UNIT EQUATIONS ON QUATERNIONS

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Abstract. A classical result about unit equations says that if $\Gamma_1$ and $\Gamma_2$ are finitely generated subgroups of $\mathbb{C}^\times$, then the equation $x + y = 1$ has only finitely many solutions with $x \in \Gamma_1$ and $y \in \Gamma_2$. We study a noncommutative analogue of the result, where $\Gamma_1, \Gamma_2$ are finitely generated subsemigroups of the multiplicative group of a quaternion algebra. We prove an analogous conclusion when both semigroups are generated by algebraic quaternions with norms greater than 1 and one of the semigroups is commutative. As an application in dynamics, we prove that if $f$ and $g$ are endomorphisms of a curve $C$ of genus 1 over an algebraically closed field $k$, and $\deg(f), \deg(g) \geq 2$, then $f$ and $g$ have a common iterate if and only if some forward orbit of $f$ on $C(k)$ has infinite intersection with an orbit of $g$.

1. Introduction

A classical result about unit equations states that the equation $f + g = 1$ has only finitely many solutions in a given finitely generated semigroup $\Gamma$ in $K^\times$, where $K$ is a field of characteristic zero. Unit equations have had important applications in many areas of mathematics, including Diophantine geometry ([9, 11]), arithmetic dynamics [4, p.291] and variants of the Mordell-Lang conjecture (for instance, see [4, p.321]). Extensions of the classical result have also been studied, for example, see [10, 13] in the characteristic $p$ setting.

In this paper we present a class of semigroups in the standard quaternion algebra over $\mathbb{R}$ for which the finiteness of solutions of the unit equation holds. This is the first analogous result in the noncommutative setting. In light of the many applications of unit equations, this raises the intriguing possibility that some of those applications might have noncommutative analogues.

Let $\mathbb{H}_a$ denote the algebra of algebraic quaternions, namely the quaternions whose coordinates are real algebraic numbers; see Section 2 for more details and properties of quaternions.

Theorem 1.1. Let $\Gamma_1, \Gamma_2$ be semigroups of $\mathbb{H}_a^\times$ generated by finitely many algebraic elements of (Euclidean) norms greater than 1, and fix $a, a', b, b' \in \mathbb{H}_a^\times$. If $\Gamma_1$ is commutative, then the equation

$$afa' + bgb' = 1$$

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has only finitely many solutions with \( f \in \Gamma_1 \) and \( g \in \Gamma_2 \).

We emphasize that even though \( \Gamma_1 \) is commutative, the semigroup \( \Gamma_2 \) need not be commutative, and that \( a, a' \) and \( \Gamma_1 \) typically will not commute with each other. The proof relies on the following result, which implies that if a certain quaternion unit equation has infinitely many solutions, then so does a different type of equation. We note that Theorem 1.2 applies in greater generality than Theorem 1.1, as Theorem 1.2 does not require \( \Gamma_1 \) to be commutative.

**Theorem 1.2.** Let \( \Gamma_1, \Gamma_2 \) be semigroups of \( \mathbb{H}_a^\times \) generated by finitely many elements of (Euclidean) norms greater than 1, and fix \( a, a', b, b' \in \mathbb{H}_a^\times \). Then the equation

\[
a f a' + b g b' = 1
\]

has only finitely many solutions with \( f \in \Gamma_1 \) and \( g \in \Gamma_2 \) such that \( |1 - a f a'| \neq |a f a'| \).

Given Theorem 1.2, in order to prove Theorem 1.1 it suffices to prove the next result which involves only the semigroup \( \Gamma_1 \):

**Theorem 1.3.** Let \( \Gamma \) be a semigroup generated by finitely many elements in \( \mathbb{H}_a \) with norms greater than 1, and fix \( a, a' \in \mathbb{H}_a^\times \). If \( \Gamma \) is commutative, then the equation

\[
|1 - a f a'| = |a f a'|
\]

has only finitely many solutions with \( f \in \Gamma \).

We remark that Theorem 1.3 is the only step in the proof of Theorem 1.1 that uses the commutativity of \( \Gamma_1 \), so any generalization of Theorem 1.3 would immediately yield a generalization of Theorem 1.1.

In light of the above results, we make the following conjecture about noncommutative unit equations:

**Conjecture 1.4.** Let \( \Gamma_1, \Gamma_2 \) be finitely generated semigroups of the multiplicative group \( A^\times \) of a finite dimensional division algebra \( A \) over \( \mathbb{Q} \). Then for any fixed \( a, a', b, b' \in A^\times \), the unit equation \( a f a' + b g b' = 1 \) has only finitely many solutions with \( f \in \Gamma_1 \) and \( g \in \Gamma_2 \).

In Section 6, we will discuss a possible \( p \)-adic approach to Conjecture 1.4, and will give a counterexample to the matrix algebra analogue of Conjecture 1.4 in Example 6.1.

Our main theorem has the following consequence about intersections of orbits of endomorphisms of a genus-1 curve in arbitrary characteristic.

**Corollary 1.5.** Let \( E \) be an elliptic curve over an algebraically closed field \( k \), and let \( f, g : E \to E \) be regular maps of degrees greater than 1. If there are points \( A, B \in E(k) \) such that the forward orbits \( O_f(A) := \{ A, f(A), f^2(A), \ldots \} \) and \( O_g(B) := \{ B, g(B), g^2(B), \ldots \} \) have infinite intersection, then \( f \) and \( g \) has a common iterate, namely, \( f^{m_0} = g^{n_0} \) for some positive integers \( m_0, n_0 \).
Analogous results have been proven in various cases in characteristic zero, in case $E$ is replaced by $A^1$ [8], a linear space [6], or a semiabelian variety [6, 7]. Furthermore, the analogue of Corollary 1.5 to arbitrary simple abelian varieties $A$ would follow from the proof of Corollary 1.5, if one could prove the relevant case of Conjecture 1.4 for the division algebras $\text{End}(A)$.

The characteristic zero case of Corollary 1.5 is an instance of the higher-rank generalization posed in [8, Question 1.6] of the dynamical Mordell-Lang conjecture [2, Chapter 3]; see also [6]. For positive characteristic, see [2, Chapter 13]. We note that the conclusions of all previous results in characteristic $p > 0$ involve the more complicated possibility of $p$-automatic sequences (e.g., [3, 5]), whereas the conclusion of Corollary 1.5 is more rigid.

The rest of the paper is organized as follows. In Section 2, we recall the standard notation about quaternions and state some known results. Then Section 3, 4 and 5 contains proofs of Corollary 1.5, Theorem 1.2 and Theorem 1.3, respectively. The proofs are independent of one another, and can be read in any order. The proof of Theorem 1.1 follows immediately from Theorem 1.2 and Theorem 1.3.

2. Preliminaries

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ denote the quaternion algebra $\mathbb{H}$ over $\mathbb{R}$, with the standard multiplication law $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. For an element $\alpha = a + bi + cj + dk \in \mathbb{H}$, where $a, b, c, d \in \mathbb{R}$, define its conjugation to be $\overline{\alpha} = a - bi - cj - dk$, its norm to be $N(\alpha) = a\overline{a} = \alpha\overline{\alpha} = a^2 + b^2 + c^2 + d^2$, and its trace $tr(\alpha) = \alpha + \overline{\alpha} = 2a$. Write $|\alpha| = \sqrt{N(\alpha)}$.

We say a quaternion number $\alpha = a + bi + cj + dk \in \mathbb{H}$ to be algebraic if all coordinates $a, b, c, d$ are algebraic over $\mathbb{Q}$. This is equivalent to that $\alpha$ satisfies a polynomial equation with coefficients in $\mathbb{Q}$, or that $\mathbb{Q}[\alpha]$ is a finite field extension of $\mathbb{Q}$. Indeed, $\alpha$ always satisfies the quadratic equation

$$X^2 - tr(\alpha)X + N(\alpha) = 0$$

and if $a, b, c, d \in \mathbb{Q}$, then so are $tr(\alpha)$ and $N(\alpha)$.

Denote by $\mathbb{H}_a$ the subalgebra of all quaternion numbers that are algebraic.

The proof of Theorem 1.2 and Theorem 1.3 makes use of the following special case of Baker’s theorem on Diophantine approximation of logarithms.

**Theorem 2.1** (Baker, Wüstholz [1]). Let $\lambda_1, ..., \lambda_r$ be complex numbers such that $e^{\lambda_i}$ are algebraic for $1 \leq i \leq r$. Then there are effectively computable constants $k, C > 0$ depending on $r$ and $\lambda_i$ such that

$$0 < |a_0 + a_1\lambda_1 + \cdots + a_r\lambda_r| \leq kH^{-C}$$

has no solutions in $a_i \in \mathbb{Z}$, where $H = \max_{i=0}^r |a_i|$. 
3. Proof of Corollary 1.5

Since \( \deg(f) > 1 \), the regular map \( f \) has a fixed point. By replacing the origin of \( E \) by a fixed point of \( f \) if necessary, we may assume that \( f \) is an endomorphism of \( E \).

Write \( g = \tau_Q \circ h \) where \( Q \) is a point on \( E \), \( \tau_Q \) is the map \( E \to E \) defined by translation by \( Q \), and \( h \) is an endomorphism of \( E \). Here \( \deg(h) = \deg(g) > 1 \), so that \( h - 1 \) is nonconstant and thus induces a surjective map \( E \to E \). Let \( R \) be a point on \( E \) such that \((h - 1)(R) = Q\). Then, for any positive integer \( n \), we have

\[
(3.1) \quad g^n = \tau_{Q+h(Q)+h^2(Q)+\ldots+h^{n-1}(Q)} \circ h^n = \tau_{(h^n-1)(R)} \circ h^n.
\]

Thus, for any positive integer \( m \), the condition \( f^m = g^n \) is equivalent to the conditions that \( f^m = h^n \) and \((h^n - 1)(R) = O\).

Pick the orbits of \( f \) and \( g \) that have infinite intersection, and let \( P \) be any point in the intersection; then the orbits \( O_f(P) \) and \( O_g(P) \) also have infinite intersection, so there are infinitely many pairs \((m, n)\) of positive integers such that

\[
(3.2) \quad (f^{m_0} - h^{n_0})(P) = (h^{n_0} - 1)(R)
\]

\[
(3.3) \quad (f^m - h^n)(P) = (h^n - 1)(R)
\]

Left-multiplying (3.2) by the dual isogeny \((h^{n_0} - 1)\) of \((h^{n_0} - 1)\), we get

\[
(h^{n_0} - 1)(f^{m_0} - h^{n_0})(P) = \deg(h^{n_0} - 1)(R)
\]

Left-multiplying further by \((h^n - 1)\), we get

\[
(h^n - 1)(h^{n_0} - 1)(f^{m_0} - h^{n_0})(P) = (h^n - 1)(h^{n_0} - 1)(R)
\]

Note that \( \deg(h^{n_0} - 1) \) is an integer, so it is in the center of \( \text{End}(E) \).

Using (3.3), we get

\[
\left( (h^n - 1)(h^{n_0} - 1)(f^{m_0} - h^{n_0}) - (f^m - h^n)\deg(h^{n_0} - 1) \right)(P) = O.
\]

Since \( O_f(P) \) is infinite, \( P \) must be a point of infinite order (otherwise, \( rP = 0 \) for some integer \( r > 0 \), so \( O_f(P) \) lies in the finite group \( E[r] \) of \( r \)-torsions).

Hence the kernel of \((h^n - 1)(h^{n_0} - 1)(f^{m_0} - h^{n_0}) - (f^m - h^n)\deg(h^{n_0} - 1)\) contains all (infinitely many) multiples of \( P \). Since the kernel of any nonzero endomorphism is a finite group, we must have

\[
(3.4) \quad (h^n - 1)(h^{n_0} - 1)(f^{m_0} - h^{n_0}) - (f^m - h^n)\deg(h^{n_0} - 1) = 0
\]

and recall that this holds for infinitely many pairs \((m, n)\).

Rewrite (3.4) as an equation in \( f^m \) and \( h^n \):
(3.5) \[ h^n(u + d) - f^m d = u \]

where \( u = (h^n_0 - 1)(f^m_0 - h^n_0), d = \deg(h^n_0 - 1). \)

Now \( \text{End}(E) \otimes \mathbb{Q} \) is either \( \mathbb{Q} \) or an imaginary quadratic field or a positive definite quaternion algebra over \( \mathbb{Q} \), all of which can be embedded into some positive definite quaternion algebra \( H \) over \( \mathbb{Q} \). View the equation (3.5) in \( H \).

If \( u \neq 0 \), then the equation \( h^n(u + d)u^{-1} - f^m du^{-1} = 1 \) has infinitely many solutions \( m, n > 0 \), a contradiction to Theorem 1.1 with \( a = b = 1, a' = (u + d)u^{-1}, b' = -du^{-1}, \Gamma_1 \) generated by \( h \), and \( \Gamma_2 \) generated by \( f \). Hence \( u = 0 \), so that \( (h^n_0 - 1)(f^m_0 - h^n_0) = 0. \)

But \( \deg h = \deg h > 1 \) implies \( h^n_0 - 1 \neq 0 \), so \( f^m_0 = h^n_0. \)

Finally, equation (3.2) implies \( (h^n_0 - 1)(R) = O \), so \( g^m_0 = h^n_0 = f^m_0 \) by (3.1).

**Remark 3.1.** In case \( k \) has characteristic 0, Corollary 1.5 is a consequence of [6, Theorem 1.4], and was also proved independently by Odesky and Zieve [12] using the classical result about unit equations. We thank Michael Zieve for suggesting the possibility of proving Corollary 1.5 by studying quaternion unit equations.

**Remark 3.2.** If \( f, g \) are endomorphisms of an elliptic curve \( E \) without translation, then Corollary 1.5 becomes trivial. For a proof, set \( P \in E(k) \) be a point in the intersection of orbits, and let \( n, m > 0 \) be such that \( f^n(P) = g^m(P). \) For any integer \( N \), we have \( Nf^n(P) = N\ x \)(\( P \)), so that \( (f^n - g^m)(NP) = O \) because \( f, g \) are endomorphisms of \( E. \) But \( P \) is of infinite order (otherwise the forward orbit of \( P \) under \( f \) would be finite), so ker(\( f^n - g^m \)) is an infinite group, and the only possibility is \( f^n - g^m = 0. \)

### 4. Proof of Theorem 1.2

Let \( \Delta \) be the set consisted of \((f, g) \in \Gamma_1 \times \Gamma_2 \) such that \( af'a' + bg'b' = 1 \) and \( |1 - af'a'| \neq |af'a'|. \) Then the goal of Theorem 1.2 is precisely to show that \( \Delta \) is a finite set.

By triangle inequality, every \((f, g) \in \Delta \) satisfies

\[ 0 < \left| |af'a'| - |bg'b'| \right| \leq 1 \]

We observe that since \( \Gamma_i \ (i = 1, 2) \) is a semigroup generated by finitely many elements with norms greater than 1, there are only finitely many elements of \( \Gamma_i \) of bounded norm.

In the rest of the proof, we will prove the claim that \( \{|f| : (f, g) \in \Delta\} \) is bounded. Given the claim, the set \( \{f : (f, g) \in \Delta\} \) is finite by the observation above. Since \( f \) determines \( g \) by \( g = b^{-1}(1 - af'a')b^{-1} \), there is only finitely many choices for \( g \) as well, and Theorem 1.2 is proved.
For contradiction, we assume that there is a solution \((f, g) \in \Delta\) with arbitrarily large \(|f|\). Using simple calculus (specifically, Lagrange’s mean value theorem), (4.1) implies

\[
(4.2) \quad 0 < \left| \log|af| - \log|bg| \right| \leq \frac{2}{|af|}
\]

for sufficiently large \(|f|\).

Let the semigroup \(\log|G_1|\) be generated by \(x_1, \ldots, x_t > 0\) and \(\log|G_2|\) by \(y_1, \ldots, y_u > 0\). Write \(\log|f| = m_1x_1 + \ldots + m_t x_t\), \(\log|g| = n_1y_1 + \ldots + n_u y_u\) for some nonnegative integers \(m_i, n_j\). Let \(c = \log|aa'/bb'|\). Then \(c, x_i, y_j\) are logarithms of real algebraic numbers, and (4.2) can be rewritten as

\[
(4.3) \quad 0 < |c + m_1x_1 + \ldots + m_t x_t - n_1y_1 - \ldots - n_u y_u| \leq \frac{2}{|a|e^{\max(x_1, \ldots, x_t, m_t)}}
\]

By Theorem 2.1 (Baker’s theorem), there are positive constants \(k, C\) such that

\[
0 < |a_0 + a_1 c + m_1 x_1 + \ldots + m_t x_t - n_1 y_1 - \ldots - n_u y_u| \leq k \max\{|a_0|, |a_1|, |m_i|, |n_j|\}^{-C}
\]

has no integer solution \((a_0, a_1, m_1, \ldots, m_t, n_1, \ldots, n_u)\). In particular, for \(a_0 = 0, a_1 = 1\) and \(m_i, n_j > 0\), the inequality

\[
(4.4) \quad 0 < |c + m_1 x_1 + \ldots + m_t x_t - n_1 y_1 - \ldots - n_u y_u| \leq k H^{-C}
\]

has no solution, where \(H = \max\{1, m_1, \ldots, m_t, n_1, \ldots, n_u\}\).

Our next goal is to bound the right hand side of (4.3) by a function of \(H\), in order to reach a contradiction with (4.4). Since \(x_i, y_j\) are positive, for \(|f|\) sufficiently large and satisfying (4.3), it is not hard to see that

\[
(4.5) \quad C_1 \max\{m_i\} < \max\{n_j\} < C_2 \max\{m_i\}
\]

for some \(C_1, C_2 > 0\) that does not depend on \(m_i, n_j\). For a proof, we note that

\[
\min\{x_i\} \max\{m_i\} \leq m_1 x_1 + \ldots + m_t x_t \leq t \max\{x_i\} \max\{m_i\}
\]

\[
\min\{y_j\} \max\{n_j\} \leq n_1 y_1 + \ldots + n_u y_u \leq u \max\{y_j\} \max\{n_j\}
\]

and (4.3) gives

\[
\frac{1}{2} (n_1 y_1 + \ldots + n_u y_u) < m_1 x_1 + \ldots + m_t x_t < 2(n_1 y_1 + \ldots + n_u y_u)
\]

for sufficiently large \(|f|\). Hence \(\max\{m_i\}, \max\{n_j\}, \log |f|\) and \(\log |g|\) are all “comparable” to each other in the sense of (4.5).

It follows that

\[
(4.6) \quad C_1 \max\{m_i\} < H \leq \max\{C_2, 1\} \max\{m_i\} =: C'_2 \max\{m_i\}
\]

where we denote \(C'_2 = \max\{C_2, 1\}\).

Now (4.3) implies

\[
(4.7) \quad 0 < |c + m_1 x_1 + \ldots + m_t x_t - n_1 y_1 - \ldots - n_u y_u| \leq \frac{2}{|a| e^{\min\{x_i\}} \max\{m_i\}} \leq \frac{2}{|a| e^{\min\{x_i\} H/C'_2}}
\]
for sufficiently large $|f|$ (or equivalently, $H$, by the “comparability” discussion above together with (4.6)).

Since the right hand side is an exponential decay in $H$, it will be less than $kH^{−c}$ for large $H$, which contradicts the lack of solution of (4.4). □

5. Proof of Theorem 1.3

First, we observe that the equation $|1−afa′| = |afa′|$ can be rewritten as $|a^{−1}a′−f| = |0 − f|$. In other words, $f$ lies on the perpendicular bisector of the line segment joining 0 and $a^{−1}a′−1$. It is a hyperplane not passing through the origin, given by

$$\{x \in \mathbb{H} : \langle a^{−1}a′−1, x \rangle = \frac{1}{2|aa′|^2}\}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{H}$.

Given the observation above, Theorem 1.3 follows from the following lemma:

Lemma 5.1. Let $\Gamma$ be a commutative semigroup of $\mathbb{H}^\times$ generated by finitely many algebraic elements of norms greater than 1, and $H$ be a hyperplane of $\mathbb{H}$ defined by

$$H = \{x \in \mathbb{H} : \Theta(x) = 1\}$$

where $\Theta : \mathbb{H} \to \mathbb{R}$ is a nonzero $\mathbb{R}$-linear functional that maps $\mathbb{H}_a$ into $\mathbb{Q} \cap \mathbb{R}$. Then $\Gamma \cap H$ is finite.

Proof of lemma. Since $\Gamma$ is commutative, it lies in a subalgebra in $\mathbb{H}$ that is isomorphic to $\mathbb{C}$. We may assume instead that $\Gamma$ is a semigroup generated by $g_1, ..., g_s \in \overline{\mathbb{Q}}^\times \subseteq \mathbb{C}$ such that $|g_j| > 1$, and $\Theta : \mathbb{C} \to \mathbb{R}$ is an $\mathbb{R}$-linear functional that maps $\overline{\mathbb{Q}}$ into $\mathbb{Q} \cap \mathbb{R}$. We need to show that $\Theta(f) = 1$ has only finitely many solutions $f \in \Gamma$.

There is no question to ask if $\Theta = 0$. In the case $\Theta \neq 0$, we may assume $\Theta$ is given by $\langle v, \cdot \rangle$ for some nonzero vector $v \in \overline{\mathbb{Q}}$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{C}$ with $\{1, i\}$ being an orthonormal basis. By rescaling, we may assume $|v| = 1$, but the equation $\Theta(f) = 1$ will become

$$\langle v, f \rangle = M$$

for some real algebraic number $M > 0$.

Write $g_j = r_jv_j$ with $r_j > 1$ and $v_j = e^{i\theta_j}$ on the unit circle, with $0 \leq \theta_j < 2\pi$. Also write $v = e^{i\theta}$ with $0 \leq \theta < 2\pi$. For $f = g_1^{n_1}...g_s^{n_s}$, the equation (5.1) becomes

$$\langle v, e^{i(n_1\theta_1+...+n_s\theta_s)} \rangle = Mr_1^{−n_1}...r_s^{−n_s}$$

(5.2)

The left hand side involves the inner product of two unit vectors, so its value is $\cos((n_1\theta_1 + ... + n_s\theta_s) − \theta)$. When $n_i$ are sufficiently large, the right hand side of 5.2 is small. But $|\cos((n_1\theta_1 + ... + n_s\theta_s) − \theta)|$ is approximately the closest distance from $(n_1\theta_1 + ... + n_s\theta_s) − \theta$ to $(m + 1/2)\pi$ for integer
If (5.2) is satisfied by infinitely many \((n_j)\)'s, then for sufficiently large solutions \((n_j)\), we have

\[
0 < \left| \left( \frac{\pi}{2} + \theta \right) + m\pi - (n_1\theta_1 + \ldots + n_s\theta_s) \right| < 2Mr_1^{-n_1} \ldots r_s^{-n_s}
\]

for some \(m \in \mathbb{Z}\).

By assumption, \(v, v_j\) are algebraic numbers, so \(\lambda := i(\frac{1}{2}\pi + \theta), \mu = i\pi\) and \(\lambda_j = i\theta_j\) are logarithms of algebraic numbers. By Theorem 2.1, there are constants \(k, C > 0\) such that the inequality

\[
0 < \left| \left( \frac{\pi}{2} + \theta \right) + m\pi - (n_1\theta_1 + \ldots + n_s\theta_s) \right| < kH^{-C}
\]

has no solution for \(m, n_j \in \mathbb{Z}, n_j \geq 0\), where

\[
H = \max\{1, |m|, n_j\}
\]

But for solutions of (5.3) with \(n_j\) large, \(m\pi\) must be close to \(n_1\theta_1 + \ldots + n_s\theta_s - (\frac{1}{2}\pi + \theta)\). Noting that

\[
n_1\theta_1 + \ldots + n_s\theta_s \leq s \max\{\theta_j\} \max\{n_j\},
\]

we have

\[
|m| \leq C' \max\{n_j\}
\]

for some constant \(C'\), and thus

\[
\max\{n_j\} \leq H = \max\{n_j, |m|\} \leq \max\{1, C'\} \max\{n_j\}
\]

It follows from (5.4) that for some constant \(k' > 0\),

\[
0 < \left| \left( \frac{\pi}{2} + \theta \right) + m\pi - (n_1\theta_1 + \ldots + n_s\theta_s) \right| < k' \max\{n_j\}^{-C}
\]

has no solution for \(m, n_j \in \mathbb{Z}, n_j \geq 0\). But for \((n_j)\) large, \(2Mr_1^{-n_1} \ldots r_s^{-n_s} < k' \max\{n_j\}^{-C}\), yielding a contradiction with (5.3).

\(\square\)

6. Future Work

We were able to arrive at the main theorem using the archimedean norm only. If we can furthermore use some version of \(p\)-adic norm on the division algebra \(A\), we can vastly improve the result by applying K. Yu’s theorem about \(p\)-adic logarithms in [14]. One possible proposal for a \(p\)-adic norm is to use the reduced norm of a division algebra over \(\mathbb{Q}_p\), which only works if \(A \otimes \mathbb{Q}_p\) is still a division algebra. Unfortunately, for each given \(A\), this only holds for finitely many \(p\).

Theorem 1.2 is potentially useful for more cases than in Theorem 1.1. For example, one can explore the analogue of Theorem 1.3 in the case where \(\Gamma\) has two or more noncommutative generators, and then apply Theorem 1.2. Even if \(\Gamma\) is replaced by its subset \(\{f_1^{n_1}f_2^{n_2} : n_1, n_2 \geq 0\}\), where \(f_1, f_2\) are noncommutative generators with norms greater than 1, the analogue of Theorem 1.3 remains open.
The following example shows that we should only consider Conjecture 1.4 where $A$ is a division algebra.

**Example 6.1.** Take $A = M_2(\mathbb{Q})$, the algebra of $2 \times 2$ matrices over $\mathbb{Q}$. Then the multiplicative semigroup generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is

$$
\Gamma := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z}, n \geq 0 \right\}.
$$

The equation $2f - g = 1_A$ has infinitely many solutions $f, g \in \Gamma$, namely all $(f, g)$ with $f \in \Gamma$ and $g = f^2$.

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