Canonical Structure of Noncommutative Quantum Mechanics as Constraint System

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Abstract

Starting with the first-order singular Lagrangian, the canonical structure in the noncommutative quantum mechanics with the noncommutativities both of coordinates and momenta is investigated. Using the projection operator method (POM) for the constraint systems and the constraint star-product, the noncommutative quantum system is constructed and the commutator algebra of projected canonically conjugate set(CC) of the system is derived in the form including all orders of the noncommutativity parameters. We discuss the alternative CCS, which obeys the ordinary noncommutative commutator algebra. The exact CCS is constructed in the framework of the POM, and which is shown to be equivalent to the CCS constructed through the Seiberg-Witten map and the Bopp shift. We further discuss the alternative Lagrangian to realize the noncommutativities both of coordinates and momenta.

1 Introduction

Since Snyder[1], Noncommutative extensions of quantum mechanics and quantum field theories have been extensively investigated until now[2, 3, 4, 5, 6, 7, 8]. It is well known that the dynamical systems with the noncommutativity among coordinates, momenta are able to be described by the constraint systems, and the dynamical models for such constraint systems have been investigated widely[9, 10, 11, 12, 13, 14, 15, 16, 17].

Using the projection operator method (POM) with the star-product quantization[18, 19, 20, 21], in this paper, we shall investigate the noncommutative quantum system with the noncommutativities both of coordinates and momenta.

For this purpose, we propose the first-order singular model Lagrangian with two kinds of Chern-Simons terms, and the initial unconstraint quantum system containing the sets of second-class constraint operators is prepared. The final constraint quantum system will be constructed through the successive projection-procedure[22]. Then, the commutator algebra of the projected CCS consisting of coordinate operators $q^i$ and momentum ones $p_i$ ($i = 1, \cdots, N$) will be shown to take the following form:

$$[q^i, q^j] = \frac{i}{\hbar} (M^{-1} \Theta M^{-1})_{ij}, \quad [q^i, p_j] = \frac{i}{\hbar} (M^{-1} \bar{M} M^{-1})_{ij}, \quad [p_i, p_j] = \frac{i}{\hbar} (M^{-1} \Xi M^{-1})_{ij},$$

(1.1)
where $M$ and $\bar{M}$ are the $N \times N$ matrices,

$$
M = I + \frac{1}{4} \Theta \Xi, \quad \bar{M} = I - \frac{1}{4} \Theta \Xi
$$

(1.2)

with the $N \times N$ unit matrix $I$, and $\Theta^{ij}$, $\Xi_{ij}$, the $N \times N$ completely antisymmetric matrices. These matrices will be defined in Sect.3 and their algebraic properties, discussed there. As shown in Ref.[22], the results of the successive projections generally depend on the order of the operations of projection operators. Because of the structure of the constraint operators in the initial system, however, the commutator algebra of the final projected system will be shown to be independent to the order of projections.

Following the POM, we will construct the CCS consisting of $Q^i$ and $P_i$ ($i = 1, \cdots, N$) in terms of the projected CCS, which satisfies

$$
[Q^i, Q^j] = 0, \quad [Q^i, P_j] = i\hbar \delta^i_j, \quad [P_i, P_j] = 0,
$$

(1.3)

and which we shall call the exact CCS. Then, it will be shown that the exact CCS provided by the POM holds the equivalent structure to the CCS obtained by the Seiberg-Witten map[2] and the Bopp shift[23]. We further discuss the alternative model Lagrangian to realize the noncommutativities both of coordinates and momenta.

This paper is organized as follows. In Sect.2, we briefly review the POM with the star-product quantization. In Sect.3, we first discuss the algebraic properties of the antisymmetric matrices $\Theta, \Xi$. We next propose the model Lagrangian and the initial unconstraint quantum system. Then, the canonical structure of the final constraint quantum system is constructed, and the alternative projected CCS is proposed. Following the POM, in Sect.4, we construct the exact CCS satisfying the commutator algebra (1.3) and the unified expression for the projected Hamiltonian of the final constraint quantum system is given. In Sect.5, we mention the alternative model Lagrangian, and some conclusions are given.

## 2 Star-product Quantization

We here present the brief review of the POM of constraint systems with the star-product quantization[18, 19].

Let $S = (\mathcal{C}, \mathcal{A}(\mathcal{C}), H(\mathcal{C}), \mathcal{K})$ be the initial unconstraint quantum system, where $\mathcal{C} = \{(q^i, p_i); i = 1, \cdots, N\} = \mathcal{C}(q, p)$ is a set of canonically conjugate operators (CCS), the commutator algebra of $\mathcal{C}$ with

\[\mathcal{A}(\mathcal{C}): [q^i, p_j] = i\hbar \delta^i_j, \quad [q^i, q^j] = [p_i, p_j] = 0, \quad (2.1)\]

and $H(\mathcal{C}) = H(q, p)$ is the Hamiltonian of the initial unconstraint system, $\mathcal{K} = \{T_\alpha(\mathcal{C}) | \alpha = 1, \cdots, 2M < 2N\}$, the set of the constraint-operators $T_\alpha(\mathcal{C})$ corresponding to the second-class constraints $T_\alpha = 0$. Starting with $S$, our goal is to construct the constraint quantum

\[\text{We shall denote } \mathcal{C} \text{ with } \mathcal{C}(q, p) \text{ and any } \mathcal{O} \text{ with } \mathcal{O}(q, p) \text{ when needed.} \]
system $S^{*} = (C^{*}, A^{*}(C^{*}), H^{*}(C^{*}))$, where $C^{*}$, which we shall call the projected CCS, is the set of $N - M$ projected canonically conjugate pairs satisfying

$$T_{\alpha}(C^{*}) = 0 \quad (\alpha = 1, \cdots, 2M). \quad (2.2)$$

The first step is to construct the associated canonically conjugate set (ACCS) from the constraint-operators $T_{\alpha}(C)$ and to prepare the projection operator $\hat{P}$ to eliminate $T_{\alpha}$ in the system, that is, $\hat{P}T_{\alpha} = 0 (\alpha = 1, \cdots, 2M)$, which we shall call the projection conditions.$^{[18]}$

Due to the Darboux’s theorem in the dynamical systems, it is possible in general to define the ACCS. Let $\{ (\xi^{a}, \pi_{a}) | \epsilon(\xi^{a}) = \epsilon(\pi_{a}) = s, a = 1, \cdots, M \}$ be the ACCS, and their symplectic forms be

$$Z_{\alpha} = \begin{cases} \xi^{a} & (\alpha = a) \\ \pi_{a} & (\alpha = a + M) \end{cases} \quad (\alpha = 1, \cdots, 2M ; \ a = 1, \cdots, M), \quad (2.3)$$

which obey the commutation relation

$$[Z_{\alpha}, Z_{\beta}] = i\hbar(-)^{s}J_{\alpha\beta} = i\hbar J^{\alpha\beta}, \quad (2.4)$$

where $s = \epsilon(\xi^{a}) = \epsilon(\pi_{a})$ is the Grassmann parity of $\xi, \pi$ and $J^{\alpha\beta}$ is the symplectic matrix and $J_{\alpha\beta}$ is the inverse of $J^{\alpha\beta}$. Then, we define the symplectic hyper-operators $\hat{Z}_{\alpha}^{(\pm)}(\alpha = 1, \cdots, 2M)$ as follows.$^{[4]}$

$$\hat{Z}_{\alpha}^{(-)} = \frac{1}{i\hbar}[Z_{\alpha}, \ ], \quad \hat{Z}_{\alpha}^{(\pm)} = \{ Z_{\alpha}, \ }, \quad (2.5)$$

which, from (2.4), obey the hyper-commutation relations

$$[\hat{Z}_{\alpha}^{(\pm)}, \hat{Z}_{\beta}^{(\pm)}] = 0,$$

$$[\hat{Z}_{\alpha}^{(\pm)}, \hat{Z}_{\beta}^{(\mp)}] = [\hat{Z}_{\alpha}^{(\mp)}, \hat{Z}_{\beta}^{(\pm)}] = J^{\alpha\beta}. \quad (2.6)$$

The projection operator $\hat{P}$ is defined by$^{[18]}$

$$\hat{P} = \exp \left[ (-1)^{s}\hat{Z}_{\alpha}^{(\pm)} \frac{\partial}{\partial \varphi_{\alpha}} \right] \exp[ J^{\alpha\beta}\varphi_{\beta}\hat{Z}_{\beta}^{(-)}] |_{\phi = 0}. \quad (2.7)$$

and the projection conditions for $\hat{P}$ are represented by

$$\hat{P}T_{\alpha}(C) = T_{\alpha}(\hat{P}C) = 0 \quad (\alpha = 1, \cdots, 2M), \quad (2.8a)$$

which we shall briefly denote as$^{[4]}$

$$\hat{P}K(C) = K(\hat{P}C) = 0. \quad (2.8b)$$

$^{[4]}$For any operators $A, B$, $[A, B] = AB - (-1)^{\epsilon(A)\epsilon(B)}BA$, $\{ A, B \} = \frac{1}{2}(AB + (-1)^{\epsilon(A)\epsilon(B)}BA)$

$^{[4]}$For a set of operators $O\{ O_{n} | n = 1, 2, \cdots \}$, we hereafter represent $\hat{P}O_{n}$, $O_{n}(C) (n = 1, 2, \cdots)$ as $\hat{P}O$, $O(C)$, respectively.
We next introduce two kinds of star-product as follows\cite{19}: For any operators $X$ and $Y$,
\[ X \star Y = \exp\left(\frac{\hbar}{2i} \hat{\Omega}^{\kappa}\right) X(\eta) Y(\zeta) \bigg|_{\eta=\zeta} \] (2.9)
and
\[ \hat{\mathcal{P}} X \star Y = \left(\hat{\mathcal{P}}(\eta) \hat{\mathcal{P}}(\zeta) \exp\left(\frac{\hbar}{2i} \hat{\Omega}_{\eta\zeta}\right) X(\eta) Y(\zeta) \right) \bigg|_{\eta=\zeta}. \] (2.10)

Here, $\hat{\Omega}_{\eta\zeta}$ is the hyper-operator defined by
\[ \hat{\Omega}_{\eta\zeta} = \hat{\xi}_{\alpha}(\eta) J^{\alpha\beta} \hat{\xi}_{\beta}(\zeta) \] (2.11a)
with the nonlocal representations for the operations of hyper-operators\cite{19}, which is also represented in terms of $\xi^a$, $\pi_a$ ($a = 1, \cdots, M$) as follows:
\[ \hat{\Omega}_{\eta\zeta} = \hat{\xi}^a(-\eta) \hat{\pi}_a(-\eta) - \hat{\pi}_a(-\eta) \hat{\xi}^a(-\eta). \] (2.11b)

Using the $\star$ and $\hat{\mathcal{P}} \star$-products, we finally define the commutator-formulas and the symmetrized product-ones under the operation of $\hat{\mathcal{P}}$ as follows:
\[ [\hat{\mathcal{P}} X, \hat{\mathcal{P}} Y] = (-1)^s [X, Y]_{\star} = (-1)^s \hat{\mathcal{P}} (X \star Y - (-1)^{\varepsilon_{X\varepsilon_{Y}} Y \star X}), \] (2.12)
\[ \{ \hat{\mathcal{P}} X, \hat{\mathcal{P}} Y \} = \hat{\mathcal{P}} \{ X, Y \}_{\star} = \hat{\mathcal{P}} \left(\frac{1}{2} (X \star Y + (-1)^{\varepsilon_{X\varepsilon_{Y}} Y \star X})\right), \]
and
\[ \hat{\mathcal{P}} [X, Y] = -(-1)^s [X, Y]_{\hat{\mathcal{P}} \star} = (-1)^s (X \hat{\mathcal{P}} \star Y - (-1)^{\varepsilon_{X\varepsilon_{Y}} Y \hat{\mathcal{P}} \star X}) \]
\[ \hat{\mathcal{P}} \{ X, Y \}_{\hat{\mathcal{P}} \star} = \frac{1}{2} (X \hat{\mathcal{P}} \star Y + (-1)^{\varepsilon_{X\varepsilon_{Y}} Y \hat{\mathcal{P}} \star X}). \] (2.13)

3 Construction of Noncommutative Quantum System

We shall consider the dynamical model to realize both of space-space and momentum-momentum noncommutativities with the constant noncommutativity-parameters. For this purpose, we propose the model Lagrangian, which is in the first-order and singular and contains two-kind of Chern-Simons like terms. Starting with this Langrangian, we shall construct the noncommutative quantum Hamiltonian system.

3.1 Noncommutativity Matrix $\Theta$, $\Xi$

Let $\Theta$ and $\Xi$ be the totally antisymmetric matrices defined as follows:
\[ \Theta = \theta \varepsilon, \quad \Xi = \eta \varepsilon, \] (3.1)
where $\theta$ is the constant parameter describing the noncommutativity of coordinates and $\eta$, that of momenta, and $\varepsilon$ is the $N \times N$ antisymmetric tensor defined as
\[
\varepsilon_{ij} = 1 \quad (i > j), \quad \varepsilon_{ji} = -\varepsilon_{ij} \quad (i, j = 1, \ldots, N).
\]
(3.2)

These matrices satisfy
\[
\Theta \Xi = \Xi \Theta, \quad (\Theta \Xi)^t = \Theta \Xi.
\]
(3.3)

In terms of $\Theta$ and $\Xi$, then, the following matrices are defined:
\[
G = \Theta \Xi = \Xi \Theta, \quad (3.4a)
\]
\[
M = I + \frac{1}{4} G, \quad (3.4b)
\]
\[
\bar{M} = I - \frac{1}{4} G, \quad (3.4c)
\]

which are symmetric and commutable with $\Theta$, $\Xi$, and therefore become commutable with each other:
\[
G^t = G, \quad M^t = M, \quad \bar{M}^t = \bar{M}, \quad (3.5a)
\]
\[
G \Theta = \Theta G, \quad G \Xi = \Xi G, \quad (3.5b)
\]
\[
M \Theta = \Theta M, \quad M \Xi = \Xi M, \quad \bar{M} \Theta = \Theta \bar{M}, \quad \bar{M} \Xi = \Xi \bar{M}. \quad (3.5c)
\]

Due to Eqs.(3.4), there exist the inverses $M^{-1}$ and $\bar{M}^{-1}$, which also satisfy the same properties as $M$, $\bar{M}$.

### 3.2 Noncommutative Quantum System

#### 3.2.1 Primary Hamiltonian System

Consider the dynamical system described by the first-order singular Lagrangian $L$
\[
L = L(x, \dot{x}, v, \dot{v}, u, \dot{u}, \lambda, \dot{\lambda})
\]
\[
= \dot{x}^i M_{ij} v_j - \lambda_i (u^i - x^i) - \frac{1}{2} \dot{\lambda}_i \Theta^{ij} v_j - \frac{1}{2} \dot{x}^i \Xi_{ij} u^j - h_0 (xvu).
\]
(3.6)

Following the canonical quantization formulation for constraint systems [18][24], then, the initial unconstraint quantum system $\mathcal{S} = (\mathcal{C}, \mathcal{A}(\mathcal{C}), H(\mathcal{C}), \mathcal{K})$ is obtained as follows:
\[
\mathcal{C} = \{(x^i, p_x^i), (v_i, \pi_v^i), (u^i, \pi_u^i), (\lambda_i, \pi_\lambda^i) | i = 1, \ldots, N\}, \quad (3.7a)
\]
\[
\mathcal{A}(\mathcal{C}) : [x^i, p_x^j] = i\hbar \delta^i_j, [v_i, \pi_v^j] = i\hbar \delta^i_j, [u^i, p_u^j] = i\hbar \delta^i_j,
\]
\[
[\lambda_i, \pi_\lambda^j] = i\hbar \delta^i_j, \quad \text{(the others)} = 0, \quad (3.7b)
\]
\[ H = \sum_{n=1}^{4} \{ \mu_i^{(n)}, \phi_i^{(n)} \} + \{ \lambda_i, \psi_i^{(1)} \} + h_0(xvu), \]  
(3.7c)

\[ \mathcal{K} = \{ \phi_i^{(1)}, \phi_i^{(2)}, \phi_i^{(3)}, \phi_i^{(4)}, \psi_i^{(1)}, \psi_i^{(2)} | i = 1, \cdots, N \} \]
with
\[ \phi_i^{(1)} = \bar{M}_{ij} v_j - p_i^x, \quad \phi_i^{(2)} = \pi_i^x + \frac{1}{2} \Theta ij v_j, \]  
(3.7d)
\[ \phi_i^{(3)} = p_i^u + \frac{1}{2} \Xi ij u^j, \quad \phi_i^{(4)} = \pi_i^u, \]
\[ \psi_i^{(1)} = u^i - x^i, \quad \psi_i^{(2)} = \lambda_i - (W^{-1}) ij \mathcal{H}_j^{(0)} (xvu). \]

Here, \( \phi^{(n)} (n = 1, \cdots, 4) \) are the constraint-operators corresponding to the primary constraints \( \phi^{(n)} \approx 0 \) due to the singularity of the Lagrangian (3.6), \( \psi^{(n)} (n = 1, 2) \), those corresponding to the secondary ones \( \psi^{(n)} \approx 0 \), and \( W = I + \bar{M}^{-1} G \bar{M}^{-1} \).

\[ \mathcal{H}_i^{(0)} (xvu) = ((\bar{M}^{-1} G \bar{M}^{-1}) ij \partial_x^j + (\bar{M}^{-1} \Xi) ij \partial_u^j - \partial_u^i) h_0 (xvu). \]
(3.9)

The commutator algebra \( \mathcal{A}(\mathcal{K}) \) is presented in Appendix A. The Lagrange multiplier operators \( \mu_i^{(n)} (n = 1, \cdots, 4) \) are determined together with the secondary constraints through the consistency conditions for the time evolution of the constraint-operators (see Appendix B).

### 3.2.2 Successive Projections of \( \mathcal{S} \)

According to the structure of the commutator algebra Eq.(A.1), it is convenient to classify \( \mathcal{K} \) into the following three subsets :

\[ \mathcal{K} = \mathcal{K}^{(A)} \oplus \mathcal{K}^{(B)} \oplus \mathcal{K}^{(C)} \]  
(3.10a)

with

\[ \mathcal{K}^{(A)} = \{ \phi^{(1)}, \phi^{(2)} \}, \quad \mathcal{K}^{(B)} = \{ \phi^{(3)}, \psi^{(1)} \}, \quad \mathcal{K}^{(C)} = \{ \phi^{(4)}, \psi^{(2)} \}. \]  
(3.10b)

As well as the Dirac bracket formalism, the POM satisfies the *iterative* property\([25, 26]\).

Starting with the initial system (3.7), we shall construct the constraint quantum system \( \mathcal{S}^* \) through the successive operations of projection operators\([22]\). For this purpose, we first rearrange the subsets (3.10b) to \( \mathcal{K}^{(n)} (n = 1, 2, 3) \), and let \( \hat{\mathcal{P}}^{(n)} \) be the projection operator associated to the subset \( \mathcal{K}^{(n)} \), that is, \( \hat{\mathcal{P}}^{(n)} \mathcal{K}^{(n)} = 0 \). Then, the successive projections of the operators of the system by \( \hat{\mathcal{P}}^{(n)} \) \( (n = 1, 2, 3) \) can be carried out through the program designated by the following diagram :

\[ \mathcal{C} \xrightarrow{\hat{\mathcal{P}}^{(1)}} \mathcal{C}^{(1)} \xrightarrow{\hat{\mathcal{P}}^{(2)}} \mathcal{C}^{(2)} \xrightarrow{\hat{\mathcal{P}}^{(3)}} \mathcal{C}^{(3)} , \]  
(3.11)

where

\[ \mathcal{C}^{(n)} = \hat{\mathcal{P}}^{(n)} \mathcal{C}^{(n-1)} \quad (n = 1, 2, 3) \]  
(3.12a)
with $C^{(0)} = C$, which satisfy
\[ K^{(n)}(C^{(n)}) = 0. \] (3.12b)
Then, $Z^{(n)}$ for the subsets $K^{(n)}$ $(n = 1, 2, 3)$ consist of the operators in $C^{(n-1)}$,
\[ Z^{(n)} = Z^{(n)}(C^{(n-1)}). \] (3.13)
From (2.7), therefore, the projection operators $\hat{P}^{(n)}$ are also represented as
\[ \hat{P}^{(n)} = \hat{P}^{(n)}(C^{(n-1)}) \quad (n = 1, 2, 3). \] (3.14)

### 3.2.3 Successive projection I

Let $K^{(n)}$ $(n = 1, 2, 3)$ be $K^{(1)} = K^{(C)}$, $K^{(2)} = K^{(B)}$ and $K^{(3)} = K^{(A)}$. Then, we shall accomplish the successive projection of operators through the following diagram:
\[ I : \hat{P}^{(1)}K^{(C)} = 0 \rightarrow \hat{P}^{(2)}K^{(B)} = 0 \rightarrow K^{(3)}K^{(A)} = 0. \] (3.15)

The ACCS $Z^{(n)}_{\alpha}$ of the projection operators $\hat{P}^{(n)}$ $(n = 1, 2, 3)$ are given as follows, respectively:

1. \[ Z^{(1)}_{\alpha} = Z^{(1)}_{\alpha}(C) = \begin{cases} \xi_{i}^{(1)} = \psi_{i}^{(2)} & (\alpha = i), \\ \pi_{i}^{(1)} = \phi_{i}^{(4)} & (\alpha = i + N), \end{cases} \]
2. \[ Z^{(2)}_{\alpha} = Z^{(2)}_{\alpha}(C^{(1)}) = \begin{cases} \xi_{i}^{(2)} = \psi_{i}^{(1)} & (\alpha = i), \\ \pi_{i}^{(2)} = \phi_{i}^{(3)} - \frac{1}{2}\xi_{j}\psi_{j}^{(1)} & (\alpha = i + N), \end{cases} \]
3. \[ Z^{(3)}_{\alpha} = Z^{(3)}_{\alpha}(C^{(2)}) = \begin{cases} \xi_{i}^{(3)} = (M^{-1})_{ij}(\phi_{j}^{(2)} + \frac{1}{2}\xi_{jk}\phi_{k}^{(2)}) & (\alpha = i), \\ \pi_{i}^{(3)} = (M^{-1})_{ij}(\phi_{j}^{(3)} - \frac{1}{2}\Theta_{jk}\phi_{k}^{(3)}) & (\alpha = i + N), \end{cases} \] (3.16)

Let $\hat{P}$ be $\hat{P} = \hat{P}^{(3)}\hat{P}^{(2)}\hat{P}^{(1)}$, then, $C^{(3)}$ is obtained as follows:
\[ C^{(3)} = \hat{P}C = C\{(\hat{P}x, \hat{P}p^{x}), (\hat{P}v, \hat{P}p_{v}), (\hat{P}u, \hat{P}p^{u}), (\hat{P}\lambda, \hat{P}\pi_{\lambda})\} \]
\[ = C^{(3)}\{(x, p^{x}), (v, p_{v}), (u, p^{u})\} \] (3.17a)
with
\[ \lambda_{i} = (W^{-1})_{ij}\hat{P}^{(3)}H^{(0)}_{j}(xvu), \]
\[ \pi_{i}^{\lambda} = 0. \] (3.17b)

Under the operation of $\hat{P}^{(3)}$ in the process I, now, the operators $x, v$ and $u$ become noncommutable with each other. For any operator $O(xvu)$, therefore, the projection
of $O(xvu)$ by $\hat{\mathcal{P}}^{(3)}$ would not always be equivalent to the operator $O$ consisting of the projections of $x, v, u$, that is,

$$\hat{\mathcal{P}}^{(3)} O(x, v, u) \neq O(\hat{\mathcal{P}}^{(3)} x, \hat{\mathcal{P}}^{(3)} v, \hat{\mathcal{P}}^{(3)} u).$$

(3.18)

The projection of $\mathcal{H}_j^{(0)}(xvu)$ in Eq.(3.17b) is thus denoted as the form of $\hat{\mathcal{P}}^{(3)}\mathcal{H}_j^{(0)}(xvu)$. From Eqs.(2.11b),(3.16), the hyper-operators $\hat{\Omega}_n^{(3\pi)}$ for $\hat{\mathcal{P}}^{(n)}$ ($n = 1, 2, 3$) are described by

$$\hat{\Omega}_n^{(3\pi)} = \xi_i^{(n)}(\eta)\pi_i^{(n)}(\zeta) - \pi_i^{(n)}(\eta)\xi_i^{(n)}(\zeta) \quad (n = 1, 2, 3),$$

(3.19)

the explicit forms of which are presented in Appendix C.

Using the commutator formulas and the symmetrized ones (2.12) and (2.13), one obtains the commutator algebra $A^{(3\pi)}$, which is presented in Appendix D. From the commutation relations (D1), thus, we shall adopt $\{(x, v)\}$ as the projected CCS $C^*$ in $\mathcal{S}^*$:

$$C^* = \{(x^i, v_i) | i = 1, \ldots, N\} = \{(x, v)\}$$

(3.20)

with

$$u^i = x^i, \quad \lambda_i = (W^{-1})_{ij}\hat{\mathcal{P}}^{(n)}\mathcal{H}_j^{(0)}(xvu(x)),$$

$$p^x_i = \bar{M}_{ij}v_j, \quad \pi^v_i = -\frac{1}{2}\Theta_{ij}v_j,$$

(3.21)

$$p^u_i = -\frac{1}{2}\bar{\Xi}_{ij}x^j, \quad \pi^\lambda_i = 0.$$

Then, the canonical structure $A^*(C^*)$ is represented as

$$A^*(C^*) : [x^i, x^j] = i\hbar(M^{-1}\Theta M^{-1})_{ij}$$

$$[x^i, v_j] = i\hbar(M^{-1}\bar{M}M^{-1})_{ij}$$

$$[v_i, v_j] = i\hbar(M^{-1}\bar{\Xi}M^{-1})_{ij},$$

(3.22)

and the projected Hamiltonian $H^*$ becomes

$$H^* = \hat{\mathcal{P}}^{(3)} h_0(xvu(x)),$$

(3.23)

where $h_0(xvu(x)) \in \mathcal{C}^{(2)}$.

Thus, we have obtained the constraint quantum system in the process I,

$$\mathcal{S}^* = (C^*, A^*(C^*), H^*(C^*))$$

(3.24)

### 3.2.4 Successive projection II

The constraint operators $\phi^{(4)}_\lambda = \pi^\lambda_i$ in $\mathcal{K}^{(c)}$ are commutable with the canonical pairs $(x, p^x), (v, \pi^v)$ and $(u, p^u)$. Therefore, it turns out that the result of the successive projection of the system does not depend on the order of the operation of $\hat{\mathcal{P}}^{(c)}$. 

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Consider, then, the projection process where the subsets $\mathcal{K}^{(A)}$ and $\mathcal{K}^{(B)}$ are interchanged in the process (3.15),

$$\Pi : \hat{P}^{(1)}\mathcal{K}^{(C)} = 0 \longrightarrow \hat{P}^{(2)}\mathcal{K}^{(A)} = 0 \longrightarrow \hat{P}^{(3)}\mathcal{K}^{(B)}. \quad (3.25)$$

The ACCS’s in the process (3.25) are given as follows:

1. $Z^{(1)}(C) = \begin{cases} \xi_i^{(1)} = \psi_i^{(2)} & (\alpha = i), \\
\pi_i^{(1)} = \phi_i^{(4)} & (\alpha = i + N), \end{cases}$

2. $Z^{(2)}(C) = \begin{cases} \xi_i^{(2)} = (\bar{M}^{-1})_{ij} \phi_j^{(1)} & (\alpha = i), \\
\pi_i^{(2)} = \phi_i^{(2)} - \frac{1}{2}(\bar{M}^{-1}\Theta)_{ij} \phi_j^{(1)} & (\alpha = i + N), \end{cases}$

3. $Z^{(3)}(C) = \begin{cases} \xi_i^{(3)} = (M^{-1})_{ij} (\bar{M} \psi_j^{(1)} + \frac{1}{2} \Xi_{jk} \phi_k^{(3)}) & (\alpha = i), \\
\pi_i^{(3)} = (M^{-1})_{ij} (\phi_j^{(3)} - \frac{1}{2}(\Xi \bar{M})_{jk} \psi_k^{(1)}) & (\alpha = i + N), \end{cases}$

and the hyper-operators $\hat{\Omega}_{n\zeta}^{(n)}$ ($n = 1, 2, 3$) for the process II is also presented in Appendix C.

As well as in the case of the successive projection I, then, we obtain the constraint quantum system $S^*$ in II, which is identical with $S^*$ in I, except that the projection of an operator $O(xvu)$ is represented as $\hat{P}^{(3)}\hat{P}^{(2)}O(xvu)$ in II.

Consequently, one can express the projected Hamiltonian $H^*(C^*)$ with the unified form as follows:

$$H^*(C^*) = H^*(xv) = \begin{cases} \hat{P}^{(3)}h_0(xvu(x)) & I \\
\hat{P}^{(3)}\hat{P}^{(2)}h_0(xvu) & II. \end{cases} \quad (3.27)$$

### 4 Exact Canonically Conjugate Set

The commutator algebra has been defined by Eq.(3.22). In order to construct the exact CCS, we first introduce

$$q^i = M_{ij}x^j, \quad p_i = M_{ij}v_j, \quad (4.1)$$

which obey the commutator algebra $\mathcal{A}(q, p)$,

$$\mathcal{A}(q, p) : \begin{align*}
[q^i, q^j] &= i\hbar\Theta^{ij}, \\
[q^i, p_j] &= i\hbar M_{ij}, \\
[p_i, p_j] &= i\hbar\Xi_{ij}. \quad (4.2)
\end{align*}$$

Let $C^*(Q, P)$ be the exact CCS with the commutator algebra

$$[Q^i, Q^j] = 0, \quad [Q^i, P_j] = i\hbar \delta^i_j, \quad [P_i, P_j] = 0, \quad (4.3)$$
we next introduce the projection operators defined by

\[ \hat{P}_q = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{Q}^{(+)n} \hat{P}^{(-)n}, \]
\[ \hat{P}_p = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{P}^{(+)}n \hat{Q}^{(-)n}, \]  

which satisfy

\[ \hat{P}_q \hat{Q}^i = \hat{P}_p \hat{P}^i = 0 \quad (i = 1, \cdots, N). \]  

Following the POM\[18, 27\], then, the exact CCS is obtained in the following way:

\[ Q^i = \hat{P}_p q^i = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{P}^{(+)}n \hat{Q}^{(-)n} q^i = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \hat{P}^{(+)}n \hat{Q}^{(-)n} q^i, \]  

\[ P_i = \hat{P}_q p_i = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{Q}^{(+)n} \hat{P}^{(-)n} p_i = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \hat{Q}^{(+)n} \hat{P}^{(-)n} p_i. \]  

From the commutator algebra (4.2), Eqs.(4.6) become

\[ Q^i = q^i + \frac{1}{2} \Theta^{ij} P_j, \]  
\[ P_i = p_i - \frac{1}{2} \Xi^{ij} Q_j. \]  

Then, one immediately obtains

\[ Q^i = M_{ij}^{-1}(q^j + \frac{1}{2} \Theta^{jk} P_k) = x^i + \frac{1}{2} \Theta^{ij} v_j, \]  
\[ P_i = M_{ij}^{-1}(p_j - \frac{1}{2} \Xi^{jk} q^k) = v_i - \frac{1}{2} \Xi^{ij} x^j, \]  

which are equivalent to the formulas in the so-called Bopp shift\[23\].

According to Eqs.(4.8), \( x^i \) and \( v_i \) become

\[ x^i = (M^{-1})_{ij}(Q^j - \frac{1}{2} \Theta^{jk} P_k), \quad v_i = (M^{-1})_{ij}(P_j + \frac{1}{2} \Xi^{jk} Q^k). \]  

The projected Hamiltonian \( H^* \) is thus represented as follows:

\[ H^* = H^*(x, v) = H^*(M^{-1}(Q - \frac{1}{2} \Theta P), M^{-1}(P + \frac{1}{2} \Xi Q)) = \tilde{H}^*(QP). \]  

which will be shown to contain the quantum corrections due to the noncommutativity among the projected CCS and the ACCS\[28\]. The projected Hamiltonian \( H^* \) is represented also in terms of \((q, p)\) defined by Eq.(4.1) as follows:

\[ H^* = H^*(M^{-1}q, M^{-1}p) = \mathcal{H}^*(q, p). \]  

Then, \( S^* \) becomes

\[ S^* = (\mathcal{C}^*(q, p), \mathcal{H}^*(q, p)), \]  

where \( \mathcal{A}^*(\mathcal{C}^*) \) is given by Eq.(4.2).
5 Discussion and Conclusions

Starting with the first-order singular Lagrangian, we have shown that the noncommutative quantum system $S^*$ is exactly constructed through the POM with the star-product quantization. The canonical structure in $S^*$ has been defined by the commutator-algebra $A^*(x, v)$, which contains all-order of the noncommutativity parameters through $M^{-1}$.

We have shown that the canonical structure $A^*$ is represented also in terms of the transformed CCS, $C^*(q, p)$, which obeys the ordinaly type (1.1) of commutator algebra.

We finally discuss the alternative model Lagrangian to realize both of space-space and momentum-momentum noncommutativities, which is proposed by

$$L' = L'(x, \dot{x}, v, \dot{v}, u, \dot{u}, \lambda, \dot{\lambda})$$

$$= \dot{x}^i v_i - \lambda_i (\bar{M}_{ij} u^j - x^i) - \frac{1}{2} \dot{v}_i \Theta^{ij} v_j - \frac{1}{2} \dot{u}^i \Xi_{ij} u^j - h_0(xvu). \quad (5.1)$$

Let $p_x^i = \partial L'/\partial \dot{x}^i$, then, we will obtain the noncommutative quantum system $S^* = (C^*(u, p_x), H^*(u, p_x))$, of which canonical structure $A^*(u, p_x)$ is equivalent to $A^*(x, v)$.

The projected Hamiltonians contain the quantum corrections due to the noncommutativity among the projected CCS and the ACCS. This problem will be investigated in near future.
Appendix

A Commutator Algebra of Constraint operators

Under the commutator algebra $A(C)$, the commutation relations among the initial constraint-operators are given by

\[ A(K) : \]
\[ [\phi_i^{(1)}, \phi_j^{(2)}] = i\hbar \bar{M}_{ij} \]
\[ [\phi_i^{(1)}, \psi_j^{(1)}] = -i\hbar \delta_{ij} \]
\[ [\phi_i^{(2)}, \psi_j^{(1)}] = -i\hbar(W^{-1})_{jk} \partial^x \mathcal{H}_k^0(xvu) \]
\[ [\phi_i^{(2)}, \phi_j^{(2)}] = i\hbar \Theta^{ij} \]
\[ [\phi_i^{(3)}, \psi_j^{(1)}] = -i\hbar \delta_{ij} \]
\[ [\phi_i^{(3)}, \phi_j^{(3)}] = i\hbar \Xi^{ij} \]
\[ [\phi_i^{(4)}, \psi_j^{(1)}] = -i\hbar \delta_{ij} \]
\[ [\phi_i^{(4)}, \phi_j^{(2)}] = -i\hbar \delta_{ij} \]
\[ (the \ others) = 0. \]

(B1)

B Lagrange multiplier operators

The Lagrange multiplier operators in the Hamiltonian (3.7c) are given as follow:

\[ \mu_{i(1)} = \mu_{i(1)}^{(1)}(xvu\lambda) = -(G^{-1}\Theta)_{ij}(\lambda_j + \partial^x h_0(xvu)) \]
\[ \mu_{i(2)} = \mu_{i(2)}^{(2)}(xvu\lambda) = (\bar{M}^{-1})_{ij}(\lambda_j - \partial^x h_0(xvu)) \]
\[ \mu_{i(3)} = \mu_{i(3)}^{(3)}(xvu\lambda) = (G^{-1}\Theta)_{ij}(\lambda_j + \partial^x h_0(xvu)) \]
\[ \mu_{i(4)} = \mu_{i(4)}^{(4)}(xvu\lambda) = (W^{-1})_{ij}(-\mu_{(1)}^k \partial^x_k + \mu_{(2)}^k \partial^x_k + \mu_{(3)}^k \partial^x_k) \mathcal{H}_j^0(xvu) \]
The representation of $\hat{\Omega}_{\eta\zeta}$

The explicit forms of hyper-operators $\hat{\Omega}_{\eta\zeta}^{(n)}$ ($n = 1, 2, 3$) in the projection processes I and II.

C.1 Projection process I

\[ \hat{\Omega}_{\eta\zeta}^{(1)} = \psi^{(2)}_{k}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta) - \hat{x}_{\lambda}^{(-)}(\eta) \hat{\psi}_{k}^{(2)}(-)(\zeta) \]

with \( \psi^{(2)}_{k} = \lambda_{k} - (W^{-1})_{kl} \mathcal{H}_{l}^{(0)}(xvu), \)

\[ \hat{\Omega}_{\eta\zeta}^{(2)} = -\Xi_{kl} \hat{x}_{\lambda}^{(-)}(\eta) \hat{\lambda}_{l}^{(-)}(\zeta) \]

\[ + (\hat{u}_{k}^{(-)}(\eta) \hat{p}_{k}^{(-)}(\zeta) - \hat{p}_{k}^{(-)}(\eta) \hat{u}_{k}^{(-)}(\zeta)) \]

\[ + (\hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta) - \hat{x}_{\lambda}^{(-)}(\eta) \hat{p}_{k}^{(-)}(\zeta)) \]

\[ + \frac{1}{2} \Xi_{kl} (\hat{x}_{\lambda}^{(-)}(\eta) \hat{u}_{l}^{(-)}(\zeta) + \hat{u}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta)) \],

\[ \hat{\Omega}_{\eta\zeta}^{(3)} = -(M^{-1}\Theta M^{-1})_{kl} \hat{x}_{\lambda}^{(-)}(\eta) \hat{p}_{l}^{(-)}(\zeta) \]

\[ - (M^{-1}\Xi M^{-1})_{kl} \hat{x}_{\lambda}^{(-)}(\eta) \hat{p}_{l}^{(-)}(\zeta) \]

\[ + \frac{1}{4} (M^{-1}G\Theta M^{-1})_{kl} \hat{x}_{\lambda}^{(-)}(\eta) \hat{p}_{l}^{(-)}(\zeta) \]

\[ + (M^{-1}\lambda M^{-1})_{kl} (\hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta) - \hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta)) \]

\[ + \frac{1}{2} (M^{-1}\Theta M^{-1})_{kl} (\hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta) + \hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta)) \]

\[ + (M^{-1}(I + \frac{1}{16}G^{2}) M^{-1})_{kl} (\hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta) - \hat{p}_{k}^{(-)}(\eta) \hat{x}_{\lambda}^{(-)}(\zeta)). \]
C.2 Projecion process II

\[
\hat{\Omega}_{\eta \zeta}^{(1)} = \hat{\psi}_{k}^{(2)(-)}(\eta)\hat{\pi}_{\lambda}^{k(-)}(\zeta) - \hat{\pi}_{\lambda}^{k(-)}(\eta)\hat{\psi}_{k}^{(2)(-)}(\zeta)
\]

with \( \hat{\psi}_{k}^{(2)} = \lambda_{k} - (W^{-1})_{kl}H_{l}^{(0)}(xvu) \),

\[
\hat{\Omega}_{\eta \zeta}^{(2)} = -(\bar{M}^{-1}(M^{-1})_{kl}\hat{p}_{k}^{x(-)}(\eta)\hat{p}_{l}^{x(-)}(\zeta) - (\bar{M}^{-1})_{kl}(\hat{p}_{k}^{x(-)}(\eta)\hat{p}_{l}^{x(-)}(\zeta) - \hat{\pi}_{\lambda}^{k(-)}(\eta)\hat{\pi}_{\lambda}^{k(-)}(\zeta) + \hat{\pi}_{\lambda}^{k(-)}(\eta)\hat{\pi}_{\lambda}^{k(-)}(\zeta))
\]

\[
\hat{\Omega}_{\eta \zeta}^{(3)} = -(\bar{M}^{-1}\bar{M}^{2}M^{-1})_{kl}\hat{\pi}_{\lambda}^{k(-)}(\eta)\hat{\pi}_{\lambda}^{k(-)}(\zeta) - (\bar{M}^{-1}\bar{M}^{2}M^{-1})_{kl}(\hat{\pi}_{\lambda}^{k(-)}(\eta)\hat{\pi}_{\lambda}^{k(-)}(\zeta) - \hat{\pi}_{\lambda}^{k(-)}(\eta)\hat{\pi}_{\lambda}^{k(-)}(\zeta))
\]
D Commutator Algebra $\mathcal{A}(\mathcal{C}^{(3)})$

\[ [x^i, x^j] = i\hbar (M^{-1}\Theta M^{-1})_{ij}, \quad [x^i, u^j] = i\hbar (M^{-1}\Theta M^{-1})_{ij}, \]
\[ [x^i, p^j_x] = i\hbar (M^{-1}\bar{M}\bar{M}^{-1})_{ij}, \quad [x^i, p^j_u] = i\hbar \frac{1}{2}(M^{-1}\Theta M^{-1})_{ij}, \]
\[ [p^i_x, p^j_x] = i\hbar (M^{-1}\bar{M}\bar{M}^{-1})_{ij}, \quad [v^i, p^j_x] = i\hbar (M^{-1}\Xi M^{-1})_{ij}, \]
\[ [v^i, v^j] = i\hbar (M^{-1}M^{-1})_{ij}, \quad [v^i, u^j] = -i\hbar (M^{-1}\bar{M}^{-1})_{ij}, \]
\[ [v^i, \pi^j_v] = i\hbar \frac{1}{2}(M^{-1}GM^{-1})_{ij}, \quad [v^i, p^j_u] = -i\hbar \frac{1}{2}(M^{-1}\Xi M^{-1})_{ij}, \]
\[ [\pi^i_v, \pi^j_v] = -i\hbar \frac{1}{4}(M^{-1}G\Theta M^{-1})_{ij}, \quad [u^i, p^j_x] = i\hbar (M^{-1}\bar{M}\bar{M}^{-1})_{ij}, \]
\[ [u^i, u^j] = i\hbar (M^{-1}\Theta M^{-1})_{ij}, \quad [u^i, \pi^j_v] = i\hbar \frac{1}{2}(M^{-1}\Theta \bar{M}^{-1})_{ij}, \]
\[ [u^i, p^j_u] = i\hbar \frac{1}{2}(M^{-1}GM^{-1})_{ij} \quad [p^i_x, \pi^j_v] = i\hbar \frac{1}{2}(M^{-1}\Theta M^{-1})_{ij}, \]
\[ [p^i_x, p^j_u] = -i\hbar \frac{1}{2}(M^{-1}\Xi GM^{-1})_{ij}, \quad [p^i_x, p^j_u] = -i\hbar \frac{1}{2}(M^{-1}\Xi \bar{M}^{-1})_{ij}, \]
\[ [x^i, v^j] = i\hbar (M^{-1}M^{-1})_{ij}, \quad [\pi^i_v, p^j_u] = i\hbar \frac{1}{4}(M^{-1}G\bar{M}^{-1})_{ij}, \]
\[ [x^i, \pi^j_v] = i\hbar \frac{1}{2}(M^{-1}\Theta M^{-1})_{ij}. \]
References

[1] H. S. Snyder, Phys.Rev. 71 (1947) 38.

[2] N. Seiberg and E. Witten, JHEP 9909 (1999) 032; String Theory and Noncommutative Geometry, [arXiv:hep-th/9908142v3].

[3] A. Connes, NonCommutative Geometry, Academic Press (1994).

[4] R. J. Szabo, Phys.Rep. 378 (2003) 207; Quantum Field Theory on Noncommutative Spaces, [arXiv:hep-th/0109162].

[5] A. Kokado, T. Okamura and T. Saito, Phys.Rev. D69 (2004)125007; Noncommutative Quantum Mechanics and Seiberg-Witten Map, [arXiv:hep-th/0401180].

[6] K. Li, J. Wang and C. Chen, Mod.Phys.Lett. A20 (2005) 2165; Representation of NoncommutativePhase Space, [arXiv:hep-th/0409234].

[7] C. Bastos and O. Bertolami, J.Math.Phys. 49 (2008) 072101; Weyl-Wigner Formulation of Noncommutative Quantum Mechanics, [arXiv:hep-th/0611257].

[8] V. G. Kupriyanov, J.Math.Phys. 54 (2013) 112105; Quantum mechanics with coordinate dependent noncommutativity, [arXiv:12044823 [math-ph]]; J.Phys.A:Math.Theor. 46 (2013) 245303.

[9] G. Jaroszkiewicz, J.Phys.A:Math. Gen. 28 (1995) L343.

[10] R. Jackiw, Nucl.Phys. Proc.Suppl. 108 (2002) 30.

[11] A. A. Deriglazov, Noncommutative quantum mechanics as a constrained system, [arXiv:hep-th/0112053].

[12] H. O. Girotti, Noncommutative Quantum Field Theories, [arXiv:hep-th/0301237].

[13] F. S. Bemfica and H. O. Girotti,Phys.Rev. D79 (2009) 125024.

[14] B. Dragovich and Z. Rakic, Noncommutative Quantum Mechanics with Path Integral, [arXiv:hep-th/0501231].

[15] P. A. Horváthy and M. S. Plyushchay, Phys.Lett.B595 (2004) 547; M. A. del Olmo and M. S. Plyushchay, Annals Phys. 321 (2006) 2830; P. D. Álvareza, J. Gomis, K. Kamimura and M. S. Plyushchay, Annals Phys. 322 (2007) 1556.

[16] A. A. Deriglazov and A. M. Pupasov-Maksimov, Lagrangian for Frenkel electron and position’s non-commutativity due to spin, [arXiv:hep-th/13126247]; Frenkel electron on an arbitrary electromagnetic background and magnetic Zitterbewegung [arXiv:hep-th/14017641].
[17] R. Banerjee, Mod.Phys.Lett. A17 (2002) 631; *A Novel Approach to Noncommutativity in Planar Quantum Mechanics*, [arXiv:hep-th/0106280].

[18] M. Nakamura and N. Mishima, Prog.Theor.Phys. 81 (1989) 514; M. Nakamura and H. Minowa, J.Math.Phys. 34 (1993) 50.

[19] M. Nakamura, *Star-product Quantization in Second-class Constraint Systems*, [arXiv:1108.4108v3 [math-ph]].

[20] M.I.Krivoruchenko, A.A.Raduta and A.Faessler, Phys.Rev. D73 (2006) 025008; *Quantum deformation of the Dirac bracket*, [arXiv:hep-th/0507049].

[21] M.I.Krivoruchenko, *Nonlinear Dynamics and Applications*. 13 (2006) 94; *Moyal dynamics of constraint systems*, [arXiv:hep-th/0610074].

[22] M. Nakamura, N. Okamoto and H. Minowa, Prog.Theor.Phys. 93 (1995) 597; M. Nakamura and H. Minowa, Nuovo Cim. 111B (1996) 521.

[23] For example,

A. Kokado, T. Okamura and T. Saito, Phys.Rev. D69 (2004) 125007; *Noncommutative Quantum Mechanics and Seiberg-Witten Map*, [arXiv:hep-th/0401180]; K.Li, J. Wang and C. Chen, K. Li, J. Wang and C. Chen, Mod.Phys.Lett. A20 (2005) 2165; *Representation of Noncommutative Phase Space*, [arXiv:hep-th/0409234]; S. Dult and K. Li, Mod.Phys.Lett. A21 (2006) 2971; *Commutator Anomaly in Noncommutative Quantum Mechanics*, [arXiv:hep-th/050860].

[24] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiba University, New York) 1969.

[25] A. G. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei Roma) 1976.

[26] M. Nakamura, and H. Minowa, Nuovo Cim. 104A (1991) 387.

[27] M. Nakamura and R. Suzuki, Prog.Theor.Phys. 64 (1980) 1086.

[28] M. Nakamura and K. Kojima, Nuovo Cim. 116B (2001) 287.