JACOB’S LADDERS AND THE QUANTIZATION OF THE HARDY-LITTLEWOOD INTEGRAL

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Abstract. We use Jacob’s ladders to solve the fine problem how to divide of the Hardy-Littlewood integral to equal parts, for example of magnitude $h = 6.6 \times 10^{-47}$ (the numerical value of elementary Planck quantum). The result of the paper cannot be obtained in known theories of Balasubramanian, Heath-Brown and Ivic.

1. The problem of dividing on equal parts

1.1. Let us remind the following facts. Titchmarsh-Kober-Atkinson (TKA) formula

\begin{equation}
\int_0^\infty Z^2(t)e^{-2\delta t}dt = \frac{c - \ln(4\pi\delta)}{2\sin\delta} + \sum_{n=1}^N c_n\delta^n + O(\delta^{N+1})
\end{equation}

(see [10], p. 141) remained as an isolated result for a period of 56 years until we have discovered the nonlinear integral equation

\begin{equation}
\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{1}{2}\pi^{1/2}t}dt = \int_0^T Z^2(t)dt
\end{equation}

(see [6]) in which the essence of the TKA formula is encoded. Namely, we have shown in [6] that the following almost exact formula for the Hardy-Littlewood integral takes place

\begin{equation}
\int_0^T Z^2(t)dt = \frac{\varphi(T)}{2} \ln \frac{\varphi(T)}{2} + (c - 2\pi)\varphi(T)\frac{\varphi(T)}{2} + c_0 + O\left(\frac{\ln T}{T}\right),
\end{equation}

where $\varphi(T)$ is the Jacob’s ladder (a solution of the nonlinear integral equation \[1.2\]).

Remark 1. Our formula \[1.3\] for the Hardy-Littlewood integral has been obtained after the time period of 90 years since this integral appeared in 1918 (see [3], pp. 122, 151-156).

Remark 2. Let us remind that

(A) The Good’s $\Omega$-theorem (see [2]) implies for the Balasubramanian formula

\begin{equation}
\int_0^T Z^2(t)dt = T\ln T + (2c - 1 - \ln 2\pi)T + R(T), \quad R(T) = O(T^{1/3+\varepsilon})
\end{equation}

(see [1]) that

$$\limsup_{T \to \infty} |R(T)| = +\infty.$$
(B) The error term in (1.3) tends to zero as \( T \) goes to infinity, namely

\[
\lim_{T \to \infty} r(T) = 0, \quad r(T) = \mathcal{O}\left(\frac{\ln T}{T}\right),
\]

i.e. our formula is almost exact (see [6]).

1.2. In this paper I consider the problem concerning the solid of revolution corresponding to the graph of the function \( Z(t), \; t \in [T_0, +\infty), \) where \( 0 < T_0 \) is a sufficiently big number.

**Problem.** To divide this solid of revolution on parts of equal volumes.

We obtain, for example, from our formula (1.3) that there exists a sequence \( \{\hat{T}_\nu\}_{\nu=0}^\infty \), for which

\[
\pi \int_{\hat{T}_\nu}^{\hat{T}_{\nu+1}} Z^2(t)dt = 6.6 \times 10^{-27},
\]

\[
\hat{T}_{\nu+1} - \hat{T}_\nu \sim \frac{6.6 \times 10^{-27}}{\pi \ln \hat{T}_\nu \tan[\alpha(\hat{T}_\nu, \hat{T}_{\nu+1})]}, \quad \nu \to \infty,
\]

where \( h = 6.6 \times 10^{-27} \text{erg} \cdot \text{sec} \) is elementary Planck quantum.

**Remark 3.** It us quite evident that the quantization rule (1.5) cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic (see, for example, [4]).

This paper is a continuation of the series of papers [6]-[8].

2. **Main result**

The following theorem holds true

**Theorem 1.** Let \( 0 < \delta < \Delta \) where \( \delta \) is an arbitrarily small and \( \Delta \) is an arbitrarily big number. Then for every \( \omega \in [\delta, \Delta] \) and every Jacob’s ladder \( \varphi(T) \) there is the sequence

\[
\{T_\nu(\omega, \varphi)\}_{\nu=0}^\infty, \quad T_\nu(\omega, \varphi) = T_\nu(\omega)
\]

for which

\[
\int_{T_\nu(\omega)}^{T_{\nu+1}(\omega)} Z^2(t)dt = \omega,
\]

\[
T_{\nu+1}(\omega) - T_\nu(\omega) = \frac{\omega + \mathcal{O}\left(\frac{\ln T_\nu}{T_\nu}\right)}{\ln \frac{\varphi[T_{\nu+1}(\omega)]}{\frac{\varphi[T_{\nu+1}(\omega)]}{2} - a} \tan[\alpha(T_\nu, T_{\nu+1})]},
\]

where \( 0 < \nu_0(\omega, \varphi) \) is a sufficiently big number, \( a = \ln 2\pi - 1 - \epsilon \) and \( \alpha = \alpha(T_\nu, T_{\nu+1}) \) is the angle of the chord binding the points

\[
\left[T_\nu, \frac{1}{2} \varphi[T_\nu(\omega)]\right], \quad \left[T_{\nu+1}, \frac{1}{2} \varphi[T_{\nu+1}(\omega)]\right]
\]

of the curve \( y = \frac{1}{2} \varphi(T) \).

We obtain from our Theorem 1
Corollary 1.

\begin{equation}
\int_{T_\nu}^{T_\nu+1(\omega)} Z^2(t) dt = [T_{\nu+1}(\omega) - T_\nu(\omega)] \ln \left( e^{-a \phi(T_\nu)} \frac{\ln T_\nu}{T_\nu} \right) + O \left( \frac{\ln T_\nu}{T_\nu} \right).
\end{equation}

Let us remind the formula (see [7], (2.1))

\begin{equation}
\int_{T}^{T+U} Z^2(t) dt = U \ln \left( e^{-a \phi(T)} \frac{\ln U}{U} \right) + O \left( \frac{1}{T_1^{1/3 - 4\epsilon}} \right).
\end{equation}

Remark 4. In the case of collection of sequences \( \{T_\nu(\omega)\} \) we obtain the essential improvement

\[ O \left( \frac{1}{T_{1/3 - 4\epsilon}} \right) \to O \left( \frac{\ln T_\nu}{T_\nu} \right) \]

(see (2.3) of the remainder term in the formula (2.4)).

Since

\[ \sum_{k=1}^{N} \int_{T_{\nu+k}(\omega)}^{T_{\nu+k-1}(\omega)} Z^2(t) dt = N \omega \]

then, in the case

\[ T_{\nu+N_0}(\omega) - T_\nu(\omega) \sim U_0 = T^{1/3 + 2\epsilon}, \]

we have (see (2.4))

\[ N_0 \sim \frac{1}{\omega} U_0 \ln T_\nu(\omega), \quad \frac{U_0}{N_0} \sim \frac{\omega}{\ln T_\nu(\omega)}. \]

Then we obtain by our Theorem 1

Corollary 2. The following asymptotic formula takes place

\begin{equation}
\frac{1}{N_0} \sum_{k=1}^{N_0} \{T_{\nu+k}(\omega) - T_{\nu+k-1}(\omega)\} \sim \frac{\omega}{\ln T_\nu(\omega)}
\end{equation}

for arithmetic mean values of \( T_{\nu+k}(\omega) - T_{\nu+k-1}(\omega), k = 1, 2, \ldots, N_0. \)

3. On transformation of the sequence \( \{T_\nu(\omega)\} \) preserving the quantization of the Hardy-Littlewood integral

3.1. If there is a sufficiently big natural number \( \bar{\nu} \) for which \( \omega \bar{\nu} = T_0 \) is fulfilled then from our Theorem 1 the resolution of our Problem follows. The complete resolution follows from the next theorem.

Theorem 2. For every \( \omega \in [\delta, \Delta], \tau \in [0, \omega) \) and every Jacob’s ladder \( \phi(T) \) there is the collection of sequences

\[ \{T_\nu(\omega, \tau; \varphi)\}_{\nu=0}^{\infty}, \quad T_\nu(\omega, \tau; \varphi) = T_\nu(\omega, \tau), \quad T_\nu(\omega, 0) = T_\nu(\omega) \]

for which

\begin{equation}
\int_{T_\nu(\omega, \tau)}^{T_{\nu+1}(\omega, \tau)} Z^2(t) dt = \omega.
\end{equation}
\begin{align*}
T_{\nu+1}(\omega, \tau) - T_\nu(\omega, \tau) &= \frac{\omega + O\left(\frac{\ln T_\nu}{T_\nu}\right)}{\left(\ln \frac{\varphi(T_\nu)}{2} - a\right) \tan[\alpha(T_\nu, T_{\nu+1})]},
\end{align*}

where \(0 < \nu_0(\omega, \varphi)\) is a sufficiently big number and \(\alpha = \alpha(T_\nu, T_{\nu+1})\) is the angle of the chord binding the points

\[
\left[T_\nu(\omega, \tau), \frac{1}{2} \varphi(T_\nu(\omega, \tau))\right], \quad \left[T_{\nu+1}(\omega, \tau), \frac{1}{2} \varphi(T_{\nu+1}(\omega, \tau))\right]
\]

of the curve \(y = \frac{1}{2} \varphi(T)\).

**Remark 5.** From (3.1), (3.2) we get full analogies of Corollaries 1., 2. and Remark 4.

**Remark 6.** The quantization (1.5) follows from the choice \(\{T_\nu(\omega, \tau)\}\) with \(\omega = \frac{\hbar}{\pi}, \bar{\nu}\omega + \tau = T_0\).

3.2. Let us remind that for Gram’s sequence \(\{t_\nu\}\) we have (see \[9\])

\begin{equation}
t_{\nu+1} - t_\nu \sim \frac{2\pi}{\ln t_\nu},
\end{equation}

and for the collection of sequences \(\{\bar{t}_\nu(\bar{\tau})\}\), \(\bar{\tau} \in [-\pi, \pi]\) defined in our paper \[5\] we have the analogue of (3.3)

\[
\bar{t}_{\nu+1}(\bar{\tau}) - \bar{t}_\nu(\bar{\tau}) \sim \frac{2\pi}{\ln t_\nu(\bar{\tau})}.
\]

**Remark 7.** Under the transformations

\(t_\nu \rightarrow T_\nu(\omega); T_\nu(\omega, \tau)\)

the individual property (3.3) is transformed to an analogous property of arithmetic means (see \[25\] and Remark 5).

4. **Proof of Theorems 1., 2.**

4.1. By (1.3) we have

\begin{align*}
\int_0^T Z^2(t) dt &= F[\varphi(T)] + O\left(\frac{\ln T}{T}\right), \quad T \geq T^{(1)}[\varphi],
\end{align*}

(4.1)

\begin{align*}
F(y) &= \frac{y}{2} \ln \frac{y}{2} + (c - \ln 2\pi) \frac{y}{2} + c_0, \quad F'(y) = \frac{1}{2} \ln \frac{y}{2} - \frac{a}{2}, \quad F''(y) = \frac{1}{2y}.
\end{align*}

(4.2)

Since the continuous function

\[
F[\varphi(T)] + O\left(\frac{\ln T}{T}\right), \quad T \geq T^{(1)}
\]

is increasing, there is a root \(T_\nu(\omega, \varphi)\) of the equation

\[
F[\varphi(T)] + O\left(\frac{\ln T}{T}\right) = \omega \nu, \quad \nu \geq \nu_0,
\]

where \(\nu_0 = \nu_0(\omega, \varphi)\) is a sufficiently big number. Thus, the sequence \(\{T_\nu(\omega; \varphi)\}_{\nu=\nu_0}^\infty\)

is constructed by equation

\begin{equation}
F[\varphi(T_\nu)] + O\left(\frac{\ln T_\nu}{T_\nu}\right) = \omega \nu.
\end{equation}

(4.3)
From (4.1) by (4.3) we obtain
\[
\int_0^{T_\nu(\omega;\varphi)} Z^2(t) dt = \omega \nu \Rightarrow \int_{T_\nu(\omega;\varphi)}^{T_{\nu+1}(\omega;\varphi)} Z^2(t) dt = \omega,
\]
i.e. (2.1).

4.2. We have by (4.2), (4.3)
\[
\omega + O\left(\frac{\ln T_\nu}{T_\nu}\right) = F[\varphi(T_{\nu+1})] - F[\varphi(T_\nu)] = \left(\frac{1}{2} \ln \frac{d}{2} - \frac{a}{2}\right) [\varphi(T_{\nu+1}) - \varphi(T_\nu)], \quad d \in (\varphi(T_\nu), \varphi(T_{\nu+1})],
\]
i.e. (see [6], (5.2); 1.9T < \varphi(T) < 2T)
\[
(4.4) \quad \varphi(T_{\nu+1}) - \varphi(T_\nu) = O\left(\frac{1}{\ln T_\nu}\right).
\]
Next we have
\[
F[\varphi(T_{\nu+1})] - F[\varphi(T_\nu)] = \left(\frac{1}{2} \ln \frac{\varphi(T_\nu)}{2} - \frac{a}{2}\right) [\varphi(T_{\nu+1}) - \varphi(T_\nu)] + 
O\left\{\frac{(\varphi(T_{\nu+1}) - \varphi(T_\nu))^2}{T}\right\} + O\left(\frac{\ln T_\nu}{T_\nu}\right) = \omega
\]
by (4.2), (4.3), i.e. (see 1.4)
\[
\frac{1}{2} [\varphi(T_{\nu+1}) - \varphi(T_\nu)] \left(\ln \frac{\varphi(T_\nu)}{2} - a\right) = \omega + O\left(\frac{\ln T_\nu}{T_\nu}\right) + O\left(\frac{1}{T_\nu \ln^2 T_\nu}\right),
\]
from which the formula (2.2) follows.

Remark 8. Proof of Theorem 2 is similar to the proof of Theorem 1.

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