Generation of circular polarization in CMB radiation via nonlinear photon-photon interaction

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Standard cosmological models do predict a measurable amount of anisotropies in the intensity and linear polarization of the Cosmic Microwave Background radiation (CMB) via Thomson scattering, even though these theoretical models do not predict circular polarization for CMB radiation. In other hand, the circular polarization of CMB has not been excluded in observational evidences. Here we estimate the circular polarization power spectrum $C_{l}^{V(S)}$ in CMB radiation due to Compton scattering and non-linear photon-photon forward scattering via Euler-Heisenberg Effective Lagrangian.

We have estimated the average value of circular power spectrum is $1(l+1)C_{l}^{V(S)}/(2\pi) \sim 10^{-4} \mu K^{2}$ for $l \sim 300$ at present time which is smaller than recently reported data (SPIDER collaboration) but in the range of the future achievable experimental data. We also show that the generation of B-mode polarization for CMB photons in the presence of the primordial scalar perturbation via Euler-Heisenberg interaction is possible however this contribution for B-mode polarization is not remarkable.

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I. INTRODUCTION

Photon-matter interactions can convert or generate the polarization states of photons in different situation such as Faraday rotation, Faraday conversion and so on. In some special cases, the measurement of circular polarization contribution provides very important tools to better understand universe. In standard scenario of cosmology, CMB anisotropies are partially linearly polarized \[ \text{[1–6]} \] while the generation of circular polarization is ignored, because there is no a notable mechanism to generate circular polarization in the recombination epoch. Note Compton (Thomson) scattering, as most important interaction of CMB radiation, cannot generate the circular polarization \[ \text{[3]} \].

In other hand, the circular polarization of CMB has not been excluded in observational evidences. For example, recently SPIDER collaboration are made maps of approximately 10% of the sky with degree-scale angular resolution in 95 and 150 GHz observing bands. Data of SPIDER group have been analyzed in \[ \text{[7]} \] and a new upper limit on CMB scattering, as most important interaction of CMB radiation, cannot generate the circular polarization \[ \text{[6]} \].

In the case of theoretical models, there are several mechanisms, almost considering new physics interactions, which discuss the possibility of the generation of circular polarization in the CMB. For instances, the conversion of the existing linear polarization into circular one in the presence of external magnetic fields of galaxy clusters \[ \text{[14]} \], the relativistic plasma remnants \[ \text{[17]} \] and magnetic fields in the primordial universe \[ \text{[16,18]} \] is discussed. Forward scattering of CMB radiation from the cosmic neutrino background \[ \text{[21]} \], and photon-photon interactions in neutral hydrogen \[ \text{[21]} \] have also been shown as potentially mechanisms for the generation of CMB circular polarization. There are some mechanisms which are postulated extensions to QED such as Lorentz-invariance violating operators \[ \text{[16,19,22]} \], axion-like pseudoscalar particles \[ \text{[23]} \], and non-linear photons interactions (through effective Euler-Heisenberg Lagrangian) \[ \text{[24]} \]. In \[ \text{[27]} \], the production of primordial circular polarization in axion inflation coupled to fermions and gauge fields, with special attention paid to reheating, have been studied. Also see a brief review of some of the mentioned mechanisms in \[ \text{[26]} \].

In this work, we focus on the generation of circular polarization due to nonlinear photon-photon interaction (via Euler-Heisenberg Lagrangian). Of course we should mention that Faraday conversion phase shift \( \Delta \phi_{\text{FC}} \) due to Euler-Heisenberg Lagrangian for CMB radiation has been estimated in \[ \text{[24]} \]. It is worthwhile to mention that one can calculate \( \Delta \phi_{\text{FC}} \) from below equation [see more detail in \[ \text{[14,29]} \]]

\[
\dot{V} = 2U \frac{d}{dt}(\Delta \phi_{\text{FC}}),
\]

where \( U \) and \( V \) are Stokes parameters which describe linear and circular polarizations respectively. Note \( \Delta \phi_{\text{FC}} \) reported in \[ \text{[24]} \] is not suitable quantity to compare with experimental data which usually reported by circular polarization power spectrum \( C^V_l \). So the main purpose of our work is to calculate \( C^V_l \) via Euler-Heisenberg effective interactions and make a comparison with recently data reported by SPIDER collaboration group.

We start by a brief discussion on Stokes parameters and their definitions in terms of density matrix elements. Then we calculate time evolution of those parameters by Euler-Heisenberg consideration. In the next two sections we solve them by some estimations to calculate dominant contribution terms. This contributions come from total intensity of CMB photon contribution in comparison with linear and circular polarizations. Finally in the last section, we compute the power spectrum and B-mode spectrum of CMB photons which are generated by Euler-Heisenberg effective Lagrangian.

II. POLARIZATION AND STOKES PARAMETERS

An ensemble of photons in a completely general mixed states is given by a normalized density matrix \( \rho_{ij} = \langle |\varepsilon_i\rangle \langle \varepsilon_j|/\text{tr} \rho \rangle \), where in the quantum mechanics description, an arbitrary polarized state of a photon with energy \( \langle |k|\rangle = |k|^2 \) propagating in the \( \hat{z} \)-direction is written as

\[
|\varepsilon\rangle = a_1 \exp(i\theta_1)|\varepsilon_1\rangle + a_2 \exp(i\theta_2)|\varepsilon_2\rangle,
\]

where \( |\varepsilon_1\rangle \) and \( |\varepsilon_2\rangle \) represent the polarization states in the \( \hat{x} \)- and \( \hat{y} \)-directions. Then the \( 2 \times 2 \) density matrix \( \rho \) of photon polarization states are given as

\[
\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix},
\]

where \( I = 1 \) is the identity matrix and \( Q = i/2 \) is the Pauli matrix.
where $I, Q, U$ and $V$ are Stokes parameters, so that $I$-parameter is the total intensity of radiation, $Q$- and $U$-parameters indicate the intensity of linear polarization of radiation, and $V$-parameter determines the intensity of circular polarization of radiation. Note $I$ and $V$ are independently physical observable quantities of the coordinate system, while $Q$- and $U$-parameters depend on the orientation of the selected coordinate system. Linear polarization can also be characterized through a vector parameter $\mathbf{P}$ which describe by $|\mathbf{P}| \equiv \sqrt{Q^2 + U^2}$ and $\alpha = \frac{1}{2} \tan^{-1} \frac{U}{Q}$.

The time evolution of each Stokes parameter can be yielded through the Quantum Boltzmann equation. To do this issue, ones can play with each polarization state of the CMB radiation as the phase space distribution function $\chi$ which can generally obey from the classical Boltzmann equation

$$\frac{d}{dt} \chi = C(\chi). \quad \text{(4)}$$

The left hand side of above equation is known as the Liouville term (containing all gravitational effects), while the right hand side one contains all possible collision terms. By considering the CMB interactions on the right hand side of Boltzmann equation, we can calculate the time evolution of the each polarization state of the photons. In the next section, we consider non-linear photon-photon forward scattering via the Euler-Heisenberg Hamiltonian to compute the time evolution of each polarization sates.

### III. THE EULER-HEISENBERG LAGRANGIAN AND THE PHOTONS POLARIZATIONS

The time evolution of $\rho_{ij}(k)$s as well as Stokes parameters are given by [see 6 for more detail],

$$(2\pi)^3 \delta^3(0)(2k^0) \frac{d}{dt} \rho_{ij}(k) = i \langle [H_1^0(t); D^0_{ij}(k)] \rangle - \frac{1}{2} \int dt \langle [H_1^0(t); [H_1^0(0); D^0_{ij}(k)]] \rangle, \quad \text{(5)}$$

where $H_1^0(t)$ is the leading order of the photon-photon interacting via Euler-Hiesenberg Hamiltonian. The first term on the right-hand side of above equation is called forward scattering term, and the second one is a higher order collision term which is in order of the ordinary cross section of photon-photon scattering. The Euler-Heisenberg Lagrangian is a low energy effectivelagrangian describing multiple photon interactions. The first order of photon-photon interacting hamiltonian via Euler-Heisenberg Lagrangian can be written as [30, 31]

$$H_1^0(t) = -\frac{\alpha^2}{90m^4} \int d^3x \left[ (F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{5} (F_{\mu\nu} F^{\mu\nu})^2 \right], \quad \text{(6)}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of electromagnetic field and $\hat{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, in which $\epsilon^{\mu\nu\alpha\beta}$ is an antisymmetric tensor of rank four [for example see 32 and 33]. Note

$$\dot{A}_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} (a_r(k) \epsilon_{r\mu}(k)e^{-ikx} + a^\dagger_r(k) \epsilon^{*}_{r\mu}(k)e^{ikx}). \quad \text{(7)}$$

where creation $a_r^\dagger$ and annihilation $a_r$ operators satisfy the canonical commutation relation as

$$[a_r(k), a^\dagger_{r'}(k')] = (2\pi)^3 2k^0 \delta_{rr'} \delta^{(3)}(k - k'). \quad \text{(8)}$$

We only compute the first order of Quantum Boltzmann Equation i.e. the first term in RHS of the Eq. (6), and neglect the second term which is in order of $\alpha^4$. In principle when first term doesn’t have any result, in any special theory, one can try to compute the second term. It is worthwhile to mention that the contribution of $(F_{\mu\nu} F^{\mu\nu})^2$ for CMB polarization is given in 24, however they have just calculated Faraday Conversion phase shift. Here we will consider both term of Euler-Heisenberg Lagrangian. After tedious but straightforward calculation, using Eq. (6), the time-evolutions of Stokes parameters Eq. (3) are obtained (find details in Appendix). First we start with $I$-parameter

$$\dot{I}(k) = 0, \quad \text{(9)}$$

$\dot{I}(k) = 0$ implies, for each ensemble of photons like CMB, the total intensity $I$ in any direction $\hat{k}$ is constant and does not change from Euler-Heisenberg forward scattering. The above result for intensity $I$ is expected, because the forward scattering cannot change momenta of photons which is necessary condition to change intensity in any direction. Note for the rest of paper, we do not consider the terms with linearly dependence of $\rho_{ij}$ on the right side.
of above equations, because we are interested in photon-photon forward scattering. The time evolution of linear and circular polarization parameters are given as following

\[
\dot{Q}(\mathbf{k}) = \frac{16\alpha^2}{45m^4k^6} V(\mathbf{k}) \int \frac{d^3p}{(2\pi)^32p^0} (p^0k^0)^2 \left[ f_1(\hat{p}, \hat{k}) U(\mathbf{p}) \right],
\]

(10)

\[
\dot{U}(\mathbf{k}) = \frac{8\alpha^2}{45m^4k^6} V(\mathbf{k}) \int \frac{d^3p}{(2\pi)^32p^0} (p^0k^0)^2 \left[ f_1(\hat{p}, \hat{k}) I(\mathbf{p}) \right],
\]

(11)

\[
\dot{V}(\mathbf{k}) = \frac{8\alpha^2}{45m^4k^6} U(\mathbf{k}) \int \frac{d^3p}{(2\pi)^32p^0} (p^0k^0)^2 \left[ f_2(\hat{p}, \hat{k}) I(\mathbf{p}) \right]
\]

(12)

where \( f_1 \)s are given in Appendix. Note in the case of CMB radiation, \( I \) can be total intensity of CMB or CMB thermal anisotropy (depending of angular dependence of \( f_1 \)s) while the contribution of \( Q, U \) and \( V \) are about or less than \( %10 \) of total CMB thermal anisotropy. As a result, to consider dominated contribution in our calculations Eqs. (10-12), we neglect terms in second order of \( Q, U \) and \( V \). As Eqs. (10-12) show, the initial circular polarization of an ensemble of photon \( V(\mathbf{k}) \) can be converted to linear one \( U(\mathbf{k}), Q(\mathbf{k}) \) and inverse due to Euler-Hisenberg interactions. To further calculate angular integrals most conveniently, we introduce the momentum and polarization vectors of incoming photons as follow

\[
\dot{\mathbf{k}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),
\]

\[
\dot{\hat{\mathbf{e}}}_1(\mathbf{p}) = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta),
\]

\[
\dot{\hat{\mathbf{e}}}_2(\mathbf{p}) = (-\sin \phi, \cos \phi, 0).
\]

(13)

The exactly same definition are correct for momentum and polarization vectors of target photons (denoted by \( \mathbf{p} \) and \( \hat{\mathbf{e}}_i(\mathbf{p}) \)) just with \( \theta \rightarrow \theta' \) and \( \phi \rightarrow \phi' \). The angular integrals in Eqs. (10-12) must be done over \( \theta' \) and \( \phi' \). As momentum and polarization vectors of photons are defined in spherical coordinate, one can expand all variables and Stokes parameters in terms of spherical harmonics \( Y_l^m \) to make angular integrals easily, so we have

\[
I(\mathbf{p}) = \sum_{l' m'} I_{l' m'}(\mathbf{p}) Y_{l'}^{m'}(\theta', \phi'),
\]

\[
(Q \pm iU)(\mathbf{p}) = \sum_{l' m'} (Q \pm iU)_{l' m'}(\mathbf{p}) Y_{l'}^{m'}(\theta', \phi'),
\]

(14)

\[
V(\mathbf{p}) = \sum_{l' m'} V_{l' m'}(\mathbf{p}) Y_{l'}^{m'}(\theta', \phi').
\]

Also we can use above equations to expand \( I(\mathbf{k}), Q(\mathbf{k}), U(\mathbf{k}) \) and \( V(\mathbf{k}) \) in terms of spherical harmonics by replacing \( \theta \rightarrow \theta' \), \( \phi \rightarrow \phi' \), \( l' \rightarrow l \) and \( m' \rightarrow m \). So by considering the time evolution of Stokes parameters given in Eqs. (10-12), using expansions in Eq. (14) and adding the Compton scattering contributions to Euler-Hisenberg contributions, we have

\[
\frac{dI}{dt} = C_{eY},
\]

\[
\frac{d}{dt}(Q \pm iU) = C_{eY} \mp i\kappa \mp \kappa V,
\]

(15)

\[
\frac{dV}{dt} = C_{eY} + \kappa U,
\]

where \( C_{eY}, C_{eY}^\pm \) and \( C_{eY}^\mp \) denote contributions of Compton scattering which their expressions could be found in [Ref. 34, 35]. The Euler-Hisenberg contribution coefficients are given as following

\[
\kappa = \frac{8\alpha^2}{45m^4k^6} \int \frac{p^2 dp dQ}{(2\pi)^32p^0} (p^0k^0)^2 \left[ (-2iU(\mathbf{p}) \pm I(\mathbf{p})) f_1(\hat{p}, \hat{k}) \right]
\]

(16)

\[
\kappa = \frac{8\alpha^2}{45m^4k^6} \int \frac{p^2 dp dU}{(2\pi)^32p^0} (p^0k^0)^2 \left[ f_2(\hat{p}, \hat{k}) I(\mathbf{p}) \right].
\]

(17)
As shown in Eq. 10, $\kappa_{\pm}$ is divided to two term which are proportional to $U(p)$ and $I(p)$. According to the earlier mentioned argument, to consider dominant contributions of Euler-Heisenberg effective Lagrangian in CMB power spectrum, we can neglect the term including $U(p)$. Then

$$\dot{\kappa}_{\pm} = \pm \frac{1}{15\pi} \sigma_T \frac{k}{m_e} \frac{l_0}{m_e} \left( \int \frac{d^3p}{(2\pi)^3} \rho f_1(\hat{p}, \hat{k}) \sum_{lm} \frac{I_{lm}(p)}{I_0} \right),$$

$$= \pm \tilde{\kappa} \left( I_0^0 + \int \frac{d^3p}{(2\pi)^3} \rho \tilde{f}_1(\hat{p}, \hat{k}) \sum_{lm} \frac{I_{lm}(p)}{I_0} \right)$$

(18)

$$\dot{\kappa}_U = \frac{1}{15\pi} \sigma_T \frac{k}{m_e} \frac{l_0}{m_e} \left( \int \frac{d^3p}{(2\pi)^3} \rho \tilde{f}_2(\hat{p}, \hat{k}) \sum_{lm} \frac{I_{lm}(p)}{I_0} \right),$$

$$= \tilde{\kappa} \left( I_0^0 + \int \frac{d^3p}{(2\pi)^3} \rho \tilde{f}_2(\hat{p}, \hat{k}) \sum_{lm} \frac{I_{lm}(p)}{I_0} \right),$$

(19)

where $\tilde{\kappa} = \frac{1}{15\pi} \sigma_T \frac{k}{m_e} \frac{l_0}{m_e}$ and here we separate $f_i(\hat{p}, \hat{k}) = f_i^0 + \tilde{f}_i(\hat{p}, \hat{k})$, note $f_i^0$ is constant part of $f_i(\hat{p}, \hat{k})$ and also

$$\int \frac{d^3p}{(2\pi)^3} \rho I(p) = I_0(\bar{p}) \simeq \bar{p} \gamma,$$

(20)

and $\bar{p} = |p|$ is the average value of the momentum of target (CMB-photons). Be ware in above equations, the term including $\tilde{f}_i(\hat{p}, \hat{k})$ is in the order of CMB temperature anisotropy $\sim \frac{\delta T}{T}$ which several order of magnitude smaller than the term including $f_i^0$. So it is reasonable to ignore the term including $\tilde{f}_i(\hat{p}, \hat{k})$ for the rest of our calculation. As a result, by considering non-linear photon-photon interaction, a linear polarization converts to circular one while crossing through an isotopic unpolarized medium beam $I_0$.

To understand the above results, we can assume that linearly polarized CMB photons encounter by an isotopic background magnetic and electric fields when they cross through the unpolarized beam. By purposing the mentioned point, we can rewrite Euler-Heisenberg Hamiltonian by replacing $F_{\mu\nu} \rightarrow B_{\mu\nu} + F_{\mu\nu}$ where $B_{\mu\nu}$ indicates background fields [for example see 23].

$$H_{7}^0(t) = \frac{\alpha^2}{90m_e^2} \int d^3x \left( \left[(F_{\mu\nu} + B_{\mu\nu})(F^{\mu\nu} + B^{\mu\nu})\right]^2 + \frac{7}{4} \left[(F_{\mu\nu} + B_{\mu\nu})(\tilde{F}^{\mu\nu} + \tilde{B}^{\mu\nu})\right]^2 \right).$$

(21)

Note in above equation, we just need terms with two $F_{\mu\nu}$ while terms including $(B_{\mu\nu}B^{\mu\nu})(F_{\mu\nu}F^{\mu\nu})$ and $(B_{\mu\nu}B^{\mu\nu})(F_{\mu\nu}\tilde{F}^{\mu\nu})$ do not affect in our results. So by using Eq. 7,.

$$H_{7}^0(t) = \frac{4\alpha^2}{90m_e^4} \int \frac{d^3p}{(2\pi)^3(2p')}^2 \sum_{ss'} \hat{a}_s(p)\hat{a}_{s'}(p)[p^\mu B_{\mu\nu}e_s^\nu p^\lambda B_{\lambda\rho}e_{s'}^\rho - \frac{7}{4} p^\mu \tilde{B}_{\mu\nu}e_s^\nu p^\lambda \tilde{B}_{\lambda\rho}e_{s'}^\rho].$$

(22)

and by substituting below equations

$$p^\mu B_{\mu\nu}e_s^\nu = \tilde{B}.(\tilde{p} \times \tilde{e}_s) + p^0 \tilde{E}.\tilde{e}_s,$$

$$p^\mu \tilde{B}_{\mu\nu}e_s^\nu = 2\tilde{E}.(\tilde{p} \times \tilde{e}_s) + 2p^0 \tilde{B}.\tilde{e}_s,$$

(23)

in Eq. 22, we obtain

$$H_{7}^0(t) = \frac{4\alpha^2}{90m_e^4} \int \frac{d^3p}{(2\pi)^3(2p')}^2 \sum_{ss'} \hat{a}_s(p)\hat{a}_{s'}(p) \left( [(\tilde{B}.(\tilde{p} \times \tilde{e}_s) + p^0 \tilde{E}.\tilde{e}_s)(\tilde{B}.(\tilde{p} \times \tilde{e}_{s'}) + p^0 \tilde{E}.\tilde{e}_{s'})] - \frac{7}{4} [(\tilde{E}.(\tilde{p} \times \tilde{e}_s) + p^0 \tilde{B}.\tilde{e}_s)(\tilde{E}.(\tilde{p} \times \tilde{e}_{s'}) + p^0 \tilde{B}.\tilde{e}_{s'})] \right).$$

(24)

At the end, we have used Eqs. 11 and 21 to obtain the time evolution of Stokes parameters, here we just discuss $V$-parameter

$$\dot{V}(\hat{k}) = \frac{4\alpha^2k_0^0}{90m_e^2} \left[ \tilde{g} Q(\hat{k}) + \tilde{f} U(\hat{k}) \right].$$

(25)
where
\[ \tilde{g} = 2\left(\tilde{B} \cdot (\dot{k} \times \dot{e}_2) \tilde{B}(\dot{k} \times \dot{e}_1) + \tilde{E} \cdot \dot{e}_2 \tilde{B} \cdot (\dot{k} \times \dot{e}_1) + \tilde{E} \cdot \dot{e}_1 \tilde{B} \cdot (\dot{k} \times \dot{e}_2) + \tilde{E} \cdot \dot{e}_1 \tilde{E} \cdot \dot{e}_2\right) \]
\[ + 14\left(\tilde{E} \cdot (\dot{k} \times \dot{e}_2) \tilde{E}(\dot{k} \times \dot{e}_1) + \tilde{B} \cdot \dot{e}_2 \tilde{B} \cdot (\dot{k} \times \dot{e}_1) + \tilde{B} \cdot \dot{e}_1 \tilde{E} \cdot (\dot{k} \times \dot{e}_2) + \tilde{B} \cdot \dot{e}_1 \tilde{B} \cdot \dot{e}_2\right) \]
(26)
and
\[ \tilde{f} = k^0\left[6((\tilde{B} \cdot \dot{e}_1)^2 - (\tilde{B} \cdot \dot{e}_2)^2) + 6((\tilde{E} \cdot \dot{e}_2)^2 - (\tilde{E} \cdot \dot{e}_1)^2) + 16((\tilde{E} \cdot \dot{e}_1)(\tilde{B} \cdot \dot{e}_2) - (\tilde{B} \cdot \dot{e}_1)(\tilde{E} \cdot \dot{e}_2))\right]. \]
(27)
Now we are ready to check the results discussed in Eq. (19). Using Eqs. (13) and considering a random direction for electric fields \( \tilde{E} = E(\sin \theta_E \cos \phi_E, \sin \theta_E \sin \phi_E, \cos \theta_E) \), we will rewrite the average value of \( \langle \tilde{f} \rangle \) and \( \langle \tilde{g} \rangle \) as following
\[ \langle \tilde{f} \rangle = 3/4(1 - \cos 2\theta) < E^2 > + < \dot{f}_1(\theta_E, \phi_E) > \]
\[ \langle \tilde{g} \rangle = 3/4(1 - \cos 2\theta) < E^2 > \]
(28)

Note in above equation \( 3/4(1 - \cos 2\theta) < E^2 > \) is independent of the direction of electric fields as well as the polarizations of radiation \( < E^2 > \). But \( \langle \tilde{g} \rangle \) does not include a term which can be independent from the direction of electric fields. In the simple word, \( \langle \tilde{f} \rangle \) has a contribution from isotropic unpolarized CMB radiation which comes from the nature of non-linear interaction between CMB photons themselves via Euler-Heisenberg Hamiltonian.

**IV. THE TIME EVOLUTION OF CMB POLARIZATIONS DUE TO EULER-HEISENBERG LAGRANGIAN AND COMPTON SCATTERING**

In present section, we consider our rest calculation in the presence of the primordial scalar perturbations indicating by \( (S) \) which we expand these perturbations in the Fourier modes characterized by a wave number \( K \). For each given wave number \( K \), it is useful to select a coordinate system with \( K \parallel \hat{z} \) and \( (\hat{e}_1, \hat{e}_2) = (\hat{e}_\theta, \hat{e}_\phi) \). Temperature anisotropy \( \Delta T(S) \), linear polarizations \( \Delta Q(S) \) and \( \Delta U(S) \) and circular polarization \( \Delta V(S) \) of the CMB radiation can be expanded in an appropriate spin-weighted basis as following [34]
\[ \Delta T(S)(K, k, \tau) = \sum_{\ell m} a_{\ell m}(\tau, K) Y_{\ell m}(n), \]
\[ \Delta Q(S)(K, k, \tau) = \sum_{\ell m} a_{\pm 2, \ell m}(\tau, K) Y_{\ell m}(n), \]
\[ \Delta U(S)(K, k, \tau) = \sum_{\ell m} a_{V, \ell m}(\tau, K) Y_{\ell m}(n), \]
(29)
(30)
(31)
where we define
\[ \Delta T(S)(K, k, \tau) = \left(4k \frac{\partial I_0}{\partial k}\right)^{-1} \Delta I(S)(K, k, t), \quad \Delta Q(S) = \left(4k \frac{\partial I_0}{\partial k}\right)^{-1} (Q(S) \pm iU(S)). \]
(32)

As usual, one can transfer the CMB temperature and polarizations \( \Delta T, P, V(\eta, K, \mu) \) in the conformal time \( \eta \) and describe them by multi-pole moments as following
\[ \Delta T, P, V(\eta, K, \mu) = \sum_{l=0}^{\infty} (2l + 1)(-i)^l \Delta_{l, P, V}(\eta, K) P(\mu) \]
(33)
where \( \mu = \hat{n} \cdot \hat{K} = \cos \theta \), the \( \theta \) is angle between the CMB photon direction \( \hat{n} = k / |k| \) and the wave vectors \( K \), and \( P(\mu) \) is the Legendre polynomial of rank \( l \). Here we should define left hand sides of Eq. (33) \( \frac{d}{d\eta} \) to take into account space-time structure and gravitational effects such as red-shift and so on. For each plane wave, each scattering and interaction can be described as the transport through a plane parallel medium [36, 37], and finally Boltzmann equations in the presence of the primordial scalar perturbations are given as
\[ \frac{d}{d\eta} \Delta T(S) + iK \mu \Delta T(S) + 4[\dot{\omega} - iK \mu \varphi] = \varphi \gamma \left[ - \Delta T(S) + \Delta U(S) + i\mu \nu_b + \frac{1}{2} P_2(\mu) \right] \]
\[ \frac{d}{d\eta} \Delta Q(S) + iK \mu \Delta Q(S) = \varphi \gamma \left[ - \Delta Q(S) + \frac{1}{2} [1 + P_2(\mu)] \right] + i a(\eta) \frac{k}{2} \Delta Q(S) \]
\[ \frac{d}{d\eta} \Delta U(S) + iK \mu \Delta U(S) = - \varphi \gamma \left[ \Delta U(S) - \frac{3}{2} \mu \Delta V(S) \right] + \frac{i}{2} k \frac{f_2}{2} (\Delta U(S) - \Delta P(S)) \]
\[ \frac{d}{d\eta} \Delta V(S) + iK \mu \Delta V(S) = - \varphi \gamma \left[ \Delta V(S) + \frac{3}{2} \mu \Delta V(S) \right] + \frac{i}{2} k \frac{f_2}{2} (\Delta U(S) - \Delta P(S)) \]
(34)
(35)
(36)
where $\dot{\tau}_{c\gamma} \equiv \frac{d\tau_{c\gamma}}{d\eta}$, which $\tau_{c\gamma}$ is Compton scattering optical depth, $a(\eta)$ is normalized scale factor and $\Pi \equiv \Delta_{l}^{2(S)} + \Delta_{p}^{2(S)} + \Delta_{0(S)}^{2(S)}$.

The values of $\Delta_{p}^{\pm(S)}(\hat{n})$ and $\Delta_{V}^{(S)}(\hat{n})$ at the present time $\eta_{0}$ and the direction $\hat{n}$ can be obtained in following general form by integrating of the Boltzmann equation (Eq’s. (34-36)) along the line of sight [34] and summing over all the Fourier modes $K$ as follows

$$\Delta_{p}^{\pm(S)}(\hat{n}) = \int d^{3}K \xi(K) e^{\pm 2i\phi K \cdot \eta_{0}} \Delta_{p}^{\pm(S)}(K, k, \eta_{0}),$$

(37)

$$\Delta_{V}^{(S)}(\hat{n}) = \int d^{3}K \xi(K) \Delta_{V}^{(S)}(K, k, \eta_{0}),$$

(38)

where $\phi_{K,n}$ is the angle needed to rotate the $K$ and $\hat{n}$ dependent basis to a fixed frame in the sky, $\xi(K)$ is a random variable using to characterize the initial amplitude of each primordial scalar perturbations mode, and also the values of $\Delta_{p}^{\pm(S)}(K, k, \eta_{0})$ and $\Delta_{V}^{(S)}(K, k, \eta_{0})$ are given as

$$\Delta_{p}^{\pm(S)}(K, \mu, \eta_{0}) \approx \int_{0}^{\eta_{0}} d\eta \dot{\tau}_{c\gamma} e^{ix(1 - \mu^{2})}\left[ \frac{3}{4} \mu \Delta_{V}^{(S)} - i \int_{0}^{\eta} \frac{k}{\dot{\tau}_{c\gamma}} \Delta_{p}^{(S)} \right],$$

(39)

and

$$\Delta_{V}^{(S)}(K, \mu, \eta_{0}) \approx \int_{0}^{\eta_{0}} d\eta \dot{\tau}_{c\gamma} e^{ix(1 - \mu^{2})}\left[ \frac{3}{4} \mu \Delta_{V}^{(S)} - i \int_{0}^{\eta} \frac{k}{\dot{\tau}_{c\gamma}} \Delta_{p}^{(S)} \right],$$

(40)

in which $x = K(\eta_{0} - \eta)$, $f_{1,2}(\dot{\tau}, \ddot{\tau})$ are defined in [17, 28] and

$$\Delta_{V}^{(S)}(K, \mu, \eta) = \int_{0}^{\eta} d\eta' \dot{\tau}_{c\gamma} e^{ix(1 - \mu^{2})}\left[ \frac{3}{4} \mu \Delta_{V}^{(S)} - i \int_{0}^{\eta} \frac{k}{\dot{\tau}_{c\gamma}} \Delta_{p}^{(S)} \right].$$

(41)

The differential optical depth $\dot{\tau}_{c\gamma}(\eta)$ and total optical depth $\tau_{c\gamma}(\eta)$ due to the Thomson scattering at time $\eta$ are defined as

$$\dot{\tau}_{c\gamma} = a n_{e} \sigma T, \quad \tau_{c\gamma}(\eta) = \int_{0}^{\eta} \dot{\tau}_{c\gamma}(\eta) d\eta.$$

(42)

V. THE CONTRIBUTION OF EULER-HEISENBERG INTERACTION FOR THE CIRCULAR POWER SPECTRUM OF CMB

In the preceding section, we have prepared all instruments to calculate different power spectra $C_{l}^{X(S)}$s of CMB radiation due to Compton scattering and photon-photon forward scattering via Euler-Heisenberg interaction. So the power spectrum $C_{l}^{X(S)}$ in the presence of primordial scalar perturbation (indicating by $(S)$) is given as

$$C_{l}^{X(S)} = \frac{1}{2l + 1} \sum_{m} \left\langle a_{X,lm}^* a_{X,lm} \right\rangle, \quad X = \{ I, E, B, V \},$$

(43)

where

$$a_{E,lm} = - (a_{2,lm} + a_{-2,lm})/2,$$

(44)

$$a_{B,lm} = i(a_{2,lm} - a_{-2,lm})/2,$$

(45)

$$a_{V,lm} = \int d\Omega Y_{lm}^* \Delta_{V}.$$

(46)

By using (39, 41), the circular power spectrum $C_{l}^{V(S)}$ of CMB radiation can be written as following

$$C_{l}^{V(S)} = \frac{1}{2l + 1} \sum_{m} \left\langle a_{V,lm}^* a_{V,lm} \right\rangle,$$

$$\approx \frac{1}{2l + 1} \int d^{3}K P_{\phi}^{(S)}(K, \eta) \sum_{m} \left| \int d\Omega Y_{lm}^* \int_{0}^{\eta_{0}} d\eta \dot{\tau}_{c\gamma} e^{ix(1 - \mu^{2})} \eta_{EH}(\eta) \Delta_{p}^{(S)} \right|^2,$$

(47)
FIG. 1: $\eta_{EH}(z)$ is plotted in terms of red-shift.

where $\eta_{EH}(\tau) = f_\tau \frac{\dot{\kappa}}{\dot{\tau}e^{\gamma}}$

$$P_{\phi}^{(S)}(\mathbf{K}, \tau) \delta(\mathbf{K}' - \mathbf{K}) = \langle \xi(\mathbf{K}) \xi(\mathbf{K}') \rangle,$$

(48)

and $P_{\phi}^{(S)}(\mathbf{K}, \tau)$ is the scalar power spectrum of primordial matter perturbations.

Furthermore as shown Eq.(47), the circular polarization cannot be generated in the scalar perturbation without considering the effects of Euler-Hiesenberg interactions. This result is in agreement with results of standard cosmology models [6]. With this knowledge that $\dot{\kappa}$ and $\dot{\tau}e^{\gamma}$ depend on red-shift, we have

$$\eta_{EH}(z) \simeq f_\tau \frac{n_0^0 (1 + z)^2}{15\pi n_e^0 \chi_e(z)} \left( \frac{T_{CMB}^0}{m_e} \right)^2,$$

(49)

where $\chi_e(z)$ is fraction of free cosmic electron, $n_0^0$ and $n_e^0$ are number densities of CMB photons and cosmic electrons at present time and $T_{CMB}^0 \simeq 2.7K$. $\eta_{EH}(z)$ is plotted in terms of red-shift in Fig.(1).

Now we can estimate $C^V_l$ in terms of the linearly polarized power spectrum $C^P_l$ and the average value of $\eta_{EH}$ as

$$C^V_l \approx (\eta_{EH}^{av})^2 C^P_l,$$

(50)

where

$$C^P_l = \frac{1}{2l+1} \int d^3K P_{\phi}^{(S)}(\mathbf{K}, \tau) \sum_m \left| \int \omega Y^*_{lm} \int_{\eta_0}^{\eta} d\eta \dot{\eta}_{e\gamma} e^{ix_\mu - \tau_{e\gamma}} \Delta_{P}^{(S)} \right|^2,$$

(51)

and

$$\eta_{EH}^{av} = \frac{1}{z_{lss}} \int_0^{z_{lss}} \eta_{EH}(z) dz \simeq 0.0002,$$

(52)

where $z_{lss}$ indicates red-shift at last scattering surface. Using the experimental value for $C^P_l$ which is in the order of $\sim \mu K^2$ and Eqs.(50)-(52), one can obtain an estimation on the range of $C^V_l \sim 10nK^2$, which is in the range of future experimental values. Note, we just make above estimation to have a sense about the contribution of Euler-Heisenberg interactions for the power spectrum of CMB circular polarization. The more precisely estimation of $l(l+1)C^V_l/(2\pi)$ is given in Fig.(2). Let’s compare our results with experimental data reported by SPIDER group [7]. Constrains of the circular power spectrum $l(l+1)C^V_l/(2\pi)$ reported by SPIDER group is in ranging from 141 to 203 $\mu K^2$ at 150 GHz for a thermal CMB spectrum and $33 < l < 307$ which is very larger than what can be found by considering non-linear photon-photon interaction. This means that if the results reported by [7] is confirmed, we have to search for another mechanisms (instead of non-linear CMB-CMB photons interaction) to satisfy them.
The Euler-Heisenberg interactions not only can generate circular polarization for CMB, but also generate the B-mode polarization in the presence of scalar metric perturbations in contrast with standard cosmology models [34, 38]. Next, one can divide the CMB linear polarization in terms of the divergence-free part (B-mode $\Delta_B^{(S)}$) and the curl-free part (E-mode $\Delta_E^{(S)}$) which are defined in terms of Stokes parameters as following

\[
\begin{align*}
\Delta_E^{(S)}(\hat{n}) &\equiv -\frac{1}{2}[\partial^2\Delta_P^{+(S)}(\hat{n}) + \bar{\partial}^2\Delta_P^{-(S)}(\hat{n})], \\
\Delta_B^{(S)}(\hat{n}) &\equiv \frac{i}{2}[\partial^2\Delta_P^{+(S)}(\hat{n}) - \bar{\partial}^2\Delta_P^{-(S)}(\hat{n})],
\end{align*}
\]

(53)

(54)

where $\partial$ and $\bar{\partial}$ indicate spin raising and lowering operators respectively [38]. As Eqs. (38), (43), (52) and (54) shown, the B-mode power spectrum $C_l^{B(S)}$ is given in terms of the circular polarization power spectrum $C_l^{V(S)}$ which can be estimated as

\[
C_l^{B(S)} \propto \hat{\eta}^2 C_l^{V(S)} \ll nK^2.
\]

(55)

Note the B-mode generating by Euler-Hiesenberg interaction is very small than $nK^2$ and so that we can neglect it.

VI. CONCLUSION AND REMARKS

In this work, we have solved the first order of the Quantum Boltzmann Equation for the density matrix of CMB radiation by considering Compton scattering and non-linear photon-photon forward scattering via the Euler-Heisenberg effective Lagrangian as collision terms. We have shown that propagating photons convert their linear polarizations to circular polarizations via the Euler-Heisenberg effective interaction. Also we have discussed that by considering non-linear CMB-CMB photons interaction, CMB linear polarization converts to circular one while crossing through CMB isotopic unpolarized medium $I_0$. The power spectrum of circular polarization in CMB radiations $C_l^{V(S)}$ in the presence of scalar perturbations is given in terms of linearly polarized power spectrum of CMB radiation $C_l^{V(S)} \sim (\eta_{EH}^\mu)^2C_l^{P(S)}$ which $\eta_{EH}$ [49] is given in terms of redshift by factor $(1+z)^2/\chi(z)$ and also $\eta_{EH}^\mu \approx 0.0002$ [52]. Also, we have estimated the average value of circular power spectrum is $1/(l+1)C_l^{V(S)}/(2\pi) \sim 10^{-4} \mu K^2$ for $l \sim 300$ at present time which is very smaller than recently reported data (SPIDER collaboration) but in the range of the future achievable experimental data. $l(l+1)C_l^{V(S)}/(2\pi)$ is plotted in Fig. [2]. As a result, it is necessary to search for another mechanisms (instead of non-linear CMB-CMB photons interaction) to satisfy SPIDER results for CMB circular polarization. We also show that the generation of B-mode polarization for CMB photons in the presence of the primordial scalar perturbation via Euler-Heisenberg interaction is possible however this contribution for B-mode polarization is not remarkable. It is shown in Eq. [55] that $C_l^{B(S)} \ll nK^2$. 

FIG. 2: The power spectrum of circular polarization $l(l+1)/2\pi C_l^{V(S)}$ is plotted in terms of $l$ and in unit $(\mu K)^2$ due to Compton scattering and photon-photon forward scattering via Euler-Heisenberg Effective Lagrangian. This file contains the LCDM power spectra that are derived from Planck (2015) parameters and also we have modified CMBquick mathematica code to make above plot.
VII. APPENDIX A

The time-evolution of the density matrix approximately obtained as

\[
(2\pi)^3\delta^3(0)2k^{\delta}d\rho_{ij}(k) \approx i[H^0_j(t), D^0_j(k)] \\
= \frac{2\alpha^2i}{45m^2}(2\pi)^3\delta^3(0) \times \int \frac{d^3p}{(2\pi)^32p^0} \left( p.k)^2[\epsilon_\nu(p)\epsilon_{\nu'}(p)\epsilon_i(k)\epsilon_{i'}(p)] \\
\times \{-5\rho_{s\nu\nu'}(p)\rho_{is}(k)\delta^{ij} + 5\rho_{s\nu\nu'}(p)\rho_{js}(k)\delta^{ii} + 4\rho_{s\nu'\nu'}(p)\rho_{ij}(k)\delta^{is} \\
- 4\rho_{s\nu'\nu'}(p)\rho_{is}(k)\delta^{ij} + 3\rho_{s\nu'\nu'}(p)\rho_{sj}(k)\delta^{ii} - 3\rho_{s\nu'\nu'}(p)\rho_{il}(k)\delta^{is} \\
+ 4\rho_{s\nu'\nu'}(p)\rho_{is}(k)\delta^{ij} - 4\rho_{s\nu'\nu'}(p)\rho_{il}(k)\delta^{ij} + 9\rho_{s\nu'\nu'}(p)\rho_{ij}(k)\delta^{ii} \\
\times \left[ \rho_{s\nu'\nu'}(p) - \rho_{s\nu'\nu'}(p)\delta^{ij} - 3\rho_{s\nu'\nu'}(p)\delta^{ij} \right] \\
\times \left[ \rho_{s\nu'\nu'}(p) + \rho_{s\nu'\nu'}(p)\delta^{ij} + 3\rho_{s\nu'\nu'}(p)\delta^{ij} \right] \right) \times \left[ \rho_{s\nu'\nu'}(p) + \rho_{s\nu'\nu'}(p)\delta^{ij} + 3\rho_{s\nu'\nu'}(p)\delta^{ij} \right] \right).
\]

where \( k \) and \( p \) indicate the energy-momentum states of photons and \( \delta^3(0) \) will be cancelled in the final expression. Here detail of abbreviated functions in Eqs. (9-12) are brought.

\[
f_1(\hat{p}, \hat{k}) = 2 \left( \hat{p}\hat{k} \right)^2 \left( \hat{e}_2(\hat{p})\hat{e}_1(\hat{k})^2 - (\hat{e}_1(\hat{k})\hat{e}_2(\hat{p}))^2 \right) \\
+ 2\left( \hat{p}\hat{k} \right)^2 \left( \hat{e}_2(\hat{p})\hat{e}_1(\hat{k})^2 - (\hat{e}_1(\hat{k})\hat{e}_2(\hat{p}))^2 \right) \\
+ 2\left( \hat{p}\hat{k} \right)^2 \left( \hat{e}_2(\hat{p})\hat{e}_1(\hat{k}) - \hat{e}_1(\hat{k})\hat{e}_2(\hat{p}) \right) \hat{k}\hat{e}_1(\hat{k}) \\
+ 2\left( \hat{p}\hat{k} \right)^2 \left( \hat{e}_2(\hat{p})\hat{e}_1(\hat{k}) - \hat{e}_1(\hat{k})\hat{e}_2(\hat{p}) \right) \hat{k}\hat{e}_2(\hat{p}) \right).
\]

\[
f_2(\hat{p}, \hat{k}) = 2 \left( \hat{p}\hat{k} \right)^2 \left( \hat{e}_2(\hat{p})\hat{e}_1(\hat{k})^2 - (\hat{e}_1(\hat{k})\hat{e}_2(\hat{p}))^2 \right) \\
+ 2\left( \hat{p}\hat{k} \right)^2 \left( \hat{e}_2(\hat{p})\hat{e}_1(\hat{k}) - \hat{e}_1(\hat{k})\hat{e}_2(\hat{p}) \right) \hat{k}\hat{e}_2(\hat{p}) \\
+ \left( \hat{p}\hat{e}_1(\hat{k})^2 - \hat{p}\hat{e}_2(\hat{k})^2 \right) \left( \hat{k}\hat{e}_1(\hat{k})^2 + \hat{k}\hat{e}_2(\hat{k})^2 \right).
\]
[1] R. B. Partridge, J. Nowakowski and H. M. Martin, Nature 331, 146 (1988).

[2] G. F. Smooth, et al. Astrophysical Journal Letters 396 (1), L1 (1992); C. L. Bennett, et al. Astrophysical Journal Letters 464, L1 (1996).

[3] R. Crittenden, R. Davis and P. Steinhardt, Ap. J. 417, L13 (1993).

[4] R. A. Frewin, A. G. Polnarev and P. Coles, Mon. Not. R. Ast. Soc. 266, L21 (1994); D. Harari and M. Zaldarriaga, Phys. Lett. B 319, 96 (1993).

[5] Kovac, J.M. et al. (2002). Nature 420 (6917): 777278.

[6] A. Kosowsky, Annals Phys. 246, 49 (1996) [arXiv:astro-ph/9601045].

[7] J. M. Nagy et al. [SPIDER Collaboration], Astrophys. J. 844, no. 2, 151 (2017), arXiv:1704.00213 [astro-ph.CO].

[8] R. Mainini, D. Minelli, M. Gervasi, et al. Journal of Cosmology and Astroparticle Physics, 033 (2013).

[9] R. Partridge, J. Nowakowski and H. Martin, Nature, 331, 146 (1988).

[10] P. Lubin, P. Melese and G. Smoot. The Astrophysical Journal, 273, L51 (1983).

[11] P. A. R. Ade et al. (BICEP Collaboration), Phys. Rev. Lett. 112, 241101 (2014); P. A. R. Ade et al. (Keck Array and BICEP2 Collaborations) Phys. Rev. D 96, 102003 (2017).

[12] P. A. R. Ade et al. (The Polarbear Collaboration), Astrophys. J. 794 (2014) no. 2, 171; P. A. R. Ade et al. (The Polarbear Collaboration), Astrophys. J. 844 (2017), arXiv:1704.00215 [astro-ph.CO].

[13] D. Hanson et. al. (SPTpol Collaboration), Phys. Rev. Lett. 111, 141301 (2013).

[14] A. Cooray, A. Melchiorri and J. Silk, Phys. Lett. B 554, 1 (2003) [arXiv:astro-ph/0205214].

[15] Soma. De, and H. Tashiro, Phys. Rev. D, 92, 123506 (2015).

[16] M. Zarei, E. Bavarsad, M. Haghighat, R. Mohammadi, I. Motie, Z. Rezaei, Phys. Rev. D 81, 084035 (2010) [arXiv:hep-th/0912.2993].

[17] R. Sawyer, Phys. Rev. D., 91, 021301 (2015).

[18] D. Colladay and V. A. Kostelecky, Phys. Rev. D, 58, 023010 (2008) [arXiv:astro-ph/0804.3380].

[19] S. Alexander, J. Ochoa and A. Kosowsky, Phys. Rev. D 79, 063524 [arXiv:astro-ph/0810.2355] (2009).

[20] R. Mohammadi, Eur. Phys. J. C 74:3102(2014), arXiv:1312.2199 [astro-ph.CO]; J. Khodagholizadeh, R. Mohammadi and S-S. Xue, Phys. Rev. D 90, 091301 (2014), arXiv:1406.6213 [astro-ph.CO]; R. Mohammadi, J. Khodagholizadeh, M. Sadegh and S. S. Xue, arXiv:1602.00237 [astro-ph.CO].

[21] R. Sawyer, Phys. Rev. D., 91, 021301 (2015).

[22] D. Colladay and V. A. Kostelecky, Phys. Rev. D, 58, 116002 (1998).

[23] F. Finelli and M. Galaverni Phys. Rev. D, 79, 063502 (2009).

[24] I. Motie and S. S. Xue, EPL 100, 17006 (2012).

[25] S. Shakeri, S. Z. Kalantari and S. S. Xue, Phys. Rev. A 95, no. 1, 012108 (2017) doi:10.1103/PhysRevA.95.012108 [arXiv:1703.10965 [hep-ph]].

[26] S. King and P. Lubin, Phys. Rev. D, 94, 023501 (2016).

[27] S. Alexander, E. McDonough and R. Sims, Phys. Rev. D 96, no. 6, 063506 (2017); arXiv:1704.00838 [gr-qc]].

[28] J. D. Jackson, Classical Electrodynamics, Wiley and Sons: New York (1998).

[29] T. W. Jones and S. L. ODell, Astrophys J. 214, 522 (1977); M. Ruszkowski and M. C. Begelman, arXiv:astro-ph/0112909 (2001).

[30] W. Heisenberg and H. Euler, Flerogerungen aus der Diracschen Theorie des Positron, Z. Phys. 98 (1936) 714; English translation: physics/0605038 H. Euler, Ann. d. Phys. 26 (1936) 398; J. Schwinger, Phys. Rev. 82 (1951) 664.

[31] V. Weisskopf, “The electrodynamics of the vacuum based on the quantum theory of the electron”, Kong. Dans. Vid. Selsk. Math.-fys. Medd. XIV No. 6 (1936);

[32] R. Ruffini, G. Verschchagin, S. S. Xue, Physical Reports 487, 1-140 (2010).

[33] G. V. Dunne, arXiv:hep-th/0406216 (2004).

[34] M. Zaldarriaga and U. Seljak, Phys. Rev. D 55, 1830 (1997) [astro-ph/9609170].

[35] W. Hu and M. J. White, New Astron 2, 323 (1997) [arXiv:astro-ph/9706147].

[36] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. 215, 203 (1992).

[37] S. Chandrasekhar, Radiative Transfer, Dover, New York, 1960.

[38] M. Zaldarriaga, D. N. Spergel and U. Seljak, Astrophys. J. 488, 1 (1997) [arXiv:9702157[astro-ph]].