A BICATEGORICAL PASTING THEOREM

NILES JOHNSON AND DONALD YAU

ABSTRACT. We provide an elementary proof of a bicategorical pasting theorem that does not rely on Power’s 2-categorical pasting theorem, the bicategorical coherence theorem, or the local characterization of a biequivalence.

1. INTRODUCTION

Bicategories and their pasting diagrams were introduced by Bénabou [Ben67]. Pasting diagrams in bicategories, such as the following,

\[
\begin{array}{ccc}
V & \to & S \\
\downarrow & & \downarrow \\
W & \to & T
\end{array}
\]

are analogous to commutative diagrams in categories. They allow one to use diagrams to express iterated vertical composites of 2-cells of the form

\[
\begin{array}{ccc}
\bullet & \to & \bullet \\
\cdots & & \cdots \\
\bullet & \to & \bullet
\end{array}
\]

with some bracketings of the top and bottom paths that are compatible with the (co)domain of the middle 2-cell. For example, the two triangle identities that define an internal adjunction in a bicategory can be compactly expressed in terms of a few pasting diagrams; see [KS74] (Section 2.1) for the 2-category case. Moreover, the definitions of monoidal bicategories, as well as their symmetric, sylleptic, and braided variants, involve a number of large pasting diagrams [McC00, Sta16], without which the long vertical composites would be very hard to read.

A pasting theorem asserts that each pasting diagram has a uniquely defined composite that is independent of the order of the vertical composites, as long as they are defined. For 2-categories, such a pasting theorem was proved by Power [Pow90]. There are basically two steps. First he defined a concept of graphs with an acyclicity condition that ensures the existence of a composite in a 2-category. Then he showed by an induction that this composite has the desired uniqueness property.

Date: 02 October, 2019.
2010 Mathematics Subject Classification. 18D05, 18A10.
Key words and phrases. Bicategories, 2-categories, pasting schemes, pasting diagrams, graphs.
For general bicategories, Verity [Ver11] proved a bicategorical pasting theorem by extending Power’s concept of graphs to include bracketings of the (co)domain of each interior face and of the global (co)domain. Briefly, he first applied the Bicategorical Coherence Theorem [MP85, Str96], which asserts each bicategory \( B \) is retract biequivalent to a 2-category \( A \). With such a biequivalence \( h : B \rightarrow A \), a pasting diagram in \( B \) yields a pasting diagram in \( A \), which has a unique composite by Power’s pasting theorem for 2-categories. Using the fact that \( h \) is locally full and faithful, a unique 2-cell composite is then obtained back in \( B \). The proof that this composite is independent of the choice of a biequivalence \( h \) also relies on the Bicategorical Coherence Theorem.

The purpose of this paper is to prove a bicategorical pasting theorem that does not rely on (i) Power’s 2-categorical pasting theorem, (ii) the Bicategorical Coherence Theorem, (iii) the local characterization of a biequivalence, or (iv) that \( \text{Bicat}(B,B) \) is a bicategory (with lax functors as objects, lax natural transformations as 1-cells, and modifications as 2-cells). In fact, our proof stays entirely within the given bicategory, and only uses the basic axioms of a bicategory.

In addition to being much more elementary, our approach yields a 2-categorical pasting theorem which is independent of Power’s theorem. Moreover, the authors were motivated by concurrent work [JY] to give a self-contained proof of the local characterization of biequivalences. Pasting diagrams are an indispensable part of such work, and therefore one requires an independent pasting theorem.

The essential difference between a 2-categorical pasting theorem and a bicategorical pasting theorem is the presence of nontrivial associators. One adds bracketings to specify the order of composition of 1-cells, but then a bracketed pasting diagram does not necessarily admit a composite. For example, the unique bracketing of the diagram in Display (1.1) does not have a well-defined composite in a general bicategory—one must extend the diagram by inserting appropriate associators. The content of this paper has three parts, as follows.

First, in Section 2 we explain the graph theoretic concepts necessary to understand pasting diagrams and their extensions by associators; these are the notions of pasting scheme (Definition 2.8) and composition scheme (Definition 2.20). The main result of this section is Theorem 2.25, which proves that every pasting scheme extends to a composition scheme.

Second, in Section 3 we apply the preceding graph theory to explain pasting diagrams and their extensions to what we call composition diagrams. Every composition diagram has a well-defined composite, as we detail in Definition 3.16. This section includes the definition of bicategory to fix notation and terminology, together with a detailed example for the diagram in Display (1.1).

Finally, in Section 4 we prove that the composites resulting from any two extensions of a given pasting diagram are equal. This is the Bicategorical Pasting Theorem 4.3. Its proof depends on a generalization of Mac Lane’s Coherence Theorem, which we explain, together with an induction argument similar to that of [Pow90]. We note that, by restricting the argument to 2-categories, we recover a pasting theorem for 2-categories which is essentially Power’s.

In the 2-categorical case, the only difference between our approach and that of [Pow90, Ver11] is in our handling of the underlying graph theory. Power and Verity consider plane graphs with a source and a sink, and bracketings in the bicategory case, that have no directed cycles. We also use plane graphs with a source
and a sink, and bracketings for all (co)domains. However, instead of the non-existence of directed cycles, our acyclicity condition is phrased as the existence of a vertical decomposition of the graph into atomic graphs, each containing one interior face like the one in Display (1.2). One advantage of this approach is that it strictly mirrors the way pasting diagrams are usually used in practice, namely, as vertical composites of 2-cells each produced by whiskering a given 2-cell with a number of 1-cells. Another is that it greatly simplifies the graph theoretic work one must do, particularly in the bracketed case.

2. PASTING SCHEMES AND COMPOSITION SCHEMES

In this section we define the graph theoretic notions of pasting scheme and composition scheme. The main result of this section is Theorem 2.25. It characterizes bracketed graphs that admit a composition scheme extension as those whose underlying anchored graphs admit a pasting scheme presentation.

**Definition 2.1.** A graph is a tuple \( G = (V_G, E_G, \psi_G) \) consisting of:
- a finite set \( V_G \) of vertices with at least two elements;
- a finite set \( E_G \) of edges with at least two elements such that \( E_G \cap V_G = \emptyset \);
- an incidence function \( \psi_G : E_G \to V_G^2 \). For each edge \( e \), if \( \psi_G(e) = (u, v) \), then \( u \) and \( v \) are called the tail and the head of \( e \), respectively, and together they are called the ends of \( e \).

Moreover:
1. The geometric realization of a graph \( G \) is the topological quotient
   \[
   |G| = \left[ \left( \biguplus_{v \in V_G} \{v\} \right) \biguplus \left( \biguplus_{e \in E_G} [0,1]_e \right) \right] / \sim
   \]
   in which:
   - \( \{v\} \) is a one-point space indexed by a vertex \( v \).
   - Each \( [0,1]_e \) is a copy of the topological unit interval \( [0,1] \) indexed by an edge \( e \).
   - The identification \( \sim \) is generated by
     \[
     u - 0 \in [0,1]_e \ni 1 \sim v \quad \text{if} \quad \psi_G(e) = (u, v).
     \]
2. A plane graph is a graph together with a topological embedding of its geometric realization into the complex plane \( \mathbb{C} \).

Each vertex \( v \) is drawn as a circle \( \bigcirc \) with the name of the vertex inside. Each edge \( e \) with tail \( u \) and head \( v \) is drawn as an arrow from \( u \) to \( v \), as in \( \bigcirc \xrightarrow{e} \bigcirc \). A plane graph is a graph together with a drawing of it in the complex plane \( \mathbb{C} \) such that its edges meet only at their ends. To simplify the notation, we will identify a plane graph \( G \) with its geometric realization \( |G| \) and with the latter’s topologically embedded image in \( \mathbb{C} \).

**Definition 2.2.** Suppose \( G = (V_G, E_G, \psi_G) \) is a graph.
1. A path in \( G \) is an alternating sequence \( v_0 e_1 v_1 \cdots e_n v_n \) with \( n \geq 0 \) of vertices \( v_i \)’s and edges \( e_i \)’s such that:
   - each \( e_i \) has ends \( \{v_{i-1}, v_i\} \);
   - the vertices \( v_i \)’s are distinct.
   This is also called a path from \( v_0 \) to \( v_n \). A path is trivial if \( n = 0 \), and is non-trivial if \( n \geq 1 \).
(2) If \( p = v_0e_1v_1\cdots e_nv_n \) is a path, then \( p^* = v_ne_n\cdots v_1e_1v_0 \) is the reversed path from \( v_n \) to \( v_0 \).

(3) A directed path is a path such that each \( e_i \) has head \( v_i \).

(4) \( G \) is connected if for each pair of distinct vertices \( \{u, v\} \), there exists a path from \( u \) to \( v \).

Using the orientation of the complex plane \( \mathbb{C} \), we identify two connected plane graphs if they are connected by a homeomorphism that preserves the orientation and the incidence relation, and that maps vertices to vertices and edges to edges.

**Definition 2.3.** Suppose \( G \) is a connected plane graph.

1. The connected subspaces of the complement \( \mathbb{C} \setminus |G| \) are called the open faces of \( G \). Their closures are called faces of \( G \). The unique unbounded face is called the exterior face, denoted by \( \text{ext}_G \). The bounded faces are called interior faces.

2. The vertices and edges in the boundary \( \partial F \) of a face \( F \) of \( G \) form an alternating sequence \( v_0e_1v_1\cdots e_nv_n \) of vertices and edges such that:
   - \( v_0 = v_n \).
   - The ends of \( e_i \) are \( \{v_{i-1}, v_i\} \).
   - Traversing \( \partial F \) from \( v_0 \) to \( v_n = v_0 \) along the edges \( e_1, e_2, \ldots, e_n \) in this order, ignoring their tail-to-head orientation, the face \( F \) is always on the right-hand side.

3. An interior face \( F \) of \( G \) is anchored if it is equipped with
   - two distinct vertices \( s_F \) and \( t_F \), called the source and the sink of \( F \), respectively, and
   - two directed paths \( \text{dom}_F \) and \( \text{cod}_F \) from \( s_F \) to \( t_F \), called the domain and the codomain of \( F \), respectively, such that \( \partial_F = \text{dom}_F \text{cod}_F^* \) with the first vertex in \( \text{cod}_F^* = t_F \) removed on the right-hand side.

4. The exterior face of \( G \) is anchored if it is equipped with
   - two distinct vertices \( s_G \) and \( t_G \), called the source and the sink of \( G \), respectively, and
   - two directed paths \( \text{dom}_G \) and \( \text{cod}_G \) from \( s_G \) to \( t_G \), called the domain and the codomain of \( G \), respectively, such that \( \partial_{\text{ext}_G} = \text{cod}_G \text{dom}_G^* \) with the first vertex in \( \text{dom}_G^* = t_G \) removed on the right-hand side.

5. \( G \) is anchored if every face of \( G \) is anchored.

6. \( G \) is an atomic graph if it is an anchored graph with exactly one interior face.

In an anchored graph, the boundary of each interior face is oriented clockwise. On the other hand, the boundary of the exterior face is oriented counterclockwise.

**Example 2.4.** Here is an atomic graph \( G \)

![Diagram of an atomic graph](https://via.placeholder.com/150)
with:

- **unique interior face** $F$ with source $s_F$, sink $t_F$, $\text{dom}_F = s_F h_1 u h_2 t_F$, and $\text{cod}_F = s_F h_3 v h_4 w h_5 t_F$;
- **exterior face** $\text{ext}_G$ with source $s$, sink $t$, $\text{dom}_G = s F t h_1 u h_2 t_F s t$, and $\text{cod}_G = s F t h_3 v h_4 w h_5 t_F s t$.

**Lemma 2.5.** If $G$ is an atomic graph with unique interior face $F$, then

$$\text{dom}_F \subseteq \text{dom}_G \quad \text{and} \quad \text{cod}_F \subseteq \text{cod}_G.$$  

**Proof.** Since $G$ only has one interior face, the boundary $\partial_{\text{ext}_G} = \text{cod}_G \text{dom}_G^*$ of the exterior face contains all of its edges. Traversing an edge $e$ in $\text{dom}_F$ from its tail to its head, $F$ is on the right-hand side, so $\text{ext}_G$ is on the left-hand side. Therefore, $e$ cannot be contained in the directed path $\text{cod}_G$. This proves the first containment. The second containment is proved similarly. □

In particular, each atomic graph $G$ consists of its unique interior face $F$, a directed path from the source $s_G$ of $G$ to the source $s_F$ of $F$, and a directed path from the sink $t_F$ of $F$ to the sink $t$ of $G$. Next we define a composition of anchored graphs that mimics the vertical composition of 2-cells in a bicategory.

**Definition 2.6.** Suppose $G$ and $H$ are anchored graphs such that $s_G = s_H$, $t_G = t_H$, and $\text{cod}_G = \text{dom}_H$. The **vertical composite** $HG$ is the anchored graph defined by the following data.

- The connected plane graph of $HG$ is the quotient

$$\frac{G \sqcup H}{\{\text{cod}_G = \text{dom}_H\}}$$  


of the disjoint union of $G$ and $H$, with the codomain of $G$ identified with the domain of $H$.
- The **interior faces** of $HG$ are the interior faces of $G$ and $H$, which are already anchored.
- The **exterior face** of $HG$ is the intersection of $\text{ext}_G$ and $\text{ext}_H$, with source $s_G = s_H$, sink $t_G = t_H$, domain $\text{dom}_G$, and codomain $\text{cod}_H$.

The following observation follows from a simple inspection.

**Lemma 2.7.** If $G$, $H$, and $I$ are anchored graphs such that the vertical composites $IH$ and $HG$ are defined, then $(IH)G = I(HG)$.

With this lemma, we will safely omit parentheses when we write iterated vertical composites of anchored graphs.

**Definition 2.8.** A **pasting scheme** is an anchored graph $G$ together with a decomposition $G = G_n \cdot \cdots \cdot G_1$, called a **pasting scheme presentation** of $G$, into vertical composites of $n \geq 1$ atomic graphs $G_1, \ldots, G_n$.

Since the horizontal composition in a bicategory is not strictly associative, we need to equip the graphs with bracketings, which we define next.

**Definition 2.9.** Bracketings are defined recursively as follows:

- The only bracketing of length 0 is the empty sequence $\varnothing$.
- The only bracketing of length 1 is the symbol $-$, called a dash.
- If $b$ and $b'$ are bracketings of lengths $m$ and $n$, respectively, then $(bb')$ is a bracketing of length $m + n$.  

A BICATEGORICAL PASTING THEOREM

5
We usually omit the outermost pair of parentheses, so the unique bracketing of length 2 is \(--\). A left normalized bracketing is either \(-\) or \((b)\) with \(b\) a left normalized bracketing.

**Definition 2.10.** For a directed path \(P = v_0 e_1 v_1 \cdots e_n v_n\) in a graph, a bracketing for \(P\) is a choice of a bracketing \(b\) of length \(n\). In this case, we write \(b(P)\), called a bracketed directed path, for the bracketed sequence obtained from \(b\) by replacing its \(n\) dashes with \(e_1, \ldots, e_n\) from left to right. If the bracketing is clear from the context, then we abbreviate \(b(P)\) to \((P)\) or even \(P\). We sometimes suppress the vertices and write \(P\) as \((e_1, \ldots, e_n)\), in which case \(b(P)\) is also denoted by \(b(e_1, \ldots, e_n)\).

**Example 2.11.** A directed path \(P = (e_1, \ldots, e_n)\) with \(0 \leq n \leq 2\) has a unique bracketing. The only bracketings of length 3 are \((-\)\(-\)\(-\)) and \((-\)\(-\))\(-\)). The five bracketings of length 4 are \((-\)\(-\)\(-\)\(-\)), \((-\)\(-\))\(-\)\(-\)), \((-\)\(-\))\(-\)\(-\)), \(-\)\(-\)\(-\)\(-\)), and \(-\)\(-\))\(-\)\(-\)). An induction shows that, for each \(n \geq 1\), there is a unique left normalized bracketing of length \(n\). If \(P = (e_1, e_2, e_3, e_4)\) is a directed path in a graph, then \(b(P)\) for the five possible bracketings for \(P\) are the bracketed sequences \(((e_1 e_2)e_3)e_4\), \((e_1 e_2)(e_3 e_4)\), \(e_1(e_2(e_3 e_4))\), \((e_1(e_2 e_3))e_4\), and \((e_1 e_2 e_3)e_4\).

**Definition 2.12.** A bracketing for an anchored graph \(G\) consists of a bracketing \(b\) for each of the directed paths \(\text{dom}_G\), \(\text{cod}_G\), \(\text{dom}_F\), and \(\text{cod}_F\) for each interior face \(F\) of \(G\). An anchored graph \(G\) with a bracketing is called a bracketed graph.

**Definition 2.13.** Suppose \(G\) and \(H\) are bracketed graphs such that:

- The vertical composite \(HG\) of underlying anchored graphs is defined as in Definition 2.6.
- \((\text{cod}_G) = (\text{dom}_H)\) as bracketed directed paths.

Then the anchored graph \(HG\) is given the bracketing determined as follows:

- \((\text{dom}_{HG}) = (\text{dom}_G)\);  
- \((\text{cod}_{HG}) = (\text{cod}_H)\);  
- Each interior face \(F\) of \(HG\) is either an interior face of \(G\) or an interior face of \(H\), and not both. Corresponding to these two cases, the directed paths \(\text{dom}_F\) and \(\text{cod}_F\) are bracketed as they are in \(G\) or \(H\).

Equipped with this bracketing, \(HG\) is called the vertical composite of the bracketed graphs \(G\) and \(H\).

**Remark 2.14.** Note that interior faces of a bracketed graph may be bracketed incompatibly; this often arises in practice as we shall see. Thus a bracketed graph may not decompose as a nontrivial composite, even if its underlying anchored graph does so.

Vertical composition of bracketed graphs is strictly associative, so we will safely omit parentheses when we write iterated vertical composites of bracketed graphs. Next is the graph theoretic version of a 2-cell whiskered with a number of 1-cells.

**Definition 2.15.** Suppose \(G\) is an atomic graph with

- unique interior face \(F\),
- \(P = (e_1, \ldots, e_n)\) the directed path from \(s_G\) to \(t_F\), and
- \(P' = (e'_1, \ldots, e'_n)\) the directed path from \(t_F\) to \(t_G\).
as displayed below with each edge representing a directed path.

\[ G = \begin{array}{ccc}
& P & \\
\circ_G & s_F & s_F' \\
\end{array} \]

A bracketing for \( G \) is **consistent** if it satisfies both

\[
\begin{align*}
\text{(dom}_G) &= b(e_1, \ldots, e_m, \text{(dom}_F), e_1', \ldots, e_n'), \\
\text{(cod}_G) &= b(e_1, \ldots, e_m, \text{(cod}_F), e_1', \ldots, e_n')
\end{align*}
\]

for some bracketing \( b \) of length \( m + n + 1 \). In \((\text{dom}_G)\), the bracketed directed path \((\text{dom}_F)\) is substituted into the \((m + 1)\)st dash in \( b \), and similarly in \((\text{cod}_G)\). An atomic graph with a consistent bracketing is called a **consistent graph**.

As we will see later, the following kind of graphs are designed for the associator and its inverse in a bicategory.

**Definition 2.17.** An **associativity graph** is a consistent graph in which the unique interior face \( F \) satisfies one of the following two conditions:

\[
\begin{align*}
\text{(dom}_F) &= (E_1E_2)E_3 \quad \text{and} \quad \text{(cod}_F) = E_1'(E_2'E_3'), \\
\text{or} \quad \\
\text{(dom}_F) &= E_1(E_2E_3) \quad \text{and} \quad \text{(cod}_F) = (E_1'E_2')E_3'.
\end{align*}
\]

Moreover, in each case and for each \( 1 \leq i \leq 3 \), \( E_i \) and \( E'_i \) are non-trivial bracketed directed paths with the same length and the same bracketing.

**Definition 2.20.** A **composition scheme** is a bracketed graph \( G \) together with a decomposition \( G = G_n \cdots G_1 \), called a **composition scheme presentation** of \( G \), into vertical composites of \( n \geq 1 \) consistent graphs \( G_1, \ldots, G_n \).

If \( G \) is a bracketed graph that admits a composition scheme presentation \( G_n \cdots G_1 \), then:

- \( G \) has \( n \) interior faces, one in each consistent graph \( G_i \) for \( 1 \leq i \leq n \).
- Each \( G_i \) has the same source and the same sink as \( G \).
- For each \( 1 \leq i \leq n-1 \), \((\text{dom}_G) = (\text{dom}_{G_{i+1}})\) as bracketed directed paths.
- \((\text{dom}_G) = (\text{dom}_{G_1})\) and \((\text{cod}_G) = (\text{cod}_{G_n})\).
- If \( 1 \leq i \leq j \leq n \), then \( G_j \cdots G_i \) is a composition scheme.

**Remark 2.21** (Composition schemes in 2-categories). In 2-category theory, associators are identities and therefore one typically does not distinguish between the notions of pasting scheme and composition scheme. However, the distinction is important in bicategory theory precisely because associators are typically nontrivial. The graphs one encounters in practice often do not admit any composition scheme presentation due to mismatched bracketings. However, they can be extended to composition schemes in the sense of the next two definitions.

**Definition 2.22.** Suppose \( G \) is a bracketed graph with a decomposition as \( G = G_2AG_1, G_2A \), or \( AG_1 \) into a vertical composite of bracketed graphs in which \( A \) is an associativity graph with unique interior face \( F \). Using the notations in Definition 2.17, the bracketed graph obtained from \( G \) by identifying each edge in \( E_i \) with its corresponding edge in \( E'_i \) for each \( 1 \leq i \leq 3 \), along with their corresponding tails and heads, is said to be obtained from \( G \) by **collapsing** \( A \), denoted by \( G/A \).
In the context of Definition 2.22:

- \((\text{dom}_{G/A}) = (\text{dom}_G)\) and \((\text{cod}_{G/A}) = (\text{cod}_G)\).
- The interior faces in \(G/A\) are those in \(G\) minus the interior face of \(A\), and their (co)domains are bracketed as they are in \(G\).
- Collapsing associativity graphs is a strictly associative operation. So we can iterate the collapsing process without worrying about the order of the collapses.
- If \(G\) originally has the form \(G_2AG_1\), then the bracketed graph \(G/A\) is not the vertical composite \(G_2G_1\) of the bracketed graphs \(G_1\) and \(G_2\) because
  \[ (\text{cod}_{G_1}) = (\text{dom}_A) \neq (\text{cod}_A) = (\text{dom}_{G_2}) \]
  as bracketed directed paths. However, forgetting the bracketings, the underlying anchored graph of \(G/A\) is the vertical composite of the underlying anchored graphs of \(G_1\) and \(G_2\).

**Definition 2.23.** Suppose \(G\) is a bracketed graph. A composition scheme extension of \(G\) consists of the following data.

1. A composition scheme \(H = H_{n}\ldots H_1\) as in Definition 2.20.
2. A proper subsequence of associativity graphs \(\{A_1, \ldots, A_j\}\) in \(\{H_1, \ldots, H_n\}\) such that \(G\) is obtained from \(H\) by collapsing \(A_1, \ldots, A_j\).

In this case, we also denote the bracketed graph \(G\) by \(H/\{A_1, \ldots, A_j\}\).

In the context of Definition 2.23:

- \((\text{dom}_G) = (\text{dom}_H)\) and \((\text{cod}_G) = (\text{cod}_H)\).
- The interior faces in \(G\) are those in \(H\) minus those in \(\{A_1, \ldots, A_j\}\), and their (co)domains are bracketed as they are in \(H\).
- The order in which the associativity graphs \(A_1, \ldots, A_j\) are collapsed does not matter.

To characterize bracketed graphs that admit a composition scheme extension, we need the following observation about moving brackets via associativity graphs.

**Lemma 2.24.** Suppose \(G\) is a bracketed atomic graph with interior face \(F\) such that:

- \((\text{dom}_G) = (\text{dom}_F)\) and \((\text{cod}_G) = (\text{cod}_F)\) as bracketed directed paths.
- \((\text{dom}_G)\) and \((\text{cod}_G)\) have the same length.

Then one of the following two statements holds.

1. \((\text{dom}_G) = (\text{cod}_G)\).
2. There exists a canonical vertical composite \(A_k\ldots A_1\) of associativity graphs such that \((\text{dom}_{A_k}) = (\text{dom}_G)\) and \((\text{cod}_{A_k}) = (\text{cod}_G)\).

**Proof.** Suppose \((\text{dom}_G)\) and \((\text{cod}_G)\) have length \(n\), and \(b^n_l\) is the left normalized bracketing of length \(n\). First we consider the case where
  \[ (\text{cod}_G) = b^n_l(e_1, \ldots, e_n) = b^n_{n-1}(e_1, \ldots, e_{n-1})e_n.\]

We proceed by induction on \(n\). If \(n \leq 2\), then there is a unique bracketing of length \(n\), so \((\text{dom}_G) = b^n_l\).

Suppose \(n \geq 3\). Then \((\text{dom}_G) = E_1E_2\) for some canonical, non-trivial bracketed directed paths \(E_1\) and \(E_2\). If \(E_2\) has length 1 (i.e., it contains the single edge \(e_n\)), then the induction hypothesis applies with \(E_1\) as the domain and \(b^n_{n-1}(e_1, \ldots, e_{n-1})\) as the codomain. Since adding an edge at the end of an associativity graph yields an associativity graph, we are done in this case.
If $E_2$ has length $>1$, then it has the form $E_2 = E_{21}E_{22}$ for some canonical, non-trivial bracketed directed paths $E_{21}$ and $E_{22}$. There is a unique associativity graph $A_1$ of the form (2.19) that satisfies

$$(\text{dom}_{A_1}) = E_1(E_{21}E_{22}) = (\text{dom}_G),$$

$$(\text{cod}_{A_1}) = (E_1E_{21})E_{22}.$$

Now we repeat the previous argument with $(\text{cod}_{A_1})$ as the new domain. This procedure must stop after a finite number of steps because $\text{dom}_G$ has finite length. When it stops, the right-most bracketed directed path $E_?$ has length 1, so we can apply the induction hypothesis as above. This finishes the induction.

An argument dual to the above shows that $\mathcal{A}(e_1, \ldots, e_n)$ and $(\text{cod}_G)$ are connected by a canonical finite sequence of associativity graphs of the form (2.18). Splicing the two vertical composites of associativity graphs together yields the desired vertical composite.

The main result of this section is the following characterization of bracketed graphs that admit a composition scheme extension.

**Theorem 2.25.** For a bracketed graph $G$, the following two statements are equivalent.

1. $G$ admits a composition scheme extension.
2. The underlying anchored graph of $G$ admits a pasting scheme presentation.

**Proof.** For the implication (1) $\Rightarrow$ (2), suppose $H = H_n \cdots H_1$ is a composition scheme. By definition, this is also a pasting scheme presentation for the underlying anchored graph of $H$ because each consistent graph $H_i$ has an underlying atomic graph. If $\{A_i\}_{1 \leq i \leq n}$ is a proper subsequence of associativity graphs in $\{H_i\}_{1 \leq i \leq n}$, then the vertical composite of the remaining underlying atomic graphs in

$$\{H_i\}_{1 \leq i \leq n} \setminus \{A_i\}_{1 \leq i \leq j}$$

is defined. Moreover, it is a pasting scheme presentation for the underlying anchored graph of the bracketed graph $H/\{A_i\}_{1 \leq i \leq j}$.

For the implication (2) $\Rightarrow$ (1), suppose $G = G_m \cdots G_1$ is a pasting scheme presentation for the underlying anchored graph of $G$. For each $1 \leq i \leq m$, let $F_i$ denote the unique interior face of $G_i$ with the consistent bracketing in which:

- $(\text{dom}_{F_i})$ and $(\text{cod}_{F_i})$ are bracketed as they are in $G$;
- $(\text{dom}_{G_i}) = \left( (P_i)(\text{dom}_{F_i}) \right) (P'_i)$;
- $(\text{cod}_{G_i}) = \left( (P_i)(\text{cod}_{F_i}) \right) (P'_i)$.

Here $(P_i)$ and $(P'_i)$ are either empty or left normalized bracketings. By Lemma 2.24:

- Either $(\text{dom}_{G_i}) = (\text{dom}_{G_{i+1}})$, or else there is a vertical composite of associativity graphs $A_{ik_i}A_{i1}$ with domain $(\text{dom}_{G_i})$ and codomain $(\text{dom}_{G_{i+1}})$.
- For each $2 \leq i \leq m$, either $(\text{cod}_{G_{i-1}}) = (\text{cod}_{G_i})$, or else there is a vertical composite of associativity graphs $A_{ik_i}A_{i1}$ with domain $(\text{cod}_{G_{i-1}})$ and codomain $(\text{dom}_{G_i})$.
- Either $(\text{cod}_{G_m}) = (\text{cod}_{G_{m+1}})$, or else there is a vertical composite of associativity graphs $A_{m+1}A_{m+1,1}$ with domain $(\text{cod}_{G_m})$ and codomain $(\text{cod}_{G_{m+1}})$.
The corresponding vertical composite

\[ H = \left( \sum_{\ell} A_{m+1, k_\ell+1} \cdots A_{m+1, 1} \right) \bigotimes_{c} \left( \sum_{r} A_{2k_r+1} \cdots A_{21} \right) \bigotimes_{c} \left( \sum_{o} A_{1k_o+1} \cdots A_{11} \right) \]

is a composition scheme. Moreover, \( G \) is obtained from \( H \) by collapsing all the associativity graphs \( A_{ij} \) for \( 1 \leq i \leq m + 1 \) and \( 1 \leq j \leq k_i \).

\[ \square \]

3. Pasting Diagrams and Composition Diagrams

In this section we apply the graph theoretic concepts in the previous section to define pasting diagrams and composition diagrams in bicategories. We begin with the definition of a bicategory. In what follows, \( 1 \) denotes the discrete category with one object \( * \). For a category \( C \), we identify the categories \( C \times 1 \) and \( 1 \times C \) with \( C \), and regard the canonical isomorphisms between them as \( \text{Id}_C \).

**Definition 3.1.** A bicategory is a tuple \((B, 1, c, a, \ell, r)\) consisting of the following data.

(i) \( B \) is equipped with a collection \( \text{Ob}(B) = B_0 \), whose elements are called objects in \( B \). If \( X \in B_0 \), we also write \( X \in B \).

(ii) For each pair of objects \( X, Y \in B \), \( B \) is equipped with a category \( B(X, Y) \), called a hom category.

- Its objects are called 1-cells, and its morphisms are called 2-cells in \( B \).
- Composition and identity morphisms in \( B(X, Y) \) are called vertical composition and identity 2-cells, respectively.
- For a 1-cell \( f \), its identity 2-cell is denoted by \( 1_f \).

(iii) For each object \( X \in B \), \( 1_X : 1 \rightarrow B(X, X) \) is a functor, which we identify with the 1-cell \( 1_X(*) \in B(X, X) \), called the identity 1-cell of \( X \).

(iv) For each triple of objects \( X, Y, Z \in B \),

\[ c_{XYZ} : B(Y, Z) \times B(X, Y) \rightarrow B(X, Z) \]

is a functor, called the horizontal composition. For 1-cells \( f \in B(X, Y) \) and \( g \in B(Y, Z) \), and 2-cells \( \alpha \in B(X, Y) \) and \( \beta \in B(Y, Z) \), we use the notations

\[ c_{XYZ}(g, f) = gf \quad \text{and} \quad c_{XYZ}(\beta, \alpha) = \beta \circ \alpha. \]

(v) For objects \( W, X, Y, Z \in B \),

\[ a_{WXYZ} : c_{WXZ}(c_{XYZ} \times \text{Id}_{B(W, X)}) \rightarrow c_{WYZ}(\text{Id}_{B(Y, Z)} \times c_{WXY}) \]

is a natural isomorphism, called the associator.

(vi) For each pair of objects \( X, Y \in B \),

\[ c_{XY}(1_Y \times \text{Id}_{B(X, Y)}) \xrightarrow{t_{XY}} \text{Id}_{B(X, Y)} \xleftarrow{r_{XY}} c_{XY}(\text{Id}_{B(X, Y)} \times 1_X) \]

are natural isomorphisms, called the left unitor and the right unitor, respectively.

The subscripts in \( c \) will often be omitted. The subscripts in \( a, \ell, \) and \( r \) will often be used to denote their components. The above data is required to satisfy the following two axioms for 1-cells \( f \in B(V, W) \), \( g \in B(W, X) \), \( h \in B(X, Y) \), and \( k \in B(Y, Z) \).
Unity Axiom: The middle unity diagram

\[
\begin{array}{c}
(g1w)f \\
\downarrow{rs1f} \quad 1g1f \\
gf
\end{array}
\]

in \(B(V,X)\) is commutative.

Pentagon Axiom: The diagram

\[
\begin{array}{c}
\phantom{a}
\phantom{a}

((kh)gf) \\
\downarrow{a_{khgf}} \\
(kh)(gf)
\end{array}
\]

\[
\begin{array}{c}
\phantom{a}
\phantom{a}

(k(hgf)) \\
\downarrow{a_{khgf}1f} \\
(k(hg)f)
\end{array}
\]

in \(B(V,Z)\) is commutative.

This finishes the definition of a bicategory.

Remark 3.3. Suppose \(B\) is a bicategory.

- We assume the hom categories \(B(X,Y)\) for objects \(X,Y \in B\) are disjoint. If not, we tacitly replace them with their disjoint union.
- For 2-cells \(\alpha : f \to f', \alpha' : f'' \to f'''\), and \(\alpha'' : f'' \to f'''\) in \(B(X,Y)\), there are equalities

\[
(\alpha''\alpha')\alpha = \alpha''(\alpha'\alpha) \quad \text{and} \quad \alpha = \alpha1_f = 1_f\alpha.
\]

- With the usual notation

\[
X \xrightarrow{f} Y
\]

for a 2-cell \(\alpha : f \to f'\), the horizontal composition \(c_{XYZ}\) is the assignment:

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \quad \xrightarrow{g\beta} X \xrightarrow{g\beta\alpha} Z
\]

- There are equalities

\[
1g1f = 1gf
\]

in \(B(X,Z)(gf,gf)\), and

\[
(\beta\beta')(\alpha'\alpha) = (\beta'\alpha')(\beta\alpha)
\]

in \(B(X,Z)(gf,gf,gf)\) for 1-cells \(f'' \in B(X,Y)\), \(g'' \in B(Y,Z)\) and 2-cells \(\alpha' : f' \to f'', \beta' : g' \to g''\).

We now apply the graph theoretic concepts above to bicategories.

Definition 3.7. Suppose \(B\) is a bicategory, and \(G\) is a bracketed graph.

1. A 1-skeletal \(G\)-diagram in \(B\) is an assignment \(\phi\) as follows.
• \( \phi \) assigns to each vertex \( v \) in \( G \) an object \( \phi_v \) in \( B \).
• \( \phi \) assigns to each edge \( e \) in \( G \) with tail \( u \) and head \( v \) a 1-cell \( \phi_e \in B(\phi_u, \phi_v) \).

(2) Suppose \( \phi \) is such a 1-skeletal \( G \)-diagram, and \( P = v_0 e_1 v_1 \cdots e_m v_m \) is a directed path in \( G \) with \( m \geq 1 \) and with an inherited bracketing \((P)\). Define the 1-cell

\[
\phi_P \in B(\phi_{v_0}, \phi_{v_m})
\]

as follows.

• First replace the edge \( e_i \) in \((P)\) by the 1-cell \( \phi_{e_i} \in B(\phi_{v_{i-1}}, \phi_{v_i}) \) for \( 1 \leq i \leq m \).
• Then form the horizontal composite of the resulting parenthesized sequence \( \phi_{v_0} \xrightarrow{\phi_{e_1}} \phi_{v_1} \xrightarrow{\phi_{e_2}} \cdots \xrightarrow{\phi_{e_m}} \phi_{v_m} \) of 1-cells.

(3) A \( G \)-diagram in \( B \) is a 1-skeletal \( G \)-diagram \( \phi \) in \( B \) that assigns to each interior face \( F \) of \( G \) a 2-cell \( \phi_F : \phi_{\text{dom} F} \to \phi_{\text{cod} F} \) in \( B(\phi_{e_i}, \phi_{e_j}) \).

(4) A \( G \)-diagram is called a composition diagram of shape \( G \) if \( G \) admits a composition scheme presentation.

(5) A \( G \)-diagram is called a pasting diagram if the underlying anchored graph admits a pasting scheme presentation. Equivalently, by Theorem 2.25, a \( G \)-diagram \( \phi \) is a pasting diagram in \( B \) if and only if \( G \) admits a composition scheme extension.

**Remark 3.9** (Pasting diagrams in 2-categories). Suppose \( B \) is a 2-category, regarded as a bicategory, and let \( B' \) denote its underlying 2-category. If \( G \) is a bracketed graph and \( \phi \) a \( G \)-diagram in \( B \), let \( G' \) denote the underlying anchored graph of \( G \) and let \( \phi' \) denote the corresponding \( G' \)-diagram in \( B' \). Then \( \phi \) is a pasting diagram in \( B \) if and only if \( \phi' \) is a pasting diagram in \( B' \).

**Definition 3.10.** Suppose \( \phi \) is a composition diagram of shape \( G \) in a bicategory \( B \) and suppose \( G_n \cdots G_1 \) is a composition scheme presentation of \( G \).

(1) For each \( 1 \leq i \leq n \), the constituent 2-cell for \( G_i \), denoted by \( \phi_{G_i} \), is defined as follows. Suppose \( G_i \) has:

• unique interior face \( F_i \);
• directed path \( P_i = (e_{i1}, \ldots, e_{ik_i}) \) from \( s_G \) to \( s_{F_i} \);
• directed path \( P_i' = (e_{i1}', \ldots, e_{ik_i}') \) from \( t_{F_i} \) to \( t_G \).

By (2.16) the bracketing of the consistent graph \( G_i \) satisfies

\[
\text{(dom}_{G_i} \text{)} = b_i(e_{i1}, \ldots, e_{ik_i} \text{ }) \ (\text{dom} F_i) \ , \ e'_{i1}, \ldots, e'_{ik_i} \text{ },
\]
\[
\text{(cod}_{G_i} \text{)} = b_i(e_{i1}, \ldots, e_{ik_i} \text{ }) \ (\text{cod} F_i) \ , \ e'_{i1}, \ldots, e'_{ik_i} \text{ },
\]

for some bracketing \( b_i \) of length \( k_i + l_i + 1 \). Then we define the 2-cell

\[
\phi_{G_i} = b_i(1, \phi_{e_{i1}}, \ldots, 1, \phi_{e_{i1}'}, \phi_{F_i}, 1, \phi_{e_{i1}'}, \ldots, 1, \phi_{e_{ik_i}'}, \phi_{F_i}, 1, \phi_{e_{ik_i}'}, \phi_{G_i}) : \phi_{\text{dom} G_i} \to \phi_{\text{cod} G_i}
\]

in \( B(\phi_{G_i}, \phi_{G_i}) \) where:

• The identity 2-cell of each \( \phi_{e_{ij}} \) is substituted for \( e_{ij} \) in \( b_i \), and similarly for the identity 2-cell of each \( \phi_{e_{ij}'} \).
• The 2-cell \( \phi_{F_i} \) is substituted for the \((k_i + 1)\)st entry in \( b_i \).
• $\phi_{G_i}$ is the iterated horizontal composite of the resulting bracketed sequence of 2-cells, with the horizontal compositions determined by the brackets in $b_i$.

(2) The composite of $\phi$ with respect to $G_n \cdots G_1$, denoted by $|\phi|$, is defined as the vertical composite

\[
\phi_{\dom G} = \phi_{\dom G_1} \xrightarrow{|\phi| = \phi_{G_n} \cdots \phi_{G_1}} \phi_{\cod G} = \phi_{\cod G'}
\]

which is a 2-cell in $B(\phi_{G'}, \phi_{G'})$.

**Example 3.13.** Suppose given a $G$-diagram $\phi$ in $B$, as displayed on the left below. The underlying anchored graph $G$ has a unique bracketing because, in all three interior faces and the exterior face, the domain and the codomain have at most two edges. The bracketed graph $G$ does not admit a composition scheme presentation.

The composite of $\phi$ is not defined in general because

\[
\begin{align*}
\cod (1 f_2 \circ \theta_1) &= f_2 (h_2 h_1) \neq (f_2 h_2) h_1 = \dom (\theta_2 \circ 1 h_1), \\
\cod (\theta_2 \circ 1 h_1) &= (g_2 h_3) h_1 \neq g_2 (h_3 h_1) = \dom (1 g_2 \circ \theta_3).
\end{align*}
\]

We can fix the mismatched bracketings by:

- expanding $G$ into a composition scheme $G'$ by inserting two associativity graphs, one of the form (2.18) and the other (2.19);
- inserting instances of the associator $a$ or its inverse $a^{-1}$ to obtain the composition diagram $\phi'$ of shape $G'$ on the right above.

The composite of $\phi$ may now be defined as the vertical composite

\[
\begin{align*}
\xymatrix{\text{f}_2 \text{f}_1 \ar@{=>}[r]^{|\phi'|} & \text{g}_2 \text{g}_1 \\
\text{f}_2 (h_2 h_1) \ar@{=>}[u]^{1 f_2 \circ \theta_1} & \text{g}_2 (h_3 h_1) \ar@{=>}[u]^{1 g_2 \circ \theta_3} \\
\text{a}^{-1} \ar@{=>}[r] & (g_2 h_2) h_1 & (g_2 h_3) h_1 \ar@{=>}[l]^{\text{a}^{-1} h_1}
}\end{align*}
\]

of 2-cells in $B(V,T)$.

The essential idea demonstrated in Example 3.13 works in general to extend a pasting diagram to a composition diagram. We explain this in the following two definitions.
Definition 3.14. Suppose $\phi$ is a 1-skeletal $A$-diagram in a bicategory $B$ for some associativity graph $A$.

1. We call $\phi$ extendable if, using the notations in Definition 2.17, for each $1 \leq i \leq 3$ and each edge $e$ in $E_i$ with corresponding edge $e'$ in $E_i'$, there is an equality of 1-cells $\phi_e = \phi_{e'}$. As defined in (3.8), this implies the equality $\phi_{E_i} = \phi_{E_i'}$ of composite 1-cells.

2. Suppose $\phi$ is extendable. The canonical extension of $\phi$ is the $A$-diagram that assigns to the unique interior face $F$ of $A$ the 2-cell

$$
\phi_{\text{dom}_F} = \phi_{E_3}(\phi_{E_2}\phi_{E_1}) \xrightarrow{\phi_F = a^{-1}} (\phi_{E_3'}\phi_{E_2'})\phi_{E_1'} = \phi_{\text{cod}_F}
$$

if $A$ satisfies (2.18), or

$$
\phi_{\text{dom}_F} = (\phi_{E_3}\phi_{E_2})\phi_{E_1} \xrightarrow{\phi_F = a} \phi_{E_3'}(\phi_{E_2'}\phi_{E_1'}) = \phi_{\text{cod}_F}
$$

if $A$ satisfies (2.19).

Example 3.15. In Example 3.13 the composition diagram $\phi'$ involves two canonical extensions of restrictions of $\phi$, one for each of $a$ and $a^{-1}$.

Definition 3.16. Suppose that $\phi$ is a pasting diagram of shape $G$ in a bicategory $B$, and suppose $H = H_n \cdots H_1$ is a composition scheme extension of $G$. The composite of $\phi$ with respect to $H = H_n \cdots H_1$, denoted by $|\phi|_H$, is defined as follows.

1. First define the composition diagram $\phi_H$ of shape $H$ by the following data:
   - The restriction of $\phi_H$ to $(\text{dom}_H)$ is $(\text{dom}_G)$; to $(\text{cod}_H)$ is $(\text{cod}_G)$; and to the interior faces in $G$, agrees with $\phi$.
   - For each $1 \leq i \leq j$, the restriction of $\phi_H$ to the associativity graph $A_i$ is extendable. The value of $\phi_H$ at the unique interior face of $A_i$ is given by the canonical extension described in Definition 3.14(2). That is, it is either a component of the associator $a$ or its inverse.

2. Now we define the 2-cell $|\phi|$ in $B((\phi_G, \phi_{G'}))$ by

$$
\phi_{\text{dom}_G} \xrightarrow{|\phi| = |\phi_H|} \phi_{\text{cod}_G'}
$$

where $|\phi_H|$ is the composite of $\phi_H$ as in (3.12) with respect to $H_n \cdots H_1$.

4. Bicategorical Pasting Theorem

In this section we prove the Bicategorical Pasting Theorem 4.3. Existence of a composite follows from Theorem 2.25 and Definition 3.16. The majority of the remaining work is to show, for a pasting diagram $\phi$ of shape $G$ in a bicategory, the composites with respect to any two composition scheme extensions of $G$ are equal. The proof of this result restricts to 2-categories and yields essentially Power’s pasting theorem for 2-categories.

We begin with an adaptation of Mac Lane’s Coherence Theorem to this context.

Theorem 4.1 (Mac Lane’s Coherence). Suppose:

1. $G = A_k \cdots A_1$ and $G' = A'_k \cdots A'_1$ are composition schemes such that:
   - All the $A_i$ and $A'_i$ are associativity graphs.
   - $(\text{dom}_G) = (\text{dom}_{G'})$ and $(\text{cod}_G) = (\text{cod}_{G'})$ as bracketed directed paths.
A BICATEGORICAL PASTING THEOREM

(2) \( \phi \) is a 1-skeletal \( G \)-diagram in \( B \) whose restriction to each \( A_i \) is extendable. With the canonical extension of \( \phi \) in each \( A_i \), the resulting composition diagram of shape \( G \) is denoted by \( \Phi \).

(3) \( \phi' \) is a 1-skeletal \( G' \)-diagram in \( B \) whose restriction to each \( A'_j \) is extendable. With the canonical extension of \( \phi' \) in each \( A'_j \), the resulting composition diagram of shape \( G' \) is denoted by \( \Phi' \).

(4) \( \phi_e = \phi'_e \) for each edge \( e \) in \( \text{dom}_G \).

Then there is an equality \( |\Phi| = |\Phi'| \) of composite 2-cells in \( B(\Phi_G, \Phi'_G) \).

Proof. The desired equality is

\[
\Phi_{A_k} \cdots \Phi_{A_1} = \Phi'_{A'_1} \cdots \Phi'_{A'_k}
\]

with

- each side a vertical composite as in (3.12), and
- \( \Phi_{A_j} \) and \( \Phi'_{A'_j} \), horizontal composites as in (3.11).

The proof that these are equal is adapted as follows from the proof of Mac Lane’s Coherence Theorem for monoidal categories in [Mac98] (p.166-168), which characterizes the free monoidal category on one object.

- Suppose the edges in \( \text{dom}_G \), and hence also in \( \text{cod}_G \), are \( e_1, \ldots, e_n \) from the source \( s_G \) to the sink \( t_G \). By hypothesis there are equalities of 1-cells:
  - \( \phi_{e_i} = \phi'_{e_i} \) for \( 1 \leq i \leq n \);
  - \( \phi_{\text{dom}_G} = \phi'_{\text{dom}_G} \) and \( \phi_{\text{cod}_G} = \phi'_{\text{cod}_G} \).

Mac Lane considered \( @ \)-words involving \( n \) objects in a monoidal category. Here we consider bracketings of the sequence of 1-cells \( (\phi_{e_1}, \ldots, \phi_{e_n}) \).

- Identity morphisms within \( @ \)-words are replaced by identity 2-cells in the ambient bicategory \( B \).
- Each instance of the associativity isomorphism \( \alpha \) in a monoidal category is replaced by a component of the associator \( a \).
- A basic arrow in Mac Lane’s sense is a \( @ \)-word of length \( n \) involving one instance of \( \alpha \) and \( n - 1 \) identity morphisms. Basic arrows are replaced by 2-cells of the forms \( \Phi_A \) or \( \Phi'_A \) for an associativity graph \( A \).
- Composites of basic arrows are replaced by vertical composites of 2-cells.
- The bifunctoriality of the monoidal product is replaced by the functoriality of the horizontal composition in \( B \).
- The Pentagon Axiom in a monoidal category is replaced by the Pentagon Axiom (3.2) in the bicategory \( B \).

Mac Lane’s proof shows that, given any two \( @ \)-words \( u \) and \( w \) of length \( n \) involving the same sequence of objects, any two composites of basic arrows from \( u \) to \( w \) are equal. With the adaptation detailed above, Mac Lane’s argument yields the desired equality (4.2). \( \square \)

Now we come to our main result, the Bicategorical Pasting Theorem.

**Theorem 4.3 (Bicategorical Pasting).** Suppose \( B \) is a bicategory. Every pasting diagram in \( B \) has a unique composite.
Proof. Suppose \( G \) is a bracketed graph whose underlying anchored graph admits a pasting scheme extension and suppose \( \phi \) is a pasting diagram of shape \( G \) in \( \mathbb{B} \). Existence of a composite follows from Theorem 2.25: \( G \) has a composition scheme extension \( H \), and \( \phi \) has a composite with respect to \( H \) as described in Definition 3.16.

Now we turn to uniqueness. Suppose we are given two composition scheme extensions of \( G \), say

- \( H = H^1_{j+n} \cdots H^1_1 \) with associativity graphs \( \{ A^1_1, \ldots, A^1_j \} \) and
- \( H' = H'^1_{k+n} \cdots H'^1_1 \) with associativity graphs \( \{ A'^1_1, \ldots, A'^1_k \} \).

We want to show that the composites of \( \phi \) with respect to \( H = H^1_{j+n} \cdots H^1_1 \) and \( H' = H'^1_{k+n} \cdots H'^1_1 \) are the same. The proof is an induction on the number \( n \) of interior faces of \( G \).

The case \( n = 1 \) follows from

(i) Lemma 2.24,
(ii) Mac Lane’s Coherence Theorem 4.1, and
(iii) the naturality of the associator \( a \) and its inverse

as follows. Suppose the unique interior face \( F \) of \( G \) appears in \( H_p \) and \( H'_q \) for some \( 1 \leq p \leq j + 1 \) and \( 1 \leq q \leq k + 1 \). Since \( H_p \) and \( H'_q \) are consistent graphs, by (2.16) there exist bracketings \( b \) and \( b' \) of the same length, say \( m \), such that

\[
\begin{align*}
\text{(dom}_{H_p}\text{)} &= b(e_1, \ldots, e_{l-1}, (\text{dom}_F), e_{l+1}, \ldots, e_m), \\
\text{(cod}_{H_p}\text{)} &= b(e_1, \ldots, e_{l-1}, (\text{cod}_F), e_{l+1}, \ldots, e_m), \\
\text{(dom}_{H'_q}\text{)} &= b'(e_1, \ldots, e_{l-1}, (\text{dom}_F), e_{l+1}, \ldots, e_m), \\
\text{(cod}_{H'_q}\text{)} &= b'(e_1, \ldots, e_{l-1}, (\text{cod}_F), e_{l+1}, \ldots, e_m).
\end{align*}
\]

There is a unique bracketed atomic graph \( C \) with interior face \( C_F \) such that

- \( \text{(dom}_C\text{)} = \text{(dom}_{C_F}\text{)} = \text{(dom}_{H_p}\text{)} \) and
- \( \text{(cod}_C\text{)} = \text{(cod}_{C_F}\text{)} = \text{(cod}_{H'_q}\text{)} \).

By (i) there exists a canonical vertical composite \( C' = C_r \cdots C_1 \) of associativity graphs \( C_1, \ldots, C_r \) such that:

- \( \text{(dom}_{C'}\text{)} = \text{(dom}_{C_r}\text{)} = \text{(dom}_{C_1}\text{)} \).
- \( \text{(cod}_{C'}\text{)} = \text{(cod}_{C_r}\text{)} = \text{(cod}_{C_1}\text{)} \).
- No \( C_i \) changes the bracketing of \( \text{(dom}_F\text{)} \).

Indeed, since the bracketed directed path \( \text{(dom}_F\text{)} \) appears as the \( l \)th entry in both \( b \) and \( b' \), we can first regard \( \text{(dom}_F\text{)} \) as a single edge, say \( e_l \), in \( C \). Applying (i) in that setting gives a vertical composite of associativity graphs with domain \( b(e_1, \ldots, e_m) \) and codomain \( b'(e_1, \ldots, e_m) \). Then we substitute \( \text{(dom}_F\text{)} \) in for each \( e_l \) in the resulting vertical composite.

The sequence of edges

\[
\{ e_1, \ldots, e_{l-1}, \text{dom}_F, e_{l+1}, \ldots, e_m \}
\]

in \( \text{dom}_{H_p} \) is the same as those in \( \text{dom}_C \) and \( \text{dom}_{H'_q} \). So the underlying 1-skeletal G-diagram of \( \phi \) uniquely determines a composition diagram \( \phi_{C'} \) of shape \( C' \), in which every interior face is assigned either a component of the associator \( a \) or its inverse, corresponding to the two cases (2.19) and (2.18). Its composite with respect to the composition scheme presentation \( C_r \cdots C_1 \) is denoted by \( |\phi_{C'}| \). Similar
remarks apply with cod\(F_p\), cod\(H_p\), cod\(G\), and cod\(H'_q\) replacing dom\(F\), dom\(H_p\), dom\(G\), and dom\(H'_q\), respectively.

Moreover, since \(n = 1\), by the definitions of \(H_p\) and \(H'_q\) there are equalities

\[
\{H_1, \ldots, H_{j+1}\} = \{A_1, \ldots, A_{p-1}, H_p, A_p, \ldots, A_j\},
\]

\[
\{H'_1, \ldots, H'_{k+1}\} = \{A'_1, \ldots, A'_{q-1}, H'_q, A'_q, \ldots, A'_k\}.
\]

Consider the following diagram in \(B(\psi_{G,\phi_{\ast C}})\).

The left-bottom boundary and the top-right boundary are the composites of \(\phi\) with respect to \(H = H_{j+1}\cdots H_1\) and \(H' = H'_{k+1}\cdots H'_1\), respectively. The top and bottom rectangles are commutative by (ii). The middle rectangle is commutative by (iii). This proves the initial case \(n = 1\).

Suppose \(n \geq 2\). We consider the two interior faces of \(G\), say \(F_1\) and \(F'_1\), that appear first in the lists

\[
\{H_1, \ldots, H_{j+n}\} \setminus \{A_1, \ldots, A_j\} \quad \text{and} \quad \{H'_1, \ldots, H'_{k+n}\} \setminus \{A'_1, \ldots, A'_k\},
\]

respectively. If \(F_1 = F'_1\), then, similar to the case \(n = 1\), the two composites of \(\phi\) are equal by (i)–(iii) and the induction hypothesis.

For the other case, suppose \(F_1 \neq F'_1\). Since \(G\) has an underlying anchored graph, by Lemma 2.5 \(F_1\) and \(F'_1\) do not intersect, except possibly for \(t_{F_1} = s_{F'_1}\) or \(t_{F'_1} = s_{F_1}\). Similar to the \(n = 1\) case, by (i)–(iii) and the induction hypothesis, we are reduced to the case with \(n = 2\), \(j = k = 0\), the underlying anchored graph of \(G\) as displayed below with each edge representing a directed path,

\[
\begin{array}{c}
  \xymatrix{\text{dom}_{F_1} & Q_1 \ar[r]^-{F_1} & t_{F_1} \ar[r]^-{F'_1} & \text{dom}_{F'_1} & Q_2 \ar[r]^-{F'_1} & s_{F'_1} \ar[r]^-{F_1} & t_{F_1} \ar[r]^-{F_1} & \text{dom}_{F_1} & Q_3 \ar[r]^-{F_1} & s_{F_1} \ar[r]^-{F'_1} & t_{F'_1}}
\end{array}
\]

and

\[
\text{(dom}_G = b''((Q_1), (\text{dom}_{F_1}), (Q_2), (\text{dom}_{F'_1}), (Q_3)),
\]

\[
\text{(cod}_G = b''((Q_1), (\text{cod}_{F_1}), (Q_2), (\text{cod}_{F'_1}), (Q_3))
\]

for some bracketing \(b''\). In this case, the equality of the two composites of \(\phi\) follows from the bicategory axioms (3.4), (3.5), and (3.6).  \(\square\)
As a corollary, we obtain essentially Power’s pasting theorem [Pow90] for 2-categories.

**Corollary 4.4.** Suppose \( \phi \) is a \( G \)-diagram in a 2-category for some anchored graph \( G \). Then the composites of \( \phi \) with respect to any two pasting scheme presentations of \( G \) are equal.

**Proof.** The proof above restricts to a proof in the 2-category case because in a 2-category the associator is the identity natural transformation. Therefore, the bracketings do not matter at all, and no associativity graphs are needed. \( \square \)

**References**

[Ben67] J. Bénabou, Introduction to bicategories, in: Lecture Notes in Math. 47, p. 1-77, Springer, Berlin, 1967. (cit. on p. 1).

[JY] N. Johnson and D. Yau, Quillen’s Theorem A and the Whitehead Theorem for bicategories, preprint available at https://nilesjohnson.net/quillen_a_bicat.pdf, 2019. (cit. on p. 2).

[KS74] G.M. Kelly and R. Street, Review of the elements of 2-categories. In: G.M. Kelly (eds) Category Seminar. Lecture Notes in Math. 420, Springer, Berlin, 1974. (cit. on p. 1).

[Mac98] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Math. 5, 2nd ed., Springer-Verlag, New York, 1998. (cit. on p. 15).

[MP85] S. Mac Lane and R. Pare, Coherence for bicategories and indexed categories, J. Pure Appl. Algebra 37 (1985), 59-80. (cit. on p. 2).

[McC00] P. McCrudden, Balanced coalgebroids, Theory Appl. Categories 7 (2000), 71-147. (cit. on p. 1).

[Pow90] A.J. Power, A 2-categorical pasting theorem, J. Algebra 129 (1990), 439-445. (cit on pp. 1, 2, 17).

[Sta16] M. Stay, Compact closed bicategories, Theory Appl. Categories 31 (2016), 755-798. (cit. on p. 1).

[Str96] R. Street, Categorical structures, in: M. Hazewinkel ed., Handbook of Algebra Vol. 1, p. 529-577, Elsevier, Amsterdam, 1996. (cit. on p. 2).

[Ver11] D. Verity, Enriched Categories, Internal Categories and Change of Base, PhD thesis, Cambridge University, Reprints in Theory Appl. Categories 20 (2011), 1-266. (cit. on p. 2).