Perron-Frobenius theorem for nonnegative multilinear forms and extensions

S. Friedland, S. Gaubert, and L. Han

July 3, 2010

Abstract

We prove an analog of Perron-Frobenius theorem for multilinear forms with nonnegative coefficients, and more generally, for polynomial maps with nonnegative coefficients. We discuss the geometric rate of convergence of the power algorithm to the unique normalized eigenvector.

2000 Mathematics Subject Classification. 15A48, 47H07, 47H09, 47H10.

Key words. Perron-Frobenius theorem for nonnegative tensors, convergence of the power algorithm.

1 Introduction

Let $f : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_d} \to \mathbb{R}$ be a multilinear form:

$$f(x_1, \ldots, x_d) := \sum_{i_1 \in [m_1], \ldots, i_d \in [m_d]} f_{i_1, \ldots, i_d} x_{i_1,1} \cdots x_{i_d,d}, \quad (1.1)$$

Here, and in the sequel, we write $[d]$ for the set $\{1, \ldots, d\}$. We assume the nontrivial case $d \geq 2, m_j \geq 2, j \in [d]$. The above form induces the tensor $F := [f_{i_1, \ldots, i_d}] \in \mathbb{R}^{m_1 \times \cdots \times m_d}$. We call $f$ nonnegative if the corresponding tensor $F$ is nonnegative, denoted by $F \geq 0$, meaning that all the entries of $F$ are nonnegative. For $u \in \mathbb{R}^m$ and $p \in (0, \infty]$ denote by $\|u\|_p := (\sum_{i=1}^m |u_i|^p)^{\frac{1}{p}}$ the $p$-norm of $u$. Let $S_{p,+}^{m-1} := \{0 \leq u \in \mathbb{R}^m, \|u\|_p = 1\}$ be the $m-1$ dimensional unit sphere in the $\ell_p$ norm restricted to $\mathbb{R}^m_+$. Let $p_1, \ldots, p_d \in (1, \infty)$ and consider a critical point $(\xi_1, \ldots, \xi_d) \in S_{p_1,+}^{m_1-1} \times \cdots \times S_{p_d,+}^{m_d-1}$ of $f|_{S_{p_1,+}^{m_1-1} \times \cdots \times S_{p_d,+}^{m_d-1}}$. It is straightforward to show that each critical point satisfies the following equality [5]:

$$\sum_{i_1 \in [m_1], \ldots, i_{j-1} \in [m_{j-1}], i_{j+1} \in [m_{j+1}], \ldots, i_d \in [m_d]} f_{i_1, \ldots, i_d} x_{i_1,1} \cdots x_{i_{j-1},j-1} x_{i_{j+1},j+1} \cdots x_{i_d,d} = \lambda x_{i_{j,j}}^{p_{j,j}-1}, \quad (1.2)$$

where $i_j \in [m_j], \quad x_j \in S_{p_{j,j},+}^{p_{j,j}-1}, \quad j \in [d].$
Note that for $d = 2$ and $p_1 = p_2 = 2$, $\xi_1, \xi_2$ are the left and right singular vectors of the nonnegative matrix $F \in \mathbb{R}^{m_1 \times m_2}$.

In the case of nonnegative matrices, irreducibility can be defined in two equivalent ways, either by requiring the directed graph associated with the matrix to be strongly connected, or by requiring that there is no non-trivial part (relative interior of a face) of the standard positive cone that is invariant by the action of the matrix, i.e., that the matrix cannot be put in upper block triangular form by applying the same permutation to its rows and columns. In the case of tensors, and more generally, of polynomial maps, both approaches can be extended, leading to distinct notions.

In the present setting, the tensor $F$ is associated with an undirected $d$-partite graph $G(F) = (V, E(F))$, the vertex set of which is the disjoint union $V = \bigcup_{j=1}^d V_j$, with $V_j = [m_j], j \in [d]$. The edge $(i_k, i_l) \in V_k \times V_l, k \neq l$ belongs to $E(F)$ if and only if $f_{i_1, i_2, \ldots, i_d} > 0$ for some $d - 2$ indices $\{i_1, \ldots, i_d\} \setminus \{i_k, i_l\}$. The tensor $F$ is called irreducible if the graph $G(F)$ is connected. We call $F$ indecomposable if for each proper nonempty subset $\emptyset \neq I \subsetneq V$, the following condition holds: Let $J := V \setminus I$. Then there exists $k \in [d], i_k \in I \cap V_k$ and $i_j \in J \cap V_j$ for each $j \in [d] \setminus \{k\}$ such that $f_{i_1, \ldots, i_d} > 0$. We will show that if $F$ is indecomposable then $F$ is irreducible. We warn the reader that in [1], the term irreducible is used in a different sense (corresponding to indecomposability).

The main result of this paper is.

**Theorem 1.1** Let $f : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_d} \to \mathbb{R}$ be a nonnegative multilinear form. Assume that $F$ is irreducible and $p_j \geq d$ for $j \in [d]$. Then the system (1.2) has a unique positive solution $(x_1, \ldots, x_d) > 0$. If $F$ is indecomposable then the system (1.2) has a unique solution $(x_1, \ldots, x_d)$, which is necessarily positive.

For $d = 2$ and $p_1 = p_2 = 2$ this theorem is a simplified version of the classical Perron-Frobenius theorem for the symmetric matrix

$$A = \begin{bmatrix} 0 & F \\ F^\top & 0 \end{bmatrix}. $$

For $d \geq 3$ and $p_1 = \ldots = p_d = d$ our theorem does not follow from the main result in [1]. We also give examples where the statement of the theorem fails if the conditions $p_j \geq d$ for $j \in [d]$ are not satisfied.

Theorem 1.1 is obtained by constructing a homogeneous monotone map of degree one and invoking the results of [3, 7]. Some of our results turn out to hold more generally for polynomial maps with nonnegative coefficients. Hence, we subsequently establish an analog of the Perron-Frobenius theorem for such maps, Theorem 4.1 below. We show the geometric convergence of the power method to the unique normalized eigenvector, under suitable conditions.

### 2 Proof of the main theorem

**Lemma 2.1** Let $f$ be a $d$-multilinear nonnegative form for $d \geq 2$. Assume that $F$ is indecomposable. Then $F$ is irreducible. For $d = 2$, $F$ is indecomposable if and only if $F$ is irreducible.
Proof. Assume to the contrary that \( \mathcal{F} \) is not irreducible. So the graph \( G(\mathcal{F}) \) is not connected. So there exists \( \emptyset \neq I \subseteq \mathbb{R} \) such that there is no edge from \( I \) to \( J = V \setminus I \). Let \( I_k := I \cap V_k \) for \( k \in [d] \). Let \( I'_k \) be defined as follows. \( I'_k = I_k \) if \( I_k \neq V_k \) and \( I'_k = [m_k - 1] \) if \( I_k = V_k \). Note that \( I'_l = I_l \) for some \( l \). Let \( I' = \bigcup_{k=1}^d I'_k \), \( J' = V \setminus I' \). Since \( \mathcal{F} \) is indecomposable there exists \( i_k \in I' \cap V_k \) and \( i_j \in J' \cap V_j \) for \( j \neq k \) such that \( f_{i_1, \ldots, i_d} > 0 \). Since \( I'_l = I_l \) it follows that \( i_l \in J \). So \((i_k, i_l) \in E(\mathcal{F})\), which contradicts our assumption.

Assume that \( d = 2 \). It is straightforward to show that if \( \mathcal{F} \) is reducible then \( \mathcal{F} \) is decomposable.

Let \( \mathbb{R}_+ := [0, \infty), \mathbb{R}_{>0} := (0, \infty) \) and \( C := (\mathbb{R}_+^m \setminus \{0\}) \times \cdots \times (\mathbb{R}_+^m \setminus \{0\}) \). Let

\[
p := \max(p_1, \ldots, p_d).
\]

For a nonnegative \( d \)-form \((1.1)\) define the following homogeneous map \( F : C \to C \) of degree one:

\[
F((x_1, \ldots, x_d))_{ij,j} = \left( x_{ij,j}^{p_j-p_d} \|x_j\|^{p_j-d} \sum_{\begin{subarray}{l}
 i_1 \in [m_1], \ldots, i_{j-1} \in [m_{j-1}], \\
i_{j+1} \in [m_{j+1}], \ldots, i_d \in [m_d]
\end{subarray}} f_{i_1, \ldots, i_d} x_{i_1,1, \ldots, x_{i_{j-1},1, \ldots, x_{i_{j-1},j-1},1, \ldots, x_{i_d,d}} \right)^{\frac{1}{p-j}},
\]

\[
i_j \in [m_j], \ j \in [d].
\]

To avoid trivial cases we assume that \( F_{ij,j} \) is not identically zero for each \( j \) and \( i_j \), i.e.

\[
\sum_{\begin{subarray}{l}
i_1 \in [m_1], \ldots, i_{j-1} \in [m_{j-1}], \\
i_{j+1} \in [m_{j+1}], \ldots, i_d \in [m_d]
\end{subarray}} f_{i_1, \ldots, i_d} > 0, \ \text{for all} \ i_j \in [m_j] \ j \in [d]. \tag{2.2}
\]

Note that if \( \mathcal{F} \) is irreducible, then this condition is satisfied.

We identify \( C^o := \mathbb{R}_{>0}^m \times \cdots \times \mathbb{R}_{>0}^m \), the interior of the cone \( C \), with \( \mathbb{R}_{>0}^{m_1+\cdots+m_d} \). Let \( y = (x_1, \ldots, x_d) \). Recall that \( F \) is called monotone if \( F(y) \leq F(z) \) for \( 0 < y < z \).

In what follows we assume the condition

\[
p_j \geq d, \ j \in [d]. \tag{2.3}
\]

Then \( F \) is monotone on \( C^o \). Recall the definition of the di-graph

\[
G(F) = (V, E(F)), \ E(F) \subset (V_1 \times \ldots \times V_d)^2
\]

associated with \( F \), as in [3]. Let \( \mathbf{1}_j = (1, \ldots, 1)^\top, \mathbf{e}_{k,j} = (\delta_{k1}, \ldots, \delta_{km_j})^\top \in \mathbb{R}^{m_j} \) for \( j \in [d] \). Then for \( i_k \in V_k, i_l \in V_l \):

\[
(i_k, i_l) \in E(F) \iff \lim_{t \to \infty} F_{i_k,k}(\mathbf{1}_1, \ldots, \mathbf{1}_d) + t(0, \ldots, 0, \mathbf{e}_{i_l,i_l}, 0, \ldots, 0) = \infty.
\]

Equivalently, there is a di-edge from \( i_k \in V_k \) to \( i_l \in V_l \) if and only if the variable \( x_{i_l,t} \) effectively appears in the expression of \( F_{i_k,k} \). The following lemma is deduced straightforwardly.

**Lemma 2.2** Let \( f \) be a nonnegative multilinear form given by \((1.1)\). Assume that the conditions \((2.2)\) and \((2.3)\) hold. Then, \((r, s)\) is a di-edge of the di-graph \( G(F) = (V, E(F)) \) if and only if at least one of the following conditions holds:
1. \( r = i_k \in V_k, s = i_l \in V_l \), with \( k \neq l \), and we can find \( d - 2 \) indices \( i_j \in V_j \), for \( 1 \leq j \leq d \), \( j \notin \{k, l\} \), such that \( f_{i_1,i_2,...,i_d} > 0 \);

2. \( r, s \) belong to \( V_k \) and \( p_k > d \);

3. \( r = s \) belongs to \( V_k \) and \( p > p_k \).

In particular, if \( F \) is irreducible then \( G(F) \) is strongly connected.

**Theorem 2.3** Let \( f \) be a nonnegative multilinear form given by (1.1). Assume that the conditions (2.2) and (2.3) hold. Let \( p = \max(p_1, \ldots, p_d) \). Suppose furthermore that \( G(F) \) is strongly connected. Then \( F \) has a unique positive eigenvector up to a positive multiple. I.e. there exist positive vectors \( 0 < \xi_j \in \mathbb{R}^{m_1}, j \in [d] \) and a positive eigenvalue \( \mu \) with the following properties.

1. \( F((\xi_1, \ldots, \xi_d)) = \mu(\xi_1, \ldots, \xi_d) \). In particular,
   \[
   f(\xi_1, \ldots, \xi_d) = \mu^{p-1}\|\xi_j\|_{p_j}^d \text{ for } j \in [d].
   \] (2.4)

2. Assume that \( F((x_1, \ldots, x_d)) = \alpha(x_1, \ldots, x_d) \) for some \((x_1, \ldots, x_d) > 0\). Then \( \alpha = \mu \) and \((x_1, \ldots, x_d) = t(\xi_1, \ldots, \xi_d) \) for some \( t > 0 \).

Let \( p_1 = \ldots = p_d = p \geq d \) and assume furthermore that \( F \) is indecomposable. Then the conditions 1-2 hold. Suppose that \( F((x_1, \ldots, x_d)) = \alpha(x_1, \ldots, x_d) \) for some \((x_1, \ldots, x_d) \geq 0\) and \( \|x_j\|_p > 0 \) for \( j \in [d] \). Then \( \alpha = \mu \) and \((x_1, \ldots, x_d) = t(\xi_1, \ldots, \xi_d) \) for some \( t > 0 \).

**Proof.** The existence a positive eigenvector \((\xi_1, \ldots, \xi_d)\) of \( F \) follows from [3, Theorem 2]. We next derive the uniqueness of \((\xi_1, \ldots, \xi_d)\) from [7, Theorem 2.5]. First recall that \( F \) is nonexpansive with respect to the Hilbert metric. Hence the condition (a) of Theorem 2.5 is satisfied. Since \( F(ty) = tF(y) \) the first condition of (b) trivially hold. We now show that the second condition of (b) also holds. Clearly each coordinate of \( F \) is a smooth function on \( C^0 \). Let \( F(y) = ty, y \in C^0 \). Denote by \( A = DF(y) \in \mathbb{R}^{(m_1+\cdots+m_d) \times (m_1+\cdots+m_d)} \) the derivative of \( F \) at \( y \). The directed graph \( G(A) \), induced by the nonnegative entries of \( A \), is equal to \( G(F) \). The assumption that \( G(F) \) is strongly connected yields the second condition of (b). Hence \( F \) has a unique positive eigenvector \( y = (x_1, \ldots, x_d) \), up to a product by a positive scalar.

It is left to show the condition (2.4) for the eigenvector \( y \). Raise the equality \( F_{i,j}(x_1, \ldots, x_d) = \mu x_{i,j} \) to the power \( p - 1 \) and divide by \( x_{i,j}^{p-j} \) to obtain the equality

\[
\|x_j\|_{p_j}^{p-j} \sum_{i_1 \in [m_1], \ldots, i_{j-1} \in [m_{j-1}]} f_{i_1, \ldots, i_2, i_{j-1}+1 \ldots i_{j-1}, j-1, i_{j+1} \ldots i_d} = \mu^{p-1}x_{i,j}^{p-j-1}.
\]

Multiply this equality by \( x_{i,j} \) and sum on \( i = 1, \ldots, m_j \) to obtain

\[
\|x_j\|_{p_j}^{p-j} f(x_1, \ldots, x_d) = \mu^{p-1}\|x_j\|_{p_j}^p, \ j \in [d],
\]

which is equivalent to (2.4). Assume that \( p_1 = \ldots = p_d = p \) and \( F \) is indecomposable. Lemma 2.1 yields that \( F \) is irreducible. Lemma 2.2 implies that \( G(F) \) is strongly connected. Hence the conditions 1-2 hold.
Assume that \( p_1 = \ldots = p_d = p \geq d \), \( F \) is indecomposable and \( F((z_1, \ldots, z_d)) = \alpha(z_1, \ldots, z_d) \) for some \((z_1, \ldots, z_d) \geq 0\), where \( \|z_j\|_p > 0 \) for \( j \in [d] \). Suppose furthermore that \((z_1, \ldots, z_d)\) is not a positive vector. Let \( \emptyset \neq I, J \subset \bigcup_{j=1}^d V_i \) be the set of indices where \((z_1, \ldots, z_d)\) have zero and positive coordinates respectively. I.e. \( i_k \in I \cap V_k \) if and only if \( z_{i_k,k} = 0 \). Since \( \|z_k\|_d > 0 \) for each \( k \in [d] \) it follows that \( I \cap V_k \neq V_k \), and for each \( i_k \in I \cap V_k \) we have the equality \( \frac{(F_{i_k}(z_1, \ldots, z_d))^{p-1}}{\|z_j\|_d^{p-1}} = 0 \). Hence \( f_{i_1, \ldots, i_d} = 0 \) for each \( i_k \in I \cap V_k \) and \( i_j \in J \cap V_j \) for each \( j \in [d] \setminus \{k\} \). This contradicts the assumption that \( F \) is indecomposable. \( \square \)

**Proof of Theorem 1.1** Apply Theorem 2.3. Normalize the positive eigenvector \((\xi_1, \ldots, \xi_d)\) by the condition \( \|\xi_1\|_{p_1} = 1 \). Then the first condition of (2.4) yields that \( F((\xi_1, \ldots, \xi_d)) = \mu^{p-1} \). The condition (2.4) for \( j > 1 \) yields that \( \|\xi_j\|_{p_j} = 1 \). Hence the equality \( F((\xi_1, \ldots, \xi_d)) = \mu(\xi_1, \ldots, \xi_d) \) implies (1.2) with \( \lambda = \mu^{p-1} \).

Assume now that (1.2) holds. Then \( F((x_1, \ldots, x_p)) = \lambda^{\frac{1}{p-1}}(x_1, \ldots, x_p) \). Hence, if \((x_1, \ldots, x_p) > 0\) it follows that \((x_1, \ldots, x_p) = (\xi_1, \ldots, \xi_d)\).

Assume now \( F \) is indecomposable. We claim that (1.2) implies that \( \lambda > 0 \) and \( x_j > 0 \) for each \( j \in [d] \). Assume to the contrary that \((x_1, \ldots, x_d)\) is not a positive vector. Let \( \emptyset \neq I, J \subset \bigcup_{j=1}^d V_i \) be the set of indices where \((x_1, \ldots, x_d)\) have zero and positive coordinates respectively. I.e. \( i_k \in I \cap V_k \) if and only if \( z_{i_k,k} = 0 \). Since \( \|x_k\|_{p_k} > 0 \) for each \( k \in [d] \) it follows that \( I \cap V_k \neq V_k \), and for each \( i_k \in I \cap V_k \) we have the equality

\[
\sum_{i_1, \ldots, i_d \in [m_1] \atop i_{k-1} \in [m_{k-1}] \atop i_{k+1} \in [m_{k+1}] \atop i_d \in [m_d]} f_{i_1, \ldots, i_d}x_{i_1,1} \ldots x_{i_{k-1},1,k-1} x_{i_{k+1},1,k+1} \ldots x_{i_d,d} = 0.
\]

Hence \( f_{i_1, \ldots, i_d} = 0 \) for each \( i_k \in I \cap V_k \) and \( i_j \in J \cap V_j \) for each \( j \in [d] \setminus \{k\} \). This contradicts the assumption that \( F \) is indecomposable. So \((x_1, \ldots, x_d)\) must be a positive vector, and so \( \lambda > 0 \). The previous arguments show that the system (1.2) has a unique solution \((x_1, \ldots, x_d)\), which is positive. \( \square \)

3 Examples and remarks

We first give numerical examples showing that the conclusion of Theorem 1.1 no longer holds for \( p < d \). Consider first the positive tensor \( \mathcal{F}_1 \in \mathbb{R}^{2 \times 2 \times 2} \) with entries

\[
f_{1,1,1} = f_{2,2,2} = a > 0; \text{ otherwise, } f_{i,j,k} = b > 0
\]  

(3.1)

So the trilinear form is

\[
f(x_1, x_2, x_3) = b(x_{1,1}+x_{2,1})(x_{1,2}+x_{2,2})(x_{1,3}+x_{2,3})+(a-b)(x_{1,1}x_{1,2}x_{1,3}+x_{2,1}x_{2,2}x_{2,3}).
\]

Clearly, the system (1.2) for \( p_1 = p_2 = p_3 = p > 1 \) has a positive solution \( x_1 = x_2 = x_3 = (0.5^{1/p}, 0.5^{1/p})^\top \). Let

\[
f_{1,1,1} = f_{2,2,2} = a = 1.2; \text{ otherwise, } f_{i,j,k} = b = 0.2.
\]  

(3.2)
For this tensor, the system (1.2) has a unique solution \( x_1 = x_2 = x_3 = (0.5^{1/p}, 0.5^{1/p})^\top \) for \( p \geq 3 \). However, for \( p = 2 \) \((< d = 3)\) (1.2), in addition to the above positive solution, has two other positive solutions

\[
x_1 = x_2 = x_3 \approx (0.9342, 0.3568)^\top
\]

and

\[
x_1 = x_2 = x_3 \approx (0.3568, 0.9342)^\top.
\]

There are irreducible tensors for which the conclusion of Theorem 1.1 can fail for \( p \) very close to \( d \). As an example, we consider the positive tensor \( F_2 \in \mathbb{R}^{2 \times 2 \times 2} \) with entries

\[
f_{1,1,1} = f_{2,2,2} = a = 1.001; \text{ otherwise, } f_{i,j,k} = b = 0.001. \tag{3.3}
\]

For tensor \( F_2 \), the system (1.2) has additional two positive solutions:

\[
x_1 = x_2 = x_3 \approx (0.9667, 0.4570)^\top
\]

and

\[
x_1 = x_2 = x_3 \approx (0.4570, 0.9667)^\top
\]

when \( p = 2.99 \).

We now show that the results in [1] do not apply in our case for \( d > 2, p = d \). For simplicity of the discussion we consider the case \( d = 3 \). The homogeneous eigenvalue problem studied in [1] is for the nonnegative tensor \( C = [c_{i,j,k}] \in \mathbb{R}_+^{n \times n \times n} \). It is of the form

\[
\sum_{j=k=1}^n c_{i,j,k} y_j y_k = \lambda y_i^2, \ i \in [n]. \tag{3.4}
\]

\( C \) is called reducible in [1] if for for some nontrivial subsets \( \emptyset \neq I \subsetneq [n], J = [n] \setminus I \) one has the equality \( c_{i,j,k} = 0 \) for \( i \in I, j, k \in J \). Equivalently, for \( y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n \) such that \( y_i = 0, y_j = 1 \) for \( i \in I, j \in J \) respectively the left-hand side of (3.4) is zero for \( i \in I \). \( C \geq 0 \) is called irreducible if it is not reducible. The Perron-Frobenius theorem in [1] is proved for irreducible tensors.

We now show that the induced tensor \( C \) by the tensor \( F \) is reducible. To this end let \( n = m_1 + m_2 + m_3 \) and define

\[
y = (x_{1,1}, \ldots, x_{m_1,1}, x_{1,2}, \ldots, x_{m_2,2}, x_{1,3}, \ldots, x_{m_3,3})^\top \in \mathbb{R}^n.
\]

Then the system (1.2) for \( d = 3, p = 3 \) can be written as the system (3.4). We claim that \( C \) is always reducible in the sense of [1]. Indeed, choose \( y \) corresponding to \( x_1 = x_2 = 0, x_3 = 1 \). Clearly, the left-hand side of (1.2) is zero for all equations. Hence the left-hand side of (3.4) is zero for all \( i \in [n] \). Hence \( C \) is reducible.

We close this section with a variation on the classical Perron-Frobenius theorem on bilinear form \( x^\top A y \), where \( A \in \mathbb{R}_+^{m \times n} \) is a nonnegative matrix. If the bipartite graph \( G(A) \) induces a connected bipartite graph then the largest singular value of \( A \), equal to \( \|A\| \), is simple, with the unique nonnegative left and right singular vectors \( \xi, \eta \) of \( A \), corresponding \( \|A\| \), of length one which are positive. This is the classical Perron-Frobenius theorem. Theorem 1.1 claims that if the induced bipartite
graph \( G(A) \) is connected then the classical Perron-Frobenius theorem holds for any \( p_1 = p_2 \geq d = 2 \). However, in the case \( p_1 = p_2 < d = 2 \) the Perron-Frobenius theorem may fail as in the case \( d = 3 \). Consider the following example.

\[
A = \begin{bmatrix}
1.0000 & 0.2000 & 0.2000 & 0.2000 \\
0.2000 & 1.0000 & 0.2000 & 0.2000 \\
0.2000 & 0.2000 & 1.0000 & 0.2000 \\
0.2000 & 0.2000 & 0.2000 & 1.0000
\end{bmatrix}
\]

When \( p_1 = p_2 = p = 1.5 < d = 2 \), the system (1.2) has three solutions:

\[
\begin{align*}
\mathbf{x} &= (0.0893, 0.9641, 0.0893)^\top, \quad \mathbf{y} = (0.0863, 0.9583, 0.0863, 0.0501)^\top, \\
\mathbf{x} &= (0.0893, 0.0893, 0.9641)^\top, \quad \mathbf{y} = (0.0863, 0.0863, 0.9583, 0.0501)^\top, \\
\mathbf{x} &= (0.9641, 0.0893, 0.0893)^\top, \quad \mathbf{y} = (0.9583, 0.0863, 0.0863, 0.0501)^\top.
\end{align*}
\]

For the same matrix, if \( p_1 = 1.2 \) and \( p_2 = 2.5 \), the system (1.2) also has three positive solutions.

### 4 Extension: Perron-Frobenius theorem for nonnegative polynomial maps

Let \( \mathbf{P} = (P_1, \ldots, P_d)^\top : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial map. We assume that each \( P_i \) is a homogeneous polynomial of degree \( d_i \geq 1 \) and that the coefficient of each monomial in \( P_i \) is nonnegative. So \( \mathbf{P} : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \). We associate with \( \mathbf{P} \) the following directed graph \( G(\mathbf{P}) = (V, E(\mathbf{P})) \), where \( V = [n] \) and \( (i, j) \in E(\mathbf{P}) \) if \( P_i \) contains a monomial (with nonzero coefficient) that contains the variable \( x_j \). We call \( \mathbf{P} \) irreducible if \( G(\mathbf{P}) \) is strongly connected. To each subset \( I \subset [n] \), is associated a part \( Q_I \), which consists of the vectors \( \mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n_+ \) such that \( x_i > 0 \) iff \( i \in I \). We say that the polynomial map \( \mathbf{P} \) is indecomposable if there is no part of \( \mathbb{R}^n_+ \) that is invariant by \( \mathbf{P} \), except the trivial parts \( Q_\emptyset \) and \( Q_{[n]} \). Observe that when \( \mathbf{P} \) is the polynomial map appearing at the left-hand side of (1.2), this definition of indecomposability coincides with the one made in Section 1. We have the following extension of Theorem 1.1.

**Theorem 4.1** Let \( \mathbf{P} = (P_1, \ldots, P_d)^\top : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial map, where each \( P_i \) a homogeneous polynomial of degree \( d_i \geq 1 \) with nonnegative coefficients. Let \( \delta_1, \ldots, \delta_n \in (0, \infty) \) be given and assume that \( \delta_i \geq d_i, i \in [n] \). Consider the system

\[
P_i(\mathbf{x}) = \lambda x_i^{\delta_i}, \quad i \in [n], \quad \mathbf{x} \geq \mathbf{0}.
\]

Assume that \( \mathbf{P} \) is irreducible. Then for each \( a, p > 0 \) there exists a unique positive vector \( \mathbf{x} > \mathbf{0} \), depending on \( a, p \), satisfying (4.1) and the condition \( \|\mathbf{x}\|_p = a \). Suppose furthermore that \( \delta_1 = \ldots = \delta_n \) and \( \mathbf{P} \) is indecomposable. Then the system (4.1) has a unique solution, depending on \( a, p \) satisfying \( \|\mathbf{x}\|_p = a \), and all the coordinates of this solution are positive.

**Proof.** Let \( \delta = \max(\delta_1, \ldots, \delta_n) \). Consider the following homogeneous monotone map \( \mathbf{F} = (F_1, \ldots, F_n)^\top : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) given by

\[
F_i(\mathbf{x}) = \left((\|\mathbf{x}\|_p)^{\delta_i-d_i} x_i^{\delta_i-d_i} P_i(\mathbf{x})\right)^{1/\gamma}, \quad i \in [n].
\]
Apply \cite[Theorem 2]{3} and \cite[Theorem 2.5]{7}, as in the proof of Theorem 2.3 above, to deduce the result. \hfill \square

If $F$ is an order preserving from $\mathbb{R}^n_+$ to itself that is positively homogeneous of degree 1, the cone spectral radius of $F$, denoted by $\rho(F)$, is the greatest scalar $\mu$ such that there exists a nonzero vector $u \in \mathbb{R}^n_+$ such that $F(u) = \mu u$. We shall refer to $\mu$ and $u$ as a nonlinear eigenvalue and eigenvector of $F$, respectively. We shall also use the following generalizations of the Collatz-Wielandt functions arising classically in Perron-Frobenius theory:
\[
\begin{align*}
cw(F) & = \inf \{ \mu \mid \exists u \in \text{int} \mathbb{R}^n_+, F(u) \leq \mu u \}, \\
cw^{-}(F) & = \sup \{ \mu \mid \exists u \in \mathbb{R}^n_+ \setminus \{0\}, F(u) \geq \mu u \}.
\end{align*}
\]
Nussbaum proved in \cite[Theorem 3.1]{6} that
\[
\rho(F) = cw(F).
\]
From this, one can deduce that
\[
cw^{-}(F) = cw(F),
\]
see \cite[Lemma 2.8]{4}. We obtain the following Collatz-Wielandt type property for nonnegative polynomial maps.

**Corollary 4.2** Let $P = (P_1, \ldots, P_d) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a polynomial map, where each $P_i$ a homogeneous polynomial of degree $d \geq 1$ with nonnegative coefficients. Assume that $P$ is irreducible. Then, the unique scalar $\lambda$ such that there is a positive vector $u$ with $P_i(u) = \lambda u_i^d$ for all $i \in [n]$ satisfies:
\[
\lambda = \inf_{x \in \text{int} \mathbb{R}^n_+} \max_{i \in [n]} \frac{P_i(x)}{x_i^d} = \sup_{x \in \mathbb{R}^n_+ \setminus \{0\}} \min_{i \in [n]} \frac{P_i(x)}{x_i^d}.
\]

**Proof.** By Theorem 4.1, $F$ has an eigenvector in the interior of $\mathbb{R}^n_+$. Let $\mu$ be the associated eigenvalue. By definition of $\rho(F)$ and $cw(F)$, we have $\rho(F) \geq \mu \geq cw(F)$. From (4.2), we deduce that $\mu = cw(F)$. Observe that
\[
cw(F) = \inf_{x \in \text{int} \mathbb{R}^n_+} \max_{i \in [n]} \frac{P_i(x)}{x_i} = \inf_{x \in \text{int} \mathbb{R}^n_+} \max_{i \in [n]} \frac{P_i(x)^{\frac{1}{d}}}{x_i^{\frac{1}{d}}}.
\]
It follows that $\lambda = \mu^d$ is given by the first expression in (4.4). A similar reasoning, this time with the lower Collatz-Wielandt type number $cw^{-}(F)$, leads to the second expression in (4.4). As an immediate consequence, we get the following analogue of the characterization of the Perron root of an irreducible nonnegative matrix as the spectral radius.

**Corollary 4.3** Let $P$, $d$ and $\lambda$ be as in Corollary 4.2. If $\nu \in \mathbb{C}$ and $v = (v_1, \ldots, v_n)^\top \in \mathbb{C}^n \setminus \{0\}$ are such that $P_i(v) = \nu v_i^d$, for all $i \in [n]$, then $|\nu| \leq \lambda$.

**Proof.** Let $u_i := |v_i|$ and $u = (u_1, \ldots, u_n)^\top$. Then, $P_i(u) \geq |\nu| u_i^d$, and so
\[
\min_{i \in [n]} \frac{P_i(u)}{u_i^d} \geq |\nu|.
\]
It follows from (4.4) that $\lambda \geq |\nu|$.
5 Algorithmic aspects

The following simple power type algorithm will allow us to compute the vector \( x \) in (4.1) in an important special case. Let \( \psi \) denote the positive linear form on \( \mathbb{R}^n_+ \), given \( \psi(x) = \sum_{i \in [n]} x_i \), and consider a sequence \( x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})^\top \) inductively defined by

\[
x_i^{(k+1)} = (\psi(F(x^{(k)})))^{-1} F_i(x^{(k)}), \quad k = 0, 1, 2, \ldots
\]

where \( x^{(0)} \) is an arbitrary vector in the interior of the cone. We shall say that \( P \) is primitive if the graph \( G(P) \) is strongly connected and if the gcd of the lengths of its circuits is equal to one. The following is readily deduced from a general result of [7].

**Corollary 5.1** Let \( P \) and \( d \) be as in Corollary 4.2, and assume in addition that \( P \) is primitive. Then, the sequence \( x^{(k)} \) produced by the power algorithm converges to the unique vector \( u \in \text{int} \mathbb{R}^n_+ \) satisfying \( P_i(u) = \lambda u_i^d \), for \( i \in [n] \), and \( \psi(u) = 1 \).

**Proof.** The map \( F \) defined above is differentiable, and its derivative at any point of the interior of \( \mathbb{R}^n_+ \) is a nonnegative matrix the graph of which is precisely \( G(P) \). Hence, this nonnegative matrix is primitive and the assumptions of Corollary 2.5 of [7] are satisfied. It follows that \( x^{(k)} \) converges to the only eigenvector \( u \) of \( F \) in the interior of \( \mathbb{R}^n_+ \) such that \( \psi(u) = 1 \). \( \square \)

The result of [7] implies that the convergence of the power algorithm is geometric, and it yields a bound on the geometric convergence rate which tends to 1 as the distance between \( x^{(0)} \) and \( u \) in Hilbert’s projective metric tends to infinity. We next estimate the asymptotic speed of convergence. To do so, it is enough to linearize \( F \) around \( u \), see for example the arguments in [2]. We give a short proof for reader’s convenience, leading to an explicit formula for the rate.

**Corollary 5.2** Let \( P, d, u \) and \( \lambda \) be as in Corollary 5.1, let \( M := F'(u) \), and let \( r \) denote the maximal modulus of the eigenvalues of \( M \) distinct from \( \lambda \). Then, the sequence \( x^{(k)} \) produced by the power algorithm satisfies

\[
\limsup_{k \to \infty} \|x^{(k)} - u\|^{1/k} \leq \lambda^{-1} r.
\]

**Proof.** For all \( \alpha > 1 \), we have \( \alpha \lambda u = F(\alpha u) = F(u) + (\alpha - 1)Mu + o(\alpha - 1) = \lambda u + (\alpha - 1)Mu + o(\alpha - 1) \), and so \( Mu = \lambda u \), which shows that \( \lambda \) is the Perron root of \( M \). Since, as observed above, \( M \) is primitive, the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( M \) can be ordered in such a way that \( \lambda = \lambda_1 > r = |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \). Consider now \( G(x) := (\psi(F(x)))^{-1} F(x) \). An elementary computation shows that

\[
G'(u) = \lambda^{-1}(M - u\psi M),
\]

where the linear form \( \psi \) is thought of as a row vector. We claim that the spectral radius of the matrix \( Q := M - u\psi M \) is equal to \( |\lambda_2| \). To see this, observe that the Perron eigenvector \( u \) of \( M \), normalized by \( \psi(u) = 1 \), is a continuous function of the entries of \( M \). By continuity of the spectrum, so does the spectral radius of \( Q \). Hence, by a density argument, we may assume that all the eigenvalues of \( M \) are distinct and that no eigenvalue of \( M \) is 0. Since \( \psi(u) = 1 \), we have \( Qu = Mu - u\psi M u = \lambda u - \lambda u = 0 \), so 0 is an eigenvalue of \( Q \). Moreover, denoting by \( \varphi_j \) an eigenvector...
of $M$ for the eigenvalue $\lambda_j$, with $2 \leq j \leq n$, we get $\lambda_j \varphi_j \mathbf{u} = \varphi_j M \mathbf{u} = \lambda \varphi_j \mathbf{u}$, and since $\lambda_j \neq \lambda$, $\varphi_j \mathbf{u} = 0$. It follows that $\lambda_j \varphi_j = Q \varphi_j$. Hence, the eigenvalues of the matrix $Q$ are precisely $0, \lambda_2, \ldots, \lambda_n$. Thus, $r = |\lambda_2|$, and the spectral radius of $G'(u)$, denoted by $\rho(G'(u))$, is equal to $\lambda^{-1} r$. Since

$$x^{(k+1)} - \mathbf{u} = G(x^{(k)}) - G(\mathbf{u}) = G'(\mathbf{u})(x^{(k)} - \mathbf{u}) + o(\|x^{(k)} - \mathbf{u}\|),$$

and since, by the previous corollary, $\|x^{(k)} - \mathbf{u}\|$ tends to zero as $k$ tends to infinity, a classical argument shows that

$$\limsup_{k \to \infty} \|x^{(k)} - \mathbf{u}\|^{1/k} \leq \rho(G'(\mathbf{u})).$$

Note that the previous corollaries apply in particular to the polynomial map appearing in our initial problem (1.2).

References

[1] K.C. Chang, K. Pearson, and T. Zhang, Perron-Frobenius theorem for non-negative tensors, Commun. Math. Sci. 6, (2008), 507-520.

[2] S. Friedland, Convergence of products of matrices in projective spaces, Linear Alg. Appl., 413(2006), 247-263.

[3] S. Gaubert and J. Gunawardena, The Perron-Frobenius theorem for homogeneous, monotone functions, Trans. Amer. Math. Soc. 356 (2004), 4931-4950.

[4] M. Akian, S. Gaubert, and A. Guterman. Tropical polyhedra are equivalent to mean payoff games. Eprint arXiv:0912.2462, 2009.

[5] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, Proc. IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP '05), 1 (2005), pp. 129-132.

[6] R. D. Nussbaum. Convexity and log convexity for the spectral radius. Linear Algebra and its Applications, 73:59–122, 1986.

[7] R. D. Nussbaum, Hilbert’s projective metric and iterated nonlinear maps, Memoirs Amer. Math. Soc., 1988, vol. 75.