On Reductions of Soliton Solutions of multi-component NLS models and Spinor Bose-Einstein condensates

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Abstract. We consider a class of multicomponent nonlinear Schrödinger equations (MNLS) related to the symmetric BD.I-type symmetric spaces. As important particular case of these MNLS we obtain the Kulish-Sklyanin model. Some new reductions and their effects on the soliton solutions are obtained by proper modifying the Zakahrov-Shabat dressing method.

Keywords: Multicomponent nonlinear Schrödinger equations, dressing method, soliton solutions, reduction group

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INTRODUCTION

Consider Bose-Einstein condensate (BEC) of alkali atoms in the $F = 1$ hyperfine state, elongated in $x$ direction and confined in the transverse directions $y, z$ by purely optical means. The dynamics of this assembly of atoms is described by a 3-component normalized spinor wave vector $\vec{\Phi} = (\Phi_1, \Phi_0, \Phi_{-1})^T(x,t)$ satisfying the multicomponent nonlinear Schrödinger (MNLS) equation [1, 2, 3], which in dimensionless coordinates can be written down as:

\begin{equation}
\begin{aligned}
&i \partial_t \Phi_\pm + \partial_x^2 \Phi_\pm + 2(|\Phi_\pm|^2 + |\Phi_0|^2)\Phi_\pm + \Phi_\mp^* \Phi_0^2 = 0, \\
&i \partial_t \Phi_0 + \partial_x^2 \Phi_0 + (2|\Phi_1|^2 + |\Phi_0|^2 + 2|\Phi_{-1}|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} = 0,
\end{aligned}
\end{equation}

The second model which describes BEC with $F = 2$ hyperfine structure is a 5-component MNLS system:

\begin{equation}
\begin{aligned}
&i \partial_t \Phi_\pm + \partial_x^2 \Phi_\pm + 2 \left((\vec{\Phi}^\dagger, \vec{\Phi}) - |\Phi_\pm|^2\right)\Phi_\pm + 2\Phi_\mp^* \Phi_+ \Phi_{-1} - \Phi_{\mp2}^* \Phi_0^2 = 0, \\
&i \partial_t \Phi_{\mp1} + \partial_x^2 \Phi_{\mp1} + 2 \left((\vec{\Phi}^\dagger, \vec{\Phi}) - |\Phi_{\mp1}|^2\right)\Phi_{\mp1} + 2\Phi_{\mp1}^* \Phi_+ \Phi_{-1} + \Phi_{\mp1}^* \Phi_0^2 = 0, \\
&i \partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2 \left((\vec{\Phi}^\dagger, \vec{\Phi}) - \frac{1}{2}|\Phi_0|^2\right)\Phi_0 + 2\Phi_0^* \Phi_+ \Phi_{-1} - 2\Phi_0^* \Phi_+ \Phi_{-2} = 0,
\end{aligned}
\end{equation}

where $\langle \vec{\Phi}^\dagger, \vec{\Phi} \rangle = \sum_{k=-2}^2 |\Phi_k|^2$. Both models allow Lax representations and therefore are integrable by the inverse scattering transform method [4, 2, 3]. The Lax pairs have natural Lie algebraic structure which relates them to the symmetric spaces $\text{BD.I} \simeq \text{SO}(n+2)/\text{SO}(n) \times \text{SO}(2)$ with $n = 3$ and $n = 5$ respectively. From algebraic point
of view this means that the potential $Q(x,t)$ of $L$ takes the form $Q(x,t) = [J, X(x,t)]$ where $X(x,t)$ is a generic element of the Lie algebra $so(n+2)$ and the constant element $J = \text{diag}(1, 0, \ldots, 0 - 1)$ is a specially chosen element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For more details see [3, 4].

The present paper extends the results of [2, 3] for the class of MNLS related to BD.I-type symmetric spaces, i.e. for any $n$. We briefly outline how the direct and inverse scattering problem for the Lax operator are reduced to a Riemann-Hilbert problem. Next we find that a simple change of variables can cast the above-mentioned MNLS into the Kulish-Sklyanin model (KSM) [6]. We also apply Mikhailov reduction group method [7] and derive several new types of MNLS interactions. We derive also the constraints on the polarization vectors in the dressing factors that are imposed by the reductions. Finally we apply a proper modification (see [2, 3]) of the Zakharov-Shabat dressing method [8, 9] and derive the soliton solutions of the MNLS and of KSM in particular. Thus we obtain several new types of integrable vector MNLS and their soliton solutions.

The majority of papers devoted to soliton equations analyze and solve the inverse scattering problem (ISP) for the relevant Lax operators using the typical (lowest dimensional) representation of the corresponding Lie algebra. At the end of our paper we briefly compare the properties of the dressing factors in two of the fundamental representations of the Lie algebra $so(2r)$. We also elucidate some additional issues considered in [3, 10] such as the structure of the soliton solutions and the effect of additional $\mathbb{Z}_2$-reductions.

**MNLS EQUATIONS FOR BD.I SERIES OF SYMMETRIC SPACES**

MNLS equations for the BD.I. series of symmetric spaces (algebras of the type $so(n+2)$ and $J$ dual to $e_1$) have the Lax representation $[L, M] = 0$ as follows

\[
L \psi(x,t, \lambda) = i \partial_t \psi + (Q(x,t) - \lambda J) \psi(x,t, \lambda) = 0, \\
M \psi(x,t, \lambda) = i \partial_t \psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J) \psi(x,t, \lambda) = 0, \\
V_1(x,t) = Q(x,t), \\
V_0(x,t) = i \text{ad}_J^{-1} \frac{dQ}{dx} + \frac{1}{2} \left[ \text{ad}_J^{-1} Q, Q(x,t) \right].
\]

where $\text{ad}_J X = [J, X]$ and $\text{ad}_J^{-1}$ is well defined on the image of $\text{ad}_J$ in $\mathfrak{g}$:

\[
Q = \begin{pmatrix}
0 & \bar{q}^T & 0 \\
\bar{p} & 0 & s_0 \bar{q} \\
0 & \bar{p}^T s_0 & 0
\end{pmatrix}, \quad J = \text{diag}(1, 0, \ldots, 0, -1).
\]

The $n$-component vectors $\bar{q}$ and $\bar{p}$ have the form

\[
\bar{q} = (q_1, \ldots, q_n)^T, \quad \bar{p} = (p_1, \ldots, p_n)^T,
\]

while the matrix $s_0 = S^{(n)}$ enters in the definition of $so(n)$:

\[
X \in so(n), \quad X + S^{(n)} X^T S^{(n)} = 0, \quad S^{(n)} = \sum_{s=1}^n (-1)^{s+1} E^{(n)}_{s,n+1-s},
\]
for \( n = 2r + 1 \) and
\[
S^{(n)} = \sum_{s=1}^{r} (-1)^{s+1} (E_{s,n+1-s}^{(n)} + E_{n+1-s,s}^{(n)})
\] (9)
for \( n = 2r \). By \( E_{sp}^{(n)} \) above we mean \( n \times n \) matrix whose matrix elements are \((E_{sp}^{(n)})_{ij} = \delta_{si}\delta_{pj}\). With the definition of orthogonality used in (8) the Cartan generators \( H_k = E_{k,k}^{(n)} - E_{n+1-k,n+1-k}^{(n)} \) are represented by diagonal matrices.

The Lax pairs, related to the symmetric spaces \( SO(n + 2)/(SO(n) \times SO(2)) \) have special algebraic properties. They are determined by choosing \( J = H_1 \) to be dual to \( e_1 \in \mathbb{E}' \). It allows one to introduce a grading in \( \mathfrak{g} \), i.e. \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) so that:
\[
[X_1,X_2] \in \mathfrak{g}_0, \quad [X_1,Y_1] \in \mathfrak{g}_1, \quad [Y_1,Y_2] \in \mathfrak{g}_0,
\] (10)
for any choice of the elements \( X_1,X_2 \in \mathfrak{g}_0 \) and \( Y_1,Y_2 \in \mathfrak{g}_1 \). The grading splits the set of positive roots of \( so(n) \) into two subsets \( \Delta^+ = \Delta_0^+ \cup \Delta_1^+ \) where \( \Delta_0^+ \) contains all the positive roots of \( \mathfrak{g} \) which are orthogonal to \( e_1 \), i.e. \( (\alpha,e_1) = 0 \); the roots in \( \beta \in \Delta_1^+ \) satisfy \( (\beta,e_1) = 1 \). For more details see [5].

In writing down the Lax pair (3) we made use of the typical \( n \times n \) representation of \( so(n) \). The Lax pair can be considered in any representation of \( so(n) \), then the potential \( Q \) will take the form:
\[
Q(x,t) = \sum_{\alpha \in \Delta_1^+} (q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha}).
\] (11)

Next we introduce \( n \)-component ‘vectors’ formed by the Weyl generators of \( so(n+2) \) corresponding to the roots in \( \Delta_1^+ \):
\[
\vec{E}_1^\pm = (E_{\pm(e_1-e_2)}, \ldots, E_{\pm(e_1-e_r)}, E_{\pm(e_1+e_r)}, \ldots, E_{\pm(e_1+e_2)}),
\] (12)
for \( n = 2r + 1 \) and
\[
\vec{E}_1^\pm = (E_{\pm(e_1-e_2)}, \ldots, E_{\pm(e_1-e_r)}, E_{\pm(e_1+e_r)}, \ldots, E_{\pm(e_1+e_2)}),
\] (13)
for \( n = 2r \). Then the generic form of the potentials \( Q(x,t) \) related to these type of symmetric spaces can be written as sum of two "scalar" products
\[
Q(x,t) = (\vec{q}(x,t) \cdot \vec{E}_1^+) + (\vec{p}(x,t) \cdot \vec{E}_1^-).
\] (14)

In terms of these notations the generic MNLS type equations connected to \( BD.I. \) acquire the form
\[
i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}, \vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} = 0,
i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q}, \vec{p})\vec{p} + (\vec{p}, s_0\vec{p})s_0\vec{q} = 0,
\] (15)

With the typical reduction \( p_k = q_k^* \) it gives:
\[
i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^*, \vec{q})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{q}^* = 0,
\] (16)
If we put \( n = 3 \) and introduce the new variables \( \Phi_{\pm 1} = q_{1,3}, \Phi_0 = q_2 \) we recover equations (1). Likewise with \( n = 5 \) and \( \Phi_{\pm 2} = q_{1,5}, \Phi_{\pm 1} = q_{2,4}, \Phi_0 = q_3 \) we find eq. (2). The Hamiltonians for the MNLS equations (13) are given by

\[
H_{\text{MNLS}} = \int_{-\infty}^{\infty} dx \left( (\partial_x \bar{p}^T, \partial_x \bar{q}) - (\bar{p}^T, \bar{q})^2 + \frac{1}{2} (\bar{p}^T, s_0 \bar{p})(\bar{q}^T, s_0 \bar{q}) \right),
\]

(17)

THE DIRECT AND THE INVERSE SCATTERING PROBLEM

The fundamental analytic solution

Herein we remind some basic features of the inverse scattering theory for the operator \( L \) (4), see [2, 3]. There we have made use of the general theory developed in [21, 11, 12, 13] and the references therein. The Jost solutions of \( L \) are defined by:

\[
\lim_{x \to -\infty} \phi(x, t, \lambda) e^{i\lambda J x} = \mathbb{I}, \quad \lim_{x \to \infty} \psi(x, t, \lambda) e^{i\lambda J x} = \mathbb{I}
\]

(18)

and the scattering matrix \( T(\lambda, t) \equiv \psi^{-1}(x, t, \lambda) \). The special choice of \( J \) and the fact that the Jost solutions and the scattering matrix take values in the group \( SO(n + 2) \) we can use the following block-matrix structure of \( T(\lambda, t) \)

\[
T(\lambda, t) = \left( \begin{array}{cc} m_1^+ & -\bar{b}^T \\bar{b}^+ - m_1^- & c_1^- \\bar{B}^T s_0 & m_1^- \end{array} \right), \quad \hat{T}(\lambda, t) = \left( \begin{array}{cc} m_1^- & \bar{B}^T \ (-\bar{B}^+ s_0 b^-) \\bar{b}^+ & \bar{b}^T s_0 \end{array} \right).
\]

(19)

where \( \bar{b}^\pm(\lambda, t) \) and \( \bar{B}^\pm(\lambda, t) \) are \( n \)-component vectors, \( T_{22}(\lambda) \) is \( n \times n \) block matrix, and \( m_1^\pm(\lambda) \), and \( c_1^\pm(\lambda) \) are scalar functions. Such parametrization is compatible with the generalized Gauss decompositions of \( T(\lambda) \) which read as follows:

\[
T(\lambda, t) = T_j^+ D_j^+ \hat{S}_j^+, \quad T(\lambda, t) = T_j^- D_j^- \hat{S}_j^-,
\]

\[
T_j^+ = e^{\pm (\bar{b}^+, \bar{\xi}_j^+)}, \quad S_j^+ = e^{\pm (\bar{\xi}^+, \bar{\xi}_j^+)}, \quad D_j^+ = \text{diag} \left( (m_1^\pm)^1, (m_2^\pm), (m_1^\pm)^1 \right).
\]

The functions \( m_1^\pm \) and \( n \times n \) matrix-valued) functions \( m_2^\pm \) are analytic for \( \lambda \in \mathbb{C}_\pm \).

We have introduced also the notations:

\[
\bar{\rho}^+ = \frac{\bar{B}^-}{m_1^+}, \quad \bar{\xi}^- = \frac{\bar{B}^+}{m_1^-}, \quad \bar{\rho}^- = \frac{\bar{b}^+}{m_1^-}, \quad \bar{\xi}^+ = \frac{\bar{b}^-}{m_1^-},
\]

\[
c_1^\pm = \frac{m_1^\pm (\bar{p}^\pm s_0 \bar{p}^\pm)}{2}, \quad c_1^- = \frac{m_1^- (\bar{\xi}^+ s_0 \bar{\xi}^+)}{2},
\]

\[
\bar{b}^- = \frac{\mu_2^- T}{m_1^-}, \quad \bar{b}^+ = \frac{\mu_2^+ T}{m_1^+}, \quad \bar{B}^+ = \frac{s_0 \mu_2^+ s_0 b^-}{m_1^+}, \quad \bar{B}^- = \frac{s_0 \mu_2^- s_0 \bar{b}^-}{m_1^-},
\]

where \( \mu^+ = m_2^+ - \bar{b}^+ b^- T/(2m_1^+) \), \( \mu^- = m_2^- - s_0 \bar{b}^+ b^- T s_0 / (2m_1^-) \). There are some additional relations which ensure that both \( \hat{T}(\lambda) \) and its inverse \( \hat{T}(\lambda) \) belong to the orthogonal group \( SO(n + 2) \) and that \( T(\lambda) \hat{T}(\lambda) = \mathbb{I} \).
Important tools for reducing the ISP to a Riemann-Hilbert problem (RHP) are the fundamental analytic solution (FAS) \( \chi^{\pm}(x,t,\lambda) \). We will introduce two pairs of FAS using the generalized Gauss decomposition of \( T(\lambda,t) \), see \[11,13,14]\:

\[
\begin{align*}
\chi^{\pm}(x,t,\lambda) &= \phi(x,t,\lambda)S_j^\mp(t,\lambda) = \psi(x,t,\lambda)T_j^\mp(t,\lambda)D_j^\mp(\lambda), \\
\chi'^{\pm}(x,t,\lambda) &= \phi(x,t,\lambda)S_j^\mp(t,\lambda)\dot{D}_j^\mp(\lambda) = \psi(x,t,\lambda)T_j^\mp(t,\lambda).
\end{align*}
\]

(20)

More precisely, this construction ensures that \( \xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)e^{i\lambda Jx} \) and \( \xi'^{\pm}(x,\lambda) = \chi'^{\pm}(x,\lambda)e^{i\lambda Jx} \) are analytic functions of \( \lambda \) for \( \lambda \in \mathbb{C}_\pm \). If \( Q(x,t) \) is a solution of the MNLS eq. (15) then the matrix elements of \( T(\lambda) \) satisfy the linear evolution equations \[2,3]\:

\[
\begin{align*}
\frac{i}{dt} \bar{B}_m^\pm &= \lambda^2 \bar{B}_m^\pm(t,\lambda) = 0, \\
\frac{i}{dt} \bar{m}_m^\pm &= \lambda^2 \bar{m}_m^\pm(t,\lambda) = 0, \\
\frac{i}{dt} \bar{m}_m^\pm &= 0, \\
\frac{i}{dt} \bar{m}_m^\pm &= 0.
\end{align*}
\]

(21)

Thus the block-diagonal matrices \( D^\pm(\lambda) \) can be considered as generating functionals of the integrals of motion. The fact that all \((2r-1)^2\) matrix elements of \( m^\pm_2(\lambda) \) for \( \lambda \in \mathbb{C}_\pm \) generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (15). We remind that \( D^\pm(\lambda) \) allow analytic extension for \( \lambda \in \mathbb{C}_\pm \) and that their zeroes and poles determine the discrete eigenvalues of \( L \).

The Riemann-Hilbert Problem

The FAS for real \( \lambda \) are linearly related \[2,3]\:

\[
\begin{align*}
\chi^+(x,t,\lambda) &= \chi^-(x,t,\lambda)G_0(\lambda,t), \\
\chi'^+(x,t,\lambda) &= \chi'^-(x,t,\lambda)G'_0(\lambda,t).
\end{align*}
\]

(22)

One can rewrite eq. (22) in an equivalent form for the FAS \( \xi^\pm(x,t,\lambda) = \chi^\pm(x,t,\lambda)e^{i\lambda Jx} \) and \( \xi'^\pm(x,t,\lambda) = \chi'^\pm(x,t,\lambda)e^{i\lambda Jx} \) which satisfy the equation:

\[
\begin{align*}
\frac{d\xi^\pm}{dx} + Q(x)\xi^\pm(x,\lambda) - \lambda [J,\xi^\pm(x,\lambda)] &= 0, \\
\frac{d\xi'^\pm}{dx} + Q(x)\xi'^\pm(x,\lambda) - \lambda [J,\xi'^\pm(x,\lambda)] &= 0
\end{align*}
\]

(23)

and the relations

\[
\begin{align*}
\lim_{\lambda \to \infty} \xi^\pm(x,t,\lambda) &= \mathbb{I}, \\
\lim_{\lambda \to \infty} \xi'^\pm(x,t,\lambda) &= \mathbb{I}
\end{align*}
\]

(24)

Then these FAS satisfy the RHP’s

\[
\begin{align*}
\xi^+(x,t,\lambda) &= \xi^-(x,t,\lambda)G_j(x,\lambda,t), \\
\xi'^+(x,t,\lambda) &= \xi'^-(x,t,\lambda)G_j(x,\lambda,t), \\
G_j(x,\lambda,t) &= e^{-i\lambda J(x+t)\lambda}G_j(\lambda,t)G_j(x,\lambda,t) = e^{-i\lambda J(x+t)\lambda}G_j(\lambda,t)e^{i\lambda J(x+t)\lambda}.
\end{align*}
\]

(25)
Obviously the sewing function $G_j(x, \lambda, t)$ (resp. $G'_j(x, \lambda, t)$) is uniquely determined by the Gauss factors $S_j^\pm (\lambda, t)$ (resp. $T_j^\pm (\lambda, t)$). In addition Zakharov-Shabat’s theorem [8] states that if sewing functions $G_j(x, \lambda, t)$ and $G'_j(x, \lambda, t)$ depend on $x$ and $t$ in the way prescribed above ensures that the corresponding FAS satisfy the linear systems (23).

Assume we have solved the RHP’s above and know the FAS $\xi^+(x,t,\lambda)$. Then the corresponding potential of $L$ is recovered by

$$Q(x,t) = \lim_{\lambda \to \infty} \lambda \left( J - \xi^+(x,t,\lambda) J \xi^+(x,t,\lambda) \right). \quad (26)$$

**REDUCTIONS OF MNLS**

The reduction group proposed by Mikhailov [7] provides four classes of reductions which are automatically compatible with the Lax representation of the corresponding MNLS eq.

The reduction group $G_R$ is a finite group which preserves the Lax representation $[L, M] = 0$, i.e. it ensures that the reduction constraints are automatically compatible with the evolution. $G_R$ must have two realizations: i) $G_R \subset \text{Aut}g$ and ii) $G_R \subset \text{Conf}\mathbb{C}$, i.e. as conformal mappings of the complex $\lambda$-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [7]:

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda), \quad (27)$$

where $C_k \in \text{Aut}g$ and $\Gamma_k(\lambda) \in \text{Conf}\mathbb{C}$ are the images of $g_k$ and $\eta_k = 1$ or $-1$ depending on the choice of $C_k$. Since $G_R$ is a finite group then for each $g_k$ there exist an integer $N_k$ such that $g_k^{N_k} = 1$. In all the cases below $N_k = 2$ and the reduction group is isomorphic to $\mathbb{Z}_2$. More specifically the automorphisms $C_k$, $k = 1, \ldots, 4$ listed above lead to the four possible classes of reductions for the matrix-valued functions

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \quad V(x,t,\lambda) = V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J, \quad (28)$$

of the Lax representation:

1) $C_1(U^+(\kappa_1(\lambda))) = U(\lambda), \quad C_1(V^+(\kappa_1(\lambda))) = V(\lambda),$

2) $C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), \quad C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \quad (29)$

3) $C_3(U^+(\kappa_1(\lambda))) = -U(\lambda), \quad C_3(V^+(\kappa_1(\lambda))) = -V(\lambda),$

4) $C_4(U(\kappa_2(\lambda))) = U(\lambda), \quad C_4(V(\kappa_2(\lambda))) = V(\lambda),$

In what follows we will examine the typical reductions of MNLS eqs. of the class 1) obtained by specifying $\kappa_1(\lambda) = \lambda^*$ and $C_1$ to be a $\mathbb{Z}_2$-automorphism of $g$ such that $C_1(J) = J$. Below we list several choices for $C_1$ leading to inequivalent reductions:

a) $C_1 = \mathbb{I}_2, \quad \bar{p}(x) = \bar{q}^*(x), \quad b) \ C_1 = K_1, \quad \bar{p}(x) = K_01 \bar{q}^*(x),$

b) $C_1 = S_{e_2}, \quad \bar{p}(x) = K_{02} \bar{q}^*(x), \quad c) \ C_1 = S_{e_3} S_{e_2}, \quad \bar{p}(x) = K_{03} \bar{q}^*(x),$ (30)

d) $C_1 = S_{e_2} S_{e_3}, \quad \bar{p}(x) = K_{03} \bar{q}^*(x)$

where

$$K_j = \text{block-diag}(1, K_{0j}, 1), \quad K_{01} = \text{diag}(\varepsilon_1, \ldots, \varepsilon_{r-1}, 1, \varepsilon_{r-1}, \ldots, \varepsilon_1), \quad (31)$$
and $\varepsilon_j = \pm 1$. The matrices $K_{02}$ and $K_{03}$ corresponding to the Weyl reflections $S_{e_2}, S_{e_2}S_{e_3}$ etc. are not diagonal; they have dimension $n \times n$ and for $n = 3, 4$ and 5 are given by:

\[
\begin{align*}
    n = 3, \quad K_{02} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
    n = 4, \quad K_{02} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
    n = 5, \quad K_{02} &= \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_{03} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Each of the above reductions impose constraints on the FAS, on the scattering matrix $T(\lambda)$ and on its Gauss factors $S_j^\pm(\lambda), T_j^\pm(\lambda)$ and $D_j^\pm(\lambda)$. These have the form:

\[
\begin{align*}
    (\phi(x,\lambda^*))^\dagger &= K_j^{-1}\phi(x,\lambda)K_j, \\
    (\psi(x,\lambda^*))^\dagger &= K_j^{-1}\psi(x,\lambda)K_j, \\
    (\chi^+(x,\lambda^*))^\dagger &= K_j^{-1}\chi^-(x,\lambda)K_j, \\
    (T(\lambda^*))^\dagger &= K_j^{-1}T(\lambda)K_j, \\
    (S^+(\lambda^*))^\dagger &= K_j^{-1}S^-(\lambda)K_j, \\
    (T^+(\lambda^*))^\dagger &= K_j^{-1}T^-(\lambda)K_j.
\end{align*}
\]

where the matrices $K_j$ are specific for each choice of the automorphism $C_1$, see eq. (31). In particular, from the last line of (33) and (31) we get:

\[
(m^+_1(\lambda^*))^* = m^-_1(\lambda),
\]

and consequently, if $m^+_1(\lambda)$ has zeroes at the points $\lambda^+_k$, then $m^-_1(\lambda)$ has zeroes at:

\[
\lambda^-_k = (\lambda^+_k)^*, \quad k = 1, \ldots, N.
\]

Below we will write down the effects of these reductions on the corresponding Hamiltonians. For the typical reduction $\bar{p} = \bar{q}^*$ we get:

\[
H_{\text{MNLS}} = \int_{-\infty}^{\infty} dx \left\{ (\partial_x \bar{q}, \partial_x \bar{q}^*) - (\bar{q}, \bar{q}^*)^2 + \frac{1}{2} (\bar{q}, s_0 \bar{q})(\bar{q}^*, s_0 \bar{q}^*) \right\},
\]

\[
H^{(j)}_{\text{MNLS}} = \int_{-\infty}^{\infty} dx \left\{ (\partial_x \bar{q} K_j \partial_x \bar{q}^*) - (\bar{q}, K_j \bar{q}^*)^2 + \frac{1}{2} (\bar{q}, s_0 \bar{q})(\bar{q}^*, s_0 \bar{q}^*) \right\},
\]

The Hamiltonian $H_{\text{MNLS}}^{(1)}$ with $K_{01}$ (31) has indefinite kinetic term. As a consequence the corresponding MNLS has singular soliton solutions which ‘blow-up’ in finite time.
The above Hamiltonians, after the change of variables can be written in more ‘aesthetic’ form. Indeed, for odd $n = 2r - 1$ we can put:

$$q_{2k-1, 2r-2k+1} = \frac{\nu_{2k-1} \pm \nu_{2r-2k+1}}{\sqrt{2}}, \quad q_{2k, 2r-2k} = \frac{i \nu_{2k} \mp \nu_{2r-2k}}{\sqrt{2}}, \quad q_r = c_{0,r} v_r;$$

with $k = 1, 2, \ldots, r-1$ and $c_{0,r} = e^{(r-1)\pi i/2}$; for $n = 2r$ we put:

$$q_{2k-1, 2r-2k+2} = \frac{\nu_{2k-1} \pm \nu_{2r-2k+2}}{\sqrt{2}}, \quad q_{2k, 2r-2k+1} = \frac{i \nu_{2k} \mp \nu_{2r-2k+1}}{\sqrt{2}}$$

(38)

with $k = 1, 2, \ldots, r$.

Inserting the above changes of variables into the Hamiltonian (36) we get

$$H_{KS} = \int_{-\infty}^{\infty} dx \left\{ \sum_{j=1}^{n} |\partial_x v_j|^2 - \left( \sum_{j=1}^{n} |v_j|^2 \right)^2 + \frac{1}{2} \left( \sum_{j=1}^{n} v_j^2 \right)^2 \right\},$$

(40)

which is the Hamiltonian of the $n$-component Kulish-Sklyanin model (KSM) [6]. Thus we have demonstrated that the Lax pairs (3), (4) can be used also for integrating the MNLS (40). In their original paper [6] Kulish and Sklyanin have used Lax pair whose potential is an element of a Clifford algebra. Later Sokolov and Svinolupov [15] discovered another class of Lax pairs for these models whose potentials take values in Jordan algebras. The above Lax pairs allowed to prove integrability of the KSM but were not convenient for solving the inverse scattering problem and constructing exact solutions. Another important property of these models is that they possess both classical [4] and quantum $R$-matrices [6].

Another way to obtain KSM is to apply the reduction of type 4) with $K_0 = \text{block-diag}(1, \varepsilon \delta_0, 1)$, where $\varepsilon = \pm 1$. For odd values of $n = 2r - 1$ this reduction means that:

$$q_k = (-1)^{k+1} \varepsilon q_{2r-k} = w_k, \quad k = 1, \ldots, r,$$

(41)

while for $n = 2r$ one gets:

$$q_k = (-1)^{k+1} \varepsilon q_{2r-k+1} = w_k, \quad k = 1, \ldots, r.$$

(42)

This reduction leads to $r$-component KSM.

Let us write down the Hamiltonians for the different reductions. Below for convenience we will split $H_{MNLS}$ into kinetic and interaction terms: $H_{MNLS} = H_{\text{kin}}^{(j)} - H_{\text{int}}^{(j)}$.

Reduction b):

$$H_{\text{kin}}^{(1)} = \int_{-\infty}^{\infty} dx \left\{ \sum_{j=1}^{r-1} \varepsilon_j (|\partial_x q_j|^2 + |\partial_x q_{2r-j}|^2) + |\partial_x q_r|^2 \right\},$$

$$H_{\text{int}}^{(1)} = \int_{-\infty}^{\infty} dx \left\{ \sum_{j=1}^{r-1} \varepsilon_j (|q_j|^2 + |q_{2r-j}|^2) + |q_r|^2 \right\} - \frac{1}{2} \left( \sum_{j=1}^{r-1} (-1)^{j+1} 2 q_j q_{2r-j} + (-1)^r q_r^2 \right).$$

(43)
One can construct other reductions, e.g. ones of type c) with reduction matrix $K_j$. Then

$$H_{\text{MNLS}}^{(j)} = \int_{-\infty}^{\infty} dx \left\{ (\partial_x \bar{q}^+ K_j \partial_x \bar{q}) - (\bar{q}^+ K_j \bar{q})^2 + \frac{1}{2} |(\bar{q}^T s_0 \bar{q})|^2 \right\},$$

(44)

Characteristic feature of the reductions involving Weyl group elements is that they lead to ‘non-diagonal’ form of the kinetic terms [16]. Making simple change of variables diagonalizing $K_j$ we can recover the diagonal form of the kinetic terms but unfortunately we cannot make it positive definite. This is related to the fact that $K_j^2 = 1$ and so has as eigenvalues both $+1$ and $-1$ with certain multiplicities.

Let us give also an important example of class 2) reductions (28). The constraints that these class of reductions impose on the FAS and on the scattering matrix $T(\lambda)$ and on its Gauss factors $S_j^\pm(\lambda)$, $T_j^\pm(\lambda)$ and $D_j^\pm(\lambda)$ take the form:

$$(\chi^+(x, \lambda))^T = K_j^{-1} \chi^-(x, \lambda) K_j'$$

$$(T(\lambda))^T = K_j^{-1} T(\lambda) K_j'$$

$$(S^\pm(\lambda))^T = K_j^{-1} S^\pm(\lambda) K_j'$$

$$(T^\pm(\lambda))^T = K_j^{-1} T^\pm(\lambda) K_j'$$

(45)

and $(D^\pm(\lambda))^T = K_j^{-1} D^\pm(\lambda) K_j'$. The explicit form of the matrices $K_j'$ is determined by the particular realization of the automorphism $C_2$. Choosing $n = 3$ and $C_2 = S_{e1}$ we obtain the constraint $q_1 = q_3$ and the reduced Hamiltonian takes the form:

$$H = \int_{-\infty}^{\infty} dx \left\{ |\partial_x v_1|^2 + |\partial_x v_2|^2 - (|v_1|^2 + |v_2|^2)^2 + \frac{1}{2} |v_1^2 - v_2^2|^2 \right\},$$

(46)

where we have put $q_1 = q_2 = \frac{1}{\sqrt{2}} v_1$ and $q_2 = v_2$. This model also been derived as relevant for $F = 1$ BEC [17].

**DRESSING METHOD AND SOLITON SOLUTIONS**

The dressing Zakharov-Shabat method [3, 9] for constructing soliton solutions of MNLS has been modified in [3] for the BD.I-type symmetric spaces. There we also analyzed the different types of soliton solutions. Below we briefly discuss the properties of the generic one-soliton solutions.

It is obtained by dressing the regular FAS $\chi_0^\pm(x, \lambda)$ of the RHP [25]. Using them we construct the singular solutions $\chi_0^\pm(x, \lambda)$ of the RHP

$$\chi^\pm(x, \lambda) = u(x, \lambda) \chi_0^\pm(x, \lambda) \hat{u}_-,$$

$$\chi'^\pm(x, \lambda) = u(x, \lambda) \chi_0^\pm(x, \lambda) \hat{u}_+,$$

$$u(x, \lambda) = \Pi + (c_1(\lambda) - 1)P_1(x) + (c_1^{-1}(\lambda) - 1)\bar{P}_1(x), \quad u_\pm = \lim_{x \to \pm\infty} u(x, \lambda).$$

(47)

For the above choice of $J$ it is enough to consider rank 1 projectors $P_1(x, t)$ and $\bar{P}_1(x, t)$ of $S_0 P_1 S_0$. Together with the constraint $P_1 \bar{P}_1 = 0$, the last condition ensures that $u(x, t) \in SO(n + 2)$. It remains to only to give the explicit form of $P_1(x, t)$. Generically it is determined by two polarization vectors $|n_{0,1}\rangle$ and $|m_{0,1}\rangle$, and the initial regular solutions:

$$P_1(x, t) = \frac{|n_1(x, t)\rangle \langle m_1(x, t)|}{\langle m_1(x, t)|n_1(x, t)\rangle}, \quad \bar{P}_1(x, t) = \frac{|m_1(x, t)\rangle \langle n_1(x, t)|}{\langle n_1(x, t)|m_1(x, t)\rangle},$$

(48)
\[ |n_1(x,t)\rangle = \chi_0^+(x,t,\lambda_1^+) |n_{0,1}\rangle, \quad \langle m_1(x,t) | = \langle m_{0,1} | \tilde{Z}_0^- (x,t,\lambda_1^-), \]
\[ |m_1(x,t)\rangle = \chi_0^-(x,t,\lambda_1^-) |m_{0,1}\rangle, \quad \langle m_1(x,t) | = \langle m_{0,1} | \tilde{Z}_0^+ (x,t,\lambda_1^+). \]  

The one soliton solution is parametrized by the two eigenvalues \( \lambda_1^\pm \) and by the polarization vectors \( |n_{0,1}\rangle \) and \( \langle m_{0,1}| \). The latter after renormalization have \( n-1 \) independent components each:
\[
|n_{0,1}\rangle = \begin{pmatrix} \sqrt{A_0} \\ \bar{v}_{0,1}/\sqrt{A_0} \end{pmatrix}, \quad \langle m_{0,1}| = \begin{pmatrix} \sqrt{B_0}, \bar{\mu}_{0,1}/\sqrt{B_0}, 1/\sqrt{B_0} \end{pmatrix},
\]

where \( A_0 = \frac{1}{2}(\bar{V}_{0,1}^T s_0 \bar{\nu}_{0,1} ) \) and \( B_0 = \frac{1}{2}(\bar{\mu}_{0,1}^T s_0 \bar{\nu}_{0,1}) \). The constraint \( P_1 \bar{P}_1 = 0 \) means that the vectors \( \bar{\mu}_{0,1} \) and \( \bar{V}_{0,1} \) must satisfy \( \bar{\mu}_{0,1}^T s_0 \bar{\nu}_{0,1} = 0 \). Therefore the one-soliton solution can be viewed as dynamical systems with \( 2n - 1 \) degrees of freedom. After some simplifications, it takes the form:
\[
q_k(x,t) = - \frac{4i\nu_1}{\Delta} e^{-imz_k} e^{i\bar{\xi}_0 k} [\cos(\delta_0) \cosh(z_0) + i \sin(\delta_0) \sinh(z_0)],
\]
\[
\Delta = 2 \cosh(2z_0) + C, \quad C = (\bar{v}_0^+ \bar{V}_0)/|A_0|, \quad \bar{z}_k = x + w_1 t - \bar{\delta}_{0,k}/\mu_1,
\]
\[
z_0 = v_1(x - u_1 t) + \bar{\xi}_0, \quad z_{0,k} = v_1 x + \bar{\xi}_{0,k}, \quad \bar{z}_{0,k} = v_1(x - v_1 t) + \bar{\xi}_{0,k},
\]
\[
\bar{\xi}_0 = \frac{1}{2} \ln |A_0|, \quad \bar{\xi}_{0,k} = \frac{1}{2} \ln |\bar{V}_0|/|\bar{V}_{0,k}|, \quad \bar{\xi}_{0,k} = \frac{1}{2} \ln |\bar{V}_{0,k}|/|\bar{V}_{0,2r-k}|/|A_0|,
\]
\[ \bar{\delta}_{0,k} = (\alpha_{0,2r-k} + \alpha_{0,k} - \alpha_0 - \pi k)/2, \quad \bar{\delta}_{0,k} = (\alpha_{0,2r-k} - \alpha_{0,k} + \alpha_0 + \pi k)/2, \]

where \( \alpha_0 = \arg A_0 \) and \( \alpha_{0,k} = \arg \bar{V}_{0,k} \).

Each of the reductions of the type \( (29) \) imposes constraints not only on \( \lambda_1^+ = (\lambda_1^-)^* \), but also on the polarization vectors:
\[
\bar{\mu}_{0,1} = K_{0,j} \bar{V}_{0,1}^*, \quad \bar{V}_{0,1}^T K_{0,j} s_0 \bar{\nu}_{0,1} = 0.
\]

As a result, after the reduction the number of independent parameters of the soliton solution becomes \( n - 1 \). The velocities \( u_1 \) and \( w_1 \) are given by \( u_1 = -2\mu_1 \) and \( w_1 = (v_1^2 - \mu_1^2)/\mu_1 \).

Special attention deserves the fact that generically all \( z_{0,k} \) are different and as a result each component \( q_k(x,t) \) has its center of mass shifted with respect to the others.

Let us now consider a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) reduction by applying simultaneously two reduction: the first is the typical one and the second is the class 2) reductions as for the model \( (46) \). The first reduction imposes the relation \( (52) \) between the two polarization vectors \( |n_{0,1}\rangle \) and \( \langle n_{0,1}| \). The second reduction imposes constraint on the vector \( |n_{0,1}\rangle \), namely:
\[
|n_{0,1}\rangle = K'_{0,j} |n_{0,1}\rangle.
\]

In particular, for \( n = 3 \) and \( C_2 = S_{e_1} \) the vector \( |n_{0,1}\rangle \) has 3 components and \( K'_{0,j} = \text{diag}(1,-1,1) \). Thus only two independent complex coefficients are enough to parametrize the corresponding polarization vector, and the corresponding soliton can be viewed as dynamical system with three degrees of freedom.
DISCUSSION AND CONCLUSIONS

One of the important consequences of the FAS is that with their help one can construct the kernel of the resolvent of \( L \) (see \([12, 18]\)) and prove the completeness relation for its eigenfunctions. From these expressions it becomes obvious that the resolvent develops poles at all points \( \lambda_k^\pm \in \mathbb{C}_\pm \) for which \( m_\pm^r(\lambda_k^\pm) = 0 \). Combining this fact with the equivalence between the solutions of the RHP and the FAS of the Lax operator we conclude that the singularities of the RHP correspond to the discrete eigenvalues of \( L \).

Quite often the general analysis of the MNLS \([10]\) is followed by simplifications which often reduce the MNLS to a single-component NLS. One way to do this was mentioned above: it is to impose the reduction \( \Phi_{+1} = \Phi_{-1} \). Another less obvious way to this is to impose this reduction on the initial conditions. Indeed, one can show that imposing \( \Phi_{+1}(x, t = 0) = \Phi_{-1}(x, t = 0) \) ensures that \( \Phi_{+1}(x, t) = \Phi_{-1}(x, t) \) for all \( t > 0 \). At the same time there is a substantial difference between the solitons of the scalar NLS or Manakov model and the solitons of MNLS \([11]\). Unlike the solitons of the Manakov model, all three components of the one-soliton solution of \([11]\) have different \( x \)-dependence; generically each component has different ‘center of mass’ position. Therefore, if one wants to demonstrate new nontrivial aspects of soliton dynamics one should use generic initial values for \( \Phi_{\pm 1}(x, t = 0) \) and \( \Phi_{-1}(x, t = 0) \).

Another still open problem is the interrelation between the solutions of the direct and inverse scattering problem for \( L \), considered in different irreducible representations (IRREP) of the corresponding Lie algebra \( \mathfrak{g} \). From the point of view of the relevant NLEE, their Lax representations have purely algebraic nature and therefore, the form of the NLEE does not depend on the choice of the IRREP of \( \mathfrak{g} \).

From the point of view of the spectral theory, the different IRREP have different dimensions; therefore changing the IRREP we change the order of the corresponding operator. Since we are dealing with simple Lie algebras whose IRREP are well known \([5]\). In particular, it is well known that the finite dimensional representations can be realized as invariant subspaces of the tensor products of the typical one. Let us assume that we are able to construct the FAS and the relevant RHP and dressing factors in the typical representation. Obviously, taking the tensors products of the FAS their analyticity properties will persist and we will get the corresponding FAS and RHP in the corresponding IRREP. However nontrivial things may take place when one considers the multiplicities of the corresponding discrete eigenvalues.

As an example I will just mention that the dressing factor can be evaluated also for the other fundamental representations of \( \mathfrak{g} \) \([19]\). If in the typical representation of \( \mathfrak{g} \simeq \text{so}(2r) \) \( u(x, \lambda) \) is given by \([17]\) then in the spinor representation it will take the form \([20]\):

\[
u(x, \lambda) = \sqrt{c_1(\lambda)} \pi_1(x, t) + \frac{1}{\sqrt{c_1(\lambda)}} \tilde{\pi}_1(x, t), \quad \tilde{\pi}_1(x, t) = \tilde{s}_0 \pi_1(x, t) \tilde{s}_0^{-1},
\]

and the projectors satisfy \( \pi_1(x, t) \tilde{\pi}_1(x, t) = 0 \) and \( \pi_1(x, t) + \tilde{\pi}_1(x, t) = \mathbb{I} \). Note the substantial change in the \( \lambda \)-dependence of \( u(x, \lambda) \), as well as the fact that now instead of having rank one projectors \( P_1(x, t) \) we get projectors \( \pi_1(x, t) \) and \( \tilde{\pi}_1(x, t) \) of rank \( r \).

We will discuss these problems in more details elsewhere.
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