Spherically symmetric solutions in a FRW background

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We impose perfect fluid concept along with slow expansion approximation to derive new solutions which, considering non-static spherically symmetric metrics, can be treated as Black Holes. We will refer to these solutions as Quasi Black Holes. Mathematical and physical features such as Killing vectors, singularities, and mass have been studied. Their horizons and thermodynamic properties have also been investigated. In addition, relationship with other related works (including mcVittie’s) are described.

I. INTRODUCTION

The Universe expansion can be modeled by the so called FRW metric

\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \right] \]  

(1)

where \( k = 0, +1, -1 \) are curvature scalars which represent the flat, closed and open universes, respectively. The WMAP data confirms a flat (\( k = 0 \)) universe \[1\]. \( a(t) \) is the scale factor and for a background which is filled by a perfect fluid with equation of state \( p = \omega \rho \), there are three classes of expanding solutions. These three solutions are

\[ a(t) = a_0 t^{\frac{\omega + 1}{3}} \]  

(2)

for \( \omega \neq 0 \) when \(-1 < \omega \) and ,

\[ a(t) = a_0 e^{Ht} \]  

(3)

for \( \omega = -1 \) (dark energy), and for the Phantom regime (\( \omega < -1 \)) is

\[ a(t) = a_0 (t_0 - t)^{\frac{-1}{\omega + 1}} \]  

(4)

where \( t_0 \) is the big rip singularity time and will be available, if the universe is in the phantom regime.

In Eq. (2), \( H(\equiv \frac{\dot{a}(t)}{a(t)}) \) is the Hubble parameter and the current estimates are \( H = 73^{+4}_{-3} \text{km s}^{-1} \text{Mpc}^{-1} \) \[1\].

Note that, at the end of the Phantom regime, everything will decompose into its fundamental constituents \[2\]. In addition, this spacetime can be classified as a subgroup of the Godel-type spacetime with \( \sigma = m = 0 \) and \( k' = 1 \) \[3\].

A signal which was emitted at the time \( t_0 \) by a co-moving source and absorbed by a co-moving observer at a later time \( t \) is affected by a redshift (\( z \)) as

\[ 1 + z = \frac{a(t)}{a(t_0)}. \]  

(5)

The apparent horizon as a marginally trapped surface, is defined as \[4\]

\[ g^{\mu\nu} \partial_\mu \xi \partial_\nu \xi = 0, \]  

(6)

which for the physical radius of \( \xi = a(t)r \), the solution will be:

\[ \xi = \frac{1}{\sqrt{H^2 + \frac{k}{a(t)^2}}}. \]  

(7)

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The surface gravity of the apparent horizon can be evaluated by:

\[ \kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \xi). \]  

(8)

Where the two dimensional induced metric is \( h_{ab} = \text{diag}(-1, a^2(t)) \). It was shown that the first law of thermodynamics is satisfied on the apparent horizon. The special case of \( \omega = -1 \) is called the dark energy, and by a suitable change of variables one can rewrite this case in the static form:

\[ ds^2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{(1 - H^2 r^2)} + r^2 d\Omega^2. \]

(9)

This metric belongs to a more general class of spherically symmetric, static metrics. For these class of spherically symmetric static metrics, the line element can be written in the form of:

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \]

(10)

where the general form of \( f(r) \) is:

\[ f(r) = 1 - 2 \frac{m}{r} + \frac{Q^2}{r^2} - H^2 r^2. \]

(11)

In the above expression, \( m \) and \( Q \) represent mass and charge, respectively. For this metric, one can evaluate redshift:

\[ 1 + z = \frac{1 - 2 \frac{m}{r_0} + \frac{Q^2}{r_0^2} - H^2 r_0^2}{1 - 2 \frac{m}{r} + \frac{Q^2}{r^2} - H^2 r^2}. \]

(12)

Where, \( r_0 \) and \( r \) are radial coordinates at the emission and the absorption points. For the horizons, the radius and the surface gravity can be found using equations:

\[ g_{tt} = f(r) = 0 \overset{\rightarrow}{\rightarrow} r_h \]

(13)

\[ \kappa = \frac{f'(r)}{2} \big|_{r_h}, \]

where \( (') \) denotes derivative with respect to the coordinate \( r \). From the thermodynamic laws of Black Holes (BHs) we know

\[ T = \frac{\kappa}{2\pi}, \]

(14)

which \( T \) is the temperature on the horizon. Validity of the first law of the thermodynamics on the static horizons for the static spherically symmetric spacetime has been shown.

The BHs with the FRW dynamic background has motivated many investigations. The first approach, which is named Swiss Cheese, includes efforts in order to find the effects of the expansion of the Universe on the gravitational field of the stars, introduced originally by Einstein and Straus (1945). In these models, authors tried to join the Schwarzschild metric to the FRW metric by satisfying the junction conditions on the boundary, which is an expanding timelike hypersurface. The inner spacetime is described by the Schwarzschild metric, while the FRW metric explains the outer spacetime. These models don’t contain dynamical BHs, Because the inner spacetime is in the Schwarzschild coordinate, hence, is static. In addition, the Swiss Cheese models can be classified as a subclass of inhomogeneous Lemabitre-Tolman-Bondi models.

Looking for dynamical BHs, some authors used the conformal transformation of the Schwarzschild BH, where the conformal factor is the scale factor of the famous FRW model. Originally, Thakurta (1981) have used this technique and obtained a dynamical version of the Schwarzschild BH. Since the Thakurta spacetime is a conformal transformation of the Schwarzschild metric, it is now accepted that its redshift radii points to the co-moving radii of the event horizon of BH. By considering asymptotic behavior of the gravitational lagrangian (Ricci scalar), one can classify the Thakurta BH and its extension to the charged BH into the same class of solutions. The Thakurta spacetime sustains an inward flow, which leads to an increase in the mass of BH. This ingoing flow comes from the back-reaction effect and can be neglected in a low density background. In fact, for the low density background, the mass will be decreased in the Phantom regime. Also, the radius of event horizon increases with the scale factor when its temperature decreases by the inverse of scale factor.
Using the Eddington-Finkelstein form of the Schwarzschild metric and the conformal transformation, Sultana and Dyer (2005) have constructed their metric and studied its properties [22]. In addition, unlike the Thakurta spacetime, the curvature scalars do not diverge at the redshift singularity radii (event horizon) of the Sultana and Dyer spacetimes. Since the Sultana and Dyer spacetimes is conformal transformation of the Schwarzschild metric, it is now accepted that the Sultana and Dyer spacetimes include dynamic BHs [16]. Various examples can be found in [16, 23–25]. Among these conformal BHs, only the solutions by McLure et al. and Thakurta can satisfy the energy conditions [10, 16]. Static charged BHs which are confined into the FRW spacetime and the dynamic, charged BHs were studied in [20, 33]. The Brane solutions can be found in [34, 36].

In another approach, mcVittie found new solutions including contracting BHs in the coordinates co-moving with the universe’s expansion [37]. Its generalization to the arbitrary dimensions and to the charged BHs can be found in [38, 39]. In these solutions, it is easy to check that the curvature scalars diverge at the redshift singularities. In this approach, authors have used the isotropic form of the FRW metric along as the perfect fluid concept and could find the physical meaning of the parameters of metric. In continue, the mcVittie like solution and its thermodynamic properties are addressed. In section 4, we generalized our debates to the charged spacetime, when the effects of the dark energy are considerable. In section 5, we summarize and conclude the results.

II. METRIC, GENERAL PROPERTIES AND BASIC ASSUMPTIONS

Let us begin with this metric:

\[ ds^2 = a(\tau)^2[-f(\tau, r)d\tau^2 + \frac{dr^2}{(1 - kr^2)f(\tau, r)} + r^2d\theta^2 + r^2\sin(\theta)^2d\phi^2]. \]  

(15)

Where \( a(\tau) \) is the arbitrary function of time coordinate \( \tau \). This metric has three Killing vectors

\[ \partial_\phi, \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \quad \text{and} \quad \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi. \]  

(16)

Now, if we define new time coordinate as

\[ \tau \rightarrow t = \int a(\tau)d\tau, \]  

(17)

we will get

\[ ds^2 = -f(t, r)dt^2 + a(t)^2\left[\frac{dr^2}{(1 - kr^2)f(t, r)} + r^2d\theta^2 + r^2\sin(\theta)^2d\phi^2\right], \]  

(18)

which possesses symmetries like as Eq. (16). From now, it is assumed that \( a(t) \) is the cosmic scale factor similar to the FRW’s. For \( f(t, r) = 1 \), Eq. (18) is reduced to the FRW metric [1]. Also, conformal BHs can be achieved by choosing \( f(t, r) = f(r) \) where, the general form of \( f(r) \) is [18]:

\[ f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \]  

(19)

Therefore, conformal BHs can be classified as a special subclass of metric [18]. \( n_\alpha = \delta^\alpha_\tau \) is normal to the hypersurface \( r = \text{const} \) and yields

\[ n_\alpha n^\alpha = g^{\tau \tau} = \frac{(1 - kr^2)f(t, r)}{a(t)^2}, \]  

(20)

which is timelike when \( (1 - kr^2)f(t, r) < 0 \), null for \( (1 - kr^2)f(t, r) = 0 \) and spacelike if we have \( (1 - kr^2)f(t, r) > 0 \). For an emitted signal at the coordinates \( t_0 \) and \( r_0 \), when it is absorbed at coordinates \( t \) and \( r \) simple calculations lead to

\[ 1 + z = \frac{\lambda}{\lambda_0} = \frac{a(t)}{a(t_0)} \left( \frac{f(t, r)}{f(t_0, r_0)} \right)^{1/2}, \]  

(21)
as induced redshift due to the universe expansion and factor $f(t, r)$. Redshift will diverge when $f(t_0, r_0)$ goes to zero or $1 + z \to \infty$. This divergence as the signal of singularity is independent of the curvature scalar ($k$), unlike the Mcvittie’s solution and its various generalizations [38, 39], which shows that our solutions are compatible with the FRW background. As a desired expectation, it is obvious that the FRW result is covered when $f(t_0, r_0) = f(t, r) = 1$. The only non-diagonal term of the Einstein tensor is

$$G^{tr} = -\frac{1 - kr^2}{f(t, r)a(t)^3r}(a(t)f(t, r) - f'(t, r)a(t)r),$$

which (’) and () are derivatives with respect to time and radius, respectively. Using $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial r}$, one gets

$$G^{tr} = -\frac{(1 - kr^2)\dot{a}(t)}{f(a(t), r)a(t)^3r}(a(t)f(a(t), r) - f'(a(t), r)r),$$

where $\dot{f}(a(t), r) = \frac{\partial f}{\partial a}$. In order to get perfect fluid solutions, we impose condition $G^{tr} = 0$ and reach to

$$f(t, r) = f(a(t)r) = \sum_{n} b_n(a(t)r)^n.$$  

Although Eq. (24) includes numerous terms, but the slow expansion approximation helps us to attribute physical meaning to the certain coefficients $b_n$. Since $G_{tr} = 0$, we should stress that here that there is no radial flow and thus, the backreaction effect is zero [19, 20], which means that there is no energy accretion in these solutions [43]. Finally and briefly, we see that the perfect fluid concept is in line with the no energy accretion condition. The only answer which is independent of the rate of expansion can be obtained by condition $b_n = \delta_{n0}$ which is yielding the FRW solution.

III. MCVITTIE LIKE SOLUTION IN THE FRW BACKGROUND

The mcVittie’s solution in the flat FRW background can be written as [16]

$$ds^2 = -\left(1 - \frac{\mathcal{M}}{2a(t)r}\right)^2dt^2 + a(t)^2(1 + \frac{\mathcal{M}}{2a(t)r})^4[d\tilde{r}^2 + \tilde{r}^2d\Omega^2].$$

This metric possess symmetries same as metric [15]. $\tilde{r}$ is isotropic radius defined by:

$$r = \tilde{r}(1 + \frac{\mathcal{M}}{2a(t)r}).$$

There is a redshift singularity at radii $\tilde{r}_h = \frac{\mathcal{M}}{2a(t)}$ which yields the radius $r_h = \frac{\mathcal{M}}{2a(t)}(1 + a(t))^2$ [51]. In addition, $\tilde{r}_h$ is a spacelike hypersurface, and can not point to an event horizon [15].

Consider $f(a(t)r) = 1 - \frac{2b_{-1}}{a(t)r}$. This assumption satisfies condition (21) and leads to

$$ds^2 = -(1 - \frac{2b_{-1}}{a(t)r})dt^2 + a(t)^2[\frac{d\tilde{r}^2}{(1 - kr^2)(1 - \frac{2b_{-1}}{a(t)r})} + r^2d\Omega^2].$$

For $b_{-1} \neq 0$, this metric will converge to the FRW metric when $r \to \infty$. The Schwarzschild metric is obtainable by putting $a(t) = 1$, $b_{-1} = \mathcal{M}$ and $k = 0$. Metric suffers from three singularities at $a(t) = 0$ (big bang), $r = 0$ and $f(a(t)r) = 0 \Rightarrow a(t)r_h = 2b_{-1}$.

Third singularity exists if $b_{-1} > 0$. In this manner, Eq. (21) will diverge at $r_0 = r_h$. In addition and in contrast to the Gao’s solutions, the radii of the redshift singularity ($r_h$) in our solutions is independent of the background curvature ($k$), while for the flat case our radius is compatible with the previous works [16, 21, 23]. Also, metric changes its sign at $r = r_h$ just the same as the schwarzschild spacetime. In addition, curvature scalars diverge at this radius as well as the mcVittie spacetime. Accordingly, this singularity point to a naked singularity which can be considered as alternatives for BHs [16, 17]. In continue, we will point to the some physical and mathematical properties of this
singularity which has the same behaviors as event horizon if one considers slow expansion approximation. The surface area integration at this radius leads to

\[ A = \int \sqrt{-g}d\theta d\phi = 4\pi r_h^2 a(t)^2 = 16\pi (b-1)^2. \]  

(29)

The main questions that arise here are: what is the nature of \( b-1 \)? and can we better clarify the meaning of \( r_h \)? For these purposes, we consider the slow expansion approximation \((a(t) \approx c)\), define new coordinate \( \eta = cr \) and get

\[ ds^2 \approx -(1 - \frac{2b-1}{\eta})dt^2 + \frac{d\eta^2}{(1 - k'\eta^2)(1 - \frac{2b-1}{\eta})} + \eta^2 d\theta^2 + \eta^2 \sin(\theta)^2 d\phi^2, \]

(30)

where \( k' = \frac{1}{2} \). In these new coordinates, \((t, \eta, \theta, \phi)\), and from Eq. (29) it is apparent that for \( b-1 > 0 \), hypersurface with equation \( \eta = \eta_h = 2b-1 \) is a null hypersurface. When our approximation is broken, then \( \eta_h \) may not be actually a null hypersurface, despite its resemblance to that. We call this null hypersurface a quasi event horizon which is signalling us an object like a BH and we refer to that as a quasi BH. From now, we assume \( b-1 > 0 \), the reason of this option will be more clear later, when we debate mass. Therefore by the slow expansion approximation, \( r_h (= \frac{2b-1}{\eta}) \) plays the role of the co-moving radius of event horizon and it is decreased with time. In order to find an answer to the first question about the physical meaning of \( b-1 \), we use Komar mass:

\[ M = \frac{1}{4\pi} \int_S n^\alpha \sigma_\beta \nabla_\alpha \xi_\beta^\alpha dA, \]

(31)

where \( \xi_\beta^\alpha \) is the timelike Killing vector of spacetime. Since the Komar mass is only definable for the stationary and asymptotically flat spacetimes \([48]\), one should consider the flat case \((k = 0)\) and then by bearing the spirit of the stationary limit in mind (the slow expansion approximation) tries to evaluate Eq. (31).

Consider \( n_\alpha = \sqrt{1 - \frac{2b-1}{a(t)} \delta_\alpha^\alpha} \) and \( \sigma_\beta = \frac{a(t)}{\sqrt{1 - \frac{2b-1}{a(t)}}} \delta_\beta^\alpha \) as the unit timelike and unit spacelike four-vectors, respectively. Now using Eq. (31) and bearing the spirit of the slow expansion approximation in mind, one gets

\[ M = \frac{1}{4\pi} \int_S n^\alpha \sigma_\beta \Gamma_\alpha^\beta \sigma_\delta dA = b-1, \]

(32)

which is compatible with the no energy accretion condition \((G_{tr} = 0)\). In addition, we will find the same result as Eq. (32), if we considered the flat case \((k = 0)\) of metric \((30)\) and use \( n_\alpha = \sqrt{1 - \frac{2b-1}{\eta} \delta_\alpha^\alpha} \) and \( \sigma_\beta = \frac{1}{\sqrt{1 - \frac{2b-1}{\eta}}} \delta_\beta^\alpha \). Since the integrand is independent of the scale factor \((a(t))\), the slow expansion approximation does not change the result of integral. But, the accessibility of the slow expansion approximation is necessary if one wants to evaluate the Komar mass for dynamical spacetimes \([48]\). Indeed, this situation is the same as what we have in the quasi-equilibrium thermodynamical systems, where the accessibility of the quasi-equilibrium condition lets us use the equilibrium formulation for the vast thermodynamical systems \([49]\). It is obvious that for avoiding negative mass, we should have \( b-1 > 0 \). Relation to the Komar mass of the mcVittie’s solution can be written as \([16, 39]\)

\[ M_{mc\text{Vittie}} = \frac{M}{a(t)}. \]

(33)

In addition, some studies show that the Komar mass is just a metric parameter in the mcVittie spacetime \([41, 42, 43]\). Indeed, Hawking-Hayward quasi-local mass satisfies \( \dot{M} = 0 \), which is compatible with \( G_{tr} = 0 \) and indicates that there is no radial flow and thus the backreaction effect, in the mcVittie’s solution \([19, 21, 43]\). In order to clarify the mass notion in the mcVittie spacetime, we consider the slow expansion approximation of the mcVittie spacetime which yields

\[ ds^2 \approx -(1 - \frac{M}{1 + \frac{2b}{\eta}})^2 dt^2 + (1 + \frac{M}{2\eta})^4 [d\eta^2 + \eta^2 d\Omega^2]. \]

(34)

This metric is signalling us that the \( M \) may play the role of the mass in the mcVittie spacetime. In addition, by defining new radii \( R \) as

\[ R(t, r) = a(t) \tilde{r} (1 + \frac{M}{2\tilde{r}})^2, \]

(35)
one can rewrite the mcVittie spacetime in the form of

\[ ds^2 = -(1 - \frac{2M}{R} - H^2R^2)dt^2 - \frac{2HR}{\sqrt{1 - \frac{2M}{R}}}dtdR + \frac{dR^2}{1 - \frac{2M}{R}} + R^2d\Omega^2, \]  

(36)

where \( H = \frac{\dot{a}}{a} \). This form of the mcVittie spacetime indicates these facts that the Komar mass is a metric parameter and \( M \) is the physical mass in this spacetime \[50\]. Finally, we see that the results of the slow expansion approximation (Eq. (34)) and Eq. (36) are in line with the result of the study of the Hawking-Hayward quasi-local mass in the mcVittie spacetime \[41, 42, 43, 50\]. For the flat case (\( k = 0 \)) of our spacetime (Eq. (27)), by considering Eq. (33) and following the slow expansion approximation, we reach at

\[ ds^2 \approx -(1 - \frac{2M}{\eta})dt^2 + \frac{d\eta^2}{(1 - \frac{2M}{\eta})} + \eta^2d\theta^2 + \eta^2\sin(\theta)^2d\phi^2. \]  

(37)

Also, if we define new radius \( R \) as

\[ r = \frac{R}{a}(1 + \frac{M}{2R^2}), \]  

(38)

we obtain

\[ ds^2 = -(\frac{1 - \frac{M}{2\eta}}{(1 + \frac{M}{2\eta})^2}) - \frac{R^2H^2(1 + \frac{M}{2\eta})^6}{(1 - \frac{2M}{\eta})^2}dt^2 - \frac{2RH(1 + \frac{M}{2R^2})^5}{(1 - \frac{2M}{\eta})^5}dtdR + (1 + \frac{M}{2R})^4[dR^2 + R^2d\Omega^2]. \]  

(39)

Both of the equations (37) and (39) as well as the no energy accretion condition suggest that, unlike the mcVittie’s spacetime, the Komar mass may play the role of the mass in our solution. From Eq. (39) it is apparent that \( R = \frac{M}{2R^2} \) points to the spacelike hypersurface where, in the metric (39), \( R = 2M \) points to the null hypersurface. In the next subsection and when we debate thermodynamics, we will derive the same result for the mass notion in our spacetime. Only in the \( a(t) = 1 \) limit (the Schwarzschild limit), Eqs. (39) and (25) will be compatible which shows that our spacetime is different with the mcVittie’s. Let us note that the obtained metric (Eq. (39)) is consistent with Eq. (36), provided we take \( M = 0 \) (the FRW limit).

**Horizons, energy and thermodynamics**

There is an apparent horizon in accordance with the FRW background which can be evaluated from Eq. (6):

\[ (1 - kr_{ap}^2)(1 - \frac{2M}{\hat{a}(t)r_{ap}})^2 - r_{ap}^2\hat{a}(t)^2 = 0. \]  

(40)

This equation covers the FRW results in the limit of \( M \rightarrow 0 \) (see Eq. (7)). In addition, one can get the Schwarzschild radius by considering \( \hat{a}(t) = 0 \), which supports our previous definition for \( b_{-1} \). Calculations for the flat case yield four solutions. The only solution which is in full agreement with the limiting situation of the FRW metric (in the limit of zero \( M \)) is

\[ r_{ap} = \frac{1 + \sqrt{1 - 8HM}}{2a}. \]  

(41)

Therefore, the physical radius of apparent horizon (\( \xi_{ap} = a(t)r_{ap} \)) is

\[ \xi_{ap} = \frac{1 + \sqrt{1 - 8HM}}{2H}, \]  

(42)

which is similar to the conformal BHs \[19\]. It is obvious that in the limit of \( M \rightarrow 0 \), the radius for the apparent horizon of the flat FRW is recovered. For the surface gravity of apparent horizon, one can use Eq. (5) and gets:

\[ \kappa = \frac{\kappa_{FRW}}{(1 - \frac{2M}{a(t)r_{ap}})^2} + \frac{M}{a(t)^2}\frac{1}{r_{ap}^2} + \frac{1}{(1 - \frac{2M}{a(t)r_{ap}})^2}(\hat{a}(t) + \frac{2\hat{a}(t)^2}{a(t)}), \]  

(43)
where \( h^{ab} = \text{diag}\left(-\frac{1}{1-\frac{2M}{a(t)^2}}, 1-\frac{2M}{a(t)^2}\right) \), \( r_{ap} \) is the apparent horizon co-moving radius \([41]\) and \( \kappa_{FRW} \) is the surface gravity of the flat FRW manifold

\[
\kappa_{FRW} = -\frac{\dot{a}(t)^2 + a(t)\ddot{a}(t)}{2a(t)\dot{a}(t)}.
\]  

(44)

The Schwarzschild limit \( (\kappa = \frac{1}{\alpha}) \) is obtainable by inserting \( a(t) = 1 \) in Eq. \((41)\). In the limiting case \( M \rightarrow 0 \), Eq. \((43)\) is reduced to the surface gravity of the flat FRW spacetime, as a desired result. The Misner-Sharp mass inside radius \( \xi \) for this spherically symmetric spacetime is defined as \([52]\):

\[
M_{MS} = \frac{\xi^2}{2}(1 - h^{ab}\partial_a \xi \partial_b \xi).
\]

(45)

Because this definition does not yield true results in some theories such as the Brans-Dicke and scalar-tensor gravities, we are pointing to the Gong-Wang definition of mass \([53]\):

\[
M_{GW} = \frac{\xi^2}{2}(1 + h^{ab}\partial_a \xi \partial_b \xi).
\]

(46)

It is apparent that, for the apparent horizon, Eqs. \((45)\) and \((46)\) yield the same result as \( M_{GW} = M_{MS} = \frac{\xi^2}{2} \). In the limit of \( M \rightarrow 0 \), the FRW’s results are recovered and we reach to \( M_{GW} = M_{MS} = \rho V \) as a desired result \([10]\). Using Eqs. \((45)\) and \((46)\) and taking the slow expansion approximation into account, we reach to \( M_{GW} = M_{MS} \approx M \) as the mass of quasi BH. Also, this result supports our previous guess about the Komar mass as the physical mass in our solution, and is in line with the result of Eqs. \((37)\) and \((39)\). For the Mcvittie metric, Eqs. \((45)\) and \((46)\) yield \( M_{GW} = M_{MS} \approx M \) as the confined mass to radius \( \xi_b = a(t)\tilde{r}_b = M \). Also, Eqs. \((32)\), \((45)\) and \((46)\) leads to the same result in the Schwarzschild’s limit \( (M = M_{GW} = M_{MS} = M) \). For the flat background, using metric \((30)\), Eq. \((13)\) and inserting results into Eq. \((13)\), one gets

\[
T \approx \frac{1}{8\pi M},
\]

(47)

for the temperature on the surface of quasi horizon. The same calculations yield similar results, as the temperature on the horizon of the Mcvittie’s solution. For the conformal Schwarzschild BH, the same analysis leads to

\[
T \approx \frac{1}{8\pi a(t)M},
\]

(48)

which shows that the \( a(t)M \) plays the role of mass, and is compatible with the energy accretion in the conformal BHs \([19, 21, 51]\). Again, we see that the temperature analysis can support our expectation from \( M \) as the physical mass in our solutions. For the area of quasi horizon, we have

\[
A = \int \sqrt{\sigma}d\theta d\phi = 4\pi a(t)^2 r_b^2 = 16\pi M^2.
\]

(49)

In the mcVittie spacetime, this integral leads to \( A = 16\pi M^2 \). In order to vindicate our approximation, we consider \( S = \frac{A}{4} \) for the entropy of quasi BH. In continue and from Eq. \((47)\), we get

\[
TdS \approx dM = dE.
\]

(50)

Whereas, we reach to \( TdS \approx dM \neq dE \) for the mcVittie spacetime. In the coordinates \( (t, \eta, \theta, \phi) \), we should remind that, unlike the mcVittie spacetime, \( E = M_{GW} = M_{MS} \approx M \) is valid for quasi BH and the work term can be neglected as the result of slow expansion approximation \((dW \sim 0)\) \([51]\). Finally and unlike the mcVittie’s horizon, we see that \( TdS \sim dE \) is valid on the quasi event horizon. This result points us to this fact that the first law of the BH thermodynamics on quasi event horizon will be satisfied if we use either the Gong-Wang or the Misner-Sharp definitions for the energy of quasi BH. \( TdS \sim dE \) is valid for the conformal Schwarzschild BH, too \([51]\). For the flat background, we see that the surface area at redshift singularity in our spacetime is equal to the mcVittie metric which is equal to the Schwarzschild metric. In continue and by bearing the slow expansion approximation in mind, we saw that the temperature on quasi horizon is like the Schwarzschild spacetime \([19]\). In addition, we saw that the quality of the validity of the first law of the BH thermodynamics on quasi event horizon is like the conformal Schwarzschild BH’s and differs from the mcVittie’s solution.
In another approach and for the mcVittie spacetime, if we use the Hawking-Hayward definition of mass as the total confined energy to the hypersurface $\tilde{r} = \frac{M}{2\sigma(r)}$, we reach to
\[ TdS \simeq dM = dE, \tag{51} \]
where we have considered the slow expansion approximation. In addition, Eq. (51) will be not valid, if one uses the Komar mass \[33\]. Finally, we saw that the first law of thermodynamics will be approximately valid in the mcVittie’s solution, if one uses the Hawking-Hayward definition of energy. Also, none of the Komar, Misner-Sharp and Gong-Wang masses can not satisfy the first law of thermodynamics on the mcVittie’s horizon.

IV. OTHER POSSIBILITIES

According to what we have said, it is obvious that there are two other meaningful sentences in expansion \[24\]. The first term is due to $n = -2$ and points to the charge, where the second term comes from $n = 2$ and it is related to the cosmological constant. Therefore, the more general form of $f(t, r)$ can be written as:
\[ f(t, r) = 1 - \frac{2M}{a(t)r} + \frac{Q^2}{(a(t)r)^2} - \frac{1}{3}\Lambda(a(t)r)^2, \tag{52} \]
where we have considered the slow expansion approximation and used these definitions $b_2 \equiv Q^2$ and $b_2 \equiv -\frac{1}{3}\Lambda$. Imaginary charge ($b_2 < 0$) and the anti De-Sitter ($\Lambda < 0$) solutions are allowed by this scheme, but these possibilities are removed by the other parts of physics. Consider Eq. (52) when $\Lambda = 0$, there are two horizons located at $r_+ = \frac{M+\sqrt{M^2-Q^2}}{a(t)}$ and $r_- = \frac{M-\sqrt{M^2-Q^2}}{a(t)}$. These radii are same as the Gao’s flat case \[39\]. In the low expansion regime ($a(t) \sim c$), these radii point to the event and the Coushy horizons, as the Riessner-Nordstrom metric \[9\]. Hence, we refer to them as quasi event and quasi Coushy horizons. The case with $Q = 0$, $M = 0$ and $\Lambda > 0$ has attractive properties. Because in the low expansion regime ($a(t) \simeq c$), one can rewrite this case as
\[ \text{This is nothing but the De-Sitter spacetime with cosmological constant } \Lambda, \text{ which points to the current acceleration era.} \]

Horizons and temperature

Different $f(t, r)$ yield apparent horizons with different locations, and one can use Eqs. \[6\] and \[8\] in order to find the location and the temperature of apparent horizon. For every $f(t, r)$, using the slow expansion regime, we get:
\[ ds^2 \approx -f(\eta)dt^2 + \frac{d\eta^2}{f(\eta)} + \eta^2d\Omega^2. \tag{54} \]
Now, the location of horizons and their surface gravity can be evaluated by using Eq. \[13\]. Their temperature is approximately equal to Eq. \[14\], or briefly:
\[ T_i \simeq \frac{f'(\eta)}{4\pi} \bigg|_{\eta_{i}}, \tag{55} \]
where $'$ is derivative with respect to radii $\eta$ and $\eta_{i}$ is the radii of $i^{th}$ horizon.

V. CONCLUSIONS

We considered the conformal form of the special group of the non-static spherically symmetric metrics, where it was assumed that the time dependence of the conformal factor is like as the FRW’s. We saw that the conformal BHs can be classified as a special subgroup of these metrics. In order to derive the new solutions of the Einstein equations, we have imposed perfect fluid concept and used slow expansion approximation which helps us to clarify the physical
meaning of the parameters of metric. Since the Einstein tensor is diagonal, there is no energy accretion and thus the backreaction effect is zero. This imply that the energy (mass) should be constant in our solutions. These new solutions have similarities with earlier metrics that have been presented by others \cite{37,39}. A related metric which is similar to the special class of our solutions was introduced by mcVittie \cite{37,39}. These similarities are explicit in the flat case (temperature and entropy at the redshift singularity), but the differences will be more clear in the non-flat case ($k \neq 0$), and we pointed to the one of them, when we debate the redshift. In addition and in the flat case, we tried to clear the some of differences between our solution and the mcVittie’s. We did it by pointing to the behavior of the redshift singularity in the various coordinates, the mass notion, and thermodynamics. Meanwhile, when our slow expansion approximation is broken then there is no horizon for our solutions. Indeed, these objects can be classified as naked singularities which can be considered as alternatives for BHs \cite{46,47}.

For our solutions and similar with earlier works \cite{37,39}, the co-moving radiuses of the redshift singularities are decreased by the expansion of universe. Also, unlike the previous works \cite{37,39}, the redshift singularities in our solutions are independent of the background curvature. By considering the slow expansion approximation, we were able to find out BH’s like behavior of these singularities. We pointed to these objects and their surfaces as quasi BHs and the quasi horizons, respectively. In continue, we introduced the apparent horizon for our spacetime which should be evaluated by considering the FRW background.

In order to compare the mcVittie’s solution with our mcVittie’s like solution, we have used the three existing definitions of mass including the Komar mass, the Misner-Sharp mass ($M_{MS}$) and the Gong-Wang mass ($M_{GW}$). We saw that the notion of the Komar mass of quasi BH differs from the mcVittie’s solution. Also, in our spacetime, we showed that the $M_{MS}$ and $M_{GW}$ masses yield the same result on the apparent horizon and cover the FRW’s result in the limiting situations. In addition, using the slow expansion approximation, we evaluated $M_{MS}$ and $M_{GW}$ on the quasi event horizon of our mcVittie’s like solution, which leads to the same result as the Komar mass. In addition, we should express that, the same as the mcVittie spacetime, the energy conditions are not satisfied near the quasi horizon.

In addition, we have proved that, unlike the mcVittie’s solution, the first law of thermodynamics may be satisfied on the quasi event horizon of our mcVittie’s like solution, if we use the Komar mass or either $M_{MS}$ or $M_{GW}$ as the confined mass and consider the slow expansion approximation. This result is consistent with previous studies about the conformal BHs \cite{51}, which shows that the thermodynamics of our solutions is similar to the conformal BHs. In order to clarify the mass notion, we think that the full analysis of the Hawking-Hayward mass for our solution is needed, which is out of the scope of this letter and should be considered as another work, but our resolution makes this feeling that the predictions by either the slow expansion approximation or using the suitable coordinates for describing the metric for mass, may be in line with the Hawking-Hayward definition of energy, and have reasonable accordance with the Komar, $M_{MS}$ and $M_{GW}$ masses of our solutions. Indeed, this final remark can be supported by the thermodynamics considerations and the no energy accretion condition ($G_{rr} = 0$). Moreover, we think that, in dynamic spacetimes, the thermodynamic considerations along as the slow time varying approximation can help us to get the reasonable assumptions for energy and thus mass. Finally, we saw that the first law of thermodynamics will be approximately valid in the mcVittie’s and our solutions if we use the Hawking-Hayward definition of the mass in the mcVittie spacetime and the Komar mass as the physical mass in our solution, respectively. In continue, the more general solutions such as the charged quasi BHs and the some of their properties have been addressed.

Results obtained in this paper may help achieving a better understanding of black holes in a dynamical background. From a phenomenological point of view, this issue is important since after all, any local astrophysical object lives in an expanding cosmological background. Finally, we tried to explore the concepts of mass, entropy and temperature in a dynamic spacetime.

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