Hall effect and inter-chain magneto-optical properties of coupled Luttinger liquids

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We consider the Hall effect in a system of weakly coupled Luttinger chains. We obtain the full conductivity tensor in the absence of dissipation along the chains. We show that while the dependence of the Hall and transverse conductivities on temperature and frequency are affected by the Luttinger interaction very strongly, the Hall resistivity is given by a simple formula corresponding to the noninteracting fermions. We compute the frequency, temperature and field dependence of the transverse conductivity. Consequences for the quasi-one-dimensional organic conductors are discussed.

I. INTRODUCTION

Interacting electrons in one dimension form a non-Fermi liquid state, usually called a Luttinger liquid (LL). There are several candidates for actual realizations of this state in real systems, including organic conductors, edge states and stripe phases in the Quantum Hall effect, carbon nanotubes, etc... In many cases however, one is dealing only with a Hall effect, carbon nanotubes, etc... In many cases however, one is dealing only with a quasi one-dimensional situation in which the chains are weakly coupled to one another. In the context of organic conductors in particular, the most relevant type of inter-chain coupling is an inter-chain hopping \( t_\perp \). The question that immediately arises is whether the Luttinger liquid effects can be observed in quasi-one dimensional systems. The most naive answer would be that such systems are essentially two-dimensional, and perhaps Fermi liquids, at energy scales lower than the interchain hopping term \( t_\perp \), but that at energies larger than this hopping the Luttinger liquid effects survive. The renormalization group calculation supports this picture, but due to the effect of the interaction between electrons, the inter-chain hopping term is strongly renormalized and the crossover between the one-dimensional Luttinger liquid and two-dimensional behavior takes place at the energy scale:

\[
\epsilon^* = t_\perp (t_\perp / \epsilon_F)^\eta/(1-\eta)
\]

In this expression, \( \epsilon_F \) is the Fermi energy of a single chain (\( \epsilon_F \propto t_\perp \), the in-chain hopping) and \( \eta/2 \) is the scaling dimension of the physical electron operator in the Luttinger liquid. For a model with spin: \( \eta = (K_\rho + 1/K_\rho)/4 - 1/2 \), with \( K_\rho \) the LL parameter in the charge sector. \( \eta \) is also the exponent associated with the singularity of \( n(k) \) at the Fermi surface\(^\dagger\). When interactions are strong \( (K_\rho \ll 1 \text{ i.e. } \eta \text{ not too small}) \), \( \epsilon^* \) can be much smaller than the naive estimate \( \epsilon^* = t_\perp \), suggesting that LL behavior could be observable down to a much smaller energy (temperature) scale than the bare \( t_\perp \).

We emphasize that the estimate (1) does not take into account the effect of an intra-chain umklapp scattering. When relevant, this coupling tends to open a Mott gap. One is thus faced with the more complicated situation of two relevant perturbations competing with each other, and this competition controls the physics of the dimensional crossover. It was recently pointed out\(^\dagger\) that this may be an important consideration for the physics of the quasi one-dimensional organic conductors, even for the metallic compounds (TMTSF)\(_2\)X . A full understanding, in these compounds, of the crossover between the high energy (temperature, frequency) phase, likely to be indeed a Luttinger liquid\(^\dagger\), and the low energy one is still lacking.

The present paper is devoted to the calculation of the inter-chain conductivity and Hall effect in a system of weakly coupled LL chains, in the presence of a magnetic field perpendicular to the chains. We work to lowest order in a perturbative expansion in the inter-chain hopping \( t_\perp \), which is justified when one of the characteristic energies associated with the temperature \( T \), the frequency \( \omega \) or the field \( H \) is larger than \( \epsilon^* \). Furthermore, the band curvature (\( \alpha \)) must also be treated as a perturbation since for a purely linear spectrum particle-hole symmetry would lead to a vanishing Hall conductance. As explained below, there are subtleties associated with these perturbative expansions, having to do with the non-commutativity of the small \( \omega \) and small \( \alpha \) or small \( t_\perp \) expansions. Because of these technical difficulties, we mainly focus in this paper on the case where there is no dissipation inside the chains (in particular, no umklapp scattering) -see, however, the remarks made in the conclusion-. Two main results are obtained in this paper: i) we show that in the absence of in-chain dissipation, the Hall resistance is a constant, independent of...
frequency and temperature (this result is exact for the model that we study) ii) using the perturbation theory in the interchain hopping we derive an explicit expression for the inter-chain conductivity as a function of temperature, frequency and magnetic field Eq. (3), which also determines the full resistivity tensor in this dissipationless limit Eqs. (28-31).

To make contact with previous works, we note that the inter-chain conductivity in zero field was considered in Refs. [19,20] and recently reexamined in Ref. [8] The Hall conductivity was first considered in Ref. [11] at zero temperature and for very large magnetic fields. Here we generalize this theory to finite temperature and arbitrary fields (including the physically important small field limit).

There are several experimental motivations for the present work. Inter-chain transport and optical measurements have been used as a probe of the in-chain physics in quasi-one dimensional organic conductors, and a critical discussion of the relevance of coupled Luttinger liquid models to these measurements has been given recently by two of the authors. Very recently, measurements of the Hall effect in the Bechgaard salt (TMTSF)$_2$PF$_6$ in two different geometries have been reported [22,23]. While in-chain dissipation may be an important ingredient for the understanding of these experiments, the results of the present work for an ideal Luttinger liquid provide a benchmark to which the experiments can be compared.

The paper is organized as follows: In Sec. II we introduce the model and define the geometry that we consider. In Sec III, we prove some general properties of the Hall effect in the absence of dissipation in the chains. In Sec IV we calculate the transverse conductivity, and the optical Hall angle. The main results are summarized in Sec V, where we also briefly discuss the possible consequences of in-chain dissipation. Technical details can be found in the appendices.

II. MODEL AND GEOMETRY

The geometry considered in this paper is depicted in Figure 1. We consider one-dimensional chains (along the x-axis), coupled by a transverse hopping (y-axis) into a two-dimensional array. A magnetic field is applied perpendicular to the array (z-axis). For simplicity we consider the case of spinless electrons, and neglect the in-chain umklapp processes (a legitimate assumption if the system is not at a commensurate filling). The main technical difficulty of this problem is that the Hall conductivity of chains with a linear electron spectrum

\[ \epsilon_{\pm} = \pm v_F (p \mp p_F) \]  

is zero due to the particle-hole symmetry. Therefore to obtain a nontrivial answer it is necessary to consider a nonlinear correction to the spectrum

\[ \epsilon_{\pm} = \pm v_F (p \mp p_F) + \alpha (p \mp p_F)^2. \]  

Thus the Hamiltonian of the problem is

\[ H = \int dx \left[ \sum_i v_F \hat{\psi}_i^\dagger \tau_3 \left(-i\partial_x \right) \hat{\psi}_i - \alpha \sum_i \hat{\psi}_i^\dagger \partial_x^2 \hat{\psi}_i + g \sum_i \hat{\psi}_i^\dagger \hat{\psi}_{i+1}^\dagger \hat{\psi}_{i-1} \right], \]  

where \( \hat{\psi} \) is a two-component vector composed from the right- and left-moving electrons \( \psi = \begin{pmatrix} \psi^+ & \psi^- \end{pmatrix} \), \( \tau_3 \) is a Pauli matrix and \( A_{ij} = \int_0^L A \cdot d\mathbf{l} \). We use the Landau gauge \( A_y = Hx \). The second term in the Hamiltonian (1) corresponds to the nonlinear correction (3) to the free electron spectrum. The model without the hopping term and \( \alpha \)-term can be solved exactly, for example by the bosonization method. It is also possible to bosonize the \( \alpha \)-term but it leads to a model with a cubic interaction between the bosons which is not exactly solvable. The perturbation theory in \( \alpha \)-term seems to be always a good approximation because the nonlinear term in the spectrum (3) is small compared with the linear one as long as the typical energy scale of the problem is less than the Fermi energy. This is true, indeed, but only for the quantities that have a regular expansion in \( \alpha \). As we show below, the components of the resistivity tensor depend regular on \( \alpha \) and therefore can be expanded in \( \alpha \). On the contrary, the components of the conductivity tensor at low frequencies depend singular on \( \alpha \) and, thus, cannot be expanded in \( \alpha \).

We consider in this paper the calculation of the anisotropic conductivity tensor (in the presence of the magnetic field), relating the current to the electric field:

\[ \begin{bmatrix} j_x \\ j_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}. \]  

The resistivity tensor is equal to the inverse of the conductivity tensor, and the Hall resistivity \( \rho_{xy} \) is in partic-
ular given by:
\[ \rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx} \sigma_{yy} + \sigma_{xy}^2}. \] (6)

In the following, we shall evaluate the conductivity and resistivity tensor using a perturbative method in both the band curvature \( \alpha \) and the inter-chain hopping \( t_\perp \). It is crucial to realize that the limits of small \( \alpha \), small \( t_\perp \) and small frequency \( \omega \) do not commute. As a result, great care must be taken in order to decide which quantity to expand. This will be fully clarified in the next section.

### III. HALL EFFECT IN THE ABSENCE OF IN-CHAIN DISSIPATION: GENERAL CONSIDERATIONS

In this section we show that, in the case where there is no dissipation in the chains, several aspects of the Hall effect can be deduced from general principles. We first consider (Sec. III A) the limit where \( \omega \) is taken to zero first, in which Galilean invariance can be used. We then turn to non-zero frequency (Sec. III B) and use the commutation relation of the in-chain current operator and Hamiltonian in order to obtain general expressions for the conductivity and resistivity tensors. These expressions involve a single non-trivial quantity, the inter-chain resistance \( \rho_{yx}(\omega, T) \) which will be explicitly evaluated (to lowest order) in Sec. IV. They also allow us to clarify which quantities have regular perturbative expansions in \( \alpha \) and \( t_\perp \). In Sec. III C we explain physically the content of these expressions.

#### A. Galilean invariance and the \( \omega \to 0 \) limit

Here we focus on the zero-frequency limit, and show that when the Hamiltonian is Galilean invariant in the direction of the chains, the Hall resistance is independent of temperature and simply given by its free electron value. The resistivity tensor is defined by
\[ \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix} \begin{bmatrix} j_x \\ j_y \end{bmatrix}. \] (7)

To find the Hall resistivity \( \rho_{yx} \) it is convenient to apply the electric field in such a way that the current will flow exactly along the chains, then \( j_y = 0 \) and (3) gives
\[ \begin{align*}
E_y &= \rho_{yx} j_x, \\
E_x &= \rho_{xx} j_x.
\end{align*} \] (8)

Let us furthermore impose that \( E_x = 0 \) (e.g. by imposing periodic conditions along \( x \)).

We now envision a different setup in which no external electric field and no current are applied, and consider a reference frame that moves with respect to this system at a velocity \( v \) along the \( x \) axis. In this moving frame, \( j_y \) is still vanishing because the transverse current is not affected by the Galilean transformation. Due to such a Galilean transformation \( E_x \) is still zero but an electric field is induced along the \( y \) axis, given by:
\[ E_y = -\frac{v}{c} \] (10)

Finally, in the moving frame, there is a current along the chains, which can be evaluated as follows. The velocity of right (left) moving modes close to the Fermi surface is given by: \( v_k = \pm v_F + 2\alpha (k \mp k_F) \), so that a constant velocity \( v \) corresponds to a momentum shift \( \Delta k = \frac{v_0}{2\alpha} \).

This in turn corresponds to a (one-dimensional) current density in a single chain given by: \( j_1D = -ve_F \Delta k \). Finally, we obtain the induced current density along \( x \) as:
\[ j_x = \frac{j_{1D}}{a_y} = -\frac{e_F}{2\pi \alpha a_y} v \] (11)

Combining the above equations for \( E_y \) and \( j_x \), we finally obtain:
\[ \begin{align*}
\rho_{xx}(\omega = 0) &= 0, \\
\rho_{yx}(\omega = 0) &= \frac{E_y}{j_x} = 2\pi \alpha \frac{H a_y}{v_F E_{ec}}.
\end{align*} \] (12)

This expression for the Hall resistivity can also be written in terms of the electron density \( n \), which is related to the in-chain Fermi momentum through the relation: \( n a_y = k_F / \pi \) (since \( na_y \) is the density per one chain). Therefore:
\[ \rho_{yx} = \frac{H}{E_{ec}} \frac{2\pi \alpha k_F}{v_F} \] (14)

In the case of a purely parabolic band \( \epsilon_k = \frac{k^2}{2m} \), one has \( 2\alpha = 1/m \) and \( v_F = k_F/m \), so that one recovers the familiar expression:
\[ \rho_{yx} = \frac{H}{E_{ec}} \] (15)

For a tight-binding dispersion along the chains \( \epsilon_k = -2t \cos k \), one has \( v_F = 2t \sin k_F \) and \( \alpha = t \cos k_F \), so that:
\[ \rho_{yx} = \frac{H}{E_{ec}} \frac{k_F}{\tan k_F} \] (16)

Hence, we have seen that Galilean invariance implies that the dc- Hall resistivity is T-independent and unchanged by interactions.

In a similar way one can use the conductivity tensor
\[ \begin{bmatrix} j_x \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} 0 \\ E_y \end{bmatrix}, \] (17)

where \( j_x \) and \( E_y \) are again given by (11) and (10). This leads to
\[ \begin{align*}
\sigma_{xy}(\omega = 0) &= \rho_{yx}(\omega = 0)^{-1}, \\
\sigma_{yy}(\omega = 0) &= 0.
\end{align*} \] (18)
Thus the Galilean invariance implies, at zero frequency
\[ \rho_{xx}(\omega = 0) = 0, \quad \sigma_{yy}(\omega = 0) = 0 \]  
\[ \sigma_{xy}(\omega = 0)^{-1} = -\rho_{xy}(\omega = 0) = 2\pi\alpha \frac{Ha_y}{v_F e c}. \]  
(21)

Note that \( \rho_{xx} = 0 \) is well in agreement with our hypothesis of a perfect conductor in the chain direction. \([21]\) also shows that at low frequency, the Hall resistivity is perturbative in the band curvature, while \( \sigma_{xy} \) is not. Finally, we emphasize that the fact that \( \sigma_{yy}(\omega = 0) = 0 \) does not provide any information on the interchain resistivity \( \rho_{yy}(\omega = 0) \) which is a finite quantity (at finite temperature) that must be calculated independently (Sec. [IV]).

\section*{B Commutation relations and general expressions for the resistivity tensor}

In order to go beyond the zero frequency limit let us compute the commutator of the current along the chains
\[ \hat{J}_x = \int dx \sum_i \hat{j}_x(x, i) \]  
\[ \hat{j}_x(x, i) = \frac{1}{a_y} \left[ ev_F \psi_i^\dagger \tau_3 \psi_i + 2\alpha \psi_i^\dagger (-i\partial_x) \psi_i \right], \]  
(23)
with the Hamiltonian \([4]\). This gives the remarkable result
\[ \dot{\hat{J}}_x = i[\hat{H}, \hat{J}_x] = \gamma \hat{J}_y, \]  
(24)
where
\[ \dot{\hat{J}}_y = \int dx \sum_i \hat{j}_y(x, i), \]  
(25)
\[ \hat{j}_y(x, i) = t_- e^{-i} \left( \psi_i^\dagger(x) \psi_{i+1}(x) e^{i\frac{\pi}{2} A_{i,i+1}} \right) - h.c., \]  
(26)
is the current perpendicular to the chains and
\[ \gamma = \frac{2\alpha H}{c}. \]  
(27)

The commutation relation \([24]\) allows to find general relations between the components of the conductivity tensor: Suppose that a short pulse of the electric field \( E_y = E_0\delta(t) \) is applied along the y-axis, then
\[ \hat{j}_x(t) = E_0 \sigma_{yy}(t) \]  
(28)
\[ \hat{j}_y(t) = E_0 \sigma_{yy}(t) \]  
(29)
and using \([24]\) we get
\[ \sigma_{xy}(t) = \gamma \sigma_{yy}(t). \]  
(30)
Using this equation we obtain the relation between \( \sigma_{yy}(\omega) \) and \( \sigma_{xy}(\omega) \):
\[ \gamma \sigma_{yy}(\omega) = \gamma \int_0^\infty \sigma_{yy}(t)e^{i\omega t} dt = \]  
\[ \int_0^\infty e^{i\omega t} \sigma_{xy}(t) = -\sigma_{xy}(t = 0^+) - i\omega \sigma_{xy}(\omega). \]  
(31)

From the Galilean invariance principle we know that \( \sigma_{yy}(\omega = 0) = 0 \) and \( \sigma_{xy}(\omega = 0) \) is finite, therefore from Eq. \([31]\) it follows that \( \sigma_{xy}(t = 0^+) = 0 \) and finally we obtain
\[ \sigma_{xy}(\omega) = \frac{\gamma}{-i\omega} \sigma_{yy}(\omega). \]  
(32)
Moreover from the Galilean invariance we have that \( \sigma_{xx}(0) = v_F ec/2\pi\alpha Ha_y \), therefore
\[ \sigma_{yy}(\omega) = \frac{v_F e^2}{\pi a_y} \frac{i\omega}{\gamma^2} \omega \rightarrow 0. \]  
(33)

A similar consideration using the commutation relation \([24]\) when a short electric pulse is applied along the x-axis leads to
\[ \sigma_{xx}(\omega) = \frac{\gamma}{-i\omega} \sigma_{xy}(\omega) - \frac{\sigma_{xx}^D}{i\omega} \]  
(34)
where
\[ \sigma_{xx}^D = \frac{e^2 v_F}{\pi a_y} \]  
(35)

which is consistent with the expected large frequency behavior of \( \sigma_{xx} \) (perfect conductor). Now, using \([22]\) \( \sigma_{xx} \) may be related with \( \sigma_{yy} \) as
\[ \sigma_{xx}(\omega) = -\frac{e^2 v_F}{\pi a_y} \frac{1}{i\omega} + \frac{\gamma^2}{\omega^2} \sigma_{yy}(\omega). \]  
(36)

According to \([33]\) we see that the singularities \( 1/\omega \) in \([33]\) are canceled out and that, in contrast to naive intuition, the conductivity \( \sigma_{xx} \) is finite at zero frequency! This is of course due to the presence of the magnetic field, and occur for frequencies smaller than the “cyclotron frequency” of the anisotropic system. The above formulas show that at low frequency the conductivity tensor is drastically affected by the presence of the magnetic field an thus does not allow for perturbative calculations. The resistivity tensor, in contrast, has a well-behaved expansion in the band curvature.

In order to derive expressions for this tensor, we use \([23]\) to obtain:
\[ \sigma_{xx}(\omega)\sigma_{yy}(\omega) + \sigma_{xy}^2(\omega) = \frac{1}{\gamma} \sigma_{xx}^D \sigma_{xy}(\omega) \]  
(37)
Therefore \( \rho_{xx} \) is given by
\[ \rho_{xx}(\omega) = \frac{\sigma_{yy}(\omega)}{\sigma_{xx}(\omega)\sigma_{yy}(\omega) + \sigma_{xy}^2(\omega)} = -\frac{i\omega}{\sigma_{xx}^D} = -\frac{\pi a_y}{e v_F c^2} i\omega. \]  
(38)

We see that \( \rho_{xx} \) is independent of magnetic field, interchain hopping and band curvature, and that \( \rho_{xx}(0) = 0 \) in agreement with the result following from Galilean invariance. We obtain \( \rho_{xy}(\omega) \) in a similar manner:
\[ \rho_{xy}(\omega) = \frac{-\sigma_{xy}(\omega)}{\sigma_{xx}(\omega)\sigma_{yy}(\omega) + \sigma_{xy}^2(\omega)} \]  
\[ = -\frac{2\pi\alpha}{v_F e c} \equiv -\frac{a_y \pi}{v_F e c^2}. \]  
(39)
This generalizes to finite frequency the result previously obtained using Galilean invariance. $\rho_{xy}(\omega)$ is seen to be independent of frequency and temperature. This result established here from general principles can also be recovered from an explicit calculation valid at high frequency, presented in Appendix A.

We note that these general considerations do not allow for the determination of the component $\rho_{yy}(\omega, T; H)$ of the resistivity tensor. This is the only non-trivial quantity that must be calculated explicitly in this dissipationless case. It is proven in Appendix [3] that this quantity has a well-behaved expansion in powers of the band curvature parameter $\alpha$, so that to lowest order one can use the value of $\rho_{yy}$ for $\alpha = 0$. To this zeroth order, the inter-chain conductivity and resistivity are simply related by:

$$\rho_{yy}(\omega, T; H)(0) = 1/\sigma_{yy}(\omega, T; H)(0)$$  (40)

In Sec. IV, an expansion of $\sigma_{yy}(\omega, T; H)(0)$ to lowest order in the inter-chain hopping is performed (i.e. to order $t^2$), which thus fully determines $\rho_{yy}(\omega, T; H)(0)$ to lowest order.

Finally, for the sake of completeness, we give the general expression of the conductivity tensor in terms of $\rho_{yy}(\omega)$:

$$\sigma_{xx}(\omega) = \frac{e^2 v_F}{\pi a_y} - \frac{1}{\omega} \frac{i \omega + \gamma^2 \omega_T^2 \rho_{yy}(\omega)}{\gamma}$$

$$\sigma_{xy}(\omega) = \frac{1}{\omega} \frac{\omega \rho_{yy}(\omega) + \gamma \omega_T^2 \gamma^2}{\gamma}$$

$$\sigma_{yy}(\omega) = \frac{1}{\rho_{yy}(\omega) - \frac{\gamma^2 \omega_T^2 \rho_{yy}(\omega)}{\gamma^2}}$$  (41)

From the equation relating $\sigma_{yy}(\omega)$ and $\rho_{yy}(\omega)$ we see that at high enough frequency the effect of curvature becomes not important and $\rho_{yy}^{-1}(\omega) = \sigma_{yy}(\omega) = \sigma_{yy}^{(0)}(\omega)$. Assuming that $\sigma_{yy}^{(0)}$ is finite at zero frequency (this is the case for any finite temperature) we estimate the crossover frequency below which the curvature effects become important

$$\omega_0 = \frac{\gamma^2 \pi a_y \sigma_{yy}^{(0)}(\omega = 0)}{v_F e^2}. \quad (42)$$

Thus for frequencies higher than $\omega_0$ one can neglect the curvature effects and use the zero curvature result for transverse conductivity calculated in Sec. IV directly. At frequencies lower than $\omega_0$ the zero curvature result may be used only for resistivity $\rho_{yy}$, since this quantity depends regularly on $\alpha$ even at low frequencies (see Appendix [3]). Thus, at frequencies lower than $\omega_0$ the conductivity tensor should be obtained by inserting in the expressions (11) the result for $\rho_{yy}(\omega, T; H)^{(0)} = 1/\sigma_{yy}(\omega, T; H)^{(0)}$ calculated in Sec. IV.

C  Physical arguments: response to a current pulse and Hall angles

Here, we rederive some of the previous results using physical arguments, and give expressions for the Hall angles of this anisotropic system. Following Ref. [13] (see also Ref. [14]), we consider the following thought experiment. A current pulse $j_y(t) = j_y^0 \delta(t)$ is sent into the system along the y-direction (i.e. perpendicular to the chains and to the magnetic field). A transient current $j_x(t)$ is thus induced along the chains, and we imagine that the electric field along the chain is maintained to $E_x = 0$. Using the definition of the conductivity tensor, the transient current $j_x(t)$ is found to be (for $t > 0$):

$$j_x(t) = j_y^0 \tan \theta_H^x(t)$$  (43)

in which $\theta_H^x(t)$ is the Fourier transform

$$\tan \theta_H^x(t) = \int \frac{d \omega}{2\pi} e^{-i\omega t} \tan \theta_H^x(\omega + i0^+)$$  (44)

of the (retarded) frequency-dependent Hall angle:

$$\tan \theta_H^x(\omega) = \frac{\sigma_{xy}(\omega)}{\sigma_{yy}(\omega)}.$$  (45)

In the absence of dissipation along the chains, we expect no time decay of the induced current $j_x(t)$, i.e $j_x(t) = j_x(t = 0) = C_x j_y^0 \theta_H^x(t)$ with $C_x$ a constant. ($C_x$ could a priori be temperature dependent, but the above reasoning using Galilean invariance shows that it is not). This implies that:

$$\tan \theta_H^x(\omega) = C_x \frac{i\omega}{\omega + i0^+}$$  (46)

and hence that $\sigma_{xy}$ and $\sigma_{yy}$ are simply proportional at all frequencies and temperatures:

$$\sigma_{xy}(\omega, T) = C_x \frac{i\omega}{\omega + i0^+} \sigma_{yy}(\omega, T)$$  (47)

one thus recovers the relation [24], which also determine the constant $C_x$ to be $C_x = 2\pi eH/c$.

One can also consider the complementary thought experiment in which a current pulse $j_x(t) = j_x^0 \delta(t)$ is generated along the chains. The relaxation of the induced Hall current perpendicular to the chains (with the constraint $E_y = 0$) is then given by:

$$j_y(t) = j_y^0 \tan \theta_H^y(t)$$  (48)

with similar notations as above, and the optical Hall angle $\theta_H^y$ given by:

$$\tan \theta_H^y(\omega) = \frac{\sigma_{yx}(\omega)}{\sigma_{xx}(\omega)} = \frac{\rho_{xy}}{\rho_{yy}(\omega)}.$$  (49)

To lowest order (in the band curvature and inter-chain hopping), we can thus relate this Hall angle to the inter-chain conductivity as follows:

$$\tan \theta_H^y(\omega) = \rho_{xy}\sigma_{yy}^{(0)}(\omega) + \cdots$$  (50)
Using the spectral representation \( \sigma(z) = \frac{i}{\pi} \int dx \frac{\text{Re}(z)}{z - \sigma(x)} \), one obtains the decay of the current pulse along \( y \) as:

\[
j_y(t) = \rho_x j_0^2 \theta(t) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \text{Re} \sigma'^{(0)}_{yy}(\omega) \tag{51}
\]

In the next section, an explicit expression will be obtained for the frequency, temperature, and magnetic field dependence of \( \sigma'^{(0)}_{yy} \), hence allowing the determination of the time decay of the current \( j_y(t) \).

IV. TRANSVERSE CONDUCTIVITY

In this section, we calculate the transverse conductivity \( \sigma'^{(0)}_{yy} \) in the absence of band curvature, to lowest order in the inter-chain hopping, as a function of frequency, temperature and magnetic field. As discussed above, this then completely determines the conductivity tensor (and Hall angle) by inserting \( \rho'^{(0)}_{yy} = 1/\sigma'^{(0)}_{yy} \) in the expressions \([1]\).

According to the Kubo formula, the conductivity is given by

\[
\sigma_{yy}(\omega) = \sigma'^{P}_{yy}(\omega) + \sigma'^{D}_{yy}(\omega), \tag{52}
\]

where \( \sigma'^{P}(\omega) \) and \( \sigma'^{D}(\omega) \) are the paramagnetic and the diamagnetic contributions respectively. The paramagnetic term is

\[
\sigma'^{P}_{yy}(\omega) = \frac{a_y}{\omega} \sum_i \int dx P_R(x, i, \omega), \tag{53}
\]

where \( P_R \) is the retarded current-current correlator

\[
P_R(x, i, \omega) = \int_0^{\infty} dt \langle e^{i\omega t} [\hat{j}_y(x, i, t), \hat{j}_y(0, 0, 0)] \rangle, \tag{54}
\]

and \( \hat{j}_y \) is the operator of the inter-chain current \([8]\).

The diamagnetic term is given by

\[
\sigma'^{D}_{yy} = \frac{-e^2 t_{\perp}^2}{i\omega} \left\langle \psi^+_0(0) \psi_1(x) e^{-ieA_n i/c} + \text{c.c.} \right\rangle. \tag{55}
\]

The retarded current-current correlator \( P_R(\omega) \) can be obtained from the Matsubara correlator

\[
P_M(x, i, \Omega) = \int_0^{\beta} d\tau e^{i\Omega\tau} \langle T_{\tau} \hat{j}_y(x, i, \tau) \hat{j}_y(0, 0, 0) \rangle \tag{56}
\]

by the analytical continuation of the Matsubara frequency \( \Omega \) to the real frequency \( \omega \)

\[
P_R(\omega) = -iP_M(\Omega)\delta(\omega+\Omega). \tag{57}
\]

To the lowest order in \( t_{\perp} \) the paramagnetic term is simply given by the expectation value of \([4]\) with respect to the single chain Hamiltonian. For the diamagnetic term one should consider the hopping term in the Hamiltonian \([4]\) to the first order. Combining the lowest order expressions for paramagnetic and diamagnetic terms we get

\[
\sigma'^{(0)}_{yy}(\omega) = \frac{2e^2 t_{\perp}^2}{i\omega} \sum_{s=\pm} \left[ \int dx \int_0^{\beta} d\tau e^{i\Omega\tau} G_s^2(x, \tau) \times \cos(\hbar x) - (\Omega = 0) \right]_{d\Omega \to \omega+i0^+} \tag{58}
\]

In this expression

\[
h = e\hbar a_y/c \tag{59}
\]

is a characteristic energy scale associated with the magnetic field, which will play an important role in the following. \( G_\pm \) are the single chain Green’s function for each chiral mode:

\[
G_\pm(x, \tau) = \frac{i}{2} \frac{T}{\sinh(\pi T(\pm x + i\tau))} \left( \frac{\pi T a_y}{\sinh(\pi T(x + i\tau))} \right)^{\eta/2} \tag{60}
\]

The integral over \( \tau \) in \([58]\) must be calculated for the Matsubara frequency \( \Omega \), and then the analytical continuation should be taken. This is performed explicitly in Appendix \([6]\) in which the following general expression is derived:

\[
\sigma'^{(0)}_{yy}(\omega') = \frac{a_y e^2 t_{\perp}^2 (2\pi T a_y)^{2\eta} \sinh(\omega/2T)}{2\pi^2} \left( \frac{2}{\omega} \right)^2 \left( \frac{\omega^2 + \hbar^2}{(\pi T)^2} \right)^{\eta/2} \left( \frac{\pi T a_y}{\sinh(\pi T(x + i\tau))} \right)^{\eta/2}. \tag{61}
\]

In the following, we examine various limiting cases of physical interest of this expression.

A. Zero temperature

At zero temperature the integrals in \([58]\) can be taken analytically giving

\[
\sigma'^{(0)}_{yy}(\omega) = A(\eta) \frac{1}{\omega} \left[ (h^2 - \omega^2)^{\eta} \frac{\hbar^2 + \omega^2}{h^2 - \omega^2} - h^{2\eta} \right], \tag{62}
\]

where the coefficient \( A(\eta) \) is

\[
A(\eta) = \frac{t_{\perp}^2 a_y e^2}{\pi} \left( \frac{a_x}{2} \right)^{2\eta} \frac{\Gamma(1 - \eta)}{\Gamma(2 + \eta)}. \tag{63}
\]

In the limit of low frequencies \( \omega \ll h \) we get

\[
\sigma'^{(0)}_{yy}(\omega) = -A(\eta)(2 - \eta)i\omega \left( \frac{e Ha_y}{c} \right)^{2\eta - 2} \tag{64}
\]

and thus a purely reactive response. For zero magnetic field \( (h = 0) \) we have

\[
\sigma'^{(0)}_{yy}(\omega)' = A(\eta) \sin(\pi\eta) |\omega|^{2\eta - 1}, \tag{65}
\]

\[
\sigma'^{(0)}_{yy}(\omega)'' = A(\eta) \cos(\pi\eta) |\omega|^{2\eta}/\omega, \tag{66}
\]
where $\sigma_{yy}^{(0)}(\omega)'$ and $\sigma_{yy}^{(0)}(\omega)''$ are the real and imaginary parts of the conductivity respectively. We see that the real and imaginary parts of conductivity have the same dependence on $\omega$, so the ratio $\sigma_{yy}^{(0)}(\omega)'/\sigma_{yy}^{(0)}(\omega)''$ is $\omega$-independent:

$$\text{sign}(\omega) \frac{\sigma_{yy}^{(0)}(\omega)'}{\sigma_{yy}^{(0)}(\omega)''} = \tan(\pi \eta) \tag{67}$$

This equation suggests an independent way to measure the exponent $\eta$ experimentally. Note that in case of Fermi liquid the limit of the above ratio at $\omega \to 0$ should be zero. Thus the conductivity dependence on $\omega$ reflects the non-Fermi liquid properties of the in-chain Luttinger liquid.

When the magnetic field is applied, according to (62), at frequencies $|\omega| < h$ the real part of the conductivity is zero and dissipation is absent. But when $|\omega| > h$ the real part is not zero and it is given by

$$\sigma_{yy}^{(0)}(\omega)' = \mathcal{A}(\eta) \frac{\sin(\pi \eta)}{|\omega|} \frac{\omega^2 + h^2}{(\omega^2 - h^2)^{1-\eta}}. \tag{68}$$

Thus the magnetic field turns off dissipation for frequencies in the region $|\omega| < h$.

### B  DC transverse conductivity

As shown in Appendix 3, the d.c. transverse conductivity can also be computed analytically. The imaginary part is zero, and the real part is given by

$$\sigma_{yy}^{(0)}(T; H) = \frac{a_y e^2 t_1^2 (2\pi T a_x)^{2n} (\frac{\pi}{2})^2 + h^2/(4\pi T)^2}{4\pi^2 T} \frac{e^{-\eta}}{\Gamma(\eta) \Gamma(\eta + 2)} |\eta/2 + \frac{h}{4\pi T}|^4. \tag{69}$$

In the absence of the magnetic field we have

$$\sigma_{yy}^{(0)}(T; H = 0) = \frac{a_y e^2 t_1^2 (2\pi a_x)^{2n} (\frac{\pi}{2})^2 \Gamma(\frac{\pi}{4})}{4\pi^2} \frac{e^{-\eta}}{\Gamma(\eta) \Gamma(\eta + 2)} T^{-1+2\eta}, \tag{70}$$

so the conductivity depends on the temperature as $T^{-1+2\eta}$, in agreement with the previous results of two of the authors in Ref. 3. If $\eta \ll 1$ one can simplify the above expression

$$\sigma_{yy}^{(0)} = \frac{a_y e^2 t_1^2}{\pi^2} \frac{1}{T \eta}. \tag{71}$$

Thus the system becomes a perfect conductor when $\eta \to 0$ as it should since the current commutes with the Hamiltonian. To plot the dependence of the DC conductivity (69) on the magnetic field it is convenient to rewrite (69) in the form

$$\sigma_{yy}^{(0)} = \frac{a_y e^2 t_1^2 (2\pi a_x)^{2n}}{16\pi^2} \frac{\eta^2 \Gamma(\frac{\pi}{2})^4}{\Gamma(\eta) \Gamma(\eta + 2)} \frac{1}{T \eta} \times T^{-1+2\eta} F(\eta, h/4\pi T), \tag{72}$$

where the function $F$ is

$$F(\eta, x) = \frac{\Gamma(\eta/2 + ix)^4}{\Gamma(\frac{\pi}{2})^4} \frac{\eta^2 + 4x^2}{\eta^2}. \tag{73}$$

The function $F(\eta, x)$ is plotted on Figure 2 for different values of $\eta$.

In case of large magnetic fields $h \gg T$ one can simplify (73) obtaining for conductivity the following expression

$$\sigma_{yy}^{(0)} = \frac{a_y e^2 t_1^2 (a_x/2)^{2n}}{16\pi^2} \frac{h^2}{T} e^{-\eta/2T}. \tag{74}$$

This is different from the result (64) obtained for the zero temperature case. Thus the answer for the conductivity depends on which limit is taken first: $\omega \to 0$ or $T \to 0$. The answer (74) corresponds to the limit $\omega \gg T, \omega, T \to 0$ and the answer (73) to $T \gg \omega, \omega, T \to 0$.

### C  Real part of the transverse AC conductivity

In the absence of magnetic field the real part of the conductivity can be written as (see Appendix 3)

$$\sigma_{yy}^{(0)}(\omega)' = \frac{a_y e^2 t_1^2 (2\pi a_x)^{2n}}{16\pi^2} \frac{\eta^2 \Gamma(\frac{\pi}{2})^4}{\Gamma(\eta) \Gamma(\eta + 2)} \times T^{-1+2\eta} \Theta(\eta, \omega/4\pi T), \tag{75}$$

where the function $\Theta$ is

$$\Theta(\eta, x) = \frac{\sinh(2\pi \eta \Theta(\frac{\pi}{2} + ix)^4}{2\pi x} \frac{\eta^2 + 4x^2}{\Gamma(\frac{\pi}{2})^4} \frac{\eta^2}{\eta^2}. \tag{76}$$

![FIG. 2: The function $F(\eta, x)$ that determines the dependence of the conductivity on the magnetic field ($x = h/4\pi T$) is plotted for different values of $\eta$: the solid line corresponds to $\eta = 0.8$, the dashed line to $\eta = 0.4$ and the dotted line to $\eta = 0.1$.](attachment:figure2.png)
This function is plotted on Figure 3 for different values of $\eta$.

In the presence of the magnetic field we shall plot the dependence of the real part of the conductivity on frequency only for a particular case $\eta = 0.5$ (The value of $\eta$ realized in quasi-one-dimensional organic conductors is believed to be close to 0.5.) Presenting the conductivity as

$$\sigma_{yy}(\omega) = \frac{a_x e^2}{2\pi a_x} \frac{(2\pi a_x)^2}{2\pi^2} S(\omega)$$

(77)

we plot the function $S$ on Figure 4 for different values of temperature. At low enough temperatures we see a definite resonance at $\omega = h$. Also at low temperatures we see the effect that was mentioned above (see end of Section IV A): At frequencies less than $h$ the dissipation is suppressed by the magnetic field. At higher temperatures the resonance becomes smeared and eventually disappears.

**V. SUMMARY AND CONCLUSION**

The main result of the present paper is that, in the absence of in-chain momentum relaxation, the Hall resistivity of weakly coupled Luttinger chains is not affected by the interactions. $\rho_{xy}$ is independent of frequency or temperature, and simply given by the free fermion expression \( \frac{a_x e^2}{2\pi a_x} \frac{(2\pi a_x)^2}{2\pi^2} S(\omega) \) (see also (16)). This result was established at zero-frequency using Galilean invariance in the chains, and generalized at finite frequency using exact relations based on current commutation relations. This method also allows us to fully determine the conductivity tensor in terms of a single quantity: the inter-chain frequency dependent resistivity Eqs. (55).

We have explicitly evaluated this quantity to lowest order in the interchain hopping, as a function of frequency, temperature and magnetic field. This expansion is valid in the Luttinger liquid regime, i.e at high enough temperature $T \gg \epsilon^*$ or frequencies $\omega \gg \epsilon^*$, or at large magnetic field. Various limiting forms of the general expression (61) have been investigated, which reflect the non-Fermi liquid aspects of the in-chain physics. Explicit expressions of the associated scaling functions of $\omega/T$ and $H/T$ have been given.

Because our calculation does not take into account the in-chain momentum relaxation processes, it is rather difficult to make definite statements concerning comparisons to the recent Hall measurements on the quasi one-dimensional organic conductor (TMTSF)$_2$PF$_6$.\(^{11,12}\) We note that both measurements yield values of $R_H$ close to the non-interacting band value at high enough temperature, in qualitative agreement with our result. As demonstrated in the present paper, this does not preclude strong electron-electron interactions to be present in the chains. Furthermore, it was shown in Ref. 3, using a generalized dynamical mean-field treatment, that the model of coupled Luttinger liquid chains considered here can lead, in the low temperature regime $T \ll \epsilon^*$ where Fermi liquid coherence has set in, to a very small Drude weight. Our results therefore show that there is no contradiction between this experimental observation and the fact that $R_H$ is close to the band value.

The two recent experimental studies of the Hall effect in (TMTSF)$_2$PF$_6$ differ by the temperature dependence observed at high temperature. In Ref. 12 with the mag-
netic field parallel to the chains, very little temperature
dependence was observed. In Ref. [11] with the field along
the least conducting axis, a significant temperature de-
pendence was measured, with \( R_H(T) \) increasing by al-
most a factor of two between 150\( K \) and 300\( K \) (where a
saturation is apparently reached). Obviously, the tem-
perature dependence of \( R_H \) can only be addressed the-
oretically once in-chain momentum relaxation pro-
cesses are included. We have not performed a detailed cal-
culation along these lines, but we conclude this paper
by making a few general remarks about what can be ex-
pected.

Momentum relaxation can be assigned to the in-
closure of an additional operator in the Hamiltonian of the
chains, in addition to the ordinary LL Hamiltonian. Let
us consider the regime in which this operator can be
treated in perturbation theory. Specifically, we have in
mind for example the case of a commensurate filling,
where an umklapp operator is generated. Perturbation
theory is valid either when the umklapp is irrelevant or,
if it is relevant, when temperature is high enough com-
pared to the energy scale \( \Delta_g \) associated with the pertur-
(bi.e. the induced Mott gap). For a model with spin
at commensurability 1/2\( n \) (\( n = 1 \) for half-filling, \( n = 2 \) for
quarter-filling), we have \( \Delta_g \sim g_3^2 \) with \( 2x = 1/(1-n^2K_p) \).
\( g_3 \) is the coupling constant and the umklapp is rele-
ant for \( x > 0 \). For \( g_3 = 0 \), each chain is a perfect conductor.
For a non-vanishing but arbitrary small \( g_3 \), the in-chain
conductivity (in zero field) obeys the scaling behavior:

\[
\sigma(\omega, T) = \frac{1}{\omega} \tilde{\sigma} \left( \frac{g_3}{\omega^{1/x}}, \frac{g_3}{T^{1/x}} \right) \tag{78}
\]

with \( \tilde{\sigma} \) a universal scaling function (taking in general
complex values). This expression can be justified from
very general scaling arguments, observing that the con-
ductivity is related to the current correlation function by
\( \sigma \sim (j_{ij})/\omega \) and has thus the dimensions of an inverse
energy (\( \nu_T = 1 \)), while \( \omega/\Delta_g \sim \omega/g_3^2 \) and
\( T/\Delta_g \sim T/g_3^2 \) are the dimensionless scaling variables. It has also been
established by a memory function calculation [12] which
allows an approximate determination of the scaling func-
tion \( \tilde{\sigma} \). Let us recall two important limiting behaviors
of this expression. At high-frequency and low tempera-
ture (\( \omega \gg \Delta_g \gg T \)), the real part \( \omega \sigma' \) becomes a scaling
function of \( g_3/\omega^{1/x} \) only, which vanishes for small argu-
ments (since at \( g_3 = 0 \) one has a perfect conductor and
hence \( \sigma \) is imaginary). This scaling function has a reg-
ular Taylor expansion in powers of \( g/\omega^{1/x} \) (which starts at
second order). Hence the above scaling expression allows
to simply predict the dominant high-frequency behavior
to be:

\[
\sigma'(\omega \gg \Delta_g) \sim \frac{1}{\omega} \left( \frac{g_3^2}{\omega^{2/x}} + \cdots \right) \tag{79}
\]

Noting that \( 1 + 2/x = 5 - 4n^2K_p \), this is the result de-

erived in Ref. [12]. In the opposite limit of low frequency,
\( T\sigma_{dc} \) becomes a scaling function of \( g/T^{1/x} \), which obvi-
ously must diverge at small \( g_3 \). Not surprisingly, it is the
scaling function associated with its inverse, the resistiv-
ity, which has a smooth Taylor expansion in this limit,
yielding:

\[
\rho_{dc} = T \tilde{\rho} \left( \frac{g_3}{T^{1/x}} \right) \sim T \left( \frac{g_3^2}{T^{2/x}} + \cdots \right) \tag{80}
\]

Again, with \( 1 - 2/x = 4n^2K_p - 3 \), this agrees with the
result of Ref. [13].

It is natural to attempt a generalization of these scaling
arguments to the Hall response. We focus here on
the low-field Hall number \( R_H \) obtained from the linear
term in the dc-Hall resistance \( \rho_{xy} = H R_H(T) + \cdots \). The
dimensional arguments lead to the following expression
for the Hall number

\[
R_H = R_H^0 \tilde{R} \left( \frac{g_3}{T^{1/x}} \right), \tag{81}
\]

where \( R_H^0 \) is a band value of the Hall resistivity and
\( \tilde{R} \) is a dimensionless scaling function. For small \( g_3 \) (or
large \( T \gg \Delta_g \sim g_3^2 \)), it is expected from the results of
the present paper that \( R_H \) tends to the band value \( R_H^0 \)
(Galilean invariance is restored in this limit). Since \( R_H \)
is obtained from a resistivity, it is natural to expect that
the scaling function \( \tilde{R} \) has a smooth Taylor expansion in
powers of \( g_3/T^{1/x} \), and therefore that the first corrections
describing the deviation from saturation as temperature
is lowered are given by:

\[
R_H(T) = R_H^0 \left( 1 + a_1 \frac{g_3}{T^{1/x}} + a_2 \frac{g_3^2}{T^{2/x}} + \cdots \right) \tag{82}
\]

It would be very interesting to confirm this expectation
from an explicit calculation and to determine the first
coefficients in this expansion. Note also that these con-
siderations suggest that a plot of \( R_H(T) \) versus \( \rho_{xy}/T \)
should define a universal scaling curve independent of
the compound or external parameters such as pressure
(as long as \( T \gg \Delta_g, \epsilon^* \)). We hope to address these is-

cues in a forthcoming work, together with a comparison of
these scaling ideas to experiments.

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APPENDIX A: EXPLICIT CALCULATION OF THE HIGH-FREQUENCY HALL CONDUCTIVITY

Considering the Hall conductivity we shall assume that the electric field is applied along the chains (x-axis) $E_x = E_0 e^{-i\omega t}$, thus the Hall conductivity $\sigma_{xy}$ relates the transverse current $j_y$ with electric field $E_x$

$$j_y(\omega) = \sigma_{yx}(\omega) E_x(\omega). \quad (A1)$$

Using Kubo formula and the perturbation theory in the hopping term $t_\perp$ for the Hall conductivity we get

$$\sigma_{yx}(\omega) = -\frac{2eit_\perp^2}{\omega} \left[ \int_0^\beta d\tau e^{i\Omega \tau} \Gamma(\tau) \right]_{\Omega = -i\omega}, \quad (A2)$$

where $\Gamma(\tau)$ is

$$\Gamma(\tau_1 - \tau_3) = \sum_{s=\pm} \int dx_3 dx_2 dt_2 \left[ \langle \psi_s(x_2, \tau_2) \psi_s^\dagger(x_1, \tau_1) \rangle \right.$$

$$\left. \langle j(x_3, \tau_3) \psi_s(x_1, \tau_1) \psi_s^\dagger(x_2, \tau_2) \rangle + (x_1, \tau_1 \leftrightarrow x_2, \tau_2) \right] \times \sin h(x_1 - x_2). \quad (A3)$$

The angular brackets in $[A3]$ represent averaging with respect to the single chain Lagrangian with density

$$\mathcal{L}^{(1)} = \psi^\dagger (-\partial_\tau + iv_F \tau_3 \partial_x + \alpha \partial_x^2) \psi$$

$$- \gamma \psi^\dagger \psi^\dagger \psi \psi - g \psi^\dagger \psi \psi \psi \psi, \quad (A4)$$

and the current $j$ is the single-chain current

$$j = ev_F \psi^\dagger \tau_3 \psi + 2e\alpha \psi^\dagger (-i\partial_x) \psi. \quad (A5)$$

Note that it contains an $\alpha-$contribution arising from the nonlinear correction to the spectrum.

The single-chain correlation functions in $[A3]$ can be calculated by the bosonization method. The functional technique allowing to find the necessary correlation functions at zero temperature was described in the previous work of one of authors. The generalization to finite temperatures is straightforward and in the following we will summaries only the results. The single-chain Lagrangian $[A4]$ in the bosonized form is

$$\mathcal{L}^{(1)} = \mathcal{L}^{(0)} + V(\Pi, \partial_x \Phi), \quad (A6)$$

where

$$\mathcal{L}^{(0)} = i \Pi \partial_\tau \Phi - \frac{1}{2} (\Pi^2 + (\partial_x \Phi)^2), \quad (A7)$$

$$V(\Pi, \partial_x \Phi) = -\frac{\alpha}{\beta^2} \tilde{\beta} \partial_x \Phi \left( \frac{\pi}{\beta^2} \Pi^2 + \frac{\tilde{\beta}^2}{\beta} (\partial_x \phi)^2 \right), \quad (A8)$$

where $\tilde{\beta}$ is related with $\eta$ by $2\eta = \tilde{\beta}^2 / \pi + \pi / \tilde{\beta}^2 - 2$.

The bosonic fields $\Phi, \Pi$ and original “fermionic” $\psi, \psi^*$ fields are related by

$$\psi_{\pm}(x, \tau) = \frac{1}{\sqrt{2\pi a_x}} e^{\mp i\Phi_{\pm}(x, \tau)}, \quad (A9)$$

$$\Phi_{\pm}(x, \tau) = \tilde{\beta} \Phi(x, \tau) \mp \frac{\pi}{\beta} \int_{-\infty}^x dx' \Pi(x', \tau). \quad (A10)$$

As was pointed out in Ref. 10, the Green function calculated by the functional method

$$G_f(x_1 - x_2, \tau_1 - \tau_2) = \langle \psi(x_1, \tau_1) \psi^* (x_2, \tau_2) \rangle \quad (A11)$$

must be multiplied by $\text{sgn}(\tau_1 - \tau_2)$ to restore the proper symmetry of the Green function arising from the fermionic anticommutation relations. The Green function corresponding to the Lagrangian $\mathcal{L}^{(0)}$ was already written above. Since the interaction $V$ will be treated as a perturbation it is very convenient to introduce the following generating functional

$$Z_{\pm}(f_0, f_1) = \langle \psi_\pm(\xi_1) \psi_\pm(\xi_2) e^{\int d^2 \xi (f_0(\xi) \Pi(\xi) + f_1(\xi) \partial_\tau \Phi(\xi))} \rangle, \quad (A12)$$

where $\xi = (x, \tau)$ and $d^2 \xi = dx d\tau$. The straightforward calculation gives

$$Z_{\pm}(f_0, f_1) = G_{\pm, f}^{(0)}(\xi_1, \xi_2)$$

$$\times e^{F_{\pm}(f_0, f_1) + \frac{i}{\pi} \int d^2 \xi_1 d^2 \xi_2 f^T(\xi_1) D(\xi_1, \xi_2) f(\xi_2)}, \quad (A13)$$

where $G_{\pm, f}^{(0)}$ is the Green function multiplied by $\text{sgn}(\tau_1 - \tau_2)$, $f = (f_0, f_1)$ is a two-vector constructed from $f_0, f_1$, and $F_{\pm}(f_0, f_1)$ is a linear functional of $f_0, f_1$

$$F_{\pm}(f_0, f_1) = \int d^2 \xi (f_0(\xi) J_0^{\pm}(\xi_1, \xi_2, \xi)$$

$$+ f_1(\xi) J_1^{\pm}(\xi_1, \xi_2, \xi)), \quad (A14)$$

with

$$J_0^{\pm}(\xi, \xi_2, \xi_3) = \frac{T}{4} \left[ \left( \mp \tilde{\beta} - \frac{\pi}{\beta} \right) \coth(z_1 - z_3) \pi T$$

$$+ \left( \pm \tilde{\beta} - \frac{\pi}{\beta} \right) \coth(z_1 - z_3^*) \pi T + (\xi_1 \leftrightarrow \xi_2) \right], \quad (A15)$$

$$J_1^{\pm}(\xi, \xi_2, \xi_3) = \frac{T}{4} \left[ \left( \pm \tilde{\beta} + \frac{\pi}{\beta} \right) \coth(z_1 - z_3) \pi T$$

$$+ \left( \pm \tilde{\beta} + \frac{\pi}{\beta} \right) \coth(z_1 - z_3^*) \pi T - (\xi_1 \leftrightarrow \xi_2) \right], \quad (A16)$$

where $z = x + i\tau, z^* = x - i\tau$. Finally, $D$ is a matrix Green function which in the momentum space is

$$D(\omega, p) = \frac{p^2}{p^2 + \omega^2} \left[ \frac{1}{2} i\frac{\omega}{p} \frac{1}{p} \right]. \quad (A17)$$
Up to the first order in the nonlinear correction $\alpha$, the expression for $\Gamma$ can be schematically presented as

$$
\langle j_0 \psi^\dagger \psi \rangle = \langle j_0 \psi^\dagger \psi \rangle + \langle j_1 \psi^\dagger \psi \rangle^{(0)} \langle \psi \psi \rangle^{(0)},
$$

where $j_0$ and $j_1$ correspond to the first and second terms in the equation for the current $\{A^2\}$ respectively. The first term in $\{A^2\}$ contains the current to the zeroth order in $\alpha$ so that the nonlinear corrections come from the Lagrangian. The second term contains no $\alpha$-corrections from the Lagrangian because $j_1$ is already proportional to $\alpha$. It was shown in Ref. [4] that the first term gives no contribution to the Hall conductivity. One can check that the same holds for finite temperatures. Thus we need to find only the contribution from the second term. The correlator

$$\int dx \langle j_1(x_3) \psi^\dagger(\xi_1) \psi(\xi_2) \rangle^{(0)},$$

where the current $j_1$ in the bosonized form is

$$j_1 = -2e\alpha \Pi \partial_x \Phi,$$

can be calculated with the help of the generating functional using Eqs. $\{A12,A13\}$

$$\int dx \langle j_1(x_3) \psi^\dagger(\xi_1) \psi(\xi_2) \rangle^{(0)}$$

$$= -2e\alpha \int dx_3 J_0^\perp(\xi_1, \xi_2, \xi_3) \psi^\dagger(\xi_1, \xi_2, \xi_3) G^{(0)}(\xi_1, \xi_2),$$

where we used that $\int dx D(x, \tau) = 0$. To obtain the Hall conductivity $\{A2\}$ we first need to take the integrals over $x_3$ and $\tau_3$ in

$$\int dx_3 d\tau_3 e^{i\Omega \tau_3} \langle j_1(x_3) \psi^\dagger(\xi_1) \psi(\xi_2) \rangle$$

$$= e\alpha T \frac{G(x, t_2)}{2\Omega} \left( \pm \beta + \frac{\pi}{\beta} \right)^2 \coth(z_3^2 \pi T)$$

$$+ \left( \pm \beta + \frac{\pi}{\beta} \right)^2 \coth(z_3 \pi T) \left( e^{-i\Omega \tau_3} - 1 \right).$$

Using Eq. $\{A13\}$ and the relation $2\eta = \beta^2 / \pi + \pi / \beta^2 - 2$ for the function $\Gamma$ we get

$$\Gamma(\Omega) = \frac{8\pi e\alpha T}{\Omega} \int_0^\beta d\tau dx G_{\perp}(x, \tau) \left[ \frac{\eta}{2} \coth[(x - i\tau)\pi T] \right.$$

$$\left. + \left( \frac{\eta}{2} + 1 \right) \coth[(x + i\tau)\pi T] \right] \sin(h x) \left. \left[ e^{i\Omega \tau} - 1 \right] \right. \left( A20 \right)$$

Using the explicit expression for the Green function $\{A14\}$ and integrating $\{A20\}$ over $x$ by parts we get

$$\sigma_{yx}(\omega) = \frac{8\pi e^2 h\alpha}{\omega^2} \int dx \int_0^\beta d\tau e^{i\Omega \tau} G_{\perp}(x, \tau)$$

$$\times \cos(h x) - (\Omega = 0) \right] \Omega \to -i\omega. \left( A21 \right)$$

To the lowest order in the curvature $\alpha$ and the interchain hopping $t_\perp$ the conductivity along the chains is given by

$$\sigma_{xx}^{(0)} = -\frac{v_F e^2}{\alpha y \pi \omega}. \left( A22 \right)$$

In addition, keeping only the leading order in the hopping $t_\perp$ we have

$$\rho_{xy} = -\frac{\sigma_{xy}}{\sigma_{xx}}. \left( A23 \right)$$

since $\sigma_{xy} \sim t_\perp^2$ at high enough frequency. Using $\{A21,A22,A24\}$ we calculate the Hall resistivity obtaining the simple result

$$\rho_{yx} = \frac{2\pi \alpha H a_y}{v_F c}. \left( A24 \right)$$

**APPENDIX B: EXPANSION OF THE INTER-CHAIN RESISTIVITY IN THE BAND CURVATURE**

In this appendix, we show that the inter-chain resistivity has a well-behaved expansion in powers of the band curvature parameter $\alpha$. We found that at low $\omega$ the conductivity $\sigma_{yy}$ behaves as

$$\sigma_{yy}(\omega) = -\frac{v_F e^2}{\pi a_y \gamma} \left[ 1 + \frac{i\Gamma}{\omega \gamma} \right], \quad \omega \to 0. \left( B1 \right)$$

This term becomes of the order of $\sigma_{xx}^{(0)}(\omega = 0)$ at the frequency of order of $\omega_0$ defined by $\{B3\}$. Therefore the next term in the expansion should have form

$$\sigma_{yy}(\omega) = -\frac{v_F e^2 i\omega}{\pi a_y \gamma^2} \left[ 1 + \frac{\omega_i}{\omega_0} \right], \quad \omega \to 0 \left( B2 \right)$$

where $\kappa$ is a coefficient of order 1. Now using the expansion $\{B3\}$ and formula $\{B1\}$ for the resistivity at $\omega \to 0$ we get

$$\rho_{yy}(\omega) = \sigma_{yy}^{-1}(\omega) + \frac{\gamma^2 a_y \pi}{i\omega v_F e^2} \kappa/\sigma_{yy}(\omega = 0). \left( B3 \right)$$

Thus the resistivity $\rho_{yy}(\omega)$ is regular in $\alpha$ even at low frequency. Also we expect that $\kappa = 1$ because the effect of curvature must be always small (as a ratio of the effective energy of the problem and Fermi energy) for quantities that have a regular dependence on $\alpha$, so the zeroth order (in $\alpha$) answer should give the correct result for the transverse resistivity $\rho_{yy}$.

**APPENDIX C: CALCULATION OF THE REAL PART OF THE TRANSVERSE CONDUCTIVITY**

The real part of the conductivity $\{B8\}$ can be calculated analytically. To do this, it is convenient to introduce a
The part of the contour integral 5 which is along the imaginary axis gives the imaginary part

$$\sigma_{yy}^{\eta}(\omega) = \frac{a_y e^{2y} T^2 \pi T a_x}{2} \int_{-\beta/2}^{\beta/2} dt \int dx e^{-i\omega t}$$

$$\left[ \sinh((x + it)\pi T) \sinh((x - it)\pi T) \right]^{-\eta} \sin^2 \left[ \pi T \right] (e^{i\omega t} - 1). \tag{C2}$$

The integral over t in Eq.(C1) can be extended to run from $-\infty$ to $\infty$ because the expression under the integral is even in t. After this one can take the integrals over x and t introducing the new variables $x - t$ and $x + t$ and get the answer

$$\sigma_{yy}^{\eta}(\omega) = a_y e^{2y} T^2 \pi T a_x \int_{-\beta/2}^{\beta/2} dt \int dx$$

$$\left[ \sinh((x + it)\pi T) \sinh((x - it)\pi T) \right]^{-\eta} \sin^2 \left[ \pi T \right] (e^{i\omega t} - 1). \tag{C3}$$

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