On a functional-differential equation with quasi-arithmetic mean value

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Abstract

In this paper we describe all differentiable functions $\varphi, \psi : E \to \mathbb{R}$ satisfying the functional-differential equation

$$[\varphi(y) - \varphi(x)] \psi'(h(x, y)) = [\psi(y) - \psi(x)] \varphi'(h(x, y)), \quad (0.1)$$

for all $x, y \in E, x < y$, where $E \subseteq \mathbb{R}$ is a nonempty open interval, $h(\cdot, \cdot)$ is a quasi-arithmetic mean, i.e. $h(x, y) = H^{-1}(\alpha H(x) + \beta H(y)), x, y \in E$, for some differentiable and strictly monotone function $H : E \to H(E)$ and fixed $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$.

1 Introduction

Given a nonempty open interval $E \subseteq \mathbb{R}$ and differentiable functions $\varphi, \psi : E \to \mathbb{R}$, the Cauchy Mean Value Theorem (MVT) states that, for any interval $(a, b) \subset E$ there exists a point $c$ in $(a, b)$ such that

$$[\varphi(b) - \varphi(a)] \psi'(c) = [\psi(b) - \psi(a)] \varphi'(c). \quad (1.1)$$

A particular case is the Lagrange MVT, in which case $\psi$ is the identity function and hence, (1.1) reads as

$$\varphi(b) - \varphi(a) = (b-a)\varphi'(c). \quad (1.2)$$

It is an interesting question to ask for which $\varphi$ and $\psi$ the mean value $c$ in (1.1) or (1.2) depends on the endpoints $a$ and $b$ in a prescribed way. This is the subject of this note. More precisely, we are concerned with the following problem.

Problem 1. Let $E \subseteq \mathbb{R}$ be a nonempty open interval, let $H : E \to H(E)$ be a differentiable and strictly monotone function and let $h : E \times E \to \mathbb{R}$ satisfy for all $x, y \in E$ that $h(x, y) = H^{-1}(\alpha H(x) + \beta H(y)), x, y \in E$, where $\alpha, \beta \in (0, 1)$ satisfy $\alpha + \beta = 1$. Find all pairs $(\varphi, \psi)$ of differentiable functions $\varphi, \psi : E \to \mathbb{R}$ which satisfy for all $x, y \in E$, with $x < y$, that

$$[\varphi(y) - \varphi(x)] \psi'(h(x, y)) = [\psi(y) - \psi(x)] \varphi'(h(x, y)). \quad (1.3)$$
For the case of the Lagrange MVT (i.e. when $\psi(x) = x$) with $H(x) = x$ and $\alpha = \beta = \frac{1}{2}$, this problem was considered first by Haruki [5] and independently by Aczél [1], who showed that the quadratic functions are the only solutions to (1.3). This particular case serve as a starting point for various functional equations, see e.g. Sahoo and Riedel [11]. More general functional equations have been considered even in the abstract setting of groups by several researchers including Kannappan [6], Ebanks [3] and Fechner & Gselmann [4]. Moreover, the result of Aczél and Haruki has been generalized for higher order Taylor expansion by Sablik [10].

The functional-differential equation (1.3) was solved by Balogh, Ibragimov & Mityagin [2] for the case $H(x) = x, E = \mathbb{R}$, under the assumption that $\varphi$ and $\psi$ are three times differentiable functions. It is worth to mention that the method of [2] applies to the case of arbitrary open interval $E$ without much additional effort. Recently, Lukasik [8] has combined the method of [2] together with an indirect approach to provide all solutions of (1.3) for the case of $H(x) = x$ and an arbitrary open interval $E \subseteq \mathbb{R}$ by only requiring the differentiability of the unknown functions. For the general case, Kiss & Páles [7] has provided all solutions of the equation (1.3) under the assumption that $\psi'$ does not vanish on $E$ and that $\psi'$ is invertible.

In this note we provide a complete solution to Problem 1 by a different and self-contained approach. Our method is heavily inspired by that of [2] and based on the tricks analogous to the ones of [7] and [8]. Our main result reads as follows.

**Theorem 1.** Assume the setting of Problem 1. If $\varphi$ and $\psi$ solves the functional-differential equation (1.3), then one of the following possibilities holds:

(a) $\{1, \varphi, \psi\}$ are linearly dependent on every open subinterval of $E$, where $\psi'$ does not vanish. Moreover, if $J = H(E)$ is a semi-infinite interval, then $\{1, \varphi, \psi\}$ are linearly dependent on $E$;

(b) $\varphi, \psi \in \text{span}\{1, H, H^2\}$ on $E$;

(c) there exists a non-zero real number $\mu$ such that

$$\varphi, \psi \in \text{span}\{1, e^{\mu H}, e^{-\mu H}\} \quad \text{on} \quad E;$$

(d) there exists a non-zero real number $\mu$ such that

$$\varphi, \psi \in \text{span}\{1, \sin(\mu H), \cos(\mu H)\} \quad \text{on} \quad E.$$

**Remark 1.** It is easy to check that any of the cases (a)–(d) of the theorem indeed provides a solution to (1.3).

**Remark 2.** For $H(x) = x^p$ with $p \in \mathbb{R}\{0\}$ and $E = [0, \infty)$, the function $h(\cdot, \cdot)$ reads as the power mean (or $p$-mean), i.e.

$$h(x, y) = \left(\alpha x^p + \beta y^p\right)^{\frac{1}{p}}, \quad x, y \in [0, \infty),$$

and Theorem 1 solves the corresponding problem mentioned in [2].
We emphasize that the even more general problem, where the derivatives of the unknown functions in (1.3) are replaced by other two unknown functions, is still an unsolved problem as mentioned in the Newsletter of the European Mathematical Society in 2016 (see [9], [11]).

The rest of the note is organized as follows. In Section 2 we prove that the functions $\varphi$ and $\psi$ which solve Problem 1 are either linearly dependent or infinitely differentiable on any subinterval of $E$ in which $\psi'$ does not vanish. In Section 3 we provide a result which will allow us to extend the linearly independence intervals to the whole set. In Section 4 we analyse the asymmetric case ($\alpha \neq \beta$). In Section 5 the symmetric case ($\alpha = \beta = 1/2$) is considered and the proof of Theorem 1 is completed.

2 Infinite differentiability of unknown functions

We start with transforming Problem 1 to a problem with a linear mean as follows. It is given that the function $H$ is strictly monotone and differentiable on $E$. Without loss of generality we may assume that $H$ is strictly increasing. Hence, the inverse of $H$ is also strictly increasing, differentiable, and has a non-vanishing derivative on $J := H(E)$. Substituting $x \mapsto H^{-1}(a)$, $y \mapsto H^{-1}(b)$ in (1.3) and denoting $F = \varphi \circ H^{-1}$ and $G = \psi \circ H^{-1}$, we get

$$[F(b) - F(a)]G'(\alpha a + \beta b) = [G(b) - G(a)]F'(\alpha a + \beta b)$$

for all $a, b \in J$. Therefore, Problem 1 reduces to the following problem.

**Problem 2.** Let $J \subseteq \mathbb{R}$ be a nonempty open interval and let $\alpha, \beta \in (0, 1)$ be fixed with $\alpha + \beta = 1$. Find all pairs $(F, G)$ of differentiable functions $F, G : J \to \mathbb{R}$ which satisfy for all $a, b \in J$ that

$$[F(b) - F(a)]G'(\alpha a + \beta b) = [G(b) - G(a)]F'(\alpha a + \beta b). \quad (2.1)$$

In this section we present certain properties of the solutions of the functional-differential equation (2.1) on the interval where the derivative of $G$ does not vanish. For convenience, we assume the following setting throughout this section.

**Setting 1.** Let $J \subseteq \mathbb{R}$ be a nonempty open interval, let $\alpha, \beta \in (0, 1)$ be fixed with $\alpha + \beta = 1$, let $F, G : J \to \mathbb{R}$ be differentiable functions, with derivatives $F' = f$, $G' = g$, satisfy for all $a, b \in J$ that

$$[F(b) - F(a)]g(\alpha a + \beta b) = [G(b) - G(a)]f(\alpha a + \beta b), \quad (2.2)$$

let $I \subseteq J$ be a nonempty interval such that for all $x \in I$ it holds that $g(x) \neq 0$, and let $v : I \to \mathbb{R}$ satisfy for all $x \in I$ that $v(x) = \frac{f(x)}{g(x)}$.

**Lemma 1.** For all $a, b \in I$ it holds that

$$\beta g(a)(v(\alpha a + \beta b) - v(a)) = \alpha g(b)(v(b) - v(\alpha a + \beta b)). \quad (2.3)$$
Proof. Using the transformation $\alpha a + \beta b \mapsto x$, $b - a \mapsto h$ the condition (2.2) reads as

$$[F(x + \alpha h) - F(x - \beta h)]g(x) = [G(x + \alpha h) - G(x - \beta h)]f(x)$$

(2.4)

for all $x \in I$ and $h \in \mathbb{R}$ such that $x + \alpha h, x - \beta h \in I$. Differentiating both sides with respect to $h$ we obtain

$$[\alpha f(x + \alpha h) + \beta f(x - \beta h)]g(x) = [\alpha g(x + \alpha h) + \beta g(x - \beta h)]f(x)$$

(2.5)

for all $x \in I$ and $h \in \mathbb{R}$ such that $x + \alpha h, x - \beta h \in I$. Applying the transformation $h \mapsto b - a$, $x \mapsto \alpha a + \beta b$, we obtain

$$[\alpha f(b) + \beta f(a)]g(\alpha a + \beta b) = [\alpha g(b) + \beta g(a)]f(\alpha a + \beta b)$$

(2.6)

or, equivalently,

$$\beta g(a)f(\alpha a + \beta b) - f(a)g(\alpha a + \beta b) = \alpha [f(b)g(\alpha a + \beta b) - g(b)f(\alpha a + \beta b)]$$

(2.7)

for all $a, b \in I$. By the definition of the function $v$ we obtain that

$$\beta g(a)(v(\alpha a + \beta b) - v(a)) = \alpha g(b)(v(b) - v(\alpha a + \beta b))$$

(2.8)

for all $a, b \in I$. The proof of Lemma 1 is thus completed.

Lemma 2. If $v$ is not a constant function on $I$, then

$$\{ x \in I: \exists c \in \mathbb{R}: v|_{(x, \sup(I))} = c \} = \emptyset.$$

Proof. We prove Lemma 2 by contradiction. For this we assume that

$$S := \{ x \in I: \exists c \in \mathbb{R}: v|_{(x, \sup(I))} = c \} \neq \emptyset,$$

then, there exists $t_0 \in S$ and let’s denote $k := \inf(S)$. Note that $k \in [\inf(I), \sup(I))$. First, assume that $k > \inf(I)$. Then, there exists $b_0 \in I$ and sufficiently small $\varepsilon > 0$ such that

$$\sup(I) > b_0 > \max\{ k + \frac{\alpha}{\beta} \varepsilon, k + \varepsilon \} \quad \text{and} \quad k - \varepsilon \in I.$$  

(2.9)

This and Lemma 1 imply for all $x \in (k - \varepsilon, k + \varepsilon)$ that

$$\beta g(x)(v(\alpha x + \beta b_0) - v(x)) = \alpha g(b_0)(v(b_0) - v(\alpha x + \beta b_0)).$$

(2.10)

From (2.9) we have for all $x \in (k - \varepsilon, k + \varepsilon)$ that

$$\sup(I) > b_0 > \alpha x + \beta b_0 > \alpha (k - \varepsilon) + \beta b_0 > k.$$  

(2.11)

and therefore, $v(b_0) = v(\alpha x + \beta b_0) = v|_{(k, \sup(I))}$. This together with (2.10) and the assumption that $\forall x \in I: g(x) \neq 0$ ensures for all $x \in (k - \varepsilon, k + \varepsilon)$ that $v(x) = v(\alpha x + \beta b_0) = v|_{(k, \sup(I))}$. Hence, we obtain $v|_{(k - \varepsilon, \sup(I))} \equiv \text{const}$ which contradicts to $k = \inf(S)$. Therefore, we get that $k = \inf(I)$. This and the fact that $S$ is connected prove that $S = I$, which, in turn, implies that $v$ is constant on $I$. This contradicts to the assumption that $v$ is not a constant function on $I$. The proof of Lemma 2 is thus completed.
Lemma 3. Let $A: I \times I \to \mathbb{R}$ be a function such that $A(a,b) = v(\alpha a + \beta b) - v(a)$ for all $(a,b) \in I \times I$ and assume that $v$ is not a constant function on $I$. Then for every $t \in I$,

(i) there exists $b_0 \in (t, \sup(I))$ such that $A(t,b_0) \neq 0$;

(ii) there exist $b_0 \in (t, \sup(I))$ and $\varepsilon > 0$ such that $t - \varepsilon, t + \varepsilon \in I,$ $b_0 \in (t + \varepsilon, \sup(I))$ and $\forall x \in (t - \varepsilon, t + \varepsilon): A(x,b_0) \neq 0$.

\textbf{Proof.} The claim in (ii) follows from the continuity of the function $A$ and the claim in (i). Therefore, it is enough to prove the claim in (i). We prove it by contradiction and for this we assume that there exists $t_0 \in I$ such that for all $b \in (t_0, \sup(I))$ it holds that

$$A(t_0,b) = v(\alpha t_0 + \beta b) - v(t_0) = 0. \quad (2.12)$$

Observe that Lemma 1 implies for all $b \in (t_0, \sup(I))$ that

$$\beta g(t_0)(v(\alpha t_0 + \beta b) - v(t_0)) = \alpha g(b)(v(b) - v(\alpha t_0 + \beta b)). \quad (2.13)$$

Since $t_0, b \in I$ and $g|_I \neq 0$, using (2.12) and (2.13) we obtain for all $b \in (t_0, \sup(I))$

$$(2.14)$$

This implies that $v(t_0) = v(b)$ for all $b \in (t_0, \sup(I))$ and thus $v|_{(t_0, \sup(I))} \equiv \text{const}$, contradicting to Lemma 2 since $v$ is not a constant function on $I$. This concludes the proof of Lemma 3. \qed

\textbf{Proposition 1.} On the interval $I$, either $\{1, F, G\}$ are linearly dependent or both $F$ and $G$ are infinitely differentiable.

\textbf{Proof.} If $\{1, F, G\}$ are linearly dependent, then (2.2) holds. Assume that $\{1, F, G\}$ are linearly independent on $I$. This implies that $v$ is not a constant function on $I$. Next note that (2.2) together with the transformation $\alpha a + \beta b \mapsto x,$ $b - a \mapsto h$ implies that

$$v(x) = \frac{f(x)}{g(x)} = \frac{F(x + \alpha h) - F(x - \beta h)}{G(x + \alpha h) - G(x - \beta h)} \quad (2.15)$$

for all $x \in I$ and $h \in \mathbb{R}$ such that $x + \alpha h, x - \beta h \in I$. The assumption that $g$ does not vanish on $I$ proves that $G$ is injective on $I$ and hence, it can be seen from (2.15) that the function $v$ is differentiable on $I$. Next, note that Lemma 3 implies that for every $t \in I$ there exists $b_0 \in (t, \sup(I))$ and $\varepsilon > 0$ such that $t - \varepsilon, t + \varepsilon \in I,$ $b_0 \in (t + \varepsilon, \sup(I))$ and $v(\alpha x + \beta b_0) - v(x) \neq 0$ for all $x \in (t - \varepsilon, t + \varepsilon).$ This and (2.3) (with $a = x$ and $b = b_0$) show that for all $x \in (t - \varepsilon, t + \varepsilon)$ it holds that

$$g(x) = \frac{\alpha}{\beta} \cdot \frac{g(b_0)(v(b_0) - v(\alpha x + \beta b_0))}{v(\alpha x + \beta b_0) - v(x)}. \quad (2.16)$$

Since $v$ is differentiable, (2.16) ensures that $g$ is differentiable at $t \in I.$ As $t$ was chosen arbitrarily in $I$ we obtain that $g|_I$ is differentiable. Hence, $f|_I = v \cdot g|_I$ is also differentiable. So we get both $F|_I$ and $G|_I$ are twice differentiable. This together with (2.15) show that $v$ is twice differentiable and, from (2.16), so is
Assume that \( \sup(\alpha, \beta) \neq 1 \) such that \( G \) is empty, i.e. \( G \) is constant on \( J \), then (3.1) holds for trivial reasons (both sides are identically zero) for any differentiable function \( F \). Similarly, if \( F \) is constant then (3.1) holds for any differentiable function \( G \). Assume therefore that \( U_g \neq \emptyset \). Then there is a sequence of mutually disjoint open intervals \( \{ I_\sigma \}_{\sigma \in \Sigma} \), \( \Sigma \subset \mathbb{N} \), such that

\[
U_g = \bigcup_{\sigma \in \Sigma} I_\sigma.
\]  

**Proposition 2.** If \( U_g \neq \emptyset \) but \( U_f \cap U_g = \emptyset \) and \( J \) is semi-infinite interval, then \( U_f = \emptyset \), i.e. \( f \equiv 0 \) on \( J \) and thus \( F \) is constant.

**Proof.** Assume that \( \sup(J) = +\infty \). Since \( U_g \neq \emptyset \), there is a non-empty interval \( (p, q) \subset U_g \) such that \( g(x) \neq 0 \) on \( (p, q) \) with \( p > \inf(J) \) and \( q < \sup(J) \) (otherwise we can choose \( (p + \varepsilon, q - \varepsilon) \) for some \( 0 < \varepsilon < \frac{q - p}{2} \)). Hence, \( f(x) = 0 \) for all \( x \in [p, q] \). Then with the change of variables \( (b - a) \mapsto h, (\alpha a + \beta b) \mapsto x \), (2.2) yields

\[
F(x + \alpha h) - F(x - \beta h) = 0,
\]  

for all \( x \in [p, q], \, h \in \mathbb{R} \) such that \( x + \alpha h, x - \beta h \in J \). We fix \( x = q \) and choose \( h \) arbitrarily in \( [0, \frac{q - p}{\alpha}] \) so that \( x - \beta h = q - \beta h \in [p, q] \) and \( x + \alpha h = q + \alpha h \in [q, q + \frac{\beta}{\alpha}(q - p)] \). Therefore, it follows from (3.4) that

\[
F|_{[p, q + \frac{\beta}{\alpha}(q - p)]} = F|_{[q, q + \frac{\beta}{\alpha}(q - p)]} = F|_{[p, q]} \equiv \text{const.}
\]  

Similarly, if we fix \( x = p \) and choose \( h \) arbitrarily in \( [0, (\frac{q}{\alpha} + \frac{1}{\beta})(q - p)] \), then \( x + \alpha h = p + \alpha h \in [p, q + \frac{\alpha}{\beta}(q - p)] \) and

\[
x - \beta h = p - \beta h \in \left( \max \left\{ \inf(J) ; p - (q - p) \left( \frac{\beta}{\alpha} + 1 \right) \right\} , p \right].
\]
This together with the last equality and (3.4) imply that
\[ F\left|_{\max\{\inf(J),p-(q-p)\left(\frac{\alpha}{\alpha+1}\right),q+\frac{\alpha}{\alpha+1}(q-p)\}} \right. = F|_{[p,q]} \equiv \text{const}. \]
It can be seen that, every time the constancy interval increases from below at least by the constant \((\frac{\alpha}{\alpha+1})(q-p)\), until we reach \(\inf(J)\), and therefore repeating this technique, we eventually obtain
\[ F|_{\inf(J),k_1]} = F|_{[p,q]} \equiv \text{const} \]
for some \(k_1 > q\). Next let us consider the set
\[ S := \{x \in (q, +\infty) : F|_{\inf(J),x]} = F|_{[p,q]} \equiv \text{const}\}. \]
Since \(k_1 \in S\), it is non-empty. Assume \(k = \sup(S) < \sup(J) = +\infty\). Then \(F|_{\inf(J),k]} = F|_{[p,q]}\) and \(f \neq 0\) in \((k, r)\) for some \(r\). By the assumption \(U_f \cap U_g = \emptyset\), we get \(g(x) = 0\) for all \(x \in [k, r]\). Using the above argument for \(g\), we obtain that \(G|_{\inf(J),l]} \equiv \text{const}\) for some \(l \geq r > k \geq q\). This, in turn, contradicts to the fact that \(g \neq 0\) on \((p, q) \subset (\inf(J), l]\). Hence, it follows that \(k = \sup(S) = +\infty\) and thus \(f \equiv 0\) on \(J\). The case \(\inf(J) = -\infty\) can be analysed analogously. \(\square\)

**Example 1.** Let \(J = (0, 1)\) and consider the functions
\[
F(x) = \begin{cases} 
  c_1, & x \in (0, \frac{1}{2}], \\
  (x - \frac{1}{2})^2 + c_1, & x \in (\frac{1}{2}, 1),
\end{cases}
\]
\[
G(x) = \begin{cases} 
  (x - \frac{3}{5})^2 + c_2, & x \in (0, \frac{3}{5}), \\
  c_2, & x \in [\frac{3}{5}, 1).
\end{cases}
\]
It is easy to check that such \(J, F, G\) satisfy the equation (3.1) for \(\alpha = \beta = \frac{1}{2}\). Moreover, \(U_g = (0, \frac{2}{5}) \neq \emptyset\) and \(U_g \cap U_f = \emptyset\) but \(U_f = (\frac{2}{5}, 1) \neq \emptyset\). Hence, the example shows that the statement of Proposition 2 does not hold if \(J\) is a finite interval.

If \(J\) is a semi-infinite interval, then Proposition 2 implies that \(U_f \cap U_g = \emptyset\) only when \(U_f = \emptyset\) or \(U_g = \emptyset\). From this fact we conclude the result below.

**Proposition 3.** Assume that
\[ U_f \cap U_g \neq \emptyset \]
and consider the representation (3.3). If \(\inf(J) = -\infty\) or \(\sup(J) = +\infty\) and \(\{F, G, 1\}\) are linearly dependent as functions on \(I_\sigma\) for every \(\sigma \in \Sigma\), then \(\{F, G, 1\}\) are linearly dependent on \(J\).

**Proof.** Assume that \(\sup(J) = +\infty\). For \(\sigma_1, \sigma_2 \in \Sigma\) with \(\sigma_1 \neq \sigma_2\), consider the intervals \(I_{\sigma_1} := (p_1, q_1)\), \(I_{\sigma_2} := (p_2, q_2)\) with
\[ p_1 < q_1 \leq p_2 < q_2, \]
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and assume that \( \{F, G, 1\} \) are linearly dependent on \( I_{\sigma_1} \) and \( I_{\sigma_2} \). Then it follows that there are constants \( c_1, c_2 \in \mathbb{R} \) such that

\[
f(x) = \begin{cases} 
c_1 g(x): & x \in I_{\sigma_1} \\
c_2 g(x): & x \in I_{\sigma_2} 
\end{cases}. \tag{3.6}
\]

With the change of the variables \((b - a) \mapsto h, (\alpha a + \beta b) \mapsto x\), (3.1) yields that

\[
[F(x + \alpha h) - F(x - \beta h)] g(x) = [G(x + \alpha h) - G(x - \beta h)] f(x)
\]

for all \( x, h \in \mathbb{R} \) such that \( x + \alpha h, x - \beta h \in J \). Using the fact that \( g \) does not vanish on \( I_{\sigma_2} \) and (3.6) we obtain for all \( x \in I_{\sigma_2}, h > 0 \) with \( x + \alpha h, x - \beta h \in J \) that

\[
F(x + \alpha h) - F(x - \beta h) = c_2 [G(x + \alpha h) - G(x - \beta h)]. \tag{3.7}
\]

Next, let us fix \( x \in I_{\sigma_2} \). Then there exists \( h > 0 \) such that \( x - \beta h \in I_{\sigma_1} \), and that \( x + \alpha h \in J \). Denoting \( x + \alpha h \) by \( y \) and differentiating (3.7) with respect to \( h \) we get

\[
f(y - h) = c_2 g(y - h). \tag{3.8}
\]

Hence, from (3.6) and (3.8) we have

\[
0 = (c_1 - c_2) g(y - h).
\]

But \( y - h \in I_{\sigma_1} \), so \( g(y - h) \neq 0 \) and thus

\[
c_2 - c_1 = 0. \tag{3.9}
\]

Since \( \sigma_1, \sigma_2 \in \Sigma \) were arbitrary, (3.9) together with (3.6) imply

\[
f(x) = cg(x) \quad \text{for some constant} \quad c \in \mathbb{R} \quad \text{and all} \quad x \in U_g. \tag{3.10}
\]

On the other hand, by changing the roles of \( F \) and \( G \) in the above analysis, we come to the conclusion that

\[
g(x) = kf(x) \quad \text{for some constant} \quad k \in \mathbb{R} \quad \text{and all} \quad x \in U_f. \tag{3.11}
\]

By (3.5) there is an \( x_0 \in U_g \cap U_f \) and hence, \( ck = 1 \). Therefore, \( c \neq 0 \) and \( k \neq 0 \). But then (3.10) implies \( U_g \subseteq U_f \) and (3.11) implies \( U_f \subseteq U_g \). Therefore, \( U_g = U_f \) and \( Z_g = Z_f \). The latter means that for \( x \in Z_g = Z_f \) we have that \( f(x) = 0 = cg(x) \) and \( g(x) = 0 = kf(x) \). Hence, with (3.10) and (3.11) these identities are valid on the entire \( J = U_f \cup Z_f = U_g \cup Z_g \). In particular, it follows that \( \{F, G, 1\} \) are linearly dependent on \( J \). The case \( \inf(J) = -\infty \) can be analysed analogously. \( \square \)
4 Main result for the asymmetric case

In this section we consider the asymmetric case, i.e. we assume in (3.1) that

\[ \alpha, \beta \in (0, 1) \quad \text{with} \quad \alpha \neq 1/2 \quad \text{and} \quad \beta = 1 - \alpha. \quad (4.1) \]

The following proposition describes all pairs \((F, G)\) of differentiable functions satisfying (3.1) in the intervals where \(g = G'\) does not vanish under the conditions (4.1).

**Proposition 4.** Let \((F, G)\) be a solution of Problem 2 with \(\alpha, \beta\) satisfying (4.1) and let \(I = (p, q) \subseteq J\) be an interval where the derivative \(G'\) does not vanish. Then \(\{F, G, 1\}\) are linearly dependent on \(I\).

Proposition 1 implies that if \(\{F, G, 1\}\) are linearly independent on \(I\), then \(F\) and \(G\) are infinitely differentiable on \(I\). Combining this with [2, Proposition 6] we conclude the proof of Proposition 4. We note that, although [2, Proposition 6] is formulated only to the case \(J = \mathbb{R}\), the result can easily be generalized to an arbitrary open interval \(J \subseteq \mathbb{R}\) just by substituting \(\mathbb{R}\) to \(J\) along the lines in the proof of [2, Proposition 6].

The following theorem is the main result of this section.

**Theorem 2.** Let \((F, G)\) be a solution of Problem 2 with \(\alpha, \beta\) satisfying (4.1). If \(\inf(J) = -\infty\) or \(\sup(J) = +\infty\), then \(\{F, G, 1\}\) are linearly dependent on \(J\), i.e. there exist constants \(c_1, c_2, c_3 \in \mathbb{R}\) such that not all of them are zero and

\[ c_1 F(x) + c_2 G(x) + c_3 = 0 \quad \text{for all} \quad x \in J. \quad (4.2) \]

*Proof.* We consider three cases separately.

Case 1: \(U_g = \emptyset\). In this case \(G\) is a constant on \(J\) and (3.1) holds for any differentiable function \(F\). Hence (4.2) holds, for example, with \(c_1 = 0, c_2 = 1, c_3 = -G\) and thus \(\{F, G, 1\}\) are linearly dependent on \(J\).

Case 2: \(U_g \neq \emptyset\) but \(U_g \cap U_f = \emptyset\). In this case Proposition 2 yields that \(F\) is a constant on \(J\) and (3.1) holds for any differentiable function \(G\). Hence (4.2) holds, for example, with \(c_1 = 1, c_2 = 0, c_3 = -F\) and thus \(\{F, G, 1\}\) are again linearly dependent on \(J\).

Case 3: \(U_g \cap U_f \neq \emptyset\). In this case Propositions 4 and 3 directly imply that \(\{F, G, 1\}\) are linearly dependent on \(J\).

**Remark 3.** If \(J\) is a bounded interval, then \(\{1, F, G\}\) do not need to be linearly dependent on the whole of \(J\). We refer the reader to the Example 1 above and also to [8, Examples 13 and 14].

5 Main result for the symmetric case and the proof of Theorem 1

In this section we consider the problem of describing all pairs \((F, G)\) of differentiable functions for which the mean value in (3.1) is taken at the midpoint of the
implies that linearly independent pairs of functions \((5.4)\) or \((5.2)\)–\((5.4)\) are real constants. Altogether, we come to the following conclusion.

Remark 4. On every interval \(I \subseteq J\) on which \(G' \neq 0\), either \(\{F, G, 1\}\) are linearly dependent, or \(G\) and thus also \(F\) has one of the forms described in \((5.2)\)–\((5.4)\).

In the sequel, we call a function \(G\) (respectively the pair \((F, G)\)) to be of quadratic, exponential or trigonometric type on \(I\) if \(G\) (respectively both of \(F\) and \(G\) have) the form \((5.2)\), \((5.3)\) or \((5.4)\), respectively.

Lemma 4. If \((F, G)\) are of quadratic type on some \(I_{\sigma} \subseteq U_{\gamma}\), then they are of quadratic type on the whole of \(J\). This statement holds also for functions of exponential and trigonometric types.

Proof. Note from Proposition 1 that linearly independent pairs of functions \((F, G)\) can be of quadratic, exponential or trigonometric type on some \(I_{\sigma} \subseteq U_{\gamma}\) only when \(\alpha = \beta = 1/2\). Let \(I = (p, q) \subseteq U_{\gamma}\) be an interval where \((F, G)\) are of quadratic type. Substituting \(\frac{p + h}{2} \mapsto x, \frac{p - h}{2} \mapsto h\), \((3.1)\) reads as

\[
[F(x + h) - F(x - h)]g(x) = [G(x + h) - G(x - h)]f(x), \tag{5.5}
\]
for all \(x \in (p,q), h \in \mathbb{R}\) such that \(x-h, x+h \in J\). Without loss of generality we may assume that \(G(x) = c_1x^2 + c_2x + c_3\) with \(c_1 \neq 0\) for \(x \in I\). Then we set from (5.1) that \(F(x) = c_4G(x) + c_5x + c_6\) for \(x \in I\). Putting these to (5.5), we obtain
\[
F(x+h) - F(x-h) = [G(x+h) - G(x-h)](c_4 + \frac{c_5}{g(x)})
\]
for all \(x \in (p,q)\) and \(h \in \mathbb{R}\) such that \(x-h, x+h \in J\). Differentiating both sides with respect to \(h\) and \(x\) simultaneously gives
\[
f(x+h) + f(x-h) = (g(x+h) + g(x-h))(c_4 + \frac{c_5}{g(x)})
\]
and
\[
f(x+h) - f(x-h) = (g(x+h) - g(x-h))(c_4 + \frac{c_5}{g(x)}) - (G(x+h) - G(x-h))g'(x)\frac{c_5}{g(x)^2}
\]
for all \(x \in (p,q)\) and \(h \in \mathbb{R}\) such that \(x+h, x-h \in J\). Using the last two equations we get
\[
2f(x-h) = 2g(x-h)(c_4 + \frac{c_5}{g(x)}) + (G(x+h) - G(x-h))g'(x)\frac{c_5}{g(x)^2}
\]
for all \(x \in (p,q)\) and \(h \in \mathbb{R}\) such that \(x-h, x+h \in J\). We fix \(0 < \varepsilon < \frac{q-p}{2}\) and choose \(h = \varepsilon\). Then \(x-h \in (p,q)\) and \(x+h \in (q,q+\varepsilon)\) for all \(x \in (q-\varepsilon, q) \subset (p,q)\).

Hence, from the last equation we obtain \(G(x+h) = c_1(x+h)^2 + c_2(x+h) + c_3\) for all \(x \in (q-\varepsilon, q) \subset (p,q)\). This together with (5.1) imply that \((F,G)\) are of quadratic type on \((p,q+\varepsilon)\). The critical points of \(F\) and \(G\) cannot disturb us because every function of quadratic type is continuous and has at most one critical point. Hence, using the above argument, we can extend the interval \((p,q)\) to the whole of \(J\).

All functions of exponential type has also at most one critical point, thus the above argument apply for them as well. Every function of trigonometric type is continuous and has a finite number of critical points in every finite subinterval \((p,q)\) of \(J\). Hence, using the above argument, we again complete the proof.

Now we are ready to give the proof of Theorem 1.

**Proof of Theorem 1.** First, we describe the solutions of Problem 2. Consider the set \(U_g\) defined in (3.2). If \(U_g = \emptyset\), then \(q \equiv 0\) on \(J\), and thus \(G\) is constant on \(J\). In this case \(F\) can be an arbitrary differentiable function on \(J\) and thus \(\{F,G,1\}\) are linearly dependent on \(J\). Next, let us assume that \(U_g \neq \emptyset\) and consider the representation (3.3). If \(\alpha \neq 1/2\), then Proposition 4 and Theorem 2 imply that \(\{F,G,1\}\) are linearly dependent on every interval \(I \subset U_g\) or if \(J\) is semi-infinite interval, they are linearly dependent on the whole of \(J\). So, we can focus on the symmetric case of \(\alpha = \beta = 1/2\). It is clear that if \(\{F,G,1\}\) are linearly independent on \(I_\sigma\) for some \(\sigma \in \Sigma\), then Remark 4 and Lemma 4 imply that \(F,G\) have only one of the quadratic, exponential or trigonometric types on whole of \(J\). Otherwise, we are left only with the case \(\{F,G,1\}\) being linearly dependent on every interval \(I \subset U_g\) which was already considered above. The solutions of the Problem 2 directly yield the proof of the Theorem 1.
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