ON DERIVATIONS WITH RESPECT TO FINITE SETS OF SMOOTH FUNCTIONS

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Abstract. The purpose of this paper is to show that functions that derivate the two-variable product function and one of the exponential, trigonometric or hyperbolic functions are also standard derivations. The more general problem considered is to describe finite sets of differentiable functions such that derivations with respect to this set are automatically standard derivations.

1. Introduction

Derivations are additive mappings of a ring into itself that possesses the so-called Leibniz Rule. More precisely, if \((R, +, \cdot)\) is a ring, then a function \(d : R \to R\) is called a derivation if, for all \(x, y \in R\),

\[
\begin{align*}
    d(x + y) & = d(x) + d(y), \\
    d(x \cdot y) & = d(x) \cdot y + x \cdot d(y).
\end{align*}
\]

In other words, derivations behave similarly to the differentiation operator which is acting on differentiable real functions. A classical example of a derivation can be constructed on the ring \(F[x]\) of polynomials over a field \(F\) as follows: given \(n \in \mathbb{N}\) and \(a_0, \ldots, a_n \in F\), define

\[
d(a_n x^n + \cdots + a_1 x + a_0) = na_n x^{n-1} + \cdots + a_1.
\]

Then, an elementary calculation shows that \(d : F[x] \to F[x]\) is a derivation.

Derivations are used in many branches of analysis and algebra. For instance, nonnegative information functions are constructed via real derivations (see Daróczy–Maksa [4], Maksa [16]). Nonconstant functions that are convex with respect to families of power means are also obtained in terms of real derivations (see Maksa–Páles [20]). Derivations are used to express the general solutions of certain functional equations (see Fechner–Gselmann [5], Gselmann [6], [7], Halter-Koch [11], [10], Jurkat [12]). Generalizations, such as higher-order derivations, bi-derivations and approximate or near-derivations were studied by Badora [1], Gselmann [8], Gselmann–Páles [9], and Maksa [17], [18].

In order to introduce the notion of a generalized derivation, first we define the classes of \(n\)-variable admissible functions as follows:

\[
A_n := \{f : \Omega \to \mathbb{R} \mid \Omega \subset \mathbb{R}^n \text{ is open, nonempty and } f \text{ is differentiable}\}
\]

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and
\[ A := \bigcup_{n=1}^{\infty} A_n. \]

The set \( \Omega \) related to \( f \) in the above definition will be called the domain of \( f \) and denoted by \( \text{dom}_f \). It would make sense to introduce admissible functions as functions that map \( \Omega \) into \( \mathbb{R}^m \), but it is not necessary in this paper. We say that a function \( d : \mathbb{R} \to \mathbb{R} \) is a derivation with respect to a subset \( A \subseteq \mathbb{R} \) (shortly, \( d \) is a derivation with respect to \( f_1, \ldots, f_k \), where \( A = \{ f_1, \ldots, f_k \} \)) if, for all \( f \in A \),
\[
d(f(x_1, \ldots, x_n)) = \partial_1 f(x_1, \ldots, x_n)d(x_1) + \cdots + \partial_n f(x_1, \ldots, x_n)d(x_n) \quad ((x_1, \ldots, x_n) \in \text{dom}_f)
\]
holds. One can immediately see that a function \( d : \mathbb{R} \to \mathbb{R} \) is a standard derivation if and only if it is a derivation with respect to \( S_2 \) and \( P_2 \), where
\[
S_2(x_1, x_2) := x_1 + x_2 \quad \text{and} \quad P_2(x_1, x_2) := x_1 x_2 \quad ((x_1, x_2) \in \mathbb{R}^2).
\]

From now on we deal with functions that map \( \mathbb{R} \) into \( \mathbb{R} \). It is very simple to see some consequences of the Leibniz Rule. In the subsequent lemmas we describe the homogeneity properties of the solutions of the two functional equations (1) and (2). We define the homogeneity set and the set of zeros of a function \( d : \mathbb{R} \to \mathbb{R} \) by
\[
\mathcal{H}_d := \{ t \in \mathbb{R} \mid d(tx) = td(x) \text{ holds for all } x \in \mathbb{R}\} \quad \text{and} \quad \mathcal{Z}_d := \{ t \in \mathbb{R} \mid d(t) = 0 \},
\]
respectively.

In the following three lemmas, we summarize the basic properties of various derivations.

**Lemma A.** We have the following two assertions.

(i) Let \( \Omega_1 \) and \( \Omega_2 \) be open subsets of \( \mathbb{R} \), let \( f : \Omega_1 \to \Omega_2 \) and \( g : \Omega_2 \to \mathbb{R} \) be differentiable functions. If \( d : \mathbb{R} \to \mathbb{R} \) is a derivation with respect to \( f \) and \( g \), then \( d \) is also a derivation with respect to the composition \( g \circ f \).

(ii) Let \( \Omega_1 \) and \( \Omega_2 \) be open subsets of \( \mathbb{R} \), let \( f : \Omega_1 \to \Omega_2 \) be a bijection such that \( f' \) does not vanish on \( \Omega_1 \). If \( d : \mathbb{R} \to \mathbb{R} \) is a derivation with respect to \( f \), then \( d \) is also a derivation with respect to the inverse function \( f^{-1} : \Omega_2 \to \Omega_1 \).

**Proof.** By the assumptions of (i), for all \( x \in \Omega_1 \) and \( y \in \Omega_2 \), we have
\[
d(f(x)) = f'(x)d(x) \quad \text{and} \quad d(g(y)) = g'(y)d(y).
\]
Therefore, with \( y := f(x) \), we get
\[
d((g \circ f)(x)) = d(g(f(x))) = g'(f(x))d(f(x)) = g'(f(x))f'(x)d(x) = (g \circ f)'(x)d(x),
\]
which yields that \( d \) is a derivation with respect to the function \( g \circ f \).

By the assumption of (ii), for all \( x \in \Omega_1 \), we have the first equality in (3). Using the substitution \( x = f^{-1}(y) \), this implies that
\[
d(f^{-1}(y)) = \frac{1}{f'(f^{-1}(y))}d(y) = (f^{-1})'(y)d(y) \quad (y \in \Omega_2).
\]
Thus, \( d \) is a derivation with respect to the inverse function \( f^{-1} \). \( \square \)
Lemma B. Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( S_2 \). Then \( \mathcal{H}_d \) is a subfield of \( \mathbb{R} \), in particular \( d \) is \( \mathbb{Q} \)-homogeneous, i.e., for all \( r \in \mathbb{Q} \) and for all \( x \in \mathbb{R} \),
\[
d(rx) = rd(x). \tag{4}
\]
Furthermore, \( \mathbb{Z}_d \) is a vector space over \( \mathcal{H}_d \). Additionally, if \( \Omega \subseteq \mathbb{R}^n \) is a nonempty open set and \( f, g : \Omega \to \mathbb{R} \) are differentiable functions such that \( d \) is a derivation with respect to \( f \) and \( g \), then, for all \( r, s \in \mathbb{Q} \), \( d \) is a derivation with respect to \( rf + sg \).

Proof. One can easily see that \( \mathcal{H}_d \) is a subring of \( \mathbb{R} \) with unit element 1. Furthermore, for all \( 0 \neq t \in \mathcal{H}_d \) and \( x \in \mathbb{R} \), \( td \left( \frac{1}{t} x \right) = d \left( t \frac{1}{t} x \right) = d(x) \), which implies that \( \frac{1}{t} \in \mathcal{H}_d \). Therefore, \( \mathcal{H}_d \) is a subfield, indeed.

To justify that \( \mathbb{Z}_d \) is a vector space over \( \mathcal{H}_d \), it suffices to observe that \( \mathbb{Z}_d \) is a subgroup of \( (\mathbb{R}, +) \), which is also closed by multiplications of elements of the field \( \mathcal{H}_d \).

Let \( f, g : \Omega \to \mathbb{R} \) be differentiable functions such that \( d \) is a derivation with respect to \( f \) and \( g \) and let \( r, s \in \mathbb{Q} \). Then, using the additivity of \( d \) and \( (4)_1 \) twice, we get
\[
d((rf + sg)(x)) = d(rf(x) + sg(x)) = d(rf(x)) + d(sg(x))
= rd(f(x)) + sd(g(x)) = r f'(x) d(x) + s g'(x) d(x) = (rf + sg)'(x) d(x).
\]
This shows that \( d \) is a derivation with respect to \( rf + sg \), indeed. \( \square \)

Lemma C. Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \). Then \( \mathcal{H}_d = \mathbb{Z}_d \) and \( \mathcal{H}_d \setminus \{0\} \) is a subgroup of \( (\mathbb{R}, \cdot) \) containing the elements \(-1, 0 \text{ and } 1\). In particular, \( d \) is odd, i.e., it is homogeneous with respect to \(-1\). Furthermore, for all \( r \in \mathbb{Q} \) and \( x \in \mathbb{R}_+ \),
\[
d(x^r) = rx^{r-1}d(x). \tag{5}
\]
Additionally, if \( \Omega \subseteq \mathbb{R}^n \) is a nonempty open set and \( f, g : \Omega \to \mathbb{R} \) are differentiable functions such that \( d \) is a derivation with respect to \( f \) and \( g \), then \( d \) is a derivation with respect to \( f \cdot g \). More generally, provided that \( f \) and \( g \) are positive on \( \Omega \), for all \( r, s \in \mathbb{Q} \), \( d \) is also a derivation with respect to \( f^r \cdot g^s \).

Proof. Let \( t \in \mathcal{H}_d \). Then, for all \( x \in \mathbb{R} \), by the Leibniz Rule, we get
\[
  td(x) = d(tx) = rd(x) + xd(t).
\]
Therefore, \( d(t) = 0 \) must be valid, proving that \( \mathcal{H}_d \subseteq \mathbb{Z}_d \). The reversed inclusion follows similarly.

With the substitutions \( x = y = 0 \), \( x = y = 1 \) and \( x = -y = 1 \) in equation \( (2) \), one can immediately see that \( d(0) = d(1) = d(-1) = 0 \). This equation also shows that if \( x, y \in \mathbb{Z}_d \), then \( xy \in \mathbb{Z}_d \). Therefore, \( \mathbb{Z}_d = \mathcal{H}_d \) is a multiplicative subsemigroup of \( \mathbb{R} \) containing \(-1, 0 \text{ and } 1\). For \( 0 \neq x \in \mathbb{Z}_d \), we get that
\[
  0 = d(1) = d \left( \frac{1}{x} x \right) = xd \left( \frac{1}{x} \right) + \frac{1}{x} d(x),
\]
hence \( d \left( \frac{1}{x} \right) = 0 \). Therefore, \( \mathcal{H}_d \setminus \{0\} \) is a multiplicative subgroup of \( \mathbb{R} \).

In this part, observe that with substitution \( x := e^u \), \( y := e^v \), the Leibniz Rule yields that the function \( a : \mathbb{R} \to \mathbb{R} \) defined by
\[
a(u) := e^{-u}d(e^u) \quad (u \in \mathbb{R})
\]
is a derivation with respect to \( S_2 \). Therefore, by Lemma B, \( a \) is \( \mathbb{Q} \)-homogeneous. Hence
\[
e^{ru}d(e^{ru}) = re^{-u}d(e^u) \quad (u \in \mathbb{R}, r \in \mathbb{Q}).
\]
Substituting $u := \ln x$ (where $x > 0$) into the above identity, it follows that (5) holds.

Finally, we prove the last assertion of the lemma. Let $f, g : \Omega \to \mathbb{R}$ be positive differentiable functions such that $d$ is a derivation with respect to $f$ and $g$ and let $r, s \in \mathbb{Q}$. Then, using the Leibniz Rule for $d$ and (5) twice, we get

$$d((f^r \cdot g^s)(x)) = d(f^r(x) \cdot g^s(x)) = g^s(x)d(f^r(x)) + f^r(x)d(g^s(x))$$

$$= rg^s(x)f^{r-1}(x)d(f(x)) + sf^r(x)g^{s-1}(x)d(g(x))$$

$$= rg^s(x)f^{r-1}(x)f'(x)d(x) + sf^r(x)g^{s-1}(x)g'(x)d(x)$$

$$= (rf^{r-1}f' \cdot g^s + sf^r \cdot sg^{s-1}g')(x)d(x) = (f^r g^s)'(x)d(x).$$

This shows that $d$ is a derivation with respect to $f^r \cdot g^s$. If $r = s = 1$, then the above argument can be applied not only for positive $f$ and $g$, hence we obtain that $d$ is a derivation with respect to the product function $f \cdot g$. \qed

The following result summarizes the most basic properties of real derivations.

**Theorem A.** (Kuczma [13] pp. 346-352.) If $d : \mathbb{R} \to \mathbb{R}$ is a derivation, then $d(x) = 0$ for every algebraic number $x \in \mathbb{R}$. On the other hand, for every non-algebraic number $t \in \mathbb{R}$, there exists a derivation $d : \mathbb{R} \to \mathbb{R}$ such that $d(t) \neq 0$. If $d : \mathbb{R} \to \mathbb{R}$ is a derivation which is upper bounded on a set of positive measure, then $d$ is identically zero.

A basic question that have been dealt with in several papers is to find conditions for additive functions which imply that this function is a standard derivation.

The following result is due to Nishiyama and Horinouchi [21] (cf. Boros–Gselmann [3]).

**Theorem B.** Let $d : \mathbb{R} \to \mathbb{R}$ be a derivation with respect to $S_2$ and let $r \in \mathbb{Q} \setminus \{0, 1\}$. Assume that $d$ satisfies the equation (5) for all $x \in \mathbb{R}_+$. Then $d$ is a standard derivation.

The particular cases $r = -1$ and $r = 2$ have been discovered by Kurepa [14], [15]. Results in the same spirit have also been established by Boros and Erdei [2].

**Theorem C.** Let $d : \mathbb{R} \to \mathbb{R}$ be a derivation with respect to $S_2$. Assume that $d$ satisfies the equation

$$d(\sqrt{1-x^2}) = -\frac{x}{\sqrt{1-x^2}}d(x)$$

for all $x \in ]-1, 1[$. Then $d$ is a standard derivation.

The following theorem was proved by Gyula Maksa [19].

**Theorem D.** Let $d : \mathbb{R} \to \mathbb{R}$ be a derivation with respect to $S_2$ and to one of the exponential, trigonometric or hyperbolic functions. Then $d$ is a standard derivation.

Motivated by the above results, the purpose of this paper is to show that functions that derivate the two-variable product function and one of the exponential, trigonometric or hyperbolic functions are also standard derivations. The more general problem considered is to describe finite subsets of $\mathcal{A}$ such that derivations with respect to this set are automatically standard derivations.

2. MAIN RESULTS

The main assumption in this section is that $d : \mathbb{R} \to \mathbb{R}$ is a derivation with respect to $P_2$. Then, under various circumstances, we prove that $d$ must be a standard derivation.
2.1. The basic lemma. The key result for our approach will be the Lemma formulated below. First, for \( t \in \mathbb{R} \setminus \{0, -1\} \), define the set \( H_t \) by

\[
H_t := \left\{ \frac{1}{t}, t, -1 - \frac{1}{t}, -\frac{t}{1 + t}, -\frac{1}{1 + t}, -1 - t \right\}.
\]

Observe that \( H_t \) is a set which is invariant with respect to the mappings \( \rho(s) := 1/s \) and \( \sigma(s) := -1 - s \). Furthermore, this set contains exactly six elements, unless \( t \in \{1, -2, -\frac{1}{2}\} \). In the latter case \( H_t \) contains exactly three elements.

**Lemma 1.** Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \) and \( U \subseteq \mathbb{R} \) be a set such that, for all \( t \in \mathbb{R} \setminus \{0, -1\} \), the set \( H_t \) intersects \( U \). Assume that

\[
d(u) = d(u + 1) \quad (u \in U).
\]

Then \( d \) is a standard derivation.

**Proof.** We are going to show that the conditions of the lemma imply that if \( x + y + z = 0 \), then \( d(x) + d(y) + d(z) = 0 \) (where \( x, y, z \in \mathbb{R} \)). If \( xyz = 0 \), then this statement is the consequence of the oddness of \( d \). Therefore, we may assume that \( xyz \neq 0 \). Using the condition \( x + y + z = 0 \), observe that

\[
\left\{ \frac{x}{y}, \frac{y}{x}, \frac{z}{x}, \frac{x}{z}, \frac{y}{z}, \frac{z}{y} \right\} = H_{\frac{1}{x}}.
\]

Therefore, by our assumption, one of the above elements belongs to \( U \). Due to the symmetry, we may assume that \( u := \frac{x}{y} \in U \). Then, by (6) and the Leibniz Rule, we obtain

\[
0 = d(u) - d(u + 1) = \frac{d(x)}{y} \left( \frac{y}{x} + \frac{z}{y} \right) = \frac{d(x)y - xd(y)}{y^2} + \frac{d(z)y - zd(y)}{y^2} = \frac{d(x) + d(y) + d(z)}{y}.
\]

This implies that \( d(x) + d(y) + d(z) = 0 \). In particular, if \( x, y \in \mathbb{R} \) are arbitrary and \( z = -x - y \), then we obtain \( d(x) + d(y) + d(z) = 0 \), which implies that \( d(x) + d(y) - d(x + y) = 0 \) and completes the proof of the additivity of \( d \). Thus, \( d \) is a standard derivation. \( \square \)

**Corollary 2.** Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \) and let \( R \subseteq ]0, 1[ \) such that one of the following conditions holds:

(i) \( 1 \notin R + R \);
(ii) \( 1 \notin (\mathbb{Z} + R)(\mathbb{Z} + R) \);
(iii) \( -1 \notin (\mathbb{Z} + R)(\mathbb{Z} + R) \).

Assume that

\[
d(u) = d(u + 1) \quad (u \in \mathbb{R} \setminus (\mathbb{Z} + R)).
\]

Then \( d \) is a standard derivation.

**Proof.** In view of Lemma 1, it suffices to show that, for all \( t \in \mathbb{R} \setminus \{0, -1\} \), the set \( H_t \) intersects \( U := \mathbb{R} \setminus (\mathbb{Z} + R) \). If condition (i) holds, then we show that \( H_t \supseteq \{ t, -1 - t \} \cap U \neq \emptyset \). Indirectly suppose that \( t \notin U \) and \( -1 - t \notin U \). Then \( t \in \mathbb{Z} + R \) and \( -1 - t \in \mathbb{Z} + R \), hence \( -1 \in \mathbb{Z} + (R + R) \), which implies that \( 1 \in R + R \) contradicting (i). Assume that condition (ii) holds. Then we prove that \( H_t \supseteq \{ t, \frac{1}{t} \} \cap U \neq \emptyset \). Similarly as before, if the previous intersection were empty, then \( t \in \mathbb{Z} + R \) and \( \frac{1}{t} \in \mathbb{Z} + R \), hence \( 1 \in (\mathbb{Z} + R)(\mathbb{Z} + R) \) would be valid, which contradicts (ii). Finally, suppose that condition (iii) holds. In this case we show that \( H_t \supseteq \{ t, -1 - \frac{1}{t} \} \cap U \neq \emptyset \).
If this were not valid, then we would obtain that $t \in \mathbb{Z} + R$ and $-1 - \frac{1}{t} \in \mathbb{Z} + R$. The last relation is equivalent to $-\frac{1}{t} \in \mathbb{Z} + R$. Thus, we would get $-1 \in (\mathbb{Z} + R)(\mathbb{Z} + R)$, which is impossible by condition (iii).

**Corollary 3.** Let $d : \mathbb{R} \to \mathbb{R}$ be a derivation with respect to $P_2$ and let further $0 \leq r < 1$ be a constant. Assume that

$$d(u) = d(u + 1) \quad (u \in \mathbb{R} \setminus (\mathbb{Z} + r)).$$

Then $d$ is a standard derivation.

**Proof.** If $0 < r \neq \frac{1}{2}$, then $1 \notin \{r\} + \{r\}$, hence condition (i) of the previous corollary holds with $R := \{r\}$, which yields the statement in this case. If $r = \frac{1}{2}$, then $1 \notin (\mathbb{Z} + \{r\})(\mathbb{Z} + \{r\})$ because otherwise, for some $n, k \in \mathbb{Z}$, we have that $1 = (n + \frac{1}{2})(k + \frac{1}{2})$, which is impossible. Thus, condition (ii) of the previous corollary holds with $R := \{r\}$ again. Finally, suppose that $r = 0$. If $t \in \mathbb{R} \setminus \mathbb{Z}$, then $t \in (\mathbb{R} \setminus \mathbb{Z}) \cap H_t$. If $t \in \mathbb{Z}$, then $\frac{1}{t} \notin (\mathbb{R} \setminus \mathbb{Z}) \cap H_t$, unless $t = 1$. In this remaining case $-\frac{1}{t} = -\frac{1}{t} \in (\mathbb{R} \setminus \mathbb{Z}) \cap H_t$, which proves that $H_t$ intersects $\mathbb{R} \setminus \mathbb{Z}$ for all $t \neq 0, -1$. Thus, the statement follows from Lemma 1.

**Corollary 4.** Let $d : \mathbb{R} \to \mathbb{R}$ be a derivation with respect to $P_2$. Denote by $I$ one of the following 6 intervals: $I_1 := \mathbb{R}$, $I_2 := [-2, -1]$, $I_3 := -1 - \frac{1}{2}$, $I_4 := [-\frac{1}{2}, 0]$, $I_5 := [0, 1]$, and $I_6 := [1, \infty]$. Assume that

$$d(u) = d(u + 1) \quad (u \in I).$$

Then $d$ is a standard derivation.

**Proof.** In view of Lemma 1 it suffices to show that, for all $t \in \mathbb{R} \setminus \{0, -1\}$, the set $H_t$ intersects $U := I$. Define the maps $\sigma : \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ by $\sigma(t) = -1 - t$ and $\rho(t) = 1/t$, respectively. One can easily see that

$$\sigma(I_1) = I_6, \quad \sigma(I_2) = I_5, \quad \sigma(I_3) = I_4, \quad \sigma(I_4) = I_3, \quad \sigma(I_5) = I_2, \quad \sigma(I_6) = I_1,$$

$$\rho(I_1) = I_4, \quad \rho(I_2) = I_3, \quad \rho(I_3) = I_2, \quad \rho(I_4) = I_1, \quad \rho(I_5) = I_6, \quad \rho(I_6) = I_5.$$

If $t \in \mathbb{R} \setminus \{-1, 0\}$, then it belongs to one of the intervals $I_i$ ($i \in \{1, \ldots, 6\}$). Then one of the elements $t, \rho(t) = 1/t, \sigma(\rho(t)) = -1 - 1/t, \rho(\sigma(\rho(t))) = -t/(1 + t), \rho(\sigma(t)) = -1/(1 + t)$, and $\sigma(t) = -1 - t$ will belong to $I$, proving that $I$ intersects $H_t$.

### 2.2. Periodic functions.

In the following result we deal with functions that are derivations with respect to $P_2$ and derive a periodic or antiperiodic function. Given two constants $0 \leq q < p$, we define the set $p\mathbb{Z} + q$ by

$$p\mathbb{Z} + q := \{pk + q \mid k \in \mathbb{Z}\}.$$

We call a function $f : \mathbb{R} \setminus (p\mathbb{Z} + q) \to \mathbb{R}$ $p$-periodic if, for all $x \in \mathbb{R} \setminus (p\mathbb{Z} + q)$,

$$f(x + p) = f(x).$$

We say that $f$ is $p$-antiperiodic if, for all $x \in \mathbb{R} \setminus (p\mathbb{Z} + q)$,

$$f(x + p) = -f(x).$$

**Theorem 5.** Let $d : \mathbb{R} \to \mathbb{R}$ be a derivation with respect to $P_2$, let $0 \leq q < p$ be constants and let $f : \mathbb{R} \setminus (p\mathbb{Z} + q) \to \mathbb{R}$ be a differentiable $p$-periodic or $p$-antiperiodic function such that $f'(x) \neq 0$ for all $x \in \mathbb{R} \setminus (p\mathbb{Z} + q)$. If

$$d(f(x)) = f'(x)d(x)$$

holds for all $x \in \mathbb{R} \setminus (p\mathbb{Z} + q)$, then $d$ is a standard derivation.
Proof. In view of Lemma C, \( d \) is odd and we have \( d(0) = d(1) = d(-1) = 0 \).

Assume first that \( f \) is \( p \)-periodic. Then also its derivative is \( p \)-periodic. Therefore, using (9) twice, for all \( x \in \mathbb{R} \setminus (p\mathbb{Z} + q) \), we get
\[
f'(x)d(x) = d(f(x)) = d(f(x + p)) = f'(x + p)d(x + p) = f'(x)d(x + p).
\]

Hence, for all \( x \in \mathbb{R} \setminus (p\mathbb{Z} + q) \),
\[
d(x) = d(x + p),
\]

(10)
i.e., \( d \) is \( p \)-periodic.

Furthermore, in each of the above cases, \( d \) is a standard derivation.

If \( f \) is \( p \)-antiperiodic, then \( f' \) is also \( p \)-antiperiodic, therefore, using the oddness of \( d \), we similarly get that
\[
f'(x)d(x) = d(f(x)) = d(-f(x + p)) = -d(f(x + p)) = f'(x + p)d(x + p) = f'(x)d(x + p).
\]

Thus, \( d \) is \( p \)-periodic in this case, too.

If \( 0 < q < p \), then \( 0 \notin p\mathbb{Z} + q \), hence (10) with \( x = 0 \) implies that \( d(p) = 0 \). We show that \( d(p) = 0 \) is also valid for all \( x \). In this case (10) holds for all \( x \in \mathbb{R} \setminus p\mathbb{Z} \). In particular, it is valid for \( x = -\frac{p}{2} \), hence
\[
d\left(-\frac{p}{2}\right) = d\left(\frac{p}{2}\right),
\]

which implies that \( d\left(\frac{p}{4}\right) = 0 \), whereby, using (10), \( d\left((2k + 1)\frac{p}{4}\right) = 0 \) holds for all \( k \in \mathbb{Z} \). By Lemma C, the zeros of \( d \) form a multiplicative subgroup in \( \mathbb{R} \), hence \( d\left(\frac{p}{4k+1}\right) = 0 \) is valid for all \( k, \ell \in \mathbb{Z} \). Now using the previous equation and substituting \( x = -\frac{p}{4} \) into (10), we obtain
\[
-d\left(\frac{p}{4}\right) = d\left(-\frac{p}{4}\right) = d\left(\frac{3p}{4}\right) = 3d\left(\frac{p}{4}\right),
\]

hence \( d\left(\frac{p}{4}\right) = 0 \). Thus,
\[
d(2) = d\left(\frac{2p}{4}\right) = 0,
\]

whence
\[
d(p) = d\left(\frac{p}{2}\right) = 2d\left(\frac{p}{4}\right) = 2d\left(\frac{p}{2}\right) + p = 0.
\]

Then, in each of the two cases, we have that \( d \) is \( p \)-homogeneous.

Let \( y \in \mathbb{R} \setminus (\mathbb{Z} + \frac{p}{4}) \) be arbitrary. Then \( py \in \mathbb{R} \setminus (p\mathbb{Z} + q) \), therefore, by (10), we obtain that
\[
d(y) = d(py) = d(py + p) = d(y + 1)
\]

for all \( y \in \mathbb{R} \setminus (\mathbb{Z} + \frac{p}{4}) \). Using Corollary 3, it follows that \( d \) is a standard derivation. \( \square \)

In the following consequence of Theorem 5, we obtain several equivalent conditions in terms of the derivation of trigonometric functions.

**Corollary 6.** Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \). Then the following statements are equivalent:

(i) \( d \) is a derivation with respect to the sine function;
(ii) \( d \) is a derivation with respect to the cosine function;
(iii) \( d \) is a derivation with respect to the tangent function;
(iv) \( d \) is a derivation with respect to the cotangent function.

Furthermore, in each of the above cases, \( d \) is a standard derivation.
Proof. Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \) and assume that condition (i) holds, that is,

\[
d(\sin(x)) = \cos(x)d(x) \quad (x \in \mathbb{R}).
\]

(11)

In order to apply Theorem 5, define \( f \) to be the restriction of sine to the set \( \mathbb{R} \setminus (\pi \mathbb{Z} + \frac{\pi}{2}) \). Then \( f \) is \( \pi \)-antiperiodic and \( f' \) does not vanish on \( \mathbb{R} \setminus (\pi \mathbb{Z} + \frac{\pi}{2}) \). The equation (11) yields that (9) also holds for \( x \in \mathbb{R} \setminus (\pi \mathbb{Z} + \frac{\pi}{2}) \). Therefore, by Theorem 5, it follows that \( d \) is a standard derivation.

Substituting \( x = \pi \) into (11), we get that \( d(\pi) = 0 \). By the additivity of \( d \), Lemma B implies that \( d(\frac{\pi}{2}) = 0 \). Therefore, using (11), for all \( x \in \mathbb{R} \), we obtain

\[
d(\cos(x)) = d\left(\sin\left(x + \frac{\pi}{2}\right)\right) = \cos\left(x + \frac{\pi}{2}\right)d\left(x + \frac{\pi}{2}\right) = -\sin(x)(d(x) + d\left(\frac{\pi}{2}\right)) = -\sin(x)d(x),
\]

which proves that condition (ii) also holds. Based on Lemma C, then \( d \) is a derivation with respect to the tangent and cotangent functions, i.e., conditions (iii) and (iv) also follow from (i).

In the second part of the proof, suppose that \( d : \mathbb{R} \to \mathbb{R} \) is a derivation with respect to \( P_2 \) and condition (ii) holds, that is,

\[
d(\cos(x)) = -\sin(x)d(x) \quad (x \in \mathbb{R}).
\]

(12)

Defining the function \( f \) as the restriction of cosine to the set \( \mathbb{R} \setminus (\pi \mathbb{Z}) \), Theorem 5 implies that \( d \) is a standard derivation. Then, following a similar train of thought as above, we can get that conditions (i), (iii), and (iv) are also valid.

In the third part, assume that \( d : \mathbb{R} \to \mathbb{R} \) is a derivation with respect to \( P_2 \) and condition (iii) holds, that is,

\[
d(\tan(x)) = \frac{d(x)}{\cos^2(x)} \quad (x \in \mathbb{R} \setminus (\pi \mathbb{Z} + \frac{\pi}{2})).
\]

(13)

Applying Theorem 5 for \( f := \tan \), it immediately follows that \( d \) is a standard derivation. Substituting \( x = \pi \) into (13), we get that \( d(\pi) = 0 \). By the additivity of \( d \), Lemma B implies that \( d(\frac{\pi}{2}) = 0 \). Therefore, using that \( d \) is a standard derivation and that (13) holds, for all \( x \in \mathbb{R} \setminus (2\pi \mathbb{Z} + \pi) \), we obtain

\[
d(\sin(x)) = d\left(\frac{2\tan(x/2)}{1 + \tan^2(x/2)}\right) = \frac{2(1 - \tan^2(x/2))}{(1 + \tan^2(x/2))^2}d(\tan(x/2)) = \frac{2(1 - \tan^2(x/2))}{(1 + \tan^2(x/2))^2} \cdot \frac{d(x/2)}{\cos^2(x/2)} = \cos(x)d(x),
\]

which proves that condition (i) is satisfied.

In the fourth part, suppose that \( d : \mathbb{R} \to \mathbb{R} \) is a derivation with respect to \( P_2 \) and condition (iv) holds, that is,

\[
d(\cot(x)) = -\frac{d(x)}{\sin^2(x)} \quad (x \in \mathbb{R} \setminus (\pi \mathbb{Z})).
\]

(14)

Using Theorem 5 for \( f := \cot \), it immediately follows that \( d \) is a standard derivation. Then, following a similar train of thought as in the third part, we can get that conditions (i) is also valid. \( \square \)
2.3. Exponential and hyperbolic functions.

**Theorem 7.** Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \). Assume that \( d(2) = 0 \). Then the following statements are equivalent:

(i) \( d \) is a derivation with respect to the sine hyperbolic function;
(ii) \( d \) is a derivation with respect to the cosine hyperbolic function;
(iii) \( d \) is a derivation with respect to the tangent hyperbolic function;
(iv) \( d \) is a derivation with respect to the cotangent hyperbolic function;
(v) \( d \) is a derivation with respect to the exponential function.

Furthermore, in each of the above cases, \( d \) is a standard derivation.

**Proof.** Let \( d : \mathbb{R} \to \mathbb{R} \) be a derivation with respect to \( P_2 \) and assume that condition (i) holds, that is,

\[
d(\sinh(x)) = \cosh(x)d(x) \quad (x \in \mathbb{R}).
\]

The property \( d(2) = 0 \) implies that \( d \) is 2-homogeneous. Using the Leibniz Rule and that \( \sinh(2x) = 2 \sinh(x) \cosh(x) \) and \( \cosh(2x) = \cosh^2(x) + \sinh^2(x) \) hold for all \( x \in \mathbb{R} \), we obtain

\[
2(\sinh(x)d(\cosh(x)) + \cosh^2(x)d(x)) = 2(\sinh(x)d(\cosh(x)) + \cosh(x)d(\sinh(x)))
\]

\[
= d(2 \sinh(x) \cosh(x)) = d(\sinh(2x))
\]

\[
= \cosh(2x)d(2x) = 2(\cosh^2(x) + \sinh^2(x))d(x).
\]

Hence

\[
\sinh(x)d(\cosh(x)) = \sinh^2(x)d(x)
\]

holds. This equality, for \( x \neq 0 \), simplifies to

\[
d(\cosh(x)) = \sinh(x)d(x),
\]

which is also valid for \( x = 0 \). Therefore, condition (i) implies condition (ii). Based on Lemma \([\text{C}]\) then \( d \) is a derivation with respect to the tangent and cotangent hyperbolic functions, i.e., conditions (iii) and (iv) also follow from (i). Since \( \exp = \sinh + \cosh \), thus \( d \) is also a derivation with respect to the exponential function, i.e., (v) holds.

In the second part, we first show that condition (ii) implies that \( d \) is a standard derivation and then that conditions (i), (iii) and (iv) also consequences of (ii). Using the assumptions and some well-known identities, we get

\[
d(2 \cosh^2(x) - 1) = d(\cosh(2x)) = \sinh(2x)d(2x)
\]

\[
= 4 \sinh(x) \cosh(x)d(x) = 4 \cosh(x)d(\cosh(x)) = d(2 \cosh^2(x))
\]

holds for all \( x \in \mathbb{R} \). Substituting \( u := 2 \cosh^2(x) - 1 \), we obtain

\[
d(u) = d(u + 1) \quad (u \geq 1),
\]

hence, in view of Corollary \([\text{D}]\) \( d \) is a standard derivation. To justify that condition (ii) implies (i), we use Lemma \([\text{C}]\) and that \( d \) is a derivation. We have that

\[
d(\sinh(x)) = d\left(\sqrt{\cosh^2(x) - 1}\right) = \frac{1}{2} \frac{d(\cosh^2(x) - 1)}{\sqrt{\cosh^2(x) - 1}} = \frac{1}{2} \frac{2 \cosh(x) \sinh(x)d(x)}{\sinh(x)} = \cosh(x)d(x)
\]

is valid for all \( x \geq 0 \). By the oddness of \( d \), the above equality also holds for \( x < 0 \) and therefore \( d \) is a derivation with respect to the tangent and cotangent hyperbolic functions, i.e., conditions
(iii) and (iv) also follow from (ii). Since \( \exp = \sinh + \cosh \), thus \( d \) is also a derivation with respect to the exponential function.

In the third part, observe that, in view of Lemma C, (iii) and (iv) are equivalent to each other. Thus, it suffices to show that condition (iv) implies that \( d \) is a standard derivation and then that conditions (i) and (ii) are also consequences of (iv). By the assumption and some well-known identities for hyperbolic functions, we obtain

\[
d\left( \frac{\coth^2(x) + 1}{2 \coth(x)} \right) = d(\coth(2x)) = \frac{1}{\sinh^2(2x)} d(2x) = \frac{1}{4 \sinh^2(x) \cosh^2(x)} 2d(x)
\]

\[
= \frac{1}{2 \cosh^2(x)} d(\coth(x)) = \frac{\coth^2(x) - 1}{2 \coth^2(x)} d(\coth(x))
\]

holds for all \( x \in \mathbb{R} \). Substituting \( t := \coth(x) \), we get

\[
d\left( \frac{t^2 + 1}{2t} \right) = \frac{t^2 - 1}{2t^2} d(t) \quad (|t| > 1).
\]

Using the Leibniz Rule again, we obtain

\[
\frac{(t^2 + 1)d(t) - td(t^2 + 1)}{t^2} = \frac{t^2 - 1}{t^2} d(t) \quad (|t| > 1).
\]

Therefore, after reduction, we have that

\[
d(t^2) = d(t^2 + 1) \quad (|t| > 1),
\]

hence with \( u := t^2 \),

\[
d(u) = d(u + 1) \quad (u > 1).
\]

The above equality also holds for \( u = 1 \), thus \( d \) is a standard derivation by Corollary 4. Using this, we show that condition (iv) implies (i). Indeed,

\[
d(\sinh(x)) = d\left( \frac{1}{\sqrt{\coth^2(x) - 1}} \right) = \frac{1}{2} \cdot \frac{2 \coth(x)}{\sqrt{\coth^2(x) - 1}} \cdot \frac{-1}{\sinh^2(x)} d(x) = \cosh(x) d(x)
\]

is valid for all \( x > 0 \). By the oddness of \( d \), the above equality also holds for \( x \leq 0 \). A similar computation yields that (ii) is also a consequence of property (iv). Since \( \exp = \sinh + \cosh \), thus it follows that \( d \) is also a derivation with respect to the exponential function.

To complete the proof, assume that condition (v) holds, i.e.,

\[
d(e^x) = e^x d(x) \quad (x \in \mathbb{R}). \tag{15}
\]

Then, using the Leibniz Rule and (15) three times, for \( x, y \in \mathbb{R} \), we obtain

\[
d(x + y) = e^{-x-y} d(e^{x+y}) = e^{-x-y} d(e^x e^y) = e^{-x-y} (e^y d(e^x) + e^x d(e^y))
\]

\[
= e^{-x-y} (e^y e^x d(x) + e^x e^y d(y)) = d(x) + d(y).
\]

Therefore, \( d \) is a derivation with respect to \( S_2 \), thus it is a standard derivation, indeed. Since \( \sinh(x) = \frac{1}{2}(e^x - e^{-x}) \), therefore \( d \) is a derivation with respect to the sine hyperbolic function, whence (i)-(iv) follow immediately. \( \square \)

**Remark 8.** In the above theorem, for those implications, where condition (v) is supposed, it is not necessary to assume that \( d(2) = 0 \). The question whether the above theorem can be proved without the condition \( d(2) = 0 \) remains open.
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