Bijective planar parameterization based on a hyperelasticity analogy

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Abstract—Isogeometric analysis (IGA) is a promising alternative for finite element analysis (FEA) with simplified design-through-analysis workflow. However, existing models in CAD software cannot be used directly for analysis. Parameterization of the input geometry is necessary to transform it into an analysis-suitable model. But it is non-trivial to produce smooth non-folding results. In this paper, we present a novel parameterization scheme based on the hyperelasticity analogy in solid mechanics. Compared with other state-of-the-art techniques, numerical examples prove that the proposed method guarantees bijective results with excellent smoothness.

1. Introduction

Isogeometric analysis (IGA) was first proposed in 2005 [1] to integrate computer aided design (CAD) with downstream analysis. By using the same set of basis functions, tedious mesh generation is omitted and the exact geometry models rather than the approximated discrete meshes are directly used for analysis, showing great advantages over traditional finite element analysis (FEA) in both accuracy and efficiency.

The most widely used basis functions in CAD industry is non-uniform rational B-spline (NURBS). In CAD systems, geometries are described by boundary surfaces, i.e., the model is only a shell with no inner parametric descriptions. However, a proper parameterization of the entire model including the inner part is essential for IGA which means that the model cannot be used directly for analysis. A parameterization needs to be performed in advance.

Parameterization refers to constructing a spline mapping from the unit square/cube (for surface or volumetric parameterization) to the computational domain. A good parameterization should be bijective and least distorted. Parameterization quality greatly influences the efficiency and accuracy of numerical solutions. It is a fundamental problem of IGA and various approaches have been proposed to address this problem, such as Coon’s patch [2], harmonic mapping [3], and variational harmonic methods [4]. But none of them guarantees bijectivity for an arbitrarily given geometry.

This paper presents a scheme to construct planar parameterization based on a solid mechanics analogy. The parameterization of the given domain is constructed by deforming a simple but well-
parameterized domain into the same shape of the given domain. Bijectivity of the mapping is guaranteed by the incompressibility of the material used.

2. Formulation of the Proposed Method

2.1 Formulation of the Proposed Method
In this paper, we aim to solve the problem stated as: Given a planar domain $x \in \mathbb{R}^2$ with four boundary curves $x(\xi, 0), x(\xi, 1), x(0, \eta), x(1, \eta)$, find a proper B-spline representation of the given domain in the form of a B-spline surface. Assume the given boundary curves are degree $p$ B-spline curves and the resulting B-spline surface can be written as

$$x(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_i^p(\xi) N_j^q(\eta) P_{i,j}$$

where $N_i^p(\xi)$ and $N_j^q(\xi)$ are B-spline basis functions and $P_{i,j}$ are the unknown control points to be solved.

2.2 Framework of the Method
Generally, the proposed method has three steps as listed below.

a) Reference domain construction. In this step, a simple but parameterized domain is constructed which will be deformed into the desired shape in subsequent steps.

b) Boundary condition enforcement. In this step, Dirichlet boundary conditions are applied on the pre-deformation domain constructed in step 1 such that its shape after deformation is the same as the given domain.

c) Solving the boundary value problem through a hyperelastic-IGA approach.

2.3 Hyperelasticity Formulation
The problem is to find the inner control points of a domain $\Omega$ enclosed by four B-spline curves. The idea is to use solid mechanics analogy to deform a well-parameterized domain $\Omega_0$ into the target domain $\Omega$. This is a boundary value problem and can be solved by an IGA approach.

In this paper, we use the traditional Lagrangian formulation. Consider the deformation from undeformed domain $\Omega_0$ with boundary $\Gamma_0$ to the target domain $\Omega$ with boundary $\Gamma$. The deformation can be described by a mapping $\Phi$ where a particle at position $X$ in $\Omega_0$ is mapped to $x$ in $\Omega$. That is

$$x = \Phi(X) = X + u(X)$$

The deformation gradient is then defined as

$$F = \frac{d\Phi}{dX}$$

According to the static equilibrium equation, we have

$$\int_{\Omega} f^b d\Omega + \int_{\Gamma} t d\Gamma = 0$$

where $f^b$ is the body force and $t$ is the surface traction.

Applying the principle of minimum potential energy, we can arrive at the weak form of a nonlinear elastic system [5]. The potential energy of an elastic system can be expressed as

$$\Pi(u) = \Pi^{\text{int}}(u) - \Pi^{\text{ext}}(u) = \int_{\Omega_0} W(E) d\Omega - \int_{\Omega_0} u^T f^b d\Omega - \int_{\Gamma_0} u^T t d\Gamma$$

where $u$ is the displacement field and $W(E)$ is the strain energy density written as

$$W(E) = \frac{1}{2} E : D : E$$

The notion ‘:’ is the contraction operator of tensors and the Lagrangian strain $E$ is then denoted by

$$E = \frac{1}{2}(C - 1)$$
\( \mathbf{C} \) is defined as a right Cauchy-Green deformation tensor \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \).

The principle of minimum potential energy tells us that the first variation of the potential energy should be 0. Assume the displacement is perturbed in the direction \( \mathbf{u} \) with size \( \tau \), i.e.,

\[
\mathbf{u}_\tau = \mathbf{u} + \tau \tilde{\mathbf{u}}
\]

(8)

The first variation of the potential energy can be defined as

\[
\Pi(\mathbf{u}, \tilde{\mathbf{u}}) \equiv \left. \frac{d}{d\tau} \Pi(\mathbf{u} + \tau \tilde{\mathbf{u}}) \right|_{\tau=0}
\]

(9)

Since the first variation of the potential energy should be 0, we can arrive at

\[
\Pi(\mathbf{u}, \tilde{\mathbf{u}}) = \int_{\Omega_0} \frac{\partial W(E)}{\partial E} : \mathbf{E}(\mathbf{u}, \tilde{\mathbf{u}}) \, d\Omega - \int_{\Omega_0} \tilde{\mathbf{u}}^T \mathbf{f}^b \, d\Omega - \int_{\Gamma_0} \tilde{\mathbf{u}}^T \mathbf{t} \, dl = 0
\]

(10)

where \( \mathbf{E}(\mathbf{u}, \tilde{\mathbf{u}}) \) is the variation of \( \mathbf{E} \) defined in the same way as Eq. (9).

The weak form of hyperelasticity problem can be then expressed as

\[
a(\mathbf{u}, \tilde{\mathbf{u}}) = \ell(\tilde{\mathbf{u}}), \forall \tilde{\mathbf{u}} \in \mathbb{Z}
\]

(11)

where

\[
a(\mathbf{u}, \tilde{\mathbf{u}}) = \int_{\Omega_0} \frac{\partial W(E)}{\partial E} : \mathbf{E}(\mathbf{u}, \tilde{\mathbf{u}}) \, d\Omega
\]

(12)

\[
\ell(\tilde{\mathbf{u}}) = \int_{\Omega_0} \tilde{\mathbf{u}}^T \mathbf{f}^b \, d\Omega + \int_{\Gamma_0} \tilde{\mathbf{u}}^T \mathbf{t} \, dl
\]

(13)

Note that in the setting of this paper, body forces and boundary tractions are all set to 0, the weak form reduces to

\[
\int_{\Omega_0} \frac{\partial W(E)}{\partial E} : \mathbf{E} \, d\Omega = 0
\]

(14)

Assuming \( \mathbb{V}^h \subset \mathbb{V} \), \( \mathbb{Z}^h \subset \mathbb{Z} \) are two discrete spaces defined by a set of B-spline basis functions. The weak form of Eq. 11 can be rewritten into a discrete form: Find \( \mathbf{u}^h \in \mathbb{V}^h \) such that for all \( \mathbf{u} \in \mathbb{Z}^h \)

\[
a(\mathbf{u}^h, \tilde{\mathbf{u}}^h) = 0
\]

(15)

When the material status can be completely described by a given total strain, the constitutive relation is called hyperelasticity. We use a neo-Hookean model for computation, defined as

\[
W(I_1) = A_{10}(I_1 - 3)
\]

(16)

where \( A_{10} \) is related to the shear modulus by \( A_{10} = \mu / 2 \) and \( I_1 \) is the trace of \( \mathbf{C} \).

**2.4 Solving the nonlinear elasticity problem**

For linear systems, Eq. (15) can be transformed to a linear system \( \mathbf{Ku}^h = 0 \). However, \( a(\mathbf{u}^h, \tilde{\mathbf{u}}^h) \) is nonlinear with respect to \( \mathbf{u}^h \) and \( \mathbf{u}^h \) can only be computed through an iterative approach. Here, the Newton-Raphson method is adopted. Suppose an approximate solution at the \( i \)-th iteration is \( \mathbf{u}^h_i \). The solution at the next iteration can be approximated by the first-order Taylor expansion, as

\[
a(\mathbf{u}^h_{i+1}, \tilde{\mathbf{u}}^h) \approx a(\mathbf{u}^h_i, \tilde{\mathbf{u}}^h) + \mathbf{K}_i(\mathbf{u}^h_i) \cdot \Delta \mathbf{u}^h
\]

(17)

where \( \mathbf{K}_i(\mathbf{u}^h_i) = \partial a(\mathbf{u}^h_i, \tilde{\mathbf{u}}^h) / \partial \mathbf{u}^h \) is the Jacobian matrix at the \( i \)-th iteration. Rearranging this equation, we arrive at

\[
\mathbf{K}_i(\mathbf{u}^h_i) \cdot \Delta \mathbf{u}^h = -a(\mathbf{u}^h_i, \tilde{\mathbf{u}}^h)
\]

(18)

Displacement increment \( \Delta \mathbf{u}^h \) can thus be solved and a new approximate solution is obtained as
\( \mathbf{u}_{i+1}^h = \mathbf{u}_i^h + \Delta \mathbf{u}_i^h \) \hspace{2cm} (19)

Usually, we define the right-hand side of Eq. (19) as residual, i.e.,
\[ R_i = -a(\mathbf{u}_i^h, \mathbf{\bar{u}}_i^h) \] \hspace{2cm} (20)
and the algorithm terminates when residual is smaller than a given tolerance.

2.5 Reference Domain Construction
The reference domain is the domain that will be deformed according to given boundary curves. We use the quadrilateral defined by four corner points of input curves as the reference domain, as shown in Figure 1. The solid curves are given boundary curves and the dashed lines are the boundaries of the reference domain.

Assume opposite boundary curves have the same knot vectors, denoted by \( \mathbf{E} \) and \( \mathbf{H} \) respectively. Using the same knot vectors, the reference domain can be represented as
\[ x_{\text{ref}} = \sum_i^n \sum_j^m P_{ij}^R N_i(\xi) N_j(\eta) \] \hspace{2cm} (21)

The control points \( P_{ij}^R \) are computed by surface fitting. We select \( n \times m \) sampling points for a unique solution of the surface to be fitted. The sampling points are uniformly distributed in the quadrilateral defined by four corner points with \( n \) rows and \( m \) columns (blue points in Figure 1(b)). Intervals between neighboring rows (or columns) correspond to equally spaced parameter intervals in the parameter domain, based on which we can calculate the parameter values of all the sampling points. The unknown control points can be computed by solving the resulting linear system.

\[ \text{Figure 1: Reference domain construction} \]

2.6 Boundary Condition Enforcement
We use displacement boundary conditions to fully describe the deformation process, which are enforced on control points of boundary curves with vectors describing movements before and after deformation.

3. Results and Comparison
We compare our method with several popular techniques, i.e., Coon’s patch, harmonic mapping, and variational harmonic method, to validate the superiority of our method.

3.1 Methods for comparison
Coons patch method \[2\] produces B-spline parameterization from boundary B-spline curves by evaluating a closed-form expression as
\[ P_{ij} = (1 - \frac{j}{m}) P_{0,j} + \frac{i}{n} P_{n,j} + (1 - \frac{i}{n}) P_{i,0} + \frac{j}{m} P_{i,m} - \left( \frac{i}{n} \right) \left[ \begin{array}{cc} P_{0,0} & P_{0,m} \\ P_{n,0} & P_{n,m} \end{array} \right] \left( \frac{1 - \frac{j}{m}}{\frac{j}{m}} \right) \] \hspace{2cm} (22)
for \( i = 1, 2, ..., n - 1 \) and \( j = 1, 2, ..., m - 1 \). \( P_{ij} \) are the inner control points.
Harmonic mapping [3] computes a parameterization by solving the following partial differential equation.

$$\nabla^2 u = 0$$  \hspace{1cm} (23)

Variational harmonic method parameterize a given domain by minimizing an objective functional for smoothing, as reported in [4].

3.2 Results and detailed comparison

Figure 2. Result comparison. Results from methods [2] [3] [4] are listed in columns 1, 2, and 3, respectively. The last column contains the results of the proposed method.

Figure 2 shows the results of different methods. The geometries all have complex boundaries and it is obvious that all other techniques have difficulties in producing non-folding results, especially at concave regions of the boundary. Meanwhile, our method produces smooth and uniform parameterizations all over the domain with no folding detected, which means bijective results in all these examples.

4. Conclusions

In this paper, a planar parameterization method is proposed which is based on a hyperelasticity analogy. The detailed formulation of the hyperelastic boundary value problem as well as its finite element formulation is discussed. Then the proposed method is compared with several popular techniques for planar parameterization. Results show that our method has great advantages over other techniques in providing smooth bijective parameterizations.
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