The String Theory Approach to Generalized 2D Yang-Mills Theory

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Abstract

We calculate the partition function of the $SU(N)$ (and $U(N)$) generalized $YM_2$ theory defined on an arbitrary Riemann surface. The result which is expressed as a sum over irreducible representations generalizes the Rusakov formula for ordinary $YM_2$ theory. A diagrammatic expansion of the formula enables us to derive a Gross-Taylor like stringy description of the model. A sum of 2D string maps is shown to reproduce the gauge theory results. Maps with branch points of degree higher than one, as well as “microscopic surfaces” play an important role in the sum. We discuss the underlying string theory.

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1 Introduction

The stringy interpretation of four dimensional $YM$ and $QCD$ theories is one of the longstanding problems of strong interactions. As a matter of fact, the string theory’s first incarnation came about as the theory of strongly interacting hadrons. Strong coupling lattice calculations as well as the large $N$ approach clearly support and hint toward a string theory representation of these gauge theories. Recently, in a beautiful series of papers\cite{1, 2, 3, 4} this problem was investigated by Gross and Taylor within the framework of $YM_2$ theory on a Riemann surface of arbitrary topology. A lattice version of 2D $YM$ ($YM_2$) theory has been known for a long time to be exactly solvable\cite{5}. Several algorithms to compute correlators of Wilson loops on the plane were written down in this framework\cite{6, 7}. More recently the partition function on an arbitrary Riemann surface was shown to have a simple expression in terms of a sum over all irreducible representations of the gauge group \cite{8}. An identical result was derived also in a continuum path-integral approach by employing a “perturbation” to the topological gauge theory \cite{9}.

By performing a large $N$ expansion of these results Gross and Taylor\cite{1, 2, 3, 4} managed to give a string interpretation to the theory. To be more precise, they have shown that the large $N$ expansion can be thought of as a string expansion with $g_{st} = \frac{1}{N}$ and with the string tension identified with $g^2 N$, where $g$ is the gauge coupling. The coefficients of this expansion have a geometrical meaning in terms of sums over maps between two dimensional orientable manifolds. They were reproduced by attributing specific weights to different classes of singular maps \cite{10, 3, 4}. Maps from non-orientable world-sheets were later shown to be related to the stringy nature of $SO(N)$ and $Sp(N)$ gauge theories \cite{11, 12}. A stringy behaviour was revealed recently also for $YM_2$ at finite $N$ \cite{13}.

One of the more challenging problems in this context is to find a string action which gives rise to the sums of maps that reproduce the gauge theory partition function. Recently, an important progress toward this goal has been made\cite{14, 15}. These works indicate that the action may very well be a topological string action. This is consistent with the observation, which was noted previously \cite{16}, that the topological string action associated with holomorphic maps correctly accounts for the leading $\frac{1}{N}$ behaviour of the partition function on a toroidal target space.

Pure $YM_2$ theory defined on a compact Riemann surface is character-
ized by its invariance under area preserving diffeomorphisms and by the fact that there are no propagating degrees of freedom. It is easy to realize that these properties are not unique to the $Tr(F^2)$ theory but rather are shared by a wide class of theories, the generalized Yang-Mills theories ($gYM_2$). An equivalent formulation of the $YM_2$ Lagrangian takes the form of $Tr(BF) + g^2 Tr(B^2)$, where $B$ is an auxiliary pseudo-scalar field in the adjoint of the group. The generalized theories correspond to a general function $\Phi(B)$ replacing the $Tr(B^2)$ term. These theories can be further coupled to fermions, thus obtaining the generalized $QCD_2$ theory \textsuperscript{[17]}. The interest in generalized $YM_2$ and $QCD_2$ theories stems from the following reasons:

- Within the framework of 2D gauge theories, they can serve as new toy models. As laboratory models for $YM_4$ and $QCD_4$ there is a-priori no special role to ordinary $YM_2$ and $QCD_2$. As remarked in [17], it is conceivable that one of the generalized 2D models will reveal features which are more relevant and more closely resemble the four dimensional theories of interest.

- The investigation of the $gYM_2$ gives information about correlators of the topological $BF$ theory and the ordinary $YM_2$ theory.

- The study of the stringy generalized $YM_2$ sheds additional light on the underlying topological string theory. This is mainly due to the singular maps of higher order branch points which do not show up in the ordinary theory and play an important role in the generalized one. These maps reside on the boundary of map space.

- In four dimensions, $YM_4$ and $QCD_4$ do share a unique position as renormalizable theories. The generalized theories in four dimensions seem to introduce non-renormalizable terms. Starting from the lattice version of these theories, such terms, which do not affect the continuum IR behaviour, can be important from a computational point of view. We refer the reader to [17] for further discussions of this point.

In the present work we show that following the same procedure used in the ordinary $YM_2$ theory all the generalized theories are exactly solvable. The partition function of the $SU(N)$ (and $U(N)$) models defined on an arbitrary Riemann surface is shown to be expressed by a formula that is a straightforward generalization of the $YM_2$ result \textsuperscript{[8,18]}, involving dimensions and higher order Casimir operators. We introduce a diagrammatic expansion
for higher Casimir operators. By using the latter we derive a Gross-Taylor like stringy description of the model. In this description a sum of 2D string maps that reproduces the gauge theory results. In terms of those maps the distinction between the various models is done by assigning different weights to branch points of degree higher than one. A concept of contracted microscopic surfaces generalizes the contracted handles of \[10, 3, 4\]. Some preliminary ideas about the action of the underlying topological string theory are sketched in relation to the Lagrangian proposed in \[15\].

The paper is organized as follows: In section (2) we review the derivation of the partition function of the ordinary (quadratic) \(YM_2\) theory. We then derive the analogous result for the \(gYM_2\) theory and sketch rules for the corresponding computations of loop averages. Section (3) is devoted to the derivation of the stringy picture of the generalized models. We start with a brief summary of the work of \[3, 4\]. We explain the necessity to express the \(SU(N)\) higher Casimir operators in terms of properties of the symmetric group \(S_n\) in order to establish the stringy picture. This is followed by a subsection where we develop in great details a diagrammatic expansion of the Casimir operators. Readers who are not interested in the derivation of the diagrammatic expansion may skip this subsection and proceed directly to the one where the rules of the cycle structure formula are stated. This then leads to the description of the partition function as a sum of maps. We encourage the reader to go over the various examples presented in subsection 3.6. Marked points and collapsed surfaces which are among the new features of the generalized model are then discussed. In section 4 we summarize the results of the work, present preliminary thoughts about the underlying topological string action and state certain open questions. In appendix (A) we show that the same scaling rules used to determine the stringy maps are obtained from a t’Hooft like analysis of a generalized \(YM_4\) or a generalized \(QCD_2\). In appendix (B) we compare the stringy calculation of certain correlators (e.g. \(tr(F^3)\)) in the \(YM_2\) theory to the corresponding correlators (e.g. \(tr(B^3)\)) derived by employing the \(gYM_2\) results.

## 2 Solution of the \(gYM_2\)

Pure \(YM_2\) theory defined on an arbitrary Riemann surface is known to be exactly solvable. In one approach the theory was regularized on the lattice\[5\]
and, using a heat kernel action, explicit expressions for the partition function
and loop averages were derived. Identical results were derived also in a
continuum path-integral approach [9]. In the following subsections we briefly
review the former derivation of the partition function and determine in a
similar way the results for the partition function and Wilson loops in the
$gYM_2$ case.

2.1 The partition function the $YM_2$ theory

The partition function for the ordinary $YM_2$ theory defined on a compact
Riemann surface $\mathcal{M}$ of genus $H$ and area $A$ is

$$Z(N, H, \lambda A) = \int [DA^\mu] \exp[-\frac{1}{4g^2} \int_\Sigma d^2 x \sqrt{\det G_{ij}} \ tr F_{ij} F^{ij}]$$

where the gauge group $G$ is taken to be either $SU(N)$ or $U(N)$, $g$ is the
gauge coupling constant, $\lambda = g^2 N$, $G_{ij}$ is the metric on $\mathcal{M}$, and $tr$ stands for
the trace in the fundamental representation. The lattice partition function
defined on an arbitrary triangulation of the surface is given by

$$Z_\mathcal{M} = \int \prod_l dU_l \prod_\triangle Z_\triangle[U_\triangle]$$

where $\prod_l$ denotes a product over all links, $U_\triangle$ is the holonomy around a
plaquette, and $Z_\triangle$ is a plaquette action. For the latter one uses a heat kernel
action [3, 8] rather than the Wilson action, i.e.

$$Z_\triangle[U] = \sum_R d_R \chi_R(U)e^{-tc_2(R)}$$

where the summation is over the irreducible representations $R$ of the group.
$d_R$, $\chi_R(U)$ and $c_2(R)$ denote the dimension, character of $U$ and the second
Cassimir operator of $R$ respectively, and $t = g^2 a$ with $a^2$ being the plaquette
area. The holonomy $U$ (its subscript $\triangle$ is omitted from here on) behaves as
$U \approx 1 - iaF$ when $a$ is small. Note that the region of validity of (3) is not
only $a \to 0$ with $F$ fixed, but actually also $a \to 0$ with $F$ going to infinity as
$a^{-1/2}$ because this is the region for which the exponential $-\frac{1}{4} a^2 Tr F^2$ is of
order unity.
We will briefly review the derivation which singles out (3) as a convenient choice among the different lattice theories which belong to the same universality class \([5, 8]\). Let us look for a function \(\Psi(U, t)\) that will replace the continuum \(e^{-\frac{1}{4}t^2 TrF^2}\).

The requirements which we impose on \(\Psi\) are

1. As \(t\) goes to zero (and, therefore, for finite \(g\) also \(a\) goes to zero) we want the holonomy to be close to 1:
   \[\Psi(U, 0) = \delta(U - 1)\]

2. For any \(V \in G\) we have
   \[\Psi(V^{-1}UV, t) = \Psi(U, t)\]
   In other words \(\Psi\) is a class-function.

3. \(\Psi\) satisfies the heat kernel equation
   \[\big\{ \frac{\partial}{\partial t} - \sum_{a, b} g_{ab} \partial^a \partial^b \big\} \Psi(U, t) = 0\]
   where \(g_{ab}\) is the inverse of the Cartan metric
   \[g^{ab} = tr(t^a t^b)\]

To see that (3) is an approximate solution to the heat-kernel equation we note that any class function is a linear combination of characters. The differentiation of a character in the direction of a Lie algebra element \(t^a\) is given by:

\[\partial^{a_1} \partial^{a_2} \cdots \partial^{a_k} \chi_R(U) = \frac{k!}{k!} \chi_R(U t^{a_1} t^{a_2} \cdots t^{a_k}) + O(U - 1)\]

The notation \(\chi_R(U t^{a_1} t^{a_2} \cdots t^{a_k})\) stands for the trace of the multiplication of the matrices which represent \(U\) and \(t^{a_1}, \ldots, t^{a_k}\) in the representation \(R\). The brackets \((\cdots)\) imply symmetrization with respect to the indices. The term \(O(U - 1)\) means that the corrections are of the order of \(U - 1 \sim aF \sim t^{1/2}\). Since

\[\sum_{a, b} g_{ab} \partial^a \partial^b \chi_R(U) \approx -\frac{1}{2} \chi_R(U \sum_{a, b} g_{ab} t^{(a b)}) = -c_2(R) \chi_R(U)\]
we see that (3) is the correct answer up to terms of the order of \(O(t^{3/2})\) which drop in the continuum limit. Using (3) as the starting point, it is straightforward to derive the partition function [5, 8]:

\[
\mathcal{Z}(N, H, \lambda A) = \sum_R d_R^{2-2H} e^{-\frac{\lambda A_{ac}(R)}{2N}}
\]  (4)

### 2.2 The Partition Function of \(gYM_2\) theories

Pure \(YM_2\) theory is in fact a special representative of a wide class of 2D gauge theories which are invariant under area preserving diffeomorphisms. These generalized \(YM_2\) theories are described by the following generalized partition function

\[
\mathcal{Z}(G, H, A, \Phi) = \int [DA] [DB] \exp \left[ \int_S d^2x \sqrt{\det G_{ij}} \text{tr}(iBF - \Phi(B)) \right]
\]  (5)

where \(F = F^{ij}\epsilon_{ij}\) with \(\epsilon_{ij}\) being the anti-symmetric tensor \(\epsilon_{12} = -\epsilon_{21} = 1\).

\(B\) is an auxiliary Lie-algebra valued pseudo-scalar field.

We wish to generalize the substitution (3) for the plaquette action (3):

\[
Z_{\Delta}[U] = \int DB e^{\text{tr}\{iaBF - t\Phi(B)\}} \rightarrow \Psi(U, t)
\]  (6)

Here \(B\) is a hermitian matrix and \(\Phi\) is an invariant function (invariant under \(B \rightarrow U^{-1}BU\) for \(U \in G\)). The quadratic case \(\Phi(X) = g^2 \text{tr}(X^2)\) obviously correspond to the \(YM_2\) theory. We will take \(\Phi\) to be of the form

\[
\sum_{\{k_i\}} a_{\{k_i\}} \prod_i \text{tr}(X^i)^{k_i}
\]  (7)

(e.g. \(\text{tr}(X^3)^2 + \text{tr}(X^6)\)) For \(SU(N) (U(N))\), \(\text{tr}(X^i)\) can be expressed for \(i \geq N\) \((i > N)\) in terms of \(\text{tr}(X^i)\) for smaller \(i\)-s. Thus the summands in (7) are not independent. This does not affect the following discussion. Moreover, in the large \(N\) limit that we will discuss in the following section, the terms do become independent.

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2In principle, we could perturb the ordinary \(YM_2\) with operators of the form \(\frac{1}{g^2} \text{tr}(F^k)\), without the need of an auxiliary field. We discuss the relation between this perturbation and the perturbation by \(\text{tr}(B^k)\) in appendix (B).
We define the general structure constants \( d_{abc...k} \) to be
\[
d_{abc...k} \overset{\text{def}}{=} g_{aa'}g_{bb'}\cdots g_{kk'}tr(t^{a'}t^{b'}t^{c'}\cdots t^{k'})
\]
For every partition \( r_1 + r_2 + \cdots + r_j \), we define the Casimir
\[
C_{\{r_1+r_2+\cdots+r_j\}} \overset{\text{def}}{=} \frac{1}{(r_1 + r_2 + \cdots + r_j)!} d^{(1)}_{a_1 \cdots a_{r_1}} d^{(2)}_{a_{r_1+1} \cdots a_{r_1+r_2}} \cdots d^{(j)}_{a_{r_1+r_2+\cdots+r_{j-1}+1} \cdots a_{r_1+r_2+\cdots+r_j}} t^{(a_1^{(1)})} \cdots t^{(a_{r_1}^{(1)})} t^{(a_2^{(2)})} \cdots t^{(a_{r_2}^{(2)})}
\]
Note that the index of \( C_{\{\cdot\}} \) will always pertain to a partition. Thus \( C_{\{p\}} \neq C_{\{r_1+r_2+\cdots+r_j\}} \) even if \( p = r_1 + r_2 + \cdots + r_j \). The brackets in the \( t \)-s mean a total symmetrization \( ( (r_1 + r_2 + \cdots + r_j)! \) terms).

\( C_{\rho} \) can easily be seen to commute with all the group elements and so, by Schur's lemma, is a constant matrix in every irreducible representation.

We claim that the correct lattice generalization of (3) is
\[
\sum_R d_R \chi_R(U) e^{-t\Lambda(R)}
\]
where
\[
\Lambda(R) = \sum_{\{k\}} a_{\{k\}} C_{\{k_1,1+k_2,2+k_3,3+\ldots\}}(R)
\]
This results from the requirements that \( \Psi(U,t) \) must satisfy:
1. \( \Psi(U,0) = \delta(U - 1) \).
2. \( \Psi \) is a class-function.
3. \( \Psi \) satisfies the equation
\[
\left\{ \frac{\partial}{\partial t} - \sum_{\{k\}} a_{\{k\}} \prod_l ((ia)^{-l}d_{a_1a_2\ldots a_l}) \frac{\partial^l}{\partial F_{a_1} \cdots \partial F_{a_l}} \right\} \Psi(U,t) + O(U - 1) = 0
\]
For the \( U \)-s that are important in the weight for a single plaquette, \( U - 1 \) is of the order of magnitude of \( aF \) which, in turn, is of the order of magnitude of \( O(t^{1/\nu}) \) where \( \nu \) is the maximal degree of \( \Phi \). Thus, the corrections to \( \Psi \) are \( O(a^{-(1+1/\nu)}) \) and drop out in the continuum limit.

The partition function for the generalized \( YM_2 \) theory is therefore:
\[
Z(G, \Sigma_H, \Phi) = \sum_R (\dim R)^{2-2H} e^{-\frac{\Lambda(R)}{2N}}
\]
where \( \Lambda(R) \) is defined in (9).
2.3 Loop Averages in the Generalized Case

The full solution of the $YM_2$ theory includes in addition to the partition function also closed expressions for the expectation values of products of any arbitrary number of Wilson loops

$$W(R_1, \gamma_1, ... R_n \gamma_n) = < \prod_{i=1}^{n} Tr R_i P e^{i \oint_{\gamma_i} A dx}>$$

where the path-ordered product around the closed curve $\gamma_i$ is taken in the representation $R_i$. Using loop equations, derived in \cite{19}, an algorithm to compute Wilson loops on the plane was written down\cite{6}. These results were derived also in \cite{7} using a different approach. Recently a prescription for computing those averages for non-intersecting loops on an arbitrary two manifold was proposed\cite{8}. Let us briefly summarize the latter. One cuts the 2D surface along the Wilson loop contours forming several connected “windows”. Each window contributes a sum over all irreducible representations of the form of (4). In addition, for each pair of neighbouring windows, a Wigner coefficient $D_{R_1...R_n} = \int dU \chi(R_1(U)) \chi(R_2(U^\dagger)) \chi_f(U)$ is attached. Altogether, one finds

$$W(R_1, \gamma_1, ... R_n \gamma_n) = \frac{1}{Z N^n} \sum_{R_1} \ldots \sum_{R_n} D_{R_1...R_n} \prod_{i=1}^{N_w} d^{2-2G_i} e^{-\frac{\lambda A_{\gamma_i} C_2(R_i)}{2N}}$$

where $N_w$ is the number of windows, $2 - 2G_i$ is the Euler number associated with the window $i$ and $D_{R_1...R_n}$ is the product of the Wigner coefficients\cite{8}. For the case of intersecting loops a set of differential equations \cite{6} provides a recursion relation by relating the average of a loop with $n$ intersections to those of loops with $m < n$ intersections.

Generalizing these results to the $gYM_2$ is straightforward. The only alteration that has to be invoked is to replace the $e^{-\frac{\lambda A_{\gamma} C_2(R)}{2N}}$ factors that show up in those algorithms with similar factor where the second Casimir operator is replaced by the generalized Casimir operator (9). For instance the expectation value of a simple Wilson loop on the plane is given by

$$\langle W(R, \gamma) \rangle = e^{-\frac{\lambda A_{\gamma} N(R)}{2N}}$$

where $A_{\gamma}$ is the area enclosed by $\gamma$.

It is interesting to note that for odd Casimir operators the expectation values of real representations (like the adjoint representation) equal unity due to the fact that the corresponding Casimirs vanish.
3 Stringy $gYM_2$ - A Gross-Taylor Type Analysis

Following the discovery of the stringy nature of pure $YM_2$ theory [1, 2, 3, 4] one may anticipate that a similar situation prevails also for the generalized $YM_2$ theories. Indeed, in this section we prove this conjecture. Several examples serve to demonstrate the nature of the maps that contribute to the $gYM_2$ and their weights. We start by reviewing the analysis of [3, 4].

3.1 Stringy $YM_2$ theory

The partition function expressed as a sum over irreducible $SU(N) \times U(N)$ representations [4] was expanded in [3, 4] in terms of powers of $\frac{1}{N}$. This involved expanding the dimension and the second Casimir operator of the various representations. Using the Frobenius relations between representations of the symmetric group $S_n$ and representations of $SU(N) \times U(N)$, the coefficients of this asymptotic expansion were written in terms of characters of $S_n$. The latter were further shown to correspond to permutations of the sheets covering the target space. The result of [4] takes the form of (using the notation of [15])

$$Z(A, H, N) \sim \sum_{n^+, i^+=0}^{\infty} \sum_{s^+_1 \cdot \cdots \cdot s^+_H} \sum_{t^-_1 \cdot \cdots \cdot t^-_H} \left( \frac{1}{N} \right)^{(n^+ + n^-)(2H-2) + (i^+ + i^-)} \frac{(-1)^{(i^+ + i^-)}}{i^+! i^-! n^+! n^-! (\lambda A)(i^+ + i^-) e^{-\frac{1}{2}((n^+ + n^-) \lambda A e^{\frac{1}{2}(n^+)^2 + (n^-)^2 - 2n^+ + n^-} \lambda A/N^2} \delta_{s^+_n, s^-_n} \left( p^+_1 \cdots p^+_i, p^-_1 \cdots p^-_i, \Omega_{n^+, n^-}^{2-2H} \prod_{j=1}^{H}[s^+_j, t^+_j] \prod_{k=1}^{H}[s^-_k, t^-_k] \right)$$

(11)

where $[s, t] = st - 1^{-1} t^{-1}$. Here $\delta$ is the delta function on the group algebra of the product of symmetric groups $S_{n^+} \times S_{n^-}$, $T_2$ is the class of elements of $S_{n^\pm}$ consisting of transpositions, and $\Omega_{n^+, n^-}$ are certain elements of the group algebra of the symmetric group $S_{n^+} \times S_{n^-}$.

The formula (11) nearly factorizes, splitting into a sum over $n^+, i^+, \cdots$ and $n^-, i^-, \cdots$. The contributions of the (+) and (−) sums were interpreted as arising from two “sectors” of a hypothetical worldsheet theory. These
sectors correspond to orientation reversing and preserving maps, respectively. One views the \( n^+ = 0 \) and \( n^- = 0 \) terms as leading order terms in a \( 1/N \) expansion. At higher orders the two sectors are coupled via the \( n^+ n^- \) term in the exponential and via terms in \( \Omega_{n^+ n^-} \).

Thus, the conventional \( YM_2 \) theory has an interpretation in terms of sums of covering maps of the target-space. Those maps are weighted by the factor of \( N^{2-2h} e^{-\frac{1}{2} n \lambda A} \) where \( h \) is the genus of the world-sheet and \( A \) is the area of the target space. The power of \( N^{2-2h} \) was obtained in [1] from the Riemann-Hurwitz formula

\[
2h - 2 = (n^+ + n^-)(2H - 2) + (i^+ + i^-)
\]

where \( B = i^+ + i^- \) is the total branching number. The number of sheets above each point in target space (the degree of the map) is \( n \) and \( \lambda = g^2 N \) is the string tension. Maps that have branch points are weighted by a factor of \( \lambda A \). The dependence on the area \( A \) results from the fact that the branch point can be at any point in the target space.

### 3.2 Toward the Stringy generalized \( YM_2 \)

Note that the stringy description of the \( YM_2 \) theory does not attribute any special weight for maps that have branch points of degree higher than one, nor is there a special weight for two (or more) branch points that are at the same point in the target space. The latter maps are counted with weight zero [3, 4], at least for a toroidal target space, since they constitute the boundary of map space (see also [5]).

The main idea behind a stringy behavior of the \( gYM_2 \) is to associate nonzero weights to those boundary maps, once one considers the general \( \Phi(B) \) case rather than the \( B^2 \) theory. In other words, we anticipate that we will have to add for the \( tr(B^3) \) theory, for example, maps that have a branch point of degree 2 and count them, as well, with a weight proportional to \( A \). From the technical point of view the emergence of the \( YM_2 \) description in terms of maps followed from a large \( N \) expansion of the dimensions and the second Casimir operators of (4). Obviously a similar expansion of the former applies also for the generalized models and therefore what remains to be done is to properly treat the Casimirs appearing in the exponents of (10).

In [1], the expansion of the second Casimir operator \( C_2(R) \) of a representation \( R \) introduced the branch points and the string tension contributions.
to the partition function. The $C_2(R)$ was expressed in terms of the eigenvalue of the sum of all the \( \frac{n(n-1)}{2} \) transpositions of \( n \) elements (permutations containing a single cycle of length 2), where \( n \) is the number of boxes in \( R \). This is the outcome of formula (2.3) of [1] which we write as

\[
C_2(R) = nN + 2\hat{P}_{\{2\}}(R) 
\]

where $\hat{P}_{\{2\}}(R)$ is the value of the scalar matrix representing the sum of transpositions $\sum_{i<j}
i j\in R$ in the representation $R$ of $S_n$ (the matrix commutes with all permutations and thus is scalar). In the partition function, $C_2(R)$ was multiplied by $\lambda A$. The resulting term $\frac{1}{2}n\lambda A$ arises from the action and is proportional to the string tension. The term $\lambda A\hat{P}_{\{2\}}$ arises from the measure and is interpreted as the contribution of branch points to the weight of a map.

Our task is, therefore, to express the generalized Casimirs $C_\rho$ of (8) in terms of $\hat{P}_{\rho'}$, the generalizations of $\hat{P}_{\{2\}}(R)$. This is expressed as

\[
C_\rho(R) = \sum_{\rho'} \alpha_\rho^{\rho'} N^{h_{\rho'}} \hat{P}_{\rho'}(R) 
\]

where $\alpha_\rho^{\rho'}$ are coefficients that are independent of $R$ and the power factors $h_{\rho'}$, as will be discussed in section 3.4, are adjusted so that a string picture is achieved.

The $\hat{P}_{\rho'}$ factors are associated with $\rho'$ which is an arbitrary partition of certain numbers, namely

\[
\rho' : \sum_i k_i \cdot i = \underbrace{1 + 1 + \cdots + 1}_{k_1} + 2 + 2 + \cdots + 2 + \cdots
\]

$\hat{P}_{\rho'}(R)$ is the product of two factors. The first is the sum of all the permutations in $S_n$ ($n$ is the number of boxes of $R$) which are in the equivalence class that is characterized by having $k_i$ cycles of length $i$ for $i \geq 2$. Just like the case of $\hat{P}_{\{2\}}(R)$, the matrix $\hat{P}_{\rho'}(R)$ commutes with all permutations and thus is a scalar. The sum is taken in the representation $R$ of $S_n$. The second factor is

\[
\binom{n - \sum_{i=2} k_i}{k_1}
\]

which will be interpreted later as the number of ways to put $k_1$ marked points on the remaining sheets that do not participate in the branch points.
3.3 Diagrammatic expansion of the Casimir operators

We now derive a diagrammatic expansion for the values of the Casimir operators for any irreducible representation $R$ of $U(N)$ and $SU(N)$ groups. The idea, following [3, 4], is to connect $C_\rho(R)$ with the characters of permutations in $S_n$. The $S_n$ representation is described by the same Young diagram as $R$.

The reader who is not interested in the details may skip this section and continue from section (3.4) where the final result is presented. (We will refer to the current section only once, in section (3.7), but this is not crucial!)

The representation space of $S_n$ that corresponds to a certain Young diagram $Y$ is a subspace of the formal algebra of $S_n$ (i.e. all combinations $\sum_{\sigma \in S_n} \lambda_\sigma \sigma$ where the $\lambda_\sigma$-s are arbitrary coefficients). It is generated by all elements of the form $\Pi R$ where $\sigma \in S_n$ is arbitrary and

$$\Pi_R = \sum_{\tau \in C_Y, \nu \in R_Y} (-)^\tau \tau \nu.$$  

$C_Y$ is the set of all permutations that do not mix the columns of $Y$ and $R_Y$ is the set of all permutations that do not mix the rows of $Y$.

The corresponding $SU(N)$ representation is constructed as follows. Tensor-multiply $n$ copies of $C_N$ which we denote by

$$W \equiv W_1 \otimes W_2 \otimes \cdots \otimes W_n.$$  

(15)

The action of $S_n$ on $W$ is

$$\sigma : w_1 \otimes w_2 \otimes \cdots \otimes w_n \mapsto w_{\sigma(1)} \otimes w_{\sigma(2)} \otimes \cdots \otimes w_{\sigma(n)}.$$  

The representation $R$ of $SU(N)$ is given by the subspace $\text{Im} \Pi_R \subset W$ (see e.g. [20]).

Next consider Casimirs $C_{\{k\}}$ with subindices which are a partition that is a single element. If $t_{(i)}^a$ denotes the matrix of the generator $t^a$ in the $i^{th}$ subspace $W_i$ then

$$t_R^a = \sum_{i=1}^n t_{(i)}^a$$  

(16)

and

$$C_{\{k\}} = \frac{1}{k!} \sum_{a_1, \ldots, a_k} Tr_F \{ t_{(1)}^{a_1} t_{(2)}^{a_2} \cdots t_{(k)}^{a_k} \} t_R^{a_1} t_R^{a_2} \cdots t_R^{a_k}$$  

(17)
where we use the normalization $tr_F(t^{a'b}) = \delta^{ab}$. Substituting (16) in (17) we get

$$C\{k\} = \sum_j \sum_{r_1, r_2, \ldots, r_j} T(r_1, r_2, \ldots, r_j)$$

where $T(r_1, r_2, \ldots, r_j)$ is the sum over all operators of the form

$$\frac{1}{k!} \sum_{a_1, a_2, \ldots, a_k} tr_F\{t^{a_1} t^{a_2} \cdots t^{a_k}\} t^{(a_1)}_{(i_1)} t^{(a_2)}_{(i_2)} \cdots t^{(a_k)}_{(i_k)}$$

Among the indices $i_1, \ldots, i_k$ several values appear more than once. Moreover, there should be exactly $j$ different values among the $k$ indices in such a way that the $l$th value (out of $j$) appears $r_l$ times. For every permutation $\sigma \in S_k$ define $P_\sigma$ which acts on

$$W_{i_1} \otimes W_{i_2} \otimes \cdots \otimes W_{i_k}$$

by

$$(P_\sigma)^{\alpha_1 \alpha_2 \cdots \alpha_k}_{\beta_1 \beta_2 \cdots \beta_k} = \delta^{\beta_1}_{\alpha_1} \delta^{\beta_2}_{\alpha_2} \cdots \delta^{\beta_k}_{\alpha_k}$$

where $\alpha_r, \beta_r$ are indices of $W_{i_r}$. For every equivalence class (given by a partition $\rho$ of $n$) we define:

$$\hat{P}_\rho = \sum_{\sigma \in \rho} P_\sigma$$

Our task is to express (18) in terms of the various $\hat{P}_\rho$-s. For this end we define

$$\Lambda_r = \sum_a t^a_F t^{a}_{(i_r)}$$

$\Lambda_r$ is a matrix in $F \otimes W_{i_r}$ where $F$ is a fixed vector space (isomorphic to the fundamental representation space). It can be checked that, in indices,

$$(\Lambda_r)^{\alpha' \beta'}_{\alpha \beta} = \delta^{\alpha'}_{\alpha} \delta^{\beta'}_{\beta} - \frac{\epsilon}{N} \delta^{\alpha'}_{\alpha} \delta^{\beta'}_{\beta}$$

where $\alpha$ and $\alpha'$ are indices of $F$, $\beta$ and $\beta'$ are indices of $W_{i_r}$ and $\epsilon = 0$ for $U(N)$ and $\epsilon = 1$ for $SU(N)$. We have

$$\sum_{a_1, \ldots, a_k} tr_F\{t^{a_1} t^{a_2} \cdots t^{a_k}\} (t^{a_1})^{\beta_1}_{(i_1)} (t^{a_2})^{\beta_2}_{(i_2)} \cdots (t^{a_k})^{\beta_k}_{(i_k)}$$

$$= \sum_{\gamma_1, \gamma_2, \ldots, \gamma_k} (\Lambda_1)^{\gamma_2 \beta_1}_{\gamma_1 \alpha_1} (\Lambda_2)^{\gamma_3 \beta_2}_{\gamma_2 \alpha_2} \cdots (\Lambda_{k-1})^{\gamma_k \beta_{k-1}}_{\gamma_{k-1} \alpha_{k-1}} (\Lambda_k)^{\gamma_2 \beta_k}_{\gamma_1 \alpha_k}$$

13
Using (19) we can describe this, graphically, as a sum of diagrams. This is demonstrated in Fig.1 for a particular term of $C_6$.

\[
\left( -\epsilon \right) \left( -\epsilon \right)
\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Diagrammatic description of the multiplication of the $\Lambda_r$ matrices.}
\end{figure}

From this picture we see that

\[
\sum_{\gamma_1,\gamma_2,\ldots,\gamma_k} (\Lambda_1)^{\gamma_2,\beta_1} (\Lambda_2)^{\gamma_3,\beta_2} \cdots (\Lambda_{k-1})^{\gamma_k,\beta_{k-1}} (\Lambda_k)^{\gamma_1,\beta_k} = \sum_{p=0}^{k} \left( -\frac{\epsilon}{N} \right)^{k-p} \left\{ \prod_{1 \leq r_1 < r_2 < \cdots < r_p \leq k} \delta_{\alpha_{r_1}}^{\beta_{r_1}} \delta_{\alpha_{r_2}}^{\beta_{r_2}} \cdots \delta_{\alpha_{r_p}}^{\beta_{r_p}} \right\} \prod_{s \notin \{r_1, \ldots, r_p\}} \delta_{\alpha_s}^{\beta_s} \right) (21)
\]

We will draw the permutation that corresponds to Fig.1 as:
From Figs. 1 and 2 we obtain immediately (see (18)): \[ T(k_1, 1, \ldots, 1) = N\left(-\frac{\epsilon}{N}\right)^k \hat{P}_{k-1} + \sum_{p=1}^{k} \frac{k!}{p!} \left(-\frac{\epsilon}{N}\right)^{k-p} \hat{P}_{1-p+(k-p)-1} \] (22)

where \{1 \cdot p + (k-p) \cdot 1\} is the partition \(p+1+1+\cdots+1\) and \(T(1,1,\ldots,1)\) was defined in eqn. (18)

We can obtain the other \(T(r_1,\ldots,r_j)\) from (21) by contracting some of the \(\alpha_{i_l}\)-s with the \(\beta_{i_m}\)-s. For example, for \(j=1\), \(T(1)\) is

\[
\sum_{i=1}^{n} \sum_{a_1,\ldots,a_k} T_{r_F}\{t_{a_1}^{a_1}, t_{a_2}^{a_2}, \ldots, t_{a_k}^{a_k}\} \left(t_{(i)}^{a_1}\right)_{\alpha_1} \left(t_{(i)}^{a_2}\right)_{\alpha_2} \ldots \left(t_{(i)}^{a_k}\right)_{\alpha_k+1}
\]

\[
= \frac{1}{(k-1)!} \sum_{i=1}^{n} \sum_{a_1,\ldots,a_k} T_{r_F}\{t_{a_1}^{a_1}, t_{a_2}^{a_2}, \ldots, t_{a_k}^{a_k}\} \sum_{\sigma\in\{\{k\}\}} \left(t_{(i)}^{a_1}\right)_{\alpha_{\sigma(1)}} \left(t_{(i)}^{a_2}\right)_{\alpha_{\sigma(2)}} \ldots \left(t_{(i)}^{a_k}\right)_{\alpha_{\sigma(k)}}
\]

(23)

The symbol \([\{k\}]\) is the equivalence class of all the \((k-1)!\) permutations in \(S_k\) that have just one cycle (of length \(k\)). Graphically, we describe contractions of the \(\alpha_{i_l}\)-s with the \(\beta_{i_m}\)-s by directed dashed lines. The diagrams that are obtained are of the form drawn in the following figure:
Fig. 3: An example of a “dashed” diagram.

There are no two incoming dashed arrows or two outcoming dashed arrows from the same vertex ‘o’.

The dashed lines can be eliminated by two rules described in Fig.4a and Fig.4b. This can be checked with the use of Fig.1.

Fig. 4a: Rules for eliminating dashed lines.
The next step is to consider trajectories of dashed lines. Suppose that such a trajectory passes through \( r \) vertices. Then there are \( r! \) different ways to draw directed dashed lines that pass through those \( r \) vertices in a single trajectory. We will first sum over all those \( r! \) dashed lines trajectories, as in Fig. 5.
Using the rules of Fig.4a − b and induction on \( r \), we easily find the rule of Fig.6.

}\[
\frac{1}{r!} \sum_{\sigma \in \{\{r\}\}} \sum_{s=1}^{r} \sigma_{\sigma(1)} \sigma_{\sigma(2)} \cdots \sigma_{\sigma(r)}
\]

Fig. 6: Graphical description of the contraction of \( r \) vertices. The notation \( \sigma \in \{\{r\}\} \) means that \( \sigma \) is a permutation with just one cycle of length \( r \). Note that the number of terms on the rhs is exactly \( r! \), since there are \((r-1)!\) permutations in the equivalence class \( \{\{r\}\} \).

At this stage the diagrammatic expansion of the Casimirs includes a sum of diagrams, each composed of loops of various sizes. Each loop of size zero should be replaced by an extra factor of \( N \) to its corresponding diagram. We now look at a specific diagram. Suppose it has \( \mu_t \) loops of size \( t \), for \( t = 1, 2, \ldots \). Let \( m = \sum_{t=2}^{\infty} t\mu_t \). The number of ways to write indices from 1, 2, \ldots, \( n \) where \( n \) is the number of boxes in the representation, or as we discuss later, the number of sheets in the string picture, is \( (n-m-\mu_1) \). If \( m + \mu_1 > n \) then, by definition, this factor is zero. Since \( m \) is independent of \( n \), and in fact depends only on \( \Phi(B) \), the stringy interpretation of this factor is of \( \mu_1 \) microscopic holes that we cut in the sheet, above the singular point.

Thus, when we discuss the weight that is to be associated with a branch point that makes a permutation of the sheets which has \( k_i \) cycles of length \( i \geq 2 \), (for example, for a simple branch point \( k_2 = 2 \) and all the other \( k_i \)-s are zero) it is natural to include \( i = 1 \) as well. We interpret such points as places where there are holes in \( k_1 \) sheets (possibly in addition to other branch points which mix other sheets, if \( k_i \neq 0 \) for other \( i \)-s).

The weight that is to be associated to such a generalized branch point is
\[ \lambda A \] times the sum of the coefficients of all the diagrams that have \( k_i \) loops of length \( i \) (after the manipulations of Fig.6), times a symmetry factor of \( \prod_{i=1}^{\infty} (i^{k_i} k_i!) \). This factor arises from the \( k_i! \) possible ways to mix the loops of size \( i \), each corresponding to a different indexing but the same permutation. Also, there are \( i \) ways to rotate each loop, which again corresponds to a different indexing (from 1, \ldots, \( n \)) but the same permutation.

Note that the factor of \( \frac{1}{r!} \) in Fig.5 arises from the fact that there is a symmetry factor of \( \frac{k!}{r_1! r_2! \cdots r_j!} \) in (18). Indeed, if we permute the subindices \( i_1, \ldots, i_k \) and the indices \( a_1, \ldots, a_k \) of (18) in such a way that we do not change the relative order of the \( a_q \)-s among \( t^{a_q}_{(i_q)} \)-s that have the same \( i_q \)-s (because they multiply each other in the same vector space \( W_{i_q} \)), we do not change the graph of the type of Fig.3 that corresponds to it. The factor of \( k! \) cancels with the factor of \( \frac{1}{r!} \) in (18).

The generalization to the non-chiral sector is done by replacing (15) by

\[ W \equiv W_1 \otimes W_2 \otimes \cdots \otimes W_{n+} \otimes W^*_{n+} \otimes W^*_{1} \otimes W^*_{2} \otimes \cdots \otimes W^*_{n-} \]  

(24)

where a generator \( t^a \) acts on \( W^*_{i} \) as the transposed matrix of \( t^a \) in the fundamental representation. The representation \( R \overline{S} \) (following \( \Pi \)) of \( SU(N) \) (or \( U(N) \)) is obtained by restricting to the kernel of all the operators that contract an index of one of the \( W^*_{i} \)-s with an index of one of the \( W_j \)-s, and also applying the two projection operators \( \Pi_R \) and \( \Pi_S \), where \( R \) is the corresponding Young diagram for \( S_{n+} \) and \( S \) is the Young diagram for \( S_{n-} \).

Putting everything together we have the following rules to obtain the cycle structure:

### 3.4 Cycle structure formula for \( U(N) \) and \( SU(N) \)

The coefficients of the various \( \hat{P}_\rho(R) \) which soon will be shown to correspond to the weight for a map with a point that has a multiple branch point (and possibly multiple marked points) is obtained diagrammatically as follows:

1. For every term in \( \Phi(B) \) of the form \( \prod_k \{ tr(B^k) \}^{\nu_k} \) draw a graph of \( \sum \nu_k \) loops so that there are \( \nu_k \) loops of size \( k \) (and each loop is oriented).
2. Divide the various loops into two groups, in all possible ways. The first group will be called the chiral group and the second group will be called the anti-chiral group. Each diagram will have a factor of

\[ (-) \text{(number of anti-chiral vertices)}. \]
3. Group the \( k\nu_k \) vertices of all the loops into groups in all possible ways with the restriction that vertices from chiral and anti-chiral loops do not belong to the same group. Each group corresponds to one ellipse as in Fig.6 (where the group is of size \( r \)).

4. Reduce each group to one vertex according to the rule of Fig.6.

5. The result is a sum of diagrams, each composed of loops of various sizes. Each loop of size zero contributes a factor \( N \) to its corresponding diagram.

For \( SU(N) \) we have to include the \(-\frac{1}{N}\) factors from Fig.7. The rules are exactly the same as for the \( U(N) \) case above, except that after we have drawn the diagrams that correspond to the terms in \( \Phi(B) \) (step (1) above) we have to add all possible diagrams that are obtained from the previous ones by changing vertices to 1-loops in all possible ways. This is represented diagrammatically as:

\[
\begin{array}{c}
\bullet \quad \Rightarrow \quad \bullet \\
\downarrow \quad \quad \quad \downarrow \\
i \quad \quad \quad \quad \quad i
\end{array}
\begin{array}{c}
\bullet \quad \Rightarrow \quad -\frac{1}{N} \{ \quad \bullet \\
\downarrow \quad \quad \quad \downarrow \\
i \quad \quad \quad \quad \quad i
\end{array}
\]

Fig. 7: The modification from \( U(N) \) to \( SU(N) \)

### 3.5 The Partition Function as a Sum of Maps in the Large \( N \) Limit

Equipped with the expansion of \( C_\rho \) in terms of the \( \hat{P}_\rho' \) factors we proceed now to deduce the full description of the partition function (10) as a sum of branched maps. Following the same steps as in [1], in the large \( N \) limit, the coefficients of the \( \hat{P}_\rho' \)-s will turn out to be the weights that are associated with a singular point in target space that has \( k_1 \) marked points, \( k_2 \) simple branch
points, $k_3$ branch points of degree 2, and so on ($k_i$ branch points of degree $i - 1$). The numbers $k_i$ are related to the partition $\rho'$ as above. The only place that is different from (11) is that the restriction $p_i^\pm, \ldots, p_i^\pm \in T_2 \subset S_n^\pm$ in (11) will be replaced by permutations that are in the equivalence class (in $S_n^\pm$) that corresponds to the partition $\rho'$ (instead of $T_2$ which corresponds to the simple partition $\{2\}$).

Using the cycle structure formula of the previous section one has to attach certain weights to the various branched maps that corresponds to the diagrams found in step (5) of section (3.4). The weight of a branch point that makes a permutation of the sheets which has $k_i$ cycles of length $i \geq 2$, and has $k_1$ marked points is $\lambda A$ times the sum of the coefficients of all the diagrams that have $k_i$ loops of length $i$ (after performing the manipulations of section 3.4), times a symmetry factor $\prod_{i=1}^\infty (i^{k_i} k_i!)$.

\[ (\lambda A) \times \left( \text{Sum of coefficients of diagrams with } k_i \text{ loops of length } i \right) \times \prod_{i=1}^\infty (i^{k_i} k_i!) \quad (25) \]

Returning to equation (14), in order to keep the overall factor of $N^{2-2h}$ which is needed for a stringy interpretation as in (11,12), the factors $h_\rho^\rho$ should be

\[ S(\rho) + N(\rho) - 2 - S(\rho') + N(\rho') \]

where we denote by $S(\rho)$ the sum of all the summands of the partition $\rho$ and by $N(\rho)$ the number of summands in $\rho$. As $\rho$ corresponds to a certain monomial in $\Phi(B)$, $S(\rho) + N(\rho)$ is the degree in $B$ of that monomial plus the total number of traces in that monomial. Moreover, we take the corresponding coefficient in $\Phi(B)$ to scale as

\[ N^{2-S(\rho)-N(\rho)} \]

(e.g. $\alpha_1 N^{-1} tr(B^2) + \alpha_2 N^{-2} tr(B^3) + \alpha_3 N^{-4} (tr(B^2)^2) \cdots$). In appendix (A) we will motivate this scaling also by considering the 't-Hooft like large $N$ limit in terms of Feynman diagrams for both generalized YM$_4$ and generalized QCD$_2$. $S(\rho') - N'(\rho')$ is the change in the Euler characteristic of the world sheet due to a generalized branch point. A $k$-cycle branch point adds $k - 1$ to $2 - 2h$. Putting all these $N$ factors together and using the Riemann-Hurwitz formula we recover the stringy $N^{2-2h}$ behaviour. As a matter of fact, in general we will obtain

\[ S(\rho) + N(\rho) - 2 - S(\rho') + N(\rho') \geq h_\rho^\rho \]

21
but we will not always have equality. In order to retain the string interpretation, this can be explained as an attachment of

$$\frac{1}{2}(S(\rho) + N(\rho) - 2 - S(\rho') + N(\rho') - h_{\rho'})$$

microscopic handles to the singular points. In the next subsection we will argue that the factors of $N$ are consistent with attaching a microscopic Riemann surface with at least $N(\rho')$ handles – each one is connected to one of the $N(\rho')$ branch points that correspond to $\hat{P}_{\rho'}$. This, as we will argue, makes the string theory local.

3.6 Examples

In this section we give a few examples for various choices of $\Phi(B)$ for both $U(N)$ and $SU(N)$ groups.

1. For $\frac{\lambda}{N} tr(B^2)$ which is the conventional $YM_2$ theory we get

$$\frac{2\lambda}{N} \hat{P}_{\{2\}} + \lambda \hat{P}_{\{1\}}$$

The first term means that we give a factor of $\frac{2\lambda}{N}$ for each branch point, and the second term means that we have a factor of $\lambda$ for each marked point (i.e. this is the string tension).

2. For $\alpha N^{-2} tr(B^3)$ in $U(N)$ we get

$$3\alpha N^{-2} \hat{P}_{\{3\}} + 3\alpha N^{-1} \hat{P}_{\{2\}} + 3\alpha N^{-2} \hat{P}_{\{1+1\}} + \frac{1}{2} \alpha \hat{P}_{\{1\}} + \frac{1}{2} \alpha N^{-2} \hat{P}_{\{1\}}$$

The first term is the contribution from branch points of degree 2 (the simple branch points are of degree 1). The next term is a modification to the weight of the usual branch points. The third is the weight of two marked points at the same point (but different sheets), which will translate into $n^+(n^+ - 1) + n^-(n^- - 1)$ in the weight of a map for which $(n^+, n^-)$ are the numbers of sheets of each orientability. The last two terms are modifications to the cosmological constant (or, in our terminology, to the weight of the single marked point).

Note that, because of the original power of $N^{-2}$ we do not get the usual $N^{2-2h}$ stringy behaviour in the partition function. We can overcome
this problem by interpreting the last term not as the usual marked point, but as a microscopic handle that is attached to the point (this is similar to the interpretations of contracted handles and tubes in \[10\]). By interpreting certain marked points as actually being microscopic handles (or higher Riemann surfaces) we can always adjust the power of \(N\) to be \(N^{2-2h}\). Similarly, we should interpret the term \(3\alpha N^{-2}\hat{P}_{(1+1)}\) as a connecting tube. We will come back to this interpretation toward the end of this section and investigate it further in the next subsection.

3. For \(N^{-2}tr(B^3)\) in \(SU(N)\) we obtain the following corrections to (26):

\[
-\frac{6}{N^3}\hat{P}_{(2+1)} - \frac{12}{N^3}\hat{P}_{(2)} + \frac{12}{N^4}\hat{P}_{(1+1+1)}
- \left( \frac{6}{N^2} - \frac{12}{N^4} \right)\hat{P}_{(1+1)} - \left( \frac{3}{N^2} - \frac{2}{N^4} \right)\hat{P}_{(1)}
\]  

(27)

These terms and the terms in the previous example (26) do not mix chiralities (i.e. sheets of opposite orientations). In the full theory (chiral and anti-chiral sectors) there is the corresponding anti-chiral term:

\[
+\frac{6}{N^3}\hat{P}_{(2+1)} + \frac{12}{N^3}\hat{P}_{(2)} - \frac{12}{N^4}\hat{P}_{(1+1+1)}
+ \left( \frac{6}{N^2} - \frac{12}{N^4} \right)\hat{P}_{(1+1)} + \left( \frac{3}{N^2} - \frac{2}{N^4} \right)\hat{P}_{(1)}
\]  

(28)

For \(SU(N)\) there are additional terms that do mix chiralities. They are

\[
-\frac{6}{N^3}(\hat{P}_{(2+1)} - \hat{P}_{(2+1)}) + \frac{12}{N^4}(\hat{P}_{(1+1+1)} - \hat{P}_{(1+1+1)})
\]

(29)

The first term is the contribution of maps that have a branch point in one orientability and a marked point in the other (at the same target space point). The second term is the contribution of maps with three marked points – two for one orientability and one for the other.

To illustrate the content of these formulae in terms of representations, we will calculate the value of the third Casimir \(\frac{1}{6}d_{abc}t^{(a}t^{b}t^{c)}\) for a totally anti-symmetric representation of \(SU(N)\) with \(k\) boxes. The term \(\hat{P}_{[3]}\) is translated into the sum of all the permutations of the \(k\) indices of a totally anti-symmetric tensor that are 3-cycles, this gives \(\frac{1}{3}k(k-1)(k-\)
2). The term $\hat{P}\{2\}$ gives the sum of all the permutations that are 2-cycles, that is $-\frac{1}{2}k(k-1)$ (a minus sign comes from anti-symmetry). All in all we get

$$C\{3\}(k) = \frac{1}{6}d_{abc}t^a t^b t^c$$

$$= k(k-1)(k-2) - \frac{3}{2}k(k-1) + \frac{3}{2}Nk(k-1) + \frac{1}{2}k + \frac{1}{2}N^2k + \frac{3}{N}k(k-1)(k-2) - 3k(k-1) + \frac{6}{N}k(k-1) - 3k + \frac{2}{N^2}k(k-1) + \frac{6}{N^2}k(k-1) + \frac{2k}{N^2}$$

$$= \frac{k}{2N^2}(N+1)(N+2)(N-k)(N-2k)$$

(30)

In appendix (B) we outline a different (much more lengthy) derivation of the stringy rules of a $tr(B^3)$ insertion, using the string-winding creation-operators formalism of Douglas [21].

4. For $N^{-3}tr(B^4)$ in $U(N)$ we get

$$4N^{-3}\hat{P}\{4\} + 6N^{-2}\hat{P}\{3\} + 6N^{-3}\hat{P}_{(2+1)} + \frac{8}{3}N^{-2}\hat{P}_{(1+1)} + \left(\frac{4}{3}N^{-1} + 6N^{-3}\right)\hat{P}_{(2)} + \left(\frac{1}{6} + \frac{5}{6}N^{-2}\right)\hat{P}_{(1)}$$

The terms that have an extra microscopic handle are

$$6N^{-3}\hat{P}_{(2+1)} + \frac{8}{3}N^{-2}\hat{P}_{(1+1)} + 6N^{-3}\hat{P}_{(2)} + \frac{5}{6}N^{-2}\hat{P}_{(1)}$$

5. For $N^{-4}(tr(B^2))^2$ in $U(N)$ we get

$$24N^{-4}\hat{P}\{3\} + 8N^{-4}\hat{P}_{(2+2)} + 4N^{-3}\hat{P}_{(2+1)}$$

$$+ \frac{16}{3}N^{-3}\hat{P}_{(2)} + (2 + \frac{8}{3}N^{-4})\hat{P}_{(1+1)} + \left(\frac{2}{3}N^{-2} + \frac{1}{3}N^{-4}\right)\hat{P}_{(1)}$$

(31)

and additional terms:

$$8N^{-4}\hat{P}_{(2+2)} + 4N^{-3}(\hat{P}_{(2+1)} + \hat{P}_{(2+1)}) + 2N^{-2}\hat{P}_{(1+1)}$$

that mix the two chiral sectors.

The meaning of the term $2N^{-2}\hat{P}_{(1+1)}$ is a factor of $N^{-2}e^{-2\alpha n^+ n^-}$, where $\alpha$ is the coefficient of the $N^{-4}(tr(B^2))^2$ term in the action. $n^+$ is the number of sheets of positive orientability and $n^-$ is the number of sheets.
of negative orientability for a given map. This extra term came from dividing the two 2-vertex loops that correspond to \((tr(B^2))^2\) into a chiral loop and an anti-chiral loop, in step (2) of section (3.4). In the next section we will interpret it, following [9], as a connecting tube.

We end this section by returning to the interpretation of the powers of \(N\) as microscopic attached surfaces. The consistency of such an interpretation stems from Fig.6 (as a matter of fact it is easier to consider Figs.4a-b, which give rise to Fig.6 by iterating them \(r-1\) times). It is straightforward to show that the reduction depicted in Fig.6 results in an even power of \(N\). Moreover, the reduction always produces a non-positive power of \(N\) which can be re-interpreted as a positive number of attached handles. We recall that contributions to the \(N\) power come from branch points, closed loops with no vertices and the original \(N\)-scaling of the coefficients of the various terms in \(\Phi(B)\). The branch points contribute \(-\sum (i-1)k_i\) (for a partition \(k_1 \cdot 1 + k_2 \cdot 2 + \cdots\) while each closed loop in the diagram (after the above steps have been performed) contributes one power of \(N\). The \(\left(\frac{1}{N}\right)^r\)-scaling of the coefficients in \(\Phi(B)\) for a term of the form \(tr(B^i)\) (the general term will be treated in the next subsection) is equal to \(i-1\) which is the number of vertices in the diagrams associated with step (1) of our construction (see section 3.4) minus 1. Each step in Figs.4a-b reduces the number of vertices by one and either increases or decreases the total number of loops by one. Thus, after each step the \(N\)-power remains the same (if the number of loops is increased by one) or decreases by 2 (in case the number of loops decreases by one).
Fig. 8: Application of the rules for $\text{tr}(B^3)$. The first row is the $U(N)$ contribution and the second and third rows are $SU(N)$ corrections. Diagrams that mix chiral and anti-chiral sectors have not been included.
3.7 Marked points and collapsed surfaces

Apart from the higher ordered branch points, it followed from sections 3.5 and 3.6 that the generalized stringy Yang-Mills theory admits another new feature – the marked points.

We can think of the $e^{-\frac{1}{2}n^2\lambda^2}$ factor as arising from the contribution of maps with marked points. A map with $k$ marked points is weighted with the factor of $\left(\frac{4\lambda^2}{k!}\right)^k$ and the $n^k$ factor comes from the fact that the marked point can be at any one of the $n$ sheets. For the ordinary Yang-Mills theory, there were only maps with two marked points at the same target space point (but different sheets) for the group $SU(N)$ and there were no marked points for $U(N)$ \[10\]. In any case, maps with more than two marked points had weight zero. The maps with two marked points were interpreted in \[10, 1\] as maps with microscopic connecting tubes. It turns out that in the generalized theory, maps with multiple marked points at a single target space point do have a nonzero weight. In what follows we show that they can be thought of as being the connecting points of microscopic surfaces of genus greater than one, thus generalizing the connecting tubes of \[10, 1\]. Note that, for $gYM_2$, microscopic surfaces appear both for the $U(N)$ case as well as for the $SU(N)$ case.

The interpretation we gave in the previous section to the term $\hat{P}_{\{\rho\}}$ leads to a problem with locality for the case in which $\rho$ has more than one cycle. For example, a term $\hat{P}_{\{a+b\}}$, corresponds to a point in target space in which there are two branch points, one of degree $(a-1)$ and the other of degree $(b-1)$. This interpretation is local in the target space, but, as it stands, is non-local in the world-sheet, and therefore cannot be interpreted as a contact term. Moreover, among the geometrical objects which we introduced to account for the large $N$ expansion, there is one object that does not have the interpretation of a local covering map i.e. the marked point.

In this section we will show that if the coefficient of the term of the form $\prod_i (tr(B_i))^k_i$ in $\Phi(B)$ scales as $N^{1-\sum k_i(i-1)}$ then we can re-interpret the marked points as microscopic connecting Riemann surfaces. As a matter of fact we will show that the power of $\frac{1}{N^2}$ which stands in front of a general term $\hat{P}_{\{\rho\}}$ is always the one needed to interpret it as a connecting microscopic Riemann surface that connects all the seemingly “world-sheet disconnected” branch-points. This generalizes the notion of a microscopic connecting tube which was first introduced in \[10\] to explain the extra $SU(N)$ terms (in
comparison to the $U(N)$ case that did not have them).

In appendix (A) we will demonstrate that the $\frac{1}{N^2}$ powers that we associate with the various monomials in $\Phi(B)$ is precisely the kind of scaling that one would adopt in the large $N$ ’t-Hooft like analysis of (generalized) $YM_4$ theory. In such an analysis we expect planar diagrams to give the leading $O(N^2)$ contribution. The same scaling behaviour is also the appropriate one for the ’t-Hooft like large $N$ analysis of generalized $QCD_2$. As we show in appendix (A), this scaling behaviour guarantees that planar diagrams with the topology of just one fermion loop are leading at large $N$.

Returning to the microscopic Riemann surfaces, we will demonstrate the interpretation for the case of

$$\Phi(B) = \frac{1}{N^{n-1}} tr(B^n),$$

for simplicity. The general case can be worked out in the same way. Suppose that after the manipulations of Figs.6-7, we end up with a partition

$$\rho = \sum_{i=1}^{k_0} k_i \cdot i \equiv 1 + \cdots + 1 + 2 + \cdots + 2 + \cdots$$

and $k_0$ closed loops with no vertices, that, according to our rules, contribute a factor of $N^{k_0}$. Then it can easily be seen from Figs.6-7 (and even more easily from Figs.4a-b) that

$$\sum_{i=0}^{\infty} k_i (i + 1) \leq n + 1$$

(Indeed, when we split a loop of $r + r' + 1$ vertices into two loops of $r$ vertices and of $r'$ vertices as in Fig.4a the LHS of (32) stays the same, whereas the step depicted in Fig.4b only decreases the LHS of (32) by two.) The factor of $N$ that goes with $\rho$ is $N^{k_0+1-n}$ and we need $N^{-\sum_{i=1}^{k_0} k_i (i-1)}$ for the branch points. Altogether, we are left with a power of

$$k_0 + \sum_{i=1}^{\infty} k_i (i - 1) + 1 - n \leq 2 - 2 \sum_{i=1}^{\infty} k_i$$

for the marked points. If there is just one marked point, with no power of $N$ attached to it, we interpret the coefficient of $\hat{P}_{(1)}$ as a modification to the
cosmological constant. For a single marked point but with additional branch points such as
\[ \hat{P}_{\{1+2+\cdots+2+\cdots\}} \]
we see from (33) that there will be, at least, a power of \( \frac{1}{N^2} \) attached to it, since \( \sum_{i=0}^{\infty} k_i \geq 2 \). Similarly, we see from (33) that a term of the form
\[ \hat{P}_{\{1+\cdots+1\}} \]
will come with a factor of at least \( \frac{1}{N^2} \). Thus, it is always possible to interpret such a term, not as \( k_1 \) marked points, but as a connecting Riemann surface of genus \( k_1 - 1 \) that connects \( k_1 \) sheets as in Fig.9.

Moreover, our problem of non-locality in the world sheet is solved, because a general term
\[ \hat{P}_{\{1+\cdots+1+2+\cdots+2+\cdots\}} \]
has \( \sum_{i=1}^{\infty} k_i \) different points in the world sheet which are the same point on the target space (branch points or marked points). We see from (33) that such a term always has a factor of at least
\[ \frac{1}{N^2 \sum_{i=1}^{\infty} k_i - 2} \]
in front of it. This is precisely the factor that is needed for a Riemann surface of \( \sum_{i=1}^{\infty} k_i \) handles that connects the \( \sum_{i=1}^{\infty} k_i \) distinct points. Therefore, we can interpret the term as a microscopic collapsed Riemann surface that connects \( \sum_{i=1}^{\infty} k_i \) points and \( \sum_{i=2}^{\infty} k_i \) of those points are themselves branch points of various degrees.

Fig. 9: A connecting Riemann surface of genus two.
4 Summary and Discussion

In this paper we studied the generalized two-dimensional Yang-Mills theory. A generalization of the exact formulae for the partition function [8, 9] and Wilson loop averages [6, 7] of the conventional YM theory were written down. These expressions are based on a replacement of the second Casimir operator with more general Casimir operators depending on the particular model. Our results agree with [9], where they were obtained by a different method, i.e. by regarding the general Yang-Mills actions as perturbations of the topological theory at zero area. Actually, the result of [9] differs by a shift of Φ, due to a normal ordering ambiguity in the action. In [9] the ambiguity was fixed by requiring the results of the instanton expansion and the large area expansions to match. We fix the ambiguity differently by a lattice regularization scheme.

Using the relations between $SU(N)$ representations and representations of the symmetric groups $S_n$, we obtained the generalizations that have to be made in the Gross-Taylor string rules for 2D Yang-Mills theory, so as to make the generalized Yang-Mills theory for $SU(N)$ or $U(N)$ a local string theory as well. The extra terms are special weights for certain maps with branch points of a degree higher than one.

An obvious extension of the present work is to consider other gauge groups. The conventional $YM_2$ theory with gauge groups $O(N)$ or $Sp(N)$ was shown [11, 12] to be related to maps from non-orientable world-sheets. A natural conjecture is that the generalized gauge theories are associated with higher order branched maps from those world sheets. Another important topic for further exploration is the issue of the phase transition distinguishing between the small and large areas behaviour, analogous to the one discovered recently in [22, 23, 24]. The coupling of the $gYM_2$ theories to fermionic matter was analyzed in [17] in the framework of ‘t Hooft’s analysis. This domain of research is far from being fully explored. A particular interesting question is to find out certain $Φ(B)$-s that lead to a special behaviour of the coupled system. This topic is under current investigation. In fact, already in the pure gauge case one may anticipate “non-universal” features for particular choices. For instance one may get a singularity in the partition function even without taking the large $N$ limit. For example, in the $U(1)$ case, the representations $R$ are labeled by an integer $n$ and for $Φ(B) = −α \log(1 + λB^2)$ we
get

\[ Z(U(1), A) = \sum_n (1 + \lambda n^2)^{-\alpha A} \]

which has a singularity for \( 2\alpha A = 1 \).

Returning to the case of \( gYM_2 \) with \( \Phi(B) = \sum_k t_k tr(B^k) \). Clearly, upon differentiation with respect to the coefficients \( t_k \), the generalized Yang-Mills theory can be used to calculate correlators of \( tr(B^k) \) operators in \( YM_2 \). This is a more efficient way to calculate such correlators than the “straight-forward” method described in appendix (B). The \( tr(B^k) \) correlators are related to \( tr(F^k) \) correlators in \( YM_2 \). The precise relation is obtained upon integration of the auxiliary \( B \) field. We demonstrate it in appendix (B) by calculating \( \langle tr(B^3) \cdots \rangle \) and comparing it to a “straight-forward” calculation of \( \langle tr(F^3) \cdots \rangle \) in the Gross-Taylor string theory. This will also demonstrate how, within the Gross-Taylor approach, contributions from maps with higher branch points result from maps with “contacts” of lower branch points.

Another interesting fact about the generalized \( YM_2 \) theories is that we can probably get rid of the \( \Omega \)-points of [1] (which were given the interpretation of the Euler character in [15]) by an appropriate choice of \( \Phi(B) \) such that \( \sum \lambda \rho C_\rho(R) \) will give \( (2 - 2G) \log(dim R) \).

One of the most challenging questions is to understand the different generalized \( \Phi \) actions in terms of string theory Lagrangians, such as the ones proposed by [14, 15]. The authors of [15] have suggested that the gravitational descendants of the area operator are related to the higher Casimirs. Indeed, in the algebraic-topological framework of [25] the \( \sigma_n \)-s of the topological gravity, impose certain constraints that pick up contributions from the boundary of the moduli space of complex structures. Those boundary terms come either from pinched world-sheets, or from maps with branch points of higher degrees. Thus, they are related to the \( \hat{P}_\rho \)-s. It is interesting to determine the exact form of the dependence (under preparation).

It seems, however, that there are much more perturbations of the form \( \Phi(B) \), for the \( YM_2 \) theory, than there are perturbations of the topological sigma model coupled to topological gravity. The \( \Phi(B) \) perturbations correspond to \( \hat{P}_\rho \) perturbations with an subindex \( \rho \) that is an arbitrary partition, whereas the index of the gravitational descendants is a single integer that probably corresponds to the degree of the degeneration of the map at a certain point. We suspect that the other perturbations (e.g. \( \langle tr(B^2)^2 \rangle \), whose leading string-theoretic analogue is \( \hat{P}_{(2+2)} \)) correspond to terms that are non-
local in the topological sigma model (e.g. $\int \sigma^{(2)}_{\mathcal{A}} \int \sigma^{(2)}_{\mathcal{A}}$ where $\mathcal{A}$ is the area operator, see [15]). This conjecture stems from the fact that every $\hat{P}_\rho$ operator can be written as a polynomial of the simple $\hat{P}_{\{n\}}$ operators. For example

$$\hat{P}_{\{2\}+2} = \frac{1}{2} \hat{P}_{\{2\}} \hat{P}_{\{2\}} - \frac{1}{2} \hat{P}_{\{3\}} - \frac{1}{2} \hat{P}_{\{1\}} \hat{P}_{\{1\}} + \frac{1}{2} \hat{P}_{\{1\}}$$

since two different branch points that coalesce (the $\hat{P}_{\{2\}} \hat{P}_{\{2\}}$ term) can form either two separate branch points at the same point (the $\hat{P}_{\{2\}+2}$ term) or a branch point of order 2 (the $\hat{P}_{\{3\}}$ term) or marked points (the $\hat{P}_{\{1\}} \hat{P}_{\{1\}} - 1$ term). Our interpretation of the general $\hat{P}_\rho$-s was in terms of collapsed microscopic surfaces, which are too at the boundary of the moduli space of complex structures and holomorphic maps. The gravitational descendant $\sigma_n(A)$ should produce a linear combination of all the possible degenerations of order $n$. (For $n = 1$ we get a combination of branch points and connecting tubes.) So, only one combination of the $\hat{P}_\rho$-s corresponds to the single local operator $\sigma_n(A)$ and the rest are probably non-local. It is interesting to check whether this operator corresponds to $tr(B^{n+1})$ and operators with more traces are non-local.

If indeed $gYM_2$ is a topological theory associated with holomorphic maps along the line of [15], we expect to find for this theory those features which are common to all topological sigma models coupled to topological gravity. In particular we should find the integrable structure [26], recursion relations[27], constraint relations [28] and some contact algebra [29]. It would be interesting to check whether the stringy interpretation advocated in this paper is consistent with such recursion relations. Work in this direction is under progress.

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Appendix A: The $O(\frac{1}{N})$ scalings of the coefficients in $\Phi(B)$

We have seen in section (3.2) that the correct $\frac{1}{N}$ scaling of the coefficient of a term $\prod_i tr(B_i)^{k_i}$ in $\Phi(B)$ should be

$$\frac{1}{N\sum k_i(i+1)-1}$$

This is the only choice that will give a stringy interpretation whose coupling constant is $\frac{1}{N}$. The purpose of this appendix is to show that the same scaling also from the 't-Hooft like large $N$ analysis in terms of Feynmann diagrams. We simply follow the arguments of [30]. It is convenient to rescale the $B$ field to $B = NE$.

We will start with generalized $YM_2$ four dimensions, since in two dimensions there are only global degrees of freedom. The Lagrangian, after rescaling the $B$ field to $B = NE$, is

$$\mathcal{L} = N tr(E^{\mu\nu} F_{\mu\nu}) + N^\alpha \Phi(E)$$

$E^{\mu\nu}$ is an anti-symmetric auxiliary field and $\Phi(E)$ is just one representative term in the general expression given in [3] and which in colour-space has the form $\prod_i tr(E^i)^{k_i}$. The function $\Phi(E)$ is a gauge invariant space-time scalar, built out of the $E^{\mu\nu}$-s. It’s precise space-time form is not relevant for the following analysis (see [17]). When we build Feynman diagrams we have propagators of the form $\langle EAA \rangle$. We will have 3-vertices of the form $\langle EAA \rangle$ and $(\sum_i ik_i)$-vertices of the form $\langle EEE \cdots E \rangle$. According to our rescaled Lagrangian, the $\langle EAA \rangle$ vertices each carry a factor of $N$. Using the usual double-line notation for fields in the adjoint representation, each closed loop gives rise to a factor of $N$. A straightforward analysis of Feynman diagrams reveals that in order to reproduce Euler’s formula:

$$2 - 2g = \text{Vertices} + \text{Faces} - \text{Edges}$$

we should take for the $\Phi(E)$ vertex in the Lagrangian

$$\alpha = 2 - \sum_i k_i = 2 - \#\{\text{traces in the expression for the vertex}\}$$

(34)
As we will see shortly, this choice guarantees that the leading $N$ behaviour will be of order $O(N^2)$, coming from planar diagrams. The $N$-dependence analysis is particularly simple for the case $\Phi(E) = tr(E^i)$ i.e. $\sum_i k_i = 1$ and, therefore, $\alpha = 1$. By drawing some simple diagrams it is easy to be convinced that Euler’s formula holds and the leading $O(N^2)$ contributions come from all the planar diagrams. When we have a more complicated vertex, with several traces, the situation is more involved. In this case, a Feynman diagram which is connected in space-time may turn out to have contributions which are disconnected in colour space. The colour disconnected contributions have relatively more closed loops and, therefore, more $N$ powers. Graphically, a given space time vertex of the form $\prod_i (tr(E^i))^{k_i}$ splits into $\sum_i k_i$ distinguished “colour-vertices” all acting at the same space time point. An example is depicted in Fig.A1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figA1.png}
\caption{Examples of various “colour-vertices”.}
\end{figure}

Let us consider a connected space-time diagram in which $q$ vertices associated with $\Phi(E)$ appear. It is easy to see that in colour space this diagram can split into at most $q \sum k_i - (q - 1)$ distinguished disconnected colour diagrams. If each disconnected colour part is a planar diagram and if one power of $N$ would be associated with each colour-vertex, then, just as in the simple one trace case, we would get a contribution of 2 from each of the disconnected planar colour diagrams to the Euler character. The total contribution would
be:
\[ 2(q \sum k_i - (q - 1)). \]

We have assigned a power \( \alpha \) to the \( N \) behaviour of \( \Phi(E) \). Thus, the \( N \) power would be instead
\[ 2(q \sum k_i - (q - 1)) + q(\alpha - \sum k_i). \]
(The second term accounts for the fact that each \( \Phi(E) \) vertex contributes \( \alpha \) and not \( \sum_i k_i \)). We choose \( \alpha \) by the requirement that the maximal total \( N \) power would not exceed 2, in accordance with Euler’s formula. From the last equation we read that this constraint leads to
\[ \alpha = 2 - \sum k_i \]  
(35)

This is the value of \( \alpha \) which we have used in section (4) and which guarantees the stringy interpretation.

The same \( \alpha \) scaling, associated with the \( \Phi(E) \) vertex, arises also in the ’t-Hooft like analysis in two dimensions. Since in 2D there are no propagating gauge degrees of freedom, we shall consider correlators of Wilson loops. Wilson are presented diagrammatically as closed solid quark lines. Rather than consider the general \( \Phi(E) \) term, we’ll take as an example \( \Phi(E) \sim N^{\alpha} tr(E^4) \).

In Fig.A2 we drew a Feynman graph contributing to the expectation value of a single Wilson loop.
For a stringy behaviour we want this diagram to behave as $O(N)$. This implies $\alpha = 1$, i.e. $\Phi(E) \sim N\text{tr}(E^4)$ (or equivalently $\Phi(B) \sim N^{-3}\text{tr}(B^4)$). For $\alpha = 1$ the leading $O(N)$ contributions are associated with all planar diagrams. (For $\alpha > 1$ it is straightforward to check that as we insert more and more $\text{tr}(E^4)$ vertices into the connected Wilson loop diagram the $N$ power will grow. For $\alpha < 1$ we will not reproduce the $O(N)$ behaviour expected of planar diagrams).

Next, we consider a $(\text{tr}(B^2))^2$ vertex. As we noted in the 4D case, space-time terms which are made of several colour-vertices (i.e. several traces) produce the largest $N$ power for diagrams which are connected in space-time but disconnected in colour-space. In our example, this situation arises for the expectation value of two Wilson loops. (For one Wilson loop the contribution is still colour-connected as depicted in Fig.A3).
We take two Wilson loops $W_1, W_2$ which are disconnected. We wish to calculate the connected part of their correlator, i.e.

$$\langle W_1 W_2 \rangle - \langle W_1 \rangle \langle W_2 \rangle$$

For a stringy behaviour this should behave as $N^0$. The leading contribution comes from diagrams which are colour disconnected as in Fig.A4:
Fig. A4: A Feynman diagram for the connected part of the expectation value of the product of two Wilson loops for a \((\text{tr}(B^2))^2\) interaction using the double-line notation.

It is easily seen that \(\alpha = 0\), i.e. \(\Phi(E) \sim (\text{tr}E)^2\) (or equivalently, \(\Phi(B) \sim \frac{1}{N^2}(\text{tr}(B^2))^2\)) produces the desired behaviour. We can confirm the general result \(\alpha = 2 - \sum_i k_i\) \((\text{B3})\) for a vertex of the form \(N^\alpha \prod_i (\text{tr}E^i)^{k_i}\) by considering a connected part of the expectation value of \(\sum k_i\) Wilson loops.

However, after this assignment of \(\alpha\), not all planar diagrams with one fermion loop, will scale as \(O(N)\). For example, the diagram of Fig.A3 scales as \(O(\frac{1}{N})\) instead of \(O(N)\). In order to restore the usual rule that the \(N\)-power of a diagram scales as the Euler characteristics of the (double-line) graph, we will postulate that a vertex of \(N^{2-\sum_i k_i} \prod_i (\text{tr}E^i)^{k_i}\) has a connecting Riemann surface with \(\sum_i k_i\) handles that connect the \(\sum_i k_i\) traces in \(\prod_i (\text{tr}E^i)^{k_i}\).
Fig. A5: We interpret a $\text{tr}(B^6)\text{tr}(B^4)\text{tr}(B^3)$ as having a connecting Riemann surface with three handles that connects the three trace points.

This also has the advantage that by now there are no disconnected colour diagrams for a diagram that is connected in space.

Appendix B

The purpose of this appendix is to discuss the relation between the operators $\text{tr}(B^k)$ (written in terms of the auxiliary $B$ field) and the operators $\text{tr}(F^k)$ in the pure YM theory.

In the body of the paper we have perturbed the $\int \text{tr}(BF) d^2 x$ action by terms like $t_k \int \text{tr}(B^{k+1}) d^2 x$. where $t_2 \neq 0$ and $t_k = 0$ for $k > 2$ corresponds to $YM_2$. Of course, by choosing infinitesimal coupling constants $t_k$ we can translate the stringy rules into rules for expectation values like:

$$G(x; C_1, C_2, \ldots, C_j) \equiv \langle W(C_1)W(C_2)\cdots W(C_j)\text{tr}(B(x)^{k+1}) \rangle$$

in the pure $YM_2$ theory, where the $W(C_i)$-s are Wilson loops and $x$ is a fixed point which is not on either of the contours of the Wilson loops. The stringy rules would be the same as in [4] for the expectation value of

$$\langle W(C_1)W(C_2)\cdots W(C_j) \rangle$$
with new restrictions that we sum over maps that have specified features in $x$. For example for $tr(B^2)$ for which we had the formula

$$2\hat{P}_{(2)} + 2\hat{P}_{(2)} + N\hat{P}_{(1)} + N\hat{P}_{(1)};$$

the maps should either have a branch point at $x$ and those are counted with a factor of 2, or should have a “marked point” (which is just a way of saying that they can be any map but they get counted with a factor of $p(x)$ – the total number of sheets at $x$) and counted with a factor of $N$.

On the other hand, we can consider expressions like:

$$\mathcal{F}(x; C_1, C_2, \ldots, C_j) \equiv \langle W(C_1)W(C_2) \cdots W(C_j)tr(F(x)^{k+1}) \rangle$$

where $F(x)$ is the $YM$ field strength. The relation between $\mathcal{G}$ and $\mathcal{F}$ is obtained by a simple integration. For example:

$$\int [DB]W(C_1)W(C_2) \cdots W(C_j)tr(B(x)^2)e^{\int_{\Sigma} tr(iBF - 4\lambda tr(B^2))d^2y} =$$

$$-W(C_1)W(C_2) \cdots W(C_j)(\frac{N^2}{4\lambda^2})tr(F(x)^2) + N\Lambda)e^{-\frac{N^2}{4\lambda^2}tr(F^2)d^2y} (36)$$

where $\Lambda$ is a cutoff, of the order of the inverse of the area of a lattice cell (when we work on a lattice). A similar result is obtained for the higher $tr(B^{k+1})$. The leading term will always be

$$\left(\frac{N}{2i\lambda}\right)^{k+1}tr(F(x)^{k+1})$$

and there will be corrections of lower $tr(F(x)^l)$ powers, which arise from (a lack of) normal ordering in the $tr(F(x)^{k+1})$. We will return to this point later on.

In this section, we will calculate the stringy rules of the $tr(F(x)^2)$ and the $tr(F(x)^3)$ insertions directly from the Gross-Taylor string theory by considering a tiny Wilson loop $C_x$ around $x$ in the fundamental representation, and expanding it in the small area $\Delta$ enclosed by $C_x$. The continuum approach gives (for a circular Wilson loop with $x$ its center):

$$Tr(Pe^{i\oint_{C_x} A_a dx^a}) \approx 1 - \frac{1}{2}\Delta^2 Tr(F(x)^2) + \frac{i}{6}\Delta^3 Tr(F(x)^3) + \ldots (37)$$

We used the fact that the matrices are in $SU(N)$ and their trace vanishes.
An Infinitesimal Simple Wilson Loop

Consider the expectation value of some operators (composed of Wilson loops) \( \langle \Phi \rangle \) and then the expectation value of \( \langle W_x \Phi \rangle \), where \( W_x \) is an infinitesimal Wilson loop surrounding the point \( x \). We can calculate this expectation value first in the string picture and then in the QCD\(_2\) picture.

We have to consider all maps with boundaries corresponding to \( \Phi \) and boundaries corresponding to \( W_x \). Let \( \Delta \) be the area of \( W_x \) which is infinitesimal. We will expand to order \( O(\Delta^2) \). According to [4] we have to put one \( \Omega \)-point in \( x \) and one \( \Omega^{-1} \)-point outside \( W_x \) (which we will assume is very close to \( W_x \). The maps fall into two classes. The first class consists of maps which are generated from the maps which calculate \( \langle \Phi \rangle \) by adding a small disk \( D \) whose boundary is \( W_x \). The disk can then be connected to the other sheets by branch points and connecting tubes which may or may not be at the \( \Omega \)-point. The second class is made up by maps that are generated from maps of \( \langle \Phi \rangle \) by cutting out a disk \( D \) from one of the sheets.

Let \( n \) and \( \tilde{n} \) be the operators of the cover numbers at the point \( x \) for a certain map that enters in the calculation of \( \langle \Phi \rangle \). For two permutations \( \sigma \in S_n \) and \( \tau \in S_{\tilde{n}} \) we define \( \langle (\sigma \otimes \tau)\Phi \rangle \) to be the sum over all maps the define \( \Phi \) according to the rules of [4] but with a slight change that at \( x \) there is a twist of \( \sigma \otimes \tau \) (and no \( \Omega \)-point at \( x \) since we calculate just \( \langle \Phi \rangle \)).

By the definition of [4], an \( \Omega \)-point is:

\[
\Omega_{n\tilde{n}} = \sum_{\sigma, \tau} \sigma \otimes \tau N^{K_\sigma + K_\tau} \prod_l V_{\sigma(l), \tau(l)} \left( \frac{l}{N^2} \right)
\]  

(38)

where

\[
V_{a,b}(x) = \sum_{v} (-1)^v v! \binom{a}{v} \binom{b}{v} x^v
\]

(39)

\( \sigma(l) \) is the number of \( l \)-cycles in \( \sigma \), and \( K_\sigma \) is the total number of cycles \( \sum_l \sigma(l) \). We will also use the notation of [4]:

\[
V_{\sigma,\tau}(\frac{1}{N}) = \prod_l V_{\sigma(l), \tau(l)} \left( \frac{l}{N^2} \right)
\]

(40)

Let us also denote by \( \rho_{ij} \) the permutation that switches \( i \) and \( j \).

We wish to expand \( \langle W_x \Phi \rangle \) as a power series in \( \Delta \) with coefficients of the form \( \langle C_n^{\sigma,\tau} \sigma \otimes \tau \Phi \rangle \).
From (41) (eq. 2.24) we see that we can treat connecting tubes and contracted handles as an effective change of the factor
\[
e^{-\frac{1}{2}(n-\tilde{n})^2\lambda A - \frac{1}{2}(\tilde{n}-n)^2\lambda A - \frac{n\tilde{n}}{N}\lambda A}
\]
into
\[
e^{-\frac{1}{2}(p-\tilde{p})^2\lambda A}
\]
with \( p = n + \tilde{n} \) and \( \tilde{p} = n - \tilde{n} \).

Let the \( \Omega \)-point be at \( x \) and the \( \Omega^{-1} \)-point be at \( y \) outside \( W_x \).

The maps of the first class with an \( \Omega \)-point at \( x \) and \( r \) branch points inside \( W_x \) and away from \( x \), and an \( \Omega^{-1} \)-point at \( y \) contribute
\[
\frac{(-\lambda\Delta)^r}{r!} N^{1-r} e^{-\frac{1}{2}\lambda\Delta} \langle e^{\frac{\lambda(p+\tilde{p})\Delta}{N^2}} \sum_{1 \leq i_1, i_2, \ldots, i_r \leq n / \sigma, \tau} (\sigma \otimes \tau) N^{K_{\rho_{i_1}(n+1)} \cdots \rho_{i_r}(n+1)\sigma + K_{\tau} - (p+1)\nu_{i_1(1)} \cdots \nu_{i_r(1)}\sigma, \tau} \Omega^{-1}(y)\Phi \rangle
\]
(42)

(\( \sigma \otimes \tau \)) is a permutation that a “distant observer” sees on \( W_x \), since for him \( W_x \) looks like a single point \( x \). The factor \( N^{1-r} \) is the contribution of \( r \) branch points and a disk to the genus. In the formula for \( K_\sigma \) and \( V_\tau \) is considered as a permutation in \( S_{n+1} \).

The maps of the second class contribute
\[
e^{\frac{1}{2}\lambda\Delta} \langle e^{\frac{\lambda(p+\tilde{p})\Delta}{N^2}} \sum_{\sigma, \tau} (\sigma \otimes \tau) N^{K_\sigma + K_{\tau} - p} V_{\sigma(1), \tau(1)} -1(N^2) \prod_{l=2}^{\infty} V_{\sigma(l), \tau(l)} \left( \frac{l}{N^2} \right) \Omega^{-1}(y)\Phi \rangle.
\]
(43)

However, the terms \( \langle (\sigma \otimes \tau) \Omega^{-1}(y)\Phi \rangle \) in (43) include also maps with a branch point inside \( W_x \) which connects the supposedly cut out sheet with some other sheet. Such maps have to be subtracted from (43), since we cannot cut away a sheet which is connected to something else. But then, we would have subtracted the contribution of maps with two branch points twice (once for each branch point) so we have to add the contribution of maps with two branch points, and so on. We obtain an “inclusion-exclusion” formula in which the contribution of \( r \) branch points is:
\[
\frac{(\lambda\Delta)^r}{r!} N^{1-r} e^{\frac{1}{2}\lambda\Delta} \langle e^{\frac{\lambda(p+\tilde{p})\Delta}{N^2}} \sum_{1 \leq j_1, j_2, \ldots, j_r \leq n / \sigma, \tau} (\sigma \otimes \rho_{i_j} \rho_{i_{j_2}} \cdots \rho_{i_j} \tau) \Omega^{-1}(y)\Phi \rangle.
\]
\[ N^{K_\sigma + (K_\tau - 1) - (p-1)} V_{\sigma^{(1)}, \tau^{(1)}}^{-1} \left( \prod_{l=2}^{\infty} V_{\sigma^{(l)}, \tau^{(l)}} \left( \frac{l}{N^2} \right) \Omega^{-1}(y) \Phi \right) \]

(44)

where \( i \) is one of the \( \tau^{(1)} \) 1-cycles of \( \tau \)

**Formulae For \( V \)**

We will obtain some formulae concerning \( V \). Following [21], we introduce the creation and annihilation operators for the winding states of the string in both orientabilities. For any function \( \psi(\sigma^{(1)}, \tau^{(1)}, \sigma^{(2)}, \tau^{(2)}, \ldots) \) we define:

\[
(a^k_l \psi)(\sigma^{(1)}, \tau^{(1)}, \sigma^{(2)}, \tau^{(2)}, \ldots) \quad \text{def} \quad \psi(\sigma^{(1)}, \tau^{(1)}, \sigma^{(2)}, \tau^{(2)}, \ldots, \sigma^{(l-1)}, \tau^{(l-1)}, \sigma^{(l)} + k, \tau^{(l)}, \ldots)
\]

\[
(\bar{a}^k_l \psi)(\sigma^{(1)}, \tau^{(1)}, \sigma^{(2)}, \tau^{(2)}, \ldots) \quad \text{def} \quad \psi(\sigma^{(1)}, \tau^{(1)}, \sigma^{(2)}, \tau^{(2)}, \ldots, \sigma^{(l)}, \tau^{(l)} + k, \sigma^{(l+1)}, \tau^{(l+1)}, \ldots)
\]

(45)

From the definition of \( V \) we easily obtain the equations:

\[
l_l \sigma^{(l)} a^{-1}_l V + N^2 \bar{a}_l V = N^2 V
\]

(46)

\[
l_l \tau^{(l)} \bar{a}^{-1}_l V + N^2 a_l V = N^2 V
\]

(47)

Multiplying (46) by \( a_l \) and (47) by \( \bar{a}_l \) and subtracting we get

\[
(l \sigma^{(l)} - l \tau^{(l)}) V = N^2 (a_l - \bar{a}_l) V
\]

(48)

Denoting \( \bar{p}^{(l)} = l \sigma^{(l)} - l \tau^{(l)} \) We obtain:

\[
l_l \sigma^{(l)} a^{-1}_l V = (N^2 + \bar{p}^{(l)} - N^2 a_l) V
\]

(49)

\[
\bar{a}_l V = (a_l + \frac{1}{N^2 \bar{p}^{(l)}}) V
\]

(50)

\[
l_l \tau^{(l)} \bar{a}^{-1}_l V = N^2 (1 - a_l) V
\]

(51)

We will denote:

\[
a^+_l \quad \text{def} \quad l \sigma^{(l)} a^{-1}_l
\]

(52)

\[
\bar{a}^+_l \quad \text{def} \quad l \tau^{(l)} \bar{a}^{-1}_l
\]

(53)

---

3The equations are the same as (21) of [21].
The multiplication of $T$ and $D$ operators is defined so that $a_l a_l^\dagger V = l(\sigma^{(l)} + 1)V$ and $a_l^\dagger a_l = l\sigma^{(l)} V$. Also $a_l^2 l^2 V = l^2 \sigma^{(l)} (\sigma^{(l)} - 1) a_l^{-2} V$.

We can use the rules (49-51) to express complicated multiples of the operators $a_l, \bar{a}_l, a_l^\dagger, \bar{a}_l^\dagger$ in terms of the $a_l$-s only (or, as will be more convenient later, in terms of the $a_l^\dagger$-s). In this procedure we have to take care of non-commuting operators (such as $a_l^\dagger$ and $a_l$ or $a_l$ and $\bar{p}^{(l)}$ etc). For example:

$$a_l^2 l^2 V = l\sigma^{(l)} a_l^{-1} (N^2 - N^2 a_l + \bar{p}^{(l)}) V = N^2 l\sigma^{(l)} a_l^{-1} V - N^2 l\sigma^{(l)} V + (\bar{p}^{(l)} - l) l\sigma^{(l)} a_l^{-1} V = (N^2 + \bar{p}^{(l)} - l)(N^2 + \bar{p}^{(l)} - N^2 a_l)V - N^2 l\sigma^{(l)} V$$

We shall need two more relations which we shall now derive. Using (53) and (47)

$$\tilde{n} V = \sum_{l=1}^{\infty} lr^{(l)} \bar{a}_l^{-1} a_l V = \sum_{l=1}^{\infty} \bar{a}_l^\dagger a_l V = \sum_{l=1}^{\infty} \bar{a}_l^\dagger (1 - \frac{1}{N^2 a_l^\dagger}) V$$  \hspace{1cm} (54)

and a similar equation for the barred sector. Thus

$$\sum_{l=1}^{\infty} \bar{a}_l^\dagger a_l^\dagger V = N^2 \sum_{l=1}^{\infty} \bar{a}_l^\dagger V - N^2 \tilde{n} V$$  \hspace{1cm} (55)

$$\sum_{l=1}^{\infty} \bar{a}_l^\dagger a_l^\dagger V = N^2 \sum_{l=1}^{\infty} a_l^\dagger V - N^2 n V$$  \hspace{1cm} (56)

Adding and subtracting we get

$$\sum_{l=1}^{\infty} \bar{a}_l^\dagger a_l^\dagger V = \frac{1}{2} N^2 \sum_{l=1}^{\infty} (a_l^\dagger + \bar{a}_l^\dagger) V - \frac{1}{2} N^2 p V$$  \hspace{1cm} (57)

$$\sum_{l=1}^{\infty} (a_l^\dagger - \bar{a}_l^\dagger) = \bar{p}$$  \hspace{1cm} (58)

The String Operators

We wish to obtain formulae for the insertion of permutations in the string picture. Working in the algebra of permutations we get

$$\sum_{i,j} \rho_{ij} \otimes 1 = \sum_{i,j} (\rho_{ij} \otimes 1) \Omega(x) \Omega^{-1}(x)$$
The structure of the cycles of $\sigma$ is changed by the multiplication of $\rho_{ij}$ according to:

$$(ij) \circ (i_1i_2\ldots i_{l-1}i)(j_1j_2\ldots j_{\nu-1}j) = (i_1i_2\ldots i_{l-1}i_1j_1j_2\ldots j_{\nu-1}j)$$  \hspace{1cm} (59)

$$(ij) \circ (i_1i_2\ldots i_{l-1}i_1j_1j_2\ldots j_{\nu-1}j) = (i_1i_2\ldots i_{l-1}i)(j_1j_2\ldots j_{\nu-1}j)$$  \hspace{1cm} (60)

Writing symbolically

$$\hat{S} \stackrel{\text{def}}{=} \sum_{\sigma,\tau} \sigma \otimes \tau N^{K_\sigma + K_\tau} \hat{S} V_{\sigma,\tau} \Omega^{-1}$$  \hspace{1cm} (62)

we obtain the relations

$$\sum_{ij} \rho_{ij} \otimes 1 = \sum_{l,l'} \left( \frac{1}{N} a_l^\dagger a_{l'}^\dagger a_{l+l'} + Na_{l+l'}^\dagger a_l a_{l'} \right)$$  \hspace{1cm} (63)

$$\sum_{ij} 1 \otimes \rho_{ij} = \sum_{l,l'} \left( \frac{1}{N} \tilde{a}_l^\dagger \tilde{a}_{l'}^\dagger \tilde{a}_{l+l'} + N\tilde{a}_{l+l'}^\dagger \tilde{a}_l \tilde{a}_{l'} \right)$$  \hspace{1cm} (64)

Finally, using (57) we obtain

$$\sum_{ij} (\rho_{ij} \otimes 1 + 1 \otimes \rho_{ij}) = \frac{1}{N} \sum_{l,l'} (a_l^\dagger a_{l'}^\dagger + \tilde{a}_l^\dagger \tilde{a}_{l'}^\dagger - 2\tilde{a}_{l'}^\dagger a_l^\dagger) + N \sum_l l(\tilde{a}_l^\dagger + a_l^\dagger) - Np$$  \hspace{1cm} (65)

The Zero Branch Points Contribution

According to (42) and (44) for $r = 0$ we get

$$Ne^{-\frac{1}{2}\lambda \Delta} \langle e^{-\frac{\lambda(\beta+\frac{1}{2})\Delta}{N^2}} \sum_{\sigma,\tau} (\sigma \otimes \tau) N^{K_\sigma + K_\tau - p} a_1 V \Omega^{-1} (y) \Phi \rangle$$

$$+ \frac{1}{N} e^{\frac{1}{2}\lambda \Delta} \langle e^{-\frac{\lambda(\beta+\frac{1}{2})\Delta}{N^2}} \sum_{\sigma,\tau} (\sigma \otimes \tau) N^{K_\sigma + K_\tau - p} \tilde{a}_1 V \Omega^{-1} (y) \Phi \rangle$$  \hspace{1cm} (66)
which according to (49-51) gives
\[
-2N \sinh \frac{1}{2} \lambda \Delta \left\langle \left( e^{\frac{\lambda (p+\frac{1}{2}) \Delta}{N^2}} \sum_{\sigma,\tau} (\sigma \otimes \tau) N^{K_\sigma + K_\tau - p} a_1 V \Omega^{-1}(y) \Phi \right) \right\rangle
+ Ne^{\frac{1}{2} \lambda \Delta} \left\langle \left( e^{\frac{\lambda (p+\frac{1}{2}) \Delta}{N^2}} \Phi \right) \right\rangle
\]

\text{(67)}

The One Branch Point Contribution

From (42) and (44) for \( r = 1 \) we get
\[
-\frac{\lambda \Delta}{N} e^{-\frac{1}{2} \lambda \Delta} \left\langle \left( e^{\frac{\lambda (p+\frac{1}{2}) \Delta}{N^2}} \sum_{\sigma,\tau} \sigma \otimes \tau N^{K_\sigma + K_\tau - p} \sum_{l=1}^{\infty} \sigma(l) a_l^{-1} a_{l+1} V \Omega^{-1}(y) \Phi \right) \right\rangle
+ \frac{\lambda \Delta}{N} e^{\frac{1}{2} \lambda \Delta} \left\langle \left( e^{\frac{\lambda (p+\frac{1}{2}) \Delta}{N^2}} \sum_{\sigma,\tau} \sigma \otimes \tau N^{K_\sigma + K_\tau - p} \sum_{l=2}^{\infty} \bar{\sigma}(l) \bar{a}_l^{-1} \bar{a}_{l-1} V \Omega^{-1}(y) \Phi \right) \right\rangle
\]

\text{(68)}

We used the fact that \( \rho_i(n+1) \) turns one \( l \)-cycle of \( \sigma \) into an \( (l + 1) \)-cycle.

Using the notation of (52-53) we write it as
\[
-\frac{\lambda \Delta}{N} e^{-\frac{1}{2} \lambda \Delta} \left\langle \left( e^{\frac{\lambda (p+\frac{1}{2}) \Delta}{N^2}} \sum_{\sigma,\tau} \sigma \otimes \tau N^{K_\sigma + K_\tau - p} \sum_{l=1}^{\infty} \sigma(l) a_l^{-1} a_{l+1} V \Omega^{-1}(y) \Phi \right) \right\rangle
+ \frac{\lambda \Delta}{N} e^{\frac{1}{2} \lambda \Delta} \left\langle \left( e^{\frac{\lambda (p+\frac{1}{2}) \Delta}{N^2}} \sum_{\sigma,\tau} \sigma \otimes \tau N^{K_\sigma + K_\tau - p} \sum_{l=2}^{\infty} \bar{\sigma}(l) \bar{a}_l^{-1} \bar{a}_{l-1} V \Omega^{-1}(y) \Phi \right) \right\rangle
\]

\text{(69)}

The Two Branch Points Contribution

In (42) we have to find the cycle structure of
\[
\rho_{i_1(n+1)} \rho_{i_2(n+1)} \sigma
= \begin{cases} 
(i_1 i_2 (n+1)) \sigma & \text{for } i_1 \neq i_2 \\
\sigma & \text{for } i_1 = i_2
\end{cases}
\]

\text{(70)}

(and a similar expression for (44)) the 3-cycle term \( (i_1 i_2 (n+1)) \) can do one of two things:

- turn an \( l \)-cycle and an \( l' \)-cycle into an \( l + l' + 1 \) cycle:
  \[
  (i_1 j_1 (n+1)) \circ (i_1 i_2 \ldots i_{l-1}) (j_1 j_2 \ldots j_{l'-1} j)(n+1) = (i_1 i_2 \ldots i_{l-1} i j_1 j_2 \ldots j_{l'-1} j)(n+1)
  \]
  \text{(71)}
• turn one \( l + l' \) cycle into two cycles, and \((l + 1)\)-cycle and an \(l'\)-cycle:

\[
(ij(n+1)) \circ (i_1i_2 \ldots i_{l-1}ij_2 \ldots j_{l' - 1}j)(n+1) = (i_1i_2 \ldots i_{l-1}(n+1))(j_1j_2 \ldots j_{l' - 1}j)
\]

(72)

Thus, we obtain for two branch points the contribution of

\[
\begin{align*}
\left(\frac{\lambda\Delta}{2!N^2}\right)^2 e^{-\frac{1}{2} \lambda \Delta \left( e^{\frac{\lambda\Delta}{N^2}} \right) } & \sum_{\sigma,\tau} \tau N K_\sigma + K_\tau - p \{ Nn_{a_1} V \\
+ & N \sum_{l,l'} a_{l+1}^\dagger a_{l+1} V + \frac{1}{N} \sum_{l,l'} a_{l+1}^\dagger a_{l+1} V \Omega^{-1}(y) \Phi \\
+ & \left(\frac{\lambda\Delta}{2!N^2}\right)^2 e^{-\frac{1}{2} \lambda \Delta \left( e^{\frac{\lambda\Delta}{N^2}} \right) } \sum_{\sigma,\tau} \tau N K_\sigma + K_\tau - p \{ \frac{1}{N}(\bar{n} - 1)\bar{a}_{l}^\dagger V \\
+ & N \sum_{l,l'} \bar{a}_{l}^\dagger a_{l+1} V + \frac{1}{N} \sum_{l,l'} \bar{a}_{l+1} a_{l+1} V \Omega^{-1}(y) \Phi \\
\end{align*}
\]

(73)

The Three Branch Points Contribution

Similarly, we obtain for three branch points the contribution of

\[
\begin{align*}
-\left(\frac{\lambda\Delta}{3!N^3}\right)^3 e^{-\frac{1}{2} \lambda \Delta \left( e^{\frac{\lambda\Delta}{N^2}} \right) } & \sum_{\sigma,\tau} \tau N K_\sigma + K_\tau - p \{ \frac{1}{N^2} \sum_{l,l',l''} a_{l+1}^\dagger a_{l+1} V \\
+ & \sum_{l,l',l''} a_{l+1}^\dagger a_{l+1} V a_{l+1} V + \sum_{l,l',l''} a_{l+1}^\dagger a_{l+1} V a_{l+1} V + \sum_{l,l',l''} a_{l+1} a_{l+1} V a_{l+1} V \\
+ & \sum_{l,l',l''} a_{l+1} a_{l+1} V a_{l+1} V + \frac{1}{N} \sum_{l,l'} a_{l+1} a_{l+1} V \Omega^{-1}(y) \Phi \\
+ & \left(\frac{\lambda\Delta}{3!N^3}\right)^3 e^{-\frac{1}{2} \lambda \Delta \left( e^{\frac{\lambda\Delta}{N^2}} \right) } \sum_{\sigma,\tau} \tau N K_\sigma + K_\tau - p \{ \frac{1}{N^2} \sum_{l,l',l''} \bar{a}_{l+1} V a_{l+1} V \\
+ & \sum_{l,l',l''} \bar{a}_{l+1} V a_{l+1} V + \sum_{l,l',l''} \bar{a}_{l+1} V a_{l+1} V + \sum_{l,l',l''} \bar{a}_{l+1} V a_{l+1} V \\
+ & \sum_{l,l',l''} \bar{a}_{l+1} V a_{l+1} V + \frac{1}{N^2} \sum_{l,l',l''} \bar{a}_{l+1} V a_{l+1} V + N \sum_{l,l'} \bar{a}_{l+1} V a_{l+1} V \\
\end{align*}
\]

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\[ + \frac{1}{N^2} \sum_{l,l'} \bar{a}^\dagger_l \bar{a}^\dagger_{l'} \bar{a}_{l+1} \bar{a}_{l'} V + (2N - 3) \sum_{l} \bar{a}^\dagger_{l+1} \bar{a}_{l} V \} \Omega^{-1}(y) \Phi \] 

(74)

We will calculate the contributions up to \(0(\Delta^3)\)

**The 0(\Delta) Order**

It is convenient to express all the \(a_l\)-s and \(\bar{a}_l\)-s in the sums in terms of \(a^\dagger_l\)-s and \(\bar{a}^\dagger_l\)-s since \(\lim_{l \to \infty} a^\dagger_l V = 0\) and likewise for \(\bar{a}^\dagger_l\). We obtain

\[ \frac{N}{2} (1 - \frac{1}{N^2}) \] 

(75)

**The 0(\Delta^2) Order**

Again it is convenient to express all the \(a_l\)-s and \(\bar{a}_l\)-s in the sums in terms of \(a^\dagger_l\)-s and \(\bar{a}^\dagger_l\)-s since \(\lim_{l \to \infty} a^\dagger_l V = 0\) and likewise for \(\bar{a}^\dagger_l\). We obtain

\[ (\lambda \Delta)^2 \{ \sum_{\sigma,\tau} \sigma \otimes \tau N^{K_\sigma + K_\tau - p} R^{(2)}_{\sigma\tau} \Omega^{-1}(y) \Phi \} \] 

(76)

Using (55) (56) (58) as well as

\[ \bar{a}^\dagger_l \bar{p} = \bar{a}^\dagger_l \sum_{l} l(\sigma^{(l)} - \tau^{(l)}) = (\bar{p} + 1) \bar{a}^\dagger_l \] 

(77)

we obtain

\[ \frac{1}{2N} p^2 - \frac{1}{2N^2} p^2 + \frac{N}{2} (\frac{1}{2} - \frac{1}{N^2}) \}^{2} + \frac{1}{2N^2} \sum_{ij} (\rho_{ij} \otimes 1 + 1 \otimes \rho_{ij}) \] 

(78)

**Comparison with the previous results**

The result we have obtained from the previous sections can be written symbolically as

\[ Tr Pe^{i \gamma, \Delta^a dx_a} = \] 

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\[ N\{1 + \lambda \Delta \left(1 - \frac{1}{N^2}\right) + \frac{1}{2}(\lambda \Delta)^2\left\{\frac{1}{2}(1 - \frac{1}{N^2})\right\}^2\} \]
\[ + (\lambda \Delta)^2\left\{\frac{1}{2N}p - \frac{1}{2N^3}b^2 + \frac{1}{2N^2}\sum_{ij}(\rho_{ij} \otimes 1 + 1 \otimes \rho_{ij})\right\} + 0(\Delta^3) \] (79)

By the way, note that, in our previous notation:

\[ \hat{P}_{\{2\}} = \sum_{ij}(\rho_{ij} \otimes 1) \]
\[ \hat{P}_{\{\bar{2}\}} = \sum_{ij}(1 \otimes \rho_{ij}) \]

We can write (79) as

\[ e^{-\frac{1}{2}\lambda \Delta C_2(F)}Tr_F Pe^{i\int \gamma A^\alpha dx^\alpha} = \]
\[ +(\lambda \Delta)^2\left\{\frac{1}{2N}p - \frac{1}{2N^3}b^2 + \frac{1}{2N^2}\sum_{ij}(\rho_{ij} \otimes 1 + 1 \otimes \rho_{ij})\right\} + 0(\Delta^3) \] (80)

where \( C_2(F) = 1 - \frac{1}{N^2} \) is the second Casimir of the fundamental representation. However, here is a subtle point. If we expand the Wilson loop in powers of \( \Delta \) we get

\[ Tr_F Pe^{i\int \gamma A^\alpha dx^\alpha} = N - \frac{1}{2}\Delta^2Tr(f^2) + 0(\Delta^3) \] (81)

If we differentiate the result of (4) for the partition function with respect to \( \lambda \) we should get the string picture interpretation of an operator insertion of \( \int Tr(f^2)dx^2 \) and performing the differentiation gives

\[ \frac{1}{2N}p - \frac{1}{2N^3}b^2 + \frac{1}{2N^2}\sum_{ij}(\rho_{ij} \otimes 1 + 1 \otimes \rho_{ij}) \] (82)

The last term, when integrated over an area, gives the number of branch points in that area.

The discrepancy between the two calculations (for example (81) has no 0(\( \Delta \)) term but (80) has one) can be explained as follows: The term that governs the fluctuations of the holonomy \( U \) in the action of a small region of area \( \Delta \) is \( e^{-Tr(f^2)\Delta} \) which means that the fluctuations go like \( \delta f^2 \Delta \sim 1 \) and thus \( U \approx 1 + Tr(f^2)\Delta^2 \sim 1 + 0(\Delta) \). We see that although almost everywhere
the field approximation to $U (U \approx 1 + A_\mu dx^\mu)$ is valid, it may not be valid at single points for which the field strength fluctuations is large (0($\Delta^3$)). We can observe from (80) that a plausible renormalization of $\text{Tr}_FPe^i\oint_\gamma A_\alpha dx^\alpha$ on a loop of area $\Delta$ is $e^{-\frac{1}{2}\lambda\Delta C_2(F)}\text{Tr}_FPe^i\oint_\gamma A_\alpha dx^\alpha$. The renormalizing factor is $e^{-\frac{1}{2}\lambda\Delta C_2(F)}$.

This factor is correct for all orders (and for other representations as well) since for the string vacuum (i.e. $n = \tilde{n} = 0$) the expectation value of a small Wilson loop is $e^{-\frac{1}{2}\lambda\Delta C_2(F)}$. (The string vacuum is projected out by taking the area of the whole target space to infinity).

The $0(\Delta^3)$ Order

Similarly to (83-84) we obtain

$$\sum_{ijk} \rho_{ijk} \otimes 1 = \sum_{l,m,k} \left( \frac{1}{N^2} a^\dagger_l a^\dagger_m a^\dagger_k a_{l+m+k} + N^2 a^\dagger_{l+m+k} a_l a_m a_k 
+ a^\dagger_l a^\dagger_m a_{l+m+k} + a^\dagger_{l+m} a^\dagger_k a_{l+m} + a^\dagger_{l+m} a^\dagger_m a_{l+k} + a^\dagger_{l+m+k} a_{l+m+k} \right)$$

$$\sum_{ijk} 1 \otimes \rho_{ijk} = \sum_{l,m,k} \left( \frac{1}{N^2} \bar{a}^\dagger_l \bar{a}^\dagger_m \bar{a}^\dagger_k \bar{a}_{l+m+k} + N^2 \bar{a}^\dagger_{l+m+k} \bar{a}_l \bar{a}_m \bar{a}_k 
+ \bar{a}^\dagger_{l+m+k} \bar{a}_l \bar{a}_{l+m+k} + \bar{a}^\dagger_{l+m} \bar{a}^\dagger_m \bar{a}_{l+k} \bar{a}_l + \bar{a}^\dagger_{l+m} \bar{a}^\dagger_{l+m+k} \bar{a}_{l+m+k} \right)$$

After a tedious calculation, we get the terms of (26,27,28,29).

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