Band-Limited Maximizers for a Fourier Extension Inequality on the Circle

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\textbf{ABSTRACT}
Among the class of functions with Fourier modes up to degree 30, constant functions are the unique real-valued maximizers for the endpoint Tomas–Stein inequality on the circle.

\textbf{KEYWORDS}
Sharp Fourier restriction theory; Bessel functions; circle

\section{1. Introduction}

We are interested in the sharp constant of the endpoint Tomas–Stein adjoint restriction inequality \cite{Tomas 75} on the circle $S^1 = \{\omega \in \mathbb{R}^2 : |\omega| = 1\}$. More precisely, we seek a maximizer for the functional $\Phi$ defined on nonzero $f \in L^2(S^1)$ by

$$\Phi(f) := \|\hat{f}\|_{L^6(\mathbb{R}^2)}^6 \|f\|_{L^2(S^1)}^2.$$  

We have written $\sigma$ for the arc length measure on the circle $S^1$, and we have used the Fourier transform

$$\hat{\sigma}(x) := \int_{S^1} f(\omega) e^{-ix\cdot\omega} d\sigma_\omega, \quad (x \in \mathbb{R}^2).$$

It is known that maximizers of $\Phi$ exist \cite{Shao 16} and are smooth \cite{Shao 16}, and that the constant function $1$ is a local maximizer of $\Phi$, see \cite[Theorem 1.1]{Carneiro et al. 17}. Moreover, real-valued maximizers of $\Phi$ are known to be nonnegative, antipodally symmetric functions, that is

$$f(\omega) \geq 0, f(-\omega) = f(\omega),$$

for every $\omega \in S^1$. It is natural to conjecture that constant functions are global maximizers of $\Phi$, in which case a complete characterization of complex-valued maximizers is given by \cite[§1, Step 6]{Carneiro et al. 17}. In this article, we report on numerical verification of a finite dimensional variant of this conjecture:

\textbf{Theorem 1.} Let $f \in L^2(S^1)$ be nonnegative and antipodally symmetric. Assume that $\hat{f}(n) = 0$ if $|n| > 30$. Then

$$\Phi(f) \leq \Phi(1),$$

with equality if and only if $f$ is constant.

We briefly elaborate on the limitation to degree thirty. We used easily accessible hardware such as standard laptops and desktops, and a high-end computing language (Mathematica). Some computations took a week to complete, using a small number of gigabytes of memory. A limiting factor for our method is the number of Bessel integrals that need to be calculated, growing with the fifth power of the maximal degree of Fourier modes. Another limiting factor is the required accuracy of each Bessel integral. We designed the numerical evaluation of Bessel integrals following our previous work \cite{Oliveira e Silva and Thiele 17}, which gives just enough accuracy for degree thirty and yet requires substantial computation time and memory.

A numerical difficulty stems from a reduction to positive semi-definiteness of an auxiliary quadratic form. Not only does the constant function produce a zero eigenvector of this form, but so does a sum of Dirac delta functions at two antipodal points on the circle. Heisenberg uncertainty allows for functions with Fourier modes up to degree 30 to localize roughly in a $\frac{2\pi}{30}$-neighborhood of these antipodal points, creating small eigenvalues for the quadratic form restricted to these Fourier modes. The shrinking spectral gap between the constant function and these near-extremizers requires growing accuracy for higher order Fourier modes.

This article continues efforts to implement Foschi’s program \cite{Foschi 15} for the 2-sphere in the case of the circle, see also \cite{Carneiro et al. 17}. The approach works through positive semi-definiteness of a certain quadratic form on the relevant finite dimensional space. It would be nice to establish positive semi-definiteness for the full space. For recent similar work on the paraboloid, see \cite{Gonçalves 17}.
Theorem 1 can be found on arXiv:1806.06605(v2).

A Mathematica notebook file containing the code for the calculations involved in the proof of Theorem 1 can be found on arXiv:1806.06605(v2).

2. Proof of Theorem 1

With \( f \) as in Theorem 1, we compute

\[
||f||^6_{L^2(\mathbb{R}^2)} = \left(2\pi\right)^2 \int_{(\mathbb{S}^1)^6} \delta \left( \sum_{j=1}^{6} \omega_j \right) \left( \prod_{j=1}^{3} f(\omega_j) \, d\sigma_{\omega_j} \right) \\
\left( \prod_{j=4}^{6} f(-\omega_j) \, d\sigma_{\omega_j} \right)
\]

\[
= 5\pi^2 \int_{(\mathbb{S}^1)^6} \delta \left( \sum_{j=1}^{6} \omega_j \right) \left( |\omega_4 + \omega_5 + \omega_6|^2 - 1 \right) \\
\left( \prod_{j=1}^{3} f(\omega_j) \, d\sigma_{\omega_j} \right) \left( \prod_{j=4}^{6} f(-\omega_j) \, d\sigma_{\omega_j} \right)
\]

\[
\leq 5\pi^2 \int_{(\mathbb{S}^1)^6} \delta \left( \sum_{j=1}^{6} \omega_j \right) \left( |\omega_4 + \omega_5 + \omega_6|^2 - 1 \right) \\
\left( \prod_{j=1}^{3} f(\omega_j)^2 \right) \left( \prod_{j=1}^{6} d\sigma_{\omega_j} \right)
\]

\[
\leq 5\pi^2 \frac{||f||^6_{L^2(\mathbb{S}^1)}}{||1||^6_{L^2(\mathbb{S}^1)}} \int_{(\mathbb{S}^1)^6} \delta \left( \sum_{j=1}^{6} \omega_j \right) \\
\left( |\omega_4 + \omega_5 + \omega_6|^2 - 1 \right) \prod_{j=1}^{6} d\sigma_{\omega_j}
\]

\[
= \Phi(1)||f||^6_{L^2(\mathbb{S}^1)}.
\]

Here the second line is simply Plancherel’s identity. The third line is Foschi’s idea to improve the situation by artificially inserting a weight, which after symmetrization over the indices \( j \) reverts to a constant, see the computation following \cite{carneiroetal17, Lemma 1.3}. The fourth line is the crucial inequality. We defer its proof for the moment. The inequality in the fifth line is an application of the main result proved in \cite{carneiroetal17, Theorem 1.2}. Equality is attained if and only if \( f \) is constant. Identification of the constant in the sixth line is then easy and was also observed in \cite{carneiroetal17}.

This proves Theorem 1, safe for verification of the crucial inequality in the third line. Note that this inequality would follow from

\[
|\omega_1 + \omega_2 + \omega_3| = |\omega_4 + \omega_5 + \omega_6|
\]

and the inequality between the arithmetic mean and the geometric mean,

\[
\prod_{j=1}^{3} f(\omega_j) \prod_{j=4}^{6} f(-\omega_j) \leq \frac{1}{2} \left( \prod_{j=1}^{3} f(\omega_j)^2 + \prod_{j=4}^{6} f(-\omega_j)^2 \right);
\]

if the weight were positive. Unfortunately, the weight is not positive. One reason to believe that the inequality still holds is that the negative part of the weight is small, and via antipodal symmetry the values of the function on the negative part of the weight have a strong correlation with the values on the positive part. However, the support of the measure

\[
\delta \left( \sum_{j=1}^{6} \omega_j \right)
\]

is not preserved under antipodal symmetry, which makes it difficult to exploit this correlation. We resort to numerical verification of the crucial inequality in the given finite dimensional space of functions.

Consider the index set

\[
Z = \{ k \in 2\mathbb{Z}, |k| \leq 30 \},
\]

and expand the band-limited function \( f \) into a Fourier series

\[
f(\omega) = \sum_{k \in Z} a_k e^{i \omega k};
\]

where we identify \( \mathbb{R}^2 \) with the complex plane and correspondingly define products and powers of elements in \( \mathbb{R}^2 \). Note that

\[
a_{-k} = \overline{a_k} \quad (2-1)
\]

for every \( k \in Z \). We write a constant multiple of the left-hand side of the crucial inequality as
Define for $m \in \mathbb{Z}^3$

$$s_m := a_{m_1} a_{m_2} a_{m_3},$$

and note that $(s_m)_{m \in \mathbb{Z}^3}$ is an element of $\text{Sym}(\mathbb{Z}^3)$, the vector space of functions on $\mathbb{Z}^3$ symmetric under permutation of the three indices. At this point we do not require the symmetry (2–1), instead we pass to a larger space allowing for a convenient orthogonal splitting later.

The crucial inequality then follows from positive semi-definiteness of the quadratic form

$$\sum_{m,n \in \mathbb{Z}^3} Q_{m,n} s_m s_n := \sum_{m,n \in \mathbb{Z}^3} \frac{1}{\pi} \sum_{\sigma \in S_3} (R_{m,n} - L_{m,n}) s_m s_n$$
on $\text{Sym}(\mathbb{Z}^3)$, where we write $S_3$ for the group of permutations of three elements and

$$n_\sigma = (n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}).$$

Note that the symmetrization over $S_3$ does not change the value of the quadratic form whenever the coefficients $s_n$ are symmetric. It merely symmetrizes the coefficients of the quadratic form, and allows to reduce the dimension of the matrix by identifying equivalent tuples. Letting $\bar{X}$ be the space of tuples in $\mathbb{Z}^3$ satisfying $m_1 \leq m_2 \leq m_3$, it therefore suffices to prove positive definiteness of the quadratic form

$$Q(s, s) = \sum_{m,n \in \bar{X}} Q_{m,n} s_m s_n$$

Note that the matrix $(Q_{m,n})_{m,n \in \bar{X}}$ is Hermitian. Moreover, for all $m \in \mathbb{Z}^3$ we have

$$R_{m,(0,0,0)} = L_{m,(0,0,0)}$$

Figure 2. $m_0 = (-24,0,24), m_0 = (-12,0,12), m_0 = (-6,0,6)$.

Figure 3. $m_0 = (-20,8,12), m_0 = (-12,6,6), m_0 = (-10,2,8)$. 

$$\sum_{k \in \mathbb{Z}^3} L_k \left( \prod_{j=1}^3 a_{k_j} \right) \left( \prod_{j=4}^6 a_{-k_j} \right),$$

$$L_k := (2\pi)^{-5} \int_{(S_3)^6} \delta \left( \sum_{j=1}^3 \omega_{k_j} \right) \left( |\omega_4 + \omega_5 + \omega_6|^2 - 1 \right) \left( \prod_{j=1}^3 \omega_{k_j} \right) \left( \prod_{j=4}^6 \omega_{-k_j} \right) \prod_{j=1}^3 \text{d}\sigma_{(\omega)}. $$

and the same multiple of the right-hand side as

$$\sum_{k \in \mathbb{Z}^3} R_k \left( \prod_{j=1}^3 a_{k_j} \right) \left( \prod_{j=4}^6 a_{-k_j} \right),$$

$$R_k := (2\pi)^{-5} \int_{(S_3)^6} \delta \left( \sum_{j=1}^3 \omega_{k_j} \right) \left( |\omega_4 + \omega_5 + \omega_6|^2 - 1 \right) \left( \prod_{j=1}^3 \omega_{k_j} \right) \left( \prod_{j=4}^6 \omega_{-k_j} \right) \prod_{j=1}^3 \text{d}\sigma_{(\omega)}. $$

and hence the Dirac delta vector $\delta_{(0,0,0)}$ corresponding to constant functions on the circle is in the kernel of the matrix $(Q_{m,n})_{m,n} \in \bar{X}$. Therefore, we replace $\bar{X}$ by $X = \bar{X} \setminus \{(0,0,0)\}$.

A change of variables

$$(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) 
\mapsto (\omega_1 \cdot \omega_1, \omega_2 \cdot \omega_2, \omega_3 \cdot \omega_3, \omega_4 \cdot \omega_4, \omega_5 \cdot \omega_5, \omega_6 \cdot \omega_6)$$

for some arbitrary $\omega$ of modulus one in the expressions for $R_k$ and $L_k$ shows that

$$Q_{m,n} = \omega^{d(m) - d(n)} Q_{m,n},$$

where we have denoted $d(m) = m_1 + m_2 + m_3$.

We conclude $Q_{m,n} = 0$ whenever $d(m) \neq d(n)$. The matrix $(Q_{m,n})_{m,n} \in X_0$ therefore has the structure of a diagonal block matrix, with blocks enumerated by $D := d(m)$. It suffices to prove positive semi-definiteness for each block $(Q_{m,n})_{m,n} \in X_0$, separately, where $X_0 = \{ m \in X : d(m) = D \}$. This will be done in the following section.

3. Numerical computations

In order to verify that the matrix $(Q_{m,n})_{m,n} \in X_0$ is positive definite, we split it into a numerically computed approximation and an error term. The smallest eigenvalue of the numerical approximation will be larger than the operator norm of the error term. We remark that the number of matrix coefficients grows with the fifth power of the degree.

Numerical approximation of the integrals $L_k$ and $R_k$ will rely on the following family of Bessel integrals for $\sum_{j=1}^{6} k_j = 0$ :

$$L_k = (2\pi)^{-5} \int_{[S]^6} \delta \left( \sum_{j=1}^{6} \omega_j \right) \left( \prod_{j=1}^{6} \omega_j^{k_j} \, d\sigma_{\omega_0} \right)$$

$$= (2\pi)^{-1} \int_{[S]} \prod_{j=1}^{6} I_k(|x|) \, dx = \int_{0}^{\infty} \prod_{j=1}^{6} I_k(r) \, dr,$$

where the Bessel function $I_k$ is defined by

$$\int_{[S]} e^{-ix\omega} \, d\sigma_{\omega} = 2\pi(-i)^k I_k(|x|) (x/|x|)^k.$$

Indeed, writing

$$|\omega_4 + \omega_5 + \omega_6|^2 - 1 = 2 + \sum_{j, k \in \{4,5,6\}, k \neq j} \omega_j \omega_k^{-1},$$

we obtain

$$L_{m,n} = 2L_{m,n} + \sum_{\sigma \in S_5} I_{m,n+1} - 1,0,0,0,$$

$$R_{m,n} = 2R_{m,n} + \sum_{\sigma \in S_5} I_{m,n+1} - 1,0,0,0,$$

Using the fact that $J_{-k} = (-1)^k J_k$ and the above representation we see that $Q_{m,n} = Q_{-m,n}$, so it suffices to consider $D \geq 0$.

To evaluate the integrals $I_k$, we follow the scheme in [Oliveira e Silva and Thiele 17]. We split the integrals into

$$I_k = \int_{0}^{\infty} \prod_{j=1}^{6} I_k(r) \, dr + \int_{0}^{\infty} \prod_{j=1}^{6} I_k(r) \, dr$$

$$+ \int_{0}^{\infty} \prod_{j=1}^{6} I_k(r) \, dr,$$

with $S = 3600$ and $R = 63000$. The first two integrals are evaluated with a Newton–Cotes quadrature rule. The step size is 0.003 for the first integral and 0.05 for the second integral. In practice, this step involved tabulating the numerical values of 61 Bessel functions at around $3 \times 10^6$ points each, with 20 digit precision obtained via the software package Mathematica [Wolfram Research 17]. This high precision lets the rounding errors be absorbed by the error estimates below.

The approximation error for the first integral in (3–1) was estimated in [Oliveira e Silva and Thiele 17, §8], independently of the vector $k$, by

$$E_{k,1} = 1.5 \times 10^{-9}.$$

The approximation error for the second integral in (3–1) was also estimated in [Oliveira e Silva and Thiele 17] by

$$E_{k,2} = C_2 \prod_{j=1}^{6} \left( 1 + \frac{k_j^2}{S} \right),$$

where

$$C_2 = 1.016 (R-S) w^8 \frac{6^3}{5} \left( \frac{2}{\pi(S-1)} \right)^3 \cosh^6(1)(R+1)$$

with $S = 3600$, $R = 36000$ and $w = 0.05$.

The third integral in (3–1) is approximated by analytic methods. Since $R = 63000$ is large when compared to $n^2 \leq 61^2$, we take advantage of the following well-known asymptotic formulae which are a simplified version of [Oliveira e Silva and Thiele 17, Corollaries 2.6 and 2.7].

We record a typo on the first formula in [Oliveira e Silva and Thiele 17, Corollary 2.7], which should read as follows:

$$|J_0(z) - \frac{2}{\pi z} \cos(\omega_0) - \frac{1}{2}(\frac{1}{z})^5 \sin(\omega_0)|$$

$$\leq \frac{9}{16\pi^2} (\frac{z}{2})^2 \cosh(\sqrt{3}/|z|)/(\pi(z^2)^{5/2}).$$
Lemma 2. Let \( \omega_n := z - \frac{\pi}{2} - \frac{m\pi}{2} \) and \( \hat{n} := \max\{1, n\} \). If \( n \geq 0 \) and \( z > \hat{n}^2 \), then

\[
\left| J_n^\pm(z) - \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cos(\omega_n) \right| \leq \left( \frac{2}{\pi |z|} \right)^{\frac{1}{2}} \frac{1}{|z|^2}, \quad (3-2)
\]

\[
\left| J_n^\pm(z) - \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cos(\omega_n) + \frac{4 nm^2 - 1}{8 z} \left( \frac{2}{\pi z} \right) \sin(\omega_n) \right| \leq \frac{1}{4} \left( \frac{2}{\pi |z|} \right)^{\frac{1}{2}} \frac{\hat{n}^4}{|z|^2}, \quad (3-3)
\]

Here the functions \( J_n^\pm \) are obtained by writing \( \cos(z t) = (e^{i z t} + e^{-i z t})/2 \) in the Poisson integral representation for \( J_n \), and as such satisfy \( J_n = (J_n^+ + J_n^-)/2 \). Using (3-2), we may split each Bessel function into a main term plus error. Applying the distributive law for each \( j \) for each \( k \), we call them cosine factor and error factor. For the main integral of the form

\[
\int_R \left( \frac{2}{\pi r} \right)^{\frac{3}{2}} \prod_{j=1}^6 \cos(\omega_{k_j}) \right) r \, dr, \quad (3-4)
\]

which can be calculated exactly, plus \( 2^6 - 1 \) further terms involving one of the two factors

\[
\left( \frac{2}{\pi r} \right)^{\frac{1}{2}} \cos(\omega_{k_j}), J_k(r) - \left( \frac{2}{\pi r} \right)^{\frac{1}{2}} \cos(\omega_{k_j})
\]

for each \( j \). We call them cosine factor and error factor. For the main integral, observe that

\[
\cos \left( r - \frac{\pi}{4} \right)^{\frac{k\pi}{2}}
\]

\[
= (-1)^{\left[ \frac{k}{2} \right]}, \quad \text{if } k \text{ is odd,}
\]

\[
= \cos \left( r - \frac{\pi}{4} \right)^{\frac{k\pi}{2}} \cos \left( \frac{\pi}{2} \right), \quad \text{if } k \text{ is even,}
\]

and so (3-4) equals a multiple of

\[
\int_R \cos^6 \left( r - \frac{\pi}{4} \right)^{\frac{k\pi}{2}} r^{-2} \, dr, \quad \text{or}
\]

\[
\int_R \cos^4 \left( r - \frac{\pi}{4} \right)^{\frac{k\pi}{2}} \sin^2 \left( r - \frac{\pi}{4} \right)^{\frac{k\pi}{2}} r^{-2} \, dr, \quad (3-5)
\]

with sign determined by the parity of \( \sum_{j=1}^3 \left( \frac{m_j}{2} \right) + \left( \frac{m_j}{2} \right) \). Mathematica calculates these expressions with any prescribed accuracy. For the further terms, consider first those consisting of an integral of a product of five cosine factors and one error factor.

To estimate these six terms, we use the finer information given by (3-3). The sine term in (3-3) leads to integrals of the type

\[
\frac{4nm^2 - 1}{8} \int_R \left( \frac{2}{\pi r} \right)^{\frac{3}{2}} \cos(\omega_{m_j}) \cos(\omega_{m_j}) \cos(\omega_{m_j}) \cos(\omega_{m_j}) \, dr
\]

and similar terms with a different cosine factor replaced by a sine factor and corresponding prefactor.

The product of the six trigonometric functions is odd about the point \( \frac{\pi}{2} \). Thus, the product integrates to 0 over each period. On the period \([R + 2nk, R + 2\pi(k + 1)]\), we may thus replace the weight \( r^{-3} \) by the difference between itself and its mean over that interval, which in turn is bounded by \( 6\pi r^{-4} \). Thus, the sum of terms arising from the sine term in (3-3) is bounded by

\[
E_{k,3} = 3\pi \sum_{j=1}^6 k_j^2 \int_R \left( \frac{2}{\pi r} \right)^{\frac{3}{2}} r^{-4} \, dr,
\]

where \( k_j := \max\{1, k_j\} \). The sum of the six terms arising from the right-hand side of (3-3) can be estimated by

\[
E_{k,4} = \frac{1}{4} \sum_{j=0}^6 k_j^2 \int_R \left( \frac{2}{\pi r} \right)^{\frac{3}{2}} r^{-4} \, dr.
\]

Next come fifteen terms of the original \( 2^6 - 1 \) terms which have four cosine factors and two error factors. It suffices to estimate these with (3-2), since they benefit from an extra integration of a negative power of \( r \). Their sum can be estimated by

\[
E_{k,5} = \sum_{i \neq j} k_i k_j^2 \int_R \left( \frac{2}{\pi r} \right)^{\frac{3}{2}} r^{-4} \, dr,
\]

where the sum is over \( \binom{6}{2} \geq 15 \) choices of distinct indices \( i, j \in \{1, 2, 3, 4, 5, 6\} \).

The remaining \( 2^6 - 1 - 6 - 15 = 42 \) terms benefit from an integration of at least the negative fifth power of \( r \), and are estimated even more crudely by
We calculated the entries of the quadratic form $Q$ numerically. A look at these entries reveals some nice patterns such as circular structures, shown in Figures 2 and 3. We do not know how to exactly describe or explain these structures independently of the numerical calculations. These structures merit further investigation. Each of the six figures shows a row of the block $D=0$. This corresponds to fixing an index $m_0$, and plotting the matrix entries corresponding to $Q_{m_0,n}$, where $n$ ranges over all admissible values. Since $n_1 + n_2 + n_3 = 0$, we may parametrize the entries of the row by $(n_1, n_2)$, which ranges in a hexagonal region in $\mathbb{Z}^2$, shown in the figures as complement of the shaded region.

We close with a remark on the central Bessel integral

$$I_{(0,0,0,0,0,0)} = \int_0^\infty J_6^6(r)r \, dr,$$

which up to a factor $(2\pi)^4$ is the conjectured sharp constant $\Phi(1)$ in the Tomas–Stein adjoint restriction inequality. This integral appears in the following related context. An $n$-step uniform random walk is a walk in the plane that starts at the origin and consists of $n$ steps of length 1 each taken into a uniformly random direction.

If $p_n$ denotes the radial density of the distance traveled after $n$ steps, then it is a classical result that $p_5(1) = I_{(0,0,0,0,0,0)}$, see e.g. [Borwein et al. 12]. In the same article, the exact value of the integral

$$p_4(1) = \int_0^\infty J_6^5(r)r \, dr = \frac{1}{2\pi^2} \sqrt{\frac{\Gamma(\frac{1}{15}) \Gamma(\frac{2}{15}) \Gamma(\frac{4}{15}) \Gamma(\frac{8}{15})}{5 \Gamma(\frac{7}{15}) \Gamma(\frac{11}{15}) \Gamma(\frac{13}{15}) \Gamma(\frac{14}{15})}}$$

is determined resorting to striking modularity properties of the function $p_n$, see [Borwein et al. 12, Theorems 4.9 and 5.1]. The corresponding problem with a sixth power remains open.

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$$E_{k,6} = \sum_{i,j,l} k_i^2 k_j^2 k_l^2 \int_R \left( \frac{2}{\pi} \right)^3 r^{-5} \, dr$$

$$+ \sum_{i,j,l,m} k_i^2 k_j^2 k_l^2 k_m^2 \int_R \left( \frac{2}{\pi} \right)^3 r^{-6} \, dr$$

$$+ \sum_{i,j,l,m,n} k_i^2 k_j^2 k_l^2 k_m^2 k_n^2 \int_R \left( \frac{2}{\pi} \right)^3 r^{-7} \, dr$$

$$+ k_1 k_2 k_3 k_4 k_5 k_6 \int_R \left( \frac{2}{\pi} \right)^3 r^{-8} \, dr,$$

where the sums are over tuples of distinct indices for a total of $\binom{6}{3} = 20$, $\binom{6}{4} = 15$, and $\binom{6}{5} = 6$ summands, respectively.

Addition of the error bounds $E_{k,1} + \cdots + E_{k,6}$ yields error bounds for $I_k$, which in turn give error bounds for the matrix coefficients $Q_{m,n}$. Applying Schur’s test to each block with a fixed $D$ individually shows that the matrix consisting of the error bounds has operator norm less than $10^{-5}$. These steps were conveniently performed in *Mathematica*. Since $10^{-5}$ is smaller than the minimal eigenvalues shown in Table 1, we can conclude that the matrix $(Q_{m,n})_{m,n \in \mathbb{X}}$ is positive definite. This completes the proof of Theorem 1.

**4. Further remarks**

We conclude our discussion with several observations.

Table 1 reveals that the smallest eigenvalues of the block $D$ is increasing in the parameter $D \geq 0$. It might be interesting to find an analytic explanation of this fact.

Zooming into the main block $D=0$, Figure 1 shows the nonzero eigenvalues of this block. There is a cluster of very small eigenvalues. The corresponding eigenvectors seem to suggest that many of these small eigenvalues are related to functions on the circle that are mainly supported in neighborhoods of two antipodally symmetric points. These functions are close competitors of constants for being maximizers. A line of attack on this problem, say for larger or infinite bandwidth, might be to understand how to estimate the spectral gap and analytically separate the effect of these antipodal functions. The remaining eigenvalues may be sufficiently far from zero to allow for crude estimation. Since we do not know how quickly the minimal eigenvalue tends to zero as the degree increases, it is not clear how to effectively estimate the precision needed for a given degree.

$$\sum_{i,j,l,m} k_i^2 k_j^2 k_l^2 k_m^2 \int_R \left( \frac{2}{\pi} \right)^3 r^{-6} \, dr$$

$$+ \sum_{i,j,l,m,n} k_i^2 k_j^2 k_l^2 k_m^2 k_n^2 \int_R \left( \frac{2}{\pi} \right)^3 r^{-7} \, dr$$

$$+ k_1 k_2 k_3 k_4 k_5 k_6 \int_R \left( \frac{2}{\pi} \right)^3 r^{-8} \, dr,$$
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