A CANONICAL DECOMPOSITION OF POSTCRITICALLY FINITE RATIONAL MAPS AND THEIR MAXIMAL EXPANDING QUOTIENTS

DZMITRY DUDKO, MIKHAIL HLUSHCHANKA, AND DIERK SCHLEICHER

ABSTRACT. We provide a natural canonical decomposition of postcritically finite rational maps with non-empty Fatou sets based on the topological structure of their Julia sets. The building blocks of this decomposition are maps where all Fatou components are Jordan disks with disjoint closures (Sierpiński maps), as well as those where any two Fatou components can be connected through a countable chain of Fatou components with common boundary points (crochet or Newton-like maps).

We provide several alternative characterizations for our decomposition, as well as an algorithm for its effective computation. We also show that postcritically finite rational maps have dynamically natural quotients in which all crochet maps are collapsed to points, while all Sierpiński maps become small spheres; the quotient is a maximal expanding cactoid. The constructions work in the more general setup of Böttcher expanding maps, which are metric models of postcritically finite rational maps.

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1. INTRODUCTION

The dynamics of a rational map is controlled in a very strong sense by its critical orbits. If all the critical orbits are finite, then the map is called postcritically finite (PCF). PCF maps are like rational points of the parameter space and have been in the focus of intense research in holomorphic dynamics.

It is often convenient to abstract from the complex structure and consider rational maps as topological branched coverings. In the PCF setting we naturally obtain a branched self-covering $f : (S^2, A) \to (S^2, A)$, where $A \subset S^2$ is a finite invariant set containing all the critical values of $f$. Topological

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maps are more amenable to classifications and applicable to many surgeries, such as decompositions and amalgams.

The *Thurston fundamental theorem of complex dynamics* characterizes those PCF branched coverings of $S^2$ that are “realized” by rational maps [DH93]. Roughly speaking, it says that $f: (S^2, A) \looparrowright$ is isotopic to a rational map if and only if $f$ does not admit a collection of disjoint annuli violating the Grötzsch inequality. A *Thurston obstruction* is an invariant multicurve composed of the core curves of such annuli. The proof of Thurston’s theorem is based on a fixed-point argument. The map $f: (S^2, A) \looparrowright$ naturally defines a pullback map $\sigma_f: \mathcal{T}_A \looparrowright$ on the Teichmüller space of $(S^2, A)$ (the space of complex structures). It follows that $f$ is isotopic to a rational map if and only if $\sigma_f$ has a fixed point. Somewhat similar ideas were used by Thurston in his work on geometry of 3-manifolds and surface homeomorphisms.

Treating rational maps as topological closely links complex dynamics to the theory of mapping class groups. Just like homeomorphisms, PCF branched coverings of $S^2$ can be decomposed by cutting the sphere along invariant multicurves. The general decomposition theory was developed by Pilgrim [Pil03], who also introduced the first *canonical decomposition* of $f: (S^2, A) \looparrowright$ along the Thurston obstruction consisting of curves that get shorter under iteration of the pullback map $\sigma_f$. These curves cut the sphere into “small” maps of three types [Sel12]: homeomorphisms, double covers of torus endomorphisms, rational maps. Recently, the *canonical Levy decomposition* was introduced in [BD18] as the smallest Levy multicurve such that all small maps in the decomposition are either homeomorphisms or Levy-free maps with deg > 1. It follows from [Sel12, SY15] that the canonical Levy decomposition is a subdecomposition of the Pilgrim canonical decomposition.

Mapping class groups naturally appear in the conjugacy problem (also known as the Thurston equivalence) between branched coverings of the sphere. As it is shown in the Bartholdi-Nekrashevych solution of the Hubbard twisting rabbit problem [BN06], the main difficulty is to understand how homeomorphisms interact with branched coverings. Computational theory developed in [BD21b] allows to effectively reduce the original conjugacy problem to the conjugacy and centralizer problems between small maps of some canonical decomposition. In particular, the conjugacy problem for PCF branched coverings of the sphere is decidable [BD17]. However, the current methods of finding canonical decompositions are not effective and do not have any complexity estimates.

1.1. Invariants in complex dynamics. There are many invariants characterizing different classes of PCF rational maps. Perhaps the most well-known is the *Hubbard tree* of a polynomial. It is a finite planar tree in the core of the filled-in Julia set [DH84]. Two polynomials are conjugate if and only if their Hubbard trees are planar conjugate. This allows to provide a *combinatorial classification* of all PCF polynomial maps [BFH92, Poi10]. A complex polynomial can also be described using invariant (or periodic in the subhyperbolic case) spiders [HS94]. This approach leads to the Poirier notion of supporting rays [Poi10]. Quadratic polynomials can also be described using kneading sequences and internals addresses. All these invariants can be converted into each other.

Given two polynomials, we can topologically glue them together along the circle at infinity and obtain a branched covering of the sphere, called the *formal mating* of the polynomials [Dou83, PM12]. If the formal mating is equivalent to a rational map, then the resulting rational map is conjugate to the *geometric mating* – the
gluing of the polynomial filled-in Julia sets along their boundaries parameterized by external angles. The mating operation attracts a lot of attention \[\text{[BEK}^*12\text{]}\] and is well-understood in the quadratic case due to M. Rees, Tan Lei, and M. Shishikura \[\text{[Ree92, Lei92, Shi00]}\]. In general, the mating does not respect the structures of polynomial Julia sets, and rational maps often can be unmated in many ways \[\text{[Mey11]}\]. However, if one of the polynomials is sufficiently simple (for example, it is the Basilica or a generalized Rabbit), then its Hubbard tree gives a useful invariant of the mating with strong parameter implications.

Newton maps arise in the root-finding problem and form the biggest well-understood non-polynomial class of rational maps. A Newton map has a unique repelling fixed point at infinity; every other fixed point is attracting. The immediate attracting basins of fixed points meet at infinity; the associated internal rays form the channel diagram that is the beginning of the puzzle theory for Newton maps \[\text{[DMRS19, DS22]}\]. The preimages of the channel diagram together with embedded Hubbard trees (for the renormalizable parts of Newton dynamics) provide the basis of the combinatorial classification of PCF Newton maps \[\text{[LMS22]}\].

The Dehn-Nielsen-Baer Theorem states that every homeomorphism \(f: (S^2, A) \rightarrow \) is uniquely characterized up to isotopy by the induced pushforward \(f_*: \pi_1(S^2, A) \rightarrow \) viewed as an outer automorphism of \(\pi_1(S^2, A)\). The theorem was extended to non-invertible branched coverings by Kameyama \[\text{[Kam01]}\] and independently by Nekrashevych \[\text{[Nek05]}\]. The group theoretical data arising from \(f: (S^2, A) \rightarrow \) is conveniently described by the \(\pi_1(S^2, A)\)-biset of \(f\). There is a bijection between branched coverings \(f: (S^2, A) \rightarrow \) considered up to isotopy rel \(A\) and sphere \(\pi_1(S^2, A)\)-bisets considered up to biset-isomorphism, see \[\text{[BD21] Theorem 2.8}\]. There are algorithms to compute the Hubbard trees and spiders out of bisets of polynomials; their implementations are available in the computer algebra system GAP \[\text{[Bar22]}\].

1.2. Expanding maps and quotients. Expansion is one of the key properties of PCF rational maps. It follows from the Schwarz lemma that \(f: (\hat{C}, A) \rightarrow \) expands the hyperbolic metric on \(\hat{C}\). (If \(|A| = 2\), then \(f(z) = z^2\) expands the Euclidean metric of the cylinder \(C\).) If some of the points from \(A\) are in the Julia set of \(f\), then it is more convenient to consider the minimal hyperbolic (or Euclidean) orbifold \((\hat{C}, orb_f)\) of \(f\). Periodic attracting cycles of \(f\) are the only removed points in \((\hat{C}, orb_f)\); everywhere else \(f\) is expanding.

A map \(f: (S^2, A) \rightarrow \) is called Böttcher expanding if it admits an expanding metric mimicking the above expansion of PCF rational maps: \(f\) is expanding everywhere except at removed critical cycles where \(f\) is attracting and where \(f\) has a Böttcher normalization (i.e., it is locally conjugate to \(z \mapsto z^2\)). By \[\text{[BD18]}\], a non-Lattès map \(f: (S^2, A) \rightarrow \) is isotopic to a Böttcher expanding map if and only if \(f\) does not possesses a Levy obstruction. Other expanding sphere maps can be obtained by collapsing Fatou attracting basins of \(f\), see \[\text{[BD18 Proposition 1.1]}\].

For an expanding map \(f: (S^2, A) \rightarrow \), there is a natural notion of the Julia \(\mathcal{J}(f)\) and Fatou \(\mathcal{F}(f)\) sets. The case \(\mathcal{J}(f) = S^2\) was studied in relation to the Cannon conjecture and quasi-symmetric geometry \[\text{[HP09, HP08, BM17]}\]. If \(\mathcal{J}(f) = S^2\), then we say that \(f: (S^2, A) \rightarrow \) is a totally expanding map. More generally, we may consider totally topologically expanding maps \(g: X \rightarrow\) on a compact metrizable space \(X\), see Section \[\text{2.3}\] for the definition.
Suppose that \( f : (S^2, A) \simeq \) is a Böttcher expanding map with non-empty Fatou set. Let us denote by \( \sim_{f} \) the smallest closed equivalence relation on \( S^2 \) generated by identifying all points in every Fatou component of \( f \). The quotient space \( S^2/\sim_{f} \) is a cactoid, that is, a continuum composed of countably many spheres and segments pairwise intersecting in at most one point. The map \( f \) naturally descends to a continuous map \( \overline{f} : S^2/\sim_{f} \rightarrow \) on the cactoid. Then the corresponding (monotone) quotient map \( \pi_{f} : S^2 \rightarrow S^2/\sim_{f} \) provides a semi-conjugacy from \( f \) to \( \overline{f} \). It is naturally characterized by the following result.

**Theorem A.** Let \( f : (S^2, A) \simeq \) be a Böttcher expanding map with \( F(f) \neq \emptyset \). Then the induced map \( \overline{f} : S^2/\sim_{f} \rightarrow \) is the maximal totally expanding quotient. That is, any other semi-conjugacy \( \pi_{g} : S^2 \rightarrow Y \) from \( f \) to a totally topologically expanding map \( g : Y \simeq \) factorizes through \( \pi_{f} : S^2 \rightarrow S^2/\sim_{f} \):

\[
\begin{array}{ccc}
S^2 & \xrightarrow{\pi_{f}} & S^2/\sim_{f} \\
\downarrow \pi_{g} & & \downarrow \\
Y & \xrightarrow{g} & Y
\end{array}
\]

1.3. Crochet decomposition. We say that a Böttcher expanding map \( f : (S^2, A) \simeq \) is a **crochet map** if there is a connected forward-invariant zero-entropy graph containing \( A \). Polynomials, Newton maps, matings where one of the polynomials has a zero-entropy Hubbard tree are examples of crochet maps. The following result provides various characterizations of crochet maps.

**Theorem B.** Let \( f : (S^2, A) \simeq \) be a Böttcher expanding map with a non-empty Fatou set. Then the following are equivalent:

(i) \( f \) is a crochet map, that is, there is a connected forward-invariant zero-entropy graph \( G \) containing \( A \);

(ii) \( S^2/\sim_{f} \) is a singleton;

(iii) every two points in \( A \) may be connected by a path \( \alpha \) with \( \alpha \cap J(f) \) being countable.

We say that a Böttcher expanding map \( f : (S^2, A) \simeq \) is a **Sierpiński map** if its Julia set is homeomorphic to the standard Sierpiński carpet, that is, the Fatou set \( F(f) \) is non-empty, Fatou components have pairwise disjoint closures, and the closure of every Fatou component is a Jordan domain. The following easily follows from Whyburn’s characterization [Why58], Moore’s theorem [Moo25, Why42], and [BD18, Section 4.5].

**Proposition 1.1.** Let \( f : (S^2, A) \simeq \) be a a Böttcher expanding map with \( F(f) \neq \emptyset \). Then the following are equivalent:

(i) \( f \) is a Sierpiński map, i.e., \( J(f) \) is homeomorphic to the standard Sierpiński carpet;

(ii) \( S^2/\sim_{f} \) is a sphere and \( \sim_{f} \) is trivial on \( A \);

(iii) every connected periodic zero-entropy graph is homotopically trivial rel. \( A \).

Let us write \( f : (S^2, A, \mathcal{C}) \simeq \) for a branched covering \( f : (S^2, A) \simeq \) with an invariant multicurve \( \mathcal{C} = f^{-1}(\mathcal{C}) \). Then \( \mathcal{C} \) splits \( (S^2, A) \) into finitely many spheres marked by \( A \) and \( \mathcal{C} \). Every small sphere of \( (S^2, A, \mathcal{C}) \) is either periodic or preperiodic under \( f \); the first return map along a periodic cycle determines the type of maps in
the cycle. The inverse operation is an amalgam producing a global map out of small maps and the gluing data [Pil03]. We remark that decompositions and amalgams are topological operations and the resulting objects are unique up to isotopy. If \( f: (S^2, A, C) \) is expanding, then every small sphere of \( f: (S^2, A, C) \) has the associated small Julia set in \( J(f) \). Small Julia sets may intersect and may even coincide with the global Julia set \( J(f) \) (for example for matings).

The next theorem is the main result of this paper providing the crochet canonical decomposition of expanding maps into crochet and Sierpiński maps.

**Theorem C.** Let \( f: (S^2, A) \) be a Böttcher expanding map with \( F(p) \neq \emptyset \). There is a unique canonical invariant multicurve \( C_{\text{dec}} \) whose small maps are Sierpiński and crochet maps such that for the quotient map \( \pi_f: S^2/\sim F(p) \to S^2 \) the following are true:

(i) small Julia sets of Sierpiński maps project onto spheres;
(ii) small Julia sets of crochet maps project to points;
(iii) different crochet Julia sets project to different points in \( S^2/\sim F(p) \).

1.4. **Bicycles and Sierpiński maps.** Let \( C \) be a multicurve consisting of periodic (up to homotopy) curves such that \( C \) is maximally strongly connected: for all \( \gamma, \delta \in C \) an iterated preimage of \( \gamma \) is homotopic to \( \delta \) and \( C \) can not be enlarge while keeping this property. We call \( C \) a bicycle if its curves replicate: there is an \( n \geq 1 \) such that every \( \gamma \in C \) is homotopic to at least two components of \( f^{-n}(\gamma) \).

The crochet decomposition can now be characterized as follows:

**Theorem D.** Let \( f: (S^2, A) \) be a Böttcher expanding map with \( F(f) \neq \emptyset \). Then maximal Sierpiński small maps are well-defined. If two bicycles have a positive geometric intersection number, then these bicycles are within a small Sierpiński map.

The crochet multicurve \( C_{\text{dec}} \) is generated by the boundaries of maximal Sierpiński small maps and the remaining bicycles.

Let us say that a Böttcher expanding map \( f: (S^2, A) \) with \( F(f) \neq \emptyset \) is Sierpiński-free if it does not contain any small Sierpiński maps with respect to any invariant multicurve. In this case \( C_{\text{dec}} \) is generated by all bicycles of \( f \).

A topological space \( X \) is a dendrite if it is a locally connected continuum that contains no simple closed curves.

**Theorem E.** Let \( f \) be a Böttcher expanding map with \( F(f) \neq \emptyset \). Then the following are equivalent:

(i) none of the small maps in the decomposition of \( f \) along the crochet multicurve \( C_{\text{dec}} \) is a Sierpiński map;
(ii) \( S^2/\sim F(f) \) is a dendrite;
(iii) the decomposition of \( f \) with respect to every invariant multicurve \( C \) does not produce a Sierpiński small map.

For a Böttcher expanding map \( f: (S^2, A) \) with \( F(f) \neq \emptyset \), let \( C_{\text{Sie}} \) denote the multicurve generated by the boundaries of maximal Sierpiński small maps. Then \( C_{\text{Sie}} \subset C_{\text{dec}} \) and both of these multicurves encode topological features of \( J(f) \):

- maximal Sierpiński small maps correspond to small spheres of \( S^2/\sim F(f) \);
- small Sierpiński-free maps of \( f: (S^2, A, C_{\text{Sie}}) \) correspond to dendrites of \( S^2/\sim F(f) \) that share at most one point with any small sphere;
Figure 1. Julia sets of PCF rational maps.

- bicycles in $C_{\text{dec}} \setminus C_{\text{Sie}}$ correspond to arcs in $S^2/\sim_{\mathcal{F}(f)}$ that share at most one point with any small sphere;
- small crochet maps correspond to points in $S^2/\sim_{\mathcal{F}(f)}$.

Let us remark that the above properties are already visible in $J(f)$, see Figure 1. Here, (a),(b) are Julia sets of crochet maps: any Fatou component may be connected to another one by a (countable) chain of touching Fatou components, that is, $S^2/\sim_{\mathcal{F}(f)}$ is a singleton. The Julia set in (c) is a Sierpiński carpet: Fatou components are Jordan domains with disjoint closures. The Julia set in (d) corresponds to a tuning, and its canonical decomposition returns (a) and (c). Note that the quotient $S^2/\sim_{\mathcal{F}(f)}$ is a sphere for (c) and (d), but $\sim_{\mathcal{F}(f)}$ is not trivial on the postcritical set for (d). For the Julia set in (e), the quotient $S^2/\sim_{\mathcal{F}(f)}$ is a segment with the quotient dynamics $\tilde{f}: S^2/\sim_{\mathcal{F}(f)} \odot$ of a Chebychev polynomial. The Fatou component of infinity corresponds to a small crochet map. We also see a Cantor set of Jordan curves in $J(f)$ separating infinity and the Fatou component in the center.

1.5. Crochet Algorithm. The proofs of Theorem C and D give an algorithm to effectively compute the crochet decomposition:

1. Compute maximal clusters of touching Fatou components and their boundary multicurve.
2. Decompose the map with respect to the boundary multicurve of the clusters.
3. Iterate Steps 1 and 2 for each small map until all small maps are crochet and Sierpiński.
4. Glue small crochet maps that correspond to the same point in $S^2/\sim_{\mathcal{F}(f)}$.

All steps in the Crochet Algorithm can be performed symbolically with the input being a sphere biset of $f : (S^2, A, C) \subset$.

1.6. Applications of the crochet decomposition. The crochet decomposition appears to be a useful tool in complex dynamics. Below we briefly discuss several natural applications in quite different contexts.

Let $f$ be a Böttcher expanding map with non-empty Fatou set, and suppose that $\mathcal{C}_{\text{Th}}$ is the Pilgrim canonical obstruction and $\mathcal{C}_{\text{dec}}$ is the canonical crochet multicurve of $f$. Using [PL98, Theorem 3.2], see also [Par20, Theorem 7.6], it follows that up to isotopy each curve $\gamma \in \mathcal{C}_{\text{Th}}$ either belongs to $\mathcal{C}_{\text{dec}}$ or to a Sierpiński small sphere (wrt. the decomposition along $\mathcal{C}_{\text{dec}}$). Since $\mathcal{C}_{\text{dec}}$ can be efficiently computed, [BD18] implies that the problem of detecting obstructions for Thurston maps is reduced to efficient localization of Levy multicurves for arbitrary Thurston maps and Thurston obstructions for Böttcher expanding maps $f$ with $\mathcal{J}(f)$ being the whole sphere $S^2$ or a Sierpiński carpet.

In [Par21], I. Park studies the Ahlfors regular conformal dimension (for short, ARConfDim) of the Julia sets of crochet maps. In particular, he proves that $\text{ARConfDim}(\mathcal{J}(f)) = 1$ for a PCF hyperbolic rational map $f$ if and only if $f$ is a crochet map. Our work (in particular, Theorems C and D) provides some ingredients for the proof of one of the directions. I. Park also conjectures that for a Böttcher expanding map $f$ with non-empty Fatou set the crochet decomposition provides a lower bound on $\text{ARConfDim}(\mathcal{J}(f))$ as the maximum of the ARConfDim's of the small Julia sets with respect to $\mathcal{C}_{\text{dec}}$ and an extremal quantity $Q(\mathcal{C}_{\text{dec}})$ associated with the invariant multicurve $\mathcal{C}_{\text{dec}}$ (see [Thu20, Section 7.5] for the definition). We refer the reader to [CYMT99, CM22] for similar results in the context of geometric group theory.

It is conjectured that the iterated monodromy groups (for short, IMGs) of PCF rational maps are amenable. Theorem [B] and [NPT, Theorem 7.1] provide the necessary ingredients for application of the amenability criterion from [JNdlS16] resulting in the following partial result: the IMGs of PCF crochet rational maps are amenable; c.f. [NPT, Corollary 7.2]. This generalizes the previous amenability results from [BKN10, Hlu17].

1.7. Organization of this paper. The paper is organized as follows. In the next section we review background. In particular, we discuss decomposition theory for Thurston maps in Section 2.7. In Section 3 we introduce the geometric amalgam operation for Böttcher expanding maps, which generalizes the notion of geometric mating. In Section 4 we introduce clusters induced by touching Fatou components and use them to build 0-entropy invariant graphs. In Section 5 we introduce the notion of a crochet multicurve, which gives the crochet decomposition, and provide the Crochet Algorithm that constructs this curve (with a proof). In Section 6 we describe how a Böttcher expanding maps with an invariant multicurve (satisfying certain natural conditions) generates a cactoid with an expanding dynamics on it. In Section 7 we prove canonicity of crochet decomposition and provide alternative characterizations.

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2. Background

2.1. Notations and conventions. The sets of positive integers, integers, and complex numbers are denoted by \( \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{C} \), respectively. We use the notation \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) for the open unit disk in \( \mathbb{C} \), \( \hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \) for the Riemann sphere, and \( S^2 \) for a topological 2-sphere.

Let \( X \) be a topological space. A curve (or a path) in \( X \) is the image of a continuous function from a closed interval of the real numbers into \( X \). By default, every curve \( \alpha \) is parametrized by \([0,1]\). As common, we will use the same notation for the curve and its parametrizing function, so that \( \alpha(0) \) and \( \alpha(1) \) denote the starting and ending point of \( \alpha \), respectively. If \( \alpha \) and \( \beta \) are two paths in \( X \) such that \( \alpha(1) = \beta(0) \), then \( \alpha \# \beta \) denotes their concatenation; i.e., the path \( \alpha \# \beta \) first runs through \( \alpha \) and then through \( \beta \).

Let \( A \subset S^2 \) be a finite set. We refer to the pair \( (S^2, A) \) as a marked sphere, and to the points in \( A \) as marked points.

A sphere map \( f: (S^2, C) \to (S^2, A) \) between marked spheres is an orientation-preserving branched covering map between the underlying spheres such that \( A \) contains \( f(C) \) as well as all critical values. If \( \alpha \) is a path in \( S^2 \setminus A \) starting at \( x \), then \( \alpha \upharpoonright_y \) denotes the unique lift of \( \alpha \) starting at \( y \in f^{-1}(x) \).

A Thurston map is a self-map \( f: (S^2, A) \to (S^2, A) \) with topological degree \( \deg(f) \geq 2 \). Note that since \( A \) contains all the critical values of \( f \) and \( f(A) \subset A \), it follows that each critical point of \( f \) has finite orbit, that is, \( f \) is postcritically finite.

A forgetful map \( i: (S^2, C) \to (S^2, A) \) is the identity map that forgets points in \( C \setminus A \); in particular \( i(C) \supset A \). Note that a non-trivial forgetful map is not a sphere map. More generally, a forgetful monotone map \( i: (S^2, C) \to (S^2, A) \) with \( C \supset A \) is a continuous monotone map that is homotopic to \( \text{id} \) rel. \( A \).

Curves in \((S^2, A)\) will be frequently considered up to homotopy rel. \( A \) and the endpoints: for two curves \( \alpha_0 \) and \( \alpha_1 \) in \( S^2 \) we write \( \alpha_0 \sim_A \alpha_1 \) if there is a path of curves \( \alpha_t: [0,1] \to S^2 \) such that \( \alpha_t^{-1}(A), \alpha_t(0), \alpha_t(1) \) are constant (i.e., do not depend on \( t \)).

By a nice curve in \((S^2, A)\) we mean a curve \( \beta: [0,1] \to S^2 \) with \( \beta(t) \notin A \) for each \( t \in (0,1) \). If a nice curve \( \beta \) starts at \( x \in S^2 \setminus A \) and \( f: (S^2, A) \to (S^2, A) \) is a Thurston map, then the lift \( \beta \upharpoonright_{f^n} \) is well-defined for each \( n \) and \( y \in f^{-n}(x) \).

2.2. Böttcher expanding maps. Let \( f: (S^2, A) \to (S^2, A) \) be a Thurston map. We denote by \( A^\infty \subset A \) the forward orbit of periodic critical points of \( f \), and by \( \text{per}(a) \) the period of each \( a \in A^\infty \).

The map \( f \) is metrically expanding if there exists a forward-invariant subset \( A' \subset A^\infty \) and a length metric on \( S^2 \setminus A' \) such that

1. for every non-trivial rectifiable curve \( \gamma: [0,1] \to S^2 \setminus A' \) the length of every lift of \( \gamma \) under \( f \) is strictly less than the length of \( \gamma \); and
(ii) \( f \) admits Böttcher normalization at all \( a \in A' \): the first return map of \( f \) at \( a \)

is locally conjugate to \( z \mapsto z^{\deg(f^p(a))} \).

In this case, we say that \( f \) is metrically expanding rel. \( A' \). We note that points in

\( A' \) are cusps or, equivalently, at infinite distance in the metric. If \( A' = A^2 \), then

\( f: (S^2, A) \to A' \) is called a Böttcher expanding map.

Metrically expanding maps \( f \) admit natural definitions of Fatou and Julia sets. We summarize the relevant notions and properties below; the reader is referred to

[BDB18] for a more detailed discussion.

In the following, let \( f \) be a metrically expanding map rel. to a forward-invariant subset \( A' \subset A^2 \). The Julia set \( J(f) \) of \( f \) is the closure of the set of repelling periodic points of \( f \); the complement \( F(f) := S^2 \setminus J(f) \) is the Fatou set of \( f \). Equivalently, \( F(f) \) is the set of points attracted by \( A' \). A Fatou component of \( f \) is a connected component of \( F(f) \). Every Fatou component \( F \) is an open topological disk. Moreover, \( F \) is eventually periodic, i.e., \( f^n(F) = f^m(F) \) for some \( n > m \geq 0 \).

Condition (ii) implies that periodic (and hence preperiodic) Fatou components enjoy Böttcher coordinates: for each Fatou component \( F \) there exists a homeomorphism \( \psi_F: \mathbb{D} \to F \) such that \( (\psi_F^{-1} \circ f \circ \psi_F)(z) = z^{\deg(f\mid_F)} \) for every \( z \in \mathbb{D} \). Every map \( \psi_F \) extends to a continuous map \( \psi_F: \overline{\mathbb{D}} \to F \). The image \( \{\psi_F(r e^{2\pi i \theta}) : r \in [0,1) \} \) of a radius in \( \overline{\mathbb{D}} \) is called the internal ray of angle \( \theta \in [0,1) \) in \( F \). The image \( \{\psi_F(r e^{2\pi i \theta}) : \theta \in [0,1) \} \) of a circle in \( \mathbb{D} \) concentric about 0 is called the equipotential of height \( r \in (0,1) \) in \( F \). Finally, the image \( \psi_F(0) \) is called the center of \( F \).

Fix a small \( r \in (0,1) \). For every \( a \in A' \), let \( U_a \) be the open disk around \( a \)

bounded by the equipotential of height \( r \) in the Fatou component centered at \( a \). Set \( W := \bigcup_{a \in A'} U_a \). Note that \( f(W) \subset W \).

Consider a nice curve \( \alpha \) in \( (S^2, A) \) parameterized by \([0,1]\). Assume that \( \alpha \) is not

in \( W \), and set \( t_0^\alpha := \inf\{t \in [0,1] : \alpha(t) \notin W \} \) and \( t_1^\alpha := \sup\{t \in [0,1] : \alpha(t) \notin W \} \).

We define the truncated length of \( \alpha \) as

\[ |\alpha|_{\approx} := \inf\{\text{length of } \alpha' : [t_0^\alpha, t_1^\alpha] \text{ of } \alpha' \approx_A \alpha\} \]

Note that if \( \alpha \) starts and ends outside \( W \) then \( |\alpha|_{\approx} \) coincides with the minimal length within the homotopy class of \( \alpha \).

The following lemmas are immediate.

**Lemma 2.1.** There are constants \( \lambda > 1 \) and \( C > 0 \) such that the following holds.

Let \( \alpha \) be a nice curve connecting a point in \( S^2 \setminus W \) to a point in \( A' \cup S^2 \setminus W \). Then

the truncated length of every lift of \( \alpha \) under \( f^n \) is at most \( \frac{1}{\lambda^n} |\alpha|_{\approx} + C \). \( \square \)

**Lemma 2.2.** For every \( M > 0 \) and every \( x, y \in A' \cup S^2 \setminus W \), there are at most

finitely many homotopy classes of curves connecting \( x \) and \( y \) with truncated length at most \( M \).

**Proof.** For every \( a \in A' \) and \( x \in \partial U_a \), there is a unique up to homotopy path in \( \overline{U}_a \)

connecting \( a \) and \( x \). This reduces the statement to the case \( x, y \in S^2 \setminus W \). \( \square \)

2.3. Topologically expanding maps. Let \( \mathcal{M}', \mathcal{M} \) be compact metrizable topological spaces with \( \mathcal{M} \subset \mathcal{M} \), and \( f: \mathcal{M}' \to \mathcal{M} \) be a continuous map (e.g., a branched covering of finite degree). Suppose \( \mathcal{U} \) is a collection of open sets in \( \mathcal{M} \).

We denote by \( \text{mesh}(\mathcal{U}) \) the supremum of all diameters of connected components of
sets in $\mathcal{U}$ (with respect to some fixed metric on $\mathcal{M}$ generating the topology of $\mathcal{M}$).

The pull-back of $\mathcal{U}$ by $f^n$ is defined as

$$f^{*n}(\mathcal{U}) := \{ V : V \text{ is a component of } f^{-n}(U), \text{ for some } U \in \mathcal{U} \}.$$ 

The partial self-cover $f : \mathcal{M}' \to \mathcal{M}$ is called topologically expanding if there exists a finite open cover $\mathcal{U}$ of $\mathcal{M}$ such that $\text{mesh}(f^{*n}(\mathcal{U})) \to 0$ as $n \to \infty$. The Julia set of $f$ is the set of points in $\mathcal{M}'$ that do not escape $\mathcal{M}'$ under iteration of $f$. If $\mathcal{M}' = \mathcal{M}$, then we say that $f : \mathcal{M} \to \mathcal{M}$ is totally topologically expanding.

Consider now a Thurston map $f : (S^2, A) \subset S^2$ and let $A'$ be a forward-invariant subset of $A^\infty$. We call $f$ topologically expanding (rel. $A'$) if there exist compact $\mathcal{M}' \subset \mathcal{M} \subset S^2$ with a topologically expanding branched covering map $f : \mathcal{M}' \to \mathcal{M}$, such that every connected component $U$ of $S^2 \setminus \mathcal{M}$ is a disk containing a unique point $\alpha \in \mathcal{A}'$, all points in $U$ are attracted to the orbit of $\alpha$, and the first return map of $f$ at $\alpha$ is locally conjugate to $z \mapsto z^{\deg_a(f^p(\alpha))}$. Every topologically expanding Thurston map $f : (S^2, A) \subset S^2$ is obtained from a Böttcher expanding map $\hat{f} : (S^2, A) \subset S^2$ that is isotopic to $f$ by collapsing grand orbits of those Fatou components that are attracted towards $A \cap (\mathcal{F}(\hat{f}) \setminus \mathcal{F}(\hat{f}))$. See [BD18] Theorem A and Proposition 1.1.

### 2.4. Monotone maps between spheres

Recall that a map $\tau : X \to Y$ between topological spaces is called monotone if $\tau^{-1}(y)$ is connected for every $y \in Y$. Every monotone self-map $\iota : S^2 \subset S^2$ arises as a uniform limit of homeomorphisms [Yon18]. In particular, the Dehn-Nielsen-Baer Theorem is applicable for monotone maps: two monotone maps $\iota_1, \iota_2 : (S^2, A) \subset S^2$ are homotopic if and only if their pushforwards $\iota_{1,*,}, \iota_{2,*,} : \pi_1(S^2, A) \subset S^2$ induce the same elements of the outer automorphism group of $\pi_1(S^2, A)$. Below we will review the pullback argument in the setting of topologically expanding maps.

#### 2.4.1. Uniform convergence

Suppose $\mathcal{U}_n = \{ U_{n,k} \}$ are finite covers of $S^2$ by open connected sets such that $\lim_{n \to \infty} \text{mesh}(\mathcal{U}_n) = 0$, that is, the maximal diameter of $U_{n,k}$ tends to 0 as $n \to \infty$. Then a sequence of homeomorphisms $i_m : S^2 \to S^2$ converges uniformly if

$$\forall n' > 1 \quad \exists N, M \geq 1 \quad \forall n \geq N \quad \forall U_{n,k} \in \mathcal{U}_n \quad \forall m \geq M$$

the image $i_m(U_{n,k})$ is within a component of $\mathcal{U}_{n'}$. In this case, the limiting self-map $\lim_m i_m : S^2 \to S^2$ exists and is a monotone map.

#### 2.4.2. Pullback argument

Let $f : (S^2, A) \subset S^2$ be a Thurston map with Böttcher normalization at a forward-invariant set $A' \subset A^\infty$, i.e., the first return map at every $a \in A'$ is locally conjugate to $z \mapsto z^{\deg_a(f^p(\alpha))}$. If $g : (S^2, A) \subset S^2$ is a topologically expanding map rel. $A'$ such that $f$ and $g$ are isotopic rel. $A$, then there is a continuous monotone map $\iota : (S^2, A) \to (S^2, A)$ semi-conjugating $f$ to $g$. Moreover, $\iota$ is unique if $|A| \geq 3$ (as follows from [BD21a Corollary 4.30]). The map $\iota$ is constructed as follows (c.f. [IS10, Section 11.1], and [BD18 Section 4.5]).

Let $i_0, i_1 : (S^2, A) \to (S^2, A)$ be homeomorphisms isotopic to id rel. $A$ that witness an equivalence of $f$ and $g$, i.e., $i_0 \circ f = g \circ i_1$ and there is an isotopy $h_0 : (S^2, A) \times [0,1] \to (S^2, A)$, with $h_{0,0} = i_0$ and $h_{0,1} = i_1$.

(Here and in the following, we adopt the convention that if $h_n$ is a homotopy then $h_{n,t}$ denotes the time-$t$ map $h_n(\cdot, t)$.) Up to modification of $i_0$, we may assume that
$i_0, i_1, h_0$ respect Böttcher coordinates (of $f$ and $g$) at $A'$, so that $i_0 = i_1 = h_{0,t}$ for each $t \in [0, 1]$ in a small neighborhood of each $a \in A'$.

By lifting, we may inductively define homeomorphisms $i_{n+1} : (S^2, A) \to (S^2, A)$ and isotopies $h_n$ between $i_n = h_{n,0}$ and $i_{n+1} = h_{n,1}$. That is, the isotopy $h_{n+1}$ is the lift of $h_n$ (so that $h_{n+1,0} = h_{n,1} = i_{n+1}$), and we have the following infinite commutative diagram:

$$\begin{array}{ccc}
\cdots & \xrightarrow{f} & (S^2, A) \\
\downarrow{h_{2,t}} & & \downarrow{h_{1,t}} \\
\cdots & \xrightarrow{g} & (S^2, A) \\
\downarrow{h_{0,t}} & & \downarrow{h_{0,t}} \\
(\cdots) & & (\cdots)
\end{array}$$

A track of $h_n$ is the curve $\gamma$ defined by

$$\gamma(t) : t \mapsto h_{n,t}(x), \quad t \in [0, 1],$$

for some fixed $x \in S^2$. Clearly, tracks of $h_{n+1}$ are lifts of tracks of $h_n$ under $g$. Tracks for concatenations $h_n \# h_{n+1} \# \ldots \# h_{n+k}$ of isotopies are defined in a similar way.

We assume that $g$ is topologically expanding with respect to a finite open cover $U_0$ of $X_0$, where $X_0$ is the sphere $S^2$ without a small forward-invariant neighborhood of $A'$ (where $i_0, i_1, h_0$ respect Böttcher coordinates). That is, we have

$$\lim_{n \to \infty} \text{mesh}(U_n) = 0,$$

where $U_n := g^n(U_0)$ is the pull-back of $U_0$ by $g^n$. Note that $U_n$ is an open cover of $X_n = g^{-n}(X_0)$.

For a curve $\gamma : [0, 1] \to X_n$, we denote by $|\gamma|_n$ the $n$-combinatorial length of $\gamma$ with respect to $U_n = \{U_{n,k}\}$:

$$|\gamma|_n := \min\{L : \gamma = \gamma_1 \# \gamma_2 \# \ldots \# \gamma_L \text{ so that } \gamma_j \subset U_{n,k(j)}\}.$$ 

Since $\lim_{n \to \infty} \text{mesh}(U_n) = 0$, there are $\lambda > 1, m \geq 1, C > 0$ such that $|\gamma|_0 \leq \frac{1}{\lambda} |\gamma|_m + C$ for all $\gamma \subset X_m$. Lifting under $g^n$ we obtain

$$|\gamma|_n \leq \frac{1}{\lambda} |\gamma|_{n+m} + C$$

for all $n \geq 1$ and $\gamma \subset X_{n+m}$.

By uniform continuity, there is a constant $T' > 0$ that bounds from above all the tracks of $h_0, \ldots, h_{m-1}$ with respect to $|\cdot|_0$. By lifting and applying a geometric series argument to $[1]$, we obtain a uniform $T \geq 1$ such that the 0-combinatorial length of all the tracks of $h_0 \# h_1 \# \ldots \# h_k$ is bounded by $T$ for all $k \geq 0$. Lifting under $g^n$, we obtain that $T$ also bounds the $s$-combinatorial length of all the tracks of $h_s \# h_{s+1} \# \ldots \# h_{s+k}$ for all $s, k$. Since $\lim_{n \to \infty} \text{mesh}(U_n) = 0$, the homotopy $h_s \# h_{s+1} \# \ldots \# h_{s+k}$ converges uniformly to the identity and $i_n$ converges uniformly to a monotone map $\iota : S^2 \to S^2$.

2.5. Multicurves on marked spheres. Let $(S^2, A)$ be a marked sphere. We consider simple closed curves $\gamma$ on $(S^2, A)$, that is, $\gamma \subset S^2 \setminus A$. Such a curve $\gamma$ is called essential if both components of $S^2 \setminus \gamma$ contain at least two marked points. Otherwise, the curve $\gamma$ is called peripheral.

A multicurve on $(S^2, A)$ is a collection $C$ of essential simple closed curves on $(S^2, A)$ that are pairwise disjoint and non-isotopic. Here and elsewhere, in the context of simple closed curves on $(S^2, A)$ isotopies are always considered rel. $A$
(unless specified otherwise). It would be also convenient to consider (multi)curves defined up to isotopy.

If \( C' \) is a finite set of pairwise disjoint simple closed curves on \((S^2, A)\), then MultiCurve\((C')\) is the multicurve obtained from \( C' \) by identifying isotopic curves and removing non-essential curves.

Let \( K \) be a compact connected subset of \( S^2 \). Then every connected component \( U \) of \( S^2 \setminus K \) is an open topological disk, and so we may choose a homeomorphism \( \rho: U \to \mathbb{D} \). For \( r \in (0, 1) \), let \( E^r = \{ z : |z| = r \} \subset \mathbb{D} \) be the circle of radius \( r \) centered at the origin. Note that \( \rho^{-1}(E^r) \) is a simple closed curve in \( S^2 \) whose isotopy class rel. \( A \) is constant for \( r \) close to 1. Moreover, this isotopy class does not depend on the choice of the homeomorphism \( \rho \). We define Curve\((K | U)\) to be such a unique up to isotopy curve \( \rho^{-1}(E^r) \). Furthermore, we set

\[
\text{MultiCurve}(K) := \text{MultiCurve} \left( \bigcup_{U \subset S^2 \setminus K} \text{Curve}(K | U) \right),
\]

where the union is taken over all connected components \( U \) of \( S^2 \setminus K \).

For a compact, but not necessarily connected, set \( K \subset S^2 \), we set

\[
\text{MultiCurve}(K) := \text{MultiCurve} \left( \bigcup_{K \subset K} \text{MultiCurve}(K') \right),
\]

where the union is taken over all connected components of \( K \). The multicurve MultiCurve\((K)\) is also defined for a countable union of pairwise disjoint compact sets.

We will write Curve\(_A(\cdot)\) and MultiCurve\(_A(\cdot)\) if we wish to underline that the (multi)curves are considered on \((S^2, A)\).

**Lemma 2.3.** Suppose \( f: (S^2, C) \to (S^2, A) \) is a sphere map and \( K \subset S^2 \) is compact. Then

\[
\text{MultiCurve}_C(f^{-1}(K)) = \text{MultiCurve}_C(f^{-1}(\text{MultiCurve}_A(K))).
\]

**Proof.** It is sufficient to prove the identity for connected \( K \). Let \( U \) be a connected component of \( S^2 \setminus K \). Then up to isotopy rel. \( C \) we have the following equality

\[
f^{-1}(\text{Curve}_A(K | U)) = \bigcup_{K' \subset U'} \text{Curve}_C(K' | U'),
\]

where the union is taken over all components \( K' \) of \( f^{-1}(K) \) and all the components \( U' \) of \( S^2 \setminus K' \) such that the \( f \)-image of \( U' \) intersected with a sufficiently small neighborhood of \( K' \) is within \( U \). The assertion of the lemma now follows from the definitions. \( \Box \)

### 2.5.1. Pseudo-multicurves.

By a pseudo-multicurve \( C \) on \((S^2, A)\) we mean a collection of pairwise disjoint and pairwise non-isotopic non-trivial simple closed curves on \( S^2 \setminus A \); i.e. peripheral curves are allowed. Naturally, pseudo-multicurves are considered up to isotopy. Pseudo-multicurves will only appear in the proof of Theorem 2.10.

Allowing peripheral curves, we define psCurve\((K | U)\) and psMultiCurve\((K)\) in the same way as Curve\((K | U)\) and MultiCurve\((K)\). Similar to Lemma 2.3 we have

\[
(2) \quad \text{psMultiCurve}_C(f^{-1}(K)) = \text{psMultiCurve}_C(f^{-1}(\text{psMultiCurve}_A(K)));
\]

where peripheral curves are allowed.
2.6. **Invariant multicurves.** In the following, let \( f: (S^2, A) \rightarrow (S^2, A) \) be a Thurston map. We say that a multicurve \( C \) on \((S^2, A)\) is invariant (under \( f \)) if the following holds:

(i) \( f^{-1}(C) \subset C \), which means that each essential component of \( f^{-1}(C) \) is isotopic to a curve in \( C \).

(ii) \( C \subset f^{-1}(C) \), which means that each curve in \( C \) is isotopic to a component of \( f^{-1}(C) \).

In other words, the multicurve \( C \) is invariant if \( \text{MultiCurve}_A(f^{-1}(C)) = C \). We will use the notation \( f: (S^2, A, C) \rightarrow (S^2, A) \) for a Thurston map \( f: (S^2, A) \rightarrow (S^2, A) \) with an invariant multicurve \( C \).

Suppose \( C' \) is any multicurve on \((S^2, A)\). If \( C' \subset f^{-1}(C') \), then there is a unique invariant multicurve \( C \) generated by \( C' \), which is given by the intersection of all invariant multicurves containing \( C' \).

Let \( C \) be an invariant multicurve. We consider the following directed graph \( \Gamma = \Gamma_C \): its vertex set is \( C \), and for every curve \( \gamma \in C \) and every essential component \( \hat{\gamma} \) of \( f^{-1}(\gamma) \) we add a directed edge in \( \Gamma \) from \( \gamma \) to the curve \( \delta \in C \) that is isotopic to \( \hat{\gamma} \).

Two vertices in \( \Gamma \) are said to be strongly connected to each other if there is a walk in \( \Gamma \) between them in each direction. Clearly, this defines an equivalence relation on \( C \). An equivalence class \( C' \subset C \) is then called a strongly connected component of \( C \), and the induced subgraph \( \Gamma[C'] \) is called a strongly connected component in \( \Gamma \).

Note that a singleton without a self-loop is never a strongly connected component in \( \Gamma \).

Strongly connected components are partially ordered: if \( C' \) and \( C'' \) are two strongly connected components of \( C \), we write \( C' < C'' \) if there is a (directed) walk in \( \Gamma \) from a curve in \( C' \) to a curve in \( C'' \). A strongly connected component \( P \subset C \) is called a primitive component of \( C \) if it is minimal for the partial order \( < \). It immediately follows from the definitions that the multicurve \( C \) is generated by its primitive components in the following sense: for each curve \( \gamma \in C \) there is a primitive component \( P \subset C \) and an iterate \( n \) such that \( \gamma \) is isotopic to a component of \( f^{-n}(P) \).

A strongly connected component \( C' \subset C \) is called a bicycle if for every \( \gamma, \delta \in C' \) there exists an \( n \in \mathbb{N} \) such that at least two (directed) walks of length \( n \) join \( \gamma \) and \( \delta \) in \( \Gamma \). Otherwise, we call \( C' \) a unicycle.

2.7. **Decompositions and amalgams of Thurston maps.** We briefly review the decomposition theory of Thurston maps developed by K. Pilgrim [Pil03]. The algebraic version of this theory was introduced in [BD21b].

Consider a Thurston map \( f: (S^2, A, C) \rightarrow (S^2, A) \). Up to homotopy, we may write \( f \) as a correspondence

\[
(S^2, A, C) \xleftarrow{\iota} (S^2, f^{-1}(A), f^{-1}(C)) \xrightarrow{\iota} (S^2, A, C).
\]

Here, the map \( f \) is the same as the original map \( f \), but it is considered now as a covering (i.e., we remove the marking sets from the domain and target). The map \( \iota \), identifying the domain and target marked spheres, is specified as follows. It first forgets all points in \( f^{-1}(A) \backslash \mathbb{A} \) and all curves in \( f^{-1}(C) \) that are not isotopic rel. \( A \) to curves in \( C \). Then it squeezes all annuli between the remaining curves in \( f^{-1}(C) \) that are isotopic, and maps them to the corresponding curve in \( C \). This uniquely defines \( \iota \) as a monotone map on \( S^2 \) up to homotopy rel. \( A \).
A small sphere $S_z$ of $(S^2, A, C)$ is a connected component of $S^2 \setminus C$. Viewing holes in $S_z$ as punctures, we obtain a sphere $\hat{S}_z$ marked by the respective subset $A_z$ of $A \cup C$. With a slight abuse of terminology, we will refer to this marked sphere as a small sphere of $(S^2, A, C)$ as well. Similarly, we introduce small spheres of $(S^2, f^{-1}(A), f^{-1}(C))$.

A small sphere $S_z$ of $(S^2, f^{-1}(A), f^{-1}(C))$ is called

(i) trivial, if $S_z$ is homotopic rel. $A$ to a point or peripheral curve in $(S^2, A)$;

(ii) annular, if $S_z$ is homotopic rel. $A$ to a curve in $C$;

(iii) essential, otherwise.

For every small sphere $S_z$ of $(S^2, f^{-1}(A), f^{-1}(C))$ there is a unique small sphere $S_{f(z)}$ of $(S^2, A, C)$ such that $f: S_z \setminus f^{-1}(A) \to S_{f(z)} \setminus A$ is a covering. Filling-in holes, we view the latter as a sphere map $f: (\hat{S}_z, A_z) \to (\hat{S}_{f(z)}, A_{f(z)})$.

Essential small spheres of $(S^2, f^{-1}(A), f^{-1}(C))$ may be canonically identified with small spheres of $(S^2, A, C)$. Namely, for every small sphere $S_z$ of $(S^2, A, C)$ there is a unique small sphere $S_{i \ast (z)}$ of $(S^2, f^{-1}(A), f^{-1}(C))$ such that $S_{i \ast (z)}$ is homotopic to $S_z$ rel. $A$ (the corresponding homotopy fills in the holes in $S_z$ associated with $f^{-1}(C) \setminus C$). This induces a forgetful map $(\hat{S}_{i \ast (z)}, A_{i \ast (z)}) \to (\hat{S}_z, A_z)$; its inverse $(\hat{S}_z, A_z) \to (\hat{S}_{i \ast (z)}, A_{i \ast (z)})$ is a sphere map.

The composition

$$ (4) \quad \hat{S}_z \to \hat{S}_{i \ast (z)} \to \hat{S}_{f(z)} \quad \text{where } f(z) = f(i \ast (z)) $$

is well-defined (see [BD21b, Lemma 4.9]) and is called a small (non-dynamical) sphere map of $f: (S^2, A, C) \hookrightarrow$. The small sphere map $(\hat{S}_z, A_z) \to (\hat{S}_{f(z)}, A_{f(z)})$ is unique up to homotopy rel. the marked points.

Note that (4) naturally induces a map on the small spheres of $(S^2, A, C)$, which we still denote by $f$ for simplicity. Clearly, every small sphere of $(S^2, A, C)$ is either periodic or strictly preperiodic. Moreover, there are only finitely many periodic cycles of small spheres. A small (self-)map of $(S^2, A, C)$ is the first return map $\hat{f} = f^k: (\hat{S}_z, A_z) \hookrightarrow$ along such a periodic cycle (with some choice of a base small sphere $\hat{S}_z$). Each such small map is either a homeomorphism or a Thurston map.

By the decomposition of $f: (S^2, A, C) \hookrightarrow$ (along the invariant multicurve $C$) we mean either

- the collection of small sphere maps (a non-dynamical decomposition); or
- the collection of small self-maps, one per every periodic cycle of small spheres. (a dynamical decomposition).

The converse procedure is called amalgam. It takes as input a collection of small sphere maps $\{f: \hat{S}_z \to \hat{S}_{f(z)}\}_{z}$, as well as an appropriate “gluing data”, and outputs a global map $f: (S^2, A, C) \hookrightarrow$; see [Pil03 §3].

3. Formal amalgam

3.1. Expanding quotients. Let $f: (S^2, A) \hookrightarrow$ be a Thurston map with Böttcher normalization at each point in $A^\infty$. We denote by $J_f$ the set of points in $S^2$ that are not attracted by $A^\infty$. We also fix a base metric on $S^2$ that induces the given topology on $S^2$.

Two points $x, y \in J_f$ are called homotopy equivalent if there is an $M > 0$ such that for every $n \geq 0$ the points $f^n(x)$ and $f^n(y)$ can be connected by a nice
curve \( \ell_n \) (see Section 2.1) with \( |\ell_n| \leq M \) such that \( \ell_n \tilde{f}_n \) ends at \( y \). Moreover, we say that \( x, y \) are strongly homotopy equivalent if the curves \( \ell_n \) additionally satisfy \( \ell_n \sim_A \ell_{n+1} \tilde{f}_n(x) \) for each \( n \geq 0 \).

**Proposition 3.1.** Let \( f : (S^2, A) \cong \) be a Böttcher normalized map and \( \tilde{f} : (S^2, A) \cong \) be a Böttcher expanding map isotopic to \( f \). Suppose that \( h : (S^2, A) \to (S^2, A) \) is a continuous monotone map with \( h \approx_A \text{id} \) that provides a semi-conjugacy between \( f \) and \( \tilde{f} \) (see Section 2.4.2). Then \( h^{-1}(\mathcal{J}_f) = \mathcal{J}_f \) and the following are equivalent for \( x, y \in \mathcal{J}_f \):

- \( h(x) = h(y) \);
- \( x, y \) are homotopy equivalent;
- \( x, y \) are strongly homotopy equivalent.

Moreover, the constant \( M \) in the definition of the strong homotopy equivalence can be taken to be uniform over all equivalence classes.

**Proof.** Since \( \mathcal{J}_f \) and \( \mathcal{J}_f \) are non-escaping sets, we have \( h^{-1}(\mathcal{J}_f) = \mathcal{J}_f \).

Clearly, strong homotopy equivalence implies homotopy equivalence. Suppose that \( x, y \) are homotopy equivalent rel. \( f \). Then \( h(x), h(y) \) are homotopy equivalent rel. \( \tilde{f} \) and we obtain \( h(x) = h(y) \).

Before proving converse, let us introduce some additional terminology. We will assume that \( f \) has hyperbolic orbifold. (The case when \( f \) has parabolic orbifold will follow in a similar way with some natural modifications.) Fix a universal orbifold covering map \( \rho : \mathbb{D} \to (S^2, A, \text{orb}_f) \). For \( X \subset S^2 \), we define \( \text{diam}_\rho(X) \in [0, \infty] \) to be the diameter (with respect to the hyperbolic metric on \( \mathbb{D} \)) of a connected component of \( \rho^{-1}(X) \).

**Lemma 3.2.** There is an \( M > 0 \) such that \( \text{diam}_\rho(h^{-1}(z)) \leq M \) for every \( z \in \mathcal{J}_f \).

**Proof.** Let us choose closed topological disks \( V_1, \ldots, V_s \) such that

\[
\mathcal{J}_f \subset \bigcup_{i=1}^s V_i \quad \text{and} \quad |V_i \cap A| \leq 1 \quad \text{for all} \ i.
\]

Choose a monotone \( \tau_i : (S^2, A) \to (S^2, A) \) such that \( \tau_i(V_i) = \{v_i\} \) and \( \tau_i \mid S^2 \setminus V_i \) is injective.

Since \( \tau_i \circ h : (S^2, A) \to (S^2, A) \) is monotone,

\[
M_i := \text{diam}_\rho(h^{-1} \circ \tau_i^{-1}(v_i)) < \infty.
\]

Then \( M := \max_{1 \leq i \leq s} M_i \) provides the desired bound. \( \Box \)

Suppose \( x, y \in Z := h^{-1}(z) \) for some \( z \in \mathcal{J}_f \). We will prove that \( x, y \) are strongly homotopy equivalent with the constant \( M \) from Lemma 3.2.

Consider

\[
f^n(x), f^n(y) \in Z_n := f^n(Z) = h^{-1} \circ \tilde{f}^n(z) \quad \text{for} \ n \geq 0.
\]

Fix a connected component \( \tilde{Z}_n \) of \( \rho^{-1}(Z_n) \). Choose a very small open neighborhood \( U_n \) of \( Z_n \), and denote by \( U \) the component of \( f^{-n}(U_n) \) containing \( Z_n \). Connect \( x, y \) by a curve \( \gamma \) in \( U \). Since \( U_n \) is a very small neighborhood of \( Z_n \), the curve \( \gamma_n := f^n(\gamma) \) has a lift \( \tilde{\gamma}_n \) connecting two lifts of \( f^n(x), f^n(y) \) in \( \tilde{Z}_n \). Let us homotope \( \tilde{\gamma}_n \) into a geodesic \( \tilde{\ell}_n \). Then \( \ell_n := \rho(\tilde{\ell}_n) \) connects \( f^n(x), f^n(y) \) and has length at most \( M \).

By construction, \( \ell_n \) has a lift \( \ell \) homotopic to \( \gamma \) rel \( A, \text{orb}_f \). \( \Box \)
3.2. Formal amalgams. In this subsection, we extend the notion of a formal mating to amalgams; compare with the notion of “trees of correspondences” from [BD18 §6.2]. This will allow us to relate the Julia set of an amalgam with the Julia sets of its small maps.

Let \( \tilde{S} \) be a finite (disjoint) union of topological spheres marked by a finite set \( A \). We assume that \( f: (\tilde{S}, A) \to \) is Böttcher expanding: it expands a length metric on \( \tilde{S}\setminus A^\infty \), where \( A^\infty \) is the forward orbit of periodic critical points, and the first return map at all \( a \in A^\infty \) is conjugate to \( z \mapsto z^{\deg_a(f_{\per(\alpha)})} \).

Given a forward-invariant set \( A_{\text{blow}} \subset A^F := A \cap \mathcal{F}(f) \), let
\[
(5) \quad f_{\text{blow}}: (\tilde{S}_{\text{blow}}, A\setminus A_{\text{blow}}) \to (\tilde{S}_{\text{blow}}, A\setminus A_{\text{blow}})
\]
be the partial branched covering obtained by blowing up every point \( a \in A_{\text{blow}} \) into a closed circle \( \delta_a \). More precisely:
- \( \tilde{S}_{\text{blow}} \) is a (disjoint) union of spheres with boundary components \( \Delta_{\text{blow}} = (\delta_a)_{a \in A_{\text{blow}}} \) together with a monotone map
\[
(6) \quad \rho_{\text{blow}}: (\tilde{S}_{\text{blow}}, A\setminus A_{\text{blow}}) \to (\tilde{S}, A)
\]
such that \( \rho_{\text{blow}}|_{\tilde{S}_{\text{blow}}\setminus \Delta_{\text{blow}}} \) is injective and \( \rho_{\text{blow}}(\delta_a) = a \) for every \( a \in A_{\text{blow}} \);
- \( \rho_{\text{blow}} \) semi-conjugates \( f_{\text{blow}} \) to \( f \) on \( \tilde{S}_{\text{blow}}\setminus A_{\text{indet}} \), where
\[
A_{\text{indet}} := f^{-1}(A_{\text{blow}}) \setminus A_{\text{blow}}.
\]

Note that the map \( f_{\text{blow}} \) is not defined on \( A_{\text{indet}} \). At the same time, the map \( f_{\text{blow}} \) is uniquely defined on each boundary circle \( \delta_a, a \in A_{\text{blow}} \), by continuity.

An annular map is a partial covering map of the form
\[
(7) \quad f: A' \to A \quad \text{with} \quad A' \subset A \quad \text{and} \quad \partial A' = \partial A
\]
where \( A \) and \( A' \) are finite unions of closed annuli and circles (i.e., degenerate annuli).

The map \( f \) is expanding if it expands a length metric on \( A \).

Consider a Thurston map \( f: (S^2, A, C) \to S^2 \), where \( A \) contains all the critical points. We say that \( f \) is a formal amalgam of expanding maps if it is obtained by
- gluing a blown up Böttcher expanding map (5) with an expanding annular map (7),
- fixing identification of the marked sets:
\[
(8) \quad \nu: [A\setminus A_{\text{blow}}] \xrightarrow{\text{gluing identification}} A \quad \text{of} \; f,
\]
- by redefining the resulting map in a small neighborhood of \( A_{\text{indet}} \) so that \( f \) is a branched covering respecting (5) and so that \( f \) has local Böttcher normalization around every \( A_{\text{indet}}^e \) \( \cap \nu(A_{\text{indet}} \cap A) \).

By construction, \( \nu \) respects the dynamics on \( A \setminus (A_{\text{blow}} \cup A_{\text{indet}}) \) but \( \nu \) may change the dynamics on \( A \cap A_{\text{indet}} \). The adjustment of \( f \) in a small neighborhood of \( A_{\text{indet}} \) is unique up to isotopy (such a neighborhood contains at most one critical point, which is necessary in \( A\setminus A_{\text{blow}} \); see [Pil03] §4). By construction, a formal amalgam has Böttcher normalization.

Naturally, we view each component of \( \tilde{S}_{\text{blow}} \) and \( A \) as a subset of \( S^2 \). Then \( \Delta_{\text{blow}} = \partial A \). Our convention is that
- \( C = \text{MultiCurve}(A) \) is the multicurve induced by \( A \).
We also note that primitive unicycles of \( C \) give rise to circle components (i.e., degenerate annuli) of \( A \). The relation between marked sets of \( f_{\text{blow}} \) and the resulting blowup is stated in Lemma 3.4.

**Lemma 3.3.** Assume \( f : (S^2, A, C) \rhd \) is a Levy-free map not doubly covered by a torus endomorphism, where \( A \) contains all the critical points of \( f \). Then \( f \) is isotopic to a formal amalgam \( f_{FA} : (S^2, A, C) \rhd \) of \( \text{(5)} \) and \( \text{(7)} \) where every (possibly degenerate) annulus of \( A \) is homotopic to a unique component of \( C \) and vice versa.

**Proof.** We can thicken \( C \) into a finite union \( A \) of closed annuli and circles and we can isotope \( f \) so that
\[
\mathcal{A}' = f^{-1}(A) \setminus \{\text{peripheral components of } f^{-1}(A)\}
\]
satisfies \( \mathcal{A}' \subset A \) and \( \partial \mathcal{A}' \supset \partial A \). Define \( S' = S^2 \setminus A \) and compactify \( S' \) into \( \tilde{S}_{\text{blow}} \) by adding boundary circles. Since \( f \) is Levy-free and not doubly covered by a torus endomorphism, so are the first return maps on periodic small spheres of the induced map \( f_{\text{blow}} \) on \( (\tilde{S}_{\text{blow}}, A) \). Therefore, we can isotope \( f_{\text{blow}} \) into an expanding map on \( (\tilde{S}_{\text{blow}}, A) \) satisfying \( \text{(5)} \). After that, we isotope \( f \) on \( \text{int}(A) \) so that the induced map \( \text{(7)} \) is expanding. \( \square \)

3.2.1. **Non-escaping sets.** Let us denote by \( K_S \) the set of points in \( \tilde{S} \setminus \Delta_{\text{blow}} \) that do not escape into the Fatou components around \( \Delta_{\text{blow}} \) under the iteration of \( \text{(5)} \). Also, let \( \mathcal{J}_A \) be the non-escaping set for the map \( \text{(7)} \); it is a collection of simple closed curves isotopic to \( C \).

**Lemma 3.4.** In a formal amalgam \( f : (S^2, A, C) \rhd \), every periodic cycle of \( f \mid A \) is either within \( K_S \) or contains a point in \( \nu(\{A \cap A_{\text{indet}}\} \setminus f_{\text{blow}}(A) \text{(5)}) \), see \( \text{(8)} \). These possibilities are mutually excluded.

**Proof.** By construction, periodic points of \( f \mid (A \cap K_S) \) are identified with periodic points of \( f_{\text{blow}} \mid A \setminus A_{\text{blow}} \text{(8)} \). The new periodic point can be created only through redefining the dynamics on \( A_{\text{indet}} \). By construction, the new periodic points are not in \( K_S \). \( \square \)

Consider a point \( a \in A_{\text{blow}} \) and an internal ray \( I \) inside the Fatou component of \( f \) centered at \( a \). An internal ray between \( K_S \) and \( A \) is the closure of \( \rho_{\text{blow}}^{-1}(\text{int}(I)) \). It is a closed arc connecting the boundary of the Fatou component around \( \delta_a \) for \( f_{\text{blow}} \) and \( \delta_a \).

3.2.2. **Iterating formal amalgams.** Consider a formal amalgam \( f : (S^2, A, C) \rhd \) of \( \text{(5)} \) and \( \text{(7)} \). Define
\[
S_{\text{blow}}^n := f^{-n}(\tilde{S}_{\text{blow}}) \subset S^2 \quad \text{and} \quad \mathcal{A}^n := f^{-n}(A) \subset S^2.
\]
By construction, every component of \( S_{\text{blow}}^n \) and \( \mathcal{A}^n \) is within a component of either \( S_{\text{blow}} \) or of \( A \). Set \( \mathcal{A}^n := f^{-1}(\mathcal{A}^n) \cap \mathcal{A}^n \). We view
\[
f|_{S_{\text{blow}}^n} : S_{\text{blow}}^n \rightarrow S_{\text{blow}}^n \quad \text{and} \quad f|_{\mathcal{A}^n} : \mathcal{A}^n \rightarrow \mathcal{A}^n
\]
as a blown-up Böttcher expanding map and an expanding annular map. This allows to represent \( f : (S^2, f^{-n}(A), f^{-n}(C)) \rhd \) as a formal amalgam of maps in \( \text{(10)} \). Clearly, components of \( S_{\text{blow}}^n \) are parametrized by small spheres of \( (S^2, f^{-n}(A), f^{-n}(C)) \).
and annuli of $A^n$ are parametrized by $C^n := f^{-n}(C)$. We denote by $K_{S^n}, J_{A^n}$ the non-escaping sets of maps in (10); they are iterated preimages of $K_S, J_A$. Clearly,

$$K_{S^n} \subset K_{S^m} \quad \text{and} \quad J_{A^n} \subset J_{A^m} \quad \text{for } n \leq m.$$  

Note also that for $n \leq m$, $J_{A^m} \setminus J_{A^n}$ contains only peripheral curves rel. $f^{-n}(A)$.

3.2.3. Gluing $K_S$ and $J_A$. Following [BD18 Definition 6.4], a pinching cycle connecting $x, y \subset K_S \cup J_A$ is a simple arc $I_1 \# I_2 \# \ldots \# I_k$ formed by a concatenation of internals rays $I_n$, where each $I_n$ connects a point in $K_{S^n}$ to a point in $\partial A^n \subset J_{A^n}$ (see Section 3.2.1) for some $n \geq 0$.

**Theorem 3.5.** Let $f : (S^2, A) \preceq$ be a formal amalgam of a blown-up Böttcher expanding map $F$ and an expanding annular map $\tilde{f}$. Suppose that $\tau$ is a semi-conjugacy from $f : (S^2, A) \preceq$ to a Böttcher expanding map $f : (S^2, A) \preceq$ as in Section 2.4.2. Then

$$\tau(x) = \tau(y) \quad \text{for} \quad x, y \in K_S \cup J_A$$

if and only if there is a pinching cycle of internal rays connecting $x$ and $y$.

Moreover, there is an $N \in \mathbb{N}$ such that $\tau|K_S \cup J_A$ identifies at most $N$ points.

**Proof.** If there is a pinching cycle $I_1 \# I_2 \# \ldots \# I_k$ of internal rays between $x, y \in J_S \cup J_A$, then $\ell_n := f^n(I_1) \# f^n(I_2) \# \ldots \# f^n(I_k)$ is a pinching cycle between $f^n(x)$ and $f^n(y)$. The curves $\ell_n$ define a homotopy equivalence between $x$ and $y$. By Proposition 3.1, $\tau(x) = \tau(y)$.

Suppose that $\tau(x) = \tau(y)$. By Proposition 3.1, the points $x, y$ are strongly homotopically equivalent. Consider a system of curves $\ell_n$ realizing the homotopy equivalence between $x, y$. By putting $\ell_n$ into the minimal position with $\tilde{f}A$, we can decompose every $\ell_n$ as

$$\ell_n = T^{(n)}_1 \# I^{(n)}_1 \# T^{(n)}_2 \# I^{(n)}_2 \ldots T^{(n)}_k \# I^{(n)}_k$$

such that

- $T^{(n)}_j$ is a curve within a small sphere or a small annulus;
- $I^{(n)}_j$ is a curve connecting a point in $K_S$ and a point in $J_A$;
- $T^{(n)}_j$ and $I^{(n)}_j$ are lifts of $T^{(n+1)}_j$ and $I^{(n+1)}_j$ respectively.

By the expansion, we may assume that $I^{(n)}_j$ are internal rays between small spheres and annuli while $T^{(n)}_j$ are trivial. This proves the first part of the theorem.

Since the length of internal rays between small spheres and annuli are bounded below, the constant $M$ from Proposition 3.1 determines the maximal size of pinching cycle of internal rays. It remains to show that for every expanding map $g : (S^2, C) \preceq$ there is $N_g$ that bounds the number of internal rays landing at any point of $x \in J_g$. By replacing $f$ with its iterate and replacing $x$ with its iterated image, we may obtain that $x$ is on the boundaries of only fixed Fatou components. For every such fixed Fatou component $F$, we can choose a basis $X_F$ for the biset of $g$ so that the associated symbolic presentation of $J_g$ includes the standard parametrization of $\partial F$ by internal rays. Then the number of internal rays of $F$ landing at $x$ is bounded by the nucleus in the basis $X_F$. This implies the required existence of $N_g$. □
3.2.4. Gluing $\cup \mathcal{K}_{S_{\gamma}}$ and $\cup \mathcal{J}_{A_{\gamma}}$. Following the setup from Section 3.2.2, for $\gamma \in \mathcal{C}^{n}$, we denote by $A_{\gamma}$ the annulus in $A^{n}$ homotopic to $\gamma$ rel. $f^{-n}(A)$. For a strongly connected component $\Sigma$ of $\mathcal{C}$ (see Section 2.6), we set $A_{\Sigma} := \bigcup_{\gamma \in \Sigma} A_{\gamma}$ and denote by $\mathcal{J}_{\Sigma}$ the non-escaping set of $f: A_{\Sigma} \cap f^{-1}(A_{\Sigma}) \to A_{\Sigma}$. If $\Sigma$ is a unicycle, then $\mathcal{J}_{\Sigma}$ is a finite periodic cycle of simple closed curves. If $\Sigma$ is a bicycle, then $\mathcal{J}_{\Sigma}$ is a Cantor bouquet of simple closed curves: topologically, $\mathcal{J}_{\Sigma}$ is a direct product between a Cantor set and $\mathbb{S}^{1}$. In all cases, every curve in $\mathcal{J}_{\Sigma}$ is isotopic to a unique curve in $\Sigma$. For $\gamma \in \Sigma$, we also write $\mathcal{J}_{\Sigma, \gamma} := \mathcal{J}_{\Sigma} \cap A_{\gamma}$.

It is easy to see that for $\delta$ in $\mathcal{C}^{n}$, the set $\mathcal{J}_{A_{\delta}} \cap A_{\delta}$ is the union of iterated preimages of the $\mathcal{J}_{\Sigma, \gamma}$ over all trajectories realizing the condition $\delta \subset f^{-n}(\gamma)$ (up to homotopy), where $\gamma$ is a periodic curve in a strongly connected component $\Sigma$.

We say that

- a simple closed curve $\beta$ in $\mathcal{J}_{A_{n}}$ is a neighbor to a component $\mathcal{K}_{S_{\gamma}, i}$ of $\mathcal{K}_{S}$ if $\beta$ is one the boundary of the component $S_{\text{blow}, i}$ containing $\mathcal{K}_{S_{\gamma}, i}$;
- two curves $\beta_{1}, \beta_{2}$ in $\mathcal{J}_{A_{n}}$ are neighbors if they have a common neighboring component $\mathcal{K}_{S_{m}, i}$ for some $m \geq n$.
- two components $\mathcal{K}_{S_{n}, i}, \mathcal{K}_{S_{m}, j}$ are neighbors if they are neighbors to a common curve in $\mathcal{J}_{A_{n}}$;
- a curve $\beta$ in $\mathcal{J}_{A_{n}}$ is a neighbor of a neighbor to a component $\mathcal{K}_{S_{m}, i}$ if they have a common neighboring component $\mathcal{K}_{S_{m}, j}$.

We remark that the notion of neighbors is independent of the embedding $\mathcal{J}_{A_{n}} \subset \mathcal{J}_{A_{m}}$ and of viewing small spheres of $f: (S^{2}, f^{-n}(A), f^{-n}(\mathcal{C})) \subset$ as small spheres of $f: (S^{2}, f^{-m}(A), f^{-m}(\mathcal{C})) \subset$ for $m \geq n$; i.e., neighbors remain neighbors if $n$ is increased.

We say that a curve $\beta$ in $\mathcal{J}_{A_{n}}$ is buried if it does not have any neighboring component $\mathcal{K}_{S_{m}, i}$ for all $m \geq n$.

Let $\tau$ be the semi-conjugacy from $f$ to $\bar{f}$ from Theorem 3.5. This theorem implies that for every buried curve $\beta$ in $\mathcal{J}_{A_{n}}$, the map $\tau|\beta$ is injective and $\tau^{-1}(\tau(\beta)) = \beta$. In particular, the image $\tau(\beta)$ is disjoint from $\tau(\mathcal{K}_{S_{m}} \cup (\mathcal{J}_{A_{n}} \setminus \beta))$ for all $m \geq n$. The gluing of neighbors is described by the pinching cycle condition in Theorem 3.5.

Lemma 3.6. Every connected component of $Y := S^{2} \setminus \bigcup_{n \geq 0} \tau(\mathcal{K}_{S_{n}} \cup \mathcal{J}_{A_{n}})$ is either a singleton or the closure of a Fatou component; this closure is a closed Jordan disk. This Fatou component is in the attracting basin of cycles intersecting $\nu([A \cap A_{\text{indet}}]_{\text{fblow}})$, see Lemma 3.4.

The set $Y$ consists of points whose $\bar{f}$-orbits do not enter $\tau(\mathcal{K}_{S} \cup \mathcal{J}_{A})$.

Proof. By construction, the set $Y$ is the $\tau$-image of the set $X$ of points in $S^{2}$ whose $f$-orbits never enter $\mathcal{K}_{S} \cup \mathcal{J}_{A} \cup \{\text{Fatou components around } \Delta_{\text{blow}}\}$. Equivalently, $x \in X$ if and only if the orbit of $x$ passes infinitely many times through Fatou components of $S^{2} \setminus \mathcal{K}_{S}$ associated with $A_{\text{indet}}$. We obtain that every component $V$ of $X$ is a nested intersection of compactly contained disks. Therefore, $f^{n}(V)$ is a continuum containing at most one point in $A$ for all $n \geq 0$. If the orbit of $V$ is disjoint from $A^{\infty}$, then all points in $V$ are homotopically equivalent and $\tau(V)$ is a singleton by Proposition 3.1. If $f^{n}(V)$ intersects $A^{\infty}$, then $\tau(V)$ contains the closure of a Fatou component $F'$ and, since $f^{n}: V \to f^{n}(V)$ is a cyclic branched covering
(i.e., it is topologically \( z \mapsto z^{D} \)), every point in \( V \) is homotopically equivalent to a point in \( \tau^{-1}(F') \). By Proposition 3.1, \( \tau(V) = F' \). Clearly, there are no non-trivial Levy arcs starting at \( A^{\infty} \cap \nu([A \cap A_{\text{indet}}]_{\text{f blow}}) \), see [8]. Therefore, \( \tau(V) = F' \) is a Jordan disk.

4. Clusters of Fatou components

4.1. Zero-entropy invariant graphs and clusters. Consider a Böttcher expanding map \( f: (S^{2}, A) \supset \) with the Julia set \( \mathcal{J} := \mathcal{J}(f) \), where \( A \) is the full preimage of the postcritical set.

A 0-entropy graph is a finite forward-invariant graph \( G \) (embedded in \( S^{2} \)) such that \( f|G \) has entropy 0. We will always assume that \( G \) intersects Fatou components along internal rays.

Consider a 0-entropy graph \( G \). Let \( \sim_{G} \subset A \times A \) be the equivalence relation such that \( a \sim_{G} b \) if and only if \( a, b \) are in the same connected component of \( G \). Let \( \{G_{i}\}_{i \in I} \) be the set of connected components of \( G \). We obtain the induced dynamics \( f: I \supset \) specified by \( f(G_{i}) \subset G_{f(i)} \).

For every \( n \geq 0 \) and \( i \in I \) we denote by \( G_{i}^{(n)} \) the connected component of \( f^{-n}(G) \) containing \( G_{i} \). It may happen that \( G_{i}^{(n)} = G_{j}^{(n)} \) for \( i \neq j \). Set

\[
K_{i}^{(n)} := \bigcup_{F \in \mathcal{J} \cap G_{i}^{(n)}} F,
\]

where the union is taken over all Fatou components that have non-empty intersection with \( G_{i}^{(n)} \); these Fatou components are centered at \( G_{i}^{(n)} \). Finally, we set

\[
K_{i} := \bigcup_{n} K_{i}^{(n)}
\]

and we denote by \( \mathcal{J}_{i} := \mathcal{J} \cap K_{i} \) the Julia part of \( K_{i} \).

We call \( K_{i} \) a (crochet) cluster. Clearly, each \( K_{i} \) is connected and \( f(K_{i}) \subset K_{f(i)} \).

Lemma 4.1. Let \( S \subset K_{i} \) be a finite set of (pre)periodic points. Then there is a 0-entropy graph \( \tilde{G} \supset G \) such that

- if \( \tilde{G}_{j} \) is a connected component of \( \tilde{G} \) containing \( G_{j} \), then \( \bigcup_{j} \tilde{G}_{j} = \tilde{G} \);
- \( \tilde{G}_{i} \supset S \).

Proof. Observe first that

\[
G^{(n)} := \bigcup_{i \in I} G_{i}^{(n)}
\]

is again a 0-entropy graph. By replacing \( G \) with \( G^{(m)} \) for a sufficiently big \( m \) and by identifying indices in \( I \), we can assume that

- \( G_{i}^{(n)} \cap G_{j}^{(n)} \) if \( i \neq j \);
- \( G^{(m)} \cap f^{-1}(G_{i}) = G_{i}^{(1)} \) for each \( m \geq 1 \).

We assume that \( S = \{x\} \) is a singleton; the general case easily follows by induction. The case \( x \in G_{i}^{(n)} \) follows by setting \( \tilde{G} := G^{(n)} \). Therefore, we assume that \( x \notin G_{i}^{(n)} \) for all \( n \geq 0 \). (Note that \( x \in G_{j}^{(n)} \) is allowed for \( j \neq i \).)

Suppose that \( x \) is on the boundary of a (pre)periodic Fatou component \( F \subset K_{i} \). Then the center of \( F \) belongs to \( G_{i}^{(n)} \) for some \( n \). We connect \( x \) to the center of
$F$ via a (pre)periodic internal ray $\ell_x$ of $F$, and set $\tilde{G} := G^{(n)} \cup \bigcup_{j \geq 0} f^j(\ell_x)$. Note that the graph $\tilde{G}$ is finite, $\ell_x$ is (pre)periodic.

From now on, we assume that $x$ is not on the boundary of any Fatou component of $K_i$. We will assume, in addition, that $x$ is periodic (the preperiodic case will follow via lifting).

Lemma 4.2. The periodic point $x$ is the landing point of a periodic bubble ray: there is a sequence of simple arcs $\ell_n \subset G_i^{(3)} \setminus G_i^{(3-1)}$, where $\ell_n$ is a concatenation of edges, such that

- $f^n(\ell_n) \subset G$ is a periodic sequence; and
- $\tilde{\ell} := \ell_1 \# \ell_2 \# \ell_3 \# \ldots$ forms a path connecting a point in $G_i$ to $x$.

Note that $\tilde{\ell}$ may have self-intersections.

Proof. By passing to an iterate of $f$, it is sufficient to prove the lemma in case $f(G_i) \subset G_i$ and $x$ is a fixed point. Let $w$ be a point in $G_i = G_i^{(0)}$.

Fix constants $\lambda > 1$ and $C > 0$ as in Lemma 2.1 and a positive $\epsilon < 1$.

For every vertex $v$ of $G_i^{(1)}$ choose a simple arc $\ell_v$ in $G_i^{(1)} \setminus G_i$ connecting $v$ and a vertex of $G_i$. (If $v \in G_i$, then $\ell_v$ is a trivial arc.) We also fix a nice curve $\beta_v$ that is pseudo-isotopic to $\ell_v$ rel $A \cup \partial v$ via $H_v : S^2 \times [0,1] \to S^2$. Let $K > 0$ be the maximal (truncated) length of the chosen $\beta_v$.

For each $v' \in f^{-1}(v) \cap G_i^{(2)}$ we fix a lift of $\beta_v$ starting at $v'$ that necessary ends in a vertex of $G_i^{(1)}$. The pseudo-isotopy $H_v$ determines a unique “lift” $\ell_{v'} = \ell_v^{\uparrow f}_{v'}$ of $\ell_v$, starting at each $v'$.

Choose a nice curve $\alpha_0$ in $(S^2, A)$ connecting $x$ and a vertex $v_0 \in G_i^{(1)}$ that is pseudo-isotopic to a curve in $K_i$ (rel $A \cup \partial \alpha_0$). Fix a sufficiently large constant $M$ that satisfy

$$M > \max \left( |\alpha_0|_\infty, \frac{K + (2C + \epsilon)\lambda}{\lambda - 1} \right).$$

We now inductively define nice curves $\alpha_n$ in $(S^2, A)$ ending in a vertex $v_n \in G_i^{(1)}$ with $|\alpha_n|_\infty < M$ in the following way. There is a unique lift of $\alpha_n - 1$ starting at $x$ (the uniqueness of the lift follows from $\deg_f(x) = 1$). This lift $\alpha_n - 1^{\uparrow f}_{v_n}$ necessary ends at a vertex $v_{n+1}$ in $G_i^{(2)}$ with $f(v_n) = v_{n+1}$. Consider the path $\tilde{\alpha}_n = \alpha_n - 1^{\uparrow f}_{v_n} \# \beta_{v_{n+1}}^{\uparrow f}_{v_n}$. Then its endpoint $v_n$ necessarily belongs to $G_i^{(1)}$. Choose a nice curve $\alpha_n$ that is homotopic to $\tilde{\alpha}_n$. We may assume that the truncated lengths of $\alpha_n$ satisfies

$$|\alpha_n|_\infty \leq |\alpha_n - 1^{\uparrow f}_{v_n}|_\infty + |\beta_{v_{n+1}}^{\uparrow f}_{v_n}|_\infty + \epsilon.$$

Now Lemma 2.1 implies:

$$|\alpha_n|_\infty \leq \frac{1}{\lambda} |\alpha_n - 1|_\infty + \frac{1}{\lambda} |\beta_{v_{n+1}}|_\infty + 2C + \epsilon \leq \frac{1}{\lambda} (|\alpha_n - 1|_\infty + K) + 2C + \epsilon$$

$$< \frac{M + K + (2C + \epsilon)\lambda}{\lambda} < M.$$

By Lemma 2.2 there are only finitely many homotopy classes $[\alpha_n]$. And since $\alpha_n$ depends only on the homotopy type of $\alpha_n - 1$, the sequence $\alpha_n$ is eventually
periodic. By shifting, we assume that \( \alpha_n \), as well as \( v_n \), is periodic, say with a period \( p \). For \( n \leq 0 \), we define \( \alpha_n \) to be \( \alpha_{n+kp} \), where \( p > 0 \) is sufficiently big.

For \( n \geq 1 \), define \( \ell_n \) to be the unique lift of \( \ell_{\alpha_{1-n}(1)} \) under \( f^{n-1} \) starting where \( \alpha_{1-n} \uparrow f^{n-1} \) ends. (such lift is constructed using the corresponding lift of the pseudoisotopy \( H_{\alpha_{1-n}(1)} \)). Then \( \ell_1 \# \ell_2 \# \ldots \) is a required periodic bubble ray. \( \square \)

Let \( p \) be an eventual period of the sequence \( f^n(\ell_n) \). This means that \( f^n(\ell_1 \# \ell_2 \# \ldots) \) coincide with \( \ell_1 \# \ell_2 \# \ldots \) in a small neighborhood of \( x \) because \( x \) is disjoint from \( \tilde{G}_i^{(n)} \).

Let \( q \) be the period of \( x \); note that \( q \mid p \). Let us consider the fundamental torus \( T \) of \( f^q \) at \( x \); i.e. if \( U \) is a closed small topological disk around \( x \) such that \( f(U) \supseteq U \), then \( T \) is obtained from \( f(U) \setminus U \) by gluing along \( f \mid \partial U \). Consider a simple arc \( \tilde{b} \subset f(U) \supseteq U \) connecting a point \( y \in \partial U \) to its image \( f(y) \in \partial f(U) \). Then \( \partial U \) and \( \tilde{b} \) project to simple closed curves \( a, b \subset T \) generating the fundamental group \( \pi_1(T) \simeq \mathbb{Z}^2 \).

Let us next project
\[
\bigcup_{k \in \{0, 1, \ldots, p/q - 1\}} f^{kq}(\ell_n \# \ell_{n+1} \# \ldots)
\]

to \( T \); we obtain a finite graph \( H \subset T \). There is a simple closed curve \( \gamma \subset H \) such that \( \gamma \) is not homotopic to \( a \). Write \( \gamma = na + mb \) in \( \pi_1(T) \) with \( b \neq 0 \). Lifting \( \tilde{\gamma} \) back to \( S^2 \) we obtain simple pairwise disjoint arcs
\[
\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{|m| - 1}
\]
emerging from \( x \) such that \( f^q \) cyclically permutes \( \tilde{\gamma}_j \) and
\[
f^q \left( \bigcup_j \tilde{\gamma}_j \right) \supseteq \bigcup_j \tilde{\gamma}_j \cup G_i^{(n)}
\]
for a sufficiently big \( n \). Observe also that every \( \tilde{\gamma}_j \) is disjoint from every \( G_j^{(n)} \neq G_i^{(n)} \). Adding the orbit of \( \tilde{\gamma} \) to \( G^{(n)} \) as well as the orbit of \( x \), we obtain a new 0-entropy graph \( \tilde{G} \) such that \( \tilde{G}_i \) contains \( x \). \( \square \)

4.2. Intersections and combinations of clusters. Let \( X \) be a compact subset of \( S^2 \). We say that a curve \( \alpha \) is essentially in \( X \) if for every neighborhood \( U \) of \( X \), the curve \( \alpha \) is homotopic (rel endpoints) to a curve in \( U \setminus A \).

Suppose \( K_i, K_j \) are periodic clusters; we allow \( i = j \). Consider two arcs \( \ell_1, \ell_2 \) of \( S^2 \setminus A \) connecting \( G_i \) and \( G_j \). We say that \( \ell_1 \) and \( \ell_2 \) are homotopic rel \( G \) if there are arcs \( \alpha \) and \( \beta \) such that
- \( \alpha \) is essentially in \( G_i \);
- \( \beta \) is essentially in \( G_j \); and
- \( \alpha \# \ell_1 \# \beta \) is homotopic to \( \ell_2 \)
up to changing the orientations of \( \ell_1, \ell_2 \).

A Levy arc between \( G_i \) and \( G_j \) is an arc \( \ell \subset S^2 \setminus A \) connecting \( G_i \) and \( G_j \) such that \( \ell \) is periodic up to homotopy rel \( G \); a certain lift \( \tilde{\ell} \) of \( \ell \) is homotopic to \( \ell \) relative \( G^n \).
Lemma 4.3. Up to enlarging the initial graph $G$ we may assume that the following holds.

Suppose $K_i \neq K_j$ are two periodic clusters generated by $G$. If $K_i \cap K_j \neq \emptyset$, then there is a Levy arc between $G_i$ and $G_j$. There are only finitely many Levy arcs between $G_i$ and $G_j$ (up to homotopy). If there is a Levy arc between $G_i$ and $G_j$, then there is a periodic point in $K_i \cap K_j$.

Every Levy arc between $G_i$ and $G_j$ is essentially in $G_i$.

Proof. We assume that every component of $G$ is non-trivial. By replacing $G$ with $G^{(m)}$ for a sufficiently big $m$ and by identifying indices in $I$, we can assume that $G_i^{(m)} = G_j^{(m)}$ if $i \neq j$.

There is a constant $M$ such that for every point in $x \in J_i$ there is an arc $\alpha_i(x)$ that is essentially in $K_i$ such that the length of $\alpha_i(x)$ is less than $M$ (see Section 2.2).

By expansion, every lift of such $\alpha_i(x)$ has length less than $M$.

For every $y \in K_i \cap K_j$ define $\gamma_y$ to be $\alpha_i(x) \# \alpha^{-1}_j(y)$. Set $N := \{ \gamma_y \mid y \in K_i \cap K_j \}$; this is a finite set up to homotopy rel $G$. We denote by $t$ the cardinality of $N$. It follows that $\gamma_{f^{m+1}(y)}$ is a Levy arc between $G_i$ and $G_j$, for the iterate $f^m$ that returns both $K_i$ and $K_j$ to themselves. This proves the first claim.

By expansion, there are at most finitely many Levy arcs between $G_i$ and $G_j$. Let $\ell$ be a Levy arc between $G_i$ and $G_j$. By definition, $\ell$ is homotopic to $\alpha \# \ell_1 \# \beta_1$, where $\ell_1$ is a lift of $\ell$ under $f^n$, and $\alpha_1, \beta_1$ are curves that essentially in $G_i, G_j$ respectively. Lifting this homotopy under $f^n k$, we obtain that $\ell$ is homotopic to $\alpha \# \ldots \# \alpha \# \ell_1 \# \beta \# \ldots \# \beta_1$, where $\alpha, \beta, \ell, \ell_1$ are lifts of $\alpha_1, \beta_1, \ell_1, \ell_1$ under $f^n$. By expansion, $\alpha \# \ldots \# \alpha \# \ldots$ lands at a periodic point in the intersections $K_i \cap K_j$.

There are at most finitely many Levy arcs between $G_i$ and $G_j$. Every such arc is realized as a concatenation of periodic bubble rays; such concatenation can be added to $G_i$, using the argument of Lemma 4.1. This finishes the proof of the lemma.

Remark 4.4. Note that Lemmas 4.2 and 4.3 allows us to “combine” intersecting clusters $K_i$ and $K_j$, that is, construct a new 0-entropy graph $\tilde{G}$ such that the union $K_i \cup K_j$ would be inside a cluster generated by $\tilde{G}$.

Definition 4.5. Let $f : (S^2, A) \supseteq$ be a Böttcher expanding map. Suppose that $G$ is a planar embedded $f$-invariant graph. We say that

1. $G$ is A-spanning if $G$ is connected and $A \subset G$;
2. $G$ is weakly A-spanning if $G$ is connected and each face of $G$ contains at most one point in $A$.

Lemma 4.6 (Spanning vs weakly spanning graphs). Let $f : (S^2, A) \supseteq$ be a Böttcher expanding map and $G$ be a planar embedded 0-entropy $f$-invariant graph. If $G$ is weakly A-spanning then there is a planar embedded 0-entropy $f$-invariant graph $\tilde{G} \supseteq G$ that is A-spanning.

Proof. Since $G$ is weakly A-spanning, $f^{-1}(G)$ is connected. Indeed, each face $U$ of $G$ is a topological disk containing at most one marked point, thus each component of $f^{-1}(U)$ is a topological disk as well. Hence, $f^{-1}(G)$ is connected.

Let $K$ be the cluster of $G$. By above, $f^{-1}(K) = K$. Let $a \in A \backslash G$. If $a \in K$, we are done using Lemma 4.1. Otherwise, by expansion, $a$ must be in the Fatou
set of $f$. Again, by expansion, $\partial F_a \cap K \neq 0$. Lemma 4.3 implies that there is a (pre)periodic point $p$ in $K \cap \partial F_a$. By Lemma 4.1 we can extend $G$ to a $0$-entropy graph that would contain $p$ and the internal ray in $F_a$ landing in $p$.

Note that we are essentially in the setup of Theorem 3.5. $f$ is a formal amalgam of $f$ with the power maps for the Fatou components $F_a$ of each $a \in (A \setminus G) \cap F(f)$. □

4.3. Maximal clusters. In the following, we fix a Böttcher expanding map $f : (S^2, A) \subset$, where $A$ is the full preimage of the postcritical set. The goal of this subsection is to define maximal clusters of touching Fatou components of $f$. The definition is based on an iterative construction (with finitely many steps).

Step 0: We denote by $F^{(0)}$ the set of all Fatou components of $f$, which we will call the pre-clusters of level $0$. The set $K^{(0)} = \{ F^0 : F^0 \in F^{(0)} \}$ of the closures of the Fatou components is called the set of clusters of level $0$.

Step n: Suppose we defined the set $K^{(n-1)}$ of clusters of level $n - 1$, or simply $(n-1)$-clusters, where $n \geq 1$. Consider an equivalence relation $\sim_{K^{n-1}}$ on $K^{(n-1)}$ defined as follows: two $(n-1)$-clusters $X^{n-1}, Y^{n-1} \in K^{(n-1)}$ are said to be equivalent if there exists a sequence $K_0^{n-1} = X^{n-1}, K_1^{n-1}, \ldots, K_m^{n-1} = Y^{n-1}$ of $(n-1)$-clusters such that $K_{j-1}^{n-1} \cap K_j^{n-1} \neq \emptyset$ for each $j = 1, \ldots, m$. We will call such a sequence of clusters an $(n-1)$-chain of length $m$. The set

$$
\bigcup_{K^{n-1} \in [X^{n-1}]_{K^{n-1}}} K^{n-1}
$$

is called a pre-cluster of level $n$. We denote by $F^{(n)}$ the set of all pre-clusters of level $n$. Then the set $K^{(n)} = \{ F^n : F^n \in F^{(n)} \}$ is called the set of clusters of level $n$.

The following statement follows from Lemmas 4.2 and 4.3 (see Remark 4.4).

Lemma 4.7. For every cluster $K^n \in K^{(n)}$ and every finite set of eventually periodic points $S \subset K^n$, there is a forward-invariant graph $G$ such that the points of $S$ are within the same connected component of $G$.

Recall that a Böttcher expanding map $f : (S^2, A) \subset$ is called a crochet map if there is a connected forward-invariant zero-entropy graph $G$ containing $A$ (i.e., $G$ is an $A$-spanning graph, see Definition 4.5).

The following easily follows definitions and Lemma 4.7.

Lemma 4.8. Let $f : (S^2, A) \subset$ be a Böttcher expanding map. Then the following are equivalent

- $f$ is crochet;
- an iterate $f^m$ is crochet for some $m \geq 1$;
- $K^n_i = S^2$ for some n-cluster $K^n_i \in K^{(n)}$ with $n \geq 1$.

The pre-clusters and clusters satisfy the following invariance properties.

Lemma 4.9. Let $F^n \in F^{(n)}$ be a pre-cluster of level $n \geq 0$ and let $K^n = F^n$ be the corresponding n-cluster. Then the following statements are true.

(i) $f(F^n) \in F^{(n)}$ and $f(F^n) = f(K^n) \in K^{(n)}$. In particular, the image of an $n$-cluster under $f$ is an $n$-cluster.

(ii) Each connected component of $f^{-1}(F^n)$ is a pre-cluster of level $n$.

(iii) Each connected component of $f^{-1}(K^n)$ is a finite union of n-clusters. If $F^n = K^n$, then each connected component of $f^{-1}(K^n)$ is an n-cluster.
Proof. We proceed by induction on \( n \). The base case \( n = 0 \) is easily to be seen.

We start with \( \text{(ii)} \). Let \( F' \) be a connected component of \( f^{-1}(F^n) \). Then \( f : F' \to F^n \) is a branching covering, and we can lift \( n \)-chains from \( F^n \) to \( F' \). More precisely, for every pair \( x, y \in F' \setminus A \), we can select a curve \( \gamma \subset S^2 \setminus A \) connecting \( x, y \) such that \( \gamma \) can be homotoped into any small neighborhood of \( F' \) (i.e., \( \gamma \) is essentially in \( F' \)). Then \( f(\gamma) \) can also be homotoped into any small neighborhood of \( F^n \).

We can select a finite chain of \((n-1)\)-pre-clusters \( F^{n-1}_1, \ldots, F^{n-1}_n \) (possibly with repetitions) such that for every \( \varepsilon > 0 \), we can homotope \( f(\gamma) \) into a concatenation \( \beta_1 \# \beta_2 \# \ldots \# \beta_i \) such that every \( \beta_i \) is in within the \( \varepsilon \) neighborhood of \( F^{n-1}_i \) for every small \( \varepsilon > 0 \). (Roughly, the \((n-1)\)-clusters \( F^{n-1}_i \) have an intersection pattern prescribed by the \( \beta_i \).) By induction, every connected component of \( f^{-1}(F^{n-1}) \) is a pre-cluster. Lifting the \( F^{n}_i \) according to \( \beta_1 \# \beta_2 \# \ldots \# \beta_i \), and taking the closure, we obtain a chain of \((n-1)\)-clusters connecting \( x, y \). So, \( F' \) is a subset of a pre-cluster of level \( n \). Since, by induction, \((n-1)\)-chains are sent over to \((n-1)\)-chains by \( f \), it follows that \( F' \) is actually a pre-cluster of level \( n \), i.e., \( \text{(ii)} \) follows.

Again, by the induction assumption, \( f(F^n) \) is a subset of a pre-cluster \( F^n_0 \) of level \( n \). Combined with \( \text{(ii)} \), we obtain \( f(F^n) = F^n_0 \). And taking the closure, we obtain \( f(K^n) = F^n_0 \) — this is \( \text{(i)} \).

Finally, property \( \text{(iii)} \) follows from \( \text{(ii)} \) because \( f \) has a finite degree. \( \square \)

We claim that the set of \( n \)-clusters eventually stabilizes, that is, \( K^{(n)} = K^{(n+1)} \) for some sufficiently large \( n \). Equivalently, this means that the \( n \)-clusters are pairwise disjoint for some sufficiently large \( n \). For \( a \in A \), let \( K^{n}_a \in K^{(n)} \) be the cluster containing \( a \); if \( a \) is not in any level-\( n \) cluster, then \( K^{n}_a = \emptyset \).

**Theorem 4.10.** There exists \( N \) such that \( K^{(n)} = K^{(n+1)} \) for each \( n \geq N \).

Moreover if \( K^{n}_a = K^{n+1}_a \) for all \( a \in A \), then \( K^{(n+1)} = K^{(n+2)} \).

The \( N \)-clusters are then called the **maximal clusters of touching Fatou components of \( f \).**

**Proof.** To prove the theorem, we consider \( |A| + 1 \) sequences of relations on the set \( A = f^{-1}(\text{post}(f)) \): \( \{\sim_{n,\text{in}}\}_{n \geq 0} \) and \( \{\sim_{n,\text{sep},a}\}_{n \geq 0} \), where \( a \in A \). The relation \( \sim_{n,\text{in}} \) is the **inclusion relation** induced by \( K^{(n)} \); two points \( a, a' \in A \) are related under \( \sim_{n,\text{in}} \) if \( a \) and \( a' \) belong to the same \( n \)-cluster. The relation \( \sim_{n,\text{sep},a} \) is the **separation relation** induced by \( K^{n}_a \): two points \( b, b' \in A \) are related under \( \sim_{n,\text{sep},a} \) if \( b \) and \( b' \) belong to the same connected component of \( S^2 \setminus K^{n}_a \); i.e., \( \sim_{n,\text{sep},a} \) is a set of pairwise disjoint subsets of \( A \), where \( b \) is absent in \( \sim_{n,\text{sep},a} \) if \( b \in K^{n}_a \). Since \((n)\)-clusters are nondecreasing it immediately follows that \( \sim_{n,\text{in}} \subseteq \sim_{n+1,\text{in}} \) and \( \sim_{n,\text{sep},a} \subseteq \sim_{n+1,\text{sep},a} \).

Our key claim:

**Lemma 4.11.** Suppose that \( \sim_{n-1,\text{in}} = \sim_{n+1,\text{in}} \) and \( \sim_{n-1,\text{sep},a} = \sim_{n+1,\text{sep},a} \) for all \( a \in A \). Then \( K^{(n)} = K^{(n+1)} \).

**Proof.** Denote by \( \tilde{K}^{n}_a \) the connected component of \( f^{-1} \circ f(K^{n}_a) \) containing \( K^{n}_a \). Consider the following set and the associated pseudo-multicurve, see Section 2.5.1:

\[
K^{n}_A := \bigcup_{a \in A} K^{n}_a, \quad C^{n} := \text{psMultiCurve}_A(K^{n}_A)
\]

**Claim.** The pseudo-multicurves \( C^{n-1} \) and \( C^{n} \) are homotopic rel \( A \).
Proof. Since $\sim_{n-1,\text{in}} = \sim_{n,\text{in}}$ we have
\[ K_{a}^{n-1} = K_{b}^{n-1} \quad \text{if and only if} \quad K_{a}^{n} = K_{b}^{n}. \]
(If $K_{a}^{n-1} \neq K_{b}^{n-1}$ but $K_{a}^{n} = K_{b}^{n}$, then $\sim_{n-1,\text{sep,a}} \sim_{n,\text{sep,a}}$.) Since $\sim_{n-1,\text{sep,a}} = \sim_{n,\text{sep,a}}$, every component $U$ of $S^{2} \setminus K_{a}^{n-1}$ with $U \cap A \neq \emptyset$ contains a unique component $U'$ of $S^{2} \setminus K_{a}^{n}$ such that $U' \cap A = U \cap A$. Since $U' \subseteq U$ are open topological disks, $\text{psCurve}(K_{a}^{n-1} | U)$ is homotopic to $\text{psCurve}(K_{a}^{n} | U')$. We obtain $C^{n-1} = C^{n} \text{ rel } A$. \hfill $\square$

It follows from the claim that $f^{-1}(C^{n-1}) = f^{-1}(C^{n}) \text{ rel } f^{-1}(A)$. Writing \( \tilde{K}_{f^{-1}(A)} = f^{-1}(K_{a}^{n}) \), we have by \([2]\):
\[ \text{psMultiCurve}_{f^{-1}(A)} \left( \tilde{K}_{f^{-1}(A)}^{n-1} \right) = \text{psMultiCurve}_{f^{-1}(A)} \left( \tilde{K}_{f^{-1}(A)}^{n-1} \right). \]

Let us now assume that $\tilde{K}_{a}^{n} \neq K_{a}^{n}$. Then $K_{a}^{n}$ intersects another cluster in $K^{(n)}$; we write this cluster as $K_{a}^{n}$, where $x \in f^{-1}(A) \sim A$ is a point in $K_{a}^{n}$.

Since $K_{a}^{n} \neq K_{a}^{n}$, the sets $\tilde{K}_{a}^{n-1}$ and $\tilde{K}_{a}^{n-1}$ are disjoint because they are in $F_{a}^{n}$ and $F_{x}^{n}$ respectively. Let $U$ be the connected component of $S^{2} \setminus \tilde{K}_{a}^{n-1}$ containing $\tilde{K}_{a}^{n-1}$. Then $\ell := \text{psCurve}_{f^{-1}(A)}(\tilde{K}_{a}^{n-1} | U)$ is a non-trivial curve because it separates $a$ and $x$. By construction, $\ell$ is in the left pseudo-multicurve of \([11]\) but not in the right. This is a contradiction. Therefore, $K_{a}^{n} = F_{a}^{n+1} = K_{a}^{n+1} = F_{a}^{n+2} = K_{a}^{n+2}$ for all $a \in A$ and Lemma \([4.3](iii)\) finishes the proof. \hfill $\square$

The first part of Theorem \([4.10]\) now immediately follows from Lemma \([4.11]\) due to the finiteness of $A$. The second part follows from Lemma \([4.3](iii)\). \hfill $\square$

5. Crochet algorithm

We start by introducing the following technical definition, having its origins in the Crochet Algorithm (see Section 1.5).

**Definition 5.1** (Pre-crochet multicurves). Let $f : (S^{2}, A, C) \to$ be a Böttcher expanding map such that each small map in the decomposition wrt. $C$ is either crochet or Sierpiński. Suppose that the set $I$ parametrizes the small spheres of $(S^{2}, A, C)$ and $f : I \to$ provides the dynamics on the small spheres (see Section 2.7).

Let us denote by $I_{\ast} \subset I$ the subset parametrizing all small spheres of $(S^{2}, A, C)$ induced by the small crochet maps. Then $I_{\circ} := I \setminus I_{\ast}$ parametrizes the small spheres arising from Sierpiński maps. The small spheres in $I_{\ast}$ and $I_{\circ}$ will be called (small) crochet spheres and Sierpiński spheres, respectively.

The invariant multicurve $C$ is called pre-crochet if there exists a partition
\[ I_{\ast} = I_{\ast}^{1} \sqcup \cdots \sqcup I_{\ast}^{n} \]
into forward-invariant sets such that the following two conditions are satisfied:

(i) The invariant multicurve $C$ is generated by the multicurve $\{ \partial S_{z}, z \in I_{\ast} \}$ consisting of the boundary curves of all crochet spheres; see Section 2.6.

(ii) For every $k \in \{1, \ldots, n\}$ and every periodic sphere $\hat{S}$ in $I_{\ast}^{k}$, the first return map $f : \hat{S} \to$ is $C^{k}$-vacant, that is, it admits a weakly spanning $0$-entropy connected invariant graph that does not pass through the Fatou components induced by $C \setminus C^{k}$ in $\partial \hat{S}$, where $C^{k}$ is the invariant multicurve generated by the boundaries of small crochet spheres in $I_{\ast}^{1} \sqcup \cdots \sqcup I_{\ast}^{k-1}$. 


The goal of this section is to show that each Böttcher expanding map $f : (S^2, A) \subset$ with non-empty Fatou set posses a pre-crochet multicurve. In fact, we will show that the Crochet Algorithm (see Section 1.5) constructs such a curve (after running the first three steps).

5.1. Gluing of crochet maps. We start by recording the following fact that follows from Theorem 3.3.

**Proposition 5.2.** Let $f : (S^2, A, C) \subset$ be a Böttcher expanding map, and $C$ be a pre-crochet multicurve. If all small maps in the decomposition of $f$ rel. $C$ are crochet and $C$ does not contain any bicycle, then $f$ is crochet.

Furthermore, if each small map in the decomposition rel. $C$ is vacant with respect to a subset $V \subset A^\infty$, then $f$ is vacant to $V$ as well.

**Proof.** By Lemma 4.8, it is sufficient to prove that an iterate of $f$ is crochet. By passing to an iteration, let us assume that all periodic spheres of $f : (S^2, A, C) \subset$ have period one and all periodic curves in $C$ have period one (the latter can be achieved due to the no-bicycles assumption). Suppose that $I_\bullet = I_1^\bullet \cup \cdots \cup I_n^\bullet$ is the partition of the set $I_\bullet$ parametrizing the small spheres of $(S^2, A, C)$ that satisfies the conditions in the definition of the pre-crochet multicurve. Below we will apply an induction on the number of small fixed spheres. The induction step is in described in the following lemma.

**Lemma 5.3.** Under the assumption of Proposition 5.2, let us assume that

- $C$ contains a unique periodic curve $\gamma$ which has period 1 (i.e., all other curves are strictly preperiodic);
- $f : (S^2, A, C) \subset$ has exactly two periodic spheres $S_1, S_2$; the period of each $S_i$ is 1 (i.e., $\gamma$ is the common boundary of $S_1, S_2$);
- $f_1$ is crochet and is vacant with respect to the Fatou components induced by $V_1 \cup \{\gamma\}$, where $V_1$ is a subset of $A_1^\infty = A^\infty \cap K_{S,1}$, see Section 3.2.2;
- $f_2$ is crochet and is vacant with respect to the Fatou components induced by $V_2$, where $V_2$ is a subset of $A_2^\infty = A^\infty \cap K_{S,2}$.

Then $f$ is crochet and is vacant with respect to the Fatou components generated by

$$V = V_1 \cup V_2 \cup (A^\infty \backslash (A_1^\infty \cup A_2^\infty)).$$

See Lemma 3.4 for how $A_1^\infty, A_2^\infty$ are related to $A$.

**Proof.** Let $f_{FA} : (S^2, A, C) \subset$ be a formal amalgam associated with $f : (S^2, A, C) \subset$, see Lemma 3.3. We denote by $\tau$ the induced semiconjugacy from $f_{FA}$ to $f$. We will show that appropriate clusters of $f_1, f_2$ get glued and produce a required cluster of $f$.

Let $K_1^\tau \subset K_{S,1}$ be the set of points that do not escape into the Fatou components induced by $V_1 \cup \{\gamma\}$. By assumption, $K_1^\tau$ is a (crochet) cluster: it supports a weakly spanning graph for $f_1$.

Let $K_2 \subset K_{S,2}$ be the set of points that do not escape into the Fatou components induced by $V_2 \cup \{\gamma\}$. Note that $K_2$ needs not to be a (crochet) cluster for $f_2$. Since $\gamma$ is a primitive unicycle, $\tau(K_1^\tau)$ contains the $\tau$-image $\Theta$ of the boundary of the Fatou component induced by $\gamma$ in $K_2$. Therefore, every lift of $\Theta$ under $f : \tau(K_2) \subset$ is glued with an appropriate lift of $\tau(K_1^\tau)$ – this is the “tuning” of $K_1^\tau$ and $K_2$. By the vacancy assumption on $f_2$, we obtain a (crochet) cluster for $f$ containing
\(\tau(K_1^+) \cup \tau(K_2)\) and separating the remaining points in \(A\); the remaining set contains \(V\). Therefore, \(f\) is vacant rel \(V\). The lemma is proven.

Let us finish the proof of Proposition 5.2. Select an “anti-primitive” unicycle \(\{\gamma\} \subset \mathcal{C}\); i.e. the multicurve \(\mathcal{C}_\gamma\) generated by \(\gamma\) does not contain any other unicycle. Then the decomposition of \(f\): \((S^2, A, \mathcal{C}_\gamma) \varnothing\) has exactly two periodic spheres \(S_1, S_2\); these spheres have period one. Let \(f_i: \tilde{S}_i \varnothing\) be the associated small maps, and let \(\mathcal{C}_i \subset \tilde{S}_i\) be the multicurve induced by \(\mathcal{C}\). Then \(\mathcal{C}_i\) is a pre-crochet multicurve for \(f_i\) (indeed, the disjoint union \(I_* = I_1^* \cup \ldots I_n^*\) would provide the respective partition for the set of small spheres). Moreover, all small maps in \(f_i: (\tilde{S}_i, \mathcal{C}_i) \varnothing\) are \(V_i\) vacant, where \(V_i\) is induced by \(V\). Finally, small sphere in either \(f_1: (\tilde{S}_1, \mathcal{C}_1) \varnothing\) or in \(f_2: (\tilde{S}_2, \mathcal{C}_2) \varnothing\) are vacant rel \(\gamma\) because such a property holds for small spheres of \(f: (S^2, A, \mathcal{C}) \varnothing\) bordering \(\gamma\). Assume that \(\gamma\) is vacant for small maps of \(f_1\).

By induction assumption, \(f_1\) is crochet and is vacant rel \(V_1 \cup \{\gamma\}\) and \(f_2\) is crochet and is vacant rel \(V_2\). Lemma 5.3 finishes the proof.

Let \(f: (S^2, A) \varnothing\) be a Böttcher expanding map and \(\mathcal{C}\) be a pre-crochet multicurve. Suppose \(\{P_k\}\) is the collection of all primitive unicycles in \(\mathcal{C}\) separating small crochet spheres of \((S^2, A, \mathcal{C})\). Consider the invariant multicurve \(\mathcal{C}' \subset \mathcal{C}\) generated by all strongly connected components of \(\mathcal{C}\) except the primitive unicycles \(\{P_k\}\), that is, glue together the clusters of small crochet spheres in \((S^2, A, \mathcal{C})\) separated by the unicycles \(P_k\). We iterate this process until there are no more primitive crochet unicycles. We will call this procedure **iterative elimination of all primitive crochet unicycles** from \(\mathcal{C}\).

**Definition 5.4** (Crochet multicurve). Let \(f: (S^2, A) \varnothing\) be a Böttcher expanding map and \(\mathcal{C}\) be a pre-crochet multicurve. Consider the invariant multicurve \(\mathcal{D} \subset \mathcal{C}\) that is generated by the boundaries of Sierpiński small spheres of \((S^2, A, \mathcal{C})\) together with bicycles in \(\mathcal{C}\). Then the multicurve \(\mathcal{D}\) is obtained by the iterative elimination of all primitive crochet unicycles from \(\mathcal{C}\) and we call it a **crochet multicurve**.

By definition, a crochet multicurve does not have any primitive crochet unicycles. It would follow that each Böttcher expanding map \(f: (S^2, A, \mathcal{D}) \varnothing\) with non-empty Fatou set has a unique crochet multicurve (up to isotopy), see Remark 7.2.

The next lemma follows from the definition and Proposition 5.2.

**Lemma 5.5.** Let \(f: (S^2, A) \varnothing\) be a Böttcher expanding map and \(\mathcal{D}\) be a crochet multicurve. Then each small map in the decomposition wrt. \(\mathcal{D}\) is either crochet or Sierpiński. Moreover, if \(\mathcal{D}\) is non-empty then either there is at least one Sierpiński small map in the decomposition or \(\mathcal{C}'\) has a primitive bicycle.

5.2. **Iterative step.** In the following, let \(f: (S^2, A) \varnothing\) be a Böttcher expanding map with a non-empty Fatou set, where \(A\) contains the full preimage of the post-critical set. We refer to Section 4.3 for the “cluster terminology”.

**Lemma 5.6.** Let \(K^N = \{K_i\}\) be the collection of maximal clusters of \(f\) (given by Theorem 4.19). Then \(\mathcal{C} := \text{MultiCurve}(K)\) is an invariant multicurve, where \(K = \bigcup K_i\) is the union of all the clusters.

Assume the multicurve \(\mathcal{C} \neq \varnothing\). Consider a non-trivial periodic \(K_i\), which means that \(\text{MultiCurve}(K_i)\) is non-empty. Then \(K_i\) is within a periodic small sphere \(S_i\) of \((S^2, A, \mathcal{C})\). The first return map on \(\tilde{S}_i\) is crochet.
Proof. Since $K \supset A \supset P_f$, $f: S^2 \setminus f^{-1}(K) \to S^2 \setminus K$ is a covering map. Let $C = \text{MultiCurve}(K)$. Lemma 4.9 immediately implies that $C \subset f^{-1}(C)$, as usual, up to homotopy. Conversely, $f^{-1}(C) \subset C$ after removal of all duplicate and peripheral curves. Thus, $C = \text{MultiCurve}(K)$ is an invariant multicurve.

Clearly, each non-trivial periodic cluster $K_i$ is within a periodic small sphere $S_i$ of $(S^2, A, C)$. Since $\hat{S}_i$ is marked by $A \cup C$, each complementary component of $\hat{S}_i \setminus K_i$ contains exactly one marked point. Theorem 3.5 and Lemma 4.8 now imply that the first return map on $\hat{S}_i$ is crochet.

\begin{lemma}[Stopping criterion] Let $K = \{K_i\}$ be the collection of maximal clusters of $f$ and $C = \text{MultiCurve}(\bigcup K_i)$ be the respective invariant multicurve, as in Lemma 5.6. Then $C$ is empty if and only if $f$ is a crochet or a Sierpiński map.
\end{lemma}

Proof. Let us suppose that the multicurve $C$ is empty. If $K$ consists of a unique cluster, then $f$ is crochet by Lemma 4.8. Otherwise, let $K = \bigcup_{i \in I} K_i$. Suppose that some cluster $K_i$ contains at least two marked points. Since $C$ is empty, each connected component $U$ of $S^2 \setminus K_i$ contains at most one point from $A$. Furthermore, since $F_f \cap U \neq \emptyset$, $a$ must be in $F_f$. Thus, there is a an internal ray joining $a$ and $K_i$ which would contradicts maximality of $K_i$.

Consequently, each cluster $K_i$ must have at most one marked point. Again, there is no Levy arc between any two Fatou points in $A$. Also, there is no periodic self-arc $\alpha$ at any Fatou point $a$ in $A$. Indeed, otherwise each component of $S^2 \setminus a$ contains a marked point $p$ and either $C$ is non-empty, or there is a Levy arc connecting $p$ and $a$ (providing a contradiction). Thus, by the characterization from [BDTS], $f$ must be a Sierpiński map.

\end{proof}

5.3. Crochet Algorithm. Consider a recursive procedure with the following step.

- Given a Böttcher expanding map $f: (S^2, A) \setminus \emptyset$ with $F_f \neq \emptyset$, we extract the maximal clusters $K_i$ and the respective invariant multicurve $C = \text{MultiCurve}(K)$, where $K = \bigcup K_i$, as in Lemma 5.6.

- If $C$ is non-empty, then we run the above step for the first return map of each periodic small sphere of $(S^2, A, C)$ that does not contain a cluster $K_i$.

- If $C$ is empty, we stop the recursive step.

Since $A$ is finite, the above recursive procedure eventually stops (completing the first three steps of the Crochet Algorithm). Let $\hat{C}$ be the union of all the invariant curves constructed during the process. We define $C_\infty$ to be the set of representatives of the isotopy classes of curves in $\cup_{n \geq 0} f^{-n}(\hat{C})$, that is, $C_\infty$ is the invariant multicurve generated by $\hat{C}$. The following lemma follows from the previous discussion.

\begin{proposition} The multicurve $C_\infty$ constructed after running the first three steps of the Crochet Algorithm is a pre-crochet multicurve for $f: (S^2, A) \setminus \emptyset$.
\end{proposition}

Proof. By construction, $C_\infty$ is an invariant multicurve. Moreover, the first return map to each periodic small sphere of $(S^2, A, C_\infty)$ is either crochet or Sierpiński (Lemmas 5.6 and 5.7). Suppose that $I_\ast$ parametrizes all small crochet spheres of $(S^2, A, C_\infty)$ induced by the small crochet maps. Partition $I_\ast$ according to the depth of the recursive step, that is, we write

$$I_\ast = I_\ast^1 \sqcup \ldots I_\ast^n.$$
where $I^*_k$, $k \in \{1, \ldots, n\}$, corresponds to the small crochet spheres induced by the small crochet maps obtained during the $k$th stage of the recursive procedure. It is now straightforward to check that the above partition satisfies the conditions of Definition 5.1. The statement follows.

After the recursive construction of $C_{\mathcal{N}}$ we do a reduction step that eliminates all primitive crochet unicycles, i.e., the primitive unicycles of $C_{\mathcal{N}}$ with crochet maps on both sides. In fact, these may only appear at different stages of the recursive procedure (due to the maximality of extracted clusters at each stage). We iterate this reduction step until no more primitive crochet unicycles are left – this is Step 4 of the Crochet Algorithm (see Section 1.5). The resulting multicurve $C_{\text{dec}}$ would be a crochet multicurve by definition; it is generated by the boundaries of Sierpiński small spheres of $(S^2, A, C_{\mathcal{N}})$ together with the bicycles of $C_{\mathcal{N}}$. That is, we proved the following result.

**Corollary 5.9.** Let $f : (S^2, A) \hookrightarrow$ be a Böttcher expanding map with non-empty Fatou set. Then the Crochet Algorithm produces a crochet multicurve $C_{\text{dec}}$.

The decomposition of $f : (S^2, A, C_{\text{dec}}) \hookrightarrow$ along the invariant multicurve $C_{\text{dec}}$ is called the **crochet decomposition of $f$**.

### 6. Cactoid maps

Let us recall that a cactoid $X$ is a continuous monotone image of $S^2$. In general, $X$ is a locally connected continuum composed of countably many spheres and segments pairwise intersecting in at most one point. We call a cactoid $X$ finite if it is a finite CW-complex, i.e., if $X$ is composed of finitely many spheres and segments. The finite cactoids $X$ we work with will usually have a natural finite marking $A \subset X$ so that each component $S$ of $X \setminus A$ is either an open arc or a punctured sphere. We refer to the closure $S$ as a **small segment of $(X, A)$** (or simply a small segment of $X$ if $A$ is understood) in the former case, and as a **small sphere** in the latter case.

Consider a Thurston map $f : (S^2, A, C) \hookrightarrow$. In this section, we discuss how to collapse $f : (S^2, A, C) \hookrightarrow$ into a totally topologically expanding map $\bar{f} : X_\infty \hookrightarrow$ on a cactoid $X_\infty$. More precisely, writing $C = C_* \cup C_-$ and $I = I_* \cup I_o$, where $I$ is an index set parametrizing small spheres of $(S^2, A, C)$, we will collapse curves in $C_*$ into points, annular neighborhoods of curves in $C_-$ into segments, small spheres of $I_*$ into points, and small spheres of $I_o$ will project into small spheres of $X_\infty$. To obtain the expansion in the quotient, we will assume that small maps associated with $I_o$ are Sierpiński maps and $C_-$ is generated by bicycles. In the proof, we will construct a finite expanding model $\bar{f}, i : X_2 \hookrightarrow X_1$, and the desired map $\bar{f} : X_\infty \hookrightarrow$ will be the inverse limit of $\bar{f}, i : X_{n+1} \hookrightarrow X_n$.

#### 6.1. The quotient map $(S^2, A, C) \rightarrow (X, \bar{A})$

Consider a marked sphere $(S^2, A)$ with a multicurve $C$, and let $I$ be an index set parametrizing the small spheres of $(S^2, A, C)$. Fix decompositions

\begin{equation}
C = C_* \cup C_- \quad \text{and} \quad I = I_* \cup I_o,
\end{equation}

which we call a **collapsing data** for $(S^2, A, C)$. We will now describe a procedure collapsing $(S^2, A, C)$ into a finite marked cactoid induced by these decompositions.

For every $\gamma \in C$, let $\Gamma_\gamma$ be either $\gamma$ (i.e., a degenerate annulus) or a thickened closed annulus $\Gamma_\gamma$ around $\gamma$. In the second case, we foliate the annulus $\Gamma_\gamma$ by curves
\( \gamma, t \in [-1, 1] \), isotopic to \( \gamma \). If \( \gamma = \Gamma_\gamma \), then we assume that \( \gamma_t = \gamma \) for all \( t \). We assume that

\begin{equation}
\gamma \subseteq \Gamma_\gamma \quad \text{if} \quad \gamma \in \mathcal{C}_-.
\end{equation}

Starting with Section 6.2.1, we will require a stronger dynamically invariant condition 19 instead of (13).

Now we perform the following collapsing procedure on \( S^2 \):

(A) for every \( \gamma \in \mathcal{C}_- \), collapse \( \Gamma_\gamma \) to a point; and

(B) for every \( \gamma \in \mathcal{C}_- \), collapse \( \Gamma_\gamma \) into a closed segment by collapsing every \( \gamma_t \) (in the foliation of \( \Gamma_\gamma \)) into a point.

After (A) and (B) we obtain a finite cactoid \( X \) whose small spheres and segments are naturally parametrized by \( I \backslash \{ \gamma_{-1} \cup \gamma_1 \} \) and \( \mathcal{C}_- \), respectively. Now we

(C) collapse every small sphere of \( X \) in \( I \backslash \{ \gamma_{-1} \cup \gamma_1 \} \) to a point.

We denote by \( X \) the resulting finite cactoid and by

\begin{equation}
\Pi_{\mathcal{C}_-, I, \bar{A}} : (S^2, A, C) \to (X, \bar{A})
\end{equation}

the associated quotient map. Here, the pair \( (X, \bar{A}) \) represents the cactoid \( X \) marked by

\[ \bar{A} := \Pi_{\mathcal{C}_-, I, \bar{A}} \left( A \cup \bigcup_{\gamma \in \mathcal{C}} \{ \gamma_{-1} \cup \gamma_1 \} \right). \]

We say that \( \Pi_{\mathcal{C}_-, I, \bar{A}} \) and \( (X, \bar{A}) \) are induced by the collapsing data 12.

The following observations are immediate from the construction: small spheres and segments of \( (X, \bar{A}) \) are naturally parameterized by \( \mathcal{C}_- \) and \( \mathcal{C}_- \), respectively; small spheres and segments intersect at marked points; every small segment connects two marked points in \( \bar{A} \).

The quotient cactoid \( (X, \bar{A}) \) is unique in the following sense.

**Lemma 6.1** (Uniqueness of \( (S^2, A, C) \to (X, \bar{A}) \)). Suppose

\[ \Pi_a : (S^2, A, C) \to (X_a, \bar{A}_a) \quad \text{and} \quad \Pi_b : (S^2, A, C) \to (X_b, \bar{A}_b) \]

are two monotone maps realizing the collapsing data 12. Then there is a homeomorphism \( h : (X_a, \bar{A}_a) \to (X_b, \bar{A}_b) \) such that \( h \circ \Pi_a \) and \( \Pi_b \) are isotopic via a continuous path of monotone maps \( p_t : (S^2, A, C) \to (X_b, \bar{A}_b) \) satisfying the above (A) (B) (C).

**Proof.** The cactoids \( (X_a, \bar{A}_a) \) and \( (X_b, \bar{A}_b) \) are parameterized by the collapsing data 12. Therefore, there is a homeomorphism \( h : (X_a, \bar{A}_a) \to (X_b, \bar{A}_b) \) respecting 12. The fibers

\[ [h \circ \Pi_a]^{-1}(x) \quad \text{and} \quad \Pi_b^{-1}(x), \quad x \in X_b \]

are described in (A) (B) (C) thus there is a homotopy of \( p_t' \) of \( (S^2, A) \) moving \( [h \circ \Pi_a]^{-1}(x) \) into \( \Pi_b^{-1}(x) \) for all \( x \). The \( p_t' \) induces a required homotopy \( p_t \) of monotone maps. \( \square \)
6.2. Cactoid correspondences. Let \( f : (S^2, A) \simeq \) be a Thurston map and \( \mathcal{C} \) be a multicurve on \( (S^2, A) \). Consider the covering
\[
(15) \quad f : (S^2, f^{-1}(A), f^{-1}(\mathcal{C})) \to (S^2, A, \mathcal{C}).
\]
We assume that the sets \( I \) and \( f^{-1}(I) \) parametrize the small spheres of \( (S^2, A, \mathcal{C}) \) and \( (S^2, f^{-1}(A), f^{-1}(\mathcal{C})) \), respectively. Given the collapsing data \((12)\), let us set
\[
(16) \quad f^{-1}(\mathcal{C}_*, C_*, I_*, I_0) = (f^{-1}(\mathcal{C}_*), f^{-1}(C_*), f^{-1}(I_*), f^{-1}(I_0)).
\]
Clearly, \((16)\) specifies a collapsing data for \((S^2, f^{-1}(A), f^{-1}(\mathcal{C}))\).

The quotient map \((14)\) induces a monotone equivalence relation \( \sim \) on \( (S^2, A, \mathcal{C}) \). Let \((X_1, A_1)\) be the corresponding marked quotient cactoid. We define \( f^* (\sim) \) to be the (monotone) equivalence relation on \((S^2, f^{-1}(A), f^{-1}(\mathcal{C}))\) whose equivalence classes are the connected components of the preimages of the equivalence classes of \( \sim \). Then \( f^* (\sim) \) defines a quotient map
\[
(17) \quad \Pi_{f^{-1}(\mathcal{C}_*, C_*, I_*, I_0)} : (S^2, f^{-1}(A), f^{-1}(\mathcal{C})) \to (X_2, A_2),
\]
induced by the collapsing data \((16)\). Furthermore, the map \( f \) induces a branched covering \( \bar{f} : (X_2, A_2) \to (X_1, A_1) \) satisfying the following commutative diagram
\[
\begin{array}{ccc}
(S^2, f^{-1}(A), f^{-1}(\mathcal{C})) & \xrightarrow{f} & (S^2, A, \mathcal{C}) \\
\Pi_{f^{-1}(\mathcal{C}_*, C_*, I_*, I_0)} \downarrow & & \downarrow \Pi_{\mathcal{C}_*, C_*, I_*, I_0} \\
(X_2, A_2) & \xrightarrow{\bar{f}} & (X_1, A_1)
\end{array}
\]

(18)

The cactoid map \( \bar{f} \) is unique in the following sense: the domain and target cactoids are unique in the sense of Lemma 6.1 and an isotopy for \( f \) induces an isotopy for \( \bar{f} \).

We note that the map \( \bar{f} : X_2 \setminus A_2 \to X_1 \setminus A_1 \) is a covering, but it may have different degrees on different components of \( X_2 \setminus A_2 \). Moreover, for every marked small sphere \((\bar{S}_2, A_{2,2})\) in \( (X_2, A_2) \), we have a branched covering
\[
\bar{f} \mid (\bar{S}_2, A_{2,2}) : (\bar{S}_2, A_{2,2}) \to (\bar{S}_{f(z)}, \bar{A}_{f(z),1}),
\]
where \((\bar{S}_{f(z)}, \bar{A}_{f(z),1})\) is a marked small sphere of \((X_1, \bar{A}_1)\).

6.2.1. Monotone maps between cactoids. Consider now a Thurston map \( f : (S^2, A, \mathcal{C}) \simeq \) and view it as a correspondence
\[
f, i : (S^2, f^{-1}(A), f^{-1}(\mathcal{C})) \rightrightarrows (S^2, A, \mathcal{C})
\]
as in \((3)\), i.e., \( f \) is a covering map (the same as the original map) and \( i \) is a forgetful monotone map (see Sections 2.1 and 2.7 for the conventions). Below we will fix \( f \) and isotope \( i \) to a forgetful monotone map \( \iota \) so that the new correspondence \( f, \iota \) projects to a cactoid correspondence (Lemma 6.2).

Let \( \mathcal{C}_{\text{np}} \) be the invariant submulticurve of \( \mathcal{C} \) generated by all its bicycles and all its non-principal unicycles. Instead of \((13)\), we will from now on require
\[
(19) \quad \gamma \subseteq \Gamma, \quad \text{if and only if} \quad \gamma \in \mathcal{C}_{\text{np}}.
\]
In this case, we say that we have dynamical collapsing data \((12)\).
Recall that the index set $I$ parametrizes the small spheres of $(S^2, A, C)$. Following the notation in Section 2.7, $f \circ i^* : I \subset$ describes the dynamics of small spheres of $(S^2, A, C)$. Let $\Pi_2 = \Pi_{f^{-1}(C_\gamma \cup A_\gamma)}$ and $\Pi_1 = \Pi_{\gamma_{\gamma_{-1}}}$ be the quotient maps as in [18]. Recall also that $I_o, C_-$ and $f^{-1}(I_o), f^{-1}(C_-)$ parametrize the small spheres and segments of the finite cactoids $X_1$ and $X_2$, respectively.

Let $\Gamma_C$ be the set of all curves $\gamma \in \Gamma_C$. For each $\gamma \in C$, let $\gamma^+, \gamma^-$ be the two curves in $f^{-1}(\Gamma_C) = \{f^{-1}(\gamma) : \gamma \in \Gamma_C\}$ such that the closed (possibly degenerate) annulus between $\gamma^+, \gamma^-$ contains all curves in $f^{-1}(\Gamma_C)$ that are isotopic to $\gamma$. Set $\bar{A}_2^{(1)} = \Pi_2(A) \cup \bigcup_{\gamma \in C} \Pi_2(\gamma^+ \cup \gamma^-)$.

Lemma 6.2. Suppose we are given a dynamical collapsing data (12) that satisfies
\begin{equation}
C_- \subset f^{-1}(C_-) \quad \text{and} \quad f \circ i^*(I_o) \subset I_o.
\end{equation}
Then the forgetful map $i : (S^2, f^{-1}(A), f^{-1}(C)) \to (S^2, A, C)$ is homotopic rel. $A$ to a forgetful monotone map $i : (S^2, f^{-1}(A)) \to (S^2, A)$ so that the following holds:

(i) $i$ projects to a forgetful monotone map $\bar{i} : (X_2, \bar{A}_2) \to (X_1, \bar{A}_1)$, that is, the following diagram commutes:

\[
\begin{array}{ccc}
(S^2, f^{-1}(A), f^{-1}(C)) & \xrightarrow{\iota} & (S^2, A, C) \\
\Pi_2 \downarrow & & \downarrow \Pi_1 \\
(X_2, \bar{A}_2) & \xrightarrow{i} & (X_1, \bar{A}_1)
\end{array}
\]

(ii) for every small sphere $\bar{S}_{z,2}$ of $X_2$, $i|\bar{S}_{z,2}$ is a homeomorphism if $\bar{S}_{z,2}$ is in $i^*(I_o)$; otherwise, it is constant.

(iii) for every small segment $T_\gamma$ of $X_2$ associated with $\gamma \in f^{-1}(C_-)$, $i|T_\gamma$ is a homeomorphism if $\gamma$ is isotopic to a curve $C_-$; otherwise, $i|T_\gamma$ is constant.

Moreover, the map $\bar{i}$ is unique up to isotopy rel. $\bar{A}_2^{(1)}$ (among maps satisfying the desired conditions).

Proof. The combinatorics uniquely determines the small spheres and segments of $X_2$ where $\bar{i}$ is a homeomorphism. This allows to define $\bar{i}$ and $i$ is then a lift of $\bar{i}$.

More precisely, every small sphere $\bar{S}_{z,1}$ of $X_1$ is identified with a unique component $S_{z,1}$ of $S^2\backslash C$. Let $S_{z,2}$ be the unique component of $S^2\backslash f^{-1}(C)$ such that $S_{z,2}$ is homotopic to $S_{z,1}$ rel. $A$, see Section 2.7. Since $f \circ i^*(I_o) \subset I_o$, the component $S_{z,2}$ is identified with a unique small sphere $\bar{S}_{z,2}$ of $X_2$. Up to isotopy, this uniquely specifies a homeomorphism $i : (\bar{S}_{z,2}, \bar{A}_2) \to (\bar{S}_{z,2}, \bar{A}_1)$.

Let $X'_2 \subset X_2$ be the union of all small spheres $\bar{S}_{z,2}$ arising in (22). It follows from $C_- \subset f^{-1}(C_-)$ that for every segment $T$ of $X_1$ there is at least one segment in $X_2$ with image in $T$. Therefore, $i : X'_2 \to X_1$ extends to a required map $\bar{i} : X_2 \to X_1$.

Since non-trivial fibers of $\Pi_2 = \Pi_{f^{-1}(C_\gamma \cup A_\gamma)}$ and $\Pi_1 = \Pi_{\gamma_{\gamma_{-1}}}$ are closed surfaces with Jordan boundaries, $\bar{i}$ lifts to a required monotone map $\iota$. 

Since $\bar{i} \circ \Pi_2 = \Pi_1 \circ \iota$, and $\iota$ is homotopic to the identity rel. $A$, we have:
Corollary 6.3. The monotone maps $\iota \circ \Pi_1, \Pi_1 : (S^2, A) \to (X_1, \hat{A}_1)$ are homotopic rel. $A$.

6.2.2. Iterating correspondences. For a Thurston map $f : (S^2, A, C) \supset$ consider the backward iteration:

$$\begin{align*}
\begin{array}{c}
\ldots \xrightarrow{f} (S^2, f^{-2}(A), f^{-2}(C)) \xrightarrow{f} (S^2, f^{-1}(A), f^{-1}(C)) \xrightarrow{f} (S^2, A, C).
\end{array}
\end{align*}$$

Pulling back the forgetful map $\iota = \iota_1$ from Lemma 6.2, we obtain the maps

$$\begin{align*}
\begin{array}{c}
\ldots \xrightarrow{\iota} (S^2, f^{-2}(A), f^{-2}(C)) \xrightarrow{\iota} (S^2, f^{-1}(A), f^{-1}(C)) \xrightarrow{\iota} (S^2, A, C)
\end{array}
\end{align*}$$

with $f \circ t_{n+1} = t_n \circ f$ such that the pair $f, t_n$ induces via the monotone maps

$$\Pi_n : (S^2, f^{-n+1}(A), f^{-n+1}(C)) \to (X_n, \hat{A}_n)$$

(iterated lifts of $\Pi_1$) the cactoid correspondence $\bar{f}, \bar{\iota} : (X_{n+1}, \hat{A}_{n+1}) \supset (X_n, \hat{A}_n)$. We obtain the inverse system:

$$\begin{align*}
\bar{f}, \bar{\iota} : \ldots \supset (X_3, \hat{A}_3) \supset (X_2, \hat{A}_2) \supset (X_1, \hat{A}_1)
\end{align*}$$

and we denote by $\bar{f} : X_\infty \supset$ the inverse limit of (25) with respect to $\bar{\iota}$.

6.3. Expanding cactoid correspondences. We begin by extending the notion of topological expansion (Section 2.3) to correspondences $\bar{f}, \bar{\iota} : (X_2, \hat{A}_2) \supset (X_1, \hat{A}_1)$.

Let $\mathcal{U} = (U_j)_{j \in J}$ be a finite cover of $(X_1, \hat{A}_1)$ by connected sets. We denote by $\bar{f}^*(\mathcal{U})$ the finite cover of $(X_2, \hat{A}_2)$ consisting of connected components of $\bar{f}^{-1}(U_j)$ over all $j \in J$. Then

$$\bar{\iota} \circ \bar{f}^*(\mathcal{U}) = \{ \bar{\iota}(U) \mid U \in \bar{f}^*(\mathcal{U}) \}$$

is a cover of $X_1$.

We say that the correspondence $\bar{f}, \bar{\iota} : (X_2, \hat{A}_2) \supset (X_1, \hat{A}_1)$ is topologically expanding if there is an open cover $\mathcal{U} = (U_s)_{s \in S}$ of $(X_1, \hat{A}_1)$ by connected open sets such that the maximal diameter of components of $\mathcal{U}_n := (\bar{\iota} \circ \bar{f}^*)^n(\mathcal{U})$ tends to zero as $n \to \infty$. Note that sets in $\mathcal{U}_n$ need not be open.

We will now specify the discussion to the case when $f : (S^2, A, C) \supset$ is a formal amalgam of a Böttcher expanding map $f_{BE} : (S^2, A, C) \supset$, see Lemma 3.3. Let us recall that $f$ is the gluing of the map $\bar{\iota}$ on a union $\hat{S}_{\text{blow}}$ of finitely many spheres and the annular map $\iota$. Moreover, connected components of $\hat{S}_{\text{blow}}$ and $A$ form a backward-invariant partition for $f$. Below we will argue that, up to isotopy of $\iota$, all relevant maps respect the partition by $\hat{S}_{\text{blow}}, \hat{A}$. Then we will construct semi-conjugacies from $f$ and $f_{BE}$ towards the limiting cactoid maps and describe the fibers of the semi-conjugacies.

Condition (19) allows us to assume that the annuli $\Gamma_{\gamma}$ in Steps [A] [B] coincide with the annuli in $A$. Therefore,

$$\Pi_1 = \Pi_\gamma, \ldots, \Pi_1 : (S^2, A, C) \to (X_1, \hat{A}_1)$$

(respects the partition by $\hat{S}_{\text{blow}}, \hat{A}$: for every sphere $S_i$ in $\hat{S}_{\text{blow}}$ (i.e., for every connected component of $\hat{S}_{\text{blow}}$) its image $\Pi_1(S_i)$ is either a singleton or a small sphere of $X_1$ and $\Pi_1(A) \subset \hat{A}_1 \cup \{\text{segments of } X_1\}$.

Since the partition by $\hat{S}_{\text{blow}}, \hat{A}$ is backward-invariant, $\Pi_2$ is the lift of $\Pi_1$, and $\iota$ respects the partition of $X_2$ (Conditions [iii] and [iii]), we obtain the following piece-wise conditions:

(PW1) for every small sphere $S_i$ in $\hat{S}_{\text{blow}}$ its image $\iota \circ \Pi_2(S_i)$ is either a singleton or a small sphere of $X_1$;
Lemma 6.4. For a formal amalgam \( f : (S^2, A, C) \supset \) as above and under Assumption (20) of Lemma 6.2 assume in addition that

(A) all small maps of \((S^2, A, C)\) parametrized by \( f \circ \iota^* : I_\circ \supset \) are Sierpiński;
(B) \( C_- \) is generated by its bicycles: if \( C_-^{\bullet} \) is the union of all bicycles in \( C_- \), then

\[ C_- \subset f^{-n}(C_-^{\bullet}) \] for \( n \gg 1 \).

Let \( \tilde{f}, \tilde{\iota} : (X_2, \tilde{A}_2) \rightrightarrows (X_1, \tilde{A}_1) \) be the correspondence between finite cactoids constructed in Section 6.2. Then, by isotyping \( \iota \), we can assume that this correspondence is topologically expanding and \( \iota, \Pi_2, \tilde{\iota} \) still satisfy Conditions (i),(ii),(iii) from Lemma 6.2 and piece-wise Conditions (PW1),(PW2),(PW3).

Observe that Assumptions (20) in Lemma 6.2 imply that every periodic cycle of \( f \circ \iota^* : I_\circ \supset \) is either in \( I_\circ \) or in \( I_* \), so Condition (A) makes sense.

Proof. We have already shown how to satisfy Conditions (i),(ii),(iii),(PW1),(PW2) (PW3). It remains to isotope \( \iota \) rel. \( \partial S_{\text{blow}} \) so that the induced correspondence is topologically expanding.

By GHMZ18, BD18, collapsing all Fatou components in a Sierpiński map \( g : (S^2, B) \supset \) results in a totally expanding map \( \bar{g} : (S^2, B) \). Assumption (A) implies that we can isotope \( \bar{\iota} \) within small spheres so that \( f \circ \iota^* \) is expanding on small spheres in \((X_1, \tilde{A}_1)\). This defines isotopy of \( \tilde{\iota} \) within \( S_{\text{blow}} \).

Let \( \gamma \) be a curve in a primitive bicycle in \( C_- \) and denote by \( T_\gamma \) the corresponding segment in the cactoid \((X_1, \tilde{A}_1)\). Assumptions of Lemma 6.2 imply that we can modify \( \bar{\iota} \) by an isotopy so that \( f \circ \iota^* \) is piecewise linear on each segment in the cactoid. Assumption (B) implies that \( f \circ \iota^* \) is expanding on each segment \( T_\gamma \). We then lift this isotopy of \( \bar{\iota} \) to an isotopy of \( \tilde{\iota} \) on \( A \) rel. \( \partial A \).

Since \( f \circ \iota^* \) is expanding on small spheres and segments, the pair \( \tilde{\iota}, \tilde{\iota} \) defines a topologically expanding correspondence.

\[ \square \]

6.3.1. A semi-conjugacy from \( f : S^2 \supset \) to \( \tilde{f} : X_\infty \supset \). Below, we follow the notation from Section 6.2.2

Lemma 6.5 (Pullback argument). Under the assumptions of Lemma 6.4, the limits

\[ \rho_m = \lim_{n \to \infty} \bar{\iota}^n \circ \Pi_{n+m} : S^2 \to X_m \]

exist as monotone maps and they semi-conjugate \( f \) to \((\tilde{f}, \tilde{\iota})\):

\[ \rho_m \circ f = \tilde{f} \circ \rho_{m+1} \quad \text{and} \quad \rho_m = \tilde{\iota} \circ \rho_{m+1}. \]

Moreover, \( \rho_1 \) is piece-wise rel. \( S_{\text{blow}}, A \): it maps spheres of \( S_{\text{blow}} \) onto spheres or singletons of \( X_1 \) and we have \( \rho_1(A) \subset A_1 \cup \{ \text{segments of } X_1 \} \).

Proof. The existence of \( \rho_m \) follows from the Pullback Argument, see Section 2.4.2. Namely, by Corollary 6.3 we have a homotopy \( h_1 \) rel. \( A \) between \( \bar{\iota} \circ \Pi_2 \) and \( \Pi_1 \). Moreover, we can assume that \( h_1 \) respects the partition by \( S_{\text{blow}}, A \) because of Conditions (i),(ii),(iii),(PW1),(PW2),(PW3). Since \( f : (S^2, \Pi^{-1}(A) \cup \Pi^{-1}(C)) \) is a covering map, we can lift \( h_1 \) into a homotopy \( h_n \) between \( \bar{\iota} \circ \Pi_{n+1} \) and \( \Pi_n \). By the expansion of \( \bar{f}, \tilde{\iota} \), the tracks of \( h_n \) decrease exponentially.
As a corollary, we obtain that $\rho_n$ induce a semi-conjugacy $\rho_\infty \coloneqq \lim_n \rho_n : S^2 \to X_\infty$ from $f : S^2 \circlearrowleft \to \bar{f} : X_\infty \circlearrowleft$:

\[
\begin{array}{ccc}
S^2 & \xrightarrow{f} & S^2 \\
\rho_\infty & \downarrow & \rho_\infty \\
X_\infty & \xrightarrow{\bar{f}} & X_\infty
\end{array}
\]

(26)

Recall from Section 3.2 that $K_i$ and $J_i$ denote the non-escaping sets of $f|S_{\text{blow}}$ and $f|A$, respectively. Moreover, $K_{S,i}$ denotes the component of $K_i$ the sphere indexed by $i$. And $J_{AS}$ denotes the set of points in $J_i$ with orbits in $A_{\Sigma}$, where $\Sigma$ is a strongly connected component of $C$.

**Lemma 6.6.** Under the assumptions of Lemma 6.4, we have

(I) for every $i \in I_\Sigma$, the map $\rho_\infty | K_{S,i}$ collapses exactly all the Fatou components of $K_{S,i}$ into a sphere of $X_1$ (i.e., there are no extra identifications);

(II) for every bicycle $\Sigma \subset C$, the map $\rho_\infty | J_{AS}$ collapses every circle into a point so that buried circles in $J_{AS}$ are collapsed into different points.

**Proof.** Recall from Lemma 6.5 that $\rho_1$ is piece-wise rel. $S_{\text{blow}}, A$. Since $\rho_1 | K_{S,i}$ is obtained by running the pullback argument on $\Pi_i | S_i$, where $S_i$ is the component of $\tilde{S}_{\text{blow}}$ indexed by $i$, the map $\rho_1$ collapses exactly the Fatou components of $K_{S,i}$ for $i \in I_\Sigma$ – this is (I). Similarly, since $\rho_1 | J_{AS}$ is obtained by running the pullback argument on $\Pi_i | A_{\Sigma}$, it follows from the properties of the cactoid correspondence on segments (see (iii)) that the limiting map $\rho_1$ satisfies (II)

6.3.2. A semi-conjugacy from $f_{BE} : S^2 \circlearrowleft \to \bar{f} : X_\infty \circlearrowleft$. Let $\tau : S^2 \to S^2$ be a continuous monotone map providing a semi-conjugacy from $f$ to $f_{BE}$, see Section 2.4.2. By Proposition 3.1, equivalence classes $\tau^{-1}(z)$ consist of homotopy equivalent points. Since $f : X_\infty \to X_\infty$ is expanding, $\rho_\infty$ is constant on homotopy equivalent points, and thus we have the induced semi-conjugacy from $f_{BE}$ to $\bar{f}$:

\[
\pi_f \coloneqq \rho_\infty \circ \tau^{-1} : S^2 \to X_\infty, \quad \pi_f \circ f_{BE} = \bar{f} \circ \pi_f.
\]

We also have the induced semi-conjugacies on cactoid correspondences:

\[
\pi_{f,m} \coloneqq [X_\infty \to X_m] \circ \pi_f, \quad \pi_{f,m} \circ f_{BE} = \bar{f} \circ \pi_{f,m+1}, \quad \pi_{f,m} = \iota \circ \pi_{f,m+1}.
\]

7. Proofs of the main results

We are now ready to prove the main results of this paper stated in the Introduction. We start with Theorem A

Let $f : (S^2, A) \circlearrowleft$ be a Böttcher expanding map with $F(f) \neq \emptyset$, and $C_{\text{dec}}$ be the crochet multicurve constructed by the Crochet Algorithm (see Section 5.3). We will assume that $T$ parametrizes the small spheres of $(S^2, A, C_{\text{dec}})$ and $f : T \circlearrowleft$ provides the dynamics on the small spheres. Set
\begin{itemize}
  \item \(C_\ast\) to be the invariant sub-multicurve of \(C_{\text{dec}}\) generated by all its bicycles;
  \item \(C_\ast = C_{\text{dec}} \setminus C_\ast\);
  \item \(I_\circ\) to be the full orbit of small spheres corresponding to Sierpinski small maps;
  \item \(I_\ast\) to be the full orbit of small spheres corresponding to small crochet maps.
\end{itemize}
By Section 6.3.2 the collapsing data \(C_{\text{dec}} = C_\ast \cup C_\ast\) and \(I = I_\ast \cup I_\circ\) induces a totally topologically expanding map \(f: X_f \supset \) on a cactoid \(X_f = X_{\text{c}}\) together with the semi-conjugacy \(\pi_f\) from \(f\) to \(\tilde{f}\). Recall also that \(\sim_{\mathcal{F}(f)}\) denotes the smallest closed equivalence relation on \(S^2\) generated by identifying all points in every Fatou component of \(f\).

**Theorem 7.1.** Let \(f: (S^2, A) \supset \) be a Böttcher expanding map with \(\mathcal{F}(f) \neq \emptyset\), and \(C_{\text{dec}}\) be the crochet multicurve constructed by the Crochet Algorithm (Corollary 5.9). Suppose that \(\tilde{f}: X_f \supset \) is the induced cactoid map together with the semi-conjugacy \(\pi_f: S^2 \rightarrow X_f\) as above. Then
\[
\pi_f(x) = \pi_f(y) \iff x \sim_{\mathcal{F}(f)} y; \text{ i.e., } X_f \cong S^2/\sim_{\mathcal{F}(f)}.
\]
Moreover, \(\pi_f\) is the maximal expanding quotient: every semi-conjugacy \(\pi_g: S^2 \rightarrow Y\) from \(f\) to a totally topologically expanding map \(g: Y \supset \) factorizes through \(\pi_f:\)
\[
\begin{array}{ccc}
T & \supset & X_f \\
\pi_f & \downarrow & \pi_g \\
& \supset & Y \\
& \pi_g & \downarrow \\
& \supset & \mathcal{F}(f)
\end{array}
\]

Clearly, Theorem 7.1 establishes Theorem A from the Introduction.

**Proof.** Let \(\sim\) be the equivalence relation on \(S^2\) induced by \(\pi_f\). We need to show that \(\sim \supset \sim_{\mathcal{F}(f)}\).

**Claim.** If \(\pi_g: S^2 \rightarrow Y\) is a semi-conjugacy from \(f\) to a totally topologically expanding map \(g: Y \supset \) then \(\pi_g\) must collapse each Fatou component of \(f\). In particular, \(\sim \supset \sim_{\mathcal{F}(f)}\).

**Proof.** Indeed, it is sufficient to show this for each periodic Fatou component \(F\). Let \(\alpha\) be a periodic internal ray in \(F\), and suppose that \(\pi_g(\alpha)\) has at least two points. Let \(U_0\) be a (small) open set that covers \(\pi_g(c_F)\) where \(c_F\) is the center of \(F\). Let \(\tilde{U}_0\) the component of \(\pi_g^{-1}(U_0)\) that covers \(c_F\), and \(\tilde{U}_n\) be the component of \(f^{-nk}(\tilde{U}_0)\) that covers \(c_F\) (here \(k = \text{per}(c_F)\) is the period of \(c_F\)). Note that \(\bigcup \tilde{U}_n\) covers \(\text{int}(\alpha)\), but the diameter of \(\pi_g(\tilde{U}_n)\) tends to 0, since \(g\) is expanding. Consequently, \(\pi_g(\alpha)\) is a singleton, and \(\pi_g\) must collapse (the closure of) \(F\) to a point. \(\square\)

Let us show the other inclusion: \(\sim \supset \sim_{\mathcal{F}(f)}\). Let \(f_{\mathcal{FA}}: (S^2, A, C_{\text{dec}}) \supset \) be the formal amalgam for \(f: (S^2, A, C_{\text{dec}}) \supset \) used in the construction of \(\pi_f\), see Section 6.3.2 and note that the maps \(f_{\mathcal{FA}}, f\) were denoted there by \(f_1, f_{BE}\), respectively. We denote by \(f_{\mathcal{FA}}: (S^2, f^{-n}(A), f^{-n}(C_{\text{dec}})) \supset \) the iteration of \(f_{\mathcal{FA}}\) and by \(K_n, J_{\mathcal{FA}}^n\) the associated sphere and annuli non-escaping sets, see Sections 3.2.2 and 3.2.4. We also denote by \(\rho = \rho_{\mathcal{FA}}\) the semi-conjugacy from \(f_{\mathcal{FA}}\) to \(\tilde{f}\) and by \(\tau\) the semi-conjugacy from \(f_{\mathcal{FA}}\) to \(f\) so that \(\pi_f = \rho \circ \tau^{-1}\).
First, we claim that \( \sim_{f(f)} \) collapses crochet components in \( K_{\tilde{S}^n} \) to points, as well as connected components of \( J_{A^n} \). Since \( C_{\text{dec}} \) is generated by the boundaries of crochet components, it is sufficient to show the claim about the crochet components.

Let us apply the Crochet Algorithm to \( f \). Suppose \( C^k \) is the invariant multicurve in \((S^2, A)\) obtained after the \( k \)-th iteration of the recursive Step 3 of the algorithm. Let \( S^1 \) be a small crochet sphere wrt. \( C^1 \) containing a maximal cluster \( K^1 \). Note that \( K^1 \) may be recognized as the set of points in the crochet sphere \( S^1 \) that do not escape to the Fatou components corresponding to the curves in \( C^1 \). By definition of \( \sim_{f(f)} \), \( K^1 / \sim_{f(f)} \) is a singleton, and thus \( J(S^1) / \sim_{f(f)} \) is a singleton as well, where \( J(S^1) \) is the small Julia set associated with \( S^1 \). The same is true for every component of \( f^{-n}(K^1), n \geq 1 \).

Consider now the multicurve \( C^2 \supset C^1 \) obtained after the second iteration of the algorithm. Suppose \( S^2 \) is a small crochet sphere wrt. \( C^2 \) that does not appear after the first iterate. Then \( S^2 \subset S^1 \) for a small sphere wrt. \( C^1 \). The sphere \( S^2 \) corresponds to a maximal cluster \( K^2 \) in \( S^1 \). Note that this cluster is constructed using Fatou components of \( f \) and the Fatou components corresponding to the curves in \( C^1 \). Note that \( \sim_{f(f)} \) collapses the boundary of each such Fatou component. Consequently, \( \sim_{f(f)} \) collapses the small Julia set \( J(S^2) \). By induction, \( \sim_{f(f)} \) collapses each crochet component wrt. \( C^k \).

By the following Claim (combined with Lemma 3.6), the fibers of \( \sim_{f(f)} \) have disjoint images in \( X_{\tilde{z}} \). This completes the proof of Theorem 7.3

Claim. For all \( n \geq 0 \), we have the following properties:

1. If \( \beta \) is a buried curve in \( J_{A^n} \), then \( \rho(\beta) \) is disjoint from \( \rho(K_{\tilde{S}^n} \cup (J_{A^n} \setminus \beta)) \).
2. Two non-neighboring disjoint components \( K_{\tilde{S}^n,i}, K_{\tilde{S}^n,j} \) of \( K_{\tilde{S}^n} \) have disjoint images \( \rho(K_{\tilde{S}^n,i}), \rho(K_{\tilde{S}^n,j}) \) unless \( K_{\tilde{S}^n,i}, K_{\tilde{S}^n,j} \) have a common crochet neighbor \( K_{\tilde{S}^n,k} \); in the exceptional case, \( \rho(K_{\tilde{S}^n,i}), \rho(K_{\tilde{S}^n,j}) \) are small spheres with a single common point \( \rho(K_{\tilde{S}^n,k}) \).
3. For a crochet component \( K_{\tilde{S}^n,i} \) of \( K_{\tilde{S}^n} \), its image \( \rho(K_{\tilde{S}^n,i}) \) is disjoint from \( \rho(J_{A^n} \setminus \{\text{neighbors of } K_{\tilde{S}^n,i}\}) \).
4. For a Sierpiński component \( K_{\tilde{S}^n,i} \) of \( K_{\tilde{S}^n} \), its image \( \rho(K_{\tilde{S}^n,i}) \) is disjoint from \( \rho(J_{A^n} \setminus \{\text{neighbors of } K_{\tilde{S}^n,i}\}) \).
5. The image \( \rho \left( S^2 \setminus \bigcup_{n \geq 0} (K_{\tilde{S}^n} \cup J_{A^n}) \right) \) is totally disconnected and is disjoint from \( \rho \left( \bigcup_{n \geq 0} \tau(K_{\tilde{S}^n} \cup J_{A^n}) \right) \).

Proof. Since every primitive unicycle is a neighbor to at least one Sierpiński sphere, Sierpiński spheres and bicycles are dense in the following sense: for every \( n \geq 0 \) iterated preimages of bicycles and Sierpiński spheres separate

- every buried curve \( \beta \) in \( J_{A^n} \) from every other curve in \( J_{A^n} \) and every component of \( K_{\tilde{S}^n} \);
- non-neighboring disjoint components \( K_{\tilde{S}^n,i}, K_{\tilde{S}^n,j} \) of \( K_{\tilde{S}^n} \) from \( \rho(J_{A^n} \setminus \{\text{neighbors of } K_{\tilde{S}^n,i}\}) \);
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• a Sierpiński component $K_{S,n,i}$ from $\mathcal{J}_A$\(\backslash\)\{neighbors of neighbors of $K_{S,n,i}$\}. Since the $\rho$-images of Sierpiński components are small spheres in $X$ and the $\rho$-images of bicycles contain Cantor sets in $X$\(\backslash\)\{small spheres of $X$\} (Lemma 6.6), we obtain the separation properties (1) (2) (3) (4) these properties also imply (5). □

Remark 7.2. Let $\mathcal{C}$ be a pre-crochet multicurve for $f$, and suppose that $\mathcal{D}$ is the associated crochet multicurve (obtained by the iterative elimination of all primitive crochet unicycles from $\mathcal{C}$; see Section 5.1). Similarly to the proof of Theorem 7.1, we may show that the collapsing data determined by $\mathcal{D}$ generates the cactoid $S^2/\sim_f$, which implies that $\mathcal{D}$ coincides with $\mathcal{C}_{\text{dec}}$ (up to isotopy), i.e., the crochet multicurve for $f$ is uniquely defined.

We now present several results that provide topological characterizations of Sierpiński maps, crochet maps, and Sierpiński-free maps (i.e., Böttcher expanding maps with non-empty Fatou sets and without Sierpiński small maps in every decomposition). First, we recall from the Introduction the following characterizations of Sierpiński maps (Proposition 1.1).

Proposition 7.3. Let $f : (S^2, A) \ni \mathcal{F}(f) \neq \emptyset$. Then the following are equivalent:

(i) $f$ is a Sierpiński map, i.e., $\mathcal{J}(f)$ is homeomorphic to the standard Sierpiński carpet;
(ii) $S^2/\sim_f$ is a sphere and $\sim_f$ is trivial on $A$;
(iii) every connected periodic zero-entropy graph is homotopically trivial rel. $A$.

Proof. Follows easily from Whyburn’s characterization [Why58], Moore’s theorem [Moo25, Why42], and [BD18, Section 4.5]. □

The next result characterizes crochet maps (Theorem B).

Theorem 7.4. Let $f : (S^2, A) \ni \mathcal{F}(f) \neq \emptyset$. Then the following are equivalent:

(i) $f$ is a crochet map, that is, there is a connected forward-invariant zero-entropy graph $G$ containing $A$;
(ii) $S^2/\sim_f$ is a singleton;
(iii) every two points in $A$ may be connected by a path $\alpha$ with $\alpha \cap J(f)$ being countable.

Proof. (i)$\Rightarrow$(ii), (iii) Let $G$ be a connected forward-invariant 0-entropy graph spanning the set of marked points $A$. Recall that $G$ is composed of periodic and preperiodic internal rays, thus (iii) follows. For each point $a \in A^\infty$, let $F_a$ be the Fatou component of $f$ centered at $a$. Consider the connected set $K := G \cup \bigcup_{a \in A} F_a$. Note that the preimage $f^{-n}(K)$ is also connected for each $n \geq 0$. By definition of $\sim_f$, all points in $K$, and thus in $f^{-n}(K)$, are equivalent to each other. Now let $V$ be any complementary component of $K$. By expansion, the diameter of each component of $f^{-n}(V)$ tends to 0 as $n \to \infty$. Since the equivalence relation $\sim_f$ is closed, the quotient $S^2/\sim_f$ is a singleton, and (ii) follows.

(i)$\Rightarrow$(ii), (iii) Suppose that $f$ is not crochet, that is, it does not admit a connected forward-invariant 0-entropy graph spanning $A$. Let us apply the Crochet
Algorithm to \( f \) producing a crochet multicurve \( \mathcal{C}_{\text{dec}} \). By Lemma 5.5 there is either at least one Sierpiński small map in the decomposition wrt. \( \mathcal{C}_{\text{dec}} \) or at least one primitive bicycle in \( \mathcal{C}_{\text{dec}} \). Consequently, the quotient cactoid \( S^2/\sim_{\mathcal{J}(f)} \) contains a small sphere or a segment, and \(-\text{(ii)}\) follows. Let \( \tau: S^2 \to S^2 \) be a continuous monotone map proving a semiconjugacy from a formal amalgam \( f_{FA} \) induced by small maps wrt. \( \mathcal{C}_{\text{dec}} \) to \( f \) (Section 2.4.2).

Suppose that the decomposition of \( f \) along \( \mathcal{C}_{\text{dec}} \) contains a Sierpiński small map \( \hat{f}: \hat{S} \subseteq \hat{J} \). Then the Julia set \( \mathcal{J}(f) \) contains the image \( \tau(\mathcal{J}(\hat{f})) \). In fact \( \tau \) embeds \( \text{int}(\mathcal{J}(\hat{f})) \) into \( \mathcal{J}(f) \) by Theorem 3.5. Suppose \( a_1, a_2 \in A \) are two marked points in distinct components of \( S^2\setminus\tau(\mathcal{J}(f)) \). Then any path \( \alpha \) connecting \( a_1 \) and \( a_2 \) intersects \( \mathcal{J}(f) \) in uncountably many points, and \(-\text{(iii)}\) follows.

Suppose that the multicurve \( \mathcal{C}_{\text{dec}} \) contains a primitive bicycle \( \Sigma \). Following the discussion in Section 3.2.4, let \( \mathcal{A}_{\Sigma} := \bigcup_{q \in \Sigma} \mathcal{A}_q \) be the union of annuli that are homotopic to curves in \( \Sigma \) in the construction of the formal amalgam \( f_{FA} \). The non-escaping set \( \mathcal{J}_\Sigma \) of \( f_{FA} : \mathcal{A}_\Sigma \cap f^{-1}(\mathcal{A}_\Sigma) \to \mathcal{A}_\Sigma \) is a direct product between a Cantor set and \( S^1 \). By Theorem 3.5, the set \( \mathcal{J}_\Sigma \) contains uncountably many buried curves that are sent by \( \tau \) injectively into \( \mathcal{J}(f) \). Choose now any two marked points \( a_1, a_2 \in A \) on different sides of a curve \( \gamma \in \Sigma \). Then any path \( \alpha \) connecting \( a_1 \) and \( a_2 \) must meet \( \tau(\mathcal{J}_\Sigma) \cap \mathcal{J}(f) \) in uncountably many points, thus \(-\text{(iii)}\) follows.

Next we observe the following separation property for crochet maps.

**Lemma 7.5.** Let \( f: (S^2, A) \to (S^2, A) \) be a crochet map. Then there exists a countable set \( Z \subseteq \mathcal{J}(f) \) that separates every two distinct points \( x, y \in \mathcal{J}(f) \), that is, \( x \) and \( y \) lie in different components of \( \mathcal{J}(f) \setminus (Z \setminus \{x, y\}) \).

**Proof.** Let \( G \) be a connected forward-invariant 0-entropy graph spanning the set of marked points \( A \). As in the proof of Theorem 7.4 we set \( K := G \cup \bigcup_{a \in A^\infty} F_a \), where \( F_a \) is the Fatou component of \( f \) centered at \( a \in A^\infty \).

Suppose now that \( x, y \) are two distinct points in \( \mathcal{J}(f) \). Choose \( n \) so that

\[
\max\{\text{diam}(V^n) : V^n \text{ is a component of } f^{-n}(S^2\setminus K)\} \leq \text{dist}(x, y)/3.
\]

Set \( G_n := f^{-n}(G) \), and let \( N_x \) be the union of all components \( W^n \) of \( S^2\setminus G_n \) with \( x \in W^n \). Then \( y \notin N_x \) and \( (N_x \cap G_n) \cap \mathcal{J}(f) \) is a countable set separating \( x \) and \( y \).

It follows that the countable set \( Z := \mathcal{J}(f) \cap \bigcup_n G_n \) separates any two points in the Julia set of \( f \).

We also note the following property of **iterated monodromy groups** of crochet maps. (We assume that the reader is familiar with the relevant terminology; see, e.g., [Nek05].)

**Lemma 7.6.** Let \( f: (S^2, A) \to (S^2, A) \) be a crochet map. Then the iterated monodromy group of \( f \) is generated by a polynomial growth automaton with respect to some groupoid basis.

**Proof.** Let \( G \) be a connected forward-invariant 0-entropy graph spanning the set of marked points \( A \), and suppose that \( E \) and \( W \) denote the sets of edges and faces of \( G \), respectively. First, for each face \( U \) of \( G \) we choose a basepoint \( t_U \in U \). Then, we choose the connecting paths in the following way: each basepoint \( t_U \) is connected to each preimage in \( f^{-1}(\{t_U : U \in W\}) \cap U \) by a path that does not intersect the graph
Proof. (i) Let \( e \) be an edge of \( G \) on the boundary of two faces \( U, U' \) (if \( e \) is on the boundary of a unique face). Choose a path \( g_e \) that connects the basepoints \( t_U \) and \( t'_{U'} \) and intersects the graph \( G \) exactly once in \( \text{int}(e) \). Let \( \mathcal{G} \) be the groupoid generated by \( \{ g_e : e \in E \} \) (which acts on the iterated preimages of the basepoints). Since \( f \mid G \) has 0-entropy, the action of \( \mathcal{G} \) is described by an automaton of polynomial activity growth. The statement follows. \( \square \)

The following result provides equivalent characterizations of Sierpiński-free maps (Theorem [E]).

**Theorem 7.7.** Let \( f : (S^2, A) \circlearrowleft \) be a Böttcher expanding map with \( \mathcal{F}(f) \not= \emptyset \). Then the following are equivalent:

(i) \( f \) is Sierpiński-free, i.e., the decomposition of \( f \) with respect to every invariant multicurve \( \mathcal{C} \) does not produce a Sierpiński small map;

(ii) none of the small maps in the decomposition of \( f \) along the crochet multicurve \( \mathcal{C}_{\text{dec}} \) is a Sierpiński map;

(iii) \( S^2 / \sim_{\mathcal{F}(f)} \) is a dendrite.

**Proof.** (i)\( \Rightarrow \) (ii) This is immediate.

(ii)\( \Rightarrow \) (iii) By Theorem 7.1, the cactoid \( S^2 / \sim_{\mathcal{F}(f)} \) coincides with the quotient \( X_f \) of \( S^2 \) under the semi-conjugacy \( \pi_f \) induced by the respective collapsing data \( \mathcal{C}_{\text{dec}} = \mathcal{C} \uplus \mathcal{C}_f \) and \( I = I_\uplus \cup I_f \). Since \( I_\uplus = \emptyset \) by (i), it follows that \( X_f \) is the inverse limit of the dendroid cactoids \( X_n \) (see Section 6), which implies (ii).

(iii)\( \Rightarrow \) (i) Suppose there is an invariant multicurve \( \mathcal{D} \) with a small Sierpiński map \( \hat{f} : \hat{S} \circlearrowleft \) in the associated decomposition. Let \( J \) be the index set parametrizing the small spheres of \((S^2, A, \mathcal{D})\), and \( J_\uplus \subset J \) be the orbit of small spheres induced by the small map \( \hat{f} : \hat{S} \circlearrowleft \). Set \( J_\uplus := J \cap J_\uplus , \mathcal{D}_\uplus := \mathcal{D} \uplus \mathcal{D}_\uplus , \mathcal{D} \uplus := \emptyset \). By the discussion in Section 6 the collapsing data \( \mathcal{D} = \mathcal{D}_\uplus \cup \mathcal{D}_\uplus \) and \( J = J_\uplus \cup J_f \) induces a totally topologically expanding map \( g : Y \circlearrowleft \) on a cactoid \( Y \) together with the semi-conjugacy \( \pi_g : S^2 \rightarrow Y \) from \( f \) to \( g \). By Theorem 7.1, \( \pi_g \) should factor through \( \pi_f : S^2 \rightarrow X_f = S^2 / \sim_{\mathcal{F}(f)} \). This is clearly not possible since \( X_f \) is a dendrite while \( Y \) contains small spheres. This finishes the proof. \( \square \)

Let \( \mathcal{C}_{\text{dec}} \) be the crochet multicurve constructed by the Crochet Algorithm. Our goal now is to provide an alternative characterization for \( \mathcal{C}_{\text{dec}} \) given by Theorem [D].

We may naturally subdivide \( \mathcal{C}_{\text{dec}} \) into two (possibly, empty) submulticurves \( \mathcal{C}_{\text{Sie}} \) and \( \mathcal{C}_{\text{bi}} \) in the following way: \( \mathcal{C}_{\text{Sie}} \) is the sub-multicurve generated by the boundaries of all small Sierpiński spheres of \((S^2, A, \mathcal{C}_{\text{dec}})\); and \( \mathcal{C}_{\text{bi}} := \mathcal{C}_{\text{dec}} \setminus \mathcal{C}_{\text{Sie}} \). Note that all primitive components of \( \mathcal{C}_{\text{bi}} \) are bicycles.

**Lemma 7.8.** Let \( f : (S^2, A) \circlearrowleft \) be a Böttcher expanding map with \( \mathcal{F}(f) \not= \emptyset \) and \( \mathcal{D} \) be an invariant multicurve. Then the following are true:

(i) If \( \tilde{S} \) is a Sierpiński sphere wrt. \( \mathcal{D} \), then \( \tilde{S} \subset S \) (up to homotopy) for a Sierpiński small sphere \( S \) wrt. \( \mathcal{C}_{\text{dec}} \).

(ii) If \( \Sigma \subset \mathcal{D} \) is a bicycle, then \( \Sigma \subset \mathcal{C}_{\text{dec}} \) or \( \Sigma \) is inside a Sierpiński small sphere wrt. \( \mathcal{C}_{\text{dec}} \) (up to homotopy).

**Proof.** Let us construct \( \mathcal{C}_{\text{dec}} \) using the Crochet Algorithm, see Section 5.3. Then, on each step of the recursive construction of the pre-crochet multicurve, the small Sierpiński Julia set (of \( \tilde{S} \)) or bicycles (of \( \mathcal{D} \)) cannot cross the 0-entropy graphs constructed during this step. Consequently, the sphere \( \tilde{S} \) is inside Sierpiński spheres.
wrt. \( \mathcal{C}_{\text{dec}} \). Similarly, every curve \( \gamma \in \Sigma \) is either in \( \mathcal{C}_{\text{dec}} \) or inside a Sierpiński small sphere wrt. \( \mathcal{C}_{\text{dec}} \).

The lemma above immediately implies Theorem 1.1 from the Introduction. We also record the following easy corollary.

**Corollary 7.9.** Let \( f \) be a Böttcher expanding map with non-empty Fatou set. Then each small sphere \( \hat{f} : \hat{S} \varsubsetneq \hat{S} \) in the decomposition of \( f \) along \( \mathcal{C}_{\text{Sie}} \) is Sierpiński-free. Furthermore, the multicurve \( \hat{\mathcal{C}}_{\text{bi}} = \mathcal{C}_{\text{bi}} \cap \hat{S} \) is the crochet multicurve for \( \hat{f} \).

**Proof.** Indeed, \( \hat{f} : \hat{S} \varsubsetneq \hat{S} \) cannot contain a Sierpiński map by Lemma 7.8. Thus, \( \hat{f} \) is Sierpiński-free and, by Theorem 1.1, the decomposing curve for \( \hat{f} \) must be generated by bicycles, that is, \( \hat{\mathcal{C}}_{\text{bi}} = \mathcal{C}_{\text{bi}} \cap \hat{S} \). The statement follows. □

Finally, we prove the following result characterizing the crochet multicurve \( \mathcal{C}_{\text{dec}} \) (Theorem C).

Let \( f : (S^2, A) \varsubsetneq (S^2, A) \) be a Böttcher expanding map with an invariant multicurve \( D \). Suppose that all maps in the decomposition of \( f \) along \( D \) are Sierpiński or crochet. Let \( f_{FA} \) be the formal amalgam for \( f : (S^2, A, D) \varsubsetneq (S^2, A, D) \) and \( \tau \) be the semiconjugacy from \( f_{FA} \) to \( f \). Let \( K_{S,i} \) be the component of the non-escaping set \( K_{S,i} \) associated with a periodic small sphere \( S_i \) of \( (S^2, A, D) \). The set \( \tau(K_{S,i}) \cap J(f) \) is called a small Julia set of the first return map to \( S_i \).

**Theorem 7.10.** Let \( f : (S^2, A) \varsubsetneq (S^2, A) \) be a Böttcher expanding map with \( F \neq \emptyset \). There is a unique canonical invariant multicurve \( \mathcal{C}_{\text{dec}} \) whose small maps are Sierpiński and crochet maps such that for the quotient map \( \pi_f : S^2 \rightarrow S^2/\sim_{F(f)} \) the following are true:

(i) small Julia sets of Sierpiński maps project onto spheres;
(ii) small Julia sets of crochet maps project to points;
(iii) different crochet Julia sets project to different points in \( S^2/\sim_{F(f)} \).

**Proof.** The multicurve \( \mathcal{C}_{\text{dec}} \) constructed by the Crochet Algorithm satisfies the required properties (by the construction of the cactoid \( X_f \equiv S^2/\sim_{F(f)} \)).
Conversely, let \( D \) be any invariant multicurve such that each small in the decomposition of \( f \) along \( D \) is either Sierpiński or crochet and it satisfies Conditions (i), (ii), (iii) of the theorem. Condition (i) implies that \( D_{\text{Sie}} = \mathcal{C}_{\text{Sie}} \) (i.e., Sierpiński spheres are maximal). So, we reduced the statement to the Sierpiński-free case. Condition (ii) and (iii) imply that the crochet Julia sets of \( D \) are within the crochet Julia set of \( \mathcal{C}_{\text{dec}} \) and thus \( D \) is isotopic to \( \mathcal{C}_{\text{dec}} \). □

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Mathematics Department, Stony Brook University, NY 11794, USA

Email address: dzmitry.dudko@stonybrook.edu

Mathematisch Instituut, Universiteit Utrecht, 3508 TA Utrecht, The Netherlands

Email address: m.hlushchanka@uu.nl

Aix-Marseille Université and CNRS, UMR 7373, Institut de Mathématiques de Marseille, 163 Avenue de Luminy Case 901, 13009 Marseille, France

Email address: dierk.schleicher@univ-amu.fr