On extensions of Leibniz algebras

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Abstract. This paper is dedicated to the study of extensions of Leibniz algebras using the annihilator approach. The extensions methods have been used earlier to classify certain classes of algebras. In the paper we first review and adjust theoretical background of the method for Leibniz algebras then apply it to classify four-dimensional Leibniz algebras over a field \( K \). We obtain complete classification of four-dimensional nilpotent Leibniz algebras. The main idea of the method is to transfer the "base change" action to an action of automorphism group of the algebras of smaller dimension on cocycles constructed by the annihilator extensions. The method can be used to classify low-dimensional Leibniz algebras over other finite fields as well.

1. Introduction

This paper describes an approach for the classification of Leibniz algebras over finite fields, which is equivalent to the Skjelbred-Sund method (see \([2, 3]\)) for the classification of Lie algebras. The approach is principally based on the actions of group automorphisms on Grassmannian of subspaces of the 2nd cohomology groups of algebras of smaller dimension.

This approach was also used in the classifications of nilpotent cases of Leibniz, Jordan and Malcev algebras over different fields (see \([4, 5, 6, 7, 8]\)).

2. Extension of Leibniz algebra via annihilator

In this section we introduce the concept of an annihilator extension of Leibniz algebras.

**Definition 2.1.** An algebra \( L \) over a field \( K \) is called a Leibniz algebra if its bilinear operation \([\cdot, \cdot]\) satisfies the identity:

\[
[[x, y], z] = [[x, z], y] + [x, [y, z]]; \quad \text{for all } x, y, z, \in L.
\]

In fact, this is the definition of right Leibniz algebra.

Herein and after the notation \( L \) will be used for a Leibniz algebra over a field \( K \).

**Definition 2.2.** For a given \( L \), the ideal

\[
\text{Ann}(L) = \{ a \in L : [L, a] = [a, L] = 0 \}
\]

describe the annihilator of a \( L \).
We define a series of ideals of $L$ by setting

$$L^1 = L, \quad L^k = [L^{k-1}, L].$$

One has

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq \cdots \supseteq L^k \supseteq L^{k+1} \supseteq \cdots$$

**Definition 2.3.** An algebra $L$ is called nilpotent, if there exists a positive integer $k \in \mathbb{N}$ such that $L^k = \{0\}$

Note that $\text{Ann}(L)$ of a nilpotent algebra $L$ is nontrivial.

**Lemma 2.1.** Let $\psi$ be an isomorphism between two Leibniz algebras $L_1$ and $L_2$. Then for any $k \in \mathbb{N}$, $\psi(\text{Ann}(L_1^k)) = \text{Ann}(L_2^k)$.

**Proof.** Considering $\psi$ an isomorphism, it implies $[\psi(a), \psi(L_1^k)] = 0$ if and only if $[a, L_1^k] = 0$. Furthermore, we have $\psi(L_1^k) = L_2^k$ for any $k \in \mathbb{N}$. Therefore, for all $k \in \mathbb{N}$, we get:

$$\psi(\text{Ann}(L_1^k)) = \psi(\{a \in L_1 : [a, L_1^k] = [L_1^k, a] = 0\}) = \{\psi(a) \in \psi(L_1) : [\psi(a), \psi(L_1^k)] = [\psi(L_1^k), \psi(a)] = 0\} = \{b \in L_2 : [b, L_2^k] = [L_2^k, b] = 0\} = \text{Ann}(L_2^k).$$

**Definition 2.4.** Let suppose that $\{L_i\}$ be a sequence of algebra $L$ and $\{\psi_i\}$ be the sequence of its homomorphisms from $L_i$ to $L_{i+1}$, then

$$\cdots \longrightarrow L_i \xrightarrow{\psi_i} L_{i+1} \xrightarrow{\psi_{i+1}} L_{i+2} \xrightarrow{\psi_{i+2}} \cdots$$

is called exact if one has

$$\text{im } \psi_i = \ker \psi_{i+1}$$

for each $i$.

**Definition 2.5.** Let $L_1$, $L_2$, and $L_3$ be Leibniz algebras. The algebra $L_2$ is called an extension of $L_3$ by $L_1$ if there exist homomorphisms $\psi_1 : L_1 \rightarrow L_2$ and $\psi_2 : L_2 \rightarrow L_3$ such that the following sequence

$$0 \rightarrow L_1 \xrightarrow{\psi_1} L_2 \xrightarrow{\psi_2} L_3 \rightarrow 0$$

is exact.

**Definition 2.6.** The sequence

$$0 \rightarrow L_1 \xrightarrow{\psi_1} L_2 \xrightarrow{\psi_2} L_3 \rightarrow 0$$

is called an annihilator extension if the kernel of $\psi_2$ is contained in the annihilator of $L_2$. That is, $\ker \psi_2 \subset \text{Ann}(L_2)$.

It is easy to see from the definition above that the annihilator extension of the algebra $L_1$ must be abelian. In the next section we construct such an extension by using 2-cocycle on $L_3$. 
3. Leibniz algebra cocycles

We introduce in this part the concept of 2-cocycle for algebras $L$ and give a few of its properties.

**Definition 3.1.** Let $L$ be a Leibniz algebra and $V$ a vector space with values in $\mathbb{K}$. A bilinear map $\theta : L \times L \to V$ satisfying:

$$\theta([x, y], z) = \theta([x, z], y) + \theta(x, [y, z]) \quad \text{where } x, y \text{ and } z \in L$$

is known as Leibniz 2-cocycle. We indicate the entire set of all 2-cocycles by $Z^2_L(L, V)$.

Consider a vector space $L_\theta = L \oplus V$. Define a multiplication $[\cdot, \cdot]$ on $L_\theta$ by

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2]_L + \theta(x_1, x_2), \quad \forall x_1, x_2 \in L \text{ and } v_1, v_2 \in V$$

Obviously $L_\theta$ is an algebra with respect to the multiplication $[\cdot, \cdot]$.

The following lemma can be easily proven.

**Lemma 3.1.** The algebra $L_\theta$ is a Leibniz algebra if and only if $\theta$ is a Leibniz 2-cocycle.

A particular type of 2-cocycles called a coboundary, define as follows.

**Definition 3.2.** Let $L$ be a Leibniz algebra and $V$ a vector space over a field $\mathbb{K}$. Furthermore, $f : L \to V$. Then, the bilinear map $\delta f : L \times L \to V$, defined as

$$\delta f(x, y) = f([x, y]),$$

is named as a coboundary. The entire set of all coboundaries is denoted by $B^2(L, V)$.

Obviously $BL^2(L, V)$ is a subspace of $ZL^2(L, V)$. Using the notion of 2-cocycles and 2-coboundaries, one defines the second cohomology group of a Leibniz algebra $L$ by $V$ as follows:

$$HL^2(L, V) = ZL^2(L, V)/BL^2(L, V)$$

**Definition 3.3.** Suppose that $L$ be a Leibniz algebra and $\theta \in ZL^2(L, V)$. The set

$$\theta^\perp = \{ x \in L : \theta(x, y) = \theta(y, x) = 0 \text{ for all } y \in L \}$$

is named the radical of $\theta$.

**Lemma 3.2.** Suppose that $\theta \in ZL^2(L, V)$. Then $Ann(L_\theta) = (\theta^\perp \cap Ann(L)) \oplus V$.

When constructing a Leibniz algebra as $L_\theta = L \oplus V$, we restrict $\theta$ such that $Ann(L_\theta) = V$. In this way we can avoid constructing the same Leibniz algebra as annihilator extension of different Leibniz algebras.

Suppose that $L$ is a algebra with a basis $\{e_1, e_2, e_3, \ldots, e_n\}$. Then, by $\Delta_{i,j}$ we describe the bilinear form $\Delta_{i,j} : L \times L \to \mathbb{K}$ by $\Delta_{i,j}(e_l, e_m) = 1$, wherever $\{i, j\} = \{l, m\}$ and 0 otherwise.
4. Analogue of the Skjelbred-Sund theorem

There is an action of $\text{Aut}(L)$ on $ZL^2(L, \mathbb{K})$ as follows: let $\phi \in \text{Aut}(L)$ and $\theta \in ZL^2(L, \mathbb{K})$ then

$$(\phi \cdot \theta)(x, y) = \theta(\phi(x), \phi(y))$$

Let assume that $G_m(\mathcal{HL}^2(L, \mathbb{K}))$ be the Grassmanian of subspaces of dimension $m$ in $\mathcal{HL}^2(L, \mathbb{K})$. The action above can be extended to $G_m(\mathcal{HL}^2(L, \mathbb{K}))$ as follows: suppose that $\phi \in \text{Aut}(L)$ and

$$T = \langle \theta_1, \theta_2, \theta_3, \cdots, \theta_s \rangle \in G_m(\mathcal{H}^2(L, \mathbb{K})).$$

We define

$$\phi \cdot T = \langle \phi \theta_1, \phi \theta_2, \phi \theta_3, \cdots, \phi \theta_s \rangle.$$

It follows that, $\phi \cdot T \in G_m(\mathcal{HL}^2(L, \mathbb{K}))$. Let express the orbit of $T \in G_m(\mathcal{HL}^2(L, \mathbb{K}))$ under the action of group automorphism $\text{Aut}(L)$ on $G_m(\mathcal{HL}^2(L, \mathbb{K}))$ as $\text{Orb}(T)$.

We have the following lemmas.

**Lemma 4.1.** Let $T_1$ and $T_2$ be two elements of $G_m(\mathcal{HL}^2(L, \mathbb{K}))$ defined by $T_1 = \langle \theta_1, \theta_2, \theta_3, \cdots, \theta_m \rangle$ and $T_2 = \langle \vartheta_1, \vartheta_2, \vartheta_3, \cdots, \vartheta_m \rangle$. If $T_1 = T_2$, then

$$\cap_{i=1}^m \theta_i^\perp \cap \text{Ann}(L) = \cap_{i=1}^m \vartheta_i^\perp \cap \text{Ann}(L).$$

As a consequence of Lemma 4.1 above, we define the subspace

$$\mathcal{W}_m(L) = \{T = \langle \theta_1, \theta_2, \theta_3, \cdots, \theta_m \rangle \in G_s(\mathcal{H}^2(L, \mathbb{K})) : \cap_{i=1}^m \theta_i^\perp \cap \text{Ann}(L) = 0 \}.$$  

**Lemma 4.2.** $\mathcal{W}_m(L)$ is stable under the action of automorphism group, $\text{Aut}(L)$.

Suppose that $L$ is a Leibniz algebra, $V$ be a $m$-dimensional vector space with a basis $\{e_1, e_2, e_3, \cdots, e_m\}$. Let $\mathcal{M}(L, V)$ denote the set of all Leibniz algebras without abelian components.

**Theorem 4.3.** (Analogue of the Skjelbred-Sund theorem). Suppose that $L$ is a Leibniz algebra with values over a field $\mathbb{K}$. Then there is a bijective correspondence between the set of $\text{Aut}(L)$-orbits on $\mathcal{W}_m(L)$ and the set of isomorphism classes of $\mathcal{M}(L, V)$.

In the next section we apply this theorem, to get an algorithm to construct all nilpotent Leibniz algebras of dimension $n$ over finite fields given those algebras of dimension $n - m$ in the following way:

(i) For a given Leibniz algebra of dimension $n - m$, we determine the annihilator $\text{Ann}(L)$, to identify the 2-cocycles satisfying $\theta^\perp \cap \text{Ann}(L) = 0$.

(ii) Determine the 2-cocycles, the 2-coboundaries and compute the quotient $\mathcal{HL}^2(L, \mathbb{K})$. To each algebra $L$ with given basis $\{e_1, e_2, e_3, \cdots, e_n\}$ the 2-cocycle $\theta$ is represented by a matrix $(c_{ij})$ as follows $\theta = \sum_{i,j=1}^n c_{ij}\Delta_{ij}$, where $\Delta_{ij}$ have been defined in the end of Section 3. When computing the 2-cocycle, we will just list all the constraints on the elements $c_{ij}$ of the matrix $(C_{ij})$.

(iii) Find the set of orbit of $\text{Aut}(L)$ on $\mathcal{W}_m(L)$.

(iv) For all orbit, construct a Leibniz algebra which corresponds to its representative.
5. Application

In this part, we apply the analogous of Skjelbred-Sund method (Theorem 4.3) to classify 4-dimensional non-Lie nilpotent Leibniz algebras over $\mathbb{K} = \mathbb{Z}_3$. We make use the list of 3-dimensional non-Lie Leibniz algebras over $\mathbb{Z}_3$ in [1]. We denote $L_{ij}$ to represent the $j$th algebra, of dimension $i$.

We have the following lemma:

Lemma 5.1. Up to isomorphism, there exist five explicit representatives of four-dimensional non-Lie nilpotent Leibniz algebras over $\mathbb{Z}_3$ given as follows:

\[
L_{4,1}(\mathbb{Z}_3) : [e_1, e_3] = e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4, \\
L_{4,2}(\mathbb{Z}_3) : [e_1, e_3] = e_4, \quad [e_2, e_2] = e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4, \\
L_{4,3}(\mathbb{Z}_3) : [e_1, e_3] = e_4, \quad [e_2, e_2] = 2e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4, \\
L_{4,4}(\mathbb{Z}_3) : [e_2, e_3] = e_4, \quad [e_3, e_2] = 2e_1, \\
L_{4,5}(\mathbb{Z}_3) : [e_1, e_3] = e_2, \quad [e_2, e_2] = e_4, \quad [e_3, e_3] = e_1.
\]

Proof.

(i) One-dimensional Annihilator Extension of $L_{3,5}$ : $[e_2, e_2] = e_1, \quad [e_3, e_3] = e_1$

$HL^2(L, \mathbb{K}) = span\{\Delta_{2,3}, \Delta_{3,3}\}$. The annihilator $Ann(L_{3,5}) = span\{e_1\}$. Consider an arbitrary subspace $\mathcal{T} \in \mathcal{W}_5(L_{3,5})$ i.e., a subspace spanned by $\theta = a\Delta_{2,3} + b\Delta_{3,2}$.

Consequently, $\theta$ has $e_1$ in its radical. Hence, there is no annihilator extension of $L_{3,5}$ with 1-dimensional annihilator. That is, $\theta^\perp \cap Ann(L_{3,5}) \neq 0$.

(ii) One-dimensional Annihilator Extension of $L_{3,6}$ : $[e_2, e_2] = e_1, \quad [e_3, e_3] = 2e_1$

Here $HL^2(L, \mathbb{K}) = span\{\Delta_{2,3}, \Delta_{3,2}\}$. Moreover, the annihilator $Ann(L_{3,6}) = span\{e_1\}$. Consider an arbitrary subspace $\mathcal{T} \in \mathcal{W}_6(L_{3,6})$ i.e., a subspace spanned by $\theta = a\Delta_{2,3} + b\Delta_{3,2}$.

However, $\theta$ has $e_1$ in its radical. Hence, there is no annihilator extension of $L_{3,6}$ with 1-dimensional annihilator. That is, $\theta^\perp \cap Ann(L_{3,6}) \neq 0$.

(iii) One-dimensional Annihilator Extension of $L_{3,7}$ : $[e_2, e_2] = e_1, \quad [e_2, e_3] = e_1, \quad [e_3, e_3] = e_1$

$HL^2(L, \mathbb{K}) = span\{\Delta_{2,3}\}$. Furthermore, the annihilator $Ann(L_{3,7}) = span\{e_1\}$. Consider an arbitrary subspace $\mathcal{T} \in \mathcal{W}_7(L_{3,7})$ i.e., a subspace spanned by $\theta = a\Delta_{3,2}$.

Thus, $\theta$ has $e_1$ in its radical. Hence, there is no annihilator extension of $L_{3,7}$ with 1-dimensional annihilator. That is, $\theta^\perp \cap Ann(L_{3,7}) \neq 0$.

(iv) One-dimensional Annihilator Extension of $L_{3,8}$ : $[e_2, e_2] = e_1, \quad [e_2, e_3] = e_1, \quad [e_3, e_3] = 2e_1$

The basis of $HL^2(L, \mathbb{K})$ is given as $\langle \Delta_{2,3} \rangle$. Moreover, the annihilator $Ann(L_{3,8}) = span\{e_1\}$. Consider an arbitrary subspace $\mathcal{T} \in \mathcal{W}_5(L_{3,5})$ i.e., a subspace spanned by $\theta = a\Delta_{3,2}$.

However, $\theta$ has $e_1$ in its radical. Hence, there is no annihilator extension of $L_{3,5}$ with 1-dimensional annihilator. That is, $\theta^\perp \cap Ann(L_{3,5}) \neq 0$. 

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(v) One-dimensional Annihilator Extension of $L_3^9$ : $[e_2, e_3] = e_1$.

The basis of $HL^2(L,K) = \text{span}\{\Delta_{1,3}, \Delta_{2,2}, \Delta_{3,2}, \Delta_{3,3}\}$. The annihilator $\text{Ann}(L_3^9) = \text{span}\{e_1\}$. By Theorem 4.3, we need to find the representatives of $\text{Aut}(L_3^9)$-orbit on $W_9(L_3^9)$. Choose an arbitrary subspace $T \in W_9(L_3^9)$.

The basis of $HL^2(L,K) = \text{span}\{\Delta_{1,3}, \Delta_{2,2}, \Delta_{3,2}, \Delta_{3,3}\}$. The annihilator $\text{Ann}(L_3^9) = \text{span}\{e_1\}$. By Theorem 4.3, we need to find the representatives of $\text{Aut}(L_3^9)$-orbit on $W_9(L_3^9)$.

\[\theta = [a, b, c, d] = a\Delta_{1,3} + b\Delta_{2,2} + c\Delta_{3,2} + d\Delta_{3,3},\] such that $\theta_{\perp} \cap \{e_1\} = 0$.

The automorphism group, $\text{Aut}(L_3^9)$ consists of matrices of the form

\[\phi = \begin{pmatrix} a_{22}a_{33} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}\] with $a_{22}a_{33} \neq 0$.

The automorphism group $\phi$ acts on $T$ as follows:

\begin{align*}
a & \rightarrow aa_{22}a_{33} \\
b & \rightarrow ba_{22}^2 \\
c & \rightarrow ca_{22}a_{33} \\
d & \rightarrow aa_{13}a_{33} + da_{33}^2
\end{align*}

We now consider the following cases:

**Case 1:** $b \neq 0$.

**Case 1.1:** Let $a = 0$. By taking $a_{33} = 1$ and $a_{22} = \frac{1}{c}$, we have $c \rightarrow 1$. To fix $c = 1$ requires $a_{22} = 1$. Then

\begin{align*}
a & \rightarrow 0 \\
b & \rightarrow \frac{b}{c^n} = \alpha \\
c & \rightarrow 1 \\
d & \rightarrow a_{13} + d \\
\end{align*}

We now set $a_{13} = -d$. In this case we obtain the representative $T_1 = [a, b, c, d] = [0, \alpha, 1, 0]$. Hence we get the algebra

\[\left[e_2, e_2\right] = \alpha e_4, \quad \left[e_2, e_3\right] = e_1, \quad \left[e_3, e_2\right] = e_4.\]  \hspace{1cm} (1)

**Case 1.2:** Let $a \neq 0$. By Taking $a_{22} = \frac{1}{a_{33}^2}$, we have $a \rightarrow 1$ and to fix it require that $a_{22} = 1$. So we have

\begin{align*}
a & \rightarrow 1 \\
b & \rightarrow b \\
c & \rightarrow a_{33} \\
d & \rightarrow \frac{a_{13}}{a_{33}} + \frac{d}{a_{33}} \\
\end{align*}

**Case 1.2.1:** Assume that $c = 0$. Setting $a_{33} = b$, we have

\begin{align*}
a & \rightarrow 1 \\
b & \rightarrow 1 \\
c & \rightarrow 0 \\
d & \rightarrow \frac{a_{13}}{b^n} + \frac{d}{b} \\
\end{align*}

Taking $\frac{d}{b} = -\frac{a_{13}}{b^n}$ we get

\begin{align*}
a & \rightarrow 1 \\
b & \rightarrow 1 \\
c & \rightarrow 0 \\
d & \rightarrow 0 \\
\end{align*}

This, give rise to the following representative $T_2 = [a, b, c, d] = [1, 1, 0, 0]$. Hence we get the algebra

\[\left[e_1, e_3\right] = e_4, \quad \left[e_2, e_2\right] = e_4, \quad \left[e_2, e_3\right] = e_1.\]  \hspace{1cm} (2)
Case 1.2.2: Assume that \( c \neq 0 \). Setting \( a_{33} = c \), we have
\[
\begin{align*}
  a &\to 1 \\
  b &\to \frac{b}{c} \\
  c &\to 1 \\
  d &\to \frac{a_{13}}{c^2} + \frac{d}{c}
\end{align*}
\]
Taking \( \frac{d}{c} = -\frac{a_{13}}{c^2} \) we have
\[
\begin{align*}
  a &\to 1 \\
  b &\to \frac{b}{c} = \alpha \\
  c &\to 1 \\
  d &\to 0
\end{align*}
\]
Thus, we get the following representative \( \mathcal{T}_3 = [a, b, c, d] = [1, \alpha, 1, 0] \). Hence we get the algebra
\[
[e_1, e_3] = e_4, \quad [e_2, e_2] = \alpha e_4, \quad [e_2, e_3] = e_1, \quad [e_4, e_2] = e_4. \tag{3}
\]

Case 2: \( b = 0 \). Setting \( a_{33} = 1, \frac{1}{a} \) we obtain \( a \to 1 \) and to fix it requires \( a_{22} = 1 \). Hence
\[
\begin{align*}
  a &\to 0 \\
  b &\to \frac{b}{c} \\
  c &\to c \\
  d &\to a_{13} + d
\end{align*}
\]
Now set \( a_{13} = -d \) we have the following representative \( \mathcal{T}_4 = [a, b, c, d] = [0, \alpha, 1, 0] \). Hence we get the algebra
\[
[e_2, e_2] = \alpha e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4. \tag{4}
\]
Algebra (1), (2) are included in (3). Similarly algebra (4) is included in (3) for \( \beta = 0, 1 \). Therefore, we cancel algebras (1), (2) and part of (4). Thus we have the following algebras:
\[
[e_1, e_3] = e_4, \quad [e_2, e_2] = \alpha e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4
\]
For \( \alpha \in \mathbb{Z}_3 \), we have the following algebras:
\[
\begin{align*}
  L_{4,1} : & \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4, \\
  L_{4,2} : & \quad [e_1, e_3] = e_4, \quad [e_2, e_2] = e_4, \quad [e_3, e_3] = e_1, \quad [e_3, e_2] = e_4, \\
  L_{4,3} : & \quad [e_1, e_3] = e_4, \quad [e_2, e_2] = 2e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = e_4
\end{align*}
\]
and for \( \beta = 2 \) in equation 4, we have
\[
L_{4,4} : [e_2, e_2] = e_4, \quad [e_2, e_3] = e_1, \quad [e_3, e_2] = 2e_4
\]
(vi) One-dimensional Annihilator Extension of \( L_{3,15} : [e_1, e_3] = e_2, \quad [e_3, e_3] = e_1 \).
Here we get the basis of \( HL^2(L, \mathbb{K}) = span \{ \Delta_{2,3} \} \). Furthermore, the annihilator \( Ann(L_{3,15}) = span \{ e_2 \} \). According to Theorem 4.3, we need to find the representatives of \( Aut(L_{3,15}) \)-orbit on \( W_{15}(L_{3,15}) \). Consider an arbitrary subspace \( T \in W_{15}(L_{3,15}) \) i.e., a subspace spanned by
\[
\theta = [a] = a\Delta_{2,3} \text{ such that } \theta^\perp \cap \{ e_1 \} = \{ 0 \}.
\]
The automorphism group, \( Aut(L_{3,6}) \) consists of matrices of the form
\[
\phi = \begin{pmatrix}
  a_{33}^2 & 0 & a_{13} \\
  a_{13}a_{33} & 1 & a_{23} \\
  0 & 0 & a_{33}
\end{pmatrix} \text{ with } a_{33} \neq 0
\]
Then, \( \phi \) acts on \( T \) as follows:
Suppose that, \( a \neq 0 \). Choosing \( a_{33} = \frac{1}{a} \), then we have \( a \rightarrow 1 \). To fix \( a = 1 \), we require \( a_{33} = 1 \). Then we get a representative \( T_1 = [a] = [1] \). So we get the following algebra:

\[
L_{4,5} : [e_1, e_3] = e_2, \ [e_2, e_3] = e_4, \ [e_3, e_3] = e_1
\]

6. Conclusion
The annihilator extensions methods have been used earlier to classify certain classes of algebras. In the paper we first review and adjust theoretical background of the method for Leibniz algebras then apply it to classify four-dimensional Leibniz algebras over a field \( \mathbb{K} \). We obtain as lemma 5.1 complete classification of four-dimensional nilpotent non-Lie Leibniz algebras over \( \mathbb{Z}_3 \).

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