An initial-boundary value problem for the coupled focusing-defocusing complex short pulse equation with a $4 \times 4$ Lax pair

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December 19, 2017

Abstract: In this paper we investigate the coupled focusing-defocusing complex short pulse equation, which describe the propagation of ultra-short optical pulses in cubic nonlinear media. Through the unified transform method, the initial-boundary value problem for the coupled focusing-defocusing complex short pulse equation with $4 \times 4$ Lax pair on the half-line are to be analyzed. Assuming that the solution $\{q_1(x,t), q_2(x,t)\}$ of the coupled focusing-defocusing complex short pulse equation exists, we show that $\{q_{1,x}(x,t), q_{2,x}(x,t)\}$ can be expressed in terms of the unique solution of a $4 \times 4$ matrix Riemann-Hilbert problem formulated in the complex $\lambda$-plane. Thus, the solution $\{q_1(x,t), q_2(x,t)\}$ can be obtained by integration with respect to $x$. Moreover, we also get that some spectral functions are not independent and satisfy the so-called global relation.

Keywords: Riemann-Hilbert problem; coupled focusing-defocusing complex short pulse equation; Initial-boundary value problem; Unified transform method

PACS numbers: 02.30.Ik, 02.30.Jr, 03.65.Nk

Mathematics Subject Classification: 35G31, 35Q51, 35Q15

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1 Introduction

One of the most important integrable systems in mathematics and physics is the following short plus (SP) equation

\[ u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \]  

(1.1)

where \( u = u(x,t) \) is a real-valued function, representing the magnitude of the electric field, the subscripts \( t \) and \( x \) denote partial differentiation, which can describe the propagation of ultra-short optical pulses in silica optical fibers. The SP equation was derived by Schäfer and Wayne [1]. Actually, the SP equation appeared first as one of Rabelo’s equations which describe pseudospherical surfaces, possessing a zero-curvature representation [2]. The SP equation is completely integrable[3], and has been studied extensively on the bi-Hamiltonian structure[4], Cauchy problem [5], integrable self-adaptive moving mesh schemes [6], soliton solutions given by the Darboux transformation (DT) method[7]. Besides, the long-time behavior for the solutions of decay initial value problem of SP equation has been analyzed by nonlinear steepest descent method/Deift-Zhou method[8], Moreover, Boutet and Monvel et al. have studied Riemann-Hilbert (RH) problem of the SP equation [9] by using the inverse scattering transform (IST) method.

However, the IST method can only be used to analyze pure initial value problems. In 1997, Fokas used IST thought to construct a new unified method, we call this method as Fokas method, this method can be used to study the initial-boundary value (IBV) problems for both linear and nonlinear integrable evolution PDEs with \( 2 \times 2 \) Lax pairs[10, 11, 12, 13, 14, 15, 16]. Just like the IST on the line, the Fokas method yields an expression for the solution of an IBV problem in terms of the solution of a RH problem. In particular, an effective way analyzed the asymptotic behaviour of the solution is based on this RH problem and by employing the nonlinear version of the steepest descent method introduced by Deift and Zhou [17]. In 2012, Lenells extended the Fokas method to study the IBV problems for integrable nonlinear evolution equations with \( 3 \times 3 \) Lax pairs [18]. After that, more and more researchers begin to pay attention to studying IBV problems for integrable evolution equations with higher-order Lax pairs, the IBV problem for the many integrable equations with \( 3 \times 3 \) or \( 4 \times 4 \) Lax pairs are studied, such as, the Degasperis-Procesi equation [19, 20], the Ostrovsky-Vakhnenko equation [21], the Sasa-Satsuma equation [22], the three wave equation [23], the spin-1 Gross-Pitaevskii equations [24] and others [25, 26, 27, 28, 29]. These authors have also done some work about integrable equations with \( 2 \times 2 \) or \( 3 \times 3 \) Lax pairs [15, 30, 31, 32, 33].

Similar to the nonlinear Schrödinger(NLS) equation, it is known that the complex-valued function has advantages in describing optical waves which have both the amplitude and phase
Following this spirit, Feng [34] proposed a complex short pulse (CSP) equation
\[ q_{xt} + q + \frac{1}{2} \epsilon (|q|^2 q_x)_x = 0, \] (1.2)
where, \( \epsilon = \pm 1 \) represents focusing- and defocusing-type, \( q(x, t) \) is the complex function. It can be derived from the Maxwell equation. The CSP equation has been studied extensively on the integrability associated with explicit form of the Lax pair [34], geometrically and algebraically [35, 36], multiply smooth soliton, loop soliton, cuspion soliton, breather soliton and rogue wave solutions given by the DT method [37, 38], periodic traveling wave solutions given by the F-expansion method [39].

To describe the propagation of optical pulses in birefringence fibers, Feng [34] also proposed a coupled complex short-pulse (CCSP) equation
\[
\begin{align*}
q_{1,xt} + q_1 + \frac{1}{2} \left[ (\epsilon_1 |q_1|^2 + \epsilon_2 |q_2|^2)q_{1,x} \right] & = 0, \\
q_{2,xt} + q_2 + \frac{1}{2} \left[ (\epsilon_1 |q_1|^2 + \epsilon_2 |q_2|^2)q_{2,x} \right] & = 0.
\end{align*}
\] (1.3)
Similarly, where \( \epsilon_i = \pm 1 \) means focusing case and defocusing case, \( \epsilon_1 = -\epsilon_2 = 1 \) means focusing-defocusing case, \( q_1(x, t) \) and \( q_2(x, t) \) are the complex functions of \( x \) and \( t \), which indicate the magnitudes of the electric fields. The CCSP equation has been studied extensively on the integrability associated with explicit form of the Lax pair, conservation laws [34], bi-Hamiltonian structure, bilinearization [36], soliton solutions given by the Hirota’s bilinear method [34, 40, 41], rogue waves solutions given by the DT method [42].

Most recently, Yang and Zhu consider the following coupled focusing-defocusing complex short pulse equation [41]
\[
\begin{align*}
q_{1,xt} + q_1 + \frac{1}{2} \left[ (|q_1|^2 - |q_2|^2)q_{1,x} \right] & = 0, \\
q_{2,xt} + q_2 + \frac{1}{2} \left[ (|q_1|^2 - |q_2|^2)q_{2,x} \right] & = 0.
\end{align*}
\] (1.4)
by Hirota’s bilinear method, the bright-bright, bright-dark, dark-dark soliton solutions and rogue waves solutions to be constructed. In this paper, we investigate the IBV problems for the following coupled focusing-defocusing complex short pulse equation on the half-line via a unified transform method. The IBV problems of system (1.4) on the interval are presented in another paper. For the focusing case \( (\epsilon_1 = \epsilon_2 = 1) \) and defocusing case \( (\epsilon_1 = \epsilon_2 = -1) \) which can be study by the same ways.

Throughout this paper, we consider the half-line domain \( \Omega \) and the IBV problems for the system (1.4) as follows

\[
\begin{align*}
\text{Half-line domain (see Figure 1): } & \Omega = \{ 0 < x < \infty, 0 < t < T \}; \\
\text{Initial values: } & u_0(x) = q_1(x, t = 0), \ v_0(x) = q_2(x, t = 0); \\
\text{Dirichlet boundary values: } & g_0(t) = q_1(x = 0, t), \ h_0(t) = q_2(x = 0, t); \\
\text{Neumann boundary values: } & g_1(t) = q_{1,x}(x = 0, t), \ h_1(t) = q_{2,x}(x = 0, t).
\end{align*}
\] (1.5)
where \( u_0(x) \) and \( v_0(x) \) lie in the Schwartz space.

The outline of the present paper is organized as follows. In section 2, we define two sets of eigenfunctions \( \{\mu_j\}_1^3 \) and \( \{M_n\}_1^7 \) of Lax pair for spectral analysis and we also get some spectral functions satisfies the so-called global relation in this part. In section 3, we show that \( \{q_1(x,t), q_2(x,t)\} \) can be expressed in terms of the unique solution of a 4 \( \times \) 4 matrix Riemann-Hilbert problem formulated in the complex \( \lambda \)-plane, and the solution \( \{q_1(x,t), q_2(x,t)\} \) of coupled focusing-defocusing complex short pulse equation can be obtained by integration with respect to \( x \). The last section is devoted to giving some conclusions and discussions.

2 The spectral analysis

The coupled focusing-defocusing complex short pulse equation (1.4) admits the 4 \( \times \) 4 Lax pair

\[
\psi_x = U\psi, \quad \psi_t = V\psi,
\]

with

\[
U = \lambda \begin{pmatrix} I_2 & Q_x \\ R_x & -I_2 \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{2}\lambda QR - \frac{1}{4\lambda} I_2 & -\frac{1}{2}\lambda QRQ_x + \frac{1}{2} Q \\ -\frac{1}{2}\lambda RQR_x - \frac{1}{2} R & -\frac{1}{2}\lambda QR + \frac{1}{4\lambda} I_2 \end{pmatrix},
\]

where \( I_2 \) is a 2 \( \times \) 2 identity matrix, and \( Q, R \) are 2 \( \times \) 2 matrices defined as

\[
Q = \lambda \begin{pmatrix} q_1 & q_2 \\ -\bar{q}_2 & \bar{q}_1 \end{pmatrix}, \quad R = \begin{pmatrix} \bar{q}_1 & -q_2 \\ -\bar{q}_2 & \bar{q}_1 \end{pmatrix},
\]

Notice that

\[
QR = RQ = (|q_1|^2 - |q_2|^2) I_2.
\]

Direct computations reveal that the zero-curvature equation \( U_x - V_t + UV - VU = 0 \) exactly gives back system (1.3).
By replacing $\lambda$ by $i\lambda$, the Lax pair equation (2.1) can be rewritten as
\[
\begin{aligned}
\psi_x &= U\psi = (i\lambda\sigma_4 + U_0)\psi, \\
\psi_t &= V\psi = (\frac{i}{4\lambda}\sigma_4 - \frac{1}{2}i\lambda V_1 + \frac{1}{2}V_0)\psi,
\end{aligned}
\tag{2.2}
\]
where $\sigma_4 = \text{diag}\{1,1,-1,-1\}$ is a $4 \times 4$ matrix, $\psi = \psi(x,t,\lambda)$ is a $4 \times 4$ matrix-valued or a $4 \times 1$ column vector-valued spectral function, the $4 \times 4$ matrix-valued functions $U_0, V_0$ and $V_1$ are defined by
\[
U_0(x,t) = \begin{pmatrix}
0 & 0 & q_{1,x} & q_{2,x} \\
0 & 0 & \bar{q}_{2,x} & \bar{q}_{1,x} \\
\bar{q}_{1,x} & -\bar{q}_{2,x} & 0 & 0 \\
-\bar{q}_{2,x} & \bar{q}_{1,x} & 0 & 0
\end{pmatrix},
V_0(x,t) = \begin{pmatrix}
0 & 0 & q_1 & q_2 \\
0 & 0 & \bar{q}_2 & \bar{q}_1 \\
-q_1 & -\bar{q}_2 & 0 & 0 \\
q_2 & -\bar{q}_1 & 0 & 0
\end{pmatrix},
\tag{2.3}
\]
\[
V_1(x,t) = (|q_1|^2 - |q_2|^2) \begin{pmatrix}
1 & 0 & q_{1,x} & q_{2,x} \\
0 & 1 & \bar{q}_{1,x} & \bar{q}_{2,x} \\
\bar{q}_{1,x} & -q_{2,x} & -1 & 0 \\
-q_{2,x} & q_{1,x} & 0 & -1
\end{pmatrix}.
\]

### 2.1 The closed one-form

We find that Lax pair Eq.(2.2) can be rewritten as
\[
\begin{aligned}
\psi_x - i\lambda\sigma_4\psi &= U_1(x,t)\psi, \\
\psi_t - \frac{1}{4\lambda}\sigma_4\psi &= U_2(x,t,\lambda)\psi,
\end{aligned}
\tag{2.4}
\]
where
\[
U_1(x,t) = U_0(x,t), \quad U_2(x,t,\lambda) = -\frac{1}{2}i\lambda V_1 + \frac{1}{2}V_0.
\tag{2.5}
\]
Introduce a new eigenfunction $\mu(x,t,\lambda)$ is defined by the transform
\[
\psi(x,t,\lambda) = \mu(x,t,\lambda)e^{i\lambda x\sigma_4 + \frac{1}{2}i\lambda \hat{\sigma}_4 t},
\tag{2.6}
\]
then the Lax pair Eq.(2.4) becomes
\[
\begin{aligned}
\mu_x - i\lambda[\sigma_4, \mu] &= U_1(x,t)\mu, \\
\mu_t - \frac{1}{4\lambda}[\sigma_4, \mu] &= U_2(x,t,\lambda)\mu,
\end{aligned}
\tag{2.7}
\]
and Eq.(2.7) leads to a full derivative form
\[
d\left(e^{-i\lambda x\hat{\sigma}_4 - \frac{1}{4\lambda}t\hat{\sigma}_4} \mu(x,t,\lambda)\right) = W(x,t,\lambda),
\tag{2.8}
\]
where the closed one-form $W(x,t,\lambda)$ defined by
\[
W(x,t,\lambda) = e^{-i(\lambda x + \frac{1}{2}\lambda t)}(U_1(x,t)dx + U_2(x,t,\lambda)dt)\mu,
\tag{2.9}
\]
and $\hat{\sigma}_4$ represents a matrix operator acting on $4 \times 4$ matrix $X$ by $\hat{\sigma}_4 X = [\sigma_4, X]$ and by $e^{x\hat{\sigma}_4} X = e^{x\sigma_4} X e^{-x\sigma_4}$ (see Lemma 2.6).
2.2 The basic eigenfunction \( \mu_j \)'s

We assume that \( \{q_1(x,t), q_2(x,t)\} \) is a sufficiently smooth function in the half-line region \( \Omega = \{0 < x < \infty, 0 < t < T\} \), and decays sufficiently when \( x \to \infty \). \( \{\mu_j(x,t, \lambda)\}_{1}^{3} \) are the \( 4 \times 4 \) matrix valued functions, based on the Volterra integral equation, we can define the three eigenfunctions \( \{\mu_j(x,t, \lambda)\}_{1}^{3} \) of Eq.(2.7) by

\[
\mu_j(x,t, \lambda) = I + \int_{t_j}^{t} e^{i\lambda x + \frac{1}{4\pi}(t-\tau)} W_j(\xi, \tau, \lambda) \, d\xi, \quad j = 1, 2, 3, \tag{2.10}
\]

where \( I = \text{diag}\{1,1,1,1\} \) is a \( 4 \times 4 \) unit matrix, \( W_j \) is determined Eq.(2.9), it is only used \( \mu_j \) in place of \( \mu \), and the contours \( \{\gamma_j\}_{1}^{3} \) are smooth curve from \( (x_j, t_j) \) to \( (x,t) \), and \( (x_1, t_1) = (0,T), (x_2, t_2) = (0,0), (x_3, t_3) = (\infty,t) \) (see Figure 2).

Thus, for the point \( (\xi, \tau) \) on each contour, we have that the following inequalities hold true

\[
\begin{align*}
\gamma_1 : & \quad x - \xi \geq 0, \quad t - \tau \leq 0; \\
\gamma_2 : & \quad x - \xi \geq 0, \quad t - \tau \geq 0; \\
\gamma_3 : & \quad x - \xi \leq 0, \quad t - \tau = 0. \tag{2.11}
\end{align*}
\]

Since the one-form \( W_j \) is closed, thus \( \mu_j \) is independent of the path of integration. If we choose the paths of integration to be parallel to the \( x \) and \( t \) axes, then the integral Eq.(2.10) becomes \( (j = 1, 2, 3) \)

\[
\mu_j(x,t, \lambda) = I + \int_{x_j}^{x} e^{i\lambda(x-\xi)} \delta_{4}(V_1 \mu_j)(\xi, t, \lambda) \, d\xi + e^{i\lambda(x-x_j)\delta_{4}} \int_{t_j}^{t} e^{\frac{1}{4\pi}(t-\tau)} \delta_{4}(V_2 \mu_j)(x_j, \tau, \lambda) \, d\tau, \tag{2.12}
\]

Let \( [\mu_j]_k \) denote the \( k \)-th column vector of \( \mu_j \), Eq.(2.11) implies that the first, second, third and fourth columns of the matrices equation(2.10) contain the following exponential term

\[
\begin{align*}
[\mu_j]_{1} : & \quad e^{-2i\lambda(x-\xi)} - \frac{1}{2\pi}(t-\tau), \quad e^{-2i\lambda(x-\xi)} + \frac{1}{2\pi}(t-\tau), \\
[\mu_j]_{2} : & \quad e^{-2i\lambda(x-\xi)} - \frac{1}{2\pi}(t-\tau), \quad e^{-2i\lambda(x-\xi)} + \frac{1}{2\pi}(t-\tau), \\
[\mu_j]_{3} : & \quad e^{2i\lambda(x-\xi)} + \frac{1}{2\pi}(t-\tau), \quad e^{2i\lambda(x-\xi)} - \frac{1}{2\pi}(t-\tau), \\
[\mu_j]_{4} : & \quad e^{2i\lambda(x-\xi)} + \frac{1}{2\pi}(t-\tau), \quad e^{2i\lambda(x-\xi)} - \frac{1}{2\pi}(t-\tau). \tag{2.13}
\end{align*}
\]
Thus, we can show that the eigenfunctions $\{\mu_j(x, t, \lambda)\}_1^3$ are bounded and analytic for $\lambda \in \mathbb{C}$ such that $\lambda$ belongs to

$$
\mu_1 \text{ is bounded and analytic for } \lambda \in \emptyset, \\
\mu_2 \text{ is bounded and analytic for } \lambda \in (D_2, D_2, D_1, D_1), \\
\mu_3 \text{ is bounded and analytic for } \lambda \in (D_1, D_1, D_2, D_2),
$$

(2.14)

where $D_1, D_2$ denote up-half plane and low-half plane, respectively, pairwisely disjoint subsets of the Riemann $\lambda$-plane shown in figure 3.

And these sets $\{D_n\}_1^2$ have the following properties:

$$
D_1 = \{\lambda \in \mathbb{C} | \text{Rel}_1 = \text{Rel}_2 < \text{Rel}_3 = \text{Rel}_4, \quad \text{Re}z_1 = \text{Re}z_2 < \text{Re}z_3 = \text{Re}z_4\}, \\
D_2 = \{\lambda \in \mathbb{C} | \text{Rel}_1 = \text{Rel}_2 > \text{Rel}_3 = \text{Rel}_4, \quad \text{Re}z_1 = \text{Re}z_2 > \text{Re}z_3 = \text{Re}z_4\},
$$

(2.15)

where $l_i(\lambda)$ and $z_i(\lambda)$ are the diagonal elements of the $4 \times 4$ matrix $i\lambda\sigma_4$ and $\frac{1}{4i\lambda}\sigma_4$.

We note that $\mu_1(x, t, \lambda)$ and $\mu_2(x, t, \lambda)$ are entire functions of $\lambda$. Moreover, in their corresponding regions of boundedness

$$
\mu_j(x, t, \lambda) = I + O\left(\frac{1}{\lambda}\right), \lambda \to \infty, \quad j = 1, 2, 3.
$$

(2.16)

### 2.3 The symmetry of eigenfunctions

For the convenience, we write a $4 \times 4$ matrix $X = (X_{ij})_{4\times4}$ as

$$
X = \begin{pmatrix}
\tilde{X}_{11} & \tilde{X}_{12} \\
\tilde{X}_{21} & \tilde{X}_{22}
\end{pmatrix}, \quad \tilde{X}_{11} = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}, \quad \tilde{X}_{12} = \begin{pmatrix}
X_{13} & X_{14} \\
X_{23} & X_{24}
\end{pmatrix},
$$

(2.17)

$$
\tilde{X}_{21} = \begin{pmatrix}
X_{31} & X_{32} \\
X_{41} & X_{42}
\end{pmatrix}, \quad \tilde{X}_{22} = \begin{pmatrix}
X_{33} & X_{34} \\
X_{43} & X_{44}
\end{pmatrix}.
$$

As $U(x, t, \lambda) = i\lambda\sigma_4 + U_0, V(x, t, \lambda) = \frac{1}{4i\lambda}\sigma_4 - \frac{i}{2}i\lambda V_1 + \frac{1}{2}V_0$. Then the symmetry properties of $U(x, t, \lambda)$ and $V(x, t, \lambda)$ imply that the eigenfunction $\mu(x, t, \lambda)$ have the symmetries

$$
(\tilde{\mu}(x, t, \lambda))_{11} = P^\beta(\tilde{\mu}(x, t, \lambda))_{22}P^\beta, \quad (\tilde{\mu}(x, t, \lambda))_{12} = \alpha(\tilde{\mu}(x, t, \lambda))_{21}^T,
$$

(2.18)
where \( P^\beta = \text{diag}(1, \beta) \) and \( \beta^2 = 1 \).

Since
\[
P_{\pm}(U(x, t, \lambda))P_{\pm} = -U(x, t, \lambda)^T, \quad P_{\pm}(V(x, t, \lambda))P_{\pm} = -V(x, t, \lambda)^T,
\]
(2.19)
where \( P_{\pm} = \text{diag}(\pm \alpha, \pm \alpha, \pm 1, \pm 1) \) and \( \alpha^2 = 1 \).

According to Eq. (2.23) (see the similar proof in Ref. [12]), we know that the eigenfunction \( \psi(x, t, \lambda) \) of the Lax pair (2.4) and \( \mu(x, t, \lambda) \) of the Lax pair (2.7) are of the same symmetric relation
\[
\psi^{-1}(x, t, \lambda) = P_{\pm}(\psi(x, t, \lambda))P_{\pm}^T, \quad \mu^{-1}(x, t, \lambda) = P_{\pm}(\mu(x, t, \lambda))P_{\pm},
\]
(2.20)
Moreover, In the domains where \( \mu(x, t, \lambda) \) is bounded, we have
\[
\mu(x, t, \lambda) = \mathbb{I} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty,
\]
(2.21)
and \( \det[\mu(x, t, \lambda)] = 1 \) since \( tr(U(x, t, \lambda)) = tr(V(x, t, \lambda)) = 0 \).

### 2.4 The adjugated eigenfunction

We also need to consider the bounded and analytical properties of the minors of the matrices \( \{\mu_j(x, t, \lambda)\}_j^3 \). We recall that the cofactor matrix \( B^A \) of a \( 4 \times 4 \) matrix \( B \) is defined by
\[
B^A = \begin{pmatrix}
  m_{11}(B) & -m_{12}(B) & m_{13}(B) & -m_{14}(B) \\
  -m_{21}(B) & m_{22}(B) & -m_{23}(B) & m_{24}(B) \\
  m_{31}(B) & -m_{32}(B) & m_{33}(B) & -m_{34}(B) \\
  -m_{41}(B) & m_{42}(B) & -m_{43}(B) & m_{44}(B)
\end{pmatrix},
\]
(2.22)
where \( m_{ij}(B) \) denote the \((ij)\)th minor of \( B \) and \( (B^A)^T B = \text{adj}(B) B = \det B \).

It follows from Eq. (2.7) that be shown that the matrix-valued functions \( \mu^A \)'s satisfies the Lax pair
\[
\begin{align*}
  \mu^A_x + i\lambda[\sigma_4, \mu^A] &= -V_1^T \mu^A, \\
  \mu^A_t + \frac{1}{4\lambda}[\sigma_4, \mu^A] &= -V_2^T \mu^A,
\end{align*}
\]
(2.23)
where the superscript \( T \) denotes a matrix transpose. Then the eigenfunctions \( \{\mu^A_j(x, t, \lambda)\}_j^3 \) are solutions solutions can be expressed as
\[
\mu^A_j(x, t, \lambda) = \mathbb{I} - \int_{x_j}^x e^{-i\lambda(x-\xi)\sigma_4}(V_1 \mu^A_j)(\xi, t, \lambda) d\xi
\]
\[
- \int_{x_j}^t e^{-i\lambda(x-x_j)\sigma_4} \int_{t_j}^t e^{-i\lambda(t-\tau)\sigma_4}(V_2 \mu^A_j)(x_j, \tau, \lambda) d\tau,
\]
(2.24)
where

\[ \mu \in \mathbb{R}, \lambda \in \mathbb{C} \]

is an eigenfunction which satisfies the following analytic properties by using the Volterra integral equations, respectively. Thus, we can obtain the adjugated equations

\[ \mu_1^A \text{ is bounded and analytic for } \lambda \in \varnothing, \]

\[ \mu_2^A \text{ is bounded and analytic for } \lambda \in (D_1, D_2, D_3), \]

\[ \mu_3^A \text{ is bounded and analytic for } \lambda \in (D_2, D_3). \]

2.5 The spectral functions and the global relation

We also define the \(4 \times 4\) matrix value spectral function \(s(\lambda), S(\lambda)\) and \(\mathcal{S}(\lambda)\) as follows

\[
\begin{align*}
\mu_3(x, t, \lambda) &= \mu_2(x, t, \lambda) e^{(i\lambda x + \frac{i}{4\pi t})\bar{\delta}_4} S(\lambda), \\
\mu_1(x, t, \lambda) &= \mu_2(x, t, \lambda) e^{(i\lambda x + \frac{i}{4\pi t})\bar{\delta}_4} S(\lambda), \\
\mu_3(x, t, \lambda) &= \mu_1(x, t, \lambda) e^{(i\lambda x + \frac{i}{4\pi t})\bar{\delta}_4} \mathcal{S}(\lambda),
\end{align*}
\]

as \(\mu_2(0, 0, \lambda) = \mathbb{I}\), we obtain

\[
\begin{align*}
s(\lambda) &= \mu_3(0, 0, \lambda), \\
S(\lambda) &= \mu_1(0, 0, \lambda) = e^{-\frac{i}{4\pi} T \bar{\delta}_4} \mu_2^{-1}(0, T, \lambda), \\
\mathcal{S}(\lambda) &= \mu_1^{-1}(0, 0, \lambda) \mu_3(0, 0, \lambda) = S^{-1}(\lambda)s(\lambda) = e^{-\frac{i}{4\pi} T \bar{\delta}_4} \mu_3^{-1}(0, T, \lambda).
\end{align*}
\]

These relations among \(\mu_j\) are displayed in Figure 4. Thus these three functions \(s(\lambda), S(\lambda)\) and \(\mathcal{S}(\lambda)\) are dependent such that we only consider two of them, e.g. \(s(\lambda)\) and \(S(\lambda)\).

According to the definition (2.12) of \(\mu_j\), Eq.(2.27) implies that

\[
\begin{align*}
s(\lambda) &= \mathbb{I} - \int_0^\infty e^{-i\lambda \xi \bar{\delta}_4} (U_1 \mu_3)(\xi, 0, \lambda) d\xi, \\
S(\lambda) &= \mathbb{I} - \int_0^T e^{-\frac{i}{4\pi} T \bar{\delta}_4} (U_2 \mu_1)(0, \tau, \lambda) d\tau \\
&= \left[ \mathbb{I} + \int_0^T e^{-\frac{i}{4\pi} T \bar{\delta}_4} (U_2 \mu_2)(0, \tau, \lambda) d\tau \right]^{-1},
\end{align*}
\]

where \(\mu_j(0, t, \lambda)(j = 1, 2)\) and \(\mu_3(x, t, \lambda), 0 < x < \infty, 0 < t < T\) satisfy the Volterra integral equations

\[
\begin{align*}
\mu_1(0, t, \lambda) &= \mathbb{I} - \int_t^T e^{\frac{i}{4\pi} (t-\tau) \bar{\delta}_4} (U_2 \mu_1)(0, \tau, \lambda) d\tau, \\
\mu_2(0, t, \lambda) &= \mathbb{I} + \int_0^T e^{\frac{i}{4\pi} (t-\tau) \bar{\delta}_4} (U_2 \mu_2)(0, \tau, \lambda) d\tau, \\
\mu_3(0, t, \lambda) &= \mathbb{I} - \int_x^\infty e^{i\lambda (x-\xi) \bar{\delta}_4} (U_1 \mu_3)(\xi, 0, \lambda) d\xi.
\end{align*}
\]
Thus, it follows from Eqs. (2.28) and (2.29) that $s(\lambda)$ and $S(\lambda)$ are determined by $U(x, 0, \lambda)$ and $V(0, t, \mu)$, that is to say, determined by the initial data $u_0(x)$, $v_0(x)$ and the boundary data $g_0(t)$, $h_0(t)$, $g_1(t)$, $h_1(t)$.

Indeed, the eigenfunctions $\mu_3(x, 0, \lambda)$ and $\mu_j(0, t, \lambda), j = 1, 2$ satisfies the $x$-part and $t$-part of the Lax pair (2.7) at $t = 0$ and $x = 0$, respectively. Then, we have

\[
\begin{align*}
\text{x - part:} & \quad \mu_x(x, 0, \lambda) - i\lambda[\sigma_4, \mu(x, 0, \lambda)] = U(x, t = 0)\mu(x, 0, \lambda), \\
& \quad \lim_{x \to \infty} \mu(x, 0, \lambda) = I, \quad 0 < x < \infty, \\
\text{t - part:} & \quad \mu_t(0, t, \lambda) - \frac{i}{4\lambda}[\sigma_4, \mu(0, t, \lambda)] = V(x = 0, t)\mu(0, t, \lambda), \quad 0 < t < T, \\
& \quad \lim_{t \to 0} \mu(0, t, \lambda) = \mu(0, 0, \lambda) = I, \\
& \quad \lim_{t \to T} \mu(0, t, \lambda) = \mu(0, T, \lambda) = I.
\end{align*}
\]

Moreover, from the properties of $\{\mu_j^3\}_1$ and $\{\mu_j^4\}_1$, we can obtain that $s(\lambda), S(\lambda), s^4(\lambda)$ and $S^4(\lambda)$ have the following bounded properties

\[
s(\lambda) \text{ is bounded for } \lambda \in (D_1, D_1, D_2, D_2), \\
S(\lambda) \text{ is bounded for } \lambda \in \emptyset, \\
s^4(\lambda) \text{ is bounded for } \lambda \in (D_2, D_2, D_1, D_1), \\
S^4(\lambda) \text{ is bounded for } \lambda \in \emptyset.
\]

The spectral functions $S(\lambda)$ and $s(\lambda)$ are not independent which is of important relationship each other. In fact, from Eq.(2.27), we have

\[
\mu_3(x, t, \lambda) = \mu_1(x, t, \lambda)e^{i\lambda x + \frac{1}{4\pi t}T\sigma_4 S^{-1}(\lambda)s(\lambda)},
\]

as $\mu_1(0, t, \lambda) = I$, when $(x, t) = (0, T)$. We can evaluate the following relationship which is the global relation as follows

\[
S^{-1}(\lambda)s(\lambda) = e^{-2i\lambda^2 T\sigma_4 c(T, \lambda)} = e^{-\frac{1}{4\pi T}T\sigma_4} \mu_3(0, T, \lambda),
\]

where $\mu_3(0, t, \lambda)$ satisfy the Volterra integral equation

\[
\mu_3(0, t, \lambda) = I - \int_0^\infty e^{-i\lambda \sigma_4 (V_1 \mu_3)(\xi, T, \lambda)} d\xi, \quad 0 < t < T, \quad \lambda \in (D_1, D_1, D_2, D_2, \ldots).
\]

### 2.6 The definition of matrix-valued functions $M_n$’s

For each $n = 1, 2$, the solution $M_n(x, t, \lambda)$ of Eq.(2.7) is defined by the following integral equation

\[
(M_n(x, t, \lambda))_{ij} = \delta_{ij} + \int_{\gamma_{ij}} e^{(i\lambda x + \frac{1}{4\pi t}T\sigma_4 W_n(\xi, \tau, \lambda))_{ij}} d\tau, \quad i, j = 1, 2, 3, 4,
\]
where $W_n(x,t,\lambda)$ is given by Eq. (2.9), it is only used $M_n$ in place of $\mu$, and the contours $\gamma^n_{ij}(n=1,2;i,j=1,2,3)$ are defined as follows

$$\gamma^n_{ij} = \begin{cases} 
\gamma_1 & \text{if } \text{Rel}_i(\lambda) < \text{Rel}_j(\lambda) \text{ and } \text{Rez}_i(\lambda) \geq \text{Rez}_j(\lambda), \\
\gamma_2 & \text{if } \text{Rel}_i(\lambda) < \text{Rel}_j(\lambda) \text{ and } \text{Rez}_i(\lambda) < \text{Rez}_j(\lambda), \text{ for } \lambda \in D_n \\
\gamma_3 & \text{if } \text{Rel}_i(\lambda) \geq \text{Rel}_j(\lambda).
\end{cases} \quad (2.37)$$

According to the definition of $\gamma^n$, we have

$$\gamma^1 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\
\gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3
\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_3 & \gamma_3 & \gamma_3 & \gamma_3 \\
\gamma_2 & \gamma_2 & \gamma_3 & \gamma_3
\end{pmatrix}. \quad (2.38)$$

Next, the following proposition guarantees that the previous definition of $M_n$ has properties, namely, $M_n$ can be represented as a RH problem.

**Proposition 2.1** For each $n = 1,2$ and $\lambda \in D_n$, the function $M_n(x,t,\lambda)$ is defined well by Eq. (2.36). For any identified point $(x,t)$, $M_n$ is bounded and analytical as a function of $\lambda \in D_n$ away from a possible discrete set of singularities $\{\lambda_j\}$ at which the Fredholm determinant vanishes. Moreover, $M_n$ admits a bounded and continuous extension to Re-$\lambda$-axis and

$$M_n(x,t,\lambda) = I + O\left(\frac{1}{\lambda}\right). \quad (2.39)$$

Proof: The associated bounded and analytical properties have been established in Appendix B in [18]. Substituting the following expansion

$$M = M_0 + \frac{M^{(1)}}{\lambda} + \frac{M^{(2)}}{\lambda^2} + \cdots, \quad \lambda \to \infty, \quad (2.40)$$

into the Lax pair Eq. (2.7) and comparing the coefficients of the same order of $\lambda$, we can obtain Eq. (2.40).

**Remark 2.2** So far, we define two sets of eigenfunctions $\{\mu_j\}_1^3$ and $\{M_n\}_1^2$. The Fokas method in [10] analyzed the $2 \times 2$ Lax pair related to two kinds of eigenfunctions $\mu_j$, which is used for spectral analysis, and the other eigenfunction is used to be shown RH problem, our definition on $M_n$ is similar to the latter eigenfunction.
2.7 The jump matrix and computations

The new spectral functions \( S_n(\lambda) (n = 1, 2) \) are defined by
\[
S_n(\lambda) = M_n(0, 0, \lambda), \quad \lambda \in D_n, \quad n = 1, 2. \tag{2.41}
\]

Let \( M(x, t, \lambda) \) be a sectionally analytical continuous function in Riemann \( \lambda \)-sphere which equals \( M_n(x, t, \lambda) \) for \( \lambda \in D_n \). Then \( M(x, t, \lambda) \) satisfies the following jump conditions
\[
M_1(x, t, \lambda) = M_2(x, t, \lambda) J(x, t, \lambda), \quad \lambda \in \mathbb{R}, \tag{2.42}
\]
where
\[
J(x, t, \lambda) = e^{(i \lambda x + \frac{1}{4 \pi t}) i 4} [S_2^{-1}(\lambda) S_1(\lambda)]. \tag{2.43}
\]

**Remark 2.3** As the integral equation (2.36) defined by \( M_n(0, 0, \lambda) \) involves only along the initial half-line \( \{0 < x < \infty, t = 0\} \) and along the boundary \( \{x = 0, 0 < t < T\} \), so \( S_n \)'s can only be determined by the initial data and boundary data, therefore, equation (2.43) represents a jump condition of RH problem. In the absence of singularity, the solution \( q_1(x, t), q_2(x, t) \) of the equation can be reconstructed from the initial data and boundary values data, but if the \( M_n \) have pole singularities at some point \( \{\lambda_j\}, \lambda_j \in \mathbb{C} \), the RH problem should be included the residue condition in these points, so in order to determine the correct residue condition, we need to introduce three eigenfunctions \( \{\mu_j(x, t, \lambda)\}_1^3 \) in addition to the \( M_n \)'s.

**Proposition 2.4** The matrix-valued functions \( S_n(x, t, \lambda), (n = 1, 2) \) defined by
\[
M_n(x, t, \lambda) = \mu_2(x, t, \lambda) e^{(i \lambda x + \frac{1}{4 \pi t}) i 4} S_n(\lambda), \quad \lambda \in D_n. \tag{2.44}
\]
can be expressed with \( s(\lambda) \) and \( S(\lambda) \) elements as follows
\[
S_1(\lambda) = \begin{pmatrix}
    s_{11} & s_{12} & 0 & 0 \\
    s_{21} & s_{22} & m_{44}(s) & 0 \\
    s_{31} & s_{32} & m_{34}(s) & m_{43}(s) \\
    s_{41} & s_{42} & m_{14}(s) & m_{13}(s)
\end{pmatrix}, \quad S_2(\lambda) = \begin{pmatrix}
    m_{22}(s) & m_{21}(s) & s_{13} & s_{14} \\
    m_{32}(s) & m_{31}(s) & m_{33}(s) & s_{23} \ s_{24} \\
    m_{42}(s) & m_{41}(s) & m_{43}(s) \ m_{44}(s) & s_{33} \ s_{34} \\
    m_{12}(s) & m_{11}(s) & m_{13}(s) & m_{14}(s)
\end{pmatrix}. \tag{2.45}
\]

where \( n_{i1,j1,i2,j2}(X) \) denotes the determinant of the sub-matrix generated by taking the cross elements of \( i_{1,2} \) th rows and \( j_{1,2} \) th columns of the \( 4 \times 4 \) matrix \( X \). that is to say
\[
n_{i1,j1,i2,j2}(X) = \begin{vmatrix}
    X_{i1,j1} & X_{i1,j2} \\
    X_{i2,j1} & X_{i2,j2}
\end{vmatrix},
\]
Proof: We set that $\gamma_3^{X_0}$ is a contour when $(X_0, 0) \rightarrow (x, t)$ in the $(x, t)$-plane, here $X_0$ is a constant and $X_0 > 0$, for $j = 3$, we introduce $\mu_3(x, t, \lambda; X_0)$ as the solution of Eq. (2.10) with the contour $\gamma_3$ replaced by $\gamma_3^{X_0}$. Similarly, we define $M_n(x, t, \lambda; X_0)$ as the solution of Eq. (2.36) with $\gamma_3$ replaced by $\gamma_3^{X_0}$. Then, by simple calculation, we can use $S(\lambda)$ and $s(\lambda; X_0) = \mu_3(0, 0, \lambda; X_0)$ to derive the expression of $S_n(\lambda, X_0) = M_n(0, 0, \lambda; X_0)$ and the Eq. (2.45) will be obtained by taking the limit $X_0 \rightarrow \infty$.

Firstly, we have the following relations:

\[
M_n(x, t, \lambda; X_0) = \mu_1(x, t, \lambda)e^{(i\lambda x + \frac{1}{4\pi t})\delta_4}R_n(\lambda; X_0),
\]

\[
M_n(x, t, \lambda; X_0) = \mu_2(x, t, \lambda)e^{(i\lambda x + \frac{1}{4\pi t})\delta_4}S_n(\lambda; X_0),
\]

\[
M_n(x, t, \lambda; X_0) = \mu_3(x, t, \lambda; X_0)e^{(i\lambda x + \frac{1}{4\pi t})\delta_4}T_n(\lambda; X_0).
\]

Secondly, we can get the definition of $R_n(\lambda; X_0)$ and $T_n(\lambda; X_0)$ as follows

\[
R_n(\lambda; X_0) = e^{-\frac{1}{4\pi}T_n}M_n(0, T, \lambda; X_0),
\]

\[
T_n(\lambda; X_0) = e^{-i\lambda X_0 T_n}M_n(X_0, 0, \lambda; X_0),
\]

then equation (2.46) mean that

\[
s(\lambda; X_0) = S_n(\lambda; X_0)T^{-1}_n(\lambda; X_0),
\]

\[
S(\lambda; X_0) = S_n(\lambda; X_0)R^{-1}_n(\lambda; X_0).
\]

These equations constitute the matrix decomposition problem of $\{s, S\}$ by use $\{R_n, S_n, T_n\}$. In fact, by the definition of the integral equation (2.36) and $\{R_n, S_n, T_n\}$, we obtain

\[
\begin{pmatrix}
(R_n(\lambda; X_0))_{ij} = 0 & if & \gamma^n_{ij} = \gamma_1,
(S_n(\lambda; X_0))_{ij} = 0 & if & \gamma^n_{ij} = \gamma_2,
(T_n(\lambda; X_0))_{ij} = \delta_{ij} & if & \gamma^n_{ij} = \gamma_3.
\end{pmatrix}
\]

Thus equation (2.47) contains 32 scalar equations for 32 unknowns. The exact solution of these system can be obtained by solving the algebraic system. In this way, we can get a similar $\{S_n(\lambda), s(\lambda)\}$ as in Eq. (2.45) which just that $\{S_n(\lambda), s(\lambda)\}$ replaces by $\{S_n(\lambda; X_0), s(\lambda; X_0)\}$ in Eq. (2.45).

Finally, taking $X_0 \rightarrow \infty$ in this equation, we obtain the Eq. (2.45).

### 2.8 The residue conditions

Because $\mu_2(x, t, \lambda)$ is an entire function, and from Eq. (2.44) we know that $M(x, t, \lambda)$ only produces singularities in $S_n(\lambda)$ where there are singular points, from the exact expression Eq. (2.45), we know that $M(x, t, \lambda)$ may be singular as follows
• \([M_1]_3\) and \([M_1]_4\) could have poles in \(D_1\) at the zeros of \(n_{11,22}(s)(\lambda)\),
• \([M_2]_1\) and \([M_2]_2\) could have poles in \(D_2\) at the zeros of \(n_{33,44}(s)(\lambda)\).

We use \(\{\lambda_j\}_1^N\) denote the possible zero point above, and assume that these zeros satisfy the following assumptions

**Assumption 2.5** Suppose that

• \(n_{11,22}(s)(\lambda)\) admits \(n_1\) possible simple zeros in \(D_1\) denoted by \(\{\lambda_j\}_1^{n_1}\),
• \(n_{33,44}(s)(\lambda)\) admits \(N - n_1\) possible simple zeros in \(D_3\) denoted by \(\{\lambda_j\}_1^{N+n_1+1}\).

And these zeros are each different, moreover assuming that there is no zero on the boundary of \(D_n\)'s \((n = 1, 2)\).

**Lemma 2.6** For a \(4 \times 4\) matrix \(X = (X_{ij})_{4 \times 4}\), \(e^{\theta_{ij}X} X e^{-\theta_{ij}X}\) is given by

\[
e^{\theta_{ij}X} X e^{-\theta_{ij}X} = \begin{pmatrix}
X_{11} & X_{12} & X_{13}e^{2\theta} & X_{14}e^{2\theta} \\
X_{21} & X_{22} & X_{23}e^{2\theta} & X_{24}e^{2\theta} \\
X_{31}e^{-2\theta} & X_{32}e^{-2\theta} & X_{33} & X_{34} \\
X_{41}e^{-2\theta} & X_{42}e^{-2\theta} & X_{43} & X_{44}
\end{pmatrix}.
\]

We can deduce the residue conditions at these zeros in the following expressions:

**Proposition 2.7** Let \(\{M_n(x, t, \lambda)\}^d\) be the eigenfunctions defined by Eq. (2.36) and assume that the set \(\{\lambda_j\}_1^N\) of singularities are as the above assumption. Then the following residue conditions hold true:

\[
\text{Res}_{\lambda = \lambda_j}[M_1(x, t, \lambda)]_k = \frac{m_{4(5-k)}(s)(\lambda_3)}{n_{11,22}(s)(\lambda_1)n_{33,44}(s)(\lambda_1)} [M_1(x, t, \lambda)]_3e^{\theta_{13}(\lambda)} + \frac{m_{3(5-k)}(s)(\lambda_3)}{n_{11,22}(s)(\lambda_1)n_{33,44}(s)(\lambda_1)} [M_1(x, t, \lambda)]_4e^{\theta_{14}(\lambda)},
\]

\(n_1 + 1 \leq j \leq n_2; \lambda_j \in D_1, k = 1, 2.\)

\[
\text{Res}_{\lambda = \lambda_j}[M_2(x, t, \lambda)]_k = \frac{m_{2(5-k)}(s)(\lambda_3)}{n_{33,44}(s)(\lambda_1)n_{13,24}(s)(\lambda_1)} [M_2(x, t, \lambda)]_1e^{\theta_{31}(\lambda)} + \frac{m_{1(5-k)}(s)(\lambda_3)}{n_{33,44}(s)(\lambda_1)n_{13,24}(s)(\lambda_1)} [M_2(x, t, \lambda)]_2e^{\theta_{32}(\lambda)},
\]

\(n_2 + 1 \leq j \leq n_3; \lambda_j \in D_2, k = 3, 4.\)

where \(\dot{f} = \frac{df}{dt}\) and \(\theta_{ij}\) defined by

\[
\theta_{ij}(x, t, \lambda) = (l_i - l_j)x - (z_i - z_j)t, \quad i, j = 1, 2, 3, 4, \quad (2.52)
\]

thus, we have

\[
\theta_{12} = \theta_{21} = \theta_{34} = \theta_{43} = 0, \\
\theta_{13} = \theta_{14} = \theta_{23} = \theta_{24} = 2i\lambda x + \frac{1}{2i\lambda}t, \\
\theta_{31} = \theta_{41} = \theta_{32} = \theta_{42} = -2i\lambda x - \frac{1}{2i\lambda}t.
\]

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Proof: The equation (2.44) mean that
\[ M_1(x, t, \lambda) = \mu_2(x, t, \lambda)e^{(i\lambda x + \frac{1}{\lambda}t)s_4}S_1, \quad (2.53) \]
\[ M_2(x, t, \lambda) = \mu_2(x, t, \lambda)e^{(i\lambda x + \frac{1}{\lambda}t)s_4}S_2, \quad (2.54) \]
In view of the expressions for \( S_1 \) given in (2.45), the four columns of Eq. (2.53) read
\[ [M_1]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} + [\mu_2]_3 s_{31}e^{\theta_{i1}} + [\mu_2]_4 s_{41}e^{\theta_{i1}}, \quad (2.55) \]
\[ [M_1]_2 = [\mu_2]_1 s_{12} + [\mu_2]_2 s_{22} + [\mu_2]_3 s_{32}e^{\theta_{i1}} + [\mu_2]_4 s_{42}e^{\theta_{i1}}, \quad (2.56) \]
\[ [M_1]_3 = [\mu_2]_3 \frac{m_{44}(s)}{n_{11,22}(s)} + [\mu_2]_4 \frac{m_{34}(s)}{n_{11,22}(s)}, \quad (2.57) \]
\[ [M_1]_4 = [\mu_2]_3 \frac{m_{43}(s)}{n_{11,22}(s)} + [\mu_2]_4 \frac{m_{33}(s)}{n_{11,22}(s)}. \quad (2.58) \]
In view of the expressions for \( S_2 \) given in (2.45), the four columns of Eq. (2.54) read
\[ [M_2]_1 = [\mu_2]_1 \frac{m_{22}(s)}{n_{33,44}(s)} + [\mu_2]_2 \frac{m_{12}(s)}{n_{33,44}(s)}, \quad (2.59) \]
\[ [M_2]_2 = [\mu_2]_1 \frac{m_{21}(s)}{n_{33,44}(s)} + [\mu_2]_2 \frac{m_{11}(s)}{n_{33,44}(s)}, \quad (2.60) \]
\[ [M_2]_3 = [\mu_2]_1 s_{13}e^{\theta_{i1}} + [\mu_2]_2 s_{23}e^{\theta_{i1}} + [\mu_2]_3 s_{33} + [\mu_2]_4 s_{43}, \quad (2.61) \]
\[ [M_2]_4 = [\mu_2]_1 s_{14}e^{\theta_{i1}} + [\mu_2]_2 s_{24}e^{\theta_{i1}} + [\mu_2]_3 s_{34} + [\mu_2]_4 s_{44}. \quad (2.62) \]
Suppose that \( \lambda_j \in D_1 \) is a simple zero of \( n_{11,22}(s)(\lambda) \). Solving Eqs. (2.57) and (2.58) for \([\mu_2]_3, [\mu_2]_4\) and substituting the result into Eqs. (2.55) and (2.56), we find
\[ [M_1]_1 = \frac{m_{44}(s)s_{42} - m_{34}(s)s_{32}}{n_{11,22}(s)n_{31,42}(s)}[M_1]_3e^{\theta_{i1}} + \frac{m_{34}(s)s_{31} - m_{44}(s)s_{41}}{n_{11,22}(s)n_{13,24}(s)}[M_1]_4e^{\theta_{i1}} + \frac{m_{24}(s)[\mu_2]_1 + m_{14}(s)[\mu_2]_2}{n_{31,42}(s)}e^{\theta_{i1}}, \quad (2.63) \]
\[ [M_1]_2 = \frac{m_{43}(s)s_{42} - m_{33}(s)s_{32}}{n_{11,22}(s)n_{31,42}(s)}[M_1]_3e^{\theta_{i1}} + \frac{m_{33}(s)s_{31} - m_{43}(s)s_{41}}{n_{11,22}(s)n_{31,42}(s)}[M_1]_4e^{\theta_{i1}} \]
Suppose that we can establish the following theorem. For all \( (x,t,\lambda) \) that its satisfies a RH problem which can be formulated in terms of the initial and boundary values of \( q_1(x,t) \), \( q_2(x,t) \) and \( s(\lambda) \) on \( \Omega \). We take the residue of this equations at \( \lambda_j \), we find condition Eqs.(2.63) and (2.64) in the case when \( \lambda_j \in D_1 \).

In the same way, suppose that \( \lambda_j \in D_2 \) is a simple zero of \( n_{33,44}(s)(\lambda) \). Solving Eqs. (2.61) and (2.62) for \([\mu_2]_1, [\mu_2]_2\) and substituting the result into Eqs. (2.59) and (2.60), we find

\[
[M_2]_3 = \frac{m_{22}(s)s_{24} - m_{12}(s)s_{14}}{n_{33,44}(s)n_{13,24}(s)}[M_2]_1 e^{\theta_{13}} + \frac{m_{12}(s)s_{13} - m_{22}(s)s_{23}}{n_{33,44}(s)n_{13,24}(s)}[M_2]_2 e^{\theta_{13}} \\
+ \frac{m_{42}(s)[\mu_2]_3 + m_{32}(s)[\mu_2]_4}{n_{13,24}(s)} e^{\theta_{13}},
\] (2.65)

\[
[M_2]_4 = \frac{m_{21}(s)s_{24} - m_{11}(s)s_{14}}{n_{33,44}(s)n_{13,24}(s)}[M_2]_1 e^{\theta_{13}} + \frac{m_{11}(s)s_{13} - m_{21}(s)s_{23}}{n_{33,44}(s)n_{13,24}(s)}[M_2]_2 e^{\theta_{13}} \\
+ \frac{m_{41}(s)[\mu_2]_3 + m_{31}(s)[\mu_2]_4}{n_{13,24}(s)} e^{\theta_{13}},
\] (2.66)

Taking the residue of this equations at \( \lambda_j \), we find condition Eqs.(2.65) and (2.66) in the case when \( \lambda_j \in D_2 \).

3 The Riemann-Hilbert problem

In section 2, we define the sectionally analytical function \( M(x,t,\lambda) \) that its satisfies a RH problem which can be formulated in terms of the initial and boundary values of \( \{q_1(x,t), q_2(x,t)\} \) and \( \{g_1(x), g_2(x)\} \) on \( \Omega \). For all \( (x,t) \), the solution of system (1.4) can be recovered by solving this RH problem. So we can establish the following theorem.

**Theorem 3.1** Suppose that \( \{q_1(x,t), q_2(x,t)\} \) is solution of system (1.4) in the half-line domain \( \Omega \), and it is sufficient smoothness and decays when \( x \to \infty \). Then the solution \( u(x,t) \) and \( v(x,t) \) of system (1.4) can be reconstructed from the initial values \( \{u_0(x), v_0(x)\} \) and boundary values \( \{g_0(t), h_0(t), g_1(t), h_1(t)\} \) defined as follows

\[
\text{Initial values: } u_0(x) = u(x,t = 0), \quad v_0(x) = v(x,t = 0); \\
\text{Dirichlet boundary values: } g_0(t) = u(x = 0,t), \quad h_0(t) = v(x = 0,t); \\
\text{Neumann boundary values: } g_1(t) = u_x(x = 0,t), \quad h_1(t) = v_x(x = 0,t).
\] (3.1)

Like Eq.(2.26) using the initial and boundary data to define the spectral functions \( s(\lambda) \) and \( S(\lambda) \), we can further define the jump matrix \( J(x,t,\lambda) \). Assume that the zero points of the
\( n_{11,22}(s)(\lambda) \) and \( n_{33,44}(s)(\lambda) \), just like in Assumption 2.5. We also have the following results

\[
\begin{align*}
q_{1,x}(x,t) &= -2i \lim_{\lambda \to \infty} (\lambda M(x,t,\lambda))_{13}, \\
q_{2,x}(x,t) &= -2i \lim_{\lambda \to \infty} (\lambda M(x,t,\lambda))_{14}.
\end{align*}
\]

(3.2)

where \( M(x,t,\lambda) \) satisfies the following 4 × 4 matrix RH problem:

- \( M(x,t,\lambda) \) is a sectionally meromorphic on the Riemann \( \lambda \)-sphere with jumps across the contours on \( \text{Re}\lambda \)-axis (see figure 3).

- \( M(x,t,\lambda) \) satisfies the jump condition with jumps across the contours on \( \text{Re}\lambda \)-axis

\[
M_2(x,t,\lambda) = M_1(x,t,\lambda)J(x,t,\lambda), \quad \lambda \in \mathbb{R}.
\]

(3.3)

- \( M(x,t,\lambda) = I + O(\frac{1}{\lambda}), \quad \lambda \to \infty. \)

- The residue condition of \( M(x,t,\lambda) \) is showed in Proposition 2.7.

Proof: We can use similar method like ref. [22] to prove this Theorem. It only remains to prove Eq. (3.2) and this equation hold true from the large \( \lambda \) asymptotic of the eigenfunctions. We omit this proof in here because of the length of this article.

Thus, the solution of the coupled focusing-defocusing complex short pulse equation \( \{q_1(x,t), q_2(x,t)\} \) can be obtained by integration with respect to \( x \).

4 Conclusions and discussions

In this paper, we consider IBV of the coupled focusing-defocusing complex short pulse equation on the half-line. Using the Fokas method for nonlinear evolution equations which taking the form of Lax pair isospectral deformations and whose corresponding continuous spectra Lax operators, assume that the solution \( \{q_1(x,t), q_2(x,t)\} \) exists, we show that it can be represented in terms of the solution of a 4 × 4 matrix RH problem formulated in the plane of the complex spectral parameter \( \lambda \). The spectral functions \( s(\lambda) \) and \( S(\lambda) \) are not independent, but related by a compatibility condition, the so-called global relation. For other integrable equations with high-order matrix Lax pair, can we construct their solution of a matrix RH problem formulated in the plane of the complex spectral parameter \( \lambda \) by the similar method? This question will be discussed in our future paper.
Acknowledgements

This work is partially supported by the National Natural Science Foundation of China under Grant Nos. 12271008 and 11601055, Natural Science Research Project of Universities of Anhui Province under Grant No.1408085QA06

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