Bäcklund loop algebras for compact and non-compact nonlinear spin models in (2 + 1) dimensions

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Abstract

The Bäcklund problem is solved for both the compact and noncompact versions of the Ishimori (2+1)-dimensional nonlinear spin model. In particular, a realization of the arising Bäcklund algebra in the form of an infinite-dimensional loop Lie algebra of the Kač–Moody type is provided.

Key words: integrable systems, nonlinear spin models, prolongation algebras, Bäcklund transformations, Bäcklund-Cartan connections.

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1 Introduction

In the study of nonlinear field equations, the Bäcklund structures method, also known as prolongation structures method (see e.g. \[1\] for a review of the procedure in some relevant applications concerning both the (1+1) and (2+1) dimensional cases) constitutes a systematic analytical procedure which enables one, in principle, to associate a linear problem with the equation under consideration. Within such a method, nonlinear (prolongation) Bäcklund algebras are related to integrable nonlinear field equations which can be expressed by means of closed differential ideals. Such algebras arise via the introduction of an arbitrary number of Bäcklund forms containing new dependent variables (called pseudopotentials), and by requiring the algebraic equivalence between the generators of the prolonged ideal and its exterior differential, i.e. by requiring an integrability condition for the prolonged differential ideal. It was pointed out that integrability properties of nonlinear field equations can be related with the existence of Bäcklund symmetries and admissible Bäcklund transformations (see e.g. \[14\]–\[15\]). In fact, Bäcklund algebras appear in form of incomplete Lie algebra structures (in the sense that not all of the commutators are known) as necessary conditions for integrability of a connection induced by

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a Bäcklund map [11], and a realization of such algebras (particularly in loop-algebras form) provides the solution of the so-called Bäcklund problem.

In more than two independent variables, the extension of the prolongation procedure is generally nontrivial and some aspects remain to be explored (see, for example, [3, 6, 11, 12]). Recently, relying on a suitable geometric description of Bäcklund transformations [14], some developments have been achieved on the theoretical side, which provide some hints for a better understanding of the whole matter [13]. The study of higher dimensional systems is a central theme in the theory of integrable systems; in [12] the general approach of generating new (2+1)-dimensional integrable systems from a given abstract algebraic structure via the extension of a Bäcklund map was tackled by showing how a connection can be induced by a Bäcklund map in the jet bundles framework and a characterization of completely integrable systems in terms of Bäcklund structures has been provided. In [13] we also proved that such a connection is an admissible Cartan connection for a suitable $\bar{K}$-structure, i.e. a subbundle of a (higher order) frame bundle.

In this note, making use of an ansatz [3, 4, 6, 12] which is in fact a suitable version of the structure equations for an admissible Bäcklund-Cartan connection [13], we associate an infinite-dimensional Bäcklund algebra structure with the Ishimori (2 + 1)-dimensional spin model in both the compact [16] and the noncompact versions [17]. Homomorphisms with quotient finite and infinite dimensional Lie algebras are worked out. In particular, in view of possible applications - e.g. to derive a whole family of (2+1)-dimensional nonlinear field equations containing the original Ishimori spin models -, a realization of the arising Bäcklund algebra in the form of an infinite-dimensional loop Lie algebra of the Kač–Moody type is provided.

2 Admissible Bäcklund transformations

In the following we shall shortly recall few basics concepts and set the notation. We shall assume the reader is familiar with the basic notions from the theory of bundles, jet prolongations, principal bundles and connections (for references and details see e.g. [15, 19]).

Let $\pi : U \to X$, $\tau : Z \to X$, be two (vector) bundles with local fibered coordinates $(x^\alpha, u^A)$ and $(x^\alpha, z^i)$, respectively, where $\alpha = 1, \ldots, m = \dim X$, $A = 1, \ldots, n = \dim U - \dim X$, $i = 1, \ldots, N = \dim Z - \dim X$. A system of nonlinear field equations of order $k$ on $U$ is geometrically described as an exterior differential system $\nu$ on $J^k U$. The solutions of the field equations are (local) sections $\sigma$ of $U \to X$ such that $(j^k \sigma)^* \nu = 0$. We shall also denote by $J^\infty \nu$ (resp. $j^\infty \sigma$) the infinite order jet prolongation of $\nu$ (resp. $\sigma$).

Let then $B$ be the infinite-order contact transformations group on $J^\infty U$. The group of (infinitesimal) Bäcklund transformations for the system $\nu$ is the closed subgroup $\bar{K}$ of $B$ which leaves invariant solution submanifolds of $J^\infty \nu$. The group of (infinitesimal) generalized Bäcklund transformations...
for the system $\nu$ is the closed subgroup $K$ of $B$ which leaves invariant $J^\infty \nu$.

Let $\pi : U \to X$, $\tau : Z \to X$, be vector bundles as the above and $\pi^1 : J^1 U \to X$, $\tau^1 : J^1 Z \to X$, the first order jet prolongations bundles, with local fibered coordinates $(x^\alpha, u^A, u^A_\alpha)$, $(x^\alpha, z^i, z^i_\alpha)$, respectively. Furthermore, let $(dx^\beta, du^A, du^A_\beta)$ and $(dx^\beta, dz^i, dz^i_\beta)$ be local bases of 1–forms on $J^1 U$ and $J^1 Z$, respectively.

**Definition 2** We define a first order Bäcklund map to be the fibered morphism over $Z$: $\phi : J^1 U \times_X Z \to J^1 Z : (x^\alpha, u^A, u^A_\alpha; z^i) \mapsto (x^\alpha, z^i, z^i_\alpha)$, with $z^i_\alpha = \phi^i_\alpha(x^\beta, u^A, u^A_\beta; z^j)$.

The fibered morphism $\phi$ is said to be an admissible Bäcklund transformation for the differential system $\nu$ if $\phi^i_\alpha = D^i_\alpha \phi^i$ - where $D^i_\alpha$ is the formal derivative on $J^1 U \times X Z$ - and the integrability conditions coincide with the exterior differential system $\nu$.

**Remark 1** By pull–back of the contact structure on $J^1 Z$, the Bäcklund morphism induces an horizontal distribution, the induced Bäcklund connection, on the bundle $(J^1 U \times_X Z, J^1 U, \pi^1_\alpha^*(\eta))$.

**Definition 3** Let $\bar{K}$ be a normal subgroup $\bar{K} \subset (\tilde{K} \cap K) \subset B$ leaving invariant (the infinite order prolongation of) $\nu$ and its solutions. We call $\bar{K}$ the group of (infinitesimal) generalized admissible Bäcklund transformations for the system $\nu$.

**Theorem 1** The following statements are equivalent.

1. $\phi$ is an admissible Bäcklund transformation for the differential system $\nu$.
2. The induced Bäcklund connection is $\bar{K}$–invariant.

### 3 Bäcklund algebras for Ishimori models

Let us now consider the nonlinear spin model originally introduced by Ishimori to extend in (2+1) dimensions the continuous isotropic Heisenberg spin model in (1+1) dimensions (for the study of the integrability properties of the latter see e.g. [20, 21, 9]). A Lax pair was provided by Ishimori and multivortex solutions were derived by means of the Hirota method thus showing that the vortex dynamics is integrable. In [17] was shown the gauge equivalence between a noncompact version of the Ishimori spin model and the Davey–Stewartson equation.

The classical continuous isotropic Heisenberg spin chain can be generalized to the following set of coupled nonlinear (2+1)-dimensional partial differential equations:

$$
\Sigma S_t = S \times (S_{xx} + \epsilon^2 S_{yy}) + \phi_y S_x + \phi_x S_y, \quad (1)
$$

$$
\phi_{xx} - \epsilon^2 \phi_{yy} = -2\epsilon^2 \Sigma S \cdot (S_x \times S_y), \quad (2)
$$
where $S = S(x, y, t)$ is a classical spin field vector, $\Sigma$ is a diagonal matrix $\text{diag} \ (1, 1, \kappa^2)$, $\kappa^2 = \pm 1$, subscripts denote partial derivatives, and the symbol $\times$ stands for the usual vector product. The spin field components $S_j, j = 1, 2, 3$ are assumed to satisfy the constraint

$$\Sigma S \cdot S = \kappa^2$$

where $\kappa^2 = \pm 1$ refers to the compact and the noncompact case, respectively. In other words, for $\kappa^2 = 1$ the quantities $S_j$ belong to the unitary sphere $SU(2)/U(1)$, while for $\kappa^2 = -1$, the $S_j$'s range over a shield of the two-fold hyperboloid $SU(1,1)/U(1)$.

Let us introduce on a (vector) bundle $J^1U$ with local coordinates $(x, y, t; S, P, Q; \phi, \alpha, \mu)$ the closed differential ideal defined by the set of (vector-valued) 3-forms:

$$\theta_1 = (dS - Pdx) \wedge dy \wedge dt,$$

$$\theta_2 = (dS - Qdy) \wedge dx \wedge dt,$$

$$\theta_3 = d\Sigma S \wedge dy \wedge dx + S \times (dP \wedge dy \wedge dt - \epsilon^2 dQ \wedge dx \wedge dt) + (\mu P + \alpha Q)dx \wedge dy \wedge dt,$$

by the scalar 3-forms:

$$\gamma_1 = (d\phi - \alpha dx) \wedge dy \wedge dt,$$

$$\gamma_2 = (d\phi - \mu dy) \wedge dx \wedge dt,$$

$$\gamma_3 = d\alpha \wedge dy \wedge dt + \epsilon^2 d\mu \wedge dx \wedge dt + 2\epsilon^2 \Sigma S \cdot (P \times Q)dx \wedge dy \wedge dt,$$

and by

$$\beta_1 = (d\Sigma S \cdot P + \Sigma S \cdot dP) \wedge dy \wedge dt,$$

$$\beta_2 = (d\Sigma S \cdot Q + \Sigma S \cdot dQ) \wedge dx \wedge dt,$$

where $\wedge$ stands for the exterior product of forms.

It is easy to verify the following.

**Proposition 1** On every integral submanifold defined by (local) sections $S = S(x, y, t), P = S_x, Q = S_y, \phi = \phi(x, y, t), \alpha = \phi_x, \mu = \phi_y$ - for which $dx \wedge dy \wedge dt \neq 0$, because of transversality of fibers - the ideal (4)–(11) is equivalent to Eqs. (1)–(2) with the constraint (3).

Now let us consider the (vector) bundle $Z$ (fibered over the same basis as the above) with local coordinates $(x, y, t; \xi^m)$ and, defined on the fibered product $J^1U \times_X Z$, the 2-forms:

$$\Omega^k = H^k dx \wedge dy + F^k dy \wedge dt + G^k dx \wedge dt + (A^k_m dx + B^k_m dy + \delta^k_m dt) \wedge d\xi^m,$$

\footnote{For convenience we made the following change of notation: $x^1 = x, x^2 = y, x^3 = t, u^1 = S, u^1_1 = P, u^1_2 = Q, u^2 = \phi, u^2_1 = \alpha, u^2_2 = \mu$; furthermore, in the sequel, $z = \xi$.}
where \( \xi = \{ \xi^m \} \), \( k, m = 1, 2, \ldots, N \) (\( N \) at this stage is arbitrary), and \( H^k, F^k \) and \( G^k \) are, respectively, the pseudopotential and functions on \( J^1U \times X \times Z \) to be determined. Furthermore, the quantities \( A_m^k \) and \( B_m^k \) denote the elements of two \( N \times N \) constant regular matrices, \( \delta_m^k \) is the Kronecker symbol and the summation convention over repeated indices is understood.

**Definition 4** The forms \( \Omega^k \) are called the Bäcklund forms associated with Eqs. \([12] - [13]\). Let \( \mathcal{I} \) be the prolonged ideal generated by the forms \( \theta_i, \gamma_i, \beta_i, i = 1, 2, 3, l = 1, 2 \) and \( \Omega^k, k = 1, \ldots, N \). We say that \( \mathcal{I} \) is closed if \( d\Omega^k \in \mathcal{I} (\theta_i, \gamma_i, \beta_i, \Omega^k) \).

**Remark 2** The reason for we call such prolongation forms ‘Bäcklund’ is that Eq. \([12]\) is in fact a specific form of the generalized ansatz introduced in \([12]\), which is assumed to be induced by an admissible Bäcklund transformation. The induced Bäcklund-Cartan connection on \( (J^1U \times X, J^1U, \pi_0^{1*}(\eta)) \) is here given by

\[
\omega^m = \Gamma^m_1 dx + \Gamma^m_2 dy + \Gamma^m_3 dt + d\xi^m ,
\]

and the closed (matrix-valued) form \( \theta \) of \([12]\) is here given by \( \theta^k_m = A^k_m dx + B^k_m dy + \delta^k_m dt \). With this notation in mind then \( H^k = \Gamma^m_1 B^k_m - \Gamma^m_2 A^k_m, F^k = \Gamma^m_2 \delta^k_m - \Gamma^m_3 B^k_m, G^k = \Gamma^m_3 \delta^k_m - \Gamma^m_1 A^k_m \). The closure condition for the ideal \( \mathcal{I} \) is an equivalent condition to Theorem 1 where \( \tilde{K} \) is here to be determined by solving the so-called Bäcklund problem. The Bäcklund map, and thus the induced Cartan connection, are completely determined by the functions \( H^k, F^k \) and \( G^k \) and the matrices \( \{ A^k_m \} \) and \( \{ B^k_m \} \).

The dependence of the functions \( H^k, F^k, G^k \) on \( (S, S_x, S_y; \phi, \phi_x, \phi_y; \xi^m) \) shall be determined by requiring the Bäcklund transformation defined by Eq. \([13]\) to be admissible for the Ishimori systems, i.e. the closure of the prolonged ideal to be satisfied.

In the following \( [G, H]^k = G^j H^k_{\xi^j} - H^j G^k_{\xi^j} \) (Lie bracket), \( H^k_{\xi^i} = \frac{\partial H^k}{\partial \xi^i} \), \( H_u = \frac{\partial H^k}{\partial \phi^u} \), and so on. Furthermore, we shall omit the indices \( k, m \) for simplicity. The Lie brackets must be understood as commutators then.

**Lemma 1** The closure condition for \( \mathcal{I} \) yields:

\[
H(S; \xi) = X(\xi) \cdot S + Y(\xi),
\]

\[
F = -(X \times S) \cdot S_x + \hat{F}(S; \xi),
\]

\[
G = c^2(X \times S) \cdot S_y + \hat{G}(S; \xi)
\]

\[
\hat{F}_S \cdot S_x - \hat{G}_S \cdot S_y + [X(\xi) \cdot S + Y(\xi), c^2(X \times S) \cdot S_y] + [X(\xi) \cdot S + Y(\xi), \hat{G}(S; \phi, \phi_y; \xi)] = 0.
\]

**Proof.** The closure condition is equivalent to the following constraints:

\[
H_\phi = H_{\phi_x} = H_{\phi_y} = G_{\phi_x} = F_{\phi_y} = 0,
\]

\[
H_{S_x} = G_{S_x} = 0, \quad F_{S_x} = -(\Sigma H S) \times S,
\]

\[
H_{S_y} = F_{S_y} = 0, \quad G_{S_y} = c^2(\Sigma H S) \times S,
\]

\[
G_{\phi_x} - c^2 F_{\phi_x} = 0.
\]
\[ [H, G] + H_S \cdot (\phi_y S_x + \phi_x S_y) + F_S \cdot S_x - G_S \cdot S_y + \]
\[ + F_\phi \cdot \phi_x - G_\phi \cdot \phi_y - 2e^2 F_\phi S \cdot (S_x \times S_y) = 0. \] (22)

From equations (15) we get
\[ H = H(S; \xi), \quad F = F(S, S_x; \phi, \phi_x; \xi), \quad G = G(S, S_y; \phi, \phi_y; \xi) \] (26)
\[ F_{S_x} = -\Sigma H_S \times S, \quad G_{S_y} = e^2 \Sigma H_S \times S, \quad G_{\phi_y} = e^2 F_{\phi_x}. \] (27)

From the last three equations above we further get
\[ F = - (\Sigma H_S \times S) \cdot S_x + \bar{F}(S; \phi, \phi_x; \xi), \] (28)
\[ G = e^2 (\Sigma H_S \times S) \cdot S_y + \bar{G}(S; \phi, \phi_y; \xi); \] (29)
\[ \bar{G}_{\phi_y} = e^2 \bar{F}_{\phi_x}; \] (30)

The first two imply
\[ H_{S, S} = 0, \] (31)
thus
\[ H(S; \xi) = X(\xi) \cdot S + Y(\xi), \] (32)
where \( X = (X_1, X_2, X_3) \).

Furthermore, we have \( \bar{F}_{\phi_x} = 0 \), so that \( \bar{F} = \bar{F}(S; \phi, \xi) \). Thus Eq. (22) become
\[ X \cdot S_x \phi_y + X \cdot S_y \phi_x + \bar{F}_S \cdot S_x - \bar{G}_S \cdot S_y + \bar{F}_\phi \phi_x - \bar{G}_\phi \phi_y + \]
\[ + [X \cdot S + Y, e^2 (X \times S) \cdot S_y + \bar{G}(S; \phi, \phi_y; \xi)] = 0. \] (33)

From the equation above we further infer \( \bar{F}_\phi = 0 \) and from Eqs. (28), \( \bar{G}_{\phi_y} = 0 \), which in turns implies that \( \bar{G}_\phi = 0 \).

So we finally get the assertion.

In the following we provide the Bäcklund algebra \( \bar{\mathfrak{g}} \) associated with the compact and non-compact Ishimori spin models.

**Proposition 2** The following incomplete Lie algebra structure \( \bar{\mathfrak{g}} \) is associated with Eqs. (15–22) and constraint (3):

\[ [Y, X_1] = [Y, X_2] = [Y, X_3] = 0, \] (34)
\[ [X_1, [X_1, X_3]] - [X_2, [X_2, X_3]] = 0, \] (35)
\[ k^2 [X_1, [X_1, X_2]] + [X_4, [X_2, X_3]] = 0, \] (36)
\[ k^2 [X_2, [X_1, X_2]] - [X_3, [X_1, X_3]] = 0, \] (37)
\[ [X_1, [X_2, X_3]] + [X_2, [X_1, X_3]] = 0, \] (38)
\[ [X_2, [X_1, X_3]] + [X_3, [X_1, X_2]] = 0, \] (39)
\[ k^2 [X_2, [X_1, X_3]] + [Z, Y] = 0. \] (40)
Proof. From the above Lemma we get

\[ [H(S; \xi), \hat{G}(S; \xi)] = 0, \]  
\[ \hat{F}_S = 0, \]  
which implies \( \hat{F} = K(\xi) \). Thus we finally have to work out the equation:

\[ \hat{G}_S \cdot S_y = \epsilon^2 [S \cdot S + Y, (X \times S) \cdot S_y]. \]  

(43)

By using the constraint (3) we easily get

\[ [Y, X_1] = [Y, X_2] = [Y, X_3] = 0, \]  
\[ \hat{G} = -S_1[X_2, X_3] + S_2[X_1, X_3] - k^2 S_3[X_1, X_2] + Z(\xi). \]  

(45)

Finally, the Bäcklund algebra \( \bar{k} \) is obtained inserting the last result in Eq. (41) and using Eqs. (28) and (32).

QED

Remark 3 The compatibility relations (23), (24) and (25) have to be satisfied by \( K(\xi) \). It is easy to realize then that commutation relations for \( K(\xi) \) also hold true.

It is possible to provide suitable realizations (i.e. homomorphisms) of the algebra \( \bar{k} \) with finite or infinite dimensional quotient Lie algebras.

Proposition 3 The Bäcklund algebra \( \bar{k} \) is homomorphic to the \( \mathfrak{sl}(2, \mathbb{C}) \) algebra

\[ [X_1, X_2] = 2i \zeta \kappa^2 X_3, \]  
\[ [X_1, X_3] = -2i \zeta X_2, \]  
\[ [X_2, X_3] = 2i \zeta X_1. \]  

(46)

where \( \zeta \) is a free parameter.

Proof. Let \( [X_1, X_2] = X_4, [X_1, X_3] = X_5, [X_2, X_3] = X_6 \) and assume that \( X_1, X_2, \) and \( X_3 \) are independent and \( X_4, X_5, X_6, Y, Z \) and \( K \) are a linear combination of the former operators. In such a way one finds \( X_4 = 2i \zeta \kappa^2 X_3, X_5 = -2i \zeta X_2, X_6 = 2i \zeta X_1, \) and \( Y = Z = K = 0 \).

QED

Proposition 4 Assume \( Y = Z = K = 0 \). Then the Bäcklund algebra \( \bar{k} \) is homomorphic to the infinite dimensional Lie algebra of the Kač-Moody type:

\[ [T^{(m)}_i, T^{(n)}_j] = i \epsilon^k_{ij} T^{(m+n)}_k, \]  

(47)

\( \epsilon^k_{ij} \) being the Ricci tensor.

Proof. In fact, let us suppose that the commutators \( [X_2, X_6], [X_3, X_5], [X_3, X_6], [X_4, X_5], [X_4, X_6] \) and \( [X_5, X_6] \) are unknown. Hence, it is easy to see that a realization of the incomplete Lie algebra \( \bar{k} \) is provided by

\[ X_1 = \kappa T^{(1)}_1, X_2 = \kappa T^{(1)}_2, X_3 = T^{(1)}_3, \]  
\[ X_4 = i \kappa^2 T^{(2)}_3, X_5 = -i \kappa T^{(2)}_2, X_6 = i \kappa T^{(2)}_1. \]  

(49)
where the vector fields $T_i^{(m)} (i = 1, 2, 3; m \in \mathbb{Z})$ satisfy the commutation relations [17].

\[
QED
\]

**Remark 4** It is easy to see that, in accordance with Theorem [11] the connection forms [13] are $\xi$-invariant; in fact we have $\Gamma_i^m = \omega_l^i X_l^m$, with $i = 1, 2, 3$ and $l = 1, \ldots, M$; $M$ equals $\text{dim} \mathfrak{k}$ or the dimension of the quotient Lie algebra taken under consideration and $\omega_l^i$ are admissible Cartan connection forms [13].

### 3.1 Conclusions

We associated an infinite-dimensional Bäcklund loop Lie algebra structure of the Kač–Moody type with the Ishimori $(2 + 1)$-dimensional spin model in both the compact [16] and the noncompact versions [17]. As a matter of principle, it would be interesting to work out homomorphisms of $\mathfrak{k}$ with quotient Lie algebras also in the case when $Y \neq 0, Z \neq 0, K \neq 0$. This subject is currently under investigation.

Notice that, generalizing to the $(2 + 1)$–dimensional case the “inverse” procedure (see e.g., [9, 10, 12]), by suitable nonlinear representations of the quotient Kač–Moody algebra [17] in terms of pseudopotentials, one could obtain a whole family of $(2+1)$-dimensional nonlinear spin field equations containing the original Ishimori spin models (see e.g., [9] for similar results concerning the $(1 + 1)$–dimensional Heisenberg spin model). This turns out to be interesting within the study of ‘hidden’ symmetries of nonlinear $\sigma$-models. Details will be the subject of a separate paper. Here we just stress that such a family should not be confused a priori with the well known hierarchy associated via the Lax pair; in fact, as it is well known, the latter comes from a linear representation of quotient Lie algebras.

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