QUASICONFORMAL DEFORMATIONS OF NONVANISHING $H^p$ FUNCTIONS AND THE HUMMEL-Scheinberg-ZALCMAN CONJECTURE

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Abstract. Recently the author proved that the 1977 Hummel-Scheinberg-Zalcman conjecture on coefficients of nonvanishing $H^p$ functions is true for all $p = 2m$, $m \in \mathbb{N}$, i.e., for the Hilbertian Hardy spaces $H^{2m}$. As a consequence, this also implies a proof of the Krzyz conjecture for bounded nonvanishing functions which originated this direction.

In the present paper, we solve the problem for all spaces $H^p$ with $p \geq 2$.

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1. INTRODUCTORY REMARKS AND MAIN RESULT

The Hummel-Scheinberg-Zalcman conjecture [7] generalizes the Krzyz conjecture for nonvanishing $H^\infty$ functions to the Hardy spaces $H^p$ of holomorphic functions $f(z)$ on the unit disk $D = \{ z : |z| < 1 \}$ with norm

$$\|f\|_p = \sup_{r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}.$$ 

It was posed in 1977 and states that Taylor’s coefficients of nonvanishing holomorphic functions $f(z) = \sum_{n=0}^\infty c_n z^n$ from $H^p$, $p > 1$, with $\|f\|_p \leq 1$ are sharply estimated by

$$|c_n| \leq \left( \frac{2}{e} \right)^{1-1/p},$$

(1)

and this bound is realized only by the functions $e_2 \kappa_n, p (\epsilon_1 z)$, where $|\epsilon_1| = |\epsilon_2| = 1$ and

$$\kappa_{n, p}(z) = \left[ \frac{(1 + z^n)^2}{2} \right]^{1/p} \left[ \frac{\exp \frac{z^n}{z^{n+1}}}{z^{n+1}} \right]^{1-1/p}.$$ 

(2)

This eminent conjecture has been investigated by many authors. A long time, the only known results here are that the conjecture is true for $n = 1$ (Brown) and $n = 2$ (Suffridge) as well as some results for special subclasses of $H^p$, see [5], [6], [14]. Brown also showed that (1) is true for arbitrary $n \geq 2$, provided $c_m = 0$ for all $m$, $1 \leq m < (n+1)/2$, and Suffridge also estimated sharply the coefficient $c_1$ for $0 < p \leq 1$.

Recently the author proved this conjecture for Hilbertian Hardy spaces $H^{2m}$ with $m \in \mathbb{N}$ in [12], applying a new approach to the coefficient problems for holomorphic functions on the disk. This approach was presented in [10], [11] and involves some fundamental results of Teichmüller space.
theory, especially the Bers isomorphism theorem for Teichmüller spaces of punctured Riemann surfaces.

In the limit as $p = 2m \to \infty$, one obtains as a consequence of (1) and (2) that the coefficients of nonvanishing $H^\infty$ functions with norm $\|f\|_\infty \leq 1$ are sharply estimated by $|c_n| \leq 2/e$, which proves the initial Krzyz conjecture.

In fact, the last estimate holds for some broader class containing also unbounded nonvanishing functions, see [12].

The aim of the present paper, which continues [12], is to prove the Hummel-Scheinberg-Zalcman conjecture for all spaces $H^p$ with $p \geq 2$. The main result states:

**Theorem 1.** The estimate (1) is valid for all spaces $H^p$ with $p \geq 2$; that is, the coefficients of any nonvanishing function $f \in H^p$, $p \geq 2$, with $\|f\|_p \leq 1$ satisfy $|c_n| \leq (2/e)^{1/p}$ for any $n > 1$; the equality in (1) is realized only on the function $f(z) = \kappa_{n,p}(z)$ and its compositions with pre and post rotations about the origin.

The proof of this general theorem is based on the same ideas as its special case $p = 2m$, $m \in \mathbb{N}$, in [12]. A new essential step is to establish that in the case of nonvanishing $H^p$ functions the needed quasiconformal deformations exist for all $p > 1$.

Another essential step in the proof of Theorem 1 relies on the fact that for any $n \geq 2$ the coefficients of the function $\kappa_{1,p}$ satisfy

$$|c_n(\kappa_{1,p})| = |c_0^0| < |c_1^0|,$$

which follows, for example, from Parseval’s equality. But this remains unknown for $f \in H^p$ with $1 < p < 2$.

The last section of the paper contains some remarks concerning the possible generalizations of this problem.

2. **QUASICONFORMAL DEFORMATIONS OF $H^p$ FUNCTIONS**

We start with establishing the existence of some special quasiconformal deformations of nonvanishing Hardy and Bergman functions.

The general result established in [9] for the generic $H^p$ functions with $p = 2m$, $m \in \mathbb{N}$, states the follows.

Consider the functions $f(z) \in H^{2m} \cap L_\infty(D)$, with

$$\sup_D |f(z)| = M > \|f\|_{2m}.$$

Let $E$ be a ring domain bounded by a closed curve $L \subset D$ containing inside the origin and by the unit circle $S^1 = \partial D$. Let, in addition,

$$d^0 = (0, 1, 0, \ldots, 0) =: (d^0_k) \in \mathbb{R}^{n+1},$$

and $|x|$ denote the Euclidean norm in $\mathbb{R}^l$.

**Proposition 1.** [9] For any holomorphic function $f(z) = \sum_{k=0}^{\infty} c_k^j z^k \in L_{2m}(E) \cap L_\infty(E)$ (with $c^0_j \neq 0$, $0 \leq j < n$ and $m \in \mathbb{N}$), which is not a polynomial of degree $n_1 \leq n$, there exists a positive number $\varepsilon_0$ such that for every point

$$d' = (d'_{j+1}, \ldots, d'_n) \in \mathbb{C}^{n-j}$$

and every $a \in \mathbb{R}$ satisfying the inequalities

$$|d'| \leq \varepsilon, \quad |a| \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$
there exists a quasiconformal automorphism \( h \) of the complex plane \( \hat{\mathbb{C}} \), which is conformal in the disk
\[
D_0 = \{ w : |w - c_0^0| < \sup_D |f_0(z)| + |c_0^0| + 1 \}
\]
(hence also outside of \( E \)) and satisfies the conditions:
(i) \( h^{(k)}(c_0^0) = k!d_k = k!(d_k^0 + d_k^1) \), \( k = j + 1, \ldots, n \) (i.e., \( d_1 = 1 + d_1^0 \) and \( d_k = d_k^0 \) for \( k \geq 2 \));
(ii) \( |h \circ f|_{2m}^2 = |f|_{2m}^2 + a \).

For any function \( f \in H^p \), we have
\[
\|f\|_p = \sup_{r < 1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p},
\]
(3)
since the mean function
\[
\mathcal{M} f(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta
\]
is a circularly symmetric subharmonic function on \( \mathbb{D} \), monotone increasing with \( r \to 1 \). Any such function is logarithmically convex with respect to \( \log r \) and has at least one-side derivative on \([0, 1]\).

Using the appropriately thin rings \( E \) adjacent to the unit circle, one derives that for any bounded in \( \mathbb{D} \) function \( f \in H^{2m} \) there exists a \( O(\varepsilon) \)-quasiconformal automorphism \( h \) of \( \hat{\mathbb{C}} \) satisfying the conditions (i) and distorting the \( H^{2m} \)-norm of \( f \) by
\[
|h \circ f|_{2m} = \|f\|_{2m} + O(\varepsilon)
\]
(4)
(where the bound of the remainder term depends on \( m \)).

The proof of Proposition 1 in [3] shows that all its assumptions are essential; the arguments do not extend to arbitrary \( p \geq 2 \) and unbounded holomorphic \( L^p \) functions.

One of the important steps in the proof of Theorem 1 is the following weakened extension of Lemma 1 to nonvanishing functions from the spaces \( H^p \) with \( p > 1 \).

Fix a natural \( n > 1 \) and consider the collections \( d = (d_0, d_1, \ldots, d_n) \in \mathbb{C}^{n+1} \).

**Proposition 2.** For every bounded nonvanishing function \( f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \in H^p \), \( p > 1 \), with \( \|f\|_p < \infty \), which is not a polynomial of degree at most \( n \), there exists \( \varepsilon_0 > 0 \) such that for any point \( d \in \mathbb{C}^{n+1} \) with \( |d| \leq \varepsilon < \varepsilon_0 \), there is a bounded nonvanishing function \( f^*(z) = c_0^* + c_1^* z + c_2^* z^2 + \cdots \in H^p \) satisfying \( c_j^* = c_j + d_j \) for all \( j = 0, 1, \ldots, n \) and
\[
\|f^*\|_p = \|f\|_p + O(\varepsilon).
\]

Note that quasiconformal deformations of such type preserve the \( L_p \)-norm of holomorphic functions, but generically increase their \( L_\infty \)-norm (an illustrating example is given in [11]).

**Proof.** For any nonvanishing function \( f(z) = \sum_{0}^{\infty} c_n z^n \in H^p, p > 1 \), the function
\[
f_{p/2}(z) := f(z)^{p/2} = e^{(p/2) \log f(z)},
\]
with a fixed branch of the logarithmic function, also is single valued, holomorphic and zero free in the unit disk \( \mathbb{D} \). We take everywhere the principal branch. Explicitly,
\[
f_{p/2}(z) = c_0^{p/2} \left( 1 + \frac{p}{2} \frac{c_1}{c_0} z + \ldots \right) = c_0(f_{p/2}) + c_1(f_{p/2})z + \ldots
\]
(5)
this function belongs to the space $H^2$. In the case of nonvanishing $f$, the correspondence $f(z) \leftrightarrow f_{p/2}(z)$ creates a biholomorphism (one-to-one and open map) between the neighborhoods of the origin in $\mathbb{C}^{n+1}$ filled by collections $d(f) = (c_0, \ldots, c_n)$ and $d(f_{p/2}) = (c_0(f_{p/2}), \ldots, c_n(f_{p/2}))$.

Since the function (5) is holomorphic, one can apply to it all arguments used in [9] in the proof of Proposition 1 for $p = 2m$ (this proof essentially requires that $f(z)^m$ is single valued). In view of the importance of Proposition 2, we outline the main steps of its proof; the details omitted are given in [9].

Fix $R \geq \sup_{\mathbb{D}} |f(z)| + |c_0| + 1$ and take the annulus

$$G_R = \{ w : R < |w - c_0^0| < R + 1 \}.$$ 

We define for $\rho \in L^p(B)$, $p \geq 2$, the operators

$$T\rho = -\frac{1}{\pi} \int_{G_R} \frac{\rho(\zeta)d\xi d\eta}{\zeta - w}, \quad \Pi\rho = \partial_w T\rho = -\frac{1}{\pi} \int_{G_R} \frac{\rho(\zeta)d\xi d\eta}{(\zeta - w)^2}$$

(the second integral exists as a principal Cauchy value). We seek the required quasiconformal automorphism $h = h^\mu$ of the form

$$h(w) = w - \frac{1}{\pi} \int_{G_R} \frac{\rho(\zeta)d\xi d\eta}{\zeta - w} = w + T\rho(w), \quad (6)$$

with the Beltrami coefficient $\mu = \mu_h$ equal to zero outside of $G_R$, with $\|\mu\|_\infty < \kappa < 1$. Substituting (6) into the Beltrami equation $\partial_\overline{w}h = \mu \partial_\overline{w}h$, we get

$$\rho = \mu + \mu \Pi \mu + \mu \Pi (\mu \Pi \mu) + \ldots.$$ 

This series is convergent in $L^p(E)$ for some $p > 2$, and on the basis of well-known properties of operators $T$ and $\Pi$, we have for any disk $\mathbb{D}_{R'} = \{ w \in \mathbb{C} : |w| < R' \}$, $0 < R' < \infty$, that

$$\|\rho\|_{L^p(\mathbb{D}_{R'})}, \quad \|\Pi\rho\|_{L^p(\mathbb{D}_{R'})} \leq M_1(\kappa, R', p)\|\mu\|_{L_\infty(\mathbb{C})}; \quad \|h\|_{C(\mathbb{D}_{R'})} \leq M_1(\kappa, R', p)\|\mu\|_\infty.$$ 

Therefore,

$$h(w) = w + T\mu(w) + \omega(w),$$

with $\|\omega\|_{C(\mathbb{D}_{R'})} \leq M_2(\kappa, R')\|\mu\|_\infty^2$. Using the pairing

$$<\nu, \varphi>_{G_R} = -\frac{1}{\pi} \int_{G_R} \nu(\zeta)\varphi(\zeta)d\xi d\eta, \quad \nu \in L_\infty(G_R), \quad \varphi \in L_1(G_R),$$

one can rewrite the above representation in the form

$$h(w) = w + \sum_{k=0}^{\infty} <\mu, \varphi_k>_E (w - c_0^0)^k + \omega(w), \quad \varphi_k(\zeta) = \frac{1}{(\zeta - c_0^0)^{k+1}}. \quad (7)$$

This equality and the condition $c_j^* = c_j + d_j$ for all $j = 0, 1, \ldots, n$, provide the first group of equalities to determine the desired Beltrami coefficient $\mu$:

$$k!d_k = <\mu, \varphi_k>_G + \omega(k)(c_0^0) = <\mu, \varphi_k>_G + O(\|\mu\|_\infty^2), \quad k = j + 1, \ldots, n. \quad (8)$$
On the other hand, (7) and the requirement of preserving $L_p$ norms give
\[ \|h \circ f\|_p^p = \|f + T \rho \circ f\|_p^p = \int_{G_R} |f(z) + T \mu \circ f(z)|^p \, dE_z + O(\|\mu\|_\infty^2) \]
\[ = \int_{G_R} \|f(z)\|^2 + 2 \text{Re} \left( \overline{f(z)} T \mu \circ f(z) \right) + |T \mu \circ f(z)|^2 \right]^{p/2} \, dx \, dy + O(\|\mu\|_\infty^2) \]
\[ = \|f\|_p^p + \frac{p}{2\pi} \text{Re} \left[ \int_{G_R} \mu(\zeta) d\xi \, d\eta \int_{G_R} \frac{|f(z)|^{p-2} \overline{f(z)}}{\zeta - f(z)} \, dx \, dy \right] + O_p(\|\mu\|_\infty^2) \]
(here $z = x + iy$). Now set
\[ \phi(\zeta) = -\frac{p}{2} \int_{G_R} \frac{|f(z)|^{p-2} \overline{f(z)}}{f(z) - \zeta} \, dx \, dy; \]
then the previous equality can be rewritten in the form
\[ \|h \circ f\|_p^p - \|f\|_p^p = \text{Re} \left( \mu, \phi \right)_{G_R} + O_p(\|\mu\|_\infty^2). \]

The function (9) is holomorphic in the disk $D_R^* = \{ w \in \hat{C} : |w - c_0^0| > R \}$ and belongs to the space $B^p_R$ formed in $B_p(B)$ by functions holomorphic in $D_R^*$; moreover, $\phi(\zeta) \neq 0$. The latter follows from the fact that for large $|c|$ we have $\phi(\zeta) = \sum_{k=0}^\infty b_k \zeta^{-k}$ with $b_2 = \frac{p}{2} \|f\|_p^p > 0$.

We shall need the following important lemma whose proof straightforwardly follows the corresponding lemma in [9].

**Lemma 1.** Under the assumptions of the theorem, the function $\phi$ is distinct from a linear combination of the fractions $\varphi_0, ..., \varphi_l$, with $l \leq n$.

According to Lemma 1, the series expansion of $\phi$ in $D_R^*$ must contain the powers $(\zeta - c_0^0)^{-k-1}$ with $k > n$, and therefore, the remainder
\[ \psi(\zeta) = \phi(\zeta) - \sum_{k=0}^n b_k (\zeta - c_0^0)^{-k-1} = \left( \sum_{k=0}^{j-1} + \sum_{s=n+1}^\infty \right) b_k (\zeta - c_0^0)^{-k-1}, \quad s \geq n + 1, \]
is distinct from zero in $D_R^*$. Let us note also that (3) implies
\[ h \circ f(z) = c_0^* + \sum_{j=0}^n c_j^* z^j, \]
with $c_0^* = c_0^0 + d_0^0$ and $c_j^* = c_j^0 d_j^0$ ($j \geq 1$). Thus,
\[ |c_j^*|^2 - |c_j|^2 = \begin{cases} 2 \text{Re}(\overline{c_j^0} d_j^0) = 2 \text{Re}(\overline{c_j^0} < \mu, \varphi_0 >) + O(\|\mu\|_\infty^2), & j = 0, \\ 2|c_j|^2 \text{Re} d_j^0 = 2|c_j|^2 \text{Re} < \mu, \varphi_0 > + O(\|\mu\|_\infty^2), & j \geq 1, \end{cases} \]
and, therefore, $b_j \neq 0$.

Let us now seek the desired Beltrami coefficient $\mu$ in the form
\[ \mu = \xi_j \varphi_j + \sum_{j+1}^n \xi_k \varphi_k + \tau \psi, \quad \mu \notin \mathbb{C} \setminus B = 0, \]
with unknown constants $\xi_j, \xi_{j+1}, ..., \xi_n, \tau$ to be determined from equalities (8) and (10).
Substituting the expression (11) into (8) and (10) and taking into account the mutual orthogonality of $\varphi_k$ on $B$, one obtains the nonlinear equations
\[ k!d_k = \xi_k r_k^2 + O(||\mu||^2), \quad k = j + 1, \ldots, n, \]
\[ ||h \circ f_0||_{2m}^2 - ||f_0||_{2m}^2 = \text{Re} < \xi_j \overline{\varphi}_j + \sum_{j+1}^n \xi_k \overline{\varphi}_k + \tau \overline{\psi}, \phi > + O(||\mu||^2) \]  
(12)
for determining $\xi_k$ and $\tau$. The only remaining equation is a relation for $\text{Re} \xi_j, \text{Im} \xi_j, \text{Re} \tau, \text{Im} \tau$. To distinguish a unique solution, we add three real equations to (12). Let us require that $\xi_j$ satisfy the equality
\[ < \xi_j \overline{\varphi}_j + \sum_{j+1}^n \xi_k \overline{\varphi}_k, \sum_{j}^n b_k \varphi_k > = 0; \]  
(13)
it is reduced to
\[ \xi_j b_j r_j^2 = - \sum_{j+1}^n \xi_k b_k r_k^2 \quad (b_j \neq 0). \]  
(14)
Then we obtain for $\tau$ the equation
\[ ||h \circ f_0||_p^2 - ||f||_p^2 = \text{Re} \overline{\tau} \overline{\psi}, \phi > + O(||\mu||\infty^2), \]
which, letting $\tau$ be real, takes the form
\[ ||h \circ f||_p^2 - ||f||_p^2 = \tau \chi + O(||\mu||\infty^2) \]  
(15)
with $\chi = \sum_k r_k^2$. The summation is taken here over all $k \neq j + 1, \ldots, n$, for which $b_k \neq 0$.

Separating the real and imaginary parts in equalities (12), (14) and adding (15), we obtain $2(n-j)+3$ real equalities, which define a nonlinear $C^1$ smooth (in fact, $\mathbb{R}$-analytic) map
\[ y = W(x) = W'(0)x + O(|x|^2), \]
of the points $x = (\text{Re} \xi_j, \text{Im} \xi_j, \text{Re} \xi_{j+1}, \text{Im} \xi_{j+1}, \ldots, \text{Re} \xi_n, \text{Im} \xi_n, \tau)$ in a small neighborhood $U_0$ of the origin in $\mathbb{R}^{2(n-j)+3}$, taking the values
\[ y = (\text{Re} d_j, \text{Im} d_j, \text{Re} d_{j+1}, \text{Im} d_{j+1}, \ldots, \text{Re} d_n, \text{Im} d_n, ||h \circ f||_p^2 - ||f_0||_p^2) \]
also near the origin of $\mathbb{R}^{2(n-j)+3}$. Its linearization $y = W'(0)x$ defines a linear map $\mathbb{R}^{2(n-j)+3} \rightarrow \mathbb{R}^{2(n-j)+3}$ whose Jacobian only differs from $r_j^2 r_{j+1}^2 \cdots r_n^2 \chi \neq 0$ by a constant factor. Therefore, $x \mapsto W'(0)x$ is a linear isomorphism of the space $\mathbb{R}^{2(n-j)+3}$ onto itself, and one can apply to $W$ the inverse mapping theorem. The latter implies the assertion of Proposition 2.

So, for any collection $d_*(f) = d(f) + O(\varepsilon)$ there exists an $O(\varepsilon)$-quasiconformal homeomorphism $h^\mu$ of $\hat{C}$ conformal on $f(\mathbb{D})$ such that $d_*(f) = (c_0(h^\mu \circ f), \ldots, c_j(h^\mu \circ f)$, and
\[ ||h^\mu \circ f||_{L_p} = ||f||_p^2. \]  
(16)
Note also that the $H^p$-norm is distorted similar to (4) via
\[ ||h^\mu \circ f||_{H^2}^2 = ||f_p/2||_{H^2}^2 = ||f||_{H^p}^2 + O(\varepsilon). \]  
(17)
The relations (6) and (7) show the difference under the actions of deformations given by Propositions 1 and 2 on the Hardy and Bergman spaces.

3. PROOF OF THEOREM 1

Step 1: Underlying lemmas. As was mentioned in the introduction, the proof of the theorem for functions from $H^p$ with $p \geq 2$ follows the lines of [12] with applying in the needed places Lemma 2.
Denote the unit ball of $H^p$ by $B_1(H^p)$ and its subset of nonvanishing functions by $B^0_1(H^p)$. It will be convenient to regard the free coefficients $c_0(f)$ also as elements of $B^0_1(H^p)$, which are constant on the disk $\mathbb{D}$. Let
\[
\hat{B}^0_1(H^p) = B^0_1(H^p) \cup \{f_0\},
\]
where $f_0(z) \equiv 0$.

We shall essentially use Brown’s result quoted above and present it as

**Lemma 2.** [5] For any $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots \in B^0_1(H^p)$, we have
\[
|c_1| \leq (2/e)^{1/p},
\]
with equality only for the rotations of function $\kappa_1(z)$ given by (2).

The following important lemma concerns one of the basic intrinsic features of nonvanishing holomorphic functions (the openness)

**Lemma 3.** [12] Every point $f \in B^0_1(H^p)$ has a neighborhood $U(f, \epsilon)$ in $H^p$, which entirely belongs to $B^0_1(H^p)$, i.e., contains only nonvanishing $H^p$ functions on the disk $\mathbb{D}$. Take the maximal balls $U(f, \epsilon)$ with such property. Then their union
\[
\mathcal{U}^p = \bigcup_{f \in B^0_1(H^p)} U(f, \epsilon)
\]
is an open path-wise connective set, hence a domain, in the space $\hat{B}^0_1(H^p)$.

Let $\mathcal{P}_n$ be the linear space of polynomials of degree less than or equal to $n$, and $\mathcal{P} = \bigcup_n \mathcal{P}_n$.

**Lemma 4.** The intersection $\mathcal{U}^p \cap \mathcal{P}$ is dense in $\mathcal{U}^p$, which means that any $f$ from the distinguished domain $\mathcal{U}^p$ is approximated in $H^p$ by nonvanishing polynomials.

We shall use in the proof of Theorem 1 somewhat different (up to a biholomorphic homeomorphism) model of the universal Teichmüller space $T$, which involves quasiconformally extendable univalent functions in the disk satisfying some non-standard prescribed normalization conditions. Their existence of such maps is ensured by the following lemma related to solutions of the Beltrami equation $\partial \bar{w} = \mu(z) \partial_z w$ on $\mathbb{C}$ with coefficients $\mu$ supported in the disk $\mathbb{D}^*$, i.e., from the ball
\[
\text{Belt}(\mathbb{D}^*)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu|\mathbb{D} = 0, ||\mu|| < 1\}.
\]

**Lemma 5.** [12] For any Beltrami coefficient $\mu \in \text{Belt}(\mathbb{D}^*)_1$ and any $\theta_0 \in [0, 2\pi]$, there exists a point $z_0 = e^{i\alpha}$ located on $S^1$ so that $|e^{i\theta} - e^{i\alpha}| < 1$ and such that for any $\theta$ satisfying $|e^{i\theta} - e^{i\alpha}| < 1$ the equation $\partial \bar{w} = \mu(z) \partial_z w$ has a unique homeomorphic solution $w = w^\mu(z)$, which is holomorphic on the unit disk $\mathbb{D}$ and satisfies
\[
w(0) = 0, \quad w'(0) = e^{i\theta}, \quad w(z_0) = z_0.
\]
This solution is holomorphic on the unit disk $\mathbb{D}$, and hence, $=w^\mu(z_*) = \infty$ at some point $z_*$ with $|z_*| \geq 1$.

**Step 2:** Holomorphic embedding of nonvanishing $H^p$ functions into Teichmüller spaces and lifting the functional $J_n(f) = c_n$. Denote by $\mathcal{B} = \mathcal{B}(\mathbb{D})$ the space of hyperbolically bounded holomorphic functions $\varphi(z)$ (regarded as holomorphic quadratic differentials $\varphi(z) dz^2$ so that $\varphi \circ h(z) h'(z)^2 = \varphi(z)$ for any conformal coordinate map $h$) on the unit disk, with norm
\[
\|\varphi\|_\mathcal{B} = \sup_{\mathbb{D}} (1 - |z|^2)^2 |f(z)|.
\]
Every \( \varphi \in \mathcal{B} \) is the **Schwarzian derivative**
\[
S_w(z) = \left( \frac{w''(z)}{w'(z)} \right)' - \frac{1}{2} \left( \frac{w''(z)}{w'(z)} \right)^2, \quad z \in \mathbb{D},
\]
of a locally univalent function \( w(z) \) in the disk \( \mathbb{D} \) determined (up to a Moebius map of the sphere \( \hat{\mathbb{C}} \)) from the nonlinear differential equation
\[
w''/w' - 3(w''/w')^2/2 = \varphi,
\]
or equivalently, as the ratio \( w = \eta_2/\eta_1 \) of two linearly independent solutions of the linear equation \( 2\eta'' + \varphi\eta = 0 \) in \( \mathbb{D} \).

The space \( \mathcal{B} \) is dual to the space \( A_1(\mathbb{D}) \) of integrable holomorphic functions on \( \mathbb{D} \) with \( L_1 \) norm.

The Schwarzians \( S_w \) of functions \( w \) univalent in the whole disk \( \mathbb{D} \) and having quasiconformal extensions to \( \hat{\mathbb{C}} \) fill a path-wise bounded domain in \( \mathcal{B} \), this domain the most applicable model of the **universal Teichmüller space** \( \mathcal{T} = \text{Teich}(\mathbb{D}) \) (with appropriate normalization of maps \( w \)).

Let \( A_p(\mathbb{D}), p \geq 1 \), be the Bergman spaces of holomorphic functions in \( \mathbb{D} \) with norm
\[
\|f\|_{A_p} = \left( \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p |dz| \right)^{1/p} (z = x + iy).
\]
For each \( f \in H^p \), we have \( \|f\|_{A_p} \leq \|f\|_{H^p} \), which yields, since \( A_p(\mathbb{D}) \subset A_1(\mathbb{D}) \subset \mathcal{B} \) and \( \|f\|_{\mathcal{B}} \leq \|f\|_{A_1(\mathbb{D})} \) for \( \varphi \in A_1(\mathbb{D}) \), that all functions \( f \in H^p \) belong to the space \( \mathcal{B} \). Therefore, these functions can be regarded as the Schwarzian derivatives of locally univalent functions in \( \mathbb{D} \).

In particular, the functions \( f \) from the ball
\[
B_p(H^p) = \{ f \in H^p : \|f\| < \rho \}
\]
with radius \( \rho = 1/2^{1/p} \) satisfy \( \|f\|_{\mathcal{B}} < 2 \), and hence are the Schwarzians of univalent functions in the whole disk \( \mathbb{D} \) admitting quasiconformal extension to the complementary disk
\[
\mathbb{D}^* = \{ z \in \hat{\mathbb{C}} : |z| > 1 \}.
\]
Therefore, such \( f \) are the points of the universal Teichmüller space \( \mathcal{T} \). This implies a **holomorphic embedding** \( \iota \) of the ball \( B_p(H^p) \) and of its open subset
\[
\frac{1}{2^p} \mathcal{U}^p = \{ \frac{1}{2^p} f : f \in \mathcal{U}^p \}
\]
into the space \( \mathcal{T} \).

Now consider the family \( \hat{S}(1) \) of quasiconformally extendable to \( \hat{\mathbb{C}} \) holomorphic univalent functions
\[
w(z) = a_1 z + a_2 z^2 + \ldots, \quad z \in \mathbb{D},
\]
with \( |a_1| = 1 \) and \( w(z_0) = z_0 \) for some point \( z_0 \in S^1 \) (depending on \( w \)), completed in the topology of locally uniform convergence on \( \mathbb{C} \). This collection is a disjunct union
\[
\hat{S}(1) = \bigcup_{-\pi \leq \theta < \pi} S_{\theta},
\]
where \( S_{\theta} \) consists of quasiconformally extendable univalent functions on \( \mathbb{D} \) with expansions
\[
w(z) = e^{i\theta} z + a_2 z^2 + \ldots
\]
having a fixed point \( z_0 \in S^1 \) (also completed in the indicated weak topology). These collections preserve conjugation with rotations \( z \mapsto e^{i\alpha} z \), i.e., contain for each \( w \) the rotated functions \( w_{\alpha}(z) = e^{-i\alpha} w(e^{i\alpha} z) \). There is often enough to deal with the class \( S_0 \) related to \( z_0 = 1 \).

The assertion of Lemma 5 is also valid for the limit functions of sequences \( \{ w_n \} \) of functions \( w_n \in \hat{S}(1) \) with quasiconformal extension, but in the general case the equality \( w(z_0) = z_0 \) must be
understand in terms of the Carathéodory prime ends. As was indicated above, any function from \( \hat{S}(1) \) with \( \theta \) chosen following Lemma 5 is holomorphic on the disk \( \mathbb{D} \) (has there no pole).

This family \( \hat{S}(1) \) is closely related to the canonical class \( S \) of univalent functions \( w(z) \) on \( \mathbb{D} \) normalized by \( w(0) = 0 \), \( w'(0) = 1 \). Every \( w(z) \in S \) has its representatives \( w_{\tau,\theta} \) in \( \hat{S}(1) \) obtained by pre and post compositions of \( w \) with rotations \( z \mapsto e^{i\tau}z \) about the origin, related by

\[
\begin{align*}
    w_{\tau,\theta}(z) &= e^{-i\theta} w(e^{i\tau}z) \quad \text{with} \quad \tau = \arg z_0, \\
    w_{\tau,\theta}(0) &= 1,
\end{align*}
\]

where \( z_0 \) is a point of the circle \( S^1 \) whose image \( w(z_0) = e^{i\theta} \) is a common point of the unit circle and the boundary of domain \( w(\mathbb{D}) \).

This is trivial for the identity map \( w(z) \equiv z \) (then one can take \( \theta = \tau = 0 \)). For any another \( w(z) \) the existence of such a point \( z_0 \) follows from the Schwarz lemma, which yields, together with the assumption \( w'(0) = 1 \), that the image \( w(\mathbb{D}) \) cannot lie entirely in \( \mathbb{D} \); hence, its boundary \( \partial w(\mathbb{D}) \) has common points with the circle \( S^1 \).

This connection also implies that the functions conformal in the closed disk \( \overline{\mathbb{D}} \) are dense in each class \( S_\theta \). Note also that the classes \( S_\theta \) and \( \hat{S}(1) \) are compact in the topology of locally uniform convergence on \( \mathbb{D} \).

The Schwarzian derivatives of \( w \) and \( w_{\tau,\theta} \) are related by

\[
S_{w,\tau,\theta}(z) = S_w(e^{i\tau}z)e^{2i\tau},
\]

which yields that for any fixed \( \theta \) the Schwarzians \( S_w \) of \( w \in S_\theta \) fill the same bounded domain in the space \( B \), which models \( T \).

In other words, the relation (9) allows us to model the universal Teichmüller space \( T \) for any fixed \( \theta \) by the Schwarzians \( S_w \), \( \varphi \) of functions \( w(z) = e^{i\theta}z + a_2 z^2 + \ldots \) from the sets \( S_\theta \).

In this case, going to the limit \( \lim_{t \to 0} \|S_w(te^{i\alpha}z)\|_B \to 0 \) along a curve \( \{S_w(te^{i\alpha}z) : 0 \leq t \leq 1\} \) with fixed nonzero \( \theta \) and \( \alpha \) one attains in the space \( T \) its base point \( \varphi = 0 \), and the corresponding function in \( S_\theta \) is the elliptic fractional linear transformation

\[
w = \frac{e^{i\alpha} z}{(1 - e^{-i\theta})z_0^{-1}z + 1}
\]

with fixed points \( 0 \) and \( z_0 = e^{i\alpha} \). For \( \alpha = \theta = 0 \), this is the identity map.

Note also that the relation (19) is compatible with existence and uniqueness of appropriate conformal and quasiconformal maps, holomorphy of their Taylor coefficients, the Teichmüller space theory, etc. Actually we deal with the classical model of Teichmüller spaces via domain in the Banach spaces of Schwarzian derivatives \( S_w \) in \( \mathbb{D} \) (or in the disk \( \mathbb{D}^* \)) of univalent holomorphic functions normalized either by fixing three boundary points on the unit circle \( S^1 \) or via \( w(0) = 0 \), \( w'(0) = 1 \), \( w(\infty) = \infty \) (often the disk is replaced by the half-plane).

An equivalent model of \( T \) is obtained by applying the inverted functions \( W(z) = 1/w(1/z) \) for \( w \in S_\theta \), which form the corresponding classes \( \Sigma_\theta \) of nonvanishing univalent functions on the disk \( \mathbb{D}^* \) with expansions

\[
W(z) = e^{-i\theta} z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots , \quad W(1/\alpha) = 1/\alpha,
\]

and \( \hat{\Sigma}(1) = \bigcup_\theta \Sigma_\theta \).

Simple computations yield that the coefficients \( a_n \) of \( f \in S_\theta \) and the corresponding coefficients \( b_j \) of \( W(z) = 1/f(1/z) \in \Sigma_\theta \) are related by

\[
b_0 + e^{2i\theta} a_2 = 0 , \quad b_n + \sum_{j=1}^{n} \epsilon_{n,j} b_{n-j} a_{j+1} + \epsilon_{n+2,0} a_{n+2} = 0 , \quad n = 1, 2, \ldots ,
\]
where \( \epsilon_{n,j} \) are the entire powers of \( e^{i\theta} \) (\( \theta \) is fixed). This successively implies the representations of \( a_n \) by \( b_j \) via

\[
a_n = (-1)^{n-1} \epsilon_{n-1,0} b_0^{n-1} - (-1)^{n-1} (n-2) \epsilon_{1,n-3} b_1 b_0^{n-3} + \text{lower terms with respect to } b_0. \quad (20)
\]

By abuse of notation, we shall denote the holomorphic embedding of \( H^p \) into the space \( T \) modelled by Schwarzians in \( D^e \) by the same letter \( \iota \). The image \( \iota H^p \) is a non-complete linear subspace in \( B \), and the image of the distinguished domain \( \frac{1}{\partial z^2} U^p \) is a complex submanifold in \( T \).

Note that the coefficients \( a_n \) of Schwarzians

\[
S_w(z) = \sum_{0}^{\infty} a_n z^n
\]

are represented as polynomials of \( n + 2 \) initial coefficients of \( w \in S_\theta \) and, in view of (10), as polynomials of \( n + 1 \) initial coefficients of the corresponding \( W \in \Sigma_\theta \) (provided that \( \theta \) and \( \alpha \) are given and fixed and the number \( e^{i\theta} \) is considered to be a constant).

We denote these polynomials by \( J_n(w) \) and \( \tilde{J}_n(W) \), respectively, and will deal with these polynomial functionals only on the union of admissible classes \( S_\theta \) or \( \Sigma_\theta \).

**Step 3: Lifting to covering space \( T_1 \) and estimating the restricted plurisubharmonic functional.** Our next step is to lift both polynomial functionals \( J_n(w) \) and \( \tilde{J}_n(W) \) onto the Teichmüller space \( T_1 \) of the punctured disk \( D_a = D \setminus \{0\} \), which covers \( T \).

Recall that the points of \( T_1 \) are the classes \([\mu]_{T_1}\) of \( T_1 \)-equivalent Beltrami coefficients \( \mu \in \text{Belt}(\mathbb{D}) \) so that the corresponding quasiconformal automorphisms \( w^\mu \) of the unit disk coincide on both boundary components (unit circle \( S^1 \) and the puncture \( z = 0 \)) and are homotopic on \( \mathbb{D} \setminus \{0\} \). This space also is a complex Banach manifold.

Due to the Bers isomorphism theorem [4], the space \( T_1 \) is biholomorphically isomorphic to the Bers fiber space

\[
\mathcal{F}(T) = \{ (\phi_T(\mu), z) \in T \times \mathbb{C} : \mu \in \text{Belt}(\mathbb{D}), z \in w^\mu(\mathbb{D}) \}
\]

over the universal space \( T \) with holomorphic projection \( \pi(\psi, z) = \psi \). This fiber space is a bounded hyperbolic domain in \( B \times \mathbb{C} \) and represents the collection of domains \( D_\mu = w^\mu(\mathbb{D}) \) as a holomorphic family over the space \( T \).

The indicated isomorphism between \( T_1 \) and \( \mathcal{F}(T) \) is induced by the inclusion map \( j : D_a \hookrightarrow D \) forgetting the puncture at the origin via

\[
\mu \mapsto (S_{w^\mu}(0), \mu) \quad \text{with} \quad \mu = j_\ast \mu := (\mu \circ j_0, \overline{\mu} / j_0),
\]

where \( j_0 \) is the lift of \( j \) to \( D \).

Now, letting

\[
\tilde{J}_n(\mu) = \tilde{J}_n(W^\mu),
\]

we lift these functionals from the sets \( S_\theta \) and \( \Sigma_\theta \) onto the ball \( \text{Belt}(\mathbb{D}) \). Then, under the indicated \( T_1 \)-equivalence, i.e., by the quotient map

\[
\phi_{T_1} : \text{Belt}(\mathbb{D}) \rightarrow T_1, \quad \mu \mapsto [\mu]_{T_1},
\]

the functional \( \tilde{J}_n(W^\mu) \) is pushed down to a bounded holomorphic functional \( J_n \) on the space \( T_1 \) with the same range domain.

Equivalently, one can apply the quotient map \( \text{Belt}(\mathbb{D}) \rightarrow T \) (i.e., \( T \)-equivalence) and compose the descended functional on \( T \) with the natural holomorphic map \( \iota_1 : T_1 \rightarrow T \) generated by the inclusion \( D_a \hookrightarrow D \) forgetting the puncture. Note that since the coefficients \( b_0, b_1, \ldots \) of \( W^\mu \in \Sigma_\theta \) are uniquely determined by its Schwarzian \( S_{W^\mu} \), the values of \( J_n \) in the points \( X_1, X_2 \in T_1 \) with \( \iota_1(X_1) = \iota_1(X_2) \) are equal.

Using the Bers isomorphism theorem, we regard the points of the space \( T_1 \) as the pairs \( X_{W^\mu} = (S_{W^\mu}, W^\mu(0)) \), where \( \mu \in \text{Belt}(\mathbb{D}) \) obey \( T_1 \)-equivalence (hence, also \( T \)-equivalence). Denote (for
simplicity of notations) the composition of $\mathcal{J}_n$ with biholomorphism $T_1 \cong F(T)$ again by $\mathcal{J}_n$. In view of (20) and (21), it is presented on the fiber space $F(T)$ by

$$\mathcal{J}(X_{W^\mu}) = \mathcal{J}(S_{W^\mu}, t), \quad t = W^\mu(0). \quad (23)$$

This yields a logarithmically plurisubharmonic functional $|\mathcal{J}_n(S_{W^\mu}, t)|$ on $F(T)$.

We have to estimate a smaller plurisubharmonic functional arising after restriction of $\mathcal{J}(S_{W^\mu}, t)$ to $S_W \in \iota\left(\frac{1}{2^p L^p}\right)$ and to $W^\mu(0)$ filling some subdomain $D_\theta$. Since our functionals are polynomials, they are defined for all $S_W \in T$ and $t$ from some domain $D_\theta$ containing $D_X\theta$. We define on $D_\theta$ the function

$$u_\theta(t) = \sup_{S_{W^\mu}} |\mathcal{J}_n(S_{W^\mu}, t)|,$$

where the supremum is taken over all $S_{W^\mu} \in T$ admissible for a given $t = W^\mu(0) \in D_\theta$.

The following basic lemma is a generalization of the corresponding result in [12]. It provides that this function inherits subharmonicity of $\mathcal{J}_n$.

**Lemma 6.** The function $u_\theta(t)$ is subharmonic on its domain $D_\theta$ filled by the admissible values of $W^\mu(0)$.

The proof of this lemma is complicated. Similar to [12], it involves the approximation of elements from $\frac{1}{2^p L^p}$ by polynomials given by Lemma 5 which provides the finite dimensional submanifolds weakly approximating $\iota\left(\frac{1}{2^p L^p}\right)$ in the underlying space $T$ (and simultaneously in the space $T_1$) in the topology of locally uniform convergence on $\mathbb{C}$.

Since the set $\iota\left(\frac{1}{2^p L^p}\right)$ is a complex submanifold in $T$, the restriction of the function $|\mathcal{J}(S_{W^\mu}, t)|$ to this submanifold and to the corresponding values of $t = W^\mu(0)$ also is plurisubharmonic. The arguments from [12] are straightforwardly extended to this restriction, giving in a similar way the corresponding maximal subharmonic function

$$u_\theta(t) = \sup_{S_{W^\mu}} |\mathcal{J}_n(S_{W^\mu}, t)|,$$

the supremum here is taken over $S_{W^\mu} \in \iota\left(\frac{1}{2^p L^p}\right)$.

One also has to extend the previous construction to the increasing unions of the quotient spaces

$$T_s = \bigcup_{j=1}^s \hat{\Sigma}_{\theta_j} / \sim = \bigcup_{j=1}^s \{(S_{W_{\theta_j}}, W^\mu_{\theta_j}(0))\} \simeq T_1 \cup \cdots \cup T_1, \quad (24)$$

where $\theta_j$ run over a dense subset $\Theta \subset [-\pi, \pi]$, the equivalence relation $\sim$ means $T_1$-equivalence on a dense subset $\hat{\Sigma}(1)$ in the union $\hat{\Sigma}(1)$ formed by univalent functions $W_{\theta_j}(z) = e^{-i\theta_j} z + b_0 + b_1 z^{-2} + \ldots$ on $\mathbb{D}^*$ with quasiconformal extension to $\hat{\mathbb{C}}$ satisfying $W_{\theta_j}(1) = 1$, and

$$W^\mu_{\theta_j}(0) := (W^{\mu_{\theta_j}}_{\theta_1}(0), \ldots, W^{\mu_{\theta_j}}_{\theta_s}(0)).$$

The Beltrami coefficients $\mu_j \in \text{Belt}(\mathbb{D})_1$ are chosen here independently. The corresponding collection $\beta = (\beta_1, \ldots, \beta_s)$ of the Bers isomorphisms

$$\beta_j : \{(S_{W_{\theta_j}}, W^{\mu_{\theta_j}}_{\theta_j}(0))\} \to F(T)$$

determines a holomorphic surjection of the space $T_s$ onto $F(T)$.

Taking in each union (24) the corresponding collection $\iota_s\left(\frac{1}{2^p L^p}\right)$, one obtains in a similar fashion the increasing sequence of maximal subharmonic functions

$$u_s(t) = \sup_{\Theta} u_{\theta_s}(t) = \sup \{ |\mathcal{J}_n(S_{W^\mu}, t)| : S_{W^\mu} \in \bigcup_{s} \iota_s\left(\frac{1}{2^p L^p}\right) \},$$

for $s$ a finite dimensional submanifolds.
whose limit
\[ u(t) = \lim_{s \to \infty} u_s(t) \]  
(25)
is determined and subharmonic on a disk
\[ D_\rho = \bigcup_{s} D_{p,\theta_s}, \]  
(26)
because the union of spaces (15) admits the circular symmetry.

**Step 4: Determination of the range domain of** \( W^\mu(0) \). Our goal now is to find the domain of admissible values of \( W^\mu(0) \), i.e. the radius of the disk (26). This requires a covering estimate of Koebe’s type given by the following lemma.

Let \( G \) be a domain in a complex Banach space \( X = \{x\} \) and \( \chi \) be a holomorphic map from \( G \) into the universal Teichmüller space \( T \) modeled as a bounded subdomain of \( B \). Consider in the unit disk the corresponding Schwarzian differential equations

\[ S_w(z) = \chi'(x) \]  
(27)
and pick their univalent solutions \( w(z) = z + a_2 + \ldots \) of the differential equation (17), the image domain \( w(\mathbb{D}) \) covers entirely the disk \( \{ |w| < 1/(2|a_2^0|) \} \).

The radius value \( 1/(2|a_2^0|) \) is sharp for this collection of functions, and the circle \( \{ |w| = 1/(2|a_2^0|) \} \) contains points not belonging to \( w(\mathbb{D}) \) if and only if \( \|a_2\| = |a_2^0| \) (i.e., when \( w \) is one of the maximizing functions).

**Lemma 7.** (1) For every indicated solution \( w(z) = z + a_2 + \ldots \) of the differential equation (17), the image domain \( w(\mathbb{D}) \) covers entirely the disk \( \{ |w| < 1/(2|a_2^0|) \} \).

The radius value \( 1/(2|a_2^0|) \) is sharp for this collection of functions, and the circle \( \{ |w| = 1/(2|a_2^0|) \} \) contains points not belonging to \( w(\mathbb{D}) \) if and only if \( |a_2| = |a_2^0| \) (i.e., when \( w \) is one of the maximizing functions).

(b) The inverted functions

\[ W(\zeta) = 1/w(1/\zeta) = \zeta - a_2^0 + b_1\zeta^{-1} + b_2\zeta^{-2} + \ldots \]

map the disk \( \mathbb{D}^* \) onto a domain whose boundary is entirely contained in the disk \( \{ |W + a_2^0| \leq |a_2^0| \} \).

Now we show that in the case of nonvanishing \( H^p \) functions this radius \( 2|a_2^0| \) is naturally connected with the extremal function \( \kappa_{1,p}(z) \) maximizing the coefficient \( |c_1| \).

Consider the collection \( \mathcal{N}_p \) \( (p > 1) \) of all nonvanishing \( H^p \) functions located in the ball \( \{ \|\varphi\| < 2 \} \) in \( B \) and denote the minimal radius of the balls in \( H^p \) containing these functions by \( r(p) \); that is

\[ r(p) = \sup\{ \|f\|_p : \|f\|_B \leq 2, \ f(z) \neq 0 \ \text{in} \ \mathbb{D} \}. \]

For any such \( f \), the solutions \( w(z) \) of the equation \( S_w = f \) are univalent holomorphic functions on the disk \( \mathbb{D} \). The set \( \bigcup_{2r(p)} H^p \) applied earlier is a proper subset of \( \mathcal{N}_p \).

**Lemma 8.** For any space \( H^p \), \( p > 1 \), and its subset \( \mathcal{N}_p \), we have the equality

\[ S_{w_0}(z) = r(p)\kappa_{1,p}(z) \]  
(29)
which means that the Schwarzian of the extremal univalent function \( w_0(z) \) maximizing the second coefficient \( a_2 \) on the set \( \mathcal{N}_p \) equals the extremal function for \( c_1 \) (hence, the maximizing function for (28) also is unique).

**Proof.** In view of Lemma 2, it is enough to establish that

\[ S'_{w_0}(0) = c_1^0 \neq 0 \]  
(30)
(in other words, that the zero set of the functional \( J_1(f) = c_1 \) is separated from the set of rotations (9) of the function \( u_0 \)). This yields that the maximal function \( (25) \) for the functional \( |J_1(f)| = |c_1| \) is defined on the whole disk \( \mathbb{D}_{2|a_2|} \), attaining its maximum on the boundary circle.

We pass to intersections
\[
B_{1,M}^0(H^p) = B_{1}^0(H^p) \cap \{ f \in L_\infty(\mathbb{D}) : \|f\|_\infty < M \},
\]
with \( M < \infty \), getting the corresponding subharmonic functions
\[
u_M(t) = \sup\{|\mathcal{J}(S_{W^p}, t)| : S_{W^p} \in t \left( 2\mathbb{D} \cap \mathcal{P} \right) \cap B_{1,M}^0(H^p) \}
\]
with \( \lim_{M \to \infty} u_M(t) = u(t) \) and the points \( f_M = S_{w_0,M} \), maximizing \( |a_2| \) on these sets. The collection \( \{f_M\} \) is weakly compact in \( H^p \).

Applying Lemma 7, one obtains similar to [12] that both maximal values \( |a_2| \) and \( |c_1| \) are obtained on the same function
\[ f_0(z) = \lim_{M \to \infty} f_M(z) = S_{w_0}, \]
and the uniqueness in Lemma 2 yields that this function must coincide with \( r(p)\kappa_{1,p} \). This completes the proof of Lemma 8.

**Step 5: Finishing the proof.** Now we can prove the assertion of the theorem. The assumption \( p \geq 2 \) insures that the boundary function \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) of any \( f \in H^p \) admits Parseval’s equality
\[
1 \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta = \sum_{n=1}^{\infty} |c_n|^2.
\]
In particular, for \( f(z) = \kappa_{1,p}(z) = \sum_{0}^{\infty} c_n^0 z^n \) we have from (2)
\[
|c_1^0|^2 = (2/e)^{2(1-1/p)} = 0.5041...^{1-1/p} > 0.5041...
\]
for all \( p > 1 \). Hence, by (31),
\[
\sum_{2}^{\infty} |c_n^0|^2 < 0.5 < |c_1^0|^2.
\]

Now take \( n = 2 \) and, letting \( f_2(z) = f(z^2) \), consider on the set \( B_{1}^0(H^p) \) the functional
\[ I_2(f) = \max \{|J_2(f)|, |J_2(f_2)|\}. \]
Since the correspondence \( f(z) \to f_2(z) \) is linear, the functional \( J_2(f_2) \) is holomorphic with respect to \( f \) in \( H^{2n} \) norm and naturally extends to a holomorphic functional on the spaces \( \mathbf{T} \). Hence, the functional \( I_2 \) is plurisubharmonic on \( \mathbf{T} \). It lifted to the covering space \( \mathbf{T}_1 \) together with \( J_2 \).

Similar to above, this lifting generates via (25) a nonconstant radial subharmonic function \( u_2(t) \) on the disk \( \{|t| < 1/2|a_2^0|\} \), \( t = W^\mu(0) \). This function is logarithmically convex, hence monotone increasing, and thus attains its maximal value at \( |t| = 2|a_2^0| \).

Taking into account the connection between the extremal value \( |a_2^0| \) and the function \( \kappa_{1,p} \) established by Lemma 8, one concludes that the maximal value of \( I_2(f) \) on \( B_{1}^0(H^p) \) is attained on the pair \( (f, f_2) \) with
\[ f(z) = \kappa_{1,p}(z), \quad f_2(z) = \kappa_{1,p}(z^2). \]
Since the set of admissible maps \( u(z) \in \tilde{S}(1) \) with \( S_u = f \) for \( J_2(f_2) \) is the same as for \( J_2(f) \), one derives from above
\[ \max_{B_{1}^0(H^p)} I_2(f) = \max \{|c_1^0|, |c_2^0|\}. \]
which by (33) is equal to $|c_1^0| = (2/e)^{-1/p}$. This yields the desired estimate (1) for $n = 2$; the extremal maximizing function is determined up to the pre and post rotations about the origin.

Now consider subsequently for $n = 3, 4, \ldots$ the functionals

$$I_n(f) = \max \{ |J_n(f_1)|, |J_n(f_2)|, \ldots, |J_n(f_n)| \}$$

with $f_n(z) = f(z^n)$. Similar to $I_2$, this does not expand the set of admissible maps $w(z) \in \hat{S}(1)$ with $S_w = f$, and therefore, $|I_n(f)|$ has the same maximum, as $|J_n(f_n)|$.

Each functional $I_n$ generates similar to above the corresponding circularly symmetric subharmonic function $u_n(t)$ on the disk $\{ |t| < 1/2 |a_2^0| \}$, $t = W^p(0)$, which provides in the same way the bound

$$\max_{B_n^1(H^p)} I_n(f) = \max \{ |c_1^n|, |c_2^n| \} = (2/e)^{-1/p},$$

with a similar description of the extremal functions. This completes the proof of Theorem 1.

4. ADDITIONAL REMARKS

4.1. Holomorphy in parameters. As was mentioned in the proof of Theorem 1, the holomorphic dependence of normalized quasiconformal maps on complex parameters is an underlying fact for the Teichmüller space theory and for many other applications. It was first established and applied by Ahlfors and Bers in [1] for maps with three fixed points on $\hat{C}$.

Another somewhat equivalent proof of holomorphy involves the variational technique for quasiconformal maps. For the maps $w$ from $S_{p,\alpha}$, this holomorphy is a consequence of the following lemma from [2], Ch. 5 combined with appropriate Möbius maps.

**Lemma 9.** Let $w(z)$ be a quasiconformal map of the plane $\hat{C}$ with Beltrami coefficient $\mu(z)$ which satisfies $\|\mu\|_\infty < \varepsilon_0 < 1$ and vanishes in the disk $\{ |z| < r \}$. Suppose that $w(0) = 0$, $w'(0) = 1$, and $w(1) = 1$. Then, for sufficiently small $\varepsilon_0$ and for $|z| \leq R < r_0(\varepsilon_0, r)$ we have the variational formula

$$w(z) = z - \frac{2^p(z - 1)}{\pi} \int_{|\zeta| > r} \frac{\mu(\zeta) d\xi d\eta}{\zeta^2(\zeta - 1)(\zeta - z)} + \Omega_{\mu}(z),$$

where $\zeta = \xi + i\eta$, $\max_{|z| \leq R} |\Omega_{\mu}(z)| \leq C(\varepsilon_0, r, R)\|\mu\|_\infty^2$, $r_0(\varepsilon_0, r)$ is a well defined function of $\varepsilon_0$ and $r$ such that $\lim_{\varepsilon_0 \to 0} r_0(\varepsilon_0, r) = \infty$, and the constant $C(\varepsilon_0, r, R)$ depends only on $\varepsilon_0$, $r$ and $R$.

4.2 Remarks on the case $1 < p < 2$. All arguments in the proof of Theorem 1, excluding the Parseval equality, applied in the last step, work for any $p > 1$. In fact, this equality was applied only to the function $\kappa_{1, p}$ maximizing $|c_1|$ and was used by estimation $|c_n|$ for comparison of the initial non-free coefficient of functions $\kappa_{1, p}(z^n)$, $1 \leq m \leq n$.

The explicit representation (2) of $\kappa_{1, p}$ shows that this function is bounded on the unit disk if $1 < p < 2$; hence belongs to $H^2$. However, since for $1 < p < 2$,

$$\|\kappa_{1, p}\|_{H^2} > \|\kappa_{1, p}\|_{H^p}$$

the needed relation (31) giving (32), (33) fails. The functionals $J_n$ and $I_n$ cannot be compared on this way.

4.3. On extremal functions in Bergman spaces. One of the interesting extensions of the Hummel-Scheinberg-Zalcman problem (also still unsolved) is to estimate the Taylor coefficients of nonvanishing holomorphic maps $f(z) = c_0 + c_1 z + \ldots$ of the unit disk $\mathbb{D}$ into other complex Banach spaces $X$. Denote by $B(X)$ the unit ball of $X$.

We illustrate here on the case of Bergman’s space $A_2$ that the features of extremal functions can be essentially different from above. Recall that the norm of $A_2$ is $\|f\| = \left( \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dx dy \right)^{1/2}$.
The collection \( B_0(A_2) \) of nonvanishing holomorphic functions \( f(z) \) mapping the disk \( \mathbb{D} \) into the closed ball \( \overline{B(A_2)} \) (i.e., with \( f(z) \neq 0 \) on \( \mathbb{D} \) and \( \|f\| \leq 1 \)) is compact in the weak topology of the locally uniform convergence in \( \mathbb{D} \). So any holomorphic coefficient functional
\[
J(f) = J(c_{m_1}, \ldots, c_{m_s}) \quad \text{with} \quad 1 \leq m_1 < m_2 < \cdots < m_s = N < \infty
\] (34)
has an extremal \( f_0 \) on which \( |J(f)| \) attains its maximum on \( B_0(A_2) \).

While the extremal functions of many problems in Hardy spaces are bounded, Proposition 2 implies, for example, that any function \( f_0 \in B_0(A_2) \) maximizing the functional (34) must be unbounded on the disk \( \mathbb{D} \) (compare with the extremal problems for nonvanishing Bergman functions investigated e.g. in [2], [3]).

This difference is caused by the fact mentioned after the proof of Proposition 2: quasiconformal deformations created by Propositions 1 and 2 preserve the norm in \( A_2 \), while the norm of the Hardy spaces can be increased.

Indeed, it follows from Proposition 2 that any extremal \( f_0 \) of \( J(f) \) on \( B_0(A_2) \) must be unbounded on \( \mathbb{D} \), unless \( f_0 \) is a zero-free polynomial
\[
p_N(z) = c_0 + c_1 z + \cdots + c_N z^N
\] (35)
(with \( c_0 \neq 0 \); otherwise, one can vary the coefficients \( c_k \) and obtain by this lemma an admissible function \( f_0 \in B_0(A_2) \) with \( |J(f_0)| > |J(f_0)| \).

It remains to establish that the polynomials (35) with \( \|p_N\|_{A_2} \leq 1 \) cannot be extremal for \( J(f) \). We pick a sufficiently small \( \varepsilon > 0 \) and consider the polynomial
\[
p_\varepsilon(z) = -\varepsilon c_0 + \varepsilon z^{N+1},
\]
for which
\[
\max_{\mathbb{D}} |p_\varepsilon(z)| < \max_{\mathbb{D}} |p_N(z)|.
\]
Then the Rouché theorem yields that the polynomial
\[
P_{N+1,\varepsilon}(z) = p_N(z) + p_\varepsilon(z) = (1 - \varepsilon)c_0 + c_1 z + \cdots + c_N z^N + \varepsilon z^{N+1}
\]
also must be, together with \( p_N \), zero-free on \( \mathbb{D} \). Its norm is estimated by
\[
\|p_{N+1,\varepsilon}\|_{A_2}^2 = (1 - \varepsilon)^2 |c_0|^2 + \frac{|c_1|^2}{2} + \cdots + \frac{|c_N|^2}{N+1} + \frac{\varepsilon^2}{N+2} = \|p_N\|_{A_2}^2 - 2\varepsilon + O(\varepsilon^2) < \|p_N\|_{A_2}^2 = 1,
\]
which implies that \( P_{N+1,\varepsilon} \) is an admissible function, with
\[
|J(p_{N+1,\varepsilon})| = |J(p_N)| = \max\{|J(f)| : f \in B_0(A_2)\}.
\] (36)
But this contradicts to Proposition 2, because this proposition allows one to construct the variations of \( p_{N+1,\varepsilon} \), which preserve its \( A_2 \)-norm and increase \( |J(p_{N+1,\varepsilon})| \), disturbing (36). This completes the proof of our claim.

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