Non linear problems in dissipative models

Nota di M.De Angelis, G. Fiore, P. Renno

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Abstract- Aim of the paper is the qualitative analysis of a quasi-linear parabolic third order equation, which describes the evolution in a large class of dissipative models. As examples of some typical boundary problems, both Dirichlet’s and Neumann’s type boundary conditions are examined. In the linear case, the related Green functions are explicitly determined, together with rigorous estimates of their behavior when the parameter of dissipation $\varepsilon$ is vanishing. These results are basic to study the integral equations to which the non linear problems can be reduced. Moreover, boundary layer estimates can be determined too.

Riassunto- Oggetto del lavoro è l’analisi qualitativa di un’equazione parabolica quasi lineare del terzo ordine, che descrive l’interazione tra propagazione ondosa e diffusione in ampie classi di modelli dissipativi. Quali esempi di problemi al contorno, vengono prese in esame condizioni sia di tipo Dirichlet che di tipo Neumann. Nel caso lineare, si determinano esplicitamente le relative funzioni di Green, insieme ad alcune stime rigorose del loro comportamento quando il parametro $\varepsilon$ della dissipazione tende a zero. Queste stime costituiscono la premessa per l’analisi delle equazioni integrali cui è possibile ridurre i problemi al contorno non lineari.

1 Facoltà di Ingegneria, Dip. Mat. Appl. "R. Caccioppoli", via Claudio 21, 80125, Napoli. E-mail: modeange@unina.it; gaetano.fiore@unina.it; renno@unina.it
1 Introduction

A great deal of models of applied sciences are described by the parabolic equation:

$$\mathcal{L}_\varepsilon u = \varepsilon u_{xxt} + c^2 u_{xx} - u_{tt} - 2au_t = -f. \quad (1.1)$$

The constants $a, c^2, \varepsilon$ are all positive and they assume various meanings according to physical problems. As for the source $f$, it can be linear or not.

For instance, the equation (1.1) is involved in the generalized Maxwell-Cattaneo system of equations [1]-[3], in problems of viscoelastic media of Kelvin-Voigt type [4], or for the study of solids at very low temperatures [5]. Further applications arise in the study of viscoelastic plates with memory, when the relaxation function is given by exponential functions. ([6] and references therein).

A typical example of the non linear case is the perturbed sine-Gordon equation which models the flux dynamics in Josephson junctions in superconductivity [7][8]. In this case, the terms $\varepsilon u_{xxt}$ and $au_t$ characterize the dissipative normal electron current flow along and across the junction.

As for the practical applications of superconductors, many areas are involved. In medicine, for instance, Magnetic Resonance Imaging (MRI) has been used since 1977 and is still improving [9]. Referring to the electric power systems, the high temperature superconductor cables are likely to lead to a lot of benefits as regards the current carrying capacity and for reducing electrical losses.[10][11].

As for typical boundary value problems related to the equation (1.1), both the Dirichlet conditions and Neumann conditions have interest for practical applications. For instance, in superconductivity, the first case can be referred to periodic conditions according to annular geometry of junction [12][13], while in the other case, the phase gradient, proportional to the magnetic field, is specified.([14]-[17]). When the source term $f$ is linear, all these problems can be explicitly solved by means of the Fourier method. The solutions are determined in sect.2-3, together with the related Green functions $G_\varepsilon, K_\varepsilon$.

When the function $f$ is non linear, then $G_\varepsilon, K_\varepsilon$ represent the kernels of the integral equations to which the above mentioned boundary value
problems can be reduced. For this, a rigorous analysis of the behavior of these kernels when \( \varepsilon \to 0 \) and \( t \to \infty \) is achieved in sect 4. At last, as first application, the influence of the dissipation on the wave behavior is estimated by an asymptotic approximation uniformly valid also for large \( t \) (sect.5).

2 Statement of the problem

If \( u_{\varepsilon}(x,t) \) is a function defined in the strip

\[
\Omega = \{(x,t) : 0 \leq x \leq \pi, \ t \geq 0\},
\]

let \( P_{\varepsilon} \) the initial-boundary value problem related to equation (1.1) with conditions

\[
\begin{align*}
\tag{2.1}
& u_{\varepsilon}(x,0) = f_0(x), \quad \partial_t u_{\varepsilon}(x,0) = f_1(x), \quad x \in [0, \pi], \\
& \tag{2.2}
& u_{\varepsilon}(0,t) = \varphi(t), \quad u_{\varepsilon}(\pi,t) = \psi(t), \quad t \geq 0,
\end{align*}
\]

where \( f_0, f_1, \psi, \varphi \) are arbitrary date.

The boundary conditions (2.2) represent only an example of the analysis we are going to apply. Equally, flux-boundary conditions or mixed-boundary conditions can be considered too. So, another example is given by the problem \( H_{\varepsilon} \) defined in \( \Omega \) by (1.1)-(2.1) together with the Neumann conditions

\[
\tag{2.3}
\partial_x u_{\varepsilon}(0,t) = \varphi_1(t), \quad \partial_x u_{\varepsilon}(\pi,t) = \psi_1(t), \quad t \geq 0.
\]

When \( \varepsilon \equiv 0 \), the parabolic equation (1.1) turns into the hyperbolic telegraph equation

\[
\tag{2.4}
\mathcal{L}_0 u_0 \equiv (c^2 \partial_{xx} - \partial_{tt} - 2a \partial_t) u_0 = -\bar{f}(x,t,u_0)
\]
and the problem $P_\varepsilon$ changes into a problem $P_0$ for $u_0(x,t)$ which has the same initial-boundary conditions (2.1) - (2.2) of $P_\varepsilon$. When the source term $\bar{f}$ of (2.4) is linear ($\bar{f} = \bar{f}(x,t)$), $P_0$ is explicitly solved by means of the well-known Green function:

$$G_0(x,\xi,t) = \frac{2}{\pi} e^{-\alpha t} \sum_{n=1}^{\infty} \frac{\sin(t \sqrt{c^2 n^2 - \alpha^2})}{\sqrt{c^2 n^2 - \alpha^2}} \sin(nx) \sin(n\xi)$$

In order to estimate the influence of the dissipative term $\varepsilon u_{xxt}$ on the wave behavior of $u_0$, the difference

$$v(x,t) = u_\varepsilon - u_0,$$

is to be evaluated and so the following problem $\Delta$ must be analyzed

$$\begin{cases}
\varepsilon v_{xxt} + c^2 v_{xx} - v_{tt} - 2av_t = -F(x,t,u_0,v) & (x,t) \in \Omega \\
v(x,0) = 0, \quad v_t(x,0) = 0, & x \in [0,\pi], \\
v(0,t) = 0, \quad v(\pi,t) = 0, & t \geq 0.
\end{cases}$$

The source term $F$ is given by

$$F = f(x,t,u_0+v) - \bar{f}(x,t,u_0) + \varepsilon u_{0xxt},$$

while, in the linear case, it is $f = \bar{f}$ and $F = \varepsilon u_{0xxt}$.

As for the problem $H_\varepsilon$, instead of (2.7)_3 the following conditions

$$v_x(0,t) = 0, \quad v_x(\pi,t) = 0, \quad t \geq 0,$$

must be specified.
3 Linear case and explicit solutions

Let $\hat{z}(s)$ the Laplace-trasform of the function $z(t)$ and let

$$\sigma(s) = \sqrt{\frac{s^2 + 2as}{\varepsilon s + c^2}}.$$  

(3.1)

When $F$ is linear and the Laplace trasform is applied to the problem $\Delta$, the transform $\hat{v}(x, s)$ of the solution $v(x, t)$ is given by

$$\hat{v}(x, s) = \int_0^\pi \hat{G}_\varepsilon(x, \xi, s) \hat{F}(\xi, s) d\xi,$$

(3.2)

where

$$\hat{G}_\varepsilon(x, \xi, s) = \frac{1}{2(\varepsilon s + c^2)} \left[ \hat{g}(|x - \xi|, \sigma) - \hat{g}(|x + \xi|, \sigma) \right]$$

(3.3)

and

$$\hat{g}(y, \sigma) = \frac{\cosh \left( (\pi - y) \sigma \right)}{\sigma \text{seh}(\pi \sigma)}.$$  

(3.4)

But, for $y \in [0, 2\pi]$, it results [18]:

$$\hat{g}(y, \sigma) = \frac{1}{\pi \sigma^2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(ny)}{n^2 + \sigma^2}$$

(3.5)

and $\cos(n|x - \xi|) - \cos(n|x + \xi|) = 2 \sin(nx) \sin(n\xi)$. So, by (3.3)- (3.5), it follows

$$\hat{G}_\varepsilon(x, \xi, s) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\xi) \sin(nx)}{s^2 + 2as + (\varepsilon s + c^2)n^2},$$

(3.6)
which represents the $L$-transform of the Green function related to problem $\Delta$.

By means of elementary formulae, one deduces

$$ G_\varepsilon(x, \xi, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} h_n(t, \varepsilon) \sin(nx) \sin(n\xi), $$

with

$$ h_n(t, \varepsilon) = e^{-(\frac{\varepsilon}{2} n^2 + a)t} \frac{\sin \left[ t \sqrt{c^2 n^2 - (\frac{\varepsilon}{2} n^2 + a)^2} \right]}{\sqrt{c^2 n^2 - (\frac{\varepsilon}{2} n^2 + a)^2}}. $$

Then, the explicit solution $v$ of the problem $\Delta$ is:

$$ v(x, t) = \int_0^t d\tau \int_0^\pi F(\xi, \tau) \ G_\varepsilon(x, \xi, t - \tau) d\xi $$

with the Green function $G_\varepsilon$ defined by (3.7),(3.8).

The formal analysis developed so far can be justified as follows. Referring to (3.9)- (3.7), the terms

$$ F_n(t) = \frac{2}{\pi} \int_0^\pi F(\xi, t) \ \sin (n\xi) \ d\xi $$

represent the Fourier coefficients of the sine series of the function $F(x, t)$:

$$ F(x, t) = \sum_{n=1}^{\infty} F_n(t) \ \sin(nx) $$

and the rapidity of pointwise convergence of this series depends, of course, on the properties of the source $F$. For instance, it can be sufficiently assumed that $F, F_x, F_{xx}$ are continuous in $(0, \pi)$ and more

$$ F(0, t) = F(\pi, t) = 0. $$
Then, the convergence of (3.11) is uniform everywhere in \([0, \pi]\) and, further, it results:

\[
F_n(t) = -\frac{1}{n^2} \frac{2}{\pi} \int_0^\pi F_{\xi\xi}(\xi, t) \sin(n\xi) \, d\xi.
\]

As consequence, if one puts:

\[
v_n(t) = h_n * F_n = \int_0^t F_n(\tau) \ h_n(t - \tau) \, d\tau,
\]

the solution (3.9),(3.7) represents the Fourier sine expansion of \(v(x, t)\):

\[
v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin(nx).
\]

**Theorem 3.1**- When \(F(x, \cdot) \in C^2(\Lambda)\) and satisfies 3.12, the solution \(v(x, t)\) of the problem \(\Delta\) can be given the form:

\[
v(x, t) = -\int_0^t d\tau \int_0^\pi F_{\xi\xi}(\xi, \tau) \ H_\varepsilon(x, \xi, t - \tau) \, d\xi,
\]

where \(H_\varepsilon\) is

\[
H_\varepsilon(\xi, x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{h_n(t, \varepsilon)}{n^2} \sin(nx) \sin(n\xi)
\]

and the convergence of the series is uniform everywhere in \([0, \pi]\).  

**Remark 3.1**- Theorem 3.1 can be applied also to the problem \(\mathcal{H}_\varepsilon\), provided that the Green function \(G_\varepsilon\) is substituted by the following function:

\[
K_\varepsilon(\xi, x, t) = \frac{h_0(t)}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} h_n(t, \varepsilon) \cos(nx) \cos(n\xi),
\]
4 Estimates and properties of the series $H_\varepsilon$

The arguments of sine functions in (3.8) are real parameters when $K_1 \leq n \leq K_2$, with

$$K_1 = \frac{c}{\varepsilon} \left(1 - \sqrt{1 - \frac{2a\varepsilon}{c^2}}\right), \quad K_2 = \frac{c}{\varepsilon} \left(1 + \sqrt{1 - \frac{2a\varepsilon}{c^2}}\right).$$

So, if $a < c$ and $N \equiv \lceil K_2 \rceil$, the $h_n$'s in (3.8) contain trigonometric functions for $1 \leq n \leq N$ and hyperbolic functions for $n \geq N + 1$. Otherwise, if $a > c$, the trigonometric case is related only to $[K_1] \leq n \leq N$. This distinction is unimportant to what we are going to demonstrate; however it holds also for the Green function $G_0$ defined by (2.5) and related to the problem $P_0$.

Let $g_n(x,\xi) = (2/\pi) \sin(nx) \sin(n\xi)$ and consider the series

$$H_0 = e^{-at} \sum_{n=1}^{\infty} \frac{\sin \left(\frac{t}{\sqrt{c^2n^2 - a^2}}\right)}{n^2 \sqrt{c^2n^2 - a^2}} g_n(x,\xi)$$

deduced from $H_\varepsilon$ setting formally $\varepsilon \equiv 0$.

In order to estimate the difference $H_\varepsilon - H_0$, let

$$A_n = a + \frac{\varepsilon}{2} n^2; \quad B_n^2 = c^2n^2 - A_n^2; \quad b_n^2 = c^2n^2 - a^2$$

and

$$r_n = \frac{h_n(t,\varepsilon)}{n^2} - \frac{h_n(t,0)}{n^2} = e^{-A_n} \frac{\sin(B_n t)}{n^2 B_n} - e^{-at} \frac{\sin(b_n t)}{n^2 b_n}$$

with $h_n(t,\varepsilon)$ defined by (3.8). It results:
\[ H_\varepsilon - H_0 = R_1 + R_2 = \sum_{n=1}^{N} r_n(t, \varepsilon) g_n + \sum_{n=N+1}^{\infty} r_n(t, \varepsilon) g_n. \] (4.5)

If \( c_0 \) denotes the Euler constant (\( c_0 \simeq 0.5773 \)) and \( k \) an arbitrary constant such that \( 0 < k < 1 \), let \( b_1^2 = c^2 - a^2 \) and

\[ \rho(t) = \frac{t}{b_1}(2 + at), \quad c_1(\varepsilon) = \frac{\varepsilon + 2c_0 c + (2c)^{2-k} \varepsilon^{k-1}}{\pi c (1 - k)}. \] (4.6)

Further, let \( c_2 \equiv (1/3) \max(1, \pi/cb_1) \). Then one has:

**Lemma 4.1** - For all \( t \geq 0, x \in [0, \pi] \), when \( \varepsilon \) is vanishing, the following estimates hold:

\[ |R_1| \leq \varepsilon c_1(\varepsilon) \rho(t) e^{-at} \] (4.7)

\[ |R_2| \leq \varepsilon c_2 [e^{-at} + \theta e^{-\frac{2}{\pi} \theta}], \] (4.8)

where \( \theta \) denotes the fast time \( t/\varepsilon \) and \( \varepsilon c_1(\varepsilon) \) vanishes with arbitrary order \( k < 1 \).

**Proof:** Referring to the trigonometric terms related to \( R_1 \), defined in (4.5), by means of the Laplace transform, by (4.4) one deduces that

\[ \hat{r}_n(s, \varepsilon) = -\frac{\varepsilon}{b_n^2} \frac{s}{(s + a)^2 + b_n^2} \] (4.9)

hence

\[ r_n(t, \varepsilon) = \frac{\varepsilon}{b_n} [e^{-at} \sin(b_n t)] * [e^{-At}(\frac{A_n}{B_n} \sin(B_n t) - \cos(B_n t))]. \] (4.10)

When this convolution is made explicit, by elementary estimates one has
\[(4.11) \quad |r_n(t, \varepsilon)| \leq \frac{\varepsilon}{n} \rho e^{-at} \quad n \in [1, N] \]

and so

\[(4.12) \quad \rho^{-1} e^{at} |R_1| \leq \frac{2}{\pi} \sum_{n=1}^{N} \frac{\varepsilon}{n} \leq \frac{2\varepsilon}{\pi} (c_0 + \frac{1}{2N} + \ln N). \]

For each positive constant \(\beta\), one has \(\ln N < \beta \, N^{1/\beta}\) so that for \(\beta = k^{-1} \) \((k < 1)\) the estimate (4.7) follows, with \(\rho\) and \(c_1\) defined by (4.6). As for \(R_2\) one has:

\[(4.13) \quad R_2 = \sum_{n=N+1}^{\infty} \frac{1}{n^2} h_n(t, \varepsilon)g_n - e^{-at} \sum_{n=N+1}^{\infty} \frac{\text{sen}(b_n t)}{n^2 b_n} g_n = R'_2 - R''_2 \]

where the terms \(h_n\) defined in (3.8) represent now hyperbolic functions \((B^2_n < 0)\). For this it results:

\[(4.14) \quad h_n(t, \varepsilon) \leq t \, e^{-\frac{\varepsilon^2 a^2}{n^2 + 2\varepsilon} t} \quad \forall n \geq N + 1 \]

and \(N + 1 \geq 2c/\varepsilon\) for \(\varepsilon < 2(c - a)\). As consequence:

\[(4.15) \quad |R'_2| \leq \frac{2t}{\pi} e^{-\frac{\varepsilon^2}{2(\varepsilon + 2at)} t} \sum_{n=1}^{\infty} \frac{1}{n^2} = (1/3) \varepsilon \, e^{-\frac{\varepsilon^2}{2(\varepsilon + 2at)} \theta}, \]

with \(\theta = t/\varepsilon\). At last, as \(b_n > b_1 n\), one deduces that

\[(4.16) \quad e^{at} |R''_2| \leq \frac{2}{\pi b_1} \sum_{n=N+1}^{\infty} \frac{1}{n^3} \leq \frac{1}{3b_1} (N + 1)^{-1} \leq \frac{\pi \varepsilon}{3c b_1}. \]

and (4.15)-(4.16) imply the estimate (4.8), \(\blacksquare\)

Referring to (4.8), let observe that
θ e^{-\frac{2}{\pi} \theta} \leq (4/c^2 e) e^{-\frac{2}{\pi} \theta}

and let $b \equiv \min(a, c^2/4\varepsilon)$. Thus, by (4.6),(4.7),(4.8) one has

$$| R_1 | + | R_2 | \leq \varepsilon^k r(t) e^{-bt}$$

with

$$r(t) = \varepsilon^{1-k} [c_1(\varepsilon) \rho(t) + c_2 (1 + 4/c^2 e)].$$

Then, the following theorem can be stated.

**Theorem 4.1** - Whatever the positive constant $k < 1$ may be, for all $t \geq 0$ and $x \in [0, \pi]$, it results:

$$| H_\varepsilon - H_0 | \leq \gamma \varepsilon^k (1 + t + t^2) e^{-bt},$$

where the constant $\gamma$ depends only on $k, a, c$.

**Remark 4.1** - The asymptotic analysis of this section and the results of theorem 4.1 can be applied also to the function $K_\varepsilon$ defined by (3.18) and related to the problem $\mathcal{H}_\varepsilon$.

## 5 Conclusions

To outline a first application and to avoid too many formulae, let consider only the term depending on the source. Then, referring to the problems $\mathcal{P}_\varepsilon$ and $\mathcal{P}_0$ and putting $\Delta H = H_\varepsilon - H_0$, it results

$$u_\varepsilon - u_0 = -\int_0^\pi d\xi \int_0^t f_{\xi \xi}(\xi, \tau) \Delta H(x, \xi, t - \tau) d\tau.$$
By assuming that \( f \in C^2(\Omega) \) and that \( f_{xx} \) is bounded also when \( t \to \infty \), let

\[
\| u_f \| = \sup_{\Omega} | f_{xx} (x, t) |. 
\] (5.2)

As consequence of theorem 4.1, when \( \varepsilon \to 0 \), the following rigorous approximation holds:

\[
u_{\varepsilon} = u_0 + \varepsilon^k \; r \quad \forall (x,t) \in \Omega
\] (5.3)

where the error \( r \) is such that

\[
| r | \leq \gamma_1 \| u_f \| \quad \forall t \geq 0
\] (5.4)

and the constant \( \gamma_1 \) depends only by \( a, c, k \). So, the error of the approximation is negligible also for large \( t \) (\( t \to \infty \)).

When \( f \) is non linear, an integral equation like

\[
v = \int_0^\pi d\xi \int_0^t \Delta H(x, \xi, t - \tau) F[\xi, \tau, u(\xi, \tau), u_t(\xi, \tau), u_{\xi}(\xi, \tau)] d\tau \] (5.5)

must be analyzed. These applications will be dealt successively.

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