A PNJL Model for Adjoint Fermions with Periodic Boundary Conditions

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Recent work on QCD-like theories has shown that the addition of adjoint fermions obeying periodic boundary conditions to gauge theories on $R^3 \times S^1$ can lead to a restoration of center symmetry and confinement for sufficiently small circumference $L$ of $S^1$. At small $L$, perturbation theory may be used reliably to compute the effective potential for the Polyakov loop $P$ in the compact direction. Periodic adjoint fermions act in opposition to the gauge fields, which by themselves would lead to a deconfined phase at small $L$. In order for the fermionic effects to dominate gauge field effects in the effective potential, the fermion mass must be sufficiently small. This indicates that chiral symmetry breaking effects are potentially important. We develop a Polyakov-Nambu-Jona Lasinio (PNJL) model which combines the known perturbative behavior of adjoint QCD models at small $L$ with chiral symmetry breaking effects to produce an effective potential for the Polyakov loop $P$ and the chiral order parameter $\bar{\psi}\psi$. A rich phase structure emerges from the effective potential. Our results are consistent with the recent lattice simulations of Cossu and D’Elia, which found no evidence for a direct connection between the small-$L$ and large-$L$ confining regions. Nevertheless, the two confined regions are connected indirectly if an extended field theory model with an irrelevant four-fermion interaction is considered. Thus the small-$L$ and large-$L$ regions are part of a single confined phase.

I. INTRODUCTION

Recent progress in the study of QCD-like gauge theories has revealed that a confined phase can exist under certain conditions when one or more spatial directions are compactified and small [1,2]. This is surprising, because a small compact direction in Euclidean time gives rise to a deconfined phase for $SU(N)$ gauge theories. It is also intriguing, because one or more small compact directions give rise to a small effective coupling constant if the theory is asymptotically free. Thus we now have four-dimensional field theories in which confinement holds, and holds under circumstances where semiclassical methods may be reliably applied. In addition, the existence of confinement with one or more compact directions suggests that these new theories may make possible the construction of large-$N$ models in small space-time models, finally realizing the potential of the Eguchi-Kawai large-$N$ reduction [3,4,5].

When one or more directions are compact, the perturbative contribution of the gauge fields to the effective potential favors the deconfined phase. This leads to the well-known deconfinement transition in finite-temperature gauge theories. Thus it is necessary to modify the gauge theory in some way to obtain confinement with small compact directions. At present, there are two methods known for achieving this. The first method directly modifies the gauge action with terms non-local in the compact direction(s) [2], while the second adds adjoint fermions with periodic boundary conditions in the compact direction(s) [1], which is our subject here.

The term adjoint QCD is often used to refer to vector gauge theories based on the gauge group $SU(N)$ where, in addition to the gauge fields, fermions in the adjoint representation are fundamental fields. We will use $N_f$ to denote the number of flavors of adjoint Dirac fermions. With this notation $N_f = 1/2$ corresponds to $\mathcal{N} = 1$ supersymmetry when the mass of the fermion is taken to zero. If $N_f \leq 2$, the theory will be asymptotically free. Here we will consider models on $R^3 \times S^1$ that give rise to a confining phase if the circumference $L$ of $S^1$ is sufficiently small and the mass $m$ of the adjoint fermions is sufficiently light.

Confinement in $SU(N)$ gauge theories is associated with an unbroken global center symmetry, which is $Z(N)$ for $SU(N)$. The deconfined phase is associated with the breaking of this $Z(N)$ symmetry, which occurs spontaneously at high temperatures [6,7]. The order parameter for $Z(N)$ breaking in the compact direction is the Polyakov loop $P$, which is the path-ordered exponential of the gauge field in the compact direction

$$P(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^L dx_4 A_4(x) \right]. \quad (1)$$

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The trace of \( R \) in a representation represents the insertion of a heavy fermion in that representation into the system. Unbroken \( Z(N) \) symmetry implies \( \langle Tr_F P \rangle = 0 \) in the confined phase, and correspondingly \( \langle Tr_F P \rangle \neq 0 \) holds in the deconfined phase where \( Z(N) \) symmetry is broken. The character \( Tr_R P \) of each irreducible representation of \( SU(N) \) transforms as \( Tr_R P \rightarrow z^k Tr_R P \) under \( P \rightarrow zP \) for some \( k \in \{0, \ldots, N-1\} \). In the confined phase, the expectation value \( \langle Tr_R P \rangle \) is 0 for all representations that transform non-trivially under \( Z(N) \), i.e., have \( k \neq 0 \). In the case where bosons have periodic boundary conditions and fermions have antiperiodic boundary conditions in the compact direction, the transfer matrix in that direction is positive-definite and the system can be interpreted as being at temperature \( T = \beta^{-1} = L^{-1} \). In gauge models where all physical fields have zero \( N \)-ality, the free energy \( F \) for a single test fermion in the fundamental representation is related to the expectation value of Polyakov loop

\[
e^{-\beta F} = \langle Tr_F P \rangle.
\]

The restoration of \( Z(N) \) symmetry for small \( L \) and small \( m \) when the adjoint fermions have periodic boundary conditions is seen from the behavior of the one-loop effective potential. This calculation is a variant of the calculation of the effective potential at finite temperature \([12, 13]\). The sum of the effective potential for the fermions plus that of the gauge bosons gives

\[
V_{1-loop}(P, L, m, N_f) = \frac{1}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{Tr_A \rho_n}{n^2} \left[ 2N_f L^2 m^2 K_2(nLm) - \frac{2}{n^2} \right].
\]

Note that the first term in brackets, due to the fermions, is positive for every value of \( n \), while the second term, due to the gauge bosons, is negative. In the limit \( m \to 0 \), this becomes

\[
V_{1-loop} \approx \sum_{j, k=1}^{N_f} (1 - \pi \delta_{jk}) \frac{2(2N_f - 1)}{2} \left[ \frac{1}{2} \left( \phi_j - \phi_k \right)^2 \right] (\phi_j - \phi_k - 2\pi)^2
\]

where we have written \( P \) as \( P_{jk} = \delta_{jk} e^{i\phi_j} \) in a gauge where the background field \( A_4 \) is diagonal and independent of \( x_4 \). If \( mL \) is sufficiently small, this effective potential has a global minimum when the Polyakov loop eigenvalues are uniformly spaced around the unit circle. This is the unique \( Z(N) \)-symmetric solution for \( P \). On the other hand, if \( mL \) is sufficiently large, the fermion contribution to \( V_{1-loop} \) is negligible compared to the gauge boson contribution, and the system will be in the deconfined phase. Interestingly, there can be intermediate phases between the confined and deconfined phases as \( m \) is varied for \( N \geq 3 \) \([12, 13]\). It is also interesting to note that the one-loop effective potential identically vanishes when \( N_f = 1/2 \) in the \( m = 0 \) limit, consistent with the supersymmetry of the model \([10, 11]\).

It is clearly crucial that the fermion mass \( m \) be sufficiently small in order for confinement to be restored at small \( L \). Experience with phenomenological models \([12, 13]\) suggests that in fact it is the constituent mass which is relevant in determining the size of the fermionic contribution to the effective potential for \( P \). The potential importance of chiral symmetry breaking and restoration is underscored by lattice results for adjoint \( SU(3) \) fermions at finite temperature, where the fermions are antiperiodic in the Euclidean timelike direction. In this case, there is a large separation between the deconfinement temperature and the chiral symmetry restoration temperature, with \( T_c/T_L \simeq 7.8(2) \) \([14, 15]\). In order to explore the interrelationship of confinement and chiral symmetry breaking, we use a generalization of Nambu-Jona Lasinio models known as Polyakov-Nambu-Jona Lasinio (PNJL) models \([13]\). In NJL models, a four-fermion interaction induces chiral symmetry breaking. There has been a great deal of work on NJL models, both as phenomenological models for hadrons and as effective theories of QCD \([16, 17]\). NJL models have been used to study hadronic physics at finite temperature, but they include only chiral symmetry restoration, and do not model deconfinement. This omission is rectified by the PNJL models, which include both chiral restoration and deconfinement. The earliest model of this type was derived from strong-coupling lattice gauge theory \([12]\), but later work on continuum models have proven to be extremely powerful in describing the finite-temperature QCD phase transition \([13, 15]\). In PNJL models, fermions with NJL couplings move in a non-trivial Polyakov loop background, and the effects of gluons at finite temperature is modeled in a semi-phenomenological way. We will develop a model of this type for both fundamental and adjoint fermions below.

These models will show why \( mL \), with \( m \) interpreted as a constituent mass, is the key parameter. Because we expect \( m \) at worst to stay constant as \( L \) is decreased, the condition for confinement that \( mL \) is small can always be met at sufficiently small \( L \). It is not \textit{a priori} obvious what sufficiently small means, given the persistence of chiral symmetry breaking in the deconfined phase. Recent lattice simulations by Cossu and D'Elia \([18]\) have confirmed the existence of the small-\( L \) confined region in \( SU(3) \) lattice gauge theory with two flavors of adjoint fermions, and we will focus on this case in our analysis. Even if the small-\( L \) confined region exists and is accessible in lattice simulations, it is not necessarily the same phase as found for large \( L \). Put slightly differently, we would like to know if the small-\( L \) and large-\( L \) confined regions are smoothly connected, and thus represent the same phase. Our main result will be a phase diagram for adjoint periodic QCD for all values of \( L \), obtained using a PNJL model. On the way to this
goal, we will use as tests of our model both standard QCD with fundamental fermions and adjoint QCD with the usual antiperiodic boundary conditions for fermions. Our principal tool will be the effective potential for the chiral symmetry order parameter $\bar{\psi}\psi$ and the deconfinement order parameter $P$.

In section II the fermionic contribution to the effective potential is derived, and section III discusses the gluonic contribution. The complete effective potential is tested with fundamental fermions in section IV. Sections V and VI discuss adjoint fermions with antiperiodic and periodic boundary conditions, respectively. A final section summarizes our conclusions.

II. FERMIONIC CONTRIBUTION TO EFFECTIVE POTENTIAL

We start our calculations by constructing the contributions to the effective potential from both fundamental and adjoint fermions. NJL models use purely fermionic interactions as a proxy for the gauge theory interactions that give rise to chiral symmetry breaking. Typically, the relation between the gauge theory and an associated NJL model is fixed by matching important hadronic parameters such as $f_a$. In the case of NJL models at finite temperature, and particularly PNJL models, it is common to assume that the NJL model parameters are fixed by $T = 0$ hadronic parameters and remain constant as $T$ is increased, at least up to the deconfinement temperature. At substantially higher temperatures, it is possible that the parameters of the NJL model might have to be adjusted to reproduce properties of the gauge theory at those higher temperatures. This possibility is particularly acute in adjoint QCD, where the scale of chiral symmetry restoration is almost a factor of ten higher than the deconfinement scale [14, 15]. Therefore, it is desirable to consider the phase diagram of the appropriate PNJL model over a range of parameters, holding open the possibility that the PNJL parameters are varied with $\beta$ or $L$.

We take the fermionic part of the Lagrangian of our PNJL model to be [13, 16, 17]

$$L_F = \bar{\psi} (i\gamma \cdot D - m_0) \psi + \frac{g_S}{2} \left[ (\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i\gamma_5 \lambda^a \psi)^2 \right] + g_D \left[ \det \bar{\psi} (1 - \gamma_5) \psi + h.c. \right]$$

(5)

where $\psi$ is associated with $N_f$ flavors of Dirac fermions in the fundamental or adjoint representation of the gauge group $SU(N)$. The $\lambda^a$'s are the generators of the flavor symmetry group $U(N_f)$; $g_S$ represents the strength of the four-fermion scalar-pseudoscalar coupling and $g_D$ fixes the strength of an anomaly-induced term. For simplicity, we take the Lagrangian mass matrix $m_0$ to be diagonal: $(m_0)_{jk} = m_0 \delta_{jk}$. If $m_0$ and $g_D$ are taken to be zero, $L_F$ is invariant under the global symmetry $U(N_f)_L \times U(N_f)_R$. With a non-zero, flavor-independent mass $m_0$ and $g_D \neq 0$, the symmetry is reduced to $SU(N_f)_V \times U(1)_V$. The covariant derivative $D_\mu$ couples the fermions to a background Polyakov loop via the component of the gauge field in the compact direction.

We will use the identity

$$\lambda^a_{ij} \lambda^b_{kl} = 2\delta_{il} \delta_{jk}$$

(6)

for the generators of $U(N_f)$ to rewrite the scalar-scalar four-fermion coupling as

$$g_S \left[ (\bar{\psi}_i \psi_j) (\bar{\psi}_j \psi_i) \right].$$

(7)

The terms with $i = j$

$$g_S \sum_i (\bar{\psi}_i \psi_i)^2$$

(8)

give rise to an interaction that does not mix flavors. These terms are of order $N^2$, as opposed to the terms with $i \neq j$, which are only of order $N$. Keeping only the terms with $i = j$ is the Hartree approximation, but the $i \neq j$ contribution to the one-loop vacuum energy is canceled by the pseudoscalar interaction in any case. We will make a similar approximation for the $g_D$ interaction.

The partition function associated with $L_F$ is

$$Z_F = \int [d\bar{\psi}] [d\psi] e^{i \int d^4x L_F}$$

(9)

and depends implicitly on the background gauge field via the covariant derivative. We introduce a set of auxiliary scalar fields, $\sigma_j$ and $\alpha_j$ for each flavor $j$, and write the partition function as

$$Z = \int [d\bar{\psi}] [d\psi] [d\sigma] [d\alpha] \exp \left[ i \int d^4x \left( L_F + \sum_j \alpha_j (\sigma_j - \bar{\psi}_j \psi_j) \right) \right]$$

(10)
where the $\alpha_j$’s provide functional $\delta$ functions that set $\sigma_j$ equal to $\tilde{\psi}_j \psi_j$. This allows us to rewrite $Z$ in the form

$$\begin{align*}
Z &= \int [d\psi] [d\bar{\psi}] [d\sigma] [d\alpha] \exp \left\{ i \int d^4x \left[ \sum_j \bar{\psi}_j (i\gamma \cdot D - m_{0j} - \alpha_j) \psi_j + \sum_j g_S \sigma_j^2 + 2g_D \prod_j \sigma_j + \sum_j \alpha_j \sigma_j \right] \right\}. \tag{11}
\end{align*}$$

We proceed by integrating over the fermion fields to obtain

$$Z = \int [d\sigma] [d\alpha] e^{iS_{eff}} \tag{12}$$

where the effective action $S_{eff}$ is given by

$$S_{eff} = -i \sum_j \ln \left( \det \left( i\gamma \cdot D - m_{0j} - \alpha_j \right) \right) + \int d^4x \left[ \sum_j g_S \sigma_j^2 + 2g_D \prod_j \sigma_j + \sum_j \alpha_j \sigma_j \right]. \tag{13}$$

We look for the stationary saddle points of the effective action regarded as a function of the $\alpha_j$’s. The expression to be minimized has the form

$$\sum_j F_j(\alpha_j) + \sum_j \alpha_j \sigma_j - V(\sigma) \tag{14}$$

whose saddle point solutions satisfy

$$\frac{\partial F_j}{\partial \alpha_j} + \sigma_j = 0 \tag{15}$$

and

$$\alpha_j - \frac{\partial V}{\partial \sigma_j} = 0 \tag{16}$$

which reduces to

$$\left[ \frac{\partial F_j}{\partial \alpha_j} \right]_{\alpha_j = \frac{\partial V}{\partial \sigma_j}} + \sigma_j = 0. \tag{17}$$

This in turn is equivalent to extremizing

$$\sum_j F_j \left( \frac{\partial V}{\partial \sigma_j} \right) + \sum_j \sigma_j \frac{\partial V}{\partial \sigma_j} - V \tag{18}$$

In this particular case, $S_{eff}$ is reduced to

$$S_{eff} = -i \sum_j \ln \left( \det \left( i\gamma \cdot D - m_{0j} + 2g_S \sigma_j + 2g_D \prod_{k \neq j} \sigma_k \right) \right)$$

$$+ \int d^4x \left[ - \sum_j g_S \sigma_j^2 - 2g_D (N_f - 1) \prod_j \sigma_j \right].$$

We see immediately that each flavor behaves as if it has a mass given by $m_j = m_{0j} - 2g_S \sigma_j - 2g_D \prod_{k \neq j} \sigma_k$. The boundary conditions in the compact direction enter the effective potential through the fermion determinant, so it is through the mass $m_j$, the constituent mass, that the Polyakov loop and the chiral order parameters $\sigma_j$ are coupled.

It is generally convenient to use the language of finite temperature to describe both the case of finite temperature, $\beta^{-1} = T > 0$, with antiperiodic boundary conditions, and the case of a periodic spatial direction, $L < \infty$. The fermionic contribution to the effective potential has both a zero-temperature and a finite-temperature contribution. The zero-temperature part consists of a potential term, given by

$$\sum_j g_S \sigma_j^2 + 2g_D (N_f - 1) \prod_j \sigma_j \tag{19}$$
as well as a contribution from the fermion functional determinant. This is formally given by

\[-2d_R \sum_{j=1}^{N_f} 2 \Lambda \sqrt{\Lambda^2 + m_j^2} (2\Lambda^2 + m_j^2) + m_j^4 \frac{[\log (m_j^2) - 2 \log (\Lambda + \sqrt{\Lambda^2 + m_j^2})]}{32\pi^2}\]

(21)

where \(\Lambda\) is the momentum space cutoff. Thus the total \(T = 0\) fermionic contribution to the effective potential is

\[V_{F0}(m, m_0, \sigma) = \sum_j g_S \sigma_j^2 + 2g_D (N_f - 1) \prod_j \sigma_j - 2d_R \sum_{j=1}^{N_f} 2 \Lambda \sqrt{\Lambda^2 + m_j^2} (2\Lambda^2 + m_j^2) + m_j^4 \frac{[\log (m_j^2) - 2 \log (\Lambda + \sqrt{\Lambda^2 + m_j^2})]}{32\pi^2}\]

(22)

In PNJL models, the finite-temperature contribution from the fermion determinant depends on the background Polyakov loop. It is convenient to work in a gauge where the temporal component of the background gauge field, \(A_t(x, t)\), is constant and diagonal. The covariant derivative then becomes \(\gamma^\mu \partial^\mu = \gamma^0 \partial^0 - i\gamma^4 A_4\). The one-loop free energy of fermions in a representation \(R\) of \(SU(N)\) gauge theory with zero chemical potential can be written as

\[V_{FL}(P, m, \sigma, \beta) = -2 \sum_j Tr_R \left[ \frac{1}{L} \int \frac{d^3k}{(2\pi)^3} \ln(1 + Pe^{-L\omega_k^{(j)}}) + h.c. \right]\]

(23)

where the minus sign is used for periodic boundary conditions and plus for antiperiodic. In the latter case, \(L\) is replaced by \(\beta\). The potential term \(V_{FT}\) has a series expansion in terms of modified Bessel functions

\[V_{FL}(P, m, \sigma, \beta) = \sum_j \frac{2m_j^2}{\pi^2 L^2} \sum_{n=1}^{\infty} \frac{(-1)^n Tr_R P^n}{n^2} K_2(n L m_j)\]

(24)

which is rapidly convergent for all values of the mass \(\beta\).

Although NJL models display chiral symmetry breaking and are plausibly related to QCD-like theories, they do not capture exactly all the features of the gauge theory to which they are related. There are many ways of associating a given NJL model with a gauge theory, and there is freedom in choosing both regularization scheme and parameters of the two theories to match. Typically, the hadronic parameters used represent hadronic physics at zero temperature, and those parameters are held fixed up to the chiral transition and perhaps beyond. For the case of \(N = 3\) models with two and three flavors in the fundamental representation, appropriate choices of \(m_0, g_S\) and \(\Lambda\) can capably model many features of hadron physics. In what follows, we will take \(N_f = 2\), and take the masses \(m_0, \sigma\) to be equal to a common mass which we also write as \(m_0\). In this case, the contribution to \(S_{eff}\) from \(g_S\) and \(g_D\) has the same form. It is convenient to take \(g_D = 0\), and also to write the common constituent mass as \(m = m_0 - 2g_S \sigma\) \([16, 17]\). In PNJL models, additional information about confinement at non-zero temperature must be added. As will be discussed below, we will use as input the deconfinement temperature \(T_d\) for the pure gauge theory.

At non-zero temperature, the observables of a given gauge theory will evolve with \(T\) according to the finite-temperature renormalization group, but the parameters of an associated NJL or PNJL model do not naturally evolve in a related way. If we follow the behavior of both the gauge theory and an associated PNJL model over a large scale of temperatures, it may be necessary or desirable to consider the parameters of the PNJL model as varying with \(T\). For example, Unsal has developed a comprehensive scenario for periodic adjoint fermions with \(m_0 = 0\) in which chiral symmetry is spontaneously broken by monopole-induced anomaly terms \([11, 19]\). If this interaction is sufficiently weak, it may not induce chiral symmetry breaking, raising the possibility of confinement without chiral symmetry breaking. There is also the possibility of modifying the strength of chiral symmetry breaking by adding additional
couplings compatible with all symmetries have been added. The easiest construction of such an extended model might be obtained by adding to a gauge theory a non-renormalizable four-fermion coupling of exactly the same form as in an NJL model, but with opposite sign for the coupling:

\[
L_{\text{gauge}} \rightarrow L_{\text{gauge}} + \frac{\delta g S}{2} \left[ (\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2 \right].
\]  

(25)

If we imagine that the gauge theory is associated with an NJL model \(L_{N\text{JL}}\) with a certain value of \(g_S\), then the additional term shifts the NJL Lagrangian as

\[
L_{N\text{JL}} \rightarrow L_{N\text{JL}} + \frac{\delta g'_S}{2} \left[ (\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2 \right].
\]  

(26)

where one expects \(\delta g'_S = \delta g_S\) for a sufficiently small perturbation. In principle, a lattice version of the extended gauge theory could be simulated, and the results combined with analytical results for the NJL model to determine \(\delta g'_S\) as a function of \(\delta g_S\). In fact, lattice gauge theories with additional non-renormalizable four-fermion terms added have already been used in the study of finite temperature gauge theories [20, 21, 22]. For the two-flavor case we discuss, it is useful to treat all parameters of the model as potentially varying.

III. GLUONIC CONTRIBUTION TO EFFECTIVE POTENTIAL

For gauge bosons, the one-loop finite temperature free energy in a background Polyakov loop is given by an expression similar to the one for fermions. The boundary conditions for the gauge bosons are periodic in all cases considered here, so \(L\) and \(\beta\) may be used equivalently in the gluonic sector. We have

\[
V_{g-1\text{loop}}(P) = 2 Tr_A \left[ \frac{1}{L} \int \frac{d^3k}{(2\pi)^3} \ln(1 - Pe^{-L\Omega}) \right]
\]  

(27)

where we have inserted a mass parameter in \(\Omega_k = \sqrt{k^2 + M^2}\) for purely phenomenological reasons explained below. If we take the zero mass limit of the series expansion, we get the standard expression for the one-loop gauge boson free energy [3, 7]

\[
- \lim_{M \rightarrow 0} \frac{M^2}{\pi^2 L^2} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^2} K_2(nLM) = - \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^4}.
\]  

(28)
This infinite series may be summed exactly, giving an expression for \( V_{g-1\text{loop}} \) proportional to \( L^{-4} \) involving the fourth Bernoulli polynomial. If we also retain the next-order term in a high-temperature expansion, proportional to \( M^2/L^2 \), we obtain a useful phenomenological model for the deconfinement transition in pure gauge theories \cite{24}. The potential takes the form

\[
V_g(P) = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{Tr A P^n}{n^4} + \frac{M^2}{2\pi^2 L^2} \sum_{n=1}^{\infty} \frac{Tr A P^n}{n^2} \tag{29}
\]

We stress that the mass parameter \( M \) should not be interpreted as a gauge boson mass, nor do we limit ourselves to \( ML \ll 1 \). The crucial feature of this potential is that for sufficiently large values of the dimensionless parameter \( ML \), the potential leads to a \( Z(N) \)-symmetric, confining minimum for \( P \) \cite{9, 24}. On the other hand, for small values of \( ML \), the pure gauge theory will be in the deconfined phase. It will be important later that \( V_g \) is a good representation of the gauge boson contribution for high temperatures; in other PNJL models, the gauge boson contribution has sometimes been chosen so as to be valid over a more narrow range of temperatures. Both of the infinite sums can be carried out exactly, giving a closed form for \( V_g \) as a function of the angles \( \phi_j \) in terms of the fourth and second Bernoulli polynomials \cite{23}.

In the Polyakov gauge, the Polyakov loop in the fundamental representation of \( SU(3) \) can be written as \( P_{jk} = \exp(i\phi_j)\delta_{jk} \) with two independent angles. With the use of \( Z(3) \) symmetry, it is sufficient to consider the case where \( \langle Tr_F P \rangle \) is real. Thus we consider only diagonal, special-unitary matrices with real trace, which may be parametrized by taking \( \phi_1 = \phi \), \( \phi_2 = -\phi \), and \( \phi_3 = 0 \), or \( P = \text{diag} [e^{i\phi}, e^{-i\phi}, 1] \) with \( 0 \leq \phi \leq \pi \). The unique set of \( Z(3) \)-invariant eigenvalues are obtained for \( \phi = 2\pi/3 \). For \( SU(3) \), \( V_g \) takes the form:

\[
V_g(P) = \left( \frac{3\phi^2}{2\pi^2} - \frac{2\phi}{\pi} + \frac{2}{3} \right) M^2 + \frac{1}{L^4} \left( -\frac{135\phi^4 - 300\pi\phi^3 + 180\pi^2\phi^2 - 16\pi^4}{90\pi^2} \right) \tag{30}\]

We will set the mass scale \( M \) by requiring that \( V_g \) yields the correct deconfinement temperature for the pure gauge theory, with a value of \( T_d \approx 270 \text{ MeV} \). This gives \( M = 596 \text{ MeV} \) \cite{23}. The pressure \( p \) is given by the value of \(-V_g\) at the minimum of the potential. The behavior obtained for the pressure, the energy density \( \epsilon \), and the so-called interaction measure \( \Delta \equiv (\epsilon - 3p)/T^4 \) are all consistent with lattice simulations for \( T > T_d \). As shown in figure 1 the order parameter \( Tr FP \) for the deconfinement transition jumps at \( T_d \), indicating a first-order deconfinement transition for \( SU(3) \).

IV. FUNDAMENTAL FERMIONS

As a test of all the components of the effective potential we have assembled, we consider the case of two flavors of fundamental fermions at finite temperature. The fermions obey antiperiodic boundary conditions, so we identify the compact circumference \( L \) with the inverse temperature \( \beta = 1/T \). A very common choice of zero-temperature parameters for two degenerate light flavors is \( m_0 = 5.5 \text{ MeV} \), \( \Lambda = 631.4 \text{ MeV} \), and \( g_s = 2 \times 5.496 \text{ GeV}^{-2} \) \cite{13, 17}. In figure 2 we show the expectation value of the Polyakov loop in the fundamental representation \( Tr_F P \) and the constituent mass \( m \) as functions of the temperature. The behavior in the crossover region is very similar to the results of Fukushima \cite{13}, and shows the explanatory power of PNJL models. The constituent mass \( m \) is heavy at low temperatures, due to chiral symmetry breaking. The larger the constituent mass, the smaller the \( Z(3) \) breaking effect of the fermions. On the other hand, a small value for \( \langle Tr_F P \rangle \) reduces the effectiveness of finite-temperature effects in restoring chiral symmetry. These synergistic effects combine in the case of fundamental representation fermions to give a single crossover temperature at which both order parameters are changing rapidly, in agreement with lattice simulations.

V. ADJOINT FERMIONS WITH ANTIPERIODIC BOUNDARY CONDITIONS

Adjoint \( SU(3) \) fermions at finite temperature show a completely different behavior in lattice simulations from fundamental fermions. Because the adjoint fermions respect the \( Z(3) \) center symmetry, there is a true deconfinement transition where \( Z(3) \) spontaneously breaks. Lattice simulations have shown that chiral symmetry is restored at a substantially higher temperature than the deconfinement temperature \cite{14, 15}. Unlike the case of fundamental fermions, there are no comprehensive \( T = 0 \) lattice results for adjoint \( SU(3) \) fermions, no established parameter sets for NJL models and, of course, no experimental results to provide guidance. We will again consider the case \( N_f = 2 \) with degenerate fermion masses. The \( T = 0 \) parameters needed are \( g_s \) and \( \Lambda \). The mass parameter \( M \) is again
Figure 2: The constituent mass \( m \) and \( \langle TrFP \rangle \) for two-flavor QCD with fundamental representation fermions with antiperiodic boundary conditions as a function of temperature. The order parameters are normalized by dividing by their values at \( T = 0 \) and \( T = \infty \), respectively.

We expect that the constituent mass \( m(T = 0) \) must be substantially larger than the corresponding value for fundamental fermions, because a large constituent mass is necessary to delay the onset of chiral symmetry restoration. On the other hand, a large constituent mass at the deconfining transitions would be expected to lead to a relatively small change in the deconfining temperature. The ratio \( m(T = 0)/\Lambda \) should be less than one in order for the cutoff theory to be meaningful. In the case of fundamental fermions, this ratio is relatively large, on the order of 0.5. We have generally found that for adjoint fermions a larger ratio of \( m(T = 0)/\Lambda \) with \( T_c/T_d \) fixed implies a larger value of \( m(T = 0) \). We will work with the representative case of \( m_0 = 0 \) and \( m(T = 0)/\Lambda = 0.1 \). This gives \( \Lambda = 23.22 \text{ GeV} \) and thus \( m(T = 0) = 2.322 \text{ GeV} \), with \( \kappa = 1.2653 \). For comparison, the critical value of \( \kappa \), \( \kappa_c \), below which \( m(T = 0) = 0 \), is \( \pi^2/8 \approx 1.234 \). In figure 3, we show the constituent mass \( m \) and Polyakov expectation value \( \langle TrFP \rangle \) as a function of temperature, normalized by dividing by their values at \( T = 0 \) and \( T = \infty \), respectively. We see that the deconfinement temperature \( T_d \) is very close to its value in the pure gauge theory, due to the large adjoint fermion constituent mass. The transition is first order. The constituent mass \( m \) has a slow decline to a second-order transition at a substantially higher temperature, as indicated by lattice simulations [14, 15].

VI. ADJOINT FERMIONS WITH PERIODIC BOUNDARY CONDITIONS

Having established that our PNJL model can provide a useful representation for the finite-temperature behavior of SU(3) with both fundamental and adjoint fermions, we turn now to our main interest, adjoint fermions on \( R^3 \times S^1 \) with periodic boundary conditions for the fermions in the compact direction. The sole change in the total effective potential from the finite-temperature case lies in the contribution of the adjoint fermions, given by \( V_{FL} \). This single change leads to an unusual and unexpected phase structure as \( L \) is decreased [2, 9].

The effective potential can be written as

\[
V_{eff} (m, P) = V_{F0} (m) + V_{FL} (m, P) + V_g (P)
\]

\[
= V_{F0} (m) + \sum_{n=1}^{\infty} \left[ \sum_j \frac{2m_j^2L^2}{\pi^2n^2} K_2 (nLm_j) + \frac{M^2L^2}{2\pi^2n^2} - \frac{2}{\pi^2n^4} \right] \frac{TrA^P}{L^4}
\]  

\[ (31) \]
which is general for the case of adjoint fermions with periodic boundary conditions in $SU(N)$. This equation contains the basic physics of the phase diagram. The contributions of the first two terms in square brackets are always positive, while the final term is always negative. Consider first the $n = 1$ term

$$\left[ \sum_j \frac{2m_j^2L^2}{\pi^2} K_2(Lm_j) + \frac{M^2L^2}{2\pi^2} - \frac{2}{\pi^2} \right] \frac{TrAP}{L^4}$$

(32)

which is generally the largest term in the series. If either $ML$ is sufficiently large, or some combination of $m_jL$’s is sufficiently small, the overall sign of the expression in square brackets will be positive. The $n = 1$ term will then favor the minimization of $TrAP$, giving $TrAP = -1$ and therefore $TrFP = 0$. The first case, with $L$ sufficiently large, can be identified with the usual low-temperature confined phase of the pure gauge theory. The second case, however, is capable of producing a confined phase at small $L$, provided the constituent mass $m$ is sufficiently small. This will certainly occur if chiral symmetry is restored, but it can also occur if $L$ is increased even if $m$ is essentially constant. On the other hand, if the sign of the coefficient of $TrAP$ in $V_{eff}$ is negative, the $n = 1$ term will favor maximizing $TrAP$, which occurs in the deconfined phase. This simple picture is sufficient for $SU(2)$. The actual phase structure, as predicted theoretically and found in lattice simulations, is more complicated when $N \geq 3$ [2, 9, 18]. In the case of $SU(3)$, there is an additional phase, the skewed phase, in which $TrFP$ is non-zero but negative. As $N$ increases, the number of possible phases increases as well, giving rise to a rich phase structure for small $L$ [9].

We consider the behavior of $m$ and $TrFP$ with periodic fermions using the same parameters we used for the antiperiodic case. Figure 4 shows the behavior of $m$ and $TrFP$ as a function of $L^{-1}$ for the $m(L = \infty)/\Lambda = 0.1$ parameter set, with $m_0 = 0$. We see that chiral symmetry breaking persists at $L^{-1} = 10$ GeV, which is much higher than the chiral restoration temperature for antiperiodic fermions. The constituent mass $m$ does fall eventually as $L^{-1}$ increases, and chiral symmetry is ultimately restored, but at a temperature on the order of $\Lambda$. In figure 4, $TrFP$ shows three distinct phase transitions as a function of $L^{-1}$. As $L^{-1}$ increases, the confined phase gives way to the deconfined phase in a first-order phase transition. Because the constituent mass of the fermions is large, the critical value of $L^{-1}$ for this transition is approximately equal to $T_d$. As $L^{-1}$ increases, there are two more first-order transitions, from the deconfined phase to the skewed phase, and then from the skewed phase to a small-$L$ confined phase we describe as reconfined.

The ordering of the phases seen in the behavior of $TrFP$ for $m_0 = 0$ persists as $m_0$ is increased. Figure 5 shows that with this parameter set the value of $L^{-1}$ for the confinement-deconfinement transition stays essentially at $T_d$ as $m_0$ increases. The only significant change in the phase diagram is the smooth growth of the extent of the skewed
Figure 4: The constituent mass $m$ and $\langle \text{Tr} P \rangle$ for two-flavor QCD with adjoint representation fermions with periodic boundary conditions as a function of $L^{-1}$ for one choice of $\kappa$ and $\Lambda$, with $m_0 = 0$. The order parameters are normalized by dividing by their values at $L = \infty$ and $L = 0$, respectively. C, D, S and R refer to the confined phase, deconfined phase, skewed phase and reconfining phase respectively.

Figure 5: The phase diagram for two-flavor QCD with adjoint representation fermions with periodic boundary conditions as a function of $L^{-1}$ and Lagrangian mass $m_0$ for one choice of $\kappa$ and $\Lambda$. C, D, S and R refer to the confined phase, deconfined phase, skewed phase and reconfining phase respectively. The Lagrangian mass is measured in MeV.

phase as $m_0$ increases. Naively, one might expect that the skewed and reconfining phases would disappear as $m_0 \to \infty$. However, at any fixed $m_0$, there will be a sufficiently large value of $L^{-1}$ such that the fermionic term in the effective potential overwhelms the gauge boson contribution, and the skewed and reconfining phases do not disappear, but move to very large values of $L^{-1}$. On the other hand, taking the limit $m_0 \to \infty$ at fixed $L^{-1}$ greater than $T_d$ always yields the deconfined phase.

PNJL models have a larger set of parameters available, namely $\{M, m_0, \kappa, \Lambda\}$, than in adjoint $SU(N)$ gauge theories, which have only $m_0$ and $\Lambda_{\text{adj}}$. To the extent that a PNJL model faithfully reproduces the physics of adjoint $SU(N)$,
we can think of adjoint $SU(N)$ as giving a two-dimensional surface in the four-dimensional space of PNJL parameters. For the case $m_0 = 0$, where the Lagrangians are chirally symmetric, we obtain a line through a three-dimensional space. This motivates us to consider the phase structure of our PNJL model in the $L^{-1} - \kappa$ plane with $\Lambda$ and $M$ fixed. Note however, that the PNJL model will not break chiral symmetry at $L^{-1} = 0$ unless $\kappa > \kappa_c = \pi^2/8 \approx 1.234$. In Figure 6, we show the phase diagram in the $L^{-1} - \kappa$ plane, with $m_0 = 0$ and $M$ and $\Lambda$ as before, obtained by numerically minimizing $V_{\text{eff}}$.  

For most values of $\kappa$ larger than $\kappa_c$, the confined large-$L$ phase and the reconfined phase at small $L$ are separated by three phase transitions. There is a transition from the confined phase to the deconfined phase, then another transition from the deconfined phase to the skewed phase, followed by a third transition from the skewed phase to the reconfined phase. All of these transitions are characterized by abrupt changes in $Tr_F P$, while the chiral order parameter shows only a slow decrease with increasing temperature. However, there is a narrow range of $\kappa$ between approximately $1.250$ and $\kappa_c \approx 1.234$ where confinement holds at all temperatures, and chiral symmetry remains broken. In this extended phase diagram, the confined and reconfined regions are smoothly connected. Although this connection appears only for small range of $\kappa$ values, the corresponding range of constituent mass values is not necessarily small. In Figure 7, we show the same phase diagram, now plotted with $m(L^{-1} = 0)$ replacing $\kappa$.

There is a remaining puzzle associated with chiral symmetry restoration. If $\kappa < \kappa_c$, there is no chiral symmetry breaking, the adjoint fermions are always light, and $Tr_F P = 0$ for all values of $L^{-1}$. The region $\kappa < \kappa_c$, is separated from the confined phase with $\kappa > \kappa_c$ by a chiral transition in which $m > 0$ for $\kappa > \kappa_c$. However, for $m_0$ light but non-zero, there is no true chiral transition across $\kappa = \kappa_c$, only a rapid crossover. The apparent existence within this model of a confining phase with unbroken chiral symmetry must give us pause. There is a long-standing argument due to Casher that suggests that under very general circumstances confining theories must also break chiral symmetry [23]. Confinement without chiral symmetry breaking will also occur in this model if $L$ is sufficiently small, even if $\kappa > \kappa_c$. It is not clear if this is due to a defect in Casher’s argument, a failure of the PNJL model, or something more subtle.

Figures 6 and 7 show that the small-$L$ and large-$L$ confining phases are connected, at least within the PNJL model. A similar smooth connection between the two regions has been observed in lattice simulations using a deformation of the pure gauge action in which a $Tr_A P$ term is added to the action [2]. However, the connection between the large-$L$ and small-$L$ regions seen here in the PNJL model may not appear in adjoint QCD without additional terms in the action. Our results bear directly on the recent work by Cossu and D’Elia [18], in which they performed lattice simulations of two-flavor $SU(3)$ gauge theory with periodic adjoint fermions. The simulations were carried out on $16^3 \times 4$ lattices at various values of the dimensionless lattice parameters, the gauge coupling $\beta$ and the dimensionless lattice fermion mass $m_l$. The size of the compact dimension $L$ in physical units is given by $L = 4a$, where $a$ is determined from $\beta$ and $m_l$ via the renormalization group. At fixed $m_l$, larger values of $\beta$ correspond to smaller values of $L$. In these simulations, clear evidence was found for confined, deconfined, skewed, and finally again confined phases.
as $\beta$ was increased, corresponding to shrinking $L$. As $m_l$ was decreased, the critical values of $\beta$ at which the three transitions occurred moved closer together. However, an extrapolation to $m_l = 0$ indicates that the two confined regions remain separated as $m_l$ goes to zero. The deconfined phase clearly persists in this limit, but the skewed phase may or may not disappear when $m_l = 0$. All lattice simulations on finite lattices have lower limits on the Lagrangian mass $m_0$, so it is a priori possible that the two confining regions seen in the simulations are connected, but for a very small range of $m_0$. The phase structure seen in these lattice simulations can also be obtained from our PNJL model if the chiral limit, $m_0 = 0$, lies near the triple point where the confined, deconfined, and skewed phases meet. For example, if we fix $m(L^{-1} = 0)$ by adjusting $\kappa$, and then decrease $L$, we will obtain either one, three, or four phases, as may be seen from Fig. 7. The figure shows that the scale over which $m(L^{-1} = 0)$ must vary in order to obtain the different phase structures is quite small, on the order of $150$ MeV, compared to the scale of $m(L^{-1} = 0)$ itself.

VII. CONCLUSIONS

We have extended the PNJL treatment of SU(3) gauge theories to the case of adjoint fermions with periodic boundary conditions on $R^3 \times S^1$. This class of models can have $Z(N)$ symmetry for small $L$ provided the constituent mass of the fermions is sufficiently light. The constituent mass rather than the Lagrangian mass is relevant because the constituent mass determines the fermionic coupling to the Polyakov loop. Previous analyses implicitly assumed that the constituent mass could be set arbitrarily, ignoring the effects of chiral symmetry breaking. Because PNJL models give a phenomenological description of both deconfinement and chiral symmetry restoration, they are natural tools for exploring adjoint fermions with periodic boundary conditions.

We have shown that a simple model reproduces the known successes of PNJL models for fundamental fermions while at the same time reproducing the expected behavior at high temperatures needed with adjoint fermions. The large separation between the deconfinement transition and the chiral symmetry restoration transition for adjoint fermion theories with antiperiodic boundary conditions requires a PNJL model which reproduces the behavior of the pure gauge theory to much smaller values of $L$ than have been considered before. We have extensively studied the case of the SU(3) gauge theory with two fermions degenerate in mass. This two-flavor case is particularly simple to work with, because the effective potential can be written as a function of a single coupling constant.

The results for our SU(3) PNJL model with two flavors of periodic adjoint Dirac fermions can be summarized in terms of two parameters: $\kappa$, which represents the dimensionless strength of four-fermion interaction, and $m_0$, the Lagrangian mass of the fermions. As with all NJL models, there is a value $\kappa_c$ below which chiral symmetry breaking does not take place, even as $L \to \infty$. Because chiral symmetry is broken at $L^{-1} = 0$ in the corresponding gauge theory, $\kappa$ must be taken greater than $\kappa_c$ at large $L$ in order to reproduce the behavior of the associated gauge theory. For any fixed values of $\kappa$ and $m_0$, there will be a value of $L$ below which the system will be in the confined phase. In most of those cases, the system will pass through the confined, deconfined, and skewed phases as $L$ is decreased,
before the confined phase is regained. There is a narrow range of parameters where only the deconfined phase lies between the two confining regions. If \( m_0 \) is set to zero, there is a small region in the \( L^{-1} - \kappa \) plane, lying above \( \kappa_c \), that connects the large-\( L \) and small-\( L \) confined regions. Because the largest contribution to the constituent mass \( m \) is from chiral symmetry breaking, this behavior will persist for some small range of non-zero \( m_0 \). Thus there is a single confining region, accessible in principle in lattice simulations via a gauge theory to which additional four-fermion terms have been added. As we have seen, our results for the PNJL model are completely compatible with the lattice simulations of Cossu and D’Elia [18], in which they performed lattice simulations of two-flavor \( SU(3) \) gauge theory with periodic adjoint fermions. The phase diagram obtained from their simulations can easily be obtained from the PNJL model with the appropriate choice of parameters. With more input from lattice simulations of adjoint fermions, particularly of basic hadronic properties for large \( L \), we could match the behavior of the PNJL model more closely to the gauge theory.

Adjoint fermions with periodic boundary conditions provide a local field theory that induces confinement in a small-\( L \) region where semiclassical arguments are valid. It appears likely that the two confinement regions, small-\( L \) and large-\( L \), can be connected if additional terms, corresponding to irrelevant operators, are added to the action. This may have implications for the use of periodic adjoint fermions in achieving volume independence in the large-\( N \) limit [26, 27, 28, 29], which requires unbroken \( Z(N) \) symmetry. Perhaps even more important is our newly-gained ability to understand confinement in a four-dimensional gauge theory, albeit under somewhat exotic and unexpected circumstances.

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