Planck-scale modification of classical mechanics

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Several quantum gravity and string theory thought experiments indicate that the Heisenberg uncertainty relations get modified at the Planck scale so that a minimal length does arises. This modification may imply a modification of the canonical commutation relations and hence quantum mechanics at the Planck scale. The corresponding modification of classical mechanics is usually considered by replacing modified quantum commutators by Poisson brackets suitably modified in such a way that they retain their main properties (antisymmetry, linearity, Leibniz rule and Jacobi identity). We indicate that there exists an alternative interesting possibility. Koopman-Von Neumann’s Hilbert space formulation of classical mechanics allows, as Sudarshan remarked, to consider the classical mechanics as a hidden variable quantum system. Then the Planck scale modification of this quantum system naturally induces the corresponding modification of dynamics in the classical substrate. Interestingly, it seems this induced modification in fact destroys the classicality: classical position and momentum operators cease to be commuting and hidden variables do appear in their evolution equations. A possible interpretation is that classicality requires the notorious hierarchy problem.

INTRODUCTION

Quantum mechanics is one of the most successful theories in science. At present no single experimental fact indicates its breakdown. On the contrary, we have every reason to believe that quantum mechanics encompasses every natural phenomena. Nevertheless some spell of mystery still accompanies quantum mechanics. Richard Feynman, worlds one of the best experts in quantum mechanics, expressed this feeling most eloquently [1]:

“We always have had (secret, secret, close the doors!) we always have had a great deal of difficulty in understanding the world view that quantum mechanics represents. At least I do, because I’m an old enough man that I haven’t got to the point that this stuff is obvious to me. Okay, I still get nervous with it. And therefore, some of the younger students . . . you know how it always is, every new idea, it takes a generation or two until it becomes obvious that there’s no real problem. It has not yet become obvious to me that there’s no real problem. I cannot define the real problem, therefore I suspect there’s no real problem, but I’m not sure there’s no real problem.”

Superposition principle, inherent of quantum mechanics in which states of quantum systems evolve according to linear Schrödinger equation, maybe is the core reason of our uneasiness with quantum mechanics. If classical mechanics is considered as a limit of quantum mechanics then the superposition principle must hold in classical mechanics too [2]. However in the classical world, as it is revealed to us by our perceptions, we never experience Schrödinger cat states (except perhaps in art, see [3]) and a widespread belief is that the environment induced decoherence explains why (see [4, 5] and references therein. For a contrary view, however see [6, 7]).

A particularly striking example of decoherence is chaotic rotational motion of Saturn’s potato-shaped moon Hyperion. The orbit of Hyperion around Saturn is fairly predictable, but the moon tumbles unpredicatably as it orbits because its rotational motion is chaotic. It was argued [8, 9] that if Hyperion were isolated from the rest of the universe, it would evolve into a macroscopic Schrödinger cat state of undefined orientation in a time period of about 20-30 years. However this never happens. Hyperion is not isolated but constantly bombarded by photons from the rest of the universe causing its quantum state to collapse into a state of definite orientation.

Gravitational interaction cannot be shielded. Therefore any object in the universe is constantly bombarded by gravitons destroying macroscopic Schrödinger cat states. It is expected, therefore, that gravity plays a prominent role in the emergence of classicality [10, 11].

However it can be argued that the Planck scale should be viewed as a fundamental boundary of validity of the classical concept of spacetime, beyond which quantum effects cannot be neglected [12]. A legitimate question then is how this expected modifications of quantum mechanics and/or gravity at the Planck scale influence the emergence of classicality. In this paper we attempt to discuss some aspects of this question in the framework of Koopman-von
Neumann theory.

The manuscript is organized as follows. In the second section a brief overview of the generalized uncertainty principle is given. The third section provides fundamentals of the Koopman-von Neumann formulation of classical mechanics. In the fourth section we describe modifications of classical mechanics expected then combining Koopman-von Neumann-Sudarshan perspective on classical mechanics with the generalized uncertainty principle. In the last section some concluding remarks are given.

GENERALIZED UNCERTAINTY PRINCIPLE

Heisenberg’s uncertainty relation, implying the non-commutativity of the quantum mechanical observables, underlines the essential difference between classical and quantum mechanics [13]. Analyzing his now famous thought experiment of measuring the position of an electron using a gamma-ray microscope, Heisenberg arrived at the conclusion that “the more precisely is the position determined, the less precisely is the momentum known, and vice versa” [14].

\[ \delta q \delta p \sim \hbar. \] (1)

Here \( \delta q \) is the uncertainty in the determination of the position of the electron \( q \), and \( \delta p \) is the perturbation in its momentum \( p \), canonically conjugate to \( q \), induced by the measurement process.

The precise meaning of “uncertainty” was not defined in Heisenberg’s paper who used heuristic arguments and some plausible measures of inaccuracies in the measurement of a physical quantity and quantified them only on a case-to-case basis as “something like the mean error” [15]. After publication of [14], “which gives an incisive analysis of the physics of the uncertainty principle but contains little mathematical precision” [16], attempts to overcome its mathematical deficiencies were soon undertaken by Kennard [17] and Weyl [18]. They proved the inequality, valid for any quantum state,

\[ (\Delta q)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4}. \] (2)

where \((\Delta q)^2\) and \((\Delta p)^2\) are the variances (the second moment about the mean value) of \( \hat{q} \) and \( \hat{p} \) defined as \((\Delta q)^2 = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2\) and similarly for \( \hat{p} \). As usual, the mean value of a quantum-mechanical operator \( \hat{A} \), in the quantum state \( |\Psi>\), is defined as follows (we are considering a one-dimensional case, for simplicity)

\[ \langle \Psi | \hat{A} | \Psi \rangle = \int dq \Psi^*(q) (\hat{A} \Psi)(q). \] (3)

Taking the standard deviation \( \Delta A \) (the square-root from the variance \( (\Delta A)^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \)) as a measure of indeterminacy (uncertainty) of the observable \( \hat{A} \) in the quantum state \( |\Psi>\) seems very natural from the point of view of the classical probability theory where the standard deviation is considered as a measure of fluctuations. Indeed, soon Ditchburn established the relation \( \Delta q = \delta q/\sqrt{2} \) between Heisenberg’s \( \delta q \) and Weyl-Kennard’s \( \Delta q \) and proved that the equality in (2) can be achieved for Gaussian probability distributions only [19, 20].

From the mathematical point of view, we can put \( \langle q \rangle = 0 \) and \( \langle p \rangle = 0 \) in (2) without loss of generality [18]. Indeed, we can achieve \( \langle q \rangle = 0 \) by suitable redefinition of the \( q \)-coordinate origin, and \( \langle p \rangle = 0 \) — by multiplying the wave function by \( \exp(-i < p > q/\hbar) \) without changing the probability density associated with it. In this case (2) becomes a mathematical statement about (normalized) square-integrable functions \( \Psi(q) \):

\[ -\left( \int_{-\infty}^{\infty} q^2 |\Psi(q)|^2 dq \right) \left( \int_{-\infty}^{\infty} \Psi^*(q) \frac{d^2 \Psi(q)}{dq^2} dq \right) = \left( \int_{-\infty}^{\infty} q^2 |\Psi(q)|^2 dq \right) \left( \int_{-\infty}^{\infty} \left| \frac{d \Psi}{dq} \right|^2 dq \right) \geq \frac{1}{4}. \] (4)

The three-dimensional sibling of (4), known as the Heisenberg inequality, is [21]

\[ \left( \int r^2 |\Psi(r)|^2 d^3r \right) \left( \int |\nabla \Psi(r)|^2 d^3r \right) \geq \frac{9}{4}. \] (5)

If an electron in the hydrogen atom is localized around the origin, then (5) tells us that its momentum (and hence kinetic energy) will be large. It is tempting to use this fact to get a lower bound on the electron’s energy in the
hydrogen atom and and thus prove its stability. Although it is a common practise to use such kind of reasoning for estimation of the hydrogen atom size and its ground state energy [22] [25], the truth is that Heisenberg inequality is too weak to ensure the hydrogen atom stability or stability of matter in general [26] [27]. The reason is that the first multiple in (3) can be very large even in the case when the main part of the wave function (its modulus squared) is localized around the origin, if only the remaining small part is localized very far away.

However mathematically the uncertainty principle in the form of Eq. (1) is just an expression of the fact from harmonic analysis that “A nonzero function and its Fourier transform cannot both be sharply localized” [16]. Heisenberg inequality (1) is just one attempt to make the above given sloppy phrase mathematically precise. But, as we have seen above for hydrogen atom, from the physics perspective the standard deviation is not always an adequate measure of localization and quantum uncertainty [28] [29]. In such situations (in particular then considering the matter stability problem [26] [27]) other uncertainty principles prove to be more useful. The examples include Hardy uncertainty principle [27] [31]

\[
\left( \int \frac{1}{r^2} |\Psi(r)|^2 d^3r \right)^{-1} \left( \int |\nabla \Psi(r)|^2 d^3r \right) \geq \frac{1}{4},
\]

or Sobolev inequality [26]

\[
\left( \int |\Psi(r)|^6 d^3r \right)^{-1/3} \left( \int |\nabla \Psi(r)|^2 d^3r \right) \geq \frac{3}{4} (4\pi^2)^{2/3},
\]

which, in some sense, is weaker than the Hardy inequality [10] [27].

From the physics side, the uncertainty principle is more than just inequalities from harmonic analysis. We can envisage at least three manifestations of uncertainty relations [15] [31]. First of all the uncertainty relations relate intrinsic spreads of two conjugate dynamical variables in a quantum state. However Heisenberg in his seminal work speaks about unavoidable disturbance that a measurement process exerts on a pair of conjugate dynamical variables. Therefore we can understand the uncertainty relation also as an assertion about a relation between inaccuracies in measurements of one dynamical variable and the ensuing disturbance in the probability distribution of the conjugated variable, or inaccuracies of a pair of conjugate dynamical variables in any joint measurements of these quantities. Although conceptually distinct, these three manifestations of uncertainty relations are closely related [31].

The first facet of the uncertainty principle can be formalized most easily by using second-order central moments of two conjugated quantum observables (but remember a caveat that the standard deviation is not always an adequate measure of quantum uncertainty [29]). For any pair of quantum observables $\hat{A}$ and $\hat{B}$ we have three independent second-order central moments of their joint quantum distributions in a quantum state $|\Psi\rangle$:

\[
[\Delta(A)]^2 = \langle \Psi | (\hat{A} - \bar{A})^2 |\Psi\rangle = \bar{A}^2 - \bar{A}^2, \quad [\Delta(B)]^2 = \langle \Psi | (\hat{B} - \bar{B})^2 |\Psi\rangle = \bar{B}^2 - \bar{B}^2,
\]

\[
\Delta(A, B) = \frac{1}{2} \langle \Psi | (\hat{A} - \bar{A})(\hat{B} - \bar{B}) + (\hat{B} - \bar{B})(\hat{A} - \bar{A}) |\Psi\rangle = \frac{1}{2} (\bar{A}\bar{B} + BA - \bar{A}\bar{B}),
\]

where overbar denotes the mean value of the corresponding observable in the state $|\Psi\rangle$. We have

\[
\hat{A}\hat{B} = \frac{1}{2} (\bar{A}\hat{B} + \hat{B}\bar{A}) + \frac{1}{2} (\hat{A}\hat{B} - \bar{A}\bar{B}),
\]

and the first Hermitian part in the r.h.s has a real mean value, while the mean value of the second anti-Hermitian part is purely imaginary. Then the Schwarz inequality $| \langle \psi | \phi \rangle |^2 \leq \langle \psi | \psi \rangle \langle \phi | \phi \rangle$, if we take $|\psi\rangle = \hat{A}|\Psi\rangle$, $|\phi\rangle = \hat{B}|\Psi\rangle$, will give

\[
\bar{A}^2 \bar{B}^2 \geq |\bar{A}\bar{B}|^2 = \left( \frac{\bar{A}\bar{B} + BA}{2} \right)^2 + \left( \frac{\bar{A}\bar{B} - BA}{2} \right)^2,
\]

because $\langle \hat{A}|\hat{A}\Psi\rangle = \bar{A}^2$, $\langle \hat{B}|\hat{B}\Psi\rangle = \bar{B}^2$ and for Hermitian observables $\hat{A}$ and $\hat{B}$, $\langle \hat{A}\hat{B}\Psi\rangle = \langle \hat{A}\hat{B}\Psi\rangle$.

If we replace $\hat{A}$ and $\hat{B}$ by operators $\hat{A} - \bar{A}$ and $\hat{B} - \bar{B}$ in the above given reasoning, we end up with the Schrödinger uncertainty relation [32]

\[
[\Delta(A)]^2[\Delta(B)]^2 \geq [\Delta(A, B)]^2 + \left( \frac{\bar{A}\bar{B} - BA}{2} \right)^2.
\]
Heisenberg uncertainty relation \( (2) \) is obtained from this more general uncertainty relation if we take \( \hat{A} = \hat{x}, \hat{B} = \hat{p} \) use the canonical commutation relations

\[
[\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij}, \quad i = 1, 2, 3.
\]

and assume that the covariance \( \Delta(x,p) \) equals to zero. Although Schrödinger uncertainty relation is more general and symmetric (it remains invariant under rotations in phase space) than the Heisenberg uncertainty relation, or its generalization due to Robertson to any pair of observables with zero covariance \( 33 \)

\[
[\Delta(A)]^2[\Delta(B)]^2 \geq \left( \frac{AB - BA}{2} \right)^2,
\]

the Schrödinger uncertainty relation is strangely ignored in almost all quantum mechanics textbooks and its usefulness was appreciated only after 50 years from its discovery in connection with the description of squeezed states in quantum optics for which the covariance \( \Delta(x,p) \) doesn’t equal zero \( 29 \).

As we see the uncertainty relations are intimately related to canonical commutation relations \( 10 \). To our best knowledge, Gleb Wataghin was the first \( 34, 35 \) who suggested that both the commutation relations \( 10 \) and the uncertainty principle \( 11 \) might be modified taking into account gravity. The crux of the Mead’s argument is that the gravitational interaction between the electron and photon in Heisenberg microscope is a source of additional uncertainty in the electron’s position.

The gravitational field of a photon was obtained in \( 37, 38 \) by boosting the Schwarzschild space-time up to the speed of light by taking the limit \( V \to c, m \to 0 \) such that the quantity \( p = mV(1 - V^2/c^2)^{-1/2} \) is held constant. The resulting space-time, for the photon with momentum \( p \) moving in the \( z \)-direction, has the metric \( 39 \)

\[
ds^2 = -2(du dv - d\zeta d\bar{\zeta}) - 4 \frac{G}{c^3} \ln \left( \frac{\zeta \bar{\zeta}}{r_0} \right) \delta(u) du^2,
\]

where \( G \) is Newton’s constant, \( u = (ct - z)/\sqrt{2} \) and \( v = (ct + z)/\sqrt{2} \) are retarded and advanced null coordinates (light-cone coordinates), while the complex coordinates \( \zeta = (x + iy)/\sqrt{2} \) and \( \bar{\zeta} = (x - iy)/\sqrt{2} \) parametrize the spatial hyperplane orthogonal to the photon’s velocity vector. The parameter \( r_0 \) in \( 13 \) is an arbitrary constant of the dimension of length which does not effect observable quantities \( 10 \).

The metric \( 13 \) describes an impulsive gravitational wave: the space-time remains flat everywhere except \( u = 0 \) null hyperplane, where it develops a delta-function singularity. This gravitational shockwave moves with the photon and when it meets with the electron within the Heisenberg microscope two physical effects take place: the timelike geodesic of the electron experiences a discontinuous jump in the null coordinate \( v \) and gets refracted in the transverse direction \( 42 \).

There are various subtleties here. The very concept of photon with sharply defined momentum (energy), existing at \( t = -\infty \), is an idealization. In reality one should take into account that the photon is produced at a finite instant of time and the corresponding light packet has a finite Fourier support \( 11 \). Besides, because \( 13 \) describes a situation when a cause (photon) and the effect (the corresponding gravitational shockwave) propagate with the same speed of light, it is not altogether clear the gravitational field is related to the photon or it arises solely in the process of emission \( 42 \). At last, to cope with the presence of ill-defined highly singular products of generalized functions in the geodesic deviation equation, precise calculation of the above mentioned physical effects of the gravitational shockwave on the test particles geodesics, requires either a suitable regularization procedure \( 43 \), or making use of the Colombeau algebra of generalized functions \( 44 \).

Anyway, for our purposes we need only an order of magnitude estimate of the additional uncertainty in Heisenberg microscope due to gravity. This was done in \( 15 \) with the result that the additional uncertainty in electron’s position due to gravitational attraction of the photon is

\[
\Delta x_G \approx \frac{Gp}{c^3} \approx l_p^2 \frac{\Delta p}{\hbar}.
\]
The second step follows from the fact that the electron momentum uncertainty \( \Delta p \) must be of order of the photon momentum \( p \). Here

\[
l_P = \sqrt{\frac{G\hbar}{c^3}}
\]  

(15)

is the Planck length.

If we add this new uncertainty linearly to the original Heisenberg uncertainty \( \Delta x_H = \hbar/\Delta p \), we get the modified uncertainty principle (the so called GUP — generalized or gravitational uncertainty principle [43])

\[
\frac{\Delta x}{l_P} \approx \frac{\hbar}{l_P \Delta p} + \frac{l_P \Delta p}{\hbar}.
\]  

(16)

In this form the uncertainty principle is invariant under momentum inversion \( \frac{\hbar}{l_P \Delta p} \rightarrow \frac{l_P \Delta p}{\hbar} \). Another remarkable property of (16) is that it predicts a minimum position uncertainty \( \Delta x_{\text{min}} = 2l_P \) at a symmetric, with respect to the above mentioned momentum inversion, point \( \Delta p = \hbar/l_P \).

Although the idea that a smallest length exists in nature can be traced back to Heisenberg and March [40], only relatively recent attempts to reconcile quantum mechanics with general relativity in string theory produced a solid foundation for it and for the generalized uncertainty principle (see, for example, [41, 42] and review articles [49, 50]).

As Robertson’s version [11] of the uncertainty principle shows, the generalized uncertainty principle [10] may imply a deformation of the usual Heisenberg algebra of canonical commutation relations (10). Various versions of this deformation have been proposed in the literature (see, for example, [51] and references therein).

As was mentioned above, Wataghin was the first to suggest modification of the canonical commutation relations at high energies. However, it was Snyder who proposed a model of noncommutative spacetime, admitting a fundamental length but nevertheless being Lorentz invariant [52], non-relativistic version of which produces a concrete form of such modification [53]:

\[
[\hat{x}_i, \hat{x}_j] = i\hbar \beta' \hat{J}_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar (\delta_{ij} + \beta' \hat{p}_i \hat{p}_j), \quad \hat{J}_{ij} = \hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i,
\]  

(17)

where \( \beta' \) is some constant, usually assumed to be of the order of \( l_P^2 / \hbar^2 \), as [11] and [10] relations do imply.

Snyder’s work was ahead of its time and its importance was not immediately recognized. Meanwhile Mead [54] and Karolyhazy [55] investigated uncertainties in measurements of space-time structure resulting from universally coupled gravity and concluded that it is impossible to measure distances to a precision better than Planck’s length. However very few took seriously the idea that the Planck length could ever play a fundamental role in physics [50, 56].

The situation changed when developments in string theory revealed the very same impossibility of resolving distances smaller than Planck’s length, and these developments inspired Adler and Santiago’s 1999 paper [45] that almost exactly reproduced Mead’s earlier arguments [54]. Various choices of deformed commutation relations have been considered in the literature beginning from the Kempf et al. landmark paper [57]. Let us mention, for example, a version that generalizes the Snyder algebra [17, 58, 59]:

\[
[\hat{x}_i, \hat{x}_j] = -i\hbar \left[ (2\beta - \beta') + \beta (2\beta + \beta') \hat{p}^2 \right], \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \left[ 2 \hat{p} \delta_{ij} + \beta' \hat{p}_i \hat{p}_j \right],
\]  

(18)

where \( \beta \) is a new constant of the same magnitude as \( \beta' \), \( \hat{p}^2 = \hat{p}_i \hat{p}^i \), and \( \hat{J}_{ij} \) was defined in (17).

A different type of modification of the canonical commutation relations was suggested by Saavedra and Utreras [61]:

\[
[\hat{x}_i, \hat{p}_j] = i\hbar \left( 1 + \frac{l_P}{c \hbar} H \right) \delta_{ij}.
\]  

(19)

One can say that in this case the configuration space becomes dynamical, much like the general relativity, because the commutation relations (19) depends on the system under study through the Hamiltonian \( H \).

As we see, commutation relations (17) and (18) imply a non-commutative spatial geometry. Mathematically this is a consequence of Jacobi identity and our tacit assumption that components of momentum operators do commute. Physical bases of this non-commutativity is a dynamical nature of space-time in general relativity: it can be argued quite generally that an unavoidable change in the space-time metric when measurement processes involve energies of the order of the Planck scale destroys the commutativity of position operators [60].

There exists a vast and partly confusing literature on the modifications of quantum mechanics and quantum field theory implied by the existence of a minimal length scale (for a review and references see, for example, [48, 50, 62, 64]).
KOOPMAN-VON NEUMANN MECHANICS

It is usually assumed that classical mechanics, in contrast to quantum mechanics, is a deterministic theory with the well-defined trajectories of underlying particles. However, if we realize the imperfect nature of classical measuring devices, which precludes the preparation of classical systems with precisely known initial data, it becomes clear that “the determinism of classical physics turns out to be an illusion, created by overrating mathematico-logical concepts. It is an idol, not an ideal in scientific research” [65]. Therefore, one can assume that a conceptually superior appropriate statistical description of classical mechanics is then given by Liouville equation (for simplicity, we consider a one-dimensional mechanical system with canonical variables \( q \) and \( p \))

\[
\frac{i}{\partial t} \rho = \hat{L} \rho = i \{ H, \rho \} = i \left( \frac{\partial H}{\partial q} \frac{\partial \rho}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial \rho}{\partial q} \right),
\]

which gives a time-evolution of the phase-space probability density \( \rho(q, p, t) \). Here \( H \) is the Hamiltonian and \( \{ \cdot, \cdot \} \) denotes the Poisson bracket.

However, classical and quantum mechanics are different not only by inherently probabilistic nature of the latter. Mathematical structures underlying these two disciplines are quite different. The mathematics underlying classical mechanics is a symplectic geometry of the phase space [66–68], while quantum mechanics is based on the theory of Hilbert spaces [69], rigged Hilbert spaces [70] or on their algebraic counterpart — the theory of \( \mathcal{C}^{\ast} \) algebras [71]. In light of this difference in the underlying mathematical structure it is surprising that it is possible to give a Hilbert space formulation for classical mechanics too, as shown long ago in classic papers by Koopman [72] and von Neumann [73] (for modern presentation, see [74] and references therein).

This translation of classical mechanics into the language of Hilbert spaces is based on the crucial observation that, because the Liouville operator

\[
\hat{L} = i \left( \frac{\partial H}{\partial q} \frac{\partial}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial}{\partial q} \right)
\]

is linear in derivatives, the square root of the probability density \( \psi(q, p, t) = \sqrt{\rho(q, p, t)} \) obeys the same Liouville equation (20):

\[
\frac{i}{\partial t} \sqrt{\rho}(q, p, t) = \hat{L} \sqrt{\rho}(q, p, t).
\]

Moreover, if we assume that \( \psi(q, p, t) \) in (22) is a complex function \( \psi(q, p, t) = \sqrt{\rho(q, p, t)} e^{iS(q, p, t)} \), then (22) implies that the amplitude and phase evolve independently through the Liouville equations:

\[
\frac{i}{\partial t} \sqrt{\rho} = \hat{L} \sqrt{\rho}, \quad \frac{\partial S}{\partial t} = \hat{L} S,
\]

and the probability density \( \rho(q, p, t) = \psi^\ast(q, p, t) \psi(q, p, t) \) also obeys the Liouville equation (20). Therefore we can introduce a Hilbert space of square integrable complex functions \( \psi(q, p, t) \), equip it with the inner product

\[
< \psi | \phi > = \int dq dp \psi^\ast(p, q, t) \phi(p, q, t),
\]

and then we recover the rules that are usually associated with quantum mechanics. Namely, observables are represented by Hermitian operators and the expectation value of an observable \( \hat{A} \) is given by

\[
\bar{A}(t) = \int dq dp \psi^\ast(q, p, t) \hat{A} \psi(q, p, t).
\]

If \( \varphi_\lambda(q, p, t) \) is an eigenstate of the observable \( \hat{A} \), \( \hat{A} \varphi_\lambda(q, p, t) = \lambda \varphi_\lambda(q, p, t) \), then the probability \( P(\lambda) \) that the outcome of a measurement of \( \hat{A} \) on a classical mechanical system with the KvN wave function \( \phi(p, q, t) \) results in the eigenvalue \( \lambda \) is given by the usual Born rule

\[
P(\lambda) = \int dq dp |\varphi_\lambda^\ast(q, p, t) \psi(q, p, t)|^2.
\]

1 Older references can be found in [73, 74]. It should be noted that, apparently independently from Koopman and von Neumann, and from each other, similar formalisms were suggested later by Schönberg [77] and by Della Riccia and Wiener [78].
There are two main differences from quantum mechanics. Firstly, and most importantly in the classical theory the operators for position and momentum do commute

\[ [\hat{q}, \hat{p}] = 0. \tag{27} \]

In the Hilbert space formalism outlined above, these operators are realized as multiplicative operators

\[ \hat{q} \psi(q,p,t) = q \psi(q,p,t), \quad \hat{p} \psi(q,p,t) = p \psi(q,p,t). \tag{28} \]

The second important difference is that the “Hamiltonian” (Liouville operator) \( \hat{L} \) that defines the time evolution of the KvN wave function is linear in spatial derivatives. This is quite unusual in quantum mechanics and such type of dynamical evolution was attributed to quantum systems that allow a genuine quantum chaos to emerge \[79\].

Thanks to the imaginary unit \( i \) (and that’s the reason why it was introduced), the Liouville operator \( \hat{L} \) is Hermitian, and thus generates a unitary evolution, with respect to the inner product \( \langle 24 \rangle \):

\[ \int dpdq \varphi^*(p,q,t) \hat{L} \psi(p,q,t) = \int dpdq (\hat{L} \varphi)^*(p,q,t) \psi(p,q,t). \tag{29} \]

This can be proved through an integration by parts under reasonable assumptions about the Hamiltonian, namely the equality of mixed derivatives \( \frac{\partial^2 H}{\partial p \partial q} = \frac{\partial^2 H}{\partial q \partial p} \). At that we assume that the wave functions \( \varphi(q,p,t) \) and \( \psi(q,p,t) \), being square integrable, vanish sufficiently fast at \( q,p \to \pm \infty \).

There are some mathematical subtleties here, however. Strictly speaking, not every square integrable function vanishes at infinity. The example is \[80\] \( f(x) = x^2 \exp(-x^8 \sin^2 x) \) which is square integrable but even not bounded at infinity. According to the Hellinger-Toeplitz theorem \[81\], everywhere defined Hermitian operator is necessarily bounded. Position and momentum operators are clearly unbounded. So is the Liouville operator. Therefore, the rigorous mathematical formulation of classical mechanics in the Hilbert space KvN formalism is not as a simple task as naively can appear. However, these mathematical subtleties and difficulties are not characteristic of only KvN mechanics and is already present in ordinary quantum mechanics \[80\]. The formalism of rigged Hilbert spaces \[70\] can provide a possible, although a rather sophisticated solution.

What the Hilbert space KvN formalism corresponds to the usual classical mechanics is most easily seen in the Heisenberg picture of time evolution. In the Schrödinger picture of evolution assumed above the operators are time-independent while the wave function evolves unitarily according to

\[ \psi(q,p,t) = e^{-iLt} \psi(q,p,0). \tag{30} \]

On the contrary, in the Heisenberg picture wave functions are assumed to be time-independent and all time dependencies of mean values of physical quantities are incorporated in the time evolution of operators according to

\[ \hat{\Lambda}(t) = e^{iLt} \hat{\Lambda}(0) e^{-iLt}. \tag{31} \]

Equation of motion that follows from \( 31 \) is

\[ \frac{d\hat{\Lambda}(t)}{dt} = i[\hat{L}, \hat{\Lambda}(t)]. \tag{32} \]

Namely, for multiplicative position and momentum operators we get

\[ \frac{dq}{dt} = i[\hat{L}, q] = \frac{\partial H(q,p,t)}{\partial p}, \quad \frac{dp}{dt} = i[\hat{L}, p] = -\frac{\partial H(q,p,t)}{\partial q}, \tag{33} \]

which are nothing but the Hamilton’s equations.

Alternatively, to show that KvN formalism corresponds to the usual Newtonian mechanics, we can apply method of characteristics in the Schrödinger picture \[82\]. Let us consider a curve \((q(\alpha), p(\alpha), t(\alpha))\) in the extended phase space, parametrized by a real parameter \( \alpha \). Along this curve

\[ \frac{d\psi}{d\alpha} \Bigg|_{\alpha = 0} = \frac{\partial \psi}{\partial t} \frac{dt}{d\alpha} + \frac{\partial \psi}{\partial q} \frac{dq}{d\alpha} + \frac{\partial \psi}{\partial p} \frac{dp}{d\alpha}, \tag{34} \]

and if the curve is chosen in such a way that

\[ \frac{dt}{d\alpha} = 1, \quad \frac{dq}{d\alpha} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\alpha} = -\frac{\partial H}{\partial q}. \tag{35} \]
we will get

\[ \frac{d\psi}{d\alpha} = -i \left( \frac{\partial \psi}{\partial t} - \hat{L}\psi \right) = 0, \]  

(36)

according to the Liouville equation (22).

As we see from (35), the parameter \( \alpha \) essentially coincides with time and the characteristics of the Liouville equation (22) are just classical Newtonian trajectories in the extended phase space. Moreover, the KvN wave function \( \psi(q, p, t) \) remains constant along these trajectories. Thus delta-function initial date, with definite initial values of \((q_0, p_0, t_0)\), will be transported along Newtonian trajectories \((q(t), p(t), t)\), as expected for a classical point particle.

In fact the Liouville operator (2) is not uniquely defined in the KvN mechanics [83]. In particular, as it is clear from (33), we can add to the Liouville operator (21) any function \( F(q, p, t) \):

\[ \hat{L}' = \hat{L} + F(q, p, t), \]  

(37)

without changing the Hamilton’s equations (33).

Of course, this gauge freedom in the choice of the Liouville operator is related to the invariance of the KvN probability density function under the phase transformations

\[ \psi'(q, p, t) = e^{ig(q, p, t)}\psi(q, p, t). \]  

(38)

Indeed, the new wave function \( \psi'(q, p, t) \) obeys the new Liouville equation

\[ i\frac{\partial \psi'}{\partial t} = \hat{L}'\psi', \]  

(39)

where

\[ \hat{L}' = e^{ig}\hat{L}e^{-ig} - \frac{\partial g}{\partial t} = \hat{L} - \frac{\partial g}{\partial t} + \{H, g\} \equiv \hat{L} + F(q, p, t). \]  

(40)

If evolution of the KvN wave function is determined by \( \hat{L}' \), then along the Newtonian trajectories we will have

\[ F(q, p, t) = -\left( \frac{\partial g}{\partial t} + \{g, H\} \right) = -\frac{dg(q, p, t)}{dt}, \]  

(41)

and

\[ \frac{d\psi'}{dt} = -i \left( \frac{\partial \psi'}{\partial t} - \hat{L}'\psi' \right) = -iF\psi' = i\frac{dg}{dt}\psi', \]  

(42)

which implies

\[ \psi'(q, p, t) = e^{i[g(q, p, t) - g(q_0, p_0, t_0)]}\psi'(q_0, p_0, t_0). \]  

(43)

That is, the KvN wave function no longer remains constant along Newtonian trajectories (along characteristics of the new Liouville equation (39)), but the change affects only the phase of the wave function, and such a change is irrelevant in the context of classical mechanics.

An obvious difference between the KvN wave function and the true quantum wave function is the number of independent variables: KvN wave function depends typically on the phase space variables \( q, p \) and time, while quantum wave function typically depends on the configuration space variables (\( q \) in our case) and time. For an interesting perspective on the importance of this difference, see [83].

There were attempts to develop operator formulation of classical mechanics based on wave functions defined over configuration space, not over phase space [84, 85]. In such attempts Hamilton-Jacobi equation, not the Liouville equation, is used as a starting point. Although interesting, we will not pursue such approach in the present work.

A very interesting perspective on the KvN mechanics was given by Sudarshan [86, 87]. Let us consider a quantum mechanical system with twice as many degrees of freedom as our initial classical mechanical system. Namely, besides \( \hat{q} \) and \( \hat{p} \) operators, let us introduce new operators \( \hat{Q} \) and \( \hat{P} \) so that \( (\hat{q}, \hat{P}) \) and \( (\hat{Q}, \hat{p}) \) form canonical pairs from the quantum mechanical point of view:

\[ [\hat{q}, \hat{P}] = i\hbar, \quad [\hat{Q}, \hat{p}] = i\hbar. \]  

(44)
Then in the \((q,p)\)-representation, where \(\hat{q}\) and \(\hat{p}\) operators are diagonal multiplicative operators, we will have
\[
\hat{P} = -i\hbar \frac{\partial}{\partial q}, \quad \text{and} \quad \hat{Q} = i\hbar \frac{\partial}{\partial p},
\] (45)
so that the Liouville operator \((21)\) takes the form
\[
\hat{L} = \frac{1}{\hbar} \left( \frac{\partial H}{\partial q} \hat{Q} + \frac{\partial H}{\partial p} \hat{P} \right),
\] (46)
and the Liouville equation \((22)\) can be rewritten as a Schrödinger equation
\[
i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi,
\] (47)
with the quantum Hamiltonian
\[
\hat{H} = \frac{\partial H}{\partial q} \hat{Q} + \frac{\partial H}{\partial p} \hat{P}.
\] (48)
The search for a hidden variable theory for quantum mechanics is a still ongoing saga \[88\]. Here, thanks to Sudarshan (for earlier thoughts in this direction see \[89\]) we have an amusing situation: classical mechanics, on the contrary, is interpreted as a hidden variable quantum theory! “If we assume that not all quantum dynamical variables are actually observable, and if we set rules for distinguishing measurable from nonmeasurable operators, it is then possible to define a classical system as a special type of quantum system for which all measurable operators commute” \[90\].

What remains is to explain how Schrödinger cat states is avoided in KvN mechanics: the superposition principle is the basic tenet of the quantum mechanics while in the classical realm the cat is either alive or dead, any superposition of these classical states does not make sense.

Of course, the fact that the amplitude and phase evolve independently, equations \((23)\), already implies the absence of any interference effects in KvN mechanics. However, this separation of the amplitude and phase is an artifact of the \((q,p)\)-representation. We can choose to work, for example, in the \((q,Q)\)-representation instead \[74, 91\]. In this representation \(\hat{q}\) and \(\hat{Q}\) are simultaneously diagonal multiplicative operators, while \(\hat{p}\) and \(\hat{P}\) are differential operators:
\[
\hat{q} \psi(q,Q,t) = q \psi(q,Q,t), \quad \hat{Q} \psi(q,Q,t) = Q \psi(q,Q,t),
\]
\[
\hat{p} \psi(q,Q,t) = -i\hbar \frac{\partial}{\partial Q} \psi(q,Q,t), \quad \hat{P} \psi(q,Q,t) = -i\hbar \frac{\partial}{\partial q} \psi(q,Q,t).
\] (49)
Wave functions in two representations are related by Fourier transform (the same symbol \(\psi\) is used for both the function and its Fourier transform for notational simplicity):
\[
\psi(q,Q,t) = \frac{1}{\sqrt{2\pi}} \int dp \, e^{ipQ/\hbar} \psi(q,p,t).
\] (50)
This follows from the following \[74\]. If \(|q,Q \rangle\) are the simultaneous eigenstates of the \(\hat{q}\) and \(\hat{Q}\) operators, while \(|q,p \rangle\) — simultaneous eigenstates of the \(\hat{q}\) and \(\hat{p}\) operators:
\[
\hat{q} |q,Q \rangle = q |q,Q \rangle, \quad \hat{Q} |q,Q \rangle = Q |q,Q \rangle,
\]
\[
\hat{q} |q,p \rangle = q |q,p \rangle, \quad \hat{p} |q,p \rangle = p |q,p \rangle,
\] (51)
then we will have
\[
q < q',p'|q,Q \rangle = < q',p'|q,Q \rangle = q' < q',p'|q,Q \rangle,
\]
\[
Q < q',p'|q,Q \rangle = < q',p'|q,Q \rangle = -i\hbar \frac{\partial}{\partial p'} < q',p'|q,Q \rangle,
\] (52)
which, together with the normalization condition
\[
< q',Q'|q,Q \rangle = \delta(q' - q)\delta(Q' - Q),
\] (53)
implies that in the \((q,p)\)-representation the \(|q,Q>\) state is given by the wave function

\[
<q',p'|q,Q> = \frac{1}{\sqrt{2\pi}} \delta(q' - q)e^{-i p'Q/\hbar}.
\] (54)

Because, like \(|q,p>\) states, \(|q,Q>\) states also form a complete set of orthonormal eigenstates in the KvN Hilbert space, we have

\[
\psi(q,Q,t) = <q,Q|\psi(t)> = \int dq'dp <q,Q|q',p> <q',p|\psi(t)> ,
\] (55)

which, in light of (54), is equivalent to (50).

In the \((q,Q)\)-representation and for the classical Hamiltonian \(H = \frac{p^2}{2m} + V(q)\), the quantum Hamiltonian (48) takes the form

\[
\mathcal{H} = \frac{dV}{dq} Q - \frac{\hbar^2}{m} \frac{\partial^2}{\partial q \partial Q}.
\] (56)

Then it follows from the Schrödinger equation (47) that the amplitude \(A(q,Q)\) and the phase \(\Phi(q,Q)\) of the KvN wave function \(\psi(q,Q) = A e^{i\Phi/\hbar}\) evolve according to the equations

\[
\begin{align*}
\frac{\partial A}{\partial t} + \frac{1}{m} \left( \frac{\partial A}{\partial Q} + \frac{\partial A}{\partial q} + A \frac{\partial^2 \Phi}{\partial Q \partial q} \right) &= 0, \\
\frac{\partial \Phi}{\partial t} + \frac{1}{m} \left( \frac{\partial \Phi}{\partial Q} - \frac{\hbar^2}{A} \frac{\partial^2 A}{\partial Q \partial q} \right) + \frac{dV}{dq} Q &= 0.
\end{align*}
\] (57)

As we see, in this representation the phase and amplitude are coupled in the equations of motion and their time evolutions become intertwined much like the ordinary quantum mechanics. This is hardly surprising because, after all, the encompassing underlying system is quantum.

According to Sudarshan \[86, 87\], it is the superselection principle \[92, 93\] which kills the interference effects in the KvN mechanics. In classical mechanics, observables are functions of the phase space variables \(q\) and \(p\). Therefore, \(\hat{q}\) and \(\hat{p}\) commute with all classical observables and thus trigger a superselection mechanism which renders the relative phase between different superselection sectors unobservable. Indeed, let

\[
|\psi> = \alpha|p,q> + \beta|p',q'>,
\] (58)

with \(|\alpha|^2 + |\beta|^2 = 1\), be a seemingly coherent superposition of different eigenstates of \(\hat{q}\) and \(\hat{p}\). As we assume that \(|p,q>\) form a complete set of orthonormal states and an observable \(\Lambda\) commutes with \(\hat{q}\) and \(\hat{p}\), \(|p,q>\) is an eigenstate of \(\Lambda\) also and thus \(<p',q' |\Lambda|p,q> = 0\). Therefore, for the mean value of the observable \(\Lambda\) in the state \(|\psi>\) we get

\[
\bar{\Lambda} = <\psi|\Lambda|\psi> = |\alpha|^2 <p,q|\Lambda|p,q> + |\beta|^2 <p',q'\Lambda|p',q'>.
\] (59)

As we see, all interference effects are gone and the mean value is the same as if we had an incoherent mixture of the states \(|p,q>\) and \(|p',q'>\) described by the diagonal density matrix

\[
\hat{\rho} = |\alpha|^2 |p,q><p,q| + |\beta|^2 |p',q'><p',q'|.
\] (60)

However, this use of the superselection principle in KvN mechanics differs from its conventional use in one essential aspect \[71, 80\]. In quantum mechanics time evolution is governed by Hamiltonian which is by itself an observable. As a result all time evolution takes place in one superselection sector and we have genuine superselection rules that the eigenvalues of the superselecting operators cannot be changed during the time evolution. Of course, in the case of KvN mechanics this would be a catastrophe because it would imply that \(q\) and \(p\) cannot change during the time evolution. Fortunately, the quantum Hamiltonian (48) is not a classical observable, because it contains unobservable hidden quantum variables \(\hat{Q}\) and \(\hat{P}\). As a result (48) does not commute with \(\hat{q}\) and \(\hat{p}\) operators and thus can generate a transition from one eigenspace of these superselection operators to the other.
MODIFICATION OF CLASSICAL MECHANICS

Modified commutation relations alone are not enough to derive physical meaning. Many attempts were made to define the dynamics of quantum systems and their observables in the presence of a minimal length, but this research field is still far from being as logically consistent and mature as the ordinary quantum mechanics is \(^{50}\). In any case, the minimal length modification of quantum mechanics entails the corresponding modification of classical mechanics, as the former is considered as the \(\hbar \to 0\) limit of the latter (see, however, \(^{94}\)).

Usually the modification of classical mechanics is obtained from the corresponding modification of quantum mechanics by replacing modified commutators with modified Poisson brackets \(^{48},^{95}\):

\[
\frac{1}{i\hbar} [\hat{x}_i, \hat{p}_j] \to \{x_i, p_j\}. \tag{61}
\]

At that the modified Poisson bracket of arbitrary functions \(F\) and \(G\) of the coordinates and momenta are defined as \(^{18}\)

\[
\{F, G\} = \left( \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_j} \right) \{x_i, p_j\} + \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \{x_i, x_j\}. \tag{62}
\]

Correspondingly, the classical equations of motion have the form

\[
\dot{x}_i = \{x_i, H\} = \{x_i, p_j\} \frac{\partial H}{\partial p_j} + \{x_i, x_j\} \frac{\partial H}{\partial x_j}, \quad \dot{p}_i = \{p_i, H\} = -\{x_i, p_j\} \frac{\partial H}{\partial x_j}. \tag{63}
\]

A number of classical mechanics problem was studied within this scenario \(^{94},^{102}\). Koopman-von Neumann mechanics, however, provides a different and in our opinion more interesting perspective on the Planck scale deformation of classical mechanics.

The main idea is the following: modification of the commutation relations, for example in the form of \(^{18}\), in the encompassing (in the Sudarshan sense) quantum system will alter classical dynamics in the \((q, p)\) classical subspace.

For simplicity, let us consider a one-dimensional classical harmonic oscillator with the Hamiltonian

\[
H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2). \tag{64}
\]

In the Sudarshan-encompassing two-dimensional quantum system we can identify \(x_1 = q, x_2 = Q, p_1 = P, p_2 = p\). Then Snyder commutation relations \(^{17}\) take the form

\[
[\hat{q}, \hat{P}] = i\hbar(1 + \beta' P^2), \quad [\hat{q}, \hat{p}] = [\hat{Q}, \hat{P}] = i\hbar\beta' \hat{p}\hat{P}, \quad [\hat{Q}, \hat{p}] = i\hbar(1 + \beta' \hat{p}^2), \quad [\hat{q}, \hat{Q}] = i\hbar\beta'(\hat{q}\hat{p} - \hat{Q}\hat{P}), \quad [\hat{p}, \hat{P}] = 0. \tag{65}
\]

The first surprise is that \(\hat{q}\) and \(\hat{p}\) cease to be commuting. According to \(^{14}\), the corresponding uncertainty relation is

\[
\Delta q \Delta p \geq \frac{\hbar\beta'}{2} (\Delta(p, P) + <p><P>). \tag{66}
\]

As we see the sharply defined classical trajectories cease to exist in the \((q, p)\) phase space, much like the quantum case. The constant \(\hbar\beta'\), that governness the fuzziness of the “classical” \((q, p)\) phase space, is induced, we believe, by the quantum gravity/string theory effects at the Planck scale. Then

\[
\hbar\beta' \sim \frac{l_p^2}{\hbar} = \frac{G}{c^3}, \tag{67}
\]

and we see that it is expected not to depend on \(\hbar\)! Classical trajectories will be lost even in the hypothetical world with \(\hbar = 0\), provided the Newton constant \(G\) is not zero and the universal velocity \(c\) is not infinity. However, Our troubles with classicality don’t end here. The quantum Hamiltonian \(^{48}\) that corresponds to \(^{64}\) has the form

\[
\mathcal{H} = \frac{1}{m} \left( \hat{p}\hat{P} + m^2\omega^2 \hat{q}\hat{Q} \right), \tag{68}
\]

indicating, according to \(^{65}\), the following equations of motion (in the Heisenberg picture) for “classical” variables \(\hat{q}\) and \(\hat{p}\):

\[
\frac{d\hat{q}}{dt} = i\frac{\hbar}{m} [\mathcal{H}, \hat{q}] = \frac{\hat{p}}{m} \left( 1 + 2\beta' \hat{P}^2 \right) + \beta' m^2\omega^2 \hat{q}(\hat{q}\hat{p} - \hat{Q}\hat{P}), \quad \frac{d\hat{p}}{dt} = i\frac{\hbar}{\hbar} [\mathcal{H}, \hat{p}] = -m^2\omega^2 \hat{q} (1 + \beta' \hat{p}^2) - \beta' m^2\omega^2 \hat{p}\hat{P} \hat{Q}. \tag{69}
\]
As one would expect from the beginning, the equations are modified. What was probably unexpected is that the additional terms depend on the hidden variables \( \hat{Q} \) and \( \hat{P} \).

Analogous equations hold for “hidden” variables \( \hat{Q} \) and \( \hat{P} \):

\[
\frac{d\hat{Q}}{dt} = \frac{i}{\hbar} [\mathcal{H}, \hat{Q}] = \left( 1 + 2\beta' \left( \frac{\hat{P}}{m} - \beta' m \omega^2 (\hat{q}\hat{p} - \hat{Q}\hat{P}) \right) \right) \frac{\hat{P}}{m}, \quad \frac{d\hat{P}}{dt} = \frac{i}{\hbar} [\mathcal{H}, \hat{P}] = -m \omega^2 \left( 1 + \beta' \frac{\hat{P}^2}{m} \right) \hat{Q} - \beta' m \omega^2 \hat{q}\hat{p} \hat{P}. \tag{70}
\]

If the modification considered emerges from the Planck scale effects, the natural scale for new phenomenological constants, like \( \beta' \), is \( \beta' \sim l_P^2/\hbar^2 = 1/p_P^2 \), where \( p_P \) is the Planck momentum. Therefore the correction terms in (65) and (69) are significant only when the momenta involved are of the order of Planck momentum. Let us suppose that this is indeed so for \((q, \dot{p})\) classical sector related momenta, while the hidden \((\hat{Q}, \dot{\hat{P}})\) sector related momenta for some reason remains much smaller, so that we can discard hidden variables \((\hat{Q}, \dot{\hat{P}})\) in (65) and (69). Then we still regain the classical sector (\(\dot{q} \) and \(\dot{p} \) will commute in this approximation), but the classical equations of motion will be modified (as \(\dot{\hat{q}} \) and \(\dot{\hat{p}} \) do commute, we write equations of motion for \(q \) and \(p \) considering them as real numbers, not operators):

\[
\frac{dq}{dt} = (1 + \beta' m^2 \omega^2 q^2) \frac{\dot{p}}{m}, \quad \frac{dp}{dt} = -m \omega^2 \left( 1 + \beta' p^2 \right) q. \tag{71}
\]

Equations (71) are non-linear oscillator equations of the type introduced in [104, 105] that model generalized one-dimensional harmonic oscillators in several important dynamical systems:

\[
\frac{dq}{dt} = f(q) p, \quad \frac{dp}{dt} = -g(p) q, \tag{72}
\]

where \(f(q)\) and \(g(p)\) functions, with the conditions \(f(0) > 0, g(0) > 0\), are assumed to be continuous with continuous first derivatives.

On the other hand, (71) can be rewritten as an second order differential equation and the result is

\[
\ddot{q} - \frac{\beta' m^2 \omega^2 q}{1 + \beta' m^2 \omega^2 q^2} \dot{q}^2 + \omega^2 (1 + \beta' m^2 \omega^2 q^2) q = 0. \tag{73}
\]

This equation is of the type of quadratic Liénard equation. The general quadratic Liénard equation, used in a vast range of applications, has the form

\[
\ddot{q} + f(q) \dot{q}^2 + g(q) = 0, \tag{74}
\]

where \(f(q)\) and \(g(q)\) are arbitrary functions that do not vanish simultaneously [106, 107].

The equation (73) (and in general the system (72) [104]) admits a first integral which can be found as follows. From (71) we have

\[
\frac{dp}{dq} = \frac{m^2 \omega^2 (1 + \beta' p^2) q}{(1 + \beta' m^2 \omega^2 q^2) p}. \tag{75}
\]

The variables \(q\) and \(p\) in this differential equation can be separated and we get after the integration

\[
\ln (1 + \beta' p^2) + \ln (1 + \beta' m^2 \omega^2 q^2) = \text{constant},
\]

which implies that

\[
(1 + \beta' p^2)(1 + \beta' m^2 \omega^2 q^2) = 1 + \beta' A, \tag{76}
\]

where \(A\) is some constant. For definiteness, let us assume the following initial values

\[
q(0) = 0, \quad p(0) = p_0 > 0. \tag{77}
\]

Then \(A = \frac{p_0^2}{2}\).

Another interesting property of the equation (73) is that it corresponds to a Lagrangian system with a position dependent mass (in fact, any quadratic Liénard equation has such a property [108]). Indeed, the Lagrangian

\[
\mathcal{L} = \frac{1}{2} m(q) \dot{q}^2 - V(q) \tag{78}
\]
leads to the Euler-Lagrange equation

$$\ddot{q} + \frac{\mu'}{2\mu} \dot{q}^2 + \frac{V'}{\mu} = 0.$$  

Here the prime denotes differentiation with respect to $q$. Comparing with (73), we get the identifications

$$\frac{\mu'}{2\mu} = -\frac{\beta' m^2 \omega^2 q}{1 + \beta' m^2 \omega^2 q^2}, \quad V' = \mu \omega^2 q (1 + \beta' m^2 \omega^2 q^2). \quad (79)$$

These differential equations can be integrated and we get

$$\mu(q) = \frac{m}{1 + \beta' m^2 \omega^2 q^2}, \quad V(q) = \frac{1}{2} m \omega^2 q^2. \quad (80)$$

Comparing the conserved energy $E = \frac{p^2}{2\mu} + V$ with (76), we see that the first integral (76) represents just the energy conservation and $A = 2mE$.

Thanks to the first integral (76), the equation of motion (73) of the Planck scale deformed harmonic oscillator can be solved in a quadrature. Namely, from (76), assuming (77) initial conditions, we get

$$p = \sqrt{p_0^2 - \frac{m^2 \omega^2 q^2}{1 + \beta' m^2 \omega^2 q^2}}.$$  

On the other hand, it follows from the first equation of (71) that

$$p = \frac{m \dot{q}}{1 + \beta' m^2 \omega^2 q^2}.$$  

Combining these two expressions of $p$, we get

$$m \frac{dq}{dt} = \sqrt{(p_0^2 - m^2 \omega^2 q^2)(1 + \beta' m^2 \omega^2 q^2)}.$$  

Introducing a new variable $z = m \omega q / p_0$, we can integrate (81) as follows:

$$\omega t = \int_0^{m \omega q / p_0} \frac{dz}{\sqrt{(1 - z^2)(1 + \beta' p_0^2 z^2)}} = \int_0^u \frac{d\theta}{\sqrt{1 + \beta' p_0^2 \sin^2 \theta}}. \quad (82)$$

where $\sin u = m \omega q / p_0$ and at the last step we have made another change of the integration variable, namely $z = \sin \theta$. The above integral is the incomplete elliptic integral of the first kind. Its amplitude $u$ satisfies the equation $\sin u = sn (\omega t, i \sqrt{\beta'} p_0)$, where $sn (\omega t, i \sqrt{\beta'} p_0)$ is the Jacobi sine elliptic function with the imaginary modulus (assuming $\beta' > 0$). Therefore

$$q(t) = \frac{p_0}{m \omega} \sn (\omega t, i \sqrt{\beta'} p_0) = \frac{p_0}{m \omega \sqrt{1 + \beta' p_0^2}} \sd \left( \sqrt{1 + \beta' p_0^2} \omega t, \sqrt{\frac{\beta'}{1 + \beta' p_0^2}} \right), \quad (83)$$

where at the last step we have used imaginary modulus transformation [109]. Period of oscillations $T$, according to (82), is given by the relation

$$\frac{\omega T}{4} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \beta' p_0^2 \sin^2 \theta}} = K(i \sqrt{\beta'}), \quad (84)$$

where $K$ is the complete elliptic integral of the first kind. Using again the imaginary modulus transformation, we get

$$T = \frac{4}{\omega \sqrt{1 + \beta' p_0^2}} K \left( \sqrt{\frac{\beta' p_0^2}{1 + \beta' p_0^2}} \right) \approx \frac{2\pi}{\omega} \left( 1 - \frac{\beta' p_0^2}{4} \right). \quad (85)$$
This reduction of the period of oscillations is similar to what was found in [102] within the framework of the one-dimensional Kempf modification of the commutation relations (a one-dimensional version of [15]) with the standard recipe of replacing commutators by Poisson brackets when considering a classical limit.

Let us also consider, as a second example, Kempf et al. modification of the commutation relations [15] with \( \beta' = 2\beta \), so that the spatial geometry remains approximately commutative (at the first order in \( \beta \)). Then we will have (for our two-dimensional quantum system the square of the momentum vector is \( p^2 + \vec{P}^2 \))

\[
[q, \vec{P}] = i\hbar [1 + \beta(p^2 + 3\vec{P}^2)], \quad [q, \vec{p}] = [\vec{Q}, \vec{P}] = i\hbar 2\beta\vec{p}\vec{P}, \quad [\vec{Q}, \vec{p}] = i\hbar [1 + \beta(3p^2 + \vec{P}^2)], \quad [\vec{q}, \vec{Q}] = 0, \quad [\vec{p}, \vec{P}] = 0. \quad (86)
\]

Again \( q \) and \( p \) cease to be commuting. Then corresponding equations of motion are

\[
\frac{dq}{dt} = \frac{\dot{p}}{m} \left[ 1 + \beta \left( p^2 + 5\vec{P}^2 \right) \right], \quad \frac{dp}{dt} = -m\omega^2 \left\{ \dot{q} \left[ 1 + \beta(3p^2 + \vec{P}^2) \right] + 2\beta \dot{\vec{p}} \vec{P} \vec{Q} \right\}, \quad (87)
\]

and

\[
\frac{d\vec{Q}}{dt} = \left[ 1 + \beta \left( 5\vec{p}^2 + \vec{P}^2 \right) \right] \vec{P}, \quad \frac{d\vec{P}}{dt} = -m\omega^2 \left\{ \left[ 1 + \beta \left( \vec{p}^2 + 3\vec{P}^2 \right) \right] \vec{Q} + 2\beta \dot{\vec{q}} \vec{P} \right\}. \quad (88)
\]

As in the previous example, hidden variables \( P \) and \( Q \) do appear in the equations of motions of the “classical” sector due to the Plank scale modification of the commutation relations.

In situations when the effects of the hidden variables \( P \) and \( Q \) can be approximately discarded, the classical equations of motion became

\[
\dot{q} = (1 + \beta p^2) \frac{p}{m}, \quad \dot{p} = -m\omega^2 (1 + 3\beta p^2) q. \quad (89)
\]

This system is no longer of the type (72). Nevertheless, it gives a second order differential equation for the variable \( p \) which is of the quadratic Liénard type:

\[
\dot{p} - \frac{6\beta p}{1 + 3\beta p^2} p^2 + \omega^2 (1 + \beta p^2) (1 + 3\beta p^2) p = 0. \quad (90)
\]

Variable mass system (in the \( p \)-space) with the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \mu(p) p^2 - V(p),
\]

which is equivalent to (91), is characterized by

\[
\mu(p) = \frac{m}{(1 + 3\beta p^2)^2}, \quad V(p) = \frac{m\omega^2}{6} \left[ p^2 + \frac{2}{3\beta} \ln \left( 1 + 3\beta p^2 \right) \right]. \quad (91)
\]

Of cause the integration constant in the potential \( V(p) \) is irrelevant and it was chosen in such a way that when \( \beta = 0 \) the Lagrangian \( \mathcal{L} \) becomes the ordinary harmonic oscillator Lagrangian. The corresponding conserved “energy” \( \frac{1}{2} \mu(p) p^2 + V(p) \) gives a first integral

\[
\frac{1}{2} \frac{m}{(1 + 3\beta p^2)^2} p^2 \left[ p^2 + \frac{2}{3\beta} \ln \left( 1 + 3\beta p^2 \right) \right] = m^2 \omega^2 E, \quad (92)
\]

where \( E \) is some constant.

Period of oscillations that follows from (92) is

\[
T = 4 \int_0^{p_0} \frac{dp}{\sqrt{\frac{2}{m} m^2 \omega^2 E - V(p)}} = \frac{4}{\omega} \int_0^{p_0} \frac{dp}{\sqrt{(1 + 3\beta p^2) \sqrt{2mE - \frac{2}{m} \left[ p^2 + \frac{2}{3\beta} \ln \left( 1 + 3\beta p^2 \right) \right]}}} \quad (93)
\]

At the first order in \( \beta \), and assuming \( \dot{p}(0) = 0, p(0) = p_0 \) initial conditions, we have \( E = \frac{p_0^2}{2m}(1 - \beta p_0^2) \) and

\[
T \approx \frac{4}{\omega} \int_0^{p_0} \frac{dp}{(1 + 3\beta p^2) \sqrt{(p_0^2 - p^2)(1 - \beta(p_0^2 + p^2))}} \approx \frac{4}{\omega} \int_0^{p_0} \frac{dp}{\sqrt{p_0^2 - p^2}} \left[ 1 - \frac{\beta}{2} \frac{5p^2 - p_0^2}{p_0^2} \right] = \frac{2\pi}{\omega} \left( 1 - \frac{3\beta p_0^2}{4} \right). \quad (94)
\]

At the final step, we have used elementary integrals

\[
\int_0^{p_0} \frac{dp}{\sqrt{p_0^2 - p^2}} = \frac{\pi}{2}, \quad \int_0^{p_0} \frac{p^2 - p^2 dp}{\sqrt{p_0^2 - p^2}} = \frac{\pi p_0^2}{4}.
\]
CONCLUDING REMARKS

In this note we have tried to combine Koopman-von Neumann-Sudarshan perspective on classical mechanics with the generalized uncertainty principle. We have considered two versions of the generalized commutation relations. The results were similar: classical position and momentum operators cease to be commuting and hidden variables show themselves explicitly in classical evolution equations. In situations then the effect of these hidden variables can be neglected in evolution equations, the modification of classical dynamics is similar (but not identical) to the modification obtained by using more traditional approach of replacement of commutators by Poisson brackets.

We suspect that the above mentioned features are common for a large class of generalized uncertainty principle based models if they are interpreted in the Koopman-von Neumann-Sudarshan framework. Therefore, from this perspective, we can conclude that Planck scale quantum gravity effects destroy classicality. However this breakdown of classicality is controlled by a small dynamical parameter $\frac{\hbar^2}{\hbar^2}$ and can be neglected for all practical purposes thanks to the huge hierarchy between the masses of ordinary particles and the Planck mass $m_P = 1.2 \times 10^{19}$ GeV/$c^2$. Usually this huge hierarchy is considered as a problem to be explained [111]. As we see, for classicality it can be beneficial. For a macroscopic body the effective deformation parameter $\beta$ is approximately $N^2$ times smaller than for its elementary constituents, where $N$ is the number of constituents [101]. Therefore macroscopic bodies, notwithstanding their large momenta, provide no advantage in observing Planck scale induced non-classical effects, as the small parameter controlling these non-classical effects for macroscopic bodies becomes $\frac{\hbar^2}{N^2 \hbar^2}$.

It may happen that the interrelations between quantum mechanics, classical mechanics and gravity are much more tight and intimate than anticipated. The imprints left by quantum mechanics in classical mechanics are more numerous than is usually believed [112, 113]. In fact the mathematical structure that allows quantum mechanics to emerge already exists in classical mechanics [114]. Particularly surprising, maybe, is that Schrödinger-Robertston uncertainty principle has an exact counterpart in classical mechanics which can be formulated using some subtle developments in symplectic topology, namely Gromov’s non-squeezing theorem and the related notion of symplectic capacity [115].

On the other hand there are unexpected and deep relations between gravity and quantum mechanics, in particular between Einstein-Rosen wormholes and quantum entanglement [116, 117].

We believe the Koopman-von Neumann formulation of classical mechanics might be useful in investigating a twilight zone between quantum and classical mechanics. “It deserves to be better known among physicists, because it gives a new perspective on the conceptual foundations of quantum theory, and it may suggest new kinds of approximations and even new kinds of theories” [82].

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