Some results on Dyck paths and Motzkin paths

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Abstract

We introduce an equivalence relation on the set of Dyck paths and some operations on them. We determine a formula for the cardinality of those equivalence classes and use this information to obtain a combinatorial formula for the number of Dyck and Motzkin paths of a fixed length.

1 Introduction

In [2], among other things, we study a certain equivalence relation on the set of Dyck paths and we compute the cardinality of the equivalence classes. We then use this information to give a combinatorial formula for the number of Dyck and Motzkin paths of a fixed length. Some of these results were obtained by [4] using continued fractions. In this paper we give a more detailed exposition of the results in [2] including some proofs there omitted.

2 Motzkin paths with horizontal steps at a single level

Definition 2.1. A Motzkin path of length $n$ is a path on the integral lattice $\mathbb{Z} \times \mathbb{Z}$ starting from $(0, 0)$ and ending in $(k, 0)$ using $n$ steps according to the vectors $(1, 1), (1, 0), (1, -1)$ and never going below the x-axis, ([1]).

For example

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (4,0) -- (5,1) -- (6,0) -- (7,1) -- (8,0) -- (9,1) -- (10,0);
\end{tikzpicture}
\end{center}
**Definition 2.2.** A Motzkin path with no \((1,0)\) steps is called a *Dyck path* ([3]).

In this section we want to count the number of Motzkin paths that have horizontal steps only at a given fixed level. For this, it is important to count how many Dyck paths have two feet, three feet, four feet, etc. defining a foot to be a point where the path touches the fixed level. For example, it is clear that there is only one Dyck path with length two and this has two feet at level zero:

\[ \)

Among the paths with length 3, one has two feet, one has three feet at level zero.

\[ \)

Among the five paths with length 6, two have three feet, two have two feet, one has four feet at level zero:

\[ \)

If we arrange the resulting numbers in a Pascal-like triangle, we have:

|       | 1-ped | 2-ped | 3-ped | 4-ped | 5-ped | 6-ped |
|-------|-------|-------|-------|-------|-------|-------|
| 0 steps | 1     | 0     | 0     | 0     | 0     | 0     |
| 2 steps | 0     | 1     | 0     | 0     | 0     | 0     |
| 4 steps | 0     | 1     | 1     | 0     | 0     | 0     |
| 6 steps | 0     | 2     | 2     | 1     | 0     | 0     |
| 8 steps | 0     | 5     | 5     | 3     | 1     | 0     |
| 10 steps | 0   | 14 | 14 | 9 | 4 | 1 |
| 12 steps | 0   | 42 | 42 | 28 | 14 | 5 |

See below for a definition of \(n\)-ped.

**Remark 2.3.** The natural conjecture one can formulate by looking at this table is that the second column is made up of the Catalan numbers, while the other columns are convolutions of the Catalan number sequence. We shall have something to say about this later on.

To count Motzkin path with horizontal level at a single level, we need to count the number of Dyck paths that at a certain level have a given number of feet. This can be done in a recursive fashion. Start by counting the number
of feet of Dyck paths at level 0. For this we use a recursive construction suggested by the following recursive formula of the Catalan numbers:

\[ C(n) = \sum_{k=0}^{n-1} C(n-1-k)C(k). \]  

(2.1)

We interpret this formula as follows. Any length 2n Dyck path can be obtained by a combination of two operations. The first operation consists in adding to each path of length 2(n − 1 − k) an up step at the beginning and a down step at the end. We call this operation lifting. The second operation consists in gluing at the end of it any path of length 2k, k = 0, . . . , n − 1.

Next, we observe that every Dyck path is obtained by a sequence of up and down steps, indicated by U and D, respectively, in such a way that the total number of U’s is equal the number of D’s and that at each step the number of U’s be not less than the number of D’s. Let’s associate to each U the number 1 and to each D the number −1. Starting from 0, let’s sum at each step the numbers 1 and −1 up to that point. For a length 2n path, the associated sequence thus obtained starts and ends with 0 and the maximum number appearing is at most n. For each integer i ≥ 0, denoted by p(i) the number of times that i appears in the sequence, we shall say that the path is p(i)-ped at the level i. As an example, the length 14 path UUDUUDDUUDUDDD has the associated sequence 012123212323210. In other words, the sequence 012123212323210 is simply the sequence of the y coordinate of the points touched by the path. The path is therefore biped at level 0, quadruped at level 1, 6-ped at level 2, 3-ped at level 3, 0-ped at higher levels. We may sometimes identify a path with its associated sequence.

For every triple of integers 2n ≥ 0, i ≥ 0, j ≥ 0, we denote by \( p_{i,j}^{2n} \) the number of Dyck paths of length 2n that are \( j \)-ped at level \( i \). Our next objective is to compute \( p_{i,j}^{2n} \). We shall do this using a recursive rule first on \( i \) and then on \( n \).

The starting point is to compute \( p_{0,j}^{2n} \), namely, how many paths of length 2n have \( j \) feet at level 0, as \( n \) grows.

**Proposition 2.4.** We have the following recursive formulas

\[ p_{0,0}^0 = 0, p_{0,1}^0 = 1, p_{0,j}^0 = 0, j > 1 \]  

(2.2)

\[ p_{0,0}^2 = 0, p_{0,1}^2 = 1, p_{0,2}^2 = 1, p_{0,j}^2 = 0, j > 2 \]  

(2.3)

\[ p_{0,0}^{2n} = 0, p_{0,1}^{2n} = 0, p_{0,2}^{2n} = \sum_{k=0}^{\infty} p_{0,k}^{2(n-1)}, n > 1 \]  

(2.4)
\[ p_{0,j}^{2n} = \sum_{i=0}^{n-2} \frac{2^{(i+1)}}{P_{0,2} \cdot P_{0,j-1}} p_{0,i}^{2(n-i-1)} \cdot j > 2, n > 1 \]  

(2.5)

Proof. There is a unique Dyck path of length 0, which we call null path and denote by 0. The corresponding numerical sequence is 0 and so this path is 1-ped at level 0. The null path clearly has a single foot and nothing else:

\[ p_{0,0}^0 = 0, p_{0,1}^0 = 1, p_{0,j}^0 = 0, j > 1. \]

The path UD, \( \bigwedge \), is the only one of length 2. The associated sequence is 010. It is therefore 2-ped at level 0, 1-ped at level 1 and 0-ped at higher levels. It follows:

\[ p_{0,0}^2 = 0, p_{0,1}^2 = 0, p_{0,2}^2 = 1, p_{0,j}^2 = 0, j > 2. \]

Before proceeding further, we observe that by lifting any path we get a 2-ped at level 0 and gluing a 2-ped at level zero to a \( j \)-ped at level 0 we get a \( (j + 1) \)-ped at level 0. Moreover, we notice that there is no path of length greater than zero that is 0-ped or 1-ped at level 0. Hence:

\[ p_{0,0}^{2n} = 0, p_{0,1}^{2n} = 0, n > 0. \]

Suppose we have determined all \( p_{0,j}^{2m} \) up to an even fixed length \( 2(n - 1) \), \( m \leq n - 1 \), let us compute \( p_{0,j}^{2n} \). All the 2-ped at level 0 of length \( 2n \) are obtained by lifting paths of length \( 2(n - 1) \) and gluing to them the null path 0. Therefore

\[ p_{0,2}^{2n} = \sum_{k=0}^{\infty} p_{0,k}^{2(n-1)}. \]  

(2.6)

The sum is actually finite since \( p_{0,k}^{2(n-1)} \) is zero for \( k > n \).

For \( j > 2 \), the \( j \)-ped paths at level 0 of length \( 2n \) are constructed by lifting paths of length \( 2i \), \( 0 \leq i \leq n - 2 \), with any number \( k \) of feet, thus becoming 2-ped, and gluing to them \( (j - 1) \)-ped paths of length \( 2(n - i - 1) \). It follows that

\[ p_{0,j}^{2n} = \sum_{i=0}^{n-2} \left( \sum_{k=0}^{\infty} p_{0,k}^{2i} P_{0,j-1}^{2(n-i-1)} \right). \]  

(2.7)

Since \( \sum_{k=0}^{\infty} p_{0,k}^{2i} = p_{0,2}^{2(i+1)} \), by (2.4), formula 2.7 can also be written as
\[ p_{2n}^{2n} = \sum_{i=0}^{n-2} p_{0.2}^{2(i+1)} p_{0,j-1}^{2(n-i-1)} \]  
(2.8)

This is the general formula for the number of \( j \)-ped paths at level zero for any length.

**Remark 2.5.** This formula proves the observation made earlier (see Remark 2.3), and it is the interpretation of formula (2.1).

**Proposition 2.6.** We have the following recursive formula

\[ p_{s,0}^0 = 1, \quad p_{s,j}^0 = 0, \quad s > 0, \quad j > 0 \]  
(2.9)

\[ p_{s,j}^{2n} = \sum_{i=0}^{n-1} \sum_{k=0}^{j} p_{s-1,k}^{2i} p_{s,j-k}^{2(n-i-1)}. \]  
(2.10)

**Proof.** Let us now proceed to compute the number \( p_{1,j}^{2n} \), for level 1. Obviously

\[ p_{1,0}^0 = 1, \quad p_{1,j}^0 = 0, \quad j > 0, \]

\[ p_{s,0}^0 = 1, \quad p_{s,j}^0 = 0, \quad s > 1, \quad j > 0 \]

Suppose we determined all \( p_{s,j}^{2n} \), \( m \leq n - 1 \), any \( s \) and any \( j \), up to a fixed even length \( 2(n-1) \). We wish to compute \( p_{s,j}^{2n} \). The \( j \)-ped paths at level \( s \) of length \( 2n \) are obtained by lifting \( k \)-ped paths at level \( s-1 \), \( 0 \leq k \leq j \), of length \( 2i \), \( 0 \leq i \leq n-1 \), and gluing to them the \((j-k)\)-ped at level \( s \) and length \( 2(n-i-1) \). Hence

\[ p_{s,j}^{2n} = \sum_{i=0}^{n-1} \sum_{k=0}^{j} p_{s-1,k}^{2i} p_{s,j-k}^{2(n-i-1)}. \]  
(2.11)

\[ \square \]

The knowledge of the number of \( j \)-ped Dyck paths at a level \( s \) allows us to compute a sort of partial binomial transform at a single level, namely counting the Motzkin paths having horizontal steps only at a prefixed level.

We now use Propositions 2.4 e 2.6 to compute the number of Motzkin paths having horizontal steps only at a level \( k \), \( k \geq 0 \) a fixed integer. We shall call such paths \( k \)-Motzkin paths and shall denote by \( m^n_k \) the number of \( k \)-Motzkin paths of length \( n \).
Theorem 2.7. The number of $k$-Motzkin paths of length $n$ is

$$m_k^{(n)} = \sum_{j=0}^{\nu} \sum_{i=0}^{\infty} p_{2j}^{k,i} \binom{n-2j+i-1}{n-2j}.$$ (2.12)

Proof. Notice that a $k$-Motzkin path of length $n$ can be obtained from a Dyck path $D$ with length $2j$, $0 \leq 2j \leq n$, by adding $n-2j$ horizontal steps at the feet of level $k$ of $D$. If $D$ is $i$-ped at level $k$, this can be done in as many ways as there are possibility to express $n-2j$ as sum of $i$ integers between 0 and $n-2j$, namely, in $\binom{n-2j+i-1}{n-2j}$ ways. Set $\nu = \left\lfloor \frac{n}{2j} \right\rfloor$.

It follows

$$m_k^{(n)} = \sum_{j=0}^{\nu} \sum_{i=0}^{\infty} p_{2j}^{k,i} \binom{n-2j+i-1}{n-2j}.$$ 

Notice also that for $n = 2j$, the binomial coefficient is 1 when $i = 0$. This is in accordance with the fact 0-ped Dyck paths at level $k$ of length $n$ are to be considered in the evaluation of $m_k^{(n)}$. Finally, notice that the infinite sum is actually finite since $p_{2j}^{0,i}$ is zero for $i > j + 1$ and $p_{2j}^{k,i}$, with $k > 0$, is zero for $i > j$.

If we give $r$ colors to the horizontal steps at level $k$, then formula (2.12), where we continue to use a self-explanatory notation, obviously becomes

Proposition 2.8.

$$m_k^{(n)} = \sum_{j=0}^{\nu} \sum_{i=0}^{\infty} p_{2j}^{k,i} \binom{n-2j+i-1}{n-2j} r^{n-2j}.$$ 

3 Frames of Dyck paths

Next, we want to add horizontal steps at more then one level. To do this it is not sufficient anymore to know the number of Dyck paths that are $j$-ped at the various levels, but we need to have more refined information, that is, we need to know the number of Dyck paths that have a fixed “frame” $(i_0, i_1, \ldots)$, according to the definition we are about to give.

Given a Dyck path $D$, with $i_k$ feet at level $k$, $k = 0, 1, \ldots$, we call the sequence $(i_0, i_1, \ldots)$, which is zero from a certain index on, the frame of $D$. In other words, the sequence tells us how many feet there are at each level. This sequence is zero from a certain index on, since, if the length of $D$ is $2n$, one has $i_{n+j} = 0$, for $j \geq 1$, namely, the maximum reachable level for a Dyck path of length $2n$ is $n$. 
Given an eventually zero sequence $I = (i_0, i_1, ...)$, we shall call the integer $-1 + \sum_{k=0}^{\infty} i_k$ the *length* of $I$.

We shall say that an eventually zero sequence is *admissible* if it is the frame of some Dyck path. For an admissible frame $I$, its length is the same as the length of a path with that frame and is necessarily even.

For example, frames of Dyck paths of length 0, 2, 4 are, respectively:

- $(1, 0, 0, ...)$,
- $(2, 1, 0, ...)$,
- $(2, 2, 1, 0, ...), (3, 2, 0, ...)$.

Notice that two distinct paths with the same length may have the same frame. For example, the two paths with length 6: $UDUUDD$ and $UUDDUD$, which give rise to the sequences $0101210$ and $0121010$ are, respectively, and have the same frame $(3, 3, 1, 0, 0, ...)$. Obviously “having the same frame” is an equivalence relation on the set of Dyck paths. Notice that lifting two equivalent paths we still get two equivalent paths. This is true also if we glue together two paths: if $p_i$ is equivalent to $q_i$, $i = 1, 2$ then gluing $p_i$ and $p_2$ gives a path that is equivalent to gluing $q_1$ and $q_2$. This observation allows us to speak of *lifting a frame* and *gluing two frames*.

It is easy to construct recursively the frames associated to Dyck paths of various lengths, using the same recursive law for the paths. Lifting the frame $(i_0, i_1, i_2, ...)$ one gets the frame $(2, i_0, i_1, i_2, ...)$, While gluing two frames $(j_0, j_1, j_2, ...)$ and $(i_0, i_1, i_2, ...)$, we get a frame $(i_0 + j_0 - 1, i_1 + j_1, i_2 + j_2, ...)$.

For the lifting and gluing operations we shall use the following symbols:

$$s(i_0, i_1, i_2, ...) = (2, i_0, i_1, i_2, ...),$$

$$(i_0, i_1, i_2, ...) \land (j_0, j_1, j_2, ...) = (i_0 + j_0 - 1, i_1 + j_1, i_2 + j_2, ...)$$

It is not difficult to check, for example, that the frames with length 4 can be obtained in this fashion from those with shorter lengths.

The gluing operation on paths is associative but not commutative: in general, $u \land v \neq v \land u$. The same operation on frames however is commutative, namely, $u \land v$ and $v \land u$ have the same frame.
Actually, any frame can be obtained by the combination of two elementary operations: lifting of frame \((i_0, i_1, i_2, \ldots)\), which turns \((i_0, i_1, i_2, \ldots)\) into \(s(i_0, i_1, i_2, \ldots) = (2, i_0, i_1, i_2, \ldots)\) and the gluing \((i_0, i_1, i_2, \ldots)\) to the frame of length 2, which we call extension, that turns \((i_0, i_1, i_2, \ldots)\) into \(a(i_0, i_1, i_2, \ldots) = (i_0, i_1, i_2, \ldots) \land (2, 1, 0, \ldots) = (i_0 + 1, i_1 + 1, i_2, \ldots)\). We have therefore:

**Theorem 3.1.** Every frame can be obtained by a combination of a lifting and a suitable number of extensions.

**Proof.** Consider any path U...D that is not a lifting, i.e., with a frame starting with \(i_0 \geq 3\). It follows that its sequence is of the form 01...101...10, with at least one subsequence 101 inside. The first triple 101 inside comes from an ordered pair DU. If we eliminate such a pair inside and add instead the pair UD at the end of the frame, we get a new path with the same frame as the previous one. Iterating such a procedure we get a Dyck path with the same frame as the starting path obtained by gluing a lifting of a suitable path and a finite number, possibly zero, of length 2 paths. \(\square\)

**Theorem 3.2.** The number of frames of length \(2n\), with \(n > 0\), is \(2^{n-1}\).

**Proof.** The frames of length \(2n\) are obtained from those of length \(2(n - 1)\) by constructing for each of them, say \(I\), the lifting \(s(I)\) or the extension \(a(I)\). The frame \(s(I)\) is different from \(a(I)\) if \(n > 1\). It follows that the number of frames with length \(2n\) is twice the number of the the frames of length \(2(n - 1)\). \(\square\)

Natural generalizations of the extension \(a\) and lifting \(s\) operations can be defined on any sequence of integers eventually zero. The operation \(a\) is a bijection on the set of such sequences. Its inverse, denoted by \(b\), is defined by:

\[
b(i_0, i_1, i_2, \ldots) = (i_0 - 1, i_1 - 1, i_2, \ldots).
\]

The operation \(s\) is injective and can be inverted on its image, via an operation \(r\) defined by:

\[
r(2, i_1, i_2, \ldots) = (i_1, i_2, \ldots).
\]

It is easy to see if a given sequence, which is eventually zero, is admissible. Indeed, we can, starting from it, trace back the elementary steps that generated it from the basic frame \((1, 0, 0, \ldots)\). The rules can be summed up as follows: every time there is a 2 we erase it by using \(r\), otherwise we subtract 1 from the first two elements, by applying \(b\).
**Example.** Consider the sequence \((3, 6, 6, 3, 1, 0, \ldots)\). We wish to see if it is admissible for a path of length \(18 = 3 + 6 + 6 + 3 + 1\). Tracing backwards we have: \((2, 5, 6, 3, 1, 0, \ldots)\), \((5, 6, 3, 1, 0, \ldots)\), \((4, 5, 3, 1, 0, \ldots)\), \((3, 4, 3, 1, 0, \ldots)\), \((2, 3, 3, 1, 0, \ldots)\), \((3, 3, 1, 0, \ldots)\), \((2, 2, 1, 0, \ldots)\), \((2, 1, 0, \ldots)\), and \((1, 0, \ldots)\). So the given sequence is admissible.

**Example.** If we take \((4, 5, 2, 3, 1, 0, \ldots)\), and we trace backwards, we get: \((3, 4, 2, 3, 1, 0, \ldots)\), \((2, 3, 2, 3, 1, 0, \ldots)\), \((3, 2, 3, 1, 0, \ldots)\), \((2, 1, 3, 1, 0, \ldots)\), \((1, 3, 1, 0, \ldots)\), and finally \((0, 2, 1, 0, \ldots)\), which is clearly not admissible since there are no Dyck paths of length 2 without feet at the zero level.

In the preceding arguments, we have implicitly used the following two lemmas whose proof is straightforward:

**Lemma 3.3.** Given an eventually zero integer sequence \(I\), with \(i_0 = 2\), \(r(I)\) is admissible if and only if \(I\) is.

**Lemma 3.4.** Given an eventually zero integer sequence \(I\), with \(i_0 \neq 2\), \(b(I)\) is admissible if and only if \(I\) is.

Given an eventually zero integer sequence \((i_0, i_1, i_2, \ldots)\), denote by \(i_f\) the last nonzero element and call \(f\) the degree of the frame.

**Theorem 3.5.** An eventually zero sequence of nonnegative integers

\[ I = (i_0, i_1, \ldots, i_f, 0, \ldots) \]

with length \(2n\), is admissible if and only if the following conditions are satisfied:

- \(i_0 = 1\), if \(f = 0\) and \(i_0 \geq 2\) if \(f > 0\);
- \(i_1 - i_0 \geq 0\) provided \(f > 1\);
- \(i_2 - i_1 + i_0 \geq 2\);
- \(i_3 - i_2 + i_1 - i_0 \geq 0\);
- \(i_4 - i_3 + i_2 - i_1 + i_0 \geq 2\);
- \(\ldots\)
- \(i_f - i_{f-1} + i_{f-2} + \cdots = -1\) when \(f\) is odd;
- \(i_f - i_{f-1} + i_{f-2} + \cdots = 1\) when \(f\) is even.
Proof. The last two conditions can be summarized in the following: \(i_0 - i_1 + i_2 - ... + (-1)^f i_f = 1\). Notice also that if \(f = 0\), then the only required condition is \(i_0 = 1\), while, in case \(f = 1\), the conditions are \(i_0 \geq 2\) and \(i_0 - i_1 = 1\).

If \(f = 0\), \(I = (i_0, 0, ...\)). Such a sequence is admissible if and only if it is the frame of the null path, namely, if and only if \(i_0 = 1\).

Suppose \(f = 1\), and use induction on the half-length \(n\). When \(n = 1\) the only admissible sequence is \((2, 1, 0, ...)\), which satisfies the conditions \(i_0 \geq 2\) and \(i_0 - i_1 = 1\). Assuming the statement true up to the half-length \((n - 1)\), we are going to show it for the half-length \(n\). Let \(I = (i_0, i_1, 0, ..., i_f, 0, ...)\) with \(i_0 + i_1 = 2n + 1\). If \(i_0 = 2\), consider \(r(I) = (i_1, 0, ..., i_f, 0, ...)\). This is admissible if and only if \(i_1 = 1\), and therefore, by Lemma 3.3, the result follows. If instead \(i_0 \neq 2\), consider \(b(I) = (i_0 - 1, i_1 - 1, 0, ..., i_f, 0, ...)\). This, being of length \(2(n - 1)\), by the induction hypothesis it is admissible if and only if \(i_0' \geq 2\) and \(i_0' - i_1' = 1\), which are equivalent to \(i_0 \geq 3\) and \(i_0 - i_1 = 1\), hence, again, Lemma 3.4 implies the result.

Thus, for degrees 0 and 1 the theorem is proved. Assume \(f \geq 2\). The result is obviously true for length \(2n = 0\). Assume the result is true up to length \(2(n - 1)\); we shall prove it for length \(2n\). We distinguish two cases.

Case A: \(i_0 = 2\). Let \(I = (2, i_1, i_2, ..., i_f, 0, ..., i_f, 0, ...)\), with \(\sum_{k=0}^{\infty} i_k = 2n + 1\). By applying \(r\), we obtain \(r(I) = (i_1, i_2, ..., i_f, 0, ..., i_f, 0, ...) = (i_0', i_1', ..., i_f', 0, ...)\). We have:

\[
\begin{align*}
i_0' &= i_1; \\
i_1' - i_0' &= i_2 - i_1 = i_2 - i_1 + i_0 - 2; \\
i_2' - i_1' + i_0' &= i_3 - i_2 + i_1 = i_3 - i_2 + i_1 - i_0 + 2; \\
i_3' - i_2' + i_1' - i_0' &= i_4 - i_3 + i_2 - i_1 = i_4 - i_3 + i_2 - i_1 - i_0 + 2; \\
&\vdots \\
i_0' - i_1' + i_2' - ... + (-1)^f i_f' &= i_1 - i_2 + ... + (-1)^{f-1} i_f \\
&= i_1 - i_2 + ... + (-1)^{f-1} i_f - i_0 + 2 \\
&= -(i_0 - i_1 + i_2 - ... + (-1)^{f} i_f) + 2
\end{align*}
\]

(3.1)
The conditions
\[
\begin{align*}
\ell'_0 & \geq 2 \\
\ell'_1 - \ell'_0 & \geq 0 \\
\ell'_2 - \ell'_1 + \ell'_0 & \geq 2 \\
\ell'_3 - \ell'_2 + \ell'_1 - \ell'_0 & \geq 0 \\
\vdots \\
\ell'_0 - \ell'_1 + \ell'_2 - \ldots + (-1)^{f'_{-1}} & = 1
\end{align*}
\] (3.2)

together with \( i_0 = 2 \) are equivalent to
\[
\begin{align*}
i_0 & = 2 \\
i_1 - i_0 & \geq 0 \\
i_2 - i_1 + i_0 & \geq 2 \\
i_3 - i_2 + i_1 - i_0 & \geq 0 \\
i_4 - i_3 + i_2 - i_1 + i_0 & \geq 2 \\
\vdots \\
i_0 - i_1 + i_2 - \ldots + (-1)^{f} & = 2 - (i'_0 - i'_1 + i'_2 - \ldots + (-1)^{f'}_{-1}) = 2 - 1 = 1
\end{align*}
\] (3.3)

Since the result is true, by the inductive hypothesis, for \( r(I) \), we conclude using Lemma 3.3. Case B: \( i_0 \neq 2 \). Let \( I = (i_0, i_1, i_2, \ldots, i_f, 0, \ldots) \), with \( \sum_{k=0}^{\infty} i_k = 2n + 1 \) and \( i_0 \neq 2 \). By applying the operation \( b \), we obtain \( b(I) = (i_0 - 1, i_1 - 1, i_2, \ldots, i_f, 0, \ldots) = (i'_0, i'_1, \ldots, i'_{f'}, 0, \ldots) \). We have:
\[
\begin{align*}
\ell'_0 & = i_0 - 1 \\
\ell'_1 - \ell'_0 & = i_1 - 1 - i_0 + 1 = i_1 - i_0 \\
\ell'_2 - \ell'_1 + \ell'_0 & = i_2 - i_1 + 1 + i_0 - 1 = i_2 - i_1 + i_0 \\
\ell'_3 - \ell'_2 + \ell'_1 - \ell'_0 & = i_3 - i_2 + i_1 - 1 - i_0 + 1 = i_3 - i_2 + i_1 - i_0 \\
\vdots \\
\ell'_0 - \ell'_1 + \ell'_2 - \ldots + (-1)^{f'}_{-1} & = i_0 - 1 - i_1 + 1 + i_2 - \ldots + (-1)^{f} \\
& = i_0 - i_1 + i_2 - \ldots + (-1)^{f}.
\end{align*}
\]
Again, in this case the conditions

\[
\begin{align*}
\iota'_0 & \geq 2 \\
\iota'_1 - \iota'_0 & \geq 0 \\
\iota'_2 - \iota'_1 + \iota'_0 & \geq 2 \\
\iota'_3 - \iota'_2 + \iota'_1 - \iota'_0 & \geq 0 \\
\vdots \\
\iota'_0 - \iota'_1 + \iota'_2 - \ldots + (-1)^{f'} \iota'_f & = 1
\end{align*}
\]  

are equivalent to

\[
\begin{align*}
\iota_0 & > 2 \\
\iota_1 - \iota_0 & \geq 0 \\
\iota_2 - \iota_1 + \iota_0 & \geq 2 \\
\iota_3 - \iota_2 + \iota_1 - \iota_0 & \geq 0 \\
\iota_4 - \iota_3 + \iota_2 - \iota_1 + \iota_0 & \geq 2 \\
\vdots \\
\iota_0 - \iota_1 + \iota_2 - \ldots + (-1)^{f} \iota_f & = 1
\end{align*}
\]  

Because of induction, the result is true for \( b(\mathbf{I}) \), and Lemma 3.4 allows us to conclude.

The following are immediate consequences of the previous theorem.

**Theorem 3.6.** If a frame \( \mathbf{I} = (i_0, i_1, i_2, \ldots, i_f, \ldots) \) with length \( 2n \) is admissible, then:

1. \( i_{f-1} > i_f \);
2. \( i_0 = i_1 + 1 \iff i_1 = i_f \);
3. \( 2 \leq i_j \leq i_{j-1} + i_{j+1} - 2 \quad (0 < j < f - 1) \)

4 **Cardinality of a frame**

Recall the two operations defined on the set of Dyck paths:

- the lifting of a path \( \mathbf{u} \) denoted by \( s(\mathbf{u}) \);
- the gluing of two paths \( \mathbf{u}, \mathbf{v} \), denote by \( \mathbf{u} \wedge \mathbf{v} \).
With these two operations one may construct any Dyck path starting from shorter Dyck paths.

Therefore, every Dyck path may be expressed as

\[ s^{j_1}(x_1) \land s^{j_2}(x_2) \land \ldots \land s^{j_t}(x_t) \]

where each \( x_i \) is again expressible in the same way, with the condition that after a finite number of steps one arrives at expressions of the form

\[ s^{k_1}(0) \land s^{k_2}(0) \land \ldots \land s^{k_r}(0). \]

For example, let's consider the path

\[ UUDUDDDUUUDDDDDD \]

with length 22:

The corresponding sequence is

\[ 01212321010123232101210 \]

and it may be expressed as

\[ s(x_1) \land s(x_2) \land s(x_3) \land s(x_4), \]

where \( s(x_1) = 012123210, s(x_2) = 010, s(x_3) = 012323210, s(x_4) = 01210, \) with \( x_1 = 0101210, x_2 = 0, x_3 = 0121210, x_4 = 010. \) One has therefore:

\[ x_1 = s(y_1) \land s(y_2), x_3 = s(y_3), x_4 = s(0), \] with \( y_1 = 0, y_2 = 010, y_3 = 01010 \)

and so \( y_2 = s(0), y_3 = s(z_1) \land s(z_2), \) with \( z_1 = 0, z_2 = 0. \)

Finally, the assigned path can be expressed as

\[ s(s(0) \land s^2(0)) \land s(0) \land s^2(s(0) \land s(0)) \land s^2(0). \]

Since a frame is an eventually zero integer sequence, we may think of it as a polynomial. The constant polynomial 1 corresponds to the null frame. Thus \( s(1) \) is the frame of the unique path with length 2, while \( s^2(1) \) and \( s(1) \land s(1) \) are the frames of the length 4 paths, and so on. It is easy to check that, denoting by \( p(x) \) a frame, one has:

**Proposition 4.1.** \( s(p(x)) = 2 + xp(x), \quad p(x) \land q(x) = p(x) + q(x) - 1. \)

The procedure to determine the admissibility of a sequence, see Examples 3, 3, can be used to determine a “canonical” representative of a frame.
Example. Consider the frame \((3, 4, 3, 1, 0, \ldots)\) and apply to it the functions \(r\) and \(b\). We obtain the sequence of frames \((2, 3, 3, 1, 0, \ldots)\), \((3, 3, 1, 0, \ldots)\), \((2, 2, 1, 0, \ldots)\), \((2, 1, 0, \ldots)\), \((1, 0, \ldots)\), thus getting to the null frame. There is, of course, only one path corresponding to the null frame: the null path.

We may now trace this procedure backwards with the operations \(s\) and \(a\) on the paths, eventually getting a desired canonical representative of the frame \((3, 4, 3, 1, 0, \ldots)\).

We start form the null path.

- this lifted gives:
- with frame \(s(1) = (2, 1, 0, \ldots)\),

- lifted gives:
- with frame \(s^2(1) = (2, 2, 1, 0, \ldots)\),

- extended gives:
- with frame \(s^2(1) \land s(1) = (3, 3, 1, 0, \ldots)\),

- lifted gives:
- with frame \(s(s^2(1) \land s(1)) = (2, 3, 3, 1, 0, \ldots)\),

- extended gives:
- with frame \(s(s^2(1) \land s(1)) \land s(1) = (3, 4, 3, 1, 0, \ldots)\)

Applying this procedure to any frame \((i_0, i_1, \ldots, i_f, 0, \ldots)\), we reach a particular path belonging to this frame. This path is called the canonical representative of the frame \((i_0, i_1, \ldots, i_f, 0, \ldots)\). In the preceding example the canonical representative is therefore \(s(s^2(0) \land s(0)) \land s(0)\).

The procedure is such that any time in a canonical representative there is a product of the form \(s^{j_1}(x_1) \land s^{j_2}(x_2) \land \ldots \land s^{j_l}(x_l)\), then \(s^{j_2}(x_2) = \ldots = s^{j_l}(x_l) = s(0)\).

In other words, a canonical path is of the form

\[s^{j_1}(s^{j_2}(\ldots(s^{j_{l-1}}(s^{j_l}(0) \land s(0) \land \ldots \land s(0)) \land s(0) \land \ldots \land s(0)) \ldots \land s(0) \land \ldots \land s(0)).\]
Another way to describe the canonical path is to observe that in the corresponding sequence of U and D, after any sequence of one or more D there is at most one U.

**Example.** Assume the frame \((3, 6, 6, 3, 1, 0, \ldots)\) is given. By the preceding remarks to reach level 4 one must necessarily begin with a sequence of 4 U. Since there is a single foot at level 4 we must go down with at least 2 D. If we go down with 3 D, we could not go back to level 3, where there should be 3 feet. Hence we go down exactly two steps D and go back up with one U. So far the sequence is UUUDDU. Since we do not need to go back to level 3 any longer, we go down with at least 2 more D’s. Again in this case, considering that at level 2 we have 6 feet, we must go down exactly with 2 D, then go up with one U, then go down with one D and climb back up with one U two more times. We thus obtain UUUDDUDUDUDU so far. Having exhausted the level 2 feet, we must go down with 2 D to level 0. Having obtained so far only 5 feet at level 1, we must still go up with U and the concluding with a D. The sequence is therefore UUUDDUDUDUDUD.

The following property is often useful in computations.

**Proposition 4.2.** \(s(p(x) \land q(x)) \land s(1) = s(p(x)) \land s(q(x)).\)

**Proof.** Use Proposition 4.1: \(s(p(x) \land q(x)) \land s(1) = s(p(x) + q(x) - 1) \land (2 + x) = (2 + x(p(x) + q(x) - 1)) \land (2 + x) = 2 + x(p(x) + q(x) - 1) + (2 + x) - 1 = 2 + xp(x) + 2 + xq(x) - 1 = s(p(x)) \land s(q(x)).\)

From this, it immediately follows:

**Corollary 4.3.** Given two Dyck paths \(x_1, x_2\), the two paths \(s(x_1) \land s(x_2), s(x_1 \land x_2) \land s(0)\) have the same frame.

**Theorem 4.4.** It is possible to obtain the canonical representative of a frame \((i_0, i_1, \ldots, i_f, 0, \ldots)\) in a finite number of steps starting from any representative path \(x\) using the commutativity property of the gluing operation and Corollary 4.3.

**Proof.** If \(x\) is a representative of the frame, suppose that \(x\) contains a product \(s(0) \land s^i(y)\), then this can be transformed into \(s^i(y) \land s(0)\). If it contains a product \(s^i(y) \land s^j(z)\), with \(s^i(y) \neq s(0), s^j(z) \neq s(0)\), this can be transformed into \(s(s^{i-1}(y) \land s^{j-1}(z)) \land s(0)\). After a finite number of steps of these two types, one gets to a canonical path when none of these two steps is any longer possible.
Using Theorem 4.4, we are now able to count the number of paths in a given frame.

**Example.** Consider the frame \((3,4,3,1,0,\ldots)\) as before. We already saw that its canonical representative is \(u = s(s^2(0) \land s(0)) \land s(0)\). From this canonical representative, using commutativity of the wedge operation, we have a total of 4 paths, including \(u\), precisely:

\[
\begin{align*}
&\text{s}(s^2(0) \land s(0)) \land s(0),
&\text{s}(s(0) \land s^2(0)) \land s(0),
&\text{s}(0) \land s(s^2(0) \land s(0)),
&\text{s}(0) \land s(s(0) \land s^2(0)).
\end{align*}
\]

Using Corollary 4.3 we have also \(v = s^3(0) \land s^2(0)\), which gives rise, by the commutativity of \(\land\), to a total of 2 paths, including \(v\):

\[
\begin{align*}
&\text{s}^3(0) \land s^2(0),
&\text{s}^2(0) \land s^3(0).
\end{align*}
\]

We have therefore a total of 6 paths belonging to the frame \((3,4,3,1,0,\ldots)\).

We wish to compute an explicit formula for the number \(p_2^n\) of paths having frame \(I = (i_0,i_1,\ldots)\) with length \(2n\).

**Definition 4.5.** We define a right frame to be any frame of the form

\[(2,i_1,i_2,i_3,\ldots)\]

and a left frame any other frame.

**Definition 4.6.** Given a frame \((i_0,i_1,i_2,\ldots)\), with \(i_0 \geq 2\), we define its right progenitor to be the frame \((2,i_1-i_0+2,i_2,\ldots)\). If the frame has \(i_0 = i_1 = \ldots = i_{t-1} = 2\) and \(i_t \neq 2\) we define its left progenitor to be \((i_t,i_{t+1},\ldots)\). If \(i_0 \neq 2\) its left progenitor is itself.

**Remark 4.7.** Every frame has a right and left progenitor except for the null frame which has only itself as a left progenitor and no right progenitor.

We can prove two propositions describing how the cardinality of a frame \(I\) is related to the one of its progenitors.

**Proposition 4.8.** The cardinality of a frame \(I\) is the same as that of its left progenitor.
Proof. It is easy to see that there is a bijection between the set of paths realizing the frame \(I\) and those realizing the frame \(s(I)\). Indeed, if \(p\) is a path of the frame \(I\) then \(s(p)\) is a path of the frame \(s(I)\), moreover \(s\) is an invertible operation. By repeatedly applying this argument we reach its left progenitor.

Obviously,

**Lemma 4.9.** The cardinality of the frame \((u + 1, u, 0, \ldots)\), \(\forall u \in \mathbb{N}\), is 1.

**Proposition 4.10.** Given a frame of the form \(X = (2 + n, a, b, \ldots)\) with \(n \geq 0\), let \(Y = (2, a - n, b, \ldots)\) be its right progenitor and \(k\) the cardinality of \(Y\). Then the cardinality of \(X\) is

\[
k \binom{a - 1}{a - n - 1}.
\]

**Proof.** If \(b = 0\) Theorem 3.6 implies that the frame is of the form \((u + 1, u, 0, \ldots)\). The right progenitor is \((2, 1, 0, \ldots)\) with cardinality \(k = 1\) and the formula holds by Lemma 4.9. If \(b > 0\) then \(a \geq 2 + n\), by Theorem 3.5. There is a bijection between the set of paths realizing the frame \(X\) and the weak compositions of \(n\) into \(a - n\) parts. Hence the formula. In fact, there are \(a - n\) positions in \(Y\), \((a - n - 2\) feet at level 1 of the path plus the two end-points at level 0) in which one can distribute the \(n\) paths \(s(0)\), using Proposition 4.2 or the commutativity property. This means that each of the \(n\) “hats” must be located in \(a - n\) positions.

In other words, any path in \(Y\) gives rise to a path in the frame \(X\) by inserting \(n\) pairs 01. These pairs may be distributed either at the beginning, in the order 01, or at the end as 10, or in the interior for every possible pair 12 as 1012. We are left to prove that if we take two different paths \(p\) and \(p'\) in the frame \(Y\) they give rise to different paths in \(X\). Notice that there are no 0’s in the middle of the associated sequences to \(p\) and \(p'\). In the first position where they differ, one has the pair \(a, a + 1\) and the other has \(a, a - 1\) where \(a \geq 2\). If \(a > 2\), a pair 01 or 10 cannot be inserted after \(a\) so the paths remain different. If \(a = 2\) then we have the sequence \(\ldots abc21\ldots\) and \(\ldots abc23\ldots\). The insertion of pairs 1,0 or 0,1 cannot turn these two sequences into equal ones.

**Theorem 4.11.** Given the frame \(I = (i_0, i_1, \ldots, i_f, 0\ldots)\), setting:

\[
j_1 = i_0 - 2,
\]
\[
j_2 = i_1 - i_0,
\]
\[
j_3 = i_2 - i_1 + i_0 - 2,
\]

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\[ j_4 = i_3 - i_2 + i_1 - i_0, \]

\[ \ldots \]

the cardinality of \( I \) is

\[
\begin{pmatrix}
    i_1 - 1 \\
    i_1 - j_1 - 1
\end{pmatrix}
\begin{pmatrix}
    i_2 - 1 \\
    i_2 - j_2 - 1
\end{pmatrix}
\cdots
\begin{pmatrix}
    i_f - 1 \\
    i_f - j_f - 1
\end{pmatrix}.
\]

**Proof.** Step \( A_1 \): subtract \( j_1 \) from the first two elements of \( I \). We get the right progenitor of \( I \):

\[ x_1 = b^{j_1}(I) = (2, i_1 - j_1, i_2, \ldots) = (2, i_1 - i_0 + 2, i_2, \ldots) = (2, 2 + j_2, i_2, \ldots). \]

Step \( B_1 \): remove the initial 2 obtaining:

\[ y_1 = r(x_1) = (2 + j_2, i_2, \ldots). \]

Notice that if \( j_2 > 0 \) then \( y_1 \) is the left progenitor of \( x_1 \).

By the previous propositions we have:

\[
| x_1 | = | y_1 |
\]

\[
| I | = \begin{pmatrix}
    i_1 - 1 \\
    i_1 - j_1 - 1
\end{pmatrix} | x_1 | = \begin{pmatrix}
    i_1 - 1 \\
    i_1 - j_1 - 1
\end{pmatrix} | y_1 |.
\]

The second identity follows from Proposition 4.10 if \( j_1 > 0 \) while it is trivial if \( j_1 = 0 \) (in this case \( x_1 = I \)).

For \( k > 1 \), step \( A_k \): subtract \( j_k \) to the first two elements of \( y_{k-1} \). We get the right progenitor of \( y_{k-1} \):

\[ x_k = b^{j_k}(y_{k-1}) = (2, i_k - j_k, i_{k+1}, \ldots) = (2, 2 + j_{k+1}, i_{k+1}, \ldots). \]

Step \( B_k \): Remove the initial 2 to obtain:

\[ y_k = r(x_k) = (2 + j_{k+1}, i_{k+1}, \ldots). \]

Notice that if \( j_{k+1} > 0 \) then \( y_k \) is the left progenitor of \( x_k \). By the previous propositions we have:

\[
| x_k | = | y_k |
\]

\[
| y_{k-1} | = \begin{pmatrix}
    i_k - 1 \\
    i_k - j_k - 1
\end{pmatrix} | x_k | = \begin{pmatrix}
    i_k - 1 \\
    i_k - j_k - 1
\end{pmatrix} | y_k |.
\]
The second identity follows from Proposition 4.10 if \( j_k > 0 \) while it is trivial if \( j_k = 0 \) (in such case \( x_k = y_{k-1} \)).

We may deduce:

\[
|I| = \left( \frac{i_1 - 1}{i_1 - j_1 - 1} \right) \cdots \left( \frac{i_k - 1}{i_k - j_k - 1} \right) |y_k|.
\]

Finally, Step \( A_{f-1} \): subtract \( j_{f-1} \) to the first two elements of \( y_{f-2} \). We get the right progenitor of \( y_{f-2} \):

\[
x_{f-1} = b^{j_{f-1}}(y_{f-2}) = (2, i_{f-1} - j_{f-1}, i_f, 0, ...) = (2, 2 + j_f, i_f, 0, ...).
\]

Step \( B_{f-1} \): Remove the initial 2 to obtain:

\[
y_{f-1} = r(x_{f-1}) = (2 + j_f, i_f, 0, ...),
\]

with \( |x_{f-1}| = |y_{f-1}| \) and

\[
|y_{f-2}| = \left( \frac{i_{f-1} - 1}{i_{f-1} - j_{f-1} - 1} \right) |x_{f-1}| = \left( \frac{i_{f-1} - 1}{i_{f-1} - j_{f-1} - 1} \right) |y_{f-1}|,
\]

hence

\[
|I| = \left( \frac{i_1 - 1}{i_1 - j_1 - 1} \right) \cdots \left( \frac{i_{f-1} - 1}{i_{f-1} - j_{f-1} - 1} \right) |y_{f-1}|.
\]

By Proposition 3.5, \( y_{f-1} \) is necessarily of the form \((u + 1, u, 0, ...),\) with right progenitor \( x_f = (2, 1, 0, ...), \) which has cardinality 1. By Proposition 4.10, we have:

\[
|y_{f-1}| = |x_f|\binom{u-1}{0} = 1.
\]

Finally, since Theorem 3.5 implies \( i_f - j_f - 1 = 0, \) we have \( \binom{i_f-1}{i_f-j_f-1} = \binom{i_f-1}{0} = 1, \) hence the conclusion. \( \qed \)

**Example.** The cardinality of the frame \( I = (5, 8, 7, 3) \) is

\[
\left( \begin{array}{c}
7 \\
4 \\
3 \\
0
\end{array} \right) = 700
\]

### 5 Colored Dyck paths

For any fixed frame \((i_0, \ldots, i_f, \ldots)\) with length \( 2n \), denote by \( v_k \) the number of U steps joining the levels \( k, k+1 \) \((k = 0, ..., f-1)\). Such number is clearly equal to the number of the D steps joining the same two levels.

**Theorem 5.1.** In the above notation:

\[
v_k = i_k - i_{k-1} + \ldots + (-1)^k i_0 + (-1)^{k+1}, \quad k = 0, ..., f - 1.
\]
Proof. Since at level 0 from the first node we have a U step, and we also have a D step in the final node, while in all the others we certainly have a U and a D, we have a total of \(2(i_0 - 1)\) steps joining level 0 and level 1. Since there are as many U as D we must have: \(v_0 = i_0 - 1\). Assume we proved the formula up to \(v_{k-1}\), we prove it for \(v_k\). From the \(i_k\) nodes at level \(k\) we have \(2i_k\) steps, half of which are U and half are D. Of these, \(2v_{k-1}\) come from the lower level. It follows that \(v_k = i_k - v_{k-1} = i_k - (i_{k-1} - ... + (-1)^{k-1}i_0 + (-1)^k)\), hence the result. \(\blacksquare\)

Notice that the number \(v_k\) depends only on the given frame and not on the particular path in that frame.

Denote by \(I_{2n}\) the set of frames with length \(2n\). Suppose that we can color with \(u_k\) colors the U steps joining level \(k\) and level \(k+1\) and with \(d_k\) colors the D steps joining the same levels \((k = 0, ..., n-1)\). The number of colored paths of a given frame \(I = (i_0, i_1, \ldots)\) with length \(2n\), is clearly \(p_I^{2n} u_0^{v_0} \cdots u_{n-1}^{v_{n-1}} d_0^{v_0} \cdots d_{n-1}^{v_{n-1}}\), recalling that \(p_I^{2n}\) denotes the number of elements in the frame \(I = (i_0, i_1, \ldots)\) with length \(2n\). We immediately have

**Theorem 5.2.** The number of Dyck paths with length \(2n\), that can be colored with \(u_k\) and \(d_k\) colors from level \(k\) to level \(k+1\) \((k = 0, ..., n-1)\) is

\[
\sum_{I \in I_{2n}} p_I^{2n} u_0^{v_0} \cdots u_{n-1}^{v_{n-1}} d_0^{v_0} \cdots d_{n-1}^{v_{n-1}}. 
\]

Notice that, the number \(v_k\) depends also on the frame \(I\). However, to explicitly indicate such relation in the notation would make for a quite cumbersome symbol such as \(v_k(p_I^{2n})\).

### 6 Colored Motzkin paths

**Proposition 6.1.** If \(m^{(n)}\) is the number of Motzkin paths with length \(n\), and we set \(\nu = \left[\frac{n}{2}\right]\), then

\[
m^{(n)} = \sum_{j=0}^{\nu} \sum_{I \in I_{2j}} p_I^{2j} \binom{n}{n-2j}. \tag{6.1}
\]

**Proof.** Analogously to what was observed in the case of Theorem 2.7, a Motzkin path with length \(n\) may be obtained from a Dyck path \(p\) with length \(2j\), \(0 \leq 2j \leq n\), by adding \(n - 2j\) horizontal steps at the feet at various levels of \(p\). If \(p\) has frame \(I = (i_0, \ldots, i_j)\), we may distribute the \(n - 2j\) horizontal
steps at each of the $2j + 1$ nodes of $p$. So this can be done in as many ways as the possibility to express $n - 2j$ as a sum of $2j + 1$ integers between 0 and $n - 2j$, that is, in $\binom{n-2j+(2j+1)-1}{n-2j} = \binom{n}{n-2j}$ ways. The statement follows. \(\square\)

The $n - 2j$ horizontal steps can be distributed as follows: $k_0$ steps at each of the $i_0$ nodes of level 0, $k_1$ steps at the $i_1$ nodes of level 1, and so on. The number of such possible arrangements are counted by weak compositions of suitable integers. If at each level $r$ one uses $h_r$ colors on the horizontal steps and denotes by $\mathcal{H} = (h_0, \ldots, h_\nu)$, the formula becomes

**Proposition 6.2.**

$$m^{(n)}_{\mathcal{H}} = \sum_{j=0}^\nu \sum_{I \in I_{2j}} p_I^{2j} \sum_{k_0 + \cdots + k_\nu = n-2j} \binom{k_0 + i_0 - 1}{k_0} \cdots \binom{k_\nu + i_\nu - 1}{k_\nu} h_0^{k_0} \cdots h_\nu^{k_\nu}.$$ 

It may happen that in the summation corresponding to a frame, the frame is too "low" to allow adding horizontal steps, for example at level $\nu$. In this case the corresponding binomial coefficient $\binom{k_\nu + i_\nu - 1}{k_\nu}$ has $i_\nu = 0$ and is therefore zero.

If we add to this the possibility of coloring the U and D steps with $u_k$ and $d_k$ colors from level $k$ to level $k + 1$ ($k = 0, \ldots, \nu - 1$), and we denote by $\mathcal{U} = (u_0, \ldots, u_{\nu-1}), D = (d_0, \ldots, d_{\nu-1})$, one gets

**Theorem 6.3.** Denoting by $m^{(n)}_{\mathcal{H}, \mathcal{U}, \mathcal{D}}$ the number of Motzkin paths colored with colors determined by $\mathcal{H}, \mathcal{U}, \mathcal{D}$, we have that $m^{(n)}_{\mathcal{H}, \mathcal{U}, \mathcal{D}}$ is equal to

$$\sum_{j=0}^\nu \sum_{I \in I_{2j}} p_I^{2j} u_0^{i_0} \cdots u_{j-1}^{i_{j-1}} d_0^{i_0} \cdots d_{j-1}^{i_{j-1}} \sum_{k_0 + \cdots + k_\nu = n-2j} \binom{k_0 + i_0 - 1}{k_0} \cdots \binom{k_\nu + i_\nu - 1}{k_\nu} h_0^{k_0} \cdots h_\nu^{k_\nu}.$$ 

(6.2)

Notice that such formula makes sense if we set $h_k = 0$ for all those levels $k$ where there are no horizontal steps provided we attribute value 1 to the expression $h_j^{k_j}$, if $h_j = 0$ and $k_j = 0$.

**Remark 6.4.** Comparing formula (6.1) with (6.2) written in the case of all $h_k = u_k = d_k = 1$ yields the following identity for binomial coefficients, where we made obvious changes of symbols:

$$\binom{m + i_0 + \cdots + i_\nu - 1}{m} = \sum_{k_0 + \cdots + k_\nu = m} \binom{k_0 + i_0 - 1}{k_0} \cdots \binom{k_\nu + i_\nu - 1}{k_\nu}.$$ 

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