A Multiplicative Formula for Structure Constants in the Cohomology of Flag Varieties

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1. Introduction

Let $G$ be a connected, simply connected, semisimple complex algebraic group and let $P \subseteq Q$ be a pair of parabolic subgroups. Consider the induced sequence of flag varieties

$$Q/P \hookrightarrow G/P \twoheadrightarrow G/Q.$$  \hspace{1cm} (1)

The goal of this paper is to give a simple multiplicative formula connecting the structure coefficients for the cohomology ring of the three flag varieties in (1) with respect to their Schubert bases. Let $W$ be the Weyl group of $G$ and let $W_P \subseteq W_Q \subseteq W$ denote the Weyl groups of $P$ and $Q$, respectively. Let $W_P$ denote the set of minimal-length coset representatives in $W/W_P$. For any $w \in W_P$, let $\bar{X}_w \subseteq G/P$ denote the corresponding Schubert variety and let $[X_w] \in H^*(G/P) = H^*(G/P, \mathbb{Z})$ denote the Schubert class of $\bar{X}_w$. It is well known that the Schubert classes $\{[X_w]\}_{w \in W_P}$ form an additive basis for cohomology. Similarly, we have Schubert classes $[X_u] \in H^*(G/Q)$ for $u \in W_Q$ and $[X_v] \in H^*(Q/P)$ for $v \in W_P \cap W_Q$. The letters $w, u,$ and $v$ will be used to denote Schubert varieties in $G/P$, $G/Q$, and $Q/P$, respectively. In Lemma 2.1 we show that, for any $w \in W_P$, there is a unique decomposition $w = uv$ with $u \in W_Q$ and $v \in W_P \cap W_Q$. Fix $s \geq 2$ and, for any $w_1, \ldots, w_s \in W_P$ such that $\sum_{k=1}^{s} \text{codim } X_{w_k} = \dim G/P$, define the associated structure coefficient (or structure constant) to be the integer $c_w$ for $[X_{w_1}] \cdots [X_{w_s}] = c_w[pt] \in H^*(G/P)$.

The following theorem is the first result of this paper.

**Theorem 1.1.** Let $w_1, \ldots, w_s \in W_P$, and let $u_k \in W_Q$ and $v_k \in W_P \cap W_Q$ be defined by $w_k = u_k v_k$. Assume that

$$\sum_{k=1}^{s} \text{codim } X_{w_k} = \dim G/P \quad \text{and} \quad \sum_{k=1}^{s} \text{codim } X_{u_k} = \dim G/Q.$$  \hspace{1cm} (2)

If $c_w, c_u, c_v \in \mathbb{Z}_{\geq 0}$ are defined by