Trivial unit conjecture and homotopy theory

Shengkui Ye

January 14, 2014

Abstract

A homotopy theoretic description is given for trivial unit conjecture in the group ring \( \mathbb{Z}G \).

1 Introduction

Let \( G \) be a torsion-free group and \( \mathbb{Z}G \) the integral group ring. The trivial unit conjecture for \( G \) says that any invertible element (unit) of \( \mathbb{Z}G \) is of the form \( \pm g \) for some \( g \in G \) (cf. [Pa], Chapter 13). For solving such a conjecture, to the author’s knowledge, almost all the approaches used are algebraic (cf. [Cp] and references therein). In this note, we give a homotopy theoretic description of such a conjecture.

Let \( X \) be a CW complex with fundamental group \( \pi_1(X) = G \). For any integer \( d \geq 2 \) and map \( f : S^d \to X \vee S^d \), we construct a CW complex \( Y_f = (X \vee S^d) \cup f e^{d+1} \). In this note, the following homotopy theoretic characterization is obtained:

**Theorem 1** Let \( G \) be a torsion-free group. The trivial unit conjecture for \( G \) is true if and only if for an Eilenberg-Mac Lane space \( X = BG \), the element \( [f] \in \pi_d(X \vee S^d, S^d) \) (the relative homotopy group of the universal covering space) vanishes for some lifting of \( S^d \) whenever the inclusion \( i_f : X \to Y_f \) is a homotopy equivalence.

All modules considered in this note are left modules. Let \( \tilde{Y}_f \) be the universal covering space of \( Y_f \) and \( C_i(\tilde{Y}_f) \) the \( i \)-th term of the cellular chain complex of \( \tilde{Y}_f \). By definition, \( C_i(\tilde{Y}_f) \) is a free \( \mathbb{Z}G \)-module spanned by the set of all \( i \)-cells. For the inclusion \( i_f : X \to Y_f \), we have a cellular map \( \tilde{i}_f : \tilde{X} \to \tilde{Y}_f \) which lifts \( i_f \). As the map \( \tilde{i}_f \) induces the identity homomorphism on fundamental groups of \( X \) and \( Y_f \), we may assume that \( \tilde{X} \) is a subspace
of $\tilde{Y}_f$. The relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ of $(\tilde{Y}_f, \tilde{X})$ is of the following form
\[ 0 \to C_{d+1}(\tilde{Y}_f, \tilde{X}) = \mathbb{Z}G \to C_d(\tilde{Y}_f, \tilde{X}) = \mathbb{Z}G \to 0. \]
This is a chain complex whose terms are all vanishing except for the $d$-th term a free $\mathbb{Z}G$-module spanned by $S^d$ and the $(d+1)$-th term a free $\mathbb{Z}G$-module spanned by $e^{d+1}$. Let $\gamma_f = \partial(1) \in \mathbb{Z}G$, the unique element determined by the boundary map $\partial$. We give a homotopy theoretic description of units in $\mathbb{Z}G$ as follows.

**Lemma 2** Let $\gamma_f \in \mathbb{Z}G$ be the element defined above. Then $\gamma_f$ is an invertible element if and only if the inclusion $i_f : X \hookrightarrow Y_f$ is a homotopy equivalence.

**Proof.** All the notations used in this proof are the same as defined before. Suppose that $\gamma_f = \partial(1)$ is an invertible element in $\mathbb{Z}G$. Then $\partial$ is both injective and surjective, which shows the relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ is acyclic. This implies that $i_f$ induces an isomorphism between the homology groups $H_i(\tilde{X})$ and $H_i(\tilde{Y}_f)$ for each $i \geq 0$. Since $\tilde{X}$ and $\tilde{Y}_f$ are both simply connected, $i_f : \tilde{X} \to \tilde{Y}_f$ is a homotopy equivalence. Since $i_f$ induces the identity homomorphism on fundamental groups, this shows that $i_f : X \to Y_f$ is a homotopy equivalence by the Whitehead theorem.

Conversely, suppose that $i_f : X \to Y_f$ is a homotopy equivalence. Then $i_f : \tilde{X} \to \tilde{Y}_f$ is a homotopy equivalence, which implies that the relative chain complex $C_*(\tilde{Y}_f, \tilde{X})$ is acyclic. This implies that $\gamma_f = \partial(1)$ has a left inverse. It is a well-known fact that in the integral group ring of a torsion-free group, one-sided invertible element is also two-sided invertible (cf. Corollary 1.9 from [Pa], p.38). This finishes the proof.  

**Proof of Theorem** Let $X = BG$, the classifying space of $G$. Suppose that the trivial unit conjecture for $G$ is true. For an integer $d \geq 2$ and a map $f : S^d \to X \vee S^d$, suppose that the CW complex $Y_f = (X \vee S^d) \cup_f e^{d+1}$ has its inclusion $i_f : X \to Y_f$ a homotopy equivalence. By Lemma 2, the element $\gamma_f$ is a unit. Therefore, $\gamma_f = \pm g$ for some element $g \in G$. As the $d$-th and $(d+1)$-th terms of the relative chain complex are free $\mathbb{Z}G$-modules, we can view them as submodules of $C_i(\tilde{Y})$ ($i = d, d + 1$ resp.). Since $\tilde{X}$ is a free $G$-CW complex and $S^d$ is simply connected, the universal covering space $\tilde{X} \vee S^d$ could be taken as the push out the following diagram
\[ \begin{array}{ccc}
G \times \text{pt} & \to & \tilde{X} \\
\downarrow & & \downarrow \\
G \times S^d & \to & \tilde{X} \vee_G (G \times S^d).
\end{array} \]
Since $X = BG$ is aspherical, $\tilde{X}$ is contractible. This implies that there is a homotopy equivalence $\tilde{X} \vee_G (G \times S^d) \simeq \vee_G S^d$, where $\vee_G S^d$ is the wedge of copies of $S^d$ indexed by $G$. For any element $h \in G$, let $p_h : \tilde{X} \vee S^d \to S^d$ be the projection onto the $h$-component of $\vee_G S^d$. Consider a lifting $\tilde{f}$ of $f$ to the universal covering space as shown in the following diagram:

\[
\begin{array}{ccc}
\tilde{X} \vee S^d & \xrightarrow{\tilde{f}} & S^d \\
\downarrow & & \downarrow \\
X \vee S^d & \xrightarrow{f} & X \vee S^d.
\end{array}
\]

This $\tilde{f}$ actually determines the $(d+1)$-th boundary map in the chain complex of $\tilde{Y}f$. By the definition of the boundary map $\partial$, the degree of the composition $S^d \tilde{f} \rightarrow \tilde{X} \vee S^d \xrightarrow{p_h} S^d$ is zero when $h \neq g$ or $\pm1$ when $h = g$. Therefore, $\tilde{f}$ is homotopic to some map $\tilde{g}$ whose image occupies only the $g$-component $S^d$. This shows that $[\tilde{f}] := [\tilde{f}] \in \pi_d(\tilde{X} \vee S^d, S^d)$ is vanishing, where $S^d$ is viewed as the $g$-component $S^d$.

Conversely, suppose that $\gamma$ is a nontrivial invertible element in $\mathbb{Z}G$. We will construct some map $f_\gamma : S^d \to X \vee S^d$ such that the inclusion $i_{f_\gamma} : X \to Y_f$ is a homotopy equivalence but $[f_\gamma] \in \pi_d(X \vee S^d, S^d)$ is not vanishing for any lifting of $S^d$. Assume that $\gamma = \sum a_g g$ for $g \in G$ and $a_g \in \mathbb{Z}$. As in the first part of this proof, the universal covering space $X \vee S^d = \tilde{X} \vee_G (G \times S^d)$ could be a free $G$-CW complex. Let $p_h : \tilde{X} \vee_G (G \times S^d) \simeq \vee_G S^d \to S^d$ be the projection onto the $h$-component. Define $\tilde{f}_\gamma : S^d \to \tilde{X} \vee_G (G \times S^d)$ as a cellular map such that the degree of the composition $S^d \tilde{f}_\gamma \rightarrow \tilde{X} \vee S^d \xrightarrow{p_h} S^d$ is $a_h$ for each $h \in G$. Denote by

$\phi_\gamma : G \times S^d \to \tilde{X} \vee_G (G \times S^d)$

the unique $G$-equivariant map determined by $\tilde{f}_\gamma$. Note that $\phi_\gamma$ is a $G$-equivariant between two free $G$-CW complexes. Passing to the quotient space, we get a map $f_\gamma : S^d \to \tilde{X} \vee_G (G \times S^d)/G = X \vee S^d$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{X} \vee S^d & \xrightarrow{\tilde{f}_\gamma} & S^d \\
\downarrow & & \downarrow \\
X \vee S^d & \xrightarrow{f_\gamma} & X \vee S^d.
\end{array}
\]
Construct a free $G$-CW complex $\tilde{Y}_\gamma = X \vee S^d \cup_{\phi_\gamma} (G \times e^{d+1})$ as the push out of the following diagram:

$$
\begin{array}{ccc}
G \times S^d & \xrightarrow{\phi_\gamma} & \tilde{X} \vee_G (G \times S^d) \\
\downarrow & & \downarrow \\
G \times e^{d+1} & \rightarrow & \tilde{Y}_f.
\end{array}
$$

This $G$-CW complex $\tilde{Y}_\gamma$ is actually the universal cover of $Y_\gamma := \tilde{Y}_\gamma / G$ (for more details on the construction, see the proof of Lemma 2.2 in [Lu1] or p.371 in [Lu2]). According to Lemma 2, the inclusion $i_f : X \rightarrow Y_\gamma$ is a homotopy equivalence, since $\gamma$ is a unit. Let $i_g : S^d \hookrightarrow X \vee S^d = \tilde{X} \vee_G (G \times S^d)$ be the inclusion of $S^d$ into the $g$-component. As $\gamma$ is nontrivial, the map $\tilde{f}_\gamma$ is not homotopic to any map $S^d \rightarrow S^d \xrightarrow{i_g} X \vee S^d = \tilde{X} \vee_G (G \times S^d)$ for any $g \in G$ by considering the degree of $p_h \tilde{f}_\gamma$ for each $h \in G$. This shows that $[f_\gamma] := [\tilde{f}_\gamma] \in \pi_d(X \vee S^d, S^d)$ is not vanishing for any lifting of $S^d$. 

**Remark 3** For zero divisor conjecture in $\mathbb{Z}G$, some necessary conditions of homotopy descriptions are given in [Iv] and [Le].

**Acknowledgement**

This note was finished when the author was a Phd student in National University of Singapore (NUS). He is grateful to his thesis advisor Professor A.J. Berrick for many discussions and constant encouragement.

**References**

[Cp] D.A. Craven and P. Pappas, On the Unit Conjecture for Supersoluble Group Rings, I, [arXiv:1010.1144](http://arxiv.org/abs/1010.1144).

[Iv] S. V. Ivanov, An asphericity conjecture and Kaplansky problem on zero divisors, J. Algebra 216 (1999), no. 1, 13–19.

[Le] I.J. Leary, Asphericity and zero divisors in group algebras, Journal of Algebra 227(2000), 362-364.

[Lu1] W. Lück, $L^2$ invariants of regular coverings of compact manifolds and CW-complexes, Handbook of geometric topology”, editors: Daverman, R.J. and Sher, R.B., Elsevier, 2002.
[Lu2] W. Lück, *L²-Invariants: theory and applications to geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 44, Springer, 2002.

[Pa] D. S. Passman, The algebraic structure of group rings. John Wiley & Sons, New York, London, Sydney, Toronto, 1977.

Mathematics and Physics Centre, Xi’an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu 215123, China.  
E-mail: Shengkui.Ye@xjtlu.edu.cn  
Mathematical Institute, University of Oxford, 24-29 St Giles’, Oxford, OX1 3LB, U.K.  
E-mail: Shengkui.Ye@maths.ox.ac.uk