Numerical solutions to lattice-refined models in loop quantum cosmology

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Received 13 November 2007, in final form 6 February 2008
Published 1 April 2008
Online at stacks.iop.org/CQG/25/085009

Abstract
In this paper, we develop an intuitive and efficient numerical technique to solve the quantum evolution equation of generic lattice-refined models in loop quantum cosmology. As an application of this method, we extensively study the solutions of the recently introduced, lattice-refined anisotropic model of the Schwarzschild interior geometry. Our calculations suggest that the results obtained from the approach are accurate, robust and in complete agreement with the expectations from the von Neumann stability analysis of the model.

PACS numbers: 04.60.Pp, 04.60.Kz, 98.80.Qc

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Loop quantum cosmology (LQC) [1] is a symmetry-reduced application of loop quantum gravity [2], a theory that leads to spacetime that is discrete at Planck scales. An important result of LQC is that it is free of singularities. This is seen as a general consequence of the quantum evolution equation, which is a ‘difference equation’ for the wavefunction and does not break down in the deep Planck regime, where the classical singularity is located [3].

Although LQC models generically appear to exhibit good behavior on the Planck scale, some of these models run into problems in the semiclassical regime [9, 10]. For example, in the semiclassical sector of the isotropic model with positive cosmological constant \( \Lambda \), the Wheeler–DeWitt equation (which should be a close approximation to the difference equation of LQC) has solutions that oscillate on scales of \( a \sqrt{\Lambda} \), where \( a \) is the scale-factor. This scale becomes smaller with the expanding universe, ultimately becoming small enough to fall under the discreteness scale of the LQC difference equation. Thus, such an LQC model would necessarily deviate from the correct semiclassical behavior [10]. Similar issues arise in the context of anisotropic LQC models, such as Bianchi I LRS [6] and the Schwarzschild interior...
geometry [7], where the evidence of the poor semiclassical behavior emerges in the form of a
generic instability in the solutions of the quantum evolution difference equation [9].

These issues relating to the semiclassical behavior of certain LQC models can be resolved
by introducing lattice-refinement [11]. The main concept of this modification is that the
underlying lattice, over which the quantum wavefunction is supported, should be refined
during the evolution of the model. In general, this leads to a new type of the quantum
evolution equation, one that no longer has a uniform step-size thus presenting new challenges
for its analysis and solution. It should be emphasized that this modification is not necessary
in the Planck regime, but only in the semiclassical regime. This is because, near the
classical singularity there are only a few action steps of the Hamiltonian that are relevant
and therefore, lattice-refinement has little impact. Research on solving and analyzing lattice-
refined difference equations is of current importance because it would play an important role in
further advancing our understanding of semiclassical LQC and also help with the development
of effective equations and dynamics. In fact, preliminary results from lattice-refined models
of LQC suggest that it plays an important role in the development of an LQG-based candidate
for dark energy [4] and also for the origin of cosmic inflation [5].

Based on full LQG, there is a heuristic way of understanding why lattice-refinement is
essential. Consider the context of a cosmological model and a state of quantum geometry
corresponding to a fiducial volume cell. As the scale-factor increases, the number of vertices of
the state will have to increase correspondingly. To compute the action of a given Hamiltonian
constraint operator on a state in the full theory, one would then need to calculate the holonomy
around the faces of an elementary cube surrounding each vertex of the graph, underlying the
state. Now, as the number of such elementary cubes contained in the specified cell grow with
the universe, the area of their faces would decrease. And for this reason, the edge length which
is used for the computation of the holonomy would decrease too. Thus, the dependence of
this edge length on the scale-factor can be thought of as an effect of the full theory on simpler
cosmological models1. This is the main idea behind lattice-refined models.

In this work, we will present an intuitive and efficient numerical method for solving the
Hamiltonian constraint of lattice-refined models in LQC. It should be noted that solving the
Hamiltonian constraint with uniform discreteness structure i.e. without refinement, is relatively
straightforward and there are a variety of analytic and numerical techniques available [8, 9].
With lattice-refinement, the naive iterative approach towards computing a solution, simply
fails. The reason is as follows: due to the non-uniform stepping, computing the future value of
the wavefunction requires you to know the past-values of the wavefunction that are different
from those computed in the previous iteration. We explain this in detail with an example in the
next section. Due to this difficulty it is simply not possible to use the large variety of techniques
available in the numerical-methods literature for solving such difference equations. However,
in our current method we overcome this issue through the intuitive approach of using a local
interpolation formula for the solution, one that will enable us to compute precisely those values
of the wavefunction that are needed, to push the evolution forward. In other words, we will use
the values of the wavefunction that were computed in the previous iteration to obtain a local
analytic approximation formula of the solution. Then we will use this formula to compute
the values of the wavefunction needed for the next iteration. In this manner, we can obtain

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1 Evolution here is understood intrinsically i.e. the edge lengths change with respect to the spatial volume if this
provides a good internal time in a given regime. Since both the behavior of the internal time and the behavior of
dge lengths are determined from states, their relation in principle can be determined for any solution of a given
Hamiltonian constraint operator. This relation then provides the specific refinement. While such a calculation is
highly involved in the full setting, possible implications of the refinement can be included more easily in reduced
models.
a complete iterative solution to the lattice-refined quantum evolution equation for any LQC model. In fact, it should be noted that this method is applicable to any system of difference equations with non-uniform stepping. Such systems also appear in contexts of other sciences and engineering and our approach then would be readily applicable to those as well.

This paper is organized as follows. In the next section, through an explicit example, we will elaborate on the challenge associated with solving a 1D lattice-refined difference equation using a simple recursive scheme. We will then introduce our local interpolation based methodology and solve the same equation, effectively and efficiently. We will also solve the same relation using a different approach by performing a change of variable that will make the relation uniformly spaced (this can always be done in the context of 1D relations, but not in general) and then make an explicit comparison between the two solutions. In particular, we will demonstrate that they agree very well in the regime of interest. Next, we will proceed to solve a 2D lattice-refined LQC Hamiltonian constraint — one that cannot be mapped over to an equi-spaced form — using the same local interpolation-based approach. We will demonstrate that the results in that case agree with our general expectations of the solution, especially with regard to its stability properties. All of the above-mentioned equations which are solved in this work are relevant to the lattice-refined LQC model of the Schwarzschild interior geometry. We will end this paper with a discussion on our results and some remarks on related work [12].

2. Solving lattice-refined models

We start this section by very briefly reviewing the lattice-refined LQC model of the Schwarzschild interior geometry. We will follow the notation and treatment as presented in [10]. Consider a lattice with \(N\) vertices that is adapted to the symmetry of the Schwarzschild interior geometry. If there are \(N_\tau\) vertices along the direction of the triad labeled by \(\tau\) and \(N_\mu^2\) vertices in spherical orbits of the symmetry group whose triad is labeled by \(\mu\), then \(N = N_\tau N_\mu^2\). Therefore, the step-size along \(\mu\) would be \(\delta/N_\mu\) and that along \(\tau\) would be \(\delta/N_\tau\). Following the process detailed in reference [10] we arrive at the following expression for the Hamiltonian constraint for this LQC model with lattice-refinement:

\[
2\delta N_\mu^{-1} \left( \sqrt{|\tau|} + 2\delta N_\tau^{-1} + \sqrt{|\tau|} \right) \left( \psi_{\mu+2\delta N_\mu^{-1}, \tau+2\delta N_\tau^{-1}} - \psi_{\mu-2\delta N_\mu^{-1}, \tau+2\delta N_\tau^{-1}} \right) \\
+ \left( \sqrt{|\tau + \delta N_\tau^{-1}|} - \sqrt{|\tau - \delta N_\tau^{-1}|} \right) \left( (\mu + 2\delta N_\mu^{-1}) \psi_{\mu+4\delta N_\mu^{-1}, \tau} - 2(1 + 2\gamma^2 \delta^2 N_\mu^{-2}) \mu \psi_{\mu, \tau} \right) \\
+ \left( \mu - 2\delta N_\mu^{-1} \right) \psi_{\mu - 4\delta N_\mu^{-1}, \tau} \\
+ 2\delta N_\mu^{-1} \left( \sqrt{|\tau - 2\delta N_\tau^{-1}|} + \sqrt{|\tau|} \right) \left( \psi_{\mu - 2\delta N_\mu^{-1}, \tau-2\delta N_\tau^{-1}} - \psi_{\mu + 2\delta N_\mu^{-1}, \tau - 2\delta N_\tau^{-1}} \right) = 0.
\]

We now make further assumptions on how exactly the lattice spacing changes with changing \(\mu\) and \(\tau\). The simplest case is that the number of vertices in each direction is proportional to the geometrical area of a transverse surface. This yields \(N_\tau \propto \sqrt{|\tau|}\) and \(N_\mu \propto \sqrt{|\mu|}\). We start with a detailed analysis of this case in this section. An alternate choice for \(N_\tau\) and \(N_\mu\), which was also considered in [10] is based on the intuition that the number of vertices in a given direction is proportional to the geometric extension of that direction. The resulting difference equation is more complex, but has improved stability properties [10] and therefore much preferable as a model for our further study. We will analyze and solve that version later in this section.
2.1. One-dimensional lattice-refined relations

In the simpler context of \( N_\tau \propto \sqrt{\tau} \) and \( N_\mu \propto \sqrt{\mu} \), the above master difference equation is variable-separable, which makes it possible to rewrite it in the form of two 1D difference equations (one with \( \mu \) as the independent variable and the other with \( \tau \)) that have non-uniform stepping. If we write the wavefunction \( \psi_{\mu,\tau} = A_\mu B_\tau \), then these two ordinary difference equations take the form shown below:

\[
\left( \sqrt{\tau + 2\delta N_\tau^{-1}} + \sqrt{\tau} \right) B_{\tau + 2\delta N_\tau^{-1}} - \left( \sqrt{\tau - 2\delta N_\tau^{-1}} + \sqrt{\tau} \right) B_{\tau - 2\delta N_\tau^{-1}} = -\lambda \left( \sqrt{\tau + \delta N_\tau^{-1}} - \sqrt{\tau - \delta N_\tau^{-1}} \right) B_\tau,
\]

\[
(\mu + 2\delta N_\mu^{-1}) A_\mu + (\mu - 2\delta N_\mu^{-1}) A_{\mu - 2\delta N_\mu^{-1}} = \lambda (A_\mu + 2\delta N_\mu^{-1}) - A_{\mu - 2\delta N_\mu^{-1}} + 2(1 + 2\gamma^2 \delta^2 N_\mu^{-2}) \mu A_\mu + (\mu - 2\delta N_\mu^{-1}) A_{\mu - 2\delta N_\mu^{-1}}.
\]

(1)

Here \( \lambda \) is the separation constant. Now, consider the \( B_\tau \) relation that appears in the above equation (1). In order to solve it, we can try using a naive recursive method, but we will demonstrate here that such an approach will be unsuccessful.

To keep our discussion simple, assume \( 2\delta = 1 \). Let us start with some initial data for \( B_\tau \): \( B_{1/\sqrt{2}} = B_0 = 0 \) and \( B_1 = 1 \). We have enough data to find the third point that is present in the \( B_\tau \) relation (1), i.e. \( B_{1+1/\sqrt{2}} = B_2 \). Now, the parameter \( \tau \) can advance only in steps of \( 1/\sqrt{2} \). Therefore, the next value that \( \tau \) can take is \( \tau + 1/\sqrt{2} = 1 + 1/\sqrt{2} = 2 \). With \( \tau = 2 \), the \( B_\tau \) relation consists of terms like \( B_{2-1/\sqrt{2}}, B_2 \) and \( B_{2+1/\sqrt{2}} \). We computed the value of \( B_2 \), but to use the above relation, in order to evaluate \( B_{2+1/\sqrt{2}} \) we need to know the value of \( B_{2-1/\sqrt{2}} \). However, there is no direct way of determining this value from the known data already i.e. \( B_0, B_1 \) and \( B_2 \). Thus, the recursive method of solving this difference equation seems to fail because the previous iteration is unable to provide us with the data points needed for the next iteration. We resolve this problem in our method by performing a local interpolation to obtain an analytic formula using the values we already know, including the one we computed just from the relation itself. Then, we evaluate this interpolation formula at the points where we need to evaluate the \( B_\tau \) before proceeding forward. More specifically, we proceed as follows: a least-squares fit is done with points \( B_0, B_1 \) and \( B_2 \) to obtain an accurate formula for \( B_\tau \). This is then used to evaluate \( B_{2-1/\sqrt{2}} \) and thus, \( B_{2+1/\sqrt{2}} \) can be evaluated. Now, this computed value can be used to get a revised fit formula that is accurate in the local neighborhood of \( \tau = 2 - 1/\sqrt{2}, 2 \) and \( 2 + 1/\sqrt{2} \). Then, this revised formula is used to evaluate \( B_\tau \) at the missing point \( B_{2+1/\sqrt{2}} \). Along with the value of \( B_{2+1/\sqrt{2}} \), we can now compute \( B_{2+1/\sqrt{2}} \) and in this manner we can iterate and obtain the entire solution.

It should be noted that our interpolation-based approach is geared towards finding only pre-classical solutions i.e. solutions that do not vary much between the neighboring points. Restricting ourselves to pre-classical solutions in the context of this work will be meaningful because our goal is to explore the semiclassical sector of LQC models. In fact, as pointed out earlier, lattice-refinement plays a significant role only in this regime.

Complete solutions for the \( B_\tau \) relation for a few different values of the separation constant \( \lambda \) appear in figure 1. An immediate concern that one may have with our approach is the sensitivity of our results to the choice of an interpolation scheme i.e. the least-squares fit. In figure 1 we plot the percent difference, \( \delta \), between various solutions of the \( B_\tau \) relation, obtained by varying the type of interpolation scheme. The results clearly indicate that our method of
Figure 1. We plot the solution of the $B_\tau$ relation for different values of the separation constant $\lambda$, using the interpolation method as presented in the text. The dashed-line corresponds to a value of $\lambda = 1.0$, the dotted line to $\lambda = 2.1$ and the solid-line is for $\lambda = -5.1$. It is interesting to note that as the value of $\tau$ increases, all plots start to appear like a solid and continuous line. This is because the density of the points in each plot increases with increasing $\tau$ (the stepping in $\tau$ decreases as $\delta/N_\tau$).

Figure 2. We plot the percent difference $d$, for the solutions of the $B_\tau$ relation for different types of interpolation schemes. The solid-line is the percent difference between the solution based on the least-squares fit that we have usually used, with the solution computed using a second-degree polynomial fit. The dotted-line is the percent difference between the solution based on the least-squares fit, with the solution computed using a spline interpolation. The differences between these solutions are at 0.1% scale. This is a clear indication of our interpolation-based method being very robust.
solution is very robust (the percent difference is at 0.1% level). In the remaining portion of this paper, we continue to use the least-squares fit as our choice of the interpolation scheme.

An alternate approach that is applicable to all 1D lattice-refined relations [10, 11], suggests that we define a change of variable, $\tau = \frac{2}{3} \tau^{3/2}$, one that can make the $B_\tau$ relation equi-spaced to linear order in $\delta$. This is because, in terms of this variable, $B_{\tau + 2\delta N^{-1}} \approx B_{\tau + 2\delta}$ and then the $B_\tau$ relation (1), to linear order in $\delta$, becomes:

$$B_{\tau + 2\delta} - B_{\tau - 2\delta} \approx -(\lambda + 2) B_\tau \delta / 3 \bar{\tau}.$$ (3)

This particular relation has been studied before at various places in the literature [8] and its solutions are well known. It appears in the separable Bianchi I LRS (LQC, without lattice-refinement) model and the asymptotic form of its solution can be written analytically as:

$$B_\tau \propto \bar{\tau}^{-(\lambda + 2)/12} \propto \tau^{-(\lambda + 2)/8}.$$ (4)

In figure 3 we plot the two solutions obtained using these different approaches together. The agreement is clearly seen to be excellent.

We now move on towards a solution of the $A_\mu$ relation (2). Note that our interpolation-based method is applicable immediately to this relation as well. Indeed, this is the main merit of the technique; it is generically applicable to lattice-refined difference equations. Solutions for a few different values of the separation parameter $\lambda$ are plotted in figure 4. Let us proceed towards finding an asymptotic, analytic solution by transforming the relation to an equi-spaced one, using the new variable: $\bar{\mu} = \frac{2}{3} \mu^{3/2}$. Using $A_{\mu + 2\delta N^{-1}} \approx A_{\bar{\mu} + 2\delta}$ and also that $\bar{\mu} \gg \delta$ the $A_\mu$ relation can be simplified and written as an approximate ODE which takes the form shown below:

$$\bar{\mu} \frac{d^2 A}{d\bar{\mu}^2} - \gamma^2 \frac{\bar{\mu}}{144} \left( \frac{\bar{\mu}}{144} \right)^{1/3} A = \frac{1}{3} \frac{dA}{d\bar{\mu}}.$$ (5)

Figure 3. We plot the solution of the $B_\tau$ relation solved using the interpolation approach alongside the approximate analytic solution of the same relation. Here the solid-line is the analytic solution. There is an excellent agreement between the two approaches.
This simplified equation, that can be assumed to be valid asymptotically, is easy to solve yielding solutions in terms of modified Bessel functions:

\[ A_\mu \propto \mu^{(\lambda+1)/4} I_{(\lambda-1)/4}(\gamma \mu/2), \mu^{(\lambda+1)/4} K_{(\lambda+1)/4}(\gamma \mu/2). \] (6)

It should be noted that these modified Bessel functions exhibit exponential behavior asymptotically, which is related to the fact that the separable Schwarzschild model under consideration (with \( N_\tau \propto \sqrt{|\tau|} \) and \( N_\mu \propto \sqrt{|\mu|} \)) exhibits generic instability [10]. In figure 5 we plot this analytic solution alongside the numerically generated solution. The agreement is seen clearly to be excellent.

2.2. Two-dimensional lattice-refined relations

In the previous section, we solved the master Schwarzschild interior LQC relation for a simple intuitive choice for \( N_\tau \) and \( N_\mu \), one that makes the relation separable in terms of variable. We noted that the solutions to that version of the relation are unstable, which implies that the model cannot be viable as a physical model for the quantum geometry of the Schwarzschild black-hole interior. This outcome is consistent with what was observed before [10] wherein von Neumann numerical analysis of the same relation yielded the same unstable characteristics. Now, let us consider the alternate choice of \( N_\tau \) and \( N_\mu \), the one that has improved stability properties: \( N_\mu \propto \sqrt{|\mu|} \) and \( N_\tau \propto \mu/\sqrt{|\tau|} \). It is very interesting that the notion of von Neumann stability that ordinarily plays a significant role in numerical analysis of partial differential equations, provides an excellent mechanism for limiting the wide range of possible choices for \( N_\tau \) and \( N_\mu \) [10].

With this choice of \( N_\tau \) and \( N_\mu \), the relation under consideration is not variable-separable. This is because the step-size of any of the two independent variables depends on the other variable. In addition, to complicate matters further, it is not possible to find two independent quantities that would make it possible for the difference equation to be transformed into a uniformly spaced version [10], like we did for the 1D relations in the previous section. Thus,
Figure 5. We plot the logarithm of the solution of the $A_\mu$ relation solved using interpolation approach alongside the approximate analytic solution of the same relation. Here the solid-line represents the analytic solution. There is an excellent agreement between the two approaches, especially for large $\mu$.

we have no choice but to solve this non-uniformly spaced, partial difference equation directly. Fortunately, the local interpolation method, that we have introduced in this paper and detailed in the previous section is readily applicable to this challenging problem.

Conceptually, our approach towards solving this 2D lattice-refined difference equation is identical to the one we took in 1D. The relation connects various points on the lattice, which are shown clearly in figure 6. In order to solve for the value $\psi_{\mu + 2N_\tau^{-1}, \tau + 2N_\mu^{-1}}$ to evolve the solution forward, we need values for the wavefunction that in general, have not been computed from previous iterations. However, these values are in close proximity to values that are already known before, therefore it is very natural to perform a local 2D interpolation using the values that are known, to compute the values that are needed. To demonstrate the success of this approach, we evolve a semiclassical wave-packet and show our results in figure 7. We explain the evolution process in some detail below.

We start by assuming a Gaussian profile for the wavefunction $\psi$, and evolve it using the interpolation approach that we developed in the previous section. First, a 2D grid for $\mu$ and $\tau$ points is set up by choosing arbitrarily an $(\mu, \tau)$ ordered pair with both $\mu$ and $\tau$ being large positive integers. The grid implements the fact that $\mu$ advances as $\mu + 1/\sqrt{\tau}$ while $\tau$ advances as $\tau + \mu/\sqrt{\tau}$. Each point on the grid is labeled by two integers $(m, n)$ which would represent the corresponding position of the point on an uniformly spaced grid. Note that although we do not have an uniformly spaced grid, these labels help with ‘book-keeping’ i.e. identifying the points being used for iterations. The first row of points have a constant $n$ label and are of the form $(\mu + 1/\sqrt{\tau}, \tau)$ i.e. the set of points (relevant to the relation) for which $\tau$ is a constant. All the other points evaluated on the grid are of the form $(\mu + 1/\sqrt{\tau}, \tau + \mu/\sqrt{\tau})$. Each of these points act as a central anchor for the other points that enter the relation. For example, referring to figure 6, it is the central point of the polygon that we evaluate while laying out the grid. All the points ‘connected’ to it via the relation, can then easily be evaluated.
Figure 6. The numerical ‘stencil’ that is used for the evolution of a wave-packet in the lattice-refined LQC model of the Schwarzschild interior. The graphic depicts all the values of the wavefunction that are connected by the evolution difference equation (Hamiltonian constraint).

Figure 7. Stable evolution of a Gaussian wave-packet in the lattice-refined LQC model of the Schwarzschild interior, based on the local 2D interpolation method as presented in this paper. The scales of the axis depicted above are based on \((m, n)\) and are arbitrary. The accompanying text gives more detail.

Now that we have explained how the 2D grid is set up, let us turn our discussion towards evaluating and evolving the wavefunction at these discrete points. As mentioned before, the
initial wavefunction is assumed to be a Gaussian wave-packet centered on $\mu$-axis. The values of the wavefunction at all the points marked as dark circles in figure 6 are evaluated using the Gaussian wave-packet expression and this data is used to find the value of the wavefunction at the location $(\mu + 1/\sqrt{\tau}, \tau + \mu/\sqrt{\tau})$, which is the white circle in the figure. In this manner, we obtain the value of the function at points in the vicinity of $(\mu + 1/\sqrt{\tau}, \tau)$ for all $\mu$. In other words (with respect to labels $m, n$), we compute values of the wavefunction for all values of $m$, for a given $n$. Then we move on to the next $n$-step (recall that $m$ and $n$ are just integer labels for points with respect to the starting point, so they advance by one along their respective axes). But, now we find ourselves in the same situation as in 1D i.e. we do not have the data points we need to make the evolution progress further than the 2D master relation. Therefore, we utilize a 2D least-squares fit to compute these unknown values using data points known from the previous $n$-step, that have the same value of $m$. In other words, to find the value of the wavefunction that is needed for the 2D relation, we find a local 2D interpolation formula for points in the neighborhood of $(m, n)$ using data from the $(m, n - 1)$ point and the points around it (see figure 6). This yields an explicit scheme for progressing the evolution forward.

The results of this approach are depicted in figure 7 (for the Schwarzschild interior model) and figure 8 (for the Bianchi I LRS model). It is clear that the evolution of the wave-packet is very smooth and stable as one would expect for the semiclassical regime. It should be noted that for both figures, we chose the domain to be such that $\mu > 4\tau$, which happens to be the unstable region of the domain for the equi-spaced case [9]. Therefore, it is clearly evident that lattice-refinement has ‘cured’ the problem of von Neumann instability in such anisotropic LQC models. This is completely consistent with the analytical stability analysis that was performed in [10]. The starting values we used for $(\mu, \tau)$ were (100000, 25000) and we performed approximately 200 iterations in both directions ($250 \times 150$ grid points). To check the robustness of these results in the 2D model, we attempted tests of a similar type to those we performed in the 1D case. The outcome of these tests is strongly indicative of the fact that
the choice of the interpolation scheme to be used does not alter results in a significant way (no more than tenth of a percent).

In the near future we shall plan on developing a high-performance numerical code based on this current proof-of-concept approach that will be able to compute on much larger grid sizes, in a very effective manner. Then we will be able to study the semiclassical wave-packet trajectories on a much larger domain and compare them with the recently published effective equations and effective dynamics [12].

3. Summary & discussion

In this paper, we have presented an effective numerical method for solving the Hamiltonian constraint in generic LQC models with lattice-refinement. The method is based on performing a local interpolation to obtain needed values from known values.

We demonstrated the success of the method by solving the lattice-refined LQC model of the Schwarzschild interior geometry using two different refinement models. In the first refinement model, the 2D quantum evolution equation becomes variable-separable, reducing down to two 1D lattice-refined equations that we then solved using our interpolation technique and also analytically, using asymptotic approximations. Using this model as a test case, we compared the accuracy and the robustness of our interpolation approach. Then we chose a refinement model that was inherently non-separable, and solved that in the context of the evolution of semiclassical wave-packets, extending the same interpolation approach in two dimensions. The results from our evolutions reinforce recently published results on the von Neumann stability analysis of the lattice-refined Schwarzschild relation [10].

One preliminary remark that we would like to make is that, based on our current results we do not see any evidence for the kind of behavior depicted in [12] which was obtained from a study of the effective semiclassical dynamics. In particular, in [12] the effective semiclassical Hamiltonian constraint is solved for the Schwarzschild interior model and those results suggest that: (a) in the equi-spaced model, there is a ‘bounce’ and the quantum dynamics matches the solution to a different black hole (note that this ‘bounce’ is not through the classical singularity, it actually avoids it!); and (b) in the lattice-refined case, one obtains an equilibrium solution asymptotically, thus avoiding the singularity again. Results from the explicit evolution of wave-packets using a quantum Hamiltonian constraint appear to be qualitatively different. For example, for the equi-spaced case, past results [8, 9] suggest that the quantum evolution equation allows for an evolution right through the classical singularity, as opposed to avoiding it. In addition, our preliminary results from the lattice-refined case show no indication of an asymptotic equilibrium, again as suggested in [12]. However, we make these preliminary remarks with caution. There are many caveats associated with the results presented in [12], as the authors themselves point out (basically, they include one quantum effect, but ignore others). In addition, the approach towards numerical evolutions of wave-packets in lattice-refined LQC that we are presenting in this paper is just a proof-of-concept demonstration, and is currently lacking any serious analysis and study. We will attempt to explore this apparent ‘discrepancy’ in the near future.

Acknowledgments

We thank Martin Bojowald and Daniel Cartin for many useful discussions related to this work, and for their comments on an early draft of this paper. We both acknowledge research support from the University of Massachusetts, Dartmouth. In addition, GK is supported by a
grant from Glaser Trust of New York and SS is supported by various grants from the UMD Foundation.

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