Fixed-Parameter Algorithms for DAG Partitioning

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Abstract

Finding the origin of short phrases propagating through the web has been formalized by Leskovec et al. [ACM SIGKDD 2009] as DAG PARTITIONING: given an arc-weighted directed acyclic graph on \( n \) vertices and \( m \) arcs, delete arcs with total weight at most \( k \) such that each resulting weakly-connected component contains exactly one sink—a vertex without outgoing arcs. DAG PARTITIONING is NP-hard.

We show an algorithm to solve DAG PARTITIONING in \( O(2^k \cdot (n + m)) \) time, that is, in linear time for fixed \( k \). We complement it with linear-time executable data reduction rules. Our experiments show that, in combination, they can optimally solve DAG PARTITIONING on simulated citation networks within five minutes for \( k \leq 190 \) and \( m \) being \( 10^7 \) and larger. We use our obtained optimal solutions to evaluate the solution quality of Leskovec et al.’s heuristic.

We show that Leskovec et al.’s heuristic works optimally on trees and generalize this result by showing that DAG PARTITIONING is solvable in \( 2^{O(t^2)} \cdot n \) time if a width-\( t \) tree decomposition of the input graph is given. Thus, we improve an algorithm and answer an open question of Alamdari and Mehrabian [WAW 2012].

We complement our algorithms by lower bounds on the running time of exact algorithms and on the effectivity of data reduction.

Keywords: NP-hard problem, graph algorithms, polynomial-time data reduction, multiway cut, linear-time algorithms, algorithm engineering, evaluating heuristics

1. Introduction

The DAG PARTITIONING problem was introduced by Leskovec et al. [26] in order to analyze how short, distinctive phrases (typically, parts or mutations of quotations, also called memes) spread to various news sites and blogs. To demonstrate their approach, Leskovec et al. [26] collected and analyzed phrases from 90 million articles from the time around the United States presidential elections in 2008; the results were featured in the New York Times [27]. Meanwhile, their approach has grown up into NIFTY, a system that allows near real-time observation of the rise and fall of trends, ideas, and topics in the internet [33].

A core component in the approach of Leskovec et al. [26] is a heuristic to solve the NP-hard DAG PARTITIONING problem. They use it to cluster short phrases, which may undergo modifications while propagating through the web, with respect to their origins. To this end, they create an arc-weighted directed acyclic graph with phrases as vertices and draw an arc from phrase \( p \) to phrase \( q \) if \( p \) presumably originates from \( q \), where the weight of an arc represents the...
support for this hypothesis: the weight assigned to an arc \((p, q)\) is chosen inversely proportional to the time difference between \(p\) and \(q\) and their Levenshtein distance when using words as tokens, whereas it is proportional to the total number of documents in the corpus that contain the phrase \(q\). A vertex without outgoing arcs is called a sink and can be interpreted as the origin of a phrase. If a phrase has directed paths to more than one sink, its ultimate origin is ambiguous and, in the model of Leskovec et al. [26], at least one of the ’’\(p\) originates from \(q\)’’ hypotheses is wrong. Leskovec et al. [26] introduced DAG PARTITIONING with the aim to resolve these inconsistencies by removing a set of arcs (hypotheses) with least support:

**DAG PARTITIONING**

*Input:* A directed acyclic graph \(G = (V,A)\) with positive integer arc weights \(\omega: A \to \mathbb{N}\) and a positive integer \(k \in \mathbb{N}\).

*Question:* Is there a set \(S \subseteq A\) with \(\sum_{a \in S} \omega(a) \leq k\) such that each weakly-connected component in \(G' = (V,A \setminus S)\) has exactly one sink?

Herein, the model of Leskovec et al. [26] exploits that a weakly-connected component of a directed acyclic graph contains exactly one sink if and only if all its vertices have directed paths to only one sink. We call a set \(S \subseteq A\) such that each weakly-connected component in \(G' = (V,A \setminus S)\) has exactly one sink a partitioning set.

Leskovec et al. [26] showed that DAG PARTITIONING is NP-hard and presented a heuristic to find partitioning sets of small weight. Alamdari and Mehrabian [1] showed that, for fixed \(\varepsilon > 0\), even approximating the minimum weight of a partitioning set within a factor of \(O(n^{1-\varepsilon})\) is NP-hard. In the absence of approximation algorithms, exact solutions to DAG PARTITIONING become interesting for the reason of evaluating the quality of known heuristics alone.

We aim for solving DAG PARTITIONING exactly using fixed-parameter algorithms—a framework to obtain algorithms to optimally solve NP-hard problems that run efficiently given that certain parameters of the input data are small [15, 28, 12]. A natural parameter to consider is the minimum weight \(k\) of the partitioning set sought, since one would expect that wrong hypotheses have little support.

**Known results.** To date, there are only few studies on DAG PARTITIONING. Leskovec et al. [26] showed that DAG PARTITIONING is NP-hard and present heuristics. Alamdari and Mehrabian [1] showed that, on \(n\)-vertex graphs, DAG PARTITIONING is hard to approximate in the sense that if \(P \neq NP\), then there is no polynomial-time factor-\(n^{1-\varepsilon}\) approximation algorithm for any fixed \(\varepsilon > 0\), even if the input graph has unit-weight arcs, maximum outdegree three, and only two sinks. Moreover, Alamdari and Mehrabian [1] showed that DAG PARTITIONING can be solved in \(2^{O(n^2)}\cdot n\) time if a width-\(t\) path decomposition of the input graph is given.

DAG PARTITIONING is very similar to the well-known NP-hard MULTIWAY CUT problem [11]: given an undirected edge-weighted graph and a subset of the vertices called terminals, delete edges of total weight at most \(k\) such that each terminal is separated from all others. DAG PARTITIONING can be considered as MULTIWAY CUT with the sinks being terminals and the additional constraint that not all arcs outgoing from a vertex may be deleted, since this would create a new sink. Xiao [34] gave an algorithm to solve MULTIWAY CUT in \(O(2^k \cdot \min(n^{2/3}, m^{1/2}) \cdot nm)\) time. Interestingly, in contrast to DAG PARTITIONING, MULTIWAY CUT is constant-factor approximable (see, e. g., Karger et al. [23]).

**Our results.** We provide algorithmic as well as intractability results. On the algorithmic side, we present an \(O(2^k \cdot (n + m))\) time algorithm for DAG PARTITIONING and complement it with linear-time executable data reduction rules. We experimentally evaluated both and, in combination, they solved instances with \(k \leq 190\) optimally within five minutes, the number of input arcs being \(10^7\) and larger. Moreover, we use the optimal solutions found by our algorithm to evaluate the quality of Leskovec et al. [26]’s heuristic and find that it finds optimal solutions for most instances that our algorithm solves quickly, but performs worse by a factor of more than two on other instances.

Also, we give an algorithm that solves DAG PARTITIONING in \(2^{O(n^2)}\cdot n\) time if a width-\(t\) tree decomposition of the input graph is given. We thus answer an open question by Alamdari and Mehrabian [1]. Since every width-\(t\) path decomposition is a width-\(t\) tree decomposition but not every graph allowing for a width-\(t\) tree decomposition allows for a width-\(t\) path decomposition, our algorithm is an improvement over the \(2^{O(t^2)}\cdot n\)-time algorithm of Alamdari and Mehrabian [1], which requires a path decomposition as input.

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1Unfortunately, Leskovec et al. [26] neither give a more precise description nor the range of their weights. All our hardness results even hold for unit weights, while all our algorithms work whenever the weight of each arc is at least one; however, we choose the weights to be positive integers to avoid representation issues.
On the side of intractability results, we strengthen the NP-hardness results of Leskovec et al. [26] and Alamdari and Mehrabian [1] to graphs of diameter two and maximum degree three and we show that our $O(2^k \cdot (n + m))$ time algorithm cannot be improved to $O(2^o(k) \cdot \text{poly}(n))$ time unless the Exponential Time Hypothesis fails. Moreover, we show that DAG PARTITIONING does not admit polynomial-size problem kernels with respect to $k$ unless NP $\subseteq$ coNP/poly.

Organization of this paper. In Section 2, we introduce necessary notation and two basic structural observations for DAG PARTITIONING that are important in our proofs.

In Section 3, we present our $O(2^k \cdot (n + m))$ time algorithm and its experimental evaluation. With the help of the optimal solutions computed by our algorithm we also evaluate the quality of a heuristic presented by Leskovec et al. [26]. Moreover, we discuss the limits of parameterized algorithms and problem kernelization for DAG PARTITIONING parameterized by $k$.

Section 4 presents our $2^{O(k^2)} \cdot n$ time algorithm. It follows that DAG PARTITIONING is linear-time solvable when at least one of the parameters $k$ or $t$ is fixed. We further show that the heuristic presented by Leskovec et al. [26] works optimally on trees.

Section 5 then shows that other parameters are not as helpful in solving DAG PARTITIONING: DAG PARTITIONING remains NP-hard even when graph parameters like the diameter or maximum degree are constants.

2. Preliminaries and basic observations

We consider finite simple directed graphs $G = (V,A)$ with vertex set $V(G) := V$ and arc set $A(G) := A \subseteq V \times V$, as well as finite simple undirected graphs $G = (V,E)$ with vertex set $V$ and edge set $E(G) := E \subseteq \{u,v\} | u,v \in V$. For a directed graph $G$, the underlying undirected graph $G'$ is the graph that has undirected edges in the places where $G$ has arcs, that is, $G' = (V(G), \{(v,w) : (v,w) \in A(G)\})$. We will use $n$ to denote the number of vertices and $m$ to denote the number of arcs or edges of a graph.

For a (directed or undirected) graph $G$, we denote by $G \setminus A'$ the subgraph obtained by removing from $G$ the arcs or edges in $A'$ and by $G - V'$ the subgraph obtained by removing from $G$ the vertices in $V'$. For $V' \subseteq V$, we denote by $G[V'] := G - (V \setminus V')$ the subgraph of $G$ induced by the vertex set $V'$.

The set of out-neighbors and in-neighbors of a vertex $v$ in a directed graph is $N^+(v) := \{u | (u,v) \in A\}$ and $N^-(v) := \{u | (v,u) \in A\}$, respectively. The outdegree, the indegree, and the degree of a vertex $v \in V$ are $d^+(v) := |N^+(v)|$, $d^-(v) := |N^-(v)|$, and $d(v) := d^+(v) + d^-(v)$, respectively. A vertex $v$ is a sink if $d^-(v) = 0$; it is isolated if $d(v) = 0$.

A path of length $\ell - 1$ from $v_1$ to $v_\ell$ in an undirected graph $G$ is a tuple $(v_1, \ldots, v_\ell) \in V^\ell$ such that $(v_{i},v_{i+1})$ is an edge in $G$ for $1 \leq i \leq \ell - 1$. An undirected path in a directed graph is a path in its underlying undirected graph. A directed path of length $\ell - 1$ from $v_1$ to $v_\ell$ in a directed graph $G$ is a tuple $(v_1, \ldots, v_\ell) \in V^\ell$ such that $(v_i,v_{i+1})$ is an arc in $G$ for $1 \leq i \leq \ell - 1$.

We say that $u \in V$ can reach $v \in V$ or that $v$ is reachable from $u$ in $G$ if there is a directed path from $u$ to $v$ in $G$. We say that $u$ and $v$ are connected if there is a (not necessarily directed) path from $u$ to $v$ in $G$. In particular, $u$ is reachable from $u$ and connected to $u$. We use connected component as an abbreviation for weakly connected component, that is, a maximal set of pairwise connected vertices. The diameter of $G$ is the maximum length of a shortest path between two different vertices in the underlying undirected graph of $G$.

Fixed-parameter algorithms. The main idea in fixed-parameter algorithms is to accept the super-polynomial running time, which is seemingly inevitable when optimally solving NP-hard problems, but to restrict it to one aspect of the problem, the parameter. More precisely, a problem $\Pi$ is fixed-parameter tractable (FPT) with respect to a parameter $k$ if there is an algorithm solving any instance of $\Pi$ with size $n$ in $f(k) \cdot \text{poly}(n)$ time for some computable function $f$ [15, 28, 12, 10]. Such an algorithm is called fixed-parameter algorithm. Since Suen et al. [33] point out that the input instances of DAG PARTITIONING can be so large that even running times quadratic in the input size are prohibitively large, we focus on finding algorithms that run in linear time if the parameter $k$ is a constant. An important ingredient of our algorithms is linear-time data reduction, which recently received increased interest since data reduction is potentially applied to large input data [32, 4, 18, 3, 22, 13].
Problem kernelization. One way of deriving fixed-parameter algorithms is (problem) kernelization [17, 7]. As a formal approach of describing efficient data reduction that preserves optimal solutions, problem kernelization is a powerful tool for attacking NP-hard problems. A kernelization algorithm consists of data reduction rules that, applied to any instance \(x\) with parameter \(k\), yield an instance \(x'\) with parameter \(k'\) in time polynomial in \(|x| + k\) such that \((x,k)\) is a yes-instance if and only if \((x',k')\) is a yes-instance, and if both \(|x'|\) and \(k'\) are bounded by some functions \(g\) and \(g'\) in \(k\), respectively. The function \(g\) is referred to as the size of the problem kernel \((x',k')\).

Note that it is the parameter that allows us to measure the effectiveness of polynomial-time executable data reduction, since a statement like “the data reduction shrinks the input by a factor \(\alpha\)” would imply that we can solve NP-hard problems in polynomial time.

From a practical point of view, problem kernelization is potentially applicable to speed up exact and heuristic algorithms to solve a problem. Since kernelization is applied to shrink potentially large input instances, recently the running time of kernelization has stepped into the focus and linear-time kernelization algorithms have been developed for various NP-hard problems [32, 4, 18, 3, 22, 13].

Two basic observations. The following easy to prove structural observations will be exploited in many proofs of our work. The first observation states that a minimal partitioning set does not introduce new sinks.

**Observation 1.** Let \(G\) be a directed acyclic graph and \(S\) be a minimal partitioning set for \(G\). Then, a vertex is a sink in \(G \setminus S\) if and only if it is a sink in \(G\).

**Proof.** Clearly, deleting arcs from a directed acyclic graph cannot turn a sink into a non-sink. Therefore, it remains to show that every sink in \(G \setminus S\) was a sink already in \(G\). Towards a contradiction, assume that there is a vertex \(s\) that is a sink in \(G \setminus S\) but not in \(G\). Then, there is an arc \((s,v)\) in \(G\) for some vertex \(v\) of \(G\). Let \(C_v\) and \(C_s\) be the connected components in \(G \setminus S\) containing \(v\) and \(s\), respectively, and let \(s_v\) be the sink in \(C_v\). Then, for \(S' := S \setminus \{(s,v)\}\), the connected component \(C_v \cup C_s\) in \(G \setminus S'\) has only one sink, namely \(s_v\). Thus, \(S'\) is also a partitioning set for \(G\), but \(S' \subseteq S\), a contradiction to \(S\) being minimal.

The second observation is that each vertex in a directed acyclic graph is connected to exactly one sink if and only if it can reach that sink.

**Observation 2.** Let \(G\) be a directed acyclic graph. An arc set \(S\) is a partitioning set for \(G\) if and only if each vertex in \(G\) can reach exactly one sink in \(G \setminus S\).

**Proof.** If \(S\) is a partitioning set for \(G\), then, by definition, each connected component of \(G \setminus S\) contains exactly one sink. Therefore, each vertex in \(G \setminus S\) can reach at most one sink. Moreover, since \(G \setminus S\) is a directed acyclic graph and each vertex in a directed acyclic graph can reach at least one sink, it follows that each vertex in \(G \setminus S\) can reach exactly one sink.

Now, assume that each vertex in \(G \setminus S\) can reach exactly one sink. We show that each connected component of \(G \setminus S\) contains exactly one sink. For the sake of contradiction, assume that a connected component \(C\) of \(G \setminus S\) contains multiple sinks \(s_1, \ldots, s_t\). For \(i \in [t]\), let \(A_i\) be the set of vertices that reach \(s_i\). These vertex sets are pairwise disjoint and, since every vertex in a directed acyclic graph reaches some sink, they partition the vertex set of \(C\).

Since \(C\) is a connected component, there are \(i, j \in [t]\) with \(i \neq j\) and some arc \((v,w)\) in \(G \setminus S\) from some vertex \(v \in A_i\) to some vertex \(w \in A_j\). This is a contradiction, since \(v\) can reach \(s_i\) as well as \(s_j\).

3. Parameter weight of the partitioning set sought

This section investigates the influence of the parameter “weight \(k\) of the partitioning set sought” on DAG PARTITIONING. First, in Section 3.1, we present an \(O(2^k \cdot (n+m))\)-time algorithm and design linear-time executable data reduction rules. In Section 3.2, we experimentally evaluate the algorithm and data reduction rules and—with the help of the optimal solutions computed by our algorithm—we also evaluate the quality of a heuristic presented by Leskovec et al. [26]. In Section 3.3, we investigate the question whether the provided data reduction rules might have a provable shrinking effect on the input instance in form of a polynomial-size problem kernel. We will see that, despite the fact that data reduction rules work very effectively in experiments, polynomial-size problem kernels for DAG PARTITIONING do not exist under reasonable complexity-theoretic assumptions. Moreover, Section 3.3 will also show that the algorithm presented in Section 3.1 is essentially optimal.
Algorithm 1: Compute a partitioning set of weight at most \( k \).

**Input:** A directed acyclic graph \( G = (V,A) \) with arc weights \( \omega \) and a positive integer \( k \).

**Output:** A partitioning set \( S \) of weight at most \( k \) if it exists; otherwise ‘no’.

1. \( (v_1, v_2, \ldots, v_n) \leftarrow \text{reverse topological order of the vertices of } G \)
2. \( L \leftarrow \text{array with } n \text{ entries} \)
3. \( \text{searchtree}(1, \emptyset) \quad \text{// start with vertex } v_1 \text{ and } S = \emptyset \)
4. \( \text{output } \text{‘no’} \quad \text{// there is no partitioning set of weight at most } k \)

**Procedure searchtree\( (i, S) \)**

// vertex counter \( i \); (partial) partitioning set \( S \)

5. \( \text{while } v_i \text{ is a sink } s \text{ or there is a sink } s \text{ such that } \forall w \in N^+(v_i) : L[w] = s \text{ do} \)
6. \( i \leftarrow i + 1 \quad \text{// continue with next vertex} \)
7. \( \text{if } i > n \text{ then} \quad \text{// all vertices have been handled, } S \text{ is a partitioning set} \)
8. \( \text{else} \quad \text{// a partitioning set of weight at most } k \text{ has been found} \)
9. \( \text{D} \leftarrow \{L[w] \mid w \in N^+(v_i)\} \quad \text{// the set of feasible sinks for } v_i \)
10. \( \text{if } |D| - 1 \leq k - \omega(S) \text{ then} \quad \text{// check whether we are allowed to delete } |D| - 1 \text{ arcs} \)
11. \( \text{foreach } s \in D \text{ do} \quad \text{// try to associate } v_i \text{ to each feasible sink } s \)
12. \( L[v_i] \leftarrow s \)
13. \( S' = S \cup \{(v_i, w) \mid w \in N^+(v_i) \text{ and } L[w] \neq s\} \)
14. \( \text{searchtree}(i + 1, S') \)

3.1. Constant-weight partitioning sets in linear time

We now present an algorithm to compute partitioning sets of weight \( k \) in \( O(2^k \cdot (n + m)) \) time. Interestingly, although both problems are NP-hard, it will turn out that DAG PARTITIONING is substantially easier to solve exactly than the closely related MULTIWAY CUT problem, for which a sophisticated algorithm running in \( O(2^k \min(n^{2/3}, m^{1/2})nm) \) time was given by Xiao [34]. This is in contrast to MULTIWAY CUT being constant-factor approximable [23], while DAG PARTITIONING is inapproximable unless \( P = NP \) [1].

The main structural advantage of DAG PARTITIONING over MULTIWAY CUT is the alternative characterization of partitioning sets given in Observation 2: we only have to decide which sink each vertex \( v \) will reach in the optimally partitioned graph. To this end, we first decide the question for all out-neighbors of \( v \). This natural processing order allows us to solve DAG PARTITIONING by the simple search tree algorithm shown in Algorithm 1, which we explain in the proof of the following theorem.

**Theorem 1.** Algorithm 1 solves DAG PARTITIONING in \( O(2^k \cdot (n + m)) \) time.

**Proof.** Algorithm 1 is based on recursive branching and computes a partitioning set \( S \) of weight at most \( k \) for a directed acyclic graph \( G = (V,A) \). It exploits the following structural properties of a minimal partitioning set \( S \): by Observation 2, a vertex \( v \) is connected to a sink \( s \) in \( G \setminus S \) if and only if it can reach that sink \( s \) in \( G \setminus S \). Thus, consider a vertex \( v \) of \( G \) and assume that we know, for each out-neighbor \( w \) of \( v \) in \( G \), the sink \( s \) that \( w \) can reach in \( G \setminus S \). We call a sink \( s \) of \( G \) feasible for \( v \) if an out-neighbor of \( v \) in \( G \) can reach \( s \) in \( G \setminus S \). Let \( D \) be the set of feasible sinks for \( v \). Since \( v \) may be connected to only one sink in \( G \setminus S \), at least \( |D| - 1 \) arcs outgoing from \( v \) are deleted by \( S \). However, \( S \) does not disconnect \( v \) from all sinks in \( D \), since then \( S \) would delete all arcs outgoing from \( v \), contradicting Observation 1. Hence, exactly one sink \( s \in D \) must be reachable by \( v \) in \( G \setminus S \). For each such sink \( s \in D \), the partitioning set \( S \) has to delete at least \( |D| - 1 \) arcs outgoing from \( v \). We simply try out all these possibilities, which gives rise to the following search tree algorithm.

Algorithm 1 starts with \( S = \emptyset \) and processes the vertices of \( G \) in reverse topological order, that is, each vertex is processed by procedure “searchtree” after its out-neighbors. The procedure exploits the invariant that, when processing
a vertex \( v \), each out-neighbor \( w \) of \( v \) has already been associated with the sink \( s \) that it reaches in \( G \setminus S \) (in terms of Algorithm 1, \( L[w] = s \)). It then tries all possibilities of associating \( v \) with a feasible sink and augmenting \( S \) accordingly, so that the invariant also holds for the vertex processed after \( v \).

Specifically, if \( v \) is a sink, then line 7 associates \( v \) with itself. If all out-neighbors of \( v \) are associated with the same sink \( s \), then line 7 associates \( v \) with \( s \). Otherwise, line 12 computes the set \( D \) of feasible sinks for \( v \). In lines 14–17, the algorithm branches into all possibilities of associating \( v \) with one of the \( |D| \) feasible sinks \( s \in D \) (by way of setting \( L[v] \leftarrow s \) in line 15) and augmenting \( S \) so that \( v \) only reaches \( s \) in \( G \setminus S \). That is, in each of the \( |D| \) branches, it adds to \( S \) the arcs outgoing from \( v \) to the out-neighbors of \( v \) that are associated with sinks different from \( s \) (the weight of \( S \) increases by at least \( |D| - 1 \)). Then, line 17 continues with the next vertex in the reverse topological order. After processing the last vertex, each vertex of \( G \setminus S \) can reach exactly one sink, that is, \( S \) is a partitioning set. If a branch finds a partitioning set with weight at most \( k \), line 10 outputs it.

We analyze the running time of this algorithm. To this end, we first bound the total number of times that procedure “searchtree” is called. To this end, we first analyze the number of terminal calls, that is, calls that do not recursively call the procedure. Let \( T(\alpha) \) denote the maximum possible number of terminal calls caused by the procedure “searchtree” when called with a set \( S \) satisfying \( \omega(S) \geq \alpha \), including itself if it is a terminal call. Note that procedure “searchtree” calls itself only in line 17, that is, for each sink \( s \) of some set \( D \) of feasible sinks with \( 1 \leq |D| - 1 \leq k - \alpha \), it calls itself with a set \( S' \) of weight at least \( \alpha + |D| - 1 \). Thus, we have

\[
T(\alpha) \leq |D| \cdot T(\alpha + |D| - 1).
\]

We now inductively show that \( T(\alpha) \leq 2^{k-\alpha} \) for \( 0 \leq \alpha \leq k \). Then, it follows that there is a total number of \( T(0) \leq 2^k \) terminal calls. For the induction base case, observe that \( T(k) = 1 \), since, if procedure “searchtree” is called with a set \( S \) of weight at least \( k \), then any recursive call is prevented by the check in line 13. Now, assume that \( T(\alpha') \leq 2^{k-\alpha} \) holds for all \( \alpha' \) with \( \alpha \leq \alpha' \leq k \). We show \( T(\alpha - 1) \leq 2^{k-(\alpha-1)} \) by exploiting \( 2 \leq |D| \leq 2|D|-1 \) as follows:

\[
T(\alpha - 1) \leq |D| \cdot T(\alpha - 1 + |D| - 1) \leq |D| \cdot 2^{k-(\alpha-1)-(|D|-1)} \leq |D| \cdot \frac{2^{k-(\alpha-1)}}{2^{|D|-1}} \leq 2^{k-(\alpha-1)}.
\]

It follows that there are at most \( T(0) = 2^k \) terminal calls to procedure “searchtree”.

In order to bound the total number of calls to procedure “searchtree”, observe the following: if each inner node of a tree has at least two children, then the number of inner nodes in a tree is at most its number of leaves. Now, since procedure “searchtree” calls itself only in line 17, that is, for each sink \( s \) of some set \( D \) of feasible sinks with \( 1 \leq |D| - 1 \leq k - \alpha \), each non-terminal call causes at least two new calls. Thus, since there are \( O(2^k) \) terminal calls, there are also \( O(2^k) \) non-terminal calls.

It follows that there are \( O(2^k) \) total calls of procedure “searchtree”. For each such call, we iterate, in the worst case, over all out-neighbors of all vertices in the graph in lines 6–12, which works in \( O(n + m) \) time. Moreover, for each call of procedure “searchtree”, we compute a set \( S' \) in line 16 in \( O(n + m) \) time. Hence, a total amount of \( O(2^k \cdot (n + m)) \) time is spent in procedure “searchtree”. Initially, Algorithm 1 uses \( O(n + m) \) time to compute a reverse topological ordering [9, Section 22.4].

The experimental results in Section 3.2 will show that Algorithm 1 alone cannot solve even moderately large instances. Therefore, we complement it by linear-time executable data reduction rules that will allow for a significant speedup. The following data reduction rule is illustrated in Figure 1.

**Reduction Rule 1.** If there is an arc \((v, w)\) such that \( w \) can reach exactly one sink \( s \neq w \) and \( v \) can reach multiple sinks, then

- if there is no arc \((v, s)\), then add it with weight \( \omega(v, w) \),
- otherwise, increase \( \omega(v, s) \) by \( \omega(v, w) \), and delete the arc \((v, w)\).

Note that, in the formulation of the data reduction rule, both \( v \) and \( w \) may be connected to an arbitrary number of sinks by an undirected path. However, we require that \( w \) can reach exactly one sink and that \( v \) can reach multiple sinks, that is, using a directed path.
**Lemma 1.** Let \((G, \omega, k)\) be a DAG PARTITIONING instance and consider the graph \(G'\) with weights \(\omega'\) output by Reduction Rule 1 applied to an arc \((v, w)\) of \(G\). Then \((G, \omega, k)\) is a yes-instance if and only if \((G', \omega', k)\) is a yes-instance.

**Proof.** First, assume that \((G, \omega, k)\) is a yes-instance and that \(S\) is a minimal partitioning set of weight at most \(k\) for \(G\). We show how to transform \(S\) into a partitioning set of equal weight for \(G'\). We distinguish two cases: either \(S\) disconnects \(v\) from \(s\) or not, where \(s\) is the only sink that \(w\) can reach.

Case 1) Assume that \(S\) disconnects \(v\) from \(s\). Note that every subgraph of a directed acyclic graph is again a directed acyclic graph and that every vertex in a directed acyclic graph is not only connected to, but also can reach some sink. Hence, by Observation 1, \(S\) cannot disconnect \(w\) from \(s\), since \(w\) can only reach \(s\) in \(G\) and would have to reach some other, that is, new sink in \(G \setminus S\). It follows that \(S\) contains the arc \((v, w)\). Now, however, \(S' := (S \setminus \{v, w\}) \cup \{(v, s)\}\) is a partitioning set for \(G'\), since \(G' \setminus S = G' \setminus S'\). Moreover, since \(\omega'(v, s) = \omega(v, s) + \omega(v, w)\), we have \(\omega'(S') = \omega(S)\), where we, for convenience, declare \(\omega(v, s) = 0\) if there is no arc \((v, s)\) in \(G\).

Case 2) Assume that \(S\) does not disconnect \(v\) from \(s\) and, for the sake of a contradiction, that \(S\) is not a partitioning set for \(G'\). Observe that \(S\) contains neither \((v, w)\) nor \((v, s)\), because it is a minimal partitioning set and does not disconnect \(v\) from \(s\). Therefore, \(G' \setminus S\) differs from \(G \setminus S\) only in the fact that \(G' \setminus S\) does not have the arc \((v, w)\) but an arc \((v, s)\) that was possibly not present in \(G \setminus S\). Hence, since \(S\) is a partitioning set for \(G\) but not for \(G'\), two sinks are connected to each other in \(G' \setminus S\) via an undirected path using the arc \((v, s)\). Thus, one of the two sinks is \(s\) and the undirected path consists of \((v, s)\) and a subpath \(p\) between \(v\) and some sink \(s'\). Then, however, \(s\) is connected to \(s'\) also in \(G \setminus S\) via an undirected path between \(s\) and \(w\) (\(S\) cannot disconnect \(s\) from \(w\) by Observation 1), the arc \((v, w)\) and the undirected path \(p\) from \(v\) to \(s'\). This contradicts \(S\) being a partitioning set for \(G\). We conclude that \(S\) is a partitioning set for \(G'\). Moreover, since \(S\) contains neither \((v, w)\) nor \((v, s)\), one has \(\omega(S) = \omega'(S)\).

Now, assume that \((G', \omega', k)\) is a yes-instance and that \(S\) is a minimal partitioning set of weight at most \(k\) for \(G'\). We show how to transform \(S\) into a partitioning set of equal weight for \(G\). Again, we distinguish between two cases: either \(S\) disconnects \(v\) from \(s\) or not.

Case 1) Assume that \(S\) disconnects \(v\) from \(s\). Then, \((v, s) \in S\). Now, \(S' := S \cup \{(v, w)\}\) is a partitioning set for \(G\), since \(G' \setminus S' = G' \setminus S\). Moreover, since \(\omega'(v, s) = \omega(v, s) + \omega(v, w)\), we have \(\omega(S') = \omega'(S)\), where we assume that \(\omega(v, s) = 0\) if there is no arc \((v, s)\) in \(G\).

Case 2) Assume that \(S\) does not disconnect \(v\) from \(s\) and, for the sake of a contradiction, assume that \(S\) is not a partitioning set for \(G\). Then, since \(S\) is minimal, \(S\) does not contain \((v, s)\). Now, observe that \(G' \setminus S\) and \(G' \setminus S'\) differ only in the fact that \(G' \setminus S\) has an additional arc \((v, w)\) and that, possibly, \((v, s)\) is missing. Hence, since \(S\) is a partitioning set for \(G'\) but not for \(G\), there is an undirected path between two sinks in \(G' \setminus S\) through \((v, w)\). Because \(S\) by Observation 1 cannot disconnect \(w\) from \(s\), one of these sinks is \(s\) and the undirected path consists of a subpath between \(s\) and \(w\), the arc \((v, w)\), and a subpath \(p\) between \(v\) and a sink \(s'\). Then, however, \(s\) and \(s'\) are also connected in \(G' \setminus S\) via the arc \((v, s)\) and the subpath \(p\) between \(v\) and \(s'\). This contradicts \(S\) being a partitioning set for \(G'\). Finally, since \(S\) does not contain \((v, s)\), one has \(\omega(S) = \omega'(S)\). \(\square\)
After applying Reduction Rule 1 exhaustively, that is, as often as it is applicable, we apply a second data reduction rule, which is illustrated in Figure 2.

**Reduction Rule 2.** If, for some sink $s$, the set $L$ of non-sink vertices that can reach only $s$ is nonempty, then delete all vertices in $L$.

**Lemma 2.** Let $G$ be a graph that is exhaustively reduced with respect to Reduction Rule 1 and $G' := G - L$ be the graph output by Reduction Rule 2 when applied to $G$ for some sink $s$. Then, any partitioning set for $G$ is a partitioning set of equal weight for $G'$ and vice versa.

**Proof.** In order to prove the lemma, we first make three structural observations about the set $L$.

i) There is no arc $(v, w)$ from a vertex $v \not\in L$ to a vertex $w \in L$ in $G$: for the sake of a contradiction, assume that such an arc exists. Then, since $v \not\in L$ and $v$ is obviously not a sink, $v$ can reach a sink $s' \neq s$. It follows that $v$ can reach two sinks: $s$ via $w$ and $s'$. This contradicts the assumption that Reduction Rule 1 is not applicable.

ii) There is no arc $(v, w)$ from a vertex $v \in L$ to a vertex $w \not\in L$ with $w \neq s$ in $G$: for the sake of a contradiction, assume that such an arc exists. Then, since $w \not\in L$ and $w \neq s$, it follows that $w$ can reach a sink $s'$ different from $s$. Then, also $v$ can reach two sinks: $s'$ via $w$ and $s$. This contradicts $v \in L$.

iii) A minimal partitioning set $S$ for $G$ does not contain any arc between vertices in $L \cup \{s\}$: this is because, by Observation 1, no minimal partitioning set $S$ can disconnect any vertex $v \in L$ from $s$, since otherwise $v$ would reach another, that is, new sink in $G \setminus S$.

Now, let $S$ be a minimal partitioning set for $G$. Then, $S$ is also a partitioning set for $G'$, since $G' \setminus S = (G \setminus S) - L$ and deleting $L$ from $G \setminus S$ cannot create new sinks.

In the opposite direction, let $S$ be a partitioning set for $G'$. Then, $S$ is also a partitioning set for $G$, since $G' \setminus S$ is just $G \setminus S$ with the vertices in $L$ and their arcs added. These, however, can reach only the sink $s$ and are only connected to vertices to which $s$ is connected. Hence, they do not create new sinks or connect distinct components of $G' \setminus S$. \qed

We now show how to exhaustively apply both data reduction rules in linear time. To this end, we apply Algorithm 2: in lines 1–4, it computes an array $L$ such that, for each vertex $v \in V$, we have $L[v] = \{s\}$ if $v$ reaches exactly one sink $s$ and $L[v] = \emptyset$ otherwise. It uses this information to apply Reduction Rule 1 in lines 6–10 and Reduction Rule 2 in lines 11 and 12.

**Lemma 3.** Given a directed acyclic graph $G$ with weights $\omega$, in $O(n + m)$ time Algorithm 2 produces a directed acyclic graph $G'$ with weights $\omega'$ such that $G'$ is exhaustively reduced with respect to Reduction Rule 1 and Reduction Rule 2. In particular, $(G, \omega, k)$ is a yes-instance if and only if $(G', \omega', k)$ is a yes-instance.
**Algorithm 2:** Apply directed Reduction Rules 1 and 2 exhaustively.

Input: A directed acyclic graph $G = (V, A)$ with arc weights $\omega$.

Output: The result of exhaustively applying Reduction Rules 1 and 2 to $G$.

1. $S \leftarrow$ sinks of $G$
2. $L \leftarrow$ array with $n$ entries
3. foreach $v \in S$ do $L[v] \leftarrow \{v\}$ // $L[v] = \{s\}$ for any sink $s \in V$ will mean that $v$ only reaches the sink $s$.
4. foreach $v \in V \setminus S$ in reverse topological order do
   5. $\quad L[v] \leftarrow \bigcap_{w \in N^+(v)} L[w]$ // Invariant: the intersection contains at most one sink.
6. foreach $v \in V$ with $L[v] = \emptyset$ do // Application of Reduction Rule 1
   7. foreach $w \in N^+(v)$ with $L[w] = \{s\}$ for some $s \in S$ and $w \notin S$ do
      8. if $(v, s) \notin A$ then add $(v, s)$ with $\omega(v, s) := 0$ to $A$
      9. $\omega(v, s) \leftarrow \omega(v, s) + \omega(v, w)$
      10. delete arc $(v, w)$
7. foreach $v \in V \setminus S$ such that $L[v] = \{s\}$ for some $s \in S$ do // Application of Reduction Rule 2
   8. delete vertex $v$
9. return $(G, \omega)$

**Proof.** We first discuss the semantics of Algorithm 2, then its running time. After line 5, $L[v] = \{s\}$ for some vertex $v$ if $v$ can reach exactly one sink $s$ and $L[v] = \emptyset$ otherwise: this is, by definition, true for all $L[v]$ with $v \in S$. For $v \in V \setminus S$ it also holds, since $v$ can reach exactly one sink $s$ if and only if all of its out-neighbors $u \in N^+(v)$ can reach $s$ and no other sinks, that is, if and only if $L[u] = \{s\}$ for all out-neighbors $u \in N^+(v)$ of $v$.

Hence, the loop in lines 6–10 applies Reduction Rule 1 to all arcs to which Reduction Rule 1 may be applied. Moreover, Reduction Rule 1 does not change which sinks are reachable from any vertex and, hence, cannot create new arcs to which Reduction Rule 1 may be applied. Hence, when reaching line 11, the graph will be exhaustively reduced with respect to Reduction Rule 1 and we do not have to update the array $L$.

The loop in lines 11 and 12 now applies Reduction Rule 2, which is allowed, since the graph is exhaustively reduced with respect to Reduction Rule 1. Moreover, an application of Reduction Rule 2 cannot create new vertices to which Reduction Rule 2 may become applicable or arcs to which Reduction Rule 1 may become applicable. Hence, line 13 indeed returns a graph that is exhaustively reduced with respect to both data reduction rules.

It remains to analyze the running time. Obviously, lines 1–3 of Algorithm 2 work in $O(n)$ time. To execute line 4 in $O(n + m)$ time, we iterate over the vertices in $V \setminus S$ in reverse topological order, which can be computed in $O(n + m)$ time [9, Section 22.4]. Hence, when computing $L[v]$ for some vertex in line 4, we already know the values $L[u]$ for all $u \in N^+(v)$. Moreover, $L[v]$ is the intersection of sets with at most one element and, therefore, also contains at most one element. It follows that we can compute $L[v]$ in $O(\lvert N^+(v) \rvert)$ time for each vertex $v \in V \setminus S$ and, therefore, in $O(n + m)$ total time for all vertices. The rest of the algorithm only iterates once over all arcs and vertices. Hence, to show that it works in $O(n + m)$ time, it remains to show how to execute lines 8 and 9 in constant time.

Herein, the main difficulty is that an adjacency list cannot answer queries of the form $\lvert \{v, s\} \in A \rvert$ in constant time. However, since we earlier required to iterate over all out-neighbors of a vertex $v$ in $O(\lvert N^+(v) \rvert)$ time, we cannot just use an adjacency matrix instead. We exploit a different trick, which, for the sake of clarity is not made explicit in the pseudo code: assume that, when considering a vertex $v \in V$ in line 6, we have an $n$-element array $A$ such that $A[x]$ holds a pointer to the value $\omega(v, s)$ if $(v, s) \in A$ and $A[x] = \bot$ otherwise. Then, we could in constant time check in line 8 whether $A[x] = \bot$ to find out whether $(v, s) \in A$ and, if this is the case, get a pointer to (and increment) the weight $\omega(v, s)$ in constant time in line 9. However, we cannot afford initializing an $n$-entry array $A$ for each vertex $v \in V$ and we cannot make assumptions on the value of uninitialized entries. Luckily, we access $A[x]$ only if there is a vertex $w \in N^+(v)$ with $L[w] = \{s\}$ for some $s$. Hence, we can create an $n$-entry array $A$ once in the beginning of the algorithm and then, between lines 6 and 7, set up $A$ for $v \in V$ as follows: for each $w \in N^+(v)$ with $L[w] = \{s\}$, set $A[v[s] := \bot$. Then, for each $w \in N^+(v)$ with $w \in S$, let $A[v[s]$ point to $\omega(v, s)$. □
Lemma 3 shows that we can exhaustively apply the two Reduction Rules 1 and 2 in linear time using Algorithm 2. A natural approach for evaluating the quality of our preprocessing would be to provide a performance guarantee in terms of small problem kernels. Unfortunately, in Section 3.3 we show that, under widely accepted complexity-theoretic assumptions, there is no problem kernel with size polynomial in $k$ for DAG PARTITIONING. Nevertheless, the next section shows that our data reduction technique performs remarkably well in empirical tests. Furthermore, we show that the running time of Algorithm 1 is significantly improved when it is applied after Algorithm 2. We achieve the largest speedup of Algorithm 1 by interleaving the application of Algorithm 2 and the branching; a technique that generally works well for search tree algorithms [30].

3.2. Experimental evaluation

In this section, we aim for giving a proof of concept of our $O(2^k \cdot (n+m))$ time search tree algorithm presented in Section 3.1 by demonstrating to which extent instances of DAG PARTITIONING are solvable within a time frame of five minutes. Moreover, using the optimal solutions found by our algorithm, we evaluate the quality of a heuristic presented by Leskovec et al. [26], which can be considered a variant of our search tree algorithm (Algorithm 1): The difference is that, while our search tree algorithm branches into all possibilities of putting a vertex into a connected component with some sink, the heuristic just puts each vertex into the connected component with the sink it would be most expensive to disconnect from. This is the strategy described by Leskovec et al. [26] as yielding the best results and is in more detail described by Suen et al. [33]. The pseudocode of the heuristics (Algorithm 3) can be found in Section 4.1, where we prove that the heuristic works optimally on trees.

Implementation details. We implemented the search tree algorithm as well as the heuristic in three variants:

1. without data reduction,
2. with initially applying the data reduction algorithm presented in Algorithm 2, and
3. with interleaving the data reduction using Algorithm 2 with the branching of Algorithm 1.

The source code uses about 1000 lines of C++ and is freely available.\textsuperscript{2} The experiments were run on a computer with a 3.6 GHz Intel Xeon processor and 64 GiB RAM under Linux 3.2.0, where the source code has been compiled using the GNU C++ compiler version 4.7.2 and using the highest optimization level (-O3).

Data. We tried to apply our algorithm to the data set described by Leskovec et al. [26]; unfortunately, its optimum partitioning sets have too large weight to be found by our algorithm.\textsuperscript{3} In order to prove the feasibility of solving large instances with small minimum partitioning sets, we generated artificial instances. Herein, however, we stick to the clustering motivation of DAG PARTITIONING and test our algorithm using simulated citation networks: vertices in a graph represent articles and if an article $v$ cites an article $w$, there is an arc $(v, w)$. Herein, we consider only directed acyclic graphs, which model that an article only cites older articles. A partitioning of such a network into connected components of which each contains only one sink can be interpreted as a clustering into different topics of which we want to identify the origins.

To simulate citation networks, we employ preferential attachment graphs—a random graph model commonly used to model citations between articles [31, 2, 21]. Preferential attachment graphs model the natural growth of citation networks, in which new articles are published over time and with high probability cite the already highly-cited articles. Indeed, Jeong et al. [21] empirically verified that in this preferential attachment model, the probability of an article being cited is linear in the number of times the article has been cited in the past.

To create a preferential attachment graph, we first choose two parameters: the number $c$ of sinks to create and the maximum outdegree $d$ of each vertex. After creating $c$ sinks, $n$ new vertices are introduced one after another. After introducing a vertex, we add to it $d$ outgoing unit-weight arcs: for each of these $d$ outgoing arcs, the target is chosen independently at random among previously introduced vertices such that each vertex is chosen as the target with a

\textsuperscript{2}http://fpt.akt.tu-berlin.de/dagpart/

\textsuperscript{3}The exact weights of the optimum partitioning sets remain unknown, since our algorithm could not compute them within several hours.
probability proportional to its indegree plus one. We do not add an outgoing arc twice, which might result in a vertex having less than $d$ outgoing arcs if the target nodes of two arcs to be added coincide.

We compared our algorithm to the heuristic of Leskovec et al. [26] on these graphs, but our algorithm could solve only instances with up to 300 arcs optimally, since the optimum solution weight grows too quickly in the sizes of the generated graphs. To show the running time behavior of our algorithm on larger graphs with small solution sizes, we employ an additional approach: we generate multiple connected components, each being a preferential attachment graph with one sink, and randomly add $k$ additional arcs between these connected components in a way so that the graph remains acyclic. Then, obviously, an optimal partitioning set cannot be larger than $k$. We call the set of $k$ randomly added arcs embedded partitioning set and it can be viewed as noise in data that clusters well.

**Experimental results.** Figure 3 compares the running time of the heuristic of Leskovec et al. [26] to the running time of our Algorithm 1 with increasing optimal partitioning set size $k$. On the left side, it can be seen that using the data reduction from Algorithm 2 slows down the heuristic. This is not surprising, since the heuristic itself is implemented to run in linear time and, hence, instead of first shrinking the input instance by Algorithm 2 in linear time, one might right away solve the instance heuristically. On the right side, one can observe that, as expected, the running time of Algorithm 1 increases exponentially in $k$. We only show the running time of the implementations with data reduction: without data reduction, we could not solve any instance in less than an hour. We can solve instances with $k \leq 190$ optimally within five minutes. This allowed us to verify that the heuristic solved all 40 generated instances optimally, regardless of the type of data reduction applied.

Figure 4 compares the running time of the heuristic of Leskovec et al. [26] to the running time of our Algorithm 1 with increasing graph size. While the heuristic shows a linear increase of running time with the graph size (on the left side), such a behavior cannot be observed for the search tree algorithm (on the right side). The reason for this can be seen in Figure 5: the data reduction applied by Algorithm 2 initially shrinks most input instances to about 2000 arcs in less than ten seconds. Thus, what we observe in the right plot of Figure 4 is, to a large extent, the running time of Algorithm 1 for constant $k = 190$ and roughly constant graph size. Our search tree algorithm allowed us to verify that the heuristic by Leskovec et al. [26] solved all 80 generated instances optimally regardless of the type of data reduction applied.

Finally, Figure 6 presents instances that could not be optimally solved by Leskovec et al. [26]’s heuristic. In the left plot, we see that in instances with large embedded partitioning sets of several hundred thousand arcs, the heuristic of Leskovec et al. [26] does not find the embedded partitioning set but an about 5‰ larger one. In all cases, the heuristic
Figure 4: Comparisons of the running time of Leskovec et al. [26]'s heuristic (left) with the running time of our search tree algorithm (right). Without interleaved data reduction, the search tree solved no instance in less than 5 minutes. The graphs were generated by adding $k = 190$ random arcs between ten connected components, each being a preferential attachment graph with outdegree twenty and a single sink. The heuristic solved all 80 instances optimally.

Figure 5: Effect (left) and running time (right) of initially running Algorithm 2 for data reduction. The graphs were generated by adding $k = 190$ random arcs between ten connected components, each being a preferential attachment graph with outdegree twenty and a single sink.
found the same partitioning sets regardless of the type of data reduction applied. Note that the plot only gives a lower bound on the deviation factor, since there might be even better partitioning sets in the instances than the embedded one; we were unable to compute the optimal partitioning sets in these instances. In the right plot of Figure 6, we used smaller preferential attachment graphs (this time without embedded partitioning sets) and see that Leskovec et al. [26]'s heuristic can be off by more than a factor of two from the optimal partitioning set. Data reduction had no effect on the quality of the partitioning sets found.

Summary. We have seen that solving large instances with partitioning sets of small weight is realistic using our algorithm. In particular, instances with more than $10^7$ arcs and $k \leq 190$ could be solved in less than five minutes. A crucial ingredient in this success is the data reduction executed by Algorithm 2; without its help, we could not solve any of our instances in less than five minutes.

However, we also observed that our algorithm works best on those instances that can already be solved mostly optimally by Leskovec et al. [26]'s heuristic and that the data reduction executed by Algorithm 2 slows down the heuristic.

Having seen that the heuristic by Leskovec et al. [26] can be off by more than a factor of two from the optimum on random preferential attachment graphs diminishes the hope that, in spite of the non-approximability results of DAG PARTITIONING by Alamdari and Mehrabian [1], the heuristic of Leskovec et al. [26] might find good approximations on naturally occurring instances. As we see, we do not have to construct adversarial instances to make the heuristic find solutions far from optimal.

3.3. Limits of data reduction and fixed-parameter algorithms

In Section 3.1, we have seen linear-time data reduction rules for DAG PARTITIONING. Moreover, the experiments in Section 3.2 have shown that, on all input instances we tested our algorithm on, the running time of our $O(2^k \cdot (n+m))$ time Algorithm 1 merely depended on $k$ because Algorithm 2 shrank our random input instances to roughly the same size.

Therefore it is natural to ask whether we can provide data reduction rules that provably shrink the size of each possible input instance to some fixed polynomial in $k$, that is, whether there is a polynomial-size problem kernel for DAG PARTITIONING. Unfortunately, in this section, we give a negative answer to this question. Specifically, we prove that DAG PARTITIONING does not admit problem kernels with size polynomial in $k$, unless NP $\subseteq$ coNP/poly. Moreover, we show that the running time $O(2^k \cdot (n+m))$ of Algorithm 1 cannot be improved to $2^{o(k)} \cdot \text{poly}(n)$, unless the Exponential Time Hypothesis fails. Herein, the Exponential Time Hypothesis as well as NP $\not\subseteq$ coNP/poly are hypotheses stronger than P $\neq$ NP, but widely accepted among complexity theorists [20, 16].
We will later exploit Lemma 4 to show our hardness results. Next, we show that arcs with non-unit weights in our constructions simple.

3.3.1. Strengthening of hardness results to unit-weight graphs

Construction 1 heavily relied on forbidding the deletion of certain arcs by giving them a high weight. The next lemma shows that we can replace these arcs by a gadget only using unit-weight arcs without changing the weight of the partitioning set sought.

Figure 7: DAG PARTITIONING instance constructed from the formula consisting only of the clause $C_1 := (x_1 \lor \bar{x}_2)$. Heavy arcs are drawn bold; dotted arcs are a partitioning set that corresponds to the satisfying assignment setting $x_1$ to true and $x_2$ to false. The variable gadgets are drawn on a gray background.

Towards proving these results, we first recall the polynomial-time many-to-one reduction from 3-SAT to DAG PARTITIONING given by Alamdari and Mehrabian [1]. The 3-SAT problem is, given a formula $\phi$ in conjunctive normal form with at most three literals per clause, to decide whether $\phi$ admits a satisfying assignment.

Construction 1 (Alamdari and Mehrabian [1]). Let $\phi$ be an instance of 3-SAT with the variables $x_1, \ldots , x_n$ and the clauses $C_1, \ldots , C_m$. We construct a DAG PARTITIONING instance $(G, \omega, k)$ with $k := 4n + 2m$ that is a yes-instance if and only if $\phi$ is satisfiable. The weight function $\omega$ will assign only two different weights to the arcs: a normal arc has weight one and a heavy arc has weight $k + 1$ and thus cannot be contained in any partitioning set of weight $k$. The remainder of this construction is illustrated in Figure 7.

We start constructing the directed acyclic graph $G$ by adding the special vertices $f, f', t,$ and $t'$ together with the heavy arcs $(f, f')$ and $(t, t')$. The vertices $f'$ and $t'$ will be the only sinks in $G$. For each variable $x_i$, introduce the vertices $x'_i, x''_i , x_i$ and $\bar{x}_i$ together with the heavy arcs $(t, x'_i)$ and $(f, x''_i)$ and the normal arcs $(x'_i, x_i), (x''_i, \bar{x}_i), (x'_i, x_i), (x''_i, \bar{x}_i), (x_i, f'), (\bar{x}_i, f'), (x_i, t'), (\bar{x}_i, t')$. For each clause $C_j$, add a vertex $C_j$ together with the heavy arc $(t, C_j)$. Finally, if some clause $C_j$ contains the literal $x_i$, then add the arc $(C_j, x_i)$; if some clause $C_j$ contains the literal $\bar{x}_i$, then add the arc $(C_j, \bar{x}_i)$.

Alamdari and Mehrabian [1] showed that, given a formula $\phi$ in 3-CNF with $n$ variables and $m$ clauses, Construction 1 outputs a graph $G$ with arc weights $\omega$ such that $\phi$ is satisfiable if and only if there is a partitioning set $S$ for $G$ that does not contain heavy arcs.

Since $G$ has only the two sinks $t'$ and $f'$, by Observation 1, a minimal such partitioning set has to partition $G$ into two connected components, one connected component containing the heavy arc $(t, t')$ and the other containing $(f, f')$. Moreover, if $\phi$ is satisfiable, then such a partitioning set has weight at most $4n + 2m$, since for each $x_i$ of the $n$ variables of $\phi$, it deletes at most one of two arcs outgoing from each of the vertices $x'_i , x''_i , x_i$, and $\bar{x}_i$, and for each $C_j$ of the $m$ clauses, it deletes at most two out of the three arcs outgoing from the clause vertex $C_j$. We thus obtain the following lemma:

**Lemma 4.** Given a formula $\phi$ in 3-CNF with $n$ variables and $m$ clauses, Construction 1 outputs a graph $G$ with arc weights $\omega$ such that $\phi$ is satisfiable if and only if there is a partitioning set $S$ for $G$ that does not contain heavy arcs.

Moreover, if $\phi$ is satisfiable, then $S$ has weight at most $4n + 2m$ and partitions $G$ into one connected component containing the constructed vertices $t$ and $t'$ and the other containing $f$ and $f'$.

We will later exploit Lemma 4 to show our hardness results. Next, we show that arcs with non-unit weights in our constructions can be simulated by arcs with unit weights. This allows us to show stronger hardness results and to keep our constructions simple.
Lemma 5. There is a polynomial-time many-one reduction from DAG PARTITIONING with polynomially bounded weights to unweighted DAG PARTITIONING that does not change the weight \( k \) of the partitioning set sought.

Proof. Let \((G, \omega, k)\) be an instance of DAG PARTITIONING. We show how to obtain an instance \((G', \omega', k)\) by replacing a single arc of weight more than one by arcs of weight one such that \((G, \omega, k)\) is a yes-instance if and only if \((G', \omega', k)\) is a yes-instance. The replacement will be done as illustrated in Figure 8. The claim then follows by repeating this procedure for every arc of weight more than one.

Consider an arc \( a = (v, w) \) in \( G \) with \( \omega(a) > 1 \). We obtain \( G' \) and \( \omega' \) from \( G \) and \( \omega \) by setting the weight \( \omega'(a) = 1 \), adding a set \( X \) of \( \omega(a) - 1 \) vertices to \( G' \), and inserting for each \( u \in X \) a weight-one arc \((v, u)\) and a weight-one arc \((u, w)\).

First, assume that \((G, \omega, k)\) is a yes-instance and that \( S \) is a minimal partitioning set for \( G \). We show how to obtain a partitioning set of weight \( k \) for \( G' \). Clearly, if \( a \not\in S \), then \( S \) is a partitioning set of equal weight for \((G', \omega', k)\). If \( a \in S \), then we get a partitioning set of equal weight for \((G', \omega', k)\) by adding the arcs between \( v \) and \( X \) to \( S \).

Second, assume that \((G', \omega', k)\) is a yes-instance and that \( S \) is a minimal partitioning set for \( G' \). We show how to obtain a partitioning set of weight \( k \) for \( G \). To this end, we consider two cases: \( v \) and \( w \) are in a common or in separate connected components of \( G' \setminus S \).

Case 1) If \( v \) and \( w \) are in one connected component of \( G' \setminus S \), then, by minimality, \( S \) does not contain \( a \) or any arc incident to vertices in \( X \). Hence, \( S \) is a partitioning set of equal weight for \( G \).

Case 2) If \( v \) and \( w \) are in separate connected components of \( G' \setminus S \), then \( a \in S \). Moreover, the vertices in \( X \) have only one outgoing arc. Hence, by Observation 1, \( S \) does not contain arcs from \( X \) to \( w \) but, therefore, contains all arcs from \( v \) to \( X \). Removing these arcs from \( S \) results in a partitioning set of equal weight for \((G, \omega, k)\).

\( \square \)

3.3.2. Limits of fixed-parameter algorithms

We now show that DAG PARTITIONING cannot be solved in \( 2^{o(k)} \text{poly}(n) \) time unless the Exponential Time Hypothesis fails. Thus, if our search tree algorithm for DAG PARTITIONING can be improved, then only by replacing the base of the exponential \( 2^k \)-term by some smaller constant.

The Exponential Time Hypothesis was introduced by Impagliazzo and Paturi [19] and states that \( n \)-variable 3-SAT cannot be solved in \( 2^{o(n)} \text{poly}(n) \) time. Using the reduction from 3-SAT to DAG PARTITIONING given by Alamdari and Mehrabian [11] (Construction 1), we can easily show the following:

Theorem 2. Unless the Exponential Time Hypothesis fails, DAG PARTITIONING cannot be solved in \( 2^{o(k)} \text{poly}(n) \) time even if all arcs have unit weight.

Proof. Construction 1 reduces an instance of 3-SAT consisting of a formula with \( n \) variables and \( m \) clauses to an equivalent instance \((G, \omega, k)\) of DAG PARTITIONING with \( k = 4n + 2m \). Thus, a \( 2^{o(k)} \text{poly}(n) \)-time algorithm for DAG PARTITIONING would yield a \( 2^{o(m)} \text{poly}(m) \)-time algorithm for 3-SAT. This, in turn, by the so-called Sparsification Lemma of Impagliazzo et al. [20, Corollary 2], would imply a \( 2^{o(n)} \text{poly}(n) \) time algorithm for 3-SAT, which contradicts the Exponential Time Hypothesis. Since the weights used in Construction 1 are polynomial in the number of created vertices and edges, we can apply Lemma 5 to transfer the result to the unit-weight case.

\( \square \)
3.3.3. Limits of problem kernelization

We now show that DAG Partitioning has no polynomial-size problem kernel with respect to the parameter \( k \)—the weight of the partitioning set sought. It follows that, despite the effectiveness of data reduction observed in experiments in Section 3.2, we presumably cannot generally shrink a DAG Partitioning instance in polynomial time to a size polynomial in \( k \).

To show that DAG Partitioning does not allow for polynomial-size kernels, we first provide the necessary concepts and techniques introduced by Bodlaender et al. [8].

**Definition 1 (Bodlaender et al. [8, Definition 3.3])**. For some finite alphabet \( \Sigma \), a problem \( L \subseteq \Sigma^* (\text{OR-})\text{-cross-composes} \) into a parameterized problem \( Q \subseteq \Sigma^* \times \mathbb{N} \) if there is an algorithm (a (OR-)cross-composition) that transforms instances \( x_1, \ldots, x_t \), of \( L \) into an instance \( (x^*, k) \) for \( Q \) in time polynomial in \( \sum_{i=1}^t |x_i| \) such that

i) \( k \) is bounded by a polynomial in \( \max_{i=1}^t |x_i| + \log s \) and

ii) \( (x^*, k) \in Q \) if and only if there is an \( i \in \{1, \ldots, s\} \) such that \( x_i \in L \).

Furthermore, the cross-composition may exploit that the input instances \( x_1, \ldots, x_t \) belong to the same equivalence class of a polynomial equivalence relation \( R \subseteq \Sigma^* \times \Sigma^* \), which is an equivalence relation such that

i) it can be decided in polynomial time whether two instances are equivalent and

ii) any finite set \( S \subseteq \Sigma^* \) is partitioned into \( \text{poly}(\max_{v \in S}|x|) \) equivalence classes.

The assumption that all instances belong to the same equivalence class of a polynomial equivalence relation can make the construction of a cross-composition remarkably easier: when giving a cross-composition from 3-SAT, we can, for example, assume that the input instances all have the same number of clauses and variables.

Cross-compositions can be used to prove that a parameterized problem has no polynomial-size kernel unless \( \text{NP} \subseteq \text{coNP/poly} \).

**Theorem 3 (Bodlaender et al. [8, Corollary 3.6])**. If some problem \( L \subseteq \Sigma^* \) is NP-hard under polynomial-time many-one reductions and \( L \) cross-composes into the parameterized problem \( Q \subseteq \Sigma^* \times \mathbb{N} \), then there is no polynomial-size problem kernel for \( Q \) unless \( \text{NP} \subseteq \text{coNP/poly} \).

In the following, we show that 3-SAT cross-composes into DAG Partitioning parameterized by \( k \), which yields the following theorem:

**Theorem 4**. Unless \( \text{NP} \subseteq \text{coNP/poly} \), DAG Partitioning does not have a polynomial-size problem kernel with respect to the weight \( k \) of the partitioning set sought even if all arcs have unit weight.

Although the proof of Theorem 4 is based on the following construction, which requires arc weights, by using Lemma 5 from Section 3.3.1, we obtain that Theorem 4 holds even on graphs with unit weights.

**Construction 2.** Let \( \varphi_1, \ldots, \varphi_s \) be instances of 3-SAT. Since we may assume \( \varphi_1, \ldots, \varphi_s \) to be from the same equivalence class of a polynomial equivalence relation, we may assume that each of the formulas \( \varphi_1, \ldots, \varphi_s \) has the same number \( n \) of variables and the same number \( m \) of clauses. Moreover, we may assume that \( s \) is a power of two; otherwise we simply add unsatisfiable formulas to the list of instances. We now construct a DAG Partitioning instance \( (G, \varnothing, k) \) with \( k := 4n + 2m + 3 \log s \) that is a yes-instance if and only if \( \varphi_i \) is satisfiable for at least one \( 1 \leq i \leq s \), where we use “log” to denote the binary logarithm. As in Construction 1, the weight function \( \varnothing \) will only assign two possible weight values: a heavy arc has weight \( k + 1 \) and thus cannot be contained in any partitioning set. A normal arc has weight one. The remainder of the construction is illustrated in Figure 9.

For each instance \( \varphi_i \), let \( G_i \) be the graph produced by Construction 1. By Lemma 4, \( G_i \) can be partitioned with \( 4n + 2m \) arc deletions if and only if \( \varphi_i \) is a yes-instance. We now build a gadget that, by means of additional \( 3 \log s \) arc deletions, chooses exactly one graph \( G_i \) that has to be partitioned.

To distinguish between multiple instances, we denote the special vertices \( f, f', t, \) and \( t' \) in \( G_i \) by \( f_i, f_i', t_i, \) and \( t_i' \). For all \( 1 \leq i \leq s \), we add \( G_i \) to the output graph \( G \) and merge the vertices \( f_1, f_2, \ldots, f_s \) into a vertex \( f \) and the vertices \( f_1', f_2', \ldots, f_s' \) into a vertex \( f' \). Furthermore, we add the vertices \( t, t' \), and \( t'' \) and the heavy arcs \( (t, t') \) and \( (t, t'') \) to \( G \).
The graphs \( G \) using this construction, we can now prove Theorem 4.

and \( f \) and \( g \) goes through some vertices \( t \) connected components, namely \( I \) incoming arc not belonging to \( \text{sink } f \) component \( P \) \( I \) in \( L \) as follows. Let \( P \) that by \( \text{and edges, we can apply Lemma 5 to transfer the result to the unit-weight case.} \)

follows from Theorem 3. Since the weights used in Construction 2 are polynomial in the number of created vertices instance if and only if at least one of the input formulas \( \phi \) is satisfiable.

Conversely, let \( S \) be a minimal partitioning set for \( G \) with \( \omega(S) \leq k \). Then, by Observation 1, \( G \setminus S \) has two connected components, namely \( P' \) with sink \( t' \) and \( P'' \) with sink \( f' \). Since \( S \) cannot contain heavy arcs, \( t \) and \( t'' \) are in \( P' \). Hence, \( t'' \) can reach \( t' \) in \( G \setminus S \), since they are in the same component of \( G \setminus S \). As every directed path from \( t'' \) to \( t' \) goes through some vertices \( t_i \) and \( t_i' \), it follows that there is an i ∈ \{1, ..., s\} such that \( t_i \) and \( t_i' \) are in \( P_i' \). Since \( f = f_i \) and \( f' = f_i' \) are in \( P_i' \), the partitioning set \( S \cap A(G_i) \) partitions \( G_i \) into two connected components: one containing \( t_i \) and \( t_i' \) and the other containing \( f = f_i \) and \( f' = f_i' \). Since \( S \) does not contain heavy arcs, from Lemma 4 it follows that \( \phi_i \) is satisfiable.

Using this construction, we can now prove Theorem 4.

**Proof (of Theorem 4).** We only have to show that the instance \((G, \omega, k)\) constructed by Construction 2 is a yes-instance if and only if at least one of the input formulas \( \phi_i \) is satisfiable. Then, the theorem for the weighted case follows from Theorem 3. Since the weights used in Construction 2 are polynomial in the number of created vertices and edges, we can apply Lemma 5 to transfer the result to the unit-weight case.

First, assume that a formula \( \phi \) is satisfiable for some \( 1 \leq i \leq s \). By Lemma 4 it follows that \( G_i \) can be partitioned by \( k' := 4n + 2m \) arc deletions into two connected components \( P_i \) and \( P_i' \) such that \( P_i \) contains \( t_i \) and \( t_i' \) and such that \( P_i' \) contains \( f_i = f \) and \( f_i' = f' \). We apply these arc deletions to \( G \) and delete \( 3 \log_4 s \) additional arcs from \( G \) as follows. Let \( L \) be the unique directed path in \( O \) from \( t'' \) to \( t_i \). Analogously, let \( L' \) be the unique directed path in \( I \) from \( t_i' \) to \( t_i \). Observe that each of these directed paths has \( \log s \) arcs. We partition \( G \) into the connected component \( P_i' = P_i \cup \{t, t', t''\} \cup V(L) \cup V(L') \) with sink \( t'' \) and into the connected component \( P_i'' = V(G) \setminus P_i' \) with sink \( f' \). To this end, for each vertex \( v \in V(L) \setminus \{t_i\} \), we remove the outgoing arc that does not belong to \( L \). Hence, exactly \( \log s \) arcs incident to vertices of \( O \) are removed. Similarly, for each vertex \( v \in V(L') \setminus \{t_i'\} \), we remove the incoming arc not belonging to \( L' \). For each vertex \( v \neq t \) of \( L' \), we remove the arc to \( f' \). Hence, exactly \( 2 \log s \) arcs incident to vertices of \( I \) are removed. Thus, in total, at most \( k = 4n + 2m + 3 \log_4 s \) normal arcs are removed to partition \( G \) into \( P_i' \) and \( P_i'' \).

Conversely, let \( S \) be a minimal partitioning set for \( G \) with \( \omega(S) \leq k \). Then, by Observation 1, \( G \setminus S \) has two connected components, namely \( P_i' \) with sink \( t_i' \) and \( P_i'' \) with sink \( f_i' \). Since \( S \) cannot contain heavy arcs, \( t' \) and \( t'' \) are in \( P_i' \). Hence, \( t'' \) can reach \( t' \) in \( G \setminus S \), since they are in the same component of \( G \setminus S \). As every directed path from \( t'' \) to \( t' \) goes through some vertices \( t_i \) and \( t_i' \), it follows that there is an \( i \in \{1, ..., s\} \) such that \( t_i \) and \( t_i' \) are in \( P_i' \). Since \( f = f_i \) and \( f' = f_i' \) are in \( P_i' \), the partitioning set \( S \cap A(G_i) \) partitions \( G_i \) into two connected components: one containing \( t_i \) and \( t_i' \) and the other containing \( f = f_i \) and \( f' = f_i' \). Since \( S \) does not contain heavy arcs, from Lemma 4 it follows that \( \phi_i \) is satisfiable. □
Algorithm 3: Leskovec et al. [26]'s heuristic to compute small partitioning sets.

**Input**: A directed acyclic graph \( G = (V, A) \) with arc weights \( \omega \).

**Output**: A partitioning set \( S \).

1. \( \{v_1, v_2, \ldots, v_n\} \leftarrow \) reverse topological order of the vertices of \( G \)
2. \( L \leftarrow \) array with \( n \) entries
3. \( S \leftarrow \emptyset \)
4. for \( i = 1 \) to \( n \) do
   5. if \( v_i \) is a sink then \( L[v_i] \leftarrow v_i \) // associate \( v_i \) with itself
   6. else for \( s \in D \) do
      7. \( D \leftarrow \{L[w] \mid w \in N^+(v_i)\} \) // the set of feasible sinks for \( v_i \)
      8. \( A_s \leftarrow \{(v_i, w) \in A \mid w \in N^+(v_i) \land L[w] = s\} \) // set of arcs to keep if \( v_i \) is associated with sink \( s \)
      9. \( s^* \leftarrow \arg \max_{s \in D} \omega(A_s) \) // cheapest sink \( s^* \) to associate \( v_i \) with
      10. \( L[v_i] \leftarrow s^* \)
      11. \( S \leftarrow S \cup \{(v_i, w) \mid w \in N^+(v_i) \land L[w] \neq s^*\} \)
5. return \( S \)

4. Parameter treewidth

In Section 3, we have seen that DAG PARTITIONING is linear-time solvable when the weight \( k \) of the requested partitioning set is constant. Alamdari and Mehrabian [1] asked whether DAG PARTITIONING is fixed-parameter tractable with respect to the parameter treewidth, which is a measure of the “tree-likeness” of a graph. We will answer this question affirmatively.

In Section 4.1, we first show that, if the input graph is indeed a tree, then the heuristic by Leskovec et al. [26] solves the instance optimally in linear time. Afterwards, in Section 4.2, we prove that this result can be generalized to graphs of bounded treewidth and thus improve the algorithm for pathwidth given by Alamdari and Mehrabian [1], since the treewidth of a graph is at most its pathwidth.

4.1. Partitioning trees

In this section, we show that the heuristic by Leskovec et al. [26] solves DAG PARTITIONING in linear time on trees. This result will be generalized in the next section, where we show how to solve DAG PARTITIONING in linear time on graphs of bounded treewidth. The heuristic by Leskovec et al. [26] is similar to our search tree algorithm presented in Algorithm 1: instead of trying all possibilities of associating a vertex with one of its feasible sinks, it associates each vertex with the sink that it would be most expensive to disconnect from. The algorithm is presented in Algorithm 3.

**Theorem 5.** Algorithm 3 solves DAG PARTITIONING optimally in linear time if the underlying undirected graph is a tree.

**Proof.** Algorithm 3 clearly works in linear time: to implement it, we only have to iterate over the out-neighbors of each vertex \( v_i \) once. In particular, all sets \( A_s \) in line 9 can be computed by one iteration over each \( w \in N^+(v_i) \) and adding the arc \((v_i, w)\) to \( A_{L[w]} \). Moreover, Algorithm 3 returns a partitioning set: each vertex \( v \) is associated with exactly one sink \( L[v] \) of \( G \) and the returned set \( S \) deletes exactly the arcs \((v, w)\) for which \( L[v] \neq L[w] \).

We show by induction on \( i \) that Algorithm 3 computes a minimum-weight partitioning set for \( G[\{v_1, \ldots, v_i\}] \). For the induction base case, observe that \( v_1 \) is a sink and, thus, \( v_1 \) only reaches the sink \( L[v_1] = v_1 \) in \( G \setminus S \) for all possible minimum-weight partitioning sets \( S \) for \( G \). Now, assume that there is a minimum-weight partitioning set \( S \) such that each \( v \in \{v_1, \ldots, v_{i-1}\} \) only reaches the sink \( L[v] \) in \( G \setminus S \). We show that there is a minimum-weight partitioning set \( S' \) such that each \( v \in \{v_1, \ldots, v_i\} \) reaches only the sink \( L[v] \) in \( G \setminus S' \). If \( v_i \) only reaches \( L[v_i] \) in \( G \setminus S \), then we are done. Otherwise, \( v_i \) reaches some sink \( s' \neq L[v_i] \) in \( G \setminus S \) and, hence, is not itself a sink. The graph \( G \) is partly illustrated in Figure 10.
We first formally define the tree decomposition of a graph and its width.

Definition 2 (Treewidth, tree decomposition). Let \( G = (V, A) \) be a directed graph. A tree decomposition \((T, \beta)\) for \( G \) consists of a rooted tree \( T = (X, E) \) and a mapping \( \beta : X \rightarrow 2^V \) of each node \( x \) of the tree \( T \) to a subset \( V_x := \beta(x) \subseteq V \), called bag, such that

i) for each vertex \( v \in V \), there is a node \( x \) of \( T \) with \( v \in V_x \),

ii) for each arc \((u, w) \in A\), there is a node \( x \) of \( T \) with \([u, w] \subseteq V_x\),

Let \( R_i \) be the set of vertices reachable from \( v_i \) that reach some sink \( s \) in \( G \). Since the underlying undirected graph of \( G \) is a tree, \( v_i \) has exactly one arc into \( R_i \) for each sink \( s \) reachable from \( v_i \). Let \((v_i, u)\) be the arc of \( v_i \) into \( R_{L[v_i]} \) and \((v_i, w)\) be the arc of \( v_i \) into \( R_{S[v_i]} \). Observe that the arc \((v_i, u)\) exists since the algorithm can set \( L[v_i] \) only to sinks reachable from \( v_i \). To show that \( S' := (S \setminus \{(v_i, w)\}) \cup \{(v_i, u)\} \) is still a partitioning set, we only have to verify that \( v_i \) and all vertices reaching \( v_i \) only reach one sink in \( G \setminus S' \). For all other vertices, this follows from \( S \) being a partitioning set.

1. The vertex \( v_i \) only reaches the sink \( L[v_i] \) in \( G \setminus S' \); this is because \( u = v_j \) for some \( j < i \) which, by induction hypothesis, reaches exactly one sink in \( G \setminus S \) and, hence, in \( G \setminus S' \).
2. A vertex \( v \) that reaches \( v_i \) in \( G \setminus S' \) reaches only the sink \( L[v_i] \) in \( G \setminus S' \); otherwise \( v \) reaches \( s' \) in \( G \setminus S' \) since \( v_i \) and, therefore, \( v \), reaches \( s' \) in \( G \setminus S \). This, however, means that \( v \) has a path to \( s' \) that bypasses \( v_i \) in \( G \setminus S' \) and hence, in \( G \setminus S \), which contradicts the undirected underlying graph of \( G \setminus S \) being a tree.

It remains to show \( \omega(S') \leq \omega(S) \), implying that \( S' \) is also a minimum-weight partitioning set. To see this, we analyze the sets \( A_{R_i} \) and \( A_{L[v_i]} \) computed in line 9 of Algorithm 3. Observe that \( A_{R_i} = \{(v_i, w)\} \) and that \( A_{L[v_i]} = \{(v_i, u)\} \). Since \( \omega(A_{R_i}) \leq \omega(A_{L[v_i]}) \) because of the choice of \( L[v_i] \) in line 10, we conclude that \( \omega(v_i, u) \leq \omega(v_i, w) \) and hence, \( \omega(S') \leq \omega(S) \). \( \square \)

4.2. Partitioning DAGs of bounded treewidth

We now give an algorithm that solves \textsc{Dag Partitioning} in linear time on graphs of bounded treewidth. In contrast to Section 3, which presented our search tree algorithm and an experimental evaluation thereof, the algorithm presented below is of rather theoretical interest: Alamdari and Mehrabian [1] asked whether \textsc{Dag Partitioning} is fixed-parameter tractable with respect to the parameter treewidth. With a dynamic programming algorithm, we can prove the following theorem, which answers their open question and is an improvement of Alamdari and Mehrabian [1]’s algorithm, since the treewidth of a graph is at most its pathwidth.

Theorem 6. Given a width-\( t \) tree decomposition of the underlying undirected graph, \textsc{Dag Partitioning} can be solved in \( 2^{O(t^2)} \cdot n \) time.

We first formally define the tree decomposition of a graph and its width.

Definition 2 (Treewidth, tree decomposition). Let \( G = (V, A) \) be a directed graph. A tree decomposition \((T, \beta)\) for \( G \) consists of a rooted tree \( T = (X, E) \) and a mapping \( \beta : X \rightarrow 2^V \) of each node \( x \) of the tree \( T \) to a subset \( V_x := \beta(x) \subseteq V \), called bag, such that

i) for each vertex \( v \in V \), there is a node \( x \) of \( T \) with \( v \in V_x \),

ii) for each arc \((u, w) \in A\), there is a node \( x \) of \( T \) with \([u, w] \subseteq V_x\),

\( 19 \)
we refer the reader that is yet inexperienced with dynamic programming on tree decompositions to introductory chapters. For a node \( x \) of \( T \) for which \( v \in V_x \) induce a subtree in \( T \).

A tree decomposition is nice if \( V_r = \emptyset \) for the root \( r \) of \( T \) and each node \( x \) of \( T \) is either

- a leaf: then, \( V_x = \emptyset \),
- a forget node: then, \( x \) has exactly one child node \( y \) and \( V_x = V_y \setminus \{v\} \) for some \( v \in V_y \),
- an introduce node: then, \( x \) has exactly one child node \( y \) and \( V_x = V_y \cup \{v\} \) for some \( v \in V \setminus V_y \), or
- a join node: then, \( x \) has exactly two child nodes \( y \) and \( z \) such that \( V_x = V_y = V_z \).

The width of a tree decomposition is one less than the size of its largest bag. The treewidth of a graph \( G \) is the minimum width of a tree decomposition for \( G \). For a node \( x \) of \( T \), we denote by \( U_x \) the union of \( V_y \) for all descendants \( y \) of the node \( x \).

For any constant \( t \), it can be decided in linear time whether a graph has treewidth \( t \) and the corresponding tree decomposition of width \( t \) can be constructed in linear time [6]. Also in \( O(tm) \) time, the tree decomposition of width \( t \) can be transformed into a nice tree decomposition with the same width and \( O(tm) \) nodes [24]. Hence, we assume without loss of generality that we are given a nice tree decomposition.

Our algorithm is based on leaf-to-root dynamic programming. That is, intuitively, we start from the leaf nodes of the tree decomposition and compute possible partial partitioning sets for each bag from the possible partial partitioning sets for its child bags. Since our algorithm for DAG PARTITIONING on graphs of bounded treewidth is relatively intricate, we refer the reader that is yet inexperienced with dynamic programming on tree decompositions to introductory chapters in corresponding text books [28, 12, 10, 24].

We will now precisely define a partial partitioning set and show that any partial partitioning set for the root bag is a partitioning set for the entire graph. The definition is illustrated in Figure 11.

**Definition 3 (Partial partitioning set).** A partial partitioning set \( S \) for \( G[U_i] \) is an arc set \( S \subseteq A(G[U_i]) \) such that

(i) no connected component of \( G[U_i] \setminus S \) contains two different sinks of \( U_i \setminus V_i \), and

(ii) every sink in a connected component of \( G[U_i] \setminus S \) that contains a vertex of \( V_i \) can be reached from some vertex of \( V_i \) in \( G[U_i] \setminus S \).

Since we assumed to work on a tree decomposition with a root \( r \) such that the bag \( V_r \) is empty, any partial partitioning set for \( G[U_i] = G \) will be a partitioning set for the entire graph \( G \). Moreover, Definition 3(i) does not require partial partitioning sets for \( G[U_i] \) to separate sinks in the bag \( V_r \). This is because vertices in \( V_r \) that are sinks in \( G[U_i] \) might be non-sinks for a supergraph, as illustrated in Figure 11. Thus, it might be unnecessary to separate the vertices in \( V_r \). However, due to Definition 2(ii and iii) of a tree decomposition, sinks in \( U_i \setminus V_i \) are sinks in all supergraphs \( G[U_i] \).
We will use a pattern \( (\mathcal{R}, \mathcal{S}, \mathcal{P}) \), where shown are the graph \( \mathcal{S} \) and a partition \( \mathcal{P} \) of its vertices into two sets \( P_1 \) and \( P_2 \). If the graph \( \mathcal{R} \) is the subgraph of \( \mathcal{S} \) induced by the vertices \( V_x \), then, in terms of Definition 5, \( \mathcal{R} = \mathcal{R}_i(S) \) and \( \mathcal{S} = \mathcal{S}_i(S) \) for the partial partitioning set \( S \) for \( G[U_x] \) shown in Figure 11. In this case, the partial partitioning set \( S \) satisfies the shown pattern. Moreover, a vertex is a sink in this figure if and only if it is a sink in \( G[U_x] \) shown in Figure 11.

for \( q \) being an ancestor node of \( x \). Definition 3(ii), by Observation 2, allows us to ensure that components containing both a sink in \( U_i \setminus V_q \) and a vertex of \( V_x \) end up with only one sink in some supergraph \( G[U_q] \). The precise purpose of Definition 3(ii) will be explained in more detail after the upcoming Definition 5.

To keep the notation free from clutter, note that Definition 3 implicitly relies on the bag \( V_x \) belonging to each set \( U_i \). Thus, when a tree decomposition has a node \( x \) and a child node \( y \) such that \( V_x \subseteq V_y \) but \( U_i = U_y \), a partial partitioning set \( S \) for \( G[U_x] \) is not necessarily a partial partitioning set for \( G[U_y] \), although \( G[U_x] \setminus S = G[U_y] \setminus S \).

Now, assume that we want to compute partial partitioning sets for \( G[U_x] \) from partial partitioning sets for child nodes of \( x \). These partial partitioning sets might, for example, disagree on which arcs to delete in the child bags or which connected components of the child bags are meant to end up in a common connected component of the entire graph: for a child node \( y \) of \( x \), multiple connected components of \( G[U_y] \setminus S \) might be one connected component of \( G[U_x] \setminus S \). To prevent such incompatibilities, we only consider those partial partitioning sets for \( G[U_y] \) that agree with partial partitioning sets for the child nodes of \( x \) on certain patterns.

On a high level, our algorithm will store for each node of the tree decomposition a table with one row for each possible pattern. The value of a row will be the minimum weight of a partial partitioning set satisfying this pattern. To compute this value, our algorithm will use the rows with corresponding patterns in the tables of the child nodes. In the following, we first formalize the terms patterns and satisfying partial partitioning sets. Then, we present our algorithm and we specify the corresponding patterns. We start by formally defining patterns, see Figure 12 for an illustration.

**Definition 4 (Pattern).** Let \( x \) be a node of a tree decomposition \( T \). A pattern for \( x \) is a triple \( (\mathcal{R}, \mathcal{S}, \mathcal{P}) \) such that

i) \( \mathcal{R} \) is a directed acyclic graph with the vertices \( V_x \).

ii) \( \mathcal{S} \) is a directed acyclic graph with the vertices \( V_x \) and at most \(|V_x|\) additional vertices such that each vertex in \( V(\mathcal{S}) \setminus V_x \) is a non-isolated sink, and

iii) \( \mathcal{P} \) is a partition of the vertices of \( \mathcal{S} \) such that each connected component of \( \mathcal{S} \) is within one set \( P_i \in \mathcal{P} \) and such that each \( P_i \) contains at most one vertex of \( V(\mathcal{S}) \setminus V_x \).

We will use a pattern \( (\mathcal{R}, \mathcal{S}, \mathcal{P}) \) for \( x \) to capture important properties of partial partitioning sets for \( G[U_x] \). Intuitively, the graph \( \mathcal{R} \) will describe which arcs between the vertices in the bag \( V_x \) a partial partitioning set \( S \) for \( G[U_x] \) will not delete. The graph \( \mathcal{S} \) will describe which vertices of \( V_x \) can reach each other in \( G[U_x] \setminus S \) and which sinks outside of \( V_x \) they can reach. The partition \( \mathcal{P} \) describes which vertices are meant to end up in the same connected component of \( G \setminus S \) for a partitioning set \( S \) of the entire graph. For this reason, the sets of the partition \( \mathcal{P} \) are allowed to contain only one vertex of \( V(\mathcal{S}) \setminus V_x \) as these vertices are sinks.

We will now explain precisely what it means for a partial partitioning set to satisfy a pattern. The following definition is illustrated in Figure 12.
**Definition 5 (Pattern satisfaction).** Let $S$ be a partial partitioning set for $G[U_x]$. A sink $s$ in $U_x \setminus V_x$ is bag-reachable in $G[U_x] \setminus S$ if some vertex in $V_x$ can reach $s$ in $G[U_x] \setminus S$. We define a canonical pattern $(\mathcal{R}_x(S), \mathcal{G}_x(S), \mathcal{P}_x(S))$ at $x$ for $S$, where

- $\mathcal{R}_x(S)$ is $G[V_x] \setminus S$,
- $\mathcal{G}_x(S)$ is the directed acyclic graph on the vertices $V_x \cup V'$, where $V'$ is the set of bag-reachable sinks in $G[U_x] \setminus S$, and there is an arc $(u,v)$ in $\mathcal{G}_x(S)$ if and only if the vertex $u$ can reach the vertex $v$ in $G[U_x] \setminus S$, and
- $\mathcal{P}_x(S)$ is the partition of the vertices of $\mathcal{G}_x(S)$ such that the vertices $u$ and $v$ are in the same set of $\mathcal{P}_x(S)$ if and only if they are in the same connected component of $G[U_x] \setminus S$.

Let $(\mathcal{R}, \mathcal{G}, \mathcal{P})$ be a pattern for $x$. We say that $S$ satisfies the pattern $(\mathcal{R}, \mathcal{G}, \mathcal{P})$ at $x$ if

i) $\mathcal{R} = \mathcal{R}_x(S)$,

ii) $\mathcal{G} = \mathcal{G}_x(S)$, and

iii) for each set $P \in \mathcal{P}_x(S)$ there exists a set $P' \in \mathcal{P}$ such that $P \subseteq P'$, that is, $\mathcal{P}$ is a coarsening of $\mathcal{P}_x(S)$.

It is easy to verify that a partial partitioning set $S$ for $G[U_x] \setminus S$ satisfies its canonical pattern $(\mathcal{R}_x(S), \mathcal{G}_x(S), \mathcal{P}_x(S))$ at node $x$: to this end, observe that $(\mathcal{R}_x(S), \mathcal{G}_x(S), \mathcal{P}_x(S))$ is indeed a pattern for $x$; for the vertex set $V_x \cup V'$ of $\mathcal{G}_x(S)$, we have $|V'| \leq |V_x|$ since each vertex in $V_x$ can reach at most one distinct sink in $V' \subseteq U_x \setminus V_x$ in $G[U_x] \setminus S$.

Note that, since $\mathcal{G}_x(S)$ contains an arc $(u,v)$ if and only if $u$ can reach $v$ instead of requiring them to be merely connected, a vertex is a sink in $\mathcal{G}_x(S)$ if and only if it is a sink in $G[U_x] \setminus S$. Herein, Definition 3(ii) ensures that any sink $s$ connected to a vertex in $V_x$ is a vertex in $\mathcal{G}_x(S)$.

While it might seem more natural to replace the condition (iii) in Definition 5 by simply $\mathcal{P} = \mathcal{P}_x(S)$, we prefer the current definition, because it allows for several connected components of $G[U_x] \setminus S$ becoming a part of one connected component of the entire graph. This greatly simplifies some parts of the algorithm.

**The Algorithm.** We now describe a dynamic programming algorithm. Starting from the leaves of the tree decomposition $T$ and working our way to its root, with each node $x$ of $T$, we associate a table $Tab_x$ that is indexed by all possible patterns for $x$. Semantically, we want that

$$Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P}) = \text{minimum weight of a partial partitioning set for } G[U_x] \text{ that satisfies the pattern } (\mathcal{R}, \mathcal{G}, \mathcal{P}) \text{ at } x.$$ 

Since we have $V_x = \emptyset$, there is exactly one pattern $(\mathcal{R}, \mathcal{G}, \mathcal{P})$ for the root $r$: $\mathcal{R} = \emptyset$ is the empty graph and $\mathcal{P} = \emptyset$. Thus, $Tab_x$ has exactly one entry and it contains the minimum weight of a partial partitioning set $S$ for $G[U_x]$, which is equivalent to $S$ being a partitioning set for $G$. It follows that once the tables are correctly filled, to decide the DAG PARTITIONING instance $(G,o,k)$, it is enough to test whether the only entry of $Tab_o$ is at most $k$.

We now present an algorithm to fill the tables and prove its correctness. First, we initialize all table entries of all tables by $\infty$. By updating the entry $Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P})$ with $m$ we mean setting $Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P}) := m$ if $m < Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P})$. For each leaf node $x$, it is obviously correct to set $Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P}) = 0$ for the only pattern $(\mathcal{R}, \mathcal{G}, \mathcal{P})$ at $x$, which has the empty graph as $\mathcal{R}$ and $\mathcal{G}$ and the empty set as $\mathcal{P}$. In the following, for each type of a node $x$ of a tree decomposition, that is, for forget nodes, introduce nodes, and join nodes, we independently show how to compute the table $Tab_x$ given that we correctly computed the tables for all children of $x$. To show that the table $Tab_x$ is filled correctly, we prove the following lemma for each node type.

**Lemma 6.**

(i) There is a partial partitioning set for $G[U_x]$ satisfying a pattern $(\mathcal{R}, \mathcal{G}, \mathcal{P})$ at $x$ with weight at most $Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P})$.

(ii) The minimum weight of a partial partitioning set for $G[U_x]$ satisfying a pattern $(\mathcal{R}, \mathcal{G}, \mathcal{P})$ at $x$ is at least $Tab_x(\mathcal{R}, \mathcal{G}, \mathcal{P})$.

We present the algorithm and the proof for Lemma 6 independently for each node type in Sections 4.2.1, 4.2.2, and 4.2.3, respectively, where we assume that all tables $Tab_y$ for child nodes $y$ of $x$ have been computed correctly.
4.2.1. Forget nodes

We use the following procedure to compute the table $\text{Tab}_y$ of a forget node $x$ under the assumption that the table $\text{Tab}_y$ for the child node $y$ of $x$ has been computed correctly.

**Procedure 1 (Forget node).** Let $x$ be a forget node with a single child $y$. Assume that $v$ is the vertex being “forgotten”, that is, $v$ is in the child bag $V_y$ but not in the current bag $V_x$. From the weights of optimal partial partitioning sets for $G[U_x]$, we want to compute the weight of optimal partial partitioning sets for $G[U_y]$.

To this end, for each pattern $\langle R, S, P \rangle$ for $y$, we distinguish four cases. In each case, we will construct a pattern $\langle R', S', P' \rangle$ for $x$ such that a partial partitioning set for $G[U_y]$ that satisfies $\langle R, S, P \rangle$ is a partial partitioning set for $G[U_x]$ and satisfies $\langle R', S', P' \rangle$. Then, we update $\text{Tab}_x(\langle R', S', P' \rangle)$ with the value of $\text{Tab}_y(\langle R, S, P \rangle)$. Herein, the following case distinction is not exhaustive. We do not take action for patterns $\langle R, S, P \rangle$ that do not satisfy any of the following conditions (for the reasons informally explained in the cases). In all cases, we set $R' := R - \{v\}$.

Case 1) If $v$ is isolated in $S$ and there is a set $\{v\}$ in $P$, then we let $S' := S - \{v\}$ and $P' := P \backslash \{\{v\}\}$ and update $\text{Tab}_x(\langle R', S', P' \rangle)$ with the value of $\text{Tab}_y(\langle R, S, P \rangle)$: an isolated vertex that is alone in its part of $P$ can simply be forgotten.

Case 2) If $v$ is a non-isolated sink in $S$ and $v \in P_1 \in P$ such that $P_1 \subseteq V_y$, then we let $S' := S$ and $P' := P$. We update $\text{Tab}_x(\langle R', S', P' \rangle)$ with the value of $\text{Tab}_y(\langle R, S, P \rangle)$; in this case, the sink $v$ “moves” from $V_y$ to $V(S') \backslash V_y$. To ensure that $\langle(R', S', P') \rangle$ is a pattern, the part $P_1$ containing $v$ cannot contain any additional sink in $V(S) \backslash V_y$, thus we require $P_1 \subseteq V_y$.

Case 3) If $v$ is not a sink in $S$ and there is no sink in $V(S) \backslash V_y$ such that $v$ is its only in-neighbor, then let $S' := S - \{v\}$ and $P'$ be the partition of the vertices of $S'$ obtained from $P$ by removing $v$ from the set it is in. Update $\text{Tab}_x(\langle R', S', P' \rangle)$ with the value of $\text{Tab}_y(\langle R, S, P \rangle)$. This the simplest case, where the vertex is somewhat unimportant to partial partitioning sets satisfying the pattern $\langle R, S, P \rangle$ at $y$, so we simply forget it.

Case 4) If there is a sink $u \in V(S) \backslash V_y$ such that $v$ is its only in-neighbor and $\{u, v\}$ is a set of $P$, then let $S' := S - \{u, v\}$ and $P'$ be the partition of the vertices of $S'$ obtained from $P$ by removing the set $\{u, v\}$. Update $\text{Tab}_x(\langle R', S', P' \rangle)$ with the value of $\text{Tab}_y(\langle R, S, P \rangle)$: If there was a sink $u$ in $V(S) \backslash V_y$ only reachable from $v$, then it would be unreachable from $V_y$ since $v$ is forgotten. Therefore, if the part $P_{1}$ of $P$ containing $u$ and $v$ contained more vertices, then we could not be sure that a partial partitioning set satisfying the pattern $\langle R, S, P \rangle$ at $y$ is a partial partitioning set for $G[U_x]$ at all. Namely, it may break Definition 3(ii).

We show that Procedure 1 fills the table $\text{Tab}_y$ associated with a forget node $x$ correctly. First, we show that there is a partial partitioning set for $G[U_y]$ satisfying a pattern $\langle R, S, P \rangle$ at $x$ and having weight at most $\text{Tab}_x(\langle R, S, P \rangle)$ as computed by Procedure 1.

**Proof (of Lemma 6(i) for forget nodes).** Let $x$ be a forget node with child node $y$ and let $v$ be the vertex “forgotten”, that is, $v$ is in the child bag $V_y$ but not in the current bag $V_x$. For any tableau entry $\text{Tab}_x(\langle R', S', P' \rangle) < \infty$, we show that there is a partial partitioning set $S$ for $G[U_y]$ satisfying $\langle R', S', P' \rangle$ and having weight at most $\text{Tab}_x(\langle R', S', P' \rangle)$. To this end, observe that, since $\text{Tab}_y(\langle R', S', P' \rangle) < \infty$ there is a pattern $\langle R, S, P \rangle$ for $y$ from which Procedure 1 generates $\langle R', S', P' \rangle$ and such that $\text{Tab}_y(\langle R', S', P' \rangle) = \text{Tab}_x(\langle R, S, P \rangle)$. Since there is a partitioning set for $G[U_y]$ that satisfies $\langle R, S, P \rangle$ and has weight at most $\text{Tab}_x(\langle R, S, P \rangle)$, it is sufficient to show that any partial partitioning set $S$ for $G[U_y]$ that satisfies the pattern $\langle R, S, P \rangle$ at $y$ is also a partial partitioning set for $G[U_x]$ that satisfies at $x$ the pattern $\langle R', S', P' \rangle$ generated in each of the cases (1)–(4) of Procedure 1.

We first argue that $S$ is a partial partitioning set for $G[U_y]$ if any of the cases (1)–(4) of Procedure 1 applies. We first verify Definition 3(i). To this end, observe that by Definition 5, a vertex $u \in V_y$ is a sink in $G[S]$ if and only if it is a sink in $G[U_x] \backslash S$. Now, assume that there is a connected component of $G[U_x] \backslash S = G[U_y] \backslash S$ that contains two different sinks $s_1, s_2$ in $U_x \backslash V_y$. Then, one of these sinks, say $s_1$, must be $v$. Since, then, $v \in S_y$ is a sink in $G[U_x] \backslash S = G[U_y] \backslash S$, it is a sink in $S = S_y(S)$ and none of the cases (3) and (4) apply. Moreover, since $s_2$ is connected to $v \in V_y$ in $G[U_x] \backslash S$, by Definition 3(ii), some vertex in $V_y$ can reach $s_2$, implying that $s_2$ is a vertex of $S = S_y(S)$. Thus, by Definition 5(iii), $s_2$ is in the same set $P_1 \in P$ as $s_1 = v$ and, hence, case (1) does not apply. Since $s_2 \notin V_y$, also (2) does not apply.

We now verify Definition 3(ii). It can only be violated if $v$ is the only vertex of $V_y$ that can reach some sink $u$ in the connected component of $v$ in $G[U_x] \backslash S$. However, then, $v$ is the only in-neighbor of $u$ in $S_y(S) = S$. Hence, only case (4) might become applicable. When this case applies, however, $\{u, v\} \in P$ implies that no vertex in $V_y \supseteq V_x$ is connected to $v$ or $u$ in $G[U_x] \backslash S = G[U_y] \backslash S$. Thus, Definition 3(ii) is satisfied.
It remains to show that $S$ satisfies the generated pattern $(R', S', P')$, that is, to verify $R' = R_1(S)$ (Definition 5(ii)), $S' = S_1(S)$ (Definition 5(ii)) and that $P'$ is a coarsening of $P_1(S)$ (Definition 5(iii)). Herein, $R' = R - \{v\} = R_1(S) - \{v\} = R_1(S)$ is trivial. To show $S' = S_1(S)$, we distinguish between the case of Procedure 1 applied.

Case 1) In this case, $v$ is not in $V_r$ and, obviously, not a bag-reachable sink in $G[U_r] \setminus S$. Hence, $v$ is not in $S_1(S)$. Moreover, $v$ is isolated in $G \setminus S$. Therefore, Procedure 1 sets $S' := S = S_1(S) - \{v\} = S_1(S)$.

Case 2) In this case, $v$ is not in $V_r$ but it is a bag-reachable sink in $G[U_r] \setminus S$, since it is not isolated in $S$. Therefore, Procedure 1 sets $S' := S = S_1(S) - \{v\} = S_1(S)$.

Case 3) In this case, $v$ is not a sink in $S$ (and thus also not in $G[U_r] \setminus S$) and, therefore, clearly does not appear in $S_1(S)$. Moreover, any sink not in $V_r$ is reachable from a vertex in $V_r \setminus S = V_r$ in $S_1(S)$ if and only if it is reachable in $S_1(S) - \{v\}$. Hence, Procedure 1 sets $S' := S = S_1(S) - \{v\} = S_1(S)$.

Case 4) In this case, $u$ was a bag-reachable sink in $G[U_r] \setminus S$ but is not bag-reachable in $G[U_r] \setminus S$. Moreover, since $\{v, u\} \in P$, no vertex of $V_r$ is connected to $v$ or $u$ in $G[U_r] \setminus S = G[U_r] \setminus S$. Hence, neither $v$ nor $u$ are vertices of $S_1(S)$. Hence, Procedure 1 sets $S' := S = S_1(S) - \{v, u\} = S_1(S)$.

Finally, we verify Definition 5(iii) by showing that $P'$ is a coarsening of $P_1(S)$. Assume the contrary, then, there are two vertices $u, w$ of $S_1(S)$ in the same set of $P_1(S)$ but in different sets of $P'$. By construction of $P'$ from $P$, they are also in different sets of $P' = P_1(S)$. It follows that $u$ and $w$ lie in the same connected component of $G[U_r] \setminus S$ but in different connected components of $G[U_r] \setminus S$. Since these two graphs are the same, we have a contradiction.

We now show that the minimum weight of a partial partitioning set for $G[U_r]$ satisfying a pattern $(R, S, P)$ at $x$ is at least $Tab_2(R, S, P)$ as computed by Procedure 1.

**Proof (of Lemma 6(ii) for forget nodes).** Let $x$ be a forget node with child node $y$. Let $v$ be the vertex “forgotten” that is, $v \in V_r$ but $v \notin V_s$. Assume that $S$ is a partial partitioning set for $G[U_r]$ satisfying the pattern $(R, S, P)$ at $x$. It is sufficient to construct a pattern $(R, S, P')$ that $S$ satisfies at $y$ and from which Procedure 1 generates exactly the pattern $(R, S, P, P')$ to update the table $Tab_2(R, S, P')$ with $Tab_2(R, S, P)$. Then, Lemma 6(ii) follows for forget nodes, because we have $Tab_2(R, S, P) \leq Tab_2(R, S, P') \leq \omega(S)$. Herein, the last inequality follows from the induction hypothesis.

We first show that $S$ is a partial partitioning set for $G[U_r]$, that is, we verify Definition 3. Definition 3(i) is easy to verify: since $S$ is a partial partitioning set for $G[U_r]$, each connected component of $G[U_r] \setminus S = G[U_r] \setminus S$ contains at most one sink in $U_r \setminus S \supseteq U_r \setminus V_r$. It remains to verify Definition 3(ii). Assume, for a contradiction, that there is a connected component $C$ in $G[U_r] \setminus S$ that contains a vertex of $V_r$ such that no vertex of $V_r \supseteq V_r$ can reach some sink $s \in C \setminus V_r$. Then, since $S$ is a partial partitioning set for $G[U_r] \setminus S$, the connected component $C$ cannot contain vertices of $V_r$ and, hence, $C \cap V_r = \{v\}$. However, since $G[U_r] \setminus S$ is a directed acyclic graph, $v$ reaches some sink in $C$. Since $v$ cannot reach $s \in C$, it follows that $C$ contains two sinks. Since $C \cap V_r = \emptyset$, this contradicts $S$ being a partial partitioning set for $G[U_r] \setminus S$. It follows that $S$ is a partial partitioning set for $G[U_r]$.

We now construct a pattern $(R, S, P)$ that $S$ satisfies at $y$. Consider $R := R_1(S)$ and $S := S_1(S)$. Note that there are at most two vertices in $S$ that are not in $S_1$: one of them is $v$, the possibly other vertex is a sink $u$ in $V(S) \setminus V_r$ only reachable from $v$. We define $P$ as a partition of the vertices of $S$ that partitions the set $V(S_1)$ in the same way as $P$. We add the possibly missing vertices $v$ and $u$ to that partition as follows: if there is a vertex $w \in V_r$ in the same connected component of $G[U_r] \setminus S = G[U_r] \setminus S$ as $v$, then we put $u$ and $v$ into same set as $w$. Otherwise, we add the set $\{v\}$ or $\{u, v\}$, respectively, to $P$. By choice of $R, S$, and $P$, the partial partitioning set $S$ clearly satisfies the pattern $(R, S, P)$ at $y$.

We have shown that $S$ satisfies the pattern $(R, S, P)$ at $y$. Moreover, if any of the cases (1)–(4) of Procedure 1 applies to $(R, S, P)$, then it generates a pattern $(R', S', P')$ with $R' = R_1$, and $S' = S_1$, since we showed in the proof of Lemma 6(i) for forget nodes that $S$ satisfies the pattern generated by Procedure 1 at $x$. Hence, it remains to show that indeed at least one of the cases (1)–(4) of Procedure 1 applies and that in all cases $P' = P$.

Case 1) If $v$ is an isolated sink in $S$, then no vertex in $V_r \supseteq V_r$ can reach $v$ in $G[U_r] \setminus S = G[U_r] \setminus S$. Hence, there is no vertex of $V_r$ in the same connected component of $G[U_r] \setminus S$ as $v$, as otherwise $v$ would be a sink in $U_r \setminus V_r$ not reachable from the vertices of $V_r$. Hence, by construction of $P$, we have $\{v\} \in P$ and case (1) of Procedure 1 applies. It sets $P' := P' \setminus \{v\} = P$.

Case 2) If $v$ is a non-isolated sink in $S$, then $v$ is a bag-reachable sink in $G[U_r] \setminus S$. Hence, it is contained in $S_1$ and we have $V(S_1) = V(S)$. By construction of $P$, we also have $P_s = P$. The set $P_i \in P$ containing $v \notin V_r$ cannot
contain any other vertex in \(V(\mathcal{S}_i) \setminus V_i\) by Definition 4(iii). Thus, \(P_i \subseteq V_i\) and case (2) of Procedure 1 applies. It sets \(P' := P = P_i\).

Case 3) If \(v\) is not a sink in \(\mathcal{S}\) and there is no sink in \(V(\mathcal{S}) \setminus V_i\) only reachable from \(v\) in \(\mathcal{S}\), then case (3) of Procedure 1 applies. Since the sink \(s\) reachable from \(v\) is also reachable from some vertex \(u \notin \{v, s\}\), and thus, connected to \(u\) in \(G[U_i] \setminus S\), the set in \(P\) containing \(v\) also contains \(u\). Procedure 1 sets \(P'\) to be \(P\) with \(v\) removed from the set it is in. This, by construction of \(P\), is exactly \(P\).

Case 4) Finally, if there is a sink \(u\) in \(V(\mathcal{S}) \setminus V_i\) only reachable from \(v\), then the connected component of \(G[U_i] \setminus S = G[U_i] \setminus S\) containing the vertex \(v\) does not contain any vertex of \(V_i\), since \(u\) is not reachable from any vertex of \(V_i\). It follows that \(\{u, v\}\) is a set of \(P\). Case (4) of Procedure 1 applies. It sets \(P' := P \setminus \{\{v, u\}\} = P\). \(\square\)

4.2.2. Introduce nodes

We use the following procedure to compute the table Tab\(_y\) of an introduce node \(x\) under the assumption that the table Tab\(_x\) for the child node \(y\) of \(x\) has been computed correctly.

Procedure 2 (Introduce node). Let \(x\) be an introduce node with a single child \(v\). Assume that \(v\) is the node being “introduced”, that is, \(v\) is not in the child bag \(V_i\) but in the current bag \(V_i\). Moreover, let \(B \subseteq A(G[U_i])\) be the set of arcs incident to \(v\). By Definition 2(ii and iii) of a tree decomposition, one actually has \(B \subseteq A(G[V_i])\).

We now try each possible subset \(B' \subseteq B\) and consider it not deleted by a partial partitioning set for the graph \(G[U_i]\). Similarly as in the case for forget nodes, we will transform each pattern \((\mathcal{R}, \mathcal{S}, P)\) for \(y\) into a pattern \((\mathcal{R}', \mathcal{S}', P')\) for \(x\) such that if a partial partitioning set \(S\) for \(G[U_i]\) satisfies \((\mathcal{R}, \mathcal{S}, P)\), then \(S \cup (B \setminus B')\) is a partial partitioning set for \(G[U_i]\) and satisfies \((\mathcal{R}', \mathcal{S}', P')\). Then, we update \(\text{Tab}_x(\mathcal{R}', \mathcal{S}', P')\) with the value of \(\text{Tab}_x(\mathcal{R}, \mathcal{S}, P) + \omega(B) - \omega(B')\).

For each pattern \((\mathcal{R}, \mathcal{S}, P)\) for \(y\) such that all vertices incident to the arcs in \(B'\) (if any) except for \(v\) are contained in the same set \(P_i \in P\), we obtain \(\mathcal{R}'\) from \(\mathcal{X}\) by adding \(v\) and the arcs in \(B'\) to \(\mathcal{X}\). Similarly, we obtain \(\mathcal{S}'\) from \(\mathcal{S}\) by adding \(v\) and the arcs in \(B'\) to \(\mathcal{S}\). Moreover, for each \(u, w \in V(\mathcal{S})\) such that \(u\) can reach \(w\) in \(\mathcal{S}'\) we add the arc \((u, w)\) to \(\mathcal{S}'\). For obtaining \(P'\), we distinguish two cases.

Case 1) If \(B' = \emptyset\), then we try all possibilities of adding \(v\) to a set in \(P\). That is, for every \(P_i \in P\), we get a set \(P'\) from \(P\) by adding \(v\) to \(P_i\) and update \(\text{Tab}_x(\mathcal{R}', \mathcal{S}', P')\) with \(\text{Tab}_x(\mathcal{R}, \mathcal{S}, P_i) + \omega(B)\). Additionally, for \(P' := P \cup \{\{v\}\}\), we update the entry \(\text{Tab}_x(\mathcal{R}', \mathcal{S}', P')\) with \(\text{Tab}_x(\mathcal{R}, \mathcal{S}, P_i) + \omega(B)\).

Case 2) If \(B' \neq \emptyset\), then let \(P_i\) be the set of \(P\) that contains all vertices incident to arcs in \(B'\) except \(v\) and let \(P'\) be obtained from \(P\) by adding \(v\) to the set \(P_i\). We update \(\text{Tab}_x(\mathcal{R}', \mathcal{S}', P')\) with \(\text{Tab}_x(\mathcal{R}, \mathcal{S}, P_i) + \omega(B) - \omega(B')\).

Note that, since \(P'\) simulates the connected components of the resulting graph, all arcs incident on \(v\) remaining in the graph must be within one set of \(P\), i.e., their endpoints different from \(v\) must be in one set of \(P\).

We show that Procedure 2 fills the table associated with an introduce node \(x\) correctly. First, we show that there is a partial partitioning set for \(G[U_i]\) satisfying a pattern \((\mathcal{R}, \mathcal{S}, P)\) at \(x\) and having weight at most \(\text{Tab}_x(\mathcal{R}, \mathcal{S}, P)\) as computed by Procedure 2.

**Proof (of Lemma 6(i) for introduce nodes).** Let \(x\) be an introduce node with child node \(y\) and let \(v\) be the vertex “introduced” that is, \(v\) is not in the child bag \(V_i\) but in the current bag \(V_i\). Let \(B \subseteq A(G[U_i])\) be the arcs incident to \(v\), \(B' \subseteq B\) and, finally, \((\mathcal{R}, \mathcal{S}, P)\) be some pattern for \(y\) such that all vertices incident to the arcs in \(B'\) (if any) except for \(v\) are contained in the same set \(P_i \in P\). For any partial partitioning set \(S\) for \(G[U_i]\) satisfying the pattern \((\mathcal{R}, \mathcal{S}, P)\) at \(y\), we show that \(S' = S \cup (B \setminus B')\) is a partial partitioning set for \(G[U_i]\) that satisfies the pattern \((\mathcal{R}', \mathcal{S}', P')\) constructed by Procedure 2. From this, since \(\alpha(S') = \alpha(S) + \alpha(B) - \alpha(B')\), Lemma 6(i) follows (as already discussed in the beginning of the proof of Lemma 6(i) for forget nodes).

We start by showing that \(S'\) is a partial partitioning set for \(G[U_i]\). First, we verify Definition 3(i). For the sake of a contradiction, assume that there is a connected component of \(G[U_i] \setminus S\) that contains two distinct sinks \(s_1, s_2\) in \(U_i \setminus V_i\). Since there is no such connected component in \(G[U_i] \setminus S = (G[U_i] \setminus S) - \{v\}\), there are vertices \(s'_1, s'_2 \in V_i\) in the same connected components of \(G[U_i] \setminus S\) as \(s_1\) and \(s_2\), respectively, that are incident to some arcs in \(B'\). By Definition 3(ii), there are vertices in \(V_i\) that can reach \(s_1\) and \(s_2\) in \(G[U_i] \setminus S\). Hence, \(s_1\) and \(s_2\) are bag-reachable and, therefore, in \(\mathcal{S} = S_y(S)\). Since \(s'_1, s'_2 \in P_i\) and \(S\) satisfies \((\mathcal{R}, \mathcal{S}, P)\), by Definition 4(iii), we also have \(s_1, s_2 \in P\). Then, however, \(P_i\) contains the two different vertices \(s_1\) and \(s_2\) of \(V(\mathcal{S}) \setminus V_i\), which contradictions Definition 4(iii).
To show that $S'$ is a partial partitioning set, it remains to verify Definition 3(ii). For the sake of contradiction, assume that some connected component of $G[U_1] \setminus S'$ contains some sink $s \in U_1 \setminus V_s$, some vertex in $V_s$, but $s$ is not reachable from any vertex in $V_s$. Then, $s$ is not reachable from any vertex in $V_s \subseteq V_s$ in the subgraph $G[U_1] \setminus S'$ either. Thus, the connected component does not contain any vertex of $V_s$ and, therefore, not of $V_s$, since the only vertex in $V_s \setminus V_s$ is $s$ and the added arcs $B'$ connect only vertices in $V_s$.

We have shown that $S'$ is a partial partitioning set for $G[U_1]$. We now show that it satisfies the pattern $(K', S', P')$ generated by Procedure 2; we verify Definition 5. Definition 5(i), that is, $K' = R'_h(S')$ is trivial by the construction of $K'$.

We verify Definition 5(ii), that is, $S'_h(S') = S'$. First, observe that $V(S'_h(S')) = V(S_h(S)) \cup \{v\} = V(S')$. We have to show that there is an arc $(u, w)$ in $S'$ if and only if $u$ can reach $w$ in $G[U_1] \setminus S'$. Let $(u, w)$ be such an arc in $S'$.

If $(u, w)$ is already in $\delta$, then $u$ can reach $w$ in $G[U_1] \setminus S = (G[U_1] \setminus S') - \{v\}$. Otherwise, $u$ reaches $w$ in $G[U_1] \setminus S'$ via some arcs $(u', v), (v, w') \in B'$, that is, $u$ can reach $u'$ and $w'$ can reach $w$ in $G[U_1] \setminus S'$. It follows that $u$ can reach $w$ in $G[U_1] \setminus S'$. Now, for the opposite direction, let $u, w$ be vertices of $S'$ such that $u$ can reach $w$ in $G[U_1] \setminus S'$. If $u$ can reach $w$ in $G[U_1] \setminus S' - \{v\} = G[U_1] \setminus S$, then the arc $(u, w)$ is already present in $\delta$. Otherwise, $u$ reaches $w$ via some arcs $(u', v), (v, w') \in B'$.

The arcs $(u', v), v, w'$ are in $\delta$, since $u', w' \in V_s$. Moreover, $u$ reaches $w' \in \delta$ and, hence, reaches $\delta$ $\delta$ via $u'$ and $w'$. By construction of $S'$, it follows that $S'$ contains the arc $(u, w)$.

Finally, we verify Definition 5(iii); we show that $P'$ is a coarsening of $P_h(S')$. For the sake of a contradiction, assume that there are two vertices $u, w$ that are in the same set of $P_h(S')$ but in different sets of $P'$. By construction of $P'$ from $P$, this implies that $u$ and $w$ are in different sets of $P$ and, therefore, in different connected components of $G[U_1] \setminus S$. Thus, in order for $u$ and $w$ to be connected in $G[U_1] \setminus S'$, there are vertices $u', w'$ in the same connected components of $G[U_1] \setminus S$ as $u$ and $w$, respectively, that are incident to arcs in $B'$ and, hence, $u', w' \in P \subseteq P'$. But then, also $u, w \in P \subseteq P$ — a contradiction.

We now show that the minimum weight of a partial partitioning set for $G[U_1]$ satisfying a pattern $(R, S, P)$ at $x$ is at least $\text{Tab} _ h (R, S, P)$ as computed by by Procedure 2.

**Proof (of Lemma 6(ii) for Introduce nodes).** Let $x$ be an introduce node with child node $y$. Let $v$ be the vertex “introduced”, that is, $v \not\in V_s$ but $v \in V_s$. Assume that $S$ is a minimum-weight partial partitioning set for $G[U_1]$ satisfying the pattern $(R, S, P)$ at $x$. Let $B$ be the set of arcs incident to $v$ in $G[U_1]$ and $B'' := B \cap S$. It is sufficient to construct a pattern $(R, S, P)$ that $S \setminus B''$ satisfies at $y$ and from which Procedure 2 generates exactly the pattern $(R, S, P)$ to update the table $\text{Tab} _ h (R, S, P)$ with $\text{Tab} _ h (R, S, P) + \alpha (B) - \alpha (B'')$. Then, Lemma 6(ii) follows for introduce nodes, since $\text{Tab} _ h (R, S, P) = \text{Tab} _ h (R, S, P) + \alpha (B) - \alpha (B'') \leq \alpha (S \setminus B') + \alpha (B) - \alpha (B'') = \alpha (S)$.

It is easy to verify that $S \setminus B''$ is a partial partitioning set for $G[U_1]$ (Definition 3), since $G[U_1] \setminus (S \setminus B'') = (G[U_1] \setminus S) \setminus \{v\}$ and $S$ is a partial partitioning set for $G[U_1]$; to this end, observe that, by Definition 2(ii and iii) of a tree decomposition, $\delta$ only has arcs $B \subseteq A(G[V_s])$ incident to vertices in $V_s$.

We now construct a pattern. Let $R = R_h(S \setminus B')$ and $S = S_h(S \setminus B')$. Let $P$ be the partition obtained from $P_h$ by removing the vertex $v$ from the set it is in or by removing the set $\{v\}$ if it exists in $P$. It is easy to verify that $S \setminus B''$ satisfies $(R, S, P)$ at $y$; Definition 5(ii) and (iii) are trivially satisfied by choice of $R$ and $S$; Definition 5(iii) holds by construction of $P$ from $P_h$, since $G[U_1] \setminus (S \setminus B'')$ is a subgraph of $G[U_1] \setminus S$.

It remains to show that Procedure 2 applies to the pattern $(R, P, S)$ and the set $B''$ in order to generate the pattern $(R, P_h, S)$ at $x$. Since $S$ satisfies $(R, S, P)$ at $x$, all vertices incident to arcs in $B''$ (if any) are contained in the same set $P$. Let $P \subseteq P_h$ and, hence, all of them except $v$ are contained in the set $P \setminus \{v\}$. Therefore, Procedure 2 applies to $B''$ and the pattern $(R, P_h, S)$, produces some new pattern $(R', S', P')$, and updates $\text{Tab} _ h (R, P, S)$ with $\text{Tab} _ h (R, P_h, S) + \alpha (B) - \alpha (B')$.

It remains to show that, for at least one of the generated patterns, $R' = R_h$, $S' = S_h$, and $P' = P_h$. If $B' \neq 0$ then $P' = P_h$ by construction of $P'$ from $P$ in Procedure 2. If $B' = \emptyset$, then $P_h$ is clearly among the partitions $P'$ generated from $P$ by Procedure 2. Moreover, we already proved in the proof of Lemma 6(i) that $S = (S \setminus B'') \cup (B \setminus B')$ satisfies the pattern generated by Procedure 2 at $x$. Hence, $S' = S_h$ and $S' = S_h$.

4.2.3. Join nodes

We use the following procedure to compute the table $\text{Tab} _ h$ of a join node $x$ under the assumption that the tables $\text{Tab} _ y$ for all child nodes $y$ of $x$ have been computed correctly.
Procedure 3 (Join node). Let $x$ be a join node with children $y$ and $z$, that is, $V_x = V_y = V_z$. For each pair of patterns $(\mathscr{R}, S_y, P_y)$ for $y$ and $(\mathscr{R}, S_z, P_z)$ for $z$ such that $P_y$ and $P_z$ partition the vertices of $V_y = V_z = V_x$ in the same way, we construct a new pattern $(\mathscr{R}, S_x, P)$ as follows.

Let $G'$ be the graph containing all vertices and arcs of $S_y$ and $S_z$, and for each $u, w \in V(G')$ such that $u$ can reach $w$ in $G'$ add the arc $(u, w)$ to $G'$. Note that by Definition 2(iii) of a tree decomposition, $S_y$ and $S_z$ have only the vertices in $V_x$ in common.

Let $P'$ be the partition of $V_x$ that partitions $V_y$ in the same way as $P_y$ and $P_z$. We extend $P'$ to a partition for the vertices of $G'$: for each $u \in V(G') \setminus V_x$ add $u$ to a set $P'$ that contains a vertex $v$ with $(v, u)$ being an arc of $G'$. Since there are no arcs between different sets of $P$ in $S_y$ or $S_z$, there is exactly one such set $P_i \in P'$.

If we created some set $P \in P'$ with more than one vertex of $V(G') \setminus V_x$ then continue with a different pair of patterns. Otherwise, we update $\text{Tab}_x(\mathscr{R}, S_y, P_y) + \text{Tab}_x(\mathscr{R}, S_z, P_z) - \omega(A(G[V_x])) + \omega(A(R))$.

We show that Procedure 3 fills the table associated with a join node $x$ correctly. First, we show that there is a partial partitioning set for $G[U_x]$ satisfying a pattern $(\mathscr{R}, S, P)$ at $x$ and having weight at most $\text{Tab}_x(\mathscr{R}, S, P)$ as computed by Procedure 3.

Proof (of Lemma 6(i) for Join Nodes). Let $x$ be a join node with child nodes $y$ and $z$, that is, $V_x = V_y = V_z$. Let $S_y$ be a partial partitioning set for $G[U_y]$ satisfying the pattern $(\mathscr{R}, S_y, P_y)$ at $y$ and let $S_z$ be a partial partitioning set for $G[U_z]$ satisfying the pattern $(\mathscr{R}, S_z, P_z)$ at $z$. We show that $S = S_y \cup S_z$ is a partial partitioning set for $G[U_x]$ that satisfies the pattern $(\mathscr{R}, S', P')$ constructed by Procedure 3. Since $\omega(S) = \omega(S_y) + \omega(S_z) - \omega(S_y \cap S_z)$, wherein $S_y \cdot S_z = A(G[V_x]) \setminus A(R)$, Lemma 6(i) follows for join nodes.

We show that $S$ is indeed a partial partitioning set for $G[U_x]$, that is, we verify Definition 3. We first verify Definition 3(ii) and then use it to verify Definition 3(i). Let $s \in U_x \setminus V_x$ be a sink such that the connected component containing $s$ in $G[U_x] \setminus S$ contains a vertex of $V_x$. Then, $s \in U_y \setminus V_y$ or $s \in U_z \setminus V_z$. Without loss of generality, let $s \in U_y \setminus V_y$. From Definition 2(ii) of a tree decomposition, we see that $U_y \cap U_z \subseteq V_x$ and, hence, $G[U_x] \setminus S_y = G[U_x] \setminus S - (U_x \setminus V_x)$. It follows that there is also a connected component of $G[U_x] \setminus S_y = G[U_x] \setminus S - (U_z \setminus V_z)$ that contains $s$ and a vertex of $V_z$ and, therefore, $s$ is reachable from some vertex $v \in V_x$ in $G[U_x] \setminus S - (U_x \setminus V_x)$ and, hence, in $G[U_x] \setminus S$. It also follows that $(v, s)$ is an arc in $S_y$ and, by construction in Procedure 3, of $G'$.

To verify Definition 3(i) for the sake of a contradiction, assume that there is a connected component of $G[U_x] \setminus S$ that contains two sinks $s_1, s_2$ in $U_x \setminus V_x$. Note that, by Definition 2(iii) of a tree decomposition, there are no arcs between $U_y \setminus V_y$ and $U_z \setminus V_z$. Hence, this connected component contains a vertex $v$ of $V_x$; otherwise, it would be a connected component with two sinks outside of $V_x$ already in either $G[U_y] \setminus S_y$ or $G[U_z] \setminus S_z$. Thus, as seen in the previous paragraph, we have arcs $(s_1', s_1)$ and $(s_2', s_2)$ with $s_1, s_2 \in V_x$. It follows by construction of $P'$ from $G'$ in Procedure 3 that $s_1$ and $s_2$ are in a set $P_1 \in P'$ and $s_1$ and $s_2$ are in a set $P_2 \in P'$. We show $i = j$, which contradicts the construction of $P'$, since then $P_1 = P_2$ contains two vertices $s_1 \notin V_x$ and $s_2 \notin V_x$.

Since $s_1$ and $s_2$ are in the same connected component of $G[U_x] \setminus S$, also $s_1'$ and $s_2'$ are, since they can reach $s_1$ and $s_2$, respectively. Hence, there is an undirected path $p$ between $s_1$ and $s_2$ in $G[U_x] \setminus S$. It consists of consecutive path segments $p'$ that only have their endpoints $u, w$ in $V_x$ (possibly, such a path segment only consists of one arc). It follows that such a path segment $p'$ is entirely contained in $G[U_x] \setminus S_y$ or $G[U_x] \setminus S_z$ and, hence, its endpoints $u$ and $w$ are in the same set of $P_1$ or $P_2$. Since $u, w \in V_x$, by construction of $P'$ in Procedure 3, $u$ and $w$ are in the same set of $P'$. It follows that $s_1'$ and $s_2'$ are in the same set of $P'$, and so are $s_1$ and $s_2$.

It follows that $S$ is indeed a partial partitioning set for $G[U_x] \setminus S$. It remains to verify that $S$ satisfies the pattern $(\mathscr{R}, S', P')$ (Definition 5). Herein, Definition 5(i), $\mathscr{R} = \mathscr{R}(S)$, is trivial. We verify (ii), that is, $\mathcal{I}_x(S) = S'$. Herein, $V(\mathcal{I}_x(S)) \subseteq V(S')$ we already verified when verifying Definition 3(ii). Now, assume that there are two vertices $u, w$ in $S'$ such that $u$ can reach $w$ in $G[U_x] \setminus S$. Since, then, $u$ is not a sink, it is in $V_x$. The directed path from $u$ to $w$ consists of consecutive subpaths, each being entirely contained in $G[U_x] \setminus S_y$ or $G[U_x] \setminus S_z$ and, thus, causing an arc in $S_y$ or $S_z$ and, therefore, in $S'$. It follows that $u$ can reach $w$ in $S'$, which therefore has an arc $(u, w)$. In the opposite direction, for every arc $(u, w)$ in $S'$ that is already in $S_y$ or $S_z$, there is an directed path in either $G[U_x] \setminus S_y$ or $G[U_x] \setminus S_z$ from $u$ to $w$ and, thus, $u$ can reach $w$ in $G[U_x] \setminus S$. For an arc $(u, w)$ in $S'$ that is neither present in $S_y$ nor $S_z$, there is an directed path in $S'$ from $u$ to $w$ consisting only of arcs that are already present in $S_y$ or $S_z$. Since we have seen that for each such arc there is a corresponding directed path in $G[U_x] \setminus S$, we have that $u$ can reach $w$ in $G[U_x] \setminus S$.

For Definition 5(iii), it has been shown above that if two vertices of $S'$ are in the same connected component of $G[U_x] \setminus S$, then they are in the same set in $P'$.
We now show that the minimum weight of a partial partitioning set for \(G[U_x]\) satisfying a pattern \((R,S,P)\) at \(x\) is at least \(Tab_x(R,S,P)\) as computed by Procedure 3.

**Proof (of Lemma 6(ii) for join nodes).** Let \(x\) be a join node with the child nodes \(y\) and \(z\), that is \(V_x = V_y = V_z\). Assume that \(S_y\) is a minimum-weight partial partitioning set for \(G[U_x] \setminus S_y\) satisfying the pattern \((R,S_y,P_y)\) at \(y\). It is sufficient to construct patterns \((R,S_z,P_z)\) and \((R,S_y,P_y)\) that are satisfied by \(S_y := S \cap A(G[U_x])\) at \(y\) and by \(S_z := S \cap A(G[U_z])\) at \(z\), respectively, such that from these patterns Procedure 3 generates exactly the pattern \((R,S_x,P_x)\) to update \(Tab_x(R,S_x,P_x)\) with

\[
Tab_x(R,S_x,P_x) + Tab_y(R,S_y,P_y) - \omega(A(G[V_y])) + \omega(A(R))
\]

\[
\leq \omega(S_y) + \omega(S_z) - \omega(S_y \cap S_z)
\]

\[= \omega(S_x).\]

We first show that \(S_y\) is a partial partitioning set for \(G[U_x] \setminus S_y\). Symmetrically, it follows that \(S_z\) is a partial partitioning set for \(G[U_z] \setminus S_z\). We first verify Definition 3(ii). Since Definition 2(iii), there are no arcs between vertices in \(U_y \setminus V_y\) and \(U_z \setminus V_z\) in \(G[U_x]\), it follows from \(G[U_x] \setminus S_y = G[U_x] \setminus S - (U_y \setminus V_y)\) that no connected component of \(G[U_x] \setminus S_y\) contains two sinks not in \(V_y = V_z\). It remains to verify Definition 3(ii). To this end, let \(u\) be a sink in \(U_y \setminus V_y\) in a connected component of \(G[U_x] \setminus S_y\) containing a vertex of \(V_y\). Then, by Definition 3(ii), the connected component of \(G[U_x] \setminus S_y\) containing \(u\) contains a directed path from some vertex in \(V_y\) to \(u\). The subpath of this directed path that contains only one vertex of \(V_y\) is preserved in \(G[U_z] \setminus S_z\). Hence, \(u\) is reachable from some vertex of \(V_y = V_z\).

We now construct the patterns. To this end, let \(S_y := S_y \setminus S_z\) and \(S_z := S_z \setminus S_y\). Moreover, we choose \(P_y\) and \(P_z\) such that they partition the set \(V_x = V_y = V_z\) in the same way as \(P_y\) and such that the vertices of \(V(S_y \setminus V_x)\) (or \(V(S_z \setminus V_x)\)) are in the same set as the other vertices of their connected components in \(G[U_x] \setminus S_y\) (or \(G[U_z] \setminus S_z\)).

We show that \(S_y\) satisfies \((R,S_y,P_y)\) at \(y\). Analogously, it then follows that \(S_z\) satisfies \((R,S_z,P_z)\) at \(z\). We verify Definition 5. Since \(R = R_y \cup R_z = R_y \cup (R_z \setminus S_z)\) and \(S_y = S_y \setminus S_z\) hold by definition, it remains to verify Definition 5(iii). To this end, observe that \(G[U_x] \setminus S_y = G[U_x] \setminus S - (U_y \setminus V_y)\). Now, assume, for the sake of a contradiction, that there are two vertices \(v,w\) of \(S_y\) in different sets of \(P_y\) but in the same connected component of \(G[U_x] \setminus S_y\). It follows that \(v\) and \(w\) are in the same connected component of \(G[U_x] \setminus S\). If \(v,w \in V_y\), then, by construction of \(P_y\) from \(P_y\), the vertices \(v\) and \(w\) are in different sets of \(P_y\), contradicting \(S_y\) satisfying \((R,S_y,P_y)\). If exactly one of \(v,w \notin V_y\), then \(v\) and \(w\) being in different sets of \(P_y\), contradicts the construction of \(P_y\). If both \(v,w \notin V_y\), then \(v\) and \(w\) are two bag-reachable sinks in \(G[U_x] \setminus S_y\), which contradicts \(v\) and \(w\) being in the same connected component of \(G[U_x] \setminus S_y\).

Hence, indeed \(S_y\) satisfies \((R,S_y,P_y)\) at \(y\) and \(S_z\) satisfies \((R,S_z,P_z)\) at \(z\). Moreover, since \(P_y\) and \(P_z\) partition \(V_x\) in the same way, Procedure 3 applies to the patterns \((R,S_y,P_y)\) and \((R,S_z,P_z)\) and produces a pattern \((R',S',P')\). If no set of \(P'_y\) contains more than one vertex of \(V(S'_y) \setminus V_x\), it indeed updates \(Tab_x(R,S',P')\).

Hence it remains to show that \(S'_y = S_y\) and \(P'_y = P_y\), as no set of \(P_y\) contains two vertices of \(V(S_y) \setminus V_x\) by Definition 4(iii). We already showed in the proof of Lemma 6(ii) for join nodes that \(S_y\) satisfies the pattern \((R',S'_y,P'_y)\) generated by Procedure 3. Hence, \(S'_y = S_y\). Finally, by construction of \(P'_y\) in Procedure 3, the vertices of \(V_x\) are partitioned the same way by \(P'_y\) and \(P_y\). For a vertex \(v \in V(S'_y) \setminus V_x\), there is a vertex \(u\) in \(V_x\) that can reach \(v\) in \(S'_y\) and, therefore, in \(G[U_x] \setminus S_y\). Hence, \(u\) and \(v\) must be in the same set of both \(P_y\) and \(P'_y\) by construction of \(P'_y\) in Procedure 3.

**4.2.4. Running time**

Having shown the correctness of the Procedures 1–3, we can finally complete the proof of Theorem 6 by analyzing the running time of the procedures.

**Proof (of Theorem 6).** Lemma 6 showed that the presented dynamic programming algorithm is correct, that is, it solves DAG PARTITIONING given a tree decomposition of the input graph. It remains to analyze the running time.

To this end, recall that each bag in a tree decomposition of width \(t\) contains at most \(t + 1\) vertices. This allows us to give an upper bound on the number of possible patterns \((R,S,P)\). There are at most \(3(t+1)\) directed acyclic graphs \(R\) on at most \(t + 1\) vertices: for each pair \((v,w)\) of vertices: there is either no arc, or an arc from \(v\) to \(w\), or an arc from \(w\) to \(v\). Similarly, there are at most \(3(t+1)^2\) directed graphs \(S\) on at most \(2t + 2\) vertices. Moreover, there are at most \((2t + 2)^{t+2}\) partitions \(P\) of at most \(2t + 2\) vertices into at most \(2t + 2\) sets. Hence, each table has at most

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\[3^{(2^{t+2}) \cdot 3^{(t+1)}} \cdot (2t + 2)^{2^{t+2}} = 3^{O(t^2 + t \log t)} = 2^{O(t^2)}\] entries and looking up entries in the tables can be implemented to run in \(O(\log 2^t)\) time, which is polynomial in \(t\).

In each leaf node, Procedure 1 iterates over the entries of the table of the child node and for each entry spends time polynomial in \(t\). Thus, it spends a total of \(2^{O(t^2)}\) time in each forget node.

To analyze the running time Procedure 2 spends in an introduce node, observe that there are at most \(t\) arcs in \(A(G[V_i])\) incident to the introduced vertex \(v\). Hence, there are at most \(2t\) subsets of them. For each of these subsets and for each entry of the child node, Procedure 2 spends time polynomial in \(t\). This makes a total of \(2^{O(t^2)}\) time spent in each introduce node.

Finally, in a join node, Procedure 3 considers every pair of patterns of its two child nodes and for each combination spends time polynomial in \(t\). Hence, the total time spent in a join node is \((2^{O(t^2)})^2 = 2^{O(t^2)}\).

Since the nice tree decomposition has \(O(tn)\) nodes, the algorithm runs in \(2^{O(t^2)} \cdot n\) time. □

5. Other parameters yield stronger NP-hardness results

In Sections 3 and 4, we have seen that DAG PARTITIONING is solvable in linear time when fixing the weight of the partitioning set sought or the treewidth of the input graph. The question whether fixed-parameter algorithms can be obtained for parameters that are smaller than the solution weight or the treewidth naturally arises [29, 25, 14]. One parameter of interest is the maximum vertex outdegree in the graph: a citation network of journal articles will, for example, have a small outdegree since journal articles seldom contain more than 50 references. In this section, however, we will show that, among others, this parameter being small will not help solving DAG PARTITIONING efficiently.

Alamdari and Mehrabian [1] already showed that DAG PARTITIONING remains NP-hard even if the input graph has only two sinks. We complement this negative result by showing that the problem remains NP-hard even if the diameter or the maximum vertex degree of the input graph are constant. In conclusion, parameters like the number of sinks, the graph diameter or maximum degree cannot lead to fixed-parameter algorithms unless \(P = NP\).

Theorem 7. DAG PARTITIONING is solvable in linear time on graphs of diameter one, but NP-complete on graphs of diameter two even if all arcs have unit weight.

Proof. On graphs of diameter one, the problem is trivial: a directed acyclic graph with diameter one is an acyclic tournament, that is, there is no pair of vertices not joined by an arc. As such, it already contains exactly one source and one sink. Hence, we just verify in linear time whether the input graph is an acyclic tournament and answer “yes” or “no” accordingly.

For graphs of diameter two, we show NP-hardness by means of a polynomial-time many-one reduction from DAG PARTITIONING, which is NP-hard even when all arcs have weight one. Therefore, we agree on all arcs in this proof having weight one.

Given an arbitrary instance \((G, \omega, k)\) of DAG PARTITIONING, we add a gadget to \(G\) to obtain in polynomial time an instance \((G', \omega', k')\) such that \(G'\) is a graph of diameter two and such that \((G, \omega, k)\) is a yes-instance if and only if \((G', \omega', k')\) is a yes-instance. We obtain a graph \(G'\) from \(G\) by adding an acyclic tournament \(T\) consisting of \(k + n + 2\) vertices and outgoing arcs from the source \(s\) of \(T\) to all vertices of \(V(G)\) in \(G'\). We set \(k' := k + n\). Since every vertex in \(G'\) is in distance one from \(s\), the constructed graph \(G'\) has diameter two.

If \((G, \omega, k)\) is a yes-instance, then let \(S\) be a partitioning set with \(k\) arcs for \(G\). A partitioning set \(S'\) with \(k'\) arcs for \(G'\) is obtained by adding to \(S\) the \(n\) arcs from the source \(s\) of \(T\) to all vertices of \(V(G)\) in \(G'\). Thus, \((G', \omega', k)\) is a yes-instance.

If \((G', \omega', k')\) is a yes-instance, then let \(S'\) be a partitioning set with \(k'\) arcs for \(G'\). By Observation 2, every vertex in \(V(G)\) reaches at least one sink in \(G' \setminus S'\). This sink cannot be the sink of \(T\), since no vertex in \(T\) is reachable from \(V(G)\). Thus, \(S'\) has to disconnect the sink of \(T\) from all vertices of \(V(G)\) in \(G'\), where all paths between the sink of \(T\) and \(V(G)\) are via the source \(s\) of \(T\). Since \(T\) has \(k + n + 2\) vertices, \(S'\) cannot disconnect \(s\) from the sink of \(T\) and thus, has to remove from \(G'\) the \(n\) arcs connecting \(s\) to the vertices of \(V(G)\). Then, the remaining \(k\) arcs in \(S'\) have to be a partitioning set for \(G'\) without the tournament \(T\), which is precisely the original graph \(G\). Thus, \((G, \omega, k)\) is a yes-instance. □
We first recall the reduction from \( M_3(k) \) where the dotted arcs are a corresponding partitioning set of size \( I \) is a yes-instance if and only if \( I \) has a solution. From the proof of Theorem 7, we agree on all arcs in this proof having weight one. We now construct the DAG \( P_3 \) replace vertices of degree greater than three by equivalent structures of lower degree.

Theorem 8. \( \text{DAG Partitioning} \) is solvable in linear time on graphs of maximum degree two, but \( \text{NP-complete on graphs of maximum degree three even if all arcs have unit weight.} \)

Proof. Any graph of maximum degree two consists of undirected cycles or undirected paths. Thus, the underlying graph has treewidth at most two. We have seen in Theorem 6 that \( \text{DAG Partitioning} \) is linear-time solvable when the treewidth of the input graph is bounded by a constant.

We prove the \( \text{NP-hardness on graphs of maximum degree three.} \) To this end, we adapt the polynomial-time many-one reduction from \( \text{Multiway Cut to DAG Partitioning} \) presented by Leskovec et al. [26]. In their reduction, we replace vertices of degree greater than three by equivalent structures of lower degree.

**Multiway Cut**

**Input:** An undirected graph \( G = (V,E) \), a weight function \( \omega : E \to \mathbb{N} \), a set \( T \subseteq V \) of terminals, and an integer \( k \).

**Question:** Is there a subset \( S \subseteq E \) with \( \sum_{e \in S} \omega(e) \leq k \) such that the removal of \( S \) from \( G \) disconnects each terminal from all the others?

We first recall the reduction from \( \text{Multiway Cut} \) to \( \text{DAG Partitioning} \). From a \( \text{Multiway Cut} \) instance \( I_1 := (G_1, \omega_1, T, k_1) \), we construct in polynomial time a \( \text{DAG Partitioning} \) instance \( I_2 := (G_2, \omega_2, k_2) \) such that \( I_1 \) is a yes-instance if and only if \( I_2 \) is. From \( I_2 \), we then obtain an instance \( I_3 \) with maximum degree three. Since \( \text{Multiway Cut} \) remains \( \text{NP-hard even for three terminals and unit weights} \) [11], we may assume \( |T| = 3 \) and, similarly as in the proof of Theorem 7, we agree on all arcs in this proof having weight one. We now construct the \( \text{DAG Partitioning} \) instance \( I_2 = (G_2, \omega_2, k_2) \) from \( I_1 = (G_1, \omega_1, T, k_1) \) as follows. The construction is illustrated in Figure 13.

1. Add three vertices \( s_1, s_2, s_3 \) to \( G_2 \), forming the vertex set \( X \),
2. add each vertex of \( G_1 \) to \( G_2 \), forming the vertex set \( Y \),
3. for each edge \( \{u,v\} \) of \( G_1 \), add a vertex \( e_{\{u,v\}} \) to \( G_2 \), forming the vertex set \( Z \),
4. for each terminal \( t_i \in T \), add the arc \( (t_i, s) \) to \( G_2 \),
5. for each vertex \( v \in Y \setminus T \), add the arcs \( (v,s_i) \) for \( i = 1, 2, 3 \) to \( G_2 \), and
6. for each edge \( \{u,v\} \) of \( G_1 \), add the arcs \( (e_{\{u,v\}}, u) \) and \( (e_{\{u,v\}}, v) \) to \( G_2 \).

Figure 13: Reduction from a \( \text{Multiway Cut} \) instance with the terminals \( t_1, t_2, \) and \( t_3 \) to \( \text{DAG Partitioning} \). The top shows an instance \( I_1 \) of \( \text{Multiway Cut} \), where the dotted edges are a multiway cut of size \( k_1 = 3 \). The bottom shows the corresponding instance \( I_2 \) of \( \text{DAG Partitioning} \), where the dotted arcs are a corresponding partitioning set of size \( k_2 = k_1 + 2(n - 3) = 7 \) (\( n \) is the number of vertices in the graph of the \( \text{Multiway Cut} \) instance). The constructed vertex sets \( X, Y, \) and \( Z \) are highlighted using a gray background.

\[ t_1 \quad v_1 \quad t_2 \quad v_2 \quad t_3 \]
Set $k_2 = k_1 + 2(n - 3)$, where $n$ is the number of vertices of $G_1$. We claim that $I_1$ is a yes-instance if and only if $I_2$ is a yes-instance.

First, suppose that there is a multiway cut $S$ of size at most $k_1$ for $G_1$. Then, we obtain a partitioning set of size at most $k_2$ for $G_2$ as follows: if a vertex $v$ belongs to the same connected component of $G_1 \setminus S$ as terminal $i$, then remove every arc $(v, s_j)$ with $j \neq i$ from $G_2$. Furthermore, for each edge $(u, v) \in S$, remove either the arc $(e_{(u, v)}, u)$ or the arc $(e_{(u, v)}, v)$ from $G_2$. One can easily check that we end up with a valid partitioning set of size $k_2 = k_1 + 2(n - 3)$ for $G_2$: we delete at most $k$ arcs from $Z$ to $Y$ and, for each of the $n - 3$ vertices in $Y \setminus T$, delete two arcs from $Y$ to $X$. There are no arcs from $X$ to $Z$.

Conversely, suppose that we are given a minimal partitioning set $S$ of size at most $k_2$ for $G_2$. Note that it has to remove at least two of the three outgoing arcs of each vertex $v_2 \in Y \setminus T$ but cannot remove all three of them: contrary to Observation 1, this would create a new sink. Thus, $S$ deletes $2(n - 3)$ arcs from $Y$ to $X$ and the remaining $k_2 - 2(n - 3) = k_1$ arcs from $Z$ to $Y$. Therefore, we can define the following multway cut of size $k_1$: for each arc $(v, s_j)$ from $G_1$, remove an edge $(u, v)$ from $G_1$ if and only if one of the arcs $(e_{(u, v)}, u)$ and $(e_{(u, v)}, v)$ is removed from $G_2$ by $S$. Again, one can easily check that we end up with a valid multway cut.

It remains to modify the instance $I_2 = (G_2, \omega_2, k_2)$ to get an instance $I_1 = (G_1, \omega_1, k_1)$ of maximum degree three. To this end, first we show how to reduce the outdegree of each vertex of $G_2$ to two. Thereafter, we show how to reduce the indegree of each vertex of $G_2$ to one by introducing gadget vertices, each having indegree two and outdegree one. The construction is illustrated in Figure 14.

Note that all vertices of $G_2$ with outdegree larger than two are in $Y$. In order to decrease the degree of these vertices, we obtain a graph $G'_2$ from $G_2$ by carrying out the following modifications (see Figure 14) to $G_2$: for each vertex $v \in Y$, with $N^+(v) = \{s_1, s_2, s_3\}$, remove $(v, s_1)$ and $(v, s_2)$ add a new vertex $v'$, and insert the three arcs $(v, v')$, $(v', s_1)$, and $(v', s_2)$.

We show that $(G'_2, \omega'_2, k_2)$ is a yes-instance if and only if $(G_2, \omega_2, k_2)$ is. To this end, for $v \in Y$, let $T_v$ be the induced subgraph $G[T_v]\{v, v', s_1, s_2, s_3\}$. In $G_v$, a minimal partitioning set removes exactly two of the outgoing arcs of $v$, since $s_1, s_2$, and $s_3$ are sinks. It is enough to show that a minimal partitioning set $S$ removes exactly two arcs in $T_v$ from $G'_2$ in such a way that there remains exactly one directed path from $v$ to exactly one of $s_1, s_2$, or $s_3$. This remaining directed path then one-to-one corresponds to the arc that a partitioning set would have removed in $G_2$ between $v$ and $s_1, s_2$, or $s_3$. Since $s_1, s_2$, and $s_3$ are sinks, $S$ indeed has to remove at least two arcs from $T_v$: otherwise, two sinks will belong to the same connected component. However, due to Observation 1, $S$ cannot remove more than two arcs from $T_v$. Moreover, again exploiting Observation 1, the two arcs removed by $S$ leave a single directed path from $v$ to exactly one of $s_1, s_2, or s_3$.

We have seen that $(G'_2, \omega'_2, k_2)$ is equivalent to $(G_2, \omega_2, k_2)$ and that all vertices of $G'_2$ have outdegree two. To shrink...
the overall maximum degree to three, it remains to reduce the indegrees. Note that the vertices newly introduced in the previous step already have indegree one. We obtain graph $G_3$ of maximum degree three from $G'_3$ as follows. For each vertex $v$ with $|N^-(v)| = |\{w_1, \ldots, w_{d^-(v)}\}| \geq 2$, do the following (see Figure 14): for $j = 2, \ldots, d^-(v)$, remove the arc $(w_j, v)$ and add a vertex $w'_j$ together with the arc $(w_j, w'_j)$. Moreover, add the arcs $(w_1, w'_2)$, $(w_{d^-(v)}, v)$, and $(w'_j, w'_{j+1})$ for each $j \in \{2, \ldots, d^-(v) - 1\}$. Now, every vertex of $V(G'_3)$ in $G_3$ has indegree one and outdegree two, while the newly introduced vertices have indegree two and outdegree one. It follows that all vertices in $G_3$ have degree at most three.

It remains to show that $(G_3, \omega_3, k_3)$ is a yes-instance if and only if $(G'_3, \omega'_3, k_3)$ is. It then follows that $(G_3, \omega_3, k_3)$ is a yes-instance if and only if $(G_2, \omega_2, k_2)$ is. To this end, note that by Observation 1, among the introduced arcs, only the arcs $w_1, w'_2$ and $(w_j, w'_j)$ can be removed by a minimal partitioning set. From this, there is a one-to-one correspondence between deleting the arc $w_1, w'_2$ or $(w_j, w'_j)$ in the graph $G_3$ and deleting the arc $(w_j, v)$ in the graph $G'_3$. \qed

6. Outlook

We have presented two fixed-parameter algorithms for DAG PARTITIONING, one with respect to the weight $k$ of the partitioning set sought and one with respect to the parameter treewidth $t$.

We demonstrated the feasibility of applying the fixed-parameter algorithm for the parameter $k$ (Algorithm 1) to large input instances with optimal partitioning sets of small weight. However, we were unable to solve the instances in the data set of Leskovec et al. [26], since the weight of optimal partitioning sets is too large. We found out that the heuristic presented by Leskovec et al. [26] finds nearly optimal partitioning sets on the instances that our algorithm can solve in reasonable time. Moreover, it remains to be seen whether there is a one-to-one correspondence between deleting the arc $w_1, w'_2$ or $(w_j, w'_j)$ in the graph $G_3$ and deleting the arc $(w_j, v)$ in the graph $G'_3$.

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References

[1] S. Alamdari and A. Mehrabian. On a DAG partitioning problem. In Proceedings of the 9th International Workshop on Algorithms and Models for the Web Graph (WAW’12), volume 7323 of Lecture Notes in Computer Science, pages 17–28. Springer, 2012. doi: 10.1007/978-3-642-30541-2_2.
[2] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. Science, 286(5439):509–512, 1999. doi: 10.1126/science.286.5439.509.
[3] R. van Bevern. Towards optimal and expressive kernelization for $d$-Hitting Set. Algorithmica, 70(1):129–147, 2014. doi: 10.1007/s00453-013-9774-3.
