MODELLING GRAVITATIONAL WAVES FROM INSPIRALLING COMPACT BINARIES

T. Damour\(^1\), B.R. Iyer\(^2\) B.S. Sathyaprakash\(^3\)

\(^1\) Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France
\(^2\) Raman Research Institute, Bangalore 560 080, India
\(^3\) Cardiff University of Wales, P.O. Box 913, Cardiff, CF2 3YB, U.K.

Abstract

Gravitational waves from inspiralling compact binaries can be reliably extracted from a noisy detector output only if the template used in the detection is a faithful representation of the true signal. In this article we suggest a new approach to constructing faithful signal models.

1 Introduction

In searching for gravitational waves from an inspiralling compact binary (ICB) we are faced with the following data analysis problem: On the one hand, we have some exact gravitational wave form \(h^X(t; \lambda_k)\) where \(\lambda_k\), \(k = 1, \ldots, n_\lambda\), are the parameters of the signal (eg., the masses \(m_1\) and \(m_2\) of the members of the emitting binary). On the other hand, we have theoretical calculations of the motion of, and gravitational radiation from, binary systems of compact bodies (neutron stars or black holes) giving the post-Newtonian (PN) expansions (expansions in powers of \(v/c\)) of an energy function \(E(v)\) and a gravitational wave luminosity function \(F(v)\). Here, the dimensionless argument \(v\) is an invariantly defined “velocity” related to the instantaneous gravitational wave frequency \(f_{GW}\) (= twice the orbital frequency) by \(v = (\pi m f_{GW}) \times\), where \(m \equiv m_1 + m_2\) is the total mass of the binary. Given the energy and flux functions one needs to compute the “phasing formula”, i.e. an accurate mathematical model for the evolution of the gravitational wave phase \(\Phi = \Phi_{GW} \times\), carrying information about the emitting binary system. The standard energy-balance equation \(dE_{\text{tot}}/dt = -F\) gives the following parametric representation of the phasing formula:

\[
t(v) = t_c + m \int_v^{v_{\text{iso}}} dv \frac{E'(v)}{F(v)}, \quad \Phi(v) = \Phi_c + \int_v^{v_{\text{iso}}} dv' v'^3 \frac{E'(v')}{F(v')},
\]

where \(t_c\) and \(\Phi_c\) are integration constants. We now turn to the discussion of what is known about the two functions \(E(v)\) and \(F(v)\) entering the phasing formula and how that knowledge can be improved.
2 New Energy and Flux Functions

Let \( E_T^n \equiv \sum_{k=0}^{n} E_k(\eta)v^n \) and \( F_T^n \equiv \sum_{k=0}^{n} F_k(\eta)v^n \), where \( \eta \equiv m_1m_2/m^2 \) is the symmetric mass ratio, denote the \( n \)th-order Taylor approximants of the energy and flux functions. For finite \( \eta \), the above Taylor approximants are known for \( n \leq 5 \). In the test mass limit, i.e. \( \eta \to 0 \), \( E(v) \) is known exactly, the exact flux is known numerically and analytically the flux is known up to the order \( n = 11 \). The problem is to construct a sequence of approximate wave forms \( h_{\lambda}(t; \lambda_k) \), starting from the PN expansions of \( E(v) \) and \( F(v) \). In formal terms, any such construction defines a map from the set of the Taylor coefficients of \( E \) and \( F \) into the (functional) space of wave forms. Up to now, the literature has only considered the standard map, say \( T \), obtained by inserting the successive Taylor approximants into the phasing formula. We propose a new map, say \( "P" \), based on two essential ingredients: (i) the introduction, on theoretical grounds, of two new, supposedly more basic and hopefully better behaved, energy-type and flux-type functions, say \( e(v) \) and \( f(v) \), and (ii) the systematic use of Padé approximants (instead of straightforward Taylor expansions) when constructing successive approximants of the intermediate functions \( e(v) \), \( f(v) \). Schematically, our procedure is:

\[
(E_T^n, F_T^n) \xrightarrow{T} h_T^n(t, \lambda_k),
\]

obtained by inserting the successive Taylor approximants into the phasing formula. We propose a new map, say \( "P" \), based on two essential ingredients: (i) the introduction, on theoretical grounds, of two new, supposedly more basic and hopefully better behaved, energy-type and flux-type functions, say \( e(v) \) and \( f(v) \), and (ii) the systematic use of Padé approximants (instead of straightforward Taylor expansions) when constructing successive approximants of the intermediate functions \( e(v) \), \( f(v) \). Schematically, our procedure is:

\[
(E_T^n, F_T^n) \rightarrow (e_T^n, f_T^n) \rightarrow (e_{P_n}, f_{P_n}) \rightarrow (E[e_{P_n}], F[e_{P_n}, f_{P_n}]) \rightarrow h_{P_n}^P(t, \lambda_k).
\]

Our new energy function \( e(x) \), where \( x \equiv v^2 \), is constructed out of the total relativistic energy \( E_{\text{tot}} \) using

\[
e(x) = \left( \frac{E_{\text{tot}}^2 - m_1^2 - m_2^2}{2m_1m_2} \right)^2 - 1.
\]

The function \( e(x) \) is symmetric in the two masses. The function \( E(x) \) entering the phasing formulas is given in terms of \( e(x) \) by

\[
E(x) = \left[ 1 + 2\eta \left( \sqrt{1 + e(x)} - 1 \right) \right]^{1/2} - 1.
\]

In the test-mass limit the exact expression for the function \( e(x) \) can be computed from its definition above which when substitute in Eq. (5) gives the well known energy function for a test mass in orbit around a Schwarzschild black hole:

\[
e(x) = -x \frac{1 - 4x}{1 - 3x}, \quad E'(x) = -\eta \sqrt{x} \frac{1 - 6x}{(1 - 3x)^{3/2}}.
\]
The test mass exact energy function $e(x)$ has a simple pole singularity while the function $E(x)$ has in addition a branch cut. Therefore the function $e(x)$ is more suitable in analysing the analytic structure. In the comparable mass case, on the grounds of mathematical continuity between the case $\eta \to 0$ and the case of finite $\eta$, one can expect the exact function $e(x)$ to admit a simple pole singularity on the real axis $\propto (x - x_{\text{pole}})^{-1}$. We do not know the location of this singularity, but Padé approximants are excellent tools for giving accurate representations of functions having such pole singularities. Indeed, it turns out that the Padé approximant of the 2PN expansion of $e(x)$ gives the exact energy function $E(x)$ entering the phasing formula instead of the standard Taylor approximants. This greatly improves the accuracy of the phasing formula.

It has been pointed out that the flux function $F(v; \eta = 0)$ has a simple pole at the light ring $v^2 = 1/3$. The light ring orbit corresponds to a simple pole $x_{\text{pole}}(\eta)$ in the new energy function $e(x; \eta)$. Let us define the corresponding (invariant) “velocity” $v_{\text{pole}}(\eta) \equiv \sqrt{x_{\text{pole}}(\eta)}$. This motivates the introduction of the following “factored” flux function, its Padé approximants $f_{P_n}$, and the corresponding flux function entering the phasing formula:

$$f(v; \eta) \equiv (1 - v/v_{\text{pole}}) F(v; \eta), \quad F_{P_n}(v; \eta) \equiv (1 - v/v_{\text{pole}})^{-1} f_{P_n}(v; \eta).$$

3 Effectual and Faithful Signal Models

In order to test whether a given approximant to the wave form is good or not we make use of the statistic used in detecting the ICB signal. We shall say that a multi-parameter family of approximate wave forms $h^{A}(t; \mu_k), k = 1, \ldots, n_{\mu}$, is an effectual model of some exact wave form $h^{X}(t; \lambda_k); k = 1, \ldots, n_{\lambda}$ (where one allows the number of model parameters $n_{\mu}$ to be different from, i.e. in practice, strictly smaller than $n_{\lambda}$) if the overlap, or normalized ambiguity function, between $h^{X}(t; \lambda_k)$ and the time-translated family $h^{A}(t - \tau; \mu_k)$,

$$A(\lambda_k, \mu_k) = \max_{\tau, \phi} \frac{\langle h^{X}(t; \lambda_k) h^{A}(t - \tau; \mu_k) \rangle}{\sqrt{\langle h^{X}(t; \lambda_k) h^{X}(t; \lambda_k) \rangle \langle h^{A}(t; \mu_k) h^{A}(t; \mu_k) \rangle}},$$

is, after maximization on the model parameters $\mu_k$, larger than some given threshold, e.g. $\max_{\mu_k} A(\lambda_k, \mu_k) \geq 0.965$. [In Eq. (8) the scalar product $\langle h, g \rangle$ denotes the usual Wiener bilinear form involving the noise spectrum $S_n(f)$.] While an effectual model may be a precious tool for the successful detection of a signal, it may do a poor job in estimating the values of the signal parameters $\lambda_k$. We shall then say that a family of approximate wave forms
$h^A(t; \lambda^A_k)$, where the $\lambda^A_k$ are now supposed to be in correspondence with (at least a subset of) the signal parameters, is a faithful model of $h^X(t; \lambda_k)$ if the ambiguity function $A(\lambda_k, \lambda^A_k)$, Eq. (8), is maximized for values of the model parameters $\lambda^A_k$ which differ from the exact ones $\lambda_k$ only by acceptably small biases. A necessary criterion for faithfulness, and one which is very easy to implement in practice, is that the “diagonal” ambiguity $A(\lambda_k, \lambda^A_k = \lambda_k)$ be larger than, say, 0.965. Using this terminology Eq. (3) defines approximants which, for practically all values of $n$ we could test, are both more effectual (larger overlaps) and more faithful (smaller biases) than the standard approximants Eq. (2). The new sequence of $P$-approximants exhibit a systematically better convergence behavior than the $T$-approximants $\S_2$. The overlaps they achieve at a fixed PN order are usually much higher. From our extensive study $\S_2$ of the formal “test-mass limit” $\eta \equiv m_1 m_2/(m_1 + m_2)^2 \to 0$, it appears that the presently known $(v/c)^5$-accurate PN results allow one to construct approximants having overlaps larger than 96.5%. Such overlaps are enough to guarantee that no more than 10% of signals may remain undetected.

Table 1: Fraction of events $F$ accessible relative to the case when the true signal is known, percentage bias in the estimation of total mass $B_m = 100(1 - m^A/m^X)$ and percentage bias in the estimation of the mass ratio $B_\eta = 100(1 - \eta^A/\eta^X)$, using wave forms $h^A_T$ and $h^A_P$, respectively. ($A$ is either $T$ or $P$, $X$ is for exact, and $n$ is the order of the approximant)

| $n$ | $F_T$ | $F_P$ | $B^T_m$ | $B^T_\eta$ | $B^P_m$ | $B^P_\eta$ |
|-----|-------|-------|---------|------------|---------|------------|
| Neutron star-black hole binaries |
| 4   | 0.928 | 0.998 | -6.96   | -3.61      | 12.2    | 5.64       |
| 5   | 0.945 | 1.000 | -97.2   | -1.11      | 69.3    | 1.83       |
| 6   | 0.967 | 0.998 | 1.00    | -0.157     | -1.56   | 0.263      |
| Black hole-black hole binaries |
| 4   | 0.920 | 0.991 | 1.40    | -1.524     | 0.282   | 0.700      |
| 5   | 0.532 | 0.998 | -23.6   | -0.205     | 23.1    | 0.042      |
| 6   | 0.988 | 1.000 | 0.391   | 0.019      | 1.44    | 0.119      |

Our results are summarized in Table 1 for two archetypal binaries involving neutron stars (NS) and black holes (BH), where we have tabulated the fraction of events which the templates constructed out of $T$- and $P$-approximants would detect relative to the total number of events that would have been detectable if we have had access to the true signal. We have also listed biases in the measurement of parameters. We clearly notice the super-
ority of the $P$-approximants.

Though we believe that the new approximants $h^P_n$ are superior over the standard ones $h^T_n$ and shows the practical sufficiency of the presently known $v^5$-accurate PN results, we still think that it is an important (and challenging) task to improve the (finite mass) PN results. Our calculations also suggest that knowing $E$ and $F$ to $v^6$ would further improve the effectualness (maximized overlap larger than 99.5%) and, more importantly, the faithfulness (diagonal overlap larger than 98%) to a level allowing a loss in the number of detectable events smaller than 1%, and significantly smaller biases (smaller than 0.5%) in the parameter estimations than the present $(v/c)^5$ results (about 1—5%).

References

1. T. Damour and N. Deruelle, Phys. Lett. **87A**, 81 (1981); and C.R. Acad. Sci. Paris **293** (II) 537 (1981); T. Damour, C.R. Acad. Sci. Paris **294** (II) 1355 (1982); and in **Gravitational Radiation**, ed. N. Deruelle and T. Piran, pp 59-144 (North-Holland, Amsterdam, 1983).
2. L. Blanchet, T. Damour, B.R. Iyer, C.M. Will and A.G. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995).
3. L. Blanchet, T. Damour and B.R. Iyer, Phys. Rev. **D51**, 5360 (1995).
4. C.M. Will and A.G. Wiseman, Phys. Rev. **D54**, 4813 (1996).
5. L. Blanchet, B.R. Iyer, C.M. Will and A.G. Wiseman, Class. Quantum. Gr. **13**, 575, (1996).
6. L. Blanchet, Phys. Rev. **D54**, 1417 (1996).
7. E. Poisson, Phys. Rev. **D52**, 5719 (1995).
8. T. Tanaka, H. Tagoshi and M. Sasaki, Prog. Theor. Phys. **96**, 1087 (1996).
9. C. Cutler, L.S. Finn, E. Poisson and G.J. Sussmann, Phys. Rev. **D47**, 1511 (1993).
10. E. Poisson, Phys. Rev. **D47**, 1497 (1993); H. Tagoshi and T. Nakamura, Phys. Rev. **D49**, 4016 (1994); M. Sasaki, Prog. Theor. Phys. **92**, 17 (1994); H. Tagoshi and M. Sasaki, Prog. Theor. Phys. **92**, 745 (1994).
11. C. Cutler et al., Phys. Rev. Lett. **70**, 2984 (1993).
12. T. Damour, B.R. Iyer, B.S. Sathyaprakash, Phys. Rev. D (submitted).
13. C.M. Bender and S.A. Orszag, *Advanced mathematical methods for scientists and engineers* (McGraw Hill, Singapore, 1984).
14. A loss in the overlap of 3.5% is equal to a reduction in the potential event rate by 10% (see eg. B.J. Owen, Phys. Rev. **D53**, 6749 (1996), for details).