A BOUNDARY VALUE PROBLEM FOR MINIMAL LAGRANGIAN GRAPHS

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1. INTRODUCTION

Consider the product $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the Euclidean metric. The product $\mathbb{R}^n \times \mathbb{R}^n$ has a natural complex structure, which is given by

$$J \frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}, \quad J \frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$  

The associated symplectic structure is given by

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k.$$  

A submanifold $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ is called Lagrangian if $\omega|_{\Sigma} = 0$.

In this paper, we study a boundary value problem for minimal Lagrangian graphs in $\mathbb{R}^n \times \mathbb{R}^n$. To that end, we fix two domains $\Omega, \bar{\Omega} \subset \mathbb{R}^n$ with smooth boundary. Given a diffeomorphism $f : \Omega \to \bar{\Omega}$, we consider its graph $\Sigma = \{(x, f(x)) : x \in \Omega\} \subset \mathbb{R}^n \times \mathbb{R}^n$. We consider the problem of finding a diffeomorphism $f : \Omega \to \bar{\Omega}$ such that $\Sigma$ is Lagrangian and has zero mean curvature. Our main result asserts that such a map exists if $\Omega$ and $\bar{\Omega}$ are uniformly convex:

**Theorem 1.1.** Let $\Omega$ and $\bar{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Then there exists a diffeomorphism $f : \Omega \to \bar{\Omega}$ such that the graph $\Sigma = \{(x, f(x)) : x \in \Omega\}$ is a minimal Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Minimal Lagrangian submanifolds were first studied by Harvey and Lawson [6], and have attracted considerable interest in recent years. Yuan [14] has proved a Bernstein-type theorem for minimal Lagrangian graphs over $\mathbb{R}^n$. A similar result was established by Tsui and Wang [10]. Smoczyk and Wang have used the mean curvature flow to deform certain Lagrangian submanifolds to minimal Lagrangian submanifolds (see [8], [9], [13]). In [1], the first author studied a boundary value problem for minimal Lagrangian graphs in $\mathbb{H}^2 \times \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the hyperbolic plane.

In order to prove Theorem 1.1, we reduce the problem to the solvability of a fully nonlinear PDE. As above, we assume that $\Omega$ and $\bar{\Omega}$ are uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Moreover, suppose that $f$
is a diffeomorphism from $\Omega$ to $\tilde{\Omega}$. The graph $\Sigma = \{(x, f(x)) : x \in \Omega\}$ is Lagrangian if and only if there exists a function $u : \Omega \to \mathbb{R}$ such that $f(x) = \nabla u(x)$. In that case, the Lagrangian angle of $\Sigma$ is given by $F(D^2u(x))$.

Here, $F$ is a real-valued function on the space of symmetric $n \times n$ matrices which is defined as follows: if $M$ is a symmetric $n \times n$ matrix, then $F(M)$ is defined by

$$F(M) = \sum_{k=1}^{n} \arctan(\lambda_k),$$

where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $M$.

By a result of Harvey and Lawson (see [6], Proposition 2.17), $\Sigma$ has zero mean curvature if and only if the Lagrangian angle is constant; that is,

$$F(D^2u(x)) = c$$

for all $x \in \Omega$. Hence, we are led to the following problem:

$$(\star) \text{ Find a convex function } u : \Omega \to \mathbb{R} \text{ and a constant } c \in (0, \frac{n\pi}{2}) \text{ such that } \nabla u \text{ is a diffeomorphism from } \Omega \text{ to } \tilde{\Omega} \text{ and } F(D^2u(x)) = c \text{ for all } x \in \Omega.$$ 

Caffarelli, Nirenberg, and Spruck [3] have obtained an existence result for solutions of (1) under Dirichlet boundary conditions. In this paper, we study a different boundary condition, which is analogous to the second boundary value problem for the Monge-Ampère equation.

In dimension 2, P. Delanoë [4] proved that the second boundary value problem for the Monge-Ampère equation has a unique smooth solution, provided that both domains are uniformly convex. This result was generalized to higher dimensions by L. Caffarelli [2] and J. Urbas [11]. In 2001, J. Urbas [12] described a general class of Hessian equations for which the second boundary value problem admits a unique smooth solution.

In Section 2, we establish a-priori estimates for solutions of (\star). In Section 3, we prove that all solutions of (\star) are non-degenerate (that is, the linearized operator is invertible). In Section 4, we use the continuity method to show that (\star) has at least one solution. From this, Theorem 1.1 follows. Finally, in Section 5, we prove a uniqueness result for (\star).

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2. A PRIORI ESTIMATES FOR SOLUTIONS OF (\star)

In this section, we prove a-priori estimates for solutions of (\star).

Let $\Omega$ and $\tilde{\Omega}$ be uniformly convex domains in $\mathbb{R}^n$ with smooth boundary. Moreover, suppose that $u$ is a convex function such that $\nabla u$ is a diffeomorphism from $\Omega$ to $\tilde{\Omega}$ and $F(D^2u(x))$ is constant. For each point $x \in \Omega$, we
define a symmetric \( n \times n \)-matrix \( A(x) = \{a_{ij}(x) : 1 \leq i, j \leq n\} \) by

\[
A(x) = [I + (D^2 u(x))^2]^{-1}.
\]

Clearly, \( A(x) \) is positive definite for all \( x \in \Omega \).

**Lemma 2.1.** We have

\[
\frac{n\pi}{2} - F(D^2 u(x)) \geq \arctan \left( \frac{\text{vol}(\Omega)^{1/n}}{\text{vol}(\bar{\Omega})^{1/n}} \right)
\]

for all points \( x \in \Omega \).

**Proof.** Since \( \nabla u \) is a diffeomorphism from \( \Omega \) to \( \bar{\Omega} \), we have

\[
\int_{\Omega} \det D^2 u(x) \, dx = \text{vol}(\bar{\Omega})
\]

Therefore, we can find a point \( x_0 \in \Omega \) such that

\[
\det D^2 u(x_0) \leq \frac{\text{vol}(\bar{\Omega})}{\text{vol}(\Omega)}.
\]

Hence, if we denote by \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) the eigenvalues of \( D^2 u(x_0) \), then we have

\[
\lambda_1 \leq \frac{\text{vol}(\bar{\Omega})^{1/n}}{\text{vol}(\Omega)^{1/n}}.
\]

This implies

\[
\frac{n\pi}{2} - F(D^2 u(x_0)) = \sum_{k=1}^{n} \arctan \left( \frac{1}{\lambda_k} \right)
\]

\[
\geq \arctan \left( \frac{1}{\lambda_1} \right)
\]

\[
\geq \arctan \left( \frac{\text{vol}(\Omega)^{1/n}}{\text{vol}(\bar{\Omega})^{1/n}} \right).
\]

Since \( F(D^2 u(x)) \) is constant, the assertion follows.

**Lemma 2.2.** Let \( x \) be an arbitrary point in \( \Omega \), and let \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) be the eigenvalues of \( D^2 u(x) \). Then

\[
\frac{1}{\lambda_1} \geq \tan \left[ \frac{1}{n} \arctan \left( \frac{\text{vol}(\Omega)^{1/n}}{\text{vol}(\bar{\Omega})^{1/n}} \right) \right].
\]
Proof. Using Lemma 2.1, we obtain
\[
    n \arctan \left( \frac{1}{\lambda_1} \right) \geq \sum_{k=1}^{n} \arctan \left( \frac{1}{\lambda_k} \right) \\
    = \frac{n \pi}{2} - F(D^2u(x)) \\
    \geq \arctan \left( \frac{\text{vol}(\Omega)^{1/n}}{\text{vol}(\tilde{\Omega})^{1/n}} \right).
\]
From this, the assertion follows easily.

By Proposition A.1, we can find a smooth function \( h : \Omega \rightarrow \mathbb{R} \) such that \( h(x) = 0 \) for all \( x \in \partial \Omega \) and
\[
    \sum_{i,j=1}^{n} \partial_i \partial_j h(x) w_i w_j \geq \theta |w|^2
\]
for all \( x \in \Omega \) and all \( w \in \mathbb{R}^n \). Similarly, there exists a smooth function \( \tilde{h} : \tilde{\Omega} \rightarrow \mathbb{R} \) such that \( \tilde{h}(y) = 0 \) for all \( y \in \partial \tilde{\Omega} \) and
\[
    \sum_{i,j=1}^{n} \partial_i \partial_j \tilde{h}(y) w_i w_j \geq \theta |w|^2
\]
for all \( y \in \tilde{\Omega} \) and all \( w \in \mathbb{R}^n \). For abbreviation, we choose a positive constant \( C_1 \) such that
\[
    C_1 \theta \sin^2 \left[ \frac{1}{n} \arctan \left( \frac{\text{vol}(\Omega)^{1/n}}{\text{vol}(\tilde{\Omega})^{1/n}} \right) \right] = 1.
\]
We then have the following estimate:

**Lemma 2.3.** We have
\[
    \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j h(x) \geq \frac{1}{C_1}
\]
for all \( x \in \Omega \).

**Proof.** Fix a point \( x_0 \in \Omega \), and let \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) be the eigenvalues of \( D^2u(x_0) \). It follows from (2) that
\[
    \sum_{i,j=1}^{n} a_{ij}(x_0) \partial_i \partial_j h(x_0) \geq \theta \sum_{k=1}^{n} \frac{1}{1 + \lambda_k^2} \geq \theta \frac{1}{1 + \lambda_1^2}.
\]
Using Lemma 2.2, we obtain
\[
    \frac{1}{1 + \lambda_1^2} \geq \sin^2 \left[ \frac{1}{n} \arctan \left( \frac{\text{vol}(\Omega)^{1/n}}{\text{vol}(\tilde{\Omega})^{1/n}} \right) \right] = \frac{1}{C_1 \theta}.
\]
Putting these facts together, the assertion follows.
In the next step, we differentiate the identity $F(D^2u(x)) = \text{constant}$ with respect to $x$. To that end, we need the following well-known fact:

**Lemma 2.4.** Let $M(t)$ be a smooth one-parameter family of symmetric $n \times n$ matrices. Then

$$\frac{d}{dt}F(M(t)) \bigg|_{t=0} = \text{tr}[(I + M(0)^2)^{-1} M'(0)].$$

Moreover, if $M(0)$ is positive definite, then we have

$$\frac{d^2}{dt^2}F(M(t)) \bigg|_{t=0} \leq \text{tr}[(I + M(0)^2)^{-1} M''(0)].$$

**Proposition 2.5.** We have

$$(4) \quad \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j \partial_k u(x) = 0$$

for all $x \in \Omega$. Moreover, we have

$$(5) \quad \sum_{i,j,k,l=1}^{n} a_{ij}(x) \partial_i \partial_j \partial_k \partial_l u(x) w_k w_l \geq 0$$

for all $x \in \Omega$ and all $w \in \mathbb{R}^n$.

**Proof.** Fix a point $x_0 \in \Omega$ and a vector $w \in \mathbb{R}^n$. It follows from Lemma 2.4 that

$$0 = \frac{d}{dt}F(D^2u(x_0 + tw)) \bigg|_{t=0} = \sum_{i,j,k=1}^{n} a_{ij}(x) \partial_i \partial_j \partial_k u(x_0) w_k.$$

Moreover, since the matrix $D^2u(x_0)$ is positive definite, we have

$$0 = \frac{d^2}{dt^2}F(D^2u(x_0 + tw)) \bigg|_{t=0} \leq \sum_{i,j,k,l=1}^{n} a_{ij}(x) \partial_i \partial_j \partial_k \partial_l u(x_0) w_k w_l.$$

From this, the assertion follows.

**Proposition 2.6.** Fix a smooth function $\Phi : \Omega \times \tilde{\Omega} \to \mathbb{R}$, and define $\varphi(x) = \Phi(x, \nabla u(x))$. Then

$$\left| \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j \varphi(x) \right| \leq C$$

for all $x \in \Omega$. Here, $C$ is a positive constant that depends only on the second order partial derivatives of $\Phi$.

**Proof.** The partial derivatives of the function $\varphi(x)$ are given by

$$\partial_i \varphi(x) = \sum_{k=1}^{n} \left( \frac{\partial}{\partial y_k} \Phi \right)(x, \nabla u(x)) \partial_i \partial_k u(x) + \left( \frac{\partial}{\partial x_i} \Phi \right)(x, \nabla u(x)).$$
This implies
\[
\partial_i \partial_j \varphi(x) = \sum_{k=1}^{n} \left( \frac{\partial}{\partial y_k} \Phi \right)(x, \nabla u(x)) \partial_i \partial_j \partial_k u(x)
\]
\[
+ \sum_{k,l=1}^{n} \left( \frac{\partial^2}{\partial y_k \partial y_l} \Phi \right)(x, \nabla u(x)) \partial_i \partial_k u(x) \partial_j \partial_l u(x)
\]
\[
+ \sum_{k=1}^{n} \frac{\partial^2}{\partial x_j \partial y_k} \Phi(x, \nabla u(x)) \partial_i \partial_k u(x)
\]
\[
+ \sum_{l=1}^{n} \frac{\partial^2}{\partial x_i \partial y_l} \Phi(x, \nabla u(x)) \partial_j \partial_l u(x)
\]
\[
+ \frac{\partial^2}{\partial x_i \partial x_j} \Phi(x, \nabla u(x)).
\]

Using (4), we obtain
\[
\sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j \varphi(x)
\]
\[
= \sum_{i,j,k,l=1}^{n} a_{ij}(x) \left( \frac{\partial^2}{\partial y_k \partial y_l} \Phi \right)(x, \nabla u(x)) \partial_i \partial_k u(x) \partial_j \partial_l u(x)
\]
\[
+ 2 \sum_{i,j,k=1}^{n} a_{ij}(x) \left( \frac{\partial^2}{\partial x_j \partial y_k} \Phi \right)(x, \nabla u(x)) \partial_i \partial_k u(x)
\]
\[
+ \sum_{i,j=1}^{n} a_{ij}(x) \left( \frac{\partial^2}{\partial x_i \partial x_j} \Phi \right)(x, \nabla u(x)).
\]

We now fix a point \( x_0 \in \Omega \). Without loss of generality, we may assume that \( D^2 u(x_0) \) is a diagonal matrix. This implies
\[
\sum_{i,j=1}^{n} a_{ij}(x_0) \partial_i \partial_j \varphi(x_0) = \sum_{k=1}^{n} \frac{\lambda_k^2}{1 + \lambda_k^2} \left( \frac{\partial^2}{\partial y_k^2} \Phi \right)(x_0, \nabla u(x_0))
\]
\[
+ 2 \sum_{k=1}^{n} \frac{\lambda_k}{1 + \lambda_k^2} \left( \frac{\partial^2}{\partial x_k \partial y_k} \Phi \right)(x_0, \nabla u(x_0))
\]
\[
+ \sum_{k=1}^{n} \frac{1}{1 + \lambda_k^2} \left( \frac{\partial^2}{\partial x_k^2} \Phi \right)(x_0, \nabla u(x_0)),
\]

where \( \lambda_k = \partial_k \partial_k u(x_0) \). Thus, we conclude that
\[
\left| \sum_{i,j=1}^{n} a_{ij}(x_0) \partial_i \partial_j \varphi(x_0) \right| \leq C,
\]
as claimed.

We next consider the function \( H(x) = \tilde{h}(\nabla u(x)) \). The following estimate is an immediate consequence of Proposition 2.6.

**Corollary 2.7.** There exists a positive constant \( C_2 \) such that
\[
\left| \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j H(x) \right| \leq C_2
\]
for all \( x \in \Omega \).

**Proposition 2.8.** We have \( H(x) \geq C_1 C_2 h(x) \) for all \( x \in \Omega \).

**Proof.** Using Lemma 2.3 and Corollary 2.7 we obtain
\[
\sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j (H(x) - C_1 C_2 h(x)) \leq 0
\]
for all \( x \in \Omega \). Hence, the function \( H(x) - C_1 C_2 h(x) \) attains its minimum on \( \partial \Omega \). Thus, we conclude that \( H(x) - C_1 C_2 h(x) \geq 0 \) for all \( x \in \Omega \).

**Corollary 2.9.** We have
\[
\langle \nabla h(x), \nabla H(x) \rangle \leq C_1 C_2 |\nabla h(x)|^2
\]
for all \( x \in \partial \Omega \).

**Proposition 2.10.** Fix a smooth function \( \Phi : \Omega \times \tilde{\Omega} \to \mathbb{R} \), and define \( \varphi(x) = \Phi(x, \nabla u(x)) \). Then
\[
|\langle \nabla \varphi(x), \nabla \tilde{h}(\nabla u(x)) \rangle| \leq C
\]
for all \( x \in \partial \Omega \). Here, \( C \) is a positive constant that depends only on \( C_1, C_2 \), and the first order partial derivatives of \( \Phi \).

**Proof.** A straightforward calculation yields
\[
\langle \nabla \varphi(x), \nabla \tilde{h}(\nabla u(x)) \rangle = \sum_{k=1}^{n} \left( \frac{\partial}{\partial x_k} \Phi \right)(x, \nabla u(x)) (\partial_k \tilde{h})(\nabla u(x))
\]
for all \( x \in \Omega \). By Corollary 2.9 we have \(|\nabla H(x)| \leq C_1 C_2 |\nabla h(x)|\) for all points \( x \in \partial \Omega \). Putting these facts together, the assertion follows.

**Proposition 2.11.** We have
\[
0 < \sum_{k,l=1}^{n} \partial_k \partial_l u(x) (\partial_k \tilde{h})(\nabla u(x)) (\partial_l \tilde{h})(\nabla u(x)) \leq C_1 C_2 \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle
\]
for all \( x \in \partial \Omega \).

**Proof.** Note that the function \( H \) vanishes along \( \partial \Omega \) and is negative in the interior of \( \Omega \). Hence, for each point \( x \in \partial \Omega \), the vector \( \nabla H(x) \) is a positive multiple of \( \nabla h(x) \). Since \( u \) is convex, we obtain

\[
0 < \sum_{k,l=1}^{n} \partial_{k} \partial_{l} u(x) (\partial_{k} \tilde{h})(\nabla u(x)) (\partial_{l} \tilde{h})(\nabla u(x))
\]

\[
= \langle \nabla H(x), \nabla \tilde{h}(\nabla u(x)) \rangle
\]

\[
= \frac{\langle \nabla h(x), \nabla H(x) \rangle}{|\nabla h(x)|^2} \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle
\]

for all \( x \in \partial \Omega \). In particular, we have \( \langle h(x), \nabla \tilde{h}(\nabla u(x)) \rangle > 0 \) for all points \( x \in \partial \Omega \). The assertion follows now from Corollary 2.9.

**Proposition 2.12.** There exists a positive constant \( C_4 \) such that

\[
\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle \geq \frac{1}{C_4}
\]

for all \( x \in \partial \Omega \).

**Proof.** We define a function \( \chi(x) \) by

\[
\chi(x) = \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle.
\]

By Proposition 2.6 we can find a positive constant \( C_3 \) such that

\[
\left| \sum_{i,j=1}^{n} a_{ij}(x) \partial_{i} \partial_{j} \chi(x) \right| \leq C_3
\]

for all \( x \in \Omega \). Using Lemma 2.3 we obtain

\[
\sum_{i,j=1}^{n} a_{ij}(x) \partial_{i} \partial_{j} (\chi(x) - C_1 C_3 h(x)) \leq 0
\]

for all \( x \in \Omega \). Hence, there exists a point \( x_0 \in \partial \Omega \) such that

\[
\inf_{x \in \Omega} (\chi(x) - C_1 C_3 h(x)) = \inf_{x \in \partial \Omega} \chi(x) = \chi(x_0).
\]

It follows from Proposition 2.11 that \( \chi(x_0) > 0 \). Moreover, we can find a nonnegative real number \( \mu \) such that

\[
\nabla \chi(x_0) = (C_1 C_3 - \mu) \nabla h(x_0).
\]

A straightforward calculation yields

\[
\langle \nabla \chi(x), \nabla \tilde{h}(\nabla u(x)) \rangle = \sum_{i,j=1}^{n} \partial_{i} \partial_{j} h(x) (\partial_{i} \tilde{h})(\nabla u(x)) (\partial_{j} \tilde{h})(\nabla u(x))
\]

\[
+ \sum_{i,j=1}^{n} (\partial_{i} \partial_{j} \tilde{h})(\nabla u(x)) \partial_{i} h(x) \partial_{j} H(x)
\]

(6)
for all $x \in \partial \Omega$. Using (2), we obtain
\begin{equation}
\sum_{i,j=1}^{n} \partial_i \partial_j h(x) (\partial_i \tilde{h})(\nabla u(x)) (\partial_j \tilde{h})(\nabla u(x)) \geq \theta |\nabla \tilde{h}(\nabla u(x))|^2
\end{equation}
for all $x \in \partial \Omega$. Since $\nabla H(x)$ is a positive multiple of $\nabla h(x)$, we have
\begin{equation}
\sum_{i,j=1}^{n} (\partial_i \partial_j \tilde{h})(\nabla u(x)) \partial_i h(x) \partial_j H(x) \geq 0
\end{equation}
for all $x \in \partial \Omega$. Substituting these inequalities into (6) gives
\begin{equation}
\langle \nabla \chi(x), \nabla \tilde{h}(\nabla u(x)) \rangle \geq \theta |\nabla \tilde{h}(\nabla u(x))|^2
\end{equation}
for all $x \in \partial \Omega$. From this, we deduce that
\begin{equation}
(C_1 C_3 - \mu) \chi(x_0) = (C_1 C_3 - \mu) \langle \nabla h(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle
\end{equation}
\begin{equation}
= \langle \nabla \chi(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle
\end{equation}
\begin{equation}
\geq \theta |\nabla \tilde{h}(\nabla u(x_0))|^2.
\end{equation}
Since $\mu \geq 0$ and $\chi(x_0) > 0$, we conclude that
\begin{equation}
\chi(x_0) \geq \frac{\theta}{C_1 C_3} |\nabla \tilde{h}(\nabla u(x_0))|^2 \geq \frac{1}{C_4}
\end{equation}
for some positive constant $C_4$. This completes the proof of Proposition 2.12.

**Lemma 2.13.** Suppose that
\begin{equation}
\sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \leq M |w|^2
\end{equation}
for all $x \in \partial \Omega$ and all $w \in T_x(\partial \Omega)$. Then
\begin{equation}
\sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \leq M \left| w - \frac{\langle \nabla h(x), w \rangle}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle} \nabla \tilde{h}(\nabla u(x)) \right|^2 + C_1 C_2 C_4 \langle \nabla h(x), w \rangle^2
\end{equation}
for all $x \in \partial \Omega$ and all $w \in \mathbb{R}^n$.

**Proof.** Fix a point $x \in \partial \Omega$ and a vector $w \in \mathbb{R}^n$. Moreover, let
\begin{equation}
z = w - \frac{\langle \nabla h(x), w \rangle}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle} \nabla \tilde{h}(\nabla u(x)).
\end{equation}
Clearly, $\langle \nabla h(x), z \rangle = 0$; hence $z \in T_x(\partial \Omega)$. This implies
\begin{equation}
\sum_{k,l=1}^{n} \partial_k \partial_l u(x) (\partial_k \tilde{h})(\nabla u(x)) z_l = \langle \nabla H(x), z \rangle = 0.
\end{equation}
From this we deduce that
\[
\sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l - \sum_{k,l=1}^{n} \partial_k \partial_l u(x) z_k z_l
\]
\[
= \frac{(\nabla h(x), w)^2}{(\nabla h(x), \nabla h(\nabla u(x)))^2} \sum_{k,l=1}^{n} \partial_k \partial_l u(x) \left( \partial_k \tilde{h}(\nabla u(x)) \right) \left( \partial_l \tilde{h}(\nabla u(x)) \right).
\]

It follows from Proposition 2.11 and Proposition 2.12 that
\[
\langle \nabla h(x), w \rangle^2 \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle^2 \leq C_1 C_2 \langle \nabla h(x), w \rangle^2 \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle^2 \leq C_1 C_2 C_4 \langle \nabla h(x), w \rangle^2.
\]

Moreover, we have
\[
\sum_{k,l=1}^{n} \partial_k \partial_l u(x) z_k z_l \leq M |z|^2
\]
by definition of \(M\). Putting these facts together, the assertion follows.

**Proposition 2.14.** There exists a positive constant \(C_9\) such that
\[
\sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \leq C_9 |w|^2
\]
for all \(x \in \partial \Omega\) and all \(w \in T_{x}(\partial \Omega)\).

**Proof.** Let
\[
M = \sup \left\{ \sum_{k,l=1}^{n} \partial_k \partial_l u(x) z_k z_l : x \in \partial \Omega, z \in T_{x}(\partial \Omega), |z| = 1 \right\}.
\]
By compactness, we can find a point \(x_0 \in \partial \Omega\) and a unit vector \(w \in T_{x_0}(\partial \Omega)\) such that
\[
\sum_{k,l=1}^{n} \partial_k \partial_l u(x_0) w_k w_l = M.
\]
We define a function \(\psi : \Omega \to \mathbb{R}\) by
\[
\psi(x) = \sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l
\]
for all \(x \in \Omega\). Moreover, we define functions \(\varphi_1 : \Omega \to \mathbb{R}\) and \(\varphi_2 : \Omega \to \mathbb{R}\) by
\[
\varphi_1(x) = \left| w - \frac{\langle \nabla h(x), w \rangle}{\eta(\langle \nabla h(x), \nabla h(\nabla u(x)) \rangle)} \nabla \tilde{h}(\nabla u(x)) \right|^2
\]
and
\[
\varphi_2(x) = \langle \nabla h(x), w \rangle^2
\]
for all $x \in \Omega$. Here, $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth cutoff function satisfying $\eta(s) = s$ for $s \geq \frac{1}{C_4}$ and $\eta(s) \geq \frac{1}{2C_4}$ for all $s \in \mathbb{R}$.

The inequality (5) implies that
\[
\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \psi(x) \geq 0
\]
for all $x \in \Omega$. Moreover, by Proposition 2.6, there exists a positive constant $C_5$ such that
\[
\left| \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \varphi_1(x) \right| \leq C_5
\]
and
\[
\left| \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \varphi_2(x) \right| \leq C_5
\]
for all $x \in \Omega$. Hence, the function
\[
g(x) = \psi(x) - M \varphi_1(x) - C_1 C_2 C_4 \varphi_2(x) + C_1 C_5 (M + C_1 C_2 C_4) h(x)
\]
satisfies
\[
(7) \quad \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} g(x) \geq 0
\]
for all $x \in \Omega$.

It follows from Proposition 2.12 that
\[
\varphi_1(x) = \left| w - \frac{\langle \nabla h(x), w \rangle}{\langle \nabla h(x), \nabla h(\nabla u(x)) \rangle} \nabla \tilde{h}(\nabla u(x)) \right|^2
\]
for all $x \in \partial \Omega$. Using Lemma 2.13, we obtain
\[
\psi(x) \leq M \varphi_1(x) + C_1 C_2 C_4 \varphi_2(x)
\]
for all $x \in \partial \Omega$. Therefore, we have $g(x) \leq 0$ for all $x \in \partial \Omega$. Using the inequality (7) and the maximum principle, we conclude that $g(x) \leq 0$ for all $x \in \Omega$.

On the other hand, we have $\varphi_1(x_0) = 1$, $\varphi_2(x_0) = 0$, and $\psi(x_0) = M$. From this, we deduce that $g(x_0) = 0$. Therefore, the function $g$ attains its global maximum at the point $x_0$. This implies $\nabla g(x_0) = \mu \nabla h(x_0)$ for some nonnegative real number $\mu$. From this, we deduce that
\[
(8) \quad \langle \nabla g(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle = \mu \langle \nabla h(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle \geq 0.
\]
By Proposition 2.10, we can find a positive constant $C_6$ such that
\[
|\langle \nabla \varphi_1(x), \nabla \tilde{h}(\nabla u(x)) \rangle| \leq C_6
\]
for all \( x \in \partial \Omega \). Hence, we can find positive constants \( C_7 \) and \( C_8 \) such that
\[
\langle \nabla g(x), \nabla \tilde{h}(\nabla u(x)) \rangle = \langle \nabla \psi(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\
- M \langle \nabla \varphi_1(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\
- C_1 C_2 C_4 \langle \nabla \varphi_2(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\
+ C_1 C_5 (M + C_1 C_2 C_4) \langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle \\
\leq \langle \nabla \psi(x), \nabla \tilde{h}(\nabla u(x)) \rangle + C_7 M + C_8
\] (9)

for all \( x \in \partial \Omega \). Combining (8) and (9), we conclude that
\[
\langle \nabla \psi(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle + C_7 M + C_8 \geq 0.
\] (10)

A straightforward calculation shows that
\[
\sum_{k,l=1}^{n} \partial_k \partial_l H(x_0) w_k w_l \\
= \sum_{i,k,l=1}^{n} (\partial_i \tilde{h})(\nabla u(x_0)) \partial_i \partial_k \partial_l u(x_0) w_k w_l \\
+ \sum_{i,j,k,l=1}^{n} (\partial_i \partial_j \tilde{h})(\nabla u(x_0)) \partial_i \partial_k u(x_0) \partial_j \partial_l u(x_0) w_k w_l.
\] (11)

Since \( H \) vanishes along \( \partial \Omega \), we have
\[
\sum_{k,l=1}^{n} \partial_k \partial_l H(x_0) w_k w_l = -\langle \nabla H(x_0), II(w, w) \rangle,
\]
where \( II(\cdot, \cdot) \) denotes the second fundamental form of \( \partial \Omega \) at \( x_0 \). Using the estimate \( |\nabla H(x_0)| \leq C_1 C_2 |\nabla h(x_0)| \), we obtain
\[
\sum_{k,l=1}^{n} \partial_k \partial_l H(x_0) w_k w_l \leq C_1 C_2 |\nabla h(x_0)||H(w, w)|.
\]

Moreover, we have
\[
\sum_{i,k,l=1}^{n} (\partial_i \tilde{h})(\nabla u(x_0)) \partial_i \partial_k \partial_l u(x_0) w_k w_l = \langle \nabla \psi(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle.
\]

Finally, it follows from (3) that
\[
\sum_{i,j,k,l=1}^{n} (\partial_i \partial_j \tilde{h})(\nabla u(x_0)) \partial_i \partial_k u(x_0) \partial_j \partial_l u(x_0) w_k w_l \\
\geq \theta \sum_{i,j,k,l=1}^{n} \partial_i \partial_k u(x_0) \partial_j \partial_l u(x_0) w_i w_j w_k w_l = \theta M^2.
\]
Substituting these inequalities into (11), we obtain
\[ C_1 C_2 \| \nabla h(x_0) \| \| II(w, w) \| \geq \sum_{k,l=1}^{n} \partial_k \partial_l H(x_0) w_k w_l \]
\[ \geq \langle \nabla \psi(x_0), \nabla \tilde{h}(\nabla u(x_0)) \rangle + \theta M^2 \]
\[ \geq \theta M^2 - C_7 M - C_8. \]

Therefore, we have \( M \leq C_9 \) for some positive constant \( C_9 \). This completes the proof of Proposition 2.14.

Corollary 2.15. There exists a positive constant \( C_{10} \) such that
\[ \sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \leq C_{10} |w|^2 \]
for all \( x \in \partial \Omega \) and all \( w \in \mathbb{R}^n \).

Proof. It follows from Lemma 2.13 that
\[ \sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \leq C_9 \left| w - \frac{\langle \nabla h(x), w \rangle}{\langle \nabla h(x), \nabla \tilde{h}(\nabla u(x)) \rangle} \nabla \tilde{h}(\nabla u(x)) \right|^2 \]
\[ + C_1 C_2 C_4 \langle \nabla h(x), w \rangle^2 \]
for all \( x \in \partial \Omega \) and all \( w \in \mathbb{R}^n \). Hence, the assertion follows from Proposition 2.12.

The interior \( C^2 \) estimate follows from Corollary 2.15 and (5):

Proposition 2.16. We have
\[ \sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \leq C_{10} |w|^2 \]
for all \( x \in \Omega \) and all \( w \in \mathbb{R}^n \).

Proof. Fix a unit vector \( w \in \mathbb{R}^n \), and define
\[ \psi(x) = \sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l. \]
The inequality (5) implies that
\[ \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j \psi(x) \geq 0 \]
for all \( x \in \Omega \). Using the maximum principle, we obtain
\[ \sup_{x \in \Omega} \psi(x) = \sup_{x \in \partial \Omega} \psi(x) \leq C_{10}. \]
This completes the proof.

Once we have a uniform $C^2$ bound, we can show that $u$ is uniformly convex:

**Corollary 2.17.** There exists a positive constant $C_{11}$ such that

$$\sum_{k,l=1}^{n} \partial_k \partial_l u(x) w_k w_l \geq \frac{1}{C_{11}} |w|^2$$

for all $x \in \Omega$ and all $w \in \mathbb{R}^n$.

**Proof.** By assumption, the map $f(x) = \nabla u(x)$ is a diffeomorphism from $\Omega$ to $\tilde{\Omega}$. Let $g : \tilde{\Omega} \to \Omega$ denote the inverse of $f$. Then $Dg(y) = [Df(x)]^{-1}$, where $x = g(y)$. Since the matrix $Df(x) = D^2 u(x)$ is positive definite for all $x \in \Omega$, we conclude that the matrix $Dg(y)$ is positive definite for all $y \in \tilde{\Omega}$. Hence, there exists a convex function $v : \tilde{\Omega} \to \mathbb{R}$ such that $g(y) = \nabla v(y)$. The function $v$ satisfies $F(D^2 v(y)) = \frac{n}{2} - F(D^2 u(x))$, where $x = g(y)$. Since $F(D^2 u(x))$ is constant, it follows that $F(D^2 v(y))$ is constant. By Proposition 2.16, the eigenvalues of $D^2 v(y)$ are uniformly bounded from above.

From this, the assertion follows.

In the next step, we show that the second derivatives of $u$ are uniformly bounded in $C^\gamma(\Omega)$. To that end, we use results of G. Lieberman and N. Trudinger [7]. In the remainder of this section, we describe how the problem $(\ast)$ can be rewritten so as to fit into the framework of Lieberman and Trudinger.

We begin by choosing a smooth cutoff function $\eta : \mathbb{R} \to [0, 1]$ such that

$$\begin{cases}
\eta(s) = 0 & \text{for } s \leq 0 \\
\eta(s) = 1 & \text{for } \frac{1}{C_{11}} \leq s \leq C_{10} \\
\eta(s) = 0 & \text{for } s \geq 2C_{10}.
\end{cases}$$

There exists a unique function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying $\psi(1) = \pi$, $\psi'(1) = \frac{1}{2}$, and $\psi''(s) = -\frac{2s}{(1+s)^2} \eta(s) \leq 0$ for all $s \in \mathbb{R}$. Clearly, $\psi(s) = \arctan(s)$ for $\frac{1}{C_{11}} \leq s \leq C_{10}$. Moreover, it is easy to see that $\frac{1}{1+4C_{10}^2} \leq \psi'(s) \leq 1$ for all $s \in \mathbb{R}$. If $M$ is a symmetric $n \times n$ matrix, we define

$$\Psi(M) = \sum_{k=1}^{n} \psi(\lambda_k),$$

where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $M$. Since $\psi''(s) \leq 0$ for all $s \in \mathbb{R}$, it follows that $\Psi$ is a concave function on the space of symmetric $n \times n$ matrices.

We next rewrite the boundary condition. For each point $x \in \partial \Omega$, we denote by $\nu(x)$ the outward-pointing unit normal vector to $\partial \Omega$ at $x$. Similarly, for each point $y \in \partial \tilde{\Omega}$, we denote by $\tilde{\nu}(y)$ the outward-pointing unit normal
vector to \( \partial \tilde{\Omega} \) at \( y \). By Proposition 2.12 there exists a positive constant \( C_{12} \) such that
\[
\langle \nu(x), \tilde{\nu}(\nabla u(x)) \rangle \geq \frac{1}{C_{12}}
\]
for all \( x \in \partial \Omega \).

We define a subset \( \Gamma \subset \partial \Omega \times \mathbb{R}^n \) by
\[
\Gamma = \{ (x, y) \in \partial \Omega \times \mathbb{R}^n : y + t \nu(x) \in \tilde{\Omega} \text{ for some } t \in \mathbb{R} \}.
\]
For each point \( (x, y) \in \Gamma \), we define
\[
\tau(x, y) = \sup \{ t \in \mathbb{R} : y + t \nu(x) \in \tilde{\Omega} \},
\]
and
\[
\Phi(x, y) = y + \tau(x, y) \nu(x) \in \partial \tilde{\Omega}.
\]
If \( (x, y) \) lies on the boundary of the set \( \Gamma \), then
\[
\langle \nu(x), \tilde{\nu}(\Phi(x, y)) \rangle = 0.
\]
We now define a function \( G : \partial \Omega \times \mathbb{R}^n \to \mathbb{R} \) by
\[
G(x, y) = \langle \nu(x), y - \chi(\langle \nu(x), \tilde{\nu}(\Phi(x, y)) \rangle [\langle \nu(x), y \rangle + \tau(x, y)]
\]
for \( (x, y) \in \Gamma \) and
\[
G(x, y) = \langle \nu(x), y \rangle
\]
for \( (x, y) \notin \Gamma \). Here, \( \chi : \mathbb{R} \to [0, 1] \) is a smooth cutoff function satisfying
\[
\chi(s) = 1 \text{ for } s \geq \frac{1}{C_{12}} \text{ and } \chi(s) = 0 \text{ for } s \leq \frac{1}{2C_{12}}.
\]
It is easy to see that \( G \) is smooth. Moreover, we have
\[
G(x, y + t \nu(x)) = G(x, y) + t
\]
for all \( (x, y) \in \partial \Omega \times \mathbb{R}^n \) and all \( t \in \mathbb{R} \). Therefore, \( G \) is oblique.

**Proposition 2.18.** Suppose that \( u : \Omega \to \mathbb{R} \) is a convex function such that \( \nabla u \) is a diffeomorphism from \( \Omega \) to \( \tilde{\Omega} \) and \( F(D^2 u(x)) = c \) for all \( x \in \Omega \). Then \( \Psi(D^2 u(x)) = c \) for all \( x \in \Omega \). Moreover, we have \( G(x, \nabla u(x)) = 0 \) for all \( x \in \partial \Omega \).

**Proof.** It follows from Proposition 2.16 and Corollary 2.17 that the eigenvalues of \( D^2 u(x) \) lie in the interval \( \left[ \frac{1}{C_{11}}, C_{10} \right] \). This implies \( \Psi(D^2 u(x)) = F(D^2 u(x)) = c \) for all \( x \in \Omega \).

It remains to show that \( G(x, \nabla u(x)) = 0 \) for all \( x \in \partial \Omega \). In order to verify this, we fix a point \( x \in \partial \tilde{\Omega} \), and let \( y = \nabla u(x) \in \partial \tilde{\Omega} \). By Proposition 2.11 we have \( \langle \nu(x), \tilde{\nu}(y) \rangle > 0 \). From this, we deduce that \( (x, y) \in \Gamma \) and \( \tau(x, y) = 0 \). This implies \( \Phi(x, y) = y \). Therefore, we have
\[
G(x, y) = \langle \nu(x), y \rangle - \chi(\langle \nu(x), \tilde{\nu}(y) \rangle) \langle \nu(x), y \rangle.
\]
On the other hand, it follows from (12) that \( \chi(\langle \nu(x), \tilde{\nu}(y) \rangle) = 1 \). Thus, we conclude that \( G(x, y) = 0 \).

In view of Proposition 2.18 we may invoke general regularity results of Lieberman and Trudinger. By Theorem 1.1 in [7], the second derivatives of
are uniformly bounded in $C^{\gamma}(\Omega)$ for some $\gamma \in (0,1)$. Higher regularity follows from Schauder estimates.

3. The linearized operator

In this section, we show that all solutions of $\star$ are non-degenerate. To prove this, we fix a real number $\gamma \in (0,1)$. Consider the Banach spaces

$$\mathcal{X} = \left\{ u \in C^{2,\gamma}(\Omega) : \int_{\Omega} u = 0 \right\}$$

and

$$\mathcal{Y} = C^{\gamma}(\Omega) \times C^{1,\gamma}(\partial\Omega).$$

We define a map $G : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ by

$$G(u,c) = \left( F(D^2 u) - c, (\tilde{h} \circ \nabla u)|_{\partial\Omega} \right).$$

Hence, if $(u,c) \in \mathcal{X} \times \mathbb{R}$ is a solution of $\star$, then $G(u,c) = (0,0)$.

**Proposition 3.1.** Suppose that $(u,c) \in \mathcal{X} \times \mathbb{R}$ is a solution to $\star$. Then the linearized operator $D G(u,c) : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ is invertible.

**Proof.** The linearized operator $B = D G(u,c)$ is given by

$$B : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad (w,a) \mapsto (Lw - a, Nw).$$

Here, the operator $L : C^{2,\gamma}(\Omega) \to C^{\gamma}(\Omega)$ is defined by

$$Lw(x) = \text{tr} \left[ (I + (D^2 u(x))^2)^{-1} D^2 w(x) \right]$$

for $x \in \Omega$. Moreover, the operator $N : C^{2,\gamma}(\Omega) \to C^{1,\gamma}(\partial\Omega)$ is defined by

$$Nw(x) = \langle \nabla w(x), \nabla \tilde{h}(\nabla u(x)) \rangle$$

for $x \in \partial\Omega$. Clearly, $L$ is an elliptic operator. Since $u$ is a solution of $\star$, Proposition 2.11 implies that $\langle \nabla \tilde{h}(x), \nabla \tilde{h}(\nabla u(x)) \rangle > 0$ for all $x \in \partial\Omega$. Hence, the boundary condition is oblique.

We claim that $B$ is one-to-one. To see this, we consider a pair $(w,a) \in \mathcal{X} \times \mathbb{R}$ such that $B(w,a) = (0,0)$. This implies $Lw(x) = a$ for all $x \in \Omega$ and $N w(x) = 0$ for all $x \in \partial\Omega$. Hence, the Hopf boundary point lemma (cf. [5], Lemma 3.4) implies that $w = 0$ and $a = 0$.

It remains to show that $B$ is onto. To that end, we consider the operator

$$\tilde{B} : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad (w,a) \mapsto (Lw, Nw + w + a).$$

It follows from Theorem 6.31 in [5] that $\tilde{B}$ is invertible. Moreover, the operator

$$\tilde{B} - B : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}, \quad (w,a) \mapsto (a, w + a)$$

is compact. Since $\tilde{B}$ is one-to-one, it follows from the Fredholm alternative (cf. [5], Theorem 5.3) that $B$ is onto. This completes the proof.
4. Existence of a solution to (∗)

In this section, we prove the existence of a solution to (∗). To that end, we employ the continuity method. Let Ω and ˜Ω be uniformly convex domains in R^n with smooth boundary. By Proposition A.1, we can find a smooth function h : Ω → R with the following properties:

- h is uniformly convex
- h(x) = 0 for all x ∈ ∂Ω
- If s is sufficiently close to inf Ω h, then the sub-level set \{x ∈ Ω : h(x) ≤ s\} is a ball.

Similarly, there exists a smooth function ˜h : ˜Ω → R such that:

- ˜h is uniformly convex
- ˜h(y) = 0 for all y ∈ ∂˜Ω
- If s is sufficiently close to inf ˜Ω ˜h, then the sub-level set \{y ∈ ˜Ω : ˜h(y) ≤ s\} is a ball.

Without loss of generality, we may assume that inf Ω h = inf ˜Ω ˜h = −1. For each t ∈ (0, 1], we define

\[ Ω_t = \{x ∈ Ω : h(x) ≤ t - 1\}, \quad ˜Ω_t = \{y ∈ ˜Ω : ˜h(y) ≤ t - 1\}. \]

Note that Ω_t and ˜Ω_t are uniformly convex domains in R^n with smooth boundary. We then consider the following problem (cf. [1]):

\[ (∗_t) \text{ Find a convex function } u : Ω → R \text{ and a constant } c ∈ (0, \frac{\pi}{2}) \text{ such that } \nabla u \text{ is a diffeomorphism from } Ω_t \text{ to } ˜Ω_t \text{ and } F(D^2 u(x)) = c \text{ for all } x ∈ Ω_t. \]

If t ∈ [0, 1) is sufficiently small, then Ω_t and ˜Ω_t are balls in R^n. Consequently, (∗_t) is solvable if t ∈ (0, 1] is sufficiently small. In particular, the set

\[ I = \{t ∈ (0, 1] : (∗_t) \text{ has at least one solution}\} \]

is non-empty. It follows from the a-priori estimates in Section 2 that I is a closed subset of (0, 1]. Moreover, Proposition 3.1 implies that I is an open subset of (0, 1]. Consequently, I = (0, 1]. This completes the proof of Theorem 1.1.

5. Uniqueness

In this final section, we show that the solution to (∗) is unique up to addition of constants. To that end, we use a trick that we learned from J. Urbas.

As above, let Ω and ˜Ω be uniformly convex domains in R^n with smooth boundary. Moreover, suppose that (u, c) and (ˆu, ˆc) are solutions to (∗). We claim that the function ˆu − u is constant.

Suppose this is false. Without loss of generality, we may assume that ˆc ≥ c. (Otherwise, we interchange the roles of u and ˆu.) For each point
$x \in \Omega$, we define a symmetric $n \times n$-matrix $B(x) = \{b_{ij}(x) : 1 \leq i, j \leq n\}$ by

$$B(x) = \int_0^1 \left[ I + (s D^2 \hat{u}(x) + (1 - s) D^2 u(x))^2 \right]^{-1} ds.$$ 

Clearly, $B(x)$ is positive definite for all $x \in \Omega$. Moreover, we have

$$\sum_{i,j=1}^n b_{ij}(x) (\partial_i \partial_j \hat{u}(x) - \partial_i \partial_j u(x))$$

$$= F(D^2 \hat{u}(x)) - F(D^2 u(x)) = \hat{c} - c \geq 0$$

for all $x \in \Omega$. By the maximum principle, the function $\hat{u} - u$ attains its maximum at a point $x_0 \in \partial \Omega$. By the Hopf boundary point lemma (see [5], Lemma 3.4), there exists a real number $\mu > 0$ such that

$$\nabla \hat{u}(x_0) - \nabla u(x_0) = \mu \nabla h(x_0).$$

Using Proposition 2.11, we obtain

$$\langle \nabla \hat{u}(x_0) - \nabla u(x_0), \nabla \hat{h}(\nabla u(x_0)) \rangle = \mu \langle \nabla h(x_0), \nabla \hat{h}(\nabla u(x_0)) \rangle > 0.$$ 

On the other hand, we have

$$\langle \nabla \hat{u}(x_0) - \nabla u(x_0), \nabla \hat{h}(\nabla u(x_0)) \rangle \leq \hat{h}(\nabla \hat{u}(x_0)) - \hat{h}(\nabla u(x_0)) = 0$$

since $\hat{h}$ is convex. This is a contradiction. Therefore, the function $\hat{u} - u$ is constant.

**Appendix A. The construction of the boundary defining function**

The following result is standard. We include a proof for the convenience of the reader.

**Proposition A.1.** Let $\Omega$ be a uniformly convex domain in $\mathbb{R}^n$ with smooth boundary. Then there exists a smooth function $h : \Omega \to \mathbb{R}$ with the following properties:

- $h$ is uniformly convex
- $h(x) = 0$ for all $x \in \partial \Omega$
- If $s$ is sufficiently close to $\inf \Omega h$, then the sub-level set $\{x \in \Omega : h(x) \leq s\}$ is a ball.

**Proof.** Let $x_0$ be an arbitrary point in the interior of $\Omega$. We define a function $h_1 : \Omega \to \mathbb{R}$ by

$$h_1(x) = \frac{d(x, \partial \Omega)^2}{4 \text{diam}(\Omega)} - d(x, \partial \Omega).$$

Since $\Omega$ is uniformly convex, there exists a positive real number $\varepsilon$ such that $h_1$ is smooth and uniformly convex for $d(x, \partial \Omega) < \varepsilon$. We assume that $\varepsilon$ is chosen so that $d(x_0, \partial \Omega) > \varepsilon$. We next define a function $h_2 : \Omega \to \mathbb{R}$ by

$$h_2(x) = \frac{\varepsilon d(x_0, x)^2}{4 \text{diam}(\Omega)^2} - \frac{\varepsilon}{2}.$$
For each point \( x \in \partial \Omega \), we have \( h_1(x) = 0 \) and \( h_2(x) \leq -\frac{\varepsilon}{2} \). Moreover, for 
\( d(x, \partial \Omega) \geq \varepsilon \), we have \( h_1(x) \leq -\frac{3\varepsilon}{4} \) and \( h_2(x) \geq -\frac{\varepsilon}{2} \).

Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a smooth function satisfying \( \Phi''(s) \geq 0 \) for all \( s \in \mathbb{R} \) and \( \Phi(s) = |s| \) for \( |s| \geq \frac{\varepsilon}{16} \). We define a function \( h : \Omega \to \mathbb{R} \) by 
\[
 h(x) = \frac{h_1(x) + h_2(x)}{2} + \Phi \left( \frac{h_1(x) - h_2(x)}{2} \right). 
\]

If \( x \) is sufficiently close to \( \partial \Omega \), then we have \( h(x) = h_1(x) \). In particular, we have \( h(x) = 0 \) for all \( x \in \partial \Omega \). Moreover, we have \( h(x) = h_2(x) \) for \( d(x, \partial \Omega) \geq \varepsilon \). Hence, the function \( h \) is smooth and uniformly convex for 
\( d(x, \partial \Omega) \geq \varepsilon \).

We claim that the function \( h \) is smooth and uniformly convex on all of \( \Omega \). To see this, we consider a point \( x \) with \( d(x, \partial \Omega) < \varepsilon \). The Hessian of \( h \) at the point \( x \) is given by
\[
\partial_i \partial_j h(x) = \frac{1}{2} \left[ 1 + \Phi' \left( \frac{h_1(x) - h_2(x)}{2} \right) \right] \partial_i \partial_j h_1(x) 
\+ \frac{1}{2} \left[ 1 - \Phi' \left( \frac{h_1(x) - h_2(x)}{2} \right) \right] \partial_i \partial_j h_2(x) 
\+ \frac{1}{4} \Phi'' \left( \frac{h_1(x) - h_2(x)}{2} \right) \left( \partial_i h_1(x) - \partial_i h_2(x) \right) \left( \partial_j h_1(x) - \partial_j h_2(x) \right). 
\]

Note that \( |\Phi'(s)| \leq 1 \) and \( \Phi''(s) \geq 0 \) for all \( s \in \mathbb{R} \). Since \( h_1 \) and \( h_2 \) are uniformly convex, it follows that \( h \) is uniformly convex.

It remains to verify the last statement. The function \( h \) attains its minimum at the point \( x_0 \). Therefore, we have \( \inf_{\Omega} h = -\frac{\varepsilon}{2} \). Suppose that \( s \) is a real number satisfying
\[
-\frac{\varepsilon}{2} < s < \frac{\varepsilon (d(x_0, \partial \Omega) - \varepsilon) - \varepsilon}{4 \text{diam}(\Omega)^2}. 
\]

Then we have \( \{ x \in \Omega : h(x) \leq s \} = \{ x \in \Omega : h_2(x) \leq s \} \). Consequently, the set \( \{ x \in \Omega : h(x) \leq s \} \) is a ball. This completes the proof of Proposition A.1.

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