Dynamic beats fixed: on phase-based algorithms for file migration*

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In the file migration problem, we are given a metric space $X$ and a large object (file) of size $D$ stored at a point of $X$. An input is a sequence of points that request access to this object, presented in online manner to an algorithm. Upon seeing a requesting point $r$, the algorithm pays the distance between $r$ and the point currently storing the object and, afterwards, it may move the object to an arbitrary point of $X$. The cost of such movement is equal to the distance travelled by the object times the object size $D$. This fundamental online problem captures the tradeoffs of efficient access to shared objects in multi-access systems, such as memory pages in multiprocessor systems or databases in networks.

In this paper, we construct a deterministic 4-competitive algorithm for this problem (for any metric space) beating the currently best 20-year old, 4.086-competitive $M_{TLM}$ algorithm by Bartal et al. (SODA 1997). Like $M_{TLM}$ our algorithm also operates in phases, but it adapts their lengths dynamically depending on the geometry of requests seen so far. We also show that if an online algorithm operates in phases of fixed length and the adversary is able to modify the graph between phases, no algorithm can beat the competitive ratio of 4.086.

1. Introduction

Consider the problem of managing a shared data item among sets of processors. For example, in a distributed program running in a network, nodes want to have access to shared files, objects or databases. Such a file can be stored in a local memory of one of the processors and when another processor wants to access (read from or write to) this file, it has to contact a processor holding the file. Such a transaction incurs a certain cost. Moreover, access patterns to this file may change frequently and unpredictably, which renders any static placement of the file inefficient. The goal is hence to minimize the total cost of communication, by moving the file in response to such accesses, so that requesting processors find the file “nearby” in the network.

The file migration problem serves as the theoretical underpinning of the application scenario described above. The problem was coined by Black and Sleator [BS89] and was initially called

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page migration, as the original motivation concerned managing a set of memory pages in a multiprocessor system. There the data item was a single memory page held at a local memory of a single processor.

Most subsequent work referred to this problem as file migration and we will stick to this convention in this paper. The file migration problem assumes non-uniform model, where the shared file is much larger than a portion accessed in a single time step. This is typical when in one step a processor wants to read a single unit of data from a file or a record from a database. On the other hand, to reduce the maintenance overhead, it is assumed that shared file is indivisible, and can be migrated between nodes only as a whole. This makes file migration much more expensive than a single access to the file. As the knowledge of future accesses is either partial or completely non-existing, the accesses to the file can be naturally modeled as an online problem, where the input sequence consists of processor identifiers, which sequentially try to access pieces of the shared file.

1.1. The Model

The studied network is modeled as an edge-weighted graph or, more generally, as a metric space \((X, d)\) whose point set \(X\) corresponds to processors and \(d\) defines the distances between them. There is a large indivisible file (historically called page) of size \(D\) stored at a point of \(X\). An input is a sequence of space points \(r_1, r_2, r_3, \ldots\) denoting processors requesting access to the file. This sequence is presented in online manner to an algorithm. More precisely, we assume that time is slotted into steps numbered from 1. Let \(\text{alg}_t\) denote the position of the file at the end of step \(t\) and \(p_0\) is the initial position of the file. In step \(t \geq 1\), the following happens:

1. A requesting point \(r_t\) is presented to the algorithm.
2. The algorithm pays \(d(\text{alg}_{t-1}, r_t)\) for servicing the request.
3. The algorithm chooses a new position \(\text{alg}_t\) for the file (possibly \(\text{alg}_t = \text{alg}_{t-1}\)) and moves the file to \(\text{alg}_t\) paying \(D \cdot d(\text{alg}_{t-1}, \text{alg}_t)\).

The algorithm has to make its decisions (where to migrate the file) after the \(t\)-th request, exclusively on the basis of the sequence up to step \(t\). To measure the performance of an online strategy, we use the standard competitive ratio metric \([BE98]\): an online deterministic algorithm \(\text{Alg}\) is \(c\)-competitive if there exists a constant \(\gamma\), such that for any input sequence \(I\), it holds that

\[
C_{\text{Alg}}(I) \leq c \cdot C_{\text{Opt}}(I) + \gamma.
\]

(1)

\(C_{\text{Alg}}\) and \(C_{\text{Opt}}\) denote the costs of \(\text{Alg}\) and Opt (optimal offline algorithm) on \(I\), respectively. The minimum \(c\) for which \(\text{Alg}\) is \(c\)-competitive is called the competitive ratio of \(\text{Alg}\).

1.2. Previous Work

The problem was stated by Black and Sleator \([BS89]\), who gave 3-competitive deterministic algorithms for the uniform metrics and trees and conjectured that 3-competitive deterministic algorithms were possible for any metric space.

Westbrook \([Wes94]\) constructed randomized strategies: 3-competitive algorithm against adaptive-online adversaries and \((1 + \phi)\)-competitive algorithm (for large \(D\)) against oblivious adversaries, where \(\phi \approx 1.618\) denotes the golden ratio. By the result of Ben-David et al. \([BBK^*94]\) this asserted the existence of a deterministic algorithm with competitive ratio at most \(3 \cdot (1 + \phi) \approx 7.854\).

The first explicit deterministic construction was the 7-competitive algorithm Move-To-Min (MtM) by Awerbuch et al. \([ABF93a]\). MtM operated in phases of length \(D\), during which the
algorithm remained at a fixed position. In the last step of a phase, \( \text{MtM} \) migrates the file to a point that minimizes the sum of distances to all requests \( r_1, r_2, \ldots, r_D \) presented in the phase, i.e., to a minimizer of the function
\[
f_{\text{MTM}}(x) = \sum_{i=1}^{D} d(x, r_i).
\]

The ratio has been subsequently improved by the algorithm \( \text{Move-To-Local-Min (MtLM)} \) by Bartal et al. [BCI01]. \( \text{MtLM} \) works similarly to \( \text{MtM} \), but changes the phase duration to \( c_0 \cdot D \) for a constant \( c_0 \), and when computing the new position for the file, it also takes the migration distance into the consideration. Namely, chooses to migrate the file to a point that minimizes the function
\[
f_{\text{MTLM}}(x) = D \cdot d(v_{\text{MTLM}}, x) + \frac{c_0 + 1}{c_0} \sum_{i=1}^{c_0 \cdot D} d(x, r_i),
\]
where \( v_{\text{MTLM}} \) denotes the point at which \( \text{MtLM} \) keeps its file during the phase. The algorithm is optimized by setting \( c_0 \approx 1.841 \) being the only positive root of the equation \( 3c^3 - 8c - 4 = 0 \). For such \( c \), the competitive ratio of \( \text{MtLM} \) is \( R_0 \approx 4.086 \), where \( R_0 \) is the largest (real) root of the equation \( R^3 - 5R^2 + 3R + 3 = 0 \). Their analysis is tight.

Note that most of the competitive ratios given above hold when \( D \) grows to infinity. In particular, for \( \text{MtLM} \) we assume that \( c_0 \cdot D \) is an integer and the ratio of \( 1 + \phi \) of Westbrook’s algorithm [Wes94] is achieved only in the limit.

Better deterministic algorithms are known only for specific graph topologies. There are 3-competitive algorithms for uniform metrics and trees [BS89], and \((3 + 1/D)\)-competitive strategies for three-point metrics [Mat15b]. Chrobak et al. [CLRW97] showed \( 2 + 1/(2D) \)-competitive strategies for continuous trees and products of trees, e.g., for \( \mathbb{R}^3 \) with \( \ell_1 \) norm. Furthermore, a \((1 + \phi)\)-competitive algorithm for \( \mathbb{R}^3 \) under any norm was also given in [CLRW97].

A straightforward lower bound of 3 was given by Black and Sleator [BS89] for deterministic algorithms and later adapted to randomized algorithms against adaptive-online adversaries by Westbrook [Wes94]. The currently best lower bound for deterministic algorithms is due to Matsubayashi [Mat15a], who showed a lower bound of \( 3 + \epsilon \) that holds for any value of \( D \), where \( \epsilon \) is a constant that does not depend on \( D \). This renders the file migration problem one of the few natural problems, where the competitive ratio of a deterministic algorithm is strictly larger than the competitive ratio of a randomized algorithm against an adaptive-online adversary. The lower bounds were improved for particular case of \( D = 1 \) to 3.148 [CLRW97] and 3.164 [Mat08].

### 1.3. Our Contribution

We propose a new deterministic algorithm that dynamically decides the length of the phase based on the geometry of requests received in the initial part of each phase. The algorithm was obtained by analyzing a linear model (factor revealing LP), which helped to find the proper selection of key parameters. We discuss this LP in Appendix B. Nevertheless, we managed to extract a human-readable combinatorial upper bound based on path-packing arguments and to obtain the following result.

**Theorem 1.** There exists a deterministic 4-competitive algorithm for the file migration problem.

We also give support to the claim that improvement of \( \text{MtLM} \) would not be possible by just selecting different parameters for a phase based algorithm operating with a fixed phase length. While the result we are presenting is not a lower bound on the actual algorithm for file migration problem, it shows that an analysis that treats each phase separately (e.g., the one employed for \( \text{MtLM} \) [BCI01]) cannot give better bounds on the competitive ratio than 4.086.

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Theorem 2. Fix any algorithm $A_{lg}$ that operates in phases of fixed length. Assume that between phases, the adversary can arbitrarily modify the graph while keeping the distance between the files of Opt and $A_{lg}$ unchanged. Then, the competitive ratio of $A_{lg}$ is at least $R_0$, where $R_0 \approx 4.086$ is the competitive ratio of algorithm $M_{tlm}$.

1.4. Other Related Work

The file migration problem has been generalized in a few directions. When we lift the restriction that the file can only be migrated and not copied, the resulting problem is called file allocation [BFR95, ABF93a, LRWY99]. It makes sense especially when we differentiate read and write requests to the file; for the former, we need to contact only one replica of the file, for the latter all copies need to be updated. The attainable competitive ratios become then worse: best deterministic algorithm is $O(\log n)$-competitive [ABF93a]; the lower bound of $\Omega(\log n)$ holding even for randomized algorithms follows by the reduction to an online Steiner tree problem [BFR95, IW91].

The file migration problem has also been extended to accommodate memory capacity constraints at nodes (when more than one file is used) [AK95, ABF98, ABF93b, Bar95], dynamically changing networks [ABF98, BBKM09], and different objective functions (minimizing congestion) [MMVW97, MVW99]. For a more systematic treatment of the file migration and related problems, see surveys [Bar96, Bie12]. For more applied approaches, see the survey [GS90] and the references therein.

2. 4-Competitive Algorithm Dynamic-Local-Min

We start with an insight concerning the hard inputs for the $M_{tlm}$ algorithm [BCI01]. We identified two classes of tight instances for $M_{tlm}$: bipartite and linear (cf. Figure 1). It can be shown that if the algorithm knew in advance on which instance it is run, it could improve its performance by changing the phase length. Namely, on bipartite instance, the longer phase would help the algorithm, whereas a shorter phase would be beneficial on linear instance.

To decide the length of the phase based on the geometry of requests, we need to measure the level of request concentration as compared to the distance from the current position of an algorithm to the center of requests. Intuitively, observing that (from some time instant) requests are concentrated around a certain point incentivizes the algorithm, to shorten the phase and quickly move to the “center of the requests”. If on the other hand, requests are scattered and the current algorithms position is essentially in the middle of the observed requests, it appears desirable for the algorithm to wait longer before it decides to move the file. This rule agrees with the desired behavior of an algorithm on linear and bipartite instances.
Turning the intuition above into an effective phase extension rule is not trivial. We present an algorithm based on a rule that we have extracted out of an optimization process applied to a natural linear model of the amortized phase-based analysis. The linear model used is quite involved and we present it in Appendix B. It can be seen as an alternative (computer-based) proof for the performance of our algorithm. Such proof technique might be interesting on its own and useful for analyzing other related online games played on a metric space.

2.1. Notation

For succinctness, we introduce the following notation. For any two points \( v_1, v_2 \in X \), let \([v_1, v_2] = D \cdot d(v_1, v_2)\). We extend this notation to sequences of points as follows:

\[
[v_1, v_2, \ldots, v_j] = [v_1, v_2] + [v_2, v_3] + \ldots + [v_{j-1}, v_j].
\]

Moreover, if \( v \in X \) is a point and \( S \subseteq X \) is a multiset of points, then

\[
[v, S] = [S, v] = D \cdot \frac{1}{|S|} \sum_{x \in S} d(v, x),
\]

i.e., \([v, S]\) is the average distance from \( v \) to a point of \( S \) times \( D \). We extend the sequence notation introduced above to sequences of points and multisets of points, e.g.

\[
[v, S, u, T] = [v, S] + [S, u] + [u, T].
\]

However, such sequences may not contain two consecutive multisets since the symbol \([S, T]\) is not defined for multisets \( S, T \). The sequence notation allows for easy expressing of the triangle inequality: \([v_1, v_2] \leq [v_1, v_3, v_2]\); we will extensively use this property. Note that the following multiset version of triangle inequality also holds: \([v_1, v_2] \leq [v_1, S, v_2]\).

2.2. Algorithm definition

We propose a new phase based algorithm that dynamically decides about the length of the current phase, which we call Dynamic-Local-Min (DLM). DLM operates in phases, but it chooses their lengths depending on the geometry of requests seen in the initial part of the phase. Roughly speaking, when it recognizes that the currently seen requests more closely resemble a linear tight example for MtLM, it ends the phase after 1.75 \( D \) steps. Otherwise, it assumes that the presented graph is more in the flavor of the bipartite construction, and only ends the phase after 2.25 \( D \) steps.

For any step \( t \), we denote the position of DLM’s file at the end of step \( t \) by \( \text{dLM}_t \) and that of \( \text{OrT} \) by \( \text{opt}_t \). We identify requests with the points where they are issued.

Assume a phase starts in step \( t + 1 \); that is \( \text{dLM}_t \) is the position of DLM at the very beginning of a phase. Within the phase DLM waits 1.75 \( D \) steps and at step \( t + 1.75 D \), it finds a node \( v_S \) that minimizes the function

\[
g(v) = [\text{dLM}_t, v, \mathcal{R}_1, v, \mathcal{R}_2] = [\text{dLM}_t, v] + 2 \cdot [v, \mathcal{R}_1] + [v, \mathcal{R}_2],
\]

where \( \mathcal{R}_1 \) is the multiset of the requests from steps \( t+1, \ldots, t+D \) and \( \mathcal{R}_2 \) is the multiset of the subsequent requests from steps \( t+D+1, \ldots, t+1.75D \).

If \( g(v_S) \leq 1.5 \cdot [\text{dLM}_t, \mathcal{R}_2] \), the algorithm moves its file to \( v_S \) which ends the current phase. Intuitively, this condition corresponds to detecting if there exists a point that is substantially closer to the first 1.75 \( D \) requests of the phase than the current position. The condition is constructed in a way to be conclusive for each of the possible outcomes. If indeed such point exists, then by
moving to this point we expect to get much closer to the position of the optimal algorithm, and hence we expect to benefit from the change of the potential function. If there is no such good point, also the optimal solution is experiencing some request servicing costs and we may afford to wait a little longer and meanwhile get more accurate estimation of the possible location of the file in the optimal solution.

If \( g(v_h) > 1.5 \cdot [dlm_t, opt] \), DLM waits the next 0.5 \( D \) steps and (in step \( t + 2.25 D \)) it moves its file to the point \( v_h \), where \( v_h \) is the minimizer of the function

\[
h(v) = [dlm_t, v] + [v, opt_1] + 1.25 \cdot [v, opt_2] + 0.75 \cdot [v, opt_3].
\]

\( R_3 \) is the multiset of the last 0.5 \( D \) requests from the prolonged phase (from steps \( t + 1.75 D + 1, \ldots, t + 2.25 D \)). Also in this case, the next phase starts right after the file movement.

Note that, the short phase consists of \( D \) requests denoted \( R_1 \) followed by 0.75 \( D \) requests denoted \( R_2 \), while the long phase consists additionally of 0.5 \( D \) requests denoted \( R_3 \). We will say that the short phase consists of two parts, \( R_1 \) and \( R_2 \), and the long phase consists of three parts, \( R_1, R_2 \) and \( R_3 \).

### 2.3. DLM Analysis

We start with a lower bound on \( \text{Opt} \). The following bound is a variant of the bound given implicitly in [BCI01]; we present its proof for completeness in the appendix.

#### Lemma 3.

Let \( \mathcal{R} \) be a subsequence of consecutive requests from the input issued at steps \( t+1, t+2, \ldots, t+|\mathcal{R}| \). Then, \( 4 \cdot C_{\text{OPT}}(\mathcal{R}) \geq (2|\mathcal{R}|/D) \cdot [\text{opt}_t, \mathcal{R}, \text{opt}_{t+|\mathcal{R}|}] + (4 - 2|\mathcal{R}|/D) \cdot [\text{opt}_t, \text{opt}_{t+|\mathcal{R}|}] \).

We define potential function at (the end of) step \( t \) as \( \Phi_t = 3 \cdot [dlm_t, \text{opt}_t] \). It suffices to show that in any (short or long) phase consisting of steps \( t + 1, t + 2, \ldots, t + \ell \) during which requests \( \mathcal{R} \) are given, it holds that

\[
C_{\text{ALG}}(\mathcal{R}) + \Phi_{t+\ell} \leq 4 \cdot C_{\text{OPT}}(\mathcal{R}) + \Phi_t.
\]

Theorem 1 follows immediately by summing the bound above over all phases of the input.

#### 2.3.1. Proof for a short phase

We consider any short phase \( \mathcal{R} \) consisting of part \( \mathcal{R}_1 \) spanning steps \( t + 1, \ldots, t + D \) and part \( \mathcal{R}_2 \) spanning steps \( t + D + 1, \ldots, t + 1.75 D \). For succinctness, we define \( \text{opt}^0 = \text{opt}_t, \text{opt}^1 = \text{opt}_{t+D} \) and \( \text{opt}^2 = \text{opt}_{t+1.75D} \). By Lemma 3,

\[
4 \cdot C_{\text{OPT}}(\mathcal{R}) + \Phi_t = 3 \cdot [dlm_t, \text{opt}^0] + 4 \cdot C_{\text{OPT}}(\mathcal{R}_1) + 4 \cdot C_{\text{OPT}}(\mathcal{R}_2) \\
\geq 3 \cdot [dlm_t, \text{opt}^0] + 2 \cdot [\text{opt}^0, \text{opt}^1] + 2 \cdot [\text{opt}^0, \mathcal{R}_1, \text{opt}^1] \\
+ 2.5 \cdot [\text{opt}^1, \text{opt}^2] + 1.5 \cdot [\text{opt}^1, \mathcal{R}_2, \text{opt}^2] .
\]

We treat the amount (3) as our budget. This is illustrated below; the coefficients are written as edge weights.

![Diagram of budget calculation](image-url)
Now, we bound $C_{\text{ALG}}(R) + \Phi_{t+1.75D}$, using the definition of $\text{ALG}$ and triangle inequality.

$$C_{\text{ALG}}(R) + \Phi_{t+1.75D} = C_{\text{ALG}}(R_1) + C_{\text{ALG}}(R_2) + 3[v_g, \text{opt}^2]$$

$$\leq [\text{dlm}_t, R_1] + 0.75 \cdot [\text{dlm}_t, R_2] + [\text{dlm}_t, v_g] + 3 \cdot [v_g, \text{opt}^2]$$

$$\leq [\text{dlm}_t, R_1] + 0.75 \cdot [\text{dlm}_t, R_2] + [\text{dlm}_t, v_g] + 2 \cdot [v_g, \text{opt}^2] + [v_g, R_2, \text{opt}^2]$$

$$= [\text{dlm}_t, R_1] + 0.75 \cdot [\text{dlm}_t, R_2] + 2 \cdot [\text{opt}^2, R_1] + [\text{opt}^2, R_2]$$

$$+ [\text{dlm}_t, v_g] + 2 \cdot [v_g, R_1] + [v_g, R_2]$$

$$= [\text{dlm}_t, R_1] + 0.75 \cdot [\text{dlm}_t, R_2] + 2 \cdot [\text{opt}^2, R_1] + [\text{opt}^2, R_2] + g(v_g)$$  \hspace{1cm} (4)

The first four summands of (4) can be bounded as follows

$$[\text{dlm}_t, R_1] + 0.75 \cdot [\text{dlm}_t, R_2] + 2 \cdot [\text{opt}^2, R_1] + [\text{opt}^2, R_2]$$

$$\leq [\text{dlm}_t, \text{opt}^0, R_1] + 0.75 \cdot [\text{dlm}_t, \text{opt}^0, R_2] + 2 \cdot [\text{opt}^2, \text{opt}^1, R_1] + [\text{opt}^2, R_2] .$$ \hspace{1cm} (5)

The final expression is depicted below.

For the last summand of (4), $g(v_g)$, we use the fact that $v_g$ is a minimizer of function $g$ (and hence $g(v_g) \leq g(\text{opt}^0)$) and the property of the short phase ($g(v_g) \leq 1.5 \cdot [\text{dlm}_t, R_2]$). Therefore,

$$g(v_g) \leq 0.5 \cdot g(\text{opt}^0) + 0.75 \cdot [\text{dlm}_t, R_2]$$

$$\leq 0.5 \cdot [\text{dlm}_t, \text{opt}^0, R_1, \text{opt}^0, R_2] + 0.75 \cdot [\text{dlm}_t, R_2]$$

$$\leq 0.5 \cdot [\text{dlm}_t, \text{opt}^0, R_1, \text{opt}^0, \text{opt}^1, R_2] + 0.75 \cdot [\text{dlm}_t, \text{opt}^0, \text{opt}^1, R_2] .$$ \hspace{1cm} (6)

By combining (4), (5) and (6) (or simply adding edge coefficients on the last two figures) we observe that the budget (edge coefficients on the first figure) is not exceeded. This implies 4-competitiveness, i.e., that (2) holds for any short phase.

**2.3.2. Proof for a long phase**

We consider any long phase $R$ consisting of part $R_1$ spanning steps $t + 1, \ldots, t + D$, part $R_2$ spanning steps $t + D + 1, \ldots, t + 1.75 \cdot D$ and part $R_3$ spanning steps $t + 1.75 \cdot D + 1, \ldots, t + 2.25 \cdot D$. Similarly to the proof for short phase, we define $\text{opt}^0 = \text{opt}_t$, $\text{opt}^1 = \text{opt}_{t+D}$, $\text{opt}^2 = \text{opt}_{t+1.75D}$, and $\text{opt}^3 = \text{opt}_{t+2.25D}$. We emphasize that the positions of Opt in long and short phase can be completely different.
By Lemma 3, we obtain a bound very similar to that for a short phase; again we depict it as coefficients on edge weights.

\[
4 \cdot C_{OPT}(R) + \Phi_s = 3 \cdot [dlm, opt^0] + 4 \cdot C_{OPT}(R_1) + 4 \cdot C_{OPT}(R_2) + 4 \cdot C_{OPT}(R_3) \\
\geq 3 \cdot [dlm, opt^0] + 2 \cdot [opt^0, opt^1] + 2 \cdot [opt^0, R_1, opt^1] \\
+ 2.5 \cdot [opt^1, opt^2] + 1.5 \cdot [opt^1, R_2, opt^2] \\
+ 3 \cdot [opt^2, opt^3] + [opt^2, R_3, opt^2].
\]

Now, we bound \( C_{ALG}(R) + \Phi_{l+2.25D} \), using the definition of \( ALG \) and the triangle inequality.

\[
C_{ALG}(R) + \Phi_{l+2.25D} = C_{ALG}(R_1) + C_{ALG}(R_2) + C_{ALG}(R_3) + 3 \cdot [v_h, opt^3] \\
\leq [dlm, R_1] + 0.75 \cdot [dlm, R_2] + 0.5 \cdot [dlm, R_3] + [dlm, v_h] + 3 \cdot [v_h, opt^3] \\
\leq [dlm, R_1] + 0.75 \cdot [dlm, R_2] + 0.5 \cdot [dlm, R_3] + [dlm, v_h] \\
+ [v_h, R_1, opt^3] + 1.25 \cdot [v_h, R_2, opt^3] + 0.75 \cdot [v_h, R_3, opt^3] \\
= [dlm, R_1] + 0.75 \cdot [dlm, R_2] + 0.5 \cdot [dlm, R_3] \\
+ [opt^3, R_1] + 1.25 \cdot [opt^3, R_2] + 0.75 \cdot [opt^3, R_3] + h(v_h).
\]

Since DLM has not migrated the file after the first two parts, \( g(v) \geq 1.5 \cdot [dlm, R_2] \) for any node \( v \). Therefore \( 0.75 \cdot [dlm, R_2] \leq 0.5 \cdot g(opt^0) = 0.5 \cdot [a_0, opt^0, R_1, opt^0, R_2] \leq 0.5 \cdot [a_0, opt^0, R_1, opt^0, opt^1, R_2] \). Using this inequality, as well as triangle inequality, the first three summands of (7) can be bounded as follows:

\[
[dlm, R_1] + 0.75 \cdot [dlm, R_2] + 0.5 \cdot [dlm, R_3] \\
\leq [dlm, opt^0, R_1] + 0.5 \cdot [dlm, opt^0, R_1, opt^0, opt^1, R_2] + 0.5 \cdot [dlm, opt^0, opt^1, opt^2, R_3].
\]

This bound is depicted below:

\[
\text{Budget} = 4 \text{ OPT} \\
+ \text{ initial potential} \\
\text{(long phase case)}
\]

The next three summands of (7) can be also bounded appropriately:

\[
[opt^3, R_1] + 1.25 \cdot [opt^3, R_2] + 0.75 \cdot [opt^3, R_3] \\
\leq [opt^3, opt^2, opt^1, R_1] + 1.25 \cdot [opt^3, opt^2, R_2] + 0.5 \cdot [opt^3, R_3].
\]

This bound is depicted below:

\[
\text{Budget} = 4 \text{ OPT} \\
+ \text{ final potential} \\
\text{(part 2, long phase)}
\]
Lastly, for bounding \( h(v_h) \), we use the fact that \( v_h \) it is a minimizer of \( h \), and hence
\[
 h(v_h) \leq h(\text{opt}^1) = [\text{opt}^1, \text{dlm}_1] + [\text{opt}^1, R_1] + 1.25 \cdot [\text{opt}^1, R_2] + 0.75 \cdot [\text{opt}^1, R_3] \\
\leq [\text{opt}^1, \text{opt}^0, \text{dlm}_1] + [\text{opt}^1, R_1] + [\text{opt}^1, R_2] + 0.25 \cdot [\text{opt}^1, \text{opt}^2, R_2] \\
+ 0.5 \cdot [\text{opt}^1, \text{opt}^2, R_3] + 0.25 \cdot [\text{opt}^1, \text{opt}^2, \text{opt}^3, R_3]
\]
(10)

Note that in (10) we split some of the paths and choose longer ones, so that the budgets on any edge are not violated. Bound (10) is depicted on the figure below.

By combining (7), (8), (9) and (10) (or simply adding edge coefficients on the last three figures), we observe that the budget (edge coefficients on the first figure) is not exceeded. This implies 4-competitiveness, i.e., that (2) holds for any long phase.

3. Lower Bound in Dynamic Graph Model

In this section, we show that, under some additional assumptions, no fixed-phase algorithm can beat the competitive ratio \( R_0 \approx 4.086 \) achieved by MTLm (see Section 1.2), where \( R_0 \) is the largest (real) root of the equation
\[
 R^3 - 5R^2 + 3R + 3 = 0 .
\]
(11)

Let \( v_{\text{opt}} \) and \( v_{\text{alg}} \) be the positions of files of Opt and Alg, respectively. Alg and Opt start at the same point of the metric. A fixed-phase algorithm chooses phase length \( c \cdot D \) and after every \( c \cdot D \) requests it makes a migration decision solely on the basis of its current position and the last \( c \cdot D \) requests. In particular, it cannot store the history of past requests beyond a window consisting of the last \( c \cdot D \) requests. Bartal et al. [BCI01] showed that no fixed-phase algorithm can achieve competitive ratio better than 3.847 (for large \( D \)).

We present our lower bound in a model that gives an additional power to the adversary. Let \( f \) denote the distance between \( v_{\text{alg}} \) and \( v_{\text{opt}} \) at the end of a phase. Then at the beginning of the next phase \( P \), the adversary removes the existing graph and creates a completely new one in which it chooses position for \( v_{\text{opt}} \). It creates a sequence of requests constituting phase \( P \) and runs Alg on \( P \). Finally, it chooses a strategy for Opt for \( P \), with the restriction that the initial distance between Opt and Alg files is exactly \( f \). We call this setting dynamic graph model.

3.1. Using known lower bound for short phases

The lower bound given for fixed-phases algorithms by Bartal et al. [BCI01] is already sufficient to show the desired lower bound for shorter phase lengths. (It can also be used for very long phases, but we do not use this property.)

**Lemma 4.** Let \( c_T = 2(R_0 + 1)/(R_0^2 - 2R_0 - 1) \approx 1.352 \). No fixed-phase algorithm using phase lengths \( c \cdot D \) with \( c \leq c_T \) can achieve competitive ratio lower than \( R_0 \).
Proof. Theorem 3.2 of [BCI01] states that no algorithm using phases of length $c \cdot D$ can have competitive ratio smaller than

$$L(c) = \inf_{a \in (0,1)} \max \left\{ \frac{a}{1-a} \frac{c + 2}{ca} + 1, c \cdot (a+1) + 1 \right\}. \quad (12)$$

Theorem 3.2 of [BCI01] also shows that $L(c) \geq 3.847$ for any $c$. However, we may also analyze this bound restricting phase lengths. In particular, we claim that when $c \leq c_T := 2(R_0 + 1)/(R_0^2 - 2R_0 - 1)$, then $L(c) \geq R_0$. To see this, consider two cases. When $a > R_0/(1 + R_0)$, the first term of (12), $a/(1-a)$, is already greater than $R_0$. Otherwise $a < R_0/(1 + R_0)$ and for $c \leq c_T$, the second term of (12), $(c + 2)/(ca) + 1$, is at least $R_0$. \qed

3.2. Construction idea

Below, we give some informal intuitions on how the input sequence is constructed and how $\text{ALG}$ can serve such sequence. In this description, we assume that the algorithm does not perform suboptimal moves; we will handle them later in the formal proof.

We describe a construction of an epoch, consisting of some number of phases. At the beginning and at the end of an epoch, $\text{ALG}$ and $\text{OPT}$ keep their files at the same node. We define three adversarial strategies, called plays: linear, bipartite and finishing. Each play consists of one or more phases. A prerequisite for each given play is a particular distance between $v_{\text{ALG}}$ and $v_{\text{OPT}}$. Each play will have some properties: it will incur some cost on $\text{ALG}$ and $\text{OPT}$ and will end with $v_{\text{ALG}}$ and $v_{\text{OPT}}$ in a specific distance.

In the first phase, when initially $v_{\text{ALG}} = v_{\text{OPT}}$, the adversary uses the linear play (the generated graph is a single edge of length 1), so that at the end of the phase, $d(v_{\text{ALG}}, v_{\text{OPT}}) = 1$. For such phase, we have $C_{\text{ALG}}(P) \geq R_0 \cdot C_{\text{OPT}}(P) - (1/(1 - 2\alpha)) \cdot D$, where $\alpha = 1/(R_0 - 1)$. Note that in this phase alone, the adversary does not enforce the desired competitive ratio, but it increases the distance between $v_{\text{OPT}}$ and $v_{\text{ALG}}$.

In each of the next $L$ phases, the adversary employs the bipartite play; the graph used is a special bipartite graph (see Figure 2 for an example). Denote by $f$ the value of $d(v_{\text{ALG}}, v_{\text{OPT}})$ at the beginning of a phase. If the algorithm plays well, then at the end of the phase this distance is equal to $2\alpha \cdot f$. Furthermore, neglecting lower order terms, for such phase $P$, it holds that $C_{\text{ALG}}(P) \geq R_0 \cdot C_{\text{OPT}}(P) + f \cdot D$. This means that this time the inequality $C_{\text{ALG}}(P) \geq R_0 \cdot C_{\text{OPT}}(P)$ holds with a slack of $f \cdot D$. The sum of these slacks over $L$ phases is $\sum_{i=0}^{L-1} (2\alpha)^i \cdot D$, which approaches $(1/(1 - 2\alpha)) \cdot D$ when $L$ grows. Hence, after $1 + L$ phases (for large $L$ and neglecting lower order terms), the cost paid by $\text{ALG}$ is at least $R_0$ times the cost paid by $\text{OPT}$ and the distance between their files is negligible.

Still, the resulting distance between $v_{\text{ALG}}$ and $v_{\text{OPT}}$ is positive, and to finish an epoch we need this distance to be zero. To this end, the adversary uses a third type of play, the finishing one. This play may consist of multiple phases and it forces the positions of $\text{ALG}$ and $\text{OPT}$ files to coincide. Note that instead of introducing finishing play, we could modify the precondition for the linear play, so that it is possible to use it for small initial distance. However, the finishing play will become useful also when the behavior of $\text{ALG}$ deviates from the one describe in the paragraphs above.

3.3. States

Our construction will be parameterized with integers $L$ and $k$; the latter is a parameter used in the bipartite play. We define

$$\varepsilon = \max \left\{ \sum_{i=L}^{\infty} (2\alpha)^i \frac{4R_0}{k+4}, \frac{(2\alpha)^L}{1 - 2\alpha}, \frac{4R_0}{k+4} \right\} = \max \left\{ \frac{(2\alpha)^L}{1 - 2\alpha}, \frac{4R_0}{k+4} \right\}.$$

10
where \( \alpha = 1/(R_0 - 1) \). Note that \( \varepsilon \) tends to zero with increasing \( L \) and \( k \). Our goal is to formally show that for any epoch \( E \) it holds that \( C_{\text{ALG}}(E) \geq (R_0 - \varepsilon) \cdot C_{\text{OPT}}(E) \). As \( \varepsilon \) can be made arbitrarily small, this will imply the lower bound of \( R_0 \).

As mentioned in our informal description above, the relation \( C_{\text{ALG}}(M) \geq (R_0 - \varepsilon) \cdot C_{\text{OPT}}(M) \) does not hold for each adversarial play \( M \). Nonetheless, we want to measure the amount \( C_{\text{ALG}}(M) - (R_0 - \varepsilon) \cdot C_{\text{OPT}}(M) \) for each play \( M \); we call this amount \textit{play gain}. For a single play \( M \), there may be multiple behaviors of \( \text{ALG} \) that result in different play gains and different final values of \( d(v_{\text{ALG}}, v_{\text{OPT}}) \).

First, we define possible \textit{states}. A state is defined between phases and depends on the distance between \( v_{\text{ALG}} \) and \( v_{\text{OPT}} \). Then we define adversarial plays; for each state there will be one play that can start at this state. For each play, we will then characterize possible outcomes: play gains and the resulting states. Finally, we analyze the total gain on any sequence of plays and we show that it can be lower-bounded by a constant. This will imply Theorem 2.

1. State \( S \): \( v_{\text{ALG}} = v_{\text{OPT}} \).
2. State \( A_{\ell} \) for \( \ell \in \{0, \ldots, L-1\} \): \( d(v_{\text{ALG}}, v_{\text{OPT}}) = (2\alpha)^{\ell} \), where \( \alpha = 1/(R_0 - 1) \).
3. State \( F \): all remaining distances between \( v_{\text{ALG}} \) and \( v_{\text{OPT}} \).

### 3.4. Starting in state \( S \): linear play

**Lemma 5.** Assume a phase starts in state \( S \). Then, the adversary may employ a single-phase (linear) play, whose gain is at least \( -\sum_{i=1}^{L-1} (2\alpha)^i \cdot D \) and which always ends in state \( A_0 \).

**Proof.** At the beginning of a linear play, \( v_{\text{ALG}} = v_{\text{OPT}} \). The created graph consists of two nodes, \( a = v_{\text{ALG}} = v_{\text{OPT}} \) and \( b \), connected with an edge of length 1. Let \( t = 1 + 1/R_0 \approx 1.245 \). Observe that \( t < c \) for all considered values of \( c \). The play consists of a single phase \( P \), whose first \( (c - t) \cdot D \) requests are given at \( a \) and the following \( t \cdot D \) requests are given at \( b \).

Note that \( \text{ALG} \) pays 1 for each of the last \( t \) requests. We consider two cases depending on the possible action of \( \text{ALG} \) at the end of \( P \).

1. \( \text{ALG} \) migrates the file to \( b \). In this case \( C_{\text{ALG}}(P) = (t + 1) \cdot D \). \( \text{OPT} \) then chooses to remain at \( a \) throughout \( P \) paying \( t \cdot D \). Then,

\[
C_{\text{ALG}}(P) - R_0 \cdot C_{\text{OPT}}(P)) = (t + 1 - R_0 \cdot t) \cdot D \\
= (1/R_0 + 1 - R_0) \cdot D .
\]

2. \( \text{ALG} \) keeps the file at \( a \). In this case \( C_{\text{ALG}}(P) = t \cdot D \). \( \text{OPT} \) then remain at \( a \) for the first \( c - t \) requests, migrates its file to \( b \), and keeps it there till the end of \( P \). Altogether, \( C_{\text{OPT}}(P) = D \). Then,

\[
C_{\text{ALG}}(P) - R_0 \cdot C_{\text{OPT}}(P) = (t - R_0 \cdot 1) \cdot D \\
= (1/R_0 + 1 - R_0) \cdot D .
\]

In both cases, the resulting state is \( A_0 \). Using the definition of \( R_0 \) (see (11)), it can be checked that \( R_0 - 1 - 1/R_0 = (R_0 - 1)/(R_0 - 3) \). Then, by the definition of \( \alpha \), we obtain \( R_0 - 1 - 1/R_0 = \)
\( (R_0 - 1)/(R_0 - 3) = 1/(1 - 2\alpha) \). Therefore, using \( C_{\text{OPT}}(P) \geq D \), we obtain that the play gain is
\[
C_{\text{CALG}}(P) - (R_0 - \varepsilon) \cdot C_{\text{OPT}}(P) = \varepsilon \cdot C_{\text{OPT}}(P) - (R_0 - 1 - 1/R_0) \cdot D \\
\geq \left( \varepsilon - \frac{1}{1 - 2\alpha} \right) \cdot D = \left( \varepsilon - \sum_{i=0}^{\infty} (2\alpha)^i \right) \cdot D \\
\geq - \sum_{i=0}^{L-1} (2\alpha)^i \cdot D .
\]

\[ \square \]

3.5. Starting in state \( A_\ell \): bipartite play

Lemma 6. Assume a phase starts in state \( A_\ell \) for \( \ell \in \{0, \ldots, L - 1\} \). Then the adversary may employ a single-phase (bipartite) play, such that one of the following conditions hold:

1. the resulting state is \( A_\ell \) and the play gain is at least zero;
2. the resulting state is \( A_{\ell+1} \) and the play gain is at least \( (2\alpha)^\ell \cdot D \);
3. the resulting state is \( F \) with distance \( 3\alpha \cdot (2\alpha)^\ell \) and the play gain is at least \( (1 + \alpha) \cdot (2\alpha)^\ell \cdot D \).

Proof. Let \( f = (2\alpha)^\ell \) denote the initial value of \( d(v_{\text{CALG}}, v_{\text{OPT}}) \). Let \( a \) denote the initial position of \( \text{CALG}' \)'s file.

Phase construction. The construction of this play will be parameterized by an integer \( k \geq 3 \). The graph created by the adversary will be bipartite with the following parts: \{\( a \}\}, set \( Q \), and set \( S \), where \( |Q| = |S| = k \). As allowed in the dynamic graph model, the exact initial position of \( v_{\text{OPT}} \) will be determined based on the behavior of \( \text{CALG} \); in any case it will be initially in set \( Q \). Node \( a \) is connected with all nodes from \( Q \) with an edge of length \( f \). The connections between \( Q \) and \( S \) constitute an almost complete bipartite graph, whose edges are of length \( \alpha \cdot f \). Namely, we number all nodes from \( Q \) and \( S \) as \( q_1, q_2, \ldots, q_k \) and \( s_1, s_2, \ldots, s_k \), respectively, and we connect \( p_i \) with \( q_j \) if and only if \( i \neq j \). An example for \( k = 3 \) is given in Figure 2. As \( k \geq 3 \), any pair of nodes from \( S \) share a common neighbor from \( Q \) and hence the distance between them is exactly \( 2\alpha \cdot f \).

All the requests are given at nodes from \( S \) in round-robin fashion (the adversary fixes an arbitrary ordering of nodes from \( S \) first). As \( D \) is sufficiently large, each node of \( S \) issues at least one request.

Cost analysis. In either case \( \text{OPT} \) remains at one node from \( Q \) for the whole phase \( P \). It pays \( \alpha \cdot f \) for any request at one of \( k - 1 \) neighboring nodes from \( Q \) and \( 3\alpha \cdot f \) for any request at the only non-incident node from \( Q \). As requests are given in round-robin fashion, the number of requests...
at that non-incident node is $m \leq \lceil cD/k \rceil \leq 2cD/k$, and the total cost of \text{Orf} is
\[
C_{\text{OPT}}(P) = \alpha \cdot f \cdot (cD - m) + 3\alpha \cdot f \cdot m
\]
\[
= \alpha \cdot f \cdot (cD + 2m)
\]
\[
\leq (1 + 4/k) \cdot \alpha \cdot f \cdot cD .
\]

By the definition of $\epsilon$, it holds that $(R_0 - \epsilon) \cdot (1 + 4/k) = R_0 + (4R_0/k - \epsilon \cdot (1 + 4/k)) \leq R_0$. Furthermore, we split the cost of \text{ALG} on $P$ into the cost of serving the requests, $C_{\text{ALG}}^R(P)$ and the migration cost $C_{\text{ALG}}^M(P)$. The former is exactly $C_{\text{ALG}}^R(P) = (1 + \alpha) \cdot f \cdot cD$. Therefore,
\[
C_{\text{ALG}}(P) - (R_0 - \epsilon) \cdot C_{\text{OPT}}(P) = C_{\text{ALG}}^M(P) + C_{\text{ALG}}^R(P) - (R_0 - \epsilon) \cdot C_{\text{OPT}}(P)
\]
\[
\geq C_{\text{ALG}}^M(P) + (1 + \alpha) \cdot f \cdot cD - R_0 \cdot \alpha \cdot f \cdot cD
\]
\[
= C_{\text{ALG}}^M(P) ,
\]
where the last equality follows as $R_0 \cdot \alpha = 1 + \alpha$ by the definition of $\alpha$. Hence, for lower-bounding the play gain is sufficient to lower-bound $C_{\text{ALG}}^M(P)$

We consider several possible migration options for \text{ALG} on the bipartite play.

1. \text{ALG} remains at $a$. In this case $C_{\text{ALG}}^M(P) = 0$, and the resulting state is still $A_{l}$. 

2. \text{ALG} migrates the file to a node $q \in Q$, paying $f \cdot D = (2\alpha)^f \cdot D$ for the migration. The adversary chooses its original position to be any node from $Q$ different from $q$. Therefore, the final distance between \text{ALG} and \text{Orf} files is exactly $2\alpha \cdot f$. The resulting state is $A_{l+1}$ and the play gain is at least $C_{\text{ALG}}^M(P) = (2\alpha)^f \cdot D$.

3. \text{ALG} migrates the file to a node $s \in S$. The adversary chooses its original position to be (the only) node from $Q$ not directly connected to $s$. The cost of migration is $(1 + \alpha) \cdot f \cdot D$ and the resulting distance between \text{ALG} and \text{Orf} files is then $3\alpha \cdot f$, i.e., the play ends in state $F$. \hfill \qed

3.6. Starting in states $F$ and $A_{L-1}$: finishing play

\textbf{Lemma 7.} Assume a phase starts in state $A_{L-1}$ or $F$ and the initial distance is $f$. Then the adversary may employ a (finishing) play, whose gain is at least $(c + 1) \cdot f \cdot D$ and which always ends in state $S$.

\textbf{Proof.} In this case the adversary uses a finishing play $M$. This is the only play that may consist of more than one phase. It is played on two nodes, connected by an edge of length $f$. Initially, one of these nodes hold the file of \text{ALG}, and the second one the file of \text{ALG}. In any phase of this play, \text{Orf} never moves and all requests are issued at $v_{\text{Orf}}$. If at the end of the phase \text{ALG} does not migrate the file, the adversary repeats the phase.

The cost of \text{Orf} in any such phase is 0. Hence, any competitive algorithm has to finally migrate to $v_{\text{Orf}}$, possibly over a sequence of multiple phases. In the first phase of $M$, \text{ALG} pays at least $c \cdot D \cdot f$ for the requests. Furthermore, within $M$, \text{ALG} migrates the file along the distance of at least $f$, paying $f \cdot D$. The final distance $d(v_{\text{ALG}}, v_{\text{Orf}})$ is zero, i.e., the resulting state is always of type $S$. The play gain is
\[
C_{\text{ALG}}(M) - (R_0 - \epsilon) \cdot C_{\text{OPT}}(M) \geq C_{\text{ALG}}(M) \geq (c + 1) \cdot f \cdot D . \hfill \qed
\]
3.7. Combining all plays

In Figure 3, we summarized the possible transitions between states, as stated in Lemma 5, Lemma 6, and Lemma 7. We may now use this graph of states to show the desired lower bound.

Theorem 2. Fix any algorithm AlgL that operates in phases of fixed length. Assume that between phases, the adversary can arbitrarily modify the graph while keeping the distance between the files of Opt and AlgL unchanged. Then, the competitive ratio of AlgL is at least R0, where R0 ≈ 4.086 is the competitive ratio of algorithm Mtlm.

Proof. We consider a sequence of state transitions that starts from state S. Basically, there are two cases. In the first case, the sequence of transitions always returns to state S and we show that the total gain on a sequence of plays starting and ending at S is non-negative. In the second case, at some point we reach state AL (for ℓ ∈ {0, ..., L − 2}) and loop there infinitely. Then, (i) the costs on the initial path from S to AL are bounded by a constant, (ii) the costs on the loop grow arbitrarily large and (iii) the total gain on this loop is non-negative. Hence, the initial path costs become negligible in the long run.

Formally, we will show that there exists a constant γ, such that for any sufficiently long input sequence I generated according to the described adversarial strategy, it holds that

\[ C_{\text{ALG}}(I) \geq (R_0 - \epsilon) \cdot C_{\text{OPT}}(I) - \gamma. \]  

As AlgL’s cost on any play is universally lower-bounded by a constant, by taking a long input sequence, the cost of AlgL can be made arbitrarily large and γ becomes negligible. Inequality (13) implies then the theorem, as ε can be made arbitrarily small.

First, we assume that a sequence of plays starting at state S at some point ends in state AL for some ℓ ∈ {0, ..., L − 2} and loops there infinitely. In this case, we split the input sequence I into I’ (transition to state AL) and I” (loop transitions at AL). The cost of Opt on I’ can be upper-bounded by a universal constant independent of I’, denoted γ’. On each loop at AL, the play gain is at least zero by Lemma 6. Hence, \( C_{\text{ALG}}(I) \geq C_{\text{ALG}}(I'') \geq (R_0 - \epsilon) \cdot C_{\text{OPT}}(I'') \geq (R_0 - \epsilon) \cdot (C_{\text{OPT}}(I) - C_{\text{OPT}}(I')) \geq (R_0 - \epsilon) \cdot (R_0 - \epsilon) \cdot \gamma', \) which implies (13) with \( \gamma = (R_0 - \epsilon) \cdot \gamma'. \)

Otherwise, a sequence of plays starting at state S always returns to state S in a finite number of plays. We then analyze any subsequence of plays starting and ending at S and we will show that the total gain on these plays is at least zero. This will imply (13). For a transition between two states S1 and S2, we denote its gain by T(S1, S2). There are two possibilities:

1. The game starts at state S, goes through states A0, A1, ..., AL−1 and ends at S. The total plays’
gain is then
\[ T(S, A_0) + \sum_{\ell=0}^{L-2} T(A_\ell, A_{\ell+1}) + T(A_{L-1}, S) \geq - \sum_{i=0}^{L-1} (2\alpha)^i \cdot D + \sum_{\ell=0}^{L-1} (2\alpha)^\ell \cdot D + 0 \geq 0. \]

2. The game starts at state \( S \), goes through states \( A_0, A_1 \) till \( A_m \) (for \( m < L - 1 \)), then to state \( F \) and returns to \( S \). Note that the distance between \( v_{alg} \) and \( v_{opt} \) at state \( A_m \) is \( (2\alpha)^m \) and therefore the distance in state \( F \) is \( 3\alpha \cdot (2\alpha)^m \). The total plays’ gain is then
\[ T(S, A_0) + \sum_{\ell=0}^{m-1} T(A_\ell, A_{\ell+1}) + T(A_m, F) + T(F, S) \]
\[ \geq - \sum_{i=0}^{L-1} (2\alpha)^i \cdot D + \sum_{\ell=0}^{m-1} (2\alpha)^\ell \cdot D + (1 + \alpha)(2\alpha)^m \cdot D + (c + 1) \cdot 3\alpha \cdot (2\alpha)^m \cdot D \]
\[ = (2\alpha)^m \cdot D \cdot \left( 1 + \alpha + (c + 1) \cdot 3\alpha - \sum_{i=0}^{L-m-1} (2\alpha)^i \right) \]
\[ \geq (2\alpha)^m \cdot D \cdot \left( 1 + \alpha + (c + 1) \cdot 3\alpha - \frac{1}{1-2\alpha} \right) \]
\[ \geq 0. \]

The last inequality can be verified numerically: for \( \alpha = 1/(R_0 - 1) \approx 0.324 \) and \( c \geq c_T = 2(R_0 + 1)/(R^2 - 2R_0 - 1) \approx 1.352 \), it holds that \( (1 + \alpha) + (c + 1) \cdot 3\alpha - 1/(1 - 2\alpha) > 0.768 > 0. \]

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A. Omitted Proofs

Lemma 3. Let $R$ be set subsequence of consecutive requests from the input issued at steps $t + 1, t + 2, \ldots, t + |R|$. Then, $4 \cdot C_{OPT}(R) \geq (2|R|/D) \cdot \left[ \text{opt}_{t}, R, \text{opt}_{t+|R|} \right] + (4 - 2|R|/D) \cdot \left[ \text{opt}_{t+1}, \text{opt}_{t+|R|} \right]$.

Proof. For simplicity of notation, we assume that $t = 0$. In these terms, $R$ corresponds to requests $r_1, r_2, \ldots, r_{|R|}$ issued at the consecutive steps. By the triangle inequality

$$\frac{|R|}{D} \cdot \left[ \text{opt}_{0}, R \right] = \sum_{i=1}^{|R|} d(\text{opt}_{0}, r_i) \leq \sum_{i=1}^{|R|} d(\text{opt}_{0}, \text{opt}_{i-1}) + \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, r_i)$$

and similarly

$$\frac{|R|}{D} \cdot \left[ \text{opt}_{|R|}, R \right] = \sum_{i=1}^{|R|} d(\text{opt}_{|R|}, r_i) \leq \sum_{i=1}^{|R|} d(\text{opt}_{|R|}, \text{opt}_{i-1}) + \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, r_i)$$

Hence,

$$\frac{|R|}{D} \cdot \left[ \text{opt}_{t}, R, \text{opt}_{t+|R|} \right] \leq \sum_{i=1}^{|R|} \left( d(\text{opt}_{0}, \text{opt}_{i-1}) + d(\text{opt}_{|R|}, \text{opt}_{i-1}) \right) + 2 \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, r_i)$$

$$\leq \sum_{i=1}^{|R|} \sum_{j=1}^{|R|} d(\text{opt}_{j-1}, \text{opt}_{j}) + 2 \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, r_i)$$

$$= |R| \sum_{j=1}^{|R|} d(\text{opt}_{j-1}, \text{opt}_{j}) + 2 \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, r_i).$$

Finally,

$$\frac{2|R|}{D} \cdot \left[ \text{opt}_{t}, R, \text{opt}_{t+|R|} \right] + \left( 4 - \frac{2|R|}{D} \right) \cdot \left[ \text{opt}_{t}, \text{opt}_{t+|R|} \right]$$

$$\leq 4 \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, \text{opt}_{i}) + 4 \sum_{i=1}^{|R|} d(\text{opt}_{i-1}, r_i)$$

$$= 4 \cdot C_{OPT}(R).$$

□

B. Linear Program for File Migration

In this section, we present a linear programming model for the analysis of both algorithm Mtlm by Bartal et al. [BCI01] and our algorithm Dilm.
B.1. LP analysis of Mtlm-like algorithms

In our approach we analyze any Mtlm-like algorithm ALG. We use our notion of distances from Section 2.1. ALG will be a variant of Mtlm parameterized with two values $\alpha$ and $\delta$. The length of its phase is $\alpha \cdot D$ and the initial point of ALG is denoted by $A_0$. We denote the set of requests within a phase by $\mathcal{R}$. At the end of a phase, ALG migrates the file to a point $A_1$ that minimizes the function

$$f(x) = [A_0, x] + \alpha \cdot [x, \mathcal{R}].$$

As in the amortized analysis of the Mtlm algorithm [BCI01], we will use a potential function equal to $\phi$ times the distance between the files of ALG and Opt, where $\phi$ is a parameter used in the analysis. Hence, the amortized cost of ALG in a single phase is $C_{\text{ALG}} = \delta \cdot [A_0, \mathcal{R}] + [A_0, A_1] + \phi([A_1, O_1] - [A_0, O_0])$.

The following factor-revealing LP tries to mimic the proof given in [BCI01]. Namely, it encodes inequalities that are true for any phase and a graph on which ALG can be run. Its goal is to maximize the ratio between $C_{\text{ALG}}$ and $C_{\text{OPT}}$; as the instance can be scaled, we set $C_{\text{OPT}} = 1$ and we maximize $C_{\text{ALG}}$. Additionally, we let $O_0$ and $O_1$ denote the initial and final position of Opt during the studied phase, respectively. Let $V = [A_0, A_1, O_0, O_1]$ and $V' = V \cup \{\mathcal{R}\}$.

minimize $C_{\text{ALG}}$

subject to $C_{\text{ALG}} = \delta \cdot [A_0, \mathcal{R}] + [A_0, A_1] + \phi \cdot ([A_1, O_1] - [A_0, O_0])$

$C_{\text{OPT}} = 1$

$C_{\text{OPT}} = C_{\text{req}} + C_{\text{move}}$

$C_{\text{move}} \geq [O_0, O_1]$

$2 \cdot C_{\text{req}} + \delta \cdot C_{\text{move}} \geq \delta \cdot [O_0, \mathcal{R}] + \delta \cdot [O_1, \mathcal{R}]$

$f(A_1) \geq f(v)$ for all $v \in V$

$0 \leq [v_1, v_3] \leq [v_1, v_2] + [v_2, v_3]$ for all $v_1, v_2, v_3 \in V'$

As $\mathcal{R}$ is a set of requests, it does not necessarily correspond to a single point in the studied metric. Nevertheless, our notion of average distances ([..]) allows writing triangle inequalities for any pair of objects from set $V \cup \{\mathcal{R}\}$.

In the LP above, $C_{\text{req}}$, $C_{\text{move}}$ denote the cost of Opt for serving the request and the cost of Opt for migrating the file, respectively. Inequality $2 \cdot C_{\text{req}} + \delta \cdot C_{\text{move}} \geq \delta \cdot [O_0, \mathcal{R}] + \delta \cdot [O_1, \mathcal{R}]$ is an analogue of Lemma 3.

For any choice of parameters $\alpha$, $\delta$ and $\phi$, the LP finds an instance that maximizes the competitive ratio of ALG. Note that such instance is not necessarily an certificate that ALG indeed performs poorly: in particular inequalities that lower-bound the cost of Opt might not be tight. However, the opposite is true: if the value of $C_{\text{ALG}}$ returned by LP is $\xi$, then for any possible instance the ratio is at most $\xi$.

Let $c_0 = 1.841$ be a phase length of Mtlm. Setting $\alpha = c_0$ and $\delta = \phi = 1 + c_0$, yields that the optimal value to the LP is $R_0 \approx 4.086$, which can be interpreted as numerical counterpart of the original analysis in [BCI01]. To obtain a formal mathematical proof, one may take a dual solution to the LP. It gives coefficients which multiplied by the corresponding LP inequalities and summed over all inequalities yield a proof that Mtlm is $R_0$-competitive.

Among other advantages, this approach allows to numerically find instances that are tight for the current analysis (cf. Section 2 and Figure 1): linear and tripartite instances can be obtained this way.
B.2. LP analysis of Dlm-like algorithms

Now we show how to adapt LP from the previous section to analyze Dlm-type of algorithms. Recall that after $1.75D$ requests, Dlm evaluates the geometry of the so-far-received requests and decides whether to continue this phase or not. Although the final parameters of Dlm are elegant numbers (multiplicities of 1/4), they were obtained by a tedious optimization process using the LP we present below. Furthermore, the LP below does not give us explicit rule for continuing the phase; it only tells that Dlm is successful in either short or in a long phase.

Recall that in a phase, Dlm considers three groups of consecutive $\delta_i \cdot D$ requests: $R_1, R_2$ and $R_3$, where $\delta_1, \delta_2$ and $\delta_3$ are parameters of Dlm. First, assume that Dlm always processes three parts and afterwards it moves the file to a point that minimizes the function

$$h(x) = [A_0, x] + a_1 \cdot [x, R_1] + a_2 \cdot [x, R_2] + a_3 \cdot [x, R_3],$$

where $a_i$ are parameters that we choose later. We use $A_3$ to denote the minimizer of function $h$, Ortl to denote the strategy of an optimal algorithm (short for Ortl-Long). Finally, $O^L_0, O^L_1, O^L_2$ and $O^L_3$ denote the trajectory of Ortl ($O^L_0$ is the initial position of Ortl’s file at the beginning of the phase, and $O^L_1$ is its position right after $i$-th part of the phase). Analogously to the previous section, we obtain the following LP.

\[
\begin{align*}
\text{minimize} & \quad C_{\text{ALGL}} \\
\text{subject to} & \quad C_{\text{ALGL}} = [A_0, A_3] + \sum_{i=1,2,3} \delta_i \cdot [A_0, R_i] + \phi \cdot ([A_3, O^L_3] - [A_0, O^L_0]) \\
& \quad C_{\text{OPTL}} = 1 \\
& \quad C_{\text{OPTL}} = \sum_{i=1,2,3} \left( C_{\text{OPTL}}^{\text{req}}(i) + C_{\text{OPTL}}^{\text{move}}(i) \right) \\
& \quad C_{\text{OPTL}}^{\text{move}}(i) \geq [O^L_{i-1}, O^L_i] \\
& \quad 2 \cdot C_{\text{OPTL}}^{\text{req}}(i) + \delta_i \cdot C_{\text{OPTL}}^{\text{move}}(i) \geq \delta_i \cdot [O^L_{i-1}, R_i] + \delta_i \cdot [O^L_i, R_i] \\
& \quad h(A_3) \geq h(v) \\
& \quad 0 \leq [v_1, v_2] \leq [v_1, v_2] + [v_2, v_3] 
\end{align*}
\]

This time $V = [A_0, A_3, O^L_0, O^L_1, O^L_2, O^L_3]$ and $V' = V \cup \{R_1, R_2, R_3\}$.

We note that such parameterization alone does not improve the competitive ratio, i.e., for any choice of parameters $\delta_i$ and $a_i$, the objective value of the LP above is at least $R_0 \approx 4.086$.

However, as stated in Section 2.1, Dlm verifies if after two phases it can migrate its file to a node $A_2$ being the minimizer of the function

$$g(x) = [A_0, x] + \beta_1 \cdot [x, R_1] + \beta_2 \cdot [x, R_2],$$

where $\beta_i$ are parameters that we choose later.

In our analysis, we gave an explicit rule whether migration to $A_2$ should take place. However, for our LP-based approach, we follow a slightly different scheme. Namely, if migration to $A_2$ guarantees that the amortized cost in the first two phases is at most 4 times the cost of any strategy for the two phases, then Dlm may move to $A_2$ and we immediately achieve competitive ratio 4 on these two phases. Otherwise, we may add additional constraints to LP, stating that the competitive ratio of an algorithm which moves to $A_2$ is at least 4 against some chosen strategy Orts. Analogously to Ortl, the trajectory of Orts is described by three points $O^S_0, O^S_1$ and $O^S_2$. 

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This allows us to add the following inequalities to LP:

\[
\begin{align*}
C_{\text{ALGS}} &= [A_0, A_2] + \sum_{i=1,2} \delta_i \cdot [A_0, R_i] + \phi \cdot ([A_2, O_2^S] - [A_0, O_0^S]) \\
C_{\text{OPTS}} &= \sum_{i=1,2} \left( C_{\text{req}}(i) + C_{\text{move}}(i) \right) \\
C_{\text{move}}(i) &\geq [O_{i-1}^S, O_i^S] \quad \text{for } i = 1, 2 \\
2 \cdot C_{\text{req}}(i) + \delta \cdot C_{\text{move}}(i) &\geq \delta_i \cdot [O_{i-1}^S, R_i] + \delta_i \cdot [O_i^S, R_i] \quad \text{for } i = 1, 2 \\
g(A_2) &\geq g(v) \quad \text{for all } v \in V \\
C_{\text{ALGS}} &\geq 4 \cdot C_{\text{OPTS}}
\end{align*}
\]

We also set \( V = \{ A_0, A_3, O_0^L, O_1^L, O_2^L, O_3^L, O_0^S, O_1^S, O_2^S \} \), both in new and in old inequalities.

When we choose \( \phi = 3 \), fix phase length parameters to be \( \delta_1 = 1 \), \( \delta_2 = 0.75 \), \( \delta_3 = 0.5 \) and parameters for functions \( g \) and \( h \) to be \( \beta_1 = 2 \), \( \beta_2 = 1 \), \( \alpha_1 = 1 \), \( \alpha_2 = 0.25 \) and \( \alpha_3 = 0.75 \), we obtain that the value of the above LP is 4. Again, this can be interpreted as a numerical argument that \( \text{DLM} \) is indeed a 4 competitive algorithm.