The Active Bijection

2.b - Decomposition of activities for oriented matroids, and general definitions of the active bijection

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Abstract

The active bijection for oriented matroids (and real hyperplane arrangements, and graphs, as particular cases) is introduced and investigated by the authors in a series of papers. Given any oriented matroid defined on a linearly ordered ground set, we exhibit one particular of its bases, which we call its active basis, with remarkable properties. It preserves activities (for oriented matroids in the sense of Las Vergnas, for matroid bases in the sense of Tutte), as well as some active partitions of the ground set associated with oriented matroids and matroid bases. It yields a canonical bijection between classes of reorientations and bases (this bijection depends only on the reorientation class of the oriented matroid, that is on the non-signed pseudosphere arrangement in terms of a topological representation). It also yields a refined bijection between all reorientations and subsets of the ground set. Those bijections are related to various Tutte polynomial expressions (in terms of usual and refined activities for bases/subsets or reorientations, in terms of beta invariants of minors). They contain various noticeable bijections involving orientations/signatures/reorientations and spanning trees/simplices/bases of a graph/real hyperplane arrangement/oriented matroid. For instance, we obtain an activity preserving bijection between acyclic reorientations and no-broken-circuit subsets.

In previous papers of this series, we defined the active bijection between bounded regions and uniactive internal bases by means of fully optimal bases (No. 1), and we defined a decomposition of activities for matroid bases by means of active filtrations (or active partitions) yielding particular sequences of minors (companion paper, No. 2.a). The present paper is central in the series. First, we define a decomposition of activities for oriented matroids, using the same sequences of minors, yielding a decomposition of an oriented matroid into bounded regions of minors. Second, we use the previous results together to provide the canonical and refined active bijections alluded to above. We also give an overview and examples of the various results of independent interest involved in the construction. They arise as soon as the ground set of an oriented matroid is linearly ordered.

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1. Introduction and overview

The active bijection for oriented matroids (and real hyperplane arrangements, and graphs, as particular cases) is the subject of several papers by the present authors [12–22]. The general setting of this set of papers is to relate orientations/signatures/reorientations and spanning trees/simplices/bases of graphs/real hyperplane arrangements/oriented matroids, and, more generally, to study oriented matroids as soon as they are defined on a linearly ordered set, in terms of structural, constructive, enumerative or bijective canonical properties.

In general, we map any oriented matroid on a linearly ordered set onto one particular of its bases, which we call its active basis. This allows us to define a canonical activity preserving correspondence between reorientations and bases of an oriented matroid, with various related bijections, constructions and characterizations. The original motivation was to provide a bijective interpretation and a structural understanding of the equality of two classical expressions of the Tutte polynomial (detailed in Section 2). The first is in terms of basis activities by Tutte [34] (extended to matroids by Crapo [5]):

\[ t(M; x, y) = \sum_{\iota, \varepsilon} b_{\iota, \varepsilon} x^\iota y^\varepsilon \]

where \( b_{\iota, \varepsilon} \) is the number of bases of \( M \) with internal activity \( \iota \) and external activity \( \varepsilon \). The second is in terms of reorientation activities by Las Vergnas [28]:

\[ t(M; x, y) = \sum_{\iota, \varepsilon} o_{\iota, \varepsilon} \left( \frac{x}{2} \right)^\iota \left( \frac{y}{2} \right)^\varepsilon \]

where \( o_{\iota, \varepsilon} \) is the number of reorientations of \( M \) with dual-activity \( \iota \) and activity \( \varepsilon \). This second expression contains various famous enumerative results from the literature, such as counting regions or acyclic reorientations, mainly by Winder [36], Stanley [33], Zaslavsky [37], and Las Vergnas [27]. Roughly, one can think of activities as situating bases and reorientations with respect to the minimal and maximal basis. Much more details, either on related results or on specific references to the literature, are given in the introduction of each section of the paper, as they deal with separate aspects of the construction. The rest of the introduction aims at presenting a practical and global overview of these various features.
Let us first situate the authors' works on the subject. The main papers are [17–20] along with the present central one. They deal with general oriented matroids and form a consistent series (details will be given further on): [17] (No. 1) deals with the bounded/uniactive case (that is, the case where \( \iota = 1 \) and \( \varepsilon = 0 \), or \( \iota = 0 \) and \( \varepsilon = 1 \)); [18] (No. 2.a) is a companion paper of the present one and deals with a decomposition of matroid bases into uniactive bases of minors; the present paper (No. 2.b) deals with a decomposition of oriented matroids into bounded minors and is central in the series as it uses the previous papers to define the active bijection in general; to be continued with [19] (No. 3) that deals with elaborations on linear programming yielding an inverse to the construction of [17]; and with [20] (No. 4) that deals with deletion/contraction constructions and broader characterizations. These papers are written in oriented matroid terms and mainly illustrated in pseudosphere/hyperplane arrangements from a topological/geometrical viewpoint (the reader may read the preliminary section of [17] for a summary on this viewpoint).

These main papers are completed with papers dealing with particular cases: [12] yields an easy case introduction to the subject by studying uniform and rank 3 oriented matroids; [14] studies the case of supersolvable real hyperplane arrangements and of particular Coxeter arrangements; and [13, 21, 22] deal with graphs. The reader specifically interested in graphs is encouraged to read [21] which summarises the main results of the whole series formulated in this structure. These papers are completed with short conference notes: [15] presents briefly the whole constructions and main results, and [16] presents briefly the elaborations on linear programming in the real case. A summary about the active bijection and related notions can also be found in [11].

Let us mention that the question of relating basis and orientation activities came from Las Vergnas in [28], following on from which, in [29], a definition for a correspondence between spanning trees and orientations of graphs was proposed. It was based on an algorithm, given with no proof\(^1\), which inspired the general decompositions of activities developed for the active bijection, but which does not yield the correspondence given by the active bijection (not for general activities, nor for the restriction to \((1, 0)\) activities, and nor with respect to duality). Also, let us mention that a different notion of activities for graph orientations had been introduced even earlier by Berman in [1], along with incorrect constructions according to [28]\(^2\). Finally, the active bijection for oriented matroids, as addressed in this series of papers, has been introduced and developed in the Ph.D. thesis [8], where most of the results from the series were given, at least in a preliminary form.

Let us now get into the substance. First, without requiring any preliminary knowledge, let us give to the reader one of the shortest definitions of the active basis (condensed yet complete, combining Definitions 5.1 and 6.2, among various equivalent definitions addressed in this paper). For any oriented matroid \( M \) on a linearly ordered set \( E \), the active basis \( \alpha(M) \) of \( M \) is determined by:

- **Fully optimal basis of a bounded region.** If \( M \) is acyclic and every positive cocircuit of \( M \) contains \( \min(E) \), then \( \alpha(M) \) is the unique basis \( B \) of \( M \) such that:
  - for all \( b \in B \setminus p \), the signs of \( b \) and \( \min(C^*(B; b)) \) are opposite in \( C^*(B; b) \);

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\(^1\) Besides the fact that no proof exist, the authors suspect that this algorithm would not yield a proper correspondence anyway if its formulation was extended beyond regular matroids. Notably, its technicalities and its non-natural behaviour with respect to duality, in contrast with the active bijection, made the authors abandon this algorithm.

\(^2\) The construction in [1] consisted in defining some active directed cycles/cocycles in a complex way, instead of active edges, and in enumerating those cycles/cocycles. It claimed to yield a Tutte polynomial formula which was formally similar to that of Las Vergnas [28] using those different activities, and a correspondence between orientations and spanning trees. According to [28, footnote page 370], those constructions were not correct.
– for all $e \in E \setminus B$, the signs of $e$ and $\min(C(B; e))$ are opposite in $C(B; e)$.

- **Duality.** $\alpha(M) = E \setminus \alpha(M^*)$.
- **Decomposition.** $\alpha(M) = \alpha(M/F) \uplus \alpha(M(F))$ where $F$ is the union of all positive circuits of $M$ whose smallest element is the greatest possible smallest element of a positive circuit of $M$.

Though the central concept can be shortly defined as above, the paper deals with separate topics and constructions of independent interest, and unifies them in a consistent framework. What we call *the active bijection* is actually a three-level construction, built from the mapping $M \mapsto \alpha(M)$ applied to reorientations of $M$. It is summarized in the diagram of Figure 1 below. In general, we get structural and bijective interpretations of several Tutte polynomial expressions, and we get various bijections involving reorientations and bases of an oriented matroid on a linearly ordered set (which apply in particular to graphs and real hyperplane arrangements). See Table 1 below for a list. Now, let us present more precisely some features and relate them to the paper sections.

We introduce some notions of *filtrations* for ordered (oriented) matroids, which are particular sequences of nested subsets (intuitively: the subsets involved in the decomposition of the above recursive definition). In a matroid, this notion yields (independently of the active bijection itself) an expression of the Tutte polynomial in terms of beta invariants of minors induced by these sequences, and a canonical decomposition of bases into uniaactive internal/external bases of minors, both detailed in [18] (No. 2.a) and recalled in Section 3. On the other hand, the same sequences yield a canonical decomposition of oriented matroids into bounded regions of minors of the primal/dual (acyclic/cyclic bipolar directed graph minors in the graph case), as detailed in Section 4. We also naturally obtain a partition of the set of reorientations into some *activity (equivalence) classes*.

Independently, the active bijection provides a canonical *uniaactive bijection* between bounded regions and uniaactive internal bases (their *fully optimal bases*, as in the above definition). This is a deep and difficult combinatorial result from [17] (No. 1), that can be seen from different manners, notably as an elaboration of linear programming optimality, as detailed in [19] (No. 3). See more details in the introduction of Section 5. This section surveys, recalls and reformulates these results in an appropriate way for this paper (see also Figure 5 for a diagram on involved duality properties).

Putting together the two previous decompositions and the previous bijection, we obtain in Section 6 the *canonical active bijection* of an ordered oriented matroid (a recursive definiton of which is given above, it can be also thought of as a sign criterion that fundamental circuits and cocircuits of one and only one basis satisfy in a given oriented matroid). It yields a canonical activity preserving (and active filtration preserving) bijection between reorientation activity classes and bases of any ordered oriented matroid, giving a bijective passage between the two Tutte polynomial expressions mentioned at the beginning of the introduction. An important feature is that the canonical active bijection depends only on the reorientation class of the oriented matroid, that is on the non-signed pseudosphere/hyperplane arrangement in terms of a topological representation (or the underlying undirected graph in the graph case).

Furthermore, each one of these two aforementioned Tutte polynomial expressions can be refined into an expansion involving *four subset activity variables*, one from bases again, the other from reorientations again, independently of each other. The result in terms of bases is known from the literature, and the result in terms of reorientation can be deduced from the above partition into acivity classes. These results are recalled and synthesized in Section 7. In our setting, the underlying structural construction is to partition the power set of the ground set, one into classical boolean intervals associated with bases, the other into reorientation activity classes, respectively.
Figure 1: Diagram of results and constructions for the active bijection. Horizontal arrows indicate in which ways the constructions or definitions apply. Vertical arrows indicate how objects are related. Dotted rectangles indicate how the Tutte polynomial is involved or transforms through the constructions. Dashed arrows concern results detailed in forthcoming papers.
| REORIENTATIONS | BASES/SUBSETS |
|----------------|--------------|
| activity classes of reorientations | bases | \(t(M; 1, 1)\) |
| activity classes of acyclic reorientations | internal bases | \(t(M; 1, 0)\) |
| activity classes of totally cyclic reorientations | external bases | \(t(M; 0, 1)\) |
| bounded reorientations (up to opposite) | uniactive internal bases | \(\beta(M) = t_{1,0}\) |
| dual bounded reorientations (up to opposite) | uniactive external bases | \(\beta(M^*) = t_{0,1}\) |

**refined active bijection w.r.t. a given reference reorientation**

| reorientations | subsets of the ground set | \(t(M; 2, 2)\) |
| reorientations with fixed orientation for active elements | independents | \(t(M; 2, 1)\) |
| reorientations with fixed orientation for dual-active elements | spanning subsets | \(t(M; 1, 2)\) |
| acyclic reorientations | no-broken-circuit subsets | \(t(M; 2, 0)\) |
| totally cyclic reorientations | supersets of external bases | \(t(M; 0, 2)\) |
| reorientations with fixed orientation for active and dual-active elements | bases | \(t(M; 1, 1)\) |
| acyclic reorientations with fixed orientation for dual-active elements | internal bases | \(t(M; 1, 0)\) |
| totally cyclic reorientations with fixed orientation for active elements | external bases | \(t(M; 0, 1)\) |

**translation for the case of real hyperplane arrangements**

| reorientations \(\sim\) signatures | bases \(\sim\) simplices |
| acyclic reorientations \(\sim\) regions |
| totally cyclic reorientations \(\sim\) dual regions |
| bounded reorientations \(\sim\) bounded regions |

**translation for the case of (connected) graphs**

| reorientations \(\sim\) orientations | bases \(\sim\) spanning trees | [13, 21, 22] |
| totally cyclic \(\sim\) strongly connected |
| bounded \(\sim\) (acyclic) bipolar |
| dual-bounded \(\sim\) cyclic-bipolar |

**even more particular cases**

| \((\text{in uniform case / general position arrangements})\) | pseudo/real linear programming |
| bounded regions | optimal vertices | [12] |
| \((\text{in graphs, for suitable orderings})\) | internal bases | [13] |
| unique sink acyclic orientations |
| \((\text{in the braid arrangement, or the complete graph})\) | increasing trees | [14] |
| permutations |
| \((\text{in the hyperoctahedral arrangement})\) | signed increasing trees | [14] |

Table 1: The two first blocks of lines list the canonical and refined active bijections along with their notable restrictions (Theorems 6.4 and 8.2). The third column gives the number of involved objects. As defined in Section 2: internal, resp. external, bases are those with external, resp. internal, activity equal to zero; uniactive bases are those with only one externally or internally active element; active, resp. dual-active, elements are smallest elements of a positive circuit, resp. cocircuit. As detailed in Definition 4.9: activity classes are obtained by reorienting arbitrarily unions of positive circuits or cocircuits with the same fixed smallest element. In the next blocks of lines, the character \(\sim\) stands for a translation. The last column gives references where these particular cases are specifically studied.
Building on what precedes, in Section 8, the \textit{refined active bijection} is defined with respect to any given reference reorientation (or signature of the pseudosphere/hyperplane arrangement, or orientation in the graph case). It maps reorientation activity classes onto intervals associated with bases, consistently with the canonical active bijection. In each class/interval couple, the reference reorientation is used to naturally fix a boolean lattice isomorphism. By this manner, the global bijection involves all reorientations/subsets, preserves the four refined activity parameters, and allows us to derive various bijections.

The paper ends with developed examples in Section 9, completing the running example and the illustrations given along the paper. We suggest that the reader could already have a glimpse at these final examples, in particular Figure 20 and its caption are intended to give a first geometrical intuition of various aspects of the construction on a simple but meaningful example. Let us also mention, as a remarkable example, that the active bijection can be seen as a far reaching generalization of the well-known bijection between permutations and increasing trees (a particular case obtained from complete graphs or from the Coxeter arrangement $A_n$ as detailed in [14, Section 5]).

Finally, let us end this overview by pointing out further features of the active bijection (the uniactive, the canonical, and the refined, as well). First, from the computational complexity viewpoint, in general, from bases to reorientations, it can be simply built by a single pass algorithm over the ground set. From reorientations to bases, the construction is more complicated, noticeably as its restriction to bounded regions of a pseudosphere/real hyperplane arrangement contains the pseudo/real linear programming problem (see [17, 19, 22]). Thus, it can be thought of, in general, as a sort of “one way function”. Second, the active bijection can also be built by deletion/contraction of the greatest element. Notably, in the bounded case, this construction can be seen as a refinement of the linear programming solving by variable/constraint deletion. More generally, one can describe a deletion/contraction framework for activity preserving correspondences among which the active bijection is uniquely and canonically determined. These constructions are detailed in [20] (No. 4) (see also [21] for a condensed version in graphs). Third, the constructions used at each of the three levels of the active bijection are independent of each other to a certain extent. One can thus use the decomposition of activities addressed in the paper to define a decomposition framework for activity preserving correspondences among which the active bijection is uniquely and canonically determined (see Remarks 6.10 and 8.6, see also [20, 21]). Fourth, at every level of its construction, the active bijection behaves nicely with respect to duality, and involves important duality properties, as witness various remarks and results in this paper and others of the series.

2. Preliminaries

2.1. Generalities

In the paper, $\subset$ denotes the strict inclusion, and $\uplus$, or $+$, denotes the disjoint union. If $\mathcal{F}$ is a set of subsets of $E$, then $\cup \mathcal{F}$ denotes the subset of $E$ obtained by taking the union of all elements of $\mathcal{F}$.

In the paper $M$ denotes a matroid or an oriented matroid on a finite set $E$. See [32] and [2] for a complete background on matroid theory and oriented matroid theory, respectively. See in particular [2, Sections 1.1 and 1.2] for the relation with graphs and hyperplane arrangements (see also Table 1 of the present paper for some translations in these two settings). See [17, Section 2] for a summary of oriented matroid combinatorial and geometrical aspects that we specifically use, and see [21, Section 2] for preliminaries similar to those below specifically using graph terminology.
A matroid or an oriented matroid $M$ on $E$ is called ordered when the set $E$ is linearly ordered. Then, the dual $M^*$ of $M$ is ordered by the same ordering on $E$. Any minor of $M$ is ordered the natural way, its ground set ordering being induced by that of $E$. A minor $M/e$, resp. $M\setminus\{e\}$, for $e \in E$, can be denoted for short $M\backslash e$, resp. $M\setminus e$. A matroid might be called loop, or isthmus, if it has a unique element and this unique element is a loop ($M = U_{1,0}$), or an isthmus ($M = U_{1,1}$), respectively. An isthmus is also called a coloop in the literature.

Let us recall some classical properties of minors in matroids. For $F \subseteq E$, it is known that: circuits of $M(F)$ are circuits of $M$ contained in $F$; cocircuits of $M(F)$ are non-empty inclusion-minimal intersections of $F$ and cocircuits of $M$; circuits of $M/F$ are non-empty inclusion-minimal intersections of $E \setminus F$ and circuits of $M$ (that is inclusion-minimal subsets obtained by removing $F$ from circuits of $M$); cocircuits of $M/F$ are cocircuits of $M$ contained in $E \setminus F$.

Let us also recall some usual matroid notions. A flat $F$ of $M$ is a subset of $E$ such that $E \setminus F$ is a union of cocircuits; equivalently: if $C \setminus \{e\} \subseteq F$ for some circuit $C$ and element $e$, then $e \in F$; and equivalently: $M/F$ has no loop. We call dual-flat $F$ of $M$ a subset of $E$ which is a union of circuits; equivalently: its complement is a flat of the dual matroid $M^*$; equivalently: if $D \setminus \{e\} \subseteq E \setminus F$ for some cocircuit $D$ and element $e$, then $e \in E \setminus F$; and equivalently: $M(F)$ has no isthmus. A cyclic-flat $F$ of $M$ is both a flat and a dual-flat of $M$; equivalently: $F$ is a flat and $M(F)$ has no isthmus; or equivalently: $M/F$ has no loop and $M(F)$ has no isthmus.

As far as oriented matroids are concerned, given an oriented matroid $M$, the underlying matroid is denoted $\underline{M}$ when a distinction is important, but it may be denoted also by the same manner $M$. Similarly, we will often make the abuse of using the same notation $C$ either for the signed element subset $(C^+, C^-)$ (oriented matroid circuit) or for its support $C = C^+ \cup C^-$ (matroid circuit). Also, we will use some typical oriented matroid technique, notably orthogonality and compositions of circuits and cocircuits, see [17, Section 2] or [2].

The set of all $-A M$ for $A \subseteq E$ is called the set of reorientations of $M$. It is very important to point out that we consider this set as isomorphic to $2^E$ (as a reorientation of $M$, $-A M$ is identified by the subset $A$). By this way, we distinguish for instance between $-A M$ and $-E \setminus A M$ as reorientations of $M$, even if the two resulting oriented matroids are equal. This is consistent with signed (real central) hyperplane arrangements and with the topological representation of oriented matroids as signed pseudosphere arrangements: the $2^E$ signatures of the underlying non-signed arrangement are obtained by reorienting any subset of $E$ from a given signature. And this is consistent with graphs: given a directed graph $G = (V, E)$, the $2^E$ orientations of the underlying undirected graph are obtained by reorienting any subset of edges from $G$. Given a reorientation $-A M$ of $M$, we call $-E \setminus A M$ its opposite reorientation. We say that $e \in E$ has a fixed orientation (with respect to $M$) in a set of reorientations of $M$ if, for every reorientation $-A M$ in this set, we have $e \notin A$.

2.2. Matroid basis activities

Let $M$ be an ordered matroid on $E$, and let $B$ be a basis of $M$. For $b \in B$, the fundamental cocircuit of $b$ with respect to $B$, denoted $C^*_M(B; b)$, or $C^*(B; b)$ for short, is the unique cocircuit contained in $(E \setminus B) \cup \{b\}$. For $e \notin B$, the fundamental circuit of $e$ with respect to $B$, denoted $C_M(B; e)$, or $C(B; e)$ for short, is the unique circuit contained in $B \cup \{e\}$. When the matroid $M$ is ordered, then, by convention, $b$ is positive in the fundamental cocircuit $C^*(B; b)$, and $e$ is positive in the fundamental circuit $C(B; e)$.
Let

$$\text{Int}(B) = \left\{ b \in B \mid b = \min \left( C^*(B; b) \right) \right\},$$

$$\text{Ext}(B) = \left\{ e \in E \setminus B \mid e = \min \left( C(B; e) \right) \right\}.$$

We might add a subscript as $\text{Int}_M(B)$ or $\text{Ext}_M(B)$ when necessary. The elements of Int($B$), resp. Ext($B$), are called internally active, resp. externally active, with respect to $B$. The cardinality of Int($B$), resp. Ext($B$) is called internal activity, resp. external activity, of $B$. We might write that a basis is $(i, j)$-active when its internal and external activities equal $i$ and $j$, respectively.

Observe that $\text{Int}(B) \cap \text{Ext}(B) = \emptyset$ and that, for $p = \min(E)$, we have $p \in \text{Int}(B) \cup \text{Ext}(B)$. If $\text{Int}(B) = \emptyset$, resp. $\text{Ext}(B) = \emptyset$, then $B$ is called external, resp. internal. If $\text{Int}(B) \cup \text{Ext}(B) = \{p\}$ then $B$ is called uniactive. Hence, a base with internal activity 1 and external activity 0, or $(1, 0)$-active for short, is called uniaactive internal, and a base with internal activity 0 and external activity 1, or $(0, 1)$-active for short, is called uniaactive external. Let us mention that internal uniaactive bases can be characterized by several ways, see [13, 17, 18]. Let us mention that exchanging the two smallest elements of $E$ yields a canonical bijection between uniaactive internal and uniaactive external bases, see [17, Proposition 5.1] up to a typing error\(^3\), see also [13, Section 4] in graphs. Also, let $B_{\min}$ be the smallest (lexicographic) base of $M$. Then, as well-known and easy to prove, we have $\text{Int}(B_{\min}) = B_{\min}$, $\text{Ext}(B_{\min}) = \emptyset$ and $\text{Int}(B) \subseteq B_{\min}$ for every base $B$. Also, let $B_{\max}$ be the greatest (lexicographic) base of $M$. Then $\text{Int}(B_{\max}) = \emptyset$, $\text{Ext}(B_{\max}) = E \setminus B_{\max}$, and $\text{Ext}(B) \subseteq E \setminus B_{\max}$ for every base $B$. Thus, roughly, internal/external activities can be thought of as situating a basis with respect to $B_{\min}$ and $B_{\max}$. Finally, we recall that internal and external activities are dual notions:

$$\text{Int}_M(B) = \text{Ext}_M(E \setminus B) \quad \text{and} \quad \text{Ext}_M(B) = \text{Int}_M(E \setminus B).$$

By [5, 34], the Tutte polynomial of $M$ is

$$t(M; x, y) = \sum_{i, \varepsilon} b_{i, \varepsilon} x^i y^\varepsilon \quad \text{("enumeration of basis activities")},$$

where $b_{i, \varepsilon}$ is the number of bases of $M$ with internal activity $i$ and external activity $\varepsilon$.

2.3. Fundamental graph/tableau of a basis

Observe that the above definitions for a basis $B$ of an ordered matroid $M$ only rely upon the fundamental circuits/cocircuits of the basis, not on the whole structure $M$. In fact all algorithms from bases to reorientations developed in the paper only rely on this local data. In particular, in [18], we give combinatorial constructions that also only depend on this local data, and thus that can be naturally expressed in terms of general bipartite graphs on a linearly ordered set of vertices. In this paper, we do not need this setting, but, for the sake of illustrations, we introduce the following definitions.

\(^3\)Let us correct here an unfortunate typing error in [17, Proposition 5.1 and Theorem 5.3]. The statement has been given under the wrong hypothesis $B_{\min} = \{p < p' < \ldots \}$ instead of the correct one $E = \{p < p' < \ldots \}$. Proofs are unchanged (independent typo: in line 10 of the proof of [17, Proposition 5.1], instead of $B' - f$, read $(E \setminus B') \setminus \{f\}$). In [13, Section 4], the statement of the same properties in graphs is correct.
Figure 2: This example of $K_4$ with elements $1 < \cdots < 6$ will serve as a running example throughout the paper. On the left: a graph representation. In the middle: a hyperplane arrangement representation (we always represent $\min(E)$ as a hyperplane at infinity, and we only represent one half of the arrangement, on a given side of $\min(E)$, see [17, Section 2] for more details on such representations). For the basis 256, we have $\Int(256) = \emptyset$ and $\Ext(256) = \{1, 3\}$. Fundamental cocircuits of the basis are written in the arrangement next to the vertices of the corresponding simplex. On the upper right and the bottom right, respectively: the fundamental bipartite graph and the fundamental tableau of the basis (see Section 2.3).

Given a basis $B$ of a matroid $M$ on $E$, the fundamental graph of $B$ in $M$, denoted $F_M(B)$ is the usual graph with set of vertices $E$, bipartite w.r.t. the couple of subsets $(B, E \setminus B)$ forming a bipartition $E = B \uplus E \setminus B$, and with edges such that for every $b \in B$, $b$ is adjacent to elements of $C^*(B; b) \setminus \{b\}$, and for every $e \in E \setminus B$, $e$ is adjacent to elements of $C(B; e) \setminus \{e\}$. Recall that

$$e \in C^*(B; b) \text{ if and only if } b \in C(B; e).$$

We call fundamental tableau $F_M(B)$ the matrix whose rows and columns are indexed by $E$, with entries in $\{\bullet, 0\}$, and such that each diagonal element indexed by $(e, e)$, $e \in E$, is non-zero and is the only non-zero entry of its row if $e \in B$, and the only non-zero entry of its column if $e \in E \setminus B$. We use the same notation for the fundamental graph and fundamental tableau since, obviously, they are equivalent structures: each non-diagonal entry of the tableau represents an edge of the corresponding bipartite graph. We choose to define both because graphs are the underlying compact combinatorial structure, whereas tableaux are better for visualization, notably for signs of the fundamental circuits/cocircuits in the oriented matroid case, and tableaux are consistent with the matrix representation used in the linear programming setting of the active bijection developed in [17, 19]. In what follows, we illustrate examples on both representations.

By the convention stated above, in an oriented matroid $M$, given a base $B$, for an element $b \in B$, $b$ is positive in the fundamental cocircuit $C^*(B; b)$, and for an element $e \notin B$, $e$ is positive in the fundamental circuit $C(B; e)$. Then, when we represent the tableau of a basis of an oriented matroid, we give signs to the entries in order to represent the fundamental cocircuits as columns and the opposites of fundamental circuits as rows (consistently with the above convention and with circuit/cocircuit orthogonality).

An example of a matroid basis, its activities, its fundamental graph and its fundamental tableau is given here in Figure 2. An example of a signed fundamental tableau is given later in Figure 7.
2.4. Oriented matroid activities

Let $M$ be an oriented matroid on $E$. A positive circuit, resp. positive cocircuit, of $M$ is a circuit, resp. cocircuit, of $M$ such that all signs of its elements are positive. The oriented matroid $M$ is acyclic if it has no positive circuit, or, equivalently, if every element belongs to a positive cocircuit. The oriented matroid $M$ is totally cyclic, if every element belongs to a positive circuit, or, equivalently, if it has no positive cocircuit.

Let $M$ be an ordered oriented matroid on a linearly ordered set $E$. Let

$$O^*(M) = \left\{ a \in E \mid a = \min (D) \text{ for a positive cocircuit } D \right\},$$

$$O(M) = \left\{ a \in E \mid a = \min (C) \text{ for a positive circuit } C \right\}.$$

The elements of $O^*(M)$, resp. $O(M)$, are called dual-active, resp. active, with respect to $M$. The cardinality of $O^*(M)$, resp. $O(M)$, is called dual-activity, resp. activity, of $M$. We might write that an ordered oriented matroid is $(i,j)$-active when its dual-activity and its activity equal $i$ and $j$, respectively. Observe that $O^*(M) \cap O(M) = \emptyset$ and that, for $p = \min(E)$, we have $p \in O^*(M) \cup O(M)$. Observe also that we have $O^*(M) = \emptyset$, resp. $O(M) = \emptyset$, if and only if $M$ is totally cyclic, resp. acyclic. Finally, observe that those two activities are dual notions:

$$O(M^*) = O^*(M) \quad \text{and} \quad O^*(M^*) = O(M).$$

An illustration is shown in Figure 3, along with a geometrical interpretation.

![Figure 3](image.png)

Figure 3: Let us detail the left part: dual-activities for acyclic reorientations (regions) of $K_4$, continuing the running example from Figure 2. The smallest element of each cocircuit is written at each corresponding vertex of the arrangement (as 1, ..., 2, ..., or 4, ...). The dual-activity of each region is written in the region (it is given by the elements written at the vertices of its border). We get that $t(K_4; x, 0) = 8.\left(\frac{1}{4}\right)^3 + 12.\left(\frac{1}{4}\right)^2 + 4.\left(\frac{1}{4}\right)$. In particular $t(K_4; 2, 0) = 24$ counts the regions of $K_4$. Observe, in general, that dual-activities indicate the positions of the regions w.r.t. the sequence of nested faces (here $1 \cap 2 \subset 1$, depicted in bold) induced by the minimal basis (here 124). Observe that the dual-activity of a region depends only on the minimal basis and on the unsigned underlying arrangement. The grey region corresponds to the directed graph shown on the left. Its dual-active elements are $\{1, 2, 4\}$, given by the positive cocircuits 124, 2345 and 456 (directed cocycles of the directed graph). The right part shows dual active elements of regions of an arrangement on 13 elements, with minimal basis $1 < 2 < 4$ (the rest of the ordering is not used). This second figure is exhaustively completed at the very end of the paper in Figure 21.
By [28], we have the following theorem enumerating reorientation activities:

\[ t(M; x, y) = \sum_{\iota, \varepsilon} o_{\iota, \varepsilon} \left( \frac{x}{2} \right)^\iota \left( \frac{y}{2} \right)^\varepsilon \]  

(“enumeration of reorientation activities”)

where \( o_{\iota, \varepsilon} \) is the number of reorientations of \( M \) with dual-activity \( \iota \) and activity \( \varepsilon \). This last formula generalizes various classical results from the literature, such as counting regions or acyclic (re)orientations [27, 33, 36, 37] (see [11] for a survey on this result and further references).

Comparing the above two expressions for \( t(M; x, y) \) we get, for all \( \iota, \varepsilon \):

\[ o_{\iota, \varepsilon} = 2^{\iota + \varepsilon} b_{\iota, \varepsilon}. \]

2.5. Beta invariant

In particular, by the above formula, we have that

\[ b_{1,0} = \frac{o_{1,0}}{2} \]

counts the number of uniactive internal bases, and counts half the number of reorientations with orientation activity 1 and dual orientation activity 0, or \((1,0)\)-active reorientations for short. This number does not depend on the linear ordering of the element set \( E \). This value

\[ \beta(M) = b_{1,0} \]

is known as the \textit{beta invariant} of \( M \), introduced by Crapo [4]. Assuming \( |E| > 1 \), it is known that \( \beta(M) \neq 0 \) if and only if \( M \) is connected. Let us recall that, for a loopless graph \( G \) with at least three vertices, the associated matroid \( M(G) \) is connected if and only if \( G \) is 2-connected. Also, we have \( \beta(M) = b_{1,0} = b_{0,1} = \beta(M^*) \) as soon as \( |E| > 1 \). Note that, if \( |E| = 1 \), we have \( \beta(M) = 1 \) if the single element is an isthmus of \( M \), and \( \beta(M) = 0 \) if the single element is a loop of \( M \).

Finally, for our constructions, we need to define the following dual slight variation \( \beta^* \) by:

\[ \beta^*(M) = \beta(M^*) = b_{0,1} = \frac{o_{0,1}}{2} = \begin{cases} \beta(M) & \text{if } |E| > 1 \\ 0 & \text{if } M \text{ is an isthmus} \\ 1 & \text{if } M \text{ is a loop.} \end{cases} \]

2.6. Bounded reorientations (or bounded regions) in oriented matroids

Let us characterize \((1,0)\)-active reorientations of an ordered oriented matroid. We say that an oriented matroid \( M \) on \( E \) is \textit{bounded with respect to} \( p \in E \) if \( M \) is acyclic and every positive cocircuit contains \( p \). In particular, if \( M \) consists in a single element \( p \) which is an isthmus, then \( M \) is bounded with respect to \( p \). In terms of a topological representation or in terms of an affine real hyperplane arrangement, \( M \) is bounded w.r.t. \( p \) if and only if it corresponds to a region of the arrangement that does not touch \( p \). Since \( p = \min(E) \) is considered as an element “at infinity”, such a region is therefore a “bounded” region in the usual sense, see details in [17, Section 2]. In terms of a directed graph \( G = (V,E) \), the associated oriented matroid \( M(G) \) is bounded w.r.t. \( p \) if and only if \( G \) is \textit{bipolar} w.r.t. \( p \), meaning that it is acyclic with a unique source and a unique sink which are the extremities of \( p \), see details in [13] or [21, Section 2]. Bounded regions with respect to a given element are counted by twice the \( \beta \)-invariant, as initially shown in [27, 37].

We say that \( M \) is \textit{dual-bounded with respect to} \( p \in E \) if \( M \) is totally cyclic and every positive circuit contains \( p \). In terms of a directed graph \( G = (V,E) \), dual-bounded is called \textit{cyclic-bipolar}
in [21, Section 2]. In particular, if $M$ consists in a single element $p$ which is a loop, then $M$ is dual-bounded with respect to $p$. Equivalently, for an oriented matroid $M$ with at least two elements, $M$ is dual-bounded w.r.t. $p$ if and only if $-pM$ is bounded w.r.t. $p$.

Therefore, for matroids with at least two elements, reorienting $p$ provides a canonical bijection between bounded reorientations with respect to $p$ and dual-bounded reorientations with respect to $p$, see [17, Proposition 5.2] (or also [13, Proposition 5] in graphs). Observe also that $M$ is bounded w.r.t. $p$ if and only if $M^*$ is dual-bounded w.r.t. $p$. Therefore bounded reorientations of $M$ with respect to $p$ correspond to dual-bounded reorientations of $M^*$ with respect to $p$.

Assuming $M$ is ordered, we get by definitions that: $M$ is bounded with respect to $p = \min(E)$ if and only if $O(M) = \emptyset$ (i.e. $M$ is acyclic, i.e. $M$ has an activity equal to zero) and $O^*(M) = \{p\}$ (i.e. it has exactly one dual-active element, i.e. $M$ has a dual-activity equal to one). Similarly, $M$ is dual-bounded if and only if $O^*(M) = \emptyset$ (i.e. $M$ is totally cyclic, i.e. $M$ has a dual-activity equal to zero) and $O(M) = \{p\}$ (i.e. it has exactly one active element, i.e. $M$ has an activity equal to one).

3. Filtrations of an ordered matroid and decomposition of matroid bases

In this section, we recall definitions and the main result from the companion paper [18], No. 2.a of the main series. Briefly, filtrations of an ordered matroid are particular sequences of nested sets (equivalent to particular partitions of the ground set). A given basis can be decomposed by its active filtration (or active partition) into a uniquely defined sequence of bases of minors, such that these bases are $(1,0)$ or $(0,1)$-active (in the sense of Tutte polynomial internal/external activities).

And all this yields a decomposition theorem for the set of all bases using all possible filtrations (Theorem 3.3). Let us recall that those bases with internal/external activities equal to 1/0 are counted by the beta invariant, as called after Crapo [4], which is equal to the coefficient of $x$ of the Tutte polynomial $t(M; x, y)$ of the matroid. Let us mention that the bijection provided by this decomposition theorem yields, numerically, an expression of the Tutte polynomial of a matroid in terms of beta invariants of minors [18, Theorem 3.5], that refines at the same time the classical expressions in terms of basis activities and orientation activities (if the matroid is oriented), and the convolution formula for the Tutte polynomial. See [18] for details and references.

**Definition 3.1** ([18, Definition 3.1]). Let $E$ be a linearly ordered finite set. Let $M$ be a matroid on $E$. We call filtration of $M$ (or $E$) a sequence $(F'_1, \ldots, F'_0, F_0, F_1, \ldots, F_i)$ of subsets of $E$ such that:

- $\emptyset = F'_i \subset \ldots \subset F'_0 = F_0 \subset \ldots \subset F_i = E$;
- the sequence $\min(F_k \setminus F_{k-1})$, $1 \leq k \leq i$ is increasing with $k$;
- the sequence $\min(F_{k-1} \setminus F'_k)$, $1 \leq k \leq \varepsilon$, is increasing with $k$.

The sequence is a connected filtration of $M$ if, in addition:

- for $1 \leq k \leq i$, the minor $M(F_k)/F_{k-1}$ is connected and is not a loop;
- for $1 \leq k \leq \varepsilon$, the minor $M(F'_k)/F'_k$ is connected and is not an isthmus.

Equivalently, a filtration of $M$ is connected if and only if

$$\left( \prod_{1 \leq k \leq i} \beta(M(F_k)/F_{k-1}) \right) \left( \prod_{1 \leq k \leq \varepsilon} \beta^*(M(F'_k)/F'_k) \right) \neq 0.$$
In what follows, we can equally use the notations \((F'_e, \ldots, F'_0, F_c, F_0, \ldots, F_i)\) or \(\emptyset = F'_e \subset \cdots \subset F'_0 = F_c = F_0 \subset \cdots \subset F_i = E\) to denote a filtration of \(M\). The \(\iota + \varepsilon\) minors involved in Definition 3.1 are said to be associated with or induced by the filtration.

Observe that filtrations of \(M\) are equivalent to pairs of partitions of \(M\) formed by a bipartition obtained from the subset \(F_c\) (with possibly one empty part, which is a slight language abuse) and a refinement of this bipartition:

\[
E = F_c \cup E \setminus F_c,
\]

\[
E = (F'_e \setminus F'_e) \cup \ldots \cup (F'_0 \setminus F'_1) \cup (F_1 \setminus F_0) \cup \ldots \cup (F_i \setminus F_{i-1}).
\]

Indeed, one can retrieve the sequence of nested subsets from the pair of partitions since the subsets in the sequence are unions of parts given by the ordering of the smallest elements of the parts.

Let us note that, by [18, Lemma 3.3], for a connected filtration of \(M\): for \(0 \leq k \leq \iota\), the subset \(F_k\) is a flat of \(M\); for \(0 \leq k \leq \varepsilon\), the subset \(F'_k\) is a dual-flat of \(M\); and the subset \(F_c\) is a cyclic-flat of \(M\), which we call cyclic flat of the (connected) filtration.

**Observation 3.2** ([18, Observation 3.4]). Let \(\emptyset = F'_e \subset \cdots \subset F'_0 = F_c = F_0 \subset \cdots \subset F_i = E\) be a connected filtration of an ordered matroid \(M\). We have:

- \(\emptyset = E \setminus F_i \subset \cdots \subset E \setminus F_0 = E \setminus F_c = E \setminus F'_0 \subset \cdots \subset E \setminus F'_i = E\) is a connected filtration of \(M^*\), for the cyclic-flat \(E \setminus F_c\) of \(M^*\);
- the minors associated with the above filtration of \(M^*\) are the duals of the minors associated with the above filtration of \(M\), indeed: for every \(1 \leq k \leq \iota\),
  \[
  (M(F_k)/F_{k-1})^* = M^*(E \setminus F_{k-1})/(E \setminus F_k),
  \]
and for every \(1 \leq k \leq \varepsilon\),
  \[
  (M(F'_{k-1})/F'_k)^* = M^*(E \setminus F'_k)/(E \setminus F'_{k-1}).
  \]

**Theorem 3.3** ([18, Theorem 4.25]). Let \(M\) be a matroid on a linearly ordered set \(E\).

\[
\left\{ \text{bases of } M \right\} = \bigcup_{\emptyset \subseteq F'_c \subseteq \ldots \subseteq F'_0 = F_c} \bigcup_{F_c = F_0 \subseteq \ldots \subseteq F_i = E} \left\{ B'_1 \cup \ldots \cup B'_c \cup B_1 \cup \ldots \cup B_i \mid \text{connected filtration of } M \right\}
\]

for all \(1 \leq k \leq \varepsilon\), \(B'_k\) base of \(M(F'_k)/F'_k\) with \(\iota(B'_k) = 0\) and \(\varepsilon(B'_k) = 1\),

for all \(1 \leq k \leq \iota\), \(B_k\) base of \(M(F_k)/F_{k-1}\) with \(\iota(B_k) = 1\) and \(\varepsilon(B_k) = 0\).

With the above notations and \(B = B'_1 \cup \ldots \cup B'_c \cup B_1 \cup \ldots \cup B_i\), we then have:

\[
\text{Int}(B) = \cup_{1 \leq k \leq \varepsilon} \min(F_k \setminus F_{k-1}) = \cup_{1 \leq k \leq \iota} \text{Int}(B_k),
\]

\[
\text{Ext}(B) = \cup_{1 \leq k \leq \varepsilon} \min(F'_k \setminus F'_{k-1}) = \cup_{1 \leq k \leq \iota} \text{Ext}(B'_k).
\]

**Definition 3.4.** Let \(M\) be a matroid on a linearly ordered set \(E\). Let \(B\) be a basis of \(M\). The active filtration of \(B\) is the unique (connected) filtration of \(M\) associated to \(B\) by Theorem 3.3.

For what follows in the paper, we only need the above definition. Two other equivalent definitions are available, they are constructive and based on the fundamental graph of the basis only. The active filtration of \(B\) can be defined:
• by applying successively the active closure to the internal/external active elements, see [18, Definition 4.17];
• by a single pass algorithm over $E$, see [18, Proposition 4.29], which is used (and formally contained) in next Theorem 6.9.

The active partition of $B$ is the partition of $E$ induced by successive differences of subsets in the active filtration of $B$, given with the cyclic flat $F_c$ of the filtration (so that the active filtration of $B$ is determined, as observed above). The active minors of $B$ are the minors induced by the active filtration of $B$.

Let us eventually recall the following observation, which is a direct consequence of Theorem 3.3, and which will yield later a remarkable constructive property.

**Observation 3.5** ([18, Observation 4.23]). Let $\emptyset = F'_c \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_e = E$ be the active filtration of a basis $B$ of $M$. Let $F$ and $G$ be two subsets in this sequence such that $F \subseteq G$. Then, $B \cap G \setminus F$ is a basis of $M(G)/F$, and its active filtration is obtained from the subsequence with extremities $F$ and $G$ (i.e. $F \subset \cdots \subset G$) of the active filtration of $B$ by subtracting $F$ from each subset of the subsequence (with $F_c \setminus F$ as cyclic flat). In particular, the subsequence ending with $F$ is the active filtration of $B \cap F$ in $M(F)$, and the subsequence beginning with $F$ yields the active filtration of $B \setminus F$ in $M/F$ by subtracting $F$ from each subset.

4. The active filtration/partition of an ordered oriented matroid, decomposition into bounded primal/dual regions of minors, and activity classes of reorientations

The decomposition addressed in this section refines the classical decomposition of an oriented matroid $M$ into an acyclic oriented matroid $M/F$ and a totally cyclic oriented matroid $M(F)$, where $F$ is the union of positive circuits of $M$ and $E \setminus F$ is the union of positive cocircuits of $M$. We use the same filtrations as in Section 3 but in an oriented matroid setting. This decomposition technique, whose idea dates back in a certain extent to [29], was developed in [8] along with the related results addressed here, and it was shortly presented in [12–15].

We show how to canonically decompose an oriented matroid on a linearly ordered set into a sequence of minors, that are induced by a connected filtration called the active filtration (Theorem 4.4). These minors are either bounded or dual-bounded with respect to their smallest element (see Section 2.6). The partition of the ground set induced by these minors is called the active partition. These minors have either $(1,0)$ or $(0,1)$ orientation activities.

This decomposition yields a decomposition theorem for the set of all reorientations of an oriented matroid (Theorem 4.6). Numerically, this decomposition theorem can be seen as the same expression of the Tutte polynomial in terms of products of beta invariants as the one from [18, Theorem 3.5] alluded to in Section 3, but this time restricted to orientable matroids.

Moreover, we define a partition of the set of reorientations into activity classes, obtained by reorienting independently all the parts of the active filtration/partition, that is, all the ground sets of the above minors (which preserves the active filtration/partition). Geometrically, active partitions and activity classes of reorientations describe the positions of regions of a pseudosphere/hyperplane arrangement with respect to the minimal flag (see Figure 6). We show in Section 7 how this partition yields a simple expression of the Tutte polynomial using four reorientation activity parameters (Theorem 7.4).
A simple example is provided by Figure 4, continuing the running example of \( K_4 \) (note also that all active partitions of orientations on \( K_4 \) are shown in Section 9.2). A more involved example is provided by Figure 5, showing on a graph the parts associated with positive circuits and the parts associated with positive cocircuits. Another more involved example is provided by Figure 6, showing the geometrical interpretation of active partitions on regions of a rank-4 hyperplane arrangement.

**Definition 4.1.** Let \( M \) be an ordered oriented matroid on \( E \) with \( \iota \) dual-orientation-active elements \( a_1 < \ldots < a_{\iota} \) and \( \varepsilon \) orientation-active elements \( a_1' < \ldots < a_{\varepsilon}' \). The active filtration of \( M \) is the sequence of subsets

\[
\emptyset = F'_\varepsilon \subset \cdots \subset F'_0 = F_c = F_0 \subset \cdots \subset F_i = E
\]

defined as follows. First,

\[
F_c = \bigcup_{C \text{ positive circuit}} C = E \setminus \bigcup_{C \text{ positive cocircuit}} C.
\]

For every \( 0 \leq k \leq \varepsilon - 1 \),

\[
F'_k = \bigcup_{min(C) \geq a'_{k+1}} C.
\]

Moreover, \( F'_\varepsilon = \emptyset \), \( F_i = E \), and, dually, for every \( 0 \leq k \leq \iota - 1 \),

\[
F_k = E \setminus \bigcup_{min(C) \geq a_{k+1}} C.
\]

The active partition of \( M \) is the partition induced by successive differences:

\[
E = (F'_{\varepsilon-1} \setminus F'_\varepsilon) \uplus \cdots \uplus (F'_0 \setminus F'_1) \uplus (F_1 \setminus F_0) \uplus \cdots \uplus (F_i \setminus F_{i-1}),
\]

with \( min(F'_{k-1} \setminus F'_k) = a'_{k} \) for \( 1 \leq k \leq \varepsilon \), and \( min(F_k \setminus F_{k-1}) = a_k \) for \( 1 \leq k \leq \iota \).

The active minors of \( M \) are the \( \iota + \varepsilon \) minors

\[
M'_k = (M|F'_k) / F'_k \text{ for } 1 \leq k \leq \varepsilon, \text{ and } M_k = (M|F_k) / F_{k-1} \text{ for } 1 \leq k \leq \iota.
\]

We assume that the active partition is always given with the set \( F_c \) (i.e. it can be thought of as a pair of partitions, one for \( F_c \), the other for \( E \setminus F_c \)). By this way, knowing the active partition, allows us to retrieve the active filtration of \( M \). Indeed, the sequence \( min(F_k \setminus F_{k-1}) \), \( 1 \leq k \leq \iota \), is increasing with \( k \), and the sequence \( min(F'_{k-1} \setminus F'_k) \), \( 1 \leq k \leq \varepsilon \), is increasing with \( k \), so the position of each part of the active partition with respect to the active filtration is identified.

Moreover, we have, for \( 1 \leq k \leq \iota \),

\[
F_k \setminus F_{k-1} = \bigcup_{D \text{ positive cocircuit}} D \setminus \bigcup_{min(D) = a_k} D, \bigcup_{min(D) > a_k} D,
\]

and, for \( 1 \leq k \leq \varepsilon \),

\[
F'_k \setminus F'_{k-1} = \bigcup_{D \text{ positive circuit}} D \setminus \bigcup_{min(D) = a'_k} D, \bigcup_{min(D) > a'_k} D.
\]
Dual-active elements: 1 and 4
Active partition: 123 ⊎ 456
Active filtration: ∅ ⊂ 123 ⊂ 123456
Active minors: M(123) and M/123
Activity class: { M, −123M, −456M, −123456M }

Figure 4: Active decomposition of an acyclic orientation of K₄. Consider M as any of the four depicted orientations of the graph K₄, or any of the two grey regions of the arrangement K₄. The activities of M are O(M) = ∅ and O*(M) = {1, 4}. By Definition 4.1, the active filtration is ∅ = Fc ⊂ 123 ⊂ 123456, and the active partition is 123 + 456, with cyclic flat Fc = ∅. The active minors are M(123456)/123, which is bounded w.r.t. 4, and M(123), which is bounded w.r.t. 1 (Theorem 4.4). Those minors are depicted as bipolar digraphs. The activity class is formed by the four depicted graphs, or by the two grey regions and their opposite (Definition 4.9).

Figure 5: Active decomposition of an ordered digraph G. The digraph G is shown on the left. The ordering of the edge set E is given by: a < b < c < · · · < q < r < s. The active edges are O(G) = {g, i, l, m} and the dual-active edges are O*(G) = {a, b, d, q} (bold edges). The active partition is given by: Fc = gno + ij + l + mprs and E \ Fc = a + bc + defhk + q (Definition 4.1). The corresponding active minors, which are bounded (or bipolar), resp. dual-bounded (or cyclic-bipolar), w.r.t. their smallest edges (Theorem 4.4), and whose edge sets are given by the active partition, are shown in the bottom right line, resp. the upper right line.
Figure 6: Active partitions for some rank 4 regions (this picture is part of a more involved picture used in [14]). The ordering is given by \(1 < 2 < 3 < 4 < a < \cdots < h\). On the left and the middle: two views of the region of the space delimited by the hyperplanes 1, 2, and 3. On the right: the active partitions of the regions of the arrangement contained (and forming a path) in this region (Definition 4.1). The minimal basis is 1234, it induces the flag of faces \(1 \cap 2 \cap 3 \subset 1 \cap 2 \subset 1\) (in shades of grey in the picture). Geometrically, dual-activities and active partitions situate regions w.r.t. the flag of faces induced by the minimal basis. Precisely: intersections of regions with this flag of faces correspond to the covectors induced by the subsets of the active filtrations (see the covectors whose supports are 34, 2abcdh, and 2afgh, drawn in bold).

Remark that each \(F_k\), \(0 \leq k \leq \iota - 1\), is the complement of the support of a positive covector of \(M\), hence it is a flat of \(M\), and that each \(F'_{k}\), \(0 \leq k \leq \varepsilon - 1\), is the support of a positive vector of \(M\), hence it is a dual-flat of \(M\). In particular, \(F_c\) is a cyclic-flat of \(M\). For convenience, we can refer to \(F_c\), or to the parts forming \(F_c\), as the cyclic part of \(M\), and to \(E \setminus F_c\), or to the parts forming \(E \setminus F_c\), as the acyclic part of \(M\).

Finally, let us point out that, in the definitions that precede and the results that follow, the particular case of acyclic oriented matroids is addressed as the case where \(F_c = \emptyset\), and the totally cyclic case is addressed as the case where \(F_c = E\). Those cases are dual to each other. Let us deepen this with the next observation (which can be compared with Observation 3.2 for filtrations in general).

**Observation 4.2.** Let \(\emptyset = F'_{\varepsilon} \subset \ldots \subset F'_{0} = F_c = F_0 \subset \ldots \subset F_\iota = E\) be the active filtration of an ordered oriented matroid \(M\). We have:

1. \(\emptyset = E \setminus F_c \subset \ldots \subset E \setminus F'_0 = E \setminus F_c = E \setminus F'_0 \subset \ldots \subset E \setminus F'_\iota = E\) is the active filtration of \(M^*\), for the cyclic-flat \(E \setminus F_c\) of \(M^*\);
2. $\emptyset = F'_0 \subset \ldots \subset F'_0 = F_0 = F_c$ is the active filtration of the totally cyclic oriented matroid $M(F_c)$, for the cyclic-flat $F_c$ of $M(F_c)$.

3. $\emptyset = \emptyset = F_0 \setminus F_c \subset \ldots \subset F_0 \setminus F_c = E \setminus F_c$ is the active filtration of the acyclic oriented matroid $M/F_c$, for the cyclic-flat $\emptyset$ of $M/F_c$.

Proof. The first observation is obvious by the definition and by properties of oriented matroid duality. The second one is direct since positive circuits of $M(F_c)$ are exactly positive circuits of $M$. The third one is dual to the second one. \[\] In the following proofs, we can arbitrarily focus either on circuits or on cocircuits. We usually focus on cocircuits, because of their natural geometrical interpretation, and we deduce the same results for circuits by duality. Some proofs are written in terms of circuits, when they imply shorter notations, and can be deduced for cocircuits by duality.

Lemma 4.3. Let $M$ be oriented matroid on a linearly ordered set $E$ with $\iota \geq 0$ dual-active elements $a_1 < \ldots < a_\iota$, with $\varepsilon \geq 0$ active elements $a'_1 < \ldots < a'_\varepsilon$, and with active filtration $\emptyset = F'_\varepsilon \subset F'_{\varepsilon-1} \subset \ldots \subset F'_0 = F_0 = F_c = F_0 \subset \ldots \subset F_{\iota-1} \subset F_\iota = E$. 

If $\iota > 0$ then, denoting $F = F_{\iota-1}$, we have:

- $M/F$ is bounded with respect to $a_\iota$,
- the active filtration of $M(F)$ is $(F'_\varepsilon, \ldots, F'_0, F_c, F_0, \ldots, F_{\iota-1})$.

If $\varepsilon > 0$ then, denoting $F' = F'_{\varepsilon-1}$, we have:

- $M(F')$ is dual-bounded with respect to $a_\varepsilon$,
- the active filtration of $M/F'$ is $(F'_{\varepsilon-1} \setminus F', \ldots, F'_0 \setminus F', F_c \setminus F, F_0 \setminus F', \ldots, F_\iota \setminus F')$.

Proof. Let us assume first that $\iota > 0$, denoting $F = F_{\iota-1}$. The cocircuits of $M/F$ are the cocircuits of $M$ contained in $E \setminus F$, where $E \setminus F$ is the union of all positive cocircuits $D$ of $M$ with smallest element $a_\iota$. Hence every element of $M/F$ belongs to a positive cocircuit, hence $M/F$ is acyclic. And $a_\iota$ belongs to a positive cocircuit of $M/F$, hence $a_\iota$ is dual-active in $M/F$. If another element was dual-active in $M/F$, then it would also be the smallest element of a positive cocircuit in $M$ and dual-active in $M$, a contradiction with $a_\iota$ being the greatest dual-active element of $M$. So we have $O^*(M/F) = \{a_\iota\}$ and $O(M/F) = \emptyset$, that is $M/F$ is bounded with respect to $a_\iota$.

As $E \setminus F$ is a union of positive cocircuits of $M$, the positive circuits of $M(F)$ are the positive circuits of $M$. Hence, $M$ and $M(F)$ have the same active elements, and the cyclic part $(F'_\varepsilon, \ldots, F'_0, F_c)$ of their active filtration is the same.

The cocircuits of $M(F)$ are exactly the non-empty inclusion-minimal intersections of intersections of $F$ and cocircuits of $M$. More precisely, the signed subsets of the form $C \cap F$, where $C$ is a cocircuit of $M$, are unions of cocircuits of $M(F)$. Since every element of $E \setminus F$ is greater than $a_\iota$ by definition of $a_\iota$, we have that $a_k \in F$ for every $1 \leq k < \iota$. A positive cocircuit $D$ of $M$ with smallest element $a_k$, for $1 \leq k < \iota$, induces a positive cocircuit contained in $D \cap F$ of $M(F)$ with smallest element $a_k$, hence $a_1, \ldots, a_{\iota-1}$ are dual-active in $M(F)$. Let $H_k = F \setminus \cup\{D \mid D \text{ positive cocircuit of } M(F), \min(D) > a_k\}$. Independently, by definition of $F_k$, we have $F_k = F \cap F_k = F \setminus \cup\{F \cap D \mid D \text{ positive cocircuit of } M, \min(D) > a_k\}$. For every positive cocircuit $D$ of $M$, $D \cap F$ is a union of positive cocircuits of $M(F)$, so we have $F \setminus F_k \subseteq F \setminus H_k$, that is $H_k \subseteq F_k$. \[\]
Now, conversely, let $e$ be an element of $F \setminus H_k$, for some $1 \leq k < \iota$. It belongs to be a positive cocircuit $D$ of $M(F)$ with smallest element $a > a_k$. We want to prove that $e$ belongs to $F \cap D'$ for some positive cocircuit $D'$ of $M$ contained in $D \cup (E \setminus F)$. The cocircuit $D$ is contained in a cocircuit $D_M$ of $M$ with $D_M \cap F = D$. Let $D'_M$ be the composition of all positive cocircuits of $M$ with smallest element $a$, whose support is $E \setminus F$ and whose signs are all positive (since given by positive cocircuits). Then $D'_M \circ D_M$ is positive, since it is positive on $E \setminus F$ as $D'_M$, and positive on $D_M \cap F = D$. And $D'_M \circ D_M$ has smallest element $a$, since $a < a_k$. By the conformal composition property of covectors in oriented matroid theory, there exists a positive cocircuit $D'$ of $M$ containing $e$ and contained in $D_M \cup (E \setminus F)$. Since every element of $E \setminus F$ is greater than $a_1$ and $a_\iota \geq a > a_k$, the smallest element of $D'$ is greater than $a$, hence strictly greater than $a_k$. Since $e$ belongs to $F \cap D'$, we get that $e \in F \setminus F_k$. We have proved $F \setminus H_k \subseteq F \setminus F_k$, that is finally $F_k = H_k$, which provides the active filtration of $M(F)$.

The second property involving $F'$ is dual to the first one involving $F$, hence its dual proof is direct, by Observation 4.2 Item 1. □

**Theorem 4.4.** Let $M$ be oriented matroid on a linearly ordered set $E$. The active filtration of $M$ is the unique (connected) filtration $(F'_0, \ldots, F'_0, F_e, F_0, \ldots, F_e)$ of $M$ (or $E$) such that the $\iota$ minors

$$M(F_k)/F_{k-1}, \text{ for } 1 \leq k \leq \iota,$$

are bounded with respect to $a_k = \min(F_k \setminus F_{k-1})$, and the $\varepsilon$ minors

$$M(F'_{k-1})/F'_k, \text{ for } 1 \leq k \leq \varepsilon,$$

are dual-bounded with respect to $a'_k = \min(F'_{k-1} \setminus F'_k)$.

**Proof.** Observe that the statement of the result is “self-dual”. Precisely, by Observation 3.2:

- a (connected) filtration $(F'_0, \ldots, F'_0, F_e, F_0, \ldots, F_e)$ of $M$ corresponds to a (connected) filtration $(E \setminus F_0, \ldots, E \setminus F_0, E \setminus F_e, E \setminus F_0, \ldots, E \setminus F_e)$ of $M^*$;

- the minors $M(F_k)/F_{k-1}$, $1 \leq k \leq \iota$, of $M$, are bounded w.r.t. $a_k = \min(F_k \setminus F_{k-1})$, if and only if the corresponding minors $M^*(E \setminus F_{k-1})/(E \setminus F_k)$ of $M^*$ are dual-bounded w.r.t. $a_k = \min((E \setminus F_{k-1}) \setminus (E \setminus F_k))$;

- the minors $M(F'_{k-1})/F'_k$, $1 \leq k \leq \varepsilon$, of $M$, are dual-bounded w.r.t. $a'_k = \min(F'_{k-1} \setminus F'_k)$, if and only if the corresponding minors $M^*(E \setminus F'_k)/(E \setminus F'_{k-1})$ of $M^*$ are bounded w.r.t. $a'_k = \min((E \setminus F'_k) \setminus (E \setminus F'_{k-1}))$.

Hence, in the following proof, we will be allowed to deduce various results by duality, applying the same reasonings to $M^*$.

First, we check that the active filtration is a filtration. By construction, we have $\emptyset = F'_0 \subset \ldots \subset F'_0 = F_e = F_0 \subset \ldots \subset F_e = E$. Assume $M$ has $\iota$ dual-active elements $a_1 < \ldots < a_\iota$, and $\varepsilon$ active elements $a'_1 < \ldots < a'_\varepsilon$. By definition of $a_k$, for $1 \leq k \leq \iota$, there exists a positive cocircuit of $M$ whose smallest element is $a_k$, hence $a_k \in F_k \setminus F_{k-1}$ according to the definition of $F_k \setminus F_{k-1}$ given above. So we have $a_k = \min(F_k \setminus F_{k-1}), 1 \leq k \leq \iota$, which is increasing with $k$ by definition of $a_k$. Similarly, for $1 \leq k \leq \varepsilon$, there exists a positive circuit of $M$ whose smallest element is $a'_k$, so we get $a'_k = \min(F'_{k-1} \setminus F'_k)$, which is increasing with $k$. Hence the result.

Second, we apply recursively Lemma 4.3. We directly get that the $\iota$ minors $M_k = M(F_k)/F_{k-1}, 1 \leq k \leq \iota$ are bounded with respect to $a_k$; and that the $\varepsilon$ minors $M'_k = M(F'_{k-1})/F'_k, 1 \leq k \leq \varepsilon$, are
dual-bounded with respect to $a'_{k}$. This proves the property stated in the result. This also proves that those minors are connected as soon as they have more than one element, which achieves the proof that the active filtration of $M$ is a connected filtration of $M$.

Now, it remains to prove the uniqueness property. Assume $(F'_{\varepsilon}, \ldots, F_{0}, F_{c}, F_{0}, \ldots, F_{t})$ is a filtration of $E$ satisfying the properties given in the result. Then it is obviously a connected filtration of $M$, by the definitions, since being bounded, resp. dual-bounded, implies being either connected or reduced to an isthmus, resp. a loop.

First, we prove that $F_c$ is the union of all positive circuits of $M$. Assume $C$ is a positive circuit of $M$, not contained in $F_c$. Let $k$ be the smallest integer such that $C \subseteq F_k$, $1 \leq k \leq t$. Then $C \setminus F_{k-1} \neq \emptyset$ (otherwise $k$ would not be minimal), so $C \setminus F_{k-1}$ contains a positive circuit of $M/F_{k-1}$. Moreover $C \setminus F_{k-1} \subseteq F_k \setminus F_{k-1}$ by definition of $k$, so $C \setminus F_{k-1}$ contains a positive circuit of $M_k = M(F_k)/F_{k-1}$, a contradiction with $M_k$ being acyclic. Hence the union of positive circuits of $M$ is contained in $F_c$. With exactly the same reasoning from the dual viewpoint, we get that the union of positive cocircuits of $M$ is contained in $E \setminus F_c$. Finally, $F_c$ contains the union of positive circuits of $M$ and has an empty intersection with the union of all positive cocircuits of $M$, so $F_c$ is exactly the union of all positive circuits of $M$.

Second, we prove the following claim: for every positive cocircuit $D$ of $M$, the smallest element of $D$ equals $a_{k+1}$, where $k$ is the greatest possible such that $D \subseteq E \setminus F_k$, $0 \leq k \leq t - 1$. Indeed, for such $D$ and $k$, we have $D \cap F_{k+1} \neq \emptyset$ (otherwise $k$ would not be maximal), so $D \cap F_{k+1}$ is a union of positive cocircuits of $M(F_{k+1})$. Moreover, $D \cap F_{k+1} \subseteq F_{k+1} \setminus F_k$ by definition of $k$, so $D \cap F_{k+1}$ is a union of positive cocircuits of $M_{k+1} = M(F_{k+1})/F_k$. By assumption that $M_{k+1}$ is bounded with respect to $a_{k+1}$, we have that $a_{k+1}$ belongs to every positive cocircuit of $M_{k+1}$, so $a_{k+1}$ is the smallest element of $D \cap F_{k+1}$. By definition of a filtration, $a_{k+1}$ is the smallest element in $E \setminus F_k$ (it is the smallest in $F_k \setminus F_{k-1}$ and the sequence $\min(F_i \setminus F_{i-1})$ is increasing with $i$), hence we have $\min(C) = a_{k+1}$. In particular, we have proved that the dual-active elements of $M$ are of type $a_k$, $1 \leq k \leq t$.

Dually, we get the following claim: for every positive circuit $C$ of $M$, the smallest element of $C$ equals $a'_{k+1}$, where $k$ is the greatest possible such that $C \subseteq F'_k$, $0 \leq k \leq \varepsilon - 1$. And in particular, we get that the active elements of $M$ are of type $a'_{k}$, $1 \leq k \leq \varepsilon$.

Third, we prove that the parts of the considered filtration are indeed the parts of the active filtration. Let us denote $F = F_{\varepsilon-1}$ and so $a_{\varepsilon} = \min(E \setminus F)$. We want to prove that $F = E \setminus \cup\{D \mid D$ positive cocircuit of $M$, $\min(D) = a_{\varepsilon}\}$. By assumption, $M_{\varepsilon} = M/F$ is bounded. So, every element of $M/F$ belongs to a positive cocircuit of $M/F$ with smallest element $a_{\varepsilon}$. The cocircuits of $M/F$ are the cocircuits of $M$ contained in $E \setminus F$. Hence, every element of $M$ belonging to $E \setminus F$ belongs to a positive cocircuit of $M$ with smallest element $a_{\varepsilon}$, which proves that $E \setminus F \subseteq \cup\{D \mid D$ positive cocircuit of $M$, $\min(D) = a_{\varepsilon}\}$. Conversely, let $D$ be a positive cocircuit of $M$ with smallest element $a_{\varepsilon}$. By the above claim, we have that $t - 1$ is the greatest possible such that $D \subseteq E \setminus F_{\varepsilon-1}$, that is $D \subseteq E \setminus F$, hence the result.

Dually, let us denote $F = F_{t+1}$ and so $a'_{t} = \min(F)$. By the same reasoning applied to $M^*$, we get that $F = \cup\{C \mid C$ positive circuit of $M$, $\min(C) = a'_{t}\}$.

Now, we can conclude by induction, assuming the result is true for minors of $M$. Assume $t > 0$ and denote again $F = F_{t-1}$, we have proved above that $F$ is indeed the largest part different from $E$ in the active filtration of $M$. It is direct to check that the sequence of subsets $(F'_{\varepsilon}, \ldots, F_{0}, F_{c}, F_{0}, \ldots, F_{t})$ is a connected filtration of $M(F)$. Moreover this filtration obviously satisfies the properties of the result for the oriented matroid $M(F)$, as the involved minors are
unchanged. Hence, this filtration is the active filtration of \( M(F) \), by induction assumption. Hence, by Lemma 4.3, we have that the subsets \( F'_0, \ldots, F'_i, F_c, F_0, \ldots, F_i - 1 \) are indeed the same subsets as in the active filtration of \( M \). Dually, assume that \( \varepsilon > 0 \) and denote again \( F' = F_{\varepsilon - 1} \), we get that the subsets \( F'_{\varepsilon - 1}, \ldots, F'_0, F_c, F_0, \ldots, F_i \) are indeed the same subsets as in the active filtration of \( M \). So, finally, we have proved that the result is true for \( M \).

The next observation is the counterpart of Observation 3.5 for oriented matroid activities.

**Observation 4.5.** Let us continue and refine Observation 4.2. Let \( \emptyset = F'_0 \subset \ldots \subset F'_0 = F_c = F_0 \subset \ldots \subset F_i = E \) be the active filtration of the ordered oriented matroid \( M \). Let \( F \) and \( G \) be two subsets in this sequence such that \( F \subseteq G \). Then, by Theorem 4.4, the active filtration of \( M(G)/F \) is obtained from the subsequence with extremities \( F \) and \( G \) (i.e. \( F \subset \cdot \cdot \cdot \subset G \)) of the active filtration of \( M \) by subtracting \( F \) from each subset of the subsequence (with \( F_c \setminus F \) as cyclic flat). In particular, the subsequence ending with \( F \) (i.e. \( \emptyset \subset \cdot \cdot \cdot \subset F \)) is the active filtration of \( M(F) \), and the subsequence beginning with \( F \) (i.e. \( F \subset \cdot \cdot \cdot \subset E \)) yields the active filtration of \( M/F \) by subtracting \( F \) from each subset.

**Theorem 4.6.** Let \( M \) be an oriented matroid on a linearly ordered set \( E \). We have

\[
\{ \text{reorientations } -A'M \text{ of } M \text{ for } A \subseteq E \} \]

\[
= \biguplus \left\{ -A'M \mid -A'M(F_k)/F_{k-1}, \ 1 \leq k \leq \iota, \ \text{bounded with respect to } \min(F_k \setminus F_{k-1}), \right. \\
\left. \text{and } -A'M(F'_{k-1})/F'_k, \ 1 \leq k \leq \varepsilon, \ \text{dual-bounded with respect to } \min(F'_{k-1} \setminus F'_k) \right\}
\]

where the disjoint union is over all connected filtrations \( (F'_0, \ldots, F'_i, F_c, F_0, \ldots, F_i) \) of \( M \). The connected filtration of \( M \) associated to a reorientation \( -A'M \) in the right-hand side of the equality is the active filtration of \( -A'M \).

**Proof.** This result consists in a bijection between all reorientations \( -A'M \) of \( M \) and sequences of reorientations of the minors involved in decomposition sequences of \( M \). It is given directly by Theorem 4.4. From the first set to the second set, the active filtration of \( M \) provides the required decomposition. Conversely, from the second set to the first set, first choose a connected filtration of \( M \). Then, for each minor of \( M \) defined by this sequence, choose a bounded/dual-bounded reorientation for this minor as written in the second set statement. This defines a reorientation \( -A'M \) of \( M \) (since every element of \( M \) appears in one and only one of these minors). Now, for this reorientation \( -A'M \), the chosen filtration satisfies the property of Theorem 4.4, hence this filtration is the active filtration of the reorientation \( -A'M \) of \( M \). Finally, the uniqueness in Theorem 4.4 ensures that the union in the second set is disjoint. \( \square \)

As mentioned above, Theorem 4.6 applies in particular to a decomposition of the set of acyclic, resp. totally cyclic, reorientations of an oriented matroid \( M \), involving only bounded, resp. dual-bounded, minors (by restriction to connected filtrations with cyclic flat \( F_c = \emptyset \), resp. \( F_c = E \)).

**Remark 4.7.** Theorem 4.6, along with the “enumeration of reorientation activities”, directly yields a proof, in orientable matroids, of the Tutte polynomial expression in terms of beta invariants of
minors stated in the companion paper No. 2.a [18, Theorem 3.5]. The proof given in [18] is available for general matroids (it uses Theorem 3.3 and the “enumeration of basis activities” in a similar way). Notice that, in graphs, both proofs can be used, as all graphs yield orientable matroids (the proof of this result given in [21] for graphs uses the orientation decomposition).

**Proposition 4.8.** Let \( M \) be an oriented matroid on a linearly ordered set \( E \), with \( \iota \) dual-active elements and \( \varepsilon \) active elements. The \( 2^{\iota+\varepsilon} \) reorientations of \( M \) obtained by reorienting any union of parts of the active partition (or filtration) of \( M \) have the same active partition (or filtration) as \( M \), and hence the same active and dual-active elements.

**Proof.** The result can be proved directly from Definition 4.1. Let us give such a proof in a condensed way (see [9] for a more detailed and more general proof). Consider the union \( A \) of all positive circuits of \( M \) whose smallest element is greater than a given element \( a \) (the same reasoning holds for cocircuits by duality). We prove the following claim: any union of all positive circuits whose smallest element is greater than a given element \( e \) is the same in \( M \) and \(-A\). Let \( C \) be a positive circuit of \( M \), with \( \min(C) \geq e \). If \( e \geq a \), then \( C \subseteq A \), then \( C \) is a positive circuit of \(-A\). Assume now \( e < a \). The set \( A \) can be considered as a positive vector of both \( M \) and \(-A\) (it is a conformal composition of positive circuits). Since \( C \) is positive on \( C \setminus A \) in \(-A\), then \( A \circ C \) is also a positive vector of \(-A\). By the generation of vectors by conformal composition, for every element \( f \) of \( C \), there exists a positive circuit \( D \) of \(-A\) containing \( f \) and contained in \( C \cup A \). Also, we have \( \min(D) \geq e \). So \( C \) is contained in the union of positive circuits of \(-A\) with smallest element greater than \( e \). So the claim is proved. The rest of the proof is straightforward: apply the claim to the subsets of the cyclic part of active filtration, and apply the same claim dually and independently to subsets built from the acyclic part of active partition.

Alternatively, the result can also be seen as a direct corollary of Theorem 4.4. Indeed, reorienting a union of parts of the active partition of \( M \) implies reorienting completely some of the active minors of \( M \). Then, by Theorem 4.4, the resulting reorientation is obtained from the same minors, that is, from the same filtration of \( M \), which is the same active filtration as that of \( M \). \( \square \)

**Definition 4.9.** Let \( M \) be an oriented matroid on a linearly ordered set, with \( \iota \) dual-active elements and \( \varepsilon \) active elements. We call activity class of \( M \) the set of \( 2^{\iota+\varepsilon} \) reorientations of \( M \) obtained by reorienting any union of parts of the active partition of \( M \).

By Proposition 4.8, activity classes of reorientations of \( M \) can be seen as equivalence classes, which partition the set of reorientations of \( M \).

We will continue to study activity classes in Section 7: from their boolean lattice structure, one can derive four refined activity parameters and a Tutte polynomial expansion in these terms. The notions of active filtration/partition and of activity classes are generalized to oriented matroid perspectives in [9]. See Figure 4 for an example of activity class (see also Figure 14).

For an ordered oriented matroid \( M \), a reorientation \(-A\) is said to be active-fixed, resp. dual-active fixed, with respect to \( M \) if no active, resp. no dual-active, element has been reoriented with respect to \( M \), that is, if \( O(-A) \cap A = \emptyset \), resp. \( O^*(-A) \cap A = \emptyset \).

**Corollary 4.10.** Let \( M \) be an ordered oriented matroid. The number of activity classes of reorientations of \( M \) with activity \( i \) and dual activity \( j \) equals \( t_{i,j} \).

Each activity class of reorientations of \( M \) contains exactly one reorientation which is active-fixed and dual-active-fixed w.r.t. \( M \). The number of such reorientations of \( M \) with activity \( i \) and dual activity \( j \) thus equals \( t_{i,j} \).
Furthermore, each activity class of reorientations of $M$ with activity $i$ and dual activity $j$ contains $2^i$ active-fixed reorientations and $2^j$ dual-active-fixed reorientations. Finally, we have the enumerations and representatives of activity classes given by Table 2.

| reorientations of $M$ / activity classes of reorientations of $M$ | number          |
|---------------------------------------------------------------|-----------------|
| active-fixed and dual-active-fixed / all                      | $t(M;1,1)$      |
| acyclic and dual-active-fixed / acyclic                       | $t(M;1,0)$      |
| active-fixed and totally cyclic / totally cyclic              | $t(M;0,1)$      |
| active-fixed (/ non-applicable)                               | $t(M;2,1)$      |
| dual-active-fixed (/ non-applicable)                          | $t(M;1,2)$      |

Table 2: Enumeration of certain reorientations based on representatives of activity classes (Corollary 4.10).

Proof. The first claims are obvious by the above construction and Proposition 4.8. Then, the enumerations of the table are obvious by the first claims and by the Tutte polynomial expression “enumeration of reorientation activities”. These enumerations are also implied by the forthcoming Tutte polynomial expression in terms of four refined orientation activities (Theorem 7.4). \qed

Remark 4.11. Let us mention properties which are specific to graphs, studied in [13], and how they generalize. As observed above, activity classes can be represented by reorientations that are active fixed and dual-active fixed (see also Section 8). An application is that, for suitable orderings of the edge set of a graph (roughly when all branches of the smallest spanning tree are increasing), there is one and only one acyclic reorientation with a unique sink in each activity class of acyclic reorientations, see [13, Section 6]. Moreover, as shown in [13, Section 7], the notion of active partition for a directed graph on a linearly ordered set of edges generalizes the notion of components of acyclic reorientations with a unique sink. This last notion relies on certain linear orderings of the vertex set. It was studied by Lass in [26] in relation with the chromatic polynomial, by Viennot in [35] in terms of heaps of pieces, and by Cartier and Foata in [3] in terms of non-commutative monoids (see also Gessel [7]). For every such vertex ordering, there exists a consistent edge ordering such that active partitions exactly match these acyclic orientation components. With respect to this construction, our generalization by means of active partitions allows us to consider any orientation, any ordering of the edge set, along with a generalization to any oriented matroid.

5. The uniactive bijection between bounded/dual-bounded reorientations and their fully optimal uniactive internal/external bases

This section mainly recalls (and also reformulates, reorganizes, or completes) definitions and important results from [17] (No. 1 of the same series of papers). Moreover, we first informally explain, as an overview, how these can be considered under different perspectives and related to other papers of the series.

We consider an ordered oriented matroid $M$ on $E$ which is bounded with respect to $p = \min(E)$ (or, in geometrical terms, a bounded region w.r.t. the element $p$ considered as a hyperplane at infinity, or, in terms of graphs, an acyclic bipolar directed graph whose unique source and unique sink are the extremities of $p$). The main result of [17] is that such an oriented matroid has a unique fully optimal basis, satisfying a simple combinatorial criterion. This directly yields a
bijection between (pairs of opposite) bounded reorientations of $M$ w.r.t. $p$ and bases of $M$ with internal/external activity equal to 1/0.

The existence and uniqueness of the fully optimal basis of a bounded region is a difficult fundamental result, that can be seen by different manners, and has important connections with duality. First, its proof essentially relies upon a topological obstruction in oriented matroids, and a tricky use of this obstruction, see [17, The Crescent Lemma 4.4 and Proposition 4.3].

Second, it witnesses a curious duality geometric property. The fully optimal basis of a bounded region induces a flag of faces adjacent to this region (“Dual-adjacency”: the successive compositions of fundamental cocircuits of the basis yield positive covectors). Dually, the complementary basis of the dual induces a flag of faces adjacent to the corresponding bounded region of the dual obtained by reorienting $p$ (“Adjacency”: the successive compositions of fundamental circuits of the initial basis yield positive vectors, up to the sign of $p$). The fully optimal basis then appears naturally, as it is the unique basis satisfying at the same time these two properties “Adjacency” and “Dual-adjacency”. See Definition 5.2 below, and see the geometrical interpretation given after [17, Proposition 3.3] for more details. See also the flag representations in Figures 20 and 21 in Section 9.

Third, it can be seen as a refinement of (pseudo-)linear programming. See [2, Chapter 10] for information on pseudo-linear programming in oriented matroids. In real hyperplane arrangements in general position or in uniform oriented matroids, building the optimal cocircuit of a real/pseudo linear program is equivalent to build the fully optimal basis [12]. In general real hyperplane arrangements or oriented matroids, we optimize a sequence of nested faces (the successive covectors obtained by composition of the fundamental cocircuits of the basis), each with respect to a sequence of objective functions (provided by the linearly ordered minimal basis of the matroid), yielding finally a unique fully optimal basis. This refines standard linear programming where just one vertex is optimized with respect to just one objective function, but this can be computed inductively using standard linear programming. Linear programming duality is then strengthened by the Active Duality property recalled in Theorem 5.9 below, and proved in [17, Section 5]4. This relation with linear programming is addressed in [17], and the construction of the fully optimal basis by this manner is detailed in [19] (see also [16] for a description in terms of real hyperplane arrangements, see also [22] for a reformulation and a simplification in the graph case).

Fourth, it is noticeable that in one direction, from bases to reorientations, the bijection is given by a simple single pass algorithm (Proposition 5.4) whereas in the other direction, from reorientations to bases, it is more difficult than real/pseudo linear programming. Hence, this bijection can be thought of as a sort of “one-way function”.

Fifth, a construction of this bijection can be made by deletion/contraction of the greatest element. It is detailed in [20] (see also [21, Section 6.1] or [22] in the graph case). This construction can be seen as an elaborated translation of the usual linear programming solving by variable/constraint deletion. It is more direct and more simple from the formal structural viewpoint than the full optimality algorithm alluded to above, but not from the computational complexity viewpoint, as it involves an exponential number of minors, whereas the previous construction involves a linear number of minors. This deletion/contraction construction can be equally made from the primal or the dual viewpoint, and this fact is non-trivial, as it is equivalent to the existence and uniqueness property of the fully optimal basis (see [20] for more details).

Finally, let us mention that the two constructions mentioned above (full optimality algorithm,

\footnote{See footnote 3 in Section 2 for a correction in the statements of [17, Proposition 5.1 and Theorem 5.3].}
which is bounded with respect to the minimal element $p$ of $E$. The fully optimal basis $\alpha(M)$ of $M$ is the unique basis $B$ of $M$ such that:

- for all $b \in B \setminus p$, the signs of $b$ and $\min(C^*(B;b))$ are opposite in $C^*(B;b)$;
- for all $e \in E \setminus B$, the signs of $e$ and $\min(C(B;e))$ are opposite in $C(B;e)$.

(Full Optimality Criterion)

**Definition 5.2** (equivalent to Definition 5.1 by [17, Proposition 3.3]). Let $M$ be an oriented matroid on a linearly ordered set $E$, which is bounded with respect to the minimal element $p$ of $E$. The fully optimal basis $\alpha(M)$ of $M$ is the unique basis $B$ of $M$ such that, denoting $B = b_1 < \cdots < b_r$ and $E \setminus B = c_1 < \cdots < c_{n-r}$:

- the maximal covector $C^*(B;b_1) \circ \cdots \circ C^*(B;b_r)$ is positive; (Adjacency)
- the maximal vector $C(B;c_1) \circ \cdots \circ C(B;c_{n-r})$ is positive on $E \setminus \{p\}$ and negative on $p$. (Dual-Adjacency)

The existence and uniqueness of a basis satisfying the criteria of Definitions 5.1 or 5.2 is the main result of [17], namely [17, Theorem 4.5]. It yields the next theorem. Notice that a bounded oriented matroid and its opposite have the same fully optimal basis. An example of fully optimal basis of a bounded region is given in Figure 7.

**Theorem 5.3** (Key theorem [17, Theorem 4.5]). Let $M$ be a matroid on a linearly ordered set $E$ with $\min(E) = p$. The mapping $M \mapsto \alpha(M)$ yields a bijection between all bounded reorientations of $M$ w.r.t. $p$, with fixed orientation for $p$, and all uniactive internal bases of $M$. Equally, it yields a bijection between all pairs of opposite bounded reorientations of $M$ w.r.t. $p$, and all uniactive internal bases of $M$.

The above mapping $M \mapsto \alpha(M)$ is called the uniactive bijection of $M$ (bounded case). A direct computation of $\alpha(M)$ for bounded oriented matroids is given in [19] by means of elaborations on linear programming (see also [16] in real hyperplane arrangements, and [22] in graphs). Moreover, $\alpha(M)$ can be built by deletion/contraction, as shown in [19] (see also [21] in graphs). See more details in the introduction of the section.

The mapping $M \mapsto \alpha(M)$ was built in [17] by its inverse, from uniactive internal bases to bounded reorientations, provided by a single pass algorithm over the base (see [17, Figure 5] for an example), or equally (dually) over its complement, or equally over the ground set, so that the criterion for element signs from Definition 5.1 is satisfied. We recall one of these algorithms below in Proposition 5.4, that we will use later in Theorem 6.9. Let us mention that internal uniactive bases can be characterized by several ways, see [13, 17, 18].

**Proposition 5.4** ([17, Proposition 4.2, Algorithm 3]). Let $M$ be an oriented matroid on a linearly ordered set of elements $E = \{e_1, \ldots, e_n\}_\prec$. For a basis $B$ with internal activity 1 and external activity 0, the two opposite reorientations of $M$ in $\alpha^{-1}(B)$ are computed by the following algorithm.

Reorient $e_1$ or not, arbitrarily.
For $k$ from 2 to $n$ do

...
if \( e_k \in B \) then
let \( a = \min(C^*(B; e_k)) \)
reorient \( e_k \) if necessary in order to have \( a \) and \( e_k \) with opposite signs
in \( C^*(B; e_k) \)

if \( e_k \not\in B \) then
let \( a = \min(C(B; e_k)) \)
reorient \( e_k \) if necessary in order to have \( a \) and \( e_k \) with opposite signs
in \( C(B; e_k) \)

Figure 7: The basis 136 is the fully optimal basis of the grey region (the arrangement is the same as in Figures 2 and 3). We add signs to the fundamental graph and fundamental tableau of the basis in order to illustrate the full optimality criterion in a practical and visual way. This will be used again in Section 6 to illustrate decompositions involving fully optimal bases of minors and to build on examples given in [18]. Let us now detail each part of the figure. In the middle: the acyclic reorientation is depicted as the grey region, the basis is depicted with bold lines, its fundamental cocircuits are written at the vertices defined by the basis. On the left: the corresponding orientation of the graph \( K_4 \), with its fully optimal spanning tree 136 in bold. On the upper right: the fundamental graph of the basis 136, where signs are added accordingly with signs of the fundamental circuits and cocircuits of the basis w.r.t. this reorientation. Precisely: by convention, elements of \( B \), resp. \( E \setminus B \), are provided with a +, resp. −, sign; and then edges are provided with a + or − sign depending on the reorientation, so that one can read signs of elements in \( C^*(B; b) \) for \( b \in B \) and in \( -C(B; e) \) for \( e \in E \setminus B \) (as well, by orthogonality). Light edges are not given signs, they are not useful for the criterion that the basis is fully optimal. On the bottom right: the fundamental tableau of the basis 136, where signs are added accordingly with signs of the fundamental circuits and cocircuits of the basis w.r.t. this reorientation. Signs of elements in \( C^*(B; b) \) for \( b \in B \), resp. in \( -C(B; e) \) for \( e \in E \setminus B \), appear in columns, resp. rows, of the tableau (by orthogonality). Precisely: by convention, diagonal elements corresponding to elements of \( B \), resp. \( E \setminus B \), are provided with a +, resp. −, sign; other non-zero elements of the tableau are given signs depending on the reorientation. They are either provided with a + or − sign, when they correspond to a minimal element of a fundamental circuit or cocircuit, or they are provided with a + or − sign, when they are not used in the full optimality criterion. Finally, the reader is invited to check on the fundamental graph or tableau that the signs for the basis 136 w.r.t. the given reorientation satisfy the full optimality criteria given by Definitions 5.1 and 5.2. On the tableau: smallest non-zero (non-diagonal) entries of rows, resp. columns, are all +

Now, let us extend by duality the above definitions and result to dual-bounded reorientations (this was not made explicitly in [17]). One can observe that Definition 5.5 below is contained in the following general Definition 6.1.
Definition 5.5. Let \( M \) be an oriented matroid on a linearly ordered set \( E \), dual-bounded with respect to the minimal element \( p \) of \( E \). Then \( M^* \) is bounded w.r.t. \( p \) and we define
\[
\alpha(M) = E \setminus \alpha(M^*). \tag{Duality}
\]

Definition 5.6 (equivalent to Definition 5.5 by Definitions 5.1 and 5.2). Let \( M \) be an oriented matroid on a linearly ordered set of elements, dual-bounded with respect to the minimal element \( p \) of \( E \). Then, \( \alpha(M) \) is the unique basis \( B \) of \( M \) such that:
- for all \( b \in B \), the signs of \( b \) and \( \min(C^*(B; b)) \) are opposite in \( C^*(B; b) \);
- for all \( e \in (E \setminus B) \setminus \{p\} \), the signs of \( e \) and \( \min(C(B; e)) \) are opposite in \( C(B; e) \).

Equivalently, \( \alpha(M) \) is the unique basis \( B \) of \( M \) such that, denoting \( B = b_1 < \cdots < b_r \) and \( E \setminus B = c_1 < \cdots < c_{n-r} \):
- the maximal covector \( C^*(B; b_1) \circ \cdots \circ C^*(B; b_r) \) is positive on \( E \setminus \{p\} \) and negative on \( p \);
- the maximal vector \( C(B; c_1) \circ \cdots \circ C(B; c_{n-r}) \) is positive.

Theorem 5.7 (dual of Theorem 5.3). Let \( M \) be a matroid on a linearly ordered set \( E \) with \( \min(E) = p \). The mapping \( M \mapsto \alpha(M) \) yields a bijection between all dual-bounded reorientations of \( M \) w.r.t. \( p \), with fixed orientation for \( p \), and all uniaactive external bases of \( M \). Equally, it yields a bijection between all pairs of opposite dual-bounded reorientations of \( M \) w.r.t. \( p \), and all uniaactive external bases of \( M \).

The above mapping \( M \mapsto \alpha(M) \) is the dual-bounded case of the the uniaactive bijection of \( M \).

Proposition 5.8. Let \( M \) be an oriented matroid on a linearly ordered set. For a basis \( B \) with internal activity 0 and external activity 1, the two opposite reorientations of \( M \) in \( \alpha^{-1}(B) \) are computed by exactly the same algorithm as in Proposition 5.4. \( \square \)

Finally, let us recall a different and more involved duality property of the active bijection, called active duality, that can be seen as a strengthening of linear programming duality (see [17, Section 5]{5}). This important property shows that the active bijection is compatible with the two canonical bijections provided by the following properties ([17, Propositions 5.1 and 5.2]{5}). For a linearly ordered set \( E \) with at least two elements:

- an oriented matroid \( M \) on \( E \) is bounded w.r.t. \( p = \min(E) \) if and only if \( -pM \) is dual-bounded w.r.t. \( p \) (if and only if \( -pM^* \) is bounded w.r.t. \( p \));
- a basis \( B \) of a matroid \( M \) on \( E \), with \( p = \min(E) \) and \( p' = \min(E \setminus \{p\}) \), is uniaactive internal if and only if \( B \setminus \{p\} \cup \{p'\} \) is a uniaactive external basis.

Combining these properties with usual duality yields the commutative diagram of Figure 5.

Theorem 5.9 ([17, Theorem 5.3]{5}). Let \( E \) be a linearly ordered set with \( |E| > 1 \). Let \( M \) be an oriented matroid on \( E \), bounded with respect to \( p = \min(E) \). Let \( p' = \min(E \setminus \{p\}) \). We have:
\[
\alpha(M) = \left(E \setminus \alpha(-pM^*)\right) \setminus \{p'\} \cup \{p\}. \tag{Active Duality}
\]

\footnote{See footnote 3 in Section 2 for a correction in the statements of [17, Proposition 5.1 and Theorem 5.3].}
**Definition 5.10** (equivalent to Definition 5.5 by Theorem 5.9). Let $E$ be a linearly ordered set with $|E| > 1$. Let $p = \min(E)$ and $p' = \min(E \setminus \{p\})$. Let $M$ be an oriented matroid on $E$, dual-bounded with respect to $p$. Then $-pM$ is bounded w.r.t. $p$, and we can define $\alpha(M)$ by:

$$\alpha(M) = \alpha(-pM) \setminus \{p\} \cup \{p'\}.$$ 

![Diagram](Figure 8: Commutative diagram of duality properties of the uniactive bijection. It involves the usual oriented matroid duality and the active duality (Theorem 5.9).

### 6. The active basis of an ordered oriented matroid, and the canonical active bijection between reorientation activity classes and matroid bases

In this section, we define the canonical active bijection by means of the two decompositions into the case of $(1, 0)$ or $(0, 1)$ activities from the previous Sections 3 (for bases) and 4 (for reorientations), along with the bijection for $(1, 0)$ activities from Section 5. One can compute this decomposition and glue together the fully optimal bases associated with the active minors in order to get the base associated with the initial oriented matroid. This canonical bijection between activity classes of reorientations and bases not only preserves activities and active elements, but also active partitions (Theorem 6.4). Conversely, a single pass algorithm can be used to compute the inverse bijection, from bases to reorientations of a given oriented matroid. It uses only fundamental circuits and cocircuits, by some propagation from the smallest to the greatest element (Theorem 6.9). The bijection and its inverse can be also built by deletion/contraction of the greatest element [20].

We call it the *canonical* active bijection since it is based on three canonical constructions, so that it finally depends only on the reorientation class of the oriented matroid, that is on the non-signed pseudosphere/hyperplane arrangement in terms of a topological representation (or the underlying undirected graph in the graph case). See Observation 6.6. Let us also point out that, at every step of the construction, important duality properties are satisfied.

Formally, for an ordered oriented matroid $M$, we will get the *active basis of $M$*, which is denoted $\alpha(M)$, and the *canonical active bijection of $M$*, which is the bijection between preimages (activity classes of reorientations of $M$) and images (bases of $M$) of the mapping $M \mapsto \alpha(M)$ applied to all
reorientations of $M$. First, we give several definitions for the active basis. Then, in Theorem 6.4 below, we state that these definitions are well-defined and equivalent.

**Definition 6.1.** For an oriented matroid $M$ on a linearly ordered set $E$, the active basis $\alpha(M)$ of $M$ satisfies the three following properties:

(F.o.b.) If $M$ is bounded with respect to $p = \min(E)$ then $\alpha(M)$ is the fully optimal basis of $M$.

(Duality) $\alpha(M^*) = E \setminus \alpha(M)$.

(Induction) $\alpha(M) = \alpha(M/F) \cup \alpha(M(F))$ where $F$ is the complement of the union of all positive cocircuits of $M$ whose smallest element $a$ is the greatest possible smallest element of a positive cocircuit of $M$ (i.e. $a$ is the greatest dual-active element of $M$).

**Definition 6.2 (equivalent variants of Definition 6.1).** In Definition 6.1, the first property (F.o.b.) can be replaced with Definition 5.6, assuming $M$ is dual-bounded w.r.t. $p = \min(E)$. In Definition 6.1, the third property (Induction) can be replaced with any of the following ones.

(Ind.* $\alpha(M) = \alpha(M/F) \cup \alpha(M(F))$ where $F$ is the union of all positive circuits of $M$ whose smallest element is the greatest possible smallest element of a positive circuit of $M$.

(Ind.+ $\alpha(M) = \alpha(M/F) \cup \alpha(M(F))$ where $F$ is the complement of the union of all positive cocircuits of $M$ whose smallest element is greater than any given element of $E$.

(Ind.*+ $\alpha(M) = \alpha(M/F) \cup \alpha(M(F))$ where $F$ is the union of all positive circuits of $M$ whose smallest element is greater than any given element of $E$.

**Definition 6.3 (equivalent to Definition 6.1).** Let $M$ be an oriented matroid on a linearly ordered set of elements, with active filtration $(F'_\epsilon, \ldots, F'_0, F_c, F_0, \ldots, F_i)$. If $\iota + \epsilon = 1$ then $\alpha(M)$ is defined by any of the equivalent definitions given in Section 5 (bounded case if $\iota = 1$, dual-bounded case if $\epsilon = 1$). Otherwise, $\alpha(M)$ is defined by

$$\alpha(M) = \bigcup_{1 \leq k \leq \iota} \alpha(M(F_k)/F_{k-1}) \cup \bigcup_{1 \leq k \leq \epsilon} \alpha(M(F'_{k-1})/F'_k).$$

**Theorem 6.4.** Let $M$ be an oriented matroid on a linearly ordered set $E$.

1. The image $\alpha(M)$ is well defined. Definitions 6.1, 6.2 and 6.3 are equivalent.
2. The image $\alpha(M)$ is a basis of $M$, and this basis has the same active filtration/partition as $M$, which implies in particular

$$\text{Int}(\alpha(M)) = O^*(M),$$
$$\text{Ext}(\alpha(M)) = O(M).$$

3. The $2^{1+\epsilon}$ reorientations of $M$ in the activity class of $M$, which has dual-activity $\iota$ and activity $\epsilon$, and are mapped onto the same basis $\alpha(M)$.
4. The mapping $M \mapsto \alpha(M)$, applied to the set of reorientations of $M$, provides a surjection onto the set of bases, and a bijection between all activity classes of reorientations of $M$ and all bases of $M$.
5. In particular, we obtain the bijections listed in Table 3.
| activity classes of reorientations | bases | \( t(M; 1, 1) \) |
|-----------------------------------|-------|-----------------|
| act. classes of acyclic reorientations | internal bases | \( t(M; 1, 0) \) |
| act. classes of totally cyclic reorientations | external bases | \( t(M; 0, 1) \) |
| bounded reorientations w.r.t. \( \min(E) \) | uniactive internal bases | \( b_{1,0} = \beta(M) \) |
| dual-bounded reorientations w.r.t. \( \min(E) \) | uniactive external bases | \( b_{0,1} = \beta^*(M) \) |

Table 3: Canonical active bijection enumeration (the third column indicates the Tutte polynomial evaluation or coefficient that counts the involved objects).

**Definition 6.5.** Let \( M \) be an ordered oriented matroid on \( E \). The bijection between activity classes of reorientations of \( M \) and bases of \( M \) provided by the mapping \( M \mapsto \alpha(M) \) applied to reorientations of \( M \), is called the canonical active bijection of \( M \).

**Observation 6.6.** It is very important to observe that the canonical active bijection of \( M \) depends only on the reorientation class of \( M \) (in the sense that the canonical active bijection of \(-X \cdot M \) for \( X \subseteq E \) is isomorphic to that of \( M \) up to symmetric difference with \( X \)). In other words: \( \alpha(M) \) depends only on the resulting oriented matroid \(-A \cdot M \), not on \( M \) and \( A \). Equivalently, in terms of a pseudosphere arrangement representation of \( M \), the canonical active bijection depends only on the non-signed arrangement. More precisely: it is a bijection between signatures and bases of the non-signed arrangement, not depending on the choice of a reference signature. In other words: it depends only on the topology of the arrangement, not on an initial signature. In particular, activity classes of regions of the non-signed arrangement are in bijection with internal bases, and bounded regions w.r.t. \( \min(E) \) of the non-signed arrangement, on one side of \( \min(E) \), are in bijection with uniactive internal bases, and these bijections are independent of any signature of the arrangement.

**Observation 6.7.** As a direct consequence of Definition 6.3, we get the following result, which puts together (in terms of the active mapping) Observation 4.5 (in terms of active filtrations of oriented matroids) and Observation 3.5 (in terms of active filtrations of bases, coming from [18, Observation 4.23]).

Let \( \emptyset = F^0_\varepsilon \subset ... \subset F^*_0 = F_c = F_0 \subset ... \subset F_c = E \) be the active filtration of the ordered oriented matroid \( M \) (or equivalently of the basis \( \alpha(M) \) by Theorem 6.4). Let \( F \) and \( G \) be two subsets in this sequence such that \( F \subseteq G \). We have:

\[
\alpha(M) = \alpha(M(F)) \uplus \alpha(M(G)/F) \uplus \alpha(M/G).
\]

**Proof of Theorem 6.4.** Throughout the proof, we may call \( \alpha \) the mapping \( M \mapsto \alpha(M) \) applied to reorientations of \( M \). Definition 6.3 is properly defined. Let us use notations from this definition. By Theorem 4.6, we have that for all \( 1 \leq k \leq \iota \), \( M(F_k)/F_{k-1} \) is bounded, and for all \( 1 \leq k \leq \varepsilon \), \( M(F'_k)/F_k \) is dual-bounded. Then, by definitions in Section 5 in the case where \( \iota + \varepsilon = 1 \), we have that for all \( 1 \leq k \leq \iota \), \( \alpha(M(F_k)/F_{k-1}) \) is an uniactive internal basis of \( M(F_k)/F_{k-1} \), and for all \( 1 \leq k \leq \varepsilon \), \( \alpha(M(F'_k)/F_k) \) is an uniactive external basis of \( M(F'_k)/F_k \). Then, by Theorem 3.3, we have that \( \alpha(M) \) is a basis of \( M \) with the same active filtration as \( M \).

Since two opposite bounded, resp. dual-bounded, reorientations are mapped onto the same spanning tree by \( \alpha \) (obvious by the definitions), we directly have by Definition 4.9 that the \( 2^{\iota + \varepsilon} \) reorientations of \( M \) in the activity class of \( M \) are mapped onto the same basis \( \alpha(M) \).

Since \( \alpha \) provides a bijection between all pairs of opposite bounded, resp. dual-bounded, reorientations of \( M \) and uniactive internal, resp. external, bases of \( M \) by Theorem 5.3, then we directly
have by Theorem 3.3 and Theorem 4.6 that $\alpha$ provides a bijection between all activity classes of reorientations of $M$ and all bases of $M$.

Now let us prove that Definitions 6.1 and 6.2 are well-defined and equivalent to Definition 6.3. First, observe that Properties (F.o.b.) and (Duality) in Definition 6.1 are consistent with Definition 5.5 of $\alpha(M)$ when $M$ is dual-bounded w.r.t. $P$. Moreover, the variant of Property (F.o.b.) is consistent, since Properties (F.o.b.) and (Duality) put together define $\alpha$ in the bounded and the dual-bounded case, as well as Definition 5.5 and Property (Duality) put together. So both Definitions 6.1 and 6.2 are well-defined and consistent in the bounded and the dual-bounded cases.

Second, observe that the variant (Ind.$^*$) of Property (Induction) put together with Property (Duality), is consistent with Property (Induction) put together with Property (Duality). Indeed, they define the same properties for $M$ and $M^*$. Precisely: if $F = F_{i-1}$ is the complementary set of the union of all positive cocircuits of $M$ with smallest element $a$, then $E \setminus F$ is the union of all positive circuits of $M^*$ with smallest element $a$, and we have $M^*/(E \setminus F) = (M(F))^*$ and $M^*(E \setminus F) = (M/F)^*$. Then, definitions are consistent as we have simultaneously:

$$\alpha(M) = \alpha(M(F)) \uplus \alpha(M/F),$$
$$E \setminus \alpha(M) = (F \setminus \alpha(M(F))) \uplus ((E \setminus F) \setminus \alpha(M/F)), $$
$$\alpha(M^*) = \alpha((M(F))^*) \uplus \alpha((M/F)^*),$$
$$\alpha(M^*) = \alpha(M^*/(E \setminus F)) \uplus \alpha(M^*(E \setminus F)).$$

Now, let us show briefly why Definition 6.1 is well-defined, assuming $M$ is not bounded w.r.t. $p$ or not dual-bounded w.r.t. $p$. If $F \neq \emptyset$, defined in Property (Induction), then $M/F$ is bounded w.r.t. its smallest element, and $M(F)$ has one dual-active element less than $M$. If $F = \emptyset$ then we apply Property (Duality) and consider the dual $M^*$ (as above), then Property (Induction) yields a set $F^* \neq \emptyset$ such that $M(F^*)$ is dual-bounded w.r.t. its smallest element, and $M/F^*$ has one active element less than $M$. Also these two constructions can be used alternatively in any order.

More precisely, first, let us assume that $\varepsilon > 0$. Observe that the set $F$ considered in Property (Induction) is the set $F_{i-1}$ of the active filtration of $M$, by Definition 4.1. Observe that the active filtration of $M(F_{i-1})$ is $(F'_i, \ldots, F'_0, F_c, F_0, \ldots, F_{i-1})$ (this has been observed in Section 4, and it is direct from Theorem 4.6). Then, applying Definition 6.3 to $M(F_{i-1})$, we directly have that $\alpha(M) = \alpha(M(F_{i-1})) \uplus \alpha(M/F_{i-1})$, which is exactly Property (Induction) in Definition 6.1. Now, assume that $\varepsilon > 0$, then we use Property (Duality) in order to apply the same reasoning in the dual $M^*$, as explained above with Property (Ind.$^*$), and we get similarly that Definition 6.1 is equivalent to Definition 6.3. At the same time, we have proved that the definition provided by the variant (Ind.$^*$) is also well-defined and equivalent to Definition 6.3.

Finally, let us observe that the variant (Ind.$_{++}$) given in Definition 6.2, which allows to consider various possible sets $F$ instead of one set $F$ in Property (Induction), is consistent. Indeed, by definition of such a set $F$, there exists $k$, $0 \leq k \leq \varepsilon - 1$ such that $F = F_k$. Hence, the active filtration of $M(F)$ is $(F'_\varepsilon, \ldots, F'_0, F_c, F_0, \ldots, F_k)$ and the active filtration of $M/F$ is $(F_k \setminus F_k, F_k \setminus F_k, \ldots, E \setminus F_k)$ (as above, this has been observed in Section 4, and it is direct from Theorem 4.6). So we have $\alpha(M) = \alpha(M(F)) \uplus \alpha(M/F)$ by Definition 6.3 applied to $M(F)$ and $M/F$. So, the variant (Ind.$_{++}$) yields an equivalent definition of $\alpha$. Dually, we obtain that the variant (Ind.$_{++}$) also yields an equivalent definition of $\alpha$.

Let us illustrate the construction by continuing the running example of $K_4$. Figures 9, 10, 11, and 12 give the details of the sign pattern that characterize the active basis, on four different repre-
sentative situations. Observe that this sign pattern depends only on the fundamental graph/tableau of the basis. Figure 13 gives the complete canonical active bijection of $K_4$.

| $C_1^*$ | $C_3^*$ | 4 | 5 | $C_6^*$ |
|---------|---------|---|---|---------|
| 1       |         |   |   |         |
| 2       |         |   |   |         |
| 3       |         |   |   |         |
| 4       |         |   |   |         |
| 5       |         |   |   |         |
| 6       |         |   |   |         |

Figure 9: Signed fundamental graph/tableau of basis 136 with activities $(1,0)$ and active partition $E = 123456$. This figures continues [18, Figure 2], and repeats the right part of Figure 7, using the same sign representation except that useless signs are replaced with the ■ symbol. A uniactive basis of $M$ equals $\alpha(M)$ if and only if its signed fundamental graph/tableau satisfies the full optimality criterion, that is satisfies the sign pattern given in the figure (Section 5). Let us recall that the cases of $(1,0)$-active and $(0,1)$-active bases can be handled the same way, up to exchanging the role of $B$ and $E \setminus B$ in the fundamental graph, or up to transposing the fundamental tableau. Also, let us recall that, given a uniactive basis, and given its (non-signed) fundamental graph/tableau, one can build a reorientation of $M$ for which this basis is fully optimal by reorienting elements one by one so that the signed fundamental graph/tableau w.r.t. the reorientation satisfies the full optimality criterion (Propositions 5.4 and 5.8).

| $C_1^*$ | $C_2^*$ | 3 | 4 | 5 | $C_6^*$ |
|---------|---------|---|---|---|---------|
| 1       |         |   |   |   |         |
| 2       |         |   |   |   |         |
| 3       |         |   |   |   |         |
| 4       |         |   |   |   |         |
| 5       |         |   |   |   |         |
| 6       |         |   |   |   |         |

Figure 10: Signed fundamental graph/tableau of basis 126 with activities $(2,0)$ and active partition $E = 1 + 23456$. This figures continues [18, Figure 3] by adding the sign pattern that the basis must satisfy to be equal to $\alpha(M)$. Observe that this sign pattern depends only on the fundamental graph/tableau, not on the whole structure. The decomposition of the (fundamental graph/tableau of) the basis from [18] yields the active partition $E = 1 + 23456$ and the active minors $M(1)$ and $M/1$. Here the first minor consists of a single isthmus. Accordingly with the construction of this section, the basis equals $\alpha(M)$ when the two bases induced in the two minors $M(1)$ and $M/1$ satisfy the full optimality criterion, meaning that the subgraphs/subtableaux induced on 1 and on 23456 satisfy the same sign pattern as illustrated in Figure 9. In this figure and the next ones, for one part of the tableau, we use circled symbols such as $\oplus$, and for the other part we use boxed symbols such as $\Box$ with the same meanings as in Figure 7. The elements of fundamental circuits or cocircuits which disappear when restricting to a part (i.e. elements that do not belong to fundamental circuits or cocircuits induced in the two minors), are depicted by dashed edges in the fundamental graph, and by the symbol $\times$ in the fundamental tableau. Given a basis, and given its (non-signed) fundamental graph or tableau, one can build a reorientation of $M$ in the preimage of this basis by $\alpha$ by reorienting elements one by one so that the signed fundamental graph/tableau w.r.t. the reorientation corresponds to the pattern given in the figure (see Theorem 8.5 for a full statement).
Figure 11: Signed fundamental graph/tableau of basis 146 with activities \((1, 1)\) and active partition \(E = 135 + 246\). This figure continues [18, Figure 4] by adding the sign pattern that the basis must satisfy to be equal to \(\alpha(M)\). Comments on Figure 10 (and Figure 9) concern this figure as well, considering that we deal here with a \((1, 0)\)-active restriction of the fundamental graph/tableau to 135, yielding \(1 = \alpha(M/246)\), and a \((0, 1)\)-active restriction of the fundamental graph/tableau to 246, yielding \(46 = \alpha(M(246))\).

Figure 12: Signed fundamental graph/tableau of basis 256 with activities \((0, 2)\) and active partition \(E = 124 + 356\). This figure continues [18, Figure 5] by adding the sign pattern that the basis must satisfy to be equal to \(\alpha(M)\). Comments on Figure 10 (and Figure 9) concern this figure as well, considering that we deal here with a \((0, 1)\)-active restriction of the fundamental graph/tableau to 124, yielding \(2 = \alpha(M/356)\), and a \((0, 1)\)-active restriction of the fundamental graph/tableau to 356, yielding \(56 = \alpha(M(356))\).

Now, let us give two constructions of the inverse of the canonical active bijection, from bases to reorientation activity classes. The first simply consists in rephrasing Definition 6.3 in the inverse way. It can be combined with Proposition 5.4 to compute the preimages of involved uniaactive bases.

**Proposition 6.8** (inverse of Definition 6.3). Let \(M\) be an ordered oriented matroid on \(E\). Let \(B\) be a basis of \(M\), with active filtration \(\emptyset = F'_1 \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_i = E\). Let us denote \(\alpha^{-1}_M(B)\) the set of reorientations of \(M\) whose active basis is \(B\). Then,

\[
\alpha^{-1}_M(B) = \prod_{1 \leq k \leq i} \alpha^{-1}_{M(F_k)}(B \cap (F_k \setminus F_{k-1})) \prod_{1 \leq k \leq e} \alpha^{-1}_{M(F'_k)}(B \cap (F'_k \setminus F_k))
\]

where \(\times\) means that the \(2^{+e}\) resulting reorientations of \(M\) are inherited from the reorientations of the involved minors the natural way (each induced basis in the involved active minors of \(B\) is uniaactive and has 2 preimages).

The second construction of the inverse consists in a simple algorithm, by a single pass over \(E\) following the ordering, signing the arrangement element after element the suitable way to fit in the active basis criterion. Furthemore, it uses only the fundamental circuits and cocircuits of the basis, not the whole oriented matroid structure. It combines the computation of the active filtration (in
Figure 13: Geometrical representation of the entire canonical active bijection for $M = K_4$, continuing and completing the running example begun with Figures 2 and 3. Since the canonical active bijection depends only on the reorientation class, that is on the non-signed arrangement, we do not specify the signature of the arrangement. Internal, resp. external, bases, associated with acyclic, resp. totally cyclic, reorientations of $M$ are written in regions of the primal, resp. dual, arrangement. Active partitions are written in italic. The minimal basis 124 is written in bold. In the dual arrangement, cobases are written in brackets (corresponding to acyclic reorientations of $M^*$). Other bases are associated with combinations of acyclic orientations of $M/F$ and acyclic orientations of $M^*/(E \setminus F)$, for cyclic flats $\emptyset \subset F \subset E$, where $F = 123, 145, 246$ or 356. We represent these minors in grey boxes with their associated bases, and we indicate with dashed lines how these cyclic flats situate geometrically in the primal and dual arrangements. Details of the construction for bases 136, 126, 146, and 256 are given by Figures 9, 10, 11, and 12, respectively. This figure will be exhaustively completed in Section 9, by Figures 16, 17, 18.
fact the active partition, see Definition 3.4) of the basis from [18, Proposition 4.29], along with the computation, in each induced minor, of the bounded (or dual-bounded) reorientation associated with a uniactive internal (or external) basis from Proposition 5.4 (or Proposition 5.8), consistently with Definition 6.3.

**Theorem 6.9** (combination of Propositions 5.4 and 5.8 with [18, Proposition 4.29]). Let $M$ be a matroid on a linearly ordered set of elements $E = e_1 < ... < e_n$. Let $B$ be a basis of $M$. In the algorithm below, the active partition of $B$ is computed as a mapping, denoted $\text{Part}$, from $E$ to $\text{Int}(B) \cup \text{Ext}(B)$, that maps an element onto the smallest element of its part in the active partition of $B$. An element is called internal, resp. external, if its image is in $\text{Int}(B)$, resp. $\text{Ext}(B)$. The set of $2^{|\text{Int}(B)|} + |\text{Ext}(B)|$ reorientations formed by the preimages of $B$ under $\alpha$, denoted here $\alpha^{-1}(B)$, is computed by doing all possible arbitrary choices to reorient or not an element during the algorithm. Equivalently, those preimages under $\alpha$ can also be retrieved from one another since we have

$$\alpha^{-1}(B) = \{ A \triangle \text{Part}^{-1}(P \cup Q) \mid P \subseteq \text{Int}(B), Q \subseteq \text{Ext}(B), A \in \alpha^{-1}(B) \}.$$ 

The algorithm consists in a single pass over $E$ and only relies upon the fundamental graph/tableau of $B$. Note that the rules when $e_k \in B$ are dual to the rules when $e_k \notin B$.

**Input:** a basis $B$ of $M$

**Output:** a reorientation of $M$ in $\alpha^{-1}(B)$ (along with the active partition of $B$)

For $k$ from 1 to $n$

if $e_k \in B$ then

if $e_k$ is internally active w.r.t. $B$ then

$e_k$ is internal

let $\text{Part}(e_k) := e_k$

reorient $e_k$ or not, arbitrarily

otherwise

(active partition computation)

it there exists $c < e_k$ external in $C^*(B; e_k)$ then

$e_k$ is external

let $c \in C^*(B; e_k)$ with $c < e_k$, $c$ external and $\text{Part}(c)$ the greatest possible

let $\text{Part}(e_k) := \text{Part}(c)$

otherwise

$e_k$ is internal

let $c \in C^*(B; e_k)$ with $c < e_k$ and $\text{Part}(c)$ the smallest possible

let $\text{Part}(e_k) := \text{Part}(c)$

(reorientation computation)

let $a$ be the smallest possible in $C^*(B; e_k)$ with $\text{Part}(a) = \text{Part}(e_k)$

reorient $e_k$ if necessary so that $e_k$ and $a$ have opposite signs in $C^*(B; e_k)$

if $e_k \notin B$ then

if $e_k$ is externally active w.r.t. $B$ then

$e_k$ is external

$\text{Part}(e_k) := e_k$

reorient $e_k$ or not, arbitrarily

or with the same sign as in $M$ if and only if $e_k \notin Q$ (to compute $\alpha_M^{-1}(X)$)

otherwise


(active partition computation)
if there exists \( c < e_k \) internal in \( C(B; e_k) \) then
\[ e_k \text{ is internal} \]
let \( c \in C(B; e_k) \) with \( c < e_k \), \( c \) internal and \( \text{Part}(c) \) the greatest possible
let \( \text{Part}(e_k) := \text{Part}(c) \)
otherwise
\[ e_k \text{ is external} \]
let \( c \in C(B; e_k) \) with \( c < e_k \) and \( \text{Part}(c) \) the smallest possible
let \( \text{Part}(e_k) := \text{Part}(c) \)
(reorientation computation)
let \( a \) be the smallest possible in \( C(B; e_k) \) with \( \text{Part}(a) = \text{Part}(e_k) \)
reorient \( e_k \) if necessary so that \( e_k \) and \( a \) have opposite signs in \( C(B; e_k) \)

\[ \text{Proof.} \] The computation of the active partition of \( B \), that is of the mapping \( \text{Part} \), is exactly the algorithm given in [18, Proposition 4.29]. By Definition 6.3, and since \( \alpha \) preserves active filtrations by Theorem 6.4, the reorientations associated with \( B \) are obtained from the reorientations of the minors induced by the active filtration of \( B \).

If \( e_k \) is internally or externally active, then it can be obviously reoriented or not, arbitrarily, as the reorientation of the associated minor in the active filtration of \( B \) will follow this initial reorientation. Each of these minors will get two possible opposite reorientations, yielding the reorientation activity class of \( M \) associated with \( B \).

Now, assume that \( e_k \in B \) is not internally active, and let \( a \) be the smallest possible in \( C^*(B; e_k) \) with \( \text{Part}(a) = \text{Part}(e_k) \). Let \( M' \) be the minor of \( M \) associated with the active filtration of \( B \) whose ground set is \( \text{Part}^{-1}(\text{Part}(a)) \). By Definition 3.4 of the active filtration of \( B \), we have \( M' = M(G)/F \) for some \( F \subseteq G \subseteq E \), and \( B' = B \cap (G \setminus F) \) is a uniaactive internal or external basis of \( M' \). Since \( B' \) is a basis of \( M' \), we have \( C^*_M(B'; e_k) = C^*_M(B; e_k) \cap (G \setminus F) \) (this is an easy matroid result, details are given in [18, Property 4.2]). So, by definition of \( a \), we have \( a = \min(C^*_M(B'; e_k)) \). This implies \( a \neq e_k \), otherwise \( e_k \) would be internally active in the basis \( B' \) of \( M' \), and hence internally active in the basis \( B \) of \( M \) (by properties of the active filtration of \( B \)), which has been forbidden by assumption. By Proposition 5.4 or Proposition 5.8, the bounded or dual-bounded reorientation of \( M' \) associated with \( B' \) by \( \alpha \) is obtained by reorienting if necessary \( e_k \) so that \( e_k \) and \( a \) have opposite signs in \( C^*_M(B'; e_k) \), that is so that \( e_k \) and \( a \) have opposite signs in \( C^*_M(B; e_k) \), which is exactly the algorithm statement.

The case where \( e_k \notin B \) is dual to the above case. Hence the algorithm is correct. \( \square \)

Before ending this section, let us mention that there exist a construction of the canonical active bijection by deletion/contraction, that builds the whole bijection in a global way from the bijections in \( M \setminus \omega \) and \( M/\omega \), where \( \omega \) is the greatest element. Actually, one can build by deletion/contraction various classes of activity preserving correspondences between reorientations and bases, among which the canonical active bijection is uniquely and canonically determined. These features are developed in [20] (see also [21, Section 6]). The next final remark shows that the decomposition technique used in this section yields another whole class of activity preserving (actually active partition preserving) correspondences between reorientations and bases, among which the canonical active bijection is also uniquely and canonically determined.
Remark 6.10 (General decomposition framework for classes of activity preserving bijections). Let us observe that the construction by decomposition of this section is rather independent of the construction of Section 5. In the construction of this section, and in particular in Definition 6.3, one can replace everywhere the notation $\alpha$ with any mapping $\psi$ from reorientations to bases, assuming it is preliminary defined for ordered bounded/dual-bounded oriented matroids and provides a bijection with uniactive internal/external bases. Then Theorem 6.4 can be directly extended, relying upon the two decompositions of Theorems 3.3 and 4.6, and this directly yields a whole class of activity preserving (and active partition preserving) bijections between reorientation activity classes and bases, without any other change needed (and from this one can also derive a whole class of refined bijections between reorientations and subsets as in the next section, see also Remark 8.6). We call this class of mappings $\psi$ the active partition preserving mapping class of $M$. See more details in [21, Section 4.4] or [20].

7. Partitions of $2^E$ into basis intervals and reorientation activity classes, and Tutte polynomial in terms of four activity parameters for subsets and for reorientations

In this section, we address results involving partitions of the power set of the ground set into boolean lattices, first from basis intervals of an ordered matroid, second from reorientation activity classes of an ordered oriented matroid. We also address related Tutte polynomial expressions in terms of refined activity parameters, first in terms of basis/subset activities, and second in terms of reorientation activities. Those partitions and expressions for bases/subsets and for reorientations are presented in this section independently of each other. They are presented in a suitable similar way so that they will be easily related to each other by the refined active bijection in the next Section 8: the canonical active bijection of the previous section can be seen as a bijection between boolean lattices of the two types, and, then, the elements of these isomorphic boolean lattices are in bijection by the refined active bijection, preserving those refined activity parameters, and thus transforming the formulas of Theorems 7.2 and 7.4 below into each other.

First, let $M$ be a matroid on a linearly ordered set $E$.

We recall the classical partition of the power set of the ground set into intervals (for inclusion), each associated with one basis with respect to basis activities, as discovered by Crapo in [4] (let us mention distinct further generalizations of this result beyond matroids: by Dawson [6] to set families, by Gordon and McMahon [23] to greedoids, by Las Vergnas [31] to matroid perspectives). Then, basis activities can be extended to subsets in such a way that four subset activities indicate the position of a subset inside its interval with respect to the associated basis (among several possible ways of understanding these subset activities, this is the viewpoint we introduce for the sake of further constructions). This directly yields a Tutte polynomial formula in terms of these four refined subset activities, as expressed by Las Vergnas [31], which is essentially a specification of a more general formula in terms of generalized activities originally discovered by Gordon and Traldi [24]. Numerous Tutte polynomial formulas can be directly derived from this general four parameter formula by specifying variables, see [24, 31]. This formula is generalized to matroid perspectives in [31] (completed in [9]). A summary about these notions can also be found in [10]. Here, we follow the notations used in [31]. We also give a very short proof of this formula, which highlights how it is the enumerative counterpart of the partition into intervals (as done in [9]).

Let $B$ be a basis of $M$. The set of subsets of $E$ containing $B \setminus \text{Int}(B)$ and contained in $B \cup \text{Ext}(B)$ will be called the interval of $B$, denoted $[B \setminus \text{Int}(B), B \cup \text{Ext}(B)]$. These sets considered for all
bases form a partition of $2^E$:

$$2^E = \bigcup_{B \text{ basis of } M} \left[ B \setminus \text{Int}(B), \ B \cup \text{Ext}(B) \right].$$

Observe that the interval of $B$ has a boolean lattice structure and can also be denoted:

$$\left[ B \setminus \text{Int}(B), \ B \cup \text{Ext}(B) \right] = \{ B \Delta (P \cup Q) \mid P \subseteq \text{Int}_M(B), \ Q \subseteq \text{Ext}_M(B) \}.$$

**Definition 7.1 ([31]).** Let $M$ be a matroid on a linearly ordered set $E$. Let $B$ be a base of $M$. Let $A$ be in the boolean interval $[B \setminus \text{Int}_M(B), B \cup \text{Ext}_M(B)]$. We denote:

$$\begin{align*}
\text{Ext}_M(A) &= \text{Ext}_M(B) \setminus A; \\
Q_M(A) &= \text{Ext}_M(B) \cap A; \\
\text{Int}_M(A) &= \text{Int}_M(B) \cap A; \\
P_M(A) &= \text{Int}_M(B) \setminus A.
\end{align*}$$

Let us mention that these four parameters can be defined directly from $A$ without using $B$. In particular, $Q_M(A)$, resp. $P_M(A)$, counts smallest elements of circuits, resp. cocircuits, contained in $A$, resp. $E \setminus A$. In particular, $Q_M(A)$, resp. $P_M(A)$, counts smallest elements of circuits, resp. cocircuits, contained in $A$, resp. $E \setminus A$. This yields $|P_M(A)| = r(M) - r(A)$ and $|Q_M(A)| = |A| - r_M(A)$ (which do not depend on the associated base).

**Theorem 7.2 ([24, 31]).** Let $M$ be a matroid on a linearly ordered set $E$. We have:

$$t(M; x + u, y + v) = \sum_{A \subseteq E} x^{\text{Int}_M(A)} (x + u)^{\text{Int}_M(B)} (y + v)^{\text{Ext}_M(B)} u^{|P_M(A)|} v^{|Q_M(A)|}.$$  

**Proof.** By the expression “enumeration of basis activities”, we have:

$$t(M; x + u, y + v) = \sum_{B \text{ basis}} (x + u)^{\text{Int}_M(B)} (y + v)^{\text{Ext}_M(B)}.$$  

By the binomial formula, this expression equals:

$$\sum_{B \text{ basis}} \left( \sum_{A' \subseteq \text{Int}_M(B)} x^{|A'|} (x + u)^{|\text{Int}_M(B) \setminus A'|} \right) \left( \sum_{A'' \subseteq \text{Ext}_M(B)} y^{|A''|} v^{|\text{Ext}_M(B) \setminus A''|} \right).$$

Since $\text{Int}_M(B) \cap \text{Ext}_M(B) = \emptyset$, one has a bijection between couples $(A', A'')$ involved in this expression and subsets $A = A' \cup A''$ of $\text{Int}_M(B) \cup \text{Ext}_M(B)$, hence this expression equals:

$$\sum_{B \text{ basis}} \left( \sum_{A \subseteq \text{Int}_M(B) \cup \text{Ext}_M(B)} x^{|\text{Int}_M(B) \cap A|} (x + u)^{|\text{Int}_M(B) \setminus A|} y^{|\text{Ext}_M(B) \setminus A|} v^{|\text{Ext}_M(B) \cap A|} \right).$$

Since $\left( B \setminus \text{Int}_M(B) \right) \cap \left( \text{Int}_M(B) \cup \text{Ext}_M(B) \right) = \emptyset$, the mapping $A \mapsto A \cup \left( B \setminus \text{Int}_M(B) \right)$ yields an isomorphism between the two boolean intervals $[\emptyset, \text{Int}_M(B) \cup \text{Ext}_M(B)]$ and $[B \setminus \text{Int}_M(B), B \cup \text{Ext}_M(B)]$, which does not change the sets $\text{Int}_M(B) \cap A$, $\text{Int}_M(B) \setminus A$, $\text{Ext}_M(B) \setminus A$, and $\text{Ext}_M(B) \cap A$. So the above expression can be equivalently written:

$$\sum_{B \text{ basis}} \left( \sum_{A \subseteq [B \setminus \text{Int}_M(B), B \cup \text{Ext}_M(B)]} x^{|\text{Int}_M(B) \cap A|} (x + u)^{|\text{Int}_M(B) \setminus A|} y^{|\text{Ext}_M(B) \setminus A|} v^{|\text{Ext}_M(B) \cap A|} \right).$$

Since $2^E = \bigcup_{B \text{ basis}} [B \setminus \text{Int}_M(B), B \cup \text{Ext}_M(B)]$, this expression equals:

$$\sum_{A \subseteq E} x^{|\text{Int}_M(B) \cap A|} (x + u)^{|\text{Int}_M(B) \setminus A|} y^{|\text{Ext}_M(B) \setminus A|} v^{|\text{Ext}_M(B) \cap A|}.$$  

Finally, by Definition 7.1, this expression equals the required one. □
Second, let $M$ be an oriented matroid on a linearly ordered set $E$.

We build on the partition of the power set of the ground set into activity classes of reorientations, introduced in Definition 4.9, and on their boolean lattice structure. We can naturally define four reorientation activity parameters that indicate the position of a reorientation inside its activity class. We obtain a short proof of a simple expression of the Tutte polynomial using these four reorientation activity parameters (Theorem 7.4). Let us mention that this result generalizes to oriented matroid perspectives: it was proposed with a rather technical proof in [30], and it is shortly proved in terms of activity classes of oriented matroid perspectives in [9] in a similar way as in the present paper.

Let us fix a reorientation $-A M$ of $M$. The active partition of $-A M$ (Definition 4.1) can be denoted as:

$$E = \biguplus_{a \in O(-A M) \cup O^*(-A M)} A_a$$

where $a = \min(A_a)$ for all $a \in O(-A M) \cup O^*(-A M)$. Then the activity class $cl(-A M)$ of $-A M$ (Definition 4.9) can be denoted the following way, highlighting its boolean lattice structure:

$$cl(-A M) = \left\{ -A' M \mid A' = A \triangle \left( \biguplus_{a \in P \cup Q} A_a \right) \text{ for } P \subseteq O^*(-A M), \ Q \subseteq O(-A M) \right\}.$$

As addressed in Section 3, activity classes of reorientations of $M$ form a partition of the set of reorientations of $M$:

$$2^E \sim \biguplus_{\text{one } -A M \text{ chosen in each activity class}} cl(-A M).$$

**Definition 7.3.** Let $M$ be an ordered oriented matroid. We define:

$$\Theta_M(A) = O(-A M) \setminus A,$$

$$\Theta_M^*(A) = O(-A M) \cap A,$$

$$\Theta_M^*(A) = O^*(-A M) \setminus A,$$

$$\Theta_M^*(A) = O^*(-A M) \cap A.$$

Hence we have $O(-A M) = \Theta_M(A) \uplus \Theta_M^*(A)$ and (dually) $O^*(-A M) = \Theta_M^*(A) \uplus \Theta_M^*(A)$.

In contrast with the definition of the activity and dual activity of a reorientation of $M$, that depends only on the resulting oriented matroid $-A M$, the definition above depends on $A$ and $M$. By this way, it refines reorientation activities into four parameters that substantially apply to reorientations of a given reference oriented matroid $M$. These parameters actually situate any reorientation in its activity class (which is independent of the reference oriented matroid, see Observation 6.6).

Precisely, consider an activity class of reorientations of $M$, and the representative $-A M$ of this class which is active-fixed and dual-active fixed with respect to the reference oriented matroid $M$ (Corollary 4.10). By definition, it satisfies:

$$\Theta_M(A) = O(-A M) \cap A = \emptyset,$$

$$\Theta_M^*(A) = O^*(-A M) \cap A = \emptyset.$$

Furthermore, the other reorientations $-A' M$ in the same activity class correspond to other possible values of $\Theta_M(A') \subseteq O(-A M)$ and $\Theta_M^*(A') \subseteq O^*(-A M)$. Using the above notation for the activity class $cl(-A M)$ as a boolean lattice, we have
\[ Q = \overline{\Theta}_M(A') \subseteq O(-A M), \]
\[ P = \overline{\Theta}_M^*(A') \subseteq O^*(-A M). \]

A way of understanding the role of the reference oriented matroid \( M \) is that it breaks the symmetry in each activity class, so that its boolean lattice structure can be expressed relatively to the aforementioned representative. This representative is noticeably used in Section 8. Other choices are possible for a representative.

Finally, we derive the following Tutte polynomial expansion formula in terms of these four parameters. A (technical) proof for Theorem 7.4 below is proposed in the preprint [30] by deletion/contraction in the more general setting of oriented matroid perspectives. This theorem can also be directly proved by means of the above construction on activity classes (as announced in [30]\(^6\), this theorem can also be seen as a direct corollary of the similar formula for subset activities from Theorem 7.2 and the refined active bijection from Theorem 8.2). We give this short proof below for completeness of the paper, though it is a translation of the proof given in [9] for oriented matroid perspectives.

**Theorem 7.4.** Let \( M \) be an oriented matroid on a linearly ordered set \( E \). We have
\[
t(M; x + u, y + v) = \sum_{A \subseteq E} x^{\Theta_M^*(A)} u^{\overline{\Theta}_M^*(A)} y^{\Theta_M(A)} v^{\overline{\Theta}_M(A)}.
\]

**Proof.** The proof is obtained by a simple combinatorial transformation. Let us start with the right-hand side of the equality, where we denote \( \theta_M^*(A) \) instead of \( |\Theta_M^*(A)| \), etc., by setting:
\[
[Exp] = \sum_{A \subseteq E} x^{\theta_M^*(A)} u^{\overline{\theta}_M^*(A)} y^{\theta_M(A)} v^{\overline{\theta}_M(A)}.
\]

Since \( 2^E \) is isomorphic to the set of reorientations, which is partitioned into activity classes of reorientations of \( M \) (Definition 4.9), and by choosing a representative for each activity class which is active-fixed an dual-active-fixed (as discussed above), we get:
\[
[Exp] = \sum_{\text{activity classes } cl(-A M) \text{ of reorientations of } M} \sum_{-A' M \in cl(-A M)} x^{\theta_M^*(A')} u^{\overline{\theta}_M^*(A')} y^{\theta_M(A')} v^{\overline{\theta}_M(A')}
\]

As discussed above, when \( -A' M \) ranges the activity class of \( -A M \), \( \overline{\Theta}_M(A') \) and \( \overline{\Theta}_M^*(A') \) range subsets of \( O(-A M) \) and \( O^*(-A M) \), respectively. So, we get the following expression (where “idem” refers to the text below the first above sum), which we then transform using the binomial formula:
\[
[Exp] = \sum_{\text{idem}} \sum_{P \subseteq O^*(-A M)} \sum_{Q \subseteq O(-A M)} x^{O^*(-A M) \setminus P} y^{O(-A M) \setminus Q} |P| |Q|
\]
\[
= \sum_{\text{idem}} \left( \sum_{P \subseteq O^*(-A M)} x^{O^*(-A M) \setminus P} \right) \left( \sum_{Q \subseteq O(-A M)} y^{O(-A M) \setminus Q} \right)
\]

\( ^6\)See footnote 7 in Section 8 for a correction on this announce as written in the preprint [30].
\[ \sum_{\text{idem}} (x + u)^{|O^*(\neg A^M)|} (y + v)^{|O(\neg A^M)|} \]

Since the activity class of \(-A^M\) has \(2^{|O(\neg A^M)|+|O^*(\neg A^M)|}\) elements with the same orientation activities, we have (denoting for short \(o(A) = |O(\neg A^M)|\) and \(o^*(A) = |O(\neg A^M)|\)):

\[ [Exp] = \sum_{\text{idem}} \frac{1}{2^{o(A)+o^*(A)}} \sum_{-A^M \in cl(\neg A^M)} (x + u)^{o^*(A')} (y + v)^{o(A')} \]
\[ = \sum_{\text{idem}} \sum_{-A^M \in cl(\neg A^M)} \left( \frac{x + u}{2} \right)^{o^*(A')} \left( \frac{y + v}{2} \right)^{o(A')} \]
\[ = \sum_{A \subseteq E} \left( \frac{x + u}{2} \right)^{o(A)} \left( \frac{y + v}{2} \right)^{o(A)} \]
\[ = t(G; x + u, y + v) \]

using at the end the “enumeration of reorientation activities” from [28] recalled in Section 2.

Obsere that Theorem 7.4 provides a proof of the enumerations of activity classes and their representatives from Corollary 4.10 and Table 2. Finally, let us mention that numerous Tutte polynomial formulas can be directly obtained from Theorem 7.4, for instance by replacing variables \((x, u, y, v)\) with \((x/2, x/2, y/2, y/2)\), or \((x+1, -1, y+1, -1)\), or \((2, 0, 0, 0)\), etc., as well as expressions for derivatives of the Tutte polynomial. These formulas are given in [9, 30] (see also [11], and see [30] for a detailed example).

8. The refined active bijection between reorientations and subsets

The present construction is a natural development of the canonical active bijection (sketchily introduced in [8, 14, 15]). Let us consider an ordered oriented matroid \(M\) and its active basis \(B = \alpha(M)\). On one hand, the activity class of \(M\) (Definition 4.9 and Section 7) obviously has a boolean lattice structure isomorphic to the power set of \(O(M) \cup O^*(M)\). On the other hand, the interval \([B \setminus \text{Int}(B), B \cup \text{Ext}(B)]\) of \(B\) (Section 7) also has a boolean lattice structure isomorphic to the power set of \(\text{Int}(B) \cup \text{Ext}(B)\). Since we have \(\text{Int}(B) \cup \text{Ext}(B) = O(M) \cup O^*(M)\) by properties of \(\alpha(M)\) (Theorem 6.4), those two boolean lattices are isomorphic. Furthermore, activity classes of reorientations of \(M\) form a partition of the set of reorientations of \(M\) (Definition 4.9), intervals of bases form a partition of the power set of \(E\) (Section 7), and activity classes of orientations are in bijection with bases under \(M \mapsto \alpha(M)\) (Theorem 6.4). Hence, selecting a boolean lattice isomorphism for each couple formed by an activity class and its active basis directly yields a bijection between all reorientations and all subsets of \(E\), which refines the canonical active bijection of \(M\), and transforms activity classes of reorientations into intervals of bases. The most natural way to select such isomorphisms (see also Remark 8.6 for variants) is to use the oriented matroid \(M\) as a reference, whose role is to “break the symmetry” in activity classes, just as in Section 7. See Figure 14 for an illustration. By this way, we shall obtain below the refined active bijection \(\alpha_M\) of \(M\), which relates the refined activities for reorientations and for subsets from Definitions 7.1 and 7.3 (as announced in [30]), giving a bijective transformation between the formulas of Theorems

\(^7\) Beware that the definition for the refined active bijection proposed at the very end of the unpublished preprint [30] in terms of the active bijection is not correct: it is not complete, and given with a wrong parameter correspondence. It is different from the present one, which is consistent with the one given in [8, 14, 15].
7.2 and 7.4:

\[ T(M; x + u, y + v) = \sum_{A \subseteq E} x^{\lvert \text{Int}_M(A) \rvert} u^{\lvert \text{P}_M(A) \rvert} y^{\lvert \text{Ext}_M(A) \rvert} v^{\lvert \Sigma_M(A) \rvert} \]

\[ = \sum_{A \subseteq E} x^{\lvert \Theta_M^*(A) \rvert} u^{\lvert \Theta_M^*(A) \rvert} y^{\lvert \Theta_M(A) \rvert} v^{\lvert \Theta_M(A) \rvert}. \]

Let us insist that, in contrast with the canonical active bijection, which depends only on the reorientation class since the active basis is intrinsically defined for an ordered oriented matroid (see Observation 6.6), the refined active bijection depends on (or is induced by the choice of) a given reference oriented matroid (that is, a reference signature in terms of a topological representation).

Figure 14: Boolean lattice isomorphism between an activity class of reorientations and the interval of the corresponding basis, figured for the activity class from Figure 4 with active partition 123 + 456 and active basis \( B = 134 \) with \( [B \setminus \text{Int}(B), B] = [3, 134] \). The layout reflects the bijection. Edges written below the graphs in the middle are those removed from \( B \), they correspond to reoriented parts in the digraphs on the left and on the regions on the right (brackets refer to the opposite regions in the opposite half of the arrangement). The reference oriented matroid \( M \) is given by any signature/orientation such that the reorientation associated with \( B \) is active-fixed and dual-active fixed w.r.t. \( M \) (e.g., simply the region/digraph associated with \( B \)).

Technically, let us take up the notations and discussion of Section 7. Let \( M \) be an oriented matroid on a linearly ordered set \( E \), thought of as the 

reference oriented matroid. Let \( A \subseteq E \). The active partition of \( -A M \) can be denoted as:

\[ E = \bigoplus_{a \in O(-A M) \cup O^*(-A M)} A_a \]

where the index of each part is the smallest element of the part. The activity class of \( -A M \) is:

\[ \text{cl}(-A M) = \left\{ -A' M \mid A' = A \bigtriangleup \left( \bigoplus_{a \in P \cup Q} A_a \right) \text{ for } P \subseteq O^*(-A M), \ Q \subseteq O(-A M) \right\}. \]

Let \( B = \alpha(-A M) \) be the active basis of \( -A M \). The interval of \( B \) can be also denoted:

\[ [B \setminus \text{Int}(B), B \cup \text{Ext}(B)] = \left\{ B' \subseteq E \mid B' = B \bigtriangleup \left( \bigoplus_{a \in P \cup Q} \{a\} \right) \text{ for } P \subseteq \text{Int}(B), \ Q \subseteq \text{Ext}(B) \right\}. \]
The above notations emphasize the two boolean lattice structures. Then, we define an isomorphism between the two by choosing that the representative of the activity class which is active-fixed and dual-active fixed w.r.t. $M$ is associated with the basis $B$. Assume $-A M$ is the representative of its class with these properties, then we formally have: $\Theta_M (A) = \cap (\Theta_M (A) = O^* (\cap) \cup \cap (A \cap O^* (A) \cup = \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup$.

\[
\begin{align*}
P &= \Theta_M (A') = P_M (B') \subseteq O^* (\cap) &= \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup
Q &= \Theta_M (A') = Q_M (B') \subseteq O^* (\cap) &= \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup
\end{align*}
\]

By this way, we naturally obtain the following definition and theorem.

**Definition 8.1.** Let $M$ be an oriented matroid on a linearly ordered set $E$. For $A \subseteq E$, we define

$$
\alpha_M (A) = \alpha(-A M) \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup

That is: $\alpha_M (A) = B \setminus P \cup Q$ with $B = \alpha(-A M)$, $P = A \cap \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup$, and $Q = A \cap \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup$.

The mapping $A \mapsto \alpha_M (A)$ from $2^E$ to $2^E$ is called the refined active bijection of $M$.

**Theorem 8.2.** Let $M$ be an oriented matroid on a linearly ordered set $E$. We have the following.

- The mapping $A \mapsto \alpha_M (A)$ from $2^E$ to $2^E$ is a bijection. It yields a bijection between reorientations $-A M$ of $M$ and subsets of $E$, which maps activity classes of $M$ onto intervals of bases of $M$ (and these restrictions are boolean lattice isomorphisms).

- For all $A \subseteq E$, with $B = \alpha(-A M)$ and $\alpha_M (A) = B \setminus P \cup Q$, we have:

\[
\begin{align*}
\cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup &= \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup
\cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup &= \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup
\cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup &= \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup
\cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup &= \cap (A \cap O^* (A) \cup \cap (A \cap O^* (A) \cup
\end{align*}
\]

- In particular, $\alpha_M (A)$ equals the active basis $\alpha(-A M)$ if and only if $-A M$ is active fixed and dual-active fixed w.r.t. $M$. Similarly, restrictions of the mapping $\alpha_M$ yield the bijections listed in Table 4.

**Proof.** The first point comes directly from Definition 8.1 and the above discussion. The second point also easily comes from this discussion. Let us precisely check the equalities of parameters in the second point anyway. In order to simplify notations, we omit subscripts $M$ of activity parameters. Let $A_B$ be the reorientation of $M$ whose image under $\alpha_M$ is the base $B$. Let $E = \cup_{A \subseteq O^* (-A M)} A_B$, with $a = \min (A_B)$, be the active partition associated with $B$ or $-A B M$.

Let $\alpha_M$ be a subset in the associated activity class, we have $A = A_B \Delta \cap (A_B \cap Q a)$ for some $P \subseteq \cap (B) = O^* (-A B M) = O^* (-A M)$ and $Q \subseteq \cap (B) = O(-A B M) = O(-A M)$ with $P \cap A_B = \emptyset$ and $Q \cap A_B = \emptyset$. By Definition 8.1, we have $\alpha_M (A) = B \setminus P \cup Q$.
reorientations | subsets | \( t(M; 2, 2) \)
acyclic reorientations | subsets of internal bases | \( t(M; 2, 0) \)
totally cyclic reorientations | suppersets of external bases | \( t(M; 0, 2) \)
dual-active-fixed acyclic reorientations | internal bases | \( t(M; 1, 0) \)
active-fixed totally cyclic reorientations | external bases | \( t(M; 0, 1) \)
active-fixed reorientations | subsets of bases | \( t(M; 2, 1) \)
dual-active-fixed reorientations | suppersets of bases | \( t(M; 1, 2) \)
active-fixed and dual-active-fixed reorientations | bases | \( t(M; 1, 1) \)

Table 4: Remarkable restrictions of the refined active bijection of \( M \), between particular types of reorientations (first column) and particular types of edge subsets (second column) enumerated by Tutte polynomial evaluations (third column). See Theorem 8.2.

By Definition 7.1, we have \( \text{Int}(\alpha_M(A)) = \text{Int}(B) \cap \alpha_M(A) \). We have \( \text{Int}(B) \cap \alpha_M(A) = \text{Int}(B) \cap (B \setminus P \cup Q) = \text{Int}(B) \setminus P \). By Theorem 6.4, we have \( \text{Int}(B) \setminus P = O^*((-A)M) \setminus P \). By properties of \( P \), we have \( O^*((-A)M) \setminus P = O^*((-A)M) \setminus (AB\Delta(\cup_{a\in P\cup Q}A_a)) = O^*((-A)M) \setminus A \). By Definition 7.3, we have \( O^*(-A)M \setminus A = \Theta^*(A) \). So finally \( \text{Int}_M(\alpha_M(A)) = \Theta^*(A) \).

On one hand, by Definition 7.1, we have \( \text{Int}(\alpha_M(A) \cup P(\alpha_M(A))) = \text{Int}(B) \). On the other hand, by Definition 7.3, we have \( \Theta^*(A) \cup \Theta^*(A) = O^*(-A)M \). By Theorem 6.4, we have \( \text{Int}(B) = O^*(-A)M \), so, by the above result, we get \( P(\alpha_M(A)) = \Theta^*(A) \).

Similarly, by Definition 7.1, we have \( \text{Ext}(\alpha_M(A)) = \text{Ext}(B) \setminus \alpha_M(A) \). We have \( \text{Ext}(B) \setminus \alpha_M(A) = \text{Ext}(B) \setminus (B \setminus P \cup Q) = \text{Ext}(B) \setminus Q \). By Theorem 6.4, we have \( \text{Ext}(B) \setminus Q = O^*(-A)M \setminus Q \). As above, by properties of \( Q \), we have \( O^*(-A)M \setminus Q = O^*(-A)M \setminus (AB\Delta(\cup_{a\in P\cup Q}A_a)) = O^*(-A)M \setminus A \). As above, by Definition 7.3, we have \( O^*(-A)M \setminus A = \Theta(A) \). So finally \( \text{Ext}(\alpha_M(A)) = \Theta(A) \). And, as above, we deduce that \( Q(\alpha_M(A)) = \Theta(A) \).

Now, let us consider the list of bijections of the third point. They are all obtained as restrictions of \( \alpha_M \). Observe that a reorientation is active-fixed, resp. dual-active-fixed, if it is obtained by \( Q = \emptyset \), resp. \( P = \emptyset \). Therefore, all these bijections are obvious by the definitions, except the two ones involving \( t(M; 1, 2) \) and \( t(M; 2, 1) \). For the first one, resp. second one, of these two, we can use that subets, resp. supersets, of bases are exactly the subsets of type \( B \setminus P \), resp. \( B \cup Q \), for some base \( B \) and \( P \subseteq \text{Int}(B) \), resp. \( Q \subseteq \text{Ext}(B) \). This result is stated separately in Lemma 8.3 below.

**Lemma 8.3.** Let \( M \) be an ordered matroid. The set of subsets of bases of \( M \) (i.e. independents) is the union of intervals \([B \setminus \text{Int}(M(B)), B]\) over all bases \( B \) of \( M \). The set of supersets of bases of \( M \) (i.e. spanning subsets) is the union of intervals \([B, B \cup \text{Ext}_M(B)]\) over all bases \( B \) of \( M \).

**Proof.** It is known that bases \( B \) of \( M \) are exactly subsets of the form \( B_i \cup B_e \) where \( B_i \) is an internal base of \( M/F \), \( B_e \) is an external base of \( M(F) \), and \( F \) is a cyclic flat of \( M \) (see details and references in [18, Corollary 4.27]). Moreover \( \text{Int}(B) = \text{Int}_{M/F}(B_i) \) and \( \text{Ext}(B) = \text{Ext}_{M(F)}(B_e) \) (for short, we omit these subscripts below).
We have \([B \setminus \text{Int}(B), B \cup \text{Ext}(B)] = [(B_e \cup B_e) \setminus \text{Int}(B), (B_e \cup B_e) \cup \text{Ext}(B)]\). Using the classical partition of \(2^E\) into basis intervals recalled Section 7 we have:

\[
2^E = \biguplus_{B \text{ base}} [B \setminus \text{Int}(B), B \cup \text{Ext}(B)] = \biguplus_{F, B_i, B_e \text{ as above}} \left[ [B_i \setminus \text{Int}(B_i), B_i] \times [B_e, B_e \cup \text{Ext}(B_e)] \right]
\]

(where \(\times\) yields all unions of a subset of the first set and a subset of the second set). So we have

\[
\biguplus_{B \text{ base}} [B \setminus \text{Int}(B), B] = \biguplus_{F, B_i, B_e \text{ as above}} \left[ [B_i \setminus \text{Int}(B_i), B_i] \times [B_e] \right]
\]

The size of the second set of the equality equals \(\sum_F t(M/F; 2,0) t(M(F); 0,1)\) by classical evaluations of the Tutte polynomial. And this number is known to be equal to \(t(M; 2,1)\) (convolution formula for the Tutte polynomial, see details and references in [18, Corollary 3.6]), which equals the number of subsets of bases (as well known). The first set of the equality is included in the set of subsets of bases, and it has the same size, hence it equals the set of subsets of bases. Dually, we get the result involving supersets of bases, whose number equals \(t(M; 2,1)\).

Now, let us give two results for building the inverse of the refined active bijection, from subsets to reorientations. They are directly obtained from the inverse constructions of the canonical active bijection. The first specifies Proposition 6.8. The second is an immediate adaptation of the single pass algorithm of Theorem 6.9.

**Proposition 8.4** (refined active bijection from subsets). Let \(M\) be an ordered oriented matroid on \(E\). Let \(A\) be a subset in the interval of a basis of \(M\) with active filtration \(\emptyset = F'_0 \subset \ldots \subset F'_t = F = F_0 \subset \ldots \subset F_t = E\). Then,

\[
\alpha^{-1}_M(A) = \biguplus_{1 \leq k \leq \ell} \alpha^{-1}_{M(F'_k/F_{k-1})}(A \cap (F'_k \setminus F_{k-1})) \uplus \biguplus_{1 \leq k \leq \ell} \alpha^{-1}_{M(F'_{k-1}/F'_k)}(A \cap (F'_{k-1} \setminus F_k)).
\]

**Proof.** This is a straightforward reformulation, in terms of Proposition 6.8, of the construction of the refined active bijection discussed above. Let us give details anyway. Consider any of the active minors \(H\), and the unactive basis \(B_H\) induced in the minor \(H\) by the basis \(B\) associated with \(A\). The inverse image of \(B_H\) under \(\alpha\) in \(H\) consists of two opposite reorientations of \(H\). Now consider the refined active bijection of \(H\), and denote \(a\) the smallest edge of \(H\). One of the two above reorientations is associated to \(B_H\) (the one for which \(a\) has not been reoriented, that is, \(a\) is active-fixed or dual-active-fixed w.r.t. \(M\)), and the other to \(B_H \triangle \{a\}\). Applying this to each minor \(H\) and to any subset \(A\) in the same interval, we always obtain a reorientation of \(M\) whose image under \(\alpha_M\) is \(A\).

**Theorem 8.5** (completing Theorem 6.9). Let \(M\) be an oriented matroid on a linearly ordered set of elements \(E = e_1 < \ldots < e_n\). Let \(X\) be a subset of \(E\). We denote \(Q = Q(X)\) and \(P = P(X)\) (Definition 7.1). We denote \(B\) the basis of \(M\) defined by \(B = X \setminus Q \cup P\) (equivalently: \(B\) is the basis such that \(X\) belongs to the interval of \(B\), that is: \(X = B \setminus P \cup Q\) with \(B\) a basis, \(P \subseteq \text{Int}(B)\) and \(Q \subseteq \text{Ext}(B)\)).

The preimage of the subset \(X\) under \(\alpha_M\) is built by applying the algorithm of Theorem 6.9, as for building the preimage of \(B\) under \(\alpha\), with the two following changes.

In the case where \(e_k \in B\) and \(e_k \in \text{Int}(B)\), replace \("'reorient\ e_k\ or\ not,\ arbitrarily'\) with: \("'reorient\ e_k\ if\ and\ only\ if\ e_k \in P'\),

In the case where \(e_k \not\in B\) and \(e_k \in \text{Ext}(B)\), replace \("'reorient\ e_k\ or\ not,\ arbitrarily'\) with: \("'reorient\ e_k\ if\ and\ only\ if\ e_k \in Q'\),

46
Proof. Let us denote $A = \alpha_M^{-1}(X)$. The computation of the reorientation class of $A$ is given by the algorithm of Theorem 6.9 applied to the basis $B$. Furthermore, by Theorem 8.2, we have $P(X) = \Theta_M^*(A) = O^*(-A) \cap A$ and $Q(X) = \Theta_M(A) = O(-A) \cap A$. So, for $e_k \in \text{Int}(B) = O^*(-A)$, we have $e_k \in A$ if and only if $e_k \in P$, and for $e_k \in \text{Ext}(B) = O(-A)$, we have $e_k \in A$ if and only if $e_k \in Q$. This is exactly the condition stated in the algorithm.

Before ending this section, let us mention that a deletion/contraction construction exists for $\alpha_M$, that is also derived directly from that of $\alpha$, see [20, 21]. And let us end with a general remark on possible variations in the construction of this section.

**Remark 8.6** (Variations of the refined active between reorientations and subsets). Let us observe that variants of $\alpha_M$ can easily be defined, again using a boolean lattice isomorphism at each activity class / basis interval. For instance, in Definition 8.1, replace $A$ with $X_B \triangle A$ for some $X_B \subseteq E$ that can vary with $B$ (i.e. the boolean lattice isomorphism can change at each considered boolean lattice). This yields other reorientations-subsets bijections refining the canonical active bijection.

By this way, for instance, one can define active-fixed and dual-active-fixed reorientations with respect to two different references reorientations respectively. Also, suitable choices of $X_B$ allow us to exchange the correspondences between the four parameter activities for bases and reorientations (i.e. make Int correspond to $\Theta^*$ instead of $\Theta^*$, and/or make Ext correspond to $\Theta$ instead of $\Theta$).

Moreover, as reorientation activity classes correspond to basis intervals, one can derive various bijections by composition with a further boolean lattice isomorphism. For instance, one can define a bijection between dual-active-fixed acyclic reorientations and minimal subsets of internal basis intervals, enumerated by $t(M; 1, 0)$, or, dually, a bijection between active-fixed totally cyclic reorientations and maximal subsets of external basis intervals, enumerated by $t(M; 1, 0)$, et caetera.

At last, let us recall (see Remark 6.10) that a general class of active partition preserving bijections can be obtained by replacing $\alpha$ with any mapping $\psi$ defined for bounded/dual-bounded reorientations and yielding a bijection with uniactive internal/external bases. The same construction as above can be applied to such a mapping $\psi$, yielding a whole class of bijections $\psi_M$ between reorientations and subsets, preserving the four parameter activities for reorientations/subsets.

9. Further examples and illustrations

In this section, we complete the paper with a few more illustrative examples.

9.1. Example of $K_3$

The canonical and refined bijections are shown in Table 5 and Figure 15. The Tutte polynomial of $K_3$ is

$$t(K_3; x, y) = x^2 + x + y.$$**

| Active filtrations | Active partitions | Reorientation activity classes | Bases |
|-------------------|------------------|-------------------------------|-------|
| $\emptyset \subset 1 \subset E$ | $1 + 23$ | $123, 123, 123$ | $12$ |
| $\emptyset \subset E$ | $123$ | $123, 123$ | $13$ |
| $\emptyset \subset E$ | $123$ | $123, 123$ | $23$ |

Table 5: Table of the canonical active bijection of $K_3$, where reorientations are written with a bar over reoriented edges w.r.t. the reference orientation given in the upper left of Figure 15. The cyclic flat of each active filtration is boxed in the first column.
\( T(K_3; x + u, y + v) = +x^2 + xu + u^2 + ux + u + v + y \)

**Figure 15:** The active bijection illustrated on the graph \( K_3 \). We have \( T(K_3; x, y) = x^2 + x + y \). The layout reflects the bijections. Each monomial corresponds to an activity class of (re)orientations in the top part and to a basis (spanning tree) in the bottom part, associated by the canonical active bijection. Each basis yields a boolean lattice of subsets (shown by bold edges). Orientations in the top part and subsets in the bottom part are associated by the refined active bijection (with respect to the orientation displayed first in the top row), consistently with the four variable formula, in the way shown by the layout.

### 9.2. Example of \( K_4 \) exhaustively completed

We consider the graph \( K_4 \). Its Tutte polynomial is

\[
t(K_4; x, y) = x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3.
\]

We consider it with the ordering \( 1 < \ldots < 6 \) as shown in Figure 2, and with the reference orientation shown in Figure 3. Here, we complete this example that served as a running example in Figures 2, 3, 4, 7, 9, 10, 11, 12, 13, 14, and in [18]. Let us mention that the canonical and refined active bijections on this example are also exhaustively listed in a graph setting in [21], whereas here we continue to present this list geometrically.

Table 6 sums up the canonical active bijection (Theorem 6.4). Figures 16, 17, and 18 provide details for Figure 13, indicating the graph orientations, and adding the refined active bijection w.r.t. the reference reorientation (extending Figure 14 to all reorientation activity classes). All reorientations are depicted (up to opposite), and all subsets are associated to them (in brackets when they correspond to the opposite reorientation).
Figure 16: Details for the primal part of Figure 13 (connected filtrations involving cyclic flat $\emptyset$ in Table 6). The reference reorientation (signature of the arrangement) is given by the dark grey region. Reorientations corresponding to the regions are written w.r.t. to this reference reorientation, they correspond to the maximal covectors of the oriented matroid. Corresponding acyclic graph orientations are also drawn in the regions. The subset associated to the reorientation by the refined active bijection is written below the basis, the subset in brackets is associated to the opposite reorientation (extending Figure 14). The dual-active-fixed representatives of activity classes of regions w.r.t. the reference orientation are shown in light grey.
Figure 17: Details for the dual part of Figure 13 (connected filtrations involving cyclic flat $E$ in Table 6). The reference reorientation is given by the dark grey region of Figure 16. Reorientations corresponding to the regions of the dual arrangement are written w.r.t. to this reference reorientation, they correspond to the maximal vectors of the oriented matroid. Corresponding totally cyclic (or strongly connected) graph orientations are also drawn in the regions. Subset associated to reorientations by the refined active bijection are indicated the same way as in Figure 16.

Figure 18: Details and refined active bijection for the active filtrations involving non-trivial cyclic flats in Figure 13 and Table 6, completing Figures 16 and 17. Those cyclic flats are shown in dashed grey circles, and geometrically situated in Figures 16 and 17, following Figure 13. They appear with dashed edges on the graphs.
| Active filtration | Active partition | Rerientation activity class | Basis |
|-------------------|------------------|-----------------------------|-------|
| $\emptyset \subset 1 \subset 123 \subset E$ | $1 + 24 + 306$ | $123456, 123456, 123456, 123456, \ldots$ | 124   |
| $\emptyset \subset 1 \subset E$ | $1 + 23456$ | $123456, 123456, \ldots$ | 126   |
| $\emptyset \subset 145 \subset E$ | $145 + 236$ | $123456, 123456, \ldots$ | 125   |
| $\emptyset \subset 123 \subset E$ | $123 + 456$ | $123456, 123456, \ldots$ | 134   |
| $\emptyset \subset E$ | $123456$ | $123456, \ldots$ | 135   |
| $\emptyset \subset E$ | $123456$ | $123456, \ldots$ | 136   |
| $\emptyset \subset 123 \subset E$ | $123 + 456$ | $123456, 123456, \ldots$ | 234   |
| $\emptyset \subset 145 \subset E$ | $145 + 236$ | $123456, 123456, \ldots$ | 245   |
| $\emptyset \subset 246 \subset E$ | $246 + 135$ | $123456, 123456, \ldots$ | 146   |
| $\emptyset \subset 356 \subset E$ | $356 + 124$ | $123456, 123456, \ldots$ | 156   |
| $\emptyset \subset E$ | $123456$ | $123456, \ldots$ | 235   |
| $\emptyset \subset E$ | $123456$ | $123456, \ldots$ | 236   |
| $\emptyset \subset 246 \subset E$ | $135 + 246$ | $123456, 123456, \ldots$ | 346   |
| $\emptyset \subset 356 \subset E$ | $124 + 356$ | $123456, 123456, \ldots$ | 256   |
| $\emptyset \subset 23456 \subset E$ | $1 + 23456$ | $123456, 123456, \ldots$ | 345   |
| $\emptyset \subset 356 \subset 23456 \subset E$ | $1 + 24 + 356$ | $123456, 123456, 123456, 123456, \ldots$ | 456   |

Table 6: Table of the canonical active bijection of $K_4$ (Theorem 6.4), where reorientations are written with a bar over reoriented elements w.r.t. the reference rerorientation (the grey region in Figure 16), and where “...” means “and opposites”. The cyclic-flat of each connected filtration is boxed in the first column. This table can be compared with [18, Table 6] for bases.

9.3. Another similar example of the canonical active bijection.

In Figure 19, we address another example of rank 3 with 6 elements, and we completely show the geometry of the canonical active bijection, involving all bases and all reorientations (up to opposite), in the primal and dual arrangements along with combinations of cyclic flats of the primal and the dual. Its Tutte polynomial is:

$$t(M; x, y) = x^3 + 2x^2 + x + x^2y + 3xy + xy^2 + y + 2y^2 + y^3.$$  

The ordering of the ground set is the natural ordering. As for Figure 13, no reference rerorientation or signature is specified. We focus on the geometry but this example is graphical and planar again, hence with a graphical dual, as shown on the figure. The reader interested in the graph setting can easily draw which acyclic and totally cyclic orientations of the graph and its dual correspond to regions of the two arrangements. The refined active bijection can also be easily deduced, as in Figures 14, 16, 17, and 18.

9.4. Example of the canonical active bijection on regions of a rank 3 (supersolvable) arrangement.

Figure 20 illustrates the canonical active bijection on regions of a rank 3 arrangement with 11 elements, highlighting its geometrical interpretation in terms of flags of faces. This example is intended to be simple and pedagogic. Explanations are given in the figure caption. The ordering is $1 < 2 < \ldots < 9 < A < B$. The restriction of the Tutte polynomial in which we are interested is:

$$t(M; x, 0) = x^3 + 8x^2 + 16x.$$  

51
Figure 19: Complete primal/dual geometrical representation of the canonical active bijection on another example (see Section 9.3). We follow the same caption as in Figure 13. Minors obtained from non-trivial cyclic flats are represented and linked to their representations in the primal and dual arrangements, in order to show how regions of these minors are involved in the construction. Namely: 35, 1345, and 246 are those cyclic flats of $M$, corresponding to the cyclic flats 1246, 26 and 135 of $M^*$. Observe for instance that the two bases 125 and 245 yield to consider the same partition $E = 14 + 26 + 35$, the same sequence of subsets $\emptyset \subset 35 \subset 1435 \subset E$, and the same minors $M/1345$, $M(1345)/35$ and $M(35)$, when one omits the associated cyclic flat, whereas on one hand the active filtration of the basis 125 is $(\emptyset, 35, 1435, E)$ with cyclic flat 35, yielding $\text{Int}(125) = 12$ and $\text{Ext}(125) = 3$ and involving an acyclic reorientation of $M(1345)/35$, and on the other hand the active filtration of the basis 245 is $(\emptyset, 35, 1435, E)$ with cyclic flat 1435, yielding $\text{Int}(245) = 2$ and $\text{Ext}(245) = 13$ and involving a totally cyclic reorientation of $M(1345)/35$ (that is an acyclic reorientation of $M^*(1246)/26$).
Figure 20: The ordering is $1 < 2 < \cdots < 9 < A < B < C < D$. In each region, we write the active basis, associated to the region by the canonical active bijection. We deal only with acyclic reorientations and internal bases (thus, we will not specify the cyclic flat of involved active filtrations as it is the empty one). The minimal basis (playing a crucial role) is $127$, it is associated with the four regions adjacent to $1$ and $1 \cap 2$ (with dual-active elements $\{1, 2, 7\}$ and active filtration $\emptyset \subset 1 \subset 123456 \subset E$). In each bounded region (with dual-active element $1$), the fully optimal basis $B = 1 < a < b$ is represented by a flag of faces: the region contains a segment of $b$ which contains a point of $a \cap b$. This sequence corresponds to the sequence of useful covectors: $C^*(B; 1) \circ C^*(B; a) \circ C^*(B; b) \supset C^*(B; 1) \circ C^*(B; a) \supset C^*(B; 1)$. Those flags of faces illustrate the Adjacency property from Definition 5.2, and they are also intended to illustrate the optimality feature of these bases (intuitively, one can imagine mobile flags in the regions, moving until they reach their fully optimal position: they are first “pushed” from $2$ towards $1$, and then from $7$ towards $1$ when a face parallel to $2$ is reached, or from $7$ towards $4 = \min(4789AB)$ in the case where the central point is reached, see details in [17, 19]). Smaller flags in non-bounded regions are intended to illustrate the same features for minors involved in the active decomposition of regions and bases. In each region adjacent to $1$ but not $1 \cap 2$ (with dual-active elements $\{1, 2\}$), the basis is of the type $B = 1 < 2 < b$. Because of the active filtration $\emptyset \subset 1 \subset E$, the basis is obtained from the fully optimal basis $2b$ for the bounded region induced on $M/1$ (added to $1$ which is the only internal basis of $M(1)$). This basis is represented by a similar flag as above, but in one dimension less for this minor (those smaller flags are intuitively “pushed” from $7$ towards $2$ along $1$). In each region adjacent to $1 \cap 2$ but not $1$ (with dual-active elements $\{1, 7\}$), the basis is of the type $B = 1 < a < 7$. Because of the active filtration $\emptyset \subset 123456 \subset E$, the basis is obtained from the fully optimal basis $1a$ for the bounded region induced on $M(123456)$ (added to $7$ which is the only internal basis of $M/123456$). This basis is represented by a similar flag as above, but in one dimension less for this minor (those smaller flags are intuitively “pushed” from $2$ towards $1$ around $1 \cap 2$). In addition, this arrangement has the specificity of being supersolvable, yielding specific constructive properties, see details in Subsection 9.4. This figure is best viewed with colors.
In addition, independently, let us mention that this arrangement is supersolvable, a case studied into the details in [14]. Briefly, we have a sequence of three arrangements $M(1) \triangleleft M(123456) \triangleleft M$, such that these arrangements have increasing ranks and the intersection of two pseudospheres in an arrangement is contained in a pseudosphere of the previous arrangement in the sequence. Then regions of the arrangement are grouped in fibers corresponding to regions in the previous arrangement and linearly ordered in these fibers. Provided some consistency with the ordering, the active bijection can be built recursively, in each fiber, using the ordering of regions in this fiber. It is related to the general deletion/contraction of the active bijection developed in [20].

In this particular example, one can see that all regions in the fiber delimited by 1 and 5, by 5 and 2, by 2 and 4, etc., are associated with bases that respectively contain 12, 15, 14, etc. Now, in each fiber, the basis associated to the two extreme regions contains 7 (the smallest of $M \setminus 123456$). For the other regions, the missing element of the associated basis is given by the element of $M \setminus 123456$ delimiting the region on the opposite side of 7.

In general supersolvable arrangements, a similar construction holds at least for all bounded regions (coming from a general property of the active bijection that, for bounded regions, the greatest element of the fully optimal basis always borders the region [20]). The fact that, in this example, this construction directly holds for every region, including all non-bounded ones, is a further particularity of this example. Here, in terms of [14, 20], we have that the active mapping equals the weak active bijection (which does not preserve active partitions in general). Let us mention that a more involved construction in rank 4 is given in [14, Figure 4], which refines Figure 6 of the present paper, and shows how the active bijection is related to active partitions in a fiber of non-bounded regions. Finally, beyond this, let us mention that the supersolvable structure of Coxeter arrangements can be used to show that the active bijections yields bijections between permutation and increasing trees (braid arrangement), and between signed permutations and signed increasing trees (hyperoctahedral arrangement). See details in [14].

9.5. Example of the canonical and refined active bijections on regions of $D_{13}$

We end in Figure 21 with regions of a more involved rank-3 example, from which the reader might foresee how the construction gets more complicated in higher dimensions. This arrangement, which we call $D_{13}$, is obtained by adding 3 points $BCD$ to a Desargue configuration on $123456789\text{A}$, see [12, Example 4.1.1] for a picture and more details. This example has been detailed in our first paper [12], devoted to the uniform case and the rank-3 case. Beware that, as a marginal change w.r.t. [12] (and [15]), here we exchanged the roles 5 and 6 (so that $M(1A7B)$ and $M(156C)$ present different shapes). The ordering is $1 < 2 < \ldots < 9 < A < B < C < D$. The restriction of the Tutte polynomial in which we are interested is:

$$t(D_{13}; x, 0) = x^3 + 10x^2 + 24x.$$
Figure 21: The ordering is $1 < 2 < \cdots < 9 < A < B$. Canonical and refined active bijections on regions of a more involved rank-3 example than Figure 20, following the same caption. Here again, in each region, we write the basis associated by the canonical active bijection, but we write in smaller size the elements that should be removed to obtain the subset associated to the region by the refined active bijection w.r.t. the grey region as reference reorientation. Opposite regions (on the other non-represented half of the sphere) are associated to the subsets in brackets (if a region with dual active elements $O^*$ is associated with the subset $A$, then the opposite region is associated with the subset $A \Delta O^*$, just as in Figure 14). This yields the bijection between regions and no-broken-circuit subsets. The pictures of flags of faces have the same meaning as in Figure 20, but more types of situations occur. For bounded regions: the construction can be intuitively understood in a similar way as in Figure 20. Let us detail non-bounded regions: the minimal basis is $124$; bases $125$, $127$, $128$, and $129$ with active elements $\{1,2\}$ are obtained from the active filtration $\emptyset \subset 1 \subset E$, implying the minor $M/1$; bases $12A$ and $12B$ with active elements $\{1,2\}$ are obtained from the active filtration $\emptyset \subset 17AB \subset E$, implying the minor $M(1A7B)$; bases $126$ and $12C$ with active elements $\{1,2\}$ are obtained from the active filtration $\emptyset \subset 156C \subset E$, implying the minor $M(156C)$; bases $13A$ and $14D$ with active elements $\{1,4\}$ are obtained from the active filtration $\emptyset \subset 123D \subset E$, implying the minor $M(123D)$. See Section 9.5 for more information on this example. This figure is best viewed with colors.
Geometrical explanations on the active bijection are provided in the caption of Figure 21, in the continuation of Figure 20. Finally, let us mention that other technical details are given in [12], where the possible situations in rank-3 arrangements are differently but exhaustively listed, and also illustrated on this example. The optimization feature is briefly explained in the caption of Figure 20. This feature was briefly addressed in [12], in order to show that, in rank-3 arrangements, the active bijection is the only bijection between bounded regions and uniaactive internal basis satisfying the Adjacency property of Definition 5.2 (illustrated here by flags of faces). As shown in [12], this uniqueness property is also true in realizable uniform oriented matroids, but is false in non-euclidean oriented matroids, where the Dual-Adjacency property of Definition 5.2 cannot be replaced by the property of having a bijection (see also [20] for a summary of properties implying the active bijection). This uniqueness property is generalized to realizable oriented matroids in [19]. Also, the optimization features, roughly introduced in Figures 20 and 21 using flags of faces, are addressed into the details in [19] (see also [16] for a brief formal presentation), where one can also find an illustration on a rank-4 bounded region.

References

[1] G. Berman, The dichromate and orientations of a graph, Canad. J. Math. 29 (1977), 947-956.
[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler, Oriented matroids 2nd ed., Encyclopedia of Mathematics and its Applications 46, Cambridge University Press, Cambridge, UK 1999.
[3] P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, Springer-Verlag, Berlin, 1969.
[4] H.H. Crapo, A higher invariant for matroids, J. Combinatorial Theory 2 (1967), 406–417.
[5] H.H. Crapo, The Tutte polynomial, Aequationes Math., 3 (1969), 211–229.
[6] J.E. Dawson, A construction for a family of sets and its application to matroids, Lect Notes in Math. (Springer) 884 (1981), 136–147, (Comb. Math. VIII, Gelong, 1980).
[7] I. M. Gessel, Acyclic orientations and chromatic generating functions, Discrete Math., 232 (2001), 119-130.
[8] E. Gioan, Correspondance naturelle entre bases et réororientations des matroïdes orientés, Ph.D. thesis, University of Bordeaux 1, 2002. Available at http://www.lirmm.fr/~gioan.
[9] E. Gioan, On Tutte polynomial expansion formulas in perspectives of matroids and oriented matroids. Submitted, preprint available at arXiv:1807.06559.
[10] E. Gioan, The Tutte polynomial of matroid perspectives, Chapter in: Handbook of the Tutte Polynomial, CRC Monographs and Research Notes in Mathematics, J. Ellis-Monaghan & I. Moffatt (Eds.), submitted.
[11] E. Gioan, The Tutte polynomial of oriented matroids, Chapter in: Handbook of the Tutte

8Let us take opportunity of this paper to make two corrections to [12]:
- in [12] page 231 line 2: instead of “acting symmetrically on 1457 with three orbits 1457 23689A BCD” read “acting symmetrically on 1357 with three orbits 1357 24689A BCD”.
- in [12] page 236 Figure 7: in the region corresponding to the basis 136, the dark angle should touch the pseudoline 6 instead of the pseudoline 3.
Polynomial, CRC Monographs and Research Notes in Mathematics, J. Ellis-Monaghan & I. Moffatt (Eds.), submitted.

[12] E. Gioan and M. Las Vergnas, 
    
    Bases, reorientations and linear programming in uniform and rank 3 oriented matroids, 
    Adv. in Appl. Math. 32 (2004), 212–238, (Special issue Workshop on Tutte polynomials, Barcelona 2001).

[13] E. Gioan and M. Las Vergnas, 
    
    Activity preserving bijections between spanning trees and orientations in graphs, 
    Discrete Math. 298 (2005), 169–188, (Special issue FPSAC 2002).

[14] E. Gioan and M. Las Vergnas, 
    
    The active bijection between regions and simplices in supersolvable arrangements of hyperplanes, 
    Electronic Journal of Combinatorics 11 (2) (2006), #R30, 39p., (Stanley Festschrift).

[15] E. Gioan and M. Las Vergnas, 
    
    Fully optimal bases and the active bijection in graphs, hyperplane arrangements, and oriented matroids, 
    Electronic Notes in Discrete Mathematics 29 (2007), 365–371, (Proceedings EuroComb 2007, Sevilla).

[16] E. Gioan and M. Las Vergnas, 
    
    A linear programming construction of fully optimal bases in graphs and hyperplane arrangements, 
    Electronic Notes in Discrete Mathematics 34 (2009), 307–311, (Proceedings EuroComb 2009, Bordeaux).

[17] E. Gioan and M. Las Vergnas, 
    
    The active bijection in graphs, hyperplane arrangements, and oriented matroids 1. The fully optimal basis of a bounded region, 
    European Journal of Combinatorics 30 (8) (2009), 1868–1886, (Special issue: Combinatorial Geometries and Applications: Oriented Matroids and Matroids).

[18] E. Gioan and M. Las Vergnas, 
    
    The active bijection 2.a - Decomposition of activities for matroid bases, and Tutte polynomial of a matroid in terms of beta invariants of minors. Companion paper, simultaneously submitted to the same journal, preprint available at arXiv:1807.06516.

[19] E. Gioan and M. Las Vergnas, 
    
    The active bijection 3. Elaborations on linear programming, in preparation.

[20] E. Gioan and M. Las Vergnas, 
    
    The active bijection 4. Deletion/contraction framework and universality results, in preparation.

[21] E. Gioan and M. Las Vergnas, 
    
    The active bijection for graphs. Submitted, preprint available at arXiv:1807.06545.

[22] E. Gioan and M. Las Vergnas, 
    
    Computing the fully optimal spanning tree of an ordered bipolar directed graph, Submitted, preprint available at arXiv:1807.06552.

[23] G. Gordon and E. McMahon, 
    
    Interval Partitions and Activities for the Greedoid Tutte Polynomial, 
    Adv. Appl. Math. 18 (1997), 33–49.

[24] G. Gordon and L. Traldi, 
    
    Generalized activities and the Tutte polynomial, 
    Disc. Math. 85 (1990), 167–176.

[25] C. Greene and T. Zaslavsky, 
    
    On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-radon partitions and orientations of graphs, 
    Trans. Amer. Math. Soc. 280 (1983), 97–126.

[26] B. Lass, 
    
    Orientations acycliques et le polynôme chromatique, 
    Europ. J. Comb 22 (2001), 1101–1123.

[27] M. Las Vergnas, 
    
    Acyclic and totally cyclic orientations of combinatorial geometries, 
    Discrete Math. 20 (1977/78), 51–61.
[28] M. Las Vergnas, *The tutte polynomial of a morphism of matroids II. Activities of orientations*, Progress in Graph Theory (J.A. Bondy & U.S.R. Murty, ed.), Academic Press, Toronto, Canada, 1984, (Proc. Waterloo Silver Jubilee Conf. 1982), pp. 367–380.

[29] M. Las Vergnas, *A correspondence between spanning trees and orientations in graphs*, Graph Theory and Combinatorics (Proc. Cambridge Combin. Conf. 1983), Academic Press, London, UK 1984, 233-238.

[30] M. Las Vergnas. The Tutte polynomial of a morphism of matroids 6. A multi-faceted counting formula for hyperplane regions and acyclic orientations. Unpublished, preliminary preprint available at arXiv 1205.5424 (2012).

[31] M. Las Vergnas. The Tutte polynomial of a morphism of matroids 5. Derivatives as generating functions of Tutte activities. *Europ. J. Comb.*, 34:1390–1405, 2013.

[32] J.G. Oxley. *Matroid Theory*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2011 (second edition).

[33] R.P. Stanley, *Acyclic orientations of graphs*, Discrete Math. 5 (1973), 171–178.

[34] W.T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80–91.

[35] X.G. Viennot, *Heaps of pieces, I : basic definitions and combinatorial lemmas*, Lect. Notes in Math. 1234 (1986), 321–350, (Combinatoire énumérative, Proc. Colloq., Montréal Can.).

[36] R.O. Winder, *Partitions of n-space by hyperplanes*, SIAM J. Applied Math. 14 (1966), 811–818.

[37] T. Zaslavsky, *Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes*, Mem. Amer. Math. Soc. 1 (1975), no. 154, issue 1.