Abstract. We give a brief account of a construction called tokens here, which is significant in algebra, analysis, combinatorics, and physics. Tokens allow to express a semigroup on one set via a semigroup convolution on another set. Therefore tokens are similar to intertwining operators but are more flexible.

Contents

1. Introduction 1
2. Newton’s Binomial Formula 2
3. Semigroups and Tokens 4
4. Examples of Tokens 5
5. Conclusion 9
Acknowledgments 9
References 9

Pure mathematics consists of tautologies, analogous to “men are men,” but usually more complicated.

Bertrand Russell History of Western Philosophy, Chap. XVI

1. Introduction

It is a fact of our specialised world that mathematics today is cut across by several borders: between pure and applied, continuous and discreet—just to name few most significant ones. Yet there are many important and vivid links which spread through those “iron curtains”.

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On leave from the Odessa University.
2. Newton’s Binomial Formula

Let us start from the fundamental binomial formula (‘Newton’s binomial’ according to Russian terminology):

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.\]

It has the immediate meaning: to express a power of a sum via powers of summands. This is of a doubtful computational benefit if applied just to real numbers. Besides that we could find several less obvious but not least important implications. It is worth first to restate the formula in a more symmetric form:

\[\frac{(x + y)^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.\]

There are its many different interpretations, see e.g. [3], but we mention just one from each combinatorics, analysis, and algebra.

2.1. Combinatorics. The expression \(x^n/n!\) counts a number of functions from a set \(N\) of \(n\) elements to a set \(X\) of \(x\) elements if we do not distinguish functions obtained by permutations of elements of \(N\). (This certainly true for an integer \(x\) but nothing could prevent a mathematician from “generalisation”). Then formula (3) reads as follows (see Figure 1):

To count number of function from a set \(N\) of \(n\) elements to a union of set \(X\) and \(Y\) of \(x\) and \(y\) elements correspondingly split \(N\) in two subsets \(N'\) and \(N''\) of \(k\) and \(n - k\) elements respectively, multiply number of functions from \(N'\) to \(X\) and \(N''\) to \(Y\) and finally sum up over all possible splittings of \(N\).

The self-evidence of the above rule could serve as a proof of the binomial formula (3).

2.2. Analysis. Let \(D\) be the derivative operator or, equivalently, a linear operator on polynomials defined by the identity:

\[D \frac{x^n}{n!} = \frac{x^{n-1}}{(n-1)!} \quad \text{or more generally} \quad D^{(k)} \frac{x^n}{n!} = \frac{x^{n-k}}{(n-k)!}.\]
Consequently we could modify the formula (2) as follows:

\[
\frac{(x + y)^n}{n!} = \sum_{k=0}^{n} \frac{x^k y^{n-k}}{k! (n-k)!} = \sum_{k=0}^{n} \frac{x^k}{k!} \left( \frac{D(k) y^n}{n!} \right)
\]

(3)

where firstly we extract \(y^n/n!\) out of sum in (3) because it is independent of \(k\) and then extend summation to infinity in (3) because \(D(k)\frac{y^n}{n!} = 0\) for \(k > n\). Comparing the first (3) and the last (3) lines we find that we got an expression for the linear operator of shift by \(x\) acting on function \(y^n/n!\) in terms of linear operator embraced in (3), which is independent of \(n\). Due to its linearity the formula is true for any linear combinations of functions \(y^n/n!\) and under suitable topological assumptions even for certain their limits. So we obtain the Taylor expansion for all functions represented by such limits:

\[
f(y + x) = \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} D(k) \right) f(y) = e^{xD} f(y).
\]

2.3. Algebra. We use the notation \(p(n, x)\) for the polynomial \(x^n/n!\). Then the formula (2) becomes:

\[
p(n, x + y) = \sum_{k=0}^{n} p(k, x) p(n - k, y),
\]

or if we define \(p(n, x) \equiv 0\) for \(k < 0\) and any \(x\) we get:

(4)

\[
p(n, x + y) = \sum_{k=-\infty}^{\infty} p(k, x) p(n - k, y).
\]

From the algebraic point of view we make the following observation.

Observation 2.1. The formula (4) expresses one algebraic operation (the sum of two real numbers in the second argument of \(p\)'s) by another algebraic operation (a convolution over integers of first arguments of \(p\)'s).

This type of polynomials deserve a special name.

Definition 2.2. \([2, 12]\) A convolution polynomials, that is, a sequence of polynomials \(p(n, x)\) in \(x\) with \(\deg p(n, x) = n\) and complex coefficients, satisfying the identities (4).

A cousin object is as follows:

Definition 2.3. \([16]\) A polynomial sequence of binomial type, that is, a sequence of polynomials \(p(n, x)\) in \(x\) with \(\deg p(n, x) = n\) and complex coefficients, satisfying the identities:

(5)

\[
p(n, x + y) = \sum_{k=-\infty}^{\infty} \binom{n}{k} p(k, x) p(n - k, y).
\]

We prefer the explicit structure of convolution in (4) vs. (5). The good news is that there is an unlimited number of such polynomials.
Theorem 2.4. [20], [12] Let \( f(t) \) be a formal power series:

\[
 f(t) = \sum_{n=1}^{\infty} c_n \frac{t^n}{n!},
\]

then the function \( e^{x f(t)} \) is a generating function for polynomials \( p_n(x) \) of the binomial type:

\[
 \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = e^{x f(t)}.
\]

and coefficients \( c_n = p'_n(0) \) in (6) are called cumulants \([14]\) for the sequence \( p_n(x) \).

Many examples of classic convolution polynomials obtained in this way are listed in [17].

3. Semigroups and Tokens

We would like to generalise the notion of convolution polynomials. In order to do that we use the Observation 2.1. A proper ground for our construction should admit an operation of convolution.

Definition 3.1. A semigroup \( C \) is called a left (right) cancellative semigroup if for any \( a, b, c \in C \) the identity \( ca = cb \) (\( ac = bc \)) implies \( a = b \). A cancellative semigroup is both left and right cancellative semigroup.

We still denote the unique-if-exist solutions to equations \( x \cdot a = b \) and \( a \cdot x = b \) by \([ba^{-1}]\) and \([a^{-1}b]\) correspondingly. Here the braces stress that both \([ba^{-1}]\) and \([a^{-1}b]\) are monosymbols and just "\( a^{-1} \)" is not defined in general.

Of course any group is a cancellative semigroup. A simple examples which are not groups are \( \mathbb{N} \) and \( \mathbb{R}_+ \) (positive real numbers). A convolution on \( \mathbb{N} \) sets was used in (4). There are less trivial examples.

Example 3.2. Let \( P \) be a poset (i.e., partially ordered set) and let \( C \) denote the subset of Cartesian square \( P \times P \), such that \( (a, b) \in C \) if \( a \leq b \), \( a, b \in P \). We can define a multiplication on \( C \) by the formula:

\[
 (a, b)(c, d) = \begin{cases} 
 \text{undefined}, & b \neq c; \\
 (a, d), & b = c.
\end{cases}
\]

One can see that \( C \) is a c-set. If \( P \) is locally finite, i.e., for any \( a \leq b \), \( a, b \in P \) the number of \( z \) between \( a \) and \( b \) \( (a \leq z \leq b) \) is finite, then we can define a measure \( d(a, b) = 1 \) on \( C \) for any \( (a, b) \in C \). With such a measure one defines the correct convolution on \( C \):

\[
 h(a, b) = \int_C f(c, d) g([(c, d)^{-1}(a, b)]) \, d(c, d) = \sum_{a \leq z \leq b} f(a, z) g(z, b).
\]

The constructed algebra is the fundamental incidence algebra in combinatorics [3].

Now we generalise the property of convolution polynomials [4].

Definition 3.3. Let \( C_1 \) and \( C_2 \) be two c-semigroups. We will say that a function \( t(c_1, c_2) \) on \( C_1 \times C_2 \) is a token \([4]\) from \( C_1 \) to \( C_2 \) if for any \( c'_1 \in C_1 \) and any

\[1\text{Please tell me if you know a better name for such a kind of objects.}\]
\(c_2, c'_2 \in C_2\) we have

\[
\int_{C_1} t(c_1, c_2) t([c_1^{-1} c'_1], c'_2) \, dc_1 = t(c'_1, c_2 c'_2).
\]

In fact there is another set of objects, called \textit{dissects} [8], which gives another generalisation of (1), but only for a discrete sets. It is not clear at the moment if all results about tokens [9] could be extended to dissects. On the other hand all examples in this paper are described by semigroups and dissects are not necessary here.

The natural question: \textit{are tokens useful}? As an answer we list different examples of tokens in the next sections.

4. Examples of Tokens

Many classic objects in various fields could be identified as instances of tokens.

4.1. Integral Kernels on Boundaries of Domains. There is a clear pattern of the same structure associated with many important integral kernels.

\textbf{Example 4.1.} Let \(C_1 = \mathbb{R}^n\) and \(C_2 = \mathbb{R}^n \times \mathbb{R}_+\) — the “upper half space” in \(\mathbb{R}^{n+1}\). For the space of harmonic function in \(C_2 = \mathbb{R}^n \times \mathbb{R}_+\), there is an integral representation over the boundary \(C_1 = \mathbb{R}^n\):

\[
f(v, t) = \int_{C_1} P(u; v, t) f(u) \, du, \quad u \in C_1, \ (v, t) \in C_2, \ v \in \mathbb{R}^n, \ t \in \mathbb{R}_+.
\]

Here \(P(u, v)\) is the celebrated Poisson kernel

\[
P(u; v, t) = \frac{2}{|S_n| (|u - v|^2 + t^2)^{(n+1)/2}}
\]

with the property usually referred as a \textit{semigroup property} [1, Chap. 3, Prob. 1]

\[
P(u; v + v', t + t') = \int_{C_1} P(u'; v', t') P(u - u'; v, t) \, du'.
\]

We meet the token in analysis.

\textbf{Example 4.2.} We preserve the meaning of \(C_1\) and \(C_2\) from the previous example and define the Weierstrass (or Gauss-Weierstrass) kernel by the formula:

\[
W(z; w, \tau) = \frac{1}{(\sqrt{2\pi \tau})^n} e^{-\frac{|z - w|^2}{2\tau}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{4}{\tau} |u|^2} e^{-(u, z - w)} \, du,
\]

where \(z \in C_1, \ (w, \tau) \in C_2\). Function \(W(z; w, \tau)\) is the fundamental solution to the heat equation [6, § 2.3]. We again have [1, Chap. 3, Prob. 1]

\[
W(z; w + w', \tau + \tau') = \int_{C_1} W(z'; w', \tau') W(z - z'; w, \tau) \, dz.
\]

Thus we again meet a token.

Two last Examples open a huge list of integral kernels [1] which are tokens of analysis.
4.2. Multiplicative Functions on Partitions. Two important examples of posets are lattices of partitions of a set and non-crossing partitions of an ordered set.

We denote by \( l(\mathbb{N}) \) the set of sequences \( f(n), n \in \mathbb{N} \), i.e. functions \( \mathbb{N} \rightarrow \mathbb{C} \). We also adopt the notion \([21], \S 5.1\) for the exponential generating function

\[
E_f(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}
\]

associated to \( f \in l(\mathbb{N}) \).

Let us define an operation on the set of sequences \( l(\mathbb{N}) \) by means of composition corresponding exponential generating functions:

\[
h = f \ast g, \text{ where } \sum_{n=0}^{\infty} h(n) \frac{x^n}{n!} = E_g(E_f(x)) = \sum_{n=0}^{\infty} g(n) \frac{(E_f(x))^n}{n!} \]

With this operation \( l(\mathbb{N}) \) is not a group: for example \( g^{-1} \) does not exist when \( E_g(x) = x^2 \) \([21], \S 5.4\). On the other hand (11) makes \( l(\mathbb{N}) \) a \( c \)-semigroup, the uniqueness of an existing \( g^{-1}f \) could be derived similarly to proof of \([21], \text{Prop. 5.4.1}\].

Let \( \Phi \) be the constructor of multiplicative functions, i.e. for any sequence of numbers \( f(n) \in l(\mathbb{N}) \) and an interval \((\sigma,\pi)\) in the lattice of partition it assign a number

\[
\Phi(f, (\sigma,\pi)) = f^{a_1}(1)f^{a_2}(2)\ldots f^{a_k}(k)\ldots,
\]

where

\[
(\sigma,\pi) \simeq \Pi^{a_1}_1\Pi^{a_2}_2\ldots \Pi^{a_k}_k \ldots.
\]

We could restate the following result \([21], \text{Th. 5.1.11}\]

\[
\Phi(f \ast g, (\sigma,\pi)) = \sum_{\sigma \leq \nu \leq \pi} \Phi(f, (\sigma,\nu))\Phi(g, (\nu,\pi))
\]

as an observation that \( \Phi \) is a token from \( c \)-semigroup of poset \( \Pi \) to a \( c \)-semigroup of \( l(\mathbb{N}) \) with the operation \([1]\).

4.3. Special Functions from Group Representations. Let we have a representation \( T \) of a group \( G \) by invertible operators in a Hilbert space \( H \).

**Definition 4.3.** The matrix elements \( t_{jk}(g) \) of a representation \( T \) of a group \( G \) (with respect to a basis \( \{e_j\} \) in \( H \)) are complex valued functions on \( G \) defined by

\[
t_{jk}(g) = \langle T(g)e_j, e_k \rangle.
\]

**Exercise 4.4.** Show that \([22], \S 1.1.3]\]

1. \( T(g)e_k = \sum_j t_{jk}(g)e_j \).
2. \( t_{jk}(g_1g_2) = \sum_n t_{jn}(g_1)t_{nk}(g_2) \).

It is well known \([14, 22]\) that many classic special functions (e.g. trigonometric functions, Legendre, Jacoby, and Hermite polynomials; Bessel, Hankel, and hypergeometric functions) appear from group representations according to the following definition.
Definition 4.5. A special function associated with a representation $T$ of a group $G$ is a matrix element $t_{ij}(g)$ of $T$.

Important addition formulae for special functions are in fact particular realisations of the simple identity:

$$t_{jk}(g_1 g_2) = \sum_n t_{jn}(g_1) t_{nk}(g_2).$$

But this identity states that the matrix coefficients $t_{jk}(g)$ generated by a representation of a group $G$ are tokens from the semigroup $\mathbb{N}^2$ (with a multiplication similar to (8)) to the group $G$.

4.4. Wavelets Refinement Equation. An orthogonal multiresolution of $L^2(\mathbb{R})$ (or wavelets analysis) is a chain of closed subspaces indexed by all integers:

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots,$$

if it has the following properties:

1. Completeness. $\lim_{n \to \infty} V_n = L^2(\mathbb{R})$ and $\lim_{n \to -\infty} V_n = \{0\}$.
2. Scale Similarity. $f_n(x) \in V_n \Leftrightarrow f_n(2x) \in V_{n+1}$.
3. Translation Invariance. $V_0$ has an orthonormal basis $\{\phi(x-n) \mid n \in \mathbb{Z}\}$ consisting of all integral translates of a single function $\phi(x)$—the mother wavelet.

Example 4.6. The classic decomposition is obtained by means of the Haar wavelet:

$$\phi(x) = \begin{cases} 1 & \text{if } x \in (0, \frac{1}{2}); \\ -1 & \text{if } x \in (\frac{1}{2}, 1); \\ 0 & \text{otherwise.} \end{cases}$$

From the above conditions we obtain the following refinement equation for the mother wavelet $\phi(x)$:

$$\phi(x) = 2 \sum_{k=-\infty}^{\infty} h_k \phi(2x - k) \quad \text{or} \quad \phi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{\infty} 2h_k \phi(x - k).$$

Let us introduce a function $h(n, j)$ which is the $j$-th power convolution over $\mathbb{Z}$ of the sequence $2h_n$ with itself. It is easy to see [9, Example 3.16] that $h(n, j)$ is a token:

$$h(n, j) = \sum_{k=-\infty}^{\infty} h(k, j)h(n - k, j').$$

From $\mathbb{Z}$ to $\mathbb{N}$. Now identity [13] just state that functions $\psi_x(n, j) = \phi(2^j x + n)$ is a dual token (the kernel of an associated delta family [9, § 4.1]) to $h(n, j)$ for any fixed $x \in \mathbb{R}$. Particularly we should have:

$$\psi_x(j, n+n') = \sum_{k=-\infty}^{\infty} \psi_x(k, n)\psi_x(j - k, n'),$$

or equivalent translation back to the function $\phi$:

$$\phi(2^j x + n + n') = \sum_{k=-\infty}^{\infty} \phi(2^k x + n)\phi(2^{j-k} x + n').$$
4.5. Quantum Mechanical Propagator. Let a physical system has the configuration space $Q$. This means that we could label states of the system at any time $t_0 \in \mathbb{R}$ by points of $Q$. For a quantum system the principal quantity is the propagator \cite{7, § 2.2}, \cite{19, § 5.1] $K(q_2, t_2; q_1, t_1)$—a complex valued function defined on $Q \times \mathbb{R} \times Q \times \mathbb{R}$. It is a probability amplitude for a transition from a state $q_1$ at time $t_1$ to $q_2$ at time $t_2$. The probability of this transition is

$$P(q_2, t_2; q_1, t_1) = |K(q_2, t_2; q_1, t_1)|^2.$$ 

Let us fix any $t_i, t_1 < t_i < t_2$. We assume that

$$K(q_2, t_2; q_1, t_1) = \int_Q K(q_2, t_2; q_i, t_i)K(q_i, t_i; q_1, t_1) \, dq_i,$$ 

i.e. there exists a measure $dq$ on $Q$ with the following property. The system could be at time $t_i$ at any point $q_i$. The transition amplitude $q_1 \to q_2$ is a result of all possible transitions amplitudes $q_1 \to q_i \to q_2$ integrated over $dq_i$.

We would like to show now that a propagator $K$ is a token in fact. To do that we consider two semigroups: the semigroup $\mathbb{R}^2$ of time intervals $[t_1, t_2]$ and the semigroup $Q^2$ of space intervals $[q_1, q_2]$. In the both cases the semigroup multiplication $*$ of two intervals is given by formulas analogous to (8). Now we consider a propagator $K([q_1, q_2], [t_1, t_2]) = K(q_2, t_2; q_1, t_1)$ as a function on $\mathbb{R}^2 \times Q^2$. Then the principal property of a quantum propagator (14) could be restated as

$$K([q_1, q_2], [t_1, t'] *[t', t_2]) = \int_{Q^2} K([q_1, q'], [t_1, t'])K([q_1, q']^{-1}[q_1, q_2], [t', t_2]) \, dq_1, q',$$ 

where $[q_1, q']^{-1}$ denotes the inverse of the interval $[q_1, q']$. This property expresses the fact that the propagator is a token in fact, as it satisfies the requirements of the token property (14) on both time and space intervals.
which is exactly yet another realisation of the defining property (10) of tokens.

There is an exciting but non-rigorous tool—the Feynman path integral—to calculate a propagator. For example it is completely strict if applied to convolution polynomials and expresses them via cumulants as

\[ p_n(x) = n! \int DkDp \exp \left( \int_0^x (-ipk' + h(p)) \, dt \right), \quad h(p) = \sum_{k=0}^{\infty} p'_k(0) \frac{e^{ipk}}{k!}. \]

where the first integration is taken over all possible paths \( k(t) : [0, x] \rightarrow \frac{1}{x} \mathbb{N} \), such that \( k(0) = 0 \) and \( k(x) = \frac{x}{\pi} \), and the path \( p(t) : [0, x] \rightarrow [-\pi, \pi] \) is unrestricted. It is enough to consider only paths \( k(t) \) with monotonic grow—other paths make the zero contributions. Indeed the identity \( p_l(x) \equiv 0 \) for \( l < 0 \) (made by an agreement) implies that contribution of all paths with \( k'(t) < 0 \) at some point \( t \) vanish. Here (and in the inner integral of (15)) \( k'(t) \) means the derivative of the path \( k(t) \) in the distributional sense, i.e. it is the Dirac delta function times \( \frac{1}{x} \) in the points where \( k(t) \) jumps from one integer \( \mod \frac{1}{x} \) value \( k(t-0) = \frac{m}{x} \) to another \( k(t+0) = \frac{m}{x} + j \). Note that (15) provides an algorithm for a quantum computation of a combinatorial quantity.

In fact the heuristic procedure defining Feynman path integral could be applied to any token with different levels of rigour however. We hope to consider this topic elsewhere.

5. Conclusion

The variety of examples of tokens from different subjects given in the previous section generates a suspicion that very little could be said about tokens in general. Fortunately it is possible to show that many important properties of convolution polynomials (or polynomial sequences of binomial type) are true for tokens in general. Therefore all mentioned and many unmentioned areas could benefit from the general theory of tokens on semi- and hypergroups. We saw that tokens are similar to intertwining operators but are more flexible. The new frontiers for intertwining operators in functional calculus and quantum mechanics were outlined in, it may be very interesting to extend those ideas from intertwining operators to tokens.

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