Gravitational fields on a noncommutative space

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Abstract

Noncommutative three-dimensional gravity can be described in terms of a noncommutative Chern-Simons theory. We extend this structure and also propose an action for gravitational fields on an even dimensional noncommutative space. The action is worked out in some detail for fields on a noncommutative $\mathbb{C}P^2$ and on $S^4$. 
1. Introduction

The dynamics of particles and fields on noncommutative spaces have been the subject of a large number of recent publications. Properties of gauge and other field theories on such spaces, their perturbative analyses, special solutions as well as the quantum mechanics of particles have been extensively studied [1]. Field theories on some intrinsically defined noncommutative space are certainly interesting but they become more relevant with the realization that noncommutative spaces arise naturally as brane configurations in the matrix model of M-theory and also as special solutions in string theory [1, 2, 3, 4]. Fluctuations of branes are generally described by gauge theories and this has been the main motivation for the study of noncommutative field theories. In this paper, we discuss the gravitational fields on a noncommutative space. In three dimensions, gravity can be described as a Chern-Simons (CS) gauge theory. Three-dimensional gravity on a noncommutative space using noncommutative CS gauge theory has been suggested in [5]. We consider CS gravity for more general odd dimensional noncommutative spaces with the time coordinate continuous and commutative. We also propose an action for even dimensional spaces where all dimensions are noncommutative. This may be relevant to noncommutative brane solutions in the matrix model for the type IIB string theory [3]. The equations of motion are an algebraic set of equations; two special cases, namely noncommutative \( \mathbb{C}P^2 \) and \( S^4 \), are considered in some detail. We also point out an intriguing possibility of dynamically changing the topologies and dimensions of the base spaces on which the theory is defined. The noncommutative branes which arise in the matrix model have an underlying Lie algebra structure and so we shall use such a framework for our presentation. This is not a real limitation. It is actually possible to go to a noncommutative space with the Heisenberg algebra as the underlying structure (such as the noncommutative plane) by taking an appropriate limit of our formulae. The construction of gravity on noncommutative \( \mathbb{R}^{2n} \) using the star product formalism has been considered before [3]. There is also the well known proposal by Connes of using noncommutative geometry for gravity [7]. It is not clear at this time if our actions are related to his proposals.

2. The data for gravity

We consider a noncommutative space with coordinates \( x_i, i = 1, 2, \ldots, d \), which are realized as \( (N \times N) \)-matrices. The commutators \( [x_i, x_j] \equiv \omega_{ij} \) can be expanded in terms of the Lie algebra \( U(N) \) of \( U(N) \). We can now consider the commutators of pairs of different \( \omega_{ij} \)'s or of \( \omega_{ij} \) with \( x_k \), to define new elements of \( U(N) \) and continue until we have a closed algebra. Then \( (x, \omega, \ldots) \) will form the \( (N \times N) \)-representation of a Lie algebra \( \mathcal{G} \) which
is naturally embedded in $U(N)$. The $x_i$’s themselves may be considered as belonging to $G - H$, describing a $G/H$-space in the commuting limit. Thus, starting with the $x_i$’s as $(N \times N)$-matrices, this is the emerging structure that we should be considering. In the following, we shall actually consider a more specialized case where $G/H$ is a symmetric space, so that $[x_i, x_j] = -\omega_{ij}$ belong to $H$. (If this is not the case, we may have to consider a larger set of $x_i$ for which this is the case and then append some algebraic conditions on the $x_i$’s.) The group $G$ is naturally embedded in $U(N)$, but in general we can find a unitary group $U(k)$ such that $G \subseteq U(k) \subseteq U(N)$. A typical example would be to consider $G = U(3) \sim SU(3) \times U(1)$ and $H = U(2) \times U(1)$ with $U(2) \subset SU(3)$ and $x_i$ represented by the matrices $t_i, t_i^\dagger \in G - H$. We can take them to be $(N \times N)$-matrices in the sequence of completely symmetric representations of $SU(3)$ with $N = \frac{1}{2}(s + 1)(s + 2)$, $s = 0, 1, ...$. In the commutative limit of large $N$ this corresponds to $CP^2$. Another example would be a noncommutative $S^4 = SO(5)/SO(4)$ described by matrices belonging to the algebra $O(5) - O(4)$. In this case, $G = SO(5) \subset U(4)$. (More details about these spaces can be found in [4, 8, 9].)

Scalar functions defined on a commutative $G/H$-space are functions on $G$ which are invariant under the subgroup $H$. Functions on a noncommutative space can be defined in a similar manner. For example, a function $f$ on the noncommutative $S^2$ can be represented as $(N \times N)$-matrices, say with elements $f_{mn}$. We may think of it as an operator $\hat{f}$ on the $N$-dimensional representation space. One presentation of such matrices, useful for large $N$ considerations, can be obtained as follows. Let $D_{mk}^{(j)}(g)$ be the Wigner $D$-functions for $SU(2)$. These are the spin-$j$ matrix representation of a group element $g = \exp(i\sigma_i \theta_i/2)$, where $\sigma_i$ are the Pauli matrices. On the $D$’s, one has the standard two sets of $SU(2)$-transformations

$$J_a \cdot D_{mk}^{(j)}(g) = \left[ D_{mk}^{(j)} \left( \frac{\sigma_a}{2} g \right) \right]_{mk}$$

$$K_a \cdot D_{mk}^{(j)}(g) = \left[ D_{mk}^{(j)} \left( g \frac{\sigma_a}{2} \right) \right]_{mk}$$

(1)

$J_a$ will correspond to the usual angular momentum. The action defined by the $K_a$’s is what is important for defining a function, derivatives, vectors, etc. In what follows, unless otherwise specified, $G, H$ will refer to these generators, the $K$’s. A function on noncommutative $S^2$ may be represented as

$$f(g, g') = \langle g | \hat{f} | g' \rangle = \sum_{mn} f_{mn} D_{mj}^{(j)}(g) D_{nj}^{(j)}(g')$$

(2)

$N = 2j + 1$ and we consider fixed value of $j$. The classical function or symbol corresponding to $\hat{f}$ is

$$f(g, g) = \langle g | \hat{f} | g \rangle = \sum_{mn} f_{mn} D_{mj}^{(j)}(g) D_{nj}^{(j)}(g)$$

(3)
Since the right index on the $\mathcal{D}$’s is fixed to be $j$, the $K_3$-value of $f$ is zero and this is invariant under $K_3$, which is $H$ in this example. $f$ may thus be considered a function on the coset $S^2 = SU(2)/U(1)$. One could also fix the right index to some other value and still have $K_3 = 0$. This means that there are different ways of representing functions on noncommutative spaces in terms of the $\mathcal{D}$-functions. We will choose (2).

On the noncommutative $S^2$ the product of functions is given by the matrix product. In terms of the representation (2), we can write this as

$$
\hat{f} \hat{h} = \int d\mu(g') \langle g|\hat{f}|g'\rangle \langle g'|\hat{h}|g''\rangle
$$

$$
= \int d\mu(g') \sum_{mnkl} f_{mn} h_{kl} \mathcal{D}_{mj}^{*}(g) \mathcal{D}_{nj}^{(j)}(g') \mathcal{D}_{kj}^{*}(g') \mathcal{D}_{lj}^{(j)}(g'')
$$

$$
= \sum_{mnl} f_{mn} h_{nl} \mathcal{D}_{mj}^{*}(g) \mathcal{D}_{lj}^{(j)}(g'')
$$

(4)

The integration measure $d\mu(g')$ is the Haar measure on $S^2 = SU(2)/U(1)$ divided by $2j + 1$. In the integration involved, one can extend this to $G = SU(2)$ since the effect of an $H$-transformation cancels out in $\mathcal{D}_{nj}^{(j)}(g') \mathcal{D}_{kj}^{*}(g')$. We can then use the orthogonality properties of $\mathcal{D}_{mk}^{(j)}(g')$ to arrive at (4).

Eventhough it is beside our main line of reasoning, it may be interesting to note that the start product on $S^2$ can be easily represented in this framework. The classical function or symbol corresponding to the product $\hat{f} \hat{h}$ is

$$(fh) = \sum_{mn} f_{mn} h_{nl} \mathcal{D}_{mj}^{*}(g) \mathcal{D}_{lj}^{(j)}(g)
$$

$$
= \sum_{mnkl} f_{mn} h_{kl} \mathcal{D}_{mj}^{*}(g) \mathcal{D}_{nk}^{*}(g) \mathcal{D}_{kr}^{*}(g) \mathcal{D}_{lj}^{(j)}(g)
$$

(5)

where we used $\mathcal{D}_{mk}^{*}(g) \mathcal{D}_{kr}^{*}(g) = \delta_{nk}$. The term $r = j$ in the summation over $r$ gives the product of the symbols for $\hat{f}$ and $\hat{h}$. The terms with $r \neq j$ may be rewritten using

$$
\mathcal{D}_{nj}^{(j)}(g) = \sqrt{\frac{(2j - s)!}{s!(2j)!}} K_s \cdot \mathcal{D}_{nj}^{(j)}(g)
$$

(6)

We can rewrite (5)

$$(fh) = \sum_{s=0}^{2j} \left[ \frac{(2j - s)!}{s!(2j)!} \right] \sum_{mn} f_{mn} \mathcal{D}_{mj}^{*}(g) K_s \mathcal{D}_{nj}^{(j)}(g) \left[ \sum_{lm} h_{kl} (K_s \mathcal{D}_{kj}^{(j)})^* (g) \mathcal{D}_{lj}^{(j)}(g) \right]
$$

(7)

This gives the star product of two functions on $S^2$.

Functions and star products on other coset spaces of unitary groups may be considered in a similar way, by writing a function as

$$f(g, g') = \langle g|\hat{f}|g'\rangle = \sum_{MN} f_{MN} \mathcal{D}_{MS}^{*}(g) \mathcal{D}_{NS}^{*}(g')
$$

(8)
where \( r \) denotes a fixed representation and the other indices are composite indices labelling the states uniquely. \( \hat{f} \) must be invariant under the \( H \)-subgroup. As mentioned before, for \( \mathbb{CP}^2 \), the representations are sequences of totally symmetric ones. For the right indices on the \( D \)'s, indicated by \( S \) above, we choose the highest weight states; \( \hat{f} \) must have with overall invariance under \( H \).

Derivatives on functions are defined by commutators with the \( K \)'s in \( G - H \), i.e.,

\[
\partial_i f = [K_i, f]
\]

(9)

The commutator of two such derivatives is given by an element of \( H \), say \([K_i, K_j] = C_{ij}^a K_a\), \( K_a \in H \), and so it vanishes on a scalar function \( f \) (which is invariant under \( H \)).

\[
(\partial_i \partial_j - \partial_j \partial_i) \cdot f = 0
\]

(10)

In describing gravitational fluctuations, we need to go beyond scalar functions to vectors, tensors, etc. These have a frame dependence and so they do not commute with elements of \( H \). They are specified by the representation according to which they transform under the \( H \)-action. They are thus of the form

\[
f^{(K)}(g, g') = \sum_{PQ} f_{PQ} D^{(r)}_{P, P'}(g) D^{(r)}_{Q, Q'}(g')
\]

(11)

where the right indices \( P', Q' \) no longer correspond to the same highest weight value but are such that \( f \) transforms as some nontrivial representation \( K \) under the action of \( H \). In taking products as in (4), the integration measure will be that of \( G/H \). In the example of the noncommutative \( S^2 \), for \( K_3 = -1 \), we have

\[
f(g, g') = \sum_{mn} f_{mn} D^{(j)}_{m, m}(g) D^{(j)}_{n, n-1}(g')
\]

(12)

The corresponding symbol may be written as

\[
f \sim f^{\alpha_1 \alpha_2 \ldots \alpha_j}_{\alpha' \alpha'' \ldots \alpha_{j-1}} \tilde{u}^{\alpha_1} \tilde{u}^{\alpha_2} \ldots \tilde{u}^{\alpha_j} u_{\beta_1} u_{\beta_2} \ldots u_{\beta_{j-1}} \tilde{u}^{\gamma} \epsilon_{\gamma \beta_j} e^{-i\theta/2}
\]

(13)

using the parametrization

\[
g = \begin{pmatrix} u_1 & u_2^* \\ u_2 & -u_1^* \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}
\]

(14)

Here \( \alpha_i, \beta_i \) take the values 1, 2. Because of the extra \( \tilde{u} \) in (13), the lowest spin value (for the action of \( J_a \)) is 1 rather than zero. \( f \) in (13) also has a natural interpretation as rectangular matrices. (Star products, derivatives and vector bundles have also been discussed in [10]). Tensor analysis on some quantum spaces was considered in [11].)
What we have given above is a description in terms of local coordinates. For some of the spaces, one can actually have a more global description by considering all the generators of $G$, but with the specification of some algebraic restrictions on them. For example, it is well known that in the case of $S^2$, we can use $SU(2)$ generators $J_i$, $i = 1, 2, 3$, with $x_i = J_i/\sqrt{2}$ obeying $x_i x_i = 1$. For $\mathbb{CP}^2$, an analogous description is given by considering eight matrices $X_A$, $A = 1, 2, \ldots, 8$, with $\sum_A X_A X_A = 2/3$ and $3\sqrt{2}L^2 - L - \sqrt{2}/3 = 0$ where $L = \sum_A X_A \lambda_A$, $\lambda_A$ being the standard Gell-Mann matrices of $SU(3)$.

Vectors and tensors, as described above, are in the the analogue of the coordinate basis. In the following, while using this coordinate basis for derivatives, etc., we shall introduce the gravitational degrees of freedom in terms of the tangent frame and the group acting on it. Towards this, introduce a covariant derivative

$$D_i = \partial_i + e_i^a(x) T^a + \Omega_i^\alpha I^\alpha \quad (15)$$

with a gauge field $A_i(x) = e_i^a(x) T^a + \Omega_i^\alpha I^\alpha$. Here $T^a$ are the analogues of the $t_i$ but acting on the tangent frames. $(e_i^a(x), \Omega_i^\alpha(x))$ are functions of $x_i$ which must also behave as vectors in the coordinate basis. In the noncommuting case, since $(e_i^a(x), \Omega_i^\alpha(x))$ are not mutually commuting functions, the gauge group must be a unitary group so that $[D_i, D_j]$ is an element of its Lie algebra. In particular, $(T^a, I^\alpha)$ must belong to the fundamental representation of this group. $T^a$ being the analogues of $t_i$, we take this group to be a copy of $U(k)$, which we denote by $U_R(k)$ to distinguish it from the $U(k) \subseteq U(N)$ discussed earlier (containing $G$) which will be denoted by $U_L(k)$ from now on. $(T^a, I^\alpha)$ then form a basis for $U_R(k)$. At this stage we are led to $U_L(k) \times U_R(k)$. The coordinates $x_i \in \mathcal{G} - \mathcal{H}$ and the tangent frame group is $U_R(k)$. Since we also have an embedding of $U_L(k)$ in $U(N)$, $D_i$ actually belong to the algebra of $U(N) \times U(k) = U(N) \times U_R(k)$.

The commutator of two covariant derivatives can be expanded as

$$[D_i, D_j] = \omega_{ij} + T_{ij}^a T^a + R_{ij}^\alpha I^\alpha \quad (16)$$

where $[\partial_i, \partial_j] = \omega_{ij} = C_{ij}^a K_a$, $T_{ij}^a$ is identified as the torsion tensor and $R_{ij}^\alpha$ is the Riemannian curvature. If we take $e_i^a = \delta_i^a$ and $\Omega_i^\alpha = 0$, then $D_i = \partial_i + T_i$. Acting on a vector of the form $V = V^k T_k$, we find

$$[D_i, D_j] \cdot V = [[T_i, T_j], T_k] V^k \quad (17)$$

where we used the fact that $V^k$ are functions and so $[[\partial_i, \partial_j], V^k] = 0$. This equation identifies the curvature as $[T_i, T_j]$, which is not zero in general. Notice that $e_i^a = \delta_i^a$, $\Omega_i^\alpha = 0$ does not correspond to flat space. The matrices $(e_i^a(x), \Omega_i^\alpha(x))$ are the gravitational degrees of freedom on the noncommuting space. (Usually the gravitational degrees of freedom are
expressed in terms of deviations from a flat space, but one can equally well specify them in terms of deviations from any chosen base space.) In contrast with the commuting limit, the tangent frame group has to be \( U(k) \) rather than the Poincaré group of appropriate dimension.

There are two sets of transformations which are of interest in terms of their action on \( \mathcal{D} \). Transformations of the form \( U(N) \otimes 1 \) do not act on the tangent frames but redefine the notion of \( \partial \). These can thus be considered as the noncommutative version of diffeomorphisms. With a chosen set of \( \partial \), one can also consider transformations of the form \( \mathcal{F} \otimes H_R \) where \( \mathcal{F} \)'s commute with \( H \) and so are functions on the noncommutative \( G/H \)-space. These correspond to the local Lorentz transformations in the usual description of gravity in terms of frame fields.

3. Two gauge theories

We must now turn to the choice of an action for the gravitational degrees of freedom. \((e^i, \Omega^i)\) form the potential of a \( U(k) \)-gauge field. The derivative of a function, \( \partial_i f \), does not necessarily commute with \( H \), but the derivatives on any function do commute. So as in the commutative limit, antisymmetrization of indices lead naturally to \( H \)-invariant quantities on the noncommutative \( G/H \)-space. The only intrinsically defined action then has to be a Chern-Simons like action or something related to it.

The CS action in \( 2n + 1 \) dimensions for gauge fields on a flat noncommutative space of dimension \( 2n \) and with one commuting time coordinate has been given in [12] as

\[
S_{2n+1} = \frac{\lambda}{\mu} \int dt \sum_{r=0}^{n} \frac{(-1)^r (n+1)!}{2r + 1 r! (n-r)!} \text{Tr} \left( \omega^{n-r} \mathcal{D}^{2r+1} \right)  
\]  

(18)

where antisymmetrization of all indices is implicit. \( \mathcal{D} \) is an antihermitian infinite-dimensional matrix, it may be taken to be of the form \( \partial_i + A_i \) for some hermitian matrix potential \( A_i \). For the flat noncommutative space \([\partial_i, \partial_j] = \omega_{ij}\) is proportional to the identity. This is not the case for the general \( G/H \)-spaces we are considering, nevertheless \([K_i, K_j] = C^\alpha_{ij} K_\alpha\) commutes with all functions on the coset space and we can use essentially the same action \( S \), with the proviso that the number of \( \partial_i \)'s has to be even. We take \( \mathcal{D} \)'s to be antihermitian \( M \times M \)-matrices which may be considered to be of the form \( \partial_i + A_i \) again, but otherwise the action is unchanged. Notice that, eventhough \([[\partial_i, \partial_j], \partial_k] = C^l_{ijk} \partial_l \) is not necessarily zero, we have \( \epsilon^{ijk} \cdots C^l_{ijk} = 0 \) by the Jacobi identity. Further, since \( A_i \) are covariant vectors in the coordinate basis, \([\omega_{ij}, A_k] = C^l_{ijk} A_l \) and so, by the previous identity, \( \epsilon^{ijk} \cdots [\omega_{ij}, A_k] = 0 \). Thus \( \epsilon^{ijk} \cdots [\omega_{ij}, \mathcal{D}_k] = 0 \). The \( \mathcal{D} \)'s in the CS action, by virtue of the antisymmetrization of indices, commute with \( \omega_{ij} \) and the relative order of \( \omega \)'s and \( \mathcal{D} \)'s in (18) is immaterial. The
constant \( \mu \) in (18) is taken to be
\[
\mu = i \frac{(n + 1) \text{Tr}(\omega^n)}{M}
\] (19)

The action may be written out in the three-dimensional and five-dimensional cases, which we use later, as
\[
S_3 = \frac{\lambda}{\mu} \int dt \text{Tr} \left( -\frac{2}{3} D^3 + 2 \omega D \right)
\]
\[
S_5 = \frac{\lambda}{\mu} \int dt \text{Tr} \left( \frac{3}{5} D^5 - 2 \omega D^3 + 3 \omega^2 D \right)
\] (20)

The \( r = 0 \) term in (18) is \( \lambda (n + 1) \text{Tr}(\omega^n A_0)/\mu \). Recall that on the plane, nontrivial \( U(1) \) gauge transformations led to the quantization of the level number \( \lambda \) of the CS term [13]. The elementary nontrivial transformation is \( A_0 \rightarrow A_0 + i \partial_0 \varphi \), with \( \varphi(t = \infty) - \varphi(t = -\infty) = 2\pi/M \). With \( \mu \) as given in (19), \( \Delta S = 2\pi \lambda \) and the singlevaluedness of \( \exp(iS) \) gives the quantization of the level number, as in the case of the flat noncommutative space.

The CS action in (18) is very general. \( D \)'s are general \( U(M) \) matrices. The only indicator of the space on which the fields are defined is in fact in the \( \omega_{ij} \) appearing in the action. This enters via the definition of the field strength \( F_{ij} = [D_i, D_j] - \omega_{ij} \). The choice of different \( \omega_{ij} \) and the corresponding definition of the \( \partial_i \)'s determine the base space on which the fields are defined and identifies the particular vacuum state of zero field strength. Once the \( \partial_i \)'s are chosen, one can introduce the splitting \( D_i = \partial_i + A_i \), identifying the potential \( A_i \). At this stage, there is still no specific choice of the gauge group made. If we take the matrices \( A_i \) to be in the algebra of \( U(N) \times U(k) \), taking \( M = Nk \), and approach the commuting limit, then we get a gauge group \( U(k) \).

We now turn to a variant of the CS action which we can use in even dimensions. The variation of the CS action gives
\[
\delta S = (n + 1) \int dt \text{Tr} (\delta D F^n)
\] (21)

We can therefore define an action in \( 2n \) dimensions by
\[
S = \alpha \text{Tr} (Q F^n)
\] (22)

where \( \alpha \) is a constant and \( Q \) is a matrix, \( \neq 1 \), which commutes with \( \omega_{ij} \). (A term with \( Q = 1 \) gives a purely “topological” action with no contribution to the equations of motion.) This is equivalent to taking the fields in the CS Lagrangian to be independent of time and assigning a value \( Q \) to the \( A_0 \). If the \( D \)'s commute with \( Q \), this term is actually zero, by cyclicity of trace;
it is thus the matrix or noncommutative version of a total derivative. \( \mathcal{D} \)'s do not commute with \( Q \) in general, and this theory is a nontrivial gauge theory in even dimensions. With the insertion of \( Q \) in (22), we have gauge invariance only under transformations which commute with \( Q \). If there are many choices for \( Q \), one can take a linear combination, interpreting the coefficients as different coupling constants. The theory depends on the choice of \( Q \), which is arbitrary except that it should commute with \( \omega_{ij} \). We may in fact consider \( Q \) as the quantity which is primarily chosen and then identify \( \omega_{ij} \) in terms of matrices which commute with it. The equations of motion for the gauge theory (22) is now given by

\[
\sum_{r=0}^{n} F^{n-1-r} [\mathcal{D}, Q] F^r = 0 \tag{23}
\]

where the antisymmetrization of indices, as in the actions (18, 22), is assumed. These equations, along with the condition \([\omega_{ij}, \mathcal{D}_k] = C^l_{ijk} \mathcal{D}_l\), give our definition of noncommutative gravity.

The gauge theory (22) is reminiscent of the gauge theory approach to constructing the gravitational action in four dimensions due to Chang, MacDowell and Mansouri (CMM) [14]. In this case, one considers \( S^4 = SO(5)/SO(4) \) with \( SO(4) \) generators \( J^{ab} \) and \( S^4 \)-translations \( P^a \) with \([P^a, P^b] = \Lambda J^{ab}\). An explicit realization of these is given by the \( 4 \times 4 \) Dirac matrices, \( P^a = i \sqrt{\Lambda} \gamma^a \) and \( J^{ab} = -[\gamma^a, \gamma^b] \equiv -\gamma^{ab} \). The covariant derivatives are

\[
\mathcal{D}_\mu = \partial_\mu + e^a_\mu i \sqrt{\Lambda} \, \gamma^a + \Omega^{ab}_\mu (-\gamma^{ab}) .
\]

The commutators give

\[
[\mathcal{D}_\mu, \mathcal{D}_\nu] = \mathcal{T}^{(i \sqrt{\Lambda} \gamma^a)}_{\mu\nu} + \mathcal{R}^{ab}_{\mu\nu} (-\gamma^{ab})
\]

\[
\mathcal{R}^{ab}_{\mu\nu} = \mathcal{R}^{ab}_{\mu\nu} + \Lambda e^a_{\mu} e^b_{\nu}
\]

\[
R^{ab}_{\mu\nu} = (\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu])^{ab} \tag{24}
\]

The action is then taken as

\[
S = \alpha \int \mathcal{R}^{ab} \mathcal{R}^{cd} \epsilon_{abcd} \tag{25}
\]

Here \( \mathcal{R}^{ab} \) is the curvature two-form with the components given (24). The leading term involving just the spin connection is a topological invariant and may be dropped as far as the classical theory is concerned. The remainder gives the Einstein-Hilbert action with a cosmological constant. If we take the limit \( \Lambda \to 0, \alpha \to \infty \) with \( \alpha \Lambda \) fixed, we get the action with zero cosmological constant. This action may be written, up to an overall constant normalization factor, as

\[
S = \alpha \int \text{Tr}(\gamma^5 \mathcal{D}^4) \tag{26}
\]

This is of the form (22) with \( Q = \gamma^5 \). Notice that \( \gamma^5 \) commutes with the subgroup \( H = SO(4) \); also we do not have any \( \omega_{ij} \)-type terms since \([\partial_\mu, \partial_\nu] = 0\). (The observation that the
trace with $\gamma^5$ could be used to obtain the $\epsilon$-tensor in (23) was also made in [15]; I thank M-I Park for bringing this paper to my attention.) In the noncommutative case, there is only one trace which covers the integration over the space as well as trace over any internal indices. We can thus regard the action (22) as a generalization of the gauge theory approach to gravity.

Eventhough the connection with the CMM approach to gravity has been mentioned, it should be noted that, as in the case of the CS action, the action (22) is very general, the only indicator of the base space being the $\omega_{ij}$ and $Q$ which commutes with them.

In adapting these actions (18, 22) for describing gravitational fluctuations, we can then say that the key step is the choice of the base space. This can be done by writing $U(M)$ matrices as $U(N) \times U_R(k)$ matrices and specifying $H \subset G \subseteq U_L(k) \subseteq U(N)$ and identifying the derivatives $\partial_i$ with elements complementary to $H$ in $G$. A similar splitting can be made in the other copy of $U(k)$, namely $U_R(k)$, with an $H_R \subset U_R(k)$. For the even dimensional case, $Q$ is chosen to be an element of $U_R(k)$ which commutes with $H_R$.

An alternate way of arriving at this action would be as follows. We choose $Q$ as a $k \times k$-matrix and then identify the closed subalgebra of $k \times k$-matrices which commute with $Q$. The algebra $H_R$ is either this algebra or a smaller closed set, for which we can find a $G$ with $G/H$ being a symmetric space. We then consider a copy of this structure, namely $U_L(k)$ and $H_L \subset G \subseteq U_L(k)$ and identify the derivatives with the elements of the algebra $G$ which are not in $H_L$. This essentially identifies the space on which the theory is to be defined. We then choose an $N$-dimensional representation of $U_L(k)$, for an appropriate $N$, with $M = Nk$ and $D$’s in the algebra of $U(N) \times U_R(k)$, to obtain the action (22). The case when $H_R$ can be taken to be the full algebra of $k \times k$-matrices commuting with $Q$ corresponds to a space whose commutative limit is $SU(k)/U(k-1) \sim \mathbb{CP}^{k-1}$. This situation is particularly natural because the gravitational fields will correspond to just those in the commuting limit, with no additional degrees of freedom, except for an overall $U(1)$ field. Notice that the choice of $Q$ is the key, the rest of the required structure is naturally determined from it.

The basic suggestion of this paper is that CS gravity in odd dimensions can be described by the CS action in (18), with $\partial_i \in G - H$, $G \subseteq U_L(k) \subseteq U(N)$, with $D_i$ belonging to the algebra of $U(N) \times U_R(k)$. If $[[\partial_i, \partial_j], \partial_k] = C_{ijk}^l \partial_l$, then the $D$’s must also obey $[[\partial_i, \partial_j], D_k] = C_{ijk}^l D_l$. For odd dimensions, there will be a quantization constraint on the gravitational coupling, since the level number of the CS action is quantized. In even dimensions, we can define a gravitational action which is (22) with the same structure for the $D$’s as given above for the odd dimensional case. Since the scale of $Q$ is arbitrary, or since the action is
gauge invariant, with no extra factors like $\text{Tr}(\omega^n A_0)$, we do not expect quantization of the coefficient. Finally, it is easy to see that the actions (18, 22) can be written down directly for the case where the underlying noncommutativity structure is that of the Heisenberg algebra. In this case $\omega_{ij}$ trivially commute with the $D$'s. Actions constructed via the star product formalism seem to involve determinants and inverses, which are evaluated by expansions via star products. Our action (22) avoids some of this awkwardness.

In a full theory of gravity, it should be possible to change the choice of $Q, G, H, \text{etc.}$, which would correspond to transitions between spaces with different topologies in the commuting limits and even between spaces of different dimensions. In fact, since $Q$ naturally determines the rest of the structure, we can say that, if the short distance theory of gravity involves noncommutative spaces, then initially we could have random values for $Q$ following some thermal distribution and as the universe cooled down, a particular choice of $Q$, perhaps randomly as in the case of vacuum orientation for spontaneously broken symmetries, was made and this determined the number of dimensions of the world. The actions we have given correspond to expansions around chosen base spaces and describe the gravitational fields which are small in the sense of not changing these structures under fluctuations. Further our actions do not involve the notion of the metric explicitly. When the coordinates do not commute among themselves, we do not even have the usual notion of a metric or distance function on the space. Therefore, a gravitational action which does not use any such additional structures is more natural, even more so than in the case of commutative spaces.

4. Special cases

Consider the case of the noncommutative $\text{CP}^2$ for which $G = SU(3) \subset U(3), H = U(2)$. We can write the covariant derivatives in the form (14) with the generators of $U_R(3)$ taken as

$$I^8 = -\frac{i}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad I^i = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}, \quad T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (27)$$

where $\sigma^i, i = 1, 2, 3$, are the Pauli matrices. $T^{\bar{i}}$ are the conjugates of the $T^i$. We also have $I^0 = -(i/\sqrt{6}) \textbf{1}$. The matrix $Q$ can be taken to be $I^8$ and we can write the action as

$$S = i\alpha \text{Tr} \left( I^8 F_{\mu\nu} F^{\alpha\beta} \right) \epsilon^{\mu\nu\alpha\beta} \quad (28)$$
The equations of motion (23) simplify as

\[ \epsilon^{\mu \nu \alpha \beta} \left\{ [D_\alpha, I^8] F_{\mu \nu} + F_{\mu \nu} [D_\alpha, I^8] \right\} = 0 \]  

(29)

The action (28) contains a topological term which does not contribute to the equations of motion. Therefore, as far as classical physics is concerned, we can simplify the expression (28). Separate the generators into \{ I^A \} which include \( I^i \), \( I^8 \) and \( I^0 \) and \( (T^i, \bar{T}^i) \). \( I^A \) commute with \( I^8 \) and also the products \( I^A T^i, I^A \bar{T}^i \) can be expanded in terms of \( T^i \) and \( T^\bar{i} \), with no \( I^A \)'s. We write

\[ D_\mu = \partial_\mu \Omega_\mu + E_\mu \equiv D_\mu + E_\mu \]

\[ D_\mu = \partial_\mu + \Omega_\mu = \partial_\mu + \Omega^A_\mu I^A \]

\[ E_\mu = \epsilon^a_\mu T^a + \epsilon^{\bar{a}}_\mu \bar{T}^{\bar{a}} \]  

(30)

The commutators can now be separated as

\[ [D_\mu, D_\nu] = \omega_{\mu \nu} + R^A_{\mu \nu} I^A \]

\[ [D_\mu, E_\nu] - [D_\nu, E_\mu] = T^i_{\mu \nu} T^\bar{i} + \bar{T}^i_{\mu \nu} T^i \]

\[ [E_\mu, E_\nu] = \Lambda^A_{\mu \nu} I^A \]  

(31)

In the action, since \( D_\mu \) commutes with \( I^8 \), one has the identity \( \epsilon^{\mu \nu \alpha \beta} \text{Tr}(I^8[D_\mu, D_\nu][D_\alpha, D_\beta]) = 0 \), which leads to \( \epsilon^{\mu \nu \alpha \beta} \text{Tr}(I^8 R_{\mu \nu} R_{\alpha \beta}) = 0 \). The term quadratic in the torsion can be simplified as

\[ \epsilon^{\mu \nu \alpha \beta} \text{Tr}(I^8(T^i_{\mu \nu} T^\bar{i} + \bar{T}^i_{\mu \nu} T^i)(T^\bar{j}_{\alpha \beta} T^\bar{j} + \bar{T}^j_{\alpha \beta} T^j)) = \epsilon^{\mu \nu \alpha \beta} \frac{i}{\sqrt{3}} \text{Tr}([D_\mu, E_\nu][D_\alpha, E_\beta]) \]

\[ = -\frac{i}{2\sqrt{3}} \epsilon^{\mu \nu \alpha \beta} \text{Tr}[D_\mu, D_\nu][E_\alpha, E_\beta] \]

\[ = -\frac{i}{2\sqrt{3}} \epsilon^{\mu \nu \alpha \beta} \text{Tr}(R_{\mu \nu} \Lambda_{\alpha \beta}) \]  

(32)

where we have used the Jacobi identity and the fact that \( \epsilon^{\mu \nu \alpha \beta}[\omega_{\mu \nu}, E_\alpha] = 0 \) by virtue of the assigned transformation law for \( E_\alpha \). Using these results, the action (28) can be written, up to total derivatives, as

\[ S = \frac{i \alpha}{4} \text{Tr} \left( J[D_\mu, D_\nu][E_\alpha, E_\beta] - J\omega_{\mu \nu}[E_\alpha, E_\beta] + I^8 [E_\mu, E_\nu][E_\alpha, E_\beta] \right) \epsilon^{\mu \nu \alpha \beta} \]

\[ J = 2I^8 - \frac{i}{2\sqrt{3}} \mathbf{1} \]  

(33)

A simple solution of these equations is given by a pure gauge form

\[ D_\mu = M^{-1}(\partial_\mu)M = \partial_\mu + M^{-1}[\partial_\mu, M] \]  

(34)
where $M$ is a matrix which is valued in $U(3)_R$ and is a function on the noncommutative $\mathbb{CP}^2$. In other words, we can write

$$M = \exp \left( \theta^i T_i + \overline{\theta}^i T^i + \varphi_A I^A \right)$$

(35)

where $\theta_i$, $\overline{\theta}_i$, $\varphi_A$ are $(N \times N)$-matrices which commute with $\omega_{\mu\nu} \in H$. When $\theta_i = \overline{\theta}_i = 0$, this corresponds to a local Lorentz-type transformation; the relevant degrees of freedom are thus in the $\theta_i$, $\overline{\theta}_i$. Expanding

$$M^{-1}[\partial_\mu, M] = e^a_\mu T^a + e^a_\mu \overline{T}^a + \Omega^A_\mu I^A$$

(36)

we can see that $\frac{1}{2}(e^a_\mu e^a_\nu + e^a_\nu e^a_\mu)$ defines the $\mathbb{CP}^2$-metric tensor in the commuting limit. The configuration (34, 35) thus corresponds to the noncommutative $\mathbb{CP}^2$ again.

We have formulated the gravitational field equations in (28) parametrizing the fields in terms of deviations from a noncommutative $\mathbb{CP}^2$. The result given above is then equivalent to saying that the noncommutative $\mathbb{CP}^2$ itself is a solution of these gravitational equations; it is in a sense the analogue of the ‘flat space’. It is well known that $\mathbb{CP}^2$ is a gravitational instanton in the standard commutative gravity with a cosmological term. What we have shown is that a noncommutative version of this statement holds as well. It would be interesting to explore other solutions of this equation; so far, we have not been able to find any other particularly interesting solutions.

Another interesting example is given by a noncommutative four-sphere defined by taking $H = SO(4)$, $G = SO(5) \subset U_L(4)$. $U_R(k)$ is $U(4)$, the algebra of which is given by the four-dimensional Dirac $\gamma$-matrices. In this case, $\omega_{ij}$ are generators of $O(4) \subset U_L(4)$. As a basis for $U_R(4)$ we can take

$$T^5 = i\gamma^5, \quad T^0 = i1$$

$$T^a = i\gamma^a, \quad \bar{T}^a = \gamma^a\gamma^5$$

$$J^{ab} = -\frac{1}{8}[\gamma^a, \gamma^b]$$

(37)

The action (22) is given by

$$S = \alpha \text{ Tr} \left( \gamma^5 F_{\mu\nu} F_{\alpha\beta} \right) \epsilon^{\mu\nu\alpha\beta}$$

(38)

$\partial_\mu$ are $4N \times 4N$-matrices of the form $\partial_\mu \otimes 1_{4 \times 4}$. We also have

$$[\omega_{\mu\nu}, \partial_\alpha] = 4(\delta_{\mu\alpha}\partial_\nu - \delta_{\nu\alpha}\partial_\mu)$$

(39)

The equations of motion for (38) are

$$\epsilon^{\mu\nu\alpha\beta} \left\{ [D_\alpha, \gamma^5] F_{\mu\nu} + F_{\mu\nu} [D_\alpha, \gamma^5] \right\} = 0$$

(40)
For $\text{CP}^2$, the relevant gauge field is a $U(3)$ gauge field, which has no additional gravitational degrees of freedom compared to the commuting limit, except for an overall $U(1)$ field. This is because $G$ is already a special unitary group, namely $SU(3)$. The situation with $S^4$ is not so nice as the case of $\text{CP}^2$. We can consider $S^4$ as $SO(5)/SO(4)$, then $G = SO(5)$ has to be embedded in $U(4)$ and a gauge theory of $U(4)$ constructed. There are therefore additional degrees of freedom. The desire to eliminate them has led to recent attempts at the so-called teleparallelism theories, where the extra degrees of freedom are set to zero \[6, 16\]. An analogous restriction can be made in our case, by only including the $e_\mu^a$, $\Omega^a_{\mu}$ components for the gauge field. There will still be components of the curvature corresponding to $T^5$, $\tilde{T}^a$ and 1 directions, but these will vanish as the commutative limit is approached. The action \[28\] can still be used, but the equations of motion are more restricted than \[40\], because there are less number of fields to vary. Restricting to $e_\mu^a$, $\Omega^a_{\mu}$, the various components of the curvature can be worked out as

$$ T_\mu^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \frac{1}{2} \left( \epsilon^b_{\mu \nu} \Omega^a_{\nu} + \Omega^a_{\mu} e_\nu^b - e_\nu^b \Omega^a_{\mu} + \Omega^a_{\mu} e_\nu^b \right) $$

$$ \tilde{T}_\mu^a = \frac{1}{4} \epsilon^{abcd} \left( [e_\mu^a, \Omega^b_{\nu}] - [e_\nu^a, \Omega^b_{\mu}] \right) $$

$$ R^0_{\mu \nu} = -i [e_\mu^a, e_\nu^a] + \frac{i}{8} [\Omega^a_{\mu}, \Omega^a_{\nu}] $$

$$ R^5_{\mu \nu} = -\frac{i}{16} \epsilon^{abcd} [\Omega^a_{\mu}, \Omega^d_{\nu}] $$

$$ R^{ab}_{\mu \nu} = \partial_\mu \Omega^a_{\nu} - \partial_\nu \Omega^a_{\mu} - \frac{1}{2} \left( \Omega^a_{\mu} \Omega^b_{\nu} - \Omega^a_{\nu} \Omega^b_{\mu} - \Omega^b_{\mu} \Omega^a_{\nu} + \Omega^b_{\nu} \Omega^a_{\mu} \right) $$

$$ \mathcal{R}^{ab}_{\mu \nu} = R^{ab}_{\mu \nu} + 4 \left( e_\mu^a e_\nu^b - e_\nu^a e_\mu^b + e_\nu^b e_\mu^a - e_\mu^b e_\nu^a \right) $$  \hspace{1cm} (41)

As in the case of $\text{CP}^2$ some of the terms in the action vanish by cyclicity of trace. The action \[38\] can then be simplified as

$$ S = 8 \alpha \text{Tr} \left\{ \epsilon^{abcd} [e_\mu^a e_\nu^b R^c_{\alpha \beta} + 8 \epsilon_\mu^a e_\nu^b e_\alpha^c e_\beta^d] + \frac{i}{4} T_{\mu \nu} \tilde{T}^a_{\alpha \beta} + \frac{i}{4} [e_\mu^a, e_\nu^a] R_5^{\alpha \beta} \right\} \epsilon^{\alpha \beta} $$  \hspace{1cm} (42)

Eventhough, the reduction of fields can be implemented as above, from the gauge theory point of view, it is more natural to keep all the additional fields corresponding to the $T^5$, $\tilde{T}^a$ and 1 directions. Perhaps a different approach might be to consider $\text{CP}^3$ which will be a solution in the analogous six-dimensional theory and which will not need additional degrees of freedom since $\text{CP}^3 = SU(4)/U(3)$. $\text{CP}^3$ can also be considered as an $S^2$ bundle over $S^4$, this is the Penrose projective twistor space for $S^4$. By some natural way of projecting out the $S^2$, this might lead to a better way to formulate $S^4$ \[3\].
I thank A. Polychronakos for comments. This work was supported in part by the National Science Foundation grant number PHY-0070883 and a PSC-CUNY-32 award.

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