Unexpected features of quantum degeneracies in a pairing model with two integrable limits

J Dukelsky\(^1\), J Okołowicz\(^2\) and M Płoszajczak\(^3\)

\(^1\) Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain
\(^2\) Institute of Nuclear Physics, Radzikowskiego 152, PL-31342 Kraków, Poland
\(^3\) Grand Accélérateur National d’Ions Lourds (GANIL), CEA/DSM—CNRS/IN2P3, BP 5027, F-14076 Caen Cedex 05, France

E-mail: dukelsky@iem.cfmac.csic.es, Jacek.Okolowicz@ifj.edu.pl and ploszajczak@ganil.fr

Received 1 April 2009
Accepted 22 May 2009
Published 1 July 2009

Online at stacks.iop.org/JSTAT/2009/L07001
doi:10.1088/1742-5468/2009/07/L07001

Abstract. The evolution pattern of level crossings and exceptional points is studied in a non-integrable pairing model with two different integrable limits. One of the integrable limits has two independent parameter-dependent integrals of motion. We demonstrate, and illustrate in our model, that quantum integrability of a system with more than one parameter-dependent integral of motion is always signaled by level crossings of a complex-extended Hamiltonian. We also find that integrability implies a reduced number of exceptional points. Both properties could uniquely characterize quantum integrability in small Hilbert spaces.

Keywords: algebraic structures of integrable models, quantum chaos
Unexpected features of quantum degeneracies in a pairing model with two integrable limits

The search for fingerprints of the chaotic/regular dynamics in the quantum regime is often focused on studies of spectral properties of the quantum systems. In this context, spectral fluctuations were intensely studied for various quantum systems. These studies lead to the BGS conjecture [1] that in the semiclassical limit the spectral fluctuations of chaotic systems are described by random matrix theory. For quantum integrable systems, Berry and Tabor [2] showed that the spectral fluctuations are well described by a Poisson statistic. While chaotic systems are characterized by level repulsion between successive levels, levels of integrable systems are uncorrelated, allowing crossings between states of the same symmetries.

Level crossings and degeneracies are important for the understanding of spectral fluctuations [3] and the onset of quantum chaos [4]. Much effort has been devoted to studies of degeneracies associated with avoided crossings in quantal spectra, focusing mainly on the topological structure of the Hilbert space and the geometric phases [5, 6]. Among these degeneracies, one finds a diabolic point where two Riemann sheets of eigenvalues touch each other [5, 7], and an exceptional point (EP) [8]–[10] where the two sheets are entangled by the square-root type of singularity. The EP appears in the complex $g$ plane of a generic Hamiltonian $H(g) = H_0 + gH_1$, where both $H_0$ and $H_1$ are Hermitian and $[H_0, H_1] \neq 0$. In many-body systems, EPs have been studied in schematic models like the Lipkin model [11] and the interacting boson model along the line connecting the dynamical symmetries $U(5)$ to $O(6)$ [12]. Both models can be considered as particular examples of two-level boson pairing models pertaining to the general class of integrable Richardson–Gaudin (RG) models [13, 14]. However, two-level pairing models have no level crossings and all degeneracies take place in the complex plane as EPs.

In this work, we will introduce a prototypical quantum integrable system, the three-level RG model, to discuss the appearance of level crossings and EPs, and their evolution both with complex parameters of the Hamiltonian and with a non-integrable complex perturbation. On this basis, we prove that integrable models with at least two parameter-dependent integrals of motion (IMs) have a level crossing in the complex-extended parameter space, providing a clear signal of their integrability. Furthermore, the inclusion of a non-integrable perturbation splits each level crossing into two EPs, transforming dramatically the topology of the Hilbert space close to the level crossing.

Let us begin by briefly reviewing the RG models, which are based on the $SU(2)$ algebra with elements $K_l^+, K_l^−, \text{ and } K_l^0$, fulfilling the commutation relations $[K_l^+, K_{l'}^-] = \delta_{ll'} K_l^0$, $[K_l^0, K_{l'}^\pm] = \pm \delta_{ll'} K_l^\pm$. The indices $l, l'$ refer to a particular copy from a set of $LSU(2)$ algebras. Each $SU(2)$ algebra possesses one quantum degree of freedom. Therefore, a quantum integrable model requires the existence of $L$ independent, global operators that commute with each other. These operators, which need not be Hermitian, are the IMs. In the following, we will work with the rational family of RG models whose IMs are [13]

$$R_l = K_l^0 + 4g \sum_{l'\neq l} \frac{1}{\varepsilon_l - \varepsilon_{l'}} \left[ \frac{1}{2} (K_l^+ K_{l'}^- + K_l^- K_{l'}^+) + K_l^0 K_{l'}^0 \right],$$

where $g$ and $\varepsilon_l$ are $L + 1$ arbitrary parameters. The IMs (1) satisfy $[R_i, R_j] = 0$ for all pairs $i, j$. 

doi:10.1088/1742-5468/2009/07/L07001
There is a profound relation between quantum integrability in finite systems and the existence of level crossings. In the context of the six-site Hubbard model, this problem has been addressed by Yuzbashyan et al [15], who showed that a level crossing implies the existence of two independent parameter-dependent IMs. Here we complete this analysis by showing that a system with at least two parameter-dependent IMs has of necessity a level crossing in the complex plane.

Let us assume a Hamiltonian \( H(g) \) depending linearly on a parameter \( g \). \( H(g) \) itself is the parameter-dependent IM. If \( g \) is complex then the Hamiltonian and eventually other parameter-dependent IMs will be non-Hermitian. The parameter-dependent IM \( Q(g) \) commuting with the Hamiltonian, \( [H(g), Q(g)] = 0 \), will be independent of the Hamiltonian if it cannot be expressed as an entire function of \( H(g) \) and \( g \), i.e., \( Q(g) \neq f(g, H(g)) \). Let us assume that \( n \) is the dimension of the Hilbert space and \( E_1(g), \ldots, E_n(g) \) and \( q_1(g), \ldots, q_n(g) \) are the corresponding eigenvalues in the basis in which both operators are diagonal. If \( Q(g) \) is an entire function of \( H(g) \), then it can be expanded for any complex \( g \) value as

\[
q_1(g) = a_n E_1^{n-1}(g) + \cdots + a_2 E_1(g) + a_1 \\
\vdots \\
q_n(g) = a_n E_n^{n-1}(g) + \cdots + a_2 E_n(g) + a_1.
\]

This set of equations always has a solution unless for some value \( g = g_0 \) a pair of equations \( \{k, k'\} \) have \( E_k(g_0) = E_{k'}(g_0) \), but \( q_k(g_0) \neq q_{k'}(g_0) \), implying a double degeneracy in the Hamiltonian but not in the second IM.

The minimal rational RG model (1) allowing for level crossings should have at least three \( SU(2) \) copies. The reason is that the sum of the IMs (1) is a global conserved symmetry: \( K^0 = \sum_{i=1}^{L} K_i^0 \), commuting with all IMs and independent of the parameter \( g \). Hence, we are left with two parameter-dependent IMs and, as shown above, this implies the existence of level crossings.

In what follows, we will use the pair representation of the \( SU(2) \) algebra leading to pairing Hamiltonians. The elementary operators in this representation are the number operators \( N_j = \sum_m a_j^m a_j^{m\dagger} \) and the pair operators \( A_j^\dagger = \sum_m a_j^m a_{j\bar{m}}^{\dagger} \), where \( j \) is the total angular momentum and \( m \) is the \( z \)-projection. The state \( j\bar{m} \) is the time reversal of \( jm \).

The relation between the operators of the pair algebra and the generators of the \( SU(2) \) algebra is \( K_j^0 = \frac{1}{2} N_j - \frac{i}{4} \Omega_j \), \( K_j^+ = (K_j^-)^\dagger = \frac{1}{2} A_j^\dagger \), where \( \Omega_j \) is the particle degeneracy level \( j \). With this correspondence, one can introduce an integrable three-level pairing Hamiltonian as

\[
H(g) = 2 \sum_i \varepsilon_i R_i(g) + C \equiv \sum_i \varepsilon_i N_i + g \sum_{ij} A_i^\dagger A_j,
\]

where \( C \) is an irrelevant constant and \( \varepsilon_i \) \( (i = 1, 2, 3) \) are the single-particle energies. One can see that \( H(g) \) itself is a parameter-dependent IM. As discussed above, the sum of the IMs (1) yields a parameter-independent IM, the particle number: \( N = 2 \sum_i R_i(g) + \frac{1}{2} \sum_i \Omega_i \). The second parameter-dependent IM can be chosen as linearly independent from the two other IMs. The simplest choice is \( R_1 \). If \( \varepsilon_1 = 0 \), the second
Unexpected features of quantum degeneracies in a pairing model with two integrable limits

parameter-dependent IM becomes

\[ Q(g) = \left[ 1 + g \left( \frac{\Omega_2}{\varepsilon_2} + \frac{\Omega_3}{\varepsilon_3} \right) \right] \frac{N_1}{2} + \frac{g\Omega_1 N_2}{\varepsilon_2} + \frac{g\Omega_1 N_3}{\varepsilon_3} \]

\[- g \left\{ \frac{1}{\varepsilon_2} \left[ \frac{1}{2}(A_1^\dagger A_2 + A_2^\dagger A_1) + N_1 N_2 \right] \right. \]

\[ + \frac{1}{\varepsilon_3} \left[ \frac{1}{2}(A_1^\dagger A_3 + A_3^\dagger A_1) + N_1 N_3 \right] \right\}. \tag{4} \]

The positions of all degeneracies in the complex-\(g\) plane are indicated by the roots of the coupled equations [16]:

\[ \det[H(g) - EI] = 0; \quad \frac{\partial}{\partial E} \det[H(g) - EI] = 0. \tag{5} \]

This is a set of two polynomial equations of degree \(n\) in the variables \(E\) and \(g\). While the first determinant defines the Hamiltonian eigenvalues, the second determinant imposes the equality of two roots of the first equation for a given value of \(g\). By eliminating \(E\) from these two equations, we are left with the discriminant \(D(g)\), a polynomial in \(g\) of degree \(M \leq n(n - 1)\). The discriminant can be written as [9]

\[ D(g) = \prod_{m < m'} [E_m(g) - E_{m'}(g)]^2, \tag{6} \]

where \(E_m(g), E_{m'}(g)\) denote the complex eigenvalues of \(H(g)\). In this form (6) the discriminant is a positive definite function of \(g\), and zero at the eigenvalue degeneracies \(E_m(g) = E_{m'}(g)\) at \(g = g_\alpha (\alpha = 1, \ldots, M)\). Therefore, these degeneracies can be found numerically by looking for sharp minima of \(D(g)\) [17]. It turns out that for real values of \(\varepsilon_j\) the roots are real or complex conjugate pairs. One finds two kinds of solutions in quantum integrable models: (i) single roots, corresponding to EPs that are common to all IMs, and (ii) double- (multiple-) root solutions, which indicate non-singular sharp crossings of two (or more) levels with two (or more) orthogonal wavefunctions, related to the existence of at least two independent IMs.

Figure 1 shows an evolution of two level crossings, the double roots of \(D(g)\), in the complex-\(g\) plane as a function of the energy of the third single-particle level \(\varepsilon_3\) (\(\varepsilon_3 > \varepsilon_2\)). The system has four pairs of fermions in a valence space with level degeneracies \(\Omega_1 = 6, \Omega_2 = 4, \Omega_3 = 2\) and \(\varepsilon_1 = 0, \varepsilon_2 = 1\). In the limit \(\varepsilon_3 \to \infty\), this system decouples effectively into the two two-level models: the first one with level \(\varepsilon_3\) occupied and the second one with \(\varepsilon_3\) empty. In this limit level crossings are forbidden and, indeed, the two level crossings that appear for finite values of \(\varepsilon_3\) move to \(\pm \infty\). With decreasing \(\varepsilon_3\), two level crossings coming from \(\pm \infty\) approach each other in the real axis up to a critical value \(\varepsilon_3^{(cr)} = 1.8499\), where they coalesce. The level crossing at this point corresponds to the quadruple root of \(D(g)\). For \(\varepsilon_3 < \varepsilon_3^{(cr)}\) this crossing splits again into the two double-root level crossings, which move into the complex-\(g\) plane. The presence of such level crossings in the complex plane is a clear signature of quantum integrability, and shows the necessity of extending the demonstration of operator independence given in (2) to the whole complex parameter space.

EPs associated with single roots of the discriminant \(D(g)\), unlike level crossings, are common to all parameter-dependent IMs including the Hamiltonian. It should be also

doi:10.1088/1742-5468/2009/07/L07001
Unexpected features of quantum degeneracies in a pairing model with two integrable limits

Figure 1. Collision and subsequent scattering of two level crossings as a function of the energy \( \varepsilon_3 \) in a complex-extended integrable three-level pairing Hamiltonian (3). Points are plotted in a descending order of \( \varepsilon_3 \). For more details, see the discussion in the text.

noted that EPs appear in the quantum integrable model even though no manifestation of level repulsion is expected and the spectral fluctuations of the Hermitian Hamiltonian obey a Poisson distribution [18]. In that sense, level repulsion may be a sufficient but not a necessary condition for the appearance of EPs.

Another feature that we found in this three-level pairing model, as well as in more general multi-level pairing models, is the reduction of the total number of discriminant roots whose maximum value is \( M_{\text{max}} = n(n - 1) \). In the particular case shown in figure 1, \( M_{\text{max}} = 20 \), but we found 16 roots consisting of two level crossings (double roots) and 12 EPs (single roots).

In the following, we generalize the three-level pairing Hamiltonian (3) to study effects of non-integrability:

\[
H(g) = \sum_i \varepsilon_i N_i + \zeta g \sum_{ij} A_i^\dagger A_j - (1 - \zeta) g \sum_i N_i^2.
\] (7)

For \( \zeta = 1 \), equation (7) corresponds to the integrable Hamiltonian (3). For \( \zeta = 0 \), the Hamiltonian (7) is also integrable with the number operators \( N_i \) playing the role of parameter-independent IMs. In the interval \( 0 < \zeta < 1 \), the Hamiltonian (7) is non-integrable, i.e. it does not possess an independent IM other than the Hamiltonian and the total number operator. Two main features characterize the emergence of non-integrability: firstly the level crossings break into pairs of EPs and secondly the missing roots of the discriminant come into play as EPs from \( \infty \).

Figure 2 shows the global pattern of level crossings and EPs as a function of the parameter \( \zeta \) (\( 0 \leq \zeta \leq 1 \)) for the three-level system of figure 1 in three regimes: \( \varepsilon_3 > \varepsilon^{(cr)}_3 \) \( (\varepsilon_3 = 7/3) \), \( \varepsilon_3 = \varepsilon^{(cr)}_3 \), and \( \varepsilon_3 < \varepsilon^{(cr)}_3 \) \( (\varepsilon_3 = 3/2) \). Energies of levels ‘1’ and ‘2’ are fixed at \( \varepsilon_1 = 0, \varepsilon_2 = 1 \). Only the lower half-plane of \( g \) is shown, where all eigenvalues are

doi:10.1088/1742-5468/2009/07/L07001
Figure 2. The evolution of level crossings and EPs of the complex-extended three-level pairing Hamiltonian (7) in the interval $0 \leq \zeta \leq 1$ depicted by a continuous line. Circles and squares denote the position of degeneracies at $\zeta = 1$ and 0, respectively. The double circles depict the double-root level crossing at $\zeta = 1$. The triple circle shows the quadruple-root level crossing corresponding to the coalescence of two double-root level crossings. For more details, see the caption of figure 1 and the discussion in the text.
Unexpected features of quantum degeneracies in a pairing model with two integrable limits

either discrete states on the real-$g$ axis or else decaying resonances. Complex conjugate degeneracies situated in the upper half-plane ($\text{Im}(g) > 0$) correspond to capturing resonances. In the limit of $\zeta = 1$, the two level crossings for $\varepsilon_3 = 7/3$ are located along the real-$g$ axis. One may also notice a quadruple-root level crossing on the real-$g$ axis at $\varepsilon_3 = \varepsilon_3^{(cr)}$ (see also figure 1). Moreover, one can see the location of six EPs of the integrable pairing Hamiltonian (3) in all regimes of $\varepsilon_3$.

With decreasing $\zeta$, one observes several distinct effects. Firstly, each double-root level crossing at $\zeta = 1$ breaks into a pair of EPs. For $\varepsilon_3 = 3/2$, two EPs resulting from this fragmentation follow independent trajectories in the complex-$g$ plane and end up in two different level crossings at $\zeta = 0$. For $\varepsilon_3 = 7/3$, the level crossing at $\zeta = 1$ breaks into two EPs symmetrically with respect to the real-$g$ axis. Since this symmetry is conserved for all $\zeta$, these EPs end up in the same level crossing for $\zeta = 0$. Secondly, roots that are missing in the integrable limit ($\zeta = 1$) appear from $g = \infty$ and end up in level crossings for $\zeta \to 0$. Thirdly, in the limit $\zeta \to 0$, all EPs either collapse in different level crossings at the real-$g$ axis or escape to infinity. The double level crossings found in this limit correspond to the two different eigenvalue degeneracies. For $\varepsilon_3 = 3/2$, one can see three EPs converging on each side of the real-$g$ axis to a single point. At this sextuple-root of the discriminant, one finds a sharp crossing of three eigenvalues.

The degree of the discriminant, which for $\zeta = 0, 1$ equals 16 in all regimes of $\varepsilon_3$, becomes $M = n(n - 1) = 20$ in the non-integrable regime. Notice the absence of EPs in the integrable case $\zeta = 0$, reflecting the fact that the Hamiltonian is diagonal in the original basis for any $g$ value.

In conclusion, we have shown that a system with two independent parameter-dependent IMs has at least one level crossing in the complex parameter space. We have used a minimal integrable pairing model consisting of three levels to exemplify this issue. Moreover, we have found that the degree of the discriminant is reduced in this integrable limit, giving rise to a lower number of EPs, but still a fraction of them persists in spite of the fact that there is no manifestation of level repulsion. If integrability is broken by the inclusion of a non-integrable perturbation, all level crossings split into EPs and other EPs come into the complex plane from $\infty$, recovering the maximal degree $M_{\max} = n(n - 1)$ of the discriminant. We conjecture that these two unexpected properties, level crossings in the complex-$g$ plane and the reduction in the number of EPs, uniquely define a quantum integrable system.

Owusu et al [19] have demonstrated recently that RG models are prototypical integrable models, i.e. a complete set of independent integrals of motion (maximally commuting Hamiltonians), determining a generic integrable model, can be mapped to the RG integrals of motion. Therefore, the conclusions we obtained from the analysis of a specific three-level RG model could be extended to generic integrable models.

If our conjecture proves to be true, then it might be particularly useful in studies of finite systems with small dimensional Hilbert spaces, where the analysis of spectral fluctuations is unreliable. More work has to be done to elucidate the relation between integrable Hamiltonians and the missing roots of the discriminant.

We acknowledge fruitful discussions with C Esebbag. This work was supported in part by the Spanish MEC under grant No FIS2006-12783-C03-01 and by the CICYT (Spanish)–IN2P3 (French) cooperation.
Unexpected features of quantum degeneracies in a pairing model with two integrable limits

References

[1] Bohigas O, Giannoni M-J and Schmit C, Characterization of quantum spectra and universality of level fluctuation laws, 1984 Phys. Rev. Lett. 52 1
[2] Berry M V and Tabor M, Level clustering in the regular system, 1977 Proc. R. Soc. Lond. A 356 375
[3] Berry M V, Quantizing a classically ergodic system: Sinai’s billiard and the KKR method, 1981 Ann. Phys., NY 131 163
[4] Berry M V, 1985 Chaotic Behaviour of Deterministic Systems (Les Houches Lectures vol XXXVI) ed R H Helleman and G Closs (Amsterdam: North-Holland)
[5] Berry M V, 1985 Quantum Chaos ed G Casati (New York: Plenum)
[6] Lauber H-M, Weidenhammer P and Dubbers D, Geometric phases and hidden symmetries in simple resonators, 1984 Phys. Rev. Lett. 72 1004
Manolopoulos D E and Child M S, Cyclic phases at an n-fold degeneracy, 1999 Phys. Rev. Lett. 82 2223
Pistolesi F and Manini N, Geometric phases and multiple degeneracies in harmonic resonators, 2000 Phys. Rev. Lett. 85 1585
Dembowski C, Gräf H-D, Harney H L, Heine A, Heiss W D, Rehfeld H and Richter A, Experimental observation of the topological structure of exceptional points, 2001 Phys. Rev. Lett. 86 787
[7] Berry M V and Wilkinson M, Diabolic points in the spectra of triangles, 1984 Proc. R. Soc. Lond. A 392 15
[8] Kato T, 1995 Perturbation Theory for Linear Operators (Berlin: Springer)
[9] Zirnbauer M R, Verbaarschot J J M and Weidenmüller H A, Destruction of order in nuclear spectra by a residual GOE interaction, 1983 Nucl. Phys. A 411 161
[10] Heiss W D and Steeb W-H, Avoided level crossings and Riemann sheet structure, 1991 J. Math. Phys. 32 3003
[11] Heiss W D and Sannino A L, Transitional regions of finite Fermi systems and quantum chaos, 1991 Phys. Rev. A 43 4159
[12] Heinzle S, Cejnar P, Jolie J and Macek M, Evolution of spectral properties along the O(6)–U(5) transition in the interacting boson model. I. Level dynamics, 2006 Phys. Rev. C 73 014306
[13] Dukelsky J, Pittel S and Sierra G, Colloquium: exactly solvable Richardson–Gaudin models for many-body quantum systems, 2004 Rev. Mod. Phys. 76 643
[14] Ortiz G, Somma R, Dukelsky J and Rombouts S, Exactly-solvable models derived from a generalized Gaudin algebra, 2005 Nucl. Phys. B 707 421
[15] Yuzbashyan E A, Altshuler B L and Shastry B S, The origin of degeneracies and crossings in the 1D Hubbard model, 2002 J. Phys. A: Math. Gen. 35 7525
[16] Heiss W D and Sannino A L, Avoided level crossing and exceptional points, 1990 J. Phys. A: Math. Gen. 23 1167
[17] Cejnar P, Heinzle S and Macek M, Coulomb analogy for non-Hermitian degeneracies near quantum phase transitions, 2007 Phys. Rev. Lett. 99 100601
[18] Relaño A, Dukelsky J, Gomez J M G and Retamosa J, Stringent numerical test of the Poisson distribution for finite quantum integrable Hamiltonians, 2004 Phys. Rev. E 70 026208
[19] Owusu H K, Wagh K and Yuzbashyan E A, The link between integrability, level crossings and exact solution in quantum model, 2009 J. Phys. A: Math. Theor. 42 035206

doi:10.1088/1742-5468/2009/07/L07001