Entanglement generation among quantum dots and surface plasmons of a metallic nanoring

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Abstract

We study the systems of two and three quantum dots positioned near a metallic nanoring having a surface plasmon in a particular mode. Interaction of surface plasmon with quantum dots generates entanglement between quantum dots as well as among the incident and reflected modes of the surface plasmons. We show that the maximum amount of entanglement is generated among two initially unentangled quantum dots separated by distance

\[ kd = (2m + 1)\pi/2 \]

and with interaction parameter \( \mu t = (2n + 1)\sqrt{2}\pi/4 \), while entanglement among two counter-propagating modes of the surface plasmons can be generated with interdot distance \( kd = 2m\pi \) and interaction parameter \( \mu t = (2n + 1)\pi/4 \), where \( k \) and \( \mu \) are the propagation vector and coupling constant, respectively. In case of three quantum dots, the maximum entanglement among the quantum dots is obtained at \( kd = m\pi/3 \) and

\[ \mu t = (\sqrt{3}/6 + 4n/7)\pi. \]

Finally, the scheme is straightforwardly generalized to generate quantum entanglement among \( N \) quantum dots as well as among two counter-propagating modes of the surface plasmon due to their interactions with quantum dots.

Keywords: entanglement generation, quantum dots, surface plasmon, metallic nanoring

(Some figures may appear in colour only in the online journal)

1. Introduction

When a photon strikes a metal surface, a surface electromagnetic wave termed as surface plasmon polariton that is coupled to plasma oscillations can be excited. Since the dynamics of such surface plasmons exhibit strong analogy with light propagating in conventional dielectrics, therefore this newly emerging and exciting field has justly been phrased as plasmonics in comparison with photonics. For example, it is now possible to confine them to sub-wavelength dimensions [1] leading to novel approaches for waveguide below the diffraction limit [2]. Furthermore, the strong coupling between surface plasmon (SP) and emitters [3] can be utilized to enhance infrared photodetectors [4], the fluorescence of quantum dots (QDs) [5] and light transmission through metal nanoarrays [6]. A variety of experimental [7–9] as well as theoretical works [10] have been focused on the properties in the nanowire-quantum dot systems. The coupling between emitters and photons is difficult but has been achieved in a
number of systems that reach the so-called ‘strong-coupling’ regime of cavity quantum electrodynamics (QED) [11, 12]. Several approaches to reach this regime on a chip at microwave frequencies have been suggested and experimentally observed [13–15]. A key feature of these approaches is the use of conductors to reduce the effective mode volume $V_{eff}$ for the photons, thereby achieving a substantial increase in the coupling strength $\mu \propto 1/\sqrt{V_{eff}}$. Realization of analogous techniques with optical photons would open the door to many potential applications in quantum communication and in addition lead to smaller mode volumes and hence faster interaction times.

In a related context, advances in quantum information science have promoted an experimental drive for physical realizations of highly entangled states. Excessively good results have been obtained with quantum-optical and atomic systems [16]. Couplings of ions with photons have also been utilized for quantum information processing [17, 18]. However, due to scalability requirements, solid-state realizations of such phenomena are equally promising [19–21]. Furthermore, while initial attention has been focused on the coupling between near by qubits with local interactions [22–24] entangling arbitrary pairs of remote qubits is still an important goal. Circuit QED, for example, is one promising candidate to couple two distant qubits via a cavity bus [25]. This photon-qubit interface provides many applications in the quantum information processing including generation and protection of entanglement between two or more qubits. Recently interactions of SPs in metal nanowire and nanoring with QDs have been studied which may result in entanglement among two quantum dots [26, 27].

Motivated by these recent developments in plasmonics and quantum information science, we study the entanglement generation and dynamics of quantum dots and counter-propagating modes of surface plasmons in a system consisting of two and three QDs coupled to a surface plasmon in a metal nanoring. A complete understanding of the dynamics of quantum systems interacting with their surroundings has become desirable, particularly with respect to applications for quantum information science [28]. For the implementation of quantum computer the main ingredient is the conditional quantum dynamics, in which one subsystem undergoes a coherent evolution depending on the state of another system. Such a dynamics gives us interaction parameters which may generate maximum entanglement among quantum dots or surface plasmons. The scheme can be generalized to a system of $N$ quantum dots interacting with a single mode SP due to metallic nanoring.

The paper is organized as follows. Section 2 gives the description of model Hamiltonian in detail. Section 3 is dedicated to entanglement analysis, which includes dynamics for two and three quantum dots interacting with single mode surface plasmon and also addresses the interaction of arbitrary number of quantum dots interacting with single mode surface plasmon. Section 4 describes the decoherence of entangled states in vacuum environment. Section 5 is the summary and conclusion of the paper.

In conclusion of the paper, we state that two quantum dots separated by a distance $d_{12}$ and are positioned near a metallic nanoring having a surface plasmon in clockwise mode ($+k$) as shown in figure 1. The anticlockwise ($-k$) mode is generated by the reflection of SP by either of the quantum dots. Such a system can be considered equivalent to two two-level atoms in a cavity having two field modes $+k$ and $-k$ along the axis of the cavity. The Hamiltonian of the system can be written as

$$H_I = H_{QD} + H_{SP} + H_I^{(2)},$$

where $H_{QD}$ and $H_{SP}$ are the free part Hamiltonian of the SP and QDs, respectively, and $H_I^{(2)}$ represents the interaction between two QDs and SP. Thus,

$$H_{QD} = \hbar \sum_{j=1,2} \omega_{0j} a_j^\dagger a_j,$$

$$H_{SP} = \hbar \sum_{k'=-k}^{k} \omega_k a_k^\dagger a_k,$$

$$H_I^{(2)} = \hbar [\mu_1 \sigma_1^+ a_1 + \mu_2 \sigma_2^+ a_1 e^{i(d_{12})}] + [\mu_1 \sigma_1^- a_{-k} + \mu_2 \sigma_2^- a_{-k} e^{-i(d_{12})}] + \text{H.C.,}$$

where $\mu_1, \mu_2$ are coupling strengths between the SP and QDs, $\sigma_j^+ (\sigma_j^-) = |e_j\rangle\langle g_j| (|g_j\rangle\langle e_j|)$ is the raising (lowering) operator for the $j$th QD with the ground and excited state being $|g_j\rangle$ and $|e_j\rangle$, respectively, $\omega_k$ is the frequency of the $k$th mode of SP, $a_k(a_k^\dagger)$ are the annihilation (creation) operator of the SP and $d_{12}$ is the separation between QD1 and QD2. In equation (4), first two terms correspond to the interaction of clockwise mode of SP with QD1 and QD2, respectively, while third and fourth term correspond to the interaction of QD1 and QD2 with reflected mode of SP. Here $L$ is the circumference of metal nanoring. Further we assume the existence of only the fundamental standing mode of SP and hence $kL = 2\pi$, so the fourth term becomes $\mu_2 \sigma_2^+ a_{-k} e^{-i(d_{12})}$. Therefore the interaction picture Hamiltonian under resonant condition and in rotating wave approximation transforms to

$$H_I^{(2)} = \hbar [\mu_1 \sigma_1^+ a_k + \mu_2 \sigma_2^+ a_k e^{i(d_{12})}] + [\mu_1 \sigma_1^- a_{-k} + \mu_2 \sigma_2^- a_{-k} e^{-i(d_{12})}] + \text{H.C.},$$

where $\mu_1, \mu_2$ are coupling strengths between the SP and QDs, $\sigma_j^+ (\sigma_j^-) = |e_j\rangle\langle g_j| (|g_j\rangle\langle e_j|)$ is the raising (lowering) operator for the $j$th QD with the ground and excited state being $|g_j\rangle$ and $|e_j\rangle$, respectively, $\omega_k$ is the frequency of the $k$th mode of SP, $a_k(a_k^\dagger)$ are the annihilation (creation) operator of the SP and $d_{12}$ is the separation between QD1 and QD2. In equation (4), first two terms correspond to the interaction of clockwise mode of SP with QD1 and QD2, respectively, while third and fourth term correspond to the interaction of QD1 and QD2 with reflected mode of SP. Here $L$ is the circumference of metal nanoring. Further we assume the existence of only the fundamental standing mode of SP and hence $kL = 2\pi$, so the fourth term becomes $\mu_2 \sigma_2^+ a_{-k} e^{-i(d_{12})}$. Therefore the interaction picture Hamiltonian under resonant condition and in rotating wave approximation transforms to

$$H_I^{(2)} = \hbar [\mu_1 \sigma_1^+ a_k + \mu_2 \sigma_2^+ a_k e^{i(d_{12})}] + [\mu_1 \sigma_1^- a_{-k} + \mu_2 \sigma_2^- a_{-k} e^{-i(d_{12})}] + \text{H.C.}.$$
3. Entanglement analysis

In earlier work by Chen et al [27], they investigated the entanglement among two quantum dots for fixed interdot separations. Here we analyze the entanglement among two and three quantum dots as well as among counter-propagating modes of surface plasmons analytically as a function of interdot distance and interaction parameters and then generalized this schemes for \( N \) quantum dots.

3.1. Interaction of two QDs with single mode SP

First we consider two QDs separated by a distance \( d_{12} = d \), initially in ground states \( |g_{1}\rangle \) and \( |g_{2}\rangle \), respectively, positioned near metallic nanoring having a surface plasmon in clockwise (+\( k \)) mode. Thus the initial state of the QDs and SP system can be written as \( |g_{1}, g_{2}, 1, 0\rangle \). The first plasmon mode represents clockwise (+\( k \)) and second represent counter-clockwise (−\( k \)) mode. The state vector of QDs and SPs considering transmission, reflection, and absorption of clockswise SP mode in QD\(_{1}\) and QD\(_{2}\) can be written as:

\[
|\psi^{(2)}(t)\rangle = A_{1}(t)|g_{1}, g_{2}, 1, 0\rangle + A_{2}(t)|g_{1}, g_{2}, 0, 1\rangle + A_{3}(t)|e_{1}, g_{2}, 0, 1\rangle + A_{4}(t)|g_{1}, e_{2}, 0, 0\rangle,
\]

where \( A_{i}(t) \) are the probability amplitudes for QDs and SPs in particular basis mentioned above.

Assuming system-environment coupling weak compared to QDs and SPs couplings, therefore, we ignore the damping terms of QDs and SPs. Equation of motion of the probability amplitudes \( A_{i}(t) \) obtained from Schrödinger equation \( |\psi^{(2)}(t)\rangle = -\frac{i}{\hbar}|H^{(2)}|\psi^{(2)}(t)\rangle \) (see appendix). The solution for the probability amplitudes subject to the initial condition \( |\psi^{(2)}(0)\rangle = |g_{1}, g_{2}, 1, 0\rangle \) are given as follows:

\[
A_{1}(t) = \frac{1}{2} \left\{ \cos \left[ 2\mu t \cos \left( \frac{kd}{2} \right) \right] + \cos \left[ 2\mu t \sin \left( \frac{kd}{2} \right) \right] \right\},
\]

\[
A_{2}(t) = \frac{1}{2} e^{ikd} \left\{ \cos \left[ 2\mu t \cos \left( \frac{kd}{2} \right) \right] - \cos \left[ 2\mu t \sin \left( \frac{kd}{2} \right) \right] \right\},
\]

\[
A_{3}(t) = -\frac{i}{2} e^{i\mu t} \left\{ \sin \left[ 2\mu t \cos \left( \frac{kd}{2} \right) \right] - \sin \left[ 2\mu t \sin \left( \frac{kd}{2} \right) \right] \right\},
\]

\[
A_{4}(t) = \frac{1}{2} e^{i\mu t} \left\{ -\sin \left[ 2\mu t \cos \left( \frac{kd}{2} \right) \right] + \sin \left[ 2\mu t \sin \left( \frac{kd}{2} \right) \right] \right\},
\]

where we have taken \( \mu_{1} = \mu_{2} = \mu \) for simplicity. By carefully examining the state vector we see that when separation \( kd = \pi/2 \) and the interaction time \( \mu t = \sqrt{2}\pi/4 \), we get the state of the two QDs as:

\[
|\psi^{(2)}_{QD}(t)\rangle = \frac{1}{\sqrt{2}} (|g_{1}, e_{2}\rangle - i|e_{1}, g_{2}\rangle) \otimes |0, 0\rangle.
\]

Thus by taking trace over the plasmonics modes we get maximally entangled states between QDs. However, if we choose \( kd = \pi \) and \( \mu t = \pi/4 \), we get

\[
|\psi^{(2)}_{QD}(t)\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle - |0, 1\rangle) \otimes |g_{1}, g_{2}\rangle
\]

By taking trace over QDs, we get

\[
\rho_{SP}(t) = \frac{1}{4} (|1, 0\rangle - |0, 1\rangle)(|1, 0\rangle - |0, 1\rangle)^{\dagger} + \frac{1}{2} |0, 0\rangle \langle 0, 0|.
\]

This shows entanglement between two modes of the surface plasmon with maximum entanglement equal to 1/2. Equations (11) and (12) represent the special cases where we get maximum entanglement in case of QDs and half maximum in case of SP modes, respectively.

To study the entanglement dynamics of the QDs as well as SP modes as a function of interdot distance \( kd \) and interaction parameter \( \mu \), a quantitative measure of entanglement is necessary. In order to quantify the amount of entanglement, we need a complete measure of entanglement contained in the quantum system. Negativity is one of the criteria, convenient for multipartite systems to be discussed later in this article, is used for the measurement of entanglement [29]. It has been reported that at least one of the eigenvalues of the partial transposed matrix with respect to any qubit of the entangled system must be negative [30]. Sum of all negative eigenvalues defines the negativity or entanglement of the system as \( E_{N}(t) = -2\sum_{\lambda} \lambda \), where \( \lambda \) are the negative eigenvalues of the partially transposed matrix of the density matrix of quantum dots \( \rho_{QD} \) or surface plasmons \( \rho_{SP} \).

Entanglement among two QDs is found by taking trace over the SP modes of the density matrix \( \rho(t) = |\psi^{(2)}(t)\rangle\langle \psi^{(2)}(t)| \) and by finding negative eigenvalues of the partial transpose of that matrix w. r. t. either of the QD. The expression of the negativity for the two QDs in terms of probability amplitudes is given by

\[
N^{(2)}_{QD} = \sqrt{4} |A_{1}(t)|^{2} |A_{2}(t)|^{2} + |A_{3}(t)|^{2} - |A_{4}(t)|^{2}.
\]

On substituting the values of \( A_{i}(t) \) we obtain from equations (7)–(10),

\[
N^{(2)}_{QD} = \frac{1}{4} \left\{ \sqrt{\chi^{2}(t) + (Y(t) - 2)^{2}} - \chi^{2}(t) \right\},
\]

where

\[
\chi(t) = \cos \left( 2\mu t \cos \left( \frac{kd}{2} \right) \right) - \cos \left( 2\mu t \sin \left( \frac{kd}{2} \right) \right),
\]

\[
Y(t) = \cos \left( 4\mu t \cos \left( \frac{kd}{2} \right) \right) + \cos \left( 4\mu t \sin \left( \frac{kd}{2} \right) \right).
\]
In figure 2, we plot the dynamics of entanglement as a function of dimensionless time $\mu t$ and interdot distance $kd$ for two QDs. The entanglement dynamics depends upon the separation between the two QDs and the coupling strengths of SP with QDs. We have assumed the coupling strengths to be equal depending upon the position of QDs in the plasmon mode and the orientation of dipole which can be altered with the help of external field [31]. Plot shows that entanglement is generated from initially separable state of QDs due to their interactions with SP. The entanglement oscillates with both the parameters $\mu t$ and $kd$. The maximum entanglement occurs at $kd = (2m + 1)\pi/2$ and $\mu t = (2n + 1)\sqrt{2}\pi/4$, with $(m, n = 0, 1, 2, \ldots)$. The minima or disentanglement occur at $(kd, \mu t) = ((2m + 1)\pi/2, \sqrt{2n}\pi/4)$ and $(m\pi, n\pi/2)$ with $(m, n = 0, 1, 2, \ldots)$. Dynamics of entanglement are symmetric about $kd = \pi/2$. Apart from these maxima and minima at particular points of $kd$ and $\mu t$, we see some other bright red and blue regions which show oscillations but the entanglement does not reach to its maximum or minimum values. Thus with the proper choice of $kd$ and $\mu t$ we get entanglement among two QDs.

Similarly entanglement between two counter-propagating modes of the SPs can be analyzed by finding the density matrix of SPs $\rho_{SP}$ by taking trace over QDs of the density matrix $\rho(t) = |\psi^{(2)}(t)\rangle \langle \psi^{(2)}(t)|$. The negativity of this SPs system is obtained as

$$N_{SP}^{(2)}(t) = \frac{1}{4} \{ \sqrt{6} - 4Y(t) + Z(t) + Y(t) - 2 \}. \tag{18}$$

It follows that on substituting the values of $A_i(t)$ from equations (7)–(10) in equation (18), that

$$N_{SP}^{(2)}(t) = \frac{1}{4} \{ \sqrt{6} - 4Y(t) + Z(t) + Y(t) - 2 \,. \tag{19}$$

where $Y(t)$ is given by equation (17)

$$Z(t) = \cos \left( 8\mu t \cos \left( \frac{kd}{2} \right) \right) + \cos \left( 8\mu t \sin \left( \frac{kd}{2} \right) \right). \tag{20}$$

This expression shows that entanglement is generated between the two counter-propagating modes of the surface plasmon. Entanglement is an oscillatory function of dimensionless time $\mu t$ and interdot distance $kd$. According to equation (18) shows that maximum value of entanglement for the case of SPs cannot be greater than 1/2, which happens at $kd = m\pi$ and $\mu t = (2n + 1)\pi/4$ with $(m, n = 0, 1, 2, \ldots)$. Figure 3 shows the entanglement dynamics of two modes of surface plasmons. In the plots bright red regions show the entanglement while blue regions show disentanglement regions. Thus with the proper choice of $kd$ and $\mu t$ we can have entanglement among two counter-propagating modes of the SP.

### 3.2. Interaction of three QDs with single mode SP

Next we consider three of quantum dots initially in their ground state $|g_i\rangle$ near a metallic nanoring having a surface plasmon in clockwise mode. The state vector of the QDs and the SPs for our initially clockwise SP mode in QDs can be written as

$$|\psi(t)\rangle = B_1(t)|g_1, g_2, g_3, 1, 0\rangle + B_2(t)|g_1, g_2, g_3, 0, 1\rangle + B_3(t)|g_1, g_2, e_3, 0, 0\rangle + B_4(t)|g_1, e_2, g_3, 0, 0\rangle + B_5(t)|e_1, g_2, g_3, 0, 0\rangle. \tag{21}$$

The interaction picture Hamiltonian in resonant interaction of SP with QDs under rotating wave approximation can be
written as
\[
H_I^{(3)} = \hbar \left[ \mu_1 \sigma_1^+ (a_k + a_{-k}) + \mu_2 \sigma_2^+ (a_k e^{i2kd} + a_{-k} e^{-i2kd}) + \mu_3 \sigma_3^+ (a_k e^{i3kd} + a_{-k} e^{-i3kd}) + \text{H.C.} \right].
\]

(22)

We assume that all three QDs are placed equally spaced, i.e.,
\[d_{12} = d_{23} = d\quad \text{and} \quad d_{13} = d_{12} + d_{23} = 2d.\]
Equations of motion of the probability amplitudes \(B_i(t)\) are obtained from Schrödinger equation \(\dot{\psi}(t) = -\frac{i}{\hbar}(H_I^{(3)})\psi(t))\), as given in the appendix. The probability amplitudes \(B_i(t)\) for the initial condition \(|\psi(0)\rangle = |g_l, g_2, g_3, 1, 0\rangle\) can be obtained as
\[
B_1(t) = \frac{1}{2} (\cos(2\mu t \sin(kd)) + \cos(\sqrt{2} \alpha \mu t e^{-2kd})),
\]
\[
B_2(t) = \frac{1}{2} e^{i2kd} [-\cos(2\mu t \sin(kd)) + \cos(\sqrt{2} \alpha \mu t e^{-2kd})],
\]
\[
B_{3,5}(t) = -\frac{i e^{-i\beta t}}{4(1 + 4 e^{i2kd} + e^{i4kd})} \{ i \sqrt{2} \alpha e^{i2kd} \times [e^{i2kd} + 2 \cos(kd) e^{i\zeta t} + e^{-i2kd}]
\]
\[
\pm i 2 e^{i2kd} \sqrt{2} \mu t e^{i2kd} (1 + 4 e^{i2kd} + e^{i4kd}) \times \sin(2\mu t \sin(kd)) \},
\]
\[
B_4(t) = -\frac{i}{\sqrt{2} \alpha} e^{i3kd} \sin(\mu \alpha t e^{-2kd}),
\]

where
\[
\alpha = \sqrt{e^{i4kd}(2 + \cos(2kd))},
\]
\[
\beta = e^{-i2kd} (e^{i3kd} + e^{-i2kd}) + e^{-i(\sqrt{2} \alpha)},
\]
\[
\zeta = e^{-i2kd} (\cos(kd) + \cos(3kd) + i4 \cos^2(kd) \sin(kd)) + i2 \sqrt{2} \alpha).
\]

Again we have taken \(\mu_1 = \mu_2 = \mu_3 = \mu\) as generally for better entanglement these coupling constants are taken equal.

An analysis of the probability amplitudes shows that at interdot distance \(kd = \pi/3\) and interaction parameter \(\mu = \sqrt{3} \pi/6\), we get \(B_1(t) = B_2(t) = 0\) and \(B_3(t) = -B_5(t) = -i B_4(t) = 1/\sqrt{3}\). These leads to the entanglement among 3 QDs as the W-state [32] of the form
\[
|W_3\rangle = \frac{1}{\sqrt{3}} (|g_l, g_2, e_3\rangle - |g_1, e_2, g_3\rangle - i|e_1, g_2, g_3\rangle).
\]

(27)

This is maximally entangled state among three QDs. In order to get other values of \(kd\) and \(\mu\) for the determination of entanglement among three QDs, we plot entanglement using negativity as entanglement criterion. The negativity for three qubits of two-dimensions each is defined as [33, 34]
\[
N^{(3)}_{\text{QD}}(t) = -\frac{1}{2} \text{tr}\{\rho^{(3)}(t)\},
\]
where \(N_1(t), N_2(t)\) and \(N_3(t)\) are negativities of the three-qubits matrix \(\rho_{\text{QD}}\) after taking transpose with respect to QD1, QD2 and QD3 respectively, as
\[
N_1(t) = \sqrt{B^2 + 4(|B_3|^2 + |B_3|^2)} - B,
\]
\[
N_2(t) = \sqrt{B^2 + 4(|B_3|^2 + |B_3|^2)} - B,
\]
\[
N_3(t) = \sqrt{B^2 + 4(|B_3|^2 + |B_3|^2)} - B.
\]

(28)

In figure 4 we show the contour plots of the entanglement as a function of \(kd\) and \(\mu\). The bright red regions show the entanglement while blue regions are for disentanglement. We see that maxima of entanglement occur at interdot distance \(kd = m \pi/3\) with \(m = 3\) or multiples of \(3\) and at interaction parameters \(\mu = (\sqrt{3}/6 + 4n/7)\pi\) with \(m = 1, 2, 4, \ldots\) and \(n = 1, 2, 3, \ldots\).

Entanglement among counter-propagating modes of the SP can also be determined by first taking the trace over the QDs of the density matrix \(\rho_{SP}(t) = \langle \Psi^{(3)}(t) | \Psi^{(3)}(t) \rangle\) to find the density matrix of surface plasmons \(\rho_{SP}\). The expression of negativity for the two modes of the SP is obtained as
\[
N^{(3)}_{\text{SP}}(t) = \sqrt{4(|B_3|^2 + |B_3|^2)} - 4(|B_3|^2 + |B_3|^2). \]

(31)

In figure 5 we show the entanglement among the two modes of the surface plasmons as a function of \(kd\) and \(\mu\). The bright red regions show the regions of entanglement. Thus by proper choice of \(kd\) and \(\mu\), the entanglement among two modes of the surface plasmons can be generated due to its interactions with three quantum dots.

### 3.3. Generalization to N quantum dots

This scheme can be generalized for the generation of entanglement among \(N\) quantum dots positioned near a metallic nanoring as shown in figure 1. Interaction picture Hamiltonian under resonance and rotating wave approximation can be
written as

\[ H^{(N)}_I = \hbar \sum_{j} \left[ \mu_j \sigma_j^+ (a_j e^{i\theta_d} + a_j^\dagger e^{-i\theta_d}) \right] + \text{H.C.} \quad (32) \]

where \( d_{ij} \) represent the separation of \( j \)th QD with the 1st QD to which the SP interacts first, therefore \( d_{11} = 0 \).

The state vector for \( N \) quantum dots, each initially in ground state \( |g_i\rangle \) and a surface plasmon in clockwise mode, can be written as

\[ |\Psi^{(N)}(t)\rangle = |g_1, g_2, \ldots, g_N\rangle \otimes (C_1(t)|1, 0\rangle + C_2(t)|0, 1\rangle) \]

\[ + (C_3(t)|g_1, g_2, \ldots, e_N\rangle + C_4(t)|g_1, g_2, \ldots, e_{N-1}, g_N\rangle \]

\[ + \ldots + C_{N+2}(t)|e_1, g_2, \ldots, g_N\rangle) \otimes |0, 0\rangle. \quad (33) \]

The time dependent probability amplitudes \( C_j(t) \) can be obtained from the solutions of equations of motions using Schrödinger equation. Proper selection of interdot distance \( kd \) and interaction parameter \( \mu \) may make the probability amplitudes \( C_1(t) = C_2(t) = 0 \) and all other probability amplitudes equal to make a W-State among \( N \) quantum dots as

\[ |W_N\rangle = \frac{1}{\sqrt{N}}(|g_1, g_2, \ldots, e_N\rangle + |g_1, g_2, \ldots, e_{N-1}, g_N\rangle \]

\[ + \ldots + |e_1, g_2, \ldots, g_N\rangle). \quad (34) \]

The striking feature of the W-State is that it remains entangled over the loss of one qubit. The nonclassical behavior of these states are stronger than the GHZ states [35]. The unique characteristics of W-states make it an ideal resource for many of the quantum information processing tasks [36–39].

![Figure 5](image)

**Figure 5.** Entanglement dynamics as given by negativity of the two modes of the surface plasmon as a function of time \( \mu t \) and interdot distance \( kd \), for the case of interaction of three quantum dots with single mode surface plasmon.

![Figure 6](image)

**Figure 6.** Decay dynamics of the entanglement of the two quantum dots and two modes of surface plasmons as a function of time \( \mu t \) for a fix interdot distance \( kd = \pi/2 \) for quantum dots and \( \pi \) for surface plasmons. System is coupled to the vacuum environment. Black solid and red dashed lines are the entanglement dynamics of two quantum dots without and with decay, respectively, while blue dashed-dotted and pink dotted lines show the entanglement dynamics of two modes of surface plasmons. Decay rates of quantum dots and surface plasmons are taken equal as \( \Gamma_1 = \Gamma_2 = \gamma_1 = \gamma_2 = \mu/20 \).

### 4. Decoherence process

While analyzing entanglement among QDs and SP modes, we neglected the system-environment coupling due to weak interaction of system with the environment as compared to the coupling between QDs and SP. However we can include the system-environment coupling if we consider environment large enough so that the memory is lost in the environment, then a Markov approximation we can be made. We can then write the master equation of the system composed of quantum dots and surface plasmons interacting with vacuum environment as [27]

\[ \frac{d}{dt}\rho(t) = \frac{1}{i\hbar}[H_I, \rho] \]

\[ = -\frac{1}{2} \sum_{k, k'} \Gamma_{k'} (b_{k'}^\dagger b_{k'} \rho(t) - 2 b_{k'}^\dagger \rho(t) b_{k'} + \rho(t) b_{k'}^\dagger b_{k'}) \]

\[ -\frac{1}{2} \sum_{j=1}^N \gamma_j (\sigma_j^x \sigma_j^z \rho(t) - 2 \sigma_j^x \rho(t) \sigma_j^z + \rho(t) \sigma_j^z \sigma_j^z), \quad (35) \]

where \( \gamma_j \) is the decay rate of \( j \)th QD, and \( \Gamma_{k'} \) is the plasmon field damping rate. In figure 6 we show the entanglement dynamics of two QDs and counter-propagating modes of SP at \( kd = \pi/2 \) and \( \pi \) for QDs and SP modes, respectively. Plots show the oscillatory behavior of entanglement as a function of time \( \mu t \). These show that the interaction parameter \( \mu t \) at which we can get the maximum entanglement between QDs or counter-propagating modes of SP. Dotted lines show the damping due to system’s interaction with the vacuum environment. This interaction does not change the entanglement behavior but only reduces the amount of entanglement with the passage of time.
5. Summary and conclusion

In summary, we have investigated entanglement generation and dynamics between $N$ quantum dots coupled to a single mode plasmon in the presence of a nano metallic ring. We note that entanglement can be generated among the initially separable quantum dots due to their interactions with the surface plasmon produced on a metallic nanoring close to the quantum dots. The schematics are executed in detail for the case with $N = 2$ and $N = 3$ and then generalized to cover any arbitrary scenario with $N$ dots placed near the nanoring. Analytical expressions are obtained for the entanglement among two and three quantum dots positioned near a metallic nanoring and interacting with a single mode surface plasmon. This interaction also generates entanglement between two counter-propagating modes of the surface plasmon. Contour plots of the entanglement between quantum dots near a metallic nanoring and interacting with a single mode surface plasmon. Thus by fixing the interdot distance $kd$ and the interaction parameter $\mu$, we can generate maximally entangled states among any number of quantum dots.

For the realization of coupling between a metal nanoring SP and two/three QDs, colloidal CdSe/ZnS QDs and a silver nanoring are ideal since the excitation energy of CdSe/ZnS QDs is around 2–2.5 eV, compatible with the saturation plasma energy of the silver nanowire [26]. Considering only the fundamental mode of SP in nanoring cavity, the radius of silver nanoring should be around 100 nm.

Concerning experimental execution, we note that all the fundamental ingredients employed in the present scheme have already been experimentally realized including a nanoring cavity with radius 100 nm, while the QDs can precisely be realized experimentally including a nanoring [8]. Ultrahigh Purcell factor ($>10^4$) of the metallic nanoring [42] and long propagation distance (2.17 mm) [43] make our scheme experimentally realizable. Measurement of entanglement in the system can be carried out by the procedure of ultrafast optical tomography [44].

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Appendix. Equation of motion for the probability amplitudes

To get the equations of motion for the probability amplitudes $A_i(t)$, we substitute the state vector $|\psi(t)\rangle$ and interaction Hamiltonian $H_I^{(2)}$ into Schrödinger equation $|\psi(t)\rangle = -\frac{i}{\hbar}(H_I^{(2)}|\psi(t)\rangle)$ and compare the amplitudes of basis in state vector $|\psi(t)\rangle$. The equations of motion of the probability amplitudes thus are

$$A_1(t) = -i(\mu_1^0 A_1(t) + \mu_2^0 A_2(t)e^{-ikd}),$$
$$A_2(t) = -i(\mu_1^0 A_2(t) + \mu_2^0 A_1(t)e^{ikd}),$$
$$A_3(t) = -i\mu_1(t_0) (A_1(t) + A_2(t)),$$
$$A_4(t) = -i\mu_2(t_0) (A_1(t)e^{ikd} + A_2(t)e^{-ikd}).$$

Similarly, equations of motion of the probability amplitudes $B_i(t)$ are also obtained by substituting the state vector for the interaction of three QDs with a single mode SP i.e. $|\psi(3)(t)\rangle$ and interaction Hamiltonian $H_I^{(3)}$ into Schrödinger equation $|\psi(3)(t)\rangle = -\frac{i}{\hbar}(H_I^{(3)}|\psi(3)(t)\rangle)$, we get

$$B_1(t) = -i(\mu_1^0 B_1(t)e^{-2ikd} + \mu_2^0 B_2(t)e^{-ikd} + \mu_3^0 B_3(t)),$$
$$B_2(t) = -i(\mu_1^0 B_2(t)e^{2ikd} + \mu_2^0 B_1(t)e^{ikd} + \mu_3^0 B_3(t)),$$
$$B_3(t) = -i\mu_1(t_0) (B_1(t)e^{-2ikd} + B_2(t)e^{-ikd}),$$
$$B_4(t) = -i\mu_2(t_0) (B_1(t)e^{2ikd} + B_2(t)e^{ikd}),$$
$$B_5(t) = -i\mu_3(t_0) (B_1(t) + B_2(t)).$$

The solutions of these equations of motion $A_i(t)$ and $B_i(t)$ are obtained by solving these two sets of coupled differential equations subject to initial conditions $|\psi(2)(t)\rangle = [\psi_0, \psi_2, 1, 0]$ and $|\psi(3)(t)\rangle = [\psi_0, \psi_2, \psi_3, 1, 0]$ for 2 and 3 quantum dots, respectively, and are produced in section 3.

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