Discreteness of time in the evolution of the universe

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In this paper, we use a deformed second quantized commutation relation to quantize the Hamiltonian constraint of general relativity. The deformed Wheeler-DeWitt equation thus constructed is solved in the minisuperspace approximation. We demonstrate that in this model, the universe evolves by taking discrete jumps rather than in a continuous manner. Thus, the deformation of the second quantized commutation relation naturally gives rise to time crystals in our universe.

I. INTRODUCTION

There are strong indications from the physics of black holes that any theory of quantum gravity should come naturally equipped with a minimum length scale of the order of Planck scale [1,2]. This is because the energy required to probe a region of spacetime below Planck scale is more than the energy required to create a mini black hole in that region of spacetime. The existence of a minimum length scale (the string scale) also occurs naturally in perturbative string theory, as strings are the smallest probes that can be used for analyzing a region of spacetime [3–7]. It is also the existence of a minimum length scale in loop quantum gravity that turns the big bang into a big bounce [8]. In fact, a minimum length scale is expected to occur in most theories of quantum gravity. However, the usual Heisenberg uncertainty principle is inconsistent with the existence of a minimum length scale, since according to it the position of a particle can be measured with an arbitrary precision as long as the momentum is not. Therefore one is required to extend this to Generalized Uncertainty Principle (GUP), which is consistent with the existence of minimum measurable length [3–7, 9–13]. This also implies a modification of the Heisenberg algebra, and a subsequent modifications of the coordinate representations of the momentum and Hamiltonian operators. The latter gives rise to corrections to all quantum mechanical systems, even low energy ones. A different kind of deformation of the Heisenberg algebra occurs in a theory called the Doubly Special Relativity (DSR) [14–16]. Apart from the velocity of light, the Planck energy is also a universal constant in DSR. It is possible to combine the deformation occurring due to GUP with the deformation occurring due to DSR into a single deformation of the Heisenberg algebra [23, 24]. Apart from the one dimensional case, this deformed Heisenberg algebra gives rise to non-local fractional derivative terms. However, it has been possible to deal with such terms in the framework of harmonic extension of functions [27, 28]. One of the most interesting consequences of this algebra is that it predicts a discrete structure for space [22]. In fact, it has been possible to obtain similar results by using the modified Dirac equation as well [20]. The deformed Poisson bracket consistent with this deformed Heisenberg algebra also turns the big bang into a big bounce [29]. All this work has been done by deforming the first quantized commutation relations. However, it is also possible to modify the second quantized commutation relations. In fact, a deformed Wheeler-DeWitt equation has been constructed by using a deformation of the second quantized commutation relations [30]. In this deformed Wheeler-DeWitt equation the big bang singularity is naturally avoided, as there is a minimum value for the scaling factor of the universe.

II. DEFORMED WHEELER-DEWITT EQUATION

In this section, we will review the deformation of the Wheeler-DeWitt equation. The combined deformation of the Heisenberg algebra incorporating both GUP and DSR is given by

\[ [x^i, p_j] = i \left[ \delta^i_j - \alpha \frac{||p||}{|\delta^i_j|} + \alpha ||p||^{-1} p^i p^j \right. \]
\[ \left. + \alpha^2 p^2 \delta^i_j + 3 \alpha^2 p^i p^j \right], \quad (1) \]

where

\[ ||p|| = \sqrt{p^i p_i}, \quad (2) \]

and \( \alpha = \alpha_0 / M_P c = \alpha_0 \ell_P / \hbar \) is the parameter measuring the strength of this deformation. We chose natural units.
and set $\hbar = c = 1$. In the one dimensional case this corresponds to the uncertainty relation given by $\Delta x \Delta p = [1 - 2\alpha < p > + 4\alpha^2 < p^2 >]$, and which in turn imply the existence of a minimum length $\Delta x \geq \Delta x_{\text{min}} \geq \alpha_0 \hbar p_1$, and a maximum momentum $\Delta p \leq \Delta p_{\text{max}} \leq \alpha_0^{-1} M_0 c$. Now the momentum in the coordinate representation can be written as
\[ p_i = \tilde{p}_i (1 - \alpha ||\tilde{p}|| + 2\alpha^2 ||\tilde{p}||^2), \]
where $\tilde{p}_i = -\partial_i$ is the coordinate representation of the original momentum.

This algebra holds true for any number of degrees of freedom $N$, and so, we can formally take the continuum limit $N \to \infty$, and write
\[ [\phi(x), \pi(y)] = i\delta(x - y) + i\alpha B(x, y) + i\alpha^2 B(x, y), \]
where
\[ A(x, y) = ||\pi||\delta(x - y) + ||\pi||^2 \pi(x) \pi(y), \]
\[ B(x, y) = ||\pi||^2 \delta(x - y) + 3\pi(x) \pi(y). \]

Here we have defined the norm of $||\pi||$ as follows,
\[ ||\pi|| = \sqrt{\int |dx\delta(x - y)\pi(x)\pi(y)|}. \]

It may be noted that this is a formal definition and has to be suitably normalized. It might be possible to define this norm in a better way by smearing it with a distribution on a compact support. This corresponds to taking the deformation for $\pi(x)$,
\[ \pi(x) = (1 - \alpha ||\tilde{\pi}|| + 2\alpha^2 ||\tilde{\pi}||^2) \tilde{\pi}(x), \]

where
\[ [\phi(x), \tilde{\pi}(y)] = i\delta(x - y). \]

Thus, $\pi(x)$ is the usual momentum density conjugate to $\phi(x)$, and it can be written in the Wheeler-DeWitt approach as,
\[ \tilde{\pi}(x) = -i\frac{\delta}{\delta\phi(x)}. \]

We can perform this deformation for every field, and so, we will now review this deformation for the gravitational field. The Hamiltonian constraint of general relativity can be written as
\[ H = \int dx \left[NH + N_i H^i\right], \]
where
\[ H = \frac{(2\kappa) G_{ijkl} \pi^i \pi^k}{2\kappa}(3 R - 2\Lambda), \]
\[ H^i = -2\nabla_j \pi^j, \]
where $\pi^{ij}$ is the momentum conjugate to $h_{ij}$, and $G_{ijkl}$ is defined by
\[ G_{ijkl} = \frac{1}{2\sqrt{\hbar}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}). \]

It may be noted that at the quantum level the classical constraints $H = 0$, becomes the Wheeler-DeWitt equation,
\[ \mathcal{H}\psi[h] = 0. \]

This is usually done by imposing standard commutation relations. However, a deformed version of the Wheeler-DeWitt equation can be constructed by using a deformed version of the commutation relations
\[ [h_{ij}(x), \pi^{kl}(y)] = (\delta^k_i \delta^l_j + \delta^l_i \delta^k_j)[i\delta(x - y) + i\alpha A(x, y) + i\alpha^2 B(x, y)], \]
where
\[ A(x, y) = ||\pi||^2 \delta(x - y) + \pi(x)\pi(y), \]
\[ B(x, y) = ||\pi||^2 \delta(x - y) + 3\pi(x) \pi(y). \]

This corresponds to the following deformation of $\pi(x)$
\[ \pi_{ij}(x) = (1 - \alpha ||\tilde{\pi}|| + 2\alpha^2 ||\tilde{\pi}||^2) \tilde{\pi}_{ij}(x). \]

Thus, $\pi_{ij}(x)$ is the usual momentum density conjugate to $h_{ij}$, and it can be written in the Wheeler-DeWitt approach as,
\[ \tilde{\pi}_{ij}(x) = -i\frac{\delta}{\delta h_{ij}(x)}. \]

\section{Minisuperspace Approximation}

In this section, we will analyze a minisuperspace approximation to the deformed Wheeler-DeWitt equation. So, now consider a closed universe filled with a vacuum of constant energy density and the radiation, $\rho(a) = \rho_v + \epsilon/a^4$, where $\rho_v$ is the vacuum energy density, $\epsilon$ is a constant characterizing the amount of radiation, and $a$ is the scale factor. We can write the Friedmann-Robertson-Walker metric for $K = 1$ as follows,
\[ ds^2 = -N^2 dt^2 + a^2(t) d\Omega^2, \]
where $d\Omega^2$ is the line element on the three sphere. Using this metric, the Lagrangian for this closed universe, in the minisuperspace approximation, can be written as
\[ \mathcal{L} = -\frac{3\pi a_0 a^2}{4G} + 3\pi a^2 - 2\pi^2 a^3 \rho(a), \]
The equation of motion obtained from this Lagrangian, can be written as
\[ -\frac{3}{4G}\dot{a}\dot{a}^2 - \frac{3}{4G}a^3 + 2\pi^2 a^3 \rho(a) = 0. \] (21)

We can also calculate the Hamiltonian from this Lagrangian,
\[ H = -\frac{G}{3\pi} p^2 - \frac{3}{4G}a^3 + 2\pi^2 a^3 \rho(a). \] (22)

Now we impose a deformed commutation relation for the momentum operator, and hence, write the deformed momentum operator for this minisuperspace model as
\[ p = \hat{p}(1 - \alpha||\hat{p}|| + 2\alpha^2||\hat{p}||^2), \] (23)
where \( \hat{p} = -i\alpha/d\alpha. \) Therefore we get
\[ p = -i\left(1 + i\alpha \frac{d}{da} - 2\alpha^2 \frac{d^2}{da^2}\right) \frac{d}{da} \] (24)

Now we can write the deformed Wheeler-DeWitt equation as follows,
\[ \frac{G}{3\pi} \frac{d^2\psi}{da^2} - 2\alpha i \frac{G}{3\pi} \frac{d^3\psi}{da^3} + M^2(a)\psi = 0, \] (25)
where \( M^2(a) = 3\pi a^2/4G + 2\pi^2 a^4 \rho(a) \) and \( k^2(a) = 3\pi M^2(a)/G. \) It may be noted that this modified quantization is consistent with the following uncertainty relation,
\[ \Delta a \Delta p = 1 - 2\alpha < p > + 4\alpha^2 < p^2 >. \] (26)

This imply the existence of a minimum scale factor for the universe \( \Delta a \geq \Delta a_{\text{min}}, \) and so, the big bang singularity is naturally avoided in this model \([30]\). Now assuming \( \psi = \exp(ma) \) we get
\[ m^2 - 2i\alpha m^3 + k^2(a) = 0. \] (27)

The solution to the leading order in \( \alpha, \) can be written as \( m = \{ik', -ik'', i/2\alpha\}, \) where \( k' = k(1 - k\alpha), \) and \( k'' = k(1 + k\alpha). \) So, we can write the solution for this equation as
\[ \psi = Ae^{ik'a} + Be^{-ik''a} + Ce^{-a/2\alpha}, \] (28)

Even though the date from type I supernova indicates that our universe is an accelerating in its expansion \([32-37]\), there are predictions from string theory that the era of rapid expansion might only be the first era in the evolution of our universe \([38-41]\). This is also consistent with predictions from loop quantum gravity \([42-43]\), where it is shown that the era of accelerated expansion will be followed by an era of contraction. It may be noted that unstable geometries can mathematically decay into bubble of nothing, which contain neither matter nor spacetime \([44, 45]\). This mathematical structure where neither matter nor spacetime is present can be viewed as a third quantized vacuum state \([46, 49]\). We can also add non-linear terms to the Wheeler-DeWitt equation, and in that case the universes can get created and annihilated in the third quantized formalism \([50, 51]\). For such models, we can impose the following boundary conditions on the wave function of the universe, \( \psi(a_0) = \psi(a_0 + \delta a) = 0. \) This corresponded to the creation of the wave function of the universe from nothing and its subsequent annihilation into nothing. Another possibility for this boundary condition is the formation the universe because of a tunneling process \([55, 56]\). Furthermore, the universe could be formed around a metastable vacuum state, and hence can tunnel to a true vacuum state \([57, 58]\). This will correspond to a spontaneous annihilation of all structure in the universe, and hence justify our boundary conditions. So, implementing the boundary conditions for the initial state of the universe, we obtain,
\[ Ae^{ik'a} + Be^{-ik''a} + Ce^{-a/2\alpha} = 0, \]
\[ Ae^{ik'(a_0 + \delta a)} + Be^{-ik''(a_0 + \delta a)} + Ce^{-a/2(\alpha + \delta a)} = 0. \] (29)

Now we let \( C = 0, \) so, we can write
\[ A + B + C = 0, \]
\[ Ae^{k'da} + Be^{-k'da} + Ce^{-\delta a/2\alpha} = 0. \] (30)

Thus, we obtain the following result,
\[ \psi = 2iA \sin(ka) + C[e^{-ka} + e^{-k/2\alpha}] + \alpha k^2 a[iCe^{ika} + 2A \sin(ka)]. \] (31)

Next implementing the boundary conditions for the final state of the universe, we obtain,
\[ 2iA \sin(\delta a) = |C|e^{-k\delta a + \theta_c} - e^{i(\delta a/2\alpha - \theta_c)} - \alpha k^2 \delta a e^{-i(\delta a/2\alpha + \theta_c)} + 2A \sin(\delta a), \] (32)
\[ \cos(\delta a/2\alpha + \theta_c) = \cos(k\delta a + \theta_c) = \cos(n\pi + \theta_c + \epsilon), \] (33)
from which it is easy to show that
\[ \delta a = 2n\pi\alpha. \] (34)

Thus, the universe evolves by taking discrete jumps rather than in an continuous manner. The above may also be an important step in the resolution of the problem of time in quantum gravity, which by virtue of the re-parametrization invariance of general relativity argues that time translation is similar to gauge transformation, and one cannot distinguish past and future wave functions of the universe \([52, 53]\). However we show here that our universe evolves by taking discrete jumps, and hence
the time steps are all distinct. Alternatively, we can still perform time re-parametrization between two points separated by distance $2\pi\alpha$, but not for points in between. Thus effectively, the universe acts like a crystal in time. It may be noted that time crystals have been studied recently for systems in which time re-parametrization is broken, just as spatial translation is broken in regular crystals [61–65]. Here we have shown that the deformation of the second quantized commutation relations naturally lead to the formation of such time crystals in the Wheeler-DeWitt equation.

IV. CONCLUSION

In this paper, we have analyzed a deformation of the second quantized commutation relations, and obtained a deformed version of the Wheeler-DeWitt equation. We analysed the solutions for this deformed Wheeler-DeWitt equation in minisuperspace approximation for a closed universe filled with a vacuum of constant energy density and the radiation. The deformation of the Wheeler-DeWitt equation caused this universe to evolve by taking discrete jumps rather than as a continuum. Thus, this deformation naturally gave rise to time crystals, which broke the time re-parametrization of the original theory. This in turn is relevant for resolution of the problem of time in quantum gravity.

It would be interesting to analyze the implication of this breaking of time re-parametrization for the full superspace Wheeler-DeWitt equation. In fact, the results of this paper can be used to motivate a GUP like modification for time. This will in turn deform the Hamiltonian for all quantum mechanical systems. In order to do that, we will need to take time as an observable, e.g. with reference to the evolution of some non-stationary quantity [66–70]. It may be noted that such a deformation for the classical field equation for quantum field theory have been analyzed [27, 28]. It would be interesting to analyze such systems, and calculate the effect on various physical processes.

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