Representability of matroids with a large
projective geometry minor

Jim Geelen and Rohan Kapadia
Department of Combinatorics and Optimization
University of Waterloo, Waterloo, Ontario, Canada

December 20, 2012; revised December 8, 2014

Abstract

We prove that, for each prime power $q$, there is an integer $n$ such that if $M$ is a 3-connected, representable matroid with a PG($n-1,q$)-minor and no $U_{2,q^2+1}$-minor, then $M$ is representable over GF($q$). We also show that for $\ell \geq 2$, if $M$ is a 3-connected, representable matroid of sufficiently high rank with no $U_{2,\ell+2}$-minor and $|E(M)| \geq 2\ell^r(M)/2$, then $M$ is representable over a field of order at most $\ell$.

1 Introduction

We recall that PG($n-1,q$) is the rank-$n$ projective geometry over GF($q$), the finite field of order $q$. We prove the following theorem.

Theorem 1.1. For each prime power $q$ there is an integer $n$ such that if $M$ is a 3-connected, representable matroid with a PG($n-1,q$)-minor, then either

- $M$ has a $U_{2,q^2+1}$-minor, or

---

1 This research was partially supported by a grant from the Office of Naval Research [N00014-10-1-0851].

† Current address: Concordia University, Montréal, Québec, Canada
• $M$ is GF($q$)-representable.

Note that $U_{2,q^2+1}$ is the longest line representable over GF($q^2$). In the $q = 2$ case, a precise version of Theorem 1.1 is known.

**Theorem 1.2** (Semple, Whittle, [11]). If $M$ is a 3-connected, representable matroid with a PG(2, 2)-minor and no $U_{2,5}$-minor, then $M$ is binary.

Semple and Whittle proved that a 3-connected, representable matroid with no $U_{2,5}$- or $U_{3,5}$-minor is binary or ternary. Theorem 1.2 follows from this result along with the fact that PG(2, 2) is not ternary and a result of Oxley [8] stating that a 3-connected matroid of corank at least three with no $U_{2,5}$-minor has no $U_{3,5}$-minor.

In the next section we exhibit counterexamples to the stronger version of Theorem 1.1 where the assumption of representability is dropped. These matroids are vertically 4-connected and have no $U_{2,q^2+2}$-minor, and we conjecture the following.

**Conjecture 1.3.** If $q$ is a prime power and $M$ is a vertically 4-connected matroid with a PG(2, $q$)-minor and no $U_{2,q^2+2}$-minor, then $M$ is GF($q$)-representable.

For a matroid $M$, we denote the simplification of $M$ by $\text{si}(M)$ and we let $\varepsilon(M) = |E(\text{si}(M))|$. For a class of matroids $\mathcal{M}$ and positive integer $k$, we define $g_\mathcal{M}(k) = \max\{\varepsilon(M) : M \in \mathcal{M}, r(M) = k\}$ or say $g_\mathcal{M}(k) = \infty$ when this maximum does not exist. The function $g_\mathcal{M}$ is called the *growth-rate function* of $\mathcal{M}$. We let $\mathcal{U}_\ell$ denote the class of matroids with no $U_{2,t+2}$-minor. A theorem of Geelen and Nelson [3] asserts that, for sufficiently large $k$, $g_{\mathcal{U}_\ell}(k) = (q^k-1)/(q-1)$ where $q$ is the largest prime power less than or equal to $\ell$, and equality is achieved only by the projective geometry PG($k-1,q$). Thus, for large $k$, the rank-$k$ matroids in $\mathcal{U}_\ell$ with the maximum number of points are representable over a field of order at most $\ell$. We prove the following extension of this fact as a corollary of Theorem 1.1 and a result of Geelen and Kabell [2].

**Theorem 1.4.** For any positive integer $\ell$, there is an integer $k$ so that if $M$ is a 3-connected, representable matroid of rank at least $k$ with no $U_{2,t+2}$-minor and $|E(M)| \geq 2^{r(M)/2}$, then $M$ is representable over a field of order at most $\ell$. 

2
2 Growth rates

We prove Theorem 1.1 in Sections 3 and 4; in this section we derive Theorem 1.4 from it and also provide an example that motivates the assumption of representability in both Theorems 1.1 and 1.4. We need the following result.

Theorem 2.1 (Geelen, Kabell, [2]). For all integers $\ell, q_0 \geq 2$ and $n$, there exists an integer $c$ such that if $M$ is a matroid with no $U_{2,\ell+2}$-minor and $\varepsilon(M) \geq c q_0^{r(M)}$, then $M$ has a PG$(n-1,q)$-minor for some prime power $q > q_0$.

We prove the following stronger version of Theorem 1.4.

Theorem 2.2. Let $\ell \geq 2$ and $q_0$ the smallest prime power greater than or equal to $\sqrt{\ell}$. There is an integer $c$ such that if $M$ is a 3-connected, representable matroid with no $U_{2,\ell+2}$-minor and $|E(M)| \geq c q_0^{r(M)}$, then $M$ is GF$(q)$-representable for some prime power $q \leq \ell$.

Proof. Applying Theorem 1.1 to every prime power $q \leq \ell$, we can choose an integer $n$ so that, for each such $q$, a 3-connected, representable matroid with a PG$(n-1,q)$-minor and no $U_{2,q^2+1}$-minor is representable over GF$(q)$. We choose $c$ as in Theorem 2.1 so that a matroid $M$ with no $U_{2,\ell+2}$-minor and $\varepsilon(M) \geq c q_0^{r(M)}$ has a PG$(n-1,q)$-minor for some prime power $q > q_0$. We let $M$ be a 3-connected, representable matroid with no $U_{2,\ell+2}$-minor and $\varepsilon(M) \geq c q_0^{r(M)}$; then $M$ has a PG$(n-1,q)$-minor for some prime power $q > q_0$. The fact that $M$ has no $U_{2,\ell+2}$-minor implies that $q \leq \ell$. Also, $q > \sqrt{\ell}$ so $M$ has no $U_{2,q^2+1}$-minor. Thus by Theorem 1.1, $M$ is GF$(q)$-representable. \qed

For any prime power $q$, we exhibit a class of matroids which provide a counterexample to the stronger versions of both Theorems 1.1 and 2.2 where the assumptions of representability are dropped (for $\ell \geq 4$ in the case of Theorem 2.2). We will use the following theorem of projective geometry, known as Pappus’s Theorem (see [1, Theorem 2.2.2]).

Theorem 2.3. Let $L_1$ and $L_2$ be lines in a plane representable over a field, with distinct points $a, b, c \in L_1 \setminus L_2$ and $d, e, f \in L_2 \setminus L_1$. If $g, h, \text{ and } i$ are points that are respectively the intersections of the lines spanned by $\{e,a\}$ and $\{f,b\}$, $\{d,a\}$ and $\{f,c\}$, and $\{d,b\}$ and $\{e,c\}$, then $g, h, \text{ and } i$ are collinear.
For each $n \geq 3$, we construct a rank-$(n + 1)$ matroid that is 3-connected, has a $\text{PG}(n - 1, q)$-minor, has no $U_{2,q+3}$-minor, and has more than $q^n$ points, but is not representable.

We recall that the rank-$n$ affine geometry $\text{AG}(n - 1, q)$ is obtained from $\text{PG}(n - 1, q)$ by deleting a hyperplane, and for any element $e$ of $\text{AG}(n - 1, q)$, $\text{si}(\text{AG}(n - 1, q)/e) \cong \text{PG}(n - 2, q)$.

For $n \geq 3$, we let $H$ be a hyperplane of $\text{PG}(n, q)$, let $C$ be a circuit of size $n + 1$ contained in $H$, let $M_n = \text{PG}(n, q) \setminus (H \setminus C)$, and let $M'_n$ be the matroid obtained from $M_n$ by relaxing the circuit-hyperplane $C$. For any element $e$ of $M'_n$, at least one of $M'_n \setminus e$ and $M'_n/e$ is $\text{GF}(q)$-representable. In particular, if $e \notin C$, then $\text{si}(M'_n/e) = \text{si}(M_n/e) \cong \text{PG}(n - 1, q)$, and if $e \in C$, then $M'_n/e = M_n/e$. However, $M'_n$ has no $U_{2,q+3}$-minor, because any non-$\text{GF}(q)$-representable, rank-2 minor $N$ of $M'_n$ is a restriction of $M'_n/X$ for some $X \subseteq C$ with $|X| = n - 1$ and so is a single-element extension of a rank-2 minor of $\text{PG}(n, q)$.

We now show that $M'_n$ is not representable. We choose a set $X \subseteq C$ with $|X| = |C| - 3$ and let $N = \text{si}(M'_n/X)$ and $N' = \text{si}(M'_n/X)$. Then $N'$ is obtained from $N$ by relaxing the circuit-hyperplane $C \setminus X$. If $q = 2$, then $N \cong \text{PG}(2, 2)$ so $N'$ is isomorphic to the non-Fano matroid, and hence $M'_n$ is not representable. If $q > 2$, we label the elements of $C \setminus X$ as $a, b,$ and $c$, and we can choose a triangle $\{d, e, f\}$ of $N$ such that $a, b, c \notin \text{cl}_N(\{d, e, f\})$. In addition, we can define $g, h$ and $i$ to be the elements of $N$ that respectively lie in $\text{cl}_N(\{e, a\}) \cap \text{cl}_N(\{f, b\})$, $\text{cl}_N(\{d, a\}) \cap \text{cl}_N(\{f, c\})$, and $\text{cl}_N(\{d, b\}) \cap \text{cl}_N(\{e, c\})$. We observe that $r_N(\{g, h, i\}) = 2$ by Pappus’s Theorem. Therefore, in $N'$ there are two triangles $\{d, e, f\}$ and $\{g, h, i\}$ that lie on distinct lines, and $a \in \text{cl}_{N'}(\{d, h\}) \cap \text{cl}_{N'}(\{e, g\})$, $b \in \text{cl}_{N'}(\{d, i\}) \cap \text{cl}_{N'}(\{f, g\})$, and $c \in \text{cl}_{N'}(\{e, i\}) \cap \text{cl}_{N'}(\{f, h\})$. If $N'$ is representable over a field, then Pappus’s Theorem asserts that $a, b$ and $c$ are collinear. But $r_{N'}(\{a, b, c\}) = 3$, so $N'$ is not representable.

### 3 Representation over a subfield

We say that a representation $A$ of a matroid $M$ is in **standard form with respect to** a basis $B$ if it has the form $A = [I | A']$ where $I$ is an identity matrix in the columns indexed by $B$. For such a representation, we index the rows by the elements of $B$ so that $A_{bh} = 1$ for all $b \in B$. When $X \subseteq B$ and $Y \subseteq E(M)$, we write $A[X, Y]$ for the submatrix of $A$ in the rows of $X$.
and the columns of $Y$. For each basis $B$ of a matroid $M$, every representation of $M$ can be converted to standard form with respect to $B$ by applying row operations and permuting the columns along with their labels.

Let $N$ be a minor of a matroid $M$ such that $N = M/C\setminus D$ for disjoint sets $C, D \subseteq E(M)$ where $C$ is independent and $D$ is co-independent. We choose a basis $B$ of $N$ and let $B' = B \cup C$, so $B'$ is a basis of $M$. Let $\mathbb{F}$ be a field and $A'$ an $\mathbb{F}$-representation of $M$ in standard form with respect to the basis $B'$. Then the matrix $A = A'[B, E(N)]$ is an $\mathbb{F}$-representation of $N$ in standard form with respect to the basis $B$. We say that $A$ is the representation of $N$ induced by $A'$, and that $A'$ is a representation of $M$ that extends the representation $A$ of $N$.

We call both row operations and column scaling projective transformations and say that two representations of a matroid over a field $\mathbb{F}$ are projectively equivalent if one can be obtained from the other by applying projective transformations and permuting columns (along with their labels).

A proof of the next result can be found in [7, Theorem 3.4].

**Theorem 3.1.** If $q$ is a prime power, $n \geq 3$, and $\mathbb{F}$ is an extension field of $\text{GF}(q)$, then each representation of $\text{PG}(n-1, q)$ over $\mathbb{F}$ is projectively equivalent to a representation with entries in $\text{GF}(q)$.

When $\mathbb{F}$ is an extension field of $\text{GF}(q)$, we say that an $\mathbb{F}$-matrix $A$ is a scaled $\text{GF}(q)$-matrix if there is a $\text{GF}(q)$-matrix obtained from $A$ by scaling rows and columns by elements of $\mathbb{F}^\times$. **Theorem 3.1** is equivalent to the fact that for $n \geq 3$, every representation of $\text{PG}(n-1, q)$ in standard form is a scaled $\text{GF}(q)$-matrix. This follows from two observations: when two projectively equivalent representations of a matroid are in standard form with respect to the same basis, then one can be obtained from the other by scaling rows and columns. Also, for $n \geq 3$, $\text{PG}(n-1, q)$ is only representable over extension fields of $\text{GF}(q)$ (see [9, p. 660]).

We will use the following theorem of Pendavingh and Van Zwam that reduces the problem of proving that a matroid $M$ with a $\text{PG}(n-1, q)$-minor $N$ is $\text{GF}(q)$-representable to checking minors of $M$ with at most $|E(N)| + 2$ elements. Suppose that $N$ is a minor of an $\mathbb{F}$-representable matroid $M$ and $\mathbb{F}'$ is a subfield of $\mathbb{F}$. We say that $N$ confines $M$ to $\mathbb{F}'$ if whenever $N'$ is a minor of $M$ isomorphic to $N$, every $\mathbb{F}$-representation of $M$ that extends an $\mathbb{F}'$-representation of $N'$ is a scaled $\mathbb{F}'$-matrix. Although Pendavingh and Van Zwam prove a theorem for representability over a generalization of fields
called partial fields [10, Theorem 1.4], we state here only a specialization of it to fields.

**Theorem 3.2** (Pendavingh, Van Zwam, [10]). If \( \mathbb{F}' \) is a subfield of a field \( \mathbb{F} \), \( M \) and \( N \) are 3-connected matroids, and \( N \) is a minor of \( M \), then either

(i) \( N \) confines \( M \) to \( \mathbb{F}' \), or

(ii) \( M \) has a 3-connected minor \( M' \) such that \( N \) does not confine \( M' \) to \( \mathbb{F}' \) and \( N \) is isomorphic to one of \( M'/x \), \( M'/y \), or \( M'/x \cup y \) for some \( x, y \in E(M') \).

4 The proof of Theorem 1.1

Before proving Theorem 1.1 we state a result from Ramsey theory and then a theorem of Tutte about matroid connectivity. The first is the following corollary of the Hales-Jewett Theorem [6]; it is also a special case of the Affine Ramsey Theorem of Graham, Leeb, and Rothschild [4], for which a proof can be found in [5, p. 42].

**Theorem 4.1.** For any finite field \( \text{GF}(q) \) and integers \( r \) and \( k \), there is an integer \( n = n_{4.1}(q, r, k) \) so that if the elements of \( \text{AG}(n-1, q) \) are \( r \)-coloured, it has a monochromatic restriction isomorphic to \( \text{AG}(k-1, q) \).

The connectivity function, \( \lambda_M \), of a matroid \( M \) is defined by \( \lambda_M(X) = r_M(X) + r_M(E(M) \setminus X) - r(M) \) for each \( X \subseteq E(M) \). For disjoint sets \( S, T \subseteq E(M) \), we define \( \kappa_M(S, T) = \min\{\lambda_M(A) : S \subseteq A \subseteq E(M) \setminus T\} \). When \( M \) is a 3-connected matroid and \( S \) and \( T \) are disjoint subsets of \( E(M) \), both of size at least two, then \( \kappa_M(S, T) \geq 2 \). The local connectivity of sets \( S \) and \( T \) in a matroid \( M \) is \( \lambda_M(S, T) = r_M(S) + r_M(T) - r_M(S \cup T) \).

**Theorem 4.2** (Tutte’s Linking Theorem, [12]). If \( M \) is a matroid and \( S, T \subseteq E(M) \) are disjoint, then \( \kappa_M(S, T) = \max\{\lambda_M(Z) : Z \subseteq E(M) \setminus (S \cup T)\} \).

Two sets \( S \) and \( T \) in a matroid \( M \) are called skew if \( \lambda_M(S, T) = 0 \). If we choose the set \( Z \) that attains the maximum in Theorem 4.2 to be minimal, then \( Z \) and \( S \) are skew, and \( Z \) and \( T \) are skew. We can now prove Theorem 1.1.
Proof of Theorem 1.1. We set $n$ to be the integer $n_{4,1}(q, q^2, 3)$ given by Theorem 4.1 such that any $q^2$-colouring of the elements AG($n - 1, q$) has a monochromatic restriction isomorphic to AG($3, q$). We let $M$ be a 3-connected, representable matroid with a PG($n - 1, q$)-minor but no $U_{2,q^2+1}$-minor. Then $M$ is representable over an extension field $F$ of GF($q$). We start with two short claims; we omit the easy proof of the first.

(1) If $P$ is a simple rank-3 matroid with an element $e$ such that $P \setminus e \cong PG(2, q)$, then $P$ has a $U_{2,q^2+1}$-minor.

(2) If $P$ is an $F$-representable matroid with an element $y$ such that $P \setminus y \cong PG(n - 1, q)$ but PG($n - 1, q$) does not confine $P$ to GF($q$), then $P$ has a $U_{2,q^2+1}$-minor.

There is a PG($n - 1, q$)-minor $N$ of $P$ and an $F$-representation $A$ of $P$, in standard form with respect to a basis $B$ of $N$, that extends a GF($q$)-representation of $N$ but is not a scaled GF($q$)-matrix. The column of $y$ is not parallel to a vector over GF($q$) so there are two elements $a, b \in B$ such that $A_{ay} \neq GF(q)$. We pick any third element $c \in B$, and let $P' = M/(B \setminus \{a, b, c\})$. Then $y$ is not in a parallel pair of $P'$ and $si(P') \setminus y \cong PG(2, q)$, so (2) follows from (1).

We apply Theorem 3.2 to $M$ with $N = PG(n - 1, q)$ and $F' = GF(q)$. If outcome (i) of this theorem holds, then it follows from Theorem 3.1 that $M$ is GF($q$)-representable. So we may assume that outcome (ii) of Theorem 3.2 holds, and there is a 3-connected minor $M'$ of $M$ such that PG($n - 1, q$) does not confine $M'$ to GF($q$) and PG($n - 1, q$) is isomorphic to either $M'/x$, $M'/y$, or $M'/x \setminus y$ for some $x, y \in E(M')$. By (2) we may assume that $M$ has a PG($n - 1, q$)-minor $N$ equal to either $M'/x$ or $M'/x \setminus y$ for some $x, y \in E(M')$.

We let $B$ be a basis of $N$ and $A$ be an $F$-representation of $M'$ in standard form with respect to the basis $B \cup \{x\}$ of $M'$. Since PG($n - 1, q$) does not confine $M'$ to GF($q$), we may assume that $A$ is not a scaled GF($q$)-matrix but it induces a GF($q$)-representation $A|B, E(N)$ of $N$. Moreover, when $N \cong M'/x \setminus y$, applying (2) to $M'/x$ lets us assume that PG($n - 1, q$) confines $M'/x$ to GF($q$) and that the induced representation $A|B, E(N) \cup \{y\}$ of $M'/x$ also has all its entries in GF($q$).

(3) There are two elements $f, g \in E(M'/x)$ such that $A_{xf}, A_{xg} \neq 0, A_{xf}^{-1}A_{xg} \notin GF(q)$, and \{f, g\} is independent in $M'/x$. 


Let $f$ and $g$ be any two distinct elements of $E(N)$ with $A_{xf}, A_{xg} \neq 0$. Then $\{f, g\}$ is independent in $M'/x$ because $N$ is simple. Therefore, we may assume that $A_{xf}^{-1}A_{xg} \in \text{GF}(q)$ for every pair $f, g \in E(N)$ with $A_{xf}, A_{xg} \neq 0$. This implies that we can scale the row and column of $x$ in $A$ to transform $A[B \cup \{x\}, E(N) \cup \{x\}]$ into a GF($q$)-matrix. But $A$ is not a scaled GF($q$)-matrix, so we may assume that we are in the case where $N = M'/x\setminus y$, that $A_{xy} \neq 0$, and that for any $f \in E(N)$ with $A_{xf} \neq 0$, we have $A_{xf}^{-1}A_{xy} \notin \text{GF}(q)$.

Note that $y$ is not a loop in $M'/x$ because $M'$ is 3-connected. If there exist two distinct elements $f, f' \in E(N)$ with $A_{xf} \neq 0$ and $A_{xf'} \neq 0$, then the fact that $N$ is simple means that at least one of $\{f, y\}$ and $\{f', y\}$ is independent in $M'/x$, and we are done. On the other hand, there is at least one element $f \in E(N)$ with $A_{xf} \neq 0$ for otherwise $A$ would be a scaled GF($q$)-matrix. So we may assume that there is precisely one element $f$ of $E(N)$ with $A_{xf} \neq 0$, and that $\{f, y\}$ is a parallel pair of $M'/x$. Now $\{f, x, y\}$ is both a circuit and a cocircuit of $M'$. Hence $\lambda_{M'}(\{f, x, y\}) = 1$, contradicting the fact that $M'$ is 3-connected. This proves (3).

We choose a pair of elements $f, g \in E(M'/x)$ as in (3), and by scaling we may assume that $A_{xf} = 1$ and $A_{xg} = \omega$ for some $\omega \notin \text{GF}(q)$. We choose some hyperplane $H$ of $M'/x$ that contains $\{f, g\}$ and choose an element $z \in E(M'/x) \setminus H$. We let $B'$ be the union of $\{z\}$ with a basis of $H$ in $M'/x$, so $B' \cup \{x\}$ is a basis of $M'$, and we let $A'$ be a representation of $M'$ in standard form with respect to $B' \cup \{x\}$. We can obtain $A'$ from $A$ by row operations without using the row of $x$, so that $A'[B', E(M')]$ has all its entries in $\text{GF}(q)$. We let $C = E(M'/x) \setminus H$, so $C$ is a cocircuit of $M'/x$ containing $z$. Then the restriction $(M'/x)|C$ is isomorphic to $\text{AG}(n-1, q)$. For each $e \in E(M'/x)$, the entry $A'_{xe}$ is non-zero if and only if $e \in C$, and by scaling columns of $A'$ we may assume that all entries in the row of $z$ are either 0 or 1. The submatrix $A'[[\{x, z\}, C]$ represents $(M'/(B'\setminus \{z\}))|C$, which has rank two. If this matrix contains a set of at least $q^2 + 1$ pairwise non-parallel columns, then $M'$, and hence $M$, has a $U_{2,q^2+1}$-minor. Otherwise, since $A'_{xe} = 1$ for all $e \in C$, there are at most $q^2$ distinct elements of $F$ that appear in $A'[[\{x\}, C]$. We can therefore $q^2$-colour the elements of $(M'/x)|C$ by assigning to each $e \in C$ the colour $A'_{xe}$. Since $(M'/x)|C \cong \text{AG}(n-1, q)$, with our choice of $n = n_{4,1}(q, q^2, 3)$ Theorem 4.1 implies that there is a monochromatic restriction of $(M'/x)|C$ isomorphic to $\text{AG}(3, q)$. We denote by $Y$ the ground set of this restriction. The entries $A'_{xe}$ for $e \in Y$ are all equal to some $\beta \in F$, so $A'[\{x\}, Y]$ is a multiple of $A'[\{z\}, Y]$ (possibly the
zero multiple) and $M'|Y$ is also isomorphic to $AG(3,q)$. Since $f,g \not\in C$, $A'_{zf} = A'_{sg} = 0$, so the row space of $A'$ contains a vector $u \in \mathbb{F}^{E(M')}$ such that $u_e = -\beta$ for all $e \in Y$ and $u_f = u_g = 0$.

As $N$ is 3-connected, $\kappa_N(\{f,g\},Y) = 2$. Also, when $N = M'/x\backslash y$, $\kappa_{M'/x}(\{f,g\},Y) = 2$ because $y$ is parallel to an element of $N$ in $M'/x$. By Theorem 4.2, there is a set $Z \subseteq E(M'/x)$ disjoint from $Y$ and $\{f,g\}$ such that $\cap_{(M'/x)/Z} (Y, \{f,g\}) = 2$, and $Z$ and $Y$ are skew. This means that $\{f,g\}$ is independent in $(M'/x)/Z$ and $f,g \in cl_{(M'/x)/Z}(Y)$. Since $Z$ and $Y$ are skew, there exists a basis $B'$ of $M'/x$ that contains $Z$ and a basis of $Y$. We apply row operations to $A'$ to get a representation $A''$ of $M'$ in standard form with respect to the basis $B'' \cup \{x\}$. The row of $x$ is the same in $A''$ and $A'$, and the vector $u$ is also in the row space of $A''$.

Consider the matrix $D$ obtained from $A''$ by adding the vector $u$ to the row of $x$ then restricting to the submatrix in rows $\{x\} \cup (B'' \cap Y)$ and columns $Y \cup \{f,g\}$. Then $D$ represents $M'' = (M'/ (B'' \setminus Y)) \cap (Y \cup \{f,g\})$ and it has the form

$$D = \begin{pmatrix} Y & f & g \\ 0 & 1 & \omega \\ D_1 & \alpha & \alpha' \end{pmatrix},$$

where $D_1$ is a $GF(q)$-representation of $AG(3,q)$ and $\alpha$ and $\alpha'$ are columns with all entries in $GF(q)$. Since $\{f,g\}$ is independent and contained in the closure of $Y$ in $(M'/x)/Z$, the vectors $\alpha$ and $\alpha'$ are both non-zero and are not parallel to each other. The minor $M''/f$ has the following representation

$$\begin{pmatrix} Y & g \\ D_1 & \alpha' - \omega \alpha \end{pmatrix}.$$
Since $\text{si}(M''/f, e)\setminus g \cong \text{PG}(2, q)$, it follows from (1) that $\text{si}(M''/f, e)$, and hence $M$, has a $U_{2,q^2+1}$-minor. \qed

References

[1] Albrecht Beutelspacher and Ute Rosenbaum, Projective Geometry: From Foundations to Applications, Cambridge University Press, 1998.

[2] Jim Geelen and Kasper Kabell, Projective geometries in dense matroids, J. Combin. Theory Ser. B 99 (2009), 1–8.

[3] Jim Geelen and Peter Nelson, The number of points in a matroid with no $n$-point line as a minor, J. Combin. Theory Ser. B 100 (2010) 625–630.

[4] R. L. Graham, K. Leeb, and B. L. Rothschild, Ramsey’s theorem for a class of categories, Adv. in Math. 8 (1972), 417–433.

[5] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, Ramsey Theory, Wiley-Interscience Series in Discrete Mathematics and Optimization, 1990.

[6] R. A. Hales and R. I. Jewett, Regularity and positional games, Trans. Amer. Math. Soc. 106 (1963), 222–229.

[7] Peter Nelson, Growth rate functions of dense classes of representable matroids, J. Combin. Theory Ser. B 103 (2013), 75–92.

[8] James G. Oxley, A characterization of certain excluded-minor classes of matroids, European J. Combin. 10 (1989), 275–279.

[9] James Oxley, Matroid Theory, Second edition, Oxford University Press, New York, 2011.

[10] R. A. Pendavingh and S. H. M. van Zwam, Confinement of matroid representations to subsets of partial fields, J. Combin. Theory, Ser. B 100 (2010), 510–545.

[11] Charles Semple and Geoff Whittle, On representable matroids having neither $U_{2,5}$- nor $U_{3,5}$-minors, in Matroid Theory, Contemporary Math. 197 (1995), 377–386.
[12] W. T. Tutte, Menger’s theorem for matroids, *J. Res. Nat. Bur. Standards Sect. B* 69B (1965), 49–53.