Regularizing dual-frame generalized harmonic gauge at null infinity

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Received 1 July 2022
Accepted for publication 16 November 2022
Published 30 December 2022

Abstract
The dual-frame formalism leads to an approach to extend numerical relativity simulations in generalized harmonic gauge (GHG) all the way to null infinity. A major setback is that without care, even simple choices of initial data give rise to logarithmically divergent terms that would result in irregular variables and equations on the compactified domain, which would in turn prevent accurate numerical approximation. It has been shown, however, that a suitable choice of gauge and constraint addition can be used to prevent their appearance. Presently we give a first order symmetric hyperbolic reduction of general relativity in GHG on compactified hyperboloidal slices that exploits this knowledge and eradicates these log-terms at leading orders. Because of their effect on the asymptotic solution space, specific formally singular terms are systematically chosen to remain. Such formally singular terms have been successfully treated numerically in toy models and result in a formulation with the desirable property that unphysical radiation content near infinity is suppressed.

Keywords: generalized harmonic gauge, null infinity, asymptotic solutions

1. Introduction
There is strong motivation coming from gravitational wave astronomy to calculate wave content at future null infinity in asymptotically flat spacetimes. The relevant geometric notions were pioneered long ago [1–5], and so it is no surprise that there are various proposals to do so. The key idea is always to compactify—to draw null infinity to a finite coordinate. Such a proposal requires picking the character of the domain on which the equations should be solved.

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The two possibilities are to use either outgoing characteristic slices [6] or hyperboloidal slices, which are spacelike but nevertheless terminate at future null infinity. For the type of domain chosen, a specific formulation of the field equations must be specified. The use of compactified coordinates necessarily means that large quantities enter the game, and need to be offset against the decay following from asymptotic flatness. Without care the resulting equations are thus too irregular to be meaningfully treated either at the partial differential equation (PDE) or numerical levels. Based on Penrose’s idea of conformal compactification, Hübner [7, 8] and Frauendiener [9] make use of the conformal Einstein field equations, which are explicitly regular, to try to solve this problem. Although this has shown remarkable results [10, 11], we have not yet figured out how to apply it to certain spacetimes of interest and in particular, to compact binary inspiral and merger. Albeit with different applications in mind, a host of different proposals have been considered for the regularization on hyperboloidal slices [12–15], and treating specifically the spherically symmetric case in [16–19]. A recurring complication is the appearance of formally singular terms which need to be treated by application of L’Hôpital’s rule at null infinity.

Yet another proposal, the subject of this paper, is the dual-frame approach [20], which consists of decoupling the coordinates from the tensor basis and carefully choosing each. This allows us to write the Einstein field equations (EFEs) in generalized harmonic gauge (GHG) and then solve them in hyperboloidal coordinates [21–24]. This is dependent on imposing a certain decay of the derivatives of the radial coordinate light speeds at null infinity, and is deeply related to the weak-null condition [22, 25], a condition on the non-linearities of quasi-linear wave equations expected to be a sufficient for small data global existence.

The dual foliation (DF) formulation of general relativity (GR), together with hyperboloidal coordinates can be used to avoid most of the formally singular terms [21]. Although it has been shown that even the simplest choices of initial data give rise to terms that diverge with log $R$ near null infinity, $R$ being a suitably defined radial coordinate. These logs create significant problems in numerical evolutions. However, in [23] the authors studied a simple system of wave equations dubbed good-bad-ugly whose non-linearities are known to mimic the ones present in the EFEs, and found that the logarithmically divergent terms can be explicitly regularized at first order through a non-linear change of variables.

Later work around the same model and a generalization to curved spacetimes resulted in a heuristic method to find asymptotic solutions and, more importantly, to find which terms may contain logs well beyond first order [26]. This idea was subsequently adapted and applied to GR in [27] to study peeling, a requirement on the decay of the components of the Weyl tensor near null infinity which is a necessary condition for the smoothness of null infinity [2]. The authors found that a naïve choice of gauge and constraint addition prevents the Weyl tensor from peeling due to the presence of logs. Furthermore, it was shown that there is special interplay between gauge and constraint addition which can be exploited to turn all of the metric components into non-radiating fields at null infinity, except the two that correspond to gravitational radiation. This idea can be used to prevent the existence of logs up to arbitrarily high order, effectively regularizing the EFEs up to that order.

Armed with this knowledge, in this work we build a formulation of GR in GHG on compactified hyperboloidal slices that provides a set of regularized equations (in a sense to be clarified) that can be implemented numerically. We start in section 2 by outlining our setup and notation. In section 3, convenient gauge source functions and constraint additions are chosen for each of the ten metric components. This involves introducing a gauge driver function that satisfies a wave equation chosen so that it asymptotes in a certain way. This is a subtle but crucial step to eradicate the divergent terms. For this reason we end up with 11 second order differential equations. In section 4 we write these equations in a concise way that is ideal to perform the
first order reduction, which is itself performed in section 5. By considering first derivatives of our fields to be evolved variables in their own right, we find a system of 55 regular first order equations whose principal part can be written in a very compact way. In section 6 we choose a radially compactified coordinate system and hyperboloidal time coordinate, and show how that alters the directional derivatives in the system of equations. Finally on section 7 we present the full system of first order differential equations and discuss the existence of formally singular terms, ways to make most of them not appear in the final equations and the way to deal with the ones that do. The final system has only two types of formally singular terms, both of which can be rendered harmless in the numerics, through the use of L'Hôpital's rule. This amounts to a first order log-free formulation of the EFEs in GHG and we expect it to serve as an alternative to the conformal field equations for the inclusion of null infinity in the computational domain. We wrap up in section 8 with a proof that the final system is symmetric hyperbolic. Concluding remarks are collected in section 9.

2. Framework

To have a self-contained discussion, in this section the general geometric set up to be used is described in broad terms. Nonetheless, for a detailed exposition of the geometric framework to be used we refer the reader to [26, 27]. Latin indices will be used as abstract tensor indices while Greek indices will be used to denote spacetime coordinate indices. \((M, g_{ab})\) will denote a four-dimensional manifold equipped with a Lorentzian metric. On \((M, g_{ab})\) the coordinate system \(X^a = (T, X^i)\) will be asymptotically Cartesian. Let \(\tilde{\nabla}\) denote a flat (and torsionless) covariant derivative with the key property that \(\tilde{\nabla}_a \partial_b^\alpha = 0\) where \(\partial^\alpha = \text{coordinate basis associated to } X^a\) (we use the obvious analogous notation with alternative coordinates). The notion of flat covariant derivative is simply a way to write partial derivatives using abstract index notation as it is used in xAct —see [28]. Since different coordinate systems will be used, we make formally identical definitions for the other coordinate systems.

The coordinates given by \(X^a' = (T', X^i') = (T, R, \theta^A)\), where \(R\) is a radial coordinate defined via \(R^2 = (X^1)^2 + (X^2)^2 + (X^3)^2\) will be called Shell coordinates. Shell coordinates are a generalization of the standard spherical polar coordinates—see for instance [21], however, for some of the calculations discussed in this paper we have used the standard coordinatization of \(S^2\) so that the Shell coordinates are simply spherical polar.

Since the difference between two connections is a tensor, the Christoffel transition tensor of a given pair of covariant derivatives is fixed by the relation,

\[
\Gamma[\nabla, \hat{\nabla}]^b_{a c} = \nabla_a v^b - \hat{\nabla}_a v^b, \tag{1}
\]

where \(v^a\) is a vector. The transition tensor \(\Gamma[\nabla, \hat{\nabla}]^b_{a c}\) is defined analogously.

We will use (1) extensively hence to have a simpler notation we define

\[
\hat{\Gamma}^b_{a c} := \Gamma[\nabla, \hat{\nabla}]^b_{a c}. \tag{2}
\]

2.1. Representation of the metric

Introduce the following vectors

\[
\psi^a = \partial_T^a + C_R^R \partial_R^a, \quad \phi^a = \partial_T^a + C_R^R \partial_R^a. \tag{3}
\]
where $C^R_+$ and $C^R_-$ are fixed by the requirement that $\psi^a$ and $\bar{\psi}^a$ are null vectors with respect to the metric $g_{ab}$. With these conventions $\psi^a$ and $\bar{\psi}^a$ correspond to outgoing and incoming null vectors respectively.

Additionally let

$$\sigma_a = e^{-\varphi}\psi_a, \quad \bar{\sigma}_a = e^{-\varphi}\bar{\psi}_a,$$

(4)

where $\varphi$ is fixed by requiring that

$$\sigma_a \partial_a R = -\bar{\sigma}_a \partial_a R = 1,$$

(5)

so we can write,

$$\sigma_a = -C^R_+ \nabla_a T + \nabla_a R + C^R_- \nabla_a \theta,$$

$$\bar{\sigma}_a = C^R_- \nabla_a T - \nabla_a R + C^R_+ \nabla_a \theta.$$  

(6)

With these elements the inverse metric is written as

$$g_{ab} = -\tau^{-1} e^{\varphi} \sigma_a \sigma_b + \hat{g}_{ab},$$

(7)

where the null vectors satisfy,

$$\sigma_a \psi_b = \bar{\sigma}_a \psi_b = 0, \quad \sigma_a \bar{\psi}_b = \bar{\sigma}_a \psi_b = -\tau,$$

(8)

where we define $\tau := C^R_- - C^R_+$. In (7), the normalization ensures that

$$\hat{g}_{ab} \sigma_b = \hat{g}_{ab} \bar{\sigma}_b = 0,$$

(9)

where $\hat{g}_{ab}$ acts as a projection operator orthogonal to these two covectors. Note that $\hat{g}_{ab}$ is not the inverse induced metric on the surfaces of constant $T$ and $R$, as it is not orthogonal to $\nabla_a T$ or $\nabla_a R$, but rather to $\sigma_a$ and $\bar{\sigma}_a$. The covariant version of metric can be written naturally as,

$$g_{ab} = -2 \tau^{-1} e^{\varphi} \sigma_a \sigma_b + \hat{g}_{ab}.$$  

(10)

Moreover, for the sector $\hat{g}_{ab}$, we introduce the following split,

$$q_{ab} = \hat{R}^2 \hat{g}_{ab}, \quad (q^{-1})^{ab} = \hat{R}^2 \hat{g}_{ab},$$

(11)

with

$$\hat{R}^2 := R^2 e^\epsilon := R^2 \sqrt{|\hat{g}|/|g|}.$$  

(12)

where $\hat{R}$ is known as the areal radius and $|\hat{g}|$ represents the determinant of the standard metric of $S^2$ of radius $R$. In [27] the areal radius is never used, however, it plays a central role in the calculations of this paper. The use of $\hat{R}$ can be thought then as a further refinement of the geometric framework of [27]. Additionally, to simplify the notation we also define $\epsilon := \frac{1}{2} \ln |\hat{g}|$. Observe that the determinant of $q$ is that of the metric of the unit $S^2$ in shell coordinates. Finally, we use a particular parameterization of the angular part of $(q^{-1})^{ab}$, which in shell coordinates reads

$$(q^{-1})^{AB} = \begin{bmatrix} e^{-h_\perp \sin \theta} & \frac{\sin h_\perp \sin h_\perp}{\sin^2 \theta} \\ \frac{\sin h_\perp \sin h_\perp}{\sin^2 \theta} & e^{h_\perp \cos h_\perp} \end{bmatrix}.$$  

(13)

Despite that no linearization is intended, in the last equation we use the symbols $h_\perp$ and $h_\parallel$ to denote the two degrees of freedom of gravitational waves.
In this work we will also use the components of $\hat{g}^a_b$ in mixed form, so we write the non-zero ones here,

$$
\hat{g}^a_A = \frac{C_A}{\tau}, \quad \hat{g}^R_b = \frac{\sigma_+ C^R_+ + \sigma_- C^R_-}{\tau}, \quad \hat{g}^B_b = \delta^B_b.
$$

The metric is thus represented by,

$$
C^R_\pm, \quad C^\pm_A, \quad \varphi, \quad \hat{R}, \quad h_+, \quad h_\times.
$$

2.1.1. The projected covariant derivative. We define the derivative $\partial_A$ for scalar functions $\phi$,

$$
\partial_A \phi := \hat{g}^b_A \partial_b \phi.
$$

We write the inverse conformal metric as $(q^{-1})^{ab}$ to stress that, in general, to raise and lower indices we do not use $q^{ab}$ but $g^{ab}$ instead. Namely, $q^{ab} = g^{ac} g^{bd} \delta_{cd} \neq (q^{-1})^{ab}$. However, for simplicity of the expressions, the index associated with this derivative and only that index will be raised with $g^{ab}$. We wrote the lower index in (16) as an angular one since $\hat{g}^b_a \partial_b \phi \neq 0$ and $\hat{g}^b_a \partial^b \phi \neq 0$. In fact $\partial_A \phi$ can be written as,

$$
\partial_A \phi = \frac{1}{\tau} \left( C^+_A \nabla^+_\psi \phi + C^-_A \nabla^-_\psi \phi \right) + \partial_A \phi.
$$

2.1.2. Representation of the connection. A calculation using

$$
\Gamma^b_{a,c} = \frac{1}{2} \hat{g}^{bd} (\hat{\nabla}_a \hat{g}_{db} + \hat{\nabla}_b \hat{g}_{ad} - \hat{\nabla}_d \hat{g}_{ab}),
$$

gives

$$
\nabla_a C^R_A = -\hat{\Gamma}^a_{\sigma} \psi, \quad \nabla_a C^\pm_A = \hat{\Gamma}^a_{\sigma} \psi,
$$

$$
\nabla_a C^+_A = 2\hat{g}^b_{ad} \hat{\Gamma}^a_{(b\sigma)} - \frac{C^+_A + C^-_A}{\tau} \hat{\Gamma}^a_{\sigma} \psi,
$$

$$
\nabla_a C^-_A = 2\hat{g}^b_{ad} \hat{\Gamma}^a_{(b\sigma)} - \frac{C^+_A + C^-_A}{\tau} \hat{\Gamma}^a_{\sigma} \psi,
$$

$$
\nabla a \psi = \hat{\Gamma}^b_{a,c} (\delta^a_c - \hat{g}^a_c), \quad \nabla a g^{AB} = -2\hat{g}^A_b \hat{g}^B_c \hat{\Gamma}^{(bc)}_a,
$$

$$
\nabla a (\epsilon + \bar{\epsilon}) = \hat{g}^a_B \hat{\Gamma}^B_{b,c}.
$$

These expressions will serve as shorthands to write the EFE in a more compact way. Similarly one can write the derivatives of metric components in terms of $\hat{\Gamma}^b_{a,c}$. For brevity these expressions are omitted. Note that these covariant derivatives are interchangeable with $\hat{\nabla}$ even when they act upon $C^R_\pm$, as $A$ is not a tensorial index but rather a coordinate one.

2.2. The good, the bad, the ugly and the stratified null forms

The term stratified null forms (SNFs) will refer to expressions involving products of terms containing at most one derivative of the evolved fields and having a faster decay than $R^{-2}$
close to $\mathcal{I}^+$. As was shown in [26, 27], this definition is important because it includes every term that cannot possibly interfere with the first order asymptotics of a good-bad-ugly system, which we introduce below. This allows us to categorize terms in an effective way according to their influence on the leading order behavior at null infinity. Throughout this work we will use a calligraphic $\mathcal{N}_\phi$ to denote SNFs, where $\phi$ is the field whose evolution equation contains $\mathcal{N}_\phi$. We introduce the good-bad-ugly model,

$$\Box g = \mathcal{N}_g,$$

$$\Box b = (\nabla_T g)^2 + \mathcal{N}_b,$$

$$\Box u = \frac{2}{3} \nabla_T u + \mathcal{N}_u,$$  \hspace{1cm} (20)

here, $\Box$ is defined as $g^{ab} \nabla_a \nabla_b$ and an analogous definition holds for $\Box$. Good fields are characterized asymptotically by a leading order term with no logs whose decay improves under derivatives along outgoing null curves and does not under derivatives along incoming ones. Bad fields have a leading term proportional to $\log R$ and behave similarly under null derivatives. Ugly fields in general have logs in subleading terms and their decay improves under derivatives of any kind. In its simplest form, this system has been studied extensively as a toy model for the EFEs because the latter can be written in such a way that its leading order non-linearities are mimicked by those present on the RHSs of (20). In fact if we consider wave operators in curved spacetimes and allow the existence of more than one good and one ugly, the EFEs can be written in exactly this form. Even in flat spacetimes and with $\mathcal{N}_\phi = 0$, this system is known to give rise to logarithmically divergent terms at null infinity. However it is shown in [27] that gauge picking and constraint addition can be used to prevent the appearance of those logs up to an arbitrarily high order. This is what we will explore throughout this work.

2.2.1. Ugly equation with $p$. It is convenient to introduce different wave operators that will make expressions more tractable throughout this work. We define the shell wave operator as,

$$\Box_p \phi = g^{ab} \nabla_a \nabla_b \phi.$$  \hspace{1cm} (21)

The ugly equation with a natural number $p$, as defined in [27], can be written in the form,

$$\Box_p u = \frac{2(p+1)}{R} \nabla_T u + \mathcal{N}_u,$$  \hspace{1cm} (22)

but it can also be written in an equivalent, yet more convenient way,

$$\Box_p u = \frac{2(p+1)e^{-\phi}}{\tau R} \nabla_\psi \nabla_\psi u + \mathcal{N}_u,$$  \hspace{1cm} (23)

by simply redefining $\mathcal{N}_u$. The shell wave operator can be expanded as,

$$\Box_p u = -\frac{2e^{-\phi}}{\tau} \nabla_\psi \nabla_\psi u + \frac{2e^{-\phi}}{\tau} (\nabla_\psi \psi)^2 \nabla_\psi u + g^{ab} \nabla_a \nabla_b u$$

$$= -\frac{2e^{-\phi}}{\tau} \nabla_\psi \nabla_\psi u + \frac{2e^{-\phi}}{\tau} \nabla_\psi \psi \nabla_\psi u - \nabla_\psi u + \hat{\Box_p} u$$  \hspace{1cm} (24)

where $\hat{\Box_p} u := g^{ab} \nabla_a \nabla_b u$. For conciseness we define the second order differential operator $\Box_p$ as,

$$\Box_p u := \Box u - \frac{2(p+1)e^{-\phi}}{\tau R} \nabla_\psi \psi \nabla_\psi u$$
\[ - \frac{2e^{-\phi}}{\tau R^{p+1}} \nabla\psi \left[ R^{p+1}\nabla\psi u \right] + \mathring{\mathcal{J}} u \]
\[ + \frac{2e^{-\phi}}{\tau^2} \nabla\psi C^R \left( \nabla\psi u - \nabla\phi u \right). \]  

(25)

Putting (22) and (24) together we get that every ugly equation with a natural number \( p \) can be written in the form,
\[ \mathring{\mathcal{J}}_p u = \mathcal{N}_u. \]  

(26)

2.2.2. Good equation. The general form of a good equation can be obtained straightforwardly from the ugly one (22) by setting \( p = 0 \). Therefore if \( g \) is good, it satisfies the following equation,
\[ - \frac{2e^{-\phi}}{\tau R} \nabla\psi \left[ R^{p} \nabla\psi g \right] + \mathring{\mathcal{J}} g = - \frac{2e^{-\phi}}{\tau^2} \nabla\psi C^R \left( \nabla\psi g - \nabla\phi g \right) + \mathcal{N}_g, \]  

(27)
or more simply,
\[ \mathring{\mathcal{J}}_0 g = \mathcal{N}_g. \]  

(28)

2.2.3. Alternative form. Another way to write the equations is to push as many SNFs as possible to the RHS and keep on the LHS only the terms which are either principal or contribute to leading order. For this we need to rewrite \( \mathring{\mathcal{J}}_0 \),
\[ \mathring{\mathcal{J}}_0 u = \frac{1}{\tau} \left( \frac{C_A}{\tau} p^A C^R_A - p^A C^+_A \right) \nabla\psi u \]
\[ - \frac{1}{\tau} \left( \frac{C_A}{\tau} p^A C^+_A + p^A C^-_A \right) \nabla\phi u + \Delta u, \]  

(29)

where \( C_A := C^+_A + C^-_A \) and \( \Delta u := D^A \partial_A u \). All terms on the RHS are SNFs except the last one, so we write (26) as,
\[ \frac{1}{R^{p+1}} \nabla\psi \left[ R^{p+1}\nabla\psi u \right] - \frac{\tau e^{\phi}}{2} \Delta u = \tilde{\mathcal{N}}_u \]  

(30)

with the RHS being a set of null forms,
\[ \tilde{\mathcal{N}}_u = - \frac{\tau e^{\phi}}{2} \mathcal{N}_u + \frac{e^{\phi}}{2} \left( \frac{C_A}{\tau} p^A C^R_A - p^A C^+_A \right) \nabla\psi u \]
\[ - \frac{e^{\phi}}{2} \left( \frac{C_A}{\tau} p^A C^+_A + p^A C^-_A \right) \nabla\phi u \]
\[ + \frac{1}{\tau} \nabla\psi C^R \left( \nabla\psi u - \nabla\phi u \right). \]  

(31)

Naturally, the version of (30) for the good fields is obtained simply by setting \( p = 0 \).

2.3. Asymptotic flatness at null infinity

Astrophysically relevant objects are modeled in GR via asymptotically flat spacetimes. Although, from a physical perspective the notion of asymptotic flatness is clear: it represents the requirement that the metric asymptotes to the Minkowski spacetime in far regions of
the spacetime, from a mathematical perspective there are several related but not necessarily equivalent definitions. Our working notion of asymptotic flatness near null infinity is that,

$$\gamma|_{I^+} = 0,$$

where we define

\begin{equation}
\varphi = \gamma_1, \quad C^\pm = \pm 1 + \gamma_2^\pm, \quad C_A^\pm = \hat{R}^\pm_{\gamma_3^\pm}, \quad \hat{R}^{-1} = \gamma_4, \quad h_+ = \gamma_5, \quad h_\times = \gamma_6.
\end{equation}

2.4. Initial data

Let $S$ denote the Cauchy surface defined by $T = T_0$, with $T_0$ constant. In [27], we considered initial data with the following fall-off at spatial infinity,

$$\gamma|_{S} = \sum_{n=1}^{\infty} \frac{m_{\gamma,n}}{R^n}, \quad \nabla_T g|_{S} = O_S(\hat{R}^{-m}),$$

where $m_{\gamma,n}$ denote functions that only depend on the angular coordinates $\theta^A$. Here the subscript $S$ is placed to stress that this decay is assumed only on the initial hypersurface and not close to null infinity. Although this data is not the most general one, this class of initial data is large enough to admit non-vanishing ADM mass and linear momentum. Moreover, it includes initial data of physically relevant spacetimes. In practice for the results in this paper this assumption could be relaxed to,

$$\gamma|_{S} = \frac{m_{\gamma,1}}{R} + O_S(\hat{R}^{-2}), \quad \nabla_T g|_{S} = O_S(\hat{R}^{-2}),$$

without further change.

2.4.1. Permissible coordinate changes. There is not one universal definition of asymptotic flatness at spatial infinity, rather there is an interplay between the field equations, the physics under consideration, and the rate at which the metric becomes flat. A weak definition thereof is,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O_p(R^{-1}),$$

where $\eta_{\alpha\beta}$ is the Minkowski metric, $p \geq 1$ and $O_p(R^{-m})$ means that its partial derivatives $\partial^n g_{\alpha\beta}$ of order $n$ decay as $R^{-n-m}$ for all $n = 1, \ldots, p$. Note that our requirements (34) satisfy this definition of asymptotic flatness. It has been shown in [29] that a coordinate transformation is permissible, meaning it preserves (36), if and only if it is asymptotically a Poincaré transformation in the following sense,

$$X^\alpha = \Lambda^\alpha_{\alpha\beta} x^\beta + c^\alpha(\theta, \phi) + O_{p+1}(\hat{r}^{-1}),$$

where $\Lambda^\alpha_{\alpha\beta}$ is a Lorenz transformation and $x^\alpha$ is another asymptotically Cartesian coordinate system with a radial coordinate $\hat{r}$ built in the usual way. It is interesting to see how a permissible coordinate change affects our initial data. In many spacetimes of interest, the functions $m_{\gamma,1} = m$, where $m$ is a constant, e.g. the Kerr metric in Boyer–Lindquist coordinates. If we boost this metric with a generic permissible coordinate change, the new functions $m_{\gamma,1}$ will pick up dependencies on the angles at leading order in $\hat{R}^{-1}$, that is, $m_{\gamma,1} = m_{\gamma,1}(m, \theta, \phi)$. Note that the initial data that one gets after such a boost, still satisfies the requirements (36) and hence can be used evolved numerically with the method we will present throughout this work.
3. Gauge and constraint addition

In [27], the vacuum EFEs were derived as a set of non-linear wave equations satisfied by the variables (15). We present them here as concisely as possible while still keeping the freedom to choose the gauge and the constraint addition in each of the equations,

\[
\begin{align*}
\Box \psi^R &= e^{-\varphi} \tilde{R} \psi + (\Gamma) \psi, \\
\Box \sigma^R &= e^{-\varphi} \tilde{R} \sigma + (\Gamma) \sigma, \\
\Box \varphi &= -\frac{2e^{-\varphi}}{\tau} \tilde{R} \psi + (\Gamma) \psi, \\
\Box C_A^+ &= -e^{-\varphi} \tilde{R} \psi \sigma + \frac{C_A}{\tau} c R^+ + (\Gamma) \psi A, \\
\Box C_A^- &= -e^{-\varphi} \tilde{R} \psi A - \frac{C_A}{\tau} c R^- + (\Gamma) \psi A, \\
\Box (\epsilon + \xi) &= -\tilde{R} + (\Gamma) \theta, \\
\Box h_+ &= -2 \tilde{g}^{\theta \theta} \tilde{R} \theta + (\Gamma) \theta^\theta, \\
\Box h_\times &= -2 \tilde{g}^{\theta \theta} \coth h_\times \tilde{R} \theta + (\Gamma) \theta^\theta.
\end{align*}
\]

Where the different components of \((\Gamma)_{ab}\) conceal the complicated non-linearities,

\[
\begin{align*}
(\Gamma)\psi &:= \frac{2}{\tau} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^a \psi - e^{-\varphi} \tilde{\Gamma}_{\psi} b \psi a \psi b - 2 \tilde{g}_{ab} \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^{ab}, \\
(\Gamma)\psi \psi &:= -\frac{2}{\tau} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^a \psi + e^{-\varphi} \tilde{\Gamma}_{\psi} b \psi a \psi b \\
&+ 2 \tilde{g}_{ab} \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^{ab}, \\
(\Gamma) \psi \varphi &:= \frac{2}{\tau} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^a \varphi + \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^a \psi \varphi - 2 \tilde{g}_{ab} \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^{ab}, \\
(\Gamma) \psi \sigma &:= \frac{2}{\tau} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^a \sigma - \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^a \psi \sigma \\
&+ \tilde{g}_{ab} \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^{ab}, \\
(\Gamma) \sigma A &:= \frac{2}{\tau} \tilde{\Gamma}_{\sigma} A \tilde{\Gamma}^a \sigma a - \tilde{\Gamma}_{\psi} \sigma a - \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^a \sigma \\
&- 2 \tilde{g}_{ab} \tilde{\Gamma}_{\sigma} A \tilde{\Gamma}^{ab}, \\
(\Gamma) \theta A &:= \frac{2}{\tau} \tilde{\Gamma}_{\varphi} \psi a \tilde{\Gamma}^a \sigma a - \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^a \varphi - \tilde{g}_{ab} \tilde{\Gamma}_{\psi} \sigma a \tilde{\Gamma}^{ab} \\
&- 2 \tilde{g}_{ab} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^{ab}, \\
(\Gamma) \theta^\theta &:= -\Box (\epsilon + \xi) - \tilde{g}^{\theta \theta} \frac{1 + \cos \theta^2}{\sin \theta^2} - \tilde{g}^{\theta \theta} \tilde{g}^{\theta \psi} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^{ab} \\
&+ 2 \cot \theta \tilde{g}^{\theta \theta} \tilde{g}^{\theta \psi} \tilde{\Gamma}_{\psi} a \tilde{\Gamma}^{ac} + \frac{4}{\tilde{g}^{\theta \theta}} \tilde{\Gamma}_{\psi} \psi a \tilde{\Gamma}^{ab} - \cot \theta \tilde{g}^{\theta \theta} \\
&+ \frac{1}{\cosh h_\times} (\Box \cosh h_\times + 2 \Box \cosh h_\times \Box h_+ + \Box h_+ \Box \cosh h_\times) \\
&+ \Box h_+ \Box \cosh h_\times.
\end{align*}
\]
\[
(\Gamma^{\theta \phi})^{ab} := -g_{\theta b} \coth h_x \left[ 2 \hat{\Gamma}^{a}_b \hat{\Gamma}^{c}_{\theta b} + g^{\theta \phi} \Box (\epsilon + \tilde{\epsilon}) \right] - 4g^{b}_{\phi c} \hat{\Gamma}^{(\theta \phi)}_{a} \hat{\Gamma}^{c}_{b} + g^{b \phi d}_{c} e^{\phi}_{e} \hat{\Gamma}^{d f}_{b} \hat{\Gamma}^{f}_{a} - g^{\theta \phi} \nabla_{a} h_x \nabla_{b} h_x \right],
\]

and \( \mathcal{R}_{ab} \) is an auxiliary tensor we introduce simply to make expressions shorter,
\[
\mathcal{R}_{ab} := R_{ab} + \nabla_{(a} f_{b)} + W_{ab},
\]
\( \mathcal{R}_{ab} \) being the reduced Ricci tensor defined as,
\[
\mathcal{R}_{ab} := R_{ab} - \nabla_{(a} Z_{b)} - W_{ab},
\]
where \( F^a \) are the gauge source functions and \( W_{ab} \) represents a generic constraint addition, by which we mean any expression homogeneous in the constraints. We do not expect the reader to be enlightened by these long expressions. However, we choose to present them nonetheless for the sake of completeness and in order to highlight the fact that the information contained in them, together with (19) and the help of computer algebra, provides a fairly quick way to write the EFEs in terms of derivatives of the ten metric functions. We will only be working in vacuum, so all the components of the reduced Ricci tensor \( \mathcal{R}_{ab} \) are zero. In order to get our final equations, we only have to pick the gauge source functions \( F^a \) and the constraint addition \( W_{ab} \).

We have seen in [27] that the special interplay between gauge choice and constraint addition can be exploited to turn the EFEs into a system of eight ugly equations with a natural number \( p \) and two good equations. These choices are highly non-unique as only the first few orders of the solutions to these equations are concerned. Here we want to find a choice that ensures this number of uglies and goods, while preventing the appearance of logs up to order \( p \), and then show the final form of the EFEs in the most convenient way.

3.1. Constraints

Up to gauge fixing, the constraints \( Z^a := \hat{\Gamma}^a + F^a \) are defined by the following,
\[
\hat{\Gamma}^\sigma := \frac{2e^{-\varphi}}{\tau} \nabla_{\psi} C_{\psi}^\sigma - \frac{2e^{-\varphi}}{\tau} \nabla_{\psi} R - \frac{C_{\psi}^\sigma}{\tau} \partial^A C_{\psi}^A + \partial^A C_{\psi}^A - \nabla_{\psi} C_{\psi}^\sigma - \frac{C_{\psi}^\sigma}{\tau} \partial^A C_{\psi}^A + \nabla_{\psi} C_{\psi}^\sigma - \frac{C_{\psi}^\sigma}{\tau} \partial^A C_{\psi}^A,
\]
\[
\hat{\Gamma}^a := \frac{2e^{-\varphi}}{R} \nabla_{\psi} R - \frac{2e^{-\varphi}}{R} \nabla_{\psi} C_{\psi}^R - \frac{C_{\psi}^R}{\tau} \partial^A C_{\psi}^A + \nabla_{\psi} C_{\psi}^R - \frac{C_{\psi}^R}{\tau} \partial^A C_{\psi}^A,
\]
\[
\hat{\Gamma}^A := \hat{R} \left( \hat{g}^{AB} \hat{q}_{BC} - \frac{1}{2} \hat{g}^{BC} \hat{q}_{AB} \right) + \partial^A \varphi
\]
\[
- \frac{\hat{g}^{AB}}{\tau} \left( \nabla_{\psi} C_{\psi}^A + \nabla_{\psi} C_{\psi}^B - \frac{C_{\psi}^R}{\tau} \nabla_{\psi} C_{\psi}^R - \frac{C_{\psi}^R}{\tau} \nabla_{\psi} C_{\psi}^R \right),
\]

where the only terms that contribute to leading order are the ones proportional to bad derivatives of \( C_{\psi}^R, R \) and \( C_{\psi}^+ \), respectively. This means that we can write,
\[
Z^\sigma \simeq \nabla_{\psi} C_{\psi}^R,
\]
\[
Z^a \simeq -\frac{2}{R} \nabla_{\psi} R,
\]
\[
Z^A \simeq -\frac{\hat{g}_{AB}^{\psi}}{2} \nabla_{\psi} C_{\psi}^+.
\]

The fact that, to leading order, each constraint is essentially a bad derivative of a metric function will allow us to use constraint addition for two separate purposes. The first is to turn four out
of ten equations into uglies with a natural number $p$, which is done by adding constraints in such a way that they will contribute to second order in $R^{-1}$. Later in this work, we will see that bad derivatives of $C_R^+$, $R$ and $C_A^+$ give rise to formally singular terms that create problems in numerical implementations. We will see that these terms appear at third order in $R^{-1}$ in some of the equations and that we can get rid of them by constraint addition. For this reason, we will separate $W_{ab}$ into a leading and a subleading contributions, $W_{ab}^{(1)}$ and $W_{ab}^{(2)}$, respectively,

$$W_{ab} = W_{ab}^{(1)} + W_{ab}^{(2)}. \quad (44)$$

In this section, because we are only worried about establishing the leading asymptotics of the metric fields and not the existence of formally singular terms, we will only fix $W_{ab}^{(1)}$ and leave $W_{ab}^{(2)}$ free for the moment.

### 3.2. Gauge

We introduce the Cartesian harmonic gauge defined by $F^a = \tilde{F}^a$ with,

$$\tilde{F}^a = g^{bc} \Gamma[\tilde{\nabla}, \tilde{\nabla}] b^c, \quad (45)$$

where $\Gamma[\tilde{\nabla}, \tilde{\nabla}] b^c$ are given functions of the coordinates. For conciseness we will use explicitly standard spherical polar coordinates as shell coordinates but in the code a different choice could be needed. For this reason, $\Gamma[\tilde{\nabla}, \tilde{\nabla}] b^c$ are just the Christoffel symbols of Minkowski spacetime in polar coordinates. Explicitly,

$$\tilde{F}^\sigma = \frac{2e^{-\epsilon}}{R} \cosh h_+ \cosh h_\times + C_R^+ \tilde{F}^A,$$

$$\tilde{F}^\varepsilon = -\frac{2e^{-\epsilon}}{R} \cosh h_+ \cosh h_\times + C_A^- \tilde{F}^A,$$

$$\tilde{F}^\theta = \frac{\cot \theta}{R^2} e^{h_+} \cosh h_\times - \frac{2}{R} g^{\theta \phi},$$

$$\tilde{F}^\phi = -\frac{2 \cot \theta}{\sin \theta R^2} \sinh h_\times - \frac{2}{R} g^{\phi \theta}. \quad (46)$$

We choose a gauge that is Cartesian harmonic to leading order, with a higher order correction $\tilde{F}^a$ which we specify later,

$$F^\sigma = \frac{2}{R} \cosh h_+ \cosh h_\times + C_R^+ \tilde{F}^A + \tilde{F}^\sigma,$$

$$F^\varepsilon = -\frac{2}{R} \cosh h_+ \cosh h_\times + C_A^- \tilde{F}^A + \tilde{F}^\varepsilon,$$

$$F^\theta = \frac{\cot \theta}{R^2} e^{h_+} \cosh h_\times - \frac{2}{R} g^{\theta \phi} + \tilde{F}^\theta,$$

$$F^\phi = -\frac{2 \cot \theta}{\sin \theta R^2} \sinh h_\times - \frac{2}{R} g^{\phi \theta} + \tilde{F}^\phi. \quad (47)$$

Notice that we in its present form, (47) does not have any explicit $\varepsilon$ or $R$. We can do this because it does not require changing the fact that the gauge is Cartesian harmonic to leading order and we choose to do it because the final expressions turn out to be simpler if these objects are replaced by $R$. 11
3.3. Ugly equations with $p$

In order to turn a wave equation into an ugly, we only need to consider the leading order contributions of the various terms. In total we have four constraints and four gauge source functions we are free to add and choose, respectively. Each of these can be used to turn one metric function into an ugly. In other words, this freedom allows us to write eight out of ten equations as uglies. In the following we explain how to do this for each of them.

3.3.1. Radial coordinate light speed $C^R_+$. The equation for $C^R_+$ can be written as,

$$\square C^R_+ = e^{-\varphi} \psi^a \nabla_\psi F_{a} + e^{-\varphi} W_{\psi \psi} + (\Gamma \Gamma)_{\psi \psi}.$$  \hspace{1cm} (48)

Plugging (47) in the first term on the RHS we get that,

$$e^{-\varphi} \psi^a \nabla_\psi F_{a} = - \left( p \nabla_\psi \dot{R} - \frac{1}{\tau} \nabla_\psi C^R_+ \right) + o^+(R^{-2}).$$  \hspace{1cm} (49)

The notation $f = o^+(h)$ means

$$\exists \epsilon > 0 : \lim_{R \to \infty} \frac{f}{hR^{-\epsilon}} = 0.$$  \hspace{1cm} (50)

or in other words, fall off faster than $f = o(h)$; more precisely, $o^+(h) = o(hR^{-\epsilon})$. Whenever it is unambiguous, we will use $\simeq$ to denote the presence of these higher order terms because it makes expressions shorter. Using computer algebra, it is possible to show that,

$$(\Gamma \Gamma)_{\psi \psi} = \frac{2}{\tau^2} (\nabla_\psi C^R_+)^2 + \frac{2}{R^2} (\nabla_\psi \dot{R})^2 + o^+(R^{-2}).$$  \hspace{1cm} (51)

As was said above, the constraint $Z^\alpha$ is essentially a bad derivative of $C^R_+$ to leading order and hence it can be used to make sure that (48) is an ugly by changing the RHS to satisfy (23). With this in mind we choose,

$$W_{\psi \psi}^{(1)} = Z^\alpha e^\varphi \left( p \nabla_\psi \dot{R} - \frac{1}{\tau} \nabla_\psi C^R_+ \right),$$  \hspace{1cm} (52)

so that the wave equation for $C^R_+$ can be written as,

$$\square C^R_+ = \frac{2(p + 1) e^{-\varphi}}{\tau R} \nabla_\psi \dot{R} \nabla_\psi C^R_+ + N_{C^R_+}.$$  \hspace{1cm} (53)

This means that $C^R_+$ is now an ugly with natural number $p$.

3.3.2. Radial coordinate light speed $C^R_-$. The equation for $C^R_-$ is the following,

$$\square C^R_- = - e^{-\varphi} \psi^a \nabla_\psi F_{a} - e^{-\varphi} W_{\psi \psi} + (\Gamma \Gamma)_{\psi \psi}.$$  \hspace{1cm} (54)

where the first term on the RHS, to leading order, behaves as,

$$- e^{-\varphi} \psi^a \nabla_\psi F_{a} \simeq - \nabla_\psi F^2 + \frac{1}{R} \nabla_\psi C^R_- + \frac{2}{R^2},$$  \hspace{1cm} (55)

and the second as,

$$(\Gamma \Gamma)_{\psi \psi} \simeq - \left( \nabla_\psi h_+ \right)^2 - \frac{1}{2} \left( \nabla_\psi h_+ \right)^2 - \frac{2}{R^2}.$$  \hspace{1cm} (56)
Because none of the constraints contains a bad derivative of \( C_R \), we cannot use constraint addition to turn this variable into an ugly. So we choose,

\[
W^{(1)}_{\psi\psi} = 0. 
\] (57)

However, a bad derivative of \( F^\sigma \) contributes to leading order, so we can choose the gauge in order to get the asymptotic behavior that we are looking for. In order to get (23), we need to make sure that our gauge choice satisfies the following condition,

\[
\nabla_\psi F^\sigma \simeq \frac{1}{2} (\nabla_\psi h_+)^2 + \frac{1}{2} (\nabla_\psi h_\times)^2 - \frac{2p}{R} \nabla_\psi C_R. 
\] (58)

We do that by separating \( F^\sigma \) in two parts,

\[
\tilde{F}^\sigma = \frac{1}{R} \tilde{F}^\sigma_T + \frac{p}{R} \left( 1 + C_R \right),
\] (59)

where \( \tilde{F}^\sigma_T \) is a function we will call gauge driver since its purpose will be to drive the asymptotics of one of the gauge source functions to a preassigned value at null infinity. The condition (58) implies that,

\[
\frac{1}{R} \nabla_\psi \tilde{F}^\sigma_T \simeq \frac{1}{2} (\nabla_\psi h_+)^2 + \frac{1}{2} (\nabla_\psi h_\times)^2,
\] (60)

which cannot be simply integrated. As other derivatives of the function \( \tilde{F}^\sigma_T \) will show up in other equations, it is possible that these will interfere with the principal part of those equations potentially spoiling hyperbolicity. The way to get around this is to treat \( \tilde{F}^\sigma_T \) as another evolved variable and choosing a wave equation for it to satisfy that forces the asymptotic condition (60). As this equation will not be an ugly, we dedicate to its treatment its own subsection at the end of this section.

3.3.3. Determinant of the metric in the T-R plane \( \varphi \). The equation for \( \varphi \) is,

\[
\Box \varphi = \frac{2e^{-\varphi}}{\tau} \psi^{(a\psi b)} (\nabla_a F_b + W_{ab}) + (\Gamma\Gamma)_{\psi\psi}. 
\] (61)

Since no constraint contains any bad derivative of \( \varphi \), we choose,

\[
W^{(1)}_{\psi\psi} = W^{(1)}_{\varphi\varphi} = 0,
\] (62)

and we must rely on gauge fixing to turn \( \varphi \) into an ugly. To leading order we have,

\[
\frac{2e^{-\varphi}}{\tau} \psi^{(a\psi b)} \nabla_a F_b \simeq \frac{1}{2} \nabla_\psi F^\sigma + \frac{1}{R} \nabla_\psi \varphi + \frac{2}{R^2},
\] (63)

and,

\[
(\Gamma\Gamma)_{\psi\psi} \simeq -\frac{2}{R}. 
\] (64)

In order to obtain (23), we must only guarantee that \( F^\sigma \) satisfies,

\[
\nabla_\psi F^\sigma \simeq \frac{2p}{R} \nabla_\psi \varphi,
\] (65)

so we make the explicit choice,

\[
\tilde{F}^\sigma = \frac{2p}{R} (e^\varphi - 1). 
\] (66)
Finally, we can write,
\[ \Box \phi = \frac{2(p + 1)e^{-\phi}}{\tau R} \nabla_\psi R \nabla_\psi \phi + \mathcal{N}_\phi. \] (67)

3.3.4. Angular coordinate light speeds \( \mathcal{C}^+_A \). The variables \( \mathcal{C}^+_A \) with wave equations,
\[ \Box \mathcal{C}^+_A = -2e^{-\phi} g^{(a} \psi^{b)} \nabla_a F_{b} + (\Gamma \Gamma)_{\psi A}, \] (68)

need to be rescaled in order to be classified as uglies, and the same goes for \( \mathcal{C}^-_A \). That is done by defining the new variables,
\[ \mathcal{R} \hat{\mathcal{C}}^\pm_A = \mathcal{C}^\pm_A, \] (69)

and requiring that \( \mathcal{R} \hat{\mathcal{C}}^\pm_A \) behave as ugly fields instead. This rescaling yields,
\[ \Box \hat{\mathcal{C}}^\pm_A \simeq \mathcal{R} \Box \hat{\mathcal{C}}^\pm_A - \frac{2e^{-\phi}}{\tau} \nabla_\psi R \nabla_\psi \hat{\mathcal{C}}^\pm_A. \] (70)

Also, it can be seen that,
\[ -2e^{-\phi} g^{(a} \psi^{b)} \nabla_a F_{b} \simeq 2 \nabla_\psi \hat{\mathcal{C}}^+_A, \] (71)

so a choice of constraint addition we can use to turn \( \hat{\mathcal{C}}^+_A \) into an ugly is,
\[ W^{(1)}_{\psi A} = 2 \mathcal{Z}^A \left( p \nabla_\psi R - 2e^\phi \right). \] (72)

Putting all this together we find,
\[ \Box \hat{\mathcal{C}}^+_A = \frac{2(p + 1)e^{-\phi}}{\tau R} \nabla_\psi R \nabla_\psi \hat{\mathcal{C}}^+_A + \mathcal{N}_\psi \hat{\mathcal{C}}^+_A. \] (73)

3.3.5. Angular coordinate light speeds \( \mathcal{C}^-_A \). The wave equation satisfied by \( \mathcal{C}^-_A \) is,
\[ \Box \mathcal{C}^-_A = -2e^{-\phi} g^{(a} \psi^{b)} \nabla_a F_{b} + (\Gamma \Gamma)_{\psi A}. \] (74)

Rescaling \( \mathcal{C}^-_A \) as in (69) yields (70). Since none of the constraints contain bad derivatives of \( \hat{\mathcal{C}}^-_A \), we set,
\[ g^{(a} \psi^{b)} W^{(1)}_{(\psi)\psi} = 0, \] (75)

and we must now use our last 2 free gauge source functions \( F_A \). We have that,
\[ -2e^{-\phi} g^{(a} \psi^{b)} \nabla_a F_{b} \simeq 2 \nabla_\psi \hat{\mathcal{C}}^-_A - \mathcal{R} \partial_\psi \mathcal{F}^A \] (76)
\[ + \cot \theta \delta_{\phi A}(2 - \nabla_\psi h_+) + 2 \cos \theta \delta_{\phi A} \nabla_\psi h_x, \]

and,
\[ (\Gamma \Gamma)_{\psi A} \simeq - \cot \theta \delta_{\phi A}(2 - \nabla_\psi h_+) - 2 \cos \theta \delta_{\phi A} \nabla_\psi h_x. \] (77)

So the condition we need \( F^A \) to satisfy is,
\[ \mathcal{R} \partial_\psi \mathcal{F}^A \simeq (1 - p) \nabla_\psi \hat{\mathcal{C}}^-_A, \] (78)

and one possible choice is,
\[ \mathcal{F}^A = \frac{1 - p}{\mathcal{R}^2} \hat{\mathcal{C}}^-_A. \] (79)
So we can write,
\[ \Box \hat{C}^-_A = \frac{2(p + 1)e^{-\varphi}}{\tau R} \nabla_\psi \hat{R} \nabla_\psi \hat{C}^-_A + N^-_A. \]  
(80)

3.4. Ugly equation with \( p = 1 \)

3.4.1. Determinant of the metric in the \( \theta - \phi \) plane \( \varepsilon \) or inverse areal radius \( \hat{R}^{-1} \). There is only one more metric function that we can possibly hope to turn into an ugly field with natural number \( p \) with this method. The field \( \varepsilon \) satisfies the following equation,
\[ \Box (\varepsilon + \hat{\varepsilon}) = -g^{ab} \nabla_a F_b - \mathcal{W} + (\mathcal{P} \Gamma), \]  
(81)
and we want to deal with it a little differently. Instead of evolving \( \varepsilon \), we want to evolve \( \hat{R}^{-1} \).

Notice that this new variable is necessarily an ugly, since both good and bad derivatives of it must improve, even if \( \varepsilon \) is just a good field. To see this we can expand the exponential in the definition of \( \hat{R}^{-1} \),
\[ \hat{R}^{-1} = \frac{1}{R} - \frac{\epsilon_{1,0}(\psi^*)}{2R^2} + \alpha^+(R^{-2}), \]  
(82)
where \( \epsilon_{1,0}(\psi^*) \) is a function that does not vary along outgoing null curves. To leading order, the quantities \( \Box (\varepsilon + \hat{\varepsilon}) \) and \( \Box \hat{R}^{-1} \) are related in the following way,
\[ \Box (\varepsilon + \hat{\varepsilon}) \simeq 2R \hat{\Box} \hat{R}^{-1} - \frac{4e^{-\varphi}}{\tau} \nabla_\psi \hat{R} \nabla_\psi \hat{R}^{-1}. \]  
(83)

Also, we have that,
\[ -g^{ab} \nabla_a F_b \simeq -\frac{2R}{\tau} F^a \nabla_\psi \hat{R}^{-1}, \]  
(84)
and,
\[ (\mathcal{P} \Gamma) = \alpha^+(R^{-1}). \]  
(85)

Setting the constraint addition \( \mathcal{W} \) to be,
\[ \mathcal{W} = \frac{\hat{R}^{(1)}}{\tau R} \nabla_\psi \hat{R}, \]  
(86)
we get that \( \hat{R}^{-1} \) is an ugly with natural number \( \hat{p} \),
\[ \hat{\Box} \hat{R}^{-1} = \frac{2(p + 1)e^{-\varphi}}{\tau R} \nabla_\psi \hat{R} \nabla_\psi \hat{R}^{-1} + \hat{N}_\hat{R}^{-1}, \]  
(87)
where we introduced the natural number \( \hat{p} \) instead of \( p \) because we want \( \hat{R}^{-1} \) to behave differently from the other fields.

3.4.2. If \( \varepsilon \) is a good, then the SNFs of \( \hat{R}^{-1} \) are \( O(R^{-4}) \). This choice of constraint addition yields a wave equation for \( \varepsilon \) of the type,
\[ \hat{\Box} \varepsilon = \frac{2\hat{p}e^{-\varphi}}{\tau R} \nabla_\psi \hat{R} \nabla_\psi \varepsilon + \hat{N}_\varepsilon. \]  
(88)

Plugging (88) in (83) then gives,
\[ \hat{\Box} \hat{R}^{-1} = \frac{2(\hat{p} + 1)e^{-\varphi}}{\tau R} \nabla_\psi \hat{R} \nabla_\psi \hat{R}^{-1} + \frac{\hat{p} - 1}{R^3} + \alpha^+(R^{-3}), \]  
(89)
or equivalently,

$$\mathcal{N}_{R^{-1}} \simeq \frac{\hat{p} - 1}{R^3}. \quad (90)$$

This means that if we choose $\hat{p} = 1$, we can ensure that the SNFs in the wave equation for $\hat{R}^{-1}$ are one order better than what would normally be expected, $o^+(R^{-3})$ rather than $o^+(R^{-2})$. According to (88), this choice makes $\varepsilon$ become a good field, while $\hat{R}^{-1}$ remains an ugly by construction, see (82). It is worth pausing here a moment to analyze this choice from the point of view of the peeling property. The strategy employed in [27] to ensure that the components of the Weyl tensor peel was to make eight metric functions be ugly with a natural number $\bar{p}$ that guarantees no logs are generated up to that order. Here, by choosing $\varepsilon$ to be a good, we are forced to have $\hat{R}^{-1}$ be an ugly with $\bar{p} = 1$, so by what we know about the subleading asymptotics of generic ugly fields, we could expect logarithmically divergent terms to appear from second order near null infinity. However $\hat{R}^{-1}$ is not a generic ugly. It is built purely from $\varepsilon$. Now we also know that good fields cannot generate logs, they can merely inherit them. So, despite $\hat{R}^{-1}$ being an ugly with a fixed $\bar{p} = 1$, it cannot generate logs at any order down in the expansion. In other words, for $p$ sufficiently large ($p = 7$, see [27]), the Weyl tensor necessarily still peels (with the strict assumptions (34) on initial data).

### 3.5. Good equations

3.5.1. $h_+$ and $h_\times$ . The wave equations satisfied by $h_+$ and $h_\times$ are,

$$\square h_+ = -\frac{2}{g^{\theta\theta}} [(\nabla F)^{\theta\theta} + W^{\theta\theta}] + (\Gamma^{\theta})^{\theta\theta},$$  
$$\square h_\times = -2g^{\theta\phi} \coth h_\times [(\nabla F)^{\theta\phi} + W^{\theta\phi}] + (\Gamma^{\phi})^{\theta\phi}. \quad (91)$$

We can see that,

$$-\frac{2}{g^{\theta\theta}}(\nabla F)^{\theta\theta} + (\Gamma^{\theta})^{\theta\theta} = \frac{1}{R} \nabla_\psi h_+ + o^+(R^{-2}),$$  
$$-2g^{\theta\phi} \coth h_\times (\nabla F)^{\theta\phi} + (\Gamma^{\phi})^{\theta\phi} = \frac{1}{R} \nabla_\psi h_\times + o^+(R^{-2}), \quad (92)$$

so, according to (28) we do not need to add constraints to get the behavior we are looking for. This means that,

$$W^{(1)\theta\theta} = W^{(1)\theta\phi} = 0, \quad (93)$$

and we can write both equations as standard good equations,

$$\square h_+ = \frac{2e^{-\varphi}}{\tau R} \nabla_\psi R \nabla_\psi h_+ + \mathcal{N}_{h_+},$$  
$$\square h_\times = \frac{2e^{-\varphi}}{\tau R} \nabla_\psi R \nabla_\psi h_\times + \mathcal{N}_{h_\times}. \quad (94)$$

### 3.6. Gauge driver

3.6.1. Gauge driver $\tilde{F}^\Box_\sigma$. To make sure that hyperbolicity remains untouched while the gauge source functions satisfy (58) we need to treat $\tilde{F}^\Box_\sigma$ as an eleventh evolved variable and require it so satisfy a wave equation that forces the asymptotic behavior (58). Let us then consider the
equation,
\[ \Box \hat{F}_\sigma^2 = \frac{2e^{-\phi}}{\tau R} \left[ (p + 1) \nabla_\psi R \nabla_\psi \hat{F}_\sigma^2 - \frac{p}{R} \mathcal{H} \right], \tag{95} \]
where \( 2\mathcal{H}/R^2 := (\nabla_\psi h_\perp)^2 + (\nabla_\psi h_\times)^2 \). We want to show that the wave equation (95) makes \( \hat{F}_\sigma^2 \) satisfy (58), so for now we are only interested in the leading order terms. Therefore, we can write
\[ \nabla_\psi \nabla_\psi f_1 \simeq -\frac{p + 1}{R} \left[ \nabla_\psi f_1 - \mathcal{H} \right], \tag{96} \]
where \( f_1 := \hat{F}_\sigma^2 \). Adding the SNF \( \nabla_\psi \mathcal{H} \) to either side, which we can do because SNFs do not influence the leading order behavior by definition, we get an ODE for the leading behavior of \( \nabla_\psi f_1 - \mathcal{H} \),
\[ \nabla_\psi \left[ \nabla_\psi f_1 - \mathcal{H} \right] \simeq -\frac{p + 1}{R} \left[ \nabla_\psi f_1 - \mathcal{H} \right]. \tag{97} \]
This can be integrated to give the asymptotics we are looking for,
\[ \nabla_\psi f_1 - \mathcal{H} \simeq \frac{\alpha (\psi^*)}{R^p}, \tag{98} \]
as long as \( p \) is a natural number. The wave equation satisfied by the gauge driver in [27] is not exactly the same as this one, but they coincide asymptotically. Therefore, the proof that the variable \( \hat{F}_\sigma^2 \) cannot introduce any logs in the system up to order \( p \) goes through in exactly the same way and we repeat it here. Note that we have assumed that the gauge driver satisfies the same requirements on initial data than the rest of the metric fields namely (34).

4. The Einstein field equations in second order form

Based on the discussion from the last section, we summarize the choices we have made for the constraint additions and gauge source functions. Then we write the resulting second order equations explicitly as concisely as possible. The choice of constraint addition encoded by the tensor \( W_{ab}^{(1)} \) is the following,
\[ W_{\psi \psi}^{(1)} = Z^\tau e^{\phi} \left( \frac{p}{R} \nabla_\psi \hat{R} - \frac{1}{\tau} \nabla_\psi \mathcal{C}_R \right), \]
\[ W_{\psi A}^{(1)} = 2Z A \left( p \nabla_\psi \hat{R} - 2e^{\phi} \right), \tag{99} \]
with all other components set to zero. Notice that up to this point we have not yet chosen \( W_{ab}^{(2)} \) for any of the equations and we do not have to, since by construction this choice bears no influence on the leading asymptotics of the metric functions. The choice of gauge is,
\[ F^\sigma = \frac{2}{R} \cosh h_\perp \cosh h_\times + C_\lambda^+ F^\lambda + \frac{2p}{R} (e^{\phi} - 1), \]
\[ F^\sigma = \frac{2}{R} \cosh h_\perp \cosh h_\times + C_\lambda^- F^\lambda - \frac{p}{R} (1 + C_{\lambda}^R) + \frac{1}{R} \hat{F}_\sigma^2, \]
\[ F^\theta = \frac{\cot \theta}{R^2} e^{A_\lambda} \cosh h_\times - \frac{2}{R} g^{\theta \phi} + \frac{1 - p}{R^2} \hat{\mathcal{C}}_{\phi}, \]
\[ F^\phi = \frac{2 \cot \theta}{\sin \theta R^2} \sinh h_\times - \frac{4}{R} g^{\phi \theta} + \frac{1 - p}{\sin^2 \theta R^2} \hat{\mathcal{C}}_{\phi}. \tag{100} \]
4.1. Einstein field equations

Here it pays off to change the variables that usually appear inside exponentials in such a way that we can clearly separate the leading ±1 from the functions we wish to evolve. This will allow us to rescale the variables directly by \( \tilde{R} \), as we will have to do later on. Let \( \tilde{\varphi} \) denote field \( e^{\varphi} - 1, \tilde{h}_+ \) denote \( e^{h_+} - 1 \) and \( \tilde{h}_x \) denote \( e^{h_x} - 1 \). Then the wave equations for these variables change like,

\[
\Box \tilde{\varphi} = \tilde{N}_\varphi (\tilde{\varphi} + 1) - \frac{1}{\tilde{\varphi} + 1} \nabla_a \tilde{\varphi} \nabla^a \tilde{\varphi},
\]

\[
\Box \tilde{h}_+ = \tilde{N}_{h_+} (\tilde{h}_+ + 1) - \frac{1}{\tilde{h}_+ + 1} \nabla_a \tilde{h}_+ \nabla^a \tilde{h}_+,
\]

\[
\Box \tilde{h}_x = \tilde{N}_{h_x} (\tilde{h}_x + 1) - \frac{1}{\tilde{h}_x + 1} \nabla_a \tilde{h}_x \nabla^a \tilde{h}_x.
\]

(101)

Although this choice of variables changes the second order equations, it does so in such a way that the leading order remains unaffected. Hence good equations remain good, ugly ones remain ugly and so on. Expanding the wave operator and using (25) we can write all 11 field equations as,

\[
\Box_{\mu} \tilde{C}_\mu^R = \sigma^a \nabla_a F_a + e^{-\varphi} W_{\varphi \psi} = \frac{2(p + 1) e^{-\varphi}}{\tau R} \nabla_\varphi \tilde{R} \nabla_\psi \tilde{C}_\psi^R + (\Gamma \Gamma)_\psi \psi,
\]

\[
\Box_{\psi} \psi \tilde{\varphi} = \frac{2e^{-\varphi}}{\tau} \nabla_a \tilde{\varphi} \nabla^a \tilde{\varphi} - \frac{2(p + 1) e^{-3\varphi}}{\tau R} \nabla_\psi \tilde{R} \nabla_\varphi \tilde{\varphi} - e^{-\varphi} \nabla_\varphi \nabla^a \tilde{\varphi},
\]

\[
\Box_{\mu} \tilde{C}_A^+ = -2e^{-\varphi} g_{\alpha \beta} \nabla_\alpha F_\beta + \frac{2e^{-\varphi}}{\tau} \nabla_\psi \tilde{R} \nabla_\varphi \tilde{C}_\psi^+ + (\Gamma \Gamma)_\psi \psi
\]

\[
\Box_{\mu} \tilde{C}_A^- = -2e^{-\varphi} g_{\alpha \beta} \nabla_\alpha F_\beta + \frac{2e^{-\varphi}}{\tau} \nabla_\psi \tilde{R} \nabla_\varphi \tilde{C}_\psi^- + (\Gamma \Gamma)_\psi \psi
\]

\[
\Box_1 \tilde{R}^{-1} = \frac{1}{2R} \left[ -g^{ab} \nabla_a F_b + \mathcal{W} + (\Gamma \Gamma) - g^{\theta \theta} (1 + \cot^2 \theta) \right] + \frac{1}{R} \nabla_\theta \tilde{R} \nabla^\theta \tilde{R},
\]

\[
\Box_\mu \tilde{h}_+ = -\frac{2e^{-h_+}}{g^{\theta \theta}} (\nabla F)^\theta \theta + e^{-h_+} (\Gamma \Gamma)^\theta \theta
\]

\[
= \frac{2e^{-\varphi - 2h_+}}{\tau R} \nabla_\psi \tilde{R} \nabla_\varphi \tilde{h}_+ - e^{-h_+} \nabla_a \tilde{h}_+ \nabla^a \tilde{h}_+,
\]

\[
\Box_\mu \tilde{h}_x = -\frac{2e^{-h_x}}{g^{\theta \theta}} (\nabla F)^\theta \theta + e^{-h_x} (\Gamma \Gamma)^\theta \theta
\]

\[
= \frac{2e^{-\varphi - 2h_x}}{\tau R} \nabla_\psi \tilde{R} \nabla_\varphi \tilde{h}_x - e^{-h_x} \nabla_a \tilde{h}_x \nabla^a \tilde{h}_x,
\]

\[
\Box_\mu \tilde{F}_1^\mu + \frac{2pe^{-\varphi}}{\tau R^2} \tilde{H} = 0.
\]

(102)
Only the LHSs of these equations can possibly contribute to leading order and all of the RHSs are SNFs. This means that the leading order asymptotics of each metric function, and therefore whether it qualifies as a good or an ugly or otherwise, is determined solely by the operator on the LHS. Therefore the 11 equations can be written in the very concise form,

\[
\Box_q \phi = N_\phi, \quad \Box_p \tilde{F}_\sigma^1 = -\frac{p+1}{R} \mathcal{H},
\]

where \(\phi \in \{C^R_{\pm A}, \hat{C}^\pm_{\pm A}, \hat{\varphi}, \hat{R}^{-1}, h_+, h_x\}, q = p \) for \(\phi \in \{C^R_{\pm A}, \hat{C}^\pm_{\pm A}, \varphi\}, q = 1\) for \(\phi = \hat{R}^{-1}\) and \(q = 0\) for \(\phi \in \{h_+, h_x\}\). The SNFs \(N_\phi\) are the RHSs of the corresponding equations in (102).

### 4.2. Alternative form

A less concise way to write the wave equations that is more useful for what we aim to do in the next section makes use of (30),

\[
\frac{1}{\hat{R}^{q+1} \hat{\psi}} \nabla_\psi \left[ \hat{R}^{q+1} \nabla_\psi \phi \right] - \frac{\tau e^\varphi}{2} \mathcal{N} \phi = \tilde{N}_\phi, \\
\frac{1}{\hat{R}^{q+1} \hat{\psi}} \nabla_\psi \left[ \hat{R}^{q+1} \nabla_\psi \tilde{F}_\sigma^1 \right] - \frac{\tau e^\varphi}{2} \mathcal{N} \tilde{F}_\sigma^1 = \tilde{N} \tilde{\mathcal{F}}^\sigma_1.
\]

This amounts to simply moving all the SNFs in \(\Box_q \phi\) to the RHS and redefining \(\tilde{N}_\phi\) so that all the principal terms and those that contribute to leading order are on the LHS.

### 5. First order reduction

In order to implement the EFEs in this framework numerically it is useful to reduce them to a system of first order differential equations. This is done by defining reduction variables, then rescaling the evolved fields and compactifying the radial coordinate. Although the equations in question are naturally more numerous and complicated, most of this goes along the lines of the work done in [23].

#### 5.1. Picking the variables

We begin by choosing the variables we will use in order to build our first order system. As was said above, it pays off to avoid special functions of variables, e.g. exponentials, in the equations because they have a leading 1 that is not explicit. We leave a more detailed explanation of this problem for the last section of this work. Let \(\phi\) denote any of the following fields,

\[
C^R_{\pm A}, \quad \hat{C}^\pm_{\pm A}, \quad e^\varphi - 1, \quad \hat{R}^{-1}, \\
e^{h_+} - 1, \quad e^{h_x} - 1, \quad \tilde{F}_\sigma^1.
\]

Moreover, let \(\phi_\psi, \phi_\varphi\) and \(\phi_A\) denote \(\nabla_\psi \phi, \nabla_\varphi \phi\) and \(\mathcal{D}_\lambda \phi\), respectively.

#### 5.2. Picking the equations

The first order reduction is done by treating \(\phi_\psi, \phi_\varphi\) and \(\phi_A\) as independent variables and the second order differential equations as first order ones for these variables. In order to relate them to the original variables \(\phi\), we will choose one (or a combination) of the definitions as an additional evolution equation and treat the others,

\[
\phi_\psi = \nabla_\psi \phi, \quad \phi_\varphi = \nabla_\varphi \phi, \quad \phi_A = \mathcal{D}_\lambda \phi,
\]
as reduction constraints. After the reduction, where we had 11 independent variables, we end up with four additional variables per original one so we will need 55 independent equations. The first set of equations is naturally the wave equations (103), which we can now write in terms of the reduction variables. This way we have,

\[
\frac{1}{R^{n+1}} \nabla_\psi \left[ R^{n+1} \phi_\psi \right] - \frac{\tau e^\psi}{2} \phi_A = \tilde{N}_\phi, \tag{107}
\]

for the uglies and goods. Finally, the equation for the derivative along \( \psi \) of the gauge driver reads,

\[
\frac{1}{R^{n+1}} \nabla_\psi \left[ R^{n+1} \tilde{F}^a \right] - \frac{\tau e^\psi}{2} \tilde{F}_{,T,A} - \frac{p}{R^2} \mathcal{H} = \tilde{N}_{\tilde{F}}. \tag{108}
\]

Replacing \( \phi \) with the appropriate fields, this gives 11 first order differential equations for the variables \( \phi_\psi \). The second and third sets of equations will be given by the torsion-free condition \( \tilde{\nabla}_a \nabla_b \phi = \tilde{\nabla}_b \nabla_a \phi \) contracted with the vectors \( \psi^a \) and \( \psi^\phi \), and \( T^a \) and \( g^a_\phi \), respectively. Let us take the first combination of vectors,

\[
\psi^a \psi^b \tilde{\nabla}_a \nabla_b \phi = \psi^a \psi^b \tilde{\nabla}_b \nabla_a \phi, \tag{109}
\]

so that \( (\tilde{\nabla}_a \nabla_b \phi)^{\psi} \) becomes,

\[
\tilde{\nabla}_a \nabla_b \phi = \tilde{\nabla}_b \nabla_a \phi = \tilde{\nabla}_b \nabla_a \phi - (\nabla_a \phi)^\psi = \tilde{\nabla}_b \nabla_a \phi - (\nabla_a \phi)^\psi = \tilde{\nabla}_b \nabla_a \phi. \tag{110}
\]

We already have an evolution equation for \( \phi_\psi \) in (107), so we want (110) to be one for \( \phi_\psi \). To do this, we expand the LHSs of (107) and (108) and plug them in (112) to get,

\[
\nabla_\psi \phi_\psi = \frac{\tau e^\psi}{2} \phi_A + \tilde{N}_\phi - (q + 1) \tilde{R} \phi_\psi (\tilde{R}^{-1})_\psi 

- \frac{1}{\tau} \left( C^R_{\psi} - C^R_{\psi, \psi} \right) \left( \phi_\psi - \phi_\psi \right). \tag{111}
\]

We are free to add any amount of the constraints we wish to these equations and it is worth doing so in (110) in order to get rid of the bad derivative of \( C^R_{\psi} \). We do this for reasons we will explain later on when we discuss the existence of formally singular terms. We add,

\[
\frac{\tau e^\psi}{2} \tilde{Z}^\psi \left( \phi_\psi - \phi_\psi \right),
\]

so that (110) becomes,

\[
\nabla_\psi \phi_\psi = \frac{\tau e^\psi}{2} \phi_A + \tilde{N}_\phi + (q + 1) \tilde{R} \phi_\psi (\tilde{R}^{-1})_\psi 

- \frac{1}{\tau} \left( C^R_{\psi} - C^R_{\psi, \psi} - \frac{\tau e^\psi}{2} \tilde{Z}^\psi \right) \left( \phi_\psi - \phi_\psi \right). \tag{112}
\]

for all metric functions except the gauge driver and,

\[
\nabla_\psi \tilde{F}^a \psi = \frac{\tau e^\psi}{2} \phi_A + \tilde{F}_{,T,A} + \frac{p}{R^2} \mathcal{H} + (p + 1) \tilde{R} \tilde{F}^a \psi (\tilde{R}^{-1})_\psi 

- \frac{1}{\tau} \left( C^R_{\psi, \psi} - C^R_{\psi, \psi, \psi} - \frac{\tau e^\psi}{2} \tilde{Z}^\psi \right) \left( \tilde{F}^a \psi - \tilde{F}^a \psi \right). \tag{113}
\]

This gives us 11 additional equations. Following the same procedure with the second combination of vectors we get an evolution equation for \( \phi_A \),
\[ \tau \nabla T \phi_{,A} - C^R \mathcal{D}_A \phi_{,\psi} + C^R \mathcal{D}_A \phi_{,\psi} = \]
\[ \phi_{,\psi} \left( \frac{C^R}{\tau} \nabla T C^R + \nabla C^R_{+} \right) \]
\[ + \phi_{,\psi} \left( -\frac{C^R}{\tau} \nabla T C^R - \nabla C^R_{-} \right), \]
\begin{equation}
(113)
\end{equation}

where \( \nabla_T \) was used merely as shorthand for \( \frac{1}{\tau} (C^R \nabla \psi - C^R - \nabla \psi) \). Again we are free to add any amount of the constraints to this equation and we choose to add,
\[ W^{(3)}_\phi = \frac{C^R R e^2}{20 R} \partial_R \delta A \left( \dot{R} \phi \right) Z^{\sigma}, \]
\begin{equation}
(114)
\end{equation}
to the LHS, to get,
\[ \tau \nabla T \phi_{,A} + W^{(3)}_\phi - C^R \mathcal{D}_A \phi_{,\psi} + C^R \mathcal{D}_A \phi_{,\psi} = \]
\[ \phi_{,\psi} \left( \frac{C^R}{\tau} \nabla T C^R + \nabla C^R_{+} \right) \]
\[ + \phi_{,\psi} \left( -\frac{C^R}{\tau} \nabla T C^R - \nabla C^R_{-} \right). \]
\begin{equation}
(115)
\end{equation}

This is not done in order to eliminate formally singular terms, but to make the proof of symmetric hyperbolicity of the final system of equations more straightforward. This gives us another 22 equations and it is valid for all the metric functions. We now have one equation for each of the reduction variables, leaving us only with the task of choosing equations that will relate these to the original variables. We pick the equations,
\[ \phi_{,\psi} = \nabla \psi \phi, \]
\[ \phi_{,A} = \mathcal{D}_A \phi. \]
\begin{equation}
(116)
\end{equation}
\begin{equation}
(117)
\end{equation}
as constraints that one needs to make sure are satisfied everywhere when evolving the system numerically.

### 5.3. Rescaling the variables

We know from previous studies on the good-bad-ugly model that the fields themselves have decay and their derivatives may enhance that decay or keep it the same depending on if the derivative is good or bad. Ideally, when implementing this model numerically, we would have regular equations for variables that we expect to have regular behavior and a finite value at null infinity. In this section we will rescale the fields and the reduction variables with powers of \( \dot{R} \) to find the best possible variables to evolve. Naturally we cannot rescale the variable \( \dot{R}^{-1} \) by powers of \( R \), so we will have to treat this particular variable differently. We begin by rescaling all the other variables in the following way,
\[ \Phi := \dot{R} \phi, \quad \Phi_{,\psi} := \dot{R} \nabla \psi \Phi, \]
\[ \Phi_{,A} := \nabla \psi \Phi, \quad \Phi_{,A} := \mathcal{D}_A \Phi. \]
\begin{equation}
(118)
\end{equation}

Having in mind the asymptotics of goods and uglies derived in [26], and of the gauge driver in [27], we can expect this rescaling to give reduction variables that asymptote to finite values
or zero at null infinity in all cases. For the variable $\dot{R}^{-1}$ we introduce a different rescaling,

$$\rho_\phi := -\dot{R} \left[ R^2 (\dot{R}^{-1})_\phi + 1 \right], \quad \rho_\psi := -\dot{R}^2 (\dot{R}^{-1})_\psi + 1,$$

$$\rho_A := -\dot{R}^2 (\dot{R}^{-1})_A,$$

which can be written alternatively as,

$$\nabla_\psi \dot{R} = 1 + \frac{\rho_\phi}{\dot{R}}, \quad \nabla_\psi \dot{R} = -1 + \rho_\psi,$$

$$\rho_\phi \dot{R} = \rho_A.$$  (120)

Plugging this into the equations derived in the last section, (107), (112), (115) and (116), we find,

$$\frac{1}{\dot{R}^{q + 1}} \nabla_\psi \left[ \dot{R}^q \Phi_\psi \right] - \frac{\tau e^{\nu}}{2R} \partial^A \Phi_A = N_\phi^{(1)},$$

$$\frac{1}{\dot{R}^{2}} \nabla_\psi \Phi_\psi - \frac{\tau e^{\nu}}{2R} \partial^A \Phi_A + \frac{q}{R^2} \Phi_\psi = N_\phi^{(2)},$$

$$\tau \nabla_A \Phi_A + W_\phi^{(3)} - C_R A \partial_\psi \Phi_\psi + \frac{C_R}{R} A \partial_\Phi_\psi = N_\phi^{(3)} \dot{R},$$

$$\nabla_\psi \Phi - \Phi_\psi = 0.$$  (121)

where we have packed all the SNFs in the definitions of $N_\phi^{(1)}, N_\phi^{(2)}$ and $N_\phi^{(3)}$ on the RHS,

$$N_\phi^{(1)} := \dot{N}_\phi - \dot{\Phi} \dot{N}_R^{\star -1} + \frac{\Phi_\psi}{R^3} (\rho_\psi - 1) - \frac{\tau e^{\nu} \Phi_A}{R^2} - \rho_A,$$

$$N_\phi^{(2)} := \frac{1}{\tau \dot{R}} \left( C^R - C^R_{\phi} - \frac{\tau e^{\nu} \Phi_A}{2 \dot{R}} \right) \left( \frac{\Phi_\psi}{R} - \Phi_\psi \right) - \frac{q}{R^3} \Phi_\psi \rho_\psi + \frac{2 \Phi_\psi}{R^3} (\rho_\psi - 1) - \frac{\tau e^{\nu} \Phi_A}{R^2} \rho_A + \dot{\Phi} \dot{N}_R^{\star -1},$$

$$N_\phi^{(3)} := \frac{1}{\tau \dot{R}} \left( - \frac{C^R}{\tau} \tau_A - \frac{C_A}{\tau} \nabla_T C^R + \nabla_T C^A - A \partial_\psi C^R \right) + \frac{1}{\tau \dot{R}^2} \Phi_\psi \left( 2 e^{\nu} \rho_A + \frac{C_R}{\tau} \tau_A + \frac{C_A}{\tau} \nabla_T C^R + \nabla_T C^A + A \partial_\psi C^R \right).$$  (122)

To obtain the equations for the gauge driver we only need to modify (121) slightly in order to include the RHS of the third equation in (103),

$$\frac{1}{\dot{R}^{q + 1}} \nabla_\psi \left[ \dot{R}^q \Phi_\psi \right] - \frac{\tau e^{\nu}}{2R} \partial^A \Phi_A - \frac{p}{R^2} \mathcal{H} = N_\tau^{(1)},$$

$$\frac{1}{\dot{R}^{2}} \nabla_\psi \Phi_\psi - \frac{\tau e^{\nu}}{2R} \partial^A \Phi_A - \frac{p}{R^2} (\mathcal{H} - \Phi_\psi) = N_\tau^{(2)},$$

$$\tau \nabla_A \Phi_A + W_\phi^{(3)} - C_R A \partial_\psi \Phi_\psi + \frac{C_R}{R} A \partial_\Phi_\psi = N_\phi^{(3)} \dot{R},$$

$$\nabla_\psi \Phi - \Phi_\psi = 0.$$  (123)

where the SNFs (122) still apply. Following the same procedure that resulted in (121) we obtain for the inverse areal radius $\dot{R}^{-1}$,
\[
\frac{1}{R^2} \nabla \psi \rho_\phi - \frac{\tau e^\varphi}{2R^2} \phi = -\tilde{N}_{k-1} - \frac{\tau e^\varphi}{R} \rho_A^A, \\
\frac{1}{R^3} \nabla \phi \rho_\phi - \frac{\tau e^\varphi}{2R^2} \phi = N^{(2)}_{k-1}, \\
\tau \nabla \psi \rho_\phi + W^\rho_\phi - C^R \phi + \frac{C^R}{R} \phi A = N^{(3)}_{k-1} \rho^A, \\
\nabla_\omega \rho^{-1} = \frac{1}{R^2} (1 - \rho_\omega), \tag{124}
\]

where the SNFs on the RHSs are defined as,
\[
N^{(2)}_{k-1} := -\tilde{N}_{k-1} - \frac{\rho_\phi (1 - \rho_\omega)}{R^2} - \frac{\tau e^\varphi \rho_A^A}{R} \\
+ \frac{1}{\tau R^2} \left( \frac{C^R}{\rho_A^A} - \frac{\tau e^\varphi}{2} \sigma \right) \left( 2 - \rho_\omega + \frac{\rho_\omega}{R} \right), \\
N^{(3)}_{k-1} = \frac{C^R}{R^3} \rho_\phi \rho_A \\
+ \frac{1}{(R^2 - \rho_\omega)} \left( \rho_A^{\rho_\omega} \frac{C^R}{\rho_A^A} \nabla_C^R - \nabla_T^R \right) - \frac{1}{R^2} \left( \rho_A^{\rho_\omega} \frac{C^R}{\rho_A^A} \nabla_C^R + \nabla_T^R \right). \tag{125}
\]

We now have all the wave equations in first order form with all variables rescaled.

5.3.1. Rescaled reduction constraints. With the exception of \( R^{-1} \) all variables must satisfy the following reduction constraints,
\[
\tilde{R} \nabla \psi \Phi - \Phi = 0, \quad \phi A = 0, \tag{126}
\]
whereas \( R^{-1} \) must satisfy,
\[
\nabla_\psi (R^{-1}) \left( 1 + \frac{\rho_\omega}{R} \right) = 0, \\
\phi_A (R^{-1}) \left( 1 + \frac{\rho_\omega}{R} \right) = 0. \tag{127}
\]

This concludes the rescaling of the variables and we are now ready to compactify the radial coordinate and move to hyperboloidal slices.

6. Hyperboloidal compactification

In order to write the evolution equations in their final form, we want to define a radially compactified hyperboloidal coordinate system in much the same way as was done in [23]. However, we want to work with the areal radius \( \tilde{R} \) as our preferred radial coordinate rather than \( R \). So as an intermediate step we do another coordinate change.

6.1. From radius to areal radius

Let us consider the coordinate system \((\tilde{T}, \tilde{R}, \tilde{\theta}^A)\) related to \((T, R, \theta^A)\) by,
\[
\tilde{T} = T, \quad \tilde{R} = R e^{\epsilon/2}, \quad \tilde{\theta}^A = \theta^A, \tag{128}
\]
and let us define the outgoing and incoming null vectors $\hat{\psi}$ and $\check{\psi}$ in a way analogous to $\psi$ and $\check{\psi}$.

$$\hat{\psi}^a = \partial_t^a + C^a_t \partial_R^t, \quad \check{\psi}^a = \partial_t^a + C^a_t \partial_R^t. \quad (129)$$

We define the null covectors $\hat{\sigma}$ and $\check{\sigma}$ analogously to $\sigma$ and $\check{\sigma}$. Similarly we define the quantities $\hat{\varphi}, \hat{\tau}, \hat{\bar{g}}^a$ and $\hat{C}_A$. We can write the vectors $\partial_t^a$ and $\partial_R^a$ in terms of $\partial_t^a$ and $\partial_R^a$ in the following way,

$$\partial_t^a = \partial_t^a + \frac{1}{T} \left( C^a_t \nabla_{\psi} \hat{R} - C^a_T \nabla_{\psi} \hat{R} \right) \partial_R^a, \quad (130)$$

and from (130) a straightforward calculation leads to,

$$\hat{\psi}^a = \frac{1}{\nabla_{\psi} \hat{R} - \nabla_{\check{\psi}} \check{R}} \left[ \left( C^a_t - \nabla_{\psi} \hat{R} \right) \psi^a + \left( \nabla_{\psi} \hat{R} - C^a_t \right) \check{\psi}^a \right], \quad (131)$$

At this point, in order to complete the expressions that relate the null vectors that correspond to either coordinate system, we need only to find an expression for the coordinate light speeds $c^R_+$ and $c^R_-$. This is given by,

$$C^R_+ = \frac{J^R_T + C^R T J^R}{J^T T + C^R T J^R} = \nabla_{\psi} \hat{R}, \quad (132)$$

Plugging (132) into (131) we find that,

$$\hat{\psi}^a = \psi^a, \quad \check{\psi}^a = \check{\psi}^a. \quad (133)$$

In other words, the expression for our incoming and outgoing null vectors are invariant under the coordinate change $(T, R, \theta^a) \rightarrow (T, R, \theta^a)$.

### 6.1.1 Angular derivatives $\mathcal{D}_a$

With (133) we already know how null derivatives transform under this coordinate change, so we are only lacking an expression that relates $\mathcal{D}_a$ to $\mathcal{D}_a$, where the latter is defined analogously to the former. We contract $\hat{\sigma}$ with $\partial_R^a$ in order to find the relation between $\varphi$ and $\hat{\varphi}$,

$$e^{\varphi - \hat{\varphi}} = \frac{\hat{\tau}}{\tau}. \quad (134)$$

Under this change of coordinates the angular basis vectors transform in the following way,

$$\partial_{\hat{t}}^a = (\partial_{t} \hat{R}) \partial_R^t + \partial_{\hat{t}}^a, \quad (135)$$

and we can use this to find the relation between $C^A_+$ and $\check{C}_A^\pm$. By recognizing that,

$$\hat{C}_A = \hat{\sigma}^a \partial_{\hat{t}}^a, \quad (136)$$
we find that,
\[
\begin{align*}
\hat{C}_A^+ &= -\partial_A \hat{R} + \frac{1}{T} (C_A^- \nabla_v \hat{R} + 2C_A^+ \nabla_v \hat{R} - C_A^+ \nabla_v \hat{R}), \\
\hat{C}_A^- &= \partial_A \hat{R} - \frac{1}{T} (2C_A^- \nabla_v \hat{R} + C_A^+ \nabla_v \hat{R} - C_A^- \nabla_v \hat{R}).
\end{align*}
\]  
(137)

We can now build the derivatives by expanding the components of \( \hat{g}^b_a \) in mixed form,
\[
\hat{D}_A \phi = \hat{g}_A^a T \partial_a \phi + \hat{g}_A^b R \partial_b \phi + \hat{g}_A^b B \partial_b \phi.
\]  
(138)

Using (14) the non-trivial components of \( \hat{g}^b_a \) can be written as,
\[
\hat{g}_A^a \hat{T} = \frac{C_A}{T}, \quad \hat{g}_A^b \hat{R} = \frac{C^A_+ \nabla_v \hat{R} + C^-_A \nabla_v \hat{R}}{T}.
\]  
(139)

A straightforward calculation then leads to,
\[
\hat{D}_A \phi = \hat{D}_A \phi,
\]  
(140)

which shows that under this coordinate transformation, all derivatives stay the same.

### 6.2. Hyperboloidal Compactification

Let us consider a third coordinate system \((t, r, \bar{\theta}^A)\) which is related to the previous one by,
\[
\bar{T} = t + H(R(r), \bar{\theta}^A), \quad \bar{R} = R(r), \quad \bar{\theta}^A = \bar{\theta}^A,
\]  
(141)

where \(H(R(r), \bar{\theta}^A)\) and \(R(r)\) are called height and compression functions, respectively. Note that this coordinate change differs from the one in [23] insofar as the height function depends on the angular coordinates. Once again we define a set of null vectors associated to (141),
\[
\xi^a = \partial_t^a + C^a_+ \partial_r^a, \quad \bar{\xi}^a = \partial_t^a + C^-_a \partial_r^a.
\]  
(142)

For this set of coordinates we define \(\bar{\sigma}, \bar{\sigma}, \bar{\varphi}, \bar{\varphi}, \bar{\phi}^{ab}\) and \(\hat{B}_A\). The relations between the first two coordinate basis vectors can be written as,
\[
\partial_{\bar{T}} = \partial_{\hat{T}}, \quad \partial_{\bar{R}} = H^{\prime} \partial_{\hat{R}} + \hat{R} \partial_{\hat{R}},
\]  
(143)

where \(H^{\prime} := \partial_{\bar{R}} H\) and \(\hat{R}^{\prime} := \partial_{\hat{T}} \hat{R}\), which yields,
\[
\xi^a = (1 + H^{\prime} \hat{R} \partial_{\hat{R}}) \partial_t^a + \hat{R} \partial_{\hat{R}}^a,
\]
\[
\bar{\xi}^a = (1 + H^{\prime} \hat{R} \partial_{\hat{R}}) \partial_{\hat{T}}^a + \hat{R} \partial_{\hat{R}}^a.
\]  
(144)

The last ingredient we need is the expressions that relate the coordinate light speeds. We find this for \(C^R_+\) as the derivation of \(C^-_+\) is completely analogous,
\[
C^R_+ = \frac{\hat{g}_t^R + C^R_+ \hat{g}_t^R}{\hat{g}_t^R + C^R_+ \hat{g}_r^R} = \frac{\nabla_v \hat{R}}{\nabla_q T} = \frac{C^R_+ \hat{R}^{\prime}}{1 + C^R_+ \hat{R}^{\prime} H^{\prime}}.
\]  
(145)

Inverting (132) we get,
\[
C^R_+ = \frac{C_+^R}{\hat{R} (1 - C_+^R H^{\prime})},
\]  
(146)
where we have also included the expression for $C^R_-$. Substituting (146) into (144) and using (129) we get,

$$
\xi^a = \frac{1}{1 - H' C^R_+} \hat{\psi}^a, \quad \xi^a = \frac{1}{1 - H' C^R_-} \hat{\psi}^a.
$$

Finally, using the invariance of the null vectors under the first coordinate change we find the relation between $\psi^a$ and $\hat{\psi}^a$, and $\xi^a$ and $\hat{\xi}^a$,

$$
\psi^a = \Omega^+ \xi^a, \quad \psi^a = \Omega^- \xi^a,
$$

where the scalar factors $\Omega^\pm$ are defined as,

$$
\Omega^+ := 1 - H' \nabla^\phi \hat{R}, \quad \Omega^- := 1 - H' \nabla^\phi \hat{R}.
$$

6.2.1. Angular derivatives $\mathcal{D}_a$. In order to finish the compactification of the coordinates we only need to find how $\mathcal{C}^\pm_\delta$ transforms. In other words, we want to find a relation between $\bar{\mathcal{C}}^\pm_\delta$ and $\hat{\mathcal{C}}^\pm_\delta$, where the former is defined in the obvious way. The steps of the procedure are the same, so we present only the relevant expressions in this derivation,

$$
e^\hat{\phi} - \phi = \frac{1}{R'} \partial_\delta (R H \partial^\mu \hat{\phi} + \partial^\mu \hat{\phi}),
$$

$$
\mathcal{C}^+_A = \frac{\bar{\mathcal{C}}^+_A - \nabla_\phi R (\partial^\delta \hat{\phi})}{R' \Omega^+}, \quad \mathcal{C}^-_A = \frac{\bar{\mathcal{C}}^-_A - \nabla_\phi R (\partial^\delta \hat{\phi})}{R' \Omega^-}.
$$

A rather lengthy calculation leads to the following result,

$$
\mathcal{D}_a \phi = \bar{\mathcal{D}}_a \phi - \hat{H} \left[ \bar{\mathcal{C}}^+_A \nabla_\xi \phi - \bar{\mathcal{C}}^-_A \nabla_\xi \phi \right],
$$

where $\bar{\mathcal{C}}^\pm_\delta$ are given in terms of our metric variables by (137).

7. Compactified EFEs in first order form

We are now equipped to present the EFEs regularized at null infinity in first order form and in radially compactified coordinates. The equations for the metric functions $C^R_\pm + 1$, $\bar{\mathcal{C}}^A_\pm$, $\phi$, $h_+$ and $h_\times$ are

$$
\Omega^+ \nabla^{\rho} \Phi_+ - \frac{\tau e^\phi}{2 R} \mathcal{D}^A \Phi_+ = N^{(1)}_\phi,
$$

$$
\Omega^- \nabla_{\xi} \Phi_\times = - \frac{\tau e^\phi}{2 R} \mathcal{D}^A \Phi_+ + \frac{q}{R^2} \Phi_\times = N^{(2)}_\phi,
$$

$$
C^R_+ \Omega^+ \nabla_{\xi} \Phi_+ - C^R_\times \Omega^+ \nabla_{\xi} \Phi_\times - C^R_\times \mathcal{D}^A \Phi_\times + \frac{C^R_\times}{R} \mathcal{D}^A \Phi_\times + W^{(1)}_\phi = R N^{(3)}_\phi,
$$

$$
\Omega^- \nabla_{\xi} \Phi_\times - \Phi_\times = 0.
$$

(152)

where $p$ is set to 0 in the cases of $h_+$ and $h_\times$ and the we use $\mathcal{D}^A$ merely as a shorthand for the expression on the RHS of (151) with raised indices. The equations for $F^\pm_\mp$ are,
\[
\frac{\Omega^+}{R^{n+1}} \nabla_\xi \left[ R^n \Phi_\omega \right] - \frac{\tau e^\varphi}{2R} D^A \Phi_A - \frac{p}{R^2} H = \mathcal{N}^{(1)}_\varphi, \\
\frac{\Omega^-}{R^2} \nabla_\xi \Phi - \frac{\tau e^\varphi}{2R} D^A \Phi_A - \frac{p}{R^2} (H - \Phi_\omega) = \mathcal{N}^{(2)}_\varphi, \\
C^R \Omega^- \nabla_\xi \Phi_A - C^R \Omega^+ \nabla_\xi \Phi_A - C^R \nabla_D A \Phi_\omega + \frac{C^R}{R} D_A \Phi_\omega + W^{(3)} = \hat{R} \mathcal{N}^{(3)}_\varphi, \\
\Omega^- \nabla_\xi \Phi - \Phi_\omega = 0,
\]

(153)

and the ones for \( \hat{R}^{-1} \) are,

\[
\frac{\Omega^+}{R^2} \nabla_\xi \rho_\omega = -\frac{\tau e^\varphi}{2R} D^A \rho_A - \frac{e^\varphi}{R^2} \rho_A \rho_A, \\
\frac{\Omega^-}{R^3} \nabla_\xi \rho_\omega = -\frac{\tau e^\varphi}{2R} D^A \rho_A = \mathcal{N}_\varphi^{(2)}, \\
C^R \Omega^- \nabla_\xi \rho_A = C^R \Omega^+ \nabla_\xi \rho_A - C^R \nabla_D A \rho_\omega + \frac{C^R}{R} D_A \rho_\omega + W^{(3)} = R^2 \mathcal{N}_\varphi^{(3)}, \\
\Omega^- \nabla_\xi \hat{R}^{-1} = \frac{1}{R^2} (1 - \rho_\omega).
\]

(154)

7.1. Compactified reduction constraints

The final form of the reduction constraints for all variables except \( \hat{R}^{-1} \) is,

\[
\hat{R} \Omega^+ \nabla_\xi \Phi - \Phi_\omega = 0, \quad D_A \Phi - \Phi_A = 0,
\]

(155)

whereas \( \hat{R}^{-1} \) must satisfy,

\[
\Omega^+ \nabla_\xi (R^{-1}) + \frac{1}{R^2} \left( 1 + \frac{\rho_\omega}{R} \right) = 0, \\
D_A (R^{-1}) + \frac{\rho_A}{R} = 0.
\]

(156)

we now have the complete set of equations written in a radially compactified coordinate system. Equations (152)-(154) are the EFEs in GHG written as a system of first order differential equations that are regular at null infinity, with reduction constraints given by (155) and (156).

7.2. Comment on the choice of reduction variables

Thus far we have written the first order equations in the third coordinate system (141), which we referred to as radially compactified coordinates. We have written the derivatives in the EFEs in terms of the compression and height functions \( \hat{R}(r) \) and \( H(\hat{R}(r), \hat{\theta}^A) \), but we have said nothing so far about how we want these functions to behave. Strictly speaking, we have not done a compactification yet. To do that we assume,

\[
\hat{R}(r) \approx R^n,
\]

(157)

where \( 1 < n \leq 2 \). The lower bound on \( n \) is necessary to make \( r \) approach a finite value as \( \hat{R} \) goes to null infinity, whereas \( 0 < n \leq 2 \) is required for numerical stability, as discussed in [30]. Moreover, we want \( C_\varphi \) \( \sim \) 1 which, together with (157) gives us the requirement on the leading order behavior of \( H \),

\[
H'(\hat{R}(r), \hat{\theta}^A) \sim 1 - \frac{m_{\xi,1}}{\hat{R}^n} = \frac{1}{R^n},
\]

(158)
where \( m_{C_{\xi}^{i_1}} \) is the leading order term of the field \( \hat{R}(C_{\xi}^{i_1} - 1) \) and depends only on the angular coordinates. Our requirements on the functional dependence of \( H \) forbid the inclusion of higher order time dependent corrections, but could include higher order corrections in \( m_{C_{\xi}^{i_1}} \).

Fortunately, provided that \( n < 2 \) the condition \( C_{\xi}^{i_1} \approx 1 \) is satisfied with (158) anyway. In the desirable \( n = 2 \) case we instead obtain \( C_{\xi}^{i_1} = O(1) \), provided that \( \rho \geq 2 \) so that no logs are present to order \( R^{-2} \), which is acceptable also. For all \( 1 < n \leq 2 \) we end up with a formally singular term in \( C_{\xi}^{i_1} \) that is easily evaluated with L’Hôpital’s rule. To obtain instead the sharper condition \( C_{\xi}^{i_1} = 1 \) over the full range of \( n \) we could instead solve the eikonal equation as proposed in [21]. Plugging this in (148) we get that the outgoing and incoming null vectors transform asymptotically as,

\[
\Omega^+ \simeq \frac{1}{R^n}, \quad \Omega^- \simeq O(1).
\]

This has a very significant influence on how far we can go in rescaling reduction variables, because replacing \( \psi^a \) with \( \xi^a \) gives an extra \( n \) powers of \( \hat{R}^{-1} \), whereas \( \psi^a \) does not. Take the first equation in (152), for example. As the SNFs are \( o^+(R^{-2}) \), we can write asymptotically,

\[
\frac{1}{\hat{R}^{q+1+n}} \nabla_{\xi} \left[ R^n \Phi_{\psi} \right] = o^+(R^{-2}),
\]

\[
\Rightarrow \frac{1}{\hat{R}^{1+n}} \nabla_{\xi} \Phi_{\psi} - q(\hat{R}^{-1})_\xi \Phi_{\psi} = o^+(R^{-2}),
\]

\[
\Rightarrow \nabla_{\xi} \Phi_{\psi} - qR^{1+n}(\hat{R}^{-1})_\xi \Phi_{\psi} = o^+(R^{-1}).
\]

According to the findings in [26], the only possibility for the error terms on the RHS of the second line to decay slower than \( O(R^{n-2}) \) is if there are terms proportional to \( \log \hat{R} \). As we know that such terms are suppressed up to order \( q \), if this number is sufficiently large we can write that,

\[
\nabla_{\xi} \Phi_{\psi} + \frac{q \Phi_{\psi}}{\hat{R}} = O(R^{n-2}).
\]

This guarantees that when we integrate numerically along integral curves of \( \xi^a \), the integral will not act upon terms that diverge at null infinity. If we had rescaled \( \Phi_{\psi} \) by another power of \( \hat{R} \), then we would have had to integrate an error term of the type \( O(\hat{R}^{n-1}) \) which always diverges. This is the reason why we cannot afford to have this particular reduction variable approach a (non-vanishing) finite value at null infinity. It is worth pausing here for a moment to analyze what (158) means.

**Remark 1.** Our hyperboloidal compactification is different from others done in the literature, for example in [24, 31], in the sense that we have allowed the height function \( H \) to depend upon the angular coordinates (141). The reason for this is that \( \hat{H} \) depends on \( m_{C_{\xi}^{i_1}} \) and we know from [27] that that function, the numerator at first order in \( \hat{R}^{-1} \) in the polyhomogeneous expansion of \( C_{\xi}^{i_1} \), is an angular function, as it would be for a general ugly field in a curved spacetime. If \( C_{\xi}^{i_1} \) were a good, then \( m_{C_{\xi}^{i_1}} \) would be a radiation field, meaning that it would be allowed to vary along integral curves of \( \psi^a \). As a consequence, \( H \) would have to vary with \( T \) as well as all other coordinates and this type of compactification (141) would not work.

### 7.3. Formally singular terms

When implementing this system numerically, it is helpful if the terms we integrate have explicitly regular limits at null infinity, as opposed to terms which can only be written as the quotient
of divergent terms, for instance $O(R)/O(R)$. These formally singular terms are implicitly regular as they acquire regular values with the asymptotics we expect for each field, but they cause problems in the implementation nonetheless. They need to be identified and carefully processed using L'Hôpital’s rule before the numerical implementation, as was done for example in [14]. This subsection is dedicated to identifying the terms in (152)–(154) that contribute to subleading order and are formally singular so they can later be treated separately, as well as to choosing the constraint addition $W^{(2)}_{ab}$ so that the null forms $\mathcal{N}_\phi$ cannot possibly have these terms.

7.3.1. $\mathcal{N}_\phi^{(i)}$ in terms of $\mathcal{N}_\phi$. For the sake of tidiness, a lot of terms in every equation have been put together into groups of SNFs throughout this work to allow us to focus on the principal terms and those that contribute to leading order. However, because the goal of this paper is ultimately to write down a set of equations that are ready to be numerically implemented, we have to be able write all terms in all equations explicitly as products of derivatives of metric functions. In order to help the reader navigate through these SNFs we gather those terms in this paragraph. The SNFs named as $\mathcal{N}_\phi^{(i)}$ are defined as,

$$
\mathcal{N}_\phi^{(1)} := \tilde{\mathcal{N}}_\phi - \Phi \tilde{\mathcal{N}}_{\tilde{R}^{-1}} + \frac{\Phi_\psi}{R^3}(\rho_\psi - 1) - \frac{\tau e^{\phi_\psi} \Phi^A}{R^2} \rho_A,
$$

$$
\mathcal{N}_\phi^{(2)} := \frac{1}{\tau R} \left( C^R R - C_{-\phi R} - \frac{\tau e^{\phi_\psi}}{2} Z^2 \right) \left( \frac{\Phi_\psi}{R} - \Phi_\psi \right)
- \frac{q}{R^3} \Phi_\psi \rho_\psi + \frac{2\Phi_\psi}{R^3}(\rho_\psi - 1) - \frac{\tau e^{\phi_\psi} \Phi^A}{R^2} \rho_A + \tilde{\mathcal{N}}_\phi - \Phi \tilde{\mathcal{N}}_{\tilde{R}^{-1}},
$$

$$
\mathcal{N}_\phi^{(3)} := \frac{1}{\tau R} \Phi_\psi \left( - \frac{C^R}{R} \frac{\Phi_\psi}{\tau - \tau_A} - \frac{C_A}{\tau} \nabla_\tau \mathcal{C}_A^R + \nabla_\tau \mathcal{C}^A_A - \mathcal{D}_A \mathcal{C}_A^R \right)
+ \frac{1}{R^2} \Phi_\psi \left( 2C^R \rho_A + \frac{C^R}{\tau} \frac{\Phi_\psi}{\tau - \tau_A} + \frac{C_A}{\tau} \nabla_\tau \mathcal{C}_A^R + \nabla_\tau \mathcal{C}^A_A \frac{\Phi_\psi}{\tau - \tau_A} \right),
$$

(162)

for all variables except $\tilde{R}^{-1}$, whereas for the latter they are,

$$
\mathcal{N}_{\tilde{R}^{-1}}^{(2)} := -\tilde{\mathcal{N}}_{\tilde{R}^{-1}} - \frac{\rho_\psi (1 - \rho_\psi)}{R^3} - \frac{\tau e^{\phi_\psi} \rho_A A^A}{R^3}
+ \frac{1}{\tau R^2} \left( C^R R - C_{-\phi R} - \frac{\tau e^{\phi_\psi}}{2} Z^2 \right) \left( 2 - \rho_\psi + \frac{\rho_\psi}{R} \right),
$$

$$
\mathcal{N}_{\tilde{R}^{-1}}^{(3)} := \frac{C^R}{R^4} \frac{\rho_\psi}{\tau - A^A}
+ \left( \frac{1}{R^2} + \frac{\rho_\psi}{R^3} \right) \left( - \frac{C^R}{\tau} \frac{\Phi_\psi}{\tau - \tau_A} - \frac{C_A}{\tau} \nabla_\tau \mathcal{C}_A^R + \nabla_\tau \mathcal{C}^A_A - \mathcal{D}_A \mathcal{C}_A^R \right)
- \left( \frac{1}{R^2} - \frac{\rho_\psi}{R^3} \right) \left( \frac{C^R}{\tau} \frac{\Phi_\psi}{\tau - \tau_A} + \frac{C_A}{\tau} \nabla_\tau \mathcal{C}_A^R + \nabla_\tau \mathcal{C}^A_A \frac{\Phi_\psi}{\tau - \tau_A} \right),
$$

(163)

The SNFs named $\tilde{\mathcal{N}}_\phi$ are defined in terms of $\mathcal{N}_\phi$ as,
\[ \mathcal{N}_\phi = -\frac{\tau e^\varphi}{2} \mathcal{N}_\phi + \frac{e^\varphi}{2} \left( \frac{C_A}{\tau} \mathcal{D}_+^\mathcal{C}_+^R - \mathcal{D}_+^\mathcal{C}_+^A \right) \nabla_{\psi \phi} \]

\[ -\frac{e^\varphi}{2} \left( \frac{C_A}{\tau} \mathcal{D}_-^\mathcal{C}_-^R + \mathcal{D}_-^\mathcal{C}_-^A \right) \nabla_{\psi \phi} + \frac{1}{\tau} \nabla_\phi \mathcal{C}_A^R (\nabla_{\psi \phi} - \nabla_{\bar{\psi} \phi}) \]  

(164)

Finally, the SNFs \( \mathcal{N}_\phi \) for each of the variables are exactly the RHSs of (102).

### 7.3.2. Identifying formally singular terms.
In order to single out which terms might cause problems, one needs to look at each equation carefully and multiply it through by whatever required power of \( R \) is in front of the leading second order term. For example, in (152), the last two equations have no \( R \) in front of them, so nothing needs to be done. The second equation needs to be multiplied through by \( R^3 \), whereas the first one should be multiplied by \( R^{1+n} \), as can be seen in (160). After that, the terms we are looking for will be those that still have some explicit positive power of \( R \) left over and contribute to subleading order. We begin with (152).

The first equation has a formally singular term on the LHS upon expanding the \( \nabla_\xi \) derivative

\[ q R^{n-1} \Phi_{\psi} \]  

(165)

On the RHS there are no such terms outside \( \mathcal{N}_\phi \) and we leave the analysis of \( \mathcal{N}_\phi \) to the next paragraph. Bear in mind that upstairs \( A \) indices have an implicit \( g_{ab} \), which contains an explicit \( R^{-2} \), see (11). The second equation in (152) contains no formally singular terms on the LHS and none on the RHS outside \( \mathcal{N}_\phi \). The third and fourth equations have no formally singular terms at all. Note that in the good equations, namely \( h_\tau \) and \( h_\nu \), terms like (165) vanish since \( q = 0 \). In (153) we find similar terms. In the first equation we have,

\[ p R^{n-1} (\Phi_{\psi} - H) \]  

(166)

Note that the reduction variable associated with a bad derivative of the gauge driver does not necessarily have decay, since the gauge driver is not an ugly, so this term seems like it diverges. However, we have built \( F_\tau^2 \) in such a way that \( (\Phi_{\psi} - H) \) decays (98). Therefore these terms are only formally singular. In (154) there are no formally singular terms outside \( \mathcal{N}_{\phi_{\nu-1}} \). The only thing left to do is then to check if any of these terms show up in the original SNFs.

### 7.3.3. Formally singular terms in \( \mathcal{N}_{\phi_\nu} \)
With the help of computer algebra we typed the wave equations in second order form (102) and substituted the gauge source functions and constraint additions with the choices that suppress logs up to order \( p \), (100) and (99). Once the EFEs were written in terms of \( (\Gamma^a)_{ab} \) functions, we replaced them with derivatives of the metric functions using all the components of \( \hat{\Gamma}^b_{\alpha c} \). Then we rescaled all the variables and their derivatives in order to make the powers of \( R^{-1} \) explicit and most of the equations have formally singular terms that arise from the fact that bad derivatives of ugly fields have an implicit \( R^{-1} \). So terms that are \( O(R^{-3}) \) but contain a bad derivative of an ugly are necessarily formally singular. However, all these bad derivatives act upon either \( C_{\nu}^R, R^{-1} \) or \( C_{\nu}^A \), precisely the fields associated with the four constraints. Because the constraints are essentially bad derivatives of these functions to leading order, we can add specific combinations of the former so the latter do not appear at third order. This is the reason why we kept the subleading constraint addition \( W_{ab}^{(2)} \) free in the equations for the ten metric variables. The constraints we have to add to each of the
equations so that the SNFs $N_\phi$ contain no formally singular terms are,

\[ W^{(2)}_{\phi\psi} = \frac{\nabla_{\phi} R}{R} Z^\sigma + \frac{\nabla_{\phi} C^R}{\tau} Z^\rho, \]

\[ W^{(2)}_{\psi\psi} = \frac{1}{\tau R} \frac{\nabla_{\psi} \phi}{\tau} Z^\sigma + \frac{\nabla_{\psi} C^A_\phi}{\tau} Z^A, \]

\[ W^{(2)}_{\phi A} = -\frac{\nabla_{\phi} C^-_A}{\tau} Z^\sigma, \]

\[ W^{(2)}_{\psi A} = \frac{\nabla_{\psi} C^-_A}{\tau} Z^\sigma + \nabla_{\psi} \phi^\tau Z^\sigma + \nabla_{\psi} C^+ \tau Z^A, \]

\[ W^{(2)}_{\theta\theta} = \frac{2}{\tau R} \frac{\nabla_{\theta} C^+_{\theta}}{\tau} Z^\rho, \]

\[ W^{(2)}_{\theta\phi} = -\frac{\nabla_{\phi} C^+_{\theta}}{\tau} Z^\phi, \]

and all the unmentioned components are zero. This means that these SNFs can now be multiplied by the $\dot{R}^{1+}$ that comes from the LHS of the first equation in (152) or the $\dot{R}^3$ in the second one and still not have diverging factors. We have now used up all the freedom we had in choosing gauge, adding constraints and picking equations and reduction variables, so the set of equations is finally complete. Equations (152)–(154) are a system of 55 first order differential equations that contain a few formally singular terms which will take finite limits on null infinity, (165) and (166).

8. Hyperbolicity

Let us define a vector $\mathbf{v}^\nu$ whose entries are each of the 55 reduction variables,

\[ \mathbf{v} = (\Phi_\psi, \Phi_\phi, \Phi_A, \Phi_\theta, \rho_\psi, \rho_\phi, \rho_A, \rho_\theta, \rho). \]

In order to show hyperbolicity we only need to look at the principal part so, by definition, we can disregard all the SNFs. Moreover, although we want to show that the system (152)–(154) is hyperbolic, we choose to work with the system in the second coordinate system (128), because it is simpler and there is a straightforward way to show that hyperbolicity gets carried over.

8.1. Hyperbolicity in second coordinate system

We take the equations before any change of coordinates, (121), (123) and (124) and see that, as the first coordinate change leaves the directional derivatives untouched, replacing the standard radius with the areal radius leaves us with essentially the same equations. Splitting the null derivatives as combinations of coordinate basis vectors we can write the principal part of our system as the following,

\[ A^T \partial_\nu \mathbf{v} = M^\nu \partial_\nu \mathbf{v} + \mathbf{S}, \]
where $S$ denotes all the non-principal terms and $M^\phi$ and $A^T$ can be written as block diagonal matrices,

$$M^\phi = \text{diag}(P^\phi, \ldots, P^\phi),$$

$$A^T = \text{diag}(Q, \ldots, Q).$$

Here all entries left blank are zero and $P^\phi$ and $Q$ can be written as,

$$P^\phi = \begin{bmatrix}
-C^R_{\phi A} & 0 & \tau e^\phi g^{AB} & 0 \\
0 & -C^R_{\phi A} & \tau e^\phi \phi g^{AB} & 0 \\
\frac{C^\phi}{\tau^2} \hat{g}^A & -\frac{C^\phi}{\tau^2} \hat{g}^A & \alpha_\phi & 0 \\
0 & 0 & 0 & -C^R_{\phi A}
\end{bmatrix},$$

(171)

and,

$$Q = \begin{bmatrix}
1 & 0 & \tau e^\phi g^{AB} \hat{C}_B & 0 \\
0 & 1 & \tau e^\phi \phi g^{AB} \hat{C}_B & 0 \\
\frac{C^\phi}{\tau^2} \hat{C}_A & \frac{C^\phi}{\tau^2} \hat{C}_A & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

(172)

where $\phi = \partial_R \phi$ and $\alpha := -(\partial_R + \frac{\kappa e^\phi}{2} \tau Z)$. We want to show that our equations are symmetric hyperbolic and that requires finding a matrix $D$ such that $DM^\phi$ and $DA^T$ are symmetric and $DA^T$ is positive definite. We try the simplest possible ansatz, a diagonal matrix and we find that a matrix of the form,

$$D = \text{diag}(\tilde{P}, \ldots, \tilde{P}),$$

(173)

and find that,

$$\tilde{P} = \begin{bmatrix}
\frac{2e^{-\phi}}{\tau^2} C^R & \frac{2e^{-\phi}}{\tau^2} C^R & \frac{\tau e^\phi g^{AB} \hat{C}_B}{\tau^2} & 0 \\
0 & \frac{2e^{-\phi}}{\tau^2} C^R & \frac{\tau e^\phi \phi g^{AB} \hat{C}_B}{\tau^2} & 0 \\
\frac{C^\phi}{\tau^2} \hat{g}^A & \frac{C^\phi}{\tau^2} \hat{g}^A & \alpha_\phi & 0 \\
0 & 0 & 0 & \frac{1}{\tau^2}
\end{bmatrix},$$

(174)

fulfills these criteria. To see this, we show the products $\tilde{P}P^\phi$ and $\tilde{P}A^T$,

$$\tilde{P}P^\phi = \begin{bmatrix}
\frac{2e^{-\phi}}{\tau^2} C^R & 0 & \frac{C^\phi}{\tau^2} \hat{g}^A & 0 \\
0 & \frac{2e^{-\phi}}{\tau^2} C^R & \frac{C^\phi}{\tau^2} \hat{g}^A & 0 \\
\frac{C^\phi}{\tau^2} \hat{g}^A & 0 & \alpha_\phi & 0 \\
0 & 0 & 0 & \frac{\tau e^\phi g^{AB} \hat{C}_B}{\tau^2}
\end{bmatrix},$$

(175)

and,

$$\tilde{P}A^T = \begin{bmatrix}
\frac{2e^{-\phi}}{\tau^2} C^R & 0 & \frac{C^\phi}{\tau^2} \hat{g}^A & 0 \\
0 & \frac{2e^{-\phi}}{\tau^2} C^R & \frac{C^\phi}{\tau^2} \hat{g}^A & 0 \\
\frac{C^\phi}{\tau^2} \hat{g}^A & \frac{C^\phi}{\tau^2} \hat{g}^A & \alpha_\phi & 0 \\
0 & 0 & 0 & \frac{1}{\tau^2}
\end{bmatrix},$$

(176)
and check that these matrices are indeed symmetric. Furthermore, using computer algebra is straightforward to compute the eigenvalues of the matrix (176) and see that it is positive-definite. We then conclude that our system of equations in the coordinates \((T, R, \theta^A)\) is symmetric hyperbolic.

8.2. Coordinate change

Changing the coordinates of (169) to the compactified ones (141) we get,

\[
A^T J_{\langle}^{\alpha} \partial_{\alpha} v = M^\beta p_{\langle}^{\alpha} \partial_{\alpha} v + S
\]

\Rightarrow X\partial_{\alpha} v = (M^\beta p_{\beta} - A^T J_{\beta})\partial_{\beta} v + S,
\]

where \(X := A^T J_{\beta} - M^\beta J_{\beta}^{\beta}\) and \(J_{\alpha}^{\beta}\) are the components of the Jacobian associated to the change \((T, R, \theta^A) \rightarrow (t, r, \bar{\theta}^A)\). We can then write that,

\[
\partial_{\beta} v = M^\beta \partial_{\beta} v + \bar{S},
\]

where \(M^\beta\) is given by,

\[
\bar{M}^\beta = X^{-1}(M^\beta J_{\beta}^{\beta} - A^T J_{\beta}).
\]

It can be seen that the Jacobian of the change of coordinates and its inverse are,

\[
J = \begin{bmatrix}
1 & 0 & 0 \\
H' R' & 0 & 0 \\
\partial_{\beta} H & 0 & 1
\end{bmatrix}, \quad J^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
-H' & 0 & 0 \\
-\partial_{\beta} H & 0 & 1
\end{bmatrix},
\]

respectively, so that,

\[
\bar{M}^\beta = X^{-1} M^\beta J_{\beta}^{\beta}.
\]

We claim that the matrix \(DX\) diagonalizes the new matrix \(M^\beta\) and to show that we only need to prove that \(DX M^\beta\) is symmetric and that \(DX\) is positive definite. It is easy to see that \(DX M^\beta\) is indeed symmetric, so let us analyze the second condition. We have that,

\[
DX = D A^T + M^\beta H' + M^\beta (\partial_{\beta} H).
\]

Once again using computer algebra to compute the eigenvalues of this matrix we can easily see that they are all positive. Therefore we conclude that our system in radially compactified coordinates, (152)–(154) is symmetric hyperbolic.

9. Conclusions

Continuing our research program on the inclusion of null-infinity in the computational domain, we studied the hyperboloidal compactification of the dual-frame GHG formulation. Our aim was to obtain the most regular form of the equations of motion with this formulation on the compactified domain. We have given a procedure which, upon careful choice of gauge and constraint addition, allows us to turn the EFEs in GHG into a set of 11 second order differential equations, ten of which fall into the categories of good and ugly, whose asymptotic solutions behave respectively like those of the wave equation or decay faster. The eleventh is a wave equation satisfied by a gauge driver, an auxiliary function that is used to sway the asymptotics of one of the metric functions without spoiling hyperbolicity. As shown in \[26\], polyhomogeneous expansions of ugly fields may have logarithmically divergent terms in subleading orders in \(R^{-1}\). Our method allows us to make sure that those terms are not generated...
below a specified order. Therefore, this approach effectively provides a way to help regularize
the equations at null infinity. As the character of the metric functions is determined solely by
their asymptotics and that of their first derivatives, the choice of gauge and constraint addition
that allows for this to happen is not unique.

Treating the first derivatives of the original variables as evolved variables in their own right
we transformed the system of 11 second order wave equations into a set of first order equations.
We then made use of the torsion free conditions to find additional equations for the remain-
ing variables and regarded the definitions of the reduction variables as reduction constraints
and transformed to a radially compactified hyperboloidal coordinate system. Formally singular
terms were identified in most of the equations and almost all of them were then canceled
through another constraint addition at subleading order, leaving the final set of equations with
only two kinds of simple formally singular terms that can be dealt with easily using L’Hôpital’s
rule. What is more, one of these two types of singular terms serve the specific purpose of
suppressing unphysical radiation fields that are otherwise present when using harmonic-like
gauges. We concluded by showing that the final system is symmetric hyperbolic. Despite the
presence of formally singular terms, one strength of the approach from the numerical perspec-
tive is that it is very close to standard formulations used in numerical relativity. We have thus
opened the possibility to use well-established numerical techniques to treat to the strong field
region (including for instance compact binaries) in conjunction with our proposed setup for
compactification.

In this paper we have focused exclusively on the evolution problem and ignored completely
the question of finding suitable constraint solved initial data. Nonetheless we note that the con-
straint equations on hyperboloidal slices have been studied, see for example [12, 32–34], and
we expect that such data can be constructed numerically and evolved using the formulation put
forward. Another shortcoming of the derivations here is that we have viewed the field equations
primarily as a set of partial differential equations rather than geometrically. Aesthetically it is
therefore appealing to give a more geometric version of the formulation, but that is again work
for the future.

Data availability statement

All data that support the findings of this study are included within the article (and any supple-
mentary files).

Acknowledgments

The Authors wish to thank Alex Vañó-Viñuales, Christian Peterson Bórquez and Shalabh
Gautam for helpful discussions. M D acknowledges support from FCT (Portugal) program
PD/BD/135511/2018, D H acknowledges support from the FCT (Portugal) IF Program
IF/00577/2015, PTDC/MAT- APL/30043/2017. J F acknowledges support from FCT (Por-
tugal) programs PTDC/MAT-APL/30043/2017, UIDB/00099/2020. E G acknowledges sup-
port from FCT (Portugal) investigator Grant 2020.03845.CEECIND.

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