Heteroscedastic Bandits with Reneging

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Abstract

Although shown to be useful in many areas as models for solving sequential decision problems with side observations (contexts), contextual bandits are subject to two major limitations. First, they neglect user “reneging” that occurs in real-world applications. That is, users unsatisfied with an interaction quit future interactions forever. Second, they assume that the reward distribution is homoscedastic, which is often invalidated by real-world datasets, e.g., datasets from finance. We propose a novel model of “heteroscedastic contextual bandits with reneging” to overcome the two limitations. Our model allows each user to have a distinct “acceptance level,” with any interaction falling short of that level resulting in that user reneging. It also allows the variance to be a function of context. We develop a UCB-type of policy, called HR-UCB, and prove that with high probability it achieves $O\left(\sqrt{T\log(T)}\right)$ regret.

1 Introduction

Multi-armed Bandits (MAB) [5] have been extensively used to model sequential decision problems with uncertain rewards. Such problems commonly arise in a large number of real-world applications such as clinical trials, search engines, online advertising, and notification systems. While in those applications, users (e.g., patients) have been modeled as being homogeneous, there is a strong motivation to enhance user experience by personalization for users and taking care of their specific demands, and thereby increase revenue with improved user experience. The model of “contextual bandits” [1] seeks to do so by proposing a MAB model for learning how to act optimally based on contexts (features) of users and arms. At the beginning of each round, the learner observes a context from the context set $\mathcal{X}$ (e.g., medical records, treatment details) and selects an arm from the arm set $\mathcal{Y}$ (e.g., different treatments). At the end of the round, the learner receives a random reward (e.g., the result of the treatment) with the mean value of its distribution depending on the observed context. The objective of the learner is to accumulate as much reward as possible within $n$ rounds. Since the parameters involved in the dependence of the mean reward on the context are unknown, the learner has to handle a trade-off between exploration (e.g., choosing new treatment with possible higher effectiveness) and exploitation (e.g., choosing the best known treatment) at each round.

While this model has been usefully applied in many areas, it is subject to two major limitations. First, it neglects the phenomenon of “reneging” that is common in real-world applications. Reneging here refers to the behavior of users cutting ties with the learner after an unsatisfactory experience, and desisting from any future interactions. This is also referred to as “churn”, “disengagement”, “abandonment”, or “unsubscribing” [13]. Since, as is well known, the acquisition cost for new users is much higher than the retention cost for existing users, handling reneging plays a critical role in business success. Reneging is common in real-world applications. For instance, in clinical trials, a patient dissatisfied with the effectiveness of a treatment quits all further trials. Search services face a similar problem; users may never again use any search engine after one returns results regarded as irrelevant. Another example is online advertising, where users stop clicking on any future advertisements, after the pursuit of one or more delivered advertisements leads to a loss of in the advertiser. Similar concerns are found in notification systems employed by content creators, where there is value in sending more e-mail notifications, but each e-mail also risks the user disabling the notification functionality, permanently eliminating any opportunity for the creator to interact with the user in the future.

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Second, previous studies have usually assumed that rewards are generated from an underlying reward distribution that is homoscedastic, i.e., its variance is independent of contexts. Unfortunately, this model is invalid due to the presence of “heteroscedasticity” in many real-world datasets, and learning algorithms based on it may be improvable. Examples abound in financial applications such as portfolio selection for hedge funds [16]. In online advertising or notification systems, the click-through rate can vary among users due to their differing spare times. Users with more spare time tend to be more tolerant to advertisements/notifications, and may continue to click on them, while users with little spare time will in most cases ignore them.

We propose a novel model of contextual bandits that addresses the challenges arising from reneging risk and heteroscedasticity. We call the model “heteroscedastic bandits with reneging.” In our model, at a round for user $t$, the learner observes a collection of contexts $(x_{t,a})_{a \in A}$, where context $x_{t,a} \in \mathbb{R}^d$ is drawn from context set $\mathcal{X}$. After observing the context, the learner selects an action $a \in A$ and receives a reward $r_{t,a}$ drawn from a reward distribution. To model heteroscedasticity, we allow for the mean $\mu(\cdot)$ and variance of the reward distribution $\sigma^2(\cdot)$ to both depend on $x_{t,a}$, i.e., $\mu(x_{t,a})$ and $\sigma^2(x_{t,a})$. To model the reneging risk, we suppose that user $t$ has a satisfaction level $\beta_t$. If $r_{t,a}$ is below level $\beta_t$, the user quits all future interactions; otherwise, the user stays. We assume that the satisfaction level for each user is fixed beforehand and does not depend on the decision of the learner. Under this model, the reneging risk associated with action $a$ of user $t$ is the probability that the observed reward is below its acceptance level, i.e., $P(r_{t,a} < \beta_t|x_{t,a})$. The parameters in $\mu(\cdot)$ and $\sigma^2(\cdot)$ are unknown and need to be learned on the fly.

Three key challenges arise in finding the optimal policy for heteroscedastic bandits with reneging. First, to estimate the unknown variance function, we have to construct a satisfactory estimator and the corresponding confidence interval. Since in statistics, there is usually no explicit way to represent the confidence interval for variance estimation, establishing regret bounds for upper confidence bound (UCB) algorithms becomes difficult. Second, the presence of reneging makes estimation of unknown functions more difficult. Each round has a non-zero probability of being the last round, and so some user-arm pair may be pulled. As a result, the conventional definition of regret needs to be modified. Moreover, since the mean and variance depend on the context, the reward distributions to be learned for one user are different from those for another user. How to transfer the knowledge accumulated on one user to another user has to be carefully handled. Third, the optimal policy needs to handle the issue of exploration vs. exploitation in terms of both rewards and risk. Intuitively, a good policy should prefer actions with high expected return and low reneging risks. This becomes difficult when there are arms that have high expected return and high risk. This work focuses on developing optimal learning algorithms that address the above challenges.

Seminal studies on contextual bandits consider linear contextual bandits [1, 10, 4], assuming that the expected reward is a linear function on contexts. Although these models have been shown to be useful in some areas, they do not address reneging and heteroscedasticity. Reneging can be handled as risk to be avoided or controlled. The risk in bandit problems has been studied for variance minimization [18] and value-at-risk maximization [21, 8, 9], and guarantees provided that outperform baselines [14, 24]. However, the risks those studies handle are different from those we are motivated by, and their models cannot be used to solve the problems of interest here. The risks they handle usually have no impact on lifetimes of bandits. Their approaches encode the consideration of risk in statistics and put them in objective functions, while in our problem, the reneging risk comes from the probability that the observed reward is below an acceptance level. Moreover, their models are restricted to homoscedastic datasets, while our model is applicable to both heteroscedastic and homoscedastic datasets. The acceptance level in our formulation has a flavor of thresholding bandits [2, 17, 12, 19]. However, the latter is based on a very different setting and assumes the distribution is context independent and homoscedastic (a more careful review and comparison are given in Section 2).

**Contributions.** Our research contributions can be summarized as follows:

1. Reward heteroscedasticity and reneging risk are common in real-world applications but not taken into account in existing bandit models. We formulate a novel model, dubbed “heteroscedastic bandits with reneging.” To the best of our knowledge, this paper is the first to address them in a bandit model.

2. To solve the proposed model, we develop a UCB-type policy, called HR-UCB, that is proved to achieve a $O \left( \sqrt{T (\log(T))^{3/2}} \right)$ regret bound with high probability. Although the proposed solution mainly applies to heteroscedastic bandits with reneging, the techniques employed here to handle heteroscedasticity can be used to solve bandits that are sensitive to variance, e.g., risk-averse bandits, thresholding bandits etc.
2 Related Work

Contextual bandits, as an approach to solve sequential decision problems with side observations (contexts) and user heterogeneity, have attracted considerable research attention recently. The most well known studies are of linear contextual bandits [1, 10, 4], where it is assumed that the expected reward is a linear function of context, an assumption also made in this paper. Although previous studies of contextual bandits have been useful in many areas, they are subject to two major limitations. First, they neglect user reneging that is commonly found in real-world applications, e.g., search engines and online advertising. That is, a user not satisfied with one interaction just drops out forever from any future interactions. Appropriately handling it has been therefore regarded by many real-world practitioners as key to their long-term viability and success [13, 3].

Second, it is usually assumed that the reward distribution is homoscedastic in contexts, which is usually invalidated by real-world datasets, e.g., datasets from financial-related applications. When the reward distribution is allowed to be context-dependent, the assumption that only the mean of the distribution depends on context restricts the applicability of those models. So motivated, in this paper we propose a novel model of contextual bandits. Differing from previous works, our model allows each user to have a distinct acceptance level, with interactions falling below it resulting in the user reneging. Moreover, our model allows the variance also to be a function of context. Modeling reneging and heteroscedasticity in contextual bandits are the salient features of this paper.

Compared to conventional contextual bandits, both the function for variance and for mean need to be learned in our model; in addition, reneging aborts future interactions and makes the learning task more complex. Moreover, diverse reward distributions make the avoidance of reneging more difficult. The objective of our paper is to propose an optimal policy that attacks those challenges. As far as we are aware, our model is the first one that addresses the two issues and achieves optimal regret.

There are two main lines of research related to our work: bandits with risk and thresholding bandits.

Bandits with Risk. Reneging can be viewed as a type of risk that the learner tries to avoid or control. The risk in bandit problems has been studied in terms of variance, quantities, and guarantees that outperform baselines. In [18] and many follow up works, mean-variance models to handle return (reward) and risk (variability) are studied, where the objective to be maximized is a linear combination of mean reward and variance. Subsequent studies [21, 8] propose a quantile (value at risk) to replace rewards and variance in evaluating which arm to select. In contrast to these works, [14, 24] control the risk by requiring that the accumulated rewards while learning the optimal policy be above those of baselines. Similarly, in [20], each arm is associated with some risk; safety is guaranteed by requiring the accumulated risk to be below a given budget. Although these studies investigate optimal policies under risk, the risks they handle are different from ours and their models cannot be used to solve our problem. The risks they handle usually have no impact on lifetime of bandits. Their approaches to handle the risk are based on more straightforward statistics, while, in our problem, the reneging risk is relatively complex, i.e., it comes from the probability that the observed reward is below an acceptance level. Moreover, their models assume homoscedasticity, while we allow the variance to depend on the context.

Thresholding Bandits. The acceptance level in our model has the flavor of thresholding bandits. However, the thresholds in the existing literature differ from our perspective. In [2], the action receives a unit payoff in the event that the sampled reward exceeds a threshold. In [17], the objective is to find the set of arms whose means are above a given threshold up to a precision. In [12], threshold is used to trigger a one-shot reward, i.e., for an arm, no rewards can be collected until the total number of successes exceeds the threshold, but once a reward is collected, the arm is removed from the interaction. Compared to the problem in this paper, the most similar one that has been studied is in [19]. However, it has a very different setting and assumes that the distribution is context independent and homoscedastic. In that paper, each arm is represented by a real number; users may abandon the program as long as the pulled arm exceeds a threshold, which measures user tolerance capability. As comparison, we consider a contextual bandit model; we allow the reward distribution to be heteroscedastic; and we capture the reneging through a probability.

As far as we are aware, only one very recent paper discusses bandits under heteroscedasticity [15]. Compared to it, our paper has two salient differences. First, we discuss heteroscedasticity under the presence of reneging. The presence of reneging makes the learning problem more challenging as the learner has to always be prepared that plans for the future may not be carried out. Second, the solution in [15] is based on information directed sampling. In contrast to that, we exhibit in this paper, a heteroscedastic UCB policy that is efficient, and easier to implement, can perfectly achieve sub-linear regret.
3 Problem Formulation

In heteroscedastic bandits with reneging, since the interaction with one user is often aborted after a finite number rounds with new users joining in the interactions afterwards, we index users by their order of interaction and conduct a regret analysis in terms of the total number of interacting users. Let \( T \) be the number of users, who are indexed by \( t = 1, 2, \ldots, T \). Let \( X := \{ x \in \mathbb{R}^d : ||x||_2 \leq 1 \} \) be the context set, where \( ||\cdot||_2 \) denotes the \( \ell_2 \)-norm. At each round for user \( t \), the learner observes a set of contexts \( X_t = \{ x_{t,a} \}_{a \in A} \subseteq X \). After observing the contexts, the learner selects an action \( a \) and receives a random reward \( r_{t,a} \) drawn from a reward distribution that satisfies:

\[
\begin{align*}
    r_{t,a} & := \theta_a^\top x_{t,a} + \varepsilon(x_{t,a}) \quad (1) \\
    \varepsilon(x_{t,a}) & \sim \mathcal{N}(0, \sigma^2(x_{t,a})) \quad (2) \\
    \sigma(x_{t,a}) & := f(\phi_a^\top x_{t,a}), \quad (3)
\end{align*}
\]

where \( \mathcal{N}(0, \sigma^2) \) denotes the Gaussian distribution with zero mean and variance \( \sigma^2 \). For the mean of the reward distribution we operate under the linear realizability assumption: that is there is an unknown \( \theta_a \in \mathbb{R}^d \) with \( ||\theta_a||_2 \leq 1 \) so that

\[
\mathbb{E}[r_{t,a}|x_{t,a}] := \theta_a^\top x_{t,a}, \quad (4)
\]

for all \( t \) and \( a \). For the variance of the reward distribution, heteroscedasticity is taken into account through a function \( f(\cdot) \)

\[
\mathbb{V}[r_{t,a}|x_{t,a}] := f^2(\phi_a^\top x_{t,a}). \quad (5)
\]

where \( f(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^+ \) is known and is required to be nonnegative, strictly increasing, and bi-Lipschitz continuous, i.e. there exists a constant \( M_f \) with \( 1 \leq M_f < \infty \) such that \( |f(z_1) - f(z_2)| \leq M_f |z_1 - z_2|_2 \), for all \( z_1, z_2 \in [-L, L] \). For example, we can choose \( f(z) = e^z \) or \( f(z) = z + L \). The parameter vector \( \phi_a \in \mathbb{R}^d \) with \( ||\phi_a||_2 \leq L \) is unknown and will be learned during interactions. Since \( \phi_a^\top x \) is bounded over all possible \( \phi_a \) and \( x \), we know that \( f(\phi_a^\top x) \) is also bounded, i.e. \( f(\phi_a^\top x) \in [\sigma_{\min}^2, \sigma_{\max}^2] \) for some \( \sigma_{\min}, \sigma_{\max} > 0 \), for all \( \phi_a \) and \( x \) defined above. This also implies that \( \varepsilon(x) \) is \( \sigma_{\max}^2 \)-sub-Gaussian, for all \( x \in X \).

The minimal expectation in an interaction of a user is characterized by its acceptance level. Denote by \( \beta_t \in \mathbb{R} \) the acceptance level of user \( t \). We assume that acceptance levels of users, like their context, are available before interacting with them. Denote by \( r_{t,i} \) the observed reward for user \( t \) at round \( i \). When \( r_{t,i} \) is below \( \beta_t \), reneging occurs and the user drops out from any future interaction. Suppose that at round \( i \), arm \( a \) is selected for user \( t \), then the risk that reneging occurs is

\[
\mathbb{P}(r_{t,i}^{(i)} < \beta_t | x_{t,a}) = \Phi \left( \frac{\beta_t - \theta_a^\top x_{t,a}}{f(\phi_a^\top x_{t,a})} \right), \quad (6)
\]

where \( \Phi(\cdot) \) is the cumulative density function (CDF) for \( \mathcal{N}(0, 1) \). Without loss of generality, we also assume that \( \beta_t \) is lower bounded by \( -B \) for some \( B > 0 \). Let \( s_t \) be the stopping time that denotes the first time that \( r_{t,i}^{(i)} \) is below the acceptance level,

\[
s_t := \min \{ i : r_{t,i}^{(i)} < \beta_t \}. \quad (7)
\]

A policy \( \pi \in \Pi \) is a rule for selecting an arm at each round of a user based on the preceding interactions with that user and other users, where \( \Pi \) denotes the set of all admissible policies. In fact, the stopping time \( s_t \) also depends on the policy that is used, so we use \( s_t^\pi \) to represent the stopping time of user \( t \) operating under policy \( \pi \). Let \( \pi_t = \{ x_{t,1}, x_{t,2}, \ldots \} \) denote the sequence of contexts that correspond to the actions of user \( t \) under policy \( \pi \). Let \( \mathcal{R}_t^\pi \) be the expected reward of user \( t \) under the action sequence \( \pi_t \). Then we have

\[
\mathcal{R}_t^\pi = \sum_{i=1}^{s_t^\pi} \left[ \theta_a^\top x_{t,i} \prod_{k=1}^{i-1} \left( 1 - \Phi \left( \frac{\beta_t - \theta_a^\top x_{t,k}}{f(\phi_a^\top x_{t,k})} \right) \right) \right], \quad (8)
\]

where \( \prod_{i=1}^{s_t^\pi-1} \left( 1 - \Phi \left( \frac{\beta_t - \theta_a^\top x_{t,k}}{f(\phi_a^\top x_{t,k})} \right) \right) \) is the probability of the event that the user \( t \) stays for at least \( i \) rounds. Then the total expected reward collected from \( T \) users can be represented by

\[
\mathcal{R}^\pi(T) = \sum_{t=1}^{T} \mathcal{R}_t^\pi. \quad (9)
\]

We are ready to define the pseudo-regret of the heteroscedastic bandits with reneging as

\[
\text{Regret}_T := \mathcal{R}^\pi(T) - \mathcal{R}^\pi(T), \quad (10)
\]

where \( \pi^* \) is the optimal policy in terms of pseudo-regret among admissible policies, i.e.,

\[
\pi^* = \arg \max_{\pi \in \Pi} \mathcal{R}^\pi(T). \quad (11)
\]

The objective of the learner is to learn a policy that achieves as minimal a regret as possible.

Illustrative examples for heteroscedasticity and reneging risk are shown in Figure 1. In Figure 1(a), the variance of the reward distribution gradually increases as the value of the one-dimensional context \( x_{t,a} \) increases. Although the mean of the reward distribution still follows the conventional formulation of being a linear function of context, and thus the ordinary least
square estimator is still unbiased, the context dependent variance makes the standard error estimates biased, and invalidates the method usually used to construct the confidence bounds. Each user-arm pair corresponds to a distribution with distinct mean and variance. Moreover, the presence of reneging risk makes every observation have a probability of being the last one, which makes the learning task more challenging. Intuitively, the optimal policy prefers the distribution that has large mean and low reneging risk. Unfortunately, it is nontrivial to follow that intuition in optimal policy construction. As shown in Figure 1(b), the reward distribution $P_1$ has mean $\mu_1$ and variance $\sigma_1^2$, correspondingly $\mu_2$ and variance $\sigma_2^2$ for $P_2$. The two correspond to the same user, but for different arms. Thus they have the same acceptance level $\beta$. A learner may prefer pulling distribution $P_2$ as its mean reward $\mu_2$ is higher than $\mu_1$. However, since the variance of $P_2$ is also higher than $P_1$, the reneging risk $P_2(r < \beta)$ (the blue shaded area) is higher than $P_1(r < \beta)$ (the red shaded area) as well. When considering which arm to pull, the learner faces an additional dilemma (beyond the exploration vs. exploitation dilemma) of choosing between receiving higher reward for one pull and staying longer to collect more future rewards. This makes the model distinct and especially difficult to solve.

4 Algorithms and Results

In this section, we present a UCB-type algorithm for heteroscedastic bandits with reneging. We start by introducing general results on heteroscedastic regression.

4.1 Heteroscedastic Regression

In this section, we consider a general regression problem with heteroscedasticity.

4.1.1 Generalized Least Squares Estimators

With a slight abuse of notation, let $\{(x_i, r_i) \in \mathbb{R}^d \times \mathbb{R}\}_{i=1}^n$ be a collection of $n$ pairs of context and reward realization that are collected sequentially. Recall from (1)-(3) that $r_i = \theta^T x_i + \varepsilon(x_i)$ and $\varepsilon(x_i) \sim \mathcal{N}(0, f(\phi_i^T x_i))$ with unknown parameters $\theta_i$ and $\phi_i$. Note that given the contexts $\{x_i\}_{i=1}^n$, $\varepsilon(x_1), \ldots, \varepsilon(x_n)$ are mutually independent. Let $r = (r_1, \ldots, r_n)^T$ and $\varepsilon = (\varepsilon(x_1), \ldots, \varepsilon(x_n))$ be the row vectors of the $n$ reward realizations and the deviations from the mean reward, respectively. Let $X_n$ be an $n \times d$ matrix in which the $i$-th row is $x_i^T$, for all $1 \leq i \leq n$. We use $\hat{\theta}_n, \hat{\phi}_n \in \mathbb{R}^d$ to denote the estimators of $\theta$ and $\phi$ based on the observations $\{(x_i, r_i)\}_{i=1}^n$, respectively. Moreover, define the estimated deviation with respect to $\hat{\theta}_n$ as

$$\hat{\varepsilon}(x_i) = r_i - \hat{\theta}_n^T x_i. \quad (12)$$

Let $\hat{\varepsilon} = (\hat{\varepsilon}(x_1), \ldots, \hat{\varepsilon}(x_n))^T$. Let $I_d$ denote the $d \times d$ identity matrix, and let $z_1 \circ z_2$ denote the Hadamard product of any two vectors $z_1, z_2$. We consider the generalized least squares estimators (GLSE) [23]

$$\hat{\theta}_n = (X_n^T X_n + \lambda I_d)^{-1} X_n^T r, \quad (13)$$

$$\hat{\phi}_n = (x_1^T x_1 + \lambda I_d)^{-1} x_1 f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon}), \quad (14)$$

where $\lambda > 0$ is some regularization parameter and $f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon}) = (f^{-1}(\hat{\varepsilon}(x_1)^2), \ldots, f^{-1}(\hat{\varepsilon}(x_n)^2))^T$ is the pre-image of the vector $\hat{\varepsilon} \circ \hat{\varepsilon}$.

Remark 1 Note that in (13), $\hat{\theta}_n$ is the conventional ridge regression estimator. On the other hand, to obtain an estimator $\hat{\phi}_n$, (14) still follows the ridge regression approach, but with two additional steps: (i) derive the estimated deviation $\hat{\varepsilon}$ based on $\hat{\theta}_n$, and (ii) apply the map $f^{-1}(-)$ on the square of $\hat{\varepsilon}$. It is known that $\hat{\phi}_n$ defined in (14) has some nice asymptotic properties (e.g. Chapter 8.2 of [23]). However, it remains unknown how to obtain non-asymptotic results regarding the confidence set for $\hat{\phi}_n$. This question will be answered rigorously in Section 4.1.2.  

4.1.2 Confidence Sets for GLSE

In this section, we discuss the confidence sets for the estimators $\hat{\theta}_n$ and $\hat{\phi}_n$ described above. To simplify notation, we define a $d \times d$ matrix $V_n$ as

$$V_n = (X_n^T X_n + \lambda I_d). \quad (15)$$

A confidence set for $\hat{\theta}_n$ was introduced in [1]. For convenience, we restate the results in the following lemma.

Lemma 1 (Theorem 2 in [1]) For all $n \in \mathbb{N}$, define

$$\alpha_n^{(1)}(\delta) = \sigma_{\text{max}}^2 \sqrt{d \log \left( \frac{n + \lambda}{\delta \lambda} \right) + \lambda^{1/2}}. \quad (16)$$

For any $\delta > 0$, with probability at least $1 - \delta$, for all $n \in \mathbb{N}$, we have

$$\|\hat{\theta}_n - \theta\|_{V_n} \leq \alpha_n^{(1)}(\delta), \quad (17)$$
where \( \|x\|_{V_n} = \sqrt{x^\top V_n x} \) is the induced vector norm of vector \( x \) with respect to \( V_n \).

**Remark 2** Note that Lemma 1 presents a uniform bound on the confidence interval and can be interpreted as follows: given any \( N_0 \in \mathbb{N} \), if there are \( N_0 \) observations obtained sequentially and the estimator \( \hat{\theta}_n \) is also updated sequentially, then with high probability (17) holds for all \( 1 \leq n \leq N_0 \).

Next, we derive the confidence set for \( \hat{\phi}_n \). Define

\[
\alpha_n^{(2)}(\delta) = \sqrt{2dL^2 \left( \left( \frac{1}{C_2} \ln \left( \frac{C_1}{\delta^2} \right) \right)^2 + 1 \right)},
\]

\[
\alpha_n^{(3)}(\delta) = \sqrt{2dL \ln \left( \frac{d}{\delta} \right)},
\]

where \( C_1 \) and \( C_2 \) are some universal constants that will be described in Lemma 3. The following is the main theorem on the confidence set for \( \hat{\phi}_n \).

**Theorem 1** For all \( n \in \mathbb{N} \), define

\[
\rho_n(\delta) = M_f \alpha_n^{(1)} \left( \frac{\delta}{3} \right) \left( 1 + 2\alpha_n^{(3)}(\delta) \right) + \alpha_n^{(2)}(\delta) + \lambda^{1/2}.
\]

For any \( \delta > 0 \), with probability at least \( 1 - \delta \), for all \( n \in \mathbb{N} \), we have

\[
\| \hat{\phi}_n - \phi_\ast \|_{V_n} \leq \rho_n \left( \frac{\delta}{n^2} \right) = O \left( \log \left( \frac{1}{\delta} \right) + \log n \right).
\]

To demonstrate the main idea behind Theorem 1, we highlight the proof procedure in the following Lemma 2-5. First, to quantify the difference between \( \hat{\phi}_n \) and \( \phi_\ast \), we start by considering the inner product of an arbitrary vector \( x \) and \( \hat{\phi}_n, \phi_\ast \) in the following lemma.

**Lemma 2** For any \( x \in \mathbb{R}^d \), we have

\[
\| x^\top \hat{\phi}_n - x^\top \phi_\ast \|_{V_n} \leq \|x\|_{V_n^{-1}} \left\{ \|\phi_\ast\|_{V_n^{-1}} + \|X_n^\top (f^{-1}(\varepsilon \circ \varepsilon) - X_n \phi_\ast)\|_{V_n^{-1}} + 2M_f \|X_n^\top (\varepsilon \circ X_n (\theta_\ast - \hat{\theta}_n))\|_{V_n^{-1}} + M_f \|X_n^\top (X_n (\theta_\ast - \hat{\theta}_n) \circ X_n (\theta_\ast - \hat{\theta}_n))\|_{V_n^{-1}} \right\}.
\]

**Proof.** The proof is provided in Appendix 6.1. \( \square \)

Based on Lemma 2, we provide upper bounds for the three terms in (23)-(25) separately as follows.

**Lemma 3** For any \( n \in \mathbb{N} \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), we have

\[
\|X_n^\top (\varepsilon \circ \varepsilon - X_n \phi_\ast)\|_{V_n^{-1}} \leq \alpha_n^{(2)}(\delta).
\]

\[
\|X_n^\top (\varepsilon \circ X_n (\theta_\ast - \hat{\theta}_n))\|_{V_n^{-1}} \leq \alpha_n^{(1)}(\delta) \cdot \alpha_n^{(3)}(\delta).
\]

\[
\|X_n^\top (X_n (\theta_\ast - \hat{\theta}_n) \circ X_n (\theta_\ast - \hat{\theta}_n))\|_{V_n^{-1}} \leq \alpha_n^{(1)}(\delta).
\]

Next, we derive an upper bound for (24).

**Lemma 4** For any \( n \in \mathbb{N} \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), we have

\[
\|X_n^\top (\varepsilon \circ X_n (\theta_\ast - \hat{\theta}_n))\|_{V_n^{-1}} \leq \alpha_n^{(1)}(\delta) \cdot \alpha_n^{(3)}(\delta).
\]

**Proof.** The main challenge is that (27) involves the product of the deviation \( \varepsilon \) and the estimation error \( \theta_\ast - \hat{\theta}_n \). Through some manipulation, we can decouple \( \varepsilon \) from \( X_n^\top (\varepsilon \circ X_n (\theta_\ast - \hat{\theta}_n))\) and apply a proper tail inequality for quadratic forms of sub-Gaussian distributions. The complete proof is provided in Appendix 6.3. \( \square \)

Next, we provide an upper bound for (25).

**Lemma 5** For any \( n \in \mathbb{N} \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), we have

\[
\|X_n^\top (X_n (\theta_\ast - \hat{\theta}_n) \circ X_n (\theta_\ast - \hat{\theta}_n))\|_{V_n^{-1}} \leq \alpha_n^{(1)}(\delta).
\]

**Proof.** Since (28) does not involve \( \varepsilon \), we can simply reuse the results in Lemma 1 through some manipulation of (28). The complete proof is provided in Appendix 6.4. \( \square \)

Now we are ready to put all the above together and prove Theorem 1.

**Proof of Theorem 1.** We use \( \lambda_{\text{min}}(\cdot) \) to denote the smallest eigenvalue of a square symmetric matrix. Recall that \( V_n = \lambda I_d + X_n^\top X_n \) is positive definite for all \( \lambda > 0 \). Then we have

\[
\|\phi_\ast\|_{V_n^{-1}} \leq \|\phi_\ast\|_{2}^2 / \lambda_{\text{min}}(V_n) \leq \|\phi_\ast\|_{2}^2 / \lambda \leq L^2 / \lambda.
\]

By (29), Lemma 2-5, we know that for a given \( n \) and a given \( \delta_n > 0 \), with probability at least \( 1 - \delta_n \), we have

\[
\|x^\top \hat{\phi}_n - x^\top \phi_\ast\|_{V_n^{-1}} \leq \|x\|_{V_n^{-1}} \cdot \rho_n(\delta_n).
\]
Note that (30) holds for any $x \in \mathbb{R}^d$. By substituting $x = V_n(\hat{\phi}_n - \phi_*)$ into (30), we have
\[
\left\| \hat{\phi}_n - \phi_* \right\|_{V_n}^2 \leq \left\| V_n(\hat{\phi}_n - \phi_*) \right\|_{V_n^{-1}} \cdot \rho_n(\delta_n). \tag{31}
\]
Since $\left\| V_n(\hat{\phi}_n - \phi_*) \right\|_{V_n^{-1}} = \left\| \hat{\phi}_n - \phi_* \right\|_{V_n}$, we know that for a given $n$ and a given $\delta_n > 0$, with probability at least $1 - \delta_n$,
\[
\left\| \hat{\phi}_n - \phi_* \right\|_{V_n} \leq \rho_n(\delta_n). \tag{32}
\]
Finally, to obtain a uniform bound, we simply choose $\delta_n = \delta/(n^2)$ and apply the union bound to (32) over all $n \in \mathbb{N}$. Note that $\sum_{n=1}^{\infty} \delta_n = \delta/n^2 = \delta(\frac{\pi}{\sqrt{2}} - 1) < \delta$. Therefore, we conclude that with probability at least $1 - \delta$, for all $n \in \mathbb{N}$,
\[
\left\| \hat{\phi}_n - \phi_* \right\|_{V_n} \leq \rho_n(\delta). \tag{33}
\]
The proof is complete. \qed

Remark 3 Note that the union bound approach utilized in (32)-(33) for deriving a uniform bound has been commonly utilized in the multi-armed bandit literature (e.g. refer to Section 2.2 of [6]).

4.2 Heteroscedastic UCB Policy

In this section, we formally introduce the proposed UCB policy based on the heteroscedastic regression discussed in Section 4.1.

4.2.1 An Oracle Policy

In this section, we consider a policy which has access to an oracle with full knowledge of $\theta_*$ and $\phi_*$. Consider $T$ users that arrive sequentially. Let $\pi^{\text{oracle}} = \{x^{i,1}_1, x^{i,2}_1, \ldots \}$ be the sequence of contexts that correspond to the actions for the user $t$ under an oracle policy $\pi^{\text{oracle}}$. The oracle policy $\pi^{\text{oracle}} = \{\pi_t^{\text{oracle}}\}$ is constructed by choosing
\[
\pi_t^{\text{oracle}} = \arg \max_{x_t \in \{x_{t,1}^{i,1}, x_{t,2}^{i,1}, \ldots \}} R^{x_t}_t, \tag{34}
\]
for each $t$. Due to the construction in (34), we know that $\pi^{\text{oracle}}$ achieves the largest possible expected reward for each user $t$, and is hence optimal in terms of pseudo-regret defined in Section 3. Based on (8) and (34), by using an one-step optimality argument, it is easy to verify that $\pi^{\text{oracle}}$ is a fixed policy for each user $t$, i.e. $x_{t,i} = x_{t,j}$, for all $i, j \geq 1$. Let $R_t^i$ denote the total expected reward of user $t$ under $\pi^{\text{oracle}}$. We have
\[
R_t^i = \theta^i x^i_*. \left( \Phi(\frac{\beta_t - \theta^i x^i_*}{f(\phi_*, x^i_*)}) \right)^{-1}. \tag{35}
\]
Next, we derive a useful property regarding (35). For any given $\beta \in [-B, \infty)$, define the function $h : [-1, 1] \times \sigma^2_{\min}, \sigma^2_{\max} \rightarrow \mathbb{R}$ as
\[
h_\beta(u, v) = u \cdot \left( \Phi\left( \frac{\beta - u}{f(v)} \right) \right)^{-1}. \tag{36}
\]
Note that for any given $x \in \mathcal{X}$, $h_\beta(\theta^\top x, \phi^\top x)$ equals the total expected reward of a single user with threshold $\beta$ if a fixed action with context $x$ is chosen under parameters $\theta_*, \phi_*$. We show that $h_\beta(\cdot, \cdot)$ has the following nice property.

Theorem 2 Let $M$ be a $d \times d$ invertible matrix. For any $\theta_1, \theta_2 \in \mathbb{R}^d$ with $\|\theta_1\| \leq 1$, $\|\theta_2\| \leq 1$, for any $\phi_1, \phi_2 \in \mathbb{R}^d$ with $\|\theta_1\| \leq L$, $\|\theta_2\| \leq L$, for any $\beta \in [-B, \infty)$, for any $x \in \mathcal{X}$,
\[
h_\beta(\theta^\top_2 x, \phi^\top_2 x) - h_\beta(\theta^\top_1 x, \phi^\top_1 x) \leq \left( C_3 \|\theta_2 - \theta_1\|_M + C_4 \|\phi_2 - \phi_1\|_M \right) \cdot \|x\|_{M^{-1}}. \tag{37}
\]
where $C_3$ and $C_4$ are some finite positive constants that are independent of $\theta_1, \theta_2, \phi_1, \phi_2$, and $\beta$.

Proof. The main idea is to apply first-order approximation under Lipschitz continuity of $h_\beta(\cdot, \cdot)$ and $f(\cdot)$. The detailed proof is provided in Appendix 6.5. \qed

4.2.2 The HR-UCB Policy

To begin with, we introduce an upper confidence bound based on the GLSE described in Section 4.1. Note that the results in Theorem 1 depend on the size of the set of context-reward pairs. Moreover, in our bandit model, the number of rounds of each user is a stopping time and can be arbitrarily large. To address this, we propose to actively maintain a regression sample set $\mathcal{S}$ through a function $\Gamma(t)$. Specifically, we let the size of $\mathcal{S}$ grow at a proper rate regulated by $\Gamma(t)$. One example is to choose $\Gamma(t) = Kt$ for some constant $K \geq 1$. Since each user will play for at least one round, we know $|\mathcal{S}|$ is at least $t$ after interacting with $t$ users. We use $\mathcal{S}(t)$ to denote the regression sample set right after the departure of user $t$. Moreover, let $X_t$ be the matrix in which the rows are composed by the contexts of all the elements in $\mathcal{S}(t)$. Similar to (15), we define $V_t = X_t^\top X_t + \lambda I_d$, for all $t \geq 1$. To simplify notation, we also define
\[
\xi_t(\delta) := C_3 \delta^2_{\mathcal{S}(t)}(\delta) + C_4 \rho_{\mathcal{S}(t)}(\delta/|\mathcal{S}(t)|^2). \tag{39}
\]
For any $x \in \mathcal{X}$, we define the upper confidence bound as follows:
\[
Q^{\text{HR}}_{t+1}(x) = h_{\beta_t}(\hat{\theta}^\top t x, \hat{\phi}^\top t x) + \xi_t(\delta) \cdot \|x\|_{V_t^{-1}}. \tag{40}
\]
Next, we show that $Q_t^{HR}(x)$ is indeed an upper confidence bound.

**Lemma 6** If the confidence set conditions (17) and (21) are satisfied, then for any $x \in \mathcal{X}$,

$$0 \leq Q_t^{HR}(x) - h_{\beta_t}(\theta_T^T x, \phi_T^T x) s \leq 2\xi_t(\delta) \|x\|_{\nu_t^{-1}}. \quad (41)$$

**Proof.** The proof is provided in Appendix 6.6.

Now, we formally introduce the HR-UCB algorithm. The complete algorithm is shown in Algorithm 1 and can be described in detail as follows:

- For each user $t$, HR-UCB observes the contexts of all available actions and then chooses an action based on the indices $Q_t^{HR}$ that depend on $\hat{\theta}_t$ and $\hat{\phi}_t$. To derive these two estimators given by (13) and (14), HR-UCB actively maintains a sample set $S$, whose size is regulated by a function $\Gamma(t)$.

- After applying an action, HR-UCB observes the corresponding reward and the reneging event if any. The current context-reward pair will be added to $S$ only if the size of $S$ is less than $\Gamma(t)$.

- Based on the regression sample set $S$, HR-UCB updates the estimators $\hat{\theta}_t$ and $\hat{\phi}_t$ right after the departure of each user.

**Algorithm 1: The HR-UCB Policy**

**Input:** action set $\mathcal{A}$, function $\Gamma(t)$, and $T$

**Initialization:** $S \leftarrow \emptyset$

for each user $t = 1, 2, \cdots, T$ do

\[ i \leftarrow 1; \]

while user $t$ stays do

\[ \pi_t^{(i)} = \arg\max_{x_t,a \in \mathcal{X}} Q_t^{HR}(x_t,a) \] (ties are broken arbitrarily); apply the action $\pi_t^{(i)}$ and observe reward $r_t^{(i)}$ and if the reneging event occurs;

if $|S| < \Gamma(t)$ then

\[ S \leftarrow S \cup \{(x_t,\pi_t^{(i)},r_t^{(i)})\}; \]

\[ i \leftarrow i + 1; \]

end

update $\hat{\theta}_t$ and $\hat{\phi}_t$ by (13)-(14) based on $S$;

end

**Remark 4** Note that under HR-UCB, the estimators $\hat{\theta}_t$ and $\hat{\phi}_t$ are updated right after the departure of each user (Line 12). Alternatively, $\hat{\theta}_t$ and $\hat{\phi}_t$ can be updated whenever $S$ is updated. While this alternative may make slightly better use of the observations, it also incurs more computation overhead. For ease of exposition, we still focus on the "lazy-update" version presented in Algorithm 1.

### 4.3 Regret Analysis

In this section, we provide the regret analysis for the proposed HR-UCB policy.

**Theorem 3** Under HR-UCB, with probability at least $1 - \delta$, the pseudo regret is upper bounded as

$$\text{Regret}_T \leq \sqrt{8\xi_T(\delta) T \cdot d \log \left( \frac{T + \lambda d}{\lambda^{1/4}d} \right)} \quad (42)$$

$$= O \left( \sqrt{T \log T \cdot \left( \log(\Gamma(T)) + \log(\frac{1}{\delta}) \right)^2} \right). \quad (43)$$

Moreover, by choosing $\Gamma(T) = K T$ for some constant $K > 0$, we have

$$\text{Regret}_T = O \left( \sqrt{T \log T \cdot \left( \log T + \log(\frac{1}{\delta}) \right)^2} \right). \quad (44)$$

**Proof.** The proof is provided in Appendix 6.7.

**Remark 5** Note that Theorem 3 presents a high-probability regret bound. To derive an expected regret bound, we can simply set $\delta = 1/T$ in (43) and get $O(\sqrt{T \log T})$.

**Remark 6** We briefly discuss the difference between our regret bound and the regret bounds of other related settings. Note that if the acceptance level $\beta_t = \infty$ for all $t$, then all the users will quit after exactly one round. This corresponds to the conventional contextual bandits setting (e.g. homoscedastic case [10] and heteroscedastic case [15]). In this degenerate case, our regret bound is $O(\sqrt{T \log T})$, which has an additional factor $\log T$ resulting from the heteroscedasticity with reneging.

### 5 Concluding Remarks

In this paper, we have studied the challenges in bandit modeling that arise from heteroscedasticity and reneging. Most existing contextual bandit algorithms suffer from neglecting them and cannot be used. These complications exist in many real-world applications, and taking them into account is economically necessary for the success of the business. To attack the above challenges, we have formulated a heteroscedastic bandit model with reneging, where the user may quit from future interactions if the reward falls below its acceptance level, and the variance of reward distribution can depend on context. We have proposed a UCB-type policy, called HR-UCB, to solve this novel model, and proved that it achieves $O(\sqrt{T (\log(T))^{3/2}})$ regret. The techniques we developed to estimate heteroscedastic variance and establish sub-linear regret under the presence of heteroscedasticity, can be extended to other variance sensitive bandit problems, such as risk-averse bandits, thresholding bandits, etc.
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6 Appendix

6.1 Proof of Lemma 2

Proof. Recall that $V_n = (X_n^T X_n + \lambda I_d)$. Note that

$$\hat{\phi}_n = (X_n^T X_n + \lambda I_d)^{-1} X_n^T f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon})$$

(45)

$$= V_n^{-1} X_n^T f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon})$$

(46)

$$= V_n^{-1} X_n^T (f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon}) - X_n \phi_s + X_n \phi_s) + \lambda V_n^{-1} \phi_s - \lambda V_n^{-1} \phi_s$$

(47)

$$= V_n^{-1} X_n^T (f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon}) - X_n \phi_s) - \lambda V_n^{-1} \phi_s + \phi_s.$$  

(48)

Therefore, for any $x \in \mathbb{R}^d$, we know

$$\left| x^T \hat{\phi}_n - x^T \phi_s \right| \leq \|x\|_{V_n^{-1}} \left( \lambda \|\phi_s\|_{V_n^{-1}} - \lambda x^T V_n^{-1} \phi_s \right)$$

(50)

$$\leq \|x\|_{V_n^{-1}} \left( \lambda \|\phi_s\|_{V_n^{-1}} - \lambda \|x\|_{V_n^{-1}} - \lambda \|x\|_{V_n^{-1}} \phi_s \right)$$

(51)

$$+ \left( \|x^T (f^{-1}(\hat{\varepsilon} \circ \hat{\varepsilon}) - X_n \phi_s)\|_{V_n^{-1}} \right).$$

(52)

Moreover, by rewriting $\hat{\varepsilon} = \varepsilon - \varepsilon + \varepsilon$, we have

$$f^{-1}(\varepsilon \circ \varepsilon)$$

(53)

$$= f^{-1}(\varepsilon \circ \varepsilon) \circ (\varepsilon - \varepsilon + \varepsilon)$$

(54)

$$\leq f^{-1}(\varepsilon \circ \varepsilon) + M_f \left( 2 \varepsilon \circ X_n (\theta_s - \hat{\theta}_n) \right)$$

(55)

$$+ \left( X_n (\theta_s - \hat{\theta}_n) \circ X_n (\theta_s - \hat{\theta}_n) \right).$$

(56)

where (56)-(57) follow from the fact that $f(\cdot)$ is bi-Lipschitz continuous and hence $f^{-1}(\cdot)$ is Lipschitz continuous as described in Section 3. Therefore, by (50)-(57) and the Cauchy-Schwarz inequality, we have

$$\left| x^T \hat{\phi}_n - x^T \phi_s \right| \leq \|x\|_{V_n^{-1}} \left\{ \lambda \|\phi_s\|_{V_n^{-1}} + \left\| X_n (f^{-1}(\varepsilon \circ \varepsilon) - X_n \phi_s)\right\|_{V_n^{-1}} \right.$$  

(58)

$$+ 2M_f \left\| X_n (\circ X_n (\theta_s - \hat{\theta}_n)) \right\|_{V_n^{-1}}$$

(59)

$$+ M_f \left\| X_n (\theta_s - \hat{\theta}_n) \circ X_n (\theta_s - \hat{\theta}_n) \right\|_{V_n^{-1}}.$$  

(60)

(61)

We use $a^\top$ to denote the conjugate transpose of $a$. Let $a = (a_1, \cdots, a_N)^\top$, let $\overline{a}$ denote the complex conjugate of $a_i$, for all $i$, and let $B = \{B_{ij}\}$ be a complex $N \times N$ matrix. Then, we have

$$P\{|a| B a - \sigma^2 \text{tr}(B)| \geq s \sigma^2 \left( \sum_{i=1}^N |B_{ii}|^2 \right)^{-1/2} \}$$

(63)

$$\leq C_1 \exp\left(-C_2 \cdot s^{1/(1+\kappa_1)}\right),$$

(64)

where $C_1$ and $C_2$ are positive constants that depend only on $\kappa_1, \kappa_2$. Moreover, for the standard $\chi^2$-distribution, $\kappa_1 = 1$ and $\kappa_2 = 2$.

Lemma 8

$$\|V_n^{-1/2} X^T\|_2 \leq 1, \forall n \in \mathbb{N}.\quad (65)$$

Proof. By the definition of induced matrix norm,

$$\|V_n^{-1/2} X^T\|_2 = \max_{\|v\|_2=1} \sqrt{v^T V_n X^T v} \leq$$

(66)

$$= \lambda_{\max} (X^T X) + \lambda \leq 1,$$

(68)

where (69) follows from the singular value decomposition and $\lambda_{\max}(X^T X) \geq 0$.  

To simplify notation, we use $X$ and $V$ as a shorthand for $X_n$ and $V_n$, respectively. For convenience, we rewrite $V^{-1/2} X^T = [v_1 \cdots v_n]$ as the matrix of $n$ column vectors $\{v_i\}_{i=1}^n$ (each $v_i \in \mathbb{R}^d$).

Lemma 9

$$\sum_{i=1}^n \|v_i\|_2^2 \leq d.$$  

(70)

Proof of Lemma 9. Recall that $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a square matrix. We know

$$\sum_{i=1}^n \|v_i\|_2^2 = \text{tr}\left((X V^{-1/2}) (V^{-1/2} X^T)\right)$$

(71)

$$= \text{tr}\left((V^{-1/2} X) (X^T V^{-1/2})\right)$$

(72)

$$\leq d \cdot \lambda_{\max}\left((V^{-1/2} X) (X^T V^{-1/2})\right).$$

(73)

where (72) follows from the trace of a product being commutative, and (73) follows since the trace is the
sum of all eigenvalues. Moreover, we have
\[
\lambda_{\text{max}} \left( (XV^{1/2})(X^T V^{-1/2}) \right) \leq \left\| (XV^{1/2})(X^T V^{-1/2}) \right\|_2 \leq \left\| XV^{1/2} \right\|_2 \left\| X^T V^{-1/2} \right\|_2 \leq 1, \tag{76}
\]
where (76) follows from the fact that the \( \ell_2 \)-norm is sub-multiplicative. Therefore, by (71)-(76), we conclude that \( \sum_{i=1}^n \|v_i\|^2_2 \leq d. \)
\[
\square
\]
We are now ready to prove Lemma 3.

**Proof of Lemma 3.** To simplify notation, we use \( X \) and \( V \) as a shorthand for \( X_n \) and \( V_n \), respectively. To begin with, we know
\[
\|X(\varepsilon \circ \varepsilon - X\phi)\|_{V^{-1}} \leq \left\| V^{-1/2}(\varepsilon \circ \varepsilon - X\phi) \right\|_2 \leq \sqrt{\left( \varepsilon \circ \varepsilon - X\phi \right)^T X^{-1} X^T (\varepsilon \circ \varepsilon - X\phi)} \tag{79}
\]
where each element in the vector \( (\varepsilon \circ \varepsilon - X\phi) \) is a centered \( \chi^2_{d} \)-distribution with a scaling of \( \phi^T x_i \). Defining \( W = \text{diag}(x_1^T \phi_1, \ldots, x_n^T \phi_n) \), we have
\[
\|X(\varepsilon \circ \varepsilon - X\phi)\|_{V^{-1}} = \left[ \left( \varepsilon \circ \varepsilon - X\phi \right)^T W^{-1} (WXX^{-1}X^TW)^{1/2} \right] \tag{80}
\]
where \( U \) is a finite sequence of fixed self-disjoint matrices of dimension \( d \times d \), and let \( \{A_k\} \) be a finite sequence of independent standard normal variables. Let \( \sigma^2 = \|\sum_k A_k\|_2 \). Then, for all \( s \geq 0 \),
\[
\mathbb{P}\left\{ \lambda_{\text{max}}(\sum_k A_k) \geq s \right\} \leq \exp(-s^2/2\sigma^2), \tag{85}
\]
where \( \lambda_{\text{max}}(\cdot) \) denotes the largest eigenvalue of a square matrix.

We use \( \eta = W^{-1}(\varepsilon \circ \varepsilon - X\phi) \) as a shorthand and define \( U = \{U_{ij}\} = WXX^{-1}X^TW \). By Lemma 7 and the fact that \( \varepsilon(x_1), \ldots, \varepsilon(x_n) \) are mutually independent given the contexts \( \{x_i\}_{i=1}^n \), we have
\[
\mathbb{P}\left\{ \eta^T U \eta - 2 \cdot \text{tr}(U) \geq 2s \left( \sum_{i=1}^n \left| U_{ii} \right|^2 \right)^{1/2} \right\} \leq C_1 \mathbb{P}\left\{ \lambda_{\text{max}}(\sum_k A_k) \geq s \right\} \tag{81}
\]
where \( \lambda_{\text{max}}(\cdot) \) denotes the largest eigenvalue of a square matrix.

6.3 Proof of Lemma 4

We first introduce a useful lemma.

**Lemma 10 (Theorem 4.1 in [22])** Consider a finite sequence \( \{A_k\} \) of fixed self-disjoint matrices of dimension \( d \times d \), and let \( \{\gamma_k\} \) be a finite sequence of independent standard normal variables. Let \( \sigma^2 = \|\sum_k A_k\|_2 \). Then, for all \( s \geq 0 \),
\[
\mathbb{P}\left\{ \lambda_{\text{max}}(\sum_k \gamma_k A_k) \geq s \right\} \leq d \cdot \exp(-s^2/2\sigma^2), \tag{82}
\]
where \( \lambda_{\text{max}}(\cdot) \) denotes the largest eigenvalue of a square matrix.

Now we are ready to prove Lemma 4.

**Proof of Lemma 4.** To simplify notation, we use \( X \) and \( V \) as a shorthand for \( X_n \) and \( V_n \), respectively. Recall that \( V^{-1/2}X^T = [v_1 \cdots v_n] \). The trace of \( U \) can be upper bounded as
\[
\text{tr}(U) = \text{tr}(WXX^{-1}X^TW) \leq \sum_{i=1}^n (x_i^T \phi_i)^2 \cdot \|v_i\|^2_2 \leq L^2 \sum_{i=1}^n \|v_i\|^2_2 \leq L^2d, \tag{84}
\]
where the last inequality in (84) follows directly from Lemma 9. Also by the commutative property of the trace operation, we have
\[
\sum_{i=1}^n |U_{ii}|^2 \leq (\sum_{i=1}^n |U_{ii}|)^2 \leq (L^2d)^2, \tag{85}
\]
where (a) follows from \( U \) being positive semi-definite (all diagonal elements are nonnegative). Therefore, (85) implies
\[
\sum_{i=1}^n |U_{ii}|^2 \leq C_1 \cdot \exp(-C_2\sqrt{s}). \tag{86}
\]
\[
\mathbb{P}\left\{ \eta^T U \eta \geq 2s \cdot (L^2d) + 2 \cdot L^2d \right\} \leq C_1 \cdot \exp(-C_2\sqrt{s}). \tag{87}
\]

By choosing \( s = \left( \frac{1}{C_2} \ln \frac{C_1}{\delta} \right)^2 \), we have
\[
\mathbb{P}\left\{ \eta^T U \eta \geq 2L^2d(\left( \frac{1}{C_2} \ln \frac{C_1}{\delta} \right)^2 + 1) \right\} \leq \delta. \tag{88}
\]
Therefore, we conclude that with probability at least \( 1 - \delta \), the following inequality holds
\[
\|X(\varepsilon \circ \varepsilon - X\phi)\|_{V^{-1}} \leq \sqrt{2L^2d(\left( \frac{1}{C_2} \ln \frac{C_1}{\delta} \right)^2 + 1)}. \tag{89}
\]
\[
\square
\]

6.3 Proof of Lemma 4

We first introduce a useful lemma.

**Lemma 10 (Theorem 4.1 in [22])** Consider a finite sequence \( \{A_k\} \) of fixed self-disjoint matrices of dimension \( d \times d \), and let \( \{\gamma_k\} \) be a finite sequence of independent standard normal variables. Let \( \sigma^2 = \|\sum_k A_k\|_2 \). Then, for all \( s \geq 0 \),
\[
\mathbb{P}\left\{ \lambda_{\text{max}}(\sum_k \gamma_k A_k) \geq s \right\} \leq d \cdot \exp(-s^2/2\sigma^2), \tag{82}
\]
where \( \lambda_{\text{max}}(\cdot) \) denotes the largest eigenvalue of a square matrix.
\[ D = \text{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n). \] Then we have:

\[
\left\| X^T (\varepsilon \circ (X(\theta_\ast - \hat{\theta})) \right\|_{V^{-1}}
\]
\[ = \left\| V^{-1/2} X^T (\varepsilon \circ (X(\theta_\ast - \hat{\theta})) \right\|_2
\]
\[ = \left\| V^{-1/2} X^T DX(\theta_\ast - \hat{\theta}) \right\|_2
\]
\[ = \left\| V^{-1/2} X^T DXV^{-1/2} V^{1/2}(\theta_\ast - \hat{\theta}) \right\|_2
\]
\[ \leq \left\| V^{-1/2} X^T DXV^{-1/2} \right\|_2 \cdot \left\| V^{1/2}(\theta_\ast - \hat{\theta}) \right\|_2
\]
\[ = \left\| V^{-1/2} X^T DXV^{-1/2} \right\|_2 \cdot \left\| \theta_\ast - \hat{\theta} \right\|_V. \] (101)

Next, the first term in (101) can be expanded into

\[
\left\| V^{-1/2} X^T DXV^{-1/2} \right\|_2
\]
\[ = \sum_{i=1}^n \varepsilon_i v_i v_i^T = \sum_{i=1}^n \frac{\varepsilon_i}{f(x_i^T \phi_\ast)} \cdot (f(x_i^T \phi_\ast) A_i). \] (103)

Note that \( \frac{\varepsilon_i}{f(x_i^T \phi_\ast)} \) is a standard normal random variable, for all \( i \). We also define a \( d \times d \) matrix \( \Sigma = \sum_{i=1}^n f(x_i^T \phi_\ast) A_i \). Then, we have

\[
\Sigma = \sum_{i=1}^n f(x_i^T \phi_\ast) (v_i v_i^T) = \sum_{i=1}^n f(x_i^T \phi_\ast) \|v_i\|^2 v_i v_i^T. \] (105)

We also know

\[
\left\| \sum_{i=1}^n A_i \right\| = \left\| \sum_{i=1}^n v_i v_i^T \right\|
\]
\[ = \left\| \left( V^{-1/2} X^T \right) (XV^{-1/2}) \right\|_2
\]
\[ \leq \left\| V^{-1/2} X^T \right\|_2 \left\| XV^{-1/2} \right\|_2 \leq 1, \] (108)

where (108) follows from Lemma 8. Moreover, we know

\[
\left\| \Sigma \right\|_2 = \left\| \sum_{i=1}^n f(x_i^T \phi_\ast) \|v_i\|^2 v_i v_i^T \right\|
\]
\[ \leq \left\| 2d \cdot L \sum_{i=1}^n v_i v_i^T \right\|
\]
\[ = 2d \cdot L \left\| \sum_{i=1}^n A_i \right\| \leq 2d \cdot L, \] (111)

where the last inequality follows directly from (108).

By Lemma 10 and the fact that \( \varepsilon(x_1), \ldots, \varepsilon(x_n) \) are mutually independent given the contexts \( \{x_i\}_{i=1}^n \), we know that

\[
\mathbb{P} \left\{ \lambda_{\text{max}} \left( \sum_{i=1}^n \varepsilon_i A_i \right) \geq \sqrt{2 \left\| \Sigma \right\|_2} s \right\} \leq d \cdot e^{-s}. \] (112)

Therefore, by choosing \( s = \ln(d/\delta) \) and the fact that \( \lambda_{\text{max}} \left( \sum_{i=1}^n \varepsilon_i A_i \right) = \| \sum_{i=1}^n \varepsilon_i A_i \| \), we obtain

\[
\mathbb{P} \left\{ \lambda_{\text{max}} \left( \sum_{i=1}^n \varepsilon_i A_i \right) \geq \sqrt{2dL \ln(\frac{d}{\delta})} \right\} \leq \delta. \] (113)

Finally, by Lemma 1, (101), and (113), we conclude that for any \( n \in \mathbb{N} \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \), we have

\[
\left\| X_n^T (\varepsilon \circ X_n(\theta_\ast - \hat{\theta}_n)) \right\|_{V^{-1}} \leq \alpha_n^{(1)}(\delta) \cdot \alpha_n^{(3)}(\delta). \] (114)

6.4 Proof of Lemma 5

We first introduce a useful lemma on the norm of the Hadamard product of two matrices.

**Lemma 11** Given any two matrices \( A \) and \( B \) of the same dimension, the following holds:

\[
\| A \circ B \|_F \leq \| A \|_2 \cdot \| B \|_2,
\]

where \( \| \cdot \| \) denotes the Frobenius norm. When \( A \) and \( B \) are vectors, the above degenerates to

\[
\| A \circ B \|_2 \leq \| A \|_2 \cdot \| B \|_2.
\]

**Proof of Lemma 5.** To simplify notation, we use \( X \) and \( V \) as a shorthand for \( X_n \) and \( V_n \), respectively. Let \( M \) be a positive definite matrix. We have

\[
\| Av \|_M = \sqrt{v^T A^T M A v}
\]
\[ = \sqrt{\left( M^{1/2} A v \right)^T \left( M^{1/2} A v \right)}
\]
\[ = \left\| M^{1/2} A \right\|_2 \leq \left\| M^{1/2} A \right\|_2 \cdot \| v \|_2,
\]

where the last inequality follows from the sub-multiplicativity of the \( \ell_2 \)-norm. Meanwhile, we also observe that

\[
\begin{align*}
\left( \theta_\ast - \hat{\theta} \right)^T X^T X \left( \theta_\ast - \hat{\theta} \right) & = \left( \theta_\ast - \hat{\theta} \right)^T V^{1/2} V^{-1/2} X^T X V^{-1/2} V^{1/2} \left( \theta_\ast - \hat{\theta} \right) \\
& = \left\| \left( \theta_\ast - \hat{\theta} \right)^T V^{1/2} V^{-1/2} X \right\|_2 \\
& \leq \left\| \left( \theta_\ast - \hat{\theta} \right)^T \right\|_2 \left\| V^{1/2} X \right\|_2 \leq \left\| \theta_\ast - \hat{\theta} \right\|_V.
\end{align*}
\]
Therefore, we know

\[ \| X^T (X(\theta_* - \bar{\theta}) \circ X(\theta_* - \bar{\theta})) \|_{V^{-1}} \leq 1 \cdot \| X(\theta_* - \bar{\theta}) \|_2^2 \leq 1 \cdot \| X(\theta_* - \bar{\theta}) \|_2 \leq \| \theta_* - \bar{\theta} \|_V \leq \alpha_h^{(i)}(\delta), \]

where (127) follows from Lemma 8 and 11, and (129) follows from Lemma 1. The proof is complete. \( \square \)

### 6.5 Proof of Theorem 2

We first need the following lemma about Lipschitz smoothness of the function \( h_\beta(\cdot, \cdot) \).

**Lemma 12** The function \( h_\beta(u, v) \) defined in (36) is (uniformly) Lipschitz smooth on its domain, i.e., there exists a finite \( M_h > 0 \) (\( M_h \) is independent of \( u, v, \) and \( \beta \)) such that for any \( \beta \) with \( |\beta| \leq B \), for any \( u_1, u_2 \in [-1, 1] \) and \( v_1, v_2 \in [\sigma_{\min}^2, \sigma_{\max}^2] \),

\[ \| \nabla h_\beta(u_1, v_1) - \nabla h_\beta(u_2, v_2) \| \leq M_h \| (u_1) - (u_2) \|_2. \]

Moreover, we have

\[ h_\beta(u_2, v_2) - h_\beta(u_1, v_1) \leq \left( u_2 - u_1 \right)^\top \nabla h_\beta(u_1, v_1) + \frac{M_h}{2} \left( v_2 - v_1 \right)^2. \]

**Proof of Lemma 12.** First, it is easy to verify that \( h_\beta(\cdot, \cdot) \) is twice continuously differentiable on its domain \([-1, 1] \times [\sigma_{\min}^2, \sigma_{\max}^2] \) and therefore is Lipschitz smooth, for some finite positive constant \( M_h \). To show that there exists an \( M_h \) that is independent of \( u, v, \beta \),

we consider the gradient and Hessian of \( h_\beta(\cdot, \cdot) \) as

\[ \frac{\partial h_\beta}{\partial u} = \phi\left(\frac{\beta - u}{v}\right) + \frac{u}{v} \cdot \phi'\left(\frac{\beta - u}{v}\right) \]

\[ \frac{\partial h_\beta}{\partial v} = \frac{u \cdot \phi'\left(\frac{\beta - u}{v}\right)}{\left(\frac{\phi\left(\frac{\beta - u}{v}\right)}{v}\right)^2} \]

\[ \frac{\partial^2 h_\beta}{\partial u^2} = \frac{-2u}{v^3} \phi'\left(\frac{\beta - u}{v^2}\right) + \frac{u}{v^4} \phi''\left(\frac{\beta - u}{v^2}\right) \]

\[ \frac{\partial^2 h_\beta}{\partial v^2} = \frac{2u}{v^3} \phi'\left(\frac{\beta - u}{v^2}\right)^2 \]

Given the facts that for all the legal \( u, v, \) and \( \beta \), we have \( \Phi\left(\frac{\beta - u}{v}\right) \in [\Phi(\frac{\beta - u}{v^2}), 1], 0 < \phi'\left(\frac{\beta - u}{v^2}\right) < 1 \), and \( |\phi''\left(\frac{\beta - u}{v^2}\right)| \leq \frac{\beta - u}{v^2} \), it is easy to verify that such an \( M_h \) indeed exists by substituting the above conditions into (133)-(138). Moreover, by Lemma 3.4 in [7], we know that (132) indeed holds. \( \square \)

**Proof of Theorem 2.** Define

\[ q_u := \sup_{u_0 \in (-1, 1)} \left| \frac{\partial h_\beta}{\partial u} \right|_{u=u_0}, \]

\[ q_v := \sup_{v_0 \in [\sigma_{\min}, \sigma_{\max}]} \left| \frac{\partial h_\beta}{\partial v} \right|_{v=v_0}. \]

By (133) and (134), we know that \( q_u \) and \( q_v \) are both bounded positive real numbers. By substituting \( u_1 = \theta_1^\top x, u_2 = \theta_2^\top x, v_1 = f(\phi_1^\top x), \) and \( v_2 = f(\phi_2^\top x) \) into
Recall that \( \pi_{\text{oracle}} \) and \( x_t^* \) denote the oracle policy and the context of the action of the oracle policy for user \( t \), respectively. Now, we can compute the pseudo regret of HR-UCB as

\[
\text{Regret}_T = \sum_{t=1}^{T} \tilde{R}_t - \tilde{R}_t^{HR} = \sum_{t=1}^{T} h_{\beta_t}(\theta_t^*, x_t^*, \phi_t^*, x_t^*) - h_{\beta_t}(\theta_t x_t, \phi_t x_t).
\]

To simplify notation, we use \( w_t \) as a shorthand for \( h_{\beta_t}(\theta_t^*, x_t^*, \phi_t^*, x_t^*) - h_{\beta_t}(\theta_t x_t, \phi_t x_t) \). Given any \( \delta > 0 \), define an event \( E_\delta \) in which (17) and (21) hold under the given \( \delta \), for all \( t \in \mathbb{N} \). By Lemma 1 and Theorem 1, we know that the event \( E_\delta \) occurs with probability at least \( 1 - 2\delta \). Therefore, with probability at least \( 1 - 2\delta \), for all \( t \in \mathbb{N} \), we have

\[
w_t \leq Q_t^{HR}(x_t) - h_{\beta_t}(\theta_t x_t, \phi_t x_t) \leq Q_t^{HR}(x_t) - h_{\beta_t}(\theta_t^* x_t, \phi_t^* x_t) \leq Q_t^{HR}(x_t) - h_{\beta_t}(\theta_t^* x_t, \phi_t^* x_t) + \xi_t(\delta) \|x_t\|_{V_t^{-1}}^{-1} - h_{\beta_t}(\theta_t^* x_t, \phi_t^* x_t) \leq 2\xi_t(\delta) \|x_t\|_{V_t^{-1}}^{-1},
\]

where (158) and (160) follow directly from the definition of the UCB index, (159) follows from the design of HR-UCB algorithm, and (162) is a direct result under the event \( E_\delta \). Define a \( d \times d \) matrix \( \nabla_t = \lambda I_d + \lambda i \sum_{i=1}^{d} x_i x_i^T \). Recall that \( V_t = \lambda I_d + X_t X_t^T \). By comparing the eigenvalues of \( \nabla_t \) and those of \( V_t \), we know that

\[
\|x_t\|_{V_t^{-1}}^{-1} \leq \|x_t\|_{\nabla_t^{-1}}^{-1} , \forall t \in \mathbb{N}.
\]

Therefore, we conclude that with probability at least \( 1 - 2\delta \), we have

\[
\text{Regret}_T = \sum_{t=1}^{T} w_t \leq \sqrt{T \sum_{t=1}^{T} w_t^2} \leq \sqrt{T \sum_{t=1}^{T} 4\xi_t^2(\delta) \|x_t\|_{V_t^{-1}}^{-1}} \leq \sqrt{\frac{8\xi_t^2(\delta) T \cdot d \log \left( \frac{+ \lambda d}{\lambda^d/ \lambda d} \right)}{}}.
\]

where (164) follows from the Cauchy-Schwarz inequality, (165) follows from the fact that \( \xi_t(\delta) \) is an increasing function in \( t \), (166) is a direct result of (163),
and (167) follows from Lemma 10 and 11 in [1]. By substituting $\xi_i(\delta)$ into (167) and using the fact that $S(T) \leq \Gamma(T)$, we know

$$\text{Regret}_T = O \left( \sqrt{T \log T \cdot \left( \log (\Gamma(T)) + \log \left( \frac{1}{\delta} \right) \right)^2} \right).$$

(168)

By choosing $\Gamma(T) = KT$ for some constant $K > 0$, we thereby have

$$\text{Regret}_T = O \left( \sqrt{T \log T \cdot \left( \log T + \log \left( \frac{1}{\delta} \right) \right)^2} \right).$$

(169)

The proof is complete. $\square$

6.8 Experiments

We evaluate the performance of HR-UCB as shown in Figure 2 and Figure 3. We follow below steps to obtain the context for an user-arm pair. The context for a user is drawn from a multi-variate Gaussian distribution and is different for different users. Each arm is assigned with a multi-variate Gaussian distribution. When a new user comes, the context of an arm is drawn from its associated distribution and is fixed while user is served. The context for an user-arm pair is obtained through concatenating the context for the user and the context for the arm. In the two figures, the multi-variate Gaussian distribution for users is $\mathcal{N}(\mu_u, \Sigma_u)$:

$$\mu_u = [12, 23]^T,$$

$$\Sigma_u = \begin{bmatrix} 10 & 7 \\ 7 & 5 \end{bmatrix}.$$  

10 arms are used in the experiments. Arm $i$ associates with multi-variate Gaussian distribution $\mathcal{N}(\mu_i, \Sigma_i)$, whose means are:

$$\mu_1 = [1, 5]^T, \quad \mu_2 = [2, 6]^T, \quad \mu_3 = [3, 7]^T,$$

$$\mu_4 = [4, 8]^T, \quad \mu_5 = [5, 9]^T, \quad \mu_6 = [6, 10]^T,$$

$$\mu_7 = [7, 15]^T, \quad \mu_8 = [8, 20]^T, \quad \mu_9 = [2, 10]^T,$$

$$\mu_{10} = [10, 20]^T.$$  

The covariance matrices are:

$$\Sigma_1 = \begin{bmatrix} 10 & 5 \\ 5 & 3 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix}, \quad \Sigma_3 = \begin{bmatrix} 6 & 7 \\ 7 & 10 \end{bmatrix},$$

$$\Sigma_4 = \begin{bmatrix} 5 & 8 \\ 8 & 16 \end{bmatrix}, \quad \Sigma_5 = \begin{bmatrix} 15 & 9 \\ 9 & 7 \end{bmatrix}, \quad \Sigma_6 = \begin{bmatrix} 16 & 10 \\ 10 & 15 \end{bmatrix},$$

$$\Sigma_7 = \begin{bmatrix} 10 & 11 \\ 11 & 15 \end{bmatrix}, \quad \Sigma_8 = \begin{bmatrix} 12 & 6 \\ 6 & 5 \end{bmatrix}, \quad \Sigma_9 = \begin{bmatrix} 7 & 4 \\ 4 & 6 \end{bmatrix},$$

$$\Sigma_{10} = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}.$$  

The values for the hyper-parameters we use in simulation are: $\lambda, C_1, C_2, C_3, C_4, M_f = 1$, $L = 1.1$, $f(x) = \sqrt{x + L}$, $\delta = 0.1$, $d = 4$, $\sigma_{\text{max}}^2 = 2.2$. The true values of parameters to be learned are:

$$\theta = [0.6, 0.1, 0.2, 0.65]^T,$$

$$\phi = [0.2, 0.4, 0.8, 0.3]^T.$$  

In Figure 2, 50,000 users are served by the HR-UCB policy. We repeat the experiments 10 times and take the average of accumulated regret. We observe that the trend of accumulated regret is consistent with the regret bound established in Theorem 3.

Figure 2: Accumulated average regret with $K = 2.5$ under the HR-UCB policy.

Figure 3: Comparison of accumulated average regrets with different values of $K$ under the HR-UCB policy.

To visualize the performance sensitivity to the choice of $K$, we conduct experiments for different $K$s. As shown in Figure 3, 50,000 users are served by the HR-UCB policy with different values of $K$. The results indicate that the performance is not sensitive to choice of $K$ when $K$ is big enough $^2$.

$^2$code available at github.com/Xi-Liu/