An Exact Algorithm for finding Maximum Induced Matching in Subcubic Graphs*

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Abstract. The Maximum Induced Matching problem asks to find the maximum $k$ such that, given a graph $G = (V, E)$, can we find a subset of vertices $S$ of size $k$ for which every vertices $v$ in the induced graph $G[S]$ has exactly degree 1. In this paper, we design an exact algorithm running in $O(1.2630n)$ time and polynomial space to solve the Maximum Induced Matching problem for graphs where each vertex has degree at most 3. Prior work solved the problem by finding the Maximum Independent Set using polynomial space in the line graph $L(G^2)$; this method uses $O(1.3139n)$ time.

1 Introduction

Most of the graph problems can be classified into either subset, permutation or partition problems. In this paper, we seek to study a subset problem known as the Maximum Induced Matching problem for subcubic graph and we approach it by designing an exact algorithm for it. A more generic problem of this is known as the Maximum $r$-Regular Induced Subgraph problem. Given a graph $G = (V, E)$, the Maximum $r$-Regular Induced Subgraph seeks to find a subset of vertices $S \subseteq V$, of maximum size such that the induced subgraph $G[S]$ is $r$-regular. When $r = 0$, then this is also known as the Maximum Independent Set problem. On the other hand, when $r = 1$, then we have the Maximum Induced Matching (MIM) problem.

MIM has a number of applications in real life. For example, it has applications in areas such as risk-free marriages [18], VLSI design and network flow problems [11]. Therefore, MIM has been studied heavily. It is known that solving MIM on special instances such as trees and interval graphs [11], chordal graphs [1], circular arc graphs [10] etc. can be done in polynomial time. On the other hand, it is also known to be NP-hard in bipartite graphs with maximum degree 4, planar 3-regular graphs, planar bipartite graphs with degree 2 vertices on one component and degree 3 vertices in the other component, Hamiltonian graphs, claw-free graphs, line graphs and regular graphs [4,13,14].

Gupta, Raman and Saurabh gave a non-trivial algorithm to solve MIM in $O^*(1.6957^n)$ time, and then improved it to $O(1.4786^n)$ time in the same paper [12] using polynomial space. Chang,
Hung and Miau also gave an $O(1.4786^n)$ time algorithm to solve MIM [3] and using polynomial space as well. Chang, Chen and Hung then gave an algorithm running in $O^*(1.4658^n)$ time to solve MIM and later improved it to $O^*(1.4321^n)$ time in the same paper [2]. Furthermore, Xiao and Tan gave first an algorithm running in $O^*(1.4391^n)$ [20] and then improved in a further paper [21] by giving two algorithms, one that runs in $O^*(1.4231^n)$ time using polynomial space, the other using $O(1.3752^n)$ time and exponential space.

In this paper, we deal with MIM for subcubic graphs (graphs with maximum degree 3). In the papers mentioned above, most of the authors deal with low degree graphs by constructing $L(G^2)$ (Line Graph of $G^2$) and then applying an algorithm solving the Maximum Independent Set problem to get the required results. If we were to do so, this allows us to solve the problem in $O(1.3139^n)$ time (more details later). Instead, we take a different approach to design a faster branch and bound algorithm, running in time $O(1.2630^n)$ time and using polynomial space; we utilise a variant of the Monien Preis bisection cut method as described below. Note that this is a polynomial space algorithm; it is known that there is an algorithm using exponential space which solves the problem in $O(1.2010^n)$; Kumar and Kumar [16, Theorems 2 and 4] provide such a type of algorithm and the algorithm for Theorem 2 uses exponential space and that for Theorem 4 directly applies that of Theorem 2; however, they maximise the subgraph which contains at least one component of size 2 and no component which is larger; this is phrased in terms of number of nodes removed. The pathwidth result of Fomin and Høie [6] can also be used to get the upper bound of Kumar and Kumar [16, Theorem 4].

2 Preliminaries

Given an undirected graph $G = (V, E)$, we let $n$ be the number of nodes $|V|$ and $m$ be the number of edge $|E|$. Given nodes $v, w$ we write $v - w$ for $v, w$ being the endpoints of an edge and call $w$ a neighbour of $v$. The degree of a node is the number of its neighbours and the degree of a graph is the maximum degree of a node in the graph. Graphs of degree up to three are called subcubic graphs. The line graph of $G$, denoted as $L(G)$, is constructed from $G$ by having the set of vertices of $L(G)$ as $E$ and they are adjacent if they are adjacent edges in $G$. The graph $G^2$ is a graph of $V$, with edges between two vertices $u$ and $v$ if there is a path of length at most 2 between $u$ and $v$ in $G$.

A set $B \subseteq E$ of edges is called a bisection cut if there are two disjoint subsets $V_1, V_2$ whose union is $V$ such that all edges in $E - B$ have either both endpoints in $V_1$ or both endpoints in $V_2$. The bisection cut is balanced iff $V_1$ and $V_2$ have each at most $n/2 + 1$ nodes.

**Theorem 1 (Monien and Preis [17]).** For any $\varepsilon > 0$, there is a value $n(\varepsilon)$ such that the bisection cut of any subcubic graph $G = (V, E)$ with $|V| > n(\varepsilon)$ is at most $(1/6 + \varepsilon)|V|$. A possible balanced bisection cut and the corresponding components $V_1, V_2$ can be found in polynomial time.

Note that the above theorem extends to graphs of maximum degree 3 [9].

**Corollary 2.** For any $\varepsilon > 0$ there is a value $\kappa \geq 3$ such that whenever the graph has $k > \kappa$ nodes of degree 3 there is a bisection cut which has on both sides of the cut have between $k/2 - 1$ and $k/2 + 1$ (inclusively) nodes of degree 3 and the bisection cut $B$ contains at most $(1/6 + \varepsilon) \cdot k$ edges.
Proof. The result is based on the following algorithm which is straight-forward to verify:

Given a connected graph \((V, E)\), let \(V'\) be the set of all nodes of degree 3 in \(V\) and let \(E' = \{(v, w) : v, w \in V' \text{ and } v \neq w \text{ and there is a sequence of edges connecting } v, w \text{ such that all nodes involved other than } v, w \text{ themselves have degree 2}\}\). Such an edge is called a “double-edge” in the case that there are two sequences of the above type. Note that there are no two neighbouring double-edges in the graph. Now one applies the Theorem of Monien and Preis for an \(\varepsilon\) chosen below in the algorithm and the corresponding \(\kappa\) to the graph \((V', E')\) and gets a bisection cut \(B'\). Whenever there is a double edge in the bisection cut, this double edge has two neighbours which are single edges. One moves now one endpoint from the larger side to the smaller side and so replaces the double edge by a single edge; the outcome of each such operation is that the number of nodes of degree 3 differ at most by 1 from \(k/2\). After this is done, no edge in the so modified \(B'\) is a double edge. Now one picks for every edge of the modified \(B'\) exactly one edge of the sequence of edges representing the edge in \(B'\) and puts this edge it into the new \(B\) of the algorithm. The so resulting \(B\) has then at most \((1/6 + \varepsilon) \cdot k\) edges. 

Remark 3. An illustration of this method is the following graph:

```
 b-b     b-b     b-b
 / \     / \     / \
 b a1-a2 a3-a4 b
 \ / \     / \     / \
 b-b b-b b-b
```

This graph \((V, E)\) has the following structure \((V', E')\):

\[
a1-a2=a3-a4
\]

Here the double edge between \(a2\) and \(a3\) is in the invocation of the Theorem of Monien and Preis considered a single edge and \(B' = \{a2 = a3\}\). Thus one moves the edge one to the side and then obtains \(B = \{a1 - a2\}\) splitting only a single edge. Both halves have 1 and 3 nodes of degree 3, respectively, but differ only by 1 from the average of 2. Here \(\kappa\) is too small to get a value near 3.

For the choice of \(\varepsilon\) and \(\kappa\), one chooses \(\varepsilon\) so small and \(\kappa\) so large, that the resulting approximation of the number of degree nodes in each half divided by the size of the bisection cut is so near to 3, that using 3 when computing the branching factors below instead of that approximation does not lead to different values. Note that all branching factors are strictly uprounded and therefore one can afford a tiny deviation below 0.0000001 or so from the actual value 3 without changing the numerical branching numbers.

We will be using Corollary 2 in the design of our algorithm.

3 Background on Branching Algorithms

In this section, we will introduce some definitions that we will use repeatedly in this paper and also the techniques needed to understand the analysis of the branch and bound algorithm.
Branch and bound algorithms are recursive in nature and have two kinds of rules associated with them: Simplification and branching rules. Simplification rules help us to simplify a problem instance or to act as a case to terminate the algorithm. Branching rules on the other hand, help us to solve a problem instance by recursively solving smaller instances of the problem. To help us to better understand the execution of a branch and bound algorithm, the notion of a search tree is commonly used. We can assign the root node of the search tree to be the original problem, while subsequent child nodes are assigned to be the smaller instances of the problem whenever we invoke a branching rule. For more information of this area, we refer to the textbooks written by Fomin and Kratsch [7] and by Gaspers [8].

Let \( \mu \) denote our parameter of complexity. To analyse the runtime of an branch and bound algorithm, one in fact just needs to bound the number of leaves generated in the search tree. This is due to the fact that the complexity of such algorithm is proportional to the number of leaves, modulo polynomial factors, that is, \( O(poly(|V|, |E|, \mu) \times \text{number of leaves in the search tree}) = O^*(\text{number of leaves in the search tree}) \), where \( poly(|V|, |E|, \mu) \) is some polynomial based on \(|V|, |E|\) and \( \mu \), while \( O^*(g(\mu)) \) is the class of all functions \( f \) bounded by some polynomial \( p(\cdot) \) times \( g(\mu) \).

Now given any branching rule, let \( r \geq 2 \) be the number of instances generated from this rule ("outcomes" or "alternatives"). Let \( t_1, t_2, ..., t_r \) be the change of measure for each instance for a branching rule, then we have the linear recurrence \( T(\mu) \leq T(\mu - t_1) + T(\mu - t_2) + ... + T(\mu - t_r) \). We can employ techniques in [15] to solve it. The number of leaves generated of this branching rule is therefore given as \( \beta \), where \( \beta \) is the unique positive root of \( x^{-t_1} + x^{-t_2} + ... + x^{-t_r} = 1 \). For ease of writing, we denote the branching factor of this branching rule as \( \tau(t_1, t_2, ..., t_r) = \beta \) and \( (t_1, t_2, ..., t_r) \) is also known as the branching vector.

If there are \( k \) branching rules in the entire branch and bound algorithm, with each having a branching factor of \( \beta_1, \beta_2, ..., \beta_k \), then the entire algorithm runs in \( O(c^\mu) \), where \( c = \max\{\beta_1, \beta_2, ..., \beta_k\} \).

Finally, correctness of branch and bound algorithms usually follows from the fact that all cases have been covered.

For Cameron’s result [1], one has of course to invoke the latest algorithm and best bound of Maximum Independent Set for a fair comparison.

As mentioned in the Introduction, we can use an algorithm solving the Maximum Independent Set on \( L(G^2) \), where the graph \( G \) is a subcubic graph. The fastest algorithm to solve Maximum Independent Set we are aware of runs in \( O^*(1.1996^n) \) time is by Xiao and Nagamochi [19].

**Theorem 4 (Cameron [1]).** Given a subcubic graph \( G \), we can solve MIM for \( G \) using an algorithm to solve Maximum Independent Set in \( O(1.3139^n) \) time.

**Proof.** Since \( G \) is subcubic, by the hand-shaking lemma, we know that \( m \leq \frac{3}{2}n \). Now we construct the graph \( L(G^2) \), where the edges of \( G \) form the nodes of the new graph and where two nodes of the new graph are connected by a new edge iff either the two edges share a node or the two edges have each some node \( v, w \), respectively, such that \((v, w)\) is an edge in \( G \). As the number of vertices in \( L(G^2) \) is then \( m \), applying Xiao and Nagamochi’s algorithm to \( L(G^2) \) will then take \( O^*(1.1996^m) \subseteq O^*(1.1996^\frac{3}{2}n) \subseteq O(1.3139^n) \) time. Note that the maximum degree
of $L(G^2)$ is 12 and therefore improved algorithms for Maximum Independent Set at degrees of small degree do not apply.

4 The Algorithm, the Measure and The Rules

This section will give the outline of the algorithm and the rules. The overall goal is to improve the bound based on the methods of Cameron [1] which reduces Maximum Induced Matching on subcubic graphs to independent set on line graphs giving complexity $O(1.3139^n)$ to the below algorithm of $O(1.2630^n)$ time.

To tighten the analysis of the algorithm, we apply the Measure and Conquer technique by Fomin, Grandoni and Kratsch [5]. Here, we will be using a standard weight based measure where we assign to each node a weight according to its degree in dependence of a parameter $s$ with $1/2 \leq s \leq 1$: Nodes of degree 3 have weight 1, nodes of degree 2 have weight $s$, all other nodes (degrees 0 and 1) have weight 0. The exact value of $s$ will be chosen later.

Let the set of nodes be $V = \{v_1, v_2, ..., v_n\}$. We assign a weight $w_i$ to node $v_i$ as outlined above. Now we define our measure $\mu = \sum_i w_i$. Note that $\mu \leq n$ by definition. Therefore, this means that if our algorithm runs in time $O^*(c^\mu)$ for some $c > 1$, then we know that our algorithm runs in time $O^*(c^n)$ because $O^*(c^\mu) \subseteq O^*(c^n)$. The design of such a measure allows us to take advantage of the fact that whenever a node is removed, then each neighbour loses some weight due to its degrees going down. In particular a neighbour of degree 2 loses weight $s$ and a neighbour of degree 3 loses at least weight $1 - s$.

The overall algorithm consists of the following ingredients: A Monien Preis subroutine to generate a bisection cut. Simplification Rules which can be applied at all positions (except of S4). Branching rules B2.1 and B2.2 which have only two possible outcomes (“alternatives”) and which are applied in the case that one side of an edge on the bisection cut satisfies the corresponding conditions. Branching rules B3.1, B3.2, B3.3 which have 3 alternatives and which again require that an edge in the bisection cut is removed in the process. Branching rule B4.1 has up to four alternatives and removes a node being the end point of two edges in the bisection cut.

The overall algorithm with original input $(V_{inp}, E_{inp})$ is recursive and each recursive step has the following inputs: Current graph $(V, E)$ where $V \subseteq V_{inp}$ and $E = \{(v, w) \in E_{inp} : v, w \in V\}$; a set $S \subseteq E_{inp}$ of edges which are already selected for the final maximum subgraph of degree 1 and all edges $(v, w) \in S$ satisfy that the neither $v$, $w$ nor their neighbours in $(V_{inp}, E_{inp})$ are in $V$; current edges $B$ in the bisection cut to be processed, $B \subseteq E$. The simplification and branching rules are listed below the algorithm. We use a recursive algorithm AlgoMIM$(V, E, S, B)$ and the initial call is AlgoMIM$(V_{inp}, E_{inp}, \emptyset, \emptyset)$.

Input: a set of vertices $V$, a set of edges $E$, a set $S$, a bisection cut $B$.
Output: A set $S$ where each node in the induced subgraph $G[S]$ has degree 1.
Function AlgoMIM$(V, E, S, B)$ selects the first case of the following which applies:

1. If $V = \emptyset$ then return $S$.
2. If $B = \emptyset$ and $V$ consists of $\ell \geq 2$ connected components $V_1, V_2, \ldots, V_\ell$ then let $E_h$ be the edges of the component given by $V_h$ and compute $S_h = \text{AlgoMIM}(V_h, E_h, \emptyset, \emptyset)$ for $h = 1, 2, \ldots, k$.
   Return the set $S \cup S_1 \cup S_2 \cup \ldots \cup S_k$. 

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3. If $B = \emptyset$ and $(V, E)$ is connected and has at least $\kappa + 1$ nodes of degree 3 then one computes a new Monien Preis bisection cut $B$ according to the bisection cut $B$ given by Corollary 2 which can be computed in polynomial time (as $\varepsilon$ and $\kappa$ are fixed). Let $S' = \text{AlgoMIM}(V, E, S, B)$ and return $S'$.

4. If one of the simplification rules S1, S2, S3, S4 applies then adjust $V$ and $S$ according to the first applying simplification rule and update $E = \{(v, w) \in E : v, w \text{ are in the new } V\}$ and $B = B \cap E$, let $S' = \text{AlgoMIM}(V, E, S, B)$ and return $S'$.

5. If $B \neq \emptyset$ then do the first of the branching rules B2.1, B2.2, B3.1, B3.2, B3.3, B4.1 which applies in this order; note that each branching rule in all outcomes reduces the size of $B$ by at least one. Now when a branching rule produces alternatives $(V_1, E_1, S_1, B_1), \ldots, (V_k, E_k, S_k, B_k)$ then compute for each alternative $S'_h = \text{AlgoMIM}(V_h, E_h, S_h, B_h)$ and return among these sets $S'_h$ one of maximum size — if there are several maximum ones, it does not matter which one is chosen.

Before giving the rules, it is noted that rules specify several alternatives and for each alternative they specify the following items: which nodes are removed from $V$ and which edges are added into $S$, note that the endpoints of each edge added to $S$ as well as their neighbours are required to be removed from $V$. $E$ and $B$ are adjusted as indicated in the above algorithm. For the verification that alternatives cover all possible cases as well for the calculation of branching numbers, the following two principles are useful.

**Subset Principle.** If a branching rule produces (among perhaps others) the alternatives $(V', E', S', B')$ and $(V'', E'', S'', B'')$ to update $(V, E, S, B)$ and if $V - V'' \subseteq V - V'$ and $|S''| \leq |S'|$ then one can omit the alternative $(V'', E'', S'', B'')$ from the choices considered.

**Proof-Sketch of Soundness of Subset Principle.** The reason is that $V'' \subseteq V'$ and $|S''| \leq |S'|$ so that the output of $\text{AlgoMIM}(V'', E'', S'', B'')$ cannot produce a larger graph than the output of $\text{AlgoMIM}(V', E', S', B')$, as at the optimal use the subset $V''$ of $V'$ does not give more new edges in the output than $V'$ and also the number of old edges in $S''$ is bounded by that of old edges in $S'$.

An application of the Subset Principle is that one can first produce a verifiably exhaustive list of alternatives and then remove some of these with the subset principle. For simplification rule S3 all but one alternatives will be removed.

**Budget Principle.** If an alternative $(V', E', S', B')$ in a branching rule allows a subsequent simplification rule then one can assume that this simplification rule is done directly after the alternative and budget the additional gains of the measure into the corresponding alternative, giving a better branching factor.

Technically, one would have to code this in the algorithm (though here it is only in the verification), this would however make the algorithm less readable, as one would need more cases to expand this into one reasoning. The budget principle is only applied in few subcases of the verification and calculation of $\tau$-numbers of the three-way branching rules.

**On Diagrams and Conventions.** In the subsequent text, nodes $a, a', a'', \ldots$ have always degree 3, nodes $b, b', b'', \ldots$ have degree 2, nodes $c, c', c'', \ldots$ have degree 1, nodes $d, d', d'', \ldots$ have
either degree 2 or degree 3 and nodes \(e, e', e'', \ldots\) are not apriori specified. Furthermore, \(a - b\) denotes that there is an edge from \(a\) to \(b\) and \(a - | - b\) denotes that there is an edge from \(a\) to \(b\) which is part of \(B\).

**Simplification Rules.** Simplification rules \(S1\) and \(S2\) handle isolated components with \(\kappa\) or less nodes of degree 3, where \(\kappa\) is a suitably chosen constant. Simplification rule \(S3\) removes various special cases which do not need to be considered in branching rules, for examples see after the rule below. Branching rule \(S4\) abstains from branching an edge as shown below but instead moves the node \(c\) onto the other side of the bisection cut and so \(S4\) avoids a bad case in branching.

\[
S1. e \quad c-c \quad c-b-c \quad c-b-b-c \quad b-b-\ldots-b-b \quad \text{All nodes}
\quad | \quad | \quad \text{have degree}
\quad c-b-b-b-c \quad c-b-\ldots-b-c \quad b-b-\ldots-b-b \quad 0, 1 \text{ or } 2.
\]

\[
S2. \text{Constantly many nodes of degree 3}
\]

\[
S3. \quad \begin{array}{cccc}
| & | & | & |
\end{array}
\quad c \quad b \quad a-c \quad a-c
\quad / \quad / \quad / \quad / \\
\quad \ldots-d-b-c \quad \ldots-d-a-c \quad \ldots-d-a-b \quad \ldots-d-b \quad \ldots-d-a-c
\]

\[
S4. \quad \ldots-d-|-c \quad (c \text{ is a node of measure } 0 \text{ on other side of bisection cut})
\]

**Simplification Rule S1.** If there is a component without nodes of degree 3, then simple computations show that degree 0 nodes cannot contribute to \(S\), an edge of the form \(c-c\) goes straight into \(S\) and a line of the form ending with \(c\) having \(3k+1, 3k+2, 3k+3\) edges contributes \(k+1\) edges to \(S\) (by having a pause of two edges between any two edges going into \(S\)) and a cycle of \(3k+3, 3k+4, 3k+5\) edges contributes \(k+1\) edges (again by having a pause of at least two edges between any two edges going into \(S\)).

**Simplification Rule S2.** If there are at most \(\kappa\) nodes of degree 3, then one can compute in time \(3^\kappa \cdot \text{Poly}(n)\) an optimal \(S'\) restricted to this component and then one replaces \(S\) by \(S \cup S'\) and removes the component from \((V, E)\).

**Simplification Rule S3.** Assume that \(d\) is a node, \(D\) a proper nonempty subset of the set of its neighbours and that \(C\) be the nodes outside \(\{d\} \cup D\) which are neighbours of the nodes in \(D\). If there is at least one edge with both endpoints in \(C \cup D\) and if all edges with an endpoint in \(C \cup D\) has one end in \(D\) and the other one in \(C \cup D \cup \{d\}\) then one puts into \(S\) a maximal and legal set of edges with both endpoints in \(C \cup D\) and removes \(\{d\} \cup C \cup D\) from \(V\).

**Simplification Rule S4.** If due to removal of other nodes it happens that there is an edge in \(B\) which connects a degree 1 node with a node on the other side of the bisection cut, then move the degree 1 node to the other side and remove the edge from \(B\).

The verification of the simplification rule \(S1\) is standard and omitted.
Possible Polynomial Time Algorithm to Implement S2. As long as there is a degree 3 node $a$, one picks a neighbour $e$ and branches with $a - e$ being an edge to go into $S$ versus only $a$ being removed from the graph versus only $e$ being removed from the graph; in the first case, $a - e$ goes into $S$ and all neighbours of $a$ and $e$ are removed together with these nodes. This algorithm processes $\kappa$ three-way branchings giving a $3^{\kappa}$-sized tree such that each leave consists of up to $3\kappa$ components without nodes of degree 3 which can be treated by using simplification rule S1. The algorithm then takes the maximum $S'$ produced by any of these $3^{\kappa}$ alternatives.

Proof of Soundness of Simplification Rule S3. Note that if $d$ is endpoint of an edge selected for $S$ then all nodes in $D$ are removed (and perhaps some in $C$) and no edge remains with an endpoint in $C$ and therefore no further edge can be put into $S$ than the one mentioned; on the other hand, there is an edge between two nodes of $D \cup C$ and thus if one selects this edge for $S$ then only the nodes in $C \cup D \cup \{d\}$ have to be removed. Thus by the subset principle, one puts this edge into $S$ and removes the set $C \cup D \cup \{d\}$ from $V$.

Note that in some few situations one can put two edges into $S$, but then $d$ is also not part of any of these edges. The graphical enumeration in S3 is not covering all arising cases.

Proof-Sketch of Soundness of S4. Note that the invariance is that on both sides of the bisection cut the number of degree 3 nodes differs at most by 2 at the start. The moving over of a degree 1 node does not destroy this balance, but reduces the duty to branch by one edge.

Examples for Simplification Rule S3. Assume the following four situations:

1. $e-d-b-c$
2. $e-d-a-c$
3. $e-d-a'-d'-b'-c'$
4. $e-d-a'-c'$

The nodes $d, d'$ are those which have in the rule S3 the name $d$. After applications of S3 only nodes $e, e'$ remain from the above displayed parts of the graph.

In the first situation, $D = \{b\}$ and $C = \{e\}$. The edge $b - c$ goes into $S$ and $d, b, c$ are removed.

In the second situation, $D = \{a\}$ and $C = \{c, c'\}$. One of the edges $a - c, a - c'$ go into $S$ and nodes $a, c, c', d$ are all removed.

In the third situation, there are two subsequent applications of S3. The immediate application uses $d'$, $D = \{b'\}$ and $C = \{c'\}$ and puts $b' - c'$ into $S$ and removes $b', c', d'$ which makes $a$ to become a degree 2 node. The next application uses $d$ has $D = \{a\}$ and $C = \{c\}$, puts $a - c$ into $S$ and removes $a, c, d$.

The fourth situation has $D = \{a, a'\}$ and $C = \{b, c, c'\}$ and the edges $a - c, a' - c'$ are put into $S$. The nodes $a, a', b, c, c', d$ are all removed from the graph.

Branching Rules. All branching rules will be set up such that in all alternatives at least one edge of the bisection cut is removed. Here the bisection cut is computed from Corollary 2, not from the original theorem. For accounting purposes, $V'$ is the set of degree 3 nodes in the current instance, then the bisection cut $B$ is set up in such a way that it contains $|V'|/6 + o(|V'|)$ edges and splits the graph into two halves such that each of them having at most $|V'|/2 + 1$ degree 3 nodes. Now the average $|V'|/2/|B|$ tends to 3 for arbitrary large $V'$. Thus the constants $\varepsilon, \kappa$
are selected such that the difference between $3$ and $(|V'|/2 - 1)/|B|$ is so small that computing the branching factors with the approximation $3$ or the latter constant does not influence the uprounded numerical value to four decimal places. In particular we analyse the branching factors from each side of the bisection cut separately and take into account the cut-off nodes as resolved. More precisely, for measure gains on the other side of the bisection cut, one only accounts the constant $3$, for the measure gains on the own side of the bisection cut, one computes according to the nodes removed or losing weight.

The branching rules are in the following situations where in the subsequent diagrams $-|-$ denotes the edge in the bisection cut. The other edges may or may not be in the bisection cut, it does not matter, as only the removal of one edge from the bisection cut in the two-way branching gives enough measure.

**Two-Way Branching Rules.** In the following three situations, one can either apply a simplification rule or resort to two-way branching which allows for good branching factors.

| S4  (see above) | B2.1 | B2.2 |
|----------------|------|------|
| d-a-|c     | d-|a-d' | d-|a-d' |
|     |      |     |      |      |
|     |      | c   |      | b    |

**Branching Rule B2.1.** One removes $a, c$ from $V$ versus putting $a - c$ into $S$ as an edge and removing $a, c, d, d'$ from $V$.

**Branching Rule B2.2.** One removes $a$ from $V$ versus putting the edge $a - b$ into $S$ and removing $d, a, b, d'$ from $V$.

**Explanation and Verification.** Note that the actual case-distinction in B2.1 is removing $a$ without putting an edge into $S$ versus putting an edge with $a$ into $S$; when $a$ is removed without putting an edge into $S$, then the node $c$ becomes a degree 0 node and can also be removed in the same action. Note that $d, d'$ have at least one additional neighbour of degree 2, as otherwise simplification rule S3 would apply.

The subset principle applies in all cases for completeness. The reason is that for node $c$ in B2.1 as well as node $b$ in B2.2 satisfy that all their neighbours are also either the respective node $a$ in the corresponding rule B2.1, B2.2 or its neighbours. As every edge in which the main node $a$ is involved requires the removal of $a$ and all its neighbours, the corresponding removed parts of $V$ include always the part which is taken when one puts into $S$ the edges $a - c / a - b$ in the respective rules. Thus in both cases one has two-way branching and each time the two-way branching removes on both sides an endpoint of the edge in the bisection cut and thus the edge in the bisection cut itself. Therefore one can always put measure 3 into the budget for removing an edge from the bisection cut and furthermore some nodes are removed or lose weight. However, in the case that both have degree 2, the loss of measure is higher, as simplification rule S3 applies after removing $a, c$ only; therefore one can assume for the worse case that the neighbour of $d$ has degree 3. On the side of $d$ the weight-loss is smaller, as $d$ and its neighbour might both be degree 2 nodes. If $d$ has degree 2 the branching factor is $\tau(3 + s, 4)$ and if $d$ has degree 3 the
branching factor is $\tau(4-s, 5-s)$, in both cases assuming that the neighbour has degree 3 for lower weight loss when downgrading the degree.

**Three-Way Branching Rules.** The situations of the rules follow the below graphics; only edges between nodes of both sides are listed. In the following diagrams, the edge $-|-$ denotes an edge in the bisection while normal lines are edges not in the bisection cut.

Three-Way Branching Rule B3.1. The endpoints $b, b'$ of the edge in the bisection cut have degree 2 and all their neighbours have either degree 2 or degree 3. In each of the 3 branches, an edge goes into $S$, namely $d - b$ or $b - b'$ or $b' - d'$; the endpoints of the edges and their neighbours will be removed from $V$ and $E$ will be updated accordingly.

Three-Way Branching Rule B3.2. One branches $b$ into $S$ versus $b$ removed as single node; thus the 3 options are to put edge $d - b$ into $S$; to put the edge $b - a$ into $S$; just to remove node $b$ from $V$. When putting an edge into $S$, its endpoints and their neighbours are removed from $V$.

Three-Way Branching Rule B3.3. One either puts the edge $a' - a$ into $S$ and removes $a, a', d, d', d'', d'''$ from $V$ or one removes only $a'$ from $V$ or only $a$ from $V$.

Coverage of all cases by the various rules. For rules B3.2 and B3.3 it is clear that the case distinction is complete: In the case of the rule B3.2, the case distinction is that no edge goes into $S$ and just $b$ is removed versus the two cases where an edge with endpoint $b$ goes into $S$; in the case of rule B3.3, the case distinction is that the edge $a' - a$ goes into $S$ versus the two cases that either $a$ or $a'$ are removed from $V$ without any edge going into $S$. For the rule B3.1, more work is required.

So assume that one looks whether the nodes $d, d'$ are part of an edge which goes into $S$ or not. If they are not, then the two nodes between them can go into $S$, that is the case that the edge selected is $b - b'$. If one of them is part of an edge going into $S$ but the other one not, then for this one edge the nodes $b, b'$ between $d, d'$ can only be part of the edge of the border node or not part of any edge going into $S$. Thus, by the subset principle, these cases are supersets of case that the selected edge is $b - b'$. The last case where edges of $d, d'$ go into $S$. If both edges do not have endpoints $b, b'$, respectively, then there is a proper subset having two edges going into $S$ which could also be removed from $V$, thus this case does not apply. So one of the edges is $d - b$ or $b' - d'$, respectively. In these cases, instead of taking out the full set, one just considers the subsets where $d - b$ or $b' - d'$ go alone to $S'$ and removes the other side $d'$ and $b$, respectively, from the nodes to be removed from $V$ to simplify the subcases. Thus the case-distinction from rule B3.1 is indeed legitimate.
Branching Numbers of Three-Way Branchings B3.1 and b-side of B3.2. A detailed analysis of the branching cases shows that for B3.1 and for the b-side of B3.2 the 3 cases do the following: the first one removes just the node $b$ (that is selecting the edge $b' - d'$ in B3.1 and just removing $b$ in B3.2); the next one removes the nodes $d, b$ which is selecting the edge $b - b'$ for $S$ in B3.1 and selecting the edge $b - a$ in B3.2; the third one removes the nodes $d, b$ as well as all not listed neighbours of $d$ on the b-side from the bisection cut. There are several cases how the side $d - b$ looks like; in cases (a) and (b), $d$ has degree 2; in the cases (c), (d), (e) and (f), $d$ has degree 3.

\[
\begin{align*}
(a) & \quad (b) & \quad (c) & \quad (d) & \quad (e) \\
\text{c-d-b-|-.. e'-e-d-b-|-..} & \quad \text{c-d-b-|-.. e'-e-d-b-|-..} & \quad \text{e-e-d-b-|-..} \\
\text{\hline c'} & \quad \text{c} & \quad \text{e'}
\end{align*}
\]

(f) Same as (e), but $e, e'$ do not have a joint edge.

Case (a) and case (c) do not occur, as simplification rule S3 covers these cases and the branching rules are not reached. Similarly, if $e'$ would have degree 1 and any further neighbours of $b, d, e$ (if any) would have degree 1 then simplification rule S3 would remove some nodes prior to branching and therefore also this case does not apply.

In the following it is shown that, using $1/2 \leq s \leq 1$, the branching number is always at least as good as $\max\{\tau(3 + 4s, 3 + 4s, 3 + 4s), \tau(4, 4 + s, 5 + s)\}$.

In cases (d), (e), (f), if one removes $b$ then this gives weight $s$ and $d$ is losing weight $1 - s$, together with the constant 3 for removing an edge from $B$, it gives 4. If one removes $d, b$ then the gain is at least $4 + s$ without taking weight loss of neighbours into account. If $e$ or $e'$ has weight 1 then the gain is at least $5 + s$. If $e$ and $e'$ have both weight $s$ then the gain is $4 + 3s$ which is above $5 + s$ since $s \geq 1/2$.

So it remains the case (b). If $e$ has degree 3 then the weight gain at removing $b$ is $3 + 2s$, the weight gain at removing $d, b$ is $3 + 2s + 1 - s = 4 + s$, the weight gain at removing $d, b, e$ is in the worst case $3 + (1 - s) + 2s = 5 + s$. As $s \geq 1/2$, $\tau(3 + 2s, 4 + s, 5 + s)$ is also better than $\tau(4, 4 + s, 5 + s)$. If $b, d, e$ all have degree 2 then removing $b$ also removes $d, e, e'$ and so one gets either $\tau(4 + 3s, 3 + 3s, 4 + 2s)$ or, if all four have degree 2, then one can apply simplification rule S3 both after removing $b$ which takes away $d, e, e'$ and after removing $b, d$ which takes away $e, e'$ and a neighbour of $e'$. The third case is that $b, d, e$ are removed and $e'$ goes down to weight 0. Note that $4 \leq 3 + 3s, 4 + s \leq 4 + 2s$ and $5 + s \leq 4 + 3s$, thus $\tau(4 + 3s, 3 + 3s, 4 + 2s)$ is better than $\tau(4, 4 + s, 5 + s)$. In the other case, the branching factor is better than $\tau(3 + 4s, 3 + 4s, 3 + 4s)$, as all four nodes are by follow-up branchings eliminated in all three alternatives of the branching.

Branching Numbers of Three-Way Branchings a-side of B3.2 and B3.3. Note that the following 3 alternatives occur on the current side of the bisection: only the edge in the bisection cut is removed; the node $a$ is removed; the nodes $a, d', d''$ are removed. The first case arises when for the rule B3.2 only $d$ is removed and when for the rule B3.3, a-side only the node $a'$ on the other side is removed. The next case occurs at B3.2 when the edge $d - b$ is selected for $S$ and thus the neighbour $a$ is removed from $V$ and for B3.3 when the node $a$ is only removed. The
third case, removal of $a, d', d''$ on the current side of the bisection cut occurs in rule $\text{B3.2}$ when
the edge $b - a$ is selected for $S$ and thus also the nodes $d', d''$ are removed and in the rule $\text{B3.3}$
when the edge $a' - a$ is selected for $S$ and thus also the neighbours $d', d''$ are removed from $V$. For
evaluating the branching number, one again has to distinguish several configurations. Cases (a),
(b) and (c) – unless $d, d'$ in (c) have both degree 3 and one of them has a further neighbour of
degree 2 – do not occur, as the corresponding simplification rules $\text{S3}$ and branching rules $\text{B2.1}$
and $\text{B2.2}$ apply.

(a)  (b)  (c)  (d)  (e)

..-|-a-c  ..-|-a-d'-..  ..-|-a-d'  ..-|-a-d'-e'  
      |    | \   | \   |
      c'   c  d''  d''-e  d''-e''

For these graphics, $d', d''$ have been replaced by $c, c'$ when their degree is 1 in cases (a) and (b); this only occurs in cases captured by prior simplification or two-way branching rules.

For (c), if both $d', d''$ have degree 3 and both have an additional neighbour of at least degree 2
then by the above mentioned exception, neither branching rule $\text{B2.2}$ nor simplification rule
$\text{S3}$ do apply, but the branching according to $\text{B3.2}$ or $\text{B3.3}$ gives either $\tau(4 - s, 6 - 2s, 8 - 2s)$
or $\tau(4 - s, 6 - 2s, 6 + s)$ where the weight-loss of the further neighbours of $d', d''$ in the last
alternative is in the worst case is either $s$ (if neighbours identical) or $1 - s$ each (if distinct
neighbours). If one of $d, d'$ has only an additional neighbour of degree 1 and the other one
has an additional neighbour of degree 2 or 3 then simplification rule $\text{S3}$ applies in the second
alternative when removing $a$ so that furthermore $d, d'$ are removed and the worst case branching
factor is $\tau(4 - s, 7 - s, 7 - s)$. The case that both have only an additional neighbour of degree 1
is eliminated by simplification rule $\text{S3}$ prior to branching.

In (d) one distinguishes several cases. First, the case that all nodes $d', d''$ have degree 2. Then
simplification rule $\text{S1}$ (if $e$ has also degree 2) or simplification rule $\text{S3}$ (if $e$ has degree 3 and is
connected to something) can apply and the corresponding gain of measure allows to evaluate the
subcase as $\tau(4 - s, 5 + 5 + 5)$. Second, it does not happen that $e$ has degree 2 and $d', d''$ have
besides $a, e$ only degree 1 neighbours (if any), as then simplification rule $\text{S3}$ would have removed
this part prior to branching. Third, the case that $d', e$ have degree 3 and $d'$ has a neighbour $c$
of degree 1. Then when removing only $a$, the simplification rule $\text{S3}$ removes $c, d', e$ which gives the
 corresponding gain of additional measure and $\tau(4 - s, 6 + s, 6 + s)$. Fourth, the case that $d'$ has
a neighbour of degree at least 2 and $e$ has degree 3. Then the case that $a, d', d''$ are all removed
reduces the weight of $e$ by 1 and brings in a further loss of measure $1 - s$ for the additional
neighbour of $d'$, resulting in the branching factor $\tau(4 - s, 5, 7 - s)$. Fifth, the case that $d', d''$ both
have besides $e$ a further neighbour $e', e''$ of degree at least 2 and that $e$ has degree 2 and $e' \neq e''$.
This gives the branching factor $\tau(4 - s, 6 - 2s, 6 + 3s)$. Sixth, the same case as fifth, but with
$e' = e''$. In that case that branching factor is $\tau(4 - s, 6 - 2s, 6 + 2s)$. All the subcases together
have $\max\{\tau(4 - s, 4 + 3s, 4 + 3s), \tau(4 - s, 5, 7 - s), \tau(4 - s, 6 - 2s, 6 + 2s)\}$ as worst case.

In (e) the degrees of $d', d'', e', e''$ are at least 2 and $e' \neq e''$ as otherwise case (d) would apply.
Again one makes a case distinction: The first case is that $d, d'$ have both degree 2; here one
assumes that $e', e''$ have degree 3 to minimise the weight-loss when removing $d, d'$. Now the
branching factor is $\tau(4 - s, 4 + 2s, 6)$ or better, as when removing $d', d''$, the neighbours $e', e''$ lose weight $1 - s$ each. The second case is as before, but $e'$ has degree 2; then when removing $a$ one can apply simplification rule S3 and remove $d', e'$ and a further node as well. The branching factor is $\tau(4 - s, 4 + 3s, 5 + 2s)$ or better and this is better than $\tau(4 - s, 4 + 2s, 6)$ from above. The third case is that $d'$ has degree 3. Then the branching factor is at least $\max\{\tau(4 - s, 5, 7 - s), \tau(4 - s, 6 - 2s, 8 - 2s)\}$ where the two cases are degree of $d''$ being 2 versus 3 and the degrees of $e', e''$ are assumed to be 3 for worst-case weight loss. In summary, the branching factor of this case is $\max\{\tau(4 - s, 4 + 2s, 6), \tau(4 - s, 5, 7 - s), \tau(4 - s, 6 - 2s, 8 - 2s)\}$.

In summary the worst case of these branching rules is $\max\{\tau(4 - s, 5, 7 - s), \tau(4 - s, 4 + 3s, 4 + 3s), \tau(4 - s, 4 + 2s, 6), \tau(4 - s, 6 - 2s, 6 + s), \tau(4 - s, 6 - 2s, 8 - 2s)\}$.

**Four-Way Branching rule B4.1.** Four-Way branching occurs only in the following situation:

B4.1. $d - | - d' - | - d''$

If $d'$ has a neighbour of degree 1 in addition to $d', d''$, a two-way branching would apply, thus one can assume that the neighbour, if it exists, has at least degree 2. As the rule S4 does not apply, the nodes $d, d''$ have also at least degree 2. Furthermore, the node $d'$ has either degree 2 or degree 3. Whenever $d'$ is removed from $V$ then two edges in the bisection cut are removed. Thus one has at least one of these branching factors (depending on whether $d'$ has degree 2 or 3): $\tau(6 + s, 6 + s, 6 + s), \tau(6 + 2s, 6 + 2s, 6 + 2s, 6 + 2s)$ where the first branching factor is better than the second due to three-way branching instead of four-way branching. Note that for the three-way branching, on the side of $d'$ of the bisection cut, only a node of value $s$ is branched together with a gain of 6 in each branch for the accounting of two edges removed from the bisection cut. For the four way branching, there are four choices of what happens with $d'$: Only removing $d'$ without adding an edge to $S$ versus adding exactly one of the edges bordering $d'$ into $S$.

**Theorem 5.** The case distinction in the algorithm is complete. The runtime $O(1.2630^n)$ where $s$ is chosen as 0.636 giving the optimal entry in the following table.

| Rule          | Formula                      | $s = 0.6$ | 0.636     | 0.7
|---------------|------------------------------|-----------|-----------|
| B2.1, B2.2    | $\tau(3 + s, 4)$, $\tau(4 - s, 5 - s)$ | 1.2004    | 1.1993    | 1.1974
| B3.1, B3.2 b-side | $\tau(3 + 4s, 3 + 4s, 3 + 4s)$, $\tau(4, 4 + s, 5 + s)$ | 1.2257    | 1.2192    | 1.2086
| B3.2 a-side, B3.3 | $\tau(4 - s, 4 + 2s, 6)$, $\tau(4 - s, 4 + 3s, 4 + 3s)$, $\tau(4 - s, 5, 7 - s)$, $\tau(4 - s, 6, 2s, 6 + s)$, $\tau(4 - s, 6 - 2s, 8 - 2s)$ | 1.2618    | 1.2615    | 1.2610
| B4.1          | $\tau(6 + 2s, 6 + 2s, 6 + 2s, 6 + 2s)$ | 1.2124    | 1.2030    | 1.2061
| Overall       | Max of Above                 | 1.2644    | 1.2630    | 1.2683

All branching factors in the table are strictly uprounded and therefore the algorithm is in $O(1.2630^n)$ without additional subexponential factors.
Proof. The runtime follows from the fact that the branching factors determine the size of the search tree and that every individual instance of a rule can be carried out in time polynomially in the number of involved nodes (without the recursive subcall). The best-possible branching factor is then 1.2630 and obtained by choosing $s$ optimally in the table given in the statement of the theorem. So the main part is to verify that always one of the rules applies, that is, that the preconditions where a rule can be applied do not leave any case undone and form a complete case-distinction of all possible cases. When several rules apply, the one applying first is done.

Completeness of Case-Distinctions. Now it is shown that as long as there are nodes in $V$, one of the four cases in the algorithm applies. If there are no edges in $B$, that is, if the previous bisection cut has been completely branched or there has not yet any been made, one checks first whether $(V, E)$ is connected. If not, then one splits the graph into the independent components and solves each of them independently. If $(V, E)$ is connected, one checks whether there are more than $\kappa$ branching nodes. If not, simplification rules $S_1$ and $S_2$ will solve the graph. If yes, then one computes according to Corollary 2 the corresponding bisection cut.

If there are edges in the bisection cut then one has to show that one of the following rules applies: Simplification rules $S_1$, $S_2$, $S_3$, $S_4$ where the first three also work when $B$ is empty and the fourth reduces $B$ by one edge in the case that on one side of the bisection cut is a node of degree 1.

If none of these apply, there are the following situations: $B_{2.1}$ and $B_{2.2}$ handle the situation that the bisection cut has on one side a node $a$ of degree 3 which is connected with a node of degree 1; $B_{3.1}$ handles the case that both sides of the bisection cut are nodes of degree 2, $B_{3.2}$ handles the case that on one side is a node of degree 2 and on the other side is a node $a$ of degree 3 which has all neighbours having at least degree 2; $B_{3.3}$ handles the case that on both sides of the bisection cut are directly nodes of degree 3 which satisfy that all neighbours have degree at least 2. $B_{4.1}$ handles the case that a node has two neighbours in the bisection cut which, by accounting, gives a better branching factors as the others, even if one needs to do four-way branching.

5 Conclusion

For various papers, the subcase of cubic graphs when solving the maximum induced matching problem was only treated by invoking the line graph argument of Cameron [1] which gives $O(1.3139^n)$. The present polynomial space algorithm gives an overall runtime of $O(1.2630^n)$; better algorithms use exponential space [6,16]. One of the ideas is not to split the graph using the Monien-Preis method such that the number of nodes is balanced, but such that the number of degree 3 nodes is balanced; for evaluating in this situation the algorithm, one looks at the branching numbers from both sides of the bisection cuts and budgets for each edge branched of the bisection cut an amortised gain of 3 degree 3 nodes which go to the other side; all branching rules considered remove at least one edge of the bisection cut.

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