ON A THEOREM OF HARER

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INTRODUCTION

Let $M_g$ be the coarse moduli space of smooth projective complex curves of genus $g$. Let $\text{Pic}(M_g)$ denote the Picard group of line bundles on $M_g$. According to Mumford, $\text{Pic}(M_g) \otimes \mathbb{Q} \cong H^2(M_g, \mathbb{Q})$ for $g \geq 2$ [9].

The purpose of this note is to give another proof of the following fundamental theorem of Harer [4].

**Theorem 1.** $\text{rk} \text{Pic}(M_g) \otimes \mathbb{Q} = 1$ for $g \geq 3$.

We will deduce Theorem 1 from the Harer stability theorem [5, 6, 8].

**Theorem 2.** $H^2(M_g, \mathbb{Q}) \cong H^2(M_{g+1}, \mathbb{Q})$ for $g \geq 6$.

Note that Theorem 2 is similar to the corresponding stability results for arithmetic groups.

**Proof of Theorem 1**

To deduce Theorem 1 from Theorem 2, it suffices to show that $\text{rk} \text{Pic}(M_g) \otimes \mathbb{Q} = 1$ for $g = 3, 4, 5, 6$. Let $V_{n,d} \subset \mathbb{P}^N$ ($N = n(n + 3)/2$) be the variety of irreducible complex plane curves of degree $n$ with $d$ nodes and no other singularities. Let $V(n, g) \subset \mathbb{P}^N$ ($N = n(n + 3)/2$) be the variety of irreducible complex plane curves of degree $n$ and genus $g = (n - 1)(n - 2)/2 - d$. Clearly $V_{n,d} \subset V(n, g)$. Let

$$
\Sigma_{n,d} \subset \mathbb{P}^N \times \text{Sym}^d(\mathbb{P}^2)
$$

be the closure of the locus of pairs $(E, \sum_{i=1}^d R_i)$, where $E$ is an irreducible nodal curve and $R_1, \ldots, R_d$ are its nodes, and $\pi_{n,d}$ the projection to the symmetric product. We fix a general point $P$ and a general line $L$ in $\mathbb{P}^2$, and consider the following two elements in $\text{Pic}(V_{n,d})$: $(CP)$ = the divisor class of curves containing the point $P$, and $(NL)$ = the divisor class of curves with a node located somewhere on $L$.

If $g \geq 3(g + 2 - n)$ then the natural map $V(n, g) \to M_g$ is a surjective morphism [3, p. 358]. Precisely, for any smooth curve $C$ of genus $g$, there is a line bundle $\mathcal{L}$ on $C$ of degree $n$ such that $h^0(C, \mathcal{L}) \geq 3$. Furthermore, Harris observed that $\text{rk} \text{Pic}(M_g) \otimes \mathbb{Q} = 1$ provided one can show that $\text{Pic}(V_{n,d})$ is a torsion group, where $d = (n - 1)(n - 2)/2 - g$ [7]. Thus Theorem 1 follows from the

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Lemma. Let $n$ and $d$ be a pair of positive integers such that one of the following conditions holds: i) $n \gg d$ or ii) $(n, d) = (5, 3), (6, 6), (6, 5)$, or $(6, 4)$. Then $\text{Pic}(V_{n,d})$ is a torsion group generated by $(CP)$ and $(NL)$.

Proof of Lemma. We will prove the lemma for $(n, d) = (6, 6)$, hence $g = 4$. The remaining cases are similar only easier. Given any 6 distinct points $q_1, \ldots, q_6$ in $\mathbb{P}^2$, no four on a line, then a general curve $C$ of the linear system $L$ of curves of degree 6 with assigned singularities at $q_1, \ldots, q_6$ has 6 nodes and no other singularities. We will calculate $\dim L$ using Castelnuovo’s method (see [1, Book IV, Chap. I, Sect. 3]).

The characteristic series determined on $C$ by $L$ is cut out by curves of $L$. Clearly one can find two curves in $L$, say $C_1$ and $C_2$, such that $(C_1 \cdot C_2)_{q_i} = 4$ for $i = 1, \ldots, 6$. It follows that the degree of the characteristic series equals $(\deg C)^2 - 6 \cdot 4 > 6 = 2g - 2$. Hence this series is non-special, the superabundance of $L$ vanishes, and $\dim L = 6(6 + 3)/3 - 6 \cdot 3$.

Let $S$ be a closed subset of $\text{Sym}^6(\mathbb{P}^2)$ which is a union of the singular locus and the cycles with at least 4 points on a line. Clearly $\text{codim } S = 2$ and $\text{Pic}(\text{Sym}^6(\mathbb{P}^2) \setminus S) = \text{Pic}(\text{Sym}^6(\mathbb{P}^2)) = \mathbb{Z}$. We set

$$\Sigma = \pi_{6,6}^{-1}(\text{Sym}^6(\mathbb{P}^2) \setminus S) \cap \Sigma_{6,6}.$$ 

By the above discussion, all the fibers of $\pi_{6,6}|\Sigma$ have the same dimension and $\pi_{6,6}|\Sigma: \Sigma \to \text{Sym}^6(\mathbb{P}^2) \setminus S$ is a projective bundle. Thus $\text{Pic}(\Sigma) = \mathbb{Z}^2$ and its generators correspond to the divisor classes $(CP)$ and $(NL)$ coming from the fiber and the base of the fibration, respectively.

We consider two divisors on $\Sigma$: (i) the closure of the locus of irreducible curves with $d - 1$ assigned nodes and one assigned cusp, and (ii) the closure of the locus of nodal curves with $d$ assigned nodes and one unassigned node. The corresponding elements of $\text{Pic}(\Sigma) \otimes \mathbb{Q}$ are linearly independent [2, Sect. 2]. There is a natural open immersion $V_{6,6} \subset \Sigma$. Hence we get an exact sequence

$$\mathbb{Z}^r \to \text{Pic}(\Sigma) \to \text{Pic}(V_{6,6}) \to 0$$

where $\mathbb{Z}^r$ is generated by the irreducible components of $\Sigma \setminus V_{6,6}$ of codimension one in $\Sigma$. Therefore $\text{Pic}(V_{6,6})$ is torsion. This proves the lemma and the theorem.

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