Wide Network Learning with Differential Privacy

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Abstract

Despite intense interest and considerable effort, the current generation of neural networks suffers a significant loss of accuracy under most practically relevant privacy training regimes. One particularly challenging class of neural networks are the wide ones, such as those deployed for NLP typeahead prediction or recommender systems.

Observing that these models share something in common—an embedding layer that reduces the dimensionality of the input—we focus on developing a general approach towards training these models that takes advantage of the sparsity of the gradients. More abstractly, we address the problem of differentially private empirical risk minimization (ERM) for models that admit sparse gradients.

We demonstrate that for non-convex ERM problems, the loss is logarithmically dependent on the number of parameters, in contrast with polynomial dependence for the general case. Following the same intuition, we propose a novel algorithm for privately training neural networks. Finally, we provide an empirical study of a DP wide neural network on a real-world dataset, which has been rarely explored in the previous work.

1 Introduction

Deep learning models are often trained on datasets that contain sensitive information, such as location, purchase history, or medical records. There is mounting evidence that, in the absence of specific measures to the contrary, deep learning models may memorize and subsequently leak some of their training samples [FJR15, CTW+20].

Differential privacy (DP) [DMNS06] was proposed as a mathematically rigorous definition of privacy, and has since become the gold standard in privacy-preserving machine learning, with applications across multiple domains. The notable examples include large technology companies, such as Google, Apple, and Microsoft that rely on differential privacy for privacy-preserving telemetry [EPK14, Dif17, DKY17] and the US Census Bureau, which committed to using differential privacy for its 2020 Census data products.

In this paper, we study the problem of deep learning with differential privacy. More broadly, we consider the problem of DP empirical risk minimization (ERM), which provides a framework of unifying various machine learning models, including regression, support vector machine (SVM), and neural network. The problem can be formulated as follows:

ERM: Given the dataset \( D = \{d_1, d_2, \ldots, d_n\} \) and a loss function \( L \), the goal is to

\[
\text{minimize } L(w; D) = \frac{1}{n} \sum_{i=1}^{n} \ell(w; d_i) \text{ over } w \in W,
\]
where the map \( \ell \) defines a loss function \( \ell(\cdot; d) \) over a parameter space \( W \) for each data point \( d \).

Our objective is to design a mechanism that satisfies \((\varepsilon, \delta)\)-differential privacy (formally defined in Definition 1) while minimizing the accuracy loss measured as the empirical risk. This problem is termed DP ERM, which is of vital importance in both academia and industry.

This problem has been considered in a line of work starting with Chaudhuri and Hsu [CH11, BST14, KJ16, JT14]. The most general approach to the problem is based on the DP stochastic gradient descent (DP-SGD) [BST14, ACG+16]. For convex and Lipschitz loss functions, Bassily et al. shows that DP-SGD achieves the risk of \( \tilde{\Theta}\left(\frac{\text{poly}(p)}{n}\right) \). For non-convex functions, subsequent work demonstrates that \( \mathbb{E}\left[\|\nabla L(w_{\text{priv}}^d; D)\|_2\right] \leq \tilde{O}\left(\frac{\text{poly}(p)}{\sqrt{n}}\right) \) [ZZMW17, WYX17, WJEG19]. In both cases, the risk depends on the dimension \( p \), in contrast to the non-private setting. This dependency becomes a significant source of accuracy loss for private algorithms, especially when the input dimensionality is large.

Other papers have studied the problem of sparse DP ERM, which optimizes over sparse outputs \( w \) in our notation [HT10, TTZ15, CWZ19]. Under this additional constraint, the loss decreases from \( \tilde{O}\left(\frac{\text{poly}(p)}{n}\right) \) to \( \tilde{O}\left(\frac{\text{poly}(\log p)}{n}\right) \). However, this assumption does not hold in many high-dimensional models. For example, in word embedding, the network parameters are evenly important, and artificially reducing the network’s dimensionality (such as the size of the input vocabulary) will negatively affect its accuracy.

This work pursues the orthogonal direction of decreasing the privacy cost of ERM. Motivated by applications to wide neural networks, which have a wide embedding layer that reduces the dimensionality of the input, we consider the setting where the input is inherently sparse. This scenario is common in machine learning tasks. For example, in wide neural networks for language models and recommendation systems [MNHZB19, HMNZ19], the input data is usually extremely sparse, with only a small fraction of parameters “active” in each update. Thus, the input sparsity usually produces a sparse gradient, which holds for a broad class of problems, as shown in Section 4.2.

The question we address in this work is the following: can we utilize the input sparsity (leading to the gradient sparsity) to improve the accuracy of DP deep learning, or broadly, DP ERM?

In this paper, we provide a positive answer to the above question. Our contributions can be summarized as follows:

1. We show that for non-convex ERM with gradient sparsity, the loss is \( O\left(\frac{\text{poly}(\log p)}{n^2}\right) \), as compared with \( O\left(\frac{\text{poly}(p)}{\sqrt{n}}\right) \) for the general case (Section 4).
2. Based on the theoretical analysis, we propose a novel algorithm that leverages the inherent gradient sparsity in wide neural networks (Section 5).
3. We provide an empirical study of DP training of a wide neural network on a real dataset. The experimental results suggest that our method can achieve a much better utility compared to the standard DP-SGD (Section 6).
4. We complement theoretical analysis of the privacy guarantees with empirical estimates of the algorithm’s privacy loss via memorization metrics. Significantly, we demonstrate that these metrics are sufficiently sensitive to inform the choice of parameters for privacy-preserving algorithms (Section 6).
2 Related work

2.1 DP neural networks

The problem of training DP deep learning models was initially addressed by [SS15], followed by [ACG+16] who proposed DP-SGD to train deep neural networks in a centralized setting. Specifically, Abadi et al. clipped each gradient in order to bound the influence of each sample and introduced the moments accountant to track privacy loss. This method has generated substantial interest, and follow-up research, focusing on improving the architecture and applying this approach to different data types, such as images and texts, see, e.g., [AZK+18, AMCDC18, BJWW+19, CXX+18, MAM+18, MRTZ18, TAM19, WBK19, PSM+18, RTM+20, VTJ19, PKP+18, LK20].

However, despite tremendous interest in privacy-preserving training for deep neural networks, there are only few works that explicitly considers wide neural networks. Perhaps the two most relevant works are [SS15] and [ZWB20]. In particular, [SS15] apply the sparse vector technique to select the subset of the gradients. Their approach is different from ours in the following ways. First, their primary motivation is the communication cost, not the model accuracy. In contrast, our objective is to utilize the sparsity in wide neural networks and improve the model accuracy. Second, Shorki and Shmatikov consider only the sparse vector technique for private selection whereas we allow any DP selection technique. Our experimental results and prior work [LSL17] show that the other selection rules (e.g., the exponential mechanism) may outperform the sparse vector techniques.

[ZWB20] consider a setting superficially similar to ours, a non-convex DP ERM over large domain and sparse gradients. However, Zhou et al. additionally require access to public samples, which we don’t assume. Furthermore, Zhou et al.’s theoretical analysis requires roughly the same amount of public and private data, which is hard to get in practice. In addition, their empirical analysis considers training a deep neural network on the MNIST dataset, while we study training a wide neural network on the Brown News corpus, which is a more challenging task.

DP wide neural networks are explicitly or implicitly used in many other previous work, e.g., for word embedding [VTJ19, RTM+20, MRTZ18]. However, these studies either assume the embedding is already available (i.e., pre-trained on a large public dataset), or by training a model with DP-SGD or its modified versions, not leveraging sparsity.

2.2 DP ERM

DP ERM has been subject of many studies [CH11, BST14, JT14, KJ16, ZZMW17, WYX17]. Particularly, [BST14] shows that the maximum risk is in the order of $\tilde{O}\left(\frac{\sqrt{p}}{\epsilon n}\right)$, for convex and Lipschitz loss functions, and $W$ contained in the unit $\ell_2$ ball. It also shows that this bound cannot be improved in general, even for the squared loss function. For non-convex functions, [ZZMW17, WCX19, WJEG19] show that $\mathbb{E}\left[\|\nabla L(w^{\text{priv}}; D)\|_2\right] \leq \tilde{O}\left(\frac{p^{\frac{1}{4}}}{\sqrt{n}}\right)$.

Other studies [HT10, TTZ15, ZZMW17, CWZ19] have evaluated the problem of sparse DP ERM, with sparse $w$ assumed. Under this additional constraint, the loss is in the order of $\tilde{O}\left(\frac{\text{poly}(\log p)}{n}\right)$ instead of $\tilde{O}\left(\frac{\text{poly}(p)}{n}\right)$.

2.3 DP selection

Our algorithm also depends on DP selection as a middle step, of which the goal is to select top $k$ items out of a set of $p$ items. There are three classic algorithms for this problem: exponential mechanism [MT07], the sparse vector algorithm [DNR+09, DR14], and report noisy max [DR14].
In [SU17], the optimal bound has been established for the database setting. [DR19a] proposes an algorithm which can be unaware of the domain, but may not always output $k$ components.

## 3 Preliminaries

### 3.1 Privacy preliminaries

A dataset $X = (X_1, \ldots, X_n) \in \mathcal{X}^n$ is a collection of points from some universe $\mathcal{X}$. We say that two datasets $X$ and $X'$ are neighboring, denoted as $X \sim X'$ if they differ in exactly one point. We first provide a formal definition of differential privacy.

**Definition 1** (Differential Privacy (DP) [DMNS06]). A randomized algorithm $A : \mathcal{X}^n \rightarrow S$ satisfies $\varepsilon, \delta$-differential privacy ($\varepsilon, \delta$-DP) if for every pair of neighboring datasets $X, X' \in \mathcal{X}^n$, and any event $S \subseteq S$,

$$\Pr(A(X) \in S) \leq e^\varepsilon \Pr(A(X') \in S) + \delta.$$

Then we introduce two important properties of differential privacy. The first is the “sampling property”, which reveals that the privacy level can be boosted by sampling.

**Lemma 1.** (Privacy amplification via sampling, Theorem 9 in [BBG18]) Over a domain of datasets $\mathcal{X}^n$, if an algorithm $A$ is $\varepsilon, \delta$-DP, where $\varepsilon \leq 1$, then for any dataset $X \in \mathcal{X}^n$, executing $A$ on uniformly random $\gamma n$ entries of $X$ ensures $\gamma \varepsilon, \gamma \delta$-DP.

Another critically important property of differential privacy is that it can be composed adaptively. By adaptive composition we mean a sequence of algorithms $A_1(X), \ldots, A_t(X)$ where the algorithm $A_t(X)$ may also depend on the outcomes of the algorithms $A_1(X), \ldots, A_{t-1}(X)$.

**Lemma 2.** (Strong composition theorem, Theorem 3.2 in [KOV15]) For any $\varepsilon > 0, \delta \in [0, 1]$, and $\tilde{\delta} \in [0, 1]$, the class of $\varepsilon, \delta$-DP mechanisms satisfies $\tilde{\varepsilon}, k \delta + \tilde{\delta}$-DP under $k$-fold adaptive composition, for

$$\tilde{\varepsilon} = k \varepsilon (e^\varepsilon - 1) + \varepsilon \sqrt{2k \log \frac{1}{\tilde{\delta}}}.$$

In their original paper, [DMNS06] provides a famous scheme for differential privacy, known as the Gaussian mechanism. This method adds Gaussian noise to a non-private output in order to make it private. We first define the sensitivity, and then state their result. Roughly speaking, the sensitivity measures the maximum difference between the outputs of the algorithm on two neighboring datasets.

**Definition 2.** The sensitivity of a non-private algorithm $f : \mathcal{X}^n \rightarrow S$ in $\ell_2$ norm is

$$s_z(f) = \max_{X, X' \text{are neighboring}} |f(X) - f(X')|_z.$$

**Lemma 3** (Gaussian mechanism [DMNS06]). For any $0 \leq \varepsilon \leq 1, 0 \leq \delta \leq 1$, and $f : \mathcal{X}^n \rightarrow \mathbb{R}^p$, $A(X) = f(X) + N(0, \sigma^2 I_p)$ satisfies $\varepsilon, \delta$-DP, where $\sigma^2 = \frac{2 \log(1.25/\delta)}{\varepsilon^2 s_2(f)^2}$.

We recall another foundational DP algorithm, Algorithm 1, known as the sparse vector technique [DNR+09, DR14]. Intuitively, the algorithm privately selects the largest coordinates (in
absolute value) of the input vector \( u \), and outputs their noisy versions.

| **Algorithm 1:** NumericSparse\((u(X), \alpha, c_1, \varepsilon, \delta)\) |
|---|
| **Input:** private vector \( u(X) \in \mathbb{R}^p \), threshold \( \alpha \), sparsity parameter \( c_1 \), privacy parameter \( (\varepsilon, \delta) \), sensitivity upper bound \( s = s_{\infty}(u(X)) \) (defined in Definition 2). |
| 1 Let \( \varepsilon_1 = 0.95\varepsilon \), \( \varepsilon_2 = 0.05\varepsilon \), and \( \sigma(\varepsilon) = \frac{s\sqrt{32c_1 \log 2}}{\varepsilon} \) |
| 2 Let \( \hat{\alpha}_0 = \alpha + \text{Lap}(\sigma(\varepsilon_1)) \), count = 0, and \( \hat{u} = 0 \) |
| 3 For \( i = 1 \) to \( p \) |
| 4 \( \begin{align*} & \text{let } v_i = \text{Lap}(2\sigma(\varepsilon_1)) \\ & \text{If } |u_i| + v_i \geq \hat{\alpha}_{\text{count}} \\ & \quad \text{Let } \hat{u}_i = u_i + \text{Lap}(2\sigma(\varepsilon_2)) \\ & \quad \text{Let count = count + 1} \\ & \quad \text{Let } \hat{\alpha}_{\text{count}} = \alpha + \text{Lap}(\sigma(\varepsilon_1)) \\ & \end{align*} \) |
| 5 If count \( \geq c_1 \) |
| 6 break |
| **Output:** \( \hat{u} \) |

We also provide its theoretical guarantees. Intuitively, the algorithm outputs at most \( c_1 \) non-zero coordinates, and with high probability, none of the coordinates with value more than \( 2\alpha \) will be output as zero.

**Lemma 4.** Algorithm 1 satisfies \((\varepsilon, \delta)\)-DP. Furthermore, let \( \alpha = \frac{20s\log p + \log \frac{4c_1}{\varepsilon}}{\varepsilon} \sqrt{c_1 \log \frac{2}{\delta}} \), and \( |\{i: |u_i| > 0\}| \leq c_1 \). Then with probability at least \( 1 - \beta \), the algorithm does not halt when \( i \leq p \). Furthermore, for all \( \hat{u}_i \neq 0 \): \( |\hat{u}_i - u_i| \leq \alpha \), and for all \( \hat{u}_i = 0 \): \( |u_i| \leq 2\alpha \).

The following lemma can be viewed as a direct corollary.

**Lemma 5.** Given all the conditions in Lemma 4, with probability at least \( 1 - \beta \), \( \|u - \hat{u}\|_2 \leq 2.5\alpha \sqrt{c_1} \).

**Proof.** Let \( p := \{1, 2, \ldots, p\} \), and \( A, B \) be subsets of \( p \), with \( A := \{i: |u_i| > 0\} \) and \( B := \{i: |u_i| = 0\} \). Note that \( |A| \leq c_1 \), then \( \|u - \hat{u}\|_2^2 = \sum_{i \in A} (\hat{u}_i - u_i)^2 + \sum_{i \in B} \hat{u}_i^2 \leq c_1 \alpha^2 + c_1 \cdot 4\alpha^2 \leq 5\alpha^2 c_1 \), where the inequality comes from Lemma 4, and the fact that the algorithm at most outputs \( c_1 \) non-zero coordinates.

### 3.2 ERM preliminaries

We introduce the definition of smooth functions.

**Definition 3.** We say a function \( \ell: \mathbb{R}^p \to \mathbb{R} \) is \( K \)-smooth, if for all \( w_1, w_2 \in \mathbb{R}^p \),

\[
|\ell(w_2) - \ell(w_1) - \langle \nabla \ell(w_1), w_2 - w_1 \rangle| \leq K \|w_2 - w_1\|_2^2.
\]

### 4 Improving DP ERM by sparsity

In Section 4.1, we provide our theoretical results for \((\varepsilon, \delta)\)-DP ERM problems under the assumption of sparse gradients. In Section 4.2, we show that for a broad class of problems, i.e., generalized linear models, sparse input leads to sparse gradients.
4.1 Private ERM with sparse gradients

We consider the following empirical risk minimization problem: given a training data set \( D = \{d_j\}_{j=1}^n \) consisting of \( n \) data points, where \( d_j \in \mathbb{R}^p \), a constraint set \( W \in \mathbb{R}^p \), and a loss function \( \ell: W \times \mathbb{R}^p \to \mathbb{R} \), we want to find \( w^* = \arg \min_{w \in W} L(w; D) = \arg \min_{w \in W} \frac{1}{n} \sum_{j=1}^n \ell(w; d_j) \) satisfying differential privacy.

To characterize the gradient sparsity, we assume the data set has the following structure: \( D \) can be evenly divided into \( m \) subsets, such that the sum of the gradients of each subset is sparse. Specifically, we let \( D = \{D_1, \ldots, D_m\} \), where \( D_1 = \{d_1, \ldots, d_{\frac{n}{m}}\}, D_2 = \{d_{\frac{n}{m}+1}, \ldots, d_{\frac{2n}{m}}\}, \ldots, \) and \( D_m = \{d_{\frac{(m-1)n}{m}+1}, \ldots, d_n\} \), such that

\[
\forall i \in [m], \quad \left\| \sum_{d_j \in D_i} \nabla \ell(w; d_j) \right\|_0 \leq c_1, \text{ for all } w \in W. \tag{1}
\]

Roughly speaking, this assumption requires that the original dataset can be partitioned into several parts, so that each part exhibits some sparsity similarity.

This assumption may appear overly strict. However, we justify that it can be satisfied in many real applications. For example, in recommender systems, samples collected from the same user usually have overlapping supports, leading to input sparsity [ISLH17]. In these scenarios, it is natural to partition the input dataset according to the user ID. Similarity also exists in NLP, where training samples are collected from different sources. In Section 4.2, we show that for a broad class of problems, sparse input always produces sparse gradients.

We note that the sparsity assumption is used to argue the utility guarantee of our algorithm for ERM problems, without being necessary for its privacy.

To this end we propose Algorithm 2. Intuitively, in each iteration the algorithm first selects the most competitive coordinates of the gradient, and only adds noise to them. Finally it updates the model according to the noisy version of the gradient. We provide the privacy guarantee and theoretical guarantees in Theorems 1 and 2.

**Algorithm 2:** Differentially private ERM with sparse gradients

| Input: | Data set \( D = \{d_j\}_{j=1}^n \), loss function \( L(w; D) \), privacy parameters \((\varepsilon, \delta)\), constraint set \( W \), learning rate \( \eta \), iteration times \( T \), and gradient \( \ell_\infty\)-norm bound \( \|\nabla \ell(w; d)\|_\infty \leq c_2 \) |
|---|---|
| 1 Initialize \( w_0 \) from an arbitrary point in \( W \) |
| 2 For \( t = 0 \) to \( T \) |
| 3 Pick \( D_t \sim_u D \) with replacement |
| 4 Compute its average gradient \( \nabla_t = \frac{m}{n} \sum_{d_j \in D_t} \nabla \ell(w_t; d_j) \) |
| 5 Let \( \Delta_t = \text{NumericSparse}(\nabla_t, \alpha, c_1, \varepsilon', \delta') \), where \( \alpha = \frac{40c_2m(\log p + \log(4c_1n))}{\sqrt{n}c_1 \log \frac{2T}{\delta'}} \), \( \varepsilon' = \frac{\varepsilon \cdot m}{2 \sqrt{2T \log \frac{2}{\delta'}}} \) and \( \delta' = \frac{\delta m}{2T} \) |
| 6 \( w_{t+1} = w_t - \eta \Delta_t \) |
| Output: \( w^{\text{priv}} = w_T \) |

**Theorem 1** (Privacy). With the assumption that \( m \leq 10\sqrt{T} \), Algorithm 2 satisfies \((\varepsilon, \delta)\)-DP.

**Proof.** We note that in each iteration, when fixing the randomness due to sampling, step 4 itself satisfies \((\varepsilon', \delta')\)-DP, where \( \varepsilon' = \frac{\varepsilon \cdot m}{2 \sqrt{2T \log \frac{2}{\delta'}}} \) and \( \delta' = \frac{\delta m}{2T} \). Then by the sampling property of differential
privacy (Lemma 1), each iteration ensures \( \frac{\varepsilon}{\sqrt{2T \log \frac{m}{2}}} \)-DP. To conclude the proof, we apply the “strong composition theorem” (Lemma 2) with \( \delta = \frac{\varepsilon}{2} \) and \( k = T \).

**Theorem 2** (Utility). We assume \( \forall d, \ell(w; d) \) is \( K \)-smooth; \( \forall w, d, \|\nabla \ell(w; d)\|_\infty \leq c_2 \) and \( \|\nabla \ell(w; d)\|_2 \leq G \). Furthermore, under Assumption (1),

1. If \( L(w^0; D) - L(w^*; D) \leq D_L \), and we set \( T = \max\left(\frac{m^2}{100}, n\right) \), then
   \[
   \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} E\left\|\nabla L(w_t)\right\|_2^2\right] = \tilde{O}\left(\frac{G(c_1c_2 + \sqrt{KD_L})}{\varepsilon}\left(\frac{m}{n} + \frac{1}{\sqrt{n}}\right)\right).
   \]

2. Assume that for all \( d \), \( \ell(w; d) \) is convex in \( w \), and \( \forall t, \|w_t - w^*\|_2 \leq D_w \). Let \( T = \max\left(\frac{m^2}{100}, n\right) \), then
   \[
   \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \left(L(w_t; D) - L(w^*; D)\right)\right] = \tilde{O}\left(\frac{D_w(G + c_1c_2)}{\varepsilon}\left(\frac{m}{n} + \frac{1}{\sqrt{n}}\right)\right).
   \]

**Proof.** First by the privacy guarantees of the sparse vector technique (Lemma 5), for each iteration with probability greater than \( 1 - \frac{1}{n} \),
\[
\|\nabla_t - \Delta_t\|_2 \leq \frac{100c_1c_2\sqrt{T}(\log p + \log(4c_1n)) \log \frac{T}{m\delta}}{n\varepsilon},
\]
where we remark that the sensitivity upper bound is \( s_\infty = \frac{2c_2m}{n} \).

Note that from the assumption \( \|\nabla_t - \Delta_t\|_2 \leq 2G \) for sure. Therefore,
\[
\mathbb{E}\left[\|\nabla_t - \Delta_t\|_2\right] \leq \frac{100c_1c_2\sqrt{T}(\log p + \log(4c_1n)) \log \frac{T}{m\delta}}{n\varepsilon} + \frac{G}{n} \leq \frac{200c_1c_2\sqrt{T}(\log p + \log(4c_1n)) \log \frac{T}{m\delta}}{n\varepsilon},
\]
where the second inequality comes from the fact that \( T \geq 1 \), and the second term dominates.

Now we need the following lemma. The first half comes from [ASY+18], and we prove the second half of the lemma in Appendix A.

**Lemma 6.** Suppose \( \forall d, \ell(w; d) \) is \( K \)-smooth, with \( \|\nabla \ell(w; d)\|_2 \leq G \). Let \( w^0 \) satisfy \( L(w^0; D) - L(w^*; D) \leq D_L \). Let \( \eta := \min\left(\frac{1}{K}, \sqrt{2DL}(\sigma \sqrt{K})^{-1}\right) \), then after \( T \) rounds,
\[
\mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \left\|\nabla L(w_t)\right\|_2^2\right] \leq \frac{2D_LK}{T} + \frac{2\sqrt{2}\sigma \sqrt{KD_L}}{\sqrt{T}} + GB.
\]

Besides, if we further assume \( \forall d, \ell(w; d) \) is convex, and \( \forall t \in [T] \), \( \|w_t - w^*\|_2 \leq D_w \). Let \( \eta := \min\left(\frac{1}{K}, D_w(\sigma \sqrt{T})^{-1}\right) \), then after \( T \) rounds.
\[
\mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \left(L(w_t; D) - L(w^*; D)\right)\right] \leq \frac{D_w^2K}{T} + \frac{D_w\sigma}{\sqrt{T}} + 2BD_w\left(1 + \frac{G}{\sigma \sqrt{T}}\right),
\]
where
\[
\sigma^2 = 2 \max_{1 \leq t \leq T} \mathbb{E} \left[ \| \nabla_t - \nabla L(w_t; D) \|^2 \right] + 2 \max_{1 \leq t \leq T} \mathbb{E} \left[ \| \nabla_t - \Delta_t \|^2 \right],
\]
and
\[
B^2 = 2 \max_{1 \leq t \leq T} \mathbb{E} \left[ \| \nabla_t - \Delta_t \|^2 \right].
\]

First we consider the non-convex setting. By the definition of \(B\) and \(\sigma\) in Lemma 6, we have
\[
B = 300 c_1 c_2 \sqrt{T (\log p + \log(4 c_1 n)) \log \frac{m}{n}}, \quad \text{and} \quad \sigma^2 = 2B^2 + 2G^2.
\]
Therefore,
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} \| \nabla L(w_t) \|^2 \right] \leq \frac{2D_LK}{T} + \frac{4 \sqrt{2} (B + G) \sqrt{KD_t}}{\sqrt{T}} + GB.
\]

Suppose \(m < 10\sqrt{n}\), where \(T = \max \left( \frac{m^2}{100}, n \right) = n\). By Lemma 6,
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} \| \nabla L(w_t) \|^2 \right] = O \left( \frac{D_LK}{m^2} + \frac{G \sqrt{K D_t}}{m} + \frac{c_1 c_2 G (\log p + \log(4 c_1 n)) \log \frac{m}{T}}{\sqrt{n \varepsilon}} \right)
\]
\[
= \tilde{O} \left( \frac{G (c_1 c_2 + \sqrt{K D_t})}{\sqrt{n \varepsilon}} \right).
\]

We then consider the case when \(m \geq 10\sqrt{n}\), where \(T = \frac{m^2}{100}\). Note that \(\frac{1}{m} \leq \frac{m}{100 n}\).
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} \| \nabla L(w_t) \|^2 \right] = O \left( \frac{D_LK}{m^2} + \frac{G \sqrt{K D_t}}{m} + \frac{c_1 c_2 m G (\log p + \log(4 c_1 n)) \log \frac{m}{T}}{n \varepsilon} \right)
\]
\[
= \tilde{O} \left( \frac{Gm (c_1 c_2 + \sqrt{K D_t})}{n \varepsilon} \right).
\]

Therefore,
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} \| \nabla L(w_t) \|^2 \right] = \tilde{O} \left( \frac{Gm (c_1 c_2 + \sqrt{K D_t})}{n \varepsilon} \right) + \tilde{O} \left( \frac{G (c_1 c_2 + \sqrt{K D_t})}{\sqrt{n \varepsilon}} \right),
\]
and we have proved the first part of Theorem 2.

Similarly, for convex loss functions,
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} (L(w_t; D) - L(w^*; D)) \right] \leq \frac{D_w^2 K}{T} + \frac{D_w (B + G)}{\sqrt{T}} + 2BD_w \left( 1 + \frac{G}{(B + G) \sqrt{T}} \right).
\]
\[
\leq \frac{D_w^2 K}{T} + \frac{D_w G}{\sqrt{T}} + 2BD_w.
\]

If we take \(T = \max \left( \frac{m^2}{100}, n \right)\), and by similar arguments,
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} (L(w_t; D) - L(w^*; D)) \right] = \tilde{O} \left( \frac{D_w (G + c_1 c_2)}{\sqrt{n \varepsilon}} + \frac{D_w m (G + c_1 c_2)}{n \varepsilon} \right).
\]
\[
\square
\]

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4.2 Sparse features lead to sparse gradients

In this section, we show that for generalized linear model (GLM), sparse input always leads to sparse gradients.

**GLM:** Let a dataset \( D = (x_j, y_j)_{j=1}^n \), where \( \forall j, x_j \in \mathbb{R}^p \), and \( y_j \in [0, 1] \). Let \( \Phi: \mathbb{R} \rightarrow \mathbb{R} \) be a cumulative generating function. The objective of GLM is to minimize \( \frac{1}{n} \sum_{i=1}^n [\Phi(\langle x_i, w \rangle) - y_i \langle x_i, w \rangle] \).

In the following lemma, we observe that for GLM, sparse input produces sparse gradients.

**Lemma 7.** For all \( x \in \mathbb{R}^p \) with \( \|x\|_0 \leq c_1 \), and \( y \in [0, 1] \),

\[
\|\nabla \ell(w; (x, y))\|_0 \leq c_1.
\]

**Proof.** Note that \( \ell(w; (x, y)) = \Phi(\langle x, w \rangle) - y \langle x, w \rangle \), and \( \nabla \ell(w; (x, y)) = (\Phi'(\langle x, w \rangle) - y_i) \cdot x_i \).

Therefore, \( \|\nabla \ell(w; (x, y))\|_0 \leq c_1. \)

As a corollary, for a group of samples \( (x_i, y_i)_{i=1}^n \), assuming the non-zero coordinates are the same for each \( x_i \) gives the condition in Equation (1).

5 Improving DP-SGD in neural networks

In this section, we move to a specific problem of privately training neural networks, which is arguably the most important application of DP-ERM. [ACG+16] put forward the DP-SGD algorithm, which has been explored in a variety of domains such as federated learning of language models [MRTZ18] or sharing of clinical data [BJWW+19]. However, DP-SGD suffers from a loss in accuracy compared to its non-private version, especially for smaller datasets and high-dimensional networks [BPS19].

Following the observations from the previous sections, a natural question is how to improve DP-SGD for tasks that exhibit input sparsity, which are ubiquitous—and practically important—in domains where neural network models excel. For example, in language models, the first layer of the neural network is usually an embedding layer, whose input is extremely sparse. Accordingly, only a tiny fraction of parameters are picked up and updated in each round of training.

For models with sparse inputs, applying DP-SGD can lead to a poor performance, since the noise has to be added to all the dimensions. However, we cannot directly apply Algorithm 2 because of the following two reasons. First, there is no upper bound of \( \|\nabla \ell\|_0 \) or \( \|\nabla \ell\|_\infty \). Second, Algorithm 2 has to aggregate mini-batches according to the feature similarity, which is impractical for training large networks. In this section, we develop a modification of the previous algorithm to
Algorithm 3: Differentially private optimization with sparse gradients

**Input:** Data set \( D = \{d_1, \ldots, d_n\} \), loss function \( L(w; D) = \frac{1}{n} \sum_{j=1}^{n} \ell(w; d_j) \), where \( w \in \mathbb{R}^p \), \((\epsilon', \delta')\)-DP selection algorithm \( M : \mathbb{R}^p \to \{0, 1\}^p \), parameters: learning rate \( \eta \), noise multiplier \( \sigma \), mini-batch size \( b \), sparsity parameter \( \gamma \), gradient norm bound \( S_1, S_2 \)

1. Initialize \( w_0 \) randomly
2. For \( t = 0 \) to \( T - 1 \)
   3. Take a random batch \( b_t \) with sampling probability \( b/n \)
   4. For each \( d_j \in b_t \), compute \( g_j = \nabla \ell(w_t; d_j) \)
   5. The first gradient clipping
      \( \hat{g}_j = g_j / \max\left(1, \frac{\|g_j\|_2}{S_1}\right) \), and \( \bar{g} = \frac{1}{b} \sum_{d_j \in b_t} \hat{g}_j \)
   6. Private selection
      Let \( M = A(\hat{g}, \gamma) \), and \( \Delta = M \odot \hat{g} \), where \( \odot \) is the Hadamard product
   7. The second gradient clipping
      \( \hat{\Delta} = \Delta / \max\left(1, \frac{\|\Delta\|_2}{S_2}\right) \)
   8. Noise addition and parameter update:
      \( \hat{\Delta} + N\left(0, \sigma^2 \min\left(\frac{S_1^2}{b^2}, S_2^2\right) \cdot I\right) \)
   9. \( w^{t+1} = w_t - \eta \left(\hat{\Delta} \odot M\right) \)

**Output:** \( w^{priv} = w_T \)

**The first gradient clipping:** Similarly to DP-SGD, our algorithm requires a bounded influence of each individual sample. We clip each gradient in the \( \ell_2 \) norm: i.e., the gradient \( g_j \) is replaced by \( \hat{g}_j = g_j / \max\left(1, \frac{\|g_j\|_2}{S_1}\right) \), which ensures that if \( \|g_j\|_2 \geq S_1 \), then \( \|g_j\|_2 \) is scaled down to \( S_1 \), else its norm is preserved. Then we aggregate the gradient of each sample and compute \( \bar{g} \), which is the average gradient of the batch.

**Private selection:** This is a new step that specifically targets input sparsity. Its objective is to select the most “competitive” coordinates from \( \hat{g} \), which will be updated in the current iteration. It is a key step in our algorithm, since it avoids adding too much noise to the parameters. In this step, \( M \) is a binary vector, indicating which coordinate is selected, with \( \|M\|_0/p \approx \gamma \). \( A \) can be any differentially private selection algorithm, such as the sparse vector technique, exponential mechanism [DR14], or the algorithm proposed in [DR19b]. It is likely that another clipping is necessary, depending on the output of the private selection algorithm. We describe the exponential
mechanism in Algorithm 4 as an example.

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{Input:} Input gradient $\hat{g} \in \mathbb{R}^p$, sparsity parameter $\gamma$, privacy parameter $\epsilon', \delta'$, gradient $\ell_\infty$ norm bound $S_0$
\State Initialize $M = 0.$
\For{$t \in [p]$}
\State $\hat{g}(t) = \hat{g}(t) / \max\left(1, \frac{\|\hat{g}(t)\|_2}{S_0}\right)$
\EndFor
\For{$t = 0 \ to \ \lfloor \gamma p \rfloor$}
\State Randomly draw dimension $k$ with probability proportional to \(\exp\left(\frac{\epsilon'' |\hat{g}(k)|}{2S_0}\right)\), where $\epsilon'' = \frac{\epsilon'}{\sqrt{2[\gamma p] \log \frac{1}{\delta'}}}$
\State $M_k = 1, \hat{g}(k) = -\infty$
\EndFor
\State \textbf{Output:} $M \in \mathbb{R}$
\end{algorithmic}
\caption{(\(\epsilon', \delta'\))-DP selection with exponential mechanism}
\end{algorithm}

The second gradient clipping: We remark that $\|\Delta\|_2$ is usually much smaller than $\|\hat{g}\|_2$, because of the impact of the private selection procedure. Applying the second gradient clipping, we can further reduce the amount of the noise, i.e., the standard deviation of the Gaussian noise. We note that this step is necessary, since we observed that it had a significant influence on the algorithm’s performance in the experiments.

Noise adding and parameter update: We first remark that $\|\Delta\|_2 \leq S_2$, so the $\ell_2$-sensitivity of $\Delta$ is upper bounded by $2 \cdot \min\left(\frac{\Delta}{\gamma p}, S_2\right)$. Second, since we have already picked up the coordinates to be updated, it is no longer necessary to add noise to all the coordinates. Instead, the noise is only added to the dimensions which are chosen by the private selection algorithm.

Finally, we give the formal privacy guarantee of our algorithm, where we defer the proof to the supplement.

\textbf{Theorem 3.} With the assumption that $\frac{b}{n} \left(\varepsilon' + 2T \cdot \log \frac{1}{\delta'}\right) \leq \frac{1}{\sqrt{T}},$ Algorithm 3 satisfies $\left(\frac{4bT \log \frac{\Delta}{2S_0}}{n}, \left(\varepsilon' + 2T \cdot \log \frac{1}{\delta'}\right), 4bT \delta'\right)$-DP.

\textbf{Proof.} In each iteration, there are two steps which incur privacy costs: private selection and noise addition. Note that the noise adding satisfies $\left(\frac{2\sqrt{2 \log (1.25/\delta')}}{\sigma}, \delta'\right)$-DP, by the privacy guarantee of the Gaussian mechanism (Lemma 3). Then by the sampling theorem (Lemma 1) and the composition theorem, each iteration satisfies $(\bar{\varepsilon}, \bar{\delta})$-DP, where $\bar{\varepsilon} = \frac{b}{n} \cdot \left(\varepsilon' + 2\sqrt{2 \log (1.25/\delta')}\right), \bar{\delta} = 2b\delta'$. Finally, by the strong composition theorem (Lemma 2), the algorithm satisfies $(\varepsilon, \delta)$-DP, where

$$\varepsilon = T\bar{\varepsilon}(e^\bar{\varepsilon} - 1) + \bar{\varepsilon} \sqrt{2T \cdot \log \frac{1}{T\bar{\delta}}}, \delta = 2T\bar{\delta}.$$  

By the assumption that $\frac{b}{n} \cdot \left(\varepsilon' + 2T \cdot \log \frac{1}{\delta'}\right) \leq \frac{1}{\sqrt{T}}, \varepsilon \leq 2\bar{\varepsilon} \sqrt{2T \log \frac{1}{T\delta}}.$

\textbf{Remark I:} The assumption in the theorem is very weak. For example, if we assume the sample rate $\frac{b}{n} = \frac{1}{10000}$, each iteration’s privacy cost is $\frac{1}{2000}$, the algorithm has to run for 400 epochs to violate the assumption!
Remark II: Except for the private selection, our privacy guarantee is roughly $\sqrt{\log \frac{T}{\delta}}$ worse than DP-SGD (Theorem 1 in [ACG+16]). However, we do not think our algorithm has a worse privacy guarantee inherently. We observe that we are using the standard adaptive composition technique. An interesting open problem is how to better characterize the privacy cost, similarly to the Rényi privacy accountant, which we leave to future work.

6 Experiments

In this section, we conduct experiments for the word embedding algorithm [MCCD13], where sparsity inherently exists in the gradients. First, we provide our implementation details, and then we present the performance for our sparse algorithm. We show that our sparse algorithm can achieve better utility at the comparable level of privacy.

6.1 Model architecture

The model we consider is the CBOW (Continuous Bag Of Words) version of Word2Vec [MCCD13], which is a popular model in the literature. However, training a CBOW model is extremely slow: all the model parameters have to be updated by every batch of the training samples. To accelerate the process and remove the waste of negligible update of all parameters, we modify the optimization objective with the technique of “Negative Sampling” [MSC+13], where only a small percentage of the parameters are updated in each training iteration. Specifically, for a pair of target and context words, we randomly pick up a set of negative examples from the vocabulary, which we denote by $N$. For each sample, which includes a target word $w_t$, a context word $w_c$, and a set of negative words $\{w_{n,i}: i \in U\}$, the loss function is defined as follows:

$$\ell(w_t, w_c, N) = -\log(\sigma(e_t^T e_c)) - \sum_{i \in N} \log(\sigma(-e_t^T e_{n,i})),$$

where $e_t, e_c, e_{n,i}$ denote embeddings of $w_t, w_c, w_{n,i}$, respectively, and $\sigma$ denotes the sigmoid function.

We run our experiments on the Brown corpus [FK79]. In the preprocessing step, we first remove the least frequent and stop words, reducing the vocabulary size to 1,000. The embedding size is set to 100 for each word. Therefore, the overall number of parameters in the model is $1K \times 100 = 100K$.

We choose a window of size 4, and set $|U|$ to be 8, which means that each sample contains one target word, one context word, and 8 negative words. Finally, our data contains training, validation, and testing datasets with sizes 200K, 100K, 200K, respectively.

6.2 Hyperparameter tuning

Hyperparameter tuning for neural networks requires training several models with various combinations of hyperparameters, which results in privacy cost increase. For simplicity, we just assume our validation dataset is public in this experiment. In other words, no additional privacy cost is incurred by tuning hyperparameters on the validation dataset.

6.3 Training process

We implement our models with Opacus [Opa20], a library for training differentially private PyTorch models. We set the batch size $b = 20$, and the clipping norm $S_1 = 15$. We train our models for

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20 epochs, by an Adam optimizer with learning rate $\eta = 0.001$. The experiment is run on a Linux server with 6 CPUs and 50GB RAM, and it takes roughly one day to complete. We did not use any GPUs in this experiment.

### 6.4 Empirical results

Figure 2: Canary rank’s distribution, when $n_c = 3$ (top) , $n_c = 9$ (middle), and $n_c = 15$ (bottom)
We compare the performance of our sparse models with DP-SGD, after hyperparameter tuning. With respect to privacy, we fix the same privacy parameters $\varepsilon = 30$ and $\delta = 10^{-5}$ for all the algorithms. For the RDP accountant [MTZ19], we choose the noise multiplier $\sigma = 0.32$ for DP-SGD. For the other sparse algorithms, we divide our privacy budget into two parts: $\varepsilon = 20$ and $\delta = 5 \times 10^{-6}$ for noise addition, $\varepsilon = 10$ and $\delta = 5 \times 10^{-6}$ for private selection. By Theorem 3, we set $\sigma = 0.5$ for our sparse algorithms. The hyperparameters for the algorithms are summarized in Table 1.

|                          | DP-SGD | DP sparse |
|--------------------------|--------|-----------|
| Batch size $b$           | 20     | 20        |
| Learning rate $\eta$     | 0.001  | 0.001     |
| Epoch                    | 20     | 20        |
| First gradient clipping norm $S_1$ | 15     | 15        |
| First gradient clipping norm $S_1$ | 15     | 15        |
| Second gradient clipping norm $S_2$ | N/A    | 1         |
| Sparsity parameter $\gamma$ | N/A    | 0.001     |
| DP selection clipping norm $S_0$ | N/A    | 0.1       |

Table 1: The summary of the hyperparameters

From Figure 1, we observe that our sparse algorithms have provided much better performance than DP-SGD, both in terms of the training error or test error. Furthermore, the private selection by the exponential mechanism slightly outperforms the sparse vector technique, which is consistent with the results in [LSL17].

One valid complaint is that our algorithms give extremely weak privacy guarantees, since $\varepsilon = 30$ is too large to be practical for most applications. We remark that this value is an upper bound on the privacy loss budget.

As mentioned in Section 5, our $\varepsilon$ computation is quite conservative, and we believe that the true $\varepsilon$ value should be much smaller. Here we justify our conjecture by an empirical method. Note that the gap between the training error and test error (generalization error) can serve as a lower bound on the privacy level, as shown in [DFH+15], and [JLN+20]. From Figure 1, we find that the generalization errors of both DP-SGD and our sparse algorithm are small that are consistent with good privacy guarantees. Furthermore, if we improve the noise multiplier of DP-SGD to the same level of our sparse algorithms ($\sigma = 0.5$), we observe that the generalization errors are still comparable between our sparse algorithms (Figure 1d and Figure 1e) and DP-SGD (Figure 1c), indicating they share similar privacy guarantees. As a benchmark, non-private algorithm has provided the best training error and test error. However, the huge gap between the training and test error indicates the model has provided almost no privacy guarantees. Besides, the periodic behaviour in the training error also indicates the model has severely memorized the training dataset.

### 6.5 Evaluation for unintended memorization

In this section, we use another method to estimate the privacy level of our model. Specifically, we follow the Secret Sharer frameworks proposed from [CLE+19], which aims to measure the unintended memorization of rarely-occurring phrases in the dataset. This method has been further explored in recent works [JE19, RTM+20].

First, we randomly generate 1,000 canaries, each containing three words. The reason we opt for inserting three-word canaries is that computing the ranks for longer canaries is time-consuming.
Each word in a canary is uniformly randomly chosen from the 1K vocabulary. This is because we want to measure unintended memorization of our models, i.e., the memorization of atypical phrases in the language model, which is in fact orthogonal to our learning task. Two examples of our canaries are “mother government opportunity” and “prices effort me”.

Next, we insert all these canaries into random positions in our original dataset, each canary appearing exactly $n_c$ times. Then we train our models as before. Note that the canaries have a trivial impact on our models, since the cumulative number of inserted phrases is relatively small relative to the size of the original dataset.

We use the Random Sampling method, as proposed in [CLE+19], to measure whether the canary is memorized by our model. Specifically, for a canary $c = \{c_0, c_1, c_2\}$, we define the log-perplexity of the model $\theta$ on $c$ as $P_\theta(c) = -\log \Pr (c_1|c_0) - \log \Pr (c_2|c_1, c_0)$. We define the rank of the canary as $\text{rank}_\theta(c) = |\{c'_1 \neq c'_2: P_\theta(\{c_0, c'_1, c'_2\}) \geq P_\theta(c)\}, c'_1, c'_2 \in V|$, where $V$ is the vocabulary. Intuitively, a high rank indicates the model highly favors the canary as compared to random chance. In other words, the model has “memorized” the canary, suggesting a privacy violation. We note that when computing each canary’s rank, it is time-consuming to enumerate all the possible phrases. Therefore, we randomly pick up 10K phrases from the domain and compute $c$’s rank in the subset instead.

In Figure 2, we show the distributions of the rank when $n_c = 3$, $n_c = 9$ and $n_c = 15$. We run the experiments as mentioned above with non-private algorithm, DP-SGD, and Algorithm 3 (instantiated with the exponential mechanism for private selection), where we use the same hyper-parameters as in the previous experiments. Note that for DP-SGD and DP Sparse, we are using exactly the same noise parameter $\sigma = 0.5$. The experiment is designed to validate the hypothesis that the trained models exhibit similar levels of unintended memorization.

As a benchmark, we randomly select 1,000 phrases that are outside of the training dataset, and we plot the histogram of the rank in Figure 2d. We find that it is very close to the uniform distribution (confirmed by the chi-squared goodness of fit test). This is indeed expected since the process is equivalent to uniformly randomly drawing one sample from an ordered set, since the phrase is independent of the trained model.

The top two rows of Figure 2 report results for small $n_c$ ($n_c = 3$ and 9, respectively). Unlike the non-private training, both DP-SGD and DP Sparse result in histograms close to the uniform distribution (Figures 2b and 2c).

We also compute the chi-squared distance, and the $p$-values of Pearson’s chi-squared tests (Table 2). Where the $p$-values are not statistically significant, the chi-squared test fails to reject the null hypothesis, i.e., that the training procedure preserves privacy. Visually, the non-private algorithm produces a histogram that is highly concentrated to the right, which indicates that the model has indeed memorized the training dataset. Therefore, we argue that our sparse algorithm gives comparable privacy guarantees as DP-SGD, which are much better than the non-private version.

Finally, this method can be part of hyperparameter tuning, where the chi-squared distance can be an excellent metric to measure privacy leakage. For the case of $n_c = 15$, we observe that Figure 2 is consistent with the empirical $\varepsilon$ being quite small (ranging between 0.07 and 0.11), following the group property of differential privacy, and the fact that the uniform distribution breaks down at some place between $n_c = 9$ and $n_c = 15$.

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| $n_c$ | non-private | DP-SGD     | DP Sparse  | random     |
|------|-------------|------------|------------|------------|
| 3    | 3.26 (.00)  | .007 (.63) | .005 (.85) | .005 (.86) |
| 9    | 3.24 (.00)  | .004 (.88) | .007 (.68) | .010 (.33) |
| 15   | 3.27 (.00)  | .011 (.31) | **.026 (.00)** | .002 (.98) |

Table 2: The chi-squared goodness of fit with the uniform distribution ($p$-value in parenthesis). Statistically significant results ($p < 0.01$) are in bold.

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A Proof of Lemma 6

The first half comes from [ASY+18]. Therefore, it is enough to prove the second half, where we use a similar proof technique with [ASY+18] and [GL13].

First observe that, for any $t = 1, \ldots, T$,
\begin{align*}
\|w_{t+1} - w^*\|_2^2 &= \|w_t - \eta\Delta_t - w^*\|_2^2 \\
&= \|w_t - w^*\|_2^2 - 2\eta\langle\Delta_t, w_t - w^*\rangle + \eta^2\|\Delta_t\|_2^2 \\
&= \|w_t - w^*\|_2^2 - 2\eta\langle\nabla L(w_t; D) + A_t, w_t - w^*\rangle + \eta^2\left(\|\nabla L(w_t; D)\|_2^2 + 2\langle\nabla L(w_t; D), A_t\rangle + \|A_t\|_2^2\right),
\end{align*}

where we define $A_t = \Delta_t - \nabla L(w_t; D)$.

By the convexity and smoothness, we have
\begin{align*}
\|\nabla L(w_t; D)\|_2^2 &\leq K\langle\nabla L(w_t; D), w_t - w^*\rangle.
\end{align*}

Combining these, for all $t = 1, \ldots, T$,
\begin{align*}
\|w_{t+1} - w^*\|_2^2 &\leq \|w_t - w^*\|_2^2 - 2\eta - K\eta^2\langle\nabla L(w_t; D), w_t - w^*\rangle - 2\eta\langle\nabla L(w_t; D), w_t - w^*\rangle - \eta\|\nabla L(w_t; D)\|_2^2 - \|\Delta_t\|_2^2 - \langle A_t, w_t - w^*\rangle - \eta^2\|\Delta_t\|_2^2 \\
&\leq \|w_t - w^*\|_2^2 - (2\eta - K\eta^2)(L(w_t; D) - L(w^*; D)) - \eta\|\nabla L(w_t; D)\|_2^2 - \|\Delta_t\|_2^2 - \langle A_t, w_t - w^*\rangle - \eta^2\|\Delta_t\|_2^2.
\end{align*}
where the last inequality uses the convexity and the fact that $\eta \leq \frac{2}{K}$.

Summing up the above inequalities and re-arranging the terms, we have

$$
(2\eta - K\eta^2) \sum_{t=1}^{T} \langle L(w_t; D) - L(w^*; D) \rangle
\leq \|w_1 - w^*\|_2^2 - \|w_{T+1} - w^*\|_2^2 - 2\eta \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, A_t \rangle + \sum_{t=1}^{T} \eta^2 \|A_t\|_2^2
\leq D_w^2 - 2\eta \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, A_t \rangle + \sum_{t=1}^{T} \eta^2 \|A_t\|_2^2.
$$

Taking the expectation on both sides, we have

$$
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} \langle L(w_t; D) - L(w^*; D) \rangle \right]
\leq \frac{1}{(2\eta - K\eta^2)T} \left( D_w^2 - 2\eta \cdot \mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, A_t \rangle \right] + \mathbb{E} \left[ \sum_{t=1}^{T} \eta^2 \|A_t\|_2^2 \right] \right).
$$

We first bound the third term in the parenthesis. According to the definition of $\sigma$ in Lemma 6, we have

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \eta^2 \|A_t\|_2^2 \right] \leq T\eta^2 \sigma^2. \tag{2}
$$

With respect to the second term,

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, A_t \rangle \right]
= \mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, (\Delta_t - \nabla_t) + (\nabla_t - \nabla L(w_t; D)) \rangle \right]
= \mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, \Delta_t - \nabla_t \rangle \right] + \sum_{t=1}^{T} \mathbb{E} \left[ \mathbb{E} \left[ \langle w_t - \eta \nabla L(w_t; D) - w^*, \nabla_t - \nabla L(w_t; D) \rangle \mid w_t = w \right] \right].
$$

Note that

$$
\mathbb{E} \left[ \langle w_t - \eta \nabla L(w_t; D) - w^*, \nabla_t - \nabla L(w_t; D) \rangle \mid w_t = w \right] = \langle w - \eta \nabla L(w; D) - w^*, \nabla_t - \nabla L(w_t; D) \rangle | w_t = w = 0.
$$

where the last equality comes from the fact that $\nabla_t$ is an unbiased estimator of $\nabla L(w_t; D)$.

Therefore,

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, A_t \rangle \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, \Delta_t - \nabla_t \rangle \right].
$$
By Cauchy-Schwarz inequality, and the definitions in Lemma 6, 

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \langle w_t - \eta \nabla L(w_t; D) - w^*, \Delta_t - \nabla \Delta_t \rangle \right] \\
\leq \mathbb{E} \left[ \sum_{t=1}^{T} \|w_t - \eta \nabla L(w_t; D) - w^*\|_2 : \|\Delta_t - \nabla \Delta_t\|_2 \right] \\
\leq T(D_w B + \eta G B). 
\]

(3)

Finally, by combining Equation (2) and (3), and note that \( K \eta \leq 1 \), 

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} (L(w_t; D) - L(w^*; D)) \right] \\
\leq \frac{D_w^2 + T \eta^2 \sigma^2}{T \eta \cdot (2 - K \eta)} + \frac{2D_w B \eta + 2 \eta^2 GB}{\eta \cdot (2 - K \eta)} \\
\leq \frac{D_w^2 + T \eta^2 \sigma^2}{T \eta} + \frac{2D_w B \eta + 2 \eta^2 GB}{\eta}. 
\]

By taking \( \eta = \min \left( \frac{1}{K}, \frac{D_w}{\sqrt{T} \sigma} \right) \), we have \( \frac{D_w^2}{T \eta} \leq \frac{D_w^2 K}{T}, \eta \sigma^2 \leq \frac{D_w \sigma}{\sqrt{T}}, \) and \( GB \eta \leq \frac{G B D_w \sigma}{\sigma \sqrt{T}} \). Therefore, by combining them, 

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} (L(w_t; D) - L(w^*; D)) \right] \leq \frac{D_w^2 K}{T} + \frac{D_w \sigma}{\sqrt{T}} + 2BD_w \left( 1 + \frac{G}{\sigma \sqrt{T}} \right). 
\]