A formal power series over a noncommutative Hecke ring and the rationality of the Hecke series for $GSp_4$

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Abstract

The present paper studies Hecke rings derived by the automorphism groups of certain algebras $L_p$ over the ring of $p$-adic integers. Our previous work considered the case where $L_p$ is the Heisenberg Lie algebra (of dimension 3) over the ring of $p$-adic integers. Although this Hecke ring is noncommutative, we showed that a formal power series with coefficients in this Hecke ring satisfies an identity similar to the rationality of the Hecke series for $GL_2$ due to E. Hecke. In the present paper, we establish an analogous result in the case of the Heisenberg Lie algebra of dimension 5 over the ring of $p$-adic integers. In this case, our identity is similar to the rationality of the Hecke series for $GSp_4$, due to G. Shimura.

1 Introduction

The present paper is a continuation of our previous works [5], [6] and [7]. We discuss identities for formal power series with coefficients in certain noncommutative Hecke rings. Our study is related to the work due to [4, Hecke], [8, Shimura] and [1, Andrianov]. Let $p$ be a fixed prime number. They showed that, for each positive integer $n$, the Hecke series $P_n(X)$ for the general symplectic group $GSp_{2n}(\mathbb{Q}_p)$ of genus $n$ over the $p$-adic field $\mathbb{Q}_p$ has rationality as follows:

**Theorem 1.1 (Rationality Theorem, [1, 4, 8]).** Denote by $R_n$ the Hecke ring associated with $GSp_{2n}(\mathbb{Q}_p)$. Then there exist elements $q^{(n)}_1, ..., q^{(n)}_{2^n}$ of $R_n$, and a polynomial $g^{(n)}(X) \in R_n[X]$ such that

$$
\sum_{i=0}^{2^n} q^{(n)}_i X^i P_n(X) = g^{(n)}(X),
$$

where $q^{(n)}_0 = 1$.

The above theorem is established by [4] for the case $n = 1$, by [8] for the case $n = 2$, and by [1] for any $n \geq 3$. Furthermore, it is known that $R_n$ is isomorphic
to the (commutative) polynomial ring of \((n + 1)\) variables. Thus \(\{q^n_i\}_i\) and \(g^{(n)}(X)\) are unique.

Throughout the present paper, algebra implies an abelian group with a biadditive product (e.g., an associative algebra, a Lie algebra). Let \(L\) be an algebra that is free of finite rank as an abelian group, and let \(L_p = L \otimes \mathbb{Z}_p\). In [6], the Hecke rings \(R_{L_p}\) derived by the automorphism groups of \(L_p\) were introduced, and in [7], the formal power series \(P_{L_p}(X)\) with coefficients in the Hecke rings were defined. We call \(P_{L_p}(X)\) and \(R_{L_p}\) the local Hecke series and the local Hecke ring associated with \(L_p\), respectively.

In the present paper, we focus on the case where \(L_p\) is the Heisenberg Lie algebra \(H^{(n)}\) over \(\mathbb{Z}\) of dimension \(2n + 1\). Here, the ring homomorphisms \(s_n : R_n \rightarrow R_{H^{(n)}}, \phi_n : R_{H^{(n)}} \rightarrow R_n\) and \(\theta_n : R_{H^{(n)}} \rightarrow R_{H^{(n)}}\) satisfying the following property are constructed. They are presented in Sections 2 and 4.

**Proposition 1.2** (Proposition 4.2, Corollary 4.4). The ring homomorphisms \(\phi_n, s_n\) and \(\theta_n\) satisfy the following properties:

1. \(\phi_n \circ s_n = id_{R_n}\).
2. \(\phi_n \circ \theta_n = \phi_n\).
3. \(P_{\phi_n}^{H^{(n)}}(X) = P_n(p^n X^{n+1})\).

We shall drop the subscript \(n\) from \(\phi_n, s_n\) and \(\theta_n\) when no confusion can occur.

Our previous work [5] shows the following identity as well as the noncommutativity of the coefficients of the local Hecke series associated with the Heisenberg Lie algebra \(H^{(1)}\) of dimension 3.

**Theorem 1.3** ([5, Theorem 7.8]). We set \(P(X) = P_{H^{(1)}}^{(1)}(X), g(X) = g^{(1)}(X)\) and \(q_i = q_i^{(1)}\) for \(i = 0, 1, 2\). Then we have the following identity.

\[
P^0(X) + q_1 Y P^0(X) + q_2 Y^2 P(X) = g^*(Y),
\]

where \(Y = pX^2\).

Our main purpose of the present paper is to extend this result. Precisely, we establish the noncommutativity of the coefficients of \(P_{H^{(2)}}^{(2)}(X)\), moreover, the following identity.

**Theorem 1.4** (Theorem 5.14). We set \(P(X) = P_{H^{(2)}}^{(2)}(X), g(X) = g^{(2)}(X),\) and \(q_i = q_i^{(2)}\) for \(0 \leq i \leq 4\). Then we have the following identity.

\[
\sum_{i=0}^{4} q_i^* Y^i P^{0^{4-i}}(X) = g^*(Y),
\]

where \(Y = p^2 X^3\).
It should be noted that Equalities (2) and (3) recover the rationality theorems of Hecke and Shimura via the morphisms $\phi_1$ and $\phi_2$, respectively. This is an immediate consequence of Proposition 1.2.

For the case $n \geq 3$, we show the noncommutativity of the coefficients of $P_{H_p}^n(X)$ (cf. Theorem 4.6), and raise the following unsolved problem.

**Problem 1.5** (Problem 4.9). We set $P(X) = P_{H_p}^n(X)$, $g(X) = g^n(X)$, and $q_i = q_i^n$ for $0 \leq i \leq 2^n$. Does $P(X)$ satisfy the following identity?

$$\sum_{i=0}^{2^n} q_i^n Y^i P^{\theta N-i}(X) = g^i(Y). \quad (4)$$

Where $Y = p^n X^{n+1}$ and $N = 2^n$.

The outline of the present paper is as follows. In Section 2, we recall the definition of Hecke rings and study the morphisms $s$, $\phi$ and $\theta$. Section 3 describes the local Hecke rings and the local Hecke series associated with algebras. In Section 4 we focus on the case of the Heisenberg Lie algebras, and then show the noncommutativity of the coefficients of $P_{H_p}^n(X)$ for all $n$. Finally, our main theorem is proved in Section 5.

## 2 Abstract Hecke rings and some morphisms

In this section, we recall the definition of Hecke rings and define the morphisms $s$, $\theta$ and $\phi$ which are used in section 4.

First, we recall the definition of Hecke rings. For more details, refer to [9, Shimura, Chapter 3]. Let $G$ be a group, $\Delta$ be a submonoid of $G$, and $\Gamma$ be a subgroup of $\Delta$. We assume that the pair $(\Gamma, \Delta)$ is a double finite pair, i.e., for all $A \in \Delta$, $\Gamma \setminus \Gamma A$ and $\Gamma A \setminus \Gamma$ are finite sets. Then, one can define the Hecke ring $R = R(\Gamma, \Delta)$ associated with the pair $(\Gamma, \Delta)$ as follows:

- The underlying abelian group is the free abelian group on the set $\Gamma \setminus \Delta / \Gamma$.
- The product of $(\Gamma A \Gamma)$ and $(\Gamma B \Gamma)$ is defined by

$$\sum_{\Gamma CT \in \Gamma \setminus \Delta / \Gamma} m_C(\Gamma CT),$$

where

$$m_C = \left| \{ \Gamma \beta \in \Gamma \setminus \Gamma B \Gamma \mid C \beta^{-1} \in \Gamma A \Gamma \} \right| = \left| \{ \alpha \Gamma \in \Gamma A \Gamma / \Gamma \mid \alpha^{-1} C \in \Gamma B \Gamma \} \right|.$$ 

Note that $m_C \neq 0$ if and only if $C \in \Gamma A \Gamma B \Gamma$.

Define the element $T_{\Gamma, \Delta}(A)$ of $R(\Gamma, \Delta)$ by $\Gamma A \Gamma$ for every $A \in \Delta$. We also define $\text{deg}_{\Gamma, \Delta}(\Gamma A \Gamma)$ by $|\Gamma \setminus \Gamma A \Gamma|$ for every $A \in \Delta$. The map $\text{deg}_{\Gamma, \Delta}$ extends
by linearity to a homomorphism from $R(\Gamma, \Delta)$ to $\mathbb{Z}$, and forms a ring homomorphism. We will write simply “$T(A)$”, “deg” and “deg $T(A)$”, for $T_{\Gamma, \Delta}(A)$, $\deg_{\Gamma, \Delta}$ and $\deg_{\Gamma, \Delta}(T_{\Gamma, \Delta}(A))$ respectively when no confusion can arise.

Fix a double finite pair $(\Gamma, \Delta)$ and a both sides $\Delta$-module $M$. Next, we construct a Hecke ring $\tilde{R}$ using $\Delta$, $\Gamma$ and $M$. We define the monoid $\tilde{\Delta} = \tilde{\Delta}(\Delta, M)$ as follows.

- The underlying set is $\Delta \times M$.
- The operation is defined by $(A, a) \cdot (B, b) = (AB, A'b + aB)$ for $A, B \in \Delta$, $a, b \in M$.

We also define $\tilde{\Gamma} = \tilde{\Gamma}(\Gamma, M)$ by the subgroup of $\tilde{\Delta}$ whose underlying set is $\Gamma \times M$. Notice that the identity element of $\tilde{\Delta}$ is $(E, 0)$, where $E$ and $0$ are the identity elements of $\Delta$ and $M$ respectively. It is easy to see $(X, x)^{-1} = (X^{-1}, -X^{-1}xX^{-1})$ for each $(X, x) \in \tilde{\Gamma}$.

From now on, we assume the following:

**Assumption 2.1.**

1. $M/(AM \cap MA)$ is a finite set for every $A \in \Delta$.
2. $M$ is a sub-$\Delta$-module of a both sides $G$-module $M'$.

Then, the monoid $\tilde{\Delta}$ is naturally a submonoid of the group $\tilde{\Delta}(G, M')$. Let us calculate the degree of each $(A, a) \in \tilde{\Delta}$ and prove the double finiteness of the pair $(\tilde{\Gamma}, \tilde{\Delta})$. We put $\Gamma_A = A^{-1} \Gamma A \cap \Gamma$ and $\Gamma^A = \Gamma \cap A \Gamma^{-1}$. Note that there are natural bijections of $\Gamma_A \setminus \Gamma$ onto $\Gamma \setminus \Gamma_A \Gamma$ and $\Gamma / \Gamma^A$ onto $\Gamma \setminus \Gamma A \Gamma$.

**Proposition 2.2.** For any two elements $(A, a)$ and $(A, b)$ of $\tilde{\Delta}$, $\tilde{\Gamma}(A, a)\tilde{\Gamma} = \tilde{\Gamma}(A, b)\tilde{\Gamma}$ if and only if there exists $X \in \Gamma^A$ such that

$$X \ast a \equiv b \mod AM + MA.$$ 

Where $X \ast a = Xa \Gamma^{-1} - A^{-1}X^{-1}A$.

**Proof.** $\tilde{\Gamma}(A, a)\tilde{\Gamma} = \tilde{\Gamma}(A, b)\tilde{\Gamma}$ if and only if there exist $(X, x), (Y, y) \in \tilde{\Gamma}$ such that $(X, x)(A, a) = (A, b)(Y, y)$, which is equivalent to

$$Y = A^{-1}X A, \quad X \ast a - b = AY^{-1} - xX^{-1}A.$$ 

This completes the proof. \qed

**Proposition 2.3.** For $(A, a) \in \tilde{\Delta}$, we have

$$\left| \tilde{\Gamma} \setminus \tilde{\Gamma}(A, a)\tilde{\Gamma} \right| = \left| \Gamma : \Gamma_A \right| [AM : AM \cap MA] \left| \Gamma^A \ast (a \mod AM + MA) \right|.$$ 

where $\Gamma^A \ast (a \mod AM)$ means the orbit of $(a \mod AM + MA)$ in $M/(AM + MA)$ under $\Gamma^A$ acting by $\ast$. Particularly, $\left| \tilde{\Gamma} \setminus \tilde{\Gamma}(A, a)\tilde{\Gamma} \right|$ is finite.
Proof. We put
\[ \tilde{\Gamma}_{(A,a)} = \left\{ (X,x) \in \tilde{\Gamma} \mid (A,a)(X,x) = (Y,y)(A,a) \text{ for some } (Y,y) \in \tilde{\Gamma} \right\}. \]

Then \[ \left| \tilde{\Gamma}(A,a) \Gamma \right| = \left| \tilde{\Gamma}(A,a) \tilde{\Gamma} \right|. \] \((X,x) \in \tilde{\Gamma}(A,a)\) if and only if \((X,x)\) has the following three conditions:
\begin{align*}
X &\in \Gamma_A, \tag{5} \\
(AXA^{-1}) * a - a &\in AM + MA, \tag{6} \\
Axx^{-1} &\in (AXA^{-1}) * a - a + MA. \tag{7}
\end{align*}

Let \(\Gamma_1\) be the image of \(\tilde{\Gamma}(A,a)\) by the canonical projection \(\tilde{\Gamma} \to \Gamma\), and put \(\mathcal{N} = \{ x \in M \mid Ax \in AM \cap MA \}\). Then we have a commutative diagram as follows.
\[
\begin{array}{c c c c c}
1 & \longrightarrow & M & \longrightarrow & \tilde{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathcal{N} & \longrightarrow & \tilde{\Gamma}(A,a) & \longrightarrow & \Gamma_1 & \longrightarrow & 1 \\
\end{array}
\]

where \(M \to \tilde{\Gamma}\) is the canonical embedding, and each of the three vertical morphisms is the inclusion. Since the two horizontal sequences are exact, we have
\[ [\tilde{\Gamma} : \tilde{\Gamma}(A,a)] = [\Gamma : \Gamma_A][\Gamma_A : \Gamma_1][\mathcal{M} : \mathcal{N}]. \]

Clearly, \(\mathcal{M}/\mathcal{N}\) is naturally isomorphic to \(AM/AM \cap MA\). Since \(\Gamma_1\) coincides with the subgroup of \(\Gamma\) consisting of elements satisfying (5) and (6), \(A\Gamma_1 A^{-1}\) is the stabilizer subgroup of \((a \mod AM + MA)\) in \(\Gamma A\). Thus we have the desired identity. \(\square\)

A similar argument shows that \(\left| \tilde{\Gamma}(A,a) \tilde{\Gamma}/\tilde{\Gamma} \right|\) is also finite. Thus we have:

Corollary 2.4. The pair \((\tilde{\Gamma}, \tilde{\Delta})\) is double finite.

We write \(\tilde{R} = R(\tilde{\Gamma}, \tilde{\Delta})\) and \(\tilde{T} = T_{\tilde{\Gamma}, \tilde{\Delta}}\). The above proposition is equivalent to the following statement:

Corollary 2.5. For \((A,a) \in \tilde{\Delta}\), we have
\[ \deg \tilde{T}(A,a) = [\Gamma : \Gamma_A][AM : AM \cap MA][\Gamma A * (a \mod AM + MA)]. \]

The following three subsections define the morphisms \(s\), \(\theta\) and \(\phi\), respectively. From now on, we assume the following:

Assumption 2.6. \(AM \supset MA\) for every \(A \in \Delta\).
2.1 The injection $s$

We define $\psi : \tilde{\Delta} \to \Delta$ and $s_0 : \Delta \to \tilde{\Delta}$ by $((C, e) \mapsto C)$ and $(C \mapsto (C, 0))$, respectively.

**Lemma 2.7.** For $A, B \in \Delta$, let

$$\tilde{\mathcal{X}} = \tilde{\Gamma}(A, 0)\tilde{\Gamma}(B, 0)\tilde{\Gamma}/\tilde{\Gamma}, \text{ and } \mathcal{X} = \Gamma(\Gamma A\Gamma B\Gamma)/\Gamma.$$ 

Let $\tilde{\psi} : \tilde{\mathcal{X}} \to \mathcal{X}$ and $s_0 : \mathcal{X} \to \tilde{\mathcal{X}}$ be the maps defined by $\psi$ and $s_0$, respectively. Then $\tilde{\psi}$ and $s_0$ are the inverse to each other.

**Proof.** It is clear that $\tilde{\psi} \circ s_0$ is the identity. We shall show $s_0 \circ \tilde{\psi}$ is also the identity. For $(X, x) \in \tilde{\mathcal{X}}$, we put $\delta = \beta(X, x) = (Bx, 0)$. Since $\Gamma(A) = (\Gamma A)\Gamma B\Gamma$, by Proposition 2.2 we have

$$\tilde{\Gamma}(A, 0)(X, x)(B, 0)\tilde{\Gamma} = \tilde{\Gamma}(AXB, 0)\tilde{\Gamma}.$$ 

This completes the proof.

**Lemma 2.8.** For $A, B, C \in \Delta$, let $\alpha = (A, 0), \beta = (B, 0), \gamma = (C, 0)$. We put

$$\tilde{\mathcal{X}} = \{\tilde{\Gamma}\delta \in \tilde{\Gamma}\beta \Gamma; \gamma \delta^{-1} \in \tilde{\Gamma} \alpha \Gamma\}$$ 

and

$$\mathcal{X} = \{\Gamma D \in \Gamma \beta \Gamma; CD^{-1} \in \Gamma \alpha \Gamma\}.$$ 

Let $\tilde{\psi} : \tilde{\mathcal{X}} \to \mathcal{X}$ and $s_0 : \mathcal{X} \to \tilde{\mathcal{X}}$ be the maps defined by $\psi$ and $s_0$, respectively. Then $\tilde{\psi}$ and $s_0$ are the inverse to each other, especially, $|\tilde{\mathcal{X}}| = |\mathcal{X}|$.

**Proof.** It is essential to prove $s_0 \circ \tilde{\psi}$ is the identity. For $(X, x) \in \tilde{\mathcal{X}}$, we put $\delta = (X, x) = (Bx, 0)$, and assume $\gamma \delta^{-1} \in \tilde{\Gamma} \alpha \Gamma$. It suffices to prove

$$\tilde{\Gamma}(X, x)(B, 0)\tilde{\Gamma} = \tilde{\Gamma}(Bx, 0)\tilde{\Gamma}.$$ 

Since $\gamma \delta^{-1} = (CX^{-1}B^{-1}, -CX^{-1}x(Bx)^{-1})$, we have $CX^{-1}B^{-1} \in \Gamma A\Gamma$. Hence $(CX^{-1}B^{-1}, 0) \in \Gamma \alpha \Gamma$. Thus

$$(CX^{-1}B^{-1}, -CX^{-1}x(Bx)^{-1}) \in \tilde{\Gamma}(CX^{-1}B^{-1}, 0)\tilde{\Gamma}.$$ 

By Proposition 2.2, $-CX^{-1}x(Bx)^{-1} \in CX^{-1}B^{-1}\mathcal{M}$, namely, $x \in B^{-1}\mathcal{M}$. Put $x = B^{-1}yBx$, then

$$\delta = (Bx, yBx) = (E, y)(Bx, 0).$$ 

This completes the proof.

By the above two lemmas, we have:

**Proposition 2.9.** The natural injection $s : R \to \tilde{R}$ defined by

$$T(A) \mapsto \tilde{T}(A, 0), \text{ for } A \in \Delta,$$

is a ring homomorphism.
2.2 The endomorphism $\theta$

Let $C_0$ be an element of the center of $\Delta$. First, we calculate the products $\tilde{T}(C_0, 0)\tilde{T}(A, a)$ and $\tilde{T}(A, a)\tilde{T}(C_0, 0)$ for each $(A, a) \in \Delta$.

**Lemma 2.10.** For all $(A, a) \in \Delta$,

$$\tilde{T}(C_0, 0)\tilde{T}(A, a) = \tilde{T}(C_0A, C_0a).$$

**Proof.** For $(X, x) \in \tilde{\Gamma}$,

$$(C_0, 0)(X, x)(A, a) = (X, 0)(C_0A, C_0a)(E, A^{-1}X^{-1}xA).$$

Hence $\tilde{T}(C_0, 0)\tilde{T}(A, a)\tilde{T} = \tilde{T}(C_0A, C_0a)\tilde{T}$. Thus there exists a positive integer $c$ such that

$$\tilde{T}(C_0, 0)\tilde{T}(A, a) = c\tilde{T}(C_0A, C_0a).$$

Hence we have

$$c = \frac{\deg \tilde{T}(C_0, 0)\deg \tilde{T}(A, a)}{\deg \tilde{T}(C_0A, C_0a)}.$$

By Corollary 2.5, we have $c = 1$, which completes the proof. \hfill \Box

**Lemma 2.11.** For all $(A, a) \in \Delta$,

$$\tilde{T}(A, a)\tilde{T}(C_0, 0) = \frac{\Gamma^A*(a \mod AM)}{\Gamma^A*(C_0^{-1}aC_0 \mod AM)}\tilde{T}(C_0A, aC_0).$$

**Proof.** For $(X, x) \in \tilde{\Gamma}$,

$$(X, x)(C_0, 0) = (X, 0)(C_0, 0)(X, C_0^{-1}xC_0).$$

Hence $\tilde{T}(A, a)\tilde{T}(C_0, 0)\tilde{T} = \tilde{T}(C_0A, aC_0)\tilde{T}$. Thus there exists a positive integer $c$ such that

$$\tilde{T}(A, a)\tilde{T}(C_0, 0) = c\tilde{T}(C_0A, aC_0).$$

By Corollary 2.5 we have

$$c = \frac{\Gamma^A*(a \mod AM)}{\Gamma^A*(C_0^{-1}aC_0 \mod AM)},$$

which completes the proof. \hfill \Box

By the lemma 2.10, the left multiplication by $\tilde{T}(C_0, 0)$ gives an injective map from the basis $\{T(\xi)\}_{\xi \in \Gamma \setminus \Delta}^r$ of $\tilde{R}$ to the same set. This implies that $\tilde{T}(C_0, 0)$ is a left nonzero divisor. Thus we can define the following element.

**Definition 2.12.** The linear endomorphism $\theta = \theta_{C_0}$ of $\tilde{R}$ is defined by the requirement that for each $(A, a) \in \Delta$

$$\tilde{T}(A, a)\tilde{T}(C_0, 0) = \tilde{T}(C_0, 0)\tilde{T}(A, a)\theta.$$
Proposition 2.14. Let $\phi$ be the canonical map $M \to M$. Put $N$ be a finite subset of $\Delta$. Hence, we can define a map $\theta : \Gamma \times C_0^{-1}aC_0 \to \tilde{T}(A, C_0^{-1}aC_0)$.

Especially, $\xi \phi = \xi$ for each $\xi \in \text{Im}(s)$.

Let $X$ be a subset of $\tilde{\Delta}$ such that $\tilde{T}(X) / \tilde{F}$ is a finite set. Then we define

$$\tilde{T}(X) = \sum_{\tilde{T}(A, a) / \tilde{F}} \tilde{T}(A, a) \cdot$$

And put

$$\tilde{T}(A, N) = \tilde{T}(A \times N),$$
$$\tilde{T}(A, N) = \tilde{T}(\{A\} \times N),$$
$$\tilde{T}(A) = \tilde{T}(A, M),$$

for a finite subset $A$ of $\Delta$, $A \in \Delta$, and a subset $N$ of $M$. By Proposition 2.12, we can define $\tilde{T}(A, a)$ for $a \in \Gamma^{A} \setminus (\Gamma^{A}N + AM) / AM$ in a natural way. Then, we easily see that $\tilde{T}(A, N) = \sum_{A} \tilde{T}(A, a)$, where $a$ runs through the set

$$\Gamma^{A} \setminus (\Gamma^{A}N + AM) / AM.$$

Proposition 2.13. For all $(A, a) \in \tilde{\Delta},$

$$\tilde{T}(A, a)^{\theta} = \frac{|\Gamma^{A} \times (a \mod AM)|}{|\Gamma^{A} \times (C_0^{-1}aC_0 \mod AM)|} \tilde{T}(A, C_0^{-1}aC_0).$$

Especially, $\xi^{\theta} = \xi$ for each $\xi \in \text{Im}(s)$.

Let $X$ be a subset of $\tilde{\Delta}$ such that $\tilde{T}(X) / \tilde{F}$ is a finite set. Then we define

$$\tilde{T}(X) = \sum_{\tilde{T}(A, a) / \tilde{F}} \tilde{T}(A, a).$$

And put

$$\tilde{T}(A, N) = \tilde{T}(A \times N),$$
$$\tilde{T}(A, N) = \tilde{T}(\{A\} \times N),$$
$$\tilde{T}(A) = \tilde{T}(A, M),$$

for a finite subset $A$ of $\Delta$, $A \in \Delta$, and a subset $N$ of $M$. By Proposition 2.12, we can define $\tilde{T}(A, a)$ for $a \in \Gamma^{A} \setminus (\Gamma^{A}N + AM) / AM$ in a natural way. Then, we easily see that $\tilde{T}(A, N) = \sum_{A} \tilde{T}(A, a)$, where $a$ runs through the set

$$\Gamma^{A} \setminus (\Gamma^{A}N + AM) / AM.$$

Proposition 2.14. Let $N$ be a subgroup of $M$ such that $\Gamma^{A}N = N$, $N + AM = N$. Put $N' = C_0^{-1}NC_0$. Let $N_A$ and $N'_A$ be the images of $N$ and $N'$ under the canonical map $M \to M / AM$, respectively. Then we have

$$\tilde{T}(A, N)^{\theta} = \frac{|N_A|}{|N'_A|} \tilde{T}(A, N').$$

Proof. Let $f : N \to N'$ be the morphism being $a \mapsto C_0^{-1}aC_0$. Let $f_A$ and $f_A$ be its derived maps $N_A \to N'_A$ and $\Gamma^{A}N_A \to \Gamma^{A}N'_A$, respectively. Let $\varphi$ be the canonical map $M / AM \to \Gamma^{A}N_A / AM$. Put $\varphi_{N} = \varphi_{N_A}, \varphi_{N'} = \varphi_{N'_A}$. Since $N_A$ and $N'_A$ are $\Gamma^{A}$-invariant sets, we have $\varphi^{-1}(a) = \varphi^{-1}_{N}(a)$ and $\varphi^{-1}(b) = \varphi^{-1}_{N'}(b)$ for each $a \in \Gamma^{A}N_A$, $b \in \Gamma^{A}N'_A$. Hence

$$\tilde{T}(A, N)^{\theta} = \sum_{a \in \Gamma^{A}N_A} \frac{|\varphi^{-1}_{N}(a)|}{|\varphi^{-1}_{N}(f_A(a))|} \tilde{T}(A, f_A(a)).$$

Hence

$$\tilde{T}(A, N)^{\theta} = \sum_{b \in \Gamma^{A}N'_A} \sum_{a \in f_A^{-1}(b)} \frac{|\varphi^{-1}_{N}(a)|}{|\varphi^{-1}_{N'}(b)|} \tilde{T}(A, b).$$
Since $\bar{f}_A \circ \varphi_N = \varphi_{N'} \circ f_A$,

$$\bar{T}(A,N)\theta = \sum_{b \in \Gamma \setminus N'_{A'}} |\varphi_{N'_{A'}}^{-1}(b)| \bar{T}(A,b).$$

Since $|\varphi_{N'_{A'}}^{-1}(b)| = |\varphi_{N_{A'}}^{-1}(b)| |\ker f_A| = |\varphi_{N_{A'}}^{-1}(b)| |N_A| / |N'_{A'}|,

$$\bar{T}(A,N)\theta = \frac{|N_A|}{|N'_{A'}|} \sum_{b \in \Gamma \setminus N'_{A'}} \bar{T}(A,b).$$

This completes the proof. \(\square\)

### 2.3 The projection $\phi$

Let $C_0$ be an element of the center of $\Delta$ satisfying the following assumption:

**Assumption 2.15.** $C_0^{-1}MC_0 \neq M$.

Since the endomorphism of $\mathcal{M}$ being $(a \mapsto C_0^{-1}aC_0)$ is injective, the sequence $\{C_0^{-n}MC_0^n\}$ is a strictly decreasing sequence of sets. Since $\mathcal{M}/AM$ is a finite set, there exists a positive integer $m$ such that $C_0^{-m}MC_0^m \subseteq AM$. Hence the sequence $\{C_0^{-n}MC_0^n \mod AM\}_n$ is stable and its limit is the identity element for each $a \in \mathcal{M}$. Thus, Proposition 2.2 and Corollary 2.13 imply the stability of the sequence $\{\bar{T}(A,a)^{\theta^n}\}$ for each $(A,a) \in \bar{\Delta}$ and that its limit is an element of the image of $s$.

**Definition 2.16.** For each $(A,a) \in \bar{\Delta}$, the limit of the sequence $\{\bar{T}(A,a)^{\theta^n}\}$ is denoted by $\bar{T}(A,a)^{\theta^\infty}$, and $\phi : \bar{R} \to R$ is defined by $s^{-1} \circ \theta^\infty$.

Note that both $\theta^\infty$ and $\phi$ are naturally ring homomorphisms.

**Proposition 2.17.** The morphism $\phi$ satisfies the following properties:

1. $\phi \circ s = \text{id}_R$.
2. $\phi \circ \theta = \phi$.
3. $\bar{T}(A)^\phi = |\mathcal{M}/AM| \bar{T}(A)$.

**Proof.** The assertions 1-2 are trivial. We shall prove the last assertion. We see that there exists a positive integer $m$ such that $C_0^{-m}MC_0^m \subseteq AM$. Put $\eta = \theta_{C_0^m}$, then $\bar{T}(A)^{\theta^\infty} = \bar{T}(A)^{\theta^m} = \bar{T}(A)^\eta$. By Proposition 2.14 we have $\bar{T}(A)^\eta = |\mathcal{M}/AM| \bar{T}(A,0)$, which completes the proof. \(\square\)
2.4 Computation of the product $\tilde{T}(A,0)\tilde{T}(B)$

In this subsection, we compute the product $\tilde{T}(A,0)\tilde{T}(B)$ for each $A, B \in \Delta$, in the case where $G$ has an anti-automorphism $(\alpha \mapsto \hat{\alpha})$ preserving $\Gamma\Lambda\Gamma$ for all $A \in \Delta$. Then it is well known that $R$ is commutative. For each $A, B, C \in \Delta$, we put

$$\mathcal{X}(A, B, C) = \{\alpha \in \Gamma \mid \hat{\alpha}^{-1}C \in \Gamma \Lambda C\}.$$  

Then, for each system of representatives $\{\alpha_i\}_i$ of $\Gamma \setminus \Gamma\Lambda\Gamma$ in $\Gamma\Lambda\Gamma$, $\{\hat{\alpha}_i\}_i$ is a system of representatives of $\Gamma\Lambda\Gamma/\Gamma$ in $\Gamma\Lambda\Gamma$. Hence we see that, for each $A, B \in \Delta$,

$$T(A)T(B) = \sum_{GCT \in \Gamma\setminus\Gamma\Lambda/G} |\Gamma\setminus\mathcal{X}(A, B, C)|T(C).$$

Similarly, we have the following formula:

**Lemma 2.18.** For $A, B, C \in \Delta$, and $c \in \mathcal{M}$, we put

$$\mathcal{Y}(A, B, C, c) = \{\alpha \in \mathcal{X}(A, B, C) \mid \hat{\alpha}^{-1}c \in \mathcal{M}\}.$$  

Then, for $A, B \in \Delta$, we have

$$\tilde{T}(A,0)\tilde{T}(B) = \sum_{(C,c)} |\Gamma\setminus\mathcal{Y}(A, B, C, c)||\tilde{T}(C,c),$$

where $(C, c)$ runs through a system of representatives of $\tilde{\Gamma}\setminus\tilde{\Gamma}$ in $\tilde{\Delta}$.

**Proof.** Let $\{\alpha_i\}_i$ be a system of representatives of $\Gamma \setminus \Gamma\Lambda\Gamma$ in $\Gamma\Lambda\Gamma$. Since $AM \supset M\Lambda$, the set $\{(\alpha_i, 0)\}$ is a system of representatives of $\tilde{\Gamma}(A,0)\tilde{\Gamma}/\tilde{\Gamma}$ in $\tilde{\Gamma}(A,0)\tilde{\Gamma}$. Hence, we have

$$\tilde{T}(A,0)\tilde{T}(B) = \sum_{(C,c)} \left\{\left|i \mid (\hat{\alpha}_i^{-1},0)(C,c) \in \cup_{b \in \mathcal{M}}\tilde{T}(B,b)\tilde{T}\right\} \right\} \tilde{T}(C,c)$$

$$= \sum_{(C,c)} \left\{\left|i \mid \hat{\alpha}_i^{-1}C \in \Gamma\Lambda\Gamma, \hat{\alpha}_i^{-1}c \in \mathcal{M}\right\} \right\} \tilde{T}(C,c)$$

$$= \sum_{(C,c)} |\Gamma\setminus\mathcal{Y}(A, B, C, c)||\tilde{T}(C,c).$$

3 Local Hecke rings and local Hecke series associated with algebras

In this section, we recall the definition of the local Hecke rings and the local Hecke series defined in [6] and [7], respectively. Let $L$ be an algebra which is free of rank $r$ as an abelian group, and fix a $\mathbb{Z}$-basis of $L$. Then $\text{Aut}_{\mathbb{Q}_p}(L \otimes \mathbb{Q}_p)$ and
The local Hecke ring $R_{L_p}$ associated with $L$ is defined by the Hecke ring with respect to the pair $(\Gamma_{L_p}, \Delta_{L_p})$.

Next, the local Hecke series $P_{L_p}(X)$ defined in [7] are recalled by using notation of Section 2. Set $T_{L_p} = T_{\Gamma_{L_p}, \Delta_{L_p}}$. For each nonnegative integer $k$, define an element $T_{L_p}(p^k)$ of $R_{L_p}$ by

$$ T_{L_p}(p^k) = \sum_{\alpha} T_{L_p}(\alpha), $$

where $\alpha$ runs through a system of representatives of $\Gamma_{L_p} \setminus \Delta_{L_p} / \Gamma_{L_p}$ and satisfies $|L_p/L_p^\alpha| = p^k$. The local Hecke series $P_{L_p}(X)$ associated with $L$ is defined to be the generating function of $\{T_{L_p}(p^k)\}_k$, that is,

$$ P_{L_p}(X) = \sum_{k \geq 0} T_{L_p}(p^k)X^k \in R_{L_p}[[X]]. $$

Example 3.1.

1. If $L$ is the ring $\mathbb{Z}^r$ of the direct sum of $r$-copies of $\mathbb{Z}$ for some positive integer $r$, then $\Gamma_{L_p} = GL_r(\mathbb{Z}_p)$ and $\Delta_{L_p} = M_r(\mathbb{Z}_p) \cap GL_r(\mathbb{Q}_p)$. The ring structure of $R_{L_p}$ and the rationality of $P_{L_p}(X)$ were shown by [4, Hecke] and [10, Tamagawa].

2. If $L$ is the Heisenberg Lie algebra of dimension 3, $R_{L_p}$ and $P_{L_p}(X)$ are treated in our previous paper [5]. $P_{L_p}(X)$ is slightly different from $D_{2,2}(X)$ defined in [5]. For the details, see Remark 4.1.

4 The local Hecke rings associated with the Heisenberg Lie algebras

Let $n$ be a positive integer. The Heisenberg Lie algebra of dimension $2n + 1$ over $\mathbb{Z}$, denoted by $H^{(n)}$, is the Lie algebra of square matrices of size $n + 2$ with entries in $\mathbb{Z}$ of the form

$$ \begin{pmatrix} 0 & a & c \\ 0 & O_n & b \\ 0 & 0 & 0 \end{pmatrix}, $$

where $a$ is a row vector of length $n$, $b$ is a column vector of length $n$, and $O_n$ is the zero matrix of size $n$. We study the local Hecke ring and the local Hecke
series associated with $\mathcal{H}^{(n)}$. Put $L = \mathcal{H}^{(n)}$. By choosing the standard basis of $L$, one can identify $G_{L_p}$ with the group of matrices of the form

$$\begin{pmatrix} A & a \\ 0_{2n} & \mu(A) \end{pmatrix},$$

where $A \in GSp_{2n}(\mathbb{Q}_p)$, $\mu(A)$ is the multiplier of $A$, $a$ is a column vector of length $2n$, and $0_{2n}$ is the zero row vector of length $2n$. It is easy to see that $\Delta_{L_p} = GL_p \cap M_{2n+1}(\mathbb{Z}_p)$ and $\Gamma_{L_p} = GL_p \cap GL_{2n+1}(\mathbb{Z}_p)$.

Put $G = GSp_{2n}(\mathbb{Q}_p)$, $\Delta = G \cap M_{2n}(\mathbb{Z}_p)$, $\Gamma = GSp_{2n}(\mathbb{Z}_p)$, and let $\mathcal{M}$ be the set of column vectors of length $2n$ with entries in $\mathbb{Z}_p$. Then $\Delta$ acts on $\mathcal{M}$ on the left naturally, and does on the right as follows: $aA = \mu(A)a$ for each $a \in \mathcal{M}$, $A \in \Delta$. Hence $\mathcal{M}$ is a both sides $\Delta$-module. Note that $X * a = \mu(X)^{-1}Xa$ for each $A \in \Delta$, $X \in \Gamma^A$, $a \in \mathcal{M}$. Let us put $\tilde{\Delta} = \tilde{\Delta}(\Delta, \mathcal{M})$, $\tilde{\Gamma} = \tilde{\Gamma}(\Gamma, \mathcal{M})$. It is well known that $(\Gamma, \Delta)$ is double finite (cf. [9]). Thus, Corollary 4.1 implies that $(\tilde{\Gamma}, \tilde{\Delta})$ is also double finite. We put $R = R(\Gamma, \Delta)$, $T = T_{\Gamma, \Delta}$, $\tilde{R} = \tilde{R}(\Gamma, \Delta)$ and $\tilde{T} = T_{\tilde{\Gamma}, \tilde{\Delta}}$. Then it is easy to see that $\Delta_{L_p}$ is isomorphic to $\tilde{\Delta}$ by the map

$$\begin{pmatrix} A & a \\ 0_n & \mu(A) \end{pmatrix} \mapsto (A, a),$$

and that the isomorphism derives the isomorphism from $R_{L_p}$ onto $\tilde{R}$. Let us indentity $R_{L_p}$ with $\tilde{R}$. Then

$$T_{L_p} \begin{pmatrix} A & a \\ 0_n & \mu(A) \end{pmatrix} = \tilde{T}(A, a).$$

Let $k$ be a nonnegative integer. We denote by $T(p^k)$ the sum of all the elements of the form $T(\Gamma \Delta)$ with $\Gamma \Delta \in \Gamma \setminus \Delta / \Gamma$ and $v_p(\mu(A)) = k$, and by $\tilde{T}(p^k)$ the sum of all the elements of the form $\tilde{T}(\tilde{\Gamma} \tilde{\Delta})$ with $\tilde{\Gamma}(A, a) \tilde{\Gamma} \in \tilde{\Gamma} \setminus \tilde{\Delta} / \tilde{\Gamma}$ and $v_p(\mu(A)) = k$. The Hecke series $P_n(X)$ associated with $GSp_{2n}$ is defined by

$$P_n(X) = \sum_{k \geq 0} T(p^k) X^k.$$

In addition, let us define the formal power series $\tilde{P}_n(X)$ with coefficients in $\tilde{R}$ by

$$\tilde{P}_n(X) = \sum_{k \geq 0} \tilde{T}(p^k) X^k.$$

Note that $\tilde{T}(p^k) = T_{L_p}(p^{(n+1)k})$ for all nonnegative integer $k$, and $\tilde{P}_n(X^{n+1}) = P_{\tilde{R}_p}(X)$.

**Remark 4.1.** $D_{2,2}(X)$ in our previous paper [9] coincides with $\tilde{P}_1(X)$. Thus $P_{\tilde{R}^{(1)}_p}(X) = D_{2,2}(X^2)$. 

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Put $C_0 = pE$. Then, the tuple $(G, \Delta, \Gamma, \mathcal{M}, C_0)$ satisfies Assumptions 2.1 and 2.15. Hence, the ring homomorphisms, $s = s_n: R \rightarrow \tilde{R}$, $\theta = \theta_n: \tilde{R} \rightarrow \tilde{R}$, and $\phi = \phi_n: \tilde{R} \rightarrow R$. are constructed. Set $(p) = T(pE), (p^2) = T(p^2E)$. Note that for each $(A, a) \in \Delta$, we have

$$\langle p \rangle s_{\tilde{T}}((A, a)) = \tilde{T}(pA, pa),$$

and

$$\tilde{T}(A, a)^\theta = \frac{[\Gamma^A * (a \mod AM)]}{[\Gamma^A * (pa \mod AM)]} \tilde{T}(A, pa).$$

By Proposition 2.17, we have the following properties:

**Proposition 4.2.** The three ring homomorphisms $\phi$, $s$ and $\theta$ satisfy the following properties:

1. $\phi \circ s = id_R$.
2. $\phi \circ \theta = \phi$.
3. $\tilde{T}(A)^\phi = p^{v_p(\mu(A))}T(A)$.

**Proof.** We need only prove the last identity, which is an immediate consequence of the facts $\frac{|\mathcal{M}/AM|}{|\det A|} = p^{v_p(\mu(A))}$ and $(\det A)^2 = \mu(A)^2$.

**Corollary 4.3.** $\tilde{T}(p^k)^\phi = p^{nk}T(p^k)$ for each nonnegative integer $k$.

**Corollary 4.4.** $\tilde{P}_n^2(X) = P_n(p^nX)$.

**Proof.** They are straightforward consequences of Property 3.

Next we show the non-commutativity of the coefficients of $\tilde{P}_n(X)$. For every positive integer $m$, we denote by $\Delta_{p^m}$, the set of elements $(A, a)$ of $\Delta$ such that $v_p(\mu(A)) = m$. To prove this, we need the following lemma.

**Lemma 4.5.** For positive integers $k$, $l$ with $k < l$. Let $c_2$ be an element of $\mathbb{Z}_{l}^n - p\mathbb{Z}_{p^{l}}^n$, and put

$$C = \begin{pmatrix} \begin{array}{cc} p^kE & O \\ O & p^lE \end{array} \end{pmatrix} \in \tilde{\Delta}, \quad c = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \in \mathbb{Z}_{p^{2n}}^n,$$

where $E$ and $O$ are respectively the identity matrix and the zero matrix of size $n$. Then we have:

$$\tilde{\Delta}_{p^k} \cdot \tilde{\Delta}_{p^l} \ni (C, c), \quad \tilde{\Delta}_{p^l} \cdot \tilde{\Delta}_{p^k} \not\ni (C, c).$$
Proof. Since 
\[
\left( \left( \begin{array}{cc} p \ E & O \\ O & E \end{array} \right), 0 \right) \in \tilde{\Delta}_{p^k}, \left( \left( \begin{array}{cc} E & O \\ O & p^l \end{array} \right), \left( \begin{array}{c} 0 \\ c_2 \end{array} \right) \right) \in \tilde{\Delta}_{p^l},
\]
we have \( \tilde{\Delta}_{p^k} \cdot \tilde{\Delta}_{p^l} \ni (C, c) \).

Let \((A, a)\) and \((B, b)\) be elements of \(\tilde{\Delta}_{p^l}\) and \(\tilde{\Delta}_{p^k}\), respectively. And assume \(AB = C\). The second component of \((A, a) (B, b)\) is contained in \(A b + p^k \mathbb{Z}_{p^2}^2\).

Since \(A = C B^{-1}\) and \(p^k B^{-1} \in \tilde{\Delta}\), we have \(A \mathbb{Z}_{p^2}^2 \subset \left( \mathbb{Z}_{p}^n / p^{l-k} \mathbb{Z}_{p}^n \right)\). Hence \(A b + p^k \mathbb{Z}_{p^2}^2\) is contained in \(\left( \mathbb{Z}_{p}^n / p^{l-k} \mathbb{Z}_{p}^n \right)\). Therefore, \(\tilde{\Delta}_{p^l} \cdot \tilde{\Delta}_{p^k} \not\ni (C, c)\).

Now we prove the non-commutativity.

**Theorem 4.6.** Notations being as the above lemma, \(\tilde{T}(p^k)\) and \(\tilde{T}(p^l)\) are not commutative to each other.

Proof. By the above lemma, the coefficients of \(\tilde{T}(C, c)\) in \(\tilde{T}(p^k) \tilde{T}(p^l)\) and \(\tilde{T}(p^l) \tilde{T}(p^k)\) are respectively not zero and zero, respectively. \(\square\)

To state the identity for \(\hat{P}_n(X)\), we recall the rationality theorem associated with \(GSp_{2n}\).

**Theorem 4.7** (Rationality Theorem, [1], [4], [8]). \(P_n(X)\) is rational, namely there exist elements \(q_1, ..., q_{2^n}\) of \(R\) and \(g(X) \in R[X]\) such that
\[
\sum_{i \geq 0} q_i X^i P_n(X) = g(X), \quad (8)
\]
where \(q_0 = 1\). Especially,

1. If \(n = 1\) then \(q_1 = -T(\text{diag}(1, p))\), \(q_2 = p\langle p \rangle\), \(g(X) = 1\).
2. If \(n = 2\) then
   
   \[
   \begin{align*}
   q_1 &= -T(\text{diag}(1, 1, p, p)), \\
   q_2 &= p T(\text{diag}(1, p, p^2, p)) + p(p^2 + 1) \langle p \rangle, \\
   q_3 &= -p^3 \langle p \rangle T(\text{diag}(1, 1, p, p)), \\
   q_4 &= p^6 \langle p^2 \rangle, \\
   g(X) &= 1 - p^2 \langle p \rangle X^2.
   \end{align*}
   \]

Where, \(\langle p \rangle = T(pE)\) and \(\langle p^2 \rangle = T(p^2 E)\).

Note that the sequence \(\{q_i\}\) and \(g(X)\) depend on \(n\). Our previous paper [5] shows the following theorem:
Theorem 4.8 ([5 Theorem 7.8]). Let us keep the notations of Theorem 4.7. If \( n = 1 \) then \( \tilde{P}_1(X) \) satisfies the following identity.

\[
\tilde{P}_1^{d_2}(X) - q_1^s Y \tilde{P}_1^d(X) + q_2^s Y^2 \tilde{P}_1(X) = 1,
\]

where \( Y = pX \).

We note that it derives the rationality of \( P_1(X) \) using Corollary 4.4. Moreover, the above identity is a solution of the following problem in the case where \( n = 1 \).

**Problem 4.9.** Let us keep the notations of Theorem 4.7. Does \( \tilde{P}_n(X) \) satisfy the following formula for each \( n \)?

\[
2^n \sum_{k=0}^{2^n} q_k^s Y^k \tilde{T}^{p^\Theta_{n-k}}(X) = g^s(Y), \tag{9}
\]

where \( Y = p^n X \).

Remark that, for each \( n \), this identity implies the rationality of \( P_n(X) \) via the morphism \( \phi_n \). In the present paper, we solve our problem for the case \( n = 2 \).

**Proposition 4.10.** Equality (9) is true modulo \( X^{2^n} \).

**Proof.** Let \( h(X) = \sum_{k=0}^{\infty} a_k X^k \) be the left hand side of (9). Then, for each \( k \leq 2^n \), \( a_k \) is a linear combination of elements of \( \{ \tilde{T}^i(p^j)_{i,j} \} \) with coefficients in the image of \( s \). Hence for each \( k \leq 2^n \), \( a_k \) is an element of the image \( s \). The proposition follows from the injectivity of \( s \) and the Theorem 4.7. \( \square \)

Thus we are reduce to proving the following problem.

**Problem 4.11.** Put \( N = 2^n \). For all \( k \geq 0 \), Is the following identity true?

\[
\sum_{i=0}^{N} p^n(N-i) q_{N-i} \tilde{T}(p^k+i)^{\Theta_i} = 0.
\]

In the rest of this section, we give a formula of the product \( \tilde{T}(A,0)\tilde{T}(p^k) \) using Lemma 2.18. Define \( \tilde{A} = \mu(A)A^{-1} \) for each \( A \in G \). Then the anti-automorphism \( (A \mapsto \tilde{A}) \) satisfies the condition in Subsection 2.4. Since

\[
\mathcal{X}(A,B,C) = \{ \alpha \in \Gamma A \Gamma \mid \alpha C \in \mu(A) \Gamma B \Gamma \}, \quad \mathcal{Y}(A,B,C,c) = \{ \alpha \in \mathcal{X}(A,B,C) \mid \alpha c \in \mu(A) \mathcal{M} \},
\]

for all \( A, B, C \in \Delta, c \in \mathcal{M} \), we have:
Lemma 4.12. For each $A, C \in \Delta, c \in \mathcal{M}$, put

\begin{align*}
\mathcal{X}(A, C) &= \{ \alpha \in \Gamma A | \alpha C \in \mu(A) \Delta \}, \\
\mathcal{Y}(A, C, c) &= \{ \alpha \in \mathcal{X}(A, C) | \alpha c \in \mu(A) \mathcal{M} \}.
\end{align*}

Then, for each $A \in \Delta$,

\[ \tilde{T}(A, 0) \tilde{T}(p^k) = \sum \left| \Gamma \setminus \mathcal{Y}(A, C, c) \right| \tilde{T}(C, c), \]

where $(C, c)$ runs through a system of representatives of $\tilde{\Gamma} \setminus \tilde{\Delta} / \tilde{\Gamma}$ and satisfies $v_p(\mu(A^{-1}C)) = k$.

5 The local Hecke series associated with the Heisenberg Lie algebra of dimension 5

In this section, we shall prove our main theorem. Let us keep the notation of Section 4 and suppose $n = 2$. First we introduce some notation.

\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \]

\[ P_1 = \text{diag}(1, 1, p, 1), \]
\[ P_3 = \text{diag}(1, p, p, p), \]
\[ A = \text{diag}(1, 1, p, p), \]
\[ B = P_1 P_3 = \text{diag}(1, p, p^2, p), \]

\[ \Gamma^A = \Gamma \cap A \Gamma A^{-1} = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ p^* & p^* & * & * \\ p^* & p^* & p^* & * \end{pmatrix} \right\} \cap \Gamma, \]

\[ \Gamma' = \Gamma^{P_1} = \Gamma^{P_3} = \left\{ \begin{pmatrix} * & * & * & * \\ p^* & * & * & * \\ p^* & p^* & * & * \\ p^* & p^* & p^* & * \end{pmatrix} \right\} \cap \Gamma. \]

Next, we provide a lemma.
Lemma 5.1. The following identities hold.

\[ \Gamma' e_3 = Z_p^4 - P_1 Z_p^4, \]  
\[ \Gamma' e_2 = P_1 Z_p^4 - P_3 Z_p^4, \]  
\[ \Gamma' e_1 = P_3 Z_p^4 - p Z_p^4, \]  
\[ \Gamma^A e_3 = Z_p^4 - A Z_p^4, \]  
\[ \Gamma^A e_1 = A Z_p^4 - p Z_p^4. \]  

Proof. It is essential to prove that the left-hand-side contains the right-hand-side for each identity. Put

\[ P(x, y, z) = \begin{pmatrix} 1 & -z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \]
\[ Q(x, y) = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix}. \]

for each \( x, y, z \in \mathbb{Z}_p \). Clearly, they are elements of \( \Gamma \). Let

\[ a = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \]

be an element of \( \mathbb{Z}_p^4 \).

(10). For \( a \in \mathbb{Z}_p^4 - P_1 Z_p^4 \), we show \( \Gamma' a = \Gamma' e_3 \). We are reduced to the case where \( c = 1 \). Then

\[ a = P(a, b, d) e_3. \]

(11). We assume \( a \in P_1 Z_p^4 - P_3 Z_p^4 \). For \( X \in SL_2(\mathbb{Z}_p) \),

\[ T_{2,3} \begin{pmatrix} E & O \\ O & X \end{pmatrix} T_{2,3} \]

is an element of \( \Gamma' \), where \( T_{2,3} \) is the permutation matrix corresponding to \( (2 \ 3) \in S_4 \). Hence we are reduced to the case where \( b = 1 \) and \( d = 0 \). Then

\[ a = Q(a, c) e_2. \]

(12). For \( a \in P_3 Z_p^4 - p Z_p^4 \), we show \( \Gamma' a = \Gamma' e_1 \). We are reduced to the case where \( a = 1 \). Then

\[ a = \begin{pmatrix} P(c, d, -b) \end{pmatrix} e_1. \]

(13). For \( a \in Z_p^4 - A Z_p^4 \), we show \( \Gamma^A a = \Gamma^A e_3 \). Since

\[ \begin{pmatrix} X & O \\ O & i X^{-1} \end{pmatrix} \in \Gamma^A \]

for each \( X \in GL_2(\mathbb{Z}_p) \), we may assume that \( c = 1, d = 0 \). Then

\[ a = P(a, b, 0) e_3. \]
For $a \in A\mathbb{Z}_p^4 - p\mathbb{Z}_p^4$, we show $\Gamma^A a = \Gamma^A e_1$. We may assume that $a = 1, b = 0$ for the same reason. Then
\[
a = ^tP(c,d,0)e_1.
\]
This completes the proof.

**Corollary 5.2.** The following identities hold.
\[
\mathbb{Z}_p^4 = \Gamma e_3 \cup \Gamma e_2 \cup \Gamma e_1 \cup p\mathbb{Z}_p^4 \quad \text{(disjoint)}.
\]
\[
\mathbb{Z}_p^4 = \Gamma^A e_3 \cup \Gamma^A e_1 \cup p\mathbb{Z}_p^4 \quad \text{(disjoint)}.
\]

**Proof.** Clear.

Next we give systems of representatives of $\Gamma \setminus \Gamma A \Gamma$ in $\Gamma A \Gamma$ and $\Gamma \setminus \Gamma B \Gamma$ in $\Gamma B \Gamma$, respectively. It is an immediate consequence of [2, Proposition 3.35] and the following lemmas. The details are left to the reader. We denote the matrix $\text{diag}(p^\alpha, p^\beta, p^{k-\alpha}, p^{k-\beta})$ by $C(\alpha, \beta, k)$ for nonnegative integers $k, \alpha, \beta$, with $0 \leq \alpha \leq \beta \leq k - \beta$.

**Lemma 5.3** ([3, Chapter 6, Lemma 5.2]). Let $M$ be an element of $\Delta$ with $v_p(\mu(M)) = 2$. $M \in \Gamma B \Gamma$ if and only if $rk_p(M) = 1$, where $rk_p(M)$ means the rank $M$ over $\mathbb{F}_p$.

**Proof.** It is a straightforward application of the “symplectic divisors theorem”. (cf. [2, Theorem 3.28]).

**Lemma 5.4.** For $D \in GL_n(\mathbb{Z}_p)$, we define the set $B(D)$ by
\[
B(D) = \{ B \in M_n(\mathbb{Z}_p) \mid ^tDB \text{ is symmetric} \}.
\]
Then $B(DD') = B(D)D'$ for each $D' \in GL_n(\mathbb{Z}_p)$.

**Proof.** Obvious.
Proposition 5.5. We put

\[
A_1 = \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
A_2(d, x) = \begin{pmatrix}
p & 0 & 0 & 0 \\
-x & 1 & 0 & d \\
0 & 0 & 1 & x \\
0 & 0 & 0 & p
\end{pmatrix} \text{ with } 0 \leq d, x < p,
\]

\[
A_3 \begin{pmatrix} a \\ b \\ d \end{pmatrix} = \begin{pmatrix}
1 & 0 & a & b \\
0 & 1 & b & d \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix} \text{ with } 0 \leq a, b, d < p,
\]

\[
A_4(d) = \begin{pmatrix}
1 & 0 & d & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \text{ with } 0 \leq d < p.
\]

Then the set of these matrices is a system of representatives of \(\Gamma \backslash \Gamma \Lambda \Gamma\) in \(\Gamma \Lambda \Gamma\).

Corollary 5.6. Let \(\alpha, \beta, k\) be nonnegative integers with \(0 \leq \alpha \leq \beta \leq k - \beta\) and \(k > 0\), then the set of the following matrices is a system of representatives of \(\Gamma \backslash \mathcal{X}(A, C(\alpha, \beta, k))\) in each case.

1. If \(\alpha = 0, \beta = 0\),

\[A_1.\]

2. If \(\alpha = 0, \beta \geq 1\),

\[A_1, \quad A_2(d, 0) \quad (0 \leq d < p).\]

3. If \(\alpha \geq 1\), all matrices in Lemma 5.5.

Proof. Clear.
Proposition 5.7. We put

\[ B_1(x) = \begin{pmatrix} p^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } 0 \leq x < p, \]

\[ B_2 = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ B_3 \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} p & 0 & a & b \\ 0 & p & b & d \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \text{ with } 0 \leq a, b, d < p, \; \text{rk}_p \begin{pmatrix} a & b \\ b & d \end{pmatrix} = 1, \]

\[ B_4(b, d, x) = \begin{pmatrix} p & 0 & 0 & pb \\ 0 & 1 & b & d \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } 0 \leq d < p^2, \; 0 \leq b, x < p, \]

\[ B_5(a, b) = \begin{pmatrix} 1 & 0 & a & b \\ 0 & p & pb & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \text{ with } 0 \leq a < p^2, \; 0 \leq b < p. \]

Then the set of these matrices is a system of representatives of \( \Gamma \bs \Gamma B \Gamma \) in \( \Gamma B \Gamma \).

Corollary 5.8. Let \( \alpha, \beta, k \) be nonnegative integers with \( \alpha \leq \beta \leq k - \beta \) and \( k \geq 3 \), then the set of the following matrices is a system of representatives of \( \Gamma \bs \mathcal{X}(B, C(\alpha, \beta, k)) \).

1. If \( \alpha = 0, \beta = 0 \), 
   \[ \emptyset. \]
2. If \( \alpha = 0, \beta \geq 1 \), 
   \[ B_1(0). \]
3. If \( \alpha = 1, \beta = 1 \), 
   \[ B_1(x) \quad (0 \leq x < p); \quad B_2; \quad B_3 \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad (0 \leq a, b, d < p, \; \text{rk}_p \begin{pmatrix} a & b \\ b & d \end{pmatrix} = 1). \]
4. If $\alpha = 1, \beta \geq 2,$
\[ B_1(x) \quad (0 \leq x < p), \]
\[ B_2, \]
\[ B_3 \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad (0 \leq a, b, d < p, \; \text{rk}_p \begin{pmatrix} a & b \\ b & d \end{pmatrix} = 1), \]
\[ B_4(b, d, 0) \quad (0 \leq d < p^2, \; 0 \leq b < p). \]

5. If $\alpha \geq 2,$ all matrices in Lemma 5.7.

Proof. Obvious.

Next we calculate $\tilde{T}(A, 0) \tilde{T}(p^{k+1})^\theta$ and $\tilde{T}(B, 0) \tilde{T}(p^{k+2})^\theta$. Let us introduce some notation. Put
\[ C_k = \{ C(\alpha, \beta, k) | 0 \leq \alpha \leq \beta \leq k - \beta \}, \; C_k^0 = \{ C(0, \beta, k) | 1 \leq \beta \leq k - \beta \}. \]

For each $D \in \Delta$ and each finite subset $C$ of $\Delta$, we put
\[ S_D(C) = \sum_{C \in C} \sum_{a | \Gamma \setminus \mathcal{Y}(D, C, a)} \tilde{T}(C, a), \]
where $a$ runs through a system of representatives of $\Gamma^C \setminus \mathbb{Z}_p^4/C\mathbb{Z}_p^4$ for each $C$.

The following lemmas are useful.

Lemma 5.9. Let $\mathcal{N}$ and $\mathcal{N}'$ be two subsets of $\mathbb{Z}_p^4$, and let $C \in \Delta$. If $\Gamma^C \mathcal{N} + C\mathbb{Z}_p^4$ and $\Gamma^C \mathcal{N}' + C\mathbb{Z}_p^4$ are disjoint, then
\[ \tilde{T}(C, \mathcal{N} \cup \mathcal{N}') = \tilde{T}(C, \mathcal{N}) + \tilde{T}(C, \mathcal{N}'). \]

Especially, if $\mathcal{N}$ is a subset of $p\mathbb{Z}_p^4$, $\mathcal{N}'$ is a subset of $\mathbb{Z}_p^4 - p\mathbb{Z}_p^4$, and $C \in p\Delta$, then the above formula holds.

Proof. Clear.

Lemma 5.10. For each $k \geq 0$, the following identities hold:
\[
\begin{align*}
\tilde{T}(pC_k, \mathbb{Z}_p^4)^\theta & = p^4(p)^\theta \tilde{T}(p^k), \quad (15) \\
\tilde{T}(pA^{k+2}, p\mathbb{Z}_p^4)^\theta & = p^2\tilde{T}(pA^{k+2}, p\mathbb{Z}_p^4)^\theta, \quad (16) \\
\tilde{T}(C_{k+4}^0, p\mathbb{Z}_p^4)^\theta & = p^3\tilde{T}(C_{k+4}^0, B\mathbb{Z}_p^4), \quad (17) \\
\tilde{T}(pC_{k+2}^0, p\mathbb{Z}_p^4)^\theta & = p^4(p)^\theta \tilde{T}(C_{k+4}^0, p\mathbb{Z}_p^4), \quad (18) \\
\tilde{T}(p^{k+2})^\theta & = p^2\tilde{T}(A^{k+2}, p\mathbb{Z}_p^4) + p^3\tilde{T}(C_{k+2}^0, p\mathbb{Z}_p^4) + p^4\tilde{T}(pC_k, p\mathbb{Z}_p^4). \quad (19)
\end{align*}
\]
Proof. They are immediate consequences of Proposition 2.2, Proposition 2.14, and the facts that $\Gamma^A \subset \Gamma^A_l$ for each $l \geq 1$ and that $\Gamma^C_1 \subset \Gamma^C_1$ for each $C \in C_l^0$ with $l \geq 2$.

Proposition 5.11. For each nonnegative integer $k$, $\bar{T}(A, 0) \bar{T}(p^{k+1})$ equals

$$\bar{T}(C^0_{k+2}; P_1 Z_4^p)^{\theta} + p^4 (1 + p)(p^1) \bar{T}(p^k) + p^3 (p^1) \bar{T}(p^k)^{\theta} + \frac{1}{p^2} \bar{T}(p^{k+2})^{g2}.$$  

Proof. By Lemma 4.12

$$\bar{T}(A, 0) \bar{T}(p^{k+1}) = S_A(\{A^{k+2}\}) + S_A(C^0_{k+2}) + S_A(pC_k).$$  

Let us put

$$X(\alpha, \beta) = X(A, C(\alpha, \beta, k + 2)),$$

and

$$Y(\alpha, \beta, a) = Y(A, C(\alpha, \beta, k + 2), a).$$

We shall calculate $|\Gamma \setminus Y(\alpha, \beta, a)|$. Notice that, for all $b \in Z_4^p$,

$$Y(\alpha, \beta, a) = Y(\alpha, \beta, a + pb).$$

The calculation will be divided into three cases.

1. If $\alpha = 0, \beta = 0$, then Corollary 5.6 implies $X(\alpha, \beta) = \Gamma A_1$. Hence it is easy to see that

$$\begin{cases}
|\Gamma \setminus Y(\alpha, \beta, a)| = 1, & \text{if } a \in A Z_4^p, \\
|\Gamma \setminus Y(\alpha, \beta, a)| = 0, & \text{otherwise}.
\end{cases}$$

Hence it is easy to see that

$$S_A(\{A^{k+2}\}) = \bar{T}(A^{k+2}, AZ_4^p).$$

By Proposition 2.2 we have

$$\bar{T}(A^{k+2}, AZ_4^p) = \bar{T}(A^{k+2}, pZ_4^p).$$

Hence, we have

$$S_A(\{A^{k+2}\}) = \bar{T}(A^{k+2}, pZ_4^p).$$

2. If $\alpha = 0, \beta \geq 1$, then

$$X(\alpha, \beta) = \left\{ \begin{pmatrix} p^1 * * * \\ p^1 * * * \\ p^1 * * * \\ p^1 * * * \end{pmatrix} \right\} \cap \Gamma A_1.$$
which is a right $\Gamma'$ invariant set. Hence for all $Y \in \Gamma'$,
\[ Y'(\alpha, \beta, a) Y = Y'(\alpha, \beta, Y^{-1}a). \]

Thus, for all $Y \in \Gamma'$,
\[ |\Gamma \setminus Y'(\alpha, \beta, a)| = |\Gamma \setminus Y'(\alpha, \beta, a)|. \]

By Corollary 5.2, we are reduced to the case where $a \in \{e_3, e_2, e_1, 0\}$. By Corollary 5.6 it is easy to see
\[ |\Gamma \setminus Y'(\alpha, \beta, e_3)| = 0, \quad |\Gamma \setminus Y'(\alpha, \beta, e_2)| = 1, \]
\[ |\Gamma \setminus Y'(\alpha, \beta, e_1)| = 1 + p, \quad |\Gamma \setminus Y'(\alpha, \beta, 0)| = 1 + p. \]

Thus, we have
\[ S_A(C_{k+2}^0) = \tilde{T}(C_{k+2}^0, \Gamma^e \cup \Gamma^e \cup p\mathbb{Z}_p^4) + p\tilde{T}(C_{k+2}^0, \Gamma^e \cup p\mathbb{Z}_p^4) \]
\[ = \tilde{T}(C_{k+2}^0, P_3^4) + p\tilde{T}(C_{k+2}^0, P_3^4). \]

By Proposition 2.2 we have
\[ \tilde{T}(C_{k+2}^0, P_3^4) = \tilde{T}(C_{k+2}^0, P_3^4). \]

Hence, we have
\[ S_A(C_{k+2}^0) = \tilde{T}(C_{k+2}^0, P_3^4) + p\tilde{T}(C_{k+2}^0, P_3^4). \]

(21)

3. If $\alpha \geq 1$, then $X'(\alpha, \beta) = \Gamma A \Gamma$ is naturally a right $\Gamma$ invariant set. Since
\[ |\Gamma \setminus Y(\alpha, \beta, e_1)| = 1 + p, \quad |\Gamma \setminus Y(\alpha, \beta, 0)| = 1 + p + p^2 + p^3, \]
we have
\[ S_A(pC_k) = (1 + p)\tilde{T}(pC_k, P_3^4) + (p^3 + p^2)\tilde{T}(pC_k, p\mathbb{Z}_p^4). \]

By (19) of Lemma 5.10
\[ S_A(pC_k) = p^4(1 + p)(p)^\theta \tilde{T}(pC_k, P_3^4) + p^3(p)^\theta \tilde{T}(pC_k, p\mathbb{Z}_p^4). \]  

(22)

Equality (19) of Lemma 5.10 and (20) − (22) imply the desired identity.

\[ \square \]

**Proposition 5.12.** For each nonnegative integer $k$,
\[ \tilde{T}(B, 0) \tilde{T}(p^{k+2}) = \tilde{T}(C_{k+4}^0, B\mathbb{Z}_p^4) + \tilde{T}(pC_{k+2}^0, P_3^4) \]
\[ + p^8(p^2)^\theta \tilde{T}(p^k) + p^7(p^3)^\theta \tilde{T}(p^k) \]
\[ + (p^2 + p - 1)(p)^\theta \tilde{T}(p^{k+2}) \]
\[ + (p)^3 \tilde{T}(p^{k+2})^\theta. \]
Proof. By Lemma 4.12, we have
\[
\hat{T} (B, 0) \hat{T} (p^{k+2}) = S_B (C_{k+4}^0) + S_B (\{pA^{k+2}\}) + S_B (pC_{k+2}^0) + S_B (p^2C_k).
\]
Note that \( S_B (\{A^{k+4}\}) = 0 \), by corollary 5.8. Let us put
\[
\mathcal{X}(\alpha, \beta) = \mathcal{X} (B, C(\alpha, \beta, k + 4)),
\]
and
\[
\mathcal{Y} (\alpha, \beta, a) = \mathcal{Y} (B, C(\alpha, \beta, k + 4), a).
\]
We shall calculate \(|\Gamma \\mathcal{Y}(\alpha, \beta, a)|\). Notice that, for all \( b \in \mathbb{Z}_p^4 \),
\[
\mathcal{Y} (\alpha, \beta, a) = \mathcal{Y} (\alpha, \beta, a + p^2b).
\]
The calculation will be divided into four cases.

1. If \( \alpha = 0, \beta \geq 1 \), then \( \mathcal{X}(\alpha, \beta) = \Gamma B_1(0) \). Hence it is easy to see that
\[
S_B (C_{k+4}^0) = \hat{T} (C_{k+4}^0, B\mathbb{Z}_p^4).
\] (23)

2. If \( \alpha = 1, \beta = 1 \), then
\[
\mathcal{X}(\alpha, \beta) = \left\{ \begin{pmatrix} p^* & p^* & * & * \\ p^* & p^* & * & * \\ p^* & p^* & * & * \\ p^* & p^* & * & * \end{pmatrix} \right\} \cap \Gamma B\Gamma.
\]
Hence \( \mathcal{X}(\alpha, \beta) \) is right \( \Gamma^A \) invariant. By Corollary 5.2, we are reduced to the case where \( a \in \{ e_3, e_1, pe_3, pe_1, 0 \} \). By cor 5.8 we have
\[
|\Gamma \\mathcal{Y}(\alpha, \beta, e_3)| = 0,
|\Gamma \\mathcal{Y}(\alpha, \beta, e_1)| = 1,
|\Gamma \\mathcal{Y}(\alpha, \beta, pe_3)| = p,
|\Gamma \\mathcal{Y}(\alpha, \beta, pe_1)| = p + p^2,
|\Gamma \\mathcal{Y}(\alpha, \beta, 0)| = p + p^2.
\]

By Lemma 5.1, \( S_B (\{pA^{k+2}\}) \) equals
\[
\hat{T} (pA^{k+2}, AZ_p^4 - p\mathbb{Z}_p^4) + p\hat{T} (pA^{k+2}, p\mathbb{Z}_p^4) + p^2\hat{T} (pA^{k+2}, pA\mathbb{Z}_p^4).
\]
By Lemma 5.9 we have
\[
\hat{T} (pA^{k+2}, AZ_p^4 - p\mathbb{Z}_p^4) = \hat{T} (pA^{k+2}, AZ_p^4) - \hat{T} (pA^{k+2}, p\mathbb{Z}_p^4).
\]
By (16) of Lemma 5.10 we have
\[
S_B (\{pA^{k+2}\})^\theta = (p^2 + p - 1)\hat{T} (pA^{k+2}, p\mathbb{Z}_p^4)^\theta
+ p^2\hat{T} (pA^{k+2}, p^2\mathbb{Z}_p^4)^\theta. \quad (24)
\]
3. If $\alpha = 1, \beta \geq 2$, then

$$\mathcal{X}(\alpha, \beta) = \begin{pmatrix} p^* & * & * \\ p^* & * & * \\ p^* & * & * \\ p^* & * & * \end{pmatrix} \cap \Gamma \mathcal{V}.$$ 

Hence $\mathcal{X}(\alpha, \beta)$ is a right $\Gamma'$ invariant set. By Corollary 5.2, we are reduced to the case where $a \in \{ e_3, e_2, e_1, p e_3, p e_2, p e_1, 0 \}$. By Lemma 5.8 we have

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, e_3)| = 0,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, e_2)| = 1,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, e_1)| = 1,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, p e_3)| = p + p^2,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, p e_2)| = p + p^2,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, p e_1)| = p + p^2 + p^3,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, 0)| = p + p^2 + p^3.$$ 

Thus $S_B(p e_3^0) = \tilde{T}(p e_3^0, p^2 \mathbb{Z}_p^4) + (p + p^2) \tilde{T}(p e_3^0, p^3 \mathbb{Z}_p^4) + p^3 \tilde{T}(p e_3^0, P P \mathbb{Z}_p^4).$

Proposition 2.2 implies $\tilde{T}(p e_3^0, P P \mathbb{Z}_p^4) = \tilde{T}(p e_3^0, p^2 \mathbb{Z}_p^4).$ Thus we have

$$S_B(p e_3^0) = \tilde{T}(p e_3^0, P_1 \mathbb{Z}_p^4) + (p^2 + p - 1) \tilde{T}(p e_3^0, p^2 \mathbb{Z}_p^4) + p^3 \tilde{T}(p e_3^0, p^2 \mathbb{Z}_p^4). \quad (25)$$

4. If $\alpha \geq 2$, then $\mathcal{X}(\alpha, \beta) = \Gamma \mathcal{V}$ is naturally right $\Gamma$ invariant set. Since

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, e_1)| = 1,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, p e_1)| = p + p^2 + p^3,$$

$$|\Gamma \setminus \mathcal{Y}(\alpha, \beta, 0)| = p + p^2 + p^3 + p^4,$$

$S_B(p e_3^0) = \tilde{T}(p e_3^0, P_1 \mathbb{Z}_p^4) + (p^3 + p^2 + p - 1) \tilde{T}(p e_3^0, p \mathbb{Z}_p^4) + p^4 \tilde{T}(p e_3^0, p^2 \mathbb{Z}_p^4).$

By Corollary 5.10

$$\tilde{T}(p^2 \mathbb{C}_k, \mathbb{Z}_p^4)^{[2]} = p^8 (p^2)^* \tilde{T}(p^k), \tilde{T}(p^2 \mathbb{C}_k, p \mathbb{Z}_p^4)^{[0]} = p^4 (p^2)^* \tilde{T}(p^k).$$

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Thus we have

\[
S_B(p^2C_k)_{\theta^2} = p^8\langle p^2 \rangle \tilde{T}(p^k) + p^7\langle p^2 \rangle \tilde{T}(p^k)\theta^2 \\
+ (p^2 + p - 1)\tilde{T}(p^2C_k, p\mathbb{Z}_p^4)_{\theta^2} \\
+ p^4\tilde{T}(p^2C_k, p^2\mathbb{Z}_p)_{\theta^2}. 
\]

(26)

Combining (19) of Lemma 5.10 and (23)–(26), we complete the proof.

By (17) and (18) of Lemma 5.10, we thus have the following identity:

**Corollary 5.13.** For all nonnegative integers \( k \),

\[
p^9\langle p \rangle T(A)_s\tilde{T}(p^{k+1})_{\theta^3} + p^2T(A)_s\tilde{T}(p^{k+3})_{\theta^3} = p^{14}\langle p^2 \rangle \tilde{T}(p^k) + p^5(T(B)_s + (p^2 + 1)\langle p \rangle)\tilde{T}(p^{k+2})_{\theta^2} + \tilde{T}(p^{k+4})_{\theta^4}.
\]

(27)

The above corollary implies the main theorem.

**Theorem 5.14.** Problem 4.9 holds for \( n = 2 \). Namely, put

\[
q_1 = -T(A), \\
q_2 = pT(B) + p(p^2 + 1)\langle p \rangle, \\
q_3 = -p^3\langle p \rangle T(A), \\
q_4 = p^6\langle p^2 \rangle, \\
g(X) = 1 - p^2\langle p \rangle X^2,
\]

then we have

\[
g^s(Y) = \tilde{P}_2^g(X) + q_1^4Y\tilde{P}_2^g(X) \\
+ q_2^4Y^2\tilde{P}_2^g(X) + q_3^4Y^3\tilde{P}_2^g(X) + q_4^4Y^4\tilde{P}_2(X),
\]

(27)

where \( Y = p^2X \).

**Remark 5.15.** Our theorem recovers Shimura’s rationality via the morphism \( \phi \).
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