OPERATOR EXTENSIONS OF HUA’S INEQUALITY

M. S. MOSLEHIAN

ABSTRACT. We give an extension of Hua’s inequality in pre-Hilbert $C^*$-modules without using convexity or the classical Hua’s inequality. As a consequence, some known and new generalizations of this inequality are deduced. Providing a Jensen inequality in the content of Hilbert $C^*$-modules, another extension of Hua’s inequality is obtained. We also present an operator Hua’s inequality, which is equivalent to operator convexity of given continuous real function.

1. Introduction

In his famous monograph “Additive theory of prime numbers”, Lo-Keng Hua [8] introduced the important inequality

$$\left( \delta - \sum_{i=1}^{n} x_i \right)^2 + \alpha \sum_{i=1}^{n} x_i^2 \geq \frac{\alpha}{n+\alpha} \delta^2,$$

where $\delta, \alpha$ are positive numbers, and $x_i (i = 1, 2, \ldots, n)$ are real numbers. The equality holds if and only if $x_i = \delta/(n + \alpha)$.

This result was generalized by C.L. Wang [20] by showing that

$$\left( \delta - \sum_{i=1}^{n} x_i \right)^p + \alpha^{p-1} \sum_{i=1}^{n} x_i^p \geq \left( \frac{\alpha}{n+\alpha} \right)^{p-1} \delta^p,$$

in which $\delta > 0$, $\alpha > 0$, $p \geq 1$ and $(x_1, \ldots, x_n)$ is a finite sequence of nonnegative real numbers with $\sum_{i=1}^{n} x_i \leq \delta$, and that the sign of inequality is reversed for $0 < p < 1$. In [19], G.-S. Yang and B.-K. Han extended this result for a finite sequence of complex numbers. C.E.M. Pearce and J.E. Pečarić [14] generalized Hua’s inequality for real convex functions; see also [11]. S.S. Dragomir and G.-S. Yang [2] extended Hua’s inequality in the setting of real inner product spaces by applying Hua’s inequality for $n = 1$. Their result was generalized by

2000 Mathematics Subject Classification. Primary 47A63; secondary 46L08, 47B10, 47A30, 47B15, 26D07, 15A60.

Key words and phrases. Hua’s inequality, operator inequality, positive operator, Hilbert $C^*$-module, $C^*$-algebra, operator convex function, Hansen–Pedersen–Jensen inequality.
There are other interpretations of Hua’s inequality; cf. [11] and references therein. An operator version of Hua’s inequality was given by R. Drnovšek [3]. Moreover, S. Radas and T. Šikić [17] generalized the Hua inequality for linear operators in real inner product spaces. Now we consider certain extensions and improvements of the above results in the setting of Hilbert $C^*$-modules and operators on Hilbert spaces. Providing a Jensen inequality in the content of Hilbert $C^*$-modules, another extension of Hua’s inequality is obtained. We also present an operator Hua’s inequality, which is equivalent to operator convexity of given continuous real function.

2. Preliminaries

The notion of Hilbert $C^*$-module is a generalization of the notion of Hilbert space. Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{X}$ be a complex linear space, which is a right $\mathcal{A}$-module satisfying $\lambda(\alpha x) = x(\lambda a) = (\lambda x)a$ for all $x \in \mathcal{X}, a \in \mathcal{A}, \lambda \in \mathbb{C}$. The space $\mathcal{X}$ is called a (right) pre-Hilbert $C^*$-module over $\mathcal{A}$ if there exists an $\mathcal{A}$-inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying

(i) $\langle x, x \rangle \geq 0$ (i.e. $\langle x, x \rangle$ is a positive element of $\mathcal{A}$) and $\langle x, x \rangle = 0$ if and only if $x = 0$;
(ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$;
(iii) $\langle x, ya \rangle = \langle x, y \rangle a$;
(iv) $\langle x, y \rangle^* = \langle y, x \rangle$;

for all $x, y, z \in \mathcal{X}, \lambda \in \mathbb{C}, a \in \mathcal{A}$.

We can define a norm on $\mathcal{X}$ by $\|x\| := \|\langle x, x \rangle\|^\frac{1}{2}$, where the latter norm denotes that in the $C^*$-algebra $\mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module is called a (right) Hilbert $C^*$-module over $\mathcal{A}$ (or a (right) Hilbert $\mathcal{A}$-module) if it is complete with respect to its norm. Any inner product space can be regarded as a pre-Hilbert $\mathbb{C}$-module and any $C^*$-algebra $\mathcal{A}$ is a Hilbert $C^*$-module over itself via $\langle a, b \rangle = a^*b (a, b \in \mathcal{A})$.

A mapping $T : \mathcal{X} \to \mathcal{Y}$ between Hilbert $\mathcal{A}$-modules is called adjointable if there exists a mapping $S : \mathcal{Y} \to \mathcal{X}$ such that $\langle T(x), y \rangle = \langle x, S(y) \rangle$ for all $x \in \mathcal{X}, y \in \mathcal{Y}$. The unique mapping $S$ is denoted by $T^*$ and is called the adjoint of $T$. It is easy to see that $T$ and $T^*$ must be bounded linear $\mathcal{A}$-module mappings. We denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of all adjointable mappings from $\mathcal{X}$ to $\mathcal{Y}$. We write $\mathcal{L}(\mathcal{X})$ for the unital $C^*$-algebra $\mathcal{L}(\mathcal{X}, \mathcal{X})$; cf. [9, p. 8]. For every $x \in \mathcal{X}$ the absolute value of $x$ is defined as the unique positive square root of $\langle x, x \rangle$, that is, $|x| = \langle x, x \rangle^{\frac{1}{2}}$. 
A Hilbert \( \mathcal{A} \)-module \( \mathcal{X} \) can be embedded into a certain \( C^* \)-algebra \( \Lambda(\mathcal{X}) \). To see this, let us denote by \( F = \mathcal{X} \oplus \mathcal{A} \), the direct sum of Hilbert \( \mathcal{A} \)-modules \( \mathcal{X} \) and \( \mathcal{A} \) equipped with the \( \mathcal{A} \)-inner product

\[
\langle (x_1, a_1), (x_2, a_2) \rangle = \langle x_1, x_2 \rangle + a_1^* a_2.
\]

Identify each \( x \in \mathcal{X} \) with \( \mathcal{A} \rightarrow \mathcal{X}, a \mapsto xa \). The adjoint of this map is \( x^*(y) = \langle x, y \rangle \). Set

\[
\Lambda(\mathcal{X}) = \left\{ \begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : a \in \mathcal{A}, x, y \in \mathcal{X}, T \in \mathcal{L}(\mathcal{X}) \right\}.
\]

\( \Lambda(\mathcal{X}) \) is a \( C^* \)-subalgebra of \( \mathcal{L}(\mathcal{F}) \), called the linking algebra of \( \mathcal{X} \). Then

\[
\mathcal{X} \simeq \begin{bmatrix} 0 & \mathcal{X} \\ 0 & 0 \end{bmatrix}, \mathcal{A} \simeq \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{A} \end{bmatrix}, \mathcal{L}(\mathcal{X}) \simeq \begin{bmatrix} \mathcal{L}(\mathcal{X}) & 0 \\ 0 & 0 \end{bmatrix}.
\]

Furthermore, \( \langle x, y \rangle \) of \( \mathcal{X} \) becomes the product \( x^*y \) in \( \Lambda(\mathcal{X}) \) and the module multiplication \( \mathcal{X} \times \mathcal{A} \rightarrow \mathcal{X} \) becomes a part of the internal multiplication of \( \Lambda(\mathcal{X}) \).

We refer the reader to [13] for undefined notions on \( C^* \)-algebra theory and to [4, 9, 18] for more information on Hilbert \( C^* \)-modules.

A continuous real valued function \( f \) on an interval \( J \) is called operator convex if for all \( \lambda \in [0, 1] \) and all self-adjoint operators \( A \) and \( B \) acting on a Hilbert space \( (\mathcal{H}, \langle ., . \rangle) \), whose spectra are contained in \( J \),

\[
f((1 - \lambda)A + \lambda B) \leq (1 - \lambda)f(A) + \lambda f(B),
\]

where \( \leq \) denotes the usual positive semi-definiteness. A function \( f : J \rightarrow \mathbb{R} \) is called operator concave if \(-f\) is operator convex. A known operator Jensen equation says that if \( A \) is a self-adjoint operator with spectrum contained in an interval \( J \) on which \( f \) is a convex function, then

\[
f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle \quad (2.1)
\]

for every unit vector \( x \); cf. [12].

By Hansen–Pedersen–Jensen’s inequality (see [5, 7]) a function \( f \) is operator convex (operator convex and \( f(0) \leq 0 \), respectively) if and only if

\[
f\left( \sum_{i=1}^{n} E_i^* A_i E_i \right) \leq \sum_{i=1}^{n} E_i^* f(A_i) E_i \quad (2.2)
\]
for all self-adjoint bounded operators $A_i$ with spectra contained in $J$ and all bounded operators $E_i$ with $\sum_{i=1}^{n} E_i^* E_i = I$ ($\sum_{i=1}^{n} E_i^* E_i \leq I$, respectively), where $I$ denotes the identity operator. The reader is referred to [6, 16] for more information on operator inequalities.

3. A Hua Type Inequality in Left Hilbert $C^*$-modules

We start our work with the following generalized Hua inequality. In our approach, we use neither convexity nor the classical Hua inequality. Throughout this section, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert modules over a $C^*$-algebra $\mathcal{A}$.

**Theorem 3.1.** Let $f : [0, \infty) \to (0, \infty)$ be a function such that $f(t) \geq t + M$ for some $M > 0$. Then

$$|y (f(c) - c)^{-1/2} - x (f(c) - c)^{1/2}|^2 + c |x|^2 \geq c f(c)^{-1} (f(c) - c)^{-1} |y|^2$$

(3.1)

for all positive central elements $c \in \mathcal{A}$ and all elements $x, y \in \mathcal{X}$. The equality holds if and only if $y = x f(c)$.

**Proof.** By the functional calculus, $f(c)$ and $f(c) - c$ are invertible positive elements of $\mathcal{A}$. Since $(f(c) - c)^{1/2}$ is a central element,

$$|y (f(c) - c)^{-1/2} - x (f(c) - c)^{1/2}|^2$$

$$= (f(c) - c)^{-1/2} \langle y, y \rangle (f(c) - c)^{-1/2} - (f(c) - c)^{-1/2} \langle y, x \rangle (f(c) - c)^{1/2}$$

$$- (f(c) - c)^{1/2} \langle x, y \rangle (f(c) - c)^{-1/2} + (f(c) - c)^{1/2} \langle x, x \rangle (f(c) - c)^{1/2}$$

$$= (f(c) - c)^{-1} \langle y, y \rangle - \langle y, x \rangle - \langle x, y \rangle + (f(c) - c) \langle x, x \rangle.$$

Due to $(f(c) - c)^{-1} - f(c)^{-1} = cf(c)^{-1} (f(c) - c)^{-1}$, we therefore get

$$|y (f(c) - c)^{-1/2} - x (f(c) - c)^{1/2}|^2 + c |x|^2 - c f(c)^{-1} (f(c) - c)^{-1} |y|^2$$

$$= f(c)^{-1} \langle y, y \rangle - \langle y, x \rangle - \langle x, y \rangle + f(c) \langle x, x \rangle$$

$$= |y f(c)^{-1/2} - x f(c)^{1/2}|^2 \geq 0.$$

Here the equality holds if and only if $y f(c)^{-1/2} = x f(c)^{1/2}$, that is, $y = x f(c)$. \qed
Example 3.2. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $B(\mathcal{H}, \mathcal{K})$ denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Then $B(\mathcal{H}, \mathcal{K})$ becomes a $B(\mathcal{K})$-module by defining $\langle A, B \rangle := A^*B$. Suppose that $f : [0, \infty) \to (0, \infty)$ is a function such that $f(t) \geq t + M$ for some $M > 0$. Replacing $x, y$ in (3.1) by $A, B$, respectively, and using this fact that the center of $B(\mathcal{H})$ is $CI$ we get

$$\left((f(c) - c)^{-1/2} B - (f(c) - c)^{1/2} A\right)^* \left((f(c) - c)^{-1/2} B - (f(c) - c)^{1/2} A\right) + c A^* A \geq c f(c)^{-1} (f(c) - c)^{-1} B^* B$$

for all positive numbers $c \in [0, \infty)$ and all $A, B \in B(\mathcal{H}, \mathcal{K})$. The equality holds if and only if $B = f(c) A$.

The following theorem is a norm extension of Hua’s inequality.

Theorem 3.3. Let $f : [0, \infty) \to (0, \infty)$ be a function such that $f(t) \geq t + M$ for some $M > 0$. Then

$$\| y (f(c) - c)^{-1/2} - T(x) (f(c) - c)^{1/2} \|^2 + \| c \| \| T \|^2 \| x \|^2 \geq \| c f(c)^{-1} (f(c) - c)^{-1} y \|^2$$

for all positive central elements $c \in \mathcal{A}$, all elements $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and all non-zero bounded linear operators $T : \mathcal{X} \to \mathcal{Y}$.

Proof. Replacing $x$ by $T(x)$ in (3.1) we get

$$\| y (f(c) - c)^{-1/2} - T(x) (f(c) - c)^{1/2} \|^2 + \| c \| \| T \|^2 \| x \|^2 \geq \| c f(c)^{-1} (f(c) - c)^{-1} y \|^2$$

utilizing the facts that $\| T x \| / \| T \| \leq \| x \|$ and $\| z \|^2 = \| z \|^2$ ($z \in \mathcal{A}$) we obtain

$$\| y (f(c) - c)^{-1/2} - T(x) (f(c) - c)^{1/2} \|^2 + \| c \| \| T \|^2 \| x \|^2 \geq \| c f(c)^{-1} (f(c) - c)^{-1} y \|^2$$

\hfill \Box

Considering the elementary operator $T = u \otimes v$ defined for given $u, v \in \mathcal{X}$ by $T(x) = u \langle v, x \rangle$ ($x \in \mathcal{X}$) and noting to the fact that $\| T \| = \| u \| \| v \|$ we get

Corollary 3.4. Let $f : [0, \infty) \to (0, \infty)$ be a function such that $f(t) \geq t + M$ for some $M > 0$. Then

$$\| y (f(c) - c)^{-1/2} - u \langle v, x \rangle (f(c) - c)^{1/2} \|^2 + \| c \| \| u \|^2 \| v \|^2 \| x \|^2 \geq \| c f(c)^{-1} (f(c) - c)^{-1} y \|^2$$

for all positive central elements $c \in \mathcal{A}$, all elements $x, y, u, v \in \mathcal{X}$.
If $\mathcal{H}$ and $\mathcal{Y}$ are assumed to be inner product spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, $A = \mathbb{B}(\mathcal{H}, \mathcal{K})$, $f(t) = t + 1$ and $c = \frac{\alpha}{\|A\|^2}$, then we deduce the main result of [17] from Theorem 3.3 as follows.

**Corollary 3.5.** Suppose that $\mathcal{H}$ and $\mathcal{K}$ are inner product spaces, $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator and $\alpha > 0$. Then

$$\|y - Ax\|^2 + \alpha \|x\|^2 \geq \frac{\alpha}{\|A\|^2 + \alpha} \|y\|^2$$

for all elements $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

**Remark 3.6.** If $\mathcal{H}$ is an inner product space, $w_i \in \mathbb{C}$ $(1 \leq i \leq n)$ and we consider the $n$-fold inner product space $\mathcal{H}^n$ and $A(x_1, \cdots, x_n) = \sum_{i=1}^{n} w_i x_i$ in Corollary 3.5 (see [17]), then $\|A\|^2 = \sum_{i=1}^{n} |w_i|^2$ and so we get

$$\left\|y - \sum_{i=1}^{n} (w_i x_i)\right\|^2 + \alpha \sum_{i=1}^{n} (|w_i|^2 \|x_i\|^2) \geq \sum_{i=1}^{n} \frac{\alpha}{|w_i|^2 + \alpha} \|y\|^2,$$

which is a generalization of the main theorem of [2] (see also [1]). The case where $\mathcal{H} = \mathbb{C}$ and $w_i = 1$ $(1 \leq i \leq n)$ gives rise to the classical Hua’s inequality.

### 4. HUA’S INEQUALITY FOR OPERATOR CONVEX FUNCTIONS IN HILBERT $C^*$-MODULES

We first generalize operator Jensen inequality (2.1) in the framework of Hilbert $C^*$-modules. In this section we assume that $\mathcal{X}$ is a Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$ with unit $e$.

**Theorem 4.1.** Let $f$ be an operator convex function on an interval $J$ containing $0$, $f(0) \leq 0$ and let $T \in \mathcal{L}(\mathcal{X})$ be self-adjoint with spectrum contained in $J$. Then

$$f(\langle x, Tx \rangle) \leq \langle x, f(T)x \rangle$$ (4.1)

for every $x$ in the closed unit ball of $\mathcal{X}$.
Proof. To prove we utilize the linking algebra $\Lambda(X)$ as a $2 \times 2$ matrix trick and the functional calculus for self-adjoint elements of the $C^*$-algebra (see [13]).

\[
\begin{bmatrix}
  f(0) & 0 \\
  0 & f(\langle x, Tx \rangle)
\end{bmatrix} = f\left(\begin{bmatrix}
  0 & 0 \\
  0 & \langle x, Tx \rangle
\end{bmatrix}\right)
\]

\[
= f\left(\begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}^* \begin{bmatrix}
  T & 0 \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}\right)
\]

\[
\leq \begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}^* f\left(\begin{bmatrix}
  T & 0 \\
  0 & 0
\end{bmatrix}\right) \begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}
\]

(by (2.2) for $n = 1$ and $E_1 = \begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}$, and by noting to $\|x\| \leq 1 \iff |x| \leq e$)

\[
= \begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}^* \begin{bmatrix}
  f(T) & 0 \\
  0 & f(0)
\end{bmatrix} \begin{bmatrix}
  0 & x \\
  0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  0 & 0 \\
  0 & \langle x, f(T)x \rangle
\end{bmatrix},
\]

whence we get (4.1). \qed

Now we are ready to establish the second Hua’s inequality in the setting of Hilbert $C^*$-modules.

**Theorem 4.2.** Let $f$ be an operator convex function on an interval $J$ containing $0$, $f(0) \leq 0$ and $S, R_i, T_i \in \mathcal{L}(X)$ such that $S$ and $T_i$ are self-adjoint and the spectrum of $S - \sum_{i=1}^n R_i^* T_i R_i$ and $T_i$ ($1 \leq i \leq n$) are contained in $J$. Then for $x$ in the closed unit ball of $X$,

\[
\left\langle x, \left[ f\left( S - \sum_{i=1}^n R_i^* T_i R_i \right) + \sum_{i=1}^n R_i^* f(T_i) R_i \right] x \right\rangle \geq k_n^{-1} f\left( k_n \left\langle x, S x \right\rangle \right),
\]

where $k_n = (1 + \sum_{i=1}^n \| R_i x \|^2)^{-1}$. 

Proof.

\[
\left\langle x, \left[ f \left( S - \sum_{i=1}^{n} R_i^* T_i R_i \right) + \sum_{i=1}^{n} R_i^* f(T_i) R_i \right] x \right\rangle
\]

\[
= \left\langle x, f \left( S - \sum_{i=1}^{n} R_i^* T_i R_i \right) x \right\rangle + \left\langle x, \sum_{i=1}^{n} R_i^* f(T_i) R_i x \right\rangle
\]

\[
= \left\langle x, f \left( S - \sum_{i=1}^{n} R_i^* T_i R_i \right) x \right\rangle + \sum_{i=1}^{n} \| R_i x \|^2 \left\langle \frac{R_i x}{\| R_i x \|}, f(T_i) \frac{R_i x}{\| R_i x \|} \right\rangle
\]

\[
\geq f \left( \left\langle x, \left( S - \sum_{i=1}^{n} R_i^* T_i R_i \right) x \right\rangle \right) + \sum_{i=1}^{n} \| R_i x \|^2 f \left( \left\langle \frac{R_i x}{\| R_i x \|}, T_i \left( \frac{R_i x}{\| R_i x \|} \right) \right\rangle \right) \quad \text{(by (4.1))}
\]

\[
= 1 \cdot f \left( \left\langle x, \left( S - \sum_{i=1}^{n} R_i^* T_i R_i \right) x \right\rangle \right) + \sum_{i=1}^{n} \| R_i x \|^2 f \left( \left\langle x, \frac{R_i^* T_i R_i x}{\| R_i x \|^2} \right\rangle \right)
\]

\[
\geq k_n^{-1} f \left( k_n \left\langle x, \left( S - \sum_{i=1}^{n} R_i^* T_i R_i \right) x \right\rangle \right) + k_n \sum_{i=1}^{n} \| R_i x \|^2 \left\langle x, \frac{R_i^* T_i R_i x}{\| R_i x \|^2} \right\rangle
\]

(by the classical weighted Jensen inequality)

\[
\geq k_n^{-1} f \left( k_n \left\langle x, \left( S - \sum_{i=1}^{n} R_i^* T_i R_i + \sum_{i=1}^{n} \| R_i x \|^2 \frac{R_i^* T_i R_i}{\| R_i x \|^2} \right) x \right\rangle \right)
\]

\[
= k_n^{-1} f \left( k_n \left\langle x, S x \right\rangle \right),
\]

where \( k_n = (1 + \sum_{i=1}^{n} \| R_i x \|^2)^{-1} \).

As a special case in which \( \mathcal{H} \) is \( \mathbb{C} \) as a \( \mathbb{C} \)-module, \( f(t) = t^2 \), \( x = 1 \), and \( S, T_i \ (1 \leq i \leq n) \) and \( R_i \ (1 \leq i \leq n) \) are given real numbers \( \delta, \alpha x_i \ (1 \leq i \leq n) \) and \( \alpha^{-1/2} \), respectively, we get the classical Hua’s inequality.

## 5. An Operator Hua’s Inequality

In this section we present an operator Hua’s inequality and show that it is equivalent to operator convexity.

**Theorem 5.1.** Let \( \mathcal{H} \) be a Hilbert space, let \( f \) be an operator convex function on an interval \( J \) and let \( C_i \ (1 \leq i \leq n) \) be arbitrary operators and \( B, A_i \ (1 \leq i \leq n) \) be self-adjoint operators operators such that the spectra of \( B - \sum_{i=1}^{n} C_i^* A_i C_i \) and \( A_i \) are contained in \( J \). Then

\[
f \left( B - \sum_{i=1}^{n} C_i^* A_i C_i \right) + \sum_{i=1}^{n} C_i^* f(A_i) C_i \geq D^{-1} f(DBD)D^{-1}, \quad (5.1)
\]
where $D = (I + \sum_{i=1}^{n} C_i^* C_i)^{-1/2}$.

Proof. It follows from

$$DD + \sum_{i=1}^{n} DC_i^* C_i D = D \left( I + \sum_{i=1}^{n} C_i^* C_i \right) D = I$$

and operator convexity of $f$ that

$$Df \left( B - \sum_{i=1}^{n} C_i^* A_i C_i \right) D + \sum_{i=1}^{n} DC_i^* f(A_i) C_i D \geq f \left( DBD - \sum_{i=1}^{n} DC_i^* A_i C_i D \right) + \sum_{i=1}^{n} DC_i^* A_i C_i D \right) = f(DBD).$$

Due to the fact that $T \geq S$ implies $R^* T R \geq R^* S R$, we deduce (5.1). □

Remark 5.2. If we use a concave operator function, then the inequality in (5.1) will be reversed.

Applying Theorem 5.1 to the convex functions $f_1(t) = t^{-1}$ and $f_2(t) = t^p$ ($1 \leq p \leq 2$ or $-1 \leq p \leq 0$), and concave functions $f_3(t) = t^p$ ($0 \leq p \leq 1$) and $f_4(t) = \log t$ on $(0, \infty)$ we get inequalities (i)-(iv) of the following Corollary, respectively. The inequalities of (i) and (iv) are operator extensions of main result of [20].

Corollary 5.3. Let $\mathcal{H}$ be a Hilbert space and let $C_i$ ($1 \leq i \leq n$) be arbitrary operators and $B$ and $A_i$ ($1 \leq i \leq n$) be self-adjoint operators acting on $\mathcal{H}$ such that $B - \sum_{i=1}^{n} C_i^* A_i C_i > 0$ and $A_i > 0$. Then

(i) $\left( B - \sum_{i=1}^{n} C_i^* A_i C_i \right)^{-1} + \sum_{i=1}^{n} C_i^* A_i^{-1} C_i \geq D^{-1}(DBD)^{-1} D^{-1}$;

(ii) $\left( B - \sum_{i=1}^{n} C_i^* A_i C_i \right)^p + \sum_{i=1}^{n} C_i^* A_i^p C_i \geq D^{-1}(DBD)^p D^{-1}$ ($p \in [-1, 0] \cup [1, 2]$);

(iii) $\left( B - \sum_{i=1}^{n} C_i^* A_i C_i \right)^p + \sum_{i=1}^{n} C_i^* A_i^p C_i \leq D^{-1}(DBD)^p D^{-1}$ ($p \in [0, 1]$);

(iv) $\log \left( B - \sum_{i=1}^{n} C_i^* A_i C_i \right) + \sum_{i=1}^{n} C_i^* \log(A_i) C_i \leq D^{-1} \log(DBD) D^{-1},$

where $D = (I + \sum_{i=1}^{n} C_i^* C_i)^{-1/2}$. 
If $\mathcal{H}$ is of dimension one, $\delta, x_1, \cdots, x_n \in \mathbb{R}$, $\alpha > 0$ and we take $C_i = \alpha^{-1/2} (1 \leq i \leq n)$, $B = \delta$ and $A_i = \alpha x_i$ ($1 \leq i \leq n$) in Theorem 5.1, then $D = (\alpha/(\alpha + \alpha))^{1/2}$ we obtain the main theorem of [14] as follows.

**Corollary 5.4.** [14 Theorem 1] Let $f : J \rightarrow \mathbb{R}$ be a convex function and let $\alpha, \delta, x_1, \cdots, x_n$ be real numbers such $\alpha > 0$ and $\delta - \sum_{i=1}^{n} x_i, \alpha x_1 \cdots, \alpha x_n \in J$. Then

$$f \left( \delta - \sum_{i=1}^{n} x_i \right) + \sum_{i=1}^{n} \alpha^{-1} f (\alpha x_i) \geq \frac{\alpha + n}{\alpha} f \left( \frac{\alpha \delta}{\alpha + n} \right). \quad (5.2)$$

Now, we shall show that our operator Hua’s inequality (5.1) implies the Hansen–Pedersen–Jensen operator inequality in the case where the value of the real function $f$ at 0 is non-positive.

**Theorem 5.5.** Let $J$ be an interval containing 0 and let $f$ be a continuous real valued function defined on $J$ with $f(0) \leq 0$. Let operator Hua’s inequality (5.1) hold, where $C_i$ ($1 \leq i \leq n$) are arbitrary operators and $B, A_i$ ($1 \leq i \leq n$) are self-adjoint operators such that the spectra of $B - \sum_{i=1}^{n} C_i A_i C_i$ and $A_i$ are contained in $J$. Then $f$ is operator convex.

**Proof.** Let $X_i$ ($1 \leq i \leq n$) be self-adjoint operators on a Hilbert space $\mathcal{H}$ with spectra contained in $J$ and $E_i$ be arbitrary operators such that $\sum_{i=1}^{n} E_i^* E_i < I$, where $< \text{ denotes the strict positivity.}$ Recall that by a strict positive operator we mean an invertible positive one.

Replacing $B$ by $\sum_{i=1}^{n} C_i A_i C_i$ in (5.1) we get

$$\sum_{i=1}^{n} C_i^* f(A_i) C_i \geq f(0) + \sum_{i=1}^{n} C_i^* f(A_i) C_i \geq D^{-1} f \left( D \sum_{i=1}^{n} C_i^* A_i C_i D \right) D^{-1},$$

that is

$$\sum_{i=1}^{n} (C_i D)^* f(A_i) C_i D \geq f \left( \sum_{i=1}^{n} (C_i D)^* A_i (C_i D) \right). \quad (5.3)$$

Set $C_i = E_i (I - \sum_{i=1}^{n} E_i^* E_i)^{-1/2}$. Then

$$\sum_{i=1}^{n} C_i^* C_i + I = \sum_{i=1}^{n} E_i^* E_i \left( I - \sum_{i=1}^{n} E_i^* E_i \right)^{-1} + I = \left( I - \sum_{i=1}^{n} E_i^* E_i \right)^{-1},$$

whence

$$D = \left( \sum_{i=1}^{n} C_i^* C_i + I \right)^{-1/2} = \left( I - \sum_{i=1}^{n} E_i^* E_i \right)^{1/2}.$$
so $C_iD = E_i$. It follows from (5.3) that

\[ n \sum_{i=1}^{n} E_i^* f(X_i) E_i \geq f \left( \sum_{i=1}^{n} E_i^* X_i E_i \right). \]

By using the same method of the proof of (iv) ⇒ (i) in [16, Theorem 1.10] with the matrix $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ replaced by the diagonal matrix $P = \begin{bmatrix} 1 - \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$ with $0 < \varepsilon < 1$ and then applying the continuity of $f$ as $\varepsilon \to 0$, we deduce that the latter inequality holds for all arbitrary operators $E_i$ with $\sum_{i=1}^{n} E_i^* E_i \leq I$ and the function $f$ is therefore operator convex.

Thus we conclude that operator Hua’s inequality (5.1) is equivalent to the Hansen–Pedersen–Jensen operator inequality. One can similarly deduce that a continuous real function satisfying (5.2) is convex.

\[ \square \]

Acknowledgement. The author would like to sincerely thank Professor Tsuyoshi Ando for their very useful comments and to express his gratitude to Professor Frank Hansen and Professor Yuki Seo for their helpful suggestions on the last paragraph of the proof of Theorem 5.5.

References

[1] S.S. Dragomir, Generalizations of Hua’s inequality for convex functions, Indian J. Math., 38 (1996), no.2, 101–109.
[2] S.S. Dragomir and G.-S. Yang, On Hua’s inequality in real inner product spaces, Tamkang J. Math., 27 (1996), no.3, 227–232.
[3] R. Drnovšek, An operator generalization of the Lo-Keng Hua inequality, J. Math. Anal. Appl., 196 (1995), no.3, 1135–1138.
[4] M. Frank, Geometrical aspects of Hilbert $C^*$-modules, Positivity 3 (1999), no. 3, 215–243.
[5] J.I. Fujii and M. Fujii, Jensen’s inequalities on any interval for operators, Nonlinear analysis and convex analysis, 29–39, Yokohama Publ., Yokohama, 2004.
[6] T. Furuta, Invitation to Linear Operators. From matrices to bounded linear operators on a Hilbert space, Taylor & Francis, Ltd., London, 2001.
[7] F. Hansen and G.K. Pedersen, Jensen’s operator inequality, Bull. London Math. Soc. 35 (2003), no. 4, 553–564.
[8] L.K. Hua, Additive theory of prime numbers, Translations of Mathematical Monographs. 13. Providence, RI: American Mathematical Society (AMS). XIII, 1965.
[9] E.C. Lance: Hilbert $C^*$-Modules, London Math. Soc. Lecture Note Series 210, Cambridge University Press, Cambridge, 1995.
[10] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and its Applications, East European Series 61, Kluwer Academic Publishers Group, Dordrecht, 1993.

[11] H. Takagi, T. Miura, T. Kanzo and S.-E. Takahasi, *A reconsideration of Hua’s inequality. II*, J. Inequal. Appl., 2006, Art. ID 21540, 8 pp.

[12] B. Mond and J.E. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math. 19 (1993), no. 3, 405–420.

[13] J. G. Murphy, *C*-Algebras and Operator Theory, Academic Press, San Diego, 1990.

[14] C.E.M. Pearce and J.E. Pečarić, *A remark on the Lo-Keng Hua inequality*, J. Math. Anal. Appl., 188 (1994), no.2, 700–702.

[15] J.E. Pečarić, *On Hua’s inequality in real inner product spaces*, Tamkang J. Math., 33 (2002), no.3, 265–268.

[16] J.E. Pečarić, T. Furuta, J.M. Hot and Y. Seo, *Mond–Pečarić method in operator inequalities. Inequalities for bounded selfadjoint operators on a Hilbert space*, Monographs in Inequalities 1. Zagreb: Element, 2005.

[17] S. Radas and T. Šikić, *A note on the generalization of Hua’s inequality*, Tamkang J. Math., 28 (1997), no.4, 321–323.

[18] I. Raeburn and D.P. Williams, *Morita Equivalence and Continuous-Trace C*-Algebras*, Mathematical Surveys and Monographs. 60. Providence, RI: American Mathematical Society (AMS), 1998.

[19] G.-S. Yang and B.-K. Han, *A note on Hua’s inequality for complex numbers*, Tamkang J. Math. 27 (1996), no. 1, 99–102.

[20] C.-L. Wang, *Lo-Keng Hua inequality and dynamic programming*, J. Math. Anal. Appl., 166 (1992), no.2, 345–350.

**Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran; Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Iran.**

**E-mail address:** moslehian@ferdowsi.um.ac.ir and moslehian@ams.org