Two-forms on four-manifolds and elliptic equations

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1 Background

Let $V$ be a four-dimensional real vector space with a fixed orientation. Then the wedge product can be viewed, up to a positive factor, as a canonical quadratic form of signature $(3, 3)$ on the six-dimensional space $\Lambda^2 V$. This gives a homomorphism from the identity component of the general linear group $GL^+(V)$ to the conformal group of the indefinite form, which is a local isomorphism. The significance of this is that geometrical structures on $V$ can be expressed in terms of the six-dimensional space $\Lambda^2 V$ with its quadratic form. Now if $X$ is an oriented 4-manifold we can apply this idea to the cotangent spaces of $X$. Many important differential geometric structures on $X$ can fruitfully be expressed in terms of the bundle of 2-forms, with its quadratic wedge product and exterior derivative. We recall some examples

- A conformal structure on $X$ is given by a 3-dimensional subbundle $\Lambda^+ \subset \Lambda^2$ on which the form is strictly positive. A Riemannian metric is specified by the addition of a choice of volume form.

- A (compatibly oriented) symplectic structure on $X$ is a closed 2-form $\omega$ which is positive at every point, $\omega \wedge \omega > 0$.

- An almost-complex structure on $X$ is given by a 2-dimensional oriented subbundle $\Lambda^{2,0} \subset \Lambda^2$ on which the form is strictly positive.

- A symplectic form $\omega$ tames an almost-complex structure $\Lambda^{2,0}$ if $\langle \omega \rangle + \Lambda^{2,0}$ is a 3-dimensional subbundle on which the form is positive, and the induced orientation agrees with a standard orientation inherited from that of $X$. The symplectic form is compatible with the almost-complex structure if $\omega$ is orthogonal to $\Lambda^{2,0}$ and in this case we say that we have an almost-Kahler structure.

- A complex-symplectic structure is given by a pair of closed two forms $\theta_1, \theta_2$ such that
  \[ \theta_1^2 = \theta_2^2, \quad \theta_1 \wedge \theta_2 = 0. \]
• A complex structure is given by an almost-complex structure $\Lambda^{2,0}$ such that in a neighbourhood of each point there is a complex-symplectic structure whose span is $\Lambda^{2,0}$.

• A Kahler structure is an almost-Kahler structure such that $\Lambda^{2,0}$ defines a complex structure.

• A hyperkahler structure is given by three closed two-forms $\theta_1, \theta_2, \theta_3$ such that
  \[ \theta_1^2 = \theta_2^2 = \theta_3^2 \quad \text{and} \quad \theta_i \wedge \theta_j = 0 \quad \text{for} \quad i \neq j. \]

A feature running through these examples is the interaction between point-wise, algebraic, constraints and the differential constraint furnished by the exterior derivative. In this article we introduce and begin the study of a very general class of questions of this nature, and discuss the possibility of various applications to four-dimensional differential geometry.

It is a pleasure to acknowledge the influence of ideas of Gromov and Sullivan on this article, through conversations in the 1980’s which in some instances may only have been subconsciously absorbed by the author at the time.

2 A class of elliptic PDE

Let $P$ be a three-dimensional submanifold in the vector space $\Lambda^2 V$, where $V$ is as in the previous section. We say that

• $P$ has negative tangents if the wedge product is a strictly negative form on each tangent space $T_P \omega$ for all $\omega$ in $P$.

• $P$ has negative chords if for all pairs $\omega, \omega' \in P$ we have $(\omega - \omega')^2 \leq 0$ with equality if and only if $\omega = \omega'$.

Clearly if $P$ has negative chords it also has negative tangents, but the converse is not true. Now we turn to our 4-manifold $X$ and consider a 7-dimensional submanifold $\mathcal{P}$ of the total space of the bundle $\Lambda^2$ such that the projection map induces a fibration of $\mathcal{P}$, so each fibre is a 3-dimensional submanifold, as considered before. We say that $\mathcal{P}$ has negative tangents, or negative chords, if the fibres do. In any case we can consider the pair of conditions, for a 2-form $\omega$ on $X$

\[ \omega \subset \mathcal{P}, \quad d\omega = 0. \quad (1) \]

Here, of course, the first condition just means that at each point $x \in X$ the value $\omega(x)$ is constrained to lie in the given 3-dimensional submanifold $\mathcal{P}_x \subset \Lambda^2_x$. The pair of conditions represent a partial differential equation over $X$. Now suppose that $X$ is compact and fix a maximal positive subspace $H^+_4 \subset H^2(X; \mathbb{R})$, for example the span of the self-dual harmonic forms for some Riemannian metric. Given a class $C \in H^2(X; \mathbb{R})$, we augment Equation (1) by the cohomological condition

\[ [\omega] \in C + H^+_4 \subset H^2(X; \mathbb{R}). \quad (2) \]
Proposition 1  

• If $\mathcal{P}$ has negative chords then there is at most one solution $\omega$ of the constraints (1), (2).

• If $\mathcal{P}$ has negative tangents and $\omega$ is a solution of the constraints (1), (2) then there is a neighbourhood $N$ of $\omega$ (in, say, the $C^\infty$ topology on 2-forms) such that there are no other solutions of the constraints in $N$. Further, if $\mathcal{P}^{(t)}$ is a smooth 1-parameter family of deformations of $\mathcal{P} = \mathcal{P}^{(0)}$ then we can choose $N$ so that for small $t$ there is a unique solution of the deformed constraint in $N$.

The proof of the first item is just to observe that if $\omega, \omega'$ are two solutions then $(\omega - \omega')^2$ is non-positive, pointwise on $X$, by the negative chord assumption. On the other hand the de Rham cohomology class of $\omega - \omega'$ lies in $H^2_+$ so

$$\int_X (\omega - \omega')^2 \geq 0.$$ 

Thus $(\omega - \omega')^2$ vanishes identically and the negative chord assumption implies that $\omega = \omega'$.

For the second item we consider, for each point $x$ of $X$ the tangent space to the submanifold $\mathcal{P}_x$ at the point $\omega(x)$. This is a maximal negative subspace for the wedge product form and so $\omega$ determines a conformal structure, on $X$ (the orthogonal complement of the tangent space is a maximal positive subspace). For convenience, we fix a Riemannian metric $g$ in this conformal class. The condition that a nearby form $\omega + \eta$ lies in $\mathcal{P}$ takes the shape

$$\eta^+ = Q(\eta),$$

where $\eta^+$ is the self-dual part of $\eta$ with respect to $g$, and $Q$ is a smooth map with $Q(\eta) = O(\eta^2)$. We choose 2-forms representing the cohomology classes in $H^2_+$, so $H^2_+$ can be regarded as a finite-dimensional vector space of closed 2-forms on $X$. Then closed forms $\omega + \eta$ satisfying our cohomological constraint can be expressed as $\omega + da + h$ where $h \in H^2_+$ and where $a$ is a 1-form satisfying the “gauge fixing” constraint $d^*a = 0$. Thus our constraints correspond to the solutions of the PDE

$$d^*a = 0, d^+a = Q(da + h) - h^+,.$$ 

where $d^+$ denotes the self-dual component of $d$. This is not quite a $1-1$ correspondence, we need to identify the solutions $a, a + \alpha$ where $\alpha$ is a harmonic 1-form on $X$. The essential point now is that the linear operator

$$d^* \oplus d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^2_+$$

is elliptic. This means that our problem can be viewed as solving a non-linear elliptic PDE and we can apply the implicit function theorem, in a standard fashion. The linearisation of the problem is represented by the linear map

$$L = d^* \oplus d^+ : \Omega^1/\mathcal{H}^1 \to \Omega^0/\mathcal{H}^0 \oplus \Omega^2_+/H^2_+,$$
where $\mathcal{H}^i$ denotes the space of harmonic $i$-forms, for $i = 0, 1$. We claim that the kernel and cokernel of $L$ both vanish. If $d^+a \in H^2_+$ we have $dd^+a = 0$ (since the forms in $H^2_+$ are closed) and then

$$0 = \int_X dd^+a \wedge a = \int_X d^+a \wedge da = \int_X |d^+a|^2,$$

so $d^+a = 0$. Now we use a fundamental identity

$$0 = \int_X da \wedge da = \int_X |d^+a|^2 - |d^-a|^2, \quad (3)$$

so $d^-a$ vanishes as well. It follows that the kernel of $L$ is trivial. The cokernel of $d^+ : \Omega^1 \to \Omega^2$ is represented by the space of self-dual harmonic forms $\mathcal{H}^2_+$. The assertion that the cokernel of $L$ is trivial is equivalent to the statement that $L^2$ projection $\pi : H^2_+ \to \mathcal{H}^2_+$ is surjective. Since these are both maximal positive subspaces for the cup product form they have the same dimension, so it is equivalent to prove that $\pi$ is injective. But if a form $h \in H^2_+$ is in the kernel of $\pi$ the cohomology class $[h]$ lies in the subspace $\mathcal{H}^2_- \subset H^2(X; \mathbb{R})$ defined by the anti-self dual harmonic forms, so $\int_X h^2 \leq 0$ and it follows that $h = 0$.

To sum up, for a given choice of $\mathcal{P}$ our constraints are represented by a system of nonlinear elliptic PDE whose linearisation is invertible. Now the assertions in the second item follow in a standard way from the implicit function theorem.

For the rest of this paper we will consider constraint manifolds $\mathcal{P}$ with negative tangents. Notice that we could generalise the set-up slightly by choosing a submanifold $Q \subset H^2(X; \mathbb{R})$ with the property that at each point the tangent space of $Q$ is maximal positive subspace for the cup product form. Then we can take in place of the constraint (2) the condition $[\omega] \in Q$. The proof above goes through without essential change. We could also express things differently by considering the moduli space $M$ of solutions to (1) with no cohomological constraint. Then the same proof shows that $M$ is a manifold of dimension $b_2^-(X)$ and that the map $\omega \mapsto [\omega]$ defines an immersion of $M$ in $H^2(X; \mathbb{R})$ whose derivative at each point takes the tangent space of $M$ to a maximal negative subspace for the intersection form.

### 3 Examples

1. We fix a Riemannian metric on $X$ and let $\mathcal{P}$ be the vector subbundle of anti-self dual forms. Then the solutions of our constraints are just the anti-self dual harmonic forms. This case is not very novel but, as we have seen in the previous section, is the model for the general situation for the purposes of local deformation theory.

2. Take $X = \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ with co-ordinates $(x_1, x_2, x_3, t)$ and consider a case where $\mathcal{P}$ is preserved by translations in the four variables, so is determined by a single 3-manifold $P \subset \Lambda^3(\mathbb{R}^4)$. (Of course $X$ is not compact here, but we only want to illustrate the local PDE aspects.) In the usual way, we write a
2-form as a pair of vector fields \((E, B)\)

\[
\omega = \sum E_i dx_i \wedge dt + \frac{1}{2} \sum \epsilon_{ijk} B_i dx_j \wedge dx_k;
\]

that is, we are identifying \(\Lambda^2 \mathbb{R}^4\) with \(\mathbb{R}^3 \oplus \mathbb{R}^3\). The condition that a submanifold \(P \subset \Lambda^2 \mathbb{R}^4\) has negative tangents implies that its tangent space at each point is transverse to the two \(\mathbb{R}^3\) factors, so locally \(P\) can be written as the graph of a smooth map \(F : \mathbb{R}^3 \to \mathbb{R}^3\). That is, locally around a given solution, we can write the constraint as \(B = F(E)\). Consider solutions which are independent of translation in the \(t\) variable. The condition that \(\omega\) is closed becomes

\[
\nabla \cdot B = 0, \quad \nabla \times E = 0.
\]

We can write \(E = \nabla u\) for a function \(u\) on \(\mathbb{R}^3\), so our constraint is a nonlinear elliptic PDE for a function \(u\) on \(\mathbb{R}^3\) of the form

\[
\nabla(F(\nabla u)) = 0.
\]

Let

\[
(H_{ij}) = \left( \frac{\partial F_i}{\partial x_j} \right),
\]

be the matrix of derivatives of \(F\). The condition that \(P\) has negative tangents becomes \(H + H^T > 0\) and the linearisation of the problem is the linear elliptic PDE

\[
\sum_{ij} \frac{\partial}{\partial x_i} (H_{ij} \frac{\partial f}{\partial x_j}) = 0.
\]

3. The next example is the central one in this article. We fix a volume form on our 4-dimensional real vector space \(V\) and also a complex structure, i.e. a 2-dimensional positive subspace \(\Lambda^{2,0} \subset \Lambda^2 V\). We define \(P\) to be the set of positive \((1,1)\)-forms whose square is the given volume form. Then \(P\) is one connected component of the set

\[
\{ \omega \in \Lambda^2 V : \omega^2 = 1, \omega \wedge \Lambda^{2,0} = 0 \},
\]

(the component being fixed by orientation requirements). It is easy to see that \(P\) has negative chords. In fact if \(\omega, \omega'\) are two points in \(P\) we can choose complex coordinates \(z_1, z_2\) so that

\[
\omega = idz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2, \quad \omega' = \lambda dz_1 \wedge d\bar{z}_1 + \lambda^{-1} dz_2 \wedge d\bar{z}_2,
\]

where \(\lambda > 0\). Then

\[
(\omega - \omega')^2 = (\lambda - 1)(\lambda^{-1} - 1) = 2 - (\lambda + \lambda^{-1}) \leq 0,
\]

with equality if and only if \(\lambda = 1\).

Thus if \(X\) is a 4-manifold with a volume form and a choice of almost-complex structure we get a constraint manifold \(\mathcal{P}\) with negative chords, and a problem
of the kind we have been considering. In particular, we can consider the case when the almost-complex structure is integrable. Then our problem becomes the renowned Calabi conjecture (in the case of two complex variables) solved by Yau: prescribing the volume form of a Kahler metric. The solution can be most easily expressed in terms of the moduli space \( \mathcal{M} \) of solutions to (1): it maps bijectively to the intersection of the Kahler cone in \( H^{1,1}(X) \) with the quadric \( [\omega]^2 = \text{Vol} \), where \( \text{Vol} \) is the integral of the prescribed volume form. Notice however that the problem formulated in terms of the cohomological constraints (2) does not always have a solution. In the most extreme case, we could take \( X \) to be a complex surface which does not admit any Kahler metric. More generally, by deforming our choice of \( C \) and \( H_2^+ \) we can deform from a case when a solution exists to a case when it does not, and understanding this phenomenon is essentially the question of understanding the boundary of the Kahler cone.

The extension of the Calabi-Yau theory to the case when the almost structure is not integrable has been considered recently by Weinkove [13]. Suppose, for simplicity, that \( b_2^+(X) = 1 \) and suppose that \( \omega_0 \) is a symplectic form compatible with the given almost-complex structure. In this case we take \( C = 0 \) and \( H_2^+ \) to be the 1-dimensional space spanned by \( \omega_0 \). Then the cohomology class of any solution of (1), (2) is fixed by the prescribed volume form and without loss of generality we can suppose it is the same as \( [\omega_0] \). Weinkove extends Yau’s a priori estimates to prove existence under the assumption that the Nijenhuis tensor of the almost complex structure is small in a suitable sense.

In general, we will say that a constraint \( \mathcal{P} \subset \Lambda^2 X \) is unimodular if there is a volume form \( \rho \) on \( X \) such that \( \omega^2 = \rho \) for any section \( \omega \subset \mathcal{P} \). So for a unimodular constraint any solution of (1) is a symplectic form.

4. Our final example is in some ways a simple modification of the previous one. We take an almost-complex structure and a fixed positive \((1,1)\)-form \( \Theta \) and we consider the submanifold \( \mathcal{P} \) of positive \((1,1)\) forms \( \omega \) with \( (\omega - \Theta)^2 \) a given volume form on \( X \). In each fibre this is a translate of the submanifold considered before, so again has negative chords. This example arises in the following way. Let \( X \) be a hyperkahler 4-manifold, with an orthogonal triple of closed forms \( \theta_1, \theta_2, \theta_3 \) and let \( \omega \) be another symplectic form on \( X \). We define three functions \( \mu_i = \mu_i(\omega) \) on \( X \) by

\[
\mu_i = \frac{\theta_i \wedge \omega}{\omega^2}.
\]

These arose in [2] as the “moment maps” for the action of the symplectomorphism group of \((X, \omega)\) with respect to the \( \theta_i \) and the triple \((\mu_1, \mu_2, \mu_3)\) can be regarded as a hyperkahler moment map. Now we ask the question: given three functions \( f_i \) on \( X \), can we find a symplectic form \( \omega \) with \( \mu_i(\omega) = f_i \)? An obvious necessary condition is that \( F = \sqrt{\sum f_i^2} \) does not vanish anywhere on \( X \) (for then the self-dual part of \( \omega \) would vanish and \( \omega^2 \) would be negative). Assuming this condition, we write \( \sum f_i \theta_i = F \sigma \) where \( \sigma \) is a unit self-dual 2-form. Then \( \sigma \) determines an almost-complex structure on \( X \) (with \( \Lambda^{2,0} \) equal to the orthogonal complement of \( \sigma \) in the span of \( \theta_1, \theta_2, \theta_3 \)) and \( \sigma \) is a positive form of type
(1,1) with respect to this structure. It is easy to check that the condition that 
\( \mu_i(\omega) = f_i \) can be expressed in the form above, with \( \Theta = \sigma/2F \) and the volume form \( \sigma^2/(4F^2) \).

4 Partial regularity theory

We have introduced a very general class of elliptic PDE problems on 4-manifolds
and shown that their behaviour with respect to deformations is straightforward.
The crux of the matter, as far as proving existence results goes, is thus to obtain
a priori estimates for solutions. Here we make some small steps in this direction, assuming an \( L^\infty \) bound. So throughout this section we assume
that \( P \) is a constraint manifold with negative tangents and \( \omega \) is a closed form
in \( P \) with \( |\omega| \leq K \) at each point. Here the norm is defined by some fixed
auxiliary Riemannian metric \( g_0 \) on \( X \). We will also use the conformal structure
determined by the pair \( \omega, P \) which for convenience we promote to another metric
\( g \), with, say, the same volume form as \( g_0 \). Since the set \( \{ \Omega \in P, |\Omega| \leq K \} \) is
compact the metrics \( g, g_0 \) are uniformly equivalent, for fixed \( K \).

Our estimates will depend on \( K \) and \( P \). More precisely, the dependence
on \( P \) will involve local quantities that could be written down explicitly and,
crucially, will be uniform with respect to continuous families of constraints \( P_t \)
(with fixed \( K \)).

Lemma 1 There exists \( C = C(K, P) \) such that
\[ \| \nabla \omega \|_{L^2} \leq C \]
(Here \( \nabla \) is the covariant derivative associated with the fixed metric \( g_0 \).)

It suffices to show that for any vector field \( v \) on \( X \) the Lie derivative \( L_v \omega \)
is bounded in \( L^2 \) by some \( C(K, P, v) \). For then we consider a cover of \( X \) by
\( D \) coordinate patches and \( 4D \) vector fields such that on each patch four of the
vector fields are the standard constant unit fields, for which the Lie derivative
becomes the ordinary derivative. Write \( \omega_v \) for \( L_v \omega \). So \( d\omega_v = 0 \), and in fact
\( \omega_v = di_v(\omega) \) is exact. Imagine first that the flow generated by \( v \) preserves the
constraint manifold \( P \). Then if we apply the Lie derivative to the condition
\( \omega \subset P \) we obtain an identity
\[ (\omega_v)^+ = 0, \]
where \( ()^+ \) denotes the self-dual part with respect to metric \( g \) defined by \( \omega \) and
\( P \). In general we will have an identity
\[ (\omega_v)^+ = \rho, \]
where the \( L^\infty \) norm of \( \rho \) can be bounded in terms of \( v, P \) and \( K \). In particular,
the \( L^2 \) norm of \( (\omega_v)^+ \) is a priori bounded (since the metrics \( g \) and \( g_0 \) are
uniformly equivalent we can take the \( L^2 \) norm with respect to either here).
Now using the metric \( g \) we have, since \( \omega_v \) is exact
\[ \|\omega_v\|^2_{L^2} = 2\| (\omega_v)^+ \|^2_{L^2} \]
just as in (3), and we obtain the desired $L^2$ bound on $\omega$.

**Lemma 2** There is a constant $c$, depending on $K$ and $P$ such that if

$$\int_{B(r)} |\nabla \omega|^2 \leq cr^2,$$

for all $r$-balls $B(r)$ in $X$ with $r \leq r_0$ then for any $p$

$$\|\nabla \omega\|_{L^p} \leq C,$$

where $C$ depends on $p, K, P, r_0$.

We use of some of the ideas developed in [3]. Fix attention on balls with a given centre, and choose local coordinates about this point such that the metric $g$ is the standard Euclidean metric at the origin. Let $\Lambda^\pm_0$ be the space of $\pm$ self-dual forms for the euclidean structure. The metric $g$ is represented by a tensor $\mu \in \text{Hom}(\Lambda^\pm_0, \Lambda^\pm_0)$, vanishing at the origin, such that the $g$-anti-self-dual 2-forms have the shape $\sigma + \mu(\sigma)$ for $\sigma$ in $\Lambda^-_0$. Now for any such tensor field $\mu$, defined over the unit ball $B$ in $\mathbb{R}^4$ say, consider the operator

$$d^+ + \mu d^- : \Omega^1 \to \Omega^1.$$

Here $d^+_\mu, d^-_\mu$ are the constant co-efficient operators defined by $\Lambda^+_0, \Lambda^-_0$. The basic point is that, for any given $p$, if $\mu$ is sufficiently small in $L^\infty$ the usual Calderon-Zygmund theory can be applied to this operator, regarded as a perturbation of $d^+_0$. More precisely, for given $p$ there is a $\delta = \delta(p)$ such that if $|\mu| \leq \delta$ then for any 1-form $\alpha$ over the ball with $d^\ast \alpha = 0$ we have

$$\|d\alpha\|_{L^p(B/2)} \leq C_p(\|d^+_\mu \alpha\|_{L^p(B)} + \|d\alpha\|_{L^2(B)}).$$

It follows that for closed 2-forms $\rho$ over $B$ we have an inequality

$$\|\rho\|_{L^p(B/2)} \leq C_p(\|\rho^+ d^+ \mu\|_{L^p(B)} + \|\rho\|_{L^2(B)}) \quad (4)$$

where $\rho^+ d^+ \mu$ denotes the self-dual part with respect to the conformal structure defined by $\mu$. (To see this we write $\rho = d\alpha$ with $d^\ast \alpha = 0$. ) To apply this idea in our situation we first fix some $p > 4$. For each $r$ we rescale the ball $B(r)$ to the unit ball in $\mathbb{R}^4$ and apply (4) to the ordinary derivatives of the form corresponding to $\omega$, defined with respect to the Euclidean coordinates, which are closed 2-forms. Since the tensor $\mu$ which arises is determined by the tangent space of $\mathcal{P}$, which varies continuously with $\omega$, there is some $\epsilon = \epsilon(p)$ such that (4) holds provided the oscillation of $\omega$ over $B(r)$ is less than $\epsilon$. (Here the “oscillation” of $\omega$ refers to the oscillation of the coefficients in the fixed coordinate system.) Transferring back to $B(r)$, and taking account of the rescaling behaviour of the quantities involved, we obtain an inequality of the form

$$r^{-4/p} \|\nabla \omega\|_{L^p(B(r/2))} \leq C_p r^{-2} \|\nabla \omega\|_{L^2(B(r))} + C'_p,$$
for constants $C_p, C'_p$ depending on $p, K, P$. This holds provided the oscillation of $\omega$ over $B(r)$ is less than $\epsilon$. On the other hand the Sobolev inequalities tell us that the oscillation of $\omega$ over $B(r/2)$ is bounded by a multiple of $r^{(p-1)/p} \| \nabla \omega \|_{L^p(B(r/2))}$. So we conclude that if the oscillation of $\omega$ over $B(r)$ is less than $\epsilon$ then the oscillation of $\omega$ over $B(r/2)$ is at most

$$C \left( r^{-2} \int_{B(r)} |\nabla \omega|^2 \right)^{1/2} + C''r \leq Cc^{1/2} + C''r.$$ 

Thus if $c$ is sufficiently small we can arrange that the oscillation of $\omega$ over the half-sized ball is less than $\epsilon/10$, say, once $r$ is small enough. Applying this to a pair of balls, we see that there is some $r_0$ such that if the oscillation of $\omega$ over all $r$-balls is less than $\epsilon$, for all $r \leq r_0$, then this oscillation is actually less than $\epsilon/2$. It follows then, by a continuity argument taking $r \to 0$, that the oscillation can never exceed $\epsilon$ over balls of a fixed small size and this gives an a priori bound on the $L^p$ norm of $\nabla \omega$ for this fixed $p$. Now we have a fixed Holder bound on $\omega$ and we can repeat the discussion, starting with this, to get an $L^p$ bound on $\nabla \omega$ for any $p$.

Let $B(r)$ be an embedded geodesic ball in $(X, g_0)$. We let $\hat{\omega}$ denote the 2-form over $B(r)$ obtained by evaluating $\omega$ at the centre of the ball and extending over the ball by radial parallel transport along geodesics.

**Lemma 3** For any $c > 0$ there is a constant $\gamma$, depending on $K, P, c$ such that if for all $\rho$-balls $B(\rho)$ (with $\rho < \rho_0$ ) we have

$$\int_{B(\rho)} |\omega - \hat{\omega}|^2 \leq \gamma \rho^4$$

then the hypothesis of Lemma 2 is satisfied, i.e.

$$\int_{B(r)} |\nabla \omega|^2 \leq cr^2$$

for all $r < r_0(\rho_0, K, P)$.

As before, we work in small balls centred on a given point in $X$ and standard coordinates about this point. There is no loss in supposing that the metric $g_0$ is the Euclidean metric in these coordinates: then the form $\hat{\omega}$ is just obtained by freezing the coefficients at the centre of the ball. Let $v$ be a constant vector field in these Euclidean coordinates. Fix a cut-off function $\beta$ equal to 1 on $[0, 1/2]$ and supported in $[0, 1)$ and let $\beta_r$ be the function on $X$ given by $\beta_r(x) = \beta(|x|/r)$ in these Euclidean coordinates. Set

$$I(r) = \int_{B(r)} \beta_r \omega_v \wedge \omega_v.$$
As in the proof of Lemma 1, it suffices to show that, when \( r \) is small, \( I(r) \leq cr^2 \) (for a different constant \( c \)). Suppose first that \( r = 1 \) and write \( \omega = \hat{\omega} + \eta \), so \( \|\eta\|^2_{L^2(B(1))} \leq \gamma \). By an \( L^2 \) version of the Poincaré inequality, as in \([12]\), we can find a 1-form \( \alpha \) over the ball with \( \eta = d\alpha \) and \( \|\nabla \alpha\|^2_{L^2(B(1))} \leq C \gamma \) for some fixed constant \( C \). Now we have \( \omega_v = \eta_v \), since the form \( \hat{\omega} \) is constant in these coordinates, and \( \omega_v = d\alpha_v \), where \( \|\alpha_v\|^2_{L^2(B(1))} \leq C \gamma \). Then

\[
I(1) = \int_{B(1)} \beta_1 \omega_v \wedge d\alpha_v = \int_{B(1)} d\beta_1 \wedge \omega_v \wedge \alpha_v,
\]

so

\[
I(1) \leq \|d\beta_1\|_{L^\infty} \|\omega_v\|_{L^2(B(1))} \|\alpha_v\|_{L^2(B(1))}.
\]

Using the \( L^2 \) bound on \( \alpha_v \) and the fact that the \( L^2 \) norm of \( \omega_v \) on \( B(1) \) is controlled by \( \sqrt{I(2)} \) we obtain

\[
I(1) \leq C \sqrt{\delta} \sqrt{I(2)}.
\]

For general \( r \), we scale the \( 2r \) ball to the unit ball and apply the same argument. Taking account of the scaling behaviour of the various quantities involved one obtains an inequality

\[
I(r) \leq (C \sqrt{\gamma}) r \sqrt{I(2r)}.
\]

It is elementary to show that this implies the desired decay condition on \( I(r) \) as \( r \) tends to 0. If we put \( J(r) = r^{-2}I(r) \) then \( J(r) \leq 2C \sqrt{\gamma} \sqrt{J(2r)} \). So if \( L_n = \log J(2^{-n}) \) we have

\[
L_{n+1} \leq \frac{L_n}{2} + \sigma
\]

where \( \sigma = \log(2C \sqrt{\gamma}) \). This gives

\[
L_n \leq 2^{-n}(L_0 - \sigma) + 2\sigma.
\]

Combined with the global \( a \) priori bound of Lemma 1, which controls \( L_0 \), this yields the desired result.

To sum up we have

**Proposition 2** Suppose \( \mathcal{P} \) is a constraint manifold with negative tangents and \( K > 0 \). There is a \( \gamma = \gamma(\mathcal{P}, K) \) with the following property. For each \( k, p, r_0 \) there is a constant \( C = C(k, K, \mathcal{P}, p, r_0) \) such that if \( \omega \) is any solution of (1) with \( \|\omega\|_{L^\infty} \leq K \) and with

\[
\int_{B(r)} |\omega - \hat{\omega}|^2 \leq \gamma r^4
\]

for all \( r \leq r_0 \) then \( \|\omega\|_{L^2} \leq C \).

For \( k = 1 \) this is a combination of the two preceding Lemmas. The extension to higher derivatives follows from a straightforward bootstrapping argument.

Of course the hypotheses of Proposition 2 are satisfied if \( \omega \) is bounded in \( L^\infty \) and has any fixed modulus of continuity, such as a Holder bound.
5 Discussion

5.1 Motivation from symplectic topology

We will outline the, rather speculative, possibility of applications of these ideas to questions in symplectic topology. Recall that the fundamental topological invariants of a symplectic form \( \omega \) on a compact 4-manifold \( X \) are the first Chern class \( c_1 \in H^2(X; \mathbb{Z}) \) and the de Rham class \([\omega] \in H^2(X; \mathbb{R})\).

**Question 1** Suppose \( X \) is a compact Kahler surface with Kahler form \( \omega_0 \). If \( \omega \) is any other symplectic form on \( X \), with the same Chern class and with \([\omega] = [\omega_0]\), is there a diffeomorphism \( f \) of \( X \) with \( f^*(\omega) = \omega_0 \)?

(McMullen and Taubes [7] have given examples of inequivalent symplectic structures on the same differentiable 4-manifold, but their examples have different Chern classes.)

A line of attack on this could run as follows. Suppose first that \( b_2^+ (X) = 1 \). Given a general symplectic form \( \omega \) we choose a unimodular constraint manifold \( \mathcal{P}_1 \) containing it and deform \( \mathcal{P}_1 \) through a 1-parameter family \( \mathcal{P}_t \) for \( t \in [0, 1] \) to a standard Calabi-Yau constraint \( \mathcal{P}_0 \) containing \( \omega_0 \). We suppose that \( \mathcal{P}_t \) are all unimodular, with the same fixed volume form. Then we choose the cohomological constraint by taking \( C = 0 \) and \( H^2_+ = \langle \omega \rangle \). Thus for any \( t \) the cohomology class of a solution is forced to be the fixed class \([\omega_0]\). At \( t = 0 \) we know that the solution \( \omega_0 \) is unique. If we imagine that we have obtained suitable \textit{a priori} estimates for solutions in the whole family of problems it would follow from the deformation result, Proposition 1, that there is a unique solution \( \omega_t \) for each \( t \), varying smoothly with \( t \), and \( \omega = \omega_1 \). Then Moser’s theorem would imply that \( \omega \) and \( \omega_0 \) are equivalent forms.

This strategy can be extended to the situation where \( b_2^+ > 1 \). We take \( H^2_+ = \langle \omega_0 \rangle + H^{2,0} \) where \( H^{2,0} \) consists of the real parts of the holomorphic 2-forms. There are two cases. If the Chern class \( c_1 \) vanishes in \( H^2(X; \mathbb{R}) \) then for any \( \theta \in H^{2,0} \) and \( s \in \mathbb{R} \) the form \( \theta + s\omega_0 \) is symplectic, provided \( \theta \) and \( s \) are not both zero. (For the zero set of \( \theta \) is either empty or a nontrivial complex curve in \( X \), and the second possibility is excluded by the assumption on \( c_1 \).) If \( c_1 \) does not vanish then the form is symplectic provided that \( s \neq 0 \). We follow the same procedure as before and (assuming the \textit{a priori} estimates) construct a path \( \omega_t \) from \( \omega_0 \) to \( \omega = \omega_1 \) with \([\omega_t] = s_t [\omega_0] + [\theta_t] \) for \( \theta_t \in H^{2,0} \). Now \([\omega_t]\) is not identically zero, so in the case \( c_1 = 0 \) we have a family of “standard” symplectic forms

\[
\tilde{\omega}_t = s_t \omega_0 + \theta_t,
\]

with \([\tilde{\omega}_t] = [\omega_t] \). Then a version of Moser’s Theorem yields a family of diffeomorphisms \( f_t \) with \( f_t^*(\omega_t) = \omega_t \). Since, by hypothesis, \( \tilde{\omega}_1 = \omega_0 \) and \( \omega_1 = \omega \), the diffeomorphism \( f_1 \) solves our problem. When \( c_1 \neq 0 \) the same argument works provided we know that \( s_t \) is never 0. But here we can use one of the deep results of Taubes [11]. If \( s_t \) were zero then there would be a class \( \theta = \theta_t \in H^{2,0} \) which is the class of a symplectic form with the same first Chern class \( c_1(X) \). But
Taubes’ inequality would give $c_1.[\theta] < 0$ whereas in our case $c_1.[\theta] = 0$ since $c_1$ is represented by a form of type $(1,1)$ and $\theta$ has type $(2,0) + (0,2)$.

### 5.2 The almost-complex case.

We have seen that, even within the standard framework of the Calabi-Yau problem on complex surfaces, solutions can blow up. In that case, everything can be understood in terms of the class $[\omega]$ and the Kahler cone. These difficulties become more acute in the nonintegrable situation. If $J_0$ is any almost complex structure on a 4-manifold $X$ we can construct a 1-parameter family $J_t$ such that there is no symplectic form, in any cohomology class, compatible with $J_1$. (This is in contrast with the integrable case, where deformations of a Kahler surface are Kahler). To do this we can simply deform $J_0$ in a small neighbourhood of a point so that $J_1$ admits a null homologous pseudo-holomorphic curve. More generally we could consider null-homologous currents $T$ whose $(1,1)$ part is positive. Thus if we form a 1-parameter family of constraints $P_t$ using these $J_t$, and any fixed volume form, solutions $\omega_t$ must blow up sometime before $t = 1$, however we constrain the cohomology class $[\omega_t]$. It seems plausible that solutions blow-up at the first time $t$ when a null-homologous $(1,1)$-current $T$ appears, and become singular along the support of $T$. It is a result of Sullivan [10] that the nonexistence of such currents implies the existence of a symplectic form taming the almost-complex structure. These considerations lead us to formulate a tentative conjecture.

**Conjecture 1** Let $X$ be a compact 4-manifold and let $\Omega$ be a symplectic form on $X$. If $P$ is a constraint manifold defined by an almost-complex structure which is tamed by $\Omega$, and any smooth volume form, then there are $C^\infty$ a priori bounds on a closed form $\omega \subset P$ with $[\omega] = [\Omega]$.

(By the results of Weinkove [13] it suffices to obtain $L^\infty$ bounds on $\omega$.)

This conjecture is relevant to the following problem.

**Question 2** If $J$ is an almost-complex structure on a compact 4-manifold which is tamed by a symplectic form, is there a symplectic form compatible with $J$?

If Conjecture 1 were true it would imply, by a simple deformation argument, an affirmative answer to Question 2 in the case when $b_2^+ = 1$, see the discussion in [13]. Such a result would be of interest even in the integrable case. It is a well-known fact that any compact complex surface with $b_2^+$ odd is Kahler. This was originally obtained from the classification theory, but more recently direct proofs have been given [11], [6]. Harvey and Lawson showed that any surface with $b_2^+$ odd admits a symplectic form taming the complex structure ([2], Theorem 26 and page 185), so a positive answer to the question above would yield another proof of the Kahler property, in the case when $b_2^+ = 1$. (Turning things around, one might hope that the techniques used in [11], [6] could have some bearing on Conjecture 1).

In connection with Question 2, note first that this is special to 4-dimensions. In higher dimensions a generic almost-complex structure does not admit any
compatible symplectic structure, even locally. In another direction, the answer is known to be affirmative in the case when the taming form is the standard symplectic form on $\mathbb{CP}^2$, by an argument of Gromov. This constructs a compatible form by averaging over the currents furnished by pseudoholomorphic spheres.

5.3 Hyperkahler structures

Recall that complex-symplectic and hyperkahler structures on 4-manifolds can be described by, respectively, pairs and triples of orthonormal closed 2-forms. We can ask

**Question 3** Let $X$ be a compact oriented 4-manifold

- Suppose there are closed two-forms $\theta_1, \theta_2$ on $X$ such that each point $\theta_i$ span a positive 2-plane in $\Lambda^2$. Does $X$ admit a complex-symplectic structure?

- Suppose there are closed two-forms $\theta_1, \theta_2, \theta_3$ on $X$ such that at each point $\theta_i$ span a positive 3-plane in $\Lambda^2$. Does $X$ admit a hyperkahler structure?

The hypotheses are equivalent to saying that the symmetric matrices $(2 \times 2$ and $3 \times 3$ in the two cases) $\theta_i \wedge \theta_j$ are positive definite, and the question asks whether we can find another choice of closed forms $\tilde{\theta}_i$ to make the corresponding matrix $\tilde{\theta}_i \wedge \tilde{\theta}_j$ the identity.

For simplicity we will just discuss the second version of the question, for triples of forms and in the case when $X$ is simply connected. The hypotheses imply that the symplectic structure $\theta_1$, say, has first Chern class zero, and a result of Morgan and Szabo [8] tells us that $b_2^+(X) = 3$. Thus the $\theta_i$ generate a maximal positive subspace $H_2^+$. For reasons that will appear soon we make an additional auxiliary assumption. We suppose that there is an involution $\sigma : X \to X$ with

$$\sigma^*(\theta_1) = \theta_1, \quad \sigma^*(\theta_2) = -\theta_2, \quad \sigma^*(\theta_3) = -\theta_3.$$  

(A model case of this set-up is given by taking $X$ to be a K3 surface double-covering the plane, with $\sigma$ the covering involution.) We show that under this extra assumption the truth of Conjecture 1 would imply a positive answer to our question. To see this we let $U_0$ be the orthogonal complement of $\theta_1$ in the span of the $\theta_i$, so $U_0$ defines an almost-complex structure on $X$, compatible with $\theta_1$. We can choose a smooth homotopy $U_t$ to the subbundle $U_1 = \text{Span}(\theta_2, \theta_3)$ giving a 1-parameter family of almost-complex structures, all tamed by $\theta_1$. We choose a 1-parameter family of volume forms equal to $\omega_t^2$ when $t = 0$ and to $\omega_t^2$ when $t = 1$. We make all these choices $\sigma$-invariant. Thus we get a 1-parameter family of constraints $P_t$ and seek $\omega_t$ with $[\omega_t]$ in $H^2_+$. The form $\theta_1$ gives a solution at time 0, so $\omega_0 = \theta_1$. The $\sigma$-invariance of the problem means that in this case we can fix the cohomology class of $\omega_t$ to be a multiple of the taming form $\theta_1$ and still solve the local deformation problem. Thus the hypotheses of Conjecture 1
are fulfilled so, assuming the conjecture, we can continue the solution over the whole interval to obtain an $\omega_1$ which satisfies

$$\omega_1^2 = \theta_2^2, \quad \omega_1 \wedge \theta_2 = \omega_1 \wedge \theta_3 = 0.$$ 

Now $\omega_1, \theta_2$ define a complex-symplectic structure on $X$, and it well known that under our hypotheses $X$ must be a K3 surface and hence hyperkahler.

The only purpose of the involution $\sigma$ in this argument is to fix the cohomology class of the form in the 1-parameter family. Of course one could hope that a suitable extension of the ideas could remove this assumption, but our discussion is only intended to illustrate how these ideas might possibly be useful.

### 5.4 Relation with Gromov’s Theory

We have seen that there is a “Calabi-Yau” constraint manifold $P$ associated to an almost complex-structure $J$ (and volume form) on a 4-manifold. On the other hand, we have the notion of a “J-holomorphic” curve, with renowned applications in global symplectic geometry due to Gromov and others. These ideas can be related, and extended, as we will now explain. Consider the Grassmannian $Gr_2(\mathbb{R}^4)$ of oriented 2-planes. It can be identified with the space of null rays in $\Lambda^2 \mathbb{R}^4$ by the map which takes a rank two 2-form to its kernel. Thus, choosing a Euclidean metric on $\mathbb{R}^4$ and hence a decomposition into self-dual and anti-self dual forms, the Grassmannian is identified with pairs $(\omega_+, \omega_-)$ where $|\omega_+|^2 = |\omega_-|^2 = 1$, or in other words with the product $S^2_+ \times S^2_-$ of the unit spheres in the 3-dimensional vector spaces $\Lambda^2_\pm$. Now there is a canonical conformal structure of signature $(2, 2)$ on $Gr_2(\mathbb{R}^4)$, induced by the wedge product form and in [4] Gromov considers embedded surfaces $T \subset Gr_2(\mathbb{R}^4)$ on which the conformal structure is negative definite. A prototype for such a surface is given by $T(\omega_+) = \{ \omega_+ \} \times S^2_-$ for some fixed $\omega_+$ in $S^2_+$. This is just the set of 2-planes which are complex subspaces with respect to the almost-complex structure defined by $\omega_+$ (with $\Lambda^{2,0}$ the orthogonal complement of $\omega_+$ in $\Lambda^2_+$). On the other hand, Gromov shows that any surface $T$ leads to an elliptic equation generalising that defining holomorphic curves in $\mathbb{C}^2$. Slightly more generally still, let $X$ be an oriented 4-manifold and form the bundle $Gr_2(X)$ of Grassmannians of 2-planes in the tangent spaces of $X$. Let $T$ be a 6-dimensional submanifold of the Grassman bundle, fibering over $X$ with each fibre $T_x$ a “negative” surface in the above sense. Then we have the notion of a “$T$-pseudoholomorphic curve” in $X$: an immersed surface whose tangent spaces lie in $T$. We say that a symplectic form $\Omega$ on $X$ tames $T$ if $\Omega(H) > 0$ for every subspace $H$ in every $T_x$. Then Gromov explains that all the fundamental results about $J$-holomorphic curves in symplectic manifolds extend to this more general context.

We will now relate this discussion to the rest of this article. Consider a negative submanifold $T \subset Gr_2(\mathbb{R}^4)$ as before and a map $f : [0, \infty) \times T \to \Lambda^2 \mathbb{R}^4$ with $f(r, \theta) = O(r^{-1})$, along with all its derivatives. Call the image of the map $(r, \theta) \mapsto r \theta + f(r, \theta)$ the $f$-deformed cone over $T$. Then we say that a 3-dimensional submanifold $P \subset \Lambda^2 \mathbb{R}^4$ is asymptotic to $T$ if there is a map $f$ as
above such that, outside compact subsets, $P$ coincides with the $f$-deformed cone over $T$. This notion immediately generalises to a pair of constraint manifolds $P, T$ over a 4-manifold. The prototype of this picture is to take the Calabi-Yau constraint defined by a volume form and almost-complex structure, which one readily sees is asymptotic to the submanifold of complex subspaces. Now suppose that $P$ lies in the positive cone with respect to the wedge-product form, has negative chords and is asymptotic to $T$. Then if $\omega \in P$ and $\theta \in T$, we have $(\omega - r\theta)^2 \leq O(r^{-1})$ for large $r$. Since $\theta^2 = 0$ and $\omega^2 > 0$ we see that $\omega \wedge \theta > 0$, which is the same as saying that $\omega$ tames $T$. In other words, just as in the Calabi-Yau case, a necessary condition for there to be a solution $\omega \subset P$ in a given cohomology class $h = [\omega]$ is that there is a taming form in $h$ for $T$.

It is tempting then to extend Conjecture 1 to this more general situation.

Conjecture 2 Let $X$ be a compact 4-manifold and let $\Omega$ be a symplectic form on $X$. If $P$ is a unimodular constraint manifold with negative chords which is asymptotic to $T$, where $T$ is tamed by $\Omega$, then there are $C^\infty$ a priori bounds on a closed form $\omega \subset P$ with $[\omega] = [\Omega]$.

(There is little hard evidence for the truth of this, so perhaps it is better considered as a question. Of course, by our results in Section 4 above, it suffices to obtain an $L^\infty$ bound and a modulus of continuity in the “BMO sense” of Proposition 2. Going a very small way in this direction, it is easy to show that a taming form for $T$ leads to an a priori $L^1$ bound on $\omega$.)

5.5 A counterexample

We can use sophisticated results in symplectic topology to obtain a negative result—showing that Conjecture 2 cannot be extended to the case where $P$ only has negative tangents.

Proposition 3 There is a simply connected 4-manifold $X$ with $b_2^+(X) = 1$ and a 1-parameter family of unimodular constraints $P_t$ ($t \in [0, 1]$) on $X$ with the following properties

1. For each $t$, $P_t$ has negative tangents and is asymptotic to the manifold $T$ associated with an almost-complex structure on $X$ which is tamed by a symplectic form $\Omega$.

2. For $t < 1$ there is a closed 2-form $\omega_t \subset P_t$ with $[\omega_t] \in \langle [\Omega] \rangle$.

3. The $\omega_t$ do not satisfy uniform $C^\infty$ bounds as $t \to 1$.

In fact we can take the almost-complex structure to be integrable, $\Omega$ to be a Kahler form, and arrange that, for each parameter value, $P_t$ coincides with the standard Calabi-Yau constraint outside a compact set.

The proof of Proposition 3 combines some simple general constructions with results of Seidel. Let $J$ be an almost-complex structure on a 4-manifold $X$ and let $\omega$ be a symplectic form on $X$. Choose an almost-complex structure
\( J' \) compatible with \( \omega \), and suppose that \( J \) and \( J' \) are homotopic, through a 1-parameter family \( J(s) \). In other words, for each \( s \) we have a 2-dimensional positive subbundle \( \Lambda^{2,0}(s) \). For convenience, suppose that \( J(s) = J' \) for \( s \) close to 0 and extend the family to all positive \( s \), with \( J(s) = J \) for \( s \geq 1 \). For fixed small \( \epsilon \) we define a subset of \( \Lambda^2 \) by

\[
\mathcal{P}^* = \{ \theta : \theta^2 = \omega^2, \theta \in \Lambda^{2,0}(\epsilon|\theta|) \},
\]

where \(|\theta|\) is the norm measured with respect to some arbitrary metric. This set \( \mathcal{P}^* \) has two connected components, interchanged by \( \theta \mapsto -\theta \), and it is easy to check that, if \( \epsilon \) is sufficiently small, one of these components is a constraint manifold \( \mathcal{P} \) with negative tangents, containing \( \omega \) and equal to the Calabi-Yau constraint defined by \( J \) and \( \omega^2 \) outside a compact set.

Now we can extend this construction to families. If \( \omega_z \) is a family of symplectic forms parametrised by a compact space \( Z \), and if the corresponding homotopy class of maps from \( Z \) to the space of almost-complex structures on \( X \) is trivial, then we can construct a family \( \mathcal{P}_z \) of unimodular constraints, equal to the fixed Calabi-Yau constraint outside a compact set and with \( \mathcal{P}_z \) containing \( \omega_z \). For our application we first take \( Z \) to be a circle, so we have a loop of symplectic forms \( \omega_z, z \in S^1 \), with fixed cohomology class \( h = [\omega_z] \). Clearly the family \( \mathcal{P}_z \) for \( z \in S^1 \) can be extended over the disc \( D^2 \). So for each \( z \in D^2 \) we have a \( \mathcal{P}_z \) satisfying the conditions of the Proposition. If the Proposition were false, then it would follow that we could extend the family of symplectic forms \( \omega_z \) over the disc, using a simple continuity argument.

The space \( S_\omega \) of symplectic forms equivalent to \( \omega \) can be identified with the quotient of the identity component of the full diffeomorphism group by the symplectomorphisms of \( (X, \omega) \) which are isotopic to the identity. So the fundamental group of \( S_\omega \) can be identified with the classes of symplectomorphisms isotopic to the identity modulo symplectic isotopy. Now in [9] Seidel gives examples of Kahler manifolds \( (X, \omega) \) satisfying our hypotheses and symplectomorphisms which are isotopic, but not symplectically isotopic, to the identity. Thus Seidel’s results assert that there are maps from the circle to \( S_\omega \) which cannot be extended to the disc (although the corresponding almost-complex structures can be). This conflict with our previous argument completes the proof of Proposition 3.

The author has not yet succeeded in understanding more explicitly the blow-up behaviour which Proposition 3 asserts must occur. Seidel’s symplectomorphisms are squares of generalised Dehn twists, associated to Lagrangian 2-spheres in \( (X, \omega) \), and it is tempting to hope that the blow-up sets should be related to these spheres in some way. It should also be noted that Seidel’s results depend crucially on fixing the cohomology class of the symplectic form.

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