Generalized Backward doubly SDEs driven by Lévy processes with discontinuous and linear growth coefficients

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Abstract

This paper deals with generalized backward doubly stochastic differential equations driven by a Lévy process (GBDSDEL, in short). Under left or right continuous and linear growth conditions, we prove the existence of minimal (resp. maximal) solutions.

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1 Introduction

Backward stochastic differential equations (BSDEs in short) are first appeared with the works of J.M. Bismut in linear form as the adjoint processes in the maximum principal of stochastic control. The non-linear BSDEs have been introduced by Pardoux and Peng in order to give a probabilistic interpretation for the solutions of both parabolic and elliptic semi-linear partial differential equations (PDEs) generalizing the well-known Feynman-Kac formula (see Pardoux and Peng and Peng). Since then, the theory of BSDEs has been developed because of its many applications in the theory of mathematical finance (El Karoui et al., in stochastic control and stochastic games (El Karoui and Hamadène, Hamadène and Lepeltier).

Furthermore, Pardoux and Zhang gave a probabilistic formula for the viscosity solution of a system of PDEs with a nonlinear Neumann boundary condition by introducing a generalized BSDEs (GBSDEs in short) which involved an integral with respect to an adapted continuous increasing process.

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T k(s, Y_s) dA_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \] (1.1)

On the other hand, in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs in short), Pardoux and Peng introduced a new class of BSDEs driven by two Brownian motions, the so called backward doubly stochastic differential equations (BDSDEs in short). Next, using this kind of BDSDEs, Bally and Matoussi gave the probabilistic representation of the weak solutions of parabolic semi linear SPDEs in Sobolev spaces.
Inspired by [21] and [22], Boufoussi et al. [7] recommended a class of generalized BDSDEs (GBDSDEs in short) and gave the probabilistic representation for stochastic viscosity solutions of semilinear SPDEs with a Neumann boundary condition.

\[
Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T k(s,Y_s)dA_s + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{1.2}
\]

In [17], Nualart and Schoutens gave a martingale representation theorem associated to Lévy process. This result allows them to establish in [18] the existence and uniqueness result for BSDEs driven by Lévy process with moments of all orders.

Motivated by the above works, especially, by [22], [17] and [18], Hu and Ren [13] showed existence and uniqueness result to GBDSDEs driven by Lévy process (GBDSDEL, in short) under Lipschitz condition on the generators and gave the probabilistic interpretation for solutions of a class of stochastic partial differential integral equations (SPDIEs, in short) with a nonlinear Neumann boundary condition.

\[
Y_t = \xi + \int_t^T f(s,Y_s^- ,Z_s)ds + \int_t^T k(s,Y_s^-)dA_s + \int_t^T g(s,Y_s^- ,Z_s)dB_s - \sum_{i=1}^{\infty} \int_t^T Z_t^{(i)}dH_s^{(i)}, \quad 0 \leq t \leq T. \tag{1.3}
\]

Recently, Aman and Owo [3] relaxed the Lipschitz condition on the generators in Hu and Ren [13]. More precisely, they established under continuous and linear growth conditions on the generators, the existence result to GBDSDEs driven by Lévy process \(L\) which have only \(m\) different jump size with no continuous part.

\[
Y_t = \xi + \int_t^T f(s,Y_s^- ,Z_s)ds + \int_t^T k(s,Y_s^-)dA_s + \int_t^T g(s,Y_s^- ,Z_s)dB_s - \sum_{i=1}^{m} \int_t^T Z_t^{(i)}dH_s^{(i)}, \quad 0 \leq t \leq T. \tag{1.4}
\]

The proof is strongly linked to the comparison theorem which does not hold in general for BDSDEs with jumps (see the counter-example in Buckdahn et al. [6]). To overcome this difficulty, they assumed an additional relation between the generator \(f\) and the jumps size of the Lévy process \(L\).

Note that, in the previous works on GBDSDEs driven by Lévy process, the generators are at least continuous (see Hu and Ren [13] for Lipschitz continuous, and Aman and Owo [3] for continuous and linear growth). But, unfortunately, the continuous conditions can not be satisfied in certain models that makes the results in [13] and [3] not applied for several applications (finance, stochastic control, stochastic games, SPDEs, etc,...). For example, let the function \(f\) be defined by \(f(t,y,z) = y1_{\{y > 1\}} + \psi(t,z)\), where \(\psi\) is a Lipschitz continuous function in \(z\). Such a function \(f\) is not continuous in \(y\). Then we can not apply the existence results in [13] and [3] to get the existence of solution of the above GBDSDEL (1.4) with such a coefficient \(f\). To correct this shortcoming, we relax in this paper the conditions on the function \(f\) in [13] and [3] by using a left or a right continuous and linear growth conditions and derive the existence of minimal (resp. maximal) solutions to GBDSDEL (1.4).

The paper is organized as follows. In section 2, we give some notations and preliminaries. Section 3 is devoted to the existence of minimal (resp. maximal) solutions result.

## 2 Preliminaries

### 2.1 Notations and Definition

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space on which are defined all the processes stated in this paper and \(T\) be a fixed final time. Throughout this paper, \(\{B_t; 0 \leq t \leq T\}\) will denote a standard
one-dimensional Brownian motion and \{L_t; 0 \leq t \leq T\} will denote a Lévy process independent of \{B_t; 0 \leq t \leq T\} corresponding to a standard Lévy measure \nu such that \int_{\mathbb{R}} (1 \wedge x)\nu(dx) < \infty. Let \mathcal{N} denote the class of \mathbb{P}\text{-}null sets of \mathcal{F}. For each \( t \in [0, T] \), we define

\[ \mathcal{F}_t \Delta \mathcal{F}_t^L \vee \mathcal{F}_t^B, \]

where for any process \{\eta_r \mid \eta_r \leq r \leq t\} \vee \mathcal{N}, \mathcal{F}_t^B = \mathcal{F}_0^B.

Note that \{\mathcal{F}_t^L, t \in [0, T]\} is an increasing filtration and \{\mathcal{F}_t^B, t \in [0, T]\} is a decreasing filtration. Thus, the collection \{\mathcal{F}_t, t \in [0, T]\} is neither increasing nor decreasing so it does not constitute a filtration.

Furthermore, we denote by \( L_t = \lim_{s \to t+} L_s \) the left limit process and by \( \Delta L_t = L_t - L_{t-} \) the jump size at time \( t \). The power jumps of the Lévy process \( L \) are defined by

\[ L_t^{(i)} = L_t \quad \text{and} \quad L_t^{(i)} = \sum_{0 < s \leq t} (\Delta L_s)^i, \quad i \geq 2. \]

The Teugels Martingale associated with the Lévy process \( L \) are defined by

\[ T_t^{(i)} = L_t^{(i)} - E(L_t^{(i)}) = L_t^{(i)} - tE(L_t), \quad i \geq 1 \]

Let \( (H^{(i)})_{i \geq 1} \) be the family of processes defined by

\[ H_t^{(i)} = c_1 T_t^{(i)} + c_{i, i-1} T_t^{(i-1)} + \ldots + c_{i, i} T_t^{(1)}, \quad i \geq 1. \]

In [18], Nualart and Schoutens proved that the coefficients \( c_{i,k} \) correspond to the orthonormalization of the polynomials \( 1, x, x^2, \ldots \) with respect to the measure \( \mu(dx) = x^2 \nu(dx) + \sigma^2 \delta_0(dx) \), i.e. \( q_i(x) = c_i x^{i-1} + c_{i,i-1} x^{i-2} + \ldots + c_{i,1} \). The martingale \( (H^{(i)})_{i \geq 1} \) can be chosen to be pair-wise strongly orthonormal martingale. That is \( [H^{(i)}, H^{(j)}], i \neq j \) and \( \{[H^{(i)}, H^{(j)}]_{t}, 0 \leq t \leq T\} \) are uniformly integrable martingale with initial value 0, i.e., for all \( i, j \), \([H^{(i)}, H^{(j)}]_{t} = \delta_{i,j}t\).

**Remark 2.1.** In the case of a Poisson process \( \mathcal{N} \) with intensity \( \lambda > 0 \), all the Teugels martingales are equal to \( T_t^{(i)} = \mathcal{N}_t - \lambda t \). Therefore, \( H_t^{(i)} = \frac{\mathcal{N}_t - \lambda t}{\sqrt{\lambda}} \) and \( H_t^{(i)} = 0 \), for all \( i \geq 2 \).

In this paper, we will consider a Lévy process that has only \( m \) different jump sizes with no continuous part, i.e.,

\[ H^{(i)} = 0, \quad \forall \ i \geq m + 1 \quad \text{and} \quad [H^{(i)}, H^{(j)}]_{t} = \delta_{i,j}t, \quad \forall t \in [0, T], i, j \in \{1, \ldots, m\}. \]

In the sequel, \( \{A_t; 0 \leq t \leq T\} \) is a \( \mathcal{F}_t \)-measurable, continuous and increasing real valued process such that \( A_0 = 0 \).

Let us introduce some spaces:

- \( \mathcal{M}^2(0, T, \mathbb{R}^m) \) the set of jointly measurable processes \( \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^m \), such that \( \varphi_t \) is \( \mathcal{F}_t \)-measurable, for any \( t \in [0, T] \), with

\[ ||\varphi||^2_{\mathcal{M}^2(\mathbb{R}^m)} = \mathbb{E}\left( \int_0^T ||\varphi_t||^2 dt \right) = \sum_{i=1}^m \mathbb{E}\left( \int_0^T ||\varphi_t^{(i)}||^2 dt \right) < \infty. \]

- \( \mathcal{S}^2(0, T) \) the set of jointly measurable continuous processes \( \varphi : \Omega \times [0, T] \rightarrow \mathbb{R} \), such that \( \varphi_t \) is \( \mathcal{F}_t \)-measurable, for any \( t \in [0, T] \), with \( ||\varphi||^2_{\mathcal{S}^2} = \mathbb{E}\left( \sup_{0 \leq t \leq T} ||\varphi_t||^2 \right) < \infty. \)

- \( \mathcal{A}^2(0, T) \) the class of \( \mathcal{F}^{dP} \otimes dA_t \) a.e. equal measurable random processes \( \varphi : \Omega \times [0, T] \rightarrow \mathbb{R} \) such that \( \varphi_t \) is \( \mathcal{F}_t \)-measurable, for any \( t \in [0, T] \), with \( ||\varphi||^2_{\mathcal{A}^2} = \mathbb{E}\left( \int_0^T ||\varphi_t||^2 dA_t \right) < \infty. \)
The space $E_m(0,T) = (\mathcal{S}^2(0,T) \cap \mathcal{A}^2(0,T)) \times \mathcal{M}^2(0,T,\mathbb{R}^m)$ endowed with norm

$$
\|(Y,Z)\|_{E_m}^2 = \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Y_s|^2 \, dA_s + \int_0^T |Z_s|^2 \, ds\right)
$$

is a Banach space.

**Definition 2.2.** A pair of $\mathbb{R} \times \mathbb{R}^m$-valued process $(Y,Z)$ is called solution of GBDSDEL $(\xi,f,g,h,A)$ driven by Lévy processes if $(Y,Z) \in E_m(0,T)$ and verifies (1.3).

**Definition 2.3.** A pair of $\mathbb{R} \times \mathbb{R}^m$-valued process $(Y,Z)$ (resp. $(\bar{Y},\bar{Z})$) is said to be a minimal (resp. maximal) solution of GBDSDEL if for any other solution $(\tilde{Y},\tilde{Z})$ of GBDSDEL, we have $\tilde{Y} \leq Y$ (resp. $Y \leq \bar{Y}$).

### 2.2 GBDSDEL with Lipschitz coefficients

First, we recall the existence and uniqueness result in Ren et al. [13] and the comparison theorem in Aman and Owo [3]. To this end, we consider the following assumptions:

- **(A1)** The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ such that for all $\lambda > 0$, $\mathbb{E}(e^{\lambda |\xi|^2}) < \infty$.
- **(A2)** Let $f,g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $k : \Omega \times [0,T] \times \mathbb{R} \rightarrow \mathbb{R}$ be jointly measurable processes, such that:
  - (i) for a.e. $t \in [0,T]$ and any $(y,z) \in \mathbb{R} \times \mathbb{R}^m$, $f(t,y,z), g(t,y,z), k(t,y)$ are $\mathcal{F}_t$-measurable;
  - (ii) there exists a constant $K > 0$ and three $\mathcal{F}_t$-measurable processes $\{f_t, g_t, k_t : 0 \leq t \leq T\}$ with values in $[1, \infty]$, such that for all $t \in [0,T], (y,z) \in \mathbb{R} \times \mathbb{R}^m$,
    $$
    \begin{align*}
    |f(t,y,z)| &\leq f_t + K\|y\| + \|z\| \\
    |g(t,y,z)| &\leq g_t + K\|y\| + \|z\| \\
    |k(t,y)| &\leq k_t + K|y|
    \end{align*}
    $$
  - (iii) there exists two constants $K > 0, 0 < \alpha < 1$ such that for all $t \in [0,T], (y,z), (y',z') \in \mathbb{R} \times \mathbb{R}^m$,
    $$
    \begin{align*}
    |f(t,y,z) - f(t,y',z')| &\leq K(|y - y'| + \|z - z'\|) \\
    |g(t,y,z) - g(t,y',z')|^2 &\leq K|y - y'|^2 + \alpha \|z - z'\|^2 \\
    |k(t,y) - k(t,y')| &\leq K|y - y'|
    \end{align*}
    $$

**Theorem 2.4 (Ren et al. [13]).** Under assumptions (A1) and (A2), the GBDSDEL (1.3) has a unique solution.

Then, considering GBDSDEL (1.4), we have the following comparison theorem established by Aman and Owo [3].

**Theorem 2.5 (Aman and Owo [3]).** Assume (A1) and (A2), and let $(Y_1,Z_1)$ and $(Y_2,Z_2)$ be the solutions of GBDSDEL (1.4) associated with $(\xi_1,f_1,g,k^1,A)$ and $(\xi_2,f_2,g,k^2,A)$ respectively. We suppose:

- $\xi_2 \geq \xi_1$, $\mathbb{P}$-a.s.,
- $f^2(t,y,z) \geq f^1(t,y,z)$ and $k^2(t,y) \geq k^1(t,y)$ $\mathbb{P}$-a.s., for all $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^m$,
- $\beta_t^{(i)} = \frac{f^1(t,y^2,z^{(i)}) - f^1(t,y^2,z^{(-1)})}{z^{(i)}_2 - z^{(-1)}_2} \mathbb{1}_{\{z^{(i)} \neq z^{(-1)}\}}\}$


where \( z^{(i)} = \left( z_{2}^{(1)}, z_{2}^{(2)}, \ldots, z_{2}^{(i)}, z_{1}^{(i+1)}, \ldots, z_{1}^{(m)} \right) \) such that
\[
\sum_{i=1}^{m} \beta_{i}^{1} \Delta H_{i}^{(i)} > -1, \quad dt \otimes d\mathbb{P}. \quad (2.1)
\]

Then, we have for all \( t \in [0, T] \), \( Y_{t}^{2} \geq Y_{t}^{1} \), a.s.

Moreover, for all \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{m}\), if \( \xi^{2} > \xi^{1} \), or \( f^{2}(t, y, z) > f^{1}(t, y, z) \), or \( k^{2}(t, y) > k^{1}(t, y) \), a.s., \( Y_{t}^{2} > Y_{t}^{1} \), a.s., \( \forall t \in [0, T] \).

Remark 2.6. - If the Lévy process \( L \) has only positive jumps size, then the inequality \((2.1)\) holds when, for each \( (t, y) \in [0, T] \times \mathbb{R} \), the function \( z \mapsto f(t, y, z) \) is non-decreasing.

- If the Lévy process \( L \) has only negative jumps size, then the inequality \((2.1)\) holds when, for each \( (t, y) \in [0, T] \times \mathbb{R} \), the function \( z \mapsto f(t, y, z) \) is non-increasing.

Remark 2.7. If the inequality \((2.1)\) does not hold, then, in general, the comparison theorem does not hold. We give now a counter-example.

Let \( N_{t} \) be a Poisson process with intensity \( \lambda > 0 \), then \( H_{t}^{(1)} = \frac{N_{t}}{\sqrt{\lambda}} \), and \( H_{t}^{(i)} = 0, \ i \geq 2 \).

Now, let \( f(t, y, z) = -(1 + \sqrt{\lambda})z \), and consider the following equations \((j = 1, 2)\) :
\[
Y_{t}^{j} = \xi^{j} + \int_{t}^{T} -(1 + \sqrt{\lambda})Z_{s}^{j}ds - \int_{t}^{T} Z_{s}^{j}dH_{s}^{(1)}, \quad 0 \leq t \leq T.
\]

If we choose
\[
\xi^{2} = \frac{N_{T}}{\sqrt{\lambda}} \quad \text{and} \quad \xi^{1} = 0,
\]
then
\[
(Y_{t}^{2}, Z_{t}^{2}) = \left( \frac{N_{T}}{\sqrt{\lambda}} + t - T, 1 \right) \quad \text{and} \quad (Y_{t}^{1}, Z_{t}^{1}) = (0, 0).
\]

Clearly, \( \xi^{2} \geq \xi^{1} \), but \( \mathbb{P}(Y_{t}^{2} < Y_{t}^{1}) = \mathbb{P}(N_{t} < \sqrt{\lambda}(T - t)) > 0 \), for all \( 0 \leq t < T \).

2.3 GBDSDEL with continuous coefficients

Furthermore, using the above comparison theorem and the well know approximation method of the functions \( f \) and \( k \), Aman and Owo \( [4] \) proved the existence of a minimal or maximal solution for GBDSDEL \((1.4)\) under the continuous and linear growth conditions. In this section, we recall this result. In this fact, we consider the following assumptions :

(H1) The terminal value \( \bar{\xi} \in L^{2}(\Omega, \mathcal{F}_{T}, \mathbb{P}, \mathbb{R}) \) such that for all \( \lambda > 0 \), \( \mathbb{E}(e^{\lambda \bar{\xi}^{2}}) < \infty \).

(H2) Let \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{m} \to \mathbb{R} \) and \( g, k : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \) be jointly measurable processes, such that :

(i) for a.e. \( t \in [0, T] \) and any \( (y, z) \in \mathbb{R} \times \mathbb{R}^{m} \), \( f(t, y, z) \), \( g(t, y) \), \( k(t, y) \) are \( \mathcal{F}_{t} \)-measurable;

(ii) there exists a constant \( K > 0 \) and three \( \mathcal{F}_{t} \)-measurable processes \( \{f_{t}, g_{t}, k_{t} : 0 \leq t \leq T\} \) with values in \([1, \infty]\), such that for all \( t \in [0, T] \), \( (y, z) \in \mathbb{R} \times \mathbb{R}^{m} \),
\[
\begin{cases}
|f(t, y, z)| \leq f_{t} + K(|y| + |z|) \\
g(t, y) \leq g_{t} + K|y|, \ \text{with} \ g(t, 0) = 0 \\
k(t, y) \leq k_{t} + K|y|
\end{cases}
\]

(iii) for all \( \mu, \lambda > 0 \), \( \mathbb{E}\left( \int_{0}^{T} e^{\mu + \lambda A_{t}} f_{t}^{2} dt + \int_{0}^{T} e^{\mu + \lambda A_{t}} g_{t}^{2} dt + \int_{0}^{T} e^{\mu + \lambda A_{t}} k_{t}^{2} dt \right) < \infty; \)

(iv) there exists a constant \( K > 0 \), such that for all \( t \in [0, T] \), \( (y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^{m} \),
\[
\begin{cases}
|f(t, y, z) - f(t, y, z')| \leq K|z - z'| \\
|g(t, y) - g(t, y')| \leq K|y - y'|^{2} \\
|k(t, y) - k(t, y')| \leq K|y - y'|^{2}
\end{cases}
\]
(v) for all \((t,z) \in [0,T] \times \mathbb{R}^m, y \mapsto f(t,y,z)\) and \(y \mapsto k(t,y)\) are continuous, for all \(\omega\) a.e.

**Theorem 2.8** (Aman and Owo [3]). Under assumptions (H1) and (H2), the GBDSDEL (1.4) has a minimal solution \((\mathcal{Y}, \mathcal{Z}) \in \mathcal{E}_m(0, T)\) (resp. a maximal solution \((\overline{\mathcal{Y}}, \overline{\mathcal{Z}}) \in \mathcal{E}_m(0, T))\).

Moreover, using the same argument as in [3], one can establish the following comparison theorem which requires at least one Lipschitz function:

**Theorem 2.9** (Comparison with at least one Lipschitz function). Let \(g\) and \(\xi^i \ (i = 1, 2)\) satisfy (H1) and (H2). Assume that GBDSDEL \((\xi^i, f^i, g, k^i, A)\) have solutions \((\mathcal{Y}^i, \mathcal{Z}^i) \in \mathcal{E}_m(0, T), i = 1, 2\), respectively. Assume moreover that

- \(\xi^2 \geq \xi^1, \mathbb{P}\)-a.s.,
- \(f^1, k^1 \) satisfy (A2) such that \(f^2(t, \mathcal{Y}^2_t, \mathcal{Z}^2_t) \geq f^1(t, \mathcal{Y}^1_t, \mathcal{Z}^1_t), \) and \(k^2(t, \mathcal{Y}^2_t) \geq k^1(t, \mathcal{Y}^1_t)\) \(\mathbb{P}\)-a.s., for all \(t \in [0, T]\), with

\[
\beta_i^t = \frac{f^1(t, \mathcal{Y}^1_t, \mathcal{Z}^1(i)) - f^1(t, \mathcal{Y}^1_t, \mathcal{Z}^1(i-1))}{\mathcal{Z}^2(i) - \mathcal{Z}^1(i)} \mathbf{1}_{\{\mathcal{Z}^2(i) \neq \mathcal{Z}^1(i)\}},
\]

where \(\mathcal{Z}^i = (\mathcal{Z}^i_1, \mathcal{Z}^i_2, \ldots, \mathcal{Z}^i_1, \mathcal{Z}^1(i+1), \ldots, \mathcal{Z}^1(m))\) \(\text{(resp. } f^2, k^2 \text{ satisfy (A2) such that } f^2(t, \mathcal{Y}^2_t, \mathcal{Z}^2_t) \geq f^1(t, \mathcal{Y}^1_t, \mathcal{Z}^1_t), \) and \(k^2(t, \mathcal{Y}^2_t) \geq k^1(t, \mathcal{Y}^1_t)\) \(\mathbb{P}\)-a.s., for all \(t \in [0, T]\), with

\[
\beta_i^t = \frac{f^2(t, \mathcal{Y}^1_t, \mathcal{Z}^1(i)) - f^2(t, \mathcal{Y}^1_t, \mathcal{Z}^1(i-1))}{\mathcal{Z}^2(i) - \mathcal{Z}^1(i)} \mathbf{1}_{\{\mathcal{Z}^2(i) \neq \mathcal{Z}^1(i)\}},
\]

where \(\mathcal{Z}^i = (\mathcal{Z}^i_1, \mathcal{Z}^i_2, \ldots, \mathcal{Z}^i_1, \mathcal{Z}^1(i+1), \ldots, \mathcal{Z}^1(m))\), such that

\[
\sum_{i=1}^{m} \beta_i^t \Delta H^i_t > -1, \ dt \otimes d\mathbb{P}\text{-a.s.} \quad (2.2)
\]

Then, \(\mathcal{Y}^2_t \geq \mathcal{Y}^1_t\) a.s., for all \(t \in [0, T]\).

### 3 GBDSDEL with left-continuous coefficients

The objective of this section is to prove an existence of minimal solution to GBDSDEL \((\xi, f, g, k, A)\) with left-continuous and linear growth coefficients. More precisely, when the following assumptions hold.

**C1** The terminal value \(\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})\) such that for all \(\lambda > 0\), \(\mathbb{E}(e^{\lambda \xi}) < \infty\).

**C2** Let \(f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}\) and \(g, k : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}\) be jointly measurable processes, such that:

- (i) for a.e. \(t \in [0, T]\) and any \((y, z) \in \mathbb{R} \times \mathbb{R}^m, f(t, y, z), g(t, y) k(t, y)\) \(\mathcal{F}_t\)-measurable;
- (ii) there exists a constant \(K > 0\) and three \(\mathcal{F}_t\)-measurable processes \(\{f_t, g_t, k_t : 0 \leq t \leq T\}\) with values in \([1, \infty]\), such that for all \(t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^m,\)

\[
\begin{cases}
|f(t, y, z)| \leq f_t + K(|y| + |z|) \\
g(t, y) \leq g_t + K|y|, \text{ with } g(t, 0) = 0 \\
k(t, y) \leq k_t + K|y|
\end{cases}
\]
(iii) for all $\mu, \lambda > 0$, \( E \left( \int_0^T e^{\mu t + \lambda A_t} f^2_t dt + \int_0^T e^{\mu t + \lambda A_t} g^2_t dt + \int_0^T e^{\mu t + \lambda A_t} k^2_t dA_t \right) < \infty \);

(iv) there exists a constant $K > 0$, such that for all $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^m$,
\[
\begin{cases}
|f(t, y, z) - f(t, y', z')| \leq K\|z - z'\| \\
g(t, y) - g(t, y') \leq K|y - y'|^2
\end{cases}
\]

(v) for all $(t, z) \in [0, T] \times \mathbb{R}^m$, $y \mapsto f(t, y, z)$ and $y \mapsto k(t, y)$ are left-continuous, for all $\omega$ a.e.

(vi) there exists two functions $h : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$, satisfying, for all $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^m$:
\[
\begin{cases}
|h(y, z)| \leq K(|y| + \|z\|) \text{ and } |p(y)| \leq K|y|, \\
h(y, z) - h(y, z') \leq K\|z - z'\|, \\
y \mapsto h(y, z) \text{ and } y \mapsto p(y) \text{ are continuous}
\end{cases}
\]
such that for all $y \geq y'$, $z, z' \in \mathbb{R}^m$, $t \in [0, T]$, we have
\[
f(t, y, z) - f(t, y', z') \geq h(y - y', z - z') \quad \text{and} \quad k(t, y) - k(t, y') \geq p(y - y').
\]

In order to reach our objective, we need some preliminary results. First, we prove the following comparison theorem which requires at least one continuous function.

**Theorem 3.1** (Comparison with at least one continuous function). Let $g$ and $\xi^i$ ($i = 1, 2$) satisfy (C1) and (C2). Assume that GBDSDEL $(\xi^i, f^i, g, k^i, A)$ have solutions $(Y^i, Z^i) \in \mathcal{E}_m(0, T)$, $i = 1, 2$, respectively. Assume moreover that

- $\xi^2 \geq \xi^1$, $\mathbb{P}$-a.s.
- $f^1$ and $k^1$ satisfy (H2) such that $f^2(t, Y^2_t, Z^2_t) \geq f^1(t, Y^1_t, Z^1_t)$, and $k^2(t, Y^2_t) \geq k^1(t, Y^1_t)$ $\mathbb{P}$-a.s., for all $t \in [0, T]$, and $(Y^1, Z^1)$ is the minimal solution, with
\[
\beta^i_t = \frac{f^1(t, Y^1_t, \overline{Z}^1_t) - f^1(t, Y^2_t, \overline{Z}^2_t)}{Z^2_t - Z^1_t} \mathbbm{1}_{\{Z^2_t \neq Z^1_t\}},
\]
where $\overline{Z}^i_t = \left( Z^2_t^{(1)}, Z^2_t^{(2)}, \ldots, Z^2_t^{(i)}, Z^1_t^{(i+1)}, \ldots, Z^1_t^{(m)} \right)$

(resp. $f^2$ and $k^2$ satisfy (H2) such that $f^2(t, Y^1_t, Z^1_t) \geq f^1(t, Y^1_t, Z^1_t)$, and $k^2(t, Y^1_t) \geq k^1(t, Y^1_t)$ $\mathbb{P}$-a.s., for all $t \in [0, T]$, and $(Y^2, Z^2)$ is the maximal solution, with
\[
\beta^i_t = \frac{f^2(t, Y^1_t, \overline{Z}^1_t) - f^2(t, Y^1_t, \overline{Z}^2_t)}{Z^1_t - Z^2_t} \mathbbm{1}_{\{Z^1_t \neq Z^2_t\}},
\]
where $\overline{Z}^i_t = \left( Z^1_t^{(1)}, Z^1_t^{(2)}, \ldots, Z^1_t^{(i)}, Z^2_t^{(i+1)}, \ldots, Z^2_t^{(m)} \right)$), such that
\[
\sum_{i=1}^m \beta^i_t \Delta H^i_t > -1, \ dt \otimes d\mathbb{P}$-a.s. \quad (3.1)
\]

Then, $Y^2_t \geq Y^1_t$ a.s., for all $t \in [0, T]$.

**Proof.** Assume that $f^1$ and $k^1$ satisfy (H2). Then, by Lemma 3.2 in [4], there exist sequences of functions $f^1_n$ and $k^1_n$ associated to $f^1$ and $k^1$, respectively, defined by:
\[
f^1_n(t, y, z) = \inf_{u \in \mathbb{Q}} \left\{ f^1(t, u, z) + n|y - u| \right\} \quad \text{and} \quad k^1_n(t, y) = \inf_{u \in \mathbb{Q}} \left\{ k^1(t, u) + n|y - u| \right\}
\]
which are Lipschitz functions such that $f_n^1 \leq f^1$ and $k_n^1 \leq k^1$, for all $n \geq K$. Furthermore, $f_n^1$ and $k_n^1$ converge suitably to $f^1$ and $k^1$, respectively. Hence, by Theorem 2.9, it follows that, for every $n \geq K$, GBDSDEL $(ξ^1, f^1, g, k^1, A)$ has a unique measurable solution $(Y^1, Z^1)$. Moreover, from the proof of the Theorem 3.1 in [3], it follows that $(Y^1, Z^1)$ converges to the minimal solution $(Y^1, Z^1)$ of GBDSDEL $(ξ^1, f^1, g, k^1, A)$. On the other hand, since $f^1(t, Y^2, Z^2) \leq f^2(t, Y^2, Z^2)$ and $k^1(t, Y^2) \leq k^2(t, Y^2)$, a.s., we have

$$f_n^1(t, Y^2, Z^2) \leq f^2(t, Y^2, Z^2) \text{ and } k_n^1(t, Y^2) \leq k^2(t, Y^2), \text{ a.s., for all } n \geq K.$$  

Moreover, by the definition of $f_n$, it follows that

$$f_n^1(t, Y^2, Z^i) - f_n^1(t, Y^2, \tilde{Z}^i(\cdot)) = f^1(t, Y^2, Z^i) - f^1(t, Y^2, \tilde{Z}^i(\cdot)),$$

and inequality (3.1) still holds. Therefore, by Theorem 2.9 we get $Y^1, Z^1 \leq Y^2, Z^2$, a.s., for all $n \geq K$. Hence, we have $Y^1 \leq Y^2$.

Next, we establish the following result, which will be useful in the sequel.

**Lemma 3.2.** Let $g, ξ$ satisfy (C1) and (C2), and $h$ and $p$ be the functions appear in assumption (C2). Assume moreover that, for any $y \in ℝ$, the rate of change of $h(y, \cdot)$:

$$α = h(y, \hat{ξ}^{(i)}) - h(y, \hat{ξ}^{(i-1)}) \mathbf{1}_{\{ξ^{(i)} \neq ξ^{(i-1)}\}},$$

where $ξ^{(i)} = \left(ξ^{(1)}, ξ^{(2)}, ..., ξ^{(i)}, ξ^{(i+1)}, ..., ξ^{(m)}\right)$, for all $ξ^{(1)}, ξ^{(2)} \in ℝ^m$ satisfies the following relation:

$$\sum_{i=1}^{m} α_i ΔH_i^{(i)} > -1, \text{ for } d\mathbb{P}-\text{a.s.} \quad (3.2)$$

Let $φ$ and $ψ$ be processes such that, for a.e. $t \in [0, T]$, $φ(t)$ and $ψ(t)$ are $ℱ_t$-measurable and $Ε\left(\int_0^T |φ(s)|^2 ds + \int_0^T |ψ(s)|^2 dA_s\right) < +∞$. We consider the following GBDSDEL:

$$Y_t = ξ + \int_t^T (h(Y_s, Z_s) + φ(s)) ds + \int_t^T (p(Y_s) + ψ(s)) dA_s + \int_t^T g(s, Y_s) d\overline{B}_s$$

$$- \sum_{i=1}^{m} \int_t^T Z_i^{(i)} dH_i^{(i)}, \quad 0 \leq t \leq T. \quad (3.3)$$

Then, the GBDSDEL $(3.3)$ has a solution $(Y, Z) \in 𝒟_m(0, T)$. Moreover, for any solution $(Y, Z)$ of GBDSDEL $(3.3)$, if $ξ ≥ 0$, and $φ(t) ≥ 0$ and $ψ(t) ≥ 0$ for any $t \in [0, T]$, we have $Y_t ≥ 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.

**Proof.** Since $h$ and $p$ are continuous and linear growth, by virtue of Theorem 2.8 the GBDSDEL $(3.3)$ has at least one solution $(Y, Z) \in 𝒟_m(0, T)$. Moreover, by Lemma 3.2 in [3], there exist non-decreasing sequences of Lipschitz functions $h_n$ and $p_n$ associated to $h$ and $p$, respectively, defined by:

$$h_n(y, z) = \inf_{u ∈ Q} \{h(u, z) + n|y - u|\} \text{ and } p_n(y) = \inf_{u ∈ Q} \{p(u) + n|y - u|\},$$

with $|h_n(y, z)| ≤ K(|y| + |z|)$ and $|p_n(y)| ≤ K|y|$. In view of Theorem 2.4 for every $n ≥ K$, the equations

$$Y_n^{(i)} = ξ + \int_t^T (h_n(Y_s^{(i)}, Z_s^{(i)}) + φ(s)) ds + \int_t^T (p_n(Y_s^{(i)}) + ψ(s)) dA_s + \int_t^T g(s, Y_s^{(i)}) d\overline{B}_s$$

$$- \sum_{i=1}^{m} \int_t^T Z_i^{(i)} dH_i^{(i)}, \quad (3.4)$$
Now, we are ready to prove the existence of solutions for GBDS DEL (1.4).

First, let $h_n(0,0) = p_n(0) = 0$, the solution $(y^n, z^n)$ of (3.5) is $(y^n, z^n) = (0,0)$.

Moreover, the solution $(Y^n, Z^n)$ of (3.3) converges to the minimal solution $(Y, Z)$ of (3.5) (see the proof of the Theorem 3.1 in [3]). Therefore, $Y \geq 0$, a.s.. Consequently, for any solution $(Y, Z)$ of GBDSDEL (3.3), we have $Y \geq 0$.

**Remark 3.3.** Lemma 3.2 still holds if $h$ and $p$ depend, as well as, on $t \in [0, T]$ with $h(t,0,0) \geq 0$ and $p(t,0) \geq 0$, for all $t \in [0, T]$, such that for a.e. $t \in [0, T]$, $h(t,0,0)$ and $p(t,0)$ are $\mathcal{F}_t$–measurable and $\mathbb{E} \left( \int_0^T |h(t,0,0)|^2 ds + \int_0^T |p(t,0)|^2 dA_t \right) < +\infty$, and $|h(t,y,z) - h(t,0,0)| \leq K(|y| + |z|)$ and $|p(t,y) - p(t,0)| \leq K|y|$. Indeed, it suffices to replace $(h, \phi)$ and $(p, \psi)$ in Lemma 3.2 by:

\[
\begin{cases}
\tilde{h}(y,z) = h(t,y,z) - h(t,0,0) \\
\tilde{\phi}(t) = \phi(t) + h(t,0,0),
\end{cases}
\]

\[
\begin{cases}
\tilde{p}(t,y) = p(t,y) - p(t,0) \\
\tilde{\psi}(t) = \psi(t) + p(t,0).
\end{cases}
\]

Now, we are ready to prove the existence of solutions for GBDSDEL (1.4).

**Theorem 3.4.** *Under the assumptions (C1) and (C2), the GBDSDEL (1.4) has a minimal solution $(Y, Z) \in \mathcal{E}_m (0, T)$.*

**Proof.** Let us construct an approximate sequence using a Picard-type iteration, by virtue of the results on GBDSDEL with continuous and linear growth coefficients, due to [3].

First, let $(Y^0, Z^0) \in \mathcal{E}_m (0, T)$ be the unique solution of

\[
Y^0 = \xi + \int_t^T \left( -f_s - K|Y^0| - K\|Z^0\| \right) ds + \int_t^T (-k_s - K|Y^0|) dA_s + \int_t^T g(s, Y^0) dB_s - \sum_{i=1}^m \int_t^T \tilde{z}^{(i)} dH^{(i)}.
\]

(3.6)

Now let $\{(Y^n, Z^n)\}_{n \geq 0}$ be a sequence in $\mathcal{E}_m (0, T)$ defined recursively by $(Y^0, Z^0)$ the solution of (3.6) and for any $n \geq 1$,

\[
Y^n = \xi + \int_t^T \left( f_s, Y^{n-1}, Z^{n-1} + h(Y^n - Y^{n-1}, Z^n - Z^{n-1}) \right) ds
\]

\[
+ \int_t^T \left( k_s, Y^{n-1} + p(Y^n - Y^{n-1}) \right) dA_s + \int_t^T g(s, Y^n) dB_s - \sum_{i=1}^m \int_t^T \tilde{z}^{(i)} dH^{(i)}.
\]

(3.7)

For $n \geq 1$ and $(Y^{n-1}, Z^{n-1}) \in \mathcal{E}_m (0, T)$, let, for any $t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^m$,

\[
\begin{cases}
\tilde{E}_n(t,y,z) = f(t, Y^{n-1}, Z^{n-1}) + h(y - Y^{n-1}, z - Z^{n-1}), \\
\tilde{K}_n(t,y) = k(t, Y^{n-1}) + p(y - Y^{n-1}).
\end{cases}
\]

Since $h$ and $p$ are continuous, $E_n$ and $K_n$ are continuous and for any $t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^m$,

\[
|E_n(t,y,z)| \leq \phi(t) + K(|y| + |z|) \quad \text{and} \quad |K_n(t,y)| \leq \psi(t) + K|y|,
\]

(3.8)
where,
\[ \phi_n(t) = f_i + 2K(|Y_n^{i-1}| + \|Z_n^{i-1}||) \] and \[ \psi_n(t) = k_i + 2K|Y_n^{i-1}|. \]
Consequently, since for \( n \geq 1 \), \((Y_n^{i-1}, Z_n^{i-1}) \in \mathcal{E}_m(0, T)\) and from (C2) (iii), we have for all \( \mu, \lambda > 0 \) and for \( n \geq 1 \)
\[ \mathbb{E} \left( \int_0^T e^{\lambda t} \phi_n^2(t) dt + \int_0^T e^{\lambda t} \psi_n^2(t) dA_t \right) < \infty. \]
Therefore, by Theorem 2.8, GBDSDEL \((\xi, \mathcal{E}_m, g, K, A)\), i.e., Eq. (3.7) has a minimal solution, which we steal denote by \((Y_n^0, Z_n^0) \in \mathcal{E}_m(0, T)\), for each \( n \geq 1 \). To complete the proof, it suffices to show that the sequence \((Y_n^0, Z_n^0)\) converges to \((Y, Z)\) which is the minimal solution of GBDSDEL \((\xi, f, g, k, A)\), i.e., Eq. (1.4).

**Step 1 :** \((Y_n^0)_{n \geq 0}\) is increasing with upper bound.

Let \((Y_0^0, Z_0^0) \in \mathcal{E}_m(0, T)\) be the unique solution of
\[ Y_t^0 = \xi + \int_t^T \left( f_s + K|Y_s^0| + K\|Z_s^0\| \right) ds + \int_t^T \left( k_s + K|Y_s^0| \right) dA_s + \int_t^T g(s, Y_s^0) dB_s - \sum_{i=1}^m \int_t^T Z_s^{(i)(n+1)} dH_s^{(i)}. \] (3.9)

Then, we have for any \( n \geq 0 \),
\[ Y_0^0 \leq Y_n^0 \leq Y_{n+1}^0 \leq Y_t^0, \quad \mathbb{P}\text{-a.s.} \quad \forall t \in [0, T]. \] (3.10)

Indeed, for \( n \geq 0 \), let
\[ (Y_{n+1}^0, Z_{n+1}^0) = (Y_{n+1}^0 - Y_n^0, Z_{n+1}^0 - Z_n^0). \]
Then, the processes \((Y_{n+1}^0, Z_{n+1}^0)\) satisfy the following GBDSDEL : for all \( t \in [0, T] \),
\[ Y_t^{n+1} = \int_t^T \left( h(Y_{s}^{n+1} + Z_{s}^{n+1}) + \phi^n(s) \right) ds + \int_t^T \left( p(Y_{s}^{n+1}) + \psi^n(s) \right) dA_s + \int_t^T g^n(s, Y_s^n + Z_s^n) dB_s - \sum_{i=1}^m \int_t^T Z_s^{(i)(n+1)} dH_s^{(i)}, \]
where, for all \( n \geq 0 \), the processes \( \phi^n \) and \( \psi^n \), and the function \( g^n \) are given by :
\[ g^n(s, y) = g(s, y + Y_s^n) - g(s, Y_s^n) \]
and for \( n = 0 \)
\[ \begin{cases} \phi^0(s) = f(s, Y_s^0, Z_s^0) + K(|Y_s^0| + \|Z_s^0\|) + f_s, \\ \psi^0(s) = k(s, Y_s^0) + K|Y_s^0| + k_s, \end{cases} \]
for \( n \geq 1 \)
\[ \begin{cases} \phi^n(s) = f(s, Y_s^n, Z_s^n) - f(s, Y_{s}^{n-1}, Z_{s}^{n-1}) - h(Y_s^n - Y_{s}^{n-1}, Z_s^n - Z_{s}^{n-1}), \\ \psi^n(s) = k(s, Y_s^n) - k(s, Y_s^{n-1}) - p(Y_s^n - Y_{s}^{n-1}). \end{cases} \]

For \( n = 0 \), since \((Y_0^0, Z_0^0) \in \mathcal{E}_m(0, T)\), and from (C2) (vi), it follows that,
\[ \phi^0(t) \geq 0, \quad \psi^0(t) \geq 0, \quad \text{for any } t \in [0, T], \quad \text{and } \mathbb{E} \left( \int_0^T |\phi^0(s)|^2 ds + \int_0^T |\psi^0(s)|^2 dA_s \right) < +\infty. \]
Moreover, by its definition, \( g^0 \) satisfies (C2) (ii) and (iv). Therefore, from Lemma 3.2, we get \( Y_t^{1,0} \geq 0 \) a.s., i.e., \( Y_0^0 \leq Y_1^0 \), a.s., for all \( t \in [0, T] \).
Now, suppose that there exists \( n \geq 1 \), such that \( Y_{n-1}^n \leq Y_n^n \). For such \( n \), and since \((Y_{n-1}^n, Z_{n-1}^n), (Y_n^n, Z_n^n) \in \mathcal{E}_m(0, T)\), one can show, by their definitions, that \( \phi^n, \psi^n \) and \( g^n \) satisfy all assumptions of Lemma 3.2. Hence, by Lemma 3.2 we have \( Y_{n+1}^{n+1} \geq 0 \) a.s., i.e., \( Y_0^0 \leq Y_1^0 \), a.s., for all \( t \in [0, T] \).
On the other hand, applying Itô’s formula to $\tilde{Y}_t^{0,0}$, we get

$$f(t, \tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}) = (\tilde{Y}_t^{0,0}, \tilde{Z}_t^{0,0}) = (\tilde{Y}_t^0 - \tilde{Y}_t^0, \tilde{Z}_t^0 - \tilde{Z}_t^0).$$

Then, it follows from (3.6) and (3.9) that for all $t \in [0,T]$,

$$\hat{Y}_t^{0,0} = \int_t^T f_s^{0} ds + \int_t^T k_s^{0} dA_s + \int_t^T Y_t^{0} ds + \int_t^T \sum_{i=1}^{m} \tilde{Z}_t^{0} dH_t^{(i)},$$

where,

$$\begin{cases}
  f_s^{0} = K\left( |Y_t^0| + |Z_t^0| + |Y_t^0| + |Z_t^0| \right) \geq 0, \\
  k_s^{0} = K\left( |Y_t^0| + |Z_t^0| \right) \geq 0.
\end{cases}$$

Moreover, since, $(Y^0, Z^0), (Y^0, Z^0) \in \mathcal{E}_m(0,T)$, $E\left( \int_0^T |f_s^{0}|^2 ds + \int_0^T |k_s^{0}|^2 dA_s \right) < +\infty$.

Therefore, by Lemma (3.2), we have $\hat{Y}_t^{0,0} \geq 0$ a.s., i.e. $Y_t^{0} \leq \bar{Y}_t^{0}$, a.s., for all $t \in [0,T]$.

Now, suppose that there exists $n \geq 0$, such that $Y_t^{n} \leq \bar{Y}_t^{0}$, and let us show that, $Y_t^{n+1} \leq \bar{Y}_t^{0}$.

Since $Y_t^{0} \geq Y_t^{n}$, it follows from assumption (C2) (vi) that

$$f(t, \tilde{Y}_t^{0}, \tilde{Z}_t^{0}) \geq f(t, Y_t^{n}, Z_t^{n}) + h(Y_t^{n} - Y_t^{n}, Z_t^{n} - Z_t^{n}) = E_{n+1}(t, Y_t^{0}, Z_t^{0})$$

and

$$k(t, Y_t^{0}) \geq k(t, Y_t^{n}) + p(Y_t^{n} - Y_t^{n}) = K_{n+1}(t, Y_t^{0}).$$

Hence,

$$f_t + K\left( \langle Y_t^{0} \rangle + \langle Z_t^{0} \rangle \right) \geq E_{n+1}(t, Y_t^{0}, Z_t^{0}) \quad \text{and} \quad k_t + K\langle Y_t^{0} \rangle \geq K_{n+1}(t, Y_t^{0}).$$

Moreover, since $E_{n+1}(t,y,z) = f(t, Y_t^{n}, Z_t^{n}) + h(y - Y_t^{n}, z - Z_t^{n})$, and $h$ satisfies inequality (3.2), then $E_{n+1}$ satisfies inequality (2.1). Consequently, from Theorem (2.5), $\bar{Y}_t^{0} \geq 10^{n+1}$. Hence, for all $n \geq 0$, $Y_t^{n} \leq \bar{Y}_t^{0}$, a.s., for all $t \in [0,T]$.

**Step 2 : A priori estimates**

There exists a constant $C > 0$ independent of $n$ such that

$$\sup_{n \geq 0} \left( \mathbb{E} \left( \sup_{0 \leq s \leq T} \left| Y_t^n \right|^2 + \int_0^T \left| Y_t^n \right|^2 (ds + dA_s) + \int_0^T \left| Z_t^n \right|^2 ds \right) \right) \leq C. \quad (3.11)$$

Indeed, first by virtue of (3.10), we derive that

$$\sup_{n \geq 0} \left( \mathbb{E} \left( \sup_{0 \leq s \leq T} \left| Y_t^n \right|^2 + \int_0^T \left| Y_t^n \right|^2 (ds + dA_s) \right) \right) \leq \mathbb{E} \left( \sup_{0 \leq s \leq T} \left( \left| Y_t^n \right|^2 + \left| \bar{Y}_t^n \right|^2 \right) + \int_0^T \left( \left| Y_t^n \right|^2 + \left| Y_t^n \right|^2 \right) (ds + dA_s) \right). \quad (3.12)$$

On the other hand, applying Itô’s formula to $e^{\mu T + \lambda T} |Y_t^n|^2$, for any $\mu, \lambda > 0$, we have

$$e^{\mu T + \lambda T} |Y_t^n|^2 + \lambda \int_t^T e^{\mu s + \lambda s} |Y_t^n|^2 dA_s + \mu \int_t^T e^{\mu s + \lambda s} |Y_t^n|^2 ds + \int_0^T e^{\mu s + \lambda s} \left| Z_t^n \right|^2 ds = e^{\mu T} + \lambda T |\xi|^2 + 2 \int_t^T e^{\mu s + \lambda s} Y_t^n E_s(s,y^n,Z_t^n) ds + 2 \int_t^T e^{\mu s + \lambda s} g(s,y^n) \tilde{B}_s \quad (3.13)$$

$$+ 2 \int_t^T e^{\mu s + \lambda s} Y_t^n K_n(s,y^n) ds + \sum_{i=1}^m \int_t^T e^{\mu s + \lambda s} Y_t^n Z_t^n dH_t^{(i)} + \int_t^T e^{\mu s + \lambda s} g(s,y^n) |dH_t^{(i)}| - \sum_{i,j=1}^m \int_t^T e^{\mu s + \lambda s} Y_t^n Z_t^n d \left[ H_t^{(i)}, H_t^{(j)} \right] s - \delta_{ij} s.$$
From, (C2)(ii), (iv), (vi) together with (3.13) and Young’s inequality, we have for any \( \sigma > 0 \) and \( \gamma > 0 \),
\[
2 \langle Y^n, E_n(s, Y^n, Z^n) \rangle \\
\leq 2|Y^n| \left( f_n + 2K|Y_{t_s}^{n-1}| + 2K\|Z^n\| + K|Y^n| + K\|Z^n\| \right) \\
\leq \left( 1 + 4K + \frac{4K^2}{\gamma} + \frac{K^2}{\sigma} \right) |Y^n|^2 + 2K|Y_{t_s}^{n-1}|^2 + \sigma\|Z_s^n\|^2 + \gamma\|Z_s^{-1}\|^2 + f_s^2,
\]
and
\[
|g(s, Y^n)|^2 \leq K|Y^n|^2.
\]
Therefore taking expectation in both side of (3.14) with the suitable \( \lambda \) and \( \mu \), and by virtue of (3.12) and assumptions (C1), (C2) (iii), there exists a constant \( c > 0 \) independent of \( n \) such that for any \( \gamma > 0 \), \( \sigma > 0 \), we derive, for any \( n \geq 1 \),
\[
\mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z^n\|^2 ds \right) \leq c + \sigma \mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z^n\|^2 ds \right) + \gamma \mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z_s^{-1}\|^2 ds \right).
\]
Consequently, choosing \( \sigma < 1 \), we obtain, for any \( \gamma > 0 \), \( n \geq 1 \),
\[
\mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z^n\|^2 ds \right) \leq \frac{c}{1 - \sigma} + \frac{\gamma}{1 - \sigma} \mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z_s^{-1}\|^2 ds \right)
\]
which, by iteration, provides, for any \( n \geq 1 \),
\[
\mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z^n\|^2 ds \right) \leq \frac{c}{1 - \sigma} \sum_{k=0}^{n-1} \left( \frac{\gamma}{1 - \sigma} \right)^k + \left( \frac{\gamma}{1 - \sigma} \right)^n \mathbb{E} \left( \int_0^T e^{\mu s + \lambda A_s} \|Z_s^0\|^2 ds \right).
\]
Now, choosing \( \gamma > 0 \), such that, \( 0 < \gamma \sigma < 1 \) and since \( \|Z^0\|_{\mathcal{L}^2(\mathbb{R}^m)} < +\infty \), we get the existence of a constant \( C > 0 \) independent of \( n \) such that
\[
\sup_{n \geq 0} \left( \mathbb{E} \int_0^T \|Z^n\|^2 ds \right) < C.
\]
Therefore, setting \( F^n_s = E_n(s, Y^n_s, Z^n_s) \) and \( K^n_s = K_n(s, Y^n_s) \), it follows from (3.13) and (3.12) that
\[
\sup_{n \geq 0} \left( \mathbb{E} \left( \int_0^T |F^n|^2 ds + \int_0^T |K^n_s|^2 dA_s \right) \right) < +\infty.
\]

**Step 3 : Convergence result**

From (3.10) and (3.12), \((Y^n)_{n \geq 0}\) is an increasing and bounded sequence in \( S^2(0, T) \cap \mathcal{A}^2(0, T) \). Then, there exists a process \( Y \) such that \( Y^n \to Y \) a.s., for all \( t \in [0, T] \). Therefore, it follows from Fatou’s lemma together with the dominated convergence theorem that
\[
\mathbb{E} \left( |Y_s|^2 + \int_0^T |Y_s|^2 ds + \int_0^T |Y_s|^2 dA_s \right) \leq C
\]
and
\[
\lim_{n \to +\infty} \mathbb{E} \left( |Y^n_s - Y_s|^2 + \int_0^T |Y^n_s - Y_s|^2 ds + \int_0^T |Y^n_s - Y_s|^2 dA_s \right) = 0. \tag{3.14}
\]
Moreover, applying again Itô’s formula to \(|Y^n_p - Y^n_s|^2\), we have

\[
|Y^n_p - Y^n_s|^2 + \int_t^T \|Z^n_p - Z^n_s\|^2 \, ds = 2 \int_t^T \langle Y^n_p - Y^n_s, F^n_s - F^n_s \rangle \, ds + 2 \int_t^T \langle Y^n_p - Y^n_s, K^n_s - K^n_s \rangle \, dA_s
\]

\[+ \int_t^T |g(s, Y^n_p) - g(s, Y^n_s)|^2 \, ds + M^n_T - M^n_T + N^n_T - N^n_T + X^n_T - X^n_T, \tag{3.15}
\]

where, \(M, N\) and \(X\) are uniformly integrable martingale defined by:

\[
\begin{align*}
M^n_t & \triangleq -2 \sum_{i=1}^m \int_0^t \langle Y^n_p - Y^n_s, Z^n_p(i) - Z^n_s(i) \rangle \, dH^n_s(i) \\
N^n_t & \triangleq 2 \int_t^T \langle Y^n_p - Y^n_s, g(s, Y^n_p) - g(s, Y^n_s) \rangle \, dB_s, \\
X^n_t & \triangleq -\sum_{i,j=1}^m \int_0^t \langle Z^n_p(i) - Z^n_s(i), Z^n_p(j) - Z^n_s(j) \rangle \, d\langle H^n(i), H^n(j) \rangle_s - \delta_{ij} \langle H^n(i), H^n(i) \rangle_s.
\end{align*}
\]

Then, taking the expectation in (3.15), we deduce by virtue of Hölder’s inequality and (C2) (ii), (iv), that

\[
\mathbb{E} \int_0^T \|Z^n_p - Z^n_s\|^2 \, ds \leq C \left( \mathbb{E} \int_0^T |Y^n_p - Y^n_s|^2 \, ds \right)^\frac{1}{2} + C \left( \mathbb{E} \int_0^T |Y^n_p - Y^n_s|^2 \, dA_s \right)^\frac{1}{2} + K \mathbb{E} \int_0^T |Y^n_p - Y^n_s|^2 \, ds.
\]

Consequently, it follows from (3.14), that \((Z^n_s)_{n \geq 0}\) is a Cauchy sequence in \(M^2(0, T, \mathbb{R}^m)\). Then, there exists a \(\mathcal{F}_t\)-jointly measurable process \(Z \in M^2(0, T, \mathbb{R}^m)\) such that

\[
\lim_{n \to +\infty} \mathbb{E} \int_0^T \|Z^n_p - Z^n_s\|^2 \, ds = 0.
\]

On the other hand, taking the supremum and the expectation in (3.15), we deduce from Burkhölder-Davis-Gundy inequality, that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y^n_p - Y^n_s|^2 \right) \leq C \left( \mathbb{E} \int_0^T |Y^n_p - Y^n_s|^2 \, ds \right)^\frac{1}{2} + \mathbb{E} \int_0^T |Y^n_p - Y^n_s|^2 \, dA_s + \mathbb{E} \int_0^T |Y^n_p - Y^n_s|^2 \, ds,
\]

from which, together with (3.14), we deduce that \(\mathbb{P}\)-almost surely, \(Y^n\) converges uniformly to \(Y\) which is continuous, such that \(\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y^n_p|^2 \right) \leq C\). And then,

\[
\|Y^n, Z\|_{E_m}^2 = \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y^n_p|^2 + \int_0^T |Y^n_s|^2 \, dA_s + \int_0^T \|Z^n_p\|^2 \, ds \right) \leq C.
\]

**Step 4:** \((Y, Z)\) verifies \(GBDSDEL\). \(\tag{1.4}\)

Since \((Y^n, Z^n) \to (Y, Z)\) in \(E_m(0, T)\), along a subsequence which we still denote \((Y^n, Z^n)\), we get

\[
(Y^n, Z^n) \to (Y, Z), \quad dt \otimes d\mathbb{P} \text{ a.e.},
\]

and there exists, \(M^2(0, T, \mathbb{R})\) such that, for all \(n \geq 1\), \(|Z^n| < \Pi\), \(dt \otimes d\mathbb{P} \text{ a.e.}\).

Therefore, from (C2) (ii), (iv), (vi), we have

\[
\mathbb{E}_n(t, Y^n, Z^n) = f(t, Y^{n-1}_t, Z^{n-1}_t) + h(t, Y^{n-1}_t, Z^{n-1}_t) \to f(t, Y_t, Z_t), \quad dt \otimes d\mathbb{P} \text{ a.e.},
\]

where, \(f, h\) are \(\mathbb{R}^d \to \mathbb{R}^d\) and \(\mathbb{R}^d \to \mathbb{R}\) Lipschitz continuous functions, respectively.
and
\[
K_n(t, Y^n) = k(t, Y^{n-1}) + p(Y^n - Y^{n-1}) \rightarrow k(t, Y_t), \ dt \otimes d\mathbb{P} \text{ a.e.,}
\]
for all \( t \in [0, T] \) as \( n \rightarrow +\infty \).

Moreover from (3.8) and (3.10), we have
\[
|E_n(t, Y^n, Z^n)| \leq \Psi(t), \ dt \otimes d\mathbb{P} \text{ a.e.,}
\]
where, \( \Psi(t) = f_t + 2K(|Y^0| + |Y^0_t|) + 2|\Pi_t| \), which, from (C2) (iii) and Cauchy-Schwarz inequality, yields, \( \mathbb{E} \int_0^T \Psi(s)ds < +\infty \). Then, it follows by the dominated convergence theorem that
\[
\mathbb{E} \int_t^T |E_n(s, Y^n, Z^n) - f(s, Y_s, Z_s)|ds \rightarrow 0, \text{ as } n \rightarrow +\infty.
\]

Also, using the same argument, we get
\[
\mathbb{E} \int_t^T K_n(s, Y^n) - k(s, Y_s)dA_s \rightarrow 0, \text{ as } n \rightarrow +\infty.
\]

On the other hand, thanks to (C2) (ii), (iv) and BDG inequality, we know that there exists a positive constant \( C > 0 \) independent of \( n \) such that,
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \int_t^T g(s, Y^n)dB_s - \int_t^T g(s, Y_s)dB_s \right|^2 \right) \leq C \mathbb{E} \int_0^T |Y^n - Y_s|^2 ds + \int_0^T |Y^n - Y_s|^2 ds \rightarrow 0 \quad n \rightarrow +\infty
\]
and
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \sum_{i=1}^m \int_t^T Z^n_s(i)dB_s - \sum_{i=1}^m \int_t^T Z(i)_s(i)dB_s \right|^2 \right) \leq C \mathbb{E} \int_0^T e^{\beta A(s)}|Z^n_s - Z_s|^2 ds \rightarrow 0 \quad n \rightarrow +\infty.
\]

Finally, passing to the limit on both sides of GBDSDEL \( (\xi, F_n, g, K_n, A) \) (3.7), we get that \( (Y, Z) \) is a solution of GBDSDEL \( (\xi, f, g, k, A) \) (1.4).

**Step 5 :** \( (Y, Z) \) is the minimal solution of GBDSDEL (1.4).

Let \( (Y, Z) \in \mathcal{E}_m(0, T) \) be any solution of GBDSDEL \( (\xi, f, g, k, A) \) (1.4) and consider GBDSDEL \( (\xi, F_n, g, K_n, A) \) (3.7) with its minimal solution \( (Y^n, Z^n) \in \mathcal{E}_m(0, T) \), for each \( n \geq 0 \).

For \( n = 0 \), let \( (\tilde{Y}_t, \tilde{Z}_t) = (Y_t - Y^0_t, Z_t - Z^0_t) \). Then,
\[
\tilde{Y}_t = \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s)ds + \int_t^T \tilde{k}(s, \tilde{Y}_s)dA_s + \int_t^T \tilde{g}(s, \tilde{Y}_s)dB_s - \sum_{i=1}^m \int_t^T \tilde{Z}_i(s)dB_s \quad 0 \leq t \leq T,
\]

where, \( \tilde{f}, \tilde{k} \) and \( \tilde{g} \) are defined by :
\[
\tilde{f}(s, y, z) = f(s, Y_s, z + Z^0_s) + K|y - Y_s| + K|Z^0_s| + f_z;
\]
\[
\tilde{k}(s, y) = k(s, Y_s) + K|y - Y_s| + k_z;
\]
\[
\tilde{g}(s, y) = g(s, y + Y^0_s) - g(s, Y^0_s),
\]
with \( \tilde{f}(0, 0) = f(s, Y_s, Z^0_s) + K|Y_s| + K|Z^0_s| + f_z \) and \( \tilde{k}(s, 0) = k(s, Y_s) + K|Y_s| + k_z \).
By their definitions, it is easy to check that,
\[ |\bar{f}(t,y,z) - \bar{f}(t,0,0)| \leq K(|y| + |z|) \quad \text{and} \quad |\bar{k}(t,y) - \bar{k}(t,0)| \leq K|y|.\]

On the other hand, from (C2)(ii), we have
\[ f(s, Y_s, Z_s^0) \geq -K|Y_s| - K|Z_s^0| - f_i \quad \text{and} \quad k(s, Y_s) \geq -K|Y_s| - k_s, \]
this implies that \( \bar{f}(s,0,0) \geq 0 \) and \( \bar{k}(s,0) \geq 0. \)

Moreover, since \( f \) is Lipschitz in \( z \) and satisfies inequality (3.1), \( \bar{f} \) is a Lipschitz function in \( (y,z) \)
and satisfies inequality (3.2). Therefore, by Lemma 3.2 and Remark 3.3 it follows that \( \bar{Y}_t \geq 0, \) i.e., \( \hat{Y}_t \geq Y_t^0, \) a.s., for all \( t \in [0, T]. \)

Now, suppose that there exists \( n \geq 1 \) such that \( Y_t \geq Y_t^n \) and let prove that \( Y_t \geq Y_t^{n+1}. \)

From (C2) (vi) and since \( Y_t \geq Y_t^n, \) it follows that
\[ f(s, Y_s, Z_s) \geq f(s, Y_s^n, Z_s^n) + h(Y_s - Y_s^n, Z_s - Z_s^n) = F_{n+1}(s, Y_s, Z_s) \]
and
\[ k(s, Y_s) \geq k(s, Y_s^n) + p(Y_s - Y_s^n) = K_{n+1}(s, Y_s). \]

Since, \( E_{n+1} \) and \( K_{n+1} \) are continuous with linear growth, and inequality (3.1) still holds, we get from Theorem 3.1 that, \( Y_t \geq Y_t^{n+1}, \) a.s., for all \( t \in [0, T]. \) Consequently, for all \( n \geq 0, \) we have \( Y_t \geq Y_t^n, \) a.s., for all \( t \in [0, T]. \) Since \( (Y^n, Z^n) \) converges to \( (Y, Z), \) we get \( Y_t \geq Y_t, \) a.s., for all \( t \in [0, T]. \)
That proves that \( (Y, Z) \) is the minimal solution of GBDSDEL \((\xi, f, g, k, A) \) (1.4).

\[ \square \]

Remark 3.5. We can prove the maximal solution result for GBDSDEL \((\xi, f, g, k, A) \) when the coefficient \( f \) and \( k \) satisfy the following: (i.e., when (v) and (vi) in (C2) are replaced by the following)

(i) for all \( (t,z) \in [0, T] \times \mathbb{R}^m, \) \( y \mapsto f(t,y,z) \) and \( y \mapsto k(t,y) \) are right-continuous, for all \( \omega \) a.e.

(ii) there exists two functions \( h : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R} \) and \( p : \mathbb{R} \rightarrow \mathbb{R}, \) satisfying, for all \( y \in \mathbb{R}, z, z' \in \mathbb{R}^m : \)
\[ \left\{ \begin{array}{l}
|h(y,z)| \leq K(|y| + |z|) \quad \text{and} \quad |p(y)| \leq K|y|,
|h(y,z) - h(y,z')| \leq K|z - z'|
\end{array} \right. \]
\[ y \mapsto h(y,z) \quad \text{and} \quad y \mapsto p(y) \text{ are continuous}, \]

such that for all \( y \geq y', z, z' \in \mathbb{R}^m, t \in [0, T], \) we have
\[ f(t,y,z) - f(t,y',z') \leq h(y - y', z - z') \quad \text{and} \quad k(t,y) - k(t,y') \leq p(y - y'). \]

To this end, we need the following approximation.

Firstly, we consider the solution \((Y^0, Z^0) \in E_m(0, T)\) of GBDSDEL \((\xi, F_0, g, K_0, A)\), where,
\[ F_0(t,y,z) = K|y| + K|z| + f_i \quad \text{and} \quad K_0(t,y) = K|y| + k_i, \]
and then define recursively a sequence \( \{Y^n, Z^n\}_{n \geq 1} \in E_m(0, T)\), by: for all \( t \in [0, T], \)
\[ Y^n_t = \xi + \int_t^T \left( f(s,Y_s^{n-1},Z_s^{n-1}) + h(Y_s^{n-1} - Y_s^{n-1},Z_s^{n-1} - Z_s^{n-1}) \right) ds \]
\[ + \int_t^T (k(s,Y_s^{n-1}) + p(Y_s^{n-1} - Y_s^{n-1})) dA_s + \int_t^T g(s,Y_s^{n-1}) dB_s + \sum_{i=1}^m \int_t^T Z_{s_i}^{n-1} dB_i. \]

For \( n \geq 1 \) and \((Y^{n-1}, Z^{n-1}) \in E_m(0, T), \) let, for any \( t \in [0, T], (y,z) \in \mathbb{R} \times \mathbb{R}^m, \)
\[ F_n(t,y,z) = f(t,Y_t^{n-1},Z_t^{n-1}) + h(y - Y_t^{n-1}, z - Z_t^{n-1}), \]
\[ K_n(t,y) = k(t,Y_t^{n-1}) + p(y - Y_t^{n-1}). \]

Then, using similar calculations, we deduce by Theorem 3.2 that GBDSDEL \((\xi, F_n, g, K_n, A)\), i.e., Eq. (3.17) has a minimal (resp. maximal) solution in \( E_m(0, T). \) Here, we consider the maximal solution, which we steal denote by \((Y^n, Z^n)\), for each \( n \geq 1. \) By similar procedures, we get the following result:
Proposition 3.6. Assume (C1) and (C2) (i)-(iv), (v'), (vi'). Then

(i) for all \( n \geq 0 \), \( Y^0_t \leq Y^{n+1}_t \leq Y^n_t, \ P\text{-a.s. } \forall t \in [0, T], \)

(ii) the sequence \( (Y^n, Z^n)_{n \geq 0} \) converges in \( E_m(0, T) \) to a limit \( (\bar{Y}, \bar{Z}) \), which is the maximal solution of GBDSDEL (1.4).

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