Exact solutions of the Grad-Shafranov equation via similarity reduction and applications to magnetically confined plasmas

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Abstract

We derive exact solutions of a linear form of the Grad-Shafranov (GS) equation, including incompressible equilibrium flow, using ansatz-based similarity reduction methods. The linearity of the equilibrium equation allows linear combinations of solutions in order to obtain axisymmetric MHD equilibria with closed and nested magnetic surfaces which are favorable for the effective confinement of laboratory plasmas. In addition, employing the same reduction methods we obtain analytical solutions for several non-linear forms of the GS equation. In this context analytic force-free solutions in both linear and nonlinear regimes are also derived.

Keywords: Grad-Shafranov Equation, Similarity Reduction, Magnetic Confinement

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1. The Grad-Shafranov equation

The GS equation governs the axisymmetric MHD equilibria of static plasmas. Essentially it is a second order elliptic, generally nonlinear PDE, whose solutions reveal the topology of the magnetic field on the poloidal cross section of a toroidal plasma. In the case of incompressible flows, parallel to the magnetic field, the equation remains identical in form [1, 2].

Also, in [1] and [2] the equation was further generalized for incompressible flows of arbitrary direction with an additional $r^2$-term associated with the equilibrium electric field (see also Eq. (19) below). In cylindrical coordinates $(r, \phi, z)$ the GS equation takes the familiar form

$$\partial_r u - r^{-1} \partial_r u + \partial_z u + f(u) + g(u) r^2 = 0$$

(1)

with $f(u), g(u)$ being free-arbitrary functions which respectively are related to the poloidal current and the plasma pressure. Specifically $g(u) = \mu_0 P_s(u)$ where $P_s$ is the pressure in the absence of macroscopic flows. In the presence of field aligned flows the total plasma pressure is given by the Bernoulli equation $P = P_s(u) - \rho v^2/2$, where $v$ is the magnitude of flow velocity and $\rho$ the mass density. The dependent variable $u$ is related to the poloidal flux function $\psi$ through the transformation

$$u(\psi) = \int_0^\psi \left[1 - M_p^2(s)\right]^{1/2} ds$$

(2)

where $M_p(s)$ is the poloidal Mach function [2]. Once the free functions $f(u)$ and $g(u)$ are specified the solutions of (1) provide us all the information about the macroscopic equilibrium properties of the magnetofluid.

So far analytical solutions of (1) have been constructed and studied extensively for choices of constant $f(u)$ and $g(u)$. The corresponding solutions describe the so-called Solovev equilibrium, which is characterized by monotonic current density and safety factor profiles. Solutions for free functions proportional to $u$, expressed in terms of Coulomb or Whittaker wave functions, have been obtained for the first time by Hernmeger [3] and independently by Maschke [4]. The work of Atanasiu et al [5] was focused on the case of free functions linear in $u$, computing the required particular solutions of the inhomogeneous PDE with a series-expansion-based technique. Wang [6] employing also linear ansatzes, reduced the GS equation into a Helmholtz equation and derived analytical solutions describing equilibria with toroidal current reversals (TCR). In addition, particularly limited progress has been made in the analytical integration of (1) for nonlinear free functions; in the works of Cicogna and Pegoraro [7, 8] classical and non-classical symmetries of special nonlinear forms of the GS equation were exploited for the derivation of analytical solutions. These solutions were implemented on the construction of nonlinear equilibria.

In this letter, exploiting the translational symmetry of the GS equation, we convert it into a system of two first order PDEs. This system contains all solutions of the GS equation and thus the original problem is reduced to the solution of the two first order PDEs. This is a quite interesting procedure since in several cases it might be more convenient to seek solutions or reduction ansatzes of the reduced equations rather than of the original PDE. In our case it wasn’t difficult to find a simple ansatz to further reduce this system. We show that the original PDE results in a single ODE in terms of a reduction variable which depends on the choice of the aforementioned ansatz. This procedure serves as a prototype for algorithmic derivation of reductions, that can possibly be exploited employing...
additional symmetries which are admitted for some particular choices of the free functions. Regarding the derived solutions, in the linear case the principle of superposition allows a linear combination of the derived solutions which results in magnetic configurations with closed and nested magnetic surfaces, essential for the confinement in fusion devices. However this is not the case for the corresponding nonlinear solutions and therefore the magnetic field configurations do not form closed surfaces.

2. Similarity reduction procedure

A similarity reduction is a procedure by which one may find a “grouping” of the independent variables so as to assume a functional form of the solution that enables the original PDE to be reduced to a simpler form, usually to an ODE. Our treatment which finds such a reduction form for the GradShafranov equation was initially inspired by the work of Anco et al [9] focused on the reduction of a certain class of semilinear reactiondiffusion equations by a method known as “group foliation”. In general a group foliation converts a PDE into an equivalent system of lower order PDEs, called group resolving system, whose independent and dependent variables correspond respectively to the invariants and the differential invariants of a given infinitesimal symmetry generator. In other words a group of point symmetry transformations induces a foliation of the solution space. Each solution of the group resolving system corresponds to a symmetry transformations induces a foliation of the solution space.

The interested reader may refer to [9] and [10].

In our case due to the arbitrariness of the free functions \( f(u) \) and \( g(u) \) the only symmetry which is admitted in general by the GS equation is the \( z \)-translational symmetry. We show that exploiting this symmetry and following the treatment described in [9] we can reduce Eq. (1) into a system of first order, coupled PDEs whose independent and dependent variables are the invariants and the differential invariants of the \( z \)-symmetry infinitesimal generator \( X_t = \partial_z \). The resulting PDEs can be split into a system of ODEs using suitable separation ansatzes. Under the assumption \( g(u) = e f(u) \) this method yields similarity reductions of the GS equation. The invariants \( I \) of \( X_t \) satisfy \( X_t(I) = \partial_z I = 0 \). In the space of independent and dependent variables \( (r, z, u) \) we can identify two such invariants namely \( \gamma = r \) and \( \vartheta = u \). The differential invariants \( \bar{I} \) of the \( z \)-symmetry infinitesimal generator are quantities invariant under the action of \( X_t \) on the extended, by the first order derivatives of \( u \), space that is \( (r, z, u, u_t, u_{tt}) \). We are making use of a mixed notation for the partial derivatives so as to underline the fact that \( u_t := \partial_t u \) and \( u_{tt} := \partial_t u_t \) serve also as new variables. The action of \( X_t \) on the extended space is realized through its first order prolongation \( p^{(1)}X_t \) which for the translation symmetry is \( X_t \) itself i.e. \( p^{(1)}X_t = \partial_z \). For a detailed description one may refer to [10] and [11]. Therefore the additional invariants provided by the action of \( X_t \) on the extended space are \( \Gamma := u_t, \Theta := u_{tt} \). Obviously the relation \( \partial_z u_t = \partial_t u_t \) must hold and the invariantized variables should also satisfy Eq. (1). Hence we obtain the following system of two coupled PDEs for the functions \( \Gamma(\gamma, \vartheta) \) and \( \Theta(\gamma, \vartheta) \)

\[
\begin{align*}
\partial_\gamma \Theta + \Gamma \partial_\vartheta \Theta - \Theta \partial_\vartheta \Gamma &= 0 \\
\partial_\gamma \Gamma + \Gamma \partial_\vartheta \Gamma - \Theta \partial_\vartheta \gamma + g(\vartheta) \gamma^2 + f(\vartheta) &= 0
\end{align*}
\]

(3) called translation group resolving system (GRS). The GRS can easily be reduced into a system of ODEs, employing the ansatz \( \Gamma = a \gamma Q(\vartheta), \Theta = bW(\vartheta) \) where \( a, b \) are constants. This choice is based upon the simple observation that if one assumes \( \Theta \) as a function of \( \vartheta \) only and \( \Gamma \) has a separable form then the first equation of the system above becomes automatically an ODE. Also the term \( -\gamma^2 \) of the second equation of the GRS becomes a function of \( \vartheta \) only if \( \Gamma \) is linear in \( \gamma \). The resulting ODEs are,

\[
\begin{align*}
Q'(\vartheta)W(\vartheta) &= W'(\vartheta)Q(\vartheta) \\
\left(W^2(\vartheta)\right)' &= -2b^{-2}f(\vartheta) \\
\left(Q^2(\vartheta)\right)' &= -2a^{-2}g(\vartheta)
\end{align*}
\]

(4)

The first equation is satisfied for \( W(\vartheta) = c_0Q(\vartheta) \), with \( c_0 = \pm 1 \) by choice, which, from the remaining equations, yields \( g(\vartheta) = e f(\vartheta) \) with \( e = a^2/b^2 \). Consequently \( Q \) is given by

\[
Q(\vartheta) = \pm \left[c - 2b^{-2}F(\vartheta)\right]^{1/2}
\]

(5)

where \( F \) is the antiderivative of \( f \) i.e. \( F(\vartheta) = \int_0^\vartheta f(s)ds \) and \( c \) is an arbitrary constant representing every constant term into the square root. The solution \( u(r, z) \) is then obtained by integrating the system:

\[
\begin{align*}
\partial_r u(r, z) &= \pm ar \left[c - 2b^{-2}F(u(r, z))\right]^{1/2} \\
\partial_z u(r, z) &= \pm b \left[c - 2b^{-2}F(u(r, z))\right]^{1/2}
\end{align*}
\]

(6)

The resulting solutions are expressed in terms of an arbitrary integration constant \( c \) which reflects the invariance of the solutions under \( z \)-translations. From system (6) one deduces that the following equation must hold:

\[
b \partial_r u(r, z) = \pm ar \partial_r u(r, z)
\]

(7)

and therefore the solutions should have the form

\[
u(r, z) = w(x), \quad \text{with} \quad x := ar^2/2 \pm bz
\]

(8)

Substituting (9) into (6) one obtains

\[
w'(x) = \pm \left[c - 2b^{-2}F(w(x))\right]^{1/2}
\]

(9)

Since \( u \) can be written as an arbitrary function of a single variable \( x \) one may search directly for solutions of the original PDE in terms of this variable. This is the concept of the direct reduction method [12][13] which in general seeks for solutions of the form \( u(r, z) = U(r, z; w(x(r, z))) \) trying to identify the appropriate forms of the function \( U \), and of the reduction variable \( x \) so as to reduce the PDE into an ODE for \( w(x) \). If we substitute \( u = w(x(r, z)) \) directly in GS equation we obtain:

\[
\begin{align*}
w'(x) \left(\partial_r x + \partial_z x - r^{-1} \partial_r x\right) + w''(x) \left[(\partial_r x)^2 + (\partial_z x)^2\right] + f(w) + g(w)r^2 &= 0
\end{align*}
\]

(10)
Substituting \( x := ar^2/2 \pm bz \) one can observe that Eq. (10) splits into two identical equations if \( g(u) = ef(u) \), where \( e = a^2/b^2 \). The function \( w(x) \) is then determined by the ODE:

\[
w''(x) + b^2 f(w(x)) = 0
\]

(11)

which can easily be transformed to Eq. (9) if we express the second order derivative of \( w(x) \) as \( w''(x) = \frac{1}{2} \left[ (w'(x))^2 \right] \). A direct reduction may not always be a convenient approach since in general it is not easy to identify the appropriate reduction variable, particularly when it assumes more complicated forms. Using the method of group foliation we established the framework for a systematic, algorithmic derivation of a reduced representation of the GS equation that is its resolving system, which along with an easily identifiable ansatz led us systematically to (9). It is also possible that additional ansatzes may be suitable for reducing the resolving system. Before closing this section note that from Eq. (10) we can also realize that for zero pressure functions containing both linear and quadratic terms in \( u \) and \( g \), the equation with free functions of the form

\[
aw^2 + bw + c = 0
\]

(12)

is then determined by the ODE:

\[
(4ax + b^2)w''(x) + 2aw'(x) + f(w(x)) = 0
\]

3. Exact solutions

3.1. Linear solutions

We first examine the case of the linearized Grad-Shafranov equation with free functions of the form \( f(u) = f_1 + f_2 u \) and \( g(u) = g_1 + g_2 u \) which correspond to poloidal and the static pressure functions containing both linear and quadratic terms in \( u \). The inclusion of quadratic terms is very interesting since equilibria with hollow toroidal current densities, which can be constructed due to the quadratic terms, are related to the formation of internal transport barriers (ITBs) [13] which reduce the transport effects and contribute in the transition to high confinement modes. Also this model can describe core-TCR scenarios which thus far have been studied mainly numerically, e.g. [15] or in the large aspect ratio limit [18]. Speaking of TCR equilibria we mention that an alternative method of spectral representation of the flux surfaces has been introduced in [19]. As we have already mentioned the methods exposed above are applicable in the case \( g(u) = ef(u) \), hence the general linearized GS equation contains 3 instead of 4 free parameters,

\[
\partial_{rr} u - r^{-1} \partial_{r} u + \partial_{z} u + f_1 + f_2 u + g_2 (f_1/f_2 + u)^2 = 0
\]

(13)

where \( f_1, f_2 \) and \( g_2 \) are the free parameters and \( e := g_2/f_2 \). The linear solutions occur by solving Eq. (9) or Eq. (11) for \( f(w) = f_0 + f_1 w \) and substituting \( x = \sqrt{e}r^2/2 \pm bz \),

\[
\begin{align*}
    u^*_1(r, z) &= \lambda_1 \cos \left( \sqrt{\frac{g_2}{2}} r^2 \pm \sqrt{f_2} z \right) \\
    + &\lambda_2 \sin \left( \sqrt{\frac{g_2}{2}} r^2 \pm \sqrt{f_2} z \right) - \frac{f_1}{f_2}
\end{align*}
\]

(14)

where \( \lambda_{1,2} \) are arbitrary constants. This solution generalizes by an inhomogeneous constant term, a solution to the homogeneous counterpart of Eq. (13), derived using a different approach in [17]. Due to the “phase” argument \( a^2/2 \pm bz \) each one of the solutions \( u^*_1 \) exhibits a parabolic topology on the \((r, z)\)-plane. However the linearity of Eq. (13) allows one to take the superposition of the two solutions, i.e.

\[
u_1(r, z) = s^* u^*_1(r, z) + s^- u^*_1(r, z)
\]

(15)

where \( s^* \) are constants satisfying \( s^* + s^- = 1 \) and which define the up-down asymmetry of the magnetic surfaces. The resulting solution displays a \((r, z)\)-plane topology in connection with configurations with non nested magnetic surfaces, i.e. having magnetic lobes (or islands) similar to those in Fig. 2 of Ref. [17]. In Fig. 1(a)-(b) are depicted two isolated sets of closed and nested magnetic surfaces which can describe equilibria of magnetic confinement systems by choosing appropriately the boundary to coincide with the outer close magnetic surface (see caption of Fig. 1). For \( s^* = s^- \) these isolated sets are up-down symmetric with respect to the plane \( z = 0 \). In this linear case we can assume \( u(r, z) = u_0(r, z) + w(x(r, z)) \), with \( u_0 \) being an arbitrary nonconstant function of \( r \) and \( z \). The function \( w(x) \) should satisfy the ODE (12) with \( f(w) = f_0 + f_1 w \) and \( u_0 \) the homogeneous counterpart of Eq. (13). Following the treatment of [20] we can write the homogeneous solution as a sum of an arbitrary number of terms,

\[
    u_0(r, z) = \sum_{j=1}^{N} \left[ \lambda_{1,j} W_{\nu,1}(q) \cos(jz) + \lambda_{2,j} W_{\nu,2}(q) \sin(jz) \right] + c.c.
\]

(16)

where \( W_{\nu,1}, W_{\nu,2} \) the Whittaker functions with \( \nu = j^2 - \frac{b^2}{4g_2} \), \( q = i \sqrt{g_2}r^2 \); \( k_{1,j}, \ldots, k_{4,j} \) with \( j = 1, \ldots N \), are arbitrary parameters and \( r^2 = -1 \). The limit \( N \) depends on the number of boundary-shaping and other equilibrium conditions which one desires to impose. Having determined \( u_0 \), the complete solution of Eq. (13) is:

\[
u_0 = u_0 + u_1
\]

(17)

We can also take into account a non-parallel flow contribution modifying our solutions as follows

\[
u_{1,2}(r, z) = u_{1,2}(r, z) - \frac{h_1}{g_2} r^2 + \frac{h_1 f_2}{g_2}
\]

(18)

where the parameter \( h_1 \) is related to the radial electric field and the function \( \tilde{u} \) is a solution of the linear generalized GS equation:

\[
\partial_{rr} u - r^{-1} \partial_{r} u + \partial_{z} u + f_0 + f_2 u + g_2 (f_1/f_2 + u)^2 + h_1 r^4 = 0
\]

(19)

with \( f_0 = f_1 - h_1 f_2 g_2^2 \). The term \( h_1 r^4 \) is due to the non-parallel flow contribution which from the Ohm’s law induces a finite electric field. Using \( \tilde{u} \) one can obtain magnetic field topologies which correspond both to compact and non-compact configurations, while exploiting \( g_2 \) we can construct equilibria with shaped boundaries. These three possibilities are discussed below.
Compact toroids. Setting the parameter \[ \beta_j \] and following the shaping method described in [23] and [24], we exploited the arbitrary parameters \[ x_{1,j}, \ldots, x_{4,j} \] with \( j = 1, \ldots, 8 \), to construct shaped equilibrium with a cross section relevant to ITER (Fig. 2).

Linear force-free equilibria. Another case of linear equilibrium solutions are those obtained in the limit \( g(u) \rightarrow 0 \) by solving Eq. (12) with \( f(u) = f_1 + f_2 u \). These solutions describe Taylor force-free relaxed states. The completely free parameters \( a, b \) and the linearity of the equilibrium equation allows general solutions expressed by means of arbitrary number of terms:

\[
 u_{\beta j}(r,z) = \sum_{j=1}^{N} \mu_{1,j} \cos \left( f_2 \left( \beta_j^2 r^2 + z^2 \pm 2\beta_j z \right) \right) \\
 + \mu_{2,j} \sin \left( f_2 \left( \beta_j^2 r^2 + z^2 \pm 2\beta_j z \right) \right) - \frac{f_1}{f_2} 
\]

where \( \beta_j \) and \( \mu_{1,j}, \mu_{2,j} \) with \( j = 1, \ldots, N \) are arbitrary parameters.

3.2. Nonlinear solutions

The reduction methods described previously are now implemented for several nonlinear choices of free functions \( f(u), g(u) \), with the reminder that \( g(u) = \epsilon f(u) \). Here we give some examples of analytically calculated solutions for several nonlinear choices of free functions. Note that employing Eq. (2) we can integrate it for \( c \neq 0 \) or \( c = 0 \). It may turn out that some solutions which belong to the former case are not reducible to solutions of the latter class just by setting \( c = 0 \), see for example the solutions denoted by the subscript 1 below. The examples presented are not exhaustive and possibly one may find additional solutions for these particular or additional free functions.

Quadratic. \( f(u) = f_0 + f_1 u^2 \). In this case we observe that Eq. (11) takes the form of a differential equation satisfied by the Weierstrass elliptic function \( \wp \). The exact solutions are given by:

\[
 u_{\wp}(r,z) = \eta \wp \left( \frac{a^2}{2} \pm bz + \hat{c} ; \frac{2f_0 \eta}{b^2} , c \right) 
\]

where \( \eta := 6^{1/3} \left( -f_1 / b^2 \right)^{-1/3} \).

Exponential. \( f(u) = f_0 e^{nu} \) with \( n \in \mathbb{R} \) and \( f_0 \) an arbitrary constant, we present two classes of solutions:

\[
 u_{\text{exp},1}(r,z) = -\frac{1}{n} \ln \left( \frac{b^2 e^{nu}}{2f_0} \right) \sech^2 \left( \frac{n \sqrt{c}}{4} \left( a^2 \pm 2bz + \hat{c} \right) \right) 
\]

\[
 u_{\text{exp},2}(r,z) = -\frac{2}{n} \ln \left[ -i \frac{\sqrt{2} f_0 m}{2b} \left( \frac{a^2}{2} \pm bz + \hat{c} \right) \right] 
\]
Integer exponent. \( f(u) = f_0 u^n \) with \( n \in \mathbb{Z} \setminus \{1\} \). In this case the solutions \( u^*_\varphi \) are obtained explicitly from \( (9) \) with \( c = 0 \).

\[
    u^*_\varphi(r, z) = 2 \sqrt{n} \left[ \frac{2f_0}{n + 1} \left( \frac{ar^2}{2} \pm bz \right) + \hat{c} \right]^{\frac{1}{n}} \tag{24}
\]

Sinusoidal. \( f(u) = f_0 \sin(nu) \) with \( n \in \mathbb{R} \). We find a first class of exact solutions given by \( (25) \), where \( am \) is the Jacobi amplitude function and a second class given by \( (26) \).

\[
    u^*_{\sin,1}(r, z) = \frac{2}{n} \frac{\sqrt{nc}}{2b} \left( \frac{ar^2}{2} \pm bz + \hat{c} \right) \left( \frac{4f_0}{c} \right) \tag{25}
\]

\[
    u^*_{\sin,2}(r, z) = (\pm) 4 \arccos \left( \exp \left( i \sqrt{\frac{4f_0}{c}} \left( \frac{ar^2}{2} \pm bz + \hat{c} \right) \right) \right) \tag{26}
\]

Force-free. \( g(u) = 0 \), \( f(u) = f_0 e^{nu} \) with \( n \in \mathbb{R} \). Solving Eq. \( (12) \) we obtain the following solutions which describe nonlinear force-free equilibria.

\[
    u^*_{\exp,ff}(r, z) = \frac{1}{n} \ln \left( \frac{2n}{4a} \sqrt{2n^2cf_0(4a^2 + az^2 \pm bz) + b^2} + \hat{c} \right) \tag{27}
\]

In Fig. 3 are depicted the level sets of solutions \( (21) \) and \( (27) \). Both of the depicted equilibria possess current sheet formations. Also, by changing the parametric values one can construct respective configurations without current sheets. The topology of the magnetic surfaces of the finite pressure-gradient equilibria displays a parabolic morphology; an example is shown in Fig. 3(a). This characteristic feature comes from the special form of the reduction ansatz.

**4. Discussion**

We remark here that due to the nonlinearity of the equilibrium equation the linear combinations of different solutions which possibly could result in closed magnetic surfaces are ruled out. However it is probable that the aforementioned similarity reduction methods can be employed using alternative reduction ansatzes leading to solutions with desirable characteristics. Particularly in the framework of the group foliation method, additional symmetries of the GS equation \( (22) \) can possibly be exploited in order to obtain alternative reductions. This possibility though, is restricted to just few particular choices of free functions in contrast with the reductions obtained by the translational symmetry, which is an intrinsic property of the GS equation, regardless the choice of free functions.

Hypothetically a nonlinear superposition principle could be used in order to combine nonlinear solutions. However it turns out that in the case of GS equation the classical method for establishing such superposition principles \( (25, 26) \), where the combination of different solutions is realized through a so-called, reduced connecting function, is of no practical use. Thus it is an open question whether any nonlinear or pseudo-linear superposition principles can be applied or established for the GS equation. If this will be proved possible, then the nonlinear solutions obtained here may give us new insights in the analytical study of axisymmetric MHD equilibria.

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