Embedded eigenvalues of the Neumann problem in a strip with a box-shaped perturbation

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Abstract

We consider the spectral Neumann problem for the Laplace operator in an acoustic waveguide \( \Pi_\varepsilon \) obtained from a straight unit strip by a low box-shaped perturbation of size \( 2l \times \varepsilon \), where \( \varepsilon > 0 \) is a small parameter. We prove the existence of the length parameter \( l_\varepsilon^k = \pi k + O(\varepsilon) \) with any \( k = 1, 2, 3, \ldots \) such that the waveguide \( \Pi_\varepsilon^k \) supports a trapped mode with an eigenvalue \( \lambda_\varepsilon^k = \pi^2 - 4\pi^2 \varepsilon^2 + O(\varepsilon^3) \) embedded into the continuous spectrum. This eigenvalue is unique in the segment \( [0, \pi^2] \) and is absent in the case \( l \neq l_\varepsilon^k \). The detection of this embedded eigenvalue is based on a criterion for trapped modes involving an artificial object, the augmented scattering matrix. The main technical difficulty is caused by corner points of the perturbed wall \( \partial \Pi_\varepsilon \) and we discuss available generalizations for other piecewise smooth boundaries.

Keywords: acoustic waveguide, Neumann problem, embedded eigenvalues, continuous spectrum, box-shaped perturbation, asymptotics

MSC: 35P05, 47A75, 49R50, 78A50.

1 Introduction

1.1 Formulation of problems

In the union \( \Pi_\varepsilon^k \), fig. 1 b and a, of the unit straight strip

\[
\Pi = \{x = (x_1, x_2) \in \mathbb{R}^2, \; x_1 \in \mathbb{R}, \; x_2 \in (0, 1)\}
\]

and a rectangle of length \( 2l > 0 \) and a small width \( \varepsilon > 0 \),

\[
\mathcal{W}_l^\varepsilon = \{x : |x_1| < l, \; x_2 \in (\varepsilon, 0]\},
\]

we consider the spectral Neumann problem

\[
-\Delta u^\varepsilon (x) = \lambda^\varepsilon u^\varepsilon (x), \quad x \in \Pi_\varepsilon^k = \Pi \cup \mathcal{W}_l^\varepsilon;
\]

\[
\partial_\nu u^\varepsilon (x) = 0, \quad x \in \partial \Pi_\varepsilon^k,
\]
where $\Delta = \nabla \cdot \nabla$ is the Laplace operator, $\nabla = \text{grad}$, $\lambda^\varepsilon$ is the spectral parameter and $\partial_{\nu} = \nu \cdot \nabla$ is the directional derivative, $\nu$ stands for the unit outward normal defined everywhere at the boundary $\partial \Pi^d_{l}$, except for corner points, i.e. vertices of the rectangle (1.2). Since a solution of the problem (1.3), (1.4) may get singularities at these points, the problem ought to be reformulated as the integral identity [28]

$$(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Pi^d_{l}} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Pi^d_{l}}, \quad \forall v^\varepsilon \in H^1(\Pi^d_{l}),$$

where $(\ , \ )_{\Pi^d_{l}}$ is the natural scalar product in the Lebesgue space $L^2(\Pi^d_{l})$ and $H^1(\Pi^d_{l})$ stands for the Sobolev space. The symmetric bilinear form on the left-hand side of (1.5) is closed and positive in $H^1(\Pi^d_{l})$ so that problem (1.3), (1.4) is associated [3 Ch 10] with a positive self-adjoint operator $A^\varepsilon$ in $L^2(\Pi^d_{l})$ whose spectrum $\varphi = \varphi_{co}$ is continuous and covers the closed positive semi-axis $\mathbb{R}_+ = [0, +\infty)$. The domain $\mathcal{D}(A^\varepsilon)$ of $A^\varepsilon$, of course, belongs to $H^1(\Pi^d_{l})$ but is bigger than $H^2(\Pi^d_{l})$ due to singularities of solutions at the corner points, see, e.g., [33, Ch.2]. The point spectrum $\varphi_{po}$ of $A^\varepsilon$ can be non-empty and the main goal of our paper is to single out a particular value of the length parameter $l$ such that the operator $A^\varepsilon$ wins an eigenvalue $\lambda_{l}^\varepsilon \in \varphi_{po}$ embedded into the continuous spectrum. The corresponding eigenfunction $u_{l}^\varepsilon \in H^1(\Pi^d_{l})$ decays exponentially at infinity and is called a trapped mode, cf. [29] and [47].

Our central result formulated below in Theorem 3, roughly speaking, demonstrates that an eigenvalue $\lambda^\varepsilon_{l}$ exists in the interval $(0, \pi^2) \subset \varphi_{co}$ for $l^\varepsilon \approx \pi k$ with $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ only. Asymptotics of $\lambda^\varepsilon$ and $l^\varepsilon$ are constructed, too.

Problem (1.3), (1.4) is a model of an acoustic waveguide with hard walls, cf. [34], but is also related in a natural way to the linear theory of surface water-waves, cf. [27]. Indeed, the velocity potential $\Phi^\varepsilon(x, z)$ satisfies the Laplace equation in the channel $\mathbb{R}^2_{l} = \Pi^d_{l} \times (-d, 0) \subset \mathbb{R}^3 \ni (x, z)$ of depth $d > 0$ with the Neumann condition (no normal flow) at its vertical walls and horizontal bottom as well as the spectral Steklov condition (the kinetic one) on the free horizontal surface

$$\partial_z \Phi^\varepsilon(x, 0) = \Lambda^\varepsilon \Phi^\varepsilon(x, 0), \quad x \in \Pi^d_{l}.$$  

After factoring out the dependence on the vertical variable $z$,

$$\Phi^\varepsilon(x, z) = u^\varepsilon(x) \left(e^{z\lambda^\varepsilon} + e^{-(z+2d)\lambda^\varepsilon}\right),$$

see, e.g., [14] and [29], the water-wave problem reduces to the two-dimensional Neumann problem (1.3), (1.3) for the function $u^\varepsilon$ in (1.6) and the parameter $\lambda^\varepsilon$ determined from the equation

$$\Lambda^\varepsilon = \lambda^\varepsilon \frac{1 - e^{-2d\lambda^\varepsilon}}{1 + e^{-2d\lambda^\varepsilon}} = \lambda^\varepsilon \tanh (d\lambda^\varepsilon).$$

We will not discuss separately this interpretation of our problem but in the next section present some asymptotic formulas for eigenvalues of the Laplace operator with either Dirichlet, or mixed boundary conditions.
1.2 Asymptotics of eigenvalues

Imposing the Dirichlet condition

$$u^\varepsilon(x) = 0, \quad x \in \partial \Pi^\varepsilon,$$

(1.7)

instead of the Neumann condition (1.4), creates the positive cut-off value $\lambda_\uparrow = \pi^2$ of the continuous spectrum $\varphi^D = [\pi^2, +\infty)$ of the Dirichlet problem (1.3), (1.7) which provides an adequate model of a quantum waveguide, cf. [17]. The interval $(0, \pi^2)$ stays now below the continuous spectrum and therefore may contain eigenvalues composing the discrete spectrum $\varphi_{di}^D$ of the problem. As follows from a result in [10], the multiplicity $\# \varphi_{di}^D$ is equal to 1 for a small $\varepsilon > 0$. Although the paper [10] deals with a regular (smooth) perturbation of the wall, it is possible to select two smooth shallow pockets boxes as in fig. 2 a and b, and to extend the existence and uniqueness result in [10] for the box-shaped perturbations by means of the max-min principle, see, e.g.,[5, Thm 10.2.2]. However, the attendant asymptotic formula

$$\lambda^\varepsilon_l = \pi^2 - 4\pi^4 \varepsilon^2 l^2 + O(\varepsilon^3), \quad \varepsilon \to 0,$$

(1.8)

cannot be supported by these results because an application in [10] of a change of variables which transforms $\Pi^\varepsilon_l$ into $\Pi$ requires certain smoothness properties of the boundary $\partial \Pi^\varepsilon_l$ which are naturally absent, fig. 1 b. In Section 8.3 we will explain how our approach helps to justify formula (1.8). Local perturbations of quantum waveguides in $\mathbb{R}^n$, $n \geq 2$, are intensively investigated during last two decades and many important results on the existence and asymptotic behavior of their discrete spectrum have been published. We mention a few of them, namely [13, 15, 30] for the slightly curved and twisted cylindrical waveguides, [4, 38, 44] for cranked waveguides and [16, 18] for the Laplacian perturbed by a small second-order differential operator with compactly supported coefficients. We also refer to [12] for non-local perturbations, fig. 3 a and to [8, 9, 10] for alternation of the Dirichlet and Neumann boundary conditions.

In the literature one finds much less results on eigenvalues embedded into the continuous spectrum, cf. the review papers [6, 7, 29]. First of all we describe an elegant method [14] which is based on imposing an artificial Dirichlet condition and had become of rather wide use in proving the existence of embedded eigenvalues but only in symmetric waveguides.
Let us consider an auxiliary mixed boundary value problem and supply the Helmholtz equation (1.3) with the Neumann condition on the lower lagged wall and the Dirichlet condition on the upper straight wall, see fig. 1 b,

\[ u^c(x_1, 1) = 0, \quad x_1 \in \mathbb{R}, \quad \partial_{\nu} u^c(x) = 0, \quad x \in \partial \Pi^c_1, \quad x_2 < 1. \] (1.9)

Problem (1.3), (1.9), has the continuous spectrum \( \varphi^M_{co} = [\pi^2/4, +\infty) \) and in Section 8.1 we will show the existence of only one eigenvalue

\[ \lambda^c_l = \frac{\pi^2}{4} (1 - \pi^2/l^2\varepsilon^2) + O(\varepsilon^3 (1 + |\ln \varepsilon|^2)), \quad \varepsilon \to +0, \] (1.10)

in the discrete spectrum \( \varphi^M_{di} \subset (0, \pi^2/4) \). Following [14] we extend the corresponding eigenfunction \( u^c_l(x_1, x_2) \) as an odd function in \( x_2 - 1 \) from \( \Pi^c_l \) onto the bigger waveguide \( \hat{\Pi}^c_l \) drawn in fig. 3 b and obtained as the union of the strip \( \mathbb{R} \times (0, 2) \) and the box \((-l, l) \times (-\varepsilon, 2 + \varepsilon) \). Owing to the Dirichlet condition in (1.9) at the midline of \( \hat{\Pi}^c_l \), this extension \( \hat{u}^c_l(x_1, x_2) \) is a smooth function everywhere in \( \hat{\Pi}^c_l \), except at corner points and inherits from \( u^c_l(x_1, x_2) \) an exponential decay as \( x_1 \to \pm \infty \). Clearly

\( -\Delta \hat{u}^c_l(x) = \lambda^c_l \hat{u}^c_l(x), \quad x \in \hat{\Pi}^c_l, \quad \partial_{\nu} \hat{u}^c_l(x) = 0, \quad x \in \partial \hat{\Pi}^c_l, \) (1.11)

and, thus, \( \hat{u}^c_l \) is an eigenfunction of the Neumann problem (1.11) while the corresponding eigenvalue (1.10) belongs to the continuous spectrum \( \hat{\varphi}_{co} = [0, +\infty) \) of this problem.

We emphasize that the method [14] requires the mirror symmetry of the waveguide and cannot be applied to the asymmetric waveguide \( \Pi^c_l \) in fig. 1 b. The detected embedded eigenvalue \( \lambda^c_l \) of the Neumann problem (1.11) is stable with respect to small symmetric perturbations of the waveguide walls but any violation of the symmetry may lead it out from the spectrum and turn it into a point of complex resonance, cf. [3] and, e.g., [41].

The intrinsic instability of embedded eigenvalues requests for special techniques to detect them as well as to construct their asymptotics. In the present paper we use a criterion for the existence of trapped modes (see [23] and Theorem 1 below) and a concept of enforced stability of eigenvalues in the continuous spectrum, cf. [39, 41].

1.3 Reduction of the problem

In view of the mirror symmetry about the \( x_2 \)-axis, notice the difference with the above mentioned assumption in [14], we truncate the waveguide \( \Pi^c_l \) and consider the Neumann problem

\( -\Delta u^c_+(x) = \lambda^c_+ u^c_+(x), \quad x \in \Pi^c_{l+}, \) (1.12)
\( \partial_{\nu} u^c_+(x) = 0, \quad x \in \partial \Pi^c_{l+}, \) (1.13)

is its right half (overshaded in fig. 1 b)

\[ \Pi^c_{l+} = \{x \in \Pi^c_l : x_1 > 0\} = \{x : x_2 \in (-\varepsilon, 0) \text{ for } x_1 \in (0, l), x_2 \in (0, 1) \text{ for } x_1 \geq l\}. \] (1.14)

Clearly, the even in \( x_1 \) extension of an eigenfunction \( u^c_+ \) of problem (1.12), (1.13) becomes an eigenfunction of the original problem (1.3), (1.4). Searching for an eigenvalue

\( \lambda^c \in (0, \pi^2), \) (1.15)

we will show in Section 7 that, first, problem (1.12), (1.13) cannot get more than one eigenvalue in \((0, \pi^2)\) and, second, the mixed boundary value problem in (1.14) with the Dirichlet condition at the
truncation segment \( \{ x : x_1 = 0, \quad x_2 \in (-\varepsilon, 1) \} \) instead of the Neumann condition as in (1.13), does not have eigenvalues (1.15). These mean that an eigenfunction of problem (1.3), (1.4) associated with the eigenvalue (1.15) is always even in the variable \( x_1 \). In this way, we will be able to describe the part \( \varphi_{po} \cap (0, \pi^2) \) of the point spectrum in the entire waveguide \( \Pi^\varepsilon_l \). In what follows we skip the subscript \( l \). Hence, we regard (1.5) as an integral identity serving for problem (1.12), (1.13) in \( \Pi^\varepsilon_{+} := \Pi^\varepsilon_{+} l \) and denote \( A^\varepsilon_{+} \) the corresponding self-adjoint operator in \( L^2(\Pi^\varepsilon_{+}) \), cf. Section 1.1.

1.4 Embedded eigenvalues

In the absence of the mirror symmetry about a midline of a waveguide the modern literature gives much less results on the existence of embedded eigenvalues. A distinguishing feature of an eigenvalue in the continuous spectrum is its intrinsic instability with respect to a variation of the waveguide shape while all eigenvalues in the continuous spectrum stay stable. In this way to detect an eigenvalue in the Neumann waveguide \( \Pi^\varepsilon_l \), see (1.3), (1.4), a fine tuning of the parameter-dependent shape is needed, namely the length \( 2l = 2l(\varepsilon) \) of the perturbation box (1.2) must be chosen specifically in dependence of its height \( \varepsilon \).

A method to construct particular waveguide shapes which support embedded eigenvalues, was developed in [21, 22, 37, 39, 41] on the basis of a sufficient condition [23] for the existence of exponentially decaying solutions trapped modes in waveguides to elliptic problems in domains with cylindrical outlets to infinity. As a result, several examples of eigenvalues in the continuous spectrum were constructed without requiring a geometrical symmetry of waveguides which are obtained from the straight unit strip by either singular [21, 22, 40], or regular [37, 39, 41] perturbations of the boundary, see fig. 1 a and b, respectively; for a non-local smooth perturbation, see [11]. To this end, a notion of the augmented scattering matrix [23] was used together with certain traditional asymptotic procedures in domains with small holes and cavities, cf. [31, Ch.4, 5 and 2], or in domains with smoothly varied boundaries, cf. [24, Ch. XII, §6.5].

It is very important in the above-mentioned approach to detect embedded eigenvalues that the asymptotic procedures in use are completed in such a way they provide both, a formal derivation of asymptotic expansions and an operator reformulation of the diffraction problem in the framework of the perturbation theory of linear operators, see, e.g., [19, 24], which helps to conclude a smooth (actually analytic) dependence of scattering matrices on geometrical parameters describing the waveguide shape. Moreover, this permits to reformulate the sufficient condition [23] for the existence of trapped modes as a nonlinear abstract equation and to fulfil the condition by means of the contraction principle, cf. Sections 2.3 and 4.1, 4.2 below.

The box-shaped perturbation (1.2) of the strip (1.1) can be regarded as a combination of regular and singular perturbations, respectively outside and inside neighborhoods of the corner points but unfortunately the authors do not know a tool to reduce the problem (1.3), (1.4) (or (1.5) in the variational form) to an abstract equation in a fixed (independent of \( \varepsilon \)) Banach space and to confirm necessary properties of the scattering matrices in the waveguide \( \Pi^\varepsilon_l = \Pi \cup \varpi^\varepsilon_l \). Thus, in order to support each step of our procedure to detect an embedded eigenvalue and to establish its uniqueness, we have to obtain a certain new result for the problems (1.3), (1.4) and/or (1.12), (1.13) which cannot
be deduced from the general perturbation theory. Although several approaches and tricks proposed in our paper work also for other shapes like in fig. 6, we focus the analysis on the particular shape in fig. 1b.

1.5 Architecture of the paper

We proceed in Section 2 with introducing different waves in $\Pi^\varepsilon$: oscillatory and exponential for $\lambda^\varepsilon \in (0, \pi^2)$ and linear in $x_1$ at the threshold $\lambda^\varepsilon = \pi^2$. Then on the basis of the Mandelstam energy principle, cf. [43 § 5.3] we perform the classification \{incoming/outgoing\} for the introduced waves and impose two types, physical and artificial, of radiation conditions at infinity. The corresponding diffraction problems give rise to two scattering matrices $s^\varepsilon$ and $S^\varepsilon$. Due to the restriction of the boundary value problem (1.3), (1.4) onto the semi-infinite waveguide (1.14) the matrix $s^\varepsilon$ reduces to classical scalar reflexion coefficient but the augmented scattering matrix $S^\varepsilon$ is of size $2 \times 2$ because the artificial radiation conditions involve the exponential waves in addition to the oscillatory waves. The above-mentioned criterion for the existence of embedded eigenvalues is formulated in terms of the matrix $S^\varepsilon$, see Theorem 1 and note that its proof is completed by Theorem 7 about solutions of the problem (1.12), (1.13) with a fast exponential decay. In Section 3 we construct formal asymptotic expansions of the augmented scattering matrix which are justified in Section 6.4. In order to detect an embedded eigenvalue in Section 4 we need the main asymptotic and first correction terms. The two-fold nature of the box-shaped perturbation manifests itself in different ansätze for the diagonal entries $S^\varepsilon_{11}$ and $S^\varepsilon_{00}$ of the matrix $S^\varepsilon$. In the first case the asymptotic procedure looks like as for a regular perturbation of the boundary, that is the boundary layer phenomenon does not influence the main asymptotic term $S^\varepsilon_{11}$ in the expansion

$$S^\varepsilon_{11} = S^0_{11} + \hat{S}^\varepsilon_{11}. \quad (1.16)$$

In the second case the correction term $S'_\infty$ in the expansion $S^\varepsilon_{00} = 1 + \varepsilon S'_0 + \hat{S}^\varepsilon_{00}$ results from the boundary layer phenomenon while the regular expansion affects higher-order terms only. It should be emphasized that the augmented scattering matrix is unitary and symmetric and the main asymptotic term in the expansion

$$S^\varepsilon_{01} = S^\varepsilon_{10} = \varepsilon^{1/2} S'_10 + \hat{S}^\varepsilon_{10} \quad (1.17)$$

of the anti-diagonal entries can be computed by means of both the approaches.

In Section 4 we first reduce the criterion $S^\varepsilon_{11} = -1$ from Theorem 1 to an abstract equation and second solve it with the help of the contraction principle. Finally we formulate Theorems 3 and 4 on the existence and uniqueness of the embedded eigenvalue. These assertions are proved in the next three sections. In Section 5 formulations of the problem (1.3), (1.4) in the Kondratiev spaces (Theorem 5) and weighted spaces with detached asymptotics (Theorem 8) are presented as well as the operator formulation of the radiation condition at infinity. At the same time, key results for the particular box-shaped perturbation (1.2), are displayed in Sections 5.3 and 5.5 where we verify the absence of trapped modes with a fast decay and clarify the dependence of majorants in a priori estimates for solutions on the small and spectral parameters $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in (0, \pi^2)$.

In Section 6 we evaluate remainders in the asymptotic formulas (1.16)-(1.17) for the augmented scattering matrix Theorem 10 while the boundary layer phenomenon brings additional powers of $|\ln \varepsilon|$ into bounds of estimates. Other necessary properties of the matrix are clarified in Section 7 where the uniqueness of the embedded eigenvalue is verified, too. Again kinks of the perturbation profile seriously complicate the analysis.

Conclusive remarks are collected in Section 8 where, in particular, we discuss essential simplifications of the analysis within the discrete spectrum and a certain hardship for detecting eigenvalues near higher thresholds of the continuous spectrum.
2 Augmented scattering matrix and a criterion for trapped modes

2.1 Classification of waves.

For the spectral problem (1.15), the limit (\(\varepsilon = 0\)) problem (1.3), (1.4) in the straight strip \(\Pi = \mathbb{R} \times (0, 1)\) has two oscillating waves

\[ w_{0}^{\pm}(x) = (2k_{\varepsilon})^{-1/2} e^{\pm ik_{\varepsilon}x_{1}}, \quad k_{\varepsilon} = \sqrt{\lambda_{\varepsilon}}. \]  

(2.1)

By the Sommerfeld principle, see, e.g., \[17, 34, 49\], the waves \(w_{0}^{+}\) and \(w_{0}^{-}\) are outgoing and incoming, respective, in the one-sided waveguide (1.14) according to their wavenumbers \(+k_{\varepsilon}\) and \(-k_{\varepsilon}\). In this way, the inhomogeneous problem (1.12), (1.13)

\[ -\Delta u_{\varepsilon}(x) - \lambda_{\varepsilon} u_{\varepsilon}(x) = f_{\varepsilon}(x), \quad x \in \Pi_{+}^{\varepsilon}, \quad \partial_{\nu} u_{\varepsilon}(x) = 0, \quad x \in \partial \Pi_{+}^{\varepsilon}, \]  

(2.2)

with, for example, compactly supported right-hand side \(f_{\varepsilon}\) is supplied with the radiation condition

\[ u_{\varepsilon}(x) = e_{0}^{\varepsilon} w_{0}^{\varepsilon+}(x) + \tilde{w}_{\varepsilon}(x), \quad \tilde{w}_{\varepsilon}(x) = O(e^{-x_{1}\sqrt{\pi^{2} - \lambda_{\varepsilon}}}). \]  

(2.3)

In Section 5 we will give an operator formulation of problem (2.2), (2.3).

At the threshold

\[ \lambda = \pi^{2} \]  

(2.4)

in addition to the oscillating waves

\[ w_{0}^{0+}(x) = (2\pi)^{-1/2} e^{\pm i\pi x_{1}} \]  

(2.5)

there appear the staying and growing waves

\[ w_{1}^{0}(x) = \cos(\pi x_{2}) \]  

which cannot be classified by the Sommerfeld principle because of zero wavenumber. However, as was observed in \[43, \S 5.3\], that, first, waves (2.1) verify the relations

\[ q_{R}(w_{p}^{\varepsilon\pm}, w_{q}^{\varepsilon\pm}) = \pm i\delta_{p,q}, \quad q_{R}(w_{p}^{\varepsilon\pm}, w_{q}^{\varepsilon\mp}) = 0 \]  

(2.6)

and, second, the linear combinations

\[ w_{1}^{0\pm}(x) = (x_{1} \mp i) \cos(\pi x_{2}) \]  

(2.7)

together with waves (2.5) fulfil formulas (2.6) at \(\varepsilon = 0\) as well. Here, \(\delta_{p,q}\) is the Kronecker symbol, \(p, q = 0\) in the first case and \(p, q = 0, 1\) in the second case, and \(q_{R}\) is a symplectic, that is sesquilinear and anti-Hermitian form

\[ q_{R}(w, v) = \int_{0}^{1} \left( v(R, x_{2}) \frac{\partial w}{\partial x_{1}}(R, x_{2}) - w(R, x_{2}) \frac{\partial v}{\partial x_{1}}(R, x_{2}) \right) dx_{2}. \]

This form emerges from the Green formula in the truncated waveguide \(\Pi_{+}^{\varepsilon}(R) = \{ x \in \Pi_{+}^{\varepsilon}, \ x_{1} \in (0, R) \}\) and therefore does not depend on the length parameter \(R > l\) for any of introduced waves and their linear combinations. Hence, we skip the subscript \(R\) in (2.6) and (2.7).

For waves (2.6), the sign of \(\operatorname{Im} q\left(w_{0}^{\varepsilon\pm}, w_{0}^{\varepsilon\mp}\right)\) coincides with the sign of the wavenumber and therefore indicates the propagation direction. Analogously, we call the wave \(w_{0}^{0+}\) outgoing and
the wave \( w_1^{0-} \) incoming in the waveguide \( \Pi_x^+ \) so that the problem (2.2) with \( \lambda^\varepsilon = \pi^2 \) ought to be supplied with the following threshold radiation condition

\[
\varepsilon(x) = c_0^e w_0^{0+}(x) + c_1^e w_1^{0+}(x) + \tilde{u}(x), \quad \tilde{u}(x) = O(e^{-\sqrt{\pi^2 - \lambda^\varepsilon}})
\]  

(2.8)

In Section 5 we will prove that this formulation of the problem at the threshold (2.4) provides an isomorphism in its operator setting.

As was demonstrated in [13, § 5.6], the form \( q \) is closely related to the Umov-Poyting vector [45, 46] so that both radiation conditions (2.3) and (2.8) arise from the Mandelstam (energy) principle, see [23, § 5.3].

2.2 Scattering matrices and exponential waves

In the case (1.15) the incoming wave in (2.1) generates the following solution of problem (1.12), (1.13):

\[
\zeta_0^\varepsilon(x) = w_0^{0-}(x) + s_{00} w_0^{0+}(x) + \tilde{\zeta}_0^\varepsilon(x)
\]  

(2.9)

Here, the remainder \( \tilde{\zeta}_0^\varepsilon \) decays as \( O(e^{-\pi\sqrt{\pi^2 - \lambda^\varepsilon}}) \) and \( s_{00} \) is the reflection coefficient which satisfies \( |s_{00}| = 1 \) due to conservation of energy.

In the same way, in the case (2.4) we can determine the solutions

\[
\zeta_p^\varepsilon(x) = w_p^{0-}(x) + s_{0p} w_0^{0+}(x) + s_{1p} w_1^{0+}(x) + \tilde{\zeta}_p^\varepsilon(x)
\]

where \( p = 1 \), \( \tilde{\zeta}_p^\varepsilon(x) = O(e^{-x_1\pi\sqrt{3\pi}}) \) and the coefficients \( s_{0p} \) form the (threshold) scattering matrix \( s^0 \) of size 2 × 2. According to the normalization and orthogonality conditions (2.6) for waves (2.5), (2.7) and the relation \( w_p^{0+}(x) = w_p^{0-}(x) \), the matrix \( s^0 \) is unitary and symmetric (cf. [34, §2]) that is

\[
\left(s^0\right)^{-1} = \left(s^0\right)^*, \quad s^0 = \left(s^0\right)^\top
\]

(2.10)

where \( \top \) stands for transposition and \( \left(s^0\right)^* = \left(s^0\right)^\top \) is the adjoint matrix.

The reflection coefficient \( s_{00} \) ought to be regarded as a scattering matrix of size 1 × 1 in view of the only one couple of waves (2.1) which are able to drive energy along the waveguide (1.14). For example, dealing with the next couple of waves

\[
v_1^{\varepsilon\pm}(x) = (k_1^\varepsilon)^{-1/2} e^{\pm k_1^\varepsilon x_1} \cos(\pi x_2), \quad k_1^\varepsilon = \sqrt{\pi^2 - \lambda^\varepsilon}
\]

(2.11)

which are decaying (−) and growing (+), one readily finds that

\[
q(v_1^{\varepsilon\pm}, v_1^{\varepsilon\pm}) = 0
\]

(2.12)

but

\[
q(v_1^{\varepsilon\pm}, v_1^{\varepsilon-}) = -q(v_1^{\varepsilon-}, v_1^{\varepsilon\pm}) = 1
\]

(2.13)

As was observed in [13, §5.6] and mentioned above, formula (2.12) annihilates the projection on the \( x_3 \)-axis of the Umov-Poyting vector and therefore waves (2.11) cannot drive energy. In the papers [21, 22] (see [23] for general elliptic systems) the linear combinations of exponential waves

\[
w_1^{\varepsilon\pm}(x) = 2^{-1/2}(v_1^{\varepsilon\pm}(x) \mp iv_1^{\varepsilon-}(x))
\]

(2.14)

were introduced. It is remarkable that, thanks to (2.12) and (2.13), waves (2.11) and (2.14) enjoy conditions (2.6) with \( p, q = 0, 1 \). The latter allows us to determine the solutions

\[
Z^\varepsilon_p(x) = w_p^{\varepsilon-}(x) + S_{0p}^{\varepsilon} w_p^{0+}(x) + S_{1p}^{\varepsilon} w_1^{0+}(x) + \tilde{Z}_p^\varepsilon(x), \quad \tilde{Z}_p^\varepsilon(x) = O(e^{-x_1\pi\sqrt{3\pi^2 - \lambda^\varepsilon}}), \quad p = 0, 1
\]

(2.15)
to compose the coefficient matrix $S^\varepsilon = (S^\varepsilon_{qp})$ and to assure its unitary and symmetry property, cf. (2.10). Moreover, since $w^\varepsilon_0^+ (x) = w^\varepsilon_0^- (x)$, this matrix is symmetric, see [39, §2] again.

In Section 5 we will give an operator formulation of the problem (2.2), at $\lambda^2 \in (0, \pi^2)$ with the radiation condition

$$
U^\varepsilon (x) = e_0^0 w_0^0 + c_1^1 w_1^+ (x) + \tilde{U}^\varepsilon (x), \quad \tilde{U}^\varepsilon (x) = O(e^{-x_1 \sqrt{4\pi^2 - \lambda^2}}).
$$

(2.16)

We recognize this condition as artificial because the right-hand side of (2.16) involves the exponentially growing wave $w_1^+ (x)$, see (2.14) and (2.11).

2.3 A criterion for trapped modes.

A reason to consider problem (2.2), (2.16) and the augmented scattering matrix $S^\varepsilon$ is explained by the following assertion.

**Theorem 1** Problem (1.12), (1.13) with the spectral parameter (1.15) has a trapped mode $u^\varepsilon \in H^1 (\Pi^\varepsilon_\varepsilon)$ if and only if

$$
S^\varepsilon_{11} = -1.
$$

(2.17)

In other words, equation (2.17) provides a criterion for the existence of a trapped mode in the one-sided waveguide (1.14).

A proof of Theorem 1 can be found, e.g., in [23, 39, Thm 2] but, since the criterion (2.17) plays the central role in our analysis, we here give the condensed proof.

The unitary property of $S^\varepsilon$ demonstrates that

$$
S^\varepsilon_{11} = -1 \quad \Rightarrow \quad S^\varepsilon_{10} = S^\varepsilon_{01} = 0.
$$

(2.18)

Thus the solution (2.15) with $p = 1$ becomes a trapped mode because formulas (2.14) and (2.11) assure that

$$
Z^\varepsilon_1 (x) = Z^\varepsilon_p (x) = w^\varepsilon_0^+ (x) - w_1^+ (x) + \tilde{Z}_p^\varepsilon (x) = -2^{1/2} iv_1^{\varepsilon-} (x) + \tilde{Z}_1^\varepsilon (x) = O(e^{-x_1 k^\varepsilon_1}).
$$

Hence, (2.17) is a sufficient condition. To verify the necessity, we first assume that the decomposition

$$
U^\varepsilon (x) = c^\varepsilon u^\varepsilon^- (x) + \tilde{U}^\varepsilon (x)
$$

(2.19)

of a trapped mode $U^\varepsilon \in H^1 (\Pi^\varepsilon_\varepsilon)$ has a coefficient $c^\varepsilon \neq 0$.

Then $U^\varepsilon$ becomes a linear combination of solutions (2.15), namely, according to (2.14), we have

$$
U^\varepsilon = C_0^\varepsilon Z_0^\varepsilon + C_1^\varepsilon Z_1^\varepsilon = C_0^\varepsilon w_0^\varepsilon^- + (S^\varepsilon_{00} C_0^\varepsilon + S^\varepsilon_{01} C_1^\varepsilon) w_0^\varepsilon^+ + 2^{-1/2} (v_1^+ - iv_1^-) C_1^\varepsilon + 2^{-1/2} (v_1^+ + iv_1^-) (S^\varepsilon_{10} C_0^\varepsilon + S^\varepsilon_{11} C_1^\varepsilon) + \tilde{U}^\varepsilon.
$$

Owing to the exponential decay of $U^\varepsilon$, coefficients of the oscillating waves $w_0^\varepsilon^+$ must vanish so that

$$
C_0^\varepsilon = 0, \quad S^\varepsilon_{01} C_1^\varepsilon = 0.
$$

Moreover, coefficients of the exponential waves $v_0^\varepsilon^+$ and $v_0^\varepsilon^-$, respectively, are $2^{-1/2} (S^\varepsilon_{11} + 1) C_1^\varepsilon = 0$ and $2^{-1/2} (S^\varepsilon_{11} - 1) C_1^\varepsilon = c^\varepsilon$. Recalling our assumption $c^\varepsilon \neq 0$, we see that $C_1^\varepsilon = -2^{-1/2} c^\varepsilon \neq 0$ and, therefore, (2.17) holds true.

If $c^\varepsilon = 0$ in (2.19), the trapped mode $U^\varepsilon (x)$ gains very fast decay rate $O(e^{-x_1 \sqrt{4\pi^2 - \lambda^2}})$. In Section 5.3 we will show with a new argument that such trapped modes do not exist for a small $\varepsilon$.

**Remark 2**. The relationship between the augmented scattering matrix and the reflection coefficient in (2.20) looks as follows:

$$
S^\varepsilon_{00} = S^\varepsilon_{01} (S^\varepsilon_{11} + 1)^{-1} S^\varepsilon_{10},
$$

(2.20)

see, e.g., [39, Thm 3]. Note that, in view of (2.18), the last term in (2.20) becomes null in the case $S^\varepsilon_{11} = -1$ when $S^\varepsilon_{00} = S^\varepsilon_{01}$. ✷
3 Formal asymptotics of the augmented scattering matrix

3.1 Step-shaped perturbation of boundaries.

In this section we derive asymptotic expansions by means of a formal asymptotic analysis and postpone their justification to Section 6.

Perturbation of the straight waveguide drawn in fig. 1b and in fig. 3b, ought to be regarded as a combination of regular and singular perturbations, see, e.g., [24, Ch.XII, § 6.5] and [31, Ch. 2 and 4], respectively. For a regular perturbation of the boundary, an appropriate change of variables, which differs from the identity in magnitude $O(\varepsilon)$ only, is usually applied in order to convert the perturbed domain into the reference one. In this way, differential operators in the problem gain small perturbations but asymptotics can be constructed with the help of standard iterative procedures like decompositions of a perturbed operator in the Neumann series.

Singular perturbations of boundaries need much more delicate analysis because they require for a description of asymptotics in the stretched coordinates which, for the domain $\Pi^{\varepsilon}_+ = \Pi^l_+$, see (1.14), take the form

$$\xi = (\xi_1, \xi_2) = \varepsilon^{-1} (x_1 - l, x_2).$$

(3.1)

Notice that the change $x \rightarrow \xi$ and setting $\varepsilon = 0$ transform $\Pi^{\varepsilon}_+$ into the upper half-plane $\mathbb{R}^2_+$ with a semi-infinite step, fig. 5a,

$$\Xi = \{ \xi \in \mathbb{R}^2 : \xi_2 > 0 \text{ for } \xi_1 \leq 0 \text{ and } \xi_2 > -1 \text{ for } \xi_1 > 0 \}.$$

(3.2)

As a result, the singular perturbation of the waveguide wall gives rise to the boundary layer phenomenon described by solutions to the following problem:

$$- \Delta_\xi v(\xi) = 0, \quad \xi \in \Xi, \quad \partial_{\nu(\xi)} v(\xi) = g(\xi), \quad \xi \in \partial \Xi.$$

(3.3)

The Laplace equation is caused by the relation $\Delta x + \lambda = \varepsilon^{-2} \Delta_\xi + \lambda$ which singles out the Laplacian as the main asymptotic part of the Helmholtz operator. The Neumann problem (3.3) with a compactly supported datum $g$ admits a solution $v(\xi) = O(|\xi|^{-1})$ as $|\xi| \rightarrow +\infty$ provided $\int_{\partial \Xi} g(\xi) \, ds_\xi = 0$. Otherwise, a solution grows at infinity like $c \ln |\xi|$ and loses the intrinsic decay property of a boundary layer so that traditional asymptotic procedures become much more sophisticated, see [31, Ch. 2, 4] and [20]. However, we will see that in our particular problem the boundary perturbation does not affect the main asymptotic term and the first correction term does not include a boundary layer.
3.2 Asymptotic procedure

We search for an eigenvalue of the problem (1.12), (1.13) in the form
\[ \lambda^\varepsilon = \pi^2 - \varepsilon^2 \mu \]  
where the correction coefficient \( \mu > 0 \) is to be found in Section 4. Recalling the normalization factors in (2.11) and (2.1)
\[ (k_1^\varepsilon)^{-1/2} = \varepsilon^{-1/2} \mu^{-1/4} + O(\varepsilon^{1/2}), \quad (2k_1^\varepsilon)^{-1/2} = (2\pi)^{-1/2} + O(\varepsilon^2), \]  
we guess at the following asymptotic ansätze for entries of the augmented scattering matrix
\[ S_{11}^\varepsilon = S_{11}^0 + \varepsilon S_{11}' + \varepsilon^2 S_{11}'' + \varepsilon^3 S_{11}''' + \varepsilon^4 S_{11}'''' \]
but aim to calculate the terms \( S_{p1}^0 \) and \( S_{p1}^0 \) only. We further estimate the remainders in Section 6.

Using definitions of waves (2.1), (2.11), (2.14), we take relations (3.4) and (3.5) into account and rewrite the decomposition (2.15) of the special solution \( Z_1^\varepsilon \) as follows:
\[ Z_1^\varepsilon (x) = \varepsilon^{-1/2} (4\mu)^{-1/4} \cos (\pi x_2) (1 + i + S_{11}^0 (1 - i) + \varepsilon (S_{11}^0 (1 - i) + x_1 \sqrt{\mu} (1 - i + S_{11}^0 (1 + i)) + ...) + \varepsilon^{1/2} (2\pi)^{-1/2} (S_{01}^0 + \varepsilon S_{01}' + ...) (e^{i\pi x_1} + ...) . \]

Here and everywhere in this section, ellipses stand for lower-order terms inessential in our formal asymptotic analysis. In (3.7), the Taylor formula
\[ e^{k_1^\varepsilon x_1} = i e^{-k_1^\varepsilon x_1} = (1 + i) + \varepsilon x_1 \sqrt{\mu} (1 + i) + O(\varepsilon^2 x_1^2) \]
was applied so that expansion (3.8) becomes meaningful under the restriction \( x_1 < R \) with a fixed \( R, \) i.e. for \( x \in \Pi^\varepsilon (R) . \)

In view of the above observation we employ the method of matched asymptotic expansions, cf. [20], in the interpretation [39] [36]. Namely, we regard (3.7) as an outer expansion and introduce the inner expansion
\[ Z_1^\varepsilon (x) = \varepsilon^{-1/2} Z_1^0 (x) + \varepsilon^{1/2} Z_1' (x) + ... \]
At the same time, coefficients of \( \varepsilon^{-1/2} \) and \( \varepsilon^{1/2} \) on the right-hand side of (3.7) exhibit a behavior at infinity of the terms \( Z_1^0 \) and \( Z_1' \) in (3.9) because the upper bound \( R \) for \( x_1 \) can be chosen arbitrary large. Thus, they must satisfy
\[ Z_1^0 (x) = (4\mu)^{-1/4} \cos (\pi x_2) (1 + i + S_{11}^0 (1 - i)) + ... \]
\[ Z_1' (x) = (4\mu)^{-1/4} \cos (\pi x_2) S_{11}' (1 - i) + x_1 \sqrt{\mu} (1 - i + S_{11}^0 (1 + i)) + S_{01}^0 (2\pi)^{-1/2} e^{i\pi x_1} + ... \]

The formal passage to \( \varepsilon = 0 \) transforms the waveguide (1.12) into the semi-infinite strip \( \Pi_+^0 = \mathbb{R} \times (0, 1) \) while due to (3.4) the Neumann problem (1.12), (1.13) converts into
\[ -\Delta u^0 (x) = \pi^2 u^0 (x), \quad x \in \Pi_+^0, \]
\[ \partial_n u^0 (x) = 0, \quad x \in \partial \Pi_+^0. \]
and, hence, shaped perturbation

Since \( \lambda \) have chapter 2 of [43], the review [35] and Section 5 of this paper. The solution (3.20) is defined up to Neumann condition becomes inhomogeneous because of the boundary perturbation. For \( p = 1 \), we have

\[
- \Delta Z_p'(x) = \pi^2 Z_p'(x), \quad x \in \Pi_+^0, \quad \partial N Z_p'(x) = g_p(x), \quad x \in \partial \Pi_+^0.
\] (3.17)

To determine the datum \( g_1 \), we observe that the function (3.10) satisfies the Neumann condition (1.13) everywhere on \( \partial \Pi^e \), except at the lower side \( \Upsilon = \{ x : x_1 \in (0, l), \quad x_2 = -\varepsilon \} \) of the box-shaped perturbation \( \varepsilon \). Furthermore, we obtain

\[
\partial_\nu Z_1'(x_1, -\varepsilon) = -\partial_\nu Z_1'(x_1, -\varepsilon) = (4\mu)^{-1/4} \pi \sin (-\pi \varepsilon) (1 + i + S_1^0 (1 - i)) = -\varepsilon (4\mu)^{-1/4} \pi^2 (1 + i + S_1^0 (1 - i)) + O(\varepsilon^3) =: -\varepsilon G_1' + O(\varepsilon^3)
\] (3.18)

and, hence,

\[
g_p(x) = \begin{cases} 0, & x \in \partial \Pi_+^0 \setminus \Upsilon_0, \\ G_p', & x \in \Upsilon_0^0. \end{cases}
\] (3.19)

Although the Neumann datum (3.19) is not smooth and has a jump at the point \( x = (l, 0) \), the problem (3.17) with \( p = 1 \) has a solution in \( H^{1, loc}(\Pi_+^0) \) such that

\[
Z_p'(x) = C_p e^{i\pi x_1} + (C_0^p + x_1 C_1^p) \cos (\pi x_2) + \tilde{Z}_p'(x), \quad \tilde{Z}_p'(x) = O(e^{-x_1 \sqrt{3\pi}}).
\] (3.20)

This fact is a direct consequence of the elliptic theory in domains with cylindrical outlets to infinity (see the key works [2, 25, 32, 33] and, e.g., the monographs [26, 43], but also may be derived by the Fourier method after splitting \( \Pi_+^0 \) into the rectangle \((0, l) \times (0, 1)\) and the semi-strip \((l, +\infty) \times (0, 1)\). A simple explanation how to apply the above-mentioned theory can be found in the introductory chapter 2 of [43], the review [35], and Section 5 of this paper. The solution (3.20) is defined up to the term \( c \cos (\pi x_2) \) and, therefore, the coefficient \( C_0^p \) can be taken arbitrary. Other coefficients in (3.20) are computed by application of the Green formula in the long \( (R \) is big) rectangle \( \Pi_+^0 (R) = (0, R) \times (0, 1) \).

Indeed, we send \( R \) to +\( \infty \) and obtain

\[
0 = \lim_{R \to +\infty} \int_{\Pi_+^0 (R)} \left( u_1^0(x) (\Delta + \pi^2) Z_1'(x) - Z_1'(x) (\Delta + \pi^2) u_1^0(x) \right) dx = \int_0^1 \cos (\pi x_2) \partial_1 Z_1'(R, x_2) dx_2 - \int_0^l \cos (\pi 0) \partial_2 Z_1'(x_1, 0) dx_1 = \frac{1}{2} C_1^p + (4\pi)^{-1/4} \pi^2 l (1 + i + S_1^0 (1 - i)).
\] (3.21)

In the same way we deal with the functions (3.14) and (3.20) that results in the equality

\[
0 = \lim_{R \to +\infty} \int_0^1 \left( \cos (\pi x_1) \partial_1 Z_1'(x) dx_2 - U_1^0(x) \partial_1 \cos (\pi x_1) \right) |_{x_1 = R} dx_2 - \int_0^l \cos (\pi x_1) \partial_2 Z_1'(x_1, 0) dx_1 = i\pi C_1 + (4\mu)^{-1/4} \pi^2 (1 + i + S_1^0 (1 - i)) \int_0^l \cos (\pi x_1) dx_1.
\] (3.22)
Comparing (3.11) and (3.20), we arrive at the relations

\[(2\pi)^{-1/2} S_{01}^0 = C_1, \quad (4\mu)^{-1/4} \sqrt{\mu} (1 - i + S_{11}^0 (1 + i)) = C_1^1, \quad S_{11}^0 (1 - i) = C_1^0 \]

which together with our calculations (3.22) and (3.23) give us the following formulas:

\[S_{01}^0 = (4\mu)^{-1/4} (2\pi)^{1/2} \pi i (1 + i + S_{11}^0 (1 - i)) \int_0^l \cos(\pi x_1) \, dx_1 = \]

\[= -(4\mu)^{-1/4} (2\pi)^{1/2} (1 - i - S_{11}^0 (1 + i)) \sin(\pi l)\]

\[\sqrt{\mu} (1 - i + S_{11}^0 (1 + i)) = -2\pi^2 l (1 + i + S_{11}^0 (1 - i)) \Rightarrow \]

\[\Rightarrow S_{11}^0 = -\frac{\sqrt{\mu} (1 - i) 2\pi^2 l (1 + i)}{\sqrt{\mu} (1 + i) 2\pi^2 l (1 - i)} = -\frac{4\pi^2 l \sqrt{\mu} + i (4\pi^4 l^2 - \mu)}{4\pi^4 l^2 + \mu}.\]

We emphasize that \(\mu = 4\pi^4 l^2 \Rightarrow S_{11}^0 = -1.\)

The necessary computations are completed. It should be mentioned that, to determine the correction terms \(S_{11}^0\) and \(S_{01}^0\) in the ansätze (3.6), one has to make another step in our asymptotic procedure, see the next section, but they are of no further use.

### 3.3 The detailed asymptotic procedure

Let us construct asymptotics of the entries

\[S_{00}^\varepsilon = S_{00}^0 + \varepsilon S_{00}^0 + \tilde{S}_{00}^\varepsilon, \quad S_{10}^\varepsilon = \varepsilon^{1/2} S_{10}^0 + \varepsilon^{3/2} S_{10}^0 + \tilde{S}_{10}^\varepsilon \]

in the augmented scattering matrix. These asymptotics are not of very importance in our analysis of eigenvalues but are much more representative than the analogous formulas (3.6) because the ansatz for \(Z_0^\varepsilon\) indeed involves the boundary layer concentrated near the edge of the box-shape perturbation in (1.14).

Using (3.5), (3.26) and (3.8), we rewrite the decomposition (2.15) of \(Z_0^\varepsilon(x)\) as follows:

\[Z_0^\varepsilon(x) = (2\pi)^{-1/2} \left( e^{-i\pi x_1} + S_{00}^0 e^{i\pi x_1} + \varepsilon S_{00}^0 e^{i\pi x_1} + \ldots \right) + \]

\[+ (4\mu)^{-1/4} \left( S_{10}^0 + \varepsilon S_{10}^0 + \ldots \right) \cos(\pi x_2) (1 - i + \varepsilon x_1 \sqrt{\mu} (1 + i) + \ldots).\]

Then we accept in a finite part of \(\Pi_+^\varepsilon\) the ansatz

\[Z_0^\varepsilon(x) = Z_0^0(x) + \varepsilon Z_0^0(x) + \ldots \]

Referring to (3.27), we fix the behavior of its terms at infinity

\[Z_0^0(x) = (2\pi)^{-1/2} \left( e^{-i\pi x_1} + S_{00}^0 e^{i\pi x_1} \right) + (4\mu)^{-1/4} S_{10}^0 (1 - i) \cos(\pi x_2) + \ldots, \]

\[Z_0'(x) = (2\pi)^{-1/2} S_{00}^0 e^{i\pi x_1} + (4\mu)^{-1/4} \cos(\pi x_2) (S_{10}^0 (1 - i) + x_1 \sqrt{\mu} S_{10}^0 (1 + i)) + \ldots \]

Comparing (3.29) with (3.14), (3.15), we conclude that

\[Z_0^0(x) = (2\pi)^{-1/2} \left( e^{-i\pi x_1} + e^{i\pi x_2} \right) + (4\mu)^{-1/4} S_{10}^0 (1 - i) \cos(\pi x_2)\]

and, hence,

\[S_{00}^0 = 1\]
Let us describe the correction terms in (3.26). The function (3.31) satisfies the equation (1.12) with \( \lambda^2 = \pi^2 \) and leaves the discrepancies

\[
\partial_v Z_0^0(x_1, -\varepsilon) = -\partial_2 Z_0^0(x_1, -\varepsilon) = -\varepsilon (4\mu)^{-1/4} \pi^2 \hat{S}_{10}^0(1 - i) + O(\varepsilon^3) = (3.33)
\]

\[
= -\varepsilon G_0^0 + O(\varepsilon^3), \quad x_1 \in (0, l)
\]

\[
\partial_v Z_0^0(l, x_2) = \partial_1 Z_0^0(l, x_2) = -(2\pi)^{1/2} \sin(\pi l), \quad x_2 \in (-\varepsilon, 0)
\]

(3.34)

in the boundary condition (1.13) on the big \( Y^\varepsilon \) and small \( v^\varepsilon = \{ x : x_1 = l, x_2 \in (-\varepsilon, 0) \} \) sides of the rectangle \( \mathbb{R}^+ \), respectively. The discrepancy (3.33) is similar to (3.18) and appears as the datum (3.19) in the problem (3.17) with \( p = 0 \). To compensate for (3.34), we need the boundary layer

\[
V_0^0(\xi) = (2\pi)^{1/2} \sin(\pi l) v(\xi)
\]

(3.35)

where \( \xi \) are stretched coordinates (3.1) and \( v \) is a solution of the Neumann problem (3.3) in the unbounded domain (3.2) with the right-hand side

\[
g(\xi) = \begin{cases} 
0, \quad \xi_1 \neq 0, \\
1, \quad \xi_1 = 0, \quad \xi_2 \in (-1, 0),
\end{cases}
\]

for \( \xi \in \partial \Xi \).

This solution, of course, can be constructed by an appropriate conformal mapping but we only need its presentation at infinity

\[
v(\xi) = (B/\pi) \ln(1/|\xi|) + c + O(1/|\xi|), \quad |\xi| \to \infty.
\]

(3.36)

The constant \( c \) is arbitrary but the coefficient \( B \) can be computed by the Green formula in the truncated domain \( \Xi(R) = \{ \xi \in \Xi : |\xi| < R \} \) with \( R \to +\infty \):

\[
0 = \lim_{R \to +\infty} R \int_{0}^{\pi + \arcsin(1/R)} \frac{\partial v}{\partial \rho}(\xi) \, d\varphi + \int_{0}^{1} \frac{\partial v}{\partial \xi_1}(0, \xi_2) \, d\xi_2 = (3.37)
\]

\[
= -\frac{B}{\pi} \int_{0}^{\pi} d\varphi + \int_{0}^{1} d\xi_2 = -B + 1 \Rightarrow B = 1.
\]

Here, \( (\rho, \varphi) \) is the polar coordinates system.

We fix \( c = -\pi^{-1} \ln \varepsilon \) in (3.36) and observe that

\[
v(\xi) = (1/\pi)(\ln(1/\rho) - \ln \varepsilon) + O(1/\rho^2) = (1/\pi) \ln(1/r) + O(\varepsilon^2/r^2)
\]

(3.38)

in the polar coordinate system \( (r, \varphi) \) centered at the point \( x = (l, 0) \in \partial \Pi^\varepsilon \), see fig. 5 b.

Applying the method of matched asymptotic expansions in the traditional manner, cf. [20, 48, 49], as well as [31] Ch. 5], [42] for ledge-shaped perturbation of domains, we consider (3.28) as an outer expansion in a finite part of the waveguide while \( \varepsilon V_0^0(\xi) \) becomes the main term of the inner expansion in the vicinity of the ledge of the box-shaped perturbation in (1.14). In view of (3.35) and (3.38), the standard matching procedure proposes \( Z_0^0 \) as a singular solution of the homogeneous problem (3.12), (3.13) with the following asymptotic condition at the point \( x = (l, 0) \in \partial \Pi^0 \):

\[
Z_0^0(x) = (2/\pi)^{1/2} \sin(\pi l) \ln(1/r) + c + O(r) \quad \text{for } r \to +0.
\]

(3.39)

Arguing in the same way as for the function \( Z_1^0 \), we conclude that the problem (3.17), (3.39) has a solution \( Z_0^0 \) which admits the representation (3.20) with \( p = 0 \) under the restriction \( x_1 \geq
$R > l$ needed due to the logarithmic singularity in $\|3.3\|$. The coefficient $C^0_0$ is arbitrary but $C^0_0$ and $C^0_1$ can be computed again by means of the Green formula in the domain $\Pi^0_\delta (R, \delta) = \{x \in \Pi^0_\delta : x_1 < R, \ r > \delta\}$ and the limit passage $R \rightarrow +\infty, \ \delta \rightarrow +0$, cf. $\|3.21\|$, $\|3.22\|$ and $\|3.37\|$. Dealing with $u^0_0$ and $Z'^0_0$, we take into account the equality $\partial_2 u^0_0 (x_1, 0) = 0$ and obtain that

$$0 = \int_{\Pi(R,\delta)} \cos (\pi x_1) \left( \Delta Z'^0_0 (x) + \pi^2 Z'^0_0 (x) \right) \, dx =$$

$$= \int_0^1 \left( \cos (\pi x_1) \frac{\partial Z'^0_0}{\partial x_1} (x) + \pi \sin (\pi x_1) Z'_0 (x) \right) \bigg|_{x_1=R} \, dx_2 -$$

$$- \delta \int_0^\pi \left( \cos (\pi x_1) \frac{\partial Z'^0_0}{\partial r} (x) - Z'_0 (x) \frac{\partial}{\partial r} \cos (\pi x_1) \right) \bigg|_{r=\delta} \, d\varphi =$$

$$= i\pi C^0_0 + \left( \frac{\pi}{2} \right)^{1/2} \sin (2\pi l) + o(1) \Rightarrow C^0_0 = i (2\pi)^{-1/2} \sin (2\pi l) .$$

Inserting $u^1_0$ and $Z'^0_0$ into the Green formula in $\Pi^0_\delta (R, \delta)$, we take into account the inhomogeneous boundary condition on $\Gamma^0$ and derive that

$$0 = \int_0^1 \cos (\pi x_2) \partial_1 Z'^0_0 (x) \bigg|_{x_1=R} \, dx_2 - \int_0^{1-\delta} \cos (\pi x_2) \partial_2 Z'_0 (x) \bigg|_{x_2=0} \, dx_1 -$$

$$- \delta \int_0^\pi \left( \cos (\pi x_2) \partial_r Z'_0 (x) - Z'_0 (x) \partial_r \cos (\pi x_2) \right) \bigg|_{r=\delta} \, d\varphi =$$

$$= \frac{1}{2} C^1_0 (4\mu)^{-1/4} \pi^2 l S^0_{10} (1 - i) + (2\pi)^{1/2} \sin (\pi l) + o (1) .$$

We now compare coefficients in the expansions $\|3.20\|$, $p = 0$, and $\|3.30\|$. According to the calculations $\|3.40\|$ and $\|3.41\|$ we derive the formulas

$$C^0_0 = (2\pi)^{-1/2} S^0_{00} \Rightarrow S^0_{00} = i \sin (2\pi l) ,$$

$$C^0_1 = (4\mu)^{-1/4} \sqrt{\pi} S^0_{10} (1 + i) \Rightarrow$$

$$S^0_{10} = - \frac{(4\mu)^{1/4} 2 (2\pi)^{1/2} \sin (\pi l)}{\sqrt{\pi} (1 + i) + 2\pi^2 l (1 - i)} = - (4\mu)^{1/4} (2\pi)^{1/2} \frac{\sqrt{\pi} (1 - i) + 2\pi^2 l (1 + i)}{4\pi^4 l^2 + \mu} \sin (\pi l) .$$

Calculation of coefficients in the ansätze $\|3.6\|$ and $\|3.26\|$ is completed. It is worth to underline that the expression $\|3.42\|$ for the main asymptotic term of $\epsilon^{-1/2} S^\epsilon_{10} = \epsilon^{-1/2} S^\epsilon_{01}$ can be derived from the cumbersome relations $\|3.21\|$ and $\|3.25\|$ as well.

## 4 Detection of a trapped mode

### 4.1 Reformulation of the criterion

We opt for the form

$$\mu = 4\pi^4 l^2 + \Delta \mu, \ \ l = \pi k + \Delta l \quad (4.1)$$

of the spectral and length parameters which support a trapped mode. Here, $k \in \mathbb{N}$ is fixed but small $\Delta \mu, \ \Delta l$ are to be determined. If $\Delta \mu = 0$ and $\Delta l = 0$, the equalities $S^0_{11} = -1$ and $S^0_{01} = 0$ hold due to $\|3.21\|$ and $\|3.22\|$.

We purpose to choose the small increments $\Delta \mu$ and $\Delta l$ in $\|4.1\|$ such that the criterion $\|2.17\|$ for the existence of a trapped mode is satisfied. Since $S^\epsilon_{11}$ is complex, the criterion furnishes two equations for two real parameters $\Delta \mu$ and $\Delta l$. It is convenient to consider the other equations

$$\text{Im} S^\epsilon_{11} = 0, \quad \text{Re} S^\epsilon_{01} = 0 \quad (4.2)$$
which, for small $\varepsilon$ and $\Delta \mu$, $\Delta l$, are equivalent to $S^{\varepsilon}_{11} = -1$. Indeed, from formulas (3.25), (3.42) and (3.32) together with estimates (6.20) it follows that

$$|S^{\varepsilon}_{11}| + 1 = O(|\varepsilon + |\Delta \mu| + |\Delta l|)^{\delta}, \quad \delta \in (0, 1).$$  \hspace{1cm} (4.3)

Since $S^{\varepsilon}$ is unitary and symmetric, see Section 2.2, the second assumption in (4.2) means that $S^{\varepsilon}_{01} = S^{\varepsilon}_{10} = i\sigma$ with some $\sigma \in \mathbb{R}$ and, furthermore,

$$0 = \overline{S^{\varepsilon}_{01}}S^{\varepsilon}_{01} + \overline{S^{\varepsilon}_{10}}S^{\varepsilon}_{10} = 2i\sigma + O\left(|\sigma| (|\varepsilon + |\Delta \mu| + |\Delta l|)^{\delta}\right).$$  \hspace{1cm} (4.4)

Hence, $\sigma = 0$ when $\Delta \mu$, $\Delta l$ and $\varepsilon > 0$ are small so that $S^{\varepsilon}_{01} = 0 \Rightarrow |S^{\varepsilon}_{11}| = -1$ due to (4.3), (4.2).

We have proved that $\Rightarrow$ (4.2) but the inverse implication $\Rightarrow$ (4.1) is obvious.

### 4.2 Solving the system of transcendental equations

By virtue of (3.25), (1.16), and (1.1), the first equation (4.2) turns into

$$\Delta \mu = -\varepsilon \left(8\pi^4 l^2 + \Delta \mu\right)\text{Im} \, \hat{S}^{\varepsilon}_{11}. \hspace{1cm} (4.5)$$

The formulas (3.42), (1.17), and a simple algebraic calculation convert the second equation (4.2) into

$$\sin l = (4\mu)^{-1/4} (2\pi)^{-1/2} \frac{4\pi^4 l^2 + \mu}{4\pi^2 l + 2\mu} \varepsilon \text{Re} \, \hat{S}^{\varepsilon}_{01}$$

and thus

$$\Delta l = \arcsin \left((-1)^k (4\mu)^{-1/4} (2\pi)^{-1/2} \frac{4\pi^2 l^2 + \mu}{2\pi^2 l + \sqrt{\mu}} \varepsilon \text{Re} \, \hat{S}^{\varepsilon}_{01}\right). \hspace{1cm} (4.6)$$

We search for a solution $(\Delta \mu, \Delta l)$ of the transcendental equations (1.5), (1.6) in the closed disk

$$\mathbb{B}_\rho = \left\{(\Delta \mu, \Delta l) \in \mathbb{R}^2 : |\Delta \mu|^2 + |\Delta l|^2 \leq \rho^2\right\} \hspace{1cm} (4.7)$$

and rewrite them in the condensed form

$$(\Delta \mu, \Delta l) = T^{\varepsilon} (\Delta \mu, \Delta l) \quad \text{in} \, \mathbb{B}_\rho \hspace{1cm} (4.8)$$

where $T^{\varepsilon}$ is a nonlinear operator involving asymptotic remainders from formulas (1.16) and (1.17) for the augmented scattering matrix $S^{\varepsilon} = S^{\varepsilon} (\Delta \mu, \Delta l)$. The estimates (6.20) and Proposition 11 below demonstrate that the operator is smooth in $\mathbb{B}_\rho$ with $\rho \leq \rho_0$, $\rho_0 > 0$, and, furthermore,

$$|T^{\varepsilon} (\Delta \mu, \Delta l)| = c_0 \varepsilon \left(1 + |\ln \varepsilon|\right)^2 \quad \text{for} \, \ (\Delta \mu, \Delta l) \in \mathbb{B}_\rho.$$  \hspace{1cm}

Hence, for any $\rho \leq \rho_0$, there exists $\varepsilon (\rho) > 0$ such that $T^{\varepsilon}$ with $\varepsilon \in (0, \varepsilon (\rho))$ is a contraction operator in the disk (4.7). By the Banach contraction principle, the abstract equation (4.8) has a unique solution $(\Delta \mu, \Delta l) \in \mathbb{B}_\rho$ and the estimate $|\Delta \mu| + |\Delta l| \leq C \varepsilon \left(1 + |\ln \varepsilon|\right)^2$ is valid. This solution depends on $\varepsilon$ and determines the spectral and length parameters (4.1) supporting a trapped mode in the perturbed waveguide (1.14) according to the criterion (2.17) from Theorem 1 reformulated as (4.2).
4.3 The main results

Based on the performed formal calculations, we will prove in the next three sections the following existence and uniqueness theorems.

**Theorem 3** Let \( k \in \mathbb{N} \). There exist \( \varepsilon_k, c_k > 0 \) and \( \Delta \mu_k(\varepsilon), \Delta l_k(\varepsilon) \), such that, for any \( \varepsilon \in (0, \varepsilon_k) \), the estimate

\[
|\Delta \mu_k(\varepsilon)| + |\Delta l_k(\varepsilon)| \leq c_k \varepsilon (1 + |\ln \varepsilon|)^2
\]

is valid and the problem \( (1.3), (1.4) \) in the waveguide \( \Pi^\varepsilon_{l(\varepsilon)} = \Pi \cup \varepsilon \overline{l_k(\varepsilon)} \) with the box-shaped perturbation \( (1.2) \) of length \( 2l_k(\varepsilon) = 2(\pi k + \Delta l_k(\varepsilon)) \) has an eigenvalue

\[
\lambda^\varepsilon_k = \pi^2 - \varepsilon^2 (4 \pi^4 (\pi k + \Delta l_k(\varepsilon))^2 + \Delta \mu_k(\varepsilon)).
\]  
(4.9)

The eigenvalue \( 4.9 \) is unique in the interval \((0, \pi^2)\).

**Theorem 4** Let \( k \in \mathbb{N} \) and \( \delta > 0 \). There exist \( \varepsilon^0_k > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon^0_k) \), the waveguide \( \Pi^\varepsilon_l \) with the length parameter

\[
l \in \left[ \pi (k - 1) + \delta, \pi (k + 1) - \delta \right]
\]  
(4.10)

does not support a trapped mode in the case \( l \neq l_k(\varepsilon) \) where \( l_k(\varepsilon) \) is taken from Theorem 3.

5 Weighted spaces with detached asymptotics

5.1 Reformulation of the problem

Let \( W^1_\beta(\Pi^\varepsilon_+) \) be the Kondratiev (weighted Sobolev) space composed from functions \( u^\varepsilon \) in \( H^1_{loc}(\Pi^\varepsilon_+) \) with the finite norm

\[
\left\| u^\varepsilon; W^1_\beta(\Pi^\varepsilon_+) \right\| = \left\| e^{\beta x}; u^\varepsilon; H^1(\Pi^\varepsilon_+) \right\|
\]  
(5.1)

where \( \beta \in \mathbb{R} \) is the exponential weight index. If \( \beta > 0 \), functions in \( W^1_\beta(\Pi^\varepsilon_+) \) decay at infinity in the semi-infinite waveguide \( (1.4) \) but in the case \( \beta < 0 \) a certain exponential growth is permitted while the rate of decay/growth is ruled by \( \beta \). Clearly, \( W^1_0(\Pi^\varepsilon_+) = H^1(\Pi^\varepsilon_+) \). The space \( C^\infty_c(\Pi^\varepsilon_+) \) of smooth compactly supported functions is dense in \( W^1_\beta(\Pi^\varepsilon_+) \) for any \( \beta \).

By a solution of the problem \( (2.2) \) in \( W^1_\sigma(\Pi^\varepsilon_+) \), \( \sigma \in \mathbb{R} \), we understand a function \( u^\varepsilon \in W^1_\sigma(\Pi^\varepsilon_+) \) satisfying the integral identity

\[
(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Pi^\varepsilon} - \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Pi^\varepsilon} = F^\varepsilon(v^\varepsilon) \quad \forall v^\varepsilon \in W^1_{-\sigma}(\Pi^\varepsilon_+)
\]  
(5.2)

where \( F^\varepsilon \in W^1_{-\sigma}(\Pi^\varepsilon_+)^* \) is an (anti)linear continuous functional on \( W^1_{-\sigma}(\Pi^\varepsilon_+) \) and \( (, )_{\Pi^\varepsilon} \) is an extension of the Lebesgue scalar product up to a duality between an appropriate couple of weighted spaces. In view of \( (5.1) \) all terms in \( (5.2) \) are defined correctly so that the problem \( (5.2) \) is associated with the continuous mapping

\[
W^1_\sigma(\Pi^\varepsilon_+) \ni u^\varepsilon \mapsto A^\varepsilon_\sigma(\lambda^\varepsilon) u^\varepsilon = F^\varepsilon \in W^1_{-\sigma}(\Pi^\varepsilon_+)^*.
\]

If \( f^\varepsilon \in L^2_\sigma(\Pi^\varepsilon_+) \) that is \( e^{\sigma z} f^\varepsilon \in L^2(\Pi^\varepsilon_+) \), then the functional

\[
v^\varepsilon \mapsto F^\varepsilon(v^\varepsilon) = (f^\varepsilon, v^\varepsilon)_{\Pi^\varepsilon_+}
\]

belongs to \( W^1_{-\sigma}(\Pi^\varepsilon_+)^* \). Clearly, \( A^\varepsilon_{-\sigma}(\lambda^\varepsilon) \) is the adjoint operator for \( A^\varepsilon_\sigma(\lambda^\varepsilon) \).
5.2 The Fredholm property, asymptotics and the index

Let us formulate some well-know results of the theory of elliptic problems in domains with cylindrical outlets to infinity (see the key papers [25, 32, 33] and, e.g., the monographs [43, 26]). This theory mainly deals with the classical (differential) formulation of boundary value problems, however as was observed in [35], passing to the weak formulation involving integral identities of type (5.2) does not meet any visible obstacle. The only disputable point, namely the dependence of constants on the small parameter \( \varepsilon \), we will be discussed in Section 5.5.

**Theorem 5** (see [25]) Let \( \lambda^\varepsilon \in (0, \pi^2] \).

1) The operator \( A^\varepsilon_\beta \) is Fredholm if and only if
\[
\beta \neq \beta_0 := 0, \quad \beta \neq \beta_{\pm j} := \pm \sqrt{\pi^2 j^2 - \lambda^\varepsilon}, \quad j \in \mathbb{N}.
\] (5.3)

In the case \( \beta = \beta_p \) with \( p \in \mathbb{Z} \) the range \( A^\varepsilon_\beta \) is not closed subspace in \( W^1_{-\beta} (\Pi_+^\varepsilon) \).

2) Let \( \gamma \in (\beta_1, \beta_2) \) and let \( u^\varepsilon \in W^1_\gamma (\Pi_+^\varepsilon) \) be a solution of problem (5.2) with the weight index \( \sigma = -\gamma \) and the right-hand side \( F^\varepsilon \in W^1_\gamma (\Pi_+^\varepsilon)^* \subset W^1_\gamma (\Pi_+^\varepsilon) \). Then the asymptotic decomposition
\[
u^\varepsilon (x) = \overline{\nu}^\varepsilon (x) + \sum_{\pm} \left( a_{\pm 0}^\varepsilon u^\varepsilon_0 (x) + b_{\pm 0}^\varepsilon w^\varepsilon_0 (x) \right)
\] (5.4)

and the estimate
\[
\left( \left\| \overline{\nu}^\varepsilon ; W^1_\gamma (\Pi_+^\varepsilon) \right\|^2 + \sum_{\pm} \left( |a_{\pm 0}^\varepsilon|^2 + |b_{\pm 0}^\varepsilon|^2 \right) \right)^{1/2} \leq c_\varepsilon \left( \left\| F^\varepsilon ; W^1_\gamma (\Pi_+^\varepsilon)^* \right\| + \left\| u^\varepsilon ; W^1_\gamma (\Pi_+^\varepsilon) \right\| \right)
\] (5.5)

are valid, where \( \overline{\nu}^\varepsilon \in W^1_\gamma (\Pi_+^\varepsilon) \) is the asymptotic remainder, \( a_{\pm 0}^\varepsilon \) and \( b_{\pm 0}^\varepsilon \) are coefficients depending on \( F^\varepsilon \) and \( u^\varepsilon \), the waves \( w^\varepsilon_0 \) are given by (2.7) and \( w^\varepsilon_\pm \) by (2.7) for \( \lambda^\varepsilon = \pi^2 \) but by (2.14) for \( \lambda^\varepsilon \in (0, \pi^2] \). The factor \( c_\varepsilon \) in (5.3) is independent of \( F^\varepsilon \) and \( u^\varepsilon \) but may depend on \( \varepsilon \in [0, \varepsilon_0] \).

As was mentioned \( A^\varepsilon_\gamma (\lambda^\varepsilon)^* = A^\varepsilon_{-\gamma} (\lambda^\varepsilon) \) and, hence, kernels and co-kernels of these operators are in the relationship
\[
\ker A^\varepsilon_{-\gamma} (\lambda^\varepsilon) = \text{coker} A^\varepsilon_{+\gamma} (\lambda^\varepsilon)
\] (5.6)

In the next assertion we compare the indexes \( \text{Ind} A^\varepsilon_{+\gamma} (\lambda^\varepsilon) = \dim \ker A^\varepsilon_{+\gamma} (\lambda^\varepsilon) - \dim \text{coker} A^\varepsilon_{+\gamma} (\lambda^\varepsilon) \); notice that \( \text{Ind} A^\varepsilon_{\gamma} (\lambda^\varepsilon) = -\text{Ind} A^\varepsilon_{-\gamma} (\lambda^\varepsilon) \) according to (5.6).

**Theorem 6** (see [43, Thm. 3.3.3, 5.1.4 (4)]) If \( \gamma \in (\beta_1, \beta_2) \), see (5.3), then
\[
\text{Ind} A^\varepsilon_{-\gamma} (\lambda^\varepsilon) = \text{Ind} A^\varepsilon_{\gamma} (\lambda^\varepsilon) + 4
\] (5.7)

We emphasize that the last 4 is nothing but the number of waves detached in (5.4). From (5.6)-(5.7), it follows that
\[
\text{Ind} A^\varepsilon_{+\gamma} (\lambda^\varepsilon) = -\text{Ind} A^\varepsilon_{-\gamma} (\lambda^\varepsilon) = 2
\] (5.8)
5.3 Absence of trapped modes with a fast decay rate.

In this section we prove that, for \( \lambda \in (0, \pi^2] \) and \( \gamma \in (\beta_1, \beta_2) \), there holds the formula

\[
\dim \ker A_\gamma^\varepsilon (\lambda) = 0 \tag{5.9}
\]

which, in particular, completes the proof of Theorem 11 cf. our assumption \( c_\varepsilon \neq 0 \) for trapped mode (2.19) while \( c_\varepsilon = 0 \) leads to \( U^\varepsilon = \tilde{U}^\varepsilon \in \ker A_\gamma^\varepsilon (\lambda) \). Clearly, \( \ker A_\gamma^0 (\lambda) = 0 \) for any \( \beta > 0 \) that is the limit problem (1.12), (1.13) in the straight semi-strip \( \Pi_0^+ \) cannot have a trapped mode. However, as was mentioned in Section 1.4 formula (5.9) does not follow by a standard perturbation argument and, moreover, \( \dim \ker A_\gamma^\varepsilon (\lambda) > 0 \) for some \( \beta \in (0, \beta_1) \) and \( \lambda \in (0, \pi^2) \).

**Theorem 7** Let \( \gamma \in (\beta_1, \beta_2) \) be fixed. There exists \( \varepsilon_0 > 0 \) such that, for \( \varepsilon \in (0, \varepsilon_0) \) and \( \lambda \in (0, \pi^2] \), formula (5.9) is valid.

**Proof.** Let us assume that, for some \( \lambda \in (0, \pi^2] \) and an infinitesimal positive sequence \( \{\varepsilon_k\}_{k \in \mathbb{N}} \), the homogeneous problem (1.12), (1.13) has a solution \( u^{\varepsilon_k} \in W^1_\gamma (\Pi_+^0) \). We denote by \( u_0^{\varepsilon_k} \) the restriction of \( u^{\varepsilon_k} \) onto the semi-strip \( \Pi_+^0 = \mathbb{R} \times (0, 1) \). Under the normalization condition

\[
\| u_0^{\varepsilon_k}; L^2 (\Pi_+^0 \setminus (2l)) \| = 1, \tag{5.10}
\]

we are going to perform the limit passage \( \varepsilon_k \to +0 \) in the integral identity

\[
(\nabla u_0^{\varepsilon_k}, \nabla v^\varepsilon)_{\Pi_+^0} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Pi_+^0}, \tag{5.11}
\]

where \( v^\varepsilon \) is obtained from a test function \( v \in C^\infty_c (\overline{\Pi_+^0}) \) by the even extension over the \( x_1 \)-axis. If we prove that

(i) \( u_0^{\varepsilon_k} \) converges to \( u_0 \in W^1_\gamma (\Pi_+^0) \) weakly in \( W^1_\gamma (\Pi_+^0) \) and, therefore, strongly in \( L^2 (\Pi_+^0 \setminus (2l)) \);

(ii) \( \| u^{\varepsilon_k}; L^2 (\Pi_+^0 \setminus (2l)) \| \to 0 ;

(iii) \( (\nabla u_0^{\varepsilon_k}, \nabla v)_{\Pi_+^0 \setminus (2l)} \to 0 \) with any smooth function \( v \) in the rectangle \([0, l] \times [-1, 1]\),

then the limit passage in (5.11) and (5.10) gives

\[
(\nabla u_0^0, \nabla v)_{\Pi_+^0} = \lambda (u_0^0, v)_{\Pi_+^0}, \quad \forall v \in C^\infty_c (\overline{\Pi_+^0}), \tag{5.12}
\]

\[
\| u_0^0; L^2 (\Pi_+^0 \setminus (2l)) \| = 1. \tag{5.13}
\]

By a density argument, the integral identity (5.12) is valid with any \( v \in W^1_\gamma (\Pi_+^0) \) and, therefore, \( u_0^0 = 0 \) because the limit problem in \( \Pi_+^0 \) cannot get a non-trivial trapped mode.

Let us confirm facts (i) – (iii). We write \( \varepsilon \) instead of \( \varepsilon_k \).

First, we apply a local estimate, see, e.g., [1], to the solution \( u^\varepsilon \) of the problem (1.12), (1.13) with \( \lambda u^\varepsilon \) as a given right-hand side:

\[
\| u^\varepsilon; H^2 (\varpi') \| \leq c \lambda \| u^\varepsilon; L^2 (\varpi'') \|. \tag{5.14}
\]

Here, \( \varpi' = (4l/3, 5l/3) \times (0, 1) \) and \( \varpi'' = (l, 2l) \times (0, 1) \) are rectangles such that \( \varpi' \subset \varpi'' \subset \Pi^\varepsilon (2l) \) and, therefore, the right-hand side of (5.14) is less than \( c \lambda \) according to (5.10).

Second, we split \( u^\varepsilon \) as follows:

\[
u_1^\varepsilon = u_1^\varepsilon + u_\infty^\varepsilon, \quad u_1^\varepsilon = (1 - \chi) u^\varepsilon, \quad u_\infty^\varepsilon = \chi u^\varepsilon \tag{5.15} \]
where \( \chi \in C^\infty (\mathbb{R}) \) is a cut-off function, \( \chi (x_1) = 1 \) for \( x_1 \geq 5l/3 \) and \( \chi (x_1) = 0 \) for \( x_1 \leq 4l/3 \). The components in (5.15) satisfy the integral identities

\[
(\nabla u_1^\varepsilon, \nabla v_1)_{\Pi_+^2 (2l)} = \lambda ((1 - \chi) u^\varepsilon, v_1)_{\Pi_+^2 (2l)} + (\nabla u^\varepsilon, v_1 \nabla \chi)_{\omega^\varepsilon} - (u^\varepsilon \nabla \chi, \nabla v_1)_{\omega^\varepsilon} \quad \forall v_1 \in H^1 \left( \Pi_+^\varepsilon (2l) \right),
\]

\[
(\nabla u_\infty^\varepsilon, \nabla v_\infty)_{\Pi_\infty^l (l)} = \lambda (u_\infty^\varepsilon, v_\infty)_{\Pi_\infty^l (l)} = F_\infty^\varepsilon (v_\infty) := (u^\varepsilon \nabla \chi, \nabla v_\infty)_{\omega^\varepsilon} - (\nabla u^\varepsilon, v_\infty \nabla \chi)_{\omega^\varepsilon} \quad \forall v_\infty \in W^{-1}_\gamma \left( \Pi_\infty^l \right).
\]

Third, inserting \( v_1 = u_1^\varepsilon \) into (5.16) and taking (5.10), (5.14) into account yield

\[
\| \nabla u_1^\varepsilon; L^2 \left( \Pi_+^\varepsilon (2l) \right) \| \leq c. \tag{5.18}
\]

The problem (5.17) needs a bit more advanced argument. It is posed in the semi-strip \( \Pi_\infty^l \) independent of \( \varepsilon \) and, thus, the following a priory estimate in the Kondratiev space, see [25] and, e.g., [43] Thm 5.1.4 (1),

\[
\| u_\infty^\varepsilon; W^{1, \gamma}_\Pi (\Pi_\infty (l)) \| \leq c_1 \left( \| F_\infty^\varepsilon; W^1_\gamma (\Pi_\infty (l)) \| + \| u_\infty^\varepsilon; L^2 (\Pi_+^\varepsilon (2l) \cap \Pi_\infty (l)) \| \right)
\]

\[
\leq c_2 \left( \| u^\varepsilon; L^2 (\omega^\varepsilon) \| + \| \nabla u^\varepsilon; L^2 (\omega^\varepsilon) \| + \| u^\varepsilon; \Pi_+^\varepsilon (2l) \| \right)
\]

involves some constants \( c_m \) independent of \( \varepsilon \). In this way, formulas (5.18) and (5.19), (5.14), (5.9) assure that

\[
\| u^\varepsilon; W^{-1, \gamma}_\Pi (\Pi_+^\varepsilon) \| \leq c. \tag{5.20}
\]

Thus, the convergence in (i) occurs along a subsequence which is still denoted by \( \{ \varepsilon_k \} \).

The last step of our consideration uses integration in \( t \in (-\varepsilon, 0) \) and \( x_1 \in (0, l) \) of the Newton-Leibnitz formula

\[
|u^\varepsilon (t, x_2)|^2 = \int_{t}^{t+1/2} \frac{\partial}{\partial x_2} \left( \chi_0 (x_2) |u^\varepsilon (x_1, x_2)|^2 \right) dx_2
\]

where \( \chi_0 \in C^\infty (\mathbb{R}) \) is a cut-off function, \( \chi_0 (x_2) = 1 \) for \( x_2 < 1/6 \) and \( \chi_0 (x_2) = 0 \) for \( x_2 > 1/3 \). As a result, we obtain the estimate

\[
\int_{\Pi_+^0 \cap \Pi_+^0} |u^\varepsilon (x)|^2 dx \leq c \varepsilon \int_{\Pi_+^0 (t)} \left( |\nabla u^\varepsilon (x)|^2 + |u^\varepsilon (x)|^2 \right) dx \leq C \varepsilon
\]

while referring to (5.20) again. This provides (ii) as well as (iii) because

\[
\int_{0}^{1} \int_{-\varepsilon}^{0} \nabla u^\varepsilon (x_1, x_2) \cdot \nabla v (x_1, x_2) dx_1 dx_2 \leq \max_{x \in \Pi_+^0 \cap \Pi_+^0} \| \nabla v (x) \| \left( \text{meas}_2 (\Pi_+^\varepsilon \cap \Pi_0^0) \right)^{1/2} \| \nabla u^\varepsilon; L^2 (\Pi_+^\varepsilon \cap \Pi_0^0) \| \leq c \varepsilon^{1/2} \| u^\varepsilon; W^{1, \gamma}_\Pi (\Pi_+^\varepsilon) \| \leq C \varepsilon^{1/2}
\]

Theorem [4] is proved. \( \heartsuit \)

### 5.4 Radiation conditions

Let \( \lambda^\varepsilon \in (0, \pi^2) \) and \( \gamma \in (\beta_1, \beta_2) \), cf. Theorem [5] The pre-image \( \mathfrak{M}^\varepsilon (\Pi_+^\varepsilon) \) of the subspace \( W^{-1, \gamma}_\Pi (\Pi_+^\varepsilon)^* \) in \( W^{1, \gamma}_\Pi (\Pi_+^\varepsilon)^* \) for the operator \( \mathcal{A}^\varepsilon_\gamma (\lambda^\varepsilon) \) consists of functions in the form (5.5). Introducing the norm \( \| u^\varepsilon; \mathfrak{M}^\varepsilon_\gamma (\Pi_+^\varepsilon) \| \) as the left-hand side of (5.5) makes \( \mathfrak{M}^\varepsilon_\gamma (\Pi_+^\varepsilon) \) a Hilbert space but this Hilbert structure is of no use in our paper.
Moreover, inserting integral identity cf. the right-hand side of (5.5). By restriction (5.24),\[\chi_{v} \in W_{\gamma}^{1} (\Pi_{+}^{\epsilon}) \] on the parameter (5.24) and the following consideration. By multiplying such problem is uniquely solvable, while its solution in the form \[\lambda_{\chi} = 0 \text{ and } \lambda_{\chi} = \lambda_{\gamma}^{\epsilon} \] (5.5) Remark on the dependence of bounds on the small parameter \(\epsilon\)

If \(\lambda_{\chi} \in [\delta, \pi^{2} - \delta]\) (5.24) with a fixed \(\delta > 0\), the coefficient in the estimate (5.5) can be chosen independent of \(\epsilon \in [0, \delta (\delta)]\) with some \(\epsilon (\delta) > 0\). This fact originates in the smooth dependence of the waves (2.1) and (2.11), (2.14) on the parameter (5.24) and the following consideration. By multiplying \(u^{\epsilon}\) with the same cut-off function \(\chi\) as in (5.15), we reduce the problem (5.24) onto the semi-strip \(\Pi_{\infty} (l)\), namely, inserting \(u^{\epsilon} = \chi v_{\infty}^{\epsilon}\) with any \(v_{\infty} \in W_{\gamma}^{1} (\Pi_{\infty} (l))\) as a test function, we obtain for \(u_{\infty}^{\epsilon} = \chi u^{\epsilon}\) the integral identity

\[
(\nabla u_{\infty}^{\epsilon}, \nabla v_{\infty})_{\Pi_{\infty} (l)} - \lambda_{\chi} (u_{\infty}^{\epsilon}, v_{\infty})_{\Pi_{\infty} (l)} = F_{\infty}^{\epsilon} (v_{\infty}) := \quad (5.25)
\]

Moreover, \[
\| F_{\infty}^{\epsilon}; W_{-1, \gamma}^{1} (\Pi_{+}^{\epsilon} (l)) \| \leq c \| u^{\epsilon}; W_{-1, \gamma}^{1} (\Pi_{\infty} (l)) \| + c_{\chi} \| u^{\epsilon}; H^{1} (\Pi_{\infty} (l) \cap \Pi_{\infty}^{\epsilon} (2l)) \| \leq \quad (5.25) \]
cf. the right-hand side of (5.5). By restriction (5.24), \(\lambda_{\epsilon}\) stays at a distance from the thresholds \(\lambda_{0}^{+} = 0\) and \(\lambda_{0}^{1} = \pi^{2}\) so that we may choose the same weight index \(\gamma\) for all legalized \(\lambda_{\epsilon}\).

Hence, a general result in [25], see also [13] § 3.2, on the basis of a perturbation argument provides a common factor \(e^{\gamma} = c_{\chi} \text{ const}\) in the estimate (5.5) for ingredients of the asymptotic representation (5.4) of the solution \(u_{\infty}^{\epsilon} = \chi u^{\epsilon}\) to the problem (5.25) in the \(\epsilon\)-independent domain \(\Pi_{\infty} (l)\).

Since the weight \(e^{\gamma} = \chi u^{\epsilon}\) is uniformly bounded in \(\Pi_{+}^{\epsilon} (2l) = \Pi_{+}^{\epsilon} \setminus \Pi_{\infty} (2l)\), the evident relation

\[
\| u^{\epsilon}; W_{\gamma}^{1} (\Pi_{+}^{\epsilon} (2l)) \| \leq \| u^{\epsilon}; W_{-1, \gamma}^{1} (\Pi_{\infty} (l) \cap \Pi_{\infty}^{\epsilon} (2l)) \| + \sum_{\pm} (| a_{\pm}^{\epsilon} | + | b_{\pm}^{\epsilon} |) \]

21
allows us to extend the above mentioned estimate over the whole waveguide \( \Pi_\varepsilon^\ast \).

Similarly, in the case (5.24) the factor \( C^\varepsilon \) in (5.28) can be fixed independent of \( \varepsilon \), too.

The desired eigenvalue (3.4) is located in the vicinity of the threshold \( \lambda_1^\varepsilon = \pi^2 \) and the above consideration becomes unacceptable. Moreover, the normalization factor \( (\pi^2 - \lambda^\varepsilon)^{-1/4} \) in (2.11) is big so that the independence property of \( e^\varepsilon \) and \( C^\varepsilon \) is surely lost. Thus, our immediate objective is to modify the estimates in order to make them homotype for all small \( \varepsilon > 0 \). We emphasize that a modification of the normalization factor does not suffice because the waves \( e^{\pm k_i x_1} \cos (\pi x_2) \) in (2.11) become equal at \( \varepsilon = 0 \).

We follow a scheme in [39 §3] and define for \( \lambda^\varepsilon \in ((0, \pi^2) \) the linear combinations of the exponential waves (2.11)

\[
\begin{align*}
   w_1^\pm (\lambda^\varepsilon; x) &= (1/2) \cos (\pi x_2) ((1/k_i^\varepsilon)(e^{k_i^\varepsilon x_1} - e^{-k_i^\varepsilon x_1}) \mp i(e^{k_i^\varepsilon x_1} + e^{-k_i^\varepsilon x_1})), \\
   w_0^\pm (\lambda^\varepsilon; x) &= w_1^0 (x) = (2k)^{-1/2} e^{\pm ik^\varepsilon x_1}, \\

\end{align*}
\]

cf. (2.14). A direct calculation demonstrates that the new waves (5.26) together with the old waves (2.1),

\[
   w_0^\pm (\lambda^\varepsilon; x) - w_1^0 (x) = O((\pi^2 - \lambda^\varepsilon) x_1), \quad w_1^\pm (\lambda^\varepsilon; x) - w_0^0 (x) = O((\pi^2 - \lambda^\varepsilon)^{1/2} x_1).
\]

In other words, the waves (5.26) and (5.27) smoothly become the waves (2.7) and (2.1) introduced in Section 2.1 at the threshold \( \lambda^\varepsilon = \pi^2 \). The first property of \( w_0^\pm (\lambda^\varepsilon; x) \) allows us to repeat considerations in Sections 5.4 2.2 and compose the space \( W_1^\gamma (\Pi_\varepsilon^\ast) \) of functions satisfying the new, so-called artificial, radiation condition

\[
   \frac{\partial}{\partial x_2} w_0^\pm (\lambda^\varepsilon; x) - w_1^\pm (\lambda^\varepsilon; x) = \tilde{w}_1^\pm (\lambda^\varepsilon; x), \quad \tilde{w}_1^\pm (\lambda^\varepsilon; x) \in W_1^\gamma (\Pi_\varepsilon^\ast)
\]

cf. (5.22), to determine the solutions \( Z_p^\varepsilon (\lambda^\varepsilon; \cdot) \in W_1^\gamma (\Pi_\varepsilon^\ast) \) of the homogeneous problem (5.2), \( \sigma = -\gamma \),

\[
   Z_p^\varepsilon (\lambda^\varepsilon; x) = \tilde{Z}_p^\varepsilon (\lambda^\varepsilon; x) + w_p^\varepsilon (\lambda^\varepsilon; x) + S_{0p}^\varepsilon (\lambda^\varepsilon) w_0^\varepsilon (\lambda^\varepsilon; x) + S_{1p}^\varepsilon (\lambda^\varepsilon) w_1^\varepsilon (\lambda^\varepsilon; x),
\]

cf. (2.15) and to detect a unitary and symmetric artificial scattering matrix \( S^\varepsilon (\lambda^\varepsilon) = (S_{qp}^\varepsilon (\lambda^\varepsilon))_{q, p=0,1} \).

At the same time, the second property of \( w_0^\pm (\lambda^\varepsilon; x) \) assures that, for a fixed \( \varepsilon \), the operator

\[
   B_\gamma^\varepsilon (\lambda^\varepsilon)_{out} : W_1^\gamma (\Pi_\varepsilon^\ast) \to W_1^\gamma (\Pi_\varepsilon^\ast)^\ast
\]

of the problem (5.24), \( \sigma = -\gamma \), with the radiation condition (5.28) depends continuously on the spectral parameter \( \lambda^\varepsilon \in (\pi^2 - \delta, \pi^2] \), \( \delta > 0 \), when the domain of \( B_\gamma^\varepsilon (\lambda^\varepsilon)_{out} \) is equipped with the norm

\[
   \| u^\varepsilon; W_1^\gamma (\Pi_\varepsilon^\ast) \| = \| \tilde{u}^\varepsilon; W_1^\gamma (\Pi_\varepsilon^\ast) \| + |a^\varepsilon| + |b^\varepsilon|
\]

of a weighted space with detached asymptotics, cf. the left-hand side of (5.24).

Recalling our reasoning in Section 5.3 and the beginning of this section, we conclude that the operator (5.30) is an isomorphism while its norm and the norm of the inverse are uniformly bounded in

\[
   \lambda \in [\pi^2 - \delta, \pi^2], \quad \varepsilon \in [0, \varepsilon_0].
\]
Furthermore, by the Fourier method, entries of the matrix \( S^e (\lambda) \) can be expressed as weighted integrals of solutions \((5.29)\), this matrix is continuous in both arguments \((5.32)\) and the limit matrix

\[
S^0 (\pi^2) = \text{diag} \{1, -1\} \quad (5.33)
\]

is nothing but augmented scattering matrix at the thresholds and its diagonal form is due to the explicit solutions \((3.14)\) and \((3.15)\) in the semi-strip \( \Pi^0 \),

\[
Z^0_0 (x) = (1/\sqrt{2\pi}) (e^{i\pi x} + e^{-i\pi x}) , \quad Z^0_0 (x_1) = \cos (\pi x_2) = (1/2i) ((x_1 + i) \cos (\pi x_2) - (x_1 - i) \cos (\pi x_2)).
\]

We resume that above consideration and find out a unique solution \( u^e \in W^1_\gamma (\Pi^e_+) \), \( W^1_\gamma (\Pi^e_+) \) of the problem \((5.2)\) with \( \sigma = -\gamma \), \( F^e \in W^1_\gamma (\Pi^e_+) \), and the artificial radiation condition \((5.28)\). Moreover, the estimate

\[
\| u^e; W^1_\gamma (\Pi^e_+) \| \leq c \| F^e; W^1_\gamma (\Pi^e_+) \| \quad (5.34)
\]

is valid, where \( c \) is independent of both parameters \((5.32)\).

We now search for a solution \( u^e \in W^1_\gamma (\Pi^e_+) \) of the same integral identity but the radiation condition from Section \( 5.4 \) in the form

\[
u^e = u^e + c_0^eZ_1^e + c_1^eZ_1^e. \quad (5.35)
\]

The unknown coefficients \( c_p^e \) should be fixed such that the decomposition \((5.22)\) is satisfied. To this end, we insert into the right-hand side of \((5.35)\) formulas \((5.28)\), \((5.29)\) and \((5.26)\), \((5.27)\), \((5.21)\), \((2.11)\) in \((5.35)\) with those in \((5.22)\) and arrive at the following systems of linear algebraic equations for the unknowns \( c_0^e \), \( c_1^e \) and \( a_+^e \), \( b_+^e \) :

\[
a_+^e = a_+^e + S^0_0 c_1^e, \quad 0 = c_0^e, \quad (5.36)
\]

\[
(2k_+^e)^{1/2} b_+^e = (1 - ik_1^e) b_+^e + ((1 + ik_1^e) + (1 - ik_1^e) S^0_0) c_1^e, \quad (5.37)
\]

\[
(2k_+^e)^{1/2} b_+^e = (1 + ik_1^e) b_+^e + ((1 - \varepsilon k_1^e) + (1 + ik_1^e) S^0_0) c_1^e.
\]

Solving the system \((5.37)\) with the help of the Cramer’s rule, a simple calculation gives the determinants

\[
(2k_+^e)^{3/2} (1 - S^0_0) = \left( 2\sqrt{\pi^2 - \lambda^e} \right)^{3/2} i (1 - S^0_0)
\]

and the estimates

\[
| b_+^e | \leq c \left( \pi^2 - \lambda^e \right)^{-1/2} | b_+^e |, \quad | c_1^e | \leq c | b_+^e | \quad (5.38)
\]

because \( 2 \geq |1 - S^0_0| \geq 1/2 \) due to \((5.33)\) and \((5.32)\). In view of the first relation in \((5.36)\) we obtain that

\[
| a_+^e | \leq c \left( | a_+^e | + | b_+^e | \right).
\]

Collecting formulas \((5.37)\), \((5.38)\) and \((5.34)\), \((5.31)\) adjusts the inequality \((5.23)\) as well as Theorem \( 8 \).

**Theorem 9** Let \( \lambda^e \in [\pi^2 - \delta, \pi^2] \), \( \varepsilon \in (0, \varepsilon_0] \) and \( \gamma \in (\beta_1, \beta_2) \). The solution \((5.22)\) of the problem \((5.2)\) with \( \sigma = -\gamma \) and \( F^e \in W^1_\gamma (\Pi^e_+) \) admits the estimate

\[
\| u^e; W^1_\gamma (\Pi^e_+) \| + \| a_+^e \| + (\pi^2 - \lambda^e)^{1/4} | b_+^e | \leq C \| F^e; W^1_\gamma (\Pi^e_+) \| \quad (5.39)
\]

where \( C \) does not depend on \( \lambda^e \), \( \varepsilon \) and \( F^e \).
6 Justification of asymptotics

6.1 The global asymptotic approximation

The reformulation \([4.12]\) of the criterion \([2.16]\) implicates the coefficients \(S^I_{11}\) and \(S^O_{10}\) in the decomposition \([2.15]\) of the special solution \(Z^*_1\) of the problem \([1.3], [1.4]\) and this section is devoted to the justification of the formal asymptotic expansions \([3.9]\). We emphasize that the similar expansions \([3.26]\) of other entries in the augmented scattering matrix \(S^e\) can be verified in the same way but actually we had used in Section 4.1 much simpler relation \([4.13]\) only.

In Section 3 we applied the method of matched asymptotic expansions and our immediate objective becomes to compose a global approximation solution from the inner and outer expansions \([3.9]\) and \([3.11]\). To this end, we employ several smooth cut-off functions:

\[
\begin{align*}
X_\varepsilon (x) &= 1 \text{ for } x_1 \leq l + 1/\varepsilon, \\
\chi_\infty (x) &= 1 \text{ for } x_1 \leq 2l, \\
\chi_\varepsilon (r) &= 1 \text{ for } r \leq 2\varepsilon,
\end{align*}
\]

where \(r = (|x_1 - l|^2 + x_2^2)^{1/2}\). We set

\[
3^e = \chi_\infty 3^{out} + X_\varepsilon 3^{in} - \chi_\varepsilon X_\varepsilon 3^{mat},
\]

\[
3^{out} (x) = w_0^\varepsilon^- (x) + S^0_{11} w_0^\varepsilon^+ (x) + \varepsilon^{1/2} S^0_{01} w_0^\varepsilon^0 (x),
\]

\[
3^{in} (x) = \varepsilon^{-1/2} Z^0_1 (x) + \varepsilon^{1/2} \left( (1 - \chi_\varepsilon (r)) \tilde{Z}^1_1 (x) + \chi_\varepsilon (r) Z^1_1 (l, 0) \right),
\]

\[
3^{mat} (x) = \varepsilon^{-1/2} (4\mu)^{-1/4} \cos (\pi x_2) \left( 1 + i + S^0_{11} (1 - i) + \varepsilon x_1 \sqrt{\mu} (1 - i + S^0_{11} (1 + i)) \right) + \varepsilon^{1/2} S^0_{01} (2\pi)^{-1/2} e^{i\pi x_1}.
\]

This construction needs explanations. First, the expressions \([6.3]\) and \([6.4]\) of the outer and inner types are multiplied with the cut-off functions \(\chi_\infty\) and \(X_\varepsilon\) whose support overlap so that the sum \([6.5]\) of terms matched in Section 3.2 attend the global approximation \([6.2]\) twice, i.e. in \(\chi_\infty 3^{out}\) and \(X_\varepsilon 3^{in}\), but we compensate for this duplication by subtracting \(\chi_\varepsilon X_\varepsilon 3^{mat}\). Moreover, the formula for commutators \([\Delta, \chi_\infty X_\varepsilon] = [\Delta, \chi_\infty] + [\Delta, X_\varepsilon]\) demonstrates that

\[
(\Delta + \lambda^e) 3^e = \chi_\infty (\Delta + \lambda^e) 3^{out} + X_\varepsilon (\Delta + \lambda^e) 3^{in} - \chi_\varepsilon X_\varepsilon (\Delta + \lambda^e) 3^{mat} + [\Delta, \chi_\infty] (3^{out} - 3^{mat}) + [\Delta, X_\varepsilon] (3^{in} - 3^{mat}) := \mathcal{F}^e = \chi_\infty \mathcal{F}^{out} + X_\varepsilon \mathcal{F}^{in} - \chi_\varepsilon X_\varepsilon \mathcal{F}^{mat} + \mathcal{F}^{oma} + \mathcal{F}^{ima}.
\]

Second, the function \(Z^0_1\) is properly defined by the formula \([3.16]\) in the whole waveguide but \(Z^1_1\) needs an extension from \(\Pi^0_+\) onto \(\Pi^0_-\) denoted by \(\tilde{Z}^1_1\) in \([6.4]\). Since the Neumann datum \([3.17]\), \(p = 1\), has a jump at the point \((0, l) \in \partial\Pi^0_+\), the solution \(Z^0_1\) gets a singular behavior near this point. A simple calculation based on the Kondratiev theory \([25]\) (see also \([43\ Ch. 2]\)) demonstrates that

\[
Z^1_1 (x) = \pi^{-1} G^*_1 r (\ln r \cos \varphi - \varphi \sin \varphi) + \tilde{Z}^1_1 (x)
\]

where \((r, \varphi) \in \mathbb{R}_+ \times (0, \pi)\) are polar coordinates in fig. 5. b and \(\tilde{Z}^1_1\) is a smooth function in the closed rectangle \(\Pi^0_+ (R)\) of any fixed length \(R\). We emphasize that the solution \(Z^1_1\) has no singularities at the corner points \((0, 0)\) and \((0, 1)\), cf. \([43\ Ch. 2]\), but third derivatives of \(\tilde{Z}^1_1\) are not bounded
when \( r \to +0 \). The extension \( \hat{Z}_1' \) in (6.7) where \( \hat{Z}_1' \) is smoothly continued through the segment \( \{ x : x_1 \in [0, l], \ x_2 = 0 \} \).

Finally, we mention that the correction term \( Z_1' \) in (3.9) was determined in Section 3.2 up to the addendum \( C_0^0 \cos (\pi x_1) \) but putting \( C_0^0 = 0 \) in the expansion (3.20) defines uniquely the function \( Z_1' \) as well as its value \( Z_1'(l, 0) \) according to (6.7). Notice that we also must take \( S_{11}' = 0 \) by virtue of (3.23). The extension \( \hat{Z}_1' \) of \( Z_1' \) is smooth everywhere in a neighborhood of \( \Pi_0^\varepsilon \) except at the point \((l, 0)\) where it inherits a singularity from (6.7). Using the partition of unity \( \{ 1 - \chi_\varepsilon, \chi_\varepsilon \} \) makes the last term in (6.3) smooth in \( \Pi_0^\varepsilon \), but produces additional discrepancy in the Helmholtz equation (1.3).

### 6.2 Estimating discrepancies

First of all, we observe that \( \mathcal{F}^\text{out} = 0 \) in \( \Pi_0^\varepsilon \) according to definition of waves in (2.11) and (2.14). In view of the factor \( \chi_\infty \) from (6.1) the first term on the right of (6.6) vanishes. Moreover, the Taylor formulas (3.5) and (3.8) assure that

\[
|Z^\text{out}(x) - Z^\text{mat}(x)| + |\nabla Z^\text{out}(x) - \nabla Z^\text{mat}(x)| \leq c\varepsilon^{3/2}
\]

on the rectangle \([3l/2, 2l] \times [0, 1]\) where supports of coefficients in the commutator \([\Delta, \chi_\infty]\) are located in. Hence

\[
|\mathcal{F}^\text{oma}(x)| \leq c\varepsilon^{3/2}, \quad \mathcal{F}^\text{oma}(x) = 0 \quad \text{for} \ x_1 \geq 2l.
\]

Let us consider the sum

\[
\mathcal{F}^\text{imm} = \mathcal{F}^\text{in} - X_\varepsilon\chi_\infty\mathcal{F}^\text{mat}.
\]

Outside the finite domain \( \Pi_\varepsilon^\varepsilon \) (3l/2) it is equal to

\[
\epsilon^{-1/2}X_\varepsilon(x)\left((\Delta + \pi^2)Z_0^0(x) + \epsilon(\Delta + \pi^2)Z_1'(x) - \chi_\infty(\Delta + \pi^2)Z^\text{mat}(x)\right) + \epsilon^{-1/2}X_\varepsilon(x)\left((\lambda - \pi^2)Z_0^0(x) + \epsilon Z_1'(x) - \chi_\infty(\Delta + \pi^2)Z^\text{mat}(x)\right) = 0 + \epsilon^{3/2}\mu(Z_0^0(x)) - \chi_\infty(x)\left(4\mu\right)^{-1/4}\cos(\pi x_2)(1+i+S_{11}^0(1-i)) - \epsilon(\Delta_x - \chi_\infty(x))\left(4\mu\right)^{-1/4}\cos(\pi x_2)x_1\sqrt{\mu}(1-i+S_{11}^0(1+i)) + S_{01}^0(2\pi)^{-1/2}e^{i\pi x_1}).
\]

where formulas (3.4) and (4.1) are taken into account. We now use the representations (3.16) and (3.20) to conclude that

\[
|\mathcal{F}^\text{imm}(x)| \leq c\varepsilon^{3/2}e^{-x_1\sqrt{3\varepsilon}} \quad \text{for} \ x_1 \geq 3l/2.
\]

Inside \( \Pi_\varepsilon^\varepsilon \) (3l/2) we have

\[
\mathcal{F}^\text{imm} = -\epsilon^2\mu Z^\text{in} + \epsilon^{1/2}(1-\chi_\varepsilon)(\Delta + \pi^2)\tilde{Z}_1' - \epsilon^{1/2}[\Delta, \chi_\varepsilon] (\tilde{Z}_1' - Z_1'(l, 0)).
\]

The inequality

\[
\epsilon^2\mu |Z^\text{in}(x)| \leq c\varepsilon^{3/2} \quad \text{in} \ \Pi_\varepsilon^\varepsilon \ (3l/2)
\]

is evident. Because of the singularity \( O(r/|ln r|) \) in (6.7), estimates of other two terms in (6.10) involve the weight function

\[
\rho(x) = r + (1 + |ln r|)
\]

cf. the Hardy inequality (6.17). Since \( \tilde{Z}_1' = Z_1' \) in \( \Pi_0^\varepsilon \) satisfies the Helmholtz equation from (3.17), we have

\[
\epsilon^{1/2}(1-\chi_\varepsilon(r))(\cdot + \pi^2)\tilde{Z}_1'(x) = 0, \quad x \in \Pi_0^\varepsilon \ (3l/2)
\]

\[
\epsilon^{1/2}
\left|
(1-\chi_\varepsilon(r))(\cdot + \pi^2)\tilde{Z}_1'(x)
\right| \leq c\varepsilon^{1/3}|x_1|(\epsilon + r)^{-2}\rho(x), \quad x \in \omega_\varepsilon = \Pi_\varepsilon^\varepsilon \setminus \Pi_0^\varepsilon
\]
we conclude that, according to the third line in (6.1), coefficients in the commutator $[\Delta, \chi_\varepsilon] = 2\nabla \chi_\varepsilon \cdot \nabla + \Delta \chi_\varepsilon$ gets orders $\varepsilon^{-1}$ and $\varepsilon^{-2}$, respectively but vanish outside the set $\mathbb{M} = \{ x \in \Pi_+^\varepsilon : 2\varepsilon < r < 3\varepsilon \}$, we conclude that

$$
\varepsilon^{1/2} [\Delta, \chi_\varepsilon] \left( \tilde{Z}_1^1 (x) - Z_1^1 (l, 0) \right) = 0, \quad x \in \Pi_+^\varepsilon (3l/2) \setminus \mathbb{M},
$$

$$
\varepsilon^{1/2} \left| [\Delta, \chi_\varepsilon (\tau)] \left( \tilde{Z}_1^1 (x) - Z_1^1 (l, 0) \right) \right| \leq c\varepsilon^{1/2} \left( \varepsilon^{-1} |\ln r| + \varepsilon^{-2} r |\ln r| \right) \leq c\varepsilon^{1/2} |x_1| (\varepsilon + r)^{-2} \rho (x), \quad x \in \mathbb{M}.
$$

Finally, we mention that the support of the term $F^{\text{ima}}$ in (6.3) belongs to the rectangle $[l + 1/\varepsilon, 2l + 1/\varepsilon] \times [0, 1]$, see the first line of (6.1), where the remainder $Z_1^1 (x)$ in (3.20) gets the exponential small order $O \left( e^{-\sqrt{3\pi/\varepsilon}} \right)$, and hence

$$
\left| [\Delta, X_\varepsilon] \left( 3^{\text{im}} (x) - 3^{\text{mat}} (x) \right) \right| = \varepsilon^{1/2} \left| [\Delta, X_\varepsilon] \tilde{Z}_1^1 (x) \right| \leq c\varepsilon^{1/2} e^{-3\sqrt{\pi}/\varepsilon}.
$$

It remains to consider discrepancies in the Neumann condition (1.4). Since the cut-off functions $X_\varepsilon$ and $\chi_\infty$ can be taken dependent on the longitudinal coordinate $x_1$ only, the asymptotic solution (6.2) satisfies the homogeneous Neumann condition everywhere on $\partial \Pi_0^\varepsilon$, except on the sides $\Sigma^\varepsilon$ and $v^\varepsilon$ of the rectangle (1.2), cf. Sections 3.2 and 3.3. Furthermore, $Z_0^1$ does not depend on $x_1$ and $Z_1^1$ is multiplied in (6.3) with the cut-off function $\chi_\varepsilon$ in the radial variable $r$. Thus, $\partial_1 3^\varepsilon = 0$ on the short side $a^\varepsilon$. Regarding the trace $G^\varepsilon$ of $\partial_1 3^\varepsilon = -\partial_2 3^\varepsilon$ on the long side $\Sigma^\varepsilon$ we take the formulas (3.17)–(3.19) into account and, similarly to (6.12) and (6.13), obtain

$$
|\mathcal{G}^\varepsilon (x_1, -\varepsilon)| \leq c\varepsilon^{3/2} (\varepsilon + r)^{-1}.
$$

We emphasize that differentiation in $x_2$ eliminates $\ln r$ in the first term of (6.7).

6.3 Comparing the approximate and true solutions.

First of all, we observe that $3^{\text{as}} (x) = 3^{\text{out}} (x)$ as $x_1 > 2l$ by virtue of the definition (6.1) of $X_\varepsilon$ and $\chi_\infty$. Thus, in view of (2.15) and (6.2), (6.3) the difference $\mathcal{R}^\varepsilon = Z^\varepsilon - 3^\varepsilon$ loses the incoming waves $w^\varepsilon_{\text{in}}$ and falls into the space $W_1^\varepsilon (\Pi_+^\varepsilon)_{\text{out}}$. Moreover, the decomposition (5.22) of $\mathcal{R}^\varepsilon (x)$ contains the coefficients $a_0^\varepsilon = \tilde{S}_{10}^\varepsilon$ and $b_1^\varepsilon = \tilde{S}_{11}^\varepsilon$ defined in (1.16) and (1.17). The integral identity (5.22) with $\sigma = -\gamma$ serving for $\mathcal{R}^\varepsilon$, involves the functional

$$
F^\varepsilon (v^\varepsilon) = (F^\varepsilon, v^\varepsilon)_{\Pi_+^\varepsilon} - (G^\varepsilon, v^\varepsilon)_{\Sigma^\varepsilon},
$$

where $F^\varepsilon$ is given in (6.6) and $G^\varepsilon = -\partial_2 3^\varepsilon$. If we prove the inclusion $F^\varepsilon \in W_1^{-\gamma} (\Pi_+^\varepsilon)^*$, then the estimate (5.39) adjusted by the weighting factor $(\pi^2 - \lambda^\varepsilon)^{1/4} = \varepsilon^{1/2} \mu^{1/4}$ demonstrates that

$$
|\tilde{S}_{10}^\varepsilon| + \varepsilon^{1/2} |\tilde{S}_{11}^\varepsilon| \leq c \left\| F^\varepsilon, W_1^{-\gamma} (\Pi_+^\varepsilon) \right\|.
$$

We fix some test function $v^\varepsilon \in W_1^\varepsilon (\Pi_+^\varepsilon)$. The classical one-dimensional Hardy inequality

$$
\int_0^l r^{-2} |\ln r|^2 |V (r)|^2 dr \leq 4 \int_0^l \left| \frac{dV}{dr} (r) \right|^2 dr, \quad V \in C_0^\infty [0, l],
$$

in a standard way, cf. [31] Ch.1, §4, leads to the relation

$$
\left\| \rho^{-1} v^\varepsilon; L^2 (\Pi_+^\varepsilon (2l)) \right\|^2 \leq c \left\| v^\varepsilon; H^1 (\Pi_+^\varepsilon (2l)) \right\|^2 \leq c_\gamma \left\| v^\varepsilon; W_1^\varepsilon (\Pi_+^\varepsilon) \right\|^2
$$

(6.17)
where $\rho$ is the weight factor (6.12). Moreover, introducing the new weight factor $\rho_1(x) = r (1 + |\ln r|)^2$, we derive the weighted trace inequality

$$
\int_{T^*} \rho_1^{-1} |v^e|^2 \, dx_1 = \int_{\Pi_+^1(l)} \frac{\partial}{\partial x_2} \left( \chi_0 \rho_1^{-1} |v^e|^2 \right) \, dx \leq c \int_{\Pi_+^1(l)} \left( \left| \frac{\partial v^e}{\partial x_2} \right| \rho_1^{-1} |v^e| + \left( 1 + \frac{\partial}{\partial x_2} \rho_1^{-1} \right) |v^e|^2 \right) \, dx \leq c \int_{\Pi_+^1(l)} \left( |\nabla v^e|^2 \rho^{-2} |v^e|^2 \right) \, dx \leq c_\gamma \|v^e; W^1_{\gamma} (\Pi_+^1) \|^2. \quad (6.18)
$$

Here, we took into account that $|\nabla \rho_1(x)^{-1}| \leq c \rho(x)^{-2}$ and used a cut-off function $\chi_0 \in C^1(\mathbb{R})$, $\chi_0(x_2) = 1$ for $x_2 \leq 1/3$ and $\chi_0(x_2) = 2$ for $x_2 \geq 2/3$. The inclusion $F^e \in W^{1,\gamma}_{1,\gamma} (\Pi_+^1)^*$ is obvious because $F^e$ has a compact support. To estimate the norm $\|F^e; W^{1,\gamma}_{1,\gamma} (\Pi_+^1)^* \|$ we apply inequalities obtained in the previous section. Since $\gamma \in (0, \sqrt{3}\pi)$, the estimates (6.14) gives

$$
\| (F^{\text{oma}}, v^e)_{\Pi_+^1} \| \leq c \varepsilon^{3/2} \|v^e; L^1 (\Pi_+^1 (3l/2)) \| \leq c \varepsilon^{3/2} \|v^e; W^{1,\gamma}_{1,\gamma} (\Pi_+^1) \|.
$$

By the formula (6.8), we have

$$
\| (F^{\text{oma}}, v^e)_{\Pi_+^1} \| \leq c \varepsilon^{3/2} \|v^e; L^1 (\Pi_+^1 (3l/2)) \| \leq c \varepsilon^{3/2} \|v^e; W^{1,\gamma}_{1,\gamma} (\Pi_+^1) \|.
$$

Recalling (6.9) and (6.11), (6.12), (6.13) yields

$$
\| (F^{\text{oma}}, v^e)_{\Pi_+^1} \| \leq \left( c \varepsilon^{3/2} \int_{\Pi_+^1 (3l/2)} e^{-x_1 \sqrt{3}\pi} \|v^e(x)| dx_1 + \varepsilon^{1/2} \int_{\Pi_+^1 (3l/2)} \frac{|x_1| \rho(x)}{(\varepsilon + r)^2} \|v^e(x)| dx_1 \right.

+ \left. \varepsilon^{1/2} \int_{\Pi_+^1 (3l/2)} \frac{\rho(x)}{(\varepsilon + r)^2} \|v^e(x)| dx_1 \right) \leq

\leq c(\varepsilon^{3/2} \left( \int_{3l/2}^{3l/2} e^{2(\gamma - \sqrt{3}\pi)} x_1 \, dx_1 \right)^{1/2} \left( \int_{\Pi_+^1 (3l/2)} e^{-2\gamma x_1} \|v^e(x)|^2 dx_1 \right)^{1/2} +

+ \varepsilon^{1/2} \left( \int_{\Pi_+^1 (3l/2)} |x_1|^2 \frac{\rho(x)^4}{(\varepsilon + r)^4} \, dx + \int_{\Pi_+^1 (3l/2)} \frac{\rho(x)^4}{(\varepsilon + r)^4} \, dx \right)^{1/2} \|v^{-1}v^e; L^1 (\Pi_+^1 (l)) \| \leq

\leq \varepsilon^{3/2} \left( 1 + |\ln r| \right)^2 \|v^e; W^{1,\gamma}_{1,\gamma} (\Pi_+^1) \|.
$$

Finally, we derive from (6.15) and (6.18) the following estimate of the last scalar product in (6.16):

$$
\langle G, v^e \rangle_{\Pi_+^1} \leq c \varepsilon^{3/2} \left( \int_{\Pi_+^1 (\varepsilon + r)^2 \rho_1 dx_1 \right)^{1/2} \left( \rho_1^{-1/2}v^e; L^2 (\Pi_+^1) \right) \leq c \varepsilon^{3/2} (1 + |\ln r|)^3/2 \|v^e; W^{1,\gamma}_{1,\gamma} (\Pi_+^1) \|.
$$

Collecting the obtained inequalities we conclude that the functional in (6.16) meets the estimate

$$
\| F^e; W^{1,\gamma}_{1,\gamma} (\Pi_+^1)^* \| \leq c \varepsilon^{3/2} (1 + |\ln r|)^2. \quad (6.19)
$$
6.4 Asymptotics of the augmented scattering matrix

We are in position to formulate the main technical result in the paper. Since the norm $|| R^\varepsilon; W^1_\gamma (\Pi^\varepsilon_{\text{out}}) ||$ in the space with detached asymptotics contains the coefficients $S^\varepsilon_{10}$ and $S^\varepsilon_{11}$ in the representation \[ (5.22) \] of $R^\varepsilon$ estimates of the asymptotic remainders in \[ (1.17) \] and \[ (1.16) \] follow directly from \[ (6.19) \] and \[ (5.39), (3.4) \]. A similar outcome for $S^\varepsilon_{00}$ can be obtained by repeating word by word calculations in the previous section based on the formal asymptotic \[ (5.23) \], \[ (3.22) \]. We only mention that the discrepancy \[ (3.34) \] on the small side $v^\varepsilon$ of the box $\varepsilon^\alpha_{\varepsilon}$ can be considered as follows:

\[
(2\pi)^{1/2} \left| \sin(\pi l) \int_{-\varepsilon}^{0} v^\varepsilon(l, x_2) \, dx_2 \right| \leq c\varepsilon^{1/2} (1 + |\ln \varepsilon|) \left\| r^{-1/2} (1 + |\ln r|)^{-1} v^\varepsilon; L^2 (v^\varepsilon) \right\| \leq c\varepsilon (1 + |\ln \varepsilon|) \left\| v^\varepsilon; W^1_\gamma (\Pi^\varepsilon_{\varepsilon}) \right\|
\]

where a weighted trace inequality of type \[ (6.18) \] is applied.

**Theorem 10** Remainders in the asymptotic forms \[ (1.16), (1.17) \] enjoy the estimate

\[
\left| \hat{S}^\varepsilon_{11} \right| + \varepsilon^{-1/2} \left| \hat{S}^\varepsilon_{01} \right| + |S^\varepsilon_{00} - 1| \leq c\varepsilon (1 - |\ln \varepsilon|)^2.
\]

We finally note that formulas \[ (1.13), (1.14) \] is the only issue in the paper which deals with $S^\varepsilon_{00}$ but requires much less accurate information.

6.5 Dependence on $\triangle l$ and $\triangle \mu$

Let $\varepsilon$ be fixed small and positive. We take $l = \pi k + \triangle l$ and make the change of coordinates

\[
x \rightarrow x = (x_1, x_2) = (x_1, x_2), \quad x_1 = (1 - \chi_k (x_1)) x_1 + \chi_k (x_1) (x_1 - \triangle l)
\]

where $\chi_k$ is a smooth cut-off function, $\chi_k (x_1) = 1$ for $|x_1 - \pi k| < \pi/3$ and $\chi_k (x_1) = 0$ for $|x_1 - \pi k| > 2\pi/3$. If $\triangle l$ is small, this change is nonsingular. Moreover, it transforms $\Pi^\varepsilon_{\varepsilon}$ into $\Pi^\varepsilon_{\varepsilon \varepsilon}$ and turns the Helmholtz operator $\triangle + \varepsilon^2 (\mu + \triangle \mu)$ into the second-order differential operator $L^\varepsilon (\triangle l, \triangle \mu; x, \nabla_x)$ whose coefficients depend smoothly on $\triangle \mu$ and $\triangle l$. Clearly, $L^\varepsilon (0, 0; x, \nabla_x) = \triangle_x + \varepsilon^2 - \varepsilon^2 \mu$. Owing the Fourier method, we can rewrite the element $S^\varepsilon_{11} = S^\varepsilon_{11} (\triangle \mu, \triangle l)$ of the augmented scattering matrix as the integral

\[
S^\varepsilon_{11} (\triangle \mu, \triangle l) = a^\varepsilon_{11} (\triangle \mu) \int_{Q_k} Z^\varepsilon_{11} (\triangle \mu, \triangle l; x) \, dx
\]

over the rectangle $Q_k = (\pi (k + 1), \pi (k + 2)) \times (0, 1)$ where $x = x$ according to \[ (6.21) \]. Due to the general result in the perturbation theory of linear operators, see, e.g., \[ [19, 24] \], the special solution $Z^\varepsilon_{11} (x) = Z^\varepsilon_{11} (\triangle \mu, \triangle l; x)$ rewritten in the coordinates $x$ depend smoothly on $\triangle \mu, \triangle l \in \mathbb{E}_\rho$. The coefficient $a^\varepsilon_{11} (\triangle \mu)$ in \[ (6.22) \] is also a smooth function whose exact form is of no need. Thus, the element \[ (6.22) \] inherits this smooth dependence while the remainder $\tilde{S}^\varepsilon_{11} (\triangle \mu, \triangle l)$ in the representation \[ (1.16) \] gets the same property according to the formula \[ (3.25) \] for $S^\varepsilon_{11} (\triangle \mu, \triangle l)$ written in the variables \[ (1.1) \].

Similar operations apply to $S^\varepsilon_{01} (\triangle \mu, \triangle l)$ and $\tilde{S}^\varepsilon_{01} (\triangle \mu, \triangle l)$.

Finally, recalling our examination in Section 5.5 and Theorem 10, we formulate the result.

**Proposition 11** The remainders in the asymptotic formulas \[ (1.16), (3.25) \] and \[ (1.17), (3.4) \] satisfy the inequality

\[
\left| \nabla (\triangle \mu, \triangle l) \tilde{S}^\varepsilon_{11} (\triangle \mu, \triangle l) \right| + \varepsilon^{-1/2} \left| \nabla (\triangle \mu, \triangle l) \tilde{S}^\varepsilon_{01} (\triangle \mu, \triangle l) \right| \leq c\varepsilon (1 + |\ln \varepsilon|^2), \quad (\triangle \mu, \triangle l) \in \mathbb{E}_\rho.
\]
7 The uniqueness assertions

7.1 Eigenvalues in the vicinity of the threshold \( \pi^2 \).

Let us adapt a trick from \([39] \S 7\) for the box-shaped perturbation \([12]\) and conclude with the uniqueness mentioned in Theorem \([3]\)

Assume that there exists an infinitesimal sequence \( \{ \varepsilon_k \}_{k \in \mathbb{N}} \) such that the problem \([1.12], [1.13]\) in the semi-infinite waveguide \( \Pi_k^\infty \) has two eigenvalues \( \lambda_{k1}^\varepsilon \) and \( \lambda_{k2}^\varepsilon \) while

\[
\varepsilon_k \to +0, \quad l_k \to l_0 > 0, \quad \lambda_{k1}^\varepsilon = \pi^2 + \hat{\lambda}_{k1}^\varepsilon \in (0, \pi^2], \quad \hat{\lambda}_{k1}^\varepsilon \to 0, \quad j = 1, 2. \tag{7.1}
\]

In what follows we write \( \varepsilon \) instead of \( \varepsilon_k \). The corresponding eigenfunctions \( u_{1}^\varepsilon \) and \( u_{2}^\varepsilon \) are subject to the normalization and orthogonality conditions

\[
\| u_{j}^\varepsilon ; L^{1}(\Pi_{+}^\varepsilon (2l)) \| = 1, \quad (u_{1}^\varepsilon, u_{2}^\varepsilon)_{\Pi_{+}^\varepsilon} = 0, \tag{7.2}
\]

cf. \((5.10)\). Repeating with evident changes our arguments in Section \(5.3\) we observe that the restrictions \( u_{j0}^\varepsilon \) of \( u_{j}^\varepsilon \) onto \( \Pi_{0}^\varepsilon \) converge to \( u_{j0}^0 \) weakly in \( W_{\gamma}^{1}(\Pi_{0}^\varepsilon) \) and strongly in \( L^{2}(\Pi_{0}^\varepsilon (2l)) \). Furthermore, the limits satisfy the formula \((5.13)\) and the following integral identity, see \((5.14)\):

\[
(\nabla u_{j0}^0, v)_{\Pi_{0}^\varepsilon} = \pi^2 (u_{j0}^0, v)_{\Pi_{0}^\varepsilon}, \quad v \in C_{c}^{\infty}(\Pi_{0}^\varepsilon). \tag{7.3}
\]

Any solution in \( W_{\gamma}^{1}(\Pi_{0}^\varepsilon) \) with \( \gamma \in (\beta_{1}, \beta_{2}) \) of the homogeneous Neumann problem \((7.3)\) in the semi-strip \( \Pi_{0}^\varepsilon = (0, +\infty) \times (0, 1) \) is a linear combination of two bounded solutions \((3.14)\) and \((3.15)\):

\[
u_{j0}^\varepsilon (x) = c_{j1} \cos (\pi x_{1}) + c_{j2} \cos (\pi x_{2}). \tag{7.4}
\]

Let us prove that \( c_{11} = c_{21} = 0 \) in \((7.4)\). Since the trapped mode \( u_{j}^\varepsilon \) has an exponential decay at infinity, the Green formula in \( \Pi_{\infty}(3l/2) \) with it and the bounded function \( e^{\pm ix_{1}\sqrt{\lambda_{j}^\varepsilon}} \) assures that

\[
\int_{0}^{1} e^{\pm ix_{1}\sqrt{\lambda_{j}^\varepsilon}} \left( \partial_{x_{1}} u_{j}^\varepsilon (x) \mp i \sqrt{\lambda_{j}^\varepsilon} u_{j}^\varepsilon (x) \right) \bigg|_{x_{1}=3l/2} dx_{2} = 0. \tag{7.5}
\]

The local estimate \((5.14)\) in \( \mathbb{R}' \equiv (3l/2, x_{2}), \ x_{2} \in (0, 1) \), and formulas in \((7.1), (7.2)\) allow us to compute the limit of the left-hand side of \((7.5)\) and obtain that

\[
e^{\pm 3l\pi/2} \int_{0}^{1} \left( \frac{\partial u_{j0}^0}{\partial x_{1}} \left( \frac{3}{2} l, x_{2} \right) \pm i \pi u_{j0}^0 \left( \frac{3}{2} l, x_{2} \right) \right) dx_{2} = 0. \tag{7.6}
\]

Inserting \((7.4)\) into \((7.6)\), we see that \( c_{j1} = 0 \), indeed.

**Remark 12.** If \( \tilde{\lambda}_{j}^\varepsilon > \pi^2 \) in \((7.7)\), one may use the Green formula in \( \Pi_{\infty}(3l/2) \) with four bounded functions \( e^{\pm ix_{1}\sqrt{\lambda_{j}^\varepsilon}} \) and \( e^{\pm ix_{1}\sqrt{\lambda_{j}^\varepsilon - \pi}} \cos (\pi x_{2}) \). In this way one derives the equalities \( c_{1j} = c_{2j} = 0 \) (see \([39], \S 7\) for details) and concludes that a small neighborhood of the threshold \( \pi^2 \) can contain only eigenvalues indicated in \((7.7)\). The same reasoning show that the problem \((1.12), (1.13)\) cannot get an eigenvalue \( \lambda^\varepsilon \to +0 \) as \( \varepsilon \to +0 \).

Since \( c_{j1} = 0 \), the limit normalization \((5.13)\) shows that \( u_{j0}^0 (x) = l^{-1/2} \cos (\pi x_{2}), \ j = 1, 2. \) Moreover, Theorem \(5\) \((2)\) applied to the trapped mode \( u_{j}^\varepsilon \in H^{1}(\Pi_{+}^\varepsilon) \subset W_{\gamma}^{1}(\Pi_{+}^\varepsilon) \) gives the formula

\[
u_{j}^\varepsilon (x) = B_{j}^\varepsilon e^{-x_{1}\sqrt{\pi^2 - \lambda_{j}^\varepsilon}} \cos (\pi x_{2}) + b_{j}^\varepsilon (x), \quad \| b_{j}^\varepsilon \| + \| \tilde{u}_{j}^\varepsilon ; W_{\gamma}^{1}(\Pi_{+}^\varepsilon) \| \leq c \| u_{j}^\varepsilon ; W_{\gamma}^{1}(\Pi_{+}^\varepsilon) \|, \tag{7.7}
\]
where $\gamma \in (\beta_1, \beta_2)$, $c$ is independent of $\varepsilon$ according to the content of Section 5.3 and the waves $u_0^{\varepsilon \pm}$ in (2.11) and $v_1^{\varepsilon \pm}$ in (2.11) do not appear in the expansion of $u_j^\varepsilon$ due to the absence of decay at infinity. Since the right-hand side of (7.7) is uniformly bounded in $\varepsilon = \varepsilon_k$, $k \in \mathbb{N}$ (see Section 5.5 again), we have

$$bB_j^\varepsilon \to l^{-1/2}, \quad \tilde{w}_j^\varepsilon \to 0$$ weakly in $W^1_\gamma (\Pi^\varepsilon_+)$

along a subsequence of $\{\varepsilon_k\}_{k \in \mathbb{N}}$. Moreover, the last equality in (7.2) turns into

$$0 = (u_1^\varepsilon, u_2^\varepsilon)_{\Pi^\varepsilon_+} = \int_{\Pi^\varepsilon_+} \cos^2 (\pi x_2) e^{-x_1 \Lambda^\varepsilon} dx + (u_1^\varepsilon - \tilde{u}_1^\varepsilon, u_2^\varepsilon)_{\Pi^\varepsilon_+} + (\tilde{u}_1^\varepsilon, u_2^\varepsilon)_{\Pi^\varepsilon_+} = \frac{1}{2} \Lambda^\varepsilon b_1^\varepsilon b_2^\varepsilon + O (1).$$

We multiply this relation with $\Lambda^\varepsilon = \sqrt{\pi^2 - \lambda_1^\varepsilon} + \sqrt{\pi^2 - \lambda_2^\varepsilon} \to 0$, as $\varepsilon \to 0$ and derive from (5.30) the absurd formula $o(1) = b_1^\varepsilon b_2^\varepsilon \to l^{-2}$. Thus, there can exist at most one eigenvalue indicated in (7.1).

### 7.2 Absence of eigenfunctions which are odd in $x_1$.

In Section 1.3 we have changed the original problem (1.3), (1.4) in $\Pi^\varepsilon$ for the Neumann problem (1.12), (1.13) in the half $\Pi^\varepsilon_+$ of the waveguide while assuming that an eigenfunction is even in $x_1$. Replacing (1.13) by the mixed boundary conditions

$$\partial_{\nu} u^\varepsilon (x) = 0, \quad x \in \partial \Pi^\varepsilon_+, \quad x_1 > 0, \quad u_1^\varepsilon (x) = 0, \quad x \in \partial \Pi^\varepsilon_+, \quad x_1 = 0, \quad (7.8)$$

we deal with the alternative, namely an eigenfunction is odd in $x_1$ and therefore vanishes at $\Gamma^\varepsilon = \{ x : x_1 = 0, \ x_2 \in (-\varepsilon, 1) \}$. The variational formulation of the problem (1.12), (7.8),

$$(\nabla u^\varepsilon_+, \nabla v^\varepsilon)_{\Pi^\varepsilon_+} = \lambda^\varepsilon \beta^\varepsilon (u^\varepsilon_+, v^\varepsilon)_{\Pi^\varepsilon_+}, \quad v^\varepsilon \in H^1_0 (\Pi^\varepsilon_+; \Gamma^\varepsilon),$$

involves a subspace of functions in $H^1_0 (\lambda)$ which are null on $\Gamma^\varepsilon$. Evident modifications of considerations in Sections 5 and 6 adapt all our results to the mixed boundary value problem (1.12), (7.8). The only but important difference is that the solutions (3.14), (3.15) of the limit Neumann problem in the half-stripe $\Pi^\varepsilon_+ = \mathbb{R}_+ \times (0, 1)$ now turn into the following ones:

$$u_0^0 (x) = i \sin (\pi x_1), \quad u_0^1 (x) = x_1 \cos (\pi x_2).$$

Thus, supposing that, for an infinitesimal sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$, the problem (1.12), (7.8) in $\Pi_+^\varepsilon$ has an eigenvalue $\lambda_1^{\varepsilon_k}$ with the properties (7.4) at $j = 1$, we obtain that a non-trivial limit $u_{01}^0$, cf. (7.4), of the corresponding eigenfunction $u_1^{\varepsilon_k}$ becomes

$$u_{10}^0 (x) = c_{11} \sin (\pi x_1) + c_{12} x_1 \sin (\pi x_2).$$

Now, in contrast to Section 7.1, we may insert $u_1^{\varepsilon_k}$ into the Green formula in $\Pi_\infty (3l/2)$ together with one of three bounded functions $e^{\pm i \sqrt{\lambda_1^{\varepsilon_k}} x_1}$ and $e^{-\sqrt{\pi^2 - \lambda_1^{\varepsilon_k}}} \cos (\pi x_2)$. Similarly to (7.5) and (7.6), these possibilities allow us to conclude that $c_{11} = 0$, $c_{12} = 0$ and, hence, $u_{10}^0 = 0$. The observed contradiction and Remark 12 which remains true for the problem (1.12), (7.8), confirm the absence of eigenvalues in a small neighborhood of the threshold $\lambda^\varepsilon = \pi^2$. 
7.3 Absence of eigenvalues at a distance from the threshold.

For any \( \lambda \in (0, \pi^2) \), the limit Neumann problem in \( \Pi_+^0 = \mathbb{R}_+ \times (0,1) \) has the solutions

\[
Z_0^{00} (\lambda, x) = (2k(\lambda))^{-1/2} \left( e^{-ik(\lambda)x_1} + e^{ik(\lambda)x_1} \right),
\]

\[
Z_1^{00} (\lambda, x) = (2k(\lambda))^{-1/2} \left( e^{k_1(x_1)} + i e^{-k_1(x_1)} \right) \cos (\pi x_2) + i \left( e^{k_1(x_1)} - i e^{-k_1(x_1)} \right) \cos (\pi x_2) =
\]

\[
= (2k(\lambda))^{-1/2} (1 + i) (e^{k_1(x_1)} + e^{-k_1(x_1)}) \cos (\pi x_2)
\]

where \( k(\lambda) = \sqrt{\lambda} \), \( k_1(\lambda) = \sqrt{\pi^2 - \lambda} \), and, thus, the augmented scattering matrix takes the form

\[
S^{00} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.
\] (7.10)

To fulfill the criterion (2.17), the perturbation \( \varpi_+^\varepsilon \) in the waveguide \( \Pi_+^\varepsilon \) has to turn the right-hand bottom element of the matrix (7.10) into \(-1\) that cannot be made, e.g., for any \( \lambda \in \left[0, \pi^2 - c\sqrt{\varepsilon}\right], c > 0 \) and a small \( \varepsilon \). The latter fact may be easily verified by either constructing asymptotics in Sections 3.2, 4.2 to construct asymptotics (1.16) with the main term \( S^{00}_{11} = -1 \) because the spectral parameter (3.4) stays too close to the threshold and the augmented scattering matrix in \( \Pi_+^0 \) is not continuous at \( \lambda = \pi^2 \), cf. Section 5.3. In the case of the mixed boundary value problem \( (1.12), (7.8) \) evident changes in solutions (7.9) give the matrix \( S^{00} = \text{diag} \{ -1, -i \} \) instead of (7.10) but our final conclusion remains the same.

7.4 Interferences on the uniqueness

The material of the previous three sections proves the last assertion in Theorem 2. The interval \((0, \pi^2)\) where the eigenvalue (1.9) is unique in the waveguide \( \Pi_+^\varepsilon \) with \( l = l_k(\varepsilon) \) and a fixed \( \varepsilon \in (0, \varepsilon_k) \) can be enlarged up to \((0, \pi^2 + c\sqrt{\varepsilon})\), \( c > 0 \), due to Remark 12. Moreover, enhancing our consideration in Section 7.3 by dealing with the exponential waves \( e^{\pm x_1 \sqrt{4\pi^2 - \lambda}} \cos (2\pi x_2) \) and the augmented scattering matrix of size \( 3 \times 3 \), cf. [39], confirms that any \( \lambda \in \left[\pi^2 + c\sqrt{\varepsilon}, 4\pi^2 - c\sqrt{\varepsilon}\right] \) cannot be an eigenvalue as well. We will discuss the higher thresholds \( \pi^2 k^2 \) with \( k = 2, 3, \ldots \) in Section 8.2.

To confirm Theorem 4 we use a similar reasoning. In this way, we recall the asymptotic formulas (1.17), (3.42) and (6.20) and observe that \( S^{01}_{01} \) cannot vanish for a small \( \varepsilon \) when the length parameter (1.10) stays outside the segment

\[
\left[ \pi k - c\varepsilon (1 + |\ln \varepsilon|)^2, \pi k + c\varepsilon (1 + |\ln \varepsilon|)^2 \right]
\] (7.11)

If \( \lambda^\varepsilon \) belongs to (7.11), the uniqueness of the solution \((\Delta \mu, \Delta l)\) of the abstract equation (4.8) which is equivalent to the criterion in Theorem 1 follows from the contraction principle.

8 Available generalizations

8.1 Eigenvalues in the discrete spectrum

Let us consider the mixed boundary value problem (1.3), (1.9). As in Section 1.3 we reduce it to the half (1.14) of the perturbed waveguide \( \Pi_+^\varepsilon = \Pi \cup \varpi_+^\varepsilon \), cf. (1.12), (1.13):

\[
-\Delta u_+^\varepsilon (x) = \lambda_+^\varepsilon u_+^\varepsilon (x), \quad x \in \Pi_+^\varepsilon, \ u_+^\varepsilon (x_1, 1) = 0, \ x_1 > 0,
\]

\[
\partial_\nu u_+^\varepsilon (x) = 0, \quad x \in \partial \Pi_+^\varepsilon, \ x_2 < 1.
\]
If $\lambda^\varepsilon \in (0, \pi^2/4)$ stays below the continuous spectrum $\varphi_{co}^M = [\pi^2/4, +\infty)$ of the problem (8.1), there is no oscillating wave but deal with the exponential waves

$$v_{1/2}^\varepsilon(x) = \left(k_{1/2}^\varepsilon\right)^{-1/2} e^{\pm k_{1/2}^\varepsilon x^1} \cos\left(\frac{\pi}{2} x^2\right), \quad k_{1/2}^\varepsilon = \sqrt{\frac{\pi^2}{4} - \lambda^\varepsilon}$$

and, similarly to (2.11), (2.14), compose the linear combinations

$$w_{1/2}^\varepsilon(x) = 2^{-1/2} \left(v_{1/2}^\varepsilon(x) \mp v_{1/2}^\varepsilon(x)\right).$$

The conditions (2.12), (2.13) and (2.6) with $p, q = 1/2$ are satisfied and we may determine the augmented scattering matrix $S^\varepsilon$ which now is a scalar. Theorem 1 remains valid and, therefore, the equality

$$S^\varepsilon = -1$$

(8.2)

states a criterion for the existence of a trapped mode. Constructing asymptotics of $S^\varepsilon$ and solving the equation (8.2) yield the relation (1.10) for an eigenvalue in the discrete spectrum of the problems (1.12) and (1.9). Repeating arguments from Sections 5 - 7 proves estimates of the asymptotic remainders as well as the uniqueness of the eigenvalue $\lambda^\varepsilon_+ \in \varphi_{d}^M$, however, for any $l > 0$. The latter conclusion requires to explain a distinction between analysis of isolated and embedded eigenvalues.

The main difference is caused by the application of the criterion (2.17) which in the case of the scalar $S^\varepsilon$ changes just into one equation

$$\text{Re } S^\varepsilon = -1$$

(8.3)

which is equivalent to (8.2) because $|S^\varepsilon| = 1$. As a result we may satisfy (8.3) by choosing $\Delta \mu$ and do not need the additional parameter $\Delta l$ in (1.11) which was used in Section 4 to solve the system (1.2) of two transcendental equations. In other words, the absence of oscillating waves crucially restricts a possible position of $S^\varepsilon_{11} = S^\varepsilon$ to the unit circle on the complex plane while the entry $S^\varepsilon_{11}$ in the previous unitary matrix $S^\varepsilon$ of size $2 \times 2$, see Section 2.2, can step aside from $S$ and a fine tuning by means of $\Delta l$ is necessary to assure the equality (2.17).

### 8.2 Higher thresholds

A straightforward modification of our approach may be used for an attempt to construct embedded eigenvalues near the thresholds $\pi^2 k^2$, $k = 2, 3, \ldots$ of the continuous spectrum $\varphi_{co}$ of the problem (1.12), (1.13) in $\Pi^\varepsilon_\perp$. At the same time, the number of oscillating outgoing waves at the threshold $\pi^2 k^2$ equals $k$ and, therefore, size of the augmented scattering matrix becomes $(k + 1) \times (k + 1)$. In this case the fine tuning needs at least $k$ free parameters, cf. 39, 41, instead of only one $\Delta l$ in Section 4. Additional parameters can be easily introduced when the perturbed wall is a broken line like in fig. 6 a, with $l, L$ and $k$. The amplification of the augmented scattering matrix does not affect the criterion (2.17) in Theorem 1 in Sections 3 and 4.

In the mirror symmetry with respect to the line $\{x : x_1 = 0\}$ is denied, see fig. 6 b, then we have to analyze the problem (1.3), (1.4) in the intact waveguide $\Pi^\varepsilon$ where the augmented scattering matrix gets rise of size even in the case $\lambda^\varepsilon \leq \pi^2$. In this sense the box-shaped perturbation is optimal because it demonstrates all technicalities but reduces the computational details to the necessary minimum. A preliminary assessment predicts that embedded eigenvalues of the problem (1.2), (1.3) in $\Pi^\varepsilon_\perp$ do not appear near any threshold $\pi^2 k^2$ with $k > 1$ but we are not able to verify this fact rigorously.
8.3 The Dirichlet boundary condition.

All procedures described above can be applied to detect eigenvalues of the Helmholtz equation (1.3) in the quantum waveguide $\Pi^\varepsilon$, cf. [10], with the Dirichlet condition (1.7). However, the asymptotic structures must be modified a bit due to the following observation. The correction term $Z'$ in the inner asymptotic expansion

$$Z^\varepsilon (x) = \sin (\pi x_2) + \varepsilon Z' (x) + \ldots,$$

cf. (3.28), must be found out from the mixed boundary value problem in the semi-strip

$$-\Delta Z' (x) = \pi^2 Z' (x), \ x \in \Pi^0_+, \quad -\partial_1 Z' (0, x_2) = 0, \ x_2 \in (0, 1),$$

$$Z' (x_1, 1) = 0, \ x_1 > 0, \quad Z' (x_1, 1) = \pi^2, \ x_1 \in (0, l), \quad Z' (x_1, 0) = 0, \ x_1 > l.$$

According to the Kondratiev theory [25], see also [43, Ch. 2], a solution of this problem admits the representation

$$Z' (x) = (C^0 + x_1 C^1) \sin (\pi x_2) + \tilde{Z}' (x)$$

where $\tilde{Z}' (x)$ has the decay $O \left(e^{-\sqrt{3} \pi x_1}\right)$, is smooth everywhere in $\overline{\Pi^0_+}$ except at the point $P = (l, 0)$ and behaves as

$$Z' (x) = \pi \varphi + O (r), \quad r \to 0,$$

while $(r, \varphi) \in \mathbb{R}_+ \times (0, \pi)$ is the polar coordinates system centered at $P$. The singularity in (8.5) leads the function $\tilde{Z}'$ out from the Sobolev space $H^1 (\Pi^0_+)$. Nevertheless, the solution $Z'$ still lives in appropriate Kondratiev space with a weighted norm so that the coefficient $C^1$ in (8.4) can be computed by inserting $Z' (x)$ and $\sin (\pi x_2)$ into the Green formula in $\Pi^0_+ (R)$. To compensate for the singularity, one may construct a boundary layer as a solution of the Dirichlet problem in the unbounded domain (3.2) in fig. 5 a.

The above commentary exhibits all changes in the asymptotic analysis in Section 3.2. As for the justification scheme in Section 6, it should be noted that, due to the Dirichlet condition (1.7), the inequality (6.17) of Hardy’s type takes the form

$$\|r^{-1} v^\varepsilon ; L^2 (\Pi^\varepsilon_+ (2l))\|^2 \leq c \|v^\varepsilon ; H^1 (\Pi^\varepsilon_+ (2l))\|^2$$

and sheds the factor $1 + |\ln r|$ in the weight function (6.12). As a result, the factor $(1 + |\ln \varepsilon|)^2$ occurring in (1.10) and (6.20) for the Neumann case, disappears from the asymptotic remainder in (1.8) for the Dirichlet condition.
References

[1] Agmon S., Douglis A., Nirenberg L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math. 12 (1959) 623-727.

[2] Agmon S., Nirenberg L., Properties of solutions of ordinary differential equations in Banach spaces, Comm. Pure Appl. Math. 16 (1963) 121–239.

[3] Aslanyan A., Parnovski L., Vassiliev D., Complex resonances in acoustic waveguides, Q.J. Mech. Appl. Math. 53 (2000) 429–447.

[4] Avishai Y., Bessis D., Giraud B.G., Mantica G., Quantum bound states in open geometries, Phys. Rev. B 44 (1991) 8028-8034.

[5] Birman M.Sh., Solomjak M.Z., Spectral theory of selfadjoint operators in Hilbert space. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[6] Bonnet-Bendhia A.-S., Starling F., Guided waves by electromagnetic gratings and nonuniqueness examples for the diffraction problem, Math. Meth. Appl. Sci. 17 (1994) 305-338.

[7] Bonnet-Bendhia A.-S., Joly P., Mathematical analysis of guided water-waves, SIAM J. Appl. Math. 53 (1993) 1507-1550.

[8] Borisov D., Bunoiu R., Cardone G., On a waveguide with frequently alternating boundary conditions: homogenized Neumann condition, Ann. Henri Poincaré 11 (2010) 1591-1627.

[9] Borisov D., Bunoiu R., Cardone G., Waveguide with non-periodically alternating Dirichlet and Robin conditions: homogenization and asymptotics, Z. Angew. Math. Phys. 64 (2013) 439-472.

[10] W. Bulla, F. Gesztesy, W. Renger, B Simon, Weakly coupled bound states in quantum waveguides, Proc. Amer. Math. Soc. 125 (1997) 1487–1495.

[11] G. Cardone, S.A. Nazarov, K. Ruotsalainen, Asymptotics of an eigenvalue in the continuous spectrum of a converging waveguide, Sbornik Mathematichs 203 (2012) 3-32.

[12] G. Cardone, S.A. Nazarov, K. Ruotsalainen, Bound states of a converging quantum waveguide, ESAIM Math. Model. Numer. Anal. 47 (2013) 305-315.

[13] Duclos P., Exner P., Curvature-induced bound states in quantum waveguides in two and three dimensions, Rev. Math. Phys. 7 (1995) 73–102.

[14] Evans D.V., Levitin M., Vasil’ev D., Existence theorems for trapped modes, J. Fluid Mech. 261 (1994) 21–31.

[15] Exner P., Seba P., Tater M., Vanek D. Bound states and scattering in quantum waveguides coupled laterally through a boundary window, J. Math. Phys. 37 (1996) 4867–4887.

[16] Gadyl’shin R.R., On local perturbations of quantum waveguides, Theoret. Math. Phys. 145 (2005) 1678–1690.

[17] Goldstein C., Scattering theory in waveguides, Scattering Theory in Mathematical Physics, D. Reide. 1974, 35-51.
[18] Grushin V.V., On the eigenvalues of a finitely perturbed Laplace operator in infinite cylindrical domains, Math. Notes 75 (2004) 331–340.

[19] Hille E., Phillips R.S., Functional analysis and semi-groups, American Mathematical Society Colloquium publications 31 (1957).

[20] Il’in A.M., Matching of asymptotic expansions of solutions of boundary value problems, Translations of Mathematical Monographs, 102. American Mathematical Society, Providence, RI, 1992.

[21] Kamotskii I.V., Nazarov S.A., Wood’s anomalies and surface waves in the problem of scattering by a periodic boundary. 1, Sbornik Math. 190 (1999) 111-141.

[22] Kamotskii I.V., Nazarov S.A., Wood’s anomalies and surface waves in the problem of scattering by a periodic boundary. 2, Sbornik Math. 190 (1999) 205-231.

[23] Kamotskii I.V., Nazarov S.A., An augmented scattering matrix and exponentially decreasing solutions of an elliptic problem in a cylindrical domain, J. Math. Sci. 111 (2002) 3657-3666.

[24] Kato T., Perturbation Theory for linear operator edition, Grundlehren der Mathematischen Wissenschaften, Band 132, Springer-New York, 1976.

[25] Kondratiev V.A., Boundary problems for elliptic equations in domains with conical or angular points, Trans. Moscow Math. Soc. 16 (1967), 227-313.

[26] Kozlov V.A., Maz’ya V.G., Rossmann J., Elliptic boundary value problems in domains with point singularities. Providence: Amer. Math. Soc., 1997.

[27] Kuznetsov N., Maz’ya V., Vainberg B., Linear Water Waves. Cambridge: Cambridge University Press. 2002.

[28] O. A. Ladyzhenskaya, The boundary value problems of mathematical physics, Nauka, Moscow 1973; Appl. Math. Sci., vol. 49, Springer-Verlag, New York 1985.

[29] Linton C.M., McIver P., Embedded trapped modes in water waves and acustics, Wave motion 45 (2007) 16–29.

[30] V.P. Maslov, An asymptotic expression for the eigenfunctions of the equation $\Delta u + k^2 u = 0$ with boundary conditions on equidistant curves and the propagation of electromagnetic waves in a waveguide, Soviet Physics Dokl. 3 (1959) 1132–1135.

[31] Maz’ya V.G., Nazarov S.A., Plamenevskij B.A., Asymptotic theory of elliptic boundary value problems in singularly perturbed domains, Operator Theory: Advances and Applications, 112, Birkhäuser Verlag, Basel (2000).

[32] Maz’ya V.G., Plamenevskii B.A., On coefficients in asymptotics of solutions of elliptic boundary value problems in a domain with conical points, Amer. Math. Soc. Transl. 123 (1984) 57-89.

[33] Maz’ya V.G., Plamenevskii B.A., Estimates in $L^p$ and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary, Amer. Math. Soc. Transl. 123 (1984) 1-56.

[34] R. Mittra, S.W. Lee, Analytical Techniques in the Theory of Guided Waves, McMillan and Company, 1971.
[35] Nazarov S.A., Properties of spectra of boundary value problems in cylindrical and quasicylindrical domains, Sobolev Spaces in Mathematics. V. II. International Mathematical Series, 9. New York: Springer, 2008, 261–309.

[36] Nazarov S.A. Variational and asymptotic methods for nding eigenvalues below the continuous spectrum threshold, Siberian Math. J. 51 (2010) 866-878.

[37] Nazarov S.A. Eigenvalues of the Laplace operator with the Neumann conditions at regular perturbed walls of a waveguide, J. Math. Sci. 172 (2011) 555-588.

[38] Nazarov S.A. The discrete spectrum of cranked, branched and periodic waveguides, St. Petersburg Math. J. 23 (2011).

[39] Nazarov S.A. Asymptotic expansions of eigenvalues in the continuous spectrum of a regularly perturbed quantum waveguide, Theoretical and mathematical physics 167 (2011) 606-627.

[40] Nazarov S. A. Asymptotics of an eigenvalue on the continuous spectrum of two quantum waveguides coupled through narrow windows, Math. Notes 93 (2013) 266–281.

[41] Nazarov S.A. Enforced stability of a simple eigenvalue in the continuous spectrum, Funct. Anal. Appl. 475 (2013) 195–209.

[42] Nazarov S.A., Olyushin M.V., Perturbation of eigenvalues of the Neumann problem due to variations of a domain’s boundary, St. Peterburg Math. J. 5 (1994) 371-387.

[43] Nazarov S.A., Plamenevsky B.A., Elliptic problems in domains with piecewise smooth boundaries, de Gruyter Expositions in Mathematics, 13. Walter de Gruyter & Co., Berlin, 1994.

[44] Nazarov S.A. Shanin A.V., Trapped modes in angular joints of 2D waveguides, Applicable Anal. 93 (2014) 572 582.

[45] Poynting J. H., On the transfer of energy in the electromagnetic field, Phil. Trans. Royal Society of London, 175 (1884) 343–361.

[46] N. A. Umov, Equations of motion of energy in bodies, Ul’rikh and Shultse Typography, Odessa 1874.

[47] Ursell F., Trapping modes in the theory of surface waves, Proc. Camb. Phil. Soc. 47 (1951) 347–358.

[48] Van Dyke M., Perturbation methods in fluid mechanics, Applied Mathematics and Mechanics, Vol. 8 Academic Press, New York-London 1964.

[49] C. H. Wilcox, Scattering theory for diffraction gratings, Appl. Math. Sci., vol. 46, Springer-Verlag, New York 1984.