On the photon mass in Very Special Relativity

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In Very Special Relativity (VSR) a gauge invariant photon mass is allowed. We explore some consequences of keeping this parameter not zero. VSR-Maxwell equations are presented. In addition, we show modifications to the Feynman rules keeping the photon mass and we apply it in the computation of the electron self energy, whose result is free from infrared divergences. A computation of the Coulomb scattering is developed and a small signal of the privileged direction predicted in the theory is present at tree level. Radiative corrections are computed and the photon mass cancels due to soft photon emission as in the standard case.

I. INTRODUCTION

The absence of any new particle in colliders allows us the possibility to explore new theories to give an answer to unsolved problems in the standard model of particles. Very Special Relativity (VSR)\cite{1} is a proposal where without new particles, it is possible to explain the neutrino mass, considering the universe invariant under a subgroup of the full Lorentz group. In the reference \cite{2} the SIM(2) group is used to explain how new terms, invariant under this group, allow the existence of neutrino mass preserving the same kinematics than special relativity because in this group there are not invariant tensors, besides the ones which are invariant under the Lorentz group. As a consequence of this model, non local terms appear and a tiny Lorentz violation parametrized by the neutrino mass is expected.

In Poincare invariant electrodynamics (QED) the photon mass is strictly zero because it is forbidden by gauge invariance. Gauge invariance is crucial to prove that unitarity and renormalizability are satisfied in QED\cite{3}. However, in VSR the possibility of a gauge invariant non-zero photon mass appears\cite{4,5}. The introduction of the mass terms for neutrino and photon in VSR theories do not affect the renormalizability of the model, because the additional SIM(2)-invariant expressions contain the non-local term $1/n \cdot \partial$, which in the large momentum limit vanishes, leaving the same ultraviolet behavior as in the Lorentz invariant theories. The model is renormalizable and unitary (some comments on this can be found in \cite{5,6}).

Experimental tests have shown the photon mass is almost zero or at least too small, for example, Bonetti et. al\cite{7} have reported an upper bound of $1.8 \cdot 10^{-14} \text{eV}/c^2$ using radio bursts, while the Particle Data Group\cite{8} in 2018 have published a bound of $10^{-18} \text{eV}$. New tests with higher precision should be carried on in the future.

Although the mass is considered to be zero, the possibility it should be different than zero without breaking gauge invariance in VSR is open. The aim of this work is to present an analysis of some consequences if we let the photon be massive within the theory, which is unitary, renormalizable and gives an answer to the neutrino mass.

The paper is organized as follows: in section II we present the VSR formalism in the QED sector considering the photon mass. In section III we derive the Feynman rules associated to the lagrangian of QED-VSR and some relations between the vertex in the theory useful to construct the Ward identities. In section IV we compute the electron self energy and the Ward identity. In section V we compute the cross section at tree level of the Coulomb Scattering. The section VI is devoted to show the cancellation of the infrared divergencies in the one loop radiative corrections. Finally in section VII we present the conclusions and some remarks.

II. QED VSR MODEL WITH PHOTON MASS

In this section we will review the references \cite{4,5} to fix the frame for the next calculations. We start with a gauge field $A_\mu$ and a scalar field $\phi$ coupled with this gauge field, whose the gauge transformations are

$$
\delta A_\mu = \tilde{\partial}_\mu \Lambda,
$$

$$
\delta \phi = i \Lambda \phi,
$$

where we have defined the wiggle operator

$$
\tilde{\partial}_\mu = \partial_\mu + \frac{1}{2} \frac{m^2}{n \cdot \partial} n_\mu,
$$

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Therefore, the lagrangian is equation (7) of equation (7) too. This fact is important because the lagrangian that we will construct below can be written in terms of equation (12) with the help of the wiggle derivative is invariant under equation (11) too. This fact is important because the lagrangian that we will construct below can be written in terms only of the standard F_{\mu\nu} as

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{2} m_\gamma^2 \left[ n_\nu \frac{1}{(n \cdot \partial)^2} \partial_\nu (n \cdot A) - n_\mu \frac{1}{(n \cdot \partial)^2} \partial_\mu (n \cdot A) \right].
\]  
(6)

We notice that a redefinition of \( A_\mu \rightarrow A_\mu - \frac{1}{2} m_\gamma^2 n_\mu ((n \cdot \partial)^{-2} (n \cdot A)) \) does not affect the observables and eliminates the modification by the mass factor. So, after this redefinition, we get the same electromagnetic tensor given by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and the same gauge transformation as in the Electroweak Standard Model given by

\[
A_\mu \rightarrow A_\mu + \partial_\mu \Lambda
\]  
(7)

However, we can define a new VSR field strength \( \tilde{F}_{\mu\nu} \) using the wiggle derivative in the way \( \tilde{F}_{\mu\nu} = \tilde{\partial}_\mu A_\nu - \tilde{\partial}_\nu A_\mu \), and we can write it in terms only of the standard F_{\mu\nu} as

\[
\tilde{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2} m_\gamma^2 \left[ n_\nu \frac{1}{(n \cdot \partial)^2} (n^\alpha F_{\mu\alpha}) - n_\mu \frac{1}{(n \cdot \partial)^2} (n^\alpha F_{\nu\alpha}) \right].
\]  
(8)

We notice that \( \tilde{F}_{\mu\nu} \) is not Lorentz but SIM(2) invariant. From its definition, we see that \( \tilde{F}_{\mu\nu} \) is invariant under equation (7) also. This fact is important because the lagrangian that we will construct below can be written in terms of \( \tilde{F}_{\mu\nu} \) and the parameter \( m_\gamma \) will be identified with the photon mass (please see equation (14)). With this we will have a theory with a photon mass coming from a term invariant under the standard gauge transformations in the equation (7) of \( A_\mu \).

We can use the VSR field strength expression to write the VSR gauge lagrangian as

\[
L_{\text{gauge}} = -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}.
\]  
(9)

Therefore, the lagrangian is

\[
L_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_\gamma^2 (n^\alpha F_{\mu\alpha})(n^\beta F_{\nu\beta}) - \frac{1}{(n \cdot \partial)^2} (n_\mu F_{\nu\beta}).
\]  
(10)

From equation (10) we get the equation of motion for \( F_{\mu\nu} \) with the help of the wiggle derivative is

\[
\partial_\mu F_{\mu\nu} + \frac{1}{2} m_\gamma^2 n_\nu \frac{1}{(n \cdot \partial)^2} \partial_\nu (n_\beta F^{\alpha\beta}) + m_\gamma^2 \frac{1}{n \cdot \partial} (n_\beta F^{\beta\nu}) = 0.
\]  
(11)

We write equation (11) in terms of the gauge field as

\[
\partial^2 A^\nu - \partial^\nu \partial_\alpha A^\alpha + \frac{1}{2} m_\gamma^2 n_\nu \frac{1}{(n \cdot \partial)^2} [\partial^2 (n \cdot A) - (n \cdot \partial)(\partial_\alpha A^\alpha)] + m_\gamma^2 \frac{1}{n \cdot \partial} (n \cdot \partial A^\nu - \partial^\nu n \cdot A) = 0.
\]  
(12)

If we contract equation (12) with \( n_\nu \) and plug it back in equation (12) we get

\[
\partial^2 A^\nu - \partial^\nu \partial_\alpha A^\alpha + m_\gamma^2 \frac{1}{n \cdot \partial} (n \cdot \partial A^\nu - \partial^\nu n \cdot A) = 0.
\]  
(13)

We will fix the gauge using the Lorenz gauge \( \partial_\mu A^\mu = 0 \) and we need to impose an additional VSR condition \( n \cdot A = 0 \) because one gauge degree of freedom still remains. So, the equation of motion is given by

\[
(\partial^2 + m_\gamma^2) A^\nu = 0.
\]  
(14)

We see from equation (14) that \( A^\nu \) is a field with mass \( m_\gamma \). Hence, in VSR we have a photon mass coming from a term that is gauge invariant, unlike the case with a term of the type \( m_\gamma^2 A^\mu A_\mu \), which is not.

Now, we use a plane wave solution for \( A^\nu \), as \( A^\nu = \epsilon^\nu e^{-ikx} \), so we have

\[
k^2 - m_\gamma^2 = 0
\]  
(15)

with the conditions \( k \cdot \epsilon = 0 \) and \( n \cdot \epsilon = 0 \). So, the gauge field has two polarizations, as in the standard case.
III. VSR MAXWELL EQUATIONS

We can use the equation of motion in (11) to get the modified Gauss law and the Ampere law in the VSR framework putting \( \mu = 0 \) and \( \mu = j \), where \( j = 1, 2, 3 \). So, using the expressions for \( \bar{E} \) and \( \bar{B} \) these two first Maxwell equations in vacuum are

\[
\nabla \cdot \bar{E} + \frac{1}{2} m_\gamma^2 \frac{n^0}{(n \cdot \partial)^2} \left[ n^0 \nabla \cdot \bar{E} + \bar{n} \cdot \left( \nabla \times \bar{B} - \frac{\partial \bar{E}}{\partial t} \right) \right] + m_\gamma^2 \frac{1}{n \cdot \partial} (\bar{n} \cdot \bar{E}) = 0, \tag{16}
\]

\[
\nabla \times \bar{B} - \frac{\partial \bar{E}}{\partial t} + \frac{1}{2} m_\gamma^2 \frac{n}{(n \cdot \partial)^2} \left[ n^0 \nabla \cdot \bar{E} + \bar{n} \cdot \left( \nabla \times \bar{B} - \frac{\partial \bar{E}}{\partial t} \right) \right] + m_\gamma^2 \frac{1}{n \cdot \partial} (\bar{n} \times \bar{B} - n^0 \bar{E}) = 0. \tag{17}
\]

We notice it is equivalent the result from the equation of motion to use \( \bar{\partial}_\mu \bar{F}^{\mu\nu} = 0 \). By the same token, we use the Bianchi identity \( \bar{\partial}_\mu \bar{F}_{\nu\rho} = 0 \), to get the last two Maxwell equations

\[
\left( 1 + \frac{1}{2} m_\gamma^2 \frac{|\bar{n}|^2}{(n \cdot \partial)^2} \right) \frac{\partial \bar{B}}{\partial t} + \left( 1 + \frac{1}{2} m_\gamma^2 \frac{(n^0)^2}{(n \cdot \partial)^2} \right) \nabla \times \bar{E} + \frac{1}{2} m_\gamma^2 \frac{1}{n \cdot \partial} (n^0 \bar{B} + \bar{n} \times \bar{E}) \]

\[
- \frac{1}{2} m_\gamma^2 \frac{1}{(n \cdot \partial)^2} \left[ n^0 (\bar{n} \cdot \nabla) \bar{B} - n^0 \bar{n} (\bar{n} \cdot \bar{B}) + \bar{n} \times \nabla (\bar{n} \cdot \bar{E}) + n^0 \bar{n} \times \nabla \left( \frac{\partial \bar{E}}{\partial t} \right) + \bar{n} \frac{\partial (\bar{n} \cdot \bar{B})}{\partial t} \right] = 0 \tag{18}
\]

\[
\left( 1 + \frac{1}{2} m_\gamma^2 \frac{|\bar{n}|^2}{(n \cdot \partial)^2} \right) \nabla \cdot \bar{B} + \frac{1}{2} m_\gamma^2 \frac{1}{n \cdot \partial} \bar{n} \cdot \bar{B} + \frac{1}{2} m_\gamma^2 \frac{1}{(n \cdot \partial)^2} \left[ n^0 \bar{n} \cdot (\nabla \times \bar{E}) - \bar{n} \cdot \nabla (\bar{n} \cdot \bar{B}) \right] = 0 \tag{19}
\]

We can observe setting \( m_\gamma^2 = 0 \) we recover the standard Maxwell equations. We remark the VSR photon mass parameter is very small to be significative in a classical experiment, so, we will not focus on this line.

IV. FEYNMAN RULES

We will review in brief the Feynman rules for the model with photon mass. In order to get them, we write the gauge lagrangian in equation (10) plus a gauge fixing term as follows

\[
L_{\text{gauge+gf}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m_\gamma^2 (n^0 F_{\mu\alpha}) \frac{1}{(n \cdot \partial)^2} (n_\beta F^{\mu\beta}) - \frac{1}{2\xi} (\partial_\mu A^\mu)^2. \tag{20}
\]

Performing some algebraic manipulations we get

\[
L_{\text{gauge+gf}} = \frac{1}{2} A_\nu \left[ (\partial^2 + m_\gamma^2) g^{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + m_\gamma^2 n^\mu n^\nu \frac{\partial^2}{(n \cdot \partial)^2} - m_\gamma^2 \frac{\partial^\mu n^\nu + \partial^\nu n^\mu}{n \cdot \partial} \right] A_\mu. \tag{21}
\]

Then, in Fourier space and with a little bit of algebra we have an expression for the propagator and we choose the Feynman gauge \( \xi = 1 \), so

\[
\Delta(p, \bar{n}) = -\frac{i}{p^2 - m_\gamma^2} \left[ g_{\mu\nu} + \frac{m_\gamma^2}{(n \cdot p) \bar{n}} n_\mu n_\nu - \frac{m_\gamma^2}{p^2 (n \cdot p) \bar{n}} (p_\mu n_\nu + p_\nu n_\mu) \right], \tag{22}
\]

where here we have assigned the subindex \( \bar{n} \) for \( n \cdot p \) in the denominators, related with the Mandelstam-Leibbrandt prescription[2][10]

\[
\frac{1}{n \cdot p} = \lim_{\epsilon \to 0} \frac{\bar{n} \cdot p}{(n \cdot p)(\bar{n} \cdot p) + i\epsilon}. \tag{23}
\]

It is important to notice, if we set \( m_\gamma = 0 \) in equation (22) we recover the standard propagator in the Feynman gauge.

If now we allow interaction with an electron, the new lagrangian will be

\[
L = L_{\text{gauge+gf}} + L_{\text{fermion}}, \tag{24}
\]
where

$$L_{\text{fermion}} = \bar{\psi} \left( i\partial - M + \frac{1}{2} m^2 \gamma \frac{1}{n \cdot D} \right) \psi,$$  \hspace{1cm} (25)

and here \( m \) is the neutrino mass, \( M \) the acquired mass of the electron after spontaneous symmetry breaking, and \( D_\mu = \partial_\mu + i e A_\mu \).

From equation [25] we can get the electron propagator

$$S_F(p, \bar{n}) = i \frac{\not{p} + M - \frac{m^2}{2} \frac{\not{n}}{(n \cdot p)_{\bar{n}_1} [n \cdot (p + q)]_{\bar{n}_2}}}{p^2 - M^2 + i \varepsilon},$$  \hspace{1cm} (26)

where \( M^2 = M^2 + m^2 \). In addition, because the nonlocality of the \((n \cdot D)^{-1}\) we can have new kind of vertex, with more external photon legs (see figure [1]).

![FIG. 1: Examples of vertex in VSR, it is possible have more photonic external legs because the nonlocal theory.](image)

We can establish a relationship between the vertex with a number of external photon legs and the vertex with one less external photon legs. We start with the standard vertex as is shown in the left diagram in figure [1]. The mathematical expression is easy to get from equation [25]. Then,

$$V_\mu([(p, \bar{n}_1), (p + q, \bar{n}_2)] = \gamma_\mu + \frac{1}{2} m^2 \frac{n_\mu}{(n \cdot p)_{\bar{n}_1} [n \cdot (p + q)]_{\bar{n}_2}},$$  \hspace{1cm} (27)

where here we have two differents \( \bar{n} \) for each momentum in the denominator with \( n \cdot p \). Our choice is general, nevertheless the Ward Identity will determine the relationships as we will see below.

If we contract the vertex with the photon external momentum \( q \) in the equation [27] and we arrange the expression we get

$$q^\mu V_\mu u = \left( \not{p} + \not{q} - M - \frac{1}{2} m^2 \frac{\not{n}}{(n \cdot (p + q))_{\bar{n}_2}} \right) - \left( \not{p} - M - \frac{1}{2} m^2 \frac{\not{n}}{(n \cdot p)_{\bar{n}_1}} \right).$$  \hspace{1cm} (28)

We recognize the inverse of the electron propagator and

$$q^\mu V_\mu [(p, \bar{n}_1), (p + q, \bar{n}_2)] = S_F^{-1}(p + q, \bar{n}_2) - S_F^{-1}(p, \bar{n}_1).$$  \hspace{1cm} (29)

In the case where \( m = 0 \), the terms with \( \bar{n} \) associated vanish, and the standard case is recovered, the expression is the Ward-Takahashi identity. Now, in VSR, as we have new kind of vertex, we can proceed for the two external photon legs vertex in the same way, giving a \( \bar{n} \) for each denominator with \( n \cdot p \), so

$$V_{\mu\nu}([(p, \bar{n}_1), (p', \bar{n}_2), (p + q_1, \bar{n}_3), (p + q_2, \bar{n}_4)] = \frac{1}{2} m^2 \frac{n_{\mu} n_{\nu}}{(n \cdot p)_{\bar{n}_1} (n \cdot p')_{\bar{n}_2}} \left( \frac{1}{[n \cdot (p + q_1)]_{\bar{n}_3}} + \frac{1}{[n \cdot (p + q_2)]_{\bar{n}_4}} \right),$$  \hspace{1cm} (30)

where \( p' = p + q_1 + q_2 \). Again, we contract with an external photon leg, for instance \( q_1 \) and after perform a little bit of algebra we get

$$q_1^\mu V_{\mu\nu} [(p, \bar{n}_1), (p', \bar{n}_2), (p + q_1, \bar{n}_3), (p + q_2, \bar{n}_4)] = V_{1\nu}([(p, \bar{n}_1), (p + q_2, \bar{n}_4)] - V_{1\nu}([(p', \bar{n}_2), (p + q_1, \bar{n}_3)] \right).$$  \hspace{1cm} (31)
For the contraction with $q_2$ the result is
\[
q_2^\mu V_{\mu\nu}[(p, \vec{n}_1), (p', \vec{n}_2), (p + q_1, \vec{n}_3), (p + q_2, \vec{n}_4)] = V_{1\mu}[(p, \vec{n}_1), (p + q_1, \vec{n}_3)] - V_{1\mu}[(p', \vec{n}_2), (p + q_2, \vec{n}_4)].
\] (32)

From this, we can write the contraction between a vertex with $n$ external photonic legs with one external momentum of a photon as the difference between a vertex with $n - 1$ external legs, without the leg whose momentum is the contracted, and another vertex with $n - 1$ external legs, whose inner fermionic leg has momentum as the sum of the original inner fermionic momentum and the momentum contracted, as is shown in the figure 2.

![FIG. 2: Vertex with $n$ external photonic legs written as difference between two diagrams with one less leg.](image)

V. ELECTRON SELF ENERGY

As the theory has a two external photon leg vertex we have another diagram, as is shown in the figure 3. Nevertheless this new diagram does not contribute. We only have the following contribution from the first diagram given by
\[
-i\Sigma(q) = -i\Sigma_{\text{prev}}(q) - i\Sigma_{\text{new}}(q),
\] (33)

where $-i\Sigma_{\text{prev}}$ is the contribution computed in [12] without photon mass, and $-i\Sigma_{\text{new}}$ is the new contribution coming from the terms with $m^\gamma$. The mathematical expressions to be computed are
\[
-i\Sigma_{\text{prev}}(q) = (-ie)^2 \int \frac{d^2\omega}{(2\pi)^2} \frac{1}{(p^2 - M^2 + i\varepsilon)((q - p)^2 - m^2 + i\varepsilon)} \left( \gamma_\mu + \frac{1}{2} m^2 \frac{n_\mu}{n \cdot q \cdot p} \right) \times \\
\times \left( p + M - \frac{1}{2} m^2 \frac{\vec{n}}{n \cdot p} \right) \left( \gamma_\nu + \frac{1}{2} m^2 \frac{n_\nu}{n \cdot q \cdot p} \right) g_{\mu\nu},
\] (34)
and
\[ -i\Sigma_{\text{new}}(q) = (-ie)^2m_\gamma^2 \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(p^2 - M_\gamma^2 + i\epsilon)((q - p)^2 - m_\gamma^2 + i\epsilon)} \left( \gamma_\mu + \frac{1}{2} m_\gamma^2 \frac{\not{p}n_\mu}{n \cdot q n \cdot p} \right) \times \]
\[ \times \left( \not{p} + 2\omega - \frac{1}{2} m_\gamma^2 \frac{\not{q}}{n \cdot q n \cdot p} \right) \left( \gamma_\nu + \frac{1}{2} m_\gamma^2 \frac{\not{p}n_\nu}{n \cdot q n \cdot p} \right) \left[ \frac{n_\mu n_\nu}{(n \cdot q n \cdot p)^2} - \frac{(q_\mu - p_\mu)n_\nu + (q_\nu - p_\nu)n_\mu}{(q - p)^2 n \cdot (q - p)} \right]. \] (35)

We use dimensional regularization and $2\omega$ is the dimension. Here we have omitted the subscripts $\tilde{n}$ only to be more readable, but they are present. If we set $m = 0$ and $m_\gamma = 0$ we recover the standard computation. In addition, in equation (34), $m_\gamma$ plays the role of the small photon mass introduced by hand in the standard case to regularize the infrared divergences, but here it is a parameter in the theory.

We use the Ward identity to determine the relationship between the $\tilde{n}$ for each term. In the figure 4 we see the three diagrams for one loop correction to the vertex.

We write the diagrams in a mathematical way as
\[
\bar{u}(p') \Lambda^\mu u(p) = \bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p' + k, \tilde{n}_2), (p', \tilde{n}_1)] S_F (p' + k, \tilde{n}_2) V_1^\nu [(p' + k, \tilde{n}_2), (p + k, \tilde{n}_3)] S_F (p + k, \tilde{n}_3) \times \]
\[ \times V_1^\sigma [(p, \tilde{n}_4), (p + k, \tilde{n}_3)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4) + \]
\[ + \bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p' + k, \tilde{n}_2), (p', \tilde{n}_1)] S_F (p' + k, \tilde{n}_2) V_2^{\nu} [(p, \tilde{n}_3), (p' + k, \tilde{n}_2), (p', \tilde{n}_1), (p + k, \tilde{n}_4)] \times \]
\[ \times \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_3) + \]
\[ + \bar{u}(p', \tilde{n}_1) V_2^{\mu} [(p - k, \tilde{n}_2), (p', \tilde{n}_1), (p, \tilde{n}_3), (p' - k, \tilde{n}_4)] S_F (p - k, \tilde{n}_2) V_1^\nu [(p, \tilde{n}_3), (p - k, \tilde{n}_2)] \times \]
\[ \times \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_3). \] (36)

We contract with the external leg $q$ and using the identities found in the last section, the first term gives
\[
\bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p' + k, \tilde{n}_2), (p', \tilde{n}_1)] S_F (p' + k, \tilde{n}_2) V_1^\nu [(p' + k, \tilde{n}_2), (p + k, \tilde{n}_3)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4) \]
\[ - \bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p' + k, \tilde{n}_2), (p', \tilde{n}_1)] S_F (p' + k, \tilde{n}_2) V_1^\nu [(p' + k, \tilde{n}_2), (p + k, \tilde{n}_3)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4), \] (37)

the second
\[
\bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p' + k, \tilde{n}_2), (p', \tilde{n}_1)] S_F (p' + k, \tilde{n}_2) V_1^\nu [(p', \tilde{n}_1), (p' + k, \tilde{n}_2)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4) \]
\[ - \bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p' + k, \tilde{n}_2), (p', \tilde{n}_1)] S_F (p' + k, \tilde{n}_2) V_1^\nu [(p', \tilde{n}_1), (p' + k, \tilde{n}_2)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4), \] (38)

and the latter after the change $k \to -k$
\[
\bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p + k, \tilde{n}_2), (p, \tilde{n}_3)] S_F (p + k, \tilde{n}_2) V_1^\nu [(p, \tilde{n}_3), (p + k, \tilde{n}_2)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4) \]
\[ - \bar{u}(p', \tilde{n}_1) V_1^{\mu} [(p + k, \tilde{n}_2), (p, \tilde{n}_3)] S_F (p + k, \tilde{n}_2) V_1^\nu [(p, \tilde{n}_3), (p + k, \tilde{n}_2)] \Delta_{\mu\nu} (k, \tilde{n}_1) u(p, \tilde{n}_4). \] (39)
To satisfy the Ward Identity, we have

\[
\begin{align*}
\tilde{n}_1 &= \bar{n}_1 = \tilde{n}_1, \\
\tilde{n}_2 &= \bar{n}_2 = \tilde{n}_4, \\
\tilde{n}_3 &= \bar{n}_4 = \tilde{n}_2, \\
\tilde{n}_4 &= \bar{n}_3 = \tilde{n}_3, \\
\end{align*}
\]

(40)

and

\[
\begin{align*}
\tilde{n}_1 &= \tilde{n}_2 = \tilde{n}_3.
\end{align*}
\]

(41)

In this way, we have the scheme shown in the figure 5 defined by the expression

\[
q_{\mu} \Lambda^\rho = V_1^\rho[(p + k, \bar{n}_3), (p, \bar{n}_4)]SF(p + k, \bar{n}_3)V_1^\nu[(p, \bar{n}_4), (p + k, \bar{n}_3)]\Delta_{\mu\nu}(k, \tilde{n}_1)
\]

\[
- V_1^\rho[(p' + k, \bar{n}_1), (p', \bar{n}_2)]SF(p' + k, \bar{n}_2)V_1^\nu[(p', \bar{n}_1), (p' + k, \bar{n}_2)]\Delta_{\mu\nu}(k, \tilde{n}_1),
\]

(42)

where we have substracted the external legs \( u \) and \( \bar{u} \).

In the reference [12] the integrals with \( \bar{n} \) are traded by a linear combination between the external legs and the null vector \( n \) to restore a SIM(2) invariant expression, because the introduction of the Mandelstam-Leibbrandt prescription breaks the VSR invariance. It was shown the way to do it is

\[
\bar{n}_\mu = -\frac{q^2}{2(n \cdot q)^2}n_\mu + \frac{q_\mu}{n \cdot q},
\]

(43)

where \( q \) is the external momentum in the diagram.

We observe in the figure 5 as \( \bar{n}_3 \) as \( \bar{n}_4 \) have the same external momentum \( p \) and \( \bar{n}_1 \) with \( \bar{n}_2 \) share the external momentum \( p' = p + k \), so \( n_1 = \bar{n}_2 \) and \( n_3 = \bar{n}_4 \). With this rule, we use the same \( \bar{n} \) to compute the integrals in equation (34) using the integration shown in [11], whose main integrals are listed in the appendix [13] then we trade the \( \bar{n} \) with the linear combination in equation (43) and we have the same result that [12].

On the other hand, because the equation (41) is satisfied, the two diagrams in the figure 5 have the same \( \bar{n} \) related to the photon propagator, but the external momenta are different. Here we cannot set \( \bar{n} \) equal to zero for this case, because it does not respect the condition \( n \cdot \bar{n} = 1 \). We need to use the equation (43) and select a common vector to both diagrams. The only one is the zero four vector and we proceed in the limit sense. Thus, we start with an arbitrary and common momentum \( P \) and then we use the limit \( P \to 0 \) to eliminate the arbitrariness. Therefore, the Mandelstam-Leibbrandt prescription in the eq. (23) after the replacement of \( \bar{n} \) using the \( P \) vector reads

\[
\frac{1}{n \cdot k} = \lim_{n \to 0} \lim_{P \to 0} \frac{-P^2n \cdot k + 2P \cdot kn \cdot P}{(n \cdot k)(-P^2n \cdot k + 2P \cdot kn \cdot P) + i\eta},
\]

(44)

where we have defined \( \eta = 2\epsilon(n \cdot P)^2 \) which satisfies \( \eta > 0 \). Notice that for \( (n \cdot P) \neq 0 \), \( \lim_{\eta \to 0} \) coincides with \( \lim_{n \to 0} \), so we use as a definition the expression (44). In the limit \( P \to 0 \) and keeping \( \eta \) not zero, the fraction \( 1/n \cdot k \) vanishes without problem. Using this rule, the final result of the self energy in equation (33) is

\[
-i\Sigma(q) = C \frac{\eta}{n \cdot q} + Dq + E,
\]

(45)
We use the frame of reference where $n$ does not introduce any new thing, as in the standard case.

\[ \theta \]

\[ A \]

with \( n \) is the deflection angle. Moreover, since the energy is conserved, \( p^0 = p'^0 = E \) and \( p^2 = M_e^2 \), then the equation \[ 50 \] is now

\[ |\mathcal{M}|^2 = \frac{Ze^4}{8|\vec{p}|^4 \sin^2 \frac{\theta}{2}} \left[ 2E^2 - 2|\vec{p}|^2 \sin^2 \frac{\theta}{2} - m^2 + \frac{1}{2} m^2 \left( \frac{n \cdot p}{n \cdot p'} + \frac{n \cdot p'}{n \cdot p} \right) - m^2 |\vec{p}|^2 \sin^2 \frac{\theta}{2} \left( \frac{n^0)^2}{n \cdot p' \cdot n \cdot p} \right) \right]. \]

We use the frame of reference where \( n = (1, 0, 0, 1) \), therefore

\[ |\mathcal{M}|^2 = \frac{Ze^4}{8|\vec{p}|^4 \sin^4 \frac{\theta}{2}} \left[ 2E^2 - 2|\vec{p}|^2 \sin^2 \frac{\theta}{2} - m^2 + \frac{1}{2} m^2 \left( E - |\vec{p}| \sin \eta \sin \phi + E - |\vec{p}| \sin \eta \sin (\phi - \theta) \right) \right] . \]
where $\eta$ is the angle between $\vec{n}$ and the normal vector to the plane where the scattering occurs. In addition, $\phi$ is the angle between the projection of $\vec{n}$ in the plane where the scattering takes place and $-\vec{p}$. With this, the cross section is given by

$$
\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4|\vec{p}|^2\beta^2 \sin^4 \left(\frac{\theta}{2}\right)} \left[ 1 - \beta^2 \sin^2 \left(\frac{\theta}{2}\right) - \frac{m^2}{2M_e^2} \left( 1 - \beta \sin \eta \sin \phi \right) + \frac{m^2}{4M_e^2} \left( 1 - \beta \sin \eta \sin (\phi - \theta) \right) \frac{1}{1 - \beta \sin \eta \sin (\phi - \theta)} \right].
$$

(54)

Here we have got a VSR Mott formula, where if set $m = 0$ we recover the classic result.

VII. INFRARED DIVERGENCES CANCELLATION

Now, we move on to the radiative corrections to the tree level Coulomb Scattering. In the standard case is well known the one loop corrections, as in the vertex and the fermionic legs, are cancelled by the bremsstrahlung process, with the soft photon emission. So, here will compute this corrections, under the assumption the photon momentum $k$ radiated is small, in the low energy. Therefore

$$
|M_{\text{brem}}|^2 = \int \frac{d^3k}{(2\pi)^2 2\sqrt{|k|^2 + m_e^2}} \sum_{\lambda=1,2} e^2 \left[ \frac{p^\mu \varepsilon_{\mu}(k)}{p' \cdot k} - \frac{p^\mu \varepsilon_{\mu}(k)}{p \cdot k} \right] \left[ \frac{p'^\nu \varepsilon^*_{\nu}(k)}{p' \cdot k} - \frac{p'^\nu \varepsilon^*_{\nu}(k)}{p \cdot k} \right] |M|^2,
$$

(55)

where the sum is over the two photon polarizations and $|M|^2$ is the tree level quantity, given by the equation (50). The sum over the polarizations in VSR is given by

$$
\sum_{\lambda=1}^2 \varepsilon_{\mu}(k)\varepsilon^*_{\nu}(k) = -g_{\mu\nu} - \frac{m_e^2}{(n \cdot k)^2} n_{\mu} n_{\nu} + \frac{1}{n \cdot k} (k_{\mu} n_{\nu} + k_{\nu} n_{\mu}).
$$

(56)

The full computation of this sum is shown in the appendix A. The algebra is straightforward, then,

$$
|M_{\text{brem}}|^2 = e^2 \int \frac{d^3k}{(2\pi)^2 2\sqrt{|k|^2 + m_e^2}} \left[ \frac{2}{p \cdot k p' \cdot k} \left( p \cdot p' + m_e^2 \frac{n \cdot p n \cdot p'}{(n \cdot k)^2} \right) - \frac{1}{(p' \cdot k)^2} \left( M_e^2 + m_e^2 \frac{(n \cdot p)^2}{(n \cdot k)^2} \right) \right] |M|^2.
$$

(57)

The vertex correction is easy to compute. Again, we work in the small $k$ approximation. To get the contributions to the $\mathcal{M}$ matrix at the order that we are working, we need the interference with the diagram at tree level. After a long computation we have got

$$
|M_{\text{mix}}|^2 = -2e^2 \int \frac{d^3k}{(2\pi)^2 2\sqrt{|k|^2 + m_e^2}} \frac{1}{p \cdot k p' \cdot k} \left( p \cdot p' + m_e^2 \frac{n \cdot p n \cdot p'}{(n \cdot k)^2} \right) |M|^2.
$$

(58)

We can observe the above expression cancel the first term in equation (57). The external legs corrections cancel the latter terms. We see that the bremsstrahlung cancel the radiative corrections at one loop as in the standard QED. As consequence, it is impossible to get a signal of the photon mass in the process to distinguish between the standard case and VSR, because the behavior is the same. Here, the photon mass is absent as usual, so, there is not difference with the standard case at this order.

VIII. CONCLUSIONS

We have reviewed the possibility in VSR to have a photon mass gauge invariant. Although the current experimental bounds tell us it should be too small, it is a new feature of the model to distinguish between the standard case and VSR. Despite $m_\gamma$ is a free parameter that we can set equal to zero without problems, we have explored the consequences to consider a non null value.
A modified Maxwell equations was presented. As the new contributions from the VSR sector are multiplied by the photon mass, which is very small, a classical experiment to measure departures from the standard results should not be useful. Nevertheless the equations are presented in any case.

We have computed the electron self energy considering the photon mass and we have seen it does not present infrared divergencies. Moreover, in the loop corrections there is not signal of this mass in the observables. We have computed the Coulomb scattering, and the photon mass is cancelled as in the standard case at one loop due to the radiative corrections in the low energy region.

In the cross section calculation we observe a small difference with the standard result, that it should be considered to measure. Although the photon mass is absent, it could be found a signal of $n$ in this observable. We expect it is small due it is proportional to $m^2/M_e^2$.

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Appendix A: Sum over polarizations

We write the sum of the polarizations as

$$\sum_{\lambda=1}^{2} \varepsilon_\mu(k)\varepsilon^*_{\nu}(k) = A g_{\mu\nu} +BK_\mu k_\nu +Cn_\mu n_\nu +D(k_\mu n_\nu + k_\nu n_\mu).$$ (A1)

We use the following conditions to get the coefficients

$$k^\mu \varepsilon_\mu = 0,$$ (A2)
$$n^\mu \varepsilon_\mu = 0,$$ (A3)
$$g^{\mu\nu}\varepsilon_\mu\varepsilon^*_{\nu} = -1.$$ (A4)

From equations (A2), (A3) and (A4) we get

$$0 = (A+Bk^2+Dn\cdot k)k_\nu + (Cn\cdot k +Dk^2)n_\nu,$$ (A5)
$$0 = Bn\cdot kk_\nu + (A+Dn\cdot k)n_\nu,$$ (A6)
$$-2 = 4A+Bk^2+2Dn\cdot k,$$ (A7)

respectively. From equation (A6) we get $B = 0$ and $A = -Dn\cdot k$. We insert it in equation (A5), from there we have a consistency relation in the first parentheses and $C = -D\frac{k^2}{n\cdot k}$. And we plug $A$ and $B$ in the equation (A7) to get $D$. Finally

$$A = -1,$$ (A8)
$$B = 0,$$ (A9)
$$C = -\frac{k^2}{(n\cdot k)^2},$$ (A10)
$$D = \frac{1}{n\cdot k}.$$ (A11)

Thus, we use $k^2 = m_\gamma^2$ in $C$ and the sum of the polarizations is

$$\sum_{\lambda=1}^{2} \varepsilon_\mu(k)\varepsilon^*_{\nu}(k) = -g_{\mu\nu} - \frac{m_\gamma^2}{(n\cdot k)^2}n_\mu n_\nu + \frac{1}{n\cdot k}(k_\mu n_\nu + k_\nu n_\mu).$$ (A12)
Appendix B: Integration with $(n \cdot p)^{-1}$

Here we quote the main integrals needed to compute our expressions from reference [11]. They are

$$
\int \frac{dp}{(p^2 + 2p \cdot q - m^2)^a (n \cdot p)^b} = (-1)^{a+b} \frac{1}{\Gamma(a+b)} \pi^{(n \cdot q)^b} \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega}},
$$

(B1)

with $\omega = d/2$.

Taking a derivative with respect to $q$ in (B1) we get

$$
\int d^4p \frac{P_\mu}{(p^2 + 2p \cdot q - m^2)^{a+1} (n \cdot p)^b} = (-1)^{a+b} \frac{1}{\Gamma(a+b)} \pi^{(n \cdot q)^b} \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} - 2 \frac{-\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} - 2 \frac{-\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} + 4 \frac{\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} - 2 \frac{-\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega}}}
$$

(B2)

With a second derivative in (B1):

$$
\int d^4p \frac{P_\mu P_\nu}{(p^2 + 2p \cdot q - m^2)^{a+2} (n \cdot p)^b} = (-1)^{a+b} \frac{1}{\Gamma(a+b)} \pi^{(n \cdot q)^b} \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} - 2 \frac{-\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} - 2 \frac{-\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} + 4 \frac{\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega} - 2 \frac{-\nu}{\Gamma(a+2)\Gamma(b)}(\bar{n} \cdot q)^b b^\mu \bar{n}_\nu \int_0^1 dt t^{a+b-1} \frac{1}{(m^2 + q^2 - 2(n \cdot q)(\bar{n} \cdot q)t)^{a+b-\omega}}}
$$

(B3)