Electrical conductivity of strongly degenerate plasma with the account of electron-electron scattering

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Abstract. The influence of electron-electron scattering on the strongly degenerate plasma conductivity is investigated with a linear response theory. In the present work the temperature dependence of the electron-electron scattering term of the electrical conductivity and further modification of the Ziman formula are discussed.

1. Introduction
It is known that electron-electron scattering makes a significant contribution to the conductivity of low-density non-degenerate plasma [1] and is completely absent at zero temperature. A temperature influence on the conductivity is usually taken into account by means of a well-known finite-temperature modification of the Ziman formula [2, 3] for electron-ion scattering. At the same time electron-electron scattering can contribute an income comparable with that due to the distribution function deviation from the Fermi step. The temperature behaviour of the electron-electron scattering influence on transport properties is interesting both for the description of experiments on laser heating of plasma [4] and the construction of interpolation formulas for the conductivity [5, 6, 7].

2. Basic approximations
A neutral two-component plasma consisting of free singly charged particles with charges $e_i$ and $e_e$ ($e_i = -e_e = e$) at temperature $T$ and density $n = n_e = n_i$, interacting via Coulombic forces is considered. Indices $i$ and $e$ denote ions and electrons, respectively. The value of the mass ratio $\gamma = m_e/m_i$ tends to zero (the adiabatic limit). The dimensionless degeneracy parameter $\Theta = (2m_e k_B T/\hbar^2)(3\pi^2 n_e)^{-2/3}$ and the electron-ion coupling constant $\Gamma = (e^2/4\pi e_0 k_B T)(4\pi n_e/3)^{1/3}$ are introduced. For the strong degenerate plasma $\Theta \ll 1$.

Within the linear response theory in the formulation of Zubarev [8], the transport properties are expressed via force-force correlation functions. The procedure of their derivation is well-known (see [9, 10, 11, 12, 13, 14]). As a result, the static electrical conductivity is related to the Onsager transport coefficients $L_{ik} = L_{ki}$ according to

$$\sigma = e^2 L_{11}$$

where

$$L_{ik} = \frac{\left((-\hbar)^{i+k-2}\right)}{\Omega \det(d)} \begin{vmatrix} 0 & \frac{k-1}{\beta \hbar} N_1 - N_0 \\ \frac{-1}{\beta \hbar} N_1 - N_0 & d \end{vmatrix},$$

(2)
where flux \[14\] and obtain for the matrix elements in Eq.(4) corresponding relevant statistical operator (see \[8\]). In the adiabatic limit we can omit the ion thermodynamic equilibrium, first Born approximation for the screened Coulomb interaction potential.

\[\epsilon = \text{thermodynamic equilibrium}, \ \text{first Born approximation for the screened Coulomb interaction potential.} \]

In (2)-(4) \(\Omega\) denotes the system volume, \(N_{mn}, d_{mn}\) are correlation functions for the thermodynamic equilibrium, \(h\) is enthalpy per one electron, \(N_e\) - the number of electrons and \(\beta = (k_B T)^{-1}\). The dimension of the matrix \(d\) coincides with the number of moments in the corresponding relevant statistical operator (see \[8\]). In the adiabatic limit we can omit the ion flux \[14\] and obtain for the matrix elements in Eq.(4)

\[d_{mn} = d_{mn}^{ei} + d_{mn}^{ee}, \]

\[N_{mn} = N_e \frac{\Gamma(m + n + 5/2)}{\Gamma(5/2)} I_{m+n+1/2}(\beta \mu_e^{id}) I_{1/2}(\beta \mu_e^{id})^{-1}, \]

with \(I_e(y) = \frac{1}{\Gamma(\nu+1)} \int_{0}^{\infty} \frac{x^{\nu-1} e^{-x}}{\nu!} \) - the Fermi integrals, \(\mu_e^{id}\) - the ideal part of the electronic chemical potential.

The correlation functions \(d_{mn}\) are evaluated using thermodynamic Green’s functions. The first Born approximation for the screened Coulomb interaction \(V(q) = e^{-2q^2\Omega_e^{-1}} [15, 13]\) with omitting the exchange term in electron-electron scattering leads to the Lenard-Balescu-type collision integrals, obtained in \[9\]:

\[d_{mn}^{ei} = \pi \beta h \sum_{k_0 p q} \int d\omega \left[ \frac{V(q)}{\epsilon(q, \omega)} \right]^2 f_{k}^{e}(1 - f_{[k+q]}^{e}) f_{p}^{e}(1 - f_{[k-p]}^{e}) \times \delta(\omega - E_{k}^{e} + E_{k}^{e}) \delta(\omega - E_{p}^{e} + E_{[p-q]}^{e}) K_{n}(\bar{k}, \bar{q}) K_{m}(\bar{k}, \bar{q}), \]

\[d_{mn}^{ee} = \frac{\pi \beta h}{2} \sum_{k_0 p q} \int d\omega \left[ \frac{V(q)}{\epsilon(q, \omega)} \right]^2 f_{k}^{e}(1 - f_{[k+q]}^{e}) f_{p}^{e}(1 - f_{[k-p]}^{e}) \times \delta(\omega - E_{k}^{e} + E_{k}^{e}) \delta(\omega - E_{p}^{e} + E_{[p-q]}^{e}) (K_{n}(\bar{k}, \bar{q}) + K_{n}(\bar{p}, \bar{q})) (K_{n}(\bar{k}, \bar{q}) + K_{n}(\bar{p}, \bar{q})) \]

\[\text{where } E_{k}^{e} = h^2 k^2 / (2m_e), K_{n}(\bar{k}, \bar{q}) = k_s(\beta E_{k}^{e})^{n} - (k_s + q_s)(\beta E_{[k+q]}^{e})^{n}, \epsilon(q, \omega) - \text{the dielectric function, } f_{k}^{e} = (\exp((E_{k}^{e} - \mu_e^{id})/\beta) + 1)^{-1} - \text{the Fermi distribution function.}\]

In the adiabatic limit

\[d_{mn}^{ei} = \frac{4m_e^2}{3\pi^2 \beta^2 h^3} \int_{0}^{\infty} dx x^{n+m+2} f_{k}^{e}(1 - f_{k}^{e}) Q_{ei}(x) \]

\[Q_{ei}(x) = \frac{\beta^2 \Omega_e^2}{16 \pi x^2} \int_{0}^{2k} V(q) \left| \epsilon_{e}(q, 0) \right|^2 S_{ii}(q) q^3 dq \]

\[\epsilon_e(q, 0) = 1 + \Omega V(q)(1 - G_e(q)) \chi_e^{(0)}(q, 0) \]

with \(x = \frac{\beta h^2 k^2}{2m_e}, Q_{ei}(x) - \text{the transport cross-section for electron-ion scattering in the first Born approximation, } \epsilon_e(q, 0) - \text{the effective static electronic dielectric function, } S_{ii}(q) - \text{the ion-ion structure factor, } \chi_e^{(0)}(q, \omega) - \text{the free-electron polarizability [19, 20], } G_e(q) - \text{the static electronic local field correction. For } d_{0}^{ei} \text{ this result was presented in [9, 21, 22].} \]
3. The limit of the strong degeneration

Expressions (9)-(10) for the electron-ion correlation functions are valid in any degeneration degree. Electron-electron correlation functions can be reduced to the two-dimension integrals in the Boltzmann limit [23]. For the strong degeneracy conductivity calculations electron-electron terms are usually omitted due to the restrictions of the Fermi statistics, and as a result the values of dc conductivity are equal to each other at any values of \( l \) in (3), (4), if the temperature in the Fermi distribution functions is taken equal to zero. With increasing temperature, the frequently used method for the calculation of the conductivity consists in substituting the non-zero temperature into distribution functions. In this case, at first, the simple proportionality between electron-ion correlation functions disappears, that leads to the dependence of the result on the dimension of determinants, even in the absence of electron-electron scattering. And, at second, the electron-electron correlation functions begin to grow.

To trace the influence of both factors, consider the simplest approximation that includes them. We restrict ourselves to \( l = 1 \) in (3), (4). For the preliminary estimation of integrals we take for the Fermi distribution function \( f(z) = 1 \) for \( z < 0 \) and \( f(z) = \exp(-z) \) for \( z > 0 \), where \( z = (E_k - \mu_d)\beta \). Introducing the dimensionless variables \( Q = h\beta/\omega/q, \nu = (\beta m_e)^{1/2}\omega/q \) we obtain that \( d_{11}^{ee} \) is proportional to the sum of the following integrals:

\[
d_{11} = \int_0^{2Q_F} \frac{dQ}{Q^2} \left[ \int_0^{Q_F} \frac{I_{1k}I_{1l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} + \int_{Q_F}^{Q_F+\frac{Q}{2}} \frac{I_{2k}I_{2l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} + \int_{Q_F+\frac{Q}{2}}^{\infty} \frac{I_{3k}I_{3l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} \right] \tag{12}
\]

\[
+ \int_0^{2Q_F} \frac{dQ}{Q^3} \left[ \int_0^{Q_F} \frac{I_{1k}I_{3l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} + \int_{Q_F}^{Q_F+\frac{Q}{2}} \frac{I_{2k}I_{2l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} + \int_{Q_F+\frac{Q}{2}}^{\infty} \frac{I_{3k}I_{3l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} \right] \tag{13}
\]

where

\[
I_{1l} = \left( \int_0^{\frac{Q_F}{2}} \mathrm{e}^{x+S_+ - \frac{Q_F}{2}x'}dx + \int_{\frac{Q_F}{2}}^{Q_F} x'dx + \int_{Q_F}^{\infty} \mathrm{e}^{-(x-S_+ + \frac{Q_F}{2}x')}x'dx \right) \tag{14}
\]

\[
I_{2l} = \left( \int_0^{\frac{Q_F}{2}} x'dx + \int_{\frac{Q_F}{2}}^{Q_F} x'dx + \int_{Q_F}^{\infty} \mathrm{e}^{-(x-S_+ + \frac{Q_F}{2}x')}x'dx \right) \tag{15}
\]

\[
I_{3l} = \int_0^{\infty} \mathrm{e}^{-x-S_+ + \frac{Q_F}{2}}x'dx \tag{16}
\]

In Eqs. (12)-(16) \( Q_F = h\beta/\omega/q \), and \( S_\pm = \frac{1}{2} \left( \nu \pm \frac{Q_F}{2} \right)^2 \).

The main contribution to \( d_{11} \) at \( \Theta \ll 1 \) comes from the first integral in (12):

\[
d_{11} = \int_0^{2Q_F} \frac{dQ}{Q^2} \int_0^{Q_F} \frac{I_{1k}I_{1l}e^{-Q\nu}d\nu}{|\epsilon(Q,\nu)|^2} \tag{17}
\]
Due to the multiplier $e^{-Q\nu}$ and the frequency dependence of the dielectric function in the form $f(\frac{\nu}{Q})$, we can neglect the frequency dependence of $\epsilon(q, \omega)$ in the overwhelming part of the domain of integration.

Bearing in mind that the non-vanishing part of $(K_1(\vec{k}, \vec{q}) + K_1(\vec{p}, \vec{-q}))^2$ due to its symmetry properties with respect to the permutation of the variables of integration $x$ and $y$ is proportional to $Q^2 ((y - x)^2 + 4Q^2\nu^2)$, and restoring dimensional multipliers, we have:

$$d_{11}^{ee} = 37.3\Theta^{7/2}k_Fd \int_0^{2k_F} \frac{dq}{q^2\epsilon^2_e(q, 0)}, \quad (18)$$

where $d = \frac{8}{3} \frac{m^{3/2}\rho_0\beta^{3/2}}{\hbar^3(4\pi\epsilon_0)^{1/2}}$.

For a more accurate determination of the numerical multiplier when integrating with the real Fermi distribution function, it is convenient to present the latest in the form of an expansion in Chebyshev polynomials [24]. Limited to three polynomials in the decomposition, we obtain:

$$d_{11}^{ee} = 17.3\Theta^{7/2}k_Fd \int_0^{2k_F} \frac{dq}{q^2\epsilon^2_e(q, 0)}, \quad (19)$$

4. Results and discussion

In the Fig. 1 the results of calculations of the electrical conductivity of the fully ionized plasma in the described approximation with and without electron-electron scattering are presented. For the local field correction $G_e(q)$ the simplest approximation of the VS type [25] is chosen. As it turned out, the results for the electrical conductivity do not depend on the choice of $G_e(q)$. For $S_{ii}(q)$ in Eq. (10) the HNC approximation is used with the numerical scheme described in [26]. The study of the dependence of the results from the model for the structure factor (see, e.g., [9, 27]) is not the subject of this work. The HNC method is quite popular and there are numerous data obtained in the Ziman approximation.

The present calculations are made for temperatures and densities corresponding to final states of systems obtained from the laser compression [27].

The boundary $\Theta = 1$ corresponds to $n_e = 10^{26} cm^{-3}$, for $n_e = 10^{27} cm^{-3}$ $\Theta$ is near 0.2. Note that the simple Ziman approximation [2, 3] ($l = 0$ in the Eqs.(3), (4)) is more close to the result.

Figure 1. Electrical conductivity of degenerate plasma dependent on electron density in the two-moment representation. $T = 10^7 K$. Solid line - with e-e scattering, dashed line - without e-e scattering, dots - data of [27] in the Ziman approximation.
for $l = 1$ with the electron-electron scattering than without it (within a few percent). The results are similar for $T = 10^6 K$ and $T = 10^8 K$. Thus, at least for the considered case of plasma with singly charged ions, the Ziman approximation for finite temperatures amazingly well describes the electrical conductivity of the degenerate plasma in a wide range of degeneration parameter values. However, with the expansion of the basis of the relevant variables in the theory, we must simultaneously take into account electron-electron scattering which also has a separate application in the study of other plasma properties (e.g., thermal conductivity, see [4]).

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