Heisenberg Uncertainty Principle for the $q$-Bessel Fourier transform

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Abstract

In this paper we uses an I.I. Hirschman-W. Beckner entropy argument to give an uncertainty inequality for the $q$-Bessel Fourier transform:

$$F_{q,v} f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2 t^{2v+1}) t^2 v^{2v+1} dt,$$

where $j_v(x, q)$ is the normalized Hahn-Exton $q$-Bessel function.

1 Introduction

I.I.Hirschman-W. Beckner entropy argument is one further variant of Heisenberg’s uncertainty principle.

Let $\hat{f}$ be the Fourier transform of $f$ defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{2i\pi xy} dy, \quad x \in \mathbb{R}.$$

If $f \in L^2(\mathbb{R})$ with $L^2$-norm $\|f\|_2 = 1$, then by Plancherel’s theorem $\|\hat{f}\|_2 = 1$, so that $|f(x)|^2$ and $|\hat{f}(x)|^2$ are probability frequency functions. The variance of a probability frequency $g$ is defined by

$$V[g] = \int_{\mathbb{R}} x^2 g(x) dx.$$

The Heisenberg uncertainty principle can be stated as follows

$$V[|f|^2] V[|\hat{f}|^2] \geq \frac{1}{16\pi^2}. \quad (1)$$

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If $g$ is a probability frequency function, then the entropy of $g$ is defined by

$$E(g) = \int_{\mathbb{R}} g(x) \log(x) dx.$$ 

With $f$ as above, Hirschman [10] proved that

$$E(|f|^2) + E(|\hat{f}|^2) \leq 0. \quad (2)$$

By an inequality of Shannon and Weaver it follows that (2) implies (1).

Using the Babenko-Beckner inequality

$$\|\hat{f}\|_{p'} \leq A(p)\|f\|_p, \quad 1 < p < 2, \quad A(p) = \left[p^{1/p}(p')^{-1/p'}\right]^{1/2},$$

in Hirschman’s proof of (2) another uncertainty inequality is deduced. For more detail the reader can consult [8,10,11].

In this paper we use I.I. Hirschman entropy argument de give an uncertainty inequality for the $q$-Bessel Fourier transform (also called $q$-Hankel transform).

Note that other versions of the Heisenberg uncertainty principle for the $q$-Fourier transform have recently appeared in the literature [1,2,6]. There are some differences of the results cited above and our result:

- In [1] the uncertainty inequality is established for the $q$-cosine and $q$-sine transform but here is established for the $q$-Bessel transform.
- In [2] the uncertainty inequality is for the $q^2$-Fourier transform but here is for the $q$-Hankel transform.
- In [6] the uncertainty inequality is established for functions in $q$-Schwartz space. In this paper the uncertainty inequality is established for functions in $L_{q,2,v}$ space.

The inequality discuss here is a quantitative uncertainty principles which give an information about how a function and its $q$-Bessel Fourier transform relate. A qualitative uncertainty principles give an information about how a function (and its Fourier transform) behave under certain circumstances. A classical qualitative uncertainty principle called Hardy’s theorem. In [4,7] a $q$-version of the Hardy’s theorem for the $q$-Bessel Fourier transform was established.

In the end, our objective is to develop a coherent harmonic analysis attached to the $q$-Bessel operator

$$\Delta_{q,v} f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - (1 + q^{2v})f(x) + q^{2v}f(qx) \right].$$
Thus, this paper is an opportunity to implement the arguments of the $q$-Bessel Fourier analysis proved before, as the Plancherel formula, the positivity of the $q$-translation operator, the $q$-convolution product, the $q$-Gauss kernel...

2 The $q$-Bessel Fourier transform

In the following we will always assume $0 < q < 1$ and $v > -1$. We denote by

$$R_q = \{ \pm q^n, \ n \in \mathbb{Z} \}, \quad R_q^+ = \{ q^n, \ n \in \mathbb{Z} \}.$$ 

For more informations on the $q$-series theory the reader can see the references [9,12,14] and the references [3,5,13] about the $q$-bessel Fourier analysis. Also for details of the proofs of the following results in this section can be fond in [3].

**Definition 1** The $q$-Bessel operator is defined as follows

$$\Delta_{q,v} f(x) = \frac{1}{x^2} \left[ f(q^{-1}x) - (1 + q^{2v})f(x) + q^{2v}f(qx) \right].$$

**Definition 2** The normalized $q$-Bessel function of Hahn-Exton is defined by

$$j_v(x,q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2v+2},q^2)_n(q^2,q^2)_n} x^{2n}. $$

**Proposition 1** The function

$$x \mapsto j_v(\lambda x,q^2)$$

is the eigenfunction of the operator $\Delta_{q,v}$ associated with the eigenvalue $-\lambda^2$.

**Definition 3** The $q$-Jackson integral of a function $f$ defined on $\mathbb{R}_q$ is

$$\int_0^\infty f(t)q^t = (1-q) \sum_{n \in \mathbb{Z}} q^n f(q^n).$$

**Definition 4** We denote by $\mathcal{L}_{q,p,v}$ the space of even functions $f$ defined on $\mathbb{R}_q$ such that

$$\|f\|_{q,p,v} = \left[ \int_0^\infty |f(x)|^p x^{2v+1}d_q x \right]^{1/p} < \infty.$$
Definition 5 We denote by $C_{q,0}$ the space of even functions defined on $\mathbb{R}_q$ tending to 0 as $x \to \pm \infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $C_{q,0}$ is complete with respect to the norm
\[\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q} |f(x)|.\]

Definition 6 The q-Bessel Fourier transform $\mathcal{F}_{q,v}$ (also called q-Hankel transform) is defined by
\[\mathcal{F}_{q,v}f(x) = c_{q,v} \int_0^\infty f(t) j_v(xt, q^2) t^{2v+1} d_q t, \quad \forall x \in \mathbb{R}_q.\]
where
\[c_{q,v} = \frac{1}{1-q} \frac{(q^{2v+2}; q^2)_\infty}{(q^2; q^2)_\infty}.\]

Proposition 2 Let $f \in \mathcal{L}_{q,1,v}$ then $\mathcal{F}_{q,v}f$ existe and $\mathcal{F}_{q,v}f \in C_{q,0}$.

Definition 7 The $q$-translation operator is given as follows
\[T_{q,x}^v f(y) = c_{q,v} \int_0^\infty \mathcal{F}_{q,v}f(t) j_v(yt, q^2) j_v(xt, q^2) t^{2v+1} d_q t, \quad \forall f \in \mathcal{L}_{q,1,v}.\]

Definition 8 The operator $T_{q,x}^v$ is said positive if $T_{q,x}^v f \geq 0$ when $f \geq 0$ for all $x \in \mathbb{R}_q$. We denote by $Q_v$ the domain of positivity of $T_{q,x}^v$
\[Q_v = \{ q \in ]0, 1[, \quad T_{q,x}^v \text{ is positive} \}.\]

In the following we assume that $q \in Q_v$.

Proposition 3 If $f \in \mathcal{L}_{q,1,v}$ then
\[\int_0^\infty T_{q,x}^v f(y) y^{2v+1} d_q y = \int_0^\infty f(y) y^{2v+1} d_q y.\]

Definition 9 The $q$-convolution product is defined as follows
\[f *_q g(x) = c_{q,v} \int_0^\infty T_{q,x}^v f(y) g(y) y^{2v+1} d_q y.\]

Proposition 4 Let $f, g \in \mathcal{L}_{q,1,v}$ then $f *_q g \in \mathcal{L}_{q,1,v}$ and we have
\[\mathcal{F}_{q,v}(f *_q g) = \mathcal{F}_{q,v}(g) \times \mathcal{F}_{q,v}(f).\]
Proposition 5 Let \( f \in L_{q,1,v} \) and \( g \in L_{q,2,v} \) then \( f \ast_q g \in L_{q,2,v} \) and we have
\[
\mathcal{F}_{q,v}(f \ast_q g) = \mathcal{F}_{q,v}(f) \times \mathcal{F}_{q,v}(g).
\]

Theorem 1 The \( q \)-Bessel Fourier transform \( \mathcal{F}_{q,v} \) satisfies
1. \( \mathcal{F}_{q,v} \) sends \( L_{q,2,v} \) to \( L_{q,2,v} \).
2. For \( f \in L_{q,2,v} \), we have
\[
\|\mathcal{F}_{q,v}(f)\|_{q,2,v} = \|f\|_{q,2,v}.
\]
3. The operator \( \mathcal{F}_{q,v} : L_{q,2,v} \to L_{q,2,v} \) is bijective and
\[
\mathcal{F}_{q,v}^{-1} = \mathcal{F}_{q,v}.
\]

Proposition 6 Given \( 1 < p \leq 2 \) and \( \frac{1}{p} + \frac{1}{\bar{p}} = 1 \). If \( f \in L_{q,p,v} \) then
\[
\mathcal{F}_{q,v}(f) \in L_{\bar{p},2,v}
\]
and
\[
\|\mathcal{F}_{q,v}(f)\|_{\bar{p},2,v} \leq B_{q,v}^{(2-1)} \|f\|_{q,p,v},
\]
where
\[
B_{q,v} = \frac{1}{1-q} \frac{(-q^{2};q^{2})_{\infty}(-q^{2v+2};q^{2})_{\infty}}{(q^{2};q^{2})_{\infty}}.
\]

Definition 10 The \( q \)-exponential function is defined by
\[
e(z,q) = \sum_{n=0}^{\infty} \frac{z^n}{(q,q)_n} = \frac{1}{(z;q)_{\infty}}, \quad |z| < 1.
\]

Proposition 7 The \( q \)-Gauss kernel
\[
G_v(x,t^2,q^2) = \frac{(-q^{2v+2}t^2,-q^{-2v}/t^2;q^2)_{\infty}}{(-t^2,-q^2/t^2;q^2)_{\infty}}e\left(-\frac{q^{-2v}}{t^2}x^2,q^2\right), \quad \forall x,t \in \mathbb{R}^+_q
\]
satisfies
\[
\mathcal{F}_{q,v}\{e(-t^2y^2,q^2)\}(x) = G_v(x,t^2,q^2),
\]
and for all function \( f \in L_{q,2,v} \)
\[
\lim_{n \to \infty} \|G_v(x,q^{2n},q^2) \ast_q f - f\|_{q,2,v} = 0.
\]
3 Uncertainty Principle

The following Lemma are crucial for the proof of our main result. First we enunciate the Jensens inequality

**Lemma 1** Let $\gamma$ be a probability measure on $\mathbb{R}_+$. Let $g$ be a convex function on a subset $I$ of $\mathbb{R}$. If $\psi : \mathbb{R}_+ \rightarrow I$ satisfies
\[
\int_0^\infty \psi(u)d\gamma(u) \in I,
\]
then we have
\[
g \left( \int_0^\infty \psi(x)d\gamma(x) \right) \leq \int_0^\infty g \circ \psi(x)d\gamma(x).
\]

**Proof.** Let
\[t = \int_0^\infty \psi(u)d\gamma(u).
\]
There exist $c \in \mathbb{R}$ such that for all $y \in I$ it holds
\[g(y) \geq g(t) + c(y - t).
\]
Now let $y = \psi(x)$ we obtain
\[g(\psi(x)) \geq g(t) + c(\psi(x) - t).
\]
Integrating both sides and using the special value of $t$ gives
\[
\int_0^\infty g(\psi(x))d\gamma(x) \geq \int_0^\infty [g(t) + c(\psi(x) - t)]d\gamma(x) = g(t).
\]
This finish the proof. □

**Lemma 2** Let $f$ be an even function defined on $\mathbb{R}_q$. Assume $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ is a convex function and $\psi \circ f \in L_{q,1,v}$. If $\varrho_n$ is a sequence of non-negative function in $L_{q,1,v}$ such that
\[
\mathcal{F}_{q,v}(\varrho_n)(0) = c_{q,v} \int_0^\infty \varrho_n(x)x^{2v+1}d_qx = 1
\]
and $\varrho_n \ast_q f \rightarrow f$ then $\psi \circ (\varrho_n \ast_q f)$ is in $L_{q,1,v}$ and
\[
\lim_{n \rightarrow \infty} \int_0^\infty \psi \circ (\varrho_n \ast_q f)(x)x^{2v+1}d_qx = \int_0^\infty \psi \circ f(x)x^{2v+1}d_qx.
\]
Proof. For a given $x$ and by Proposition 3 we have

$$c_{q,v} \int_0^\infty T_{q,x}^v \rho_n(y)y^{2v+1}d_qy = 1$$

From the positivity of $T_{q,x}^v$ we see that

$$c_{q,v}T_{q,x}^v \rho_n(y)y^{2v+1}d_qy$$

is a probability measure on $\mathbb{R}^+$. The following holds by Jensens Inequality

$$\psi \circ (\rho_n *_q f)(x) = \psi \left[ c_{q,v} \int_0^\infty f(y)T_{q,x}^v \rho_n(y)y^{2v+1}d_qy \right]$$

$$\leq c_{q,v} \int_0^\infty \psi \circ f(y)T_{q,x}^v \rho_n(y)y^{2v+1}d_qy$$

$$= \rho_n *_q \psi \circ f(x).$$

By the use of the Fatou’s Lemma and Proposition 4 we obtain

$$\int_0^\infty \psi \circ f(x)x^{2v+1}d_qx$$

$$= \int_0^\infty \liminf_{n \to \infty} \psi \circ (\rho_n *_q f)(x)x^{2v+1}d_qx$$

$$\leq \liminf_{n \to \infty} \int_0^\infty \psi \circ (\rho_n *_q f)(x)x^{2v+1}d_qx$$

$$\leq \limsup_{n \to \infty} \int_0^\infty \psi \circ (\rho_n *_q f)(x)x^{2v+1}d_qx$$

$$\leq \lim_{n \to \infty} \int_0^\infty \rho_n *_q \psi \circ f(x)x^{2v+1}d_qx$$

$$= \frac{1}{c_{q,v}} \lim_{n \to \infty} \mathcal{F}_q,v(\rho_n)(0) \times \mathcal{F}_q,v(\psi \circ f)(0)$$

$$= \int_0^\infty \psi \circ f(x)x^{2v+1}d_qx.$$

This finish the proof. ■

Definition 11 For a positive function $\phi$ define the entropy of $\phi$ to be

$$E(\phi) = \int_0^\infty \phi(x) \log \phi(x)x^{2v+1}d_qx.$$

$E(\phi)$ can have values in $[-\infty, \infty]$. 

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Remark 1 For a given $c \in \mathbb{R}_q^+$ let

$$d\gamma(x) = k_c^{-1} \exp\left(-|cx|^a\right) x^{2v+1} d_q x$$

where

$$\sigma_a = \int_0^\infty \exp\left(-|x|^a\right) x^{2v+1} d_q x, \quad k_c = \frac{\sigma_a}{c^{2v+2}}.$$ 

Then $d\gamma(x)$ is a probability measure on $\mathbb{R}_q^+$.

Lemma 3 Let $a > 0$. For a positive function $\phi \in \mathcal{L}_{q,1,v}$ such that

$$\|\phi\|_{q,1,v} = 1$$

and

$$M_a(\phi) = \left(\int_0^\infty |x|^a \phi(x)x^{2v+1} d_q x\right)^\frac{1}{a}$$

is finite, we have

$$-E(\phi) \leq \log k_c + c^a M_a^a(\phi). \quad (3)$$

Proof. Indeed, defining

$$\psi(x) = k_c \exp\left(|cx|^a\right) \phi(x),$$

From Remark 1 we see that

$$\int_0^\infty \psi(x)d\gamma(x) = 1.$$ 

According to the fact that $g : t \mapsto t \log t$ is convex on $\mathbb{R}_q^+$, so Jensen’s inequality gives

$$g \left[\int_0^\infty \psi(x)d\gamma(x)\right] \leq \int_0^\infty g \circ \psi(x)d\gamma(x).$$ 

Hence,

$$0 = \left[\int_0^\infty \psi(x)d\gamma(x)\right] \log \left[\int_0^\infty \psi(x)d\gamma(x)\right] \leq \int_0^\infty \psi(x) \log \psi(x)d\gamma(x).$$

This implies

$$0 \leq \int_0^\infty \phi(x) \log\left[k_c \exp\left(|cx|^a\right) \phi(x)\right] x^{2v+1} d_q x$$

$$= \int_0^\infty \phi(x) [\log k_c + |cx|^a + \log \phi(x)] x^{2v+1} d_q x.$$ 

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\[ 0 \leq \log k_c + c^a \int_0^\infty |x|^a \phi(x)x^{2v+1}d_qx + \int_0^\infty \phi(x) \log \phi(x)x^{2v+1}d_qx. \]

In the end
\[ 0 \leq \log k_c + c^a M_a^a(\phi) + E(\phi). \]

This finish the proof. \( \blacksquare \)

**Lemma 4** Let \( f \in L_{q,1,v} \cap L_{q,2,v} \) then we have
\[
E \left( |f|^2 \right) + E \left( |\mathcal{F}_{q,v} f|^2 \right) \leq 2\|f\|^2_{q,v,2} \log \left( B_{q,v} \|f\|^2_{q,v,2} \right). \tag{4}
\]

**Proof.** Hölder inequality implies that \( f \) will be in \( L_{q,p,v} \) for \( 1 < p \leq 2 \). With
\[
\frac{1}{p} + \frac{1}{p'} = 1,
\]
Hausdorff-Young’s inequality (Proposition 6) tells us that \( \mathcal{F}_{q,v} f \) is in \( L_{q,p',v} \). So we can define the functions
\[
A(p) = \int_0^\infty |f(x)|^p d_qx \quad \text{and} \quad B(\overline{p}) = \int_0^\infty |\mathcal{F}_{q,v} f(x)|^{\overline{p}} x^{2v+1}d_qx.
\]
Now define
\[
C(p) = \log \|\mathcal{F}_{q,v} f\|_{q,p,v} - \log \left( B_{q,v}^{\frac{2}{p-1}} \|f\|_{q,p,v} \right) = \frac{1}{p} \log B(\overline{p}) - \frac{1}{p} \log A(p) - \left( \frac{2}{p} - 1 \right) \log B_{q,v}.
\]
By Hausdorff-Young’s inequality
\[
C(p) \leq 0, \text{ for } 1 < p < 2,
\]
and by Plancherel equality (Theorem 1 part 2)
\[
C(2) = 0.
\]
Then
\[
C'(2^-) \geq 0.
\]
On the other hand for \( 1 < p < 2 \) we have
\[
C'(p) = \frac{\overline{p}}{p} B'(\overline{p}) + \frac{\overline{p}'}{p'} \log B(\overline{p}) - \frac{1}{p} A'(p) + \frac{1}{p'} \log A(p) + \frac{2}{p^2} \log B_{q,v}.
\]
The derivative of $\varphi$ with respect to $p$ is

$$\varphi' = -\frac{1}{(p-1)^2}.$$  

For a given $x > 0$ we have

$$\lim_{p \to 2^\pm} \frac{x^p - x^2}{p - 2} = x^2 \log x.$$  

Then

$$A'(2^-) = \lim_{p \to 2^-} \frac{A(p) - A(2)}{p - 2} = \frac{1}{2} E \left( |f|^2 \right),$$

$$B'(2^+) = \lim_{p \to 2^+} \frac{B(p) - B(2)}{p - 2} = \frac{1}{2} E \left( |F_{q,v}f|^2 \right).$$

Since

$$p \mapsto \frac{x^p - x^2}{p - 2}$$

is an increasing function, the exchange of the signs limit and integral is valid sense. On the other hand

$$\lim_{p \to 2^-} A(p) = \|f\|_{q,v,2}^2, \quad \lim_{p \to 2^+} B(p) = \|F_{q,v}f\|_{q,v,2}^2 = \|f\|_{q,v,2}^2.$$  

So

$$C'(2^-) = \lim_{p \to 2^-} \frac{C(p) - C(2)}{p - 2} = -\frac{1}{2\|f\|_{q,v,2}^2} \left[ A'(2^-) + B'(2^+) \right] + \frac{1}{2} \log \left( B_{q,v}\|f\|_{q,v,2}^2 \right).$$

Therefore

$$A'(2^-) + B'(2^+) - \|f\|_{q,v,2}^2 \log \left( B_{q,v}\|f\|_{q,v,2}^2 \right) \leq 0,$$

and then

$$E \left( |f|^2 \right) + E \left( |F_{q,v}f|^2 \right) \leq 2\|f\|_{q,v,2}^2 \log \left( B_{q,v}\|f\|_{q,v,2}^2 \right).$$

This finish the proof. □

**Lemma 5** Let $f \in L_{q,2,v}$ then we have

$$E \left( |f|^2 \right) + E \left( |F_{q,v}f|^2 \right) \leq 2\|f\|_{q,v,2}^2 \log \left( B_{q,v}\|f\|_{q,v,2}^2 \right). \quad (5)$$
Proof. Assume that $E(|f|^2)$ and $E(|\mathcal{F}_{q,v}f|^2)$ are defined and then approximate $f$ by functions in $L_{q,1,v} \cap L_{q,2,v}$. Let
\[ h_n(x) = e(-q^{2n}x^2, q^2). \]
The function $h_n$ is in $L_{q,2,v}$ then $h_nf \in L_{q,1,v}$. On the other hand $h_n \in C_{q,0}$ then $h_nf \in L_{q,2,v}$. We obtain
\[ h_nf \in L_{q,1,v} \cap L_{q,2,v}. \]

The following holds by (2)
\[ E \left( |h_nf|^2 \right) + E \left( |\mathcal{F}_{q,v}(h_nf)|^2 \right) \leq 2\|h_nf\|_{q,2,v}^2 \log \left( \frac{B_{q,v}^2}{q^{2n}h_nf} \right). \quad (6) \]

One can see by the Lebesgue Dominated Convergence Theorem that
\[ \lim_{n \to \infty} \|h_nf\|_{q,2,v} = \|f\|_{q,2,v} \quad (7) \]
and
\[ \lim_{n \to \infty} E \left( |h_nf|^2 \right) = E \left( |f|^2 \right). \quad (8) \]

By the use of Proposition 5 and the inversion formula (Theorem 1 part 3) we see that
\[ \mathcal{F}_{q,v}(h_nf) = \mathcal{F}_{q,v}h_n \ast_q \mathcal{F}_{q,v}f. \]

We will prove that
\[ \lim_{n \to \infty} E \left( |\mathcal{F}_{q,v}h_n \ast_q \mathcal{F}_{q,v}f|^2 \right) = E \left( |\mathcal{F}_{q,v}f|^2 \right). \]

The functions
\[ \phi_1(x) = x^2 \log^+ |x| \quad \text{and} \quad \phi_2(x) = x^2 \left( -\log^- |x| + \frac{3}{2} \right), \]
are convex on $\mathbb{R}$, where
\[ \log^+ x = \max \{0, \log x\} \quad \text{and} \quad \log^- x = \min \{0, \log x\}. \]

Note that
\[ 2\phi_1(x) - 2\phi_2(x) + 3x^2 = x^2 \log |x|^2. \]

Since
- From the inversion formula we see that
\[ c_{q,v} \int_0^\infty \mathcal{F}_{q,v}h_n(t)t^{2v+1}d_qt = h_n(0) = 1. \]
• The function $F_{q,v,h_n} \geq 0$.
• The functions $\phi_i$ are convex on $\mathbb{R}$.
• $E(F_{q,v,f})$ is finite then $\phi_i(F_{q,v,f})$ is in $L_{q,1,v}$.
• From Proposition 7 we have
  \[
  \lim_{n \to \infty} F_{q,v,h_n} * q F_{q,v,f}(x) = F_{q,v,f}(x)
  \]
we deduce that $F_{q,v,h_n}$ and $\phi_i$ satisfy the conditions of Lemma 2. Then we obtain
  \[
  \lim_{n \to \infty} \int_0^\infty \phi_i(F_{q,v,h_n} * q F_{q,v,f})(x)x^{2v+1}d_q x = \int_0^\infty \phi_i(F_{q,v,f})(x)x^{2v+1}d_q x, \quad i = 1, 2.
  \]
It also hold
  \[
  E \left( |F_{q,v,f}|^2 \right) = 2 \int_0^\infty \phi_1(F_{q,v,f})x^{2v+1}d_q - 2 \int_0^\infty \phi_2(F_{q,v,f})x^{2v+1}d_q x + 3 \|F_{q,v,f}\|^2_{q,2,v},
  \]
and
  \[
  E \left( |F_{q,v,h_n} * q F_{q,v,f}|^2 \right) = 2 \int_0^\infty \phi_1(F_{q,v,h_n} * q F_{q,v,f})x^{2v+1}d_q x
  \]
  \[
  \quad - 2 \int_0^\infty \phi_2(F_{q,v,h_n} * q F_{q,v,f})x^{2v+1}d_q x + 3 \|F_{q,v,h_n} * q F_{q,v,f}\|^2_{q,2,v}.
  \]
Then
  \[
  \lim_{n \to \infty} E \left( |F_{q,v,h_n} * q F_{q,v,f}|^2 \right) = E \left( |F_{q,v,f}|^2 \right).
  \]
With (6) and the limits (7), (8) and (9) we complete the proof of (5).

Note that these limits also hold in the case where $E(|f|^2)$ and $E(|F_{q,v,f}|^2)$ are $\infty$ or $-\infty$. ■

Now we are in position to state and prove the uncertainty inequality for the $q$-Bessel Fourier transform.

**Theorem 2** Given $a, b > 0$. Then for all $c, d \in \mathbb{R}_q^+$ satisfying
  \[
  0 < B_{q,v}^2 \frac{\sigma_a \sigma_b}{(cd)^{2v+2}} < 1,
  \]
the following hold for any function $f \in L_{q,2,v}$
  \[
  c^a \left\| x^{a/2} f \right\|^2_{q,2,v} + d^b \left\| x^{b/2} F_{q,v,f} \right\|^2_{q,2,v} \geq - \log \left( B_{q,v}^2 \frac{\sigma_a \sigma_b}{(cd)^{2v+2}} \right) \| f \|^2_{q,2,v}.
  \]
Proof. Assume that \( \| f \|_{q,v} = 1 \). By (3) we can write

\[
- E(|f|^2) \leq \log k_c + c^a \left\| x^{a/2} f \right\|^2_{q,v} \\
- E \left( |\mathcal{F}_{q,v} f|^2 \right) \leq \log k_d + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|^2_{q,v}.
\]

Which implies with (5)

\[
-2 \log B_{q,v} \leq - E \left( |f|^2 \right) - E \left( |\mathcal{F}_{q,v} f|^2 \right) \leq \log (k_c k_d) + c^a \left\| x^{a/2} f \right\|^2_{q,v} + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|^2_{q,v}.
\]

By replacing \( f \) by \( \frac{f}{\|f\|_{q,v}} \) we get

\[
c^a \left\| x^{a/2} f \right\|^2_{q,v} + d^b \left\| x^{b/2} \mathcal{F}_{q,v} f \right\|^2_{q,v} \geq - \log (B_{q,v}^2 k_c k_d) \|f\|^2_{q,v}.
\]

This finish the proof. \( \blacksquare \)

**Corollary 1** There exist \( k > 0 \) such that for any function \( f \in \mathcal{L}_{q,v} \) we have

\[
\| xf \|_{q,v} \| x \mathcal{F}_{q,v} f \|_{q,v} \geq k \| f \|_{q,v}^2.
\]

**Proof.** Let \( a = b = 2 \) and \( c = d \) then by Theorem 3

\[
\| xf \|_{q,v}^2 + \| x \mathcal{F}_{q,v} f \|_{q,v}^2 \geq - \frac{1}{c^2} \log \left( B_{q,v}^2 \frac{\sigma_2^2}{c^4(v+1)} \right) \| f \|_{q,v}^2,
\]

where

\[
0 < \left( B_{q,v}^2 \frac{\sigma_2^2}{c^4(v+1)} \right) < 1.
\]

Now put \( f_t(x) = f(tx), \ t \in \mathbb{R}_q^+ \),
then

\[
\mathcal{F}_{q,v} f_t(x) = \frac{1}{t^{2v+2}} \mathcal{F}_{q,v} f(x/t), \ \| x \mathcal{F}_{q,v} f_t \|_{q,v}^2 = \frac{1}{t^{2v}} \| \mathcal{F}_{q,v} f \|_{q,v}^2,
\]

and

\[
\| f_t \|_{q,v}^2 = \frac{1}{t^{2v+2}} \| f \|_{q,v}^2, \ \| x f_t \|_{q,v}^2 = \frac{1}{t^{2v+4}} \| x f \|_{q,v,2}^2.
\]
which gives
\[ t^4 \| x \mathcal{F}_{q,v} f \|^2_{q,2,v} + t^2 \frac{1}{c^2} \log \left( B_{q,v}^2 \frac{\sigma^2}{c^4(v+1)} \right) \| f \|^2_{q,v,2} + \| x f \|^2_{q,2,v} \geq 0, \]
and then
\[ \| x f \|_{q,2,v} \| x \mathcal{F}_{q,v} f \|_{q,2,v} \geq \psi(c) \| f \|^2_{q,2,v}, \]
where
\[ \psi(c) = \frac{v + 1}{\lceil \sigma^2 B_{q,v} \rceil^{v+1}} |z_c \log(z_c)|, \quad z_c = \frac{\lceil \sigma^2 B_{q,v} \rceil^{1/2}}{c^2}, \quad 0 < z_c < 1. \]

One can see that
\[ \sup_{0 < z_c < 1} \psi(c) = \psi(q^\alpha), \quad \alpha = \frac{\log(\sigma^2 B_{q,v})}{2(1 + v) \log q} + \frac{1}{2 \log q}. \]

Let
\[ n_1 = \lfloor \alpha \rfloor, \quad n_2 = \lceil \alpha \rceil, \]
where \( \lfloor . \rfloor \) and \( \lceil . \rceil \) are respectively the floor and ceiling functions. Now the constant \( k \) is given as follows
\[ k = \psi(q^{n_1}), \quad \text{if} \quad \lfloor \alpha \rfloor \geq \alpha - \frac{1}{2 \log q} \]
and
\[ k = \max\{ \psi(q^{n_1}), \psi(q^{n_2}) \}, \quad \text{if} \quad \lceil \alpha \rceil < \alpha - \frac{1}{2 \log q}. \]
This finish the proof.

References

[1] N. Bettaibi, A. Fitouhi and W. Binous, Uncertainty principles for the q-trigonometric Fourier transforms, Math. Sci. Res. J. 11 (2007).

[2] N. Bettaibi, Uncertainty principles in \( q^2 \)-analogue Fourier analysis, Math. Sci. Res. J. 11 (2007).

[3] L. Dhaouadi, A. Fitouhi and J. El Kamel, Inequalities in q-Fourier Analysis, Journal of Inequalities in Pure and Applied Mathematics, Volume 7, Issue 5, Article 171, 2006.
[4] L. Dhaouadi, Hardy’s theorem for the q-Bessel Fourier transform, arXiv: 0707.2346 v1 [math.CA].

[5] A. Fitouhi, M. Hamza and F. Bouzeffour, The $q - j_\alpha$ Bessel function J. Appr. Theory. 115, 144-166 (2002).

[6] A. Fitouhi, N. Bettaibi, W. Binous and H.B. Elmonser, Uncertainty principles for the basic Bessel transform, Ramanujan J., in press.

[7] A. Fitouhi, N. Bettaibi and R. Bettaieb, On Hardy’s inequality for symmetric integral transforms and analogous, Appl. Math. Comput. 198 (2008).

[8] G.B. Folland and A. Sitaram. The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl., 3(3):207–238, 1997.

[9] G. Gasper and M. Rahman, Basic hypergeometric series, Encyclopedia of mathematics and its applications 35, Cambridge university press, 1990.

[10] I.I. Hirschman, Jr. A note on entropy. Amer. J. Math., 79:152156, 1957.

[11] H.P. Heinig, and M. Smith, Extensions of the Heisenberg-Weyl inequality. Internat. J. Math. Math. Sci. 9 (1986), no. 1, 185–192.

[12] F. H. Jackson, On a q-Definite Integrals, Quarterly Journal of Pure and Application Mathematics 41, 1910, 193-203.

[13] T. H. Koornwinder and R. F. Swarttouw, On q-Analogues of the Hankel and Fourier Transform, Trans. A. M. S. 1992, 333, 445-461.

[14] R. F. Swarttouw, The Hahn-Exton q-Bessel functions PhD Thesis The Technical University of Delft (1992).