EXISTENCE AND STABILITY OF STRONG SOLUTIONS TO THE ABELS-GARCKE-GRÜN MODEL IN THREE DIMENSIONS

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ABSTRACT. This work is devoted to the analysis of strong solutions to the Abels-Garcke-Grün (AGG) model in three dimensions. First, we prove the existence of local-in-time strong solutions originating from an initial datum \((u_0, \phi_0) \in H^1_\sigma \times H^2(\Omega)\) such that \(\mu_0 \in H^1(\Omega)\) and \(|\phi_0| \leq 1\). For the subclass of initial data that are strictly separated from the pure phases, the corresponding strong solutions are locally unique. Finally, we show a stability estimate between the solutions to the AGG model and the model H. These results extend the analysis achieved by the author in Calc. Var. (2021) 60:100 to three dimensional bounded domains.

1. INTRODUCTION

Given a domain \(\Omega \subset \mathbb{R}^3\), we study the Abels-Garcke-Grün (AGG) model in \(\Omega \times (0, T)\)

\[
\begin{align*}
\partial_t (\rho(\phi) u) + \nabla \cdot (u \otimes (\rho(\phi) u + \bar{J})) - \nabla \cdot (\nu(\phi) \mathbb{D} u) + \nabla P &= -\nabla \phi \otimes \nabla \phi, \\
\nabla u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi &= \Delta \mu, \\
\mu &= -\Delta \phi + \Psi'(\phi),
\end{align*}
\]

completed with the following boundary and initial conditions

\[
\begin{align*}
\begin{cases}
  u = 0, & \partial_n \phi = \partial_n \mu = 0 \quad \text{on } \partial \Omega \times (0, T), \\
  u(\cdot, 0) = u_0, \quad \phi(\cdot, 0) = \phi_0 \quad \text{in } \Omega.
\end{cases}
\end{align*}
\]

Here, \(n\) is the unit outward normal vector on \(\partial \Omega\), and \(\partial_n\) denotes the outer normal derivative on \(\partial \Omega\). In the system, \(u = u(x, t)\) represents the volume averaged velocity, \(P = P(x, t)\) is the pressure of the mixture, and \(\phi = \phi(x, t)\) is the difference of the fluids concentrations. The operator \(\mathbb{D}\) is the symmetric gradient \(\frac{1}{2}(\nabla + \nabla^T)\). The flux term \(\bar{J}\), the density \(\rho\) and the viscosity \(\nu\) of the mixture are defined as

\[
\begin{align*}
\bar{J} &= -\frac{\rho_1 - \rho_2}{2} \nabla \mu, \\
\rho(\phi) &= \rho_1 \frac{1 + \phi}{2} + \rho_2 \frac{1 - \phi}{2}, \\
\nu(\phi) &= \nu_1 \frac{1 + \phi}{2} + \nu_2 \frac{1 - \phi}{2},
\end{align*}
\]

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Theorem 1.1. Let $\Omega$ be a bounded domain of class $C^3$ in $\mathbb{R}^3$. Assume that $\mathbf{u}_0 \in H^1_\sigma$ and $\phi_0 \in H^2(\Omega)$ such that $\|\phi_0\|_{L^\infty} \leq 1$, $|\vec{\phi}_0| < 1$, $\mu_0 = -\Delta \phi_0 + \Psi'(\phi_0) \in H^1(\Omega)$, and $\partial_n \phi_0 = 0$ on $\partial \Omega$. Then, there exist $T_0 > 0$,
depending on the norms of the initial data, and (at least) a strong solution \((u, P, \phi)\) to system (1.1)-(1.2) on \((0, T_0)\) in the following sense:

(i) The solution \((u, P, \phi)\) satisfies the properties
\[ u \in C([0, T_0]; H^1) \cap L^2(0, T_0; H^2) \cap W^{1,2}(0, T_0; L^2), \quad P \in L^2(0, T_0; H^1(\Omega)), \]
\[ \phi \in L^\infty(0, T_0; W^{2,0}(\Omega)), \quad \partial_\tau \phi \in L^\infty(0, T_0; (H^1(\Omega))') \cap L^2(0, T_0; H^1(\Omega)), \]
\[ \phi \in L^\infty(\Omega \times (0, T_0)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T_0), \]
\[ \mu \in L^\infty(0, T_0; H^1(\Omega)) \cap L^2(0, T_0; H^3(\Omega)), \quad F''(\phi) \in L^\infty(0, T_0; L^6(\Omega)). \]

(ii) The solution \((u, P, \phi)\) fulfills the system (1.1) almost everywhere in \(\Omega \times (0, T_0)\) and the boundary conditions \(\partial_n \phi = \partial_n \mu = 0\) almost everywhere in \(\partial \Omega \times (0, T_0)\).

Furthermore, if additionally \(\|\phi_0\|_{L^\infty} = 1 - \delta_0\), for some \(\delta_0 > 0\), then the solution is locally unique. This is, there exists a time \(T_1 : 0 < T_1 < T_0\), depending only on the norm of the initial data and \(\delta_0\), such that the solution is unique on the time interval \([0, T_1]\).

Before proceeding with our second result, it is worth mentioning that the proof of Theorem 1.1, although still based on a semi-Galerkin approximation, differs from the one of [21, Theorem 3.1] for several aspects. First, the proof of [21, Theorem 3.1] exploited the continuity of the chemical potential and the regularity of its time derivative, which are properties available for the strong solutions of the convective Cahn-Hilliard equation in two dimensions. Since these are still an open question in three dimensions, we overcome this issue by employing an approximation procedure involving the convective viscous Cahn-Hilliard equation (see Appendix A), together with an appropriate regularization of the initial datum. Such approximations are crucial to rigorously justify the higher-order Sobolev estimates obtained for the approximate solutions. Secondly, due to the lack of global-in-time separation property in three dimensions, we show local uniqueness of solutions departing from a subclass of initial data such that \(\|\phi_0\|_{L^\infty} < 1\). For such class of solutions, the separation property holds on a (possible short) time interval by embedding in Hölder spaces. Notice that the argument proposed in [22] based on estimates in dual spaces cannot be used due to the non-constant density. Moreover, the separation property (or, at least, \(L^p\)-estimates of \(\Psi'(\phi)\) and \(\Psi''(\phi)\)) seems to be necessary to control the additional term \(\rho'(\phi)(\nabla \mu \cdot \nabla)u\). Furthermore, the proof of the uniqueness relies on estimates of higher-order Sobolev spaces compared to the argument in [21, Theorem 3.1], which is due to the above mentioned novel term \(\rho'(\phi)(\nabla \mu \cdot \nabla)u\) in (1.1).

Next, we prove a stability result between the strong solutions to the AGG model and the model \(H\) departing from the same initial datum in terms of the density values.

**Theorem 1.2.** Let \(\Omega\) be a bounded domain of class \(C^3\) in \(\mathbb{R}^3\). Given an initial datum \((u_0, \phi_0)\) as in Theorems 1.1, we consider the strong solution \((u, P, \phi)\) to the AGG model with density (1.3) and the strong solution \((u_H, P_H, \phi_H)\) to the model \(H\) with constant density \(\overline{\rho} > 0\), both defined on \([0, T_0]\). Then, there exists a constant \(C\), that depends on the norm of the initial data, the time \(T_0\) and the parameters of the systems, such that
\[ \sup_{t \in [0, T_0]} \|u(t) - u_H(t)\|_{(H^1)^*} + \sup_{t \in [0, T_0]} \|\phi(t) - \phi_H(t)\|_{(H^1)^*} \leq C \left( \left| \frac{\rho_1 - \rho_2}{2} \right| + \left| \frac{\rho_1 + \rho_2}{2} - \overline{\rho} \right| \right). \]

**Remark 1.3.** Assuming that \(\rho_1 = \overline{\rho}\) and \(\rho_2 = \overline{\rho} + \varepsilon\), for (small) \(\varepsilon > 0\), the stability estimate (1.8) reads as
\[ \sup_{t \in [0, T_0]} \|u(t) - u_H(t)\|_{(H^1)^*} + \sup_{t \in [0, T_0]} \|\phi(t) - \phi_H(t)\|_{(H^1)^*} \leq C\varepsilon. \]
Theorem 1.2 justifies the model H as the constant density approximation of the AGG model when the two viscous fluids have negligible densities difference. To make a comparison with [21, Theorem 3.5], we notice that the estimate holds in dual Sobolev spaces. Indeed, the main idea is to write the momentum equation for the solutions difference \((u - u_H, \phi - \phi_H)\) as Navier-Stokes equations with constant density and exploit the uniqueness argument introduced in [22].

Plan of the paper. We report in Section 2 the preliminaries for the analysis. Sections 3 and 4 are devoted to the proof of Theorem 1.1, in particular, the local existence of strong solutions and their uniqueness, respectively. In Section 5 we prove the stability result contained in Theorem 1.2. The Appendix A is concerned with well-posedness results for the convective Viscous Cahn-Hilliard equation.

2. Notation and Functional Spaces

Let \(X\) be a real Banach space. Its norm is denoted by \(\| \cdot \|_X\) and the symbol \(\langle \cdot, \cdot \rangle_{X', X}\) stands for the duality between \(X\) and its dual space \(X'\). We assume that \(\Omega\) is a bounded domain in \(\mathbb{R}^3\) with boundary \(\partial\Omega\) of class \(C^3\). For \(p \in [1, \infty]\), let \(L^p(\Omega)\) denote the Lebesgue space with norm \(\| \cdot \|_{L^p}\). The inner product in \(L^2(\Omega)\) is denoted by \((\cdot, \cdot)\). For \(s \in \mathbb{N}, \; p \in [1, \infty]\), \(W^{s,p}(\Omega)\) is the Sobolev space with norm \(\| \cdot \|_{W^{s,p}}\). If \(p = 2\), we use the notation \(W^{s,p}(\Omega) = H^s(\Omega)\). For every \(f \in H^1(\Omega)'\), we denote by \(\overline{f}\) the generalized mean value over \(\Omega\) defined by \(\overline{f} = |\Omega|^{-1} \int_{\Omega} f \, dx\). By the generalized Poincaré inequality, there exists a positive constant \(C\) such that

\[
\|f\|_{H^1} \leq C \left( \|\nabla f\|_{L^2}^2 + |\overline{f}|^2 \right)^{\frac{1}{2}}, \quad \forall f \in H^1(\Omega). \tag{2.1}
\]

We recall the Ladyzhenskaya, Agmon and Gagliardo-Nirenberg inequalities in three dimensions

\[
\|f\|_{L^3} \leq C \|f\|_{L^2}^{\frac{1}{3}} \|f\|_{H^1}^{\frac{2}{3}}, \quad \forall f \in H^1(\Omega), \tag{2.2}
\]

\[
\|f\|_{L^\infty} \leq C \|f\|_{H^1}^3 \|f\|_{H^2}^{\frac{1}{3}}, \quad \forall f \in H^2(\Omega), \tag{2.3}
\]

\[
\|\nabla f\|_{L^3} \leq C \|f\|_{L^\infty}^{\frac{1}{3}} \|f\|_{H^2}^{\frac{2}{3}}, \quad \forall f \in H^2(\Omega), \tag{2.4}
\]

\[
\|f\|_{W^{1,4}} \leq C \|f\|_{H^1}^{\frac{5}{3}} \|f\|_{W^{2,6}}^{\frac{2}{3}}, \quad \forall f \in W^{2,6}(\Omega). \tag{2.5}
\]

Next, we introduce the Hilbert spaces of solenoidal vector-valued functions. In the case of a bounded domain \(\Omega \subset \mathbb{R}^3\), we define

\[
L^2_\sigma = \{u \in L^2(\Omega) : \text{div}u = 0 \text{ in } \Omega, \; u \cdot n = 0 \text{ on } \partial\Omega\},
\]

\[
H^1_\sigma = \{u \in H^1(\Omega) : \text{div}u = 0 \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega\}.
\]

We also use \((\cdot, \cdot)\) and \(\| \cdot \|_{L^2}\) for the inner product and the norm in \(L^2_\sigma\). The space \(H^1_\sigma\) is endowed with the inner product and norm \((u, v)_{H^1_\sigma} = (\nabla u, \nabla v)\) and \(\|u\|_{H^1_{\sigma}} = \|\nabla u\|_{L^2}\), respectively. We report the Korn inequality

\[
\|\nabla u\|_{L^2} \leq \sqrt{2} \|\mathcal{D} u\|_{L^2}, \quad \forall u \in H^1_{\sigma}, \tag{2.6}
\]

which implies that \(\|\mathcal{D} u\|_{L^2}\) is a norm on \(H^1_{\sigma}\) equivalent to \(\|u\|_{H^1_{\sigma}}\). We introduce the space \(H^2_\sigma = H^2(\Omega) \cap H^1_{\sigma}\) with inner product \((u, v)_{H^2_\sigma} = (Au, Av)\) and norm \(\|u\|_{H^2_\sigma} = \|Au\|_{L^2}\), where \(A = \mathbb{P}(-\Delta)\) is the Stokes operator and \(\mathbb{P}\) is the Leray projection from \(L^2(\Omega)\) onto \(L^2_\sigma\). We recall that there exists a positive constant \(C > 0\) such that

\[
\|u\|_{H^2} \leq C \|u\|_{H^2_\sigma}, \quad \forall u \in H^2_\sigma. \tag{2.7}
\]
We denote by $A^{-1} : (H^1_0)^\prime \to H^1_0$ the inverse map of the Stokes operator. That is, given $f \in (H^1_0)^\prime$, there exists a unique $u = A^{-1}f \in H^1_0$ such that $(\nabla A^{-1}f, \nabla v) = (f, v)$, for all $v \in H^1_0$. As a consequence, it follows that $\|f\|_2 := \|\nabla A^{-1}f\| = (f, A^{-1}f)^{1/2}$ is an equivalent norm on $(H^1_0)^\prime$.

Throughout this paper, we will use the symbol $C$ to denote a generic positive constant whose value may change from line to line. The specific value depends on the domain $\Omega$ and the parameters of the system, such as $\rho_\ast, \rho_\ast^*, \nu_\ast, \nu_\ast^*, \theta$ and $\theta_0$. Further dependencies will be specified when necessary.

3. **Proof of Theorem 1.1. Part one: Existence of Solutions**

In the sequel we will use the following notation

$$\rho_\ast = \min\{\rho_1, \rho_2\}, \quad \rho_\ast^* = \max\{\rho_1, \rho_2\}, \quad \nu_\ast = \min\{\nu_1, \nu_2\}, \quad \nu_\ast^* = \max\{\nu_1, \nu_2\}. $$

3.1. **Approximation of the Initial Datum.** First of all, we approximate the initial concentration $\phi_0$ following the argument introduced in [22]. For $k \in \mathbb{N}$, there exists a sequence of functions $(\phi_{0,k}, \tilde{\mu}_{0,k})$ such that

$$
\begin{align*}
-\Delta \phi_{0,k} + F'(\phi_{0,k}) &= \tilde{\mu}_{0,k} \quad \text{in } \Omega, \\
\partial_n \phi_{0,k} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(3.1)

where $\tilde{\mu}_{0,k} = h_k \circ \tilde{\mu}_0$, $h_k$ is a cut-off function and $\tilde{\mu}_0 = -\Delta \phi_0 + F'(\phi_0)$. It follows that $\tilde{\mu}_0 \in H^1(\Omega)$, and

$$\|\tilde{\mu}_{0,k}\|_{H^1} \leq \|\tilde{\mu}_0\|_{H^1}. $$

(3.2)

There exists a unique solution $\phi_{0,k}$ to (3.1) such that $\phi_{0,k} \in H^2(\Omega)$, $F'(\phi_{0,k}) \in L^2(\Omega)$, which satisfies (3.1) almost everywhere in $\Omega$ and $\partial_n \phi_{0,k} = 0$ almost everywhere on $\partial \Omega$. In addition, there exist $\bar{m} \in (0, 1)$, which is independent of $k$, and $\overline{k}$ sufficiently large such that

$$\|\phi_{0,k}\|_{H^1} \leq 1 + \|\phi_0\|_{H^1}, \quad |\bar{m}| < 1, \quad \|\phi_{0,k}\|_{H^2} \leq C(1 + \|\tilde{\mu}_0\|), \quad \forall k > \overline{k}. $$

(3.3)

Furthermore, since

$$\|F'(\phi_{0,k})\|_{L^\infty} \leq \|\tilde{\mu}_{0,k}\|_{L^\infty} \leq k. $$

As a byproduct, there exists $\delta = \delta(k) > 0$ such that

$$\|\phi_{0,k}\|_{L^\infty} \leq 1 - \delta. $$

(3.4)

As a consequence, due to $F'(\phi_{0,k}) \in H^1(\Omega)$, it is easily seen that $\phi_{0,k} \in H^3(\Omega)$. Finally, observing that $\tilde{\mu}_{0,k} \to \tilde{\mu}_0$ in $L^2(\Omega)$, it follows that $\phi_{0,k} \to \phi_0$ in $H^1(\Omega)$.

3.2. **Definition of the Approximate Problem.** Let us consider the family of eigenfunctions $\{w_j\}_{j=1}^\infty$ and eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of the Stokes operator $A$. For any integer $m \geq 1$, let $V_m$ denote the finite-dimensional subspaces of $L^2_0$ defined as $V_m = \text{span}\{w_1, \ldots, w_m\}$. The finite-dimensional spaces $V_m$ are endowed with the norm of $L^2_0$. The orthogonal projection on $V_m$ with respect to the inner product in $L^2_0$ is denoted by $\mathbb{P}_m$. Recalling that $\Omega$ is of class $C^3$, the regularity theory of the Stokes operator yields that $w_j \in H^3(\Omega) \cap H^1_0$ for all $j \in \mathbb{N}$. As a consequence, the following inverse Sobolev embedding inequalities hold for all $v \in V_m$

$$\|v\|_{H^1} \leq C_m \|v\|_{L^2}, \quad \|v\|_{H^2} \leq C_m \|v\|_{L^2}, \quad \|v\|_{H^3} \leq C_m \|v\|_{L^2}. $$

(3.5)
Let us set $T > 0$. For any $k > 0, \alpha \in (0, 1)$ and $m \in \mathbb{N}$, we claim that there exists an approximate solution $(u_m, \phi_m)$ to the system (1.1) - (1.2) in the following sense:

$$u_m \in C^1([0, T]; V_m),$$
$$\phi_m \in L^\infty(0, T; H^3(\Omega)), \quad \partial_t \phi_m \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \quad \phi_m \in L^\infty(\Omega \times (0, T)) : |\phi_m(x, t)| \leq 1 - \delta \text{ a.e. in } \Omega \times (0, T),$$
$$\mu_m \in L^\infty(0, T; H^2(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)), \quad (3.6)$$

for some $\delta > 0$, such that

$$\begin{align*}
(\rho(\phi_m) \partial_t u_m, w) + (\phi_m (u_m \cdot \nabla) u_m, w) + (\nu(\phi_m) D u_m, \nabla w) \\
- \frac{\rho_1 - \rho_2}{2} ((\nabla \mu_m \cdot \nabla) u_m, w) = (\mu_m \nabla \phi_m, w),
\end{align*} \quad (3.7)$$

for all $w \in V_m$ and $t \in [0, T]$,

$$\partial_t \phi_m + u_m \cdot \nabla \phi_m = \Delta \mu_m, \quad \mu_m = \alpha \partial_t \phi_m - \Delta \phi_m + \Psi'(\phi_m) \quad \text{a.e. in } \Omega \times (0, T), \quad (3.8)$$

together with

$$\begin{cases}
u_m = 0, \quad \partial_n \phi_m = \partial_n \mu_m = 0 & \text{on } \partial \Omega \times (0, T), \\
\tilde{u}_m(x, 0) = \mathbb{P}_m u_0, \quad \phi_m(x, 0) = \phi_{0,k} & \text{in } \Omega.
\end{cases} \quad (3.9)$$

3.3. **Existence of Approximate Solutions.** We exploit a fixed point argument to show the existence of $(u_m, \phi_m)$ satisfying (3.6)-(3.9). For this purpose, we fix $v \in W^{1,2}(0, T; V_m)$. We consider the convective Viscous Cahn-Hilliard system

$$\begin{align*}
\partial_t \phi_m + v \cdot \nabla \phi_m = \Delta \mu_m \\
\mu_m = \alpha \partial_t \phi_m - \Delta \phi_m + F'(\phi_m) - \theta_0 \phi_m
\end{align*} \quad \text{in } \Omega \times (0, T), \quad (3.10)$$

which is equipped with the boundary and initial conditions

$$\partial_n \phi_m = \partial_n \mu_m = 0 \quad \text{on } \partial \Omega \times (0, T), \quad \phi_m(x, 0) = \phi_{0,k} \quad \text{in } \Omega. \quad (3.11)$$

Thanks to Theorem A.1, there exists a unique solution $\phi_m$ to (3.10)-(3.11) such that

$$\phi_m \in L^\infty(0, T; \dot{H}^3(\Omega)), \quad \partial_t \phi_m \in L^\infty(0, T; \dot{H}^1(\Omega)) \cap L^2(0, T; \dot{H}^2(\Omega)),$$
$$\phi_m \in L^\infty(\Omega \times (0, T)) : |\phi_m(x, t)| \leq 1 - \tilde{\delta} \text{ a.e. in } \Omega \times (0, T),$$
$$\mu_m \in L^\infty(0, T; H^2(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)), \quad (3.12)$$

for some $\tilde{\delta}$ depending on $\alpha$ and $k$. We report the following estimates for the system (3.10)-(3.11):

[1.] $L^2$ estimate: for any $T > 0$

$$\sup_{t \in [0, T]} \left( \|\phi_m(t)\|_{L^2}^2 + \alpha \|\nabla \phi_m(t)\|_{L^2}^2 \right) + \int_0^T \|\Delta \phi_m(t)\|_{L^2}^2 \, dt \leq \|\phi_{0,k}\|_{L^2}^2 + \alpha \|\nabla \phi_{0,k}\|_{L^2}^2 + \theta_0^2 \|\Omega\| T.$$
3.12 Energy estimate: for any $T > 0$

$$
\sup_{t \in [0,T]} E_{\text{free}}(\phi(t)) + \frac{1}{2} \int_0^T \|\nabla \mu_m(\tau)\|_{L^2}^2 \, d\tau + \alpha \int_0^T \|\partial_t \phi_m(\tau)\|_{L^2}^2 \, d\tau
\leq E_{\text{free}}(\phi_{0,k}) + \frac{1}{2} \int_0^T \|\nu(\tau)\|_{L^2}^2 \, d\tau.
$$

(3.13)

We now make the ansatz

$$
u_m(x, t) = \sum_{j=1}^m a_j^m(t)w_j(x)
$$

as solution to the Galerkin approximation of (1.1) that reads as

$$
(\rho(\phi_m) \partial_t u_m, w_l) + (\rho(\phi_m)(v \cdot \nabla) u_m, w_l) + (\nu(\phi_m) D u_m, \nabla w_l)
- \frac{\rho_1 - \rho_2}{2} ((\nabla \mu_m, v) u_m, w_l) = (\mu_m \nabla \phi_m, w_l), \quad \forall l = 1, \ldots, m,
$$

(3.14)

such that $u_m(\cdot, 0) = \mathbb{P}_m u_0$. Setting $A^m(t) = (a_1^m(t), \ldots, a_m^m(t))^T$, (3.14) is equivalent to the system of differential equations

$$
M^m(t) \frac{d}{dt} A^m + L^m(t) A^m = G^m(t),
$$

(3.15)

where the matrices $M^m(t)$, $L^m(t)$ and the vector $G^m(t)$ are defined as

$$
(M^m(t))_{l,j} = \int_\Omega \rho(\phi_m) w_l \cdot w_j \, dx,
(L^m(t))_{l,j} = \int_\Omega \left( \rho(\phi_m)(v \cdot \nabla) w_j \cdot w_l + \nu(\phi_m) D w_j : \nabla w_l - \left( \frac{\rho_1 - \rho_2}{2} \right) (\nabla \mu_m, v) w_j \cdot w_l \right) \, dx,
(G^m(t))_l = \int_\Omega \mu_m \nabla \phi_m \cdot w_l \, dx,
$$

and $A^m(0) = ((\mathbb{P}_m u_0, w_1), \ldots, (\mathbb{P}_m u_0, w_m))^T$. The regularity properties (3.12) imply the continuity of $\phi_m \in C([0,T]; W^{1,4}(\Omega))$ and $\mu_m \in C([0,T]; H^1(\Omega))$. In turn, we have $\rho(\phi_m), \nu(\phi) \in C(\Omega \times [0, T])$. Moreover, we observe that $v \in C([0,T]; L^2)$. Thus, we infer that $M^m$ and $L^m$ belong to $C([0,T]; \mathbb{R}^{m \times m})$, and $G^m \in C([0,T]; \mathbb{R}^m)$. Since the matrix $M^m(\cdot)$ is definite positive on $[0,T]$ (see [23, Appendix A]), the inverse $(M^m)^{-1} \in C([0,T]; \mathbb{R}^{m \times m})$. Thus, the existence and uniqueness theorem for system of linear ODEs guarantees that there exists a unique solution $A^m \in C^1([0,T]; \mathbb{R}^m)$ to (3.15) on $[0,T]$. As a result, the problem (3.14) has a unique solution $u_m \in C^1([0,T]; V_m)$.

Next, multiplying (3.14) by $a_i^m$ and summing over $l$, we find

$$
\int_\Omega \rho(\phi_m) \partial_t \left( \frac{|u_m|^2}{2} \right) \, dx + \int_\Omega \rho(\phi_m) v \cdot \nabla \left( \frac{|u_m|^2}{2} \right) \, dx + \int_\Omega \nu(\phi_m)|D u_m|^2 \, dx
- \frac{\rho_1 - \rho_2}{2} \int_\Omega \nabla \mu_m \cdot \nabla \left( \frac{|u_m|^2}{2} \right) \, dx = \int_\Omega \mu_m \nabla \phi_m \cdot u_m \, dx.
$$

Integrating by parts, we obtain

$$
\frac{d}{dt} \int_\Omega \rho(\phi_m) \frac{|u_m|^2}{2} \, dx = \int_\Omega \left( \partial_t \rho(\phi_m) + \text{div} (\rho(\phi_m)v) \right) \frac{|u_m|^2}{2} \, dx + \int_\Omega \nu(\phi_m)|D u_m|^2 \, dx
$$
Recalling that $\rho'(\phi_m) = \frac{\rho_1 - \rho_2}{2}$ and $\text{div } v = 0$, by using (3.10)\textsubscript{1}, we have

$$- \int_{\Omega} \left( \partial_t \rho(\phi_m) + \text{div} \left( \rho(\phi_m)v \right) \right) \frac{|u_m|^2}{2} \, dx + \frac{\rho_1 - \rho_2}{2} \int_{\Omega} \Delta \mu_m \frac{|u_m|^2}{2} \, dx = 0.$$  

Thus, we infer that

$$\frac{d}{dt} \int_{\Omega} \rho(\phi_m) \frac{|u_m|^2}{2} \, dx + \int_{\Omega} \nu(\phi_m) \|D u_m\|^2 \, dx = \int_{\Omega} \phi_m \nabla \mu_m \cdot u_m \, dx. \quad (3.16)$$

By using (3.12)\textsubscript{2} and the Poincaré inequality, we get

$$\int_{\Omega} \phi_m \nabla \mu_m \cdot u_m \, dx \leq \|\phi_m\|_{L^\infty} \|\nabla \mu_m\|_{L^2} \|u_m\|_{L^2} \leq \frac{\nu_s}{2} \|D u_m\|_{L^2}^2 + \frac{1}{\lambda_1 \nu_s} \|\nabla \mu_m\|_{L^2}^2,$$

so, we find the differential inequality

$$\frac{d}{dt} \int_{\Omega} \rho(\phi_m) \frac{|u_m|^2}{2} \, dx + \frac{\nu_s}{2} \int_{\Omega} \|D u_m\|^2 \, dx \leq \frac{1}{\lambda_1 \nu_s} \|\nabla \mu_m\|_{L^2}^2. \quad (3.17)$$

Integrating the above inequality on $[0, s]$, with $s \in [0, T]$, and using (3.13), it follows that

$$\int_{\Omega} \frac{\rho^*}{2} |u_m(s)|^2 \, dx \leq \int_{\Omega} \rho(\phi_{0,k}) \frac{|p_m u_0|^2}{2} \, dx + \frac{2}{\lambda_1 \nu_s} E_{\text{free}}(\phi_{0,k}) + \frac{1}{\lambda_1 \nu_s} \int_0^s \|v(\tau)\|_{L^2}^2 \, d\tau, \quad (3.18)$$

which, in turn, entails that

$$\|u_m(s)\|_{L^2}^2 \leq \frac{\rho^*}{\rho_s} \|u_0\|_{L^2}^2 + \frac{4}{\lambda_1 \rho_s \nu_s} E_{\text{free}}(\phi_{0,k}) + \frac{2}{\lambda_1 \rho_s \nu_s} \int_0^s \|v(\tau)\|_{L^2}^2 \, d\tau. \quad (3.19)$$

At this point, setting

$$C_1 = \frac{\rho^*}{\rho_s} \|u_0\|_{L^2}^2 + \frac{4}{\lambda_1 \rho_s \nu_s} E_{\text{free}}(\phi_{0,k}), \quad C_2 = \frac{2}{\lambda_1 \rho_s \nu_s},$$

and assuming

$$\int_0^t \|v(\tau)\|_{L^2}^2 \, d\tau \leq C_3 e^{C_2 t}, \quad t \in [0, T], \quad (3.20)$$

where $C_3 = C_1 T$, we deduce that

$$\int_0^t \|u_m(s)\|_{L^2}^2 \, ds \leq C_3 + C_2 \int_0^t \int_0^s \|v(\tau)\|_{L^2}^2 \, d\tau \, ds \leq C_3 e^{C_2 t}, \quad \forall \, t \in [0, T]. \quad (3.21)$$

Furthermore, thanks to (3.19) and (3.20), we also infer that

$$\sup_{t \in [0, T]} \|u_m(t)\|_{L^2} \leq \left( C_1 + C_3 C_2 e^{C_2 T} \right)^{\frac{1}{2}} =: K_0. \quad (3.22)$$

Next, we control the time derivative of $u_m$. Multiplying (3.14) by $\frac{d}{dt} a^m_l$ and summing over $l$, we find

$$\rho_s \|\partial_t u_m\|_{L^2}^2 \leq - (\rho(\phi_m)(v \cdot \nabla) u_m, \partial_t u_m) - (\nu(\phi_m) D u_m, \nabla \partial_t u_m) + \frac{\rho_1 - \rho_2}{2} ((\nabla \mu_m \cdot \nabla) u_m, \partial_t u_m) + \phi_m \nabla \mu_m \cdot \partial_t u_m).$$
By exploiting (3.5), we obtain
\[
\rho_s \| \partial_t u_m \|_{L^2}^2 \leq \rho^* \| v \|_{L^2} \| \nabla u_m \|_{L^\infty} \| \partial_t u_m \|_{L^2} + \nu^* \| \mathbb{D} u_m \|_{L^2} \| \nabla \partial_t u_m \|_{L^2} \\
+ \left[ \frac{\rho_1 - \rho_2}{2} \right] \| \nabla u_m \|_{L^\infty} \| \nabla \mu_m \|_{L^2} \| \partial_t u_m \|_{L^2} + \| \phi_m \|_{L^\infty} \| \nabla \mu_m \|_{L^2} \| \nabla \partial_t u_m \|_{L^2} \\
\leq \rho^* C \| v \|_{L^2} \| u_m \|_{H^3} \| \partial_t u_m \|_{L^2} + \nu^* C_m^2 \| u_m \|_{L^2} \| \partial_t u_m \|_{L^2} \\
+ C \left[ \frac{\rho_1 - \rho_2}{2} \right] \| u_m \|_{H^3} \| \nabla \mu_m \|_{L^2} \| \partial_t u_m \|_{L^2} + C_m \| \nabla \mu_m \|_{L^2} \| \partial_t u_m \|_{L^2} \\
+ C_m \left[ \frac{\rho_1 - \rho_2}{2} \right] \| u_m \|_{L^2} \| \nabla \mu_m \|_{L^2} \| \partial_t u_m \|_{L^2} + C_m \| \nabla \mu_m \|_{L^2} \| \partial_t u_m \|_{L^2}.
\]

Then, by using (3.13), (3.20), (3.21) and (3.22), we infer that
\[
\int_0^T \| \partial_t u_m (\tau) \|_{L^2}^2 \, d\tau \leq 4 \left( \frac{\rho^*}{\rho_s} C_m K_0 \right)^2 \int_0^T \| v(\tau) \|_{L^2}^2 \, d\tau + 4 \left( \frac{\nu^* C_m^2}{\rho_s} \right) C_3 e^{C_2 T} \\
+ 4 \left( \frac{C_m}{\rho_s} \left[ \frac{\rho_1 - \rho_2}{2} \right] K_0 \right)^2 + \frac{C_m^2}{\rho_s^2} \int_0^T \| \nabla \mu_m (\tau) \|_{L^2} \, d\tau \\
\leq 4 \left( \frac{\rho^*}{\rho_s} C_m K_0 \right)^2 + \left( \frac{\nu^* C_m}{\rho_s} \right)^2 \left( 2 E_{\text{free}}(\phi_{0,k}) + C_3 e^{C_2 T} \right) =: K_1^2,
\]

where $K_1$ depends only on $\rho_s$, $\rho^*$, $\nu^*$, $\theta_0$, $\| u_0 \|_{L^2}$, $E_{\text{free}}(\phi_0)$, $T$, $\Omega$, $m$.

Now we define the setting of the fixed point argument. We introduce the set
\[
S = \left\{ u \in W^{1,2}(0, T; V_m) : \int_0^t \| u(\tau) \|_{L^2}^2 \, d\tau \leq C_3 e^{C_2 t}, \quad t \in [0, T], \quad \| \partial_t u \|_{L^2(0, T; V_m)} \leq K_1 \right\},
\]
which is a subset of $L^2(0, T; V_m)$. We define the map
\[
\Lambda : S \to L^2(0, T; V_m), \quad \Lambda(u) = u_m,
\]
where $u_m$ is the solution to the system (3.14). In light of (3.21) and (3.23), we deduce that $\Lambda : S \to S$. It is easily seen that $S$ is convex and closed. Furthermore, $S$ is a compact set in $L^2(0, T; V_m)$. We are left to prove that the map $\Lambda$ is continuous. This is done by adapting the argument in [21, Proof of Theorem 3.1] to the viscous case. Let us consider a sequence $\{ v_n \} \subset S$ such that $v_n \to \bar{v}$ in $L^2(0, T; V_m)$. By arguing as above, there exists a sequence $\{ (\psi_n, \mu_n) \}$ and $\bar{\psi}$ that solve the convective viscous Cahn-Hilliard equation (3.10)-(3.11), where $v$ is replaced by $v_n$ and $\bar{v}$, respectively. Repeating the uniqueness argument in the proof of Theorem A.1, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla A^{-1}(\psi_n - \bar{\psi}) \|_{L^2}^2 + \alpha \| \psi_n - \bar{\psi} \|_{L^2}^2 \right) + \| \nabla (\psi_n - \bar{\psi}) \|_{L^2}^2 \\
\leq \int_{\Omega} \psi_n (v_n - \bar{v}) \cdot \nabla A^{-1}(\psi_n - \bar{\psi}) \, dx + \int_{\Omega} (\psi_n - \bar{\psi}) \bar{v} \cdot \nabla A^{-1}(\psi_n - \bar{\psi}) \, dx + \theta_0 \| \psi_n - \bar{\psi} \|_{L^2}^2,
\]
where the operator $A$ is the Laplace operator $-\Delta$ with homogeneous Neumann boundary conditions. Since $\tilde{v}$ belong to $S$, we infer that
\[
\frac{1}{2} \frac{d}{dt} f(t) + \frac{1}{2} \| \nabla (\psi_n - \tilde{\psi}) \|^2_{L^2} \leq C f(t) + \| \nu_n - \tilde{\nu} \|^2_{L^2},
\]
where $f(t) = \| \nabla A^{-1}(\psi_n(t) - \tilde{\psi}(t)) \|^2_{L^2} + \alpha \| \psi_n(t) - \tilde{\psi}(t) \|^2_{L^2}$, for some constant $C$ depending on $C_1, C_2, K_1$ and $\theta_0$. Observing that $\psi_n(0) - \tilde{\psi}(0) = 0$, by the Gronwall lemma we obtain
\[
\| \psi_n - \tilde{\psi} \|_{L^\infty(0,T;L^2(\Omega))} \leq e^{CT} \int_0^T \| \nu_n(\tau) - \tilde{\nu}(\tau) \|^2_{L^2} d\tau \to 0, \quad \text{as } n \to \infty. \tag{3.24}
\]
On the other hand, using that $\{\nu_n\}$ and $\tilde{v}$ belong to $S$, the continuous embedding $W^{1,2}(0, T; V_n) \hookrightarrow Y_T$ (see Appendix A for the definition of $Y_T$ and the properties of the initial condition $\phi_{0,k}$ (cf. $\phi_{0,k} \in H^3(\Omega)$ and (3.4)) it follows from Theorem A.1 that
\[
\| \partial_t \psi_n \|_{L^\infty(0,T;H^1(\Omega))} + \| \partial_t \psi_n \|_{L^2(0,T;H^2(\Omega))} \leq C, \tag{3.25}
\]
\[
\| \partial_t \tilde{\psi} \|_{L^\infty(0,T;H^1(\Omega))} + \| \partial_t \tilde{\psi} \|_{L^2(0,T;H^2(\Omega))} \leq C, \tag{3.26}
\]
for some $C$ independent of $n$. Moreover, we also have
\[
\| \mu_n \|_{L^\infty(0,T;H^1(\Omega))} + \| \nu_n \|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{3.27}
\]
\[
\| \tilde{\mu} \|_{L^\infty(0,T;H^1(\Omega))} + \| \tilde{\psi} \|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{3.28}
\]
\[
\| \partial_t \mu_n \|_{L^2(0,T;L^2(\Omega))} \leq C, \quad \| \partial_t \tilde{\mu} \|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{3.29}
\]
and
\[
\max_{(x,t) \in \Omega \times (0,T)} | \psi_n(x,t) | \leq 1 - \delta^*, \quad \max_{(x,t) \in \Omega \times (0,T)} | \tilde{\psi}(x,t) | \leq 1 - \delta^*, \tag{3.30}
\]
for some positive $C$ and $\delta^* \in (0, 1)$, which are independent of $n$. In light of the above estimates, we first observe that $\mu_n - \tilde{\mu} \to \mu^*$ in $L^\infty(0,T;L^2(\Omega))$. Our goal is to show that $\mu^* = 0$. To this aim, we use the equation
\[
\mu_n - \tilde{\mu} = \varepsilon \partial_t (\psi_n - \tilde{\psi}) - \Delta (\psi_n - \tilde{\psi}) + \Psi'(\psi_n) - \Psi'(\tilde{\psi}).
\]
By standard interpolation, we deduce from (3.24), (3.27) and (3.28) that
\[
\| \psi_n - \tilde{\psi} \|_{L^\infty(0,T;H^2(\Omega))} \to 0, \quad \text{as } n \to \infty. \tag{3.31}
\]
As a consequence, thanks to (3.30), $\| \Psi'(\psi_n) - \Psi'(\tilde{\psi}) \|_{L^\infty(0,T;L^2(\Omega))} \to 0$, as $n \to \infty$. On the other hand, it follows from (3.24), (3.25) and (3.26) that $\partial_t (\psi_n - \tilde{\psi}) \to 0$ weakly in $L^2(0,T;H^2(\Omega))$. Thus, by uniqueness of the weak limit, we can conclude that
\[
\| \mu_n - \tilde{\mu} \|_{L^\infty(0,T;L^2(\Omega))} \to 0, \quad \text{as } n \to \infty. \tag{3.32}
\]
We now define $u_n = \Lambda(\nu_n) \in S$, for any $n \in \mathbb{N}$, and $\tilde{u} = \Lambda(\tilde{\nu}) \in S$. We consider $u = u_n - \tilde{u}$, $\psi = \psi_n - \tilde{\psi}$, $\nu = \nu_n - \tilde{\nu}$, and $\mu = \mu_n - \tilde{\mu}$ that solve
\[
(\rho(\psi_n) \partial_t u, w) + ((\rho(\psi_n) - \rho(\tilde{\psi})) \partial_t \tilde{u}, w) + (\rho(\psi_n)(\nu_n \cdot \nabla) u_n - \rho(\tilde{\psi})(\nu \cdot \nabla) \tilde{u}, w)
\]
\[
+ (\nu(\psi_n) \nabla u, \nabla w) + ((\nu(\psi_n) - \nu(\tilde{\psi})) \nabla \tilde{u}, \nabla w)
\]
\[
- \frac{p_1 - p_2}{2} ((\nabla \mu_n \cdot \nabla) u_n - (\nabla \nu \cdot \nabla) \tilde{u}, w) = (\mu_n \nabla \psi_n - \tilde{\mu} \nabla \tilde{\psi}, w),
\]
for all \( w \in V_m \), for all \( t \in [0, T] \). Taking \( w = u \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho(\psi_n) |u|^2 \, dx + \int_{\Omega} \nu(\psi_n) |D u|^2 \, dx = \frac{\rho_1 - \rho_2}{4} \int_{\Omega} \partial_t \psi_n |u|^2 \, dx - \frac{\rho_1 - \rho_2}{2} \int_{\Omega} \psi(\partial_t \tilde{u} \cdot u) \, dx
\]
\[
- \int_{\Omega} \left( \rho(\psi_n)(v \cdot \nabla)u_n - \rho(\tilde{\psi})(\tilde{v} \cdot \nabla)\tilde{u} \right) \cdot u \, dx - \frac{\nu_1 - \nu_2}{2} \int_{\Omega} \psi(D \tilde{u} : D u) \, dx
\]
\[
+ \frac{\rho_1 - \rho_2}{2} \int_{\Omega} \left( (\nabla \mu_n \cdot \nabla)u_n - (\nabla \tilde{\mu} \cdot \nabla)\tilde{u} \right) \cdot u \, dx + \int_{\Omega} \left( \mu_n \nabla \psi_n - \tilde{\mu} \nabla \tilde{\psi} \right) \cdot u \, dx.
\]
Thanks to (2.6) and (3.25), we have
\[
\frac{\rho_1 - \rho_2}{4} \int_{\Ω} \partial_t \psi_n |u|^2 \, dx \leq C \| \partial_t \psi_n \|_{L^6} \| u \|_{L^2} \| u \|_{L^3} \leq \frac{\nu_s}{10} \| D u \|_{L^2}^2 + C \| u \|_{L^2}^2,
\]
and
\[
- \frac{\rho_1 - \rho_2}{2} \int_{\Ω} \psi(\partial_t \tilde{u} \cdot u) \, dx \leq C \| \psi \|_{L^\infty} \| \partial_t \tilde{u} \|_{L^2} \| u \|_{L^2} \leq C \| u \|_{L^2}^2 + C \| \partial_t \tilde{u} \|_{L^2} \| \psi \|_{H^2}^2.
\]
Noticing that \( v_n, \tilde{v}, u_n \in S \), by exploiting (2.6) and (3.5), we find
\[
- \int_{\Omega} \left( \rho(\psi_n)(v \cdot \nabla)u_n - \rho(\tilde{\psi})(\tilde{v} \cdot \nabla)\tilde{u} \right) \cdot u \, dx
\]
\[
= - \frac{\rho_1 - \rho_2}{2} \int_{\Omega} \psi((v \cdot \nabla)u_n) \cdot u \, dx - \int_{\Omega} \rho(\tilde{\psi})((\tilde{v} \cdot \nabla)u_n) \cdot u \, dx - \int_{\Omega} \rho(\tilde{\psi})((\tilde{v} \cdot \nabla)u_n) \cdot u \, dx
\]
\[
\leq C \| \psi \|_{L^\infty} \| v \|_{L^\infty} \| \nabla u_n \|_{L^2} \| u \|_{L^2} + C \| v \|_{L^2} \| \nabla u_n \|_{L^2} \| u \|_{L^2} + C \| \tilde{v} \|_{L^\infty} \| \nabla u \|_{L^2} \| u \|_{L^2}
\]
\[
\leq C_m \| \psi \|_{H^2} \| u \|_{L^2} + C_m \| v \|_{L^2} \| u \|_{L^2} + C \| \nabla u \|_{L^2} \| u \|_{L^2}
\]
\[
\leq \frac{\nu_s}{10} \| D u \|_{L^2}^2 + C_m \| u \|_{L^2}^2 + C_m \| \psi \|_{H^2}^2 + C_m \| v \|_{L^2}^2.
\]
In addition, we deduce that
\[
- \frac{\nu_1 - \nu_2}{2} \int_{\Omega} \psi(D \tilde{u} : D u) \, dx \leq C \| \psi \|_{L^\infty} \| D \tilde{u} \|_{L^2} \| D u \|_{L^2} \leq \frac{\nu_s}{10} \| D u \|_{L^2}^2 + C_m \| \psi \|_{H^2}^2,
\]
and
\[
\frac{\rho_1 - \rho_2}{2} \int_{\Omega} \left( (\nabla \mu_n \cdot \nabla)u_n - (\nabla \tilde{\mu} \cdot \nabla)\tilde{u} \right) \cdot u \, dx
\]
\[
= - \frac{\rho_1 - \rho_2}{2} \int_{\Ω} (\mu_n \Delta u_n - \tilde{\mu} \Delta \tilde{u}) \cdot u \, dx - \frac{\rho_1 - \rho_2}{2} \int_{\Ω} (\mu_n \nabla u_n - \tilde{\mu} \nabla \tilde{u}) : \nabla u \, dx
\]
\[
= - \frac{\rho_1 - \rho_2}{2} \int_{\Ω} (\mu_n \Delta u_n + \tilde{\mu} \Delta \tilde{u}) \cdot u \, dx - \frac{\rho_1 - \rho_2}{2} \int_{\Ω} (\mu \nabla u_n + \tilde{\mu} \nabla \tilde{u}) : \nabla u \, dx
\]
\[
\leq C \| \mu \|_{L^\infty} \| \Delta u_n \|_{L^2} \| u \|_{L^\infty} + C \| \tilde{\mu} \|_{L^\infty} \| \Delta \tilde{u} \|_{L^2} \| u \|_{L^3}
\]
\[
+ C \| \mu \|_{L^\infty} \| \nabla u_n \|_{L^6} \| \nabla u \|_{L^3} + C \| \tilde{\mu} \|_{L^\infty} \| \nabla \tilde{u} \|_{L^6} \| \nabla u \|_{L^3}
\]
\[
\leq C_m \| \mu \|_{L^2} \| \nabla u \|_{L^2} + C_m \| \nabla u \|_{L^2} \| u \|_{L^2}
\]
Finally, by (3.27)-(3.28), we have
\[
\int_\Omega (\mu_n \nabla \psi_n - \bar{\mu} \nabla \tilde{\psi}) \cdot \mathbf{u} \, dx \leq (\|\mu\|_{L^2} \|
abla \psi_n\|_{L^2} + \|\bar{\mu}\|_{L^2} \|\nabla \psi\|_{L^2}) \|\mathbf{u}\|_{L^2}
\leq C (\|\mu\|_{L^2} + \|\psi\|_{H^2}) \|
abla \mathbf{u}\|_{L^2}
\leq \frac{\nu_*}{10} \|\nabla \mathbf{u}\|_{L^2}^2 + C \|\mu\|_{L^2}^2 + C \|\nabla \psi\|_{H^2}^2.
\]
Combining the above inequalities, we are led to the differential inequality
\[
\frac{d}{dt} \int_\Omega \rho(\psi_n) |\mathbf{u}|^2 \, dx \leq h_1(t) \int_\Omega \rho(\psi_n) |\mathbf{u}|^2 \, dx + h_2(t),
\]
where
\[
h_1(t) = C_m \left(1 + \|\partial_t \psi_n(t)\|_{H^1}^2\right)
\]
and
\[
h_2(t) = C_m \left(\|\nabla \mathbf{u}^\prime(t)\|_{L^2}^2 \|\psi(t)\|_{H^2}^2 + \|\psi(t)\|_{H^2}^2 + \|\mathbf{v}(t)\|_{L^2}^2 + \|\mu(t)\|_{L^2}^2\right).
\]
Thus, the Gronwall lemma entails
\[
\sup_{t \in [0,T]} \|\mathbf{u}(t)\|_{L^2}^2 \leq \frac{1}{\rho_*} e^{\int_0^t h_1(\tau) d\tau} \int_0^T h_2(\tau) d\tau.
\]
On account of (3.25), (3.31), (3.32), and the convergence \(v_n \to \tilde{\nu}\) in \(L^2(0, T; V_m)\), we deduce that \(u_n \to \tilde{\mathbf{u}}\) in \(L^\infty(0, T; V_m)\), implying that the map \(\Lambda\) is continuous. Finally, we are in the position to apply the Schauder fixed point theorem and conclude that the map \(\Lambda\) has a fixed point in \(S\), which gives the existence of the approximate solution \((\mathbf{u}_m, \phi_m)\) on \([0, T]\) satisfying (3.6)-(3.9) for any \(m \in \mathbb{N}\).

3.4. Uniform estimates independent of the approximation parameters. First, integrating (3.8) over \(\Omega\)
\[
\int_\Omega \phi_m(t) \, dx = \int_\Omega \phi_{0,k} \, dx, \quad \forall \, t \in [0, T].
\]
Owing to (3.3), for \(k > \overline{k}\), \(|\phi_m(t)| \leq \overline{m} < 1\) for all \(t \in [0, T]\). Taking \(w = \mathbf{u}_m\) in (3.7) and integrating by parts, we have (cf. (3.16))
\[
\frac{d}{dt} \int_\Omega \frac{1}{2} \rho(\phi_m) |\mathbf{u}_m|^2 \, dx + \int_\Omega \nu(\phi_m) |\nabla \mathbf{u}_m|^2 \, dx = \int_\Omega \mu_m \nabla \phi_m \cdot \mathbf{u}_m \, dx.
\]
Multiplying (3.10) by \(\mu_m\), integrating over \(\Omega\) and exploiting the definition of \(\mu_m\), we find
\[
\frac{d}{dt} \left(\int_\Omega \frac{1}{2} |\nabla \phi_m|^2 + \Psi(\phi_m) \, dx\right) + \int_\Omega |\nabla \mu_m|^2 + \alpha |\partial_t \phi_m|^2 \, dx + \int_\Omega \mathbf{u}_m \cdot \nabla \phi_m \mu_m \, dx = 0.
\]
By summing (3.35) and (3.36), we reach
\[
\frac{d}{dt} E(\mathbf{u}_m, \phi_m) + \int_\Omega \nu(\phi_m) |\nabla \mathbf{u}_m|^2 \, dx + \int_\Omega |\nabla \mu_m|^2 \, dx = 0.
\]
An integration in time on \([0, t]\), with \(0 < t \leq T\), yields
\[
E(\mathbf{u}_m(t), \phi_m(t)) + \int_0^t \int_\Omega \nu(\phi_m) |\nabla \mathbf{u}_m|^2 \, dx + \int_0^t \int_\Omega |\nabla \mu_m|^2 \, dx = E(\mathbb{P}_m \mathbf{u}_0, \phi_{0,k}), \quad \forall \, t \in [0, T].
\]
Thanks to (3.3) and (3.4), we observe that
\[
E(\mathbb{P}_m u_0, \phi_{0,k}) \leq \frac{\rho^*}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} \|\phi_0\|_{H^1}^2 + \theta_0 \left(1 + |\Omega| \max_{s \in [-1,1]} |\Psi(s)| \right).
\]
Since \( \phi_m \in L^\infty(\Omega \times (0,T)) : |\phi_m(x,t)| < 1 \) almost everywhere in \( \Omega \times (0,T), \) we obtain
\[
\|u_m\|_{L^\infty(0,T;L^2)} + \|u_m\|_{L^2(0,T;H^1)} \leq C, 
\]
(3.38)
\[
\|\phi_m\|_{L^\infty(0,T;H^1(\Omega))} \leq C, 
\]
(3.39)
\[
\|\nabla \mu_m\|_{L^2(0,T;L^2(\Omega))} \leq C, 
\]
(3.40)
\[
\sqrt{\alpha} \|\partial_t \phi_m\|_{L^2(0,T;L^2(\Omega))} \leq C, 
\]
(3.41)
where the constant \( C \) depends on \( \|u_0\|_{L^2} \) and \( \|\phi_0\|_{H^1}, \) but is independent of \( m, \alpha \) and \( k. \) Multiplying (3.10) by \( -\Delta \phi_m, \) integrating over \( \Omega \) and using (3.12), we get
\[
\|\Delta \phi_m\|_{L^2}^2 + \int_\Omega F''(\phi_m)|\nabla \phi_m|^2 \, dx = \alpha \int_\Omega \partial_t \phi_m \Delta \phi_m \, dx + \int_\Omega \nabla \mu_m \cdot \nabla \phi_m \, dx + \theta_0 \|\nabla \phi_m\|_{L^2}^2. 
\]
(3.42)
Since \( F''(s) > 0 \) for \( s \in (-1,1), \) by using (3.39), we have
\[
\|\Delta \phi_m\|_{L^2}^2 \leq C \left(1 + \|\nabla \mu_m\|_{L^2}^2 + \alpha^2 \|\partial_t \phi_m\|_{L^2}^2\right) , 
\]
for some \( C \) independent of \( m. \) Then, it follows from (3.40) and (3.41) that
\[
\|\phi_m\|_{L^2(0,T;H^2(\Omega))} \leq C. 
\]
(3.43)
We now recall the well-known inequality (see [28])
\[
\int_\Omega |F'(\phi_m)| \, dx \leq C \int_\Omega F'(\phi_m)(\phi_m - \bar{\phi}_{0,k}) \, dx + C, 
\]
(3.44)
where the constant \( C \) depends only on \( \bar{\phi}_{0,k}, \) thereby it is independent of \( k \) (for \( k \) large). Then, multiplying (3.8) by \( \phi_m - \bar{\phi}_{0,k} \) (cf. (3.34)), we find
\[
\int_\Omega |\nabla \phi_m|^2 \, dx + \int_\Omega F'(\phi_m)(\phi_m - \bar{\phi}_{0,k}) \, dx \\
= -\alpha \int_\Omega \partial_t \phi_m (\phi_m - \bar{\phi}_{0,k}) \, dx + \int_\Omega (\mu - \bar{\mu}) \phi_m \, dx + \theta_0 \int_\Omega \phi_m (\phi_m - \bar{\phi}_{0,k}) \, dx.
\]
By the Poincaré inequality and (3.39), we obtain
\[
\left| \int_\Omega F'(\phi_m)(\phi_m - \bar{\phi}_{0,k}) \, dx \right| \leq C \left(1 + \|\nabla \mu_m\|_{L^2} + \alpha \|\partial_t \phi_m\|_{L^2}\right). 
\]
(3.45)
Since \( \bar{\mu}_m = F'(\bar{\phi}_m) - \theta_0 \bar{\phi}_{0,k} \), we infer from (3.44) and (3.45) that
\[
\|\bar{\mu}_m\| \leq C \left(1 + \|\nabla \mu_m\|_{L^2} + \alpha \|\partial_t \phi_m\|_{L^2}\right). 
\]
Thanks to (2.1), we have
\[
\|\mu_m\|_{H^1} \leq C \left(1 + \|\nabla \mu_m\|_{L^2} + \alpha \|\partial_t \phi_m\|_{L^2}\right). 
\]
(3.46)
As a direct consequence, we deduce that
\[
\|\mu_m\|_{L^2(0,T;H^1(\Omega))} \leq C, 
\]
(3.47)
for some constant $C$ independent of $m$, $\alpha$ and $k$. In addition, using the boundary conditions (3.9) and (3.38), we find
\[
\| \partial_t \phi_m \|_{(H^1)'(\Omega)} \leq C (1 + \| \nabla \mu_m \|_{L^2}) ,
\]
which, in turn, implies that
\[
\| \partial_t \phi_m \|_{L^2(0,T;(H^1(\Omega))')} \leq C.
\]
Next, taking $w = \partial_t u_m$ in (3.7), we find
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \nu(\phi_m) \| \nabla \phi_m \|^2 \, dx + \int_\Omega \rho(\phi_m) \| \partial_t \phi_m \|^2 \, dx
\]
\[
= - \int_\Omega \rho(\phi_m) (u_m \cdot \nabla) \phi_m \cdot \partial_t u_m \, dx + \frac{\nu_1 - \nu_2}{2} \int_\Omega \partial_t \phi_m \| \nabla \phi_m \|^2 \, dx
\]
\[
+ \frac{\rho_1 - \rho_2}{2} \int_\Omega (\nabla u_m) \cdot \partial_t u_m \, dx + \int_\Omega \mu \nabla \phi_m \cdot \partial_t u_m \, dx. \tag{3.49}
\]
Thanks to the regularity of $\mu$ (cf. (3.12)), we multiply (3.8) by $\partial_t \mu_m$ and integrate over $\Omega$
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \| \nabla \mu_m \|^2 \, dx + (\partial_t \mu_m, \partial_t \phi_m) + (\partial_t \mu_m, u_m \cdot \nabla \phi_m) = 0.
\]
Direct computations give that
\[
(\partial_t \mu_m, \partial_t \phi_m) = \alpha (\partial_t \mu_m, \partial_t \phi_m) + \| \nabla \partial_t \phi_m \|^2_{L^2} + \int_\Omega F''(\phi_m) |\partial_t \phi_m|^2 \, dx - \theta_0 \| \partial_t \phi_m \|^2_{L^2}
\]
and
\[
(\partial_t \mu_m, u_m \cdot \nabla \phi_m) = \frac{d}{dt} \left( \int_\Omega \mu \nabla u_m \cdot \nabla \phi_m \, dx \right) - \int_\Omega \mu \partial_t u_m \cdot \nabla \phi_m \, dx - \int_\Omega \mu u_m \cdot \nabla \partial_t \phi_m \, dx.
\]
As a result, we find
\[
\frac{d}{dt} \left( \int_\Omega \frac{1}{2} \| \nabla \mu_m \|^2 \, dx + \int_\Omega \frac{\alpha}{2} |\partial_t \phi_m|^2 \, dx + \int_\Omega \mu u_m \cdot \nabla \phi_m \, dx \right) + \| \nabla \partial_t \phi_m \|^2_{L^2}
\]
\[
\leq \theta_0 \| \partial_t \phi_m \|^2_{L^2} + \int_\Omega \mu \partial_t u_m \cdot \nabla \phi_m \, dx + \int_\Omega \mu u_m \cdot \nabla \partial_t \phi_m \, dx. \tag{3.50}
\]
By summing (3.49) and (3.50), we arrive at
\[
\frac{d}{dt} H_m + \rho_s \| \partial_t u_m \|^2_{L^2} + \| \nabla \partial_t \phi_m \|^2_{L^2}
\]
\[
\leq - \int_\Omega \rho(\phi_m) (u_m \cdot \nabla) u_m \cdot \partial_t u_m \, dx + \frac{\nu_1 - \nu_2}{2} \int_\Omega \partial_t \phi_m \| \nabla \phi_m \|^2 \, dx
\]
\[
+ \frac{\rho_1 - \rho_2}{2} \int_\Omega (\nabla u_m) \cdot \partial_t u_m \, dx + 2 \int_\Omega \mu \nabla \phi_m \cdot \partial_t u_m \, dx + \theta_0 \| \partial_t \phi_m \|^2_{L^2} + \int_\Omega \mu u_m \cdot \nabla \partial_t \phi_m \, dx \tag{3.51}
\]
\[
= \sum_{k=1}^{6} R_k,
\]
Similarly, it is easily seen that for some $C$ where
\[ |\int_{\Omega} \mu_m u_m \cdot \nabla \phi_m \, dx| \leq \mu_m ||u_m||_{L^6} ||\nabla \phi_m||_{L^2} \]
\[ \leq C \left(1 + ||\nabla \mu_m||_{L^2} + \alpha ||\partial_t \phi_m||_{L^2}\right) ||\nabla u_m||_{L^2}^\frac{3}{2} \]
\[ \leq \frac{1}{4} \int_{\Omega} \nu(\phi_m) ||\nabla u_m||^2 \, dx + \frac{1}{4} ||\nabla \mu_m||_{L^2}^2 + \frac{\alpha}{4} ||\partial_t \phi_m||_{L^2}^2 + C_0, \]
for some $C_0$ independent of $m$, $\alpha$ and $k$. Thus, it follows that
\[ H_m \geq \frac{1}{4} \int_{\Omega} \nu(\phi_m) ||\nabla u_m||^2 \, dx + \frac{1}{4} ||\nabla \mu_m||_{L^2}^2 + \frac{\alpha}{4} ||\partial_t \phi_m||_{L^2}^2 - C_0. \] (3.52)
Similarly, it is easily seen that
\[ H_m \leq \int_{\Omega} \nu(\phi_m) ||\nabla u_m||^2 \, dx + ||\nabla \mu_m||_{L^2}^2 + \alpha ||\partial_t \phi_m||_{L^2}^2 + \tilde{C}_0, \] (3.53)
for some $\tilde{C}_0$ independent of $m$, $\alpha$ and $k$. Before proceeding with the estimate of the terms $R_i$, $i = 1, \ldots, 7$, we need to control the norms $||Au_m||_{L^2}$ and $||\mu_m||_{H^3}$. To this aim, taking $w = Au_m$ in (3.14), we have
\[ -\frac{1}{2} \nu(\phi_m) \Delta u_m, Au_m = - (\rho(\phi_m) \partial_t u_m, Au_m) - (\rho(\phi_m) (u_m \cdot \nabla)u_m, Au_m) \]
\[ + \frac{\rho_1 - \rho_2}{2} (\nabla \mu_m \cdot \nabla)u_m, Au_m) + (\mu_m \nabla \phi_m, Au_m) \] (3.54)
+ \[ \nu_1 - \nu_2 \frac{1}{2} (\nabla u_m \nabla \phi_m, Au_m). \]
By arguing as in [22] (see also [21]), there exists $\pi_m \in C([0, T]; H^1(\Omega))$ such that $-\Delta u_m + \nabla \pi_m = Au_m$ almost everywhere in $\Omega \times (0, T)$ and satisfies
\[ ||\pi_m||_{L^2} \leq C ||\nabla u_m||_{L^2}^\frac{3}{2} ||Au_m||_{L^2}^\frac{1}{2}, \quad ||\pi_m||_{H^1} \leq C ||Au_m||_{L^2}, \] (3.55)
where $C$ is independent of $m$, $\alpha$ and $k$. Therefore, we obtain
\[ \frac{1}{2} \nu(\phi_m) Au_m, Au_m = - (\rho(\phi_m) \partial_t u_m, Au_m) - (\rho(\phi_m) (u_m \cdot \nabla)u_m, Au_m) \]
\[ + \frac{\rho_1 - \rho_2}{2} (\nabla \mu_m \cdot \nabla)u_m, Au_m) + (\mu_m \nabla \phi_m, Au_m) \]
\[ + \frac{\nu_1 - \nu_2}{2} (\nabla u_m \nabla \phi_m, Au_m) - \frac{\nu_1 - \nu_2}{4} (\nabla \pi_m \nabla \phi_m, Au_m) \] (3.56)
\[ = \sum_{i=7}^{12} R_i. \]
On the other hand, taking the gradient of (3.8), multiplying it by $\nabla \Delta \mu$ and integrating over $\Omega$, we find
\[ ||\nabla \Delta \mu_m||_{L^2}^2 = (\nabla \partial_t \phi_m, \nabla \Delta \mu_m) + (\nabla (u_m \cdot \nabla \phi_m), \nabla \Delta \mu_m). \] (3.57)
Then, in light of (3.8)$_1$ and (3.9)$_1$, it follows that
\[ \|\mu_m\|^2_{H^3} \leq C \left( \|\mu_m\|^2_{H^1} + \|\nabla\Delta\mu_m\|^2_{L^2} \right), \]
which, in turn, by (3.52) gives that
\[ \|\mu_m\|^2_{H^3} \leq C \left( 1 + \|\nabla\mu_m\|^2_{L^2} + \alpha^2\|\partial_t\phi_m\|^2_{L^2} + (\nabla\partial_t\phi_m, \nabla\Delta\mu_m) + (\nabla(u_m \cdot \nabla\phi_m), \nabla\Delta\mu_m) \right) \]
\[ = C \left( 1 + C_0 + H_m \right) + \sum_{i=13}^{14} R_i, \tag{3.58} \]
where $C$ is independent of $m$, $\alpha$ and $k$. Now, multiplying (3.56) and (3.58) by two positive constants $\varpi_1$ and $\varpi_2$ (which will be chosen later on), respectively, and summing them to (3.51), we obtain
\[ \frac{d}{dt} H_m + \rho_s\|\partial_t u_m\|^2_{L^2} + \|\nabla\partial_t\phi_m\|^2_{L^2} + \frac{\nu_s\varpi_1}{2}\|A_u_m\|^2_{L^2} + \varpi_2\|\mu_m\|^2_{H^3} \]
\[ \leq C \left( 1 + \varpi_2 \right) \left( 1 + C_0 + H_m \right) + \sum_{i=1}^{6} R_i + \varpi_1 \sum_{i=7}^{12} R_i + \varpi_2 \sum_{i=13}^{14} R_i. \tag{3.59} \]
Let us proceed with the estimate of the terms $R_i$, $i = 1, \ldots, 14$. In the sequel the generic constant $C$ may depend on $\varpi_1$ and $\varpi_2$. Exploiting (2.2), (2.6), (3.38) and (3.52), we have
\[ \left| - \int_{\Omega} \rho(\phi_m)(u_m \cdot \nabla)u_m \cdot \partial_t u_m \, dx \right| \leq \rho_s\|u_m\|^6_{L^\infty} \|\nabla u_m\|^3_{L^3} \|\partial_t u_m\|^2_{L^2} \]
\[ \leq \frac{\rho_s}{8} \|\partial_t u_m\|^2_{L^2} + C \|\nabla u_m\|^3_{L^2} \|A_u_m\|_{L^2} \]
\[ \leq \frac{\rho_s}{8} \|\partial_t u_m\|^2_{L^2} + \frac{\nu_s\varpi_1}{32} \|A_u_m\|^2_{L^2} + C \|\nabla u_m\|^6_{L^2} \]
\[ \leq \frac{\rho_s}{8} \|\partial_t u_m\|^2_{L^2} + \frac{\nu_s\varpi_1}{32} \|A_u_m\|^2_{L^2} + C \left( C_0 + H_m \right)^3. \]
By Sobolev embedding, (2.2) and (3.52), we obtain
\[ \left| \frac{\nu_1 - \nu_2}{2} \int_{\Omega} \partial_t\phi_m \|\nabla u_m\|^2 \, dx \right| \leq C \|\partial_t\phi_m\|_{L^6} \|\nabla u_m\|_{L^3} \|\nabla u_m\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla\partial_t\phi_m\|^2_{L^2} + C \|A_u_m\|^3_{L^2} \|\nabla u_m\|_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla\partial_t\phi_m\|^2_{L^2} + \frac{\nu_s\varpi_1}{32} \|A_u_m\|^2_{L^2} + C \|\nabla u_m\|^3_{L^2} \]
\[ \leq \frac{1}{8} \|\nabla\partial_t\phi_m\|^2_{L^2} + \frac{\nu_s\varpi_1}{32} \|A_u_m\|^2_{L^2} + C \left( C_0 + H_m \right)^3. \]
By Sobolev interpolation, (2.3) and (3.46), we get
\[
\left| \frac{\rho_1 - \rho_2}{2} \int_\Omega (\nabla \mu_m \cdot \nabla) u_m \cdot \partial_t u_m \, dx \right| \leq C \| \nabla \mu_m \|_{L^\infty} \| \nabla u_m \|_{L^2} \| \partial_t u_m \|_{L^2}
\]
\[
\leq C \| \nabla \mu_m \|_{H^1} \| \mu_m \|_{H^3} \| \nabla u_m \|_{L^2} \| \partial_t u_m \|_{L^2}
\]
\[
\leq \frac{\rho_s}{8} \| \partial_t u_m \|_{L^2}^2 + C \| \nabla \mu_m \|_{H^1} \| \mu_m \|_{H^3} \| \nabla u_m \|_{L^2}^2
\]
\[
\leq \frac{\rho_s}{8} \| \partial_t u_m \|_{L^2}^2 + \frac{\omega_2}{6} \| \mu_m \|_{H^3}^2 + C \| \nabla \mu_m \|_{L^2} \| \nabla u_m \|_{L^2}^8
\]
\[
\leq \frac{\rho_s}{8} \| \partial_t u_m \|_{L^2}^2 + \frac{\omega_2}{6} \| \mu_m \|_{H^3}^2 + C (C_0 + H_m)^5.
\]

Exploiting (3.42), (3.46),(3.48) and (3.52), we find
\[
\left| \int_\Omega \mu_m \nabla \phi_m \cdot \partial_t u_m \, dx \right| \leq 2 \| \mu_m \|_{L^6} \| \nabla \phi_m \|_{L^4} \| \partial_t u_m \|_{L^2}
\]
\[
\leq \frac{\rho_s}{8} \| \partial_t u_m \|_{L^2}^2 + C \| \phi_m \|_{H^1} \| \mu_m \|_{H^3}^2
\]
\[
\leq \frac{\rho_s}{8} \| \partial_t u_m \|_{L^2}^2 + C \left( 1 + \| \nabla \mu_m \|_{L^2}^2 + \alpha^2 \| \partial_t \phi_m \|_{L^2}^2 \right)^2
\]
\[
\leq \frac{\rho_s}{8} \| \partial_t u_m \|_{L^2}^2 + C \left( 1 + C_0 + H_m \right)^2,
\]
\[
\theta_0 \| \partial_t \phi_m \|_{L^2}^2 \leq C \| \partial_t \phi_m \|_{H^1} \| \nabla \partial_t \phi_m \|_{L^2}
\]
\[
\leq \frac{1}{8} \| \nabla \partial_t \phi_m \|_{L^2}^2 + C \left( 1 + C_0 + H_m \right),
\]

and
\[
\left| \int_\Omega \mu_m u_m \cdot \nabla \partial_t \phi_m \, dx \right| \leq \| \mu_m \|_{L^6} \| u_m \|_{L^4} \| \nabla \partial_t \phi_m \|_{L^2}
\]
\[
\leq \frac{1}{8} \| \nabla \partial_t \phi_m \|_{L^2}^2 + C \| \nabla u_m \|_{L^2}^2 \left( 1 + \| \nabla \mu_m \|_{L^2}^2 + \alpha^2 \| \partial_t \phi_m \|_{L^2}^2 \right)
\]
\[
\leq \frac{1}{8} \| \nabla \partial_t \phi_m \|_{L^2}^2 + C \left( 1 + C_0 + H_m \right)^2.
\]

By Young’s inequality, we have
\[
\left| - \int_\Omega \rho(\phi_m) \partial_t u_m \cdot Au_m \, dx \right| \leq \omega_1 \rho_s \| \partial_t u_m \|_{L^2} \| Au_m \|_{L^2}
\]
\[
\leq \frac{\rho_s}{8 \omega_1} \| \partial_t u_m \|_{L^2}^2 + \frac{2 (\rho_s)^2 \omega_1}{\rho_s} \| Au_m \|_{L^2}^2.
\]
By using (2.2), (2.3), (2.6) and (3.52), we find

$$\left| - \int_{\Omega} \rho(\phi_m)(u_m \cdot \nabla) u_m \cdot Au_m \, dx \right| \leq \rho^* \| u_m \|_{L^6} \| \nabla u_m \|_{L^6} \| Au_m \|_{L^2}$$

$$\leq C \| \mathbb{D} u_m \|_{L^2} \| Au_m \|_{L^2}$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + C \| \mathbb{D} u_m \|_{L^2}^6$$

and

$$\left| \frac{\rho_1 - \rho_2}{2} \int_{\Omega} (\nabla \mu_m \cdot \nabla) u_m \cdot Au_m \, dx \right| \leq C \| \nabla \mu_m \|_{L^\infty} \| \nabla u_m \|_{L^2} \| Au_m \|_{L^2}$$

$$\leq C \| \nabla \mu_m \|_{L^1} \| \mu_m \|_{H^3} \| \nabla u_m \|_{L^2} \| Au_m \|_{L^2}$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + \frac{\nu_s^2}{6 \omega_1} \| \mu_m \|_{H^3}^2 + C \| \nabla \mu_m \|_{L^2}^2 \| \mathbb{D} u_m \|_{L^2}^8$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + \frac{\nu_s^2}{6 \omega_1} \| \mu_m \|_{H^3}^2 + C \| \nabla \mu_m \|_{L^2}^2 \| \mathbb{D} u_m \|_{L^2}^8$$

In light of (3.42) and (3.46), we have

$$\left| \int_{\Omega} \mu_m \nabla \phi_m \cdot Au_m \, dx \right| \leq \| \mu_m \|_{L^6} \| \nabla \phi_m \|_{L^6} \| Au_m \|_{L^2}$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + C \| \mu_m \|_{H^3} \| \phi_m \|_{H^2}^2$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + C \left( 1 + \| \nabla \mu_m \|_{L^2}^2 + \alpha^2 \| \partial_t \phi_m \|_{L^2}^2 \right)^2$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + C \left( 1 + C_0 + H_m \right)^2,$$

and

$$\left| \frac{\nu_1 - \nu_2}{2} \int_{\Omega} \mathbb{D} u_m \nabla \phi_m \cdot Au_m \, dx \right| \leq C \| \mathbb{D} u_m \|_{L^1} \| \nabla \phi_m \|_{L^6} \| Au_m \|_{L^2}$$

$$\leq C \| \mathbb{D} u_m \|_{L^2} \| Au_m \|_{L^2} \| \phi_m \|_{H^2}$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + C \| \mathbb{D} u_m \|_{L^2} \| \phi_m \|_{H^2}^4$$

$$\leq \frac{\nu_s}{32} \| Au_m \|_{L^2}^2 + C \left( 1 + C_0 + H_m \right)^2.$$. 
Finally, by exploiting (2.2), (2.3), (2.6), (3.42) and (3.52), we infer that

\[
\int_\Omega \nabla \partial_t \phi_m \cdot \nabla \Delta \phi_m \, dx \leq C \left( \| \nabla \phi_m \|_{L^2}^2 + \| \nabla^2 \phi_m \|_{L^2} \right) \| \nabla \Delta \phi_m \|_{L^2}.
\]

Combining (3.59) with the above estimates, we arrive at

\[
\frac{d}{dt} H_m + \frac{\rho_s}{2} \| \partial_t u_m \|_{L^2}^2 + \frac{1}{2} \| \nabla \partial_t \phi_m \|_{L^2}^2 + \left( \frac{\nu_s \varpi_1}{4} - \frac{2 (\rho^*)^2 \varpi_2}{\rho_s} \right) \| \Delta u_m \|_{L^2}^2
\]

\[
+ \left( \frac{\varpi_2}{2} - 2 \varpi_2 \right) \| \mu_m \|_{H^3}^2 \leq C \left( 1 + C_0 + H_m \right)^5,
\]

where the positive constant \( C \) depends on \( \varpi_1 \) and \( \varpi_2 \), but is independent of \( m \), \( \alpha \) and \( k \). Therefore, by setting

\[
\varpi_1 = \frac{\rho_s \nu_s}{16 (\rho^*)^2}, \quad \varpi_2 = \frac{1}{8},
\]

we deduce the differential inequality

\[
\frac{d}{dt} H_m + F_m \leq C \left( 1 + C_0 + H_m \right)^5,
\]

where

\[
F_m(t) = \frac{\rho_s}{2} \| \partial_t u_m(t) \|_{L^2}^2 + \frac{1}{2} \| \nabla \partial_t \phi_m(t) \|_{L^2}^2 + \frac{\varpi_1 \nu_s}{8} \| \Delta u_m(t) \|_{L^2}^2 + \frac{1}{32} \| \mu_m(t) \|_{H^3}^2,
\]

and the constant \( C \) is independent of the approximation parameters \( \alpha \), \( m \) and \( k \). Hence, whenever \( \tilde{T} > 0 \) satisfies

\[
1 - 4 C \tilde{T} \left( 1 + C_0 + H_m(0) \right)^4 > 0,
\]

we have

\[
H_m(t) \leq C \left( 1 + C_0 + H_m(0) \right)^5.
\]
we infer that
\[ C_0 + H_m(t) \leq \frac{1 + C_0 + H_m(0)}{(1 - 4Ct (C_1 + H_m(0))^4)^{\frac{1}{4}}}, \quad \forall t \in [0, \tilde{T}]. \quad (3.62) \]

To deduce an estimate of \( H_m \) which is independent of \( m, \alpha \) and \( k \), we are left to control \( \alpha \| \partial_t \phi_m(0) \|_{L^2} \) (cf. definition of \( H_m \) and (3.53)). To this aim, we first observe that \( \partial_t \phi_m \in C([0, T]; H^1(\Omega)) \), \( \mu_m \in C([0, T]; H^1(\Omega)) \) due to the regularity in Theorem A.1. By comparison in (3.8)_2, it follows that \( -\Delta \phi_m + \Psi'(\phi_m) \in C([0, T]; H^1(\Omega)) \). Now, multiplying (3.8)_2 by \( \partial_t \phi_m \) and integrating over \( \Omega \), we have
\[ \alpha \| \partial_t \phi_m \|^2_{L^2} + (-\Delta \phi_m + \Psi'(\phi_m), \partial_t \phi_m) = (\mu_m, \Delta \phi_m - \nabla \phi_m). \]

By using (3.8)_1, we find
\[ \alpha \| \partial_t \phi_m \|^2_{L^2} + \| \nabla \mu_m \|^2_{L^2} = (\nabla(-\Delta \phi_m + \Psi'(\phi_m)), \nabla \mu_m - \phi_m u_m) + (\nabla \mu_m, \phi_m u_m). \]

Integrating by parts, we arrive at
\[ \alpha \| \partial_t \phi_m \|^2_{L^2} + \| \nabla \mu_m \|^2_{L^2} = (\nabla(-\Delta \phi_{0,k} + \Psi'(\phi_{0,k})), \nabla \mu_m - \phi_{0,k} u_m(0)) + (\nabla \mu_m(0), \phi_{0,k} u_m(0)), \]

which, in turn, implies that
\[ \alpha \| \partial_t \phi_m(0) \|^2_{L^2} + \| \nabla \mu_m(0) \|^2_{L^2} \leq C \| \nabla(-\Delta \phi_{0,k} + \Psi'(\phi_{0,k})) \|^2_{L^2} + C \| u_m(0) \|^2_{L^2}. \]

Thus, we conclude from (3.1), (3.2), (3.3) and (3.53) that
\[ H_m(0) \leq C \left( 1 + \| u_0 \|^2_{H^3} + \| -\Delta \phi_0 + F'(\phi_0) \|^2_{H^1} + \| \phi_0 \|^2_{H^1} \right) + \tilde{C}_0 := \tilde{K}_0, \]

where the constant \( C \) is independent of \( m, \alpha \) and \( k \). Therefore, setting \( \tilde{T}_0 = \frac{1}{4C(C_1 + \tilde{K}_0)^2} \), it yields that
\[ 0 \leq C_0 + H_m(t) \leq \frac{1 + C_0 + \tilde{K}_0}{(1 - 4Ct (C_1 + \tilde{K}_0)^4)^{\frac{1}{4}}}, \quad \forall t \in [0, \tilde{T}_0). \]

Notice that \( \tilde{T}_0 \) is independent of \( m, \alpha \) and \( k \). Let us now fix \( T_0 \in (0, \tilde{T}_0) \). Thanks to (3.52), we infer that
\[ \sup_{t \in [0, T_0]} \| \nabla u_m(t) \|_{L^2} + \sup_{t \in [0, T_0]} \| \nabla \mu_m(t) \|_{L^2} + \sup_{t \in [0, T_0]} \sqrt{\alpha} \| \partial_t \phi_m(t) \|_{L^2} \leq K_1, \quad (3.64) \]

where \( K_1 \) is a positive constant that depends on \( E(u_0, \phi_0) \), \( \| u_0 \|_{H^3} \), \( \| \mu_0 \|_{H^1} \), and the parameters of the system, but is independent of \( m, \alpha \) and \( k \). Recalling (3.42) and (3.46), we immediately obtain
\[ \sup_{t \in [0, T_0]} \| \phi_m(t) \|_{H^2} + \sup_{t \in [0, T_0]} \| \mu_m(t) \|_{H^1} + \sup_{t \in [0, T_0]} \| F'(\phi_m(t)) \|_{L^2} \leq K_2. \]

Integrating (3.59) on \([0, T_0]\), we deduce that
\[ \int_0^{T_0} \| \partial_t u_m(\tau) \|^2_{L^2} + \| \nabla \partial_t \phi_m(\tau) \|^2_{L^2} + \| A u_m(\tau) \|^2_{L^2} + \| \mu_m(\tau) \|^2_{H^3} \, d\tau \leq K_3. \]

\[ \text{(3.66)} \]
Finally, in light of the regularity properties (3.64) and (3.66) of the velocity, we observe that the separation property (3.12)\textsubscript{2} (cf. Theorem A.1) only depends on $\alpha$ and $k$, but it independent of $m$, namely

\[
\phi_m \in L^\infty (\Omega \times (0, T)) : |\phi_m(x, t)| \leq 1 - \delta \text{ a.e. in } \Omega \times (0, T_0)
\]  

(3.67) for some $\tilde{\delta} = \tilde{\delta}(\alpha, k)$.

3.5. **Passage to the Limit and Existence of Strong Solutions.** Thanks to the above estimates (3.64)-(3.66), we deduce the following convergences (up to a subsequence) as $m \to \infty$

\[
\begin{align*}
    u_m &\to u_\alpha \quad \text{weak-star in } L^\infty (0, T_0; H^1_\sigma), \\
    u_m &\to u_\alpha \quad \text{weakly in } L^2 (0, T_0; H^2) \cap W^{1,2} (0, T_0; L^2_\sigma), \\
    \phi_m &\to \phi_\alpha \quad \text{weak-star in } L^\infty (0, T_0; H^2 (\Omega)), \\
    \phi_m &\to \phi_\alpha \quad \text{weakly in } W^{1,2} (0, T_0; H^1 (\Omega)), \\
    \mu_m &\to \mu_\alpha \quad \text{weak-star in } L^\infty (0, T_0; H^1 (\Omega)), \\
    \mu_m &\to \mu_\alpha \quad \text{weakly in } L^2 (0, T_0; H^3 (\Omega)).
\end{align*}
\]  

(3.68)

The strong convergences of $u_m$ and $\phi_m$ are recovered through the Aubin-Lions lemma, which implies that

\[
\begin{align*}
    u_m &\to u_\alpha \quad \text{strongly in } L^2 (0, T_0; H^1_\sigma), \\
    \phi_m &\to \phi_\alpha \quad \text{strongly in } C ([0, T_0]; W^{1,p} (\Omega)), \quad \forall \ p \in [2, 6].
\end{align*}
\]  

(3.69)

As a consequence, we infer that

\[
\rho (\phi_m) \to \rho (\phi_\alpha), \quad \nu (\phi_m) \to \nu (\phi_\alpha) \quad \text{strongly in } C ([0, T_0]; W^{1,p} (\Omega)),
\]

(3.70)

for all $p \in [2, 6]$. Additionally, we have

\[
\phi_\alpha \in L^\infty (\Omega \times (0, T)) : |\phi_\alpha(x, t)| \leq 1 - \delta \text{ a.e. in } \Omega \times (0, T_0)
\]  

(3.71)

for some $\delta = \delta(\alpha, k)$. The above properties entail the convergence of the nonlinear terms in (3.7) and of the logarithmic potential $\Psi(\phi)$ in (3.8), thereby we pass to the limit in the Galerkin formulation as $m \to \infty$ in (3.7)-(3.8). The limit solution $(u_\alpha, \phi_\alpha)$ satisfies

\[
\begin{align*}
    \rho (\phi_\alpha) \partial_t u_\alpha + (\rho (\phi_\alpha) (u_\alpha \cdot \nabla) u_\alpha, w) - (\partial_t \nu (\phi_\alpha) \delta u_\alpha, w) - \rho (\phi_\alpha) (\nabla \mu_\alpha \cdot \nabla) u_\alpha, w) - \rho (\phi_\alpha) \nabla \mu_\alpha, w) &= 0,
\end{align*}
\]  

(3.72)

for all $w \in L^2_\sigma$, $t \in [0, T_0]$, and

\[
\partial_t \phi_\alpha + u_\alpha \cdot \nabla \phi_\alpha = \Delta \mu_\alpha, \quad \mu_\alpha = \alpha \partial_t \phi_\alpha - \Delta \phi_\alpha + \Psi'(\phi_\alpha) \quad \text{a.e. in } \Omega \times (0, T_0).
\]

(3.73)

Moreover, we have

\[
\begin{align*}
    \begin{cases}
        u_\alpha &= 0, \quad \partial_n \phi_\alpha = \partial_n \mu_\alpha = 0 \quad \text{a.e. on } \partial \Omega \times (0, T), \\
        u_\alpha (\cdot, 0) &= u_0, \quad \phi_\alpha (\cdot, 0) = \phi_{0,k} \quad \text{in } \Omega.
    \end{cases}
\end{align*}
\]  

(3.74)

Next, we proceed with the vanishing viscosity limit in the Cahn-Hilliard equation. Thanks to the lower semicontinuity of the norm, we obtain from (3.64)-(3.66) that

\[
\begin{align*}
    \text{ess sup}_{t \in (0, T_0)} \| \nabla u_\alpha (t) \|_{L^2} + \text{ess sup}_{t \in (0, T_0)} \| \mu_\alpha (t) \|_{H^1} + \text{ess sup}_{t \in (0, T_0)} \sqrt{\alpha} \| \partial_t \phi_\alpha (t) \|_{L^2} &\leq K_1, \\
    \text{ess sup}_{t \in [0, T_0]} \| \phi_\alpha (t) \|_{H^2} + \text{ess sup}_{t \in [0, T_0]} \| F'(\phi_\alpha (t)) \|_{L^2} &\leq K_2.
\end{align*}
\]  

(3.75) (3.76)
and
\[ \int_0^{T_0} \|\partial_t \mathbf{u}_\alpha(\tau)\|^2_{L^2} + \|\nabla \partial_t \phi_\alpha(\tau)\|^2_{L^2} + \|A \mathbf{u}_\alpha(\tau)\|^2_{L^2} + \|\mu_\alpha(\tau)\|^2_{H^3} \, d\tau \leq K_3. \] (3.77)
Therefore, we can infer that
\[ \mathbf{u}_\alpha \rightharpoonup \mathbf{u}_k \quad \text{weak-star in } L^\infty(0, T_0; H^1_\sigma), \]
\[ \mathbf{u}_\alpha \to \mathbf{u}_k \quad \text{weakly in } L^2(0, T_0; H^2) \cap W^{1,2}(0, T_0; L^2_\sigma), \]
\[ \phi_\alpha \rightharpoonup \phi_k \quad \text{weak-star in } L^\infty(0, T_0; H^2(\Omega)), \]
\[ \phi_\alpha \to \phi_k \quad \text{weakly in } W^{1,2}(0, T_0; H^1(\Omega)), \] (3.78)
\[ \mu_\alpha \rightharpoonup \mu_k \quad \text{weak-star in } L^\infty(0, T_0; H^1(\Omega)), \]
\[ \mu_\alpha \to \mu_k \quad \text{weakly in } L^2(0, T_0; H^3(\Omega)). \]

In a similar manner as above, we have
\[ \mathbf{u}_\alpha \to \mathbf{u}_k \quad \text{strongly in } L^2(0, T_0; H^1_\sigma), \]
\[ \phi_\alpha \to \phi_k \quad \text{strongly in } C([0, T_0]; W^{1,p}(\Omega)), \]
\[ \rho(\phi_\alpha) \to \rho(\phi_k) \quad \text{strongly in } C([0, T_0]; W^{1,p}(\Omega)), \]
\[ \nu(\phi_\alpha) \to \nu(\phi_k) \quad \text{strongly in } C([0, T_0]; W^{1,p}(\Omega)), \] (3.79)
for all \( p \in [2, 6) \). In order to pass to the limit in \( F' \), we observe that
\[ \phi_\alpha \in L^\infty(\Omega \times (0, T_0)) : |\phi_\alpha(x, t)| < 1 \, \text{a.e. in } \Omega \times (0, T_0). \]
Thanks to (3.79)_2, it follows that \( \phi_\alpha \to \phi_k \) almost everywhere in \( \Omega \times (0, T) \), and thereby
\[ \phi_k \in L^\infty(\Omega \times (0, T_0)) : |\phi_k(x, t)| < 1 \, \text{a.e. in } \Omega \times (0, T_0). \]
Then, we have that \( F'(\phi_\alpha) \to F'(\phi_k) \) almost everywhere in \( \Omega \times (0, T) \) and, by Fatou Lemma, \( F'(\phi_k) \in L^2(\Omega \times (0, T)) \). Owing to this, and by (3.76), we conclude that
\[ F'(\phi_\alpha) \to F'(\phi_k) \quad \text{weakly in } L^\infty(0, T; L^2(\Omega)). \]
Thus, letting \( \alpha \to 0 \) in (3.73)-(3.72), we obtain
\[ (\rho(\phi_k)\partial_t \mathbf{u}_k, \mathbf{w}) + (\rho(\phi_k)(\mathbf{u}_k \cdot \nabla)\mathbf{u}_k, \mathbf{w}) - (\text{div} (\nu(\phi_k) \nabla \mathbf{u}_k), \mathbf{w}) - (\rho'(\phi_k)(\nabla \mu_k \cdot \nabla)\mathbf{u}_k, \mathbf{w}) - (\mu_k \nabla \phi_k, \mathbf{w}) = 0, \] (3.80)
for all \( \mathbf{w} \in L^2_\sigma, \ t \in [0, T_0], \) and
\[ \partial_t \phi_k + \mathbf{u}_k \cdot \nabla \phi_k = \Delta \mu_k, \quad \mu_k = -\Delta \phi_k + \Psi'(\phi_k) \quad \text{a.e. in } \Omega \times (0, T_0), \] (3.81)

\[ \begin{cases} \mathbf{u}_k = 0, & \partial_n \phi_k = \partial_n \mu_k = 0 \quad \text{a.e. on } \partial \Omega \times (0, T), \\ (\mathbf{u}_k(\cdot, 0) = \mathbf{u}_0, \ \phi(\cdot, 0) = \phi_{0,k} & \text{in } \Omega. \end{cases} \] (3.82)

Finally, since the estimates (3.75)-(3.77) are independent of \( k \), we can further pass to the limit as \( k \to \infty \). The argument readily follows the one above, and so it left to the reader. As a result, we obtain
\[ (\rho(\phi)\partial_t \mathbf{u} + \rho(\phi)(\mathbf{u} \cdot \nabla)\mathbf{u} - \text{div} (\nu(\phi) \nabla \mathbf{u}) - \rho'(\phi)(\nabla \mu \cdot \nabla)\mathbf{u} - \mu \nabla \phi, \mathbf{w}) = 0, \] (3.83)
for all \( w \in L^2_\sigma, \ t \in [0, T_0], \) and
\[
\partial_t \phi + u \cdot \nabla \phi = \Delta \mu \quad \mu = -\Delta \phi + \Psi' (\phi) \quad \text{a.e. in } \Omega \times (0, T_0),
\]
(3.84)

together with
\[
\begin{cases}
u = 0, \quad \partial_{nn} \phi = \partial_{nn} \mu = 0 \quad \text{a.e. on } \partial \Omega \times (0, T), \\\nu (\cdot, 0) = \nu_0, \quad \phi (\cdot, 0) = \phi_0 \quad \text{in } \Omega.
\end{cases}
\]
(3.85)

Recalling the well-known relation
\[
\mu \nabla \phi = -\operatorname{div} (\nabla \phi \otimes \nabla \phi) + \nabla \left( \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \right),
\]

in a classical way, there exists \( P \in L^2 (0, T_0; H^1 (\Omega)) \), \( \bar{P} (t) = 0 \) (see, e.g., [18]) such that
\[
\nabla P = -\rho (\phi ) \partial_t u - \rho (\phi ) (u \cdot \nabla u) + \operatorname{div} (\nu (\phi ) \nabla u) + \rho ' (\phi ) \nabla u \nabla \mu - \operatorname{div} (\nabla \phi \otimes \nabla \phi).
\]

Moreover, exploiting the regularity theory of the Cahn-Hilliard equation with logarithmic potential (see [1, Lemma 2] or [22, Theorem A.2]), we deduce that \( \phi \in L^\infty (0, T; W^{2,6} (\Omega)) \) and \( F' (\phi ) \in L^\infty (0, T; L^6 (\Omega)) \).

4. PROOF OF THEOREM 1.1. PART TWO: UNIQUENESS

Let \((u_1, P_1, \phi_1)\) and \((u_2, P_2, \phi_2)\) be two strong solutions to system (1.1)-(1.2) defined on the interval \([0, T_0]\) as stated in Theorem 1.1. We define \( u = u_1 - u_2, \ P = P_1 - P_2 \) and \( \phi = \phi_1 - \phi_2 \), which solve
\[
\begin{align*}
\rho (\phi_1) \partial_t u + (\rho (\phi_1) - \rho (\phi_2)) \partial_t u_2 + (\rho (\phi_1) (u_1 \cdot \nabla) u_1 - \rho (\phi_2) (u_2 \cdot \nabla) u_2) \\
- \frac{\rho_1 - \rho_2}{2} \left( (\nabla \mu_1 \cdot \nabla) u_1 - (\nabla \mu_2 \cdot \nabla) u_2 \right) - \operatorname{div} (\nu (\phi_1) \nabla u) - \operatorname{div} ((\nu (\phi_1) - \nu (\phi_2)) \nabla u_2) \\
+ \nabla P = -\operatorname{div} (\nabla \phi_1 \otimes \nabla \phi_2 - \nabla \phi_2 \otimes \nabla \phi_2),
\end{align*}
\]
(4.1)
\[
\begin{align*}
\partial_t \phi + u_1 \cdot \nabla \phi + u_2 \cdot \nabla \phi_2 = \Delta \mu, \\
\mu = -\Delta \phi + \Psi' (\phi_1) - \Psi' (\phi_2),
\end{align*}
\]
(4.2)

almost everywhere in \( \Omega \times (0, T_0) \). We recall that
\[
\| \phi_i \|_{L^\infty (0, T_0; W^{2,6} (\Omega))} + \| \partial_t \phi_i \|_{L^2 (0, T_0; H^1 (\Omega))} \leq K, \quad i = 1, 2,
\]
(4.3)

where \( K \) is a positive constant only depending on \( E (u_0, \phi_0), \| u_0 \|_{H^5}, \| \mu_0 \|_{H^1} \) and \( T_0 \). As a consequence, we claim that
\[
\| \phi_i \|_{C\left([0,T_0];C(\overline{\Omega})\right)} \leq CK, \quad i = 1, 2,
\]
for some constant \( C \) depending only on \( \Omega \). Indeed, by (2.5), we have
\[
\| \phi_i (t_1) - \phi_i (t_2) \|_{C(\overline{\Omega})} \leq C \| \phi_i (t_1) - \phi_i (t_2) \|_{W^{1,4}}
\]
\[
\leq C \| \phi_i (t_1) - \phi_i (t_2) \|_{H^1} \| \phi_i (t_1) - \phi_i (t_2) \|_{W^{2,6}}^{\frac{3}{5}}
\]
\[
\leq CK^{\frac{2}{5}} \left( \int_{t_1}^{t_2} \| \partial_t \phi_i (\tau) \|_{H^1} \, d\tau \right)^{\frac{3}{5}}
\]
\[
\leq CK^{\frac{2}{5}} \| \partial_t \phi_i \|_{L^2(0,T_0;H^1(\Omega))} |t_1 - t_2|^{\frac{3}{10}}, \quad \forall t_1, t_2 \in [0, T_0], \ i = 1, 2.
\]
In light of the assumption \( \| \phi_0 \|_{L^\infty} = 1 - \delta_0 \) for some \( \delta_0 > 0 \), we infer that

\[
\| \phi(t) \|_{L^\infty} \leq 1 - \frac{\delta_0}{2}, \quad \forall t \in [0, T_1], \quad \text{where} \quad T_1 = \left( \frac{\delta_0}{2CK} \right)^{\frac{4}{5}}. \tag{4.4}
\]

Owing to (4.4), it is possible to deduce by elliptic regularity that \( \phi \in L^2(0, T_1; H^5(\Omega)) \) and \( \partial_t \mu \in L^2(0, T_1; (H^1(\Omega))^\prime) \).

Next, multiplying (4.1) by \( u \) and integrating over \( \Omega \), we find

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho(\phi_1)|u|^2 \, dx + \int_\Omega \nabla(\phi_1)||\nabla u||^2 \, dx
\]

\[
= - \int_\Omega \left( \frac{\rho(\phi_1) - \rho(\phi_2)}{2} \right) \partial_t u_2 \cdot u \, dx - \int_\Omega \rho(\phi_1)(u_2 \cdot \nabla)u_2 \cdot u \, dx
\]

\[
- \int_\Omega \left( \frac{\rho(\phi_1) - \rho(\phi_2)}{2} \right) (u_2 \cdot \nabla)u_2 \cdot u \, dx + \frac{\rho_1 - \rho_2}{2} \int_\Omega ((\nabla \mu \cdot \nabla)u_2) \cdot u \, dx
\]

\[
- \int_\Omega \left( \frac{\rho(\phi_1) - \rho(\phi_2)}{2} \right) \nabla u_2 : \nabla u \, dx + \int_\Omega \left( \nabla \phi_1 \otimes \nabla \phi + \nabla \phi \otimes \nabla \phi_2 \right) \cdot \nabla u \, dx
\]

\[
= \sum_{i=1}^{6} Z_i.
\]

Here we have used that

\[
- \int_\Omega \partial_t \rho(\phi_1) \frac{|u|^2}{2} \, dx + \int_\Omega \rho(\phi_1)u_1 \cdot \nabla \frac{|u|^2}{2} \, dx - \frac{\rho_1 - \rho_2}{2} \int_\Omega \nabla \mu_1 \cdot \nabla \frac{|u|^2}{2} \, dx = 0.
\]

Taking the gradient of (4.2), multiplying by \( \nabla \Delta \phi \) and integrating over \( \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \Delta \phi \|_{L^2}^2 + \| \Delta^2 \phi \|_{L^2}^2 = \int \nabla u_1 \cdot \nabla \phi \Delta^2 \phi \, dx + \int \nabla u_2 \cdot \nabla \phi_2 \Delta^2 \phi \, dx + \int \Delta(\Psi'(\phi_1) - \Psi'(\phi_2)) \Delta^2 \phi \, dx
\]

\[
= \sum_{i=7}^{9} Z_i.
\]

Therefore, we arrive at

\[
\frac{d}{dt} \left( \frac{1}{2} \int_\Omega \rho(\phi_1)|u|^2 \, dx + \frac{1}{2} \| \Delta \phi \|_{L^2}^2 \right) + \int_\Omega \nu(\phi_1)\|\nabla u\|^2 \, dx + \| \Delta^2 \phi \|_{L^2}^2 = \sum_{i=1}^{9} Z_i.
\]

Arguing in a similar way as in [21, Section 6], it is easily seen that

\[
|Z_1 + Z_2 + Z_3 + Z_5 + Z_6| \leq \frac{\nu}{2} \| \nabla u \|_{L^2}^2 + C \left( 1 + \| u_2 \|_{L^2}^2 + \| \partial_t u_2 \|_{L^2}^2 \right) \left( \| u \|_{L^2}^2 + \| \Delta \phi \|_{L^2}^2 \right).
\]

By (4.3) and (4.4), together with Sobolev embeddings, we find

\[
|Z_4| \leq \int \left| (\nabla \Delta \phi \cdot \nabla) u_2 - u_2 \cdot \partial_t \phi \right| \, dx + \int \left| (\nabla (\Psi'(\phi_1) - \Psi'(\phi_2)) \cdot \nabla) u_2 \cdot u_2 \cdot u \right| \, dx
\]

\[
\leq \| \nabla \Delta \phi \|_{L^6} \| \nabla u_2 \|_{L^3} \| u_2 \|_{L^2} + \| \Psi''(\phi_1) \|_{L^\infty} \| \nabla \phi \|_{L^6} \| \nabla u_2 \|_{L^3} \| u \|_{L^2}
\]

\[
+ \| \Psi''(\phi_1) - \Psi''(\phi_2) \|_{L^\infty} \| \phi \|_{L^6} \| \nabla \phi_2 \|_{L^6} \| \nabla u_2 \|_{L^2} \| u \|_{L^2}.
\]
As to the remaining terms, by using (4.3) and (4.4) once more, we have
\[
\begin{align*}
|Z_7 + Z_8| &\leq \|u_1\|_{L^6} \|\nabla \phi\|_{L^6} \|\Delta^2 \phi\|_{L^2} + \|u\|_{L^2} \|\nabla \phi_2\|_{L^\infty} \|\Delta^2 \phi\|_{L^2} \\
&\leq \frac{1}{6} \|\Delta^2 \phi\|_{L^2}^2 + C (\|\Delta \phi\|_{L^2}^2 + \|\Delta \phi\|_{L^2}^2),
\end{align*}
\]
and
\[
|Z_9| \leq \int_{\Omega} \left| \left( \Psi''(\phi_1) \Delta \phi + (\Psi''(\phi_1) - \Psi''(\phi_2)) \Delta \phi_2 \right) \Delta^2 \phi \right| \, dx \\
+ \int_{\Omega} \left| \left( \Psi'''(\phi_1) \left( |\nabla \phi_1|^2 - |\nabla \phi_2|^2 \right) + (\Psi'''(\phi_1) - \Psi'''(\phi_2)) |\nabla \phi_2|^2 \right) \Delta^2 \phi \right| \, dx \\
\leq C \|\Delta \phi\|_{L^2} \|\Delta^2 \phi\|_{L^2} + C \left( \|\Psi''(\phi_1)\|_{L^\infty} + \|\Psi''(\phi_2)\|_{L^\infty} \right) \|\Delta \phi_2\|_{L^2} \|\Delta^2 \phi\|_{L^2} \\
+ C \left( \|\nabla \phi_1\|_{L^\infty} + \|\nabla \phi_2\|_{L^\infty} \right) \|\Delta \phi\|_{L^2} \|\Delta^2 \phi\|_{L^2} \\
\leq \frac{1}{6} \|\Delta^2 \phi\|_{L^2}^2 + C \|\Delta \phi\|_{L^2}^2.
\]
In conclusion, we find the differential inequality
\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \rho(\phi_1) |u|^2 \, dx + \frac{1}{2} \|\Delta \phi\|_{L^2}^2 \right) + \frac{\nu}{2} \|\Delta \phi\|_{L^2}^2 + \frac{1}{2} \|\Delta^2 \phi\|_{L^2}^2 \\
\leq C(K) \left( 1 + \|u_2\|_{H^2}^2 + \|\partial_t u_2\|_{L^2}^2 \right) \left( \|\Delta \phi\|_{L^2}^2 + \|\Delta \phi\|_{L^2}^2 \right).
\]
An application of the Gronwall lemma implies the desired uniqueness of strong solutions on the time interval \([0, T_1]\).

5. PROOF OF THEOREM 1.2: STABILITY

Let \( (u, P, \phi) \) and \( (u_H, P_H, \phi_H) \) be the strong solutions to the AGG model with density \( \rho(\phi) \) and to the model H with constant density \( \overline{\rho} \), respectively, defined on a common interval \([0, T_0]\). We recall that the existence of \( (u_H, P_H, \phi_H) \) fulfilling the same regularity properties of \( (u, P, \phi) \), as stated in Theorem (1.1), has been proven in [22, Theorem 5.1]. For simplicity, we assume that the viscosity function is given by \( \nu(s) = \nu_1 \frac{\nu_1}{2} + \nu_2 \frac{\nu_2}{2} \) (cf. (1.3)) for both systems. We define \( v = u - u_H, Q = P - P_H, \varphi = \phi - \phi_H, \) and the difference of the chemical potentials \( w = \mu - \mu_H \). They clearly solve the problem
\[
\begin{align*}
\frac{\partial_1 + \rho_2}{2} \partial_t v + \left( \frac{\rho_1 - \rho_2}{2} \right) \partial_t u + \left( \frac{\rho_1 + \rho_2}{2} - \overline{\rho} \right) \partial_t u_H + (\rho(\phi)(u \cdot \nabla)u - \overline{\rho} (u_H \cdot \nabla)u_H) \\
- \left( \frac{\rho_1 - \rho_2}{2} \right) (\nabla \mu \cdot \nabla)u - \nabla (\nu(\phi) \nabla v) - \nabla ((\nu(\phi) - \nu(\phi_H)) \nabla u_H) \\
+ \nabla Q = -\nabla (\nabla \phi \otimes \nabla \varphi - \nabla \phi_H \otimes \nabla \phi_H),
\end{align*}
\]
(5.1)
\[
\partial_t \varphi + u \cdot \nabla \varphi + v \cdot \nabla \varphi_H = \Delta w,
\]
(5.2)
\[
w = -\Delta \varphi + \Psi'(\phi) - \Psi'(\phi_H),
\]
almost everywhere in $\Omega \times (0, T_0)$. In addition, we have the boundary and initial conditions

$$v = 0, \quad \partial_n \varphi = \partial_n w = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \quad v(\cdot, 0) = 0, \quad \varphi(\cdot, 0) = 0 \quad \text{in} \quad \Omega.$$  \hfill (5.3)

Multiplying (5.1) by $A^{-1}v$ and integrating over $\Omega$, we obtain

$$\left(\frac{\rho_1 + \rho_2}{4}\right) \frac{d}{dt} \|v\|^2 + \int_{\Omega} \nu(\phi) \nabla v : \nabla A^{-1}v \, dx = - \int_{\Omega} \left(\frac{\rho_1 - \rho_2}{2}\right) \partial_t u \cdot A^{-1}v \, dx
- \int_{\Omega} \left(\frac{\rho_1 + \rho_2}{2} - \overline{\rho}\right) \partial_t u_H \cdot A^{-1}v \, dx - \int_{\Omega} (\rho(\phi)(u \cdot \nabla)u - \overline{p}(u_H \cdot \nabla)u_H) \cdot A^{-1}v \, dx
+ \int_{\Omega} \int_{\Omega} \left(\frac{\rho_1 - \rho_2}{2}\right) ((\nabla \mu \cdot \nabla)u) \cdot A^{-1}v \, dx - \int_{\Omega} (\nu(\phi) - \nu(\phi_H)) \nabla u_H : \nabla A^{-1}v \, dx
+ \int_{\Omega} \nabla \phi \otimes \nabla \phi - \nabla \phi_H \otimes \nabla \phi_H : \nabla A^{-1}v \, dx.$$  \hfill (5.4)

Following [22, proof of Theorem 3.1], we infer that

$$\int_{\Omega} \nu(\phi) \nabla v : \nabla A^{-1}v \, dx \geq \frac{\nu_*}{2} \|u\|^2_{L^2} - \int_{\Omega} \nu'(\phi) \nabla A^{-1}v \nabla \phi \cdot v \, dx + \frac{1}{2} \int_{\Omega} \nu'(\phi) \nabla \phi \cdot v \Pi \, dx,$$  \hfill (5.5)

where $\Pi \in L^\infty(0, T_0; H^1(\Omega))$ is such that $-\Delta A^{-1}v + \nabla \Pi = v$ a.e. in $\Omega \times (0, T_0)$. In addition, it fulfills the estimates

$$\|\Pi\|_{L^2} \leq C\|\nabla A^{-1}v\|_{L^2}^{\frac{1}{2}}\|v\|_{L^2}^{\frac{1}{2}}, \quad \|\Pi\|_{H^1} \leq C\|v\|_{L^2}.$$  \hfill (5.6)

Therefore, we are led to

$$\left(\frac{\rho_1 + \rho_2}{4}\right) \frac{d}{dt} \|v\|^2 + \frac{\nu_*}{2} \|v\|^2 = - \int_{\Omega} \left(\frac{\rho_1 - \rho_2}{2}\right) \partial_t u \cdot A^{-1}v \, dx
- \int_{\Omega} \left(\frac{\rho_1 + \rho_2}{2} - \overline{\rho}\right) \partial_t u_H \cdot A^{-1}v \, dx
- \int_{\Omega} (\rho(\phi)(u \cdot \nabla)u - \overline{p}(u_H \cdot \nabla)u_H) \cdot A^{-1}v \, dx
- \int_{\Omega} \int_{\Omega} \left(\frac{\rho_1 - \rho_2}{2}\right) ((\nabla \mu \cdot \nabla)u) \cdot A^{-1}v \, dx
+ \int_{\Omega} \nabla \phi \otimes \nabla \phi - \nabla \phi_H \otimes \nabla \phi_H : \nabla A^{-1}v \, dx
+ \frac{1}{2} \int_{\Omega} \nu'(\phi) \nabla \phi \cdot v \Pi \, dx.$$  \hfill (5.7)

On the other hand, multiplying (5.2) by $A^{-1}\varphi$, where $A$ is the Laplace operator with homogeneous Neumann boundary conditions, and integrating over $\Omega$, we get (see [22, Proof of Theorem 3.1] for more details)

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \frac{1}{2} \|\nabla A^{-1}\varphi\|^2_{L^2} \leq C\|\varphi\|^2_{L^2} + \int_{\Omega} \varphi u \cdot \nabla A^{-1}\varphi \, dx + \int_{\Omega} \phi_H v \cdot \nabla A^{-1}\varphi \, dx.$$  \hfill (5.8)

We proceed with the estimate of the terms on the right-hand side of (5.6) and (5.7). To this aim, we will exploit the following bounds on the solution

$$\|u \cdot u_H\|_{L^\infty(0, T_0; H_{\text{reg}}^1(\Omega))} \leq K_0,$$
$$\|\phi \cdot \phi_H\|_{L^\infty(0, T_0; W^{2,6}(\Omega))} \leq K_0,$$
$$\|\nabla \mu\|_{L^\infty(0, T_0; L^2(\Omega))} \leq K_0.$$  \hfill (5.9)
where $K_0$ is a constant depending on the norms of the initial conditions. Exploiting this estimates, we have

$$\int_{\Omega} \left( \frac{\rho_1 - \rho_2}{2} \right) \partial_t u \cdot A^{-1} v \, dx \leq \left| \frac{\rho_1 - \rho_2}{2} \right| \| \phi \|_{L^{\infty}} \| \partial_t u \|_{L^2} \| A^{-1} v \|_{L^2}$$

$$\leq C \| v \|_2^2 + C \left| \frac{\rho_1 - \rho_2}{2} \right|^2 \| \partial_t u \|_{L^2}^2,$$

and

$$\int_{\Omega} \left( \frac{\rho_1 + \rho_2 - p}{2} \right) \partial_t u_H \cdot A^{-1} v \, dx \leq C \| v \|_{1_+}^2 + C \left| \frac{\rho_1 + \rho_2 - p}{2} \right|^2 \| \partial_t u_H \|_{L^2}^2.$$

By Sobolev embedding, we find

$$\int_{\Omega} (\rho(\phi)(u \cdot \nabla)u - \rho(u_H \cdot \nabla)u_H) \cdot A^{-1} v \, dx$$

$$\leq \left| \int_{\Omega} \rho(\phi)(v \cdot \nabla)u \cdot A^{-1} v \, dx \right| + \left| \int_{\Omega} \rho(\phi)(u_H \cdot \nabla)A^{-1} v \, dx \right| + \left| \int_{\Omega} (\rho(\phi) - p)(u_H \cdot \nabla)A^{-1} v \, dx \right|$$

$$\leq \rho^* \| v \|_{L^2} \| \nabla u \|_{L^6} \| A^{-1} v \|_{L^3} + \int_{\Omega} \rho(\phi)(u_H \cdot \nabla)A^{-1} v \cdot v \, dx + \int_{\Omega} \rho(\phi)(\nabla \phi \cdot u_H) (v \cdot A^{-1} v) \, dx$$

$$+ \| \rho(\phi) - p \|_{L^{\infty}} \| u_H \|_{L^6} \| \nabla u_H \|_{L^2} \| A^{-1} v \|_{L^3}$$

$$\leq \frac{\nu_*}{10} \| v \|_{L^2}^2 + C (1 + \| u \|_{H^2}^2) \| v \|_{L^2}^2 + \rho^* \| \nabla A^{-1} v \|_{L^2} \| u_H \|_{L^{\infty}} \| v \|_{L^2}$$

$$+ \frac{\rho_1 - \rho_2}{2} \| \nabla \phi \|_{L^{\infty}} \| u_H \|_{L^6} \| v \|_{L^2} \| A^{-1} v \|_{L^3} + C(K_0) \left( \left| \frac{\rho_1 - \rho_2}{2} \right|^2 + \left| \frac{\rho_1 + \rho_2 - p}{2} \right|^2 \right)$$

$$\leq \frac{\nu_*}{8} \| v \|_{L^2}^2 + C(K_0) (1 + \| u \|_{H^2}^2 + \| u_H \|_{H^2}^2) \| v \|_{L^2}^2 + C(K_0) \left( \left| \frac{\rho_1 - \rho_2}{2} \right|^2 + \left| \frac{\rho_1 + \rho_2 - p}{2} \right|^2 \right),$$

and

$$\int_{\Omega} \left( \frac{\rho_1 - \rho_2}{2} \right) (\nabla \mu \cdot \nabla) u \cdot A^{-1} v \, dx \leq \left| \frac{\rho_1 - \rho_2}{2} \right| \| \nabla \mu \|_{L^2} \| \nabla u \|_{L^3} \| A^{-1} v \|_{L^6}$$

$$\leq C \| v \|_{1_+}^2 + C(K_0) \left| \frac{\rho_1 - \rho_2}{2} \right|^2 \| \nabla u \|_{L^3}^2.$$

In a similar way as in [22, Proof of Theorem 5.1], we obtain

$$\int_{\Omega} (\nu(\phi) - \nu(\phi_H)) \nabla A^{-1} v \, dx \leq C \| \phi \|_{L^6} \| \nabla u_H \|_{L^2} \| \nabla A^{-1} v \|_{L^2}$$

$$\leq \frac{1}{6} \| \nabla \phi \|_{L^2}^2 + C \| u_H \|_{H^2}^2 \| v \|_{L^2}^2,$$

$$\int_{\Omega} (\nabla \phi \otimes \nabla \phi - \nabla \phi_H \otimes \nabla \phi_H) : \nabla A^{-1} v \, dx \leq (\| \nabla \phi \|_{L^{\infty}} + \| \nabla \phi_H \|_{L^{\infty}}) \| \nabla \phi \|_{L^2} \| \nabla A^{-1} v \|_{L^2}$$

$$\leq \frac{1}{6} \| \nabla \phi \|_{L^2}^2 + C(K_0) \| v \|_{L^2}^2,$$
\[
\left| \int_{\Omega} \nu'(p) \mathbb{D} A^{-1} \nabla \phi \cdot \mathbf{v} \, dx \right| \leq C \| \mathbb{D} A^{-1} \mathbf{v} \|_{L^2} \| \nabla \phi \|_{L^\infty} \| \mathbf{v} \|_{L^2} \leq \frac{\nu_s}{8} \| \mathbf{v} \|_{L^2}^2 + C(K_0) \| \mathbf{v} \|_{L^2}^2
\]

\[
\left| \frac{1}{2} \int_{\Omega} \nu'(p) (\nabla \phi \cdot \mathbf{v}) \, dx \right| \leq C \| \nabla \phi \|_{L^\infty} \| \mathbf{v} \|_{L^2} \| \Pi \|_{L^2} \leq \frac{\nu_s}{8} \| \mathbf{v} \|_{L^2}^2 + C(K_0) \| \mathbf{v} \|_{L^2}^2
\]

\[
\int_{\Omega} \phi \mathbf{u} \cdot \nabla A^{-1} \phi \, dx \leq \frac{1}{6} \| \nabla \phi \|_{L^2}^2 + C \| \mathbf{u} \|_{H^2(\Omega)}^2 \| \phi \|_{L^\infty}^2
\]

\[
\int \phi_H \mathbf{v} \cdot \nabla A^{-1} \phi \, dx \leq \frac{\nu_s}{8} \| \mathbf{v} \|_{L^2}^2 + C \| \phi \|_{L^\infty}^2
\]

Collecting the above estimates together, we find the differential inequality

\[
\frac{d}{dt} \left( \frac{\rho_1 + \rho_2}{4} \| \mathbf{v} \|_{L^2}^2 + \frac{1}{2} \| \phi \|_{L^\infty}^2 \right) \leq f_1(t) (\| \mathbf{v} \|_{L^2}^2 + \| \phi \|_{L^\infty}^2) + f_2(t) \left( \left| \frac{\rho_1 - \rho_2}{2} \right|^2 + \left| \frac{\rho_1 + \rho_2}{2} - \rho_0 \right|^2 \right),
\]

where

\[
f_1(t) = C(K_0) \left( 1 + \| \mathbf{u_H} \|_{H^2}^2 + \| \mathbf{u} \|_{H^2}^2 \right),
\]

\[
f_2(t) = C(K_0) \left( 1 + \| \partial_t \mathbf{u_H} \|_{L^2}^2 + \| \mathbf{u_H} \|_{H^2}^2 + \| \partial_t \mathbf{u} \|_{L^2}^2 + \| \mathbf{u} \|_{H^2}^2 \right).
\]

Here, the positive constant \( C \) depends on the norm of the initial data and the time \( T_0 \). By using the Gronwall lemma, together with the initial conditions (5.3), we infer that

\[
\| \mathbf{v}(t) \|_{L^2}^2 + \| \phi(t) \|_{L^\infty}^2 \leq \frac{\left( \left| \frac{\rho_1 - \rho_2}{2} \right|^2 + \left| \frac{\rho_1 + \rho_2}{2} - \rho_0 \right|^2 \right)}{\min \left\{ \frac{\rho_1 + \rho_2}{4}, \frac{1}{2} \right\}} \int_0^t e^{\int_s^t f_1(r) \, dr} f_2(s) \, ds, \quad \forall t \in [0, T_0].
\]

Thus, the above inequality implies that

\[
\| \mathbf{u}(t) - \mathbf{u_H}(t) \|_{H^\gamma} + \| \phi(t) - \phi_H(t) \|_{H^\gamma} \leq \frac{C(K_0)}{\min \{ \sqrt{\rho_1, 1} \} \left( \left| \frac{\rho_1 - \rho_2}{2} \right| + \left| \frac{\rho_1 + \rho_2}{2} - \rho_0 \right| \right)} , \quad \forall t \in [0, T_0],
\]

where the positive constant \( C(K_0) \) depends on the norm of the initial data, the time \( T_0 \) and the parameters of the systems.

**APPENDIX A. ON THE CONVECTIVE VISCOS Cahn-Hilliard SYSTEM**

Given \( \alpha > 0 \) and an incompressible velocity field \( \mathbf{u} \), we consider the convective Viscous Cahn-Hilliard (cVCH) system

\[
\partial_t \phi + \mathbf{u} \cdot \nabla \phi = \Delta \mu, \quad \mu = \alpha \partial_t \phi - \Delta \phi + \Psi'(\phi) \quad \text{in} \, \Omega \times (0, T), \quad \text{(A.1)}
\]

with boundary and initial conditions

\[
\partial_n \phi = \partial_n \mu = 0 \quad \text{on} \, \partial \Omega \times (0, T), \quad \phi(\cdot, 0) = \phi_0 \quad \text{in} \, \Omega. \quad \text{(A.2)}
\]

We observe that (A.1) can be rewritten as

\[
\partial_t (\phi - \alpha \Delta \phi) + \mathbf{u} \cdot \nabla \phi = \Delta (-\Delta \phi + F'(\phi) - \theta_0 \phi) \quad \text{in} \, \Omega \times (0, T).
\]

We state well-posedness and regularity results for system (A.1). The aim of this Appendix is to extend the analysis performed in [28] to the convective case under minimal assumptions on the velocity field. In particular, we focus on the regularity of the chemical potential.
Theorem A.1. Assume that \( u \in L^\infty(0, T; L^2(\Omega) \cap L^3(\Omega)), \phi_0 \in H^1(\Omega) \cap L^\infty(\Omega) \) such that \( \|\phi_0\|_{L^\infty} \leq 1 \) and \( |\phi_0| < 1 \). Then, there exists a unique a weak solution to (A.1)-(A.2) such that

\[
\phi \in L^\infty(0, T; H^1(\Omega) \cap L^\infty(\Omega)) : |\phi(x, t)| < 1 \text{ a.e. in } \Omega \times (0, T),
\phi \in L^2(0, T; H^2(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)),
\mu \in L^2(0, T; H^2(\Omega)),
\]

which satisfies (A.1) almost everywhere in \( \Omega \times (0, T) \), (A.2) almost everywhere on \( \partial \Omega \times (0, T) \) and \( \phi(\cdot, 0) = \phi_0(\cdot) \) in \( \Omega \). In addition, the following regularity results hold:

(R1) If \( -\Delta \phi_0 + F'(\phi_0) \in L^2(\Omega) \) and \( \partial_t u \in L^{2^*}(0, T; L^2(\Omega)) \), we have

\[
\partial_t \phi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\phi \in L^\infty(0, T; H^2(\Omega)),
\mu \in L^\infty(0, T; H^2(\Omega)).
\]

(R2) Let the assumptions of (R1) hold. Suppose that \( \|\phi_0\|_{L^\infty} \leq 1 - \delta_0 \), for some \( \delta_0 \in (0, 1) \). Then, there exists \( \delta > 0 \) such that

\[
\max_{(x, t) \in \Omega \times (0, T)} |\phi(x, t)| \leq 1 - \delta,
\]

and

\[
\phi \in L^2(0, T; H^3(\Omega)).
\]

(R3) Let the assumption of (R2) hold. Suppose that \( \phi_0 \in H^3(\Omega) \) such that \( \partial_\nu \phi = 0 \) on \( \partial \Omega \), and \( \partial_t u \in L^2(0, T; L^{3^*}(\Omega)) \), we have

\[
\partial_t \phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\phi \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)),
\partial^2_t \phi \in L^2(0, T; L^2(\Omega)),
\partial_t \mu \in L^2(0, T; L^2(\Omega)).
\]

Proof. The proof is divided in several parts. We notify the reader that the estimates herein proved are not independent of the viscous parameter \( \alpha \).

Existence. The existence of a weak solution satisfying (A.3) is proved in a classical way\(^1\). We proceed here by proving the basic energy estimates. First, we observe that, by integrating (A.1)\(_1\) over \( \Omega \) and using the boundary conditions, we have

\[
\overline{\phi}(t) = \overline{\phi_0} \quad \text{and} \quad \partial_t \phi(t) = 0 \quad \forall \ t \in [0, T].
\]

Multiplying (A.1)\(_1\) by \( \mu \), integrating over \( \Omega \), using the boundary conditions (A.2) and [30, Lemma 4.3, Ch. IV], we find

\[
\frac{d}{dt} \left( \int_\Omega \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \, dx \right) + \|\nabla \mu\|_{L^2}^2 + \alpha \|\partial_t \phi\|_{L^2}^2 = \int_\Omega \phi u \cdot \nabla \mu \, dx.
\]

By the Hölder inequality and the boundedness of \( \phi \), we simply obtain

\[
\frac{d}{dt} \left( \int_\Omega \frac{1}{2} |\nabla \phi|^2 + \Psi(\phi) \, dx \right) + \frac{1}{2} \|\nabla \mu\|_{L^2}^2 + \alpha \|\partial_t \phi\|_{L^2}^2 \leq \frac{1}{2} \|u\|_{L^2}^2.
\]

\(^1\)The interested reader might exploit the combination of the Galerkin method with the approximation of the logarithmic potential by smooth potentials.
Thus, integrating over $[0,T]$ and using the continuity of $\Psi$, we have
\[
\|\nabla \phi\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \mu\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t \phi\|_{L^2(0,T;L^2(\Omega))} 
\leq C_{\alpha} \left( \sqrt{\mathcal{E}_{\text{free}}(\phi_0)} + \|u\|_{L^2(0,T;L^2(\Omega))} \right).
\] (A.6)

In light of (2.1) and (A.5), we infer that
\[
\|\phi\|_{L^\infty(0,T;H^1(\Omega))} \leq C_{\alpha} \left( \sqrt{\mathcal{E}_{\text{free}}(\phi_0)} + \|u\|_{L^2(0,T;L^2(\Omega))} + \|\phi_0\|_{L^2(\Omega)} \right).
\] (A.7)

Now, multiplying (A.1) by $-\Delta \phi$ and integrating over $\Omega$, we get
\[
\int_0^T \frac{\alpha}{2} \frac{d}{dt} \|\nabla \phi\|_{L^2}^2 + \|\Delta \phi\|_{L^2}^2 + \int_\Omega -F'(\phi) \Delta \phi \, dx = \int_\Omega \nabla \mu \cdot \nabla \phi \, dx + \theta_0 \|\nabla \phi\|_{L^2}^2.
\]

The second term on the left-hand side is clearly positive by monotonicity. Then, using (A.7) we obtain
\[
\int_0^T \|\Delta \phi(\tau)\|_{L^2}^2 \, d\tau \leq \frac{\alpha}{2} \|\nabla \phi_0\|_{L^2}^2 + C_{\alpha} (1 + T) \left( \sqrt{\mathcal{E}_{\text{free}}(\phi_0)} + \|u\|_{L^2(0,T;L^2(\Omega))} \right)^2,
\] (A.8)

which entails that
\[
\|\phi\|_{L^2(0,T;H^2(\Omega))} \leq C_{\alpha} \left( 1 + \|\nabla \phi_0\|_{L^2} + \sqrt{1 + T} \left( \sqrt{\mathcal{E}_{\text{free}}(\phi_0)} + \|u\|_{L^2(0,T;L^2(\Omega))} \right) \right).
\] (A.9)

Next, we control the total mass of the chemical potential. Arguing as for the Cahn-Hilliard equation, we multiply (A.1) by $\phi - \overline{\phi}$ and integrate over $\Omega$. We find
\[
\int_\Omega |\nabla \phi|^2 \, dx + \int_\Omega F'(\phi)(\phi - \overline{\phi}) \, dx = \int_\Omega \mu(\phi - \overline{\phi}) \, dx + \theta_0 \|\phi - \overline{\phi}\|_{L^2}^2 - \alpha \int_\Omega \partial_t \phi(\phi - \overline{\phi}) \, dx.
\]

By using the Poincaré inequality and (A.3), we find
\[
\int_\Omega F'(\phi)(\phi - \overline{\phi}) \, dx \leq C_{\alpha} \left( 1 + \|\nabla \mu\|_{L^2} + \|\partial_t \phi\|_{L^2} \right),
\]

for some $C_{\alpha}$ depending on $\Omega$, $\theta_0$ and $\alpha$. We are now in position to control a full Sobolev norm of $\mu$. Thanks to [28, Proposition A.1], there exist two positive constants $C_1, C_2$ (only depending on $\overline{\phi_0}$) such that
\[
\int_\Omega |F'(\phi)| \, dx \leq C_1 \int_\Omega F'(\phi)(\phi - \overline{\phi_0}) \, dx + C_2,
\]

thus we infer that
\[
\|F'(\phi)\|_{L^1} \leq C_{\alpha} \left( 1 + \|\nabla \mu\|_{L^2} + \|\partial_t \phi\|_{L^2} \right).
\]

Since $\overline{\mu} = F'(\phi) - \theta_0 \overline{\phi_0}$, the above control yields
\[
\|\overline{\mu}\| \leq C_{\alpha} \left( 1 + \|\nabla \mu\|_{L^2} + \|\partial_t \phi\|_{L^2} \right).
\] (A.10)

As a result, it immediately follows that
\[
\|\mu\|_{L^2(0,T;H^1(\Omega))} \leq C_{\alpha} \left( \sqrt{T} + \sqrt{\mathcal{E}_{\text{free}}(\phi_0)} + \|u\|_{L^2(0,T;L^2(\Omega))} \right).
\] (A.11)

In addition, by using (A.1) we observe that
\[
\|\Delta \mu\|_{L^2} \leq \|\partial_t \phi\|_{L^2} + \|u\|_{L^2} \|\nabla \phi\|_{L^6}.
\]

Then, combining the elliptic regularity with (A.6) and (A.9), we find
\[
\|\mu\|_{L^2(0,T;H^2(\Omega))} \leq C_{\alpha} \mathcal{E}_{\text{free}}(\phi_0) \left( 1 + \|u\|_{L^\infty(0,T;L^3(\Omega))} \right) \left( 1 + \|u\|_{L^2(0,T;L^2(\Omega))} \right).
\] (A.12)
By comparison in (A.1)2, a similar estimate can be obtained for $F'(\phi)$ in $L^2(0,T; L^2(\Omega))$.

**Uniqueness.** Let $\phi_1, \phi_2$ be two weak solutions. We define the solutions difference $\psi = \phi_1 - \phi_2$ which solves

$$\partial_t \psi + u \cdot \nabla \psi = \Delta (\alpha \partial_t \psi - \Delta \psi + \Psi'(\phi_1) - \Psi'(\phi_2)) \quad \text{in } \Omega \times (0,T).$$

Since $\overline{\psi}(t) = 0$ for all $t \in [0,T]$, multiplying by $A^{-1}\psi$, where the operator $A$ is the Laplace operator $-\Delta$ with homogeneous Neumann boundary conditions, and integrating over $\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \| \nabla A^{-1}\psi \|_{L^2}^2 + \alpha \| \psi \|_{L^2}^2 \right) + \| \nabla \psi \|_{L^2}^2 \leq \int_{\Omega} \psi u \cdot \nabla A^{-1}\psi \, dx + \theta_0 \| \psi \|_{L^2}^2.$$

Here we have used that $F'$ is a monotone function. Observing that

$$\left| \int_{\Omega} \psi u \cdot \nabla A^{-1}\psi \, dx \right| \leq \| \psi \|_{L^2} \| u \|_{L^2} \| \nabla A^{-1}\psi \|_{L^2} \leq C \| u \|_{L^2} \| \psi \|_{L^2}^2,$$

it is easily seen that

$$\frac{1}{2} \frac{d}{dt} \left( \| \nabla A^{-1}\psi \|_{L^2}^2 + \alpha \| \psi \|_{L^2}^2 \right) \leq C \left( 1 + \| u \|_{L^2} \right) \| \psi \|_{L^2}^2.$$

An application of the Gronwall lemma yields

$$\| \nabla A^{-1}\psi(t) \|_{L^2}^2 + \alpha \| \psi(t) \|_{L^2}^2 \leq \left( \| \nabla A^{-1}\psi(0) \|_{L^2}^2 + \alpha \| \psi(0) \|_{L^2}^2 \right) e^{C \int_0^t (1 + \| u(\tau) \|_{L^2}) \, d\tau}$$

for all $t \in [0,T]$, which implies the uniqueness of the solution.

**Regularity 1.** For $h \in (0,1)$, we define the notation $\partial_t^h f(\cdot,t) = \frac{1}{h}(f(\cdot,t+h) - f(\cdot,t))$. We observe that $\phi \in C([0,T]; H^1(\Omega))$ and $u \in C([0,T]; L^1(\Omega))$, thereby we can extend both $\phi$ and $u$ on $[0,T+1]$ by $\phi(t) = \phi(T)$ and $u(t) = u(T)$ for $t \in (T,T+1)$. It follows from (A.1) that

$$\partial_t \partial_t^h \phi + \partial_t^h u \cdot \nabla \phi(\cdot+h) + u \cdot \nabla \partial_t^h \phi = \Delta (\varepsilon \partial_t \partial_t^h \phi - \Delta \partial_t^h \phi + \partial_t^h \Psi'(\phi)) \quad \text{in } \Omega \times (0,T). \quad (A.13)$$

We multiply the above equation by $A^{-1} \partial_t^h \phi$ and integrate over $\Omega$. Exploiting the monotonicity of $F'$, the boundary condition of $u$ and the Agmon inequality (2.3), we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \| \nabla A^{-1} \partial_t^h \phi \|_{L^2}^2 + \alpha \| \partial_t^h \phi \|_{L^2}^2 \right) + \| \nabla \partial_t^h \phi \|_{L^2}^2$$

$$\leq \int_{\Omega} \phi(\cdot+h) \partial_t^h u \cdot \nabla A^{-1} \partial_t^h \phi \, dx + \int_{\Omega} \partial_t^h \phi u \cdot \nabla A^{-1} \partial_t^h \phi \, dx + \theta_0 \| \partial_t^h \phi \|_{L^2}^2$$

$$\leq \| \partial_t^h u \|_{L^1} \| \nabla A^{-1} \partial_t^h \phi \|_{L^\infty} + \| \partial_t^h \phi \|_{L^2} \| u \|_{L^1} \| \nabla A^{-1} \partial_t^h \phi \|_{L^2} \leq \theta_0 \| \partial_t^h \phi \|_{L^2} + C(1 + \| u \|_{L^1}) \| \partial_t^h \phi \|_{L^2}$$

$$\leq \frac{1}{2} \| \nabla \partial_t^h \phi \|_{L^2}^2 + C \| \partial_t^h u \|_{L^1} \left( 1 + \| \partial_t^h \phi \|_{L^2}^2 \right) + C(1 + \| u \|_{L^2}) \| \partial_t^h \phi \|_{L^2}^2.$$

The Gronwall lemma entails

$$\alpha \| \partial_t^h \phi(t) \|_{L^2}^2 + \int_0^t \| \nabla \partial_t^h \phi(\tau) \|_{L^2}^2 \, d\tau \leq \left( \| \nabla A^{-1} \partial_t^h \phi(0) \|_{L^2}^2 + \alpha \| \partial_t^h \phi(0) \|_{L^2}^2 \right. + C \int_0^t \| \partial_t^h u(\tau) \|_{L^1}^\frac{3}{2} \, d\tau \right) e^{\int_0^t (1 + \| u(\tau) \|_{L^2}) \, d\tau} \quad (A.14)$$
for all $t \in [0, T]$, where $g(\tau) = C_\alpha \left(1 + \|u\|_{L^3} + \|\partial_t^3 u\|_{L^1}^4\right)$. In order to control the right-hand side, we compute

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla^{-1}(\phi - \phi_0)\|_{L^2}^2 + \alpha \|\phi - \phi_0\|_{L^2}^2\right) = (\alpha \partial_t \phi - \mu, \phi - \phi_0) + (\phi u, \nabla A^{-1}(\phi - \phi_0))$$

$$= (\Delta \phi - \Psi'(\phi), \phi - \phi_0) + (\phi u, \nabla A^{-1}(\phi - \phi_0))$$

$$= (\Delta(\phi - \phi_0) - (F'(\phi - F'\phi_0)), \phi - \phi_0) + (\Delta \phi_0 - F'(\phi_0), \phi - \phi_0) + \theta_0(\phi, \phi - \phi_0)$$

$$+ (\phi u, \nabla A^{-1}(\phi - \phi_0)).$$

Therefore, we have

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla^{-1}(\phi - \phi_0)\|_{L^2}^2 + \alpha \|\phi - \phi_0\|_{L^2}^2\right) \leq C_\alpha \left(1 + \|\Delta \phi_0 - F'(\phi_0)\|_{L^2} + \|u\|_{L^2}\right)\|\phi - \phi_0\|_{L^2}.$$

Thanks to [30,Lemma 4.1, Chap. IV], we arrive at

$$\|\nabla^{-1}(\phi(t) - \phi_0)\|_{L^2}^2 + \alpha \|\phi(t) - \phi_0\|_{L^2}^2 \leq \left(C_\alpha \left(1 + \|\Delta \phi_0 - F'(\phi_0)\|_{L^2} + \|u\|_{L^2}\right) t + C_\alpha \int_0^t \|u(\tau)\|_{L^2} d\tau\right)^2,$$

for all $t \in [0, T]$. By choosing $t = h$, we deduce that

$$\|\nabla^{-1}\partial_t^3 \phi(0)\|_{L^2}^2 + \alpha \|\partial_t^3 \phi(0)\|_{L^2}^2 \leq C_\alpha \left(1 + \|\Delta \phi_0 - F'(\phi_0)\|_{L^2}^2 + \|u\|_{L^2(0,T;L^2(\Omega))}\right).$$

Since $\|\partial_t^3 u\|_{L^4(0,T;L^1(\Omega))} \leq \|\partial_t u\|_{L^4(0,T;L^1(\Omega))}$, by combining (A.14) and (A.15), we obtain

$$\alpha \|\partial_t^3 \phi(t)\|_{L^2}^2 + \int_0^t \|\nabla \partial_t^3 \phi(\tau)\|_{L^2}^2 d\tau$$

$$\leq C_\alpha \left(1 + \|\Delta \phi_0 - F'(\phi_0)\|_{L^2}^2 + \|u\|_{L^2(0,T;L^2(\Omega))}^2 + \|\partial_t u\|_{L^4(0,T;L^1(\Omega))}^4\right)e^{G(T)},$$

for all $t \in [0, T]$, where $G(T) = \int_0^T C_\alpha \left(1 + \|u(\tau)\|_{L^2}\right) d\tau + C_\alpha \int_0^T \|\partial_t u(\tau)\|_{L^1}^4 d\tau$. In light of the convergence $\partial_t^3 \phi \rightarrow \partial_t \phi$ in $L^2(0,T;L^2(\Omega))$ as $h \rightarrow 0$, we infer that

$$\|\partial_t \phi\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t \phi\|_{L^2(0,T;H^1(\Omega))} \leq C(\alpha, T, \|\Delta \phi_0 - F'(\phi_0)\|_{L^2}, \|u\|_{X_T}),$$

where $X_T = L^\infty(0,T;L^2(\Omega)) \cap W^{1,4}(0,T;L^1(\Omega))$. Next, we derive further regularity properties on $\phi$ and $\mu$. By the incompressibility constraint, we recall that $\|\nabla \mu\|_{L^2} \leq C \left(\|\partial_t \phi\|_{L^2} + \|u\|_{L^2}\right)$. Then, thanks to (A.10) and (A.17), we easily have

$$\|\mu\|_{L^\infty(0,T;H^1(\Omega))} \leq C(\alpha, T, \|\Delta \phi_0 - F'(\phi_0)\|_{L^2(\Omega)}, \|u\|_{X_T}).$$

As a consequence, by [22, Theorem A.1] we get

$$\|\phi\|_{L^\infty(0,T;H^2(\Omega))} + \|F'(\phi)\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\alpha, T, \|\Delta \phi_0 - F'(\phi_0)\|_{L^2(\Omega)}, \|u\|_{X_T}).$$

Finally, since $u \in L^\infty(0,T;L^2(\Omega))$ and $\nabla \phi \in L^\infty(0,T;L^6(\Omega))$, by comparison in (A.1), we also find

$$\|\mu\|_{L^\infty(0,T;H^2(\Omega))} \leq C(\alpha, T, \|\Delta \phi_0 - F'(\phi_0)\|_{L^2(\Omega)}, \|u\|_{X_T}).$$
Regularity 2. Let us now write (A.1) as follows
\[ \alpha \partial_t \phi - \Delta \phi + F'(\phi) = h \quad \text{in } \Omega \times (0, T), \]  
where \( h = \mu + \theta_0 \phi \). Thanks to (A.20), \( h \in L^\infty(0, T; L^\infty(\Omega)) \). Next, we consider the ODEs problems
\[
\begin{aligned}
\alpha \partial_t U + F'(U) &= \mathcal{H}, \\
U(0) &= 1 - \delta_0 \\
\alpha \partial_t V + F'(V) &= H, \\
V(0) &= -1 + \delta_0
\end{aligned}
\]  
in \( (0, T) \), (A.22)

where \( \mathcal{H} = \|h\|_{L^\infty} \) and \( H = -\|h\|_{L^\infty} \). It is not difficult to show that there exist two unique solutions \( U, V \in C([0, T]) \) with \( U_t, V_t \in L^\infty(0, T) \). In particular, since \( \lim_{s \to \pm 1} F'(s) = \pm \infty \) and \( \mathcal{H}, H \in L^\infty(0, T) \), a simple comparison argument entails that there exists \( \delta > 0 \) such that
\[ -1 + \delta \leq V(t) \leq U(t) \leq 1 - \delta, \quad \forall t \in [0, T]. \]

More precisely, it can be checked that \( 1 - \delta \leq \max\{1 - \delta_0, (F')^{-1}(\|H\|_{L^\infty(0, T)})\} \). We are left to show that \( V(t) \leq \phi(x, t) \leq U(t) \) in \( \Omega \times [0, T] \). To this aim, we use the Stampacchia method. We define \( w = \phi - U \) and consider the problem
\[
\begin{aligned}
\alpha \partial_t w + u \cdot \nabla \phi - \Delta \phi + F'\phi - F'(U) &= h - \mathcal{H} \\
w(0) &= \phi_0 - 1 + \delta_0
\end{aligned}
\]  
in \( \Omega \times (0, T) \), (A.23)

Multiplying the equation by \( w^+ = \max\{\phi - U, 0\} \) and integrating over \( \Omega \), and using that \( \nabla \phi = \nabla w^+ \) on the set \( \{x \in \Omega : \phi \leq 0\} \), we find
\[
\frac{\alpha}{2} \frac{d}{dt} \|w^+\|_{L^2}^2 + \int_{\Omega} (u \cdot \nabla w^+) w^+ \, dx + \|\nabla w^+\|_{L^2}^2 + \int_{\Omega} (F'(\phi) - F'(U)) w^+ \, dx = \int_{\Omega} (h - \mathcal{H}) w^+ \, dx.
\]

By the monotonicity of \( F' \), it follows that
\[
\frac{d}{dt} \|w^+\|_{L^2}^2 \leq 0 \quad \Rightarrow \quad \|w^+(t)\|_{L^2}^2 \leq \|w^+(0)\|_{L^2}^2 = 0, \quad \forall t \in [0, T],
\]
which, in turn, gives the desired result, namely \( \phi(x, t) \leq U(t) \) in \( \Omega \times [0, T] \). A similar argument entails that \( V(t) \leq \phi(x, t) \) in \( \Omega \times [0, T] \). Therefore, we obtain by continuity the separation property
\[ \max_{(x, t) \in [0, T]} |\phi(x, t)| \leq 1 - \delta. \]  
(A.24)

As a consequence, it follows from (A.19) that \( \Psi'(\phi) \in L^\infty(0, T; H^1(\Omega)) \). Then, we deduce by comparison in (A.1) and by elliptic regularity that
\[ \|\phi\|_{L^2(0, T; H^3(\Omega))} \leq C(\alpha, T, \delta, \|\Delta \phi_0 - F'(\phi_0)\|_{L^2}, \|u\|_{L^2}. \]

Regularity 3. Thanks to the above regularity, we rewrite (A.13) as follows
\[ \int_{\Omega} \partial_t \partial_t^h \phi \cdot v + \alpha \nabla \partial_t \partial_t^h \phi \cdot \nabla v \, dx + \int_{\Omega} \partial_t^h \partial_t^h \phi \cdot \nabla v \, dx = \int_{\Omega} (\nabla \Delta \partial_t^h \phi - \nabla \partial_t^h \Psi'(\phi)) \cdot \nabla v \, dx \]  
(A.25)

for all \( v \in H^1(\Omega) \). Taking \( v = \partial_t^h \phi \) and exploiting the boundary conditions of \( \phi \) and \( u \), we find
\[
\frac{1}{2} \frac{d}{dt} \left( \|\partial_t^h \phi\|_{L^2}^2 + \alpha \|\nabla \partial_t^h \phi\|_{L^2}^2 \right) + \int_{\Omega} |\Delta \partial_t^h \phi|^2 \, dx
\]
\[ = \int_{\Omega} \partial_t^h (u \phi) \cdot \nabla \partial_t^h \phi \, dx + \int_{\Omega} \partial_t^h F'(\phi) \Delta \partial_t^h \phi \, dx + \theta_0 \|\nabla \partial_t^h \phi\|_{L^2}^2
\]

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\[ \leq \| \partial_t^h u \|_{L^6} + \| \Delta \partial_t^h \phi \|_{L^6} + \| \partial_t^h \phi \|_{L^6} + \| \nabla \partial_t^h \phi \|_{L^6} + C \| \partial_t^h \phi \|_{L^2} \| \Delta \partial_t^h \phi \|_{L^2} + \theta_0 \| \nabla \partial_t^h \phi \|_{L^2}^2 \]

\[ \leq \frac{1}{2} \| \Delta \partial_t^h \phi \|_{L^2}^2 + C \| \partial_t^h \phi \|_{L^2}^2 + C (1 + \| u \|_{L^1}) \| \nabla \partial_t^h \phi \|_{L^2}^2 + C \| \partial_t^h \phi \|_{L^2}^2. \]

Here we have used the separation property (A.24) and the inequality \( \| \partial_t^h \phi \|_{H^2} \leq C \| \Delta \partial_t^h \phi \|_{L^2} \). Then, we infer from the Gronwall lemma that

\[ \| \partial_t^h \phi(t) \|_{L^2}^2 + \alpha \| \nabla \partial_t^h \phi(t) \|_{L^2}^2 + \int_0^t \| \Delta \partial_t^h \phi(\tau) \|_{L^2}^2 \, d\tau \]

\[ \leq \left( \| \partial_t^h \phi(0) \|_{L^2}^2 + \alpha \| \nabla \partial_t^h \phi(0) \|_{L^2}^2 + C \int_0^t \| \partial_t^h u(\tau) \|_{L^6}^2 \, d\tau \right) e^{\bar{G}(T)} \tag{A.26} \]

for all \( t \in [0, T] \), where \( \bar{G}(T) = C_o \int_0^T (1 + \| u(\tau) \|_{L^3}) \, d\tau \). Since \( \partial u \phi_0 = 0 \) on \( \partial \Omega \) by assumption, we observe that

\[ \frac{1}{2} \frac{d}{dt} \left( \| \phi - \phi_0 \|_{L^2}^2 + \alpha \| \nabla (\phi - \phi_0) \|_{L^2}^2 \right) \]

\[ = \int_{\Omega} \phi u \cdot \nabla (\phi - \phi_0) \, dx + \int_{\Omega} \nabla (\Delta \phi - F'(\phi) + \theta_0 \phi) \cdot \nabla (\phi - \phi_0) \, dx \]

\[ = \int_{\Omega} \phi u \cdot \nabla (\phi - \phi_0) \, dx - \| \Delta (\phi - \phi_0) \|_{L^2}^2 + \int_{\Omega} \nabla \Delta \phi_0 \cdot \nabla (\phi - \phi_0) \, dx \]

\[ + \int_{\Omega} \nabla (-F'(\phi) + \theta_0 \phi) \cdot \nabla (\phi - \phi_0) \, dx. \]

Thus, we obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \| \phi - \phi_0 \|_{L^2}^2 + \alpha \| \nabla (\phi - \phi_0) \|_{L^2}^2 \right) \leq C (1 + \| u \|_{L^2} + \| \phi_0 \|_{H^3}) \| \nabla (\phi - \phi_0) \|_{L^2}. \]

By using [30, Lemma 4.1, Chap. IV] and taking \( t = h \), we arrive at

\[ \| \partial_t^h \phi(0) \|_{L^2}^2 + \alpha \| \nabla \partial_t^h \phi(0) \|_{L^2}^2 \leq C_o \left( 1 + \| \phi_0 \|_{H^3}^2 + \| u \|_{L^6(0,T)}^2 \right). \tag{A.27} \]

Combining the above inequality with (A.26), we are led to

\[ \| \partial_t^h \phi(t) \|_{L^2}^2 + \alpha \| \nabla \partial_t^h \phi(t) \|_{L^2}^2 + \int_0^t \| \Delta \partial_t^h \phi(\tau) \|_{L^2}^2 \, d\tau \]

\[ \leq C_o \left( 1 + \| \phi_0 \|_{H^3}^2 + \| u \|_{L^6(0,T)}^2 + \| \partial u \|_{L^6(0,T)}^2 + \| u \|_{L^6(0,T)}^2 \right) e^{C \int_0^T (1 + \| u(\tau) \|_{L^3}) \, d\tau} \]

for all \( t \in [0, T] \), which, in turn, implies

\[ \| \partial_t \phi \|_{L^6(0,T;H^1(\Omega))} + \| \partial_t \phi \|_{L^2(0,T;H^2(\Omega))} \leq C(\alpha, T, \delta, \| \phi_0 \|_{H^3}, \| u \|_{Y_T}), \tag{A.28} \]

where \( Y_T = L^6(0, T; L^3(\Omega)) \cap W^{1,2}(0, T; L^6(\Omega)) \). As an immediate consequence, in light of (A.19), (A.20) and (A.24), we infer by comparison in (A.1) that

\[ \| \phi \|_{L^6(0,T;H^3(\Omega))} + \| \phi \|_{L^2(0,T;H^4(\Omega))} \leq C(\alpha, T, \delta, \| \phi_0 \|_{H^3(\Omega)}, \| u \|_{Y_T}), \tag{A.29} \]
Next, we take \( v = A^{-1} \partial_t^h \partial_t \phi \) in (A.25). Exploiting (A.24) and (A.28), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \partial_t^h \phi \|_{L^2}^2 + \| \nabla A^{-1} \partial_t^h \partial_t \phi \|_{L^2}^2 + \alpha \| \partial_t^h \partial_t \phi \|_{L^2}^2 \leq \int_{\Omega} \partial_t^h (\phi u) \cdot \nabla A^{-1} \partial_t^h \partial_t \phi \, dx - \int_{\Omega} \partial_t^h \Psi' (\phi) \partial_t^h \partial_t \phi \, dx
\]
\[
\leq C \| \partial_t \mu \|_{L^6} \| \partial_t \partial_t \phi \|_{L^2} + C \| \mu \|_{L^3} \| \partial_t \partial_t \phi \|_{L^2} \| \nabla A^{-1} \partial_t^h \partial_t \phi \|_{L^6} + C \| \partial_t^h \phi \|_{L^2} \| \partial_t^h \partial_t \phi \|_{L^2}
\]
\[
\leq \frac{1}{2} \| \partial_t^h \partial_t \phi \|_{L^2}^2 + C \left( 1 + \| \partial_t \mu \|_{L^6}^3 + \| \mu \|_{L^3}^2 \right).
\]

By recalling (A.27), the Gronwall lemma entails
\[
\int_0^T \| \partial_t^h \partial_t \phi \|_{L^2}^2 \, dt \leq C (\alpha, T, \delta, \| \phi_0 \|_{H^3}, \| \mu \|_{Y_T}),
\]
which, in turn, gives that there exists \( \partial_t^2 \phi \in L^2(0, T; L^2(\Omega)) \) such that
\[
\| \partial_t^2 \phi \|_{L^2(0, T; L^2(\Omega))} \leq C (\alpha, T, \delta, \| \phi_0 \|_{H^3}, \| \mu \|_{Y_T}).
\]
Thus, by comparison in (A.1), we conclude that there exists \( \partial_t \mu \in L^2(0, T; L^2(\Omega)) \) such that
\[
\| \partial_t \mu \|_{L^2(0, T; L^2(\Omega))} \leq C (\alpha, T, \delta, \| \phi_0 \|_{H^3(\Omega)}, \| \mu \|_{Y_T}).
\]

The proof is complete. \( \square \)

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