RIEMANN SURFACE OF THE RIEMANN ZETA FUNCTION

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Abstract. In this paper we treat the classical Riemann zeta function as a function of three variables: one is the usual complex 1-dimensional, customly denoted as $s$, another two are complex infinite dimensional, we denote it as $b = \{b_n\}_{n=1}^\infty$ and $z = \{z_n\}_{n=1}^\infty$. When $b = \{1\}_{n=1}^\infty$ and $z = \{\frac{1}{n}\}_{n=1}^\infty$ one gets the usual Riemann zeta function. Our goal in this paper is to study the meromorphic continuation of $\zeta(b, z, s)$ as a function of the triple $(a, z, s)$.

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1. Introduction.

1.1. Euler-Riemann zeta function. For a real $\sigma$ let $H_\sigma := \{s \in \mathbb{C} : \text{Re}s > \sigma\}$ be the half-plane. For a complex variable $s \in H_1$ the classical Riemann or, better Euler-Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \left(\frac{1}{n}\right)^s = \zeta\left(\frac{1}{n}, s\right).$$

(1.1)

In the right hand side of (1.1) we interpret $\frac{1}{n} := \{\frac{1}{n}\}_{n=1}^\infty$ as the distinguished element of the following complex sequence space

$$l_\frac{1}{n} := \{z = \{z_n\}_{n=1}^\infty : \|z\|_\frac{1}{n} := \sup_{n \in \mathbb{N}} (n|z_n|) < \infty\}.$$  

(1.2)

We shall also use the notation $n^{-1}$ for $\frac{1}{n}$ as well as $l_{n^{-1}}$ for $l_\frac{1}{n}$. More generally:

Definition 1. Let $r = \{r_k\}_{k \in \mathbb{N}}$ be a sequence of positive reals. The following space of complex sequences

$$l_r := \left\{z = \{z_n\}_{n=1}^\infty : \|z\|_r := \sup_{n \in \mathbb{N}} \frac{|z_n|}{r_n} < \infty\right\}.$$  

(1.3)

we shall call the $r$-sequence space or, simply an $r$-space.
l₁ is a Banach space with respect to the norm \( \| \cdot \|_r \). In fact it is isometrically isomorphic to \( l^\infty \). Actually \( l^\infty \) appears to be \( l₁ \) according to this definition, here \( 1 := \{1\}^\infty_{n=1} \). By \( B_r^\infty(z^0,\beta) \) denote the open ball of radius \( \beta > 0 \) in the Banach space \( l₁ \), centered at \( z^0 \).

Notice that for \( \mathbf{z} = \{z_n\} \in B^\infty_{n=1}(n^{-1},1) \) we have that \( \Re z_n > 0 \) and therefore we can fix that \( \text{Arg} \ z_n \in [-\pi/2,\pi/2] \).

For given \( b = \{b_n\}^\infty_{n=1} \in l^\infty \), \( \mathbf{z} = \{z_n\}^\infty_{n=1} \in B^\infty_{n=1}(n^{-1},1) \) and \( s \in H_1 \) set (with arguments of \( z_n \) taken as above):

\[
\zeta^{ER}(b,\mathbf{z},s) = \sum_{n=1}^\infty b_n z_n^s
\]

and call it still the Euler-Riemann zeta function. It is not difficult to observe, see Lemma 2.4, that \( \zeta^{ER} \) is holomorphic as the function of the triple \( (b,\mathbf{z},s) \) on \( l^\infty \times B^\infty_{n=1}(n^{-1},1) \times H_1 \).

Variable \( \mathbf{z} \) we shall call a vectorial variable \( \zeta^{ER} \) and \( b \) a (complex) parameter. \( \zeta^{ER}(b,\mathbf{z},s) \) will be denoted simply as \( \zeta(b,\mathbf{z}) \) whenever this will not lead to a confusion. Notice that \( \zeta(b,\mathbf{z},s) \) is certainly not well defined on the coordinate cross \( \mathcal{C}_{n=1}^\infty = \{ \mathbf{z} = \{z_n\}^\infty_{n=1} \in \mathcal{L}_{n=1} : z_n = 0 \text{ for some } n \} \) at least for a general \( b \in l^\infty \). The closure of \( \mathcal{C}_{n=1}^\infty \) in \( \mathcal{L}_{n=1} \) is

\[
\bar{\mathcal{C}}_{n=1}^\infty = \left\{ z \in \mathcal{L}_{n=1} : \inf_{n \in \mathbb{N}} |z_n| = 0 \right\},
\]

see Lemma 2.1. We prove in Lemma 2.5 that \( \zeta \) can be continued to a (multivalued) analytic function on \( l^\infty \times (\mathcal{L}_{n=1} \setminus \bar{\mathcal{C}}_{n=1}^\infty) \times H_1 \), again as a function of the triple \( (b,\mathbf{z},s) \).

**Remark 1.** One should be a bit careful when speaking about multivalued analyticity and universal covers in infinite dimensions. In our case the space \( \mathcal{L}_{n=1} \setminus \bar{\mathcal{C}}_{n=1}^\infty \) admits the universal cover since it is semi-locally simply connected, see Corollary 14 in [Sp].

Denote by \( \mathcal{M} \) the maximal set over \( l^\infty \times (\mathcal{L}_{n=1} \setminus \bar{\mathcal{C}}_{n=1}^\infty) \) which consists of such \( (b,\mathbf{z}) \) that \( \zeta(b,\mathbf{z},\cdot) \) can be analytically continued to a meromorphic function on \( \mathbb{C} \). By the classical theorem of Riemann \( (1,n^{-1}) \in \mathcal{M} \), here \( 1 \) and \( n^{-1} \) are sequences defined as above. Furthermore denote by \( \mathcal{Z} \) the maximal set over \( l^\infty \times (\mathcal{L}_{n=1} \setminus \bar{\mathcal{C}}_{n=1}^\infty) \times \mathbb{C} \) to which \( \zeta(b,\mathbf{z},s) \) can be analytically continued as a meromorphic function, i.e., \( \mathcal{Z} \) is the “Riemann surface” of \( \zeta \). Notice that \( \mathcal{Z} \supset \mathcal{M} \times \mathbb{C} \). In this paper we are interested in the following:

**Question:** What can be said about the structure of \( \mathcal{M} \) and \( \mathcal{Z} \)?

Our first result states that \( \mathcal{M} \) is “quite thick” and, moreover, possesses a certain Banach analytic structure. Denote by \( \mathcal{L}_{e^{-n}} \) the sequence space defined by the sequence \( e^{-n} := \{e^{-n}\}^\infty_{n=1} \), i.e.,

\[
\mathcal{L}_{e^{-n}} := \{ \{z_n\}^\infty_{n=1} : \|z\|_{\mathcal{L}_{e^{-n}}} := \sup_{n \in \mathbb{N}} (e^n |z_n|) < \infty \}.
\]

(1.5)

Notice that \( \mathcal{L}_{e^{-n}} \) is obviously densely imbedded into \( \mathcal{L}_{n=1} \). We prove the following:

**Theorem 1.** For every \( (b^0,\mathbf{z}^0) \in \mathcal{M} \) the following holds:

i) \( (b^0,\mathbf{z}^0 + B^\infty_0(0,\|\mathbf{z}^0\|_{\mathcal{L}_{n=1}})) \subset \mathcal{M} \), or in other words \( \{(b^0,\mathbf{z}^0 + B^\infty_0(0,\|\mathbf{z}^0\|_{\mathcal{L}_{n=1}}))\} \times \mathbb{C} \subset \mathcal{Z} \);

ii) the restriction \( \zeta(b^0,\cdot,\cdot) \) of \( \zeta \) to \( \{b^0\} \times \{\mathbf{z}^0 + B^\infty_0(0,\|\mathbf{z}^0\|_{\mathcal{L}_{n=1}})\} \times \mathbb{C} \) can be analytically continued as a multivalued meromorphic function to \( \{b^0\} \times (\{\mathbf{z}^0 + \mathcal{L}_{e^{-n}}\} \setminus \bar{\mathcal{C}}_{n=1}^\infty) \times \mathbb{C} \).

**Remark 2.** 1. First of all let’s make clear that for \( \mathbf{z}^0 \in \mathcal{L}_{n=1} \) one has

\[
B^\infty_0(0,\|\mathbf{z}^0\|_{\mathcal{L}_{n=1}}) = \{ z \in \mathcal{L}_{e^{-n}} : \|z\|_{\mathcal{L}_{e^{-n}}} < \|\mathbf{z}^0\|_{\mathcal{L}_{n=1}} \}.
\]

2. When writing \( (b^0,\mathbf{z}^0) \in \mathcal{M} \) we mean, in particular, that \( (b^0,\mathbf{z}^0) \in l^\infty \times (\mathcal{L}_{n=1} \setminus \bar{\mathcal{C}}_{n=1}^\infty) \).

Therefore due to Lemma 2.3 we have that \( \{\mathbf{z}^0 + \mathcal{L}_{e^{-n}}\} \setminus \bar{\mathcal{C}}_{n=1}^\infty = \{\mathbf{z}^0 + \mathcal{L}_{e^{-n}}\} \setminus \mathcal{C}_{n=1}^\infty \) and,
moreover, in the same lemma we prove that \( \{ z^0 + l_{e-a} \} \cap \mathbb{C}_{n-1}^\infty \) is a countable union of hyperplanes such that every bounded subset of the affine plane \( \{ z^0 + l_{e-a} \} \) meets only finitely many of them. Therefore an analytic continuation within its complement makes well sense.

3. Let \( L_n := \{ z = \{ z_m \}_{m=1}^\infty \in l_{e-a} : z_n + z^0_n \neq 0 \} \) be one of these hyperplanes. Then it is easy to see that the monodromy of \( \zeta(b^0, \cdot, s) \) around this hyperplane along a loop \( \gamma_n \) in \( z_n \)-plane starting at point \( z^0_n + z_n \neq 0 \) and going in the positive direction is \( b^0_n(z_n^0 + z_n)^s(e^{2\pi i s} - 1) \). Since \( \{ z^0 + l_{e-a} \} \cap \mathbb{C}_{n-1}^\infty = \bigcup_n L_n \) the fundamental group of \( \{ z^0 + l_{e-a} \} \setminus \mathbb{C}_{n-1}^\infty \) is generated by such \( \gamma_n \)-s and is abelian. This shows that \( \zeta(b^0, \cdot, \cdot) \) lifts to the universal cover of \( \{ z^0 \} \times (\{ z^0 + l_{e-a} \} \setminus \mathbb{C}_{n-1}^\infty) \times \mathbb{C} \) and separates points in the fiber (for a non-integer \( s \)).

4. Since \( \{ z^0 + l_{e-a} \} \times \mathbb{C} \) is a dense affine subspace of \( l_{n-1} \times \mathbb{C} \) this statement means that the Riemann surface of the function \( \zeta \) contains a huge quantity of domains spread over parallel dense affine subspaces of the total space \( l^\infty \times l_{n-1} \times \mathbb{C} \). Indeed, apart from \( (b^0, z^0) = (1, n^{-1}) \) we know quite a number of pairs \( (b^0, z^0) \in \mathcal{M} \). If, for example, \( \alpha = \sqrt{D} \) is a quadratic irrationality then \( \{ (n\alpha)_{n=1}^\infty, n^{-1} \} \in \mathcal{M} \) due to results of Hecke and Hardy-Littlewood, see [He, HL]. More examples can be found in [Pa], [KL] and many other places.

5. Notice as well that the result of this theorem can be reformulated in terms of the general Dirichlet series, see subsection 2.5.

6. It should be noticed that the analytic continuation in this theorem preserves the poles of \( \zeta(b^0, z^0, s) \).

1.2. Zeta function in the form of Lerch-Lipschitz. In order to produce new pairs \( (b, z) \in \mathcal{M} \) let us make the following substitution to \( \zeta \):

\[
\begin{cases}
    b_n = e^{2\pi i a_n} \\
    z_n = \frac{1}{n + \xi_n}
\end{cases}
\]  

We obtain the function \( \zeta \) in the form of Lerch-Lipschitz

\[
\zeta^{LL}(a, z, s) = \sum_{n=1}^\infty \frac{e^{2\pi i a_n}}{(n + z_n)^s}. 
\]  

Variable \( \xi \) we immediately renamed as \( z \), but parameter \( a \) will distinguish the Lerch-Lipschitz form of the function zeta from that of Euler-Riemann and \( \zeta^{LL} \) will be renamed back to \( \zeta \). This function is certainly well defined and holomorphic for \( a \in l^\infty_+ := \{ a \in l^\infty : \text{Im} a_n > 0 \}, z = \{ z_n \} \in l^\infty_{Re+:} = \{ z \in l^\infty : \text{Re} z_n > 0 \} \) and \( s \in H_1 \), see Corollary 3.1.

Consider the following “\( k \)-dimensional cut” \( \zeta_k \) of function \( \zeta \):

\[
\zeta_k(a, z, s) = \sum_{n=1}^\infty \frac{e^{2\pi i a_n}}{(n + z_n)^s} = \frac{e^{2\pi i a_1}}{(1 + z_1)^s} + \frac{e^{2\pi i a_2}}{(1 + z_2)^s} + \frac{e^{2\pi i a_k}}{(k + 1 + z_1)^s} + \cdots 
\]  

Here \( a = (a_1, \ldots, a_k) \in \mathbb{C}_{l_{m^+}}^{k}, z = (z_1, \ldots, z_k) \in \mathbb{C}_{Re^+}^{k} \) and \( [m]_k \) is the remainder of the division of \( m \) by \( k \). I.e., \( \zeta_k \) is, in fact, \( \zeta \) itself but restricted to the finite dimensional subspace of “\( k \)-periodic sequences” \( \{ (a_{[n-1]_k + 1}, z_{[n-1]_k + 1}) \}_{n=1}^\infty \). Our second result is the following:

Theorem 2. Function \( \zeta_k \) admits an analytic continuation to a single-valued meromorphic function \( \tilde{\zeta}_k \) on

\[
\tilde{\zeta}_k := \tilde{\zeta}_{k,ab}^{\mathbb{C}} z^{[1/k]}_\mathbb{Z} \times \tilde{\zeta}_{k,ab}^{\mathbb{C}} z^{-} \times \mathbb{C}.
\]
Here we denote by \( \mathbb{Z}[1/k] \) the subring of \( \mathbb{Q} \) which consists from rationals of the form \( \{ l/k : l \in \mathbb{Z} \} \) for a fixed \( k \). \( \overline{\mathbb{C}}_{\mathbb{Z}[1/k]}^k \) stands for the abelian cover of \( \mathbb{C}_{\mathbb{Z}[1/k]}^k := (\mathbb{C} \setminus \mathbb{Z}[1/k])^k \). 

\( \overline{\mathbb{C}}_{\mathbb{Z}[1/k]}^k \) is the abelian cover of \( \mathbb{C}^{-\infty} := (\mathbb{C} \setminus \mathbb{Z}^-) \times \ldots \times (\mathbb{C} \setminus \mathbb{Z}^-_k) \), where \( \mathbb{Z}^- = \mathbb{Z}_{<0} \) is the set of negative integers and \( \mathbb{Z}^-_k := \{ z \in \mathbb{Z}^- : [z]_k = -j \} \). The statement of the theorem means that the monodromy of \( \zeta_k \) along any loop generating the commutator of \( \pi_1(\mathbb{C}^k_{\mathbb{Z}[1/k]} \times \mathbb{C}^{k-1}_-^\infty) \) is trivial and that \( \tilde{\mathbb{Z}}_k \) is the genuine Riemann surface of \( \zeta_k \). Moreover, this theorem shows that for any \( k \) the set \( \mathcal{M} \) is infinitely sheeted over the space of \( k \)-periodic sequences. This provides us a huge quantity of new pairs \( (a, z) \in \mathcal{M} \).

**Remark 3.** Periods of \( a \)-s and \( z \)-s in Theorem 2 can be taken different, say \( k_1 \) and \( k_2 \). Then setting \( k = \text{lcm}(k_1, k_2) \) we can apply our theorem.

1.3. The structure of the paper and proofs. We organize the material of this paper as follows. First, in section 2 we recall and prove few topological properties of sequence spaces in order to clarify the statement of Theorem 1. After that we prove this theorem. The needed material on complex and pluricomplex analysis on sequence spaces is moved to the Appendix. In particular, the following infinite dimensional version of Hartogs-Siciak theorem will be proved there (i.e., in Appendix). Given complex sequence spaces \( l_{1_2} \) and \( l_{1_2} \).

**Theorem 3.** Let a holomorphic function \( f \) on \( B_{l_{1_2}}^\infty(0,1) \times B_{l_{1_2}}^\infty(0,1) \) be such that for every \( z \) in some plurithick subset \( E \) of \( B_{l_{1_2}}^\infty(0,1) \) the restriction \( f(z,\cdot) \) can be continued to a holomorphic function on \( l_{r_2} \). Then \( f \) can be continued to a holomorphic function on \( B_{l_{1_2}}^\infty(0,1) \times l_{r_2} \).

Second, we prove Theorem 2 in section 3 it doesn’t requires any infinite dimensional analysis and is accessible right away, i.e., without section 2. Our proof follows the main lines of the original approach of Riemann [Rm] and then of Lerch [Le] and, finally of Lagarias and Li in [LL], where the case \( k = 1 \) of Theorem 2 was proved. The particularity of our approach is again the use of (finite dimensional) Hartogs-Siciak theorem which makes the exposition considerably shorter and accessible when \( k > 1 \).

2. Vectorial variable in the Riemann zeta function

We start this section recalling the notion of a sequence space and pointing out few facts about their vectorial topology. They are absolutely needed for the very understanding of the statement of Theorem 1 from the Introduction as well as for its proof. After that we turn to the Riemann zeta function and to the proof of the mentioned theorem. Needed facts from complex and pluricomplex analysis on sequence spaces are placed to the Appendix.

2.1. Vectorial topology on sequence spaces. Throughout this paper we consider various complex vector spaces of infinite sequences of complex numbers \( z = \{ z_k \} \), they are customly called the sequence spaces. If, for some reasons, we shall need to consider real sequences in the complex sequence space \( l_r \), then the corresponding real vector space will be denoted as \( l_{r,\mathbb{R}} \). The standard basis of any sequence space is \( e_1 = (1,0,...,0,...),...,e_k = (0,...,0,k,0,...) \) etc. The ball of radius \( \beta \) centered at \( z \) in \( l_r \) will be denoted as \( B_r^\infty(z,\beta) \). The corresponding topology will be called the standard one or, an r-topology.

A polyradius is a sequence \( \rho = \{ \rho_k \} \) such that all \( \rho_k > 0 \). If \( \rho = \{ \rho_k \} \in l_r \) we call it an r-polyradius. Fix an polyradius \( \rho = \{ \rho_k \}_{k=1}^\infty \). A polydisk of polyradius \( \rho \) is the set
\(\Delta^\infty(z^0, \rho) := \{z \in l_r : |z_k - z_k^0| < \rho_k \text{ for all } k\}\). It is not an open subset of \(l_r\) in general. The “closed” polydisk is \(\overline{\Delta^\infty}(z^0, \rho) := \{z \in l_r : |z_k - z_k^0| \leq \rho_k\}\) and it is a closed subset of \(l_r\).

For a real positive \(\beta\) and a polyradius \(\rho\) we set \(\beta \rho = \{\beta \rho_n\}_{n=1}^\infty\). Note that for the special polyradius \(r\) we have that \(\Delta^\infty(0, \beta r) \subset \overline{B_r^\infty}(0, \beta) \subset \Delta^\infty(0, (\beta + \varepsilon) r)\) for any \(\varepsilon > 0\). We also denote \(\Delta^\infty(0, \beta \frac{1}{n})\) as \(\Delta^\infty(0, \frac{\beta}{n})\). We call the polydisk \(\Delta^\infty(0, \beta r)\) congruent to \(\Delta^\infty(0, r)\), the same for \(\Delta^\infty(0, \beta r) = z^0 + \Delta^\infty(0, \beta r)\) with \(\Delta^\infty(z^0, r) = z^0 + \Delta^\infty(0, r)\).

By \(n^{-1}\) or by \(\frac{1}{n}\) we denoted the sequence/polyradius \(\{\frac{z_n}{r_n}\}_{n=1}^\infty\) and by \(l_{n-1}\) or by \(\overline{l}_n\) the corresponding sequence space. We set \(l := \{1\}_{n=1}^\infty\), and therefore \(l^\infty\) (\(= l_1\)) is the sequence space of bounded complex sequences with the norm \(\|z\|_\infty := \sup_n |z_n|\). We shall also consider polyradii \(r^n := \{r^n_n\}_{n=1}^\infty\) and \(\beta r^n := \{\beta r^n_n\}_{n=1}^\infty\) for \(\beta, n > 0\). Notice that these polyradii are not in \(l_1\) in general.

Now let us turn to the coordinate cross \(C^\infty_r := \left\{\{z_n\}_{n=1}^\infty \in l_r : z_n = 0 \text{ for some } n\right\}\) in \(l_r\).

**Lemma 2.1.** The closure of the cross \(C^\infty_r\) in \(r\)-topology is equal to

\[
\overline{C^\infty_r} = \left\{z \in l_r : \inf_{n \in \mathbb{N}} \frac{|z_n|}{r_n} = 0 \right\},
\]

and the open set \(l_r \setminus \overline{C^\infty_r}\) is pathwise connected.

**Proof.** Let \(\overline{C^\infty_r}\) be the set defined by \((2.1)\). Take some \(z \in l_r \setminus \overline{C^\infty_r}, \text{ i.e., such that}\ \varepsilon := \inf_{n \in \mathbb{N}} \frac{|z_n|}{r_n} > 0\). The ball \(B^\infty(z, \varepsilon)\) obviously doesn’t intersect \(\overline{C^\infty_r}\) and this proves that the complement to \(\overline{C^\infty_r}\) is open. On the other hand take any \(z \in \overline{C^\infty_r}\). Then for any \(\beta > 0\) there exists \(n\) such that \(\frac{|z_n|}{r_n} < \beta\). That means that the point \(z^0 := z - z_n e_n \in \overline{C^\infty_r} \cap B^\infty(z, \beta)\). This proves that \(\overline{C^\infty_r}\), defined by \((2.1)\) is the closure of \(C^\infty_r\) in \(l_r\).

To prove the connectivity of \(l_r \setminus \overline{C^\infty_r}\) take some other \(w \in l_r \setminus \overline{C^\infty_r}\) with \(\delta := \inf_{n \in \mathbb{N}} \frac{|w_n|}{r_n} > 0\). We need to construct a continuous path from \(w\) to \(z\) in \(l_r \setminus \overline{C^\infty_r}\). Consider the following path: in every coordinate go from \(w_n\) to \(|w_n|\) along the circle of radius \(|w_n|\) in the clockwise direction, then to \(|z_n|\) along the real axis and then along the circle of radius \(|z_n|\) to \(z_n\) in the counterclockwise direction. I.e., \(\gamma = \gamma_0 + \gamma_1 + \gamma_2\), where \(\gamma_0(t) = \{|w_n|e^{i(t-t_i)\text{Arg} w_n}\}\), \(\gamma_1(t) = \{|w_n| + t(|z_n| - |w_n|)\}_{n=1}^\infty\) and \(\gamma_2(t) = \{|z_n|e^{it\text{Arg} z_n}\}\). In all cases \(t \in [0, 1]\) and \(\text{Arg}\) is supposed to take its values in \([0, 2\pi]\). Remark that for every \(t \in [0, 1]\) one has that

\[
\frac{|\gamma_0^n(t)|}{r_n} = \frac{|w_n|}{r_n} \geq \delta > 0,
\]

which shows that \(\gamma_0\) is a path in \(l_r \setminus \overline{C^\infty_r}\). The same for \(\gamma_2\). As for \(\gamma_1\) we see that

\[
\frac{|\gamma_1^n(t)|}{r_n} = \frac{(1-t)|w_n| + t|z_n|}{r_n} \geq \min\{\varepsilon, \delta\} > 0,
\]

which, again proves that \(\gamma_1\) is a path in \(l_r \setminus \overline{C^\infty_r}\). To check the continuity of \(\gamma_0\) take some \(t_0 \in [0, 1]\) and \(t_1 \in [0, 1]\) and write

\[
\left|\{|w_n|e^{i(t-t_1)\text{Arg} w_n} - |w_n|e^{i(t-t_0)\text{Arg} w_n}\}\right| \leq \frac{3}{2}|w_n||t_1 - t_0|2\pi.
\]

This implies that \(\|\gamma_0(t_1) - \gamma_0(t_0)\|_r \leq 3\pi \|w\|_r |t_1 - t_0|\), i.e., that \(\gamma_0\) is \(r\)-continuous. The same work for \(\gamma_2\). As for \(\gamma_1\) write

\[
\frac{|\gamma_1^n(t_1) - \gamma_1^n(t_0)|}{r_n} = \frac{|(1-t_1)|w_n| + t_1|z_n| - (1-t_0)|w_n| - t_0|z_n|}{r_n} = \frac{|t_0 - t_1|(w_n + z_n)}{r_n}.
\]
This shows that \( \| \gamma_1(t_1) - \gamma_1(t_0) \|_r \leq |t_1 - t_0| (\| z \|_r + \| w \|_r) \), i.e., that \( \gamma_1 \) is continuous. 

It is easy to see that paths appeared in the proof of this lemma have \textit{bounded argument}. By saying this we mean the following:

**Definition 2.1.** We say that a continuous path \( \gamma(t) = \{ \gamma_n(t) : t \in [0,1] \}_{n \in \mathbb{N}} \) in \( l_r \setminus C_r^\infty \) is a path with a bounded argument if for every \( n \in \mathbb{N} \) there exists a continuous choice of \( \text{Arg} \gamma_n(t) \) such that

\[
\Pi := \sup_{t \in [0,1]} \{ |\text{Arg} \gamma_n(t)| \} < +\infty. \tag{2.2}
\]

**Lemma 2.2.** Every continuous path in \( l_r \setminus \overline{C_r}^\infty \) is a path with a bounded argument.

**Proof.** First let us make a remark about paths in the complement of the cross. Notice that for a point \( p = \{ p_n \} \in l_r \setminus \overline{C_r}^\infty \) one has that

\[
\text{dist} (p, \overline{C_r}^\infty) = \text{dist} (p, C_r^\infty) = \inf_{n \in \mathbb{N}} \left\{ \frac{|p_n|}{r_n} \right\}. \tag{2.3}
\]

Indeed,

\[
\text{dist} (p, C_r^\infty) = \inf_{n \in \mathbb{N}} \text{dist}(p, \{ z_n = 0 \}) = \inf_{n \in \mathbb{N}} \{ \inf \{ \| p - z \|_r : z_n = 0 \} \} = \inf_{n \in \mathbb{N}} \left\{ \frac{|p_n|}{r_n} \right\}.
\]

Therefore for a continuous path \( \gamma \) in \( l_r \setminus \overline{C_r}^\infty \) the distance from \( \gamma(t) \) to \( \overline{C_r}^\infty \) is equal to

\[
\frac{|\gamma_n(t)|}{r_n} \geq d \quad \text{for all } n. \tag{2.5}
\]

Now for a continuous path \( \gamma : [0,1] \rightarrow l_r \setminus \overline{C_r}^\infty \) fix the argument of every \( \gamma_n(0) \) in the interval \([0,2\pi]\) and then extend it continuously in \( t \in [0,1] \) for every \( n \). Since \( \gamma \) is uniformly continuous we see that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\left| \frac{\gamma_n(t_1)}{r_n} - \frac{\gamma_n(t_2)}{r_n} \right| \leq \varepsilon \quad \text{for all } n, \tag{2.6}
\]

provided \( |t_1 - t_2| \leq \delta \). Rewrite this as

\[
\left| \frac{\gamma_n(t_1)}{r_n} e^{i\text{Arg} \gamma_n(t_1)} - \frac{\gamma_n(t_2)}{r_n} e^{i\text{Arg} \gamma_n(t_2)} \right| \leq \varepsilon \quad \text{for all } n. \tag{2.7}
\]

Via (2.5) we see that

\[
\varepsilon \geq \left| \frac{\gamma_n(t_1)}{r_n} e^{i\text{Arg} \gamma_n(t_1)} - \frac{\gamma_n(t_2)}{r_n} e^{i\text{Arg} \gamma_n(t_2)} \right| = \left| \frac{\gamma_n(t_1)}{r_n} \right| \left| e^{i\text{Arg} \gamma_n(t_1)} - e^{i\text{Arg} \gamma_n(t_2)} \right| \geq d \left| e^{i\text{Arg} \gamma_n(t_1)} - e^{i\text{Arg} \gamma_n(t_2)} \right|.
\]

Which shows that \( |\text{Arg} \gamma_n(t_1) - \text{Arg} \gamma_n(t_2)| \leq \frac{2\pi}{2\pi d} = \frac{\pi}{d} \) for all \( n \) provided \( |t_1 - t_2| \leq \delta \). This proves the assertion of the lemma.
Remark 2.1. a) The (non-open) set $l \setminus \mathbb{C}^\infty$ is connected as well. To see this take for a given points $z, w \in l \setminus \mathbb{C}^\infty$ the line $l$ joining them. $l \cap (l \setminus \mathbb{C}^\infty)$ is either empty (i.e., $l$ is contained in $\mathbb{C}^\infty$) or, is a complement to an at most countable set $l \cap \mathbb{C}^\infty$. In our case $l$ is not contained in $\mathbb{C}^\infty$. Therefore $l \cap (l \setminus \mathbb{C}^\infty)$ is connected. This implies the connectivity (by paths) of $l \setminus \mathbb{C}^\infty$.

b) At the same time not every continuous path in $l \setminus \mathbb{C}^\infty$ is a path with a bounded argument. For example $\gamma(t) = \left(\frac{1}{n^{1+t}}\right)_{n=1}^\infty$ is a $C^\infty$-path in $l_{n-1} \setminus \mathbb{C}^\infty$ with an unbounded argument.

c) The intersection $l \cap \mathbb{C}^\infty$, where $l$ is a complex line such that $l \not\subset \mathbb{C}^\infty$, can be an arbitrary countable set in the topology of $l$. Consider, for example, the following line $l$ in $l$

$$z_n = e^{i\theta_n}r_n^2 - r_n\lambda, \quad \lambda \in \mathbb{C}, \text{ and } \theta_n \in [0, 2\pi) \text{ are fixed.}$$

Here we assume that the sequence $\{r_n\}$ is bounded. Then $l \cap \mathbb{C}^\infty = \{r_n e^{i\theta_n}\}_{n=1}^\infty$ in the topology of $l$.

d) Notice also that $\mathbb{C}^\infty$ contains the line $l_1 = \{(\lambda, \{r_n^2 \lambda\}_{n=2}^\infty) : \lambda \in \mathbb{C}\}$ entirely (provided $r_n \to 0$), but $l_1 \cap \mathbb{C}^\infty = \{0\}$.

Lemma 2.3. Let $\rho = \{\rho_n\}_{n=1}^\infty$ and $r = \{r_n\}_{n=1}^\infty$ be two polyradii such that $\frac{\rho_n}{r_n} \to 0$. Then for a point $z^0 = \{z^0_n\}_{n=1}^\infty \in l \setminus \mathbb{C}^\infty$ one has

$$\{z^0 + l_\rho\} \cap \mathbb{C}^\infty = \{z^0 + l_\rho\} \cap \mathbb{C}^\infty. \quad (2.8)$$

Moreover, the set on the right-hand side of (2.8) is a countable union of complex hyperplanes leaving any bounded in the topology of $l_\rho$ subset of the affine plane $z^0 + l_\rho$.

Proof. For a given $z^0 = \{z^0_n\} \in l \setminus \mathbb{C}^\infty$ let $z = \{z_n\}_{n=1}^\infty \in l_\rho$ be such that $z^0 + z \in \mathbb{C}^\infty$, i.e., we have that

$$\inf_{n \in \mathbb{N}} \frac{|z^0_n + z_n|}{r_n} = 0, \quad \text{but} \quad \inf_{n \in \mathbb{N}} \frac{|z^0_n|}{r_n} = : \varepsilon > 0. \quad (2.9)$$

Since

$$\beta := \sup_{n \in \mathbb{N}} \frac{|z_n|}{\rho_n} < +\infty \quad (2.10)$$

we get

$$\frac{|z^0_n + z_n|}{r_n} \geq \frac{|z^0_n|}{\rho_n} \frac{\rho_n}{r_n} \geq \varepsilon - \beta \frac{\rho_n}{r_n}. \quad (2.11)$$

Due to the assumed $\rho_n/r_n \to 0$ we see that

$$\frac{|z^0_n + z_n|}{r_n} \geq \frac{\varepsilon}{2} \quad \text{for} \quad n \gg 1. \quad (2.12)$$

Therefore the first infimum in (2.11) is achieved for a finite $n$, i.e., $z^0_n + z_n = 0$ for some $n$. This proves that $z^0 + z \in \mathbb{C}^\infty$. The opposite inclusion being trivial this proves the identity (2.8).

Setting $L_n := \{z = \{z_m\}_{m=1}^\infty \in l_\rho : z_n + x^0_n = 0\}$ we see that

$$\{z^0 + l_\rho\} \cap \mathbb{C}^\infty = \bigcup_{n \in \mathbb{N}} L_n,$$

is a countable union of hyperplanes. Suppose that an infinite number of $L_n$-s do intersect some bounded in the topology of $l_\rho$ subset of the affine hyperplane $z^0 + l_\rho$. That means that $\exists z^0 + z^k \in L_{nk}$ with $n_k \to \infty$ such that $\|z^k\|_\rho \leq C$ for some constant $C$ independent of
Proof. Let \( z^0 + z^k \in L_{nk} \) means that \( z^k_{nk} = -z^0_{nk} \). Therefore \(|z^0_{nk}|/\rho_{nk} \leq C\). At the same time \(|z^0_{nk}|/r_{nk} \geq \varepsilon > 0\) because it was assumed that \( z^0 \in l_\varepsilon \subset \bar{C}_\varepsilon\). This is a contradiction, indeed
\[
\frac{|z^0_{nk}|}{r_{nk}} = \frac{|z^0_{nk}|}{\rho_{nk}} \frac{\rho_{nk}}{r_{nk}} \leq C \frac{\rho_{nk}}{r_{nk}} \to 0.
\]
\( \square \)

2.2. Riemann zeta function, definition. For \( \sigma \in \mathbb{R} \) we had set \( H_\sigma := \{ s \in \mathbb{C} : \text{Re} \, s > \sigma \} \). In particular \( H_1 = \{ z \in \mathbb{C} : \text{Re} \, z > 1 \} \), \( H := H_0 = \{ s \in \mathbb{C} : \text{Re} \, s > 0 \} \), as well as \( H_+ = \emptyset \) and \( H_- = \mathbb{C} \). For \( b \in l^\infty \), \( z = \{ z_n \} \in B_{n-1}^\infty (n^{-1}, 1) \) and \( s \in H_1 \) we had set
\[
\zeta(b, z, s) = \sum_{n=1}^\infty b_n z_n^s.
\]
(2.11)

For the usual Riemann zeta function \( \zeta \) one has \( \zeta(s) = \zeta(1, n^{-1}, s) \) in our notations.

**Lemma 2.4.** Function \( \zeta \) is well defined and holomorphic on \( l^\infty \times B_{n-1}^\infty (n^{-1}, 1) \times H_1 \) as a function of the triple \((b, z, s)\).

**Proof.** Remark that for \( z = \{ z_n \}_{n=1}^\infty \in B_{n-1}^\infty (n^{-1}, 1) \) one has \( \text{Re} \, z_n > 0 \) for all \( n \) and therefore the principal determinations \( \text{Ln} z_n = \ln |z_n| + i \text{Arg} z_n \) with \( \text{Arg} z_n \in (-\pi/2, \pi/2) \) are well defined. Write now
\[
\sum_{n=1}^\infty |b_n z_n^s| = \sum_{n=1}^\infty |b_n| e^{s \cdot \text{Ln} z_n} = \sum_{n=1}^\infty |b_n| e^{\text{Re} \, s \cdot \text{log} |z_n| - \text{Im} \, s \cdot \text{Arg} z_n}.
\]

Take an exhaustive sequence of compacts in \( H_1 \), say
\[
K = \{ s : 1 + 1/k \leq \text{Re} \, s \leq k \text{ and } |\text{Im} \, s| \leq k \}.
\]
(2.12)

Since \( |\text{Arg} z_n| \leq \pi/2 \) we see that \( e^{-\text{Im} \, s \cdot \text{Arg} z_n} \leq e^{k \pi/2} \) for all \( n \in \mathbb{N} \) and all \( s \in K \). From this fact and from the observation that \( |z_n| < \frac{2}{n} \) for \( z \in B_{n}^\infty (n^{-1}, 1) \) we obtain
\[
\sum_{n=1}^\infty |b_n z_n^s| \leq \| b \| e^{k \pi/2} \sum_{n=1}^\infty \left( \frac{2}{n} \right)^{1+1/k}
\]
(2.13)

uniformly on \( B \times B_{n}^\infty (n^{-1}, 1) \times K \) for any bounded \( B \subset l^\infty \). This implies the needed statement. Indeed, the partial sums
\[
\sum_{n=1}^N b_n z_n^s = \sum_{n=1}^N b_n e^{s \cdot \text{Ln} z_n}
\]
(2.14)

of (2.11) are functions of finitely many variables, which are obviously holomorphic, and by what was just proved, converge to \( \zeta(b, z, s) \) uniformly on \( B \times B_{n}^\infty (n^{-1}, 1) \times K \) for every bounded \( B \subset l^\infty \) and compact \( K \subset H_1 \). Therefore \( \zeta(b, z, s) \) is holomorphic, see Proposition 0.2 in the Appendix.

\( \square \)

Let us give a definition, though obvious, of the notion of analytic continuation in \( l_\varepsilon \) (in any complex Banach space in fact). Let \( \gamma : [0, 1] \to l_\varepsilon \) be a continuous path and let \( f_0 \) be a holomorphic function (an element) in some open connected \( V_0 \ni \gamma(0) \).
Definition 2.2. An analytic continuation of the element $f_0$ along $\gamma$ is a collection of open connected neighborhoods $V_t$ of $\gamma(t)$ and $r$-holomorphic on $V_t$ functions $f_t$ such that for $|t_1 - t_2|$ small enough one has

$$f_t|_{V_t \cap V_{t_2}} = f_{t_2}|_{V_t \cap V_{t_2}},$$

(2.15)

Lemma 2.5. Function $\zeta$ admits an analytic continuation along any continuous path in $l^\infty \times (l_{n-1} \setminus \bar{C}_{n-1}) \times H_1$.

(2.16)

Proof. Along the proof of the lemma we shall assume that $b$ is fixed in some bounded open $B \subset l^\infty$ and $s$ is fixed in $K := \{ 1 + 1/k \leq \Re s \leq k, |\Im s| \leq k \}$. Take a path $\gamma = \{ \gamma(t) : t \in [0,1] \}$ from the distinguished point $n^{-1}$ to some $z = \{ z_n \}$ in $l_{n-1} \setminus \bar{C}_{n-1}$, with the argument bounded by $\Pi$ as in Definition 2.1. Due to (2.4) the distance from $\gamma(t)$ to $\bar{C}_{n-1}$ is separated from zero, say by $\varepsilon > 0$. Therefore

$$|\gamma_n(t)| \geq \frac{\varepsilon}{n} \quad \text{for all} \quad n \in \mathbb{N}. \quad (2.17)$$

Step 1. Continuation along $\gamma$. As open sets take $V_t := B \times B_{n-1}^\infty(\gamma(t), \varepsilon) \times H_1$. One obviously has that $V_t \subset l_{n-1} \setminus \bar{C}_{n-1}$ for every $t$ due to (2.17), as well as $V_{t_1} \cap V_{t_2} \neq \emptyset$ for $t_1, t_2 \in [0,1]$ close enough. Function $\zeta(b, w, s)$ for $(b, w, s) \in V_t$ define naturally as follows. Notice first that every $w = \{ w_n \} \in B^\infty(\gamma(t), \varepsilon)$ can be uniquely written in the form

$$w_n = \gamma_n(t) \left[ 1 + r_n e^{i\theta_n} \right] \quad \text{for} \quad n = 1, \ldots$$

(2.18)

with some $\theta_n \in [0,2\pi)$ and some $0 \leq r_n < 1$. The latter is because one should have $|n r_n \gamma_n(t)| < \varepsilon$. We set

$$\zeta_t(b, w, s) = \sum_{n=1}^{\infty} b_n w^n = \sum_{n=1}^{\infty} b_n e^{s \ln w_n}, \quad (2.19)$$

were the arguments are defined as $\text{Arg} w_n = \text{Arg} \gamma_n(t) + \text{Arg} \left[ 1 + r_n e^{i\theta_n} \right]$. The first summand was predefined by the boundedness of argument condition imposed on $\gamma$ and the second varies in $]-\pi/2, \pi/2[$ and is determined uniquely by $r_n$ and $\theta_n$. Remark that series (2.19) converge normally on $B \times B_{n-1}^\infty(\gamma(t), \varepsilon) \times K$. Indeed, for $b \in B$, $w \in B_{n-1}^\infty(\gamma(t), \varepsilon)$ and $s \in K := \{ 1 + 1/k \leq \Re s \leq k, |\Im s| \leq k \}$ our series can be estimated as follows

$$\sum_{n=1}^{\infty} |b_n \gamma_n(t)^s \left[ 1 + r_n e^{i\theta_n} \right]^s| \leq \|b\|_\infty e^{(\Pi+\pi/2)k} \sum_{n=1}^{\infty} |\gamma_n(t)| \left[ 1 + r_n e^{i\theta_n} \right]^{\Re s} \leq \|b\|_\infty e^{(\Pi+\pi)k} \sum_{n=1}^{\infty} \frac{1}{n^{\Re s}}. \quad (2.20)$$

Here we used the identity $|a^s| = |a|^{\Re s} e^{-\Im s \text{Arg} a}$ and the bound (2.2) imposed on the argument of $\gamma$. Remark that the estimate (2.20) is uniform along $B \times B_{n-1}^\infty(\gamma(t), \varepsilon) \times K$ because the $l_{n-1}$-norm of $\gamma(t)$ is bounded. This proves that $\zeta_t$ is holomorphic in $V_t$.

Step 2. The fact that $\zeta_{t_1}|_{V_{t_1} \cap V_{t_2}} = \zeta_{t_2}|_{V_{t_1} \cap V_{t_2}}$ is obvious since both are defined by the same series (2.19) and the arguments of $w_n$ do not depend on $t_1 \sim t_2$. The latter is again due to the possibility to choose each $\text{Arg} \gamma_n(t)$ continuously and uniformly bounded on $n$. Lemma is proved. □
Remark 2.2. We know that every \( z \in l_{n-1} \) can be joined with \( n^{-1} \) by a path with bounded argument. Moreover, consider the following path from \( \frac{1}{n} \) to \( z \): in every coordinate go from \( \frac{1}{n} \) first to \( |z_n| \) along the real axis and then along the circle of radius \( |z_n| \) in the counterclockwise direction. I.e., \( \gamma = \gamma_1 + \gamma_2 \), where \( \gamma_1(t) = \{ \frac{1}{n} + t(|z_n| - \frac{1}{n}) \}_{n=1}^{\infty} \) and \( \gamma_2(t) = \{|z_n|e^{it\text{Arg}z_n} \} \), in both cases \( t \in [0, 1] \). \( \text{Arg} \) can be supposed to vary in \([0, 2\pi)\) only.

2.3. Taylor expansion of function zeta. For \( (b, z, s) \in l^\infty \times B_{n-1}^\infty(n^{-1}, 1) \times H_1 \) we have

\[
\zeta(b, z, s) = \sum_{n=1}^{\infty} b_n z_n^s,
\]

and therefore we can easily compute all partial derivatives of \( \zeta \) with respect to \( z \) to get

\[
\frac{\partial^m \zeta}{\partial z^m}(b, z, s) = s(s-1)...(s-m+1)b_n z_n^{s-m} \quad \text{for all } n, m \in \mathbb{N}.
\]

Corresponding \( m \)-homogeneous polynomial in the Taylor expansion of \( \zeta \) at \( z \) is

\[
P_m(b, z, w, s) = \frac{s(s-1)...(s-m+1)}{m!} \sum_{n=1}^{\infty} b_n z_n^{s-m} w_n^m.
\] (2.21)

This gives the Taylor expansion of zeta function at \( (b, z, s) \in l^\infty \times B_{n-1}^\infty(n^{-1}, 1) \times H_1 \) with respect to the variable \( z \):

\[
\zeta(b, z + w, s) - \zeta(b, z, s) = \sum_{m=1}^{\infty} \frac{s(s-1)...(s-m+1)}{m!} \sum_{n=1}^{\infty} b_n z_n^{s-m} w_n^m.
\] (2.22)

It occurs that the right hand side of (2.22) makes sense and converge for \( z = \{z_n\} \in l_{n-1} \setminus C_{n-1}^\infty \) with the silent assumption that arguments of \( z_n \) are chosen to be bounded, i.e.,

\[
\Pi := \sup \{ |\text{Arg} z_n| \} < \infty.
\] (2.23)

For example \( z \) is an endpoint of a path with bounded argument.

Lemma 2.6. For the series on the right hand side of (2.22) the following holds true.

i) For every \((b, z, s) \in l^\infty \times (l_{n-1} \setminus C_{n-1}) \times H_1\), all polynomials \( P_m(b, z + w, s) \) are \( n^{-1} \)-bounded and the series (2.22) uniformly converge on \( \{ \text{bounded sets in } l^\infty \} \times \bar{B}_{n-1}^\infty(0, \beta) \times \{ \text{compacts of } H_1 \} \) for every \( 0 < \beta < ||z||_{n-1} \).

ii) For every \( 0 \leq \eta < +\infty \) and every \( 0 < \beta < ||z||_{n-1} \), the power series expansion (2.22) converge uniformly on \( \{ \text{bounded sets in } l^\infty \} \times \bar{\Delta}_{\infty} \left( 0, \frac{\beta}{n^{1-\eta}} \right) \times \{ \text{compacts of } H_{1-\eta} \} \).

iii) For all \( 0 < \eta < +\infty \) and \( 0 < \beta < ||z||_{n-1} \), the power series expansion (2.22) converge uniformly on \( \{ \text{bounded sets in } l^\infty \} \times \bar{\Delta}_{\infty} \left( 0, \frac{\beta e^{-m}}{n} \right) \times \{ \text{compacts of } C \} \).

Proof. Here, as usual, \( H_{1-\eta} := \{ \text{Res} > 1 - \eta \} \) and \( \tilde{\beta} \) is the polyradius \( \{ \beta \}_{n=1}^{\infty} \). By \( \frac{\beta}{n^{1-\eta}} \) we denote the polyradius \( \{ \tilde{\beta} \}_{n=1}^{\infty} \) and by \( \frac{\beta e^{-m}}{n} \) we denote the polyradius \( \{ \beta e^{-m} \}_{n=1}^{\infty} \).

(i) As for the \( n^{-1} \)-boundedness of \( P_m \) fix \((b, z) \in l^\infty \times (l_{n-1} \setminus C_{n-1}) \), \( w \in \bar{B}_{n-1}^\infty(0, \beta) \) and write, taking into account that \( |z_n| \leq ||z||_{n-1}/n \) and \( |w_n| < \beta/n \) for all \( n \), that

\[
\sum_{n=1}^{\infty} b_n z_n^{s-m} w_n^m \leq \sum_{n=1}^{\infty} |b_n| |z_n|^{s-m} e^{k\Pi} |w_n|^m \leq ||b||_\infty e^{k\Pi} \sum_{n=1}^{\infty} \beta e^{-m} |z_n||n^{-1}}^{s-m} \leq \sum_{n=1}^{\infty} \frac{\beta}{n^{1-\eta}} \frac{e^{k\Pi}}{n^{1-\eta}} |w_n|^m \leq (2.24)
\]
\[ \leq \|b\|_\infty e^{k_{\Pi}} \|z\|_{n-1}^{\Re s - m} \beta^m \sum_{m=1}^{\infty} \frac{1}{m^{\Re s}} = e^{k_{\Pi}} \|b\|_\infty \|z\|_{n-1}^{\Re s - m} \beta^m \zeta(\Re s). \]

Here \( s \) is taken in the compact \( \{1 + \frac{k}{2} \leq \Re s \leq k, -k \leq \Im s \leq k\} \). When \( \beta = 1 \) this gives the \( n^{-1} \)-boundedness of \( P_m(b, z, w, s) \). As for the convergence observe first the following Taylor expansion of the holomorphic function \( \lambda^s \) at 1

\[ \lambda^s = 1 + \sum_{m=1}^{\infty} \frac{s(s-1)...(s-m+1)}{m!} (\lambda - 1)^m. \]  

(2.25)

Since \( \lambda^s \) is holomorphic in \( H = \{\Re \lambda > 0\} \) the radius of convergence of series in (2.25) is 1. In particular series

\[ \sum_{m=1}^{\infty} \frac{|s(s-1)...(s-m+1)|}{m!} \]

converge absolutely for any \( |t| < 1 \). For \( w \in \Delta_\infty(0, \beta_n) \) we get from (2.24) that in (2.22) one has

\[ \sum_{m=1}^{\infty} |P_m(b, z, w, s)| \leq \zeta(\Re s) \|b\|_\infty \sum_{m=1}^{\infty} \frac{|s(s-1)...(s-m+1)|}{m!} \beta^m \|z\|_{n-1}^{\Re s - m} = \]

\[ = \zeta(\Re s) \|b\|_\infty \|z\|_{n-1}^{\Re s - m} \sum_{m=1}^{\infty} \frac{|s(s-1)...(s-m+1)|}{m!} \left( \frac{\beta}{\|z\|_{n-1}} \right)^m. \]  

Estimate (2.27) implies, provided \( \beta < \|z\|_{n-1} \), the normal convergence of (2.22) on \{ bounded sets in \( \mathbb{C}_h \times \mathbb{C}_n \) \} as claimed.

(\( ii \)) Every \( w = \{w_n\}^{\infty}_{n=1} \in \Delta_\infty(0, \beta_n) \) can be written as \( w_n = \frac{\beta_n}{\eta n^{1+\eta}} e^{i\theta_n} \), where \( 0 \leq r_n \leq 1 \) and \( \theta_n \in [0, 2\pi) \). Now (2.22) writes as

\[ \zeta(b, z + w, s) - \zeta(b, z, s) = \sum_{m=1}^{\infty} \frac{s(s-1)...(s-m+1)}{m!} \sum_{n=1}^{\infty} b_n n^{s-m} w_n^m = \]

\[ = \sum_{m=1}^{\infty} \frac{s(s-1)...(s-m+1)}{m!} \beta^m \sum_{n=1}^{\infty} b_n n^{s-m} e^{i\theta_n} n^{(1+\eta)m} z_n^{s-m}. \]  

(2.28)

Since \( z \in \mathbb{C}_{n-1} \setminus \bar{C}_{n-1} \), we see that for \( \Re s > 1 - \eta \) and \( w \in \Delta_\infty(0, \frac{\beta}{n^{1+\eta}}) \) (2.28) can be estimated as follows

\[ \sum_{m=1}^{\infty} \frac{|s(s-1)...(s-m+1)|}{m!} \beta^m \sum_{n=1}^{\infty} |b_n| \|z\|_{n-1}^{\Re s - m} e^{k_{\Pi}} \]

\[ = e^{k_{\Pi}} \|z\|_{n-1}^{\Re s} \sum_{m=1}^{\infty} \frac{|s(s-1)...(s-m+1)|}{m!} \beta^m \sum_{n=1}^{\infty} |b_n| \|z\|_{n-1}^{\Re s - m} e^{k_{\Pi}}. \]

Since \( \Re s + m\eta > 1 \) for \( m \geq 1 \) the term (\( ii \)) can be majored by \( \|b\|_\infty \zeta(\Re s + m\eta) \). Term (\( i \)) can be estimated by the remark about power series (2.26) since it was supposed that \( \beta < \|z\|_{n-1} \). Therefore sums (\( i \)) and (\( ii \)) converge on compacts in \( H_{1-\eta} \) when \( 0 < \beta < \|z\|_{n-1} \).
(iii) Every \( w \) in the polydisk \( \Delta^\infty \left( 0, \frac{\beta e^{-\eta n}}{n} \right) \) writes as \( w_n = \frac{\beta r_n e^{-\eta n}}{n} e^{i\theta_n} \) with \( 0 \leq r_n \leq 1 \), \( \theta_n \in [0, 2\pi) \). Therefore (2.22) writes as

\[
\zeta(b, z + w, s) - \zeta(b, z, s) = \sum_{m=1}^{\infty} s(s-1)\ldots(s-m+1) \frac{b_m^m e^{im\theta_n}}{n^m} e^{-\eta n m} z_n^{s-m}.
\]  

(2.29)

Since \( |z_n| \leq \|z\|_{n-1} / n \) the right-hand side of (2.28) can be estimated by

\[
\|z\|_{n-1} \sum_{m=1}^{\infty} \frac{|s(s-1)\ldots(s-m+1)|}{m!} \left( \frac{\beta}{\|z\|_{n-1}} \right)^m \sum_{n=1}^{\infty} e^{-\eta n m}.
\]  

(2.30)

And the second term in (2.30) converge normally on compacts in \( \mathbb{C} \) since

\[
\sum_{n=1}^{\infty} \frac{1}{n^\mathbb{R} e^{-\eta n m}} \leq \sum_{n=1}^{\infty} \frac{1}{n^\mathbb{R} e^{-\eta n}}
\]

for all \( m \geq 1 \). Convergence of (2.30) follows now from (2.26).

\[\square\]

2.4. Proof of Theorem 1

Now we turn to the proof of Theorem 1 from the Introduction. First let us state the item (iii) of Lemma 2.6 in a suitable form. Notice once more that writing that \( (b, z) \in \mathcal{M} \) we mean not only that \( \zeta(b, z, \cdot) \) extends meromorphically to \( \mathbb{C} \) but also that \( z \in \Omega_{-1} \setminus \mathbb{C}_{n-1} \).

**Corollary 2.1.** For every \( (b, z) \in \mathcal{M} \) the power series expansion

\[
\zeta(b, z + w, s) - \zeta(b, z, s) = \sum_{m=1}^{\infty} s(s-1)\ldots(s-m+1) \frac{b_m^m e^{im\theta_n}}{n^m} w_n^m
\]  

(2.31)

converge uniformly on \( B_e^{-\infty}(0, \beta) \times \{ \text{compacts of } \mathbb{C} \} \) for any \( 0 < \beta < \|z\|_{n-1} \). Moreover, this expansion provides the analytic continuation of \( \zeta(b, \cdot, \cdot) \) to \( \{ z + B_e^{-\infty}(0, \beta) \} \times \mathbb{C} \).

**Proof.** Indeed, notice that for \( 0 < \beta < \|z\|_{n-1} \) we have that

\[
B_e^{-\infty}(0, \beta) \subset \Delta^{\infty} \left( 0, \frac{\beta e^{-\eta n}}{n} \right)
\]  

(2.32)

in the topology of \( l_{-n} \). Therefore item (iii) of Lemma 2.6 gives us the uniform convergence of (2.31) on \( B_e^{-\infty}(0, \beta) \times \{ \text{compacts of } \mathbb{C} \} \). This results to the analytic continuation of the power expansion in question to \( B_e^{-\infty}(0, \beta) \times \mathbb{C} \). Since \( \zeta(b, z, \cdot) \) was supposed to extend meromorphically to \( \mathbb{C} \) we see that for every \( z + w \in \{ z + B_e^{-\infty}(0, \beta) \} \) function \( \zeta(b, z + w, \cdot) \) extends meromorphically to \( \mathbb{C} \).

Now recall that \( \zeta(b, \cdot, \cdot) \) is holomorphic on \( \left( \Omega_{-1} \setminus \mathbb{C}_{n-1} \right) \times H_1 \supset \{ z + B_e^{-\infty}(0, \|z\|_{n-1}) \} \times H_1 \). Remark that (a local) restriction of a \( n^{-1} \)-holomorphic function to \( l_{-n} \) is \( e^{-n} \)-holomorphic. This follows from the obvious Gâteaux differentiability and continuity of such restriction. Therefore we can apply Lemma 0.3 to \( \zeta(b, z + w, s) - \zeta(b, z, s) \) and conclude that it is holomorphic on \( \{ z + B_e^{-\infty}(0, \|z\|_{n-1}) \} \times \mathbb{C} \) as a function of a couple \( (z, s) \). This gives us the meromorphy of \( \zeta(b, z + w, s) \) on \( \{ z + B_e^{-\infty}(0, \beta) \} \times \mathbb{C} \) as stated.

\[\square\]

We continue the proof of Theorem 1. Let a point \( (b, z_0) \in \mathcal{M} \) be given, i.e., \( \zeta(b, z_0, s) \) extends in \( s \) to a meromorphic function on \( \mathbb{C} \). Then by Corollary 2.1 the difference \( \zeta(b, z, s) - \zeta(b, z_0, s) \) extends holomorphically to \( \{ z_0 + B_e^{-\infty}(0, \beta) \} \times \mathbb{C} \) for any \( 0 < \beta < \)}
Dirichlet series

Let us interpret the item (iii) of Lemma 2.6 in terms of the general Dirichlet series

\[
D(a, \lambda, s) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} \frac{a_n}{e^{\lambda_n s}},
\]

where \(a = \{a_n\}_{n=1}^{\infty} \in l^\infty\) and \(\lambda = \{\lambda_n\}_{n=1}^{\infty}\) are complex. We require that \(\sup |ne^{-\lambda_n}| < \infty\), i.e., that \(\{e^{-\lambda_n}\} \in l_{-1}\). If, for example, \(a_n = 1\) then we deal with a Dirichlet series tout court

\[
D(\lambda, s) := \sum_{n=1}^{\infty} e^{-\lambda_n s} = \sum_{n=1}^{\infty} \frac{1}{e^{\lambda_n s}}.
\]

Now (iii) means that for \(z_n = e^{-\lambda_n}\) one has that series (2.34) meromorphically extend to the whole of \(\mathbb{C}\) provided that for some 0 < \(\eta < \infty\) and 0 < \(\beta < 1\) one has

\[
|e^{-\lambda_n} - \frac{1}{n}| < \frac{\beta e^{-\eta n}}{n} \quad \text{for all} \quad n.
\]

Even better:

**Corollary 2.2.** Suppose that (2.33) extends meromorphically to \(\mathbb{C}\) for a given \(\{\lambda_n\}_{n=1}^{\infty}\). Then

\[
D(a, \mu, s) := \sum_{n=1}^{\infty} a_n e^{-\mu_n s} \tag{2.36}
\]

will extend to \(\mathbb{C}\) for all \(\{\mu_n\}_{n=1}^{\infty}\) such that for some 0 < \(\beta, \eta < \infty\) one has

\[
|e^{-\lambda_n} - e^{-\mu_n}| < \frac{\beta e^{-\eta n}}{n} \quad \text{for all} \quad n. \tag{2.37}
\]

I.e., the set of \(\lambda\) such that \(D(a, \lambda, s)\) is meromorphic on \(\mathbb{C}\) is open (and non-empty) in an appropriate sequence space. Remark, in addition, that according to (2.38) the abscissa of convergence of (2.34) is 1 for the classical case when \(\lambda_n = \ln n\).

**Remark 2.3.** Recall that for a general Dirichlet series (2.34) the abscissa of convergence is given by

\[
\sigma_c = \limsup_{n \to \infty} \frac{\ln |a_1 + \ldots + a_n|}{\lambda_n} \quad \text{or} \quad \sigma_c = \limsup_{n \to \infty} \frac{\ln |a_{n+1} + a_{n+2} + \ldots|}{\lambda_n} \tag{2.38}
\]

depending on either \(\sum_{n=1}^{\infty} a_n\) diverge or converge.
Remark 2.4. 1. Remark that the item (ii) of Lemma 2.6 is uniform on \( z \in \Delta^\infty \left( \frac{1}{n}, \frac{\beta}{n^2} \right) \).
Therefore the function \( \zeta_0(z, s) \) is continuous as a function of all variables on \( \Delta^\infty \left( \frac{1}{n}, \frac{\beta}{n^2} \right) \times H \)
in the standard sense.

2. When \( z \) moving in \( \Delta^\infty \left( \frac{1}{n}, \frac{\alpha}{n^2} \right) \) the pole \( s = 1 \) of \( \zeta(z, s) \) doesn’t move. But it moves in general. Let \( \eta \sim 1 \) then
\[
\zeta \left( \frac{1}{n\eta}, s \right) = \zeta(n, \eta s) = \frac{1}{\eta s - 1} + \zeta_0 \left( \frac{1}{n}, s \right).
\]
I.e., the pole of \( \zeta \left( \frac{1}{n\eta}, s \right) \) is \( \frac{1}{\eta} \) with residue \( \frac{1}{\eta} \).

3. Zeta function in the form of Lerch-Lipschitz

3.1. Zeta function in the form of Lerch-Lipschitz. By \( l^\infty_{\text{lm}+} = \{ \{a_n\} \in l^\infty, \text{Im} a_n \geq 0 \} \) let us denote the closure of \( l^\infty_{\text{lm}+} \), and by \( \overline{l^\infty_{\text{Re}+}} = \{ \xi_n \} \in l^\infty : \text{Re} \xi_n \geq 0 \} \), the closure of \( l^\infty_{\text{Re}+} \). Having made the substitution in zeta function \( \zeta(b, z, s) \) as in (1.6) we obtained the following Lerch-Lipschitz form of zeta function
\[
\zeta^{\text{LL}}(a, \xi, s) = \sum_{n=1}^{\infty} e^{2\pi ina_n} \frac{1}{(n+\xi_n)^s}. \tag{3.1}
\]
One immediately sees that this series converge absolutely for \( (a, \xi) \in l^\infty_{\text{lm}+} \times l^\infty_{\text{Re}+} \) and \( s \in H_1 \). Therefore \( \zeta^{\text{LL}} \) is well defined, holomorphic for \( (a, \xi, s) \in l^\infty_{\text{lm}+} \times l^\infty_{\text{Re}+} \times H_1 \) and continuous up to \( l^\infty_{\text{lm}+} \times l^\infty_{\text{Re}+} \times H_1 \). Indeed, we see that \( b = \{b_n\} \in l^\infty \) if \( a \in l^\infty_{\text{lm}+} \). As for \( \xi = \{\xi_n\} \in \overline{l^\infty_{\text{Re}+}} \) we see that
\[
|z_n| \leq \frac{1}{n+\text{Re} \xi_n} \leq \frac{1}{n},
\]
i.e., \( z = \{z_n\} \in l^\infty_{\text{lm}+} \). Therefore \( \zeta^{\text{LL}} \) is obtained as a composition of \( \zeta \) with mapping (1.6) on the whole domain in question. Notice that the “cross”
\[
C^\infty := \{ z \in l^\infty : z_n = -n \text{ for some } n \} = \bigcup_{n=1}^{\infty} \{ z_n = -n \}, \tag{3.2}
\]
is a closed subset of \( l^\infty \). Set \( l^\infty_{\text{lm}^-} := l^\infty \setminus C^\infty \). Let us call a set \( R \) in \( l^\infty_{\text{lm}^-} \) bounded if there exist constants \( E, \varepsilon > 0 \) such that for every \( \xi = \{\xi_n\} \in R \) one has
\[
\frac{n}{E} \leq |n+\xi_n| \leq \frac{n}{\varepsilon} \quad \forall n. \tag{3.3}
\]
This relation is equivalent to
\[
\frac{\varepsilon}{n} \leq |n+\xi_n| \leq \frac{E}{n} \quad \forall n, \tag{3.4}
\]
i.e., to the fact that (1.6) maps \( \xi \) to \( z \) in a bounded subset of \( l^\infty_{\text{lm}+} \setminus \overline{C^\infty_{\text{lm}}} \). As the result from Lemma 2.5 we obtain the following:

Corollary 3.1. Function \( \zeta^{\text{LL}} \) admits a multivalued analytic continuation to any bounded open subset of
\[
l^\infty_{\text{lm}+} \times l^\infty_{\text{lm}^-} \times H_1. \tag{3.5}
\]
The ramification takes place in variable \( z \) at points of \( \mathbb{Z}^- \), more precisely around hypersurfaces of the kind \( l^\infty \times \{-n\} \times \mathbb{C} \), here \( n \in \mathbb{N} \). With respect to the parameter \( a \in l^\infty_{\text{lm}+} \) function \( \zeta^{\text{LL}} \) is continuous up to \( l^\infty_{\text{lm}+} \).
Vector from Corollary 3.1, which will be proved later, that \( \zeta \) depends on \( k \) converges on compacts of \( \{ 3.4 \} \) we see that the sum under the integral on the right hand side of (3.6) uniformly converges on compacts of \( \{ \Re(s+1) > 1 \} \), i.e., on \( H_0 \). Due to the assumption of our lemma \( \zeta(a^0, z^0, s) \) can be meromorphically continued to \( H_0 \). Therefore the same holds for \( \zeta(a^0, z_1, s) \) whatever \( z^1 \) in a convex neighborhood of \( z^0 \) in \( \l_\infty^{-} \). Is if \( z^1 \in \l_\infty^{-} \) is arbitrary we can joint it with \( z^0 \) by an appropriate polygonal curve and finish the proof in a finite number of steps.

\[ \zeta(a^0, z^1, s) - \zeta(a^0, z^0, s) = \int_0^1 \frac{d\zeta(a^0, z(t), s)}{dt} dt = -s \int_0^\infty \sum_{n=1}^\infty e^{2\pi ina^0_n(z_n^1 - z_n^0)} (z_n^1 - z_n^0)^{-s-1} dt. \] (3.6)

In more details \( \zeta_k(a, z, s) \) is the sum of blocks indexed by \( p = 0, 1, \ldots \). Each of these blocks depends on \( k \) variables \( z_1, \ldots, z_k \) and \( k \) parameters \( a_1, \ldots, a_k \) as follows

\[ \zeta_k(a, z, s) = \sum_{n=1}^\infty e^{2\pi ina_{n-1,k+1}} + \sum_{p=0}^{k-1} \sum_{j=1}^k e^{2\pi i(pk+j)a_j} (1 + z_j)^s + \sum_{j=1}^k e^{2\pi i(pk+j)a_j} (1 + z_j)^s + \ldots + \sum_{j=1}^k e^{2\pi i(pk+j)a_j} (1 + z_j)^s. \] (3.7)

When \( k \to \infty \) functions \( \zeta_k(\{a_j\}_{j=1}^k, \{z_j\}_{j=1}^k, s) \) in some sense approximate the function \( \zeta(a, z, s) = \zeta(\{a_j\}_{j=1}^\infty, \{z_j\}_{j=1}^\infty, s) \). From (3.8) it is well seen that \( \zeta_k \) is well defined for \( z = (z_1, \ldots, z_k) \) with \( z_j \in \mathbb{C} \setminus \{z \in \mathbb{Z}^\ast : \Re(z) < -j\} = \mathbb{C} \setminus \mathbb{Z}^- \) and \( s \in H_1 \). It is clear as well that \( \zeta_k \) is continuous in parameter \( a \) up to \( \Im a_j = 0 \).

Function \( \zeta_k \) can be obtained from the Euler-Riemann form of \( \zeta \) by the substitution

\[ b_n = e^{2\pi ina_{n-1,k+1}}, \quad z_n = \frac{1}{n + z_{n-1,k+1}}. \] (3.9)

where \( \Im a_j > 0 \) and \( \Re z_j > 0 \) for \( j = 1, \ldots, k \), and \( n \in \mathbb{N} \). Therefore it follows also from Corollary 3.1, which will be proved later, that \( \zeta_k \) admits a multivalued analytic...
continuation to \( \mathbb{C}_{lm}^k \times \mathbb{C}_{Z^-} \times H_1 \) continuous in parameter \( a \) up to \( \mathbb{C}_{lm}^k \). We shall not stop on this here.

### 3.3. Analytic continuation with respect to variables \( s \) and \( z \)

For a fixed \( a \in \mathbb{C}_{lm}^k \) and fixed \( z \in \mathbb{C}_{Z^-} \) let us fulfill an analytic continuation of \( \zeta_k(a,z,s) \) in \( s \)-variable from \( H_1 \) to \( \mathbb{C} \). First we restrict \( z \)-variable to \( \mathbb{R}^k_+ \), i.e., \( z = x \in \mathbb{R}^k_+ := (\mathbb{R}^+)^k \), where \( \mathbb{R}^+ = \{ t \in \mathbb{R} : t \geq 0 \} \).

**Lemma 3.1.** For every couple \( (a,x) \in \mathbb{C}_{lm}^k \times \mathbb{R}^k_+ \) function \( \zeta_k(a,x,s) \) can be analytically continued in \( s \) to a meromorphic function on \( \mathbb{C} \). Moreover:

i) if \( a_j \notin \mathbb{Z}[1/k] \) for all \( j \in \{1,...,k\} \) then \( \zeta_k(a,x,s) \) is holomorphic on \( \mathbb{C} \);

ii) if \( a_j \in \mathbb{Z}[1/k] \) for some \( j \in \{1,...,k\} \) then \( \zeta_k(a,x,s) \) has a simple pole at \( s = 1 \) with the residue equal to

\[
\text{Res}(\zeta_k,1) = \frac{1}{k} \sum_{a_j \in \mathbb{Z}[1/k]} e^{2\pi i a_j}.
\] (3.10)

**Proof.** The proof follows the classical line of arguments, which are due to Riemann, see [Rm]. Making the substitution \( t \to (pk+j+x_j)t \) to the Euler’s gamma function

\[
\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = (pk+j+x_j)^s \int_0^\infty t^{s-1} e^{-(pk+j+x_j)t} dt
\]

we get from (3.7)

\[
\zeta_k(a,x,s) = \frac{1}{\Gamma(s)} \sum_{p=0}^\infty \sum_{j=1}^k \int_0^\infty t^{s-1} e^{2\pi i(pk+j)x_j} e^{-(pk+j+x_j)t} dt =
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{p=0}^\infty \sum_{j=1}^k e^{2\pi i(pk+j)x_j} e^{-(pk+j+x_j)t} dt
\] (3.11)

due to the absolute convergence of this series for \( s \in H_1 \) and \( \text{Im} a_j \geq 0 \). We can rewrite (3.11) as follows

\[
\zeta_k(a,x,s) = \frac{1}{\Gamma(s)} \sum_{j=1}^k e^{2\pi i a_j} \int_0^\infty t^{s-1} e^{-(j+x_j)t} \sum_{p=0}^\infty e^{2\pi ipk a_j} e^{-p t} dt =
\]

\[
= \frac{1}{\Gamma(s)} \sum_{j=1}^k e^{2\pi i a_j} \int_0^\infty t^{s-1} \sum_{p=0}^\infty (e^{2\pi i p k a_j - t})^p dt =
\]

\[
= \frac{1}{\Gamma(s)} \sum_{j=1}^k e^{2\pi i a_j} \int_0^\infty t^{s-1} \frac{e^{-(j+x_j)t}}{1 - e^{2\pi i p k a_j - t}} dt.
\] (3.12)

The integral on the right hand side of (3.12) converges absolutely for \( \text{Re} s > 1 \) and \( (a,x) \in \mathbb{C}_{lm}^k \times \mathbb{R}^k_+ \). For every \( j = 1,...,k \) and small \( \rho > 0 \) consider the integral

\[
I_j^\rho(a_j,x_j,s) := \int_{C_\rho} \lambda^{s-1} \frac{e^{-(j+x_j)\lambda}}{1 - e^{2\pi i p k a_j - k\lambda}} d\lambda,
\] (3.13)
where $\lambda = t + i\tau$ and $C_\rho$ is the contour defined below, see Figure 1(a). Let us explain that $I_\rho^j(s)$ doesn’t depend on $\rho > 0$. Indeed, the only problem could arise with denominator if $2\pi i a_j - k \lambda \in 2\pi i \mathbb{Z}$, say equals to $2\pi i l$ for some $l \in \mathbb{Z}$. This means that

$$\lambda = 2\pi i \left(a_j - \frac{l}{k}\right). \quad (3.14)$$

In the case under discussion we take $\rho > 0$ small enough in order that the contour $C_\rho$ avoids points as in (3.14) for all $j = 1, \ldots, k$ and, moreover, the disk bounded by $C(\rho)$ doesn’t contain these points except, possibly, the origin.

This implies holomorphy of $I_\rho^j(s)$ on the whole of $\mathbb{C}$. At the same time for $s \in H_1$ one has

$$I_\rho^j(a_j, x_j, s) = \lim_{\rho \to 0} \int_{C_\rho} \lambda^{s-1} \frac{e^{-(j+x_j)\lambda}}{1 - e^{2\pi i a_j - k\lambda}} d\lambda = (e^{2\pi is} - 1) \int_0^{\infty} t^{s-1} \frac{e^{-(j+x_j)t}}{1 - e^{2\pi i a_j - kt}} dt. \quad (3.15)$$

Therefore from (3.12) we see that for $s \in H_1$ one has

$$(e^{2\pi is} - 1) \Gamma(s) \zeta_k(a, x, s) = \sum_{j=1}^{k} e^{2\pi i a_j} I_\rho^j(a_j, x_j, s), \quad (3.16)$$

with the right hand side being holomorphic on $\mathbb{C}$. This gives the analytic continuation of $\zeta_k(a, x, s)$ to a meromorphic function on $\mathbb{C}$ with eventual poles at points of $\mathbb{Z}_{>0} := \{n \in \mathbb{Z} : n > 0\}$ at most.
Case 1. \( a_j \not\in \mathbb{Z}[1/k] \) for all \( j \in \{1, \ldots, k\} \). In this case \( \zeta_k(a, x, s) \) is an entire function. Indeed, observe that for \( s \in \mathbb{Z} \) one has

\[
I^j_\rho(a_j, x_j, s) = \int_{C_\rho} \frac{\lambda^{-1} e^{-(j+x_j)\lambda}}{1-e^{2\pi i k a_j-k\lambda}} d\lambda = \int_{C(\rho)} \frac{\lambda^{-1} e^{-(j+x_j)\lambda}}{1-e^{2\pi i k a_j-k\lambda}} d\lambda = (3.17)
\]

\[
= \begin{cases} 
0 & \text{if } s \in \mathbb{Z} \setminus \{0\}, \\
\frac{2\pi i}{1-e^{2\pi i k a_j}} & \text{if } s = 0.
\end{cases}
\]

This implies that \( \zeta_k(a, x, s) \) is an entire function which vanishes when \( s \) is a negative integer, i.e., \( s \in \mathbb{Z}_{<0} := \{-1, -2, \ldots\} \). It satisfies the following relation on the whole of \( \mathbb{C} \)

\[
\zeta_k(a, x, s) = \frac{1}{(e^{2\pi i s} - 1)\Gamma(s)} \sum_{j=1}^{k} e^{2\pi i j a_j} I^j_\rho(a_j, x_j, s).
\]

Case 2. \( a_j \in \mathbb{Z}[1/k] \) for some \( j \in \{1, \ldots, k\} \). In that case for such \( j \) and \( s \in \mathbb{Z} \) we have

\[
I^j_\rho(a_j, x_j, s) := \int_{C_\rho} \frac{\lambda^{-1} e^{-(j+x_j)\lambda}}{1-e^{-k\lambda}} d\lambda = \int_{C(\rho)} \frac{\lambda^{-1} e^{-(j+x_j)\lambda}}{1-e^{-k\lambda}} d\lambda = \begin{cases} 
0 & \text{if } s \in \mathbb{Z} \setminus \{1\}, \\
\frac{2\pi i}{k} & \text{if } s = 1.
\end{cases}
\]

The same relation \((3.16)\) or, better \((3.18)\), provides us the meromorphic continuation of \( \zeta_k(a, x, s) \) to \( \mathbb{C} \) having a simple pole at \( s = 1 \) with residue

\[
\text{Res}(\zeta_k, 1) = \lim_{s \to 1} \left[ \frac{s-1}{e^{2\pi i s} - 1} \right] \left[ \frac{1}{\Gamma(s)} \right] \left[ \sum_{j=1}^{k} e^{2\pi i j a_j} I^j_\rho(s) \right] = \frac{1}{k} \sum_{a_j \in \mathbb{Z}[1/k]} e^{2\pi i j a_j}.
\]

Recall that a subset \( R \) of an open set \( D \) in a complex manifold called plurithick if for every sequence \( \{u_n\} \) of uniformly bounded from above plurisubharmonic functions on \( D \) condition

\[
\limsup_{n \to \infty} |u_n|_R \equiv -\infty \quad \text{implies} \quad \limsup_{k \to \infty} u_k \equiv -\infty.
\]

Lemma 3.2. For every \( a \in \mathbb{C}^k_{lm+} \) function \( \zeta_k(a, \cdot, \cdot) \) can be analytically continued along any path in \( \mathbb{C}^k_{2-} \times \mathbb{C} \). Moreover:

i) if \( a_j \notin \mathbb{Z}[1/k] \) for all \( j \in \{1, \ldots, k\} \) then \( \zeta_k(a, z, \cdot) \) is holomorphic on \( \mathbb{C} \);

ii) if \( a_j \in \mathbb{Z}[1/k] \) for some \( j \in \{1, \ldots, k\} \) then \( \zeta_k(a, z, \cdot) \) has a simple pole at 1 with the residue equal to

\[
\text{Res}(\zeta_k, 1) = 1/k \cdot \sum_{a_j \in \mathbb{Z}[1/k]} e^{2\pi i j a_j}.
\]

iii) In addition for every \( j = 1, \ldots, k \) and every \( p \in \mathbb{Z}^- \) the monodromy of \( \zeta_k(a, \cdot, s) \) around the hyperplane \( z_j = -pk-j \) is equal to

\[
\frac{e^{2\pi i (pk+j)a_j}}{(pk+jz_j)^s} \left( e^{-2\pi i s} - 1 \right).
\]

Proof. (i) Take any path \( \gamma : [0, 1] \to \mathbb{C}^k_{2-} \) starting at \( 1 = (1, \ldots, 1) \in \mathbb{R}^k_+ \). Let \( B_t \) be the ball \( B_t(\gamma(t), \varepsilon) \) in \( \mathbb{C}^k \) with an appropriate \( \varepsilon > 0 \). We need to extend \( \zeta_k(a, \cdot, \cdot) \) to \( B_t \times \mathbb{C} \) for all \( t \). Since \( \mathbb{R}^k_+ \cap B_0 \) is a plurithick subset of \( B_0 \), as well as \( B_{t_1} \cap B_t \) in \( B_t \) for every \( t_1 < t \) close to \( t \) the needed statement follows from Corollary 1.2.
(ii) Repeat the same argument for the function \((s-1)\zeta_k(a,\cdot,\cdot)\) to get its holomorphic extension. This provides the meromorphic extension of \(\zeta_k\) itself with the simple pole at 1. As for the residual remark that

\[
\text{Res}(\zeta_k(a,z,\cdot),1) = \frac{1}{2\pi i} \int_{|a-1|=\varepsilon} \zeta_k(a,z,s) ds
\]

holomorphically depends on \(z\) and therefore is constant, because it is constant on \(\mathbb{R}^k\).

(iii) The monodromy of \(\zeta_k(a,\cdot,s)\) around the hyperplane \(z_j = -pk - j\) in the positive direction is, in fact, the monodromy along the closed path in \(z_j\)-plane starting at \(z_j \neq -pk - j\) and going around \(-pk - j\) in the counterclockwise direction of the summand

\[
e^{2\pi i(pk+j)a_j} \frac{1}{(pk+j+z_j)^s},
\]

which is obviously equal to \(e^{2\pi i(pk+j)a_j} (e^{-2\pi is} - 1).\)

\[\square\]

Remark 3.1. Notice that item (iii) of this lemma implies that \(\zeta(a,\cdot,s)\) descends to the abelian cover of \(C^k_{\mathbb{Z}} \times \mathbb{C}, \) i.e., to \(\tilde{C}^{k,ab}_{\mathbb{Z}} \times \mathbb{C}\) and separates points over \(C^k_{\mathbb{Z}} \times \mathbb{C}\) as stated in Theorem 2. Indeed, the fundamental group of \(C^k_{\mathbb{Z}}\) is generated by simple loops starting from \(1 = (1,\ldots,1)\) and going around points \(-pk - j\) on the \(z_j\)-plane for all \(p \in \mathbb{N}\) and \(1 \leq j \leq k\). Let \(\gamma_1\) be such loop around \(-p_k k - j\) on the \(z_j\)-plane and \(\gamma_2\) around \(-p_2 k - r\) on the \(z_r\)-plane. Then the monodromy of \(\zeta_k(a,\cdot,s)\) along \(\gamma_1 \cdot \gamma_2^{-1} \cdot \gamma_1 \cdot \gamma_2\) is obviously zero if \(r \neq j\). If \(r = j\) then it is zero too. Therefore the monodromy along any loop generating the commutator of \(\pi_1(C^k_{\mathbb{Z}})\) is trivial.

3.4. Analytic continuation with respect to the parameter \(a\). Now let us turn our attention to the parameter \(a\). Up to now it was aloud to vary in \(\tilde{C}^{k}_{\mathbb{Im}+}\). We shall prove the following

Lemma 3.3. Function \(\zeta_k\) admits an analytic continuation to a meromorphic function on

\[
\tilde{\mathcal{Z}}_k := \tilde{\mathcal{C}}^{k,ab}_{\mathbb{Z}[1/k]} \times \tilde{\mathcal{C}}^{k,ab}_{\mathbb{Z}^-} \times \mathbb{C},
\]

where \(\tilde{\mathcal{C}}^{k,ab}_{\mathbb{Z}[1/k]}\) is the abelian cover of \(C^k_{\mathbb{Z}[1/k]} := (\mathbb{C} \setminus \mathbb{Z}[1/k])^k\). The monodromy of \(\zeta_k(\cdot,z,s)\) around the hyperplane \(\{a_j = l/k\}\) is equal to

\[
[2\pi(a_j - \frac{l}{k})]^{-(s-1)} e^{i\pi(s-1)} e^{-2\pi i(j/l)(a_j-z_j)}.
\]

Proof. This issue will be proved by studying integrals from the right hand part of (3.15). We follow \[\square\]. For a fixed \(s \in H_1\) and \(x \in \mathbb{R}^k_+\) write as in (3.15) \(I^j_k(a_j, x_j, s) = (e^{2\pi is} - 1)J^j_k(a_j, x_j, s)\), i.e.,

\[
J^j_k(a_j, x_j, s) := \int_0^\infty t^{s-1} e^{-(j+x_j)t} \frac{e^{-i\pi ak_j-klt}}{1-e^{2\pi iak_j-klt}} dt.
\]

One readily sees that this integral is well defined and holomorphic with respect to \(a_j\) unless

\[
a_j - \frac{l}{k} \in \frac{i}{2\pi} \mathbb{R}^+ \quad \text{for some } l \in \mathbb{Z}.
\]
In other words \( J^1_p(a_j, x_j, s) \) is holomorphic in \( a_j \) on \( \mathbb{C} \setminus \{ \mathbb{Z}[1/k] - i\mathbb{R}^+ \} \). Fix some \( u \in \mathbb{R}^+ \)

\[
\begin{array}{c|c|c}
0 & R^+ & \lambda \text{- variable} \\
\hline
l/k & 0 & u \\
\hline
l/k - \imath u + \imath v & & \end{array}
\]

\[
\begin{array}{c|c|c}
0 & R & \lambda \text{- variable} \\
\hline
l/k & 0 & u \\
\hline
l/k - \imath u + \imath v & & \end{array}
\]

**Figure 2.** (a) Integration along \( \mathbb{R}^+ \) gives us zeta function defined for \( a_j \) on \( \mathbb{C} \) minus vertical halph-lines starting at points of \( \mathbb{Z}[1/k] \). (b) Integration along \( L_{u,\varepsilon} \) produces extension to bumps as on the right. To find monodromy of \( \zeta_k \) one should take the difference between the integral along \( \mathbb{R}^+ \) and \( L_{u,\varepsilon} \) for the values of \( a_j = \frac{l}{k} - \imath u + \imath v \), which is marked by a bold point on the both Pictures (a) and (b).

and deform \( \mathbb{R}^+ \) to a contour \( L_{u,\varepsilon} = [0, u - \varepsilon] \cup S^+_{k}(u) \cup [u + \varepsilon, +\infty) \) as on the Figure 2. Here \( S^+_{k}(u) \) is the upper part of the circle of radius \( 0 < \varepsilon < u/2 \) centered at \( u \). We have that

\[
\int_0^\infty t^{s-1} \frac{e^{-(j+x_j)t}}{1 - e^{2\pi ik a_j - k t}} dt = \int_{L_{u,\varepsilon}} \lambda^{s-1} \frac{e^{-(j+x_j)\lambda}}{1 - e^{2\pi ik a_j - k \lambda}} d\lambda. \tag{3.27}
\]

Indeed, for \( \lambda = u + \varepsilon e^{i\theta}, 0 \leq \theta \leq \pi \), the condition (3.14) reads as

\[
a_j - \frac{l}{k} = -i(u + \varepsilon \cos \theta) + \frac{\varepsilon \sin \theta}{2\pi}, \tag{3.28}
\]

and this is inconsistent for \( \Im a_j \geq 0 \) and the choice of \( \varepsilon \) made. Deforming the contour of integration from \( \mathbb{R}^+ \) to \( L_{u,\varepsilon} \) we extend \( J^1_p(s) \) analytically in the variable \( a_j \) to an \( \varepsilon \)-neighborhood of each point \( \frac{l}{k} - iu \) from the left, see Figure 2(b), i.e., to a point of the form \( \frac{l}{k} - iu + v \) with \( v > 0 \). The monodromy calculated at such point \( a_j = \frac{l}{k} - iu + v \) with \( v > 0 \) is equal to the difference between the left-hand and the right-hand sides of (3.27). Taking into account that \( u + iv = i(a_j - l/k) \) this monodromy is equal to i.e., to

\[
\int_{S^+_{k}(u) + [u - \varepsilon, u + \varepsilon]} \lambda^{s-1} \frac{e^{-(j+x_j)\lambda}}{1 - e^{2\pi ik (a_j - l/k) - k \lambda}} d\lambda = \int_{S^+_{k}(u) + [u - \varepsilon, u + \varepsilon]} \lambda^{s-1} \frac{e^{-(j+x_j)\lambda}}{1 - e^{2\pi ik (u + iv) - \lambda/2\pi}} d\lambda = \lim_{\lambda \to 2\pi i(a_j - \frac{l}{k})} \frac{(\lambda - 2\pi i(a_j - \frac{l}{k}))\lambda^{s-1} \frac{e^{-(j+x_j)\lambda}}{1 - e^{2\pi i(a_j - \frac{l}{k}) - \lambda}}}{\frac{2\pi i(a_j - \frac{l}{k})}{k}} = \frac{2\pi i(a_j - \frac{l}{k})^{s-1} e^{-2\pi i(a_j - \frac{l}{k}) (j+x_j)}}{k} = \frac{2\pi i(a_j - \frac{l}{k})^{s-1} e^{i\pi (s-1)} e^{-2\pi i(a_j - \frac{l}{k}) (j+x_j)}}{k}. \tag{3.29}
\]
Here $\text{Arg}(a_j - \frac{1}{k}) = \text{Arg}(u + iv)$ should be taken in $[0, \pi/2)$, and therefore $\text{Arg}(a_j - \frac{1}{k}) \in [-\pi/2, 0)$. Therefore for $(x, s) \in \mathbb{R}_+^k \times H_1$ function

$$(e^{2\pi is} - 1) \Gamma(s) \zeta^{LL}_k(a, x, s) = \sum_{j=1}^k e^{2\pi is} a_j I_j^s(s)$$

extends in variable $a$ analytically from $\mathbb{C}_k \text{Im} +$ to the abelian cover $\tilde{\mathbb{C}}_{k,ab} \mathbb{Z}^{-} \times \mathbb{C}$ of $\mathbb{C}_k \mathbb{Z}^{-} \mathbb{Z}[1/k] = (\mathbb{C} \setminus \mathbb{Z}[1/k])^k$. Since $\mathbb{R}_+^k \times H_1$ is plurithick in $\tilde{\mathbb{C}}_{k,ab} \mathbb{Z}^{-} \times \mathbb{C}$ we obtain that the same is true for all $(z, s) \in \tilde{\mathbb{C}}_{k,ab} \mathbb{Z}^{-} \times \mathbb{C}$. Lemma and Theorem 2 are proved. 

\[\square\]

\section*{Appendix 1 Pluricomplex analysis on complex sequence spaces}

We recall here some rudiments of pluricomplex analysis on sequence spaces which we used along the text. We need to do this since the Hartogs-Kazaryan type results we used for the proof of the principal results of this paper seem not to be readily available in the literature in the infinite dimensional case. Our exposition will cover finite and infinite dimensional cases simultaneously.

\subsection*{A1.5. Holomorphic functions.}

We shortly recall a few needed notions concerning Banach holomorphicity in order to fix the notations adapted to our sequence spaces. For details we refer to [Mu]. For a natural $m \geq 1$ an $m$-linear form on $l_r$ is a mapping

$$A : l_r \times \ldots \times l_r \to \mathbb{C}, \quad m\text{-times}$$

which is linear on each variable. Mapping $A$ is called r-bounded if

$$\|A\|_r := \sup \{|A(z_1, \ldots, z_m)| : z_j \in B_r^\infty(0, 1)\} < \infty. \quad (0.30)$$

We denote by $\mathcal{L}_m^b(l_r)$ the linear space of r-bounded $m$-linear forms on $l_r$.

\begin{definition}
An r-bounded homogeneous polynomial of degree $m$ on $l_r$ is such mapping $P : l_r \to \mathbb{C}$ that

$$P(z) = A(z, \ldots, z) \quad m\text{-times} \quad (0.31)$$

for some $A \in \mathcal{L}_m^b(l_r)$. The

An $m$-linear form $A$ defining $P$ can be supposed to be symmetric because its symmetrization

$$A^\text{Sym}(z_1, \ldots, z_m) = \frac{1}{m!} \sum_{\sigma \in S_m} A(z_{\sigma(1)}, \ldots, z_{\sigma(m)}).$$

defines the same polynomial $P$ and, moreover, $A^\text{Sym}$ is r-bounded if such is $A$. From the polarization formula for a symmetric $A$, i.e., from

$$A(z_1, \ldots, z_m) = \frac{1}{m!2^m} \sum_{\epsilon_j = \pm 1} \epsilon_1 \ldots \epsilon_m A(z_0 + \epsilon_1 z_1 + \ldots + \epsilon_m z_m, \ldots, z_0 + \epsilon_1 z_1 + \ldots + \epsilon_m z_m) \quad (0.32)$$
for every couple \( z_0, \ldots, z_m \) of vectors from \( l_r \), see \([Mu]\), one obtains that such \( A \) is uniquely defined by \( P \). This uniquely for \( P \) defined symmetric form let us denote as \( \hat{P} \). Again from the polarization formula one can see that

\[
\| P \|_r \leq \| \hat{P} \|_r \leq \frac{m^m}{m!} \| P \|_r ,
\]

(0.33)

where \( \| \hat{P} \|_r \) is defined as in (0.30) and the norm of a polynomial \( P \) is defined as

\[
\| P \|_r := \sup \{|P(z)| : z \in \bar{B}^\infty_r(0,1)\}.
\]

(0.34)

**Remark 0.2.** Notice that an \( r \)-bounded \( m \)-homogeneous polynomial \( P \) is bounded and \( r \)-continuous on \( \bar{B}^\infty_r(0,\beta) \) for any \( 0 < \beta < +\infty \). Boundedness is simple and follows from the relation

\[
P(\lambda z) = \lambda^m P(z),
\]

(0.35)
i.e., on \( B^\infty_r(0,\beta) \) our polynomial \( P \) is bounded by \( \beta^m \| P \|_r \). As for the continuity take some \( z^1, \ldots, z^m, \ldots, w^m \in B^\infty_r(0,\beta) \) such that \( z^j - w^j \in B^\infty_r(0,\varepsilon) \) for \( j = 1, \ldots, m \). Then

\[
\begin{align*}
|\hat{P}(z^1, \ldots, z^m) - \hat{P}(w^1, \ldots, w^m)| &\leq |\hat{P}(z^1, \ldots, z^m) - \hat{P}(w^1, z^2, \ldots, z^m)| + \ldots + \\
|\hat{P}(w^1, \ldots, w^{m-1}, z^m) - \hat{P}(w^1, \ldots, w^m)| &\leq \sum_{j=1}^m |\hat{P}(w^1, \ldots, w^j, z^j - w^j, z^{j+1}, \ldots, z^m)| \\
&\leq m \varepsilon \beta^{m-1} \| \hat{P} \|_r ,
\end{align*}
\]

which proves the assertion.

**Definition 0.2.** Let \( D \subset l_r \) be an \( r \)-open set. Function \( f : D \rightarrow \mathbb{C} \) is called \( r \)-holomorphic or, simply holomorphic if \( r \) is clear from the context, if the following holds.

i) \( f \) is \( r \)-bounded, i.e., for every \( \beta > 0 \) such that \( B^\infty_r(z^0,\beta) \subset D \) function \( f \) is bounded on \( B^\infty_r(z^0,\beta) \) for any \( 0 < \beta_1 < \beta \).

ii) \( f \) is Gâteaux differentiable, i.e., for every \( z \in l_r \) the function of one complex variable \( f(z^0 + tz) \) is holomorphic in \( t \) near the origin.

**Proposition 0.1.** Function \( f \) defined on an \( r \)-open set \( D \) is \( r \)-holomorphic if and only if for every \( B^\infty_r(z^0,\beta) \subset D \) there exists a sequence of \( r \)-bounded \( m \)-homogeneous polynomials \( \{P_m\}_{m=1}^\infty \) such that

\[
f(z^0 + z) = f(z^0) + \sum_{m=1}^\infty P_m(z)
\]

(0.36)
normally on \( \bar{B}^\infty_r(0,\beta_1) \) for every \( 0 < \beta_1 < \beta \). The latter means that

\[
\sum_{m=1}^\infty \beta_1^m \| P_m \|_r < \infty.
\]

(0.37)

**Proof.** Polynomials \( P_m \) depend also on \( z^0 \) of course and it would be more colloquial to write (0.36) as

\[
f(z^0 + z) = f(z^0) + \sum_{m=1}^\infty P_m(z^0, z).
\]

(0.38)

\( \Leftarrow \) Take \( z^0 \in D \) and \( \beta > 0 \) such that \( \bar{B}^\infty_r(z^0,\beta) \subset D \). Due to the assumption there exist \( r \)-bounded \( m \)-homogeneous polynomials \( P_m \) such that (0.36) and (0.37) hold on
$\bar{B}^\infty(0, \beta_1)$ whatever $0 < \beta_1 < \beta$ is. This immediately implies that $f$ is bounded on all such $\bar{B}^\infty_r(z^0, \beta_1)$. As for the Gâteaux differentiability of $f$ take any $z \in B^\infty_r(0, \beta_1)$ and write

$$f(z^0 + tz) = f(z^0) + \sum_{m=1}^{\infty} P_m(tz) = f(z^0) + \sum_{m=1}^{\infty} P_m(z)t^m. \quad (0.39)$$

Series (0.39) converge uniformly on $|t| \leq 1$ due to (0.37), thus providing the holomorphicity of $f(z^0 + tz)$ near the origin.

$\Rightarrow$ Let again $\bar{B}^\infty_r(z^0, \beta) \subset D$. Fix $0 < \beta_1 < \beta$. For $z \in B^\infty_r(0, \beta_1)$ denote by $\Delta_z f(z^0)$ the derivative at $t = 0$ of $f(z^0 + tz)$. Take some $z^1, \ldots, z^m \in B^\infty_r(0, \beta_1)$. Then for any couple $(t_1, \ldots, t_m)$ in a neighborhood $U$ of zero of $\mathbb{C}^m$ vector $t_1 z^1 + \ldots + t_m z^m \in B^\infty(0, \beta_1)$. Neighborhood $U$ being fixed for a choice of $z^1, \ldots, z^m$ made. By the finite dimensional Hartogs’ separate analyticity theorem function $f(z^0 + t_1 z^1 + \ldots + t_m z^m)$ is holomorphic in $U$. Therefore we can set

$$A_m(z_1, \ldots, z_m) := \Delta_{z^1} \circ \ldots \circ \Delta_{z^m} f(z^0) = \frac{\partial^m f(z^0 + t_1 z^1 + \ldots + t_m z^m)}{\partial t_1 \ldots \partial t_m} \bigg|_{t_1=\ldots=t_m=0}. \quad (0.40)$$

Function $A_m$ is obviously symmetric and $m$-linear, this is a finite dimensional argument. Also from one complex variable applied to the holomorphic function $f(z^0 + tz)$, where $z \in B^\infty_r(0, \beta_1)$ and $|t| \leq \frac{\beta}{\beta_1}$, we get that

$$f(z^0 + z) = f(z^0) + \sum_{m=1}^{\infty} \frac{1}{m!} A_m(z, \ldots, z), \quad (0.41)$$

which is nothing else but the Taylor expansion of $f(z^0 + tz)$ at $t = 1$. Set $P_m(z) = \frac{1}{m!} A_m(z, \ldots, z)$. Since $f$ is bounded, say by $M_{\beta_2}(f)$, on $B^\infty_r(z^0, \beta_2)$ for some $\beta_1 < \beta_2 < \beta$ we see from Cauchy formula and the fact that $f(z^0 + tz)$ is defined for $|t| \leq \frac{\beta}{\beta_1}$ that

$$|A_m(z, \ldots, z)| = \left| \frac{\partial^m f(z^0 + tz)}{\partial t^m} \bigg|_{t=0} \right| \leq \frac{m! \beta_1^m}{\beta_2^m} \times M_{\beta_2}(f). \quad (0.42)$$

Therefore $P_m(z)$ is bounded by $(\beta_1/\beta_2)^m M_{\beta_2}$ on $\bar{B}^\infty_r(0, \beta_1)$, note that $\beta_1/\beta_2 < 1$. This provides $r$-boundedness of $P_m$ and the normal on $\bar{B}^\infty_r(0, \beta_1)$ convergence of the series

$$\sum_{m=1}^{\infty} P_m(z), \quad \text{because} \quad \sum_{m=1}^{\infty} |P_m(z)| \leq M_\beta \sum_{m=1}^{\infty} (\beta_1/\beta_2)^m = M_{\beta_2}(f) \frac{\beta_1/\beta_2}{1 - \beta_1/\beta_2}. \quad (0.43)$$

In the following propositions we list the elementary properties of $r$-holomorphic functions.

**Proposition 0.2.** One has the following:

i) A uniform on bounded open sets limit of $r$-holomorphic functions is $r$-holomorphic.

ii) Let $f$ be $r$-holomorphic on an $r$-open $D \subset \mathbb{C}$ and let $r_1$ be a polyradius lexicographically smaller than $r$. Then $D_1 := D \cap 1_{r_1}$ is $r_1$-open and $f|_{D_1}$ is $r_1$-holomorphic.

**Proof.** i) The question being local it is sufficient to consider two balls $\bar{B}^\infty_r(0, \beta_1) \subset B^\infty(0, \beta_2)$, i.e., $0 < \beta_1 < \beta_2$ and a sequence $\{f_n\}$ of holomorphic on $B^\infty(0, \beta_2)$ function uniformly on $B^\infty(0, \beta_2)$ converging to $f$. $r$-boundedness of the limit $f$ is obvious. The Gâteaux holomorphicity of $f$ follows from the Weierstrass theorem.
ii) When saying that $r_1 = \{r_1,n\}$ is smaller than $r = \{r_n\}$ lexicografically we mean that $r_1,n \leq r_n$ for all $n$. Remark that $B^{\infty}_r(0,1) \subset B^{\infty}_r(0,1)$ in this case. And this implies the $r_1$-openness of $D_1$. Continuity of $f|_{D_1}$ follows from the fact that $\|\cdot\|_r \leq \|\cdot\|_{r_1}$. Its Gâteaux holomorphicity is obviously follows from that of $f$.

\[\square\]

**Proposition 0.3.** An $r$-holomorphic function on an $r$-open connected set $D$ possesses the following properties.

i) If $f$ vanishes on some $r$-open subset of $D$ then $f$ vanishes identically. If $|f|$ achieves its maximum at some interior point of $D$ then $f \equiv \text{const}$.

ii) $r$-holomorphic functions possess the following uniqueness property: let $z^0 = x^0 + iy^0 \in D$ and $\beta > 0$ be such that $B^{\infty}_r(z^0,\beta) \subset D$. If the restriction of $f$ to $iy^0 + B^{\infty}_{r,R}(x^0,\beta r)$ vanishes then $f$ vanishes identically.

**Proof.** Item (i) can be proved by considering restrictions $f(z^0 + tz)$ for $z \in B^{\infty}_r(0,\beta_1)$. (ii) First we shall prove that $f$ vanishes on $B^{\infty}_r(z^0,\beta)$. Without loss of generality we can assume that $z^0 = 0$. Write the Taylor expansion \((0.36)\) for the real values of variables

\[
f(x) = \sum_{m=1}^{\infty} P_m(x), \tag{0.44}\]

which converges on $B^{\infty}_r(x^0,\beta_2)$ for any $0 < \beta_1 < \beta_2 < \beta$. Since for real $t$ one has that $0 \equiv f(tx) = \sum_{m=1}^{\infty} P_m(x)t^m$ we deduce that $P_m(x) = 0$ for all $x \in B^{\infty}_{r,R}(0,\beta_2)$. Let us prove that $P_m(z) = 0$ also for complex $z \in B^{\infty}_r(0,\beta_1)$. Fix such $z = x + iy$ and remark that $x \in B^{\infty}_{r,R}(0,\beta_1)$. Therefore the function of one complex variable $p(\lambda) := P_m(\lambda x)$, which is well defined for $|\lambda| \leq \beta_1/\beta_2$, vanishes for all $|\lambda| \leq \beta_1/\beta_2$. In particular it vanishes at $\lambda_0$ which is such that $\lambda_0 x = z$. We proved that $P_m$ vanishes on $B^{\infty}_r(0,\beta_1)$. This means that $P_m(z) \equiv 0$ in $B^{\infty}_r(0,\beta r)$ for all $\beta > 0$. Therefore $f$ vanishes on every $B^{\infty}_r(0,\beta_1)$ such that $0 < \beta_1 < \beta$ and $B^{\infty}_r(0,\beta) \subset D$. The uniqueness property (ii) implies the needed statement.

\[\square\]

**A1.6. Plurisubharmonic functions.** Let $D$ be a $r$-open subset in $l_r$.

**Definition 0.3.** Function $u : D \to \mathbb{R} \cup \{-\infty\}$ is called $r$-plurisubharmonic, $r$-psh for short, if it is upper-semicontinuous in $r$-topology and for every $z^0 \in D$ and every $z \in l_r$ the restriction $u|_{\Omega}$ of $u$ to $\Omega := D \cap \{z^0 + tz : t \in \mathbb{C}\}$ is subharmonic.

If $r$ is clear from the context we say simply that $u$ is psh. The set of $r$-psh functions on $D$ we shall denote as $\mathcal{P}_r(D)$.

**Definition 0.4.** An $r$-open set we call $\mathcal{P}_r$-hyperconvex if there exists a negative $u \in \mathcal{P}_r(D)$ such that $u(z) \to 0$ when $z \to \partial D$ in $r$-topology.

Negative here means that $u$ is strictly $< 0$ on the $r$-interior of $D$.

**Example 0.1.** The ball $B^{\infty}_r(z^0,\beta)$ is $\mathcal{P}_r$-hyperconvex for any $\beta > 0$. It is sufficient to prove this for $\beta = 1$. To see this let us prove that the function

\[
u(z) := \ln \left( \sup_{n \in \mathbb{N}} \left\{ \frac{|z - z_n^0|}{r_n} \right\} \right) = \ln \|z - z_0\|_r, \tag{0.45}\]

is $r$-plurisubharmonic.
is r-psh on $B_r^\infty(z^0, 1)$. $u$ is an increasing limit of plurisubharmonic functions of finitely many variables

$$u_N(z) = \ln \left( \max \left\{ \frac{|z_n - z^0_n|}{r_n} \right\}^N \right) = \max \left\{ \ln \left( \frac{|z_n - z^0_n|}{r_n} \right) \right\}^N.$$  

Therefore it is sufficient to prove that it is upper-semicontinuous. For this it is sufficient to prove the upper-semicontinuity of $u_0(z) := e^{u(z)} = \|z - z_0\|_r$. But the letter is continuous in fact.

**Remark 0.3.** Remark that function $v_0 = u_0 - 1$ is r-psh on $B_r^\infty(z^0, 1)$, non-positive on the interior, $\geq -1$ everywhere, and $\equiv 0$ on the boundary.

For an r-open set $D$ denote by $P_r^{-}(D)$ the set of non-positive $u \in P_r(D)$.

**Definition 0.5.** A $P_r$-measure of a subset $E$ of a $P_r$-hyperconvex r-open set $D$ is the upper regularization $\omega^*$ of the function

$$\omega(z, E, D) = \sup \{u(z) : u \in P_r^{-}(D), u|_E \leq -1\}, \quad (0.46)$$

i.e., $\omega^*(z, E, D) := \limsup_{n \to \infty} \omega(\xi, E, D)$.

Remark that $\omega^*$ is r-psh in $D$.

**Definition 0.6.** $E$ is called globally $P_r$-polar if there exist $u \in P_r(D)$, which is not $\equiv -\infty$, such that $E \subset \{z \in D : u(z) = -\infty\}$.

**Example 0.2.** For any $\eta > 1$ polydisk $\Delta_\infty^0(0, \frac{1}{n^{1/\eta}})$ is a pluripolar subset of $\Delta_\infty^0(0, \frac{1}{n})$. Let’s start with proving that function $v$ defined for $z \in \Delta_\infty^0(z^0, r)$ as

$$v(z) := \ln \left( \limsup_{n \to \infty} \left\{ \frac{|z_n - z^0_n|}{r_n} \right\} \right) \quad (0.47)$$

is r-psh. Indeed, $v$ is a decreasing limit of the r-psh, by the Example functions

$$v_N(z) := \ln \left( \sup_{n \geq N} \left\{ \frac{|z_n - z^0_n|}{r_n} \right\} \right).$$

And therefore is r-psh itself. Now remark that for $z \in \Delta_\infty^0(0, \frac{1}{n^{1/\eta}})$ one has that $e^{\eta N(z)} \leq e^{\eta z|z_n|} / r_n \to 0$ as $N \to \infty$. Therefore $v_N(z) \searrow -\infty$ for such $z$. I.e., $v|_{\Delta_\infty^0(0, \frac{1}{n^{1/\eta}})} = -\infty$. At the same time for $z_\beta = \{\beta r_n\}_{n=1}^\infty$ one has that $v(z_\beta) = \ln \beta = -\infty$, i.e., $v$ is not $\equiv -\infty$.

**Definition 0.7.** Let $E$ be a subset of a $P_r$-hyperconvex open set $D$. $E$ is called locally $P_r$-regular if for every accumulation point $z^0$ of $E$ in $D$ and every $\beta > 0$ such that $B_r^\infty(z^0, \beta) \subset D$ one has

$$\omega^*(z^0, E \cap B_r^\infty(z^0, \beta), B_r^\infty(z^0, \beta)) = -1. \quad (0.48)$$

**Example 0.3.** (a) The set (real part) $B_r^\infty(0, \beta)$ is locally $P_r$-regular in $B_r^\infty(0, \beta)$.

(b) For $0 < \beta_2 < \beta_1 < \infty$ the congruent ball $B_r^\infty(z^0, \beta_2)$ is a $P_r$-regular subset of the ball $B_r^\infty(0, \beta_1)$ provided $\|z^0\|_r < \beta_1 - \beta_2$.

The following statement is the so called Two Constants Theorem.

**Proposition 0.4.** Let $D$ be an r-open set in $l_r$ and let $u \in P_r(D)$ be bounded by $M$ on the whole of $D$. Suppose that for a subset $E \subset D$ one has that $u|_E \leq m$. Then for all $z \in E$ the following inequality holds

$$u(z) \leq M \left[ 1 + \omega^*(z, E, D) \right] - m \omega^*(z, E, D). \quad (0.49)$$
Proof. Let $\beta \in \mathcal{P}_r^{-1}(D)$ and $v(z) := \frac{u(z) - M}{M - m}$ belongs to $\mathcal{P}_r^{-1}(D)$ and $v|_{E} \leq -1$. Therefore $v(z) \leq \omega^{*}(z, E, D)$ for all $z \in D$, and this implies (0.49).

**Definition 0.8.** Call a subset $E \subset D$ of an r-open set $\mathcal{P}_r$-thick (plurithick) if for any sequence $\{u_k\}$ of uniformly bounded from above r-psh functions on $D$

$$\limsup_{k \to \infty} u_k|_{E} = -\infty \implies \limsup_{k \to \infty} u_k \equiv -\infty. \quad (0.50)$$

**Example 0.4.** (a) For every $z^0 \in B_r^\infty(0, \beta)$ and every $0 < \beta_1 < \beta - \|z^0\|$, the set $B_r^\infty(z^0, \beta_1)$ is $\mathcal{P}_r$-thick in $B_r^\infty(0, \beta)$. By the pathwise connectivity of the ball it is sufficient to prove this when $z^0 = 0$, i.e., that $B_r^\infty(0, \beta_1)$ is $\mathcal{P}_r$-thick in $B_r^\infty(0, \beta)$ for $0 < \beta_1 < \beta$. Take any $z \in B_r^\infty(0, \beta)$ with $\|z\| = \beta$ and consider the disk $\Delta = \{t \in B_r^\infty(0, \beta)\} = \{t : |t| < 1\}$, where $i : t \to tz$ is the obvious imbedding of $\Delta$ to $B_r^\infty(0, \beta)$. Disk $\Delta$ contains a subdisk $\Delta_{\beta_1} := \{t : |t| < \beta_1\} = \{t \in B_r^\infty(0, \beta_1)\}$, which is a thick subset of $\Delta$ in the classical (i.e., one variable) sense. In particular, since $\limsup_{k \to \infty} u_k|_{\Delta_{\beta_1}} = -\infty$, it follows that $\limsup_{k \to \infty} u_k|_{\Delta} = -\infty$, and we conclude that $\limsup_{k \to \infty} u_k(z) \equiv -\infty$.

(b) The set $\mathbb{R}^k \cap B^k(0, \beta)$ is plurithick in $B^k(0, \beta)$. For $k = 1$ this follows from the Poisson formula for harmonic functions. The general case can be easily proved by induction on $k$.

**Lemma 0.4.** Let $f$ be a holomorphic function on $B_r^\infty(z^0, \beta) \times \Delta$. Suppose that for every $z$ in some $\mathcal{P}_r$-thick subset $E$ of $B_r^\infty(z^0, \beta)$ the restriction $f_z(s) := f(z, s)$ extends holomorphically to $\mathbb{C}$. Then $f$ extends to a holomorphic function on $B_r^\infty(z^0, \beta) \times \mathbb{C}$.

**Proof.** Write the Taylor expansion of $f$ in $B_r^\infty(z^0, 1) \times \Delta$ in the form

$$f(z^0 + z, s) = f(z^0, 0) + \sum_{m=1}^{\infty} a_k(z)s^k, \quad (0.51)$$

where $a_k$ are r-holomorphic in $B_r^\infty(0, 1)$. To get this from (0.36) represent vectors $w$ from $l_1 \times \mathbb{C}$ as $w = z + se_0$, where $e_0$ spans the subspace $\{0\} \times \mathbb{C}$ of $l_1 \times \mathbb{C}$. With $w^0 = z^0 + 0e_0 = z^0$ write (0.36) in coordinates $w$ as

$$f(w^0 + w) - f(w^0) = \sum_{m=1}^{\infty} P_m(w) = \sum_{m=1}^{\infty} \hat{P}_m(w, ..., w) = \sum_{m=1}^{\infty} \hat{P}_m(z + se_0, ..., z + se_0) =$$

$$= \sum_{m=1}^{\infty} \sum_{k=0}^{m} C_k^m \hat{P}_m(s, se_0, ..., se_0, z, ..., z) = \sum_{k=0}^{\infty} \left( \sum_{m=k}^{\infty} C_k^m \hat{P}_m(e_0, ..., e_0, z, ..., z) \right) s^k, \quad (0.52)$$

which gives us (0.51). To estimate the coefficients

$$a_k(z) = \sum_{m=k}^{\infty} C_k^m \hat{P}_m(e_0, ..., e_0, z, ..., z) \quad (0.53)$$

and justify the change of summation in (0.52) remark that from (0.42) we get that

$$|\hat{P}_m(w, ..., w)| \leq \left( \frac{\beta_1}{\beta_2} \right)^m M_{\beta_2}(f) \quad \text{on} \quad \Delta^\infty(0, \beta_1 r) \times \Delta(\beta_1)$$
for every choice of \(0 < \beta_1 < \beta_2 < 1\). This implies the uniform on \(\Delta^\infty(0, \beta_1 r)\) estimate
\[
|a_k(z)| \leq 
\frac{1}{\beta_2^r} \sum_{m=k}^{\infty} \left(\frac{\beta_1}{\beta_2}\right)^m M_{\beta_2} = \beta_2^k \sum_{m=k}^{\infty} \left(\frac{\beta_1}{\beta_2}\right)^{m-k} M_{\beta_2},
\] (0.54)
which implies the normal convergence of the series \((0.52)\). Therefore
\[
\ln \sqrt{|a_k(z)|} \leq 1/k \ln \left(\frac{\beta_2}{1 - \beta_2(\beta_2 - \beta_1)} M_{\beta_2}\right) \quad \text{on} \quad \Delta^\infty(0, \beta_1 r).\] (0.55)
Functions \(\ln \sqrt{|a_k(z)|}\) are r-psh on \(B^\infty_r(0, 1)\) and tend to \(-\infty\) on \(E\). Due to the assumption of our lemma they tend to \(-\infty\) also on \(\Delta^\infty(0, \beta_1 r)\). We conclude that series \((0.51)\) converge on \(\mathbb{C}\) for every \(z \in B^\infty_r(0, 1)\). This gives us an extension of \(f\) to \(z \in B^\infty_r(z_0, 1) \times \mathbb{C}\).

To see that this extension (denoted as \(\tilde{f}\)) is Gâteaux differentiable take any two vectors \(w^0 = z^0 + s^0 e_0, w = z + se_0 \in B^\infty_r(0, 1) \times \mathbb{C}\), take \(0 < \beta < 1\) such that \(z^0, z \in B^\infty_r(0, \beta)\) and consider the two (or one)-dimensional subspace \(L\) of \(l^1\) spanned by \(z^0\) and \(z\). For \(t\) in some neighborhood of zero vectors \(z^0 + tz\) belong to \(L \cap B^\infty_r(0, 1)\). Applying the classical Hartogs theorem to the restriction of \(\tilde{f}\) to \((L \cap B^\infty_r(0, 1)) \times \mathbb{C}\) we get the holomorphicity of this restriction. In particular \(\tilde{f}(z^0 + tz)\) is holomorphic in \(t\). Gâteaux differentiability is proved.

Our extension \(\tilde{f}\) is Gâteaux differentiable on \(B^\infty_r(z_0, 1) \times \mathbb{C}\) and holomorphic in \(B^\infty_r(z_0, 1) \times \Delta\), therefore by Theorem 36.5 from [Mu] \(\tilde{f}\) is holomorphic everywhere. Lemma is proved.

We need also the following form of the previous result.

**Corollary 0.2.** Let \(f\) be a holomorphic function on \(B^\infty_r(z^0, \beta) \times H_0\). Suppose that for every \(z\) in some \(P_r\)-thick subset \(E\) of \(B^\infty_r(z^0, \beta)\) the restriction \(f_\Delta(z) := f(z, s)\) extends holomorphically to \(\mathbb{C}\). Then \(f\) extends to a holomorphic function on \(B^\infty_r(z^0, \beta) \times \mathbb{C}\).

**Proof.** Fractional linear trasformation \(\varphi(s) = \frac{s - 1}{s + 1}\) sends \(H_0\) to \(\Delta\) and \(\infty\) to 1. Composing \(f\) with \(\varphi^{-1}\) we get a function \(g\) that for every \(z \in E\) extends from \(\Delta\) to \(\mathbb{C} \setminus \{1\}\). Let \(\pi : \mathbb{C} \to \mathbb{C} \setminus \{1\}\) be the universal covering map. Then \(\pi^{-1} \circ g\) satisfies the assumptions of Lemma 0.4. Therefore it extends to \(B^\infty_r(z^0, \beta) \times \mathbb{C}\). And thus \(g\) extends to \(B^\infty_r(z^0, \beta) \times (\mathbb{C} \setminus \{1\})\). Which gives the desired extension of \(f\).

**Proof of Theorem 3** Now let us give the proof of this theorem. For every \(z \in B_{r_2}(0, 1)\) we can apply Lemma 0.3 to extend our \(f\) to \(B_{r_1} \times (\mathbb{C} \setminus z)\). This gives us an extension \(\tilde{f}\) of \(f\) to \(B_{r_1} \times l_{r_2}\). In view of already cited Theorem 36.5 from [Mu] all we need to prove is the Gâteaux differentiability of this extension. For this take any pair of vectors \(z_0, z_1 \in B_{r_1}(0, 1)\) and any \(w_0, w_1 \in l_{r_2}\). Consider the subspace \(L\) of \(l_{r_1} \times l_{r_2}\) generated by \(z_0, z_1, w_0, w_1\). The classical Hartogs theorem gives us holomorphicity of \(\tilde{f}_L\). And therefore we obtain differentiability of \(\tilde{f}(z_0 + tz_1, w_0 + \tau w_1)\) with respect to \(t, \tau\). Theorem is proved.

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Appendix A1

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