Kardar-Parisi-Zhang Equation And Its Critical Exponents Through Local Slope-Like Fluctuations.

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Growth of interfaces during vapor deposition are analyzed on a discrete lattice. For a rough surface, relation between the roughness exponent $\alpha$, and corresponding step-step (slope-slope) couplings is obtained in (1+1) and (2+1) dimensions. From the discrete form and the symmetries of the growth problem, the step-step couplings can be determined. Thus $\alpha$ can be obtained. The method is applied to the linear EW as well as to the non linear growth equation, the Kardar-Parisi-Zhang equation in all dimensions. Further, exact exponents for the fourth order linear and nonlinear terms relevant to the growth with non conserved noise are obtained in (1+1) and (2+1) dimensions.

It has been shown recently [1] that the kinetic considerations on a growing surface can lead to various growth terms in a growth equation representing a conserved growth from vapor. A growth equation helps in understanding the roughening of a growing interface. It represents dynamics of height fluctuations due to the deposition noise. The non-linear behavior in stochastic equations is of general interest. The mode coupling due to the non-linearity can lead to the nontrivial physical effects that are crucial in understanding many stochastic processes [2,3]. However, in the presence of non-linearity, absence of analytical solutions in many cases does not allow a conclusive understanding of the roughening phenomenon. In the present work, slope components are discretized (i.e. steps). The discrete form depends upon the height difference between the nearest neighbor sites. It represents step at a given site and is proportional to the local slope. In the presence of non conserved noise the height difference fluctuates within the correlation length. We refer to these fluctuations as the step fluctuations. Coupling conditions between the step fluctuations can be determined from the discretization of the relevant terms in the growth equation and the symmetry associated with the growth problem. These conditions determine the exact asymptotic behavior of the growth. In particular, in a growth, roughness along the interface is measured in terms of the exponent $\alpha$ and the time evolution of the height-height ($h-h$) correlation in terms of the exponent $z$. Knowledge of $\alpha$ leads to the determination of $z$, in the case of conserved growth from the scaling behavior of noise and in the case of KPZ equation from the Galilean invariance [3]. The method is further applied to fourth order linear and non linear terms considered to be important in understanding the growth phenomenon. Since we obtain exact results the renormalization behaviour of non linear terms can be obtained without ambiguity.

In the following we develop the method for obtaining the critical exponents for the second order growth equation, both conserved and non conserved. Same method is later extended for fourth order terms.

A linear equation representing interface motion normal to the surface can be obtained in the frame of reference moving with the interface velocity by considering interplaner hopping of adatoms on the interface with a bias for downward or in-plane hopping toward step edge [1]. It has the form

$$\frac{\partial h}{\partial t} = \nu_0 \nabla^2 h + \eta$$

where, $\nu_0$ explicitly depends upon $F$, and $\eta$ is the noise due to the randomness in the deposition flux. It has the correlation given by $\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t')$. The angular brackets denote the average of the contents. Eq. (1) is known as Edward-Wilkinson (EW) equation [4]. The lowest ordered non-linear correction to EW equation was introduced by Kardar, Parisi, and Zhang [5]. The resulting equation,

$$\frac{\partial h}{\partial t} = \nu_0 \nabla^2 h + \lambda (\nabla h)^2 + \eta$$

is known as KPZ equation. This is a non-conservative equation.

The KPZ equation has large number of applications in growth [2,3]. Its variants are also useful in analyzing various physical situations [6]. The directed polymer representation of the KPZ equation [7] is one of the most studied equation for understanding the non-linear dynamics of the interfaces. Being non-linear an exact solution of the KPZ equation is not possible analytically. However, many physically relevant quantities are obtainable from the analysis of the equation that can be measured experimentally. In particular the scaling exponents that characterize the growth can be measured experimentally. By applying scaling transformations, $x \rightarrow bx$, $t \rightarrow b^2t$, and $h \rightarrow b^z h$ to the KPZ equation and demanding that the equation remains invariant under these transformations one obtains, $z = 2$, $z + \alpha = 2$, and $z = 2\alpha + d$ for the equation in $d$ dimensions. Since KPZ equation is obtainable from the noisy Burger’s equation [8,2,3] using the transformation $v \rightarrow \nabla h$, where $v$ is the velocity field,
Galilean invariance is implied in the KPZ equation. The coefficient $\lambda$ explicitly appears in the Galilean transformation rendering it constancy under renormalization [3]. As a result,

$$z + \alpha = 2 \quad (3)$$

is the relation valid in all dimensions. One can measure $\alpha$ from the height-height (h-h) correlations,

$$G(x, t) = \frac{1}{N} \sum_{x'} (h(x + x', t) - h(x', t))^2$$

$$= x^{2\alpha} f \left( \frac{x}{\xi(t)} \right) \quad (4)$$

where, correlation length $\xi(t) \sim t^{1/z}$. In the limit $x \to 0, f \to 1$. Time exponent $\beta$, where $z = \alpha/\beta$ can be obtained by measuring the width over a substrate of length $L$ as, $w_2 = \frac{1}{N} \sum_x (h(x, t) - \bar{h})^2 = L^{2\alpha} \bar{g} \left( \frac{1}{\xi(t)} \right)$. It can be shown that [2] for small times $w_2 \sim t^{2\beta}$.

In 1+1- dimensions $\alpha$ has been determined exactly using perturbation expansion [5] and also using Fokker-Planck equation with fluctuation - dissipation theorem [2]. In 2+1- dimensions perturbation theory fails since renormalization process indicates existence of only strong coupling regime. The dimension $d_c = 2$ is a critical dimension for KPZ equation. For $d > 2$, renormalization reduces $\lambda$ to zero, signaling EW behavior in the weak coupling regime. The strong coupling regime has a different behavior than EW equation. Thus, for $d > 2$, a phase transition is observed for KPZ equation [2,3]. To obtain various exponents in dimensions $d \geq 2$, non-perturbative methods are employed. There are computer simulations [9-11] and theoretical methods [12-14]. Most of the theoretical approaches suggest that there is an upper critical dimension beyond which strong coupling vanishes indicating weak coupling EW like behavior. Simulations [9] suggest that there is no upper limit on the dimension. Using effective large distance field theory subject to a few phenomenological constraints, it is shown that $\alpha = 2/5$ and $2/7$ in 2+1 and 3+1 dimensions respectively. This claim is not supported by large scale computer studies [10]. It is shown [10] that the values of $\alpha$ are not rational numbers as predicted in reference [12]. Using a pseudo spectral method [11], numerical solution of the KPZ equation in 2+1 dimensions is obtained. Based on this method, $\alpha$ is obtained from the saturated width $(0.37 \pm 0.02)$, height-height correlation $(0.38 \pm 0.02)$ and structure factor $(0.40 \pm 0.02)$. In the present work we obtain exact values of the exponents of the KPZ equation. This is done by discretization of the KPZ equation and from the symmetry requirements, evaluating the step-step couplings. A step fluctuation at site $i$ is defined to be $(h_i - h_{i+1})$. We will refer to it simply as `step’. It is proportional to the local discrete slope at site $i$.

Consider a one dimensional substrate with a lattice constant $a$ such that $a > l_d$, where $l_d$ is the diffusion length. The physical lattice constant will be $\leq a$. We consider growth on this substrate where atoms are depositing on the physical lattice and follow the relaxation rules as depicted by a conservative growth equation. In 1+1- dimensions such an equation can be written as,

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} J + \eta \quad (5)$$

where, $J$ is the local particle current at $x$. Eq.(5) is written in the frame of reference moving with the average velocity of the interface defined by the incident flux $F$.

In 1+1- dimensions, the (h-h) correlations in the discrete form are

$$G(n) = \langle (h_i - h_{i+n})^2 \rangle \quad (6)$$

We assume that the correlation length $\xi$ is very large compared to the length $na$. We define step at $i$ as

$$\delta x_i = h_i - h_{i+1} \quad (7)$$

The local slope is then $-\delta x_i/a$. In terms of steps we have

$$G(2) = \langle \delta x_i^2 + \delta x_{i+1}^2 + 2\delta x_i\delta x_{i+1} \rangle . \quad (8)$$

Let $\delta x_i^2 \geq \delta x_{i+1}^2 \geq \delta^2$ and , $\delta x_i\delta x_{i+1} \geq s\delta^2$ where $s$ is the coupling between the steps around $i$. The distribution for $\delta x_i$ is always symmetric around zero and time independent [15] for an ensemble average.

In the limit $\xi \to \infty$ Eq. (4) reduces to $G(x) = cx^{2\alpha}$ where constant $c = G(1)$. Hence Eq. (8) can be written as

$$2^{2\alpha} = 2 + 2s \quad (9)$$

where $G(1) = \delta^2$ in the discretized case. Coupling $s$ uniquely determines $\alpha$. Thus, for $s=-1/2,0, and 1$, $\alpha$ is $0, 0.5$, and 1 respectively.

We analyze the growth equations in 1+1- dimensions in the discrete form to obtain $\alpha$ for various terms. Consider EW equation,

$$\frac{\partial h}{\partial t} = \nu_0(\delta x_{i-1} - \delta x_i) + \eta_i \quad (10)$$

The first term on the right side represents second derivative in terms of local steps. In order that the steady growth continues , at any instant the surface must be characterized by the fluctuations in the second term i.e. various values of second derivative. These must be consistent with the requirement that $\delta x_i^2 \geq \delta x_{i+1}^2 \geq \delta^2$ and that corresponding distribution of $\delta x_i^2$ must be scalable. From the analysis of the linear second order growth equation [2,3] $\alpha = 0.5$. Hence $s$ must be zero for the EW equation. $s = 0$ correspond to Gaussian distribution of slopes with average fluctuation of $\delta^2$. The average squared difference between consecutive slopes is
\[
\langle (\delta x_i - \delta x_{i-1})^2 \rangle = 2(1-s)\delta^2 \text{ and the average squared sum is } \langle \left(\frac{\delta x_i + \delta x_{i+1}}{2}\right)^2 \rangle = \frac{1}{2}(1+s)\delta^2. \text{ For given } \delta^2, \text{ the volume in the parameter space formed with these two parameters, squared difference and sum of consecutive slopes, decides the number of configurations available for the second derivative in Eq. (10). This volume is maximum for } s = 0. \text{ This result can be generalized to any higher order term by noting that the steps in the present case are proportional to the particle current corresponding to the EW term. Thus by following above argument for any other conservative term in } 1+1- \text{ dimensions, the condition for maximum number of configurations is same as } <J_iJ_{i+1}> = 0. \text{ Thus, when all the values } -1/2 \leq s \leq 1 \text{ are scalable in } 1+1- \text{ dimensions, in the presence of non conserved white noise the EW term drives the surface morphology to maximize the configurations for the linear second order term.}

For KPZ equation, the scaling transformations lead to \( \alpha = 1/3, \text{ i.e. } s = -0.206299. \) Renormalization is expected as a result of the non linearity. The KPZ exponents in \( 1+1- \) dimensions are obtained without resorting to perturbation approach [2]. Here we obtain by referring to the step-step coupling. For a scalable surface \( \delta^2 \) is constant. For a given \( \delta^2 \), each scalable value of \( s \) will correspond to certain distributions of slopes on the surface. Any value of \( s \) lying between \( -1/2 \) and \( 1 \) is possible under renormalization since Eq. (2) does not indicate any condition that will prefer any particular value of \( s \). Under these conditions, the surface is expected to exhibit the maximum range of fluctuations in the local slope \( \delta x_i \) consistent with the average of \( \delta^2 \). Fig. 1 displays the distributions of slopes for \( s = 0 \) and \( s = 1 \) with \( \delta^2 = 1 \). Distribution for \( s = 1 \) is sharp and symmetric on both sides and for \( s = 0 \) it is Gaussian. For non zero value of \( s \), the average squared difference and sum of consecutive slopes constrain the fluctuations at a given site. e.g. for \( s = 1 \) the average slope difference is zero while sum is \( \delta^2 \) giving the sharp distribution as in Fig. 1. For any scalable \( s \neq 0 \), a double peaked distribution is obtained. However, the Gaussian distribution provides the maximum range of slopes. Hence, for KPZ equation the renormalization drives the system toward \( s = 0 \) with \( \alpha = 0.5 \).

In \( 2+1- \) dimensions, we need to consider the steps in two directions. We define, \( \delta x_{i,j} = h(i,j) - h(i+1,j), \delta y_{i,j} = h(i,j) - h(i,j+1), \delta x_{i-1,j} = h(i-1,j) - h(i,j), \delta y_{i,j-1} = h(i,j-1) - h(i,j) \ldots \) over a square lattice. We consider isotropic situation so that, \( <\delta x^2_{i,j}> <\delta y^2_{i,j}> = \delta^2 \) and the linear coupling constant in the two directions is same, \( s \). In \( 2+1- \) dimensions, apart from the linear couplings in the two directions, cross-coupling \( <\delta x_{i,j}\delta y_{i,j}> \) also contributes. \( \alpha \) can be expressed in terms of the cross coupling through the following relation,

\[
<\delta x(i,j)\delta y(i,j)\ldots> <\delta x(i,j)\delta y(i,j)\ldots> = \alpha \]

This can be written as,

\[
G(\sqrt{2}) = <\delta x^2_{i,j-1}> + <\delta y^2_{i,j}> + 2<\delta x_{i,j-1}\delta y_{i,j}> \]

This leads to

\[
2^\alpha = 2 + 2q \tag{11}
\]

Consider a set of \( N \) number of pairs of slopes such that the average cross coupling over this set is \( q\delta^2 \). One can characterize such a set by a distribution of slopes, along with a distribution of difference and sum of the slopes. All these distributions are needed simultaneously to obtain the same coupling \( q \) by averaging. Same is true in the case of linear coupling. In the analysis to be followed we will consider set of pairs of steps while averaging over cross or linear couplings.

To analyze various terms in \( 2+1- \) dimensions, we consider a two dimensional substrate with \( N \) number of sites on a space renormalizable lattice. \( N \) is large enough for averaging purpose. The growth is assumed to have reached steady state with correlations developed over large distances. Various configurations are developed on the surface in statistically significant number so that the spatial averaging can be considered to be equivalent to the ensemble averaging. In \( 2+1- \) dimensions given a reference height, morphology can be defined by specifying the \( N \) pairs of slopes, one at every lattice point. In order to specify it in terms of linear couplings, a pair in \( x \)-direction and a pair in \( y \)-direction are necessary. Thus a quadruplet of slopes at a given site is needed. The morphology defined by these quadruplets for a scalable surface is such that the average over linear step coupling is \( s \) and that over the cross coupling is \( q \) consistent with Eq. (12). However, the relevant growth term in the growth equation demands certain relationship between \( s \) and \( q \), which helps in obtaining the growth exponents. We analyze the EW equation in \( 2+1- \) dimensions. The EW term in discrete form is \( (\delta x_{i-1,j} - \delta x_{i,j} + \delta y_{i,j-1} - \delta y_{i,j}) \). Steps in this term form a quadruplet of steps at a given site \( i \). Thus there are \( N \) quadruplets defining the frozen EW surface. This surface represents solution of the Eq. (1). This set of steps can be considered to form \( 2N \) cross couplings \( (\delta x_{i,j}\delta y_{i,j}) \) and \( (\delta x_{i-1,j}\delta y_{i,j-1}) \). The morphology of the surface is defined through the specification of set of quadruplet defined at every site, i.e. by specifying the
value of $\nabla^2 h$ at every point. Let the cross coupling constant be $q_{ew}$. This surface can also be described by linear couplings defined over the same set of quadruplets over $N$ sites. Let $s_{ew}$ be the linear coupling characterizing this set of $2N$ linear couplings $(\delta x_{i,j}, \delta x_{i-1,j})$ and $(\delta y_{i,j}, \delta y_{i,j-1})$ over $N$ number of sites. Let $2s_{ew}/q_{ew} = 1 + p$, where $-1 < p$ so that $\delta x_{i-1,j} > \delta x_{i,j}$ then $\delta y_{i,j-1} > \delta y_{i,j}$. The subscript $N$ denotes the averaging is over $N$ sites and the negative sign ensures the directions in which the slopes are measured are consistent. This implies that, the linear coupling $s_{ew}$ over the given surface representing solution of EW equation is same as cross coupling $q_{ew}$ on another surface where slopes from the quadruplets defined over the original surface are scaled by $\sqrt{1+p}$ while keeping the $\delta^2$ constant. These two conditions cannot be simultaneously satisfied over the same original surface for any non zero $p$. Hence $p$ must be zero. This implies $s_{ew} = q_{ew}$. From Eqs. (11), (12) we get $\alpha = 0$ for the EW term in 2+1- dimensions. Thus in 2+1- dimensional case, the surface morphology is characterized by scalable $\nabla^2 h$ term for the condition $s = q$.

Consider KPZ term in 2+1- dimensions. The discrete form is $(\delta^2 x_{i,j}) + (\delta^2 y_{i,j})$. This form suggests that on the $N$ sites there are $N$ pairs of slopes $(\delta x_{i,j}, \delta y_{i,j})$. These pairs will be characterized by a cross coupling $q_{kpz}$. This set of pairs will have the appropriate rotational symmetry. The surface morphology is defined over $N$ sites with this set of $N$ pairs. Thus in 2+1- dimensions, over $N$ sites it is necessary and sufficient to have only $N$ pairs of slopes to define the morphology of the KPZ surface as compared to $2N$ in the case of EW surface. Corresponding to this surface there will be $2N$ linear couplings $(\delta x_{i-1,j}, \delta x_{i,j})$ and $(\delta y_{i,j-1}, \delta y_{i,j})$ each being defined over $N$ sites and characterized by $s_{kpz}$. We will consider set of $N$ quadruplets $(\delta x_{i-1,j}, \delta x_{i,j}, \delta y_{i,j-1}, \delta y_{i,j})$ defined over the $N$ sites (same as in the case of EW term) over which these linear couplings are defined. The set of cross couplings is also defined over the same set of quadruplets since the symmetry of the problem allows any of the cross coupled pairs to be chosen from a given quadruplet. Let $2s_{kpz}/q_{kpz} = 1 + p$. As is discussed in the case of EW equation, we identity linear couplings $2s_{kpz}$ same as cross coupling $q$ on a surface where steps are scaled by a factor of $\sqrt{1+p}$ and $\delta^2$ remains unchanged. This is possible only for $p = 0$. Hence, $s_{kpz} = q_{kpz}/2$. Using this relation, Eqs. (11) and (12) we obtain

$$\alpha = 0.35702 \tag{13}$$

for this surface, accurate up to 5 digits. One can compute it with arbitrary accuracy. Note that for KPZ equation, the surface can also be defined using $2N$ pairs of slopes as in the case of EW surface. However, for KPZ surface this $2N$ pairs of slopes are sufficient to define a morphology but not necessary. This will lead to the well known result of EW surface for KPZ case in the weak coupling (in this case small $\lambda$ in Eq. (2)) limit.

In 3+1- dimensions, we consider a cubic lattice. For the additional degree of freedom we define $\delta z_{i,j,k} = h(i,j,k) - h(i,j,k+1)$ as the step in the $z$ direction. The discrete form of the KPZ term is $(\delta^2 x_{i,j,k}) + (\delta^2 y_{i,j,k}) + (\delta^2 z_{i,j,k})$. There is a triplet of slopes at every site on the surface. It is required that average cross coupling over $N$ sites for any two directions must result in to same $q$ value, $q_{kpz}$. We can choose a set of $N$ pairs of cross coupling between $\delta x_{i,j,k}$ and $\delta y_{i,j,k}$ characterized by $q_{kpz}$. The slope $\delta z_{i,j,k}$ must be chosen so that its cross coupling with $\delta x_{i,j,k}$ or $\delta y_{i,j,k}$ is $q_{kpz}\delta^2$. To satisfy this condition we consider at every site the average value $\frac{1}{2}(\delta x_{i,j,k} + \delta y_{i,j,k})$ as one of the slopes in the cross coupling. The other slope is $\delta z_{i,j,k}$ chosen in such a way that corresponding coupling will result in to a set of cross couplings that averages over $N$ sites to $q_{kpz}\delta^2$. This ensures the requirement that cross coupling between slopes in any two given directions is same, $q_{kpz}$. Thus, $2N$ pairs characterized by $q_{kpz}$ are required to define the morphology of a surface whose growth dynamics is governed by KPZ equation on a three dimensional surface. Consider the linear couplings defined over this three dimensional surface. There are $3N$ linear couplings on the surface. As has been argued in the case of 2+1- dimensions, in the present case, on this KPZ surface only $2N$ cross couplings are characterized by $q_{kpz}$. Both the couplings are defined over the set of sextet of steps defined at $N$ sites. Following the arguments as in the case of (2+1) dimensional KPZ growth one arrives at the relation $s_{kpz} = \frac{2}{3}q_{kpz}$ on a three dimensional surface. $s = (2/3)q$ gives $\alpha = 0.28125$.

By continuing the argument for 2+1- and 3+1- dimensional growth it is possible to obtain the relation between $s_{kpz}$ and $q_{kpz}$ in any dimension. On a $d$ dimensional surface, $\alpha$ can be determined from the condition $s_{kpz} = \frac{d-1}{2}q_{kpz}$ which provides next higher value of $\alpha$. Thus for KPZ equation, $\alpha$ can be determined in any dimension. This implies that there is no upper critical dimension for the KPZ equation. For EW term on a surface with dimension $d > 1$, the condition is always $s = q$ giving $\alpha = 0$.

Next we consider the application to fourth order terms. It has been shown [16] that the relevant non linearities to this order are $\nabla^2(\nabla h)^3$ and $\nabla \cdot (\nabla h)^3$. The corresponding linear term is $\nabla^4 h$, referred as Mullin’s term [17]. The application of perturbation up to second order to $\nabla^2(\nabla h)^2$ term shows that its coefficient is not renormalized [18]. In the case of $\nabla \cdot (\nabla h)^3$ it is shown to generate EW term [19,20].

In 1+1- dimensions $\alpha$ is unity for fourth order linear and $\nabla^2(\nabla h)^2$ term [16,18]. In the later case it implies that the corresponding coefficient in the growth equation is not renormalized. It is 0.5 for $\nabla \cdot (\nabla h)^3$ term [19]. Consider the quantity $<(\delta x_{i-1,j} - \delta x_{i,j})^2 = 2 - 2s$. It represents average local second derivative on the surface
and the difference in the local particle current for the EW term. If this difference is not zero on the average, then corresponding term contributes in the growth evolution of the surface. In the present case it means that EW term is present if this difference is not zero. Hence for any other conservative term to be exclusively present in the growth equation, this current difference must be zero. It is zero for $s = 1$ giving $\alpha = 1$. Thus in 1+1 dimensions for $\nabla^2(\nabla h)^2$ and $\nabla^4 h$ terms $s = 1$ [2,3]. The term $\nabla \cdot (\nabla h)^3$ is similar to EW term in that the current for this non linear term is related to EW current and preserves the sign of the current locally. Thus condition for maximum number of configurations for EW term is same as that for the non linear term. Thus for the $\nabla \cdot (\nabla h)^3$ term $s = 0$ or $s = 1$ in 1+1 dimensions. Thus in 1+1 dimensions with non conserved noise, $s = 0$ and $s = 1$ are the only two possibilities defining the surface morphology.

In 2+1 dimensions as before we must consider linear and other than linear (otl) couplings. So far we have encountered cross coupling as one of the otl couplings. The main difference in the analysis of the higher order terms using these couplings compared to second order terms is that, 1) more than one lattice points are to be considered for defining the discrete form of the term and 2) otl couplings other than cross couplings are needed. The roughness exponent is obtained by equating the total minimum linear couplings required to define the local morphology to the total minimum otl couplings along with the Eq. (12).

Consider the term $\nabla^2(\nabla h)^2$ or ldv term, in 2+1 dimensions. In this case the corresponding discrete form is $\delta^2x_{i+1,j} + \delta^2x_{i-1,j} - 4\delta^2x_{i,j} + \delta^2x_{i,j+1} + \delta^2x_{i,j-1} + \delta^2y_{i+1,j} + \delta^2y_{i,j+1} + \delta^2y_{i,j-1} - 4\delta^2y_{i,j} + \delta^2y_{i+1,j} + \delta^2y_{i,j+1} + \delta^2y_{i,j-1}$. There are ten steps required to define the local morphology of the surface. As can be seen from Fig.2, these steps are spread over total five lattice points. The morphology at every lattice point must be defined over five lattice points associated with the given lattice point. To define the local morphology in terms of linear couplings, we note that linear couplings at $(i-1,j)$, $(i+1,j)$, $(i,j+1)$ and $(i,j-1)$ are necessary. These contain total of 16 steps. However, the very nature of linear couplings is such that minimum two pairs of steps in orthogonal directions are needed at a given lattice point to allocate it the correct relative height. Thus the number of steps involved are in excess to the number required to define the local morphology. This situation is also encountered in the case of KPZ equation previously. The contribution of linear couplings is 8s. Let’s consider otl couplings. Since the linear couplings are added, in the explicit form, all the steps, defining the morphology appear in the expression involving sum of eight different linear couplings in terms of the steps. Hence, while otl couplings are considered, the relative sign of all the couplings therein must be same. This ensures the presence of all the steps in the explicit expression for otl couplings. This means that otl couplings be formed between steps that are parallel or following. The example of a following type coupling is $< \delta y_{i-1,j}\delta x_{i,j} >$ where the coupling is formed between steps that are following (arrows) as in Fig. 2. This coupling is $q\delta^2$. To discriminate from the other cross coupling such as $< \delta y_{i,j}\delta x_{i,j} >$ which is $-q\delta^2$, we will call this coupling proportional to $q$ as dispersing one. We have used this later type in arriving at the results for EW and KPZ equations. In the present case following otl couplings are considered. $< \delta y_{i-1,j}\delta y_{i,j} >$, $< \delta x_{i+1,j}\delta y_{i,j} >$, $< \delta x_{i-1,j}\delta x_{i,j} >$, $< \delta x_{i,j}\delta x_{i,j+1} >$, $< \delta x_{i,j}\delta x_{i,j+1} >$. There is another possibility of addition of five dispersing cross couplings from Fig. 2. These are $< \delta y_{i+1,j}\delta x_{i+1,j} >$, $< \delta x_{i,j-1}\delta y_{i,j-1} >$, $< \delta x_{i,j+1}\delta y_{i,j+1} >$, $< \delta x_{i,j-1}\delta y_{i,j-1} >$, and $< \delta x_{i,j}\delta y_{i,j} >$. These two possibilities are chosen because they are additive involving all the steps needed to define the $\nabla^2(\nabla h)^2$ term. The couplings also are distributed to render correct weightages to the lattice points involved. Thus from Fig. 2 it is seen that points at $(i+1,j)$ and $(i,j+1)$ contribute steps each while $(i,j-1)$ and $(i,j+1)$ contribute two each. Thus in both the sets, ratio of the couplings at $(i+1,j)$ and $(i,j+1)$ to those at $(i-1,j)$ and $(i-1)$ is 3:2. Any other additive possibilities do not render this ratio. It is further required that in the otl couplings height at $(i,j)$ must appear. This requirement is that of locality of the term $\nabla^2(\nabla h)^2$. Consider e.g. $< \delta x_{i+1,j}\delta y_{i,j+1} >$. This coupling does not contain $h(i,j)$. Such a coupling will independently couple with the steps at $(i+1,j)$ leading to second order term. We will illustrate this point in connection with fourth order linear term. This requirement eliminates set of five dispersive couplings. As can be verified from the Fig. 2, all the couplings in other set involve $h(i,j)$.

In order to determine surface corresponding to the otl couplings preserving the locality, we need to find parallel coupling of the form $< \delta x_{i,j-1}\delta x_{i,j} >$. This is easily obtained by noting that $0 = h(i,j) - h(i+1,j) + h(i+1,j-1) - h(i,j-1) + h(i,j+1) - h(i,j)$. Squaring this expression and writing in terms of steps gives the desired result,$$
< \delta x_{i,j-1}\delta x_{i,j} >= (1 + 2q)\delta^2
$$
We denote this coupling as $q_p$. $< \delta x_{i-1,j}\delta y_{i,j+1} >$ can be obtained by noting that $5\alpha = (5\delta x_{i-1,j} + \delta y_{i,j} + \delta y_{i,j+1})^2/\delta^2$. This leads to
$$
< \delta x_{i-1,j}\delta y_{i,j+1} >= (5\alpha - 3 - 2q - 2s)/2
$$
We denote this coupling as $q_s$. We note that $\sqrt{5} = 2.2360694$. The relation between the linear and otl couplings is obtained from,
$$
8s = 4q_p + 2q_s
$$
Using Eq. (11), (12) and (14) we obtain $q = -0.206$ giving $\alpha = 2/3$ with possible numerical error beyond 4th digit.
Thus the contribution from linear couplings is $4$ the morphology are at $(i,j)$ and $(i+1,j+1)$ lattice points. The steps included in its discrete form are

\[ \delta y_{i,j} \]

This results in to $\delta x_{i,j}$ and $\delta x_{i,j+1}$. This will contribute $4\delta x_{i,j}$ to the term $\nabla \cdot (\nabla h)^2$. From the scaling transformations, in $2+1$ dimensions, the EW, KPZ and ldv are the only relevant terms up to fourth order. Further non linearities if relevant must contain higher number of derivatives than in ldv term such as $\nabla^4 (\nabla h)^2$ etc.. Experimental values of exponents other than the terms discussed here are likely to be the result of correlated noise or transients in the growth.

In conclusion, we have shown that using the discrete form of the terms in a growth equation and the use of linear and cross coupling between steps at a site, $\alpha$ can be exactly determined for various terms in $1+1$- dimensions, and in $2+1$- dimensions. In particular for KPZ term, the existence of $\alpha$ in all the higher dimensions shows that there is no upper critical dimension for the KPZ equation. The method can be extended to any non linear stochastic equation with non conserved noise. Exact results for fourth order non linearities are obtained and show that the non renormalization for ldv term is a consequence of the coincidence of the exact value of $\alpha$ with that obtained from the scaling transformations. This study indicates that in the experiments the results are likely to be affected by the presence of the different non linear terms, the correlated noise and the transients.

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We will now consider all the non linearities derivable from the non linear terms considered above. Consider terms of the form $\nabla \cdot (\nabla h)^{2n+1}$ with $n=1, 2, 3, \ldots$. The relevant diagrams representing these terms in the discrete form are exactly same as that corresponding to the term with $n=1$. Hence, all such non linearities will have same behaviour i.e. like EW term [20] and with additional surface possible having $\alpha = 0.58497$. Consider terms $\nabla^2 (\nabla h)^{2n}$ with $n=1, 2, 3, \ldots$ In this case also, we find that the diagrams representing the discrete form are all same as that corresponding to $n=1$. Thus the behaviour of all such non linear terms is same as that corresponding to the ldv term. For non conservative terms such as $(\nabla h)^{2n}$ with $n=1, 2, 3, \ldots$ we find that all such terms will behave as the KPZ term asymptotically. These results show that in the conserved or non conserved growth, the EW, KPZ and ldv are the only relevant terms up to fourth order.

[1] S.V. Ghaisas, Phys. Rev. E 68, 011605 (2003).
[2] A.L. Barabasi and H.E. Stanley, Fractal Concepts in Surface Growth, (Cambridge University Press, New York, 1995).
[3] J. Krug, Adv. Phys. 46, 141 (1997).
[4] S.F. Edwards and D.R. Wilkinson, Proc. R. Soc. London A381, 17 (1982).
[5] M. Kardar, G. Parisi and, Y.C. Zhang, Phys. Rev. Lett. 56, 889 (1986).
[6] I. Sivashinsky, Ann. Rev. Fluid Mech., 15, 179 (1983);
REFERENCES

[1] C. Misbah, H. Mullar-Krumbhaar, and D. E. Temkin, J. Phys. (France) I, 1, 585 (1991).
[2] E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A 39, 3053 (1989).
[3] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A 16, 732 (1977).
[4] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. 62, 2289 (1989). D. E. Wolf and J. Kertesz, Europhys. Lett. 4, 651 (1987); T. Ala-Nissila, T. Hjelt, J. M. Kosterlitz and O. Ventooinen, J. Stat. Phys. 72, 207 (1993).
[5] E. Marinari, A. Pagnani and, G. Parisi, J. Phys. A, Math. Gen. 33, 8181 (2000).
[6] L. Giada, A. Giacometti, and M. Rossi, Phys. Rev. E 65, 036134 (2002).
[7] M. Lassig, Phys. Rev. Lett. 80, 2366 (1998).
[8] * Other references are available from the review articles like references 2) and 3) above.
[9] C. Castellano, M. Mrsi, M. A. Munoz, and L. Pietronero, Phys. Rev. E 59, 6460 (1999).
[10] * In the cases where \( \alpha > 1 \), \( \delta^2 \) can vary in time. This is referred as super rough growth and is discussed in references 3) above and references therein.
[11] D. E. Wolf and J. Villain, Europhys. Lett. 13, 389 (1990). Z. W. Lai and S. Das Sarma, Phys. Rev. Lett. 66, 2348 (1991).
[12] W. W. Mullin, J. Appl. Phys. 30, 77 (1959).
[13] S. Das Sarma and R. Kotlyar, Phys. Rev. E 50, R4275 (1994).
[14] J. M. Kim and S. Das Sarma, Phys. Rev. E 51, 1889 (1995).
[15] A. K. Kshirsagar and S. V. Ghaisas, Phys. Rev. E, 53 R1 (1996).

FIG. 1. Slope distributions corresponding to \( s = 0 \) (dotted curve) and \( s = 1 \) (solid lines).

FIG. 2. Steps contributing to the term \( \nabla^2(\nabla h)^2 \). Each step is represented by an arrow as per the definition given in the text.

FIG. 3. Steps contributing to the term \( \nabla^4h \)

FIG. 4. Steps contributing to the term \( \nabla \cdot (\nabla h)^3 \)