Exact g-function flow between conformal field theories

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Abstract

Exact equations are proposed to describe $g$-function flows in integrable boundary quantum field theories which interpolate between different conformal field theories in their ultraviolet and infrared limits, extending previous work where purely massive flows were treated. The approach is illustrated with flows between the tricritical and critical Ising models, but the method is not restricted to these cases and should be of use in unravelling general patterns of integrable boundary flows between pairs of conformal field theories.
1 Introduction

Since work by A.B. Zamolodchikov more than 20 years ago [1], many examples of two-dimensional quantum field theories which flow between different conformal field theories in their short and long distance limits have been found. If such theories are placed on manifolds with one or more boundaries, then the corresponding boundary conditions must also flow, between conformal boundary conditions appropriate to the conformal field theories sitting at the two limits. To figure out the resulting pattern of combined bulk and boundary flows is an interesting problem, with potential relevance to a variety of issues in condensed matter physics and string theory.

Zamolodchikov’s original paper (see also [2]) concerned the bulk perturbation of the unitary minimal model $M_{p,p+1}$ by its $\phi_{13}$ operator. For $p$ large, a perturbative calculation of the $c$-function [3] enabled him to show that, for one sign of the coupling, the resulting renormalisation group flow interpolates between $M_{p,p+1}$ in the ultraviolet and $M_{p-1,p}$ in the infrared. The generalisation of this approach to the boundary situation is surprisingly tricky, but has recently been achieved in a paper by Fredenhagen, Gaberdiel and Schmidt-Colinet [4], where the $g$-function or boundary entropy [5, 6] was used to identify the destination boundary conditions, again for large $p$. (Even more recently, the same pattern of flows has been shown to hold on fluctuating surfaces with boundaries [7].)

The calculations of [4] are perturbative in $1/p$, and they do not give reliable information about flows near the bottom of the minimal series. In fact, even at large $p$ the authors of [4] had to borrow some non-perturbative information about pure-boundary flows in order to obtain a full picture. In the absence of boundaries, Al.B. Zamolodchikov showed how such problems could be circumvented in integrable situations through the use of exact, non-perturbative equations of Thermodynamic Bethe Ansatz (TBA) type [8]. These equations encode the evolution of a quantity called the effective central charge, $c_{\text{eff}}$, during renormalisation group flows, where $c_{\text{eff}}$ is an off-critical generalisation of the central charge of a conformal field theory, which agrees with the $c$-function used in [1] at fixed points of the renormalisation group. The purpose of this paper is to show that a similarly-exact description of bulk flows with boundaries is possible, at least in cases where the combined bulk and boundary theory is integrable. Our starting-point is the exact off-critical $g$-function for massive integrable quantum field theories that was introduced in [9] and further studied in [10]. After some background material in section 2, the proposed massless variant of the TBA-inspired exact $g$-function of [9] is introduced in section 3 together with some numerical illustrations of its implications. These results are backed by exact calculations of limiting $g$-function values in section 4 where we also report some simple perturbative checks of our proposal. Finally section 5 contains some conclusions.

In cases where the bulk remains critical, the use of equations of TBA type to evaluate $g$-functions has a long history, dating back at least to work on the Kondo problem [5]. In this respect the main novelty of our result is the demonstration that, for off-critical interpolating flows, bulk-induced changes to $g$-functions can also be accounted for, exactly, through the TBA approach. Some motivation for our specific proposal came from a consideration of Al.B. Zamolodchikov’s staircase model [11] (see also [12–15]). The full set of flows implied by this connection is rather rich, and we postpone its discussion to another occasion. We have also limited the treatment in this paper to flows between the tricritical Ising and Ising models (noting, though, that these cases are of particular interest, being the furthest possible from the perturbative limit studied previously). Generalisation to other cases

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1 Though we should mention a previous attempt to use the staircase TBA to study bulk- and boundary-induced $g$-function flows between conformal field theories, reported in [16]. However the equations developed there do not fully account for the effects of an off-critical bulk on the $g$-functions, and do not agree with our results.
appears to be straightforward, and we plan to present a more detailed analysis elsewhere.

2 Background

2.1 The bulk flow

The TBA system found in [8] for the tricritical Ising to Ising flow encodes the ground state energy $E(R)$ of the interpolating theory on a circle of circumference $R$ via a pair of integral equations for two pseudoenergies $\epsilon_1(\theta)$ and $\epsilon_2(\theta)$. Using the symmetry $\epsilon_2(\theta) = \epsilon_1(-\theta)$ these boil down to a single equation, for $\epsilon(\theta) \equiv \epsilon_1(\theta) = \epsilon_2(-\theta)$:

$$\epsilon(\theta) = \frac{1}{2} r e^\theta - \int_R \phi(\theta + \theta') L(\theta') d\theta'. \quad (2.1)$$

Here $L(\theta) = \ln(1 + e^{-\epsilon(\theta)})$, $\phi(\theta) = \frac{1}{2\pi \cosh(\theta)}$, and $r = MR$ with $M$ a parameter with the dimensions of mass which sets the (inverse) crossover scale. Then

$$E(R) = -\frac{\pi}{6R} c_{\text{eff}}(r) \quad (2.2)$$

where

$$c_{\text{eff}}(r) = \frac{3}{\pi^2} \int_R r e^\theta L(\theta) d\theta. \quad (2.3)$$

The limiting values $c_{\text{eff}}(0) = 7/10$ and $c_{\text{eff}}(\infty) = 1/2$ can be calculated exactly [8], and match the central charges of the tricritical Ising and Ising models. Later, the form of $L(\theta)$ in these two limits will be important. As $r \rightarrow 0$, three regions develop where $L(\theta)$ is approximately constant:

$$L(\theta) \sim \ln(2) = 0.6931\ldots \quad \text{for } \theta \ll -\ln(1/r); \quad (2.4)$$

$$L(\theta) \sim \ln((3+\sqrt{5})/2) = 0.9624\ldots \quad \text{for } -\ln(1/r) \ll \theta \ll \ln(1/r); \quad (2.5)$$

$$L(\theta) \sim 0 \quad \text{for } \theta \gg \ln(1/r). \quad (2.6)$$

In the opposite limit, $r \rightarrow \infty$, there are instead just two regions:

$$L(\theta) \sim \ln(2) \quad \text{for } \theta \ll -\ln(r); \quad (2.7)$$

$$L(\theta) \sim 0 \quad \text{for } \theta \gg -\ln(r). \quad (2.8)$$

These behaviours are illustrated in figure 1.

![Figure 1](image.png)

Figure 1: $L(\theta)$ for various values of $r$. From the uppermost to the lowermost curve, the values of $\ln(r)$ run from $-20$ to $20$ in equal steps.
2.2 The conformal boundary conditions

With the addition of boundaries, the endpoints of the interpolating flows become boundary conformal field theories: the boundary tricritical Ising model in the ultraviolet, and the boundary (critical) Ising model in the infrared. The basic (Cardy) boundary states for these two models follow from [17], and their physical interpretations were discussed in [17] (for Ising) and [18, 19] (for tricritical Ising). For Ising, there are three possibilities, corresponding to the boundary spins being fixed up (+), fixed down (−), or free (f).

Written in terms of Ishibashi states [20] \(|0\rangle\), |ε⟩ and |σ⟩] they are [17]

\[
|(+\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle + \frac{1}{\sqrt{2}}|\sigma\rangle
\]
\[
|(-\rangle) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle - \frac{1}{\sqrt{2}}|\sigma\rangle
\]
\[
|(f\rangle) = |0\rangle - |\varepsilon\rangle
\]

The inner products of these states with the vacuum |0⟩ give the corresponding values of the conformal g-function [6]. On Ishibashi states |α⟩ we have \langle 0 | α⟩ = δ₀α, so

\[
\ln g_{(+)}|_{\text{Ising}} = \ln g_{(-)}|_{\text{Ising}} = \ln \frac{1}{\sqrt{2}} = -0.3465... \tag{2.10}
\]
\[
\ln g_{(f)}|_{\text{Ising}} = \ln 1 = 0. \tag{2.11}
\]

We will also treat the superposition of (+) and (−) boundaries, (+)\&(−), for which

\[
\ln g_{(+\&(-)}|_{\text{Ising}} = \ln 2g_{(+)}|_{\text{Ising}} = 0.3465... \tag{2.12}
\]

For the tricritical Ising model there are instead six options, each labelled, roughly speaking, by the value (or values) available to the order parameter \(\langle \sigma \rangle\) at that boundary, taken from \{−, 0, +\} [18]. (In the conformal field theory, \(\sigma\) becomes the leading spin field, with dimensions (3/80, 3/80).) These are (−), (0), (+), (−0), (0+), and (−0+), though the last of these is traditionally labelled as (d), with \(d\) standing for ‘degenerate’. The corresponding boundary states are[2]

\[
|(+\rangle) = C\left[|0\rangle + \eta|\frac{1}{10}\rangle + \eta|\frac{3}{5}\rangle + |\frac{3}{2}\rangle + \sqrt{2}|\frac{1}{16}\rangle + \sqrt{2}\eta|\frac{3}{80}\rangle\right]
\]
\[
|(-\rangle) = C\left[|0\rangle + \eta|\frac{1}{10}\rangle + \eta|\frac{3}{5}\rangle + |\frac{3}{2}\rangle - \sqrt{2}|\frac{1}{16}\rangle - \sqrt{2}\eta|\frac{3}{80}\rangle\right]
\]
\[
|(0\rangle) = \sqrt{2}C\left[|0\rangle - \eta|\frac{1}{10}\rangle + \eta|\frac{3}{5}\rangle - |\frac{3}{2}\rangle\right]
\]
\[
|(0+\rangle) = C\left[\eta^2|0\rangle - \eta^{-1}|\frac{1}{10}\rangle - \eta^{-1}|\frac{3}{5}\rangle + \eta^2|\frac{3}{2}\rangle + \sqrt{2}\eta^2|\frac{1}{16}\rangle + \sqrt{2}\eta^{-1}|\frac{3}{80}\rangle\right]
\]
\[
|(−0\rangle) = C\left[\eta^2|0\rangle - \eta^{-1}|\frac{1}{10}\rangle - \eta^{-1}|\frac{3}{5}\rangle + \eta^2|\frac{3}{2}\rangle + \sqrt{2}\eta^2|\frac{1}{16}\rangle - \sqrt{2}\eta^{-1}|\frac{3}{80}\rangle\right]
\]
\[
|(d\rangle) = \sqrt{2}C\left[\eta^2|0\rangle + \eta^{-1}|\frac{1}{10}\rangle - \eta^{-1}|\frac{3}{5}\rangle - \eta^2|\frac{3}{2}\rangle\right]\tag{2.13}
\]

where

\[
C = \frac{\sin(\pi/5)}{\sqrt{5}} = \left(\frac{1}{8} - \frac{1}{8\sqrt{5}}\right)^{1/4}; \quad \eta = \sqrt{2\cos(\pi/5)} = \sqrt{(1+\sqrt{5})/2}. \tag{2.14}
\]

[2] Note, our assignments of the |(+\rangle) and |(-\rangle) states, and of the |(0+\rangle) and |(−0\rangle) states, are opposite to those in [18]. This is to ensure that the one-point function of the spin field \(\sigma\) is positive in the presence of the (+) boundary, and negative in the presence of the (−) boundary, which is more natural, and matches the convention adopted in [21] for the Ising boundary states. We’ve also corrected a typo in the |(+\rangle) and |(-\rangle) boundary states as given in [18]; our states match those given in, for example, [21].
Again adding in the (+)&(−) superposition, the $g$-function values we will need, ordered by increasing $g$ and expressed in ways that will be useful for comparisons later on, are

\[
\begin{align*}
\ln g(+)\big|_{\text{tricrit}} &= \ln g(−)\big|_{\text{tricrit}} = \ln(C) = \frac{1}{4} \ln\left(\frac{1}{8} - \frac{1}{8\sqrt{5}}\right) = -0.6680 \ldots \\
\ln g(0)\big|_{\text{tricrit}} &= \ln(\sqrt{2}C) = \frac{1}{4} \ln\left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right) = -0.3214 \ldots \\
\ln g(0+)\big|_{\text{tricrit}} &= \ln g(−0)\big|_{\text{tricrit}} = \ln(\eta^2C) = \frac{1}{4} \ln\left(\frac{1}{4} + \frac{1}{2\sqrt{5}}\right) = -0.1868 \ldots \\
\ln g(+)\&(−)\big|_{\text{tricrit}} &= \ln 2g(+)\big|_{\text{tricrit}} = \ln g(0)\big|_{\text{tricrit}} + \frac{1}{2} \ln 2 = 0.0250 \ldots \\
\ln g(d)\big|_{\text{tricrit}} &= \ln(\sqrt{2}\eta^2C) = \ln g(0+)\big|_{\text{tricrit}} + \frac{1}{2} \ln 2 = 0.1597 \ldots
\end{align*}
\]

### 2.3 The boundary flows

The flows which occur when the two models are perturbed at the boundary alone are well-understood. In Ising, the $(f)$ boundary admits a single relevant boundary field $\phi_{13}$, with dimension $1/2$. This breaks the $\mathbb{Z}_2$ symmetry of the bulk and can be interpreted as a boundary magnetic field. Depending on the sign of the perturbation, a flow is induced to the $(−)$ or to the $(+)$ boundary, as shown in figure 2.

\[
\begin{array}{c}
\text{(−)} \quad \text{(f)} \quad \text{(+)}
\end{array}
\]

Figure 2: Flows from the $(f)$ boundary in the Ising model.

The $(+)$ and $(−)$ boundaries have no relevant boundary fields, but one can also consider their superposition, $(+)&(−)$. The boundary-condition changing operators correspond to $\phi_{13}$ again, and generate the flow illustrated in figure 3 to the free boundary condition [22, 23]:

\[
\begin{array}{c}
(+)&(−) \quad \text{(f)}
\end{array}
\]

Figure 3: The flow from the $(+)&(−)$ boundary in the Ising model.

For the tricritical Ising model the structure is richer [18, 19] (see also [24–26]). Including the superposition $(+)&(−)$, the full map is shown in figure 4.

\[
\begin{array}{c}
(−) \quad (−0) \quad (0) \quad (0+) \quad (+)
\end{array}
\]

Figure 4: Boundary flows in the tricritical Ising model. Solid lines (red online) show flows induced by a $\phi_{13}$ boundary field. The dashed lines (green online) are induced by $\phi_{12}$ (for the lower two lines) or $\phi_{11}$ (for the upper two). Finally, the dotted lines (blue online) are induced by a combination of $\phi_{12}$ and $\phi_{13}$.
With the bulk conformal, the flows induced by $\phi_{13}$, $\phi_{12}$ and $\phi_{11}$ (the solid and dashed lines on figure 4) are all integrable. However, for integrability to survive when the bulk is also perturbed, it is not enough for the bulk and boundary perturbations to be separately integrable – they must also be compatible with each other. For the $\phi_{13}$ bulk perturbation which leads to the interpolating flow to Ising, this is thought to hold if the boundary perturbing operator is also $\phi_{13}$ [27], and so it is these combined bulk-and-boundary flows that we should aim to treat using the exact $g$-function.

3 Exact $g$-functions for the interpolating flow

In [9, 10], exact equations were proposed for the off-critical $g$-function in certain massive integrable boundary theories. To give our proposal for the interpolating bulk and boundary tricritical Ising to Ising flows, we first set

$$
(x)(\theta) = \frac{\sinh \left( \frac{\theta}{2} + \frac{i \pi x}{2} \right)}{\sinh \left( \frac{\theta}{2} - \frac{i \pi x}{2} \right)}, \quad \phi(x)(\theta) = -\frac{i}{2\pi} \frac{d}{d\theta} \ln (x)(\theta) = -\frac{\sin(\pi x)/(2\pi)}{\cosh(\theta) - \cos(\pi x)}, \quad (3.1)
$$

so that the kernel function $\phi(\theta)$ in the bulk TBA equation (2.1) is equal to $-\phi_{(1/2)}(\theta)$, and

$$
\int_\mathbb{R} \phi(x)(\theta) d\theta = -(1 - |x|) \text{sgn}(x). \quad (3.2)
$$

Now let $\epsilon(\theta)$ solve the bulk TBA equation (2.1) for a system on a cylinder of circumference $r$, and suppose a boundary is placed at the end of that cylinder with a boundary condition which depends on a further parameter $\theta_b$. We will propose the following expression for the logarithm of a $g$-function $\ln g(r)$:

$$
\ln g(r) = \ln g_0(r) + \ln g_b(r) \quad (3.3)
$$

where

$$
\ln g_0(r) = \sum_{j=1}^\infty \frac{1}{2j-1} \int_{\mathbb{R}^{2j-1}} \frac{d\theta_1}{1 + e^{\epsilon(\theta_1)}} \cdots \frac{d\theta_{2j-1}}{1 + e^{\epsilon(\theta_{2j-1})}} \phi(\theta_1 + \theta_2) \phi(\theta_2 + \theta_3) \cdots \phi(\theta_{2j-1} + \theta_1) \quad (3.4)
$$

and

$$
\ln g_b(r) = -\frac{1}{2} \ln(2) + \int_\mathbb{R} (\phi_b(\theta) - \phi(2\theta)) L(\theta) d\theta \quad (3.5)
$$

with $L(\theta) = \ln(1 + e^{-\epsilon(\theta)})$ as before, and

$$
\phi_b(\theta) = \phi_{(3/4)}(\theta) - \phi_{(1/2)}(\theta - \theta_b). \quad (3.6)
$$

(Note, the normalisation of $\phi_b$ here differs by a factor of 2 from that in [9,10].)

The expression (3.3) has the same general structure as the exact massive $g$-function introduced in [9], with $g_0(r)$ containing the boundary-condition specific parts of the $g$-function, while $g_b(r)$ is a ‘universal’ piece which incorporates the effects of the bulk perturbation on the boundary entropy. However, the new formula involves some significant changes too – in particular, the infinite series in (3.3) contains only odd terms, and all rapidity combinations in the kernel functions $\phi(\theta_1 + \theta_{i+1})$ appear as sums. (This second aspect is related to the fact that (3.4) has been written in terms of the single function $\epsilon(\theta) = \epsilon_1(\theta)$, rather than $\epsilon_1(\theta)$ and $\epsilon_2(\theta) = \epsilon_1(-\theta)$.)

The infinite series gives an expansion for $\ln g_0(r)$ about $r = \infty$, but it converges rapidly for all values of $r$, and can be summed exactly at $r = 0$, and in various intermediate double-scaling limits. Before giving these details, figures 5, 6 and 7 show numerically-obtained
plots of $\ln g(r)$ for $\theta_b = -15, 0$ and $+15$. The plots were obtained using the first five terms from the series (3.4), though truncating to just three terms would have given visually indistinguishable results. For all three values of $\theta_b$, $\ln g(r)$ tends to $\ln g_{(0+)}|_{\text{tricrit}}$ in the far ultraviolet, to $\ln g_{(+)}|_{\text{Ising}}$ in the far infrared, and undergoes a transition at $\ln r \approx 0$, which is where the bulk crossover occurs. For $\theta_b = -15$, there are two further transitions, at $\ln r \approx \pm 15$, while for $\theta_b = +15$ there is one, at $\ln r \approx -15$.

Figure 5: The exact $g$-function flow for $\theta_b = -15$. The flow of $c_{\text{eff}}(r)/2$, running from 0.35 down to 0.25, is also shown, to indicate the location and duration of the bulk crossover. Tricritical Ising $g$-function values are $g_{0+}$, $g_0$ and $g_+$; critical Ising values are $g_{\text{free}}$ and $g_{\text{fixed}}$.

Figure 6: The exact $g$-function flow for $\theta_b = 0$. Labelling as for figure 5.

Note, though, that these are equally the values of $\ln g_{(0-)}|_{\text{tricrit}}$ and $\ln g_{(-)}|_{\text{Ising}}$ respectively. We will mostly leave this ambiguity implicit in the following, but we will return to it briefly later in this section.
The natural interpretation of the plot for $\theta_b = -15$ is that the corresponding renormalisation group flow starts with a pure-boundary transition at $\ln r \approx -15$ from the $(0+)_{\text{tricritical}}$ boundary to the vicinity of the $(0)_{\text{tricritical}}$ boundary, with the bulk remaining near to the tricritical Ising fixed point, then undergoes a bulk-and-boundary transition with the bulk flowing from tricritical Ising to Ising while the boundary moves from the neighbourhood of $(0)_{\text{tricritical}}$ to the neighbourhood of $(f)_{\text{Ising}}$, before finally making a further boundary transition, at $\ln r \approx 15$, to $(+)^{\text{Ising}}$. For $\theta_b = 0$, there is a single combined bulk-and-boundary transition, from $(0+)_{\text{tricritical}}$ to $(+)^{\text{Ising}}$, at $\ln r \approx 0$. The absence of an independent boundary transition suggests that this case corresponds to the boundary perturbation being zero, and we will give further evidence for this claim in the next section. Finally, for $\theta_b = 15$ there is a pure-boundary transition at $\ln r \approx -15$, from $(0+)_{\text{tricritical}}$ to the neighbourhood of $(+)_{\text{tricritical}}$, followed by a bulk-and-boundary transition to $(+)^{\text{Ising}}$ at $\ln r \approx 0$, and no further transitions.

These results combine to give the picture sketched in figure 8, which matches the predictions made on the basis of large-$p$ perturbative calculations in [4]. Furthermore,
taking the limits $\theta_b \to -\infty$ and $\theta_b \to +\infty$ shows that in addition to the $\theta_b = 0$ flow from $(0+)|_{\text{tricrit}}$ to $(+)|_{\text{Ising}}$, there should be bulk-induced flows from $(0)|_{\text{tricritical}}$ to $(f)|_{\text{Ising}}$, and from $(+)|_{\text{tricritical}}$ to $(+)|_{\text{Ising}}$; as explained in [4], these claims match the results of [28].

One caveat, though: as mentioned above, strictly speaking our results cannot distinguish between $(+)$ and $(-)$, nor between $(0+)$ and $(-0)$, as the $g$-functions do not distinguish between these pairs of boundary conditions. Physically it is clear that the picture given in figure 8 and its image under a global swap of $+$ for $-$, must be correct, but to resolve the issue within the context of exact $g$-function flows alone, one would have to track the evolution of the inner products of states other than the ground state with the boundary state. We expect that this will be possible using pseudenergies which solve excited-state TBA equations [29,30], but we shall leave the further exploration of this issue to future work.

Finally, we need a proposal for the off-critical deformations of the $\mathbb{Z}_2$-symmetric $\phi_{13}$ flows which run from $(d)$ up to $(+)&(-)$ and down to $(0)$ in figure 4. We claim that these flows are captured by replacing the formula (3.5) for $\ln g_b(r)$ by

$$\ln g_b(r) = \int_\mathbb{R} (\phi_b(\theta) - \phi(2\theta)) \ln(1 + e^{-\epsilon(\theta)}) \, d\theta.$$  

(3.7)

In other words, we simply add $1/2 \ln 2 = 0.3465\ldots$ to the logarithm of the previous exact $g$-function. The graphs in figures 5, 6 and 7 are then shifted upwards by this constant, and the transitions occur at the same values of $r$ as before, but between a different set of conformal boundary conditions, as summarised in figure 9. Again, this matches the extrapolation of the predictions of [4] down to $p = 4$.

![Figure 9: Combined bulk and boundary flows predicted by (3.3), (3.4) and (3.7). The labelling convention for renormalisation group fixed points is as for figure 8. The flows marked d, e and f correspond to $\theta_b = -15$, 0 and 15 respectively.](image-url)

### 4 Exact and numerical tests of the proposal

#### 4.1 Exact limiting values of the g-function

We first deal with the universal factor $\ln g_0(r)$ defined by equation (3.4). From (3.2), $\int_\mathbb{R} \phi(\theta) \, d\theta = 1/2$, and so

$$\int_{\mathbb{R}^{2j-1}} d\theta_1 \ldots d\theta_{2j-1} \phi(\theta_1 + \theta_2)\phi(\theta_2 + \theta_3) \ldots \phi(\theta_{2j-1} + \theta_1) = \frac{1}{2^{2j}}$$  

(4.1)

with the product of the kernel functions $\phi$ tending to zero exponentially outside a region of order 1 about the origin. In the infrared, only this latter property is needed: from (2.8), each factor $1/(1+\epsilon(\theta_i)) \to 0$ for $\theta_i \gg -\ln(r)$. In particular this holds in the neighbourhood of the origin where the product of the kernel functions is significantly different from zero. Hence all terms in the series (3.4) tend to zero as $r \to \infty$, and $g_0(r) \to 0$. 

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In the ultraviolet, via (2.5), the pseudoenergies tend to constants in the central region
\[ -\ln(1/r) \ll \theta \ll \ln(1/r), \]
with \( e^{-\epsilon(\theta)} \to (1+\sqrt{5})/2, 1/(1+e^{\epsilon(\theta)}) \to (\sqrt{5}-1)/2 \).
Combined with (4.1), this implies
\[ \int_{R} d\theta_{1} d\theta_{2j-1} \frac{\phi(\theta_{1} + \theta_{2}) \phi(\theta_{2} + \theta_{3}) \ldots \phi(\theta_{2j-1} + \theta_{1})}{1 + \epsilon(\theta_{2j-1})} = \frac{1}{2} x^{2j-1} \]
where \( x = (\sqrt{5}-1)/4 \), and so
\[ \lim_{r \to 0} \ln g_{0}(r) = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{2j-1} x^{2j-1} = \frac{1}{4} \ln \left( \frac{1+x}{1-x} \right) = \frac{1}{4} \ln \left( 1 + \frac{2}{\sqrt{5}} \right) = 0.15972912974 \ldots . \]

The boundary condition dependent piece can be treated by rewriting (3.5) so as to split \( \ln g_{b}(r) \) into three parts: a constant \( \ln g_{b1} \), a parameter-independent piece \( \ln g_{b2}(r) \), and a \( \theta_{b} \)-dependent piece \( \ln g_{b3}(r, \theta_{b}) \):
\[ \ln g_{b}(r) = \ln g_{b1} + \ln g_{b2}(r) + \ln g_{b3}(r, \theta_{b}) \]
where
\[ \ln g_{b1} = -\frac{1}{2} \ln 2, \]
\[ \ln g_{b2}(r) = \int \left( \phi_{(3/4)}(\theta) - \phi(2\theta) \right) L(\theta) d\theta, \]
\[ \ln g_{b3}(r, \theta_{b}) = -\int \phi_{(1/2)}(\theta - \theta_{b}) L(\theta) d\theta. \]

Note also, from (3.2), that
\[ \int \left( \phi_{(3/4)}(\theta) - \phi(2\theta) \right) d\theta = -\frac{1}{2}, \]
\[ -\int \phi_{(1/2)}(\theta - \theta_{b}) d\theta = \frac{1}{2}. \]

The decay properties of \( \phi_{(3)}(\theta) \) mean that the support for the integral (4.8) is concentrated near to \( \theta = 0 \), while that for (4.9) is concentrated near to \( \theta = \theta_{b} \). Combined with the asymptotic behaviours of \( L(\theta) \) recorded in equations (2.4) – (2.8), these results allow the various limiting values of \( \ln g(r) \) to be computed.

1. In the far infrared limit \( \{ r \to \infty, \theta_{b} \text{ fixed} \} \), \( \ln g_{b}(r) \to 0 \), and \( L(\theta) \to 0 \) in the regions where the integrals (4.6) and (4.7) can receive contributions, so
\[ \ln g(r) \to \ln g_{b1} = -\frac{1}{2} \ln 2 = \ln g_{(+)_{\text{Ising}}}. \]

2. In the far ultraviolet limit \( \{ r \to 0, \theta_{b} \text{ fixed} \} \), \( L(\theta) \) acquires a constant value in the whole region where the integrands in (4.6) and (4.7) are significantly different from zero. The integrals (4.6) and (4.7) therefore cancel in the limit, and
\[ \ln g(r) \to \ln g_{b}(0) + \ln g_{b1} = \frac{1}{4} \ln \left( 1 + \frac{2}{\sqrt{5}} \right) - \frac{1}{2} \ln 2 = \frac{1}{4} \ln \left( 1 + \frac{1}{2\sqrt{5}} \right) = \ln g_{(+)_{\text{tricrit}}}. \]

3. If \( \theta_{b} \ll 0 \) and \( \theta_{b} \ll r \ll 0 \), then for \( \theta \approx \theta_{b} \), \( L(\theta) \approx \ln 2 \) from (2.4), while for \( \theta \approx 0 \), \( L(\theta) \approx \ln((3+\sqrt{5})/2) \) from (2.5). Hence \( \ln g_{b3} \approx \frac{1}{2} \ln 2 \) and \( \ln g_{b2} \approx -\frac{1}{2} \ln((3+\sqrt{5})/2) \), and
\[ \ln g(r) \to \frac{1}{4} \ln \left( 1 + \frac{2}{\sqrt{5}} \right) - \frac{1}{2} \ln 2 - \frac{1}{2} \ln((3+\sqrt{5})/2) + \frac{1}{2} \ln 2 = \ln g_{(+)_{\text{tricrit}}}. \]
4. If \( \theta_b < 0 \) and \( 0 < \ln r < -\theta_b \), then for \( \theta \approx \theta_b \), \( L(\theta) \approx \ln 2 \) from (2.7), while for \( \theta \approx 0 \), \( L(\theta) \approx 0 \) from (2.8). Hence \( \ln g_{b3} \approx \frac{1}{2} \ln 2 \) and \( \ln g_{b2} \approx 0 \), and

\[
\ln g(r) \rightarrow -\frac{1}{2} \ln 2 + \frac{1}{2} \ln 2 = 0 = \ln g(+)\big|_{\text{Ising}}. \tag{4.13}
\]

5. If \( \theta_b > 0 \) and \( -\theta_b < \ln r < 0 \), then for \( \theta \approx 0 \), \( L(\theta) \approx \ln(3+\sqrt{5})/2 \) from (2.5), while for \( \theta \approx \theta_b \), \( L(\theta) \approx 0 \) from (2.6). Hence \( \ln g_{b2} \approx -\frac{1}{2} \ln(3+\sqrt{5})/2 \) and \( \ln g_{b3} \approx 0 \), and

\[
\ln g(r) \rightarrow \frac{1}{2} \ln(1+\frac{2}{\sqrt{5}}) - \frac{1}{2} \ln 2 - \frac{1}{2} \ln(3+\sqrt{5})/2 = \ln g(+)|_{\text{tricrit}}. \tag{4.14}
\]

To make the statements of 3, 4 and 5 precise, they should be considered as double-scaling limits: for example, for 3 one could fix two constants \( \bar{\theta}_b \) and \( \tilde{r} \) with \( \bar{\theta}_b < \ln \tilde{r} < 0 \), and set \( r = \bar{\tilde{r}} \), \( \theta_b = \rho \bar{\theta}_b \); then (4.12) holds in the limit \( \rho \to \infty \). If instead \( \theta_b \) is kept fixed and \( r \) is varied from 0 to \( \infty \), then the case \( \theta_b \ll 0 \), figure 6, by 1, 2; and the case \( \theta_b \gg 0 \), figure 7, is covered by 1, 3, 4, 2; the case \( \theta \approx 0 \), figure 5, is covered by 1, 3, 4, 2; the case \( \theta \approx 0 \), figure 6, by 1, 2; and the case \( \theta_b > 0 \), figure 7, is covered by 1, 3, 4, 2.

4.2 Comparisons with conformal perturbation theory

The bulk perturbation which induces the flow from the tricritical Ising model to the Ising model corresponds to the addition of a term \( \lambda \int \phi_{13}(x, \bar{x}) \, d^2 x \) to the action of the tricritical model, where the bulk coupling \( \lambda \) has dimension (mass)\(^{4/5}\). If the unperturbed conformal boundary condition \( (\alpha) \) supports the boundary field \( \phi_{13}(x) \), the addition of a boundary perturbation \( \mu \int \phi_{13}(x) \, dx \) can also be considered, where \( \mu \) is the boundary coupling, with dimension (mass)\(^{2/5}\). (For the tricritical Ising conformal boundary conditions featured on figure 4 (–0), (d) and (0+) do support this field, while (–), (0), (+) and (+)\( \kappa \) do not.) A \( g \)-function as evaluated in conformal perturbation theory should therefore have the expansion

\[
\ln \mathcal{G}(\lambda, \mu, R) = \sum_{m,n=0}^{\infty} c_{m,n}^{(\alpha)}(\mu R^{2/5})^m (\lambda R^{4/5})^n. \tag{4.15}
\]

In general this is a regular series in powers of \( R^{2/5} \), reducing to a series in \( R^{4/5} \) when \( \mu = 0 \). At large \( R \), the function defined by (4.15) will typically develop a linear behaviour, with \( \ln \mathcal{G}(h, \lambda, R) \sim -fMR \) where \( f \) is a free energy per unit length, which we choose to measure in units of the inverse crossover scale \( M \). Our exact equations, by contrast, yield ‘subtracted’ \( g \)-functions from which this term is absent in the infrared, and instead reappears as an irregular term in the ultraviolet [32, 33]. They are also expressed in terms of \( M \) and the boundary parameter \( \theta_b \), rather than \( \lambda \) and \( \mu \). The relation between \( \lambda \) and \( M \) is known [8, 31]:

\[
\lambda = \kappa M^{4/5}, \quad \kappa = \frac{1}{2\sqrt{2}(3\pi)^{1/5}} \sqrt{\frac{\Gamma(7/10)}{\Gamma(3/10)}} = 0.14869551611 \ldots \tag{4.16}
\]

and on dimensional grounds it must be possible to write \( \mu \) as

\[
\mu = \nu(\theta_b)M^{2/5} \tag{4.17}
\]

where \( \nu \) is some dimensionless function of \( \theta_b \). The \( g \)-function defined by (3, 3) should thus have the following expansion about \( r \equiv MR = 0 \):

\[
\ln g(r, \theta_b) = \ln g_0(r) + \ln g_b(r) = fr + \sum_{m,n=0}^{\infty} c_{m,n}^{(\alpha)}(\nu R^{2/5})^m (\kappa R^{4/5})^n \tag{4.18}
\]
where \( c^{(\alpha)}_{0,0} \) is equal to the logarithm of the conformal \( g \)-function \( g^{(\alpha)} \) for the \( (\alpha) \) boundary condition. For all values of \( \theta_b \) we expect

\[
(c^{(\alpha)}_{1,0}) = 0
\]  

(4.19)

since \( c^{(\alpha)}_{1,0} \) is proportional to the one-point function of the perturbing boundary operator on a disk with the vacuum field at its centre [33], and vanishes in a unitary theory such as this one [9]. Previous examples suggest that \( \ln g^{(0)}(r) \) will not contribute to the irregular term \( f_r \) [9,10,33]; assuming that this holds true here too, the value of \( f \) can be calculated from \( \langle \alpha | \rangle \) as in [32,33], with the result

\[
f(\theta_b) = \frac{1}{2} e^{-\theta_b} - \frac{1}{2 \sqrt{2}}.
\]  

(4.20)

Finally, the first bulk-induced coefficient in the expansion of \( \ln g^{(\alpha)}(r) \) is [10]

\[
c^{(\alpha)}_{0,1} = -\frac{B(1-x_{\phi}, x_{\phi}/2)}{2(2\pi)^{1-x_{\phi}}} \langle \phi |(\alpha) \rangle \langle 0 |(\alpha) \rangle
\]  

(4.21)

where \( B(x,y) = \Gamma(x) \Gamma(y)/\Gamma(x+y) \) is Euler’s beta function, \( \phi \) is the bulk perturbing field, \( \phi_{13} \) in this case, and \( x_{\phi} \) is its scaling dimension, here equal to \( 6/5 \). The inner products \( \langle \phi |(\alpha) \rangle \) and \( \langle 0 |(\alpha) \rangle \) can be read from \( \langle 2.13 \rangle \), bearing in mind that the Ishibashi states in those formulae have been labelled by the conformal dimensions of their Virasoro representations, which are half the scaling dimensions of the corresponding bulk fields.

The bulk TBA equation \( \langle 2.21 \rangle \) was solved numerically for 101 evenly-spaced values of \( r^{4/5} \) running from 0.0005 to 0.1255, discretising the \( \theta \) axis to 1520 points between \( \theta = -50 \) and \( \theta = 50 \) and using extended (20 decimal digit) precision in GNU Fortran 95. The resulting estimates for the pseudoenergy \( \epsilon(\theta) \) were then used to compute \( \ln g^{(\alpha)}(r) \) via \( \langle 3.3 \rangle \), summing the series \( \langle 3.4 \rangle \) for \( \ln g^{(0)}(r) \) to 12 terms, and evaluating the \( \theta_b \)-dependent part \( \ln g^{(\alpha)}(r) \) from \( \langle 3.5 \rangle \) for values of \( \theta_b \) ranging from \(-2.5\) and \(2.5\). (Were accurate results to be required for a larger range of \( \theta_b \), care would have to be taken to decrease the values of \( r^{4/5} \) used for the fits, to avoid their contamination by the intermediate plateau values of \( \ln g^{(\alpha)}(r) \) which appear as \( |\theta_b| \) increases, as on figures \( 5 \) and \( 7 \)).

As a first check of our numerics, we made a least-squares fit of the function \( c_{\text{eff}}^{(\alpha)}(r) \) defined by \( \langle 2.3 \rangle \) to a regular expansion in powers of \( r^{4/5} \) plus a single ‘antibulk’ term proportional to \( r^2 \), finding coefficients which matched those reported in \( \langle 8 \rangle \) to the full accuracy claimed there.

Then, for each value of \( \theta_b \), the numerically-obtained \( \ln g^{(\alpha)}(r; \theta_b) \) was fitted to a series in \( r^{2/5} \) plus a single term proportional to \( r \), as in \( \langle 4.18 \rangle \):

\[
\ln g^{(\alpha)}(r; \theta_b) = \sum_{k=0}^{\infty} d_k(\theta_b) r^{2k/5} + e(\theta_b) r.
\]  

(4.22)

If the match with conformal perturbation theory is to hold, we should have

\[
d_k(\theta_b) = \sum_{l=0}^{[k/2]} c^{(\alpha)}_{k-2l,l} \nu^{k-2l} \kappa^l, \quad e(\theta_b) = f(\theta_b).
\]  

(4.23)

The constant term \( d_0(\theta_b) \) obtained from the fits matched the value predicted by \( \langle 4.11 \rangle \), namely \( \ln g^{(0)}(1+) \) to at least 10 digits for the whole range of \( \theta_b \). Furthermore, \( d_1(\theta_b) \)

\footnote{Note, if \( c_{1,0} \) did not vanish, then the \( g \)-theorem, which states that \( g \) decreases for all pure-boundary flows in unitary models [6,34], would be violated for one or other sign of the boundary coupling. Conversely, the non-vanishing of \( c_{0,1} \) [10] is an easy way to see that the \( g \)-theorem can be violated when the bulk flows, even in a unitary theory (see also [35]).}
was zero to the same accuracy, in line with (4.19). In figure 10a, the values of $e(\theta_b)$ found from our fits are compared with the exact predictions from (4.20); the good agreement supports our claim that $\ln g_0(r)$ does not contribute to this linear term.

Figure 10a: A comparison of the exact prediction (4.20) for $f(\theta_b)$ (dotted line) with values of $e(\theta_b)$ from the fit (4.22) (points).

Figure 10b: Estimates of $d_3(\theta_b)$ from the fit (4.22). The estimate at $\theta_b = 0$ is $5.24 \times 10^{-8}$, consistent with 0 to our numerical accuracy.

Next, in figure 10b, we show the values of $d_3(\theta_b)$. The apparent zero of this function at $\theta_b = 0$ suggests that this point should correspond to $\mu = 0$ in (4.18), where all odd terms in the regular series should vanish. This is consistent with the fit at $\theta_b = 0$, which is

$$
\ln g(r, 0) = -0.1868444605395000 + 0.1464465945456 r - 8.429 \times 10^{-12} r^{2/5} \\
- 0.2038867755577 r^{4/5} + 5.24 \times 10^{-8} r^{6/5} - 0.008541571 r^{8/5} \\
+ 3.68 \times 10^{-6} r^2 - 0.00209 r^{12/5} + \ldots
$$

(4.24)

Supposing that the odd terms are exactly zero for $\theta_b = 0$, a more-constrained fit to a regular series in powers of $r^{4/5}$ plus a term linear in $r$ gives the result

$$
\ln g(r, 0) = -0.1868444605395363 + 0.1464466094005 r - 0.2038867770734 r^{4/5} \\
- 0.008541178 r^{8/5} - 0.0020624 r^{12/5} + 0.00151 r^{16/5} - 0.0004 r^4 + \ldots
$$

(4.25)

For this case, $d_2$ is known exactly, since with $\mu$ and hence $\nu = 0, d_2 = c^{(a)}_{0,1} \kappa$ and can be calculated from (4.21) and (4.16). For the (0+) boundary with $\theta_b = 0$ we thus have the exact predictions

$$
d_0 = \ln g_{(0+)} = -0.1868444605395326 \ldots
$$

(4.26)

$$
d_1 = 0
$$

(4.27)

$$
d_2 = B(-1/5, 3/5) \kappa \\
= -0.2038867770751855 \ldots
$$

(4.28)

$$
d_3 = 0
$$

(4.29)

$$
e = f(0) = 0.1464466094067 \ldots
$$

(4.30)

all of which are reproduced very well by the fits (4.24) and (4.25).

The limits $\theta_b \to -\infty$ and $\theta_b \to +\infty$ admit similarly-simple checks. A consideration of figure 8 and the results from subsection 4.1 shows that if these limits are taken first, keeping $r$ finite, and $r$ is only then allowed to vary, then the resulting equations should describe the bulk-induced flows (0)$|_{\text{tricrit}} \to (f)$|_{Ising} and (+)$|_{\text{tricrit}} \to (+)$|_{Ising} respectively. Neither of the UV boundary conditions for these flows admit a $\phi_{13}$ boundary field, so the logarithms of their $g$-functions should have regular expansions about $r = 0$ in powers of $r^{4/5}$, with a coefficient $d_2$ of $r^{4/5}$ that can be predicted from (4.21).
In the limit $\theta_b \to -\infty$, making use of (4.9) and (2.7), equation (3.5) reduces to

$$\ln g_b(r)|_{\theta_b=-\infty} = \int_R (\phi(3/4)(\theta) - \phi(2\theta))L(\theta) \, d\theta$$

while equation (3.4) for $\ln g_0(r)$ is unchanged. With the numerical work as before, the fit to a series in powers of $r^{2/5}$, together with a linear term, was

$$\ln g(r)|_{\theta_b=-\infty} = -0.3214826953191443 - 0.353553993721 r - 5.178 \times 10^{-12} r^{2/5} + 0.533782513495 r^{4/5} + 3.04 \times 10^{-8} r^{6/5} - 0.01741761 r^{8/5} + 1.98 \times 10^{-6} r^2 + 0.0132 r^{12/5} + \ldots$$

and the more-constrained fit to a regular series in powers of $r^{4/5}$ plus a linear term gave

$$\ln g(r)|_{\theta_b=-\infty} = -0.3214826953191671 - 0.353553905994 r + 0.5337825122412 r^{4/5} - 0.017417394 r^{8/5} + 0.0133024 r^{12/5} - 0.00130 r^{16/5} - 0.0008 r^4 + \ldots$$

These results can be compared with the exact predictions for the first few coefficients for the bulk-induced flow from the (0) boundary:

$$d_0 = \ln g_0 = -0.3214826953191634 \ldots$$

$$d_1 = 0$$

$$d_2 = -B(-1/5, 3/5) \frac{\kappa \eta}{2(2\pi)^{-1/5}} = 0.533782512395085 \ldots$$

$$d_3 = 0$$

$$e = -\frac{1}{2\sqrt{2}} = -0.353553905932 \ldots$$

Again, the agreement is very good. It is also straightforward to check analytically that this $g$-function interpolates between the desired values – the argument is essentially covered by cases 3 and 4 of the last subsection.

For the $(\+)tricrit \to (\+)ising$ flow expected to arise in the $\theta_b \to +\infty$ limit the story is very similar. Since, by (2.8), $L(\theta) \to 0$ as $\theta \to +\infty$, there is this time no modification to the constant term in (3.5), which becomes

$$\ln g_b(r)|_{\theta_b=+\infty} = -\frac{1}{2} \ln(2) + \int_R (\phi(3/4)(\theta) - \phi(2\theta))L(\theta) \, d\theta$$

with the expression for $\ln g_0(r)$ again unchanged. The fits for the expansion coefficients (apart from the constant term) are therefore the same as before, and it is straightforward to check that these match expectations from perturbation theory for this situation. The same also holds for the set of $\mathbb{Z}_2$-symmetric flows predicted by our second proposal, (3.7), and so we will leave the details to the reader.

5 Conclusions

In this paper we have proposed an extension of the exact off-critical $g$-function equations of [9,10] to cover situations where the bulk field theory retains massless degrees of freedom even in the far infrared, and therefore interpolates between two different conformal field theories. While our proposals are still conjectural, they have passed a number of non-trivial checks against perturbation theory, leaving us in little doubt that they are correct. Nevertheless, a first-principles derivation from field-theoretic considerations would be valuable,
as would an understanding via lattice models, as obtained for the TBA equations for the bulk interpolating flow of $c_{\text{eff}}(r)$ in [36]. However this may be a hard task, and indeed previous attempts to derive exact equations for exact $g$-function flows have not been entirely successful [37]. It would also be interesting to see whether a process of analytic continuation could be used to relate massive and massless $g$-functions, though again this might be delicate, given that the functions are initially defined via a sub-leading term in the asymptotic behaviour of cylinder partition functions.

One feature of our results is the exact equality, up to a constant factor, of the $g$-functions for various a-priori different flows. It seems likely that this can be understood through an extension of the defect-related work of [38] to theories off-critical in the bulk and it would be interesting to explore this further.

The tricritical to critical Ising flow is interesting in its own right (see for example [8,39]), but the main reason for concentrating on this particular case in this paper has been its relative simplicity, which has allowed us to make a detailed check of the feasibility of our approach, and to illustrate the main ideas without too many distracting complications. As mentioned in the introduction, we expect that the general method will be of much wider applicability, and we hope to return to its further applications in the future.

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See the comments at the end of section 5 of [9].

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