Four-operator splitting algorithms for solving monotone inclusions

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Abstract. Monotone inclusions involving the sum of three maximally monotone operators or more have received much attention in recent years. In this paper, we propose three splitting algorithms for finding a zero of the sum of four monotone operators, which are two maximally monotone operators, one monotone Lipschitz operator, and one cocoercive operator. These three splitting algorithms are based on the forward-reflected-Douglas-Rachford splitting algorithm, backward-forward-reflected-backward splitting algorithm, and backward-reflected-forward-backward splitting algorithm, respectively. As applications, we apply the proposed algorithms to solve the monotone inclusions problem involving a finite sum of maximally monotone operators. Numerical results on the Projection on Minkowski sums of convex sets demonstrate the effectiveness of the proposed algorithms.

Key words: Maximally monotone operators; Lipschitz operator; Cocoercive operator; Operator splitting algorithms.

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1 Introduction

Monotone inclusions play an important role in studying various convex minimization problems, which arise in signal and image processing, medical image reconstruction and machine learning, etc. The traditional operator splitting algorithms are used to solve monotone inclusions involving the sum of two maximally monotone operators, where one of them is cocoercive or monotone Lipschitz. The most popular operator splitting algorithms include the Douglas-Rachford splitting algorithm [1, 2], the forward-backward splitting algorithm [1], and the forward-backward-forward splitting algorithm [3]. These operator splitting algorithms and their variants have been extensively studied. See, e.g., [4–11] and references therein. To deal with monotone inclusions of the sum of three maximally monotone operators or more, several new operator splitting algorithms have been proposed. See, for example [12–24].

Let \( \mathcal{H} \) be a real Hilbert space and \( m \geq 1 \) be an integer. For each \( i \in \{1, \cdots, m\} \), let \( A_i : \mathcal{H} \to 2^{\mathcal{H}} \) be maximally monotone. Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be maximally monotone, \( B : \mathcal{H} \to \mathcal{H} \) be monotone and \( L \)-Lipschitz with \( L > 0 \), and \( C : \mathcal{H} \to \mathcal{H} \) be \( \beta \)-cocoercive, for some \( \beta > 0 \). Raguet et al. [12] first proposed a generalized forward-backward splitting algorithm for solving the following monotone inclusions problem:

\[
\text{find } x \in \mathcal{H} \text{ such that } 0 \in \sum_{i=1}^{m} A_i x + C x. \tag{1.1}
\]

In particular, the generalized forward-backward splitting algorithm reduces to the classical forward-backward splitting algorithm when \( m = 1 \). When the cocoercive operator \( C \) is replaced by a monotone Lipschitz operator \( B \), Banert [13] considered the following monotone inclusions problem:

\[
\text{find } x \in \mathcal{H} \text{ such that } 0 \in \sum_{i=1}^{m} A_i x + B x. \tag{1.2}
\]
As a consequence, Banert \cite{13} proposed a relaxed forward-backward splitting algorithm. On the other hand, Davis and Yin \cite{14} introduced a so-called three-operator splitting algorithm for solving the following monotone inclusions

\[
\text{find } x \in H \text{ such that } 0 \in A_1x + A_2x + Cx.
\] (1.3)

Briceño-Arias \cite{16} studied the three-operator monotone inclusions \cite{13}, where one of the maximally monotone operators is the normal cone of a closed vector subspace. It is worth mentioning that the generalized forward-backward splitting algorithm \cite{12} could be recovered by both of the operator splitting algorithms proposed by \cite{14} and \cite{16}.

In recent years, many authors studied the following three-operator monotone inclusions problem.

\[
\text{find } x \in H \text{ such that } 0 \in Ax + Bx + Cx.
\] (1.4)

In particular, Briceño-Arias and Davis \cite{18} first proposed a so-called forward-backward-half-forward splitting algorithm, which combined the classical forward-backward splitting algorithm and Tseng’s forward-backward-forward splitting algorithm. Recently, a semi-forward-reflected-backward splitting algorithm has been proposed by Malitsky and Tam \cite{19}, and a semi-reflected-forward-backward splitting algorithm has been proposed by Cevher and Vă \cite{20,25}. In addition, Yu et al. \cite{23} introduced an outer reflected forward-backward splitting algorithm to solve this problem as well.

If the cocoercive operator of \cite{13} is relaxed to monotone Lipschitz operator, then the problem \cite{13} is reformulated as the following monotone inclusions:

\[
\text{find } x \in H \text{ such that } 0 \in A_1x + A_2x + Bx.
\] (1.5)

Ryu and Vă \cite{21} proposed a so-called forward-reflected-Douglas-Rachford splitting algorithm, which is defined by,

\[
\begin{aligned}
x_{n+1} &= J_{\gamma A_2}(x_n - \gamma u_n + \gamma (2Bx_n - Bx_{n-1})) \\
y_{n+1} &= J_{\lambda A_1}(2x_{n+1} - x_n + \lambda u_n) \\
u_{n+1} &= u_n + \frac{1}{\lambda}(2x_{n+1} - x_n - y_{n+1}),
\end{aligned}
\] (1.6)

where \(\lambda > 0\) and \(\gamma \in (0, \frac{\beta}{1 + 2\beta L})\). Besides, Rieger and Tam \cite{22} proposed two splitting algorithms to solve the problem \cite{13}. One is the Backward-forward-reflected-backward splitting algorithm, which combines the Forward-reflected-backward splitting algorithm and the Douglas-Rachford splitting algorithm. The iterative scheme is given by

\[
\begin{aligned}
x_{n+1} &= J_{\gamma A_1}z_n \\
y_{n+1} &= J_{\gamma A_2}(2x_{n+1} - z_n - 2\gamma By_n + \gamma By_{n-1}) \\
z_{n+1} &= z_n + y_{n+1} - x_{n+1},
\end{aligned}
\] (1.7)

where \(\gamma \in (0, \frac{1}{8L})\). The other is called a Backward-reflected-forward-backward splitting algorithm, which combines the Reflected-forward-backward splitting algorithm and the Douglas-Rachford splitting algorithm. The iterative scheme is given by

\[
\begin{aligned}
x_{n+1} &= J_{\gamma A_1}z_n \\
y_{n+1} &= J_{\gamma A_2}(2x_{n+1} - z_n - \gamma B(2y_n - y_{n-1})) \\
z_{n+1} &= z_n + y_{n+1} - x_{n+1},
\end{aligned}
\] (1.8)

where \(\gamma \in (0, \frac{\beta}{1 + 2\beta L})\). In summary, we summarize existing algorithms for solving monotone inclusions containing three-operator and beyond in Table 1.

In this paper, we consider the following four-operator monotone inclusions:

\[
\text{find } x \in H \text{ such that } 0 \in A_1x + A_2x + Bx + Cx
\] (1.9)
Table 1: Operator splitting algorithms for solving three-operator monotone inclusions and beyond.

| Monotone inclusions | Operator splitting algorithms |
|---------------------|-----------------------------|
| \(0 \in \sum_{i=1}^m A_i x + C x\) (1.1) | Generalized forward-backward splitting algorithm [12] |
| \(0 \in \sum_{i=1}^m A_i x + B x\) (1.2) | Relaxed forward-backward splitting algorithm [13] |
| \(0 \in A_1 x + A_2 x + C x\) (1.3) | Three-operator splitting algorithm [14] |
| \(0 \in A x + B x + C x\) (1.4) | Forward-backward-half-forward splitting algorithm [18] |
| \(0 \in A_1 x + A_2 x + B x\) (1.5) | Semi-forward-reflected-backward splitting algorithm [19] |
| \(0 \in A_1 x + A_2 x + B x\) (1.6) | Semi-reflected-forward-backward splitting algorithm [20] |
| \(0 \in A_1 x + A_2 x + B x\) (1.7) | Outer reflected forward-backward splitting algorithm [23] |
| \(0 \in A_1 x + A_2 x + B x\) (1.8) | Forward-reflected-Douglas-Rachford splitting algorithm [21] |
| \(0 \in A_1 x + A_2 x + B x\) (1.9) | Backward-forward-reflected-backward splitting algorithm [22] |
| \(0 \in A_1 x + A_2 x + B x\) (1.10) | Backward-reflected-forward-backward splitting algorithm [22] |

Although the four-operator monotone inclusions (1.9) could be viewed as special cases of the three-operator monotone inclusions (1.4) or (1.5), there are some drawbacks:

(i) Let \(A = A_1 + A_2\), then (1.9) is a special case of (1.4). Therefore, we can employ the three-operator splitting algorithms [18–20, 23] to solve (1.9). However, we have to compute the resolvent \(J_{\lambda A} = J_{\lambda(A_1 + A_2)}\), \(\lambda > 0\), which usually doesn’t have a closed-form solution.

(ii) Let \(\bar{B} = B + C\). Since \(C\) is cocoercive, \(\bar{B}\) is monotone and \(L + \frac{1}{\beta}\)-Lipschitz. Then, we can use the three-operator splitting algorithms (1.6)-(1.8) to solve (1.9). It is obvious that we do not make full use of the cocoercive property of \(C\).

To overcome these drawbacks, in this paper, we introduce and analyze three new splitting algorithms for solving the monotone inclusions problem (1.9). These algorithms are established on the foundation of the Forward-reflected-Douglas-Rachford splitting algorithm [21], the Backward-forward-reflected-backward splitting algorithm [22] and the Backward-reflected-forward-backward splitting algorithm [22]. As applications, we study composite monotone inclusions involving a finite sum of maximally monotone operators.

The paper is organized as follows. In Section 2, we introduce notations and preliminary results in monotone operator theory. In Section 3, we present three splitting algorithms for solving monotone inclusions involving four operators and prove their convergence. In Section 4, the proposed algorithms are applied to solve the monotone inclusions problem involving the sum of a finite of maximally monotone operators. In Section 5, we conduct some numerical experiments on the Projection problem onto the Minkowski sum of convex sets. Finally, we will give some conclusions.

2 Preliminaries

Throughout this paper, let \(H\) be a real Hilbert space, and its scalar product and the associated norm are denoted by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\), respectively. The symbols \(\rightharpoonup\) and \(\rightarrow\) denote weak and strong convergence, respectively. \(N\) denotes the set of natural numbers. Let \(2^H\) be the power set of \(H\).

Let \(A : H \to 2^H\) be a set-valued operator. The graph of \(A\) is defined by \(graA = \{(x, u) \in H \times H | u \in Ax\}\), the effective domain of \(A\) is defined by \(domA = \{x \in H | Ax \neq \emptyset\}\), and the sets of zeros of \(A\) is defined by \(zerA = \{x \in H | 0 \in Ax\}\). The inverse of \(A\) is \(A^{-1} : u \mapsto \{x \in Ax\}\). Let \(C\) be a nonempty closed convex set of \(H\), the normal cone to \(C\) is defined by

\[ NC : H \to 2^H : x \mapsto \begin{cases} \{u \in H | (\forall y \in C) \langle y - x, u \rangle \leq 0\}, & \text{if } x \in C; \\ \emptyset, & \text{otherwise.} \end{cases} \] (2.1)
Definition 2.1. (Maximally monotone operator) A set-valued operator \( A : \mathcal{H} \to 2^{\mathcal{H}} \) is monotone if
\[
(\forall (x, u), (y, v) \in \text{gra}A) \quad \langle x - y, u - v \rangle \geq 0,
\]
and it is said to be maximally monotone if there exists no monotone operator \( B : \mathcal{H} \to 2^{\mathcal{H}} \) such that \( \text{gra}B \) properly contains \( \text{gra}A \).

Definition 2.2. (Resolvent) Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be a set-valued operator. The resolvent \( J_A : \mathcal{H} \to 2^{\mathcal{H}} \) of \( A \) is
\[
J_A = (\text{Id} + A)^{-1},
\]
where \( \text{Id} \) denotes the identity operator. When \( A \) is maximally monotone, \( J_A \) is single-valued.

Definition 2.3. (Lipschitz continuous) A single-valued operator \( B : \mathcal{H} \to \mathcal{H} \) is \( L \)-Lipschitz for some \( L > 0 \) if
\[
(\forall x, y \in \mathcal{H}) \quad \|Bx - By\| \leq L\|x - y\|.
\]
In particular, it is nonexpansive if the operator \( B \) is 1-Lipschitz.

Definition 2.4. (Cocoercive operator) A single-valued operator \( C : \mathcal{H} \to \mathcal{H} \) is \( \beta \)-cocoercive for some \( \beta > 0 \) if
\[
(\forall x, y \in \mathcal{H}) \quad \langle x - y, Cx - Cy \rangle \geq \beta\|Cx - Cy\|^2.
\]

Definition 2.5. (Parallel sum) The parallel sum \( A_1 \square \cdots \square A_m : \mathcal{H} \to 2^{\mathcal{H}} \) of the operators \( A_i, i = 1, \cdots, m \) is
\[
A_1 \square \cdots \square A_m = (A_1^{-1} + \cdots + A_m^{-1})^{-1}.
\]

We present some useful properties of maximally monotone operators.

Lemma 2.1. (Maximally monotone operator) Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be maximally monotone, and let \( \gamma > 0 \). Then \( J_\gamma A : \mathcal{H} \to \mathcal{H} \) is firmly nonexpansive, that is,
\[
(\forall x, y \in \mathcal{H}) \quad \|J_\gamma A x - J_\gamma A y\|^2 + \|(\text{Id} - J_\gamma A)x - (\text{Id} - J_\gamma A)y\|^2 \leq \|x - y\|^2.
\]

Lemma 2.2. (Resolvent) (i) Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be maximally monotone, let \( r, y \in \mathcal{H} \), and let \( \alpha > 0 \). Then \( A^{-1} \) and \( \alpha A(x - r) + y \) are maximally monotone.

(ii) Let \( \mathcal{H} \) and \( \mathcal{G} \) be real Hilbert spaces. Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) and \( B : \mathcal{G} \to 2^{\mathcal{G}} \) be maximally monotone. Then \( A \times B \) is maximally monotone.

Lemma 2.3. (Maximally monotone operator) Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be maximally monotone and \( B : \mathcal{H} \to \mathcal{H} \) be monotone and Lipschitz with \( \text{dom}B = \mathcal{H} \). Then \( A + B \) is maximally monotone.

We shall make use of the following Opial’s lemma to prove the convergence.

Lemma 2.4. (Opial’s lemma) Let \( Z \) be a nonempty subset of \( \mathcal{H} \) and \( \{z_n\} \) be a sequence in \( \mathcal{H} \). Suppose the following conditions hold:
(i) For every \( z \in Z \), \( \lim_{n \to \infty} \|z_n - z\| \) exists;
(ii) Every weak cluster point of \( \{z_n\} \) belongs to \( Z \).
Then \( \{z_n\} \) converges weakly to a point in \( Z \).

3 Main algorithms and convergence theorems

In this section, we present three main algorithms and prove their convergence theorems.
3.1 Backward-Semi-Forward-Reflected-Backward splitting algorithm

Now, we are ready to introduce the first splitting algorithm.

\[
\begin{aligned}
  x_{n+1} &= J_{\gamma A_1} z_n \\
y_{n+1} &= J_{\gamma A_2}(2x_{n+1} - z_n - 2\gamma By_n + \gamma By_n - \gamma Cy_n) \\
z_{n+1} &= z_n + y_{n+1} - x_{n+1}
\end{aligned}
\]  

(3.1)

Remark 3.1. The following iterative algorithms are particular cases of (3.1).

(i) Semi-forward-reflected-backward splitting algorithm [19]: if \( A_1 = 0 \), (3.1) reduces to

\[
z_{n+1} = J_{\gamma A_2}(z_n - 2\gamma Bz_n + \gamma Bz_{n-1} - \gamma Cz_n).
\]  

(3.2)

(ii) Backward-forward-reflected-backward splitting algorithm [22]: if \( C = 0 \), (3.1) becomes

\[
\begin{aligned}
x_{n+1} &= J_{\gamma A_1} z_n \\
y_{n+1} &= J_{\gamma A_2}(2x_{n+1} - z_n - 2\gamma By_n + \gamma By_{n-1}) \\
z_{n+1} &= z_n + y_{n+1} - x_{n+1}.
\end{aligned}
\]  

(3.3)

Therefore, we call the iterative algorithm (3.1) a backward-semi-forward-reflected-backward splitting algorithm.

Lemma 3.1. Suppose that there exist \( x, z \in H \) such that \( z - x \in \gamma A_1 x \) and \( x - z \in \gamma (A_2 + B + C)x \). Let the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) be defined in (3.1). Then, for all \( n \in N \), we have

\[
\|z_{n+1} - z\|^2 + 2\gamma \langle By_{n+1} - By_n, x - y_{n+1} \rangle + \|z_{n+1} - z_n\|^2 \\
\leq \|z_n - z\|^2 + 2\gamma \langle By_n - By_{n-1}, x - y_n \rangle + 2\gamma \langle By_n - By_{n-1}, y_n - y_{n+1} \rangle + 2\gamma \langle Cy_n - Cx, x - y_{n+1} \rangle.
\]  

(3.4)

Proof. Since \( A_1 \) is monotone, we have

\[
0 \leq \langle (z - x) - (z_n - x_{n+1}), x - x_{n+1} \rangle.
\]  

(3.5)

Combining the monotonicity of \( A_2 \) and (3.5), it follows that

\[
0 \leq \langle (x - z - \gamma (B + C)x) - (2x_{n+1} - z_n - y_{n+1} - 2\gamma By_n + \gamma By_{n-1} - \gamma Cy_n), x - y_{n+1} \rangle \\
= \langle (x - z) - (x_{n+1} - z_n), x - x_{n+1} \rangle + \langle z_{n+1} - z_n, z - z_{n+1} \rangle + \gamma \langle By_n - Bx, x - y_{n+1} \rangle \\
+ \gamma \langle By_n - By_{n-1}, x - y_{n+1} \rangle + \gamma \langle Cy_n - Cx, x - y_{n+1} \rangle \\
\leq \langle z_{n+1} - z_n, z - z_{n+1} \rangle + \gamma \langle By_n - Bx, x - y_{n+1} \rangle + \gamma \langle By_n - By_{n-1}, x - y_{n+1} \rangle \\
+ \gamma \langle Cy_n - Cx, x - y_{n+1} \rangle.
\]  

(3.6)

Using the monotonicity of \( B \) yields

\[
\gamma \langle By_n - Bx, x - y_{n+1} \rangle = \gamma \langle By_n - By_{n+1}, x - y_{n+1} \rangle + \gamma \langle By_{n+1} - Bx, x - y_{n+1} \rangle \\
\leq \gamma \langle By_{n+1} - By_{n+1}, x - y_{n+1} \rangle.
\]  

(3.7)

By substituting the estimate (3.7) into (3.6), and using the identity

\[
\langle z_{n+1} - z_n, z - z_{n+1} \rangle = \frac{1}{2}(\|z_n - z\|^2 - \|z_{n+1} - z_n\|^2 - \|z_{n+1} - z\|^2),
\]  

(3.8)

the inequality (3.6) can be expressed as

\[
0 \leq \|z_n - z\|^2 - \|z_{n+1} - z_n\|^2 - \|z_{n+1} - z\|^2 + 2\gamma \langle By_n - By_{n+1}, x - y_{n+1} \rangle + 2\gamma \langle Cy_n - Cx, x - y_{n+1} \rangle.
\]  

(3.9)

which implies the claimed inequality (3.4) holds. \( \square \)
Theorem 3.1. Let $A_i : \mathcal{H} \to 2^\mathcal{H}$, $i = 1, 2$ be maximally monotone, let $B : \mathcal{H} \to \mathcal{H}$ be monotone and $L$-Lipschitz continuous, let $C : \mathcal{H} \to \mathcal{H}$ be $\beta$-cocoercive, and assume that $\text{zer}(A_1 + A_2 + B + C) \neq \emptyset$. Let $\gamma \in \left(0, \frac{\beta}{2(1+4\beta L)}\right)$. Let $z_0$, $y_0$, $y_{-1} \in \mathcal{H}$ and consider the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ defined in (3.1). Then, for all $n \in N$, the following hold:

(i) The sequence $\{z_n\}$ converges weakly to a point $\bar{z} \in \mathcal{H}$.

(ii) The sequences $\{x_n\}$, $\{y_n\}$ converge weakly to a point $\bar{x} \in \mathcal{H}$.

(iii) $\bar{x} = J_{\gamma A_1}(\bar{z}) \in \text{zer}(A_1 + A_2 + B + C)$.

Proof. We define the set $Z$ in Lemma 2.3 by

$$Z := \{z \in \mathcal{H} : J_{\gamma A_1} z \in \text{zer}(A_1 + A_2 + B + C), (J_{\gamma A_1} - \text{Id})z \in \gamma(A_2 + B + C)(J_{\gamma A_1} z)\}.$$  (3.10)

Next, we need to prove that $Z$ is nonempty. Note that $\text{zer}(A_1 + A_2 + B + C) \neq \emptyset$, thus there exist $x \in \text{zer}(A_1 + A_2 + B + C)$ and $z \in \mathcal{H}$ such that $z - x \in \gamma A_1 x$ and $x - z \in \gamma(A_2 + B + C)x$. Using the two inclusions, it gives

$$x = J_{\gamma A_1} z \quad \text{and} \quad (J_{\gamma A_1} - \text{Id})z \in \gamma(A_2 + B + C)(J_{\gamma A_1} z),$$  (3.11)

which imply $z \in Z$, that is, the set $Z$ is nonempty.

Let $z \in Z$ and define $x := J_{\gamma A_1} z$. By Lemma 3.1 for all $n \in N$, we have

$$\|z_{n+1} - z\|^2 + 2\gamma \langle By_{n+1} - By_n, x - y_{n+1} \rangle + \|z_{n+1} - z_n\|^2$$

$$\leq \|z_n - z\|^2 + 2\gamma \langle By_n - By_{n-1}, x - y_n \rangle + 2\gamma \langle By_n - By_{n-1}, y_n - y_{n+1} \rangle + 2\gamma \langle C y_n - C x, x - y_{n+1} \rangle.$$  (3.12)

By applying Young’s inequality and using the nonexpansivity of $\text{Id} - J_{\gamma A_1}$, which follows from the firm nonexpansivity of $J_{\gamma A_1}$, we obtain

$$\|y_n - y_{n-1}\|^2 = \|(z_n - z_{n-1} + x_n) - (z_{n-1} - z_{n-2} + x_{n-1})\|^2$$

$$\leq (1 + a)\|z_n - z_{n-1}\|^2 + (1 + \frac{1}{a})\|(x_n - z_{n-1}) - (x_{n-1} - z_{n-2})\|^2$$

for some $a > 0$. Combining the Lipschitz property of $B$ and (3.13), the second-last term in (3.12) can be estimated by

$$2\gamma \langle By_n - By_{n-1}, y_n - y_{n+1} \rangle$$

$$\leq \gamma L(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)$$

$$\leq (1 + \frac{1}{a})\gamma L\|z_{n-1} - z_{n-2}\|^2 + (2 + a + \frac{1}{a})\gamma L\|z_n - z_{n-1}\|^2 + (1 + a)\gamma L\|z_{n+1} - z_n\|^2.$$  (3.14)

Combining the cocoercivity of $C$ and (3.13), for all $\varepsilon > 0$, the last term in (3.12) can be estimated by

$$2\gamma \langle Cy_n - Cx, x - y_{n+1} \rangle$$

$$= 2\gamma \langle Cy_n - Cx, x - y_n \rangle + 2\varepsilon \langle \frac{2}{\varepsilon} (Cy_n - Cx), y_n - y_{n+1} \rangle$$

$$\leq - 2\gamma^2 \|Cy_n - Cx\|^2 + \frac{\varepsilon}{\varepsilon} \|Cy_n - Cx\|^2 + \varepsilon \|y_n - y_{n+1}\|^2 - \varepsilon \|\frac{2}{\varepsilon} (Cy_n - Cx) - (y_n - y_{n+1})\|^2$$

$$\leq (1 + a)\varepsilon \|z_{n+1} - z_n\|^2 + (1 + \frac{1}{a})\varepsilon \|z_n - z_{n-1}\|^2 - \frac{\gamma^2}{\varepsilon} (2\beta \varepsilon - \gamma) \|Cy_n - Cx\|^2.$$  (3.15)

By substituting the estimates (3.13), (3.14) into (3.12), it follows that

$$\|z_{n+1} - z\|^2 + 2\gamma \langle By_{n+1} - By_n, x - y_{n+1} \rangle + \|z_{n+1} - z_n\|^2$$

$$\leq \|z_n - z\|^2 + 2\gamma \langle By_n - By_{n-1}, x - y_n \rangle + (1 + a)(\gamma L + \varepsilon)\|z_{n+1} - z_n\|^2$$

$$+ ((2 + a + \frac{1}{a})\gamma L + (1 + \frac{1}{a})\varepsilon)\|z_n - z_{n-1}\|^2 + (1 + \frac{1}{a})\gamma L\|z_{n+1} - z_{n-2}\|^2 - \frac{\gamma^2}{\varepsilon} (2\beta \varepsilon - \gamma) \|Cy_n - Cx\|^2.$$  (3.16)
For convenience, we denote
\[ V_n := \|z_n - z\|^2 + 2\gamma\langle By_n - By_{n-1}, x - y_n \rangle + \left((3 + a + \frac{2}{a})\gamma L(1 + \frac{1}{a})\epsilon\right)\|z_n - z_{n-1}\|^2 + (1 + \frac{1}{a})\gamma L\|z_{n-1} - z_{n-2}\|^2. \]

Hence, (3.16) can be simply expressed as
\[ V_{n+1} + (1 - (2 + a + 1)\epsilon)(\xi + 2\gamma L))\|z_{n+1} - z_n\|^2 + \frac{\gamma}{\epsilon}(2\beta\epsilon - \gamma)\|Cy_n - Cx\|^2 \leq V_n. \]

In order to ensure that the sequence \( \{V_n\} \) is nonincreasing, the second and third terms in (3.18) shall be positive, from which we derive two upper bounds of \( \gamma \): \( \gamma < \frac{1 - (2 + a + 1)\epsilon}{2L(2 + a + 1)\epsilon} \) and \( \gamma < 2\beta\epsilon \). To get the largest interval for \( \gamma \), taking these two bounds equal yields \( \epsilon = \frac{1}{(2 + a + 1)(1 + 4\beta\epsilon + 1)} \). Therefore, in the particular case when \( a = 1 \), we obtain the desired result of \( \gamma \). In addition, (3.18) gives
\[ V_{n+1} + \epsilon'\|z_{n+1} - z_n\|^2 \leq V_n, \]
which implies that
\[ V_{n+1} + \epsilon'\sum_{i=0}^n \|z_{i+1} - z_i\|^2 \leq V_0, \]
where \( \epsilon' = 1 - \frac{1}{1 + 4\beta\epsilon} - 8\gamma L \). Next, we need to show that \( \{V_{n+1}\} \) has a lower bound. By the nonexpansivity of \( J_{\gamma A_1} \), the Lipschitz property of \( B \) and (3.13), it follows that
\[ 2\gamma\langle By_{n+1} - By_n, x - y_{n+1} \rangle \leq \gamma L\|y_{n+1} - y_n\|^2 + 2\gamma L\|z_{n+1} - z_{n-1}\|^2 + 2\gamma L\|z_{n+1} - (z_{n-1} + x_{n+1})\|^2. \]

This gives
\[ V_{n+1} = \|z_{n+1} - z\|^2 + 2\gamma\langle By_{n+1} - By_n, x - y_{n+1} \rangle + (6\gamma L + 2\epsilon)\|z_{n+1} - z_n\|^2 + 2\gamma L\|z_{n+1} - z_{n-1}\|^2 \]
\[ \geq (1 - 2\gamma L)\|z_{n+1} - z\|^2 + (2\gamma L + 2\epsilon)\|z_{n+1} - z_n\|^2 \]
\[ \geq (6\gamma L + 2\epsilon)\|z_{n+1} - z\|^2 \geq 0. \]

Consequently, \( \{V_n\} \) converges, together with (3.20), (3.22), which implies \( \|z_{n+1} - z_n\| \to 0 \) and \( \{z_n\} \) is bounded. Due to the fact that \( x_{n+1} = J_{\gamma A_1}z_n \) and \( x = J_{\gamma A_1}z \), the nonexpansivity of \( J_{\gamma A_1} \) yields \( \|x_{n+1} - x_n\| \to 0 \) and \( \{x_n\} \) is bounded. From the identity \( y_{n+1} = z_{n+1} - z_n + x_{n+1} \), we deduce that \( \|y_{n+1} - y_n\| \to 0 \) and \( \{y_n\} \) is bounded. Hence, we obtain that
\[ \lim_{n \to \infty} V_n = \lim_{n \to \infty} \|z_n - z\|^2. \]

On the other hand, let \( z \in H \) be a weak cluster point of \( \{z_n\} \). By the boundedness of \( \{x_k\} \), it follows that there exists \( x \in H \) such that \( (x, z) \) is a weak cluster point of \( \{(x_{nk}), (z_{nk})\}_{k \in N} \). Using the definition of the resolvent operator, (3.11) can be expressed as
\[ (z_{nk} - z_{nk+1}) - \gamma (By_{nk} - By_{nk+1}) \leq 0 \quad \gamma (z_{nk} - z_{nk+1}) \leq 0 \quad \gamma (Cy_{nk} - Cy_{nk+1}) \leq 0. \]

where \( A = \begin{bmatrix} (\gamma A_1)^{-1} & 0 \\ 0 & \gamma (A_2 + B + C) \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{bmatrix} \). Since \( (\gamma A_1)^{-1} \) and \( \gamma (A_2 + B + C) \) are maximally monotone by Lemma [2.2] and [2.3], \( A + B \) is also maximally monotone. Thus, its graph is closed in the weak-strong topology on \( H \times H \). By taking the limit along the subsequence in (3.24), it yields
\[ \begin{cases} 0 \in (\gamma A_1)^{-1}(z - x) - x \\ 0 \in z - x + (A_2 + B + C)x \end{cases} \Rightarrow x = J_{\gamma A_1}z \]
\[ (J_{\gamma A_1} - \text{Id})z \in \gamma (A_2 + B + C)(J_{\gamma A_1}z). \]
Hence, we deduce that \( z \in Z \). According to Lemma 2.4 it follows that \( \{z_n\} \) converges weakly to \( \bar{z} \in Z \). Let \( x \in H \) be a weak cluster point of \( \{x_n\} \). Taking the limit along the subsequence \( (\{x_n\}, \{z_n\}) \) of the inclusion \( z_n - x_{n+1} \in \gamma A_1 x_{n+1} \) and yield \( x = J_{\gamma A_1} \bar{z} \). Therefore, the sequence \( \{x_n\} \) has the unique weak cluster point and converges weakly to \( \bar{x} = J_{\gamma A_1} \bar{z} \). Finally, Combining the identity \( y_{n+1} = z_{n+1} - z_n + x_{n+1} \) and the convergence of the sequence \( \{x_n\} \) and \( \{z_n\} \), we derive that \( \{y_n\} \) converges weakly to a point \( \bar{x} \in H \).

**Remark 3.2.** It is worth mentioning that the problem (1.9) can also be solved by applying the Backward-forward-reflected-backward splitting algorithm (1.7) when \( \beta \in \gamma \) and yield (3.30). Meanwhile, we derive that the stepsize \( \gamma \) in the algorithm (1.7) satisfies \( \gamma \in \left(0, \frac{\beta}{8(1+\beta L)}\right) \). Due to the fact that \( \frac{\beta}{8(1+\beta L)} < \frac{\beta}{2(1+\beta L)} \), the stepsize \( \gamma \) in the algorithm 3.1 has a slight improvement.

### 3.2 Backward-Semi-Reflected-Forward-Backward splitting algorithm

We propose the second splitting algorithms as follows.

\[
\begin{aligned}
x_{n+1} &= J_{\gamma A_1} z_n \\
y_{n+1} &= J_{\gamma A_2} (2x_{n+1} - z_n - \gamma B(2y_n - y_{n-1}) - \gamma C y_n) \\
z_{n+1} &= z_n + y_{n+1} - x_{n+1}
\end{aligned}
\]  

(3.26)

**Remark 3.3.** The following iterative algorithms are special cases of (3.26).

(i) Semi-reflected-forward-backward splitting algorithm [20]: if \( A_1 = 0 \), (3.26) reduces to

\[
\begin{aligned}
z_{n+1} &= J_{\gamma A_2} (z_n - \gamma B(2z_n - z_{n-1}) - \gamma C z_n).
\end{aligned}
\]  

(3.27)

(ii) Backward-reflected-forward-backward splitting algorithm [22]: if \( C = 0 \), (3.26) becomes

\[
\begin{aligned}
x_{n+1} &= J_{\gamma A_1} z_n \\
y_{n+1} &= J_{\gamma A_2} (2x_{n+1} - z_n - \gamma B(2y_n - y_{n-1})) \\
z_{n+1} &= z_n + y_{n+1} - x_{n+1}.
\end{aligned}
\]  

(3.28)

Therefore, we call the iterative algorithm (3.26) a backward-semi-reflected-forward-backward splitting algorithm. For convenience, we first give the following notations.

\[
\begin{aligned}
\hat{x}_n &= 2x_n - x_{n-1}, \hat{y}_n = 2y_n - y_{n-1}, \hat{z}_n = 2z_n - z_{n-1}
\end{aligned}
\]  

(3.29)

**Lemma 3.2.** Suppose that there exist \( x, z \in H \) such that \( z - x \in \gamma A_1 x \) and \( x - z \in \gamma (A_2 + B + C)x \). Let the sequences \( \{x_n\} \), \( \{y_n\} \) and \( \{z_n\} \) be defined in (3.26). Then, for all \( n \in N \), we have

\[
\begin{aligned}
&\|z_{n+1} - z\|^2 + 2\gamma \langle B\hat{y}_n - Bx, y_{n+1} - y_n \rangle + 2\|z_{n+1} - z_n\|^2 + \|z_{n+1} - \hat{z}_n\|^2 \\
&\leq \|z_n - z\|^2 + 2\gamma \langle B\hat{y}_{n-1} - Bx, y_n - y_{n-1} \rangle + \|z_n - z_{n-1}\|^2 + 2\gamma \langle B\hat{y}_n - B\hat{y}_{n-1}, \hat{y}_n - y_{n+1} \rangle \\
&+ 2\gamma \langle C y_{n-1} - C y_n, y_{n+1} - y_n \rangle + 2\gamma \langle C y_n - C x, x - y_{n+1} \rangle.
\end{aligned}
\]  

(3.30)

**Proof.** Since \( A_1 \) is monotone, we have

\[
0 \leq \langle (z - x) - (z_n - x_{n+1}), x - x_{n+1} \rangle.
\]  

(3.31)

Combining the monotonicity of \( A_2 \) and (3.5), it follows that

\[
\begin{aligned}
0 &\leq \langle (x - z - \gamma (B + C)x) - (2x_{n+1} - z_n - y_{n+1} - \gamma B\hat{y}_n - \gamma C y_n), x - y_{n+1} \rangle \\
&= \langle (x - z) - (x_{n+1} - z_n), x - x_{n+1} \rangle + \langle z_{n+1} - z_n, z - z_{n+1} \rangle + \gamma \langle B\hat{y}_n - Bx, x - y_{n+1} \rangle \\
&+ \gamma \langle C y_{n-1} - C x, x - y_{n+1} \rangle \\
&\leq \langle z_{n+1} - z_n, z - z_{n+1} \rangle + \gamma \langle B\hat{y}_n - Bx, x - y_{n+1} \rangle + \gamma \langle C y_n - C x, x - y_{n+1} \rangle.
\end{aligned}
\]  

(3.32)
Using the monotonicity of $B$ yields

$$
\gamma(B\hat{y}_n - Bx, x - y_{n+1}) = \gamma(B\hat{y}_n - Bx, x - \hat{y}_n) + \gamma(B\hat{y}_n - Bx, \hat{y}_n - y_{n+1})
\leq \gamma(B\hat{y}_n - Bx, \hat{y}_n - y_{n+1}).
$$

(3.33)

By substituting the estimate (3.33) into (3.32) and using the identity (3.13), the inequality (3.32) can be expressed as

$$
0 \leq \|z_n - z\|^2 - \|z_{n+1} - z_n\|^2 - \|z_{n+1} - z\|^2 + 2\gamma(B\hat{y}_n - B\hat{y}_{n+1}, \hat{y}_n - y_{n+1}) + 2\gamma(B\hat{y}_{n-1} - Bx, \hat{y}_n - y_{n+1}) + 2\gamma(Cy_n - Cx, x - y_{n+1}).
$$

(3.34)

Next, we need to estimate the second-last term in (3.34). Since $A_1$ is monotone, we obtain

$$
0 \leq \langle (z_n - x_{n+1}) - (z_{n-1} - x_n), x_{n+1} - x_n \rangle.
$$

(3.35)

Combining the monotonicity of $A_2$ and (3.35), it follows that

$$
0 \leq \langle (2x_{n+1} - z_n - y_{n+1} - \gamma B\hat{y}_n - \gamma Cy_n) - (2x_n - z_{n-1} - y_n - \gamma B\hat{y}_{n-1} - \gamma Cy_{n-1}), y_{n+1} - y_n \rangle
= \langle (x_{n+1} - z_{n-1} - \gamma B\hat{y}_n - \gamma Cy_n) - (x_n - z_n - \gamma B\hat{y}_{n-1} - \gamma Cy_{n-1}), y_{n+1} - y_n \rangle
= \langle (x_{n+1} - z_{n-1}) - (x_n - z_n), x_{n+1} - x_n \rangle + \langle z_{n-1} - z_n, z_{n+1} - \hat{z}_n \rangle + \gamma(Bx - B\hat{y}_n, y_{n+1} - y_n)
+ \gamma(B\hat{y}_{n-1} - Bx, y_{n+1} - y_{n-1}) + \gamma(B\hat{y}_n - Bx, y_{n+1} - \hat{y}_n) + \gamma(Cy_{n-1} - Cy_n, y_{n+1} - y_n)
\leq \langle z_{n-1} - z_n, z_{n+1} - \hat{z}_n \rangle + \gamma(Bx - B\hat{y}_n, y_{n+1} - y_n) + \gamma(B\hat{y}_{n-1} - Bx, y_{n+1} - y_{n-1})
+ \gamma(B\hat{y}_n - Bx, y_{n+1} - \hat{y}_n) + \gamma(Cy_{n-1} - Cy_n, y_{n+1} - y_n).
$$

(3.36)

By using the identity

$$
\langle z_{n-1} - z_n, z_{n+1} - \hat{z}_n \rangle = \frac{1}{2}(\|z_n - z_{n-1}\|^2 - \|z_{n+1} - z_n\|^2 - \|z_{n+1} - \hat{z}_n\|^2),
$$

(3.37)

and reorganizing, we get

$$
2\gamma(B\hat{y}_{n-1} - Bx, \hat{y}_n - y_{n+1}) \leq \|z_{n-1} - z_n\|^2 - \|z_{n+1} - z_n\|^2 - \|z_{n+1} - \hat{z}_n\|^2 + 2\gamma(Bx - B\hat{y}_n, y_{n+1} - y_n)
+ 2\gamma(B\hat{y}_n - Bx, y_n - y_{n-1}) + 2\gamma(Cy_{n-1} - Cy_n, y_{n+1} - y_n).
$$

(3.38)

Substituting the estimate (3.38) into (3.34), the claimed inequality (3.30) holds.



\textbf{Theorem 3.2.} Let $A_1 : H \to 2^H$, $i = 1, 2$ be maximally monotone, let $B : H \to H$ be monotone and $L$-Lipschitz continuous, let $C : H \to H$ be $\beta$-cocoercive, and assume that $\operatorname{zer}(A_1 + A_2 + B + C) \neq \emptyset$. Let $\gamma \in \left(0, \frac{\beta}{5\varepsilon(10+2)L}\right)_+$, where $\beta = 17\beta L + 10(17\beta L + 10 + 144\varepsilon^2 L^2)$. Let $z_0, y_0, y_{-1} \in H$ and consider the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined in (3.20). Then, for all $n \in N$, the following hold:

(i) The sequence $\{z_n\}$ converges weakly to a point $z \in H$.

(ii) The sequences $\{x_n\}, \{y_n\}$ converge weakly to a point $x \in H$.

(iii) $x = J_{A_1}(z) \in \operatorname{zer}(A_1 + A_2 + B + C)$.

\textbf{Proof.} Consider the nonempty set $Z$ defined as (3.10). Let $z \in Z$ and define $x := J_{A_1}z$. By Lemma 3.2, for all $n \in N$, we have

$$
\|z_n - z\|^2 + 2\gamma(B\hat{y}_n - Bx, y_{n+1} - y_n) + 2\|z_{n+1} - z_n\|^2 + \|z_{n+1} - \hat{z}_n\|^2
\leq \|z_n - z\|^2 + 2\gamma(B\hat{y}_{n-1} - Bx, y_n - y_{n-1}) + \|z_{n} - z_{n-1}\|^2 + 2\gamma(B\hat{y}_n - B\hat{y}_{n-1}, \hat{y}_n - y_{n+1})
+ 2\gamma(Cy_{n-1} - Cy_n, y_{n+1} - y_n) + 2\gamma(Cy_n - Cx, x - y_{n+1}).
$$

(3.39)

From the firm nonexpansivity of $J_{A_1}$, it follows that

$$
\|(z_{n-1} - \hat{x}_n) - (z_n - x_{n+1})\|^2 \leq 2\|(z_{n-1} - x_n) - (z_n - x_{n+1})\|^2 + 2\|(z_{n-1} - x_n) - (z_{n-2} - x_{n-1})\|^2
\leq 2\|z_n - z_{n-1}\|^2 + 2\|z_{n-1} - z_{n-2}\|^2.
$$

(3.40)
and
\[ \|y_n - y_{n-1}\|^2 = \|(z_n - z_{n-1} + x_n) - (z_{n-1} - z_{n-2} + x_{n-1})\|^2 \]
\[ \leq 2\|z_n - z_{n-1}\|^2 + 2\|(x_n - z_{n-1}) - (x_{n-1} - z_{n-2})\|^2 \] (3.41)
\[ \leq 2\|z_n - z_{n-1}\|^2 + 2\|z_{n-1} - z_{n-2}\|^2. \]

By applying Young’s inequality and (3.39), we obtain
\[ \|\hat{y}_n - y_{n+1}\|^2 = \|(\hat{z}_n - \hat{z}_{n-1} + \hat{x}_n) - (z_{n+1} - z_n + x_{n+1})\|^2 \]
\[ \leq (1 + a)\|z_{n+1} - z_n\|^2 + (1 + \frac{1}{a})\|\hat{z}_{n-1} - \hat{x}_n\| - \|z_n - x_{n+1}\| \] (3.42)
\[ \leq (1 + a)\|z_{n+1} - z_n\|^2 + 2(1 + \frac{1}{a})(\|z_n - z_{n-1}\|^2 + \|z_{n-1} - z_{n-2}\|^2) \]
for some \( a > 0 \). Combining (3.41) and (3.42), we derive that
\[ \|\hat{y}_n - \hat{y}_{n-1}\|^2 \leq 2\|y_n - y_{n-1}\|^2 + 2\|y_n - \hat{y}_{n-1}\|^2 \]
\[ \leq 4\|z_n - z_{n-1}\|^2 + 4(2 + \frac{1}{a})\|z_n - z_{n-1}\|^2 + 4(1 + \frac{1}{a})\|z_{n-1} - z_{n-2}\|^2 + 2(1 + a)\|z_n - \hat{z}_{n-1}\|^2. \] (3.43)

By using the inequality (3.32) and (3.33), it gives
\[ 2\gamma (B\hat{y}_n - B\hat{y}_{n-1}, \hat{y}_n - y_{n+1}) \]
\[ \leq \gamma L\|\hat{y}_n - \hat{y}_{n-1}\|^2 + \gamma L\|\hat{y}_n - y_{n+1}\|^2 \]
\[ \leq 2(3 + \frac{1}{a})\gamma L\|z_n - z_{n-1}\|^2 + 2(5 + \frac{3}{a})\gamma L\|z_{n-1} - z_{n-2}\|^2 + 4(1 + \frac{1}{a})\gamma L\|z_{n-2} - z_{n-3}\|^3 \]
\[ + (1 + a)\gamma L\|z_{n+1} - \hat{z}_n\|^2 + 2(1 + a)\gamma L\|z_n - \hat{z}_{n-1}\|^2. \] (3.44)

Note that \( C \) is \( \beta \)-cocoercive, thus \( C \) is \( \frac{1}{\beta} \)-Lipschitz. By the Lipschitz property of \( C \) and (3.31), the second-last term in (3.39) can be estimated by
\[ 2\gamma (Cy_{n-1} - Cy_n, y_{n+1} - y_n) \]
\[ \leq \frac{\gamma}{\beta} (\|y_{n+1} - y_n\|^2 + \|y_n - y_{n-1}\|^2) \]
\[ \leq \frac{\gamma}{\beta} (2\|z_{n+1} - z_n\|^2 + 4\|z_n - z_{n-1}\|^2 + 2\|z_{n-1} - z_{n-2}\|^2). \] (3.45)

Using the cocoercivity of \( C \) and (3.31), the last term in (3.39) can be estimated by
\[ 2\gamma (Cy_n - Cx, x - y_n) = 2\gamma (Cy_n - Cx, x - y_n) + 2\varepsilon (\frac{\gamma}{\varepsilon} (Cy_n - Cx), y_n - y_{n+1}) \]
\[ \leq -2\gamma \beta \|Cy_n - Cx\|^2 + \frac{\gamma^2}{\varepsilon} \|Cy_n - Cx\|^2 + \varepsilon \|y_{n+1} - y_n\|^2 \]
\[ - \varepsilon \|\frac{\gamma}{\varepsilon} (Cy_n - Cx) - (y_n - y_{n+1})\|^2 \]
\[ \leq \varepsilon \|y_{n+1} - y_n\|^2 - \frac{\gamma}{\varepsilon} (2\beta \varepsilon - \gamma) \|Cy_n - Cx\|^2 \]
\[ \leq 2\varepsilon \|z_{n+1} - z_n\|^2 + 2\varepsilon \|z_n - z_{n-1}\|^2 + \frac{\gamma}{\varepsilon} (2\beta \varepsilon - \gamma) \|Cy_n - Cx\|^2. \] (3.46)

Substituting the estimates (3.41), (3.45) and (3.46) into (3.39), it follows that
\[ \|z_{n+1} - z\|^2 + 2\gamma (B\hat{y}_n - Bx, y_{n+1} - y_n) + 2\|z_{n+1} - z_n\|^2 + \|z_{n+1} - \hat{z}_n\|^2 \]
\[ \leq \|z_n - z\|^2 + 2\gamma (B\hat{y}_{n-1} - Bx, y_n - y_{n-1}) + (2\varepsilon + \frac{2\gamma}{\beta}) \|z_{n+1} - z_n\|^2 + (1 + 2\varepsilon + \frac{4\gamma}{\beta} + (6 + \frac{2}{a})\gamma L) \|z_n - z_{n-1}\|^2 \]
\[ + (\frac{2\gamma}{\beta} + (10 + \frac{6}{a})\gamma L) \|z_{n-1} - z_{n-2}\|^2 + 4(1 + \frac{1}{a})\gamma L \|z_{n-2} - z_{n-3}\|^3 + (1 + a)\gamma L \|z_{n+1} - \hat{z}_n\|^2 \]
\[ + 2(1 + a)\gamma L \|z_{n+1} - \hat{z}_n\|^2 - \frac{\gamma}{\varepsilon} (2\beta \varepsilon - \gamma) \|Cy_n - Cx\|^2. \] (3.47)
For convenience, we denote
\[
V_n := \|z_n - z\|^2 + 2\gamma(B\hat{y}_{n-1} - Bx, y_n - y_{n-1}) + (1 + 2\varepsilon + \frac{6\gamma}{\beta} + (20 + \frac{12}{a})\gamma L)\|z_n - z_{n-1}\|^2
+ (\frac{2\gamma}{\beta} + (14 + \frac{10}{a})\gamma L)\|z_{n-1} - z_{n-2}\|^2 + 4(1 + \frac{1}{a})\gamma L\|z_{n-2} - z_{n-3}\|^2 + 2(1 + a)\gamma L\|z_n - \hat{z}_{n-1}\|^2.
\]
Hence, (3.47) can be simply expressed as
\[
V_{n+1} + (1 - 4\varepsilon - \frac{8\gamma}{\beta} - (20 + \frac{12}{a})\gamma L)\|z_{n+1} - z_n\|^2 + (1 - 3(1+a)\gamma L)\|z_n - \hat{z}_{n-1}\|^2 + \frac{2}{\varepsilon}(2\beta\varepsilon - \gamma)\|Cy_n - Cx\|^2 \leq V_n. \tag{3.49}
\]
In order to ensure that the sequence \(\{V_n\}\) is nonincreasing, the second, third and fourth terms in (3.49) shall be positive, from which we derive three upper bounds of \(\gamma\): \(\gamma < \frac{1 - 4\varepsilon - \frac{8\gamma}{\beta} - (20 + \frac{12}{a})\gamma L}{1 - 3(1+a)\gamma L}\), \(\gamma < \frac{1}{\frac{1}{\varepsilon} + 2\beta\varepsilon - \gamma}\), and \(\gamma < 2\beta\varepsilon\). To get the largest interval for \(\gamma\), first taking the first and third bound equal yields \(\varepsilon = \frac{1}{\frac{1}{\varepsilon} + 2\beta\varepsilon - \gamma}\) > 0 and then setting the last two bounds equal yields \(\alpha = \frac{17\beta L + 10\sqrt{(17\beta L + 10)^2 + 144\beta^2 L^2}}{63L}\) > 0. Therefore, we obtain the desired result of \(\gamma\).
In addition, (3.49) gives
\[
V_{n+1} + \varepsilon'\|z_{n+1} - z_n\|^2 \leq V_n, \tag{3.50}
\]
which implies that
\[
V_{n+1} + \varepsilon'\sum_{i=0}^{n} \|z_{i+1} - z_i\|^2 \leq V_0, \tag{3.51}
\]
where \(\varepsilon' = 1 - 4\varepsilon - \frac{8\gamma}{\beta} - (20 + \frac{12}{a})\gamma L\) and the value of \(\alpha, \varepsilon\) as the above. Next, we need to estimate the lower bound of the sequence \(\{V_{n+1}\}\). From the firm nonexpansivity of \(J_{\gamma A_1}\), it follows that
\[
\|\hat{y}_n - x\|^2 = \|(\hat{z}_n - \hat{z}_{n-1} + \hat{x}_n) - (z - z + x)\|^2
\leq 2\|\hat{z}_n - z\|^2 + 2\|\hat{x}_n - \hat{z}_{n-1} + (z - z)\|^2
\leq 2\|\hat{z}_n - z_{n+1}\| + \|z_{n+1} - z\|\|\hat{z}_n - z_n\|^2 + 2\|\hat{z}_{n-1} - z\|^2
\leq 4\|z_{n+1} - z\|^2 + 4\|z_{n+1} - \hat{z}_n\|^2 + 4\|z_{n+1} - z\|^2 + 4\|z_{n+1} - z_{n-2}\|^2,
\]
which combined the inequality
\[
\|z_{n+1} - z\|^2 = \|(z_{n+1} - z) - (z_{n+1} - z_n + z_n - z_{n-1})\|^2
\leq 2\|z_{n+1} - z\|^2 + 2\|(z_{n+1} - z_n) + (z_n - z_{n-1})\|^2
\leq 2\|z_{n+1} - z\|^2 + 2\|z_{n+1} - z_n\|^2 + 4\|z_{n+1} - \hat{z}_n\|^2,
\]
it implies
\[
\|\hat{y}_n - x\|^2 \leq 12\|z_{n+1} - z\|^2 + 16\|z_{n+1} - z_n\|^2 + 16\|z_n - z_{n-1}\|^2 + 4\|z_{n+1} - z_{n-2}\|^2 + 4\|z_{n+1} - \hat{z}_n\|^2. \tag{3.54}
\]
Hence, by using the Lipschitz property of \(B\), Young’s inequality, (3.41) and (3.54), we conclude that
\[
2\gamma(B\hat{y}_n - Bx, y_{n+1} - y_n)
\leq \frac{1}{2}\gamma L\|\hat{y}_n - x\|^2 + 2\gamma L\|y_{n+1} - y_n\|^2
\leq 6\gamma L\|z_{n+1} - z\|^2 + 12\gamma L\|z_{n+1} - z_n\|^2 + 12\gamma L\|z_n - z_{n-1}\|^2 + 2\gamma L\|z_{n+1} - z_{n-2}\|^2 + 2\gamma L\|z_{n+1} - \hat{z}_n\|^2.
\]
which yields
\[
V_{n+1} \geq (1 - 6\gamma L)\|z_{n+1} - z\|^2 + (1 + 2\varepsilon + \frac{6\gamma}{\beta} + (8 + \frac{12}{a})\gamma L)\|z_{n+1} - z_{n-1}\|^2
+ (\frac{2\gamma}{\beta} + (2 + \frac{10}{a})\gamma L)\|z_n - z_{n-1}\|^2 + (2 + \frac{4}{a})\gamma L\|z_{n-1} - z_{n-2}\|^2 + 2\alpha\gamma L\|z_{n+1} - \hat{z}_n\|^2
\geq (4\varepsilon + \frac{8\gamma}{\beta} + (14 + \frac{12}{a})\gamma L)\|z_{n+1} - z\|^2 \geq 0.
\]
Consequently, \( \{V_n\} \) converges, together with (3.51), (3.56), which implies \( \|z_{n+1} - z_n\| \to 0 \) and \( \{z_n\} \) is bounded. Due to the fact that the first and third lines of (3.1) and (3.20) is identical, we deduce that \( \|x_{n+1} - x_n\| \to 0 \) and \( \{x_n\} \) is bounded, \( \|y_{n+1} - y_n\| \to 0 \) and \( \{y_n\} \) is bounded. Hence, we obtain that

\[
\lim_{n \to \infty} \|z_n - z\|^2 = \lim_{n \to \infty} \|V_n\|
\]  

exists. On the other hand, (3.26) can be expressed as

\[
\begin{pmatrix}
z_{nk} - z_{nk+1} \\
\frac{z_{nk} - z_{nk+1}}{\lambda} - \gamma \left( Bg_{nk} - Bg_{nk+1} \right) - \gamma \left( Cy_{nk} - Cy_{nk+1} \right)
\end{pmatrix} \in (A + B) \begin{pmatrix} z_{nk+1} - z_{nk} \\
- z_{nk+1} + x_{nk+1} \end{pmatrix},
\]

where \( A = \begin{pmatrix} (\gamma A_1)^{-1} & 0 \\
0 & \gamma (A_2 + B + C) \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & -\Id \\
\Id & 0 \end{pmatrix} \). The remainder of the proof is analogous to Theorem 3.1.

3.3 Semi-Forward-Reflected-Douglas-Rachford splitting algorithm

The third splitting algorithm is presented below.

\[
\begin{cases}
x_{n+1} = J_{\gamma A_2}(x_n - \gamma u_n - \gamma (2Bx_n - Bx_{n-1}) - \gamma Cx_n) \\
y_{n+1} = J_{\lambda A_1}(2x_{n+1} - x_n + \lambda u_n) \\
u_{n+1} = u_n + \frac{1}{\alpha}(2x_{n+1} - x_n - y_{n+1})
\end{cases}
\]  

(3.59)

Remark 3.4. The following iterative algorithms are special cases of (3.59).

(i) Semi-forward-reflected-backward splitting algorithm [19]: if \( A_1 = 0 \), (3.59) reduces to

\[
z_{n+1} = J_{\gamma A_2}(z_n - 2\gamma Bz_n + \gamma Bz_{n-1} - \gamma Cz_n).
\]

(3.60)

(ii) Forward-reflected-Douglas-Rachford splitting algorithm [21]: if \( C = 0 \), (3.59) becomes

\[
\begin{cases}
x_{n+1} = J_{\gamma A_2}(x_n - \gamma u_n - \gamma (2Bx_n - Bx_{n-1})) \\
y_{n+1} = J_{\lambda A_1}(2x_{n+1} - x_n + \lambda u_n) \\
u_{n+1} = u_n + \frac{1}{\alpha}(2x_{n+1} - x_n - y_{n+1})
\end{cases}
\]

(3.61)

Therefore, we call the iterative algorithm (3.59) a semi-forward-reflected-Douglas-Rachford splitting algorithm.

Theorem 3.3. Let \( A_i : \mathcal{H} \to 2^{\mathcal{H}}, i = 1, 2 \) be maximally monotone, let \( B : \mathcal{H} \to \mathcal{H} \) be monotone and \( L \)-Lipschitz continuous, let \( C : \mathcal{H} \to \mathcal{H} \) be \( \beta \)-cocoercive, and assume that \( \text{zer}(A_1 + A_2 + B + C) \neq \emptyset \). Let \( \lambda > 0 \) and \( \gamma \in \left( 0, \frac{\lambda}{\beta + M(2L + 1)} \right) \). Let \( x_0, x_{-1}, u_0 \in \mathcal{H} \) and consider the sequences \( \{x_n\}, \{u_n\} \) defined in (3.59). Then, for all \( n \in \mathbb{N} \), the sequences \( \{x_n\} \) converges weakly to a point \( \bar{x} \in \text{zer}(A_1 + A_2 + B + C) \).

Proof. Let us first define the Hilbert space \( \mathcal{K} = \mathcal{H} \times \mathcal{H} \) with inner product and associated norm

\[
\langle (x, u), (y, v) \rangle_{\mathcal{K}} := \frac{1}{\gamma} \langle x, y \rangle - \langle x, v \rangle - \langle y, u \rangle + \lambda \langle u, v \rangle, \\
\| (x, u) \|^2_{\mathcal{K}} := \frac{1}{\gamma} \|x\|^2 - 2 \langle x, u \rangle + \lambda \|u\|^2.
\]

(3.62)

The inner product and norm are proved to be valid by simple calculation.
Let \( \bar{x} \in \text{ker}(A_1 + A_2 + B + C) \) and \( \bar{u} \in A_1 \bar{x} \). Combining these two inclusions yields \( -\bar{u} \in (A_2 + B + C)\bar{x} \).

Next, we define
\[
A_1 y_{n+1} := u_n + \frac{1}{\lambda} (2x_{n+1} - x_n - y_{n+1}) \in A_1 y_{n+1}, \\
A_2 x_{n+1} := \frac{1}{\gamma} (x_n - x_{n+1}) - u_n - 2Bx_n + Bx_{n-1} - Cx_n \in A_2 x_{n+1}, \\
A_1 \bar{x} := \bar{u} \in A_1 \bar{x}, \quad A_2 \bar{x} := -\bar{u} - B\bar{x} \in A_2 \bar{x}.
\]

By the monotonicity of \( A_1 \) and \( A_2 \), it follows that
\[
\left\| (x_{n+1}, u_{n+1}) - (\bar{x}, \bar{u}) \right\|_K^2
= \left\| (x_n, u_n) - (\bar{x}, \bar{u}) \right\|_K^2 - 2\left\langle A_2 x_{n+1} - A_2 \bar{x}, x_{n+1} - \bar{x} \right\rangle
- 2\left\langle A_1 y_{n+1} - A_1 \bar{x}, y_{n+1} - \bar{x} \right\rangle
- 2\left\langle B \bar{x} - B x_n, \bar{x} - x_{n+1} \right\rangle + 2\left\langle B \bar{x} - B x_{n-1}, x_n - x_{n+1} \right\rangle
+ 2\left\langle B x_n - B x_{n-1}, x_n - x_{n+1} \right\rangle + 2\left\langle B x_n - B x_{n-2}, \bar{x} - x_n \right\rangle
- 2\left\langle B \bar{x} - B x_n, \bar{x} - x_{n+1} \right\rangle
+ 2\left\langle B \bar{x} - B x_{n-1}, \bar{x} - x_n \right\rangle
- 2\left\langle C \bar{x} - C x_n, x_n - x_{n+1} \right\rangle
\]
(3.64)

\begin{align*}
-2\left\langle B \bar{x} - B x_n, \bar{x} - x_{n+1} \right\rangle &= -2\left\langle B \bar{x} - B x_{n+1}, \bar{x} - x_{n+1} \right\rangle - 2\left\langle B x_{n+1} - B x_n, \bar{x} - x_{n+1} \right\rangle \\
&\leq -2\left\langle B x_{n+1} - B x_n, \bar{x} - x_{n+1} \right\rangle. \\
\end{align*}
(3.65)

Using the Lipschitz continuity of \( B \) yields
\[
2\left\langle B x_n - B x_{n-1}, x_n - x_{n+1} \right\rangle \leq L \left( \left\| x_n - x_{n-1} \right\|^2 + \left\| x_{n+1} - x_n \right\|^2 \right). \\
\]
(3.66)

In addition, from the cocoercivity of \( C \), for all \( \varepsilon > 0 \), we derive that
\[
-2\langle C x_n - C \bar{x}, x_n - \bar{x} \rangle
= -2\langle C x_n - C \bar{x}, x_n - \bar{x} \rangle - 2\varepsilon \left\langle C x_n - C \bar{x}, x_{n+1} - x_n \right\rangle \\
\leq -2\beta \|Cx_n - C\bar{x}\|^2 + \frac{1}{\varepsilon} \|Cx_n - C\bar{x}\|^2 + \varepsilon \|x_n - x_{n+1}\|^2 - \varepsilon \left\langle \frac{1}{\varepsilon} (C x_n - C \bar{x}) - (x_n - x_{n+1}) \right\|_K^2 \\
\leq \varepsilon \|x_n - x_{n+1}\|^2 - (2\beta - \frac{1}{\varepsilon}) \|Cx_n - C\bar{x}\|^2. \\
\]
(3.67)

Substituting the estimates \(3.65\), \(3.66\) and \(3.67\) into \(3.64\), it follows that
\[
\left\| (x_{n+1}, u_{n+1}) - (\bar{x}, \bar{u}) \right\|_K^2
\leq \left\| (x_n, u_n) - (\bar{x}, \bar{u}) \right\|_K^2 + 2\left\langle B x_n - B x_{n-1}, \bar{x} - x_n \right\rangle + \frac{1}{2} \left\| (x_n, u_n) - (x_n, u_n) \right\|_K^2 \\
- \frac{1}{2} \left\| (x_{n+1}, u_{n+1}) - (x_n, u_n) \right\|_K^2 + \left\| (x_n, u_n) - (x_n, u_n) \right\|_K^2 \\
+ (L + \varepsilon) \|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2 - (2\beta - \frac{1}{\varepsilon}) \|Cx_n - C\bar{x}\|^2. \\
\]
(3.68)

Note that Young’s inequality implies
\[
0 \leq \frac{\lambda \gamma}{\lambda - \gamma} \left( \frac{1}{\lambda} \|x_n - x_{n+1}\|^2 - 2\langle x_{n+1} - x_n, u_{n+1} - u_n \rangle + \lambda \|u_{n+1} - u_n\|^2 \right), \\
0 \leq \frac{\lambda \gamma}{\lambda - \gamma} \left( \frac{1}{\lambda} \|x_n - x_{n-1}\|^2 - 2\langle x_{n-1} - x_n, u_n - u_{n-1} \rangle + \lambda \|u_n - u_{n-1}\|^2 \right). \\
\]
(3.69)

Summing \(3.68\) and \(3.69\), and denoting
\[
V_n := \left\| (x_n, u_n) - (\bar{x}, \bar{u}) \right\|_K^2 + 2\left\langle B x_n - B x_{n-1}, \bar{x} - x_n \right\rangle + \frac{1}{2} \left\| (x_n, u_n) - (x_{n-1}, u_{n-1}) \right\|_K^2, \\
\]
(3.70)
we obtain

\[ V_{n+1} + \frac{\lambda - \gamma - 2\lambda \gamma (L + \varepsilon)}{2(\lambda - \gamma)} \| (x_{n+1}, u_{n+1}) - (x_n, u_n) \|_K^2 \]

\[ + \| (x, u) - (x_{n-1}, u_{n-1}) \|_K^2 + (2\beta - \frac{1}{\varepsilon}) \| C x - C \bar{x} \|^2 \leq V_n. \]  

(3.71)

To ensure that the sequence \( \{V_n\} \) is nonincreasing, we have \( \gamma < \frac{\lambda}{\lambda + 2\beta (L + \varepsilon)} \) and \( \varepsilon < \frac{1}{2\gamma} \). Combining these two inequalities yields \( \gamma < \frac{\lambda}{2 \varepsilon} \). In addition, (3.71) gives

\[ V_{n+1} + \varepsilon \left(\| (x_{n+1}, u_{n+1}) - (x, u) \|_K^2 + \| (x, u) - (x_{n-1}, u_{n-1}) \|_K^2\right) \leq V_n, \]

(3.72)

which implies that

\[ V_{n+1} + \varepsilon \sum_{i=0}^{n} \left(\| (x_{i+1}, u_{i+1}) - (x_i, u_i) \|_K^2 + \| (x_i, u_i) - (x_{i-1}, u_{i-1}) \|_K^2\right) \leq V_0, \]

(3.73)

where \( \varepsilon' = \frac{\beta (\lambda - \gamma - 2\lambda \gamma (2\beta + 1)(L + \varepsilon))}{2\beta (\lambda - \gamma)} \).

On the other hand, we need to estimate the lower bound of the sequence \( \{V_{n+1}\} \). Note that the Lipschitz property of \( B \) yields

\[ 2(Bx - Bx_{n-1}, \bar{x} - x_n) \leq L(\| x_n - x_{n-1} \|^2 + \| \bar{x} - x_n \|^2), \]

(3.74)

and it follows from Young’s inequality that

\[ \langle x_n - \bar{x}, u_n - \bar{\bar{u}} \rangle \leq \frac{1}{2\lambda} \| x_n - \bar{x} \|^2 + \frac{\lambda}{2} \| u_n - \bar{\bar{u}} \|^2, \]

(3.75)

\[ \langle x_n - x_{n-1}, u_n - u_{n-1} \rangle \leq \frac{1}{2\lambda} \| x_n - x_{n-1} \|^2 + \frac{\lambda}{2} \| u_n - u_{n-1} \|^2. \]

By combining (3.71), (3.75) and \( \gamma < \frac{\lambda}{2\varepsilon} \), we deduce that

\[ V_{n+1} \geq \frac{1}{2} \| (x_{n+1}, u_{n+1}) - (x, \bar{u}) \|_K^2 + \frac{1}{2\gamma} \| x_{n+1} - \bar{x} \|^2 - \langle x_{n+1} - \bar{x}, u_{n+1} - \bar{u} \rangle + \frac{\lambda}{2} \| u_{n+1} - \bar{u} \|^2 \]

\[ + \frac{1}{2\gamma} \| x_{n+1} - x_n \|^2 - \langle x_{n+1} - x_n, u_{n+1} - u_n \rangle + \frac{\lambda}{2} \| u_{n+1} - u_n \|^2 - L(\| x_{n+1} - x_n \|^2 + \| \bar{x} - x_{n+1} \|^2) \]

\[ \geq \frac{1}{2} \| (x_{n+1}, u_{n+1}) - (x, \bar{u}) \|_K^2 + \left(\frac{1}{2\gamma} - \frac{1}{2\lambda} - L\right) \| x_{n+1} - \bar{x} \|^2 + \left(\frac{1}{2\gamma} - \frac{1}{2\lambda} - L\right) \| x_{n+1} - x_n \|^2 \]

\[ \geq \frac{1}{2} \| (x_{n+1}, u_{n+1}) - (x, \bar{u}) \|_K^2 \geq 0. \]

(3.76)

Hence, \( \{V_n\} \) converges, together with (3.73) and (3.76), which implies \( x_{n+1} - x_n \rightarrow 0, u_{n+1} - u_n \rightarrow 0 \) and the sequence \( \{(x_n), (u_n)\} \) is bounded. From the identity \( u_{n+1} = u_n + \frac{1}{\lambda} (2x_{n+1} - x_n - y_{n+1}) \), we have \( x_{n+1} - y_{n+1} \rightarrow 0 \). Consequently, applying the above conclusions and the Lipschitz continuity of \( B \), we obtain

\[ \lim_{n \rightarrow \infty} \| (x_n, u_n) - (\bar{x}, \bar{u}) \|_K^2 = \lim_{n \rightarrow \infty} V_n = V \]

(3.77)

exists. On the other hand, let (\( \bar{x}, \bar{u} \)) be a weak cluster point of a subsequence \( \{(x_{n_k}), (u_{n_k})\}_{k \in N} \) of \( \{(x_n), (u_n)\} \). By the definition of resolvent operator, (3.59) can be expressed as

\[
\begin{align*}
\left(\frac{y_{n_{k+1}} - x_{n_{k+1}}}{\lambda (x_{n_{k+1}} - y_{n_{k+1}})}\right) + \left(\frac{1}{\lambda} - \frac{1}{\gamma}\right) x_{n_{k+1}} - x_n + \left(\frac{0}{Bx_{n_{k+1}} - Bx_n}\right) - \left(\frac{0}{By_{n_{k+1}} - By_{n_{k-1}}}\right)
\end{align*}
\]

(3.78)

where \( A = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2 + B + C \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{bmatrix} \). Since \( A_1^{-1} \) and \( A_2 + B + C \) are maximally monotone by Lemma 2.2 and 2.3, \( A + B \) is also maximally monotone. Thus, its graph is closed in the weak-strong topology on \( \mathcal{H} \times \mathcal{H} \). By taking the limit along the subsequence in (3.78), it yields

\[ 0 \in A_1^{-1} \bar{u} - \bar{x}, \quad 0 \in (A_2 + B + C)\bar{x} + \bar{u}. \]

(3.79)
Hence, we deduce that \( \bar{x} \in \text{zer}(A_1 + A_2 + B + C) \). According to Lemma 2.4 it follows that \( \{x_n, \{u_n\}\} \) converges weakly to \( (\bar{x}, \bar{u}) \), which implies that the conclusion of Theorem 3.3 holds.

4 Applications

In this section, we study the problem of finding a zero of the sum of \( m \) maximally monotone operators, a monotone Lipschitz operator, and a cocoercive operator. In detail, we consider the following monotone inclusion problem.

Problem 4.1 Let \( H \) be a real Hilbert space, let \( A_i : H \to 2^H \) be maximally monotone, let \( B : H \to H \) be monotone and \( L \)-Lipschitz, and let \( C : H \to H \) be \( \beta \)-cocoercive. The problem is to

\[
\text{find } x \in H \text{ such that } 0 \in \sum_{i=1}^{m} A_i x + B x + C x, \tag{4.1}
\]

under the assumption that the set of solutions is nonempty.

Next, we will present the relationship between (1.9) and (4.1). Let \( (\omega_i)_{1 \leq i \leq m} \) be real numbers in (0, 1] such that \( \sum_{i=1}^{m} \omega_i = 1 \). Let \( H = H^m \) be the Hilbert direct sum of \( H \), and its scalar product and associated norm are defined as \( \langle x|y \rangle = \sum_{i=1}^{m} \omega_i \langle x_i|y_i \rangle \) and \( \|x\| = \sqrt{\sum_{i=1}^{m} \omega_i \|x_i\|^2} \), respectively, where \( x = (x_i)_{1 \leq i \leq m} \) and \( y = (y_i)_{1 \leq i \leq m} \) are elements of \( H \).

Lemma 4.1. Let \( H, B, C, (A_i)_{1 \leq i \leq m} \) be as in Problem 4.1, define

\[
\begin{align*}
V & := \{x = (x_i)_{1 \leq i \leq m} \in H : x_1 = \cdots = x_m\}, \\
\mathcal{J} & : H \to V \subset H : x \mapsto (x, \cdots, x), \\
A & : H \to 2^H : x \mapsto \frac{1}{\omega_1} A_1 x_1 \times \cdots \times \frac{1}{\omega_m} A_m x_m, \\
B & : H \to H : x \mapsto (B x_1, \cdots, B x_m), \\
C & : H \to H : x \mapsto (C x_1, \cdots, C x_m).
\end{align*}
\]

Then the following statements hold:

(i) For any \( x = (x_i)_{1 \leq i \leq m} \in H \), \( P_V x = \mathcal{J}(\sum_{i=1}^{m} \omega_i x_i) \).

(ii) \( N_V(x) = \begin{cases} V^\perp = \{x = (x_i)_{1 \leq i \leq m} \in H : \sum_{i=1}^{m} \omega_i x_i = 0\}, & \text{if } x \in V; \\
\emptyset, & \text{otherwise}. \end{cases} \)

(iii) \( A \) is maximally monotone, and for any \( \gamma > 0 \), \( J_{\gamma A} : (x_i)_{1 \leq i \leq m} \mapsto (J_{\gamma A_i/\omega_i} x_i) \).

(iv) \( B \) is monotone and \( L \)-Lipschitz.

(v) \( C \) is \( \beta \)-cocoercive.

(vi) For any \( x \in H \), \( x \) is a solution to Problem 4.1 if and only if \( j(x) \in \text{zer}(A + B + C + N_V) \).

Proof. (i) This follows from Proposition 26.4 (iii) of [20].

(ii) Combining Proposition 26.4 (i) and (ii) of [20], we can obtain it.

(iii) See Proposition 23.18 of [20].

(iv) & (v) They follow from easy computations by combining (4.2) and the properties of \( B \) and \( C \).

(vi) For all \( x \in H \), we have

\[
0 \in \sum_{i=1}^{m} A_i x + B x + C x \iff (\exists (y_i)_{1 \leq i \leq m} \in \times_{i=1}^{m} A_i x) 0 = \sum_{i=1}^{m} y_i + B x + C x
\]

\[
\iff (\exists (y_i)_{1 \leq i \leq m} \in \times_{i=1}^{m} A_i x) 0 = \sum_{i=1}^{m} \omega_i (-y_i/\omega_i - B x - C x)
\]

\[
\iff (\exists (y_i)_{1 \leq i \leq m} \in \times_{i=1}^{m} A_i x) \langle -y_1/\omega_1, \cdots, -y_m/\omega_m \rangle - j(B x) - j(C x) \in V^\perp
\]

\[
\iff 0 \in A(j(x)) + B(j(x)) + C(j(x)) + N_V(j(x))
\]
\[ j(x) \in \text{zer}(A + B + C + N_V). \] (4.3)

Let \( A_1 = N_V \) and \( A_2 = A \) in (4.9). Therefore, we can apply the proposed splitting algorithms in Section 3 to solve the problem (4.1).

**Theorem 4.1.** Consider the Problem 4.1. Let \((z_{i,0})_{1 \leq i \leq m}, (y_{i,0})_{1 \leq i \leq m}, (y_{i,-1})_{1 \leq i \leq m} \in H^m\) and set

\[
\begin{align*}
    x_{n+1} &= \sum_{j=1}^{m} \omega_j z_{j,n} \\
    y_{i,n+1} &= J_{\gamma_A}(2x_{n+1} - z_{i,n} - 2\gamma B y_{i,n} + \gamma B y_{i,n-1} - \gamma C y_{i,n}) \\
    z_{i,n+1} &= z_{i,n} + y_{i,n+1} - x_{n+1}
\end{align*}
\]

where \( \gamma \in \left(0, \frac{\beta}{2(1+4\sqrt{\lambda_1})} \right) \). Then \( \{x_n\} \) converges weakly to a solution of Problem 4.1.

**Proof.** Let \( x_n = j(x_n) \), \( y_n = (y_{i,n})_{1 \leq i \leq m} \) and \( z_n = (z_{i,n})_{1 \leq i \leq m} \). By Lemma 4.1 (i), \( P_V z_n = j(\sum_{i=1}^{m} \omega_i z_{i,n}) \).

Hence, it follows from Lemma 4.1 that (4.4) can be written as

\[
\begin{cases}
    x_{n+1} = P_V z_n \\
    y_{n+1} = J_{\gamma_A}(2x_{n+1} - z_{n} - 2\gamma B y_{n} + \gamma B y_{n-1} - \gamma C y_{n}) \\
    z_{n+1} = z_{n} + y_{n+1} - x_{n+1}
\end{cases}
\]

Therefore, the conclusions of Theorem 4.1 follows directly from Theorem 3.1.

**Theorem 4.2.** Consider the Problem 4.1. Let \((z_{i,0})_{1 \leq i \leq m}, (y_{i,0})_{1 \leq i \leq m}, (y_{i,-1})_{1 \leq i \leq m} \in H^m\) and set

\[
\begin{align*}
    x_{n+1} &= \sum_{j=1}^{m} \omega_j z_{j,n} \\
    y_{i,n+1} &= J_{\gamma_A}(2x_{n+1} - z_{i,n} - \gamma B(2y_{i,n} - y_{i,n-1}) - \gamma C y_{i,n}) \\
    z_{i,n+1} &= z_{i,n} + y_{i,n+1} - x_{n+1}
\end{align*}
\]

where \( \gamma \in \left(0, \frac{\beta}{5+10\sqrt{2}+\sqrt{17\beta L+10}+144\beta^2 L^2} \right) \). Then \( \{x_n\} \) converges weakly to a solution of Problem 4.1.

**Proof.** Let \( x_n = j(x_n) \), \( y_n = (y_{i,n})_{1 \leq i \leq m} \) and \( z_n = (z_{i,n})_{1 \leq i \leq m} \). By Lemma 4.1 (i), \( P_V z_n = j(\sum_{i=1}^{m} \omega_i z_{i,n}) \).

Hence, it follows from Lemma 4.1 that (4.6) can be written as

\[
\begin{cases}
    x_{n+1} = P_V z_n \\
    y_{n+1} = J_{\gamma_A}(2x_{n+1} - z_{n} - \gamma B(2y_{n} - y_{n-1}) - \gamma C y_{n}) \\
    z_{n+1} = z_{n} + y_{n+1} - x_{n+1}
\end{cases}
\]

Therefore, the conclusions of Theorem 4.2 follows directly from Theorem 3.2.
Theorem 4.3. Consider the Problem 4.1. Let $x_0, x_{-1} \in H, (u_i,0)_{1 \leq i \leq m} \in H^m$ and set
\[
\begin{align*}
x_{n+1} &= \sum_{j=1}^{m} \omega_j (x_n - \gamma u_{j,n} - \gamma (2Bx_n - Bx_{n-1}) - \gamma Cx_n) \\
For i &= 1, \ldots, m \\
y_{i,n+1} &= J_{\frac{\lambda}{\gamma} A_i} (2x_{n+1} - x_n + \lambda u_{i,n}) \\
u_{i,n+1} &= u_{i,n} + \frac{1}{\lambda} (2x_{n+1} - x_n - y_{i,n+1})
\end{align*}
\]
where $\lambda > 0$ and $\gamma \in \left(0, \frac{\lambda^2}{\beta + \lambda (2\beta + 1)}\right)$. Then $\{x_n\}$ converges weakly to a solution of Problem 4.1. 

Proof. Let $x_n = j(x_n), y_n = (y_{i,n})_{1 \leq i \leq m},$ and $u_n = (u_{i,n})_{1 \leq i \leq m}$. By Lemma 4.1 (i), $P_{\gamma}(x_n - \gamma u_n - \gamma (2Bx_n - Bx_{n-1}) - \gamma Cx_n) = \gamma (\sum_{i=1}^{m} \omega_i (x_n - \gamma u_{i,n} - \gamma (2Bx_n - Bx_{n-1}) - \gamma Cx_n)).$ Hence, it follows from Lemma 4.1 that (4.8) can be written as
\[
\begin{align*}
x_{n+1} &= P_{\gamma}(x_n - \gamma u_n - \gamma (2Bx_n - Bx_{n-1}) - \gamma Cx_n) \\
y_{n+1} &= J_{\frac{\lambda}{\gamma} A_i} (2x_{n+1} - x_n + \lambda u_{i,n}) \\
u_{i,n+1} &= u_{i,n} + \frac{1}{\lambda} (2x_{n+1} - x_n - y_{i,n+1})
\end{align*}
\]
Therefore, the conclusions of Theorem 4.3 follows directly from Theorem 3.3.

5 Numerical experiments

In this section, we perform numerical experiments to illustrate the performance of the proposed algorithms. All experiments are conducted on a Lenovo Laptop with an AMD Ryzen 5 2500U CPU (2.00 GHZ) and 8.00 GB RAM.

Suppose that $M_1, \cdots, M_k$ are nonempty, closed and convex sets, the Minkowski sum is defined as follows:
\[
M_1 + \cdots + M_k = \{m_1 + \cdots + m_k | m_1 \in M_1, \cdots, m_k \in M_k\}.
\]
We can show that the Minkowski sum of convex sets is convex by simple calculation. The problem of finding the projection of a point $f \in H$ onto the Minkowski sum $M_1 + \cdots + M_k$ is defined by
\[
\min \{\|f - x\| | x \in M_1 + \cdots + M_k\}.
\]
Let $x$ be a solution of (5.2), according to Lemma 4.3.1 of [13], we have
\[
0 \in \partial \delta_{M_1}(0) + \cdots + \partial \delta_{M_k}(x),
\]
where $\partial \delta_{M_1}(0) + \cdots + \partial \delta_{M_k}$ denotes the infimal convolution of the indicator functions $\delta_{M_1}, \cdots, \delta_{M_k}$. Let $y \in (\partial \delta_{M_1}(0) + \cdots + \partial \delta_{M_k})(x)$, and define
\[
C : (x, y) \mapsto (x - f, 0),
\]
\[
B : (x, y) \mapsto (y - x),
\]
\[
A_i : (x, y) \mapsto (0, (\partial \delta_{M_i})^{-1}y), i = 1, \cdots, k.
\]
Hence, $x$ satisfies (5.3) is equivalent to $(x, y) \in zE(\sum_{i=1}^{k} A_i + B + C)$. In particular, for each $i = 1, \cdots, k, A_i$ is maximally monotone, $B$ is 1-Lipschitz continuous, and $C$ is 1-cocoercive. Therefore, we can employ the proposed algorithms in Section 4 to solve (5.3). Due to $B$ is linear, the proposed algorithms (4.4) and (4.8) are equivalent.

Example 5.1 ([13]) Let $H = R^2$ and consider the problem (5.2) with the sets $M_1 = [-2, 2] \times \{0\}, M_2 = \{0\} \times [-1, 1] \text{ and } M_3 = \{(x, y) \in R^2 ||(x, y)|| \leq 1\}$. 

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Notice that the projection on the single sets are as follows:

\[
P_{M_1}(x, y) = \begin{cases} 
(-2, 0) & \text{if } x < -2, \\
(x, 0) & \text{if } -2 \leq x \leq 2, \\
(2, 0) & \text{if } x > 2,
\end{cases}
\]

(5.5)

\[
P_{M_2}(x, y) = \begin{cases} 
(0, -1) & \text{if } y < -1, \\
(0, y) & \text{if } -1 \leq y \leq 1, \\
(0, 1) & \text{if } y > 1,
\end{cases}
\]

(5.6)

\[
P_{M_3}(x, y) = \begin{cases} 
(x, y) & \text{if } \|(x, y)\| \leq 1, \\
\frac{1}{\|(x, y)\|} (x, y) & \text{if } \|(x, y)\| > 1.
\end{cases}
\]

(5.7)

Let \( f = (1, -4), f = (2, 7), \) and \( f = (6, -4), \) then the projection of \( f \) is \( \bar{x} = (1, -2), \) \( \bar{x} = (2, 2), \) and \( \bar{x} = (2.8, -1.6), \) respectively. We terminate the algorithm when \( \|x_n - \bar{x}\| \leq \varepsilon, \) where \( \varepsilon = 10^{-6}. \) We use “Iter” to denote the iteration numbers and ”Time(s)” to denote the elapsed CPU time (in seconds). The obtained results are reported in Table 2. It can be seen from Table 2 that the larger the iteration parameter, the faster the proposed iterative algorithms (4.4) and (4.8) converge. For the iterative algorithm (4.8), when the parameter \( \lambda \) increases, although the range of the parameter \( \gamma \) increases, the number of iteration numbers increases instead. For the choice of \( \lambda = 0.5, \) it can be seen from the results that the algorithm (4.8) is better than the algorithm (4.4).

| The proposed algorithms | \( \lambda \) | \( \gamma \) | \( f = (6, -4) \) | \( f = (1, -4) \) | \( f = (2, 7) \) |
|------------------------|------|------|----------------|----------------|----------------|
|                        |      |      | Iter | Time(s) | Iter | Time(s) | Iter | Time(s) |
| (4.4)                  | 0.02 | -0.06| 941  | 0.71    | 946  | 0.73    | 1110 | 0.72    |
|                        | 0.04 | 0.08 | 564  | 0.71    | 566  | 0.73    | 558  | 0.71    |
|                        | 0.06 | 0.1  | 378  | 0.72    | 379  | 0.87    | 374  | 0.72    |
|                        | 0.08 | -0.1 | 285  | 0.74    | 240  | 0.77    | 282  | 0.72    |
|                        | 0.1  | -     | 229  | 0.72    | 193  | 0.74    | 226  | 0.73    |
| (4.8)                  | 0.05 | 0.05 | 457  | 0.90    | 456  | 0.93    | 457  | 0.88    |
|                        | 0.1  | 0.1  | 250  | 0.90    | 250  | 0.92    | 250  | 0.89    |
|                        | 0.15 | 0.15 | 180  | 0.91    | 149  | 0.91    | 179  | 0.90    |
|                        | 0.2  | 0.2  | 143  | 0.90    | 142  | 0.90    | 166  | 0.88    |
| (5.7)                  | 0.05 | 0.05 | 718  | 0.93    | 889  | 0.91    | 1306 | 0.91    |
|                        | 0.1  | 0.1  | 501  | 0.90    | 446  | 0.90    | 592  | 0.90    |
|                        | 0.15 | 0.15 | 317  | 0.93    | 360  | 0.91    | 383  | 0.89    |
|                        | 0.2  | 0.2  | 189  | 0.91    | 276  | 0.92    | 293  | 0.92    |
|                        | 0.25 | 0.25 | 226  | 0.88    | 228  | 0.90    | 250  | 0.89    |
|                        | 0.28 | 0.28 | 208  | 0.91    | 213  | 0.92    | 226  | 0.91    |
|                        | 0.31 | 0.31 | 1759 | 0.92    | 1691 | 0.91    | 1756 | 0.91    |
|                        | 0.31 | 0.31 | 1209 | 0.89    | 946  | 0.90    | 914  | 0.91    |
|                        | 0.31 | 0.31 | 738  | 0.89    | 806  | 0.91    | 797  | 0.91    |
|                        | 0.31 | 0.31 | 678  | 0.89    | 679  | 0.89    | 670  | 0.89    |
|                        | 0.31 | 0.31 | 581  | 0.92    | 547  | 0.90    | 621  | 0.91    |
|                        | 0.31 | 0.31 | 481  | 0.89    | 510  | 0.93    | 531  | 0.89    |

6 Conclusions

In this paper, we considered the monotone inclusions with a sum of four operators, in which two of them are maximally monotone, one is monotone Lipschitz, and one is cocoercive. We introduced three new splitting
algorithms to solve it and analyzed their convergence. As applications, we considered composite monotone inclusion problems. To solve this monotone inclusion, we transformed it into the formulation of (4.3) by the technique of product space. We evaluated the performance of the proposed algorithms on the Projection on the Minkowski sums of convex sets problem (5.2). Numerical experiments demonstrated the effectiveness and efficiency of the proposed algorithms.

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