$k$-spectrum of decaying, aging and growing passive scalars in Lagrangian chaotic fluid flows

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Abstract. We derive the $k$-spectrum of decaying passive scalars in Lagrangian chaotic fluid flows. In the case of exponentially decaying scalar particles, this is a power law, the exponent of which depends on the scalar decay rate, as well as on the dimensionality and compressibility of the flow. In the case of aging scalar particles, the $k$-spectrum departs from a power law. We express analytically it in terms of the scalar decay function, and provide calculations in the particular case of constant life-time scalar particles.

1. Introduction

While majority of the studies of the passive scalar turbulence deal with non-decaying tracers, several problems of high practical importance involve decaying or growing (multiplying) scalar particles. Fluorescent dyes, evaporating and/or decaying pollution on water surface, evaporating suspension particles, and dying micro-organisms (plankton) serve as a non-exhaustive list of decaying particles. As for growing and multiplying admixtures, one can mention chemical instabilities (e.g. auto-catalytic reactions), plankton blooming, c.f. Abraham (1998)), and growth of droplet nuclei in warm clouds, c.f. Falkovich et al. (2002).

The chaotic mixing of exponentially decaying passive scalars by incompressible flows has been studied by Nam et al. (1999), who showed that the resulting $k$-spectrum departs from the Batchelor’s law. Here we provide a simple explanation to this phenomenon, which allows us to calculate the $k$-spectra for a significantly broader class of tracer particles. In particular, we derive $k$-spectra for

(a) non-exponentially decaying scalar particles (such as aging micro-organisms and chemical material decaying via a complex chain of reactions, c.f. Abraham (1998));
(b) exponentially or non-exponentially growing particles (such as multiplying bacteria or growing droplets in warm clouds);
(c) tracers mixed by compressible flows (such as the two-dimensional flows on the free-slip surface of a turbulent fluid).

2. Evolution equation of the $k$-spectrum

In what follows, we consider a $n$-dimensional turbulent velocity field $\mathbf{v}(x,y,t)$, with $n = 2$ or $n = 3$ (the results can be easily generalized to cover geometries with $n > 3$). Furthermore, we assume that the velocity field is smooth, i.e.

$$\langle |\mathbf{v}(\mathbf{r}, t) - \mathbf{v}(\mathbf{r}', t)|^2 \rangle \propto |\mathbf{r} - \mathbf{r}'|^{2h}$$ (1)
It can be shown that for isotropic velocity fields, the non-constant part of $D_{ij}(\mathbf{r})$ is given by

$$d_{ij}(\mathbf{r}) = D_1[(d - 1 + \xi - \varphi \xi) \delta_{ij} r^\xi + \xi (\varphi d - 1) r_i r_j r^{\xi - 2}],$$

where the compressibility $\varphi \equiv \langle (\partial v_i / \partial r_j)^2 / (\partial v_i / \partial r_j)^2 \rangle$, $D_1$ is a constant, $\xi$ is the smoothness exponent, and $d - \text{dimensionality of the flow}$, c.f. Falkovich et al. (2001); in what follows we assume $\xi = 2$, which provides a match to the stretching statistics of real smooth flows ($h = 1$) [more detailed discussion of this matching is provided e.g. in Kalda (2007)].

For delta-correlated flows, it has been shown [c.f. Le Jan (1985); Falkovich et al. (2001)] that the average values of the Lyapunov exponents of a vector drawn between two material points, arranged in the decreasing order, can be found as

$$\lambda_i = D_1 \{ d(d - 2i + 1) - 2\varphi[d + (d - 2)i] \} ,$$

and that the dispersion of the largest Lyapunov exponent is

$$\Delta = 2D_1(d - 1)(1 + 2\varphi).$$

The smallest Lyapunov exponent describes the approaching rate of two surfaces, e.g., two close surfaces of constant density of the tracer: the logarithm of the distance decreases as $\lambda_d t$ (note that $\lambda_d < 0$), with a variance of $\Delta t$. In the absence of the molecular diffusivity, the tracer gradient evolves proportionally to the reciprocal of that distance. It is important that in the case of smooth chaotic velocity fields, the tracer isodensity surfaces (isodensity lines in the case of two-dimensional geometry) become almost flat (perpendicular to the eigenvector of the smallest Lyapunov exponent), c.f. Kalda (2000), and the small-scale tracer density fluctuations obtain the form of almost plane waves (note that in the case of non-smooth velocity fields, the isodensity surfaces are fractal; hence, the small-scale fluctuations cannot be approximated as plane waves, and thus the ideas presented below cannot be applied). Therefore, using a logarithmic scale, the probability density function $p(\ln g, t)$ of the tracer gradient $g \equiv |\nabla \psi|$ evolves according to the diffusion equation

$$\frac{\partial p}{\partial t} - \lambda_d \frac{\partial p}{\partial \ln g} = \frac{\Delta}{2} \frac{\partial^2 p}{\partial (\ln g)^2}.$$

Using a rescaled time, this equation can be rewritten as

$$\frac{\partial p}{\partial \hat{t}} + u \frac{\partial p}{\partial \ln g} = \frac{\partial^2 p}{\partial (\ln g)^2}.$$
where
\[ u \equiv 2\lambda_{d}/\Delta = (d - 4\wp)/(1 + 2\wp). \] (7)

As long as we can neglect the molecular diffusivity \( \kappa \), the spectral power \( \Phi \equiv kE_k \) associated with a wave vector \( k \) is transferred along the spectrum of wave vectors according to the growth of the wave vector. Here, \( E_k \) stands for the \( k \)-spectrum of the tracer field:

\[ E_k = \int_{|k|=k} E_k d^{n-1}k, \]

where
\[ E_k = \int \langle \psi_k \psi_k' \rangle d^n k, \]

and \( \psi_k \) is the Fourier transform of the tracer density field \( \psi(r,t) \). The growth of the wave vector \( k \) is caused by the stretching of material lines, and is directly proportional to the growth of the tracer gradient \( g \); hence, the quantity \( \Phi \) (considered as a function of \( \ln k \) and \( t \)) evolves in the same way as \( p(\ln g, t) \). As mentioned above, at the limit of large Peclet’ numbers, the small-scale tracer fluctuations are in the form of almost plane waves. The diffusive decay of such quasi-one-dimensional fluctuations can be easily analyzed in the Lagrangian coordinates; the diffusive decay rate of the spectral power appears to be given by \( \kappa k^2 \). Therefore, upon denoting
\[ \ln k \equiv \sigma, \]

the evolution equation of the power spectrum can be written as

\[ \frac{\partial \Phi}{\partial t} + u \frac{\partial \Phi}{\partial \sigma} = \frac{\partial^2 \Phi}{\partial \sigma^2} - 2\kappa \exp(2\sigma)\Phi + g(\sigma, t), \] (8)

where \( g(\sigma, t) \) describes the effect of a statistically stationary scalar source at the forcing scale \( \sigma = 0 \) (i.e. we assume that there is a constant source of scalar, with a wave vector \( k = 1 \)).

Note that this equation has been derived for two-dimensional incompressible flows in Kalda (2000), both for Kraichnan flows and real flows at the limit of long-time evolution. Here, we have generalized it to three-dimensional, possibly compressible flows; however, the derivation has been based on the assumption of delta-correlation in time. Therefore, we need to discuss the applicability of Eq. (8) to real flows.

In the context of turbulent mixing, delta-correlation in time means that the characteristic time-scale of the scalar spectrum evolution needs to be much longer than the correlation time of the velocity field. The former, however, increases with time. Indeed, Eq. (8) shows that the \( k \)-spectrum evolves according to a diffusion equation, so that the characteristic time scale can be estimated as the time elapsed since the beginning of the process. In other words, at the long-time limit, the characteristic time-scale of the velocity field becomes much shorter than that of the \( k \)-spectrum: the velocity field is effectively delta-correlated in time. So, we can conclude that the basic form of Eq. (8) remains valid even for real flows.

However, the question which remains, is whether one can still use Eq. (7) for determining the parameter \( u \). As long as we are dealing with incompressible two-dimensional flows, there is a simple and fundamental reason for Eq. (7) to remain valid. Indeed, in that case, the tracer gradient is directly proportional to the distance between two material points. Hence, in that case, the PDF of inter-particle distance is also described by Eq. (6), assuming that the parameter \( \ln g \) is substituted by \( \ln R \) (\( R \) being the inter-particle distance). At the long-time limit, this PDF tends to the stationary solution \( p \propto \exp(u \ln R) \). Comparing this with the observation that in a well-mixed state, the probability of finding the other particle within a circle of radius \( R \) is proportional to the area of the circle, we can conclude that \( u = 2 \).

While these arguments cannot be applied to three-dimensional incompressible flows (the gradient is no longer directly related to the inter-particle distance), there are more complicated ways of showing that Eq. (7) still holds at the long-time limit of the mixing process [which
Figure 1. The differential scaling exponent $\alpha$ of the power spectrum of a finite life-time aging passive scalar is plotted versus the wave vector $k$. The numerical value $u \approx 0.24$ has been used. Different curves correspond to a different life-time $\tau$ of the scalar.

involve deriving expressions analogous to Eqns (4) and (5) for statistically isotropic flows with finite correlation times. However, situation is completely different in the case of compressible flows. In fact, one can easily understand that for compressible flows Eq. (7) can be violated. Indeed, the definition of compressibility takes into account only relative weight of solenoidal and potential components of the velocity field. Meanwhile, the stretching action of the field depends also on the correlation time: longer correlations lead to faster stretching. Therefore, the stretching effect of the compressible (potential) component of the velocity field can be increased by increasing its correlation time and keeping the velocity amplitude intact. To conclude, in the case of compressible flows, Eq. (7) can be used only if the solenoidal and potential components of the velocity field have identical correlation times.

3. Decaying scalar fields

Equation (8) can be interpreted as follows. Turbulent flow leads to a certain average stretching rate of fluid elements; this net stretching implies a passive scalar spectral flux towards small scales and is described by the term $u \partial \Phi / \partial \sigma$. The stretching rate fluctuations result in the spectral diffusion $\partial^2 \Phi / \partial \sigma^2$. Finally, at the high spatial frequencies, there is an exponential decay of the passive scalar fluctuations, due to the smoothing effect of the molecular diffusivity, which is described by the term $-2\kappa \exp(2\sigma) \Phi$. Meanwhile, neglecting the molecular diffusivity, the density of an exponentially decaying tracer would be proportional to $\exp(-\gamma t)$. Therefore, exponentially decaying scalars can be conveniently analyzed by substituting the diffusive decay rate $\kappa \exp(2\sigma)$ in Eq. (1) by a scale-independent decay rate $\gamma$:

$$\frac{\partial \Phi}{\partial t} + u \frac{\partial \Phi}{\partial \sigma} = \frac{\partial^2 \Phi}{\partial \sigma^2} - 2\gamma \Phi + \delta(\sigma);$$

here $g = \delta(\sigma)$ corresponds to a stationary tracer source at the unit wave vector $k = 1$. Note that exponentially multiplying scalar particles can be described by negative values of the exponent $\gamma$. Then, the stationary solution of Eq. (3) is $\Phi \propto \exp[-4\gamma \sigma/(u + \sqrt{u^2 + 8\gamma})]$. Hence, the power spectrum is given by $E \propto k^{-\alpha}$, where

$$\alpha = 1 + 4\gamma/(u + \sqrt{u^2 + 8\gamma}).$$

Non-exponentially decaying or growing tracer particles can be analysed by making use of the Green function $G(t)$. The Green function of a non-decaying tracer [Eq. (3), $\gamma = 0$] is
\(G_0 = \exp[-(\sigma - ut)^2/t/(\pi t)^{1/2}]\). If the tracer particles decay (or grow) proportionally to some function \(f(t)\), then the Green function will be given by \(G = f(t)G_0\). The function \(f(t)\) can describe several scenarios: (a) disappearing tracer particles, in which case it provides the probability \(p(t)\) that a particle launched at the moment \(t = 0\) survives at least till moment \(t\); (b) growing (or decreasing in size) particles, in which case it provides the mass of the particles as a function of time; (c) multiplying particles (bacteria), in which case it provides the mathematical expectation of the number of surviving descendants (including the original) of the particle launched at \(t = 0\). The stationary solution of this equation is given by

\[
\Phi \propto \int_0^\infty f(t) \exp[-(\sigma - ut)^2/t]t^{-1/2}dt.
\] (11)

In the simplest case of an aging tracer, all the tracer particles disappear after achieving a fixed dying age \(\tau\): \(f(t) = \theta(\tau - t)\), where \(\theta\) is the Heaviside function. Then, Eq. (4) is easily integrated,

\[
\Phi \propto 1 - \text{erf}\left(\frac{\sigma - ut\tau}{\sqrt{\tau}}\right)\left[1 + \text{erf}\left(\frac{\sigma - ut\tau}{\sqrt{\tau}}\right)\right].
\] (12)

The scaling properties of Eq. (5) can be studied by calculating the differential scaling exponent of the power spectrum, \(\alpha = -d \ln E_k/d \ln k\), or equivalently,

\[
\alpha = 1 - d \ln \Phi/d \sigma.
\] (13)

These curves are plotted in Fig. 1 for several values of \(\tau\). Asymptotically (for \(k \to \infty\)),

\[
\alpha = 1 + 2(\ln k - u\tau)/\tau.
\] (14)

It can be shown that such a linear asymptotic dependence between \(\alpha\) and \(\ln k\) is characteristic to all tracers which disappear completely after achieving a certain age. Indeed, the asymptotical behaviour of Eq. (12) can be easily analyzed by the saddle-point method. The same technique can be used to show that for tracers decaying faster than exponentially, \(\alpha\) is an increasing function of \(\ln k\).

4. Conclusions

According to the classical Batchelors law, chaotic mixing of a tracer with a constant large-scale source leads to the spectral power of the tracer field being inversely proportional to the wave vector (with a high-\(k\)-cutoff due to the molecular diffusivity). However, in the case of exponentially decaying or growing tracers, the spectrum follows still a power law; depending on the values of the parameters, the exponent can be both larger or smaller than 1, c.f. Eq. (10). In the case of aging tracer particles (which decay non-exponentially), the spectrum is no longer a power law.

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