Limits on entropic uncertainty relations for 3 and more MUBs

Andris Ambainis*

Abstract

We consider entropic uncertainty relations for outcomes of the measurements of a quantum state in 3 or more mutually unbiased bases (MUBs), chosen from the standard construction of MUBs in prime dimension. We show that, for any choice of 3 MUBs and at least one choice of a larger number of MUBs, the best possible entropic uncertainty relation can be only marginally better than the one that trivially follows from the relation by Maassen and Uffink (PRL, 1987) for 2 bases.

1 Introduction

Uncertainty relations quantify the amount of uncertainty in the outcomes of quantum measurements. The most famous uncertainty relation is due to Heisenberg [11] who showed that either the position or the momentum of the particle has at least a certain amount of uncertainty.

For finite-dimensional state spaces, the uncertainty relations are most often stated in terms of the entropy of the measurement outcomes [4, 5, 13, 14, 18]. Entropic uncertainty relations have several applications, from locking information in quantum states [9] to quantum cryptography in the bounded-storage model [4]. (For more details on those, we refer the reader to the survey by Wehner and Winter [18].)

Let $P_1, P_2$ be the probability distributions obtained by measurements with respect to two orthonormal bases $M_1, M_2$ and let $c$ be the maximum of $|\langle \psi_1 | \psi_2 \rangle|$, over all $|\psi_1 \rangle$ from $M_1$ and $|\psi_2 \rangle$ from $M_2$. Then, as shown by Maassen and Uffink [13],

$$H(P_1) + H(P_2) \geq -2 \log c. \quad (1)$$

The lower bound is maximized if $M_1$ and $M_2$ are mutually unbiased. Then $|\langle \psi_1 | \psi_2 \rangle| = \frac{1}{\sqrt{N}}$ and we get a lower bound of $\log N$ on the sum of the two entropies. This bound is optimal: if we measure a state $|\psi \rangle$ from one of the bases, the outcome has an entropy of 0 in that basis and an entropy of $\log N$ in the other basis.

---

*Faculty of Computing, University of Latvia, Raina bulv. 19, Riga, LV-1586, Latvia, ambainis@lu.lv. Supported by University of Latvia Research Grant ZB01-100 and Marie Curie International Reintegration Grant (IRG).
In contrast, when we try to quantify the sum of entropies for three or more bases, fairly little is known. Most of the research on this subject considers the case when each two of the measurement bases are mutually unbiased.

There are two known constructions of mutually unbiased bases (MUBs). The first and the most commonly used construction is based on generalized Pauli matrices [21, 2]. The second construction [20] is based on Latin squares.

For either of those constructions, we trivially have
\[ H(P_1) + \ldots + H(P_k) \geq \frac{k}{2} \log N, \quad (2) \]
which follows from dividing the bases into pairs and applying (1) to each pair. Better bounds are known for the case when the number of measurement bases is large (i.e. we use the full collection of \( d + 1 \) MUBs in dimension \( d \) or a large subset of it [12, 13, 16]). But, for the case when we consider a small number of measurements, only two partial results are known, one for each of the two constructions of MUBs.

For the first construction, computer simulations by DiVincenzo et al. [9] (for the number of bases \( k \) from 3 to 29) indicate
\[ H(P_1) + \ldots + H(P_k) \approx ck \log N, \]
with \( c \) scaling as \( 1 - \epsilon - \frac{1}{k} \) where \( \epsilon \) is between 0.10 and 0.15. For the second construction, Ballester and Wehner [1] show that (2) is essentially optimal and no better bound can be achieved, even when we use the maximum number of MUBs provided by the Latin square construction.

Thus, it seems that the two constructions display a significantly different behaviour: one provides better and better uncertainty relations as we increase the number of bases (which is good for applications such as locking of correlations in quantum states [9]) while the second does not.

In this note, we provide some new results which show that the first constructions of MUBs also fails to give better uncertainty relations in some situations:

1. For any 3 bases from this construction, we can find a state \( |\psi\rangle \) with
\[ H(P_1) + H(P_2) + H(P_3) \leq \left( \frac{3}{2} + o(1) \right) \log N. \]

   Thus, the trivial bound (2) is nearly tight in this case.

2. For any \( k \leq n' \), we can select \( k \) MUBs in dimension \( n \) so that
\[ H(P_1) + \ldots + H(P_k) \leq (1 + \epsilon + o(1)) \frac{k}{2} \log N. \]

   Our results do not rule out the possibility of good uncertainty relations for \( k \geq 4 \) MUBs but indicate that a careful choice of the set of MUBs may be necessary to obtain such relations.
2 MUBs in prime dimensions

In this section, we first describe the Wootters-Fields \cite{21} construction of mutually unbiased bases and then analyze its symmetry properties. The results on symmetry properties will be used to prove our bound on entropic uncertainty relations for 3 MUBs in section 3.1.

The Wootters and Fields \cite{21} construction for prime dimension \( p \) is as follows. The first MUB, \( M_c \), just consists of the computational basis states \( |0\rangle, |1\rangle, \ldots, |p-1\rangle \). The other \( p \) MUBs are denoted \( M_0, \ldots, M_{p-1} \), with \( M_j \) consisting of states \( |\psi_{j,0}\rangle, \ldots, |\psi_{j,p-1}\rangle \) defined by

\[
|\psi_{j,k}\rangle = \sum_{l \in \{0,1,\ldots,p-1\}} w^j l^2 + k l |l\rangle
\]

where \( w = e^{2\pi i/p} \).

We say that two triplets of MUBs \( (M'_1, M'_2, M'_3) \) and \( (M''_1, M''_2, M''_3) \) are equivalent if there is unitary \( U \) that maps \( M'_1, M'_2, M'_3 \) to \( M''_1, M''_2, M''_3 \) (in some order).

\textbf{Lemma 1} Let \( d \) be the smallest quadratic nonresidue mod \( p \). Any set of 3 different MUBs selected from \( M_c, M_0, \ldots, M_{p-1} \) is equivalent to either the set \( M_c, M_0, M_1 \) or the set \( M_c, M_0, M_d \).

\textbf{Proof:} Let \( M'_1, M'_2, M'_3 \) be three different MUBs (selected from \( M_c, M_0, \ldots, M_{p-1} \)). We consider the following two unitary transformations that permute the MUBs:

- The unitary transformation \( W |l\rangle = w^l |l\rangle \) leaves \( M_c \) unchanged and maps \( M_j (j \in \{0,1,\ldots,p-1\}) \) to \( M_{(j+1) \mod p} \).
- Quantum Fourier transform

\[
F |l\rangle = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} w^{lj} |j\rangle
\]

maps \( M_c \) and \( M_0 \) one to another and permutes \( M_1, \ldots, M_{p-1} \) in some way.

We can map \( M'_1, M'_2, M'_3 \) to \( M_c, M_0 \) and \( M_k \) (for some \( k \in \{0,1,\ldots,p-1\} \)) as follows:

1. We repeatedly apply \( W \) until one of \( M'_i \) is mapped to \( M_0 \).
2. We then apply \( F \), mapping \( M_0 \) to \( M_c \).
3. We then repeatedly apply \( W \) until one of the other MUBs is mapped to \( M_0 \).

Next, let \( a \in \{1,\ldots,p-1\} \). Define \( U_a |j\rangle = |(a^{-1}j) \mod p \rangle \) (where \( a^{-1} \) is the inverse of \( a \) in \( Z_p \)). The transformation \( U_a \) leaves \( M_c \) unchanged (permuting the basis states in this basis). For the basis \( M_j \), we have

\[
U_a |\psi_{j,k}\rangle = \sum_{l \in \{0,1,\ldots,p-1\}} w^{jl^2+k(a^{-1}l) \mod p} |l\rangle
\]
\[ \sum_{l \in \{0, 1, \ldots, p-1\}} w^{j^2 + k} |l\rangle = |\psi_{(a^2j) \mod p, (ak \mod p)} \rangle. \]

Thus, the basis \( M_j \) is mapped to \( M_{(a^2j) \mod p} \). In particular, this means that \( M_0 \) is mapped to itself.

If we have a set \( M_c, M_0, M_j \) with \( j \) being a quadratic residue, then \( j^{-1} \) is a quadratic residue as well. Let \( a \) be a solution of \( x^2 \equiv j^{-1} \pmod{p} \). Then, \( U_a \) leaves \( M_c \) and \( M_0 \) unchanged and maps \( M_j \) to \( M_{(a^2j) \mod p} = M_1 \).

If we have a set \( M_c, M_0, M_j \) with \( j \) being a quadratic non-residue, then \( j^{-1} \) is a quadratic non-residue and \( j^{-1}d \) is a quadratic residue (modulo a prime, a product of two quadratic non-residues is a quadratic residue). Let \( a \) be a solution of \( x^2 \equiv j^{-1}d \pmod{p} \). Then, \( U_a \) leaves \( M_c \) and \( M_0 \) unchanged and maps \( M_j \) to \( M_{(a^2j) \mod p} = M_d \). □

**Lemma 2** The sets of MUBs \( M_c, M_0, M_1 \) and \( M_c, M_0, M_d \) are equivalent if and only if the prime \( p \) is of the form \( p = 4k + 3 \), \( k \in \mathbb{Z} \).

**Proof:** If \( p = 4k + 3 \), then \(-1\) is a quadratic non-residue \( \text{mod } p \) \[19\]. As shown in the proof of Lemma \[14\], \( M_c, M_0, M_d \) is then equivalent to \( M_c, M_0, M_{-1} \). Applying the unitary transformation \( W \) from Lemma \[14\] maps \( M_c, M_0, M_{-1} \) to \( M_c, M_1 \) and \( M_0 \).

Next, we consider the case when \( p = 4k + 1 \). Then, \(-1\) is a quadratic residue \[19\]. We first show

**Claim 1** Assume that \(-1\) is a quadratic residue \( \text{mod } p \). Then, any permutation of \( M_c, M_0 \) and \( M_1 \) can be implemented by a unitary transformation.

**Proof:** It suffices to show that we can implement the following two permutations:

\[ M_c \rightarrow M_0, M_0 \rightarrow M_c, M_1 \rightarrow M_1 \]
\[ M_c \rightarrow M_c, M_0 \rightarrow M_1, M_1 \rightarrow M_0 \]

because any permutation can be expressed as a product of those. Those two transformations can be implemented as follows:

1. The quantum Fourier transform \( F \) transforms bases in a following way: \( F(M_c) = M_0, F(M_0) = M_c, F(M_1) = M_d \) where \( a \) is the unique element of \( \{0, 1, \ldots, p-1\} \) satisfying \( 4a \equiv -1 \pmod{p} \). We can then transform these bases to \( M_0, M_c, M_{-1} \) by applying the transformation \( U_2 \) defined in the proof of Lemma \[14\].

Since \(-1\) is a quadratic residue \( \text{mod } p \), there exists \( x \) such that \( x^2 \equiv -1 \pmod{p} \). Applying \( U_x \) maps \( M_0, M_c, M_{-1} \) to \( M_0, M_c, M_1 \).

2. We first apply \( U_x \) mapping \( M_c, M_0, M_1 \) to \( M_c, M_0, M_{-1} \). We then apply \( U|l\rangle = e^{2\pi i l^2/p} |l\rangle \) which maps those to \( M_c, M_{1}, M_0 \).

Therefore, if we have a unitary transformation \( U \) that transforms \( M_c, M_0, M_1 \) to \( M_c, M_0, M_d \), we can assume that it implements the following map

\( M_c \rightarrow M_c, M_0 \rightarrow M_0, M_1 \rightarrow M_d \).

Since \( U \) fixes \( M_c, U \) is of the form

\[ U|i\rangle = \lambda(i)|f(i)\rangle, \]
where $f(0), \ldots, f(p-1)$ is a permutation of $0, \ldots, p-1$ and $\lambda(i)$ are complex numbers of absolute value 1. Without a loss of generality, we can assume that $\lambda(0) = 1$. The other $\lambda(i)$ all must be powers of $w$ (otherwise, vectors in $\mathcal{M}_0$ (whose coefficients are powers of $w$) would not be mapped to vectors in $\mathcal{M}_0$). Let

$$U|0\rangle = |i_0\rangle, U|1\rangle = w^{k_1}|i_1\rangle.$$  

We claim that this implies

$$U|j\rangle = w^{k_1j}|i_0 + j(i_1 - i_0)\rangle.$$  

To show that, we first assume

$$U|j\rangle = w^{k_1j}|i_0 + j(i_1 - i_0)\rangle.$$  

We consider the state $|\psi_{0,0}\rangle = \sum_{i=0}^{p-1} \frac{1}{\sqrt{d}} |i\rangle$ which belongs to the basis $\mathcal{M}_0$. It must get mapped to a state in $\mathcal{M}_0$ and the only possibility that is consistent with (3) is

$$U|\psi_{0,0}\rangle = w^{-k_0}|\psi_{0,k}\rangle$$  

where $k = \frac{k_1}{i_1 - i_0}$ (with all operations modulo $p$). Then, we must have

$$U|j\rangle = w^{k_1(i_j - i_0)}|i_j\rangle.$$  

Similarly, the state $|\psi_{0,1}\rangle = \sum_{i=0}^{p-1} \frac{1}{\sqrt{d}} |i\rangle$ must also get mapped to a state in $\mathcal{M}_0$ and the only possibility consistent with (3) is

$$U|\psi_{0,1}\rangle = w^{-k_0}|\psi_{0,k}\rangle$$  

where $k' = \frac{k_1 + 1}{i_1 - i_0}$. Then, we must have

$$U|j\rangle = w^{k'(i_j - i_0) - j}|i_j\rangle.$$  

Since $i_j$ must have the same coefficients in (5) and (6), we have

$$\frac{k_1}{i_1 - i_0}(i_j - i_0) = \frac{k_1 + 1}{i_1 - i_0}(i_j - i_0) - j$$

and

$$j = \frac{i_j - i_0}{i_1 - i_0}$$

which is equivalent to $i_j = i_0 + j(i_1 - i_0)$. The coefficient of $|i_j\rangle$ in (3) is

$$w^{k(i_j - i_0)} = w^{\frac{k_1}{i_1 - i_0}(i_j - i_0)} = w^{k_1j}.$$  

This implies (4).

Next, a transformation of the form (4) can be expressed as a product of three transformations:

1. $|j\rangle \rightarrow w^{bj}|j\rangle$ for some $b \in \{0, 1, \ldots, p - 1\}$;
2. $|j\rangle \rightarrow |cj\rangle$ for some $c \in \{1, \ldots, p - 1\}$;
3. $|j\rangle \rightarrow |j + d\rangle$ for some $d \in \{0, 1, \ldots, p - 1\}$.
The second transformation maps \( \mathcal{M}_1 \) to \( \mathcal{M}_{c2} \). The first and the third transformation just permute the vectors within each \( \mathcal{M}_i \). Therefore, we can map \( \mathcal{M}_1 \) to \( \mathcal{M}_{c2} \) but not to \( \mathcal{M}_d \) where \( d \) is a quadratic non-residue.

For our result, we also need an upper bound on the smallest quadratic non-residue \( d \). It is known that:

- If \( p = 8k + 5 \), \( k \)-integer, then \( d = 2 \).
- For \( p = 8k+1 \), then \( d = O(\log^2 p) \) for all \( p \), assuming the generalized Riemann hypothesis is true \([10]\).
- Although \( d \) is small for most primes \( p \), no good bound without the use of GRH is known \([10,17]\).

### 3 Limit on entropic uncertainty relations

#### 3.1 Measurement in 3 bases

As shown in the previous section, any set of 3 MUBs is equivalent to \( \mathcal{M}_c,\mathcal{M}_0,\mathcal{M}_1 \) or \( \mathcal{M}_c,\mathcal{M}_0,\mathcal{M}_d \). We first consider the case of measurements \( \mathcal{M}_c,\mathcal{M}_0,\mathcal{M}_1 \).

**Theorem 1** Let \( E(\psi) \) be the average of the entropies of probability distributions obtained by measuring \( \psi \) in the bases \( \mathcal{M}_c,\mathcal{M}_0,\mathcal{M}_1 \). There exists a state \( |\psi\rangle \) such that

\[
E(\psi) \leq \frac{1}{2} \log p + \frac{1}{6} \log \log p + c
\]

for some constant \( c \).

**Proof:** Let \( |\psi\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j\rangle \), where

\[
m = \left\lfloor \frac{p}{4\pi \log p} \right\rfloor. \quad (7)
\]

Measuring \( |\psi\rangle \) in \( \mathcal{M}_c \) produces one of values 0, 1, \ldots, \( m-1 \) with probability \( \frac{1}{m} \) each. This probability distribution has the entropy of \( \log m \).

**Lemma 3** Measuring \( |\psi\rangle \) in \( \mathcal{M}_0 \) produces a probability distribution with an entropy of at most \( \log p - \log m + 10 \).

**Proof:** Let \( |k\rangle = \min(k, p - k) \). Measuring \( |\psi\rangle \) in \( \mathcal{M}_0 \) gives the value \( k \) with probability

\[
\frac{1}{mp} \sum_{j=0}^{m-1} e^{2\pi i \frac{jk}{p}} \leq \frac{1}{mp} \left| \frac{e^{2\pi i \frac{km}{p}} - 1}{e^{2\pi i \frac{k}{p}} - 1} \right|^2
\]

\[
\leq \frac{4}{mp} \frac{1}{\left| e^{2\pi i \frac{k}{p}} - 1 \right|^2} \leq \frac{\pi^2 p}{|k|^2 m}, \quad (8)
\]

where the last inequality follows from \( |e^{ix} - 1| \geq \frac{2|x|}{\pi} \) being true for all \( x \in [-\pi, \pi] \).

Let \( t = \left\lfloor \frac{8 \pi^2 p}{|k|^2} \right\rfloor \). Let \( S \) be the set of all \( k \) with \( |k| < t \) and let \( S_i \) (for \( i = 0, 1, \ldots \)) be the set of all \( k \) with \( 2^i t \leq |k| < 2^{i+1} t \).
Claim 2 Let \( p_i \) be the probability of measuring \( k \in S_i \). Then,
\[
p_i \leq \frac{1}{2^{i+2}}.
\]

**Proof:** If \(|k| \geq 2^t \), the probability (8) is at most \( \frac{\pi^2 p}{2^i t^2 m} \). Since there are \( 2^{i+1} t \) values \( k \in S_i \), we have
\[
p_i \leq 2^{i+1} t \frac{\pi^2 p}{2^i t^2 m} = \frac{\pi^2 p}{2^{i-1} tm} \leq \frac{1}{4},
\]
with the last inequality following from the definition of \( t \).

This claim also implies that
\[
\sum_i p_i \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+2}} = \frac{1}{2}.
\]

The entropy of the probability distribution of outcomes of \( \mathcal{M}_0 \) is upper-bounded by the entropy of the probability distribution in which each element of \( S_i \) has a probability \( \frac{p_i}{|S_i|} \) and each element of \( S \) has a probability \( \frac{p_0}{|S|} \), where \( p_0 = 1 - \sum_{i \geq i_0} p_i \). The entropy of this probability distribution is
\[
-|S| \frac{p_0}{|S|} \log \frac{p_0}{|S|} - \sum_{i \geq i_0} |S_i| \frac{p_i}{|S_i|} \log \frac{p_i}{|S_i|} =
\]
\[
-p_0 \log \frac{p_0}{|S|} - \sum_{i \geq i_0} p_i \log \frac{p_i}{|S_i|} =
\]
\[
(p_0 \log |S| + \sum_{i \geq i_0} p_i \log |S_i|) - (p_0 \log p_0 + \sum_{i \geq i_0} p_i \log p_i).
\]

Since \( |S| \leq 2t \) and \( |S_i| \leq 2^{i+1} t \), we can upperbound the first component by
\[
p_0 (1 + \log t) + \sum_{i \geq 0} p_i (i + 1 + \log t) = (1 + \log t) + \sum_{i \geq 0} p_i i
\]
\[
\leq (1 + \log t) + \sum_{i \geq 0} \frac{1}{2^{i+2} t} i \leq \log t + \frac{3}{2}.
\]

For the second component, we have
\[
-(p_0 \log p_0 + \sum_{i \geq i_0} p_i \log p_i) \leq -\frac{1}{2} \log \frac{1}{2} - \sum_{i \geq 2} \frac{1}{2^i} \log \frac{1}{2^i} \leq \frac{1}{2} + \sum_{i \geq 2} \frac{1}{2^i} \leq 2.
\]

Therefore, the entropy is at most
\[
\log t + \frac{7}{2} \leq \log m - \log p + 10,
\]
with the last inequality following from the definition of \( t \).

**Lemma 4** Assume that \( m^2 \leq \frac{p}{4\pi \log p} \). Let \( H_0 \) and \( H_1 \) be the entropies of the probability distributions obtained by measuring \( |\psi\rangle \) in \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \). Then,
\[
H_1 \leq H_0 + 1.
\]
Proof: Measuring $|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m} |j\rangle$ in the basis $M_1$ produces the same probability distribution as measuring $|\psi'\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m} e^{-2\pi i j^2/p} |j\rangle$ in the basis $M_0$. We have 
$$|e^{-2\pi i j^2/p} - 1| \leq 2\frac{j^2}{p}.$$
Therefore, $\|\psi - \psi'\| \leq 2\pi \max_j \frac{j^2}{p} = 2\pi \frac{m^2}{p}$. The variational distance between the probability distributions obtained by measuring $|\psi\rangle$ and $|\psi'\rangle$ is at most $2||\psi - \psi'|| \leq 4\pi \frac{m^2}{p}$. Because of the definition of $m$, this is at most $\frac{1}{\log p}$. Lemma 4 now follows from the lemma below.

**Lemma 5** [6] Let $P, P'$ be probability distributions over a $p$ element set and $|P - P'| \leq \delta$. Then,
$$|H(P) - H(P')| \leq H(\delta) + \delta \log(p - 1).$$

By combining all the bounds on the entropies (the trivial $\log m$ bound on the entropy of the measurement in $M_c$, Lemma 3 and Lemma 4), the average of entropies must be at most 
$$\frac{2}{3} \log p - \frac{1}{3} \log m + 7 + o(1).$$
Substituting (7) instead of $m$ completes the proof of the theorem. \]

For the case when the set of 3 MUBs consists of $M, M_0$ and $M_d$, a similar proof gives
$$E(\psi) \leq \frac{1}{2} \log p + \frac{1}{6} \log \log p + \frac{1}{6} \log d + c.$$
(The main difference is that we have to take
$$m = \left\lfloor \sqrt{\frac{p}{4\pi d \log p}} \right\rfloor$$
instead of (4).) As discussed at the end of section 2, generalized Riemann hypothesis (GRH) implies $d = O(\log^2 p)$ and $\log d \leq 2 \log \log p + O(1)$ for all $p$. Thus, we have

**Theorem 2** Let $M'_1, M'_2, M'_3$ be an arbitrary subset of $M_c, M_0, \ldots, M_{p-1}$. Let $E(\psi)$ be the average of the entropies of probability distributions obtained by measuring $\psi$ in $M_0, M'_1, M'_2$. we have:

1. If $p$ is not of the form $p = 8k + 1$, there exists a state $|\psi\rangle$ such that
$$E(\psi) \leq \frac{1}{2} \log p + \frac{1}{6} \log \log p + c$$
for some constant $c$.

2. If $p = 8k + 1$ and GRH is true, there exists a state $|\psi\rangle$ such that
$$E(\psi) \leq \frac{1}{2} \log p + \frac{1}{2} \log \log p + c$$
for some constant $c$. 

8
3.2 Measurement in a larger number of bases

**Theorem 3** Let $E(\psi)$ be the average of the entropies of probability distributions obtained by measuring $\psi$ in the bases $M_c, M_0, \ldots, M_{\lfloor p/\epsilon \rfloor}$. There exists state $|\psi\rangle$ such that

$$E(\psi) \leq \frac{1 + \epsilon}{2} \log p + o(\log p)$$

for some constant $c$.

**Proof:** As in the proof of Theorem 1, we take $|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j\rangle$. But we now choose

$$m = \left\lfloor \frac{p^{1 - \epsilon}}{\sqrt{4\pi \log p}} \right\rfloor. \quad (9)$$

Then, the entropy of measuring $|\psi\rangle$ in the basis $M_c$ is $\log m$ and the entropy of measuring $|\psi\rangle$ in the basis $M_0$ is $\log p - \log m + 10$ (by Lemma 3).

We now bound the entropy of measuring $|\psi\rangle$ in a basis $M_k$, $k \in \{1, \ldots, \lfloor n/\epsilon \rfloor\}$.

Similarly to the proof of Lemma 3, measuring $|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} |j\rangle$ in the basis $M_k$ produces the same probability distribution as measuring $|\psi'\rangle = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} e^{-2\pi i k j^2 / p} |j\rangle$ in the basis $M_0$. We have

$$|e^{-2\pi i k j^2 / p} - 1| \leq 2\pi k j^2 / p.$$ 

We can upperbound $k$ by its maximum value, $p^\epsilon$ and $j$ by its maximum value, $m$. By summing over all $j \in \{0, 1, \ldots, m - 1\}$, we get

$$\|\psi - \psi'\| \leq 2\pi p^\epsilon m^2 / p \leq \frac{1}{2\log p}.$$ 

The variational distance between the probability distributions obtained by measuring $|\psi\rangle$ and $|\psi'\rangle$ is at most $2\|\psi - \psi'\| \leq \frac{1}{\log p}$. By Lemma 5 this means that the entropies of the two probability distributions differ by at most $1 + o(1)$. Since the entropy of the distribution obtained by measuring $|\psi\rangle$ in $M_0$ is $\log p - \log m + 10$, this means that the entropy of the distribution obtained by measuring $|\psi\rangle$ in $M_k$ is at most

$$\log p - \log m + 11 + o(1).$$

By substituting (9) instead of $m$, this is at most

$$\frac{1 + \epsilon}{2} \log p + o(\log p).$$

This upperbounds the entropy for $M_0, M_1, \ldots, M_{\lfloor p/\epsilon \rfloor}$. For $M_c$, the entropy is $\log m \leq \frac{1}{2\epsilon} p$. Therefore, the theorem follows.
References

[1] M. Ballester, S. Wehner. Entropic uncertainty relations and locking: tight bounds for mutually unbiased bases. Physical Review A, 75:022319, 2007. Also quant-ph/0606244

[2] S. Bandyopadhyay, S. P. Boykin, V. Roychowdhury, F. Vatan. A new proof for the existence of mutually unbiased bases. Algorithmica, 34:512-528, 2002. Also quant-ph/0103162

[3] E. Bernstein, U. Vazirani. Quantum complexity theory. SIAM Journal on Computing, 26(5): 1411-1473 (1997)

[4] I. Bialynicki-Birula, J. Mycielski. Uncertainty relations for information entropy. Communications in Mathematical Physics, 44:129, 1975.

[5] S. Brierley, S. Weigert, I. Bengtsson, All mutually unbiased bases in dimensions two to five. arXiv:0907.4097

[6] I. Csiszar, J. Körner. Information Theory: Coding Theorems for Discrete Memoryless Systems. Academic Press, New York, 1981.

[7] I. Damgaard, S. Fehr, L. Salvail, C. Schaffner, Cryptography in the bounded quantum-storage model. Proceedings of FOCS’2005, pp. 449-458. Also quant-ph/0508222

[8] D. Deutsch. Uncertainty in quantum measurements. Phys. Rev. Lett., 50:631-633, 1983.

[9] D. DiVincenzo, M. Horodecki, D. Leung, J. Smolin, B. Terhal. Locking classical correlations in quantum states. Phys. Rev. Lett. 92:067902, 2004. Also quant-ph/0303088

[10] T. Gowers. A Tricki issue. Weblog post, January 20, 2009. http://gowers.wordpress.com/2009/01/20/a-tricki-issue/

[11] W. Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. Zeitschrift für Physik, 43:172-198, 1927.

[12] I. D. Ivanovic. Geometrical description of quantum state determination. Journal of Physics A: Mathematical and General, 25:3241-3245, 1992.

[13] H. Maassen, J. Uffink. Generalized entropic uncertainty relations. Phys. Rev. Lett., 60:1103-1106, 1988.

[14] V. Majernik, L. Richterek. Entropic uncertainty relations. European Journal of Physics, 18:79-89, 1997.

[15] J. Sanchez, Entropic uncertainty and certainty relations for complementary observables, Physics Letters A, 173:223-239, 1993.

[16] J. Sanchez-Ruiz, Improved bounds in the entropic uncertainty and certainty relations for complementary observables, Physics Letters A, 201:125-131, 1995.

[17] T. Tao, The least quadratic non-residue and the square root barrier. Weblog post, http://terrytao.wordpress.com/2009/08/18/the-least-quadratic-nonresidue-and-the-square-root-barrier/, August 18, 2009.
[18] S. Wehner, A. Winter. Entropic uncertainty relations - a survey. arXiv:0907.3704.

[19] Weisstein, Eric W. "Quadratic Nonresidue." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/QuadraticNonresidue.html

[20] P. Wocjan, T. Beth. New construction of mutually unbiased bases in square dimensions. Quantum Information and Computation, 5:93-101, 2005. Also quant-ph/0407081

[21] W. Wootters, B. Fields. Optimal state-determination by mutually unbiased measurements. Annals of Physics, 191:363-381, 1989.