order $T$. However, the light mode is no longer massless, but its mass is of order $gT$. Hence, we conclude that we have two widely separated mass scales at high temperature, which are $T$ and $gT$. The Appelquist-Carrazone decoupling theorem \[17\] then suggests that the heavy modes decouple on the scale $gT$. Moreover, modern developments in renormalization theory guarantee that static correlators of the full theory can be reproduced to any desired accuracy by an effective three-dimensional field theory of static mode at long distances $R \gg 1/T$ by tuning the parameters of the effective Lagrangian as functions of temperature and the parameters in the underlying theory. The effective Lagrangian is nonrenormalizable and contains infinitely many terms. The coupling constants of $\mathcal{L}_{\text{eff}}$ encode the physics on the scale $T$. This process of going from a full four dimensional theory to an effective three-dimensional Lagrangian is called \textit{dimensional reduction}. This is the key observation and the starting point for the construction of effective field theories at finite temperature.

Let us now move on to massless scalar electrodynamics. The Euclidean Lagrangian of SQED is

$$\mathcal{L}_{\text{SQED}} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger (D_\mu \Phi) + \frac{\lambda}{6} (\Phi^\dagger \Phi)^2 + \mathcal{L}_{gf} + \mathcal{L}_{gh}.$$  \hspace{1cm} (8)

The covariant derivative is $D_\mu = \partial_\mu + ieA_\mu$. In the present work all calculations are carried out in the Feynman gauge, but we emphasize that the parameters in the effective theory are gauge independent. The gauge fixing term is

$$\mathcal{L}_{gf} = \frac{1}{2} (\partial_\mu A_\mu)^2,$$  \hspace{1cm} (9)

If $\eta$ denotes the ghost field, the ghost term is then

$$\mathcal{L}_{gh} = (\partial_\mu \eta)(\partial_\mu \eta),$$  \hspace{1cm} (10)

and is thus decoupled from the rest of the Lagrangian.

In $g^2\Phi^4$ theory we have seen that we have the two momentum scales $T$ and $gT$. In scalar electrodynamics we have the scale $T$ as well as the scale of electric screening and the scale of scalar screening. In the rest of the paper we shall assume that $e^2 \sim \lambda \ll 1$ so that counting of powers of the coupling constants is simplified.

We call the effective three-dimensional field theory electrostatic scalar electrodynamics (ESQED), in analogy with the definitions introduced by Braaten and Nieto in the case of QCD \[21\]. The first task is to identify the appropriate fields and the symmetries in
ESQED. The fields can be identified (up to normalizations) with the zero-frequency modes of the original fields. $\mathcal{L}_{\text{ESQED}}$ contains a real massive scalar field, which we denote by $\rho$. This field is identified with the $n = 0$ mode of the timelike component of the gauge field in full SQED. There is a complex massive scalar field, $\phi$, which is identified with $\Phi$. Finally, there is a massless gauge field, which is denoted by $A_{i}^{3d}$ and is identified with the spatial component of the gauge field in the underlying theory. Since the fields of ESQED are identified with the static modes of SQED, we have to a first approximation the following relations:

$$
\rho = \frac{1}{\sqrt{T}} A_0, \quad \phi = \frac{1}{\sqrt{T}} \Phi, \quad A_{i}^{3d} = \frac{1}{\sqrt{T}} A_i.
$$

(11)

Now, $\mathcal{L}_{\text{ESQED}}$ must be a gauge invariant function of the spatial fields $A_i$, up to the usual gauge fixing terms. This symmetry follows from the corresponding symmetry in the full theory and the Ward-Takahashi identity in the high temperature limit [16]. Since SQED is an Abelian gauge theory, there will be no magnetic mass [25]. The fact that the fields $\rho$ and $\Phi$ are massive reflects the screening of static electric and static scalar fields, respectively. There is rotational symmetry for $\rho$ as well a $Z_2$-symmetry for the fields $\phi$ and $\rho$. The effective Lagrangian then has the general form

$$
\mathcal{L}_{\text{ESQED}} = \frac{1}{4} F_{ij} F_{ij} + (D_i \phi) \phi^i D_i \phi + M^2(\Lambda) \phi^\dagger \phi + \frac{1}{2} (\partial_i \rho)^2 +
\frac{1}{2} m^2(\Lambda) \rho^2 + h E(\Lambda) \phi^\dagger \phi \rho^2 + \frac{\lambda_3}{6} (\phi^\dagger \phi)^2 + \mathcal{L}_{g\phi} + \mathcal{L}_{gh} + \delta \mathcal{L}.
$$

(12)

Here, $\Lambda$ is the ultraviolet cutoff in the effective theory. The $\Lambda$-dependence of the parameters is necessary in order to cancel the dependence of $\Lambda$ which arises in perturbative calculations in electrostatic scalar electrodynamics. The scale $\Lambda$ can be thought of as an arbitrary factorization scale separating the scale $T$ and $eT$. Furthermore, $\delta \mathcal{L}$ represents all local terms that can be constructed out of $A_i$ and $\rho$, which respect the symmetries of the theory. This includes renormalizable terms, such as $g E(\Lambda) \rho^6$, as well as nonrenormalizable ones like $h E(\Lambda) (F_{ij} F_{ij})^2$.

It is not too difficult to find the order in the couplings at which a certain operator in $\mathcal{L}_{\text{ef}}$ starts to contribute to a physical quantity. Take e.g. $\lambda E(\Lambda) \rho^4$; The coefficient in front of the quartic coupling at leading order in $e^2$, can be found by considering the one-loop contribution to the four-point function for timelike photons at zero external momenta. Thus, it goes like $e^4$. Moreover, this operator contributes to the free energy first at the two-loop level (the double bubble), and each loop is proportional to $m E(\Lambda)$. Since the mass goes like $e T$, this implies that the leading order contribution is of order $e^6$. Other operators are analyzed in a similar manner.

In (12), we did not include the unit operator. The coefficient of the unit operator, which we denote $f_{\text{ESQED}}(\Lambda)$, gives the contribution to the free energy from the momentum
scale $T$. So if we are interested in calculating the pressure we must determine it, as we determine other coefficients in ESQED. Generally, this coefficient also depends on the renormalization scale $\Lambda$. By including $f_{\text{ESQED}}(\Lambda)$ in $L_{\text{ESQED}}$, we have two equivalent ways of writing the partition function in SQED in terms of its path integral representation. In the full theory we have

$$Z = \int D\pi D\eta DA_{\mu} D\bar{\psi} D\psi \exp \left[ -\int_0^\beta d\tau \int d^3x \mathcal{L} \right], \quad (13)$$

where $\eta$ denotes the ghost field. The result using the effective three-dimensional theory is

$$Z = e^{-f_{\text{ESQED}}(\Lambda)V} \int D\pi D\eta DA_i D\rho \exp \left[ -\int d^3x \mathcal{L}_{\text{ESQED}} \right]. \quad (14)$$

The free energy may then be written as the sum of two terms:

$$F = T \left[ f_{\text{ESQED}}(\Lambda) - \ln Z_{\text{ESQED}}/V \right], \quad (15)$$

The first term represents the contribution from the scale $T$ and the second term represents the contribution from the scale $eT$, and may be calculated perturbatively using ESQED.

Let us close this section by pointing out a difference between Abelian and Non-Abelian gauge theories. In QCD, we have three momentum scales. The scale $T$, which is a typical momentum of a particle, the scale $gT$ which is the scale of colour electric screening, and $g^2T$, which is the scale of colour magnetic screening. So, it is convenient to construct a sequence of two effective field theories in order to unravel the contribution to the free energy from the three momentum scales $T$, $gT$ and $g^2T$, as demonstrated by Braaten and Nieto [21]. Since the gauge field in ESQED is massless, one could construct a second effective field theory only involving $A_{3d}^i$ by integrating out the massive fields $\rho$ and $\Phi$. However, below we shall argue that this is in fact not necessary.

We call this effective field theory magnetostatic scalar electrodynamics, (MSQED), and it is valid on distance scales $R \gg 1/eT$. The Lagrangian consists of all terms made up by $A_{3d}^i$ which respect gauge invariance and the other symmetries of MSQED. At leading order in the couplings of full SQED, MSQED is simply free Maxwell theory in three dimensions. However, the only ultraviolet divergences one would encounter doing perturbative calculations in MSQED are powerlike. The coefficients in front of these depend upon the regulator, and are thus artifacts of the regulator. Because of the unphysical character of these divergences, it is convenient to use a regulator where the these divergences are subtracted. Dimensional regularization provides such a procedure, since power divergences by definition are set to zero. This implies that there will be no contribution to the free energy from perturbative calculations in MSQED. This fact reflects that there is no magnetic screening in Abelian gauge theories and no contribution from the scale $e^2T$ to the free energy. Hence, there is no need to construct MSQED.
3 The Coefficients in the Effective Lagrangian

In this section we shall determine the parameters of $L_{\text{ESQED}}$ to the order in $\lambda$ and $e^2$ which is required for obtaining the free energy to order $\lambda^2, \lambda e^2$ and $e^4$. We know from the previous discussion that static correlators in SQED at long distances $R \gg 1/T$ can be reproduced in ESQED by tuning the parameters as functions of $\lambda, e^2, T$ and the renormalization scale $\Lambda$. We determine the coefficients in $L_{\text{ESQED}}$ by the use of strict perturbation theory. This way of determining the parameters of the effective three-dimensional theories were first carried out by Braaten and Nieto in Ref. [20]. Strict perturbation theory is simply ordinary perturbation theory and is therefore plagued with infrared divergences, which become more and more severe as we go to higher orders in the loop expansion. Accordingly, we split the Lagrangian of SQED into a free piece and an interacting piece:

\begin{align}
(L_{\text{SQED}})_0 &= \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \Phi) (\partial_\mu \Phi) + L_{gf} + L_{gh}, \\
(L_{\text{SQED}})_{\text{int}} &= e^2 \Phi^\dagger \Phi A_\mu^2 - ie A_\mu (\Phi^\dagger \partial_\mu \Phi - \Phi \partial_\mu \Phi^\dagger) + \frac{\lambda}{6} (\Phi^\dagger \Phi)^2. \tag{16}
\end{align}

Although strict perturbation theory breaks down on the scale $eT$, we can nevertheless use it as a tool for determining the parameters of ESQED. The idea is that the coefficient of the effective Lagrangian are sensitive to the scale $T$, but insensitive to the scale $eT$. We only have to be sure that we make the same incorrect assumptions using the effective theory in matching calculations. The infrared divergences which occur in perturbative calculations in the effective theory will then be exactly the same as those encountered in the full theory, and so they cancel against each other in the matching procedure. In ESQED, these incorrect assumptions amounts to treating the mass parameters as interactions. The partition of the Lagrangian of ESQED is

\begin{align}
(L_{\text{ESQED}})_0 &= \frac{1}{4} F_{ij} F^{ij} + (\partial_i \phi) (\partial_i \phi) + \frac{1}{2} (\partial_i \rho)^2 + L_{gf} + L_{gh}, \\
(L_{\text{ESQED}})_{\text{int}} &= \frac{1}{2} m^2_E(\Lambda) \rho^2 + M^2(\Lambda) \phi^\dagger \phi + e^2_E(\Lambda) \phi^\dagger \phi A_i A_i^3d + h^2_E(\Lambda) \phi^\dagger \phi \rho^2 \\
&\quad - ie_E(\Lambda) A_i^3d(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger) + \frac{\lambda_3(\Lambda)}{6} (\phi^\dagger \phi)^2 + \delta L. \tag{17}
\end{align}

Since the strict perturbative expansion given by (16) and (17) is naive perturbation theory, it should be clear that the coupling constants of ESQED can be written as power series in $\lambda$ and $e^2$.

In full SQED, wiggly, dashes and dotted lines denote the propagators of photons, charged scalars and ghosts, respectively. In ESQED, the same conventions apply. Moreover, solid lines denote the propagators of the real scalar field $\rho$.
3.1 Coupling Constants

In the present work, one needs the coupling constants $\lambda_3^2(\Lambda)$, $e_E^2(\Lambda)$ and $h_E^2(\Lambda)$ to leading order in $\lambda$ and $e^2$. At this order one can simply read off these coefficients from the Lagrangian in full SQED. We have previously found that $\rho = \frac{1}{\sqrt{T}} A_0$, $\phi = \frac{1}{\sqrt{T}} \Phi$ and $A_i^{3d} = \frac{1}{\sqrt{T}} A_i$ at leading order. Substituting these expressions into (8) and comparing it with (12), we conclude that

$$\lambda_3(\Lambda) = \lambda T.$$  \hspace{1cm} (18)

Similarly, one finds

$$e_E^2(\Lambda) = e^2 T, \quad h_E^2(\Lambda) = e^2 T.$$  \hspace{1cm} (19)

There is no dependence on the renormalization scale $\Lambda$.

The above parameters have been obtained at next-to-leading order in $\lambda$ and $e^2$ in e.g. Ref. [26] by considering the appropriate static correlators.

3.2 Mass parameters

We shall now determine the mass parameters $m_E^2(\Lambda)$ and $M^2(\Lambda)$ appearing in the effective Lagrangian. These parameters are needed to leading order in $\lambda$ and $e^2$. The mass parameter $m_E^2(\Lambda)$ has previously been calculated by the present author in the two-loop approximation using strict perturbation theory [22]. The mass parameter $M^2(\Lambda)$ has also been calculated to two-loop order in Ref. [26] using the effective potential. Nevertheless, we include the calculations here for completeness.

The mass parameters have the following interpretation; they represent the contribution from the scale $T$ to the electric and scalar screening mass, respectively. These screening masses give information of the screening of static electric and scalar fields in the plasma [27]. Now, the scalar screening mass cannot be calculated beyond leading order due to a mass shell singularity, as discussed by Blaizot and Iancu in Ref. [28]. The problem is the same as in QCD, namely a scalar field interacting with a massless gauge field in three dimensions. In QCD, this singularity may be screened by a magnetic mass of nonperturbative origin. In SQED the magnetic mass is absent since it is an Abelian theory [25], and so the problem cannot be solved this way. We shall not discuss this any further, but refer to Ref. [28] where a nonperturbative definition of the scalar screening mass is discussed in detail. Nevertheless, these infrared problems do not prevent us from determining the mass parameter $M^2(\Lambda)$ using the strict perturbation expansion.

There are several ways of determining the mass parameters, and the simplest way to do so is to match the screening masses in the two theories. The screening mass is defined
as the pole position of the propagator at spacelike momentum \([20]\). Let us denote the static self-energy function of the zeroth component of the gauge field by \(\Pi(\mathbf{k})\). Similarly, we denote the static scalar self-energy function by \(\Sigma(\mathbf{k})\). The corresponding screening masses are \(m_s^2\) and \(M_s^2\), respectively. In the full theory, one then finds that the screening masses are solutions of the following set of equations:

\[
\begin{align*}
 k^2 + \Pi(\mathbf{k}) &= 0, & k^2 &= -m_s^2, \\
 k^2 + \Sigma(\mathbf{k}) &= 0, & k^2 &= -M_s^2.
\end{align*}
\]

The self-energy functions can be expanded in powers of the external momentum \(\mathbf{k}\) and also in the number of loops in the loop expansion. The \(n\)th order contribution to the self-energy functions are then denoted by \(\Pi^{(n)}(\mathbf{k})\) and \(\Sigma^{(n)}(\mathbf{k})\), respectively. At leading order in the couplings, we simply substitute the one-loop approximations to the self-energy functions into (20) and (21). The solutions are

\[
\begin{align*}
 m_s^2 &\approx \Pi^{(1)}(0), & \quad M_s^2 &\approx \Sigma^{(1)}(0).
\end{align*}
\]

The symbol \(\approx\) is a reminder that the expressions are obtained using strict perturbation theory. Since strict perturbation theory treat infrared effects incorrectly, the mass parameters generally do not coincide with the physical screening masses \([20]\). At leading order, however, they normally do coincide.

In the full theory the Feynman graphs in Figs. 1 and 2 display the contributions at the one-loop level to the the self-energy function of \(\rho\) and \(\Phi\), respectively. We find

\[
\begin{align*}
 \Pi^{(1)}(0) &\approx 2e^2 \int \frac{1}{P^2} - 4e^2 \int \frac{p^2}{P^4}, \\
 \Sigma^{(1)}(0) &\approx (d - 1)e^2 \int \frac{1}{P^2} + \frac{2\lambda}{3} \int \frac{1}{P^2}.
\end{align*}
\]

Here, \(d = 4 - 2\epsilon\). In ESQED, the self-energy function of the real scalar field \(\rho\) is denoted by \(\Pi_{\text{eff}}(\mathbf{k}, \Lambda)\), while the corresponding self-energy function of \(\phi\) is \(\Sigma_{\text{eff}}(\mathbf{k}, \Lambda)\). In the effective theory the screening masses are by the following set of equations, where the superscript “(1)” indicates that the self-energies are taken in the one-loop approximation:

\[
\begin{align*}
 k^2 + m^2(\Lambda) + \Pi_{\text{eff}}^{(1)}(0, \Lambda) &= 0, & k^2 &= -m_s^2, \\
 k^2 + M^2(\Lambda) + \Sigma_{\text{eff}}^{(1)}(0, \Lambda) &= 0, & k^2 &= -M_s^2.
\end{align*}
\]

In the effective theory, all diagrams vanish identically; Since all fields are massless in strict perturbation theory, there is no scale in the loop integrals when the external momentum \(\mathbf{k}\) is set to zero. The integrals are set to zero in dimensional regularization. Hence, \(\Pi_{\text{eff}}^{(1)}(0, \Lambda) = 0\) and \(\Sigma_{\text{eff}}^{(1)}(0, \Lambda) = 0\). The matching conditions then imply that \(m^2_E(\Lambda) =\)
\[ \Pi^{(1)}(0) \text{ and } M^2(\Lambda) = \Sigma^{(1)}(0). \] Using (A.3)–(A.5) of Appendix A, we obtain the mass parameters at leading order in \( \lambda \) and \( e^2 \):

\[ m_E^2(\Lambda) = \frac{e^2 T^2}{3}, \]  
\[ M^2(\Lambda) = \frac{e^2 T^2}{4} + \frac{\lambda T^2}{18}. \]

To leading order in the couplings, then, the mass parameters are independent of \( \Lambda \).

### 3.3 Coefficient of the Unit Operator

In this subsection we shall determine the coefficient of the unit operator to next-to-leading order in \( \lambda \) and \( e^2 \). We shall consider the one, two and three-loop contributions separately. The matching condition we use to determine \( f_{\text{ESQED}}(\Lambda) \) follows from the two path integral representations of the partition function

\[ \ln Z = -f_{\text{ESQED}}(\Lambda)V + \ln Z_{\text{ESQED}}. \]

Let us first focus on the SQED. The one-loop graphs in the underlying theory are depicted in Fig. 3 and the corresponding contribution reads

\[ \frac{d}{2} \int \frac{\ln P^2}{P^2} = -\frac{2\pi^2}{45} T^4. \]

This is simply the free energy of a noninteracting gas of photons and charged scalars.

The two-loop diagrams are shown Fig. 4. After some purely algebraic manipulations, they factorize into products of simpler one-loop sum-integrals. Using Appendix A, we obtain

\[ Z_{\lambda} \lambda \int \frac{1}{P^2 Q^2} + (d - \frac{3}{2}) Z_{e^2} e^2 \int \frac{1}{P^2 Q^2} = \left( \frac{T^2}{12} \right)^2 \left[ \frac{\lambda}{3} + \frac{5e^2}{2} \right]. \]

Here, \( Z_{\lambda} \) and \( Z_{e^2} \) are the renormalization constants for the quartic coupling and the gauge coupling, respectively.

Let us now turn to the three-loop diagrams. These are displayed in Fig. 5. Here, the shaded blob means insertion of the one-loop photon polarization tensor \( \Pi_{\mu\nu}(k_0, k) \), while the black blob implies insertion of the scalar self-energy function \( \Sigma(k_0, k) \), also in the one-loop approximation.

The first four diagrams can expressed entirely in terms of the bosonic basketball, which is defined in Appendix A. After some purely algebraic manipulations involving
several changes of variables, one finds:

\[- \frac{\lambda^2}{18} + (d - 13/4)e^4 \int_{PQK} \frac{1}{P^2Q^2K^2(P + Q + K)^2}.\] (32)

The fifth diagram gives a contribution

\[- \frac{1}{4} \int_{P} \frac{1}{P^4} \left[ \Pi_{\mu\nu}(p_0, P) \right]^2 = \frac{e^4}{4} \int_{PQK} \frac{1}{P^2Q^2K^2(P + Q + K)^2} \]
\[- e^4 \int_{PQK} \frac{1}{P^2Q^2K^2(P - Q)^2} \]
\[- (d - 6)e^4 \int_{PQK} \frac{1}{P^2Q^2K^2}.\] (33)

The last graph reads

\[- \frac{1}{2} \int_{P} \frac{1}{P^4} \left[ \Sigma(p_0, P) \right]^2 = - \left[ \frac{2\lambda^2}{9} + \frac{2(d-1)\lambda e^2}{3} + \frac{(d-1)^2e^4}{2} \right] \int_{PQK} \frac{1}{P^2Q^2K^2} \]
\[- 2e^4 \int_{PQK} \frac{1}{P^2Q^2K^2(P + Q + K)^2}.\] (34)

Note that the fifth as well as the sixth diagram are infrared divergent. It stems from the facts that both the zeroth component of the static photon polarization tensor, \(\Pi_{00}(0, k)\), and the static scalar self-energy function, \(\Sigma(0, k)\), are nonzero in the infrared limit \(k \rightarrow 0\).

These diagrams are the first in an infinite series of infrared graphs which are called ring diagrams or plasmon diagrams. They are obtained from the two diagrams considered above, by one or more insertions of \(\Pi_{\mu\nu}(k_0, k)\) and \(\Sigma(k_0, k)\), respectively.

Renormalization of the quartic coupling is carried out by substituting \(Z_\lambda\) into (31). To order \(\lambda^2, \lambda e^2\) and \(e^4\), the renormalization constant reads

\[Z_\lambda \lambda = \lambda + \frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{48\pi^2\epsilon}.\] (35)

Similarly, we renormalize the gauge coupling by substituting \(Z_{e^2}\) into (31). To order \(e^4\), \(Z_{e^2}e^2\) is given by

\[Z_{e^2} = e^2 + \frac{e^4}{48\pi^2\epsilon}.\] (36)

The result in full SQED is now given by the sum of (30-34). After we have carried out coupling constant renormalization, we are still left with terms proportional to \(1/\epsilon\):

\[- \frac{T \ln Z}{V} \approx - \frac{2\pi^2 T^4}{45} + \left( \frac{T^2}{12} \right)^2 \left[ \frac{\lambda}{3} + \frac{5e^2}{2} \right].\]
\[-\frac{\lambda^2}{16\pi^2}\left(\frac{T^2}{12}\right)^2 \left[ \frac{10}{9}\ln \frac{\Lambda}{4\pi T} + \frac{4}{9} \gamma_E + \frac{31}{45} - \frac{2}{3} \frac{\zeta'(-3)}{\zeta(-3)} + \frac{4}{3} \frac{\zeta'(-1)}{\zeta(-1)} \right]
\]  
\[-\frac{\lambda e^2}{16\pi^2}\left(\frac{T^2}{12}\right)^2 \left[ \frac{4}{\epsilon} + 20\ln \frac{\Lambda}{4\pi T} + 4\gamma_E + \frac{44}{3} + \frac{16}{\zeta(-1)} \right]
\]  
\[-\frac{e^4}{16\pi^2}\left(\frac{T^2}{12}\right)^2 \left[ \frac{18}{\epsilon} + \frac{365}{3}\ln \frac{\Lambda}{4\pi T} + 13\gamma_E + \frac{236}{3} - \frac{110}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right]
\]  
\[+\frac{436 \zeta'(-1)}{3 \zeta(-1)}. \quad (37)\]

Again, the sign \(\approx\) indicates that the result is obtained using strict perturbation theory.

Let us next consider ESQED. The contributions to \(\ln Z_{\text{ESQED}}\) come from ordinary one, two, and three-loop diagrams, as well as one-loop graphs with one or two mass insertions. The loop-integrals all vanish in the strict perturbation expansion, since there is no mass scale. However, note that some of the three-loop diagrams as well the one-loop graphs with two mass insertions and the two-loop graphs with a single mass insertion are linearly divergent in the infrared. These divergences are identical to those encountered in full SQED (the fifth and sixth diagrams in Fig. 5).

The vanishing of \(\ln Z_{\text{ESQED}}\) implies that the free energy in the strict perturbation expansion is given by

\[-\frac{T}{V} \ln Z \approx T \left[f_{\text{ESQED}}(\Lambda) + \delta f_{\text{ESQED}}(\Lambda)\right]. \quad (38)\]

Here, \(\delta f_{\text{ESQED}}(\Lambda)\) is the counterterm of the unit operator. According to [21], \(\delta f_{\text{ESQED}}(\Lambda)\) can be determined by calculating the logarithmic ultraviolet divergences in the effective theory. Generally, \(\delta f_{\text{ESQED}}(\Lambda)\) is given by a polynomial in the parameters of ESQED. At leading order it turns out that it is given by

\[\delta f_{\text{ESQED}}(\Lambda) = -\frac{e^2 M^2}{2(4\pi)^2} \frac{1}{\epsilon} \quad (39)\]

which follows from a two-loop calculation in the next section. Since the mass \(M\) is multiplied by \(1/\epsilon\), it is necessary to expand it to first order in \(\epsilon\) when expressing \(\delta f_{\text{ESQED}}(\Lambda)\) in terms of \(e^2, \lambda\) and \(T\). From (23) one finds

\[\frac{\partial M^2}{\partial \epsilon} \bigg|_{\epsilon=0} = \frac{e^2 T^2}{12} \left[ 6 \ln \frac{\Lambda}{4\pi T} + 4 + 6 \frac{\zeta'(-1)}{\zeta(-1)} \right] + \frac{\lambda T^2}{18} \left[ 2 \ln \frac{\Lambda}{4\pi T} + 2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right]. \quad (40)\]

This implies that

\[\delta f_{\text{ESQED}}(\Lambda) T = -\frac{e^4}{(4\pi)^2} \left(\frac{T^2}{12}\right) \left[ \frac{18}{\epsilon} + 36 \ln \frac{\Lambda}{4\pi T} + 24 + 36 \frac{\zeta'(-1)}{\zeta(-1)} \right]
\]  
\[-\frac{\lambda e^2}{(4\pi)^2} \left(\frac{T^2}{12}\right) \left[ \frac{4}{\epsilon} + 8 \ln \frac{\Lambda}{4\pi T} + 8 + 8 \frac{\zeta'(-1)}{\zeta(-1)} \right]. \quad (41)\]
Putting our results together, one finally obtains

$$f_{\text{ESQED}}(\Lambda)T = -\frac{2\pi^2 T^4}{45} + \left(\frac{T}{12}\right)^2 \left[\frac{\lambda}{3} + \frac{5e^2}{2}\right]$$

$$-\frac{\lambda^2}{16\pi^2} \left(\frac{T}{12}\right)^2 \left[\frac{10}{9} \ln \frac{\Lambda}{4\pi T} + \frac{4}{9} \gamma_E + \frac{31}{45} - \frac{2}{3} \zeta(-3) + \frac{4}{3} \zeta(-1)\right]$$

$$-\frac{\lambda e^2}{16\pi^2} \left(\frac{T}{12}\right)^2 \left[-4 \ln \frac{\mu}{4\pi T} + 16 \ln \frac{\Lambda}{4\pi T} + 4 \gamma_E + \frac{20}{3} + 8 \frac{\zeta(-1)}{\zeta(-1)}\right]$$

$$-\frac{e^4}{16\pi^2} \left(\frac{T}{12}\right)^2 \left[\frac{41}{3} \ln \frac{\mu}{4\pi T} + 72 \ln \frac{\Lambda}{4\pi T} + 13 \gamma_E + \frac{164}{3} - \frac{110}{3} \frac{\zeta(-3)}{\zeta(-1)} + \frac{328 \zeta(-1)}{3} \zeta(-1)\right].$$

(42)

We have here used the renormalization group equations,

$$\frac{dc^2}{d\mu} = \frac{e^4}{24\pi^2},$$

(43)

$$\frac{d\lambda}{d\mu} = \frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{24\pi^2},$$

(44)

to change the scale from $\Lambda$ to $\mu$. The remaining logarithm of $\Lambda$ shows that $f_{\text{ESQED}}(\Lambda)$ depends explicitly on $\Lambda$. This logarithmic dependence on $\Lambda$ is necessary to cancel the $\Lambda$-dependence which arises in calculations in ESQED, as we shall see in section IV. This is contrast with spinor QED [23], where the unit operator is independent of the factorization scale to order $e^4$.

### 3.4 Renormalization Group Equations in ESQED

Above, we have seen that some of the parameters in ESQED do not depend explicitly on the cutoff $\Lambda$ to the order which we calculated them, while others do. More generally, the coefficients in $L_{\text{ESQED}}$ satisfy a set of renormalization group equations. The solutions to these equations can be used to sum up leading logarithms of the couplings constants from higher orders in the perturbation expansion, as discussed in detail in Ref. [20]. Although we shall not need these evolution equations in the present work, we include a short discussion for completeness. They will be relevant when calculations are pushed to higher orders.

The renormalization group equations follow from the requirement that physical quantities such as the electric screening mass and the free energy be independent of this arbitrary factorization scale. If $C_n(\Lambda)$ denotes an arbitrary operator, one can write

$$\Lambda \frac{dC_n(\Lambda)}{d\Lambda} = \beta_n(C(\Lambda)).$$

(45)
The beta functions $\beta_n(C(\Lambda))$ can be written as power series in the coupling constants of the effective theory. Using dimensional arguments we can infer the general structure. Consider first the coefficient of the unit operator $f_{\text{ESQED}}(\Lambda)$ and let us restrict ourselves to the superrenormalizable terms which are explicitly displayed in \[^12\]. The unit operator has dimension three. The beta function then contains terms in the form $e_1^2m_1^2$, $h_1^2M_1^2$, $\lambda_3m_3^2$, $e_2^2M_2^2$, $h_2^2M_2^2$ and $\lambda_3M_3^2$, as well as a cubic polynomial in the three coupling constants $e_1^2$, $h_2^2$ and $\lambda_3$. To order $\lambda e^2$, $\lambda e^2$ and $e^4$ only the first kind of terms can appear. The coefficients in front of these terms are determined by calculating the logarithmic ultraviolet divergences in ESQED. This is done in the next section, and the result is

$$\Lambda \frac{df_{\text{ESQED}}(\Lambda)}{d\Lambda} = \frac{2e_1^2M_1^2}{(4\pi)^2} + \mathcal{O}(e^6). \quad (46)$$

Using the fact that $e_1^2 = e^2T$ at leading order and the leading order result for $M_2^2$, we can write the equation for $f_{\text{ESQED}}(\Lambda)$ in terms of $\lambda$ and $e^2$:

$$\Lambda \frac{df_{\text{ESQED}}(\Lambda)}{d\Lambda} = \frac{T^3}{(4\pi)^2}\left[\frac{e^4}{2} + \frac{\lambda e^2}{9}\right] + \mathcal{O}(e^6). \quad (47)$$

This form of the renormalization group equation could of course have been obtained simply by differentiating \[^12\] with respect to $\Lambda$.

Consider next the mass parameter $m_2^2(\Lambda)$ and $M_2^2(\Lambda)$. The beta functions for these parameters must be quadratic polynomials in the $\lambda_3$, $e_1^2$, $h_2^2$ and $\lambda_3$ (the latter is the coefficient in front of $\rho^2$). We have already noted their independence of $\Lambda$ at leading order in the couplings. In Ref. \[^22\], I have calculated $m_1^2(\Lambda)$ to next-to-leading in $\lambda$ and $e^2$. The mass parameter turns out to be independent of the factorization scale also to this order in the couplings:

$$\Lambda \frac{dm_1^2(\Lambda)}{d\Lambda} = \mathcal{O}(e^6). \quad (48)$$

This is in contrast with the corresponding result for the scalar mass parameter $M_2^2(\Lambda)$. In Ref. \[^26\], this parameter has also been calculated at next-to-leading order in $\lambda$ and $e^2$, and the beta function reads

$$\Lambda \frac{dM_2^2(\Lambda)}{d\Lambda} = \frac{1}{(4\pi)^2}\left[6e_1^4 - \frac{4\lambda_3 e_1^2}{3} + \frac{2\lambda_3^2}{9}\right] + \mathcal{O}(e^6). \quad (49)$$

Finally, we make the remark that the $\beta$ functions for the couplings $\lambda_3(\Lambda)$, $e_1^2(\Lambda)$, $h_2^2(\Lambda)$ and $\lambda_3(\Lambda)$ vanish to order $\lambda^2$, $\lambda e^2$ and $e^4$ \[^26\]. This property actually holds to all orders in superrenormalizable interactions.

In summary, there are two parameters in ESQED whose evolution are relevant up to order $e^6$. These are $f_{\text{ESQED}}(\Lambda)$ and $M_2^2(\Lambda)$. The solutions to the renormalization group
equations (46) and (49) are, respectively

\[ f_{\text{ESQED}}(\Lambda) = f_{\text{ESQED}}(\Lambda') - \frac{2e_2^2 M^2}{(4\pi)^2} \ln \frac{\Lambda}{\Lambda'}, \]  
\[ M^2(\Lambda) = M^2(\Lambda') + \frac{1}{(4\pi)^2} \left[ 6e_4^4 - \frac{4 \lambda_3 e_2^2}{3} + \frac{2 \lambda_2^2}{9} \right] \ln \frac{\Lambda}{\Lambda'}. \]

4 Calculations in ESQED

Having calculated the coefficients in ESQED to the order required, we are now ready to calculate the free energy in SQED to order \( \lambda^2, \lambda e^2 \) and \( e^4 \). The free energy is given by the two terms in (15), where \( f_{\text{ESQED}}(\Lambda)T \) is given by (42). The remaining term \( \ln Z_{\text{ESQED}} \) is given by perturbative calculations using \( L_{\text{ESQED}} \). In order to obtain this term, it is necessary that we take screening effects properly into account. This is done by including the mass parameters \( m_2^2(\Lambda) \) and \( M^2(\Lambda) \) into the free part of the Lagrangian. We then include the effects of the mass parameters to all orders, while all other operators are treated as perturbations. Thus, we split \( L_{\text{ESQED}} \) according to:

\[ (L_{\text{ESQED}})_0 = \frac{1}{4} F_{ij} F_{ij} + (\partial_i \phi^\dagger)(\partial_i \phi) + M^2(\Lambda) \phi^\dagger \phi + \frac{1}{2} (\partial_i \rho)^2 + \frac{1}{2} m^2(\Lambda) \rho^2 + L_{gf} + L_{gh}, \]

\[ (L_{\text{ESQED}})_{\text{int}} = e_2^2(\Lambda) \phi^\dagger \phi A_i^{3d} A_i^{3d} + h_2^2(\Lambda) \phi^\dagger \phi \rho^2 - i e_3(\Lambda) A_i^{3d}(\phi^\dagger \partial_i \phi - \phi \partial_i \phi^\dagger) + \frac{\lambda_3(\Lambda)}{6} (\phi^\dagger \phi)^2 + \delta L. \]

The one and two-loop contributions to the pressure are depicted in Fig. 4. By adding the counterterm \( \delta f_{\text{ESQED}}(\Lambda) \), we obtain the contribution to the free energy from the momentum scale \( eT \).

\[- \frac{T \ln Z_{\text{ESQED}}}{V} = \frac{1}{2} T \int \ln(p^2 + m_2^2) + T \int \ln(p^2 + M^2) + \frac{1}{2} (d - 2) T \int \ln p^2 \]

\[- \frac{1}{2} e_2^2 e_4^4 \int \left( \frac{(\mathbf{p} + \mathbf{q})^2}{(p^2 + M^2)(q^2 + M^2)(\mathbf{p} - \mathbf{q})^2} + \frac{d e_2^2 e_4^4}{T} \int \frac{1}{(p^2 + m_2^2)(q^2 + m_2^2)} \right) \frac{1}{p^2 + M^2) (q^2 + m_2^2)} \]

\[- \frac{1}{3} T \int \frac{1}{p^2 + M^2) (q^2 + m_2^2)} + \frac{\lambda_3}{3} T \int \frac{1}{p^2 + M^2) (q^2 + M^2)} + T \delta f_{\text{ESQED}}(\Lambda). \]

Here, \( d = 3 - 2 \epsilon \). The two-loop contributions may be reduced to products of one-loop integrals and a two-loop integral, which are tabulated in Appendix B. This yields

\[- \frac{T \ln Z_{\text{ESQED}}}{V} = -\frac{m_2^2 T}{12 \pi} - \frac{M^2 T}{6 \pi} + \frac{e_2^2 M^2}{32 \pi^2} + \frac{e_2^2 M^2}{32 \pi^2} \left( 1 + 2 + 4 \ln \frac{\Lambda}{2 M} \right) + e_2^2 M m_2^2 \frac{1}{16 \pi^2}. \]
\[ \lambda_3 \frac{M^2}{48\pi^2} + \delta f_{\text{ESQED}}(\Lambda). \]  

The logarithmic ultraviolet divergence stems from the theta-diagram. This pole in \( \epsilon \) is then canceled by the counterterm of the unit operator, \( \delta f_{\text{ESQED}}(\Lambda) \), which is given by \((53)\). Using the expressions for the the parameters in ESQED in terms of the couplings in full SQED as well as the factorization scale \( \Lambda \) and temperature, we get the contribution to the free energy from the scale \( eT \):

\[ -T \ln \frac{Z_{\text{ESQED}}}{V} = -\frac{e^3 T^4}{36\pi \sqrt{3}} - \frac{M^3 T}{6\pi} + \frac{e^3 M T^3}{16\pi^2 \sqrt{3}} + \frac{\lambda^2}{16\pi^2} \left( T^2 \right)^2 \left[ \frac{8}{3} \right] + \frac{\lambda e^2}{16\pi^2} \left( T^2 \right)^2 \left[ 16 \ln \frac{\Lambda}{M} + 24 \right] + \frac{e^4}{16\pi^2} \left( T^2 \right)^2 \left[ 72 \ln \frac{\Lambda}{M} + 54 \right]. \]  

(54)

Adding \((52)\) and \((55)\), we finally obtain the free energy of massless scalar electrodynamics at high temperature through order \( \lambda^2, \lambda e^2 \) and \( e^4 \):

\[ F = -\frac{2\pi^2 T^4}{45} + \left( \frac{T^2}{12} \right)^2 \left[ \frac{\lambda^2}{3} + \frac{5e^2}{2} \right] - \frac{e^3 T^4}{36\pi \sqrt{3}} - \frac{M^3 T}{6\pi} + \frac{e^3 M T^3}{16\pi^2 \sqrt{3}} - \frac{\lambda^2}{16\pi^2} \left( T^2 \right)^2 \left[ \frac{10}{9} \ln \frac{\Lambda}{4\pi T} + \frac{4}{9} \gamma_E - \frac{89}{45} - \frac{2}{3} \zeta(-3) + \frac{4}{3} \zeta(-1) \right] - \frac{\lambda e^2}{16\pi^2} \left( T^2 \right)^2 \left[ -4 \ln \frac{\mu}{4\pi T} + 16 \ln \frac{2M}{4\pi T} + 4\gamma_E - \frac{52}{3} + \frac{8}{3} \zeta(-1) \right] - \frac{e^4}{16\pi^2} \left( T^2 \right)^2 \left[ \frac{41}{3} \ln \frac{\mu}{4\pi T} + 72 \ln \frac{2M}{4\pi T} + 13\gamma_E + \frac{2}{3} - \frac{110}{3} \zeta(-3) \right] + \frac{328}{3} \zeta(-1) \].  

(56)

This represents the main result of the present paper, and at this point some comments are in order. Firstly, we note that the two-loop contribution in the gauge sector is exactly the same as the one appearing in spinor QED \((13)\). The terms which go like \( e^3 \) and \( \lambda^{3/2} \) are non-analytic in the coupling constants, and therefore demonstrate that one receives contributions from all order in naive perturbation theory. The above terms represent the leading order contributions from the infinite string of infrared divergent ring diagrams. We also note that the \( \Lambda \)-dependence cancels to next-to-leading order in \( \lambda \) and \( e^2 \), leaving a result which is renormalization group invariant to order \( \lambda^2, \lambda e^2 \) and \( e^4 \). This can be easily checked by using the renormalization group equations, \((13)\) and \((14)\), for the running coupling constants.

The effective field theory approach also explains the occurrence of logarithms of \( T/M \) in the expression for the free energy. We have seen that \( f_{\text{ESQED}}(\Lambda) \) depends explicitly upon the factorization scale \( \Lambda \) and such terms appear as logarithms of \( \Lambda/T \). In ESQED,
logarithms in the form $\Lambda/M$ occur in perturbative calculations. In order to cancel the dependence of $\Lambda$ in the final answer, these logarithms must match the logarithms from $f_{\text{ESQED}}(\Lambda)$. We are then left with logarithms of $M/T$. Hence these logarithms are connected to the renormalization of $f_{\text{ESQED}}(\Lambda)$.

A similar term $g^4 \ln(m_E/T)$ has also been found by Braaten and Nieto in the case of QCD [21]. No term of order $g^4 \ln(m/T)$ arises in $g^2 \Phi^4$-theory, since $f(\Lambda)$ does not run at next-to-leading order in $g^2$ [20]. This remark also applies to QED [23].

5 Concluding Remarks

In the present work we have calculated the free energy in SQED to order $\lambda^2$, $\lambda e^2$ and $e^4$ by effective field theory methods. The major advantage of the effective field theory approach is that it allows one to work with a single scale at a time. This makes perturbative calculations simpler, since the scales $T$ and $e T$ are not intertwined in already complicated sum-integrals in the full theory. However, we would also like to emphasize that effective field theory have some limitations. At present it is not clear how to apply it in the computations of dynamical quantities. Here, the resummation program of Braaten and Pisarski or extensions thereof are still essential ingredients of high temperature field theory.

We would also like to outline the calculations necessary to push the calculations of the free energy to one more order in the coupling constants. The coefficient of the unit operator are written as a power series in $\lambda$ and $e^2$, and so we already know this parameter to sufficient precision. The free energy at one-loop in the effective theory goes like $M^3(\Lambda) T$ and $m_E^3(\Lambda) T$. This implies that we need to know the mass parameters in ESQED at next-to-leading order. The former has been calculated by Farakos et al. in Ref. [26] while the present author has computed the latter in Ref. [22]. Furthermore, there are no new operators which contribute at this order, so all that remains is calculating the three-loop diagrams in ESQED. As we have already noted both the coefficient of the unit operator and the scalar mass parameter depend explicitly on the scale $\Lambda$. We realize that, upon expanding $M^2(\Lambda)$ in powers of coupling constants, we get terms proportional to $e^5 \ln \Lambda/T$. This allows us to predict terms in the form $e^5 \ln M/T$ or $e^5 \ln m_E/T$ in the pressure of scalar electrodynamics. This also explains the absence of such terms in QCD [21] as well as QED [23], since the corresponding mass parameter $m_E^2(\Lambda)$ is independent of $\Lambda$ to fourth order in the gauge coupling.

Finally, we would like to briefly comment upon other applications of the present approach. This method is perhaps the most transparent way of doing effective field theory at finite temperature, and it would certainly be worthwhile extending it to e.g. the study
of phase transitions. This approach would then represent an alternative to the methods which explicitly construct the effective field theory of the light mode by integrating over the heavy modes [19]. Work along these lines is in progress [29].

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A Sum-integrals in the Full Theory

In this appendix we give the necessary details for the sum-integrals used in present work. Throughout the work we use the imaginary time formalism, where the four-momentum is \( P = (p_0, \mathbf{p}) \) with \( P^2 = p_0^2 + \mathbf{p}^2 \). The Euclidean energy takes on discrete values, \( p_0 = 2n\pi T \) for bosons. Dimensional regularization is used to regularize both infrared and ultraviolet divergences by working in \( d = 4 - 2\epsilon \) dimensions, and we apply the \( \overline{\text{MS}} \) renormalization scheme. We use the following shorthand notation for the sum-integrals that appear below:

\[
\sum \int f(P) \equiv \left( \frac{\gamma_E \mu^2}{4\pi} \right)^\epsilon T \sum_{p_0 = 2n\pi T} \int \frac{d^{3-2\epsilon} k}{(2\pi)^{3-2\epsilon}} f(P). \tag{A.1}
\]

Then \( \mu \) coincides with the renormalization scale in the \( \overline{\text{MS}} \) renormalization scheme.

Arnold and Zhai [12] have developed a new machinery to deal with complicated sum-integrals. The methods are completely analytic and represent significant progress in perturbative calculations at finite temperature. They have calculated and listed all the sum-integrals needed in the present work, except for the one in (A.8). This sum-integral has been evaluated by Braaten and Nieto [20] using the methods of Ref. [12]. We reproduce them below for the convenience of the reader.

The specific one-loop sum-integrals needed are

\[
\sum \int \ln P^2 = -\frac{\pi^2 T^4}{45} [1 + O(\epsilon)], \tag{A.2}
\]

\[
\sum \int \frac{1}{P^2} = \frac{T^2}{12} [1 + \left(2 \ln \frac{\mu}{4\pi T} + 2 + 2\zeta(-1)\right)\epsilon + O(\epsilon^2)], \tag{A.3}
\]

\[
\sum \int \frac{1}{(P^2)^2} = \frac{1}{(4\pi)^2} \left[\frac{1}{\epsilon} + 2 \ln \frac{\mu}{4\pi T} + 2\gamma_E + O(\epsilon)\right], \tag{A.4}
\]

20
\[
\sum \int \frac{p_0^2}{(P^2)^2} = -\frac{T^2}{24} \left[ 1 + \left( 2 \ln \frac{\mu}{4\pi T} + 2 \zeta'(-1) \right) \epsilon + O(\epsilon^2) \right].
\] (A.5)

All of the two-loop sum-integrals encountered in this work factorize into products of the one-loop sum-integrals listed above. The only two-loop sum-integral needed is:

\[
\sum \int \frac{1}{P^2 Q^2 (P + Q)^2} = 0.
\] (A.6)

Some of the three-loop sum-integrals factorize into products of three one-loop sum-integrals, while others factorize into a product of two one-loop sum-integrals and the two-loop sum-integral in (A.6). The specific three-loop sum-integrals needed in the present work are

\[
\sum \int \frac{1}{P^2 Q^2 K^2 (P + K + Q)^2} = \frac{1}{(4\pi)^2} \frac{T^4}{144} + 6 \ln \frac{\mu}{4\pi T} - 12 \zeta'(-3) \zeta(-3)
\]
\[
+ 48 \zeta'(-1) \zeta(-1) + \frac{182}{5} + O(\epsilon)
\] (A.7)

\[
\sum \int \frac{(P - Q)^4}{P^2 Q^2 K^2 (P + K + Q)^2} = \frac{1}{(4\pi)^2} \frac{T^4}{144} + \frac{22}{3} + 8\gamma_E + 64 \zeta'(-1) - \frac{20}{3} \zeta'(-3) \zeta(-3) + O(\epsilon).
\] (A.8)

The first of these three-loop sum-integrals is the bosonic basketball \[12\], while the second is denoted by \(\mathcal{M}_{2,-2}\) in Ref \[21\].

### B Integrals in the Effective Theory

In the effective three-dimensional theory we use dimensional regularization in \(3 - 2\epsilon\) dimensions to regularize infrared and ultraviolet divergences. In analogy with Appendix A, we define

\[
\int_p f(p) \equiv \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right) \epsilon \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}} f(p).
\] (B.1)

Again \(\mu\) coincides with the renormalization scale in the modified minimal subtraction renormalization scheme.

All the integrals needed in the present work have been evaluated and listed by Braaten and Nieto in Ref \[20\]. We list them below for the convenience of the reader.
In the effective theory we need the following one-loop integrals

\[
\int p \ln(p^2 + m^2) = -\frac{m^3}{6\pi} \left[ 1 + \epsilon \left( 2 \ln \frac{\mu}{2m} + \frac{8}{3} \right) + O(\epsilon^2) \right], \quad (B.2)
\]

\[
\int p \frac{1}{p^2 + m^2} = -\frac{m}{4\pi} \left[ 1 + \left( 2 \ln \frac{\mu}{2m} + 2 \right) \epsilon + O(\epsilon^2) \right], \quad (B.3)
\]

\[
\int p \frac{1}{(p^2 + m^2)^2} = \frac{1}{8\pi m} \left[ 1 + \left( 2 \ln \frac{\mu}{2m} \right) \epsilon + O(\epsilon^2) \right]. \quad (B.4)
\]

The two-loop integrals in the effective theory also factorize into products of one-loop integrals plus a single two-loop integral. More specifically one needs

\[
\int_{pq} \frac{1}{(p^2 + m^2)(q^2 + m^2)(p - q)^2} = \frac{1}{(4\pi)^2} \left[ \frac{1}{4\epsilon} + \frac{1}{2} + \ln \frac{\mu}{2m} + O(\epsilon) \right].
\]

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Figure 1: One-loop scalar self-energy diagrams in SQED.

Figure 2: One-loop self-energy diagrams in the full theory.

Figure 3: One-loop vacuum diagrams in SQED.
Figure 4: Two-loop diagrams for the free energy in SQED.

Figure 5: Three-loop diagrams contributing to the free energy in the underlying theory.
Figure 6: One and two-loop diagrams contributing to the free energy in ESQED.