Chapter 1

Algebraic Generating Functions for Gegenbauer Polynomials

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It is shown that several of Brafman’s generating functions for the Gegenbauer polynomials are algebraic functions of their arguments, if the Gegenbauer parameter differs from an integer by one-fourth or one-sixth.

Two examples are given, which come from recently derived expressions for associated Legendre functions with octahedral or tetrahedral monodromy. It is also shown that if the Gegenbauer parameter is restricted as stated, the Poisson kernel for the Gegenbauer polynomials can be expressed in terms of complete elliptic integrals. An example is given.

1. Introduction

For any \( \lambda \in \mathbb{C} \), the Gegenbauer polynomials \( C_\lambda^N(x) \), \( n = 0, 1, 2, \ldots \), where \( n \) is the degree, are defined by

\[
\sum_{n=0}^{\infty} C_\lambda^N(x)t^n = (1 - 2xt + t^2)^{-\lambda}.
\]

That is, they have \( R^{-2\lambda} := (1 - 2xt + t^2)^{-\lambda} \) as their ordinary generating function. (In the sequel, \( R \) will signify \( (1 - 2xt + t^2)^{1/2} \), with \( R = 1 \) when \( t = 0 \).) When \( \lambda = 1/2 \), the \( C_\lambda^N(x) \) become the Legendre polynomials \( P_n(x) \).

They are specializations themselves: \( C_\lambda^N(x) \) is proportional to the ultraspherical polynomial \( P_\lambda^{(\lambda-1/2,\lambda-1/2)}(x) \), which is the \( \alpha = \beta = \lambda - 1/2 \) case of the Jacobi polynomial \( P_\lambda^{(\alpha,\beta)}(x) \). If \( \lambda \neq -1/2, -3/2, \ldots \), one can write

\[
C_\lambda^N(x) = \frac{(2\lambda)_n}{n!} 2F_1\left(\begin{array}{c}-n, n + 2\lambda \\ \lambda + 1/2 \end{array}\right| \frac{1-x}{2} \),
\]

where \( (a)_n := a(a+1)\ldots(a+n-1) \) is the Pochhammer symbol and \( 2F_1 \) is the Gauss hypergeometric function.
We were intrigued by a remark of Mourad Ismail, in Sec. 4.3 of Ref. 1, to the effect that not many generating functions for Jacobi polynomials are known, which are algebraic functions of their arguments (denoted \(t, x\) here). In this chapter, we show that for the Gegenbauer polynomials with the parameter \(\lambda\) differing by one-fourth or one-sixth from an integer, there are several distinct non-ordinary generating functions that are algebraic, and can be expressed in closed form. (A similar result in the more familiar case when \(\lambda\) is an integer or a half-odd-integer was previously known.) The simplest example is

\[
\sum_{n=0}^\infty \frac{(-1/12)^n}{(1/2)_n} C_n^{1/4}(x) t^n = 2^{-1/4} R^{1/12} \left[ \cosh(\xi/3) + \sqrt{\frac{\sinh \xi}{3 \sinh(\xi/3)}} \right]^{1/4},
\]

\[
e^\xi := R^{-1} \left[ 1 - \left( x - \sqrt{x^2 - 1} \right) t \right],
\]

which holds when \(|x| > 1\) as an equality between power series in \(t\). Because \(\sinh \xi, \sinh(\xi/3)\), and \(\cosh(\xi/3)\) are algebraic functions of \(e^\xi\), the right-hand side is an algebraic function of \(t, x\) that can be expressed using radicals.

Why it is more easily written in a trigonometric form will become clear.

The algebraic generating functions derived below are specializations of two of Brafman’s non-ordinary (and in general, non-algebraic) generating functions, which appear in Theorems 1 and 6, and their respective extensions, which appear in Theorems 10 and 11. For additional light on his two generating functions, see Chap. 17 of Rainville; also Chap. III, Sec. 4 of Ref. 5; and Ref. 6. The extensions come with the aid of an identity given as Eq. (4) in Ref. 3, which is generalized and placed in context by Srivastava (cf. Sec. 4.1 of Srivastava and Manocha). The generating functions of Brafman are usually expressed in terms of \(2F_1\), having been derived by series rearrangement, but can also be written in terms of the associated Legendre functions \(P_{\nu}^\mu\), or their Ferrers counterparts \(P_{\nu}^\mu\).

There are cases in which the associated Legendre function \(P_{\nu}^\mu\), of degree \(\nu\) and order \(\mu\), can be written in closed form; such as when \(\mu = 0\) and \(\nu = n \in \mathbb{Z}\), in which case \(P_{\nu}^\mu\) and \(P_{\nu}^\mu\) equal \(P_n\). There are others. It has been known since the early work of Schwarz on the algebraicity of \(2F_1\) that if the ordered pair \((\nu, \mu)\) \(\in \mathbb{C}^2\) differs from \((\pm 1/6, \pm 1/4), (\pm 1/4, \pm 1/3), \) or \((\pm 1/6, \pm 1/3)\) by an element of \(\mathbb{Z}^2\), the functions \(P_{\nu}^\mu\) and \(P_{\nu}^\mu\) will be algebraic. (For an exposition focused on \(2F_1\), see Chap. VII of Poole; also Sec. 2.7.2 of Ref. 11 and Ref. 12.) But simple, non-parametric representations of these algebraic functions were not known.
We recently obtained explicit trigonometric formulas for the functions $P_{-1/6}^{-1/4}$, $P_{-1/6}^{-1/4}$, and the second-kind Legendre function $Q_{-1/4}^{-1/3}$. In the Appendix, the resulting simple formulas for $P_{-1/6}^{1/4}$, $P_{-1/6}^{1/4}$, and $P_{-1/4}^{1/3}$, $P_{-1/4}^{1/3}$ are given. In Secs. 2 and 3, Brafman’s results are specialized with the aid of these representations, and yield novel algebraic generating functions for the set $\{C_n^\lambda(x)\}_{n=0}^\infty$ when $\lambda \in \mathbb{Z} \pm 1/4$ and $\lambda \in \mathbb{Z} \pm 1/6$. That specializing Brafman’s two generating functions yields interesting identities has been pointed out by Viswanathan but the present focus on algebraicity is new.

For the Gegenbauer polynomials $C_n^\lambda(x)$, the cases $\lambda \in \mathbb{Z} \pm 1/4$ and $\lambda \in \mathbb{Z} \pm 1/6$ have long been recognized as special. For instance, $C_n^{-1/4}(x)$ and $C_n^{-1/6}(x)$ have been expressed in terms of elliptic functions. Also, the polynomials $C_n^{1/6}(x)$ and $C_n^{7/6}(x)$ have recently been used in applied mathematics, in the modeling of wave scattering in the exterior of what is locally a right-angled wedge. In Sec. 4, we point out that the Poisson kernel for each of the sets $\{C_n^{1/4}(x)\}_{n=0}^\infty$ and $\{C_n^{1/6}(x)\}_{n=0}^\infty$ is also special: it can be expressed in terms of the first and second complete elliptic integral functions, $K = K(m)$ and $E = E(m)$. (A similar elliptic formula in the case $\lambda = 1/2$, i.e., for a kernel arising from the Legendre set $\{P_n(x)\}_{n=0}^\infty$, was obtained by Watson.) Some final remarks appear in Sec. 5.

2. The First Gegenbauer Generating Function

The following theorem presents Brafman’s first \(2F_1\)-based generating function:

**Theorem 1.** For any $\lambda \in \mathbb{C} \setminus \{0, -1/2, -1, \ldots\}$ and $\gamma \in \mathbb{C}$, one has

$$
\sum_{n=0}^\infty \frac{\gamma_n}{(2\lambda)_n} C_n^\lambda(x)^n = R^{-\gamma} 2F_1 \left( \begin{array}{c} \gamma, 2\lambda - \gamma \\ \lambda + 1/2 \end{array} \left| \frac{R - 1 + xt}{2R} \right. \right) \quad (4a)
$$

$$
= (1 - xt)^{-\gamma} 2F_1 \left( \begin{array}{c} \gamma/2, \gamma/2 + 1/2 \\ \lambda + 1/2 \end{array} \left| 1 - \frac{R}{1 - xt} \right. \right)^2, \quad (4b)
$$

as equalities between power series in $t$.

**Note.** Version (4b) is Brafman’s, rewritten; (4a) follows by a quadratic transformation of $2F_1$. When $\gamma = -N$, $N = 0, 1, 2, \ldots$, the series terminate, and the $2F_1$ in (4a) is proportional to $C_N^{\lambda}(1 - xt)/R$, by (2).

The $2F_1$ in (4a) can be rewritten as an associated Legendre (or Ferrers) function, with the aid of the formula (A.1); as is true of the $2F_1$ in (4b).
if $\gamma = 2\lambda - 1/2$. It must be remembered that $P_\nu^\mu(z)$, $P_\nu^-\mu(z)$ are defined if, respectively, $z \notin (-\infty, 1]$ and $z \notin (-\infty, -1] \cup [1, \infty)$. The rewriting yields

**Theorem 2.** For any $\mu \notin \{1/2, 1, 3/2, \ldots\}$, one has

$$\sum_{n=0}^{\infty} \frac{(-\nu - \mu)^n}{(1 - 2\mu)n} C_n^{1/2-\mu}(x)t^n$$

$$= 2^{-\mu} \Gamma(1 - \mu) R^{\nu+\mu} \left[(z^2 - 1)^{\mu/2} P_{\nu}^\mu(z)\right] \bigg|_{z=\frac{1-xt}{R}}$$

for any $\nu \in \mathbb{C}$. Moreover,

$$\sum_{n=0}^{\infty} \frac{(1/2 - 2\mu)n}{(1 - 2\mu)n} C_n^{1/2-\mu}(x)t^n$$

$$= 2^{-\mu} \Gamma(1 - \mu)(1 - xt)^{2\mu-1/2} \left[(z^2 - 1)^{\mu/2} P_{\nu}^{\mu}_{-1/4}(z)\right] \bigg|_{z=2\left(\frac{1-xt}{R}\right)^2 - 1}\,.$$  

These hold as equalities between power series in $t$. For real $z$, they hold as stated when $z > 1$, and hold when $z \in (-1, 1)$ if the Legendre function $P_\nu^\mu$ and $z^2 - 1$ are replaced by the Ferrers function $P_\nu^{\mu}$ and $1 - z^2$.

**Note.** By examination, if $x$ is real and $t$ is real and of sufficiently small magnitude, then $z > 1$ (implying that the Legendre case is applicable) in (5a) when $|x| > 1$ and in (5b) when $|x| < 1$. Conversely, $z \in (-1, 1)$ (implying that the Ferrers case is applicable) in (5a) when $|x| < 1$ and in (5b) when $|x| > 1$. For the $|x| < 1$ case of (5b), cf. Thm. 3 of Ref. 17.

Specializing parameters $\nu, \mu$ in this theorem yields a number of interesting identities. As is summarized in the Appendix, the associated Legendre function $P_\nu^\mu$ and Ferrers function $P_\nu^{\mu}$ can be written in terms of elementary functions in several cases. They are referred to here as the reducible case (i), the quasi-algebraic cases (ii[a]), (ii[b]), and the algebraic cases (iii), (iv[a]), (iv[b]).

In the reducible case (i), $\nu = -\mu + N$, $N = 0, 1, 2, \ldots$. The functions $P_{\nu}^\mu = P_{-\mu+N}^\mu$ and $P_{\nu}^-\mu = P_{\mu+N}^{\mu}$ can then be expressed in terms of the polynomial $C_N^{\mu-1/2}$ (see (A.3)). This leads to

**Theorem 3.** For any $\lambda \in \mathbb{C} \setminus \{0, -1/2, -1, \ldots\}$ and $N = 0, 1, 2, \ldots$, one
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has
\[ \sum_{n=0}^{N} \frac{(-N)^n}{(2\lambda)^n} C_n^\lambda(x) t^n = \frac{N^!}{(2\lambda)^N} R^N C_N^\lambda \left( \frac{1 - xt}{R} \right), \] (6)
\[ \sum_{n=0}^{\infty} \frac{(2\lambda + N)^n}{(2\lambda)^n} C_n^\lambda(x) t^n = \frac{N^!}{(2\lambda)^N} R^{-2\lambda-N} C_N^\lambda \left( \frac{1 - xt}{R} \right), \] (7)
as equalities between power series in \( t \).

**Proof.** To obtain (6), substitute (A.3) into (5a); and for (7) do the same, first using \( P_{\mu + \nu} = P_{\mu - \nu} \) (or \( P_{\mu - \nu} = P_{\mu + \nu} \)).

The finite sum identity (6) has been derived Lie-theoretically by Miller (see Eq. (4.11) of Ref. 18 and p. 204 of Ref. 19), and (7) is also known. The \( N = 0 \) case of (7) is of course the defining generating function (1) for the Gegenbauer polynomials. There are also identities resembling (6),(7) that come from (5b).

In the quasi-cyclic case (ii[a]), the degree \( \nu \) is an integer, and \( P_{\mu}^\nu \) is therefore expressible in closed form (see (A.4)). For instance, \( P_{\mu - 1}^\nu (\coth \xi) = \Gamma(1 - \mu)^{-1} e^{\mu \xi} \), or equivalently
\[ P_{\mu - 1}^\nu (z) = P_{\mu}^\nu (z) = \Gamma(1 - \mu)^{-1} [(z + 1)/(z - 1)]^{\mu/2}. \] (8)
Setting \( \nu = -1, 0 \) in (8a) accordingly yields the pair
\[ \sum_{n=0}^{\infty} \frac{(\lambda + 1/2)^n}{(2\lambda)^n} C_n^\lambda(x) t^n = R^{-1} \left( \frac{1 + R - xt}{2} \right)^{1/2 - \lambda}, \] (9a)
\[ \sum_{n=0}^{\infty} \frac{(\lambda - 1/2)^n}{(2\lambda)^n} C_n^\lambda(x) t^n = \left( \frac{1 + R - xt}{2} \right)^{1/2 - \lambda}. \] (9b)
The identity (9a) is a well-known alternative generating function for the Gegenbauer polynomials. But its companion (9b) is less well known, though a generalization to Jacobi polynomials was found by Carlitz; it is the fact that a closed form can be computed whenever the coefficient \((\lambda + 1/2)_n / (2\lambda)_n\) is replaced by \((\lambda + k + 1/2)_n / (2\lambda)_n\), with \( k \in \mathbb{Z} \).

In the quasi-dihedral case (ii[b]), the order \( \mu \) is a half-odd-integer, which in (5a) means that the Gegenbauer parameter \( \lambda = 1/2 - \mu \) must be taken to be an integer. This is the fairly straightforward trigonometric (e.g., Chebyshev) case, and the resulting identities are not given here.

The focus here is on the octahedral case (iii), when \((\nu, \mu) \in \mathbb{Z}^2 + (\pm 1/6, \pm 1/4)\), and the tetrahedral subcases (iv[a]) and (iv[b]), when \((\nu, \mu) \in \mathbb{Z}^2 + (\pm 1/3, \pm 0)\).
The functions $P^0_{\nu}(z), P^0_{\nu}(z)$ are then algebraic in $z$ (see the Appendix).

**Theorem 4.** The Gegenbauer generating function

$$
\sum_{n=0}^{\infty} \frac{(\gamma)_n}{(2\lambda)_n} C_\lambda^\mu(z)t^n
$$

is algebraic (1) if $\lambda \in \mathbb{Z} \pm 1/4$ with $\gamma - \lambda \in \mathbb{Z} \pm 1/3$; or (2) if $\lambda \in \mathbb{Z} \pm 1/6$ with $\gamma - \lambda \in \mathbb{Z} \pm \{1/3, 1/4\}$.

**Proof.** Claim 1 comes by restricting (5a) to the octahedral case (iii), and claim 2 by restricting (5a) to the tetrahedral cases (iv[a]),(iv[b]).

For the octahedral case (iii) and the tetrahedral case (iv[a]), the fundamental algebraic formulas are (A.7) and (A.9), where $P^{\pm 1/4}_{\nu}(\cosh \xi)$, $P^{\pm 1/4}_{\nu}(\cos \theta)$ and $P^{\pm 1/3}_{\nu}(\cot \xi)$, $P^{\pm 1/3}_{\nu}(\tanh \xi)$ are given in terms of trigonometric functions: hyperbolic ones of $\xi$ and circular ones of $\theta$. In effect, they are given in terms of $e^\xi$ or $e^{i\theta}$. But in Theorem 2 the Legendre/Ferrers argument $z$ equals $(1 - xt)/R$ (in (5a)) or $2[R/(1 - xt)]^2 - 1$ (in (5b)). The following table adapts (A.7),(A.9) to the needs of Theorem 2.

| $z$ | $(1 - xt)/R$ |
|-----|---------------|
| $z = \cosh \xi$ | $e^\xi = R^{-1}[1 - (x - \sqrt{x^2 - 1})t]$ |
| $z = \cos \theta$ | $e^{i\theta} = R^{-1}[1 - (x - i\sqrt{1 - x^2})t]$ |
| $z = \coth \xi$ | $e^\xi = t\sqrt{x^2 - 1}/(1 - R + xt)$ |
| $z = \tanh \xi$ | $e^\xi = t\sqrt{1 - x^2}/(-1 + R + xt)$ |

| $z$ | $2[R/(1 - xt)]^2 - 1$ |
|-----|------------------|
| $z = \cosh \xi$ | $e^{i\xi} = (1 - xt)/R - t\sqrt{1 - x^2}$ |
| $z = \cos \theta$ | $e^{i\theta}/2 = (1 - xt)/[R - i t\sqrt{x^2 - 1}]$ |
| $z = \coth \xi$ | $e^\xi = R/\sqrt{1 - x^2}$ |
| $z = \tanh \xi$ | $e^\xi = R/(t\sqrt{x^2 - 1})$ |

By combining the $(\nu, \mu) = (-1/6, 1/4)$ case of (5a) with (A.7a), aided by the first line of this table, one readily derives an explicit, octahedrally algebraic generating function for the set of polynomials $\{C_{\nu/4}^\mu(x)\}_{\nu=0}^{\infty}$. It appeared in the Introduction as Eq. (3). One also derives a tetrahedrally...
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algebraic generating function for \( \{C_{1/6}^\lambda(x)\}_{n=0}^\infty \), namely

\[
\sum_{n=0}^\infty \frac{(-1/12)_n}{(1/3)_n} C_{1/6}^\lambda(x)^n
\]

\[
= 2^{-7/12} 3^{-3/8} R^{1/12} (\sinh \xi)^{-1/3} \left[ \sqrt{\sqrt{3} + 1} f_+ + \sqrt{\sqrt{3} - 1} f_- \right],
\]

\[e^\xi := t \sqrt{x^2 - 1/(1 - R - xt)} ;\]

\[
= 2^{-7/12} 3^{-3/8} R^{1/12} (\cosh \xi)^{-1/3} \left[ \sqrt{\sqrt{3} + 1} g_+ + \sqrt{\sqrt{3} - 1} g_- \right],
\]

\[e^\xi := t \sqrt{1 - x^2/(-1 + R + xt)} .\]

This comes by combining the \((\nu, \mu) = (-1/4, 1/3)\) case of (5a) with (A.9a) and (A.9b), aided by the third and fourth lines of the table. Here, the functions \(f_\pm = f_\pm (\coth \xi)\) and \(g_\pm = g_\pm (\tanh(\xi))\) are algebraic in \(e^\xi\) and are defined in (A.8a), (A.8b). The two right-hand sides of (10) are equivalent, but when the argument \(x\) is real, they are most useful when, respectively, \(|x| > 1\) and \(|x| < 1\). Additional explicitly algebraic generating functions that arise from (5a) or (5b) can be worked out.

The preceding results stemmed from Theorem 1 but can be generalized, because Brafman’s first generating function can be hypergeometrically extended. The extension uses the identities appearing in his Ref. 3 as Eqs. (4) and (11). The latter identity, which is a specialization of the former to Gegenbauer polynomials, can be restated as follows.

**Lemma 5.** For any \(\lambda \in \mathbb{C}\), and parameters \(c_1, \ldots, c_p, d_1, \ldots, d_q\) and \(u \in \mathbb{C}\) for which the below \(p+1\) \(F_q\) coefficients are defined, one has

\[
\sum_{n=0}^\infty \binom{-n, c_1, \ldots, c_p}{d_1, \ldots, d_q} u^n C_n^\lambda(x) t^n = R^{-2\lambda} \sum_{n=0}^\infty \frac{(c_1)_n \cdots (c_p)_n}{(d_1)_n \cdots (d_q)_n} \binom{x - t}{R} \binom{-tu}{R}^n ,
\]

as an equality between power series in \(t\).

**Note.** In Ref. 3 the argument of the \(C_n^\lambda\) on the right is written as \(w\), which is defined in Eq. (1) of that work to equal \(2(x - t)/R\). The ‘2’ is easily seen to be erroneous, and has been removed. This identity is proved by series rearrangement, once each \(C_n^\lambda\) has been expressed hypergeometrically: not as in [2], but in a form that incorporates a quadratic transformation.
By applying the \( p = q = 1 \) case of this lemma to the statement of Theorem 1, one readily obtains the following corollaries of (4a) and (4b). Here and below, \( U \) signifies \([1 - 2(1-u)xt + (1-u)^2t^2]^{1/2}\), with \( U = 1 \) when \( t = 0 \). Hence, \( U \) interpolates between \( R \) at \( u = 0 \) and unity at \( u = 1 \). Similarly, \( R^2 + u(x-t)t \) interpolates between \( R^2 \) and \( 1 - xt \). It should be noted that to make (11b) resemble (4b) as closely as possible, Euler’s transformation of \( _2F_1 \) has been applied to its right-hand side.

**Theorem 6.** For any \( \lambda \in \mathbb{C} \setminus \{0, -1/2, -1, \ldots \} \) and \( \gamma \in \mathbb{C} \), and arbitrary \( u \), one has

\[
\sum_{n=0}^{\infty} _2F_1 \left( -n, \frac{2\lambda - \gamma}{2\lambda}; u \right) \binom{\lambda}{n} t^n
= U^{\gamma - 2\lambda} R^{-\gamma} _2F_1 \left( \gamma, \frac{2\lambda - \gamma}{\lambda + 1/2}; UR - \left[ R^2 + u(x-t)t \right]/2UR \right)
\]

(11a)

\[
= U^{2\gamma - 2\lambda} [R^2 + u(x-t)t]^{-\gamma} _2F_1 \left( \frac{\gamma}{2}, \frac{\gamma}{2} + 1 + \frac{1}{2}; \lambda + 1/2; UR - \left[ R^2 + u(x-t)t \right]/2UR \right)
\]

(11b)

where the equalities are between power series in \( t \).

**Note.** When \( \gamma = -N, N = 0, 1, 2, \ldots, \) the right-hand series terminate, and the \( _2F_1 \) in (11a) is proportional to \( C^\lambda_N \left( [R^2 + u(x-t)t]/UR \right) \).

Theorem 6 is not merely a corollary of Theorem 1, but an extension. It reduces to Theorem 1 when \( u = 1 \), because the left-hand \( _2F_1 \) then equals \( (\gamma)_n/(2\lambda)_n \), by the Chu–Vandermonde formula. Rewriting each right-hand \( _2F_1 \) in Theorem 6 as an associated Legendre function yields the following, which is an extension of Theorem 2. (In (11b), the rewriting is possible only if \( \gamma = 2\lambda - 1/2 \).)

**Theorem 7.** For any \( \mu \notin \{1/2, 1, 3/2, \ldots \} \), and arbitrary \( u \), one has

\[
\sum_{n=0}^{\infty} _2F_1 \left( -n, \frac{\nu - \mu + 1}{1 - 2\mu}; u \right) \binom{1/2-\mu}{n} t^n
= 2^{-\mu} \Gamma(1 - \mu) U^{-\nu+\mu-1} R^{\nu+\mu} \left[ (z^2 - 1)^{\mu/2} P^\mu(z) \right] \bigg|_{z = R^2 + u(x-t)t}/UR \]

(12a)
for any \( \nu \in \mathbb{C} \). Moreover,

\[
\sum_{n=0}^{\infty} 2F1\left(\begin{array}{c}
-n, 1/2 \\
1 - 2\mu
\end{array} \mid u\right) C^{1/2-\mu}_n(x)t^n
\]

\[
= 2^{-\mu} \Gamma(1 - \mu) U^{-2\mu} \times \left[R^2 + u(x - t)t\right]^{2\mu-1/2} \left[(z^2 - 1)^{\mu/2} P_{-1/4}^\mu(z)\right] \bigg|_{z=2\left[R^2 + u(x - t)t\right]} - 1.
\]

These hold as equalities between power series in \( t \). For real \( z \), they hold as stated when \( z > 1 \), and hold when \( z \in (-1, 1) \) if \( P_{-1/4}^\mu \) and \( z^2 - 1 \) are replaced by \( P_{1/4}^\mu \) and \( 1 - z^2 \).

By specializing Theorem 7, one can immediately extend the preceding results on closed-form generating functions and algebraicity to include the free parameter \( u \). The following specializations of (12a), based on (A.3), are extensions of the identities (6) and (7) of Theorem 3; for the latter, cf. Eq. (4.14) of Viswanathan\textsuperscript{6} and Eq. (5.121) of Miller\textsuperscript{19}.

**Theorem 8.** For any \( \lambda \in \mathbb{C} \setminus \{0, -1/2, -1, \ldots\} \) and \( N = 0, 1, 2, \ldots \), and arbitrary \( u \), one has

\[
\sum_{n=0}^{\infty} 2F1\left(\begin{array}{c}
-n, 2\lambda + N \\
2\lambda
\end{array} \mid u\right) C^\lambda_n(x)t^n
\]

\[
= \frac{N!}{(2\lambda)_N} U^{-2\lambda - N} R^N C^\lambda_N \left(\frac{R^2 + u(x - t)t}{UR}\right),
\]

\[
\sum_{n=0}^{\infty} 2F1\left(\begin{array}{c}
-n, -N \\
2\lambda
\end{array} \mid u\right) C^\lambda_n(x)t^n
\]

\[
= \frac{N!}{(2\lambda)_N} U^N R^{-2\lambda - N} C^\lambda_N \left(\frac{R^2 + u(x - t)t}{UR}\right),
\]

as equalities between power series in \( t \).

A \( u \)-dependent extension of Theorem 4, which was implied by (5a), can also be obtained; it is an immediate consequence of the \( u \)-dependent extension (12b) of (A.3), and is

**Theorem 9.** The Gegenbauer generating function

\[
\sum_{n=0}^{\infty} 2F1\left(\begin{array}{c}
-n, 2\lambda - \gamma \\
2\lambda
\end{array} \mid u\right) C^\lambda_n(x)t^n
\]

is algebraic, for arbitrary \( u \), (1) if \( \lambda \in \mathbb{Z} \pm 1/4 \) with \( \gamma - \lambda \in \mathbb{Z} \pm 1/3 \); or (2) if \( \lambda \in \mathbb{Z} \pm 1/6 \) with \( \gamma - \lambda \in \mathbb{Z} \pm \{1/3, 1/4\} \).
Explicit expressions for these \(u\)-dependent generating functions, which are algebraic in \(u\) as well as in \(t, x\), can be computed with some effort from Theorem 7 if one exploits the fundamental formulas (A.7), (A.9).

3. The Second Gegenbauer Generating Function

The following theorem presents Brafman’s second \(2F_1\)-based generating function.

**Theorem 10.** For any \(\lambda \in \mathbb{C} \setminus \{0, -1/2, -1, \ldots\}\) and \(\gamma \in \mathbb{C}\), one has

\[
\sum_{n=0}^{\infty} \frac{(\gamma)_n(2\lambda - \gamma)_n}{(2\lambda)_n(\lambda + 1/2)_n} C_\lambda^n(x) t^n
\]

\[
= \frac{2F_1}{2F_1}
\]

\[
\left(\begin{array}{c}
\gamma, 2\lambda - \gamma \\
\lambda + 1/2
\end{array} \right) \frac{1 - R - t}{2} \frac{1 - R + t}{2}
\]

\[
= (1 - 2xt)^{-\gamma}
\]

\[
\times \frac{2F_1}{2F_1}
\]

\[
\left(\begin{array}{c}
\gamma/2, \gamma/2 + 1/2 \\
\lambda + 1/2
\end{array} \right) 1 - \frac{1}{(R + t)^2}
\]

\[
\times \frac{2F_1}{2F_1}
\]

\[
\left(\begin{array}{c}
\gamma/2, \gamma/2 + 1/2 \\
\lambda + 1/2
\end{array} \right) 1 - \frac{1}{(R - t)^2}
\]

(13a)

(13b)

as equalities between power series in \(t\).

**Note.** Version (13a) is Brafman’s; (13b) follows by quadratically transforming each \(2F_1\). When \(\gamma = -N,\ N = 0, 1, 2, \ldots\), the series terminate, and the right-hand side of (13a) is proportional to \(C^\lambda_N(R + t)C^\lambda_N(R - t)\). Irrespective of \(\gamma\), the Legendre (i.e., \(\lambda = 1/2\)) case is of special interest. The \(C_\lambda^n\) then reduces to the Legendre polynomial \(P_n\), and each \(2F_1\) in (13a) is proportional to the Legendre function \(P_{\gamma-\lambda}\).

By applying the \(p = q = 2\) case of Lemma 5 to the statement of Theorem 10, one readily obtains the following corollaries of (13a) and (13b); for the former, cf. Brafman

**Theorem 11.** For any \(\lambda \in \mathbb{C} \setminus \{0, -1/2, -1, \ldots\}\) and \(\gamma \in \mathbb{C}\), and arbi-
trary $u$, one has
\[
\sum_{n=0}^{\infty} {}_{3}F_{2}\left( \begin{array}{c} -n, \gamma, 2\lambda - \gamma \\ 2\lambda, \lambda + 1/2 \end{array} \right| u \right) C_n^\lambda(x) t^n
\]
\[
= R^{-2\lambda} {}_{2}F_{1} \left( \begin{array}{c} \gamma, 2\lambda - \gamma \\ \lambda + 1/2 \end{array} \right| \frac{R - U + ut}{2R} \right) 2\Gamma \left( \begin{array}{c} \gamma, 2\lambda - \gamma \\ \lambda + 1/2 \end{array} \right| \frac{R - U - ut}{2R} \right)
\]
\[
= (U^2 - u^2 t^2)^{-\gamma} R^{2\lambda - 2\gamma} \times \left( \frac{R^2}{(U - ut)^2} \right) \times \left( \frac{R^2}{(U + ut)^2} \right)
\]
where the equalities are between power series in $t$.

**Note.** When $\gamma = -N$, $N = 0, 1, 2, \ldots$, the series terminate, and the right-hand side of (14a) is proportional to $C_N^\lambda((U - ut)/R) C_N^\lambda((U + ut)/R)$.

The Legendre case of (14a), i.e., that of $\lambda = 1/2$ with $\gamma$ unrestricted, was derived before Brafman by Rice\(^{21}\) who attributed its $u = 1$ sub-case to Bateman\(^{22}\) (Cf. Rice’s Eqs. (2.11) and (2.14), and Bateman’s (4.3).)

Theorem 11 does not reduce easily to Theorem 10 (for instance, by setting $u = 1$, which reduces Theorem 6 to Theorem 1). It is better described as a corollary than as an extension.

But the $\, {}_{2}F_{1} \,$ functions in Theorems 10 and 11 are familiar: they have the same parameters as in Theorems 1 and 6, to which Theorems 10 and 11 are analogous. Rewriting Theorem 10 in terms of associated Legendre or Ferrers functions yields the following, which is analogous to Theorem 2.

(For (15a), cf. Thm. 2 of Cohl and MacKenzie\(^{17}\))

**Theorem 12.** For any $\mu \notin \{1/2, 1, 3/2, \ldots\}$, one has
\[
\sum_{n=0}^{\infty} \frac{(-\nu - \mu)_n (1 + \nu - \mu)_n}{(1 - 2\mu)_n (1 - \mu)_n} C_n^{1/2-\mu}(x) t^n
\]
\[
= 2^{-2\mu} \Gamma(1 - \mu)^2 \mathcal{F}^\mu_t(R + t) \mathcal{F}^{-\mu}_t(R - t),
\]
for any $\nu \in \mathbb{C}$. Moreover,
\[
\sum_{n=0}^{\infty} \frac{(1/2 - 2\mu)_n (1 - 2\mu)_n}{(1 - 2\mu)_n (1 - \mu)_n} C_n^{1/2-\mu}(x) t^n
\]
\[
= 2^{-2\mu} \Gamma(1 - \mu)^2 (1 - 2xt)^{2\mu - 1/2} \mathcal{F}^{-\mu}_{1/4}(2/(R - t)^2 - 1) \mathcal{F}^{-\mu}_{-1/4}(2/(R + t)^2 - 1).
\]
These hold as equalities between power series in $t$. The function $F_{\mu \nu}(z)$ is defined as $(z^2 - 1)^{\mu/2} P_{\mu}^\nu(z)$ or $(1 - z^2)^{\mu/2} P_{\mu}^\nu(z)$, which are equivalent. (For real $z$, the Legendre definition should be used if $z > 1$ and the Ferrers definition if $z \in (-1, 1)$.)

The rewriting of Theorem 11 in terms of associated Legendre functions proceeds similarly to the rewriting of Theorem 10. By specializing parameters in the two rewritten theorems, one can derive a number of interesting identities. For example, in the reducible case (i), when $\nu = -\mu + N$, $N = 0, 1, 2, \ldots$, one can derive analogues of Theorems 3 and 8.

The focus here is on the octahedral and tetrahedral algebraic cases. From the algebraic formulas in the Appendix, substituted into the rewritten theorems, one deduces the following from their first halves; again, as in the last section.

**Theorem 13.** The Gegenbauer generating functions

$$
\sum_{n=0}^{\infty} \frac{(\gamma)_{n}(2\lambda - \gamma)_{n}}{(2\lambda)_{n}(\lambda + 1/2)_{n}} C_{n}^\lambda(x) t^n,
$$

$$
\sum_{n=0}^{\infty} \binom{-n, \gamma, 2\lambda - \gamma}{2\lambda, \lambda + 1/2} C_{n}^\lambda(x) t^n,
$$

are algebraic (1) if $\lambda \in \mathbb{Z} \pm 1/4$ with $\gamma - \lambda \in \mathbb{Z} \pm 1/3$; or (2) if $\lambda \in \mathbb{Z} \pm 1/6$ with $\gamma - \lambda \in \mathbb{Z} \pm 1/3$. In the latter generating function, $u$ is arbitrary; the algebraicity is in $x, t$ and $u$.

4. The Poisson Kernel

The Poisson kernel for a set of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ plays a major role in approximation theory. It is a bilinear generating function of the form

$$
K_t(x, y) = \sum_{n=0}^{\infty} h_n p_n(x)p_n(y) t^n, \quad (16)
$$

where the normalization coefficients $h_n$ would be absent if the polynomials were orthonormal and not merely orthogonal. The Poisson kernel for the Gegenbauer polynomials can be expressed in terms of $\text{}_2F_1$, as can a slightly simpler companion function.\cite{23,24} Let $x = \cos \theta$ and $y = \cos \phi$, which are appropriate when $x, y \in (-1, 1)$, and define

$$
\bar{z} = \frac{-4 t \sin \theta \sin \phi}{1 - 2t \cos(\theta - \phi) + t^2}, \quad (17a)
$$

$$
z = \frac{4 t^2 \sin^2 \theta \sin^2 \phi}{(1 - 2t \cos \theta \cos \phi + t^2)^2}, \quad (17b)
$$
which are related by $z = \tilde{z}/(2 - \tilde{z})^2$. The Poisson kernel for $\{C_\lambda^x(x)\}_{n=0}^\infty$ is

$$
\sum_{n=0}^\infty \frac{\lambda + n}{\lambda} \frac{n!}{(2\lambda)_n} C_\lambda^x(x)C_\lambda^y(y) t^n = \frac{1 - t^2}{[1 - 2t \cos(\theta - \phi) + t^2]^{\lambda + 1}} 2F_1\left(\frac{\lambda, \lambda + 1}{2\lambda}\left|\tilde{z}\right\rangle\right),
$$

and its companion is

$$
\sum_{n=0}^\infty \frac{n!}{(2\lambda)_n} C_\lambda^x(x)C_\lambda^y(y) t^n = \frac{1}{[1 - 2t \cos(\theta \cos \phi + t^2)]^{\lambda + 1}} 2F_1\left(\frac{\lambda + 1}{2}, (\lambda + 2)/2 \left|\lambda + 1/2\right\rangle\right)
$$

In each of (18) and (19), the two right-hand sides are related by a quadratic hypergeometric transformation. Equation (18) can be obtained from Eq. (19) by applying the operator $\lambda^{-1} t^{-\lambda+1} \frac{d}{dt} \circ t^\lambda$ to both sides.

When the parameter $\lambda$ is an integer (e.g., in the Chebyshev case), the Poisson kernel and its companion are elementary functions. When $\lambda = 1/2$, so that $\{C_\lambda^x(x)\}_{n=0}^\infty$ are the Legendre polynomials, Watson expressed the companion in terms of the first complete elliptic integral function, $K = K(m)$. In principle, this can be done when $\lambda$ is any half-odd-integer.

It does not seem to have been remarked that when $\lambda$ differs from an integer by one-fourth or one-sixth, expressions in terms of complete elliptic integrals can also be obtained. This is implied by the pattern of hypergeometric parameters in (19), as will be explained. The focus here is on the cases $\lambda = 1/4$ and $\lambda = 1/6$; the general $\lambda \in \mathbb{Z} \pm 1/4$ and $\lambda \in \mathbb{Z} \pm 1/6$ cases can be handled by applying the contiguity relations of $2F_1$.

The Gauss hypergeometric ODE satisfied by the function $2F_1(a, b; c; z)$ has singular points at $z = 0, 1, \infty$, with respective characteristic exponents $0, 1 - c, 0, c - a - b$; and $a, b$. The respective exponent differences are $1 - c, c - a - b$, and $b - a$, and differences are significant only up to sign. It is well known that if a Gauss ODE has an unordered set of (up-to-sign) exponent differences $\{\delta, \delta_1, \delta_2\}$, it admits a quadratic transformation to one with differences $\{\delta_1, \delta_1, 2\delta_2\}$, and the solutions of the two ODEs will correspond.
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(For instance, in each of (18) and (19) the first right-hand side comes from the second in this way.) One may write \( \{ 1, \delta_1, \delta_2 \} \sim \{ \delta_1, \delta_1, 2 \delta_2 \} \). There are hypergeometric transformations of higher order than the quadratic (see Sec. 25 of Ref. [10] inter alia). In particular, there are sextic ones that arise as compositions of quadratic and cubic ones, the action of which is summarized by \( \{ \frac{1}{2}, \frac{1}{2}, \delta \} \sim \{ \frac{3}{2}, \frac{1}{2}, 2 \delta \} \sim \{ 2 \delta, 2 \delta, 2 \delta \} \).

The exponent differences for the \( 2F_1 \)'s in the two right-hand sides of (19) are respectively \( 1 - 2 \lambda, 0, 0 \) and \( 1/2 - \lambda, 0, 1/2 \). If \( \lambda = 1/6 \), the second of these triples (when unordered) is \( \{ \frac{1}{2}, \frac{1}{2}, 0 \} \sim \{ 0, 0, 0 \} \). If \( \lambda = 1/4 \), the first is \( \{ \frac{1}{2}, 0, 0 \} \sim \{ 0, 0, 0 \} \). In other words, a sextic and a quadratic transformation will respectively convert the \( 2F_1 \)'s in the two right-hand sides of (19) to solutions of a Gauss ODE with exponent differences \( 0, 0, 0 \). This is the Gauss ODE with parameters \( a = b = 1/2, c = 1 \), one of the solutions of which is \( 2F_1(1/2, 1/2; 1; z') \). (Here, \( z' \) signifies the new independent variable, which is determined by the hypergeometric transformation.) But \( 2F_1(1/2, 1/2; 1; z') \) equals \( (2/\pi)K(\tilde{w}') \), and the full solution space of the ODE is spanned by \( K(z') \) and \( K'(z') := K(1 - z') \).

The expressions resulting from the just-described reduction procedure are somewhat inelegant, but better ones can be obtained heuristically. In fact, one can begin with the Poisson kernel itself, rather than its companion. The case \( \lambda = 1/4 \) is illustrative. When \( \lambda = 1/4 \), the \( 2F_1 \) in the first right-hand side of (18) is of the form \( 2F_1(1/4, 5/4; 1/2; w) \), where \( w = \tilde{z} \). An explicit formula for this function can be found in the database of Roach, which is currently available at www.planetquantum.com. It is

\[
\frac{\Gamma(1/4)^2}{2\sqrt{\pi}} 2F_1 \left( \begin{array}{c} 1/4, 5/4 \\ 1/2 \end{array} \right) \left( \begin{array}{c} w \end{array} \right) = \frac{2\sqrt{w}}{1 - w} E(\tilde{w}_+ - \frac{2\sqrt{w}}{1 - w} E(\tilde{w}_-)} + \frac{1}{1 + \sqrt{w}} K(\tilde{w}_+) + \frac{1}{1 - \sqrt{w}} K(\tilde{w}_-),
\]

where \( \tilde{w}_\pm = (1 \pm \sqrt{w})/2 \). The presence of the second complete elliptic integral, \( E = E(m) \), for which the exponent differences are \( 1, 1, 0 \) and not \( 0, 0, 0 \), can be attributed to an application of the contiguous relations of \( 2F_1 \).

5. Final Remarks

It has been shown that if the Gegenbauer parameter \( \lambda \) differs by one-fourth or one-sixth from an integer, there are several generating functions for the Gegenbauer polynomials \( \{ C_n^\lambda \}_{n=0}^\infty \) that are algebraic. These are special cases of Brafman’s generating functions, including the extended ones with
an additional free parameter (denoted \( u \) here). Gegenbauer polynomials with \( \lambda \) restricted as stated were shown to be special in another way: the Poisson kernel computed from them can be expressed with the aid of hypergeometric transformations in terms of complete elliptic integrals.

The results on algebraicity are consequences of Schwarz’s classification of the algebraic cases of the Gauss function \( {}_2F_1 \), and the explicit examples of algebraic generating functions came from recently developed closed-form expressions for certain algebraic \( {}_2F_1 \)’s with octahedral and tetrahedral monodromy; i.e., octahedral and tetrahedral associated Legendre functions. This is because the \( {}_2F_1 \)’s in Brafman’s generating functions admit quadratic transformations, so that in essence, they are Legendre functions.

Generalizations can be considered. One matter worthy of investigation is the relevance of icosahedral \( {}_2F_1 \)’s, which though algebraic cannot be expressed in terms of radicals. Some parametric formulas for them are known, and may yield manageable parametrizations of the consequent algebraic generating functions for Gegenbauer polynomials.

The generalization from Gegenbauer to Jacobi polynomials is also worth pursuing. It follows readily from Schwarz’s classification that certain special cases of Brafman’s second generating function, generalized to non-Gegenbauer Jacobi polynomials but still expressed in terms of \( {}_2F_1 \), are algebraic functions of their arguments. However, the Jacobi-polynomial generalization of his first generating function is known to involve the Appell function \( F_4 \), i.e., a bivariate hypergeometric function. A full classification of the algebraic cases of the first generating function, generalized to Jacobi polynomials, will require results on the algebraicity of Appell functions; and the same is true of the Poisson kernel.

Finally, it should be mentioned that Brafman’s extension procedure, leading to identities parametrized by \( u \), is not the only one that can be applied to Gegenbauer generating functions. By exploiting the connection formula for Gegenbauer polynomials, Cohl and collaborators have obtained novel extensions of the defining relation, as Eq. (11) of their Ref. and of Brafman’s second identity, as Eq. (26) of their Ref. For suitably chosen parameter values, such extensions will be algebraic.
Appendix. Associated Legendre Functions in Closed Form

The associated Legendre function $P_{\mu}^{\nu}(z)$ of degree $\nu \in \mathbb{C}$ and order $\mu \in \mathbb{C}$ is defined in terms of the Gauss function $2F_1$ by

$$P_{\mu}^{\nu}(z) = \frac{2^{\mu}}{\Gamma(1-\mu)}(z^2-1)^{-\mu/2} 2F_1\left(\begin{array}{c} -\nu-\mu, 1+\nu-\mu \\ 1-\mu \end{array}\bigg| \frac{1-z}{2}\right). \quad (A.1)$$

The Ferrers function $P_{\mu}^{\nu}$ is defined similarly, with $1-z^2$ replacing $z^2-1$. By convention, $P_{\mu}^{\nu}(z)$ and $P_{\mu}^{\nu}(z)$ are defined and analytic on the complex $z$-plane, with the respective omissions of the cut $(-\infty, 1]$ and the cut-pair $(-\infty, -1] \cup [1, \infty)$. When $\mu = 1, 2, \ldots$, $P_{\mu}^{\nu}(z)$ must be taken in a limiting sense. In the singular case when $\nu = 0, 1, 2, \ldots$ and $\mu - \nu$ is a positive integer, $P_{\mu}^{\nu}$ and $P_{\mu}^{\nu}$ are identically zero.

On their respective domains, $P_{\mu}^{\nu}(z)$ and $P_{\mu}^{\nu}(z)$ span the two-dimensional solution space of the associated Legendre ODE, except when $(\nu, \mu) \in \mathbb{Z}^2$. This space can also be viewed as the span of $P_{\mu}^{\nu}$ and $Q_{\mu}^{\nu}$, resp. $P_{\mu}^{\nu}$ and $Q_{\mu}^{\nu}$, where $Q_{\mu}^{\nu}$ and $Q_{\mu}^{\nu}$ are the associated Legendre and Ferrers functions of the second kind. (Again, singular cases are excepted.) The function $P_{\mu}^{\nu}(z)$ is singled out as an element $f(z)$ of the solution space with

$$f(z) \sim \frac{2^{\mu/2}}{\Gamma(1-\mu)}(z-1)^{-\mu/2}, \quad z \to 1 \quad (A.2)$$

as asymptotic behavior.

For all $\nu, \mu \in \mathbb{C}$, it follows from $P_{\nu}^{\mu} = P_{-\nu}^{\mu}$ and $P_{-\nu}^{\mu} = P_{\nu}^{\mu}$. Also, the ordered pair $(\nu, \mu)$ can be displaced by any element of $\mathbb{Z}^2$, for either $P_{\nu}^{\mu}$ or $P_{\mu}^{\nu}$, by applying an appropriate differential operator. (See Sec. 6 of Ref. [13].) Such ‘ladder operators,’ which increment and decrement $\nu$ and/or $\mu$, come from the contiguity relations of $2F_1$.

There are several cases when the functions $P_{\nu}, P_{\mu}^{\nu}$ are elementary; or to put it more broadly, when all solutions of the associated Legendre ODE can be reduced to quadratures. These include the case when the ODE, or the equivalent ODE satisfied by the $2F_1$ in (A.1), is ‘reducible’ (see Sec. 2.2 of Ref. [11]), and certain algebraic cases, when the ODE has a finite projective monodromy group (see Sec. 2.7.2 of Ref. [11] and Chap. VII of Ref. [10]). In the algebraic cases, this group as a subgroup of the M"obius group may be cyclic, dihedral, octahedral, tetrahedral, or icosahedral, but the last of these possibilities does not lead to radical expressions. The other four algebraic cases are numbered (ii)[a], (ii)[b], (iii), (iv) here.

The reducible case is numbered (i) here. It is the case when $\nu = -\mu + N$, $N = 0, 1, 2, \ldots$, and is also called the degenerate or Gegenbauer case. It
follows from (2) and (A.1) that when $\mu \neq \frac{1}{2}, 1, \frac{3}{2}, \ldots,$

$$P_{\mu-N}^\mu(z) = \frac{2^\mu N!}{\Gamma(1-\mu)} \frac{(z^2 - 1)^{-\mu/2} C_N^{1/2-\mu}(z)}{(1-2\mu)_N},$$  \hspace{1cm} (A.3)

with the same holding if $P_{\mu+N}^\mu$ and $z^2 - 1$ are replaced by $P_{\mu-N}^\mu$ and $1 - z^2$.

Of the four non-icosahedral algebraic cases, the simplest is (ii[a]): the cyclic case, when the degree $\nu$ is an integer. The basic formulas are

$$P^\mu_0(\coth \xi) = \Gamma(1-\mu)^{-1} e^{\mu \xi},$$ \hspace{1cm} (A.4a)

$$P^\mu_0(\tanh \xi) = \Gamma(1-\mu)^{-1} e^{\mu \xi}. \hspace{1cm} (A.4b)$$

Actually, $P^\mu_0(z)$ are algebraic in $z$ only if $\mu$ is rational; for general $\mu$, the term ‘quasi-cyclic’ will be used. For any nonzero $\nu \in \mathbb{Z}$, $P^\mu_0, P^\mu_1$ are computed from $P^\mu_{1/2}, P^\mu_{1/2}$ by applying ladder operators that shift the degree.

There is also (ii[b]): the dihedral case, when the order $\mu$ is a half-odd-integer. The basic formulas are

$$P^{1/2}_\nu^\mu(\cosh \xi) = \sqrt{\frac{2}{\pi}} \frac{\cosh (\nu + 1/2) \xi}{\sinh \xi},$$ \hspace{1cm} (A.5a)

$$P^{1/2}_\nu^\mu(\cos \theta) = \sqrt{\frac{2}{\pi}} \frac{\cos (\nu + 1/2) \xi}{\sin \theta}. \hspace{1cm} (A.5b)$$

which define algebraic functions $P^{1/2}_\nu^\mu, P^{1/2}_1$ only if $\nu$ is rational; for general $\nu$, the term ‘quasi-dihedral’ will be used. For any half-odd-integer $\mu$ other than $1/2$, $P^\mu_0, P^\mu_1$ are computed from $P^{1/2}_\nu^\mu, P^{1/2}_1$ by applying ladder operators that shift the degree.

The algebraic cases recently examined include (iii): the octahedral case, when $(\nu, \mu) \in \mathbb{Z}^2 + (\pm 1/6, \pm 1/4)$. For this, define algebraic functions $h_\pm, k_\pm$ trigonometrically by

$$h_\pm(\cosh \xi) = \left\{ (\sinh \xi)^{-1} \left[ \pm \cosh(\xi/3) + \sqrt{\frac{\sinh \xi}{3 \sinh(\xi/3)}} \right] \right\}^{1/4}, \hspace{1cm} (A.6a)$$

$$k_\pm(\cos \theta) = \left\{ (\sin \theta)^{-1} \left[ \cos(\theta/3) \pm \sqrt{\frac{\sin \theta}{3 \sin(\theta/3)}} \right] \right\}^{1/4}. \hspace{1cm} (A.6b)$$

Then, the basic formulas are

$$P_{-1/4}^{\pm 1/4}(\cosh \xi) = 3^{(3/8)(1+1)} \Gamma(1 \mp 1/4)^{-1} h_\pm(\cosh \xi), \hspace{1cm} (A.7a)$$

$$P_{-1/6}^{\pm 1/4}(\cos \theta) = 3^{(3/8)(1+1)} \Gamma(1 \mp 1/4)^{-1} k_\pm(\cos \theta). \hspace{1cm} (A.7b)$$
The plus formulas were derived in Ref. [13] and the minus formulas follow from them, the normalization factors coming from the condition (A.2). Ladder operators can be applied to these basic formulas, as needed.

The other algebraic case recently examined [13] is (iv[a]): the first subcase of the tetrahedral case, when \((\nu, \mu) \in \mathbb{Z}^2 + (\pm 1/4, \pm 1/3)\). For this, define algebraic functions \(f_\pm, g_\pm\) trigonometrically by

\[
 f_\pm (\coth \xi) = \left\{ \sinh \xi \left[ \pm \cosh(\xi/3) + \sqrt{\frac{\sinh \xi}{3 \sinh(\xi/3)}} \right] \right\}^{1/4}, \quad (A.8a)
\]

\[
 g_\pm (\tanh \xi) = \left\{ \cosh \xi \left[ \pm \sinh(\xi/3) + \sqrt{\frac{\cosh \xi}{3 \cosh(\xi/3)}} \right] \right\}^{1/4}. \quad (A.8b)
\]

Then, the basic formulas are

\[
P_{-1/4}^\pm (\coth \xi) = 2^{1/2} \pi^{3/4} 3^{-3/8} \Gamma(1 \mp 1/3)^{-1} \times \left[ \sqrt{3} \pm 1 f_+ \pm \sqrt{3} \mp 1 f_- \right] (\coth \xi), \quad (A.9a)
\]

\[
P_{-1/4}^\pm (\tanh \xi) = 2^{1/2} \pi^{3/4} 3^{-3/8} \Gamma(1 \mp 1/3)^{-1} \times \left[ \pm \sqrt{3} \pm 1 g_+ + \sqrt{3} \mp 1 g_- \right] (\tanh \xi). \quad (A.9b)
\]

(It was shown in Ref. [13] that \(Q_{-1/4}^{-1/3}\)) is a multiple of \(f_-\), and \(f_+\) is an independent solution of the same associated Legendre ODE; so \(P_{-1/4}^{\pm 1/3}\) must be linear combinations of \(f_+, f_-\), and the coefficients shown in (A.9) can be deduced with some effort from the condition (A.2).) Ladder operators can be applied to these basic formulas, as needed.

There remains (iv[b]): the second subcase of the tetrahedral case, when \((\nu, \mu) \in \mathbb{Z}^2 + (\pm 1/6, \pm 1/3)\). This subcase is related to the first tetrahedral one by a quadratic hypergeometric transformation, but the resulting formulas are complicated and are not given here.

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