QUANTUM SU(2)-INVARIANTS FOR THREE-MANIFOLDS ASSOCIATED WITH NON-TRIVIAL COHOMOLOGY CLASSES MODULO TWO

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ABSTRACT. We show an integrality of the quantum SU(2)-invariant associated with a non-trivial first cohomology class modulo two.

For a given Lie group $G$ and an integer (level) $k$, E. Witten gave an ‘invariant’ of three-manifolds by using the path integral $\[33\]$. Mathematically rigorous proof of its existence was given by several people. See for example $\[26, 7, 4, 32, 29\]$. There are also some refinements for these invariants corresponding to various structures of the three-manifolds, such as cohomology classes and spin structures $\[30, 7, 9, 18, 3, 2\]$.

Some topological properties of the invariants were given especially for $G = SU(2)$. The first striking result was found by R. Kirby and P. Melvin stating that the quantum SU(2)-invariant with level two splits into the sum of the invariants associated with spin structures and each summand can be described in terms of the $\mu$-invariant. They and X. Zhang obtained a topological interpretation for the quantum SU(2)-invariant with level four $\[8\]$. In this case the invariant splits into the sum of the invariants associated with the first cohomology classes modulo two and each summand can be described in terms of the Wit invariant and the cohomologies of both the three-manifold and its double cover defined by the cohomology class. Similarly the author gave a topological interpretation for the quantum SU(3)-invariant with level three $\[18\]$. It also decomposes into the sum of the invariants associated with the first cohomology classes modulo three and each summand can be described in terms of the cohomologies of the manifold and its triple cover.

If $G = SU(n)$ and $n$ and the level $k$ are coprime, then the invariant splits into the product of the $U(1)$-invariant $\[22, 21\]$ and $PSU(n)$-invariant $\[1, 10\]$. If $n + k$ is odd prime and the three-manifold is a rational homology sphere, the invariant is a cyclotomic integer $\[17, 19, 15, 14, 23\]$. Similar integrality holds for a three-manifold with positive first Betti number. Moreover one can define the perturbative invariant from the $PSU(n)$-invariant for rational homology three-spheres $\[22, 24\]$. Recently it is proved by T.T.Q. Le that every coefficient of the perturbative invariant for integral homology three-spheres is of finite type in T. Ohtsuki’s sense $\[28\]$. (For $n = 2$ case, it is proved in by A. Kricker and B. Spence $\[11\]$). In particular its first non-trivial coefficient coincides with (a multiple of) the Casson invariant $\[11\]$. (For rational homology three-spheres, Le showed in $\[22\]$ that this coincides with a...

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multiple of the Casson-Walker invariant \[\frac{2}{3}\]. See also [23] for recent developments of these invariants; especially their relation with the LMO invariant [3].

In this paper we consider the quantum $SU(2)$-invariant with level divisible by four. By V. Turaev, and Kirby and Melvin [8, 30] it decomposes into the sum of the invariants associated with the cohomology classes modulo two. We will consider the case where $(n + 2)/p$ is an odd prime.

Let $p$ be an odd prime and $M$ a $\mathbb{Z}/p\mathbb{Z}$-homology three-sphere. For a cohomology class $\theta \in H^1(M; \mathbb{Z}/2\mathbb{Z})$, let $\tau_{2p}^{SU(2)}(M, \theta)$ be the quantum $SU(2)$-invariant with level $2p - 2$ associated with the $2p$th root of unity $q = \exp(\pi\sqrt{-1}/p)$. In the previous paper [20] the author proved

**Theorem 1.** $\tau_{2p}^{SU(2)}(M, 0) \in \mathbb{Z}[1/2, \xi]$ and the ‘coefficient’ of $(\xi - 1)$ is congruent to the Casson-Walker invariant modulo $p$, where $\xi = \exp(2\pi\sqrt{-1}/p)$.

In this paper we will show a weaker result for non-trivial cohomology classes. We will show

**Theorem 2.**

$$\tau_{2p}^{SU(2)}(M, \theta) \in \begin{cases} (\xi - 1)\mathbb{Z}[1/2, \xi] & \text{if } \theta \cup \theta \cup \theta = 0, \\ \sqrt{-1}(\xi - 1)\mathbb{Z}[1/2, \xi] & \text{if } \theta \cup \theta \cup \theta \neq 0. \end{cases}$$

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1. PROOF

We put $p = 4u + \varepsilon$ with $\varepsilon = \pm 1$. Put $\xi = \exp(2\pi\sqrt{-1}/p)$, $q = \exp(\pi\sqrt{-1}/p)$ and $s = \exp(\pi\sqrt{-1}/2p)$. Therefore $\xi = q^2 = s^4$.

Suppose that a closed three-manifold $M$ is given by an algebraically split, framed link $\mathcal{L}$ with $m$ components. We also assume that the framing of $\mathcal{L}$ is $(f_1, f_2, \ldots, f_m)$ with $f_i \neq 0 \mod p$. Moreover we assume that $f_i \equiv 0 \mod 2$ for $i \leq b$ and $f_i \equiv 1 \mod 2$ for $b < i \leq m$. Let $\theta \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ be presented by the sublink $\mathcal{L}_\theta = L_1 \cup L_2 \cup \ldots \cup L_b$. We put $f_i = 2g_i$ for $1 \leq i \leq b$.

By [8] (8.32) Theorem, $\tau_{2p}^{SU(2)}(M, \theta) = \alpha(\mathcal{L})\Sigma(\mathcal{L})$ where $\alpha(\mathcal{L})$ and $\Sigma(\mathcal{L})$ is given as follows.

$$\alpha(\mathcal{L}) = \left(\frac{\pi}{p}\sin\left(\frac{\pi}{2p}\right)\right)^m \exp\left(-\frac{3(p - 1)}{8p} \times 2\pi\sqrt{-1}\right)^{\sigma(\mathcal{L})}$$

$$\Sigma(\mathcal{L}) = \sum_{0 < k_i \leq p, \mathcal{L}_k = \mathcal{L}_\theta} 2^{T_k}[k]J(\mathcal{L}, k),$$

where $J(\mathcal{L}, k)$ is the colored Jones polynomial of $\mathcal{L}$ with color $k = (k_1, k_2, \ldots, k_m)$ ($0 < k_i \leq p$), $\mathcal{L}_k$ is the sublink of $\mathcal{L}$ consisting of components with the color $k_i$ even, $|T_k|$ is the number of $k_i$ which are less than $p$ and $\sigma(\mathcal{L})$ is the number of positives minus that of negatives in $\{f_1, f_2, \ldots, f_m\}$.

After some calculations we can simplify $\alpha(\mathcal{L})$ and $\Sigma(\mathcal{L})$ as follows.

**Lemma 1.1** (Lemma 2.2 in [23]).

$$\alpha(\mathcal{L}) = \frac{(\xi^u + \xi^{-u})^m}{(2G(\xi))^{m/2}}(-1)^{\mu(\mathcal{L}) + \mu_{\Sigma}(\mathcal{L})/2} \xi^{3\times\bar{\Theta}_0}(\mathcal{L}),$$

where $G(\xi)$ is the Euler polynomial of the link $\mathcal{L}$.
where $G(\xi)$ is a Gaussian sum $\sum_{k=0}^{p-1} k^2 \xi^k$, $\overline{8}$ is the inverse of 8 in $\mathbb{Z}/p\mathbb{Z}$ and $\sigma_-(L)$ is the number of negatives in $\{f_1, f_2, \ldots, f_m\}$. Note that $G(\xi)$ is of the form $\gamma(\xi - 1)^{(p-1)/2}$ with $\gamma$ invertible in $\mathbb{Z}[\xi]$.

**Lemma 1.2** (see Lemma 2.3 in [20]).

\[ \Sigma(L) = 2^c \sum_{n_1, n_2, \ldots, n_m = 1}^{(p-1)/2} (-1)^{n_1} V(L^{(2n_1-1,2n_2-1,\ldots,2n_c-1,2n_{c+1},2n_{c+2},\ldots,2n_m)}) \times \prod_{i=1}^{c} C_{n_i}(g_i) \prod_{i=c+1}^{m} S_{n_i}(f_i) \]

with

\[ S_n(f) = \sum_{k=n}^{p-n-1} (-1)^k [2k + 1] \xi^{(k^2+k)/2} \binom{k+n}{k-n} \]

and

\[ C_n(g) = \sum_{k=n}^{(p-1)/2} (-1)^k [2k] \xi^{(4k^2-1)/2} \binom{k+n-1}{k-n} \]

where $L^{(d_1,d_2,\ldots,d_m)}$ is the (unframed) link obtained from $L$ by replacing its component with $d_i$ parallels respecting the zero-framing. For a link $L$, $V(L)$ is the Jones polynomial defined by the skein relation $qV(L_{+}) - q^{-1}V(L_{-}) = (s - s^{-1})V(L_0)$ for the usual skein triple $(L_{+}, L_{-}, L_0)$ and by $V(\text{trivial knot}) = 1$.

**Proof.** The proof is similar to [20, Lemma 2.3] and so details are omitted. \qed

To prove Theorem 2, we prepare three lemmas.

**Lemma 1.3** (Lemma 2.4 in [20]). $S_n(f) \in \mathbb{Z}[\xi]$ is divisible by $(\xi - 1)^{(p-1)/2-n}$ in $\mathbb{Z}[\xi]$.

**Lemma 1.4.** $\sqrt{-1}^{p+1}C_n(g) \in \mathbb{Z}[\xi]$ is divisible by $(\xi - 1)^{(p-1)/2-n+1}$ in $\mathbb{Z}[\xi]$.

**Proof.** First note that $s = \sqrt{-1} \xi^{-n}$. Therefore an easily calculation shows that

\[ C_n(g) = \frac{\sqrt{-1} \xi^{(g+1)} \xi} {\xi^{-u} + \xi^{-u}} \sum_{k=n}^{(p-1)/2} \xi^{-2\epsilon u k} \xi^{2\epsilon u k} \xi^{g k^2} \binom{k+n-1}{k-n} \]

Now the proof follows from [14, Lemma 6.2] (see also [22]). \qed

The following lemma is a generalization of a result of M. Sakuma [27, Theorems 1 and 2], who proves the case where $c = 0$.

**Lemma 1.5.** $\sqrt{-1}^{p+1} V(L^{(2n_1-1,2n_2-1,\ldots,2n_c-1,2n_{c+1},2n_{c+2},\ldots,2n_m)}) \in \mathbb{Z}[\xi]$ is divisible by $(\xi - 1)^{\sum_{i=1}^{c} n_i - c}$.

**Proof.** We will show more generally that for a link $L = L_1 \cup L_2$, $V(L_1 \cup L_2)$ is divisible by $(\xi - 1)^{z(L_2)}$, where $L_2^*$ is the two-parallel of $L_2$ with respect to the zero-framing and $z(L_2)$ is the number of components in $L_2$. 

We will use L.H. Kauffman’s bracket polynomial \( \langle L \rangle \) for a framed link \( L \) defined by
\[
\langle \begin{array}{c}
\circlearrowleft \\
\circlearrowright
\end{array} \rangle = A \langle \begin{array}{c}
\circlearrowleft \\
\circlearrowright
\end{array} \rangle + A^{-1} \langle \begin{array}{c}
\circlearrowright \\
\circlearrowleft
\end{array} \rangle
\]
and
\[
\langle O \rangle = 1,
\]
where \( O \) is the trivial knot with zero-framing. Since
\[
V(L) = (-1)^{s(L)+1} \langle L \rangle \bigg|_{A = s^{-1/2}},
\]
we only have to prove that \( \langle L_1 \cup L_2 \rangle \) is divisible by \( (A^2 + A^{-2})^{\sharp(L_2)} \) (note that \( s + s^{-1} = \sqrt{-1} (\xi^{-4c_{\nu}} - \xi^{4c_{\nu}}) \)). Here \( L = L_1 \cup L_2 \) is the zero-framed link obtained from \( L = L_1 \cup L_2 \).

By simple calculations we have
\[
\langle \begin{array}{c}
\circlearrowright \\
\circlearrowleft \\
\circlearrowright
\end{array} \rangle - \langle \begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\circlearrowleft
\end{array} \rangle = (A^2 - A^{-2})(A^2 + A^{-2}) \left\{ \langle \begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\circlearrowleft
\end{array} \rangle - \langle \begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\circlearrowleft
\end{array} \rangle \right\},
\]
where dotted lines indicate the connectivity. By induction on \( \sharp(L_2) \), we see that the divisibility of \( \langle L_1 \cup L_2 \rangle \) does not change by double crossing change between a single strand and a double strand. This shows that the divisibility of \( \langle L_1 \cup L_2 \rangle \) is greater than or equal to that of \( \langle L_1 \rangle \langle L_2 \rangle \) which is greater than or equal to that of \( \langle L_2 \rangle \), which is greater than or equal to \( \sharp(L_2) \) by [20, Lemma 4.1]. This completes the proof. \( \square \)

**Proof of Theorem 3.** We only prove the case where \( M \) is obtained from an algebraically split, framed link since the general case follows from T. Ohtsuki’s diagonalizing lemma [24, Lemma 2.4].

Now we assume that \( \theta \in H^1(M; \mathbb{Z}/2\mathbb{Z}) \) is given as described before. From the lemmas above, we easily see that \( \sqrt{-1} \sum_{i=1}^c g_i \alpha(L) \Sigma(L) \) belongs to \( \mathbb{Z}[1/2, \xi] \) and divisible by \( (\xi - 1) \) (Note that \( \sqrt{-1} [2] \in \mathbb{Z}[\xi] \) and divisible by \( (\xi - 1) \) and that we need \( 1/2 \) from \( \alpha(L) \)).

Now from [3], \( \theta \cup \theta \cup \theta \equiv \sum_{i=1}^c g_i \mod 2. \) Therefore \( \tau^{SU(2)}_{2p}(M, \theta) \in \mathbb{Z}[\xi] \) if \( \theta \cup \theta \cup \theta \equiv 0 \mod 2 \) and \( \sqrt{-1} \mathbb{Z}[\xi] \) otherwise, completing the proof. \( \square \)

**References**

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