Computing Horn Rewritings of Description Logics Ontologies

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Abstract

We study the problem of rewriting an ontology $O_1$ expressed in a DL $L_1$ into an ontology $O_2$ in a Horn DL $L_2$ such that $O_1$ and $O_2$ are equisatisfiable when extended with an arbitrary dataset. Ontologies that admit such rewritings are amenable to reasoning techniques ensuring tractability in data complexity. After showing undecidability whenever $L_1$ extends $ALCF$, we focus on devising efficiently checkable conditions that ensure existence of a Horn rewriting. By lifting existing techniques for rewriting Disjunctive Datalog programs into plain Datalog to the case of arbitrary first-order programs with function symbols, we identify a class of ontologies that admit Horn rewritings of polynomial size. Our experiments indicate that many real-world ontologies satisfy our sufficient conditions and thus admit polynomial Horn rewritings.

1 Introduction

Reasoning over ontology-enriched datasets is a key requirement in many applications of semantic technologies. Standard reasoning tasks are, however, of high worst-case complexity. Satisfiability checking is 2NEXPTime-complete for the description logic (DL) $SROIQ$ underpinning the standard ontology language OWL 2 and NEXPTime-complete for $SHOIN\!^\mathcal{R}$, which underpins OWL DL [Kazakov, 2008]. Reasoning is also co-NP-hard with respect to data complexity—a key measure of complexity for applications involving large amounts of instance data [Hustadt et al., 2005].

Tractability in data complexity is typically associated with Horn DLs, where ontologies correspond to first-order Horn clauses [Ortiz et al., 2011; Hustadt et al., 2005]. The more favourable computational properties of Horn DLs make them a natural choice for data-intensive applications, but they also come at the expense of a loss in expressive power. In particular, Horn DLs cannot capture disjunctive axioms, i.e., statements such as “every $X$ is either a $Y$ or a $Z$”. Disjunctive axioms are common in real-world ontologies, like the NCI Thesaurus or the ontologies underpinning the European Bioinformatics Institute (EBI) linked data platform.

In this paper we are interested in Horn rewritability of description logic ontologies; that is, whether an ontology $O_1$ expressed in a DL $L_1$ can be rewritten into an ontology $O_2$ in a Horn DL $L_2$ such that $O_1$ and $O_2$ are equisatisfiable when extended with an arbitrary dataset. Ontologies that admit such Horn rewritings are amenable to more efficient reasoning techniques that ensure tractability in data complexity.

Horn rewritability of DL ontologies is strongly related to the rewritability of Disjunctive Datalog programs into Datalog, where both the source and target languages for rewriting are function-free. [Kaminski et al., 2014b] characterised Datalog rewritability of Disjunctive Datalog programs in terms of linearity: a restriction that requires each rule to contain at most one body atom that is IDB (i.e., whose predicate also occurs in head position in the program). It was shown that every linear Disjunctive Datalog program can be rewritten into plain Datalog (and vice versa) by means of program transposition—a polynomial transformation in which rules are “inverted” by shuffling all IDB atoms between head and body while at the same time replacing their predicates by auxiliary ones. Subsequently, [Kaminski et al., 2014a] proposed the class of markable Disjunctive Datalog programs, where the linearity requirement is relaxed so that it applies only to a subset of “marked” atoms. Every markable program can be polynomially rewritten into Datalog by exploiting a variant of transposition where only marked atoms are affected.

Our contributions in this paper are as follows. In Section 3 we show undecidability of Horn rewritability whenever the input ontology is expressed in $ALCF$. This is in consonance with the related undecidability results by [Bienvenu et al., 2014] and [Lutz and Wolter, 2012] for Datalog rewritability and non-uniform data complexity for $ACF$ ontologies.

In Section 4 we lift the markability condition and the transposition transformation in [Kaminski et al., 2014a] for Disjunctive Datalog to arbitrary first-order programs with function symbols. We then show that all markable first-order programs admit Horn rewritings of polynomial size. This result is rather general and has potential implications in areas such as theorem proving [Robinson and Voronkov, 2001] and knowledge compilation [Darwiche and Marquis, 2002].

The notion of markability for first-order programs can be seamlessly adapted to ontologies via the standard FOL translation of DLs [Baader et al., 2003]. This is, however, of limited practical value since Horn programs ob-

1http://www.ebi.ac.uk/rdf/platform
tained via transposition may not be expressible using standard DL constructors. In Section 3 we introduce an alternative satisfiability-preserving translation from $\text{ALCHIF}$ ontologies into first-order programs and show in Section 6 that the corresponding transposed programs can be translated back into Horn-$\text{ALCHIF}$ ontologies. Finally, we focus on complexity and show that reasoning over markable $\mathcal{L}$-ontologies is $\text{EXPTIME}$-complete in combined complexity and $\text{PTIME}$-complete w.r.t. data for each DL $\mathcal{L}$ between $\mathcal{E}\mathcal{L}\mathcal{U}$ and $\text{ALCHIF}$. All our results immediately extend to DLs with transitive roles (e.g., $\mathcal{SHLF}$) by exploiting standard transitivity elimination techniques [Baader et al., 2003].

We have implemented markability checking and evaluated our techniques on a large ontology repository. Our results indicate that many real-world ontologies are markable and thus admit Horn rewritings of polynomial size.

The proofs of all our results are delegated to the appendix.

2 Preliminaries

We assume standard first-order syntax and semantics. We treat the universal truth $\top$ and falsehood $\bot$ symbols as well as equality ($\approx$) as ordinary predicates of arity one ($\top$ and $\bot$) and two ($\approx$), the meaning of which will be axiomatised.

Programs A first-order rule (or just a rule) is a sentence

$$\forall \bar{x}\forall \bar{z}. [\varphi(\bar{x}, \bar{z}) \rightarrow \psi(\bar{x})]$$

where variables $\bar{x}$ and $\bar{z}$ are disjoint, $\varphi(\bar{x}, \bar{z})$ is a conjunction of distinct atoms over $\bar{x} \cup \bar{y}$, and $\psi(\bar{x})$ is a disjunction of distinct atoms over $\bar{x}$. Formula $\varphi$ is the body of $r$, and $\psi$ is the head. Quantifiers are omitted for brevity, and safety is written on the left-hand side of Table 1.

O such that $R_1 \sqsubseteq^* R_2$ and $\text{inv}(R_1) \sqsubseteq^* \text{inv}(R_2)$ hold whenever $R_1 \sqsubseteq R_2$ is an axiom in $O$.

We refer to the DL where only axioms of type T1-T3 are available and the use of inverse roles is disallowed as $\mathcal{E}\mathcal{L}\mathcal{U}$. The logic $\text{ALC}$ extends $\mathcal{E}\mathcal{L}\mathcal{U}$ with axioms T4. We then use standard naming conventions for DLs based on the presence of inverse roles ($\mathcal{I}$), axioms T5 ($\mathcal{H}$) and axioms T6 ($\mathcal{F}$). Finally, an ontology is $\mathcal{E}\mathcal{L}\mathcal{U}$ if it is both $\mathcal{E}\mathcal{L}\mathcal{U}$ and Horn.

Table 1 also provides the standard translation $\pi$ from normalised axioms into first-order rules, where $\mathtt{at}(R, x, y)$ is defined as $R(x, y)$ if $R$ is named and as $S(y, x)$ if $R = S^{-}$. We define $\pi(O)$ as the smallest program containing $\pi(\alpha)$ for each axiom $\alpha$ in $O$. Given a dataset $D$, we say that $O \cup D$ is satisfiable iff so is $\pi(O) \cup D$ in first-order logic.

3 Horn Rewritability

Our focus is on satisfiability-preserving rewritings. Standard reasoning tasks in description logics are reducible to unsatisfiability checking [Baader et al., 2003], which makes our results practically relevant. We start by formulating our notion of rewriting in general terms.

Definition 1. Let $\mathcal{F}$ and $\mathcal{F}'$ be sets of rules. We say that $\mathcal{F}'$ is a rewriting of $\mathcal{F}$ if it holds that $\mathcal{F} \cup D$ is satisfiable iff so is $\mathcal{F}' \cup D$ for each dataset $D$ over predicates from $\mathcal{F}$. ⊥

We are especially interested in computing Horn rewritings of ontologies—that is, rewritings where the given ontology $O_1$ is expressed in a DL $\mathcal{L}_1$ and the rewritten ontology $O_2$ is in a Horn DL $\mathcal{L}_2$ (where preferably $\mathcal{L}_2 \subseteq \mathcal{L}_1$). This is not possible in general: satisfiability checking is co-NP-complete in data complexity even for the basic logic $\mathcal{E}\mathcal{L}\mathcal{U}$ [Krisnadhi and Lutz, 2007], whereas data complexity is tractable even for highly expressive Horn languages such as Horn-$\mathsf{SROTQ}$ [Ortiz et al., 2011]. Horn rewritability for DLs can be formulated as a decision problem as follows:

Definition 2. The ($\mathcal{L}_1$, $\mathcal{L}_2$)-Horn rewritability problem for DLs $\mathcal{L}_1$ and $\mathcal{L}_2$ is to decide whether a given $\mathcal{L}_1$-ontology admits a rewriting expressed in Horn-$\mathcal{L}_2$. ⊥

Our first result establishes undecidability whenever the input ontology contains at-most cardinality restrictions and thus equality. This result is in consonance with the related undecidability results by Bienvenu et al. [2014] and Lutz and Wolter [2012] for Datalog rewritability and non-uniform data complexity for $\text{ALCF}$ ontologies.

Theorem 3. ($\mathcal{L}_1$, $\mathcal{L}_2$)-Horn rewritability is undecidable for $\mathcal{L}_1 = \text{ALCF}$ and $\mathcal{L}_2$ any DL between $\mathcal{E}\mathcal{L}\mathcal{U}$ and $\text{ALCHIF}$. This result holds under the assumption that $\text{PTIME} \neq \text{NP}$.

Intractability results in data complexity rely on the ability of non-Horn DLs to encode co-NP-hard problems, such as non-3-colourability [Krisnadhi and Lutz, 2007] [Hustadt et al., 2005]. In practice, however, it can be expected that ontologies do not encode such problems. Thus, our focus from now onwards will be on identifying classes of ontologies that admit (polynomial size) Horn rewritings.

4 Program Markability and Transposition

In this section, we introduce the class of markable programs and show that every markable program can be rewritten into...
a Horn program by means of a polynomial transformation, which we refer to as transposition. Roughly speaking, transposition inverts the rules in a program \( P \) by moving certain atoms from head to body and vice versa while replacing their corresponding predicates with fresh ones. Markability of \( P \) ensures that we can pick a set of predicates (a marking) such that, by shuffling only atoms with a marked predicate, we obtain a Horn rewriting of \( P \). Our results in this section generalise the results by [Kaminski et al., 2014a] for Disjunctive Datalog to first-order programs with function symbols.

To illustrate our definitions throughout this section, we use an example program \( P_{ex} \) consisting of the following rules:

\[
\begin{align*}
A(x) &\rightarrow B(x) & B(x) &\rightarrow C(x) \lor D(x) \\
C(x) &\rightarrow \bot(x) & D(x) &\rightarrow C(f(x))
\end{align*}
\]

**Markability.** The notion of markability involves a partitioning of the program’s predicates into Horn and disjunctive. Intuitively, the former are those whose extension for all datasets depends only on the Horn rules in the program, whereas the latter are those whose extension may depend on a disjunctive rule. This intuition can be formalised using the standard notion of a dependency graph in Logic Programming.

**Definition 4.** The dependency graph \( G_P = (V, E, \mu) \) of a program \( P \) is the smallest edge-labelled digraph such that:

(i) \( V \) contains all predicates in \( P \); (ii) \( r \in \mu(P, Q) \) whenever \( r \in \mathcal{P}, P \) is in the body of \( r \), and \( Q \) is in the head of \( r \); and (iii) \( (P, Q) \in E \) whenever \( \mu(P, Q) \neq \emptyset \). A predicate \( Q \) depends on \( r \in \mathcal{P} \) if \( G_P \) has a path ending in \( Q \) and involving an \( r \)-labeled edge. Predicate \( Q \) is Horn if \( Q \) depends only on Horn rules; otherwise, \( Q \) is disjunctive.

For instance, predicates \( C, D, \) and \( \bot \) are disjunctive in our example program \( P_{ex} \), whereas \( A \) and \( B \) are Horn. We can now introduce the notion of a marking—a subset of the disjunctive predicates in a program \( P \) ensuring that the transposition of \( P \) where only marked atoms are shuffled between head and body results in a Horn program.

**Definition 5.** A marking of a program \( P \) is a set \( M \) of disjunctive predicates in \( P \) satisfying the following properties, where we say that an atom is marked if its predicate is in \( M \):

(i) each rule in \( P \) has at most one marked body atom; (ii) each rule in \( P \) has at most one unmarked head atom; and (iii) if \( Q \in M \) and \( P \) is reachable from \( Q \) in \( G_P \), then \( P \in M \).

We say that a program is markable if it admits a marking.

Condition (ii) in Def. [\ref{def:markability}] ensures that at most one atom is moved from body to head during transposition. Condition (ii) ensures that all but possibly one head atom are moved to the body. Finally, condition (iii) requires that all predicates depending on a marked predicate are also marked. We can observe that our example program \( P_{ex} \) admits two markings:

\[
M_1 = \{C, \bot\} \text{ and } M_2 = \{C, D, \bot\}.
\]

Markability can be efficiently checked via a 2-SAT reduction, where we assign to each predicate \( Q \) in \( P \) a propositional variable \( X_Q \) and encode the constraints in Def. [\ref{def:markability}] as 2-clauses. For each rule \( \phi \land \bigwedge_{i=1}^n P_i(t_i) \rightarrow \bigvee_{j=1}^m Q_j(t_j) \), with \( \phi \) the conjunction of all Horn atoms in the rule head, we include clauses \( (i) \neg X_{P_i} \lor \neg X_{P_j} \) for all \( 1 \leq i < j \leq n \), which enforce at most one body atom to be marked; \( (ii) X_{Q_i} \lor X_{Q_j} \) for \( 1 \leq i < j \leq m \), which ensure that at most one head atom is unmarked; and \( (iii) \neg X_{P_i} \lor X_{Q_i} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), which close markings under rule dependencies. Each model of the resulting clauses yields a marking of \( P \).

**Transposition.** Before defining transposition, we illustrate the main intuitions using program \( P_{ex} \) and marking \( M_1 \).

The first step to transpose \( P_{ex} \) is to introduce fresh unary predicates \( C, \top \), which stand for the negation of the marked predicates \( C \) and \( \bot \). To capture the intended meaning of these predicates, we introduce rules \( X(x) \rightarrow \top(x) \) for \( X \in \{A, B, C, D\} \) and a rule \( \top(x) \rightarrow \top(f(x)) \) for the unique function symbol \( f \) in \( P_{ex} \). The first rules mimic the usual axiomatisation of \( \top \) and ensure that an atom \( \top(z) \) holds in a Herbrand model of the transposed program whenever \( X(z) \) also holds. The last rule ensures that \( \top \) holds for all terms in the Herbrand universe of the transposed program—an additional requirement that is consistent with the intended meaning of \( \top \), and critical to the completeness of transposition in the presence of function symbols. Finally, a rule \( \top(z) \land C(z) \land \top(C(x)) \rightarrow \bot(z) \) ensures that the fresh predicate \( C \) behaves like the negation of \( C \) (\( C(z) \) is added for safety).

The key step of transposition is to invert the rules involving the marked predicates by shuffling marked atoms between head and body while replacing their predicate with the corresponding fresh one. In this way, rule \( B(x) \rightarrow C(x) \lor D(x) \) yields \( B(x) \lor \top(C(x)) \rightarrow D(x) \), and \( C(x) \rightarrow \bot(x) \) yields \( \top(x) \rightarrow C(x) \). Additionally, rule \( D(x) \rightarrow C(f(x)) \) is transposed as \( \top(z) \land D(x) \land \top(C(f(x))) \rightarrow \bot(z) \) to ensure safety. Finally, transposition does not affect rules containing only Horn predicates, e.g., rule \( A(x) \rightarrow B(x) \) is included unchanged.

**Definition 6.** Let \( M \) be a marking of a program \( P \). For each disjunctive predicate \( P \) in \( P \), let \( \mathcal{P} \) be a fresh predicate of the same arity. The \( M \)-transposition of \( P \) is the smallest program \( \Xi_M(P) \) containing every rule in \( P \) involving only Horn predicates and all rules 1–6 given next, where \( Q \) is the conjunction of all Horn atoms in a rule, \( \phi \rightarrow \top \) is the least conjunction of \( \top \)-atoms making a rule safe and all \( P_i, Q_i \) are disjunctive:

1. \( \phi \land Q_i(t_i) \land \bigwedge_{j=1}^m P_j(t_j) \rightarrow Q_i(t_i) \) for each rule in \( P \) of the form \( \phi \land Q(t) \land \bigwedge_{j=1}^m Q_j(t_j) \rightarrow \bigvee_{i=1}^n P_i(s_i) \)

| T1. | \( \bigwedge_{i=1}^n \bigcup_{j=1}^m A_i \rightarrow \bigvee_{j=1}^m C_j \) \( A_i(x) \rightarrow \bigvee_{j=1}^m C_j(x) \) |
| T2. | \( \exists R.A \subseteq C \) \( at(R, x, y) \land A(y) \rightarrow C(x) \) |
| T3. | \( A \subseteq \exists R.B \) \( A(x) \rightarrow \bigwedge_{j=1}^m B_j(t_j) \lor A(x) \rightarrow B(f(x)) \) |
| T4. | \( \forall R.C \) \( A(x) \land \bigwedge_{j=1}^m B_j(t_j) \rightarrow C(y) \) |
| T5. | \( S \subseteq R \) \( S(x, y) \rightarrow \bigwedge_{j=1}^m B_j(t_j) \) |
| T6. | \( A \subseteq \leq 1 R.B \) \( A(z) \land \bigwedge_{i=1}^m B_i(t_i) \land \bigwedge_{j=1}^m B_j(t_j) \land B(x_1) \rightarrow \bot(x_1) \approx \bot(x_2) \) |
where $Q(\bar{t})$ is the only marked body atom;

2. $\overline{(x)} \land \varphi \land \bigwedge_{j=1}^{m} Q_{j}(\bar{t}_{j}) \land \bigwedge_{i=1}^{n} P_{i}(\bar{s}_{i}) \rightarrow \bot(x)$, where $x$ a fresh variable, for each rule in $\mathcal{P}$ of the form $\varphi \land \bigwedge_{j=1}^{m} Q_{j}(\bar{t}_{j}) \land \bigwedge_{i=1}^{n} P_{i}(\bar{s}_{i})$, with no marked body atoms and no unmarked head atoms;

3. $\varphi \land \bigwedge_{j=1}^{m} Q_{j}(\bar{t}_{j}) \land \bigwedge_{i=1}^{n} \overline{P_{i}(\bar{s}_{i})} \rightarrow \overline{P(\bar{t})}$ for each rule in $\mathcal{P}$ of the form $\varphi \land \bigwedge_{j=1}^{m} Q_{j}(\bar{t}_{j}) \land \bigwedge_{i=1}^{n} \overline{P_{i}(\bar{s}_{i})}$ where $P(\bar{t})$ is the only unmarked head atom;

4. $\overline{\mu(\bar{z})} \land P(\bar{x}) \land \overline{P(\bar{y})} \rightarrow \bot(\bar{z})$ for marked predicate $P$;

5. $P(x_{1}, \ldots, x_{n}) \rightarrow \bot(x_{i})$ for each $P$ in $\mathcal{P}$ and $1 \leq i \leq n$;

6. $\overline{\mu(\bar{x})} \land \cdots \land \overline{\mu(\bar{x})} \rightarrow \overline{f(x_{1}, \ldots, x_{n})}$ for each $n$-ary function symbol $f$ in $\mathcal{P}$.

Note that rules of type 1 in Def. 6 satisfy $\{P_{1}, \ldots, P_{n}\} \subseteq M$ since $Q \in M$, while for rules of type 3 we have $\{Q_{1}, \ldots, Q_{m}\} \cap M = \emptyset$ since $P \notin M$.

Clearly, $\mathcal{P}_{ex}$ is unsatisfiable when extended with fact $A(\alpha)$. To show that $\mathcal{P}\mu(\mathcal{P}) \cup \{A(\alpha)\}$ is also unsatisfiable, note that $B(\alpha)$ is derived by the unchanged rule $A(x) \rightarrow B(x)$. Fact $\overline{C}(\alpha)$ is derived using $A(x) \rightarrow \overline{C}(x)$ and the transposed rule $\overline{\mu}(\bar{x}) \rightarrow \overline{C}(x)$. We derive $D(\alpha)$ using $B(x) \land \overline{C}(\alpha) \rightarrow D(x).$

But then, to derive a contradiction we need to apply rule $\overline{\mu}(\bar{x}) \land \overline{\mu}(\bar{x}) \land \overline{\mu}(f(x)) \rightarrow \bot(\bar{z})$, which is not possible unless we derive $\overline{C}(f(\bar{a}))$. For this, we first use $\overline{\mu}(\bar{x}) \rightarrow \overline{\mu}(f(x))$, which ensures that $\overline{\mu}$ holds for $f(\bar{a})$, and then $\overline{\mu}(\bar{x}) \rightarrow \overline{C}(x)$.

Transposition yields quadratically many Horn rules. The following theorem establishes its correctness.

Theorem 7. Let $\mathcal{M}$ be a marking of a program $\mathcal{P}$. Then $\mathcal{M}(\mathcal{P})$ is a polynomial-size Horn rewriting of $\mathcal{P}$.

It follows that every marking set of non-Horn clauses $\mathcal{N}$ can be polynomially transformed into a set of Horn clauses $\mathcal{N}'$ such that $\mathcal{N}' \subseteq \mathcal{D}$ and $\mathcal{N}' \subseteq \mathcal{D}$ are equisatisfiable for every set of facts $\mathcal{D}$. This result is rather general and has potential applications in first-order theorem proving, as well as in knowledge compilation, where Horn clauses are especially relevant.

5 Markability of DL Ontologies

The notion of markability is applicable to first-order programs and hence can be seamlessly adapted to ontologies via the standard translation $\pi$ in Table 1. This, however, would be of limited value since the Horn programs resulting from transposition may not be expressible in Horn-$\mathcal{ALCIF}$. Consider any ontology with an axiom $\exists R.A \subseteq B$ and any marking $M$ involving $R$. Rule $R(x, y) \land A(y) \rightarrow B(x)$ stemming from $\pi$ would be transposed as $\overline{B(x)} \land A(y) \rightarrow \overline{R(x, y)}$, which cannot be captured in $\mathcal{ALCIF}$.

To address this limitation we introduce an alternative translation $\xi$ from DL axioms into rules, which we illustrate using the example ontology $\mathcal{O}_{ex}$ in Table 2. The key idea is to encode existential restrictions in axioms $T_{3}$ as unary atoms over functional terms. For instance, axiom $\alpha_{2}$ in $\mathcal{O}_{ex}$ would yield $B(x) \rightarrow D(f_{R.D}(x))$, where the “successor” relation between an instance $b$ of $B$ and some instance of $D$ in a Herbrand model is encoded as a term $f_{R.D}(b)$, instead of a binary atom of the form $R(b, q(b))$. This encoding has an immediate impact on markings: by marking $B$ we are only forced to also mark $D$ (rather than both $R$ and $D$). In this way, we will be able to ensure that markings consist of unary predicates only.

To compensate for the lack of binary atoms involving functional terms in Herbrand models, we introduce new rules when translating axioms $T_{2}$, $T_{4}$, and $T_{6}$ using $\xi$. For instance, $\xi(\alpha_{3})$ yields the following rules in addition to $\pi(\alpha_{3})$: a rule $D(f_{R.D}(x)) \rightarrow D(x)$ to ensure that all objects $e$ with an $R$-successor $f_{R.D}(e)$ generated by $\xi(\alpha_{2})$ are instances of $D$; a rule $D(f_{R.B}(x)) \rightarrow D(x)$, which makes sure that an object whose $R$-successor generated by $\xi(\alpha_{3})$ is an instance of $D$ is also an instance of $D$. Finally, axioms $\alpha_{3}$ and $\alpha_{4}$, which involve no binary predicates, are translated as usual.

Definition 8. Let $\mathcal{O}$ be an ontology. For each concept $\exists R.B$ in an axiom of type $T_{3}$, let $f_{R.B}$ be a unary function symbol, and $\Phi$ the set of all such symbols. We define $\xi(\mathcal{O})$ as the smallest program containing $\pi(\alpha)$ for each axiom $\alpha$ in $\mathcal{O}$ of type $T_{1}$-$T_{2}$ and $T_{4}$-$T_{6}$, as well as the following rules:

- $A(x) \rightarrow B(f_{R.B}(x))$ for each axiom $T_{3}$;
- $A(f_{R.Y}(x), y) \rightarrow C(x)$ for each axiom $T_{2}$ and $R'$ and $Y$ s.t. $f_{R.Y} \in \Phi$ and $R' \subseteq R$;
- $A(f_{inv(R')}.Y(x)) \rightarrow C(x)$ for each axiom $T_{4}$ and $R'$ and $Y$ s.t. $f_{inv(R')}.Y \in \Phi$ and $R' \subseteq R$;
- $A(x) \land Y(f_{inv(R')}.Y(x)) \rightarrow C(f_{inv(R')}.Y(x))$ for each axiom $T_{2}$ and $R'$ and $Y$ s.t. $f_{inv(R')}.Y \in \Phi$ and $R' \subseteq R$;
- $A(x) \land (f_{inv(R')}.Y(x)) \rightarrow C(f_{inv(R')}.Y(x))$ for each axiom $T_{4}$ and $R'$ and $Y$ s.t. $f_{inv(R')}.Y \in \Phi$ and $R' \subseteq R$;
- $A(x) \land B(f_{R.Y}(x)) \land \text{at}(R,z,x) \land B(x) \rightarrow f_{R.Y}(x) \approx x \forall y$ for each axiom $T_{6}$ and $R'$ and $Y$ s.t. $f_{inv(R')}.Y \in \Phi$ and $R' \subseteq R$.

Note that, in contrast to the standard translation $\pi$, which introduces at most two rules per DL axiom, $\xi$ can introduce linearly many rules in the size of the role hierarchy induced by axioms of type $T_{5}$.

The translation $\xi(\mathcal{O}_{ex})$ of our example ontology $\mathcal{O}_{ex}$ is given in the second column of Table 2. Clearly, $\mathcal{O}_{ex}$ is unsatisfiable when extended with $A(\alpha)$ and $E(\alpha)$. We can check that $\xi(\mathcal{O}_{ex}) \cup \{A(\alpha), E(\alpha)\}$ is also unsatisfiable. The following theorem establishes the correctness of $\xi$.

Theorem 9. For every ontology $\mathcal{O}$ and dataset $\mathcal{D}$ over predicates in $\mathcal{O}$ we have that $\mathcal{O} \cup \mathcal{D}$ is satisfiable if and only if $\xi(\mathcal{O}) \cup \mathcal{D}$.

This translation has a clear benefit for markability checking: in contrast to $\pi(\mathcal{O})$, binary predicates in $\xi(\mathcal{O})$ do not belong to any minimal marking. In particular, $M_{ex} = \{B, D, \bot\}$ is the only minimal marking of $\xi(\mathcal{O}_{ex})$.

Proposition 10. (i) If $\approx$ is Horn in $\xi(\mathcal{O})$ then so are all binary predicates in $\xi(\mathcal{O})$. (ii) If $\xi(\mathcal{O})$ is markable, then it has a marking containing only unary predicates.
Thus, we define markability of ontologies in terms of \( \pi \) rather than in terms of \( \tau \). We can check that \( \pi(O_{ex}) \) is not markable, whereas \( \xi(O_{ex}) \) admits the marking \( M_{ex} \).

**Definition 11.** An ontology \( O \) is markable if so is \( \xi(O) \).

We conclude this section with the observation that markability of an ontology \( O \) can be efficiently checked by first computing the program \( \xi(O) \) and then exploiting the 2-SAT encoding sketched in Section 4.

### 6 Rewriting Markable Ontologies

It follows from the correctness of transposition in Theorem 7 and \( \xi \) in Theorem 9 that every ALC\( H \)IF ontology \( O \) admitting a marking \( M \) has a Horn rewriting of polynomial size given as the program \( \Xi_M(\xi(O)) \). In what follows, we show that this rewriting can be expressed within Horn-ALC\( H \).

Let us consider the transposition of \( \xi(O_{ex}) \) via the marking \( M_{ex} \), which is given in the third column of Table 2. The transposition of \( \alpha_1 \) and \( \alpha_2 \) corresponds directly to DL axioms via the standard translation in Table 1. In contrast, the transposition of all other axioms leads to rules that have no direct correspondence in DLs. The following lemma establishes that the latter rules are restricted to the types T7-T20 specified on the left-hand side of Table 3.

**Lemma 12.** Let \( O \) be an ontology and \( M \) a minimal marking of \( \xi(O) \). Then \( \Xi_M(\xi(O)) \) contains only Horn rules of type T1-T2 and T4-T6 in Table 2 as well as type T7-T20 in Table 3.

We can now specify a transformation \( \Psi \) that allows us to translate rules T7-T20 in Table 3 back into DL axioms.

**Definition 13.** We define \( \Psi \) as the transformation mapping (i) each Horn rule \( r \) of types T1-T2 and T4-T6 in Table 2 to the DL axiom \( \pi^{-1}(r) \) (ii) each rule T7-T20 on the left-hand side of Table 3 to the DL axioms on the right-hand side of Table 3.

Intuitively, \( \Psi \) works as follows: (i) Function-free rules are “rolled up” as usual into DL axioms (see e.g., T7). (ii) Unary atoms \( A(f_{R,Y}(x)) \) with \( A \neq \bot \) involving a functional term are translated as either existentially or universally quantified concepts depending on whether they occur in the body or in the head (e.g., T10, T11); in contrast, atoms \( \exists \) in rules \( \exists \) in rules \( T_{15}-T_{18} \), which involve \in \ in the head and roles \( R \) and \( R' \) in the body, are rolled back into axioms of type T6 over the “union” of \( R \) and \( R' \), which is captured using fresh roles and role inclusions.

The ontology obtained by applying \( \Psi \) to our running example is given in the last column of Table 2. Correctness of \( \Psi \) and its implications for the computation of Horn rewritings are summarised in the following lemma.

**Lemma 14.** Let \( O \) be a markable ALC\( H \)IF ontology and let \( M \) be a marking of \( O \). Then the ontology \( \Psi(\Xi_M(\xi(O))) \) is a Horn rewriting of \( O \).

A closer look at our transformations reveals that our rewritings do not introduce constructs such as inverse roles and cardinality restrictions if these were not already present in the input ontology. In contrast, fresh role inclusions may originate from cardinality restrictions in the input ontology. As a result, our approach is language-preserving: if the input \( O_1 \) is in a DL \( \mathcal{L} \) between ALC and ALC\( H \), then its rewriting \( O_2 \) stays in the Horn fragment of \( \mathcal{L} \); furthermore, if \( \mathcal{L} \) is between ALC\( IF \) and ALC\( H \), then \( O_2 \) may contain fresh role inclusions (H). A notable exception is when \( O_1 \) is an EL\( U \) ontology, in which case axioms T2 and T3 in \( O_1 \) may yield axioms of type T4 in \( O_2 \). The following theorem follows from these observations and Lemma 14.

**Theorem 15.** Let \( \mathcal{L} \) be a DL between ALC and ALC\( H \). Then every markable \( \mathcal{L} \) ontology is polynomially rewritable into a Horn-\( \mathcal{L} \) ontology. If \( \mathcal{L} \) is between ALC\( IF \) and ALC\( H \), then every markable \( \mathcal{L} \) ontology is polynomially rewritable into Horn-\( \mathcal{L} \). Finally, every markable EL\( U \) ontology is polynomially rewritable into Horn-ALC.

### 7 Complexity Results

We next establish the complexity of satisfiability checking over markable ontologies.

We first show that satisfiability checking over markable EL\( U \) ontologies is Exptime-hard. This implies that it is not
possible to polynomially rewrite every markable $\mathcal{ELIU}$ ontology into $\mathcal{EL}$. Consequently, our rewriting approach is optimal for $\mathcal{ELIU}$ in the sense that introducing universal restrictions (or equivalently inverse roles) in the rewriting is unavoidable.

**Lemma 16.** Satisfiability checking over markable $\mathcal{ELIU}$ ontologies is EXPTime-hard.

All Horn DLs from $\mathcal{ALC}$ to $\mathcal{ALCHIF}$ are EXPTime-complete in combined complexity and PTIME-complete in data complexity [Krotzsch et al., 2013]. By Theorem 15 the same result holds for markable ontologies in DLs from $\mathcal{ALC}$ to $\mathcal{ALCHIF}$. Finally, Lemma 16 shows that these complexity results also extend to markable $\mathcal{ELIU}$ ontologies.

**Theorem 17.** Let $\mathcal{L}$ be in-between $\mathcal{ELIU}$ and $\mathcal{ALCHIF}$. Satisfiability checking over markable $\mathcal{L}$-ontologies is EXPTime-complete and PTIME-complete w.r.t. data.

### 8 Related Work

Horn logics are common target languages for knowledge compilation [Darwiche and Marquis, 2002]. Selman and Kautz [1996] proposed an algorithm for compiling a set of propositional clauses into a set of Horn clauses s.t. their Horn consequences coincide. This approach was generalised to FOL by Del Val [2005], without termination guarantees. Bienvenu et al. [2014] showed undecidability of Datalog rewriting for $\mathcal{ALCF}$ and decidability in NEXPTIME for $\mathcal{SHIT}$. Cuenca Grau et al. [2013] and Kaminski et al. [2014a] proposed practical techniques for computing Datalog rewritings of $\mathcal{SHIT}$ ontologies based on a two-step process. First, $\mathcal{O}$ is rewritten using a resolution calculus $\Omega$ into a Disjunctive Datalog program $\Omega(\mathcal{O})$ of exponential size [Hustadt et al., 2007]. Second, $\Omega(\mathcal{O})$ is rewritten into a Datalog program $\mathcal{P}$. For the second step, Kaminski et al. [2014a] propose the notion of markability of a Disjunctive Datalog program and show that $\mathcal{P}$ can be polynomially computed from $\Omega(\mathcal{O})$ using transposition whenever $\Omega(\mathcal{O})$ is markable. In contrast to our work, Kaminski et al. [2014a] focus on Datalog as target language for rewriting (rather than Horn DLs). Furthermore, their Datalog rewritings may be exponential w.r.t. the input ontology and cannot generally be represented in DLs.

Gottlob et al. [2012] showed tractability in data complexity of fact entailment for the class of first-order rules with singleton atoms. It is sufficient to capture most DLs in the DL-Lite family [Artale et al., 2009]. Lutz and Wolter [2012] investigated (non-uniform) data complexity of query answering w.r.t. fixed ontologies. They studied the boundary of PTIME and co-NP-hardness and established a connection with constraint satisfaction problems. Finally, Lutz et al. [2011] studied model-theoretic rewritability of ontologies in a DL $\mathcal{L}_1$ into a fragment $\mathcal{L}_2$ of $\mathcal{L}_1$. These rewritings preserve models rather than just satisfiability, which severely restricts the class of rewritable ontologies; in particular, only ontologies that are “semantically Horn” can be rewritten. For instance, $\mathcal{O} = \{ A \sqsubseteq B \sqcup C \}$, which is rewritable by our approach, is not Horn-rewritable according to Lutz et al. [2011].

### 9 Proof of Concept

To assess the practical implications of our results, we have evaluated whether real-world ontologies are markable (and hence also polynomially Horn rewritable). We analysed 120 non-Horn ontologies extracted from the Protege Ontology Library, BioPortal (http://biopote.bioontology.org), the corpus by Gardiner et al. [2006], and the EBI linked data platform (http://www.ebi.ac.uk/rdf/platform). To check markability, we have implemented the 2-SAT reduction in Section 4.
and a simple 2-SAT solver.

We found that a total of 32 ontologies were markable and thus rewritable to a Horn ontology, including some ontologies commonly used in applications, such as ChEMBL (see http://www.ebi.ac.uk/rdf/services/chembl/) and BioPAX Reactome (http://www.ebi.ac.uk/rdf/services/reactome/). When using $\pi$ as first-order logic translation, we obtained 30 markable ontologies—a strict subset of the ontologies markable using $\xi$. However, only 27 ontologies were rewritable to a Horn DL since in three cases the marking contained a role.

10 Conclusion and Future Work

We have presented the first practical technique for rewriting non-Horn ontologies into a Horn DL. Our rewritings are polynomial, and our experiments suggest that they are applicable to widely-used ontologies. We anticipate several directions for future work. First, we would like to conduct an extensive evaluation to assess whether the use of our rewritings can significantly speed up satisfiability checking in practice. Second, we will investigate relaxations of markability that would allow us to capture a wider range of ontologies.

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A Proofs for Section 3

Theorem 3. \((L_1, L_2)\)-Horn rewritability is undecidable for \(L_1 = ALC\mathcal{F}\) and \(L_2\) any DL between \(\mathcal{ELU}\) and \(ALC\mathcal{HF}\). This result holds under the assumption that PTIME \(\neq\) NP.

Proof. We adapt the undecidability proof for datalog-rewritability of \(ALC\mathcal{F}\) in [Bienvenu et al., 2014]. Given an instance \(\Pi\) of the undecidable finite rectangle tiling problem, [Bienvenu et al.] give an \(ALC\mathcal{F}\) ontology \(O_1\), signature \(\Sigma\) and concept name \(E\) such that the following three conditions are equivalent:

1. \(\Pi\) admits a tiling
2. there is a dataset \(D\) over \(\Sigma\) such that \(O_1 \cup D\) is satisfiable and \(O_1 \cup D \models E(a)\) for some \(a\) in \(D\);
3. there is a dataset \(D\) over \(\Sigma\) such that \(O_1 \cup D\) is satisfiable and \(O_1 \cup D \models \exists x. E(x)\).

Let \(S, S'\) be fresh role names and \(P_1, P_2, P_3\) fresh concept names. Let \(O_2\) be an extension of \(O_1\) by the following axioms.

\[
\begin{align*}
\top &\subseteq \exists S. E \\
\exists S'. \top &\subseteq P_1 \cup P_2 \cup P_3 \\
(\exists S'. \top) \cap P_i \cap P_j &\nsubseteq \bot \\
(\exists S'. \top) \cap P_i \cap \exists S'. P_i &\nsubseteq \bot
\end{align*}
\]

We now show that \(\Pi\) admits a tiling if and only if \(O_2\) is not rewritable to Horn-\(L_2\). First, suppose \(\Pi\) admits a tiling. Then there is a dataset \(D_1\) over \(\Sigma\) such that \(O_1 \cup D_1\) is satisfiable and \(O_1 \cup D_1 \models E(a)\) for some \(a\) in \(D_1\). Given a connected undirected graph \(G\), let \(D_G = \{ S'(d, d') \mid \{d, d'\} \text{ edge in } G \}\) and \(D_2 = D_1 \cup \exists S'. P\) \(\because\) \(S\) \(\subseteq\) \(\{S(d, c) \mid d \text{ occurs in } D_G \cup D_1, c \text{ occurs in } D_1\}\). Then \(O_2 \cup D_2\) is consistent if and only if \(G\) is 3-colourable. Therefore, since 3-colourability is NP-complete in data whereas satisfiability checking w.r.t. Horn-\(ALC\mathcal{HF}\) ontologies is tractable in data, \(O_2\) is not rewritable to Horn-\(L_2\) unless PTIME = NP.

Now suppose \(\Pi\) does not admit a tiling. Then \(O_2 \cup D\) is unsatisfiable for every \(D\) and hence the ontology \(\{\top \subseteq \bot\}\) is a Horn-\(\mathcal{ELU}\) rewriting of \(O_2\).

B Proofs for Section 4

Reasoning w.r.t. programs can be realised by means of the hyperresolution calculus. In our treatment of hyperresolution we treat disjunctions of atoms as sets and hence we do not allow for duplicated atoms in a disjunction. Let \(\rho\) be an MGU of each \(\beta_1, \alpha_i\). Then the disjunction of atoms \(\varphi' = \varphi_\sigma \lor \chi_1 \lor \cdots \lor \chi_n\) is a hyperresolvent of \(\varphi\) and \(\phi_1, \ldots, \phi_n\). Let \(P\) be a program, let \(D\) be a dataset, and let \(\varphi\) be a disjunction of atoms. A (hyperresolution) derivation of \(\varphi\) from \(P \cup D\) is a pair \(\rho = (T, \lambda)\) where \(T\) is a tree, \(\lambda\) a labeling function mapping each node in \(T\) to a disjunction of atoms, and the following properties hold for each \(v \in T\):

1. \(\lambda(v) = \varphi\) if \(v\) is the root;
2. \(\lambda(v) \in P \cup D\) if \(v\) is a leaf; and
3. if \(v\) has children \(w_1, \ldots, w_n\), then \(\lambda(v)\) is a hyperresolvent of a rule \(r \in P\) and \(\lambda(w_1), \ldots, \lambda(w_n)\).

We write \(P \cup D \vdash \varphi\) to denote that \(\varphi\) has a derivation from \(P \cup D\). Hyperresolution is sound and complete in the following sense: \(P \cup D\) is unsatisfiable iff \(P \cup D \vdash \square\). Furthermore, if \(P \cup D\) is satisfiable then \(P \cup D \vdash \alpha\) iff \(P \cup D \models \alpha\) for every atom \(\alpha\).

Hyperresolution derivations satisfy the following property.

Proposition 18. Let \(P\) be a program, \(D\) a dataset, and \(\rho\) a derivation from \(P \cup D\). Then every node in \(\rho\) is labeled by either a single Horn atom or a (possibly empty) disjunction of disjunctive atoms.

Proof. The claim follows by a straightforward induction on \(\rho\).

Proposition 19. Let \(P\) be a program, \(M\) a marking of \(P\), and \(D\) a dataset over the predicates in \(P\). Then \(\Xi_M(P) \cup D \models \Xi(s)\) for every ground term \(s\) over the signature of \(P \cup D\).

Proof. The claim is a straightforward consequence of the axiomatisation of \(\Xi\) in \(\Xi_M(P)\).

Theorem 7. Let \(M\) be a marking of a program \(P\). Then \(\Xi_M(P)\) is a polynomial-size Horn rewriting of \(P\).

Proof. We proceed in two steps, which together imply the theorem. We fix an arbitrary markable program \(P\), a marking \(M\) of \(P\), and a dataset \(D\). W.l.o.g. we assume that \(D\) only contains predicates in \(P\).
1. We show that $P \cup D \models \Box$ implies $\Xi_M(P) \cup D \models \Box$. For this, we consider a derivation $\rho$ of $\Box$ from $P \cup D$ and show that for every disjunctive atom $Q(s)$ in the label of a node in $\rho$, we have $\Xi_M(P) \cup D \models Q(s)$ if $Q \in M$ and otherwise $\Xi_M(P) \cup D \models \neg Q(s)$. This claim, in turn, is shown by first showing a more general statement and then instantiating it with $\rho$.

2. We show that $\Xi_M(P) \cup D \models \Box$ implies $P \cup D \models \Box$. Again, we first show a general claim that holds for any derivation from $\Xi_M(P) \cup D$ and then instantiate the claim with a derivation of $\Box$.

In both steps we use that $P$ and $\Xi_M(P)$ entail the same facts over Horn predicates for every dataset. We now detail the two steps formally.

**Step 1.** Suppose $P \cup D \models \Box$. We show $\Xi_M(P) \cup D \models \Box$. We begin by showing the following claim.

Claim ($\Box$). Let $\varphi = Q_1(s_1) \lor \cdots \lor Q_n(s_n)$ be a non-empty disjunction of facts satisfying the following properties: (i) $\Xi_M(P) \cup D \models \neg \Box_i(s_i)$ for each $Q_i \in M$. (ii) $\varphi$ is derivable from $P \cup D$. Then, for each derivation $\rho$ of $\varphi$ from $P \cup D$ and each atom $R(\bar{t})$ with $R$ disjunctive in the label of a core node in $\rho$ we have $\Xi_M(P) \cup D \models R(\bar{t})$ if $R \in M$ and $\Xi_M(P) \cup D \models \neg R(\bar{t})$ otherwise.

We show the claim by induction on $\rho = (T, \lambda)$. W.l.o.g., the root $v$ of $T$ has a disjunctive predicate in its label (otherwise, the claim is vacuous since the core of $\rho$ contains no disjunctive nodes).

For the base case, suppose $\rho$ has no children labeled with disjunctive predicates. We then distinguish two subcases:

- $\varphi \in D$. Then $\varphi$ is a fact, i.e., $\varphi = Q(\bar{a})$ for some $Q$ and $\bar{a}$. If $Q \in M$, the claim is immediate by assumption (i). If $Q \not\in M$, the claim follows as $D \models Q(\bar{a})$.
- $\varphi$ is obtained by a rule $\psi \rightarrow \varphi'$ in $P$ where $\psi$ is a conjunction of Horn atoms and, for some $\sigma$, $\varphi = \varphi' \sigma$ and $P \cup D \models \psi \sigma$. If $\{Q_1, \ldots, Q_n\} \subseteq M$, the claim is immediate by assumption (i), so let us assume w.l.o.g. that $Q_1 \notin M$. By the definition of a marking, we then have $\{Q_2, \ldots, Q_n\} \subseteq M$, and hence it suffices to show $\Xi_M(P) \cup D \models \neg Q_1(s_1)$. This follows since $\psi \land \bigwedge_{i=2}^n \Box_i(s_i) \rightarrow Q_1(s_1)$ is $\Xi_M(P)$ (where $s_i' = s_1$, $\Xi_M(P) \cup D \models \bigwedge_{i=2}^n \Box_i(s_i)$ by assumption (i), $\Xi_M(P) \cup D \models \psi \sigma$ since $\Xi_M(P) \cup D \models \psi$ and $\Xi_M(P) \cup D \models \neg R(\bar{t})$ otherwise.)

For the inductive step, suppose $\rho$ has children $w_1, \ldots, w_m$ in $T$ that are labeled with disjunctive predicates. W.l.o.g., there is a rule $r = \psi \land \bigwedge_{i=1}^m R_i(\bar{t}_i) \rightarrow \bigvee_{j=1}^k Q_j(s_j')$ in $P$ (with $\psi$ a conjunction of Horn atoms, $0 \leq k \leq n$, and all $R_i$ disjunctive in $P$) such that $\lambda(v)$ is obtained by a hyperrepetition step using $r$ from $\psi \sigma$ and $\lambda(w_1), \ldots, \lambda(w_m)$ where $\sigma$ is a substitution mapping every atom $R_i(\bar{t}_i)$ to a disjunct in $\lambda(w_i)$. In particular, we have $s_i' = \bar{s}_i$, $R_i(\bar{t}_i') \in \lambda(w_i)$, and $P \cup D \models \psi \sigma$. We distinguish three cases:

- $\{Q_1, \ldots, Q_k\} \subseteq M$ and $\{R_1, \ldots, R_m\} \cap M = \emptyset$. Then, for every $i \in [1, m]$, every marked atom in $\lambda(w_i)$ also occurs in $\lambda(v)$; furthermore, every unmarked atom in $\lambda(v)$ occurs in $\lambda(w_i)$ for some $i \in [1, m]$. By the latter statement, it suffices to show the claim for the subderivations rooted at $w_1, \ldots, w_m$. Let $i \in [1, m]$. By the fact that every marked atom in $\lambda(w_i)$ also occurs in $\lambda(v)$ and assumption (i), we have $\Xi_M(P) \cup D \models S(\bar{u})$ for every marked disjunct $S(\bar{u})$ in $\lambda(w_i)$. Then, we can apply the inductive hypothesis to the subderivation root at $w_i$ and the claim follows.

- $\{Q_1, \ldots, Q_k\} \subseteq M$, $R_1 \in M$, and $\{R_2, \ldots, R_m\} \cap M = \emptyset$ (note that $R_1 \in M$ implies $\{R_2, \ldots, R_m\} \cap M = \emptyset$ since $M$ is a marking). Then (a) for every $i \in [1, m]$, every marked atom in $\lambda(w_i)$ except for possibly $R_i(\bar{t}_i')$ in $\lambda(w_i)$ also occurs in $\lambda(v)$, and (b) every unmarked atom in $\lambda(v)$ occurs in $\lambda(w_i)$ for some $i \in [1, m]$. We also have (c) $\varphi \land \psi \land \bigwedge_{i=2}^m R_i(\bar{t}_i) \land \bigwedge_{j=1}^k Q_j(s_j') \rightarrow R_i(\bar{t}_i') \in \Xi_M(P)$. As in the preceding case, by (b), it suffices to show the claim for the subderivations rooted at $w_1, \ldots, w_m$. For $w_2, \ldots, w_n$, we proceed as follows. Let $i \in [2, m]$. By (a) and assumption (i), we have $\Xi_M(P) \cup D \models S(\bar{u})$ for every marked disjunct $S(\bar{u})$ in $\lambda(w_i)$. Thus, we can apply the inductive hypothesis to the subderivation root at $w_i$. In particular, we obtain $\Xi_M(P) \cup D \models R_i(\bar{t}_i')$. In the case of $w_1$, we need to show $\Xi_M(P) \cup D \models R_1(\bar{t}_i')$ in order to apply the inductive hypothesis. This follows by (c) and assumption (i) since $\Xi_M(P) \cup D \models \psi \sigma, Q_1(s_1')$ is $\lambda(v)$, $\Xi_M(P) \cup D \models R_i(\bar{t}_i')$ for $i \in [2, m]$, and $\Xi_M(P) \cup D \models \varphi \land \psi \land \bigwedge_{i=2}^m R_i(\bar{t}_i) \land \bigwedge_{j=1}^k Q_j(s_j') \rightarrow R_1(\bar{t}_i')$. Finally, note that $\psi \land \bigwedge_{i=1}^m R_i(\bar{t}_i) \land \bigwedge_{j=2}^k Q_j(s_j') \rightarrow Q_1(s_1')$ is $\Xi_M(P)$ (since $r \in P$). Then, $\Xi_M(P) \cup D \models Q_1(s_1')$ follows from $\Xi_M(P) \cup D \models \psi \sigma$, the inductive hypothesis (which implies $\Xi_M(P) \cup D \models R_i(\bar{t}_i')$), and the assumption (i) (which implies $\Xi_M(P) \cup D \models \neg Q_1(s_1')$).
We next instantiate (♦) to show the claim in Step 1. Let \( \varphi = \bot(s) \). We have assumed in Step 1 that \( \mathcal{P} \cup \mathcal{D} \vdash \Box \) so \( \bot(s) \) is derivable from \( \mathcal{P} \cup \mathcal{D} \) for some \( s \) (as \( \Box \) can only be derived by the rule \( \bot(x) \to \Box \), and hence condition (ii) in (♦) holds. Furthermore, if \( \bot \in M \), we have \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \bot(s) \); hence, condition (i) in (♦) also holds.

Now, let \( \rho = (T, \lambda) \) be a derivation of \( \bot(s) \) from \( \mathcal{P} \cup \mathcal{D} \). We exploit (♦) applied to \( \rho \) to show that \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \bot(s) \). We distinguish two cases:

- \( \bot \notin M \). Since \( \bot(s) \) labels the root of \( \rho \) we can apply (♦) to obtain \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \bot(s) \); the claim follows.

- \( \bot \in M \). Then there is a core node \( v \) in \( \rho \) such that: \( \lambda(v) \) contains only marked atoms and \( v \) has no successor \( w \) in \( T \) such that all atoms in \( \lambda(w) \) are marked. We distinguish two cases.

  If \( \lambda(v) \in \mathcal{D} \), then \( \lambda(v) = Q(i) \) for some \( Q \) and \( i \). Moreover, by (♦), we have \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \Box Q(i) \). The claim follows since \( \Box(z) \land Q(\bar{x}) \land \Box(\bar{x}) \rightarrow \bot(z) \in \Xi_M(\mathcal{P}) \) and \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \bot(s) \).

  If \( \lambda(v) \notin \mathcal{D} \), then \( v \) has successors \( v_1, \ldots, v_n \) \( (n \geq 0) \) in \( T \) such that \( \lambda(v) \) is a hyperresolvent of \( \lambda(v_1), \ldots, \lambda(v_n) \) and a rule in \( \mathcal{P} \) of the form \( \Lambda_{i=1}^n Q_i(s_i) \rightarrow \bigwedge_{j=1}^m R_j(t_j) \), where the atoms \( Q_i(s_i) \) are resolved with \( \lambda(v_i) \). Since \( \lambda(v_i) \) contains only marked atoms but \( \lambda(v_1), \ldots, \lambda(v_n) \) all contain Horn or unmarked atoms, all \( Q_i \) must be Horn or unmarked and all \( R_j \) must be marked. Hence, \( \Xi_M(\mathcal{P}) \) contains a rule \( r = \Box(x) \land (\bigwedge_{i=1}^k Q_i(s_i)) \land (\bigwedge_{i=k+1}^n Q_i(s_i)) \land \bigwedge_{i=1}^m R_i(t_i) \rightarrow \bot(x) \) where, w.l.o.g., \( Q_1, \ldots, Q_k \) are Horn and \( Q_{k+1}, \ldots, Q_n \) are disjunctive and unmarked. Let \( \sigma \) be the substitution that is used in the hyperresolution step deriving \( \lambda(v) \). By (♦), we then have \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash Q_i(s_i) \sigma \) for every \( i \in [k+1, n] \) and \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash R_i(t_i) \sigma \) for every \( j \in [1, m] \). Moreover, we have \( \lambda(v_i) = Q_i(s_i) \sigma \) and hence \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash Q_i(s_i) \sigma \) for every \( i \in [1, k] \). Finally, we have \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \bot(s) \). The claim follows with \( r \).

**Step 2.** Let \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash \Box \). Then there is a derivation \( \rho \) of \( \bot(s) \) for some \( s \) from \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \). The fact that \( \mathcal{P} \cup \mathcal{D} \vdash \bot(s) \) follows directly from Statement 1 in Claim (♣), which we show next.

Claim (♣). Let \( \rho \) be a derivation from \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \), and let \( v \) be the root of \( \rho \). Then:

1. If \( \lambda(v) = Q(i) \), then \( \mathcal{P} \cup \mathcal{D} \vdash Q(i) \).
2. If \( \lambda(v) = \overline{Q(i)} \), then \( \mathcal{P} \cup \mathcal{D} \vdash \overline{Q(i)} \).

We show the two claims by simultaneous induction on \( \rho \). For the base case, suppose \( v \) is the only node in \( \rho \). We distinguish two cases:

- \( \lambda(v) \in \mathcal{D} \). Then \( \mathcal{D} \vdash \lambda(v) \) and the claim is immediate.

- \( \lambda(v) = Q(i) \) where \( Q \) is Horn in \( \mathcal{P} \) and \( r = (\rightarrow Q(i)) \in \Xi_M(\mathcal{P}) \). Then \( r \in \mathcal{P} \) and the claim follows.

For the inductive step, suppose \( v \) has children \( v_1, \ldots, v_n \) and, \( \lambda(v) \) is a hyperresolvent of \( \lambda(v_1), \ldots, \lambda(v_n) \) and a rule \( r \in \Xi_M(\mathcal{P}) \). We distinguish five cases:

- \( r \) contains no disjunctive predicates, in which case the claim follows since \( \mathcal{P} \cup \mathcal{D} \) and \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \) entail the same facts over a Horn predicate.

- \( r = \Box(z) \land P(\bar{x}) \land \overline{P(\bar{x})} \rightarrow \bot(z) \). Then \( \lambda(v) = \bot(s) \) for some \( s \). Since, by the inductive hypothesis, \( \mathcal{P} \cup \mathcal{D} \vdash P(\bar{i}) \land \overline{P(\bar{i})} \) for some \( \bar{i} \), \( \mathcal{P} \cup \mathcal{D} \) is inconsistent, and hence \( \mathcal{P} \cup \mathcal{D} \vdash \bot(s) \).

- \( r = \Box(z) \land P(\bar{x}) \land \overline{P(\bar{x})} \rightarrow \bot(z) \). Then \( \lambda(v) = \bot(s) \) for some \( s \). For some \( \sigma \), we have \( \mathcal{P} \cup \mathcal{D} \vdash \varphi \sigma, \top \sigma = \top \sigma \) and, for each \( i, j, \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash P_i(s_i) \sigma \) and \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash Q_j(t_j) \sigma \). Then, by the inductive hypothesis, \( \mathcal{P} \cup \mathcal{D} \vdash \top \sigma \) and \( \mathcal{P} \cup \mathcal{D} \vdash Q_j(t_j) \sigma \). With \( \top \sigma \), we obtain \( \mathcal{P} \cup \mathcal{D} \vdash \top \sigma \).

- \( r = \Box(z) \land P(\bar{x}) \land \overline{P(\bar{x})} \rightarrow \bot(z) \). Then \( \lambda(v) = \bot(s) \) for some \( s \). For some \( \sigma \), we then have \( \mathcal{P} \cup \mathcal{D} \vdash \varphi \sigma \) and, for each \( i, j, \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash P_i(s_i) \sigma \) and \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash Q_j(t_j) \sigma \). Then, by the inductive hypothesis, \( \mathcal{P} \cup \mathcal{D} \vdash \overline{P_i(s_i)} \sigma \) and \( \mathcal{P} \cup \mathcal{D} \vdash Q_j(t_j) \sigma \). With \( \overline{P_i(s_i)} \sigma \), we obtain \( \mathcal{P} \cup \mathcal{D} \vdash \bot(s) \).

- \( r = \Box(z) \land P(\bar{x}) \land \overline{P(\bar{x})} \rightarrow \bot(z) \). Then \( \lambda(v) = \bot(s) \) for some \( s \). For some \( \sigma \) we then have \( \mathcal{P} \cup \mathcal{D} \vdash \varphi \sigma, \overline{P_i(s_i)} \sigma = \overline{P_i(s_i)} \sigma \) and, for each \( i, j, \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash P_i(s_i) \sigma \) and \( \Xi_M(\mathcal{P}) \cup \mathcal{D} \vdash Q_j(t_j) \sigma \). Then, by the inductive hypothesis, \( \mathcal{P} \cup \mathcal{D} \vdash \overline{P_i(s_i)} \sigma \) and \( \mathcal{P} \cup \mathcal{D} \vdash Q_j(t_j) \sigma \). With \( \overline{P_i(s_i)} \sigma \), we obtain \( \mathcal{P} \cup \mathcal{D} \vdash \bot(s) \).

\( \square \)
C  Proofs for Section 5

Theorem 9. For every ontology $\mathcal{O}$ and dataset $\mathcal{D}$ over predicates in $\mathcal{O}$ we have that $\mathcal{O} \cup \mathcal{D}$ is satisfiable iff so is $\xi(\mathcal{O}) \cup \mathcal{D}$.

Proof. For the direction from left to right, suppose $\mathcal{I}$ is a model of $\mathcal{O} \cup \mathcal{D}$. We define the interpretation $\mathcal{J}$ such that

- the domain of $\mathcal{J}$ extends the domain of $\mathcal{I}$ by one additional individual $u$;
- $\mathcal{J}$ coincides with $\mathcal{I}$ on every concept name, role name and constant in $\mathcal{O} \cup \mathcal{D}$, and $\simeq^\mathcal{J} = \simeq^\mathcal{I}\cup\{(u,u)\}$;
- $f^\mathcal{J}_{R,A}(v) \in \{w \in A^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$ if the set $\{w \in A^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$ is nonempty and otherwise $f^\mathcal{J}_{R,A}(v) = u$ (if $R = S^\mathcal{I}$ for a role name $S$, we write $R^\mathcal{I}$ for $(S^\mathcal{I})^{-1}$).

We show that $\mathcal{J}$ is a model of $\xi(\mathcal{O}) \cup \mathcal{D}$. Clearly, $\mathcal{J}$ satisfies $\mathcal{D}$ and every rule in $\xi(\mathcal{O})$ of type T1-T2 and T4-T6, so it suffices to show that $\mathcal{J}$ satisfies the rules introduced by $\xi$. So, let $r = (\xi(\alpha)) \setminus \pi(\alpha)$ for some $\alpha \in \mathcal{O}$. We distinguish the following cases:

1. $r = A(x) \rightarrow B(f_{R,B}(x))$ and $\alpha = A \sqsubseteq \exists R.B$. Let $v \in A^\mathcal{J}$. It suffices to show that $f^\mathcal{J}_{R,B}(v) \in B^\mathcal{J}$. Since $\mathcal{I}$ satisfies $\alpha$, $v$ has an $R^\mathcal{I}$-successor that is in $B^\mathcal{I}$, and hence $f^\mathcal{J}_{R,B}(v) \in B^\mathcal{J} = B^\mathcal{I}$.

2. $r = A\left(f^\mathcal{J}_{R,Y}(x)\right) \rightarrow C(x)$, $\alpha = \exists R.A \sqsubseteq C$, and $R^\mathcal{J} \sqsubseteq^* R$. Let $v \in A^\mathcal{J}$. It suffices to show $v \in C^\mathcal{J}$. By construction, we have $f^\mathcal{J}_{R,Y}(v) \neq u$ and hence $f^\mathcal{J}_{R,Y}(v) \in \{w \in A^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$. Since $R^\mathcal{J} \subseteq^* R$, it follows that $f^\mathcal{J}_{R,Y}(v) \in \{w \in A^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$ and $\mathcal{J}$ satisfies $\alpha$, we conclude $v \in C^\mathcal{J} = C^\mathcal{I}$.

3. $r = A(x) \land Y(f^\mathcal{I}_{inv(R)}(y,x)) \rightarrow C(f^\mathcal{I}_{inv(R)}(y,x))$, $\alpha = \exists R.A \sqsubseteq C$ and $R^\mathcal{J} \sqsubseteq^* R$. Let $v \in A^\mathcal{J}$ and $f^\mathcal{J}_{inv(R)}(y,x) \in Y^\mathcal{J}$. It suffices to show $f^\mathcal{J}_{inv(R)}(y,x) \in C^\mathcal{J}$. By construction, we have $f^\mathcal{J}_{inv(R)}(y,x) \neq u$ and hence $f^\mathcal{J}_{inv(R)}(y,x) \in \{w \in Y^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$. Since $R^\mathcal{J} \subseteq^* R$, it follows that $f^\mathcal{J}_{inv(R)}(y,x) \in \{w \in Y^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$. Since $v \in A^\mathcal{J} = A^\mathcal{I}$, we have $f^\mathcal{J}_{inv(R)}(y,x) \in \{w \in A^\mathcal{I} \mid (v,w) \in R^\mathcal{I}\}$. Since $\mathcal{J}$ satisfies $\alpha$, we conclude $f^\mathcal{J}_{inv(R)}(y,x) \in C^\mathcal{J} = C^\mathcal{I}$.

4. $r = A(f_{R,Y}(x),B(x)) \rightarrow C(x)$, $\alpha = A \sqsubseteq \forall R.C$, and $R^\mathcal{J} \subseteq^* R$. The claim follows similarly to Case 2.

5. $r = A(x) \land Y(f^\mathcal{J}_{R,Y}(x)) \rightarrow C(f^\mathcal{J}_{R,Y}(x))$, $\alpha = A \sqsubseteq \forall R.C$, and $R^\mathcal{J} \subseteq^* R$. The claim follows similarly to Case 3.

6. $r = A(z) \land B(f^\mathcal{J}_{R,Y}(z)) \land at(R,z,x) \land B(x) \rightarrow f^\mathcal{J}_{R,Y}(z) \approx x$, $\alpha = A \sqsubseteq^1 R.B$, and $R^\mathcal{J} \subseteq^* R$. Let $v \in A^\mathcal{J}$, $f^\mathcal{J}_{R,Y}(v) \in B^\mathcal{J}$, $(v,w) \in R^\mathcal{J}$ and $w \in B^\mathcal{J}$. It suffices to show $f^\mathcal{J}_{R,Y}(v) \approx_w w$. By construction, we have $f^\mathcal{J}_{R,Y}(v) \in \{w' \mid (v,w') \in R^\mathcal{I}\} \subseteq \{w' \mid (v,w') \in R^\mathcal{I}\}$. The claim follows since $A^\mathcal{J} = A^\mathcal{I}$, $B^\mathcal{J} = B^\mathcal{I}$, $R^\mathcal{J} = R^\mathcal{I}$ and $\mathcal{I}$ satisfies $\alpha$.

7. $r = A(z) \land B(f_{R_1,Y_1}(z)) \land B(f_{R_2,Y_2}(z)) \rightarrow f^\mathcal{J}_{R_1,Y_1}(z) \approx f^\mathcal{J}_{R_2,Y_2}(z)$, $\alpha = A \sqsubseteq^1 R.B$, and $R_i^\mathcal{J} \subseteq^* R$. The claim follows similarly to Case 6.

8. $r = A(z) \land B(f_{R_1,Y_1}(z)) \land B(f_{R_2,Y_2}(z)) \rightarrow f^\mathcal{J}_{R_1,Y_1}(z) \approx f^\mathcal{J}_{R_2,Y_2}(z)$, $\alpha = A \sqsubseteq^1 R.B$, and $R_i^\mathcal{J} \subseteq^* R$. The claim follows similarly to Case 7.

For the direction from right to left, suppose $\mathcal{J}$ is a minimal Herbrand model of $\xi(\mathcal{O}) \cup \mathcal{D}$. We define the interpretation $\mathcal{I}$ such that

- $\mathcal{I}$ coincides with $\mathcal{J}$ on its domain as well as on every concept name and every constant in $\mathcal{O} \cup \mathcal{D}$;
- $R^\mathcal{I} = R^\mathcal{J} \cup \{ (v,f^\mathcal{J}_{R,Y}(v)) \mid f^\mathcal{J}_{R,Y}(v) \in \Phi, v \in A^\mathcal{J}, f^\mathcal{J}_{R,Y}(v) \in Y^\mathcal{J}, R_i^\mathcal{J} \subseteq^* R \} \cup \{ (f^\mathcal{J}_{inv(R)}(Y),v) \mid f^\mathcal{J}_{inv(R)}(Y) \in \Phi, v \in A^\mathcal{J}, f^\mathcal{J}_{inv(R)}(Y) \in Y^\mathcal{J}, R_i^\mathcal{J} \subseteq^* R \}$.

We show that $\mathcal{I}$ is a model of $\mathcal{O} \cup \mathcal{D}$. Clearly, $\mathcal{I}$ satisfies $\mathcal{D}$ and every axiom in $\mathcal{O}$ of type T1, so it suffices to show that $\mathcal{I}$ satisfies axioms of type T2-T6, which we do next.

- Let $\exists R.A \sqsubseteq C \in \mathcal{O}$. W.l.o.g., let $v \in (\exists R.A)^{\mathcal{I}} \setminus (\exists R.A)^{\mathcal{J}}$ (if $v \in (\exists R.A)^{\mathcal{J}}$ the claim is immediate since $\pi(\exists R.A \sqsubseteq C) \in \xi(\mathcal{O})$). It suffices to show $v \in C^\mathcal{I}$. By construction of $R^\mathcal{I}$, there exists $R_i^\mathcal{J} \subseteq^* R$ and $Y$ such that either $f^\mathcal{I}_{R,Y}(v) \in Y^\mathcal{I}$ or $f^\mathcal{J}_{inv(R)}(Y) \in \Phi$ and there is some $w \in A^\mathcal{J}$ such that $v = f^\mathcal{J}_{inv(R)}(Y) \in Y^\mathcal{J}$ and $w \in A^\mathcal{I} = A^\mathcal{J}$. In the former case, $v \in C^\mathcal{J} = C^\mathcal{I}$ follows since $A(f^\mathcal{J}_{R,Y}(x)) \rightarrow C(x) \in \xi(\mathcal{O})$. In the latter case, $v \in C^\mathcal{I}$. Since $A(x) \land Y(f^\mathcal{J}_{R,Y}(x)) \rightarrow C(f^\mathcal{J}_{inv(R)}(Y)) \in \xi(\mathcal{O})$.  

Let $A \subseteq \exists R.B \in \mathcal{O}$ and let $v \in A^\mathcal{T}$. It suffices to show $v \in (\exists R.B)^\mathcal{T}$. Since $A^\mathcal{T} = A^\mathcal{D}$ and $A(x) \rightarrow B(f_{R,B}(x)) \in \xi(\mathcal{O})$, we have $f_{R,B}^\mathcal{D}(v) \in B^\mathcal{D} = B^\mathcal{T}$. Hence it suffices to show $(v, f_{R,B}^\mathcal{D}(v)) \in R^\mathcal{T}$, which follows since $f_{R,B} \in \Phi$, $f_{R,B}^\mathcal{D}(v) \in B^\mathcal{D}$ and $R \subseteq R$.

Let $A \subseteq \forall R.C \in \mathcal{O}$. The claim follows analogously to the case for $\exists R.A \subseteq C \in \mathcal{O}$.

Let $S \subseteq R \in \mathcal{O}$. W.l.o.g., let $(v, w) \in S^\mathcal{T} \setminus S^\mathcal{D}$ (if $(v, w) \in S^\mathcal{D}$ we immediately obtain $(v, w) \in R^\mathcal{D}$ since $\pi(S \subseteq R) \in \xi(\mathcal{O})$). We show $(v, w) \in R^\mathcal{T}$. By construction of $S^\mathcal{T}$, there exists some $R^* \subseteq S^\mathcal{D}$ and $Y$ such that either $f_{R^*,Y} \in \Phi$ and $w = f_{R^*,Y}(v) \in Y^\mathcal{D}$ or $f_{inv(R^*)}^\mathcal{D}, Y) \in \Phi$ and $v = f_{inv(R^*)}^\mathcal{D}, Y)(w) \in Y^\mathcal{D}$. In both cases we obtain $(v, w) \in R^\mathcal{D}$ since $R^* \subseteq R$ and $S \subseteq R \in \mathcal{O}$ implies $R^* \subseteq R$.

Let $A \subseteq \leq 1 R.B \in \mathcal{O}$. Let $v \in A^\mathcal{T}$, $u \in B^\mathcal{T}$, and $(v, w), (v, u) \in R^\mathcal{T}$. We show $w \approx \mathcal{O} u$. We distinguish the following subcases:

- $\{(v, u), (v, w)\} \subseteq R^\mathcal{T}$. Then the claim is immediate since $\pi(A \subseteq \leq 1 R.B) \in \xi(\mathcal{O})$.
- $(v, u) \in R^\mathcal{T}$ and $(v, w) \in R^\mathcal{T} \setminus R^\mathcal{D}$. By construction of $R^\mathcal{T}$, there exists some $R^* \subseteq R$ and $Y$ such that either $f_{R^*,Y} \in \Phi$ and $w = f_{R^*,Y}(v) \in Y^\mathcal{D}$ or $f_{inv(R^*)}^\mathcal{D}, Y) \in \Phi$ and $v = f_{inv(R^*)}^\mathcal{D}, Y)(w) \in Y^\mathcal{D}$. In the former case, $w \approx \mathcal{O} u$ follows since $A(z) \land B(f_{R^*,Y}(z)) \land \mathsf{at}(R^*, z, x) \land B(x) \rightarrow f_{R^*,Y}(z) \approx x \in \xi(\mathcal{O})$. In the latter case, the claim follows since $A(f_{inv(R^*)}, Y(x)) \land B(x) \land \mathsf{at}(R^*, f_{inv(R^*)}, Y)(x, y) \land B(y) \rightarrow x \approx y \in \xi(\mathcal{O})$.
- $\{(v, u), (v, w)\} \subseteq R^\mathcal{D} \setminus R^\mathcal{T}$. By construction of $R^\mathcal{T}$, there are some $R^1, R^2 \subseteq \mathcal{O}^\mathcal{R}$ and $Y_1, Y_2$ such that we have one of the three following cases:
  1. $\{f_{R^1, Y_1}, f_{R^2, Y_2}\} \subseteq \Phi, u = f_{R^1, Y_1}(v) \in Y_1^\mathcal{D}$ and $w = f_{R^2, Y_2}(v) \in Y_2^\mathcal{D}$. Then the claim follows since $A(z) \land B(f_{R^1, Y_1}(z)) \land B(f_{R^2, Y_2}(z)) \rightarrow f_{R^1, Y_1}(z) \approx f_{R^2, Y_2}(z) \in \xi(\mathcal{O})$.
  2. $\{f_{inv(R^1)}, f_{R^1, Y_2}\} \subseteq \Phi, v = f_{inv(R^1)}(u) \in Y_1^\mathcal{D}$ and $w = f_{R^2, Y_2}(f_{inv(R^1)}^\mathcal{D}, Y_1(x)) \in Y_2^\mathcal{D}$. Then the claim follows since $A(f_{inv(R^1)}, Y_1(x)) \land B(x) \land B(f_{R^2, Y_2}(f_{inv(R^1)}, Y_1(x))) \rightarrow x \approx f_{R^2, Y_2}(f_{inv(R^1)}, Y_1(x)) \in \xi(\mathcal{O})$.
  3. $\{f_{inv(R^1)}, f_{R^1, Y_2}\} \subseteq \Phi, v = f_{inv(R^1)}(u) \in Y_1^\mathcal{D}$ and $w = f_{R^2, Y_2}(u) \in Y_2^\mathcal{D}$. The claim then follows since, as $J$ is a Herbrand model, we must have $u = u$ and $\approx \mathcal{O}$ is reflexive.

Proposition 10. (i) If $\approx$ is Horn in $\xi(\mathcal{O})$ then so are all binary predicates in $\xi(\mathcal{O})$. (ii) If $\xi(\mathcal{O})$ is markable, then it has a marking containing only unary predicates.

Proof. Note that all non-Horn rules in $\xi(\mathcal{O})$ are of type T1, i.e., have unary predicates in the head. Both claims follow from this observation and the fact that $\xi(\mathcal{O})$ contains no rules with unary predicates in the body and binary predicates in the head except for rules of type T6. Thus, whenever a binary predicate $P$ is disjunctive in $\xi(\mathcal{O})$ (resp., is part of a minimal marking of $\xi(\mathcal{O})$), this is due to an axiom $P(x, y) \land x \approx z \rightarrow P(z, y)$ or $P(x, y) \land y \approx z \rightarrow P(x, z)$ in $\xi(\mathcal{O})^\text{simplified}$ where $\approx$ is disjunctive (resp., marked) in $\xi(\mathcal{O})$. However, predicate $\approx$ cannot be part of any marking since then the transitivity rule $x \approx y \land y \approx z \rightarrow x \approx z$ in $\xi(\mathcal{O})^\text{simplified}$ would have two marked body atoms.

D Proofs for Section 6

Lemma 12. Let $\mathcal{O}$ be an ontology and $M$ a minimal marking of $\xi(\mathcal{O})$. Then $\Xi_M(\xi(\mathcal{O}))$ contains only Horn rules of type T1-T2 and T4-T6 in Table 7 as well as type T7-T20 in Table 3.

Proof. The claim follows by a simple case analysis over the possible rule types in $\xi(\mathcal{O})$ (as given in Definition 8) as well as the possible minimal markings for each rule type. The analysis exploits that minimal markings involve no binary predicates (Proposition 10(ii)).

Lemma 14. Let $\mathcal{O}$ be a markable $\mathcal{ALCHI}^\mathcal{F}$ ontology and let $M$ be a marking of $\mathcal{O}$. Then the ontology $\Psi(\Xi_M(\xi(\mathcal{O})))$ is a Horn rewriting of $\mathcal{O}$.

Proof. By Theorems 7 and 9 it suffices to show that $\Psi(\mathcal{P})$ is a rewriting of $\mathcal{P}$ whenever $\mathcal{P} = \Xi_M(\xi(\mathcal{O}))$ for some $\mathcal{O}$ and $M$. So let $\mathcal{P}$ be as required and let $\mathcal{D}$ be a dataset over the predicates in $\mathcal{P}$. We show that $\mathcal{P} \cup \mathcal{D}$ is satisfiable if and only if so is $\pi(\Psi(\mathcal{P})) \cup \mathcal{D}$.

For the direction from left to right, let $\mathcal{I}$ be a minimal Herbrand model of $\mathcal{P}$. We define the interpretation $\mathcal{J}$ such that

- $\mathcal{J}$ coincides with $\mathcal{I}$ on its domain as well as on every concept name, role name, and individual constant in $\mathcal{P} \cup \mathcal{D}$;
- $R^\mathcal{J}_1 = \{(v, f_{R^1, Y}(v)) \mid v \in A^\mathcal{T}\}$ for each function $f_{R^1, Y}$ in $\mathcal{P}$;
- $R^\mathcal{J}_2 = (R^\mathcal{D}_1)^{-1}$ for each role $R^\mathcal{D}_1$ in $\Psi(\mathcal{P})$;
- $S^\mathcal{J}_{R^1, R^2} = R^\mathcal{J}_1 \cup R^\mathcal{J}_2$ for each role $S_{R^1, R^2}$ in $\Psi(\mathcal{P})$. 


We next show that $\mathcal{J}$ is a model of $\pi(\Psi(\mathcal{P})) \cup \mathcal{D}$. By construction, $\mathcal{J}$ satisfies axioms of type T1–T2 and T4–T6, so it suffices to show that $\mathcal{J}$ satisfies axioms of type T7–T20:

**T7** Let $\prod(z) \wedge B(x) \wedge R(x, y) \wedge A(y) \rightarrow \bot(z) \in \mathcal{P}$ and $B \sqcap \exists R.A \sqsubseteq \bot \in \Psi(\mathcal{P})$. Let $v \in B^{\mathcal{J}} \cap (\exists R.A)^{\mathcal{J}} = B^{\mathcal{J}} \cap (\exists R.A)^{\mathcal{J}}$. By Proposition 19, we also have $v \in \prod^{\mathcal{J}}$, and hence $v \in \bot^{\mathcal{J}} = \bot^{\mathcal{J}}$.

**T8** Let $\prod(z) \wedge A(x) \wedge B(x) \rightarrow \bot(z) \in \mathcal{P}$ and $B \sqcap \exists R.Y.A \sqsubseteq \bot \in \Psi(\mathcal{P})$. Let $v \in B^{\mathcal{J}} \cap (\exists R.Y.A)^{\mathcal{J}}$. Then $v \in B^{\mathcal{J}}$ and $f_{R,Y}^{\mathcal{J}}(v) \in A^{\mathcal{J}}$. Moreover, by Proposition 19, we have $v \in \prod^{\mathcal{J}}$, and hence $v \in (\exists R.Y)^{\mathcal{J}}$.

**T9** Let $\prod(x) \rightarrow \prod(x) \wedge A(x) \in \mathcal{P}$ and $\bot \sqsubseteq \exists R.Y.\bot \in \Psi(\mathcal{P})$. Let $v \in \prod^{\mathcal{J}} = \prod^{\mathcal{J}}$. Then $f_{R,Y}^{\mathcal{J}}(v) \in \prod^{\mathcal{J}}$, and hence $v \in (\exists R.Y)^{\mathcal{J}}$.

**T10** Let $B(x) \rightarrow A(x) \in \mathcal{P}$ and $B \sqsubseteq \forall R.Y.A \in \Psi(\mathcal{P})$. Let $v \in B^{\mathcal{J}} = B^{\mathcal{J}}$. Then $f_{R,Y}^{\mathcal{J}}(v) \in A^{\mathcal{J}}$, and hence $v \in (\exists R.Y.A)^{\mathcal{J}}$. Moreover, since $R_{Y}^{\mathcal{J}}$ is functional by definition, we have $v \in (\forall R.Y.A)^{\mathcal{J}}$.

**T11** Let $B(f_{R,Y}(x)) \rightarrow A(x) \in \mathcal{P}$ and $\exists R.Y.B \sqsubseteq A \in \Psi(\mathcal{P})$. Let $v \in (\exists R.Y.B)^{\mathcal{J}}$. Then $f_{R,Y}^{\mathcal{J}}(v) \in B^{\mathcal{J}}$, and hence $v \in A^{\mathcal{J}} = A^{\mathcal{J}}$.

**T12** Let $A(x) \wedge B(f_{R,Y}(x)) \rightarrow C(f_{R,Y}(x)) \in \mathcal{P}$ and $A \sqcap \exists R.Y.B \sqsubseteq \forall R.Y.C \in \Psi(\mathcal{P})$. Suppose $v \in A^{\mathcal{J}} \cap (\exists R.Y.B)^{\mathcal{J}}$. Then $v \in A^{\mathcal{J}}$ and $f_{R,Y}^{\mathcal{J}}(v) \in B^{\mathcal{J}}$. Consequently, $f_{R,Y}^{\mathcal{J}}(v) \in C^{\mathcal{J}} = C^{\mathcal{J}}$, and hence $v \in (\exists R.Y.C)^{\mathcal{J}}$. Moreover, since $R_{Y}^{\mathcal{J}}$ is functional by definition, we have $v \in (\forall R.Y.C)^{\mathcal{J}}$, as required.

**T13** Let $\prod(z) \wedge A(x) \wedge B(f_{R,Y}(x)) \rightarrow C(f_{R,Y}(x)) \rightarrow \bot(z) \in \mathcal{P}$ and $A \sqcap \exists R.Y(B \sqcap C) \sqsubseteq \bot \in \Psi(\mathcal{P})$. The claim follows similarly to Case T8.

**T14** Let $B(f_{R,Y}(x)) \wedge C(f_{R,Y}(x)) \rightarrow A(x) \in \mathcal{P}$ and $\exists R.Y(B \sqcap C) \sqsubseteq A \in \Psi(\mathcal{P})$. The claim follows similarly to Case T11.

**T15** Let $A(z) \wedge B(f_{R,Y}(z)) \rightarrow \bot(z) \wedge A(z) \in \mathcal{P}$ and $\exists R_{Y} \sqsubseteq S_{(R_{Y}, R)} \cdot R \sqsubseteq S_{(R_{Y}, R)} \cdot A \sqsubseteq \leq \mathcal{I}_{S_{(R_{Y}, R)} \cdot B} \cdot \Psi(\mathcal{P})$ where $R$ occurs in $\mathcal{O}$. It suffices to show that $\mathcal{J}$ satisfies $A \sqsubseteq \leq \mathcal{I}_{S_{(R_{Y}, R)} \cdot B}$. Let $v \in A^{\mathcal{J}} = A^{\mathcal{J}}$, $\{u, w\} \subset B^{\mathcal{J}} = B^{\mathcal{J}}$, and $\{(v, u), (u, w)\} \subset S_{(R_{Y}, R)}^{\mathcal{J}}$. We show $u \approx \mathcal{J} w$. We distinguish three cases:

- $\{(v, u), (w, v)\} \subset R^{\mathcal{J}} \approx R^{\mathcal{J}}$. Then the claim follows since $A(z) \wedge \mathcal{I}(R, z, x) \wedge A(z) \in \mathcal{P}$ and $\{(R_{Y}, R) \equiv \mathcal{I}(R_{Y}, R)\} \subset \Psi(\mathcal{P})$ where $R$ occurs in $\mathcal{O}$. It suffices to show that $\mathcal{J}$ satisfies $A \sqsubseteq \leq \mathcal{I}_{S_{(R_{Y}, R)} \cdot B}$. Let $v \in A^{\mathcal{J}} = A^{\mathcal{J}}$, $\{u, w\} \subset B^{\mathcal{J}} = B^{\mathcal{J}}$, and $\{(v, u), (u, w)\} \subset S_{(R_{Y}, R)}^{\mathcal{J}}$. We show $u \approx \mathcal{J} w$. We distinguish three cases:

  - $\{(v, u), (w, v)\} \subset R^{\mathcal{J}} \approx R^{\mathcal{J}}$. Then the claim follows since $A(z) \wedge \mathcal{I}(R, z, x) \wedge A(z) \in \mathcal{P}$ and $\{(R_{Y}, R) \equiv \mathcal{I}(R_{Y}, R)\} \subset \Psi(\mathcal{P})$ where $R$ occurs in $\mathcal{O}$. It suffices to show that $\mathcal{J}$ satisfies $A \sqsubseteq \leq \mathcal{I}_{S_{(R_{Y}, R)} \cdot B}$. Let $v \in A^{\mathcal{J}} = A^{\mathcal{J}}$, $\{u, w\} \subset B^{\mathcal{J}} = B^{\mathcal{J}}$, and $\{(v, u), (u, w)\} \subset S_{(R_{Y}, R)}^{\mathcal{J}}$. We show $u \approx \mathcal{J} w$. We distinguish three cases:

    - $\{(v, u), (w, v)\} \subset R^{\mathcal{J}} \approx R^{\mathcal{J}}$. Then the claim follows since $A(z) \wedge \mathcal{I}(R, z, x) \wedge A(z) \in \mathcal{P}$ and $\{(R_{Y}, R) \equiv \mathcal{I}(R_{Y}, R)\} \subset \Psi(\mathcal{P})$ where $R$ occurs in $\mathcal{O}$. It suffices to show that $\mathcal{J}$ satisfies $A \sqsubseteq \leq \mathcal{I}_{S_{(R_{Y}, R)} \cdot B}$. Let $v \in A^{\mathcal{J}} = A^{\mathcal{J}}$, $\{u, w\} \subset B^{\mathcal{J}} = B^{\mathcal{J}}$, and $\{(v, u), (u, w)\} \subset S_{(R_{Y}, R)}^{\mathcal{J}}$. We show $u \approx \mathcal{J} w$. We distinguish three cases:
Let $\{v, u\}, (v, w) \subseteq \hat{R}_Y^T$. Since $I$ is a Herbrand model, $f^T_{R_Y}$ is injective and hence $\hat{R}_Y^T$ is functional. Therefore, we have $u = w$ and hence $u \approx_J w$ by reflexivity of $\approx_J$.

$(v, u) \in \hat{R}_Y^T$ and $(v, w) \in \hat{R}_Z^T$. Then, by construction, $v = f_{R,Z}(u)$ and $w = f_{R,Z}(f_{R,Y}(u))$. The claim follows since $A(f_{R,Y}(x)) \land B(x) \land B(f_{R,Y}(f_{R,Y}(x))) \rightarrow x \equiv f_{R,Z}(f_{R,Y}(x)) \in \Psi$ and $\approx_J = \approx_I$.

$(v, u), (v, w) \subseteq \hat{R}_Y^T$ since $\hat{R}_Y^T$ is functional by definition, we have $u = w$ and hence $u \approx_J w$ by reflexivity of $\approx_J$.

T19 Let $R(x, y) \rightarrow \overline{I}(x) \in \Psi$ and $\exists R. \overline{I} \subseteq \overline{I} \in \Psi(\mathcal{P})$. Let $v \in (\exists R. \overline{I})^J$. Then, for some $w$, $(v, w) \in \overline{R}^J = \overline{R}^2$. Consequently, $v \in \overline{I}^J = \overline{I}^J$.

T20 Let $R(x, y) \rightarrow \overline{I}(y) \in \Psi$ and $\overline{I} \subseteq \forall R. \overline{I} \in \Psi(\mathcal{P})$. Let $(v, w) \in \overline{R}^J = \overline{R}^2$. Then $w \in \overline{I}^J = \overline{I}^J$.

For the direction from right to left, let $J$ be a minimal Herbrand model of $\pi(\Psi(\mathcal{P})) \cup \mathcal{D}$. We define the interpretation $\mathcal{I}$ such that

- $\mathcal{I}$ coincides with $\mathcal{J}$ on its domain as well as on every concept name, role name, and individual constant in $\mathcal{P} \cup \mathcal{D}$;
- $\overline{I}^J(v, w)$ for each function $f_{R,Y}$ in $\mathcal{P}$ (note that by Proposition 19 and the fact that $\overline{I} \subseteq \overline{R}$. $\overline{I} \in \Psi(\mathcal{P})$, the set $\{v \in \overline{I}^J | (v, w) \in \overline{R}^J\}$ is nonempty for every $v \in \Delta^J$).

We show that $I$ is a model of $\mathcal{P} \cup \mathcal{D}$. By construction, $I$ satisfies rules of type T1–T2 and T4–T6, so it suffices to show that $\mathcal{J}$ satisfies rules of type T7–T20:

T7 Let $\overline{z}(z) \land B(z) \land R(x, y) \land A(y) \rightarrow \overline{\bot}(z) \in \mathcal{P}$ and $B \sqsubseteq \exists R.A \sqsubseteq \bot \in \Psi(\mathcal{P})$. Let $w \in \overline{I}^J$, $v \in B^J$, $v' \in A^J$, and $(v, v') \in \overline{R}^J$. Then $v \in (B \sqsubseteq \exists R.A)^J$ and hence $v \in \overline{\bot}^J$. Since $\overline{\bot}(x) \rightarrow \square \in \pi(\Psi(\mathcal{P}))$, this means that $\mathcal{J}$ is not a model of $\pi(\Psi(\mathcal{P})) \cup \mathcal{D}$, so the claim holds vacuously.

T8 Let $\overline{z}(z) \land A(f_{R,Y}(x)) \land B(x) \rightarrow \overline{\bot}(z) \in \mathcal{P}$ and $B \sqsubseteq \exists R.Y.A \sqsubseteq \bot \in \Psi(\mathcal{P})$. Let $w \in \overline{I}^J$, $v \in B^J$, and $f^J_{R,Y}(v) \in A^J$. Then $(v, f^J_{R,Y}(v)) \in \overline{R}^J$. Thus, $v \in (B \sqsubseteq \exists R.Y.A)^J$, and so $v \in \overline{\bot}^J$. Since $\overline{\bot}(x) \rightarrow \square \in \pi(\Psi(\mathcal{P}))$, this means that $\mathcal{J}$ is not a model of $\pi(\Psi(\mathcal{P})) \cup \mathcal{D}$, so the claim holds vacuously.

T9 Let $\overline{z}(x) \rightarrow \overline{I}(f_{R,Y}(x)) \in \mathcal{P}$ and $\overline{I} \subseteq \exists R.Y.\overline{I} \in \Psi(\mathcal{P})$. Let $v \in \overline{I}^J = \overline{I}^J$. Then $v \in (\exists R.Y. \overline{I})^J$, hence the set $\{v \in \overline{I}^J | (v, w) \in \overline{R}^J\}$ is nonempty and $f^J_{R,Y}(v) \in \overline{I}^J = \overline{I}^J$.

T10 Let $B(x) \rightarrow A(f_{R,Y}(x)) \in \mathcal{P}$ and $B \subseteq \forall R.Y.A \subseteq \Psi(\mathcal{P})$. Let $v \in B^J = B^J$. Then $v \in (\forall R.Y.A)^J$. Since $f^J_{R,Y}(v) \in \overline{I}^J$, it follows that $f^J_{R,Y}(v) \in A^J = A^J$.

T11 Let $B(f_{R,Y}(x)) \rightarrow A(x) \in \mathcal{P}$ and $\exists R.Y. A \sqsubseteq A \in \Psi(\mathcal{P})$. Let $f^J_{R,Y}(v) \in B^J = B^J$. Since $(v, f^J_{R,Y}(v)) \in \overline{R}^J$, we have $v \in (\exists R.Y. B)^J$, and hence $v \in A^J = A^J$.

T12 Let $A(x) \land B(f_{R,Y}(x)) \rightarrow C(f_{R,Y}(x)) \in \mathcal{P}$ and $A \sqsubseteq \exists R.Y.B \subseteq \forall R.Y.C \subseteq \Psi(\mathcal{P})$. Let $v \in A^J = A^J$ and $f^J_{R,Y}(v) \in B^J = B^J$. Then $(v, f^J_{R,Y}(v)) \in \overline{R}^J$, and consequently $v \in (\exists R.Y.B)^J$. Since $v \in (\exists R.Y.B)^J$, this implies $v \in (\forall R.Y.C)^J$ and $(v, f^J_{R,Y}(v)) \in \overline{R}^J$, we then obtain $f^J_{R,Y}(v) \in C^J = C^J$, as required.

T13 Let $\overline{z}(z) \land A(x) \land B(f_{R,Y}(x)) \land C(f_{R,Y}(x)) \rightarrow \overline{\bot}(z) \in \mathcal{P}$ and $A \sqsubseteq \exists R.B \sqsubseteq \bot \in \Psi(\mathcal{P})$. The claim follows similarly to Case T8.

T14 Let $B(f_{R,Y}(x)) \land C(f_{R,Y}(x)) \rightarrow A(x) \in \mathcal{P}$ and $\exists R.B \sqsubseteq A \in \Psi(\mathcal{P})$. The claim follows similarly to Case T11.

T15 Let $A(x) \land B(f_{R,Y}(x)) \land \text{at}(R, z, x) \land B(x) \land f_{R,Y}(x) \equiv x \in \mathcal{P}$ and $\{R_Y \subseteq S_{\hat{R}_Y}^T, R \subseteq S_{\hat{R}_Y}^T, A \subseteq 1S_{\hat{R}_Y}^T, B \subseteq \Psi(\mathcal{P})\}$. Let $v \in A^J = A^J$, $f^J_{R,Y}(v) \in B^J = B^J$, $(v, w) \in R^2 = R^J$, and $w \in B^J = B^J$. We show $f^J_{R,Y}(v) \approx_I w$. By construction, we have $(v, f^J_{R,Y}(v)) \in R^J$. Since $\{R_Y \subseteq S_{\hat{R}_Y}^T, R \subseteq S_{\hat{R}_Y}^T\} \subseteq \Psi(\mathcal{P})$, we thus have $(v, f^J_{R,Y}(v)), (v, w) \subseteq \overline{S}_{\hat{R}_Y}^J$. Since $A \subseteq 1S_{\hat{R}_Y}^T, B \subseteq \Psi(\mathcal{P})$, we conclude $f^J_{R,Y}(v) \approx_J w$ and hence $f^J_{R,Y}(v) \approx_I w$.

T16 Let $A(f_{R,Y}(x)) \land B(x) \land \text{at}(R, f_{R,Y}(x), y) \rightarrow x \equiv y \in \mathcal{P}$ and $\hat{R}_Y \subseteq S_{\hat{R}_Y}^T, R \subseteq S_{\hat{R}_Y}^T, A \subseteq 1S_{\hat{R}_Y}^T, B, \hat{R}_Y \equiv \text{inv}(R_Y) \subseteq \Psi(\mathcal{P})$. Let $f_{R,Y}(v) \in A^J = A^J$, $v \in B^J = B^J$, $(f_{R,Y}(v), w) \in R^2 = R^J$, and $w \in B^J = B^J$. We show $v \approx_J w$. By construction, we have $(v, f^J_{R,Y}(v)) \in R^J$, and hence $(f^J_{R,Y}(v), v) \in \hat{R}_Y^T$. Since $\hat{R}_Y \subseteq S_{\hat{R}_Y}^T, R \subseteq S_{\hat{R}_Y}^T \subseteq \Psi(\mathcal{P})$, we thus have $(f^J_{R,Y}(v), v), (f^J_{R,Y}(v), w) \subseteq \overline{S}_{\hat{R}_Y}^J$. Since $A \subseteq 1S_{\hat{R}_Y}^T, B \subseteq \Psi(\mathcal{P})$, we conclude $v \approx_J w$ and hence $v \approx_I w$.
Theorem 17. Equivalently, Proof.

\( T_18 \) Let \( A(f_{R,Y}(x)) \wedge B(x) \wedge B(f_{R,Z}(f_{R,Y}(x))) \rightarrow x = f_{R,Z}(f_{R,Y}(x)) \in \mathcal{P} \) and \( \{ \tilde{R}_Y \subseteq \mathcal{S}_{(R,Y)}, R_Z \subseteq \mathcal{S}_{(R,Z)} \} \subseteq \Psi(\mathcal{P}) \). Let \( v \in A^T = A^\mathcal{J} \) and \( \{ f_{R,Y}(v) \}, f_{R,Y}(v) \} \subseteq B^T = B^\mathcal{J} \). We show \( f_{R,Y}(v) \approx_{\mathcal{J}} f_{R,Z}(v) \).

By construction, we have \( (v, f_{R,Y}(v)) \in R^T \) and \( (v, f_{R,Z}(v)) \in R^T \). Since \( \{ R_Y \subseteq \mathcal{S}_{(R,Y)}, R_Z \subseteq \mathcal{S}_{(R,Z)} \} \subseteq \Psi(\mathcal{P}) \), we thus have \( \{(v, f_{R,Y}(v)), (v, f_{R,Z}(v))\} \subseteq S^\mathcal{J}_{(R,Y,Z)} \). Since \( A \subseteq \mathcal{S}_{(R,Y),B} \) \( B \in \Psi(\mathcal{P}) \), we conclude \( f_{R,Y}(v) \approx_{\mathcal{J}} f_{R,Z}(v) \) and hence \( f_{R,Y}(v) \approx_{\mathcal{J}} f_{R,Z}(v) \).

T19 Let \( R(x, y) \rightarrow T(x) \in \mathcal{P} \) and \( \exists R.T \sqsubseteq \exists T \in \Psi(\mathcal{P}) \). Let \( v, w \in R^T \). Since \( R(x, y) \rightarrow T(x) \in \pi(\Psi(\mathcal{P})) \), we then have \( v \in (\exists R.T)^\mathcal{J} \) and hence \( v \in \exists T = \mathcal{I}^T \).

T20 Let \( R(x, y) \rightarrow T(y) \in \mathcal{P} \) and \( \forall R.T \sqsubseteq \forall T \in \Psi(\mathcal{P}) \). Let \( v, w \in R^T \). Since \( R(x, y) \rightarrow T(x) \in \pi(\Psi(\mathcal{P})) \), we then have \( v \in \mathcal{I}^T \) and hence \( v \in (\forall R.T)^\mathcal{J} \). Since \( v, w \in R^T \), we thus obtain \( w \in R^\mathcal{J} = T^\mathcal{J} \).

\textbf{Theorem 15.} Let \( \mathcal{L} \) be a DL between \( \text{ALC} \) and \( \text{ALCHI} \). Then every markable \( \mathcal{L} \) ontology is polynomially rewritable into a Horn-\( \mathcal{L} \) ontology. If \( \mathcal{L} \) is between \( \text{ALCF} \) and \( \text{ALCHIF} \), then every markable \( \mathcal{L} \) ontology is polynomially rewritable into Horn-\( \mathcal{L} \). Finally, every markable \( \mathcal{L} \) ontology is polynomially rewritable into Horn-\( \text{ALC} \).

\textbf{Proof.} The claims follow from the observation that \( \psi(\Xi_M(\xi(0))) \) only introduces new axioms of type T1-T2, T4, T7-T14 and T19-T20 to \( \xi(0) \) unless \( O \) contains functionality assertions. Moreover, none of the axioms in \( \psi(\Xi_M(\xi(0))) \setminus \xi(0) \) contains inverse roles unless so does \( \xi(0) \). Thus, all axioms in \( \psi(\Xi_M(\xi(0))) \setminus \xi(0) \) are

- in Horn-\( \text{ALC} \) if \( O \) is between \( \text{EL} \) and \( \text{ALCH} \);
- in Horn-\( \text{ALCI} \) if \( O \) is in \( \text{EL} \) or \( \text{ALCHI} \);
- in Horn-\( \text{ALCH} \) if \( O \) is in \( \text{ALCF} \) or \( \text{ALCHIF} \);
- in Horn-\( \text{ALCH} \) if \( O \) is in \( \text{ALCIF} \) or \( \text{ALCHIF} \).

The claims immediately follow.

\section{Proofs for Section \ref{section:hardness}}

\textbf{Lemma 16.} Satisfaction checking over markable \( \mathcal{L} \) ontologies is ExPTIME-hard.

\textbf{Proof.} We prove the claim by adapting the ExPTIME-hardness argument for Horn-\( \text{ALC} \) by \cite{Kroetzsch13}. Given a polynomially space-bounded alternating Turing machine \( \mathcal{M} \) and a word \( w \), \cite{Kroetzsch13} construct a Horn-\( \text{ALC} \) ontology \( O_{\mathcal{M},w} \) such that \( \mathcal{M} \) accepts a word \( w \) if and only if \( O_{\mathcal{M},w} \models I_w \subseteq A \) for a concept name \( A \) and a conjunction \( I_w \) that encodes \( w \). Equivalently, \( \mathcal{M} \) accepts \( w \) if and only if \( O_{\mathcal{M},w} \cup \{ I_w \cap A \subseteq \bot \} \) is unsatisfiable.

Ontology \( O_{\mathcal{M},w} \) is in \( \mathcal{L} \) except for axioms of the form \( H \cap C \subseteq \forall S.C \). We will now encode all such axioms into \( \mathcal{L} \). Let \( \text{not}_C \) be a fresh concept name for every atomic concept \( C \) in \( O_{\mathcal{M},w} \), and let \( O'_{\mathcal{M},w} \) be obtained from \( O_{\mathcal{M},w} \) by replacing every axiom of the form \( H \cap C \subseteq \forall S.C \) by the axioms \( H \cap C \cap \forall S.\text{not}_C \subseteq \bot \), \( C \cap \text{not}_C \subseteq \bot \) and \( \forall T \subseteq C \cup \text{not}_C \). Clearly, \( O'_{\mathcal{M},w} \cup \{ I_w \cap A \subseteq \bot \} \) is in \( \mathcal{L} \) and \( O'_{\mathcal{M},w} \cup \{ I_w \cap A \subseteq \bot \} \) is satisfiable if and only if \( O_{\mathcal{M},w} \cup \{ I_w \cap A \subseteq \bot \} \). The claim follows since the set \( \{ \text{not}_C \mid T \subseteq C \cup \text{not}_C \} \) is a marking of \( O'_{\mathcal{M},w} \cup \{ I_w \cap A \subseteq \bot \} \).

\textbf{Theorem 17.} Let \( \mathcal{L} \) be in-between \( \mathcal{L} \) and \( \text{ALCHIF} \). Satisfaction checking over markable \( \mathcal{L} \)-ontologies is ExPTIME-complete and PTIME-complete w.r.t. data.

\textbf{Proof.} The claim follows by Theorem \ref{thm:horn-alc} Lemma \ref{lem:horn-alchi} and the results for logics between Horn-\( \text{ALC} \) and Horn-\( \text{ALCHIF} \) in \cite{Kroetzsch13}.

\qed