MODELING THE INFLUENCE OF HUMAN POPULATION AND HUMAN POPULATION AUGMENTED POLLUTION ON RAINFALL

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ABSTRACT. Worldwide, human population is increasing continuously and this has magnified the level of pollutants in the environment. Pollutants affect the human population as well as the environmental ecology including rainfall. Here, we formulate a mathematical model comprising ordinary differential equations to see the effect of human population and pollution caused by human population on the dynamics of rainfall. In the modeling process, it is assumed that the augmentation in the density of human population increases the concentration of pollutants; however, decreases the rate of formation of cloud droplets. It is also assumed that pollutants have negative impact on human population and affect the precipitation. The feasibility of all equilibrium and their stability properties are discussed. Further, to capture the effect of environmental randomness, the proposed model is also analyzed by incorporating white noise terms. For the proposed stochastic model, we have established the existence and uniqueness of global positive solution. It is also shown that system possesses a unique stationary distribution with some restrictions. The model analysis reveals that rainfall may decrease or increase due to the anthropogenic emission of pollutants in the atmospheric environment. Finally, for the validation of analytical findings, numerical simulations are presented.

1. Introduction. Rainfall plays an important role in recycling of water on the planet Earth and fulfills the water’s requirement of all living beings. During past few decades, floods and drought like situations have been occurred in several parts of the world [25, 33]. The continuous increase in human population has given the birth of several problems, e.g. deforestation, food scarcity, sea level rise, increase in greenhouse gases, pollution, etc. [3, 17, 26]. The increase in pollution level affects the human population, forestry resources, etc. on the Earth’s surface and

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it also affects the dynamics of rainfall when pollutants reach into the atmospheric environment [10, 22].

During the last few decades, level of atmospheric pollution has increased due to its emission from Industries, vegetation burning, etc. and this has impacted the cloud dynamics and thus precipitation [4, 23, 32]. Aerosols present in the atmosphere serve as cloud condensation nuclei (CCN) and affect cloud properties and initiation of rainfall [34]. The tiny cloud droplets condense around pollution particles and float in the air with low probability of collision to each other and merging into raindrops, which results to less sleet and snowfall. Thus, the pollutants present in the atmosphere prevent cloud droplets condensing into raindrops and stifle the rain [23, 32]. Also, the data collected by the Tropical Rainfall Measuring Mission (TRMM) satellite indicates that cloud droplet coalescence as well as ice precipitation formation both are inhibited in polluted clouds [23, 24]. It is also pointed out that human activities (emission of pollutants due to Industries) may alter the clouds and rainfall globally [23]. It is further reported that high concentration of aerosols in the atmosphere caused by human activities may decrease or increase the rainfall due to their radiative and CCN activities. This is because of the fact that in the case of pristine tropical clouds having less CCN concentrations, the rainfall increases as aerosols increase, while in the other case, the increment in aerosols loading causes the evaporation of much of water from polluted clouds before precipitation and thus decreases the rainfall [12, 25].

The increase in the atmospheric pollutants has several adverse effects on human health and increases the rate of mortality. Recent report on pollution indicates that approximately 15% of all deaths on the planet Earth occur due to pollution [21]. To clean the atmosphere, rain plays a prominent role and in this regard, Sundar et al. [30] have proposed a deterministic mathematical model to assess the effect of rain on the removal of atmospheric pollutants. This model has been further studied by Huang et al. [9] in stochastic settings by incorporating environmental fluctuations as white noise terms. In this study, it is concluded that for less intensified white noise, solution fluctuates around the mean. Keeping in mind the importance of rain in the removal of pollutants and agriculture, some mathematical models are formulated to make rain artificially by introducing conducive aerosols in the regional atmosphere [18, 19, 27, 28, 29]. On analyzing these models, it is concluded that an increment in the introduction rate of aerosols increases the precipitation; however, the solution trajectories remain in the closed neighborhood of the deterministic model provided the environmental fluctuations are small. It may be noted that 90% cloud droplets in the atmosphere are formed due to evaporation of water from Oceans, open land water sources, which includes lakes, ponds, rivers; however 10% is contributed through transpiration from forests and crops, etc. To see the effect of transpiration from forests on rainfall, Lata and Misra [13] have proposed a mathematical model to see the effect of forestry trees on rainfall. The study reveals that augmentation in forestry trees increases the cloud droplets in the atmosphere and thus precipitation.

The increase in human population decreases the formation rate of cloud droplets as the evaporation of water from the land surface area decreases due to development activities (making house complexes, roads, clearing forests, increase in water pollution, etc.)[5]. Also when the pollutants reach into the atmospheric environment, they impact the cloud dynamics and thus precipitation. To address this important problem, we formulate a model to study the effect of human population
and its associated pollution on precipitation. We also propose a stochastic model by incorporating environmental fluctuations as white noise terms in the formulated deterministic model.

2. Model formulation. Here, we formulate our model to study the effects of human population and pollution on the dynamics of rainfall. Let in the region under consideration, at any time $t$, the densities of human population, cloud droplets, rain drops and concentration of pollutants are denoted by $N(t)$, $C_d(t)$, $C_r(t)$ and $P(t)$, respectively. In the modeling process, it is assumed that density of human population follows logistic growth with intrinsic growth rate $r$ and carrying capacity $r/r_0$, here $r_0$ is the intra-specific competition among human population due to resource limitation. Further, it is assumed that the per capita growth rate of human population decreases in proportion to the concentration of pollutants at a rate $\theta$. For the cloud droplets’ dynamics, we have assumed that cloud droplets are naturally formed in the atmosphere at a rate $Q_d$, which decreases with a fraction proportional to human population, following Holling type-II functional response $\phi N/(m + N)$. Here, the constant $\phi$ represents the maximum fraction of reduction in the rate of formation of cloud droplets and $m$ is the half saturation constant, i.e. maximum reduced fraction becomes half when the density of human population arrives at $m$. It is also assumed that the density of cloud droplets depletes naturally (conversion into rain drops) proportional to its density at a rate $\alpha$ and it also decreases at a rate proportional to the concentration of pollutants and cloud droplets’ density (i.e. $k_1 C_d P$), here $k_1$ is the decay rate in cloud droplets due to pollutants. The density of rain drops is assumed to be increased naturally at a rate proportional to the natural depletion rate of cloud droplets (i.e. $\lambda \alpha C_d$), here the proportionality constant $\lambda \in (0, 1)$ represents the conversion of cloud droplets into rain drops. The pollutants present in the atmosphere affect the condensation of cloud droplets and thus we have considered that the growth rate in the density of rain drops due to pollutants is proportional to the depletion rate of cloud droplets (i.e. $\pi_1 k_1 C_d P$), where $\pi_1 \in [0, 1]$ is a proportionality constant. Here, $\pi_1 = 0$ represents the situation where $k_1 C_d P$ cloud droplets’ density per unit time in the atmosphere does not participate in the process of condensation and agglomeration to form rain drops. We have also considered that the density of rain drops depletes naturally at a rate $\delta_r$. We also assume that pollutants enter at a constant rate $Q_p$ (natural emission) in the atmosphere and are enhanced in proportional to the density of human population at a rate $\eta$ due to anthropogenic emissions. These pollutants deplete naturally as well as due to their usage in condensation of cloud droplets and removal during rainfall. The depletion rates of pollutants in the atmosphere are assumed to be $\mu_0$ (natural), $k_1$ (proportional to $C_d$) due to their use in condensation and $\mu_r$ (proportional to $C_r$) due to removal during rainfall.
Keeping above facts in mind, we have derived the following mathematical model comprising non-linear ordinary differential equations:

\[
\begin{align*}
\frac{dN}{dt} &= rN - r_0N^2 - \theta NP, \\
\frac{dC_d}{dt} &= Q_d \left(1 - \frac{\phi N}{m + N}\right) - \alpha C_d - k_1 C_d P, \\
\frac{dC_r}{dt} &= \lambda \alpha C_d + \pi_1 k_1 C_d P - \delta_r C_r, \\
\frac{dP}{dt} &= Q_p + \eta N - \mu_0 P - k_1 C_d P - \mu_r C_r P,
\end{align*}
\]

where \(N(0) \geq 0\), \(C_d(0) > 0\), \(C_r(0) > 0\), \(P(0) > 0\). Since the model system (1) governs the dynamics of densities of human population, cloud droplets, rain drops and concentration of pollutants, therefore we assume that all the involved parameters are non-negative and \(r - \theta P > 0\) for all time \(t > 0\).

**Lemma 2.1.** For the model system (1), the region of attraction is contained in the bounded set

\[
\mathcal{D} := \{(N, C_d, C_r, P) \in \mathbb{R}_+^4 : 0 \leq N \leq \max \{N(0), r/r_0\}; 0 \leq C_d \leq \max \{C_d(0), C_{dm}\}; 0 \leq C_r \leq \max \{C_r(0), C_{rm}\}; 0 \leq P \leq \max \{P(0), P_m\}\}
\]

where \(C_{dm} = Q_d/\alpha\), \(C_{rm} = (\lambda \alpha + \pi_1 k_1 P_m)C_{dm}/\delta_r\) and \(P_m = (Q_P + \eta r/r_0)/\mu_0\).

3. **Feasibility of equilibria.** For the formulated system (1), \(E_1(0, C_{d1}, C_{r1}, P_1)\) and \(E^*(N^*, C^*_d, C^*_r, P^*)\) are only two feasible equilibria. The first equilibrium \(E_1\) depicts the case when the effect of density of human population and its associated pollution on rainfall is not considered; however, in the latter equilibrium both these effects will be visible and thus this is an interesting equilibrium.

**Feasibility of \(E_1(0, C_{d1}, C_{r1}, P_1)\):** In this equilibrium, the first component, i.e. \(N = 0\) and the other three components may be computed by analyzing the following equations:

\[
\begin{align*}
Q_d - \alpha C_d - k_1 C_d P &= 0, \quad (2) \\
\lambda \alpha C_d + \pi_1 k_1 C_d P - \delta_r C_r &= 0, \quad (3) \\
Q_p - \mu_0 P - k_1 C_d P - \mu_r C_r P &= 0. \quad (4)
\end{align*}
\]

From equations (2) and (3), we get,

\[
\begin{align*}
C_d &= \frac{Q_d}{\alpha + k_1 P}, \quad (5) \\
C_r &= \frac{\lambda \alpha + \pi_1 k_1 P}{\delta_r} \frac{Q_d}{\alpha + k_1 P}. \quad (6)
\end{align*}
\]

Now, using \(C_d\) and \(C_r\) from equations (5) and (6) in equation (4), we obtain a quadratic equation in \(P\) as follows:

\[
aP^2 + bP - c = 0,
\]

where \(a = \mu_0 \delta_r k_1 + \mu_r \pi_1 k_1 Q_d, b = \mu_0 \delta_r \alpha + \mu_r \lambda \alpha Q_d + \delta_r k_1 Q_d - \delta_r k_1 Q_p\) and \(c = \delta_r \alpha Q_p\).

Clearly, equation (7) possesses a unique positive root \(P = (-b + \sqrt{b^2 + 4ac})/2a = P_1\) (say). Now, using this value of \(P = P_1\) in equations (5) and (6), we get the positive value of \(C_d\) and \(C_r\) as \(C_{d1}\) and \(C_{r1}\), respectively.
Feasibility of $E^*(N^*, C^*_d, C^*_r, P^*)$: To show the feasibility of interior equilibrium $E^*$, we analyze the following equations in detail:

$$r - r_0 N - \theta P = 0,$$  
$$Q_d \left(1 - \frac{\phi N}{m + N}\right) - \alpha C_d - k_1 C_d P = 0,$$  
$$\lambda \alpha C_d + \pi_1 k_1 C_d P - \delta r C_r = 0,$$  
$$Q_p + \eta N - \mu_0 P - k_1 C_d P - \mu_r C_r P = 0.$$  

Now, using the value of $N$ from equation (8) in equation (9), we obtain an equation in $C_d$ and $P$ as

$$C_d = \frac{Q_d}{\alpha + k_1 P} \frac{mr_0 + (1 - \phi)(r - \theta P)}{mr_0 + r - \theta P}.$$  

Further, using the value of $N$ from equation (8) and the value of $C_r$ from equation (10), we get another equation in $C_d$ and $P$ as follows

$$C_d = \left[\frac{[(r_0 Q_p + \eta r) - (\eta \theta + \mu_0 r_0) P]}{\delta r r_0 k_1 + \mu_r r_0 (\lambda \alpha + \pi_1 k_1 P)}\right].$$  

We make a plot of the isoclines (12) and (13) for the feasibility of $E^*$. The analysis of the isocline (12), yields that

(i) At $P = \frac{r}{\theta} + \frac{mr_0}{\pi_1 k_1 - \eta r} = P_a$ (say), $C_d = 0$, (ii) $P = -\frac{r}{\theta}$ and $P = \frac{r}{\theta} + \frac{mr_0}{\pi_1 k_1 - \eta r} = P_b$ (say) are the asymptotes, and (iii) as $P \to -\frac{r}{k_1} \text{ or } P \to P_b$, $C_d \to +\infty$ and $P \to P_b$, $C_d \to -\infty$. Here, it may be noted that $P_a > P_b$.

Similarly, from equation (13), we may note that

(i) At $P = \frac{r_0 Q_p + \eta r}{\eta \theta + \mu_0 r_0} = P_c$ (say), $C_d = 0$, (ii) $P = 0$ and $P = -\frac{\delta r k_1 C_d + \lambda \alpha \mu_r r_0}{\pi_1 k_1 \mu_r r_0}$ are the asymptotes, and (iii) $\frac{dC_d}{dP}$ is negative.

Now, with the above information, we can plot the isoclines (12) and (13) and observe that these two will intersect in the interior of the positive quadrant before $P = P_b$ at a unique point $(P^*, C^*_d)$. However, there may or may not be another intersection in the interior of first quadrant depending on the condition whether $P_c > P_a$ or $P_c < P_a$, respectively [See Figure (1)]. This another point of intersection (if exists) will not be of our interest as for this $P^* > P_b > r/\theta$. Finally, using the value of $P^*$ and $C^*_d$ in equations (8) and (10), we get the positive value of $N = N^*$ (if $r - \theta P^* > 0$) and $C_r = C^*_r$, respectively.

4. Stability analysis. In the previous section, we have shown the feasibility of all the equilibria and now we discuss the local asymptotic stability behavior of these equilibria. This stability behavior of these equilibria will provide the information regarding the movement of solution trajectories starting near to, but not precisely at equilibrium. The results regarding the local stability behavior of both the equilibria in the form of theorem is stated below.

**Theorem 4.1.** (i) The equilibrium $E_1(0, C_{d1}, C_{r1}, P_1)$ is unstable,

(ii) the necessary and sufficient condition for local asymptotic stability of the interior equilibrium $E^*(N^*, C^*_d, C^*_r, P^*)$ is

$$D_1 D_2 D_3 - D_2^2 D_4 > 0,$$  

where $D_i$'s are given in the proof.
For proof of above theorem see Appendix - A.

Now, we perform the global stability analysis of the interior equilibrium \( E^*(N^*, C^*_d, C^*_r, P^*) \). In the following theorem, we provide the sufficient condition for the global stability of interior equilibrium \( E^* \).

**Theorem 4.2.** The interior equilibrium \( E^*(N^*, C^*_d, C^*_r, P^*) \) is globally asymptotically stable inside the region of attraction \( \mathcal{D} \), if the following inequality is satisfied:

\[
\max \left\{ \frac{2\theta k_d^2 P^*}{\eta \mu_0 \alpha}, \frac{\theta \mu_r P^* (\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 k_1 \eta \alpha \delta \pi_i C_d^*} \right\} < \min \left\{ \frac{r_0 \alpha (m + N^*)}{Q_d^2 \delta^2}, \frac{\theta \mu_0 \alpha}{2 \eta k_d^2 C_d^*} \right\}. \tag{15}
\]

For proof of above theorem see Appendix - B.

5. **Stochastic model.** It is noteworthy that environmental fluctuations, like seasonal variation, drought, flood, pollution, hot gases, etc. affect our ecological system and hence are not ignorable to get more realistic results. Thus, to assess the impact of environmental fluctuations, we convert deterministic model system (1) into stochastic by introducing environmental noise term into each equation [2, 9, 11, 14, 15, 31]. For this, we consider that intrinsic growth rate of human population (\( r \)), natural depletion rate of cloud droplets (\( \delta \)), natural depletion rate of rain drops (\( \sigma \)) and depletion rate coefficient of pollution (\( \mu_0 \)) fluctuates due to environmental fluctuations. Thus, by perturbing \( r \rightarrow r + \sigma_1 \xi_1(t) \), \( \alpha \rightarrow \alpha - \sigma_2 \xi_2(t) \), \( \delta \rightarrow \delta_r - \sigma_3 \xi_3(t) \) and \( \mu_0 \rightarrow \mu_0 - \sigma_4 \xi_4(t) \), the randomness is introduced into the deterministic model system (1). Now, we obtain the following system:

\[
\begin{align*}
\frac{dN}{dt} &= (r + \sigma_1 \xi_1(t))N(t) - r_1 N^2(t) - \theta N(t)P(t), \\
\frac{dC_d}{dt} &= Q_d \left( 1 - \frac{\phi N(t)}{m + N(t)} \right) - (\alpha - \sigma_2 \xi_2(t))C_d(t) - k_1 C_d P, \\
\frac{dC_r}{dt} &= \lambda \alpha C_d(t) + \pi_1 k_1 C_d(t)P(t) - (\delta_r - \sigma_3 \xi_3(t))C_r(t), \\
\frac{dP}{dt} &= Q_p + \eta N(t) - (\mu_0 - \sigma_4 \xi_4(t))P(t) - k_1 C_d(t)P(t) - \mu_r C_r(t)P(t),
\end{align*}
\]  \tag{16}

where, \( \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t) \) are white noise terms and are mutually independent. These are described as

\[
\langle \xi_1(t) \rangle = \langle \xi_2(t) \rangle = \langle \xi_3(t) \rangle = \langle \xi_4(t) \rangle = 0 \text{ and } \langle \xi_i(t) \xi_j(t) \rangle = \delta_{ij} \delta(t - t_1).
\]

Where, \( \delta_{ij} \) is Kronecker delta, \( \delta(\cdot) \) is Dirac-\( \delta \) function and \( \sigma_i^2 > 0 \), \( i = 1, 2, 3, 4 \) denote the intensities of environmental fluctuations. Hence, we derive the following stochastic model system as follows:

\[
\begin{align*}
\frac{dN}{dt} &= (r N(t) - r_1 N^2(t) - \theta N(t)P(t))dt + \sigma_1 N(t)dB_1(t), \\
\frac{dC_d}{dt} &= \left( Q_d \left( 1 - \frac{\phi N(t)}{m + N(t)} \right) - \alpha C_d(t) - k_1 C_d P(t) \right)dt + \sigma_2 C_d(t)dB_2(t), \\
\frac{dC_r}{dt} &= (\lambda \alpha C_d(t) + \pi_1 k_1 C_d(t)P(t) - \delta_r C_r(t))dt + \sigma_3 C_r(t)dB_3(t), \\
\frac{dP}{dt} &= (Q_p + \eta N(t) - \mu_0 P(t) - k_1 C_d P(t) - \mu_r C_r P(t))dt + \sigma_4 P(t)dB_4(t),
\end{align*}
\]  \tag{17}

where, \( B_1, B_2, B_3 \) and \( B_4 \) denote the one-dimensional independent standard Brownian motion. Here, \( dB_i = \xi_i(t)dt, i = 1, 2, 3, 4 \), denote the relation between Brownian motion and white noise terms [8].
6. Preliminaries. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is complete probability space and satisfying the usual conditions [16]. Further, it is also considered that $\text{Int}(\mathbb{R}_+^n) = \{y \in \mathbb{R}^n : y_i > 0, \; i = 1, 2, \ldots, n\}$. Let us consider $n-$ dimensional stochastic differential equation (for detail, see [15])

$$dy(t) = f(y(t), t)dt + g(y(t), t)dB(t) \text{ for } t \geq t_0,$$

(18)

with $y(t_0) = y_0 \in \mathbb{R}^n$. Here, $B(t)$ represents the $l-$ dimensional Brownian motion over the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider $Z(y, t)$ be a $C^{2,1}$ function and defined on $\mathbb{R}^l \times [0, \infty]$. $L$ is the differential operator associated with equation (18) and defined as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(y, t) \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{n} [g^T(y, t)g(y, t)]_{ij} \frac{\partial^2}{\partial y_i \partial y_j}.$$

If $L$ acts on a function $Z$, then

$$LZ = Z_t(y, t) + Z_y(y, t)f(y, t) + \frac{1}{2} \text{trace}[g^T(y, t)Z_{yy}(y, t)g(y, t)],$$

where, $Z_t = \frac{\partial Z}{\partial t}, Z_y = (\frac{\partial Z}{\partial y_1}, \frac{\partial Z}{\partial y_2}, \ldots, \frac{\partial Z}{\partial y_n}), Z_{yy} = (\frac{\partial^2 Z}{\partial y_i \partial y_j})_{n \times n}$.

Applying Itô’s formula,

$$dZ(y(t), t) = LZ(y(t), t)dt + Z_y(y(t), t)g(y(t), t)dB(t).$$

Further, the criteria in the context of stationary distribution is given in the next (for detail see [7]).

We assume that $\mathcal{Y}(t)$ be a homogeneous Markov process and defined in $\mathbb{R}^n$ ($n-$ dimensional Euclidean space) described as

$$d\mathcal{Y}(t) = e(\mathcal{Y}(t))dt + \sum_{r=1}^{l} g_r(\mathcal{Y})dB_r(t).$$

The diffusion matrix is

$$A(y) = (a_{ij}(y)), \; a_{ij} = \sum_{r=1}^{l} g_{cr}^e(y)g_{cr}^e(y).$$

Here, we consider that with the regular boundary $\Gamma$, the bounded domain $D \subset \mathbb{R}^n$ exists with the following properties:

$H_1$: The lowest eigenvalue of the diffusion matrix $A(y)$ is bounded away from zero in the domain $D$ and some neighborhood thereof.

$H_2$: If $y \in \mathbb{R}^n \setminus D$, the mean time $\tau$ at which a path issuing from $y$ reaches the set $D$ is finite and $\sup_{y \in M} E_y \tau < \infty$ for every compact subset $M \subset \mathbb{R}^n$.

Lemma 6.1. If $H_1$ and $H_2$ are satisfied, then the Markov process $\mathcal{Y}(t)$ has a stationary distribution $\theta(\cdot)$. Consider $h(\cdot)$ be a integrable function with respect to the measure $\theta$. Then

$$P_y \left\{ \lim_{t \to \infty} \frac{1}{T} \int_0^T h(\mathcal{Y}(t))dt = \int_{\mathbb{R}^n} h(y)\theta(dy) \right\} = 1, \; \forall \; y \in \mathbb{R}^n.$$
6.1. **Existence and uniqueness of global positive solution.** Here, by using change of variables, the existence of unique positive solution of model system (17) is shown and further by using Lyapunov’s method [1, 2, 6], we prove that this solution is global.

**Lemma 6.2.** There is a unique positive local solution \((N(t), C_d(t), C_r(t), P(t))\) of model system (17) with initial values \((N(0), C_d(0), C_r(0), P(0)) \in \text{Int}(\mathbb{R}_+^4)\) for \(t \in [0, \tau_e)\) almost surely, where \(\tau_e\) is the explosion time.

For proof of above lemma see **Appendix - C.**

Further, to prove this solution is global, we have the following theorem.

**Theorem 6.3.** There is a unique positive solution \((N(t), C_d(t), C_r(t), P(t))\) of model system (17) on \(t \geq 0\) with any given initial value \((N(0), C_d(0), C_r(0), P(0)) \in \text{Int}(\mathbb{R}_+^4)\), and this solution will remain in \(\text{Int}(\mathbb{R}_+^4)\) with probability 1.

For proof of above theorem see **Appendix - D.**

6.2. **Boundedness.** To show that the ultimately boundedness of model system (17) in mean, we have following theorem.

**Theorem 6.4.** Model system (17) is stochastically ultimately bounded.

For proof of above theorem see **Appendix - E.**

6.3. **Asymptotic behavior of model system (17).** In this section, we discuss about the asymptotic behavior of stochastic model system (17) around the interior equilibrium \(E^*\) of model system (1).

**Theorem 6.5.** If the following inequalities hold
\[
4\sigma_2^2 < \alpha, \quad \sigma_3^2 < \delta_r, \quad 2\sigma_4^2 < \mu_0, \quad \max\{W_1, W_2\} < \min\{W_3, W_4\},
\]
then the solution of stochastic model system (17) with any given initial value \((N(0), C_d(0), C_r(0), P(0)) \in \text{Int}(\mathbb{R}_+^4)\), has the property
\[
\lim_{t \to \infty} \sup_t \frac{1}{t} E \int_0^t \left( (N(s) - N^*)^2 + (C_d(s) - C_d^*)^2 + (C_r(s) - C_r^*)^2 + (P(s) - P^*)^2 \right) ds \
\leq \frac{K_x}{K},
\]
where, the values of \(W_1, W_2, W_3, W_4, K\) and \(K_x\) are given in the proof.

For proof of above theorem see **Appendix - F.**

**Remark 1.** From the above theorem, it is easily noted that solution of the model system (17) fluctuates around the interior equilibrium \(E^*\) of model system (1). Moreover, smaller fluctuations arise if the intensities of noise terms are small. This indicates that the solution of the stochastic model system (17) is near to the interior equilibrium \(E^*\) as the stochastic perturbations are small.

**Theorem 6.6.** Stochastic model system (17) has a unique stationary distribution \(\theta(\cdot)\) with Ergodic property, provided condition (19) is satisfied.

For proof of above theorem see **Appendix - G.**
7. **Numerical simulation.** In the previous sections, we have presented the analysis of the formulated deterministic model (1) and its stochastic version (17). Now, to validate the obtained analytical results, we simulate these models by taking parameter values as given below:

\[
\begin{align*}
    r &= 0.1, \quad r_0 = 0.00001, \quad \theta = 0.0005, \quad Q_d = 1, \quad \phi = 0.1, \quad m = 500, \quad \alpha = 0.8, \quad k_1 = 0.5, \\
    \lambda &= 0.5, \quad \pi_1 = 0.09, \quad \delta_r = 0.4, \quad Q_p = 0.02, \quad \eta = 0.00002, \quad \mu_0 = 0.2, \quad \mu_r = 0.02. \\
\end{align*}
\]

(21)

For the parameter values given in (21), the components \(N^*, C_d^*, C_r^*, P^*\) of \(E^*\) are found as:

\[
\begin{align*}
    N^* &= 9984, \quad C_d^* = 0.9435, \quad C_r^* = 0.9772, \quad P^* = 0.3177.
\end{align*}
\]

Moreover, for these parameter values, eigenvalues of the Jacobian matrix \(J^* (J, \text{evaluated at } E^*)\) are computed as:

\[
\begin{align*}
    -0.10004, \quad -0.3888, \quad -0.5343, \quad -1.1268.
\end{align*}
\]

Here, it is noted that all the four eigenvalues are negative and so the equilibrium \(E^*\) is locally asymptotically stable. It may be further noted that for the chosen parameter values, the condition for global stability given in Theorem (4.2) is also satisfied, this implies that inside region of attraction, \(E^*\) is globally asymptotically stable. This result is graphically shown in Figure (2) by plotting the solution trajectories initiating from different points in \(N - C_r - P\) and \(N - C_d - C_r\) spaces. From figure (2), we can see that all the solution trajectories move towards their equilibrium values in their respective spaces irrespective of their starting point inside the region of attraction. This guarantees the global asymptotic stability of the interior equilibrium \(E^*\) for the chosen parameter values in these spaces.

Furthermore, to explore the efficacy of the parameter related to the augmentation rate of atmospheric pollution due to anthropogenic emission, i.e. \(\eta\) on the rainfall pattern, we have plotted Figure (3). From this figure, it is apparent that as pollution increases in the atmosphere, the rainfall may decrease or increase depending on the types of clouds and pollutants. For some clouds, the cloud droplets condense on the pollution particles and float in the atmosphere without collision (for less value of \(\pi_1\) in comparison to the natural formation of raindrops, i.e. \(\lambda\)) resulting to decrease in the rainfall; however, in the case of other clouds with less CCN concentrations, cloud droplets condense on pollution particles to form raindrops (for large value of \(\pi_1\) in comparison to natural formation of raindrops, i.e. \(\lambda\)) resulting to increase in rainfall. This indicates that the increase in the level of atmospheric pollution may lead to drought/flood like situations in several parts of the Earth’s surface and this has been also observed in several parts of the world.

Now we apply Milstein’s method, [8] to solve stochastic model (17) numerically for the chosen parameter values and for this we discretize the equations of system (17)
as given below:

\[
N_{i+1} - N_i = (r N_i - r_1 N_i^2 - \theta N_i P_i) \Delta t + \sigma_1 N_i \sqrt{\Delta t} \zeta_i + \frac{\sigma_2^2}{2} N_i (\zeta_i^2 - 1) \Delta t,
\]

\[
C_{di+1} - C_{di} = (Q_d \left(1 - \frac{\phi N_i}{m + N_i}\right) - \alpha C_{di} - k_1 C_{di} P_i) \Delta t + \sigma_2 C_{di} \sqrt{\Delta t} \zeta_i
+ \frac{\sigma_3^2}{2} C_{di} (\zeta_i^2 - 1) \Delta t,
\]

\[
C_{ri+1} - C_{ri} = (\lambda \alpha C_{di} + \pi_1 k_1 C_{di} P_i - \delta_r C_{ri}) \Delta t + \sigma_3 C_{ri} \sqrt{\Delta t} \zeta_i
+ \frac{\sigma_4^2}{2} C_{ri} (\zeta_i^2 - 1) \Delta t,
\]

\[
P_{i+1} - P_i = (Q_p + \eta N_i - \mu_0 P_i - k_1 C_{di} P_i - \mu_r C_{ri} P_i) \Delta t + \sigma_4 P_i \sqrt{\Delta t} \zeta_i
+ \frac{\sigma_5^2}{2} P_i (\zeta_i^2 - 1) \Delta t,
\]

here the independent Gaussian random variable \(N(0,1)\) is represented by \(\zeta_i (i = 1, 2, 3, \cdots, n)\).

In order to simulate model system (17), we take the values of \(\sigma_1 = 0.0001, \sigma_2 = 0.006, \sigma_3 = 0.004\) and \(\sigma_4 = 0.008\), along with the parameter values as provided in (21). It is noted that for these parameter values, the conditions given in theorem (6.5) are satisfied, thus model (17) has a unique stationary distribution. Furthermore, in Figure (4), we have shown that for small values of the intensities of noise terms, small fluctuations arise. This indicates that the solution of model (17) oscillates around the components of \(E^*\) of model (1) with small amplitude of oscillations. Similar results are found in Figure (5) with larger amplitude of oscillations on increasing the values of intensities to \(\sigma_1 = 0.08, \sigma_2 = 0.2, \sigma_3 = 0.09\) and \(\sigma_4 = 0.1\). Moreover, Figures (6), (7) and (8) are drawn to stimulate the effect of little bit changes in the intensities of noise on the stationary distribution of human population, cloud droplets, raindrops and pollution. From Figure (6), it is clear that for \(\sigma_1 = 0.0001, \sigma_2 = 0.006, \sigma_3 = 0.004\) and \(\sigma_4 = 0.008\), the stationary distribution of human population, cloud droplets, raindrops and pollution around the mean values 9984, 0.9435, 0.9772 and 0.3177 is normally distributed. Moreover, from Figures (7) and (8), it is manifested that an increase in the intensities of noise to \(\sigma_1 = 0.08, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.1\) and \(\sigma_1 = 0.09, \sigma_2 = 0.3, \sigma_3 = 0.1, \sigma_4 = 0.2\), respectively reflects the small change in the mean values and skewness of the distribution. More precisely, stationary distribution of human population, cloud droplets, raindrops and pollution follows standard normal distribution for \(\sigma_1 = 0.0001, \sigma_2 = 0.006, \sigma_3 = 0.004, \sigma_4 = 0.008\) and positively skewed as the intensities of noise increase to \(\sigma_1 = 0.08, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.1\) and \(\sigma_1 = 0.09, \sigma_2 = 0.3, \sigma_3 = 0.1, \sigma_4 = 0.2\).

8. Conclusion. Continuous increase in pollution is a major threat to the Government of several developing countries across the globe. The main reason behind the augmentation of environmental pollution is the increase in human population and their activities. This has given the birth of various problems, such as deterioration in human health, change in rainfall pattern, ecological imbalance, etc. It indicates that there is a relationship between human population, rainfall and pollution; therefore to understand this relationship, we have formulated a mathematical model comprising non-linear ordinary differential equations and its stochastic version.
In the modelling process, it is considered that cloud droplets are naturally formed in the atmosphere and their rate of formation decreases in proportion to human population with saturated type function. It is also assumed that pollution poses negative effect on human population and when the pollution particles reach in the atmosphere, they change the dynamics of CCN concentrations and thus precipitation. Stability theory of differential equations is used to perform the qualitative analysis of proposed deterministic model. For the proposed model, it is found that two equilibria are feasible; namely, boundary equilibrium and interior equilibrium. It is proved that boundary equilibrium is always unstable. Under condition (14), the local asymptotic stability of the interior equilibrium is established and further inside the region of attraction, this equilibrium is globally asymptotically stable provided the condition stated in Theorem (4.2) is satisfied.

Further, for the stochastic version of the proposed ODE model, using Lyapunov’s method, we have shown that positive solution exists globally and is also unique. We have also shown the boundedness of this unique positive solution. Further using Ergodic theory, under certain conditions, the uniqueness of stationary distribution of the stochastic model is also established. We have also observed that if the values of the intensities of white noise terms are small then equilibrium solutions of the stochastic model are close to the equilibrium values of $E^*$ with small fluctuations.

Moreover, numerical simulations yield that the augmentation in the concentration of pollutants due to anthropogenic emissions may decrease or increase the rainfall. It is noted that if the intensities of white noise are small, then the stationary distribution of all dynamical variables follow normal distribution around their average values. The model analysis reveals that the reduction and growth in rainfall depend on the types of clouds and pollutant particles emitted in the environment. This result is inline with the observations of several regions of the world. This implies that the pollutants emitted in the environment may create drought/flood like situations in several regions of the Earth and both of them are harmful to the Earth’s environment. Thus, it is important to reduce the anthropogenic emissions of pollutants in the atmospheric environment and further studies are required in this area by considering different kinds of clouds and pollutants.

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Proof. To prove this theorem, we have obtained the Jacobian matrix \( J \) for the model system (1) as

\[
J = \begin{bmatrix}
  r - 2r_0N - \theta P & 0 & 0 & -\theta N \\
  -\frac{Q_d\phi\eta}{(m+\eta)N} & -(\alpha + k_1P) & 0 & -k_1C_d \\
  0 & (\lambda_0 + \pi_1k_1P) & -\delta_r & \pi_1k_1C_d \\
  \eta & -k_1P & -\mu_rP & -(\mu_0 + k_1C_d + \mu_rC_r)
\end{bmatrix}.
\] (22)

(i) First,we evaluate the Jacobian matrix \( J \) at the equilibrium \( E_1 \) (i.e. \( J_1 \)) and note that one eigenvalue of \( J_1 \) is \( r - \theta P_1 \), which is positive for the feasibility of model system (1) and rest eigenvalues are the roots of the cubic equation

\[
\psi^3 + A_1\psi^2 + A_2\psi + A_3 = 0,
\] (23)

where, \( A_1 = (\delta_r + \mu_0 + k_1C_{d1} + \mu_rC_{r1}) + (\alpha + k_1P_1) \),
\( A_2 = \delta_r(\mu_0 + k_1C_{d1} + \mu_rC_{r1}) + \mu_0\pi_1k_1C_{d1}P_1 + (\alpha + k_1P_1)(\delta_r + \mu_0 + \mu_rC_{r1}) + \alpha k_1C_{d1} \)
and \( A_3 = \alpha\delta_r(\mu_0 + k_1C_{d1} + \mu_rC_{r1}) + (\alpha + k_1P_1)\mu_r\pi_1k_1C_{d1}P_1 + \delta_r\mu_0k_1P_1 \).

Here, it is easy to note that \( A_1 > 0, A_3 > 0 \) and some manipulation yields that \( A_1A_2 - A_3 > 0 \). Now using Routh-Hurwitz criterion, it can be said that all the roots of equation (23) are either negative or with negative real part and hence the equilibrium \( E_1(0, C_{d1}, C_{r1}, P_1) \) has unstable manifold locally in \( N \)-direction and stable manifold locally in \( C_d - C_r - P \) space.

(ii) The characteristic polynomial for the Jacobian matrix \( J^* \) (\( J \), evaluated at \( E^* \)) is

\[
\Psi^4 + D_1\Psi^3 + D_2\Psi^2 + D_3\Psi + D_4 = 0,
\] (24)

where \( D_1 = r_0N^* + (\alpha + k_1P^*) + \delta_r + (\mu_0 + k_1C^*_d + \mu_rC^*_r) \),
\( D_2 = r_0N^* + r_0N^* + (\alpha + \lambda_0 + \pi_1k_1P^*) + \delta_r + k_1P^* + \mu_rC^*_r + \pi_1\mu_rC^*_d + (r_0N^* + \alpha + \lambda_0 + \mu_0) + k_1C^*_d + \mu_rC^*_r \),
\( D_3 = r_0N^* + (\alpha + k_1P^*)\delta_r + \mu_0 + \mu_rC^*_r + \pi_1k_1\mu_rC^*_dP^* + (r_0N^* + \alpha + k_1P^*)\mu_0 + k_1C^*_d + \mu_rC^*_r \delta_r + k_1\mu_0\delta_rP^* + (\alpha + k_1P^*)\pi_1\mu_rC^*_dP^* + \theta\eta(\delta_r + \alpha + k_1P^*)N^* \),
and \( D_4 = r_0N^* \left\{ (\alpha\delta_r(\mu_0 + k_1C^*_d + \mu_rC^*_r) + (\alpha + k_1P^*)\pi_1k_1\mu_rC^*_dP^* + \eta(\delta_r + \alpha + k_1P^*)N^* \right\} \).

It can be easily seen that all \( D_i \)'s are positive and thus by using Routh-Hurwitz criterion, it can be stated that the roots of the characteristic polynomial (24) will be either negative or with negative real part iff the condition (14) is satisfied. \( \square \)

Appendix B Proof of Theorem 4.2

Proof. For this, we consider a Lyapunov’s function, \( W : \mathbb{R}^4 \to \mathbb{R} \) as:

\[
W = \left( N - N^* \right) + \frac{1}{2}m_1(C_d - C^*_d)^2 + \frac{1}{2}m_2(C_r - C^*_r)^2 + \frac{1}{2}m_3(P^* - P^*)^2,
\]
where, $m_1, m_2$ and $m_3$ are positive constants, which will be determined appropriately. Differentiating above equation ‘$W$’ with respect to time ‘$t$’ along the solutions of model system (1), we have

$$
\frac{dW}{dt} = (N - N^*) \frac{dN}{dt} + m_1(C_d - C_d^*) \frac{dC_d}{dt} + m_2(C_r - C_r^*) \frac{dC_r}{dt} + m_3(P - P^*) \frac{dP}{dt},
$$

$$
= -r_0(N - N^*)^2 - m_1(\alpha + k_1 P)(C_d - C_d^*)^2 - m_2\delta r(C_r - C_r^*)^2
- m_3(\mu_0 + k_1 C_d + \mu_r C_r)(P - P^*)^2 - k_1(m_1 C_d^* + m_3 P^*)(P - P^*)(C_d - C_d^*)
- \frac{m_1 Q_d \phi_m}{(m + N^*)(m + N)} (C_d - C_d^*)(N - N^*) - (\theta - m_3 \eta)(P - P^*)(N - N^*) + m_2(\lambda \alpha + \pi_1 k_1 P)(C_r - C_r^*)(C_d - C_d^*) + (m_2 \pi_1 k_1 C_d^* - m_3 \mu_r P^*)(P - P^*)(C_r - C_r^*).
$$

Choosing $m_3 = \frac{\theta}{\eta}$, $m_2 = \frac{\theta \mu_r P^*}{\pi_1 \eta C_d^*}$ and making a simple algebraic manipulation, we obtain

$$
\frac{dW}{dt} \leq - \left( r_0 - \frac{m_1 Q_d^2 \phi^2}{(m + N^*)\alpha} \right) (N - N^*)^2 - \left( \frac{m_1}{4} \alpha - \frac{\theta k_1^2 P^*^2}{2\eta \mu_0} \right) (C_d - C_d^*)^2
- \frac{\theta \mu_r P^*}{\pi_1 \eta C_d^*} \left( \delta r - \frac{\theta \mu_r P^*(\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 \eta C_d^* \alpha m_1} \right) (C_r - C_r^*)^2
- \left( \frac{\theta \mu_0}{2\eta} - \frac{m_1 k_1^2 C_d^*^2}{\alpha} \right) (P - P^*)^2,
$$

(25)

here, it may be noted that $\frac{dW}{dt}$ is negative definite inside $\mathcal{D}$, if the following inequalities are satisfied:

$$
r_0 > \frac{m_1 Q_d^2 \phi^2}{(m + N^*)\alpha},
$$

(26)

$$
\frac{m_1}{4} \alpha > \frac{\theta k_1^2 P^*^2}{2\eta \mu_0},
$$

(27)

$$
m_1 \delta r > \frac{\theta \mu_r P^*(\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 \eta C_d^* \alpha m_1},
$$

(28)

$$
\frac{\theta \mu_0}{2\eta} > \frac{m_1 k_1^2 C_d^*^2}{\alpha}.
$$

(29)

From inequalities (26)-(29), we can choose a positive value of $m_1$ as

$$
\max \left\{ \frac{2\theta k_1^2 P^*^2}{\eta \mu_0 \alpha}, \frac{\theta \mu_r P^*(\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 \kappa \eta \alpha \delta r C_d^*} \right\} < m_1 < \min \left\{ \frac{r_0 \alpha (m + N^*)}{Q_d^2 \phi^2}, \frac{\theta \mu_0 \alpha}{2\eta k_1^2 C_d^*^2} \right\}.
$$

From the above inequality, we can say that $\frac{dW}{dt}$ is negative definite if the condition (15) is satisfied. This proves the theorem.

\begin{appendices}

\section*{Appendix C Proof of Lemma 6.2}
\end{appendices}
Proof. Let \( v_1(t) = \log N(t), v_2(t) = \log C_d(t), v_3(t) = \log C_r(t), v_4(t) = \log P(t) \). Now, using Itô’s formula, we have the following system:

\[
\begin{align*}
d v_1(t) &= \left[ r - \frac{\sigma_1^2}{2} - r_0 e^{v_1(t)} - \theta e^{v_4(t)} \right] dt + \sigma_1 dB_1(t), \\
d v_2(t) &= \left[ \frac{Q_d}{e^{v_2(t)}} \left( 1 - \frac{\theta e^{v_4(t)}}{m + e^{v_1(t)}} \right) - \frac{\sigma_2^2}{2} - k_1 e^{v_4(t)} \right] dt + \sigma_2 dB_2(t), \\
d v_3(t) &= \left[ \frac{\lambda}{e^{v_3(t)}} - \frac{\sigma_2^2}{2} - \delta_r + \frac{\pi k_1 e^{v_3(t)+v_4(t)}}{e^{v_3(t)}} \right] dt + \sigma_3 dB_3(t), \\
d v_4(t) &= \left[ \frac{Q_p}{e^{v_4(t)}} + \eta e^{v_1(t)} - \mu_0 - k_1 e^{v_2(t)} - \mu_r e^{v_3(t)} - \frac{\sigma_4^2}{2} \right] dt + \sigma_4 dB_4(t),
\end{align*}
\]

where, \( v_1(0) = \log N(0), v_2(0) = \log C_d(0), v_3(0) = \log C_r(0), v_4(0) = \log P(0), \)
on \( t \geq 0 \). Here, it may be noted that there exists a unique local solution \( v_1(t), v_2(t), v_3(t), v_4(t) \) on \([0, \tau_e] \) because all the coefficients of stochastic model system (17) satisfy the local Lipschitz condition. Hence for initial value \( N(0) > 0, C_d(0) > 0, C_r(0) > 0, P(0) > 0 \) positive local solutions \( N(t) = e^{v_1(t)}, C_d(t) = e^{v_2(t)}, C_r(t) = e^{v_3(t)}, P(t) = e^{v_4(t)} \) of model system (17) exists.

Appendix D Proof of Theorem 6.3

Proof. We choose non-negative large number \( \varrho_0 > 0 \), so that \( N(0), C_d(0), C_r(0) \) and \( P(0) \) lies within the interval \( \left[ \frac{1}{\varrho_0}, \varrho_0 \right] \). We can define the sequence of stopping times for each integer \( \varrho \geq \varrho_0 \) as follows:

\[
\tau_\varrho = \inf \left\{ t \in [0, \tau_e) : (N(t), C_d(t), C_r(t), P(t)) \notin \left( \frac{1}{\varrho}, \varrho \right) \right\},
\]

where \( \inf \emptyset = \infty (\emptyset \) denotes the empty set). Evidently, as \( \varrho \to \infty, \tau_\varrho \) is increasing. Further, we consider \( \tau_\infty = \lim_{\varrho \to \infty} \tau_\varrho \) then \( \tau_\infty \leq \tau_e \) a.s.. To prove \( \tau_e = \infty \), we only need to prove that \( \tau_\infty = \infty \) a.s.. If this claim is false, then there exists a constant \( T > 0 \) and \( \epsilon \in (0, 1) \) such that \( \mathbb{P}\{\tau_\infty \leq T\} > \epsilon \). Therefore, there exists an integer \( \varrho_1 \geq \varrho_0 \) such that

\[
\mathbb{P}\{\tau_\varrho \leq T\} > \epsilon \text{ for all } \varrho \geq \varrho_1.
\]

Now, consider a \( C^2 \)-function \( X : \text{Int}(\mathbb{R}^+_1) \to \text{Int}(\mathbb{R}_+) \) as follows:

\[
X(N, C_d, C_r, P) = (N + 1 - \log N) + (C_d + 1 - \log C_d) + (C_r + 1 - \log C_r) + (P + 1 - \log P),
\]

this function is non-negative by

\[
v + 1 - \log v \geq 0 \text{ for } v > 0.
\]

By Itô’s formula, we get

\[
dX(N, C_d, C_r, P) = \left[ \left( 1 - \frac{1}{N} \right) (rN - r_1 N^2 - \theta NP) + \left( 1 - \frac{1}{C_d} \right) \left( Q_d \left( 1 - \frac{\dot{B}N}{m + N} \right) \right) - \alpha C_d - k_1 C_d P + \left( 1 - \frac{1}{C_r} \right) \left( \lambda C_d + \pi k_1 C_d - \delta_r C_r \right) + \left( 1 - \frac{1}{P} \right) (Q_p + \eta N - \mu_0 P - k_1 C_d P - \mu_r C_r P) + \frac{1}{2} \sum_{i=1}^2 \sigma_i^2 \right] dt
\]
\[
+ \sigma_1 (N - 1) dB_1(t) + \sigma_2 (C_d - 1) dB_2(t) + \sigma_3 (C_r - 1) dB_3(t) + \sigma_4 (P - 1) dB_4(t)
\]

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Hence, we can write

\[ dX = \left[ (rN - r_0N^2 - \theta NP - r + r_0N + \theta) + \left( \frac{Q_d^2}{m + N} - \alpha \right) 
- k_1C_dP - \frac{Q_d^2}{C_d} + \frac{Q_d^2\alpha + \alpha}{(m + N)C_d} + (\alpha + k_1P) \right] dt 
- \frac{\lambda_0C_r - \frac{\lambda_0k_1C_dP}{C_r} + \delta_r}{C_r} \left( Q_p + \eta N - \mu_0P - k_1C_dP \right) 
- \frac{\mu_rC_rP - \frac{Q_p}{P} - \eta N}{P} + k_1C_d + \mu_rC_r \right] dt 
+ \frac{4}{2} \sum_{i=1}^{2} \sigma_i^2 \right] dt \]

Thus

\[ dX(N, C_d, C_r, P) \leq \left[ C_1 + C_2X(N, C_d, C_r, P) \right] dt + \sigma_1(N - 1)dB_1(t) 
+ \sigma_2(C_d - 1)dB_2(t) + \sigma_3(C_r - 1)dB_3(t) 
+ \sigma_4(P - 1)dB_4(t), \]

where, \( C_1 = \frac{1}{r_0}(r + r_0 + \eta)^2 + \left( \frac{Q_d^2 + Q_p + \alpha + \delta_r + \mu_0 + \frac{1}{2} \sum_{i=1}^{4} \sigma_i^2} \right) \),

and \( C_2 = \max\{ (r_0 + r + \eta), \mu_r, (\theta + k_1) \} \).

Hence, we can write

\[ dX(N, C_d, C_r, P) \leq \left[ C_1 + C_2X(N, C_d, C_r, P) \right] dt + \sigma_1(N - 1)dB_1(t) 
+ \sigma_2(C_d - 1)dB_2(t) + \sigma_3(C_r - 1)dB_3(t) + \sigma_4(P - 1)dB_4(t). \]

Let \( C_3 = \max\{ C_1, C_2 \} \), we have

\[ dX(N, C_d, C_r, P) \leq C_3(1 + X(N, C_d, C_r, P)) dt + \sigma_1(N - 1)dB_1(t) 
+ \sigma_2(C_d - 1)dB_2(t) + \sigma_3(C_r - 1)dB_3(t) + \sigma_4(P - 1)dB_4(t). \]

If \( t_1 \leq T \). Integrating inequality (32) between 0 and \( \tau_\theta \wedge t_1 \), we get

\[ \int_0^{\tau_\theta \wedge t_1} dX(N, C_d, C_r, P) \leq C_3 \int_0^{\tau_\theta \wedge t_1} (1 + X(N, C_d, C_r, P)) dt 
+ \sigma_1 \int_0^{\tau_\theta \wedge t_1} (N - 1)dB_1(t) + \sigma_2 \int_0^{\tau_\theta \wedge t_1} (C_d - 1)dB_2(t) 
+ \sigma_3 \int_0^{\tau_\theta \wedge t_1} (C_r - 1)dB_3(t) + \sigma_4 \int_0^{\tau_\theta \wedge t_1} (P - 1)dB_4(t). \]

Further, on taking expectation of the above inequality, we have

\[ E[X(N(\tau_\theta \wedge t_1), C_d(\tau_\theta \wedge t_1), C_r(\tau_\theta \wedge t_1), P(\tau_\theta \wedge t_1))] \]
\[ \leq E[X(N(0), C_d(0), C_r(0), P(0))] + C_3E \int_0^{\tau_\theta \wedge t_1} (1 + X(N, C_d, C_r, P)) dt, \]
\[ \leq X(N(0), C_d(0), C_r(0), P(0)) + C_3T 
+ C_3 \int_0^{t_1} EX(N(\tau_\theta \wedge t), C_d(\tau_\theta \wedge t), C_r(\tau_\theta \wedge t), P(\tau_\theta \wedge t)) dt. \]

By Gronwall’s inequality [20]

\[ E[X(N(\tau_\theta \wedge T), C_d(\tau_\theta \wedge T), C_r(\tau_\theta \wedge T), P(\tau_\theta \wedge T))] \leq C_4, \quad (33) \]

where, \( C_4 = (X(N(0), C_d(0), C_r(0), P(0)) + C_3T) e^{C_3T}. \quad (34) \]
Consider $\Omega_\varrho = \{ \omega : \tau_\varrho \leq T \} \ \forall \ \varrho \geq \varrho_1$, then by (31), $P(\Omega_\varrho) \geq \epsilon$. Thus $\forall \ \omega \in \Omega_\varrho$, there is at least one of $N(\tau_\varrho, \omega), C_d(\tau_\varrho, \omega), C_r(\tau_\varrho, \omega), P(\tau_\varrho, \omega)$ equals either $\varrho$ or $\varrho/2$. So $X(N(\tau_\varrho, \omega), C_d(\tau_\varrho, \omega), C_r(\tau_\varrho, \omega), P(\tau_\varrho, \omega))$ is no less than either
$$\varrho + 1 - \log \varrho \ \text{or} \ \frac{1}{\varrho} + 1 - \log \frac{1}{\varrho} = \frac{1}{\varrho} + 1 + \log \varrho.$$ Hence
$$X(N(\tau_\varrho, \omega), C_d(\tau_\varrho, \omega), C_r(\tau_\varrho, \omega), P(\tau_\varrho, \omega)) \geq (\varrho + 1 - \log \varrho) \wedge \left( \frac{1}{\varrho} + 1 + \log \varrho \right).$$ Further, from inequalities (31) and (34), we get
$$C_4 \geq \epsilon \left[ (\varrho + 1 - \log \varrho) \wedge \left( \frac{1}{\varrho} + 1 + \log \varrho \right) \right],$$ where, $1_{\Omega_\varrho}$ denotes the indicator function of $\Omega_\varrho$. Now taking $\varrho \rightarrow \infty$, we have
$$\infty > C_4 = \infty,$$ is contradiction. Hence, we must have $\tau_\infty = \infty \ a.s.$

**Appendix E** Proof of Theorem 6.4

**Proof.** Let $H(N(t), C_d(t), C_r(t), P(t)) = N(t) + C_d(t) + C_r(t) + P(t)$. Using Itô’s formula, we get
$$dH(t) = LHdt + \sigma_1 NdB_1(t) + \sigma_2 Cd dB_2(t) + \sigma_3 Cr dB_3(t) + \sigma_4 P dB_4(t)$$

$$LH = rN - r_0 N^2 - \theta NP + Q_d \left( 1 - \frac{\alpha N}{m + N} \right) - \alpha C_d - k_1 C_d P + \lambda \alpha C_d$$

$$+ \pi_1 k_1 C_d P - \delta_r C_r + \varrho P + \eta N - \mu_0 P - k_1 C_d P - \mu_r C_r P,$$

$$\leq 2(r + \eta)N - r_0 N^2 - (r + \eta)N + Q_d + \varrho P - \alpha(1 - \lambda_1)C_d - \delta_r C_r - \mu_0 P,$$

$$\leq \left( Q_d + \varrho P + \frac{(r + \eta)^2}{r_0} \right) - (r + \eta)N - \alpha(1 - \lambda_1)C_d - \delta_r C_r - \mu_0 P.$$ Therefore,
$$dH \leq \left( Q_d + \varrho P + \frac{(r + \eta)^2}{r_0} \right) - (r + \eta)N - \alpha(1 - \lambda_1)C_d - \delta_r C_r - \mu_0 P$$

$$+ \sigma_1 Nd B_1(t) + \sigma_2 Cd dB_2(t) + \sigma_3 Cr dB_3(t) + \sigma_4 P dB_4(t).$$ By applying generalized Itô’s formula and then on taking expectation, we obtain
$$e^{\beta t}E[H((N(t), C_d(t), C_r(t), P(t)), t)] = E[H(N(0), C_d(0), C_r(0), P(0))]$$

$$+ E \int_0^t e^{\beta s} \left( (\beta H(N(s), C_d(s), C_r(s), P(s)), t)) ds \right)$$

$$+ LH((N(s), C_d(s), C_r(s), P(s)), s) dr$$

$$\leq H(N(0), C_d(0), C_r(0), P(0))$$

$$+ E \int_0^t e^{\beta s} \left( \left( Q_d + \varrho P + \frac{(r + \eta)^2}{r_0} \right) \right)$$

$$- (r + \eta - \beta)N(s) - \alpha(1 - \lambda - \beta)C_d(s)$$

$$- (\delta_r - \beta)C_r(s) - (\mu_0 - \beta)P(s) ds,$$

here, the parameter $\beta$ is positive constant and chosen as
$$\beta = \min\{(r + \eta), \alpha(1 - \lambda), \delta_r, \mu_0\}.$$
Then
\[ e^{\beta t}E[H((N(t), C_d(t), C_r(t), P(t)), t)] = H(N(0), C_d(0), C_r(0), P(0)) + \left( Q_d + Q_p + \frac{(r + \eta)^2}{r_0} \right) \int_0^t e^{\beta s} ds. \]

Hence
\[ \lim_{t \to \infty} \sup E[H((N(t), C_d(t), C_r(t), P(t)), t)] \leq \frac{((Q_d + Q_p) r_0 + (r + \eta)^2}{r_0 \beta}, \]
as required.

\section*{Appendix F Proof of Theorem 6.5}

\textit{Proof.} Let \( x = (N, C_d, C_r, P)^T \) and \( C^2 \)- function \( X : \text{Int}(\mathbb{R}^4_+) \to \text{Int}(\mathbb{R}_+) \). Then
\[ X(x) = \left( N - N^* - N^* \log \frac{N}{N^*} \right) + M_1 \left( C_d - C_d^* \right)^2 + \frac{M_2}{2} \left( C_r - C_r^* \right)^2 + M_3 (P - P^*)^2, \]
where \( M_1, M_2, M_3 > 0 \) and will be chosen appropriately. Using Itô's formula, we obtain
\[ dX(x) = LX dt + \left( 1 - \frac{N^*}{N} \right) \sigma_1 N dB_1(t) + M_1 (C_d - C_d^*) \sigma_2 C_d dB_2(t) \]
\[ + M_2 (C_r - C_r^*) \sigma_3 C_d dB_3(t) + M_3 (P - P^*) \sigma_4 P dB_4(t). \]

On using model system (1) and doing simple calculations, we have
\[ LX(x) = -r_0 (N - N^*)^2 - M_1 \left( C_d - C_d^* \right)^2 - M_2 \delta_1 (C_r - C_r^*)^2 \]
\[ - M_3 (\mu + k_1 C) (C_d - C_d^*)^2 - k_2 (M_1 C_d^* + M_3 P^*) (P - P^*) (C_d - C_d^*) \]
\[ - \frac{M_1 Q_d \delta_1 m}{(m + N^*) (m + N)} (C_d - C_d^*) (N - N^*) - (\theta - M_3 \eta) (P - P^*) (N - N^*) \]
\[ + M_2 \left( \lambda + \pi_1 k_1 P \right) (C_r - C_r^*) (C_d - C_d^*) + \frac{1}{2} \sigma_1^2 N^* + \frac{M_1}{2} \sigma_2^2 C_d^2 \]
\[ + (M_2 \pi_1 C_d^* - M_3 \mu P^*) (P - P^*) (C_r - C_r^*) + \frac{M_2}{2} \sigma_3^2 C_d^2 + \frac{M_3}{2} \sigma_4^2 P^2. \]

Here, we can choose \( M_3 = \frac{\theta}{\eta}, M_2 = \frac{\theta \mu - P^*}{\pi_1 k_1 C_d^*} \) and using them in the above expression, we get
\[ LX(x) \leq -r_0 (N - N^*)^2 - M_1 \alpha (C_d - C_d^*)^2 - \frac{\theta \mu - P^*}{\pi_1 k_1 \eta C_d} (C_r - C_r^*)^2 \]
\[ - \frac{\theta \mu \eta}{\eta} (P - P^*)^2 + \frac{\theta \mu \eta}{\pi_1 k_1 \eta C_d^*} (\lambda + \pi_1 k_1 P) (C_r - C_r^*) (C_d - C_d^*) \]
\[ - \eta_1 \left( M_1 C_d^* + \frac{\theta P^*}{\eta} \right) (P - P^*) (C_d - C_d^*) - \frac{M_1 Q_d \delta_1 m}{(m + N^*) (m + N)} (C_d - C_d^*) (N - N^*) \]
\[ + \frac{1}{2} \sigma_1^2 N^* + \frac{M_1}{2} \sigma_2^2 C_d^2 + \frac{\theta \mu \eta}{2 \pi_1 k_1 \eta C_d} \sigma_3^2 C_d^2 + \frac{\theta}{2 \eta} \sigma_4^2 P^2. \]
Putting these values into the inequality (37), we have

\[
LX(x) \leq - \left( r_0 - \frac{M_1 Q_d^2 \delta^2}{(m + N^*) \alpha} \right) (N - N^*)^2 - \left( \frac{M_1 (\alpha - 4 \sigma^2)}{4} - \frac{\theta k_1^2 P^*}{2 \eta \mu_0} \right) (C_d - C_d^*)^2 \\
- \frac{\theta \mu_r P^*}{\pi_1 k_1 \eta C_d^*} \left( \delta_r - \frac{\theta \mu_r P^* (\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 k_1 \eta C_d^* \alpha M_1} \right) (C_r - C_r^*)^2 \\
- \left( \frac{\theta \mu_0}{2 \eta} - \frac{M_1 k_1^2 C_d^2}{\alpha} \right) (P - P^*)^2 + \frac{1}{2} \sigma_1^2 N^* + \frac{M_1}{2} \sigma_2^2 C_d^2 \\
+ \frac{\theta \mu_r P^*}{2 \pi_1 k_1 \eta C_d^*} \sigma_3^2 C_r^* + \frac{\theta \sigma_3^2 P^*}{2 \eta \gamma^2}.
\]  

(37)

Set,

\[
C_d^2 = \left( C_d - C_d^* + C_d^* \right)^2 \leq 2((C_d - C_d^*)^2 + (C_d^*)^2), \\
C_r^2 = \left( C_r - C_r^* + C_r^* \right)^2 \leq 2((C_r - C_r^*)^2 + (C_r^*)^2), \\
P^2 = (P - P^* + P^*)^2 \leq 2((P - P^*)^2 + (P^*)^2).
\]

Putting these values into the inequality (37), we have

\[
LX(x) \leq - \left( r_0 - \frac{M_1 Q_d^2 \delta^2}{(m + N^*) \alpha} \right) (N - N^*)^2 - \left( \frac{M_1 (\alpha - 4 \sigma^2)}{4} - \frac{\theta k_1^2 P^*}{2 \eta \mu_0} \right) (C_d - C_d^*)^2 \\
- \frac{\theta \mu_r P^*}{\pi_1 k_1 \eta C_d^*} \left( \delta_r - \frac{\theta \mu_r P^* (\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 k_1 \eta C_d^* \alpha M_1} \right) (C_r - C_r^*)^2 \\
- \left( \frac{\theta \mu_0}{2 \eta} - \frac{M_1 k_1^2 C_d^2}{\alpha} \right) (P - P^*)^2 + \frac{1}{2} \sigma_1^2 N^* + \frac{M_1}{2} \sigma_2^2 C_d^2 \\
+ \frac{\theta \mu_r P^*}{\pi_1 k_1 \eta C_d^*} \sigma_3^2 C_r^* + \frac{\theta \sigma_3^2 P^*}{2 \eta \gamma^2}.
\]  

(38)

We can write the inequality (38) as

\[
LX(x) \leq -K_1 (N - N^*)^2 - K_2 (C_d - C_d^*)^2 - K_3 (C_r - C_r^*)^2 - K_4 (P - P^*)^2 + K_\sigma.
\]

where,

\[
K_1 = \left( r_0 - \frac{M_1 Q_d^2 \delta^2}{(m + N^*) \alpha} \right), \\
K_2 = \left( \frac{M_1 (\alpha - 4 \sigma^2)}{4} - \frac{\theta k_1^2 P^*}{2 \eta \mu_0} \right), \\
K_3 = \frac{\theta \mu_r P^*}{\pi_1 k_1 \eta C_d^*} \left( \delta_r - \frac{\theta \mu_r P^* (\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 k_1 \eta C_d^* \alpha M_1} \right), \\
K_4 = \left( \frac{\theta \mu_0}{2 \eta} - \frac{M_1 k_1^2 C_d^2}{\alpha} \right) \\
K_\sigma = \frac{1}{2} \sigma_1^2 N^* + M_1 \sigma_2^2 C_d^2 + \frac{\theta \mu_r P^*}{\pi_1 k_1 \eta C_d^*} \sigma_3^2 C_r^* + \frac{\theta \sigma_3^2 P^*}{2 \eta \gamma^2}.
\]
Here, it is noticeable that $K_1, K_2, K_3$ and $K_4$ are positive provided following inequalities are satisfied:

\[
\begin{align*}
\tau_0 &> \frac{M_1Q_2^2\sigma^2}{(m+N^*)\alpha}, \\
\frac{M_1(\alpha - 4\sigma^2)}{4} &> \frac{\theta k_1^2 P^2}{2\eta \mu_0}, \\
M_1(\delta_r - \sigma^2) &> \frac{\theta \mu_r P^*(\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 k_1 \eta C_d \alpha}, \\
\frac{\theta}{2\eta} (\mu_0 - 2\sigma^2) &> \frac{M_1 k_2^2 C_d^2 \sigma^2}{\alpha}.
\end{align*}
\]

Thereafter, from inequalities (39)-(42), we can choose the positive value of $M_1$ as

\[
\max \{W_1, W_2\} < M_1 < \min \{W_3, W_4\},
\]

where, $W_1 = \frac{2\theta (\mu_0 - 2\sigma^2)}{k_1^2 C_d}, \quad W_2 = \frac{\theta \mu_r P^*(\lambda \alpha + \pi_1 k_1 P_m)^2}{\pi_1 k_1 \eta C_d \alpha}, \quad W_3 = \frac{\tau_0 (m+N^*)}{Q_2^2 \sigma^2}$ and $W_4 = \frac{\theta (\mu_0 - 2\sigma^2)}{k_1^2 C_d}$. Therefore, $K_1, K_2, K_3$ and $K_4$ are positive under condition (19).

Further, on integrating (44) between 0 to $t$ and then taking expectations, we have

\[
0 \leq E[X(x(t))] \leq X(x_0) - E \left[ \int_0^t \left\{ K_1 (N(s) - N^*)^2 + K_2 (C_d(s) - C_d^*)^2 + K_3 (C_r(s) - C_r^*)^2 + K_4 (P(s) - P^*)^2 \right\} ds + K_\sigma t \right].
\]

\[
E \left[ \int_0^t \left\{ K_1 (N(s) - N^*)^2 + K_2 (C_d(s) - C_d^*)^2 + K_3 (C_r(s) - C_r^*)^2 + K_4 (P(s) - P^*)^2 \right\} ds \right] \leq X(x_0) + K_\sigma t.
\]

Now, both sides of (45) are multiplying by $\frac{1}{t}$ and taking limit $t \to \infty$, we have

\[
\lim_{t \to \infty} \sup \frac{1}{t} E \left[ \int_0^t \left\{ (N(s) - N^*)^2 + (C_d(s) - C_d^*)^2 + (C_r(s) - C_r^*)^2 + (P(s) - P^*)^2 \right\} ds \right] \leq \frac{K_\sigma}{K},
\]

where, $K = \min\{K_1, K_2, K_3, K_4\}$.

\[
\square
\]

**Appendix G Proof of Theorem 6.6**

**Proof.** The diffusion coefficient $g(x)$ of the model system (17) is

\[
g(x) = \begin{pmatrix}
\sigma_1 N & 0 & 0 & 0 \\
0 & \sigma_2 C_d & 0 & 0 \\
0 & 0 & \sigma_3 C_r & 0 \\
0 & 0 & 0 & \sigma_4 P
\end{pmatrix}, \quad x = (N, C_d, C_r, P)^T \in \text{Int}(\mathbb{R}_+^4).
Here, it may be easily noted that the rank of $g(x)$ is 4 and positive definite diffusion matrix in $\text{Int}(\mathbb{R}^4_+)$ is

$$A(x) = g(x)g(x)^T = \begin{pmatrix} \sigma_1^2 N^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 C_d^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 C_r^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 P^2 \end{pmatrix}.$$ 

Hence $g(x)$ is continuous in $x$. Therefore in any compact set $M \subset \text{Int}(\mathbb{R}^4_+)$, $A(x)$ is uniformly elliptical.

In order to complete the proof, it is sufficient to find a Lyapunov function $X(t)$ and a compact set $M \subset \text{Int}(\mathbb{R}^4_+)$ such that $LX(x) \leq -d$, $d > 0$ and $x \in \text{Int}(\mathbb{R}^4_+)$ \ $M$.

Furthermore, the Lyapunov function $X : \text{Int}(\mathbb{R}^4_+) \rightarrow \text{Int}(\mathbb{R}_+)$ is given by equation (35) and the expression for $LX(x)$ from the proof of Theorem (6.5) is extracted as

$$LX(x) \leq -K_1(N - N^*)^2 - K_2(C_d - C_d^*)^2 - K_3(C_r - C_r^*)^2 - K_4(P - P^*)^2 + K_\sigma.$$ 

Thus

$$K_1(N - N^*)^2 + K_2(C_d - C_d^*)^2 + K_3(C_r - C_r^*)^2 + K_4(P - P^*)^2 = K_\sigma,$$

lies entirely in $\text{Int}(\mathbb{R}^4_+)$. Hence, there a positive constant $d$ and a compact set $M \subset \text{Int}(\mathbb{R}^4_+)$ exist, such that for any $x \in \text{Int}(\mathbb{R}^4_+) \setminus M$,

$$K_1(N - N^*)^2 + K_2(C_d - C_d^*)^2 + K_3(C_r - C_r^*)^2 + K_4(P - P^*)^2 \geq K_\sigma + d.$$ 

Thus,

$$LX(x) \leq -d < 0, \text{ for any } x \in \text{Int}(\mathbb{R}^4_+) \setminus M.$$ 

Hence the proof.

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Figure 2. Global stability of \((N^*, C^*_r, P^*)\) and \((N^*, C^*_d, C^*_r)\) in \(N - C_r - P\) and \(N - C_d - C_r\) spaces, respectively.

Figure 3. Variation of \(C_r(t)\) with time for different values of \(\eta\) (a) for \(\pi_1 = 0.09\) and (b) for \(\pi_1 = 0.9\).
Figure 4. Path of $N(t), C_d(t), C_r(t)$ and $P(t)$ for stochastic model (17) with $\sigma_1 = 0.0001, \sigma_2 = 0.006, \sigma_3 = 0.004, \sigma_4 = 0.008$ as well as deterministic model (1).

Figure 5. Path of $N(t), C_d(t), C_r(t)$ and $P(t)$ for stochastic model (17) with $\sigma_1 = 0.08, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.1$ as well as deterministic model (1).
Figure 6. The stationary distribution of $N(t), C_d(t), C_r(t)$ and $P(t)$ obtained at $t = 100$ from 10,000 simulation run for the stochastic model (17), for $\sigma_1 = 0.0001, \sigma_2 = 0.006, \sigma_3 = 0.004, \sigma_4 = 0.008$.

Figure 7. The stationary distribution of $N(t), C_d(t), C_r(t)$ and $P(t)$ obtained at $t = 100$ from 10,000 simulation run for the stochastic model (17), for $\sigma_1 = 0.08, \sigma_2 = 0.2, \sigma_3 = 0.09, \sigma_4 = 0.1$. 
Figure 8. The stationary distribution of $N(t), C_d(t), C_r(t)$ and $P(t)$ obtained at $t = 100$ from 10,000 simulation run for the stochastic model (17), for $\sigma_1 = 0.09, \sigma_2 = 0.3, \sigma_3 = 0.1, \sigma_4 = 0.2$. 