The Lax pairs for the Holt system.

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Abstract

By using known non-canonical transformation between the Holt system and the Henon-Heiles system the Lax pairs for all the integrable cases of the Holt system are constructed from the known Lax representations for the Henon-Heiles system.

1 Introduction

The Holt system is defined by the Hamilton function

\[ H = \frac{1}{2} (p_X^2 + p_Y^2) + \alpha X^{-2/3} \left( \frac{3}{4} \beta X^2 + Y^2 + \gamma \right). \] (1.1)

Only three integrable cases are known [1, 2]

(i) \( \beta = 1 \), (ii) \( \beta = 6 \), (iii) \( \beta = 16 \), (1.2)

while the remaining parameters \( \alpha \) and \( \gamma \) be an arbitrary constants. These parameters were isolated by the singular analysis [2], although the second integrals may be obtained directly [1, 3].

By integrability here we mean existence of a second independent integral of motion \( K \), and in this case the Liouville theorem implies that the problem can be solved by quadratures. This, however, can be done only after the finding special new variables which separate the associated Hamilton-Jacobi equation. Recall, for the Holt system the additional second integrals \( K \) are the polynomials of the third, fourth and sixth order in momenta [1, 3], respectively. Therefore, it seems that the Hamiltonians (1.1) cannot be separable in the standard curvilinear coordinate systems.

But at \( \beta = 1, 6 \) the Holt system belongs to the family of the Stäckel systems and the separation variables are related to the usual curvilinear coordinates according to [1]. In fact, rescaling constant \( \alpha \) and \( \gamma \) in (1.1)

\[ \alpha \rightarrow 4 \left( \frac{3}{2} \right)^{1/3} \alpha, \quad \gamma \rightarrow \frac{\gamma}{3\alpha} \]
and performing the canonical change of variables at first proposed in [3]

\[ X = \frac{2}{3} x^{3/2}, \quad p_X = p_x \sqrt{x}, \]
\[ Y = \frac{1}{2 \sqrt{3 \alpha}} p_y, \quad p_Y = 2 \sqrt{3 \alpha} y. \]

the Hamilton function (1.1) becomes

\[ H = \frac{p_x^2 + p_y^2}{2 x} + 2 \alpha \left( \beta x^2 + 3 y^2 \right) + \frac{2 \gamma}{x}, \quad (1.3) \]

Note, that at \( \beta = 1, 6 \) the second additional integral of motion \( K \) is a quadratic polynomial in momenta \( \{p_x, p_y\} \), what related with separability of the Hamilton-Jacobi equation in rotated cartesian coordinates for \( (i) \) and in parabolic coordinates for \( (ii) \) [4].

According to [4], integrals of motion for the Holt system may be transformed by the rule

\[ H \rightarrow \tilde{H} = x H, \quad K \rightarrow \tilde{K} = K - \frac{1}{3} n H, \quad n = \left[ \sqrt{\beta} \right] = 1, 2, 4, \quad (1.4) \]

into the integrals of motion for the Henon-Heiles system

\[ \tilde{H} = \frac{p_x^2 + p_y^2}{2} + 2 \alpha \left( \beta x^2 + 3 y^2 \right) + 2 \gamma = T + V. \quad (1.5) \]

Here the ratio of the Hamiltonians \( H \) and \( \tilde{H} \) are equal to the ratio of the determinants of the associated Stäckel matrices, which is independent on the potential parts \( V \) (1.3) of the Hamiltonians [3]. On the other hand this duality may be considered as the coupling-constant metamorphosis between integrable systems [3] with respect to the constant \( \gamma \) in the potential \( V \).

The purpose of this letter is to show as the non-canonical transformation (1.4) acts on the Lax matrices \( L(\lambda) \) and \( A(\lambda) \) in the Lax equation

\[ \frac{dL(\lambda)}{dt} = \{H, L(\lambda)\} = [A(\lambda), L(\lambda)] \quad (1.6) \]

and on the corresponding spectral curve

\[ C(z, \lambda) : \quad \det(z I + L(\lambda)) = 0. \quad (1.7) \]

The Lax representations for all the integrable cases of the Henon-Heiles system was constructed in [8] by using connection with stationary flows of some known integrable PDEs. Namely these Lax pairs we shall use to discuss the Lax representations for the Holt systems by exploiting transformation (1.4). In [7] the separability and another Lax pairs for the Henon-Heiles system have been considered. We shall use these results to construct non-canonical transformation of the Hamiltonian (1.1) into the Stäckel form at \( \beta = 16 \) (iii) .
2 Results

We begin with the known Lax matrices \( \tilde{L}(\lambda) \) and \( \tilde{A}(\lambda) \) for the Henon-Heiles system \([8]\) to construct the new Lax representations for all the Holt system. For our present purposes it is more convenient to use a different version of the Lax representation obtained from that presented in \([8, 9]\). Each of the Lax matrices \( \tilde{L}(\lambda) \) will be presented as a sum of the two standard matrices on the loop subalgebras \( L(sl(2)) \) in \( L(sl(3)) \) and the third term may be associated with the outer automorphism of the whole loop algebra \([10]\).

Case (i).

Let us begin with the Lax pair for the Henon-Heiles system at \( \beta = 1 \) in (1.5). Namely, for brevity put \( \alpha = a^2/3 \) and introduce the Lax matrices \([8, 9]\)

\[
\tilde{L}(\lambda) = \begin{pmatrix} 6 x & 0 & 9 \lambda - \frac{3}{2} a p_x \\ 9 \lambda + \frac{2}{a^3} p_x & -3 x \lambda & 0 \\ 0 & 9 \lambda^3 & -3 x \lambda \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -p_y y & y^2 \\ 0 & -p^2 y & p y \end{pmatrix}
\]

\[
- a (3 x^2 + y^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad a = \sqrt{3} \alpha,
\]

\[(2.1)\]

\[
\tilde{A}(\lambda) = \begin{pmatrix} 0 & 2 a \lambda & 0 \\ 0 & 0 & 1 \\ 2 a \lambda & -4 a^2 x & 0 \end{pmatrix}
\]

in the Lax equation \([1.6]\) for the Henon-Heiles system. The spectral curve \([1.7]\) of the Lax matrix \( \tilde{L}(\lambda) \) (2.1)

\[
\tilde{C}_1: \quad \alpha z^3 + 729 \lambda^7 - 162 \tilde{H} \lambda^3 + 324 \gamma \lambda^3 + \frac{\tilde{K}^2}{\lambda} = 0,
\]

\[(2.2)\]

is a third order cyclic covering of the line. Note, it is a very particular case in the class of generic trigonal algebraic curves.

Now we turn to the Holt system and non-canonical transformation \([1.4]\). Namely, the Lax matrices for the Holt system at \( \beta = 1 \) and \( \alpha = a^2/3 \) read as

\[
L(\lambda) = \tilde{L}(\lambda) + \frac{3}{2 a} \tilde{H} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(\lambda) = \frac{1}{x} \tilde{A}(\lambda).
\]

The spectral curve of this matrix \( L(\lambda) \) takes the following trigonal form

\[
C_1: \quad \alpha z^3 - 54 \tilde{H} z \lambda^2 + 729 \lambda^7 + 324 \gamma \lambda^3 + \frac{\tilde{K}^2}{\lambda} = 0.
\]

3
After point transformation
\[ u = \frac{1}{2} (x + y), \quad v = \frac{1}{2} (x - y), \tag{2.3} \]
the integrals of motion for the Henon-Heiles system became
\[ \tilde{H} = p_u^2 + p_v^2 + \alpha (u^3 + v^3) + 2 \gamma, \tag{2.4} \]
\[ \tilde{K} = p_u^2 - p_v^2 + \alpha (u^3 - v^3). \]

The same change of the variables for the Holt systems leads to
\[ H = 2 \frac{p_u^2 + p_v^2 + \alpha (u^3 + v^3) + 2 \gamma}{u + v}, \tag{2.5} \]
\[ K = 2 \frac{v(p_u^2 + \alpha u^3 + \gamma) - u(p_v^2 + \alpha v^3 + \gamma)}{u + v}. \]

For both system \( u, v \) are separation variables and these systems belong to the Stäckel set of the integrable systems \([4]\). In these separation variables the Henon-Heiles dynamics splitting on two tori. Thus, according to \([11]\), we can construct another \( 2 \times 2 \) Lax representation for the Henon-Heiles system with hyperelliptic spectral curve. Notice, the non-canonical transformation \( C \rightarrow \tilde{C} \) rearranges moduli \( H \) and \( \tilde{H} \) and preserves the genus of the corresponding spectral curves. Therefore, by using a slightly different covering of the two tori the \( 2 \times 2 \) Lax representation for the Holt system at \( \beta = 1 \) may be constructed as well.

**Case (ii).**

Here we shall use the general construction of the Lax representations for the dual Stäckel systems proposed in \([1]\). For the Henon-Heiles system the Lax matrices are given by
\[ \tilde{L}(\lambda) = \begin{pmatrix} p_x/2 & \lambda - x \\ 0 & -p_x/2 \end{pmatrix} + \frac{1}{4 \lambda} \begin{pmatrix} p_y y & -y^2 \\ p_y^2 & -p_y y \end{pmatrix} - 6 \alpha \left[ \lambda^2 + x \lambda + \frac{4 x^2 + y^2}{4} \right] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.6} \]
\[ \tilde{A}(\lambda) = \begin{pmatrix} 0 & 1 \\ -6 \alpha (\lambda + 2 x) & 0 \end{pmatrix}. \]

The spectral curve \((2.7)\) of the Lax matrix \( \tilde{L}(\lambda) \) \((2.6)\)
\[ \tilde{C}_2 : \quad z^2 + 6 \alpha \lambda^3 - \frac{1}{2} \tilde{H} + \gamma + \frac{\tilde{K}}{4 \lambda} = 0 \tag{2.7} \]
is a second order covering of the line or a hyperelliptic curve.

For the Holt system the associated Lax matrices have the form

\[ L(\lambda) = \tilde{L}(\lambda) + \frac{1}{2} H \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad A(\lambda) = \frac{1}{x} \tilde{A}(\lambda). \]

The corresponding spectral curve

\[ C_2 : \quad z^2 + 6 \alpha \lambda^3 - \frac{1}{2} H \lambda + \gamma + \frac{K}{4 \lambda} = 0 \]

remains the the same genus hyperelliptic curve.

According to [4], by using the standard parabolic coordinates

\[ u = \frac{x - \sqrt{x^2 + y^2}}{2}, \quad p_u = p_x - \sqrt{x^2 + y^2} + x p_y, \]
\[ v = \frac{x + \sqrt{x^2 + y^2}}{2}, \quad p_v = p_x + \sqrt{x^2 + y^2} - x p_y, \]

we can transform integrals of motion for the Holt and for the Henon-Heiles systems at \( \beta = 6 \) into the Stäckel form

\[
H = \frac{u (p_u^2 + u^3) - v (p_v^2 + v^3)}{u^2 - v^2}, \\
K = \frac{v^2 u (p_u^2 + u^3) - u^2 v (p_v^2 + v^3)}{u^2 - v^2}, \\
\tilde{H} = (u + v) H, \quad \tilde{K} = K + \frac{u v}{u + v} \tilde{H}.
\]

So, the Holt system at \( \beta = 6 \) belongs to the Stäckel family of integrable systems.

**Case (iii).**

The Lax representation for the Henon-Heiles system at \( \beta = 16 \) in (1.5) takes the form (see [8, 9])

\[
\tilde{L}(\lambda) = \left( \begin{array}{ccc} 12 x & 0 & \frac{3}{8} \alpha \\ 9 \lambda + 3 p_x & -6 x & 0 \\ 0 & 9 \lambda - 3 p_x & -6 x \end{array} \right) + \frac{1}{4 \lambda} \left( \begin{array}{ccc} -2 p_y y & y^2 & 0 \\ -2 p_y^2 & 0 & y^2 \\ 0 & -2 p_y^2 & 2 p_y y \end{array} \right) \\
+ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 6 \alpha x y^2 & 0 & 0 \\ -24 \alpha (\frac{24 x^2 + y^2}{2} + \frac{x y p_y}{\lambda}) & -6 \alpha x y^2 & 0 \end{array} \right). 
\]
\( \tilde{A}(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24\alpha (\lambda - p_x) & -48\alpha x & 0 \end{pmatrix} \).

As above, the spectral curve of the Lax matrix (2.8) \( \tilde{C}_3 \):
\[
512\alpha z^3 + \frac{9}{8}\lambda^2 - \frac{1}{4}\tilde{H} + \frac{\gamma}{2} + \frac{\tilde{K}^2}{\lambda^2} = 0
\]

is a third order cyclic covering of the line.

For the Holt system at \( \beta = 16 \) in (1.3) the Lax matrices are equal to
\[
L(\lambda) = \tilde{L}(\lambda) + 3H \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(\lambda) = \frac{1}{x} \tilde{A}(\lambda).
\]

The spectral curve (1.7) of \( L(\lambda) \) takes the following trigonal form
\[
C_3 : 512\alpha z^3 - Hz + \frac{9}{8}\lambda^2 + \frac{\gamma}{2} + \frac{\tilde{K}^2}{\lambda^2} = 0
\]

The second integrals of motion \( \tilde{K} \) and \( K \) in (2.9-2.11) are the square root of the non-factorable polynomial of the fourth order in momenta. Nevertheless, both these integrals are the rational functions in separation variables.

Recall, that according to [7], the separation variables for the Henon-Heiles may be written as [12]
\[
u = -\frac{\tilde{K}}{\alpha y^2} - \frac{p_y^2}{2\alpha y^2} + x, \quad \nu_u = \frac{p_x}{2} + \frac{p_y}{2y} \left( \frac{p_y^2}{\alpha y^2} - 6x + \frac{2\tilde{K}}{\alpha y^2} \right),
\]
\[
u = \frac{\tilde{K}}{\alpha y^2} - \frac{p_y^2}{2\alpha y^2} + x, \quad \nu_v = \frac{p_x}{2} + \frac{p_y}{2y} \left( \frac{p_y^2}{\alpha y^2} - 6x - \frac{2\tilde{K}}{\alpha y^2} \right).
\]

Here integral of motion \( K \) as yet unspecified functions of the new variables \( (x, p_x, y, p_y) \). Now we have to substitute the separation variables (2.12) into the definition of the second integrals \( \tilde{K} \) (2.4) and to solve the resulting equation (2.12). Thus, substituting an explicit value of the integral \( \tilde{K} \) into (2.12) we get canonical change of variables, which transforms the Henon-Heiles integrals (2.4) into the following form
\[
\tilde{H} = \frac{p_x^2}{2} + \frac{p_y^2}{2} - \alpha x \left( 16x^2 + 3y^2 \right) + 2\gamma,
\]
\[
\tilde{K}^2 = p_y^4 + 4\alpha y^3 p_x p_y - 12\alpha x y^2 p_y^2 - 12\alpha^2 x^2 y^4 - 2\alpha^2 y^6,
\]

For the Holt system we can also substitute new variables (2.12) into the definition of the corresponding second integrals \( K \) (2.7) and solve the resulting
equation. Thus one gets change of variables (2.12), which transform integrals of motion (2.5) into the desired form

\[ H = \frac{p_x^2}{2x} + \frac{p_y^2}{2x} - \alpha \left( 16x^2 + 3y^2 \right) + \frac{2\gamma}{x}, \quad K^2 = \tilde{K}^2 + \frac{y^4}{3}H, \quad (2.14) \]

In contrast with the Henon-Heiles case, this change of the variables is a non-canonical transformation.

So, at the third case we have a non-canonical change of variables

\[ (t, x, y, p_x, p_y) \rightarrow (\tilde{t}, u, v, p_u, p_v), \]

which transforms integrals of motion (2.14) into the Stäckel form. Of course, such transformations are not new. As an example, the complete Kolosoff transformation [13] connects the Stäckel system with the Kowalewski top, which is an integrable but non-Stäckel system. By using such transformations we can construct the separated equations in the Lagrangian variables \((u, \dot{u}, v, \dot{v})\) and get solutions of the equations of motion in theta-functions. Till now, in the quantum mechanics we can not construct a counterpart of this transformation for the Holt system.

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