Consistency of the Plug-In Estimator of the Entropy Rate for Ergodic Processes

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Abstract—A plug-in estimator of entropy is the entropy of the distribution where probabilities of symbols or blocks have been replaced with their relative frequencies in the sample. Consistency and asymptotic unbiasedness of the plug-in estimator can be easily demonstrated in the IID case. In this paper, we ask whether the plug-in estimator can be used for consistent estimation of the entropy rate $h$ of a stationary ergodic process. The answer is positive if, to estimate block entropy of order $k$, we use a sample longer than $2^{h(1+\epsilon)}$, whereas it is negative if we use a sample shorter than $2^{h-1}$. In particular, if do not know the entropy rate $h$, it is sufficient to use a sample of length $(|X|+\epsilon)^k$ where $|X|$ is the alphabet size. The result is derived using $k$-block coding. As a by-product of our technique, we also show that the block entropy of a stationary process is bounded above by a nonlinear function of the average block entropy of its ergodic components. This inequality can be used for an alternative proof of the known fact that the entropy rate a stationary process equals the average entropy rate of its ergodic components.

I. Results

Nonparametric entropy estimation is a task that requires a very large amount of data. This problem has been studied mostly in the IID case, see a review of literature in [1]. Moreover, the novel results of [1] state that it is impossible to estimate entropy of a distribution with a support size $S$ using an IID sample shorter than of order $S/\log S$, whereas it is possible for a sample longer than of order $S/\log S$, and a practical estimator achieving this bound has been exhibited. Earlier, in [2], another entropy estimator was proposed which, for a finite alphabet, has a bias exponentially decreasing with the sample length. The exponential decay of the bias is, however, too slow to beat the $S/\log S$ sample bound.

In this paper we would like to pursue the more difficult and less recognized question of entropy estimation for general stationary ergodic processes, cf. [3]. For a stationary process $(X_i)_{i=-\infty}^{\infty}$ over a finite alphabet $X$, consider the blocks of random symbols $X_{i}^{k} = (X_i)_{k \leq i \leq l}$. Consider then the true block distribution

$$p_k(w) = P(X_{i+1}^{i+k} = w)$$

and the empirical distribution

$$p_k(w, X_1^n) = \frac{1}{[n/k]} \sum_{i=1}^{[n/k]} 1\{X_{i+k}^{i(k-1)+1} = w\}.$$  \hspace{1cm} (2)

Having denoted the entropy of a discrete distribution

$$H(p) = - \sum_{w: p(w) > 0} p(w) \log p(w),$$ \hspace{1cm} (3)

let the block entropy be

$$H(k) = H(p_k)$$ \hspace{1cm} (4)

with the associated entropy rate

$$h = \lim_{n \to \infty} H(k)/k.$$ \hspace{1cm} (5)

As shown in [4], for the variational distance

$$|p - q| := \sum_w |p(w) - q(w)| ,$$ \hspace{1cm} (6)

we have

$$\lim_{k \to \infty} \left| p_k - p_k(\cdot, X_1^{n(k)}) \right| = 0,$$ \hspace{1cm} (7)

if we put $n(k) \geq 2^{h(1+\epsilon)}$ for IID processes as well as for irreducible Markov chains, for functions of irreducible Markov chains, for $\psi$-mixing processes, and for weak Bernoulli processes. This result suggests that sample size $n(k) \geq 2^{h(1+\epsilon)}$ may be sufficient for estimation of block entropy $H(k)$.

Let us state our problem formally. The plug-in estimator of the block entropy is

$$H(k, X_1^n) = H(p_k(\cdot, X_1^{n(k)})),$$ \hspace{1cm} (8)

as considered e.g. by [5]. Since

$$E p_k(w, X_1^n) = P(X_1^{n(k)} = w)$$ \hspace{1cm} (9)

then, applying the Jensen inequality, we obtain

$$E H(k, X_1^n) \leq H(k)$$ \hspace{1cm} (10)

so the plug-in estimator is a biased estimator of $H(k)$. The bias of the plug-in estimator can be quite large since by inequality $p_k(w, X_1^n) \geq [n/k]^{-1}$ for $p_k(w, X_1^n) > 0$ we also have

$$H(k, X_1^n) \leq \log [n/k].$$ \hspace{1cm} (11)

The plug-in estimator $H(k, X_1^n)$ depends on two arguments: the block length $k$ and the sample $X_1^n$. If we fix the block length $k$ and let the sample size $n$ tend to infinity, we obtain a consistent and asymptotically unbiased estimator of the block
entropy $H(k)$. Namely, for a stationary ergodic process,

$$\lim_{n \to \infty} H(k, X^n_1) = H(k) \text{ a.s.}$$

(12)

by the ergodic theorem and hence

$$\lim_{n \to \infty} E H(k, X^n_1) = H(k)$$

(13)

by inequality ([10]) and the Fatou lemma. These results generalize what is known for the IID case ([5]).

Now the question arises what $n(k)$ we should choose so that $H(k, X^n_1(k))/k$ be a consistent estimator of the entropy rate. Using a technique based on source coding, which is different than used in [4], we may establish some positive result in a more general case than considered in [4]:

**Theorem 1:** Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary ergodic process over a finite alphabet $\mathcal{X}$. For any $\epsilon > 0$ and

$$n(k) \geq 2^{k(h+\epsilon)},$$

(14)

we have

$$\lim_{k \to \infty} E [H(k, X^n_1(k))/k] = h,$$

(15)

$$\liminf_{k \to \infty} H(k, X^n_1(k))/k = h \text{ a.s.},$$

(16)

$$\forall \eta > 0 \lim_{k \to \infty} P \left( H(k, X^n_1(k))/k - h > \eta \right) = 0.$$  

(17)

According to Theorem 1 for the sample size (14) the plug-in estimator $H(k, X^n_1(k))/k$ of the entropy rate $h$ is consistent in probability. In contrast, applying inequality ([11]) for $\epsilon > 0$ and

$$n(k) \leq 2^{k(h-\epsilon)}$$

(18)

yields

$$\limsup_{k \to \infty} H(k, X^n_1(k))/k \leq h - \epsilon \text{ a.s.}$$

(19)

Hence the sample size (13) is insufficient to obtain a consistent estimate of the entropy rate $h$ using the plug-in estimator.

Let us observe that in general there are two different kinds of random entropy bounds for stationary processes:

1) Random upper bounds $K(X^n_1)$ based on universal coding ([6], [7]) or universal prediction ([8]). For these bounds, we have Kraft inequality $\sum_{x \in \mathcal{X}} 2^{-K(x^n_1)} \leq 1$. Therefore, for a stationary process, we have the source coding inequality $E K(X^n_1) \geq H(n)$ and the Barron inequality

$$P(K(X^n_1) + \log P(X^n_1) \leq -m) \leq 2^{-m}$$

(20)

[9], Theorem 3.1. Moreover, for a stationary ergodic process, we have

$$\lim_{k \to \infty} E [K(X_1^n)/n] = h,$$

(21)

$$\lim_{n \to \infty} K(X_1^n)/n = h \text{ a.s.}$$

(22)

In particular, these conditions hold for

$$K(X^n_1) = \min_k K(k, X^n_1),$$

(23)

where $K(k, X^n_1)$ is the length of the code which will be considered for proving Theorem 1 in Section 3, cf. [7].

2) Random lower bounds, such as the plug-in estimator $H(k, X^n_1(k))$: As we have seen, for a stationary process, we have (10), whereas for a stationary ergodic process, we have ([15], [17]).

Both quantities $K(X^n_1)$ and $H(k, X^n_1(k))$ can be used for estimation of the entropy rate $h$.

When applying $H(k, X^n_1(k))$ for the estimation of entropy rate, we are supposed not to know the exact value of $h$. Therefore, the choice of minimal admissible $n(k)$ is not so trivial.

According to Theorem 1 we may put $n(k) = (|X| + \epsilon)^k$. This bound is, however, pessimistic, especially for processes with a vanishing entropy rate $h = 0$, cf. [10]. Having a random upper bound of the block entropy $K(X^n_1)$, we may also put

$$n(k) = 2E K(X^n_1) + k.$$  

(24)

A question arises whether we can improve Theorem 1. Thus, let us state three open problems:

1) Does the equality

$$\lim_{k \to \infty} H(k, X^n_1(k))/k = h \text{ a.s.}$$

(25)

hold true in some cases? In other words, is the plug-in estimator $H(k, X^n_1(k))/k$ an almost surely consistent estimator of the entropy rate?

2) What happens for $\lim_{n \to \infty} k^{-1} \log n(k) = h$? In particular, can Theorem 1 be strengthened by setting $n(k)$ equal to some random stopping time, such as

$$n(k) = 2K(X^n_1),$$

(26)

where $K(X^n_1)$ is a length of a universal code for $X^n_1$?

3) The plug-in estimator is not optimal in the IID case. Can we propose a better estimator of the entropy rate also for an arbitrary stationary ergodic process?

Another class of less clearly stated problems concerns comparing the entropy estimates $K(X^n_1)$ and $H(k, X^n_1(k))$. Although the gap between these estimates is closing when divided by $k$, i.e.,

$$\lim_{k \to \infty} E \left[ K(X^n_1) - H(k, X^n_1(k)) \right] / k = 0,$$

(27)

difference $K(X^n_1) - H(k, X^n_1(k))$ can be arbitrarily large. To see it, let us note that inequalities $E K(X^n_1) \geq H(k)$ and $H(k) \geq E H(k, X^n_1)$ hold for any stationary process, regardless whether it is ergodic or not. Hence, by the ergodic decomposition ([11]), we have $E K(X^n_1) \geq H(X^n_1)$ and $H(X^n_1|\mathcal{I}) \geq E H(k, X^n_1)$, where $\mathcal{I}$ is the shift-invariant algebra of a stationary process $(X_i)_{i=\infty}^{-\infty}$, $H(X^n_1) = H(k)$ is the entropy of $X^n_1$, and $H(X^n_1|\mathcal{I})$ is the conditional entropy of $X^n_1$ given $\mathcal{I}$. Consequently,

$$E \left[ K(X^n_1) - H(k, X^n_1) \right] \geq H(X^n_1) - H(X^n_1|\mathcal{I})$$

(28)

$$= I(X^n_1; \mathcal{I}),$$

(29)

where $I(X^n_1; \mathcal{I})$ is the mutual information between block $X^n_1$ and the shift-invariant algebra $\mathcal{I}$. In fact, for an arbitrary stationary process, the mutual information $I(X^n_1; \mathcal{I})$ can grow.
as fast as any sublinear function, cf. [12], [10].

Whereas there is no universal sublinear upper bound for mutual information $I(X^k_i;I) = H(X^k_i) - H(X^k_i|I)$, we may ask whether there is an upper bound for entropy $H(X^k_i)$ in terms of a function of conditional entropy $H(X^k_i|I)$ for an arbitrary stationary process and $n \geq k$. Using the code from the proof of Theorem [1] we can provide this bound:

Theorem 2: For a stationary process $(X_i)_{i=-\infty}^{\infty}$, natural numbers $p$ and $k$, $n = pk$, and a real number $m \geq 1$,
\[
\frac{H(X^k_i)}{n} - \frac{H(X^k_i|I)}{k} \leq \frac{2}{k} + \frac{2}{n} \log k + 3 \log |X| \times \left(1 + \frac{1}{m} + \left(1 - \frac{1}{m}\right)\sigma\left(mH(X^k_i|I) - \log \frac{n}{k}\right) + \frac{k}{n}\right),
\]
(28)
where $\sigma(y) = \min(\exp(y), 1)$.

Theorem 2 states that the block entropy of a stationary process is bounded above by a nonlinear function of the average block entropy of its ergodic components. We suppose that this inequality can be strengthened if there exists a better estimator of the block entropy than the plug-in estimator. A simple corollary of Theorem 2 is that
\[
\lim_{k \to \infty} \frac{H(X^k_i)}{k} = h,
\]
(29)
a fact usually proved by the ergodic decomposition [11]. To derive (29) from (28), we first put $n \to \infty$ and next $n \to \infty$ and $k \to \infty$.

In the following, in Section II we prove Theorem 1 whereas in Section III we prove Theorem 2.

II. PROOF OF THEOREM 1

Our proof of Theorem 1 applies source coding. To be precise, it rests on a modification of the simplistic universal code by Neuhoff and Shields [7]. The Neuhoff-Shields code is basically a k-block code with parameter $k$ depending on the string $X^n_i$. In the following, we will show that the plug-in estimator $H(k, X^n_i)$ multiplied by $n/k$ is the dominating term in the length of a modified k-block code for $X^n_i$ by the results of [7], [13]. This length cannot be shorter than $nh$ so the expectation of $H(k, X^n_i)/k$ must tend to $h$.

The idea of a k-block code is that we first describe a code book, i.e., we enumerate the collection of blocks $w$ of length $k$ contained in the compressed string $X^n_i$ and their frequencies $n_p(w, X^n_i)$, and then we apply the Shannon-Fano coding to $X^n_i$ partitioned into blocks from the code book. Let $D(k, X^n_i)$ be the number of distinct blocks of length $k$ contained in the compressed string $X^n_i$. Formally,
\[
D(k, X^n_i) = \left\{ w \in X^k : \exists i \in 1, \ldots, \lfloor n/k \rfloor X^k_{(i-1)k+1} = w \right\}.
\]
(30)

To fully describe $X^n_i$ in terms of a k-block code we have to specify, cf. [7]:

1) what $k$ is (description length $2 \log k$),
2) what $D(k, X^n_i)$ is (description length $\log \lfloor n/k \rfloor$),
3) what the code book is (we have to specify the Shannon-Fano code word for each k-block, hence the description length is $\leq (k \log |X| + 2 \log \lfloor n/k \rfloor)D(k, X^n_i)$),
4) what the Shannon-Fano code words for block $X^{k\lfloor n/k \rfloor}_1$ are (description length $\leq \lfloor n/k \rfloor (H(k, X^n_i) + 1)$),
5) what the remaining block $X^{k\lfloor n/k \rfloor+1}_i$ is (description length $\leq k \log |X|$).

Hence quantity
\[
2 \log k + \frac{n}{k} (H(k, X^n_i) + 1) + \left( k \log |X| + 2 \log \frac{n}{k} \right) (D(k, X^n_i) + 1)
\]
(31)
is an upper bound for the length of the k-block code.

For our application, the k-block code has a deficiency that very rare blocks have too long codewords, which leads to an unwanted explosion of term $2 \log \frac{n}{k}$ in the upper bound of the code length for $n \to \infty$. Hence let us modify the k-block code so that a k-block is Shannon-Fano coded if and only if its Shannon-Fano code word is shorter than $k \log |X|$, whereas it is left uncoded otherwise. In the coded sequence, to distinguish between these two cases, we have to add some flag, say 0 before the Shannon-Fano code word and 1 before the uncoded block. In this way, to fully describe $X^n_i$ in terms of the modified k-block code we have to specify:

1) what $k$ is (description length $2 \log k$),
2) what the number of used distinct Shannon-Fano code words is (description length $k \log |X|$),
3) what the code book is (we have to specify the Shannon-Fano code word for each coded k-block, hence the description length is $\leq 3k \log |X|(D(k, X^n_i))$,)
4) what the sequence of code words for block $X^{k\lfloor n/k \rfloor}_1$ is (description length $\leq \lfloor n/k \rfloor (H(k, X^n_i) + 2)$),
5) what the remaining block $X^{k\lfloor n/k \rfloor+1}_i$ is (description length $\leq k \log |X|$).

In view of this, quantity
\[
K(k, X^n_i) = 2 \log k + \frac{n}{k} (H(k, X^n_i) + 2) + 3k \log |X|(D(k, X^n_i) + 1)
\]
(32)
is an upper bound for the length of the modified k-block code.

Since the k-block code is an instantaneous code, the upper bound for its length satisfies Kraft inequality $\sum_{k} 2^{−K(k, X^n_i)} \leq 1$. Therefore, we have $\mathbf{E}K(k, X^n_i) ≥ H(n)$, whereas the Barron inequality
\[
P(K(k, X^n_i) + \log P(X^n_i)) ≤ -m) ≤ 2^{−m}
\]
(33)
\[\text{Theorem 3.1], the Borel-Cantelli lemma, and the Shannon-McMillan-Breiman theorem}
\]
\[\lim_{n \to \infty} \frac{−\log P(X^n_i)}{n} = h \text{ a.s.}
\]
(34)
\[\text{[14], we obtain}
\]
\[\liminf_{k \to \infty} \frac{K(k, X^{n(k)}_i) + \log P(X^{n(k)}_i)}{n(k)} ≥ \lim_{k \to \infty} \frac{−\log P(X^{n(k)}_i)}{n(k)} = h \text{ a.s.}
\]
(35)
According to [13, Theorem 2], for each $\delta > 0$ almost surely
there exists $k_0$ such that for all $k \geq k_0$ and $n > 2^k h$ we have
\[ D(k, X_1^n) \leq 2^{k(h+\delta)} + \frac{n}{k} \delta. \] (36)
Hence for $n(k) \geq 2^{k(h+\epsilon)}$ and $\delta < \epsilon$, we have almost surely
\[ h \leq \liminf_{k \to \infty} \frac{K(k, X_1^n)}{n(k)} \]
\[ = \liminf_{k \to \infty} \left( \frac{3k}{n(k)} \log |X| D(k, X_1^n) + \frac{H(k, X_1^n)}{k} \right) \]
\[ \leq \liminf_{k \to \infty} \left( 3 \log |X| \left( k2^{-k(\epsilon-\delta) + \delta} + \frac{H(k, X_1^n)}{k} \right) \right) \]
\[ = 3 \log |X| \delta + \liminf_{k \to \infty} \frac{H(k, X_1^n)}{k}. \] (37)
Since $\delta$ can be chosen arbitrarily small then
\[ \liminf_{k \to \infty} \frac{H(k, X_1^n)}{k} \geq h \text{ a.s.} \] (38)
In contrast, inequality (10) implies
\[ \limsup_{k \to \infty} E \left[ H(k, X_1^n) \right] \leq h. \] (39)
Hence, by the Fatou lemma and inequality (38), we have
\[ h = E \liminf_{k \to \infty} \frac{H(k, X_1^n)}{k} \geq \liminf_{k \to \infty} E \frac{H(k, X_1^n)}{k}, \] (40)
i.e., equality (15) is established. By inequality (38) and equality (40), we also obtain equality (16).

The proof of statement (17) requires a few additional steps. Denoting $X^+ = X1\{X > 0\}$ and $X^- = -X1\{X < 0\}$, we obtain from Markov inequality, inequality (10), and inequality (X + Y) \leq X^- + Y^- that
\[ \eta P \left( \frac{H(k, X_1^n)}{k} - h > \eta \right) \leq E \left[ \frac{H(k, X_1^n)}{k} - h \right] + \left[ \frac{H(k, X_1^n)}{k} - h \right] \]
\[ = E \left[ \frac{H(k, X_1^n)}{k} - h \right] + \left[ \frac{H(k, X_1^n)}{k} - h \right] \]
\[ \leq \left[ \frac{H(k)}{k} - h \right] + \left[ \frac{H(k, X_1^n)}{k} - \frac{K(k, X_1^n)}{n(k)} \right] \]
\[ + \left[ \frac{K(k, X_1^n)}{n(k)} + \log P(X_1^n) \right] \]
\[ + \left[ \frac{-\log P(X_1^n)}{n(k)} - h \right]. \] (41)

Now we will show that all four terms on the RHS of (41) tend to 0, which is sufficient to establish (17). First,
\[ \lim_{k \to \infty} \left[ \frac{H(k)}{k} - h \right] = 0 \] (42) by the definition of the entropy rate. Second,
\[ \lim_{k \to \infty} E \left[ \frac{H(k, X_1^n)}{k} - K(k, X_1^n) \right] = 0 \] (43) since
\[ \lim_{k \to \infty} \frac{k}{n(k)} E D(k, X_1^n) = 0 \] (44) by the result of (7) Eq. (8)]. Third,
\[ \lim_{n \to \infty} \frac{1}{n} E \left[ K(k, X_1^n) + \log P(X_1^n) \right] = 0 \] (45) since
\[ E \left[ K(k, X_1^n) + \log P(X_1^n) \right] \leq \sum_{m=0}^{\infty} m2^{-m} < \infty \] (46) by the Barron inequality (33). Fourth,
\[ \lim_{n \to \infty} E \left[ \frac{-\log P(X_1^n)}{n} - h \right] = 0 \] (47) since the Shannon-McMillan-Breiman theorem (34) implies convergence in probability, i.e., for all $\epsilon > 0$, we have
\[ \lim_{n \to \infty} P \left( \frac{-\log P(X_1^n)}{n} - h < -\epsilon \right) = 0. \] (48) Hence
\[ E \left[ \frac{-\log P(X_1^n)}{n} - h \right] \]
\[ \leq h P \left( \frac{-\log P(X_1^n)}{n} - h < -\epsilon \right) + \epsilon P \left( \frac{-\log P(X_1^n)}{n} - h \geq -\epsilon \right) \] (49) tends to a value smaller than $\epsilon$, where $\epsilon$ is arbitrarily small.

III. PROOF OF THEOREM 2

For the code from the proof of Theorem 1 we have
\[ \frac{H(X_1^n)}{n} - H(X_1^n|X) \leq E \left[ \frac{K(k, X_1^n)}{n} - \frac{H(k, X_1^n)}{k} \right] \]
\[ = \frac{2}{k} + \frac{2}{n} \log k + \frac{3k}{n} \log |X| (E D(k, X_1^n) + 1) \] (50)
Then, following the idea of (3) Theorem 1, we may express the number of distinct $k$-blocks as
\[ D(k, X_1^n) = \sum_{w \in X^k} \sum_{i=1}^{n/k} 1 \left\{ X_{(i-1)k+1}^i = w \right\} \geq 1 \] (51)
Hence by the Markov inequality,

\[
\mathbb{E} \left( D(k, X^n_i) | I \right) = \sum_{w \in X^k} P \left( \sum_{i=1}^{n/k} 1 \{ X^{i-1}_k, k + 1 = w' \} \geq 1 \mid I \right) \leq \sum_{w \in X^k} \min \left[ 1, \mathbb{E} \left( \sum_{i=1}^{n/k} 1 \{ X^{i+k}_k, (i-1)_k + 1 = w \} \mid I \right) \right] = \sum_{w \in X^k} \min \left[ 1, \frac{n}{k} P(X^k_i = w \mid I) \right] = \frac{n}{k} \mathbb{E} \left( \min \left( \frac{n}{k} P(X^k_i \mid I) \right)^{-1}, 1 \mid I \right). \tag{52}
\]

In consequence,

\[
k \frac{n}{k} D(k, X^n_i) \leq \mathbb{E} \left( -\log P(X^k_i \mid I) - \log \frac{n}{k} \right), \tag{53}
\]

where \( \mathbb{E} \left[ -\log P(X^k_i \mid I) \right] = H(X^k_i \mid I) \). Therefore, using another Markov inequality

\[
P \left( -\log P(X^k_i \mid I) \geq mH(X^k_i \mid I) \right) \leq \frac{1}{m} \tag{54}
\]

for \( m \geq 1 \), we further obtain from (53) that

\[
k \frac{n}{k} D(k, X^n_i) \leq \frac{1}{m} + \left( 1 - \frac{1}{m} \right) \mathbb{E} \left( mH(X^k_i \mid I) - \log \frac{n}{k} \right). \tag{55}
\]

Inserting (55) into (50) yields the requested bound.

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