Consistency analysis of Kaluza-Klein geometric sigma models

M. Vasilić*
Institute of Physics, P.O.Box 57, 11001 Beograd, Yugoslavia
(October 26, 2018)

Abstract

Geometric σ-models are purely geometric theories of scalar fields coupled to gravity. Geometrically, these scalars represent the very coordinates of space-time, and, as such, can be gauged away. A particular theory is built over a given metric field configuration which becomes the vacuum of the theory. Kaluza-Klein theories of the kind have been shown to be free of the classical cosmological constant problem, and to give massless gauge fields after dimensional reduction. In this paper, the consistency of dimensional reduction, as well as the stability of the internal excitations, are analyzed. Choosing the internal space in the form of a group manifold, one meets no inconsistencies in the dimensional reduction procedure. As an example, the $SO(n)$ groups are analyzed, with the result that the mass matrix of the internal excitations necessarily possesses negative modes. In the case of coset spaces, the consistency of dimensional reduction rules out all but the stable mode, although the full vacuum stability remains an open problem.

*E-mail address: mvasilic@phy.bg.ac.yu
I. INTRODUCTION

Geometric $\sigma$-models have originally been proposed as an attempt to explain the pure geometric origin of fermionic matter. Indeed, it has been shown in [1] that scalar matter can be coupled to gravity in such a way that two goals are achieved. First, the theory possesses a kink solution with topologically nontrivial scalar sector which allows for the fermionic type of quantization. Second, using diffeomorphism invariance, all the scalar fields can be gauged away giving the theory a purely geometric form. The possibility of geometrizing fermionic matter has a solid basis in the early work of Finkelstein and Rubinstein [2] who realized the role of multiply-connected configuration spaces for the existence of fermions. An example of the kind is the configuration space of four-dimensional gravitational kinks of Finkelstein and Misner [3]. Its double-connectedness enables the existence of double-valued wave functions but, unfortunately, it is in no way related to the spin of the system. The Ref. [1], however, uses as its role model the 't Hooft-Polyakov monopole solution of the $SO(3)$ gauge theory spontaneously broken to $U(1)$ by a Higgs triplet [4]. It has been shown [5] that these monopoles admit both half-integer spin and fermion statistics in the sense of Finkelstein and Rubinstein. The necessary multiple connectedness of the configuration space, however, stems from the Higgs triplet exclusively, and is not directly related to the gauge fields. Using this idea, the same goal has been achieved in [1] by identifying the coordinates of space-time with the components of a set of scalar fields. The resulting theory has a form of a nonlinear $\sigma$-model coupled to gravity, and necessarily possesses a solution very much similar to the 't Hooft-Polyakov monopole. The difference is that the scalar fields of this solution have a pure geometric meaning—they are just the coordinates of our space-time, and can be gauged away.

The actual procedure of constructing geometric $\sigma$-models begins by specifying a fixed metric field configuration $\hat{g}_{\mu \nu}$ which later becomes the vacuum of the theory. The dynamics is chosen from a variety of possibilities. The simplest one is given by the Einstein like equations of the form $R_{\mu \nu} = \hat{R}_{\mu \nu}$, where the fixed function $\hat{R}_{\mu \nu}$ stands for the Ricci tensor of the vacuum metric $\hat{g}_{\mu \nu}$. These are the non-covariant field equations whose non-vanishing right-hand side actually defines matter, and which, by construction, possess the classical solution $g_{\mu \nu} = \hat{g}_{\mu \nu}$. The covariantization of the theory is achieved by employing a new set of coordinates, say $\Omega^i = \Omega^i(x)$, to fix the Ricci tensor on the right-hand side. Then, the equations of motion take the form of a nonlinear $\sigma$-model coupled to gravity, with the scalar sector consisting of as many scalar fields $\Omega^i(x)$ as the number of space-time dimensions. By choosing the gauge $\Omega^i = x^i$, the field equations are brought back to their non-covariant but purely geometric form $R_{\mu \nu} = \hat{R}_{\mu \nu}$. The multiple-connectedness of the configuration space is a consequence of the topologically nontrivial one-to-one mapping $\Omega^i = x^i$.

The idea of geometric $\sigma$-models has further been developed in [6], and applied to Kaluza-Klein theories. Using the fact that geometric $\sigma$-models are built over freely chosen ground states, one can build a Kaluza-Klein theory of the kind by specifying the vacuum geometry in the form of the direct product of the 4-dimensional Minkowski space $M^4$ with the internal $d$-dimensional space $B^d$. The resulting theory will necessarily possess the classical solution $M^4 \times B^d$, and, therefore, be free of the classical cosmological constant problem. An action functional of this kind has already been discussed in literature. The authors of Refs. [7] and [8] have employed scalar fields in the form of a nonlinear $\sigma$-model to trigger the compacti-
fication, but failed to obtain massless gauge fields. In [6], however, this problem has been solved by abandoning the simple dynamics given by $R_{MN} = \hat{R}_{MN}$ in favor of $R^{MN} = \hat{R}^{MN}$, the functions on the right-hand side standing for the Ricci tensor of the Kaluza-Klein vacuum metric $\hat{G}_{MN}$ of the needed form $M^4 \times B^d$. Owing to the non-covariant form of the above equations, the two respective theories are inequivalent. When modified by adding terms proportional to $(G_{MN} - \hat{G}_{MN})$, the equations of motion allow for the construction of a simple Lagrangian. It is this Lagrangian which will be the subject of our analysis in the subsequent sections.

The lay-out of the paper is as follows. In Section II, we shall define our model, and analyze its symmetry properties. The theory turns out to have a gauge symmetry bigger than pure diffeomorphism invariance. Owing to this, we shall be able to demonstrate how the gauge fixing of the complete scalar sector still leaves us with the standard 4-dimensional gauge invariance. The basic results of Ref. [6] are then recollected with the emphasis on the unsolved stability problem. In Section III, we shall study some consistency aspects of the dimensional reduction procedure. In particular, the Lagrangian constraints of the theory are recognized in the gauge fixed matter field equations, and their response to the dimensional reduction ansatz is analyzed. In the case of $B^d = S^2$, we shall find that all five unstable modes [6] are ruled out by the consistency requirements. On the other hand, choosing $B^d$ in the form of a group manifold is shown to be consistent with the complete set of equations of motion. In Section IV, we shall calculate the mass matrix of the internal excitations of the $SO(n)$ group manifolds, with the result that all of them contain negative modes.

Section V is devoted to concluding remarks.

II. LAGRANGIAN AND SYMMETRIES

The model we are going to explore consists of Einstein gravity in 4+$d$ dimensions coupled to 4+$d$ scalar fields $\Omega^A(X)$, as given by the following action functional:

$$I = -\kappa^2 \int d^{4+d}X \sqrt{-G} \left[ R + F^{AB}(\Omega) \frac{\partial X^M}{\partial \Omega^A} \frac{\partial X^N}{\partial \Omega^B} G_{MN} - V(\Omega) \right].$$

(2.1)

The target metric $F^{AB}(\Omega)$ and the potential $V(\Omega)$ are defined through

$$F^{AB}(\Omega) \equiv \hat{R}^{AB}(\Omega), \quad V(\Omega) \equiv 2\hat{R}(\Omega),$$

where $\hat{R}^{MN}(X)$ and $\hat{R}(X)$ stand for the vacuum values of the Ricci tensor and scalar curvature, respectively. As explained in the introduction, the vacuum metric $\hat{G}_{MN}$ is fixed in advance, and we choose it to be the direct product of the 4-dimensional Minkowski space $M^4$ and the internal $d$-dimensional compact space $B^d$:

$$\hat{G}_{MN} \equiv \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \phi_{mn}(y) \end{pmatrix},$$

(2.2)

Here, $\phi_{mn}(y)$ stands for the metric of $B^d$, and the coordinates $X^M \equiv (x^\mu, y^m)$ are decomposed into 4-dimensional $x^\mu$, and internal $y^m$. 

3
The theory given by (2.1) differs from the conventional $\sigma$-models by employing the inverse of $\Omega^A_M$ rather than $\Omega^A_M$ itself. We shall see, however, that this non-polynomial dependence on the scalar field derivatives is a pure gauge, and can easily be removed. This is why we postpone the inspection of the full set of field equations until the symmetry analysis is done. Let us only notice that

$$G_{MN} = \hat{G}_{MN}, \quad \Omega^A = X^A$$

is easily checked to be a solution of the equations of motion. It illustrates the geometric origin of the scalar fields $\Omega^A(X)$, and is a consequence of the construction procedure described in [6].

The covariant form of the action functional (2.1) tells us that our theory is invariant under general coordinate transformations $X^M \to X^M + \xi^M(X)$. The scalar fields $\Omega^A$ and the metric $G_{MN}$ transform in the usual way:

$$\Omega^M = \Omega^M - \xi^M \Omega^A_M, \quad G_{MN}' = G_{MN} - \xi^L_M G_{LN} - \xi^L_N G_{LM} - \xi^L G_{MN,L},$$

where $\xi^M(X)$ are arbitrary functions of all $4+d$ coordinates. The full symmetry of the action is, however, not exhausted by the general coordinate transformations. Owing to our special choice of the vacuum metric $\hat{G}_{MN}$, the corresponding Ricci tensor $\hat{R}^{MN}$ is block diagonal with $\hat{R}^{mn}(y)$ the only non-zero components. This means that our target metric $F^{AB}(\Omega)$ and the potential $V(\Omega)$ are independent of $\Omega^\mu$, and that only $F^{mn}$ components survive in (2.1). If, in addition, we choose our internal space $B^d$ to be symmetric, with $m$ Killing vectors $K^I_a(y), a = 1, \ldots, m$, the action functional (2.1) will have an extra internal symmetry of the form

$$\Omega^I = \Omega^I + \epsilon^I(\Omega^\nu), \quad \Omega^m = \Omega^m + \epsilon^a(\Omega^\nu) K^m_a(\Omega^n)$$

where $\epsilon^I$ and $\epsilon^a$ are arbitrary functions of $\Omega^\mu$. When applied to small excitations of the vacuum, let us say

$$\Omega^A(X) \equiv X^A + \omega^A(X),$$

the scalar part of the transformation laws (2.4) and (2.3) takes the form

$$\omega^I = \omega^I + \epsilon^I(x), \quad \omega^m = \omega^m + \epsilon^a(x) K^m_a(y) - \xi^m(x,y).$$

We see that it is possible to fix the gauge $\omega^A = 0$, or equivalently

$$\Omega^A(X) = X^A,$$

thereby reducing the action (2.1) to a non-covariant but pure metric form

$$I = -\kappa^2 \int d^{4+d}X \sqrt{-G} \left[ R - \hat{R} + \hat{R}^{MN} \left( G_{MN} - \hat{G}_{MN} \right) \right].$$

The gauge condition (2.7) constrains our gauge parameters to satisfy

$$\xi^I(x,y) = \epsilon^I(x), \quad \xi^m(x,y) = \epsilon^a(x) K^m_a(y).$$
Therefore, the gauge fixed theory \((2.8)\) is still invariant under the restricted general coordinate transformations, as given by \((2.9)\). Notice that this is exactly the form of symmetry obtained in the standard Kaluza-Klein treatments. Owing to this, the effective 4-dimensional theory will have the well known structure of a non-Abelian Yang-Mills theory coupled to gravity.

Let us now recollect the basic results of Ref. [6]. The equations of motion obtained by varying the gauge fixed action \((2.8)\) have the form

\[
R^{MN} = \tilde{R}^{MN} - \frac{2}{2 + d} G^{MN} \tilde{R}^{LR} \left( G_{LR} - \tilde{G}_{LR} \right),
\]

and coincide with the gauge fixed equations of motion of the covariant action \((2.1)\). Indeed, the matter field equations of \((2.1)\) boil down to

\[
\left( \tilde{R}^{LM} G_{LA} \right)_{,M} + \Gamma^{N}_{NM} \tilde{R}^{LM} G_{LA} + \frac{1}{2} \tilde{R}^{LM} G_{LM} - \tilde{R}_{,A} = 0
\]

when the gauge condition \((2.7)\) is imposed, and are easily shown not to be independent equations of motion. Instead, they follow from the Bianchi identities applied to \((2.10)\), and represent the standard constraint equations of generally covariant theories. Now, we see that the vacuum metric \(\tilde{G}_{MN}\) is an obvious solution to the equations of motion. When chosen in the form \((2.2)\), it gives the Kaluza-Klein theory free of the classical cosmological constant problem.

The dimensional reduction procedure begins with the standard \(4 + d\) decomposition [9]

\[
G_{MN} \equiv \left( g_{\mu\nu} + B^{k}_{\mu} B^{l}_{\nu} B_{kli} B_{\mu kn} \right),
\]

where \(g_{\mu\nu}\) and \(u_{mn}\) are further decomposed as

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad u_{mn} = \phi_{mn} + \varphi_{mn}.
\]

The internal manifold \(B^{d}\) is supposed to be a homogeneous space with \(m\) Killing vectors \(K^{l}_{a}(y)\) which form a (generally overcomplete) basis in \(B^{d}\). By projecting the metric components on this basis, let us say

\[
B^{m}_{\nu} = K^{m}_{a} A^{a}_{\nu}, \quad \varphi_{mn} = K_{am} K_{bn} \varphi^{ab},
\]

we obtain the set of field variables suitable for dimensional reduction. The Latin indices \(m, n, \ldots\) are raised and lowered by the internal vacuum metric \(\phi_{mn}\). The dimensional reduction ansatz is defined through the constraints

\[
g_{\mu\nu} = g_{\mu\nu}(x), \quad A^{a}_{\mu} = A^{a}_{\mu}(x), \quad \varphi^{ab} = \varphi^{ab}(x).
\]

This ansatz is applied to the linearized field equations \((2.10)\) which are then averaged over the internal coordinates. To simplify the analysis, we choose our internal space to be an Einstein manifold

\[
\tilde{R}_{mn} = \lambda \phi_{mn}
\]
with $\lambda < 0$ in accordance with the adopted conventions ($R^M_{\text{NLR}} = \Gamma^M_{\text{NLR}} - \cdots$, $\text{diag}(G_{MN}) = (-, +, \ldots, +)$). Then, the effective 4-dimensional equations become

$$R_{\mu\nu} + \frac{1}{2} \varphi_{,\mu\nu} + \frac{2\lambda}{d+2} \eta_{\mu\nu} \varphi = O(2),$$

(2.16a)

$$\gamma_{ab} \partial_\mu F^{b\mu\nu} = O(2),$$

(2.16b)

$$\sigma_{abcd} \Box \varphi^{cd} + \mu_{abcd} \varphi^{cd} = O(2).$$

(2.16c)

Here, $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$, $\Box$ is the corresponding d’Alembertian, and $F_{\mu\nu}^a \equiv A_{\mu,\nu}^a - A_{\nu,\mu}^a + O(2)$ is the gauge field strength for the gauge fields $A_{\mu}^a$. The coefficients in (2.16) are expectation values of products of the Killing vectors and their covariant derivatives, as explicitly shown in [6]. In particular,

$$\gamma_{ab} \equiv \langle K^m_{a} K^{bm} \rangle$$

is used to raise and lower the group indices. The scalar field $\varphi \equiv \varphi_a^a$ is shown to satisfy

$$\left( \Box - \frac{8\lambda}{d+2} \right) \varphi = O(2).$$

(2.17)

independently of the choice of the Einstein manifold $B^d$. We see that the conventional choice $\lambda < 0$ ensures the correct sign for its mass term. As for the first of the equations (2.16), it is easily brought to the standard Einstein form by the rescaling

$$g_{\mu\nu} \rightarrow \left( 1 + \frac{\varphi}{2} \right) g_{\mu\nu} + O(2).$$

The classical linear stability of our effective theory rests upon the signature of the mass matrix $\mu_{abcd}$. Unlike their trace mode, the traceless components of the scalar excitations $\varphi^{ab}$ have masses which depend on the particular choice of $B^d$. In the case of $B^d = S^2$, for example, it has been shown in [3] that all five traceless modes have the same negative mass square equaling $\frac{4\lambda}{7}$. In the subsequent sections, we shall try to clarify some consistency aspects of the search for a stable internal manifold $B^d$.

### III. DIMENSIONAL REDUCTION

The spectral analysis of Kaluza-Klein theories, especially in the internal sector, crucially depends on the consistency of the dimensional reduction ansatz. The constraints (2.15), as all the other constraints in our theory, should be preserved in time when governed by the field equations (2.10). In addition, the new constraints (2.13) should be compatible with the ones already present in the theory, such as (2.11). This means that no further reduction of the number of degrees of freedom is expected. To see how this works, we shall first rewrite the constraint equations (2.11) for our special case of the vacuum metric $M^4 \times B^d$, with $B^d$ an Einstein symmetric space. Thus, we find
\((\phi^{ln} G_{IM})_{,n} + (\ln \sqrt{-G})_{,n} \phi^{ln} G_{IM} + \frac{1}{2} \phi^{ln}_{,M} G_{ln} = 0\),

with \(G_{lm} = u_{lm}, G_{l\mu} = B_k^\mu u_{kl}\) and \(G = gu\), as follows from the decomposition (2.12). It is not difficult to check the effect of dimensional reduction on the constraints (3.1). When the ansatz (2.15) is used, these become

\[
\left(\ln \sqrt{\frac{u}{\phi}}\right)_{,n} (\delta^n_m + \phi^n_m) - \frac{1}{2} \phi^n_{n,m} = 0,
\]

(3.2)

\[
\left[\left(\ln \sqrt{\frac{u}{\phi}}\right)_{,n} (\delta^n_m + \phi^n_m) - \frac{1}{2} \phi^n_{n,m}\right] B^m_{\mu} = 0.
\]

(3.3)

As we can see, the first of the above equations implies the second, and, consequently, it is only the constraint (3.2) we shall be occupied with in what follows. Remember that the variables \(\varphi_{mn}\) and \(B^n_{\mu}\) have the dimensionally reduced form (2.14), with \(K^m_a(y)\) the Killing vectors of the symmetric space \(B^d\), and \(y\)-independent related coefficients. To make use of this fact, we shall continue our analysis by perturbative methods. Expanding the logarithm, \(\ln \sqrt{\frac{u}{\phi}} = \frac{1}{2} \varphi^m_m - \frac{1}{4} \varphi^n_m \varphi^m_n + \cdots\),

one immediately finds the constraint (3.2) to lack the linear part. Explicitly,

\[
[(K^m_a K^m_e) (K^m_b K^m_g) f_{de}^g + a \leftrightarrow e] \varphi^{ab} \varphi^{cd} = O(3),
\]

(3.4)

where \(f^{ab}_{\ c}\) are the structure constants of the isometry group of \(B^d\), as defined by

\[
K^m_a K^l_{b,m} - K^m_b K^l_{a,m} = \alpha f^{c}_{\ ab} K^l_{c}.
\]

(3.5)

The parameter \(\alpha\) has the dimension of the inverse length, and is introduced to make the structure constants \(f^{ab}_{\ c}\) dimensionless. The expression in square brackets has a generic nontrivial \(y\)-dependence. This means that, depending on the particular \(B^d\), the number of \(y\)-independent fields \(\varphi^{ab}(x)\) may strongly be reduced. For example, in the case of a two-sphere, one finds that (3.4) forces all but the trace component of \(\varphi^{ab}\) to vanish. This is why we have to be careful with the interpretation of the results involving the non-vanishing traceless components of \(\varphi^{ab}\). An example of the kind is the result of Ref. [6] which states that \(B^d = S^2\) effective 4-dimensional theory has unstable traceless modes in the scalar sector. Now, we see that the reliability of this result is ruled out by the inconsistency of the dimensional reduction used in its derivation. The trace mode alone, however, is consistent with (3.4), and has positive mass square. In fact, the scalar excitations of the form

\[
\varphi_{mn} \sim \varphi(x) \phi_{mn}(y)
\]

satisfy the constraints (3.2) in all orders and for any choice of the internal manifold \(B^d\). This is, however, not enough to ensure the full consistency of the ansatz (2.15) supplemented by (3.6). Apart from the compatibility with the constraints of the theory, one should also check
if the ansatz is preserved in time when governed by the full set of field equations. In linear approximation, the equations of motion (2.10) have the form

\[ \mathcal{R}_{\mu\nu} + \frac{1}{2} \varphi_{,\mu\nu} + \frac{2\lambda}{d+2} \eta_{\mu\nu} \varphi + \frac{1}{2} h_{\mu\nu,l} = O(2), \tag{3.7a} \]

\[ F^{\mu\nu}_{\mu\nu} + (B^\mu_{l:;n} + B^\mu_{n:l})^{;l} - h^{\mu\nu,\nu n} + \frac{3}{2} (h + \varphi)^{,\mu}_n = O(2), \tag{3.7b} \]

\[ \Box \varphi_{mn} - 2\lambda \varphi_{mn} - 2 \tilde{R}_{kmln} \varphi^{kl} + \varphi_{mn;l} + \frac{4\lambda}{d+2} \phi_{mn} \varphi + 2 (\varphi + h);mn - (B^\mu_{m:n} + B^\mu_{n:m})_{,\mu} = O(2), \tag{3.7c} \]

where \( F^{\mu\nu}_{\mu\nu} \equiv B^\mu_{\mu\nu} - B^\nu_{\mu\nu} + O(2), h \equiv h^\mu_{\mu} \) and \( \varphi \equiv \varphi^m_m \). In addition, the linearized constraints (2.11) read

\[ B^n_{\mu;n} = O(2), \quad (\varphi + h);m + 2 \varphi^m_{m;n} = O(2). \tag{3.8} \]

Now, it is easy to see that the ansatz defined by (2.13) and (3.6) brings the field equations (3.7) into a form free of \( y \)-dependent coefficients. Hence, the linearized theory can consistently be reduced to four dimensions, and the resulting effective theory turns out to be stable against small fluctuations of the vacuum. The inclusion of the interaction terms, however, spoils the nice character of this result. Although the linear part of the field equations (2.10) contains no \( y \)-dependent coefficients, the higher order terms do. The restriction (3.6) then produces additional, unphysical constraints in the theory. Therefore, the correct treatment of the generic internal excitations must take care of their full \( y \)-dependence. This is, in particular, true for the internal manifolds which have the form of coset spaces. The full harmonic analysis of higher dimensional geometric \( \sigma \)-models will be done elsewhere. Here, we turn our attention to group manifolds.

When our internal manifold is chosen to have the structure of a semi simple Lee group, the Killing vectors \( K^l_a(y), a = 1, \ldots, d \), form a non-degenerate basis in \( B^d \). It holds then,

\[ \phi^{mn} = K^m_a K^n_b \gamma_{ab}, \quad K^m_a K^n_b \phi_{mn} = \gamma_{ab}, \tag{3.9} \]

where

\[ \gamma_{ab} \equiv -\frac{1}{2} f_{ac}^d f_{bd}^c \tag{3.10} \]

is the Cartan metric of the group, and \( f_{ac}^d \) are the corresponding structure constants. We shall use \( \phi_{mn} \) and \( \gamma_{ab} \) to raise and lower the world and group indices, respectively. If we now apply the ansatz (2.15) to our constraint equations (3.1), and analyze the resulting expression (3.4), we shall find that it is identically satisfied. Therefore, no further reduction of the number of degrees of freedom occurs. We do not need the additional constraint (3.6), and this holds true in all orders, as is seen by the inspection of the non-perturbative expression (3.2). Similarly, one can analyze the very equations of motion. By projecting them on the Killing basis, we shall obtain the equations for our \( y \)-independent variables.
\[ g_{\mu\nu}(x), A^a_\mu(x) \text{ and } \varphi^{ab}(x), \] with the coefficients consisting of \(d\)-scalar combinations of the Killing vectors and their covariant derivatives. Now, we notice that, for group manifolds, it holds

\[ K_{am;\,n} = \frac{\alpha}{2} f_{abc} K^b_m K^c_n. \tag{3.11} \]

Therefore, the Killing vector derivatives are fully expressed in terms of the Killing vectors themselves. Being \(d\)-scalars, the coefficients of our field equations boil down to completely contracted products of the Killing vectors. It follows from (3.9) then that these coefficients contain only constant group tensors \(\gamma_{ab}\) and \(f_{abc}\). The resulting field equations have no \(y\)-dependent coefficients, and we conclude that \textit{dimensional reduction of group manifolds is a consistent procedure.}

We now want to analyze the mass spectrum of the corresponding effective 4-dimensional theory. When \(B^d = \text{a group manifold,}\) the linearized equations of motion (3.7) become

\[ R_{\mu\nu} + \frac{1}{2} \varphi_{,\mu\nu} + \frac{2\lambda}{d + 2} \eta_{\mu\nu} \varphi = O(2), \tag{3.12a} \]

\[ F^{\mu\nu}_a}_{b,\nu} = O(2), \tag{3.12b} \]

\[ \Box \varphi_{ab} + \mu_{abcd} \varphi^{cd} = O(2). \tag{3.12c} \]

with the mass matrix \(\mu_{abcd}\) given by

\[ \mu_{abcd} \equiv 2\lambda \left( \frac{2}{2 + d} \gamma_{ab} \gamma_{cd} - f_{ac}^\ e f_{bd}^\ e \right). \tag{3.13} \]

The parameter \(\lambda\) is related to the coupling constant \(\alpha\) by

\[ \lambda \equiv -\frac{\alpha^2}{2}, \]

as follows from the well known form of the curvature tensor for group manifolds:

\[ R^o_{abcd} = -\frac{\alpha^2}{4} f_{ab}^\ e f_{cde}. \]

The first two equations (3.12) describe the well known Einstein and Yang-Mills sectors of the theory, and will not be examined further. The stability of the Klein-Gordon sector, however, crucially depends on the signature of the mass matrix \(\mu_{abcd}\), and, thus, on the particular group considered. One immediately sees, for example, that 3-dimensional groups necessarily carry negative modes. Indeed, owing to \(f_{abc} \sim \varepsilon_{abc}\) and \(\gamma_{ab} \sim \delta_{ab}\), the traceless modes of \(\varphi^{ab}\) are easily seen to have negative mass terms. In particular, the groups \(SU(2)\) and \(SO(3)\) define unstable theories. Similarly, the group \(SO(4)\), being locally isomorphic to \(SU(2) \times SU(2)\), gives an equally unattractive result. In the next section, we shall analyze the spectrum of the generic \(SO(n)\) groups.
IV. SO(N) GROUP MANIFOLDS

The $SO(n)$ group generators are commonly denoted by $M_{ij} \equiv -M_{ji}$, $i, j = 1, \ldots, n$, and are subject to the commutation relations

$$[M_{ij}, M_{kl}] = \frac{\alpha}{2} f_{ijkl} m_{mn} M_{mn},$$

with

$$f_{ijkl} m_{mn} = 2 \left( \delta_i^m \delta_j^n \delta_k^l - m \leftrightarrow n \right).$$

In this section, the indices $i, j, \ldots$ are taken to run from 1 to $n$, and should not be confused with internal indices of the preceding sections. The group indices $a, b, \ldots$ are seen as anti-symmetric pairs of indices $i, j, \ldots$. In accordance with (3.10), the Cartan metric of the group is found to be

$$\gamma_{ijkl} = \frac{n-2}{2} \delta_{i[k} \delta_{j]}.$$

Our task in this section is to analyze the mass spectrum of the scalar sector of the theory, as given by (3.12c). To this end, we shall decompose the variables $\varphi_{ab} \equiv \varphi_{ijkl}$ ($a \rightarrow ij$, $b \rightarrow kl$) into irreducible components which diagonalize the mass matrix (3.13). Thus, we obtain

$$\varphi_{ijkl} = \tilde{\varphi}_{ijkl} + \frac{4}{n-2} \delta_{i[k} \tilde{\varphi}_{j]} + \frac{2}{n(n-1)} \delta_{i[k} \delta_{j]} \varphi,$$

where $\tilde{\varphi}_{ijkl}$ and $\varphi_{ij} = \tilde{\varphi}_{ji}$ are traceless, and

$$\varphi_{ij} = \varphi_{ij} - \frac{1}{n} \delta_{ij} \varphi, \quad \varphi_{ij} = \delta^{kl} \varphi_{ikjl}, \quad \varphi = \delta^{ij} \varphi_{ij}.$$

The traceless component $\tilde{\varphi}_{ijkl}$ is still reducible, and can further be decomposed into totally antisymmetric part and the rest:

$$\tilde{\varphi}_{ijkl} = A_{ijkl} + \frac{2}{3} (S_{ikjl} - S_{iljk}),$$

where

$$A_{ijkl} \equiv \frac{1}{3} (\tilde{\varphi}_{ijkl} + \tilde{\varphi}_{iklj} + \tilde{\varphi}_{iljk}), \quad S_{ijkl} \equiv \frac{1}{2} (\tilde{\varphi}_{ikjl} + \tilde{\varphi}_{iljk}).$$

The quantity $A_{ijkl}$ is totally antisymmetric, while $S_{ijkl}$, in addition to being traceless and symmetric with respect to $i \leftrightarrow j$, $k \leftrightarrow l$ and $ij \leftrightarrow kl$, satisfies the cyclic identity

$$S_{ijkl} + S_{iklj} + S_{iljk} = 0.$$

The components $\varphi, \varphi_{ij}, A_{ijkl}$ and $S_{ijkl}$ are irreducible components of $\varphi_{ijkl}$. The dimensions of the corresponding irreducible subspaces ($n > 2$) are:
\[ n + 1 \quad \frac{1}{2} - 1, \quad \frac{n}{4} \quad \text{and} \quad \frac{n - 3}{2} \left( \frac{n + 2}{3} \right), \]

respectively. Their sum gives the total number of \( \frac{1}{2} d(d + 1) \) independent components \( \varphi_{ab} \), where \( d = \frac{1}{2} n(n - 1) \) for \( SO(n) \) group manifolds.

Now, we shall apply the decomposition (4.4), (4.6) to the Klein-Gordon sector of the field equations (3.12). We have already seen that the trace component \( \varphi \equiv \varphi_{\alpha}^{\alpha} \) has positive mass square independently of the choice of \( B^d \). Indeed, our \( \tilde{\varphi} \), being proportional to \( \varphi \), is explicitly found to satisfy

\[ \left( \Box - \frac{8\lambda}{d+2} \right) \tilde{\varphi} = O(2), \quad (4.8) \]

with \( \lambda \equiv -\frac{1}{2} \alpha^2 \) ensuring the correct sign of the mass term. A similar result is obtained for the totally antisymmetric irreducible component \( A_{ijkl} \). The cumbersome, but otherwise simple, calculations lead to

\[ \left( \Box - \frac{8\lambda}{n-2} \right) A_{ijkl} = O(2). \quad (4.9) \]

The content of these equations is nontrivial only for \( n > 3 \) because the totally antisymmetric \( A_{ijkl} \) otherwise vanishes. Therefore, the corresponding mass term is positive in all nontrivial cases.

This is not quite so in the case of \( \tilde{\varphi}_{ij} \) components. Evaluating the mass matrix (3.13) for the \( SO(n) \) structure constants (4.2), and diagonalizing it by (4.4), one finds

\[ \left( \Box - 2\lambda \frac{n-4}{n-2} \right) \tilde{\varphi}_{ij} = O(2). \quad (4.10) \]

We see that \( SO(3) \) group manifolds contain 5 unstable modes in the Klein-Gordon sector, which restrains us to consider only \( n \geq 4 \) cases. The stable massive modes, having masses of the order of the Planck mass, do not appear in the effective low energy theory. An exception is the \( SO(4) \) theory which accommodates 9 zero-mass modes. In any case, the irreducible components \( \tilde{\varphi}, \tilde{\varphi}_{ij} \) and \( A_{ijkl} \) are all stable if \( n \geq 4 \).

Finally, the irreducible component \( S_{ijkl} \) is found to satisfy the Klein-Gordon equation of the form

\[ \left( \Box + \frac{4\lambda}{n-2} \right) S_{ijkl} = O(2). \quad (4.11) \]

As we can see, the mass term of (4.11) is negative in all nontrivial cases. The \( SO(3) \) theory, which does not accommodate either \( A_{ijkl} \) or \( S_{ijkl} \) modes, is ruled out by the presence of unstable \( \tilde{\varphi}_{ij} \) components in (4.10). Therefore, neither of \( SO(n) \) group manifolds can be used as a stable internal space of our Kaluza-Klein geometric \( \sigma \)-model.

One could try to improve this situation by imposing additional constraints (such as \( S_{ijkl} = 0 \)) to the theory. However, losing kinetic terms for some field components can lead to the appearance of new, unphysical constraints stemming from the interaction part of field equations. Moreover, even if we find a consistent set of constraints to define our
dimensional reduction, we cannot be sure how infinitesimal perturbations of the ansatz itself affect the whole scheme. In other words, not only the consistency, but also the stability of the dimensional reduction prescription is needed. To be more specific, if the constraints (2.15) are perturbed by adding small $y$-dependent terms, these may evolve to considerably change the initial ansatz. To prevent this, one must take into consideration the full $y$-dependence of the theory. Only the complete $(4 + d)$-dimensional stability of the vacuum can lead to a correctly reduced effective theory. Once the masses of all higher dimensional excitations are proven positive, the effective theory is obtained by discarding heavy mass modes. The influence of higher modes in the harmonic decomposition of fields in Kaluza-Klein $\sigma$-models will be considered elsewhere.

V. CONCLUDING REMARKS

The analysis of the preceding sections has mainly been devoted to the consistency of dimensional reduction of higher dimensional geometric $\sigma$-models. The motivation came from the failure of Ref. [6] to provide an example of a stable $M^4 \times B^d$ vacuum of the theory. In particular, the excitations of the internal manifold $S^2$ were shown to possess unstable modes. In this paper, the search for a stable $B^d$ has been required to respect a consistent scheme of dimensional reduction. With this in mind, the constraints of the theory were analyzed in detail in Section III. When dimensionally reduced, these constraints have been shown to confine the number of independent excitations of a generic internal manifold $B^d$. In particular, the consistently reduced $M^4 \times S^2$ theory turned out not to accommodate the traceless scalar excitations of the vacuum. The trace mode alone, however, has been proven stable independently of the specific $B^d$ used. The restriction of the internal excitations to their trace mode has also been shown to respect the full set of linearized field equations. Unfortunately, the inclusion of higher order terms leads to the appearance of new, unphysical constraints in the theory, which brings us to the conclusion that the correct treatment of generic vacuum excitations must take care of their full $y$-dependence. In the last part of Section III, we have analyzed the role of group manifolds in defining the internal spaces of our Kaluza-Klein geometric $\sigma$-models. It has been demonstrated how the full interactive theory can be consistently reduced to 4 dimensions.

In Section IV, we have examined the example of a generic $SO(n)$ group. Decomposing the scalar fields $\varphi_{ab}$ into their irreducible components, the mass matrix of the Klein-Gordon sector of the theory has successfully been diagonalized. It turned out, however, that no $SO(n)$ group led to a positive definite mass matrix.

The solution of the problem could hardly be found in additional restrictions of the theory (such as rejecting the unstable modes), since losing some of the modes usually leads to the appearance of new, unphysical constraints. It has been argued that it is not enough to have a consistent set of constraints which define dimensional reduction, but also that the effective theory should be stable against small perturbations of the ansatz itself. In this respect, having found a group manifold leading to a stable 4-dimensional theory, one is still left with the task to check the influence of higher harmonics. Only the full higher dimensional stability of the vacuum can lead to a correctly reduced effective theory. The harmonic analysis of internal manifolds of higher dimensional geometric $\sigma$-models, especially $d$-spheres, will be considered elsewhere.
REFERENCES

[1] M. Vasilić, Class. Quantum Grav. 15, 29 (1998).
[2] D. Finkelstein and J. Rubinstein, J. Math. Phys. 9, 1762 (1968).
[3] D. Finkelstein and C. W. Misner, Ann. Phys. NY 6, 230 (1959).
[4] P. Goddard and D. I. Olive, Rep. Prog. Phys. 41, 1357 (1978).
[5] G. A. Ringwood and L. M. Woodward, Phys. Rev. Lett. 47, 625 (1981).
[6] M. Vasilić, Phys. Rev. D 60, 25003 (1999).
[7] C. Omero and R. Percacci, Nucl. Phys. B 165, 351 (1980).
[8] M. Gell-Mann and B. Zwiebach, Phys. Lett. 141B, 333 (1984).
[9] A. Salam and J. Strathdee, Ann. Phys. NY 141, 316 (1982).