Fuzzy AdS-conifold $Y^6_{AdS_F}$ and Dirac operator of principal fibration $X^5_{AdS_F} \to AdS^2_F \times AdS^2_F$

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Abstract

It has been constructed fuzzy AdS-conifold $Y^6_{AdS_F}$ on the base $AdS^3_F \times AdS^2_F$ which is topologically homeomorphic with the total space of the fibration $X^5_{AdS_F} \to AdS^2_F \times AdS^2_F$. After the projective module description of this bundle, the pseudo fuzzy Dirac and chirality operators on fuzzy $AdS^2_F \times AdS^2_F$ have been studied. Using the fuzzy Ginsparg-Wilson algebra, it has been studied the gauged fuzzy Dirac and chirality operators in instanton sector. It has been showed that they have correct commutative limit in the limit case when noncommutative parameter $l_\alpha$ tends to infinity.

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1 Introduction

Conifolds are singular complex spaces that are $C^\infty$ except in a number of isolated conical singularities. In the neighbourhood of these singularities the conifolds are described by

$$\sum_{n=1}^{n} z_n^2 = 0.$$ 

We consider the singular point in the origin of $\mathbb{C}^n$. It has been showed that the first Chern class of these spaces are zero [1]. The $(2n-2)$-dimensional conifold $Y^{2n-2}$ is a noncompact Calabi-Yau manifold with $SO(n) \times U(1)$ as its symmetry group. The base manifold of the conifold $Y^6$ is a $(2n-3)$-dimensional manifold $X^{2n-3}$ which is a compact Einstein manifold and satisfies $R_{ij} = (2n-4)g_{ij}$. The manifold $X^{2n-3}$ is the intersection of $Y^{2n-2}$ with the sphere $S^{2n-1}$. The conifold $Y^{2n-2}$ is a cone over $X^{2n-3}$. The $n = 3$ fuzzy case is the four-dimensional Fuzzy conifold $Y^4_F$ with $X^3_F$ as its base. The manifold $X^3_F$ is the intersection of $Y^4_F$ with the sphere $S^5_F$: $X^3_F = Y^4_F \cap S^5_F$. The fuzzy version of these spaces and also the $U(1)$ principal monopole bundle $X^5_F \to S^2_F$ has been studied in [2]. Another important case, is the case $n = 6$. The fuzzy compact manifold $X^8_F$ is the intersection of the fuzzy ten-dimensional conifold $Y^{10}_F$ with $S^5_F \times S^5_F$. $X^8_F$ is the base of $Y^{10}_F$. Authors in [3] have studied monopoles and Dirac operator of the principal fibration $X^5_F \to S^3_F \times S^3_F$. The most important case is the case $n = 4$ [4-7].

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The fuzzy compact manifold \(X_F^5\) is the intersection of the fuzzy six-dimensional conifold \(Y_F^6\) with \(S^2_F \times S^2_F\). The manifold \(X_F^5\) is the base of \(Y_F^6\). In [4] it has been constructed fuzzy conifold \(Y_F^6\) and its fuzzy base \(X_F^5\). Also, it has been showed that \(X_F^5\) is a \(U(1)\) principal fibration over fuzzy \(S^2 \times S^2\). In [5], \(U(n)\) gauge theory on fuzzy \(S^2 \times S^2\) as a matrix model has been studied. Authors have showed that this gauge theory reduces to Y-M gauge theory on \(S^2 \times S^2\) in commutative limit. Quantum effective potential for \(U(1)\) fields on fuzzy \(S^2 \times S^2\) has been studied in [6]. In [7], it has been studied the construction of a topological charge on fuzzy \(S^2 \times S^2\) via a Ginsparg-Wilson relation. In present paper, it has been constructed fuzzy AdS-conifold \(X_{AdS_F}^5\) on the base \(AdS^2_F \times AdS^2_F\) which is topologically homeomorphic with the total space of the fibration \(X_{AdS_F}^5 \rightarrow AdS^2_F \times AdS^2_F\).

Dirac and chirality operators are two important self-adjoint operators for the Connes-Lott approach to noncommutative geometry[8,9]. There are three types of Dirac and chirality operators. Ginsparg-Wilson Dirac operator, \(D_{GW}\) [10-18], Watamura-Watamura Dirac operator \(D_{WW}\) [19-21] and Grosse-Klimcik-Presnajder Dirac operator \(D_{GKP}\) [22, 23]. These three types of Dirac operators are compared with each other in [24].

In this paper we generalize Ginsparg-Wilson algebra to pseudo fuzzy Ginsparg-Wilson algebra on \(AdS^2_F \times AdS^2_F\) and construct its pseudo Dirac and chirality operators. This paper is organized as follows: In section 2 we study the fuzzy AdS-conifold \(Y_{AdS}^6\) and fibre bundle structure of the fibration \(X_{AdS_F}^5 \rightarrow AdS^2_F \times AdS^2_F\). Projective module description of the fibration \(X_{AdS_F}^5 \rightarrow AdS^2_F \times AdS^2_F\) has been studied in section 3. In section 4, spin \(\frac{1}{2}\) projectors of the pseudo projective \(A(AdS^2_F \times AdS^2_F)\) module has been constructed. Fuzzy Ginsparg-Wilson algebra and its Spin \(\frac{1}{2}\) fuzzy pseudo Dirac and chirality operators on \(AdS^2_F \times AdS^2_F\) have been studied in section 5. In section 6, gauged pseudo Dirac and chirality operators was constructed. In section 7, instanton coupling and in section 8, gauging the pseudo fuzzy Dirac operator in instanton sector have been studied, respectively.

## 2 Fuzzy AdS-conifold \(Y_{AdS}^6\) and fibre bundle structure of the fibration

\[X_{AdS_F}^5 \rightarrow AdS^2_F \times AdS^2_F\]

The AdS-conifold \(Y_{AdS}^6\) is a six-dimensional manifold embedded in a four-dimensional complex space \(\mathbb{C}^4\) with four complex coordinates \(w_i (i = 1, ..., 4)\) satisfying:

\[w_1^2 + w_2^2 - w_3^2 - w_4^2 = w_1\eta^{ij}w_j = 0, \quad i, j = 1, 2, 3, 4, \quad (2-1)\]

where \(\eta_{ij} = diag(+1, +1, -1, -1)\) and we have used the Einstein summation convention. From (2-1) one can calculate the base of the \(Y_{AdS}^6\) by intersecting the space of solutions of (2-1) with the manifold \(AdS^7\) of radius \(r\) in \(\mathbb{C}^4\)

\[|w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = w_1\eta^{ij}w_j = -r^2. \quad (2-2)\]

If we break up \(w_i\) into its real and imaginary parts, \(w_i = x_i + y_i\) then it is easy to see that from (2-1) and (2-2) we have:

\[x_1\eta^{ij}x_j = x_1^2 + x_2^2 - x_3^2 - x_4^2 = -\frac{r^2}{2}, \quad y_1\eta^{ij}y_j = y_1^2 + y_2^2 - y_3^2 - y_4^2 = -\frac{r^2}{2}, \quad x_1\eta^{ij}y_j = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 = 0. \quad (2-3)\]

The first of these equations defines an \(AdS^3\) with radius \(r/\sqrt{2}\). The other two equations define an \(AdS^2\) principal \(U(1)\) fibration over \(AdS^3\)

\[U(1) \leftrightarrow AdS^3 \rightarrow AdS^2. \quad (2-4)\]
Because all such bundles are trivial, the base of $Y^6_{AdS}$ has a topology of $AdS^2 \times AdS^3$. The 3-dimensional anti de Sitter space $AdS^3$ is a maximally symmetric space with constant negative curvature. It is the hyperboloid $AdS^3 \hookrightarrow \mathbb{R}^{2,2}$. The isometry group of $AdS^3$ is $SO(2, 2) \simeq SU(1, 1) \times SU(1, 1) \simeq AdS^2 \times AdS^2$ with the Lie algebra $so(2, 2) \simeq su(1, 1) \oplus su(1, 1)$. The base of the $Y^6_{AdS}$ is $X^5_{AdS}$ which is the intersection of $Y^6_{AdS}$ with the hyperboloid $AdS^7(\xi \eta^{ij} z_j = -r^2)$ i.e. $X^5_{AdS} = Y^6_{AdS} \cap AdS^7$ for $r = cons > 0$, the manifold $X^5_{AdS}$ is a differentiable five-dimensional manifold. The conform $Y^6_{AdS}$ has a class of manifolds $T^{p,q}_{AdS}$ (p and q are integers) as its base. These $T^{p,q}_{AdS}$’s are topologically homeomorphic with the manifold $AdS^3 \times AdS^2$, but they are not geometrically equivalent i.e. they are not diffeomorphic. All of $T^{p,q}_{AdS}$’s are $U(1)$ principal bundles over the cross manifold $AdS^2 \times AdS^2$. The manifold $AdS^2 \times AdS^2$ can be defined by the coordinates $x_1$ and $x_2$ as:

$$x_1 = x_{1i}(\theta_1, \xi_1), \quad x_2 = x_{2i}(\theta_2, \xi_2), \quad x_{1i} \eta^{ij} x_{j1} = -r^2_1,$$
$$x_{2i} = x_{2j}(\theta_2, \xi_2), \quad x_{2j} \eta^{ij} x_{ij} = -r^2_2, \quad i, j = 1, 2, 3.$$  

(2-5)

As $SU(1, 1)$ is isomorphic with $AdS^3$ so we have:

$$AdS^3 \times AdS^2 = SU(1, 1) \times \frac{SU(1, 1)}{U(1)}.$$  

(2-6)

For the special case when $p = q = 1$ we have $X^5_{AdS}(= T^{1,1})$. For the manifold $X^5_{AdS}$ it is not important the $U(1)$ is quotiented from which $SU(1, 1)$. So one can write:

$$X^5_{AdS} \cong \frac{SU(1, 1) \times SU(1, 1)}{U(1)}.$$  

(2-7)

It is clear that both $X^5_{AdS}(= T^{1,1}_{AdS})$ and $AdS^3 \times AdS^2(= T^{1,0}_{AdS})$ are principal $U(1)$ bundles over $AdS^2 \times AdS^2$.

As $Y^6_{AdS}$ has the $SO(2, 2) \times U(1)$ symmetry, the symmetry group of $X^5_{AdS}$ is $SO(2, 2)$. $SO(2, 2)$ is a Lie group generated by $\{M_1, M_2, M_3, T_1, T_2, T_3\}$ satisfying:

$$[M_i, M_j] = iC_{ij}^k M_k, \quad [M_i, T_j] = iC_{ij}^k T_k, \quad [T_i, T_j] = iC_{ij}^k M_k, \quad i, j, k = 1, 2, 3.$$  

(2-8)

where $C_{ij}^k$ are determined as $C_{ij}^k = \eta^{kl}C_{lj}^i$, in which $C_{123} = 1$ and $C_{ij}^k$ are completely antisymmetric. The Minkowskian metric $\eta^{ij} = \eta_{ij} = diag(1, 1, -1, -1)$ raises and lowers the indexes. The structure constants $C_{ij}^k$ satisfy the following relation:

$$C_{im}^k \eta^{ij} C_{jl}^n = \eta_{im} \eta_{kn}^l - \eta_{ml} \eta_{kn}^i.$$  

(2-9)

The generators of $su(1, 1)$ Lie algebra are pseudo Hermitian with respect to $\Lambda$:

$$M_i^\dagger = \Lambda M_i \Lambda^{-1}, \quad T_i = \Lambda T_i \Lambda^{-1},$$  

(2-10)

where the operator $\Lambda$ satisfies $\Lambda^\dagger = \Lambda, \Lambda^2 = 1$ and $\Lambda^{-1} = \Lambda^\dagger$.

Let us define the new operators $Q_i$ and $\bar{Q}_i$ as:

$$Q_i = \frac{1}{2}(M_i + T_i), \quad \bar{Q}_i = \frac{1}{2}(M_i - T_i).$$  

(2-11)

Using the new operators, the Lie algebra $so(2, 2)$ decomposes into two disjoint $su(1, 1)$ algebra:

$$[Q_i, Q_j] = iC_{ij}^k Q_k, \quad [Q_i, \bar{Q}_j] = iC_{ij}^k \bar{Q}_k, \quad [Q_i, \bar{Q}_j] = 0.$$  

(2-12)

To see the relation of $SO(2, 2)$ with two $SU(1, 1)$’s, let us define:

$$A_i = U^\dagger Q_i U = \frac{1}{2} I_2 \otimes \Sigma_i, \quad B_i = U^\dagger \bar{Q}_i U = \frac{1}{2} \Sigma_i \otimes I_2,$$  

(2-13)

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where

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & -i & i & 0 \end{pmatrix}, \tag{2-14} \]

and \( \Sigma_1 = i \sigma_1, \Sigma_2 = i \sigma_2 \) and \( \Sigma_3 = \sigma_3 \) (\( \sigma_i \) are Pauli matrices.) are the generators of \( su(1, 1) \) Lie algebra

\[ [\Sigma_i, \Sigma_j] = iC_{ij}^{\;\;k} \Sigma_k, \tag{2-15} \]

One can easily check that:

\[ [A_i, A_j] = iC_{ij}^{\;\;k} A_k, \quad [B_i, B_j] = iC_{ij}^{\;\;k} B_k, \quad [A_i, B_j] = 0. \tag{2-16} \]

Now let us define the fibre projection map \( \pi : X_{AdS}^5 \to AdS^2 \times AdS^2 \) as:

\[ x_{i_1} = z^1 A_{i_1} z, \quad x_{i_2} = z^1 B_{i_2} z, \quad i = 1, 2, 3, \tag{2-17} \]

with \( z = (z_1, z_2, z_3, z_4)^\ell \in \mathbb{C}^4 \). It is obvious that \( x_{i_1} \) and \( x_{i_2} \) are real and satisfies the following relation:

\[ x_{i_1} \eta_{i_1 j_1} x_{j_1} = x_{i_2} \eta_{i_2 j_2} x_{j_2} = -\frac{1}{2} \varepsilon_{12}. \tag{2-18} \]

which is the space \( AdS^2 \times AdS^2 \) with radius \( r/\sqrt{2} \). So \( X_{AdS}^5 \) is a principal \( U(1) \) bundle over \( AdS^2 \times AdS^2 \):

\[ U(1) \xrightarrow{\text{right } U(1) - \text{action}} X_{AdS}^5 \xrightarrow{\pi} AdS^2 \times AdS^2. \tag{2-19} \]

Noncommutative geometry is a pointless geometry. In this geometry instead of the coordinates \((x_{i_1}, x_{i_2})\) of \( AdS^2 \times AdS^2 \), the \( SU(1, 1) \times SU(1, 1) \) angular momentum generators in the unitary irreducible \( l_1, l_2 \)-representation spaces have the role of the points of the fuzzy \( AdS^2 \times AdS^2 \) i.e. \( AdS^2_\ell \times AdS^2_\ell \). Let us consider \( X_{i_1} = \mu_1 L_{i_1} \) and \( X_{i_2} = \mu_2 L_{i_2} \) (here we use \( L_{i_1} \) and \( L_{i_2} \) as the generators of the first and second \( SU(1, 1) \), respectively). \( L_{i_1, 2} \) satisfies the \( su(1, 1) \) Lie algebra:

\[ [L_{i_1}, L_{j_1}] = iC_{i_1 j_1}^{\;\;k_1} L_{k_1}, \quad [L_{i_2}, L_{j_2}] = iC_{i_2 j_2}^{\;\;k_2} L_{k_2}, \quad [L_{i_1}, L_{i_2}] = 0. \tag{2-20} \]

\( \mu_{1, 2} \) determined by the value of \( su(1, 1) \) Casimir operator

\[ \frac{1}{\mu_1^2} = C_{su(1, 1)} = -l_1(l_1 - 1), \quad \frac{1}{\mu_2^2} = C_{su(1, 1)} = -l_2(l_2 - 1). \tag{2-21} \]

Then, the noncommutative coordinates are

\[ X_{i_1} = \frac{L_{i_1}}{2\sqrt{l_1(1 - l_1)}}, \quad X_{i_2} = \frac{L_{i_2}}{2\sqrt{l_2(1 - l_2)}}, \tag{2-22} \]

which satisfy the \( su(1, 1) \) Lie algebra

\[ [X_{i_1}, X_{j_1}] = \frac{i}{2\sqrt{l_1(1 - l_1)}} C_{i_1 j_1}^{\;\;k_1} X_{k_1}, \quad [X_{i_2}, X_{j_2}] = \frac{i}{2\sqrt{l_2(1 - l_2)}} C_{i_2 j_2}^{\;\;k_2} X_{k_2}, \quad [X_{i_1}, X_{i_2}] = 0. \tag{2-23} \]

In the principal fibration \( X_{AdS}^5 \xrightarrow{U(1)} AdS^2 \times AdS^2 \), the module of sections is \( C(AdS^2 \times AdS^2) \)-module \( \Gamma^\infty(AdS^2 \times AdS^2, E^{(n)}) \) in which \( C(AdS^2 \times AdS^2) \) is the commutative algebra of functions on the manifold \( AdS^2 \times AdS^2 \). In the fuzzy case, this algebra is a noncommutative algebra and therefore, left and right
modules are not isomorphic. In this case to each angular momentum operator $L_{i,2}$, we associate two linear operators $L_{i,2}^L$ and $L_{i,2}^R$ with the left and right actions on the fuzzy pseudo Hermitian matrix algebra $A_{i,2} = \{ \psi \in M(2l_1+1)(2l_2+1)(\mathbb{C}) \}$:

$$L_{i,2}^L \psi = L_{i,a} \psi, \quad L_{i,2}^R \psi = \psi L_{i,a}, \quad \forall \psi \in A_{i,2}, \alpha = 1, 2,$$

(2-24)

where the right action satisfies the $su(1,1)$ algebra with minus sign $-L_{i,a}^R$. These left and right operators commute with each other:

$$[L_{i,2}^L, L_{i,2}^R] = 0, \quad \alpha = 1, 2.$$  

(2-25)

The $L_{i,2}^L$ and $L_{i,2}^R$ have the same $su(1,1)$ algebra. The coordinates $(x_{i1}, x_{i2})$ of commutative $AdS^2 \times AdS^2$ can be obtain as the limit case

$$x_{i,a} = \lim_{l_a \to \infty} \frac{L_{i,a}^{L,R}}{l_{i,a}(1-l_{i,a})} = \lim_{l_a \to \infty} L_{i,a}^{L,R}.$$  

(2-26)

We use $L_{i,2}^L$, $L_{i,2}^R$, to define the fuzzy version of orbital momentum operators $L_\alpha$ on the fuzzy $AdS^2 \times AdS^2$. We define $L_\alpha$ by the adjoint action of $L_i$ on the $A_i$:

$$L_i \psi = (L_i^{L,R}) \psi = ad \psi L_i = [L_i, \psi], \quad \alpha = 1, 2.$$  

(2-27)

It is easy to see that the algebra of $L_i$ is $su(1, 1)$ Lie algebra:

$$[L_i, L_j] = iC_{i,j,a} L_{k,a}, \quad [L_i, L_{i2}] = 0.$$  

(2-28)

In the commutative limit we have the following angular momentum operators on commutative anti-deSitter space:

$$\lim_{l_a \to \infty} (L_i^{L,R}) = iC_{i,j,a} x_{j,a} \frac{\partial}{\partial x_{k,a}}, \quad \alpha = 1, 2.$$  

(2-29)

3 Projective module description of the fibration $X^5_{AdS} \to AdS^2 \times AdS^2$

Consider the $U(1)$ principal fibration $\pi$ with $X^5_{AdS} \cong SU(1, 1) \times SU(1, 1)$ as total space:

$$U(1) \xrightarrow{\text{right } U(1)-\text{action}} X^5_{AdS} \xrightarrow{\pi} AdS^2 \times AdS^2,$$

(3-1)

over the four-dimensional cross manifold $AdS^2 \times AdS^2$. The total manifold is $X^5_{AdS}$ which is the base of the conifold $Y^5_{AdS}$. Let $B_C = C^\infty(X^5_{AdS}, \mathbb{C})$ and $A_C = C^\infty(AdS^2 \times AdS^2, \mathbb{C})$ denote the algebras of C-valued smooth functions on the total manifold $X^5_{AdS}$ and base manifold $AdS^2 \times AdS^2$ under point-wise multiplication, respectively. The irreducible representations of the group $U(1)$ are labeled by an integer $n$. The elements of $B_C$ can be classified into the right modules,

$$C^\infty_{(\pm n)}(X^5_{AdS}, \mathbb{C}) = \{ \varphi_{(\pm n)} : X^5_{AdS} \to \mathbb{C}, \quad \varphi_{(\pm n)}(p \cdot \omega) = \omega^{(\pm n)} \varphi(p), \quad \forall p \in X^5_{AdS}, \forall \omega \in U(1) \},$$

(3-2)

over the pull back of the $A_C$. The left actions of the group $U(1)$ on $\mathbb{C}$ are labeled by an integer $n$ which characterizes the bundle. The Serre-Swan theorem [25] states that for a compact smooth manifold $M$, there is a complete equivalence between the category of vector bundles over that manifold and bundle maps, and the category of finitely generated projective modules over the algebra $C(M)$ of functions over $M$ and module morphisms. In algebraic $K$-theory, it is well known that corresponds to these bundles, there are projectors $P_n$ [25] such that, for the associated vector bundle

$$E^{(n)} = X^5_{AdS} \times U(1) \mathbb{C} \xrightarrow{\pi} AdS^2 \times AdS^2,$$

(3-3)
right $A_c$-module of sections $\Gamma(\text{AdS}^2 \times \text{AdS}^2, E^{(n)})$ which is isomorphic with $C^{\infty}_{(n)}(X^5_{\text{AdS}}, \mathbb{C})$ is equivalent to the image in the free module $(A_c)^{(2n+1)} = C^{\infty}(\text{AdS}^2 \times \text{AdS}^2, \mathbb{C}) \otimes \mathbb{C}^{2n+1}$ of a projector $P_n$. $\Gamma(\text{AdS}^2 \times \text{AdS}^2, E^{(n)}) = P_n(A_c)^{2n+1}$. The projector $P_n$ is a $\Lambda$-pseudo Hermitian operator of rank 1 over $\mathbb{C}$$$
abla_{\alpha} P_n = P_n, \quad P_n^\dagger = \Lambda P_n \Lambda^{-1}, \quad Tr P_n = 1.
$$ (3-4)

where $Tr$ is trace and $1$ is the constant function. For the right $A_c$-module of sections $\Gamma(\text{AdS}^2 \times \text{AdS}^2, E^{(n)})$ there exist $n + 1$ projectors $P_1, P_2, \ldots, P_{n+1}$ having the same rank 1. Therefore, the free module $(A_c)^{2n+1}$ can be written as a direct sum of the projective $A_c$-modules,$$(A_c)^{2n+1} = \bigoplus_{i=1}^{2n+1} P_i(A_c)^{2n+1}.
$$ (3-5)

4 Spin $\frac{1}{2}$ pseudo-projectors of the pseudo-projective

$A(\text{AdS}^2_F \times \text{AdS}^2_F)$—module

According to the Serre-Swan’s theorem, in noncommutative geometry, the study of the principal fibration

\[X^5_{\text{AdS}_F} \xrightarrow{U(1)} \text{AdS}^2_F \times \text{AdS}^2_F,\]

replaces with the study of noncommutative finitely generated pseudo-projective $A(\text{AdS}^2_F \times \text{AdS}^2_F)$—module of its sections. To build the left and right pseudo-projective modules we should construct the fuzzy pseudo-projectors of these modules. The pseudo $\Lambda$—projectors for left pseudo-projective module can be written as:

\[P_{L,\alpha}^{\lambda,\omega} = \frac{1}{2} \left\{ 1 + \left( \frac{\alpha \cdot X_{\alpha} - \frac{\mu_\omega}{\omega}}{\sqrt{1 - \frac{\mu_\omega^2}{\omega^2}}} \right) \right\}, \quad (P_{L,\alpha}^{L,\omega})^\dagger = \Lambda P_{L,\alpha}^{L,\omega} \Lambda^{-1}, \alpha = 1, 2,
\]

(4-1)

where $\alpha = 1$ and $\alpha = 2$ are for the first and second fuzzy anti-deSitter space in $\text{AdS}^2_F \times \text{AdS}^2_F$, respectively and $\Sigma_{\alpha,1}, \Sigma_{\alpha,2}$ and $\Sigma_{\alpha,3}$ are the generators of $\text{su}(1, 1)$ Lie algebra. Substituting (2-21) in (4-1) we can write:

\[P_{L,\alpha}^{L,\omega} = \frac{1}{2} \left[ 1 + \frac{\Sigma_{\alpha} \cdot L_{\alpha} - 1}{\sqrt{l_{\alpha}(1 - l_{\alpha}) - 1}} \right],
\]

(4-2)

which couples left angular momentum and spin $\frac{1}{2}$ to its maximum and minimum values $l_{\alpha} \pm \frac{1}{2}$, respectively. It is easy to see that

\[P_{L,\alpha}^{L,\omega} + P_{L,\alpha}^{L,\omega} = 1(2l_{\alpha+1}(2l_{\alpha+1}).
\]

(4-3)

These are the pseudo-projectors of our left projective $A(\text{AdS}^2_F)$—module and we have

\[(A(\text{AdS}^2_F))^2 = (A(\text{AdS}^2_F))^2 P_{L,\alpha}^{L,\omega} \oplus (A(\text{AdS}^2_F))^2 P_{L,\alpha}^{L,\omega}.
\]

(4-4)

One can expand these operators to act on $A(\text{AdS}^2_F \times \text{AdS}^2_F)$—module. Let us define $P_L := (P_{L,1}, P_{L,2})$. Then we have

\[(A(\text{AdS}^2_F \times \text{AdS}^2_F))^2 = (A(\text{AdS}^2_F \times \text{AdS}^2_F))^2 P_{L,\alpha}^{L,\omega} \oplus (A(\text{AdS}^2_F \times \text{AdS}^2_F))^2 P_{L,\alpha}^{L,\omega}.
\]

(4-5)

Using (4-3) we can define the corresponding $\Lambda$—pseudo idempotents as:

\[\Gamma_{L,\alpha}^{L,\omega} = 2P_{L,\alpha}^{L,\omega} - 1 = \frac{\Sigma_{\alpha} \cdot L_{\alpha} - 1}{\sqrt{l_{\alpha}(1 - l_{\alpha}) - 1}}, \quad (\Gamma_{L,\alpha}^{L,\omega})^\dagger = \Lambda \Gamma_{L,\alpha}^{L,\omega} \Lambda^{-1}.
\]

(4-6)
The pseudo-projectors $P^{R,\alpha}_{(l_a, \pm \frac{1}{2})}$ coupling the right momentum and spin $\frac{1}{2}$ to its maximum and minimum values $l_a \pm \frac{1}{2}$, respectively, are obtained by changing $L^R_{\alpha}$ to $-L^R_{\alpha}$ in the above expression

$$P^{R,\alpha}_{(l_a, \pm \frac{1}{2})} = \frac{1}{2}[1 \pm \frac{\Sigma_\alpha \cdot L^R_\alpha + 1}{\sqrt{l_a(1 - l_a) - 1}}], \quad (P^{R,\alpha}_{(l_a, \pm \frac{1}{2})})^\dagger = \Lambda P^{R,\alpha}_{(l_a, \pm \frac{1}{2})}\Lambda^{-1}. \quad (4-7)$$

These are the pseudo-projectors of our right projective $A(AdS^2_F)$—module

$$(A(AdS^2_F))^2 = (A(AdS^2_F))^2 P^{R,\alpha}_{(l_a, \pm \frac{1}{2})} \oplus (A(AdS^2_F))^2 P^{R,\alpha}_{(l_a, \mp \frac{1}{2})} \quad (4-8)$$

One can expand these operators to act on $A(AdS^2_F \times AdS^2_F)$—module. Let us define $P^R := (P^{R,1}, P^{R,2})$. Then we have

$$(A(AdS^2_F \times AdS^2_F))^2 = (A(AdS^2_F \times AdS^2_F))^2 P^R_{(l \mp \frac{1}{2})} \oplus (A(AdS^2_F \times AdS^2_F))^2 P^R_{(l \mp \frac{1}{2})} \quad (4-9)$$

The corresponding $\Lambda$—pseudo idempotents are

$$\Gamma^{R,\alpha}_{(l_a, \pm \frac{1}{2})} = 2P^{R,\alpha}_{(l_a, \pm \frac{1}{2})} - 1 = \pm \frac{\Sigma_\alpha \cdot L^R_\alpha + 1}{\sqrt{l_a(1 - l_a) - 1}}, \quad (\Gamma^{R,\alpha}_{(l_a, \pm \frac{1}{2})})^\dagger = \Lambda \Gamma^{R,\alpha}_{(l_a, \pm \frac{1}{2})}\Lambda^{-1}. \quad (4-10)$$

5. **Fuzzy pseudo Ginsparg-Wilson algebra and its Spin $\frac{1}{2}$ fuzzy Dirac and chirality operators on $AdS^2_F \times AdS^2_F$**

The fuzzy Ginsparg-Wilson algebra $\mathcal{A}$ is the $\Lambda$—pseudo $\dagger$—algebra over $\mathbb{C}$, generated by two $\Lambda$—pseudo $\dagger$—invariant involution $\Gamma_\alpha$ and $\Gamma'_\alpha$:

$$\mathcal{A}_\alpha = \{ \Gamma_\alpha, \Gamma'_\alpha \} : \quad (\Gamma_\alpha)^2 = (\Gamma'_\alpha)^2 = I_{(2l_\alpha + 1)(2l_\alpha + 1)}, \quad (\Gamma_\alpha) = \Lambda \Gamma_\alpha \Lambda^{-1}, \quad (\Gamma'_\alpha) = \Lambda \Gamma'_\alpha \Lambda^{-1}, \alpha = 1, 2, \quad (5-1)$$

where $\alpha = 1, 2$ denote the Ginsparg-Wilson algebra associated to first and second anti-deSitter in $AdS^2_F \times AdS^2_F$. Each representation of (5-1) is a realization of the Ginsparg-Wilson algebra. Now, consider the following two elements constructed out of the generators $\Gamma_\alpha$ and $\Gamma'_\alpha$ of the pseudo fuzzy Ginsparg-Wilson algebra $\mathcal{A}_\alpha$:

$$\Gamma_{\alpha,1} = \frac{1}{2}(\Gamma_\alpha + \Gamma'_\alpha), \quad (\Gamma_{\alpha,1})^\dagger = \Lambda \Gamma_{\alpha,1} \Lambda^{-1}, \quad (\Gamma_{\alpha,1}) = \frac{1}{2}(\Gamma_\alpha - \Gamma'_\alpha), \quad (\Gamma_{\alpha,2})^\dagger = \Lambda \Gamma_{\alpha,2} \Lambda^{-1}. \quad (5-2)$$

So that, $\Gamma_{\alpha,1}$ and $\Gamma_{\alpha,2}$ anticommute with each other:

$$\{ \Gamma_{\alpha,1}, \Gamma_{\alpha,2} \} = 0. \quad (5-3)$$

Identifying $\Gamma^{L,\alpha}_{(l_a, \pm \frac{1}{2})}$ and $\Gamma^{R,\alpha}_{(l_a, \pm \frac{1}{2})}$ with $\Gamma_\alpha$ and $\Gamma'_\alpha$, we get:

$$\Gamma^{\pm,\alpha}_{\alpha,1} = \pm \frac{\Sigma_\alpha \cdot L^F_\alpha - 1}{\sqrt{l_a(1 - l_a) - 1}}, \quad \Gamma^{\pm,\alpha}_{\alpha,2} = \pm \frac{\Sigma_\alpha \cdot (L^L_\alpha + L^R_\alpha)}{\sqrt{l_a(1 - l_a) - 1}}. \quad (5-4)$$

Now, let us define the $\Lambda$—pseudo fuzzy Dirac operator on each fuzzy anti-deSitter spaces $AdS^2_F$ as:

$$D^\pm_{\alpha,F} = \sqrt{l_a(1 - l_a) - 1} \Gamma^{\pm,\alpha}_{\alpha,1} = \pm(\Sigma_\alpha \cdot L^F_\alpha - 1) = \pm(\Sigma_\alpha \eta^{\alpha,j} L^F_{\alpha,j}), \quad (D^\pm_{\alpha,F})^\dagger = \Lambda D^\pm_{\alpha,F} \Lambda^{-1}. \quad (5-5)$$

In the limit case (5-5)becomes the Dirac operator on each commutative $AdS^2$:

$$\lim_{l_a \to \infty} D^\pm_{\alpha,F} = \pm(\Sigma_\alpha \cdot L_\alpha - 1). \quad (5-6)$$
Also, we can define pseudo-chirality operators on $AdS^2_{\alpha,F}$ as $\gamma^\pm_{\alpha,F} = \Gamma^\pm_{\alpha,2}$ which in the commutative limit they become:

$$\lim_{l_\alpha \to \infty} \gamma^\pm_{\alpha,F} = \pm \Omega_{\alpha} \cdot x_\alpha = \pm \Omega_{\alpha} \eta^\alpha j^\alpha x_\alpha.$$  \hspace{1cm} (5-7)

Also, it is easy to see that

$$\lim_{l_\alpha \to \infty} \{D^\pm_{\alpha,F}, \gamma^\pm_{\alpha,F}\} = 0 \hspace{1cm} (5-8)$$

which we expect from Dirac and chirality operators on each $AdS^2$. Now, let us define the Ginsparg-Wilson algebra for $AdS^2_F \times AdS^2_F$ and then construct its corresponding pseudo Dirac and chirality operators. This algebra is a $\Lambda$—pseudo—$\dagger$—invariant algebra over $\mathbb{C}$ which can be defined as:

$$\mathcal{A}_{1,2} = \langle \Gamma = \Gamma_1 \Gamma_2, \qquad \Gamma^\dagger = \Gamma_1^\dagger \Gamma_2^\dagger \quad \Gamma^2 = \Gamma^\dagger = 1, \quad \Gamma^\dagger = \Lambda \Gamma \Lambda^{-1}, \quad \Gamma^\dagger = \Lambda \Gamma^\dagger \Lambda^{-1} \rangle.$$  \hspace{1cm} (5-9)

where $\Gamma_{1,2}$ and $\Gamma^\dagger_{1,2}$ are the generators of the Ginsparg-Wilson algebra of each $AdS^2$ which are given in (5-4). Also, they satisfy the following commutation relations

$$[\Gamma_1, \Gamma_2] = [\Gamma_1, \Gamma_2^\dagger] = [\Gamma_1^\dagger, \Gamma_2] = 0.$$  \hspace{1cm} (5-10)

We consider the radii of the anti-deSitter spaces equal. We can chose one of the generators of the Ginsparg-Wilson algebra (5-9) as chirality operator on $AdS^2_F \times AdS^2_F$ because in the commutative limit both of them become the same chirality operators $\gamma = \gamma_1 \gamma_2$ on the commutative $AdS^2 \times AdS^2$. Let us define the $\Lambda$—pseudo fuzzy Dirac operator as:

$$D_F = \frac{\Gamma_1 \Gamma_2 - \Gamma_1^\dagger \Gamma_2^\dagger}{2\sqrt{l_1(1 - l_1) - 1}\sqrt{l_2(1 - l_2) - 1}}, \quad D_F^\dagger = \Lambda D_F \Lambda^{-1}.$$  \hspace{1cm} (5-11)

It is easy to see that

$$\Gamma_1 \Gamma_2 - \Gamma_1^\dagger \Gamma_2^\dagger = \frac{1}{2}[(\Gamma_1 - \Gamma_1^\dagger)(\Gamma_2 + \Gamma_2^\dagger) + (\Gamma_1 + \Gamma_1^\dagger)(\Gamma_2 - \Gamma_2^\dagger)].$$  \hspace{1cm} (5-12)

Now, using the following definitions

$$D_1^F = \frac{(\Gamma_1 - \Gamma_1^\dagger)(\Gamma_2 + \Gamma_2^\dagger)}{2\sqrt{l_1(1 - l_1) - 1}\sqrt{l_2(1 - l_2) - 1}}, \quad D_2^F = \frac{(\Gamma_1 + \Gamma_1^\dagger)(\Gamma_2 - \Gamma_2^\dagger)}{2\sqrt{l_1(1 - l_1) - 1}\sqrt{l_2(1 - l_2) - 1}}.$$  \hspace{1cm} (5-13)

which satisfies

$$[D_1^F, D_2^F] = 0, \quad (D_1^F)\dagger = \Lambda D_1^F \Lambda^{-1}.$$  \hspace{1cm} (5-14)

The pseudo fuzzy Dirac operator (5-11) on $AdS^2_F \times AdS^2_F$ can be written as:

$$D_F = D_1^F + D_2^F.$$  \hspace{1cm} (5-15)

In the commutative limit (5-15) tends to

$$\lim_{l_{1,2} \to \infty} D_F = D_1 + D_2 = (\Sigma_1 \cdot L_1 + 1)(\Sigma_2 \cdot x_2) + (\Sigma_1 \cdot x_1)(\Sigma_2 \cdot L_2 + 1).$$  \hspace{1cm} (5-16)

## 6 Fuzzy gauged pseudo Dirac operator ( no instanton fields)

Let us denote by $A^L = (A^L_1, A^L_2)$ the connection 1—form associated with the pseudo-projector $P$ on $AdS^2_F \times AdS^2_F$. $A^L_1$ and $A^L_2$ are connection 1—forms on the first and second $AdS^2$, respectively.

$$A^L \in \text{End}_\mathbb{C}(C^\infty(X^5_{AdS_F}), \mathbb{C}) \otimes_\mathbb{C} \Omega^1((X^5_{AdS_F}), \mathbb{C})$$  \hspace{1cm} (6-1)
The components of this $U(1)$ gauged field according to our principal fibration are given by

$$A = (dx_{i_1}, A_{i_1}, dx_{i_2} A_{i_2}). \quad (6-2)$$

The $\hat{\dagger}$-invariant fuzzy gauge field $A^L$ acts on $\xi = (\xi_i, \xi_\bar{i})$, $\xi_i \in AdS_2^\beta(2l + 1)$ as:

$$[(A^L_{i_1}, A^L_{i_2})(\xi_i, \xi_{\bar{i}})]_m = ((A_{i_1})_m (\xi_i)_n, (A_{i_2})_m (\xi_{\bar{i}})_n). \quad (6-3)$$

The $\Lambda$-pseudo $\hat{\dagger}$-invariant condition on $A^L_{i_1 i_2}$ is:

$$(A^L_{i_1 i_2})^{\dagger} = \Lambda A^\dagger_{i_1 i_2} \Lambda^{-1}. \quad (6-4)$$

The corresponding curvature 2-form $F^L$ on $X^5_{AdS^p}$

$$F^L \in \text{End}_C(C^\infty(X_{AdS^p}^5), \mathbb{C}) \otimes_\mathbb{C} \Omega^2((X^5_{AdS^p}), \mathbb{C}) \quad (6-5)$$

is given by

$$F_{i_1 j_1}^L = i([L_{i_1}, A^L_{j_1}] + [A^L_{i_1}, L^L_{j_1}] + [A^L_{i_1}, A^L_{j_1}]),$$

$$F_{i_2 j_2}^L = i([L_{i_2}, A^L_{j_2}] + [A^L_{i_2}, L^L_{j_2}] + [A^L_{i_2}, A^L_{j_2}]),$$

$$F_{i_2 j_2}^L = i([L_{i_1}, A^L_{j_1}] + [A^L_{i_1}, L^L_{j_2}] + [A^L_{i_1}, A^L_{j_2}]). \quad (6-6)$$

The components of curvature 2-form also satisfies:

$$[(L + A)^L_{i_1}, (L + A)^L_{j_1}] = iC^L_{i_1 j_1} (L + A)^L_{k_1} + iF_{i_1 j_1}^L,$$

$$[(L + A)^L_{i_2}, (L + A)^L_{j_2}] = iC^L_{i_2 j_2} (L + A)^L_{k_2},$$

$$[(L + A)^L_{i_2}, (L + A)^L_{j_2}] = iC^L_{i_2 j_2} (L + A)^L_{k_2} + iF_{i_2 j_2}^L. \quad (6-7)$$

The fuzzy gauge field $A^L$ on the commutative $AdS_2 \times AdS_2$ becomes a commutative field $a = (a_1, a_2)$ and its components $a_{i_1 i_2}$ satisfy the following condition:

$$x \cdot a = (x_{i_1} a_{i_1 j_1}, x_{i_2} a_{i_2 j_2}) = (0, 0). \quad (6-8)$$

We need a condition to get the above result for large $l$. One of the conditions of such a nature on each $AdS_2$ is:

$$(L^L_{a_1} + A^L_{a_1}) \cdot (L^L_{a_2} + A^L_{a_2}) = L^L_{a_1} \cdot L^L_{a_2} = L^L_{a_1} \eta_\alpha \eta_\beta L^L_{a_2} = L_\alpha (1 - L_\alpha), \quad \alpha = 1, 2. \quad (6-9)$$

The expansion of (6-9) is:

$$L^L_{a_1} \eta_\alpha \eta_\beta A^L_{a_2} + L^L_{a_2} \eta_\alpha \eta_\beta A^L_{a_1} = 0. \quad (6-10)$$

When the parameter $l_\alpha$ tends to infinity, $\frac{A^L_{a_1}}{l_\alpha}$ tends to zero. Also, in this limit $L^L_{i_1}$, $L^L_{i_2}$ and $A^L_{i_1}$ tends to $x_{i_1}$, $x_{i_2}$ and $a_{i_1}$, respectively. So we have the condition $x_{a_1} \cdot a_1 = 0$ on each $AdS_2$.

Now, we can introduced the $\Lambda$-pseudo gauged Ginsparg–Wilson system on each $AdS_2$ as follow: For each $\alpha = 1, 2$, we can set:

$$A^\pm_\alpha (A^L_{a_1}) = (\Gamma^\pm_\alpha (A^L), \Gamma^\pm_\alpha (A^L) = \Gamma^\pm_\alpha = 1, \quad \Gamma^\pm_\alpha (A^L) = \Lambda \Gamma^\pm_\alpha (A^L) \Lambda^{-1}, \quad \Gamma^\pm_\alpha = \Lambda \Gamma^\pm_\alpha \Lambda^{-1}), \quad (6-11)$$

where

$$\Gamma^\pm_\alpha (A^L) = \frac{\Sigma_\alpha \cdot (L^L_{a_1} + A^L_{a_1}) - 1}{\Sigma_\alpha \cdot (L^L_{a_1} + A^L_{a_1}) - 1}, \quad \Gamma^\pm_\alpha (A^L) = \Gamma^\pm_\alpha (0) = \pm \frac{\Sigma_\alpha \cdot L^L_{a_1} + 1}{\Sigma_\alpha \cdot L^L_{a_1} + 1} \quad (6-12)$$
They are involutory and $\Lambda$—pseudo $\dagger$—invariant operators:

$$\Gamma_\alpha(A_L^\alpha)^2 = 1, \quad \Gamma_\alpha(A_L^\alpha) = \Lambda \Gamma_\alpha(A_L^\alpha) \Lambda^{-1}. \quad (6-13)$$

The gauged involution (6-12), reduces to (5-4) for zero $A^L$. We put $\Gamma_\alpha = \Gamma_\alpha(A_L^\alpha = 0)$.

Also, we can define the second gauged involution as:

$$\Gamma_\alpha'(A_L^\alpha) = \Gamma_\alpha'(0) = \Gamma_\alpha'. \quad (6-14)$$

We put $\Gamma_\alpha' = \Gamma_\alpha'(A_L^\alpha = 0)$. Notice that, the operators $L^1,R$ do not have continuity limit as their squares $l_\alpha(1-l_\alpha)$ diverge as $l_\alpha$ tends to infinity. In contrast, $L_\alpha$ and $A_L^\alpha$ do have continuity limits.

It is easy to see that up to the first order (6-12) becomes:

$$\Gamma_\alpha^\pm(A_L^\alpha) = \mp \frac{\Sigma_\alpha \cdot (L^2_L + A^L_L) - 1}{\sqrt{l_\alpha(1-l_\alpha) - 1}}. \quad (6-15)$$

and

$$\Gamma_\alpha'^\pm(A_L^\alpha) = \mp \frac{\Sigma_\alpha \cdot L^R + 1}{\sqrt{l_\alpha(1-l_\alpha) - 1}}. \quad (6-16)$$

Using (6-15) and (6-16) we can construct the following $\Lambda$—pseudo $\dagger$—invariant operators:

$$\Gamma_{1,\alpha}^\pm(A_L^\alpha) = \frac{1}{2}(\Gamma_{1,\alpha}^\pm(A_L^\alpha) + \Gamma_{2,\alpha}^\pm), \quad (\Gamma_{1,\alpha}^\pm)^\dagger = \Lambda \Gamma_{1,\alpha}^\pm \Lambda^{-1}, \quad (6-17)$$

$$\Gamma_{2,\alpha}^\pm(A_L^\alpha) = \frac{1}{2}(\Gamma_{2,\alpha}^\pm(A_L^\alpha) - \Gamma_{2,\alpha}^\pm), \quad (\Gamma_{2,\alpha}^\pm)^\dagger = \Lambda \Gamma_{2,\alpha}^\pm \Lambda^{-1}. \quad (6-17)$$

Now, let us define the gauged pseudo fuzzy Dirac and chirality operators on each $AdS_2^\alpha$ as:

$$D_{F,\alpha}(A_L^\alpha) = \mp \frac{\Sigma_\alpha \cdot (L^L_L + L^R + A^L_L) - 1}{\sqrt{l_\alpha(1-l_\alpha) - 1}}. \quad (6-18)$$

and for the chirality operator:

$$\gamma_{F,\alpha}(A_L^\alpha) = \Gamma_{2,\alpha}^\pm(A_L^\alpha) = \mp \frac{\Sigma_\alpha \cdot (L^L_L + L^R + A^L_L) - 1}{\sqrt{l_\alpha(1-l_\alpha) - 1}}. \quad (6-19)$$

In the commutative limit when $l_\alpha$ tends to infinity (6-18) and (6-19) become:

$$\lim_{l_\alpha \to \infty} D_{F,\alpha}(A_L^\alpha) = \pm (\Sigma_\alpha \cdot (\mathcal{L} + A^L_L) - 1), \quad \lim_{l_\alpha \to \infty} \gamma_{F,\alpha}(A_L^\alpha) = \mp \Sigma_\alpha \cdot x_\alpha. \quad (6-20)$$

These are the correct pseudo gauged Dirac and chirality operators on each commutative $AdS^2$.

Now, let us construct fuzzy Ginsparg–Wilson algebra for $AdS_2 \times AdS_2$. It is a $\Lambda$—pseudo $\dagger$—invariant algebra over $\mathbb{C}$,

$$A_{l_1,l_2}(A^L) = (\Gamma(A^L) = \Gamma_1(A^L_1)\Gamma_2(A^L_2), \quad \Gamma' = \Gamma_1^\prime \Gamma_2', \quad \Gamma^2 = \Gamma^2_1 = 1, \quad \Gamma^\dagger = \Lambda \Gamma \Lambda^{-1}, \quad \Gamma'^\dagger = \Lambda \Gamma' \Lambda^{-1}). \quad (6-21)$$

where $\Gamma_1,\Gamma_2$ are the generators of the Ginsparg–Wilson algebra of each $AdS_2$ which are given in (6-15) and (6-16). Let us define the fuzzy $\Lambda$—pseudo gauged Dirac operator on $AdS_2 \times AdS_2$ as:

$$D_F(A^L) = \frac{\Gamma_1(A^L_1)\Gamma_2(A^L_2) - \Gamma_1'(A^L_2)\Gamma_2'(A^L_2)}{2\sqrt{l_1(1-l_1) - 1}\sqrt{l_2(1-l_2) - 1}} \quad D_F(A^L)^\dagger = \Lambda D_F(A^L) \Lambda^{-1}. \quad (6-22)$$

It is easy to see that

$$\Gamma_1(A^L_1)\Gamma_2(A^L_2) - \Gamma_1'(A^L_2)\Gamma_2'(A^L_2) = \frac{1}{2}[(\Gamma_1(A^L_1) - \Gamma_1')(\Gamma_2(A^L_2) + \Gamma_2') + (\Gamma_1(A^L_1) + \Gamma_1')(\Gamma_2(A^L_2) - \Gamma_2])] \quad (6-23)$$
Now, using the following definitions
\[
D_F^L(A^L_\alpha) = \frac{(\Gamma_1(A^L_\alpha) - \Gamma'_1)(\Gamma_2(A^L_\alpha) + \Gamma'_2)}{2\sqrt{l_1(1-l_1) - 1}\sqrt{l_2(1-l_2) - 1}}, \quad D_F^L(A^L_\pm) = \frac{(\Gamma_1(A^L_\pm) + \Gamma'_1)(\Gamma_2(A^L_\pm) - \Gamma'_2)}{2\sqrt{l_1(1-l_1) - 1}\sqrt{l_2(1-l_2) - 1}},
\]
which satisfies
\[
[D_F^L(A^L_\alpha), D_F^L(A^L_\pm)] = 0,
\]
the fuzzy Dirac operator (6-22) on \(AdS^2_F \times AdS^2_F\) can be written as:
\[
D_F(A^L) = D_F^L(A^L_\alpha) + D_F^L(A^L_\pm).
\]

In the commutative limit (6-26) tends to
\[
\lim_{l_1, l_2 \to \infty} D_F(A^L) = D_1(A_1) + D_2(A_2) = (\Sigma_1 \cdot (L_1 + A_1) - 1)(\Sigma_2 \cdot l_{\pm} + 1)\Sigma_2 \cdot (L_2 + A_2) - 1).
\]

7 Fuzzy pseudo Dirac and chirality operators on \(AdS^2_F \times AdS^2_F\) in instanton sector

As we mentioned in section 2, according to the Serre-Swan’s theorem, in noncommutative geometry, the study of the principal fibration \(X^5_{AdS_F} \rightarrow \frac{U(1)}{AdS^2_F \times AdS^2_F}\), replaces with the study of noncommutative finitely generated projective \(\mathcal{A}(AdS^2_F \times AdS^2_F)\)-module of its sections. To build the pseudo-projective module, let introduce \(C^{2l_1+1} \times C^{2l_2+1}\) carrying the \(t_1, t_2\)-representations of angular momentum of \(su(1,1) \times su(1,1)\). Here, the algebra \(su(1,1) \times su(1,1)\) is generated by elements \(T_{t_1}\) and \(T_{t_2}\) satisfying the following relations:
\[
[T_{t_1}, T_{t_2}] = iC_{t_1 t_2} k_1 T_{k_1}, \quad [T_{t_1}, T_{t_2}] = iC_{t_1 t_2} k_2 T_{k_2}, \quad [T_{t_1}, T_{t_2}] = 0.
\]

Also, let \(P_{t_1 t_2}^l\) be the pseudo-projector coupling left angular momentum operator \(L^l_{t_1}\) with \(T_{t_2}\) to produce maximum angular momentum \(l_\alpha + t_\alpha\) on each \(AdS^2_F\). We know that the image of a projector on a free module is a projective module. Then, as \(Mat(2l_1 + 1)^{2l_2+1} = Mat(2l_1 + 1) \otimes C^{2l_2+1}\) is a free module, therefore, \(P_{t_1 t_2}^l\) is the fuzzy version of \(U(1)\) bundle on each \(AdS^2_F\). Also, we can use the pseudo-projector \(P_{t_1 t_2}^l\) to produce the projective module \(P_{t_1 t_2}^l\) to introduce the least angular momentum \((l_\alpha - t_\alpha)\).

The \(\Lambda\)-pseudo fuzzy projectors \(P_{t_1 t_2}^{t_\alpha \pm t_\alpha}\) corresponding to \((t_\alpha \pm t_\alpha)\)-representations of \(su(1,1)\) can be written as:
\[
P_{t_1 t_2}^{t_\alpha \pm t_\alpha} = \frac{1}{2}(1 + \frac{\sum_{\alpha} \cdot (L^L_\alpha + T_{t_\alpha}) - 1}{(l_\alpha \pm t_\alpha)(1 - l_\alpha \mp t_\alpha)}), \quad P_{t_1 t_2}^{t_\alpha \pm t_\alpha} = \Lambda P_{t_1 t_2}^{l_\alpha \pm t_\alpha} \Lambda^{-1}.
\]
\[
Mat(2l_1 + 1) \otimes C^{2l_2+1} = (Mat(2l_1 + 1) \otimes C^{2l_2+1})P_{t_1 t_2}^{l_\alpha \pm t_\alpha} \otimes (Mat(2l_1 + 1) \otimes C^{2l_2+1})P_{t_1 t_2}^{l_\alpha \pm t_\alpha}.
\]

To set the fuzzy Ginsparg-Wilson system in instanton sector to each \(AdS^2_F\), we choose the following \(\Lambda\)-pseudo \(\dagger\)-invariant involution \(\Gamma_{t_\alpha}\) for the highest and lowest weights \(l_\alpha \pm t_\alpha\):
\[
\Gamma_{t_\alpha}^\dagger(T_{t_\alpha}) = 2P_{t_1 t_2}^{l_\alpha \pm t_\alpha} - 1 = \frac{\sum_{\alpha} \cdot (L^L_\alpha + T_{t_\alpha}) - 1}{(l_\alpha \pm t_\alpha)(1 - l_\alpha \mp t_\alpha) - 1}, \quad (\Gamma_{t_\alpha}^\dagger(T_{t_\alpha}))^2 = 1, \quad \Gamma_{t_\alpha}^{\dagger}(T_{t_\alpha}) = \Lambda \Gamma_{t_\alpha}(T_{t_\alpha}) \Lambda^{-1}.
\]

We choose \(\Gamma_{t_\alpha}\) as in (6-16). It is clear that \(\Gamma_{t_\alpha}^\dagger(T = 0) = \Gamma_{t_\alpha}\). On the module \((Mat(2l_1 + 1) \otimes C^{2l_2+1})P_{t_1 t_2}^{l_\alpha \pm t_\alpha}\) we have:
\[
(L^L_\alpha + T_{t_\alpha})^2 = (l_\alpha \pm t_\alpha)(1 - l_\alpha \pm t_\alpha).
\]
Now, we can introduce our pseudo fuzzy Ginsparg-Wilson system in instanton sector for each $AdS^2_F$: as:

\[
A_\alpha^\pm(T_\alpha) = \langle \Gamma_\alpha^\pm(T_\alpha), \Gamma_\alpha' : \Gamma_\alpha^\pm(T_\alpha) = \Gamma_\alpha^2 = 1, \quad \Gamma_\alpha^{\dagger 1}(T_\alpha) = A \Gamma_\alpha^2(T_\alpha) A^{-1}, \quad \Gamma_\alpha^{\dagger 1} = \Lambda \Gamma_\alpha^{\dagger -1} \rangle. \tag{7-6}
\]

Using the definitions (5-2), (5-5) and (7-4) one can calculate $\Lambda-$pseudo fuzzy Dirac and chirality operators on each $AdS^2_F$ which in the commutative limit they become:

\[
\lim_{l_\alpha \to \infty} D^\pm_{F,\alpha}(T_\alpha) = \pm(\Sigma_\alpha \cdot (L_\alpha + T_\alpha) - 1), \quad \lim_{l_\alpha \to \infty} \gamma^\pm_{F,\alpha}(T_\alpha) = \mp \Sigma_\alpha \cdot x_\alpha. \tag{7-7}
\]

These are the correct pseudo Dirac and chirality operators on each commutative $AdS^2$. It is obvious that the Dirac operator (7-7) is $\Lambda-$pseudo $\dagger-$invariant:

\[
D^{(\pm)\dagger}_{F,\alpha}(T_\alpha) = \Lambda D^{(\pm)}_{F,\alpha}(T_\alpha) \Lambda^{-1}, \tag{7-8}
\]

which we expect from commutative Dirac operator in instanton sector on each $AdS^2$.

Now, let us construct pseudo fuzzy Ginsparg-Wilson algebra for $AdS^2_F \times AdS^2_F$ in instanton sector. It is a $\Lambda-$pseudo $\dagger-$invariant algebra over $\mathbb{C}$,

\[
\mathcal{A}_{1,2}(T) = \langle \Gamma(T) = \Gamma_1(T_1) \Gamma_2(T_2), \quad \Gamma' = \Gamma_1' \Gamma_2', \quad \Gamma^2 = \Gamma^{\dagger 2} = 1, \quad \Gamma^{\dagger 1} = A \Gamma^2 \Lambda^{-1}, \quad \Gamma^{\dagger 1} = A \Gamma^{\dagger -1} \rangle. \tag{7-9}
\]

where $\Gamma_{1,2}$ and $\Gamma_{1,2}'$ are the generators of the Ginsparg-Wilson algebra of each $AdS^2_F$ in instanton sector, which are given in (7-4). Let us define the $\Lambda-$pseudo fuzzy gauged Dirac operator on $AdS^2_F \times AdS^2_F$ in instanton sector as:

\[
D_F(T) = \frac{\Gamma_1(T_1) \Gamma_2(T_2) - \Gamma_1' \Gamma_2'}{2\sqrt{(l_1 \pm t_1)(1 - l_1 \mp t_1) - 1}\sqrt{(l_2 \pm t_2)(1 - l_2 \mp t_2) - 1}}, \quad D_F(T)\dagger = \Lambda D_F(T) \Lambda^{-1}. \tag{7-10}
\]

It is easy to see that

\[
\Gamma_1(T_1) \Gamma_2(T_2) - \Gamma_1' \Gamma_2' = \frac{1}{2}[(\Gamma_1(T_1) - \Gamma_1')(\Gamma_2(T_2) + \Gamma_2') + (\Gamma_1(T_2) + \Gamma_1')(\Gamma_2(T_2) - \Gamma_2')] \tag{7-11}
\]

Now, using the following definitions

\[
D^1_F(T_\alpha) = \frac{(\Gamma_1(T_1) - \Gamma_1')(\Gamma_2(T_2) + \Gamma_2')}{2\sqrt{(l_1 \pm t_1)(1 - l_1 \mp t_1) - 1}\sqrt{(l_2 \pm t_2)(1 - l_2 \mp t_2) - 1}}, \tag{7-12}
\]

\[
D^2_F(T_\alpha) = \frac{(\Gamma_1(T_1) + \Gamma_1')(\Gamma_2(T_2) - \Gamma_2')}{2\sqrt{(l_1 \pm t_1)(1 - l_1 \mp t_1) - 1}\sqrt{(l_2 \pm t_2)(1 - l_2 \mp t_2) + 1}}, \tag{7-13}
\]

which satisfies

\[
[D^1_F(T_\alpha), D^2_F(T_\alpha)] = 0 \tag{7-14}
\]

the fuzzy Dirac operator (7-10) on $AdS^2_F \times AdS^2_F$ in instanton sector can be written as:

\[
D_F(T) = D^1_F(T_\alpha) + D^2_F(T_\alpha). \tag{7-15}
\]

in the commutative limit (7-15) tends to

\[
\lim_{l_{1,2} \to \infty} D_F(T) = D_1(T_\alpha) + D_2(T_\alpha) = (\Sigma_1 \cdot (L_1 + T_1) - 1)(\Sigma_2 \cdot x_{t_2}) + (\Sigma_1 \cdot x_{t_1})(\Sigma_2 \cdot (L_2 + T_2) - 1). \tag{7-16}
\]
8 Gauging the pseudo fuzzy Dirac operator in instanton sector

The derivation \( \mathcal{L}_{1,1} \) does not commute with the projectors \( P_F^{(l_\alpha \pm t_\alpha)} \) and then has no action on the modules \( Mat(2l_\alpha + 1)P_F^{(l_\alpha \pm t_\alpha)} \). But \( J_{l_\alpha} = \mathcal{L}_{1,1} + T_{l_\alpha} \) does commute with \( P_F^{(l_\alpha \pm t_\alpha)} \) on each \( AdS^2_F \). Here, \( J_{l_\alpha} \) has been considered as the total angular momentum on each \( AdS^2_F \). Now, we need to gauge \( J_{l_\alpha} \). When \( T = 0 \), the gauge fields \( A_\alpha \) were function of \( L^L_\alpha \). Here, we consider \( A^L_\alpha \) to be a functions of \( L^L_\alpha + T_\alpha \), because \( A^L_\alpha \) does not commute with \( P_F^{(l_\alpha \pm t_\alpha)} \). Let us introduce the covariant derivative as:

\[
\nabla_\alpha = J_\alpha + A^L_\alpha. \quad (8-1)
\]

In this case the limiting transversality of \( L^L_\alpha + T_\alpha \) can be guaranteed by imposing the condition:

\[
(L^L_\alpha + A^L_\alpha + T_\alpha) \cdot (L^L_\alpha + A^L_\alpha + T_\alpha) = (L^L_\alpha + T_\alpha) \cdot (L^L_\alpha + T_\alpha) = (l_\alpha \pm t_\alpha)(1 - l_\alpha \mp t_\alpha), \quad (8-2)
\]

The expansion of (8-2) is:

\[
(L^L_\alpha + T_\alpha) \cdot A^L_\alpha + A^L_\alpha \cdot (L^L_\alpha + T_\alpha) + A_\alpha \cdot A_\alpha = 0. \quad (8-3)
\]

When the parameter \( l_\alpha \) tends to infinity, \( \frac{A^L_\alpha}{l_\alpha} \) and \( \frac{T_\alpha}{l_\alpha} \) tend to zero and \( (L_1^L, L_2^L) \) tend to \((x_1, x_2)\). Then, for large \( l_\alpha \), the (8-3) tends to the condition \( x_\alpha \cdot a_\alpha = 0 \) on each \( AdS^2 \). Now, we can construct the gauged pseudo fuzzy Ginsparg-Wilson system in instanton sector and its corresponding fuzzy Dirac and chirality operators on each \( AdS^2_F \) as follow:

\[
A^\dagger_\alpha(T_\alpha, A^L_\alpha) =
\]

\[
\Gamma^\pm_\alpha(T_\alpha, A^L_\alpha), \Gamma^\prime_\alpha : \Gamma^\pm_\alpha(T_\alpha, A^L_\alpha) = \Gamma^\prime_\alpha = \Gamma^\alpha = 1, \quad \Gamma^\pm(T_\alpha, A^L_\alpha) = \alpha^\dagger(T_\alpha) \Gamma^\pm, \quad \Gamma^\prime = \Gamma^\alpha = \Gamma^\prime \Gamma^\alpha = \Gamma^\prime \Gamma^\alpha = \Gamma^\alpha \Gamma^\prime - 1. \quad (8-4)
\]

We introduce the involutory \( \Lambda \)–pseudo \( \dagger \)–invariant generators of the Ginsparg-Wilson system as:

\[
\Gamma^\pm_\alpha(T_\alpha, A^L_\alpha) = \frac{\Sigma_\alpha \cdot (L^L_\alpha + T_\alpha + A^L_\alpha) - 1}{[\Sigma_\alpha \cdot (L^L_\alpha + T_\alpha + A^L_\alpha) - 1]}, \quad \Gamma^\prime_\alpha = \pm \frac{\Sigma_\alpha \cdot L^L_\alpha + 1}{[\Sigma_\alpha \cdot (L^L_\alpha) + 1]} \quad (8-5)
\]

Now, up to the first order (8-5) becomes:

\[
\Gamma^\pm_\alpha(T_\alpha, A^L_\alpha) = \frac{\Sigma_\alpha \cdot (L^L_\alpha + T_\alpha + A^L_\alpha) - 1}{\sqrt{(l_\alpha \pm t_\alpha)(1 - l_\alpha \mp t_\alpha) - 1}}, \quad \Gamma^\prime_\alpha = \pm \frac{\Sigma_\alpha \cdot L^L_\alpha + 1}{\sqrt{l_\alpha (1 - l_\alpha) - 1}} \quad (8-6)
\]

Using the definitions (5-2), (5-5) and (8-6) one can calculate \( \Lambda \)–pseudo fuzzy Dirac and chirality operators on each \( AdS^2_F \), which in the commutative limit they become:

\[
\lim_{l_\alpha \to \infty} D^\pm_\alpha(T_\alpha, A^L_\alpha) = \pm(\Sigma_\alpha \cdot (L_\alpha + T_\alpha + A^L_\alpha) - 1), \quad \lim_{l_\alpha \to \infty} \gamma^\pm_\alpha(T_\alpha, A^L_\alpha) = \pm \Sigma_\alpha \cdot x_\alpha, \quad (8-7)
\]

which we expect from commutative gauge Dirac and chirality operators in instanton sector.

Now, let us construct gauged pseudo fuzzy Ginsparg-Wilson algebra for \( AdS^2_F \times AdS^2_F \) in instanton sector. It is a \( \Lambda \)–pseudo \( \dagger \)–invariant algebra over \( \mathbb{C} \),

\[
\mathcal{A}_{l_1, l_2}(A^L, T) =
\]

\[
\langle \Gamma(A^L, T) = \Gamma_1(A^L_1, T_1) \Gamma_2(A^L_2, T_2), \quad \Gamma^\prime = \Gamma^\prime_1 \Gamma^\prime_2, \quad \Gamma^2 = \Gamma^\prime 2 = 1, \quad \Gamma^\dagger = \Lambda \Gamma\Lambda^{-1}, \quad \Gamma^\dagger = \Lambda \Gamma^\dagger \Lambda^{-1} \rangle \quad (8-8)
\]

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where $\Gamma_{1,2}$ and $\Gamma'_{1,2}$ are the generators of the gauged Ginsparg-Wilson algebra of each $AdS_F^2$ in instanton sector, which are given in (8-6). Let us define the fuzzy $\Lambda$–pseudo gauged Dirac operator on $AdS_F^2 \times AdS_F^2$ in instanton sector as:

$$D_F(A^L, T) = \frac{\Gamma_1(A^L_1, T_1)\Gamma_2(A^L_2, T_2) - \Gamma'_1\Gamma'_2}{2\sqrt{(l_1 \pm t_1)(1-l_1 \mp t_1) - l_1\sqrt{(l_2 \pm t_2)(1-l_2 \pm t_2) - 1}}} ,$$

$$D_F(A^L, T)^\dagger = \Lambda D_F(A^L, T)\Lambda^{-1}$$

(8-9)

It is easy to see that

$$\Gamma_1(A^L_1, T_1)\Gamma_2(A^L_2, T_2) - \Gamma'_1\Gamma'_2 =$$

$$\frac{1}{2}[(\Gamma_1(A^L_1, T_1) - \Gamma'_1)(\Gamma_2(A^L_2, T_2) + \Gamma'_2) + (\Gamma_1(A^L_1, T_2) + \Gamma'_1)(\Gamma_2(A^L_2, T_2) - \Gamma'_2)]$$

(8-10)

Now, using the following definitions

$$D_1^F(A^L_{\alpha}, T_\alpha) = \frac{(\Gamma_1(A^L_1, T_1) - \Gamma'_1)(\Gamma_2(A^L_2, T_2) + \Gamma'_2)}{2\sqrt{(l_1 \pm t_1)(1-l_1 \mp t_1) - l_1\sqrt{(l_2 \pm t_2)(1-l_2 \pm t_2) - 1}}}$$

(8-11)

$$D_2^F(A^L_{\alpha}, T_\alpha) = \frac{(\Gamma_1(A^L_1, T_1) + \Gamma'_1)(\Gamma_2(A^L_2, T_2) - \Gamma'_2)}{2\sqrt{(l_1 \pm t_1)(1-l_1 \mp t_1) - l_1\sqrt{(l_2 \pm t_2)(1-l_2 \pm t_2) - 1}}}$$

(8-12)

which satisfies

$$[D_1^F(A^L_{\alpha}, T_\alpha), D_2^F(A^L_{\alpha}, T_\alpha)] = 0 ,$$

(8-13)

the fuzzy Dirac operator (8-9) on $AdS_F^2 \times AdS_F^2$ in instanton sector can be written as:

$$D_F(A^L, T) = D_1^F(A^L_{\alpha}, T_\alpha) + D_2^F(A^L_{\alpha}, T_\alpha).$$

(8-14)

in the commutative limit (8-14) tends to

$$\lim_{t_{1,2} \to \infty} D_F(A^L, T) = D_1(A_{\alpha}, T_\alpha) + D_2(A_{\alpha}, T_\alpha) =$$

$$(\Sigma_1 \cdot (\mathcal{L}_1 + A_1 + T_1) - 1)(\Sigma_2 \cdot x_{i_2}) + (\Sigma_1 \cdot x_{i_1})(\Sigma_2 \cdot (\mathcal{L}_2 + A_2 + T_2) - 1).$$

(8-15)

9 Conclusion

In this paper, using the $\Lambda$–pseudo projectors and idempotents of the finitely generated projective $A(AdS_F^2 \times AdS_F^2)$–module of the principal fibration $X^{5}_{AdS_F} \to AdS_F^2 \times AdS_F^2$ it has been constructed the generators of the gauged fuzzy Ginsparg-Wilson algebra in instanton sector. It has been constructed gauged fuzzy Dirac operator in instanton sector using the fuzzy Ginsparg-Wilson algebra. The importance of this Dirac operator is that it has correct commutative limit.

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