Majorana fermions in T-shaped semiconductor nanostructures

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Abstract

We investigate the Majorana fermions in a T-shaped semiconductor nanostructure with the Rashba spin–orbit coupling and a magnetic field in the proximity of an s-wave superconductor. It is found that the properties of the low-energy modes (including the Majorana and near-zero-energy modes) at the ends of this system are similar to those in the Majorana nanowire. However, very distinct from the nanowire, one Majorana mode emerges at the intersection of the T-shaped structure when the number of the low-energy modes at each end \( N \) is odd, whereas neither Majorana nor near-zero-energy mode appears at the intersection for even \( N \). We also discover that the intersection Majorana mode plays an important role in the transport through the above T-shaped nanostructure with each end connected with a normal lead. Due to the presence of the intersection mode, the deviation of the zero-bias conductance from the ideal value in the long-arm limit \( N \epsilon^2 / h \) is more pronounced in the regime of odd \( N \) compared to the one of even \( N \). Furthermore, when the magnetic field increases from the regime of odd \( N \) to the one of even \( N + 1 \), the deviation from the ideal value tends to decrease. This behavior is also very distinct from that in a nanowire, where the deviation always tends to increase with the increase of magnetic field.

Keywords: Majorana fermion, nanostructure, transport

(Some figures may appear in colour only in the online journal)

1. Introduction

Since Majorana fermions, particles which are their own antiparticles, are proposed theoretically in topological superconductors [1–16], the search of Majorana fermions has attracted much attention in the condensed matter community. Apart from the importance for fundamental physics, Majorana modes in topological superconductors are of great use for quantum computation due to their non-Abelian exchange statistics [2, 17–19].

A promising proposal for engineering Majorana quasi-particles is based on a semiconductor nanowire with both spin–orbit coupling (SOC) and magnetic field in the proximity of an s-wave superconductor [8–11]. In a long one-dimensional nanowire, in which only the lowest subband is involved, Majorana modes emerge as one pair of zero-energy states located at the two ends of the nanowire in the parameter regime satisfying the condition \( |V_Z| > \sqrt{\Delta^2 + \mu^2} \) [8]. Here \( V_Z \), \( \Delta \) and \( \mu \) represent the Zeeman splitting induced by the magnetic field, proximity-induced superconducting gap and chemical potential, respectively. A characteristic feature of Majorana states is the conductance peak at zero bias [20–39]. The value of this zero-bias peak is predicted to be quantized, i.e., \( 2e^2 / h \) for a normal–superconductor surface [20–26] and \( e^2 / h \) for a normal–superconductor–normal (NSN) structure, as the NSN structure consists of two normal–superconductor surfaces [27, 28]. When the length of wire is comparable to or shorter than the coherence length of Majorana modes, the interaction between the two end Majorana modes becomes important and leads to an energy splitting of these states [23, 24, 40]. This effect can reduce the value of the zero-bias peak and can even make this peak split into two peaks at finite bias when the splitting is large enough [23–25].
There are also a few works on Majorana fermions in multi-subband nanowires [24, 26, 27, 41–46]. This system supports multiple low-energy modes at each end. The number of these modes $N$ is determined by the $Z_2$ topological invariant, which comes from the approximate chiral symmetry [11, 26, 42]. In the weak superconducting-pairing limit, $N$ is approximately equal to the number of subbands in which only the states with one kind of spin are occupied [11, 41, 42]. Without considering the SOC between different subbands, the chiral symmetry is weakly broken, and most of the low-energy modes are split off and become the near-zero-energy modes. The number of Majorana modes left is determined by the $Z_2$ topological invariant, which corresponds to the parity of $N$ [8, 11, 42]. In the nontrivial (trivial) topological regime with odd (even) $N$, there is one (no) Majorana mode at each end. The presence of the Majorana mode for odd $N$ is also according to the restriction from the particle–hole symmetry of the Bogoliubov–de Gennes (BdG) Hamiltonian. When the splittings of these low-energy modes are still negligible compared with the energy broadening from the leads, the near-zero-energy modes behave the same as the Majorana modes. Thus, the conductance still shows a peak at zero bias with peak value being $N e^2/h$ in an NSN structure. However, when the splitting induced by the inter-subband SOC becomes important, the behavior of the conductance becomes complex due to the interference between different low-energy modes and the zero-bias conductance can vary between 0 and $N e^2/h$ [27].

So far, most works in this field focus on the two-terminal Majorana nanowire. However, there are few works on the multi-terminal Majorana nanostructure [19, 47–50], especially on the three-terminal T-shaped semiconductor nanostructure with the Rashba SOC and magnetic field in the proximity of superconductor. In fact, the three-terminal T-shaped structure built with normal conductor has been extensively investigated and shows very distinct electric and transport properties. In particular, a localized state appears at the intersection of the three-terminal structure and induces the Fano line shape in the bias dependence of the conductance [51–58]. Thus, it is expected that there are also some interesting features in the T-shaped Majorana nanostructure. Furthermore, the unique properties of this system can be shown in a simple view. We take the case where both the main-arm and side-arm are one-dimensional chains as an example. Due to similarity of the Hamiltonian around the ends of the T-shaped Majorana nanostructure and nanowire, the Majorana modes should appear at all three ends of the T-shaped nanostructure in the nontrivial topological regime. Thus, the total number of zero-energy end states is three. However, it is known that Majorana modes must emerge in pairs, since one Majorana fermion only contains half the degrees of freedom of a normal fermion [15, 16]. Therefore, an additional unknown Majorana mode must exist. The main purpose of this work is to identify this additional Majorana mode and reveal its influence on the transport through the T-shaped Majorana nanostructure.

In this paper, we investigate the low-energy spectrum and transport properties of a T-shaped Majorana nanostructure. We discover that, distinct from the behavior in nanowires, where all Majorana modes appear at the two ends, a Majorana mode shows up at the intersection of the three-terminal T-shaped structure in the case with odd low-energy modes (including the zero-energy and near-zero-energy modes) at the three ends. However, neither the Majorana nor the near-zero-energy mode appears at the intersection in the case with even low-energy modes at the three ends. We further show that the presence of the intersection Majorana mode enhances the deviation of the zero-bias conductance from its ideal value in the long-arm limit. Also considering that the regime with the intersection Majorana mode can appear at lower magnetic field compared to that without it, the deviation from the ideal value tends to decrease with the increase of magnetic field in this case. This behavior is also distinct from the transport properties through a Majorana nanowire, where the deviation always shows an increasing trend with increasing magnetic field [27].

The paper is organized as follows. In section 2, we set up the tight-binding Hamiltonian of a T-shaped Majorana nanostructure and calculate the low-energy spectrum and identify the Majorana states. In section 3, we derive the formula of current between different ends of the T-shaped structure and present the numerical results of electric conductance. Finally, we conclude in section 4.

2. Hamiltonian and energy spectrum

2.1. Hamiltonian

We consider a T-shaped semiconductor nanostructure with the Rashba SOC and proximity-induced superconducting pairing in the presence of a magnetic field perpendicular to the plane of this structure, as sketched in figure 1. Note that the leads plotted in this figure are excluded in this section but included in calculating the transport properties in section 3. The tight-binding Hamiltonian of this structure can be written as [41, 44, 46]

$$H_{\text{eff}} = H_0 + H_{\text{SC}},$$

(1)
\( H_0 = \sum_{\sigma} (\sigma V_Z + V_i - \mu) c_{i\sigma} \gamma_{i\sigma} - \sum_{(i,j)\sigma} t c_{i\sigma}^\dagger c_{j\sigma} \)
+ \( iE_R \sum_{(i,j)\sigma\sigma'} (v^x_{ij} \gamma_{i\sigma} - v^y_{ij} \gamma_{i\sigma'}) c_{i\sigma}^\dagger \gamma_{j\sigma'} \), \hspace{1cm} (2)
\[ H_{SC} = \sum_i \Delta c_{i+}^\dagger c_{i-}^\dagger + H.c. \] \hspace{1cm} (3)

Here \( t \) represents the hopping energy; \( V_i = 4t \) denotes the on-site energy; \( (i,j) \) represents a pair of the nearest neighbors; \( \sigma^l \) for \( l = x, y \) are the Pauli matrices; \( v^l_{ij} = e_i \cdot d_{ij} \) with \( d_{ij} = (r_i - r_j)/|r_i - r_j| \); \( E_R \) represents the Rashba SOC constant.

It is convenient to rewrite the above Hamiltonian into the following form [59]
\[ H_{\text{eff}} = \frac{i}{2} \sum_{ij} \Phi^\dagger_{ij} H_{\text{BdG}}(i, j) \Phi_{ji}, \] \hspace{1cm} (4)
where \( \Phi^\dagger_{ij} = (c_{i\uparrow}^\dagger, c_{i\downarrow}^\dagger, c_{i\downarrow}, -c_{i\uparrow}) \) denotes the Nambu spinor and
\[ H_{\text{BdG}}(i, j) = \begin{pmatrix} H_0(i, j) & \Delta \delta_{ij} \\ \Delta^* \delta_{ij} & -\gamma_{x} H_0^\dagger(i, j) \gamma_{y} \end{pmatrix} \] \hspace{1cm} (5)
represents the BdG Hamiltonian [59]. By diagonalizing the matrix form of the BdG Hamiltonian (labeled by \( \tilde{H}_{\text{BdG}} \)), one obtains the energy spectrum and eigenstates of this system.

### 2.2. Low-energy spectrum and Majorana states

In this subsection, we present the numerical results of the low-energy spectrum and eigenstates. We choose \( \Delta = E_R = 0.2t \) unless otherwise specified. We first discuss the simplest T-shaped Majorana nanostructure with \( W = W_s = 1 \). Evidently, there is only one subband in this case. As mentioned in section 1, due to the similarity of this system and the one-dimensional nanowire, the Majorana modes are expected to appear at all three ends of the T-shaped nanostructure in the nontrivial topological regime, i.e., \( \sqrt{\Delta^2 + (\mu - E_0)^2} < |V_Z| < \sqrt{\Delta^2 + (\mu - E_0 - 4t)^2} \), where \( E_0 \) represents the energy of bulk states at \( k_1 = 0 \) with \( k_2 \) being the momentum along the free propagating direction\(^{1}\). Nevertheless, there must be additional Majorana mode(s) to ensure that the total number of Majorana modes is even. To confirm the above claim, we plot the low-energy spectrum of this structure with \( L = L_s = 100 \) and \( 200 \) for \( \mu = E_0 \) and \( V_Z = 0.4t \), which belongs to the nontrivial topological regime, in figure 2(a). Here only the ones in the positive eigenenergy regime are shown due to the particle–hole symmetry of the BdG Hamiltonian. It is seen that there are two eigenstates with extremely small energy and their energies decrease almost exponentially with the increase of the arm length. This indicates that both states are strict zero-energy states in the long-arm limit. Considering one-zero-energy fermion state corresponds to one pair of Majorana fermions [15, 16], one finds that there are four (two pairs of) Majorana modes in total. This means that there is

\(^{1}\) Note that the magnitude of the Zeeman splitting of the nontrivial topological regime has an upper limit in the tight-binding model [46], which is distinct from the continuum model.
the Majorana modes (equations (A.1)–(A.6)) and following a similar method to [39], we confirm the odd-frequency Cooper pairing [14, 39] in the present situation.

We then turn to the multi-subband T-shaped Majorana nanostructure. The number of low-energy modes $N$ at each end of this structure can be obtained following a similar approach to that in the multi-subband nanowire [8, 11, 26, 42]. It is determined by the $Z$ topological invariant from the approximate chiral symmetry [11, 26, 42]. In most of the parameter regimes investigated here, the superconducting pairing is much weaker than the chemical potential and Zeeman splitting. In these cases, $N$ is approximately equal to the number of subbands in which only the states with one kind of spin are occupied. However, as the inter-subband SOC weakly breaks the chiral symmetry, most of these low-energy modes are the near-zero-energy modes instead of the Majorana modes [11, 42]. The number of Majorana modes is determined by the $Z_2$ topological invariant, which corresponds to the parity of $N$ [8, 11, 42]. One (no) Majorana mode appears at each end in the nontrivial (trivial) topological regime with odd (even) $N$.

In figure 3, we plot the phase diagram of the T-shaped Majorana nanostructure with $W = W_s = 4$. Here the regions with the same color share the same $N$. The solid curves represent the transition points of the $Z_2$ topological invariant, which are obtained from the gap closing condition of the bulk energy spectrum at $k_y = 0$ or $\pi/a$ [8, 46]. It is observed that these solid regime boundaries show some anti-crossings, e.g., between the regimes with $N = 0$ and 2. This effect comes from the anti-crossings between the energy spectra of bulk states in different subbands, induced by the inter-subband SOC. This kind of anti-crossing is also observed in the phase diagram in the quasi-1D nanowire with magnetic field perpendicular to the nanowire plane [11, 44]. Around these anti-crossings, the inter-subband SOC cannot be treated perturbatively, thus the $Z$ topological invariant and $N$ cannot be well defined. This indicates that there are no strict boundaries between the relevant regimes. Here we only plot dashed curves at the positions where the bulk gap at $k_y = 0$ or $\pi/a$ reaches a finite minimum as rough boundaries to separate these regimes inside these anti-crossings.

Now we examine the existence of the Majorana or near-zero-energy modes in different regimes for $W = W_s = 4$. We choose $\mu = 0.4t$ and $V_Z = 0.8t$, 1.2$t$ and 2.3$t$, which belong to the regimes with $N = 1$, 2 and 3, respectively, as indicated by the squares in figure 3. The energy spectra in these three cases are plotted in figures 4(a), (c) and (e), respectively. In the insets of these figures, we also schematically plot the occupation of the four lowest spin bands, in which the $m \sigma$ band represents the spin-majority ($\sigma = -$) or -minority ($\sigma = +$) band of the $n$th transverse subband. We first focus on the case with $N = 1$, where only the spin-majority band of the lowest subband (1−) is occupied. It is seen that both the energy spectrum (figure 4(a)) and the wavefunctions of the low-energy states (the one with $n = 2$ for $L = L_s = 200$ is shown in figure 4(b)) in this case are similar to those in the T-shaped nanostructure with $W = W_s = 1$ discussed above: there are four (two pairs of) Majorana modes in total and one appears at the intersection.

![Figure 3. Phase diagram of an isolated T-shaped Majorana nanostructure with $W = W_s = 4$ as a function of the Zeeman splitting $V_Z$ and the chemical potential $\mu$. $N$ represents the number of low-energy modes at each end. The black squares indicate the chemical potential and Zeeman splitting used in figure 4.](image-url)

Then we turn to the case with $N = 2$ (curves with dots in figure 4(c)), where the spin-majority bands of the lowest two subbands (1− and 2−) are occupied. It is shown that there are three low-energy eigenstates. In the long-arm limit, their energies become very close to each other and all saturate to the order of $10^{-3}t$. This indicates that they are only the near-zero-energy states but not the Majorana states. It is also seen that, after removing the inter-subband SOC (curves with squares)², the energies of all the low-energy states decrease with the increase of arm length and hence recover the behavior of the Majorana modes. This further demonstrates that the small splitting of these near-zero-energy states is due to the inter-subband SOC, consistent with the above discussions based on the topological invariant. The absence of the Majorana modes also agrees with the fact that the regime with even $N$ belongs to the trivial topological phase. Moreover, we also plot the wavefunction of the eigenstate with $n = 2$ for $L = L_s = 200$ in figure 4(d). The wavefunctions of the other two low-energy eigenstates are similar to this one. One finds that all low-energy eigenstates are constructed by the near-zero-energy modes at the ends and hence the intersection near-zero-energy mode does not appear.

The behavior for $N = 3$ (curves with dots in figure 4(e)), in which the 1−, 2− and 3− bands are occupied, is more complex. It is found that there are five low-energy eigenstates. The lowest two tend to be zero energy with increasing length and the other three saturate to the order of $10^{-3}t$, indicating two zero-energy eigenstates and three near-zero-energy ones in the long-arm limit. The presence of the Majorana fermions is consistent with the fact that the regime with odd $N$ belongs to the nontrivial topological phase. The finite splitting of the

²The inter-subband SOC in the T-shaped structure contains the terms including $v_{ij}^y$ in the Hamiltonian of the main-arm (see equation (2)) and the terms including $v_{ij}^y$ in the Hamiltonian of the side-arm.
Figure 4. Isolated T-shaped Majorana nanostructures with \( W = W_s = 4 \). \( \mu = 0.4t \); \( V_Z = 0.8t \) (a), (b), 1.2t (c), (d), 2.3t (e), (f). (a), (c), (e) Low-energy spectra for different arm lengths \( L = L_s \). The dots and squares represent the results with and without the inter-subband SOC, respectively. The curves are only plotted as a guide for the eye. In the insets, we also schematically plot the occupation of the four lowest spin bands. (b), (d), (f) Magnitude of the wavefunctions of the low-energy eigenstates with \( n = 2 \) for \( L = L_s = 200 \).

near-zero-energy states also comes from the inter-subband SOC, as indicated by the comparison of the energy spectra with (curves with dots) and without the inter-subband SOC (curves with squares). We further examine the wavefunctions of the zero-energy eigenstates and plot the one with \( n = 2 \) in figure 4(f). It is shown that there is also a Majorana mode at the intersection, just similar to the case with \( N = 1 \). In addition, we verify that there is no near-zero-energy mode at the intersection in this case. Based on the above discussions, one can conclude that one intersection Majorana mode appears in the case with odd \( N \), while there is neither a Majorana nor a near-zero-energy mode at the intersection in the case with even \( N \).

3. Electric conductance

3.1. Formalism

In this section, we discuss the transport properties through the T-shaped Majorana nanostructure and, more importantly, identify the role of the intersection Majorana modes in it. Here we connect each end of this structure with a normal lead, as indicated in figure 1. We also add barriers between the leads and the T-shaped structure to reduce the broadening of energy level and hence avoid the states above the bulk gap contributing to the low-energy transport. The Hamiltonian of the leads (including the barriers) is similar to \( H_0 \) in the T-shaped structure and can be written as

\[
H_\eta = \sum_{i,\sigma,\sigma'} (\sigma V_Z + V_l - \mu) d_{i,\sigma}^{\dagger} d_{i,\sigma'} - \sum_{i,j,\sigma,\sigma'} t d_{i,\sigma}^{\dagger} d_{j,\sigma'} + i E_R \sum_{i,j,\sigma,\sigma'} (v_{ij}^{x} \sigma_{x}^{\sigma} \sigma_{\sigma'}^{\sigma'} - v_{ij}^{y} \sigma_{y}^{\sigma} \sigma_{\sigma'}^{\sigma'}) d_{i,\sigma}^{\dagger} d_{j,\sigma'},
\]

where \( \eta = 1, 2, 3 \) represents the left, central and right leads (see figure 1), and the on-site energy \( V_l \) is chosen to be \( 4t + V_b \) with \( V_b \) being the barrier height in the barrier region and \( 4t \) otherwise. The hopping between the leads and the T-shaped structure is described by
The presence of the superconducting gap and hence only the right-hand side of the conductance between terminals 1 and 3 when no current flows through terminal 1, cannot be obtained through a simple analytic formula but can be determined self-consistently using the Green’s functions in the Nambu spinor basis (see appendix B) and following a similar way deriving the current through normal mesoscopic nanostructures [60], one obtains

\[ I_n(t) = e\hbar \int \text{d}\varepsilon \sum_{\eta,\beta} P^{\eta\beta}_{\eta\eta}(\varepsilon) \{ f_{\eta\varepsilon}(\varepsilon) - f_{\eta\varepsilon}'(\varepsilon) \}, \]  

(10)

in which

\[ P^{\eta\beta}_{\eta\eta}(\varepsilon) = \text{Tr} \left\{ \hat{G}^r(\varepsilon) \hat{G}^a(\varepsilon) \hat{G}^a(\varepsilon) \hat{G}^r(\varepsilon) \right\}. \]  

(11)

Here \( \hat{G}^r(\varepsilon) \) and \( \hat{G}^a(\varepsilon) \) are the retarded and advanced Green’s functions in the T-shaped nanostructure connected with leads (see equation (B.2)); \( \Gamma^{\eta\eta}(\varepsilon) \) is the self-energy of the electric (\( \alpha = e \)) or hole part (\( \alpha = h \)) in the lead \( \eta \) (see equation (B.3)). It is noted that this formula of current is just equivalent to the one obtained from the transfer matrix approach [27, 61].

At zero temperature, equation (10) can be reduced into

\[ I_n = \frac{e}{\hbar} \sum_{\eta,\beta} \int_{-\mu_n}^{\mu_n} \text{d}\varepsilon \int_{-\mu_n}^{\mu_n} \text{d}\varepsilon P^{\eta\beta}_{\eta\eta}(\varepsilon), \]  

(12)

where \( \chi_{\beta} = 1 \) \( (-1) \) for \( \beta = e \) (h) and \( \mu_\eta \) is the chemical potential in lead \( \eta \). The differences between chemical potentials in different leads are determined by the bias, \( \varepsilon V_1 = \mu_1 - \mu_3 \) and \( \varepsilon V_2 = \mu_2 - \mu_3 \). In this investigation, we focus on two quantities: (i) the differential conductance between terminals 1 and 3 when no current flows through terminal 2, i.e., \( G_1 = \frac{\partial I_n}{\partial \varepsilon}_{I_2 = 0} \); (ii) the differential conductance between terminals 2 and 3 when no current flows through terminal 1, i.e., \( G_2 = \frac{\partial I_n}{\partial \varepsilon}_{I_1 = 0} \). Generally speaking, these two quantities cannot be obtained through a simple analytic formula but can only be calculated through a self-consistent numerical scheme.

We take \( G_1 \) as an example to explain this scheme: (1) for certain \( V_1, \mu_3 \) and \( I_n \), \( G_1 \) can be determined self-consistently using the conditions \( I_2 = 0 \) and \( \sum_{\eta} I_n = 0 \); (2) for the bias slightly deviating from \( V_1 \), termed as \( V_1' \), one obtains the corresponding current \( I_n' \) in a similar way; (3) the differential conductance is obtained from \( (I_n' - I_n)/(V_1' - V_1) \).

When all arms of the T-shaped Majorana nanostructure are very long, the transmissions between different leads (i.e., \( P^{\eta\beta}_{\eta\eta} \) for \( \eta \neq \eta' \)) become negligible around zero energy due to the presence of the superconducting gap and hence only the Andreev reflection contributes to the transport. In this case, equation (12) becomes simpler,

\[ I_n = \frac{e}{\hbar} \int_{-\mu_n}^{\mu_n} \text{d}\varepsilon P^{\eta\beta}_{\eta\eta}(\varepsilon). \]  

(13)

Further using \( P^{\eta\beta}_{\eta\eta}(\varepsilon) = P^{\eta\beta}_{\eta\eta}(\varepsilon) \), which comes from the left-right symmetry of this structure, one obtains the conductance \( G_1 \)

\[ G_1 = e^2/2h \left[ P^{\eta\beta}_{\eta\eta}(\varepsilon V_1/2) + P^{\eta\beta}_{\eta\eta}(-\varepsilon V_1/2) \right]. \]  

(14)

Nevertheless, the conductance \( G_2 \) in this case still needs to be obtained through a self-consistent scheme.

### 3.2. Numerical results

In this subsection, we present the numerical results of the conductance through the T-shaped Majorana structure in various parameter regimes. We choose \( W = W_s = 4 \) and \( \mu = 0.4t \), just as figure 4. We also set the barrier width \( W_b = 2 \) throughout this subsection. We first discuss the bias dependence of conductance for \( V_Z = 0.8t \), which belongs to the regime with \( N = 1 \), as shown in figure 3. The conductance \( G_1 \) is plotted against bias with different arm lengths for the barrier height \( V_b = 0.8t \) in figure 5(a). The behavior of \( G_2 \) is similar to this one and not shown here. It is seen that the conductance exhibits a Lorentzian peak at zero bias with the peak value being \( e^2/h \) for \( L = L_s = 200 \). This is just the typical behavior of Majorana fermion-assisted transport [27], indicating the arm has been long enough so that the interaction between different Majorana modes becomes negligible. The behaviors with shorter arms are more interesting. One observes a sharp valley at zero bias and double peaks at finite bias for \( L = L_s = 100 \) and 50. Note that these behaviors are very distinct from those in nanowires, which are plotted in figure 5(b) with the same parameters as the previous ones except \( L_s = 0 \) (i.e., the side-arm is removed). In that figure, one observes that the conductance shows the double-peak structure only for extremely short length \( L = 15 \). Obviously, the double-peak behavior appears at a much longer length in the T-shaped structure compared with that in the nanowire.

Two reasons lead to the above distinct behaviors in these two structures. The first one is straightforward: due to the presence of the intersection Majorana mode, the distance of the adjoining Majorana modes \( L \) in the T-shaped structure is only about one half of that in the corresponding nanowire, whose total length is \( 2L + W_s \). This enhances the interaction between the adjoining Majorana modes and makes the split-peak structure appear at longer length. The second reason is more subtle: all the Majorana modes in the nanowire are located at the ends and hence their self-energies from the leads are large, whereas the intersection Majorana mode in the T-shaped structure has a very small self-energy. The influence of this factor can be seen clearly in the limit where the self-energy of the intersection Majorana mode is negligible compared with all the other quantities. In this limit, the conductance can be described by equation (C.3) in appendix C with \( \Gamma_1 = \Gamma_{1L} \) and \( \Gamma_2 = \Gamma_{2L} = 0 \) (the Majorana modes at the left end and the intersection are numbered 1 and 2, respectively)

\[ G_1(V_1) = \frac{e^2}{h} (\varepsilon V_2 + 4|\beta|^2)^2 + \Gamma_1^2 e^2 V_1^2. \]  

(15)

From this formula, one finds that \( G_1(V_1) \) takes its minimum value 0 at zero bias and reaches its maximum value \( e^2/h \) at
In the adjoining Majorana modes. This indicates that the conductance always shows the double-peak structure in this limit. In fact, the self-energy of the intersection Majorana mode is not so small in most cases and the behavior of the conductance in the T-shaped structure is usually between the above limit and the Lorentzian-peak behavior. Nevertheless, the small self-energy of the intersection mode still facilitates the formation of the split-peak behavior and makes it appear at longer length.

Then we turn to the bias dependence of conductance for $V_Z = 1.2t$, which corresponds to $N = 2$, i.e., there are two near-zero-energy modes at each end. As discussed in section 2, the splitting of these modes is mainly from the inter-subband SOC and hence insensitive to the arm length as long as the arm is not too short. Thus, here we do not change the length as in the previous case with $N = 1$, instead, we fix the length $L = L_s = 100$ and change the barrier height to show the typical transport behavior in this situation. We again only plot $G_1$ due to the similar behaviors between the conductances $G_1$ and $G_2$. The results are plotted as curves in figure 5(b). It is seen that the conductance shows a peak at zero bias with the peak value being close to $2e^2/h$ for low barrier height $V_b = 0.8t$, whereas it exhibits double peaks at finite bias when the barrier height is large enough, e.g., $V_b = 2.4t$. The underlying physics is as follows. For low barrier height, the self-energies of the near-zero-energy states are larger than the splitting induced by the inter-subband SOC and hence all the near-zero-energy modes just act the same as the Majorana modes. In this case, the conductance can be described by equation (C.3) with $|\epsilon_{12}| = 0$, $\Gamma_1 = \Gamma_{1L}$ and $\Gamma_2 = \Gamma_{2L}$.

$$G_1(V_1) = \frac{e^2}{h} \left( \frac{\Gamma_1^2}{\epsilon_1^2 V_1^2 + \Gamma_1^2} + \frac{\Gamma_2^2}{\epsilon_2 V_1^2 + \Gamma_2^2} \right). \quad (16)$$

Evidently, $G_1$ in this case is just the summation of two Lorentzian functions with height $e^2/h$. For high barrier height, the self-energy of one of the low-energy modes at the end is much smaller than the splitting. Thus, the conductance can be described by equation (15) and shows the split-peak behavior. Moreover, we also plot the conductances through nanowires in the corresponding cases as dots. It is seen that they almost coincide with the corresponding ones in T-shaped structures. This is because in the case with $N = 2$, there is no low-energy state at the intersection and the properties of the low-energy states at the three ends in T-shaped structures are similar to those in nanowires. In addition, we also investigate the conductance for higher $N$ (not shown) and find that the above phenomena in the case with $N = 1$ (2) also appear in the case with $N$ being another odd (even) number. Note that the above distinct behaviors for odd and even $N$ are consistent with the odd–even parity effect in the isotropic junction of multiple Majorana nanowires [48]. This is because if the inter-subband SOC is neglected, the T-shaped structure with $N$ low-energy modes at each end can be seen as a junction formed by $3N$ Majorana nanowires.

The unique transport properties in the T-shaped Majorana nanostructure can be seen more clearly in the magnetic-field dependence of the linear conductance (i.e., at zero bias). Since the effect of the inter-subband SOC on the transport in the T-shaped structure is similar to that in the nanowire addressed in the literature [27], here we focus on the case with $V_b = 0.8t$. 

**Figure 5.** Conductance $G_1$ in T-shaped Majorana nanostructures (a) and nanowires (b) versus bias $V_1$ with different lengths for the Zeeman splitting $V_Z = 0.8t$ (corresponding to $N = 1$) and barrier height $V_b = 0.8t$. (c) Conductance $G_1$ in T-shaped Majorana nanostructures (curves) and nanowires (dots) versus bias $V_1$ with different barrier heights for $V_Z = 1.2t$ (corresponding to $N = 2$) and $L = 100$. $eV_1 = \pm 2|\epsilon_{12}|$ with $|\epsilon_{12}|$ representing the interaction between the adjoining Majorana modes. This indicates that the conductance always shows the double-peak structure in this limit.
where the splitting of the near-zero-energy states induced by the inter-subband SOC is unimportant compared with their self-energies, as shown in figure 5(c). We plot $G_1(0)$ and $G_2(0)$ as function of magnetic field for different arm lengths in figure 6(a). We first discuss the case in the long-arm limit, i.e., $L = 800$. As $G_1$ and $G_2$ coincide in this case, only $G_1$ is shown. It is seen that the linear conductance is very close to the ideal value $Ne^2/h$ in all parameter regimes investigated in this work. This indicates that the splitting of the relevant low-energy modes is negligible compared with their self-energies.

The behavior becomes more interesting for shorter arm length. In the case with $L = 200$, both $G_1$ and $G_2$ take the ideal value $Ne^2/h$ in the regimes for $N = 1$, 2 and 4, however, they deviate much from their ideal value for $N = 3$. This phenomenon can be understood as follows. As said above, in the regime for odd $N$, the intersection Majorana mode emerges, which enhances the discrepancy between the zero-bias conductance and its ideal value. It is also known that, with the increasing magnetic field, the coherence length of the low-energy modes tends to increase, and hence the splitting of these states tends to increase [40]. Therefore, the pronounced deviation from the ideal value appears in the regime for $N = 3$, in which $N$ is odd and the corresponding magnetic field is high. It is also shown that the deviation in $G_2$ is larger than $G_1$. This can be understood by considering that the magnitude of the wavefunction of the intersection Majorana mode in the side-arm is larger than that in the main-arm, as shown in its analytic solution with $W = W_s = 1$ (equations (A.4)-(A.6)). Note that the deviation from the ideal value in the regime for $N = 3$ is even larger than that for $N = 4$, although the latter one appears at higher magnetic field. A similar phenomenon is observed in the case with $L = 50$. In that case, one finds that the derivation for $N = 1$ is larger than that for $N = 2$. The above behaviors are very distinct from those in Majorana nanowires. In that system, as shown in figure 6(b), the deviation from the ideal value always tends to increase with the increase of magnetic field due to the decrease of the coherence length.

4. Conclusion

In conclusion, we have investigated the Majorana fermions in a T-shaped semiconductor nanostructure with the Rashba SOC and proximity-induced superconducting pairing in the presence of a magnetic field perpendicular to the plane of this structure. We first discuss the low-energy spectrum of this system. We find that the properties of the low-energy modes (including the Majorana and near-zero-energy modes) at the ends of the T-shaped structure are similar to those in the Majorana nanowire. The number of low-energy modes at each end $N$ is approximately equal to the number of subbands in which only the states with one kind of spin are occupied and the number of Majorana modes at each end is one (zero) for odd (even) $N$. Moreover, very distinct from the nanowire, it is discovered that one Majorana mode appears at the intersection of the T-shaped structure in the case with odd $N$ to ensure that the total number of Majorana modes is even. However, there is neither a Majorana nor a near-zero-energy mode at the intersection for even $N$.

We also investigate the transport properties through the above T-shaped nanostructure with each end connected with a normal lead. It is found that the deviation of the zero-bias conductance from its ideal value in the long-arm limit $Ne^2/h$ is more pronounced in the regime for odd $N$ compared to the one for even $N$. This is because the presence of the intersection Majorana mode reduces the distance between the adjoining Majorana modes and the self-energy of this intersection mode from the leads is very small. Moreover, the regime for odd $N$ can appear at lower magnetic field than that for the even one. Therefore, around the boundary between these two regimes, the deviation from the ideal value tends to decrease with increasing magnetic field. This behavior is also very distinct from that in the nanowire, where the deviation from the ideal value always tends to increase with increasing magnetic field due to the decrease of the coherence length of the low-energy modes.

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Appendix A. Wave functions of Majorana modes in the one-dimensional T-shaped structure

In this appendix, we present the derivation of the wavefunctions of the Majorana modes in the one-dimensional T-shaped structure, i.e., \( W = W_s = 1 \). It is known that, in the nontrivial topological regime, one Majorana mode appears at each end of the one-dimensional nanowire. This indicates that there is one solution \( \Phi_0(x) \) satisfying the BdG equation \( H_{\text{BdG}}(x', x) \Phi_0(x) = 0 \) and the boundary condition \( \Phi_0(0) = 0 \). Also from the particle–hole symmetry of the BdG Hamiltonian, any zero-energy solution can be written into the form \( \Phi_0(x) = (u_0(x), i\tilde{\sigma}_z u_0^\dagger(x))^T \) [40]. Thus, \( u_0(x) \) corresponds to the above solution \( \Phi_0(x) \). After performing the translation and rotation, one obtains the normalized wavefunction of the Majorana mode \( u_\eta(x, y) \) at the end \( \eta \) of the T-shaped nanostructure (see figure 1):

\[
\begin{align*}
  u_1(x, y) &= u_0(x + L)\delta_{y, 0}, \\
  u_2(x, y) &= \sqrt{2} (\hat{I} + i\tilde{\sigma}_z) u_0(-y + L)\delta_{x, 0}, \\
  u_3(x, y) &= i\tilde{\sigma}_z u_0(-x + L)\delta_{y, 0}.
\end{align*}
\]

Generally speaking, the wavefunction of the intersection Majorana mode \( u_4(x, y) \) cannot be constructed in this way. However, for \( W = W_s = 1 \), the exact numerical calculation gives \( u_4(x = 0, y = 0) = 0 \) within the computational accuracy. Thus one obtains the form of the intersection Majorana mode,

\[
\begin{align*}
  u_4(x > 0, y = 0) &= \frac{A}{2} u_0(x), \\
  u_4(x < 0, y = 0) &= \frac{iB}{2} \tilde{\sigma}_z u_0(-x), \\
  u_4(x = 0, y > 0) &= \frac{C}{2} (\hat{I} - i\tilde{\sigma}_z) u_0(y).
\end{align*}
\]

It can be verified that the above solution \( u_4(x, y) \) indeed satisfies the BdG equation when \( A = B = -C = 1 \).

Appendix B. Green’s functions in the Nambu spinor basis

Here we briefly discuss the Green’s functions in the Nambu spinor basis [8, 41, 44, 46]. We first define the contour-ordered Green’s functions in an isolated superconducting nanostructure in this basis as

\[
\tilde{c}_{\sigma \alpha}(t, t') = -i \langle T_\varepsilon \tilde{c}_{\sigma \alpha}(t) \tilde{c}_{\sigma \alpha}^\dagger(t') \rangle
\]

with \( \tilde{c}_{\sigma \alpha} \equiv c_{\sigma \alpha} \) and \( \tilde{c}_{\sigma \alpha}^\dagger \equiv \sigma c_{\sigma \alpha}^\dagger \). After connecting this superconducting structure with normal leads, the contour-ordered Green’s functions can be obtained through the Dyson equation

\[
\hat{G}^\varepsilon(t, t') = \delta^\varepsilon(t, t') + \int d_1 d_2 \hat{G}^\varepsilon(t_1, t_2) \hat{\Sigma}^\varepsilon(t_1, t_2) \hat{G}^\varepsilon(t_2, t_1).
\]

Here symbols with hat (\( \hat{\cdot} \)) represent the corresponding quantities in the lattice and Nambu spinor space; \( \hat{\Sigma}^\varepsilon(t_1, t_2) \) denotes the total self-energy from all leads

\[
\Sigma^\varepsilon_{\eta, j\alpha,j'\sigma_2}(t_1, t_2) = \sum_{j_1, j_2} T_{\eta, j_1,j_2\sigma_1}^{\delta,\alpha} T_{j_2,j_1\sigma_2}^{\eta,\alpha} F_{j_1,j_2,j_3,j_4}^{\eta,\delta,\alpha}(t_1, t_2) \delta_{\eta,\gamma_3,j_4\sigma_2},
\]

in which

\[
F_{j_1,j_2,j_3,j_4}^{\eta,\delta,\alpha}(t_1, t_2) = -i \langle T_\varepsilon d_{\eta, j_3,j_4} \varepsilon(t) d_{\eta, j_1,j_2}^\dagger \rangle,
\]

\[
\eta^{\eta,\alpha}_{j_3,j_4} = \begin{cases} T_{\eta, j_3,j_4}^{\eta,\alpha} & \alpha = \varepsilon \\ T_{\eta, j_3,j_4}^{\eta,\alpha} & \alpha = h. \end{cases}
\]

Similar to equation (B.1), one can define the retarded, advanced, lesser and greater Green’s functions in the isolated superconducting nanostructure \( \hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(t, t') \). Further performing the Fourier transformation, one obtains \( \hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) \) (\( \varepsilon \)). It can be demonstrated that these Green’s functions satisfy

\[
(\varepsilon - \hat{H}_{\text{BdG}} + i0^+) \hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) = 1,
\]

\[
(\varepsilon - \hat{H}_{\text{BdG}}) \hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) = 0.
\]

Since the investigated system is finite, the infinitesimal in equation (B.6) can be neglected [62, 63]. Thus,

\[
[\hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon)]^{-1} - [\hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(0)]^{-1} = 0,
\]

\[
[\hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon)]^{-1} - \hat{g}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) = 0.
\]

Performing the Langreth rules [60] and the Fourier transformation on equation (B.2) and further exploiting equations (B.8) and (B.9), one obtains

\[
\hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) = \hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) \hat{\Sigma}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) \hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon),
\]

\[
\hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) - \hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) = \hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) \hat{\Sigma}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) \hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon).
\]

Note that the above relations are in the same form as those well-known relations for the Green’s functions in the normal conductor [60, 62]. This indicates that the formula of current through the superconducting mesoscopic nanostructure can be derived following a similar method to the current through the normal nanostructure [60].

Appendix C. Approximate formula of conductance induced by two interacting Majorana modes

When only two Majorana modes contribute to the low-energy transport, all Green’s functions and self-energies can be reduced into the small space formed by these two modes. Then one obtains

\[
\hat{G}^\varepsilon_{\eta, \gamma_1, \gamma_2}(\varepsilon) = \left[ \varepsilon - \left( \begin{array}{cc} 0 & \varepsilon \gamma_2 \\ \varepsilon \gamma_2 & 0 \end{array} \right) - i \frac{1}{2} \left( \begin{array}{cc} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{array} \right) \right]^{-1},
\]
Here $\xi_{12}$ represents the coupling between the two Majorana modes; $\Gamma_1$ and $\Gamma_{1L}$ stand for the total self-energy and the one from the left lead of the $\eta$th Majorana mode, respectively. Substituting equations (C.1) and (C.2) into equations (11) and (14), one obtains

$$G_1(V_i) = \frac{e^2}{h} \left( \Gamma_1^2 + \Gamma_{1L}^2 \right)^{-1} e^2 V_i^2 + \frac{\Gamma_1^2}{2} \Gamma_{1L}^2 + \frac{\Gamma_{1L}^2}{2} \Gamma_1^2 + \frac{8}{\hbar} \Gamma_{1L} \Gamma_{2L} |\xi|^2 \left[ (e^2 V_i^2 - 4|\xi|^2)^2 - (1 - \Gamma_1^2 \Gamma_2^2) \right] \right) \left( \Gamma_1^2 + \Gamma_{1L}^2 \right)^{-1}.

(C.3)

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