UNIQUENESS OF $E_\infty$ STRUCTURES FOR CONNECTIVE COVERS

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Abstract. We refine our earlier work on the existence and uniqueness of $E_\infty$ structures on $K$-theoretic spectra to show that at each prime $p$, the connective Adams summand $\ell$ has a unique structure as a commutative $S$-algebra. For the $p$-completion $\ell_p$ we show that the McClure-Staffeldt model for $\ell_p$ is equivalent as an $E_\infty$ ring spectrum to the connective cover of the periodic Adams summand $L_p$. We establish a Bousfield equivalence between the connective cover of the Lubin-Tate spectrum $E_n$ and $BP(n)$.

Introduction

The aim of this short note is to establish the uniqueness of $E_\infty$ structures on connective covers of certain periodic commutative $S$-algebras $E$, most prominently for the connective $p$-complete Adams summand. It is clear that the connective cover of an $E_\infty$ ring spectrum inherits a $E_\infty$ structure; there is even a functorial way of assigning a connective cover within the category of $E_\infty$ ring spectra [9, VII.3.2]. But it is not obvious in general that this $E_\infty$ multiplication is unique.

Our main concern is with examples in the vicinity of $K$-theory; we apply our uniqueness theorem to real and complex $K$-theory and their localizations and completions and to the Adams summand and its completion.

The existence and uniqueness of $E_\infty$ structures on the periodic spectra $KU$, $KO$ and $L$ was established in [5] by means of the obstruction theory for $E_\infty$ structures developed by Goerss-Hopkins [8] and Robinson [12]. Note however, that obstruction theoretic methods would fail in the connective cases. Let $e$ be a commutative ring spectrum. If $e$ satisfies some Künneth and universal coefficient properties [12, proposition 5.4], then the obstruction groups for $E_\infty$ multiplications consist of André-Quillen cohomology groups in the context of differential graded $E_\infty$ algebras applied to the graded commutative $e_\ast$-algebra $e_\ast e$. Besides problems with non-projectivity of $e_\ast e$ over $e_\ast$, the algebra structures of $ku_\ast ku$, $ko_\ast ko$ and $\ell_\ast \ell$ are far from being étale and therefore one would obtain non-trivial obstruction groups. One would then have to identify actual obstruction classes in these obstruction groups in order to establish the uniqueness of the given $E_\infty$ structure – but at the moment, this seems to be an intractable problem. Thus an alternative approach is called for.

In Theorem 1.3 we prove that a unique $E_\infty$ structure on $E$ gives rise to a unique structure on the connective cover if $E$ is obtained from some connective spectrum via a process of Bousfield localization. In particular, we identify the $E_\infty$ structure on the $p$-completed connective Adams summand $\ell_p$ provided by McClure and Staffeldt in [10] with the one that arises by taking the unique $E_\infty$ structure on the periodic Adams summand $L = E(1)$ developed in [5] and taking its connective cover.

Our Theorem applies as well to the connective covers of the Lubin-Tate spectra $E_n$ and we prove in section 2 that these spectra are Bousfield equivalent to the truncated Brown-Peterson spectra $BP(n)$. Unlike other spectra that are Bousfield equivalent to $BP(n)$, such as the connective cover of the completed Johnson-Wilson spectrum, $\tilde{E}(n)$, the connective cover of

2000 Mathematics Subject Classification. 55P43; 55N15.

We are grateful to John Rognes who suggested to exploit the functoriality of the connective cover functor to obtain uniqueness of $E_\infty$ structures. The first author thanks the Max-Planck Institute and the mathematics department in Bonn. The second author was partially supported by the Strategisk Universitetsprogram i Ren Matematikk (SUPREMA) of the Norwegian Research Council.
Let us first make explicit what we mean by uniqueness of $E_\infty$ structures. We admit that this is an \textit{ad hoc} notion, but it suffices for the examples we want to consider.

\textbf{Definition 1.1.} In the following, we will say that an $E_\infty$ structure on some homotopy commutative and associative ring spectrum $E$ is unique if whenever there is a map of ring spectra $\varphi: E' \to E$ from some other $E_\infty$ ring spectrum $E'$ to $E$ which induces an isomorphism on homotopy groups, then there is a morphism in the homotopy category of $E_\infty$ ring spectra $\varphi': E' \to E$ such that $\pi_*(\varphi) = \pi_*(\varphi')$.

If $E$ and $F$ are spectra whose $E_\infty$ structure was provided by the obstruction theory of Goerss and Hopkins \cite{G-H II}, then we can compare our uniqueness notion with theirs. Note that examples of such $E_\infty$ ring spectra include $E_n$ \cite[7.6]{G-H I}, $KO$, $KU$, $L$ and $E(n)$ \cite{S}. In such cases the Hurewicz map

\begin{equation}
\text{Hom}_{E_\infty}(E, F) \xrightarrow{h} \text{Hom}_{F_*-\text{alg}}(F_*E, F_*)
\end{equation}

is an isomorphism. Assume that we have a mere ring map $\varphi$ as above between $E$ and $F$. This gives rise to a map of $F_*$-algebras from $F_*E$ to $F_*$ by composing $F_*(\varphi)$ with the multiplication $\mu$ in $F_*F$. The left hand side in (1.1) denotes the derived space of $E_\infty$ maps from $E$ to $F$. In presence of a universal coefficient theorem we have $\text{Hom}_{F_*-\text{hom}}(F_*E, F_*) = [E, F]$, therefore the element $\mu \circ F_*(\varphi)$ gives rise to a homotopy class of maps of ring spectra $\tilde{\varphi}$ from $E$ to $F$. We can assume that we have functorial cofibrant replacement $Q(-)$, hence we obtain a ring map $Q(\tilde{\varphi})$ from $Q(E)$ to $Q(F)$. Via the isomorphism (1.1) this gives a map of $E_\infty$ spectra from $Q(E)$ to $Q(F)$, $\Phi$, therefore we obtain a zigzag

\[
\begin{array}{ccc}
Q(E) & \xrightarrow{\Phi} & Q(F) \\
\sim & & \sim \\
E \xrightarrow{\varphi} & F
\end{array}
\]

of weak equivalences of $E_\infty$ spectra from $E$ to $F$. Thus in such cases our definition agrees with the uniqueness notion that is natural in the Goerss-Hopkins setting.

For the rest of the paper we assume the following.

\textbf{Assumption 1.2.} Let $E$ be a periodic commutative $S$-algebra with periodicity element $v \in E_*$ of positive degree. We will view $E$ as being obtained from a connective commutative $S$-algebra $e$ by Bousfield localization at $e[v^{-1}]$ in the category of $e$-modules. Furthermore we assume that the localization map induces an isomorphism between the homotopy groups of $e$ and the homotopy groups of the connective cover of $E$.

Let us denote the connective cover functor from \cite[VII.3.2]{H} by $c(-)$. For any $E_\infty$ ring spectrum $A$, there is a weakly equivalent commutative $S$-algebra $B(\mathbb{P}, \mathbb{P}, \mathbb{L})(A)$, with equivalence

\[
\lambda: B(\mathbb{P}, \mathbb{P}, \mathbb{L})(A) \xrightarrow{\sim} A,
\]

in the $E_\infty$ category \cite[XII.1.4]{H}. Here $B(\mathbb{P}, \mathbb{P}, \mathbb{L})$ is a bar construction with respect to the monad associated to the linear isometries operad $L$ and the monad for commutative monoids in the category of $S$-algebras $\mathbb{P}$. We will denote the composite $B(\mathbb{P}, \mathbb{P}, \mathbb{L}) \circ c$ by $\bar{c}$. For a commutative $S$-algebra $R$ and an $R$-module $M$, let $L_M^R(-)$ denote Bousfield localization at $M$ in the category of $R$-modules and we denote the localization map by $\sigma: E \to L_M^R(E)$ for any $R$-module $E$.

\textbf{Theorem 1.3.} Assume that we know that the $E_\infty$ structure on $E$ is unique. Then the $E_\infty$ structure on $c(E)$ is unique.
Proof. Each commutative \( \mathcal{S} \)-algebra can be viewed as an \( E_\infty \) ring spectrum. Let \( e' \) be a model for the connective cover \( c(E) \), i.e., \( e' \) is an \( E_\infty \) ring spectrum with a map of ring spectra \( \varphi \) to \( c(E) \), such that \( \pi_* \varphi \) is an isomorphism. Write \( v \in e'_p \) for the isomorphic image of \( v \) under the inverse of \( \pi_* \varphi \). As \( \varphi \) is a ring map it will induce a ring map on the corresponding Bousfield localizations. But as the \( E_\infty \) structure on \( E \) is unique by assumption, this map can be replaced by an equivalent equivalence, \( \xi \), of \( E_\infty \) ring spectra. We abbreviate \( \mathcal{B}(\mathcal{P}, \mathcal{P}, \mathcal{L})(e') \) to \( B(e') \).

We consider the following diagram whose dotted lines provide a zigzag of \( E_\infty \) equivalences and hence a map in the homotopy category of \( E_\infty \) ring spectra.

\[
\begin{array}{c}
\mathcal{B}(e') & \xrightarrow{c} & c(B(e')) \\
\sigma & \xrightarrow{c} & c(e) \\
L_{B(e')}^{B(e')} & \xrightarrow{c} & E \\
\sigma & \xrightarrow{c} & c(\xi) \\
\end{array}
\]

\(\square\)

Real and complex \( K \)-theory, \( ko \) and \( ku \), have \( E_\infty \) structures obtained using algebraic \( K \)-theory models \( [3] \) VIII, §2. The connective Adams summand \( \ell \) has an \( E_\infty \) structure because it is the connective cover of \( E(1) \). In the following we will refer to these models as the standard ones. The \( E_\infty \) structures on \( KO \) and \( E(1) \) are unique by \( [3] \) theorems 7.2, 6.2. In all of these cases, the periodic versions are obtained by Bousfield localization \( [7] \) VIII.4.3.

**Corollary 1.4.** The \( E_\infty \) structures on \( ko \), \( ku \) and \( \ell \) are unique.

In \( [10] \), McClure and Staffeldt construct a model for the \( p \)-completed connective Adams summand using algebraic \( K \)-theory of fields. Let \( \hat{\ell} = K(\mathbb{F}) \), the algebraic \( K \)-theory spectrum of \( \mathbb{F} = \bigcup_i \mathbb{F}_{q^i} \), where \( q \) is a prime which generates the \( p \)-adic units \( \mathbb{Z}_p^\times \). Then the \( p \)-completion of \( \hat{\ell} \) is additively equivalent to the \( p \)-completed connective Adams summand \( \ell_p \) \( [10] \) proposition 9.2. For further details see also \( [2] \) §1. Note that the \( p \)-completion \( \ell_p \) inherits an \( E_\infty \) structure from \( \ell \) because \( p \)-completion is Bousfield localization with respect to \( HF_p \) and therefore preserves commutative \( \mathcal{S} \)-algebras \( [7] \) VIII.2.2. An a priori different model for the \( p \)-completion of the connective Adams summand can be obtained by taking the connective cover of the \( p \)-complete periodic version \( L = E(1) \). This is consistent with the statement of Corollary 1.4 because \( p \)-completion and Bousfield localization of \( \ell \) in the category of \( \ell \)-modules with respect to \( L \) are compatible in the following sense. Consider \( \ell = \tilde{c}(L) \) and its \( p \)-completion

\[\lambda_L: \ell \to \ell_p = (\tilde{c}(L))_p.\]

The \( p \)-completion map \( \lambda \) is functorial in the spectrum, therefore the following diagram of solid arrows commutes.

\[
\begin{array}{c}
\ell = \tilde{c}(L) & \xrightarrow{\lambda_L} & \ell_p = \tilde{c}(L)_p & \xrightarrow{\lambda} & \tilde{c}(L)_p \\
L & \xrightarrow{\lambda_L} & L_p
\end{array}
\]

The universal property of the connective cover functor ensures that there is a map in the homotopy category of commutative \( \mathcal{S} \)-algebras from \( \ell_p \) to \( \tilde{c}(L_p) \) which is a weak equivalence. In the following we will not distinguish \( \ell_p \) from \( \tilde{c}(L_p) \) anymore and denote this model simply by \( \ell_p \).
Proposition 1.5. The McClure-Staffeldt model $\tilde{\ell}_p$ of the $p$-complete connective Adams summand is equivalent as an $E_\infty$ ring spectrum to $\ell_p$.

Remark 1.6. If $E$ is a commutative $S$-algebra with naive $G$-action for some group $G$, then neither the connective cover functor $\tilde{c}(-)$ nor Bousfield localization of $E$ has to commute with taking homotopy fixed points. As an example, consider connective complex $K$-theory $ku$ with the conjugation action by $C_2$. The homotopy fixed points $ku^{hc_2}$ are not equivalent to $ko$, but on the periodic versions we obtain $KU^{hc_2} \simeq KO$.

of Proposition 1.6 Consider the algebraic $K$-theory model for connective complex $K$-theory, $ku = K(k)$, with $k = \bigcup_i \mathbb{F}_{q^i(p-1)}$. The canonical inclusions $\mathbb{F}_{q^i} \hookrightarrow \mathbb{F}_{q^i(p-1)}$ assemble into a map $j: k' \to k$. The Galois group $C_{p-1}$ of $k$ over $k'$ acts on $k$ and induces an action on algebraic $K$-theory. As $k'$ is fixed under the action of $C_{p-1}$ there is a factorization of $K(j)_p$ as

$$K(k')_p \xrightarrow{i} (K(k)_p)^{hc_{p-1}} \to K(k)_p \to KU_p.$$  

and $i$ yields a weak equivalence of commutative $S$-algebras, where $K(k)_p^{hc_{p-1}}$ is a model for the connective $p$-complete Adams summand which is weakly equivalent to $\ell_p$ (see [2, §1]).

Consider the composition of the following chain of maps between commutative $S$-algebras:

$$K(k')_p \xrightarrow{i} (K(k)_p)^{hc_{p-1}} \to K(k)_p \to KU_p.$$  

The target $KU_p$ is as well the target of the map $\tilde{c}(KU_p) \to KU_p$. Note that the universal property of $\tilde{c}(-)$ yields a zigzag $\varsigma: K(k)_p \leftrightarrow \tilde{c}(KU_p)$ of equivalences between $K(k)_p$ and $\tilde{c}(KU_p)$ in the category of commutative $S$-algebras.

As $KU_p$ is the Bousfield localization of $K(k)_p$ in the category of $K(k)_p$-modules with respect to the Bott element,

$$KU_p = L_{K(k)_p[\beta^{-1}]}^K K(k)_p,$$

it inherits the $C_{p-1}$-action on $K(k)_p$. The functoriality of the connective cover lifts this action to an action on $\tilde{c}(KU_p)$.

The connective cover functor is in fact a functor in the category of commutative $S$-algebras with multiplicative naive $G$-action for any group $G$. To see this we have to show that the map $\tilde{c}(A) \to A$ is $G$-equivariant if $A$ is a commutative $S$-algebra with an underlying naive $G$-spectrum. The functor $B(\mathbb{P}, \mathbb{P}, L)$ does not cause any problems. Proving the claim for the functor $c$ involves chasing the definition given in [3, VII, §3].

The prespectrum underlying $c(A)$ applied to an inner product space $V$ is defined as $T(A_0)(V)$, where $A_0$ is the zeroth space of the spectrum $A$ and $T$ is a certain bar construction involving suspensions and a monad consisting of the product of a fixed $E_\infty$ operad with the partial operad of little convex bodies $\mathcal{K}$. For a fixed $V$ the suspension $\Sigma V$ and the operadic term $\mathcal{K}_V$ are used. As the $G$-action is compatible with the $E_\infty$ and the additive structure of $A$, the evaluation map $T(A_0)(V) \to A(V)$ is $G$-equivariant. For varying $V$, these maps constitute a map of prespectra and its adjoint on the level of spectra is $c(A) \to A$. As the spectification functor preserves $G$-equivariance, the claim follows. Therefore the resulting zigzag $\varsigma: K(k)_p \leftrightarrow \tilde{c}(KU_p)$ is $C_{p-1}$-equivariant and we obtain an induced zigzag on homotopy fixed points,

$$\varsigma^{hc_{p-1}}: (K(k)_p)^{hc_{p-1}} \leftrightarrow (\tilde{c}(KU_p))^{hc_{p-1}}.$$  

As $\varsigma$ is an isomorphism in the homotopy category and is $C_{p-1}$-equivariant, $\varsigma^{hc_{p-1}}$ yields an isomorphism as well. □
Goerss and Hopkins proved in \[3\] that the Lubin-Tate spectra \(E_n\) with
\[
(E_n)_* = W(F_p^n)[[u_1, \ldots, u_{n-1}]][u^{-1}]
\]
possess unique \(E_\infty\) structures for all primes \(p\) and all \(n \geq 1\). The connective cover \(c(E_n)\) has coefficients
\[
(c(E_n))_* = W(F_p^n)[[u_1, \ldots, u_{n-1}]][u^{-1}] \quad \text{with} \quad |u_i| = 0 \quad \text{and} \quad |u| = -2.
\]
Of course \(c(E_n)((u^{-1})^{-1}) \sim E_n\).

The spectra \(BP(n)\) can be built from the Brown-Peterson spectrum \(BP\) by killing all generators of the form \(v_m\) with \(m > n\) in \(BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots]\). Using for instance Angeltveit’s result \[1\] theorem 4.2 one can prove that the \(BP\) spectra are \(A_\infty\) spectra and from \[3\] it is known that this \(S\)-algebra structure can be improved to an \(MU\)-algebra structure. On the other hand, Strickland showed in \[13\] that \(BP(n)\) with \(n \geq 2\) is not a homotopy commutative \(MU\)-ring spectrum for \(p = 2\). We offer \(c(E_n)\) as a replacement for the \(p\)-completion \(BP(n)_p\) of \(BP(n)\).

We also need to recall that in the category of \(MU\)-modules, \(E(n)\) is the Bousfield localization of \(BP(n)\) with respect to \(BP(n)[v_n^{-1}]\), hence by \[7\] it inherits the structure of an \(MU\)-algebra and the natural map \(BP(n) \to E(n)\) is a morphism of \(MU\)-algebras. Furthermore, the Bousfield localization of \(E(n)\) with respect to the \(MU\)-algebra \(K(n)\) is the \(I_n\)-adic completion \(\widehat{E(n)}\), which was shown to be a commutative \(S\)-algebra in \[5\], and the natural map \(\widehat{E(n)} \to E(n)\) is a morphism of commutative \(S\)-algebras, see for example \[6\] example 2.2.6. Thus there is a morphism of ring spectra \(BP(n) \to E_n\) which lifts to a map \(BP(n) \to c(E_n)\).

**Proposition 2.1.** The spectra \(BP(n)\) and \(BP(n)_p\) are Bousfield equivalent to \(c(E_n)\).

**Proof.** On coefficients, we obtain a ring homomorphism \((BP(n)_p)_* \to (c(E_n))_*\) which on homotopy is given by
\[
v_k \mapsto \begin{cases} u_1^{-p^k}u_k & \text{for } 1 \leq k \leq n-1, \\ u_1^{-p^n} & \text{for } k = n. \end{cases}
\]
extending the natural inclusion of the \(p\)-adic integers \(\mathbb{Z}_p = W(F_p)\) into \(W(F_p^n)\). This homomorphism is induced by a map of ring spectra.

Recall from \[3\] that \(E(n)\) and \(\widehat{E(n)}\) are Bousfield equivalent as \(S\)-modules, and it follows that \(E_n\) is Bousfield equivalent to these since it is a finite wedge of suspensions of \(\widehat{E(n)}\).

If \(X\) is a \(p\)-local spectrum with torsion free homotopy groups then its \(p\)-completion \(X_p\) is Bousfield equivalent to \(X\), i.e., \(\langle X_p \rangle = \langle X \rangle\). This follows using the cofibre triangles (in which \(M(p)\) is the mod \(p\) Moore spectrum and the circled arrow indicates a map of degree one)

\[
\begin{array}{ccc}
X & \overset{p}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
X \wedge M(p) & & X_p \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \overset{p}{\longrightarrow} & X_p \\
\downarrow & & \downarrow \\
X \wedge M(p) & & X_p \\
\end{array}
\]

together with the fact that the rationalization \(p^{-1}X\) is a retract of \(p^{-1}(X_p)\). In particular, we have \(\langle BP(n)_p \rangle = \langle BP(n) \rangle\) and \(\langle E(n)_p \rangle = \langle E(n) \rangle\).

From \[11\] theorem 2.1, the Bousfield class of \(BP(n)\) is
\[
\langle BP(n) \rangle = \langle E(n) \rangle \vee \langle HF_p \rangle.
\]

There is a cofibre triangle
\[
\Sigma^2 c(E_n) \overset{c}{\longrightarrow} \longrightarrow (c(E_n) \\
\downarrow & & \downarrow \\
HW(F_p^n)[[u_1, \ldots, u_{n-1}]] & & \}
\]
in which \( \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_{n-1}]] \) is the Eilenberg-MacLane spectrum. More generally we can construct a family of Eilenberg-MacLane spectra with \( W(\mathbb{F}_p^n)[[u_1, \ldots, u_k]] \) as coefficients for \( k = 0, \ldots, n - 1 \) which are related by cofibre triangles

\[
\text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_k]] \xrightarrow{u_k} \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_{k-1}]] \]

such that for \( k = 0 \) we obtain \( \text{HW}(\mathbb{F}_p^n) \). With the help of these cofibre sequences we can identify \( \langle \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_k]] \rangle \) with \( \langle \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_{k-1}]] \rangle \lor \langle \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_k]][u_k^{-1}] \rangle \).

In general, if \( R \) is a commutative ring, then the ring of finite tailed Laurent series \( R((x)) \) is faithfully flat over \( R \) and therefore we have

\[
\langle \text{HR}((x)) \rangle = \langle \text{HR} \rangle.
\]

Using this auxiliary fact we inductively get that

\[
\langle \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_k]] \rangle = \langle \text{HW}(\mathbb{F}_p^n)[[u_1, \ldots, u_{k-1}]] \rangle.
\]

This reduces the Bousfield class of \( c(E_n) \) to \( (E_n) \lor (\text{HW}(\mathbb{F}_p^n)) \). As \( W(\mathbb{F}_p^n) \) is a finitely generated free \( \mathbb{Z}_p \)-module and as \( \langle H\mathbb{Z}_p \rangle = \langle H\mathbb{Q} \rangle \lor \langle H\mathbb{F}_p \rangle \) this leads to

\[
\langle c(E_n) \rangle = \langle E(n) \lor H\mathbb{Q} \lor H\mathbb{F}_p \rangle = \langle E(n) \lor H\mathbb{F}_p \rangle = \langle BP\langle n \rangle \rangle.
\]

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