USING LAGRANGIAN PERTURBATION THEORY FOR PRECISION COSMOLOGY

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ABSTRACT

We explore the Lagrangian perturbation theory (LPT) at one-loop order with Gaussian initial conditions. We present an expansion method to approximately compute the power spectrum LPT. Our approximate solution has good convergence in the series expansion and enables us to compute the power spectrum in LPT accurately and quickly. Non-linear corrections in this theory naturally satisfy the law of conservation of mass because the relation between matter density and the displacement vector of dark matter corresponds to the conservation of mass. By matching the one-loop solution in LPT to the two-loop solution in standard perturbation theory, we present an approximate solution of the power spectrum which has higher order corrections than the two-loop order in standard perturbation theory with the conservation of mass satisfied. With this approximation, we can use LPT to compute a non-linear power spectrum without any free parameters, and this solution agrees with numerical simulations at $k = 0.2 \, h \, \text{Mpc}^{-1}$ and $z = 0.35$ to better than 2%.

Key words: dark matter – large-scale structure of universe

Online-only material: color figures

1. INTRODUCTION

Since the first measurement of the baryon acoustic oscillation (BAO) in the SDSS LRG survey (Eisenstein et al. 2005) and the 2dF Galaxy survey (Cole et al. 2005), various other large-scale structure surveys have measured the galaxy power spectrum and the position of the baryon acoustic peak with ever increasing precision (Tegmark et al. 2006; Percival et al. 2007, 2010; Kazin et al. 2010; Beutler et al. 2011; Blake et al. 2010, 2011a, 2011b). In the coming decade, we anticipate that new ground-based surveys, such as the Prime Focus Spectrograph and Big BOSS, and space-based surveys, such as Euclid and WFIRST, will make even more accurate measurements of the galaxy power spectrum. Therefore, predicting the precise non-linear behavior of the galaxy power spectrum using analytical approaches is an essential step in interpreting these data and in elucidating the nature of dark energy.

The past decade has seen the development of a plethora of perturbation approaches to the non-linear matter power spectrum: standard perturbation theory (SPT; Bernardeau et al. 2002; Fry 1984; Goroff et al. 1986; Suto & Sasaki 1991; Makino et al. 1992; Jain & Bertschinger 1994; Scoccimarro & Frieman 1996a, 1996b; Jeong & Komatsu 2006; Sugiyama & Komatsu 2006, 2008; Sugiyama & Futamase 2012, 2013), closure theory (Taruya & Hiramatsu 2008; Taruya et al. 2009), renormalized perturbation theory (RPT; Crocce & Scoccimarro 2006b, 2006a, 2008), closure theory (Taruya & Hiramatsu 2008; Taruya et al. 2009), multi-point propagator method (the $\Gamma$-expansion method; Bernardeau et al. 2008, 2012), regularized multi-point propagator method (RegPT; Bernardeau et al. 2012; Taruya et al. 2012; Taruya et al. 2013), the Wiener Hermite expansion method (Sugiyama & Futamase 2012), as well as other techniques (Pajer & Zaldarriaga 2013; Tassev & Zaldarriaga 2012; Valageas et al. 2013; Gil-Marín et al. 2012; Wang & Szalay 2012; Carlson et al. 2013; Tassev et al. 2013; Wang et al. 2013).

In this paper, we explore Lagrangian perturbation theory (LPT). At the linear order, LPT reduces to the well-studied Zel’’dovich approximation (e.g., Taylor & Hamilton 1996), but at higher order it has not been calculated. This is because there are numerical difficulties in computing the power spectrum in LPT, even though some approximate methods in the Lagrangian description have been proposed (Matsubara 2008; Wang et al. 2013; Carlson et al. 2013). We present an expansion method to approximately compute the LPT power spectrum. Our approximate solution has good convergence in the series expansion and enables us to compute the LPT power spectrum accurately and quickly. The main goal of the present work is to explore LPT at the one-loop order and give higher order correction terms than the two-loop SPT solution.

The main result of this paper is

$$P(z, k) = D^2 P_{\text{lin}}(k) + D^4 P_{1\,\text{-loop}}(k) + D^6 P_{2\,\text{-loop}}(k) + \sum_{n=3}^{\infty} P_{n\,\text{-loop}, \Gamma}(z, k),$$

where $z$ and $D$ are the redshift and the linear growth function, and $P_{\text{lin}}$, $P_{1\,\text{-loop}}$, and $P_{2\,\text{-loop}}$ are the SPT solutions at the linear, one-loop, and two-loop order, respectively. The last term $\sum_{n=3}^{\infty} P_{n\,\text{-loop}, \Gamma}(z, k)$ is the correction terms computed in the one-loop LPT that have higher order than the two-loop SPT. As we will show in Sections 8 and 9, this works and agrees very well with the numerical simulations in Figure 9.

This paper is organized as follows. Section 2 reviews LPT. Section 3 gives the motivation for extending LPT to higher order. Section 4 computes the correlation functions of the displacement vector. In Section 5, we investigate how the LPT solutions reproduce the SPT solutions. Section 6 presents an expansion method to approximately compute the LPT power spectrum and computes the LPT power spectrum in the linear and one-loop order. Section 7 shows a simple relation between LPT and the $\Gamma$-expansion method. Section 8 presents an approximate non-linear power spectrum which has the two-loop solution in SPT as well as higher order terms
than the two-loop SPT computed in the one-loop LPT. Section 9 compares the predicted power spectra in LPT and N-body simulation results, and a final section summarizes our findings.

The cosmological parameters we used are presented by the Wilkinson Microwave Anisotropy Probe five-year release (Komatsu et al. 2009): Ω_m = 0.279, Ω_λ = 0.721, Ω_b = 0.046, h = 0.701, n_s = 0.96, and σ_8 = 0.817. We used the publicly available code RegPT (Taruya et al. 2012)\(^1\) to compute the two-loop power spectrum in SPT.

2. GENERAL FORMULA OF THE LAGRANGIAN PERTURBATION THEORY

In the Lagrangian description, the spatial coordinates \( \mathbf{x} \) are transformed as

\[
\mathbf{x} = \mathbf{q}_1 + \Psi(z, \mathbf{q}_1),
\]

where \( \Psi \) is the displacement vector of dark matter particles. Conservation of mass implies that the density perturbation \( \delta \) can be described as a function of the displacement vector in real and Fourier spaces, respectively:

\[
\delta(z, \mathbf{x}) = \int d^3q_1 \delta_D(\mathbf{x} - \mathbf{q}_1 - \Psi(z, \mathbf{q}_1)) - 1,
\]

\[
\delta(z, \mathbf{k}) = \int d^3q_1 e^{-i\mathbf{k}\cdot\mathbf{q}_1} (e^{-i\mathbf{k}\cdot\Psi(z, \mathbf{q}_1)} - 1)
\]

\[
= \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}_{[1,n]}) [\mathbf{k} \cdot \Psi(z, \mathbf{k_1})] \cdots [\mathbf{k} \cdot \Psi(z, \mathbf{k_n})],
\]

where \( \mathbf{k}_{[1,n]} \equiv \mathbf{k}_1 + \ldots + \mathbf{k}_n \). In LPT, the displacement vector field is expanded in a perturbation series in the linear growth function \( D \) in Fourier space (Bernardeau et al. 2002; Rampf 2012):

\[
\Psi(z, \mathbf{k}) = \sum_{n=1}^{\infty} D^n \frac{1}{n!} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_n}{(2\pi)^3} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{p}_{[1,n]}) L_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) \delta_{\text{lin}}(\mathbf{p}_1) \cdots \delta_{\text{lin}}(\mathbf{p}_n),
\]

where \( \delta_{\text{lin}} \) is the linearized density perturbation at \( z = 0 \) and the \( n \)th order of the kernel function in LPT \( L_n \) is given by Rampf (2012). The linear displacement vector \( \Psi_{\text{lin}}(\mathbf{p}) = i\mathbf{p} \delta_{\text{lin}}(\mathbf{p})/p^2 \), called the “Zel’dovich approximation,” leads to

\[
\delta(z, \mathbf{k}) = \sum_{n=1}^{\infty} D^n \frac{1}{n!} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_n}{(2\pi)^3} (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{p}_{[1,n]}) F_n|_{\text{ZA}}(\mathbf{p}_1, \ldots, \mathbf{p}_n) \delta_{\text{lin}}(\mathbf{p}_1) \cdots \delta_{\text{lin}}(\mathbf{p}_n),
\]

where

\[
F_n|_{\text{ZA}}(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \frac{1}{n!} \left( \frac{\mathbf{k} \cdot \mathbf{p}_1}{p_1^2} \right) \cdots \left( \frac{\mathbf{k} \cdot \mathbf{p}_n}{p_n^2} \right).
\]

The power spectrum is given by

\[
P(z, k) = \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \left\{ (e^{-i\mathbf{k}\cdot(\Psi(z, \mathbf{q}) - \Psi(z, \mathbf{q}_1))}) - 1 \right\}
\]

\[
= \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \left\{ \exp \left[ \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} (\mathbf{k} \cdot \Psi(z, \mathbf{q}) - \mathbf{k} \cdot \Psi(z, 0))^n \right] - 1 \right\}
\]

\[
= \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \left\{ e^{\Sigma(z, \mathbf{k}, \mathbf{q}) - \Sigma(z, \mathbf{k})} - 1 \right\},
\]

where the integration variable \( \mathbf{q} \) is the relative coordinate between the initial positions of dark matter particles: \( \mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2 \). In the second line, we used the translation symmetry in the ensemble average, and \( \langle \cdots \rangle_\Sigma \) denotes the cumulant. The functions \( \Sigma \) and \( \bar{\Sigma} \) are defined as

\[
\Sigma(z, \mathbf{k}, \mathbf{q}) \equiv \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{(-i)^n(-1)^m}{m!(n-m)!} \langle (\mathbf{k} \cdot \Psi(z, \mathbf{q}))^{n-m}(\mathbf{k} \cdot \Psi(z, 0))^m \rangle_\Sigma,
\]

\[
\bar{\Sigma}(z, k) \equiv \Sigma(z, \mathbf{k}, \mathbf{q} = 0) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \langle (\mathbf{k} \cdot \Psi(z, 0))^{2n} \rangle_\Sigma.
\]

\(^1\) http://www-utap.phys.s.u-tokyo.ac.jp/~ataruya/regpt_code.html
These functions $\Sigma$ and $\bar{\Sigma}$ are the same as Equations (9) and (10) in Matsubara (2008). The relation $\bar{\Sigma}(z, k = 0) = 0$ recasts Equation (6) as

$$P(z, k) = e^{-\bar{\Sigma}(z, k)} \int d^3q e^{-ikq} \{e^{\Sigma(z, k, q)} - 1\},$$

(8)

where we used $\int d^3q e^{-ikq} e^{-\bar{\Sigma}(z, k)} = \int d^3q e^{-ikq}$. Furthermore, we expand $\Sigma$ in Legendre polynomials as

$$\Sigma(z, k, q) = \sum_{\ell=0}^{\infty} i^\ell \Sigma_\ell(z, k, q) L_\ell(\mu),$$

(9)

where $\mu = \hat{k} \cdot \hat{q}$. Note that $\bar{\Sigma}$ comes from the monopole term $\bar{\Sigma}(z, k) = \Sigma_0(z, k, q = 0)$. In other words, the other $\Sigma_\ell$ functions for $\ell \geq 1$ become zero at $q = 0$: $\Sigma_{\ell \geq 1}(z, k, q = 0) = 0$. For the functions $\Sigma_\ell$ to be real, the imaginary number should appear in the Legendre expansion. Thereby, odd terms in the expansion behave like the changing Lagrangian spatial coordinates $q$ in Equation (8). Finally, we arrive at the general expression of the power spectrum in LPT:

$$P(z, k) = e^{-\bar{\Sigma}(z, k)} \int d^3q \frac{1}{2}(e^{-ikq}e^{\Sigma(z, k, q)} + e^{ikq}e^{\Sigma(z, k, -q)}) - e^{-\bar{\Sigma}(z, k)} \int d^3q \frac{1}{2}(e^{-ikq} + e^{ikq})$$

$$= 2\pi e^{-\bar{\Sigma}(z, k)} \int_0^\infty dq q^2 \int_0^{4\pi} d\mu \left\{ \cos(kq L_1(\mu) - \sum_{\ell=1}^{\infty} (-1)^\ell \Sigma_{2\ell+1}(z, k, q)L_{2\ell+1}(\mu)) - \cos(kq L_1(\mu)) \right. + \left. \cos(kq L_1(\mu) - \sum_{\ell=1}^{\infty} (-1)^\ell \Sigma_{2\ell+1}(z, k, q)L_{2\ell+1}(\mu)) (e^{\sum_{\ell=0}^{\infty}(-1)^\ell \Sigma_{2\ell}(z, k, q)L_{2\ell}(\mu)} - 1) \right\},$$

(10)

where we used $L_{2\ell+1}(\mu) = -\bar{\Sigma}_{2\ell+1}(\mu)$ and $L_{2\ell}(\mu) = L_{2\ell}(\mu)$.

3. WHAT IS THE MOTIVATION FOR CONSIDERING LPT?

The relation between the matter density and the displacement vector (Equation (2)) corresponds to the law of mass conservation. Therefore, the non-linear solutions in LPT naturally guarantee mass conservation (see Section 8). The law is related to various properties of the matter density perturbation. From the expression in Equation (2), the space-independent displacement vector $\Psi(z)$ does not yield the matter perturbation:

$$\delta(z, x) \rightarrow \int d^3q \delta_\Sigma(x - q - \Psi(z)) - 1 = 0.$$

(11)

This implies that dark matter particles which globally move in the same way throughout the universe do not yield the matter density perturbation. This fact corresponds to the Galilean invariance (Scoccimarro & Frieman 1996b; Peloso & Pietroni 2013; Kehagias & Riotto 2013; Bernardeau et al. 2013; Sugiyama & Spergel 2013; Blas et al. 2013). In other words, conservation of mass guarantees Galilean invariance. In connection with this, in calculating the power spectrum, the integrand in Equation (6) converges to zero at $q = q_1 - q_2 = 0$, where $q$ is the relative coordinates between the initial positions of dark matter particles, and the power spectrum has no contribution at this point. As discussed in Sections 4.2 and 5, this feature is related to the well-known cancellation of the high-$k$ limit solutions and the IR divergence problem in SPT (Sugiyama & Spergel 2013; Scoccimarro & Frieman 1996b; Pajer & Zaldarriaga 2013; Carrasco et al. 2013), because $q = |q_1 - q_2| \to 0$ means the small-scale limit. Furthermore, as shown in Section 7, the power spectrum in LPT has a simple relation to the $\Gamma$-expansion (Bernardeau et al. 2008, 2012) and RegPT (Bernardeau et al. 2012; Taruya et al. 2012; Taruya et al. 2013). Thus, LPT has various interesting properties, and this is the reason we explore LPT.

4. CORRELATION FUNCTIONS OF THE DISPLACEMENT VECTOR

To obtain the power spectrum in LPT, we have to compute the correlation function of the displacement vector $\Sigma$ in Equation (7). In this section, we investigate the properties of $\Sigma$ at the linear and one-loop orders, where the $n$-loop in LPT means $\Sigma_n = O(P^n_{\text{lin}})$:

$$\Sigma(z, k, q) = D^2\Sigma_{\text{lin}}(k, q) + D^4\Sigma_{1\text{-loop}}(k, q),$$

$$\bar{\Sigma}(z, k) = \Sigma_0(z, k, q = 0) = D^2\bar{\Sigma}_{\text{lin}}(k) + D^4\bar{\Sigma}_{1\text{-loop}}(k).$$

(12)

4.1. Multipole Expansion of $\Sigma$

In the Zel’dovich approximation, Equation (7) leads to

$$\Sigma_{\text{lin}}(k, q) = \int \frac{d^3p}{(2\pi)^3} e^{i\hat{q} \cdot \hat{p}} [\hat{k} \cdot L_1(\hat{p})]^2 P_{\text{lin}}(p)$$

$$= \delta_{0,\text{lin}}(k, q) L_0(\hat{k} \cdot \hat{q}) - \Sigma_{2,\text{lin}}(k, q) L_2(\hat{k} \cdot \hat{q}).$$

(13)
Thus, the Zel’dovich approximation has monopole and quadrupole terms. The linear correlation functions of the displacement vector \( \Sigma_{\ell,\text{lin}} \) only involve the spherical Bessel functions \( j_\ell \) in their integrals. Note that the non-linear scale dependence of the Zel’dovich solution only comes from the non-linearity of the law of mass conservation, where at the linear order the mass conservation shows \( \Psi_{\text{lin}}(k) = i k \delta_{\text{lin}}(k)/k^2 \). On the other hand, the linear equation of motion of the displacement vector provides the linear growth function \( D \). The factors 1/3 and 2/3 in front of the monopole and quadrupole terms result from isotropy and anisotropy, respectively. Since \( |j_2(pq)| \sim |j_0(pq)| \) at large scales satisfy \( pq \gg 1 \), the quadrupole term has two times greater amplitude than the monopole term at these scales. The limiting small-scale \( q = 0 \) leads to \( \Sigma_{0,\text{lin}} \to \Sigma_{\text{lin}} \) and \( \Sigma_{2,\text{lin}} \to 0 \) due to \( j_0(0) = 1 \) and \( j_2(0) = 0 \).

At the one-loop order in LPT, we decompose \( \Sigma_{\ell,\text{lin}} \) into two parts as in the one-loop SPT:

\[
\Sigma_{\ell,\text{lin}}(k) = \Sigma_{0,\ell,\text{lin}}(k) + \tilde{\Sigma}_{\ell,\text{lin}}(k).
\]

Equation (7) leads to

\[
\Sigma_{22}(k, q) = \frac{1}{2} \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} e^{i p_1 \cdot k} e^{i p_2 \cdot q} \left[ |k \cdot L_2(p_1, p_2)|^2 P_{\text{lin}}(p_1) P_{\text{lin}}(p_2) \right]
+ \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} e^{i p_1 \cdot k} e^{i p_2 \cdot q} \left[ k \cdot L_1(p_1) k \cdot L_1(p_2) \right] P_{\text{lin}}(p_1) P_{\text{lin}}(p_2)
= \sum_{\ell=0}^3 i^\ell \Sigma_{\ell,22}(k, q) L_\ell(\hat{k} \cdot \hat{q}),
\]

and

\[
\Sigma_{13}(k, q) = \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} e^{i p_1 \cdot k} e^{i p_2 \cdot q} \left[ k \cdot L_1(p_1) k \cdot L_3(p_1, p_2, -p_2) \right] P_{\text{lin}}(p_1) P_{\text{lin}}(p_2)
- 2 \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} e^{i p_1 \cdot k} e^{i p_2 \cdot q} \left[ k \cdot L_1(p_1) k \cdot L_2(p_1, p_2) \right] P_{\text{lin}}(p_1) P_{\text{lin}}(p_2)
= \sum_{\ell=0}^3 i^\ell \Sigma_{\ell,13}(k, q) L_\ell(\hat{k} \cdot \hat{q}).
\]

Then, the multipole terms in the one-loop LPT are given by

\[
\Sigma_{\ell,22}(k, q) = \int_{-1}^{1} \int_{-1}^{1} \mu \nu j_\ell(|\mu + \nu|) K_{\ell,22}(k, p_1, p_2, \mu) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),
\]

\[
\tilde{\Sigma}_{22}(k) = \int_{-1}^{1} \int_{-1}^{1} \mu \nu j_1(|\mu + \nu|) K_{\ell,22}(k, p_1, p_2, \mu) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),
\]

\[
\Sigma_{\ell,13}(k, q) = \int_{-1}^{1} \int_{-1}^{1} \mu \nu j_\ell(|\mu + \nu|) K_{\ell,13}(k, p_1, p_2) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),
\]

\[
\tilde{\Sigma}_{13}(k) = \int_{-1}^{1} \int_{-1}^{1} \mu \nu j_1(|\mu + \nu|) K_{\ell,13}(k, p_1, p_2) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),
\]

where \( \mu \equiv p_1 \cdot \hat{p}_2 \) and \( \nu \equiv p_2 \cdot \hat{p}_1 \). Appendix A.1 summarizes the definitions of the kernel functions \( K_{\ell,22} \) and \( K_{\ell,13} \). Note that \( \tilde{\Sigma}_{22}(k) \) and \( \tilde{\Sigma}_{13}(k) \) are given by \( \tilde{\Sigma}_{22}(k) = \Sigma_{0,22}(k, q = 0) \) and \( \tilde{\Sigma}_{13}(k) = \Sigma_{0,13}(k, q = 0) \) and satisfy \( \Sigma_{22}(k) = (27/140) \tilde{\Sigma}_{13}(k) \).

We find that the one-loop LPT has monopole, dipole, quadrupole, and octupole terms, where the dipole and octupole terms come from \( \langle \Psi \Psi \rangle \) in Equation (7). The subscripts 13 and 22 in \( \Sigma_{\ell,13} \) and \( \Sigma_{\ell,22} \) mean that they make the correction terms \( P_{13} \) and \( P_{22} \).
in the one-loop SPT (for details, see Section 5.2). Unlike the Zel’dovich approximation, $\Sigma_{\ell,22}$ and $\Sigma_{\ell,13}$ have the kernel functions $K_{\ell,22}$ and $K_{\ell,13}$ in their integrals which come from the non-linear dynamics of dark matter. Similarly to the case in the Zel’dovich approximation, at $q = 0$, the dipole, quadrupole, and octupole terms become zero: $\Sigma_{\ell \geq 1,22}(k, q = 0) = \Sigma_{\ell \geq 1,13}(k, q = 0) = 0$.

Here, we define the quantities $\sigma_\ell$ which have the dimension of length [Mpc/ $h$]:

\[
\sigma_0(z, k, q) = \frac{k^2 \sigma_0^2(z, q)}{2}, \quad \sigma_1(z, k, q) = \frac{k^3 \sigma_1^3(z, q)}{2},
\]
\[
\sigma_2(z, k, q) = \frac{k^2 \sigma_2^2(z, q)}{2}, \quad \sigma_3(z, k, q) = \frac{k^3 \sigma_3^3(z, q)}{2},
\]
\[
\sigma_{0,\text{lin}}(k, q) = \frac{k^2 \sigma_{0,\text{lin}}^2(q)}{2}, \quad \sigma_{2,\text{lin}}(k, q) = \frac{k^2 \sigma_{2,\text{lin}}^2(q)}{2}, \quad 
\sigma_{1,13}(k, q) = \frac{k^3 \sigma_{1,13}^3(q)}{2}, \quad \sigma_{3,13}(k, q) = \frac{k^3 \sigma_{3,13}^3(q)}{2}, 
\]
\[
\sigma_{0,22}(k, q) = \frac{k^2 \sigma_{0,22}^2(q)}{2}, \quad \sigma_{1,22}(k, q) = \frac{k^3 \sigma_{1,22}^3(q)}{2}, \quad 
\sigma_{2,22}(k, q) = \frac{k^2 \sigma_{2,22}^2(q)}{2}, \quad \sigma_{3,22}(k, q) = \frac{k^3 \sigma_{3,22}^3(q)}{2}.
\]

The left panel of Figure 1 shows $\sigma_\ell$. At large scales ($q \geq 100$ Mpc/ $h$), the linear contributions to the monopole and quadrupole terms are dominant and the amplitude of the quadrupole is twice larger than that of the monopole. On the other hand, at small scales the dipole, quadrupole, and octupole terms become zero. In the right panel of Figure 1, we find that the linear contributions are larger than the non-linear ones at large scales: $|\sigma_{0,\text{lin}}| > |\sigma_{\ell,22}|$ and $|\sigma_{\ell,13}|$. These features of $\sigma_\ell$, $\sigma_{0,\text{lin}}$, $\sigma_{\ell,22}$, and $\sigma_{\ell,13}$ are indeed what we expected.

### 4.2. IR Divergence

In the Zel’dovich approximation, $\tilde{\Sigma}_{\text{lin}}$ has the following integral (Equation (15)):

\[
\int_0^\infty dp P_{\text{lin}}(p).
\]

For the power-law initial power spectrum $P_{\text{lin}}(p) \propto p^n$, this integral has IR divergence for $n < -1$ and UV divergence for $n > -1$ (Scoccimarro & Frieman 1996b). However, computing the power spectrum alleviates the condition of IR divergence because the above integral appears as a combination of the monopole terms as follows

\[
\Sigma_{0,\text{lin}}(k, q) - \tilde{\Sigma}_{\text{lin}}(k) \rightarrow -\frac{k^2 q^2}{36 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0,
\]
\[
\Sigma_{2,\text{lin}}(k, q) \rightarrow \frac{k^2 q^2}{45 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0,
\]
\[
\Sigma_{1,13}(k, q) \rightarrow \frac{k^3 q^3}{90 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0,
\]
\[
\Sigma_{3,13}(k, q) \rightarrow \frac{k^3 q^3}{90 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0,
\]
\[
\Sigma_{0,22}(k, q) \rightarrow \frac{k^2 q^2}{54 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0,
\]
\[
\Sigma_{1,22}(k, q) \rightarrow \frac{k^3 q^3}{54 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0,
\]
\[
\Sigma_{2,22}(k, q) \rightarrow \frac{k^2 q^2}{54 \pi^2} \int_0^\infty dp p^2 P_{\text{lin}}(p) \quad \text{for } p \to 0.
\]
where the terms proportional to \( j_0(x) = 1 - x^2/6 \) and \( j_2(x) = x^2/15 \) for \( x \ll 1 \). Therefore, the Zel’dovich power spectrum has IR divergence for \( n \leq -3 \) (Taylor & Hamilton 1996). This is the result of the law of conservation of mass because the non-linear \( k \)-dependence of the power spectrum in the Zel’dovich approximation results only from the non-linear equation of mass conservation.

At the one-loop order in LPT, the asymptotic behaviors of the kernel functions \( K_{\ell,22} \) and \( K_{\ell,13} \) (see Appendix A.1) lead to those of \( \Sigma_{\ell,22} \) and \( \Sigma_{\ell,13} \): for \( p_2/p_1 \ll 1 \),

\[
\Sigma_{\ell,22}(k, q) \propto \Sigma_{\ell,13}(k, q) \propto k^2 \int_0^\infty dp_1 j_1(p_1 q) P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2), \quad \text{for } \ell = 0 \text{ and } 2,
\]

\[
\Sigma_{\ell,22}(k, q) \propto \Sigma_{\ell,13}(k, q) \propto k^3 \int_0^\infty dp_1 j_1(p_1 q) P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2), \quad \text{for } \ell = 1 \text{ and } 3, \quad (24)
\]

and for \( y = p_2/p_1 \gg 1 \),

\[
\Sigma_{\ell,22}(k, q) \propto k^2 \int_0^\infty dp_1 p_1^2 j_1(p_1 q) P_{\text{lin}}(p_1) \int_0^\infty dp_2 P_{\text{lin}}(p_2), \quad \text{for } \ell = 0 \text{ and } 2,
\]

\[
\Sigma_{\ell,13}(k, q) \propto k^3 \int_0^\infty dp_1 p_1^2 j_1(p_1 q) P_{\text{lin}}(p_1) \int_0^\infty dp_2 P_{\text{lin}}(p_2), \quad \text{for } \ell = 1 \text{ and } 3, \quad (25)
\]

where \( \Sigma_{\ell,22}(y = p_2/p_1 \gg 1) \) are given by replacing \( p_1 \) with \( p_2 \) in Equation (24) due to the symmetry of \( K_{\ell,22} \) about \( p_1 \) and \( p_2 \). For details, see Appendix A.2. Thus, we find that for the power-law initial power spectrum \( P_{\text{lin}}(p) \propto p^n \) with \(-3 < n < -1\), \( \Sigma_{\ell,22} \) and \( \Sigma_{\ell,13} \) have no IR and UV divergences in both cases: (1) \( p_1 \to \infty \) and \( p_2 \to 0 \), and (2) \( p_1 \to 0 \) and \( p_2 \to \infty \).

5. POWER SPECTRUM IN SPT

In this section, we investigate how the solutions of LPT reproduce those of SPT, where \( n \)-loop in SPT means \( P_n = \mathcal{O}(P_{\text{lin}}^{n+1}) \). 

5.1. At the Linear Order in SPT

Expanding the exponential factor in Equation (6) and using Equation (14), the monopole and quadrupole terms yield \((1/3)P_{\text{lin}}\) and \((2/3)P_{\text{lin}}\) respectively:

\[
\begin{align*}
P_{\text{lin}}(k) &= \int d^3 q e^{-ik\cdot q}(\Sigma_{\text{lin}}(k, q) - \bar{\Sigma}_{\text{lin}}(k)) \\
&= 4\pi \int_0^\infty dq q^2 j_0(kq)[\Sigma_{0,\text{lin}}(k, q) + j_2(kq)\Sigma_{2,\text{lin}}(k, q)] - (2\pi)^3\delta_D(k)\bar{\Sigma}_{\text{lin}}(k) \\
&= \frac{1}{3} P_{\text{lin}}(k) + \frac{2}{3} P_{\text{lin}}(k),
\end{align*}
\]

(26)

where we used the mathematical formula \( e^{-ik\cdot q} = \sum_{\ell=0}^{\infty} (-i)^\ell j_\ell(kq)\mathcal{L}_\ell(\hat{k} \cdot \hat{q}) \), \( \int_{-1}^1 d\mu\mathcal{L}_\ell(\mu)\mathcal{L}_\ell(\mu) = 2\delta_{\ell,0} \), and \( j_0^\infty dq q^2 j_a(kq)j_a(pq) = (\pi/2k^2)\delta_D(k - p) \). In the second line, the last term \((2\pi)^3\delta_D(k)\bar{\Sigma}_{\text{lin}}(k)\) is zero because \( \bar{\Sigma}_{\text{lin}}(k = 0) = 0 \).

5.2. At the One-loop Order in SPT

Substituting Equation (12) into Equation (6) and expanding the exponential factor in Equation (6), we obtain the one-loop correction term \( P_{1,\text{loop}} = P_{22} + P_{13} \) in SPT as follows:

\[
P_{1,\text{loop}}(k) = \int d^3 q e^{-ik\cdot q}\left\{(\Sigma_{1,\text{loop}}(k, q) - \bar{\Sigma}_{1,\text{loop}}(k)) + \frac{1}{2}(\Sigma_{\text{lin}}(k, q) - \bar{\Sigma}_{\text{lin}}(k))^2\right\}
\]

\[
= 4\pi \sum_{\ell=0}^{3} \int_0^\infty dq q^2 j_\ell(kq)[\Sigma_{\ell,22}(k, q) + \Sigma_{\ell,13}(k, q)]
\]

\[
+ \int d^3 q e^{-ik\cdot q}\frac{1}{2}[(\Sigma_{\text{lin}}(k, q))^2 - 2\Sigma_{\text{lin}}(k, q)\bar{\Sigma}_{\text{lin}}(k)]
\]

\[
= \sum_{\ell=0}^{3} \{P_{\ell,22}(k) + P_{\ell,13}(k)\} + P_{1,\text{loop}}|_{\text{ZA}}(k),
\]

\[
= P_{22}(k) + P_{13}(k),
\]

(27)

where the terms proportional to \( \delta_D(k) \) become zero due to \( \bar{\Sigma}_{\text{lin}}(k = 0) = \bar{\Sigma}_{1,\text{loop}}(k = 0) = 0 \). Thus, \( P_{1,\text{loop}} \) is decomposed into the multipole terms \( \sum_{\ell=0}^{3}(P_{\ell,22} + P_{\ell,13}) \) and the contribution from the Zel’dovich approximation \( P_{1,\text{loop}}|_{\text{ZA}} = P_{22}|_{\text{ZA}} + P_{13}|_{\text{ZA}} \), defined as

\[
P_{\ell,22}(k) \equiv 4\pi \int_0^\infty dq q^2 j_\ell(kq)\Sigma_{\ell,22}(k, q), \quad P_{\ell,13}(k) \equiv 4\pi \int_0^\infty dq q^2 j_\ell(kq)\Sigma_{\ell,13}(k, q),
\]

\[
P_{22}|_{\text{ZA}}(k) \equiv \int d^3 q e^{-ik\cdot q}\frac{1}{2}(\Sigma_{\text{lin}}(k, q))^2, \quad P_{13}|_{\text{ZA}}(k) \equiv -\bar{\Sigma}_{\text{lin}}(k)P_{\text{lin}}(k).
\]

(28)
positions of dark matter particles: 2013; Carrasco et al. 2013),

In the final line of Equation (27), Figure 2. The Astrophysical Journal 788:63 (24pp), 2014 June 10

The effects of the displacement vector are suitable for perturbation quantities

The contributions from the one-loop LPT are represented as the multipole terms $P_{\ell}$, as mentioned in Section 3. Furthermore, it is known that the high-

amplitude of $P_{\ell}$, high-$k$ and $P_{\ell,13}$ for $\ell = 0, 1, 2, 3$. This figure shows that $P_{\ell,22}/P_{\text{low}} < 1$ and $P_{\ell,13}/P_{\text{low}} < 1$ over the range of $k \lesssim 1.0 \, [h \, \text{Mpc}^{-1}]$ even at $z = 0$, and they are suitable for perturbation quantities.

(A color version of this figure is available in the online journal.)

In the final line of Equation (27), $P_{22}$ and $P_{13}$ are given by

$$
P_{22}(k) = \sum_{\ell=0}^{3} P_{\ell,22} + P_{22}|_{\text{ZA}}, \quad P_{13}(k) = \sum_{\ell=0}^{3} P_{\ell,13} + P_{13}|_{\text{ZA}}. \quad (29)
$$

The specific expressions of $P_{\ell,22}$, $P_{\ell,13}$, and $P_{\ell,22}|_{\text{ZA}}$ are summarized in Appendix B.

For $p/k \ll 1$, $P_{22}$ and $P_{13}$ cancel out each other (Sugiyama & Spergel 2013; Scocciarmo & Frieman 1996b; Pajer & Zaldarriaga 2013; Carrasco et al. 2013),

$$
P_{22,\text{high-}k}(k) \rightarrow \bar{\Sigma}_{\text{lin}}(k) P_{\text{lin}}(k), \quad P_{13,\text{high-}k}(k) \rightarrow -\bar{\Sigma}_{\text{lin}}(k) P_{\text{lin}}(k), \quad (30)
$$

where $P_{22,\text{high-}k}$ and $P_{13,\text{high-}k}$ are called the high-$k$ solutions of $P_{22}$ and $P_{13}$, such that $P_{1\text{-loop}}$ is proportional to $\int dp p^2 P_{\text{lin}}(p)$ but not to $\int dp P_{\text{lin}}(p)$ at $p \rightarrow 0$. In other words, the one-loop SPT alleviates the IR divergence problem (Section 4.2). In the context of LPT, the high-$k$ (small-scale) limit corresponds to the limit of $q \rightarrow 0$, because $q$ is the relative distance between the initial positions of dark matter particles: $q = |\mathbf{q}_1 - \mathbf{q}_2|$. In Equation (27), $\bar{\Sigma}_{\ell,\text{loop}}(k, \mathbf{q}) - \bar{\Sigma}_{\ell,\text{loop}}(k)$ and $(\sigma_{\text{lin}}(k, \mathbf{q}) - \sigma_{\text{lin}}(k))^2$ by definition become zero at $q = 0$. Therefore, the cancellation at the high-$k$ limit naturally occurs. Specifically, for $p/k \ll 1 (q \rightarrow 0)$, we can show $P_{22}|_{\text{ZA}}(k) \rightarrow \bar{\Sigma}_{\text{lin}}(k) P_{\text{lin}}(k)$ and $P_{22,22} \propto P_{\ell,13} \propto P_{\text{lin}}(k) \int dp p^2 P_{\text{lin}}(p)$ (for details, see Appendix B.) Thus, the cancellation of the high-$k$ solutions in the one-loop SPT comes from the Zel’dovich approximation and is understood to be the result of the mass conservation, because the fact that the power spectrum has no contribution at $q = 0$ is derived only from the law of mass conservation as mentioned in Section 3. Furthermore, it is known that the high-$k$ solutions $P_{22,\text{high-}k}$ and $P_{13,\text{high-}k}$ have considerable contributions even at low-$k$ regions in each term of $P_{22}$ and $P_{13}$ despite their complete cancellation (Sugiyama & Spergel 2013). As a result, the amplitude of $P_{1\text{-loop}}$ is substantially different from those of $P_{22}$ and $P_{13}$ even at large scales. Because of all these reasons, we focus on the following quantities:

$$
\Delta P_{22}(k) \equiv P_{22}(k) - P_{22,\text{high-}k}(k) = \sum_{\ell=0}^{3} P_{\ell,22}(k) + P_{1\text{-loop}}|_{\text{ZA}}(k),
$$

$$
\Delta P_{13}(k) \equiv P_{13}(k) - P_{13,\text{high-}k}(k) = \sum_{\ell=0}^{3} P_{\ell,13}(k), \quad (31)
$$

where $P_{1\text{-loop}} = P_{22} + P_{13} = \Delta P_{22} + \Delta P_{13}$. In our previous work (Sugiyama & Spergel 2013), we called $\Delta P_{22}$ and $\Delta P_{13}$ short-wavelength terms. The Zel’dovich approximation only contributes to $\Delta P_{22}$: $P_{1\text{-loop}}|_{\text{ZA}} = \Delta P_{22}|_{\text{ZA}}$ and $\Delta P_{13}|_{\text{ZA}} = 0$.

Figure 2 shows how each term of $P_{22}$, $P_{13}$, and $P_{1\text{-loop}}|_{\text{ZA}}$ contributes to $\Delta P_{22}$ and $\Delta P_{13}$ at $z = 0$. We find that the non-linear effects of the displacement vector are suitable for perturbation quantities $P_{22}/P_{\text{low}} < 1$ and $P_{13}/P_{\text{low}} < 1$ over the range of $k \lesssim 1.0 \, [h \, \text{Mpc}^{-1}]$.

5.3. At the Two-loop Order in SPT

Because of the non-linearity of the relation between the matter density and the displacement vector, the one-loop LPT solution has non-linear correction terms that have the same order as the two-loop SPT.
The solutions in the two-loop SPT have the following four terms: \( P_{2\text{-loop}} = P_{15} + P_{24} + P_{33a} + P_{33b} \), where

\[
P_{15}(k) = 30 P_{\text{lin}}(k) \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} F_5(k, p_1, -p_1, p_2, -p_2) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),
\]

\[
P_{33a}(k) = \frac{(P_{13}(k))^2}{4 P_{\text{lin}}(k)},
\]

\[
P_{24}(k) = 24 \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \delta_D(k - k_{1,2}) F_2(k_1, k_2) F_3(k, k_2, p, -p) P_{\text{lin}}(p) P_{\text{lin}}(k_1) P_{\text{lin}}(k_2),
\]

\[
P_{33b}(k) = 6 \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \int \frac{d^3 k_3}{(2\pi)^3} \delta_D(k - k_{1,3}) \{ F_3(k_1, k_2, k_3) \}^2 P_{\text{lin}}(k_1) P_{\text{lin}}(k_2) P_{\text{lin}}(k_3).
\]

(32)

Here, the high-\( k \) solutions in the two-loop SPT are given by (Sugiyama & Spergel 2013)

\[
P_{33a,\text{high-}k}(k) = -\frac{1}{2} \tilde{\Sigma}_{\text{lin}}(k) P_{13}(k) - \frac{1}{4} (\tilde{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k),
\]

\[
P_{33b,\text{high-}k}(k) = \tilde{\Sigma}_{\text{lin}}(k) P_{22}(k) - \frac{1}{2} (\tilde{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k) + \tilde{\Sigma}_{22} P_{\text{lin}}(k),
\]

\[
P_{24,\text{high-}k}(k) = -\tilde{\Sigma}_{\text{lin}}(k) P_{22}(k) + \tilde{\Sigma}_{\text{lin}}(k) P_{13}(k) + (\tilde{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k) + \tilde{\Sigma}_{13} P_{\text{lin}}(k),
\]

\[
P_{15,\text{high-}k}(k) = -\frac{1}{2} \tilde{\Sigma}_{\text{lin}}(k) P_{13}(k) - \frac{1}{4} (\tilde{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k) - \tilde{\Sigma}_{13} P_{\text{lin}}(k).
\]

(33)

Similar to the one-loop SPT, we define the following quantities:

\[
\Delta P_{33a}(k) = P_{33a}(k) - P_{33a,\text{high-}k}(k),
\]

\[
\Delta P_{33b}(k) = P_{33b}(k) - P_{33b,\text{high-}k}(k),
\]

\[
\Delta P_{24}(k) = P_{24}(k) - P_{24,\text{high-}k}(k),
\]

\[
\Delta P_{15}(k) = P_{15}(k) - P_{15,\text{high-}k}(k).
\]

(34)

Note that \( P_{2\text{-loop}} = \Delta P_{15} + \Delta P_{24} + \Delta P_{33a} + \Delta P_{33b} \) and \( \Delta P_{33a} = (\Delta P_{13})^2 / (4 P_{\text{lin}}) \).

Similar to the one-loop SPT, \( \Delta P_{33a}, \Delta P_{33b}, \Delta P_{24}, \) and \( \Delta P_{15} \) are derived from combinations that become zero at \( q = 0 \) in Equation (6). The Zel’dovich approximation only contributes to \( \Delta P_{33b} \):

\[
P_{2\text{-loop}}|_{ZA}(k) = \Delta P_{33b}|_{ZA}(k) = \frac{1}{3!} \int d^3 q e^{i k \cdot q} (\Sigma_{\text{lin}}(k, q) - \bar{\Sigma}_{\text{lin}}(k))^3,
\]

\[
= P_{33b}|_{ZA}(k) - \bar{\Sigma}_{\text{lin}}(k) P_{22}|_{ZA}(k) + \frac{1}{2} (\tilde{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k),
\]

(35)

and \( \Delta P_{33a}|_{ZA} = \Delta P_{24}|_{ZA} = \Delta P_{15}|_{ZA} = 0 \). On the other hand, we derive the following expressions corresponding to the SPT solutions from the one-loop LPT:

\[
\Delta P_{33b}|_{\text{LPT,1-loop}}(k) = \Delta P_{33b}|_{ZA}(k) + \int d^3 q e^{-i k \cdot q} \{ (\Sigma_{22}(k, q) - \bar{\Sigma}_{22}(k)) (\Sigma_{\text{lin}}(k, q) - \bar{\Sigma}_{\text{lin}}(k)) \},
\]

\[
= P_{33b}|_{\text{LPT,1-loop}}(k) - P_{33b,\text{high-}k}(k),
\]

\[
\Delta P_{24}|_{\text{LPT,1-loop}}(k) = \int d^3 q e^{-i k \cdot q} \{ (\Sigma_{13}(k, q) - \bar{\Sigma}_{13}(k)) (\Sigma_{\text{lin}}(k, q) - \bar{\Sigma}_{\text{lin}}(k)) \},
\]

\[
= P_{24}|_{\text{LPT,1-loop}}(k) - P_{24,\text{high-}k}(k),
\]

(36)

and \( \Delta P_{33a}|_{\text{LPT,1-loop}} = \Delta P_{15}|_{\text{LPT,1-loop}} = 0 \). The specific expressions of \( P_{33b}|_{ZA}, P_{33b}|_{\text{LPT,1-loop}}, \) and \( P_{24}|_{\text{LPT,1-loop}} \) are given in Appendix C. Figure 3 compares the two-loop solutions in SPT with the approximate ones computed in the linear and one-loop LPT at \( z = 0 \). Around \( k \approx 0.2 [h \text{ Mpc}^{-1}] \), the validity of the approximate solutions in the one-loop LPT is violated. This is because of the lack of non-linear dynamics. The one-loop LPT has the third order displacement vector in the perturbation series, but we need the fifth order displacement vector to completely reproduce the two-loop SPT solutions. The limitation scale of the validity of the one-loop LPT is estimated as \( |P_{2\text{-loop}} - P_{2\text{-loop}}|_{\text{LPT,1-loop}}| > |P_{2\text{-loop}}| \). The scale where this relation is satisfied is \( k \gtrsim 0.2 [h \text{ Mpc}^{-1}] \). In other words, at these scales, the one-loop SPT solution is better than the one-loop LPT solution. This behavior of the one-loop LPT solution is independent of redshifts. Thus, we can theoretically check the validity of the one-loop LPT solution without using N-body simulations.

6. POWER SPECTRUM IN LPT

Generally, the power spectrum is represented as (Crocce & Scoccimarro 2008)

\[
P(z, k) = G^2(z, k) P_{\text{lin}}(k) + P_{\text{MC}}(z, k),
\]

(37)
where $G$ and $P_{MC}$ are referred to as “propagator” and “mode-coupling term” in the context of RPT. While we can compute the propagator with relative ease, it is difficult to explicitly compute the mode-coupling term in LPT, even for the Zel’dovich approximation. In this section, we decompose the LPT power spectrum into these two parts and present an expansion method to approximately compute the mode-coupling term in LPT. Our approximate solution has good convergence in the series expansion and enables a computation of the LPT power spectrum accurately and quickly.

6.1. At the Linear Order (Zel’dovich Approximation)

We derive the Zel’dovich power spectrum from Equations (8) and (14) as follows:

$$P(z, k) = e^{-D^2\Sigma_{lin}(k)} \int d^3q e^{-i\mathbf{k} \cdot \mathbf{q}} \{ D^2\Sigma_{lin}(k, q) + (e^{D^2\Sigma_{lin}(k, q)} - 1 - D^2\Sigma_{lin}(k, q)) \}$$

$$\quad = G^2(z, k) P_{lin}(k) + P_{MC}(z, k),$$

where

$$G^2(z, k) = e^{-D^2\Sigma_{lin}(k)} D^2,$$

$$P_{MC}(z, k) = 2\pi e^{-D^2\Sigma_{lin}(k)} \int_0^\infty dq q^2 \int_{-1}^1 d\mu \cos(kq\mu)$$

$$\times \{ e^{D^2\Sigma_{0,lin}(k, q) - D^2\Sigma_{2,lin}(k, q)\zeta_2(\mu)} - 1 - (D^2\Sigma_{0,lin}(k, q) - D^2\Sigma_{2,lin}(k, q)\zeta_2(\mu)) \} \quad (39)$$

with $\mu = \mathbf{k} \cdot \mathbf{q}$, and we used $\int d^3q e^{-i\mathbf{k} \cdot \mathbf{q}} \Sigma_{lin}(k, q) = P_{lin}(k)$.

We naturally find the exponential damping behavior of the propagator in the Zel’dovich approximation, even though the damping behavior in the high-$k$ limit has been obtained in several previous works (for one of the latest works, see Bernardeau et al. 2014). The
exponential damping behavior of the propagator is the result of mass conservation, because the non-linear scale dependence of the Zel’dovich power spectrum comes only from the non-linearity of the law of mass conservation.

It is difficult to numerically compute the mode-coupling term in the Zel’dovich approximation, because the integrand in the mode-coupling term has a complicated oscillatory behavior caused by \(\cos(k q_{\mu})\). Therefore, here we present an approximation method to reproduce well the Zel’dovich power spectrum. Note that the first term \(G^2 P_{\text{lin}}\) mainly contributes at large scales because of the contribution from the linear order in SPT \(P_{\text{lin}}\), while the mode-coupling term is dominant at small scales. Since Figure 1 shows \(\Sigma_0 \gg \Sigma_2\) at small scales, we expand the exponential factor in the mode-coupling term (Equation (39)) provided that \(\Sigma_0 \gg \Sigma_2\), obtaining the following approximate mode-coupling term:

\[
P_{\text{MC}}(z, k) = \sum_{n=0}^{\infty} P_{\text{MC}}^{(n)}(z, k),
\]

where

\[
P_{\text{MC}}^{(0)} \equiv 4 \pi e^{-D^2 \Sigma_{\text{lin}}(k)} \int_0^\infty dq q^2 j_0(k q)(e^{D^2 \Sigma_{\text{lin}}(k, q)} - 1 - D^2 \Sigma_{\text{lin}}(k, q)),
\]

\[
P_{\text{MC}}^{(1)} \equiv 4 \pi e^{-D^2 \Sigma_{\text{lin}}(k)} \int_0^\infty dq q^2 j_2(k q)D^2 \Sigma_{\text{lin}}(k, q)(e^{D^2 \Sigma_{\text{lin}}(k, q)} - 1),
\]

\[
P_{\text{MC}}^{(n)} \equiv 4 \pi e^{-D^2 \Sigma_{\text{lin}}(k)} \int_0^\infty dq q^2 J^{(n)}(z, k, q)e^{D^2 \Sigma_{\text{lin}}(k, q)} \quad \text{for } n \geq 2,
\]

with

\[
J^{(n)}(z, k, q) \equiv \frac{(D^2 \Sigma_{\text{lin}}(k, q))^n}{n!} \sum_{\ell=0}^{2n} (-i)^{\ell} j_\ell(k q) \left(\frac{2\ell + 1}{2}\right) \int_{-1}^{1} d\mu \mathcal{L}_\ell(\mu)(-\mathcal{L}_\ell(\mu))^n.
\]

Analytical calculations of the \(\mu\) integral in \(J^{(n)}\) (Equation (D1)) enable computation of the mode-coupling term quickly and safely.

We have another theoretical reason for our approximation method (Equation (40)). As mentioned in Section 3, we should keep a combination of \(\Sigma_0(z, k, q) - \bar{z}(z, k)\) to satisfy the fact that the power spectrum has no contribution at \(q = 0\) and to respect the law of mass conservation. This is also related to the IR divergence problem and the cancellation of the high-\(k\) solutions in SPT (see Sections 4.2 and 5). Here, note that the propagator and the mode-coupling term each have the integral \(\int dp P_{\text{lin}}(p)\) in the limit of \(p \to 0\). However, the total Zel’dovich power spectrum does not have \(\int dp P_{\text{lin}}(p)\), but \(\int dp^2 P_{\text{lin}}(p)\) in the limit as shown in Section 4.2. The same thing also occurs in the one-loop SPT (Section 5.2). The propagator and mode-coupling term are described in the one-loop SPT as \(G^2(z, k)P_{\text{lin}}(k) = D^2 P_{\text{lin}}(k) + D^4 P_{13}(k)\) and \(P_{\text{MC}}(z, k) = D^4 P_{12}(k)\). Each of their terms is proportional to \(\int dp P_{\text{lin}}(p)\) in the limit of \(p \to 0\) (in the high-\(k\) limit), but completely cancels out each other. Thus, to satisfy this cancellation at all orders in SPT, we should not expand the exponential factor for the monopole term \(e^{-\Sigma}\) when we do not expand the exponential damping factor \(e^{-z}\) in the mode-coupling term. This idea is the first main result of this paper.

6.2. At the One-loop Order

From Equations (10), (12), and (31), the propagator term is given by

\[
G^2(z, k)P_{\text{lin}}(k) = e^{-\tilde{z}(z, k)} \int d^3 \mathbf{q} e^{-i \mathbf{k} \cdot \mathbf{q}}(D^2 \Sigma_{\text{lin}}(\mathbf{k}, \mathbf{q}) + D^4 \Sigma_{13}(\mathbf{k}, \mathbf{q}))
\]

\[
= e^{-D^2 \tilde{z}(k) - D^4 \tilde{z}_{\text{lin}}(k)} \left(1 + D^2 \Delta P_{13}(k) / P_{\text{lin}}(k)\right) D^2 P_{\text{lin}}(k).
\]

Compared to the Zel’dovich solution of the propagator, the additional factors \(e^{-D^4 \tilde{z}_{\text{lin}}(k)}\) and \((1 + D^2 \Delta P_{13}/P_{\text{lin}})\) appear in the one-loop LPT. They come from the non-linear equation of the motion of the displacement vector, the kernel functions \(\mathbf{L}_2\) and \(\mathbf{L}_3\) in Equation (3).

We obtain the mode-coupling term from

\[
P_{\text{MC}}(z, k) = e^{-\tilde{z}(z, k)} \int d^3 \mathbf{q} e^{-i \mathbf{k} \cdot \mathbf{q}}(e^{\tilde{z}(z, k, \mathbf{q})} - 1 - D^2 \Sigma_{\text{lin}}(\mathbf{k}, \mathbf{q}) - D^4 \Sigma_{13}(\mathbf{k}, \mathbf{q}))
\]

\[
= e^{-\tilde{z}(z, k)} D^4 (P_{22}(k) - P_{22}|Z_A(k)\rangle) + \tilde{P}_{\text{MC}}(z, k),
\]

where

\[
\tilde{P}_{\text{MC}}(z, k) = e^{-\tilde{z}(z, k)} \int d^3 \mathbf{q} e^{-i \mathbf{k} \cdot \mathbf{q}}(e^{\tilde{z}(z, k, \mathbf{q})} - 1 - \tilde{z}(z, k, \mathbf{q})).
\]
Figure 4. Two contributions to the power spectrum \(G^2 P_{\text{lin}}/P_{\text{aw}}\) and \(\sum_{n=0}^{n_{\text{MC}}} P_{\text{MC}}^{(n)}/P_{\text{aw}}\) for \(n_{\text{MC}} = 0, 1, 2, 3, 4, 5\) are plotted in the Zel’dovich approximation and the one-loop LPT at \(z = 0\). This figure shows the performance of our method of expanding the mode-coupling term presented in Equations (40) and (46). In particular, the bottom panels imply that the fourth and fifth orders in the expansion \((P_{\text{MC}}^{(4)}, P_{\text{MC}}^{(5)})\) contribute less than 1% compared to the linear power spectrum. In other words, the approximate mode-coupling term has good convergence in the series expansion, and we only have to compute up to the third order of the expansion \(P_{\text{MC}} = \sum_{n=0}^{3} P_{\text{MC}}^{(n)}\) with inaccuracy of less than 1% until \(k = 1 [h \text{ Mpc}^{-1}]\). At large scales \(k \lesssim 0.2 [h \text{ Mpc}^{-1}]\), \(P_{\text{MC}} = \sum_{n=0}^{3} P_{\text{MC}}^{(n)}\) presents a good approximate solution with an inaccuracy of less than 1%.

(A color version of this figure is available in the online journal.)

In computing the mode-coupling term, we can use the same analysis as in the Zel’dovich approximation. Provided \(\Sigma_0 \gg \Sigma_1, \Sigma_2, \) and \(\Sigma_3\), the mode-coupling term \(P_{\text{MC}}\) is approximated as follows:

\[
P_{\text{MC}}(z, k) = \sum_{n=0}^{\infty} P_{\text{MC}}^{(n)}(z, k) = e^{-2 \bar{\Sigma}(z,k)} (D^4 P_{22}(k) - D^2 P_{22}|z(k)) + \sum_{n=0}^{\infty} \bar{P}_{\text{MC}}^{(n)}(z, k),
\]

where

\[
\bar{P}_{\text{MC}}^{(0)}(z, k) = 4 \pi e^{-2 \bar{\Sigma}(z,k)} \int_0^{\infty} dq q^2 j_0(qk)(e^{2 \bar{\Sigma}(z,k,q)} - 1 - \bar{\Sigma}(z,k,q)),
\]

\[
\bar{P}_{\text{MC}}^{(1)}(z, k) = 4 \pi e^{-2 \bar{\Sigma}(z,k)} \int_0^{\infty} dq q^2 \sum_{l=1}^{3} j_l(qk) \bar{\Sigma}(z,k,q)(e^{2 \bar{\Sigma}(z,k,q)} - 1),
\]

\[
\bar{P}_{\text{MC}}^{(n)}(z, k) = 4 \pi e^{-2 \bar{\Sigma}(z,k)} \int_0^{\infty} dq q^2 J^{(n)}(z,k,q) e^{2 \bar{\Sigma}(z,k,q)} \quad \text{for} \; n \geq 2,
\]

with

\[
J^{(n)}(z, k, q) \equiv \frac{1}{n!} \sum_{l=0}^{3n} (-i)^l j_l(qk) \frac{2l + 1}{2} \int_{-1}^{1} d \mu \mathcal{L}_l(\mu) \left( \sum_{\ell' = 1}^{3} i^{\ell'} \bar{\Sigma}(z, k, q) \mathcal{L}_{\ell'}(\mu) \right)^n.
\]

Note that \(\bar{\Sigma}(z, k, q) = D^2 \Sigma_{\ell=1}(k, q) + D^2 \Sigma_{\ell=2}(k, q) + D^4 \Sigma_{\ell=3}(k, q)\). The analytical calculation of the \(\mu\) integral in \(J^{(2)}\) is given in Equation (D2).

Figure 4 shows the performance of our method for expanding the mode-coupling term. The fourth and fifth orders in the expansion of the mode-coupling term, \(P_{\text{MC}}^{(4)}\) and \(P_{\text{MC}}^{(5)}\), contribute less than 1% over the range of \(k \leq 1 [h \text{ Mpc}^{-1}]\) at \(z = 0\). Therefore, the
perturbation series. Therefore, at high \( z \), the LPT solution can well describe the non-linear evolution of dark matter, and the ratio \( P_{\text{LPT}}/P_{\text{nw}} \) becomes larger than unity. On the other hand, at low \( z \), because of lack of non-linear dynamics of dark matter, the ratio \( P_{\text{LPT}}/P_{\text{nw}} \) becomes less than unity.

Figure 5 shows the Zel’’dovich and one-loop LPT power spectra at various redshifts \((z = 0, 0.35, 0.5, 1.0, 3.0, \text{and } 5.0)\). The LPT solution has full non-linear effects from the law of conservation of mass, but its non-linear equation of the motion of dark matter (the equation of motion of the displacement vector) is solved in the perturbation series. Therefore, at high \( z \), the LPT solution can well describe the non-linear evolution of dark matter, and the ratio \( P_{\text{LPT}}/P_{\text{nw}} \) becomes larger than unity. On the other hand, at low \( z \), because of lack of non-linear dynamics of dark matter, the ratio \( P_{\text{LPT}}/P_{\text{nw}} \) becomes less than unity.

(A color version of this figure is available in the online journal.)

approximate mode-coupling term has good convergence in the series expansion, and we only have to compute up to the third order of the expansion \( P_{\text{MC}} = \sum_{n=0}^{3} P_{\text{MC}}^{(n)} \) to compute the mode-coupling term with an inaccuracy of less than 1% until \( k = 1 \ h \text{ Mpc}^{-1} \). The approximate solution \( P = G^{2} P_{\text{lin}} + \sum_{n=0}^{3} P_{\text{MC}}^{(n)} \) works well at any redshift because \( P_{\text{MC}}^{(n)} (n \geq 4) \) are non-linear effects and become progressively smaller at high \( z \). When we focus only on large scales \( k \lesssim 0.2 \ h \text{ Mpc}^{-1} \), \( P_{\text{MC}} = \sum_{n=0}^{2} P_{\text{MC}}^{(n)} \) is enough to reproduce the LPT power spectrum with an inaccuracy of less than 1%.

Figure 5 shows the Zel’dovich and one-loop LPT power spectra at various redshifts \((z = 0, 0.35, 0.5, 1.0, 3.0, \text{and } 5.0)\). The LPT solution has full non-linear effects from the law of conservation of mass, but its non-linear equation of the motion of dark matter (the equation of motion of the displacement vector) is solved in the perturbation series. Therefore, at high \( z \), the LPT solution can describe well the non-linear evolution of dark matter, and the ratio \( P_{\text{LPT}}/P_{\text{nw}} \) becomes larger than unity. On the other hand, at low \( z \), the third order of the displacement vector in the perturbation series is not enough to accurately describe the non-linear growth of dark matter, and the ratio \( P_{\text{LPT}}/P_{\text{nw}} \) becomes less than unity.

7. COMPARISON WITH THE \( \gamma \)-EXPANSION METHOD

To clarify the relation between LPT and existing works, in this section we will show that the expansion method used in LRT (Matsubara 2008) corresponds to the \( \gamma \)-expansion (Bernardeau et al. 2008, 2012; Taruya et al. 2012; Sugiyama & Futamase 2012), leading to the solution of RegPT. The LRT solution is derived from expanding \( e^{\Sigma_{z,k,q}} \) in Equation (8) as

\[
P(z, k) = e^{-\Sigma(z,k)} \sum_{n=1}^{\infty} \frac{1}{n!} \int d^{3}q e^{i k \cdot q} \langle \Sigma(z, k, q) \rangle^{n}, \tag{49}\]

and truncating at a finite order of \( n \). Unlike our expansion method (Section 6), LRT (the \( \gamma \)-expansion method) expands the exponential factor including the monopole term \( e^{\Sigma_{z,k}} \).

7.1. Review of the \( \Gamma \) Expansion

The \( \Gamma \) expansion is only used to obtain information on the power spectrum at large-scale regions. The higher order terms of the \( \Gamma \) expansion have information on smaller scales. In the \( \Gamma \)-expansion method, the full non-linear power spectrum is described as

\[
P(z, k) = G^{2}(z, k) P_{\text{lin}}(k) + \sum_{r=2}^{\infty} P_{r}^{(r)}(z, k). \tag{50}\]

Therefore, the mode-coupling term is \( P_{\text{MC}} = \sum_{r=2}^{\infty} P_{r}^{(r)} \), where \( P_{r}^{(r)} \) is the \( r \)-th order contribution to the power spectrum in the \( \Gamma \) expansion, defined as

\[
P_{r}^{(r)}(z, k) \equiv r! \int \frac{d^{3}k_{1}}{(2\pi)^{3}} \cdots \int \frac{d^{3}k_{r}}{(2\pi)^{3}} (2\pi)^{3} \delta_{D}(k - k_{1,\ldots,r}) [\Gamma^{(r)}(z, k_{1}, \ldots, k_{r})]^{2} P_{\text{lin}}(k_{1}) \cdots P_{\text{lin}}(k_{r}) \tag{51}\]

with

\[
\Gamma^{(r)}(z, k_{1}, \ldots, k_{r}) \equiv \sum_{n=0}^{\infty} D^{r+2n}(r+2n)! (2\pi)^{3} \int \frac{d^{3}p_{1}}{(2\pi)^{3}} \cdots \int \frac{d^{3}p_{n}}{(2\pi)^{3}} \times F_{r+2n}(k_{1}, \ldots, k_{r}, p_{1}, \ldots, p_{n}, -p_{n}) P_{\text{lin}}(p_{1}) \cdots P_{\text{lin}}(p_{n}). \tag{52}\]
The propagator is defined as

\[ G(z, k) = P^{(1)}_{\Gamma}(z, k) \equiv \frac{\langle \delta(z, k) \delta_{\text{lin}}(z = 0, k') \rangle}{\langle \delta_{\text{lin}}(z = 0, k) \rangle} = \left( 1 + \sum_{n=1}^{\infty} D^{2n} \frac{P_{\Gamma}(2n+1)(k)}{2P_{\text{lin}}(k)} \right) D. \] (53)

### 7.2. Zel’dovich Approximation

In the Zel’dovich approximation, the LRT expansion method provides \( P^{(r)}_{\Gamma} \) as

\[
P^{(r)}_{\Gamma}(z, k) = e^{-D^2 \Sigma_{\text{lin}}(k)} D^2 r! \int d^3 q e^{-ik \cdot q} \left\{ \Gamma(k, q) \right\}^r
\]

\[
= e^{-D^2 \Sigma_{\text{lin}}(k)} D^2 r! \int d^3 q e^{-ik \cdot q} \left\{ \int \frac{d^3 p}{(2\pi)^3} e^{ip \cdot q} \left( \frac{k \cdot p}{p^2} \right)^2 P_{\text{lin}}(p) \right\}^r
\]

\[
= e^{-D^2 \Sigma_{\text{lin}}(k)} D^2 r! \int \frac{d^3 k_1}{(2\pi)^3} \cdots \int \frac{d^3 k_r}{(2\pi)^3} \delta_D(k - k_1, \cdots, k_r) \left| F_{r|ZA}(k_1, \cdots, k_r) \right|^2 P_{\text{lin}}(k_1) \cdots P_{\text{lin}}(k_r). \] (54)

It is worth noting that while we need a \( 3(r - 1) \)-dimensional integral to compute the expression using \( F_{r|ZA} \) in the final line, in the first line we only need a two-dimensional integral for any \( r \) as follows:

\[
P^{(r)}_{\Gamma}(z, k) = 4\pi e^{-D^2 \Sigma_{\text{lin}}(k)} \frac{1}{r!} \int_0^\infty dq d^2 q \sum_{\ell=0}^{\infty} (-i)^\ell j_\ell(kq) \frac{2\ell + 1}{2}
\]

\[
\times \int_{-1}^1 d\mu \mathcal{L}_\ell(\mu) (D^2 \Sigma_{\text{lin}}(k, q) - \mathcal{L}_2(\mu) D^2 \Sigma_{\text{lin}}(k, q))^\ell. \] (55)

### 7.3. LPT at the One-loop Order

Similar to the case of the Zel’dovich approximation, we obtain \( P^{(n)}_{\Gamma} \) in the one-loop LPT:

\[
P^{(n)}_{\text{MC}}(z, k) = \sum_{r=2}^{\infty} P^{(r)}_{\Gamma}(z, k) = \sum_{n=2}^{\infty} P^{(n)}_{\Gamma-1}(z, k) + \sum_{n=1}^{\infty} P^{(2n)}_{\Gamma-2}(z, k) + \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} P^{(n+m)}_{\Gamma-3}(z, k), \] (56)

where

\[
P^{(n)}_{\Gamma-1}(z, k) = 4\pi e^{-\Sigma_{\text{lin}}(z, k)} \frac{1}{n!} \int_0^\infty dq d^2 q \sum_{\ell=0}^{\infty} (-i)^\ell j_\ell(kq) \frac{2\ell + 1}{2}
\]

\[
\times \int_{-1}^1 d\mu \mathcal{L}_\ell(\mu) (D^2 \Sigma_{\text{lin}}(k, q) + D^4 \Sigma_{13}(k, q))^\ell. \]

\[
P^{(2n)}_{\Gamma-2}(z, k) = 4\pi e^{-\Sigma_{\text{lin}}(z, k)} \frac{1}{n!} \int_0^\infty dq d^2 q \sum_{\ell=0}^{\infty} (-i)^\ell j_\ell(kq) \frac{2\ell + 1}{2}
\]

\[
\times \int_{-1}^1 d\mu \mathcal{L}_\ell(\mu) (D^4 \Sigma_{32}(k, q))^\ell. \]

\[
P^{(n+m)}_{\Gamma-3}(z, k) = 4\pi e^{-\Sigma_{\text{lin}}(z, k)} \frac{1}{n!} \int_0^\infty dq d^2 q \sum_{\ell=0}^{\infty} (-i)^\ell j_\ell(kq) \frac{2\ell + 1}{2}
\]

\[
\times [D^2 \Sigma_{\text{lin}}(k, q) + D^4 \Sigma_{13}(k, q)]^{m} [D^4 \Sigma_{32}(k, q)]^m. \] (57)

Specifically, we have the following expressions up to the third order of the \( \Gamma \) expansion:

\[
P^{(2)}_{\Gamma}(z, k) = e^{-D^2 \Sigma_{\text{lin}}(k, z)} D^2 \left( D^4 P_{24}|_{\text{LPT, 1-loop}}(k) + \tilde{\Sigma}_{\text{lin}}(k) P_{22}(k) \right) + \cdots,
\]

\[
P^{(3)}_{\Gamma}(z, k) = e^{-D^2 \Sigma_{\text{lin}}(k, z)} D^2 \left( D^6 P_{33}|_{\text{LPT, 1-loop}}(k) \right) + \cdots. \] (58)

Note that \( P^{(2)}_{\Gamma} \) and \( P^{(3)}_{\Gamma} \) correspond to those in the original two-loop RegPT, even though the approximate solutions \( P_{24}|_{\text{LPT, 1-loop}} \) and \( P_{33}|_{\text{LPT, 1-loop}} \) are used in the one-loop LPT. Figure 6 gives a demonstration of how the coefficients of the \( \Gamma \) expansion reproduce the LPT power spectrum, where they were computed up to the fifth order, namely, \( \Sigma_{\Gamma=5} P^{(r)}_{\Gamma} \).

Truncating the \( \Gamma \) expansion at the second order and ignoring some non-linear effects in the one-loop LPT (Equation (58)), we have the one-loop LRT solution:

\[
P_{\Gamma|\text{LRT, 1-loop}}(z, k) = P^{(1)}_{\Gamma}(z, k) + P^{(2)}_{\Gamma}(z, k)
\]

\[
= e^{-D^2 \Sigma_{\text{lin}}(k)} \left( 1 + D^2 \Delta P_{13}(k) \frac{P_{\text{lin}}(k)}{P_{\text{lin}}(k)} \right) D^2 P_{\text{lin}}(k) + e^{-D^2 \Sigma_{\text{lin}}(k)} D^4 P_{22}(k)
\]

\[
= e^{-D^2 \Sigma_{\text{lin}}(k)} (D^2 P_{\text{lin}}(k) + D^4 (P_{1\text{-loop}} + \tilde{\Sigma}_{\text{lin}}(k) P_{\text{lin}}(k))). \] (59)
where we ignored $\bar{\Sigma}_{13}$, $\bar{\Sigma}_{22}$, $P_{24}|_{\text{LPT,1-loop}}$, and so on. On the other hand, the original one-loop RegPT solution is given by

$$P|_{\text{RegPT,1-loop}}(z, k) = P(1)(z, k) + P(2)(z, k)$$

$$= e^{-D^2\bar{\Sigma}_{13}(k)} \left( 1 + D^2 \frac{\Delta P_{13}(k)}{2 P_{\text{lin}}(k)} \right) D^2 P_{\text{lin}}(k) + e^{-D^2\bar{\Sigma}_{22}(k)} D^4 P_{22}(k)$$

$$= P|_{\text{LRT,1-loop}}(z, k) + e^{-D^2\bar{\Sigma}_{13}(k)} D^6 \Delta P_{33}(k).$$

Since the one-loop LPT solution does not have the two-loop correction term $\Delta P_{33a}$ in SPT which comes from the two-loop LPT solution (see Section 5.3), the one-loop LPT solution does not completely reproduce the one-loop RegPT solution. However, the term $e^{-D^2\bar{\Sigma}_{13}(k)} D^6 \Delta P_{33a}$ is small enough to be ignored and we can actually regard as $P|_{\text{RegPT,1-loop}} \simeq P|_{\text{LRT,1-loop}}$. Clearly, the two-loop LPT includes the one-loop RegPT.

Finally, we present the general expression of the propagator in LPT:

$$\langle \delta(z, \mathbf{k}) \delta_{\text{lin}}(z = 0, \mathbf{k}') \rangle = \int d^3 q_1 \int d^3 q_2 e^{-i\mathbf{k} \cdot \mathbf{q}_1} e^{-i\mathbf{k}' \cdot \mathbf{q}_2} \langle e^{-i\mathbf{k} \cdot \Psi(z, \mathbf{q}_1)}(-i\mathbf{k}' \cdot \Psi_{\text{lin}}(z = 0, \mathbf{q}_2)) \rangle$$

$$= (2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}') \langle e^{-i\mathbf{k} \cdot \Psi(z, 0)} \rangle \int d^3 q e^{-i\mathbf{k} \cdot \mathbf{q}} \langle e^{-i\mathbf{k} \cdot \Psi(z, \mathbf{q})}(i\mathbf{k} \cdot \Psi_{\text{lin}}(z = 0, 0)) \rangle_c,$$

(61)

where $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$. This implies

$$G(z, k) = \exp \left( -\frac{\bar{\Sigma}(z, k)}{2} \right) \left( 1 + \sum_{n=1}^{\infty} D^{2n} \frac{\Delta P_{1(2n+1)}(k)}{2 P_{\text{lin}}(k)} \right) D,$$

(62)

where we used $\langle (-i\mathbf{k} \cdot \Psi(z, 0))^n \rangle_c = \langle (i\mathbf{k} \cdot \Psi(z, 0))^n \rangle_c$, and

$$\int d^3 q e^{-i\mathbf{k} \cdot \mathbf{q}} \langle e^{-i\mathbf{k} \cdot \Psi(z, \mathbf{q})}(i\mathbf{k} \cdot \Psi_{\text{lin}}(z = 0, 0)) \rangle_c = \left( 1 + \sum_{n=1}^{\infty} D^{2n} \frac{\Delta P_{1(2n+1)}(k)}{2 P_{\text{lin}}(k)} \right) D P_{\text{lin}}(k).$$

(63)

In other words, the above relation is the definition of $\Delta P_{1(2n+1)}$. For example, Equation (63) leads to

$$\Delta P_{13}(k) = P_{\text{lin}}(k) \int \frac{d^3 p}{(2\pi)^3} \left[ k \cdot L_3(k, p, -p) - k \cdot L_2(k, p) k \cdot L_1(p) \right] P_{\text{lin}}(p).$$

$$= P_{13}(k) + \bar{\Sigma}_{\text{lin}}(k) P_{\text{lin}}(k).$$

(64)

This expression is the same as Equation (31). Furthermore, at the one-loop order, the square of Equation (62) leads to Equation (43) with the term $\Delta P_{33a}$ ignored.

8. BEYOND THE TWO-LOOP SOLUTION IN SPT

Our main goal is to obtain non-linear information on the matter perturbation beyond the two-loop SPT. While the exact two-loop solution in SPT has been well studied (Section 5.3), it is too computationally expensive to compute solutions of higher order than
the two-loop in SPT. Therefore, we want approximate information on the three- and higher loop orders in SPT. For that purpose, we have so far solved the one-loop LPT solution, which is described in the standard perturbation series as follows:

$$P(z, k)_{\text{LPT, 1-loop}}(k) = D^2 P_{\text{lin}}(k) + D^4 P_{\text{1-loop}}(k) + D^6 P_{\text{2-loop}}(k) + \sum_{n=3}^{\infty} D^{2n+2} P_{n-\text{loop}}_{\text{LPT, 1-loop}}(k).$$  \hspace{1cm} (65)

Note that solutions of higher order than the two-loop order in SPT come from the non-linearity of the conservation of mass. The one-loop LPT solution has the exact one-loop correction in SPT because of the third order of the displacement vector in the perturbation expansion. In this section, we focus on the three- and more order terms in SPT computed in the one-loop LPT $\sum_{n=3}^{\infty} D^{2n+2} P_{n-\text{loop}}_{\text{LPT, 1-loop}}$ and investigate how they behave.

Figure 7 shows the behavior of $P_{2-\text{loop}}$ and $\sum_{n=3}^{\infty} D^{2n+2} P_{n-\text{loop}}_{\text{LPT, 1-loop}}$ at various redshifts ($z = 0$, 0.35, 0.5, 1.0, 2.0, and 3.0). This figure implies the limitation of the validity of the solutions in SPT at the one- and two-loop order. For example, at $z = 3.0$ the two-loop correction in SPT is small enough to be ignored until $k \lesssim 0.5 \, h \, \text{Mpc}^{-1}$ with an inaccuracy of less than 1%; and the one-loop solution in SPT, therefore, works well until this scale. On the other hand, at $z = 0$ the validity of the two-loop solution in SPT is violated around $k \simeq 0.1 \, h \, \text{Mpc}^{-1}$ because the approximate higher loop solutions have a considerable contribution of more than 1% around these scales. In other words, we expect that around $k \simeq 0.1 \, h \, \text{Mpc}^{-1}$ and $z = 0$ the two-loop SPT solution will be too small to predict the precise non-linear power spectrum. A rough estimate of the scales where the one- and two-loop solutions in SPT are valid with an inaccuracy less than 1% is summarized in Table 1. These predictions of the behavior of the SPT solutions are confirmed by comparing N-body simulations in Section 9.

Let us mention the exact three-loop solution in SPT recently computed by Blas et al. (2014). The converging properties of the three- and higher loop order corrections computed in the one-loop LPT differ from the exact three-loop results. The origin of this difference is higher order of the displacement vector than the third order, because we need up to the seventh order of the displacement vector in the perturbation series to reproduce the exact three-loop SPT solutions. We leave the investigation of how their non-linear corrections affect the power spectrum for our future work. At least, we find that the third-order displacement vector and the full non-linear law of conservation of mass yield good converging properties.

We end this section by presenting the following approximate solution of the non-linear power spectrum:

$$P(z, k) = D^2 P_{\text{lin}}(k) + D^4 P_{\text{1-loop}}(k) + D^6 P_{\text{2-loop}}(k) + \sum_{n=3}^{\infty} D^{2n+2} P_{n-\text{loop}}_{\text{LPT, 1-loop}}(k).$$  \hspace{1cm} (66)

This is the second main result of this paper. Since we already have the exact two-loop solution in SPT, we do not need to use the approximate two-loop solution in the one-loop LPT $P_{2-\text{loop}}_{\text{LPT, 1-loop}}$. Therefore, by matching the solutions at the three- and higher loop orders in SPT computed in the one-loop LPT to the two-loop SPT, we can obtain more information on the non-linearity of the law of conservation of mass and a better approximate non-linear power spectrum than the two-loop SPT solution.
9. COMPARISON WITH N-BODY SIMULATION: POWER SPECTRUM

In this section, we compare the analytical predicted power spectra and N-body simulation results. We use two N-body simulation results created by the public N-body codes GADGET2 and 2LPT (Springel 2005; Crocce et al. 2006) with low and high resolutions presented in Taruya et al. (2009) and Valageas & Nishimichi (2011), respectively. The high-resolution N-body simulations are computed by combining the results with different box sizes, called L11-N11 and L12-N11. We summarize our sets of N-body simulation in Table 2.

9.1. One-loop Order

In Figure 8, we plot the analytically predicted power spectra at the one-loop order (SPT in Equations (29), RegPT in Equation (60), and LPT in Equations (43) and (44)), and the N-body simulations. The top and bottom panels show the N-body simulations with the low and high resolutions, respectively, while the analytical predictions are the same. First, let us recall that the one-loop SPT solution should be correct until \( k \simeq 0.5 \, [h \text{ Mpc}^{-1}] \) at \( z = 3 \) within accuracy less than 1\% until \( k \simeq 0.4 \, [h \text{ Mpc}^{-1}] \) at \( z = 2 \) within accuracy less than 2\%. Furthermore, the top panels in Figure 8 show that the low-resolution N-body simulations do not agree with the one-loop SPT result. This inconsistency implies that the low-resolution N-body simulations underestimate true values at \( z = 2.0 \) and \( z = 3.0 \). This fact is not surprising. It is well known that this underestimation happens due to difficulty of describing small fluctuations of dark matter at high-\( z \). In fact, the N-body simulations with the high resolutions are in excellent agreement with the one-loop SPT result at \( z = 3.0 \) in the bottom panels. Second, as expected, the one-loop LPT solution is better than the one-loop SPT solution at relatively low-\( z \): \( z = 1.0, z = 0.5, \) and \( z = 0.35 \). This is because the two-loop contribution becomes large enough to not be ignored at \( k \lesssim 0.2 \, [h \text{ Mpc}^{-1}] \) at these redshifts.

9.2. Two-loop Order and More

In Figure 9, we compare the one- and two-loop SPT solutions (Equations (29) and (32)) and the two-loop LPT solution in addition to the approximate solutions at the three- and higher loop orders computed in the one-loop LPT (Equation (66)). Similarly to the case in the last subsection, there is no disagreement between the analytical results and the low-resolution N-body simulations at \( z = 2.0 \) and \( z = 3.0 \). At \( z = 1.0 \), the two-loop SPT result agrees well with the N-body result as expected from Figure 7. Furthermore, higher order contributions than that from the two loops in SPT, which come from the non-linearity of the law of conservation of mass \( \sum_{n=3}^{\infty} P_{n \text{-loop} \mid LPT, 1 \text{-loop}} \), indeed improve the two-loop SPT solution at \( z = 0.35 \) and \( z = 0.5 \). (see the magenta and green symbols in the right panel of Figure 9). Although the two-loop LPT would give better solutions, the calculations are left to our future works.

10. CONCLUSION

We calculated the LPT power spectrum at the one-loop order. In LPT, the full non-linear law of conservation of mass is naturally satisfied by the relation between the matter density and the displacement vector. The conservation of mass relates various properties of the matter density perturbation: Galilean invariance, cancellation of high-\( k \) solutions in SPT, and IR divergence problem. Furthermore, the LPT solution has a simple relation to the \( \Gamma \)-expansion method.

Although it is difficult to explicitly compute the LPT power spectrum even using the Zel’dovich approximation, we presented an expansion method to approximately compute the LPT power spectrum. Our approximate solution has good convergence in the series expansion and enables us to compute the LPT power spectrum accurately and quickly.

The one-loop LPT solution has full non-linear information on the conservation of mass. Therefore, by matching the one-loop LPT solution to the two-loop SPT solution, we can obtain a better approximate solution of the power spectrum than the two-loop SPT without any free parameter. This solution agrees with the N-body simulation at \( k = 0.2 \, [h \text{ Mpc}^{-1}] \) and \( z = 0.35 \) to better than 2\%.

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Figure 8. Comparison between the N-body simulation results with the low and high resolutions and various analytical predictions at the one-loop order are shown. The top panels and bottom panels plot the N-body simulations with the low and high resolutions, respectively, even though the analytical predictions are the same. Left panels: ratios of the predicted non-linear power spectra and the no-wiggle linear power spectrum $P/P_{nw}$ are plotted: one-loop SPT (blue), one-loop RegPT (brown), one-loop LPT (magenta), and N-body simulations (black symbols). Right panels: fractional differences $\text{Diff}[\%] \equiv \left[ \frac{P_{\text{N-body}} - P}{P_{nw}} \right] \times 100$ are plotted. (A color version of this figure is available in the online journal.)

APPENDIX A
NON-LINEAR CORRECTION TERMS IN LPT

The LPT power spectrum is described as

$$P(z, k) = \int d^3q e^{-i\mathbf{k} \cdot \mathbf{q}} \{ e^{\Sigma(z, \mathbf{k}, \mathbf{q})} - 1 \},$$

where in the one-loop LPT, $\Sigma$ is given by

$$\Sigma(z, \mathbf{k}, \mathbf{q}) = \sum_{\ell=0}^3 i^\ell (D^2 \epsilon_{\ell,\text{lin}}(k, q) + D^4 \epsilon_{\ell,22}(k, q) + D^4 \epsilon_{\ell,13}(k, q)) \mathcal{L}_\ell(\hat{k} \cdot \hat{q}),$$

$$\tilde{\Sigma}(z, k) = D^2 \tilde{\epsilon}_{\text{lin}}(k) + D^4 \tilde{\epsilon}_{22}(k) + D^4 \tilde{\epsilon}_{13}(k),$$

with

$$\epsilon_{0,\text{lin}}(k, q) = \frac{1}{3} k^2 \int \frac{d\mathbf{p}}{2\pi^2} j_0(pq) P_{\text{lin}}(p), \quad \epsilon_{2,\text{lin}}(k, q) = \frac{2}{3} k^2 \int \frac{d\mathbf{p}}{2\pi^2} j_2(pq) P_{\text{lin}}(p),$$

$$\epsilon_{22}(k, q) = \int_{-\infty}^{\infty} \frac{dp_1 p_1^2}{2\pi^2} \int_{-\infty}^{\infty} \frac{dp_2 p_2^2}{2\pi^2} \int_{-1}^1 d\mu j_4(\mu |p_1 + p_2| q) K_{\ell,22}(k, p_1, p_2, \mu) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),$$

$$\epsilon_{13}(k, q) = \int_{-\infty}^{\infty} \frac{dp_1 p_1^2}{2\pi^2} \int_{-\infty}^{\infty} \frac{dp_2 p_2^2}{2\pi^2} j_4(p_1 q) K_{\ell,13}(k, p_1, p_2) P_{\text{lin}}(p_1) P_{\text{lin}}(p_2),$$

and $\tilde{\epsilon}_{\text{lin}}(k) = \epsilon_{0,\text{lin}}(k, q = 0), \tilde{\epsilon}_{22}(k) = \epsilon_{0,22}(k, q = 0),$ and $\tilde{\epsilon}_{13}(k) = \epsilon_{0,13}(k, q = 0).$
Figure 9. Same as Figure 8: predicted power spectra (the one-loop SPT in Equation (29), the two-loop SPT in Equation (32), and our main result in Equation (66)) are plotted as blue, green, and magenta lines. As expected from Figure 7, at $z = 3.0$, the one-loop SPT solution works well until $k = 0.4 \, h \, \text{Mpc}^{-1}$, and at $z = 1.0$, the two-loop SPT solution is in extremely good agreement with the N-body simulation result until $k = 0.3 \, h \, \text{Mpc}^{-1}$. At $z = 0.35$, the two-loop SPT solution is not enough to describe the non-linear power spectrum at $k = 0.2 \, h \, \text{Mpc}^{-1}$ and our main result is indeed better than the two-loop SPT solution at this scale. Our result agrees with numerical simulations at $k = 0.2 \, h \, \text{Mpc}^{-1}$ and $z = 0.35$ to better than 2%.

(A color version of this figure is available in the online journal.)

### A.1. Kernel Functions $K_{\ell,22}$ and $K_{\ell,13}$

In Equation (A3), the kernel functions $K_{\ell,13}$ and $K_{\ell,22}$ are given by

$$K_{0,22}(k, p_1, p_2, \mu) = k^2 \frac{3}{196} \left( 1 - \mu^2 \right)^2 \frac{\langle p_1 + p_2 \rangle^2}{\langle p_1 \rangle^2 \langle p_2 \rangle^2},$$

$$K_{1,22}(k, p_1, p_2, \mu) = k^3 \frac{3}{70} \left( 1 - \mu^2 \right)^2 \left( \frac{\langle p_1^2 \rangle + \langle p_2^2 \rangle}{\langle p_1 \rangle \langle p_2 \rangle} \right) + 4 \mu^2 + 2 \mu^2 + 2,$$

$$K_{2,22}(k, p_1, p_2, \mu) = 2 K_{0,22}(k, p_1, p_2, \mu),$$

$$K_{3,22}(k, p_1, p_2, \mu) = k^3 \frac{3}{70} \left( 1 - \mu^2 \right)^2 \left( 2 \mu \left( \frac{\langle p_1^2 \rangle + \langle p_2^2 \rangle}{\langle p_1 \rangle \langle p_2 \rangle} \right) + \mu^2 + 3 \right), \quad (A4)$$

and

$$K_{0,13}(k, p_1, p_2) = k^2 \frac{5}{1008} \frac{1}{p_1} \frac{1}{y^3} \left( y^2 - 1 \right) \left( y^2 - 1 \right) \left( y^2 + 1 \right) \ln \left( \frac{1 + y}{1 - y} \right) - \frac{2}{3} y (3 y^6 - 11 y^4 - 11 y^2 + 3),$$

$$K_{1,13}(k, p_1, p_2) = k^3 \frac{3}{560} \frac{1}{p_1^3} \frac{1}{y^3} \left( y^2 - 1 \right) y (2 y^2 + 4) \ln \left( \frac{1 + y}{1 - y} \right) - \frac{2}{3} y (6 y^6 - 4 y^4 + 26 y^2 - 12),$$

$$K_{2,13}(k, p_1, p_2) = 2 K_{0,13}(k, p_1, p_2),$$

$$K_{3,13}(k, p_1, p_2) = k^3 \frac{3}{560} \frac{1}{p_1^3} \frac{1}{y^3} \left( y^2 - 1 \right) y (3 y^2 + 1) \ln \left( \frac{1 + y}{1 - y} \right) - \frac{2}{3} y (9 y^6 - 21 y^4 - y^2 - 3), \quad (A5)$$

where $y = p_2/p_1$ and $\mu = \hat{p}_1 \cdot \hat{p}_2$. 

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For \( y = p_2/p_1 \ll 1 \), \( K_{\ell,22} \) and \( K_{\ell,13} \) become

\[
K_{0,22}(k, p_1, p_2, \mu) \rightarrow k^2 \frac{3}{196} \left( 1 - \mu^2 \right)^2 \frac{p_1^3}{p_1^2},
\]
\[
K_{1,22}(k, p_1, p_2, \mu) \rightarrow k^3 \frac{3}{70} \left( 1 - \mu^2 \right) \left( 3\mu \frac{p_1}{p_2} + 4\mu^2 + 2 \right) \frac{p_1}{p_1^2},
\]
\[
K_{0,22}(k, p_1, p_2, \mu) \rightarrow k^2 \frac{6}{196} \left( 1 - \mu^2 \right)^2 \frac{p_1^3}{p_1^2},
\]
\[
K_{0,22}(k, p_1, p_2, \mu) \rightarrow k^3 \frac{3}{70} \left( 1 - \mu^2 \right) \left( 2\mu \frac{p_1}{p_2} + \mu^2 + 3 \right) \frac{p_1}{p_1^2}, \quad (A6)
\]

and

\[
K_{0,13}(k, p_1, p_2) \rightarrow \frac{16k^2}{189p_1^3} - \frac{16k^2}{241p_1^1} \frac{y^2}{p_1^2} + \frac{16k^2}{3969p_1^1} \frac{y^4}{p_1^2},
\]
\[
K_{1,13}(k, p_1, p_2) \rightarrow - \frac{4k^3}{175p_1^3} \frac{12k^3}{44k^3} \frac{y^2}{p_1^2} + \frac{44k^3}{3675p_1^1} \frac{y^4}{p_1^2},
\]
\[
K_{2,13}(k, p_1, p_2) \rightarrow \frac{32k^2}{189p_1^3} \frac{32k^2}{44p_1^1} \frac{y^2}{p_1^2} + \frac{32k^2}{3969p_1^1} \frac{y^4}{p_1^2},
\]
\[
K_{3,13}(k, p_1, p_2) \rightarrow \frac{24k^3}{175p_1^3} \frac{24k^3}{44p_1^1} \frac{y^2}{p_1^2} + \frac{8k^3}{525p_1^1} \frac{y^4}{p_1^2}. \quad (A7)
\]

On the other hand, for \( y = p_2/p_1 \gg 1 \), \( K_{\ell,22} \) are given by replacing \( p_1 \) with \( p_2 \) in Equation (A6) due to the symmetry of \( K_{\ell,22} \) about \( p_1 \) and \( p_2 \), and \( K_{\ell,13} \) are given by

\[
K_{0,13}(k, p_1, p_2) \rightarrow \frac{16k^2}{189p_1^3} \frac{1}{y^2} - \frac{16k^2}{441p_1^1} \frac{1}{y^4},
\]
\[
K_{1,13}(k, p_1, p_2) \rightarrow - \frac{4k^3}{175p_1^3} \frac{1}{y^2} + \frac{156k^3}{1225p_1^3} \frac{1}{y^4},
\]
\[
K_{2,13}(k, p_1, p_2) \rightarrow \frac{32k^2}{189p_1^3} \frac{1}{y^2} - \frac{32k^2}{441p_1^1} \frac{1}{y^4},
\]
\[
K_{3,13}(k, p_1, p_2) \rightarrow \frac{8k^3}{175p_1^3} \frac{1}{y^2} + \frac{24k^3}{1225p_1^3} \frac{1}{y^4}. \quad (A8)
\]

A.2. Asymptotic Expressions of \( \Sigma_{\ell,22} \) and \( \Sigma_{\ell,13} \)

The asymptotic behaviors of \( K_{\ell,22} \) and \( K_{\ell,13} \) (Equations (A6), (A7), and (A8)) lead to those of \( \Sigma_{\ell,22} \) and \( \Sigma_{\ell,13} \): for \( p_2/p_1 \ll 1 \),

\[
\Sigma_{0,22}(k, q) - \tilde{\Sigma}_{22}(k) \rightarrow \frac{k^2}{245\pi^4} \int_0^\infty dp_1 \left[ j_0(p_1q) - 1 \right] P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2),
\]
\[
\Sigma_{1,22}(k, q) \rightarrow \frac{k^3}{25\pi^4} \int_0^\infty dp_1 \frac{j_1(p_1q)}{p_1} P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2),
\]
\[
\Sigma_{2,22}(k, q) \rightarrow \frac{2k^2}{245\pi^4} \int_0^\infty dp_1 j_2(p_1q) P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2),
\]
\[
\Sigma_{1,22}(k, q) \rightarrow \frac{8k^3}{175\pi^4} \int_0^\infty dp_1 \frac{j_3(p_1q)}{p_1} P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2), \quad (A9)
\]

and

\[
\Sigma_{0,13}(k, q) - \tilde{\Sigma}_{13}(k) \rightarrow \frac{4k^2}{189\pi^4} \int_0^\infty dp_1 \left[ j_0(p_1q) - 1 \right] P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2),
\]
\[
\Sigma_{1,13}(k, q) \rightarrow - \frac{k^3}{175\pi^4} \int_0^\infty dp_1 \frac{j_1(p_1q)}{p_1} P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2),
\]
\[
\Sigma_{2,13}(k, q) \rightarrow \frac{8k^2}{189\pi^4} \int_0^\infty dp_1 j_2(p_1q) P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2),
\]
\[
\Sigma_{3,13}(k, q) \rightarrow \frac{6k^3}{175\pi^4} \int_0^\infty dp_1 \frac{j_3(p_1q)}{p_1} P_{\text{lin}}(p_1) \int_0^\infty dp_2 p_2^2 P_{\text{lin}}(p_2). \quad (A10)
\]
and for $y = p_{2}/p_{1} \gg 1$,
\[\Sigma_{0,22}(k, q) - \tilde{\Sigma}_{22}(k) \rightarrow \frac{k^{2}}{245\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{2}\mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2}[j_{0}(p_{2}q) - 1] \mu_{\text{lin}}(p_{2}),\]
\[\Sigma_{1,22}(k, q) \rightarrow \frac{k^{3}}{25\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{3}\mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2}\frac{j_{1}(p_{2}q)}{p_{2}} \mu_{\text{lin}}(p_{2}),\]
\[\Sigma_{2,22}(k, q) \rightarrow \frac{2k^{2}}{245\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{2}\mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2}j_{2}(p_{2}q) \mu_{\text{lin}}(p_{2}),\]
\[\Sigma_{1,22}(k, q) \rightarrow \frac{8k^{3}}{175\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{2}\mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2}\frac{j_{3}(p_{2}q)}{p_{2}} \mu_{\text{lin}}(p_{2}),\] (A11)

and
\[\Sigma_{0,13}(k, q) - \Sigma_{13}(k) \rightarrow \frac{4k^{2}}{189\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{2}[j_{0}(p_{1}q) - 1] \mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2} \mu_{\text{lin}}(p_{2}),\]
\[\Sigma_{1,13}(k, q) \rightarrow -\frac{k^{3}}{25\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{2}\frac{j_{1}(p_{1}q)}{p_{1}} \mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2} \mu_{\text{lin}}(p_{2}),\]
\[\Sigma_{2,13}(k, q) \rightarrow \frac{8k^{2}}{189\pi^{4}} \int_{0}^{\infty} dp_{1}p_{1}^{2}\mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2} \mu_{\text{lin}}(p_{2}),\]
\[\Sigma_{4,13}(k, q) \rightarrow \frac{2k^{3}}{175} \int_{0}^{\infty} dp_{1}p_{1}^{2}\mu_{\text{lin}}(p_{1}) \int_{0}^{\infty} dp_{2} \mu_{\text{lin}}(p_{2}).\] (A12)

APPENDIX B

NON-LINEAR CORRECTIONS IN THE ONE-LOOP SPT

In LPT, the one-loop correction term in SPT, $P_{1\text{-loop}} = P_{22} + P_{13}$, is described as
\[P_{22}(k) = \sum_{\ell=0}^{3} P_{\ell,22}(k) + P_{22}|_{ZA}(k)\quad \text{and} \quad P_{13}(k) = \sum_{\ell=0}^{3} P_{\ell,13}(k) + P_{13}|_{ZA}(k).\] (B1)

The contribution from the Zel’dovich approximation, $P_{1\text{-loop}}|_{ZA} = P_{22}|_{ZA} + P_{13}|_{ZA}$, is given by
\[P_{1\text{-loop}}|_{ZA}(k) = \frac{1}{2} \int d^{3}q e^{-i\hat{k} \cdot \hat{q}} \left\{ \sum_{\ell=0}^{2} i^{\ell} \mu_{\ell,\text{lin}}(k, q) \mathcal{L}_{\ell}(\hat{k} \cdot \hat{q}) - \tilde{\Sigma}_{\text{lin}}(k) \right\}^{2},\]
\[P_{22}|_{ZA}(k) = \frac{1}{2} \int d^{3}q e^{-i\hat{k} \cdot \hat{q}} \left\{ \sum_{\ell=0}^{2} i^{\ell} \mu_{\ell,\text{lin}}(k, q) \mathcal{L}_{\ell}(\hat{k} \cdot \hat{q}) \right\}^{2}\]
\[= 4\pi \int_{0}^{\infty} dq q^{2} \left\{ j_{0}(kq) \left( \frac{\Sigma_{0,\text{lin}}(k, q)}{2} \right)^{2} + j_{2}(kq) \mu_{2,\text{lin}}(k, q) \right\}\]
\[+ \left( \frac{18}{35} j_{4}(kq) - \frac{2}{7} j_{2}(kq) + \frac{1}{5} j_{0}(kq) \right) \left( \frac{\Sigma_{2,\text{lin}}(k, q)}{2} \right)^{2}\]
\[P_{13}|_{ZA}(k) = -\tilde{\Sigma}_{\text{lin}}(k) \int d^{3}q e^{-i\hat{k} \cdot \hat{q}} \left\{ \sum_{\ell=0}^{2} i^{\ell} \mu_{\ell,\text{lin}}(k, q) \mathcal{L}_{\ell}(\hat{k} \cdot \hat{q}) \right\}\]
\[= -\tilde{\Sigma}_{\text{lin}}(k) \mu_{\text{lin}}(k).\] (B2)

Here, note that $P_{22}|_{ZA}$ is also represented as
\[P_{22}|_{ZA}(k) = \frac{1}{4} \int_{0}^{\infty} dp \mu^{2} \int_{-1}^{1} d\mu \frac{\mu^{2}(1 - r\mu)^{2}}{r^{2}(1 - 2r\mu + r^{2})^{2}} \mu_{\text{lin}}(p) \mu_{\text{lin}}(|k - \mu|).\] (B3)

where $r \equiv p/k$ and $\mu \equiv \hat{k} \cdot \hat{p}$.

The non-linear correlation functions of the displacement vector $\Sigma_{\ell,22}$ and $\Sigma_{\ell,13}$ (Equation (A3)) yield $P_{\ell,22}$ and $P_{\ell,13}$ as follows:
\[P_{\ell,22}(k) = 4\pi \int_{0}^{\infty} dq q^{2} j_{\ell}(kq) \Sigma_{\ell,22}(k, q)\quad \text{and} \quad P_{\ell,13}(k) = 4\pi \int_{0}^{\infty} dq q^{2} j_{\ell}(kq) \Sigma_{\ell,13}(k, q),\] (B4)
Thus, at the high-

The asymptotic behaviors of

Finally, we can show the following well-known results:

and

Finally, we can show the following well-known results:

B.1. Asymptotic Behaviors of $P_{t,22}$ and $P_{t,13}$

For $r = p/k < 1$, $P_{t,22}|_{ZA}$ (Equation (B3)) becomes

Thus, at the high-$k$ limit, the one-loop contributions in SPT from the Zel’dovich approximation cancel out each other,

The asymptotic behaviors of $\Sigma_{t,22}$ and $\Sigma_{t,13}$ (Equations (A9) and (A10)) lead to those of $P_{t,22}$ and $P_{t,13}$: for $r = p/k \ll 1$,
\[ P_{0,13}(k) \rightarrow \frac{8}{189\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p), \]
\[ P_{1,13}(k) \rightarrow -\frac{2}{175\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p), \]
\[ P_{2,13}(k) \rightarrow \frac{16}{189\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p), \]
\[ P_{3,13}(k) \rightarrow \frac{12}{175\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p). \]  
\hspace{1cm} (B11)

Similarly, for \( r = p/k \gg 1 \), Equation (A12) leads to
\[ P_{0,13}(k) \rightarrow \frac{8k^2}{189\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p), \]
\[ P_{1,13}(k) \rightarrow -\frac{2k^2}{25\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p), \]
\[ P_{2,13}(k) \rightarrow \frac{16k^2}{189\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p), \]
\[ P_{3,13}(k) \rightarrow \frac{4k^2}{175\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p). \]  
\hspace{1cm} (B12)

These show
\[ P_{13}(k) = \sum_{\ell=0}^3 P_{\ell,13}(k) + P_{13}|_{ZA}(k) \rightarrow -\frac{61k^2}{630\pi^2} P_{\text{lin}}(k) \int_0^\infty dp P_{\text{lin}}(p) \quad \text{for} \quad p/k \gg 1. \]  
\hspace{1cm} (B13)

APPENDIX C

NON-LINEAR CORRECTIONS IN THE TWO-LOOP SPT

The SPT two-loop solution is described as \( P_{2,\text{loop}} = P_{33b} + P_{33c} + P_{24} + P_{15} \). The Zel’dovich approximation leads to
\[ P_{2,\text{loop}}|_{ZA}(k) = \frac{1}{3!} \int d^3q e^{-ikq} \left\{ \sum_{\ell=0}^2 i^\ell \Sigma_{\ell,\text{lin}}(k, q) L_\ell(\hat{k} \cdot \hat{q}) - \bar{\Sigma}_{\text{lin}}(k) \right\}^3 \]
\[ = P_{33b}|_{ZA}(k) - \bar{\Sigma}_{\text{lin}}(k) P_{22}|_{ZA}(k) + \frac{1}{2} (\bar{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k), \]  
\hspace{1cm} (C1)

where
\[ P_{33b}|_{ZA}(k) = \frac{1}{3!} \int d^3q e^{-ikq} \left\{ \sum_{\ell=0}^2 i^\ell \Sigma_{\ell,\text{lin}}(k, q) L_\ell(\hat{k} \cdot \hat{q}) \right\}^3 \]
\[ = 4\pi \int_0^\infty dq q^2 \left\{ j_0(kq) \frac{(\Sigma_{0,\text{lin}}(k, q))^3}{3!} + j_2(kq) \frac{(\Sigma_{0,\text{lin}}(k, q))^2}{2!} (\Sigma_{2,\text{lin}}(k, q)) \right. \]
\[ + \left. \left( \frac{18}{35} j_2(kq) - \frac{2}{7} j_0(kq) + \frac{2}{7} j_2(kq) \right) \frac{(\Sigma_{2,\text{lin}}(k, q))^2}{2!} (\Sigma_{0,\text{lin}}(k, q)) \right\}. \]  
\hspace{1cm} (C2)

The one-loop LPT leads to
\[ P_{2,\text{loop}|_{\text{LPT,1-loop}}}(k) = P_{2,\text{loop}}|_{ZA}(k) + \int d^3q e^{-ikq} \left\{ \sum_{\ell=0}^3 i^\ell (\Sigma_{\ell,22}(k, q) + \Sigma_{\ell,13}(k, q)) L_\ell(\hat{k} \cdot \hat{q}) - \bar{\Sigma}_{22}(k) + \bar{\Sigma}_{13}(k) \right\} \]
\[ \times \left\{ \sum_{\ell=0}^2 i^\ell \Sigma_{\ell,\text{lin}}(k, q) L_\ell(\hat{k} \cdot \hat{q}) - \bar{\Sigma}_{\text{lin}}(k) \right\} \]
\[ = P_{33b}|_{\text{LPT,1-loop}} + P_{24}|_{\text{LPT,1-loop}} \]
\[ - \bar{\Sigma}_{\text{lin}}(k) P_{13}(k) - \frac{1}{2} (\bar{\Sigma}_{\text{lin}}(k))^2 P_{\text{lin}}(k) - (\bar{\Sigma}_{13}(k) + \bar{\Sigma}_{22}(k)) P_{\text{lin}}(k). \]  
\hspace{1cm} (C3)
where

\[
P_{33b}\mid_{\text{LPT, 1-loop}}(k) = \frac{1}{3!} \int d^3 q e^{-ik \cdot q} \left\{ \sum_{\ell=0}^2 i^{\ell} \Sigma_{\ell, \text{lin}}(k, q) \mathcal{L}_\ell(k \cdot \hat{q}) \right\}^3 \\
+ \int d^3 q e^{-ik \cdot q} \left\{ \sum_{\ell=0}^3 i^{\ell} \Sigma_{\ell,22}(k, q) \mathcal{L}_\ell(k \cdot \hat{q}) \right\} \left\{ \sum_{\ell=0}^2 i^{\ell} \Sigma_{\ell, \text{lin}}(k, q) \mathcal{L}_\ell(k \cdot \hat{q}) \right\} \right\}
\]

\[
= P_{33b}\mid_{\text{Za}}(k) + 4\pi \int_0^\infty dq q^2 \left\{ j_0(kq) \Sigma_{0, \text{lin}}(k, q) \Sigma_{0,22}(k, q) \\
+ j_1(kq) \Sigma_{1,22}(k, q) \Sigma_{0, \text{lin}}(k, q) + j_2(kq) \Sigma_{2,22}(k, q) \Sigma_{0, \text{lin}}(k, q) \\
+ j_3(kq) \Sigma_{3,22}(k, q) \Sigma_{0, \text{lin}}(k, q) + j_2(kq) \Sigma_{2, \text{lin}}(k, q) \Sigma_{0,22}(k, q) \\
+ \left( -\frac{2}{5} j_1(kq) + \frac{3}{5} j_3(kq) \right) \Sigma_{1,22}(k, q) \Sigma_{2, \text{lin}}(k, q) \\
+ \left( \frac{1}{10} j_0(kq) - \frac{1}{7} j_2(kq) + \frac{9}{35} j_4(kq) \right) \Sigma_{2,22}(k, q) \Sigma_{2, \text{lin}}(k, q) \\
+ \left( \frac{9}{35} j_1(kq) - \frac{4}{15} j_3(kq) + \frac{10}{21} j_5(kq) \right) \Sigma_{3,22}(k, q) \Sigma_{2, \text{lin}}(k, q) \right\}.
\]

\section*{APPENDIX D}

\textit{J}^{(N)}

\subsection*{D.1. Zel’dovich Approximation}

The specific forms of \(J^{(n)}\) are given from \(n = 2\) to \(n = 4\) as follows:

\[
J^{(2)}(z, k, q) = \frac{(D^2 \Sigma_{2, \text{lin}}(k, q))^2}{2!} \left\{ \frac{18}{35} j_4(kq) - \frac{2}{7} j_2(kq) + \frac{1}{5} j_0(kq) \right\},
\]

\[
J^{(3)}(z, k, q) = \frac{(D^2 \Sigma_{2, \text{lin}}(k, q))^3}{3!} \left\{ \frac{18}{77} j_6(kq) - \frac{108}{385} j_4(kq) + \frac{3}{7} j_2(kq) - \frac{2}{35} j_0(kq) \right\},
\]

\[
J^{(4)}(z, k, q) = \frac{(D^2 \Sigma_{2, \text{lin}}(k, q))^4}{4!} \left\{ \frac{72}{715} j_8(kq) - \frac{72}{385} j_6(kq) + \frac{1836}{5005} j_4(kq) - \frac{20}{77} j_2(kq) + \frac{3}{35} j_0(kq) \right\}.
\]

\subsection*{D.2. LPT at the One-loop Order}

\[
J^{(2)}(z, k, q) = \frac{1}{21} \left\{ j_0(kq) \left( -\frac{1}{3} \Sigma^2(z, k, q) + \frac{1}{5} \Sigma^2(z, k, q) - \frac{1}{7} \Sigma^2(z, k, q) \right) \\
+ j_1(kq) \left( -\frac{4}{5} \Sigma^2(z, k, q) \Sigma_3(z, k, q) + \frac{18}{35} \Sigma_2(z, k, q) \Sigma_3(z, k, q) \right) \right\}
\]
\[ j_2(kq) \left( \frac{2}{3} \Sigma_1(z, k, q) - \frac{2}{7} \Sigma_2(z, k, q) + \frac{4}{21} \Sigma_3(z, k, q) - \frac{6}{7} \Sigma_4(z, k, q) \right) + j_3(kq) \left( \frac{6}{5} \Sigma_1(z, k, q) \Sigma_2(z, k, q) - \frac{8}{15} \Sigma_2(z, k, q) \Sigma_3(z, k, q) \right) + j_4(kq) \left( \frac{18}{35} \Sigma_2(z, k, q) - \frac{18}{77} \Sigma_2(z, k, q) + \frac{8}{7} \Sigma_1(z, k, q) \Sigma_3(z, k, q) \right) + j_5(kq) \left( \frac{20}{21} \Sigma_2(z, k, q) \Sigma_3(z, k, q) \right) \]
\]

\[ \text{(D2)} \]