Controllability function as time of motion. I *

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Abstract

The admissible positional control problem for the canonical system with geometrical restrictions on the control is considered. The investigation is performed with the help of the controllability function method. We obtain controllability functions which are the time of motion from an arbitrary initial condition to the origin. We also reveal a set of controls which solves this problem.

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1 Introduction

The problem of synthesis with bounded controls (SBC) of the time optimal control [1] is one of the well–known optimal control problems. The problem of SBC is stated as follows:

Given a system of differential equations,
\[ \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^r, \quad 0 \in \text{int } \Omega. \]  
(1.1)

It is required to find a control \( u \) of the form \( u = u(x) \), with values in the set \( \Omega \), such that the trajectory of the system
\[ \dot{x} = f(x, u(x)), \]  
(1.2)

starting at an arbitrary point \( x_0 \), terminates at a given point \( x_1 \) in minimal time.

In this case, the control \( u(x) \) is time optimal and satisfies the Bellman equation [2]:
\[ \min_{u \in \Omega} \sum_{i=1}^{n} \frac{\partial T(x)}{\partial x_i} f_i(x, u) = -1, \]  
(1.3)

where \( T(x) \) is the time of motion along the trajectory of system (1.2), which corresponds to the control \( u(x) \). Let us denote by \( \dot{T}(x) \), \( \dot{T}(x) \) the time derivative along system (1.1), (1.2) respectively. Hence equality (1.3) takes the form
\[ \min_{u \in \Omega} \dot{T}(x) = \dot{T}(x) = -1, \]  
which means that the derivative by virtue of system (1.2) of the time optimal \( T(x) \) from an arbitrary point \( x \) to a given point \( x_1 \) is equal to \(-1\). Obviously this equality holds at points where the derivative exists.

If we renounce the time optimality, we would consider the admissible positional synthesis problem, which consists of the construction of the positional control \( u = u(x) \), which satisfies a given restriction, i.e. \( u \in \Omega \). As a result, the trajectory of system (1.2), starting at an arbitrary point \( x_0 \), terminates at a given point \( x_1 \) at some finite time \( T(x_0) \). Furthermore, we assume that \( x_1 = 0 \) and \( f(0, 0) = 0 \).

For solving this problem, the method of the controllability function was introduced by V.I. Korobov [3]. This method is based in the construction of the controllability function \( \Theta(x) \) \( (\Theta(x) > 0 \text{ for } x \neq 0 \text{ and } \Theta(0) = 0) \), as well as, positional control \( u(x) = \tilde{u}(x, \Theta(x)) \), such that inequality
\[ \sum_{i=1}^{n} \frac{\partial \Theta(x)}{\partial x_i} f_i(x, u(x)) \leq -\beta \Theta^{1-1/\alpha}(x) \]  
(1.4)
is satisfied for some $\beta > 0$, $\alpha > 0$. The realization of this condition guarantees that the trajectory arrives to the origin at finite time. We assume that the controllability function $\Theta(x)$ is continuously differentiable at $x \neq 0$. In case that inequality (1.4) holds for $\alpha = \infty$, the function $\Theta(x)$ is a Lyapunov function.

In [4] a general method of constructing a controllability function and a control, which is solution of the admissible positional synthesis, was introduced for the linear control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^r, \quad (1.5)$$

where $A$ and $B$ are $(n \times n)$ and $(n \times r)$ constant matrices, respectively. In particular, the function $\varphi(s) = 1 - s$ for $s \in [0, 1]$, $\varphi(s) = 0$ for $s > 1$ generates a controllability function $\Theta_\varphi(x)$, which is the time of motion (1.5) with the control $u_\varphi(x)$ from an arbitrary point $x$ to the origin according to the system (1.5).

We are interested in finding a wider set of pairs of functions: the function of controllability $\Theta(x)$, which is the time of motion and the control which solves the synthesis problem.

In the first part of the work, we consider in detail this problem for the canonical system. The second part is devoted to the solution of the positional synthesis control problem for the completely controllable system (1.5) with a restriction of the form $u \in \Omega = \{u : \|u\| \leq d\}$, $d > 0$. The controllability functions $\Theta(x)$, which are the time of motion from the initial point $x$ to the origin, satisfy the equality $T(x) = \Theta(x)$, and the derivative of the controllability function by virtue of the system

$$\dot{x} = Ax + Bu(x) \quad (1.6)$$

is equal to $-1$: $\dot{\Theta}(x)|_{(1.6)} = -1$.

In the present work, we use the controllability function method which was first introduced in [3].

## 2 Preliminary results

We consider the canonical control system

$$\dot{x} = A_0x + b_0u, \quad (2.1)$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

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with restrictions on control \(|u| \leq d\). We will choose a control as a function depending on states coordinates \(u = u(x)\) such that the trivial solution of system (2.1) for \(u = u(x)\) would be asymptotically stable. We will refer to this control as the auxiliary control. Real numbers \(a_1, a_2, \ldots, a_n\) will be chosen in such a way that \(\lambda^n - a_n \lambda^{n-1} - \cdots - a_1 = 0\) would have roots with the negative real part. In addition, set \(u(x) = \sum_{i=1}^{n} a_i x_i = (a, x)\), where \(a = (a_1, \ldots, a_n)^*\). In this case, system (2.1) has the form

\[
\dot{x} = A_1 x,
\]

where

\[
A_1 = \begin{pmatrix}
a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_n \\
1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix};
\]

its trivial solution is asymptotically stable. In this way, the auxiliary control solves the stabilizability problem for system (2.1) in all the phase space, but does not satisfy the given restriction. Since \(A_1\) is stable, we can find a positive definite quadratic form \(V(x) = (Fx, x)\) which is a Lyapunov function. The derivative of \(V\) by virtue of system (2.2) represents a negative definite quadratic form \((-Wx, x)\), with positive definite matrices \(F\) and \(W\). Since

\[
\frac{d}{dt} (Fx, x) = (F \dot{x}, x) + (Fx, \dot{x}) = ((FA_1 + A_1^* F)x, x),
\]

this equation is reduced then to the Lyapunov matrix equation

\[
FA_1 + A_1^* F = -W,
\]

which for given matrices \(A_1\) and \(W\) has a unique solution.

Let \(F\) be a positive definite matrix, the form of which will be specified later. Denote by \(D(\Theta), H\) diagonal matrices of the form

\[
D(\Theta) = \text{diag} \left( \Theta^{-\frac{2i-1}{2}} \right)_{i=1}^{n}, \quad H = \text{diag} \left( -\frac{2i-1}{2} \right)_{i=1}^{n}.
\]

Let \(a_0\) be a positive number which will be determined later. For each \(x \neq 0\), define the controllability function \(\Theta(x)\) as a solution of the following equation:

\[
2a_0 \Theta = (D(\Theta)FD(\Theta)x, x).
\]
Set $\Theta(0) = 0$. This equality and equation (2.3) determine the function $\Theta(x)$ which is continuous for all $x$ and continuously differentiable for $x \neq 0$.

We will look for control $u(x)$ in the form

$$u(x) = \Theta^{-\frac{1}{2}}(x)a^*D(\Theta(x))x = \sum_{k=1}^{n} \frac{a_k x_k}{\Theta^k(x)}$$

where $F$ and the vector $a$ will be selected in such a way that the controllability function $\Theta(x)$ must be the time of motion from the point $x$ to the origin. In other words, the time derivative $\dot{\Theta}(x)$ by virtue of system (2.1) with the control $u(x)$ (2.4) satisfies equality

$$\dot{\Theta}(x) = -1.$$  (2.5)

**Lemma 2.1.** Let the controllability function $\Theta(x)$ satisfy equality (2.5). Thus,

$$a_1 = -\frac{n(n+1)}{2}.$$  (2.6)

**Proof** Set $y(\Theta, x) = D(\Theta)x$. The controllability function $\Theta(x)$ for $x \neq 0$ then satisfies equality

$$2a_0 \Theta(x) = (Fy(\Theta(x), x), y(\Theta(x), x)),$$  (2.7)

and control (2.4) has the form

$$u(x) = \Theta^{-\frac{1}{2}}(x)a^*y(\Theta(x), x).$$  (2.8)

In view of equalities

$$D(\Theta)A_0D^{-1}(\Theta) = \Theta^{-1}A_0, \quad D(\Theta)b_0 = \Theta^{-\frac{1}{2}}b_0$$

the derivative of the function $y(\Theta(x), x)$ by virtue of the system (2.1) with this control has the form

$$\dot{y}(\Theta(x), x) = \Theta^{-1}(x) \left( \dot{\Theta}(x)H + A_0 + b_0a^* \right) y(\Theta(x), x).$$

Therefore, from equality (2.7) we find that the derivative of the controllability function by virtue of system (2.1) with control $u(x)$ (2.8) is given by equality

$$\dot{\Theta}(x) = \frac{((F(A_0 + b_0a^*) + (A_0 + b_0a^*)F)y(\Theta(x), x), y(\Theta(x), x))}{((F - HF - FH)y(\Theta(x), x), y(\Theta(x), x))}.$$
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Hence, by the conditions of the lemma we have

\[ F \left( A_0 + b_0 a^* + \frac{1}{2} I - H \right) + \left( A_0 + b_0 a^* + \frac{1}{2} I - H \right)^* F = 0, \quad (2.9) \]

where \( I \) is the identity matrix. From this equality, we conclude that the matrix

\[ F \left( A_0 + b_0 a^* + \frac{1}{2} I - H \right) F - \frac{1}{2} \]

is a skew-symmetric matrix, therefore its characteristic equation has the form

\[ \det \left( \left( A_0 + b_0 a^* + \frac{1}{2} I - H \right) - \lambda I \right) = 0, \quad (2.10) \]

which is equivalent to

\[ \det \left( A_0 + b_0 a^* + \frac{1}{2} I - H \right) - \lambda I \right) = 0. \]

In addition, it has eigenvalues with zero real part. Let us write equality \((2.10)\) in the following form:

\[ 0 = \det \left( \begin{array}{cccc}
-1 & -a_2 & \cdots & -a_{n-1} & -a_n \\
\lambda - 2 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -1 & \lambda - n
\end{array} \right) = \]

\[ = \prod_{j=1}^{n} (\lambda-j) - \sum_{j=1}^{n-1} a_j \prod_{i=j+1}^{n} (\lambda-i) - a_n = \lambda^n - (a_1 + n(n+1)/2) \lambda^{n-1} - \ldots. \]

From the fact that the roots of the last equality have zero real part and are complex conjugate, we have that \( a_1 = -n(n+1)/2 \).

Lemma 2.2. Let \( F = (f_{ij})_{i,j=1}^{n} \) be a positive definite matrix. Thus, the matrix

\[ G(\xi) = \begin{pmatrix}
\xi & f_{12} & \cdots & f_{1n} \\
f_{21} & f_{22} & \cdots & f_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
f_{n1} & f_{n2} & \cdots & f_{nn}
\end{pmatrix} \]

is a positive definite matrix and \( \det G(\xi_0) = 0 \) for \( \xi > \xi_0 \).
Proof. The determinant det $G(\xi)$ is a linear function on $\xi$, i.e., $G(\xi) = a\xi + b$, hence by Sylvester criteria $a > 0$. Therefore, if det $G(f_{11}) = a f_{11} + b > 0$, then for any $\xi \geq f_{11}$ we obtain that det $G(\xi)$. Moreover, there exists a unique $\xi_0$ such that det $G(\xi_0) = a \xi_0 + b = 0$, and obviously, for any $\xi > \xi_0$ we have that det $G(\xi) > 0$.

Lemma 2.3.

i) The $(s \times s)$ matrices

$$P_{s,k} = \left( \frac{1}{i + j + k - 2} \right)_{i,j=1}^s = \left( \begin{array}{cccc} \frac{1}{s+1} & \frac{1}{s+2} & \cdots & \frac{1}{s+k-1} \\ \frac{1}{s+2} & \frac{1}{s+3} & \cdots & \frac{1}{s+k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{s+k-1} & \frac{1}{s+k} & \cdots & \frac{1}{2s+k-2} \end{array} \right), \quad k \in \mathbb{N},$$

are positive definite ones.

ii) For every $k \in \mathbb{N}$, we have

$$\Delta_{s,k} = \det \left( \frac{1}{i + j + k - 2} : \frac{1}{i + j + k - 1} \right)_{i,j=1}^s > 0.$$

iii) The determinant of the $(n \times n)$ matrix $P$ of the form

$$P = \begin{pmatrix} 1 - \frac{n(n+1)}{2} & 1 & \cdots & 1/n(n+1) \\ -1 & 2/3 & \cdots & (n+1)(n+2) \\ 0 & 3/4 & \cdots & (n+2)(n+3) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1/(2n-1)2n \end{pmatrix}.$$ (2.11)

is equal to zero, and its rank is equal to $(n-1)$.

Proof. i) The positive definiteness of the matrices $P_{s,k}$ follows from the positive definiteness of the Hilbert matrix $\left( \frac{1}{i+j-1} \right)_{i,j=1}^s$.

The proof of this fact readily follows from the integral representation

$$P_{s,k} = \int_{0}^{1} \left( \begin{array}{c} 1 \\ t \\ \vdots \\ t^{s-1} \end{array} \right) (1, t, \ldots, t^{s-1}) d^{\frac{k}{n}}.$$ 

The matrix $P_{s,k}$ is then positive definite one [5].

ii) Let us represent the determinant $\Delta_{s,k}$ in the form

$$\Delta_{s,k} = \det \left( \frac{1}{i + j + k - 2} : \frac{1}{i + j + k - 1} \right)_{i,j=1}^s.$$
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Add to the $i$-th row ($i \geq 1$) all the next rows, that is, the rows with numbers $i+1$, ..., $s$, and extract from every $i$-th row ($i = 1, \ldots, s$) of the obtained determinant the value $(s-i+1)$ and from every $j$-th column the value $1/(s+j+k-1)$, $j = 1, \ldots, s$. We obtain

\[
\Delta_{s,k} = \frac{s!}{(s+k)\ldots(2s+k-1)} \det P_{s,k}.
\]

From there and in view of i), it follows that $\Delta_{s,k} > 0$, $k \in \mathbb{N}$.

iii) Let us write the determinant of the matrix $P$ in the form

\[
\begin{vmatrix}
1 - \frac{n(n+1)}{2} & \frac{1}{n} - \frac{1}{3} & \cdots & \frac{1}{n+1} - \frac{1}{n+2} \\
-1 & \frac{1}{3} - \frac{1}{4} & \cdots & \frac{1}{n+2} - \frac{1}{n+3} \\
\vdots & \cdots & \cdots & \cdots \\
0 & \frac{1}{(n+1)2n} & \cdots & \frac{1}{(2n-2)2n} - \frac{1}{(2n-1)2n}
\end{vmatrix}.
\]

We transform this determinant in two different ways. In the beginning, we add to every $j$-th column ($j \geq 2$) of this determinant all the next ones; the columns with numbers $j+1$, ..., $n$. We have

\[
\det P = \frac{(n-1)!}{(n+1)\ldots2n} \begin{vmatrix}
(n+1) \left(1 - \frac{n(n+1)}{2}\right) & \frac{1}{n} & \cdots & \frac{1}{n+1} & \frac{1}{n+2} \\
-(n+2) & \frac{1}{n+1} & \cdots & \frac{1}{n+2} & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
0 & \frac{1}{n+2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-1}
\end{vmatrix}.
\]

On the other side, by adding the rows with numbers from $i+1$ till $n$ to the $i$-th row ($i \geq 1$) of determinant (2.12), we obtain

\[
\det P = \frac{n!}{(n+2)\ldots2n} \begin{vmatrix}
\frac{1}{(n+1)(n+2)} & \frac{1}{n+2} & \cdots & \frac{1}{n+1} & \frac{1}{n+2} \\
-1 & \frac{n+2}{n+1} & \frac{n+1}{n+2} & \cdots & \frac{n+1}{n} & \frac{n+2}{n+1} \\
\vdots & \cdots & \cdots & \cdots \\
0 & \frac{1}{(n+1)(n+2)} & \cdots & \frac{1}{(2n-2)(2n-1)} & \frac{1}{(2n-1)2n}
\end{vmatrix}.
\]
Denote by $B_1$ and $B_2$ the complementary minors with numbers (1,1) and (1,2), respectively, which appear in the RHS of (2.13) and (2.14). Thus
\[
\det P = \left(1 - \frac{n(n+1)}{2}\right) B_1 + \frac{n+2}{n+1} B_2 + \frac{(n-1)!}{(n+1)\cdots 2n},
\]
\[
\det P = \left(- \frac{n(n+1)}{2}\right) B_1 + \frac{n}{n-1} B_2 + \frac{(n-1)!}{(n+1)\cdots 2n},
\]
and
\[
B_1 = \frac{2}{n^2 - 1} B_2, \quad \det P = 0.
\]
Since $B_1 \neq 0$, then $\text{rank}(P) = n - 1$. The Lemma is proved.

We determine matrix $F$, which appears in equation (2.3), and the vector $a$, which appears in control of form (2.4). From equality (2.9), we have
\[
\left(A_0 + b_0 a^* + \frac{1}{2}I - H\right) F^{-1} + F^{-1} \left(A_0 + b_0 a^* + \frac{1}{2}I - H\right)^* = 0.
\]

Denote $D_n = \text{diag}\left((-1)^{i-1}/(i-1)!\right)_{i=1}^n$. We must find the matrix $F^{-1}$ with the form $F^{-1} = D_n C D_n$, where the matrix $C$ is a Hankel matrix $C = (c_{i+j})_{i,j=0}^{n-1}$. We find the matrix $C$. To this end we write equality (2.15) in the form
\[
\left(\frac{1}{2}I - H + D_n^{-1} A_0 D_n + D_n^{-1} b_0 a^* D_n\right) C +
\]
\[
+ C \left(\frac{1}{2}I - H + D_n^{-1} A D_n + D_n^{-1} b_0 a^* D_n\right)^* = 0.
\]
Denote by $Q = (q_{ij})_{i,j=1}^n$ the matrix in the LHS of equality (2.16) and write it in the following form
\[
Q = \begin{pmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{pmatrix},
\]
where $Q_{11} = q_{11}$, $Q_{12} = (q_{12}, \ldots, q_{1n})$, $Q_{21} = Q_{12}^*$, $Q_{22} = (q_{ij})_{i,j=2}^n$. Hence the elements of the matrix $Q$ are given by the following equality:
\[
\begin{align*}
q_{11} &= 2(1 + \tilde{a}_1)c_0 + 2 \sum_{j=1}^{n-1} \tilde{a}_{j+1} c_j, \\
q_{1i} &= -(i - 1)c_{i-2} + (1 + \tilde{a}_1 + i)c_{i-1} + \sum_{j=1}^{n-1} \tilde{a}_{j+1} c_{i+j-1}, \quad i = 2, \ldots, n, \\
q_{ij} &= -(i + j - 2)c_{i+j-3} + (i + j)c_{i+j-2}, \quad i = 2, \ldots, n, \quad j = 2, \ldots, n,
\end{align*}
\]
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where $\tilde{a}_1, \ldots, \tilde{a}_n$ are the components of the vector $\tilde{a} = D_n a$. Since $Q$ is a zero matrix, then from the equalities $Q_{22} = 0$ we obtain

$$c_j = \frac{(2n - 1)2n}{(j + 1)(j + 2)} c_{2n - 2}, \quad j = 1, \ldots, 2n - 3,$$

and equality (2.17) has the form

$$\left(1 + \tilde{a}_1\right)c_0 + \sum_{j=1}^{n-1} \frac{\tilde{a}_{j+1}}{(j + 1)(j + 2)} = 0,$$

$$\left(\frac{3}{2} + \tilde{a}_1\right)c_0 + \sum_{j=1}^{n-1} \frac{\tilde{a}_{j+1}}{(j + 2)(j + 3)} = 0,$$

$$\left(\frac{4}{3} + \tilde{a}_1\right)c_0 + \sum_{j=1}^{n-1} \frac{\tilde{a}_{j+1}}{(j + 3)(j + 4)} = 0,$$

$$\left(\frac{n + 1}{n} + \tilde{a}_1\right)c_0 + \sum_{j=1}^{n-1} \frac{\tilde{a}_{j+1}}{(j + n)(j + n + 1)} = 0,$$

which in vector form can be written as

$$Py = y_0,$$

where $P$ is a $(n \times n)$ matrix defined by equality (2.11). In addition, $y, y_0$ are vectors of the form

$$y = \left(\frac{c_0}{(2n - 1)2nc_{2n - 2}}, \tilde{a}_2, \ldots, \tilde{a}_n\right)^*, \quad y_0 = \left(0, -\frac{3 + \tilde{a}_1}{2 \cdot 3}, -\frac{\tilde{a}_1}{3 \cdot 4}, \ldots, -\frac{\tilde{a}_1}{n(n + 1)}\right)^*.$$

This system represents a subsystem, since the matrix $C = \left(\frac{1}{(i + j)(i + j)}\right)_{i,j=1}^n$ and the vector $a^* = -b_0^* D_n^{-1} C^{-1} D_n^{-1}$ satisfy the equality (2.16). Actually, the equality (2.16) for such a selection of the vector $a$ has the form

$$\left(\frac{1}{2} I - H + D_n^{-1} A_0 D_n\right) C + C \left(\frac{1}{2} I - H + D_n^{-1} A_0 D_n\right)^* = b_0 b_0^*.$$

This equality holds if one inserts the matrix $C$ in the last equality.

Therefore, $\text{rank}(P) = \text{rank}(P, y_0)$; by virtue of iii) and Lemma 2.3 we have that $\text{rank}(P) = n - 1$. 

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Let us find all the solutions of system (2.19). Consider matrix
\((n-1)\times(n-1)\), which has the form
\[
\tilde{P} = \begin{pmatrix}
-1 & \frac{1}{3 \cdot 4} & \cdots & \frac{1}{n(n+1)} \\
0 & \frac{1}{4 \cdot 5} & \cdots & \frac{1}{(n+1)(n+2)} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & \frac{1}{(2n-2)(2n-1)}
\end{pmatrix}.
\]
This matrix, by because of ii) of Lemma 2.3 for \(k = 4\) and \(s = n-2\), is a nondegenerate matrix. Denote by \(\tilde{y}_0 = a_n d' + d''\), where
\[
d' = \left(-\frac{1}{(n+1)(n+2)}, -\frac{1}{(n+2)(n+3)}, \cdots, -\frac{1}{(2n-1)2n}\right)^{\ast},
\]
\[
d'' = \left(-\frac{3 + a_1}{2 \cdot 3}, -\frac{a_1}{3 \cdot 4}, \cdots, -\frac{a_1}{n(n+1)}\right)^{\ast}.
\]
Furthermore, by \(\tilde{y}\) we denote the \((n-1)\)-dimensional vector
\[
\tilde{y} = \left(\frac{c_0}{(2n-1)2n c_{2n-2}}, \tilde{a}_2, \ldots, \tilde{a}_{n-1}\right)^{\ast}
\]
and consider a system of equations \(\tilde{P}\tilde{y} = \tilde{y}_0\) with respect to \(\tilde{y}\). This system has a unique solution \(\tilde{y} = \tilde{P}^{-1}\tilde{y}_0\):
\[
\frac{c_0}{(2n-1)2n c_{2n-2}} = \frac{1}{\Delta} \left(\Delta'_1\tilde{a}_n + \Delta''_1\right) = \frac{1}{\Delta} \left(\Delta'_1\frac{(-1)^n}{(n-1)!} a_n + \Delta''_1\right),
\]
\[
\tilde{a}_j = \frac{1}{\Delta} \left(\Delta'_j\tilde{a}_n + \Delta''_j\right) = \frac{1}{\Delta} \left(\Delta'_j\frac{(-1)^{n-1}}{(n-1)!} a_n + \Delta''_j\right), \quad i = 2, \ldots, n-1,
\]
where \(\Delta = \det \tilde{P}\), and \(\Delta'_j, \Delta''_j\) \((j = 1, \ldots, n-1)\) are the determinants of the matrix \(\tilde{P}\), in which instead of its the \(j\)-th the columns, vectors \(d', d''\) are inserted. By virtue of condition \(\text{rank}(P) = \text{rank}(P, y_0) = n-1\) and relations (2.20) describe all the solutions of the system (2.19).

Therefore, the next lemma follows.

**Lemma 2.4.** Matrix
\[
C = \begin{pmatrix}
\frac{1}{\Delta} \left(\Delta'_1\frac{(-1)^{n-1}}{(n-1)!} a_n + \Delta''_1\right) & \frac{1}{2 \cdot 3} & \cdots & \frac{1}{n(n+1)} \\
\frac{1}{2 \cdot 3} & \frac{1}{4 \cdot 5} & \cdots & \frac{1}{(n+1)(n+2)} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{1}{n(n+1)} & \frac{1}{(n+1)(n+2)} & \cdots & \frac{1}{(2n-2)(2n-1)}
\end{pmatrix}
\]
(2.21)
Controllability function as time of motion. I

and the vector

\[ a = \left( -\frac{n(n+1)}{2}, -\frac{1}{\Delta} \left( \Delta'_2 \left( -\frac{1}{n-1} \right)^{n-1} a_n + \Delta''_2 \right), \ldots, \right. \]

\[ \ldots, (-1)^{n-2} (n-2)! \frac{1}{\Delta} \left( \Delta'_{n-1} \left( -\frac{1}{n-1} \right)^{n-1} a_n + \Delta''_{n-1} \right), a_n \left. \right)^* \]

(2.22)
give all the solutions of equation (2.16); moreover for

\[ c_{2n-2} > 0, \quad \frac{1}{\Delta} \left( \Delta'_1 \left( -\frac{1}{n-1} \right)^{n-1} a_n + \Delta''_1 \right) > \xi_0, \]  

(2.23)

where \( \xi_0 \) is a root of the equation

\[ \left| \begin{array}{cccccccc}
\xi_0 & \frac{1}{2} & \ldots & \frac{1}{n(n+1)} \\
\frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{(n+1)(n+2)} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{1}{n(n+1)} & \frac{1}{(n+1)(n+2)} & \ldots & \frac{1}{(2n-1)2n} \\
\end{array} \right| = 0, \]  

(2.24)

matrix \( C \) is a positive definite one.

Proof. Let \( c_{2n-2} > 0 \), and consider matrix

\[ \tilde{C}(z) = \left( \begin{array}{cccc}
 \frac{z}{n(n+1)} & \frac{1}{2} & \ldots & \frac{1}{n(n+1)} \\
 \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{(n+1)(n+2)} \\
 \ldots & \ldots & \ldots & \ldots \\
 \frac{1}{n(n+1)} & \frac{1}{(n+1)(n+2)} & \ldots & \frac{1}{(2n-1)2n} \\
\end{array} \right) (2n - 1)2n c_{2n-2}. \]

This matrix is positive definite for \( z = 1/2 \), by virtue of ii) of the Lemma 2.3, its principal minors \( \Delta_s \), \( s = 1, \ldots, n \), are positive: the Sylvester criterion holds. Because of Lemma 2.2, matrix \( \tilde{C}(z) \) is positive definite for \( z > \xi_0 \), where \( \xi_0 \) is a root of the equation \( \det \tilde{C}(\xi_0) = 0 \). From where we obtain that for the parameters \( c_{2n-2} \ a_n \), which satisfy condition (2.23), matrix \( C \) is positive definite.

Remark 2.1. Observe that equality (2.24) is reduced to the form

\[ \left| \begin{array}{cccc}
(1 + \frac{1}{n}) \xi_0 + \frac{1}{2} - \frac{1}{2n} & \frac{1}{2} & \ldots & \frac{1}{n(n+1)} \\
\frac{1}{3} & \frac{1}{3} & \ldots & \frac{1}{n+1} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{1}{n} & \frac{1}{n+1} & \ldots & \frac{1}{2n-1} \\
\end{array} \right| = 0. \]  

(2.25)

For establishing this fact, to each \( i \)-th row we add all the remaining rows; from the obtained rows and columns of the new determinant we then extract common factors. For the cases \( n = 2, 3, 4, 5, 6 \) and \( 7 \) the roots of \( \xi_0 \) is equal to \( 1/3, 5/12, 9/20, 7/15, 10/21, \) and \( 27/56 \) respectively.
Lemma 2.5. Matrix $C^1 = C - CH - HC$ for the form

$$C^1 = \begin{pmatrix}
\frac{2}{\Delta} \left( \Delta_1' \frac{(-1)^{n-1}}{(n-1)!} a_n + \Delta_1'' \right) & \frac{1}{n} & \cdots & \frac{1}{n+1} \\
\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n+1} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n+1} \\
\end{pmatrix} (2n - 1) 2n c_{2n-2}
$$

is positive definite for

$$c_{2n-2} > 0, \quad \frac{1}{\Delta} \left( \Delta_1' \frac{(-1)^{n-1}}{(n-1)!} a_n + \Delta_1'' \right) > \left( \frac{1}{2} + \frac{1}{2n} \right) \xi_0 + \frac{1}{4} \frac{1}{4n},$$

where $\xi_0$ is a root of equation (2.25).

Proof. By i) of Lemma 2.3 for $s = n$ and $k = 1$, matrix $C^1$ is positive definite if the parameter $a_n$ satisfies the condition

$$\frac{2}{\Delta} \left( \Delta_1' \frac{(-1)^{n-1}}{(n-1)!} a_n + \Delta_1'' \right) = 1.$$  

From Lemma 2.2 for $c_{2n-2} > 0$, matrix

$$\tilde{C}^1(z) = \begin{pmatrix}
z & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{n+1} \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{n+1} \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{n+1} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{n+1} \\
\end{pmatrix} (2n - 1) 2n c_{2n-2} \quad (2.27)$$

is positive definite for all

$$z > \xi,$$  

where $\xi$ is a root of equality $\det \tilde{C}^1(\xi) = 0$. By equality (2.25), we have

$$0 = \det \tilde{C}^1(\xi) = \det \tilde{C}^1 \left( \left( 1 + \frac{1}{n} \right) \xi_0 + \frac{1}{2} - \frac{1}{2n} \right),$$

and consequently

$$\xi = \left( 1 + \frac{1}{n} \right) \xi_0 + \frac{1}{2} - \frac{1}{2n}. \quad (2.29)$$

Thus, the proof readily follows by comparing (2.26) and (2.27), as well as (2.29) and the condition (2.28).
3 Synthesis of the bounded controls for the canonical system

The solution of the synthesis problem for the canonical system (2.1) when the controllability function is time of motion, gives the following solution

Theorem 3.1. Let

\[ c_{2n-2} > 0, \quad \frac{1}{\Delta} \left( \Delta' \frac{(-1)^{n-1}}{(n-1)!} a_n + \Delta'' \right) > \max \left\{ \xi_0, \left( \frac{1}{2} + \frac{1}{2n} \right) \xi_0 + \frac{1}{4} - \frac{1}{4n} \right\}, \]

where \( \xi_0 \) is a root of equation (2.25). Let the matrix \( C \) and the vector \( a \) be as in (2.21) and (2.22), respectively. The number \( a_0 \) satisfies the condition

\[ 0 < a_0 \leq \frac{d^2}{2(F^{-1}a, a)}; \quad F^{-1} = D_n C D_n, \]

and the controllability function \( \Theta(x) \) for \( x \neq 0 \) is defined by equality (2.3), where \( F = D_n^{-1} C^{-1} D_n^{-1} \), and \( \Theta(0) = 0 \).

Thus, control \( u(x) \) (2.4) transfers the arbitrary initial point \( x \in \mathbb{R}^n \) to the origin along the trajectory \( \dot{x} = A_0 x + b_0 u(x) \) in time \( T(x) = \Theta(x) \) and satisfies the restriction \( |u(x)| \leq d \).

Proof. By Lemma 2.4 and condition 3.1, matrix \( F^{-1} = D_n C D_n \), and consequently matrix \( F \) are positive definite.

If matrix \( (F-FH-HF) \) is positive definite, equation (2.3) for \( x \neq 0 \) has a unique continuously differentiable solution \( \Theta(x) \); see [3].

We establish for which parameters \( a_n \) and \( c_{2n-2} \) is matrix \( (F-FH-HF) \) positive definite. The positive definiteness of this matrix will follow from the positive definiteness of matrix \( (F^{-1}-HF^{-1}-F^{-1}H) \). Since matrix \( F^{-1} = D_n C D_n \), and matrices \( D_n \) and \( H \) commute with each other, then the following equality is valid

\[ F^{-1}-HF^{-1}-F^{-1}H = D_n(C-HC-CH)D_n. \]

By Lemma 2.3 and condition 3.1, matrix \( (C-HC-CH) \) is positive definite; hence matrix \( (F-FH-HF) \) is also positive definite.

Therefore, under conditions 3.1 equation (2.3) for \( x \neq 0 \) has a unique continuously differentiable solution \( \Theta(x) \). Set \( \Theta(0) = 0 \), and \( \Theta(x) \) becomes a continuous function for all \( x \).
Now let us establish that the control is bounded. We estimate the expression \(a^*y(\Theta, x)\Theta^{-\frac{1}{2}}\). To this end, we fix \(\Theta\) and solve the problem of finding the extremal of the function \(a^*y(\Theta, x)\Theta^{-\frac{1}{2}}\) under the condition
\[
(Fy(\Theta, x), y(\Theta, x)) - 2a_0\Theta = 0. \tag{3.3}
\]
This problem we solve with the help of the Lagrange multipliers. The Lagrange function has the form
\[
a^*y(\Theta, x)\Theta^{-\frac{1}{2}} - \lambda(Fy(\Theta, x), y(\Theta, x)) + 2\lambda a_0\Theta.
\]
Let \(y_0\) be the extremal point. The necessary condition gives \(a\Theta^{-\frac{1}{2}} - 2\lambda Fy_0 = 0\), from where we have \(y_0 = 1/(2\lambda)\Theta^{-\frac{1}{2}}F^{-1}a\). Setting \(y_0\) in restriction (3.3), we have
\[
|u(x)| \leq \sqrt{2a_0(F^{-1}a, a)} \tag{3.4}
\]
Selecting the number \(a_0\) from condition (3.2), from inequalities (3.4) it follows that \(u(x)\) satisfies the given restriction in all the phase space.

Hence from Theorem 1 [3], control \(u(x)\) solves the synthesis problem of bounded controls in all the phase space, and the time of motion from point \(x\) to the origin is equal to the function of controllability at \(x\):
\[
T(x) = \Theta(x).
\]

**Example.** Let us consider the synthesis problem for the system
\[
\dot{x}_1 = u, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_2, \tag{3.5}
\]
with a restriction on the control of the form \(|u| \leq 1\).

Since \(\Delta = -1/20\), \(\Delta_1 = 1/3600\), and \(\Delta_2 = -1/60\), then vector \(a\) (2.22) matrix \(C\) (2.21) and matrix \(D_3\) take the form
\[
a = \begin{pmatrix}
-6 \\
-a_3
\end{pmatrix}, \quad C = \begin{pmatrix}
10 - \frac{a_3}{12} & 5 - \frac{5}{2} & \frac{5}{4} \\
\frac{5}{2} & \frac{5}{2} & \frac{5}{4} \\
\frac{5}{4} & \frac{5}{4} & 1
\end{pmatrix}c_4, \quad D_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}.
\]

Thus, the matrix \(F^{-1} = D_3CD_3\) and its inverse matrix has the form
\[
F^{-1} = \begin{pmatrix}
10 - \frac{a_3}{12} & -5 & \frac{5}{4} \\
-5 & 5 & -\frac{5}{4} \\
\frac{5}{4} & -\frac{5}{4} & 1
\end{pmatrix}c_4, \quad F = c_4c_4^T + \begin{pmatrix}
12 & 60 & 120 \\
60 & 180 - 4a_3 & 240 - 12a_3 \\
120 & 240 - 12a_3 & -40a_3
\end{pmatrix}.
\]
Controllability function as time of motion. I

From (2.24) and we obtain that 
\[ \xi_0 = \frac{5}{12}. \]
Condition (3.1) then has the form
\[ \frac{a_4}{2} > 0, \quad \frac{a_3}{360} > \frac{4}{9}, \]
according to it, we set \( c_4 = 1, \quad a_3 = -45. \) As a result,
\[
\begin{pmatrix}
-6 \\
-25 \\
-45
\end{pmatrix},
F^{-1} = \frac{1}{4}
\begin{pmatrix}
55 & -20 & 5 \\
-20 & 10 & -3 \\
5 & -3 & 1
\end{pmatrix},
F = 4
\begin{pmatrix}
1/5 & 1 & 2 \\
1 & 6 & 13 \\
2 & 13 & 30
\end{pmatrix}.
\]

We select the number \( a_0 \) of equality (2.3) from conditions (3.2), which in this case has the form \( 0 < a_0 \leq \frac{2}{205}. \)
We determine the controllability function \( \Theta(x) \) for \( x \neq 0 \) as a positive solution of equation (2.3) (which is unique). This equation in this case has the form
\[
\Theta^6 = 4101x_1^2 + 4100\Theta^3x_1x_2 + 820\Theta^2x_1x_3 + 1230\Theta^2x_2^2 + 5330\Theta x_2x_3 + 6150x_3^2.
\]
(3.6)
The control \( u(x) \) (2.4), which solves the global synthesis problem for system (3.5) and satisfies the restriction \(|u(x)| \leq 1\), is given by the formula
\[
u(x) = -\frac{6}{\Theta(x)}x_1 - \frac{25}{\Theta^2(x)}x_2 - \frac{45}{\Theta^3(x)}x_3.
\]
(3.7)
Let us find the trajectory of the system (3.5), which corresponds to the control \( u = u(x) \) of form (3.7) and starts at the point \( x(0) = x_0 \in \mathbb{R}^3 \).
This trajectory represents a solution of the system
\[
\dot{x}_1 = -\frac{6}{\Theta(x)}x_1 - \frac{25}{\Theta^2(x)}x_2 - \frac{45}{\Theta^3(x)}x_3, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_2.
\]
Since \( \Theta(x) \) is the time motion from point \( x \) to the origin, that is, the equality \( \Theta(x(t)) = -1 \) is satisfied, thus, \( \Theta(x(t)) = \Theta_0 - t \), where \( \Theta_0 \) is a positive root of equation (3.6) for \( x = x_0 \). Consequently, the trajectory of the solution is the solution of the Cauchy problem of the form
\[
\dot{x}_1 = -\frac{6}{\Theta_0 - t}x_1 - \frac{25}{(\Theta_0 - t)^2}x_2 - \frac{45}{(\Theta_0 - t)^3}x_3, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_2,
\]
\[
x_1(0) = x_0^0, \quad x_2(0) = x_0^0, \quad x_3(0) = x_0^0.
\]
This equation is reduced to the differential equation of the form
\[
(\Theta_0 - t)^3x_3^{(3)} + 6(\Theta_0 - t)^2\ddot{x}_3 + 25(\Theta_0 - t)\dot{x}_3 + 45x_3 = 0,
\]
with initial conditions \( x_3(0) = x_0^0, \quad \dot{x}_3(0) = x_0^0, \quad \ddot{x}_3(0) = x_0^0. \) With the change of variables \( t = \Theta_0 - e^\tau \), this Euler-type differential equation is reduced to a differential equation with constant coefficients with respect to
the function $y(\tau) = x_3(\Theta_0 - e^\tau)$, which has the form $y''' - 9y'' + 33y' - 45y = 0$. Thus, we have

$$y(\tau) = e^{3\tau} \left( c_1 + c_2 \cos \sqrt{6} \tau + c_3 \sin \sqrt{6} \tau \right),$$

where the constants $c_1$, $c_2$, and $c_3$ are found from the conditions

$$y(\tau_0) = x_3^0, \quad y'(\tau_0) = -\Theta_0 x_2^0, \quad y''(\tau_0) = \Theta_0^2 x_1^0, \quad (\tau_0 = \ln \Theta_0)$$

and are equal to

$$c_1 = \frac{1}{6\Theta_0} \left( \frac{15}{\Theta_0} x_3^0 + \frac{5}{\Theta_0} x_2^0 + x_1^0 \right),$$

$$c_2 = \xi_1 \cos \left( \sqrt{6} \ln \Theta_0 \right) - \xi_2 \sin \left( \sqrt{6} \ln \Theta_0 \right),$$

$$c_3 = \xi_1 \sin \left( \sqrt{6} \ln \Theta_0 \right) + \xi_2 \cos \left( \sqrt{6} \ln \Theta_0 \right).$$

Here

$$\xi_1 = - \frac{1}{6\Theta_0} \left( \frac{9}{\Theta_0} x_3^0 + \frac{5}{\Theta_0} x_2^0 + x_1^0 \right), \quad \xi_2 = - \frac{1}{\sqrt{6} \Theta_0^2} \left( \frac{3}{\Theta_0} x_3^0 + x_2^0 \right).$$

Since $x_3(t) = y(\ln(\Theta_0 - t))$, and the functions $x_2(t)$ and $x_1(t)$ are found by taking the derivative of the function $x_3(t),$

$$x(t) = \begin{pmatrix}
(\Theta_0 - t) \left( 6c_1 + 5\sqrt{6} \xi_2 \cos \alpha(t) - 5\sqrt{6} \xi_1 \sin \alpha(t) \right)

(\Theta_0 - t)^2 \left( -3c_1 - (3\xi_1 + \sqrt{6} \xi_2) \cos \alpha(t) + (-3\xi_1 + \sqrt{6} \xi_2) \sin \alpha(t) \right)

(\Theta_0 - t)^3 \left( c_1 + \xi_1 \cos \alpha(t) + \xi_2 \sin \alpha(t) \right)
\end{pmatrix},$$

where $\alpha(t) = \sqrt{6} \ln (1 - t/\Theta_0)$. Obviously, $x(t) \to 0$ for $t \to \Theta_0$.

The control $u(x)$ on the trajectory $x(t)$ has the form

$$u(x(t)) = -6c_1 + 5(6\xi_1 - \sqrt{6} \xi_2) \cos \alpha(t) + 5(\sqrt{6} \xi_1 + 6\xi_2) \sin \alpha(t).$$

For simplicity we solve the problem of attaining the origin from the points belonging to the curve $x_1 > 0, x_2 = -41x_1^2/121$, and $x_3 = 0$. In this case, from $[5,6]$, the time of motion $\Theta_0$ from $x_0 = (x_1^0, -41(x_1^0)^2/121, 0)^*$ is equal to $41x_1^0/11$. The trajectory which begins in this point has the form

$$x(t) = \begin{pmatrix}
\frac{41x_1^0 - 11t}{451} \left( 6 + 5 \cos \alpha(t) + 5\sqrt{6} \sin \alpha(t) \right)

\frac{(41x_1^0 - 11t)^2}{9922} \left( -6 + 4 \cos \alpha(t) - 3\sqrt{6} \sin \alpha(t) \right)

\frac{(41x_1^0 - 11t)^3}{327426} \left( 6 - 6 \cos \alpha(t) + \sqrt{6} \sin \alpha(t) \right)
\end{pmatrix},$$

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where \( \alpha(t) = \sqrt{6} \ln(1 - 11t/(41x_0^6)) \). The control \( u(x) \) on this trajectory is determined by the relation

\[
u(t) = -\frac{1}{41} (6 + 35 \cos \alpha(t)).
\]

Clearly, this control satisfies the restriction.

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