ON THE FUNDAMENTAL DOMAIN OF AFFINE SPRINGER FIBERS

ZONGBIN CHEN

Abstract. For $G$ a connected reductive group, $\gamma \in g(F)$ semisimple regular unramified, we introduce a fundamental domain $F_\gamma$ for the affine Springer fibers $X_\gamma$. There is a beautiful way to reduce the purity conjecture of $X_\gamma$ to that of $F_\gamma$, we call it the Arthur-Kottwitz reduction. It turns out that these fundamental domains behave well in family. We formulate a rationality conjecture about a generating series of their Poincaré polynomials. We then study them in detail for the group $GL_3$. In particular, we pave them in affine spaces and we prove the rationality conjecture.

1. Introduction

Let $k = \mathbb{F}_q$, we fix an algebraic closure $\bar{k}$ of $k$. Let $F = k((\epsilon))$ be the field of Laurent series with coefficients in $k$, $\mathcal{O} = k[[\epsilon]]$ the ring of integers of $F$, $\mathfrak{p} = \epsilon k[[\epsilon]]$ the maximal ideal of $\mathcal{O}$. Let $\text{val} : F^\times \to \mathbb{Z}$ be the discrete valuation normalized by $\text{val}(\epsilon) = 1$.

Let $G$ be a connected reductive algebraic group split over $k$, we make the assumption that $\text{char}(k) > \text{rk}(G)$, where $\text{rk}(G)$ is the semisimple rank of $G$. Let $T$ be a split maximal torus of $G$, let $B$ be a Borel subgroup of $G$ containing $T$. Their Lie algebras will be denoted by the corresponding Gothic letters. Let $K = G(\mathcal{O})$ be the standard maximal compact subgroup of $G(F)$. The affine grassmannian $\mathcal{X} = G(F)/K$ is the ind-$k$-scheme such that $\mathcal{X}(\mathbb{F}_q^n) = G(F_q^n((\epsilon)))/G(F_q^n[[\epsilon]])$, $\forall n \in \mathbb{N}$.

For a regular element $\gamma \in t(\mathcal{O})$, the affine Springer fiber $\mathcal{X}_\gamma$ at $\gamma$

$$\mathcal{X}_\gamma = \{ g \in G(F)/K \mid \text{Ad}(g^{-1})\gamma \in g(\mathcal{O}) \}$$

has been introduced by Kazhdan and Lusztig [KL]. It has been used by Goresky, Kottwitz and Macpherson [GKM] to prove the fundamental lemma of Langlands-Shelstad in the unramified case, under the following hypothesis:

Conjecture 1.1 (Goresky-Kottwitz-Macpherson). The cohomology of $\mathcal{X}_\gamma$ is pure in the sense of Deligne, i.e. the eigenvalues of the action of Frobenius $\text{Fr}_q$ on $H^i(\mathcal{X}_\gamma, \mathbb{Q}_l)$ have an absolute value of $q^{i/2}$ for any embedding $\mathbb{Q}_l \to \mathbb{C}$.

This conjecture has been proved in some particular cases, see [GKM2], [Lu], [C], where the authors have found affine pavings of $\mathcal{X}_\gamma$.

The affine Springer fibers have a large symmetry group. The free abelian group $\Lambda$ generated by $\chi(\epsilon), \chi \in X_*(T)$ acts simply and transitively on the irreducible components of $\mathcal{X}_\gamma$. It is desirable to use this symmetry to reduce the study of $\mathcal{X}_\gamma$ to that of its irreducible components. But the condition of irreducibility is difficult to explore. Instead, we construct...
a fundamental domain $F_\gamma$ of $\mathcal{X}_\gamma$ with respect to the action of $\Lambda$, which should be exactly one of the irreducible components of $\mathcal{X}_\gamma$. Our first main result is the following:

**Theorem 1.1.** For any $\gamma \in t(\mathcal{O})$, suppose that $F_\gamma^M$ is cohomologically pure for any Levi subgroup $M$ of $G$ containing $T$. Then $\mathcal{X}_\gamma$ is cohomologically pure if and only if $F_\gamma$ is.

The conjecture of Goresky, Kottwitz and Macpherson can be restated as:

**Conjecture 1.2.** The cohomology of $F_\gamma$ is pure in the sense of Deligne.

It is believed that $F_\gamma$ is an irreducible component of $\mathcal{X}_\gamma$, and that it is also the normalization of $\Lambda \setminus \mathcal{X}_\gamma$. For example, this follows from the purity conjecture. These observations lead us to restate the conjectures of Goresky, Kottwitz, Macpherson and of Laumon [L] in the following way, which opens the door to a possible proof of the purity conjecture by deformation.

**Conjecture 1.3.** Let $C$ be a projective geometrically integral algebraic curve over $k$. Suppose that all the singularities of $C$ are planar. Then the normalization of the compactified Jacobian $\text{Jac}_C$ of $C$ is cohomologically pure.

Now we restrict to the group $G = \text{GL}_d$. Let $T$ be the maximal torus of diagonal matrices, let $B$ be the Borel subgroup of $G$ of the upper triangular matrices. Let $\Phi = \{\alpha_{i,j}\}$ be the root system of $G$ with respect to $T$, let $\alpha_i = \alpha_{i,i+1}$, $i = 1, \ldots, d - 1$, be the simple roots with respect to $B$. Let $\gamma \in t(\mathcal{O})$ be regular, it is said to be in minimal form if

$$\text{val}(\alpha_{i,j}(\gamma)) = \min \{\text{val}(\alpha_l(\gamma))\}, \quad \forall i < j.$$ 

In this case, we say that the root valuation of $\gamma$ is $(\text{val}(\alpha_1(\gamma)), \ldots, \text{val}(\alpha_{d-1}(\gamma)))$. We can always conjugate $\gamma$ such that it is in minimal form. For $n = (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1}$, we can find $\gamma \in t(\mathcal{O})$ in minimal form with root valuation $n$. It is believed that the topology of $F_\gamma$ only depends on its root valuation. Let $P_n(t)$ be its Poincaré polynomial.

**Conjecture 1.4.** The power series

$$\sum_{n_1=1}^{+\infty} \cdots \sum_{n_{d-1}=1}^{+\infty} P_{(n_1, \ldots, n_{d-1})}(t) T_1^{n_1} \cdots T_{d-1}^{n_{d-1}} \in \mathbb{Z}[[t, T_1, \cdots, T_{d-1}]]$$

is a rational function, i.e. it is an element of $\mathbb{Z}(t, T_1, \cdots, T_{d-1})$.

The second main result of our article is the following:

**Theorem 1.2.** Let $G = \text{GL}_3$, let $n = (n_1, n_2) \in \mathbb{N}^2$, $n_1 \leq n_2$, let $\gamma \in t(\mathcal{O})$ be in minimal form with root valuation $n$. The fundamental domain $F_\gamma$ can be paved in affine spaces, and the paving only depends on $n$. Its Poincaré polynomial is

$$P_n(t) = \sum_{i=1}^{n_1} i(t^{4i-2} + t^{4i-4}) + \sum_{i=2n_1}^{n_1+n_2-1} (2n_1 + 1)t^{2i} + \sum_{i=n_1+n_2}^{2n_1+n_2-1} 4(2n_1 + n_2 - i)t^{2i} + t^{4n_1+2n_2}.$$ 

The rationality conjecture for $\text{GL}_3$ is an easy consequence of the theorem.
Notations. Let $\Phi = \Phi(G, T)$ be the root system of $G$ with respect to $T$, let $W$ be the Weyl group of $G$ with respect to $T$. For any subgroup $H$ of $G$ which is stable under the conjugation of $T$, we note $\Phi(H, T)$ for the roots appearing in $\text{Lie}(H)$. Let $\Delta_B$ be the set of simple roots with respect to $B$, let $(\varpi_\alpha)_{\alpha \in \Delta_B}$ be the corresponding fundamental weights. To an element $\alpha \in \Delta_B$, we have a unique maximal parabolic subgroup $P_\alpha$ of $G$ containing $B$ such that $\Phi(N_{P_\alpha}, T) \cap \Delta_B = \alpha$, where $N_{P_\alpha}$ is the unipotent radical of $P_\alpha$. This gives a bijective correspondence between the simple roots in $\Delta_B$ and the maximal parabolic subgroups of $G$ containing $B$. Any maximal parabolic subgroup $P$ of $G$ is conjugate to certain $P_\alpha$ by an element $w \in W$, the element $w \cdot \varpi_\alpha$ doesn’t depend on the choice of $w$, we denote it by $\varpi_P$.

We use the $(G, M)$ notation of Arthur. Let $F(T)$ be the set of parabolic subgroups of $G$ containing $T$, let $L(T)$ be the set of Levi subgroups of $G$ containing $T$. For every $M \in L(T)$, we denote by $P(M)$ the set of parabolic subgroups of $G$ whose Levi factor is $M$. For $P \in P(M)$, we denote by $P^-$ the opposite of $P$ with respect to $M$. Let $X^*(M) = \text{Hom}(M, \mathbb{G}_m)$ and $a_M^* = X^*(M) \otimes \mathbb{R}$. The restriction $X^*(M) \to X^*(T)$ induces an injection $a_M^* \hookrightarrow a_T^*$. Let $(a_M^*)^*$ be the subspace of $a_T^*$ generated by $\Phi(M, T)$. We have the decomposition in direct sums

$$a_T^* = (a_M^*)^* \oplus a_M^*.$$

The canonical pairing

$$X_*(T) \times X^*(T) \to \mathbb{Z}$$

can be extended linearly to $a_T \times a_T^* \to \mathbb{R}$, with $a_T = X_*(T) \otimes \mathbb{R}$. For $M \in L(T)$, let $a_M^* \subset a_T$ be the subspace orthogonal to $a_M^*$, and $a_M \subset a_T$ be the subspace orthogonal to $(a_M^*)^*$, then we have the decomposition

$$a_T = a_M \oplus a_M^*,$$

let $\pi_M$, $\pi^M$ be the projections to the two factors.

We identify $X_*(T)$ with $T(F)/T(O)$ by sending $\chi$ to $\chi(\varepsilon)$. With this identification, the canonical surjection $T(F) \to T(F)/T(O)$ can be viewed as

$$T(F) \to X_*(T).$$

We use $\Lambda_G$ to denote the quotient of $X_*(T)$ by the coroot lattice of $G$ (the subgroup of $X_*(T)$ generated by the coroots of $T$ in $G$). We have a canonical homomorphism

$$G(F) \to \Lambda_G,$$

which is characterized by the following properties: it is trivial on the image of $G_{sc}(F)$ in $G(F)$ ($G_{sc}$ is the simply connected cover of the derived group of $G$), and its restriction to $T(F)$ coincides with the composition of $[1]$ with the projection of $X_*(T)$ to $\Lambda_G$. Since the morphism $[2]$ is trivial on $G(O)$, it descends to a map

$$\nu_G : \mathcal{X} \to \Lambda_G,$$

whose fibers are the connected components of $\mathcal{X}$.

Finally, we suppose that $\gamma \in \mathfrak{t}(O)$ satisfies $\gamma \equiv 0 \mod \varepsilon$ to avoid unnecessary complications.
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2. The fundamental domain

2.1. Truncated affine Springer fibers. For $M \in \mathcal{L}(T)$, the natural inclusion of $M(F)$ in $G(F)$ induces a closed immersion of $\mathcal{X}^M$ in $\mathcal{X}^G$. For $P = MN \in \mathcal{F}(T)$, we have the retraction

$$f_P : \mathcal{X} \rightarrow \mathcal{X}^M$$

which sends $gK = nmK$ to $mM(O)$, where $g = nmk$, $n \in N(F)$, $m \in M(F)$, $k \in K$ is the Iwasawa decomposition. More generally we can define $f_L^P : \mathcal{X}^L \rightarrow \mathcal{X}^M$ for $L \in \mathcal{L}(T)$, $L \supset M$ and $P_L \in \mathcal{P}^L(M)$. These retractions satisfy the transition property: Suppose that $Q \in \mathcal{P}(L)$ satisfy $Q \supset P_L$, then

$$f_P = f_L^P \circ f_Q.$$ 

For $P \in \mathcal{F}(T)$, we have the function $H_P : \mathcal{X} \rightarrow a^G_M = a_M/a_G$ which is the composition

$$H_P : \mathcal{X} \xrightarrow{f_P} \mathcal{X}^M \xrightarrow{\mu_M} \Lambda_M \rightarrow a^G_M.$$ 

Proposition 2.1 (Arthur). Let $B', B'' \in \mathcal{P}(T)$ be two adjacent Borel subgroups, let $\alpha^\vee_{B', B''}$ be the coroot which is positive with respect to $B'$ and negative with respect to $B''$. Then for any $x \in \mathcal{X}$, we have

$$H_{B'}(x) - H_{B''}(x) = n(x, B', B'') \cdot \alpha^\vee_{B', B''},$$

with $n(x, B', B'') \in \mathbb{Z}_{\geq 0}$.

Proof. Let $P$ be the parabolic subgroup generated by $B'$ and $B''$, let $P = MN$ be the Levi factorization. The application $H_{B'}$ factor through $f_P$, i.e. we have commutative diagram

and similarly for $H_{B''}$. Since $M$ has semisimple rank 1, the proposition is thus reduced to $G = SL_2$. In this case, let $T$ be the maximal torus of the diagonal matrices, $B' = \left( \begin{array}{cc} * & * \\ * & * \end{array} \right)$, $B'' = \left( \begin{array}{cc} * & * \\ -* & -* \end{array} \right)$, and we identify $a^G_T$ with the line $H = \{(x, -x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ in the usual way. By the Iwasawa decomposition, any point $x \in \mathcal{X}$ can be written as $x = \left( \begin{array}{cc} a \\ b \\ d \end{array} \right) K$. Let $m = \min\{\text{val}(a), \text{val}(b)\}$, $n = \text{val}(d)$, then $m + n \leq \text{val}(a) + \text{val}(d) = 0$ and

$$H_{B'}(x) = (-n, n), \quad H_{B''}(x) = (m, -m).$$
So
\[ H_{B'}(x) - H_{B''}(x) = -(n + m), n + m) = -(n + m) \cdot \alpha_{B', B''}, \]
and the proposition follows.

For any point \( x \in \mathcal{X}_\gamma \), we denote by \( \text{Ec}(x) \) the convex envelope in \( a_x^G \) of the \( H_{B'}(x), B' \in \mathcal{P}(T) \). For any \( P \in \mathcal{F}(T) \), we denote by \( \text{Ec}^P(x) \) the face of \( \text{Ec}(x) \) whose vertices are \( H_{B'}(x), B' \in \mathcal{P}(T), B' \subset P \).

**Definition 2.1.** A family \( D = (\lambda_{B'})_{B' \in \mathcal{P}(T)} \) of elements in \( a_x^G \) is called \((G, T)\)-orthogonal if it satisfies
\[ \lambda_{B'} - \lambda_{B''} \in \mathbb{R}_{\geq 0} \cdot \alpha_{B', B''}, \]
for any two adjacent Borel subgroups \( B', B'' \in \mathcal{P}(T) \).

Let \( D = (\lambda_{B'})_{B' \in \mathcal{P}(T)} \) be a \((G, T)\)-orthogonal family, we also denote by \( D \) the convex envelope of \( \lambda_{B'}, B' \in \mathcal{P}(T) \) in \( a_x^G \). For \( P = MN \in \mathcal{F}(T) \), parallel to \( \text{Ec}^P(x) \), we denote by \( D^P \) the face of \( D \) whose vertices are \( \lambda_{B'}, B' \in \mathcal{P}(T), B' \subset P \). With the projection \( \pi^M \), it will also be seen as a \((M, T)\)-orthogonal family.

Following Chaudouard and Laumon [CL], we define the **truncated affine grassmannian** \( \mathcal{X}(D) \) to be
\[ \mathcal{X}(D) = \{ x \in \mathcal{X} \mid \text{Ec}(x) \subset D \}, \]
and the **truncated affine Springer fiber** \( \mathcal{X}_\gamma(D) \) to be the intersection \( \mathcal{X}_\gamma \cap \mathcal{X}(D) \). The connected components of \( \mathcal{X}(D) \) and \( \mathcal{X}_\gamma(D) \) are projective \( k \)-schemes. It should be noted that there is slight difference between different components of both \( \mathcal{X}(D) \) and \( \mathcal{X}_\gamma(D) \).

**2.2. The fundamental domain.** A point \( x = gK \in \mathcal{X}_\gamma \) is said to be regular if the image of \( \text{Ad}(g^{-1})_\gamma \) under the reduction \( g(O) \to g(k) \) is regular. It can be proved that the subvariety \( \mathcal{X}_\gamma^{\text{reg}} \) of regular points is open dense in \( \mathcal{X}_\gamma \).

**Proposition 2.2** (Bezrukavnikov). The group \( T(F) \) acts transitively on \( \mathcal{X}_\gamma^{\text{reg}} \).

The reader is referred to [B] for the proof. As a consequence, the abelian group \( \Lambda \) generated by \( \chi(e) \in T(F), \chi \in X_\ast(T) \) acts freely and transitively on the irreducible components of \( \mathcal{X}_\gamma \).

Goresky, Kottwitz and Macpherson [GKM3] have given the following description of regular points in \( \mathcal{X}_\gamma \). We reproduce their proof here.

**Lemma 2.3** (Goresky-Kottwitz-Macpherson). A point \( x \in \mathcal{X}_\gamma \) is regular if and only if for any Levi subgroup \( M \in \mathcal{L}(T) \) of semisimple rank 1, the point \( f_P(x) \in \mathcal{X}_\gamma^M \) is regular for any \( P \in \mathcal{P}(M) \).

**Proof.** For \( x = gK \in \mathcal{X}_\gamma \), the image of \( \text{Ad}(g^{-1})_\gamma \) under the reduction \( g(O) \to g(k) \) is well defined up to conjugacy, we denote it by \( u_G(x) \). For any \( P = MN \in \mathcal{F}(T), g = pk, p \in P(F), k \in K \), then \( \text{Ad}(p^{-1})_\gamma \in \mathfrak{p}(F) \cap g(O) = p(O) \). Its image in \( \mathfrak{p}(k) \) under the reduction is well defined up to conjugacy, we will denote it by \( u_P(x) \). It is obvious that \( u_P(x) \) goes to \( u_M(f_P(x)) \) under the projection \( \mathfrak{p} \to \mathfrak{m} \). So if \( u_G(x) \) is regular, then
Proposition 2.4 (Goresky-Kottwitz-Macpherson). Let \( x \in \mathcal{X}_\gamma \).

1. For any two adjacent Borel subgroups \( B', B'' \in \mathcal{P}(T) \), we have
   \[
   n(x, B', B'') \leq \text{val}(\alpha_{B', B''}(\gamma)),
   \]
   where \( \alpha_{B', B''} \) is the root associated with the coroot \( \alpha_{B', B''}^\vee \).

2. The point \( x \) is regular if and only if the above equality holds for any two adjacent Borel subgroups.

Proof. First of all, observe that for any \( x, y \in \mathcal{X} \) such that \( y \) lies in the closure of the orbit \( T(O) \cdot x \), we have \( Ec(y) \subset Ec(x) \). Now that \( \mathcal{X}_\gamma^{\text{reg}} \) is dense open in \( \mathcal{X}_\gamma \), it suffices to prove the second assertion. By lemma 2.3, it suffices to prove the proposition for \( G = \text{GL}_2 \). This follows from proposition 2.5 where we have picked a particular regular point \( x_0 \in \mathcal{X}_\gamma^{\text{reg}} \) and calculated that
   \[
   H_B(x_0) = (\text{val}(\alpha(\gamma)), 0), \quad H_{B^-}(x_0) = (0, \text{val}(\alpha(\gamma))).
   \]
   It is obvious that \( H_B(x_0) - H_{B^-}(x_0) = \text{val}(\alpha(\gamma)) \cdot \alpha^\vee \).

The above results motivate the following definition.

Definition 2.2. Take a regular point \( x \in \mathcal{X}_\gamma^{\text{reg}} \). Let \( F_\gamma \) be the connected component of the truncated affine Springer fiber \( \mathcal{X}_\gamma((H_B(x))_{B' \in \mathcal{P}(T)}) \) which contains \( x \), it is the fundamental domain of \( \mathcal{X}_\gamma \) with respect to the action of \( \Lambda \).

It is clear that different choice of \( x \in \mathcal{X}_\gamma^{\text{reg}} \) gives rise to isomorphic fundamental domain.

2.3. Examples for \( \text{GL}_d \). For the group \( \text{GL}_d \), \( \gamma \in t(O) \) regular, we have a particular choice of a regular point \( x_0 \) on \( \mathcal{X}_\gamma \). Let \( x_0 \in \mathcal{X}_\gamma^{\text{GL}_d} \) be the point representing the lattice \( O[\gamma] \) sitting inside \( F[\gamma] \cong F[X]/(X-\gamma_1) \oplus \cdots \oplus F[X]/(X-\gamma_d) \cong F^d \), where \( \gamma_i \) are the eigenvalues of \( \gamma \). Taking \( \{1, \gamma, \cdots, \gamma^{d-1}\} \) as a basis of \( F[\gamma] \), we check easily that \( x_0 \) is a regular point.

Proposition 2.5. For \( \sigma \in \mathcal{S}_d \), we have
   \[
   H_{\sigma B^-}(x_0) = \sigma^{-1} \left[ \bigoplus_{i=1}^d \text{val}(\alpha_{\sigma(i), \sigma(j)}(\gamma)) \right].
   \]
Proof. Let \( \{e_1, \cdots, e_d\} \) be the natural basis of \( F^d \), the vector \( \gamma^s \in O[\gamma] \) corresponds to the vector \( \sum_{i=1}^{d} \gamma_i^s e_i \) in \( F^d \). Let \( g \) be the matrix

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
\gamma_1 & \gamma_2 & \cdots & \gamma_d \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_1^{d-1} & \gamma_2^{d-1} & \cdots & \gamma_d^{d-1}
\end{bmatrix},
\]

then \( O[\gamma] = gO^d \). From this expression, we see that it suffices to prove the proposition for the standard \( B^- \).

After certain elementary operations on the columns, the matrix \( g \) can be put in lower triangular form with \( 1, \gamma_2 - \gamma_1, (\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1), \ldots, \prod_{i=1}^{d-1} (\gamma_d - \gamma_i) \) on the diagonal from top to bottom, from which the claim for \( H_{B^-} \) (4) follows easily. \( \square \)

Let \( \gamma \in \mathfrak{t}(O) \) be regular in minimal form, suppose that its valuation data \( (n_1, \cdots, n_{d-1}) \) satisfies \( n_1 \leq n_2 \leq \cdots \leq n_{d-1} \), then the fundamental domain \( F_\gamma \) can be written as the intersection of \( \mathcal{X}_\gamma \) with two affine Schubert varieties. First of all, we identify \( X_\gamma(T) \) with \( \mathbb{Z}^d \) in the natural way. For \( \mu \in X_\gamma(T) \), we denote by \( \text{Sch}(\mu) \) the affine Schubert variety \( K\varepsilon K/K \). We fix

\[
\mu = \left( 0, n_1, n_1 + n_2, \cdots, \sum_{i=1}^{d-1} n_i \right),
\]

\[
\lambda = \left( (d-1)n_1, n_1 + (d-2)n_2, n_1 + n_2 + (d-3)n_3, \cdots, \sum_{i=1}^{d-1} n_i, \sum_{i=1}^{d-1} n_i \right).
\]

Observe that

\[
\mu_i = \sum_{j=1}^{i-1} \text{val}(\alpha_{j,i}(\gamma)), \quad \lambda_i = \sum_{j=1}^{d} \text{val}(\alpha_{j,i}(\gamma)).
\]

Proposition 2.6. In the above setting, the fundamental domain \( F_\gamma \) is the intersection

\[
F_\gamma = \mathcal{X}_\gamma \cap [\text{Sch}(\mu) \cap \varepsilon^\lambda \cdot \text{Sch}(\varepsilon^{-\mu})].
\]

Proof. Let \( D_1, D_2 \) be the convex envelope of the \((G,T)\)-orthogonal family \((w \cdot \mu)_{w \in W}\) and \((\lambda - w \cdot \mu)_{w \in W}\) respectively. By Bruhat-Tits decomposition, we have

\[
\mathcal{X}^{[\mu]}(D_1) = \text{Sch}(\mu), \quad \mathcal{X}^{[\mu]}(D_2) = \varepsilon^\lambda \cdot \text{Sch}(\varepsilon^{-\mu}),
\]

where \( \mathcal{X}^{[\mu]} \) is the connected component containing \( e^\mu \). So we only need to prove that \( \text{Ec}(x_0) = D_1 \cap D_2 \). Using proposition 2.5 on verifies that \( H_{B'}(x_0) \in D_1 \cap D_2, \forall B' \in \mathcal{P}(T) \), so \( \text{Ec}(x_0) \subset D_1 \cap D_2 \). Given a \((G,T)\)-orthogonal family \( D \), given \( P = MN \in \mathcal{F}(T) \) maximal, let \( d_P(D) \) be the distance between the two opposite faces \( D^P \) and \( D^{P'} \). It suffices then to prove that

\[
d_P(\text{Ec}(x_0)) = d_P(D_1 \cap D_2).
\]
Choose a minimal gallery of Borel subgroups $B_1, \cdots, B_{l+1}$ such that $B_1 \in P, B_{l+1} \in P^-$, then $\alpha_{B_i, B_{i+1}}, i = 1, \cdots, l$ runs through $\Phi(N, T)$ exactly once. So we have

$$d_P(Ec(x_0)) = \varpi_P(H_{B_1}(x_0) - H_{B_{l+1}}(x_0)) = \sum_{i=1}^{l} \varpi_P(H_{B_i}(x_0) - H_{B_{i+1}}(x_0)).$$

To calculate $d_P(D_1 \cap D_2)$, let $a_P$ be the distance of $D_1^P$ and $D_2^P$, then

$$d_P(D_1 \cap D_2) = \min\{d_P(D_1), a_P\}.$$

Let $P_i = M_i N_i \in \mathcal{F}(T)$ be the maximal parabolic subgroup associated to the simple root $\alpha_i$. For $P$ conjugate to $P_i$ or $P_{d-i}, i \leq d/2$, we calculate that

$$d_P(D_1) = \sum_{j=1}^{i} \sum_{l=1}^{d-i+j-1} \text{val}(\alpha_{l,d-i+j}(\gamma)) = \sum_{j=1}^{i} \sum_{l=1}^{d-i+j-1} n_l.$$

We calculate $a_{P_i}$ to be

$$a_{P_i} = 2\varpi_{P_i}(\mu) - \varpi_{P_i}(\lambda) = -2\varpi(\mu) + \varpi(\lambda)$$

$$= 2 \left( \mu_i + \cdots + \mu_d - \frac{d-i}{d}(\mu_1 + \cdots + \mu_d) \right) - \left( \lambda_i + \cdots + \lambda_d - \frac{d-i}{d}(\lambda_1 + \cdots + \lambda_d) \right)$$

$$= \sum_{\alpha \in \Phi(N_i, T)} \text{val}(\alpha(\gamma)),$$

here we use equation (3) in the last equality. Conjugate the above calculation by $\sigma \in S_d$, we found that

$$a_P = \sum_{\alpha \in \Phi(N, T)} \text{val}(\alpha(\gamma)).$$

It is easy to verify that $a_P \leq d_P(D_1)$, so we have $d_P(D_1 \cap D_2) = a_P = d_P(Ec(x_0)).$ 

\[\Box\]

3. ARTHUR-KOTTWITZ REDUCTION

Fix a regular point $x_0 \in X^{\text{reg}}$. Take $\xi \in a_G^\gamma$ such that $\alpha(\xi)$ is positive but almost equal to 0 for any $\alpha \in \Delta_B$. Let $D_0 = (\lambda_{B'})_{B' \in P(T)}$ be the $(G, T)$-orthogonal family given by

$$\lambda_{B'} = H_{B'}(x_0) + w' \cdot \xi,$$
where \( u' \in W \) is taken such that \( B' = u' \cdot B \). For \( P = MN \in \mathcal{F}(T) \), define \( R_P \) to be the subset of \( a^G_T \) satisfying conditions

\[
\pi^M(a) \subset D_0^P; \\
\alpha(\pi_M(a)) \geq \alpha(\pi_M(\lambda_{B'})), \forall \alpha \in \Phi(N, T), \forall B' \in \mathcal{P}(T), B' \subset P.
\]

Notice that \( R_G = D_0 \). We get a partition

\[
a^G_T = \bigcup_{P \in \mathcal{F}(T)} R_P.
\]

The figure 1 gives an illustration of the partition for the group \( GL_3 \). The partition (4) induces a disjoint partition of \( X_T^\gamma \) via the map \( X_T^\gamma \rightarrow a^G_T \), since we have perturbed the \((G, T)\)-family \((H_{B'}(x_0))_{B' \in \mathcal{P}(T)}\) with \( \xi \).

**Figure 1. Partition of \( a^G_T \) for GL3.**

**Lemma 3.1.** For any \( x \in \mathcal{X}_\gamma \), there exists a unique \( P \in \mathcal{F}(T) \) such that \( \text{Ec}^P(x) \subset R_P \).

**Proof.** By proposition 2.4, the convex polytope \( \text{Ec}(x) \) is contained in a translation of \( \text{Ec}(x_0) \) by some \( \lambda \in \Lambda \). The uniqueness is clear for such a translation, from which the uniqueness for general case follows.

Now we prove the existence. For a maximal parabolic \( P \in \mathcal{F}(T) \), let \( R_P = \bigcup_{P' \in P} R_{P'} \). Notice that \( \varpi_P(H_{B'}(x)) \) doesn’t depend on the choice of \( B' \in \mathcal{P}(T), B' \subset P \). Let \( P_0 \) be the maximal parabolic such that \( \varpi_P(H_{B'}(x) - H_{B'}(x_0)) \) is maximal among all the maximal parabolic subgroups \( P \). It follows that \( \text{Ec}^{P_0}(x) \subset R_{P_0} \), the basic reason is that in a right triangle with sides \( a, b, c \), we always have \( c > a, b \). Now we can use the retraction \( f_{P_0} : \mathcal{X}_\gamma \rightarrow \mathcal{X}_\gamma^{M_{P_0}} \) to find the required parabolic subgroup \( P \) inductively. \( \square \)
With this lemma, we define \( S_P := \{ x \in \mathcal{X}_\gamma \mid \text{Exp}(x) \subseteq R_P \} \). Notice that \( F_\gamma \) is one connected component of \( S_G = \mathcal{X}_\gamma(D_0) \), the other connected components of \( S_G \) have slight difference from \( F_\gamma \). We get a disjoint partition

\[
\mathcal{X}_\gamma = \mathcal{X}_\gamma(D_0) \cup \bigcup_{P \in \mathcal{F}(T) \setminus \mathcal{F}^T} S_P.
\]

Consider the restriction of the retraction \( f_P : \mathcal{X} \rightarrow \mathcal{X}^M \) to \( S_P \), its image is \( S_P \cap \mathcal{X}^M \).

Recall that the connected components of \( \mathcal{X}^M \) are fibers of the map \( \nu_M : \mathcal{X}^M \rightarrow \Lambda_M \). For \( \nu \in \Lambda_M \), let \( \mathcal{X}^{M,\nu} \) be its fiber at \( \nu \). Let \( S_P^\nu = f_P^{-1}(S_P \cap \mathcal{X}^{M,\nu}) \), it is easy to verify that

\[
S_P \cap \mathcal{X}^{M,\nu} = \mathcal{X}^{M,\nu}_\gamma(D_0^P).
\]

**Proposition 3.2.** The retraction \( f_P : S_P^\nu \rightarrow \mathcal{X}^{M,\nu}_\gamma(D_0^P) \) is an iterated fibration in affine spaces.

**Proof.** We follow the proof of Kazhdan-Lusztig [KL], §5. By assumption, \( \text{char}(k) > \text{rk}(G) \), the exponential map \( \exp : \mathfrak{n} \rightarrow N \) is well defined. The group \( N \) has the decreasing filtration by normal subgroups

\[
N_0 = N \supset N_1 = [N, N] \supset \cdots \supset N_i = [N_{i-1}, N] \supset \cdots \supset N_{\text{rk}(G)} \supset 1.
\]

The exponential map induces an isomorphism \( \mathfrak{n}_i/\mathfrak{n}_{i+1} \rightarrow N_i/N_{i+1} \) which sends \( n_i \) to \( 1 + n_i \).

Since \( f_P(ux) = f_P(x), \forall u \in N, x \in \mathcal{X} \), by the definition of \( S_P^\nu \), we have the identity

\[
S_P^\nu = [N(F) \cdot \mathcal{X}^{M,\nu}_\gamma(D_0^P)] \cap \mathcal{X}_\gamma.
\]

So the fiber of \( f_P : S_P^\nu \rightarrow \mathcal{X}^{M,\nu}_\gamma(D_0^P) \) at \( mM(O) \) is

\[
\left\{ umM(O) \mid u \in N(F), \text{Ad}(u^{-1}) \gamma \in \text{Ad}(m)(\mathfrak{g}(O)) \right\}.
\]

We’ll prove that they form a family which is an iterated fibration in affine spaces.

Let \( K \) be the \( K \)-equivariant fiber bundle \( G(F) \times_K K \) on \( \mathcal{X} \), let \( \mathfrak{k} \) be the \( K \)-equivariant vector bundle \( G(F) \times_K \mathfrak{g}(O) \) on \( \mathcal{X} \), where \( K \) acts on \( K \) and \( \mathfrak{g}(O) \) by conjugaison. Let \( \tilde{N}_i \) be the constant fiber bundle \( \mathcal{X} \times N_i(F) \), let \( \mathfrak{n}_i \) be the constant vector bundle \( \mathcal{X} \times \mathfrak{n}_i(F) \).

We denote also \( \tilde{N} = \tilde{N}_0 \).

To begin with, observe that with the retraction \( f_P \), the locally closed subvariety \( f_P^{-1}(\mathcal{X}^{M,\nu}) \) of \( \mathcal{X} \) is isomorphic to the restriction of the fiber bundle \( \tilde{N}/\tilde{N} \cap K \) over \( \mathcal{X}^{M,\nu} \), we will identify them in the following. For \( i = 0, \cdots, \text{rk}(G) + 1 \), let \( S_i \) be the sub bundle of \( \tilde{N}_i \backslash \tilde{N} \cap K \) restricted to \( \mathcal{X}^{M,\nu}_\gamma(D_0^P) \), whose fiber at \( mM(O) \) is given by

\[
\left\{ u \in N_i(F) \backslash N(F) \cap N(M(F)) | \text{Ad}(u)^{-1} \gamma \in \text{Ad}(m)(\mathfrak{g}(O)) + \mathfrak{n}_i(F) \right\}.
\]

Let \( p_i : S_{i+1} \rightarrow S_i \) be the natural projection, we get a tower of projections

\[
S^\nu_P \cong S_{\text{rk}(G)+1} \rightarrow S_{\text{rk}(G)} \rightarrow \cdots \rightarrow S_0 \cong \mathcal{X}^{M,\nu}_\gamma(D_0^P).
\]
The last isomorphism is due to the equivalence of the equations \( \gamma \in \text{Ad}(m)g(O) + n(F) \) and \( \gamma \in \text{Ad}(m)g(O) \) since \( \text{Ad}(m)^{-1} \gamma \in m(F) \). We will prove that each \( S_{i+1} \) is a homogeneous space under a vector bundle over \( S_i \), this will end the proof of the proposition.

Given \( gK \in S_i \), we have

\[
\gamma \in \text{Ad}(g)g(O) + n_i(F).
\]

Let \( u = 1 + n \in N_{i+1}(F) \setminus N_i(F) \), with \( n \in n_{i+1}(F) \setminus n_i(F) \), then

\[
u gK \in S_{i+1} \quad \iff \quad \text{Ad}(u^{-1}) \gamma \in \text{Ad}(g)g(O) + n_{i+1}(F)
\]

Using the isomorphism

\[
\frac{\text{Ad}(g)g(O) + n_i(F)}{\text{Ad}(g)g(O) + n_{i+1}(F)} \cong \frac{n_i(F)/n_i(F) \cap \text{Ad}(g)g(O)}{n_{i+1}(F)/n_{i+1}(F) \cap \text{Ad}(g)g(O)},
\]

let \( \tilde{\gamma} \) be the image of \( \gamma \) under the isomorphism, then the equation \([5]\) means that \( n \) should satisfy the equation \( \text{ad}(\gamma)n = -\tilde{\gamma} \) in the above quotient. Consider the endomorphism \( \text{ad}(\gamma) \) of the restriction of the vector bundle

\[
\frac{\tilde{n}_i/\tilde{n}_i \cap R}{\tilde{n}_{i+1}/\tilde{n}_{i+1} \cap R}
\]
on \( S_i \). It is surjective since \( \text{ad}(\gamma) : n_i(F) \to n_i(F) \) is. This means that there is always \( n \) such that equation \([5]\) is satisfied, i.e. \( p_i \) are surjective for all \( i \). Further more, let \( V_i \) be kernel of the endomorphism \( \text{ad}(\gamma) \) of the vector bundle \([6]\), then \( V_i \) is a vector bundle on \( S_i \). The above calculation shows that \( S_{i+1} \) is a homogeneous space over \( S_i \) under the vector bundle \( V_i \).

\[\qed\]

**Proposition 3.3.** The stratas \( S_P^\nu \) are locally closed sub varieties of \( \mathcal{X}_\gamma \). Further more, in the decomposition

\[
\mathcal{X}_\gamma = \mathcal{X}_\gamma(D_0) \cup \bigcup_{P \in \mathcal{F}(T) \setminus T} \bigcup_{\nu \in \Lambda_{M_p} \cap R_P} S_P^\nu,
\]

we can order the strata \( S_P^\nu \) such that at each step we have a closed sub variety of \( \mathcal{X}_\gamma \).

**Proof.** To begin with, \( \mathcal{X}_\gamma(D) \) is a closed sub variety of \( \mathcal{X}_\gamma \) for any \( (G,T) \)-orthogonal family \( D \). Now we prove by induction. Let \( P_0 = M_0N_0 \in \mathcal{F}(T) \) be a maximal parabolic subgroup containing \( P \). For \( P' = M'N' \in \mathcal{F}(T) \), \( P' \subseteq P_0 \), let \( p_{M_0}^{M'} \) be the natural projection \( \Lambda_{M'} \to \Lambda_{M_0} \), let \( \nu_0 = p_{M_0}^{M} (\nu) \). Consider

\[
Z_{P_0}^{\nu_0} \overset{:=}{=} \bigcup_{P'' \subseteq P_0} \bigcup_{p_{M_0}^{M''} (\nu')} = \nu_0 S_{P''}^\nu.
\]

Firstly, \( Z_{P_0}^{\nu_0} \) can be written as a difference \( \mathcal{X}_\gamma(D) \setminus \mathcal{X}_\gamma(D') \) for two \( (G,T) \)-orthogonal family \( D, D' \). Secondly, observe that

\[
S_{P_0}^{\nu'} = \left[ N_0(F) \cdot (S_{P_0}^{\nu'} \cap \mathcal{X}_{M_0,\nu_0}) \right] \cap \mathcal{X}_\gamma,
\]
the same proof as that of proposition 3.2 shows that the retraction
\[ f_{P_0} : Z_{F_0}^{10} \to \mathcal{A}_{M_0}^{M_0}(D_{F_0}) \]
is an iterated fibration in affine spaces. Now the claim follows by induction. \(\square\)

By proposition 3.2 each strata \(S'_{P_0}\) has an iterated affine fibration onto \(\mathcal{A}_{\gamma}^{M_0}(D_{F_0})\), so the study of \(\mathcal{A}_{\gamma}\) is reduced to that of \(F_{\gamma}\). We call the decomposition (7) the Arthur-Kottwitz reduction. We remark that this is not a reduction of Harder-Narasimhan type.

**Lemma 3.4.** Suppose that \(F_{\gamma}^{M}\) is cohomologically pure for any Levi subgroup \(M\) of \(G\) containing \(T\). Suppose that \(F_{\gamma}\) is pure, then the truncated affine Springer fiber \(\mathcal{A}_{\gamma}(D_{0})\) is pure for all \(\nu \in \Lambda_G\).

**Proof.** After certain translation on \(\mathcal{X}\) by \(\Lambda\), we have \(F_{\gamma} = \mathcal{X}_{\gamma}(D_{0} + \nu)\), where \(\nu\) is a minuscule coweight, and \(D_{0} + \nu\) is the translation of \(D_{0}\) by \(\nu\). It is easy to see that \(\mathcal{A}_{\gamma}(D_{0}) \subset F_{\gamma}\). Applying the reduction of Arthur-Kottwitz, the open sub variety \(F_{\gamma} \setminus \mathcal{A}_{\gamma}^{\nu}(D_{0})\) is naturally stratified into finite unions of \(S'_{P_0} \cap F_{\gamma}\). Since the two truncation parameters differ by a minuscule coweight, we have an inclusion of \(T\)-fixed points \((S'_{P_0})^T \subset (F_{\gamma})^T\) and \((S'_{P_0})^T\) all lies on the faces of \(E_c(x_0)\). By proposition 2.3 we have \(S'_{P_0} \subset F_{\gamma}\), so \(S'_{P_0} \cap F_{\gamma} = S'_{P_0}\).

Now we prove by induction. Suppose that the lemma is proved for all the Levi subgroups \(M \in \mathcal{L}(T)\), then \(\mathcal{X}_{\gamma}^{M',\nu}(D_{0}^P)\) are all cohomologically pure for all \(P \in \mathcal{P}(M)\) and all \(\nu \in \Lambda_M\). By proposition 3.2 and 3.3 we see that \(F_{\gamma} \setminus \mathcal{A}_{\gamma}^{\nu}(D_{0})\) is cohomologically pure. Now the long exact sequence
\[ \cdots \to H^{i-1}(\mathcal{A}_{\gamma}^{\nu}(D_{0})) \to H^{i}(F_{\gamma} \setminus \mathcal{A}_{\gamma}^{\nu}(D_{0})) \to H^{i}(F_{\gamma}) \to H^{i}(\mathcal{A}_{\gamma}^{\nu}(D_{0})) \to \cdots \]
will split into short exact sequence
\[ 0 \to H^{i}(F_{\gamma} \setminus \mathcal{A}_{\gamma}^{\nu}(D_{0})) \to H^{i}(F_{\gamma}) \to H^{i}(\mathcal{A}_{\gamma}^{\nu}(D_{0})) \to 0, \]
because \(H^{i-1}(\mathcal{A}_{\gamma}^{\nu}(D_{0}))\) is of weight at most \(i - 1\) by [Weil II]. The claim then follows from the above short exact sequence. \(\square\)

Now we come to the proof of theorem 1.1. We will prove a slightly stronger result. A \((G, T)\)-orthogonal family \(D = (\mu_{B'})_{B' \in \mathcal{P}(T)}\) is said to be regular with respect to \(D_0\) if \(\mu_{B'} \in R_{B'}, \forall B' \in \mathcal{P}(T)\).

**Theorem 3.5.** Suppose that \(F_{\gamma}^{M}\) is cohomologically pure for any Levi subgroup \(M\) of \(G\) containing \(T\). Let \(D\) be a \((G, T)\)-orthogonal family regular with respect to \(D_{0}\). Then \(F_{\gamma}\) is cohomologically pure if and only if the truncated affine Springer fiber \(\mathcal{A}_{\gamma}(D)\) is.

**Proof.** The complication that some connected components of \(\mathcal{A}_{\gamma}(D)\) doesn’t contain \(F_{\gamma}\) is already treated in lemma 3.4 so we can suppose that every connected component of \(\mathcal{A}_{\gamma}(D)\) contains a translation of \(F_{\gamma}\). Applying the Arthur-Kottwitz reduction to every connected component \(\mathcal{A}_{\gamma}^{\nu}(D)\), we get a stratification of \(\mathcal{A}_{\gamma}^{\nu}(D) \setminus F_{\gamma}\) into finite union of \(S'_{P_0} \cap \mathcal{A}_{\gamma}^{\nu}(D)\).
The hypothesis that $D$ is regular with respect to $D_0$ implies that each $S_{P_0}^{\nu'}$ is either contained in $\mathcal{X}_\gamma^{\nu'}(D)$ or disjoint from it. Applying lemma 3.4 to the Levi subgroups $M \in \mathcal{L}(T)$, we see that all the truncated affine Springer fibers $\mathcal{X}_\gamma^{M,\nu'}(D_0^P), P \in \mathcal{P}(M)$ are cohomologically pure, which implies that $\mathcal{X}_\gamma^{\nu'}(D) \setminus F_\gamma$ is cohomologically pure by proposition 3.2 and 3.3. Now the theorem follows from the same argument as the last part of the proof of lemma 3.4.

4. Rationality conjecture for $GL_3$

Let $G = GL_3$, let $n = (n_1, n_2) \in \mathbb{N}^2$, $n_1 \leq n_2$, let $\gamma \in t(O)$ be in minimal form with root valuation $n$. Let $F_{\gamma}$ be the fundamental domain of $\mathcal{X}_\gamma$ containing $x_0$, where $x_0$ is the regular point defined in §2.3.

**Proposition 4.1.** The fundamental domain $F_\gamma$ is the intersection of $\mathcal{X}_\gamma$ with

$$\text{Sch}(2n_1 + n_2, 0, 0) \cap \text{diag}(e^{-n_2}, e^{-n_1}, e^{-n_1}) \cdot \text{Sch}(2n_1 + n_2, 2n_1 + n_2, 0).$$

**Proof.** By proposition 2.5 $Ec(x_0)$ is the hexagon with vertices marked as indicated in figure 2.

![Figure 2](image-url)  

**Figure 2.** Hexagon as intersection of two triangles.

This hexagon can also be represented as the intersection of two triangles as indicated also in the figure. Let $\Delta, \nabla$ be the upward and the downward triangle in the figure. By
Bruhat-Tits decomposition, we see that $\text{Ec}(x) \in \emptyset$ if and only if $x \in \text{Sch}(2n_1 + n_2, 0, 0)$. We notice that $\triangle$ is the translation by $(-n_2, -n_1, -n_1)$ of the triangle $\triangle'$ with vertices

$$(2n_1 + n_2, 2n_1 + n_2, 0), \quad (2n_1 + n_2, 0, 2n_1 + n_2), \quad (0, 2n_1 + n_2, 2n_1 + n_2).$$

Again by Bruhat-Tits decomposition, we see that $\text{Ec}(x) \in \emptyset$ if and only if $x \in \text{Sch}(2n_1 + n_2, 2n_1 + n_2, 0)$, the result follows directly from these considerations.

\[ \square \]

4.1. **Affine paving.** We can pave $F_\gamma$ in affine spaces, the strategy is the following: By proposition 4.1, we have

$$F_\gamma = \mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0) \cap \text{diag}(\epsilon^{-n_2}, \epsilon^{-n_1}, \epsilon^{-n_1}) \cdot \text{Sch}(2n_1 + n_2, 2n_1 + n_2, 0).$$

So we firstly pave the intersection of the two affine schubert varieties in affine spaces, but this paving doesn't induce an affine paving of $F_\gamma$, we need to regroup the resulting pavements and do a second nonstandard paving. Our main result in this section is:

**Theorem 4.2.** The fundamental domain $F_\gamma$ admits an affine paving, which only depends on the root valuation of $\gamma$.

**Proof.** Let $I$ be the standard Iwahori subgroup, i.e. it is the inverse image of the Borel subgroup $B$ under the reduction $G(\mathcal{O}) \to G(k)$. Let $I' = \text{Ad}(\text{diag}(\epsilon^{n_1}, \epsilon^{n_2}, \epsilon^{n_2}))I$. By \cite{C} corollary 2.3, we have the affine paving

$$\text{Sch}(2n_1 + n_2, 0, 0) \cap \text{diag}(\epsilon^{-n_2}, \epsilon^{-n_1}, \epsilon^{-n_1}) \cdot \text{Sch}(2n_1 + n_2, 2n_1 + n_2, 0)$$

$$= \bigcup_{\mu \in (F_\gamma)^T} \text{Sch}(2n_1 + n_2, 0, 0) \cap I\epsilon^\mu K/K$$

$$= \bigcup_{\mu \in (F_\gamma)^T} \left[ \begin{array}{ccc} \mathcal{O} & p^a & p^b \\ p^{n_2-n_1+1} & \mathcal{O} & \mathcal{O} \\ p^{n_2-n_1+1} & p & \mathcal{O} \end{array} \right] \epsilon^\mu K/K,$$

where $a = \max\{n_1 - n_2, -\mu_2\}$, $b = \max\{n_1 - n_2, -\mu_3\}$. We denote by $C(\mu)$ the resulting pavement containing $\epsilon^\mu$.

To pave $F_\gamma$, we cut it into 4 parts. Let $(\mu_1', \mu_2', \mu_3') = (\mu_1 - n_1, \mu_2 - n_2, \mu_3 - n_2)$, and

$$R_1 = \{ \mu \in (F_\gamma)^T \mid \mu_1' \leq \mu_2', \mu_3' \},$$

$$R_1' = \{ \mu \in (F_\gamma)^T \mid \mu_1' \geq \mu_2', \mu_3' ; \mu_2 \leq n_2 - n_1 ; \mu_3 \leq n_2 - n_1 \},$$

$$R_2 = \{ \mu \in (F_\gamma)^T \mid \mu_2 < \mu_1', \mu_3' ; \mu_3 > n_2 - n_1 \},$$

$$R_3 = \{ \mu \in (F_\gamma)^T \mid \mu_3 < \mu_1', \mu_2' ; \mu_2 > n_2 - n_1 \}.$$

Although $R_1$ and $R_1'$ may intersect at one point, it doesn't cause trouble to the paving. Figure 3 gives an idea of the cutting. Let $V_i = \bigsqcup_{\mu \in R_i} C(\mu), i = 1, 2, 3$. For $l \in \mathbb{Z}$, let $R_{i,l} = \{ \mu \in R_i \mid \mu_i = l \}$ and $V_{i,l} = \bigsqcup_{\mu \in R_{i,l}} C(\mu)$. Similar notations for $R_1'$.
Figure 3. Nonstandard paving.

We use the Iwahori subgroup $I'$ to pave $V_1 \cap \mathcal{X}_\gamma$. Since we have

$$C(\mu) = \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ p^{n_2-n_1+1} & p & p \end{bmatrix} e^{\mu} K/K,$$

we see easily that $C(\mu) \cap \mathcal{X}_\gamma$ is isomorphic to an affine space.

We also use $I'$ to pave $V_1' \cap \mathcal{X}_\gamma$. We have

$$C(\mu) = \begin{bmatrix} \mathcal{O} & p^{-\mu_2} & p^{-\mu_3} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ p & p & \mathcal{O} \end{bmatrix} e^{\mu} K/K.$$

It is easily checked that $C(\mu) \cap \mathcal{X}_\gamma$ is isomorphic to an affine space.

We need a second nonstandard paving in order to pave $V_2 \cap \mathcal{X}_\gamma$ and $V_3 \cap \mathcal{X}_\gamma$. Since they are symmetric, we only give details for $V_3 \cap \mathcal{X}_\gamma$. Since $V_3 = \bigsqcup_{l \in \mathbb{Z}} V_{3,l}$, we only need to pave $V_{3,l} \cap \mathcal{X}_\gamma$. Let $I'_l = \text{Ad}(\text{diag}(1, e^l, e^l)) I'$, we claim that

$$V_{3,l} \cap \mathcal{X}_\gamma = \bigsqcup_{\mu \in R_{3,l}} V_{3,l} \cap I'_l e^{\mu} K/K \cap \mathcal{X}_\gamma$$

is an affine paving. Since we have

$$C(\mu) = \begin{bmatrix} \mathcal{O} & p^{n_1-n_2} & p^{-\mu_3} \\ p^{n_2-n_1+1} & \mathcal{O} & \mathcal{O} \\ p & \mathcal{O} & \mathcal{O} \end{bmatrix} e^{\mu} K/K,$$
we see easily that $V_{3,l}$ admits an affine fibration onto the closed subvariety
\[ \bigcup_{\mu \in R_{3,l}} \left[ \begin{array}{ccc} \mathcal{O} & p^{n_2-n_1+1} & p^{n_2-n_1} \\ p & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{array} \right] \mathcal{E}^{\mu K/K} \]
of $\mathcal{X}^{GL_2 \times GL_1}$. This implies
\[ V_{3,l} \cap I'_1 \mathcal{E}^{\mu K/K} = \left[ \begin{array}{ccc} \mathcal{O} & p^c & p^{-l} \\ p & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{array} \right] \mathcal{E}^{\mu K/K}, \]
with $c = \max(n_1 - n_2 - l, -\mu_2)$. With this equality, it is easily checked that $V_{3,l} \cap I'_1 \mathcal{E}^{\mu K/K} \cap \mathcal{X}_\gamma$ is isomorphic to an affine space.

It remains to precise the order of the paving. First of all, the Bruhat-Tits order with respect to $I'$ induces an ordering of $V_{1,l}'$ and $V_{3,l}$, $i = 1, 2, 3, l \in \mathbb{Z}$. On $V_{1,l}$ and $V_{1,l}'$, we use the Bruhat-Tits order with respect to $I'$, while on $V_{2,l}$ and $V_{3,l}$, we use the Bruhat-Tits order with respect to $I'_1$.

\[ \square \]

4.2. **Rationality conjecture.** To calculate the Poincaré polynomial of $F_\gamma$, we proceed by an indirect way in order to avoid the combinatorial complexity. Our strategy is the following: we calculate firstly the Poincaré polynomial of $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0)$, then we calculate the Poincaré polynomial of the complementary of $F_\gamma$, their difference gives what we want. It turns out that the complementary of $F_\gamma$ can be paved in affine spaces.

**Theorem 4.3.** The Poincaré polynomial of $F_\gamma$ is
\[ P_n(t) = \sum_{i=1}^{n_1} i (t^{4i-2} + t^{4i-4}) + \sum_{i=2n_1}^{n_1+n_2-1} (2n_1 + 1)t^{2i} + \sum_{i=n_1+n_2}^{2n_1+n_2-1} 4(2n_1 + n_2 - i)t^{2i} + t^{4n_1+2n_2}. \]

Taking into account the fact that $F_{n_2,n_1}$ has the same Poincaré polynomial as $F_{n_1,n_2}$, we get the precise expression for the generating series.

**Corollary 4.4.** The power series
\[ \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} P(n_1,n_2)(t) T_1^{n_1} T_2^{n_2} \in \mathbb{Z}[t][[T_1, T_2]] \]
equals the rational function
\[ \frac{(t^2 + 1)T_1 T_2}{(1 - T_2)(1 - T_1 T_2)(1 - t^4 T_1 T_2)^2} + \frac{t^4 T_1 T_2(3 - t^4 T_1 T_2)}{(1 - T_2)(1 - t^2 T_2)(1 - t^4 T_1 T_2)^2} \]
\[ + \frac{4t^4 T_1 T_2}{(1 - t^2 T_2)(1 - t^4 T_1 T_2)^2(1 - t^6 T_1 T_2)} + \frac{t^6 T_1 T_2}{(1 - t^2 T_2)(1 - t^6 T_1 T_2)} \]
\[ - \left[ \frac{(t^2 + 1)T_1 T_2}{(1 - T_1 T_2)(1 - t^4 T_1 T_2)^2} + \frac{4t^4 T_1 T_2}{(1 - t^4 T_1 T_2)^2(1 - t^6 T_1 T_2)} + \frac{t^6 T_1 T_2}{1 - t^6 T_1 T_2} \right] \]
Proof of the theorem \[4.3\] To pave $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0)$, we use the same idea as the proof of theorem 3.11 in [C]. We can pave $\text{Sch}(2n_1 + n_2, 0, 0)$ in affine spaces with the Iwahori subgroup

$$I' = \text{Ad}(\text{diag}(e^{2n_1+n_2}, 1, 1))I.$$ 

Let $C(\mu) = \text{Sch}(2n_1 + n_2, 0, 0) \cap I'e^K/K$, then we have the affine paving

$$\text{Sch}(2n_1 + n_2, 0, 0) = \bigcup_{\mu \in \text{Sch}(2n_1+n_2,0,0)^T} C(\mu),$$

with

$$C(\mu) = \left[ \begin{array}{ccc} \mathcal{O} & \mathcal{O} & e^K/K \\ {p^{-\mu_1}} & {p^{-\mu_1}} & p \end{array} \right].$$

Then we prove with the same method that $C(\mu) \cap \mathcal{X}_\gamma$ is an affine space of dimension

$$\min\{n_1, \mu_2\} + \min\{n_1, \mu_3\} + \min \left\{ n_2, |\mu_2 - \mu_3| + \frac{\text{sign}(\mu_2 - \mu_3) - 1}{2} \right\}.$$ 

It suffices to count the number of affine pavements of each dimension to get the Poincaré polynomial. To facilitate the work, we cut $\text{Sch}(2n_1 + n_2, 0, 0)^T$ into 7 parts, as indicated in figure \[4\] where

\[
\begin{align*}
R_1 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_2 - \mu_3 > n_2 \}, \\
R'_1 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_3 - \mu_2 > n_2 \}, \\
R_2 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_2 - \mu_3 \leq n_2, \mu_3 < n_1, \mu_2 > n_1 \}, \\
R'_2 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_3 - \mu_2 \leq n_2, \mu_2 < n_1, \mu_3 > n_1 \}, \\
R_3 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_3 \geq n_1, \mu_2 \geq n_1 \}, \\
R_4 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_3 \leq n_1, \mu_2 \leq n_1, n_2 < \mu_1 \leq n_1 + n_2 \}, \\
R'_4 & = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T \mid \mu_3 < n_1, \mu_2 < n_1, n_1 + n_2 < \mu_1 \leq 2n_1 + n_2 \},
\end{align*}
\]

Now we count the contribution of each part. We first sum over each blue lines as indicated in figure \[4\] then we add up all blue lines. We use the notation $\sum_{\mu = \nu}^{\nu'}$ to mean summation over the line having ends in $\nu, \nu'$. Since the Poincaré polynomial is a polynomial in $t^2$, we use $q := t^2$ to simplify the notation.

1. The contribution of $C(\mu) \cap \mathcal{X}_\gamma, \mu \in R_1$ to the Poincaré polynomial of $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0)$ is

$$\sum_{i=0}^{n_1-1} \sum_{\mu=(0,2n_1+n_2-i,i)} q^{n_1+n_2+i} = 2 \sum_{i=1}^{n_1} i q^{2n_1+n_2-i}.$$ 

2. The contribution of $C(\mu) \cap \mathcal{X}_\gamma, \mu \in R'_1$ is the same as $R_1$. 
\begin{enumerate}
\item The contribution of \( C(\mu) \cap \mathcal{X}_\gamma, \mu \in R_2 \) is

\[
\sum_{i=0}^{n_1-1} \sum_{\mu=(2n_1-2i, n_2+i)} q^{i+n_1+\mu_2-\mu_3} n_1 + n_2 - i, n_1 + 1, i \]

\[
= n_1 \sum_{i=2n_1+1}^{n_1+n_2} q^i + \sum_{i=1}^{n_1-1} (n_1 - i) q^{n_1+n_2+i}.
\]

\item The contribution of \( R_2' \) is

\[
\sum_{i=0}^{n_1-1} \sum_{\mu=(2n_1-2i, n_2+i)} q^{i+n_1+\mu_3-\mu_2-1} n_1 + n_2 - i, n_1 + 1, i \]

\[
= n_1 \sum_{i=2n_1+1}^{n_1+n_2} q^{i-1} + \sum_{i=1}^{n_1-1} (n_1 - i) q^{n_1+n_2+i-1}.
\]
\end{enumerate}

\textbf{Figure 4.} Partition of the triangle.
The contribution of $R_3$ is
\[
\sum_{i=0}^{n_2} \sum_{\mu=(i,n_1+n_2-i,n_1)} q^{2n_1+|\mu_2-\mu_3|+\frac{\text{sign}(\mu_2-\mu_3)-1}{2}} = \sum_{i=0}^{n_2} q^{2n_1}(1 + q + q^2 + \cdots + q^{n_2-i}) = q^{2n_1} \sum_{i=0}^{n_2} (n_2 + 1 - i)q^i.
\]

The contribution of $R_4$ is
\[
\sum_{i=0}^{n_1-1} \sum_{\mu=(n_1+n_2-i,n_1+1,n_1)} q^{n_1+i+|\mu_2-\mu_3|+\frac{\text{sign}(\mu_2-\mu_3)-1}{2}} = \sum_{i=0}^{n_1-1} q^{n_1+i}(1 + q + \cdots + q^{n_1-i}) = n_1q^{2n_1} + \sum_{i=0}^{n_1-1} (i + 1)q^{n_1+i}.
\]

The contribution of $R'_4$ is
\[
\sum_{i=0}^{n_1-1} \sum_{\mu=(2n_1+n_2-i,0,n_1)} q^{i+|\mu_2-\mu_3|+\frac{\text{sign}(\mu_2-\mu_3)-1}{2}} = \sum_{i=0}^{n_1-1} q^i(1 + q + \cdots + q^i).
\]

The complementary of $F_\gamma$ in $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0)$ can be paved in affine spaces in the following way: Observe that $F_\gamma$ is contained in the intersection $\mathcal{X}_\gamma \cap \text{Sch}(n_1 + n_2, n_1, 0)$, whose complementary in $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0)$ can be paved in affine spaces using the standard Iwahori subgroup $I$. It suffices to pave the complementary of $F_\gamma$ in $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, n_1, 0)$, which can be done by using the Iwahori subgroup
\[
I'' = \text{Ad}(\text{diag}(\epsilon^{n_1}, \epsilon^{n_2}, \epsilon^{n_2}))I.
\]

We cut the complementary of $F_\gamma^T$ in $\text{Sch}(2n_1 + n_2, 0, 0)^T$ as indicated in figure 5, where
\[
T_1 = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T | \mu_1 \geq n_1 + n_2 + 1 \},
T_2 = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T | \mu_2 \geq n_1 + n_2 + 1 \},
T_3 = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T | \mu_3 \geq n_1 + n_2 + 1 \},
T_4 = \{ \mu \in \text{Sch}(2n_1 + n_2, 0, 0)^T | 2n_1 + 1 \leq \mu_1 \leq n_1 + n_2 \}.
\]
Figure 5. Complementary of $F_\gamma$.

The complementary of $\mathcal{X}_\gamma \cap \text{Sch}(n_1 + n_2, n_1, 0)$ in $\mathcal{X}_\gamma \cap \text{Sch}(2n_1 + n_2, 0, 0)$ is

$$\bigsqcup_{\mu \in T_1 \cup T_2 \cup T_3} I e^\mu K / K \cap \mathcal{X}_\gamma.$$ 

It is easy to verify that this is an affine paving. To calculate its Poincaré polynomial, in each region we first sum over the vertices on the blue lines as indicated in figure 5 then we sum over all the lines.

1. The contribution of $T_1$ is

$$\sum_{i=0}^{n_1 - 1} \sum_{\mu = (2n_1 + n_2 - i, 0, i)} q^{2n_1 + |\mu_2 - \mu_3| + \frac{\text{sign}(\mu_2 - \mu_3) - 1}{2}} = \sum_{i=0}^{n_1 - 1} q^{2n_1 (1 + q + \cdots + q^i)}.$$ 

2. The contribution of $T_2$ is

$$\sum_{i=0}^{n_1 - 1} \sum_{\mu = (0, 2n_1 + n_2 - i, i)} q^{n_1 + n_2 + i} = \sum_{i=1}^{n_1} i q^{2n_1 + n_2 - i}.$$ 

3. The contribution of $T_3$ is the same as that of $T_2$. 


It remains to calculate the Poincaré polynomial of the complementary of $F_{\gamma}$ in $\mathcal{X}_{\gamma} \cap \text{Sch}(n_1 + n_2, n_1, 0)$. By proposition 4.1, it is the union

$$
\bigcup_{\mu \in T_2} I^{\mu} e^{\mu} K/K \cap \text{Sch}(n_1 + n_2, n_1, 0) \cap \mathcal{X}_{\gamma}.
$$

By proposition 2.3, points in $I e^{\mu} K/K \cap \mathcal{X}_{\gamma}$, $\mu \in T_2 \cup T_3$ don’t belong to any $B e^{\nu} K/K \cap \mathcal{X}_{\gamma}$, $\nu \in T_1$, for any $B \in \mathcal{F}(T)$. The above intersection is thus equal to

$$
\bigcup_{\mu \in T_1} I^{\mu} e^{\mu} K/K \cap \text{Sch}(2n_1 + n_2, 0, 0) \cap \mathcal{X}_{\gamma},
$$

which is easily verified to be an affine space of dimension

$$
2n_1 + |\mu_2 - \mu_3| + \frac{\text{sign}(\mu_2 - \mu_3) - 1}{2},
$$

using the equality

$$
I^{\mu} e^{\mu} K/K \cap \text{Sch}(2n_1 + n_2, 0, 0) = \begin{bmatrix}
O & p^a & p^b \\
p^a & O & O \\
p & O & O
\end{bmatrix} e^{\mu} K/K,
$$

where $a = \max\{n_1 - n_2, -\mu_2\}, b = \max\{n_1 - n_2, -\mu_3\}$.

Summing up the contributions of all the pavements in $T_1$ in the order as for the region $T_1$, we find the Poincaré polynomial of the complementary of $F_{\gamma}$ in $\mathcal{X}_{\gamma} \cap \text{Sch}(n_1 + n_2, n_1, 0)$ to be

$$
\sum_{i=n_1}^{n_2-1} \sum_{\mu=(2n_1+n_2-i,0)}^{n_2} q^{2n_1 + |\mu_2 - \mu_3| + \text{sign}(\mu_2 - \mu_3) - 1}.
$$

Now taking into account all the above calculations, we get the result as claimed in the theorem. \[\square\]

**Remark 4.1.** Observe that in the above proof we actually give an affine paving of the complementary of $F_{\gamma}$ in $\mathcal{X}_{\gamma} \cap \text{Sch}(2n_1 + n_2, 0, 0)$, and this paving can also be obtained by the Arthur-Kottwitz reduction. With more efforts, the same method can be generalized to $\text{GL}_4$ using pavings in [C].

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EPFL SB Mathgeom/Geom, MA B1 447, Station 8, CH-1015, Lausanne, Switzerland
E-mail address: zongbin.chen@gmail.com