EXISTENCE OF STEADY VERY WEAK SOLUTIONS TO NAVIER-STOKES EQUATIONS WITH NON-NEWTONIAN STRESS TENSORS

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ABSTRACT. We provide existence of very weak solutions and new a-priori estimates for steady flows of non-Newtonian fluids when the right-hand sides are not in the natural existence class. To obtain the a-priori estimates we make use of a newly developed solenoidal Lipschitz truncation that preserves zero boundary values. We also estimate in (Muckenhoupt) weighted spaces which permit us to regain a duality pairing. Our estimates are valid even in the presence of the convective term. They are obtained via a newly developed comparison method that allows to "cut out" the singularities of the right hand side such that the skew symmetry of the convective term can be used for large parts of the right hand side.

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1. Introduction

In this work we are concerned with the existence and regularity of models prescribing the motion of an incompressible non-Newtonian fluid under singular forcing. Throughout the paper we assume that \( \Omega \subset \mathbb{R}^3 \) is a bounded Lipschitz domain and \( p \in (1, \infty) \). We consider the following steady system of Navier-Stokes equations

\[
\begin{aligned}
\text{div} (u(x) \otimes u(x)) - \text{div} A(x, \varepsilon u(x)) + \nabla \pi(x) &= -\text{div} f(x) & \text{in } \Omega \\
\text{div} u &= 0 & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(1.1)

Here the unknowns are the velocity \( u : \Omega \to \mathbb{R}^d \) and the pressure \( \pi : \Omega \to \mathbb{R} \). The force is \( f \in L^d(\Omega; \mathbb{R}^{3 \times 3}) \) and \( \varepsilon u := \frac{1}{2} (\nabla u + \nabla u^T) \) is the symmetric gradient. The prescribed tensor \( A : \Omega \times \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \) is a Carathéodory mapping; this means it is measurable in the first variable and continuous in the second variable. Additionally, we assume coercivity, boundedness and monotonicity on \( A \), that is: for all \( z_1, z_2 \in \mathbb{R}^{3 \times 3} \) and almost all \( x \in \Omega \) the following relations hold:

\[
A(x, z_1) : z_1 \geq C_1 |z_1|^p - C_3, \text{ coercivity}
\]  

(1.2)

\[
|A(x, z_1)| \leq C_2 |z_1|^{p - 1} + \frac{C_3}{p}, \text{ boundedness}
\]  

(1.3)

\[
(A(x, z_1) - A(x, z_2)) : (z_1 - z_2) \geq 0, \text{ monotonicity}.
\]  

(1.4)

Observe that in case \( A(x, z) \equiv \frac{1}{2} z \), with \( \nu \) being the constant viscosity the system (1.1) becomes the steady Navier-Stokes equation:

\[
\begin{aligned}
\text{div}(u(x) \otimes u(x)) - \nu \Delta u + \nabla \pi(x) &= -\text{div} f(x) & \text{in } \Omega \\
\text{div} u &= 0 & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(1.5)

Once \( q = 2 \) the existence theory is standard and follows by monotone operator theory. For the Navier-Stokes equation (1.5) in case \( q < 2 \) the existence can be achieved by approximating \( f \) with functions in \( L^2 \) provided some a-priori estimates are satisfied in a space that embeds compactly in \( L^2 \). To the best of our knowledge the lower bound of exponents \( q \) for which an existence theory for (1.5) is available (in three space dimensions) is \( q \geq \frac{3}{2} \); see [13, 19, 25] and the references therein. These results can not be transferred to non-linear stress tensors \( A(x, \varepsilon(u)) \) directly since they relay on the linearity of the Stokes operator. In this paper we develop an independent methodology that is suitable for non-Newtonian fluids. However, the lower bounds on \( q \) are larger. In case \( p = 2 \) and under the additional hypothesis (2.5) our methods do imply the existence of solutions for exponents \( q \geq \frac{12}{7} (> \frac{3}{2} = \frac{3}{2}) \) (see Theorem 2.4 below).
In case of \( p \neq 2 \) we recover the non-Newtonian fluids of Stokes type; in particular so-called \( p \)-fluids where \( A(x,z):=|z|^{p-2}z \) which were introduced by Ladizenskaya and Lions in the late 60s [20, 22]. We point out that this is when the viscosity \( \nu \) depends on the shear rate \( |e\nu| \) as \( v(t) \equiv t^{p-2} \); it can become shear thinning if \( p < 2 \) or shear thickening if \( p > 2 \). The results introduced here are new even for the following non-linear Stokes type system:

\[
\begin{aligned}
  -\text{div}A(x,e\nu(x)) + \nabla \pi &= -\text{div} f & \text{in } \Omega \\
  \text{div} u &= 0 & \text{in } \Omega \\
  u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]  

(1.6)

The existence theory is motivated by the model case of \( p \)-fluids which are minimizers of the functional

\[
\mathcal{F} : W^{1,p}_{0,\text{div}}(\Omega) \ni \nu \mapsto \int_{\Omega} \frac{|e\nu|^p}{p} \, dx - \int_{\Omega} f \cdot \nabla \nu \, dx \in \mathbb{R},
\]

since the respective Euler Lagrange equation is

\[
\int_{\Omega} |e\nu|^{p-2} e\nu \cdot e\phi \, dx = \int_{\Omega} f \cdot \nabla \phi \, dx \quad \text{for all } \phi \in C^0_{0,\text{div}}(\Omega).
\]

(1.7)

This existence approach fails in case \( q < p' \) since in this case we cannot guarantee for the coercivity of the functional. That relates to the fact that in this case the class of test functions has to be restricted severely. In particular we cannot use the solution \( u \) as a test function if \( u \in W^{1,q}_{0,\text{div}}(\Omega) \) with \( q < p \), only. Hence we introduce the following definition of what we call a very weak solution for non-linear PDEs [17].

**Definition 1.1.** We say a function \( u \in W^{1,1}_{0,\text{div}} \cap L^2(\Omega) \) such that \(|e\nu|^{p-1} \in L^1(\Omega)\) is a very weak solution to (1.1) if \( f \in L^q(\Omega) \) with \( q \in [1,p') \) and

\[
\int_{\Omega} (-u \otimes u) \cdot \nabla \phi + A(\cdot,e\nu) \cdot e\phi - \pi \text{div}(\phi) \, dx = \int_{\Omega} f \cdot \nabla \phi \, dx \quad \text{for all } \phi \in C^0_{0,\text{div}}(\Omega).
\]

(1.8)

The definition for very weak solutions to (1.6) is analogous.

2. **Main results.**

2.1. **Existence and a-priori estimates for the p-Stokes system.** One of the aims of the current paper is to show that for \( q \) close enough to \( p' \) and \( f \in L^q(\Omega) \) there exists a very weak solution \( u \in W^{1,q(p-1)}_{0,\text{div}}(\Omega) \) to (1.6) provided that (1.2)–(1.4) are satisfied.

**Theorem 2.1.** Let \( p \in (1,\infty) \) and let \( \Omega \) be a bounded, open and Lipschitz domain. If \( A \) satisfies (1.2)–(1.4), then there exists an \( \varepsilon_0 > 0 \) depending on \( \Omega, C_1, C_2, C_3 \) and \( p \) such that if \( \nu \in [p' - \varepsilon_0, p') \) and \( f \in L^q(\Omega,\mathbb{R}^{3 \times 3}) \) there exists a weak solution \( (u, \pi) \in W^{1,q(p-1)}_{0,\text{div}}(\Omega;\mathbb{R}^{3 \times 3}) \times L^q(\Omega;\mathbb{R}) \) to (1.6). Furthermore

\[
\int_{\Omega} |e\nu|^{q(p-1)} + |\pi|^q \, dx \leq c(C_1, C_2, C_3, p, q, \Omega) \left( \int_{\Omega} |f|^q \, dx + 1 \right)
\]

(2.1)

and

\[
\int_{\Omega} |e\nu|^M(|f|+1)^{q-p'} + |\pi|^p \, dx \leq c(C_1, C_2, C_3, p, q, \Omega) \left( \int_{\Omega} |f|^q \, dx + 1 \right)
\]

(2.2)

for some positive constant \( c(C_1, C_2, C_3, p, q, \Omega) > 0 \).

The existence theory follows densely the approach of [11] and [8]. The idea is to obtain a sequence of approximate solutions and then to pass to (weak) limits in appropriate Sobolev spaces. The main point, which was already remarked in [10] is that \( L^q \) estimates (2.1) are not enough to identify the non-linearity; further estimates are needed, namely weighted estimates as (2.2). Then it follows from the weighted a-priori estimates, by using the compactness result contained in [8, Theorem 1.9] (and mentioned here as Theorem 3.9), that we can pass to the limit in the sequence of approximations. Both estimates are new for \( p \neq 2 \). The respective weighted estimates for the p-Laplacean was shown in [11]. The \( L^q \) estimates for the classical p-Laplacean system are already known for some time, see [13, 21].

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1 For a definition of \( M \) see [5, 3].
2.2. Existence and a-priori estimates for the p-Navier-Stokes system. The second aim of this paper is to obtain a-priori estimates for the Navier-Stokes system (1.1). We wish to point out that the a-priori estimates for the p-Navier-Stokes system do not follow in a straightforward manner from the respective estimates of the Stokes system. Unfortunately, we were unable to extend the existence theory for the Navier-Stokes regime (1.1) in the case when \( p < 2 \), since the scaling of the convective term is then overwhelming the scaling of the diffusion term. In case \( p > 2 \), we have replaced (1.2) by the following stronger assumption. We assume that for all \( z_1, z_2 \in \mathbb{R}^{3\times 3} \)

\[
(A(x, z_1) - A(x, z_2)) \cdot (z_1 - z_2) \geq C_1 |z_1 - z_2|^p - C_3.
\]

(2.3)

Please observe that (2.3) is satisfied by the model case (1.7) as well as by many other stress laws. The result for \( p > 2 \) is the following:

**Theorem 2.2.** Assume that \( \Omega \) is a bounded Lipschitz domain. Let \( p \in (2, \infty) \) and \( A \) satisfying (1.3), (1.4) and (2.3), then there exists an \( \varepsilon_0 > 0 \) depending on \( \Omega \) and \( p \) such that if \( q \in [p' - \varepsilon_0, p'] \) and \( f \in L^q(\Omega, \mathbb{R}^{3\times 3}) \) there exists a weak solution \((u, \pi) \in W^{1,q(p-1)}_{0, \text{div}}(\Omega; \mathbb{R}^3) \times L^q_{\Omega}(\Omega; \mathbb{R})\) to (1.1).

Furthermore, we find

\[
\int_{\Omega} |\pi|^q + |\pi|^p \, dx + \int_{\Omega} |\nabla u|^q(p-1) + |\nabla u|^p M(|f| + 1)^{q-p'} \, dx \leq c(C_1, C_2, C_3, p, q, \Omega) \left( \int_{\Omega} |f|^q \, dx + 1 \right)^{\frac{1}{q}} \alpha(s) \left( \frac{C_3}{\varepsilon_0} + 1 \right) \tag{2.4}
\]

where

\[
\alpha(s) = s - \frac{2}{p-2} \text{ if } p \in (2, 3) \text{ and } s = \max \left\{ s - \frac{2}{p-2}, \frac{p-3}{p-2} \right\} \text{ if } p > 3.
\]

**Remark 2.3.** Please observe that in case \( p > 3 \) and \( \varepsilon_0 \) small enough the estimate for the Navier-Stokes equation (1.1) is the same as for the Stokes equation (1.6). This is natural due to the scaling of the convective term which can be overwhelmed exactly when \( p > 3 \).

In case \( p = 2 \) much more can be shown provided that we know for large shear speeds that the stress-tensor becomes diagonal. The additional assumption here has been introduced in [8], where the respective Stokes theory has been developed. We can extend the theory of [8] to the non-linear Navier-Stokes case (with convective term). We assume that we call the linear at infinity condition which says that there is a viscosity at infinity \( \nu \), such that

\[
\lim_{|z| \to \infty} \frac{|A(x, z) - \nu z|}{|z|} = 0 \text{ and } \lim_{|z| \to \infty} |\partial_z A(x, z)[y] - \nu y| = 0
\]

uniformly in \( x \in \Omega \) and \( y \in \mathbb{R}^{3\times 3} \). For such stresses we have the following theorem:

**Theorem 2.4.** Let \( \Omega \) be a bounded, open and \( C^1 \) domain and \( A \) satisfying (1.2), (1.3), (1.4) and (2.5), then for \( q \in [p^*, 2] \) and \( f \in L^q(\Omega, \mathbb{R}^{3\times 3}) \) there exists a weak solution \((u, \pi) \in W^{1,q}_{0, \text{div}}(\Omega; \mathbb{R}^3) \times L^q_{\Omega}(\Omega; \mathbb{R})\) to (1.1).

Furthermore we find

\[
\int_{\Omega} |\pi|^q + |\pi|^p \, dx + \int_{\Omega} |\nabla u|^q(p-1) + |\nabla u|^p M(|f| + 1)^{q-p'} \, dx \leq C
\]

for some positive constant \( C \) that depends on \( \int_{\Omega} |f|^q \, dx, C_1, C_2, C_3 \) and the linear at infinity condition.

In order to achieve the result we introduce here a new decoupling method that divides the estimate by splitting the right hand side into a large part which is in the dual space (and hence the skew symmetry of the convective term may be used) and a small singular part. It is then possible to use the smallness of the mass of the singular part to quantify the difference of the Navier-Stokes solution to the Stokes solution.

**Remark 2.5.** In case \( q \in [2, \infty) \) the existence of solutions to (1.1) follows by monotone operator theory and fixed point methods. The a-priori estimates (i.e. showing that \( \nabla u \in L^q(\Omega) \) for \( q > 2 \)) then follows by [8] Theorem 1.4] using \( \text{div}(u \odot u) \) as part of the right hand side.

2.3. A solenoidal Lipschitz truncation with zero boundary values. The main tool in order to get the announced a-priori estimates for (1.6) is called solenoidal relative truncation. Let us say a few words about the development of this tool.

Suppose we are given a Sobolev function \( u \in W^{1,1}(\Omega) \) where \( \Omega \) is an open set of \( \mathbb{R}^n \). A Lipschitz truncation of \( u \) is a function \( u_\lambda, \lambda > 0 \) that is Lipschitz continuous and such that \( \{|u_\lambda \neq u| \} \to 0 \) as \( \lambda \to \infty \). This is done
by modifying the function $u$ on the level set where the Hardy-Littlewood maximal function of $\nabla u$ is greater than $\lambda$. To our knowledge, this was first achieved by Acerbi and Fusco in [1], [2] and [3].

The Lipschitz truncation method was successfully applied in many areas of analysis such as:

- Calculus of variations: weak lower semicontinuity for Lipschitz functions imply weak lower semicontinuity for Sobolev functions [1], [2], [3].
- Fluid dynamics: existence of non-Newtonian fluids [4], [5], [6] and the references therein.
- Very weak solutions: a-priori estimates for p-Laplacian [7], existence and uniqueness issues [8], [9], [10], non-linear flows [11]. See also the recent parabolic results [12, 13].

A self-contained survey on Lipschitz truncations with applications to fluid dynamics and some more references can also be found in the recent book [14].

The main tool for the a-priori estimates in weighted spaces is the use of a divergence free truncation that is chosen relative to the weight. The technique is closely related to the so-called Lipschitz truncation method.

Our method include the following new refinement of the solenoidal Lipschitz truncation method introduced in [7].

**Theorem 2.6.** Let $p \in (1, \infty)$, let $\Omega \subset \mathbb{R}^3$ be an open, bounded subset with Lipschitz boundary and let $u \in W^{1,p}_{0,div}(\Omega)$. Then there exists a set $\mathcal{O} \subset \Omega$, with

$$|\mathcal{O}| \lesssim \lambda^{-p} \int_\Omega |\nabla u|^p \, dx$$

and a function $u_\lambda \in W^{1,\infty}_{0,div}(\Omega)$, such that $u(x) = u_\lambda(x)$ for all $x \in \mathcal{O}^c$. Additionally

$$\|\nabla u_\lambda\|_{L^\infty(\Omega)} \leq c\lambda$$

almost everywhere,

$$\int_\mathcal{O} |\nabla (u - u_\lambda)|^q \, dx \lesssim \lambda^{q-p} \int_\Omega |\nabla u|^p \, dx$$

and

$$\|u_\lambda - u\|_{L^q(\mathcal{O}^c \setminus \Omega)} \lesssim \|u\|_{L^q(\Omega)}$$

for all $q \in [1, p]$. The constants depend on $p$ and the domain only. Moreover, if $\nabla u \in L^p(\omega)$ with $\omega \in A_p$ we find

$$\int_\mathcal{O} |\nabla (u - u_\lambda)|^p \, \omega \, dx \lesssim c \int_\Omega |\nabla (u - u_\lambda)|^p \, \omega \, dx$$

with $c$ depending on the $A_p$ and the domain only.

**Remark 2.7.** We point out that, in contrast to earlier versions of the Lipschitz truncation, our truncation inherits both the solenoidality and the zero trace property of the Sobolev function. In addition to the usual $L^q$ estimates we provide weighted $L^p$ estimates as well. We mention that our result uses the techniques recently introduced in [14, 15, 16]. Roughly speaking we improve an inverse curl operator introduced in [5] with weighted estimates. Then its divergence will be zero and the appropriate estimates will be available. We think this improved weighted inverse curl might be of interest as well and its complete formulation is presented as Theorem 4.1.

3. **List of notations and basic tools**

3.1. **List of notations.** In the present work we use the following notations:

1. If $E \subset \mathbb{R}^n$ then $\chi_E$ denotes the characteristic function of $E$ that assigns 1 to each element of $E$ and otherwise is 0.
2. If $E$ is Lebesgue measurable we denote by $|E|$ its Lebesgue measure.
3. for a measurable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}_+$ and a measurable set $\Omega$: $\int_\Omega f(x) \, dx$ is the integral with respect to the Lebesgue measure and

$$\frac{1}{|\Omega|} \int_\Omega f(x) \, dx =: \int_\Omega f(x) \, dx =: \langle f \rangle_\Omega.$$

4. For a function $u : \Omega \to \mathbb{R}^n$ we define its symmetric gradient by

$$e_u := \frac{\nabla u + (\nabla u)^T}{2}.$$
(5) Throughout the paper we usually use the letter $B$ for a ball and $Q$ for a cube with sides parallel to the axis.

3.2. Basic tools. We state below basic notions and results that are further needed in the proofs of our results.

**Young’s inequality with $\varepsilon$.** The following (elementary) inequality will be used intensively:

$$ab \leq \varepsilon \frac{a^p}{p} + C(\varepsilon) \frac{b^{p'}}{p'} \text{ for all } a, b \geq 0, \varepsilon > 0, p > 1$$

where $p'$ denotes the H"older exponent associated to $p$, that is $\frac{1}{p'} := 1 - \frac{1}{p}$ and $C(\varepsilon)$ is a positive constant depending on $\varepsilon$.

**The Whitney covering.** We introduce the Whitney covering, which is a decomposition of a proper, open, nonempty set $\mathcal{O} \subset \mathbb{R}^d$ as a countable union of closed dyadic cubes. We use here a version which is present in [17] and then slightly modified in [11] and [7].

**Proposition 3.1.** Let $\mathcal{O}$ be an open and proper subset of $\mathbb{R}^d$. Then there exists a countable family $Q_i$ of closed, dyadic cubes such that:

(a) $\bigcup Q_i = \mathcal{O}$ and all the cubes $Q_i$ have disjoint interiors.

(b) $\operatorname{diam}(Q) < \operatorname{dist}(Q, \mathcal{O}^c) \leq 4\operatorname{diam}(Q)$.

(c) If $\emptyset \neq Q_i \cap Q_j$, then $\operatorname{diam}(Q_i) \in [\frac{1}{2}, 2]$.

(d) For given $Q_i$, there exists at most $4^d - 2^d$ cubes $Q_j$ touching $Q_i$ (boundaries intersect, but not the interiors).

(e) The family of cubes $\{\frac{1}{2}Q_i\}_{i \in \mathbb{N}}$ has finite intersection. The family can be split in $4^d - 2^d$ disjoint families.

(f) There is a partition of unity, $\psi_i \in C^\infty_c(\mathbb{R}^d)$, such that $\chi_{\frac{1}{2}Q_i} \leq \psi_i \leq \chi_{\frac{1}{4}Q_i}$ and $\operatorname{diam}(Q_i) \sqrt{\psi_i} \leq c(d)$ uniformly.

**Proof.** We shall first fix the notation; namely, for any $m \in \mathbb{Z}$ we denote by:

- $\mathcal{D}_m$ the set of all dyadic cubes of length $2^{-m}$
- $\mathcal{O}_m := \{x \in \mathcal{O}: 2\sqrt{2}^{-m} < \operatorname{dist}(x, \mathcal{O}^c) \leq 4\sqrt{2}^{-m}\}$
- $\mathcal{F}_m := \{Q \in \mathcal{D}_m: Q \cap \mathcal{O}_m \neq \emptyset\}$
- $\mathcal{F} := \bigcup_{m \in \mathbb{Z}} \mathcal{F}_m$

It is immediate to see that $\bigcup_{m \in \mathbb{Z}} \mathcal{O}_m = \mathcal{O}$ and, consequently $\bigcup_{m \in \mathbb{Z}} \mathcal{F} = \mathcal{F}$. The property (b) holds for any $Q \in \mathcal{F}'$ since if $Q \in \mathcal{F}$ and $x \in Q \cap \mathcal{O}_m$, then

$$\operatorname{diam}(Q) < \operatorname{dist}(x, \mathcal{O}^c) - \operatorname{diam}(Q) \leq \operatorname{dist}(Q, \mathcal{O}^c) \leq \operatorname{dist}(Q, \mathcal{O}^c) \leq 4\operatorname{diam}(Q).$$

In order to fulfill the condition a) we need to choose a subfamily of $\mathcal{F}'$ such that any two cubes in this new subfamily have disjoint interiors. Since two dyadic cubes have disjoint interiors or one contains the other we can define, for any $Q \in \mathcal{F}'$, $Q^{\text{max}}$ to be the maximal cube of $\mathcal{F}'$ that contains $Q$. Now set $\mathcal{F} := \{Q^{\text{max}}: Q \in \mathcal{F}'\}$. Then any two cubes in $\mathcal{F}$ have disjoint interiors by maximality. Now the conditions a) and b) are fulfilled.

To prove c), consider $Q, Q' \in \mathcal{F}$ with $Q \cap Q' \neq \emptyset$. We have

$$\operatorname{diam}(Q) < \operatorname{dist}(Q, \mathcal{O}^c) \leq \operatorname{dist}(Q, Q') + \operatorname{dist}(Q', \mathcal{O}^c) \leq 4\operatorname{diam}(Q')$$

If $l(Q) = 2^{-k}$ and $l(Q') = 2^{-l}$ we have that $-k < -l + 2$ or $-k \leq -l + 1$ and thus $\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \in [\frac{1}{2}, 2]$.

Now, following c) we notice that given $Q \in \mathcal{D}_m$ from $\mathcal{F}$, any cube from $\mathcal{F}$ that touches $Q$ contains at least one cube in $\mathcal{D}_{m+1}$ that touches $Q$. Thus the number of neighbours of $Q$ is at least $4^d - 2^d$ which is the number of cubes in $\mathcal{D}_{m+1}$ that touch $Q$.

Using the condition b) we can show that $\frac{1}{2}Q_i \subset \mathcal{O}$; otherwise there would exist $x \in \frac{1}{2}Q_i \cap \mathcal{O}^c$ and therefore

$$\operatorname{diam}(Q_i) < \operatorname{dist}(Q_i, \mathcal{O}^c) \leq \operatorname{dist}(Q_i, x) \leq \frac{1}{2} \operatorname{diam}(Q_i)$$

which is a contradiction. Thus $\bigcup_{i \in \mathbb{N}} \frac{1}{2}Q_i = \mathcal{O} = \bigcup_{i \in \mathbb{N}} Q_i$. We see that $\frac{1}{2}Q_i$ only intersects its neighbours-by-c) and therefore each $x \in Q_i$ is covered by at most $4d - 2^d$ elements of $\frac{1}{2}Q_k$. By c) we also see that $\frac{1}{2}Q_i$ and $\frac{1}{2}Q_j$ do not intersect if $i \neq j$. Therefore consider $\psi$ a smooth function such that it equals 1 on $[-1/2, 1/2]^d := Q$ and 0 outside $[-9/8, 9/8]^d := Q'$. Then we can define for any $k \in \mathbb{N}$, $\psi_k := \psi((x - c_k)/l(Q_k))$, where $c_k$ is the center of the cube $Q_k$ and $l(Q_k)$ its side-length. Finally we consider $\psi_\sigma := \sum_k \psi_k$. Since $\psi$ is a smooth function with compact support, it is uniformly bounded (and the same applies for any $\partial \psi$). This is the partition of unity we wanted.
Maximal function. The Hardy-Littlewood maximal operator is defined by
\[
L^1_{\text{loc}}(\Omega) \ni f \mapsto Mf(x) := \sup_{B(x; r)} \frac{1}{|B|} \int_B |f(y)| \chi_\Omega(y) \, dy \in [0, \infty]
\]
where the supremum is considered over all the open balls that contain \(x\). This definition is extended for vector-valued functions \(v \in L^1_{\text{loc}}(\Omega; \mathbb{R}^n)\) by \(M(v)(x) := M(|v|(x))\). Some basic properties of the maximal operator are contained in the next lemma and they can be found for example in [24].

Lemma 3.2. Let \(v \in L^1_{\text{loc}}(\mathbb{R}^n)\) and \(\lambda > 0\). Then the level-set
\[
\{x \in \mathbb{R}^n; |M(v)(x)| > \lambda\}
\]
is open:
(a) For all \(1 < p \leq \infty\) the following holds
\[
\|M(v)\|_{L^p(\mathbb{R}^n)} \leq c_p \|v\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in L^p(\mathbb{R}^n);
\]
(b) The following weak-type estimate holds
\[
\|\{x \in \mathbb{R}^n; |M(v)(x)| > \lambda\}\|_{L^{p,\infty}(\mathbb{R}^n)} \leq c_p \frac{\|v\|_{L^p(\mathbb{R}^n)}}{\lambda} \quad \text{for all } v \in L^p(\mathbb{R}^n).
\]

Weights and weighted spaces. The following notions that involve weights and weighted spaces are well known and we closely follow their exposure from [10, Section 3]. A function \(\omega : \mathbb{R}^n \rightarrow \mathbb{R}\) is called a weight if it is measurable, positive and finite almost everywhere. Given a weight \(\omega\) we can define the space
\[
L^p_\omega(\mathbb{R}^n) := \left\{ u : \Omega \rightarrow \mathbb{R}^n; \|f\|_{L^p_\omega} := \left( \int_\Omega |u(x)|^p \omega(x) \, dx \right)^{1/p} < \infty \right\}
\]
with \(1 \leq p < \infty\). Similarly we can define the following weighted Sobolev space:
\[
W^{k,q}_\omega(\mathbb{R}^n) := \left\{ u \in L^q_\omega(\mathbb{R}^n); \|u\|_{W^{k,q}_\omega(\Omega)} := \sum_{l=0}^k \|\nabla^l u\|_{L^q_\omega(\Omega)} < \infty \right\}
\]
As \(W^{k,q}_\omega(\mathbb{R}^n)\) we denote the closure of \(C_0^\infty(\mathbb{R}^n)\) with respect to the respective weighted Sobolev norm.

Muckenhoupt weights. We say that a weight belongs to the Muckenhoupt class \(A_p\) if and only if for every ball \(B \subset \mathbb{R}^n\) we have that
\[
\left( \int_B \omega(x) \, dx \right) \left( \int_B \omega^{-\alpha'(p'-1)}(x) \, dx \right)^{1/(p'-1)} \leq A \quad \text{if } p \in (1, \infty)
\]
or \(M \omega(x) \leq A \omega(x)\) if \(p = 1\). The smallest constant \(A\) for which these inequalities hold is called the Muckenhoupt constant and is denoted by \(A_\omega(\omega)\). One of the special features of these weights is contained in the seminal result due to B. Muckenhoupt [24]: if \(1 < p < \infty\) we have that \(\omega \in A_p\) if and only if there exists a constant \(A'\) such that for any \(f \in L^p(\mathbb{R})\) it follows that
\[
\int |Mf|^p \omega \, dx \leq A' \int |f|^p \omega \, dx.
\]
The following two lemmas contain useful properties for the Muckenhoupt weights that we will also need.

Lemma 3.3. [22, p. 5-6:] Let \(\omega \in A_p\) for some \(p \in [1, \infty)\). Then \(\omega \in A_q\) for all \(q \geq p\). Also \(\omega \in A_p\) is equivalent to \(\omega^{-\alpha'(p'-1)} \in A_{p'}\).

Lemma 3.4. [22, p. 5-6:] Let \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) such that \(Mf < \infty \) almost everywhere in \(\mathbb{R}^n\). Then for all \(\alpha \in (0, 1)\) we have that \((Mf)^\alpha \in A_1\). Furthermore, for all \(\alpha \in (0, 1)\), we have \((Mf)^{-\alpha(p-1)} \in A_p\).

Korn and Poincaré’s inequalities.

Theorem 3.5 (Korn, Poincaré). Let \(q \in (1, \infty)\) and \(\Omega\) be a bounded Lipschitz domain and \(u \in W^{1,p}_{0,\text{loc}}(\Omega; \mathbb{R}^d)\). We have
\[
\|u\|_{W^{1,\frac{p}{q}}(\Omega)} \leq c_1 \|\nabla u\|_{L^p(\Omega)} \leq c_2 \|\nabla u\|_{L^p(\Omega)}
\]
where \(c_1\) and \(c_2\) depend only on \(\Omega\) and \(q\).
An embedding theorem. The following result is contained, not explicitly however, in [27] at p. 25.

Theorem 3.6 (Muckenhoupt, Wheeden). If \( 1 < p < 3 \) we define the Sobolev exponent (in dimension 3) by \( p^* := \frac{3p}{3-p} \). Suppose \( \omega \in A_p^{\alpha'/\alpha} + 1 \). Then if \( u \in W_0^{1,3} (\Omega) \) with \( \nabla u \in L^{p^*} (\omega^{3-p/3}) (\Omega) \) we have

\[
\left( \int_{\Omega} |u|^p \omega \, dx \right)^{1/p} \leq c \left( p, A_p^{\alpha'/\alpha} (\omega) \right) \left( \int_{\Omega} |\nabla u|^{p^*} \omega^{3-p/3} \, dx \right)^{1/p}.
\]

Moreover, we find by [13, Theorem 5.1] for \( \Omega \) Lipschitz and \( \nabla u \in L^p (\Omega) \), that

\[
\int_{\Omega} |u - (u)_{\Omega}|^p \omega \, dx \leq c \left( p, A_p (\omega) \right) \int_{\Omega} |\nabla u|^p \omega \, dx.
\]

Weighted Korn inequality. We record here the following weighted version of Korn’s inequality. It appears in [13, Theorem 5.1].

Theorem 3.7. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain, \( q \in (1, \infty) \) and \( \omega \in A_q \). Then for all \( u \in W_0^{1,q} (\Omega) \) the following inequality holds:

\[
\|\nabla u\|_{L^q (\Omega)} \leq c \|\epsilon u\|_{L^q (\Omega)}
\]

where \( c \) only depends on \( q \) and \( A_q (\omega) \).

The Bogovski operator. The following theorem will be essential for proving the existence of the pressure, in Theorem 3.1. See [4] and [13, Theorem 5.2].

Theorem 3.8 (Bogovski). Let \( q \in (1, \infty) \) and \( \Omega \) be a bounded Lipschitz domain and \( \omega \in A_q \). Then there exists a bounded linear operator

\[
\text{Bog} : L^q_{0,\omega} (\Omega) \rightarrow W_0^{1,q} (\Omega)
\]

such that the system

\[
\begin{align*}
\text{div} \text{Bog} (g) &= g & \text{in } \Omega, \\
g &= 0 & \text{on } \partial \Omega
\end{align*}
\]

has a weak solution and for which we also have

\[
\|\text{Bog} (g)\|_{W^{1,q}_{0,\omega} (\Omega)} \leq c \|g\|_{L^q_{0,\omega} (\Omega)}
\]

for some positive constant \( c \) that only depends on \( p \) and \( A_q \). Here

\[
L^q_{0,\omega} (\Omega) := \left\{ f \in L^q_{0,\omega} (\Omega) : \int_{\Omega} f \, dx = 0 \right\}.
\]

Solenoidal, weighted, biting div–curl lemma. Finally, for the existence of solutions we need the following fundamental result that can be found in [8, Theorem 1.9].

Theorem 3.9 (solenoidal, weighted, biting div–curl lemma). Let \( \Omega \subset \mathbb{R}^n \) denote an open, bounded set. Assume that for a given \( q \in (1, \infty) \) and \( \omega \in A_q \), there is a sequence of measurable, tensor-valued functions \( a^k, s^k : \Omega \rightarrow \mathbb{R}^{n \times n}, k \in \mathbb{N} \), such that \( k \)-uniformly

\[
\|a^k\|_{L^q(\Omega)} + \|s^k\|_{L^q(\Omega)} \leq C.
\]

Furthermore, assume that for every bounded sequence \( \{a^k\}_{k=1}^\infty \) in \( W_0^{1,\infty} (\Omega) \) and for every bounded solenoidal sequence \( \{a^k\}_{k=1}^\infty \) in \( W_0^{1,\infty} (\Omega) \) such that

\[
\nabla a^k \rightharpoonup 0 \quad \text{weakly}^* \quad \text{in } L^\infty (\Omega),
\]

\[
\nabla d^k \rightharpoonup 0 \quad \text{weakly}^* \quad \text{in } L^\infty (\Omega)
\]

one has

\[
\lim_{k \to \infty} \int_{\Omega} \hat{s} : \nabla d^k \, dx = 0,
\]

\[
\lim_{k \to \infty} \int_{\Omega} a^k_{ij} \partial_{ij} c^k - a^k_{ij} \partial_{ij} c^k \, dx = 0 \quad \text{for all } i, j = 1, \ldots, n,
\]

and that

\[
\text{tr}(a^k) \quad \text{converges pointwisely almost everywhere in } \Omega.
\]
Then, there exists a (non-relabeled) subsequence \((a^k, b^k)\) and a non-decreasing sequence of measurable subsets \(\Omega_j \subset \Omega\), with \(|\Omega \setminus \Omega_j| \to 0\) as \(j \to \infty\), such that

\[
\begin{align*}
(a^k &\to a) \quad \text{weakly in } L^1(\Omega), \\
(s^k &\to s) \quad \text{weakly in } L^1(\Omega), \\
(a^k \cdot s^k \omega &\to a \cdot s \omega) \quad \text{weakly in } L^1(\Omega_j) \quad \text{for all } j \in \mathbb{N}.
\end{align*}
\]

4. Lipschitz Truncations & Relative Truncations

In the following section we construct a relative truncation \(u_p\) which is solenoidal and has zero boundary values. We follow the approach in [11]. If not specified otherwise we use the trivial extension by 0 of all functions to the whole-space without any further notice.

Let \(\Omega \subset \mathbb{R}^3\) be an open, bounded and Lipschitz domain. We consider \(W^{1,p}_0(\Omega)\), the closure in \(W^{1,p}(\Omega)\) of the set

\[
\{ \varphi | \varphi \in C_c^\infty(\Omega, \mathbb{R}^3), \text{div} \, \varphi = 0 \}.
\]

We will provide the following theorem of the inverse curl operator for weighted spaces. It seems to be an improvement to Lipschitz domains even in Lebesgue spaces [3 Corollary 2.3].

**Theorem 4.1.** Let \(p \in (1, \infty)\), let \(\Omega \subset \mathbb{R}^3\) be a bounded subset of \(\mathbb{R}^3\) with Lipschitz boundary and let \(u \in W^{1,p}_0(\Omega)\). There exist \(w \in W^{2,1}_0(\Omega)\) such that \(\text{curl} \, (w) = u\). In particular

\[
\int_\Omega |\nabla^2 w|^p \, dx \leq c \int_\Omega |\nabla u|^p \, dx,
\]

where \(c\) depends on \(p, n\) and the domain.

Moreover, if \(\omega \in \mathcal{A}_p\) and \(\nabla u \in L^p_0(\Omega)\), then there exists \(w \in W^{2,1}_0(\Omega)\) such that \(\text{curl} \, (w) = u\) and \(\nabla^2 w \in L^p_0(\Omega)\). In particular

\[
\int_\Omega |\nabla^2 w|^p \, d\omega \leq c \int_\Omega |\nabla u|^p \, d\omega,
\]

where \(c\) depends on \(p, n\), the \(\mathcal{A}_p\)-constant and the domain.

The proof relies on the following extension results [27 Theorem 2.1.13].

**Theorem 4.2.** Let \(p \in (1, \infty)\), let \(\Omega \subset \mathbb{R}^3\) be an open, bounded and Lipschitz domain. Let \(\omega \in \mathcal{A}_p\). Assume that \(\nabla^k g \in L^p_0(\Omega^c)\), then the there exists an extension \(E(g) \in W^{k,p}_0(\Omega)\), such that

\[
E(g) = g \text{ in } \Omega^c \quad \text{and } \|E(g)\|_{W^{k,p}_0(\Omega)} \leq c \|\nabla^k g\|_{L^p_0(\Omega^c)}.
\]

**Proof of Theorem 4.2.** We begin with the same strategy as in [3]. Let first \(u \in C^{0,0}_0(\Omega)\) (the general results follows then by density). We extend \(u\) by 0 to the whole space and take the global solution of the inverse curl of \(u\)

\[
\hat{w}(y) = \int_{\mathbb{R}^3} \frac{\text{curl}(u)(z)}{|y - z|} \, dz.
\]

The mapping \(\text{curl}(u) \mapsto \nabla^2 \hat{w}\) is a singular integral operator and hence (cf. [27])

\[
\|\nabla^2 \hat{w}\|_{L^p_0(\mathbb{R}^3)} \leq c \|\nabla u\|_{L^p_0(\Omega)}.
\]

It is easy to check that \(\text{curl} \, \hat{w} = u \chi_\Omega\). However \(\nabla \hat{w} \neq 0\) on \(\partial \Omega\). We will correct the boundary value with another singular operator on \(\Omega\). First, since \(\text{curl} \, \hat{w} = 0\) on \(\Omega^c\) the Helmholtz decomposition implies that there is a \(z \in W^{3,1}_{\text{loc}}(\mathbb{R}^3 \setminus \Omega)\) satisfying \(\nabla z(x) = \hat{w}(x)\) for all \(x \in \Omega^c\). Obviously

\[
\|\nabla^3 z\|_{L^p_0(\mathbb{R}^3 \setminus \Omega)} \leq \|\nabla^2 \hat{w}\|_{L^p_0(\mathbb{R}^3)} \leq c \|\nabla u\|_{W^{2,p}_0(\Omega)}.
\]

Since \(\Omega\) is bounded there exists \(R > 0\) such that \(\Omega \subset B_R(0)\). We consider a smooth function \(\eta\) that equals 1 in \(B_R(0)\), is 0 outside \(2B_R(0)\) and \(0 \leq \eta \leq 1\). Thus \(\eta z \in W^{3,1}_0(\Omega^c)\). Now we apply the extension operator from [27] so that \(E(\eta z) \in W^{3,1}_0(\mathbb{R}^3)\) and

\[
\|E(\eta z)\|_{W^{3,1}_0(2B_R(0))} \leq c \|\nabla u\|_{W^{2,p}_0(\Omega)}.
\]

We now set \(w := \hat{w} - \nabla E(\eta p)\). Consequently \(w = \nabla w = 0\) on \(\partial \Omega\). Furthermore, \(\text{curl}(w) = \text{curl}(\hat{w}) = u \chi_\Omega\) and \(w \in W^{2,1}_0(\Omega)\) by the above estimates.
We consider a function $u \in W^{1,p}_0(\Omega)$ and a domain $\Omega \subset \mathbb{R}^3$ which is open, bounded, with Lipschitz boundary. Then according to Theorem 4.1 there is a function $w \in W^{1,p}_0(\Omega)$ for which $\text{curl} w = u$. Without further notice we extend $w$ by zero outside $\Omega$.

Given $\mathcal{O}$ an open and proper set we can consider a Whitney covering and a related partition of unity as in the Proposition 3.1. Then we define

$$w_\mathcal{O} := \begin{cases} w(x) & x \in \mathbb{R}^3 \setminus \mathcal{O} \\ \sum \varphi_i w_i & x \in \mathcal{O} \end{cases}$$

where

$$w_i := \begin{cases} (\nabla w)_{\frac{1}{2}Q_i}(x-x_i) + \left(w - (\nabla w)_{\frac{1}{2}Q_i}(x-x_i)\right)_{\frac{1}{2}Q_i} & \text{if } \frac{1}{2}Q_i \subset \Omega \\ 0 & \text{else} \end{cases}$$

and

$$|\varphi_i| + r_i |\nabla \varphi_i| + r_i^2 |\nabla^2 \varphi_i| \leq c \quad \text{where } r_i := \text{diam}(Q_i).$$

Finally we define the set of neighbors of $Q_i$ as $A_i := \{j : \{\varphi_j(x) > 0 : x \in Q_i\} \neq \emptyset\}$. Such that

$$w_\mathcal{O}(x) = \sum_{j \in A_i} \varphi_i w_j \text{ for } x \in Q_i.$$

We now introduce the stability estimates for the relative truncation introduced above: In this section we prove the following theorem.

**Theorem 4.3 (Relative Truncation).** Let $\mathcal{O}$ a nonempty, open and proper subset of $\mathbb{R}^3$. Let $u \in W^{1,p}_{0,\text{div}}(\Omega)$ where $\Omega$ is an open, bounded and Lipschitz subset of $\mathbb{R}^3$. Let $\omega \in \mathcal{A}_p$ be a Muckenhoupt weight. Then there exists a function which we denote $u_\mathcal{O} \in W^{1,p}_{0,\text{div}}(\Omega)$ with the following properties:

\begin{align}
(4.1) & \quad u_\mathcal{O} = u \quad \text{on } \mathbb{R}^3 \setminus \mathcal{O} \\
(4.2) & \quad \int_{\Omega} |\nabla (u - u_\mathcal{O})|^p \, dx \leq c(p) \int_{\Omega} |\nabla u|^p \, dx \\
(4.3) & \quad \int_{\Omega} |\nabla (u - u_\mathcal{O})|^p \omega \, dx \leq c(p,A_p(\omega)) \int_{\Omega} |\nabla u|^p \omega \, dx \\
& \quad \text{and for } q < p \text{ and } \omega \in A_q \text{ we find} \\
(4.4) & \quad \int_{\Omega} |\nabla (u - u_\mathcal{O})|^q \omega \, dx \leq c(q,A_q(\omega)) \omega(\mathcal{O})^\frac{p}{p-q} \left(\int_{\Omega} |\nabla u|^p \omega \, dx\right)^\frac{q}{p}.
\end{align}

The theorem is a consequence of the following lemma.

**Lemma 4.4.** The following relations hold for $\omega \in A_p$

\begin{enumerate}[(a)]
\item \[ \int_{\frac{1}{2}Q_i} \frac{|w-w_j|^p}{r_i^p} \omega dx + \int_{\frac{1}{2}Q_i} \frac{|\nabla(w-w_j)|^p}{r_i} \omega dx \leq c(p,A_p) \int_{\frac{1}{2}Q_i} |\nabla^2 w|^p \omega dx. \]
\item If $|Q_i \cap Q_j| \neq 0$ then
\[ \int_{Q_i \cap Q_j} \frac{|w-w_j|^p}{r_j^p} \omega dx + \int_{Q_i \cap Q_j} \frac{|\nabla(w-w_j)|^p}{r_j} \omega dx \leq c(p,A_p) \int_{\frac{1}{2}Q_i} |\nabla^2 w|^p \omega dx + c(p,A_p) \int_{\frac{1}{2}Q_j} |\nabla^2 w|^p \omega dx. \]
\item If $|Q_i \cap Q_j| \neq 0$ then
\[ \left\|\frac{|w_j-w_i|}{r_i^2}\right\|_{L^p(Q_i \cap Q_j)} + \left\|\frac{|\nabla(w_j-w_i)|}{r_i^2}\right\|_{L^p(Q_i \cap Q_j)} \leq c \int_{\frac{1}{2}Q_i} |\nabla^2 w| \, dx + c \int_{\frac{1}{2}Q_j} |\nabla^2 w| \, dx. \]
\end{enumerate}

**Proof.** (a) We just apply weighted Poincare’s inequality (5.3) twice to obtain
\[ \int_{\frac{1}{2}Q_i} \frac{|w-w_j|^p}{r_i^p} \omega dx \leq c(p) \int_{\frac{1}{2}Q_i} \frac{|\nabla(w-w_j)|^p}{r_i} \omega dx. \]

(b) Follows by applying the triangle’s inequality and (a).
We now consider \( u_\varepsilon := \text{curl} w_\varepsilon \). To conclude the proof of Theorem 4.3 we have to check that the properties (4.1)–(4.3) are fulfilled. To this end notice that (4.1) is immediate as the relation \( \text{div} u_\varepsilon \equiv 0 \) follows by construction. The zero boundary values are due to the fact that \( w_\varepsilon \equiv 0 \) outside \( \Omega \). To prove (4.2) we estimate

\[
\int_{\Omega} |\nabla (u - u_\varepsilon)|^p \, dx = \int_{\Omega \cap \varepsilon} |\nabla \text{curl} (w - w_\varepsilon)|^p \, dx \\
= \sum_i \int_{\Omega_i} |\nabla \text{curl} \left( w - \sum_{j \in A_i} w_j \phi_j \right)|^p \, dx \\
= \sum_i \int_{\Omega_i} |\nabla \text{curl} \left( \sum_{j \in A_i} (w - w_j) \phi_j \right)|^p \, dx \\
\leq c(p) \sum_i \int_{\Omega_i} |\nabla^2 \left( \sum_{j \in A_i} (w - w_j) \phi_j \right)|^p \, dx \\
\leq c(p) \sum_i \sum_{j \in A_i} \int_{\Omega \cap \varepsilon Q_j} |\nabla^2 ((w - w_j) \phi_j)|^p \, dx.
\]

Further, we estimate pointwisely

\[
|\nabla^2 ((w - w_j) \phi_j)|^p \leq \left( |\nabla^2 w| \cdot |\phi_j| + |\nabla (w - w_j) \cdot | \nabla \phi_j| + |w - w_j| \cdot |\nabla^2 \phi_j| \right)^p
\]

(4.5)

\[
\leq c(p) \left( |\nabla^2 w|^p + \frac{|\nabla (w - w_j)|^p}{r_j^p} + \frac{|w - w_j|^p}{r_j^p} \right).
\]

Then we find by the previous lemma to obtain

\[
\int_{\Omega} |\nabla (u - u_\varepsilon)|^p \, dx \leq \sum_i \sum_{j \in A_i} c(p) \left( \int_{\Omega \cap \varepsilon Q_j} |\nabla^2 w|^p \, dx + \int_{\varepsilon Q_j} |\nabla^2 w|^p \, dx \right) \\
\leq c(p) \int_{\varepsilon \cap \Omega} |\nabla^2 w|^p \, dx \\
\leq c(p) \int_{\Omega} |\nabla u|^p \, dx.
\]

By using the weighted estimates we find in the same manner

\[
\int_{\Omega} |\nabla (u - u_\varepsilon)|^p \, \omega \, dx \leq c(p) \int_{\varepsilon \cap \Omega} |\nabla^2 w|^p \, \omega \, dx \\
\leq c(p, A_p(\omega)) \int_{\Omega} |\nabla u|^p \, \omega \, dx
\]

which concludes (4.3) for (4.4) we take \( q < p \) and find by the previous and Hölder’s inequality that

\[
\int_{\Omega} |\nabla (u - u_\varepsilon)|^q \, \omega \, dx \leq \int_{\varepsilon \cap \Omega} |\nabla^2 w|^q \, \omega \, dx \\
\leq \omega(\varepsilon)^{\frac{p-q}{p}} c(q) \left( \int_{\varepsilon \cap \Omega} |\nabla^2 w|^p \, \omega \, dx \right)^{\frac{q}{p}}
\]

which ends the proof.

**Remark 4.5.** The previous result leads to the following: Let \( u \in W^{1, p}_{0, \text{div}}(\Omega) \) be a weak solution to the system from the Theorem 4.1, i.e.

\[
\int_{\Omega} A(\varepsilon u) \cdot \varepsilon \phi \, dx = \int_{\Omega} f \cdot \nabla \phi \quad \text{for all} \ \phi \in W^{1, p}_{0, \text{div}}(\Omega).
\]
Then we can use $w_{\lambda}$ as a test function for any open and proper set $\Omega$. A suitable choice for the set $\Omega$ will be made for the a-priori estimates.

4.1. **Proof of Theorem 2.6.** We will follow the approach from [7, p. 27-28]. Now for $\lambda > 0$ we define

$$\{ M (\nabla^2 w_{\Omega}) > \lambda \} =: \Omega_{\lambda}$$

where $M$ is the Hardy-Littlewood maximal operator. The set $\Omega_{\lambda}$ is the so-called "bad" set, where the singularities if the function $w$ are contained. We define $w_{\lambda} := w_{\Omega_{\lambda}}$ defined via Theorem 4.3 and the solenoidal Lipschitz truncation of $u$ as

$$u_{\lambda} := \text{curl} (w_{\lambda}).$$

**Proof of Theorem 2.6.** We have that div $u_{\lambda} = \text{div curl} (w_{\lambda}) = 0$ in $\Omega$. According to Theorem 4.3 we have that $\nabla w_{\lambda} = 0$ on $\partial \Omega$, so $u_{\lambda} = 0$ on $\partial \Omega$.

For $j \in \mathbb{N}$ we find that (by Proposition 3.1) $Q_j \subset 16 Q_j$ and $16 Q_j \cap \partial \Omega_{\lambda} \neq \emptyset$. It follows that $\int_{16 Q_j} |\nabla^2 w| \, dx \leq \lambda$ and then $\int_{2 Q_j} |\nabla^2 w| \, dx \leq c \lambda$. Hence for $x \in \partial \Omega_{\lambda}$ using the assumptions on $\psi_j$ and Lemma 4.4 that

$$|\nabla^2 w(x)| \leq \left| \sum_{\lambda \in \lambda_j} \nabla^2 (\psi_j (w_j - w)) \right| \leq c \lambda.$$ 

Hence

$$|\nabla u_{\lambda}| \leq 2 |\nabla w_{\lambda}| \leq c \lambda.$$ 

Now due to the weak $L^1$ estimate for Maximal functions and the $L^p$ bounds of $\nabla^2 w$ we find that

$$|\partial \Omega| \lesssim \lambda^{-p} \int_\Omega |u|^p \, dx.$$ 

Hence the gradient bounds in $L^q_\Omega$ and in $L^q$ follow directly from Theorem 4.3. Finally we would like to prove that $\|u_{\lambda}\|_{L^q(\Omega)} \leq c \|u\|_{L^q(\Omega)}$ if $u \in L^q(\Omega)$. In order to prove this, notice that the only relevant situation is to prove the estimates under $L^q (\partial \Omega_{\lambda} \cap \Omega)$. Now because $\sum_j \psi_j = 1$ it is immediate to check that $u_{\lambda} - u = \sum_{j \in \mathbb{N}} \text{curl} (\psi_j (w_j - w))$ and then

$$|u_{\lambda} - u| \leq \sum_j \left| \text{curl} (\psi_j (w_j - w)) \right| \leq \sum_j \left| \nabla (\psi_j (w_j - w)) \right|$$

$$\leq \sum_j \left( |\nabla \phi_j| |w_j - w_j - w_j| + |\phi_j| |\nabla (w_j - w_j)| \right)$$

$$\leq \sum_j \left( \frac{|w_j - w_j|}{r_j} + |\nabla (w_j - w_j)| \right)$$

If we integrate the last inequality we will obtain that

$$\|u_{\lambda} - u\|_{L^q_\Omega} \lesssim \sum_j \int_{Q_j \cap \Omega} \frac{|w_j - w_j|}{r_j} + |\nabla (w_j - w_j)| \, dx \lesssim \sum_j \int_{Q_j \cap \Omega} |\nabla w_j| \, dx \lesssim \|u\|_{L^q(\Omega)};$$

so $\|u_{\lambda}\|_{L^q(\Omega)} \leq c \|u\|_{L^q(\Omega)}$.

5. **A-priori estimates**

5.1. **A-priori estimates for (1.6).** In this section we introduce a-priori bounds of solutions in $L^q$ spaces, with the $q$ below the natural duality exponent $p$. Since we follow the approach of [1,1] in the non-Newtonian setting, a-priori estimates in $L^q$ spaces are not sufficient to prove existence. Indeed, in order to apply the existence machinery developed in [3] estimates in weigthed $L^p$ spaces are necessary:

**Theorem 5.1.** Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain which is open, and bounded, $A$ satisfying (1.2)-(1.4) and $h \in L^1(\Omega)$ be an a.e. positive function. Then there exists $\varepsilon_0 > 0$ depending on $\Omega, C_1, C_2, C_3$ and $p$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the following holds: If $f \in L^{p'}(\Omega) \cap L^{p'}((Mh)^{1 - \varepsilon})$ and $u \in W^{1, p}_{0, \text{div}}(\Omega)$ is a weak solution for (1.6)
then necessarily $\nabla u \in L^p_{(M\delta)^{1-c}}(\Omega)$ and moreover

$$\int_{\Omega} |\nabla u|^p \, dx \leq c(p, \Omega, C_1, C_2, C_3) \int_{\Omega} \frac{|f|^p + C_1(p, \Omega)}{(M\delta)^{c}} \, dx.$$  

Proof. We set $g := h\chi_{\Omega} + \delta$; this function will approximate $h$ but for $g$ we have that $Mg > \delta$ and so $f, \nabla u \in L^p_{(M\delta)^{1-c}}(\Omega)$ a priori. This fact will be very important in the end of the proof. For any $\lambda > 0$ we define $\mathcal{O}(\lambda) := \{ x \in \mathbb{R}^d : Mg(x) > \lambda \}$. This is an open set because of the sub-linearity of the Maximal operator and if $\mathcal{O}(\lambda) = \mathbb{R}^d$ then we set $u_{\Theta(\lambda)} = 0$. But since $\mathcal{O}(\lambda)$ is a proper open set we may construct the relative truncation $u_{\Theta(\lambda)} \in W^{1,p}_{0, \text{div}}(\Omega)$ by Theorem 4.3 and use it as a test function. We obtain

$$\int_{\Omega} A(\epsilon u) \cdot \epsilon u_{\Theta(\lambda)} \, dx = \int_{\Omega} f \cdot \nabla u_{\Theta(\lambda)} \, dx$$

hence by (1.2)

$$c \int_{\{Mg \leq \lambda\}} |\epsilon u|^p \, dx \leq \int_{\{Mg \leq \lambda\}} f \cdot \nabla u \, dx + \int_{\{Mg > \lambda\}} f \cdot \nabla u_{\Theta(\lambda)} \, dx - \int_{\{Mg > \lambda\}} A(\epsilon u) \cdot \epsilon u_{\Theta(\lambda)} \, dx$$

which implies (using the elementary pointwise inequality $|\epsilon u| \leq |\nabla u|$ and (1.3)) that

$$\int_{\{Mg \leq \lambda\}} |\epsilon u|^p \, dx \leq c \int_{\{Mg \leq \lambda\}} |f| \nabla u \, dx + c \int_{\{Mg > \lambda\}} |f| \nabla u_{\Theta(\lambda)} \, dx + c \int_{\{Mg > \lambda\}} (|\nabla u|^p + 1) |\nabla u_{\Theta(\lambda)}| \, dx.$$  

Now let $G \in L^p(\Omega)$. Then

\begin{align*}
\langle 1 \rangle := & \int_{\{Mg > \lambda\}} |G| \nabla u_{\Theta(\lambda)} \, dx \\
& \leq 2 \sum_i \sum_{j \in A_i} \int_{Q_i} |G| |\nabla (\varphi_j w_j)| \, dx \\
& \leq 2 \sum_i \sum_{j \in A_i} \int_{Q_i} |G| \nabla^2 ((w_j - w_i) \varphi_j) \, dx \\
\end{align*}

where on the last inequality we used the partition of unity property. Now for $x \in Q_i$, we find

\begin{align*}
|\nabla^2 ((w_j - w_i) \varphi_j)| & \leq |\nabla (w_j(x) - w_i(x))| |\nabla \varphi_j(x)| + |w_j(x) - w_i(x)| |\nabla \varphi_j^2(x)| \\
& \leq \frac{\|\nabla (w_j - w_i)\|_{L^2(Q_i \cap Q_j)}}{r_j^2} + \frac{|w_j - w_i|_{L^2(Q_i \cap Q_j)}}{r_j^2} \\
\end{align*}

to pointwise and if we integrate this on $Q_i$ we have by Lemma 4.4 H"older’s inequality and Proposition 3.1 (c)

\begin{align*}
\langle 1 \rangle \leq & c \sum_i \sum_{j \in A_i} |Q_i| \int_{Q_i} |G| \, dx \left( \int_{Q_i} \nabla^2 w \, dx + \int_{Q_i} \nabla^2 w \, dx \right) \\
= & c \sum_i \sum_{j \in A_i} |Q_i| \int_{Q_i} \left| G \right| ((Mg)^p)^{\frac{1}{p}} \left( \int_{Q_i} \frac{|\nabla^2 w|^p}{(Mg)^p} \, dx \right)^{\frac{1}{p}} \left( \int_{Q_i} \frac{|\nabla^2 w|^p}{(Mg)^p} \, dx \right)^{\frac{1}{p}} \\
= & c \sum_i \sum_{j \in A_i} |Q_i| \left( \int_{Q_i} \left| G \right| \frac{1}{(Mg)^p} \, dx \right)^{\frac{1}{p}} \left( \int_{Q_i} \frac{|\nabla^2 w|^p}{(Mg)^p} \, dx \right)^{\frac{1}{p}} \\
& \times \left( \int_{Q_i} \frac{|\nabla^2 w|^p}{(Mg)^p} \, dx \right)^{\frac{1}{p}} \\
\end{align*}

We now estimate $\int_{Q_i} (Mg)^{\alpha} \, dx$. By Lemma 3.3 it follows that $(Mg)^{\alpha} \in A_1$ and hence $M(Mg)^{\alpha} \leq c(\alpha)(Mg)^{\alpha}$. Since by the Whitney covering (Proposition 3.1 (b)) we find $8Q_i \cap \mathcal{O} = \emptyset$ it follows that if $x_0$ belongs to this intersection then

$$\int_{9Q_i} (Mg)^{\alpha} \, dx \leq (Mg)^{\alpha} (x_0) \leq c(\alpha) \lambda^{\alpha}$$

and thus

$$\int_{Q_i} (Mg)^{\alpha} \, dx \leq c(\alpha) \lambda^{\alpha}.$$
Observe that by Young’s inequality for any $i \in \mathbb{N}$ and any $j \in A_i$,
\[
\lambda^\alpha |Q_i| \left( \int \frac{|G|^p}{Q_i} \frac{d\lambda}{(M G)^{\frac{ap}{p}}} dx \right)^\frac{1}{p} \leq c(\alpha) |Q_i| \left( \int \frac{|G|^p}{Q_i} \frac{\lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} dx + \int \frac{|\nabla W|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} dx \right) \]
which implies by summing over all such $i$ and $j$ that
\[
\int_{\{ M < \lambda \}} |G| |\nabla u_{\theta(\lambda)}| \, dx \leq c(\Omega, \alpha) \int_{\{ M > \lambda \} \cap \Omega} \frac{|G|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} + \frac{|\nabla W|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} \, dx.
\]
By choosing $G \in \{ |f| + 1, |u|^p \}$ we obtain
\[
\int_{\{ M < \lambda \}} (|f| + 1) |\nabla u_{\theta(\lambda)}| \, dx \leq c(\Omega, \alpha) \int_{\{ M > \lambda \} \cap \Omega} \frac{(1 + |f|^p) \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} + \frac{|\nabla W|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} \, dx
\]
and
\[
\int_{\{ M > \lambda \}} |u|^p |\nabla u_{\theta(\lambda)}| \, dx \leq c(\Omega, \alpha) \int_{\{ M > \lambda \} \cap \Omega} \frac{|\nabla u|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} + \frac{|\nabla W|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} \, dx.
\]
If we add them and use (5.11), it follows that
\[
\int_{\{ M < \lambda \}} |u|^p \, dx \leq c \int_{\{ M < \lambda \}} |f| |\nabla u| + 1 \, dx + c(\Omega, \alpha) \int_{\{ M > \lambda \} \cap \Omega} \frac{(1 + |f|^p + |\nabla u|^p) \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} + \frac{|\nabla W|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} \, dx.
\]
Let us set $(p - 1) := \min \{ p - 1, (p - 1)^{-1} \}$. We multiply the above inequality by $\lambda^{-1-\varepsilon}$ with $\varepsilon \in (0, p - 1)$ and integrate over $\lambda \in (0, \infty)$ to obtain
\[
(5.2) \quad I_0(\varepsilon) \leq I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon)
\]
where
\[
I_0(\varepsilon) := \int_0^\infty \int_{\{ M < \lambda \}} |u|^p \, dx \, d\lambda \lambda^{-1-\varepsilon} \, d\lambda
\]
\[
I_1(\varepsilon) := c \int_0^\infty \int_{\{ M < \lambda \}} |f| |\nabla u| + 1 \, dx \, d\lambda \lambda^{-1-\varepsilon} \, d\lambda
\]
\[
I_2(\varepsilon) := c(\Omega, \alpha) \int_0^\infty \int_{\{ M > \lambda \} \cap \Omega} \left( 1 + |f|^p + |\nabla u|^p \right) \lambda^{\frac{ap}{p}} \lambda^{-1-\varepsilon} \, d\lambda \, d\lambda
\]
\[
I_3(\varepsilon) := c(\Omega, \alpha) \int_0^\infty \int_{\{ M > \lambda \} \cap \Omega} \frac{|\nabla W|^p \lambda^{\frac{ap}{p}}}{(M G)^{\frac{ap}{p}}} \lambda^{-1-\varepsilon} \, d\lambda \, d\lambda.
\]
We apply Fubini’s theorem several times to obtain
\[
I_0(\varepsilon) = \int_\Omega \int_{M(\varepsilon)} \lambda^{-1-\varepsilon} \, d\lambda \, |\nabla u|^p \, dx = \frac{1}{\varepsilon} \int_\Omega \frac{|\nabla u|^p}{(M G)^{\frac{ap}{p}}} \, dx
\]
and
\[
I_1(\varepsilon) = \frac{c}{\varepsilon} \int_\Omega \frac{1 + |f| |\nabla u|}{(M G)^{\frac{ap}{p}}} \, dx.
\]
Also
\[
I_2(\varepsilon) = c(\Omega, \alpha) \frac{1}{\varepsilon} \int_\Omega \frac{1 + |f|^p + |\nabla u|^p}{(M G)^{\frac{ap}{p}}} \, dx
\]
and

\[ I_3 (\epsilon) = c (\Omega, \alpha) \frac{1}{\frac{p}{p'} - \epsilon} \int_\Omega |\nabla^2 w|^p dx. \]

Therefore (5.2) becomes

\[ \int_\Omega \frac{|\nabla u|^p}{(Mg)^{p-\epsilon}} dx \leq c \int_\Omega \frac{1 + |f||\nabla u|}{(Mg)^{p-\epsilon}} dx + \epsilon c (\Omega, \alpha, \alpha, \alpha, p) \left( \int_\Omega \frac{1 + |f|^p + |\nabla u|^p}{(Mg)^{p-\epsilon}} dx \right) \]

after applying the weighted Korn inequality (Theorem 5.2) and the fact that \( \epsilon < (p-1) \). We point out that \( (Mg)^{-\epsilon} = (Mg)^{-(p-1)-\epsilon} \in \mathcal{A}_p \) by using Lemma [4.0] since \( \frac{\alpha}{p-1} \in (0, 1) \). Further, by applying Young’s inequality we obtain

\[ \int_\Omega \frac{|\nabla u|^p}{(Mg)^{p-\epsilon}} dx \leq c (p, \epsilon) \int_\Omega \frac{1 + |f|^p + |\nabla u|^p}{(Mg)^{p-\epsilon}} dx \]

and we notice that the term involving \(|\nabla u|^p\) can be absorbed into the left hand side if \( \epsilon_0 \) is chosen small enough. Actually this is possible, since \( \lim_{\epsilon \to 0} \epsilon c (\Omega, a, p) = 0 \). The argument is concluded by taking the limit when \( \delta \to 0 \) and applying the Monotone Convergence Theorem.

Please observe that the assumption \( u \in W^{1,p}(\Omega) \) is only formal.

**Corollary 5.2.** Let \( p \in (1, \infty) \), \( \Omega \) a bounded, open and Lipschitz and \( A \) satisfying (1.2)–(1.4), then there exists an \( \epsilon > 0 \) depending on \( \Omega, C_1, C_2, C_3 \) and \( p \) such that if \( q \in [p', q') \) and \( f \in L^q \cap L^p (\Omega, \mathbb{R}^{3 \times 3}) \) and \( (u, \pi) \in W^{1,p}_0 (\Omega; \mathbb{R}^3) \times L^p_0 (\Omega; \mathbb{R}) \) a solution to (1.6). Then

\[ \int_\Omega |\nabla u|^q (p-1) dx \leq c (C_1, C_2, C_3, p, q, \Omega) \left( \int_\Omega |f|^q dx + 1 \right) \]

and

\[ \int_\Omega |\nabla u|^p M (|f| + 1)^{q - p'} dx \leq c (C_1, C_2, C_3, p, q, \Omega) \left( \int_\Omega |f|^q dx + 1 \right) \]

for some positive constant \( c (C_1, C_2, C_3, p, q, \Omega) > 0 \).

**Proof.** We apply Theorem 5.1 for \( \epsilon = p - q \in (0, \epsilon_0) \) and \( \omega := M (|f| + 1)^{-\epsilon} \)

\[ \int_\Omega |\nabla u|^p w dx \leq c \int_\Omega \left( |f|^p + 1 \right) w dx \]

\[ \leq c \int_\Omega |f|^p + M (|f|)^{-\epsilon} + c_1 \cdot 1 dx \]

\[ \leq c \int_\Omega M (|f|)^q + c_1 dx \]

\[ \leq c \int_\Omega |f|^q + c_1 dx \]

where the last inequality makes use of the continuity of the maximal operator. On the other hand we can apply Young’s inequality with \( \frac{q}{p'} + \frac{q'}{p} = 1 \) and the continuity of the maximal operator to obtain

\[ \int_\Omega |\nabla u|^q (p-1) dx = \int_\Omega \frac{|\nabla u|^q (p-1)}{M (|f| + 1)^{\frac{q}{p'}}} M (|f| + 1)^{\frac{q'}{p'}} dx \]

\[ \leq \int_\Omega M (|f| + 1)^\frac{q}{p'} dx + \int_\Omega M (|f| + 1)^q dx \]

\[ \leq c \int_\Omega |f|^q dx + c c_2. \]
5.2. A-priori estimates for (1.1)—the case \( p > 2 \).

**Proposition 5.3.** Let \( p \in (2, \infty) \), \( \Omega \) be bounded, open and Lipschitz and \( A \) satisfying (1.3), (1.4) and (2.4), then there exists an \( \varepsilon > 0 \) depending on \( \Omega \) and \( p \) such that if \( q \in [p' - \varepsilon, p'] \) and \( f \in L^{q'}(\Omega; \mathbb{R}^3) \) and \((u, \pi) \in W_{0, \text{div}}^{1,p}(\Omega; \mathbb{R}^3) \times L^p_{0}(\mathbb{R}; \mathbb{R})\) a solution to (1.1). Then

\[
\int |\nabla u|^q \, dx < c \int |f|^q + 1 \, dx
\]

for \( \alpha \) defined in (2.4).

**Proof.** The basic idea is to split the problem into a Stokes part which is contained in the error.

Next we observe that formally

\[
(5.5)
\]

Next we solve the following auxiliary problem (which exists due to the assumption that \( f \in L^{p'} \)

\[
\begin{cases}
-\text{div} A(x, \varepsilon v(x)) + \nabla \pi(x) = -\text{div} f(x) & \text{in } \Omega \\
\text{div} \, v = 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega
\end{cases}
\]

For \( q \in (p' - \varepsilon, p') \) we consider

\[
\omega = M(||f||_{\varepsilon \Omega} + 1)^{q-p'}.
\]

Corollary 5.2 implies the existence of a \( v \) that satisfies

\[
(5.4)
\]

Next we observe, that formally

\[
(5.6)
\]

we find by (2.3) that

\[
\begin{align*}
\int |\varepsilon u - \varepsilon v|^p \, dx &\lesssim \int (A(x, \varepsilon u) - A(x, \varepsilon v)) \cdot (u - v) \, dx + 1 \\
&= \int u \otimes u \cdot \nabla (u - v) \, dx + 1
\end{align*}
\]

This implies by the structure of the convective term, Young’s inequality and Korn’s inequality (Theorem 3.7) we find for \( \delta \in (0, 1) \)

\[
\begin{align*}
\langle u \otimes u, \nabla (u - v) \rangle &= \langle u \otimes (u - v), \nabla (u - v) \rangle + \langle u \otimes v, \nabla (u - v) \rangle \\
&= \int u \cdot \nabla |u - v|^2 \, dx + \langle u \otimes v, \nabla (u - v) \rangle \\
&\leq \delta \int |u - v|^p \, dx + c_\delta \int |u \otimes v|^p' \, dx.
\end{align*}
\]

This implies (by absorption) that

\[
\int |\varepsilon u - \varepsilon v|^p \, dx \lesssim \int |u \otimes v|^{p'} \, dx.
\]

Let \( \frac{2p'}{p'} = 1 - \gamma > 0 \). We now want to apply Theorem 3.6. For this, we would need to have \( \omega^{(3-p)/3} \in A_{p^*/p'+1} \). Here we need to notice that the weight \( \omega \) is defined via the Hardy-Littlewood maximal function and using Lemma 3.4 it is enough to check if

\[
(q - p') \frac{3-p}{3} = -\alpha \frac{p^*}{p'} \quad \text{for some } \alpha \in (0, 1).
\]

But by a simple computation we obtain

\[
\alpha = (p' - q) \frac{3-p}{3} = \frac{p'}{p' - 1} \in (0, 1).
\]
Consequently it follows that

\begin{equation}
\left( \int_{\Omega} |u|^b \omega^{3/(3-p)} \, dx \right)^{1/p} \leq c \left( p, A_{p'/(p'+1)} \left( \omega^{3/(3-p)} \right) \right) \left( \int_{\Omega} |\nabla u|^p \omega \, dx \right)^{1/p}.
\end{equation}

Now we define \( b = p \) and so \( b^* = p^* = \frac{3p}{p} \) for \( p < 3 \) and \( b = 2 \) and so \( b^* = 2 = 6 > 2p' \) for \( p \geq 3 \) and \( \gamma \) by \( \frac{2p'}{p^*} + \gamma = 1 \). Please observe that this is possible since for \( p < 3 \)

\[ \frac{2p'}{p^*} = \frac{2p(3-p)}{3p(p-1)} < 1 \] (which is possible for all \( p > \frac{9}{5} \)).

By Hölder’s inequality and Theorem 3.6 we obtain

\begin{equation}
\int_{\Omega} |u \otimes v|^{p'} \, dx = \int_{\Omega} |u \otimes v|^{p} \frac{\omega^{\frac{2p'}{p}}}{\omega^{\frac{p}{p'}}} \, dx \\
\leq \left( \int_{\Omega} |u|^b \omega^\frac{3}{3-p} \, dx \right)^{\frac{p'}{2p}} \left( \int_{\Omega} |v|^b \omega^\frac{3}{3-p} \, dx \right)^{\frac{p}{2p}} \left( \int_{\Omega} \omega^{\frac{2p'}{p}} \, dx \right)^{\gamma}.
\end{equation}

(5.8)

Since \( p > 2 \) we assume that \( p' - q \) is small enough, such that \( \frac{2(p' - q)p'}{p} \leq q \), in which case we find that

\[ \left( \int \omega^{\frac{2p'}{p}} \, dx \right)^{\gamma} \leq C \left( \int (M(|f|\chi_{\Omega} + 1))^{\frac{2(p' - q)p'}{p}} \, dx \right)^{\gamma} \leq C \left( \int |f|^q + 1 \, dx \right)^{\frac{2(p' - q)p'}{p}}, \]

with \( C \) depending on \( p, q \) and \( \Omega \) but independent of \( f \). Hence (for \( p > 2 \)) we find by (5.5) and Young’s inequality that

\[ \int_{\Omega} |u \otimes v|^{p'} \, dx \leq \delta \int_{\Omega} |\nabla u|^{p} \omega \, dx + \delta \int_{\Omega} |\nabla |f|^q + 1 \, dx \]

\[ \leq \delta \int_{\Omega} |\nabla u|^{p} \omega \, dx + \delta \int_{\Omega} |f|^q + 1 \, dx \]

And so, by Korn’s inequality, Young’s inequality Theorem 3.7 (2.4) and (5.5)

\[ \int_{\Omega} |\nabla u|^{p} \omega \, dx \leq \int_{\Omega} |\mathbf{e}_n - \mathbf{v}|^{p} \, dx + \int_{\Omega} |\nabla |f|^q | \omega \, dx \]

\[ \leq \int_{\Omega} |u \otimes v|^{p'} \, dx + \int_{\Omega} |f|^q + 1 \, dx \]

\[ \leq \delta \int_{\Omega} |\nabla u|^{p} \omega \, dx + \delta \int_{\Omega} |f|^q + 1 \, dx \]

\[ \geq \delta \int_{\Omega} |\nabla u|^{p} \omega \, dx + c \delta \int_{\Omega} |f|^q + 1 \, dx \]

\[ \geq \delta \int_{\Omega} |\nabla u|^{p} \omega \, dx + c \delta \int_{\Omega} |f|^q + 1 \, dx \]

\[ \geq \delta \int_{\Omega} |f|^q + 1 \, dx \]

This implies the uniform bound by absorption. Finally the \( W^{1,q/(p-1)}(\Omega) \)-bound follows as in (5.3).

\[ \text{\underline{5.3. A-priori estimates for (1.1) -- the case } p = 2.} \]

\[ \text{Proposition 5.4. Let } p = 2, \Omega \text{ a bounded, open and } C^1 \text{ and } A \text{ satisfying (1.4) and (5.3) (with } p = 2) \text{ and (2.5), then for } q \in [\frac{1}{4}, 2) \text{ and } f \in L^q \cap L^2(\Omega, \mathbb{R}^{3 \times 3}) \text{ and } (u, \pi) \in W^{1,2}_{0,dx} (\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}) \text{ a solution to (1.1). Then} \]

\[ \int_{\Omega} |\nabla u|^q + |\nabla u|^2 M(|f| + 1)^{q-2} \, dx \leq C \]

\[ \text{with } C' \text{ depending on } \| f \|_{L^q(\Omega)}, C_1, C_2, C_3 \text{ and the linear at infinity condition.} \]

\[ \text{\footnote{Please observe that we may assume in case } p > 3 \text{ that } p' - q \text{ is small enough such that } \omega \in A_2.} \]
Proof. The basic idea is to split the problem into a large part which is contained in $W^{1,p}(\Omega)$ and a small very-weak part. Let $\delta \in (0, \frac{1}{2})$. By the assumption (2.5) there is a $K > 2$, such that
\[ |vz - A(x,z)| + |v - D_z A(x,z)| \leq \delta \text{ for all } |z| \geq K \text{ and all } x \in \Omega. \]
We define $\varphi \in C^2([0,\infty), [0,1])$, such that $\chi_{(K/2,\infty)} \leq \varphi \leq \chi_{(K,\infty)}$ and $\varphi' \leq 1$
\[ \bar{A}(x,z) = vz + \varphi(|z|) (A(x,z) - vz) \]
Please observe that $\bar{A}$ satisfies (1.3) with $p = 2$ and (2.5). Moreover it satisfies (2.3) for $p = 2$ and $C_3 = 0$, since in case $|z_1| \geq |z_2|$
\[ (\bar{A}(x,z_1) - \bar{A}(x,z_2)) \cdot (z_1 - z_2) = |v| z_1 - z_2|^2 + \left( \varphi(|z_1|)(A(x,z_1) - vz_1) - \varphi(|z_2|)(A(x,z_2) - vz_2) \right) \cdot (z_1 - z_2) \]
\[ = |v| z_1 - z_2|^2 + \varphi(|z_1|) (A(x,z_1) - vz_1 - A(x,z_2) + vz_2) \cdot (z_1 - z_2) \]
\[ + \left( \varphi(|z_1|) - \varphi(|z_2|) \right) (A(x,z_1) - vz_1) \cdot (z_1 - z_2) \]
\[ = |v| z_1 - z_2|^2 + (I) + (II) \]
Due to the support of $\varphi$ and the fact that $\varphi(|u|) - \varphi(|v|) \leq 1$ for all $u, v \in \mathbb{R}$, we find that that
\[ |(I)| \leq \delta |\varphi(|z_1|) - \varphi(|z_2|)||z_1 - z_2| \]
\[ \leq \frac{\sqrt{2}}{4} |z_1 - z_2|^2 \]
Similarly we find
\[ |(I)| = \varphi(|z_2|) \left[ \sum_{i,j=1}^{3} \int \frac{\partial_j A(x,\xi)}{\partial_i} - \delta_{ij} v d\xi (z_1^j - z_2^j) \right] \leq \delta \varphi(|z_2|)|z_1 - z_2|^2 \leq \frac{\sqrt{2}}{4} |z_1 - z_2|^2 \]
And so
\[ (\bar{A}(x,z_1) - \bar{A}(x,z_2)) \cdot (z_1 - z_2) \geq |v| z_1 - z_2|^2 - |(I)| - |(II)| \geq \frac{\sqrt{2}}{2} |z_1 - z_2|^2. \]
We split
\[ f = g_k + b_k := f\chi_{|f| < k} + f\chi_{|f| > k}. \]
Next we solve the following auxiliary Stokes problem:
\begin{equation}
\begin{cases}
-\text{div}(\bar{A}(\epsilon v)) + \nabla \pi_2(x) = -\text{div} b_k(x) & \text{in } \Omega \\
\text{div } v = 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}
(5.9)
Since $f \in L^2(\Omega)$ the existence follows by monotone operator theory.
Moreover, we find that in $\Omega$ (as $\Delta v = 2\text{div}(\epsilon v)$)
\begin{equation}
\frac{\epsilon}{2} \Delta v + \nabla \pi_2(x) = -\text{div}(b_k(x) + \epsilon v v - \bar{A}(x,\epsilon v)) = -\text{div}(b_k(x) + \varphi(|\epsilon v|) (A(x,\epsilon v) - \epsilon v v))
\end{equation}
(5.10)
We consider
\[ \omega = M(|f|\chi_{\Omega} + 1)^{q-2} \in A_2 \text{ as } q - 2 \in (-1,0). \]
Now (8, Lemma 3.2) and (2.5) in combination with the support of $\varphi$ implies that
\[
\int_{\Omega} |\nabla v|^2 \omega dx \leq c \int_{\Omega} |b_k|^q dx + c \int_{\Omega} |\varphi(\epsilon v)| (A(x,\epsilon v) - \epsilon v v)^2 \omega dx
\]
\[ \leq c \int_{\Omega} |b_k|^q dx + c \delta \int_{\Omega} |v\epsilon v|^2 \omega dx\]
using that $\text{supp}(\varphi) \subset [K,\infty)$. Choosing $\delta$ small enough implies
\[
\int_{\Omega} |\nabla v|^2 \omega dx \leq c \int_{\Omega} |b_k|^q dx.
\]
By Fubini theorem
\[ ||f||_{L^q_{\omega}(\Omega)}^q \sim \int_0^\infty \omega(|f| > l) l^q dl, \]
which implies that for every $\beta > 0$ there exists a $k$, such that
\[ ||b_k||_{L^q_{\omega}(\Omega)}^q = \int_k^\infty \omega(|f| > l) l^q dl = \beta. \]
Hence we find
\begin{equation}
\int_{\Omega} |\nabla v|^2 \omega \, dx \leq c \int_{\Omega} |b_k|^q \, dx \leq c \beta.
\end{equation}

Next we observe, that that
\begin{equation}
- \frac{\nu}{2} \Delta (u - v) + \nabla (\pi - \pi_2)(x) = - \text{div}(\nu (\varepsilon (u - v)) - (A(\cdot, \varepsilon u) - \tilde{A}(\cdot, \varepsilon v)) + (u(x) \otimes u(x)) + g_k(x)).
\end{equation}

By testing (5.12) with $u - v$ and using Young’s inequality we find for $\delta > 0$ the estimate (using $p \geq 2$)
\begin{equation}
\frac{\nu}{2} \int_{\Omega} |\nabla u - \nabla v|^2 \, dx = \langle u \otimes u, \nabla (u - v) \rangle + (\nu (\varepsilon (u - v)) - (A(\cdot, \varepsilon u) - \tilde{A}(\cdot, \varepsilon v)) + g_k, \nabla (u - v))
\end{equation}

This implies by the structure of the convective term and the symmetry of the convective term that
\begin{equation}
\langle u \otimes u, \nabla (u - v) \rangle = \langle u \otimes v, \nabla (u - v) \rangle
\end{equation}

\begin{align*}
&\leq \delta \int_{\Omega} |\nabla u - \nabla v|^2 \, dx + c_\delta \int_{\Omega} |u \otimes v|^2 \, dx \\
&\leq \int_{\Omega} |\nabla (u - v)|^2 \, dx + c_\delta \int_{\Omega} |u \otimes v|^2 \, dx.
\end{align*}

And finally
\begin{align*}
&\langle \nu (\varepsilon (u - v)) - (A(\cdot, \varepsilon u) - \tilde{A}(\cdot, \varepsilon v)), \nabla (u - v) \rangle \leq \delta \int_{\Omega} |\nabla (u - v)|^2 \, dx \\
&+ c_\delta \int_{\Omega} |\nu (\varepsilon (u - v)) - (A(\cdot, \varepsilon u) - \tilde{A}(\cdot, \varepsilon v))|^2 \, dx.
\end{align*}

But now
\begin{align*}
&|\nu (\varepsilon (u - v)) - (A(\cdot, \varepsilon u) - \tilde{A}(\cdot, \varepsilon v))| \leq cK + |A(\cdot, \varepsilon u) - \nu \varepsilon u| X_{(|\varepsilon| \leq 2K)}X_{(|\varepsilon| \geq 4K)} \\
&+ |A(\cdot, \varepsilon u) - A(\cdot, \varepsilon v) - \nu \varepsilon (u - v)| X_{(|\varepsilon| \leq 2K)}X_{(|\varepsilon| \geq 4K)} \\
&\leq cK + 3\delta |\varepsilon (u - v)| + \sum_{i,j} \int_{|\varepsilon| \leq 2K} \delta d\xi
\end{align*}

This implies (by choosing $\delta = \frac{1}{6}$ and $\delta \leq \frac{\delta}{3|\gamma|}$ and absorption) that
\begin{equation}
\int_{\Omega} |\varepsilon u - \varepsilon v|^2 \, dx \leq \int_{\Omega} |u \otimes v|^2 + g_k^2 \, dx + K \leq \int_{\Omega} |u \otimes v|^2 + k^{2-q} f^q \, dx + K.
\end{equation}

In order to estimate the convective term we use (5.8) and take $p = 2$, $p^* = 6$ and $\gamma = \frac{1}{4}$, which implies that
\begin{equation}
\frac{-2p'}{p \gamma} = \frac{-2}{\frac{3}{4}} = -6
\end{equation}

and so
\begin{equation}
\int_{\Omega} |u \otimes v|^2 \, dx \leq \int_{\Omega} |\nabla u|^2 \omega \, dx \int_{\Omega} |\nabla v|^p \omega \, dx \left( \int_{\Omega} \omega^{-6} \, dx \right)^{\frac{1}{p'}}
\end{equation}

Since
\begin{equation}
\omega^{-6} = M |\varepsilon| X_{\Omega} + 1)^{(2-q)\delta},
\end{equation}

and
\begin{equation}
(2-q)6 \leq q \text{ by the assumption that } q \in \left[\frac{12}{7}, 2\right],
\end{equation}

we find that
\begin{equation}
\int_{\Omega} |u \otimes v|^2 \, dx \leq \beta \left( \int_{\Omega} |f|^q + 1 \, dx \right)^{\frac{1}{4}} \int_{\Omega} |\nabla u|^2 \omega \, dx.
\end{equation}
Hence we can estimate
\[\int_\Omega |\nabla u|^2 \omega \, dx \lesssim \int_\Omega |\nabla (u - v)|^2 \, dx + \int_\Omega |\nabla v|^2 \omega \, dx \]
\[\lesssim \int_\Omega |u \otimes v|^2 + k^{2-q}|f|^q \, dx + \beta + K \]
\[\lesssim \int_\Omega |\nabla u|^2 \omega \, dx \beta \left( \int_\Omega |f|^q + 1 \, dx \right)^{\frac{1}{q}} + C(\|f\|_{L^q(\Omega)}) \]
By choosing \(\beta\) small enough (meaning \(k\) large enough) we can absorb and find
\[\int_\Omega |\nabla u|^p \omega \, dx \leq C(\|f\|_{L^q(\Omega)}), \]
and so (using Hölder’s inequality as in Corollary 5.2) we find
\[\int_\Omega |\nabla u|^q \, dx \leq C(\|f\|_{L^q(\Omega)}). \]

\[
\begin{align*}
\int_\Omega |\nabla u|^p \omega \, dx + \int_\Omega |\nabla u|^q (p-1) \, dx &\leq C.
\end{align*}
\]

Using these estimates and the reflexivity of the spaces
\[u_k \rightharpoonup u \quad \text{weakly in } W^{1,q(\rho-1)}_{0,\text{div}}(\Omega) \]
\[\nabla u_k \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega) \cap L^{q(\rho-1)}(\Omega) \]
\[A(\cdot, \varepsilon u_k) \rightharpoonup \overline{A} \quad \text{weakly in } L^p(\Omega) \cap L^q(\Omega). \]

and if we pass to the limit as \(k \to \infty\) we end up with the following a priori estimate:
\[\int_\Omega |\nabla u|^p |\overline{A}|^{\rho} \omega \, dx + \int_\Omega |\nabla u|^{q(\rho-1)} + |\overline{A}|^q \, dx \leq C. \]

Let us prove now that \(u\) is a weak solution. Notice that we can pass to the limit (in case of (6.4)) using (6.1) and (6.2) to find
\[\int_\Omega \overline{A} \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in W^{1,p}_{0,\text{div}}(\Omega), \]
and in case of (6.3) we use the fact that \(W^{1,q(\rho-1)}(\Omega)\) compactly embeds into \(L^2(\Omega)\) and find
\[\int_\Omega (\overline{A} - (u \otimes u)) \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in W^{1,p}_{0,\text{div}}(\Omega), \]
6.2. Establishing the non-linearity. In this subsection we aim to show that

\[ (6.8) \]  
\[ A = A(\cdot, \varepsilon u). \]

This will be achieved by using Theorem 3.9. Indeed, we choose \( d^k := \nabla u_k \) \( s^k := A(\cdot, \varepsilon u_k), q := p \) and \( n = N = 3 \) and \( \omega \) as before. By applying (6.5) we can see that

\[ \| d^k \|_{L^p(\Omega)} + \| s^k \|_{L^p(\Omega)} \leq c \| \nabla u_k \|_{L^p(\Omega)} \]

which means that (3.4) is fulfilled. Then we have that with \( g_k = f_k \) in case of (1.1) and \( g_k = f_k + u_k \otimes u_k \) in case of (1.1)

\[ \lim_{k \to \infty} \int_{\Omega} s^k \cdot \nabla d^k \, dx = \lim_{k \to \infty} \int_{\Omega} g_k \cdot \nabla d^k \, dx = 0 \]

using the equation (6.1) and the hypothesis on \( d^k \); this implies (3.5). Last but not least (3.6) and (3.7) follow by the fact that \( d^k \) is a gradient. So we apply Theorem 3.9 to get a sequence of measurable sets \( \Omega_j \subset \Omega \) with \( |\Omega \setminus \Omega_j| \to 0 \) as \( j \to \infty \) so that

\[ (6.9) \]
\[ A(\cdot, \varepsilon u_k) \cdot \nabla u_k \omega \to A \cdot \nabla u \omega \text{ weakly in } L^1(\Omega_j). \]

Now notice that for any \( B \in L^p_\omega(\Omega) \) we obtain

\[ (6.10) \]
\[ (A(\cdot, \varepsilon u_k) - A(\cdot, B^j)) \cdot (\nabla u_k - B) \omega \to (A - A(\cdot, B^j)) \cdot (\nabla u - B) \omega \text{ weakly in } L^1(\Omega_j) \]

where we denoted \( B^j := \frac{B \gamma_j}{p+q} \). Denote \( A(Q) := \{Q^{p-2}Q \} Q : \Omega \to \mathbb{R}^{3 \times 3} \). Then \( (A(Q') - A(P')) \cdot (Q - P) \geq 0 \). Thus

\[ \int_{\Omega_j} (A - A(\cdot, B^j)) \cdot (\nabla u - B) \omega \, dx \geq 0 \]

which can be rewritten as

\[ \infty > \int_{\Omega} (A - A(\cdot, B^j)) \cdot (\nabla u - B) \omega \, dx \geq \int_{\Omega_j} (A - A(\cdot, B^j)) \cdot (\nabla u - B) \omega \, dx. \]

Since \( |\Omega \setminus \Omega_j| \to 0 \) as \( j \to \infty \) we can apply the dominated convergence theorem and obtain that

\[ \int_{\Omega} (A - A(\cdot, B^j)) \cdot (\nabla u - B) \omega \, dx \geq 0. \]

We can now choose \( B := \nabla u - \delta G \) with \( G \in L^\infty(\Omega) \) and \( \delta > 0 \). The last relation becomes

\[ \int_{\Omega} (A - A(\cdot, \varepsilon u - \delta G^j)) \cdot G \omega \, dx \geq 0. \]

We let \( \delta \to 0^+ \) and we use again Dominated Convergence Theorem to obtain

\[ \int_{\Omega} (A - A(\cdot, \varepsilon u)) \cdot G \omega \, dx \geq 0 \text{ for all } G \in L^\infty(\Omega). \]

Finally, we choose

\[ G := -\frac{A - A(\varepsilon u)}{|A - A(\varepsilon u)| + 1} \]

and we conclude that (6.3) is true.

The proof is concluded, once the existence and the estimates for the pressure \( \pi \) are shown:

6.3. Existence & estimates for pressure. We start by noticing that the following holds for with \( g = f \) in case of (1.6) and \( g = f + u \otimes u \) in case of (1.1). Since in case of (1.1) we have that \( 2q \leq 4 \) and so \( L^{2q} \Omega \subset W^{1,q(p-1)}(\Omega) \) we find that

\[ \int_{\Omega} A(\cdot, \varepsilon u) \cdot \nabla \varphi \, dx = \int_{\Omega} g \cdot \nabla \varphi \, dx \text{ for all } \varphi \in W^{1,q}_{0,\text{div}}(\Omega). \]

The weak formulation for our system of equations can also be rewritten as

\[ \int_{\Omega} A(\cdot, \varepsilon u) \cdot \nabla \varphi \, dx - \int_{\Omega} \pi \text{div} \varphi \, dx = \int_{\Omega} g \cdot \nabla \varphi \, dx \text{ for all } \varphi \in W^{1,q}_{0,\text{div}}(\Omega). \]

or equivalently

\[ \int_{\Omega} \pi \text{div} \varphi \, dx = \int_{\Omega} A(\cdot, \varepsilon u) \cdot \nabla \varphi \, dx - \int_{\Omega} g \cdot \nabla \varphi \, dx \text{ for all } \varphi \in W^{1,q}_{0,\text{div}}(\Omega). \]
We consider the following mapping $\mathcal{F} : L^q_w(\Omega) \rightarrow \mathbb{R}$ given by

$$a \mapsto \mathcal{F}(a) := \int_{\Omega} A(\cdot,\varepsilon u) \cdot \mathbf{e} \text{Bog}(a) \ dx - \int_{\Omega} g \mathbf{\nabla} \text{Bog}(a) \ dx.$$ 

It is standard to apply Hölder’s inequality and to conclude that $\mathcal{F}$ is linear and continuous. Therefore there exists

$$\pi \in L^q_w(\Omega) \cong \left( L^q_w(\Omega) \right)'$$

such that

$$\int_{\Omega} \pi dx = \int_{\Omega} A(\cdot,\varepsilon u) \cdot \mathbf{e} \text{Bog}(a) \ dx - \int_{\Omega} g \mathbf{\nabla} \text{Bog}(a) \ dx.$$ 

Since $\varphi \in W^{1,q}_w(\Omega) \implies \text{div} \varphi \in L^q_w(\Omega)$ and since $\text{div} \varphi = \text{divBogdiv} \varphi$ we obtain :

$$\int_{\Omega} \pi \text{div} \varphi dx = \int_{\Omega} A(\cdot,\varepsilon u) \cdot \mathbf{e} \text{Bog}(\text{div} \varphi) \ dx - \int_{\Omega} g \mathbf{\nabla} \text{Bog}(\text{div} \varphi) \ dx$$

$$= \int_{\Omega} A(\cdot,\varepsilon u) \varphi \ dx - \int_{\Omega} g \mathbf{\nabla} \varphi dx \quad \text{for all } \varphi \in W^{1,q}_w(\Omega)$$

where the last equality is due to the fact that $\varphi - \text{Bogdiv} \varphi \in W^{1,q}_{0,\text{div}}(\Omega)$ can be used as a test function. By duality, we have

$$\| \pi \|_{L^q_w(\Omega)} = \| \mathcal{F} \|_{(L^q_w(\Omega))'} \leq \| A(\cdot,\varepsilon u) \|_{L^q_w(\Omega)} + \| g \|_{L^q_w(\Omega)}$$

And in exactly the same manner we obtain the estimate

$$\| \pi \|_{L^q_w(\Omega)} \leq c \| g \|_{L^q_w(\Omega)} + c \| \mathbf{\nabla} v^{p-1} \|_{L^q_w(\Omega)}$$

using now the fact

$$\left( L^q_w(\Omega) \right)' \cong L^{q'}_w(\Omega), \quad \omega' := \omega^\frac{1}{p-1}$$

and since $w \mapsto w^{p-1} \geq 1$ we find

$$\| \pi \|_{L^q_w(\Omega)} \leq c \| g \|_{L^q_w(\Omega)} + c \| \mathbf{\nabla} v^{p-1} \|_{L^q_w(\Omega)}$$

This now ends the proof of the a-priori bounds of the Theorem 2.1 and Theorem 2.2 and Theorem 2.4.

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