Mixed multidimensional integral operators with piecewise constant kernels and their representations

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Abstract

We consider the algebra of mixed multidimensional integral operators. In particular, Fredholm integral operators of the first and second kind belongs to this algebra. For the piecewise constant kernels we provide an explicit representation of the algebra as a product of simple matrix algebras. This representation allows us to compute the inverse operators (or to solve the corresponding integral equations) and to find the spectrum explicitly. Moreover, explicit traces and determinants are also constructed. So, roughly speaking, the analysis of integral operators is reduced to the analysis of matrices. All the qualitative characteristics of the spectrum are preserved since only the kernels are approximated.

Keywords: operator algebras, traces and determinants, periodic lattice with defects, guided and localized waves

1. Introduction

Let me describe briefly the results of the paper. The classical 1D Fredholm operators

$$A u(k) = \int_0^1 A(k, x) u(x) dx$$

are well studied, see, e.g., the famous pioneer work [1]. In particular, if

$$A(k, x) = \sum a_{ij} p \chi_i(k) \chi_j(x)$$

is a piecewise constant (p-step, see [3]) kernel with $a_{ij} \in \mathbb{C}$ then the integral operators form an algebra which is isomorphic to the matrix algebra $\mathbb{C}^{p \times p}$, see, e.g., [2]. The isomorphism is $A \leftrightarrow (a_{ij})_{i,j=1}^p$. But because the algebra of integral operators does not contain the standard identity operator, the invertibility in $\mathbb{C}^{p \times p}$ does not mean the invertibility in the large algebra of bounded operators. The simplest extension that devoid of this shortcoming consists of operators of the form

$$A u(k) = a u(k) + \int_0^1 A(k, x) u(x) dx, \quad a \in \mathbb{C}.$$
In this case, the corresponding algebra is isomorphic to \( \mathbb{C} \times \mathbb{C}^{p \times p} \), the isomorphism is \( \mathcal{A} \leftrightarrow (a, (a_{ij} + a\delta_{ij})_{i,j=1}^p) \) (\( \delta \) is the Kronecker \( \delta \)). The invertibility in \( \mathbb{C} \times \mathbb{C}^{p \times p} \) means also the invertibility in the algebra of bounded operators. The result becomes more substantial if \( a \) in (3) is a \( p \)-step function. In the current paper, we consider the algebra of all mixed integral operators of different dimensions with \( p \)-step kernels that extend the algebras mentioned above. As it is known, various spectral problems in multidimensional case are much more complex than in one-dimensional case. The main motivation for the paper is to show that discrete analogues of multidimensional integral operators admit an explicit spectral analysis. We give an explicit representation of the algebra as a direct product of simple matrix algebras. This representation leads to explicit procedures of finding inverse operators and the spectra, and allows us to construct explicit multidimensional traces and determinants. The algebra of mixed multidimensional integral operators has many physical applications. In particular, the \( p \)-step approximations of periodic operators acting on the structures with crossing defects of various dimensions belong to this algebra, see, e.g. [3, 4, 5] for operators with defects and [6] for classical periodic operators without defects. So, the studying of the guided, surface, and other Rayleigh waves propagating in such structures are based on the corresponding determinants and inverse operators. Note that, due to the Stone-Weierstrass theorem, only step functions allow us to use all the power of the theory of finite-dimensional algebras. At the same time, \( p \)-step kernels can approximate continuous (or other class) kernels with arbitrary precision when \( p \to \infty \). All this shows that \( p \)-step kernels are noteworthy, see also [7].

Before introducing the algebra of multidimensional integral operators with step kernels let us define some auxiliary objects. We fix some \( p \in \mathbb{N} \). Let \( \alpha \) be some subset of integer numbers. The set of multi-indices \( P_\alpha \) is defined by

\[
P_\alpha = \{ \mathbf{m} = (m_n)_{n \in \alpha} : m_n \in \{1, ..., p\} \},
\]

where the components of \( \mathbf{m} \) are arranged in the order of increasing indices \( n \). Further, we always assume that the components of any vectors are arranged in the order of increasing indices. Let \( \alpha, \beta \) be two disjoint sets consisting of integer numbers and let \( \mathbf{x}_\alpha = (x_n)_{n \in \alpha}, \mathbf{y}_\beta = (y_n)_{n \in \beta} \) be two vectors. It is convenient to use the following notation

\[
\mathbf{y}_\beta \diamond \mathbf{x}_\alpha = \mathbf{x}_\alpha \diamond \mathbf{y}_\beta = \mathbf{z}_{\alpha \cup \beta}, \quad \text{where} \quad z_n = \begin{cases} 
x_n, & n \in \alpha, \\
y_n, & n \in \beta.
\end{cases}
\]

Also, for \( \mathbf{x} = (x_n)_{n \in \gamma} \) and some sets \( \alpha \subset \gamma \) we denote \( \mathbf{x}_\alpha = (x_n)_{n \in \alpha} \). We fix a positive integer \( N \) and denote the set \( \eta = \{1, ..., N\} \). Let \( L^2 \) be the Hilbert space of square-integrable scalar functions acting on the cube \([0,1)^N\). We consider the operators \( \mathcal{A} : L^2 \to L^2 \) of the form

\[
\mathcal{A} u(k) = \sum_{\alpha \subset \eta} \int_{[0,1)^{|\alpha|}} A_\alpha(k, x_\alpha) u(kr \diamond x_\alpha) dx_\alpha, \quad u \in L^2,
\]

where \( \mathbf{x} = \eta \setminus \alpha \) denotes the complement to the set \( \alpha \), the vector \( \mathbf{x} = (x_n)_{n \in \eta} \), the differential \( dx_\alpha = \prod_{n \in \alpha} dx_n \) corresponds to the Lebesgue measure, and \(|\alpha|\) is the number of elements
in $\alpha$. If $\alpha = \emptyset$ is the empty set then the corresponding term in (6) is $A_0(k)u(k)$. The $p$-step kernels $A_\alpha$ have the following form

$$A_\alpha(k, x_\alpha) = \sum_{(i, m) \in P_\eta \times P_\alpha} p^{\alpha} a_\alpha(i, m) \prod_{j \in \eta} \chi_{i_j}(k_j) \prod_{n \in \alpha} \chi_{m_n}(x_n),$$  

(7)

where $a_\alpha(i, m) \in \mathbb{C}$ and the step functions $\chi$ are defined by

$$\chi_r(y) = \begin{cases} 1, & y \in [(r-1)/p, r/p), \\ 0, & \text{otherwise}. \end{cases}$$  

(8)

The operators $A$ (6) form an algebra $\mathcal{L}$ which is a subalgebra of the algebra of bounded operators acting in $L^2$. In fact, this algebra is generated by the following $pN + p$ elementary operators

$$\mathcal{L} = \text{Alg}\left(\left\{\chi_i(k_j) \cdot, \int_0^1 dx_j \right\}_{i=1, j=1}^{p, N} \right),$$  

(9)

where $\cdot$ denotes the place of the operator argument $u \in L^2$. Note that $\mathcal{L}$ contains the identical operator from the large algebra of bounded operators. Hence, the invertibility in $\mathcal{L}$ means also the invertibility in the algebra of bounded operators and vice versa. Because $\mathcal{L}$ is a finite dimensional Von Neumann algebra, it can be represented as a direct product of simple algebras. Our goal is to find this representation explicitly. There are probably various ways to do that, one based on composition series is preferred for us since it allows us to simplify some of calculations. We denote the simple matrix algebras as $\mathbb{C}^{n \times n}$, $n \in \mathbb{N}$ ($\mathbb{C}^{1 \times 1} = \mathbb{C}$). For any $\alpha \subset \eta$, $i \in P_\eta$, $m \in P_\alpha$ introduce the following functions

$$b_\alpha(i, m) = \sum_{\beta \subset \alpha} \delta(i, m_{\alpha \setminus \beta}) a_\beta(i, m_\beta),$$  

(10)

where the Kronecker delta satisfies: $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ if $x \neq y$. Next, for any $\alpha \subset \eta$ and $i \in P_\eta$ introduce the matrices

$$B_\alpha(i) = (b_\alpha(i \diamond m, n))_{m, n \in P_\alpha} \in \mathbb{C}^{p^{\alpha} \times p^{\alpha}}.$$  

(11)

Introduce the following mapping

$$\sigma : \mathcal{L} \to \prod_{n=0}^{N} (\mathbb{C}^{p^n \times p^n})^\binom{N}{n}_p, \quad \sigma(A) = ((B_\alpha(i))_{i \in P_\eta})_{\alpha \subset \eta},$$  

(12)

where $A$, $B$ are defined in (6)-(7), (10)-(11) and $\binom{N}{n}$ are the binomial coefficients. The identities (11)-(12) allow us to compute the inverse mapping $\sigma^{-1}$ explicitly by using

$$a_\alpha(i, m) = \sum_{\beta \subset \alpha} (-1)^{\alpha \setminus \beta} \delta(i, m_{\alpha \setminus \beta}) b_\beta(i, m_\beta).$$  

(13)

The following theorem is our main result.

**Theorem 1.1.** The mapping $\sigma$ is an algebra isomorphism.
We immediately obtain the next corollary which is useful in physical and numerical applications.

**Corollary 1.2.** i) The operator $A$ is invertible if and only if all matrices $B_{\alpha}(i)$ are invertible. In this case

$$A^{-1} = \sigma^{-1}((B_{\alpha}^{-1}(i))_{i \in \mathcal{I}_\alpha})_{\alpha \subset \eta}.$$  

ii) The spectrum of $A$ consists of all eigenvalues of the matrices $B_{\alpha}(i)$. The eigenvalues of $B_{\eta}$ form a discrete spectrum, other eigenvalues belong to essential spectrum.

iii) The multidimensional trace $\tau$ and the determinant $\pi$ can be defined as follows

$$\tau, \pi : \mathcal{L} \rightarrow \prod_{n=0}^{N} \mathbb{C}((N)p^{N-n}) = \mathbb{C}((p+1)^N),$$

They satisfy the usual properties

$$\tau(\lambda A + \mu B) = \lambda \tau(A) + \mu \tau(B), \quad \tau(AB) = \tau(BA), \quad \pi(AB) = \pi(A)\pi(B), \quad \pi(e^A) = e^{\tau(A)},$$

where $\lambda, \mu \in \mathbb{C}$ and $A, B \in \mathcal{L}$. The operator is invertible iff all determinants are nonzero.

**Remark.** All the results can be easily extended to the case where $A$ acts on $\bigoplus_{m=1}^{M} L^2$ and the kernels $A_{\alpha} (7)$ are $M \times M$ $p$-step matrices (this means that all entries of these matrices are $p$-step functions and, hence, the coefficients $a_{\alpha} (7)$ are $M \times M$ constant matrices). The integral operators with $p$-step $M \times M$ matrix-valued kernels $A_{\alpha}$ form an algebra $\mathcal{L}_M$. It can be shown that

$$\mathcal{L}_M \simeq \prod_{n=0}^{N} \left( \mathbb{C}^{Mp^n \times Mp^n} \right)_{n=0}^{N}$$

and the corresponding isomorphism has the same form as $\sigma$ but with $a_{\alpha}$ instead of $a_{\alpha}$.

2. **Proof of Theorem 1.1**

Introduce the following operators

$$A_{ij} = \chi_i(k_j), \quad B_{ij} = p \int_{0}^{1} \chi_i(x_j) \cdot dx_n.$$  

**Lemma 2.1.** The operators $A, B$ are commute if their second indices are different. Moreover, for any $i, j, r$ the following identities hold true

$$A_{ij}A_{rj} = \delta(i, r)A_{ij}, \quad B_{ij}A_{rj} = \delta(i, r)B_{ij}, \quad B_{ij}B_{rj} = B_{rj}.$$
The case to the commutativity following from Lemma 2.1, we have

These operators are connecting through the next equations.

Using (20) we obtain the following identities

\[ \mathcal{C}_\alpha(i, m) = \prod_{j \in \alpha} A_{ij} \prod_{j \in \beta} B_{mj} \quad \mathcal{D}_\alpha(i, m) = \mathcal{C}_\alpha(i, m) \prod_{j \in \tau}(1 - B_{ij}). \] (21)

Proof. The direct calculations and Fubini theorem give the result. ■

For any \( \alpha \subset \eta, i \in P_\eta, m \in P_\alpha \) introduce the following operators

\[ \mathcal{C}_\alpha(i, m) = \sum_{\beta \supset \alpha} \mathcal{D}_\beta(i, m \diamond i_{\beta \setminus \alpha}), \quad \mathcal{D}_\alpha(i, m) = \sum_{\beta \supset \alpha} (-1)^{|\alpha \setminus \beta|} \mathcal{C}_\beta(i, m \diamond i_{\beta \setminus \alpha}). \] (22)

Proof. Using the commutativity from Lemma 2.1 and (21) we deduce that

\[ \mathcal{C}_\alpha(i, m) = \mathcal{C}_\alpha(i, m) \prod_{j \in \tau}(1 - B_{ij} + B_{ij}) = \sum_{\beta \supset \alpha} \mathcal{D}_\beta(i, m \diamond i_{\beta \setminus \alpha}) \] (23)

and

\[ \mathcal{D}_\alpha(i, m) = \mathcal{C}_\alpha(i, m) \prod_{j \in \tau}(1 - B_{ij}) = \sum_{\beta \supset \alpha} (-1)^{|\alpha \setminus \beta|} \mathcal{C}_\beta(i, m \diamond i_{\beta \setminus \alpha}). \] (24)

While \( \mathcal{C}_\alpha(i, m) \) is a standard basis in \( \mathcal{L} \) (see (5)-(7)), the basis \( \mathcal{D}_\alpha(i, m) \) is "orthogonal" that will be proved in the next lemma.

Lemma 2.3. The following identities hold true

\[ \mathcal{D}_\alpha(i, m) \mathcal{D}_\beta(p, r) = \begin{cases} 0, & \alpha \neq \beta, \\ 0, & \alpha = \beta, \quad i_\tau \neq p_\tau, \\ \delta(m, p, r) \mathcal{D}(i, r), & \alpha = \beta, \quad i_\tau = p_\tau. \end{cases} \] (25)

Proof. The case \( \alpha \neq \beta \). There are two possibilities. The first one, there is \( i_j \in \tau \setminus \beta \). Due to the commutativity following from Lemma 2.1 we have

\[ \mathcal{D}_\alpha(i, m) \mathcal{D}_\beta(p, r) = \mathcal{E} A_{ij}(1 - B_{ij}) A_{p,j} B_{r,j}, \] (26)

where \( \mathcal{E} \) is a product of elements \( A, B, 1 - B \) which have second indices not equal to \( j \). Using (20) we obtain the following identities

\[ A_{ij}(1 - B_{ij}) A_{p,j} B_{r,j} = \delta(i_j, p_j) A_{ij}(1 - B_{ij}) B_{r,j} = \delta(i_j, p_j) A_{ij} B_{r,j} - B_{r,j} = 0. \] (27)

The second possibility, there is \( p_j \in \beta \setminus \tau \). Due to the commutativity following from Lemma 2.1 we have

\[ \mathcal{D}_\alpha(i, m) \mathcal{D}_\beta(p, r) = \mathcal{E} A_{ij} B_{m,j} A_{p,j}(1 - B_{p,j}), \] (28)

where \( \mathcal{E} \) is a product of elements \( A, B, 1 - B \) which have second indices not equal to \( j \). Using (20) we obtain the following identities

\[ A_{ij} B_{m,j} A_{p,j}(1 - B_{p,j}) = \delta(m_j, p_j) A_{ij} B_{m,j}(1 - B_{p,j}) = \delta(m_j, p_j) A_{ij} B_{m,j} - B_{p,j} = 0. \] (29)
We have proved the first identity in (25).

The case $\alpha = \beta$, $i_\pi \neq p_\pi$. Then there is $j \in \pi = \overline{\beta}$ such that $i_j \neq p_j$. Due to the commutativity (see Lemma 2.1), we have

$$D_\alpha(i, m)D_\beta(p, r) = E A_{i,j}(1 - B_{i,j})A_{p,j}(1 - B_{p,j}),$$

where $E$ is a product of elements $A, B, 1 - B$ which have second indices not equal to $j$. Using (20) we obtain

$$A_{i,j}(1 - B_{i,j})A_{p,j}(1 - B_{p,j}) = \delta(i, p_j)A_{i,j}(1 - B_{i,j})(1 - B_{p,j}) = 0. \quad (31)$$

We have proved the first identity in (25).

The case $\alpha = \beta$, $i_\pi = p_\pi$. Due to the commutativity (see Lemma 2.1), we have

$$D_\alpha(i, m)D_\beta(p, r) = \prod_{j \in \pi}(A_{i,j}(1 - B_{i,j})A_{p,j}(1 - B_{p,j})) \prod_{j \in \alpha}(A_{i,j}B_{m,j}A_{p,j}B_{r,j}). \quad (32)$$

Using (20) and $i_j = p_j$ for $j \in \pi$ we obtain

$$A_{i,j}(1 - B_{i,j})A_{p,j}(1 - B_{p,j}) = A_{i,j}(1 - B_{i,j})(1 - B_{i,j})$$

$$= A_{i,j}(1 - B_{i,j} - B_{i,j} + B_{i,j}) = A_{i,j}(1 - B_{i,j}). \quad (33)$$

Using (20) we obtain also

$$A_{i,j}B_{m,j}A_{p,j}B_{r,j} = \delta(m, p_j)A_{i,j}B_{r,j}. \quad (35)$$

Substituting (33)-(35) into (32) leads to

$$D_\alpha(i, m)D_\beta(p, r) = \prod_{j \in \alpha}(1 - B_{i,j}) \prod_{j \in \pi}(A_{i,j}(1 - B_{i,j})) \prod_{j \in \alpha}(A_{i,j}B_{r,j})$$

$$= \delta(m, p_\alpha)C(i, r) \prod_{j \in \pi}(1 - B_{i,j}) = \delta(m, p_\alpha)D(i, r), \quad (37)$$

where we have used the commutativity from Lemma 2.1. Now (25) is completely proved. □

**Proof of Theorem 1.11** The identities (25) mean that

$$\text{Alg}(\bigcup_{\alpha \subset \eta} \bigcup_{i \in P_\eta} \bigcup_{m \in P_\eta} \{D_\alpha(i, m)\}) = \prod_{\alpha \subset \eta} \text{Alg}(\bigcup_{i \in P_\eta} \bigcup_{m \in P_\eta} \{D_\alpha(i, m)\})$$

$$= \prod_{\alpha \subset \eta} \prod_{p \in P_\pi} \text{Alg}(\bigcup_{r \in P_\eta} \bigcup_{m \in P_\eta} \{D_\alpha(p \circ r, m)\}) = \prod_{\alpha \subset \eta} \prod_{p \in P_\pi} \mathbb{C}^{p[|\alpha|] \times p[|\alpha|]} = \prod_{n=0}^{N} (\mathbb{C}^{p \times p})^{(N)_n} p^{N-n}, \quad (39)$$

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where $D_{\alpha}(p \circ r, m)$ corresponds to the elementary matrix $(\delta(r, m))_{r, m \in P_{\alpha}}$ in the algebra $C[P_{\alpha}] = C^{p(\alpha) \times p(\alpha)}$. Let $A \in \mathcal{L}$ be some operator of the form (10)-(13). Using (21), (22) we deduce that

$$A = \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} a_{\alpha}(i, m) C_{\alpha}(i, m) = \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} a_{\alpha}(i, m) \sum_{\beta \supset a} D_{\beta}(i, m \circ i_{\beta \setminus a})$$ (40)

$$= \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} (\sum_{\beta \subseteq \alpha} \delta(\alpha \setminus \beta, m_{\alpha \setminus \beta}) a_{\beta}(i, m_{\beta})) D_{\alpha}(i, m) = \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} b_{\alpha}(i, m) D_{\alpha}(i, m)$$ (41)

which give us the form of matrices $B_{\alpha}$ (11) in the basis $D_{\alpha}$. To calculate the inverse mapping $\sigma^{-1}$ we take the operator $A$ in the basis $D_{\alpha}$ and, using (22), write it in the standard basis $C_{\alpha}$, i.e.

$$A = \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} b_{\alpha}(i, m) D_{\alpha}(i, m) = \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} b_{\alpha}(i, m) (\sum_{\beta \supset a} (-1)^{|\alpha \setminus \beta|} C_{\beta}(i, m \circ i_{\beta \setminus a}))$$ (42)

$$= \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} (\sum_{\beta \subseteq \alpha} (-1)^{|\alpha \setminus \beta|} \delta(\alpha \setminus \beta, m_{\alpha \setminus \beta}) b_{\beta}(i, m_{\beta})) C_{\alpha}(i, m) = \sum_{a \in \eta} \sum_{i \in P_{\eta}} \sum_{m \in P_{\eta}} a_{\alpha}(i, m) C_{\alpha}(i, m)$$ (43)

which give us (13). 

3. Examples

We consider the simplest case $p = 1$

$$\mathcal{L} = \text{Alg} \left( 1, \int_0^1 \cdot dx_1, \ldots, \int_0^1 \cdot dx_N \right) \cong \mathbb{C}^{2^N}. \quad (44)$$

By Corollary 1.2 the operator

$$Au(k) = \sum_{a \subseteq \{1, \ldots, N\}} a_{\alpha} \int_{[0,1]^{|\alpha|}} u(k) d\mathbf{k}_{\alpha}, \quad u \in L^2_N$$ (45)

is invertible if and only if the numbers $b_{\alpha} = \sum_{\beta \subseteq \alpha} a_{\beta}$ are all non-zero and then (see (13))

$$A^{-1}u(k) = \sum_{a \subseteq \{1, \ldots, N\}} \left( \sum_{\beta \subseteq \alpha} (-1)^{|\alpha \setminus \beta|} b_{\beta}^{-1} \right) \int_{[0,1]^{|\alpha|}} u(k) d\mathbf{k}_{\alpha}, \quad u \in L^2_N. \quad (46)$$

In particular, for $N = 2$ we have

$$\left( a \cdot + b \int_0^1 \cdot dk_1 + c \int_0^1 \cdot dk_2 + d \int_0^1 \int_0^1 \cdot dk_1 \cdot dk_2 \right)^{-1} = a^{-1} \cdot - \quad (47)$$
\[
\frac{b}{a(a+b)} \int_0^1 \cdot dk_1 - \frac{c}{a(a+c)} \int_0^1 \cdot dk_2 + \frac{(2a+b+c+d)bc-a^2d}{a(a+b)(a+c)(a+b+c+d)} \int_0^1 \int_0^1 \cdot dk_1 dk_2. \quad (48)
\]

Identity (48) leads also to the following example

\[
\left( \sum_{\alpha \subset \{1,\ldots,N\}} \int_{[0,1)^{\alpha}} \cdot dk_\alpha \right)^{-1} = \sum_{\alpha \subset \{1,\ldots,N\}} (-1/2)^{|\alpha|} \int_{[0,1)^{\alpha}} \cdot dk_\alpha. \quad (49)
\]

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