Necessary and sufficient condition for quantum state-independent contextuality

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We solve the problem of whether a set of quantum tests reveals state-independent contextuality and use this result to identify the simplest set of minimal dimension. We also show that identifying state-independent contextuality graphs [R. Ramanathan and P. Horodecki, Phys. Rev. Lett. 112, 040404 (2014)] is not sufficient for revealing state-independent contextuality.

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Introduction.—Contextuality, i.e., that the result of a measurement does not reveal a preexisting value that is independent of the set of concomensurate measurements jointly realized (i.e., the context of the measurement), is one of the most striking features of quantum theory and has been recently identified as a critical resource for quantum computing [1–3]. The earliest manifestation of contextuality in quantum theory is the Kochen-Specker theorem [4, 5], which states that, if the dimension $d$ of the quantum system is greater than $2$, there exists a finite set of elementary tests (represented by rank-one projectors in quantum theory) such that a value $1$ or $0$ (representing true or false, respectively) cannot be assigned to each of them respecting that: (i) result $1$ cannot be assigned to two mutually exclusive tests (represented in quantum theory by mutually orthogonal projectors), and (ii) result $1$ must be assigned to exactly one of $d$ mutually exclusive tests. Sets of elementary tests in which this assignment is impossible are called Kochen-Specker sets [6].

Assumptions (i) and (ii) are not needed for detecting contextuality. It can be revealed by the violation of correlation inequalities satisfied by any model with noncontextual results. These inequalities are called noncontextuality (NC) inequalities [7]. Bell inequalities [8] are a special case of them.

Remarkably, there are NC inequalities which are violated by any quantum state for a fixed set of measurements [9]. A NC inequality with this property is called a state-independent NC (SI-NC) inequality, whereas a set of elementary tests which can be used for such a state-independent violation is called a state-independent contextuality (SIC) set.

Every Kochen-Specker set is a SIC set [10], but there are SIC sets that are not Kochen-Specker sets [11, 12]. This observation, together with the experimental implementation of SIC sets for testing SI-NC inequalities [13–18] and the emergence of applications of SIC sets (e.g., device-independent secure communication [19], local contextuality-based nonlocality [20], Bell inequalities revealing full nonlocality [21], state-independent quantum dimension witnessing [22], and state-independent hardware certification [23]) motivated the interest in the problem of identifying SIC sets.

In some cases, one can guess that a given set of elementary quantum tests is a SIC set. Then, to prove it, it is sufficient to construct a SI-NC inequality violated by these tests. For example, the set of elementary quantum tests associated to the Peres-Mermin square [24, 25] violates a SI-NC inequality [9], therefore it is a SIC set. However, in general, one cannot follow this strategy and it is convenient to adopt a more general point of view and consider not a specific set of elementary quantum tests, but all sets of elementary quantum tests with a given exclusivity graph. In this graph, vertices correspond to tests and edges occur when two tests are mutually exclusive. Since elementary tests are represented by rank-one projectors and two of them are mutually exclusive if and only if the corresponding projectors are orthogonal, the exclusivity graph is equivalent to the orthogonality graph of the corresponding projectors. This approach using graphs has been very successful in investigating the general properties of quantum contextuality [26–28] and the separation between quantum theory and other hypothetical theories [29–31].

An open question is when, for a given orthogonality graph, there exists a realization of the graph which is a SIC set. Unfortunately, it has been notoriously difficult to answer this question [32]. The aim of this Letter is to provide a versatile tool that allows to approach this problem.

Recently, Ramanathan and Horodecki (RH) [33] have presented a solution to a relaxation of the problem of identifying SIC sets, namely of identifying “SIC graphs.” That is, whether a given graph admits, for any given state, a realization as a set of projectors (with orthogonality relations corresponding to edges in the graph) such that the correlations of such projectors on that state violate some NC inequality. This definition fits neither with the definition of SIC set above nor with most of the previous literature (cf. Refs. [9–12, 20–23, 32]). As far as we know, the only work where a similar definition has been used is Ref. [34]. Moreover, the definition of a SIC set in Ref. [33] is not state-independent on an operational level. The issue is that, according to this definition, the realization of a SIC graph may depend on the state; the
set of measurements that violate the NC inequality may be different for different initial states. Therefore, the definition is not state-independent on an operational level. To make an analogy, adopting a similar definition one will reach to the conclusion that a pentagon is a “SIC graph for pure states,” since any pure state will violate the Klyachko-Can-Bincioğlu-Shumovsky NC inequality [35] for some five rank-one projectors whose orthogonality graph is a pentagon. In contrast, the problem of identifying SIC sets not only has a long tradition (see, e.g., Refs. [6, 11, 12, 36, 37]), but an immediate experimental translation (see Refs. [16–18, 23]).

To prove that the result in Ref. [33] does not solve the problem of identifying SIC sets, we begin by showing that there exists a SIC graph for which no realization violates a NC inequality for every quantum state (Theorem 1). After that, we present a solution to the problem of identifying SIC sets (Theorem 3). Finally, we use it to prove a conjecture formulated by Yu and Oh in Ref. [11] on the operational state dependence of a SIC graph as defined in Ref. [33] is apparent in the following theorem.

**Theorem 1.** There exists a SIC graph for which no realization is a SIC set.

**Proof.** In Ref. [33] it is proven that a necessary and sufficient condition for a graph G with a [d, r]-realization (i.e., a realization in dimension d by means of rank-r projectors) to be a SIC graph is that the fractional chromatic number \( \chi_f(G) \) is strictly larger than \( d/r \).

However, consider the 13-vertex graph of Yu and Oh [11], \( G_{YO} \). This graph has a [3, 1]-realization and its fractional chromatic number \( \chi_f(G_{YO}) = 35/11 \). Now consider the 14-vertex graph \( G_{YO+1} \) constructed by adding one vertex to \( G_{YO} \) and linking this new vertex with the 13 vertices of \( G_{YO} \). Clearly, this graph has a [4, 1]-realization and \( \chi_f(G_{YO+1}) = 35/11 + 1 > 4 \). It is true that, for any state in \( d = 4 \), there is a realization which violates a NC inequality. However, whatever the realization, when the system is in the eigenstate corresponding to the new vertex, there is an obvious noncontextual assignment of results.

Now we will address the problem of identifying SIC sets. We first recall a result from Ref. [38], that helps us to identify sets of (not necessarily rank-one) projectors for which there is a SI-NC inequality.

**Theorem 2.** A set of dichotomic observables \( \{A_1, \ldots, A_n\} \) with \( A_k^2 = 1 \) and their corresponding contexts \( C \) (i.e., the set of sets of comeasurable observables) violates a SI-NC inequality of the form

\[
\sum_{c} \lambda_C \prod_{k \in c} A_k \leq \eta \quad (1)
\]

with 0 \( \leq \eta < 1 \) if and only if

\[
\sum_{c} \lambda_C \prod_{k \in c} A_k \leq \eta \quad (2)
\]

where the entries in \( a = (a_1, \ldots, a_n) \) take values \( \pm 1 \).

Then, the necessary and sufficient condition for a set of rank-one projectors to constitute a SIC set is given by the following.

**Theorem 3.** A set of rank-one projectors \( S = \{\Pi_1, \ldots, \Pi_n\} \) is a SIC set if and only if there are nonnegative numbers \( w = (w_1, w_2, \ldots) \) and a number \( 0 \leq y < 1 \) such that

\[
\sum_{j \in \mathcal{I}} w_j \leq y \quad \text{for all } \mathcal{I} \quad \text{and} \quad \sum_{i} w_i \Pi_i \geq 1, \quad (3)
\]

where \( \mathcal{I} \) is any set such that \( i, j \in \mathcal{I} \) implies \( \Pi_i \Pi_j \neq 0 \) (i.e., \( \mathcal{I} \) is any independent set of the orthogonality graph of \( S \)).

In particular, \( w \) gives rise to the SI-NC inequality

\[
\sum_{i} w_i (\Pi_i) - \sum_{i} w_i \sum_{j \in \mathcal{N}(i)} \langle \Pi_i, \Pi_j \rangle \leq y \quad (4)
\]

where \( \mathcal{N}(i) = \{ j \mid \Pi_i \Pi_j = 0 \} \) is the orthogonality neighborhood of \( i \).

**Proof.** Given \((y, w)\), we first show that inequality (4) is violated for every state. For that, it is enough to realize that the best noncontextual assignment of values 1 or 0 for the left-hand side of inequality (4) is one that respects the orthogonality conditions, i.e., two orthogonal projectors cannot have both assigned the value 1. Let \( \Pi_i \) and \( \Pi_j \) be orthogonal projectors and \( p \in \{0, 1\}^n \) be any noncontextual assignment such that \( p_i = 1 \) but \( p_j = 0 \). By changing the value of \( p_j \), we get an extra contribution \( w_j \) from the first term and

\[
- \sum_{k \in \mathcal{N}(j)} (w_j + w_k) p_k \leq -w_j \quad \text{from the second term, decreasing the total value of inequality (4).}
\]

For proving the converse, we start from the most general NC inequality,

\[
\sum_{c} \lambda_C \prod_{k \in c} \Pi_k \leq \eta, \quad (5)
\]

where the sum is over all cliques \( C \) different from the empty set in the orthogonality graph of \( S \) and \( \lambda_C \) are real numbers. Since inequality (5) holds in particular for all assignments respecting orthogonality, we have

\[
\sum_{k \in \mathcal{I}} \lambda_{\{k\}} \Pi_k \leq \eta \quad \text{for any independent set } \mathcal{I}.
\]

At the same time, we assume a state-independent violation and hence, without loss of generality, \( \sum_{k \in \mathcal{I}} \lambda_{\{k\}} \Pi_k \geq 1 \) and \( \eta < 1 \). (In general we have \( \sum_{k \in \mathcal{I}} \lambda_{\{k\}} \Pi_k \geq \xi \Pi \) and \( \eta < \xi \). But the assignment \( p \equiv (0, 0, \ldots) \) yields \( 0 \leq \eta < \xi \), which allows to rescale \( \lambda_C \rightarrow \lambda_C/\xi \) and \( \eta \rightarrow \eta/\xi \). Eventually, we identify \( w_i = \max\{0, \lambda_{\{i\}}\} \) and \( y = \eta \).
Indeed, inequality (5) has to hold for any assignment $p=(p_1,\ldots,p_n)$ respecting orthogonality and having $p_k=0$ for all $\lambda_{i(k)}<0$. This way, the condition in Eq. (3) is obeyed by that identification.

We mention that the condition in Theorem 2 as well as Theorem 3 can be verified by means of a semi-definite program. Semi-definite programs are a class of optimization problems that can be solved numerically with a certificate of optimality [39].

At this point, it is interesting to point out the relation between Theorem 3 and the results in Ref. [33]. According to Ref. [33], to conclude that a graph of orthogonality is a SIC graph, it is sufficient to check the expectation value of $\sum_j w_j \Pi_j$ on the maximally mixed state $\rho=I/d$. Assuming rank-one projectors, we can substitute the condition $\sum_i w_i \Pi_i \geq I$ with $\frac{1}{2} \sum_i w_i \geq 1$. The condition in Eq. (3) can be formulated in terms of the existence of a solution greater than $d$ for the linear program

$$\begin{align*}
\text{maximize:} & \quad \sum_i w_i \\
\text{subject to:} & \quad \sum_{j \in I} w_j \leq 1 \text{ for all } I, \\
& \quad w_i \geq 0 \text{ for all } i.
\end{align*}$$

Every $(w,y)$ obeying Eq. (3) with $y<1$ can be used to achieve $\sum_i w_i > d$ by rescaling all the weights by $1/y$.

The maximum of the linear program in Eq. (6) is the fractional clique number, and it is the dual problem of the fractional chromatic number $\chi_f(G)$ of the orthogonality graph $G$ [40]. Together with the fact that the chromatic number $\chi(G)$ is lower bounded by the fractional chromatic number $\chi_f(G)$ [40], we have the following.

**Theorem 4.** Necessary conditions for a set of rank-one projectors in dimension $d$ to be a SIC set are that for the orthogonality graph $G$, (i) $\chi_f(G) > d$ and (ii) $\chi(G) > d$.

Condition (i) is also a direct consequence of the results in Ref. [33], where it was in addition demonstrated that in general condition (ii) is strictly weaker than condition (i). However, condition (ii) has the advantage of being solvable exactly by simple integer arithmetics, while condition (i) is the solution to a linear program.

The minimal dimension in which SIC sets exist is $d=3$ [5]. Therefore, identifying the smallest SIC set in $d=3$ is a problem of fundamental importance. Using the previous results we can prove a conjecture from Ref. [11].

**Theorem 5.** In dimension $d=3$ there exists no SIC set with less than 13 projectors. The set provided by Yu and Oh in Ref. [11] is therefore the simplest for $d=3$.

**Proof.** The orthogonality graph of a SIC set has to obey at least the following necessary conditions: (a) that the graph has a $[3,1]$-representation, and (b) that the graph has a fractional chromatic number greater than 3.

From condition (a) it follows that the graph must be square-free, because for a projector represented by a vertex of the square, the other two connected to it must be in the orthogonal plane, and the fourth is orthogonal to both, so it must be the same as the first.

We use the utility geng from the software package nauty v2.5r9 [41] to generate all $143129$ nonisomorphic connected square-free graphs with 12 or less vertices and then calculate the chromatic number for each graph. There is only one such graph, which is depicted in Fig. 1 (c). By direct calculation one finds that the fractional chromatic number of this graph is $\chi_f(G)=3$.

In addition, we have numerical evidence that the only SIC set with 13 projectors is Yu and Oh’s graph $G_{YO}$ depicted in Fig. 1 (a). There are in total eight square-free graphs with 13 vertices and $\chi(G)>3$ [42], and out of these eight graphs, up to numerical precision, only three have $\chi_f(G) > 3$ [43]. Two of them are depicted in Fig. 1, respectively, (a) $G_{YO}$ and (b) $G_{YO}$ minus one edge, together with the 12-vertex graph (c), which is a common induced subgraph of all remaining 13-vertex graphs with $\chi(G)>3$. As can be checked by a polynomial optimization problem, out of those three graphs only $G_{YO}$ can be the orthogonality graph of a SIC set in dimension $d=3$.

**Conclusion.**—We have started arguing that the definition of “state-independent contextuality scenario” used in Ref. [33] is inconsistent with almost all the previous literature on the topic and is not state-independent on an operational level because the realization of the scenario depends on the state. Then we have shown that the criterion proposed in Ref. [33] does not solve the problem of whether or not a set of quantum tests reveals state-independent contextuality in the sense defined in most of the literature, including all experimental implementations and applications. Then we have presented a solution to this problem and explained the connection between this solution and the results in Ref. [33]. Finally, we have used our result to prove that Yu and Oh’s set is the simplest set of elementary quantum tests revealing state-independent contextuality in dimension 3. Our results clarify the structure of state independent contextuality and—as we demonstrated on an example—enables the systematic investigation of state independent contex-
tuality sets.

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[1] M. Howard, J. J. Wallman, V. Veitch, and J. Emerson, Nature (London) 510, 351 (2014).
[2] R. Raussendorf, Phys. Rev. A 88, 022322 (2013).
[3] N. Delfosse, P. A. Guerin, J. Bian, and R. Raussendorf, Phys. Rev. Lett. 102, 010401 (2009).
[4] E. P. Specker, Dialectica 14, 239 (1960). English translation: arXiv:1103.4537.
[5] S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).
[6] M. Pavičić, J.-P. Merlet, B. D. McKay, and N. D. Megill, J. Phys. A: Math. Gen. 38, 1577 (2005); J. Phys. A: Math. Gen. 38, 3709 (2005).
[7] R. W. Spekkens, D. H. Buzacott, A. J. Keehn, Ben Toner, and G. J. Pryde, Phys. Rev. Lett. 103, 210404 (2009).
[8] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
[9] A. Cabello, Phys. Rev. Lett. 101, 210401 (2008).
[10] P. Badziag, I. Bengtsson, A. Cabello, and I. Pitowsky, Phys. Rev. Lett. 103, 050401 (2009).
[11] S. Yu and C. H. Oh, Phys. Rev. Lett. 108, 030402 (2012).
[12] I. Bengtsson, K. Blanchfield, and A. Cabello, Phys. Lett. A 376, 374 (2012).
[13] G. Kirchmair, F. Zähringer, R. Gerritsma, M. Kleinmann, O. Gühne, A. Cabello, R. Blatt, and C. F. Roos, Nature (London) 460, 494 (2009).
[14] E. Amselem, M. Rädmack, M. Bourennane, and A. Cabello, Phys. Rev. Lett. 103, 160405 (2009).
[15] O. Moussa, C. A. Ryan, D. G. Cory, and R. Laflamme, Phys. Rev. Lett. 104, 160501 (2010).
[16] C. Zfu, Y.-X. Wang, D.-L. Deng, X.-Y. Chang, K. Liu, P.-Y. Hou, H.-X. Yang, and L.-M. Duan, Phys. Rev. Lett. 109, 154001 (2013).
[17] V. D’Ambrosio, I. Herbut, E. Amselem, E. Nagali, M. Bourennane, F. Sciarrino, and A. Cabello, Phys. Rev. X 3, 011012 (2013).
[18] G. Caiaias, S. Etcheverry, E. Gómez, C. E. Saavedra, G. B. Xavier, G. Lima, and A. Cabello, Phys. Rev. A 90, 012119 (2014).
[19] K. Horodecki, M. Horodecki, P. Horodecki, R. Horodecki, M. Pawłowski, and M. Bourennane, arXiv:1006.0468.
[20] A. Cabello, Phys. Rev. Lett. 104, 220401 (2010).
[21] L. Aolita, R. Gallego, A. Acín, A. Chiuri, G. Vallone, P. Mataloni, and A. Cabello, Phys. Rev. A 85, 032107 (2012).
[22] O. Gühne, C. Budroni, A. Cabello, M. Kleinmann, and J.-A. Larsson, Phys. Rev. A 89, 062107 (2014).
[23] G. Caiaias, M. Arias, S. Etcheverry, E. Gómez, A. Cabello, G. B. Xavier, and G. Lima, Phys. Rev. Lett. 113, 090404 (2014).
[24] A. Peres, Phys. Lett. A 151, 107 (1990).
[25] N. D. Mermin, Phys. Rev. Lett. 65, 3373 (1990).
[26] A. Cabello, S. Severini, and A. Winter, arXiv:1010.2163.
[27] A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. 112, 040401 (2014).
[28] A. Acín, T. Fritz, A. Leverrier, and A. B. Sainz, to appear in Comm. Math. Phys.; arxiv:1212.4084.
[29] A. Cabello, Phys. Rev. Lett. 110, 060402 (2013).
[30] T. Fritz, A. B. Sainz, R. Augusiak, J. Bohr Brask, R. Chaves, A. Leverrier, and A. Acín, Nat. Commun. 4, 2263 (2013).
[31] A. Cabello, Phys. Rev. A 90, 062125 (2014).
[32] A. Cabello, arXiv:1112.5149v1.
[33] T. Fritz, A. B. Sainz, R. Augusiak, J. Bohr Brask, R. Chaves, A. Leverrier, and A. Acín, Nat. Commun. 4, 2263 (2013).
[34] A. Cabello, Phys. Rev. A 90, 062125 (2014).
[35] A. Cabello, Phys. Rev. Lett. 112, 040404 (2014).
[36] P. Kurzyński and D. Kaszlikowski, Phys. Rev. A 86, 042125 (2012).
[37] A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. 101, 200403 (2008).
[38] A. Cabello, J. M. Esteban Aranz. and G. García-Alcaine, Phys. Lett. A 212, 183 (1996).
[39] P. Lisonek, Badziag, J. R. Portillo, and A. Cabello, Phys. Rev. A 89, 042101 (2014).
[40] M. Kleinmann, C. Budroni, J.-A. Larsson, O. Gühne, and A. Cabello, Phys. Rev. Lett. 109, 250402 (2012).
[41] L. Vandenbergh and S. Boyd., SIAM Rev., 38, 49 (2001).
[42] E. R. Scheinerman and D. H. Ullman, Fractional Graph Theory: A Rational Approach to the Theory of Graphs (Dover, New York, 2011).
[43] B. D. McKay and A. Piperno, J. Symbolic Computation 60, 94 (2014).
[44] In the graph6 encoding ([http://cs.anu.edu.au/~ebdm/data/formats.html](http://cs.anu.edu.au/~ebdm/data/formats.html)), they are: “L?AE?oDDIQSUS”, “L?AE?OFCDHISPS”, “L?ABA_oO_oREJa”, “L?ABAgf0bWgHc”, “L?ABEagE’gH’/c” (which is $G_{YO}$ minus one edge), “L?AB?v0LDPHa’a” (which is $G_{YO}$), “L?BDA_gEREHcac”, and “L?bCUCbDgWc”.
[45] In the graph6 encoding, “L?ABEagE’gH’/c”, having $\chi_f = 19/6$, “L?AB?v0LDPHa’a”, having $\chi_f = 35/11$, and “L?bCUCbDgWc”, having $\chi_f = 13/4$. 