Viscosity Solutions to First Order Path-Dependent HJB Equations

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Abstract

In this article, a notion of viscosity solutions is introduced for first order path-dependent Hamilton-Jacobi-Bellman (HJB) equations associated with optimal control problems for path-dependent differential equations. We identify the value functional of the optimal control problems as a unique viscosity solution to the associated HJB equations. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property.

Key Words: Path-dependent Hamilton-Jacobi-Bellman equations; Viscosity solutions; Optimal control; Path-dependent differential equations

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1 Introduction

In the early 1980’s, Crandall and Lions [2] introduced the notion of viscosity solutions to first order Hamilton-Jacobi-Bellman (HJB) equations. Lions [16] applied this notion to deterministic optimal control problems. From then on, a large number of papers have been published developing the theory of viscosity solutions. We refer to the survey paper of Crandall, Ishii and Lions [1]. Soon afterwards, Crandall and Lions [3], [4], [5], [6] and [7] systematically introduced the corresponding theory for viscosity solutions in infinite dimensional Hilbert spaces. Then, Li and Yong [15] studied the general unbounded first-order HJB equations in infinite dimensional Hilbert spaces.

For the path-dependent case, the theory of viscosity solutions is more difficult. Luyakonov [17] developed a theory of viscosity solutions to fully non-linear path-dependent first order Hamilton-Jacobi equations. The existence and uniqueness theorems are proved when Hamilton function $H$ is $d_p$-locally Lipschitz continuous in the path function. For the stochastic

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path-dependent case, the notion of viscosity solutions was introduced by Ekren, Keller, Touzi and Zhang [10] in the semilinear context, and further extended to the fully nonlinear case by Ekren, Touzi and Zhang [11], [12] and [13], Ekren [9] and Ren [18]. The uniqueness results for the fully nonlinear case are only valid if the Hamilton function $H$ is uniformly nondegenerate. Ren, Touzi and Zhang [19] studied the degenerate case and established the comparison principle when the Hamilton function $H$ is $d_p$-uniformly continuous with respect to the path function.

In this paper, we consider the following controlled path-dependent differential equation:

$$\begin{cases} 
    dX_{s}^{\gamma_t,u}(s) = F(X_{s}^{\gamma_t,u},u(s))ds, & s \in [t,T], \\
    X_{t}^{\gamma_t,u} = \gamma_t \in \Lambda_t.
\end{cases} \tag{1.1}$$

In the equation above, $\Lambda_t$ denotes the set of all continuous $\mathbb{R}^d$-valued functions defined over $[0,t]$ and $\Lambda = \bigcup_{t \in [0,T]} \Lambda_t$; the unknown $X_{s}^{\gamma_t,u}(s)$, representing the state of the system, is an $\mathbb{R}^d$-valued function; the control process $u$ takes values in a metric space $(U,d)$ and the coefficient $F$ is assumed to satisfy Lipschitz condition with respect to $d_\infty$-metric in $\Lambda$. There exists a unique function $X_{s}^{\gamma_t,u}(s)$, $s \in [t,T]$, solution to (1.1).

We wish to minimize a cost functional of the form:

$$J(\gamma_t,u) := \int_{t}^{T} q(X_{s}^{\gamma_t,u},u(\sigma))d\sigma + \phi(X_{T}^{\gamma_t,u}), \quad (t,\gamma_t) \in [0,T] \times \Lambda, \tag{1.2}$$

over all admissible controls $U[t,T]$. Here $q$ and $\phi$ are given real functionals on $\Lambda \times U$ and $\Lambda_T$, respectively. We define the value functional of the optimal control problem as follows:

$$V(\gamma_t) := \inf_{u \in U[t,T]} J(\gamma_t,u), \quad (t,\gamma_t) \in [0,T] \times \Lambda. \tag{1.3}$$

The goal of this article is to characterize this value functional $V$. We assume that $q$ and $\phi$ satisfy suitable conditions and consider the following path-dependent HJB equation:

$$\begin{cases} 
    \partial_t V(\gamma_t) + H(\gamma_t, \partial_x V(\gamma_t)) = 0, & (t,\gamma_t) \in [0,T] \times \Lambda, \\
    V(\gamma_T) = \phi(\gamma_T), & \gamma_T \in \Lambda_T,
\end{cases} \tag{1.4}$$

where

$$H(\gamma_t,p) = \inf_{u \in U} [(p,F(\gamma_t,u))_{\mathbb{R}^d} + q(\gamma_t,u)], \quad (t,\gamma_t,p) \in [0,T] \times \Lambda \times \mathbb{R}^d.$$ 

Here we let $(\cdot,\cdot)_{\mathbb{R}^d}$ denote the scalar product of $\mathbb{R}^d$. The definitions of $\partial_t$ and $\partial_x$ will be introduced in subsequent section.

The primary objective of this article is to develop the concept of viscosity solutions to equation (1.4) (see Definition 3.2 for details). We shall show the value functional $V$ defined in (1.3) is a unique viscosity solution to the equation given in (1.4) when the coefficients $F$, $q$ and $\phi$ only satisfy $d_\infty$-Lipschitz conditions with respect to the path function.

The main difficulty for our case lies in both facts that the path space $\Lambda_T$ is an infinite dimensional Banach space, and that the maximal norm $\| \cdot \|_0$ is not smooth. In order to study the path-dependent HJB equations defined in path space $\Lambda$, we need to give a suitable definition to ensure that the value functional is a viscosity solution of the path-dependent
HJB equations. It is more important to guarantee the uniqueness of the solutions. Since the value functional is only $d_\infty$-Lipschitz continuous in the path function, the auxiliary function in the proof of uniqueness should include the term $|| \cdot ||_0^2$ or a functional which is equivalent to $|| \cdot ||_0^2$. The lack of smoothness of $|| \cdot ||_0^2$ makes the definition of viscosity solutions more complex.

Our main contributions are as follows. We want to extend the results in [17] to $d_\infty$-Lipschitz continuous case. This extension is nontrivial since the maximal norm $|| \cdot ||_0$ is not smooth. First, we find a functional $S$ which is equivalent to $|| \cdot ||_0^2$ belongs to $C^1(\Lambda^t)$ (see Lemma 2.3) and study its properties (see Lemma 2.4). The two Lemmas are the key to prove the uniqueness of viscosity solutions. Second, we give a functional formula which only requires the functional belong to $C^1(\Lambda^t)$ (see Theorem 2.6). It will be used to prove the existence of viscosity solutions. Third, we give a definition of viscosity solutions in a sequence of bounded and uniformly Lipschitz continuous functions spaces $C^0_{\Lambda^t}$ which are compact subsets of $\Lambda$, and prove that the value functional is a solution by functional formula. Finally, we define an auxiliary function $\Psi$ which includes the square of norm of the difference between the two elements in $\Lambda^t$ (see Step 1 in the proof of Theorem 4.1) and use the properties of $S$ to prove the uniqueness of viscosity solutions.

The remaining of this article is organized as follows. In the following section, we introduce preliminary results on path-dependent optimal control problems, and prove Theorem 2.6 and Lemmas 2.3 and 2.4 which are the key of the existence and uniqueness results of viscosity solutions. In Section 3, we introduce our notion of viscosity solutions to equation (1.4) and prove that the value functional $V$ defined by (1.3) is a viscosity solution. We also show the consistency with the notion of classical solutions and the stability result. Finally the uniqueness of viscosity solutions for equation (1.4) is proven in Section 4.

### 2 Preliminary work

Let $T > 0$ be a fixed number. For each $t \in [0,T]$, define $\hat{\Lambda}_t := D([0,t]; R^d)$ as the set of càdlàg $R^d$-valued functions on $[0,t]$. We denote $\hat{\Lambda}^t = \bigcup_{s \in [t,T]} \hat{\Lambda}_s$ and let $\hat{\Lambda}$ denote $\hat{\Lambda}^0$.

A very important remark on the notations: as in [8], we will denote elements of $\hat{\Lambda}$ by lower case letters and often the final time of its domain will be subscripted, e.g. $\gamma \in \hat{\Lambda}_t \subset \hat{\Lambda}$ will be denoted by $\gamma_t$. Note that, for any $\gamma \in \hat{\Lambda}$, there exists only one $t$ such that $\gamma \in \hat{\Lambda}_t$. For any $0 \leq s \leq t$, the value of $\gamma_t$ at time $s$ will be denoted by $\gamma_t(s)$. Moreover, if a path $\gamma_t$ is fixed, the path $\gamma_t \|_{[0,s]}$, for $0 \leq s \leq t$, will denote the restriction of the path $\gamma_t$ to the interval $[0,s]$. We also point out that the space $\hat{\Lambda}^t$ does not possess an algebraic structure since $\gamma_s + \eta_t$ is not well defined for each $\gamma_s, \eta_t \in \hat{\Lambda}^t$ when $s \neq t$.

For convenience, define for $x \in R^d, \gamma_t, \gamma_{\tilde{t}} \in \hat{\Lambda}, 0 \leq t \leq \tilde{t} \leq T,

\[\gamma^x_t(s) := \gamma_t(s)1_{[0,t]}(s) + (\gamma_t(t) + x)1_{\{t\}}(s), \quad s \in [0,t];\]

\[\gamma_{t,\tilde{t}}(s) := \gamma_t(s)1_{[0,t]}(s) + \gamma_{\tilde{t}}(t)1_{[t,\tilde{t}]}(s), \quad s \in [0,\tilde{t}].\]

We define a norm and metric on $\hat{\Lambda}$ as follows: for any $0 \leq t \leq \tilde{t} \leq T$ and $\gamma_t, \gamma_{\tilde{t}} \in \hat{\Lambda}$,

\[||\gamma||_0 := \sup_{0\leq s \leq t} |\gamma_t(s)|, \quad d_\infty(\gamma_t, \gamma_{\tilde{t}}) := |t - \tilde{t}| + \sup_{0 \leq s \leq \tilde{t}} |\gamma_{t,\tilde{t}}(s) - \gamma_{\tilde{t}}(s)|.\]

(2.1)
It is clear that $(\hat{\Lambda}_t, || \cdot ||_0)$ is a Banach space for every $t \in [0, T]$. Moreover, from Lemma 5.1 in the Appendix, it follows that $(\Lambda_t, d_\infty)$ is a complete metric space. Following Dupire [8], we define spatial derivatives of $u : \hat{\Lambda} \to R$, if exist, in the standard sense: for the basis $e_i$ of $R^d$, $i = 1, 2, \ldots, d$,

$$\partial_{x_i} u(\gamma_s) := \lim_{h \to 0, h > 0} \frac{1}{h} \left[ u(\gamma_{s+h}^{e_i}) - u(\gamma_s) \right], \quad i = 1, 2, \ldots, d, \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda}. \quad (2.2)$$

and the right time-derivative of $u$, if exists, as:

$$\partial_t u(\gamma_s) := \lim_{h \to 0, h > 0} \frac{1}{h} \left[ u(\gamma_{s,h}^{s+h}) - u(\gamma_s) \right], \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda}. \quad (2.3)$$

For the final time $T$, we define

$$\partial_T u(\gamma_T) := \lim_{s \downarrow T, s \uparrow T} \partial_t u(\gamma_T | [0,s]), \quad \gamma_T \in \hat{\Lambda}.$$

We take the convention that $\gamma_s$ is column vector, but $\partial_x u$ denotes row vector.

**Definition 2.1.** Let $t \in [0, T)$ and $u : \hat{\Lambda} \to R$ be given.

(i) We say $u \in C^0(\hat{\Lambda}^t)$ if $u$ is continuous in $\gamma_s$ on $\hat{\Lambda}^t$ under $d_\infty$.

(ii) We say $u \in C^1(\hat{\Lambda}^t) \subset C^0(\hat{\Lambda}^t)$ if $\partial_{x_1} u$, $\partial_{x_2} u$, $\ldots$, $\partial_{x_d} u$ and $\partial_t u$ exist on $\hat{\Lambda}^t$ and are in $C^0(\hat{\Lambda}^t)$.

Let $\Lambda_t := C([0,t], R^d)$ be the set of all continuous $R^d$-valued functions defined over $[0, t]$. We denote $\Lambda^t = \bigcup_{s \in [t,T]} \Lambda_s$ and let $\Lambda$ denote $\Lambda^0$. Clearly, $\Lambda := \bigcup_{t \in [0,T]} \Lambda_t \subset \hat{\Lambda}$, and each $\gamma \in \Lambda$ can also be viewed as an element of $\hat{\Lambda}$. $(\Lambda_t, || \cdot ||_0)$ is a Banach space, and $(\Lambda^t, d_\infty)$ is a complete metric space. $u : \Lambda \to R$ and $\hat{u} : \hat{\Lambda} \to R$ are called consistent on $\Lambda$ if $u$ is the restriction of $\hat{u}$ on $\Lambda$. For every $t \in [0, T]$, $\mu > 0$ and $M_0 > 0$, we also define $C^\mu_{t, M_0}$ by

$$C^\mu_{t, M_0} := \left\{ \gamma_s \in \Lambda^t : || \gamma_s ||_0 \leq M_0, \sup_{0 \leq t \leq s} \frac{|\gamma_s(t) - \gamma_s(r)|}{|t - r|} \leq \mu(1 + M_0) \right\}.$$  

For simplicity, we let $C^\mu_{M_0}$ denote $C^\mu_{t, M_0}$ when $t = 0$.

**Definition 2.2.** Let $t \in [0, T)$ and $u : \Lambda^t \to R$ be given.

(i) We say $u \in C^0(\Lambda^t)$ if $u$ is continuous in $\gamma_s$ on $\Lambda^t$ under $d_\infty$.

(ii) We say $u \in C^1(\Lambda^t) \subset C^0(\Lambda^t)$ if there exists $\hat{u} \in C^1(\hat{\Lambda}^t)$ which is consistent with $u$ on $\Lambda^t$.

Let $(U, d)$ be a metric space. An admissible control $u = \{ u(r), r \in [t, s] \}$ on $[t, s]$ (with $0 \leq t \leq s \leq T$) is a measurable function taking values in $U$. The set of all admissible controls on $[t, s]$ is denoted by $U[t, s]$, i.e.,

$$U[t, s] := \{ u(\cdot) : [t, s] \to U \mid u(\cdot) \text{ is measurable} \}.$$  

Now, we describe some continuous properties of solutions of state equation (1.1) and value functional (1.3). First we assume that functionals $F : \Lambda \times U \to R^d$, $q : \Lambda \times U \to R$ and $\phi : \Lambda_T \to R$ satisfy the following assumption.
Hypothesis 2.3. (i) For every fixed \((t, \gamma_t) \in [0, T] \times \Lambda\), \(F(\gamma_t, \cdot)\) and \(q(\gamma_t, \cdot)\) are continuous in \(u\).

(ii) There exists a constant \(L > 0\) such that, for all \((t, \gamma_t, u), (t', \gamma_{t'}, u) \in [0, T] \times \Lambda \times U\),
\[
|F(\gamma_t, u) - F(\gamma_{t'}, u)| \leq Ld_{\infty}(\gamma_t, \gamma_{t'}), \quad |F(\gamma_t, u)| \leq L(1 + ||\gamma_t||_0).
\]

(iii) There exists constant \(L > 0\) such that, for all \((t, \gamma_t, u), (t', \gamma_{t'}, u) \in [0, T] \times \Lambda \times U\) and \(\eta_T, \eta'_T \in \Lambda_T\),
\[
|q(\gamma_t, u) - q(\gamma_{t'}, u)| \leq Ld_{\infty}(\gamma_t, \gamma_{t'});
\]
\[
|\phi(\eta_T) - \phi(\eta'_T)| \leq L||\eta_T - \eta'_T||_0;
\]
\[
|q(\gamma_t, u)| \leq L(1 + ||\gamma_t||_0);
\]
\[
|\phi(\eta_T)| \leq L(1 + ||\eta_T||_0).
\]

The following theorem is standard (see Theorem 2.3 on page 42 of [14] for details).

**Theorem 2.1.** Assume that Hypothesis 2.3 holds. Then for every \(u \in U[t, T]\) and \((t, \gamma_t) \in [0, T] \times \Lambda\), equation (1.1) admits a unique solution \(X^{\gamma_t, u}\). Moreover,
\[
\sup_{s \in [0, T]} |X^{\gamma_t, u}(s)| \leq C_1(1 + ||\gamma_t||_0), \tag{2.4}
\]
where the constant \(C_1\) depends only on \(L\) and \(T\).

Let us now consider the continuous dependence of the solution \(X^{\gamma_t, u}(\cdot)\) to equation (1.1) on the initial condition, the property will be used in the proof of Theorem 2.4.

**Theorem 2.2.** Assume that Hypothesis 2.3 holds. Then, constant \(C_2 > 0\) exists that depend only on \(L\) and \(T\), such that, for every \(0 \leq t_1 \leq t_2 \leq T\), and \(\gamma_{t_1}^1, \gamma_{t_2}^2 \in \Lambda\),
\[
\sup_{u \in U[t, T]} ||X_T^{\gamma_{t_1}^1, u} - X_T^{\gamma_{t_2}^2, u}||_0 \leq C_2(||\gamma_{t_1}^1 - \gamma_{t_2}^2||_0 + (1 + ||\gamma_{t_1}^1||_0)(t_2 - t_1)). \tag{2.5}
\]

**Proof.** For any \(0 \leq t_1 \leq t_2 \leq T\) and \(\gamma_{t_1}^1, \gamma_{t_2}^2 \in \Lambda\), let \(X^{u, i}_s\) denote \(X^{\gamma_s^i, u}\) for \(s \in [t_i, T]\), where \(i = 1, 2\). Thus, we obtain
\[
||X^{u, 1}_t - X^{u, 2}_t||_0 \leq ||\gamma_{t_1, t_2}^1 - \gamma_{t_1, t_2}^2||_0 + L(1 + ||X^{u, 1}_{t_2}||_0)(t_2 - t_1) + L \int_{t_2}^t ||X^{u, 1}_{\sigma} - X^{u, 2}_{\sigma}||_0 d\sigma.
\]
Using the Gronwall-Bellman inequality, by (2.4), we obtain the following result, for a constant \(C_2 > 0\) depending only on \(L\) and \(T\),
\[
||X_T^{u, 1} - X_T^{u, 2}||_0 \leq C_2(||\gamma_{t_1, t_2}^1 - \gamma_{t_1, t_2}^2||_0 + (1 + ||\gamma_{t_1}^1||_0)(t_2 - t_1)).
\]

Applying the supremum i.e., \(\sup_u \{d(t, T)\}\), to both sides of the previous inequality, we get (2.5). \(\Box\) The following theorem show that the solution \(X^{\gamma_t, u}(\cdot)\) to equation (1.1) is Lipshitz continuous with respect to the time \(s \in [t, T]\) even if the initial value \((t, \gamma_t)\) belongs to \([0, T] \times \Lambda\). The result will be used to prove the existence of viscosity solutions in Theorem 3.2.
Theorem 2.3. Assume that Hypothesis 2.3 holds. Then, constant $C_3 > 0$ exists that depend only on $L$ and $T$, such that, for every $(t, \gamma_t) \in [0, T] \times \Lambda$,

$$
\sup_{u \in \mathcal{U}[t, T]} |X^{\gamma_t, u}(s_2) - X^{\gamma_t, u}(s_1)| \leq C_3(1 + ||\gamma_t||_0)|s_2 - s_1|, \quad t \leq s_1 \leq s_2 \leq T. \quad (2.6)
$$

Proof. For any $0 \leq t \leq s \leq T$ and $\gamma_t, \eta_t, \gamma'_s \in \Lambda$, by (2.4), we obtain the following result:

$$
|X^{\gamma_t, u}(s_2) - X^{\gamma_t, u}(s_1)| \leq L(1 + C_1(1 + ||\gamma_t||_0))|s_2 - s_1|.
$$

Taking the supremum in $\mathcal{U}[t, T]$, we obtain (2.6). \quad \Box

Our first result about the value functional is the local boundedness and two kinds of continuities.

Theorem 2.4. Suppose that Hypothesis 2.3 holds true. Then, there exists a constant $C_4 > 0$ such that, for every $0 \leq t \leq s \leq T$ and $\gamma_t, \eta_t, \gamma'_s \in \Lambda$,

$$
|V(\gamma_t)| \leq C_4(1 + ||\gamma_t||_0); \quad |V(\gamma_t) - V(\eta_t)| \leq C_4||\gamma_t - \eta_t||_0; \quad (2.7)
$$

$$
|V(\gamma_t) - V(\gamma'_s)| \leq C_4(1 + ||\gamma_t||_0 \lor ||\gamma'_s||_0)d_\infty(\gamma_t, \gamma'_s). \quad (2.8)
$$

Proof. By Hypothesis 2.3, (2.4) and (2.5), for any $u \in \mathcal{U}[t, T]$, we have

$$
|J(\gamma_t, u) - J(\gamma'_s, u)|
\leq L \int_t^s (1 + ||X^{\gamma_t, u}||_0)d\sigma + L(T + 1)||X^{\gamma_t, u}_T - X^{\gamma'_s, u}_T||_0
\leq L(T + 1)C_2(||\gamma_{t,s} - \gamma'_{s}||_0 + (1 + ||\gamma_t||_0)(s - t)) + L(1 + C_1(1 + ||\gamma_t||_0))(s - t).
$$

Thus, taking the infimum in $u \in \mathcal{U}[t, T]$, we can find a constant $C_4 > 0$ such that (2.8) holds. By the similar procedure, we can show (2.7) holds true. The theorem is proved. \quad \Box

Secondly, we present the dynamic programming principle (DPP) for optimal control problems (1.1) and (1.3).

Theorem 2.5. Assume the Hypothesis 2.3 holds true. Then, for every $(t, \gamma_t) \in [0, T] \times \Lambda$ and $s \in [t, T]$, we have that

$$
V(\gamma_t) = \inf_{u \in \mathcal{U}[t, T]} \left[ \int_t^s q(X^{\gamma_t, u}_\sigma, u(\sigma))d\sigma + V(X^{\gamma_t, u}_s) \right]. \quad (2.9)
$$

The proof is very similar to the case without path-dependent (see Theorem 2.1 in page 160 of [21]). For the convenience of readers, here we give its proof.

Proof. First of all, for any $u \in \mathcal{U}[s, T]$, $s \in [t, T]$ and any $u \in \mathcal{U}[t, s]$, by putting them concatenatively, we get $u \in \mathcal{U}[t, T]$. Let us denote the right-hand side of (2.9) by $\overrightarrow{V}(\gamma_t)$. By (1.3), we have

$$
V(\gamma_t) \leq J(\gamma_t, u) = \int_t^s q(X^{\gamma_t, u}_\sigma, u(\sigma))d\sigma + J(X^{\gamma_t, u}_s, u), \quad u(\cdot) \in \mathcal{U}[t, T].
$$
Thus, taking the infumum over \( u(\cdot) \in U[s, T] \), we obtain

\[
V(\gamma_t) \leq \int_t^s q(X_s^{\gamma, u}, u(\sigma)) d\sigma + V(X_s^{\gamma, u}).
\]

Consequently,

\[
V(\gamma_t) \leq \overline{V}(\gamma_t).
\]

On the other hand, for any \( \varepsilon > 0 \), there exists a \( u^\varepsilon \in U[t, T] \) such that

\[
V(\gamma_t) + \varepsilon \geq J(\gamma_t, u^\varepsilon) = \int_t^s q(X_s^{\gamma, u^\varepsilon}, u^\varepsilon(\sigma)) d\sigma + J(X_s^{\gamma, u^\varepsilon}, u^\varepsilon) \\
\geq \int_t^s q(X_s^{\gamma, u^\varepsilon}, u^\varepsilon(\sigma)) d\sigma + V(X_s^{\gamma, u^\varepsilon}) \geq \overline{V}(\gamma_t).
\]

Hence, (2.9) follows. □

The following theorem is needed to prove the existence of viscosity solutions.

**Theorem 2.6.** Suppose \( X \) is a continuous function on \([0, T]\) and an absolutely continuous function on \([\hat{t}, T]\), and \( u \in C^1(\Lambda^i) \). Then for any \( t \in [\hat{t}, T] \):

\[
u(X_t) = u(X_t) + \int_t^\hat{t} \partial_t u(X_s) ds + \int_t^\hat{t} \partial_x u(X_s) dX(s).\]  (2.10)

**Proof.** Denote \( X^n = X_{0,t} + \sum_{i=0}^{2^n-1} X(t_{i+1})1_{[t_i, t_{i+1}]} + X(t)1_{[t, \hat{t}]} \) which is a càdlàg piecewise constant approximation of \( X \). Here \( t_i = \hat{t} + \frac{i(t-\hat{t})}{2^n} \). For every \((s, \gamma_s) \in [0, T] \times \hat{\Lambda}\), define \( \gamma_{s-} \in \hat{\Lambda} \) by

\[
\gamma_{s-}(\theta) = \gamma_s(\theta), \quad \theta \in [0, s), \quad \text{and} \quad \gamma_{s-}(s) = \lim_{\theta \uparrow s} \gamma_s(\theta).
\]

We start with the decomposition

\[
u(X^n_{t_{i+1}-}) - u(X^n_{t_i+}) = u(X^n_{t_{i+1}-}) - u(X^n_{t_i}) + u(X^n_{t_i}) - u(X^n_{t_i-}).\]  (2.11)

Let \( \psi(l) = u(X^n_{t_{i+1}} - X_{t_i}) \), we have \( u(X^n_{t_{i+1}-}) - u(X^n_{t_i}) = \psi(h) - \psi(0) \), where \( h = \frac{t_{i+1} - t_i}{2^n} \). Since \( u \in C^1(\Lambda^i) \), the right derivative of \( \psi \) denoted by \( \psi_{t+} \) is continuous, therefore,

\[
u(X^n_{t_{i+1}-}) - u(X^n_{t_i}) = \psi(h) - \psi(0) = \int_0^h \psi_{t+}(l) dl = \int_{t_i}^{t_{i+1}} \partial_t u(X^n_{t_{i+1}}) dl.
\]

The term \( u(X^n_{t_i}) - u(X^n_{t_i-}) \) in (2.11) can be written \( \phi(X(t_{i+1}) - X(t_i)) - \phi(0) \), where \( \phi(l) = u(X^n_{t_i} + l1_{\{t_i\}}) \). Since \( u \in C^1(\Lambda^i) \), \( \phi \) is a \( C^1 \) function and \( \phi'(l) = \partial_x u(X^n_{t_i} + l1_{\{t_i\}}) \). Thus, we have that:

\[
\phi(X(t_{i+1}) - X(t_i)) - \phi(0) = \int_{t_i}^{t_{i+1}} \partial_x u(X^n_{t_i-} + (X(s) - X(t_i))1_{\{t_i\}}) dX(s).
\]
Then it is clear that Lemma 2.3. For every fixed \((\hat{t}, a_{\hat{t}})\), define

\[ u(\gamma_t) := ||\gamma_t - a_{\hat{t}, t}||^2_{\hat{H}}, \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}, \]

where

\[ ||\gamma_t||^2_{\hat{H}} = \int_0^t |\gamma_t(s)|^2 ds, \quad (t, \gamma_t) \in [0, T] \times \hat{\Lambda}. \]

We also define, for every \((t, \gamma_t), (t, \gamma'_t) \in [0, T] \times \hat{\Lambda},\)

\[ S(\gamma_t, \gamma'_t) = \begin{cases} \frac{(||\gamma_t - \gamma'_t||^2_0 - ||\gamma(t) - \gamma'_t(t)||^2_0)^2}{||\gamma_t - \gamma'_t||^2_0}, & ||\gamma_t - \gamma'_t||_0 \neq 0; \\ 0, & ||\gamma_t - \gamma'_t||_0 = 0. \end{cases} \]

For simplicity, we let \(S(\gamma_t)\) denote \(S(\gamma_t, \gamma'_t)\) when \(\gamma'_t(t) \equiv 0\) for all \(t \in [0, t].\)

**Lemma 2.2.** \(u \in C^1(\hat{\Lambda}).\)

**Proof.** It is clear that \(u \in C^0(\hat{\Lambda})\) and \(\partial_x u(\gamma_s) = 0\) for all \((s, \gamma_s) \in [\hat{t}, T] \times \hat{\Lambda}\). Now we consider \(\partial_t u\). For every \((s, \gamma_s) \in [\hat{t}, T) \times \hat{\Lambda},\)

\[ \partial_t u(\gamma_s) = \lim_{h \to 0, h > 0} \frac{u(\gamma_{s+h}) - u(\gamma_s)}{h} = \lim_{h \to 0, h > 0} \frac{\int_s^{s+h} |\gamma(s) - a_{\hat{t}, \hat{t}}(s)|^2 ds}{h} = |\gamma_s(s) - a_{\hat{t}, \hat{t}}(s)|^2. \]

For \(\gamma_T \in \hat{\Lambda},\)

\[ \partial_t u(\gamma_T) := \lim_{s \to T, s > T} \partial_t u(\gamma_T|_{[0, s]}) = |\lim_{s \to T} \gamma_T(s) - a_{\hat{t}, \hat{t}}(s)|^2. \]

It is clear that \(\partial_t u \in C^0(\hat{\Lambda}).\) Thus, we show that \(u \in C^1(\hat{\Lambda}).\) \(\square\)

**Lemma 2.3.** For every fixed \((\hat{t}, a_{\hat{t}}) \in [0, T] \times \hat{\Lambda},\) define \(S^{a_{\hat{t}}}: \hat{\Lambda} \to R\) by

\[ S^{a_{\hat{t}}}(\gamma_t) := S(\gamma_t, a_{\hat{t}, t}), \quad (t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}. \]

Then \(S^{a_{\hat{t}}} \in C^1(\hat{\Lambda}).\) Moreover,

\[ 3 - 5\frac{t}{2} \leq S(\gamma_t) + |\gamma_t(t)|^2 \leq 2|\gamma_t|^2_0, \quad (t, \gamma_t) \in [0, T] \times \hat{\Lambda}. \] (2.12)
Proof. First, by the definition of $S^{a_i}$, it is clear that $S^{a_i} \in C^0(\Lambda^i)$ and $\partial_iS^{a_i}(\gamma_t) = 0$ for all $(t, \gamma_t) \in [\hat{t}, T] \times \Lambda^i$. Second, we consider $\partial_xS^{a_i}$. For every $(t, \gamma_t) \in [\hat{t}, T] \times \Lambda^i$,

$$\partial_xS^{a_i}(\gamma_t) = \lim_{h \to 0} \frac{S^{a_i}(\gamma_t^h) - S^{a_i}(\gamma_t)}{h} = \lim_{h \to 0} \frac{S(\gamma_t^h, a_t) - S(\gamma_t, a_t)}{h} = \lim_{h \to 0} \frac{\frac{\|\gamma_t^h - a_t\|_0^2 - \|\gamma_t(t) + h(e_i - a_t(\hat{t}))\|^2}{\|\gamma_t^h - a_t\|_0^2} - \frac{\|\gamma_t - a_{\hat{t},t}\|_0^2 - \|\gamma_t(t) - a_t(\hat{t})\|^2}{\|\gamma_t - a_{\hat{t},t}\|_0^2}}{h}.$$

For every $(t, \gamma_t) \in [0, T] \times \Lambda$, let $\|\gamma_t\|_0 = \sup_{0 \leq s \leq t} |\gamma_t(s)|$ and $\gamma_t(t) = \gamma_t(t)e_i$, $i = 1, 2, \ldots, d$. Then, if $|\gamma_t(t) - a_t(\hat{t})| < |\gamma_t - a_{\hat{t},t}|_0$,

$$\partial_xS^{a_i}(\gamma_t) = \lim_{h \to 0} \frac{|\gamma_t - a_{\hat{t},t}|_0^2 - |\gamma_t(t) + h(e_i - a_t(\hat{t}))|^2 - (|\gamma_t - a_{\hat{t},t}|_0^2 - |\gamma_t(t) - a_t(\hat{t})|^2)^2}{h|\gamma_t - a_{\hat{t},t}|_0^2} = -4\frac{(\|\gamma_t - a_{\hat{t},t}\|_0^2 - |\gamma_t(t) - a_t(\hat{t})|^2)(\gamma_t(t) - a_t(\hat{t}))}{h^2} \leq 0; \quad (2.13)$$

if $|\gamma_t - a_t(\hat{t})| > |\gamma_t - a_{\hat{t},t}|_0$,

$$\partial_xS^{a_i}(\gamma_t) = 0; \quad (2.14)$$

if $|\gamma_t - a_t(\hat{t})| = |\gamma_t - a_{\hat{t},t}|_0 = 0$, since

$$\|\gamma_t^h - a_{\hat{t},t}\|_0^2 - |\gamma_t(t) + h(e_i - a_t(\hat{t}))|^2 = \begin{cases} 0, & |\gamma_t(t) + h(e_i - a_t(\hat{t}))| \geq |\gamma_t(t) - a_t(\hat{t})|, \\ |\gamma_t(t) - a_t(\hat{t})|^2 - |\gamma_t(t) + h(e_i - a_t(\hat{t}))|^2, & |\gamma_t(t) + h(e_i - a_t(\hat{t}))| < |\gamma_t(t) - a_t(\hat{t})|, \end{cases}$$

we have

$$0 \leq \lim_{h \to 0} \left| \frac{S^{a_i}(\gamma_t^h) - S^{a_i}(\gamma_t)}{h} \right| \leq \lim_{h \to 0} \left| \frac{h^2(2(\gamma_t(t) - a_t(\hat{t}))^2)}{h(\|\gamma_t^h - a_{\hat{t},t}\|_0^2)} \right| = 0; \quad (2.15)$$

if $|\gamma_t - a_t(\hat{t})| = |\gamma_t - a_{\hat{t},t}|_0 = 0$,

$$\partial_xS^{a_i}(\gamma_t) = 0. \quad (2.16)$$

From (2.13), (2.14), (2.15) and (2.16) we obtain that, for all $(t, \gamma_t) \in [\hat{t}, T] \times \Lambda^i$,

$$\partial_xS^{a_i}(\gamma_t) = \begin{cases} -4\frac{(\|\gamma_t - a_{\hat{t},t}\|_0^2 - |\gamma_t(t) - a_t(\hat{t})|^2)(\gamma_t(t) - a_t(\hat{t}))}{\|\gamma_t - a_{\hat{t},t}\|_0^2}, & |\gamma_t - a_{\hat{t},t}|_0 \neq 0, \\ 0, & |\gamma_t - a_{\hat{t},t}|_0 = 0. \end{cases}$$

It is clear that $\partial_xS^{a_i} \in C^0(\Lambda^i)$. Thus, we have show that $S^{a_i} \in C^1(\Lambda^i)$.

Now we prove (2.12). It is clear that

$$S(\gamma_t) + |\gamma_t(t)|^2 \leq 2\|\gamma_t\|_0^2, \quad (t, \gamma_t) \in [0, T] \times \Lambda.$$
On the other hand, for every \((t, \gamma_t) \in [0, T] \times \hat{\Lambda}\),
\[
S(\gamma_t) + |\gamma_t(t)|^2 \geq \frac{3 - \frac{5}{2}}{2}||\gamma_t||_0^2, 
\] if \(||\gamma_t||_0^2 - |\gamma_t(t)||^2 \leq \frac{5}{2} - \frac{1}{2}||\gamma_t||_0^2,\n\]
and
\[
S(\gamma_t) + |\gamma_t(t)|^2 \geq \left(\frac{5}{2} - 1\right)||\gamma_t||_0^2 + \frac{3 - \frac{5}{2}}{2}||\gamma_t||_0^2, 
\] if \(||\gamma_t||_0^2 - |\gamma_t(t)||^2 \geq \frac{5}{2} - 1||\gamma_t||_0^2.\n\]
Thus, we have (2.12) holds true. The proof is now complete. □

For every constant \(M > 0\), define
\[
\Upsilon^M(t) := S(\gamma_t) + M|\gamma_t(t)|^2, \quad (t, \gamma_t) \in [0, T] \times \hat{\Lambda}.
\]

**Lemma 2.4.** For \(M \geq 2\), we have that
\[
2\Upsilon^M(\gamma_t) + 2\Upsilon^M(\gamma'_t) \geq \Upsilon^M(\gamma_t + \gamma'_t), \quad (t, \gamma_t, \gamma'_t) \in [0, T] \times \hat{\Lambda} \times \hat{\Lambda}. \tag{2.17}
\]

**Proof.** If one of \(||\gamma_t||_0, ||\gamma'_t||_0\) and \(||\gamma_t + \gamma'_t||_0\) is equal to 0, it is clear that (2.17) holds. Then we may assume that all of \(||\gamma_t||_0, ||\gamma'_t||_0\) and \(||\gamma_t + \gamma'_t||_0\) are not equal to 0. By the definition of \(\Upsilon^M\), we get, for every \((t, \gamma_t, \gamma'_t) \in [0, T] \times \hat{\Lambda} \times \hat{\Lambda},
\]
\[
\Upsilon^M(\gamma_t + \gamma'_t) = ||\gamma_t + \gamma'_t||_0^2 + \frac{|\gamma_t(t) + \gamma'_t(t)|^4}{||\gamma_t + \gamma'_t||_0^2} + (M - 2)|\gamma_t(t) + \gamma'_t(t)|^2.
\]
Letting \(x := ||\gamma_t + \gamma'_t||_0^2\) and \(a := |\gamma_t(t) + \gamma'_t(t)||^2\), we have
\[
\Upsilon^M(\gamma_t + \gamma'_t) = f(x, a) := x + \frac{a^2}{x} + (M - 2)a.
\]
By
\[
f_x(x, a) = 1 - \left(\frac{a}{x}\right)^2 \geq 0, \quad f_a(x, a) = 2\frac{a}{x} + M - 2 \geq 0,
\]
and
\[
||\gamma_t + \gamma'_t||_0^2 \leq 2||\gamma_t||^2 + 2||\gamma'_t||_0^2, \quad |\gamma_t(t) + \gamma'_t(t)||^2 \leq 2|\gamma_t(t)||^2 + 2|\gamma'_t(t)||^2,
\]
we obtain that
\[
\frac{1}{2}\Upsilon^M(\gamma_t + \gamma'_t) \leq ||\gamma_t||^2 + ||\gamma'_t||_0^2 + \frac{(|\gamma_t(t)||^2 + |\gamma'_t(t)||^2)^2}{||\gamma_t||^2 + ||\gamma'_t||_0^2} + (M - 2)(|\gamma_t(t)||^2 + |\gamma'_t(t)||^2).
\]
Combining with
\[
\Upsilon^M(\gamma_t) + \Upsilon^M(\gamma'_t) = ||\gamma_t||_0^2 + ||\gamma'_t||_0^2 + \frac{|\gamma_t(t)||^4}{||\gamma_t||_0^2} + \frac{|\gamma'_t(t)||^4}{||\gamma'_t||_0^2} + (M - 2)(|\gamma_t(t)||^2 + |\gamma'_t(t)||^2),
\]
we have
\[
\Upsilon^M(\gamma_t) + \Upsilon^M(\gamma'_t) - \frac{1}{2}\Upsilon^M(\gamma_t + \gamma'_t) \geq \frac{|\gamma(t)||^4}{||\gamma_t||_0^2} + \frac{|\gamma'_t(t)||^4}{||\gamma'_t||_0^2} - \frac{(|\gamma_t(t)||^2 + |\gamma'_t(t)||^2)^2}{||\gamma_t||^2 + ||\gamma'_t||_0^2}.
\]
Let \(c = \frac{|\gamma_t(t)||^2}{||\gamma_t||_0^2}, \quad b = \frac{|\gamma'_t(t)||^2}{||\gamma'_t||_0^2}, \quad z = ||\gamma_t||_0^2\) and \(y = ||\gamma'_t||_0^2\), we get that
\[
(|\gamma_t||_0^2 + ||\gamma'_t||_0^2)[\Upsilon^M(\gamma_t) + \Upsilon^M(\gamma'_t) - \frac{1}{2}\Upsilon^M(\gamma_t + \gamma'_t)] \geq (z + y)(c^2z + b^2y) - (cz + by)^2 = (c - b)^2z^2y \geq 0.
\]
Thus we obtain (2.17) holds true. The proof is now complete. □
3 Viscosity solutions to HJB equations: Existence theorem.

In this section, we consider the first order path-dependent HJB equation \( (1.4) \). As usual, we start with classical solutions.

**Definition 3.1.** (Classical solution) A functional \( v \in C^1(\Lambda) \) is called a classical solution to the path-dependent HJB equation \( (1.4) \) if it satisfies equation \( (1.4) \) point-wisely.

We shall get that the value functional \( V \) defined by \( (1.3) \) is a viscosity solution of equation \( (1.4) \). We give the following definition for the viscosity solutions.

For every \( M_0 > 0 \), \( \mu > 0 \), \( (t, \gamma_t) \in [0,T] \times \Lambda \) and \( w \in C^0(\Lambda) \), define
\[
J_{\mu,M_0}^+(\gamma_t, w) := \left\{ \varphi \in C^1(\Lambda^t) : 0 = (w - \varphi)(\gamma_t) = \sup_{\eta_s \in C^0_t,M_0} (w - \varphi)(\eta_s) \right\},
\]

and
\[
J_{\mu,M_0}^-(\gamma_t, w) := \left\{ \varphi \in C^1(\Lambda^t) : 0 = (w + \varphi)(\gamma_t) = \inf_{\eta_s \in C^0_t,M_0} (w + \varphi)(\eta_s) \right\}.
\]

**Definition 3.2.** Let \( w \in C^0(\Lambda) \).

(i) For any \( \mu > 0 \), \( w \) is called a viscosity \( \mu \)-subsolution (resp., \( \mu \)-supersolution) of equation \( (1.4) \) if the terminal condition \( w(\gamma_T) \leq \phi(\gamma_T) \) (resp., \( w(\gamma_T) \geq \phi(\gamma_T) \)), \( \gamma_T \in \Lambda_T \) is satisfied, and for every \( M_0 > 0 \), whenever \( \varphi \in J_{\mu,M_0}^+(\gamma_t, w) \) (resp., \( \varphi \in J_{\mu,M_0}^-(\gamma_t, w) \)) with \( (s, \gamma_s) \in [0,T] \times C^0_s \) and \( |\gamma_s(s)| < M_0 \), we have
\[
\partial_t \varphi(\gamma_s) + H(\gamma_s, \partial_x \varphi(\gamma_s)) \geq 0,
\]

(resp., \( -\partial_t \varphi(\gamma_s) + H(\gamma_s, -\partial_x \varphi(\gamma_s)) \leq 0 \)).

(ii) \( w \) is called a viscosity subsolution (resp., supersolution) of equation \( (1.4) \) if there exists a \( \mu_0 > 0 \) such that, for all \( \mu \geq \mu_0 \), \( w \) is a viscosity \( \mu \)-subsolution (resp., \( \mu \)-supersolution) of equation \( (1.4) \).

(iii) \( w \in C^0(\Lambda) \) is said to be a viscosity solution of equation \( (1.4) \) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 3.1.** Assume that the coefficients \( F(\gamma_t, u) = F(t, \gamma_t(t), u) \), \( q(\gamma_t, u) = q(t, \gamma_t(t), u) \), \( \phi(\eta_T) = \phi(\eta_T(T)) \) and \( V(\gamma_t) = V(t, \gamma_t(t)) \) for all \( (t, \gamma_t, u) \in [0,T] \times \Lambda \times U \) and \( \eta_T \in \Lambda_T \). Then path-dependent HJB equation \( (1.4) \) reduces to the following HJB equation:

\[
\begin{cases}
\dot{V}_t(t,x) + \overline{H}(t,x, \nabla_x V(t,x)) = 0, & (t,x) \in [0,T] \times R^d, \\
V(T,x) = \overline{\phi}(x), & x \in R^d;
\end{cases}
\]

where
\[
\overline{H}(t,x,p) = \inf_{u \in U} [(p, F(t,x,u))_{R^d} + q(t,x,u)], \quad (t,x,p) \in [0,T] \times R^d \times R^d.
\]

Here and in the sequel, \( \nabla_x \) denotes the standard first order derivative with respect to \( x \). However, slightly different from the HJB literature, \( \nabla_{t+} \) denotes the right time-derivative of \( V \).
The following theorem shows that our definition of viscosity solutions to path-dependent HJB equation (1.4) is a natural extension of classical viscosity solution to HJB equation (3.1).

**Theorem 3.1.** Consider the setting in Remark 3.1. Assume that $V$ is a viscosity solution of path-dependent HJB equation (1.4) in the sense of Definition 3.2. Then $\overline{V}$ is a viscosity solution of HJB equation (3.1) in the standard sense (see Definition 2.4 on page 165 of [21]).

**Proof.** Without loss of generality, we shall only prove the viscosity subsolution property. First, from $V$ is a viscosity subsolution of equation (1.4), it follows that, for every $x \in R^d$, $\nabla(T, x) = V(\gamma_T) \leq \phi(\gamma_T) = \phi(x)$, where $\gamma_T \in \Lambda$ with $\gamma_T(T) = x$.

Next, let $\overline{V} \in C^1([0, T] \times R^d)$ and $(t, x) \in [0, T] \times R^d$ such that
\[
0 = \nabla(\overline{V} - \overline{\varphi})(t, x) = \sup_{(s, y) \in [0, T] \times R^d} \nabla(\overline{V} - \overline{\varphi})(s, y).
\]

Define $\varphi : \hat{\Lambda} \to R$ by
\[
\varphi(\gamma_s) = \overline{\varphi}(s, \gamma_s(s)), \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda},
\]
and define $\hat{\gamma}_t \in \Lambda_t$ by
\[
\hat{\gamma}_t(s) = x, \quad s \in [0, t].
\]

It is clear that
\[
\partial_t \varphi(\gamma_s) = \overline{\varphi}_t(s, \gamma_s(s)), \quad \partial_x \varphi(\gamma_s) = \nabla_x \overline{\varphi}(s, \gamma_s(s)), \quad (s, \gamma_s) \in [0, T] \times \hat{\Lambda},
\]
and
\[
\partial_t \varphi(\gamma_T) = \lim_{s \nearrow T} \partial_t \varphi(\gamma_T|_{[0, s]}) = \sup_{(s, y) \in [0, T] \times R^d} \nabla(\overline{V} - \overline{\varphi})(s, y).
\]

Thus we have $\varphi \in C^1(\Lambda) \subset C^1(\Lambda')$. Let $M_0 > 0$ be large enough such that $|x| < M_0$, since $\hat{\gamma}_t \in C^\mu_{t, M_0}$ for all $\mu > 0$, by the definitions of $V$ and $\varphi$, we get that, for all $\mu > 0$,
\[
0 = (V - \varphi)(\hat{\gamma}_t) = (\nabla - \overline{\varphi})(t, x) = \sup_{(s, y) \in [0, T] \times R^d} (\nabla - \overline{\varphi})(s, y) = \sup_{\gamma_s \in C^\mu_{t, M_0}} (V - \varphi)(\gamma_s).
\]

Therefore, for all $\mu > 0$, we have $\varphi \in J^+_{\mu, M_0}(\hat{\gamma}_t, V)$ with $(t, \hat{\gamma}_t) \in [0, T] \times C^\mu_{M_0}$ and $|\hat{\gamma}_t(t)| < M_0$.

Since $V$ is a viscosity subsolution of path-dependent HJB equation (1.4), there exists a $\mu_0 > 0$ such that, for all $\mu \geq \mu_0$,
\[
\partial_t \varphi(\hat{\gamma}_t) + \mathbf{H}(\hat{\gamma}_t, \partial_x \varphi(\hat{\gamma}_t)) \geq 0.
\]

Thus,
\[
\overline{\varphi}_t(t, x) + \overline{\mathbf{H}}(t, x, \nabla_x \overline{\varphi}(t, x)) \geq 0.
\]

By the arbitrariness of $\overline{V} \in C^1([0, T] \times R^d)$, we see that $\overline{V}$ is a viscosity subsolution of HJB equation (3.1), and thus completes the proof. \qed

We are now in a position to give the existence proof for the viscosity solutions.
Theorem 3.2. Suppose that Hypothesis 2.3 holds. Then the value functional $V$ defined by (1.3) is a viscosity solution to equation (1.4).

Proof. First, for every $M_0 > 0$, let $\mu_0 = C_3 > 0$. For every $\mu \geq \mu_0$, we let $\varphi \in J_{\mu_0}^+(\gamma_t, V)$ with $(t, \gamma_t) \in [0, T) \times C_{M_0}^\mu$ and $|\gamma_t(t)| < M_0$. For fixed $u \in U$, by Theorem 2.3 we can let $\delta > 0$ be small enough such that $t + \delta \leq T$, $\|X_t^{\gamma_t, u}\| \leq M_0$ and

$$\sup_{t \leq s_1 < s_2 \leq t + \delta} \frac{|X_t^{\gamma_t, u}(s_2) - X_t^{\gamma_t, u}(s_1)|}{|s_2 - s_1|} \leq C_3(1 + |\gamma_t|) \leq \mu(1 + M_0).$$

Combining with $\gamma_t \in C_{M_0}^\mu$, we have $X_t^{\gamma_t, u} \in C_{t, M_0}^\mu$. Then by the DPP (Theorem 2.5), we obtain that

$$\varphi(\gamma_t) = V(\gamma_t) \leq \int_t^{t+\delta} q(X^{\gamma_t, u}_\sigma, u(\sigma))d\sigma + V(X_t^{\gamma_t, u}) \leq \int_t^{t+\delta} q(X^{\gamma_t, u}_\sigma, u(\sigma))d\sigma + \varphi(X_t^{\gamma_t, u}).$$

Applying Theorem 2.6 we show that

$$0 \leq \lim_{\delta \to 0} \frac{1}{\delta} \int_t^{t+\delta} q(X^{\gamma_t, u}_\sigma, u(\sigma))d\sigma + \frac{1}{\delta}[\varphi(X_t^{\gamma_t, u}) - \varphi(\gamma_t)]$$

$$= q(\gamma_t, u) + \partial_t \varphi(\gamma_t) + (\partial_x \varphi(\gamma_t), F(\gamma_t, u))_{R^d}.$$

Taking the minimum in $u \in U$, we have

$$0 \leq \partial_t \varphi(\gamma_t) + H(\gamma_t, \partial_x \varphi(\gamma_t)).$$

On the other hand, it is clear that $V(\gamma_T) \leq \phi(\gamma_T)$ for all $\gamma_T \in \Lambda_T$. Then $V$ is a viscosity $\mu$-subsolution of equation (1.4) for all $\mu \geq \mu_0$. Thus $V$ is a viscosity subsolution of equation (1.4).

Next, for every $M_0 > 0$, let $\mu_0 = C_3 > 0$. For every $\mu \geq \mu_0$, we let $\varphi \in J_{\mu_0}^-(\gamma_t, V)$ with $(t, \gamma_t) \in [0, T) \times C_{M_0}^\mu$ and $|\gamma_t(t)| < M_0$. By Theorem 2.3 we can let $\delta > 0$ be small enough such that $t + \delta \leq T$, $\sup_{u \in U[t, T]} \|X_t^{\gamma_t, u}\| \leq M_0$ and

$$\sup_{u \in U[t, T]} \sup_{t \leq s_1 < s_2 \leq t + \delta} \frac{|X_t^{\gamma_t, u}(s_2) - X_t^{\gamma_t, u}(s_1)|}{|s_2 - s_1|} \leq C_3(1 + |\gamma_t|) \leq \mu(1 + M_0).$$

Combining with $\gamma_t \in C_{t, M_0}^\mu$, we have $X_t^{\gamma_t, u} \in C_{t, M_0}^\mu$ for all $u \in U[t, T]$. Then, for any $\varepsilon > 0$, by the DPP (Theorem 2.5), one can find a control $u^\varepsilon(\cdot) \equiv u^\varepsilon(\cdot) \in U[t, T]$ such that

$$\varepsilon \delta \geq \int_t^{t+\delta} q(X^{\gamma_t, u^\varepsilon}_\sigma, u^\varepsilon(\sigma))d\sigma + V(X_t^{\gamma_t, u^\varepsilon}) - V(\gamma_t) \geq \int_t^{t+\delta} q(X^{\gamma_t, u^\varepsilon}_\sigma, u^\varepsilon(\sigma))d\sigma - \varphi(X_t^{\gamma_t, u^\varepsilon}) + \varphi(\gamma_t).$$

Then, by Theorem 2.6 we obtain that

$$\varepsilon \geq \frac{1}{\delta} \int_t^{t+\delta} q(X^{\gamma_t, u^\varepsilon}_\sigma, u^\varepsilon(\sigma))d\sigma - \frac{\varphi(X_t^{\gamma_t, u^\varepsilon}) - \varphi(\gamma_t)}{\delta}$$

$$= -\partial_t \varphi(\gamma_t) + \frac{1}{\delta} \int_t^{t+\delta} q(\gamma_t, u^\varepsilon(\sigma)) - (\partial_x \varphi(\gamma_t), F(\gamma_t, u^\varepsilon(\sigma)))_{R^d}d\sigma + o(1)$$

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By Theorem 2.6, the inequality above implies that

\[ 0 \geq -\partial_t \varphi(\gamma_t) + \inf_{u \in \mathcal{U}} [q(\gamma_t, u) - (\partial_x \varphi(\gamma_t), F(\gamma_t, u))_{R^d}] + o(1). \]

Letting \( \delta \downarrow 0 \) and \( \varepsilon \to 0 \), we show that

\[ 0 \geq -\partial_t \varphi(\gamma_t) + \inf_{u \in \mathcal{U}} [q(\gamma_t, u) - (\partial_x \varphi(\gamma_t), F(\gamma_t, u))_{R^d}] + o(1). \]

Moreover, we also have \( V(\gamma_T) \geq \phi(\gamma_T) \) for all \( \gamma_T \in \Lambda_T \). Therefore, \( V \) is also a viscosity \( \mu \)-supsolution of (1.4) for all \( \mu \geq \mu_0 \). Thus \( V \) is a viscosity supsolution of equation (1.4). This completes the proof. \( \square \)

Now, let us give the result of classical solutions, which show the consistency of viscosity solutions.

**Theorem 3.3.** Let \( V \) denote the value functional defined by (1.3). If \( V \in C^1(\Lambda) \), then \( V \) is a classical solution of equation (1.4).

**Proof.** First, using the definition of \( V \) yields \( V(\gamma_T) = \phi(\gamma_T) \) for all \( \gamma_T \in \Lambda_T \). Next, for fixed \((t, \gamma_t, u) \in [0, T) \times \Lambda \times U\), from the DPP (Theorem 2.6), we obtain the following result:

\[ 0 \leq \int_t^{t+\delta} q(X^{\gamma_t, u}_s, u)ds + V(X^{\gamma_t, u}_{t+\delta}) - V(\gamma_t), \quad 0 < \delta < T - t. \] (3.3)

By Theorem 2.6 the inequality above implies that

\[
0 \leq \lim_{\delta \to 0^+} \frac{1}{\delta} \left[ \int_t^{t+\delta} q(X^{\gamma_t, u}_s, u)ds + V(X^{\gamma_t, u}_{t+\delta}) - V(\gamma_t) \right]
= \partial_t V(\gamma_t) + (F(\gamma_t, u), \partial_x V(\gamma_t))_{R^d} + o(\delta).
\]

Taking the minimum in \( u \in U \), we have that

\[ 0 \leq \partial_t V(\gamma_t) + H(\gamma_t, \partial_x V(\gamma_t)). \] (3.4)

On the other hand, let \((t, \gamma_t) \in [0, T) \times \Lambda \) be fixed. Then, by (2.9) and Theorem 2.6 there exists an \( \tilde{u} \equiv u^{\varepsilon, \delta} \in U[t, T] \) for any \( \varepsilon > 0 \) and \( 0 < \delta < T - t \) such that

\[
\varepsilon \delta \geq E \left[ \int_t^{t+\delta} q(X^{\gamma_t, \tilde{u}}_s, \tilde{u}(s))ds + V(X^{\gamma_t, \tilde{u}}_{t+\delta}) \right] - V(\gamma_t)
= \partial_t V(\gamma_t) + \int_t^{t+\delta} q(\gamma_t, \tilde{u}(s))ds + \left( \partial_x V(\gamma_t), \int_t^{t+\delta} F(\gamma_t, \tilde{u}(\sigma))d\sigma \right)_{R^d} + o(\delta)
\geq \partial_t V(\gamma_t) + H(\gamma_t, \partial_x V(\gamma_t))\delta + o(\delta).
\]

Then, dividing through by \( \delta \) and letting \( \delta \to 0^+ \), we obtain that

\[ \varepsilon \geq \partial_t V(\gamma_t) + H(\gamma_t, \partial_x V(\gamma_t)). \]

The desired result is obtained by combining the inequality given above with (3.4). \( \square \)

We conclude this section with the stability of viscosity solutions.
Theorem 3.4. Let $\mu > 0$, $F, q, \phi$ satisfy Hypothesis 2.3, and $v \in C^0(\Lambda)$. Assume
(i) for any $\varepsilon > 0$, there exist $F^\varepsilon, q^\varepsilon, \phi^\varepsilon$ and $v^\varepsilon \in C^0(\Lambda)$ such that $F^\varepsilon, q^\varepsilon, \phi^\varepsilon$ satisfy Hypothesis 2.3 and $v^\varepsilon$ is a viscosity $\mu$-subsolution (resp., $\mu$-supersolution) of equation (1.4) with generators $F^\varepsilon, q^\varepsilon, \phi^\varepsilon$;
(ii) as $\varepsilon \to 0$, $(F^\varepsilon, q^\varepsilon, \phi^\varepsilon, v^\varepsilon)$ converge to $(F, q, \phi, v)$ uniformly in the following sense:
\[
\lim_{\varepsilon \to 0} \sup_{(t, \gamma, u)\in [0, T] \times \Lambda \times \mathbb{R}} \sup_{\eta \in \Lambda} |(F^\varepsilon - F) + |q^\varepsilon - q|| \gamma, u + \phi^\varepsilon - \phi| \eta + |v^\varepsilon - v| \gamma| = 0. \tag{3.5}
\]
Then $v$ is a viscosity $\mu$-subsolution (resp., $\mu$-supersolution) of equation (1.4) with generators $F, q, \phi$.

Proof. Without loss of generality, we shall only prove the viscosity subsolution property. First, from $v^\varepsilon$ is a viscosity $\mu$-subsolution of equation (1.4) with generators $F^\varepsilon, q^\varepsilon, \phi^\varepsilon$, it follows that
\[
v^\varepsilon(\gamma_T) \leq \phi^\varepsilon(\gamma_T), \quad \gamma_T \in \Lambda_T.
\]
Letting $\varepsilon \to 0$, we have
\[
v(\gamma_T) \leq \phi(\gamma_T), \quad \gamma_T \in \Lambda_T.
\]
Next, for every $M_0 > 0$, we let $\varphi \in J_{F, M_0}(\hat{\gamma}_t, v)$ with $(\hat{t}, \hat{\gamma}_t) \in [0, T] \times C^\mu_{t, M_0}$ and $|\hat{\gamma}(\hat{t})| < M_0$. Denote $\varphi_1(\gamma_t) := \varphi(\gamma_t) + |t - \hat{t}|^2 + ||\gamma_t, x - \hat{\gamma}_t, x||^2_H$ for all $(t, \gamma_t) \in [0, T] \times \Lambda$. By Lemma 2.2, we have $\varphi_1 \in C^1(\Lambda^t)$. For every $\varepsilon > 0$, from Lemma 2.1 it follows that there exists $(t_{\varepsilon}, \gamma_{t_{\varepsilon}}) \in [\hat{t}, T] \times C^\mu_{t, M_0}$ such that
\[
(v^\varepsilon - \varphi_1)(t_{\varepsilon}, \gamma_{t_{\varepsilon}}) = \sup_{\gamma \in C^\mu_{t, M_0}} (v^\varepsilon - \varphi_1)(\gamma_t).
\]
We claim that $d_{\infty}(\gamma_{t_{\varepsilon}}, \hat{\gamma}_t) \to 0$ as $\varepsilon \to 0$. Indeed, if not, by Lemma 2.1 we may assume there exist $(\hat{t}, \hat{\gamma}_t) \in [\hat{t}, T] \times C^\mu_{t, M_0}$ and a subsequence of $(t_{\varepsilon}, \gamma_{t_{\varepsilon}})$ still denoted by themselves such that $(\hat{t}, \hat{\gamma}_t) \neq (\hat{t}, \hat{\gamma}_t)$ and $d_{\infty}(\gamma_{t_{\varepsilon}}, \hat{\gamma}_t) \to 0$ as $\varepsilon \to 0$. Thus
\[
(v - \varphi)(\hat{\gamma}_t) = \lim_{\varepsilon \to 0} (v - \varphi)(\gamma_{t_{\varepsilon}}) \leq (v - \varphi)(\hat{\gamma}_t) = (v - \varphi_1)(\hat{\gamma}_t)
\]
\[
= \lim_{\varepsilon \to 0} [(v - v^\varepsilon)(\hat{\gamma}_t) + (v^\varepsilon - \varphi_1)(\gamma_{t_{\varepsilon}})] \leq \lim_{\varepsilon \to 0} [(v - v^\varepsilon)(\hat{\gamma}_t) + (v^\varepsilon - \varphi_1)(\gamma_{t_{\varepsilon}})]
\]
\[
= (v - \varphi)(\gamma_t) - |\hat{t} - \hat{t}|^2 + ||\gamma_t - \hat{\gamma}_t||^2_H,
\]
contradicting $|\hat{t} - \hat{t}|^2 + ||\gamma_t - \hat{\gamma}_t||^2_H > 0$. Then, for any $\rho > 0$, by (3.5) there exists $\varepsilon > 0$ small enough such that
\[
\hat{t} \leq t_{\varepsilon} < T, \quad |\gamma_{t_{\varepsilon}}(t_{\varepsilon})| < M_0, \quad 2|t_{\varepsilon} - \hat{t}| + |\gamma_{t_{\varepsilon}}(t_{\varepsilon}) - \gamma_{\hat{t}}(\hat{t})|^2 \leq \frac{\rho}{4},
\]
and
\[
|\partial_t \varphi(\gamma_{t_{\varepsilon}}) - \partial_t \varphi(\hat{\gamma}_t)| \leq \frac{\rho}{4}, \quad |I| \leq \frac{\rho}{4}, \quad |II| \leq \frac{\rho}{4},
\]
where
\[
I = H^{\varepsilon}(\gamma_{t_{\varepsilon}}, \partial_t \varphi(\gamma_{t_{\varepsilon}})) - H(\gamma_{t_{\varepsilon}}, \partial_t \varphi(\gamma_{t_{\varepsilon}})),
\]
\[ II = H(\gamma^\epsilon_t, \partial_x \varphi(\gamma^\epsilon_t)) - H(\hat{\gamma}_t, \partial_x \varphi(\hat{\gamma}_t)), \]

and
\[ H^\epsilon(\gamma_t, p) = \inf_{u \in U^\epsilon} [(p, F^\epsilon(\gamma_t, u))_{R^d} + q^\epsilon(\gamma_t, u)], \quad (t, \gamma_t, p) \in [0, T] \times \Lambda \times \mathbb{R}^d. \]

Since \( v^\epsilon \) is a viscosity \( \mu \)-subsolution of equation (1.4) with generators \( F^\epsilon, q^\epsilon, \phi^\epsilon \), we have
\[ \partial_t \varphi_1(\gamma^\epsilon_t) + H^\epsilon(\gamma^\epsilon_t, \partial_x \varphi_1(\gamma^\epsilon_t)) \geq 0. \]

Thus
\[
0 \leq \partial_t \varphi(\gamma^\epsilon_t) + 2(t_\epsilon - \hat{\epsilon}) + |\gamma^\epsilon_t(t_\epsilon) - \hat{\gamma}_t(\hat{\epsilon})|^2 + H(\hat{\gamma}_t, \partial_x \varphi(\hat{\gamma}_t)) + H^\epsilon(\gamma^\epsilon_t, \partial_x \varphi(\gamma^\epsilon_t)) - H(\hat{\gamma}_t, \partial_x \varphi(\hat{\gamma}_t)) \leq \partial_t \varphi(\hat{\gamma}_t) + H(\hat{\gamma}_t, \partial_x \varphi(\hat{\gamma}_t)) + \rho.
\]

Letting \( \rho \downarrow 0 \), we show that
\[ \partial_t \varphi(\hat{\gamma}_t) + H(\hat{\gamma}_t, \partial_x \varphi(\hat{\gamma}_t)) \geq 0. \]

Since \( \varphi \in C^1(\Lambda^\hat{\epsilon}) \) is arbitrary, we see that \( v \) is a viscosity \( \mu \)-subsolution of equation (1.4) with generators \( F, q, \phi \), and thus completes the proof. \( \square \)

4 Viscosity solutions to HJB equations: Uniqueness theorem.

This section is devoted to a proof of uniqueness of viscosity solutions to equation (1.4). This result, together with the results from the previous section, will be used to characterize the value functional defined by (1.3).

We now state the main result of this section.

**Theorem 4.1.** Suppose Hypothesis 2.3 holds. Let \( W_1 \in C^0(\Lambda) \) (resp., \( W_2 \in C^0(\Lambda) \)) be a viscosity subsolution (resp., supsolution) to equation (1.4) and let there exist constants \( L > 0 \) and \( m > 0 \), such that, for any \( (t, \gamma_t), (s, \gamma_s') \in [0, T] \times \Lambda \),
\[
|W_1(\gamma_t)| \vee |W_2(\gamma_s)| \leq L(1 + ||\gamma_t||_0). \tag{4.1}
\]
\[
|W_1(\gamma_t) - W_1(\gamma_s')| \vee |W_2(\gamma_t) - W_2(\gamma_s')| \leq L(1 + ||\gamma_t||_0 \vee ||\gamma_s'||_0)d_\infty(\gamma_t, \gamma_s'). \tag{4.2}
\]

Then \( W_1 \leq W_2 \).

Theorems 3.2 and 4.1 lead to the result (given below) that the viscosity solution to the path-dependent HJB equation given in (1.4) corresponds to the value functional \( V \) of our optimal control problem given in (1.1) and (1.3).

**Theorem 4.2.** Assume that Hypothesis 2.3 holds. Then the value functional \( V \) defined by (1.3) is the unique viscosity solution to equation (1.4) in the class of functionals satisfying (4.1) and (4.2).
**Proof.** Theorem 3.2 shows that \( V \) is a viscosity solution to equation (1.4). Thus, our conclusion follows from Theorems 2.1 and 4.1. □

Next, we prove Theorem 4.1. Let \( W_1 \) be a viscosity solution of equation (1.4). We note that for \( \delta > 0 \), the functional defined by \( \bar{W} := W_1 - \frac{\delta}{\delta} \) is a subsolution for

\[
\begin{aligned}
\partial_t \bar{W} + H(\gamma_t, \partial_x \bar{W}(\gamma_t)) &= \frac{\delta}{\delta^2}, \quad (t, \gamma_t) \in [0, T) \times \Lambda,
\bar{W}(\gamma_T) &= \phi(\gamma_T), \quad \gamma_T \in \Lambda_T.
\end{aligned}
\]

As \( W_1 \leq W_2 \) follows from \( \bar{W} \leq W_2 \) in the limit \( \delta \downarrow 0 \), it suffices to prove \( W_1 \leq W_2 \) under the additional assumption given below:

\[
\partial_t W_1(\gamma_t) + H(\gamma_t, \partial_x W_1(\gamma_t)) \geq c, \quad (t, \gamma_t) \in [0, T) \times \Lambda, \quad c := \frac{\delta}{T^2}.
\]

**Proof of Theorem 4.1.** The proof of this theorem is rather long. Thus, we split it into several steps.

*Step 1.* Definitions of auxiliary functions.

By the definition of viscosity solutions, there exists a \( \mu_0 > 0 \) such that \( W_1 \) (resp., \( W_2 \)) is a viscosity \( \mu \)-subsolution (resp., \( \mu \)-supsolution) to equation (1.4) for all \( \mu \geq \mu_0 \).

We only need to prove that \( W_1(\gamma_i) \leq W_2(\gamma_i) \) for all \( (t, \gamma_i) \in [T - \bar{a}, T) \times \Lambda \). Here,

\[
\bar{a} = \frac{1}{96L} \wedge T.
\]

Then, we can repeat the same procedure for the case \( [T - i\bar{a}, T - (i-1)\bar{a}) \). Thus, we assume the converse result that \( (\bar{t}, \bar{\gamma}_i) \in [T - \bar{a}, T) \times \Lambda \) exists such that \( 2\bar{m} := W_1(\bar{\gamma}_i) - W_2(\bar{\gamma}_i) > 0 \). Because \( \bigcup_{\mu > 0, M > 0} C_{\mu, M}^\mu \) is dense in \( \Lambda \), by (1.1) there exist \( \mu \geq \mu_0, M_0 > 0, \bar{t} \in (T - \bar{a}, T) \) and \( \bar{\gamma}_i \in C_{\mu, M_0}^{\mu} \) such that \( W_1(\bar{\gamma}_i) - W_2(\bar{\gamma}_i) > \bar{m} \).

Let \( \nu = 1 + \frac{1}{96TL} \), and consider that \( \varepsilon > 0 \) is a small number such that

\[
W_1(\bar{\gamma}_i) - W_2(\bar{\gamma}_i) - 2\varepsilon \frac{\nu T - \bar{t}}{\nu T} (S(\bar{\gamma}_i) + |\bar{\gamma}_i|) - \frac{\varepsilon}{\bar{t} - T + \bar{a}} > \frac{\bar{m}}{2},
\]

and

\[
\frac{\varepsilon}{\nu T} \leq \frac{c}{4}. \quad (4.3)
\]

Next, we define for any \( (t, \gamma_1^i, \gamma_2^i) \in (T - \bar{a}, T] \times \Lambda \times \Lambda \),

\[
\Psi(\gamma_1^i, \gamma_2^i) = W_1(\gamma_1^i) - W_2(\gamma_2^i) - \frac{\alpha}{2} d(\gamma_1^i, \gamma_2^i) - \frac{\varepsilon}{(t - T + \bar{a})} - \varepsilon \frac{\nu T - \bar{t}}{\nu T} (S(\gamma_1^i) + S(\gamma_2^i) + |\gamma_1^i| + |\gamma_2^i|),
\]

where

\[
d(\gamma_1^i, \gamma_2^i) = 2|\gamma_1^i - \gamma_2^i| + S(\gamma_1^i, \gamma_2^i) + ||\gamma_1^i - \gamma_2^i||_H.
\]

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Finally, for the fixed $\hat{\mu} \geq \mu_0$ and every $M \geq M_0$, we can apply Lemma 2.1 to find $(\hat{t}, \hat{\gamma}_1^1, \hat{\gamma}_1^2) \in (T - \bar{a}, T] \times C_M^\mu \times C_M^\mu$ such that

$$\Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \geq \Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) > \frac{\bar{m}}{2} \quad \text{and} \quad \Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \geq \Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2), \ (t, \gamma_1^1, \gamma_1^2) \in (T - \bar{a}, T] \times C_M^\mu \times C_M^\mu.$$  

We should note that the point $(\hat{t}, \hat{\gamma}_1^1, \hat{\gamma}_1^2)$ depends on $\alpha, \hat{\mu}, \varepsilon, M$.

**Step 2.** For the fixed $\hat{\mu} \geq \mu_0$ and every $M \geq M_0$, the following result holds true:

$$a \bar{d}(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \leq |W_1(\hat{\gamma}_1^1) - W_1(\hat{\gamma}_1^2)| + |W_2(\hat{\gamma}_1^1) - W_2(\hat{\gamma}_1^2)| \rightarrow 0 \text{ as } \alpha \rightarrow +\infty. \quad (4.4)$$

Let us show the above. By the definition of $(\hat{\gamma}_1^1, \hat{\gamma}_1^2)$, we have

$$2\Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \geq \Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) + \Psi(\hat{\gamma}_1^2, \hat{\gamma}_1^2). \quad (4.5)$$

This implies that

$$a \bar{d}(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \leq |W_1(\hat{\gamma}_1^1) - W_1(\hat{\gamma}_1^2)| + |W_2(\hat{\gamma}_1^1) - W_2(\hat{\gamma}_1^2)| \leq 2L(2 + ||\hat{\gamma}_1^1||_0 + ||\hat{\gamma}_1^2||_0) \leq 4L(1 + M). \quad (4.6)$$

Letting $\alpha \rightarrow \infty$, we get

$$\bar{d}(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \rightarrow 0 \text{ as } \alpha \rightarrow +\infty.$$ 

Then from (2.12) it follows that

$$||\hat{\gamma}_1^1 - \hat{\gamma}_1^2||_0 \rightarrow 0 \text{ as } \alpha \rightarrow +\infty. \quad (4.7)$$

Combining (1.2), (4.6) and (4.7), we see that (1.4) holds.

**Step 3.** For the fixed $\hat{\mu} \geq \mu_0$, there exist $\hat{M} \geq M_0$ and $N \geq 0$ such that $\hat{t} \in (T - \bar{a}, T)$, $\hat{\gamma}_1^1, \hat{\gamma}_1^2 \in C_M^{\hat{\mu}}$ and $||\hat{\gamma}_1^1(\hat{t})|| \lor ||\hat{\gamma}_1^2(\hat{t})|| < \hat{M}$ for all $\alpha \geq N$.

First, noting $\frac{\varepsilon_0 T - t}{\gamma_1^1} \geq \frac{\varepsilon_0}{1 + 96\bar{d}}$, by the definition of $\Psi$, there exists an $\hat{M} \geq M_0$ that is sufficiently large that $\Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) < 0$ for all $t \in (T - \bar{a}, T]$ and $||\gamma_1^1(t)|| \lor ||\gamma_1^2(t)|| \geq \hat{M}$. Thus, we have $||\hat{\gamma}_1^1(\hat{t})|| \lor ||\hat{\gamma}_1^2(\hat{t})|| < \hat{M}$.

Next, for the fixed $\hat{M} > 0$, by (1.7), we can let $N > 0$ be a large number such that

$$L(1 + \hat{M})||\hat{\gamma}_1^1 - \hat{\gamma}_1^2||_0 \leq \frac{\bar{m}}{4},$$

for all $\alpha \geq N$. Then we have $\hat{t} \in (T - \bar{a}, T)$ for all $\alpha \geq N$. Indeed, if say $\hat{t} = T$, we will deduce the following contradiction:

$$\frac{\bar{m}}{2} \leq \Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \leq \phi(\hat{\gamma}_1^1) - \phi(\hat{\gamma}_1^2) \leq L(1 + \hat{M})||\hat{\gamma}_1^1 - \hat{\gamma}_1^2||_0 \leq \frac{\bar{m}}{4}.$$ 

**Step 4.** Completion of the proof.

From above all, for the fixed $\hat{\mu} \geq \mu_0$ in step 1 and the fixed $\hat{M} \geq M_0$ and $N > 0$ in step 3, we find $\hat{\gamma}_1^1, \hat{\gamma}_1^2 \in C_M^{\hat{\mu}}$ satisfying $\hat{t} \in (T - \bar{a}, T)$ and $||\hat{\gamma}_1^1(\hat{t})|| \lor ||\hat{\gamma}_1^2(\hat{t})|| < \hat{M}$ for all $\alpha \geq N$ such that

$$\Psi(\hat{\gamma}_1^1, \hat{\gamma}_1^2) \geq \Psi(\gamma_1^1, \gamma_1^2), \ (t, \gamma_1^1, \gamma_1^2) \in (T - \bar{a}, T] \times C_M^{\hat{\mu}} \times C_M^{\hat{\mu}}.$$  

(4.8)
We put, for \((t, \gamma_t^1, \gamma_t^2) \in (T - \bar{a}, T] \times \Lambda \times \Lambda\),
\[
W'_1(\gamma_t^1) &= W_1(\gamma_t^1) - \epsilon \frac{\nu T - t}{\nu T} (S(\gamma_t^1) + |\gamma_t^1(t)|^2) - \epsilon (|t - \hat{t}|^2 + ||\gamma_t^1 - \hat{\gamma}_t^1||_H^2) - \frac{\epsilon}{t - T + \bar{a}}, \\
W'_2(\gamma_t^2) &= W_2(\gamma_t^2) + \epsilon \frac{\nu T - t}{\nu T} (S(\gamma_t^2) + |\gamma_t^2(t)|^2) + \epsilon (|t - \hat{t}|^2 + ||\gamma_t^2 - \hat{\gamma}_t^2||_H^2).
\]

Define \(\bar{O}_M := \{x \in R^d : |x| \leq \hat{M}\}\). Now we can define, for \((t, x_0, y_0, \gamma_t^1, \gamma_t^2) \in (T - \bar{a}, T] \times \bar{O}_M \times \bar{O}_M \times \Lambda \times \Lambda\),
\[
\tilde{W}_1(\gamma_t^1, x_0) = \sup_{\xi^l \in \mathcal{C}_{\hat{t}, M}^e, \xi^l(t) = x_0} \left[ W'_1(\xi^l_1) - \alpha d(\gamma_t^1, \xi^l_1) \right], \\
\tilde{W}_2(\gamma_t^2, y_0) = \inf_{\xi^2 \in \mathcal{C}_{\hat{t}, M}^e, \xi^2(t) = y_0} \left[ W'_2(\xi^l_2) + \alpha d(\gamma_t^2, \xi^l_2) \right].
\]

Then, by Lemma 2.4 we obtain that, for \((t, x_0, y_0, \xi_t) \in (T - \bar{a}, T] \times \bar{O}_M \times \bar{O}_M \times \Lambda\),
\[
\tilde{W}_1(\xi_t, x_0) - \tilde{W}_2(\xi_t, y_0) = \sup_{\gamma_t^1, \gamma_t^2 \in \mathcal{C}_{\hat{t}, M}^e, \gamma_t^1(t) = x_0, \gamma_t^2(t) = y_0} \left[ W'_1(\gamma_t^1) - \alpha d(\gamma_t^1, \xi_t) - W'_2(\gamma_t^2) - \alpha d(\gamma_t^2, \xi_t) \right] \\
\leq \sup_{\gamma_t^1, \gamma_t^2 \in \mathcal{C}_{\hat{t}, M}^e, \gamma_t^1(t) = x_0, \gamma_t^2(t) = y_0} \left[ W'_1(\gamma_t^1) - W'_2(\gamma_t^2) - \frac{\alpha}{2} d(\gamma_t^1, \gamma_t^2_2) \right] \\
\leq W'_1(\hat{\gamma}_t^1) - W'_2(\hat{\gamma}_t^2) - \frac{\alpha}{2} d(\hat{\gamma}_t^1, \hat{\gamma}_t^2), \quad (4.9)
\]

where the last inequality becomes equality if and only if \(t = \hat{t}, \gamma_t^1 = \hat{\gamma}_t^1, \gamma_t^2 = \hat{\gamma}_t^2\) and \(x_0 = \hat{\gamma}_t^1(\hat{t}), y_0 = \hat{\gamma}_t^2(\hat{t})\). The previous inequality becomes equality if and only if \(\xi_t = \frac{\hat{\gamma}_t^1 + \hat{\gamma}_t^2}{2}\).

Then we obtain that
\[
\tilde{W}_1(\xi_t, x_0) - \tilde{W}_2(\xi_t, y_0) \leq W'_1(\hat{\gamma}_t^1) - W'_2(\hat{\gamma}_t^2) - \frac{\alpha}{2} d(\hat{\gamma}_t^1, \hat{\gamma}_t^2), \quad (t, x_0, y_0, \xi_t) \in (T - \bar{a}, T] \times \bar{O}_M \times \bar{O}_M \times \Lambda,
\]
and the equality only holds at \(\hat{t}, \hat{\gamma}_t^1(\hat{t}), \hat{\gamma}_t^2(\hat{t}), \hat{\xi}_t = \frac{\hat{\gamma}_t^1 + \hat{\gamma}_t^2}{2}\).

Define, for every \((t, x_0, y_0) \in [0, T] \times \bar{O}_M \times \bar{O}_M\),
\[
\tilde{W}_1(t, x_0) = \begin{cases} 
\tilde{W}_1(\hat{\xi}_t, x_0) - (\hat{t} - t)^\frac{1}{2}, & t \in [0, \hat{t}), \\
\tilde{W}_1(\hat{\xi}_{t, t}, x_0), & t = \hat{t}, \\
\tilde{W}_1(t, x_0), & t \in [\hat{t}, T], 
\end{cases} \\
\tilde{W}_2(t, y_0) = \begin{cases} 
\tilde{W}_2(\hat{\xi}_t, y_0) + (\hat{t} - t)^\frac{1}{2}, & t \in [0, \hat{t}), \\
\tilde{W}_2(\hat{\xi}_{t, t}, y_0), & t = \hat{t}, \\
\tilde{W}_2(t, y_0), & t \in [\hat{t}, T]. 
\end{cases} \quad (4.10)
\]

Thus \(\tilde{W}_1(t, x_0) - \tilde{W}_2(t, y_0)\) has a maximum at \((\hat{t}, \hat{\gamma}_t^1(\hat{t}), \hat{\gamma}_t^2(\hat{t}))\) on \([0, T] \times \bar{O}_M \times \bar{O}_M\). Then by Lemmas 1.1 and 4.2, Theorem 8.3 in [1] can be used to obtain sequences \(t_k, s_k \in (0, T), \gamma_k^1, \gamma_k^2 \in \bar{O}_M\) such that \((t_k, x_k^0) \to (\hat{t}, \hat{\gamma}_t^1(\hat{t})), (s_k, y_k^0) \to (\hat{t}, \hat{\gamma}_t^2(\hat{t}))\) as \(k \to +\infty\) and the sequences of functions \(\varphi_k, \psi_k \in C^{1,2}((0, T) \times R^d)\) such that
\[
\tilde{W}_1(t, x_0) - \varphi_k(t, x_0) \leq 0, \quad \tilde{W}_2(t, y_0) + \psi_k(t, y_0) \geq 0,
\]

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equalities only hold true at \((t_k, x_0^k), (s_k, y_0^k)\), respectively,
\[
(\varphi_k)_t(t_k, x_0^k) \rightarrow b_1, \quad (\psi_k)_t(s_k, y_0^k) \rightarrow b_2, \\
\nabla_x \varphi_k(t_k, x_0^k) \rightarrow 0, \quad -\nabla_x \psi_k(s_k, y_0^k) \rightarrow 0,
\]
and
\[
b_1 + b_2 = 0. \tag{4.12}
\]

We claim that \(t_k, s_k \in [\hat{t}, T)\) for \(k\) to be large enough. Indeed, if not, for example, there exists a subsequence \(\{t_{k_l}\}_{l \geq 0}\) of \(\{t_k\}_{k \geq 0}\) such that \(\{t_{k_l}\}_{l \geq 0} < \hat{t}\). Since \(\hat{W}_1(t, x_0) - \varphi_{k_l}(t, x_0)\) has a maximum at \((t_{k_l}, x_0^{k_l})\) on \((0, T) \times R^d\), we obtain that
\[
(\varphi_{k_l})_t(t_{k_l}, x_0^{k_l}) = \frac{1}{2}(\hat{t} - t_{k_l})^{-\frac{1}{2}} \rightarrow +\infty, \quad \text{as } l \rightarrow +\infty,
\]
which is contradict to \((4.11)\) and \((4.12)\).

Now, for every \(k > 0\), we consider the functional, for \((t, \gamma_1^k), (s, \gamma_2^k) \in [\hat{t}, T] \times \Lambda,
\]
\[
\Gamma_k(\gamma_1^k, \gamma_2^k) = W_1^{\prime}(\gamma_1^k) - W_2^{\prime}(\gamma_2^k) - \alpha(d(\gamma_1^k, \hat{\xi}_{t,l}) + d(\gamma_2^k, \hat{\xi}_{s,l})) - \varphi_k(t, \gamma_1^k) - \psi_k(s, \gamma_2^k) \tag{4.13}
\]
By Lemma 2.4, it has a maximum at some point \((\hat{t}_k, \hat{s}_k, \hat{\gamma}_1^{1,k}, \hat{\gamma}_2^{2,k})\) in \([\hat{t}, T] \times [\hat{t}, T] \times C_{\hat{t}} \times C_{\hat{t}}\).

Then by Lemma 4.3 we get that
\[
\hat{t}_k = t_k, \quad \hat{s}_k = s_k, \quad \hat{\gamma}_1^{1,k} = x_0^k, \quad \hat{\gamma}_2^{2,k} = y_0^k, \tag{4.14}
\]
and
\[
\hat{\gamma}_1^{1,k} \rightarrow \hat{\gamma}_1^1, \quad \hat{\gamma}_2^{2,k} \rightarrow \hat{\gamma}_2^2 \quad \text{in } (\Lambda, d_{\infty}) \quad \text{as } k \rightarrow +\infty. \tag{4.15}
\]
On the other hand, by \((4.14)\), we can let \(\alpha > N\) be large enough such that
\[
2\alpha(L + 1)(|\hat{\gamma}_1^1 - \hat{\gamma}_2^2|^2 + ||\hat{\gamma}_1^1 - \hat{\gamma}_2^2||_0^2) + L(1 + ||\hat{\gamma}_1^1||_0 + ||\hat{\gamma}_2^2||_0)\|\hat{\gamma}_1^1 - \hat{\gamma}_2^2||_0 \leq \frac{c}{4}. \tag{4.16}
\]
For the fixed \(\alpha > N\), from \((t_k, x_0^k) \rightarrow (\hat{t}, \hat{\gamma}_1^1(\hat{t}))\), \((s_k, y_0^k) \rightarrow (\hat{t}, \hat{\gamma}_2^2(\hat{t}))\) as \(k \rightarrow +\infty\), \(\hat{t} < T\) and \(|\hat{\gamma}_1^1| \vee |\hat{\gamma}_2^2| < M\), it follows that constant \(K > 0\) exists such that
\[
|t_k| \vee |s_k| < T, \quad |x_0^k| \vee |y_0^k| < \hat{M}, \quad \text{for all } k \geq K.
\]
Since \(W_1\) (resp., \(W_2\)) is a viscosity \(\hat{\mu}\)-subsolution (resp., \(\hat{\mu}\)-supersolution) to equation \((4.4)\), from Lemmas 2.2 and 2.3 it follows that
\[
(\varphi_k)_t(\hat{t}_k, \hat{\gamma}_1^{1,k}(\hat{t}_k)) - \frac{\varepsilon}{\nu T}(S(\hat{\gamma}_1^{1,k}(\hat{t}_k)) + |\hat{\gamma}_1^{1,k}(\hat{t}_k)|^2) + \varepsilon|\hat{\gamma}_1^{1,k}(\hat{t}_k) - \hat{\gamma}_1^1(\hat{t})|^2 - \varepsilon(\hat{t}_k - T + \alpha)^{-2}
+ 2\varepsilon(\hat{t}_k - \hat{t}) + \alpha|\hat{\gamma}_1^{1,k}(\hat{t}_k) - \hat{\xi}_t(\hat{t})|^2 + H(\hat{\gamma}_1^{1,k}, \nabla_x(\varphi_k)(\hat{t}_k, \hat{\gamma}_1^{1,k}(\hat{t}_k)) + \frac{\nu T - \hat{t}_k}{\nu T}(\partial_x S(\hat{\gamma}_1^{1,k}(\hat{t}_k))
+ 2\hat{\gamma}_1^{1,k}(\hat{t}_k)) + 4\alpha(\hat{\gamma}_1^{1,k}(\hat{t}_k) - \hat{\xi}_t(\hat{t})) + \alpha\partial_x S(\hat{\gamma}_1^{1,k}(\hat{t}_k)) \geq c; \tag{4.17}
\]
and
\[-(\psi_k)(s_k, \gamma^{2,k}_s(k)) + \frac{\varepsilon}{\nu T}(S(\gamma^{2,k}_s) + |\gamma^{2,k}_s(k)|^2) - 2\varepsilon(s_k - \hat{t}) - \varepsilon|\gamma^{2,k}_s(k)|^2 \]
\[-\alpha|\gamma^{2,k}_s(k) - \hat{t}|^2 + H(\gamma^{2,k}_s, -\nabla_x(\psi_k)(s_k, \gamma^{2,k}_s(k))) - \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{2,k}_s)) \]
\[+ 2\alpha|\gamma^{2,k}_s(k) - \hat{t}|^2 - 4\alpha(\gamma^{2,k}_s(k) - \hat{t})(\partial_x S(\gamma^{2,k}_s)) \leq 0. \quad (4.18)\]

Combining (4.17) and (4.18), and letting \( k \to \infty \), we obtain
\[
c + \frac{\varepsilon}{\nu T}(S(\gamma^{1}_l) + |\gamma^{1}_l(\hat{t})|^2 + S(\gamma^{2}_l) + |\gamma^{2}_l(\hat{t})|^2) \leq \frac{\alpha}{2}|\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})|^2 + H(\gamma^{1}_l, 2\alpha(\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})) + \varepsilon \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{1}_l) + 2\gamma^{1}_l(\hat{t})) + \alpha \partial_x S(\gamma^{2}_l(\hat{t}))) \]
\[-H(\gamma^{2}_l, 2\alpha(\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})) - \varepsilon \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{2}_l) + 2\gamma^{2}_l(\hat{t})) - \alpha \partial_x S(\gamma^{2}_l(\hat{t}))) \leq \sup_{u \in U}(J_1 + J_2), \quad (4.19)\]

On the other hand, by a simple calculation we obtain
\[
H(\gamma^{1}_l, 2\alpha(\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})) + \varepsilon \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{1}_l) + 2\gamma^{1}_l(\hat{t})) + \alpha \partial_x S(\gamma^{2}_l(\hat{t}))) \]
\[-H(\gamma^{2}_l, 2\alpha(\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})) - \varepsilon \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{2}_l) + 2\gamma^{2}_l(\hat{t})) - \alpha \partial_x S(\gamma^{2}_l(\hat{t}))) \leq \sup_{u \in U}(J_1 + J_2), \]

where
\[
J_1 = (F(\gamma^{1}_l, u), 2\alpha(\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})) + \varepsilon \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{1}_l) + 2\gamma^{1}_l(\hat{t})) + \alpha \partial_x S(\gamma^{2}_l(\hat{t}))) \]
\[-(F(\gamma^{2}_l, u), 2\alpha(\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})) - \varepsilon \frac{\nu T - \hat{t}}{\nu T}(\partial_x S(\gamma^{2}_l) + 2\gamma^{2}_l(\hat{t})) - \alpha \partial_x S(\gamma^{2}_l(\hat{t}))) \]
\[\leq 4\alpha L|\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})| \times ||\gamma^{1}_l - \gamma^{2}_l||_0 + 6\varepsilon \frac{\nu T - \hat{t}}{\nu T}L|\gamma^{1}_l(\hat{t})|(1 + ||\gamma^{1}_l||_0) \]
\[+ 6\varepsilon \frac{\nu T - \hat{t}}{\nu T}L|\gamma^{2}_l(\hat{t})|(1 + ||\gamma^{2}_l||_0); \quad (4.21)\]

\[
J_2 = q(\gamma^{1}_l, u) - q(\gamma^{2}_l, u) \leq L(1 + ||\gamma^{1}_l||_0 \vee ||\gamma^{2}_l||_0)||\gamma^{1}_l - \gamma^{2}_l||_0. \quad (4.22)\]

Combining (4.19)-(4.22), we obtain
\[
c \leq -\frac{\varepsilon}{\nu T}(S(\gamma^{1}_l) + |\gamma^{1}_l(\hat{t})|^2 + S(\gamma^{2}_l) + |\gamma^{2}_l(\hat{t})|^2) + \frac{\alpha}{2} + 2\alpha L)|\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})|^2 + 2\alpha L||\gamma^{1}_l - \gamma^{2}_l||^2_0 \]
\[+ L(1 + ||\gamma^{1}_l||_0 \vee ||\gamma^{2}_l||_0)||\gamma^{1}_l - \gamma^{2}_l||_0 + 12\varepsilon \frac{\nu T - \hat{t}}{\nu T}L(1 + ||\gamma^{1}_l||^2_0 + ||\gamma^{2}_l||^2_0). \quad (4.23)\]

Recalling \( \nu = 1 + \frac{1}{\sqrt{6}T_L} \) and \( \bar{a} = \sqrt{\frac{1}{6}\bar{T}_L} \), by (2.72), we have
\[
c \leq 2\alpha(L + 1)|\gamma^{1}_l(\hat{t}) - \gamma^{2}_l(\hat{t})|^2 + ||\gamma^{1}_l - \gamma^{2}_l||^2_0 + L(1 + ||\gamma^{1}_l||_0 \vee ||\gamma^{2}_l||_0)||\gamma^{1}_l - \gamma^{2}_l||_0 + \frac{\varepsilon}{\nu T}. \]
Then, by (4.3) and (4.16), the following contradiction is induced:

\[ c \leq \frac{c}{2} \]

The proof is now complete. \(\square\)

To complete the previous proof, it remains to state and prove the following three lemmas.

**Lemma 4.1.** The functions \( \bar{W}_1 \) and \( \bar{W}_2 \) defined by (4.10) are continuous in \( (t, x) \in [0, T] \times \bar{O}_M \).

**Proof.** We shall only prove the result for \( \bar{W}_1 \), the proof for \( \bar{W}_2 \) being similar. For every \( \hat{t} \leq t \leq \hat{s} \leq T \) and \( x_0, y_0 \in \bar{O}_M \), there exists a constant \( C_5 > 0 \) independent of \( t, s, x_0 \) and \( y_0 \) such that

\[
\bar{W}_1(t, x_0) - \bar{W}_1(s, y_0) = \sup_{\gamma \in \bar{O}_M, \gamma(t) = x_0} \left[ W'_1(\gamma_t) - \alpha d(\gamma_t, \hat{\xi}_{t,\hat{t}}) \right] - \sup_{\eta \in \bar{O}_M, \eta(s) = y_0} \left[ W'_1(\eta_s) - \alpha d(\eta_s, \hat{\xi}_{t,\hat{t}}) \right] \\
\leq \sup_{\gamma \in \bar{O}_M, \gamma(t) = x_0} \left[ W'_1(\gamma_t) - \alpha d(\gamma_t, \hat{\xi}_{t,\hat{t}}) - W'_1(\gamma_s) + \alpha d(\gamma_s, \hat{\xi}_{t,\hat{t}}) \right] \\
\leq C_5(1 + \hat{M}^2)(|s - t| + |x_0 - y_0|),
\]

where

\[
\gamma'_s(\sigma) = \begin{cases} 
\gamma_t(\sigma) + y_0 - x_0, & |\gamma_t(\sigma) + y_0 - x_0| \leq \hat{M}, \\
(\gamma_t(\sigma) + y_0 - x_0) \frac{\hat{M}}{|\gamma_t(\sigma) + y_0 - x_0|}, & |\gamma_t(\sigma) + y_0 - x_0| > \hat{M},
\end{cases} \\
\gamma'_t(\sigma) = \begin{cases} 
\gamma_s(\sigma) + x_0 - \gamma_t(\sigma), & |\gamma_s(\sigma) + x_0 - \gamma_t(\sigma)| \leq \hat{M}, \\
(\gamma_s(\sigma) + x_0 - \gamma_t(\sigma)) \frac{\hat{M}}{|\gamma_s(\sigma) + x_0 - \gamma_t(\sigma)|}, & |\gamma_s(\sigma) + x_0 - \gamma_t(\sigma)| > \hat{M},
\end{cases}
\]

It is clear that \( \gamma'_s(\sigma) \in C_{s,\hat{t}}^{\hat{M}} \) and \( \gamma'_s(s) = y_0 \). Moreover,

\[
W_1(s, y_0) - W_1(t, x_0) = \sup_{\gamma \in \bar{O}_M, \gamma(s) = y_0} \left[ W'_1(\gamma_s) - \alpha d(\gamma_s, \hat{\xi}_{t,\hat{t}}) \right] - \sup_{\eta \in \bar{O}_M, \eta(t) = x_0} \left[ W'_1(\eta_t) - \alpha d(\eta_t, \hat{\xi}_{t,\hat{t}}) \right] \\
\leq \sup_{\gamma \in \bar{O}_M, \gamma(s) = y_0} \left[ W'_1(\gamma_s) - \alpha d(\gamma_s, \hat{\xi}_{t,\hat{t}}) - W'_1(\gamma_t) + \alpha d(\gamma_t, \hat{\xi}_{t,\hat{t}}) \right] \\
\leq C_5(1 + \hat{M}^2)(|x_0 - y_0| + ((1 + \hat{M})\hat{\mu} + 1)|s - t|);
\]

It is clear that \( \gamma'_t(\sigma) \in C_{t,\hat{M}}^{\hat{M}} \) and \( \gamma'_t(t) = x_0 \). Thus,

\[
|W_1(s, y_0) - W_1(t, x_0)| \leq C_5(1 + \hat{M}^2)(|x_0 - y_0| + ((1 + \hat{M})\hat{\mu} + 1)|s - t|), \quad \hat{t} \leq t \leq \hat{s} \leq T, \ x_0, y_0 \in \bar{O}_M.
\]

Then, by the definition of \( \bar{W}_1 \), we have that \( \bar{W}_1 \) is continuous in \( (t, x) \in [0, T] \times \bar{O}_M \). The proof is now complete. \(\square\)
Lemma 4.2. The functions $\hat{W}_1$ and $-\hat{W}_2$ defined in (4.10) satisfy condition (8.5) of Theorem 8.3 in [1].

**Proof.** We only prove $\hat{W}_1$ satisfies condition (8.5) of Theorem 8.3 in [1]. The same result for $-\hat{W}_2$ can be obtained by a symmetric way.

Set $r = M - |\hat{\gamma}_T(\hat{t})|$, for a given $N > 0$, let $\varphi \in C^{1,2}((T - \hat{a}, T) \times R^d)$ be a function such that $\hat{W}_1(t, x_0) - \varphi(t, x_0)$ has a maximum at $(\hat{t}, \hat{x}_0) \in (T - \hat{a}, T) \times R^d$, moreover, the following inequalities hold true:

$$\begin{align*}
|\hat{t} - \hat{\hat{t}}| + |\hat{x}_0 - \hat{\gamma}_T(\hat{t})| &< r, \\
|\hat{W}_1(t, \hat{x}_0)| + |\nabla_x \varphi(\hat{t}, \hat{x}_0)| + |\nabla^2_x \varphi(\hat{t}, \hat{x}_0)| &\leq N.
\end{align*}$$

If $\hat{t} < \hat{\hat{t}}$, we have $b = \varphi(\hat{t}, \hat{x}_0) = \frac{1}{\nu T} (\hat{t} - \hat{\hat{t}})^{\frac{\hat{t}}{2}} \geq 0$. If $\hat{t} \geq \hat{\hat{t}}$, we consider the functional

$$\Upsilon(\gamma_t) = W_1^j(\gamma_t) - \alpha d(\gamma_t, \hat{\xi}_{t,t}) - |t - \hat{\hat{t}}| - |\gamma_t(t) - \hat{x}_0|^2 - \varphi(t, \gamma_t(t)) \text{, } (t, \gamma_t) \in [\hat{t}, T) \times \Lambda.$$ 

By Lemma 2.1 and the proof procedure of the following Lemma 4.3, we may assume that

$$\Upsilon(\gamma_t) \geq \Upsilon(\gamma_t), \ (t, \gamma_t) \in [\hat{t}, T) \times C^\mu_M,$$

with equality at $(\hat{\hat{t}}, \hat{\gamma}_{\hat{t}})$ for some $\hat{\hat{t}} \in [\hat{t}, T)$ and $\hat{\gamma}_{\hat{t}} \in C^\mu_M$, satisfying $\hat{\gamma}_{\hat{t}}(\hat{t}) = \hat{x}_0$.

Since $|\hat{x}_0| < |\hat{\gamma}_T(\hat{t})| + (\hat{M} - |\hat{\gamma}_T(\hat{t})|)$, we get $|\hat{\gamma}_{\hat{t}}(\hat{t})| = |\hat{x}_0| < \hat{M}$. Thus, the definition of the viscosity $\hat{\mu}$-subsolution can be used to obtain the following result:

$$\begin{align*}
\varphi(\hat{t}, \hat{\gamma}_{\hat{t}}(\hat{t})) - \frac{\nu T}{\nu T} (S(\hat{\gamma}_{\hat{t}})) + |\hat{\gamma}_{\hat{t}}(\hat{t})|^2 &+ \varepsilon |\hat{\gamma}_{\hat{t}}(\hat{t}) - \hat{\gamma}_{\hat{t}}(\hat{t})|^2 \\
+ 2\varepsilon (t - \hat{\hat{t}}) + \alpha \hat{\gamma}_{\hat{t}}(\hat{t}) - \hat{\xi}_{t,t}(\hat{t})^2 - \varepsilon (t - T + \hat{a})^{-2} + H(\hat{\gamma}_{\hat{t}}, \nabla \varphi(\hat{t}, \hat{\gamma}_{\hat{t}})) \\
+ \frac{\nu T}{\nu T} (\partial_s S(\hat{\gamma}_{\hat{t}})) &+ 2\hat{\gamma}_{\hat{t}}(\hat{t}) \geq 0. \quad (4.24)
\end{align*}$$

By the definition of $H$, it follows that there exists a constant $\hat{c}$ such that $b = \varphi(\hat{t}, \hat{x}_0) \geq \hat{c}$. The proof is now complete. □

Lemma 4.3. The maximum points $(\hat{t}_k, \hat{s}_k, \hat{z}_{1,k}^j, \hat{z}_{2,k}^j)$ of $\Gamma_k$ defined by (4.13) in $[\hat{t}, T] \times \hat{t}, T] \times C^\mu_M \times C^\mu_M$ satisfy conditions (4.14) and (4.15).

**Proof.** By the definitions of $\hat{W}_1, \hat{W}_2$, we get that, for every $(t, \gamma_t), (s, \gamma_s) \in [\hat{t}, T] \times C^\mu_M,$

$$\begin{align*}
W_1(\hat{t}_k, \hat{z}_{1,k}^j(\hat{t}_k)) - W_2(\hat{s}_k, \hat{z}_{2,k}^j(\hat{s}_k)) - \varphi_k(\hat{t}_k, \hat{z}_{1,k}^j(\hat{t}_k)) + \psi_k(\hat{s}_k, \hat{z}_{2,k}^j(\hat{s}_k)) \\
\geq W_1(\gamma^1_{\hat{t}_k}) - W_2(\gamma^2_{\hat{s}_k}) - \alpha d(\hat{z}_{1,k}^j, \hat{\xi}_{t,t}(\hat{t}_k)) - \alpha d(\hat{z}_{2,k}^j, \hat{\xi}_{t,t}(\hat{t}_k)) - \varphi_k(\hat{t}_k, \hat{z}_{1,k}^j(\hat{t}_k)) + \psi_k(\hat{s}_k, \hat{z}_{2,k}^j(\hat{s}_k)) \\
\geq W_1(\gamma^1_t) - W_2(\gamma^2_s) - \alpha d(\hat{z}_{1,k}^j, \hat{\xi}_{t,t}) - \alpha d(\hat{z}_{2,k}^j, \hat{\xi}_{t,t}) - \varphi_k(t, \gamma^1_t(t)) + \psi_k(s, \gamma^2_s(s))
\end{align*}$$

Taking the supremum over $\gamma^1_{\hat{t}_k}, \gamma^2_{\hat{s}_k} \in C^\mu_M$, and $\gamma^1_{\hat{t}_k}(t_k) = x^k_0, \gamma^2_{\hat{s}_k}(s_k) = y^k_0$, we have that

$$\begin{align*}
W_1(\hat{t}_k, \hat{z}_{1,k}^j(\hat{t}_k)) - W_2(\hat{s}_k, \hat{z}_{2,k}^j(\hat{s}_k)) - \varphi_k(\hat{t}_k, \hat{z}_{1,k}^j(\hat{t}_k)) + \psi_k(\hat{s}_k, \hat{z}_{2,k}^j(\hat{s}_k)) \\
\geq W_1(t_k, x^k_0) - W_2(s_k, y^k_0) - \varphi_k(t_k, x^k_0) + \psi_k(s_k, y^k_0).
\end{align*}$$

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This shows that (4.14) holds true and
\[ W_1'(\gamma_{1,k}) - W_2'(\gamma_{2,k}) - \alpha d(\dot{\gamma}_{1,k}, \dot{\xi}_{t,i,k}) - \alpha d(\dot{\gamma}_{2,k}, \dot{\xi}_{t,i,k}) = \dot{W}_1(t_k, x_0^k) - \dot{W}_2(s_k, y_0^k). \] (4.25)

Letting \( k \to \infty \), by (4.9), we show that
\[
\lim_{k \to \infty} \left[ W_1'(\gamma_{1,k}) - W_2'(\gamma_{2,k}) - \alpha d(\dot{\gamma}_{1,k}, \dot{\xi}_{t,i,k}) - \alpha d(\dot{\gamma}_{2,k}, \dot{\xi}_{t,i,k}) \right] = \dot{W}_1(t, \gamma_{1,t}) - \dot{W}_2(t, \gamma_{2,t}) - \frac{\alpha}{2} d(\dot{\gamma}_{1,t}, \dot{\gamma}_{2,t}).
\]

On the other hand, since \( \gamma_{1,k}, \gamma_{2,k} \in C_{\hat{m}}^\epsilon \), we may assume \( \gamma_{1,k} \to \gamma_{1,t}, \gamma_{2,k} \to \gamma_{2,t} \) for some \( \gamma_{1,t}, \gamma_{2,t} \in C_{\hat{m}}^\epsilon \). Then we have that
\[
\lim_{k \to \infty} \left[ W_1'(\gamma_{1,k}) - W_2'(\gamma_{2,k}) - \alpha d(\dot{\gamma}_{1,k}, \dot{\xi}_{t,i,k}) + d(\dot{\gamma}_{2,k}, \dot{\xi}_{t,i,k}) \right] \leq W_1'(\gamma_{1,t}) - W_2'(\gamma_{2,t}) - \frac{\alpha}{2} d(\dot{\gamma}_{1,t}, \dot{\gamma}_{2,t}).
\]

Therefore, \( \gamma_{1,t} = \gamma_{1,t}, \gamma_{2,t} = \gamma_{2,t} \). The proof is now complete. \( \Box \)

## 5 Appendix

In this Appendix, we prove \((\hat{\Lambda}^t, d_\infty)\) is a complete metric space.

**Lemma 5.1.** \((\hat{\Lambda}^t, d_\infty)\) is a complete metric space for every \( t \in [0, T) \).

**Proof.** Assume \( \{\gamma_{1,n}\}_{n \geq 0} \) is a cauchy sequence in \((\hat{\Lambda}^t, d_\infty)\), then for any \( \varepsilon > 0 \), there exists \( N(\varepsilon) > 0 \) such that, for all \( m, n \geq N(\varepsilon) \), we have
\[
d_\infty(\gamma_{1,m}, \gamma_{1,n}) = |t_m - t_n| + \sup_{0 \leq s \leq T} |\gamma_{1,m}(s \wedge t_m) - \gamma_{1,n}(s \wedge t_n)| < \varepsilon.
\]

Therefore, there exists \( \hat{t} \in [t, T] \) such that \( \lim_{n \to \infty} t_n = \hat{t} \). Moreover, for all \( s \in [0, T] \),
\[
|\gamma_{1,n}(s \wedge t_n) - \gamma_{1,m}(s \wedge t_m)| < \varepsilon, \quad (\forall m, n \geq N(\varepsilon)). \tag{5.1}
\]

For fixed \( s \in [0, T] \), we see that \( \{\gamma_{1,n}(t_n \wedge s)\} \) is a cauchy sequence, thereby the limit \( \lim_{n \to \infty} \gamma_{1,n}(t_n \wedge s) \) exists and denoted by \( \gamma_T(s) \). Letting \( m \to \infty \) in (5.1), we obtain that
\[
|\gamma_T(s) - \gamma_{1,n}(s \wedge t_n)| \leq \varepsilon, \quad (\forall n \geq N(\varepsilon)).
\]

Taking the supremum over \( s \in [0, T] \), we get
\[
\sup_{s \in [0, T]} |\gamma_T(s) - \gamma_{1,n}(s \wedge t_n)| \leq \varepsilon, \quad (\forall n \geq N(\varepsilon)). \tag{5.2}
\]

We claim that \( \gamma_T(s) = \gamma_T(\hat{t}) \) for all \( s \in [\hat{t}, T] \). In fact, if there exists a subsequence \( \{t_{n_i}\}_{i \geq 0} \) of \( \{t_n\}_{n \geq 0} \) such that \( \{t_{n_i}\}_{i \geq 0} \leq \hat{t} \), then we have, for every \( s \in [\hat{t}, T] \),
\[
\gamma_T(s) = \lim_{n \to \infty} \gamma_{1,n}(s \wedge t_n) = \lim_{n \to \infty} \gamma_{1,n}(t_n) = \lim_{l \to \infty} \gamma_{1,n}(t_{n_l}) = \lim_{l \to \infty} \gamma_{1,n}(t_{n_l} \wedge \hat{t}) = \lim_{n \to \infty} \gamma_{1,n}(t_n \wedge \hat{t}) = \gamma_T(\hat{t}).
\]
Otherwise, we may assume \( \{ t_n \}_{n \geq 0} > \hat{t} \). Letting \( s = t_m \) and \( m \to \infty \) in (5.1), we obtain, for all \( s \in (\hat{t}, T] \),

\[
|\gamma^n_{t_n}(\hat{t}) - \gamma_T(s)| \leq \varepsilon, \quad (\forall n \geq N(\varepsilon)).
\]

Letting \( n \to \infty \), we have

\[
|\gamma_T(\hat{t}) - \gamma_T(s)| \leq \varepsilon, \quad \text{for all } s \in (\hat{t}, T].
\]

Then, by (5.2) we obtain

\[
d_{\infty}(\eta_\hat{t}, \gamma^m_{t_n}) \to 0 \text{ as } n \to \infty.
\]

Here we let \( \eta_\hat{t} \) denote \( \gamma_T|_{(0, \hat{t}]} \). Now we prove \( \eta_\hat{t} \in \hat{A}' \). First, we prove \( \eta_\hat{t} \) is right-continuous. For every \( 0 \leq s < \hat{t} \) and \( 0 < \delta \leq \hat{t} - s \), we have

\[
|\eta_\hat{t}(s+\delta) - \eta_\hat{t}(s)| \leq |\gamma_T(s+\delta) - \gamma^m_{t_n}((s+\delta) \land t_n)| + |\gamma^m_{t_n}((s+\delta) \land t_n) - \gamma^m_{t_n}(s \land t_n)| + |\gamma^m_{t_n}(s \land t_n) - \gamma_T(s)|.
\]

For every \( \varepsilon > 0 \), by (5.2), there exists \( n > 0 \) be large enough such that

\[
|\gamma_T(s + \delta) - \gamma^m_{t_n}((s + \delta) \land t_n)| + |\gamma^m_{t_n}(s \land t_n) - \gamma_T(s)| < \frac{\varepsilon}{2}.
\]

For the fixed \( n \), since \( \gamma^m_{t_n} \in \hat{A}' \), there exists a constant \( 0 < \Delta \leq \hat{t} - s \) such that, for all \( 0 \leq \delta < \Delta \),

\[
|\gamma^m_{t_n}((s + \delta) \land t_n) - \gamma^m_{t_n}(s \land t_n)| < \frac{\varepsilon}{2}.
\]

Then \( |\eta_\hat{t}(s + \delta) - \eta_\hat{t}(s)| < \varepsilon \) for all \( 0 \leq \delta < \Delta \). Next, let us prove \( \eta_\hat{t} \) has left limit in \((0, \hat{t}]\). For every \( 0 < s \leq \hat{t} \) and \( 0 \leq s_1, s_2 < s \), we have

\[
|\eta_\hat{t}(s_1) - \eta_\hat{t}(s_2)| \leq |\gamma_T(s_1) - \gamma^m_{t_n}(s_1 \land t_n)| + |\gamma_T(s_2) - \gamma^m_{t_n}(s_2 \land t_n)| + |\gamma^m_{t_n}(s_1 \land t_n) - \gamma^m_{t_n}(s_2 \land t_n)|.
\]

For every \( \varepsilon > 0 \), by (5.2), there exists \( n > 0 \) be large enough such that

\[
|\gamma_T(s_1) - \gamma^m_{t_n}(s_1 \land t_n)| + |\gamma_T(s_2) - \gamma^m_{t_n}(s_2 \land t_n)| < \frac{\varepsilon}{2}.
\]

For the fixed \( n \), if \( t_n < s \), we can let \( \Delta > 0 \) be small enough such that \( t_n < s - \Delta \), then for all \( s_1, s_2 \in [s - \Delta, s) \)

\[
|\gamma^m_{t_n}(s_1 \land t_n) - \gamma^m_{t_n}(s_2 \land t_n)| = |\gamma^m_{t_n}(t_n) - \gamma^m_{t_n}(t_n)| = 0;
\]

if \( t_n \geq s \), since \( \gamma^m_{t_n} \in \hat{A}' \), there exists a constant \( \Delta > 0 \) such that, for all \( s_1, s_2 \in [s - \Delta, s) \),

\[
|\gamma^m_{t_n}(s_1 \land t_n) - \gamma^m_{t_n}(s_2 \land t_n)| = |\gamma^m_{t_n}(s_1) - \gamma^m_{t_n}(s_2)| < \frac{\varepsilon}{2}.
\]

Then there exists a constant \( \Delta > 0 \) such that \( |\eta_\hat{t}(s_1) - \eta_\hat{t}(s_2)| < \varepsilon \) for all \( s_1, s_2 \in [s - \Delta, s) \).

The proof is now complete. \( \square \)
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