RATIONAL HOMOTOPY THEORY AND NONNEGATIVE CURVATURE

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Abstract. In this note, we answer positively a question by Belegradek and Kapovitch about the relation between rational homotopy theory and a problem in Riemannian geometry which asks that total spaces of which vector bundles over compact nonnegative curved manifolds admit (complete) metrics with nonnegative curvature.

1. Introduction

Given a Riemannian manifold $M$ with metric 

$$<>: TM \times TM \rightarrow TM$$

an affine connection is a bilinear map

$$\nabla: Vec(M) \times Vec(M) \rightarrow Vec(M)$$

which satisfies the following

- $\nabla fV W = f \nabla V W$
- $\nabla_V (fW) = (V f)W + f \nabla_V W$

where $f \in C^\infty(M), V, W \in Vec(M)$

An affine connection is called Levi-Civita connection if it satisfies

also the following

- $X <> V, W >= < \nabla_X V, W > + < V, \nabla_X W >$
- $\nabla_V (W) - \nabla_W V - [V, W] = 0$

where $[V, W]f = (XY - YX)f$ is the Lie bracket.

A fundamental result in Riemannian geometry asserts that

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Theorem 1.1. For each Riemannian metric, there exists a unique Levi-Civita connection.

Given a Riemannian manifold $M$ with Levi-Civita connection, there is defined a curvature operator

$$R : \text{Vec}(M) \times \text{Vec}(M) \times \text{Vec}(M) \to \text{Vec}(M)$$

defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

From it one arrives at an important geometric invariant which is called sectional curvature defined by

$$K(\sigma) = \frac{< R(v, w)w, v >}{< v \wedge w, v \wedge w >}$$

where $\sigma \subset T_pM$ is a tangent plane at $p \in M$ and $v, w \in \sigma$ span it.

It is well known that $K(\sigma)$ does not depend on the choice of spanning vectors.

A well known question in Riemannian geometry is

Question 1.2. Does the restriction on curvature imply the restriction on topology and vice versa?

In particular, how does the positive (nonnegative) curvature restrict the topology of the underlying manifold?

A Riemannian manifold is called positively (or nonnegatively) curved if, for any $\sigma$, $K(\sigma) > 0$ (or $K(\sigma) \geq 0$).

For compact manifold, we have the following classical

Theorem 1.3. Let $M$ be a compact Riemannian manifold with positive curvature. Then

$$\pi_1(M) = \begin{cases} 
\text{finite group} & \text{if dim } M \text{ is odd} \\
0 & \text{if dim } M \text{ is even and } M \text{ is orientable} \\
\mathbb{Z}_2 & \text{if dim } M \text{ is even and } M \text{ is nonorientable}
\end{cases}$$

The main concern of this note is on noncompact manifold. In this case there is the following
Theorem 1.4. Let $M$ be a complete noncompact Riemannian manifold with nonnegative curvature. Then $M$ is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold which is called the soul.

Another central question in Riemannian geometry is to what extent the converse is true, or in other words

Question 1.5. Total spaces of which vector bundles over compact nonnegatively curved manifolds admit (complete) metrics with nonnegative curvature?

Previously, obstructions to the existence of nonnegatively curved metrics on vector bundles were only known for a flat soul [7]. No obstructions are known when the soul is simply-connected. In [2] an approach to the reduction of the problem to the vector bundle over simply connected manifold was initiated. The start point is another result of Cheeger and Gromoll [4] that a finite cover of any closed nonnegatively curved manifold (throughout the paper by a nonnegatively curved manifold we mean a complete Riemannian manifold of nonnegative sectional curvature) is diffeomorphic to a product of a torus and a simply-connected closed nonnegatively curved manifold. It turns out that a similar statement holds for open complete nonnegatively curved manifolds which is the basis of their analysis.

Lemma 1.6. [2] Let $(N, g)$ be a complete nonnegatively curved manifold. Then there exists a finite cover $N'$ of $N$ diffeomorphic to a product $M \times T^k$ where $M$ is a complete open simply connected nonnegatively curved manifold. Moreover, if $S'$ is a soul of $N'$, then this diffeomorphism can be chosen in such a way that it takes $S'$ onto $C \times T^k$ where $C$ is a soul of $M$.

By using this and characteristic classes technique, they proved that, in various case, the total spaces of rank $k$ vector bundles over $C \times T$ admit no nonnegatively curved metric if they do not become the pullback of a bundle over $C$ in a finite cover. The following is such an example
Corollary 1.7. [1] Let $B$ be a closed nonnegatively curved manifold. If $\pi_1(B)$ contains a free abelian subgroup of rank four (two, respectively), then for each $k \geq 2$ (for $k = 2$, respectively) there exists a finite cover of $B$ over which there exist infinitely many rank $k$ vector bundles whose total spaces admit no nonnegatively curved metrics.

Belegradek and Kapovitch [2] are thus led to the following

Definition 1.8. Given a closed smooth simply connected manifold $C$, a torus $T$, and a positive integer $k$, we say that a triple $(C, T, k)$ is splitting rigid if any rank $k$ vector bundle over $C \times T$ with nonnegatively curved total space splits, after passing to a finite cover, as the product of a rank $k$ bundle over $C$ and a rank zero bundle over $T$.

Let $\mathcal{H}$ be the class of simply-connected CW-complexes whose rational cohomology algebra is finite dimensional, as a rational vector space, and has no nonzero derivations of negative degree (see [2] for the reason to choose such a class $\mathcal{H}$). For example, $\mathcal{H}$ contains any compact simply-connected Kähler manifold [6].

A natural question is

Question 1.9. [2] Let $C \in \mathcal{H}$ be a closed smooth manifold. Is $(C, T, k)$ splitting rigid for any $T$ and $k$?

The main result in this note is a positive answer to this question

Theorem 1.10. Let $C \in \mathcal{H}$ be a closed smooth manifold. Then $(C, T, k)$ is splitting rigid for any $T$ and $k$.

In this paper, all (co)homology groups have rational coefficients, all manifolds and vector bundles are smooth; all topological spaces are homotopy equivalent to connected CW-complexes. $[X, Y]$ will be the based homotopy classes of based maps between them. $\text{map}(X, Y)$ is the space of maps from $X$ to $Y$ and $\text{map}(X, Y)_f$ is the connected component of $\text{map}(X, Y)$ which contains the map $f : X \to Y$.

2. A splitting criterion

Given a finite cell complex $C$, define $\text{Char}(k, C)$ to be the subspace of $H^*(C)$ which is the direct sum of $\bigoplus_{i=1}^{[(k-1)/2]} H^{4i}(C)$ and the subspace
equal to $H^k(C)$ if $k$ is even, and to $H^{4k/2}(C)$ if $k$ is odd. Note that any rational characteristic class of a rank $k$ vector bundle over $C$ lies in the subalgebra of $H^*(C)$ generated by $\text{Char}(k, C)$.

Belegradek and Kapovitch transform the problem of a triple $(C, T, k)$ being splitting rigid into a homotopy problem as follows.

**Proposition 2.1.** Let $C$ be a closed simply-connected manifold, $T$ be a torus, $k$ be a positive integer. If any self-homotopy equivalence of $C \times T$ maps $\text{Char}(k, C)$ to itself, then the triple $(C, T, k)$ is splitting rigid.

We are thus led to compute the group of homotopy classes of self homotopy equivalences $\text{Aut}(C \times T)$ of $C \times T$.

Before we can do this, let’s recall a work by Booth and Heath.

Given spaces with base point $(X, x_0)$ and $(Y, y_0)$, there is a natural map

$$\varphi : \text{map}(X \times Y, X \times Y) \to \text{map}(X, X) \times \text{map}(Y, Y)$$

defined by $\varphi(f) = (g, h)$ where $g(x) = \pi_X \circ f(x, y_0)$, $h(y) = \pi_Y \circ f(x_0, y)$ and $\pi_X : X \times X, \pi_Y : X \times Y \to Y$ are projections to the factors $X$ and $Y$ respectively.

**Definition 2.2.** Let $X$ and $Y$ be two spaces. We say $X$ and $Y$ have the induced equivalence property (IEP) if whenever $f$ is a homotopy equivalence, then $g, h$ defined above are homotopy equivalences.

**Remark 2.3.** Let $X$ and $Y$ be such that for each $i > 0$, at least one of $\pi_i(X)$ and $\pi_i(Y)$ is zero. Then they satisfy the IEP by Whitehead theorem.

With the above notion, we can quote the following.

**Theorem 2.4.** Let $X$ and $Y$ be two spaces having IEP. Suppose further that $[X, \text{map}(Y, Y)]_{\text{id}} = 0$, then there is a short exact sequence of groups and homomorphisms

$$1 \to [Y, \text{map}(X, X)]_{\text{id}} \xrightarrow{\theta} \text{Aut}(X \times Y) \to \text{Aut}(X) \times \text{Aut}(Y) \to 1$$

which splits by a homomorphism $\sigma : \text{Aut}(X) \times \text{Aut}(Y) \to \text{Aut}(X \times Y)$ given by $\sigma(g, h) = g \times h$.
Let $X = C$ and $Y = T$ where $C, T$ be as in Theorem 1.10. Then $X$ and $Y$ have IEP by the remark following the definition 2.2. On the other hand, $\left[ C, \text{map}(T, T)_{id} \right] = 0$ since it is well known that $\text{map}(T, T)_{id} = T$ and $C$ is 1-connected and thus first cohomology of $C$ is trivial.

Now given $f \in \text{Aut}(C \times T)$, to prove that the induced homomorphism in cohomology maps $\text{Char}(k, C)$ to itself, it suffices to assume that $f \in \text{Im}([T, \text{map}(C, C)_{id}])$ by the exact sequence above. Recall that the map $\theta : [T, \text{map}(C, C)_{id}] \to \text{Aut}(C \times T)$ is given by $\theta(f)(x, y) = (f(y)(x), y)$. The above argument gives the following

**Corollary 2.5.** Let $C$ be a closed simply-connected manifold, $T$ be a torus, $k$ be a positive integer. Then the triple $(C, T, k)$ is splitting rigid if, for any map $f : T \to \text{map}(C, C)_{id}$, the adjoint $\tilde{f} : T \times C \to C$ induces a homomorphism in cohomology given by $\tilde{f}^*(u) = 1 \otimes u$ for any $u \in H^*(C)$.

### 3. The Proof of Theorem 1.10

**Proof of Theorem 1.10.** Let $T = (S^1)^s$. By Corollary 2.5, to prove Theorem 1.10, it suffices to prove that, for map $f : T \times C \to C$ such that $f(y_0, \cdot)$ homotopic to $id$, it induces a homomorphism in cohomology given by $f^*(u) = 1 \otimes u$ for any $u \in H^*(C)$. Given such $f$, for any $u \in H^*(C)$,

$$f^*(u) = 1 \otimes u + \sum_k \sum_{i_1 \cdots i_k} \lambda_{i_1 \cdots i_k}(u) \otimes t_{i_1} \cdots t_{i_k}$$

where the first sum is taken over $k$ from 1 to $s$ and the second sum is taken over all $(i_1 \cdots i_k)$'s such that $0 < i_1 < \cdots < i_k < s+1$. Thus we get a sequence of maps $\lambda_{i_1 \cdots i_k} : H^n(C) \to H^{n-k}(C)$ where $0 < k < s+1$ and $0 < i_1 < \cdots < i_k < s+1$.

To prove that $f^*(u) = 1 \otimes u$, it suffices to prove that $\lambda_{i_1 \cdots i_k} = 0$ where $0 < k < s+1$ and $0 < i_1 < \cdots < i_k < s+1$ while for the proof of later we need to study the behaviour of these maps with respect to the cup product of cohomology.

If $u, v \in H^*(C)$, then

$$f^*(uv) = 1 \otimes uv + \sum_k \sum_{i_1 \cdots i_k} \lambda_{i_1 \cdots i_k}(uv) \otimes t_{i_1} \cdots t_{i_k}$$
On the other hand $f^*(uv) = f^*(u)f^*(v)$. Using the formula for $f^*(uv), f^*(u), f^*(v)$ and comparing the terms associated with $\iota_{i_1} \cdots \iota_{i_k}$, we find the following equations

$$
\lambda_{i_1 \cdots i_k}(uv) = \lambda_{i_1 \cdots i_k}(u)v + (-1)^{|v|}u\lambda_{i_1 \cdots i_k}(v) \pm \sum \lambda_{j_1 \cdots j_p}(u)\lambda_{l_1 \cdots l_q}(v) \otimes \iota_{i_1} \cdots \iota_{i_k}
$$

where $p + q = k$ with $p > 0, q > 0$ and the sum is taken over all partitions of $i_1, \cdots, i_k$ into $j_1 < \cdots < j_p$ and $l_1 < \cdots < l_q$.

Let $k = 1$. Then, the above formula implies that $\lambda_{i_1}$ is a derivation of degree $-1$ which is trivial by the condition of the Theorem 1.10. The above formula in case $k = 2$ implies that $\lambda_{i_1, i_2}$ is a derivation of degree $-2$ modulo products of derivations of degree $-1$ which are trivial. It follows that $\lambda_{i_1, i_2}$ is a derivation of degree $-2$ and thus is trivial by the condition of the Theorem 1.10. Inductively, we can prove that all $\lambda_{i_1 \cdots i_k}$ are trivial which completes the proof of the Theorem 1.10.

\[\square\]

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