Gauge-Yukawa Unification (GYU) is obtained in GUTs by searching for renormalization group invariant relations among gauge and Yukawa couplings beyond the unification scale. Of particular interest are two supersymmetric GUTs, the finite and the minimal $SU(5)$ models. Both models provided us, among others, with predictions of the top quark mass which so far have passed successfully the tests of progressively more accurate measurements.

1 Introduction

An outstanding question of the theory of Elementary Particle Physics is the plethora of free parameter of the Standard Model (SM). The traditional way of reducing the number of free parameters of the SM is to require that the theory is more symmetric at higher scales. This approach has been applied, e.g. in GUTs, with a certain success. In fact the LEP data seem to suggest that we should require $N = 1$ supersymmetry in addition to the minimal $SU(5)$ in order to obtain a successful prediction for one of the low energy gauge couplings.

However, this attractive possibility has its limitations. As it is well known increasing the gauge symmetry of a GUT (e.g. $SO(10)$, $E_6$) does not lead to a more predictive theory for the low energy parameters. This is due to the fact that an enlarged symmetry requires also further breakings, which in general require additional free parameters. Alternatively we have suggested that a natural gradual extension of the GUT philosophy, in the prospect of increasing the predictability of the low energy parameters of the theory, is to attempt to relate the couplings of the gauge and Yukawa sectors, i.e. to achieve Gauge-Yukawa Unification. Searching for a symmetry that could provide GYU one is naturally lead to consider $N = 2$ supersymmetric theories which however proved to have more serious phenomenological problems than the SM. The same criticism holds also for superstring theories and composite models which could in principle lead to relations among the gauge and Yukawa couplings.

2 Gauge-Yukawa Unification

In our recent studies we have considered the GYU which is based on the principles of reduction of couplings and in addition finiteness. These principles, which are formulated in perturbation theory, are not explicit symmetry principles, although they might imply symmetries. The former principle is based on the existence of renormalization group invariant (RGI) relations among couplings which preserve perturbative renormalizability. Similarly, the latter one is based on the fact that it is possible to find RGI relations among couplings that keep finiteness in perturbation theory, even to all orders. Applying these principles, one can relate the gauge and Yukawa couplings, thereby improving the predictive power of a model. In what follows, we briefly outline the basic tool of this GYU scheme and its application to the most promising models.

A RGI relation among couplings can be ex-
pressed in an implicit form
\[ \Phi(g_1, \cdots, g_N) = 0 , \tag{1} \]
which has to satisfy the partial differential equation (PDE)
\[ \mu \frac{d\Phi}{d\mu} = \sum_{i=1}^{N} \beta_i \frac{\partial \Phi}{\partial g_i} = 0 , \]

where \( \beta_i \) is the \( \beta \)-function of \( g_i \). There exist \((N-1)\) independent \( \Phi \)'s, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations
\[ \beta_g \frac{dg_i}{dg} = \beta_i , \ i = 1, \cdots, N , \tag{2} \]

where \( g \) and \( \beta_g \) are the primary coupling and its \( \beta \)-function, and \( i \) does not include \( g \). Using all the \((N-1)\) \( \Phi \)'s to impose RGI relations, one can in principle express all the couplings in terms of a single coupling \( g \). The complete reduction, which formally preserve perturbative renormalizability, can be achieved by demanding a power series solution
\[ g_i = \sum_{n=0}^{\infty} \kappa_i^{(n)} g^{2n+1} . \tag{3} \]

The uniqueness of such a power series solution can be investigated at the one-loop level. The completely reduced theory contains only one independent coupling with the corresponding \( \beta \)-function. In supersymmetric Yang-Mills theories with a simple gauge group, something more drastic can happen; the vanishing of the \( \beta \)-function to all orders in perturbation theory, if all the one-loop anomalous dimensions of the matter fields in the completely and uniquely reduced theory vanish identically.

This possibility of coupling unification is attractive, but it can be too restrictive and hence unrealistic. To overcome this problem, one may use fewer \( \Phi \)'s as RGI constraints. This is the idea of partial reduction, and the power series solution (3) becomes in this case
\[ g_i = \sum_{n=0}^{N'} \kappa_i^{(n)} (g_a/g) g^{2n+1} , \tag{4} \]

\[ i = 1, \cdots, N' , \ a = N' + 1, \cdots, N . \]

The coefficient functions \( \kappa_i^{(n)} \) are required to be unique power series in \( g_a/g \) so that the \( g_a \)'s can be regarded as perturbations to the completely reduced system in which the \( g_a \)'s identically vanish. In the following, we would like to consider two very interesting models which are also representative of the two mentioned possibilities.

3 Gauge-Yukawa Unified Models

3.1 The \( SU(5) \) Finite Unified Theory

This is a \( N = 1 \) supersymmetry Yang-Mills theory based on \( SU(5) \) which contains one 24, four pairs of \((5 + 5)\)-Higgses and three \((5 + 10)\)'s for three fermion generations. It has been done a complete reduction of the dimensionless parameters of the theory in favour of the gauge coupling \( g \) and the unique power series solution corresponds to the Yukawa matrices without intergenerational mixing, and yields in the one-loop approximation
\[ g_i^2 = g_{e}^2 = g_{a}^2 = \frac{8}{5} g^2 , \tag{5} \]
\[ g_{b}^2 = g_{s}^2 = g_{d}^2 = \frac{6}{5} g^2 , \tag{6} \]
\[ g_{r}^2 = g_{\mu}^2 = g_{\tau}^2 = \frac{6}{5} g^2 , \tag{7} \]

where \( g_i \)'s stand for the Yukawa couplings. At first sight, this GYU seems to lead to unacceptable predictions of the fermion masses. But this is not the case, because each generation has an own pair of \((5 + 5)\)-Higgses so that one may assume that after the diagonalization of the Higgs fields the effective theory is exactly MSSM, where the pair of its Higgs supermultiplets mainly stems from the \((5 + 5)\) which couples to the third fermion generation. (The Yukawa couplings of the first two generations can be regarded as free parameters.) The predictions of \( m_t \) and \( m_b \) for various \( m_{SU(5)} \) are given in Table 1.

3.2 The minimal supersymmetric \( SU(5) \) model

The field content is minimal. Neglecting the CKM mixing, one starts with six Yukawa and two Higgs couplings. We then require GYU to occur among the Yukawa couplings of the third generation and the gauge coupling. We also require the theory to be completely asymptotically free. In the one-loop approximation, the GYU yields
\[ g_{i,b}^2 = \sum_{m,n=1}^{\infty} \kappa_{i,b}^{(m,n)} h^m f^n g^2 \ (h \text{ and } f \text{ are related to the Higgs couplings}) . \]

Where \( h \) is allowed
to vary from 0 to 15/7, while $f$ may vary from 0 to a maximum which depends on $h$, and vanishes at $h = 15/7$. As a result, we obtain  

$$0.97 g^2 \lesssim g_t^2 \lesssim 1.37 g^2,$$

$$0.57 g^2 \lesssim g_B^2 = g_t^2 \lesssim 0.97 g^2. \quad (8)$$

We found that consistency with proton decay requires $g_t^2$, $g_B^2$ to be very close to the left hand side values in the inequalities. In Table 2 we give the predictions for representative values of $m_{\text{SUSY}}$.

In all of the analyses above, we have used the RG technique and regarded the GYU relations the boundary conditions holding at the unification scale $M_{\text{GUT}}$. We have assumed that it is possible to arrange the susy mass parameters along with the soft breaking terms in such a way that the desired symmetry breaking pattern really occurs, all the superpartners are unobservable at present energies, there is no contradiction with proton decay, and so forth. To simplify our numerical analysis we have also assumed a unique threshold $m_{\text{SUSY}}$ for all the superpartners.

Using the updated experimental data on the SM parameters, we have re-examined the $m_t$ prediction of the two GYU $SU(5)$ models described above. They predict

$$m_t = (183 + \delta_{m_t}^{\text{MSSM}} \pm 5) \text{ GeV} \quad (9)$$

Min. SUSY $SU(5)$:

$$m_t = (181 + \delta_{m_t}^{\text{MSSM}} \pm 3) \text{ GeV} \quad (10)$$

where $\delta_{m_t}^{\text{MSSM}}$ stands for the MSSM threshold corrections. One obtains an idea about the magnitude of the correction by considering the case that all superpartners have the same mass $m_{\text{SUSY}}$ and $m_{\text{SUSY}} \gg \mu_H$, where $\mu_H$ describes the mixing of the two Higgs doublets in the superpotential. In that case we found $\delta_{m_t}^{\text{MSSM}} \sim -1\%$.

### 4 Discussion and Conclusions

As a natural extension of the unification of gauge couplings provided by all GUTs and the unification of Yukawa couplings, we have introduced the idea of Gauge-Yukawa Unification. GYU is a functional relationship among the gauge and Yukawa couplings provided by some principle. In our studies GYU has been achieved by applying the principles of reduction of couplings and finiteness. The consequence of GYU is that in the lowest order in perturbation theory the gauge and Yukawa couplings above $M_{\text{GUT}}$ are related in the form

$$g_i = \kappa_i g_{\text{GUT}}, \ i = 1, 2, 3, e, \cdots, \tau, b, t, \quad (11)$$

where $g_i$ ($i = 1, \cdots, t$) stand for the gauge and Yukawa couplings, $g_{\text{GUT}}$ is the unified coupling, and we have neglected the Cabibbo-Kobayashi-Maskawa mixing of the quarks. So, Eq. (11) exhibits a boundary condition on the the renormalization group evolution for the effective theory below $M_{\text{GUT}}$, which we have assumed to be the MSSM. We have shown that it is possible to construct some supersymmetric GUTs with GYU in the third generation that can predict the bottom and top quark masses in accordance with the recent experimental data. This means that the top-bottom hierarchy could be explained in these models, in a similar way as the hierarchy of the gauge couplings of the SM can be explained if one assumes the existence of a unifying gauge symmetry at $M_{\text{GUT}}$.

It is clear that the GYU scenario is the most predictive scheme as far as the mass of the top quark is concerned. It may be worth recalling the predictions for $m_t$ of ordinary GUTs, in particular of supersymmetric $SU(5)$ and $SO(10)$. The MSSM with $SU(5)$ Yukawa boundary unification allows $m_t$ to be anywhere in the interval between 100-200 GeV for varying $\tan \beta$, which is now a free parameter. Similarly, the MSSM with $SO(10)$ Yukawa boundary conditions, i.e. $t-b-\tau$ Yukawa Unification gives $m_t$ in the interval 160-200 GeV.

| $m_{\text{SUSY}}$ [GeV] | $\alpha_3(M_Z)$ | $\tan \beta$ | $M_{\text{GUT}}$ [GeV] | $m_b$ [GeV] | $m_t$ [GeV] |
|------------------------|----------------|--------------|---------------------|------------|------------|
| 200                    | 0.123          | 53.7         | $2.25 \times 10^{16}$ | 5.2        | 184.0      |
| 500                    | 0.118          | 54.2         | $1.45 \times 10^{16}$ | 5.1        | 184.4      |
We have analyzed the infrared quasi-fixed-point behaviour of the \( m_t \) prediction in some detail. In particular we have seen that the infrared value for large \( \tan \beta \) depends on \( \tan \beta \) and its lowest value is \( \sim 188 \) GeV. Comparing this with the experimental value \( m_t = (176.8 \pm 6.5) \) GeV we may conclude that the present data on \( m_t \) cannot be explained from the infrared quasi-fixed-point behaviour alone.

Clearly, to exclude or verify different GYU models, the experimental as well as theoretical uncertainties have to be further reduced. One of the largest theoretical uncertainties in FUT results from the not-yet-calculated threshold effects of the superheavy particles. Since the structure of the superheavy particles is basically fixed, it will be possible to bring these threshold effects under control, which will reduce the uncertainty of the \( m_t \) prediction. We have been regarding \( \delta_{\text{MSSM}} m_t \) as unknown because we do not have sufficient information on the superpartner spectra. Recently, however, we have demonstrated how to extend the principle of reduction of couplings in a way as to include the dimension full parameters. As a result, it is in principle possible to predict the superpartner spectra as well as the rest of the massive parameters of a theory.

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References

1. U. Amaldi, W. de Boer and H. Fürstenau, \textit{Phys. Lett.} B 260, 447 (1991).
2. D. Kapetanakis, M. Mondragón and G. Zoupanos, \textit{Z. Phys.} C 60, 181 (1993); M. Mondragón and G. Zoupanos, \textit{Nucl. Phys.} B 37C, 98 (1995).
3. J. Kubo, M. Mondragón and G. Zoupanos, \textit{Nucl. Phys.} B 424, 291 (1994).
4. J. Kubo, M. Mondragón, N.D. Tracas and G. Zoupanos, \textit{Phys. Lett.} B 342, 155 (1991).
5. J. Kubo, M. Mondragón, S. Shoda and G. Zoupanos, \textit{Nucl. Phys.} B 469, 3 (1996).
6. P. Fayet, \textit{Nucl. Phys.} B 149, 134 (1979).
7. W. Zimmermann, \textit{Commun. Math. Phys.} 97, 211 (1985); R. Oehme and W. Zimmermann, \textit{Commun. Math. Phys.} 97, 569 (1985); J. Kubo, K. Sibold and W. Zimmermann, \textit{Nucl. Phys.} B 259, 331 (1985).
8. A.J. Parkes and P.C. West, \textit{Phys. Lett.} B 138, 99 (1984); \textit{Nucl. Phys.} B 256, 340 (1985); D.R.T. Jones and A.J. Parkes, \textit{Phys. Lett.} B 160, 267 (1985); D.R.T. Jones and L. Mezincescu, \textit{Phys. Lett.} B 136, 242 (1984); \textit{Phys. Lett.} B 138, 293 (1984); A.J. Parkes, \textit{Phys. Lett.} B 156, 73 (1985); I. Jack and D.R.T. Jones, \textit{Phys. Lett.} B 333, 372 (1994).
9. S. Hamidi and J.H. Schwarz, \textit{Phys. Lett.} B 147, 301 (1984); D.R.T. Jones and S. Raby, \textit{Phys. Lett.} B 143, 137 (1984); J.E. Björkman, D.R.T. Jones and S. Raby, \textit{Nucl. Phys.} B 259, 503 (1985); J. León et al, \textit{Phys. Lett.} B 156, 66 (1985).
10. A.V. Ermashev, D.I. Kazakov and O.V. Tarasov, \textit{Nucl. Phys.} B 281, 72 (1987); D.I. Kazakov, \textit{Mod. Phys. Lett.} A 2, 663 (1987); \textit{Phys. Lett.} B 179, 352 (1986); C. Lucchesi, O. Piguet and K. Sibold, \textit{Helv. Phys. Acta.} 61, 321 (1988).
11. J. Kubo, M. Mondragón, M. Olechowski and G. Zoupanos, \textit{hep-ph/9510279}, Proc. of the Int. Europhysics Conf. on HEP, Brussels 1995.
12. J. Kubo, M. Mondragón, M. Olechowski and G. Zoupanos, \textit{Testing Gauge-Yukawa-Unified Models by \( M_t \)}, \textit{hep-ph/9512435}, to be published in \textit{Nucl. Phys.} B.

\begin{table}[h]
\centering
\caption{The predictions of the minimal SUSY SU(5)}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\text{\( m_{\text{SUSY}} \) [GeV]} & \( g_2^2/g_1^2 \) & \( g_5^2/g_1^2 \) & \( \alpha_3(M_Z) \) & \( \tan \beta \) & \( M_{\text{GUT}} \) [GeV] & \( m_b \) [GeV] & \( m_t \) [GeV] \\
\hline
\hline
300 & 0.97 & 0.57 & 0.120 & 47.7 & \( 1.8 \times 10^{16} \) & 5.4 & 179.7 \\
500 & 0.97 & 0.57 & 0.118 & 47.7 & \( 1.39 \times 10^{16} \) & 5.3 & 178.9 \\
\hline
\end{tabular}
\end{table}
13. J. Kubo, M. Mondragón and G. Zoupanos, *Perturbative Unification of Soft Susy Breaking Terms*, Preprint MPI-PhT/96-71, [hep-ph/9609218](http://arxiv.org/abs/hep-ph/9609218); see also references therein. To be published in *Phys. Lett.* B