ON THE STRUCTURE OF WEAK HOPF ALGEBRAS

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ABSTRACT. We study the group of group-like elements of a weak Hopf algebra and derive an analogue of Radford’s formula for the fourth power of the antipode $S$, which implies that the antipode has a finite order modulo a trivial automorphism. We find a sufficient condition in terms of $\text{Tr}(S^2)$ for a weak Hopf algebra to be semisimple, discuss relation between semisimplicity and cosemisimplicity, and apply our results to show that a dynamical twisting deformation of a semisimple Hopf algebra is cosemisimple.

1. Introduction

Weak Hopf algebras were introduced by G. Böhm and K. Szlachányi in [2] and [7] (see also their joint work [1] with F. Nill) as a generalization of ordinary Hopf algebras and groupoid algebras. A weak Hopf algebra is a vector space that has both algebra and coalgebra structures related to each other in a certain self-dual way and that possesses an antipode. The main difference between ordinary and weak Hopf algebras comes from the fact that the comultiplication of the latter is no longer required to preserve the unit (equivalently, the counit is not required to be an algebra homomorphism) and results in the existence of two canonical subalgebras playing the role of “non-commutative bases” in a “quantum groupoid”. The axioms of a weak Hopf algebra are self-dual, which ensures that when $H$ is finite dimensional, the dual vector space $H^*$ has a natural structure of a weak Hopf algebra.

For the foundations of the weak Hopf algebra theory we refer the reader to [1] and to the recent survey [15].

The initial motivation to study weak Hopf algebras was their connection with the theory of algebra extensions. It was explained in [13, 14] that weak Hopf $C^*$-algebras naturally arise as symmetries of finite depth von Neumann subfactors. A purely algebraic analogue of this result was proved in [6], where it was shown that a depth two Frobenius extension of algebras $A \subset B$ with a separable centralizer $C_B(A)$ comes from a smash product with a semisimple and cosemisimple weak Hopf algebra $H$, i.e., $(A \subset B) \cong (A \subset A\# H)$.

Another important application of weak Hopf algebras is that they provide a natural framework for the study of dynamical twists in Hopf algebras. It was proved in [4] that every dynamical twist in a Hopf algebra gives rise to a weak Hopf algebra. Also in [4] a family of dynamical quantum groups (weak Hopf algebras corresponding to dynamical twists in quantum groups at roots of unity)
was constructed. These weak Hopf algebras were shown to be quasi-triangular with non-degenerate $R$-matrices and, therefore, self-dual.

It turns out that many important properties of ordinary Hopf algebras have "weak" analogues. For example, the category $\text{Rep}(H)$ of finite rank left modules over a weak Hopf algebra $H$ is a rigid monoidal category (the category $\text{Rep}(H)$ was defined and studied in [8] for weak Hopf $C^*$-algebras and in [14] for general weak Hopf algebras). The importance of weak Hopf algebra representation categories can be seen from the result of V. Ostrik [16], who proved that every semisimple rigid monoidal category with finitely many (classes of) simple objects is equivalent to $\text{Rep}(H)$ for some semisimple weak Hopf algebra $H$.

Also, the theory of integrals for weak Hopf algebras developed in [1] is essentially parallel to that of ordinary Hopf algebras. Using it, one can prove an analogue of Maschke's theorem [1, Theorem 3.13] for weak Hopf algebras and show that semisimple weak Hopf algebras are finite dimensional.

Despite these similarities, the structure of weak Hopf algebras is much more complicated than that of usual Hopf algebras, even in the semisimple case. For instance, the antipode of a semisimple weak Hopf algebra $H$ over $\mathbb{C}$, the field of complex numbers, may have an infinite order and quantum dimensions of irreducible $H$-modules can be non-integer (this is clear from the relation between weak Hopf algebras and semisimple monoidal categories mentioned above). Also, weak Hopf algebras of prime dimension can be non-commutative and non-cocommutative, quite in contrast with the usual Hopf algebra theory, cf. [21]. An example of an indecomposable semisimple weak Hopf algebra of dimension 13 having such properties was given in [2] (see also [14, Appendix]).

In this paper we start an investigation of the structure of finite dimensional weak Hopf algebras and prove weak Hopf algebra analogues of several classical Hopf algebra results obtained in [8] and [18].

After giving necessary definitions and discussing basic properties of weak Hopf algebras in Section 2, we classify all minimal weak Hopf algebras (i.e., those generated by the identity element) in Section 3. We show that every weak Hopf algebra can be obtained as a deformation of a weak Hopf algebra with the property that the square of the antipode acts as the identity on the minimal weak Hopf subalgebra.

Next, in Section 4 we define the group $G(H)$ of group-like elements of a weak Hopf algebra $H$ and show that it contains a normal subgroup $G_0(H)$ of trivial group-like elements that belong to the minimal weak Hopf subalgebra of $H$. An adjoint action of any group-like element defines a weak Hopf algebra automorphism of $H$. Note that even in finite dimensional case both $G(H)$ and $G_0(H)$ are usually infinite. However, when $H$ is finite dimensional, we show that the quotient group $\tilde{G}(H) = G(H)/G_0(H)$ is finite. This $\tilde{G}(H)$ turns out to be the correct weak Hopf algebra analogue of the group of group-like elements in an ordinary Hopf algebra.

We introduce the notion of a distinguished coset of group-like elements of $H$, that "measures the difference" between left and right integrals in $H^*$.

In Section 5 we extend to weak Hopf algebras the result of D. Radford [18] stating that the antipode of a finite dimensional Hopf algebra has a finite order. Namely, we establish a formula for the fourth power of the antipode analogous to [18, Proposition 6] and show that the order of the antipode of a finite dimensional Frobenius weak Hopf algebra is finite modulo the group of trivial automorphisms of $H$. As in [18], the proof uses non-degenerate integrals and distinguished group-like elements of $H$ and $H^*$. 
Finally, we extend the Larson-Radford formula for $\text{Tr}(S^2)$ and use it to establish a sufficient condition for a weak Hopf algebra being both semisimple and cosemisimple. We also prove that semisimplicity of a weak Hopf algebra with coinciding bases is equivalent to its cosemisimplicity. As an application, we show that a dynamical twisting deformation of a semisimple Hopf algebra \cite{4} is a semisimple and cosemisimple weak Hopf algebra (dynamical twists are closely related to the dynamical quantum Yang-Baxter equation of Felder \cite{6}).

2. Preliminaries

Throughout this paper $k$ denotes a field. We use Sweedler’s notation for a comultiplication: $\Delta(hg) = \Delta(h)\Delta(g)$, $h,g \in H$, (1)

(i) The comultiplication $\Delta$ is a (not necessarily unit-preserving) homomorphism of algebras:

(ii) The unit and counit satisfy the following identities:

(iii) There is a linear map $S: H \to H$, called an antipode, such that

\begin{align*}
(1,2,3) & : A \text{ weak Hopf algebra is a vector space } H \text{ with the structures of an associative algebra } (H, m, 1) \text{ with a multiplication } m : H \otimes_k H \to H \text{ and unit } 1 \in H \text{ and a coassociative coalgebra } (H, \Delta, \epsilon) \text{ with a comultiplication } \Delta : H \to H \otimes_k H \text{ and counit } \epsilon : H \to k \text{ such that:}
\end{align*}

\begin{enumerate}
\item $\Delta(hg) = \Delta(h)\Delta(g), \quad h,g \in H,$
\item The unit and counit satisfy the following identities:
\item There is a linear map $S : H \to H$, called an antipode, such that
\item Axioms (2) and (3) above are analogous to the usual bialgebra axioms of $\Delta$ being a unit preserving map and $\epsilon$ being an algebra homomorphism. Axioms (4) and (5) generalize the properties of the antipode with respect to the counit. Also, it is possible to show that given (2) - (5), axiom (6) is equivalent to $S$ being both anti-algebra and anti-coalgebra map.
\item A morphism between weak Hopf algebras $H_1$ and $H_2$ is a map $\alpha : H_1 \to H_2$ which is both algebra and coalgebra homomorphism preserving $1$ and $\epsilon$ and which intertwines the antipodes of $H_1$ and $H_2$, i.e., $\alpha \circ S_1 = S_2 \circ \alpha$. The image of a morphism is clearly a weak Hopf algebra.
When \( \dim_k H < \infty \), there is a natural weak Hopf algebra structure on the dual vector space \( H^* = \text{Hom}_k(H, k) \) given by

\[
\langle \phi \psi, h \rangle = \langle \phi \otimes \psi, \Delta(h) \rangle,
\]
\[
\langle \Delta(\phi), h \otimes g \rangle = \langle \phi, hg \rangle,
\]
\[
\langle S(\phi), h \rangle = \langle \phi, S(h) \rangle,
\]
for all \( \phi, \psi \in H^* \), \( h, g \in H \). The unit of \( H^* \) is \( \epsilon \) and the counit is \( \phi \mapsto \langle \phi, 1 \rangle \).

In what follows we use the Sweedler arrows for the dual actions, writing

\[
h \mapsto \phi = \phi(1)\langle \phi(2), h \rangle, \quad \phi \mapsto h = \langle \phi(1), h \rangle\phi(2).
\]

for all \( h \in H, \phi \in H^* \).

**Example 2.3.** Let \( G \) be a groupoid (a small category with inverses) with finitely many objects, then the groupoid algebra \( kG \) (generated by morphisms \( g \in G \) with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise) is a weak Hopf algebra via:

\[
\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}, \quad \text{for all } g \in G.
\]

In fact, when \( G \) is finite and \( k \) is algebraically closed and has characteristic 0, this is the most general example of a cocommutative weak Hopf algebra [1].

If \( G \) is finite, then the dual weak Hopf algebra \((kG)^*\) is isomorphic to the algebra of functions on \( G \), i.e., it is generated by idempotents \( p_g, g \in G \) such that \( p_gp_h = \delta_{g,h}p_g \), with the following structure operations

\[
\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \epsilon(p_g) = \delta_{g,gg^{-1}} p_{g^{-1}}, \quad S(p_g) = p_{g^{-1}}.
\]

This is the most general example of a finite dimensional commutative weak Hopf algebra over an algebraically closed field of characteristic 0 [1].

More interesting examples of weak Hopf algebras arise as dynamical twisting deformations of Hopf algebras [4] and as symmetries of finite depth subfactors [13], [14].

We refer the reader to the original article [1] and recent survey [15] for a detailed introduction to the theory of weak Hopf algebras and its applications.

**Counital maps and bases.** The linear maps defined in (4) and (5) are called target and source counital maps and are denoted \( \varepsilon_t \) and \( \varepsilon_s \) respectively:

\[
\varepsilon_t(h) = \epsilon(1_{(1)}h)1_{(2)}, \quad \varepsilon_s(h) = 1_{(1)}\epsilon(h1_{(2)}),
\]

for all \( h \in H \). The images of the counital maps

\[
H_t = \varepsilon_t(H), \quad H_s = \varepsilon_s(H)
\]

are separable subalgebras of \( H \), called target and source bases or counital subalgebras of \( H \). These subalgebras commute with each other; moreover

\[
H_t = \{ (\phi \otimes \text{id})\Delta(1) \mid \phi \in H^* \} = \{ h \in H \mid \Delta(h) = \Delta(1)(h \otimes 1) \},
\]
\[
H_s = \{ (\text{id} \otimes \phi)\Delta(1) \mid \phi \in H^* \} = \{ h \in H \mid \Delta(h) = (1 \otimes h)\Delta(1) \},
\]

i.e., \( H_t \) (respectively, \( H_s \)) is generated by the right (respectively, left) tensor factors of \( \Delta(1) \) in the shortest possible representation of \( \Delta(1) \) in \( H \otimes H \).
Integrals and Frobenius weak Hopf algebras. The notion of an integral in a weak Hopf algebra is a generalization of that of an integral in a usual Hopf algebra [2, 2.1.1].

**Definition 2.4 ([1])**. A left (right) integral in $H$ is an element $\ell \in H$ ($r \in H$) such that
\begin{align}
\,(15)\, h\ell = \varepsilon_t(h)\ell, \quad (r h) = r \varepsilon_s(h) \quad \text{for all } h \in H.
\end{align}

We denote by $\int_l^l H$ (respectively, by $\int_r^r H$) the space of left (right) integrals in $H$.

Clearly, $\int_l^l H$ is a left ideal of $H$ and $\int_r^r H$ is a right ideal of $H$.

An integral in a finite dimensional weak Hopf algebra $H$ (left or right) is called non-degenerate if it defines a non-degenerate functional on $H^*$. A left integral $\ell$ is called normalized if $\varepsilon_t(\ell) = 1$. Similarly, $r \in \int_r^r H$ is normalized if $\varepsilon_s(r) = 1$.

Any left integral $\lambda \in \int_l^l H^*$ satisfies the following invariance property :
\begin{align}
\,(16)\, g((1))\langle \lambda, h g((2)) \rangle = S(h((1)))\langle \lambda, h((2)) g \rangle, \quad g, h \in H.
\end{align}

Similarly, a right integral $\rho \in \int_r^r H^*$ satisfies
\begin{align}
\,(17)\, \langle \rho, g((1)) h g((2)) \rangle = \langle \rho, g h((1)) S(h((2)) \rangle, \quad g, h \in H.
\end{align}

The fundamental theorem for weak Hopf modules [1, 3.9] implies that if $H$ is finite dimensional then $\int_l^l H$ generates $H$ as a dual left $H^*$-module. More precisely, there is an isomorphism $H \cong \int_l^l H \otimes_{H^*} H^*$ of right $H^*$-Hopf modules given by
\begin{align}
\ell \otimes \phi \mapsto S(\phi) \twoheadrightarrow \ell,
\end{align}

where $\ell \in \int_l^l H$ and $\phi \in H^*$. Note that, unlike in the usual Hopf algebra case, this does not immediately imply existence of non-degenerate integrals in $H$. However, it follows from the arguments of [1, Section 3] that $H$ is automatically a Frobenius algebra when its bases are commutative. It was also shown in [1, 3.16] that $H$ is a Frobenius algebra if and only if there exists a non-degenerate integral in $H$, if and only if $\dim \int_l^l H = \dim H$, and if and only if $H^*$ is a Frobenius algebra.

**Note 2.5.** In what follows we always work with Frobenius weak Hopf algebras.

Maschke’s theorem for weak Hopf algebras, proved in [1, 3.13] states that a weak Hopf algebra $H$ is semisimple if and only if $H$ is separable and if and only if there exists a normalized left integral in $H$. In particular, every semisimple weak Hopf algebra is finite dimensional. Note that since $\varepsilon_t(\int_l^l H) \subset Z(H) \cap H_t$, it follows that $H$ is semisimple if and only if there exists $\ell \in \int_l^l H$ such that $\varepsilon_t(\ell)p \neq 0$ for all primitive idempotents $p \in Z(H) \cap H_t$.

For a Frobenius $H$, there is a useful notion of duality between left integrals in $H$ and $H^*$ [1, 3.18]. If $\ell \in \int_l^l H$ is a left integral and there exists $\lambda \in H^*$ such that $\lambda \rightarrow \ell = 1$, then such $\lambda$ is unique, it is a left integral in $H^*$, and both $\ell$ and $\lambda$ are non-degenerate. Moreover, $\ell \rightarrow \lambda = \varepsilon$. Such a pair of integrals is called a pair of dual integrals.

### 3. Minimal weak Hopf algebras

Here we classify all weak Hopf algebras generated by their bases. These are probably the most trivial (although in general non-commutative and non-cocommutative)
weak Hopf algebras, as every weak Hopf algebra contains a weak Hopf subalgebra with this property.

Let $H$ be a weak Hopf algebra. Observe that a subalgebra $H_{\min} := H_t H_s$ is a finite dimensional weak Hopf subalgebra of $H$ (this follows from the fact that bases are finite dimensional and commute with each other). Moreover, since $H_t$ and $H_s$ are generated by the tensor factors in the shortest possible representation of $\Delta(1)$, $H_{\min}$ is the minimal weak Hopf subalgebra of $H$ containing 1.

**Definition 3.1.** A weak Hopf algebra is called **minimal** if it has no proper weak Hopf subalgebras.

**Remark 3.2.** The only minimal usual Hopf algebra is $k1$.

We have

$$H_{\min} \cong H_t \otimes_{H_t \cap H_s} H_s \cong H_t \otimes_{H_t \cap H_s} H_t^{\text{op}},$$

as algebras (with the obvious multiplication in the relative tensor product), where $H_t \cap H_s \subseteq Z(H_t) \cap Z(H_s)$.

Recall that for an algebra $A$ an element $E \in A \otimes A$ is called a **separability element** if $(a \otimes 1)E = E(1 \otimes a)$ for all $a \in A$ and $m(E) = 1$, where $m$ is the product in $A$. An algebra $A$ for which there exists such an element is called separable; it is a standard fact in the theory of associative algebras that a separable algebra over a field is finite dimensional and semisimple. A separability element $E$ is two-sided if $E(a \otimes 1) = (1 \otimes a)E$; if a two-sided separability element exists, it is unique.

**Remark 3.3.** Note that $H_{\min}$ is a separable algebra with a separability element $E = 1 \otimes \pi(1)$ and $E(1 \otimes \pi(2))$. In particular, any minimal weak Hopf algebra is semisimple.

Assume that $k$ is an algebraically closed field of characteristic 0. The next Proposition gives an explicit description and complete classification of minimal weak Hopf algebras over $k$.

**Proposition 3.4.** Every minimal weak Hopf algebra $H$ is completely determined by the following data : $(B, A, g)$, where $B$ is a finite dimensional semisimple algebra, $A \subseteq Z(B)$ is a commutative subalgebra, and $g \in B$ is an invertible element such that $\text{Tr}(\pi(g)) = \deg \pi$ for any finite dimensional representation $\pi$ of $B$, as follows.

As an algebra, $H \cong B \otimes_A B^{\text{op}}$, i.e., $H$ is generated by the elements $b \in B$, $\overline{c} \in B^{\text{op}}$ (where $c \mapsto \overline{c}$ is the canonical algebra anti-isomorphism from $B$ to $B^{\text{op}}$) and relations $b \overline{c} = \overline{c}b$, $a = \overline{a}$, $a \in A$. The coalgebra structure is given by

$$\Delta(b \overline{c}) = b(ge^{(1)}) \otimes e^{(2)} \overline{c},$$ (18)

$$e(b \overline{c}) = \text{Tr}_{\text{reg}}(g^{-1}eb),$$ (19)

where $e = e^{(1)} \otimes e^{(2)}$ is the unique 2-sided separability element of $B$ and $\text{Tr}_{\text{reg}}$ is the trace of the regular representation of $B$, and the antipode is given by

$$S(b \overline{c}) = g^{-1}cg.$$

Minimal weak Hopf algebras defined by $(B, A, g)$ and $(B', A', g')$ are isomorphic if and only if there exists an algebra isomorphism $\tau : B \rightarrow B'$ such that $\tau(A) = A'$ and $\tau(g) = g'$. 
Proof. Let $B = H_t$ and $A = H_t \cap H_s \subseteq Z(B)$, then we clearly have $H \cong B \otimes_A B^{\text{op}}$ as algebras (where $B^{\text{op}} \cong H_s$). We identify the map $b \mapsto \overline{b}$ with the antipode map of $H$ restricted to $H_t$.

It can be easily deduced from the axioms of a weak Hopf algebra that for all $b, c \in B$ we have $\epsilon(bc) = \epsilon(cS^2(b))$ and that $\epsilon|_B$ is non-degenerate. Therefore, there exists an invertible element $g \in B$ such that $\epsilon(b) = \text{Tr}_{\text{reg}}(g^{-1}b)$. Hence, $S^2(b) = g^{-1}bg$ and $S(b\overline{c}) = S^2(c)S(b) = g^{-1}cg\overline{b}$. Since $\epsilon_{|H_s} = S|_{H_s}$, we have:

$$\epsilon(b\overline{c}) = \epsilon(bS(c)) = \epsilon(bS^2(c)) = \epsilon(bg^{-1}cg) = \text{Tr}_{\text{reg}}(g^{-1}cb).$$

Since $S(1) \otimes 1 = 1$, where $h \in B$ is such that $\epsilon h = 1$, i.e., $\text{Tr}_{\text{reg}}(\pi(h)) = \deg \pi$ for any irreducible representation $\pi$ of $B$. The counit and antipode properties imply that $h = g$, since

$$1 = 1 \delta(1) = S^{-1}(\delta(1))\text{Tr}(g^{-1}h\delta(2)) = S^{-1}(g^{-1}h).$$

We compute:

$$\Delta(1) = S^{-1}(\delta(1)) \otimes \delta(2) = S(S^{-2}(\delta(1))) \otimes \delta(2)$$

$$= S\delta(1)g^{-1} \otimes \delta(2) = (\delta(1)g^{-1}) \otimes \delta(2) = \delta(1) \otimes \delta(2).$$

The comultiplication on bases is determined by the value of $\Delta(1)$, therefore

$$\Delta(b\overline{c}) = b\overline{c} \otimes \delta(2), \quad b \in B, \overline{c} \in B^{\text{op}},$$

which completely defines the structure of $H$. One can check that the above operations indeed define a weak Hopf algebra.

Let $H'$ be the minimal weak Hopf algebra defined by $(B', A', g')$ and $\tau : H \to H'$ be an isomorphism of weak Hopf algebras. Then

$$\tau(B) = B' \quad \text{and} \quad \tau(A) = \tau(H_t \cap H_s) = H'_t \cap H'_s = A'.$$

Since $\tau$ preserves the regular trace of $H_t$ and $\epsilon$ we have $\tau(g) = \tau(g')$.

\[ \square \]

Remark 3.5. Examples of minimal weak Hopf algebra structures on the algebra $B \otimes_k B^{\text{op}}$ appeared in [1, Appendix].

We will denote the minimal weak Hopf algebra defined by the data $(B, A, g)$ by $H_{\text{min}}(B, A, g)$.

Remark 3.6.  
(i) The square of the antipode of $H_{\text{min}}(B, A, g)$ is given by the adjoint action of $g^{-1}S(g)$. In particular, in any weak Hopf algebra with commutative bases, $S^2$ is trivial on the minimal weak Hopf subalgebra.

(ii) For any given $A$ and $B$ as above, $H_{\text{min}}(B, A, 1)$ is a unique, up to an isomorphism, minimal weak Hopf algebra, for which $H_t \cong B$, $H_t \cap H_s \cong A$, and $S^2 = \text{id}$.

(iii) For any minimal idempotent $p \in A$, the space $pH_{\text{min}}(B, A, g)$ is a simple subcoalgebra of $H_{\text{min}}(B, A, g)$; in particular, any minimal weak Hopf algebra is cosemisimple. Duals of weak Hopf algebras $H_{\text{min}}(B, k, 1)$, which are simple as algebras, were considered in [12].
Remark 3.7. Let us explain that every weak Hopf algebra is a deformation of a weak Hopf algebra $H$ with the property that

$$S^2 = \text{id}$$
onumber
on the minimal weak Hopf subalgebra $H_{\text{min}}$ of $H$.

Explicitly, if $H$ is a weak Hopf algebra and $q \in H_t$ is an invertible element such that $S^2(q) = q$ and $S(1_{(1)})q1_{(2)} = 1$ then there is a deformation weak Hopf algebra $H_q$ with the underlying algebra $H$ and the structure operations

$$\Delta'(h) = \Delta(h)(1 \otimes q), \quad \epsilon'(h) = \epsilon(hq^{-1}), \quad \text{and} \quad S'(h) = q^{-1}S(h)q, \quad h \in H.$$

This deformation can be understood in terms of twisting, cf. [15, 6.1.4]. In particular, if the minimal weak Hopf subalgebra of $H$ is $H_{\text{min}}(H_t,H_t \cap H_s,g)$, then $H_{g^{-1}}$ has property (21). Thus, problems regarding general weak Hopf algebras can be translated to problems regarding those with the regularity property (21).

Note that if $H$ satisfies (21) then so does $H^*$. This appears to be a natural property, since it is satisfied by dynamical quantum groups [4] and weak Hopf algebras arising as symmetries of Jones-von Neumann subfactors [13], [14].

The next Corollary shows that, in contrast with usual semisimple Hopf algebras (cf. [19]), there can be infinitely (even uncountably) many non-isomorphic semisimple weak Hopf algebras with the same algebra structure.

Corollary 3.8. Let $B$ be a non-commutative finite dimensional semisimple algebra, and $g_1, g_2 \in B$ be invertible elements with different spectra. Then weak Hopf algebras $H_{\text{min}}(B,k_1,g_1)$ and $H_{\text{min}}(B,k_1,g_2)$ are not isomorphic.

Proof. It is clear that no automorphism of $B$ can map $g_1$ to $g_2$. \qed

Conjecture 3.9. The number of non-isomorphic weak Hopf algebras $H$ with the property (21) is finite in any given dimension.

4. Group-like elements in a weak Hopf algebra

Definition and properties. Let $H$ be a weak Hopf algebra. The notion of a group-like element in a weak Hopf algebra was introduced in [1].

Definition 4.1. An element $g \in H$ is said to be group-like if it is invertible and satisfies

$$(22) \quad \Delta(g) = (g \otimes g)\Delta(1) \quad \text{and} \quad \Delta(g) = \Delta(1)(g \otimes g).$$

Group-like elements of $H$ form a group which we will denote by $G(H)$.

Lemma 4.2. For any $g \in G(H)$ we have $\epsilon_s(g) = \epsilon_s(g) = 1$ and the element $S(g) = g^{-1}$ is group-like.

Proof. The identities for counital maps follow from applying $\epsilon$ to (22). That $S(g) = g^{-1}$ follows from the uniqueness of the inverse element and antipode. \qed

Lemma 4.3. If $H$ is finite dimensional, then $\gamma \in G(H^*)$ if and only if it is invertible and satisfies the following two conditions :

$$\langle \gamma, hg \rangle = \langle \gamma, h1_{(1)} \rangle \langle \gamma, S(1_{(2)})g \rangle,$$
$$\langle \gamma, hg \rangle = \langle \gamma, hS(1_{(1)}) \rangle \langle \gamma, 1_{(2)}g \rangle.$$  

for all $h, g \in H$. 

Proof. This is a straightforward dualization of (22). Equation (23) is equivalent to
\( \Delta(\gamma) = (\gamma \otimes \gamma) \Delta(\epsilon) \) and equation (24) is equivalent to \( \Delta(\gamma) = \Delta(\epsilon)(\gamma \otimes \gamma) \).

Remark 4.4. If \( H \) is an ordinary Hopf algebra, then the notion of a group-like element coincides with the usual one (cf. [9], [20]).

Proposition 4.5. Any group-like element in a minimal weak Hopf algebra \( H_{\text{min}} \) has the form

\[
g = S(y)y^{-1},
\]

where \( y \in H_s \) is such that \( S^2(y) = y \).

Proof. Without loss of generality we may assume that \( H_t \cap H_s = k1 \), as every minimal weak Hopf algebra is a direct sum of weak Hopf algebras with this property.

Let \( g = \sum_{i=1}^n y_i z_i \) be a group-like element in \( H_{\text{min}} \), where \( \{y_i\} \subset H_s \) and \( \{z_i\} \subset H_t \) are linearly independent sets. We may assume that each \( y_i \) is invertible. Then

\[
\sum_{i=1}^n (z_i \otimes y_i)\Delta(1) = \sum_{ij=1}^n (y_i z_i \otimes y_j z_j)\Delta(1) = \sum_{ij=1}^n (z_i \otimes y_j z_j S^{-1}(y_i))\Delta(1),
\]

whence \( y_i = \left( \sum_{ij=1}^n y_j z_j \right) S^{-1}(y_i) \) by linear independence. Therefore, \( n = 1 \) and \( g = y_1 S^{-1}(y_1)^{-1} \). It is straightforward to check that this \( g \) satisfies the second equality of (22) if and only if \( S^2(y_1) = y_1 \).

Definition 4.6. Elements of the form \( g = S(y)y^{-1}, \ y \in H_s \) are called trivial group-like elements.

Group-like elements give rise to weak Hopf algebra automorphisms.

Proposition 4.7. If \( g \in H \) is a group-like element, then the map \( h \mapsto ghg^{-1} \), where \( h \in H \), is a weak Hopf algebra automorphism. If \( \gamma \in H^* \) is a group-like element, then the map \( h \mapsto (\gamma \mapsto h \mapsto \gamma^{-1}) \) is a weak Hopf algebra automorphism.

Proof. A direct computation.

Proposition 4.8. If \( \xi \in H^*_s \) is invertible and \( y = (\xi^{-1} \rightharpoonup 1) \in H_s \) then

\[
(S(\xi)\xi^{-1}) \rightharpoonup h \leftarrow (S(\xi)^{-1} \xi) = S(y)y^{-1} h S(y)^{-1} y.
\]

Proof. It follows from the properties of bases of a weak Hopf algebra that

\[
(S(\xi)\xi^{-1}) \rightharpoonup h \leftarrow (S(\xi)^{-1} \xi) =
\]

\[
= (S(\xi) \rightharpoonup 1)(1 \leftarrow S(\xi)^{-1} \xi) h (\xi^{-1} \rightharpoonup 1)(1 \leftarrow \xi)
\]

\[
= (\xi^{-1} \rightharpoonup 1) S(\xi^{-1} \rightharpoonup 1) h S(\xi^{-1} \rightharpoonup 1)^{-1} (\xi^{-1} \rightharpoonup 1),
\]

for all \( \xi \in H^*_s \).

Definition 4.9. We will call a weak Hopf algebra automorphism of \( H \) trivial if it is defined by (23).

Trivial weak Hopf algebra automorphisms of \( H \) form a normal subgroup \( \text{Aut}_0(H) \) of the group \( \text{Aut}(H) \) of all weak Hopf algebra automorphisms of \( H \). Let \( \text{Aut}(H) \) denote the corresponding quotient group.
The group of group-like elements. Let us denote

\[
G_1(H) := \{ g \mid \Delta(g) = (g \otimes g)\Delta(1) \}\backslash \{0\},
\]

\[
G_2(H) := \{ g \mid \Delta(g) = \Delta(1)(g \otimes g) \}\backslash \{0\},
\]

then the group \(G(H)\) of group-like elements of \(H\) consists of invertible elements of \(G_1(H) \cap G_2(H)\).

The group \(G_0(H)\) of all trivial group-like elements is a normal subgroup in \(G(H)\). Define \(\tilde{G}(H) = G(H)/G_0(H)\), the quotient group of \(G(H)\) by \(G_0(H)\). Let \(g \mapsto \tilde{g}\) denote the canonical projection from \(G(H)\) to \(\tilde{G}(H)\).

It turns out that \(\tilde{G}(H)\) plays more important role than \(G(H)\), as it possesses properties extending those of the group of group-like elements of a usual Hopf algebra.

**Remark 4.10.** We have \(G_0(H) = \{1\}\) if and only if the bases of \(H\) coincide.

**Remark 4.11.** For any \(g \in G(H)\) the map \(x \mapsto gx\) is an isomorphism between cosemisimple coalgebras \(H_{\text{min}} = H_1H_s\) and \(H_g := gH_{\text{min}}\). For \(g,h \in G(H)\) we have \(H_g = H_h\) if and only if \(\tilde{g} = \tilde{h}\) in \(\tilde{G}(H)\).

**Corollary 4.12.** If \(H\) is finite dimensional then \(\tilde{G}(H)\) is finite.

**Proof.** By the previous remark, to every \(g \in G(H)\) there corresponds a cosemisimple subcoalgebra \(H_g\) of \(\text{Corad}(H)\), the coradical of \(H\); and \(H_g = H_h\) if and only if \(g\) and \(h\) define the same coset. But there are only finitely many mutually non-equal cosemisimple subcoalgebras of \(\text{Corad}(H)\), so the quotient group is necessarily finite.

**Modules associated with group-like elements.** In the case of a usual Hopf algebra \(H\), the group-like elements of \(H^*\) are precisely the homomorphisms from \(H\) to the ground field \(k\) (i.e., \(H\)-module structures on \(k\)). In this subsection we present an analogous correspondence for weak Hopf algebras.

Let \(H\) be a finite dimensional weak Hopf algebra. For every \(\gamma \in G_1(H^*)\) we define a linear map \(\varepsilon_\gamma^*: H \to H_s\) by setting

\[
\varepsilon_\gamma^*(h) := \langle \gamma, h\rangle_{(1)}S(1_{(2)}).
\]

Similarly, for every \(\gamma \in G_2(H^*)\) we define a linear map \(\varepsilon^*_\gamma : H \to H_t\) by

\[
\varepsilon^*_\gamma(h) := S(1_{(1)})\langle \gamma, 1_{(2)}h\rangle, \quad h \in H.
\]

It follows from Lemma 4.2 that \(\varepsilon_\gamma^*\) and \(\varepsilon_\gamma^*_t\) are projections, i.e., \(\varepsilon_\gamma^* \circ \varepsilon_\gamma^* = \varepsilon_\gamma^*\) and \(\varepsilon_\gamma^*_t \circ \varepsilon_\gamma^*_t = \varepsilon_\gamma^*_t\). These projections are generalizations of counital maps, since we have \(\varepsilon^*_1 = \varepsilon_t\) and \(\varepsilon^*_s = \varepsilon_s\).

**Lemma 4.13.** For every \(\gamma \in G_1(H^*)\) the source counital subalgebra \(H_s\) becomes a right \(H\)-module via

\[
y \cdot h := \varepsilon_\gamma^*(yh), \quad y \in H_s, h \in H.
\]

Similarly, for every \(\gamma \in G_2(H^*)\) the target counital subalgebra \(H_s\) becomes a left \(H\)-module via

\[
h \cdot z := \varepsilon^*_\gamma(hz), \quad z \in H_t, h \in H.
\]
Proof. We will only prove the first statement since the proof of the second one is completely similar. We have

\[ y \cdot 1 = \langle \gamma, y1(1) \rangle S(1(2)) = \epsilon(y1(1))S(1(2)) = y, \]

for all \( y \in H_s \), using Lemma 4.2 and properties of the counit. Also, for all \( g, h \in H \) we compute, using Lemma 4.3:

\[
(y \cdot g) \cdot h = \langle \gamma, yg1(1) \rangle (S(1(2)) \cdot h) = \langle \gamma, yg1(1) \rangle \langle \gamma, S(1(2))h1'(1) \rangle S(1'(2)) = \langle \gamma, ygh1'(1) \rangle S(1'(2)) = y \cdot (gh),
\]

where \( 1' \) stands for the second copy of 1.

Let us denote \( H^*_s \) (respectively, \( H^*_t \)) the \( H \)-module from Lemma 4.13.

**Proposition 4.14.** The correspondence \( \gamma \mapsto H^*_s \) is a bijection between \( G_1(H^*) \) and the set of all right \( H \)-module structures on \( H_s \) that restrict to the regular \( H \)-module. Moreover, if \( \gamma_1, \gamma_2 \in G_1(H^*) \), then \( H^*_1 \cong H^*_2 \) if and only if \( \gamma_2 = \gamma_1 S(\xi)\xi^{-1} \) for some \( \xi \in H^*_s \).

There is also a similar bijection between \( G_2(H^*) \) and the set of all right \( H \)-module structures on \( H_t \) that restrict to the regular \( H_1 \)-module.

Analogous statements also hold for left \( H \)-modules.

**Proof.** If \( H_s \) has a structure of a right \( H \)-module via \( y \otimes h \mapsto y \cdot h \), such that its restricted \( H_s \)-module is the regular \( H_s \)-module, then the functional

\[ \gamma : h \mapsto \epsilon(1 \cdot h), \quad h \in H \]

(30) belongs to \( G_1(H^*) \) by Lemma 4.3, since

\[ \langle \gamma, hg \rangle = \epsilon((1 \cdot h) \cdot g) = \epsilon((1 \cdot h)1(1))\epsilon(S(1(2)) \cdot g) = \langle \gamma, h1(1) \rangle \langle \gamma, S(1(2))g \rangle, \]

for all \( h, g \in H \). Clearly, formulas (28) and (30) establish a bijective correspondence.

Next, \( H^*_1 \) and \( H^*_2 \) are isomorphic if and only if there is an invertible element \( v \in H_s \) such that

\[ vS(1(2)) \langle \gamma_1, h1(1) \rangle = S(1(2)) \langle \gamma_2, hv1(1) \rangle, \]

for all \( h \in H \). This is equivalent to

\[ \gamma_2 = S^{-2}(v) \gamma_1 S^{-1}(v) = S^{-1}(v) \gamma_1 S^{-1}(v) = \epsilon \gamma_1 S^{-1}(v) = \epsilon \gamma_1 S^{-1}(v), \]

i.e., \( \gamma_2 = \gamma_1 S(\xi)\xi^{-1} \), where \( \xi = S^{-1}(v^{-1}) \epsilon \in H^*_s \). Hence, \( \gamma_1 = \gamma_2 \) in \( \tilde{G}(H^*) \).

Conversely, if \( \gamma_2 = \gamma_1 S(\xi)\xi^{-1} \) for some \( \xi \in H^*_s \), then the map \( y \mapsto vy \), where \( v = S(\xi^{-1}) \epsilon \), is an isomorphism between \( H^*_1 \) and \( H^*_2 \).

In a similar way one can show that if \( z \otimes h \mapsto z \circ h \) is a right \( H \)-module structure on \( H_t \) that restricts to the regular \( H_1 \)-module, then

\[ \gamma : h \mapsto \epsilon(1 \circ h), \quad h \in H \]

(31) defines an element of \( G_2(H) \). This correspondence is also bijective and \( H \)-modules corresponding to \( \gamma_1, \gamma_2 \) are isomorphic if and only if \( \gamma_2 = \gamma_1 S(\xi)\xi^{-1} \) for some \( \xi \in H^*_s \).
Remark 4.15. Suppose that we have both $H$-module structures on $H_s$ and $H_t$ described in Proposition 4.14 with actions of $H$ denoted by $\cdot$ and $\circ$ respectively. Let $\gamma$ and $\gamma'$ be the corresponding group-like elements:

$$\langle \gamma, h \rangle = \epsilon(1 \cdot h), \quad \langle \gamma', h \rangle = \epsilon(1 \circ h), \quad h \in H.$$

If $\epsilon(1 \cdot h) = \epsilon(1 \circ h)$ for all $h \in H$, then $\gamma = \gamma'$ is a group-like element in $H^*$. Indeed, by Proposition 4.14 it satisfies both conditions of (22). Note that $\varepsilon_s(\gamma) = \epsilon$ since

$$\langle \varepsilon_s(\gamma), h \rangle = \epsilon(1 \cdot \varepsilon_s(h)) = \epsilon(h), \quad h \in H,$$

and, therefore, $S(\gamma)\gamma = \varepsilon_s(\gamma) = \epsilon$, i.e., $\gamma$ is invertible.

The next Proposition describes the space of self-intertwiners of $H_s^\gamma$, $\gamma \in G(H^*)$.

Proposition 4.16. The map $T \mapsto T(1)$ is an isomorphism between the algebras $\text{End}(H_s^\gamma)_H$ and $Z(H) \cap H_s$. These algebras are also isomorphic to $H_s^* \cap H_s^\gamma$.

Proof. Since $H_s^\gamma$ restricts to the right regular $H_s$-module, every self-intertwiner of $H_s^\gamma$ is of the form $y \mapsto T(1)y$. The condition that $T \circ h = h \circ T$ in $\text{End}(H_s^\gamma)_H$ for all $h$ is equivalent to

$$\langle \gamma, hS^{-2}(T(1)) \rangle = \langle \gamma, T(1)h \rangle \quad \text{for all } h \in H.$$

This, in turn, is equivalent to the identity

$$T(1) \rightarrow \epsilon = \epsilon \rightarrow T(1).$$

This means that $\epsilon \rightarrow T(1) \in H_s^* \cap H_s^\gamma$. Observe that $\zeta \mapsto (\zeta \rightarrow 1)$ is an algebra isomorphism between $H_s^* \cap H_s^\gamma$ and $Z(H) \cap H_s$ that maps $\epsilon \rightarrow T(1)$ to $T(1)$. Clearly, $T \mapsto T(1)$ is an algebra isomorphism.

5. The Radford formula for $S^4$

Distinguished group-like elements. We use the idea of Radford [18] to define a canonical coset of group-like elements in the quotient group $\bar{G}(H) = G(H)/G_0(H)$. Let $H$ be a finite dimensional Frobenius weak Hopf algebra such that $S^4|_{H_{\text{min}}} = \text{id}$, and let $\ell$ be a non-degenerate left integral in $H$.

Lemma 5.1 ([1, 3.17]). The element $\ell$ is cyclic and separating for the right $H_s$-module (respectively, $H_t$-module) $\int_H^\ell$, where $H_s$ (respectively, $H_t$) acts by the right multiplication.

Proof. The map $\xi \mapsto (\xi \rightarrow 1)$ is a linear isomorphism between $H_s^\star$ and $H_s$. For all non-zero $\xi \in H_s^\star$ we have $0 \neq \xi \rightarrow \ell = \ell(\xi \rightarrow 1)$, whence for all $y \in H_s$ we have $\ell y = 0$ if and only if $y = 0$, i.e., $\ell$ is separating. It is cyclic since $\dim \int_H^\ell = \dim H_s$. The proof of the statement regarding $H_t$ is similar and uses that $\ell \leftarrow \xi = \ell(1 \leftarrow \xi)$ for all $\xi \in H_s^\star$.

By Lemma 5.1 any non-degenerate $\ell \in \int_H^\ell$ gives rise to a right $H$-module structures on $H_s$ and $H_t$ given by

$$\ell y h = \ell(y \cdot h) \quad \text{and} \quad \ell z h = \ell(z \circ h)$$

for all $y \in H_s$, $z \in H_t$, and $h \in H$. These modules are well-defined and restrict to the regular $H_s^\star$ and $H_t^\star$-modules.

Lemma 5.2. We have $\epsilon(1 \cdot h) = \epsilon(1 \circ h)$ for all $h \in H$. 

Proof. What we need to show is that the equality \( \ell y = \ell z \) for \( y \in H_s \) and \( z \in H_t \) implies that \( \epsilon(y) = \epsilon(z) \).

By Lemma 5.3, there exists a unique well-defined linear map \( T : H_t \rightarrow H_s \) such that \( \ell T(z) = \ell z \) for all \( z \in H_t \). Clearly, \( T \) is an algebra anti-homomorphism and the composition \( ST \) is an automorphism of \( H_t \). Since \( \epsilon|_{H_t} = \text{Tr}_{\text{reg}} \) when \( S^2|_{H_{\text{min}}} = \text{id} \) by Proposition 3.4, we have \( \epsilon(T(z)) = \epsilon(ST(z)) = \epsilon(z) \), because any automorphism of a semisimple algebra preserves the regular trace.

**Corollary 5.3.** The above modules associated to any non-degenerate \( \ell \in \mathcal{I}_H \) canonically define a group-like element \( \hat{\gamma}_\ell \in G(H^*) \).

**Proof.** Follows from Remark 4.13 and Lemma 5.2.

Let us denote by \( W_\ell \) the right \( H \)-module structure on \( H_s \) corresponding to \( \ell \).

**Lemma 5.4.** Let \( \ell' \) be another non-degenerate left integral in \( H \) and \( \gamma_{\ell'} \in G(H^*) \) be the corresponding group-like element as above. Then \( \hat{\gamma}_\ell = \hat{\gamma}_{\ell'} \) in \( \tilde{G}(H) \), i.e., \( \gamma \) and \( \gamma_{\ell'} \) define the same coset in \( G(H) \).

**Proof.** Since both \( \ell \) and \( \ell' \) are non-degenerate, we have \( \ell' = \ell v \) for some invertible \( v \in H_s \), and the action \( \bullet \) of \( H \) on \( W_\ell \) is related to the action \( \cdot \) of \( H \) on \( W_\ell \) by \( y \bullet h = v^{-1}(vy \cdot h) \) for all \( y \in H_s \). This means that right \( H \)-modules \( W_\ell \) and \( W_{\ell'} \) are isomorphic via \( y \mapsto vy \). Therefore, by Proposition 4.14, the corresponding group-like elements \( \gamma \) and \( \gamma' \) define the same coset in \( \tilde{G}(H^*) \).

**Corollary 5.5.** Let \( H \) be a finite dimensional Frobenius weak Hopf algebra. There is a canonically defined coset \( \hat{\alpha} \in \tilde{G}(H) \) such that \( W_\ell \cong H_{\alpha}^* \) for any non-degenerate integral \( \ell \in \mathcal{I}_H \) and any representative \( \alpha \) of \( \hat{\alpha} \).

**Definition 5.6.** We will call \( \hat{\alpha} \) the canonical coset of group-like elements and any representative \( \alpha \) of this coset a canonical group-like element.

**Dual pairs of non-degenerate integrals.** We continue to assume that \( S^2|_{H_{\text{min}}} = \text{id} \). Recall from Section 2 that left integrals \( \ell \in \mathcal{I}_H \) and \( \lambda \in \mathcal{I}_H^* \) are dual to each other if \( \lambda \rightarrow \ell = 1 \), in which case they are both non-degenerate and also satisfy \( \ell \rightarrow \lambda = \epsilon \) (this was shown in \( \tilde{G}(H^*) \)).

It turns out that such pairs are closely related to distinguished group-like elements in \( H \) and \( H^* \). In this subsection we study this connection, following [38].

Recall the projections \( \varepsilon_{1}^\gamma \) and \( \varepsilon_{1}^r \) from (26) and (27). For all \( \gamma \in G(H^*) \) define

\[
L_\gamma := \{ h \in H \mid gh = \varepsilon_1^\gamma(g)h \text{ for all } g \in H \},
\]

\[
R_\gamma := \{ h \in H \mid hg = h\varepsilon_1^\gamma(g) \text{ for all } g \in H \}.
\]

Also, for all \( g \in G(H) \) define

\[
L_g := \{ \phi \in H^* \mid \psi\phi = \varepsilon_1^g(\psi)\phi \text{ for all } \psi \in H^* \},
\]

\[
R_g := \{ \phi \in H^* \mid \phi\psi = \phi\varepsilon_1^g(\psi) \text{ for all } \psi \in H^* \}.
\]

Clearly, \( L_g, R_g \) are subspaces of \( H^* \) and \( L_\gamma, R_\gamma \) are subspaces of \( H \). Note that \( L_1 = \mathcal{I}_{H^*}, R_1 = \mathcal{I}_H^* \), and \( L_\ell = \mathcal{I}_{H^*}^\ell, R_\ell = \mathcal{I}_H^\ell \).
Remark 5.7. By Corollary 5.5, every non-degenerate $\ell \in \int^l_H$ belongs to some $R_{\alpha}$ and also every non-degenerate $\rho \in \int^r_H$ belongs to $L_{\alpha}$, where $\alpha$ is a distinguished group-like element in $G(H^*)$. Similarly, every non-degenerate $\lambda \in \int^l_{H^*}$ belongs to some $R_{\alpha}$ and every non-degenerate $\rho \in \int^r_{H^*}$ belongs to $L_{\alpha}$, where $\alpha$ is a distinguished group-like element in $G(H^*)$.

Proposition 5.8. Let $g, h \in G(H)$. The map $\phi \mapsto (h \mapsto \phi)$ is a linear isomorphism between $L_g$ and $L_{gh^{-1}}$.

Proof. Take $\phi \in L_g, \psi \in H^*$ and compute
\[
\psi(h \mapsto \phi) = h \mapsto ((h^{-1} \mapsto \psi)\phi) = h \mapsto (\varepsilon^g(h^{-1} \mapsto \psi)\phi) = \varepsilon^g(h^{-1} \mapsto \psi)(h \mapsto \phi) = S(\varepsilon(1)) \langle \varepsilon(2), h \mapsto \phi \rangle(h \mapsto \phi) = \varepsilon^{gh^{-1}}(\psi)(h \mapsto \phi).
\]
Here we used the definition of $\varepsilon^g$ and that $\Delta(\zeta) = \varepsilon(1)\zeta \otimes \varepsilon(2)$ for all $\zeta \in H^*$. Thus, $(h \mapsto \phi) \in L_{gh^{-1}}$, which proves the claim.

The following Lemma was proved in [18, 2.2] for ordinary Hopf algebras. It is valid for weak Hopf algebras as well, we give a proof for the sake of completeness.

Lemma 5.9. Let $\ell, \lambda$ be a dual pair of integrals as above. Then for all $\phi \in H^*$ we have
\[
S(\phi) = (\ell \leftarrow \phi) \rightarrow \lambda.
\]

Proof. We compute, using the invariance property (16) of left integrals:
\[
(\ell \leftarrow \phi) \rightarrow \lambda = \langle \phi(1), \ell(1) \rangle \ell(2) \rightarrow \lambda = \lambda(1) \langle \phi \lambda(2), \ell \rangle = S(\phi(1)) \langle \phi(2) \lambda, \ell \rangle = S(\phi(1)) \langle \phi(2) \lambda, \ell \mapsto \ell \rangle = S(\phi),
\]
for all $\phi \in H^*$.

Lemma 5.10. We have $y_{\ell} = 1$.

Proof. First, we note that $S(\ell) = \alpha \mapsto ((y_{\ell}) = (y_{\ell} \mapsto \alpha) \rightarrow \ell$, by the definition of $y_{\ell}$. On the other hand,
\[
S(\ell) = (\lambda \leftarrow \ell) \rightarrow \ell
\]
by Lemma 5.9. Since \( \ell \) is non-degenerate, we have
\[
y_\ell \rightarrow \alpha = \lambda \leftarrow \ell.
\]
Applying the target counital map to the both sides of the last identity we get
\[
\varepsilon_t(\lambda \leftarrow \ell) = \langle \lambda(1), \ell \rangle \varepsilon_t(\lambda(2)) = \langle \varepsilon(1)\lambda, \varepsilon(2)\ell \rangle = \varepsilon \alpha S(y_\ell \rightarrow \epsilon)\alpha^{-1},
\]
therefore \( y_\ell \rightarrow \epsilon = \epsilon \), i.e., \( y_\ell = 1 \).

**Corollary 5.11.** For any pair of left integrals \( \ell \) and \( \lambda \) such that \( \ell \rightarrow \lambda = \epsilon \) and \( \lambda \leftarrow \ell = 1 \) there exist distinguished group-like elements \( \alpha \in G(H^*) \) and \( a \in G(H) \) such that
\[
\lambda \leftarrow \ell = \alpha \quad \text{and} \quad \ell \leftarrow \lambda = a.
\]
Furthermore, \( S(\ell) = \alpha \leftarrow \ell \) and \( S(\lambda) = a \rightarrow \lambda \).

**Proof.** The statements concerning \( \alpha \) follow from (34) in the the proof of Lemma 5.10 and those concerning \( a \) follow by duality.

**Formula for \( S^4 \).** We extend the argument of [18, Section 3] to establish a formula for the fourth power of the antipode in terms of distinguished group-like elements. We keep notation of the previous subsection. Let \( \alpha \) and \( a \) be distinguished group-like elements from Corollary 5.11.

Define linear isomorphisms \( \ell_L, \ell_R : H^* \rightarrow H \) and \( \lambda_L, \lambda_R : H \rightarrow H^* \) by
\[
\ell_L(\phi) = \phi \rightarrow \ell, \quad \lambda_L(h) = h \rightarrow \lambda,
\]
\[
\ell_R(\phi) = \ell \leftarrow \phi, \quad \lambda_R(h) = \lambda \leftarrow h,
\]
for all \( \phi \in H^* \) and \( h \in H \).

**Proposition 5.12.** We have the following relations
\[
\ell_L \circ \lambda_R(h) = S(h),
\]
\[
\ell_L \circ \lambda_L(h) = S^{-1}(\alpha \rightarrow h),
\]
\[
\ell_R \circ \lambda_R(h) = S^{-1}(a^{-1}h),
\]
\[
\ell_R \circ \lambda_L(h) = S((h \leftarrow \alpha)a^{-1}),
\]
for all \( h \in H \).

**Proof.** We establish these relations by a series of direct computations that use the invariant properties of integrals and Corollary 5.11:
\[
\ell_L \circ \lambda_R(h) = \langle \lambda(1), h \rangle \ell_L(\lambda(2))
\]
\[
= \langle \lambda(1), h\ell(2) \rangle
\]
\[
= S(h(1))\langle \lambda, h(2)\ell \rangle = S(h),
\]
\[
\ell_L \circ \lambda_L(h) = \langle \lambda(1), \ell(2) \rangle
\]
\[
= S^{-1}(h(1))\langle \lambda, \ell h(2) \rangle
\]
\[
= S^{-1}(h(1))\langle \alpha, h(2) \rangle = S^{-1}(\alpha \rightarrow h),
\]
\[
\ell_R \circ \lambda_R(h) = \langle \lambda, h\ell(1) \rangle \ell(2)
\]
\[
= \langle a^{-1} \rightarrow S(\lambda), h\ell(1) \rangle \ell(2)
\]
\[
= S(\lambda, h\ell(1)a^{-1})(\ell(2)a^{-1})a
\]
\[ = \langle S(\lambda), h_{(1)} a^{-1} S^{-1}(h_{(2)}) a \rangle \]
\[ = \langle \lambda, h_{(1)} a^{-1} S^{-1}(h_{(2)}) a \rangle \]
\[ = S^{-1}(h)a = S^{-1}(a^{-1}h), \]
\[ \ell_R \circ \lambda_L(h) = \langle \lambda, (\ell_{(1)}(h))_{(2)} \rangle \]
\[ = \langle a^{-1} \rightarrow S(\lambda), \ell_{(1)}(h)_{(2)} \rangle \]
\[ = \langle S(\lambda), \ell_{(1)}(h)a^{-1} \rangle_{(2)} \]
\[ = \langle S(\lambda), \ell_{(1)}(h)a^{-1} S(h_{(2)})a^{-1} \rangle \]
\[ = \langle \lambda, \ell_{(1)}(h)a^{-1} S(h_{(2)})a^{-1} \rangle \]
\[ = \langle \alpha, h_{(1)} S(h_{(2)})a^{-1} \rangle \]
\[ = \langle \alpha, h_{(1)} S(h_{(2)})a^{-1} \rangle \]
\[ = S((h \leftarrow \alpha)a^{-1}), \]
for all \( h \in H. \)

**Theorem 5.13.** For all \( h \in H \) we have
\[ S^4(h) = a^{-1}(\alpha \rightarrow h \leftarrow \alpha^{-1})a. \]

**Proof.** From Proposition 5.12 we deduce
\[ \ell_L \circ \lambda_L(S^2(h)) = S^{-1}(\alpha \rightarrow (S^2(h))) \]
\[ = S(h_{(1)})\langle \alpha, S^2(h_{(2)}) \rangle \]
\[ = S(\alpha \rightarrow h) \]
\[ = \ell_L \circ \lambda_R(\alpha \rightarrow h), \]
\[ \ell_R \circ \lambda_R(S^2(h)) = S^{-1}(a^{-1}S^2(h)) \]
\[ = S(a^{-1}h) \]
\[ = S((g \leftarrow \alpha)a^{-1}) \]
\[ = \ell_R \circ \lambda_L(g), \]
for all \( h \in H, \) where \( g = (a^{-1}ha) \leftarrow \alpha^{-1}. \) Since the maps \( \ell_L \) and \( \ell_R \) are bijective, it follows that
\[ \lambda_L(S^2(h)) = \lambda_R(\alpha \rightarrow h) \quad \text{and} \quad \lambda_R(S^2(h)) = \lambda_L((a^{-1}ha) \leftarrow \alpha^{-1}). \]
Combining these identities we get
\[ \lambda_L(S^4(h)) = \lambda_R(\alpha \rightarrow S^2(h)) \]
\[ = \lambda_R(S^2(\alpha \rightarrow h)) \]
\[ = \lambda_L((a^{-1}(\alpha \rightarrow h)a) \leftarrow \alpha^{-1}). \]
Replacing \( h \) by \( h \leftarrow \alpha \) we get the desired formula. 

**Remark 5.14.** If \( S^2|_{H_{\min}} \neq \text{id} \), then according to Remark 3.7 there exists a deformation \( H_g \) of \( H \) for which the antipode satisfies
\[ S^2(h) = g^{-1}S(g)S^2(h)gS(g)^{-1}, \quad h \in H, \]
and \( S^2|_{H_{\min}} = \text{id} \), where \( g \in H_t, \) \( S^2(g) = g. \)

**Corollary 5.15.** For any finite dimensional Frobenius weak Hopf algebra there exists an integer \( n \geq 1 \) such that \( S^{4n} \) is a trivial weak Hopf algebra automorphism of \( H. \) In other words, \( S^2 \) has a finite order modulo a trivial automorphism.
Proof. By the previous remark we may assume that $S^2|H_{\text{min}} = \text{id}$. Since any weak Hopf algebra automorphism of $H$ preserves the class of distinguished group-like elements, one can see that weak Hopf algebra automorphisms
\[ h \mapsto (a^{-1}(\alpha \mapsto h \mapsto \alpha) a) \quad \text{and} \quad h \mapsto (\alpha \mapsto (a^{-1} ha) \mapsto \alpha) \]
(38)
differ by a trivial automorphism, i.e., $\text{Ad}_{a^{-1}}$ and $\text{Ad}_{a}^*$ commute in $\widetilde{\text{Aut}}(H)$. Since both $a$ and $\alpha$ have finite order modulo trivial group-like elements, the claim follows.

Corollary 5.16. If bases of $H$ coincide, or if bases of $H^*$ coincide (i.e., $H_t = H_s$, or $H^*_t = H^*_s$), then $\tilde{G}(H) = G(H)$ and $S$ has finite order in $\text{Aut}(H)$.

Proof. Under the given assumption, all trivial group-like elements are equal to the identity element 1.

Remark 5.17. In the special case when both $H$ and $H^*$ have non-degenerate 2-sided integrals, it was shown in [1, 3.23] that $S^4$ is a trivial automorphism of $H$. This result also follows from Theorem 5.13 above since in this case distinguished group-like elements of $H$ and $H^*$ are trivial.

6. Trace formula and semisimplicity

Let $k$ be an algebraically closed field of characteristic 0.

Formula for $\text{Tr}(S^2)$. In this Section we derive a weak Hopf algebra analogue of the Larson-Radford formula for $\text{Tr}(S^2)$ [8]. As in the case of usual Hopf algebras, this formula turns out to be closely related with the semisimplicity of the corresponding weak Hopf algebra and its dual.

It follows from (13) that for a non-degenerate integral $\ell$ of $H$ with the dual integral $\lambda$, the element $(\ell(2) \mapsto \lambda) \otimes S^{-1}(\ell(1)) \in H^* \otimes H$ is the dual bases tensor. This implies that for every $T \in \text{End}_k H$ we have
\[ \text{Tr}(T) = (\lambda, \ell(1)T(\ell(2))). \]
(39)
In particular, for $T = S^2$, we get the following analogue of [8, Theorem 2.5(a)], with counits replaced by counital maps.

Proposition 6.1. In any finite dimensional $H$ we have
\[ \text{Tr}(S^2) = \langle \varepsilon_s(\lambda), \varepsilon_s(\ell) \rangle. \]
(40)
Proof. Follows from (39) for $T = S^2$.

Remark 6.2. When $H$ is a finite dimensional Hopf algebra, an immediate consequence of the above formula is that $H$ and $H^*$ both are semisimple if and only if $\text{Tr}(S^2) \neq 0$ [8, Theorem 2.5(b)]. Note that the situation is more complicated for weak Hopf algebras, since one can have $\varepsilon_s(l) = 0$ while $\varepsilon(l) \neq 0$ (e.g., for minimal weak Hopf algebras with non-commutative bases, see Section 3).

Recall from Proposition 4.16 that for any weak Hopf algebra $H$ the algebras $H_s \cap Z(H)$ and $H^*_t \cap H^*_s$ are isomorphic via $y \mapsto (y \mapsto \varepsilon)$.

Definition 6.3. If the $H_s \cap Z(H) = k1$, or, equivalently, $H^*_s \cap H^*_t = ke$, we say that $H$ is connected. If both $H$ and $H^*$ are connected, we say that $H$ is biconnected.
Below we always assume that $S^2|_{H_{\text{min}}} = \text{id}$.

**Proposition 6.4.** Let $I$ be the set of primitive idempotents of $H_s \cap Z(H)$. If $\text{Tr}(S^2|_{pH}) \neq 0$ for all $p \in I$, then $H$ is semisimple.

**Proof.** Fix a non-degenerate left integral $\ell$ of $H$. Using the dual bases tensor (39) we see that $\text{Tr}(S^2|_{pH}) \neq 0$ implies that $y_p = p \varepsilon_s(\ell) \neq 0$.

Let $\alpha$ be a group-like element of $H^*$ such that $S(\ell) = \alpha \mapsto \ell$. Such an $\alpha$ exists by Corollary 6.7 and depends on $\ell$. Using the definition of integrals we compute:

$$y_p \ell = S(y_p) \ell = S(\varepsilon_s(\ell)) \ell p = \varepsilon_s(S(\ell)) \ell p = S(\ell) \ell p = S(\ell) y_p,$$

whence

$$y_p \ell = (\alpha \mapsto \ell) y_p \quad \text{for all} \ p \in I.$$  

Let $\pi = p \mapsto 1 \in H_s^\ast \cap H_1^\ast$ and let $\xi_p \in \pi H_s^\ast$ be such that $y_p = \xi_p \mapsto 1$. For all $h \in H$ we have $y h = \xi_p \mapsto h$ and $hy = S^{-1}(\xi_p) \mapsto h$. Due to this observation and non-degeneracy of $\ell$, we can rewrite equation (41) as

$$\xi_p = S^{-1}(\xi_p) \alpha.$$  

By Remark 3.6(iii), $H_{\text{min}}^\ast = H_s^\ast H_s^\ast$ is a cosemisimple subcoalgebra of $H^*$ and $\pi H_{\text{min}}^\ast$, is a simple subcoalgebra of $H_{\text{min}}^\ast$ (since the element $\pi$ is a primitive idempotent in $H_s^\ast \cap H_1^\ast$).

Recall from Remark 4.1 that $\iota_\alpha : \phi \mapsto \phi \alpha$ is a coalgebra automorphism of $H^*$ which preserves $H_{\text{min}}^\ast$ only if $\alpha$ is trivial. This is exactly the case here since the relation (42) insures that $\iota_\alpha$ maps $\pi H_{\text{min}}^\ast$ to itself for all primitive idempotents $\pi \in H_s^\ast \cap H_1^\ast$. Thus, one can choose $\ell$ in such a way that $\alpha = \iota_\alpha$, i.e., one can assume that $\ell = S(\ell)$ (in which case $\ell$ is a two-sided integral). Then (42) implies that $\xi_p \in H_s^\ast \cap H_1^\ast$, therefore, $y_p \in H_s \cap Z(H)$ is a non-zero multiple of $p$. We conclude that $y = \varepsilon_s(\ell) = \sum p y_p$ is an invertible central element of $H$. Hence, $\ell' = y^{-1} \ell$ is a normalized right integral. By Maschke’s theorem, $H$ is semisimple.

**Corollary 6.5.** Let $H$ be a finite dimensional weak Hopf algebra. Suppose that $\text{Tr}(S^2|_{H^* \pi}) \neq 0$ for every primitive idempotent $\pi \in H_s^\ast \cap H_1^\ast$. Then $H$ is semisimple.

**Proof.** Note that $p H_s \cap Z(H)$ is a primitive idempotent, is naturally identified with the dual space of $H^* \pi$, where $\pi = p \mapsto 1$, so that $\text{Tr}(S^2|_{pH}) = \text{Tr}(S^2|_{H^* \pi})$.  

**Corollary 6.6.** Let $H$ be a connected finite dimensional weak Hopf algebra. If $\text{Tr}(S^2) \neq 0$, then $H$ is semisimple.

**Corollary 6.7.** Let $H$ be a biconnected finite dimensional weak Hopf algebra. If $\text{Tr}(S^2) \neq 0$, then $H$ and $H^*$ are semisimple.

**Semisimplicity and cosemisimplicity.** It is a well-known theorem due to Larson and Radford [8] that a semisimple Hopf algebra over $k$ is automatically cosemisimple. The proof of this theorem uses the finiteness of the order of the antipode and the formula for $\text{Tr}(S^2)$.

Here we prove a similar result for weak Hopf algebras with coinciding bases.

**Lemma 6.8.** Let $H$ be a semisimple weak Hopf algebra. Then $\text{Tr}(S^2|_{pHp}) \neq 0$ for all primitive central idempotents of $H_{\text{min}}$. 

Proof. Note that since $H$ is semisimple, $S^2$ is an inner algebra automorphism of $H$ by [1, 3.22]. For all $p \in H_{\text{min}}$ we have
\begin{equation}
\text{Tr}(S^2|_{pH_p}) = \text{Tr}(S^2|_{pH_0p}) + \sum_j \text{Tr}(S^2|_{pJp}),
\end{equation}
where $H_0$ is the minimal weak Hopf quotient of $H$ (viewed as a minimal ideal of $H$) and the sum is taken over the rest of minimal two-sided ideals of $H$ (so that each $pJp$ is a simple algebra). Since $p$ is a primitive idempotent of $H_{\text{min}}$, it follows from Corollary 5.15 that $S^n|_{pH_p} = \text{id}$ for some $n > 0$, so that $\text{Tr}(S^2|_{pJp}) \geq 0$ by the same reasoning as in [8, Lemma 3.2]. Since $S^2 = \text{id}$ on $H^*_{\text{min}}$, we have $S^2|_{H_0} = \text{id}_{H_0}$. Therefore, equation (43) implies that
\begin{equation}
\text{Tr}(S^2|_{pH_p}) > 0 \quad (44)
\end{equation}
for all primitive idempotents $p \in H_{\text{min}}$.\qed

**Theorem 6.9.** Let $H$ be a semisimple weak Hopf algebra such that $H_t = H_s$. Then $H^*$ is semisimple.

Proof. By Corollary 5.10, we have $S^n = \text{id}$ for some $n > 0$. Also, the bases are necessarily commutative, so that $S^2|_{H_{\text{min}}} = \text{id}_{H_{\text{min}}}$. By Lemma 6.8 we have $\text{Tr}(S^2|_{pH_p}) \neq 0$ for all primitive idempotents $p$ of $H_s = H_{\text{min}}$. For any dual pair $(\ell, \lambda)$ of non-degenerate integrals, the element $(\ell(2) \rightarrow \lambda) \otimes pS^{-1}(\ell(2))p$ is the dual bases tensor for $pHp$, therefore,
\begin{equation}
0 \neq \text{Tr}(S^2|_{pH_p}) = \langle \varepsilon_t(\lambda), p \varepsilon_s(p\ell) \rangle.
\end{equation}
Let $\rho = (\ell \leftrightarrow p) \in H^*_t$, then $\varepsilon_t(\lambda)p \neq 0$. Since the latter holds for all primitive idempotents $p \in H^*_t$, we conclude that $\varepsilon_t(\lambda) \in Z(H^*) \cap H^*_s$ is invertible. Therefore, $\lambda = \lambda^{-1} \varepsilon_t(\lambda)^{-1}$ is a normalized left integral and, hence, $H^*$ is semisimple by Maschke’s theorem.\qed

**Cosemisimplicity of weak Hopf algebras arising from dynamical twisting.**
As an application of the above results we prove below that weak Hopf algebras obtained from dynamical twisting of semisimple Hopf algebras (such as, e.g., finite group algebras) are semisimple.

A *twist* for a weak Hopf algebra $H$ [4, 3.1.1] is a pair $(\Theta, \bar{\Theta})$ with
\begin{equation}
\Theta \in \Delta(1)(H \otimes H), \quad \bar{\Theta} \in (H \otimes H)\Delta(1), \quad \text{and} \quad \Theta \bar{\Theta} = \Delta(1)
\end{equation}
such that $\Delta_\Theta : H \rightarrow H \otimes H$ defined by
\begin{equation}
\Delta_\Theta(h) = \bar{\Theta} \Delta(h) \Theta, \quad h \in H,
\end{equation}
is a coassociative comultiplication on the algebra $H$ that makes it a new weak Hopf algebra, which we will denote by $H_\Theta$. If we write $\Theta = \Theta^{(1)} \otimes \Theta^{(2)}$ and $\bar{\Theta} = \bar{\Theta}^{(1)} \otimes \bar{\Theta}^{(2)}$, where a summation is understood, then the antipode of $H_\Theta$ is given by
\begin{equation}
S_\Theta(h) = v^{-1}S(h)v, \quad h \in H,
\end{equation}
where $v = S(\Theta^{(1)})\Theta^{(2)}$ and $v^{-1} = \bar{\Theta}^{(1)}S(\bar{\Theta}^{(2)})$. The counital maps of $H_\Theta$ were computed in [4, 3.1.2] :
\begin{equation}
\varepsilon_{t\Theta}(h) = \varepsilon(\Theta^{(1)}h)\Theta^{(2)}, \quad \varepsilon_{s\Theta}(h) = \bar{\Theta}^{(1)}\varepsilon(h\bar{\Theta}^{(2)}),
\end{equation}
for all \( h \in H_{\Theta} \); in particular, the twisting may deform the bases of a weak Hopf algebra.

Recall the definition of a dynamical twist in a Hopf algebra from \( \text{[4]} \). Let \( U \) be a Hopf algebra and \( A \) be a finite Abelian subgroup of the group of group-like elements of \( H \). Let \( A^* \) be the group of characters of \( A \), and let \( P_\mu \), \( \mu \in A^* \) be the minimal idempotents in \( kA \).

**Definition 6.10.** A function \( J : A^* \to U \otimes U \) with invertible values and such that \( J(\lambda)\Delta(a) = \Delta(a)J(\lambda) \), for all \( \lambda \in A^* \) and \( a \in A \), is called a **dynamical twist** for \( U \) if it satisfies the following functional equations:

\[
(\Delta \otimes \text{id})J(\lambda)(J(\lambda + h^{(3)}) \otimes 1) = (\text{id} \otimes \Delta)J(\lambda)(1 \otimes J(\lambda)),
\]

\[
(\varepsilon \otimes \text{id})J(\lambda) = (\text{id} \otimes \varepsilon)J(\lambda) = 1.
\]

Here \( J(\lambda + h^{(3)}) = \sum_\mu J(\lambda + \mu) \otimes P_\mu \in U \otimes U \otimes A \).

We refer the reader to \( \text{[5]} \) for an introduction to the theory of dynamical quantum Yang-Baxter equations and dynamical twists.

Let us consider the tensor product weak Hopf algebra \( H = M_{|A|}(k) \otimes U \), where the matrix algebra \( M_{|A|}(k) \) is a weak Hopf algebra (in fact, a groupoid algebra) with the basis consisting of matrix units \( \{E_{\lambda \mu}\}_{\lambda, \mu \in A^*} \) and with the structure operations

\[
\Delta(E_{\lambda \mu}) = E_{\lambda \mu} \otimes E_{\lambda \mu}, \quad \varepsilon(E_{\lambda \mu}) = 1, \quad S(E_{\lambda \mu}) = E_{\mu \lambda}.
\]

Then for any dynamical twist \( J : A^* \to U \otimes U \) the pair

\[
\Theta = \sum_{\lambda, \mu} E_{\lambda \mu} J^{(1)}(\lambda) \otimes E_{\lambda \lambda} J^{(2)}(\lambda) P_\mu,
\]

\[
\bar{\Theta} = \sum_{\lambda, \mu} E_{\lambda \mu} \lambda J^{-1}(\lambda) \otimes E_{\lambda \mu} P_\mu J^{-2}(\lambda),
\]

is a twist for \( H \) \( \text{[4, 4.2.4]} \), where \( J = J^{(1)} \otimes J^{(2)} \) and \( J^{-1} = J^{-1} \otimes J^{-2} \).

By \( \text{[17]} \), the counital maps of the corresponding twisted weak Hopf algebra \( H_{\Theta} \) are given by

\[
\varepsilon_{\Theta}(E_{\alpha \beta} h) = \varepsilon(h) \sum_{\lambda} E_{\lambda \lambda} P_{\lambda - \alpha}, \quad \varepsilon_{\Theta}(E_{\alpha \beta} h) = \varepsilon(h) E_{\beta \beta}, \quad h \in U.
\]

Thus, the bases of \( H_{\Theta} \) are

\[
(H_{\Theta})_t = \text{span}\{\sum_{\lambda} E_{\lambda \lambda} P_{\lambda - \alpha} \mid \alpha \in A^*\}
\]

\[
(H_{\Theta})_s = \text{span}\{E_{\beta \beta} \mid \beta \in A^*\}.
\]

Note that the bases do not depend on \( J \), and that \( H_{\Theta} \) is bicontinuous in the sense of Definition 5.3.

**Proposition 6.11.** Let \( U \) be a semisimple Hopf algebra, and \( J : A^* \to U \otimes U \) be a dynamical twist for \( U \). The weak Hopf algebra \( H_{\Theta} \) obtained by the dynamical twisting of \( H = M_{|A|}(k) \otimes U \) is cosemisimple, i.e., its dual is semisimple.

**Proof.** By Corollary 6.7, it suffices to show that \( \text{Tr}(S^2_{\Theta}) \neq 0 \) in \( \text{End}_k(H_{\Theta}) \). From \( \text{[49]} \) we see that

\[
S^2_{\Theta}(h) = g^{-1}hg, \quad \text{where } g = S(v)^{-1}v, \quad h \in H_{\Theta}.
\]
Therefore, $\text{Tr}(S_0^g) = \sum_\pi \text{Tr}(\pi(g))\text{Tr}(\pi(g^{-1}))$, where the summation is taken over all irreducible representations of the semisimple algebra $H$. From the formulas for $v$ and $v^{-1}$ and explicit formulas (51) and (52) we have:

$$g = \sum_{\lambda \mu} E_{\lambda \mu} P_{\mu} S(J^{(1)}(\lambda))J^{(2)}(\lambda)J^{-2}(\lambda)S(J^{-1}(\lambda))P_{\mu},$$

$$g^{-1} = \sum_{\lambda \mu} E_{\lambda+\mu \lambda+\mu} P_{-\mu} S(J^{(2)}(\lambda))J^{(1)}(\lambda)J^{-1}(\lambda)S(J^{-2}(\lambda))P_{-\mu},$$

whence we can compute

$$\text{Tr}(\pi(g)) = \sum_{\lambda} \text{Tr}(\pi(E_{\lambda \mu} S(J^{(1)}(\lambda))J^{(2)}(\lambda)J^{-2}(\lambda)S(J^{-1}(\lambda)))) = \deg \pi,$$

and, similarly, $\text{Tr}(\pi(g^{-1})) = \deg \pi$ so that $\text{Tr}(S_0^g) = \dim(H_0) \neq 0$. 

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