A SUPERMARTINGALE ARGUMENT FOR CHARACTERIZING THE FUNCTIONAL HILL PROCESS WEAK LAW FOR SMALL PARAMETERS

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Abstract. The law of the functional Hill process is guided by sums of independent random variables (rv) when the distribution function (df) of the data is in the Frechet or Weibull extremal domain of attraction and the Kolmogorov Theorem for centred rv’s is then used. But for df’s in the Weibull domain, the limiting laws is a sum of dependent rv’s. We show in this paper that such laws are derived from this following process
\[ \sum_{j=1}^{k-1} (f(j) - f(j-1)) (\exp(-\gamma \sum_{h=j}^{k-1} E_h/h) - E \exp(-\gamma \sum_{h=j}^{k-1} E_h/h)), \]
where \( E_1, E_2, \ldots \) are independent standard rv’s, \( \gamma > 0 \), \( k \) is a positive integer and \( f(j) \) is an increasing function of the integer \( j \geq 0 \) and \( f(0) = 0 \). We use martingale results to characterize the asymptotic law of this process and next apply the findings to determine the asymptotic behavior the functional Hill process for small parameters within the Extreme Value Theory (EVT) field. Simulations, statistical tests and software packaging conclude the paper.

1. Introduction

We are interested, in this paper, by the functional stochastic processes based on extreme values of independent and identically distributed rv’s \( X_1, X_2, \ldots \) defined as follows. Let, for each \( n \geq 1 \), \( X_{1,n} \leq \cdots \leq X_{n,n} \) the related order statistics, let \( k(n) \) be a sequence of integers satisfying \( 1 \leq k(n) < n \), and let finally \( f(j) \) be a real and increasing function of \( j \in \mathbb{N} \) such that \( f(0) = 0 \). The following empirical process, named after the functional Hill process,

\[ \sum_{j=1}^{k(n)} f(j) (\log(X_{n-j+1,n}) - \log(X_{n-j,n})) / k(n), \]

was introduced by Deme et al.(2012) [4] as a generalization of the Diop et al. continuous generalization of the Hill statistic for \( f(j) = j^\tau \), for \( j \leq 1 \) and \( f(0) = 0 \), \( \tau > 0 \) (See Diop and Lo (1994 [6] and 2009 [7]), that is the Diop and Lo generalization of Hill’s estimator : \n
\[ \sum_{j=1}^{k(n)} j^\tau (\log(X_{n-j+1,n}) - \log(X_{n-j,n})) / k(n), \tau > 0, \]

2000 Mathematics Subject Classification. Primary 62E20, 62F12, 60F05. Secondary 60B10, 60F17.

Key words and phrases. Asymptotic laws; Supermartingale; Functional Hill process; Extreme Value Theory; Simulation Studies; Statistical tests.
These statistics are closely related to Kernel-Type estimators like the csörgő et al. (2),

$$\left\{ \sum_{j=1}^{k(n)} j K(j/k(n)) \left( \log(X_{n-j+1,n}) - \log(X_{n-j,n}) \right) / k(n) \right\} / \left\{ \sum_{j=1}^{k(n)} K(j/k(n)) \right\},$$

where K is Kernel function and analogue ones (See Gogebeur et al. [10] and Groeneboom [11]). All these statistics are generalizations of the Hill estimator corresponding to K=1 in (1.3), $\tau = 1$ in (1.2). This latter plays a pivotal role in Extreme Value Theory (UEVT).

This theory has its foundations in finding the asymptotic law of the maximum observation $X_{n,n} = \max(X_1, ..., X_n)$. It is said that the underlying df $F$ of the observations is attracted to some df $H$ if for some sequences $(a_n > 0)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, we have for any continuity point $x \in \mathbb{R}$ of $H$,

$$\lim_{n \to \infty} P\left( \frac{X_{n,n} - b_n}{a_n} \leq x \right) = \lim_{n \to \infty} F^n(a_n x + b_n) = H(x).$$

It is known that, when it is nondegenerated, $H$ can be parametrized as $G_{\tau}(x) = \exp(- (1 + \gamma x)^{-1/\gamma})$, $1 + \gamma x > 0$, $\gamma \in \mathbb{R}$, where $G_{\tau}(x)$ is interpreted as $\exp(- \exp(-x))$ for $\gamma = 0$, named as the Generalized Extremal Value (GEV) distribution. It is said that $F$ is in the domain of attraction of $G_{\tau}$, hereby denoted as : $F \in D(G_{\tau})$.

The reader is referred to de Haan and Feirreira [12], Resnick [15], Galambos [9], Beirlant et al. [11] and Emberechts et al. [8] for a modern account of UEVT.

Although the parameter $\gamma$ in the GEV is continuous, the three cases ($\gamma < 0$, $\gamma = 0$ and $\gamma > 0$, respectively named after the Weibull, Gumbel and Frechet cases, may behave radically differently. But in all the cases, the Hill statistic is used to estimate what is called the extreme value index in the following sense with $\tau = 1$ : For $\gamma \geq 0$, ref. [15] converges in probability to $\gamma$ as $n \to +\infty$ and $k/n \to 0$; for $\gamma < 0$, then the upper endpoint of $G(x) = F(e^x)$ defined by $y_0 = \log\sup\{x \in \mathbb{R}, F(x) < 1\}$, is finite and ref. [15] when normalized by $y_0 - G^{-1}(1 - k(n)/n)$ converges to $(1 - \gamma)^{-1}$ as $n \to +\infty$ and $k(n)/n \to 0$ and $G^{-1}$ stands for the generalized inverse function of $G$.

The Diop and Lo generalization of Hill’s estimator (1.2) has been introduced in [6] and studied in [7] where its asymptotic normality was proved for any $\gamma$ but for $\tau > 1/2$. The Hungarian Gaussian Approximation used in this paper could not allow to find the asymptotic law for $\tau \leq 1/2$. Recently, the functional form (1.1) which generalizes (1.5) for $f_{\tau}(j) = j^\tau$, has been extensively studied for Frechet and Gumbel cases by Demir et al. [14] who proved this : (1.1) has a Gaussian limiting process when $A(2,f) = \sum_{j=1}^{+\infty} f(j)^2/j^2 = +\infty$ and

$$B_n(f) = \max\{f(j)/j, 1 \leq j \leq k\}/(\sum_{j=1}^{k} f(j)^2/j^2)^{1/2} \to 0$$

as $n \to \infty$. 


It has a non Gaussian limiting process when \( A(2, f) < +\infty \). When particularized for \( f_\tau \), we get that the asymptotic normality holds for \( \tau \geq 1/2 \) and not for \( 0 < \tau < 1/2 \). Their results are based on sums of independent rv's, and then on Kolmogorov’s type theorems (see [14]). When put together, for the class of functions \( f_\tau \), we remark that the behavior of (1.2) is known for any \( \gamma \) in the whole extremal domain except for the Weibull domain and for \( 0 < \tau \leq 1/2 \), that if for small parameters \( \tau \)'s.

This problem remained unsolved, may be, by the fact that it depends of sums of dependent data and that we did not have the appropriate setting. It is also worth mentioning that our methods are more general and may be used for Kerner-type statistics. This will be done further.

We intend to use here a supermartingale argument to provide a definitive tool for solving the just described problem. In the sequel, we get rid of the quotient \( k(n) \) in (1.1) and (1.2) to directly study

\[
T_n(f) = \sum_{j=1}^{k(n)} f(j) \log(X_{n-j+1,n}) - \log(X_{n-j,n}),
\]

and its particular form

\[
D_n(\tau) = \sum_{j=1}^{k(n)} j^{\tau} (\log X_{n-j+1,n} - \log X_{n-j,n}), \quad \tau > 0,
\]

Our best achievement is the asymptotic characterization of the leading part that guides the asymptotic law of (1.4) when \( F \in D(G_{-1/\gamma}), \quad <\gamma < 1/20 \), by providing its general law, and its specific law for \( f = f_\tau \). This non Gaussian law will be described and its f.d. computed with the help of a computer package.

The rest of the paper is organized as follows. In Section 2, we study a special process based on a sequence of iid standard exponential rv’s whose limiting law will be found with martingale techniques. In section 3, we apply the results of Section 2 to our problem. In section 3, we perform simulations and make statistical tests in the extreme values domain.

2. A supermartingale tool

Let \( E_1, E_2, \ldots \) be iid standard exponential rv’s, \( k \geq 1, \gamma > 0, \) and \( f(j) = f(j) - f(j - 1) \) for \( j \geq 1 \). Define the sequence in \( k \in \mathbb{N} \)

\[
W_k(f) = \sum_{j=1}^{k} f(j) (\exp(-\gamma \sum_{h=j}^{k} E_h/h) - \mathbb{E} \exp(-\gamma \sum_{h=j}^{k} E_h/h)).
\]

Consider the filtration \( \mathcal{F}_k = \sigma(E_1, \ldots, E_k), k \geq 1 \) and remark the sequence \( (W_k)_{k \geq 1} \) is adapted with respect to \( (\mathcal{F}_k)_{k \geq 1} \). We have the following intermediate results.
Theorem 1. The sequence $W_k(f)$ is a supermartingale with respect to $\mathcal{F}_k$. Furthermore, it converges almost-surely (a.s) to random variable $W_\infty(f)$ with finite expectation whenever

\[(K1) \limsup_{k \to +\infty} k^{-\gamma} \sum_{j=L}^{k-1} \mathcal{T}(j)j^{\gamma-1/2} < +\infty\]

holds.

Corollary 1. For $f(j) = f_\tau(j) = j^\tau, 0 < \tau < 1/2, W_k(f_\tau)$ converges almost surely to a finite random variable $W_\infty(\tau)$.

Proof. Now denote

\[S_{j,k} = \exp(-\gamma \sum_{h=j}^{k} E_h/h)\]

\[V_k = \exp(-\gamma E_k/k)\]

for $k \geq 1$. We have for $k \geq 1$

\[W_{k+1} = \sum_{j=1}^{k} \mathcal{T}(j)(S_{j,k}V_{k+1} - E(S_{j,k}V_{k+1})) + \mathcal{T}(k+1)(V_{k+1} - E(V_{k+1}))\]

Let us use the following three facts. First $V_{k+1}$ and $S_{j,k}$ are independent for $1 \leq j \leq k$. Secondly $V_{k+1}$ is independent of $\mathcal{F}_k$. Finally, $E(V_{k+1}) = \gamma(k+1) = (1 + \gamma/(k + 1))^{-1}$ by using the formula of the generating function of a standard exponential rv's. By combining these facts, we get

\[E((V_{k+1} - E(V_{k+1}))/\mathcal{F}_k) = 0\]

and next

\[E(W_{k+1}/\mathcal{F}_k) = E(V_{k+1}) \sum_{j=1}^{k+1} \mathcal{T}(j)(S_{j,k} - E(S_{j,k})) = \gamma(k+1)W_k.\]

Since the function $\gamma(k)$ is increasing in $k$, we arrive at

\[E(W_{k+1}/\gamma(k+1)/\mathcal{F}_k) = \frac{W_k}{\gamma(k)} \times \frac{\gamma(k+1)}{\gamma(k)} \leq \frac{W_k}{\gamma(k)}\]

We conclude that $W_{k+1}\gamma(k+1)^{-1}$ is a supermartingale. A sufficient condition of a.s. convergence of $W_k/\gamma(k)$ to a random variable with finite expectation is

\[\limsup_{k \to +\infty} E(|W_k|) \leq +\infty\]

since $\gamma(k)$ tends to the unity (1) when $k$ becomes infinite. Now by Cauchy-Schwarz’s inequality and next by Minkowski’s one, we have

\[E|W_k| \leq \langle EW_k \rangle^{1/2} = \|W_k\|_2 \leq \sum_{j=1}^{k} \|\mathcal{T}(j)(S_{j,k} - s_{j,k})\|_2\]

\[\leq \sum_{j=1}^{k-1} \mathcal{T}(j)(\text{Var}(S_{j,k}))^{1/2}.\]

In the appendix Section [5], we provide moments computations of the $S_{j,k}$’s especially their expectations, variances and covariances. These computations themselves are based on integral calculations given in the subsection [5.2].
Var(S_{j,k}) is bounded by the unity for any 1 ≤ j ≤ k. We combine this with (5.4) and fix ε such that 0 < ε ≤ 1. Then for an enough large integer L, we get for k − 1 > L,

\[ E |W_k| \leq \sum_{j=1}^{L} \mathcal{J}(j)(Var(S_{j,k}))^{1/2} + \frac{(2(1 + ε)(ε + 1/2))^{1/2} \gamma^{-1/2}}{k^{\gamma}} \sum_{j=L}^{k-1} \mathcal{J}(j)j^{\gamma-1/2} \]

\[ \leq \sum_{j=1}^{L} \mathcal{J}(j) + \frac{4\gamma k^{-\gamma}}{k^{\gamma}} \sum_{j=L}^{k-1} \mathcal{J}(j)j^{\gamma-1/2} \]

Since the first term is bounded for a fixed L, we see that the supremum limit of E |W_k| is finite whenever (K1) holds. This proves the Theorem. To prove the corollary, we remark that for large values of j, \( \mathcal{J}(j) \sim \tau j^{\gamma - 1} \) and the condition becomes equivalent to the boundedness of

(2.3) \[ k^{-\gamma} \sum_{j=L}^{k-1} j^{\gamma + \gamma - 3/2} \]

Now fix 0 < \( \tau \leq 1/2 \). Consider the four possible cases: \( \tau + \gamma - 3/2 = -1 \), \( \tau + \gamma - 3/2 < -1 \), \( \tau + \gamma - 3/2 \neq -1 \), \( \tau + \gamma - 3/2 = 0 \) and \( \tau + \gamma - 3/2 > 0 \). By using (5.6), we get that (2.3) is less than \( k^{-\gamma} (\log(k-1) - \log L) + (k-1)^{-1} \) for the first case, than \( k^{-\gamma}((k-1)^{\gamma + (\tau - 1/2)} - L^{\gamma + (\tau - 1/2)})/(\gamma + \tau - 1/2) + (k-1)^{-1} \) for the second case. Now for the third one, (2.3) is exactly \( k^{-\gamma}(k - 1 - L) = (k^{\gamma - 1/2} - (1 + L)k^{\gamma - 3/2}) \). Finally in the last case, (2.3) is less than \( k^{-\gamma}((k - 1)^{\gamma + (\tau - 1/2)} - L^{\gamma + (\tau - 1/2)})/(\gamma + \tau - 1/2) + L^{\gamma + (\tau - 3/2)} \) by (5.7). In all these four cases, we get that \( \limsup_{k \to \infty} k^{-\gamma} \sum_{j=L}^{k-1} j^{\gamma + \gamma - 3/2} < \infty \). This achieves the proof of the corollary.

\[ \square \]

3. MAIN RESULTS

We resume to the extreme value problem. We will suppose without any loss of generality that the observations \( X_i \) are greater than one so that

(3.1) \[ T_n(f) = \sum_{j=1}^{k(n)} f(j) \log(X_{n-j+1,n}) - \log(X_{n-j,n}) \]

Now in the sequel we lessen the notation of \( k(n) \) and simply put \( k \). Denote by \( G(y) = F(e^y) \) the df of log \( X_i \). Remind that \( G \in D(G_{-1/\gamma}) \) if and only if \( F \in D(G_{-1/\gamma}) \). As promised, this paper is devoted to find out the leading part in the distributional theory of (3.1) when \( F \in D(G_{-1/\gamma}), \gamma < 1/2 \). We then start with the pure and simplest case of functions \( F \in D(G_{-1/\gamma}) \), that is

(3.2) \[ y_0 - G^{-1}(1 - u) = u^{1/\gamma}, 0 \leq u \leq 1 \]

where \( y_0 \) is the upper endpoint of \( G \). We use here the index \( -\gamma < 0 \) instead of \( \gamma < 0 \). Whence this law is set, it should be the same for any \( F \in D(G_{-1/\gamma}) \), \( O < \gamma < \) if some further conditions are fullfiled as illustrated in Subsection 4.1 of Section 4.

We are going to characterize the asymptotic law \( T_n(f) \) under the condition (K1).
Theorem 2. Let \( X_1, X_2, \ldots \) be a sequence of iid rv's with common df \( G \) defined in (3.2). Let \( f(j) \) be an increasing function of the integer \( j \geq 1 \) such that (K1) holds and let for any \( 1 \leq k \leq n, \)

\[
A_{k,n}(f) = f(k-1) - \sum_{j=1}^{k-1} (f(j) - f(j-1)) \exp \left( - \sum_{h=j}^{k-1} \log(1 + \gamma/h) \right).
\]

Then

\[
W_{k-1,n}(f) = A_{k,n}(f) - T_n(f)/(y_0 - Y_{n-k+1,n})
\]

converges in distribution to the finite random variable \( W_\infty(f) \) defined in Theorem 3. Further if \( f(j) = f_\tau(j) = j^\tau, \) \( 0 < \tau \leq 1/2, \) then \( W_{k-1,n}(f_\tau) \) converges distribution to \( W_\infty(\tau) \) defined in Corollary 2.

3.1. Proof of Theorem 3. We use in this proof the classical representation of the \( Y_j = \log X_j \) associated with the df \( G(x) = F(e^x) \) through a sequence of independent standard uniform rv’s \( U_1, U_2, \ldots, \) that is

\[
\{Y_j, j \geq 1\} = d \{G^{-1}(1 - U_j), j \geq 1\}
\]

and then

\[
\{Y_{n-j+1,n}, 1 \leq j \leq n\} = d \{G^{-1}(1 - U_{j,n}), 1 \leq j \leq n\} = d \{G^{-1}(1 - U_{j,n}), 1 \leq j \leq n\}, n \geq 1\}.
\]

This gives

\[
T_n(f)/(y_0 - Y_{n-k+1,n}) = \sum_{j=1}^{k} f(j) \frac{(\log X_{n-j+1,n} - \log X_{n-j,n})}{(y_0 - Y_{n-k+1,n})}
\]

\[
= \sum_{j=1}^{k} f(j) \frac{(y_0 - \log X_{n-j,n}) - (y_0 - \log X_{n-j+1,n})}{(y_0 - Y_{n-k+1,n})}
\]

\[
= d \sum_{j=1}^{k} f(j) U_{k,n}((U_{j+1,n}/U_{k,n})^{\gamma} - (U_{j,n}/U_{k,n})^{\gamma}).
\]

We have for \( 1 \leq j \leq k - 1, \)

\[
(U_{j,n}/U_{k,n})^{\gamma} = \prod_{h=j}^{k-1} (U_{h,n}/U_{h+1,n})^{\gamma} = \exp(-\gamma \sum_{h=j}^{k-1} \frac{1}{h} \log(U_{h+1,n}/U_{h,n})^{h})
\]

\[
\equiv \exp(-\gamma \sum_{h=j}^{k-1} E_{h}^{(n)}/h).
\]

By the Malmquist representation (see ([12]), p. 336), the rv’s \( E_{h}^{(n)}, 1 \leq h \leq n, \) are independent and standard exponential rv’s. We arrive at

\[
T_n(f)/(y_0 - Y_{n-k+1,n})
\]

\[
= \sum_{j=1}^{k} f(j) \left\{ \exp(-\gamma \sum_{h=j+1}^{k-1} E_{h}^{(n)}/h) - \exp(-\gamma \sum_{h=j}^{k-1} E_{h}^{(n)}/h) \right\}.
\]
An easy computation yields
\[ T_n(f)/(y_0 - Y_{n-k+1,n}) = f(k - 1) - \left( \sum_{j=1}^{j=k-1} (f(j) - f(j - 1)) \exp(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h) \right). \]

We put
\[ S_{j,k,n} = \exp(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h) \]
and remark that for each \( n \geq 1 \), \( S_{j,k,n} \) and \( S_{j,k} \) (defined in (2.2)) has the same law. Also put, by (5.1)
\[ s_{j,k,n} = \mathbb{E}(S_{j,k,n}) = \exp(-\sum_{h=j}^{k-1} \log(1 + \gamma/h)). \]

Then we have
\[ A_{k,n}(f) = f(k - 1) - \sum_{j=1}^{j=k-1} f(j)s_{j,k,n}. \]

This yields
\[ W_{k-1,n}^*(f) = \sum_{j=1}^{j=k-1} f(j)(S_{j,k,n} - s_{j,k,n}). \]

At this step, we compare (2.1) and (3.4) and remark that for any \( n \geq 1 \),
\[ W_{k(n)-1,n}^*(f) = d W_{k(n)-1}. \]

Then \( W_{k(n)-1,n}^*(f) \) converges in distribution to \( W_\infty(f) \) whenever \( W_{k(n)-1} \) converges almost surely to \( W_\infty(f) \). This achieves the proof.

4. Application to Extreme value Theory

4.1. Asymptotic results in the Weibull case. We indeed remark that for this simple case in the Weibull case, the law of the functional Hill process is found for \( 0 < \tau < 1/2 \). For the general case, we have the following Karamata representation when \( F \) is in the Weibull case of parameter \( \gamma > 0 : x_0(F) < \infty \) and

\[ \log x_0 - F^{-1}(1 - u) = (1 + p(u))u^\gamma \exp(\int_u^1 b(t)t^{-1}dt), \]

where \( (p(u), b(u)) \to (0, 0) \) as \( u \to 0 \). In a coming paper, we will determine general conditions on \( b \) and \( p \) under which \( T_n^*(f) \) behaves as \( W_{k,n}^* \) as in the present case.

Nevertheless, we will include in the statistical tests some models with specific forms of \( b(\cdot) \) as shown in Table 1 and used in Subsection 4.3.
We proceed as follows. Fix $\tau$, $0 < \tau < 1/2$, $\gamma > 0$ and $k \geq 2000$. At each step $B$ from 1 to $B_0 = 1000$, we generate standard exponential samples $E_1(B), ..., E_k(B)$ and compute $W_k^\ast$ denoted by $W_k^\ast(B)$. We finally consider the empirical $df$, denoted by $G_k$, based on $W_k^\ast(1), ..., W_k^\ast(B_0)$. Since $G_k$ is stable in the sense that it does not significantly change from $k = 2000$, we do approximate the $df$ $G_\infty$ of $W_\infty(f_\tau)$ by $G_k$ for $k$ large enough.

As an example, we illustrate in Figure 1 the $df$ $G_k$ for $k = 250, 500, 750, 1000, 2000, 500$ for $\gamma = 1$ and $\tau = 1/4$. Here for instance, we infer that the support of $G_\infty$ is $[-0.5, 0.5]$. On the whole, the figures clearly establish stability and support our proposal. For users interested to use our method, we provide an executable file located at:

http://www.ufrsat.org/lerstad/resources/lmhfw1.exe

for the computation of $P(W_\infty(f_\tau) \leq x) = G_\infty(x)$ and $P(|W(f_\tau)| \leq |x|) = G_\infty(|x|) - G_\infty(-|x|)$ for $x \in \mathbb{R}$.

4.3. Statistical tests. Let us illustrate here how $G_\infty$ may be used to test the hypothesis that $F \in D(G_{-1/\gamma})$. We use here the following approximation:

$$T_n^\ast(f) = T_n(f) / (g_0 - \log X_{n-k+1,n}) \approx T_n(f) / (\log X_{n,n} - \log X_{n-k+1,n}).$$

We consider here the statistical test (H) : $F \in D(G_{-1/\gamma})$, and compute the p-values for the models as described in Table 1. The seven first $df$'s are in the Weibull domain with $\gamma = 1$. The first (Weibull 1) is the one we used in the paper. In the six others (Weibull 2), we introduce a shift of order $(1 + u)^q$ and inspect the influence of $q$. We put here $n = 300$ and $k = 200$. Here are our results :

(1) : The pure model is accepted with large p-values around 68%. (2) : For a shift parameter $q$ less than 5, the model accepted for $6 \leq q \leq$, and rejected for $q \leq 5$. (3) : The exponential and Pareto cases are rejected as expected.
Figure 1. Illustration the distribution functions of $W_{k,n}(1/4)$ for different values of $k$

This is conceivable since, as we pointed out above, the convergence depends on the functions $b$ and $p$ in (4.1) that are here $p(u) = 0$ and $p(u) = (1 + q)u^q$ and $c = 1$. This dependence of the results on the auxiliary functions will be studied in a coming paper.

5. Appendix

This section is devoted to the computations of the moments of

$$S_{j,k} = \exp(-\gamma \sum_{h=j}^{k-1} E_h/h)$$

where the $E_h^j$ s are independent standard exponential, and their approximations for large values of $j$. We begin to give a particular and useful tool for the the
expansion of the logarithm function.

**Fact 1.** Let \( \varepsilon > 0 \) be fixed for once. There exists \( 0 < u_0 \) such that
\[
0 < u < u_0, \quad \log(1 + u) = u + \theta(\varepsilon, u)u^2,
\]
where \( \theta(\varepsilon, u) \in [-\varepsilon - 1/2, \varepsilon - 1/2] \equiv A(\varepsilon) = [a_1(\varepsilon), a_2(\varepsilon)] \). For any integer \( m \geq 1 \), let \( J_0(m) \) such that \( J_0(m) \geq \gamma/(mu_0) \) so that
\[
j \geq J_0(m) \implies \log(1 + \gamma/j) = u + \theta_j u^2 \quad \text{with} \quad \theta_j \in A(\varepsilon).
\]
In the remainder, we concentrate on the moment computations.

### 5.1. Moment estimation.

#### 5.1.1. Exact values.

We have for any integer \( m \geq 1 \),
\[
E(S^m_{j,k}) = E(\exp(-m\gamma \sum_{h=j}^{k-1} E_h/h)) = \prod_{h=j}^{k-1} E(-m\gamma E_h/h) = \prod_{h=j}^{k-1} (1 + m\gamma/h)^{-1}
\]
(5.1)
\[
E(S^m_{j,k}) = \exp \left( -\sum_{h=j}^{k-1} \log(1 + m\gamma/h) \right).
\]

Now for \( j \geq J_0(m) \),
\[
E(S^m_{j,k}) = \exp \left( -m\gamma \sum_{h=j}^{k-1} (1/h) - m^2\gamma^2 \sum_{h=j}^{k-1} \theta_h/h^2 \right).
\]
Then for any \( j \) an \( k \), \( \text{Var}(S_{j,k}) \) is
\[
\exp \left( -2 \sum_{h=j}^{k-1} \log(1 + 2\gamma/h) \right) - \exp \left( -2 \sum_{h=j}^{k-1} \log(1 + \gamma/h) \right) \leq 1,
\]
since this is difference of two points of \([0, 1]\).

#### 5.1.2. Approximated values for moments.

We have by
\[
\left| m^2\gamma^2 \sum_{h=j}^{k-1} \theta_h/h^2 \right| \leq |a_1(\varepsilon)| m^2\gamma (\frac{1}{j} - \frac{1}{k-1} - \frac{1}{(k-1)^2}) \leq \frac{|a_1(\varepsilon)| m^2\gamma}{j}.
\]
For
\[
\frac{|a_1(\varepsilon)| m^2\gamma}{J_1(\varepsilon, m)} \leq \varepsilon,
\]
we have
\[
j \geq J_1(\varepsilon, m) \vee J_0(m) \implies \exp \left( -m^2\gamma^2 \sum_{h=j}^{k-1} \theta_h/h^2 \right) \leq e^\varepsilon.
\]

Next by [5.8],
\[
\exp(-m\gamma \sum_{h=j}^{k-1} (1/h)) = \left( \frac{j}{k-1} \right)^{m\gamma} \exp(-m\gamma \left\{ \sum_{h=j}^{k-1} \frac{1}{h} - \log((k-1)/j) \right\})
\]

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with
\[\exp(-1/j) \leq \exp(-m\gamma \left( \sum_{h=j}^{k-1} \frac{1}{h} - \log((k-1)/j) \right)) \leq \exp(-1/(k-1)).\]

We finally have for \(j \geq J_1(\varepsilon, m) \lor J_0(m),\)
\[(5.3)\quad E(S_{j,k}^m) = \left( \frac{j}{k-1} \right)^{m\gamma} B(1, m, j) B(2, m, j),\]
with
\[0 \leq B(1, j) = 1 + O\left( \frac{|a_1(\varepsilon)| m^2 \gamma}{j} \right) \quad \text{and} \quad B(2, j) = 1 + O(j^{-1}).\]

5.1.3. Approximated values for variances. We have for \(j > J_0(2)\)
\[E(S_{j,k}^2) = \exp\left( -2\gamma \sum_{h=j}^{k-1} \frac{1}{h} - 4^2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(1)}{h^2} \right),\]
and for \(j > J_0(1)\)
\[E(S_{j,k}^2) = \left( -\gamma \sum_{h=j}^{k-1} \frac{1}{h} - \gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(2)}{h^2} \right)^2 = \exp\left( -2\gamma \sum_{h=j}^{k-1} \frac{1}{h} - 2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(2)}{h^2} \right).\]

Thus
\[\text{Var}(S_j^*) = \exp(2\gamma \sum_{h=j}^{k-1} \frac{1}{h}) \times \left\{ \exp(-4^2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(1)}{h^2}) - \exp(-2\gamma^2 \sum_{h=j}^{k-1} \frac{(2\theta_h(1) - \theta_h(2))}{h^2}) \right\}.
\]

Since \(x = 4^2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(1)}{h^2}\) and \(y = 2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(2)}{h^2}\) are both nonnegative, we have \(|e^x - e^y| \leq |x - y|\). Thus
\[0 \leq \exp(-4^2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(1)}{h^2}) - \exp(-2\gamma^2 \sum_{h=j}^{k-1} \frac{\theta_h(2)}{h^2}) \leq 2\gamma^2 \sum_{h=j}^{k-1} \frac{|2\theta_h(1) - \theta_h(2)|}{h^2} \leq \frac{2\gamma^2 |a_1(\varepsilon)|}{j},\]
by (5.9). Hence
\[(5.4)\quad \text{Var}(S_{j,k}) = \left( \frac{j}{k-1} \right)^{2\gamma} V(1, j) V(2, j)\]
with
\[|V(1, j)| = 1 + O(j^{-1}) \quad \text{and} \quad 0 \leq V(2, j) \leq \frac{2\gamma^2 |a_1(\varepsilon)|}{j}.\]
5.1.4. Covariance approximate values. Let $\ell > 1$ and consider $\sigma_{j,j+\ell} = \text{cov}(S_{j+\ell,k}, S_{j,k})$. We have

$$E(S_{j,k}) = \exp(\sum_{h=j}^{k-1} - \log(1 + \gamma/h))$$

$$= \exp(\sum_{h=j}^{j+\ell-1} - \log(1 + \gamma/h)) \exp(\sum_{h=j+\ell}^{k-1} - \log(1 + \gamma/h))$$

$$= E(S_{j+\ell,k}) \exp(\sum_{h=j}^{j+\ell-1} - \log(1 + \gamma/h)).$$

Also

$$S_{j,k}S_{j+\ell,k} = \exp(-\gamma \sum_{h=j}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j+\ell}^{k-1} E_h/h)$$

$$= \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h - \gamma \sum_{h=j+\ell}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h)$$

$$= \exp(-2\gamma \sum_{h=j+\ell}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h) = S_{j+\ell,k}^2 \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h).$$

Hence

$$E(S_{j,k}S_{j+\ell,k}) = E(S_{j+\ell,k}^2) \exp(\sum_{h=j}^{j+\ell-1} - \log(1 + \gamma/h)).$$

For $j \geq J_0(1) \vee J_0(2)$,

$$\text{cov}(S_{j,k}S_{j+\ell,k}) = \text{Var}(S_{j+\ell,k}) \exp(\sum_{h=j}^{j+\ell-1} - \log(1 + \gamma/h))$$

and

$$\text{cov}(S_{j,k}, S_{j+\ell,k}) = \text{Var}(S_{j+\ell,k}) \exp(-\gamma \sum_{h=j}^{j+\ell-1} 1/h - \gamma^2 \sum_{h=j}^{j+\ell-1} \theta_h/h^2)$$

(5.5)

5.2. Integrals’ computations. Let $b \geq 1$, we get by comparing the area under the curve $x \rightarrow x^{-b}$ from $j$ to $k - 1$ and those of the rectangles based on the intervals $[h, h+1], j = 1, \ldots, k - 2$, we get

$$\sum_{h=j}^{k-2} h^{-b} \leq \int_{j}^{k-1} x^{-b}dx \leq \sum_{h=j+1}^{k-1} h^{-b},$$

that is

$$\int_{j}^{k-1} x^{-b}dx + j^{-b} \leq \sum_{h=j}^{k-1} h^{-b} \leq \int_{j}^{k-1} x^{-b}dx + (k - 1)^{-b}.$$

(5.6)
As well, by comparing the area under the curve $x \mapsto x^b$ from $j$ to $k-1$ and those of the rectangles based on the intervals $[h, h + 1]$, $j = 1, \ldots, k - 2$, we also get

\[ \int_j^{k-1} x^b dx + (k-1)^b \leq \sum_{h=j}^{k-1} h^{-b} \leq \int_j^{k-1} x^b dx + j^b. \]

For $b = 1$, (5.6) yields

\[ \frac{1}{k-1} \leq \log((k-1)/j) - \left( \sum_{h=j}^{k-1} \frac{1}{h} \right) \leq \frac{1}{j}. \]

and for $b = 2$, it implies

\[ \frac{1}{j} - \frac{1}{k-1} - \frac{1}{j^2} \leq \sum_{h=j}^{k-1} h^{-2} \leq \frac{1}{j} - \frac{1}{k-1} - \frac{1}{(k-1)^2}, \]

that is

\[ \frac{1}{(k-1)^2} \leq \frac{1}{j} \left( 1 - \frac{j}{k-1} \right) - \sum_{h=j}^{k-1} h^{-2} \leq \frac{1}{j^2}. \]

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