Testing of quantum nonlocal correlations under constrained free will and imperfect detectors

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In this work, we deal with the relaxation of two central assumptions in standard locally realistic hidden variable (LRHV) inequalities: free will in choosing measurement settings, and the presence of perfect detectors at the measurement devices. Quantum correlations violating LRHV inequalities are called quantum nonlocal correlations. In principle, in an adversarial situation, there could be a hidden variable introducing bias in the selection of measurement settings, but observers with no access to that hidden variable could be unaware of the bias. In practice, however, detectors do not have perfect efficiency. A main focus of this paper is the introduction of the framework in which given a quantum state with nonlocal behavior under constrained free will, we can determine the threshold values of detector parameters (detector inefficiency and dark counts) such that the detectors are robust enough to certify nonlocality. We also introduce a class of LRHV inequalities with constrained free will, and we discuss their implications in the testing of quantum nonlocal correlations.

I. INTRODUCTION

The randomness acts as a primitive resource for various desirable information processing and computational tasks, e.g., cryptography [1–6], statistical sampling [7, 8], probabilistic algorithms [9–11], genetic algorithms [12–14], simulated annealing algorithms [15–17], neural networks [18–20], etc. Generation and certification of private randomness also form major goal of cryptographic protocols [4–6, 21–23]. It is known that the experimental verification of Bell’s inequality requires additional assumptions which could lead to incurring loopholes such as locality loophole [34], freedom-of-choice (measurement independence or free will) loophole [35, 36], fair-sampling loophole (detection loophole) [35, 37]. In recent major breakthroughs [38–41], incompatibility of quantum mechanics with LRHV theories has been demonstrated by considerably loophole-free experiments showing violation of Bell’s inequality by quantum states with quantum nonlocal correlations.

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The free will assumption [46] in the Bell-type inequality states that the parties (users) can choose the measurement settings freely or, use uncorrelated random number generators. In an experimental demonstration presented in [41] human random decisions were used to choose the measurement settings in an attempt to close this loophole. In practice, the detectors are imperfect which introduce dark counts [47] and detection inefficiency [47]. In this paper, we discuss implications of bias in the freedom-of-choice and non-ideal detectors on the LRHV tests of quantum nonlocal correlations. To the best of our knowledge, the assumption of free will was relaxed in [48] using

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A main focus of this work is to study the implications of imperfect detectors and constrained free will on the test of quantum nonlocal correlations. We adapt the approach discussed in [55] to model imperfect detectors for the Bell experiment as a sequential application of a perfectly working inner box followed by a lossy outer box. The inner box contains a quantum source whose behavior is nonlocal under constrained free will, i.e., violates certain measurement dependent LRHV inequality. The outer box separately introduces detector inefficiency and dark counts for each party. Using this model, we determine the threshold values of the detector parameters that make detectors robust for testing of quantum nonlocality under constrained free will (e.g., see Fig. 1 with details in Section IV). Next for the scenario of perfect detectors, we compare the implications two different approaches presented in [51] and [48] to quantify measurement dependence (a) by bounding the probability of choosing the measurement settings $x$ (for Alice’s side) and $y$ (for Bob’s side) conditioned on a hidden variable $\lambda$ to be in the range $[l, 1 - 3\ell]$ [51] and (b) by using a distance measure $M$ to quantify measurement settings distinguishability [48]. This comparison is made in the 2 (party) - 2 (measurement settings per party) - 2 (outcome per measurement) scenario and their effects on the certification of the nonlocality. We also introduce a new set of measurement dependent LRHV (MDL) inequalities by introducing distance-based measurement dependent quantity in adapted AMP tilted Bell inequality [21] and discuss implications and trade-off between measurement dependence parameters and tilted parameters for the certification of quantum nonlocal correlations.

Throughout this paper we limit our discussions to LRHV for bipartite quantum systems. The structure of this paper is as follows. In Section II, we introduce the framework of locality and measurement dependence, which is a constraint that limits the free will of the user. We present a model where the user is tricked by an adversary into thinking that they have freedom-of-choice for the measurement. In Section III, we consider and compare two different approaches to quantify the measurement dependence in terms of parameters $I$ and $M$ mentioned earlier. We determine the critical values of these parameters necessary for the certification of nonlocality in the measurement dependence settings and compare them with amount of violation obtained for the Bell-CHSH inequality, tilted Hardy relations and the tilted Bell inequalities. In Section IV, We determine the threshold values of the detector parameters namely inefficiency and dark count probability such that the detectors are robust enough to certify nonlocality in presence of constrained free will. In Section V, we introduce a new set of LRHV inequalities adapted from the AMP tilted Bell inequality using distance measures based measurement dependence quantities. We use these inequalities to observe the effect of relaxing the free will assumption for either parties on the certification of quantum nonlocal correlations.

II. THE ADVERSERIAL ROLE IN CHOICE OF MEASUREMENT SETTINGS

Formally a quantum behavior of a device with bipartite state $\rho_{AB}$ is given by its quantum representation $\{p(ab|xy)\}$, where $p(ab|xy) = \text{Tr} [\Gamma^x_a \otimes \Gamma^y_b \rho_{AB}]$ with $\{\Gamma^x_a\}$ and $\{\Gamma^y_b\}$ being Positive Operator Valued Measures (POVMs) for each input $(x, y)$ to the device, for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. If a state $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ is pure, then we may simply represent it as a ket $|\psi\rangle_{AB}$ instead of density operator $\rho_{AB}$. Henceforth, the abbreviation MDL inequalities would stand for measurement depen-
dent LRHV inequalities.

We consider the Bell scenario where two parties, Alice and Bob, share a bipartite quantum state $\rho_{AB}$. Each party can choose to perform one of two measurements available to them, i.e., $|x\rangle = 2 = |y\rangle$. We attribute POVM $\{\Lambda^x_x\}_x$ to Alice and $\{\Lambda^y_y\}_y$ to Bob. Each of these measurements can have two outcomes. We denote measurement outcomes for Alice and Bob by $a$ and $b$, respectively. The statistics of the measurement outcomes in the experiment can then be described by the probability distribution $P$.

In an adversarial scenario, Alice and Bob can believe that they are choosing all the settings with equal probability, i.e., $p(xy) = 1/4$ for each pair $(x, y)$ while an adversary biases their choice in the scale $\lambda$. The adversary can distribute the settings chosen by Alice and Bob according to

$$p(xy) = \sum_{\lambda \in \Lambda} p(xy|\lambda)p(\lambda). \tag{2}$$

In Eq. (2), let $\lambda$ take two values, $\lambda_1$ and $\lambda_2$, whose probability distributions are given as $p(\lambda_1) = \sin^2(\theta_1)$ and $p(\lambda_2) = \cos^2(\theta_1)$, respectively. In the simplest scenario $x$ and $y$ can each take values 0 or 1 and there are four possible ways in which the measurement settings can be chosen by Alice and Bob, i.e., $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Let the probability of choosing the measurement setting $(x, y)$ conditioned on the hidden variable be distributed according to Table I. To observe the effect of the conditional probability distribution for choice of $x$ and $y$ from Table I on $p(xy)$, consider the following examples:

(i) For the case of $\theta = 0.6847$, $\phi = 1.2491$, $\phi_1 = 0.8861$, $\delta = 1.2491$, $\theta_1 = 0.785398$, $\theta_1 = 0.50413$, and $\delta_1 = 0.175353$. It can be seen that this choice of parameters set $p(xy) = 0.25 \forall (x, y)$.

(ii) For the case of $\theta = 0.5796$, $\phi = 1.2491$, $\phi_1 = 0.6847$, $\delta = 0.75$, $\theta_1 = 0.5796$, $\theta_1 = 0.793732$, and $\delta_1 = 1.06395$. It can be seen that this choice of parameters set $p(xy) = 0.25 \forall (x, y)$.

The above examples show that by proper choice of parameters in Table I, the adversary can trick Alice and Bob into thinking that they have free will in choosing the measurement settings.

A. Review of MDL inequalities from prior works

We review approach discussed in [51, 52] to quantify measurement dependence by bounding the probability distributions of Eve.

| Joint Setting | Distribution 1 ($\lambda_1$) | Distribution 2 ($\lambda_2$) |
|---------------|------------------------------|------------------------------|
| $p(0, 0|\lambda)$ | $\cos^2(\phi_{s_1})$ | $\cos^2(\phi_{s_2})$ |
| $p(0, 1|\lambda)$ | $\sin^2(\theta_{s_1})\sin^2(\phi_{s_1})$ | $\sin^2(\theta_{s_2})\sin^2(\phi_{s_2})$ |
| $p(1, 0|\lambda)$ | $\cos^2(\delta_{s_1})\cos^2(\theta_{s_1})\sin^2(\phi_{s_1})$ | $\cos^2(\delta_{s_2})\cos^2(\theta_{s_2})\sin^2(\phi_{s_2})$ |
| $p(1, 1|\lambda)$ | $\sin^2(\delta_{s_1})\cos^2(\theta_{s_1})\sin^2(\phi_{s_1})$ | $\sin^2(\delta_{s_2})\cos^2(\theta_{s_2})\sin^2(\phi_{s_2})$ |

TABLE I. Probability distribution for choice of settings by Alice and Bob based on the hidden variable $\lambda$. An adversary can use this distribution and by suitably choosing the parameters of the table trick Alice and Bob into thinking they have free will in choosing the measurement settings.
of choice of measurement settings conditioned on a hidden variable to be in a specific range (Section III A 1). Then we review approach discussed in [48–50] to quantify measurement dependence using a distance measure (Section III A 2).

1. Bound on the probability of choosing the measurement settings

In the works [51, 52], we observe that the probability of Alice and Bob to choose measurement settings $x$ and $y$ conditioned on $\lambda$ can be bounded as

$$l \leq p(xy|\lambda) \leq h,$$  \hspace{1cm} (5)

where $0 \leq l \leq p(xy|\lambda) \leq h \leq 1$. If Alice and Bob each chooses from two possible values of measurement settings, then $l = h = 0.25$ corresponds to the complete measurement independence; other values of $l$ and $h$ represent bias in the choice of measurement settings.

In the 2 (user) - 2 (measurement settings per user) - 2 (outcome per measurement) scenario with $a, b \in \{+, -\}$ and $x, y \in \{0, 1\}$, it was shown in [51, 52] that all the measurement dependent local correlations satisfy the PRBLG MDL inequality

$$lp(+00) - h(p(+01) + p(-10) + p(+11)) \leq 0.$$  \hspace{1cm} (6)

A two-dimensional slice in the non-signalling space is shown in Fig. 2 (figure from [51]). In the figure, the quantum set is bounded by the green line and the set of non-signalling correlations lie within the black triangle. In Fig. 2, the red dotted line corresponds to Eq. (6) with $h = 1 - 3l$. If we set $h = 1 - 3l$, the PRBLG MDL inequality tilts from the Bell-CHSH inequality ($l = 0.25$) to the non-signalling border ($l = 0$). For $h = 1 - 3l$, Eq. (6) is expressed as

$$lp(+00) - (1 - 3l)(p(+01) + p(-10) + p(+11)) \leq 0.$$  \hspace{1cm} (7)

We note that if Alice and Bob believes they have complete measurement dependence, i.e., $p(xy) = 0.25 \forall (x, y)$, then Eq. (7) reduces to

$$lp(+00) - (1 - 3l)(p(+01) + p(-10) + p(+11)) \leq 0.$$  \hspace{1cm} (8)

It follows from [53] that invoking the PRBLG MDL inequality (7) in the tilted Hardy test [53], we obtain the ZRLH MDL inequality. The ZRLH MDL inequality is expressed as

$$lp(+00) + wp(-00) - \max\{0, w\} - (1 - 3l)\{p(+01) + p(-10) + p(+11)\} \leq 0,$$  \hspace{1cm} (9)

where $w$ is the tilting parameter taking real numbers in the range, $w \in (-0.25, 1)$. We call the quantum behaviors that violate Eqs. (8) and (9) as quantum nonlocal in the presence of measurement dependence.

2. Distance measure to quantify measurement distinguishability

We discussed in Section II that the experimental statistics described by the joint probability distribution $p(ab|xy)$ can be explained by $\lambda \in \Lambda$ in the following form,

$$p(ab|xy) = \int d\lambda p(ab|xy, \lambda)p(\lambda|xy).$$  \hspace{1cm} (10)

The assumption of the measurement independence constrains the probability distribution of measurement settings via

$$p(\lambda|xy) = p(\lambda).$$  \hspace{1cm} (11)

Eq. (11) implies that no extra information about $\lambda$ can be obtained from the knowledge of $x$ and $y$. This is equivalent to saying

$$p(xy|\lambda) = \frac{p(\lambda|xy)p(xy)}{p(\lambda)} = p(xy).$$  \hspace{1cm} (12)

Eq. (12) implies that Alice and Bob has complete freedom in choosing the measurement settings $x$ and $y$ respectively. If $x \in U \equiv \{x_1, x_2\}$ and $y \in V \equiv \{y_1, y_2\}$, measurement dependence implies $p(\lambda|x_1, y_1) \neq p(\lambda|x_2, y_2)$. Distinguishability between $p(\lambda|x_1, y_1)$ and $p(\lambda|x_2, y_2)$ can be quantified using a distance measure defined in [48],

$$M = \int d\lambda |p(\lambda|x_1, y_1) - p(\lambda|x_2, y_2)|.$$  \hspace{1cm} (13)
We can express the probability to successfully distinguish \( p(\lambda|x_1, y_1) \) and \( p(\lambda|x_2, y_2) \) based on the knowledge of \( \lambda \) as

\[
F = \frac{1}{2} \left( 1 + \frac{M}{2} \right). \tag{14}
\]

When \( M = 0 \), we have \( F = \frac{1}{2} \), which suggests that no additional information about the hidden variable \( \lambda \) can be obtained from knowing the choice of measurement settings.

We observe in Fig. 3 that for a fixed value of \( \alpha \), the range of the allowed values of \( \phi \) from Table II that violates the PRBLG MDL inequality given by Eq. (8) increases with the increase in \( l \). Also as \( \alpha \) increases, for a particular value of \( l \), there is a decrease in the range of the allowed values of \( \phi \) from Table II that violates the AMP tilted Bell inequality. It was shown in [53] that the state \( \rho_g = |\psi_g\rangle \langle \psi_g| \) (18) and measurement settings \( \{A^g_0, A^g_1, B^g_0, B^g_1\} \) can be used to obtain close to 1.6806 bits of global randomness (at \( \theta \approx 1.13557 \)), where

\[
|\psi_g\rangle = \cos \left( \frac{\theta}{2} \right) |00\rangle - \sin \left( \frac{\theta}{2} \right) |11\rangle, \tag{18}
\]

\[
A^g_0 = B^g_0 = \frac{-\sqrt{2} \sin(\theta) \sin(\theta)}{(2 - \sin(\theta)) \sqrt{\sin(\theta)} + 1} \sigma_z - \frac{-(\sin(\theta) + 2) \sqrt{1 - \sin(\theta)}}{(2 - \sin(\theta)) \sqrt{\sin(\theta)} + 1} \sigma_z, \tag{19}
\]
settings

\{t \text{ tum behavior specified by the state}

\{\text{Observation 1.}

\}

\text{respectively.}

\(\xi = \sin^{-1}(3 - \sqrt{4w + 5})\) and \(A^x_B, B^y_B\) for \(x, y \in \{0, 1\}\) denote measurement settings for Alice and Bob, respectively.

\text{Observation 1. Let us consider the quantum behavior given by the state } \rho_g \text{ and measurement settings } \{A^x_B, A^y_B, B^x_B, B^y_B\}. \text{ For such a behavior, the PRBLG MDL inequality given by Eq. (8) reduces to}

\[
\frac{1}{2(\xi - 1)} (2l (-4w + 7\xi - 15) + (1 - 3l) \sqrt{\xi - 2} \sin(2 \sin^{-1}(3 - \xi)) \sqrt{4 - \xi} + 8w - 10\xi + 22) - l \max(0, w) \leq 0. \tag{22}
\]

\text{On inspection we see that for all } w \in (-0.25, 1), \text{ Eq. (22) reduces to } l \leq 0.

\text{That is, the quantum behavior given by state } \rho_g \text{ and measurement settings } \{A^x_B, A^y_B, B^x_B, B^y_B\} \text{ violates both PRBLG and ZRLH MDL inequalities and its quantum nonlocality can be certified for all possible } l > 0.

\text{IV. IMPERFECT DETECTOR AND CONSTRAINED FREE WILL}

\text{We model the detection units of Alice and Bob using a two-box approach following [55]. There is an inner box containing a quantum source generating bipartite quantum states whose behavior is nonlocal under constrained free will but assuming that detectors are perfect. Nonlocality of the quantum behavior in the inner box is tested by violation of a given MDL inequality. The output of the inner box is quantum nonlocal behavior that violates a given MDL inequality. An outer box introduces the detector imperfections, namely the detection inefficiency and dark counts. The quantum nonlocal behavior obtained from the inner box gets mapped at the outer box to the output behavior with detector imperfection parameters. The output behavior then undergoes an LRHV test based on which we are able to determine threshold values of detection inefficiency and dark counts such that the given quantum nonlocal behavior can still be certified to be nonlocal with imperfect detectors. A deviation from a two-box approach in [55] is the introduction of the measurement dependence assumption to the working of the inner box. Alice and Bob has access only to the input settings and the outputs of the outer box. We assume that either party (Alice and Bob) has access to two identical detectors that can distinguish between the orthogonal outputs.}

\text{The measurement outcomes for the inner box are labeled as } a^{id} \text{ and } b^{id} \text{ respectively. We note that } a^{id} \text{ and } b^{id} \text{ can each take values from the set } \{+, -\}. \text{ Introducing non-unit detection efficiency, } 0 \leq \eta \leq 1, \text{ and non-zero dark count probability, } 0 \leq \delta \leq 1 \text{ in the outer box, the ideal two outcome scenario becomes a 4-outcome scenario with the addition of no-detection event } \Phi, \text{ and the dark-count event } \chi. \text{ These events are defined in the following way:}

\[\Phi: \text{One particle is sent to the party and none of the two detectors of that party click.}\]

\[\chi: \text{One particle is sent to the party and both the detectors of the party click.}\]
We label the measurement outcomes for Alice and Bob obtained from the outer box as \(a_{ob}\) and \(b_{ob}\) respectively. We note that \(a_{ob}\) and \(b_{ob}\) can each take values from the set \(\{+, -, \Phi, \chi\}\). Furthermore, we assume that Alice and Bob’s detection units have identical values for \(\eta\) and \(\delta\). The conditional probability of observing the outcome \(t_{ob}\) from the outer box conditioned on observing \(t_{id}\) in the ideal scale is given by \(p(t_{ob}|t_{id})\) with \(t_{ob} \in \{a_{ob}, b_{ob}\}\) and \(t_{id} \in \{a_{id}, b_{id}\}\). The observed joint probabilities can then be expressed as [55]

\[
p(a_{ob}b_{ob}|xy) = \sum_{a_{id}, b_{id}} p(a_{ob}|a_{id})p(b_{ob}|b_{id})p(a_{id}b_{id}|xy).
\]

(23)

We relax the free will assumption in the inner box by introducing the hidden variable \(\lambda \in \Lambda\). Considering this assumption, \(p(a_{id}b_{id}|xy)\) is expressed as

\[
p(a_{id}b_{id}|xy) = \sum_{\lambda \in \Lambda} p(a_{id}b_{id}|xy\lambda)p(\lambda|xy).
\]

(24)

The hidden variable, \(\lambda\) (distributed according to \(p(\lambda)\)) provides an explanation of the observed experimental statistics of the inner box. The distribution of settings that are chosen by Alice and Bob depends on \(\lambda\) via the following relation,

\[
p(xy) = \sum_{\lambda \in \Lambda} p(xy|\lambda)p(\lambda).
\]

(25)

If we impose the locality condition from Eq. (3) on the experimental statistics of the inner box, we arrive at the following factorisability constraint,

\[
p(a_{id}b_{id}|xy\lambda) = p(a_{id}|x\lambda)p(b_{id}|y\lambda).
\]

(26)

Also, if we impose the measurement independence assumption from Eq. (4), we arrive at the following constraint,

\[
p(xy|\lambda) = p(xy)\) or equivalently, \(p(\lambda|xy) = p(\lambda).
\]

(27)

The output statistics of the outer box for imperfect detectors can depend on the output of the inner box in the following four ways [55]:

1. No particle is detected on either of the detectors and no dark count detection event takes place. We can then write the following:

\[
p(a_{ob}|a_{id}) = (1 - \eta)(1 - \delta)^2.
\]

(28)

2. No particle is detected by the detector that should have detected it and a dark count takes place in the other detector. We can then write the following:

\[
p(a_{ob}|a_{id}) = (1 - \eta)(1 - \delta)\delta.
\]

(29)

3. Either the particle is detected by one of the detectors and a dark count takes place in the other detector or the particle is not detected and dark counts take place in both the detectors. We can then write the following:

\[
p(a_{ob}|a_{id}) = \eta\delta + (1 - \eta)\delta^2 = \delta[1 - (1 - \eta)(1 - \delta)].
\]

(30)

4. Either the particle is detected and no dark count takes place or the particle is not detected and a dark count takes place in the detector in which the particle should have been registered. We can then write the following:

\[
p(a_{ob}|a_{id}) = \eta(1 - \delta) + (1 - \eta)\delta(1 - \delta) = (1 - \delta)[1 - (1 - \eta)(1 - \delta)].
\]

(31)

The quantum nonlocal behavior obtained from the inner box after getting mapped to the output behavior with detector imperfection parameters remain nonlocal if the behavior obtained from the outer box violates the inequality [55]

\[
p(+ |00) + p(+ |01) + p(+) |10) - p(+ |11) - pA(+ |0) - pB(+ |0) \leq 0,
\]

(32)

where \(pA(a|s)\) and \(pB(b|s)\) are the probabilities of Alice and Bob to obtain the outcome \(s\) on measuring \(s\).

At first, let us assume there is a quantum source in the inner box that is generating a bipartite quantum state whose behavior \(\{p(a_{id}, b_{id}|xy)\}\) violates the PRBLG MDL inequality given by Eq. (8) assuming that detectors are perfect. The quantum behavior \(\{p(a_{id}, b_{id}|xy)\}\) obtained from the inner box gets mapped at the outer box to the behavior \(\{p(a_{ob}, b_{ob}|xy)\}\) with the introduction of the detector inefficiency \(\eta\) and dark count probability \(\delta\). The behavior \(\{p(a_{ob}, b_{ob}|xy)\}\) is then inserted in Eq. (32) to obtain the critical detector parameters using Algorithm 1. For a fixed value of \(\delta\) we obtain the minimum value of \(\eta\) that violates Eq. (32) using Algorithm 1. We abbreviate left-hand side as LHS.

**Algorithm 1**

1. **Initialize:**
   1. \(l \leftarrow \) parameter of Eq. (8)
2. **for** \(\delta\) in range \((0, 1)\) **do**
3. **for** \(\eta\) in range \((0, 1)\) **do**
4. \(P(a_{id}, b_{id}|xy) \leftarrow\) inner box quantum behavior
5. \(P(a_{ob}, b_{ob}|xy) \leftarrow f(P(a_{id}, b_{id}|xy), \delta, \eta)\)
6. \(\text{obj} \leftarrow\) LHS of Eq. (32) for \(P(a_{ob}, b_{ob}|xy)\)
7. \(\text{MDL} \leftarrow\) LHS of Eq. (8) for \(P(a_{id}, b_{id}|xy)\)
8. **maximize:** \(\text{obj}\)
9. **such that:** MDL > 0
10. \(\text{opt} \leftarrow\) max value of \(\text{obj}\)
11. **if** \(\text{opt} > 0\) **then**
12. **print:** \(\delta, \eta\)
13. **Break**
We plot the critical values of $\eta$ and $\delta$ obtained using Algorithm 1 in Fig. 1.

**Observation 2.** From Fig. 1, we see that the minimum value of $\eta$ for a given $\delta$ takes the highest value for $l = 0$ and decreases as $l$ increases. For a fixed value of $l$, the minimum value of $\eta$ monotonically with the increase in dark count probability. We note that for $\delta = 0$, we have $\eta \approx 0.667$ for all the values of $l$.

We next assume there is a quantum source in the inner box that is generating a bipartite quantum state whose behavior $\{p(a^{id}, b^{id}|xy)\}$ violates the ZRLH MDL inequality given by Eq. (9) assuming that detectors are perfect. The quantum behavior $\{p(a^{id}, b^{id}|xy)\}$ obtained from the inner box gets mapped at the outer box to the behavior $\{p(a^{ob}, b^{ob}|xy)\}$ with the introduction of the detector inefficiency $\eta$ and dark count probability $\delta$. The behavior $\{p(a^{ob}, b^{ob}|xy)\}$ is then inserted in Eq. (32) to obtain the critical parameter values using Algorithm 2. For a fixed value of $\delta$ we obtain the minimum value of $\eta$ that violates Eq. (32) using Algorithm 2. We abbreviate left-hand side as LHS.

**Algorithm 2**

1: Initialize:
   $w \leftarrow$ parameter of Eq. (9)
   $l \leftarrow$ parameter of Eq. (9)
2: for $\delta$ in range $(0, 1)$ do
3:     for $\eta$ in range $(0, 1)$ do
4:         $P(a^{id}, b^{id}|xy) \leftarrow$ inner box quantum behavior
5:         $P(a^{ob}, b^{ob}|xy) \leftarrow f(P(a^{id}, b^{id}|xy), \delta, \eta)$
6:         $\text{obj} \leftarrow$ LHS of Eq. (32)
7:         $\text{MDL} \leftarrow$ LHS of Eq. (9)
8:         maximize: $\text{obj}$
9:     such that: $\text{MDL} > 0$
10:    opt $\leftarrow$ max value of obj
11:   if opt $> 0$ then
12:       print: $\delta, \eta$
13:   Break

We plot in Fig. 1, the critical values of $\eta$ and $\delta$ from Algorithm 2 for $w = 0$.

**Observation 3.** The ZRLH MDL inequality given by Eq. (9) and the PRBLG MDL inequality given by Eq. (8) reduce to the same inequality for $l = 0$ and is independent of $w$. For $l = 0$, Eqs. (9) and (8) cannot be violated [57]. For $w = 0$, Eqs. (9) and (8) reduce to the same inequality and their violation can happen only when $l > 0$ [58]. For the case $w = 0$, we observe same dependence for the threshold detector parameters as can be seen from Fig. 1. We observe from Fig. 1 that for a fixed value of $l$, the minimum detection efficiency increases monotonically with the dark count probability.

We comment here that using Algorithm 2, one can calculate the detector requirements for other values of $(w, l)$ not mentioned in this section. Next, we consider the state $\rho_g$ and measurement settings $\{A^g_0, A^g_1, B^g_0, B^g_1\}$ with $\theta \approx 1.13557$. This choice of state and measurement settings was shown to ensure 1.6806 bits of global randomness in [53].

**Observation 4.** Consider the quantum source in the inner box generates bipartite quantum state $\rho_g$. Let Alice and Bob choose the measurement settings $\{A^g_0, A^g_1, B^g_0, B^g_1\}$, with $\theta \approx 1.13557$. With this choice of state and measurement settings we obtain the quantum behavior $\{p(a^{ob}, b^{ob}|xy)\}$ assuming that the detectors are perfect. The output behavior $\{p(a^{ob}, b^{ob}|xy)\}$ obtained from the outer box is evaluated using Eq. (23). The behavior $\{p(a^{ob}, b^{ob}|xy)\}$ is nonlocal if we have a violation of the inequality

$$\delta \eta \left[ (9.50424 \delta - 4 \delta^2 - 9.00848) + 4.25636 \right] + \eta^2 \left[ (2 \delta^3 - 5.50424 \delta^2 + 5.82473 \delta - 3.13675) + 0.816258 + 2 \delta (\delta - 1)(\delta^2 - \delta + 1) \right] - 0.752119 \eta \leq 0.$$  

(33)

The allowed values of the pair $(\eta, \delta)$ for which $p(a^{ob}, b^{ob}|xy)$ is nonlocal are shown in Fig. 4. In Fig. 4 we observe that for $\delta = 0$ the minimum detection efficiency $\eta_{\text{crit}} \approx 0.92$ is required to ensure $p(a^{ob}, b^{ob}|xy)$ is quantum nonlocal. We also observe that $\eta_{\text{crit}}$ increases monotonically with increase in $\delta$.

![FIG. 4. In this figure we plot the detector parameters $\eta$ and $\delta$ for which the behavior $\{p(a^{ob}, b^{ob}|xy)\}$ produced in the outer box is quantum nonlocal when the quantum behavior $\{p(a^{id}, b^{id}|xy)\}$ is produced in the inner box using the state $\rho_g$ and measurement settings $\{A^g_0, A^g_1, B^g_0, B^g_1\}$ for $\theta \approx 1.13557$. (Color Online).](image-url)
V. THE TILTED BELL INEQUALITY

The AMP tilted Bell inequality is given as [21]
\[ I^3_\alpha := \beta \langle x_1 \rangle + \alpha \langle x_1 y_1 \rangle + \alpha \langle x_1 y_2 \rangle + \langle x_2 y_1 \rangle - \langle x_2 y_2 \rangle \leq \beta + 2\alpha. \] (34)

It plays an important role in (a) demonstrating the inequivalence between the amount of certified randomness and the amount of nonlocality [21], (b) self-testing of all bipartite pure entangled states [59], (c) protocol for device-independent quantum random number generation with sublinear amount of quantum communication [60], (d) unbounded randomness certification from a single pair of entangled qubits with sequential measurements [61].

In the following proposition, we obtain the bound on \( I^3_\alpha \) when the measurement independence assumption is relaxed (see Appendix A for proof).

**Proposition 1.** The AMP tilted Bell expression \( I^3_\alpha \) in the presence of locality and the relaxed measurement independence is bounded by
\[ I^3_\alpha \leq \beta + 2\alpha + \min\{\alpha(M_1 + \min\{M_1, M_2\}) + M_2, 2\}, \] (35)
where \( M_1 \) and \( M_2 \) are the measurement dependence parameters for Alice and Bob.

### A. Comparison of one-sided and two-sided measurement dependence to ensure quantum representation

Let Alice and Bob have free will in choosing the measurement settings, then the maximum violation of \( I^3_\alpha \) obtained by quantum nonlocal behaviors is given by [21]
\[ I^3_\alpha \leq 2\sqrt{(1 + \alpha^2)(1 + \frac{\beta^2}{4})}, \] (36)

As direct consequences of Proposition 1, we have following corollaries.

**Corollary 1.** When \( M_1 = M_2 = M \), the quantum nonlocal behaviors that maximally violate Eq. (34) with the amount of violation given by the RHS of Eq. (36), remains nonlocal for
\[ M < \frac{-2\alpha - \beta + \sqrt{(1 + \alpha^2)(4 + \beta^2)}}{1 + 2\alpha}. \] (37)

For \( \alpha = 1 \) and \( \beta = 0 \) (the Bell-CHSH inequality), Eq. (37) reduces to \( M < \frac{2}{3}(\sqrt{2} - 1) \approx 0.276 \) and is consistent with the observation in [49].

**Corollary 2.** When \( M_1 = 0 \) and \( M_2 = M \), the quantum nonlocal behaviors that maximally violate Eq. (34) with the amount of violation given by the RHS of Eq. (36), remains nonlocal for
\[ M < -2\alpha - \beta + \sqrt{(1 + \alpha^2)(4 + \beta^2)}. \] (38)

For \( \beta = 0 \) and \( \alpha = 1 \) (the Bell-CHSH inequality), Eq. (38) reduces to \( M < 2(\sqrt{2} - 1) \approx 0.828 \) and is consistent with the observation in [49].

**Corollary 3.** When \( M_1 = M \) and \( M_2 = 0 \), the quantum nonlocal behaviors that maximally violate Eq. (34) with the amount of violation given by the RHS of Eq. (36), remains nonlocal for
\[ M < \frac{1}{\alpha}(-2\alpha - \beta + \sqrt{(1 + \alpha^2)(4 + \beta^2)}). \] (39)

For \( \beta = 0 \) and \( \alpha = 1 \) (the Bell-CHSH inequality), Eq. (39) reduces to \( M < 2(\sqrt{2} - 1) \approx 0.828 \) and is consistent with the observation in [49].

In Fig. 5, we plot an upper bound on \( M \) as a function of \( \alpha \) for MDL violating behaviors when \( \beta = 0 \) and \( \alpha > 1 \). The values of \( M \) and \( \alpha \) from Fig. 5 ensures that the quantum nonlocal behaviors that maximally violate Eq. (34) remains nonlocal in the presence of measurement dependence. We observe in Fig. 5 that for \( \beta = 0 \) and \( \alpha > 1 \), the introduction of one-sided measurement dependence allows for higher values of \( M \) (implying lower degree of freedom-of-choice) as compared to introducing both-sided measurement dependence. Also, for the case of one-sided measurement dependence, Alice can have higher values of \( M \) as compared to Bob.
B. Bounds on the measurement dependence for testing nonlocality

We observe in Proposition 1 that in the presence of relaxed measurement dependence, the behaviors \( p(ab|xy) \), that violates Eq. (40) are nonlocal.

\[
I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha(M_1 + \min\{M_1, M_2\}) + M_2, 2\} \tag{40}
\]

In the following, we discuss some of the cases where Eq. (40) can be violated.

a. We consider a situation when both Alice and Bob have the same measurement dependence, we have \( M_1 = M_2 = M \). For this case, violating Eq. (40) requires

\[
I_\alpha^\beta > \beta + 2\alpha + \min\{(2\alpha + 1)M, 2\}. \tag{41}
\]

If \((2\alpha + 1)M \geq 2\), \(I_\alpha^\beta\) reaches the no-signalling boundary. Whereas, if \((2\alpha + 1)M < 2\), then Eq. (41) reduces to inequality

\[
M < \frac{I_\alpha^\beta - \beta - 2\alpha}{2\alpha + 1}. \tag{42}
\]

b. We consider the situation when only Bob has measurement dependence, we have \( M_1 = 0, M_2 = M \). For this case, violating Eq. (40) requires

\[
I_\alpha^\beta > \beta + 2\alpha + \min\{M, 2\}. \tag{43}
\]

If \( M = 2 \), \(I_\alpha^\beta\) reaches the no-signalling boundary. Whereas, if \( M < 2 \), then Eq. (43) to the inequality

\[
M < I_\alpha^\beta - \beta - 2\alpha. \tag{44}
\]

c. We consider the situation when only Alice has measurement dependence, we have \( M_1 = M, M_2 = 0 \). For this case, violating Eq. (40) requires

\[
I_\alpha^\beta > \beta + 2\alpha + \min\{\alpha M, 2\}. \tag{45}
\]

If \( \alpha M \geq 2 \), \(I_\alpha^\beta\) reaches the no-signalling boundary. Whereas if \( \alpha M < 2 \), then Eq. (45) reduces to the inequality

\[
M < \frac{I_\alpha^\beta - \beta - 2\alpha}{\alpha}. \tag{46}
\]

We note that for \( \alpha = 1 \) and \( \beta = 0 \), i.e., for the Bell-CHSH inequality, the values of \( M_1 \) and \( M_2 \) that violates Eq. (40) are in agreement with that obtained in [49].

Consider there exists some behavior \( P' = \{p'(ab|xy)\} \) that violates Eq. (34) with the amount of violation given by \( I_\alpha^\beta = I'(\alpha, \beta, P') \). For \( \alpha = 1, \beta = 8 \), we show in Fig. 6 the possible values of measurement dependence parameters \( M_1 \) and \( M_2 \) for which the amount of violation of Eq. (34) given by \( I' \) cannot be described by a deterministic measurement dependent local model. The plots for the other combinations of \( (\alpha, \beta) \) is given in Appendix B.

VI. DISCUSSION

In this paper, two different approaches in quantifying measurement dependence via parameters \( l \) and \( (M_1, M_2) \) from standard literature is presented and a bound on these parameters for certifying nonlocality of different behaviors is obtained. It is observed that the quantum behavior certifying close to 2 bits of randomness remains measurement dependent nonlocal only in the limit of complete measurement independence, i.e., in the limit of \( l \rightarrow 0.25 \). The quantum behavior that provides 1.6806 bits of global randomness from the violation of the tilted Hardy relations is measurement dependent nonlocal for arbitrarily small values of \( l \). This motivates further study on obtaining an inequality that provides close to two bits of global randomness in the limit of arbitrarily low measurement dependence, a problem we leave for future study. Deviating from the conventional approach of assuming perfect detectors, we present a framework to determine the threshold values of the detector parameters that are robust enough to certify nonlocality.
of given quantum behaviors. This is an important step towards experimentally obtaining nonlocality in the presence of relaxed measurement independence. For an illustration, we presented the critical requirements for generating 1.6806 bits of tilted-Hardy certified global randomness. The detector parameters obtained from this study is expected to have important applications in experimentally implementing various information processing tasks that rely on quantum nonlocality as a resource.

The modified analytical bound on the AMP tilted Bell inequality in terms of $M_1$ and $M_2$ has been obtained. Using the analytical bound, the bounds on $M_1$ and $M_2$ to ensure quantumness and nonlocality are observed. It is observed that one-sided measurement dependence is more advantageous from the point of view of the user as compared to two-sided measurement dependence. The analytical bound obtained is expected to have applications in self-testing of quantum states and other device-independent information processing protocols like randomness generation and secure communication.

For future work, it would be interesting to see implications of measurement dependence in multipartite Bell-type inequalities [27, 28] for the certification of multipartite nonlocality and device-independent conference keys [62–64].

Note—We noticed the related work “Quantum nonlocality in presence of strong measurement dependence” by Ivan Šupić, Jean-Daniel Bancal, and Nicolas Brunner recently posted as arXiv:2209.02337.

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Appendix A: Calculating the modified local bound for the AMP tilted Bell inequality

We consider the AMP tilted-Bell inequality introduced in [21] and given by,

$$I_α^β = β⟨x_1⟩ + α⟨x_1y_1⟩ + α⟨x_1y_2⟩ + ⟨x_2y_1⟩ − ⟨x_2y_2⟩ \quad (A1)$$

The determinism assumption states that the measurement outcomes are deterministic functions of the choice of settings and the hidden variable $λ$, i.e., $a = A(x, λ)$ and $b = B(x, λ)$ with $p(a|x, λ) = δ_α A(x, λ)$ and $p(b|y, λ) = δ_β B(y, λ)$. If we assume that the choice of the measurement settings on the side of Alice and Bob can depend on some hidden variable, $λ$, the correlations of Alice and Bob can be expressed as

$$⟨xy⟩ = ∫ dλp(λ|x, y)A(x, λ)B(y, λ). \quad (A2)$$

The correlation function of Alice can also be written as

$$⟨x⟩ = ∫ dλp(λ|x)A(x, λ). \quad (A3)$$

Applying Eq. (A2) and Eq. (A3) to Eq. (A1) we have

$$I_α^β = β ∫ dλp(λ|x_1)A(x_1, λ) + α ∫ dλp(λ|x_1, y_1)A(x_1, λ)B(y_1, λ) + α ∫ dλp(λ|x_1, y_2)A(x_1, λ)B(y_2, λ) + ∫ dλp(λ|x_2, y_1)A(x_2, λ)B(y_1, λ) − ∫ dλp(λ|x_2, y_2)A(x_2, λ)B(y_2, λ). \quad (A4)$$

Adding and subtracting the terms

$$α ∫ dλp(λ|x_1, y_2)A(x_1, λ)B(y_1, λ)$$

and

$$∫ dλp(λ|x_2, y_2)A(x_2, λ)B(y_1, λ)$$

to Eq. (A4) we obtain

$$I_α^β = β ∫ dλp(λ|x_1)A(x_1, λ) + α ∫ dλA(x_1, λ)B(y_1, λ) \left[p(λ|x_1, y_1) − p(λ|x_1, y_2)\right] + ∫ dλA(x_2, λ)B(y_1, λ) \left[p(λ|x_2, y_1) − p(λ|x_2, y_2)\right] + α ∫ dλp(λ|x_1, y_2) \left[A(x_1, λ)B(y_2, λ) + A(x_1, λ)B(y_1, λ) − A(x_2, λ)B(y_1, λ) − A(x_2, λ)B(y_2, λ)\right]. \quad (A5)$$

We observe that Eq. (A5) is bounded by $I_α^β ≤ \max[T_1] + \max[T_2] + \max[T_3]$. We also note that as $p(λ|x_1)$ is a normalised probability distribution, $∫ dλp(λ|x_1) = 1$. This implies that the maximum value of the quantity
\[ \beta \int d\lambda p(\lambda|x_1)A(x_1, \lambda) \text{ is given by } \beta \text{ when } A(x_1, \lambda) \text{ is set to } 1. \text{ With these observations, } T_1 \text{ can be simplified as} \]
\[ T_1 = \beta \int d\lambda p(\lambda|x_1)A(x_1, \lambda) + \alpha \int d\lambda A(x_1, \lambda)B(y_1, \lambda) \left[ p(\lambda|x_1, y_1) - p(\lambda|x_1, y_2) \right] \leq \beta + \alpha M_2. \] (A6)

where we have set \( B(y_1, \lambda) = 1 \). To evaluate the maximum value of \( T_2 \) we set \( A(x_2, \lambda) = 1 \) and obtain
\[ T_2 = \int d\lambda A(x_2, \lambda)B(y_1, \lambda) \left[ p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2) \right] \leq M_2. \] (A7)

We evaluate \( T_3 \) as follows,
\[ T_3 = \alpha \int d\lambda p(\lambda|x_1, y_2) \left[ A(x_1, \lambda)B(y_2, \lambda) + A(x_1, \lambda)B(y_1, \lambda) \right] \]
\[ - \int d\lambda p(\lambda|x_2, y_2) \left[ A(x_2, \lambda)B(y_2, \lambda) - A(x_2, \lambda)B(y_1, \lambda) \right] \]
\[ = \int d\lambda A(x_1, \lambda)B(y_2, \lambda) \left[ \alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \frac{A(x_2, \lambda)}{A(x_1, \lambda)} \right] \]
\[ + \int d\lambda A(x_1, \lambda)B(y_1, \lambda) \left[ \alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2) \frac{A(x_2, \lambda)}{A(x_1, \lambda)} \right]. \] (A8)

To get the maximum value of \( T_3 \), we set the values of \( A(x_1, \lambda), B(y_1, \lambda), A(x_1, \lambda), B(y_2, \lambda) \) to one. This then implies,
\[ T_3 = \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] \]
\[ + \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2) \right]. \] (A9)

We evaluate the first term of the Eq. (A9) as
\[ \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] \]
\[ = \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) - \alpha p(\lambda|x_2, y_2) + \alpha p(\lambda|x_2, y_2) \right. \]
\[ \left. - p(\lambda|x_2, y_2) \right] \]
\[ = \int d\lambda \left[ \alpha \left( p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right) \right. \]
\[ \left. + (\alpha - 1)p(\lambda|x_2, y_2) \right] \]
\[ = \alpha \int d\lambda \left[ p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] + (\alpha - 1) \int d\lambda p(\lambda|x_2, y_2). \] (A12)

In Eq. (A12) we note that
\[ \alpha \int d\lambda \left( p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right) \leq \alpha M_1. \]

We note that as \( p(\lambda|x_2, y_2) \) is a normalised probability distribution, \( \int d\lambda p(\lambda|x_2, y_2) = 1 \). This then implies
\[ \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] \leq \alpha M_1 + \alpha - 1, \] (A13)

which on simplification reduces to
\[ \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] \leq \alpha(M_1 + 1) - 1. \] (A14)

We proceed in the same way for the second term in Eq. (A9) as follows,
\[ \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2) \right] \]
\[ = \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) + \alpha p(\lambda|x_2, y_2) - \alpha p(\lambda|x_2, y_2) \right. \]
\[ \left. + p(\lambda|x_2, y_2) \right] \]
\[ = \int d\lambda \left[ p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2) \right] \]
\[ - (\alpha - 1) \int d\lambda p(\lambda|x_2, y_2) \]
\[ = \alpha \int d\lambda \left[ p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2) \right] \]
\[ + (1 - \alpha) \int d\lambda p(\lambda|x_2, y_2). \] (A17)

We note that as \( p(\lambda|x_1, y_2), p(\lambda|x_2, y_2) \) are normalised probability distributions, \( \int d\lambda p(\lambda|x_1, y_2) = 1 \) and
Adding and subtracting the terms

\[ \int d\lambda p(\lambda|x_1, y_2) = 1. \]  
This then implies

\[ \int d\lambda \left[ \alpha p(\lambda|x_1, y_2) + p(\lambda|x_2, y_2) \right] = \alpha + 1. \quad \text{(A18)} \]

If we insert Eq. (A14) and Eq. (A18) in Eq. (A9), we obtain

\[ T_3 \leq \alpha(M_1 + 2). \quad \text{(A19)} \]

Now combining Eq. (A6), Eq. (A7) and Eq. (A19), we have the bound on \( I_\alpha^2 \) as

\[ I_\alpha^2 \leq T_1 + T_2 + T_3 \leq \beta + \alpha M_2 + M_2 + \alpha(M_1 + 2) \leq \beta + 2\alpha + \alpha M_1 + (\alpha + 1)M_2 \leq \beta + 2\alpha + \alpha M_1 + M_2. \quad \text{(A20)} \]

We again start by considering Eq. (A4) as

\[ I_\alpha^2 = \int d\lambda p(\lambda|x_1) A(x_1, \lambda) \]

\[ + \alpha \int d\lambda p(\lambda|x_1, y_1) A(x_1, \lambda) B(y_1, \lambda) \]

\[ + \alpha \int d\lambda p(\lambda|x_1, y_2) A(x_1, \lambda) B(y_2, \lambda) \]

\[ + \int d\lambda p(\lambda|x_2, y_1) A(x_2, \lambda) B(y_1, \lambda) \]

\[ - \int d\lambda p(\lambda|x_2, y_2) A(x_2, \lambda) B(y_2, \lambda). \quad \text{(A21)} \]

Adding and subtracting the terms \( \alpha \int d\lambda p(\lambda|x_2, y_1) A(x_1, \lambda) B(y_1, \lambda) \) and \( \alpha \int d\lambda p(\lambda|x_2, y_2) A(x_1, \lambda) B(y_2, \lambda) \) to Eq. (A4) we obtain

\[ I_\alpha^2 = \int d\lambda p(\lambda|x_1) A(x_1, \lambda) \]

\[ + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) \left[ p(\lambda|x_1, y_1) - p(\lambda|x_2, y_1) \right] \]

\[ + \alpha \int d\lambda A(x_1, \lambda) B(y_2, \lambda) \left[ p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] \]

\[ + \int d\lambda A(x_2, \lambda) B(y_1, \lambda) \left[ p(\lambda|x_2, y_1) - \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right] \]

\[ + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) \left[ p(\lambda|x_2, y_1) + \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right] \]  

\[ + \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \]. \quad \text{(A22)} \]

We observe that Eq. (A25) is bounded by \( I_\alpha^2 \leq \max[t_1] + \max[t_2] + \max[t_3] \). We also note that as \( p(\lambda|x_1) \) is a normalised probability distribution, \( \int d\lambda p(\lambda|x_1) = 1 \). This implies that the maximum value of the quantity \( \beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda) \) then takes the value of \( \beta \) when \( A(x_1, \lambda) \) is set to 1. With these observations, \( t_1 \) can be simplified as,

\[ t_1 = \beta \int d\lambda p(\lambda|x_1) A(x_1, \lambda) \]

\[ + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) \left[ p(\lambda|x_1, y_1) - p(\lambda|x_2, y_1) \right] \]

\[ \leq \beta + \alpha M_1. \quad \text{(A26)} \]

Similarly, \( t_2 \) can be simplified as

\[ t_2 = \alpha \int d\lambda A(x_1, \lambda) B(y_2, \lambda) \left[ p(\lambda|x_1, y_2) - p(\lambda|x_2, y_2) \right] \leq \alpha M_1, \quad \text{(A27)} \]

and for \( t_3 \), we have the expression

\[ t_3 = \int d\lambda A(x_2, \lambda) B(y_1, \lambda) \left[ p(\lambda|x_2, y_1) - \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right] + \alpha \int d\lambda A(x_1, \lambda) B(y_1, \lambda) \left[ p(\lambda|x_2, y_1) + \frac{B(y_2, \lambda)}{B(y_1, \lambda)} p(\lambda|x_2, y_2) \right]. \quad \text{(A28)} \]

We set the values of \( A(x_1, \lambda), B(y_1, \lambda), A(x_2, \lambda), B(y_2, \lambda) \) to one and obtain

\[ t_3 = \int d\lambda \left[ p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2) \right] \]

\[ + \alpha \int d\lambda \left[ p(\lambda|x_2, y_1) + p(\lambda|x_2, y_2) \right]. \quad \text{(A29)} \]

The first term in Eq. (A29) is bounded by

\[ \int d\lambda \left[ p(\lambda|x_2, y_1) - p(\lambda|x_2, y_2) \right] \leq M_2. \quad \text{(A30)} \]

We note that as \( p(\lambda|x_2, y_1) \) and \( p(\lambda|x_2, y_2) \) are normalised probability distribution, \( \int d\lambda p(\lambda|x_2, y_1) = 1 \) and \( \int d\lambda p(\lambda|x_2, y_2) = 1 \). Eq. (A29) is then expressed as

\[ t_3 \leq M_2 + 2\alpha. \quad \text{(A31)} \]
Now combining the values of \( t_1, t_2 \) and \( t_3 \), we have the bound on \( I_\alpha^\beta \) as
\[
I_\alpha^\beta \leq t_1 + t_2 + t_3 \quad \text{(A32)}
\]
\[
\leq \beta + \alpha M_1 + \alpha M_1 + M_2 + 2\alpha \quad \text{(A33)}
\]
\[
\leq \beta + 2\alpha + 2\alpha M_1 + M_2. \quad \text{(A34)}
\]
We note that the bound on \( I_\alpha^\beta \) must be the minimum of Eqs. (A23) and (A34) and is expressed as
\[
I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha (M_1 + \min\{M_1, M_2\}) + M_2, 2\} \quad \text{(A35)}
\]
We observe that for the case of \( \beta = 0 \) and \( \alpha = 1 \),
\[
I_1^\beta \leq 2 + M_1 + M_2 + \min\{M_1, M_2\}, \quad \text{(A36)}
\]
which is in agreement with that obtained in [49]. We note that the maximum value that \( I_\alpha^\beta \) [Eq. (A1)] can take is \( \beta + 2\alpha + 2 \) (We get this bound by setting \( \langle x_1 \rangle = 1, \langle x_1 y_1 \rangle = 1, \langle x_1 y_2 \rangle = 1, \langle x_2 y_1 \rangle = 1, \langle x_2 y_2 \rangle = -1 \)) and arrive at
\[
I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha (M_1 + \min\{M_1, M_2\}) + M_2, 2\}. \quad \text{(A37)}
\]

**Appendix B: Bounds on the measurement dependence to certify nonlocality**

We observe in Proposition 1 that in the presence of relaxed measurement dependence, the behaviors \( \{p(ab|xy)\} \) that violates Eq. (B1) are nonlocal.
\[
I_\alpha^\beta \leq \beta + 2\alpha + \min\{\alpha (M_1 + \min\{M_1, M_2\}) + M_2, 2\} \quad \text{(B1)}
\]
Consider there exists some behavior \( P' = \{p'(ab|xy)\} \) that violates Eq. (34) with the amount of violation given by \( I_\alpha^\beta = I'(\alpha, \beta, P') \). We show in Fig. 7 the possible values of measurement dependence parameters \( M_1 \) and \( M_2 \) for which the amount of violation of Eq. (34) given by \( I' \) cannot be described by a deterministic measurement dependent local model. We observe that Fig. (7) is in agreement with that obtained in [49].

![FIG. 7.](image) In this figure, we plot the values of the measurement dependence parameters \( M_1 \) and \( M_2 \) for which the violation of Eq. (34) given by \( I_\alpha^\beta = I'(\alpha = 1, \beta = 0, P') \in \{4.00, 3.42, 2.83, 2.42\} \) cannot be described by a deterministic MDL model. The correlations violating Eq. (34) with (a) \( I_\alpha^\beta = 4.00 \) belong to the no-signalling boundary (shown enclosed by red) (b) \( I_\alpha^\beta = 2.83 \) belong to the quantum boundary (shown enclosed by dashed yellow line) (c) \( I_\alpha^\beta = 2.42 \) belong to the quantum set (shown enclosed by blue dotted), and (d) \( I_\alpha^\beta = 3.42 \) belong to the no-signalling set (shown enclosed by dot-dashed purple line). The black line in the figure denotes equal values of \( M_1 \) and \( M_2 \) in the regions (a), (b), (c), and (d). (Color Online)
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that is independent of . The probabilities are always
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violated.

[58] For , violation of Eqns. (9) and (8) requires

\[ t > \frac{p(+|01) + p(+|10) + p(+|11)}{p(+|00) + 3[p(+|01) + p(+|00) + p(+|11)]} \]
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