New matrix formalism for heterotic string theory on a torus

Oleg V. Kechkin

Institute of Nuclear Physics,
M.V. Lomonosov Moscow State University,
Vorob’jovy Gory, 119899 Moscow, Russia,
e-mail: kechkin@depni.npi.msu.su

Abstract

A new Lagrange formalism based on the use of a single scalar $(d+1) \times (d+1+n)$ matrix potential is developed for the low–energy heterotic string theory with $n$ $U(1)$ gauge fields compactified from $d+3$ to 3 dimensions on a torus. This formalism also includes three pairs on–shell defined scalar and vector matrix potentials of the dimensions $(d+1) \times (d+1)$, $(d+1) \times (d+1+n)$ and $(d+1+n) \times (d+1+n)$. All these potentials undergo linear transformations when the group of charging symmetries acts.
1 Introduction

A successful choice of the variables often plays the crucial role in the study of a nonlinear theory. The most powerful tools for this study are related to use of the symmetry methods. A form of the finite symmetry transformations depends on the variables choice; the simplest and the most convenient symmetry representation has the linear form. This form allows one to deal with the general group transformation and to obtain the results which cannot be generalized again by the help of additional transformations from this symmetry group. In contrast to this situation, often in real practice one needs in the series of special transformations to obtain more or less general result in the case of the nonlinear symmetry representation. Moreover, at the end of this step–by–step special symmetry applications it is necessary to establish what kind of generality had been obtained.

In this paper we continue to study the heterotic string theory using the symmetry based methods. At low energies this theory becomes the nonlinear field theory of its massless modes living in the multidimensional space–time [1]. This effective field theory becomes the nonlinear $\sigma$–model after its compactification to three dimensions on a torus [2]–[3]. Below we develop a new representation for this $\sigma$–model based on the use of a single scalar matrix potential which has the lowest possible matrix dimension. This scalar matrix potential transforms linearly under the action of the group of charging symmetries, which preserve trivial spatial asymptotics of the all $\sigma$–model fields [4]. In fact, this representation provides the compact and convenient tool for the study of asymptotically flat fields and for the generation of asymptotically flat solutions of the heterotic string theory.

The paper is organized as follows: in the next section we “derive” one special $\sigma$–model which can be considered as the simplest nonlinear modification of the linear theory, possessing the same charging symmetry group of transformations. Both the linear and the nonlinear $\sigma$–models are given in terms of the single scalar nonquadratic matrix potential. In the next section we show how the same nonlinear $\sigma$–model arises in the framework of heterotic string theory. After that, in the next section, we establish the relation between our and chiral matrix formalisms. In fact we express the chiral matrix in terms of the “main” scalar matrix potential found in the previous section. Finally, in the last section, starting from the conserving chiral current we construct on shell three pairs of the scalar and vector matrix potentials defined in terms of the “main” scalar matrix potential. It is shown that all these scalar and vector potentials transform linearly under the action of charging symmetry group of transformations. In “Conclusions” we discuss the results obtained and their possible applications to the problem of generation of new exact solutions of the heterotic string theory.


2 \( \sigma \)-model derivation

Let us consider a class of \( \sigma \)-models possessing two commuting groups of symmetry transformations isomorphic to \( O(p_1, q_1) \) and \( O(p_2, q_2) \) as the “underlying” symmetry. Let us choose the real matrix potential \( Z \) of the dimension \( (p_1 + q_1) \times (p_2 + q_2) \) as the dynamical variable for these \( \sigma \)-models. Then, the underlying symmetry can be realized as the linear transformation

\[
Z \rightarrow C_1 Z C_2,
\]

where

\[
C_a^T \Sigma_a C_a = \Sigma_a, \quad a = 1, 2,
\]

and \( \Sigma_a \) are the diagonal matrices with \( p_a \) “+1” and \( q_a \) “−1” on their main diagonals. Thus, \( C_a \in O(p_a, q_a) \), where \( a = 1, 2 \).

Our class of \( \sigma \)-models is not empty; the Lagrangian of its simplest representative has the form

\[
L_0 = K \, Tr \left( \Sigma_1 \nabla Z \Sigma_2 \nabla Z^T \right),
\]

where \( K = \text{const.} \). This simplest theory is linear, its Euler–Lagrange equation is the Laplace equation for the matrix potential \( Z \):

\[
\nabla^2 Z = 0.
\]

A set of the nonlinear modifications of the theory \( [3] \) can be obtained using the replacement \( \Sigma_a \rightarrow \hat{\Sigma}_a^{-1} \), where the matrix fields \( \hat{\Sigma}_a(Z) \) must transform accordingly to the relations

\[
\hat{\Sigma}_1 \rightarrow C_1 \hat{\Sigma}_1 \hat{C}_1^T \quad \text{and} \quad \hat{\Sigma}_2 \rightarrow C_2^T \hat{\Sigma}_2 \hat{C}_2,
\]

when the transformation \( [1] \rightarrow [2] \) acts. In this case the Lagrangian

\[
L = K \, Tr \left( \hat{\Sigma}_1^{-1} \nabla Z \hat{\Sigma}_2^{-1} \nabla Z^T \right)
\]

remains invariant under the action of the transformation \( [1] \rightarrow [2] \), i.e., it belongs to the class under consideration.

It is not difficult to establish the explicit realization of the matrix fields \( \hat{\Sigma}_a(Z) \). The result can be expressed in terms of the following formal series:

\[
\hat{\Sigma}_1 = \kappa_1 \Sigma_1 + \kappa_{11} Z \Sigma_2 Z^T + \kappa_{12} Z \Sigma_2 Z^T \cdot \Sigma_1 \cdot Z \Sigma_2 Z^T + \cdots,
\]

\[
\hat{\Sigma}_2 = \kappa_0 \Sigma_2 + \kappa_{21} Z^T \Sigma_1 Z + \kappa_{22} Z^T \Sigma_1 Z \cdot \Sigma_2 \cdot Z^T \Sigma_1 Z + \cdots.
\]
Here the coefficients \( \kappa_{ak} = \kappa_{ak}(Z) \) \( (k = 0, 1, \ldots) \) are the invariants of the transformation (1)–(2); they can be taken as the arbitrary functions of the invariants \( I_l = Tr[(\Sigma_1 Z \Sigma_2 Z^T)^l] \), \( l = 1, 2, \ldots \). Thus, we have obtained the infinite set of the nonlinear \( \sigma \)--models possessing the “underlying” symmetry.

Now our main goal is to choose the single special nonlinear \( \sigma \)--model from this set using some additional principles. In fact we must specify the coefficient functions \( \kappa_{ak}(Z) \). Our general principle is the maximal possible similarity between \( L \) and \( L_0 \), i.e., between the nonlinear and the linear \( \sigma \)--models. We suppose, in particular, that \( L_0 \) is the zero term in the expansion of \( L \) into the perturbation series with the center at \( Z = 0 \). From this it follows that \( \kappa_{a0}(0) = 1 \). To obtain more information let us consider the special case of \( p_1 + q_1 = p_2 + q_2 \), when all the matrices under consideration are quadratic. We demand \( L \) invariance under the inversion

\[
Z \rightarrow Z^{-1}. \tag{8}
\]

This immediately gives \( \kappa_{ak} = \text{const} \) for any \( k \) and, hence, \( \kappa_{a0} = 1 \). Moreover, one obtains, that \( \kappa_{ak} = 0 \) for \( k \geq 2 \), and that \( \kappa_{11} = \kappa_{21} = \sigma_1 = \pm 1 \). Also one obtains that \( \Sigma_2 = \sigma_2 \Sigma_1 \), where \( \sigma_2 = \pm 1 \). The resulting form of the nonlinear Lagrangian \( L \) reads:

\[
L = K Tr \left[ (\Sigma + \sigma \Sigma \Sigma^T)^{-1} \nabla Z (\Sigma + \sigma \Sigma^T \Sigma Z)^{-1} \nabla Z^T \right], \tag{9}
\]

where \( \Sigma = \Sigma_1 \), \( \sigma = \sigma_1 \sigma_2 = \pm 1 \) and \( \sigma_1 K \) is redefined as \( K \). The corresponding form of the linear Lagrangian is

\[
L_0 = K Tr \left[ \Sigma \nabla Z \Sigma \nabla Z^T \right]. \tag{10}
\]

It is easy to see that \( L_0 \) does not possess the inversion (8) as the symmetry for the arbitrary nondegenerated matrix \( Z \). However, if one considers \( Z \) restricted by the relation

\[
\Sigma Z^T \Sigma = c^2 \Sigma^{-1}, \tag{11}
\]

where \( c \) is the arbitrary constant, this symmetry actually takes place. This relation means that \( c^{-1} Z \in O(p_1, q_1) \); the Lagrangians \( L \) and \( L_0 \) calculated on this group subspace read:

\[
L = \frac{1}{(1 + \sigma c^2)^2} L_0 = \frac{c^2 K}{(1 + \sigma c^2)^2} Tr \left[ \nabla Z \nabla Z^{-1} \right]. \tag{12}
\]

Let us demand the numerical identity \( L = L_0 \) on the group subspace (11); it is easy to see that it is possible only in the case of \( c^2 = 2 \) and \( \sigma = -1 \). This last opportunity fixes the sign in Eq. (9) as negative.

4
The last step is to generalize Eq. (9) with $\sigma = -1$ to the case of nonquadratic matrix potential $Z$. We make it in the following natural way:

$$L = K Tr \left[ \left( \Sigma - Z \Xi Z^T \right)^{-1} \nabla Z \left( \Xi - Z^T \Sigma Z \right)^{-1} \nabla Z^T \right],$$

(13)

where

$$\Xi = \begin{pmatrix} \Sigma & 0 \\ 0 & \tilde{\Sigma} \end{pmatrix}$$

(14)

and $\tilde{\Sigma}$ is a diagonal matrix with $\pm 1$ on the main diagonal. In this generalization we have supposed, for definiteness, that $p_2 \geq p_1$ and $q_2 \geq q_1$; it is easy to see that in the special case of $p_2 = p_1$ and $q_2 = q_1$ we regress to the special situation considered above.

The constructed $\sigma$–model is actually nonlinear: its Euler–Lagrange equation can be written in one of two equivalent forms:

$$\nabla^2 Z + 2 \nabla Z \Xi Z^T \left( \Sigma - Z \Xi Z^T \right)^{-1} \nabla Z = 0$$

(15)

or

$$\nabla^2 Z + 2 \nabla Z \left( \Xi - Z^T \Sigma Z \right)^{-1} Z^T \Sigma \nabla Z = 0.$$  

(16)

The matrix field $Z$ is defined under the coordinate space $x^\mu$. Our consideration cannot fix the dimensionality and signature of this space–time as well as the value of the parameters $K$, $p_\alpha$ and $q_\alpha$. All it will be done in the next section where the $\sigma$–model (13) will be used for the representation of the toroidally compactified heterotic string theory.

### 3 $\sigma$–model in heterotic string theory

At low energies the bosonic sector of heterotic string theory becomes the field theory of its massless modes. These modes live in $3 + d$–dimensional space–time (with the coordinates $X^M$, $M = 1, \ldots, 3 + d$) of the signature $- + \cdots +$ and include the dilaton field $\Phi$, the Kalb–Ramond field $B_{MN} = - B_{NM}$, the set of $n$ $U(1)$ gauge fields $A_{iM}^I$ ($I = 1, \ldots, n$) and the metric field (graviton) $G_{MN} = G_{NM}$. The corresponding action reads [4]:

$$S_{3+d} = \int d^{3+d}X \sqrt{- \det G_{MN}} e^{-\Phi} \left( R_{3+d} + \Phi_M \Phi^M - \frac{1}{12} H_{MNP} H^{MNP} - \frac{1}{4} F_{MN}^{I} F^{I MN} \right),$$

(17)
where $H_{M N K} = \partial_M B_{N K} - \frac{1}{2} A_M^I F_{N K}^I + \text{c.p. of } \{M N K\}$ and $F_{M N}^I = \partial_M A_N^I - \partial_N A_M^I$.

Let us perform the toroidal compactification of the first $d$ space–time dimensions, i.e., let us consider the fields independent on $X^M$ with $M = m = 1, ..., d$ and possessing the functional dependence on the coordinates $X^M$ with $M = d + \mu$, $\mu = 1, 2, 3$. In this case the field components can be classified in respect to transformations of the three–dimensional coordinates $x^\mu \equiv X^{d+\mu}$. Namely, one has (18):

1) three scalar matrices $G$, $B$ and $A$ of the dimensions $d \times d$, $d \times d$ and $d \times n$ with the components $G_{mk}, B_{mk}$ and $A_{mI} = A_I^m$ correspondingly, and also the scalar function $\phi = \Phi - \ln \sqrt{-\det G}$;

2) three vector columns $\vec{A}_1, \vec{A}_2$ and $\vec{A}_3$ of the dimensions $d \times 1$, $d \times 1$ and $n \times 1$; their components read:

$$
(\vec{A}_1)_{m\mu} = (G^{-1})_{mk} G_{k,d+\mu},
$$

$$
(\vec{A}_2)_{m\mu} = B_{m,d+\mu} - B_{pq}(\vec{A}_1)_{m\mu} + \frac{1}{2} A_{mI} (A_3)_{I\mu},
$$

$$
(\vec{A}_3)_{I\mu} = -A^I_{d+\mu} + A^I_{m}(\vec{A}_1)_{m\mu}.
$$

(18)

and

3) two tensor fields

$$
h_{\mu\nu} = e^{-2\phi} \left[ G_{\mu\nu} - G_{mk}(\vec{A}_1)_{m\mu} (\vec{A}_1)_{k\nu} \right],
$$

$$
b_{\mu\nu} = B_{\mu\nu} - B_{mk}(\vec{A}_1)_{m\mu} (\vec{A}_1)_{k\nu} - \frac{1}{2} \left[ (\vec{A}_1)_{m\mu} (A_3)_{m\nu} - (A_1)_{m\nu} (\vec{A}_2)_{m\mu} \right].
$$

(19)

In three dimensions the field $b_{\mu\nu}$ is nondynamical; following (3) we put $b_{\mu\nu} = 0$ without contradiction with the motion equations. Moreover, in three dimensions it is possible to introduce pseudoscalar fields $u, v$ and $s$ accordingly the relations (3, 4, 5)

$$
\nabla \times \vec{A}_1 = e^{2\phi} G^{-1} \left[ \nabla u + (B + \frac{1}{2} AA^T) \nabla v + A \nabla s \right],
$$

$$
\nabla \times \vec{A}_2 = e^{2\phi} G \nabla v - (B + \frac{1}{2} AA^T) \nabla \times \vec{A}_1 + A \nabla \times \vec{A}_3.
$$

$$
\nabla \times \vec{A}_3 = e^{2\phi} (\nabla s + A^T \nabla v) + A^T \nabla \times \vec{A}_1.
$$

(20)

Finally, the bosonic sector of the heterotic string theory toroidally compactified to three dimensions is equivalent on shell to the effective three–dimensional theory which describes
the interacting scalar fields $G$, $B$, $A$ and $\phi$ and the pseudoscalar ones $u$, $v$ and $s$ coupled to the metric $h_{\mu \nu}$.

For our purposes it is useful to embed all the scalar and pseudoscalar fields into the following matrices $\mathcal{X}$ and $\mathcal{A}$ of the dimensions $(d+1) \times (d+1)$ and $(d+1) \times n$ correspondingly:

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T A s + \frac{1}{2} s^T s & v^T X - u^T \\ X v + u + A s & X \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix},$$

(21)

where $X = G + B + \frac{1}{2} A A^T$ (11), (12). In terms of these matrices the action of the effective three–dimensional theory of heterotic string reads:

$$S_3 = \int d^3 x \sqrt{h} \{-R_3 + L_3\},$$

(22)

where

$$L_3 = \text{Tr} \left[ \frac{1}{4} \left( \nabla \mathcal{X} \right) G^{-1} \left( \nabla \mathcal{X}^T - A \nabla A^T \right) G^{-1} + \frac{1}{2} \nabla A^T G^{-1} \nabla A \right],$$

(23)

$$G = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - A A^T \right),$$

(24)

$h = \text{det} \, h_{\mu \nu}$ and the scalar curvature $R_3$ is constructed using the metric $h_{\mu \nu}$. This action describes the nonlinear $\sigma$–model coupled to the three–dimensional gravity and has the form of the action of the stationary Einstein–Maxwell theory [6]. In Eq. (23) the matrices $\mathcal{X}$ and $\mathcal{A}$ play the same role as the complex Ernst potentials in Einstein–Maxwell theory [7] and can be named as “matrix Ernst potentials” [5].

The main statement of this chapter is the following: the heterotic string theory $\sigma$–model given by the Lagrangian $L_3$ is the special one from the class of $\sigma$–models given by Eq. (13). Actually, let us put in Eq. (13) $K = 1$, $\Sigma = \text{diag} \left( -1, -1, +1, \cdots, +1 \right)$ (two “$-1$” and $d - 1$ “$+1$” on the main diagonal, so $\Sigma$ is the $(d + 1) \times (d + 1)$ matrix) and $\Sigma = 1_n$ (where $1_n$ is the $n \times n$ unit matrix). Let us also introduce for the $(d + 1) \times (d + 1 + n)$ matrix $\mathcal{Z}$ the following parametrization:

$$\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2),$$

(25)

where the $(d + 1) \times (d + 1)$ and $(d + 1) \times n$ matrices $\mathcal{Z}_1$ and $\mathcal{Z}_2$ read:

$$\mathcal{Z}_1 = 2 (\mathcal{X} + \Sigma)^{-1} - \Sigma, \quad \mathcal{Z}_2 = \sqrt{2} (\mathcal{X} + \Sigma)^{-1} A$$

(26)
(the Einstein–Maxwell analogy of these potentials one can find in [8]). Then, after some nontrivial algebraical calculations one can prove that in this special case \( L = L_3 \). Thus,

\[
L_3 = \text{Tr} \left[ \left( \Sigma - Z \Xi Z^T \right)^{-1} \nabla Z \left( \Xi - Z^T \Sigma Z \right)^{-1} \nabla Z^T \right],
\]

where the matrices \( \Sigma \) and \( \Xi \) (see Eq. (14)) are defined as it was explained above. We see that the heterotic string theory \( \sigma \)–model can be represented in terms of the matrix potential \( Z \). As the straightforward consequence of this fact we conclude (see Eqs. (1)–(2)) that \( L_3 \) is invariant under the transformation

\[
Z \rightarrow Z' = C_1 Z C_2,
\]

where the matrices \( C_1 \) and \( C_2 \) satisfy the \( O(2, d - 1) \) and \( O(2, d - 1 + n) \) group relations

\[
C_1^T \Sigma C_1 = \Sigma \quad \text{and} \quad C_2^T \Xi C_2 = \Xi.
\]

Then, the matter part of the heterotic string theory equations is given by Eq. (15) or (16), whereas the Einstein equation reads:

\[
R_{3 \mu \nu} = \text{Tr} \left[ \left( \Sigma - Z \Xi Z^T \right)^{-1} \nabla_{(\mu} Z \left( \Xi - Z^T \Sigma Z \right)^{-1} \nabla_{\nu)} Z^T \right].
\]

In [4] it was shown that Eqs. (28)–(29) gives the total group of charging symmetries for the theory under consideration. The charging symmetries form the subgroup of the complete symmetry group of the \( \sigma \)–model; the transformations from this subgroup preserve trivial values of the all three–dimensional “matter” fields. These trivial values are defined as the zero ones for the fields \( B, A, \phi, u, v, s \) and as the \( d \times d \) matrix \( \text{diag}(-1, +1, \cdots, +1) \) for \( G \). From the \( (d + 3) \)–dimensional point of view this field configuration (together with \( h_{\mu \nu} = \delta_{\mu \nu} \)) describes the empty Minkowskian space–time in the Carthesian coordinates. From Eq. (24) it follows that \( \mathcal{X} = \Sigma, \mathcal{A} = 0 \) for the trivial fields and, therefore (see Eqs. (23)–(24)), \( Z = 0 \) in this case. The zero \( Z \)–value is preserved by the transformation (28) with the arbitrary matrices \( C_1 \) and \( C_2 \); however only the transformation with \( C_1 \) and \( C_2 \) restricted by the relations (29) belongs to the group of \( \sigma \)–model symmetries.

A class of the asymptotically flat solutions of the theory under consideration is defined as the class containing all solutions trivial at the spatial infinity [4], [6], [9]. Thus, at the spatial infinity all the solutions from this class become the trivial field configuration discussed above. It is clear that the charging symmetry subgroup contains all the transformations which can be applied for generation of the new asymptotically flat solutions from the known ones. The
noncharging symmetries change the three–dimensional field asymptotics and provide trivial action on the solutions; so the charging symmetry subgroup includes all the resultative symmetry transformations. The linear form of the transformation (28) extremely simplifies the solution generation procedure; its straightforward application allows one to obtain the most general possible generation results for the asymptotically flat solutions. To compare our solution generation approach with the previously known ones one can use [10].

4 Chiral matrix and related structures

As it has been established above, the effective three–dimensional $\sigma$–model of the heterotic string theory compactified on a torus can be represented in two different but algebraically related forms: in terms of the potential matrix pair $X, A$ and in terms of the single matrix potential $\mathcal{Z}$. Now we consider the chiral matrix representation of the same theory, which can be originally given in terms of the pair $X, A$. Actually, it is easy to prove that the $[2(d + 1) + n] \times [2(d + 1) + n]$ matrix

$$
\mathcal{N} = \begin{pmatrix}
G^{-1} & G^{-1}X & G^{-1}A \\
X^T G^{-1} & X^T G^{-1}X & X^T G^{-1}A \\
A^T G^{-1} & A^T G^{-1}X & A^T G^{-1}A
\end{pmatrix},
$$

satisfies the relation

$$
\mathcal{N} \mathcal{L} \mathcal{N} = 2 \mathcal{N},
$$

where the matrix $\mathcal{L}$ reads

$$
\mathcal{L} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
$$

Then the matrix

$$
\mathcal{M} = \mathcal{N} - \mathcal{L}
$$

satisfies the relations

$$
\mathcal{M}^T \mathcal{L} \mathcal{M} = \mathcal{L}, \quad \mathcal{M}^T = \mathcal{M}
$$

which mean that $\mathcal{M} \in O(d + 1, d + 1 + n)/O(d + 1) \times O(d + 1 + n)$. The complete list of $\sigma$–models possessing the matrix representations one can find in [11]. The coset matrix
representation for the heterotic string theory compactified to three dimensions on a torus was established at the first time by A. Sen in [3]. Our matrix $M$ differs from the Sen’s one and related to the matrix Ernst potential formulation.

The Lagrangian $L_3$ can be rewritten in terms of this coset matrix in the following “chiral form” ([3], [5]):

$$L_3 = \frac{1}{8} \text{Tr}(\vec{J}^2),$$

where

$$\vec{J} = \nabla M \cdot M^{-1}.$$ \hspace{1cm} (37)

From this it immediately follows that the total symmetry group for the theory is $O(d+1, d+1+n)$, because $L_3$ remains invariant under the action of the transformation

$$M \rightarrow M' = \hat{C}^T M \hat{C},$$ \hspace{1cm} (38)

where

$$\hat{C}^T \mathcal{L} \hat{C} = \mathcal{L},$$ \hspace{1cm} (39)

i.e. if $\hat{C} \in O(d+1, d+1+n)$. The group of charging symmetries forms a subgroup of this total symmetry group; the symmetry matrix $\hat{C}$ belongs to this subgroup if it preserves the trivial field configuration defined in the previous section. This trivial field configuration corresponds to the matrix

$$M_0 = \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ \hspace{1cm} (40)

so the charging symmetry matrix $\hat{C}$ must satisfy the restriction

$$\hat{C}^T M_0 \hat{C} = M_0$$ \hspace{1cm} (41)

in addition to the relation (39). Eq. (11) shows that this matrix also belongs to the group $O(2, 2d+n)$, so the matrix of the charging symmetry subgroup belongs to the intersection of the groups $O(d+1, d+1+n)$ and $O(2, 2d+n)$. We have the explicit realization of this symmetry in terms of the potential $\mathcal{Z}$ (see Eqs. (28)–(31)). To obtain its chiral realization (in the form of Eq. (38) with $\hat{C}$ restricted by Eqs. (39) and (41)) it is useful to express
the chiral matrix $\mathcal{M}$ in terms of the matrix potential $\mathcal{Z}$. To do it, let us introduce the $(d + 1) \times (d + 1 + n)$ matrix

$$
\mathcal{Y} = (\mathcal{X}, \mathcal{A}). \tag{42}
$$

Then, for the matrix $\mathcal{N}$ one has:

$$
\mathcal{N} = \begin{pmatrix} \mathcal{G}^{-1} & \mathcal{G}^{-1} \mathcal{Y} \\ \mathcal{Y}^{T} \mathcal{G}^{-1} & \mathcal{Y}^{T} \mathcal{G}^{-1} \mathcal{Y} \end{pmatrix}. \tag{43}
$$

Here the matrix $\mathcal{G}$ can also be written by the help of $\mathcal{Y}$:

$$
\mathcal{G} = \frac{1}{2} \left[ \mathcal{Y} l^{T} + l \mathcal{Y} + \mathcal{Y} (1 - l^{T} l) \mathcal{Y}^{T} \right], \tag{44}
$$

where the matrix

$$
l = (1 \ 0) \tag{45}
$$

is constructed from the $(d + 1) \times (d + 1)$ and $(d + 1) \times n$ blocks. Now one can express the matrices $\mathcal{Y}$ and $\mathcal{G}$ in terms of the matrix potential $\mathcal{Z}$ using Eqs. (25) and (26); the result is:

$$
\mathcal{Y} = (\Sigma + \mathcal{Z} l^{T})^{-1} (l + \mathcal{Z} P), \quad \mathcal{G} = (\Sigma + \mathcal{Z} l^{T})^{-1} (\Sigma - \mathcal{Z} \Xi \mathcal{Z}^{T}) (\Sigma + l \mathcal{Z}^{T})^{-1}, \tag{46}
$$

where

$$
P = \begin{pmatrix} -\Sigma & 0 \\ 0 & \sqrt{2} \end{pmatrix}. \tag{47}
$$

Using these relations, one establishes that

$$
\mathcal{N} = \Psi^{T} H^{-1} \Psi, \tag{48}
$$

where $H = \hat{\Sigma}_{1} = \Sigma - \mathcal{Z} \Xi \mathcal{Z}^{T}$ and $\Psi = \Psi(\mathcal{Z})$ is the linear $(d + 1) \times [2(d + 1) + n]$ matrix function

$$
\Psi = \mathcal{A} + \mathcal{Z} \mathcal{B} \tag{49}
$$

with the proportionality coefficient matrices

$$
\mathcal{A} = (\Sigma, l), \quad \mathcal{B} = (l^{T}, P). \tag{50}
$$
Thus, for the explicit form of $\mathcal{M}(Z)$ one obtains:

$$\mathcal{M} = (A + ZB)^T H^{-1}(A + ZB) - \mathcal{L}. \quad (51)$$

As it can be easily verified, the matrices $A$ and $B$ yield the following remarkable properties:

\[
\begin{align*}
(a) & \quad A\mathcal{L}A^T = 2\Sigma, \\
(b) & \quad A\mathcal{L}B^T = 0, \\
(c) & \quad B\mathcal{L}B^T = -2\Sigma.
\end{align*}
\quad (52)
\]

Now using them and Eq. (51) we are ready to calculate the matrices $\hat{C}_a$ $(a = 1, 2)$ which define the transformations (38) of the chiral matrix $\mathcal{M}$ corresponding to the ones of the potential $Z$ given by the matrices $C_a$ (see Eqs. (28) and (29)). Actually, from Eqs. (39) and (51) it follows that

$$M' = \hat{C}_a^T \left[ (A + ZB)^T H^{-1}(A + ZB) - \mathcal{L} \right] \hat{C}_a; \quad (53)$$

whereas from Eqs. (28) and (51) one obtains that

$$M' = (A + Z'B)^T H'^{-1}(A + Z'B) - \mathcal{L}, \quad (54)$$

where

$$Z' = C_1 Z \quad \text{and} \quad Z' = Z C_2 \quad (55)$$

for $a = 1$ and $a = 2$ correspondingly. From comparison of Eq. (53) and Eqs. (54)–(55) it is easy to obtain the relations defining the matrices $\hat{C}_a$. So, for $a = 1$ one has ($H' = C_1 HC_1^T$)

\[
\begin{align*}
(a) & \quad B\hat{C}_1 = B, \\
(b) & \quad A\hat{C}_1 = C_1^{-1} A, \\
(c) & \quad \hat{C}_1^T \mathcal{L}\hat{C}_1 = \mathcal{L}.
\end{align*}
\quad (56)
\]

From Eqs. (56-a) and (52-b) it follows that $\hat{C}_1 = 1 + \mathcal{L}A^T \xi$, where the matrix $\xi = \frac{1}{2}\Sigma(C_1^{-1} - 1)A$ in view of Eq. (56-b). Thus, the matrix $\hat{C}_1$ reads:

$$\hat{C}_1 = 1 + \frac{1}{2} \mathcal{L}A^T \Sigma(C_1^{-1} - 1)A; \quad (57)$$
the condition (58-c) is satisfied in view of Eqs. (29) and (52-a). Then, for \( a = 2 \) the relations defining \( \hat{C}_2 \) relations read \((H' = H)\):

\[
\begin{align*}
(a) & \quad \mathcal{A}\hat{C}_2 = \mathcal{A}, \\
(b) & \quad B\hat{C}_2 = C_2B, \\
(c) & \quad \hat{C}_2^T \mathcal{L}\hat{C}_2 = \mathcal{L}.
\end{align*}
\] (58)

From Eqs. (58-a) and (52-b) it follows that \( \hat{C}_1 = 1 + \mathcal{L}B^T\eta \), where the matrix \( \eta = \frac{1}{2} \Xi (1 - C_2)B \) in view of Eq. (58-b). Thus, the matrix \( \hat{C}_2 \) reads:

\[
\hat{C}_2 = 1 + \frac{1}{2} \mathcal{L}B^T\Xi (1 - C_2)B; \quad (59)
\]

the condition (58-c) is satisfied in view of Eqs. (29) and (52-c). It is easy to see that from Eq. (52-b) it also follows that

\[
[\hat{C}_1, \hat{C}_2] = 0, \quad (60)
\]

i.e., the charging symmetry subgroups ‘1’ and ‘2’ commute; this fact also follows from Eq. (28) which gives the action of the same subgroups in the \( \mathcal{Z} \)–representation. Moreover, from Eqs. 57 and (59) it follows that the matrices \( \mathcal{A} \) and \( \mathcal{B} \) provide the transition from the \( \mathcal{Z} \)–representation to the chiral one for the charging symmetry subgroups ‘1’ and ‘2’ correspondingly.

### 5 Vector matrix potentials

It is possible to express the Lagrangian \( L_3 \) only in terms of the matrix function \( \Psi \). Actually, using the relations (52) one can prove that

\[
H = \frac{1}{2} \Psi \mathcal{L}\Psi^T, \quad (61)
\]

so

\[
\mathcal{M} = 2\Psi^T(\Psi \mathcal{L}\Psi^T)^{-1}\Psi - \mathcal{L}. \quad (62)
\]

Then, using the relations (32) and (37) it is easy to show that

\[
\vec{j} = \vec{J}\mathcal{L} = \left(1 - \frac{1}{2} \Psi^T H^{-1}\Psi \mathcal{L}\right) \nabla \Psi^T H^{-1}\Psi - \Psi^T H^{-1}\nabla \Psi \left(1 - \frac{1}{2} \mathcal{L}\Psi^T H^{-1}\Psi\right), \quad (63)
\]
and from Eq. (36) for the three–dimensional Lagrangian $L_3$ one obtains:

$$L_3 = -\frac{1}{2} \text{Tr} \left[ \nabla \Psi \left( \mathcal{L} - \frac{1}{2} \mathcal{L} \Psi^T H^{-1} \Psi \mathcal{L} \right) \nabla \Psi^T T^{-1} \right].$$

(64)

Now, using Eqs. (49) and (52) it is easy to check that Eq. (27) actually takes place as the consequence of Eq. (64) and, hence, Eqs. (23) and (36) (it is not difficult to establish the equivalence of $L_3$ defined by Eqs. (23) and (36)).

Then, from Eqs. (38), (39) and (63) it follows that the transformation low for the matrix current $\vec{j}$ reads:

$$\vec{j} \rightarrow \vec{j}' = \hat{C}^T \vec{j} \hat{C},$$

(65)

so it has the same form as the one for the chiral matrix $\mathcal{M}$. The matrix current $\vec{j}$ is antisymmetric ($\vec{j} = \nabla \mathcal{M} \mathcal{L} \mathcal{M} = -\mathcal{M} \mathcal{L} \nabla \mathcal{M} = -\vec{j}^T$); this property is preserved by the transformation (65). In this section we study the action of the charging symmetry group of transformations on the vector matrix potentials which can be introduced as the potentials related to the divergence–free vector field $\vec{j}$. Namely, from the chiral form (36) of the Lagrangian $L_3$ it follows that the matter part of motion equations reads:

$$\nabla \vec{j} = 0,$$

(66)

so one can introduce on shell the vector matrix potential $\vec{\Omega}$ accordingly the relation

$$\nabla \times \vec{\Omega} = \vec{j}.$$  

(67)

From Eqs. (63) and (64) one obtains that the matrix potential $\vec{\Omega}$ also transforms under the action of the general symmetry group as the chiral matrix $\mathcal{M}$,

$$\vec{\Omega} \rightarrow \vec{\Omega}' = \hat{C}^T \vec{\Omega} \hat{C}.$$  

(68)

The matrix potential $\vec{\Omega}$ is antisymmetric, this property is preserved by the transformation (68). We would like to extract from $\vec{\Omega}$ the set of independent vector potentials, to express these potentials in terms of $\mathcal{Z}$ and to establish the action of the charging symmetry group of transformations on this set.

To realize this program let us express the current $\vec{j}$ in terms of the potential $\mathcal{Z}$ using Eqs. (51)–(52); the result reads:

$$\vec{j} = -A^T \vec{j}_1 B + B^T \vec{j}_1^T A + A^T \vec{j}_2 A + B^T \vec{j}_3 B,$$

(69)
where

\[
\begin{align*}
\vec{j}_1 &= H^{-1} \nabla Z - H^{-1} \left( Z \Xi \nabla Z^T - \nabla Z \Xi Z^T \right) H^{-1} Z, \\
\vec{j}_2 &= H^{-1} \left( Z \Xi \nabla Z^T - \nabla Z \Xi Z^T \right) H^{-1}, \\
\vec{j}_3 &= \nabla Z^T H^{-1} Z - Z^T H^{-1} \nabla Z + Z^T H^{-1} \left( Z \Xi \nabla Z^T - \nabla Z \Xi Z^T \right) H^{-1} Z.
\end{align*}
\]

(70)

Thus, the current \( \vec{j} \) is a linear combination of three currents \( \vec{j}_i \), \( i = 1, 2, 3 \). These currents are obviously linear independent and expressed in terms of \( Z \); also they are divergent–free. To prove this fact, let us introduce the ‘proection’ operators

\[
\Pi_1 = \frac{1}{2} \mathcal{L} A^T \Sigma \quad \text{and} \quad \Pi_2 = -\frac{1}{2} \mathcal{L} B^T \Xi.
\]

(71)

From the relations (52) it follows that

\[
\begin{align*}
\mathcal{A} \Pi_1 &= 1, \quad \mathcal{B} \Pi_1 = 0, \\
\mathcal{A} \Pi_2 &= 0, \quad \mathcal{B} \Pi_2 = 1,
\end{align*}
\]

(72)

so

\[
\vec{j}_1 = -\Pi_2^T \vec{j} \Pi_1, \quad \vec{j}_2 = \Pi_1^T \vec{j} \Pi_1, \quad \vec{j}_3 = \Pi_2^T \vec{j} \Pi_2.
\]

(73)

Thus, from Eq. (66) it follows that \( \nabla \vec{j}_i = 0 \). Let us now define three vector matrix potentials \( \vec{\Omega}_i \) accordingly the relations

\[
\nabla \times \vec{\Omega}_i = \vec{j}_i;
\]

(74)

from Eq. (70) one concludes that \( \vec{\Omega}_1 \), \( \vec{\Omega}_2 \) and \( \vec{\Omega}_3 \) have the matrix dimensions \((d + 1) \times (d + 1 + n)\), \((d + 1) \times (d + 1)\) and \((d + 1 + n) \times (d + 1 + n)\) correspondingly.

The matrix vector potentials \( \vec{\Omega}_i \) are algebraically independent. The action of the group of charging symmetry transformations on the set of \( \vec{\Omega}_i \) can be established using Eqs. (57), (59) and (68). However, the same result can be easily obtained from Eqs. (28) and (70); it reads:

\[
\vec{\Omega}_1 \rightarrow \vec{\Omega}_1' = C_1^T \vec{\Omega}_1 C_2, \quad \vec{\Omega}_2 \rightarrow \vec{\Omega}_2' = C_1^T \vec{\Omega}_2 C_1^{-1}, \quad \vec{\Omega}_3 \rightarrow \vec{\Omega}_3' = C_2^T \vec{\Omega}_3 C_2.
\]

(75)

Thus, the charging symmetry transformations do not ‘mix’ the potentials \( \vec{\Omega}_i \), and these vector potentials (as well as the scalar one \( Z \)) transform as singlets of the group of charging symmetry transformations.
Now let us note that the matrices $\mathcal{M}$ and $\vec{\Omega}$ has the very similar structure in terms of their linearly independent components. Actually, for the matrix $\vec{\Omega}$ one has (see Eqs. (67) and (69)):

$$
\vec{\Omega} = -A^T \vec{\Omega}_1 B + B^T \vec{\Omega}_2 A + A^T \vec{\Omega}_2 A + B^T \vec{\Omega}_3 B,
$$

whereas for the matrix $\mathcal{M}$ one obtains from Eq. (51) that

$$
\mathcal{M} = A^T \mathcal{M}_1 B + B^T \mathcal{M}_1 A + A^T \mathcal{M}_2 A + B^T \mathcal{M}_3 B - L,
$$

where

$$
\mathcal{M}_1 = H^{-1} Z, \quad \mathcal{M}_2 = H^{-1}, \quad \mathcal{M}_3 = 2^T H^{-1} Z.
$$

Both the sets $\vec{\Omega}_i$ and $\mathcal{M}_i$ can be extracted from their linear combinations $\vec{\Omega}$ and $\mathcal{M}$ (Eqs. (70) and (77)) in the same manner as it had been done for the currents $\vec{j}_i$ extracted from Eq. (69). Thus, for the components $\mathcal{M}_i$ one has:

$$
\mathcal{M}_1 = \Pi_1^T \mathcal{M} \Pi_2, \quad \mathcal{M}_2 = \Pi_1^T \mathcal{M} \Pi_1 + \frac{1}{2} \Sigma, \quad \mathcal{M}_3 = \Pi_2^T \mathcal{M} \Pi_2 - \frac{1}{2} \Xi
$$

(where the relations

$$
\Pi_1^T L \Pi_2 = 0, \quad \Pi_1^T L \Pi_1 = \frac{1}{2} \Sigma, \quad \Pi_2^T L \Pi_2 = -\frac{1}{2} \Xi
$$

following from Eq. (82) had been used).

It is easy to see that the matrices $\mathcal{M}_i$ transform exactly as the matrices $\vec{\Omega}_i$ under the action of charging symmetry group, i.e.,

$$
\mathcal{M}_1 \rightarrow \mathcal{M}_1' = C_1^{-1} \mathcal{M}_1 C_2, \quad \mathcal{M}_2 \rightarrow \mathcal{M}_2' = C_1^{-1} \mathcal{M}_2 C_1^{-1}, \quad \mathcal{M}_3 \rightarrow \mathcal{M}_3' = C_2^T \mathcal{M}_3 C_2.
$$

Thus, we have established two sets of the linearly independent matrix potentials, the scalar $\mathcal{M}_i$ and the vector $\vec{\Omega}_i$ ones. These scalar and vector potentials with equal indeces have the same matrix dimensions and the same transformation properties. They can be combined into three pairs of potentials with one scalar and one vector matrix potentials in any pair. All these potentials are defined in terms of the “main” matrix potential $Z$; using Eq. (67) it is possible to relate vector and scalar matrix potentials.

At the end of this chapter let us come back to the linear theory described by the Lagrangian $L_0$. The heterotic string theory $\sigma$–model had been constructed as some “minimal”
nonlinear generalization of this linear theory, so one can wait that the properties of the nonlinear theory, being linearized in the appropriate way, take place for the linear one. We would like to establish the sets of the potentials $\mathcal{M}_i$ and $\vec{\Omega}_i$ for the linear theory. To do it let us replace

$$H = \hat{\Sigma}_1 \to \Sigma_1 = \Sigma.$$  \hfill (82)

This immediately gives the following relations for the components $\mathcal{M}_i$ (see Eq. (78)):

$$\mathcal{M}_1 = \Sigma Z, \quad \mathcal{M}_2 = \Sigma, \quad \mathcal{M}_3 = Z^T \Sigma Z.$$  \hfill (83)

It is easy to see that the charging symmetry transformations act on this set of $\mathcal{M}_i$ exactly as on the nonlinear one. Then, if one also removes the nonlinear terms of the third and fourth square from Eq. (70), one obtains the set of matrix currents

$$\vec{j}_1 = \Sigma \nabla Z,$$

$$\vec{j}_2 = \Sigma \left( Z \Xi \nabla Z^T - \nabla Z \Xi Z^T \right) \Sigma,$$

$$\vec{j}_3 = \nabla Z^T \Sigma Z - Z^T \Sigma \nabla Z.$$  \hfill (84)

which conserve on shell (i.e., have the zero divergences), when the motion equation (4) of the linear theory is satisfied. After that we define the set of $\vec{\Omega}_i$ accordingly Eq. (74) and see, that these vector potentials also transform as their analogies constructed for the heterotic string theory. We can go even more far and to define the matrices $\vec{j}_i$, $\mathcal{M}_i$ and $\vec{\Omega}_i$ for the linear theory accordingly Eqs. (69), (76) and (77), where the sets $\vec{j}_i$, $\mathcal{M}_i$ and $\vec{\Omega}_i$ must be taken in their linearized form. It is clear that, for example, Eq. (66) takes place again, but now $\mathcal{L} \mathcal{M} \mathcal{L} \neq \mathcal{M}^{-1}$ and $\nabla \mathcal{M} \mathcal{M}^{-1} \mathcal{L} \neq \vec{j}$. Thus, it is impossible to rewrite the linear theory (3) in the same chiral form as the heterotic string theory. However this fact is not really surprising, because in the opposite case one can make the false conclusion about the equivalence of these essentially different theories.

6 Conclusion

In this paper we have developed a new formalism for the heterotic string theory compactified to three dimensions on a torus. This formalism is based on the use of the single real matrix potential $Z$, which has the lowest possible matrix dimension possessing by the theory. Actually, $Z$ is the $(d + 1) \times (d + 1 + n)$ matrix, whereas the number of components of the scalar fields for the effective three–dimensional heterotic string theory is exactly $(d + 1) \cdot (d + 1 + n)$.
The new formalism is naturally related to two sets of three scalar and three vector matrix potentials $\mathcal{M}_i$ and $\vec{\Omega}_i$ (see some analogies for the case of General Relativity in [12]). Both these sets, as well as the potential $\mathcal{Z}$, undergo linear transformations under the action of the charging symmetries.

The new formalism can be effectively applied for generation of the asymptotically flat solutions of the heterotic string theory. The straightforward application of the relations (75) and (78) allows one to obtain the complete set of components for the vector and scalar fields of the generated solution using only the algebraical (not differential) calculations. The result of this generation procedure will be nongeneralizable by means of the three–dimensional charging symmetries in the case when one uses for generation the matrices $C_\alpha$ restricted only by Eq. (29).

The developed formalism can be used for straightforward construction of the new solutions. In the forthcoming publications we hope to present some charging symmetry complete classes of the asymptotically flat solutions constructed using the $\mathcal{Z}$–representation of the theory. Here we would like to note that the potentials $\mathcal{M}_i$ and $\vec{\Omega}_i$ provide the compact and convenient tool for the study of monopole and dipole characteristics of any asymptotically flat field configuration. Moreover, in this analysis one can use the linearized variants of the matrix potentials, because the higher nonlinear terms correspond to the higher multipole moments.

Acknowledgments

This work was supported by RFBR grant $N^0$ 00 02 17135.

References

[1] E. Kiritsis, “Introduction to superstring theory”, Lectures presented at the Catholic University of Leuven and University of Padova, 1996-1997. Leuven, Belgium: Leuven Univ. Pr. (1998) 315 p, (Leuven notes in mathematical and theoretical physics. B9).

[2] J. Maharana and J.H. Schwarz, Nucl. Phys. B390 (1993) 3.

[3] A. Sen, Nucl. Phys. B434 (1995) 179.

[4] A. Herrera–Aguilar and O.V. Kechkin, Phys. Rev. D59 (1999) 124006.

[5] A. Herrera Aguilar, O.V. Kechkin, Int. J. Mod. Phys. A13 (1998) 393.
[6] D. Kramer, H. Stephani, M. MacCallum, E. Herlt, “Exact solutions of the Einstein field equations”, Deutcher Verlag der Wissenschaften, Berlin, 1980.

[7] F. J. Ernst, Phys. Rev. 168 5 (1968) 1415; Phys. Rev. 167 5 (1968) 1175.

[8] P.O. Mazur, Acta Phys. Pol., B14 (1983) 219.

[9] A. Herrera–Aguilar, O.V. Kechkin, Mod. Phys. Lett. A13 (1998) 1907.

[10] D. Yuom, Phys. Rept. 316 (1999) 1.

[11] P. Breitenlohner, D. Maison G. W. Gibbons Commun. Math. Phys. 120 (1988) 225; P. Breitenlohner, Dieter Maison Commun. Math. Phys. 209 (2000) 785.

[12] W. Kinnersley, J. Math. Phys. 14 (1973) 651.