Discrepancy of arithmetic progressions in grids

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Abstract
We prove that the discrepancy of arithmetic progressions in the $d$-dimensional grid $\{1, \ldots, N\}^d$ is within a constant factor depending only on $d$ of $N^{\frac{d}{2d+2}}$. This extends the case $d = 1$, which is a celebrated result of Roth and of Matoušek and Spencer, and removes the polylogarithmic factor from the previous upper bound of Valkó from about two decades ago. We further prove similarly tight bounds for grids of differing side lengths in many cases.

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1  |  INTRODUCTION

Given a finite set $\Omega$, a coloring of $\Omega$ is a map $\chi : \Omega \to \{1, -1\}$, and $\chi(A) = \sum_{x \in A} \chi(x)$. For a family $\mathcal{A}$ of subsets of $\Omega$, the discrepancy of $\mathcal{A}$ is defined to be

$$\text{disc}(\mathcal{A}) := \min_{\chi} \max_{A \in \mathcal{A}} |\chi(A)|,$$

where the minimum is over all colorings of $\Omega$. Let $\mathcal{A}_1$ be the family of arithmetic progressions contained in $[N] := \{1, \ldots, N\}$. Roth [9] showed using Fourier analysis that there is an absolute constant $c > 0$ such that

$$\text{disc}(\mathcal{A}_1) \geq cN^{\frac{1}{2}}.$$

In the other direction, Beck [2] proved that

$$\text{disc}(\mathcal{A}_1) \leq CN^{\frac{1}{2}}(\log N)^{\frac{3}{2}}.$$
for some absolute constant $C$, thereby showing that Roth’s lower bound is sharp up to a polylogarithmic factor. Finally, Matoušek and Spencer [8] removed the polylogarithmic factor and resolved this problem of determining the discrepancy up to a constant factor.

It is natural to study the generalization of this problem to higher dimensions. An arithmetic progression in $d$ dimensions is a set of the form

$$\text{AP}_d(a, b, l) := \{a + ib : i = 0, 1, \ldots, l - 1\},$$

where $a, b \in \mathbb{Z}^d$ with $b \neq 0$, and $l \in \mathbb{N}$. Here, $b$ is the common difference of the arithmetic progression. Let $\mathcal{A}_d$ be the set of arithmetic progressions in $d$ dimensions that are subsets of $[N]^d$. The quantity we are interested in is

$$\text{disc}(\mathcal{A}_d) := \min_{\chi : [N]^d \to \{1, -1\}} \max_{A \in \mathcal{A}_d} |\chi(A)|,$$

where $\chi(A) = \sum_{x \in A} \chi(x)$. Valkó [10] proved that there exist constants $c = c(d), C = C(d)$ such that

$$cN^{\frac{d}{2d+2}} \leq \text{disc}(\mathcal{A}_d) \leq CN^{\frac{d}{2d+2}}(\log N)^{\frac{5}{2}}.$$

Valkó’s proof of the lower bound extends Roth’s proof, while the upper bound adapts Beck’s proof. The problem of estimating the discrepancy of higher dimensional arithmetic progressions is further discussed in [4]. In this paper, we remove the polylogarithmic factor in the upper bound and thus determine the quantity up to a constant factor dependent on $d$.

**Theorem 1.1.** For all positive integers $N$ and $d$, we have

$$\text{disc}(\mathcal{A}_d) = \Theta_d(N^{\frac{d}{2d+2}}).$$

The general proof strategy is similar to that in the paper by Matoušek and Spencer [8]. However, new ideas are needed to make the strategy work. In particular, we need to overcome some difficulties arising from geometric aspects that require delicate analysis and tools like Minkowski’s theorem and the Lenstra–Lenstra–Lovász basis reduction algorithm.

It is natural to study the generalization of the problem to the discrepancy of arithmetic progressions in grids of side lengths that are not necessarily equal. Given positive integers $N_1, \ldots, N_d$, let $\Omega = [N_1] \times \cdots \times [N_d] \subseteq \mathbb{Z}^d$ and $\mathcal{A}_N$ be the set of arithmetic progressions in $d$ dimensions that are subsets of $\Omega$, where $N = (N_1, \ldots, N_d)$. The discrepancy is defined in a similar way,

$$\text{disc}(\mathcal{A}_N) := \min_{\chi : \Omega \to \{1, -1\}} \max_{A \in \mathcal{A}_N} |\chi(A)|.$$

In the proof of Theorem 1.1, we shall see that we will have to consider more generally grids with side lengths of comparable size (see (1.1)).
**Theorem 1.2.** For any positive integer $d$ and positive integers $N_1, \ldots, N_d$, if $\delta > 0$ satisfies that

$$N_1 \cdots N_d \leq \left( \min_{1 \leq i \leq d} N_i \right)^{d+1-\delta},$$

(1.1)

then there exist positive constants $c_d, C_d$ such that for $\mathbf{N} = (N_1, \ldots, N_d)$,

$$c_d(N_1 \cdots N_d)^{\frac{1}{2d+2}} \leq \text{disc}(\mathbf{N}) \leq C_d \cdot \frac{1}{\delta} (N_1 \cdots N_d)^{\frac{1}{2d+2}}.$$  

We remark that Theorem 1.2 implies Theorem 1.1 by choosing $\delta = 1$ and $N_1 = N_2 = \cdots = N_d = N$.

The lower bound in Theorem 1.2 holds even without condition (1.1). Theorem 1.3 gives a more general lower bound, and a matching upper bound up to a sub-logarithmic factor for general grids of differing side lengths. The lower bound in Theorem 1.3 implies the lower bound in Theorem 1.2 by taking $I = [d]$ in the maximum. The proof of lower bound uses Fourier analytic tools.

**Theorem 1.3.** For any positive integer $d$ and $\mathbf{N} = (N_1, \ldots, N_d)$ where the $N_i$’s are positive integers whose product is at least three, there exist positive constants $c_d, C_d$ such that

$$c_d \max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{\frac{1}{2|I|+2}} \leq \text{disc}(\mathbf{N}) \leq C_d \cdot \frac{\log(N_1 \cdots N_d)}{\log \log(N_1 \cdots N_d)} \cdot \max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{\frac{1}{2|I|+2}}. \quad (1.2)$$

Here by convention if $I = \emptyset$, then $\prod_{i \in I} N_i = 1$.

**Remark.** As disc($\mathbf{A}_\mathbf{N}$) does not depend on the order of the $N_i$’s, we may assume without loss of generality that $N_1 \geq N_2 \geq \cdots \geq N_d$. In this case,

$$\max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{\frac{1}{2|I|+2}} = \max_{1 \leq k \leq d} \left( \prod_{i=1}^{k} N_i \right)^{\frac{1}{2k+2}}.$$  

**Organization.** In Section 2, we show how to efficiently decompose arithmetic progressions into “canonical” arithmetic progressions and provide an upper bound on the number of such canonical arithmetic progressions with a given size. This implies that a coloring that has low discrepancy in canonical arithmetic progressions of each possible size also achieves low discrepancy for all arithmetic progressions (see Lemma 2.1 for details). We prove the upper bound in Theorem 1.2 in Section 3 by showing the existence of a coloring that has low discrepancy in canonical arithmetic progressions of each possible size. In the proof we use an improved bound on the number of canonical arithmetic progressions with each given size, the proof of which is deferred to Section 4. We finally study the case of grids with different side lengths in the last three sections. We prove the lower bound of Theorem 1.3 in Section 5 and the upper bound of Theorem 1.3 in Section 6. Finally, we have some concluding remarks in Section 7, including a conjecture that the lower bound for the discrepancy for grids in Theorem 1.3 is tight up to the constant factor.

**Notations.** Throughout the paper, all logarithms are base $e$ unless specified. We generally assume that $d$ is fixed, except in Section 4 where the proof relies on an induction argument on $d$. We use
symbols $c, c_1, c_2, C_0, C, C^*$ to denote positive absolute constants, and $c_d, C_d$ to denote those that only depend on $d$. We use notation $f = O_d(g)$ if there exists a positive constant $C_d$ so that $f \leq C_d g$. We write $\chi(A) = \sum_{a \in A} \chi(a)$ for any $\chi : X \to \mathbb{R}$ and $A \subseteq X$.

## 2 DECOMPOSITION

Let $N_1, \ldots, N_d$ be positive integers, $\Omega = [N_1] \times \cdots \times [N_d]$, and $\mathbb{N} = (N_1, \ldots, N_d)$.

To find a coloring giving low discrepancy, the general idea is to apply a partial coloring lemma (specifically Lemma 3.1) to repeatedly partially color $\Omega$ until we get a full coloring of $\Omega$ with low discrepancy. At each stage, we color a constant fraction of the remaining uncolored elements, until we get a full coloring of $\Omega$ with low discrepancy. To accomplish this, we show that for any $X \subseteq \Omega$, there is a partial coloring of $X$ with low discrepancy. Once we have this statement, we may apply this with $X$ as $\Omega$ in the initial iteration to get a partial coloring of $\Omega$, and then pick $X$ as the set of uncolored elements of $\Omega$ in later iterations. Hence, more generally the set family we need to consider is $(X, A_X)$ where $A_X := \{ A \cap X : A \in \mathcal{A}_X \}$.

The family of sets $A_X$ is too large if we want to apply the partial coloring lemma on $(X, A_X)$ directly. Instead we apply it to a small subfamily $C_X \subseteq A_X$ so that any set in $A_X$ can be efficiently decomposed into sets in $C_X$. For each $b \in \mathbb{Z}^d \setminus \{0\}$, we may partition elements in $X$ into congruence classes modulo $b$. For each congruence class $I = \{ x \in X : x \equiv a \pmod{b} \}$, as distinct elements in $I$ differ by nonzero multiples of $b$, and their dot products with $b$ differ by nonzero multiples of $\|b\|^2 \neq 0$, we may order elements in $I$ by their dot products with $b$. Write $I = \{ x_1, x_2, \ldots, x_l \}$, where the subscripts respect the ordering and $l = |I|$. Now any set in $A_X$ can be written as $\{ x_u : i \leq u \leq j \}$ for some $(b, I, i, j)$. We use the following decomposition. For each $b \in \mathbb{Z}^d \setminus \{0\}$ and congruence class $I$ modulo $b$, we consider sets of the form $\{ x_u : (j - 1)s + 1 \leq u \leq js \}$ where $s = 2^t$ is a power of $2$, and $1 \leq j \leq |I/s|$. Let $C_X$ be the collection of such sets for all $(b, I)$. All sets in $C_X$ are of sizes powers of 2. The following lemma relates the discrepancy with respect to $C_X$ and that with respect to $A_X$. This uses idea from [8, Lemma 3.3].

**Lemma 2.1.** Let $b : \mathbb{N} \to (0, \infty)$ be a function. If $\chi : X \to \mathbb{R}$ satisfies that

$$|\chi(S)| \leq b(|S|)$$

for all $S \in C_X$, then

$$|\chi(A)| \leq 2 \sum_{s : s = 2^t} b(s)$$

for all $A \in A_X$.

**Proof.** For any $A \in A_X$, we know that it can be written as $A_0 \cap X$ for some arithmetic progression $A_0 \in \mathcal{A}_X$. Let $b$ be the common difference of $A_0$, and let $I$ be intersection of $X$ and the congruence class $I$ modulo $b$. Then $A$ is a subset of $I$. Moreover, as we describe in the procedure above, if we order elements in $I = \{ x_1, x_2, \ldots, x_l \}$ by their dot product with $b$, we know that $A$ must be in the form $\{ x_u : i \leq u \leq j \}$. We write $A = A_1 \setminus A_2$ where $A_1 = \{ x_u : 1 \leq u \leq j \}$ and $A_2 = \{ x_u : 1 \leq u \leq i - 1 \}$. Then we know that $A_1$ can be written as a disjoint union of sets in $C_X$ of different sizes $A_1 = S_1 \cup S_2 \cup \cdots \cup S_t$ using the binary representation of $j$, where $t$ is the number
of digits 1 in the representation. Also note that all sets in $C_X$ are of sizes powers of 2, so we have

$$|\chi(A_1)| = \left| \sum_{k=1}^{t} \chi(S_k) \right| \leq \sum_{k=1}^{t} |\chi(S_k)| \leq \sum_{s:s=2^t} b(s).$$

We may prove the similar inequality for $\chi(A_2)$ by replacing $j$ with $i - 1$. Combining them we get

$$|\chi(A)| = |\chi(A_1) - \chi(A_2)| \leq 2 \sum_{s:s=2^t} b(s).$$

To apply the partial coloring lemma, Lemma 3.1, to $(X, C_X)$, we need to estimate the number of sets of each size, and pick each $\Delta_S$ appropriately. Let $s = 2^t$ be any power of 2, we define $f(s, X)$ to be the number of sets of size $s$ in $C_X$. Note that $f(1, X) = |X|$.

For a positive integer $s$, a finite set $X \subseteq \mathbb{Z}^d$ and a vector $b \in \mathbb{Z}^d \setminus \{0\}$, let $U^d(X, b, s)$ denote the set of all $x \in X$, whose residue class mod $b$ contains at least $s$ elements in $X$, or formally $\{x' \in X : x' \equiv x \pmod{b}\}$ is of size at least $s$. The following inequality shows how these sets $U^d$ relate to the quantity $f(s, X)$.

**Lemma 2.2.** Let $X \subseteq \Omega$, and $s$ be a power of 2. Then

$$f(s, X) \leq \frac{1}{s} \sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^d(X, b, s)|.$$  \hfill (2.1)

**Proof.** We estimate the number of sets in $C_X$ of size $s$. For each $b \in \mathbb{Z}^d \setminus \{0\}$, we partition $X$ into the congruence classes modulo $b$ which we denote by $I_1, ..., I_t$. By definition, $U^d(X, b, s)$ is the disjoint union of all $I_k$ that contains at least $s$ elements. Each set of size $s$ in $C_X$ lies entirely in some $I_k$ for some appropriate choice of $b$ and $I_k$. The number of such sets in $I_k$ is at most $|I_k|/s$. Therefore, if we sum over all congruence classes, the number of sets in $C_X$ of size $s$ for any fixed $b$ is

$$\sum_{k=1}^{t} \left\lfloor |I_k|/s \right\rfloor \leq \sum_{k:|I_k|\geq s} \left\lfloor |I_k|/s \right\rfloor = \frac{|U^d(X, b, s)|}{s}.$$ 

Summing over all possible common differences $b \in \mathbb{Z}^d \setminus \{0\}$, we obtain (2.1). \hfill $\Box$

Hence, we need to estimate the sum of the numbers of elements in these $U^d$ sets. We have the following simple upper bound.

**Lemma 2.3.** For any $s \geq 2$ and $X \subseteq [N_1] \times \cdots \times [N_d]$, we have

$$\sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^d(X, b, s)| \leq |X| \cdot \prod_{i=1}^{d} \left( \frac{4N_i}{s} + 1 \right).$$
Proof. We focus on those \( b \) for which \( U^d(X, b, s) \) is nonempty. If \( U^d(X, b, s) \) is nonempty, then we know that for each \( i \), the \( i \)th coordinate of \( b \) is in the interval \(( -\frac{N_i}{s-1}, \frac{N_i}{s-1} )\). Therefore, the number of nonempty \( U^d(X, b, s) \) is at most

\[
\prod_{i=1}^{d} \left( \frac{2N_i}{s-1} + 1 \right) \leq \prod_{i=1}^{d} \left( \frac{N_i}{s} + 1 \right).
\]

Applying the trivial bound \( |U^d(X, b, s)| \leq |X| = m \), we get the desired inequality. \( \square \)

By combining Lemmas 2.2 and Lemma 2.3, the following upper bound on \( f(s, X) \) holds.

**Corollary 2.4.** For any \( 1 \leq s \leq \min_{1 \leq i \leq d} N_i \), we have

\[
f(s, X) \leq 5^d \frac{N_1 \cdots N_d}{s^{d+1}} |X|.
\]

**Proof.** When \( s = 1 \), we have \( f(1, X) = |X| \) and the inequality clearly holds. In the remaining cases, \( 2 \leq s \leq \min_{1 \leq i \leq d} N_i \), we would like to apply Lemmas 2.2 and 2.3. As \( s \leq N_i \) for each \( 1 \leq i \leq d \), it follows that

\[
f(s, X) \leq \frac{1}{s} \cdot |X| \prod_{i=1}^{d} \left( \frac{N_i}{s} + 1 \right) \leq 5^d \frac{N_1 \cdots N_d}{s^{d+1}} |X|.
\]

\( \square \)

We remark that in Section 4 we prove Lemma 3.5 that, together with Lemma 2.2, gives a better upper bound on \( f(s, X) \) than Corollary 2.4 for a certain range of \( s \).

### 3 | PROOF OF THE UPPER BOUND IN THEOREM 1.2

We use the following version of the partial coloring lemma proved by Lovett and Meka in [6].

**Lemma 3.1** [6, Theorem 2.1]. Let \( (\Omega, C) \) be a set system on \( n \) elements, and let a number \( \Delta_S \geq 0 \) be given for each set \( S \in C \) and let \( \delta > 0 \) be a real number. Suppose that

\[
\sum_{S \in C : S \neq \emptyset} \exp \left( - \frac{\Delta_S^2}{16 |S|} \right) \leq \frac{n}{16}.
\]  

Then there is an efficient algorithm that, given any \( \chi_0 : \Omega \rightarrow [-1, 1] \), finds \( \chi : \Omega \rightarrow [-1, 1] \) that has \( |\chi(x)| \geq 1 - \delta \) for at least \( n/2 \) values of \( x \in \Omega \) and satisfies \( |(\chi - \chi_0)(S)| \leq \Delta_S \) for each \( S \in C \).

**Remark.** The algorithm in this lemma is probabilistic, and runs in \( O((|C| + |\Omega|)^{3\delta^{-2}} \log(|\Omega||C|/\delta)) \) time with success probability at least \( \frac{1}{10} \). Compared to the nonconstructive version of the partial coloring lemma that uses the entropy method (see [7, 8]), the
inequality (3.1) takes a simpler form and helps simplify later computations. For more details about the differences between the two versions of the partial coloring lemma, see [6, Section 3].

The next lemma gives an estimate that is helpful in applying the previous lemma.

**Lemma 3.2.** Let \( d \in \mathbb{N} \) and \( c \geq 2d + 4 \). Let \( K \) be a positive real number. Then for \( b \) defined as

\[
b(s) = \begin{cases} 
0 & \text{if } s \geq cK \frac{1}{d+1} \\
\frac{c^2 s^2 K}{4d+4} & \text{if } cK \frac{1}{d+1} > s < cK \frac{1}{d+1}, 
\end{cases}
\]  

we have

\[
\sum_{i=0}^{\infty} \frac{K}{2^{i(d+1)}} \exp \left( -\frac{b(2i)^2}{16 \cdot 2^i} \right) \leq 3c^{-d-1}
\]

**Proof.** We shall estimate the sum over the ranges of \( b \). In the summation below, we assume that \( s \) is taken over powers of 2.

\[
\sum_{s \geq cK \frac{1}{d+1}} \frac{K}{s^{d+1}} \leq 2 \frac{K}{cK \frac{1}{d+1}^{d+1}} = \frac{2}{c^{d+1}}.
\]

Now we parameterize \( s \) by \( \tau = s(cK \frac{1}{d+1})^{-1} \). When \( s < cK \frac{1}{d+1} \), we see that \( \tau < 1 \) and

\[
\frac{K}{s^{d+1}} \exp \left( -\frac{b(s)^2}{16s} \right) = \frac{1}{c^{d+1} \tau^{d+1}} \exp \left( -c^2 \frac{1}{2} s \frac{1}{2} K \frac{1}{\tau^{d+2}} \right) = \frac{e^{-c\tau^{-\frac{1}{2}}}}{c^{d+1} \tau^{d+1}}.
\]

Because \( e^x \geq x^k/k! \) for any \( x > 0 \), by taking \( k = 2d + 4 \) and \( x = c\tau^{-\frac{1}{2}} \), we have \( e^{-c\tau^{-\frac{1}{2}}} \leq (2d + 4)^{\frac{1}{d+2}} \frac{1}{c^{d+5}} \). Noting that \( \{ \tau : \tau < 1 \} \) can be expressed as a geometric series with common ratio 1/2, we have

\[
\sum_{s < cK \frac{1}{d+1}} \frac{K}{s^{d+1}} \exp \left( -\frac{b(s)^2}{16s} \right) = \sum_{\tau < 1} \frac{e^{-c\tau^{-\frac{1}{2}}}}{c^{d+1} \tau^{d+1}} \leq \sum_{\tau < 1} \frac{(2d + 4)^{\frac{1}{d+2}} \frac{1}{c^{d+5}}}{c^{d+5}} < \frac{2(2d + 4)!}{c^{3d+5}} \cdot \tau < \frac{2(2d + 4)!}{c^{3d+5}}.
\]

Recall that \( c \geq 2d + 4 \), so \( 2(2d + 4)! \leq (2d + 4)^{2d+4} \leq c^{2d+4} \). Hence, \( \frac{2(2d + 4)!}{c^{3d+5}} \leq \frac{1}{c^{d+1}} \). Thus,

\[
\sum_{s} \frac{K}{s^{d+1}} \exp \left( -\frac{b(s)^2}{16s} \right) = \sum_{s \geq cK \frac{1}{d+1}} \frac{K}{s^{d+1}} + \sum_{s < cK \frac{1}{d+1}} \frac{K}{s^{d+1}} \exp \left( -\frac{b(s)^2}{16s} \right) \leq 3c^{-d-1},
\]

as desired. \( \square \)
We apply Lemma 3.1 to the set system \((X, C_X)\) defined in Section 2. To illustrate how we are going to apply the lemma, we first prove a result (Corollary 3.4) that is slightly weaker than Theorem 1.1. As a first step in this direction, we have the following proposition.

**Proposition 3.3.** Let \(d \in \mathbb{N}\). Suppose that \(N_1, \ldots, N_d \in \mathbb{N}\) satisfy \(N_1 \cdots N_d \leq \left(\min_{1 \leq i \leq d} N_i\right)^{d+1}\). For any \(X \subseteq [N_1] \times \cdots \times [N_d]\) and \(X_0 : X \to [-1, 1]\), there exists \(\chi : X \to [-1, 1]\) that has \(|\chi(x)| \geq 1 - \frac{1}{\max_{1 \leq i \leq d} N_i}\) for at least \(|X|/2\) values of \(x \in X\) and satisfies

\[
\max_{A_0 \in \mathcal{A}_d} |(\chi - X_0)(A_0 \cap X)| \leq 200d^{d/4}(N_1 \cdots N_d)^{1/2d+2}.
\]

**Proof.** Without loss of generality, we may assume that \(N_1 \leq \cdots \leq N_d\). We want to apply Lemma 3.1 to the sets system \((X, \mathcal{C}_X)\) and \(\delta = 1/N_d\). We first need to find a function \(b : \mathbb{N} \to [0, +\infty)\) such that the values of \(\Delta_S = b(|S|)\) for \(S \in \mathcal{C}_X\) satisfy the assumption (3.1) in Lemma 3.1. By definition of \(\mathcal{C}_X\), we know that \(s = |S|\) only takes values in powers of 2. The assumption (3.1) may be written as

\[
\sum_{s : s = 2^t \leq N_d} f(s, X) \exp\left(-\frac{b(s)^2}{16s}\right) \leq \frac{|X|}{16}.
\]

(3.3)

Here, we sum only over \(s \leq N_d\) as there is no congruence class of \(X \subseteq [N_1] \times \cdots \times [N_d]\) of length greater than \(N_d\). Once we have this inequality, we can set \(\Delta_S = b(|S|)\) for \(S \in \mathcal{C}_X\) and the assumption in Lemma 3.1 would be satisfied.

To estimate \(f(s, X)/|X|\), we know that Corollary 2.4 gives an upper bound for \(1 \leq s \leq N_1\). For \(N_1 < s \leq N_d\), we apply Lemmas 2.2 and get that

\[
\frac{f(s, X)}{|X|} \leq \frac{1}{s} \prod_{i=1}^d \left(\frac{N_i}{s} + 1\right) \leq \frac{1}{s} \cdot \frac{5}{N_1} \cdot \frac{d-1}{5} \cdot \frac{N_d}{N_1} = \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \cdot \frac{1}{s^2}.
\]

(3.4)

Here we use \(4 \cdot \frac{N_d}{s} + 1 \leq 5 \cdot \frac{N_d}{s}\) as \(s \leq N_d\), and \(4 \cdot \frac{N_1}{s} + 1 \leq 5 \cdot \frac{N_1}{N_1}\) as \(s > N_1\) and \(N_1 \geq N_i\). When \(N_1 < s \leq 8 \cdot 5^{\frac{d}{2}} N_1\), we have

\[
\frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \cdot \frac{1}{s^2} \leq \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \left(\frac{s}{N_1}\right)^{d-1} \leq \frac{8^d 5^d}{s^{d+1}} \frac{N_1 \cdots N_d}{N_1^{d-1}}.
\]

In summary,

\[
\frac{f(s, X)}{|X|} \leq \begin{cases} \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \cdot \frac{1}{s^2} & \text{if } 8 \cdot 5^{\frac{d}{2}} N_1 < s \leq N_d, \\ \frac{8^d 5^d}{s^{d+1}} \frac{N_1 \cdots N_d}{N_1^{d-1}} & \text{if } 1 \leq s \leq 8 \cdot 5^{\frac{d}{2}} N_1. \end{cases}
\]

(3.5)

Recall that \(s\) takes values in powers of 2. For sets in the first range in (3.5), we have

\[
\sum_{8 \cdot 5^{\frac{d}{2}} N_1 < s \leq N_d} \frac{f(s, X)}{|X|} \leq \sum_{8 \cdot 5^{\frac{d}{2}} N_1 < s \leq N_d} \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \cdot \frac{1}{s^2} \leq \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \cdot \frac{2}{\left(8 \cdot 5^{\frac{d}{2}} N_1\right)^2} \leq \frac{1}{32}.
\]

(3.6)
Here in the last inequality we use that $N_1 \cdots N_d \leq N_1^{d+1}$. For the remaining sets, we apply Lemma 3.2 to $K = 8^d 5^{d(d+1)/2} N_1 \cdots N_d$ and $c = 10d$ and derive that, if we choose

$$b(s) = \begin{cases} 
0 & \text{if } s \geq cK^{1/(d+1)} \\
\frac{1}{c^2 s^2} K^{\frac{1}{4d+4}} & \text{if } s < cK^{1/(d+1)},
\end{cases} \tag{3.7}$$

then we have

$$\sum_{s \leq 8 \cdot 5^{\frac{d}{2}} N_1} f(s, X) \exp \left( -\frac{b(s)^2}{16s} \right) \leq \sum_{s} K|X| \exp \left( -\frac{b(s)^2}{16s} \right) \leq 3c^{-d-1}|X| \leq \frac{3|X|}{100} < \frac{|X|}{32}. \tag{3.8}$$

Hence, combining (3.6) and (3.8) we have

$$\sum_{s} f(s, X) \exp \left( -\frac{b(s)^2}{16s} \right) \leq \sum_{s \leq 8 \cdot 5^{\frac{d}{2}} N_1} f(s, X) + \sum_{s > 8 \cdot 5^{\frac{d}{2}} N_1} f(s, X) < \frac{|X|}{32} + \frac{|X|}{32} = \frac{|X|}{16}.$$

Therefore, (3.3) is satisfied for this specific choice of $b$. We shall apply Lemma 3.1 and get that for any given $X_0$, there exists $\chi : X \to [-1, 1]$ that has $|\chi(x)| \geq \frac{N_{d-1}}{N_d}$ for at least $|X|/2$ values of $x \in X$, and satisfies that for any $S \in C_X$,

$$|\chi - \chi_0)(S)| \leq b(|S|),$$

where $b$ is as in (3.7). Now applying Lemma 2.1 to set $X$ and function $(\chi - \chi_0)$, we have

$$|(\chi - \chi_0)(A)| \leq \sum_{s} b(s) = \sum_{s < cK^{1/(d+1)}} \frac{3}{c^2 s^2} K^{\frac{1}{4d+4}} \leq \frac{cK^{\frac{1}{d+1}}}{1 - (1/2)^{\frac{1}{2}}} = \frac{10d8^{d/2} 5^{d/2}}{1 - (1/2)^{\frac{1}{2}}} (N_1 \cdots N_d)^{\frac{1}{4d+2}}.$$

for any $A \in A_X = \{A_0 \cap X : A_0 \in A_N\}$, as desired. Here we use that $\{s^{\frac{1}{2}} : s \leq cK^{1/(d+1)}\}$ can be written as a geometric sequence with initial term at most $cK^{1/(d+1)}$ and ratio $\sqrt{1/2}$. Finally, note that $\sqrt{8}/(1 - \sqrt{1/2}) < 20$, so the constant claimed in the statement is valid.

Iteratively applying this lemma, we can get the following corollary.

**Corollary 3.4.** For $d, N \in \mathbb{N}$, we have $\text{disc}(A_d) \leq 200d^2 5^{d/4} N^{\frac{d}{4d+2}} \log_2(2N) + 1$.

**Proof.** Let $\Omega = [N]^d$. Given a map $\chi : \Omega \to [-1, 1]$, we call an element $x \in \Omega$ is frozen in $\chi$ if $|\chi(x)| \geq \frac{N_{d-1}}{N_d}$, and free otherwise.

Start with $X_0$ that maps everything from $\Omega$ to 0, we show that there exists a sequence $X_0 = 0, X_1, \ldots, X_i$ such that the number of free elements in $X_{i+1}$ is at most a half of that in $X_i$, and

$$\max_{A_0 \in A_d} |(X_{i+1} - X_i)(A_0)| \leq cN^{\frac{d}{4d+2}}$$
for $i \geq 0$ where $c = 200d^{d/4}$. We stop until we get $\chi_T$ where all elements are frozen.

We now show the existence of $\chi_{i+1}$ given $\chi_i$ for $i < T$. Let $X_i$ be the set of free elements in $\chi_i$. We apply Proposition 3.3 to the set $X_i \subseteq \Omega$ and the given $\chi_i|_{X_i} : X_i \to [-1, 1]$. The proposition says that there exists $\hat{\chi}_i : X_i \to [-1, 1]$ such that at least a half of elements $x \in X_i$ has $|\hat{\chi}_i(x)| \geq \frac{N-1}{N}$ and

$$\max_{A_0 \in \mathcal{A}_d} |(\hat{\chi}_i - \chi_i|_{X_i})(A_0 \cap X_i)| \leq cN^{d/3+2}.$$  

We shall define $\chi_{i+1}(x) = \hat{\chi}_i(x)$ if $x \in X_i$ and $\chi_{i+1}(x) = \chi_i(x)$ otherwise. Then for $x \in \Omega \setminus X_i$, because $x$ is frozen in $\chi_i$, it is also frozen in $\chi_{i+1}$. Therefore, the number free elements in $\chi_{i+1}$ is at most half of $|X_i|$, as desired. Moreover, we know that $\chi_{i+1} - \chi_i$ is zero outside $X_i$. Hence, for any $A_0 \in \mathcal{A}_d$,

$$|\chi_{i+1} - \chi_i(A_0)| = |\chi_{i+1} - \chi_i(A_0 \cap X_i)| = |\hat{\chi}_i - \chi_i|_{X_i}(A_0 \cap X_i)| \leq cN^{d/3+2}.$$  

As the number of free elements halves each round, the number of rounds $T \leq \log_2(2N^d) \leq d \log_2(2N)$. For the final $\chi_T$, we have that for any $A_0 \in \mathcal{A}_d$,

$$|\chi_T(A_0)| = |\chi_T - \chi_0(A_0)| \leq \sum_{i=0}^{T-1} |\chi_{i+1} - \chi_i(A_0)| \leq cN^{d/3+2}T \leq cN^{d/3+2} \cdot d \log_2(2N).$$  

To get a full coloring $\chi : \Omega \to \{1, -1\}$, we take $\chi(x)$ to be $\chi_i(x)$ rounded up or down, depending on whether it is closer to 1 or $-1$. Then as $|\chi(x) - \chi_T(x)| \leq 1/N$ for all $x \in \Omega$, we have $|\chi(A_0) - \chi_T(A_0)| \leq |A_0|/N \leq 1$ as any arithmetic progression $A_0 \in \mathcal{A}_d$ has length at most $N$. By triangle inequality we conclude that $\chi$ gives the desired upper bound on discrepancy of $\mathcal{A}_d$.  

Not surprisingly, to improve on the bound above, we need to improve on Corollary 2.4. In particular, we will show in Section 4 that the following holds.

**Lemma 3.5.** There exists an absolute constant $C$ such that the following holds. Let $d$ be a positive integer. Given positive integers $N_1, N_2, \ldots, N_d$ satisfying $N_1 \cdots N_d \leq (\min_{1 \leq i \leq d} N_i)^{d+1-\delta}$ for some $\delta \in (0, 1]$, suppose that $X \subseteq [N_1] \times \cdots \times [N_d]$ is of size $m$. Letting $\rho = m^{1/d} N_1 \cdots N_d$, if integer $s$ satisfies

$$(N_1 \cdots N_d)^{1/d+1} \rho^{\min(\delta, (d+1)/2)} \leq s \leq (\min_{1 \leq i \leq d} N_i)^\beta$$

for some $\beta \in (0, 1/2)$, then

$$\sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^d(X, b, s)| \leq C 2^{d^2} s^d m N_1 N_2 \cdots N_d \cdot \rho^{\min(\delta, d+1)}.$$  

Assuming Lemma 3.5, we prove the following improvement on Proposition 3.3.

**Proposition 3.6.** Let $d$ be a positive integer. There exist constants $C_d$ and $c_d$ such that the following holds. Let $N_1, N_2, \ldots, N_d$ be positive integers satisfying $N_1 \cdots N_d \leq (\min_{1 \leq i \leq d} N_i)^{d+1-\delta}$ for some $\delta \in (0, 1]$. For any nonempty $X \subseteq [N_1] \times \cdots \times [N_d]$, given any $\chi_0 : X \to [-1, 1]$, there exists $\chi : X \to$
[−1, 1] that has |χ(x)| ≥ 1 − \( \frac{1}{\max_{1 \leq i \leq d} N_i} \) for at least |X|/2 values of \( x \in X \), and satisfies that

\[
\max_{A_0 \in \mathcal{A}_N} |(\chi - \chi_0)(A_0 \cap X)| \leq C_d(N_1 \cdots N_d)^{\frac{1}{d+2}} \cdot \left( \frac{|X|}{N_1 \cdots N_d} \right)^{c_d \delta}.
\]

**Proof.** The general proof strategy is the same as in the proof of Proposition 3.3. Without loss of generality, we may assume \( N_1 \leq \cdots \leq N_d \). Moreover, compared with Proposition 3.3, we may assume

\[
\max_{A_0 \in \mathcal{A}_N} |(\chi - \chi_0)(A_0 \cap X)| \leq C_d(N_1 \cdots N_d)^{\frac{1}{d+2}} \cdot \left( \frac{|X|}{N_1 \cdots N_d} \right)^{c_d \delta} < 200d^{d/4},
\]

or otherwise we already have the statement with Proposition 3.3. In particular, we may assume that \((N_1 \cdots N_d)^{\delta} \geq \rho^{-\delta}\) are both sufficiently large with respect to \( d \) by choosing \( C_d \) and \( c_d \) appropriately.

As in the proof of Proposition 3.3, we would like to find a function \( b : \mathbb{N} \to (0, \infty) \) such that

\[
\sum_{s: s=2^t \leq N_d} f(s, X) \exp\left(-\frac{b(s)^2}{16s}\right) \leq |X|/16.
\]

(3.11)

From now on, we take \( s \) to be a power of 2. To estimate \( f(s, X)/|X| \), we know that Corollary 2.4 gives an upper bound for \( 1 \leq s \leq N_1 \). For \( N_1 < s \leq N_d \), as in the proof of Proposition 3.3, we apply Lemmas 2.2 and 2.3 and get that

\[
\frac{f(s, X)}{|X|} \leq \frac{1}{s} \prod_{i=1}^{d} \left( \frac{4N_i}{s} + 1 \right) \leq \frac{1}{s} \cdot 5 \frac{N_d}{s} \cdot \prod_{i=1}^{d-1} 5 \frac{N_i}{N_1} = \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} \cdot \frac{1}{s^2}.
\]

(3.12)

Take \( \beta = \frac{\delta}{4(d+1)^2} \). Now combining with Lemma 3.5, we derive the following inequality:

\[
\frac{f(s, X)}{|X|} \leq \begin{cases} 
\frac{1}{s} \frac{5^d N_1 \cdots N_d}{N_1^{d-1}} & \text{if } N_1 < s \leq N_d \\
\frac{5^d N_1 \cdots N_d}{s^{d+1}} & \text{if } 1 \leq s < (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\frac{\delta}{d^2(d+2)} \text{ or } N_1 \rho^\delta < s \leq N_1 \\
C_0 N_1 \cdots N_d \rho^{\frac{\delta}{d^2(d+2)}} & \text{if } (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\frac{\delta}{d^2(d+1)} \leq s \leq N_1 \rho^\delta
\end{cases}
\]

(3.13)

for \( \rho := \frac{|X|}{N_1 \cdots N_d} \). Here we applied (3.12) on the first range, Corollary 2.4 on the second, and Lemma 3.5 on the third with \( C_0 = C2^{d^2}5^d \) for some absolute constant \( C \).

We now show that (3.11) is satisfied for \( c = 100d \) if we take \( b \) to be

\[
b(s) = \begin{cases} 
0 & \text{if } (N_1 \cdots N_d)^{\frac{1}{d+1}} < s \\
\frac{1}{c s^{\frac{\delta}{d+2}}} (C_0 N_1 \cdots N_d)^{\frac{1}{d+2}} \rho^{\frac{\delta}{d^2(d+2)}} & \text{if } (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\frac{\delta}{d^2(d+1)} \leq s \leq (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\frac{\delta}{d^2(d+1)} \\
\frac{1}{c s^{\frac{\delta}{d+2}}} (5^d N_1 \cdots N_d)^{\frac{1}{d+2}} & \text{if } 1 \leq s < (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\frac{\delta}{d^2(d+1)}
\end{cases}
\]

(3.14)
We consider summation over each range. Applying (3.13), we have (noting that \( \rho \geq (N_1 \cdots N_d)^{-1} \))

\[
\sum_{N_1 < s} \frac{f(s, X)}{|X|} \leq \sum_{N_1 < s} \frac{N_1 \cdots N_d}{s^2 N_1} \leq 2 \cdot \frac{N_1 \cdots N_d}{(N_1 \cdots N_d)^{\frac{\delta}{d+1} - \frac{\delta}{d+1}}} < \frac{1}{300},
\]

\[
\sum_{N_1 \rho^\delta < s \leq N_1} \frac{f(s, X)}{|X|} \leq \sum_{N_1 \rho^\delta < s \leq N_1} \frac{N_1 \cdots N_d}{s^{d+1}} \leq 2 \cdot \frac{N_1 \cdots N_d}{(N_1 \cdots N_d)^{\frac{\delta}{d+1} - \frac{\delta}{d+1}}} < \frac{1}{300},
\]

In the last inequalities, we use \((N_1 \cdots N_d)^{\delta} \geq \rho^{-\delta}\) are both sufficiently large. Using \(b(s) = 0\) we get

\[
\sum_{N_1 \rho^\delta < s \leq N_1} \frac{f(s, X)}{|X|} \exp\left(-\frac{b(s)^2}{16s}\right) < \frac{1}{100}. \tag{3.15}
\]

For the second range, we apply Lemma 3.2 to \(c = (100d)^{\frac{4}{3}}\) and \(K_1 = C_0N_1 \cdots N_d \rho^{\frac{\beta}{d^{(d+1)}}}\), and let \(b_1\) be as defined in (3.2). Then \(b \geq b_1\) on \((N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\delta \leq s \leq (N_1 \cdots N_d)^{\frac{1}{d+1}}\). Therefore, we have

\[
\sum_{(N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\delta < s \leq (N_1 \cdots N_d)^{\frac{1}{d+1}}} \frac{f(s, X)}{|X|} \exp\left(-\frac{b(s)^2}{16s}\right) \leq \sum_{s} K_1 \frac{s^{d+1}}{s^{d+1}} \exp\left(-\frac{b_1(s)^2}{16s}\right) < \frac{3}{c^{d+1}}. \tag{3.16}
\]

Similarly, for the third range, we apply Lemma 3.2 to \(c = (100d)^{\frac{4}{3}}\) and \(K_2 = 5^d N_1 \cdots N_d\). Let \(b_2\) be as defined in (3.2). Then \(b \geq b_2\) on \(1 \leq s < (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^\delta\), so we have

\[
\sum_{1 \leq s < (N_1 \cdots N_d)^{\frac{1}{d+1}}} \frac{f(s, X)}{|X|} \exp\left(-\frac{b(s)^2}{16s}\right) \leq \sum_{s} K_2 \frac{s^{d+1}}{s^{d+1}} \exp\left(-\frac{b_2(s)^2}{16s}\right) < \frac{3}{c^{d+1}}. \tag{3.17}
\]

In summary, combining (3.15), (3.16), and (3.17), and noting that \(3c^{-d-1} < \frac{1}{100}\) and \(\frac{3}{100} < \frac{1}{16}\), we conclude that (3.11) is satisfied. By Lemma 3.1, there exists \(\chi : X \rightarrow [-1, 1]\) that has \(|\chi(x)| \geq N_0^{-1} N_d\) for at least \(|X|/2\) values of \(x \in X\), and satisfies that for any \(S \subseteq C_X\), \(|(\chi - \chi_0)(S)| \leq b(|S|)\) where \(b\) is as in (3.14). Applying Lemma 2.1 to set \(X\) and function \((\chi - \chi_0)\), we have that for any \(A \in \mathcal{A}_X\), \(|(\chi - \chi_0)(A)| \leq \sum b(s)\). Note that \(b(s)\) is a geometric sequence with ratio \(\sqrt{2}\) on the second and the third range. On the second range and the third range, the sums are at most

\[
\sum_{s \leq (N_1 \cdots N_d)^{\frac{1}{d+1}}} c \frac{s^{\frac{1}{d+1}} (C_0N_1 \cdots N_d)^{\frac{1}{d+4}}} \rho^{\frac{\beta}{d^{(d+1)(d+2)}}} \leq \frac{100d}{1 - (1/2)^{1/4}} C_0^{\frac{1}{d+4}} (N_1 \cdots N_d)^{\frac{1}{d+4}} \rho^{\frac{\beta}{d^{(d+1)(d+2)}}},
\]

\[
\sum_{s \leq (N_1 \cdots N_d)^{\frac{1}{d+1}}} c \frac{s^{\frac{1}{d+4}} (5^d N_1 \cdots N_d)^{\frac{1}{d+4}}} \rho^{\frac{\beta}{d^{(d+1)}}} \leq \frac{100d}{1 - (1/2)^{1/4}} 5^d \frac{d}{4^{d+4}} (N_1 \cdots N_d)^{\frac{1}{d+4}} \rho^{\frac{\beta}{d^{(d+1)(d+2)}}}.
\]
Therefore, we may take $C_d \geq 2000d^{\frac{1}{4+d+2}}$ and $c_d \leq \frac{1}{4^{d+2}(d+1)^3(d+2)!}$ so that $\sum_s b(s)$ is at most the right-hand side of the desired inequality.

The next corollary improves on Corollary 3.4. The proof follows the same routine, but replaces Proposition 3.3 by Proposition 3.6.

**Corollary 3.7.** Let $d$ be a positive integer. There exist constants $C_d$ and $c_d$ such that the following holds. Let $N_1, N_2, \ldots, N_d$ be positive integers satisfying $N_1 \cdots N_d \leq (\min_{1 \leq i \leq d} N_i)^{d+1-\delta}$ for some $\delta \in (0,1]$. For any nonempty $X \subseteq [N_1] \times \cdots \times [N_d]$, there exists a coloring $\chi : X \to \{-1,1\}$ with

$$\max_{A_0 \in A_N} |\chi(A_0 \cap X)| \leq C_d^{\frac{1}{\delta}} (N_1 \cdots N_d)^{1 \frac{1}{4+d+2}} \cdot \left( \frac{|X|}{N_1 \cdots N_d} \right)^{c_d \delta}.$$

**Proof.** Let $\Omega = [N_1] \times [N_2] \times \cdots \times [N_d]$. Recall for a given a map $\chi : \Omega \to [-1,1]$, we call an element $x \in \Omega$ is frozen in $\chi$ if $|\chi(x)| \geq \frac{N-1}{N}$, and free otherwise. Start with $\chi_0$ that maps everything from $\Omega$ to 0, we show that there exists a sequence $\chi_0 = 0, \chi_1, \ldots, \chi_T$ such that the number of free elements in $\chi_{i+1}$ is at most a half of that in $\chi_i$, and

$$\max_{A_0 \in A_N} |(\chi_{i+1} - \chi_i)(A_0 \cap X)| \leq C_d^{\frac{1}{\delta}} (N_1 \cdots N_d)^{1 \frac{1}{4+d+2}} \cdot \left( \frac{|X|}{N_1 \cdots N_d} \right)^{c_d \delta}$$

for $i \geq 0$. We stop until we get $\chi_T$ where all elements are frozen.

We now show the existence of $\chi_{i+1}$ given $\chi_i$ for $i < T$. Let $X_i$ be the set of free elements in $\chi_i$. We apply Proposition 3.6 to the set $X_i \subseteq \Omega$ and the given $\chi_i|X_i : X_i \to [-1,1]$. The proposition says that there exists $\hat{\chi}_i : X_i \to [-1,1]$ such that at least a half of elements $x \in X_i$ has $|\hat{\chi}_i(x)| \geq \frac{N-1}{N}$ and

$$\max_{A_0 \in A_d} |(\hat{\chi}_i - \chi_i|X_i)(A_0 \cap X_i)| \leq C_d (N_1 \cdots N_d)^{\frac{1}{4+d+2}} \cdot \left( \frac{|X_i|}{N_1 \cdots N_d} \right)^{c_d \delta}.$$

We shall define $X_{i+1}(x) = \hat{\chi}_i(x)$ if $x \in X_i$ and $X_{i+1}(x) = \chi_i(x)$ otherwise. Then for $x \in \Omega \setminus X_i$, because $x$ is frozen in $\chi_i$, it is also frozen in $\chi_{i+1}$. Therefore, the number free elements in $\chi_{i+1}$ is at most half of $|X_i|$, as desired. Moreover, we know that $\chi_{i+1} - \chi_i$ is zero outside $X_i$. Hence, for any $A_0 \in A_d$,

$$|(\chi_{i+1} - \chi_i)(A_0)| = |(\chi_{i+1} - \chi_i)(A_0 \cap X_i)|$$

$$= |(\hat{\chi}_i - \chi_i|X_i)(A_0 \cap X_i)|$$

$$\leq C_d (N_1 \cdots N_d)^{\frac{1}{4+d+2}} \cdot \left( \frac{|X_i|}{N_1 \cdots N_d} \right)^{c_d \delta}.$$ (3.18)
For the final $\chi_T$, we have that for any $A_0 \in A_d$,

$$|\chi_T(A_0)| = |(\chi_T - \chi_0)(A_0)| \leq \sum_{i=0}^{T-1} |(\chi_{i+1} - \chi_i)(A_0)| \leq C_d(N_1 \cdots N_d)^{\frac{1}{2d+2}} \left( \frac{|X|}{N_1 \cdots N_d} \right)^{\epsilon_d \delta} \sum_i \left( \frac{1}{2} \right)^{\epsilon_d \delta},$$

where the inner sum converges and with abuse of notation, and we absorb it into the constant $C_d$. To get a full coloring $\chi : \Omega \to \{1, -1\}$, we take $\chi(x)$ to be $\chi_T(x)$ rounded up or down, depending on whether it is closer to 1 or $-1$. Then as $|\chi(x) - \chi_T(x)| \leq 1/N$ for all $x \in \Omega$, we $|\chi(A_0) - \chi_T(A_0)| \leq |A_0|/N \leq 1$ because any arithmetic progression $A_0 \in \mathbb{A}_d$ has length at most $N$. By triangle inequality, we conclude that $\chi$ gives the desired upper bound on discrepancy of $A_d$. \hfill $\Box$

**Proof of Theorem 1.2.** By using Corollary 3.7 and taking $X = [N_1] \times \cdots \times [N_d]$, we have $\text{disc}(A_N) \leq C_d \left( \frac{N_1 \cdots N_d}{\delta} \right)^{\frac{1}{2d+2}}$ for some $C_d$ only depends on $d$ that proves Theorem 1.2. \hfill $\Box$

### 4 A BETTER ESTIMATE ON THE NUMBER OF SETS IN THE DECOMPOSITION

In this section, we prove Lemma 3.5, which improves upon Lemma 2.3. Recall that $U_d(X, b, s)$ is the set of elements in $X$ for which there are at least $s$ elements of $X$ in the same residue class mod $b$.

To prove Lemma 3.5, we induct on $d$. We need the following two results regarding lattice points.

**Lemma 4.1** Minkowski’s theorem, see, e.g., [3, section III.2.2]. Let $X \subseteq \mathbb{R}^d$ be a point set of volume $V(X)$ that is symmetric about the origin and convex. Let $\Gamma \subseteq \mathbb{R}^d$ be a $d$-dimensional lattice of determinant $\det(\Gamma)$. If $V(X) > 2^d \det(\Gamma)$, then $X \cap \Gamma$ contains a pair of distinct points $\pm x$.

**Lemma 4.2** Lenstra–Lenstra–Lovász Basis Reduction [5]. Let $\Gamma \subseteq \mathbb{R}^d$ be a $d$-dimensional lattice of determinant $\det(\Gamma)$. There exists a basis $x_1, \ldots, x_d$ of $\Gamma$ such that

$$\det(\Gamma) \leq \prod_{i=1}^n \|x_i\|_2^2 \leq 2^{d(d-1)/4} \det(\Gamma).$$

**Remark.** The inequality above is true for any reduced basis (see [5, Proposition 1.6]); the existence of which is guaranteed by an algorithm that transforms any given basis to a reduced one (see [5, Proposition 1.26]). It will not be important for us what the definition of a reduced basis is.
Corollary 4.3. Let $\Gamma \subseteq \mathbb{R}^d$ be a $d$-dimensional lattice. Suppose that $V_0 > 0$ is a real number such that the following holds: for every set $P \subseteq \mathbb{R}^d$ of volume $V_d(P) > V_0$ that is symmetric about the origin and convex, $P \cap \Gamma$ contains a nonzero point. Then there exists a basis $x_1, \ldots, x_d$ of $\Gamma$ such that

$$\prod_{i=1}^n \|x_i\|_2 \leq 2 \frac{(d-1)^d}{4} - d V_0.$$  

Proof. Let $y_1, \ldots, y_d$ be a basis of $\Gamma$. Note that the fundamental parallelepiped of $\Gamma$ defined as $\{a \cdot y : a \in (0,1)^d\}$ contains no nonzero vector of $\Gamma$. By translation, there is no nonzero vector of $\Gamma$ in $P := \{a \cdot y : a \in (-1,1)^d\}$, and from the condition we know that $V_d(P) \leq V_0$. Note that $V_d(P) = 2^d \det(\Gamma)$, so $\det(\Gamma) \leq 2^{-d} V_0$. By Lemma 4.2, there exists a basis $x_1, \ldots, x_d$ such that

$$\prod_{i=1}^n \|x_i\|_2 \leq 2 \frac{(d-1)^d}{4} \det(\Gamma) \leq 2 \frac{(d-1)^d}{4} - d V_0,$$

which completes the proof. \hfill $\square$

Before we start to prove Lemma 3.5, we introduce some standard notation. For a map $\phi : X \to Y$ and subsets $A \subseteq X$ and $B \subseteq Y$, let $\phi(A)$ be the set of images $\{\phi(a) : a \in A\} \subseteq Y$, and let $\phi^{-1}(B)$ be the set of preimages $\{x \in X : \phi(x) \in B\} \subseteq X$.

We next prove a geometric lemma which shows that, given a vector $b \in \mathbb{Z}^d$ whose coordinates have greatest common divisor 1, there is a linear map from $\mathbb{Z}^d$ to $\mathbb{Z}^{d-1}$ that has full rank with null space generated by $b$, and maps a grid into another grid with similar size.

Lemma 4.4. Let $d \geq 2$ be a positive integer, and $N_1, N_2, \ldots, N_d$ be positive integers. Suppose that $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ is a nonzero vector satisfying that $\gcd(b_1, \ldots, b_d) = 1$ and $|b_i| \leq N_i$ for all $1 \leq i \leq d$. Then there exists an affine map $f_b : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$, so that the following two conditions hold.

1. For any $x_1, x_2 \in \mathbb{Z}^d$, $f_b(x_1) = f_b(x_2)$ if and only if $x_1 - x_2 = kb$ for some $k \in \mathbb{Z}$.
2. There exist positive integers $N_1^*, N_2^*, \ldots, N_d^* \geq \min_{1 \leq i \leq d} N_i$, so that

$$\frac{1}{2} N_1 \cdots N_d \cdot \left( \max_{1 \leq i \leq d} \frac{|b_i|}{N_i} \right)^2 \leq N_1^* \cdots N_d^* \leq 2^{d^2} N_1 \cdots N_d \cdot \left( \max_{1 \leq i \leq d} \frac{|b_i|}{N_i} \right)^2 \quad (4.1)$$

and $f_b([N_1] \times \cdots \times [N_d]) \subseteq [N_1^*] \times \cdots \times [N_d^*]$.

Proof. We may write $f_b(x) = Mx + v$ for some $M \in \mathbb{Z}^{(d-1) \times d}$ and $v \in \mathbb{Z}^{d-1}$ to be chosen later. Condition (1) says that $M$ is full rank, with null space generated by $b$.

We regard the rows of $M$ as vectors $r_1, \ldots, r_{d-1} \in \mathbb{Z}^d$. For each $1 \leq j \leq d - 1$, if we write $r_j = (r_{j,1}, \ldots, r_{j,d})$, then we define $r_j^* := (r_{j,1} N_1, \ldots, r_{j,d} N_d) \in \Lambda$ where $\Lambda := N_1 \mathbb{Z} \times \cdots \times N_d \mathbb{Z}$. Condition (1) is equivalent to saying that the vectors $r_j$ in $\mathbb{Z}^d$ for $1 \leq j \leq d - 1$ are linearly independent and $r_j \cdot b = 0$ for all $1 \leq j \leq d - 1$. In terms of $r_j^*$, this is equivalent to $r_j^* \cdot b^* = 0$ for $b^* := \left( \frac{b_1}{N_1}, \ldots, \frac{b_d}{N_d} \right)$, and $r_1^*, \ldots, r_{d-1}^*$ are linearly independent vectors. The following claim allows us to find these vectors whose product of $\ell_2$-norms is small.
Claim 1. There exists linearly independent vectors \( \mathbf{r}_1^*, \ldots, \mathbf{r}_{d-1}^* \in \Lambda \) that satisfy \( \mathbf{r}_j^* \cdot \mathbf{b}^* = 0 \) for each \( 1 \leq j \leq d-1 \), and

\[
\prod_{j=1}^{d-1} \| \mathbf{r}_j^* \|_2 \leq 2^{ \frac{(d-1)(d-2)}{4} } \cdot \| \mathbf{b}^* \|_2 N_1 N_2 \cdots N_d, \tag{4.2}
\]

Proof of Claim 1. Consider the subspace of \( \mathbb{R}^d \) defined by \( \langle \mathbf{b}^* \rangle^\perp := \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{b}^* = 0 \} \) that has dimension \( d-1 \). The intersection \( \Lambda^* := \Lambda \cap \langle \mathbf{b}^* \rangle^\perp \) is a lattice in \( \langle \mathbf{b}^* \rangle^\perp \). As \( \mathbf{b} \in \mathbb{Z}^d \) is a nonzero vector with integer entries, there exist linearly independent vectors \( \mathbf{r}_1, \ldots, \mathbf{r}_{d-1} \in \mathbb{Z}^d \) that satisfy \( \mathbf{r}_j \cdot \mathbf{b} = 0 \) for each \( 1 \leq j \leq d-1 \). Hence, we can find \( d-1 \) linearly independent vectors \( \mathbf{r}_j^* \in \Lambda \cap \langle \mathbf{b}^* \rangle^\perp \) defined by \( \mathbf{r}_j^* = (\mathbf{r}_j N_1, \ldots, \mathbf{r}_j N_d) \), where \( \mathbf{r}_j = (\mathbf{r}_j N_1, \ldots, \mathbf{r}_j N_d) \) for \( 1 \leq j \leq d-1 \). The linear independence of \{\( \mathbf{r}_j^* \)\}_{j=1}^{d-1} follows from the linear independence of \{\( \mathbf{r}_j \)\}_{j=1}^{d-1}, while \( \mathbf{r}_j^* \cdot \mathbf{b}^* = \mathbf{r}_j \cdot \mathbf{b} = 0 \) for each \( 1 \leq j \leq d-1 \). Thus, we conclude that \( \Lambda^* \) is a \((d-1)\)-dimensional lattice.

We next consider some geometric properties of \( \Lambda^* \) as a subset of the \((d-1)\)-dimensional Euclidean space \( \langle \mathbf{b}^* \rangle^\perp \). Let \( P \) be a subset of \( \langle \mathbf{b}^* \rangle^\perp \) that is symmetric about the origin and convex. Now we consider the set \( X := \{ \mathbf{x} + a \mathbf{b}^* : \mathbf{p} \in P, a \in (-1/\|\mathbf{b}\|_2^2, 1/\|\mathbf{b}\|_2^2) \} \). We could equivalently phrase this as: \( X \) contains all the point \( \mathbf{x} \) that satisfies \( \mathbf{x} \cdot \mathbf{b}^* \in (-1, 1) \), and the projection of \( \mathbf{x} \) onto the hyperspace \( \langle \mathbf{b}^* \rangle^\perp \) is in \( P \). Therefore, from this geometric interpretation we know that \( V_d(X) = \frac{2}{\|\mathbf{b}\|_2} V_{d-1}(P) \) where \( V_d \) denotes the \( d \)-dimensional volume. Meanwhile, we see that as \( P \) and \{\( a\mathbf{b}^* : a \in (-1/\|\mathbf{b}\|_2^2, 1/\|\mathbf{b}\|_2^2) \)\} are both convex and symmetric about the origin, their Minkowski sum \( X \) is also convex and symmetric about the origin. By Minkowski’s theorem, Lemma 4.1, if \( V_d(X) > 2^d \det(\Lambda) \), then there is a nonzero point \( \mathbf{x} \in \Lambda \cap X \). Note that any point \( \mathbf{x} \in \Lambda \) satisfies that \( \mathbf{x} \cdot \mathbf{b}^* \) is an integer, while any point \( \mathbf{y} \in X \) satisfies that \( \mathbf{y} \cdot \mathbf{b}^* \in (-1, 1) \), we know that if \( \mathbf{x} \in \Lambda \cap X \), then \( \mathbf{x} \in \langle \mathbf{b}^* \rangle^\perp \). Note that \( \Lambda^* = \Lambda \cap \langle \mathbf{b}^* \rangle^\perp \) and \( P = X \cap \langle \mathbf{b}^* \rangle^\perp \), this means \( \mathbf{x} \in \Lambda^* \cap P \). In summary, if \( V_{d-1}(P) > 2^{d-1} \|\mathbf{b}\|_2 \det(\Lambda) = 2^{d-1} \|\mathbf{b}\|_2 N_1 \cdots N_d \), then \( P \cap \Lambda^* \) contains a nonzero point.

Therefore, we may apply Corollary 4.3 with dimension \( d-1 \), lattice \( \Lambda^* \), and \( V_0 = 2^{d-1} \|\mathbf{b}\|_2 N_1 \cdots N_d \), and we obtain that there exists a basis \( \mathbf{r}_1^*, \ldots, \mathbf{r}_{d-1}^* \) of \( \Lambda^* \) such that

\[
\prod_{j=1}^{d-1} \| \mathbf{r}_j^* \|_2 \leq 2^{ \frac{(d-1)(d-2)}{4} } \cdot \| \mathbf{b}^* \|_2 N_1 N_2 \cdots N_d,
\]

so we have these linearly independent vectors as expected. \( \square \)

From the set of vectors \{\( \mathbf{r}_j^* \)\}_{j=1}^{d-1} whose existence is guaranteed by Claim 1, we obtain the set of vectors \{\( \mathbf{r}_j \)\}_{j=1}^{d-1} which are the row vectors of \( M \). Then condition (1) is satisfied, as they are \( d-1 \) linearly independent vectors in \( \mathbb{Z}^d \) and satisfy that \( \mathbf{r}_j^* \cdot \mathbf{b}^* = 0 \) for each \( 1 \leq j \leq d-1 \).

For each \( 1 \leq j \leq d-1 \), we know that for any \( \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d \), \( (M\mathbf{x})_j = \mathbf{r}_j \cdot \mathbf{x} \). Note that \( \mathbf{r}_j = (r_{j1}, \ldots, r_{jd}) \). We have that whenever \( \mathbf{x} \in [N_1] \times \cdots \times [N_d] \),

\[
|\mathbf{r}_j \cdot \mathbf{x}| = \sum_{i=1}^{d} r_{ji} x_i \leq \sum_{i=1}^{d} |r_{ji}| N_i = \| \mathbf{r}_j^* \|_1, \tag{4.3}
\]
Let \( N_j^* = 3\|r_j^*\|_1 \) and \( \mathbf{v} = (2\|r_j^*\|_1)_{j=1}^{d-1} \). Observe that this choice of parameters together with (4.3) ensures that \( f_{\mathbf{v}}([N_1] \times \cdots \times [N_d]) \subseteq [N_1^*] \times \cdots \times [N_{d-1}^*] \). For each \( 1 \leq j \leq d-1 \), as \( r_j \in \mathbb{Z}^d \) is nonzero, we have \( r_{ji} \) is nonzero for some \( i \), and hence \( N_j^* \geq \|r_j^*\|_1 \geq |r_{ji}|N_i \geq N_i \geq \min_{1 \leq i \leq d} N_i \).

Also, by condition (1), elements in \( f_{\mathbf{v}}^{-1}([N_1] \times \cdots \times [N_d]) \) differ by multiples of \( \mathbf{v} \). Note that there are at most \( \frac{1}{\|\mathbf{v}\|_{\infty}} + 1 \leq \frac{2}{\|\mathbf{v}\|_{\infty}} \) such elements in \([N_1] \times \cdots \times [N_d]\) for each fixed \( x^* \in [N_1^*] \times \cdots \times [N_{d-1}^*] \). We have

\[
\frac{2}{\|\mathbf{v}\|_{\infty}} N_1^* \cdots N_{d-1}^* \geq N_1 \cdots N_d.
\]

It remains to show the other half of the inequality in (4.1). With \( N_j^* \) as above, we have

\[
N_1^* \cdots N_{d-1}^* = \prod_{j=1}^{d-1} 3\|r_j^*\|_1 \leq \prod_{j=1}^{d-1} 3\sqrt{d}\|r_j^*\|_2 \leq 3^{d-1}(\sqrt{d})^{d-1}2^{\frac{(d-1)(d-2)}{4}}\|\mathbf{v}\|_2 N_1 N_2 \cdots N_d.
\]

Using that \( \|\mathbf{v}\|_2 \leq \sqrt{d}\|\mathbf{v}\|_{\infty} \), we have

\[
3^{d-1}(\sqrt{d})^{d-1}2^{\frac{(d-1)(d-2)}{4}}\|\mathbf{v}\|_2 N_1 N_2 \cdots N_d \leq 2d^2 N_1 N_2 \cdots N_d\|\mathbf{v}\|_{\infty},
\]

so condition (2) is also satisfied. Here, we use that \( 3^{d-1}(\sqrt{d})^{d-1}2^{\frac{(d-1)(d-2)}{4}} \leq 2d^2 \) for \( d \geq 2 \).

For any \( \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d \setminus \{0\} \) whose entries are not coprime, we may apply the lemma above to \( \mathbf{b}/\gcd(b_1, \ldots, b_d) \) instead. This gives the following corollary.

**Corollary 4.5.** Let \( d \geq 2 \) be an integer, and \( N_1, N_2, \ldots, N_d \) be positive integers. Suppose that \( \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d \) is a nonzero vector satisfying that \( \lambda = \lambda(\mathbf{b}) := \max_{1 \leq i \leq d} \frac{|b_i|}{\gcd(b_1, \ldots, b_d) N_i} \leq 1 \). Then there exists an affine map \( f_{\mathbf{b}} \in \mathbb{Z}^{d(1 \times d)} \) so that the following two conditions holds.

1. For any \( x_1, x_2 \in \mathbb{Z}^d \), \( f_{\mathbf{b}}(x_1) = f_{\mathbf{b}}(x_2) \) if and only if \( x_1 - x_2 = k\mathbf{b} \) for some \( k \in \frac{1}{\gcd(b_1, \ldots, b_d)} \mathbb{Z} \).
2. There exist positive integers \( N_1^*, N_2^*, \ldots, N_{d-1}^* \geq \min_{1 \leq i \leq d} N_i \) so that \( \frac{1}{2} \leq N_1^* \cdots N_{d-1}^* \lambda N_1 \cdots N_d \leq 2d^2 \) and \( f_{\mathbf{b}}([N_1] \times \cdots \times [N_d]) \subseteq [N_1^*] \times \cdots \times [N_{d-1}^*] \).

The affine map in the corollary above is the main tool for reduction from \( \mathbb{Z}^d \) to \( \mathbb{Z}^{d-1} \). We have the following simple relation between \( f_{\mathbf{b}} \) and \( U_d(X, \mathbf{b}, s) \).

**Lemma 4.6.** Let \( d \geq 2 \) be an integer, and \( N_1, \ldots, N_d \) be positive integers. Let \( X \subseteq [N_1] \times \cdots \times [N_d] \), let \( \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d \setminus \{0\} \) with \( \lambda = \lambda(\mathbf{b}) := \max_{1 \leq i \leq d} \frac{|b_i|}{\gcd(b_1, \ldots, b_d) N_i} \leq 1 \). Let \( s \leq \min \{N_i\} \) be a positive integer. Suppose that \( f_{\mathbf{b}} : \mathbb{Z}^d \to \mathbb{Z}^{d-1} \) is an affine map satisfying (1) in Corollary 4.5. Then each element in \( f_{\mathbf{b}}(U_d(X, \mathbf{b}, s)) \) has at least \( s \) and at most \( \frac{2}{\lambda} \) preimages under \( f_{\mathbf{b}} \) in \( U_d(X, \mathbf{b}, s) \).

**Proof.** As \( f_{\mathbf{b}} \) satisfies (1), we know that if \( f_{\mathbf{b}}(x_1) = f_{\mathbf{b}}(x_2) \), then \( x_1 - x_2 \) is a multiple of \( \mathbf{b}/\gcd(b_1, \ldots, b_d) \). From the definition of \( \lambda \), we see that each element in \( \mathbb{Z}^{d-1} \) has at most \( \frac{1}{\lambda} + 1 \leq \frac{2}{\lambda} \) preimages in \( U_d(X, \mathbf{b}, s) \subseteq [N_1] \times \cdots \times [N_d] \).
Note that each element in $U^d(X, b, s)$ is in a residue class mod $b$ of size at least $s$. Again by condition (1), elements in the residue class mod $b$ get mapped to the same element by $f_b$. Therefore, every element in $f_b(U^d(X, b, s))$ has at least $s$ preimages.

Using Lemma 4.6, we derive the following lemma, which bounds the size of the intersection of two $U^d$ sets.

Lemma 4.7. Let $d \geq 2$ be a positive integer, and $N_1, \ldots, N_d$ be positive integers. Let $X \subseteq [N_1] \times \cdots \times [N_d]$, let $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d \setminus \{0\}$ with $\lambda = \lambda(b) := \max_{1 \leq i \leq d} \frac{|b_i|}{\gcd(b_1, \ldots, b_d) N_i} \leq 1$. Let $s^* \leq s \leq \min N_i$ be positive integers. Suppose that $f_b : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ satisfies condition (1) in Corollary 4.5, and $b' \in \mathbb{Z}^d$ satisfies that $f_b(b') \neq f_b(0)$. Then

\[ |U^d(X, b, s) \cap U^d(X, b', s)| \leq s^* - 1 \frac{s}{s^*} |U^d(X, b', s)| + 2 \frac{\lambda}{\lambda} |U^{d-1}(f_b(U^d(X, b, s)), f_b(b') - f_b(0), s^*)|. \] (4.4)

Proof. For simplicity let $X_1 = U^d(X, b, s)$. Note that $f_b(X_1)$ is a subset of $\mathbb{Z}^{d-1}$.

Partition the set $U^d(X, b', s)$ into nonempty residue classes mod $b'$. By definition, we know that each such residue class contains at least $s$ elements, so there are at most $\frac{1}{s} |U^d(X, b', s)|$ such residue classes. For each such residue class, there are two cases:

- the residue class contains at most $s^* - 1$ elements in $X_1 \cap U^d(X, b', s)$;
- the residue class contains at least $s^*$ elements in $X_1 \cap U^d(X, b', s)$.

We next upper bound the size of $X_1 \cap U^d(X, b', s)$. The number of elements in $X_1 \cap U^d(X, b', s)$ contained in a residue class of the first case is at most $\frac{s^* - 1}{s} |U^d(X, b', s)|$. It remains to estimate the number of elements contained in a residue class of the second case.

We first show that, if $x$ is an element of $X_1 \cap U^d(X, b', s)$, whose residue class mod $b'$ contains at least $s^*$ elements in $X_1 \cap U^d(X, b', s)$, then

\[ f_b(x) \in U^{d-1}(f_b(X_1), f_b(b') - f_b(0), s^*). \] (4.5)

Let $I = \{x_1, \ldots, x_k\}$ be the residue class mod $b'$ of $X_1 \cap U^d(X, b', s)$ with $x = x_1$. Suppose that $I$ contains at least $s^*$ elements. Note that if $x_i = tb' + x_j$ for some integer $t \neq 0$, then $f_b(x_i) = f_b(x_j) + t(f_b(b') - f_b(0))$. As $f_b(b') \neq f_b(0)$, we know that $f_b(I) = \{f_b(x_i) : 1 \leq i \leq k\}$ consists of $k \geq s^*$ distinct elements, whose pairwise differences are multiples of $(f_b(b') - f_b(0))$. In particular, there are at least $k \geq s^*$ elements in $f_b(X_1)$ that are congruent to $f_b(x)$ mod $(f_b(b') - f_b(0))$. This proves (4.5).

Note that by Lemma 4.6, each element in $U^{d-1}(f_b(X_1), f_b(b') - f_b(0), s^*)$ has at most $\frac{2}{\lambda}$ preimages in $X_1 \cap U^d(X, b', s)$. Therefore, the number of elements contained in a residue class of the second case is at most $\frac{2}{\lambda} |U^{d-1}(f_b(X_1), f_b(b') - f_b(0), s^*)|$. Putting these together, we have (4.4). \qed
\( s^* \leq s \) be a positive integer. Then we have

\[
\sum_{b' \in B} |U^d(X, b, s) \cap U^d(X, b', s)| \leq \frac{4}{s\lambda_0} |U^d(X, b, s)| + \frac{s^* - 1}{s} \sum_{b' \in B} |U^d(X, b', s)| + \frac{12}{s\lambda_0^2} \sum_{b' \in \mathbb{Z}^{d-1} \setminus \{0\}} |U^{d-1}(f_b(U^d(X, b, s)), b^*, s^*)|.
\] (4.6)

**Proof.** Let us denote \( b = (b_1, \ldots, b_d) \). We first give an upper bound on the number of elements \( b' \in B \) with \( f_b(b') = f_b(0) \). By condition (1), we know that all such \( b' \) are given by \( k b / \gcd(b_1, \ldots, b_d) \) for \( k \in \mathbb{Z} \). If \( k b / \gcd(b_1, \ldots, b_d) \) is contained in \( B \subseteq [-\frac{N_1}{s-1}, \frac{N_1}{s-1}] \times \cdots \times [-\frac{N_d}{s-1}, \frac{N_d}{s-1}] \), then we know that \( \frac{|k|}{\gcd(b_1, \ldots, b_d)} \leq \frac{N_i}{s-1} \) for all \( i \). Hence, each choice of \( k \) with \( k b / \gcd(b_1, \ldots, b_d) \in B \) satisfies that

\[
|k| \leq \min_{1 \leq i \leq d} \frac{N_i \gcd(b_1, \ldots, b_d)}{|b_i|(s-1)} \leq 1 \frac{1}{s-1} \lambda_0 \leq \frac{2}{s\lambda_0}.
\]

Therefore, the number of such \( b' \in B \) is at most \( \frac{4}{s\lambda_0} \) (noting that \( k \neq 0 \)). Thus, we have

\[
\sum_{b' \in B : f_b(b') = f_b(0)} |U^d(X, b, s) \cap U^d(X, b', s)| \leq \sum_{b' \in B : f_b(b') = f_b(0)} |U^d(X, b, s)| \leq \frac{4}{s\lambda_0} |U^d(X, b, s)|.
\] (4.7)

Now we consider the summation over all \( b' \in B \) with \( f_b(b') \neq f_b(0) \). For each such \( b' \), by Lemma 4.7,

\[
|U^d(X, b, s) \cap U^d(X, b', s)| \leq \frac{s^* - 1}{s} |U^d(X, b', s)| + \frac{2}{\lambda} |U^{d-1}(f_b(U^d(X, b, s)), f_b(b') - f_b(0), s^*)|.
\] (4.8)

Observe that

\[
\sum_{b' \in B : f_b(b') \neq f_b(0)} |U^d(X, b', s)| \leq \sum_{b' \in B} |U^d(X, b', s)|.
\]

Now, note that \( b' \in B \subseteq [-\frac{N_1}{s-1}, \frac{N_1}{s-1}] \times \cdots \times [-\frac{N_d}{s-1}, \frac{N_d}{s-1}] \). Thus, by a similar argument as above, for each \( b^* \in \mathbb{Z}^{d-1} \setminus \{0\} \), the number of \( b' \in B \) with \( f_b(b') - f_b(0) = b^* \) is at most \( \frac{2}{(s-1)\lambda_0^2} + 1 \leq \frac{6}{s\lambda_0} \). Therefore, we have

\[
\sum_{b' \in B : f_b(b') \neq f_b(0)} |U^{d-1}(f_b(U^d(X, b, s)), f_b(b') - f_b(0), s^*)| \leq \frac{6}{s\lambda_0} \sum_{b' \in \mathbb{Z}^{d-1} \setminus \{0\}} |U^{d-1}(f_b(U^d(X, b, s)), b^*, s^*)|.
\]
We sum (4.8) over all $b' \in B$ with $f_b(b') \neq f_b(0)$. Combining it with (4.7), we have
\[
\sum_{b' \in B} |U^d(X, b, s) \cap U^d(X, b', s)| = \sum_{b' \in B: f_b(b') = f_b(0)} |U^d(X, b, s) \cap U^d(X, b', s)| \\
+ \sum_{b' \in B: f_b(b') \neq f_b(0)} |U^d(X, b, s) \cap U^d(X, b', s)| \\
\leq \frac{4}{s^2 \lambda_0} |U^d(X, b, s)| + \frac{s^s - 1}{s} \sum_{b' \in B} |U^d(X, b', s)| \\
+ \frac{12}{s^2 \lambda_0} \sum_{b^* \in \mathbb{Z}^d \setminus \{0\}} |U^{d-1}(f_b(U^d(X, b, s)), b^*, s^*)|.
\]
(4.9)

This establishes the desired inequality (4.6). \qed

Lemma 4.8 is a useful bound for those $b$ for which $\lambda(b)$ (as defined in Lemma 4.7) is not too small. The following lemma shows that there are not many choices of $b$ for which $\lambda(b)$ is small.

**Lemma 4.9.** Let $d \geq 2$ be a positive integer, $n_1, \ldots, n_d \in \mathbb{N}$ and $\epsilon \in (0, 1)$. If $\frac{1}{\epsilon} \leq n_i$ for all $1 \leq i \leq d$, then there are at most $6^d e n_1 \cdots n_d$ nonzero points $(b_1, b_2, \ldots, b_d) \in [-n_1, n_1] \times \cdots \times [-n_d, n_d]$ with $|b_j / \gcd(b_1, \ldots, b_d)| \leq \epsilon n_i$ for all $1 \leq i \leq d$.

**Proof.** For each $i$, the number of tuples with $b_i = 0$ is given by
\[
\prod_{j \neq i} (2n_j + 1) \leq 3^d n_1 \cdots n_d \cdot \frac{1}{n_i} \leq 3^d \epsilon n_1 \cdots n_d.
\]

Hence, the number of such tuples with at least one zero entry is at most $3^d d \epsilon n_1 \cdots n_d$.

We may next only consider the tuples with nonzero entries. Suppose that $\gcd(b_1, \ldots, b_d) = k$. For any given $k$, we know that $b'_j = b_j / k$ satisfies that $|b'_j| \leq n_j / k$. From the problem condition, we further know that $|b'_j| \leq \epsilon n_j$. Hence, when $k$ is fixed, the number of such tuples is at most $(2 \epsilon)^d n_1 \cdots n_d$ if $k \leq \frac{1}{\epsilon}$, and $(2 \epsilon/k)^d n_1 \cdots n_d$ if $k > \frac{1}{\epsilon}$. Thus, summing over $k$, we know that the number of such tuples $(b_1, b_2, \ldots, b_d)$ is at most
\[
\sum_{1 \leq k \leq \frac{1}{\epsilon}} 2^d \epsilon^d n_1 \cdots n_d + \sum_{\frac{1}{\epsilon} < k} 2^d \frac{n_1 \cdots n_d}{k^d} \leq 2^d \epsilon^{d-1} n_1 \cdots n_d + 2^d \cdot 2 \epsilon^{d-1} n_1 \cdots n_d = 3 \cdot 2^d \epsilon^{d-1} n_1 \cdots n_d.
\]

Therefore, the total number of such tuples is at most
\[
3^d d \epsilon n_1 \cdots n_d + 3 \cdot 2^d \epsilon^{d-1} n_1 \cdots n_d \leq 6^d \epsilon n_1 \cdots n_d.
\]
\[\square\]

If $s \geq 2$ and $U^d(X, b, s)$ is nonempty, then $b \in [-N_1 \frac{s-1}{s-1}, N_1 \frac{s-1}{s-1}] \times \cdots \times [-N_d \frac{s-1}{s-1}, N_d \frac{s-1}{s-1}]$. In the lemma above we pick $n_i = N_i \frac{s-1}{s-1}$. Then the lemma above gives an upper bound on the number of nonzero points $b$ whose $\lambda$ value (as defined in Lemma 4.7) is at most $\frac{\epsilon}{s-1}$.

Finally, we need the following lemma.
Lemma 4.10. Let $X$ be a set of size $m > 0$. Let $\{A_i\}_{i \in I}$ be a family of subsets of $X$ over indices $i \in I$. Then we have

$$\sum_{i,j \in I} |A_i \cap A_j| \geq \frac{1}{m} \left( \sum_{i \in I} |A_i| \right)^2.$$

Proof. We count the number of tuples in $T = \{(x, i, j) : x \in X \times I \times I : x \in A_i \cap A_j\}$. Note that if we fix $i$ and $j$, the number of choices for $x$ is exactly $|A_i \cap A_j|$. Hence, we have $|T| = \sum_{i,j \in I} |A_i \cap A_j|$.

For each $x \in X$, let $m_x = |\{i \in I : x \in A_i\}|$, the number of sets $A_i$ that contain $x$. First we see that $\sum_{x \in X} m_x = \sum_{i \in I} |A_i|$.

Moreover, when counting $X$, once we fix $x \in X$, the number of choices for $(i, j)$ is $m_x^2$, so we have $|T| = \sum_{x \in X} m_x^2$. By the Cauchy–Schwarz inequality, we have

$$m \sum_{i,j \in I} |A_i \cap A_j| = m|T| = |X| \left( \sum_{x \in X} m_x^2 \right) \geq \left( \sum_{x \in X} m_x \right)^2 = \left( \sum_{i \in I} |A_i| \right)^2.$$

This gives the desired inequality. \qed

We next prove Lemma 3.5. The proof is by induction on $d$. The following proposition handles the base case $d = 1$ and is due to Matoušek and Spencer [8].

Proposition 4.11 [8, Proposition 4.1]. There exists an absolute constant $C$ such that the following holds. For positive integers $N$ and $m$, if $X \subseteq [N]$ is a subset of size $m$ and $s \geq 5 \sqrt{m}$, then

$$\sum_{b \in \mathbb{Z} \setminus \{0\}} |U^1(X, b, s)| \leq C N^{\frac{1}{2}} m^{\frac{3}{2}} s.$$

Now we have all the tools to set up the proof of Lemma 3.5, which is restated below for convenience. It gives under certain conditions an upper bound on the sum of $|U^d(X, b, s)|$ over all nonzero $b \in \mathbb{Z}^d$, where $X \subseteq [N_1] \times \cdots \times [N_d]$. This improves upon the simple bound given in Lemma 2.3 by a factor $\epsilon = \rho^\gamma$ where $\rho := \frac{|X|}{N_1 \cdots N_d}$ is the density of $X$ inside the grid it lies and $\gamma$ is a small constant dependent on $d$ ($\gamma$ is the exponent of $\rho$ in (3.9)). We first describe the proof idea.

In estimating the sum, we will break up the sum into three parts. As Lemma 4.9 says that only $O_d(\epsilon)$-fraction of $b$’s satisfy $\lambda(b) < \frac{\epsilon}{s}$, we can easily bound the contribution of those $b$ with $\lambda(b) < \frac{\epsilon}{s}$, or $U^d(X, b, s)$ is small. This allows us to only focus on $b \in B$, which consists of $b$ with $\lambda(b) \geq \frac{\epsilon}{s}$, and $|U^d(X, b, s)| \geq \epsilon |X|$.

Let us consider a fixed $d \geq 2$ with the induction hypothesis that the statement holds for $d - 1$. As we are to run an induction, the crux of the proof is to apply the induction hypothesis. Corollary 4.5 enables us to project, for each $b \in B$, the $d$-dimensional set $U^d(X, b, s)$ to a $(d - 1)$-dimensional set $X^* \subseteq [N_{1}^*] \times \cdots \times [N_{d-1}^*]$ for some set $X^*$ and integers $N_j^*$ for $1 \leq j \leq d - 1$. Let

$$\rho^* = \frac{|X^*|}{N_1^* \cdots N_{d-1}^*}.$$

It follows that we can estimate both $N_j^* \cdots N_{d-1}^*$ and the size of $X^* = f_b(U^d(X, b, s))$ within a factor of $\rho^{O_d(\gamma)}$ by applying Corollary 4.5 and Lemma 4.6, respectively. Thus, we can estimate $\rho^*$ within a factor of $\rho^{O_d(\gamma)}$. Finally, we apply Lemma 4.8 and combine that with Lemma 4.10 and the induction hypothesis in $d - 1$ dimension to get the desired bound. Together these two lemmas give us an upper bound on the sum of the sizes of the $U^d(X, b, s)$ over all $b \in B$. 


In Lemma 4.8, the bound we get in (4.6) is nontrivial and useful. In particular, we sum (4.6) over \( b \in B \), and the left-hand side can be further lower bounded by \( \frac{1}{m} \left( \sum_{b \in B} |U^d(X, b, s)| \right)^2 \) using Lemma 4.10. It remains to get an upper bound on the sum of the right-hand side of (4.6). First, it is not hard to see that the first term on the right-hand side is of lower order compared to the desired upper bound. By choosing \( s^* = s \rho \gamma \), we improve upon the simple bound by a factor of \( \rho \gamma \) in the second term. For the third term, we apply the induction hypothesis to get an improvement of a factor of \( (\rho^* \gamma^*)^{d-1} \) (where the bound is expressed in terms of \( |X^*|, N_1^* \cdots N_{d-1}^* \) and \( \rho^* \)). As we can estimate each of them within a factor of \( \rho^{d-1}(\gamma) \) in the third term. We can make \( \gamma \) small enough so that \( \gamma^* - O_d(\gamma) \geq \gamma \). In summary we improve on the simple bound by a factor of \( O_d(\rho \gamma) \) in all three terms, which is exactly what we need in Lemma 3.5.

**Lemma 3.5.** There exists an absolute constant \( C \) such that the following holds. Let \( d \) be a positive integer. Given positive integers \( N_1, N_2, \ldots, N_d \) satisfying \( N_1 \cdots N_d \leq (\min_{1 \leq i \leq d} N_i)^{d+1-\delta} \) for some \( \delta \in (0, 1] \), suppose that \( X \subseteq [N_1] \times \cdots \times [N_d] \) is of size \( m \). Letting \( \rho = \frac{m}{N_1 \cdots N_d} \), if integer \( s \) satisfies

\[
\frac{1}{d+1} \rho \frac{\delta}{4d(d+1)} \leq s \leq (\min_{1 \leq i \leq d} N_i) \rho^{\beta},
\]

then

\[
\sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^d(X, b, s)| \leq C \frac{2^d m N_1 N_2 \cdots N_d s^d}{\rho^{\min(\beta, \delta)}} \cdot \rho^{\frac{\min(\beta, \delta)}{4d(d+2)!}} \cdot (d+2)!.
\]

**Proof of Lemma 3.5.** Let \( C > 1 \) be an absolute constant that satisfies Proposition 4.11. We prove by induction on \( d \) that the statement holds for \( C_0 = C \cdot 5^d \cdot 2^d \). We may assume \( N_1 = \min_{1 \leq i \leq d} N_i \).

If \( m = 0 \) or if \( N_1 = \cdots = N_d = 1 \), then the statement trivially holds. Hence, we may assume that \( m \geq 1 \) and \( N_1 \cdots N_d > 1 \). Therefore, noting that \( \frac{1}{d+1} - \frac{\delta}{4d(d+1)} > 0 \), we have that \( s \) satisfies

\[
s \geq (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho \frac{\delta}{4d(d+1)} = (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho \frac{\min(\beta, \delta)}{4d(d+1)} \geq (N_1 \cdots N_d)^{\frac{1}{d+1}} \frac{\delta}{4d(d+1)}.
\]

It follows that \( s \geq 2 \). Also note that \( \rho \leq 1 \). We know that \( 2 \leq s \leq \min_{1 \leq i \leq d} N_i \). Hence, we have

\[
\sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^d(X, b, s)| \leq |X| \cdot \prod_{i=1}^d \left( 4 \frac{N_i}{s} + 1 \right) \leq m \cdot \prod_{i=1}^d 5 \frac{N_i}{s} = 5^d \frac{m N_1 \cdots N_d}{s^d}
\]

by Lemma 2.3. Therefore, the statement holds if \( C \rho \frac{\min(\beta, \delta)}{4d(d+2)!} \cdot 2^d \geq 1 \). Hence, we may assume that \( \rho \frac{\min(\beta, \delta)}{4d(d+2)!} \cdot 2^d \geq \frac{1}{C} \). We prove the base case \( d = 1 \) using Proposition 4.11. Note that \( \frac{\delta}{4d(d+1)} \leq \frac{\delta}{8} \leq \frac{1}{8} \) and \( \frac{\min(\beta, \delta)}{4d(d+2)!} \leq \frac{1}{24} \). As we assumed that \( s^* = s \rho \gamma^* \geq 5 \), we have \( \sqrt{N_1 \rho} \frac{\delta}{4d(d+1)} \geq \sqrt{N_1 \rho} \frac{\gamma^*}{8} \geq 5 \sqrt{m} \).

If \( s \geq \sqrt{N_1 \rho} \frac{\delta}{4d(d+1)} \geq 5 \sqrt{m} \), then by Proposition 4.11 we have the desired inequality for \( d = 1 \)

\[
\sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^1(X, b, s)| \leq C \frac{m N_1}{s} \rho \frac{1}{2} \leq C_0 \frac{m N_1}{s} \rho \frac{\min(\beta, \delta)}{4d(d+2)!}.
\]

We next show the desired bound for \( d \geq 2 \), assuming the induction hypothesis for \( d^* = d - 1 \). Let \( \gamma = \frac{\min(\beta, \delta)}{4d(d+2)!} \) and \( \epsilon = \rho^\gamma \). As \( |U^d(X, b, s)| \) is zero if \( b \) is not a nonzero integer point in
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We estimate each term on the right-hand side of (4.12). As $s \leq N_1\rho^{\beta} \leq N_i\rho^{\beta}$, we have $n_i := \frac{N_i}{s-1} \geq \rho^{-\beta} \geq \rho^{-\gamma} = \frac{1}{\epsilon}$. Hence, we may apply Lemma 4.9 and conclude that

$$|B_1| \leq 5\epsilon n_1 \cdots n_d \leq 5\epsilon \frac{N_1 \cdots N_d}{s-1} \leq \frac{12\epsilon}{s} \frac{N_1 \cdots N_d}{s^d} = 12\rho^{-\gamma} \frac{N_1 \cdots N_d}{s^d}.$$ 

For each $b \in B_1$, we have $U_d(X, b, s) \subseteq X$, so $|U_d(X, b, s)| \leq m$. It follows that

$$\sum_{b \in B_1} |U_d(X, b, s)| \leq |B_1| m \leq 12\rho^{-\gamma} \frac{mN_1 \cdots N_d}{s^d}.$$

As $B_0 \subseteq [-N_{s-1}, N_{s-1}] \times \cdots \times [-N_{s-1}, N_{s-1}]$ and $s \leq N_1\rho^{\beta} \leq N_i$, we have

$$|B_0| \leq \prod_{i=1}^d \left( \frac{N_i}{s-1} + 1 \right) \leq \prod_{i=1}^d \left( \frac{4}{s} + 1 \right) \leq 5\frac{N_1 \cdots N_d}{s^d}.$$ 

Therefore, as $B_2 \subseteq B_0$, we have that

$$\sum_{b \in B_2} |U_d(X, b, s)| \leq |B_2| \cdot \epsilon m \leq 5\frac{N_1 \cdots N_d}{s^d} \cdot \epsilon m = 5\rho^{-\gamma} \frac{mN_1 \cdots N_d}{s^d}.$$ 

Observe that $B \subseteq B_0$, and it follows

$$|B| \leq |B_0| \leq 5\frac{N_1 \cdots N_d}{s^d}.$$ 

Here (4.14) gives an upper bound on the second term in (4.12). Finally, we bound the third term. By Lemma 4.10, noting that $\{U_d(X, b, s)\}_{b \in B}$ is a family of subsets of $X$, we have

$$\sum_{b \in B} \sum_{b' \in B} |U_d(X, b, s) \cap U_d(X, b', s)| \geq \frac{1}{m} \left( \sum_{b \in B} |U_d(X, b, s)| \right)^2.$$ 

We next give an upper bound on $\sum_{b' \in B} |U_d(X, b, s) \cap U_d(X, b', s)|$ for fixed $b \in B$ by Lemma 4.8. Before we can apply it, we need to make a few preparations to ensure that the conditions are satisfied.

As we have excluded elements in $B_1$, for any $b = (b_1, \ldots, b_d) \in B$ there exists some index $i$ for which $|b_i/\gcd(b_1, \ldots, b_d)| > \epsilon n_i = \epsilon \frac{N_i}{s-1}$. This implies that

$$\lambda_b := \max_{1 \leq i \leq d} \frac{|b_i|}{\gcd(b_1, \ldots, b_d)N_i} \geq \frac{\epsilon}{s-1} > \frac{\epsilon}{s}.$$
Meanwhile, as \(|b_i| \leq \frac{N_i}{s-1}\) for each \(i\), we know that \(\lambda_b \leq \frac{1}{s-1} \leq \frac{2}{s}\). Therefore, \(\lambda_b \in (\frac{\varepsilon}{s}, \frac{2}{s}]\) for all \(b \in B\). Hence, \(B\) satisfies the conditions in Lemma 4.8 for \(\lambda_0 := \frac{\varepsilon}{s}\).

We fix an arbitrary \(b \in B\). By Corollary 4.5, there exists an affine map \(f_b : \mathbb{Z}^d \to \mathbb{Z}^d-1\) that satisfies conditions (1) and (2). Let \(s^* = [\varepsilon s]\). Now we know that \(b, f_b, \) and \(s^*\) satisfy the conditions in Lemma 4.8. We apply it and get

\[
\sum_{b' \in B} |U^d(X, b, s) \cap U^d(X, b', s)| \leq \frac{4|U^d(X, b, s)|}{s\lambda_0} + \frac{s^* - 1}{s} \sum_{b' \in B} |U^d(X, b', s)|
\]

\[
+ \frac{12}{s^2 \lambda_0^2} \sum_{b^* \in \mathbb{Z}^{d-1}\setminus\{0\}} |U^{d-1}(f_b(U^d(X, b, s)), b^*, s^*)|.
\]

(4.18)

For the third term on the right-hand side of (4.18), we would like to apply the induction hypothesis. To do this, we need to verify the various conditions in the statement by proving following claims.

As \(f_b\) satisfies condition (2) in Corollary 4.5, there exist positive integers \(N^*_1, \ldots, N^*_{d-1}\) such that \(f_b([N_1] \times \cdots \times [N_d]) \subseteq [N^*_1] \times \cdots \times [N^*_{d-1}]\),

\[
\frac{1}{2} \lambda_b N_1 \cdots N_d \leq N^*_1 \cdots N^*_{d-1} \leq 2^{d^2} \lambda_b N_1 \cdots N_d,
\]

(4.19)

Let \(M := \min_{1 \leq i \leq d-1} N^*_i \geq N_1\). As \(\lambda_b \in (\frac{\varepsilon}{s}, \frac{2}{s}]\), we have

\[
\frac{\varepsilon}{2s} N_1 \cdots N_d \leq N^*_1 \cdots N^*_{d-1} \leq \frac{2^{d+1}}{s} N_1 \cdots N_d.
\]

(4.20)

Let \(X^* = f_b(U^d(X, b, s))\). As \(f_b\) satisfies (2) in Corollary 4.5, we have \(X^* \subseteq [N^*_1] \times \cdots \times [N^*_{d-1}]\). Let \(m^* := |X^*|\) and \(\rho^* := m^*/(N^*_1 \cdots N^*_{d-1})\).

The following claims allow us to apply the induction hypothesis to \((N^*_j)_{j=1}^{d-1}, X^*,\) and \(s^*\).

Claim 2. \(N^*_1 \cdots N^*_{d-1} \leq M^{d-\delta/3}\).

Proof of Claim 2. Note that \(N_1 \leq M\), so we have that

\[
N_1 \cdots N_d \leq N_1^{d+1-\delta} \leq M^{d+1-\delta}.
\]

(4.21)

As \(m \geq 1\), we know that \(M^{d+1} \geq N_1 \cdots N_d \geq \frac{1}{c} > \frac{2^{(d+2)!d^3}}{\min(\beta, \delta^3)}\) and so \(M \geq \frac{2^{16d+2d^3}}{\min(\beta, \delta^3)} > \frac{2^{12d^2+1}}{s}\). Combining (4.21) with (4.10) and (4.20), we have

\[
N^*_1 \cdots N^*_{d-1} \leq 2^{d^2} \lambda_b N_1 \cdots N_d \leq 2^{d^2+1} (N_1 \cdots N_d)^{\frac{d+\delta}{d+1} + \frac{\delta}{4d}} \leq 2^{d^2+1} M^{\frac{d+1-\delta}{d+1} \cdot \frac{d+\delta}{d+1}} \leq 2^{d^2+1} M^{\frac{d+\delta}{d+1}}.
\]

(4.22)

For the exponent of \(M\) on the right-hand side in (4.22), we know that

\[
\frac{d + 1 - \delta}{d + 1} \left( d + \frac{\delta}{4d} \right) = d + \frac{\delta}{4d} - \delta \frac{d + \frac{\delta}{4d}}{d + 1} \leq d + \frac{\delta}{16} - \frac{\delta}{2} < d - \frac{\delta}{3} - \frac{\delta}{12}.
\]

Therefore, we can simplify (4.22) and get

\[
N^*_1 \cdots N^*_{d-1} \leq 2^{d^2+1} M^{\frac{d+\delta}{d+1} \cdot \frac{\delta}{4d}} \leq M^{d+\delta \cdot \frac{\delta}{4d}} \leq M^{d+\delta \cdot \frac{\delta}{12}}
\]

Therefore, we can simplify (4.22) and get

\[
N^*_1 \cdots N^*_{d-1} \leq 2^{d^2+1} M^{d+\frac{\delta}{12}} \leq M^{d+\frac{\delta}{3}}
\]
as expected, where in the last inequality we use that \( M \geq 2^{-\frac{12(d^2+1)}{s}} \).

By Claim 2, we can apply the induction hypothesis to \( X^* \subseteq [N_1^*] \times \cdots \times [N_{d-1}^*] \) and \( \delta^* = \delta/3 \). It remains to show that \( s^* \) is also in the desired range.

**Claim 3.** \((N_1^* \cdots N_{d-1}^*)^{\frac{1}{d}}(\rho^*)^{\frac{\delta}{d(d+1)}} \leq s^* \leq M(\rho^*)^\beta.\)

**Proof of Claim 3.** By Lemma 4.6, we have

\[
\frac{\lambda_b}{2} |U^d(X, b, s)| \leq m^* \leq \frac{1}{s} |U^d(X, b, s)|. \tag{4.23}
\]

Note that \( \varepsilon = \rho^\gamma \) and that \( \rho = \frac{m}{N_1 \cdots N_d} \). As \( |U^d(X, b, s)| \leq m \), combining with (4.20), we have

\[
\rho^* = \frac{m^*}{N_1^* \cdots N_{d-1}^*} \leq \frac{m}{s} \leq \frac{m}{\varepsilon N_1 \cdots N_d} = 2\rho^{1-\gamma}. \tag{4.24}
\]

Recall that \( \gamma = \min(\beta, \delta) \leq \frac{\delta}{12 \cdot 4d} \), and so

\[
\frac{\delta}{3 \cdot 4d} (1-\gamma) > \frac{\delta}{3 \cdot 4d} - \gamma = \frac{\delta}{4d} + \frac{\delta}{3 \cdot 4d} - \gamma \geq \frac{d+1}{d} \cdot \frac{\delta}{4d(d+1)} + \gamma + \frac{\delta}{6 \cdot 4d}.
\]

As a result, raising both sides of \((N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^{\frac{\delta}{d(d+1)}} \leq s \) to the \( \frac{d+1}{d} \)-th power, we have

\[
s\rho^\gamma \geq s^{\frac{1}{d}} \cdot (N_1 \cdots N_d)^{\frac{1}{d+1}} \rho^{\frac{\delta}{d(d+1)}} \geq 2^{-\frac{d^2+1}{d}} \cdot \rho^{\frac{\delta}{d(d+1)}} (N_1^* \cdots N_{d-1}^*)^{\frac{1}{d}}(\rho^*)^{\frac{\delta}{d(d+1)}} \geq (N_1^* \cdots N_{d-1}^*)^{\frac{1}{d}}(\rho^*)^{\frac{\delta}{d(d+1)}}. \tag{4.25}
\]

Here in the last inequality we use that \( \rho^{\frac{\delta}{d(d+1)}} \geq \rho^{-2\gamma} \geq 2^{d^2+1} \geq \frac{\delta}{4d^2} \). Note that from our choice of \( b \in B \), \( |U^d(X, b, s)| \) is at least \( m \). Therefore, combining with (4.23) we have \( m^* \geq \frac{\varepsilon}{2} \lambda_b m \). Combining this with (4.19) and \( \varepsilon = \rho^\gamma \), we have

\[
\rho^* = \frac{m^*}{N_1^* \cdots N_{d-1}^*} \geq \frac{\varepsilon}{2} \lambda_b m \geq 2^{-d^2+1} \cdot \varepsilon \rho = 2^{-d^2+1} \rho^{1+\gamma}. \tag{4.26}
\]

By (4.10), we know that \( s^* \geq \rho^* \cdot (m/\rho)^{\frac{1}{d+1}} \geq \rho^{-\frac{1}{d+1}} + \frac{\delta}{d(d+1)} + \gamma \). As \( \frac{\delta}{d(d+1)} + \gamma < \frac{1}{d+1} \) and \( \rho < 1 \), we have \( s^* > |s^*| \leq 2s \). Note that \( s \leq N_1 \rho^\beta \leq M \rho^\beta \). Therefore, we have

\[
s^* \leq 2s \leq 2\rho^* M \rho^\beta = 2^{1+\beta(d^2+1)} \rho^{-\gamma-\beta\gamma} (2-\rho^\gamma)^\beta \leq M(\rho^*)^\beta, \tag{4.27}
\]

where in the last inequality we use (4.26) and that \( \rho^{-\gamma-\beta\gamma} \geq \rho^{-2\gamma} > 2^{d^2/2} > 2^{1+\beta(d^2+1)} \) for \( d \geq 2 \). \qed
We conclude that the conditions for the induction hypothesis are satisfied by $d^* = d - 1$, $(N_i^*)_{i=1}^{d-1} = \delta^*/3$, $\beta^* = \beta^*$, $X^* = f_b(U^d(X, b, s))$, and $s^* = \lceil \varepsilon s \rceil$. Applying the induction hypothesis we get
\[
\sum_{b^* \in \mathbb{Z}^{d-1} \setminus \{0\}} |U^{d-1}(f_b(U^d(X, b, s)), b^*, s^*)| \leq C^* \frac{N_1^* \cdots N_{d-1}^* m^*}{(s^*)^{d-1}} (\rho^*)^{\min(\beta/\delta/3)} (1-\gamma)^s |U^d(X, b, s)|,
\]
where $C^* = C \cdot 5^{d-1} \cdot 2^{(d-1)}$. Using (4.20), (4.23), (4.24), and that $s^* \geq \rho^' s$, we have
\[
C^* \frac{N_1^* \cdots N_{d-1}^* m^*}{(s^*)^{d-1}} (\rho^*)^{\gamma s^*} \leq C^* \frac{2^{d+1} N_1 \cdots N_d}{\rho^{d-1} \gamma s^{d+1}} \rho^{(1-\gamma)s^*} |U^d(X, b, s)|,
\]
where for simplicity we denote $\gamma^* := \frac{\min(\beta, \delta/3)}{4d-1(d+1)!} \geq \frac{\min(\beta, \delta)}{3d-1(d+1)!} = \frac{4(d+2)}{3} > \gamma$. Note that exponent of $\rho$ on the right-hand side of (4.29) satisfies $-(1-\gamma)(d-1)\gamma = \gamma^* - (d-1 + \gamma^*)\gamma > 3\gamma$. Combining this with the inequalities (4.28) and (4.29), we have
\[
\sum_{b^* \in \mathbb{Z}^{d-1} \setminus \{0\}} |U^{d-1}(f_b(U^d(X, b, s)), b^*, s^*)| \leq 2^{d+1} C^* \rho^{3\gamma} \frac{N_1 \cdots N_d}{s^{d+1}} |U^d(X, b, s)|.
\]
Put this into (4.18). Note that $\lambda_0 = \frac{\varepsilon}{s} = \rho^'/s$ and that $\frac{s^*-1}{s} \leq \frac{\varepsilon s}{s} = \rho^'$. We have
\[
\sum_{b^* \in B} |U^d(X, b, s) \cap U^d(X, b', s)| \leq 4\rho^{-\gamma} |U^d(X, b, s)| + \rho^' \sum_{b^* \in B} |U^d(X, b', s)|
\]
\[
+ 12 \cdot 2^{d+1} C^* \rho^' \frac{N_1 \cdots N_d}{s^d} |U^d(X, b, s)|.
\]
We sum (4.31) over $b \in B$. Recall that $\gamma = \frac{\min(\beta, \delta)}{4d-1(d+1)!} < \frac{\beta}{4}$. We have
\[
\frac{N_1 \cdots N_d}{s^d} \geq \frac{N_1^* \cdots N_d^*}{s^d} \geq \rho^{(-d)\beta} > \rho^{-2\gamma},
\]
and it follows that $\rho^{-\gamma} < \rho^' \frac{N_1^* \cdots N_d^*}{s^d}$. Using this and (4.15), we have
\[
\sum_{b, b' \in B} |U^d(X, b, s) \cap U^d(X, b', s)| \leq \left( 4\rho^{-\gamma} + \rho^' |B| + 12 \cdot 2^{d+1} C^* \rho^' \frac{N_1 \cdots N_d}{s^d} \right) \sum_{b \in B} |U^d(X, b, s)|
\]
\[
\leq \left( 4 + 5^d + 12 \cdot 2^{d+1} C^* \right) \rho^' \frac{N_1 \cdots N_d}{s^d} \sum_{b \in B} |U^d(X, b, s)|.
\]
Combine this with (4.16), we have
\[
\sum_{b \in B} \sum_{b' \in B} |U^d(X, b, s)| \leq \left( 4 + 5^d + 12 \cdot 2^{d+1} C^* \right) \rho^' \frac{N_1 \cdots N_d}{s^d} m.
\]
Substituting in the bounds (4.13), (4.14), and (4.32) into (4.12), we have
\[
\sum_{b \in \mathbb{Z}^d \setminus \{0\}} |U^d(X, b, s)| \leq \left( 12^d + 5^d + 4 + 5^d + 12 \cdot 2^{d+1} C^* \right) \rho^' \frac{mN_1 \cdots N_d}{s^d}.
\]
Finally, we show that the sum of the additive constant above is at most $C_0$. Recall that $C^* = C \cdot 5^{d-1} \cdot 2^{(d-1)^3}$ and $C_0 = C \cdot 5^d \cdot 2^{d^3}$, we know that $24 \cdot 2^d C^* \leq \frac{3}{5} \cdot 5 \cdot 2^{3d^2-3d+1} C^* = \frac{3C_0}{5}$ as $d \geq 2$.

For other terms, we have $4 \leq 5^d \leq 12^d \leq 2^{d^3} = C_0$. Therefore, we conclude that the statement holds for $d$. Therefore, we conclude that the statement holds for all positive integer $d$. □

5 | A PROOF OF THE LOWER BOUND IN THEOREM 1.3

We prove the following result, which is the lower bound on the discrepancy in Theorem 1.3.

**Theorem 5.1.** For any positive integer $d$, there exists a constant $c_d > 0$ such that the following holds. For positive integer $N_1, \ldots, N_d$, letting $N = (N_1, \ldots, N_d)$, we have

$$
\text{disc}(\mathcal{A}_N) \geq c_d \max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{1 \over 2|I|+2}.
$$

Here by convention if $I = \emptyset$ then $\prod_{i \in I} N_i = 1$.

Roth [9] proved the case $d = 1$, and Valkó [10] proved the case that the $N_i$’s are equal. Similar to these previous results, our proof uses Fourier analysis. We first set up some notations. Let $f, g : \mathbb{Z}^d \to \mathbb{C}$ be two functions that each has finite support. The Fourier transform $\hat{f} : [0,1]^d \to \mathbb{C}$ is given by $\hat{f}(\mathbf{r}) = \sum_{\mathbf{x} \in \mathbb{Z}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \mathbf{r}}$. The convolution $f * g : \mathbb{Z}^d \to \mathbb{C}$ is given by $f * g(\mathbf{x}) = \sum_{\mathbf{r} \in \mathbb{Z}^d} f(\mathbf{r}) g(\mathbf{x} - \mathbf{r})$, which also has finite support. With these notations, we have the convolution identity $\hat{f} * \hat{g} = \hat{f} \cdot \hat{g}$ and Parseval’s identity

$$
\sum_{\mathbf{x} \in \mathbb{Z}^d} f(\mathbf{x}) g(\mathbf{x}) = \int_{[0,1]^d} \hat{f}(\mathbf{r}) \hat{g}(\mathbf{r}) \, d\mathbf{r}.
$$

In the proof of Theorem 5.1, for a vector $\mathbf{x} \in \mathbb{Z}^d$, we let $x_i$ denote the $i$th coordinate of $\mathbf{x}$.

**Proof of Theorem 5.1.** We take $c_d = \frac{6^{-d/2}}{2}$. Let $\Omega = [N_1] \times \cdots \times [N_d] \subseteq \mathbb{Z}^d$. Fix any $\chi : \Omega \to \{1, -1\}$.

Let $T = \max_{A \in \mathcal{A}_N} |\chi(A)|$. It suffices to show that $T \geq c_d \max_{I \subseteq [d]} (\prod_{i \in S} N_i)^{1 \over 2|I|+2}$.

For $\chi : \Omega \to \{1, -1\}$, we may extend it to a function $\chi : \mathbb{Z}^d \to \{-1, 0, 1\}$ by assigning 0 to $\mathbb{Z}^d \setminus \Omega$. Clearly $\chi$ takes nonzero values on $N_1 \cdots N_d$ points. Hence, we may apply Parseval’s identity and get

$$
\int_{[0,1]^d} |\hat{\chi}(\mathbf{r})|^2 = \sum_{\mathbf{x} \in \mathbb{Z}^d} |\chi(\mathbf{x})|^2 = N_1 \cdots N_d. \tag{5.1}
$$

Let $L$ be a positive integer and $D_1, \ldots, D_d$ be nonnegative integers to be determined later. For each $\mathbf{b} \in \mathbb{Z}^d \setminus \mathbf{0}$ satisfying that $\mathbf{b}_i \in [-D_i, D_i]$ for $1 \leq i \leq d$, let $g_{\mathbf{b}} : \mathbb{Z}^d \to \mathbb{C}$ be the indicator function of the set $\{0, \mathbf{b}, \ldots, (L-1)\mathbf{b}\}$. Now we have for each $\mathbf{x} \in \mathbb{Z}^d$,

$$
g_{\mathbf{b}} * \chi(\mathbf{x}) = \sum_{t=0}^{L-1} \chi(\mathbf{x} - t\mathbf{b}) = \chi(\Omega \cap \{\mathbf{x} - t\mathbf{b} : 0 \leq t < L\}).
$$
As $\Omega \cap \{ x - t \mathbf{b} : 0 \leq t < L \}$ is an arithmetic progression contained in $\Omega$, it is a set in $A_N$, so $|g_b \ast \chi(x)| \leq T$. This is true for all $x \in \mathbb{Z}^d$. Also note that $|g_b \ast \chi(x)|$ is nonzero only when $x - t \mathbf{b} \in \Omega$ for some $0 \leq t < L$. In this case we have $x_i \in [1 - LD_i, N_i + LD_i]$ for each $1 \leq i \leq d$. Therefore, $g_b \ast \chi$ is nonzero on at most $\prod_{i=1}^d (N_i + 2LD_i)$ points in $\mathbb{Z}^d$. We have

$$\sum_{x \in \mathbb{Z}^d} g_b \ast \chi(x) g_b \ast \chi(x) = \sum_{x \in \mathbb{Z}^d} |g_b \ast \chi(x)|^2 \leq T^2 \prod_{i=1}^d (N_i + 2LD_i). \quad (5.2)$$

By Parseval’s identity and the convolution identity, we have

$$\sum_{x \in \mathbb{Z}^d} |g_b \ast \chi(x)|^2 = \int_{[0,1]^d} |\hat{g_b}(\mathbf{r}) \chi(\mathbf{r})|^2 d\mathbf{r} = \int_{[0,1]^d} \hat{g_b}(\mathbf{r}) \hat{\chi}(\mathbf{r}) d\mathbf{r} \quad (5.3)$$

Combining (5.2) and (5.3), we get that for any nonzero $\mathbf{b} \in [-D_1, D_1] \times \cdots \times [-D_d, D_d]$

$$\int_{[0,1]^d} \left| \hat{g_b}(\mathbf{r}) \right|^2 |\hat{\chi}(\mathbf{r})|^2 d\mathbf{r} \leq T^2 \prod_{i=1}^d (N_i + 2LD_i). \quad (5.4)$$

Let $A$ be the set of integer points in $[0, D_1] \times \cdots \times [0, D_d]$ and $B$ be the set of nonzero integer points in $[-D_1, D_1] \times \cdots \times [-D_d, D_d]$. Clearly any two distinct points in $A$ have their difference in $B$. The number of points in $B$ is at most $\prod_{i=1}^d (2D_i + 1)$. Hence, if we sum over $\mathbf{b} \in B$ in (5.4), we get

$$\int_{[0,1]^d} \left( \sum_{\mathbf{b} \in B} |\hat{g_b}(\mathbf{r})|^2 \right) |\hat{\chi}(\mathbf{r})|^2 d\mathbf{r} \leq T^2 \prod_{i=1}^d (N_i + 2LD_i)(2D_i + 1) \quad (5.5)$$

Fix any $\mathbf{r} \in [0, 1]^d$. By the pigeonhole principle, there exists two distinct $\mathbf{a}, \mathbf{a}' \in A$ such that the fractional parts of $\mathbf{a} \cdot \mathbf{r}$ and $\mathbf{a}' \cdot \mathbf{r}$ differ by at most $1/|A|$. Hence, for any $\mathbf{r}$ we can find $\mathbf{b}' \in B$ (we shall take $\mathbf{b}' = \mathbf{a} - \mathbf{a}'$ or $\mathbf{b}' = \mathbf{a}' - \mathbf{a}$) such that the fractional part of $\mathbf{b}' \cdot \mathbf{r}$ is in $[0, 1/|A|]$. If $L \leq |A|/2 = \frac{1}{2} \prod_{i=1}^d (D_i + 1)$, then for any $\mathbf{r} \in [0, 1]^d$,

$$\sum_{\mathbf{b} \in B} |\hat{g_b}(\mathbf{r})|^2 \geq |\hat{g_{\mathbf{b}'}}(\mathbf{r})|^2 = \left| \sum_{t=0}^{L-1} e^{-2\pi i \mathbf{b}' \cdot \mathbf{r}} \right|^2 \geq \frac{4}{\pi^2} L^2.$$

Put this into (5.5) and combine with (5.1). We conclude that for any positive integer $L$ and nonnegative integers $D_1, \ldots, D_d$ such that $L \leq \frac{(D_1+1) \cdots (D_d+1)}{2}$, then

$$T^2 \prod_{i=1}^d (N_i + 2LD_i)(2D_i + 1) \geq \frac{4}{\pi^2} L^2 \prod_{i=1}^d N_i. \quad (5.6)$$

Let $R = \max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{1/|I|+1}$. If $R \leq 2$, the statement is trivial as $c_d = \frac{6^{-d/2}}{2} \leq 1/2$. Therefore, we may assume that $R > 2$ and the maximum in the definition of $R$ is achieved by some nonempty $I \subseteq [d]$. For each $j \in I$, we have $R \geq \left( \prod_{i \in I \setminus \{j\}} N_i \right)^{1/|I|}$, so $N_j \geq R^2$. With these properties, we may now choose the values of $L$ and $D_1, \ldots, D_d$. We set $L = [R^2/2], D_i = \left\lfloor \frac{N_i}{R^2} \right\rfloor$ for $i \in I$, and $D_j = 0$ for
each $j \notin I$. As 
\[
\frac{(D_1 + 1) \cdots (D_d + 1)}{2} \geq \frac{\prod_{i \in I} N_i}{2R^{2|I|}} = \frac{R^2}{2} \geq L,
\]
we can apply (5.6) to these variables. For $j \notin I$, as $D_j = 0$, we have $(N_j + 2LD_j)(2D_j + 1) = N_j$.
For $i \in I$, as $N_i \geq R^2$, we have $N_i/R^2 \geq D_i \geq 1$, so
\[
(N_i + 2LD_i)(2D_i + 1) \leq \left( N_i + 2 \cdot \frac{R^2}{2} \cdot \frac{N_i}{R^2} \right) \cdot \frac{N_i}{R^2} = \frac{N_i^2}{R^2}.
\]
Put these into (5.6). Note that $L \geq \frac{R^2}{2}$. We have
\[
T^2 \prod_{i \in I} \frac{N_i^2}{R^2} \cdot \prod_{j \notin I} N_j \geq \frac{R^4}{4} \prod_{i=1}^d N_i.
\]
Also note that $\prod_{i \in I} N_i = R^{2|I|+2}$. We conclude that $T \geq \frac{6}{\sqrt{2} R} \geq c_d R$. \qed

6 \hspace{1cm} \textbf{A PROOF OF THE UPPER BOUND IN THEOREM 1.3}

In this section, we aim to generalize the upper bound in Theorem 1.2 to all grids of differing side lengths. The following lemma allows us to remove dimensions of short side lengths.

\textbf{Lemma 6.1.} Let $d \geq 2$ be a positive integer and $N_1, \ldots, N_d$ be positive integers. Then for $N = (N_1, \ldots, N_d)$ and $N' = (N_1, \ldots, N_{d-1})$, we have
\[
\text{disc}(\mathcal{A}_N) \leq \max \left( \text{disc}(\mathcal{A}_{N'}), \sqrt{6N_d \log(2N_1 \cdots N_d)} \right).
\]

\textbf{Proof.} First, we may choose an optimal coloring $\chi'$ for the grid $[N_1] \times \cdots \times [N_{d-1}]$ that achieves discrepancy $\text{disc}(\mathcal{A}_{N'})$.

We extend this coloring to a coloring $\chi : [N_1] \times \cdots \times [N_d] \to \{1, -1\}$ by the following procedure. Take $N_d$ independent and identically distributed Rademacher random variables $v(i)$ for $1 \leq i \leq N_d$ (i.e., $\Pr(v(i) = 1) = \Pr(v(i) = -1) = \frac{1}{2}$). Now we set $\chi(x_1, \ldots, x_d) = \chi'(x_1, \ldots, x_{d-1})v(x_d)$ for any $(x_1, \ldots, x_d) \in [N_1] \times \cdots \times [N_d]$.

Now we analyze $\chi(S)$ for $S \in \mathcal{A}_N$. Let $(k_1, \ldots, k_d)$ be the common difference of arithmetic progression $S$. If $k_d = 0$, then all elements in $S$ share the same $d$th coordinate $x_d$, so we can write $S$ as $S' \times \{x_d\}$, where $S'$ is also an arithmetic progression with common difference $(k_1, \ldots, k_{d-1})$. By our construction of $\chi$, we have $|\chi(S)| = |\chi'(S')v(x_d)| \leq \text{disc}(\mathcal{A}_{N'})$.

Otherwise if $k_d \neq 0$, then all elements in $S$ have distinct $d$th coordinates, and $|S| \leq N_d$. As $\chi'$ is deterministic, we know that $\chi(S)$ is a summation of $|S|$ independent and identically distributed Rademacher random variables. Now by the Chernoff bound (e.g., see [1, Theorem A.1.1 and Corollary A.1.2]), we have
\[
\Pr(|\chi(S)| > \sqrt{6N_d \log(2N_1 \cdots N_d)}) \leq 2e^{-\frac{6N_d \log(2N_1 \cdots N_d)}{2|S|}} \leq \frac{1}{4}(N_1 \cdots N_d)^{-3} < (N_1 \cdots N_d)^{-3},
\]
where in the last inequality we use that $|S| \leq N_d$. Finally, we apply the union bound on all $S$ with $k_d \neq 0$. Clearly there are $N_1 \cdots N_d$ ways to pick the first element in the arithmetic progression, and at most $N_1 \cdots N_d$ ways to pick the last element, and at most $N_d$ ways to choose $|S|$ (as $1 \leq |S| \leq N_d$). Once these are chosen, then clearly $S$ is determined as the common difference in the last coordinate is determined. Hence, the total number of distinct $S$ in $A_N$ with $k_d \neq 0$ is at most $(N_1 \cdots N_d)^3$. By union bound, we conclude that there exists a choice of $\nu$ such that $|\chi(S)| \leq \sqrt{6N_d \log(2N_1 \cdots N_d)}$ for all $S \in A_N$ with distinct $d$th coordinates.

In summary, we conclude that there is a choice of $\chi : [N_1] \times \cdots \times [N_d] \to \{1, -1\}$ so that

$$\max_{S \in A_N} |\chi(S)| \leq \max \left( \text{disc}(A_{N'}) \cdot \sqrt{6N_d \log(2N_1 \cdots N_d)} \right).$$

Note that $\text{disc}(A_N)$ is defined as the minimum over all $\chi$, so we have the desired inequality. □

**Proof of the upper bound in Theorem 1.3.** Suppose that $N_1 \geq N_2 \geq \cdots \geq N_d \geq 1 = N_{d+1}$. Assume that $N_1$ is sufficiently large to avoid triviality.

Let $R_i = \left( \prod_{j=1}^{i} N_j \right)^{\frac{1}{i+1}}$ for $1 \leq i \leq d$. Clearly,

$$\max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{\frac{1}{|I|+1}} = \max_{1 \leq i \leq d} R_i.$$

Now we take $t$ to be the first index $1 \leq i \leq d$ such that $R_i > \frac{N_{i+1}}{(\log(N_1 \cdots N_d))^{\frac{1}{t}}}$. By repeatedly applying Lemma 6.1, for $N' = (N_1, \ldots, N_t)$, we have

$$\text{disc}(A_N) \leq \max \left( \text{disc}(A_{N'}), 4 \sqrt{N_{t+1} \log(N_1 \cdots N_d)} \right). \tag{6.1}$$

By our choice of $t$, we have $4 \sqrt{N_{t+1} \log(N_1 \cdots N_d)} \leq 4 \sqrt{R_t \log(N_1 \cdots N_d)}^{\frac{1}{t}}$.

Also we have $N_t \geq R_{t-1} \left( \log(N_1 \cdots N_d) \right)^{\frac{1}{t}}$, so $N_t \geq R_t \left( \log(N_1 \cdots N_d) \right)^{\frac{1}{2(t+1)}} \geq R_t \left( \log(N_1 \cdots N_d) \right)^{\frac{1}{t}}$. Consequently, we may pick $\delta = O \left( \frac{\log(N_1 \cdots N_d)}{\log \log(N_1 \cdots N_d)} \right)$ so that $N_t^{t+1-\delta} = R_t^{t+1}$. By Theorem 1.2, we have

$$\text{disc}(A_{N'}) = O_d \left( \sqrt{\frac{\log(N_1 \cdots N_d)}{\log \log(N_1 \cdots N_d)}} \right).$$

This completes the proof by invoking (6.1). □

**Remark.** The above proof gives that we can take $C_d = 2^{O(d^3)}$ in Theorem 1.3.

### 7 CONCLUDING REMARKS

Theorem 1.3 determines $\text{disc}(A_N)$ up to a constant factor for many $N$’s. However, even when $d = 2$, there is a regime where the upper and lower bounds are not within a constant factor. As a special case, let $N = (N, \sqrt{N}(\log N)^k)$ for $k \geq \frac{3}{2}$ and large $N$. Theorem 1.3 yields a lower bound of $\Omega \left( N^{\frac{1}{2}} (\log N)^{k+1} \right)$ and an upper bound of $O \left( N^{\frac{1}{2}} (\log N)^{k+1} (\log \log N)^{-1} \right)$. If we apply Lemma 6.1 and the Matoušek–Spencer theorem in one dimension [8], we get a weaker upper
bound of $O(N^{\frac{1}{4}}(\log N)^{\frac{k+1}{2}})$. In some other regimes, such as when $0 < k < \frac{3}{2}$ in the above example, Lemma 6.1 and [8] gives a better upper bound than Theorem 1.3, yet it is still not within a constant factor from the lower bound.

It is interesting to know if the sub-logarithmic factor in the upper bound of Theorem 1.3 can be removed or not. We conjecture that it can be and the lower bound is tight.

**Conjecture 7.1.** For any integer $d \geq 1$, let $\mathbf{N} = (N_1, N_2, \ldots, N_d)$ where $N_1, \ldots, N_d$ are positive integers. Then

$$\text{disc}(\mathcal{A}_\mathbf{N}) = \Theta_d \left( \max_{I \subseteq [d]} \left( \prod_{i \in I} N_i \right)^{\frac{1}{2d+2}} \right).$$

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