On matrix models for anomalous dimensions of super Yang–Mills theory

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Abstract

We consider the matrix model approach to the anomalous dimension matrix in $\mathcal{N} = 4$ super Yang–Mills theory. We construct the path integral representation for the anomalous dimension density matrix and analyze the resulting action. In particular, we consider the large $N$ limit, which results in a classical field theory. Since the same limit leads to spin chains, we propose to consider the former as an alternative description of the latter. We consider also the limit of small $N$, which corresponds to the restriction to the diagrams of maximal topological genus.

1 Introduction

The large $N$ approach \cite{1} provides a description of the quantized gauge models in terms of a topological expansion. This expansion is very similar to

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the string perturbative expansion in terms of the geometrical genus. This old idea to describe the strongly coupled gauge model as a string model has found its so far best realization in the AdS/CFT conjecture [2, 3]. As a true duality, the AdS/CFT correspondence relates the weakly coupled theory on the one hand, with the strongly coupled on the other. Once proven, this theory would become a powerful tool for the description of both super Yang–Mills (SYM) theory and string theory. The same property, however, prevents all possible proofs from being easy. The first approaches to the problem relied heavily on the supersymmetry properties of the theories (see [4] for a review).

A substantial progress was achieved when it was realized that various limits of the correspondence could be considered. In particular, a Penrose limit of the AdS geometry [5, 6] was found to correspond to the pp-wave geometry, in which the string theory is solvable [7]. In SYM theory this corresponds to operators with a large $R$-charge $J$. Various spinning string solutions were found in [8, 9, 10, 11, 12] and the respective SYM sectors identified (see [13] for a review and a complete list of references).

On the other hand, the extensive study of the SYM anomalous dimensions revealed the dilatation operator for one and higher loops [14, 15], leading as well to the discovery of integrability in the planar limit [16, 17]. Integrable structures discovered in SYM were compared to those of string theory and a pretty good match was found [18, 19, 20].

At the same time, the nonplanar regime of SYM theory and AdS/CFT correspondence was paid much less attention to. In the two-impurity sector of the BMN limit it was shown [21] that nonplanar corrections correspond to splitting/joining of string-like configurations. The corresponding spin description with the dynamical chain formation of the complete one-loop dilatation operator was introduced in [22, 23]. Such a nonplanar analysis was then further pursued at the two- [24] and higher-loop [25] levels.

In this paper we are returning a little bit back and consider the anomalous dimension operator of [14] without going to the spin description. As noted in [26], this operator can be regarded as the Hamiltonian of a matrix model in the Schrödinger picture\(^1\). We adopt this point of view and analyze the respective model. In particular, we propose a path integral formula for the anomalous dimension density and analyze the large, as well as the small $N$ limit of the path integral. The model is described in terms of a gauge theory on a compact noncommutative space. The obtained model, however, is different from the noncommutative Yang–Mills theory, due to a modified phase structure. This modification can be regarded as an additional noncommutativity in the space.

\(^1\)For an earlier matrix model approach in BMN limit see [27]
of gauge field configurations, modulo gauge symmetry. Next, we probe the extremal limits of $N$. The limit of large $N$ in the spin description corresponds to the integrable spin chain. In the case of the matrix model description, this is the semiclassical limit, $1/N$ playing the role of the Planck constant $\hbar$. As a result, physical quantities of the quantum spin chain, such as the partition function or correlation functions, can be computed in terms of the matrix model, as integrals over the moduli space of classical solutions.

A much less studied case (if studied at all) is the $N \to 0$ limit of a gauge theory. In the topological expansion it corresponds to counting the contribution of diagrams with the maximal topological genus. This limit cannot be reached directly within SYM theory, since one cannot reach a vanishing value of $N$ continuously, because $N$, being the rank of the gauge group, is restricted to integer values. The noncommutative field theory description allows one to go beyond this limitation and extend the description to arbitrary real values of $N$, which is compatible with the $N \to 0$ limit. The analysis of the star product expansion in this limit unveils its strongly non local character. An alternative approach in terms of spin bit model confirms this conclusion, the resulting model being given by a nonlocal spin model.

The plan of the paper is as follows. In the next section we introduce the anomalous one-loop dimension operator and find the corresponding matrix model, by passing to the path integral description of the partition function or the Fourier transform of the eigenvalue density. A gauge invariant matrix model of the reduced Yang–Mills type emerges, as a result of taking into consideration the symmetries of the system. Then, we find it suitable to represent the matrix model in terms of a gauge model on a compact noncommutative space. We choose the simplest case of the noncommutative torus, but other options like the fuzzy sphere are also possible. Finally, we consider the large/small $N$ limit of the obtained model and draw our conclusions.

## 2 Matrix model for anomalous dimensions

In this paper we consider the SU(2)-invariant sector of SYM operators which is generated by all gauge invariant polynomials of two complex combinations of SYM scalars. The one-loop anomalous dimension matrix for this sector was found in \[28\]. It can be written in a compact form as follows:

$$H_{(2)} = -\frac{g_{YM}^2}{16\pi^2} : \text{tr}[\Phi^a, \Phi^b][\dot{\Phi}_a, \dot{\Phi}_b] :,$$

where $\Phi^a \ a = 1, 2$ are the mentioned complex combinations of the SU($N$) scalar field: $\Phi^1 = \phi_5 + i\phi_6$ and $\Phi^2 = \phi_1 + i\phi_2$ and checked letters correspond
The operators under consideration are polynomials in $\Phi^a$ invariant under the SU($N$) gauge transformation

$$\Phi^a \rightarrow U^{-1} \Phi^a U, \quad U \in \text{SU}(N).$$

(2.2)

Generally they can be imagined as a product of traces of products of $\Phi^a$’s. An alternative parameterization can be introduced using a permutation group element [22, 23].

Therefore, the “physical states”, i.e. the states on which the operator (2.1) is allowed to act, are given by gauge invariant polynomials of “rising operators” $\Phi^a$. Formally, this corresponds to projecting onto the gauge invariant sector of the Hilbert space, by imposing the following condition:

$$G |\Psi\rangle = 0,$$

(2.3)

where $G$ is the generator of gauge transformations

$$G = - : [\Phi^a, \hat{\Phi}_a] :,$$

(2.4)

i.e.

$$[\text{tr} uG, \Phi^a] = [\Phi^a, u], \quad u \in \text{su}(N),$$

(2.5)

where the fat commutator $[,]$ is the one defined over the Hilbert space for which

$$[\hat{\Phi}_{a,i}^j, \Phi^b_k] = \delta^b_a \delta^i_j \delta^l_k,$$

(2.6)

and, in contrast to the usual one $[,]$, it denotes the alternation in matrix matrix products.

As noted in [26], the operator (2.1) can be regarded as the quantized Hamiltonian in the Schrödinger picture of a matrix model given by the classical Hamiltonian

$$H_{cl} = -\frac{g_{\text{YM}}^2}{16\pi^2} \text{tr}[X^a, X^b][\bar{X}_a, \bar{X}_b],$$

(2.7)

the canonical Poisson bracket,

$$\{X^a_{i,j}, \bar{X}^k_{l,i}\} = i\delta^a_i \delta^k_l \delta^j_l,$$

(2.8)

\footnote{It coincides with the Hamiltonian of the matrix model counting the combinatorial factors of Feynman diagrams proposed in [21].}
and the constraint
\[ G \equiv -[X^a, \bar{X}^a] \approx 0. \] (2.9)

The relation with the classical model given by (2.7) and (2.8), on the one
side, and the operator (2.1) on the other, can be easily checked by quantizing
the former and going to the Schrödinger picture. A little less trivial approach
is to go back and construct the path integral representation for the operator
(2.1), by considering the partition function
\[ Z_\tau = \text{Tr} e^{i \tau H(2)}. \] (2.10)

The meaning of the partition function (2.10) in the dual theory is clear:
under given boundary conditions it describes respective string amplitudes. A
less obvious fact is that it has an important meaning also in the original SYM
theory. In fact, up to a multiplicative factor, the partition function gives
the Hamiltonian \( H(2) \) eigenvalue density function, or the SYM anomalous
dimension distribution. Indeed,
\[ \rho(\lambda) = \frac{1}{2\pi} \int d\tau \text{tr} e^{i \tau (H(2) - \lambda)} = \sum_k \delta(\lambda_k - \lambda), \] (2.11)
where the sum over the eigenvalues \( \lambda_k \) should be understood in a broad
sense, including both summation over the discrete set and integration over
the continuous one.

Let us describe briefly the derivation of the path integral. As usual, one
should split the time interval \( \tau \) in \( L \) smaller pieces \( \Delta = \tau/L \) and write the
exponential under the trace in (2.10) as a product over these pieces
\[ Z_\tau = \text{Tr}(e^{-\Delta H(2)})^L. \] (2.12)

Now we should employ the oscillator coherent states
\[ |X\rangle = \exp \text{tr}(\bar{X}\phi - X\check{\phi}) \cdot \exp \text{tr}(\bar{Y}Z - X\check{Z}) |0\rangle = e^{-\text{tr} \frac{1}{2}(\bar{X}X + \bar{Y}Y)} e^{\bar{X}\phi + \bar{Y}Z} |0\rangle. \] (2.13)

Inserting the unity operator decomposition
\[ I = \int d^4X |X\rangle \langle X| \] (2.14)
between each factor in (2.12) and taking the limit \( L \to \infty \), one gets the
partition function
\[ Z_\tau = \int [dX \ d\bar{X}] e^{i S(X, \bar{X})}, \] (2.15)
\[ ^3\text{See [29] for an exhaustive introduction to coherent states.} \]
where the action $S(X, \bar{X})$ is given by

$$S_t(X, \bar{X}) = \int dt \left( \text{tr} \frac{i}{2}(\dot{X}_a \dot{\bar{X}}^a - \dot{\bar{X}}_a X^a) + \frac{g_{\text{YM}}^2}{16\pi^2} \text{tr}[X^a, X^b][\bar{X}_a, \bar{X}_b] \right), \quad (2.16)$$

with $X^a$ and their Hermitian conjugate $\bar{X}_a$ being $N \times N$ time-dependent matrices in the adjoint representation of SU($N$).

The action (2.16) is manifestly invariant, with respect to constant unitary conjugation

$$X^a \to U^{-1} X^a U, \quad (2.17)$$

$$\bar{X}_a \to U^{-1} \bar{X}_a U, \quad U \in \text{SU}(N). \quad (2.18)$$

In fact, the full theory is invariant with respect to a bigger group of time-dependent ($\dot{U} \neq 0$) gauge transformations. Indeed, an infinitesimal transformation of the action (2.16) with $U = e^u \approx 1 + u$ produces the following term:

$$\delta_{\text{gauge}} S_t = - \text{tr} \dot{u}[X^a, \bar{X}_a]. \quad (2.19)$$

The latter is proportional to the quantity $G(X) = -[X, \bar{X}]$, which for the physical states is identically zero, since

$$G(X) = \langle X | : [\Phi^a, \bar{\Phi}_a] : | X \rangle = [X^a, \bar{X}_a] \equiv 0. \quad (2.20)$$

Therefore, the path integral (2.15) can be regarded as the one corresponding to the gauge invariant action

$$S = S_t + \frac{i}{2} \text{tr}(\bar{X}_a [A_0, X^a] - [A_0, \bar{X}_a] X^a), \quad (2.21)$$

where $A_0$ is the time (and unique) component of the SU($N$) gauge field, in the temporal gauge $\dot{A}_0 = 0$.

The gauge fixed non-invariant form of the action appeared, due to the gauge non-invariance of the resolution of the unity operator in (2.14). Indeed, the action of the generator of the infinitesimal gauge transformation $\text{tr} u G(\Phi, \bar{\Phi})$, $u \in \text{SU}(N)$ on a coherent state $| X \rangle$ yields

$$\text{tr} u G | X \rangle = | U^{-1} X U \rangle - | X \rangle = | X + [X, u] \rangle - | X \rangle \neq 0. \quad (2.22)$$

4or the action for a system with the constraint $G \approx 0$; in this case $iA_0$ is a Lagrange multiplier.
One can improve this situation by restricting the integration solely over the physical coherent states satisfying (2.20). This can be implemented by introducing, in place of the unity, the projector to the physical sector

$$
\Pi = \int d^4X \delta(G(X)) |X\rangle \langle X| = \Pi = \int d^4X dA_0 e^{i tr A_0 G} |X\rangle \langle X|. \quad (2.23)
$$

In spite of the manifest gauge invariance of the procedure, the expression for the projector (2.23) cannot be considered yet completely satisfactory. In fact, now there is “too much” gauge invariance, since both the integrand and the measure are gauge invariant. Therefore, there is an extra dummy integration over the gauge group $SU(N)$ in (2.23). Normally, as $SU(N)$ is compact, this is not a big trouble but, since we are inserting this integration an infinite number of times, this could be a source of potential divergencies. This situation is completely equivalent to that in any theory with gauge invariance and it is solved in a similar way. Namely, one can explicitly break the gauge invariance by introducing a gauge fixing term $F_{gf}$. In this case the physical state projector looks like

$$
\Pi = \int d^4X dA_0 dX \Delta_{FP}(X) e^{i tr A_0 G + i tr X F_{gf}} |X\rangle \langle X|, \quad (2.24)
$$

where $\chi$ is the Lagrange multiplier for the gauge fixing constraint $F_{gf} = 0$ and $\Delta_{FP}(X)$ is the famous Faddeev–Popov determinant\(^5\), defined as

$$
\Delta_{FP}(X) = \int dU \delta(F_{gf}(U^{-1}XU)).
$$

Now it is not difficult to see that, using the gauge fixed projector to the physical sector (2.24) instead of unity, one gets the BRST invariant form of the path integral,

$$
Z_\tau = \int [dX dA_0 d\lambda d\bar{c} d\bar{c}] e^{i S_{gf}(X,A) + i tr \lambda F_{gf}(X,A) + i tr \bar{c} M_{FP}(X,A) \bar{c}}, \quad (2.25)
$$

where

$$
S_{gf}(X, A) = \int d\tau \left( \frac{i}{2} \{ \bar{X}_a \nabla_0 X^a - \nabla_0 \bar{X}_a X^a \} + \frac{g_Y^2}{16\pi^2} \text{tr}[X^a, X^b][\bar{X}_a, \bar{X}_b] \right) \quad (2.26)
$$

\(^5\)For details regarding admissible gauge fixings, Faddeev–Popov determinants and BRST invariance in a gauge theory, we refer the reader to the classical reference [30].
is the gauge invariant action, $\nabla_0 X = \partial_0 X + [A_0, X]$, $\lambda$ is the Lagrange multiplier implementing the gauge fixing condition $F_{gf} = 0$, and $M_{FP}$ is the Faddeev–Popov operator defined by

$$\delta_{\text{gauge}} F_{gf} = M_{FP}(X, A)u. \quad (2.27)$$

Obviously, the gauge transformation of the gauge field $A_0$ is given by

$$A_0 \to U^{-1} A_0 U + U^{-1} \dot{U}. \quad (2.28)$$

### 3 Noncommutative torus representation

The gauge invariant classical action (2.26) resembles a lot a Yang–Mills-type model. Drawn by this, we will rewrite in this section the action (2.26) in terms of Yang–Mills theory on a two-dimensional noncommutative torus. In fact, the choice of the two dimensional torus is not special, rather it is dictated just by the simplicity of the space. In general, using the compact form of the maps of noncommutative gauge theories considered in [31, 32, 33] (see also [34]), one can pass among different theories, within the class of Morita equivalent noncommutative spaces [35].

The two dimensional noncommutative torus is defined by the “coordinate operators” $U$ and $V$ subject to the following commutation relations:

$$UV = q VU. \quad (3.1)$$

This algebra can be embedded into a usual Heisenberg algebra

$$U = e^{2\pi i x^1}, \quad V = e^{2\pi i x^2}, \quad [x^1, x^2] = i\theta, \quad (3.2)$$

with $q = e^{4\pi^2 i \theta}$. On the other hand, when $q$ is a $N$-th root of unity: $q^N = 1$, i.e. when $\theta = 1/2\pi N$, the dimensionality of the irreducible representation of the algebra is finite and equal to $N$. Indeed, $U$ and $V$ can be represented in terms of the following $N \times N$ unitary matrices:

$$U_{mn} = \delta_{m+1,n}, \quad V_{mn} = e^{2\pi i m/N} \delta_{mn}, \quad (3.3)$$

where no summation is assumed over repeating indices and indices are periodic in $N$, $m + N \sim m$.

We leave to the reader the proof that any arbitrary $N \times N$ matrix $F$ can be expressed as a Weyl ordered polynomial of degree up to $N - 1$ in respectively $U$ and $V$

$$F = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f_{mn} W^{mn}, \quad (3.4)$$
where $W^{mn}$ is the Weyl ordered product of $V^m$ and $U^n$. In terms of the Heisenberg algebra embedding, one has

$$W^{mn} = e^{2\pi mx^1 + 2\pi nx^2}.$$  

(3.5)

Based on eq.(3.4), one can construct a one-to-one map from $N \times N$ matrices to functions on the unit two-dimensional torus

$$F \mapsto F(x, y) = \sum_{mn} f_{mn}e^{2\pi imx + 2\pi iny}.$$  

(3.6)

The matrix product under this map is replaced by the noncommutative star product

$$F \cdot G \mapsto F \star G(x, y) = F(x, y)e^{i\theta(\bar{\partial}_x \partial_y - \bar{\partial}_y \partial_x)}G(x, y),$$  

(3.7)

where the left/right arrow indicates that the derivative acts on $F(x, y)$ or $G(x, y)$ respectively.

Some other useful properties are that (i) the trace of a matrix is given by the integral over the torus of the corresponding function

$$\text{tr} F = N \int_{T^2} dx \, dy \, F(x, y)$$  

(3.8)

and (ii) commutators of a noncommutative torus function with $x$ and $y$ correspond to the derivative over, respectively, $y$ and $x$

$$[x, F] \mapsto i\theta \partial_y F(x, y), \quad [y, F] \mapsto -i\theta \partial_x F(x, y),$$  

(3.9)

which allow one to express the derivatives of a function in an algebraic way and, viceversa, to rewrite algebraic expressions as derivatives.

Using all of the above properties, one can rewrite the gauge invariant action of the matrix model in terms of fields on the noncommutative torus

$$S_{\text{gl}}(X, A) =$$

$$N \int dt \, dx \, dy \left( \frac{i}{2}(\bar{X}_a \nabla_0 X^a - \nabla_0 \bar{X}_a X^a) + \frac{g_{\text{YM}}^2}{16\pi^2} [X^a, X^b]_*[\bar{X}_a, \bar{X}_b]_* \right),$$  

(3.10)

where $X^a$ and $\bar{X}_a$ are now functions on the torus and the star-commutators are defined using the star product (3.7)

$$[F, G]_* = F \star G - G \star F.$$  

(3.11)
The action \( (3.10) \) is gauge invariant with respect to local time dependent star-gauge transformations

\[
X^a \to U^{-1} \ast X^a \ast U \tag{3.12}
\]

\[
\bar{X}_a \to U^{-1} \ast \bar{X}_a \ast U \tag{3.13}
\]

\[
A_0 \to U^{-1} \ast A_0 \ast U + U^{-1} \ast \partial_0 U, \tag{3.14}
\]

where \( U \equiv U(x, y, t) \) is a local time dependent \( U(1) \) gauge transformation

\[
U^* \ast U \equiv U^{-1} \ast U = 1. \tag{3.15}
\]

If the time derivative term in the action \( (3.10) \) were of the second order, i.e. \( \nabla_0 \bar{X} \nabla_0 X \), rather than of the first one, we could rewrite \( (3.10) \) shifting the fields \( X, \bar{X} \) and up to total derivative terms, in terms of a Yang–Mills type of action

\[
S_{gi} = -\frac{16\pi^2 N}{g_{YM}^2} \int d^3x \, \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}, \tag{3.16}
\]

with the gauge field strength defined as

\[
\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3.17}
\]

and the spatial part of the gauge field \( A_a, a = 1, 2 \) defined through the relation

\[
X^a = \frac{4\pi}{g_{YM}} (i\theta^{-1}\epsilon_{ab}x_b + A_a), \quad \theta = 1/2\pi N, \tag{3.18}
\]

where \( \epsilon_{ab} \) is the two dimensional antisymmetric tensor with the only non-zero components \( \epsilon_{xy} = -\epsilon_{yx} = 1 \). Eq. \( (3.18) \) gives the splitting of the matrix field \( X^a \) into the partial derivative and the gauge field parts. The first order time derivative of the action makes it not only impossible rewriting the action in terms of the Yang–Mills model, but it makes the action non-invariant with respect to Lorentz boosts, apart from the fact that this symmetry is broken by the noncommutativity.

The case at hand can be regarded as a sort of Landau limit of the Yang–Mills type model, where the symplectic structure of the type \( dp_i \wedge dq^j \) is replaced with the “noncommutative” one, of the type \( \theta_{ij}^{-1}dx^i \wedge dx^j \).

**String interpretation**

It is very tempting to relate the perturbative SYM anomalous dimension matrix with a nonperturbative string dynamics given in terms of branes. Let us try to find the meaning of the obtained matrix model in this context.
Let us consider the BFSS type matrix model describing the dynamics of $N$ zero-branes \[36\]. It is given by the action

$$S_{\text{BFSS}} = \int dt \, \text{tr} \left( \frac{1}{2} (\nabla_0 X_i)^2 + \frac{g}{4} [X_i, X_j]^2 \right), \quad (3.19)$$

where $g$ is the string coupling and $X_i$, $i = 1, \ldots, 9$ are $N \times N$ Hermitian matrices. The eigenvalues of the matrices $X_i$ have the meaning of 0-brane coordinates. A modification of BFSS model describing the dynamics of holomorphic branes in two-dimensional complex space will be formulated in terms of $\text{sl}(N)$ matrices rather than the $\text{su}(N)$ ones of \[3.19\], which correspond to real coordinates. The modified action in this case takes the form

$$S_c = \int dt \, \text{tr} \left( \frac{1}{2} \nabla_0 X_a \nabla_0 X^a - \frac{g}{4} [X_a, X_b] [X^a, X^b] \right), \quad (3.20)$$

where $a, b = 1, 2$ and the bar stands for the Hermitian conjugate quantity. This model is almost our matrix model \[2.26\] except for the kinetic term.

This model is invariant, with respect to the time dependent $\text{SU}(N)$ gauge transformations \[2.17\]. This symmetry of the zero branes reflects the Chan–Patton gauge invariance of open strings. It also means that zero branes are charged, with respect to some $\text{SU}(N)$ gauge field. The interaction with an external “matrix (electro-)magnetic field” $\mathcal{A}_i$ can be introduced by adding to the action the following term:

$$\Delta S = ie \int dt \, \text{tr} \, \mathcal{A}_i (X) \dot{X}_i, \quad (3.21)$$

where $e$ is the unit charge of a zero brane. In particular a “constant” magnetic field is given by

$$\mathcal{A}_i = \frac{1}{2} \mathcal{F}_{ij} X_j. \quad (3.22)$$

Taking the value of the magnetic field such that it respects the holomorphic structure, i.e. $\mathcal{F}_{ab} = \mathcal{F}_{\bar{a}\bar{b}} = 0$, $\mathcal{F}_{\bar{a}b} = f \delta_{a\bar{b}}$ will lead to the following modification of the action \[3.20\]:

$$S_{\text{cm}} = \int dt \, \text{tr} \left( \frac{1}{2} |\nabla_0 X^a|^2 + ie f (\bar{X}_a \dot{X}^a - \dot{X}_a X^a) - \frac{g}{4} ||X^a, X^b||^2 \right), \quad (3.23)$$

Now, rescaling $X, \bar{X} \to \sqrt{ef} X, \sqrt{ef} \bar{X}$ and taking $ef \to \infty$, one gets

$$S_{\text{Landau}} = \int dt \, \text{tr} \left( \frac{1}{2} (\bar{X} \dot{X}^a - \dot{X}_a X^a) - \frac{g}{4} ||X^a, X^b||^2 \right), \quad (3.24)$$

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where \( g^* = g/(ef)^2 \) is the modified string coupling, which can be put into correspondence with the analogous factor in (2.26)

\[
g^* = \frac{g}{(ef)^2} = \frac{g_{YM}^2}{4\pi^2}.
\]

(3.25)

As it can be noted, the string coupling \( g \) in this limit should be very large, in order to keep \( g^* \) fixed.

The limit which we described is similar to the one yielding the noncommutative description for open strings [37, 38]. It is remarkable that here one ends up with a model which is noncommutative in both moduli space and space-time.

4 Extremal cases

In this section we discuss the extremal cases for the value of \( N \). Since the rank of the gauge group \( N \) is a free parameter of the model, one may hope to get simplifications, when it goes to some particular extremal values. The best studied case is of course the planar \( N \to \infty \) limit, where the SYM coupling \( g_{YM} \) scales according to \( \lambda_{pl} = g_{YM}^2 N = \) fixed. As we see immediately below, this limit results also in a great simplification of our matrix model.

On the SYM side, in the planar limit, only topologically trivial SYM Feynman diagrams survive. The number of contributing diagrams is drastically reduced and the instanton contribution is vanishing, which allows one to expect that the perturbation theory is exact and analytic in \( \lambda_{pl} \). Via AdS/CFT correspondence, the planar limit corresponds to taking the limit of free strings on \( \text{AdS}_5 \times S^5 \), while the expansion in powers of \( 1/N \) corresponds to the topological expansion in the theory of interacting strings.

Generally, a contribution of a SYM Feynman diagram with \( V_3 \) triple vertices, \( V_4 \) quadruple vertices and \( H \) holes comes with a factor [1]

\[
(g_{YM}^2 N)^F N^{2-2H},
\]

where \( F = V_4 + \frac{3}{2} V_3 \). For a fixed value of \( F \), the maximal number of holes in the diagram is bounded by \( 2H = F + 2 \), since there is no contribution with a negative power of \( N \) (for fixed \( g_{YM} \)). Thus, at any loop level, which is controlled by the power of \( g_{YM} \), the topological class of the diagram is bounded from both below and above, the planar limit describing the lowest part of this expansion. The natural question which can be addressed is whether there is an effective theory describing the opposite limit of the expansion. Formally, this limit is achieved when \( N \) goes to zero, keeping \( g_{YM} \) fixed at the same time.
As \( N \) is finite, this results in two different choices in the description of the same model; therefore one may conjecture that these two limits result in dual models.

Before going to the detailed description of the limits, let us make the following remark. While there is no problem with achieving the planar limit \( N \to \infty \), from the point of view of SYM theory the range of the gauge group \( N \) is always a positive integer and the limit \( N \to 0 \) cannot be reached smoothly. The same remains true for the dimensions of matrices in the matrix model description. Fortunately, the noncommutative torus description allows one to overcome this handicap. Since \( N \) enters as the commutativity parameter, one can continue it to arbitrary real analytical values. For an arbitrary real noncommutativity parameter, however, the representation of the algebra of the noncommutative torus becomes infinite dimensional and, in some sense, this limit is similar to the \( N \to \infty \) limit.

### 4.1 Planar limit \( (N \to \infty) \)

Let us fix the 't Hooft coupling to be \( \lambda_{\text{pl}} = g_{\text{YM}}^2 N \) and make the following rescaling of the fields:

\[
X \mapsto (2\pi/g_{\text{YM}})X.
\]

(4.1)

The action (3.10) then takes the following form:

\[
S_{\text{g.i.}} = \frac{(2\pi N)^2}{\lambda_{\text{pl}}} \int_{\mathbb{R}^1 \times T^2} d^3x \left( i\bar{X}\nabla_0 X + \frac{i}{4} |[X, X]|^2 \right),
\]

(4.2)

where we dropped the indices \( a, b, \ldots \) of the matrices \( X_a, X_b, \ldots \) etc. The integration is performed over time times the unit torus \( 0 \leq x_1, x_2 < 1 \).  

In the limit \( N \to \infty \) the noncommutativity parameter \( \theta = 1/2\pi N \) vanishes and the star product in the action (4.2) can be approximated by the leading terms in \( 1/N \)

\[
A \star B \equiv A e^{i\theta \partial_x \partial_y} B \approx AB + \frac{i}{4\pi N} \{ A, B \}, \quad [A, B] \star \approx \frac{i}{2\pi N} \{ A, B \}
\]

(4.3)

where \( \{ \cdot, \cdot \} \) denotes the Poisson bracket defined as

\[
\{ A, B \} = \partial_x A \partial_y B - \partial_y A \partial_x B.
\]

(4.4)

Making another rescaling of the fields similar to \( X \mapsto (1/2\pi N)X \), one arrives to the following form of the action:

\[
(2\pi N)^4 S_{\text{pl}} = \frac{(2\pi N)^4}{\lambda_{\text{pl}}} \int_{\mathbb{R}^1 \times T^2} d^3x \left( i\bar{X}\nabla_0 X + \frac{i}{4} |\{X, X\}|^2 \right),
\]

(4.5)
where the fields $X$ are functions on the ordinary (commutative) torus. This action describes a charged membrane in a strong magnetic field.

The dependence on $N$ is reduced to a diverging factor $(2\pi N)^4$ in front of the action. This factor is analogous to the factor $1/\hbar$ in the standard definition of the path integral

$$\int e^{i\frac{1}{\hbar}S}.$$

Therefore, the limit $N \to \infty$ corresponds to the semiclassical limit $\hbar \to 0$ in ordinary quantum mechanics. In other words, the diverging factor in the exponential of the path integral restricts it to the configurations with minimal action, i.e. to the classical ones.

Indeed, in the large $N$ limit the path integral (2.15) is reduced to the following expression:

$$Z_{\tau} = \int [dX] e^{i(2\pi N)^4S_{pl}(X)} = \int dX_0 [dX_\perp] e^{i(2\pi N)^4(S_{pl}(X_0)+S''_{pl}(X_0)X_\perp^2+...)}$$

$$= \int dX_0 \det [S''(X_0)] e^{i(2\pi N)^4S_{pl}(X_0)}, \quad (4.6)$$

where the integration in the last line is performed over the moduli space of classical solutions with the measure $dX_0$. Thus, if a classical solution continuously depend on $D_M$ parameters $y_i$, $i = 1, \ldots, D_M$, the measure $dX_0$ can be expressed as

$$dX_0 = \prod_{i=1}^{D_M} dy_i \sqrt{\det ij \int d^3x \partial_i \bar{X} \partial_j X}, \quad (4.7)$$

where $\partial_i = \partial/\partial y_i$ are partial derivatives, with respect to the solution parameters. As we expect, the moduli space of the solutions has more than just one connected component, therefore the integration over continuous parameters should be supplemented with the summation of the connected components. In this case the classical action is constant on each connected component, while it may vary from component to component.

The study of the structure of the moduli space of the solution of the system (4.5) and the comparison with the results obtained, e.g. via Bethe Ansatz in the spin chain approach, goes beyond the scope of the present work and we leave it for a future research.
4.2 Anti-planar limit \((N \to 0)\)

As we discussed above, the limit in which Feynman diagrams with maximal topological genus dominate, formally corresponds to taking a fixed small \(g_{YM}\) and \(N \to 0\). The analytic extension for achieving this limit is obtained using the noncommutative torus representation of the matrix model (2.26), with \(\theta = 1/2\pi N\) as the noncommutativity parameter. The representation of the noncommutative torus algebra depends in a complicated manner on whether \(N\) is rational or not. In what follows, we avoid these subtleties and just continue the definition of the action (3.10) to arbitrary values of \(N\), using the fact that it depends on \(N\) only through the star product definition and as an overall factor of the action, both allowing non-integer values.

In contrast to the planar case, the limit of small \(N\) has two complicating effects. The small overall factor of the action indicates that the integration in the partition function, as \(N\) goes to zero, is spread over arbitrary field configurations, irrelevant for the value of their classical action. Since the domain of the field values is non-compact, the path integral diverges in each point. The situation is similar to the strong coupling limit of Yang–Mills theory. In the latter case one can get finite answers evaluating the model on the lattice, where the gauge fields are represented as compact group valued variables, in contrast to noncompact algebra valued continuous fields. In this case one can compute the partition function or some other correlation functions, in order to see e.g. that they correspond to a confined system (for details see \cite{39}). Lattice discretizations of gauge models on noncommutative tori were considered a few years ago, in connection with the twisted Eguchi–Kawai model\(^6\).

A different approach to the problem can be based on the fact that the star product (3.4) can be equivalently written in a “dual” form

\[
A \ast B(x) = \frac{1}{\det(\pi \theta)} \int dz dy \, e^{2iz^a \theta^{-1}_a y^b} A(x+y)B(x+z). \tag{4.8}
\]

Indeed, the kernel acting on the product of \(A\) and \(B\) can be represented as the following Gaussian type integral:

\[
\frac{1}{\det(\pi \theta)} \int dz dy \, e^{2iz^a \theta^{-1}_a y^b + z^a \partial_a + y^b \partial'_b} = e^{i \partial a \partial'_b}, \tag{4.9}
\]

where \(\partial\) acts only on \(A(x)\), while the primed derivative \(\partial'\) acts only on \(B(x)\). The formal manipulation of (4.9) is given a precise meaning to, in terms of the Fourier modes of \(A(x)\) and \(B(x)\). Note, however that the integration in

\(^6\)See \cite{40} and the related references for a recent review.
variables $y$ and $z$ should be performed over an infinite range: $-\infty < y, z < +\infty$, in order to have the Gaussian integral. The infinite range of integration can be split into the toric integration and summation over the widening modes in the following way:

$$y^a = n^a + \tilde{y}^a, \quad z^a = m^a + \tilde{z}^a,$$

(4.10)

where $n^a \in \mathbb{Z}$ and $m^a \in \mathbb{Z}$ are the winding modes and the tilded variables $\tilde{y}^a \in [0, 1)$, $\tilde{z}^a \in [0, 1)$ are the toric variables. Then, the star product can be takes the following form:

$$A \ast B(x) = \int_{T^2 \times T^2} d^2\tilde{y} d^2\tilde{z} K(y, z; \theta) A(x + y) B(x + z),$$

(4.11)

where the integration is now performed over the tori and the kernel $K(y, z; \theta)$ is given by the sum over the winding modes

$$K(y, z; \theta) = \frac{1}{\det(\pi \theta)} \sum_{m,n} e^{2i(z^a + m^a)\theta^{-1}_{ab}(y^a + n^a)},$$

(4.12)

and we dropped the tildas from the toric variables $y$ and $z$.

In order to evaluate the $N \to 0$ limit of the matrix model, it suffices to take the expansion of the kernel (4.12) in the powers of $\theta^{-1}$. Thus, for the zero the winding mode one has

$$A \ast B(x) = \int_{T^2 \times T^2} d^2\tilde{y} d^2\tilde{z} K(y, z; \theta) A(x + y) B(x + z) = \int_{T^2 \times T^2} d^2\tilde{y} d^2\tilde{z} (1 + 2i[z^a]\theta^{-1}_{ab}[y^b]) A(x + y) B(x + z) = \int d^2z B(z) \int d^2y B(y) + 2i\theta^{-1}_{ab} \int d^2z [z^a - x^a] B(z) \int d^2y [y^b - x^b] A(a),$$

(4.13)

where $[\ldots]$ is the see-saw function, defined as

$$[x] = x - n, \quad \text{for } n \leq x < n + 1, \quad n \in \mathbb{Z},$$

(4.14)

which takes into account the periodicity of the variables. The last term in the expansion (4.13) can be rewritten, using the properties of the see-saw function, in the following form:

$$2i\theta^{-1}_{ab} \int d^2z ([z^a] - [x^a] + \epsilon^a(x, z)) B(z) \int d^2z ([y^b] - [x^b] + \epsilon^b(x, y)) A(y),$$

(4.15)
where \( \epsilon^a(x, y) \) is the step function

\[
\epsilon^a(x, y) = \begin{cases} 
1, & 0 \leq y^a < x^a \\
0, & \text{otherwise.}
\end{cases}
\]  

(4.16)

The result of the expansion of the kernel (4.12) leads to a rather unpleasant conclusion: even the leading terms of the star product in this limit are highly nonlocal, containing terms which are integrals of \( A, B, y^a A \) and \( z^a B \) over the torus, as well as indefinite integrals of \( A \) and \( B \), to which it is difficult to attribute any meaning.

### 4.2.1 Spin bit approach

A way out of this apparently hopeless situation, in the attempt to understand the antiplanar limit of the dilatation operator, is provided by the spin bit approach [22, 23].

As we discussed earlier, the limit \( N \to 0 \) corresponds to the strongly coupled limit of the matrix model (2.26). In the strongly coupled regime the path integral formulation does not offer a big advantage with respect to the “operator picture” of (2.1) since one can evaluate the path integral explicitly. One can start with the operator picture form (2.1) of the dilatation operator, in order to map it to a spin system.

The map is constructed as follows [22, 23]. Let us choose the vacuum of the model, which satisfies the constraint (2.3) and is cancelled by both \( \hat{\Phi}_a, a = 1, 2 \). The physical states of the model are created by acting by \( U(N) \)-invariant polynomials of \( \Phi^a \),

\[
|\Psi\rangle = P(\Phi) |\Omega\rangle, \quad \hat{\Phi}_a |\Omega\rangle = 0, \quad P(U^{-1} \Phi U) = P(\Phi).
\]  

(4.17)

The problem of classification of gauge invariant states is thus reduced to the problem of the classification of invariant polynomials \( P(\Phi) \). An invariant polynomial of the order \( L \) is given by the product of traces of \( \Phi_{a_1}, \Phi_{a_2}, \ldots, \Phi_{a_L} \), where \( a_k = 1, 2 \). This can be encoded in a state \( |\{a_1, \ldots, a_L\}, \gamma\rangle \) “encoding” the spin data described by the labels \( \{a_1, \ldots, a_L\} \) and the chain structure data described by the permutation \( \gamma \in S_L \), where \( S_L \) is the permutation group of the labels 1, 2, \ldots, \( L \). In order to complete the correspondence one

---

7Here we are ignoring the issue related to the trace identities in Lie algebras, which is due to the fact that not all polynomials of traces are algebraically independent. Normally, the true Hilbert space of the model is the one factorised over such identities. As soon as we deal with the analytic continuation over \( N \), ignoring the trace identities is equivalent to the analytic continuation from large \( N \).
should identify the “physically identical” states

\[ \{|a_{\sigma_1}, \ldots, a_{\sigma_L}\}, \sigma^{-1} \gamma \sigma \rangle \sim |\{a_1, \ldots, a_L\}, \gamma \rangle, \tag{4.18} \]

where \( \sigma \in S_L \) is an arbitrary permutation and \( \sigma_k \equiv \sigma(k) \). The equivalence \((4.18)\) reflects the invariance of the physical state with respect to relabellings.

The polynomial which corresponds to the state \(|\{a_1, \ldots, a_L\}, \gamma \rangle\) is given by

\[ P[\{a_{\sigma_1}, \ldots, a_{\sigma_L}\}, \gamma](\Phi) = \Phi^a_{i_1 \gamma_1} \Phi^a_{i_2 \gamma_2} \ldots \Phi^a_{i_L \gamma_L}. \tag{4.19} \]

It is clear that, due to the fact that \( \gamma \) is a permutation, each matrix index appears in \((4.19)\) exactly twice: once as the left and once as the right index.

The dilatation operator in the above representation can be found by the direct evaluation of \((2.1)\) on the polynomials of the type of \( P[\{a_{\sigma_1}, \ldots, a_{\sigma_L}\}, \gamma] \).

As a result we have

\[ H(2) = \frac{g_{YM}^2}{16\pi^2} \sum_{kl} H_{kl}(N\delta_{k\gamma_l} + \Sigma_{k\gamma_l}), \tag{4.20} \]

where the spin part of the Hamiltonian is given by the two site Hamiltonian of the Heisenberg spin chain

\[ H_{kl} = 2(\mathbb{I} - P_{kl}), \tag{4.21} \]

and the chain part is given by the chain splitting/joining operator \( \Sigma_{kl} \) defined as

\[ \Sigma_{kl} |\{a_{\sigma_1}, \ldots, a_{\sigma_L}\}, \gamma \rangle = (1 - \delta_{kl}) |\{a_{\sigma_1}, \ldots, a_{\sigma_L}\}, \gamma_{\sigma_{kl}} \rangle \tag{4.22} \]

As one can see, the \( N \) dependence of the dilatation operator in the spin form is extremely simple: \( N \) appears only as the coupling to the planar part of the Hamiltonian \((4.20)\). Therefore, taking the limit \( N \to 0 \) with \( g_{YM}^2 \) fixed, results just in the elimination of the planar part of the dilatation operator! The resulting expression reads

\[ H_{ap} = \frac{g_{YM}^2}{16\pi^2} \sum_{kl} H_{kl}\Sigma_{k\gamma_l}. \tag{4.23} \]

Clearly, in the model describing by the Hamiltonian \((4.23)\) there is no local structure: each spin bit is interacting equally with any other spin bits, having no preferable neighbor, a situation which is completely opposite to the planar limit.

\footnote{Note a slight difference in notations with \[22, 23\].}
5 Discussion

In this paper we considered the matrix interpretation of the SYM anomalous dimension operator in the SU(2) sector of the theory. We constructed a matrix path integral representation of the trace of the exponential of the anomalous dimension operator, which in the dual theory has the meaning of the partition function, while in the original SYM model it gives the Fourier transform of the anomalous dimension density.

The matrix model we obtained has a potential part very similar to the one of BFSS matrix model, the difference being the first order kinetic term in our case. Such a term can be obtained effectively by placing the BFSS type matrix model in a strong magnetic field. This class of models could be interpreted, from the physical viewpoint, as describing the dynamics of the zero branes in such a magnetic field. If this interpretation is correct, our approach gives the relation between the Yang–Mills coupling, string coupling and background magnetic field.

To the best of our knowledge, this type of models has not been studied in the literature before, so they can serve as a topic for a future research, as we expect them to have interesting properties.

We hope that the matrix model representation will be useful in the semi-classical study of anomalous dimensions in the nonplanar sector, by analyzing the corresponding solutions to the equations of motion. This analysis in some cases is much simpler than the diagonalization of the Hamiltonian in \[22\] [23]. This seems to be an alternative/dual approach to that of the sigma model description in the spirit of [41,42,43] which includes nonplanar effects. (The noncommutative target space seems to be a common feature in both approaches.) As we introduce the noncommutative space parametrization for our model, this becomes the path integrals over the space of noncommutative function, as obtained by the canonical quantization [44,45].

The path integral and noncommutative field theory representation turns out to be useful also for the analysis of various extremal limits, e.g. when the parameter $N$ is either large or small. As for the large $N$ limit, the model corresponding to it is well known: it is the integrable Heisenberg $\text{XXX}_{1/2}$ spin chain (or, better to say, the direct sum of the spin chains corresponding to all lengths of the chains $L^9$). In our approach it corresponds to the semi-classical limit of the matrix model. Thus, knowing all the classical solutions to the obtained model, one can compute various quantities in the quantum Heisenberg model.

Another extremal case we analyze is the limit of small $N$. Unfortunately,
the expansion in inverse powers of the noncommutative parameter does not lead to any nice physical model. The problem is caused by highly non-smooth and nonlocal limit of the star product. There may exist a hope for the analysis using a regularized version of the large $\theta$ star product. So far this appears technically difficult. On the other hand, this limit is facilitated in the spin approach, if one neglects the trace identity issue. The surprising result is that the antiplanar limit corresponds to just the elimination of the planar contribution from the dilatation operator. This appears to be possible, since the only $N$ dependence of the model comes through the trace of unity, which we make vanish.

Another point we would like to mention is the relation of our matrix model to the matrix models describing various brane systems. There is a temptation to make such an identification. In principle, our matrix model can be obtained as a limit of holomorphic brane dynamics in a strong magnetic field. Perhaps there is another possibility to find it, in the limit of fast rotating branes. In this context, it is interesting if one meets there the situation with different noncommutative phases, similar to the one which can be found in noncommutative quantum mechanics (see [46, 47]; see also [48] for applications to physical processes).

As a future development, beyond the already mentioned directions, it would be interesting to extend the analysis to the whole SYM spectrum and beyond one loop. In particular, it would be interesting to apply the matrix model approach to the study of doubling effects in the presence of fermions [49, 50, 51, 52, 28]. (Let us note that the doubling problem in the context of matrix models was already addressed in [53, 54, 55]).

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