Necessity of Hyperbolic Absolute Risk Aversion for the Concavity of Consumption Functions

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Abstract

Carroll and Kimball (1996) have shown that, in the class of utility functions that are strictly increasing, strictly concave, and have nonnegative third derivatives, hyperbolic absolute risk aversion (HARA) is sufficient for the concavity of consumption functions in general consumption-saving problems. This paper shows that HARA is necessary, implying the concavity of consumption is not a robust prediction outside the HARA class.

Keywords: concavity, consumption function, hyperbolic absolute risk aversion, robust predictions.

JEL codes: C65, D11, D14.

1 Introduction

The notion that the marginal propensity to consume decreases with wealth, or that the consumption function is concave, dates back at least to Keynes (1936). This observation is important in macroeconomics because the effect of a fiscal transfer of one dollar to a wealthy household is smaller than that to a poor household, implying that fiscal policies need to account for household heterogeneity. In an important contribution, Carroll and Kimball (1996) have shown that in the class of utility functions that are strictly increasing, are strictly concave, and have nonnegative third derivatives, hyperbolic absolute risk aversion (HARA) is sufficient for the concavity of consumption functions in finite-horizon consumption-saving problems without liquidity constraints. Their result has been extended in several directions: Carroll and Kimball (2001, Section 5) obtain concavity under finite-horizon, HARA utility, and liquidity constraints; Nishiyama and Kato (2012) obtain concavity under infinite-horizon, quadratic utility, and liquidity constraints; Ma et al. (2020, Proposition 2.5 and Remark 2.1) obtain concavity under infinite horizon, constant relative risk aversion (CRRA) utility, and liquidity constraints.1

In all of these theoretical papers, the utility function is restricted to be HARA. Thus a natural question is whether it is possible to obtain the concavity

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1It is clear from the proof technique of Ma et al. (2020) that concavity obtains in finite- or infinite-horizon, with or without liquidity constraints, and HARA utility if the domain of the utility function is appropriately modified. See Section 3 for more discussion.
of consumption functions under weaker assumptions: does concavity hold in a larger class than HARA, or is concavity a non-robust prediction that fails to hold outside the HARA class? This paper provides a definitive answer to this question, which is the latter. More precisely, I show that if the utility function is strictly increasing, is strictly concave, has a positive third derivative, but is not HARA, then there exists a finite-horizon (in fact, one period) consumption-saving problem such that the consumption function is not concave. Combined with the earlier results on the concavity of consumption functions just cited, my result shows that in the class of natural utility functions, HARA is both necessary and sufficient for the concavity of consumption functions in general consumption-saving problems.

2 Main result

Following Carroll and Kimball (1996), I consider a general consumption-saving problem with a finite horizon. The problem can be informally described as follows. The agent is endowed with initial wealth $w_0 > 0$ and lives for $T$ periods. The period utility function is $u : (0, \infty) \to \mathbb{R}$, which is strictly increasing and concave. The agent receives income $Y_t \geq 0$ at the beginning of time $t$. The gross return on wealth between time $t-1$ and $t$ is $R_t > 0$. The agent discounts utility between time $t-1$ and $t$ using the discount factor $\beta_t > 0$, where $\beta_0 \equiv 1$. Letting $c_t > 0$ be the consumption at time $t$ and $w_t \geq 0$ the financial wealth at the beginning of time $t$, the agent’s objective is to solve

$$\max_{c_t} \mathbb{E}_0 \sum_{t=0}^{T} \left( \prod_{s=0}^{t-1} \beta_s \right) u(c_t)$$

subject to

$$w_{t+1} = R_{t+1}(w_t - c_t) + Y_{t+1} \geq 0,$$

where the initial wealth $w_0 = w > 0$ is given and (1b) is the budget and borrowing constraint. The discount factor $\beta_t$, return on wealth $R_t$, and income $Y_t$ can all be stochastic.

If a solution to the consumption-saving problem (1) exists, the time $t$ consumption $c_t$ can be viewed as a function of the current wealth $w_t$, which we call the consumption function. The main result of this paper is that if the utility function satisfies some regularity conditions and the consumption function is always concave regardless of the specification of the process $(\beta_t, R_t, Y_t)_{t=1}^{T}$, then $u$ must exhibit hyperbolic absolute risk aversion (HARA).

We now make this statement more precise. If the consumption function is concave regardless of the specification of the process $(\beta_t, R_t, Y_t)_{t=1}^{T}$, then in particular the consumption function in a one period problem ($T = 1$) must always be concave. We can then rewrite (1) as the static maximization problem

$$\max_c u(c) + \mathbb{E}[\beta u(R(w - c) + Y)]$$

where $\beta, R, Y > 0$ are arbitrary random variables and $c$ satisfies $R(w-c)+Y \geq 0$. Furthermore, if $\beta, R, Y$ are arbitrary, in particular we may restrict attention to problems in which $\beta, R, Y$ take finitely many values. In this case the expectation in (2) is a finite sum and always well-defined.
The following lemma shows that under standard monotonicity, concavity, and Inada conditions, the consumption and saving functions can be unambiguously defined, which are differentiable and strictly increasing.

**Lemma 1.** Suppose that (i) the utility function $u : [0, \infty) \to \mathbb{R} \cup \{ -\infty \}$ is twice continuously differentiable on $(0, \infty)$ and satisfies $u' > 0$, $u'' < 0$, and $\lim_{x \to 0^+} u'(x) = \infty$, and (ii) the random vector $(\beta, R, Y)$ has finite support. Then for any initial wealth $w > 0$, the consumption-saving problem (2) has a unique solution, and the consumption function $c = c(w)$ and saving function $s(w) = w - c(w)$ satisfy the Euler equation

$$u'(c(w)) = E[\beta Ru'(Rs(w) + Y)].$$

Furthermore, $c, s$ are continuously differentiable and $c'(w), s'(w) \in (0, 1)$.

In what follows, we maintain the assumptions of Lemma 1. Since the saving function $s$ is continuously differentiable and strictly increasing, its range $S := s((0, \infty))$ is an open interval of $\mathbb{R}$. Since $u'$ is strictly decreasing, by (3) for each $s \in S$ there exists a unique $c \in (0, \infty)$ such that $u'(c) = E[\beta Ru'(Rs + Y)]$. Therefore we can unambiguously define the function $g : S \to (0, \infty)$ by

$$g(s) := (u')^{-1}(E[\beta Ru'(Rs + Y)]).$$

The function $g$ in (4) returns the consumption level $c > 0$ that implies the saving $s \in S$. Since $u'$ is strictly decreasing, $g$ is clearly strictly increasing. Ma et al. (2020, Proposition 2.5) show that the concavity of $g$ is sufficient for the concavity of the consumption function. The following lemma shows that the concavity of $g$ is actually necessary.

**Lemma 2.** If $c$ is concave, then so is $g$ in (4).

The following lemma, which is closely related to Hardy et al. (1952, Section 3.16), plays a crucial role in the proof of the main result.

**Lemma 3.** Let $I \subseteq \mathbb{R}$ be an open interval and $\phi : I \to \mathbb{R}$ be a twice differentiable function such that $\phi(I) = (0, \infty)$, $\phi' < 0$, and either $\phi'' > 0$ on $I$ or $\phi'' \equiv 0$ on $I$. For $p, x, v \in \mathbb{R}^N$ with $p, v \geq 0$ and $x \in I^N$, let

$$g(s; p, x, v) := \phi^{-1}\left(\sum_{n=1}^{N} p_n \phi(x_n + v_n s)\right),$$

which is well-defined in the neighborhood of $s = 0$. Then $g$ is concave in $s$ for arbitrary $p, x, v$ if and only if $\phi$ and $I$ take one of the following forms:

- $\phi(x) = c(ax + b)^{-1/a}$, $-1 < a < 0$, $I = (-\infty, -b/a)$,
- $\phi(x) = -b e^{-x/b}$, $b > 0$, $I = \mathbb{R}$, or
- $\phi(x) = c(ax + b)^{-1/a}$, $a > 0$, $I = (-b/a, \infty)$,

where $c > 0$ is arbitrary.

Note that in either case in Lemma 3, log-differentiating $\phi$, we obtain

$$\frac{\phi'(x)}{\phi(x)} = -\frac{1}{ax + b},$$

where $a > -1$ with $I = \{ x \in \mathbb{R} : ax + b > 0 \}$. We can now state the main result.
**Theorem 4.** Suppose that (i) the utility function \( u : [0, \infty) \to \mathbb{R} \cup \{-\infty\} \) is three times differentiable on \((0, \infty)\) and satisfies \( u' > 0, \ u'' < 0, \ u''' > 0, \ \lim_{x \to 0^+} u'(x) = \infty, \) and \( \lim_{x \to \infty} u'(x) = 0, \) and (ii) the random vector \((\beta, R, Y)\) has finite support. If the consumption function in Lemma 1 is concave for arbitrary distribution of \((\beta, R, Y)\), then \( u \) exhibits constant relative risk aversion (CRRA). Conversely, if \( u \) is CRRA, then the consumption function of the consumption-saving problem (1) is concave.

**Proof.** Let \( c, s \) be the consumption and saving functions. By Lemma 1, \( c, s \) are continuously differentiable, strictly increasing, and \( S = s((0, \infty)) \) is an open interval of \( \mathbb{R} \). Let us show \( 0 \in S \). If \( 0 \not\in S \), then either \( S \subset (0, \infty) \) or \( S \subset (-\infty, 0) \).

If \( S \subset (0, \infty) \), then \( s(w) > 0 \) for all \( w \). Since \( c(w) + s(w) = w \), we have \( c(w) < w \). Using the Euler equation (3) and \( u'' < 0 \), we obtain

\[ u'(w) < u'(c(w)) = E[\beta Ru'(Rs(w) + Y)] \leq E[\beta Ru'(Y)] < \infty. \]

Letting \( w \downarrow 0 \), we obtain a contradiction because \( u'(0) = \infty \).

If \( S \subset (-\infty, 0) \), then \( s(w) < 0 \) for all \( w \). Since \( c(w) + s(w) = w \), we have \( c(w) > w \). Again by (3) and \( u'' < 0 \), we obtain

\[ u'(w) > u'(c(w)) = E[\beta Ru'(Rs(w) + Y)] \geq E[\beta Ru'(Y)] > 0. \]

Letting \( w \uparrow \infty \), we obtain a contradiction because \( u'(\infty) = 0 \).

The above argument shows that \( 0 \in S \) regardless of the specification of \((\beta, R, Y)\). Since \((\beta, R, Y)\) has finite support, we can write \( g(s) \) in (4) as

\[ g(s) = (u')^{-1} \left( \sum_{n=1}^{N} \pi_n \beta_n R_n u'(R_n s + Y_n) \right), \]

where \( \pi_n > 0 \) is the probability of state \( n \). Since \((\beta_n, R_n, Y_n) \ni 0 \) is arbitrary, we can apply Lemma 3 for \( \phi = u' \), \( I = (0, \infty) \), \( p_n = \pi_n \beta_n R_n \), \( x_n = Y_n \), and \( v_n = R_n \), yielding \( u'(x) = \phi(x) = c(ax + b)^{-1/a} \) for some \( a, c > 0 \) and \( b = 0 \).

Since the multiplicative constant \( c \) does not affect the ordering of utility, we may assume \( u'(x) = x^{-\gamma} \) with \( \gamma = 1/a > 0 \), so \( u \) is CRRA.

The sufficiency of CRRA for the concavity of consumption can be shown by the same argument as in Ma et al. (2020, Proposition 2.5 and Remark 2.1).

## 3 Discussion

Theorem 4 essentially shows that for the consumption functions in general consumption-saving problems to be concave, constant relative risk aversion (CRRA) is necessary and sufficient. This statement may appear to be in conflict with Carroll and Kimball (1996), who have shown that hyperbolic absolute risk aversion (HARA) is sufficient for concavity. However, there is no contradiction between the two results. This is because in Theorem 4, to avoid unnecessary complications, I have restricted the utility function to have domain \((0, \infty)\) and satisfy appropriate Inada conditions. If the domain is changed to \((-b/a, \infty)\), then it is straightforward to adopt the proof of Theorem 4 to show that \( u \) is HARA, as we can see from (5) with \( \phi = u' \). This argument shows that for the
concavity of consumption functions, HARA is necessary and sufficient (among
the class of utility functions with $u' > 0$, $u'' < 0$, and $u''' \geq 0$).

Next, we discuss some subtleties regarding the concavity of consumption
functions in the existing literature.

When describing the consumption-saving problem, Carroll and Kimball (1996)
are unclear about the domain of the utility function, Inada conditions, and bor-
rowing constraints. Since the proof of their Lemma 2 uses first-order condi-
tions (that hold for interior solutions), they implicitly assume that consump-
tion can be any value in the domain of the HARA utility (which trivially satisfies
the Inada conditions) and that the agent can save or borrow freely. In the
consumption-saving problems (1), (2), and Lemma 1, these assumptions were
made explicit. However, note that the possibility of borrowing is used in Theo-
rem 4 to show the necessity of HARA for the concavity of consumption functions;
the possibility of borrowing is not required for the sufficiency of HARA for con-
cavity, and in fact Ma et al. (2020, Remark 2.1) show that HARA is sufficient
even in the presence of borrowing constraints.\footnote{Strictly speaking, Ma et al. (2020, Remark 2.1) concerns CRRA utility, but this is because the domain of the utility function is restricted to $(0, \infty)$. It is straightforward to handle HARA utility by shifting the domain.}

Theorem 4 and the subsequent discussion suggest that we cannot expect the
concavity of consumption functions unless the stochastic process $\{\beta_t, R_t, Y_t\}_{t=1}^T$
is somehow restricted. Indeed, Gong et al. (2012) show that in a finite-horizon,
deterministic consumption-saving problem in which the discount factor $\beta_t$ and
the gross risk-free rate $R_t$ satisfy $\beta_t R_t \geq 1$, the concavity of the absolute risk
tolerance $-u'(x)/u''(x)$ is sufficient for the concavity of consumption functions.
This condition is much weaker than HARA, as the absolute risk tolerance is
affine when $u$ is HARA. However, note that the condition $\beta R \geq 1$ is quite
restrictive, and in fact not satisfied in stationary general equilibrium models
with infinitely-lived agents (see the discussion in the proof of Theorems 8 of
Stachurski and Toda, 2019).

Finally, in the more recent literature, several authors have studied consump-
tion-saving problems with hyperbolic discounting (Cao and Werning, 2018; Morris and Postlewaite, 2020). Although the concavity of consumption functions has not been discussed
in this context, since exponential discounting is a special case of hyperbolic dis-
counting, the result in this paper immediately implies that we cannot expect
concavity unless the utility function is HARA.

A Proof of lemmas

Proof of Lemma 1. Fix $w > 0$. Suppose $(\beta, R, Y)$ takes the value $(\beta_n, R_n, Y_n)$
with probability $\pi_n > 0$, where $n = 1, \ldots, N$. Define

$$\bar{c} = \sup \{ c > 0 : (\forall n) R_n (w - c) + Y_n \geq 0 \} = w + \min_n \frac{Y_n}{R_n} \geq w > 0,$$

which is well-defined because $R_n, Y_n > 0$. Define $f : (0, \bar{c}) \to \mathbb{R}$ as the objective function in (2). Since

$$f'(c) = u'(c) - \mathbb{E}[\beta R u'(R (w - c) + Y)],$$
$$f''(c) = u''(c) + \mathbb{E}[\beta^2 R^2 u''(R (w - c) + Y)] < 0$$
implies the Euler equation (2). Letting \(s(w) = w - c(w)\), the first-order condition \(f'(c(w)) = 0\) implies the Euler equation (3).

To show the continuous differentiability of \(c\), let
\[
F(w, c) = u'(c) - E[\beta Ru'(R(w - c) + Y)].
\]
Since \(u'' < 0\), we obtain
\[
\frac{\partial F}{\partial w} = -E[\beta R^2 u''(R(w - c) + Y)] > 0,
\]
\[
\frac{\partial F}{\partial c} = u''(c) + E[\beta R^2 u''(R(w - c) + Y)] < 0.
\]
By the implicit function theorem, \(c\) is continuously differentiable and
\[
c'(w) = \frac{\partial F/\partial w}{\partial F/\partial c} = \frac{-E[\beta R^2 u''(R(w - c) + Y)]}{-u''(c) - E[\beta R^2 u''(R(w - c) + Y)]} \in (0, 1).
\]
Then \(s'(w) = 1 - c'(w) \in (0, 1)\).

**Proof of Lemma 2.** By the discussion preceding Lemma 2, we have
\[
g(s(w)) = c(w)
\]
for all \(w > 0\). If \(g\) is not concave, we can take \(s_1, s_2 \in S\) and \(\alpha \in [0, 1]\) such that
\[
g((1 - \alpha)s_1 + \alpha s_2) < (1 - \alpha)g(s_1) + \alpha g(s_2).
\]
Let \(w_j = s^{-1}(s_j)\) for \(j = 1, 2\). Then \(s_j = s(w_j)\) and \(g(s_j) = c(w_j)\) by (6). Define \(\bar{c} := (1 - \alpha)c(w_1) + \alpha c(w_2)\) and \(\bar{w} := (1 - \alpha)w_1 + \alpha w_2\). Since by assumption \(c\) is concave, we have
\[
c(\bar{w}) = c((1 - \alpha)w_1 + \alpha w_2) \geq (1 - \alpha)c(w_1) + \alpha c(w_2) = \bar{c}.
\]
Therefore
\[
c(\bar{w}) \geq \bar{c} = (1 - \alpha)c(w_1) + \alpha c(w_2) \quad (\because (8))
\]
\[
= (1 - \alpha)g(s_1) + \alpha g(s_2) > g((1 - \alpha)s_1 + \alpha s_2) \quad (\because (6), (7))
\]
\[
= g((1 - \alpha)(w_1 - c(w_1)) + \alpha(w_2 - c(w_2)))
\]
\[
= g(\bar{w} - c) \geq g(\bar{w} - c(\bar{w})) \quad (\because (8), g increasing)
\]
\[
= g(s(\bar{w})) = c(\bar{w}),
\]
which is a contradiction.

**Proof of Lemma 3.** The proof is similar to Hardy et al. (1952, Section 3.16), who assume \(\phi' > 0\) and the goal is to characterize the convexity of \(g\). Since these authors do not cover all the fine details, I reproduce their argument.

To simplify the notation, write \(g = g(s; p, x, v)\) and \(\sum_i = \sum_{n=1}^{\infty}\). Since \(x_n \in I\) for all \(I\), \(I\) is open, \(\phi(I) = (0, \infty)\), and \(\phi\) is monotonic, \(g\) is well-defined
in a neighborhood of 0. Differentiating both sides of \( \phi(g(s)) = \sum \phi(x_n + v_n s) \) with respect to \( s \) twice, we obtain

\[
\phi'(g) g' = \sum p_n v_n \phi'(x_n + v_n s),
\]
\[
\phi''(g)(g')^2 + \phi'(g) g'' = \sum p_n v_n^2 \phi''(x_n + v_n s).
\]

Eliminating \( g' \), we obtain

\[
\phi'(g)^2 g'' = \phi'(g)^2 \sum p_n v_n^2 \phi''(x_n + v_n s) - \phi''(g) \left( \sum p_n v_n \phi'(x_n + v_n s) \right)^2.
\]

Since by assumption \( \phi' < 0 \), we have \( g''(s) \leq 0 \) if and only if

\[
\phi'(g)^2 \sum p_n v_n^2 \phi''(x_n + v_n s) - \phi''(g) \left( \sum p_n v_n \phi'(x_n + v_n s) \right)^2 \geq 0.
\]

If \( \phi'' \equiv 0 \) on I, (9) is trivial. In this case \( \phi \) is quadratic and (because \( \phi > 0 \), \( \phi' < 0 \), and \( \phi(I) = (0, \infty) \)) we must have \( \phi(x) = c(ax + b)^{-1/a} \) for \( a = -1/2 \) and \( c > 0 \).

Therefore it remains to consider the case \( \phi'' > 0 \) on I. If (9) holds for all \( s \) in a neighborhood of 0 for arbitrary \( x \in I^N \) and \( p, v \gg 0 \), in particular letting \( s = 0 \), we obtain

\[
\phi'(g)^2 \sum p_n v_n^2 \phi''(x_n) - \phi''(g) \left( \sum p_n v_n \phi'(x_n) \right)^2 \geq 0
\]

\[
\frac{\phi'(g)^2}{\phi''(g)} \geq \frac{\left( \sum p_n v_n \phi'(x_n) \right)^2}{\left( \sum p_n v_n^2 \phi''(x_n) \right)},
\]

where \( \phi_n = \phi(x_n) \) and \( \phi'_n, \phi''_n \) are defined analogously. Applying the Cauchy-Schwarz inequality, we obtain

\[
\left( \sum p_n v_n \phi'_n \right)^2 = \left( \sum \sqrt{p_n \phi'_n} v_n \right)^2 \leq \left( \sum p_n v_n^2 \phi''(x_n) \right) \left( \sum \frac{p_n v_n^2 \phi'_n}{\phi''_n} \right),
\]

with equality achieved when \( v_n = k \phi'_n / \phi''_n \) for some \( k < 0 \) (so that \( v_n > 0 \)). Therefore (10) for all \( x \in I^N \) and \( p, v \gg 0 \) is equivalent to

\[
\frac{\phi'(g)^2}{\phi''(g)} \geq \sum \frac{p_n v_n^2 \phi'_n}{\phi''_n} \quad \text{for all } x \in I^N \text{ and } p \gg 0.
\]

Now define \( y_n = \phi(x_n) \) and

\[
\Phi(y) := \frac{[\phi'(\phi^{-1}(y))]^2}{\phi''(\phi^{-1}(y))}.
\]

Noting that \( \phi(I) = (0, \infty) \), (11) is equivalent to

\[
\Phi\left( \sum p_n y_n \right) \geq \sum p_n \Phi(y_n) \quad \text{for all } p, y \in \mathbb{R}^N_{++}.
\]

If we take \( N = 2 \), \( y_1 = x \), \( y_2 = y \), \( p_1 = y/2x \), and \( p_2 = 1/2 \) in (13), we obtain

\[
\Phi(y) \geq \frac{y}{2x} \Phi(x) + \frac{1}{2} \Phi(y) \iff \Phi(y) \geq \frac{y}{2x} \Phi(x).
\]
Interchanging the role of $x, y$, it follows that $\Phi(y)/y$ is constant, and hence $\Phi(y) = ky$ for some constant $k > 0$ (because $\phi'' > 0$). Using the definition of $\Phi$ in (12) and letting $x = \phi^{-1}(y)$, we obtain
\[
\frac{\phi'(x)^2}{\phi''(x)} = k\phi(x) \iff \frac{\phi(x)\phi''(x) - \phi'(x)^2}{\phi'(x)^2} = a \quad \text{for } x \in I,
\]
where $a = 1/k - 1 > -1$. Integrating both sides, we obtain
\[
\frac{\phi(x)}{\phi'(x)} = ax + b \iff \frac{\phi'(x)}{\phi(x)} = -\frac{1}{ax + b},
\]
where $a > -1$ and $b$ is such that $ax + b > 0$ (because $\phi > 0$ and $\phi' < 0$). The forms of $\phi$ and $I$ in Lemma 3 follow by integrating both sides with respect to $x$ and considering each case $-1 < a < 0$, $a = 0$, and $a > 0$ separately.

Conversely, if $\phi$ and $I$ take one of the forms in Lemma 3, then we have $\Phi(y) = \frac{y}{\varphi(x)}$ in (12), and the inequality (13) is trivial. Then we can go back the argument to show that $g$ is concave.

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