Strict deformation quantization and local spin interactions

N. Drago\textsuperscript{a} and C. J. F. van de Ven\textsuperscript{b}

\textsuperscript{a}Dipartimento di Matematica, Università di Trento and INFN-TIFPA and INdAM, Via Sommarive 14, I-38123 Povo, Italy
\textsuperscript{b}Julius Maximilian University of Würzburg, Department of Mathematics Chair of Mathematics X (Mathematical Physics), Emil-Fischer-Straße 31, 97074 Würzburg, Germany

January 25, 2024

Abstract

We define a strict deformation quantization which is compatible with any Hamiltonian with local spin interaction (e.g. the Heisenberg Hamiltonian) for a spin chain. This is a generalization of previous results known for mean-field theories. The main idea is to study the asymptotic properties of a suitably defined algebra of sequences invariant under the group generated by a cyclic permutation. Our point of view is similar to the one adopted by Landsman, Moretti and van de Ven \cite{Landsman}, who considered a strict deformation quantization for the case of mean-field theories. However, the methods for a local spin interaction are considerably more involved, due to the presence of a strictly smaller symmetry group.

Contents

1 Introduction 2
2 The algebra of $\gamma$-sequences 6
  2.1 Definition of $\gamma$-sequences .......................... 6
  2.2 Asymptotic properties of $\gamma$-sequences .................. 8
  2.3 The algebra generated by $\gamma$-sequences ............... 10
3 Strict deformation quantization of $\gamma$-sequences 15
  3.1 The continuous bundle of $C^\ast$-algebras $[B]_\gamma$ associated with $[B]_\infty$ ................. 15
  3.2 Canonical representative of $[q_N]_N \in [B]_\infty$ .................. 22
  3.3 The Poisson structure of $[B]_\infty$ .......................... 32
A Characterization of $\tilde{B}$-irreducible elements 36
1 Introduction

In this paper we provide a rigorous $C^*$-algebraic framework for the study of the semi-classical properties of any Hamiltonian with local spin interaction for a spin chain. This covers, for example, the Heisenberg Hamiltonian. This result is achieved by means of a suitable strict deformation quantization, whose construction is the main result of this paper — cf. Theorem 24.

Strict deformation quantization originates with Berezin [3] and Bayen et al. [1, 2] and it is based on the idea of “deforming” a given commutative Poisson algebra representing a classical system into a given non-commutative algebra modelling the associated quantized system. In Rieffel’s approach [17] the deformed algebras are $C^*$-algebras. Notably, the “classical-to-quantum” interpretation of a strict deformation quantization is not the unique point of view which can be taken. In Landsman’s approach [10, 11] the starting point of a strict deformation quantization is often taken to be a continuous field of $C^*$-algebras. The latter models an increasingly larger sequence of quantum physical systems, whose limit defines a macroscopic classical theory. The advantage of this point of view is that it leads to a rigorous notion of the classical limit of quantum theories [11]. This in turn yields a mathematically sound description of several physically interesting emergent phenomena, e.g. symmetry breaking [13, 18, 19]. This paper is considering this “micro-to-macro” point of view on strict deformation quantization.

From a technical point of view a strict deformation quantization is defined by the following data:

1. A commutative Poisson $C^*$-algebra $\mathcal{A}_\infty$, namely a commutative $C^*$-algebra $\tilde{\mathcal{A}}_{\infty}$ equipped with a Poisson structure $\{\ ,\ \} : \tilde{\mathcal{A}}_{\infty} \times \tilde{\mathcal{A}}_{\infty} \to \tilde{\mathcal{A}}_{\infty}$ defined on a dense $*$-subalgebra $\tilde{\mathcal{A}}_{\infty} \subseteq \mathcal{A}_{\infty}$ — cf. Section 3.3.

2. A continuous bundle of $C^*$-algebras $\prod_{N \in \mathbb{N}} \mathcal{A}_N$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$; (Thorough the whole paper we will stick to the case of continuous bundle of $C^*$-algebras over $\overline{\mathbb{N}}$, see [11] for the generic case.)

3. A family of linear maps, called quantization maps, $Q_N : \tilde{\mathcal{A}}_{\infty} \to \mathcal{A}_N$, $N \in \overline{\mathbb{N}}$, such that
   (a) $Q_{\infty} = \text{Id}_{\tilde{A}_{\infty}}$ and $Q_N(a)^* = Q_N(a^*)$ for all $a \in \tilde{\mathcal{A}}_{\infty}$. Moreover, the assignment
   $\overline{\mathbb{N}} \ni N \mapsto Q_N(a) \in \mathcal{A}_N,$
   defines a continuous section of the bundle $\prod_{N \in \overline{\mathbb{N}}} \mathcal{A}_N$.
   (b) For all $a, a' \in \tilde{\mathcal{A}}_{\infty}$ it holds
   \[ \lim_{N \to \infty} \|Q_N(\{a, a'\}) - iN[Q_N(a), Q_N(a')]\|_{\mathcal{A}_N} = 0. \] (1)
   (c) For all $N \in \mathbb{N}$, $Q_N(\tilde{\mathcal{A}}_{\infty})$ is a dense $*$-subalgebra of $\mathcal{A}_N$. 2
The algebra $A_\infty$ represents the classical (macroscopic) observables of the physical system. Likewise, the fibers $A_N$, $N \in \mathbb{N}$, of the bundle $\prod_{N \in \mathbb{N}} A_N$ recollect the quantum observables of the (increasingly larger) quantum system.

A relevant example is the strict deformation quantization described in [12, 16] for the $C^*$-algebra $[B]_\pi^\infty$ of (equivalence classes of) symmetric sequences — cf. Section 2 for further details. In this scenario the role of the commutative $C^*$-algebra $A_\infty$ is played by $[B]_\pi^\infty = C(S(B))$ — here $B = M_\kappa(\mathbb{C})$, $\kappa \in \mathbb{N}$, while $S(B)$ denotes the states space over $B$. The continuous bundle of $C^*$-algebras $\prod_{N \in \mathbb{N}} B_N^\pi$ is such that, for $N \in \mathbb{N}$, $B_N^\pi \subseteq B^N$ is the $N$-th symmetric tensor product of $B$.

From a physical point of view the ensuing quantization maps $Q_N$ are of particular interest as they relate to mean-field theories like the Curie-Weiss model [9, §2], for which the interaction between $N$ spin sites is described by

$$H_{cw,N} := - \frac{J}{2N} \sum_{j_1+j_2=N-2} I^{j_1} \otimes \sigma_3 \otimes I^{j_2} \otimes \sigma_3 - h \sum_{j=1}^{N-1} I^{N-1-j} \otimes \sigma_1 \otimes I^j ,$$

where $\sigma_3, \sigma_1 \in M_2(\mathbb{C})$ denote the Pauli’s matrices while $I \in M_2(\mathbb{C})$ is the identity matrix and $I^j := I^{\otimes j}$. Here $J \in \mathbb{R}$ represents the strength of the spin interaction whereas $h \in \mathbb{R}$ models an external magnetic field acting on the system. As observed in [12] one may recognize that

$$H_{cw,N}/N = Q_N(h_{cw}) + R_N ,$$

where $h_{cw} \in C(S(B))$ while $R_N \in B_N^\pi$ is such that $\|R_N\|_N = O(1/N)$.

The physical interpretation is that $C(S(B))$ is the algebra containing macroscopic observables, i.e. observables of an infinite quantum system describing classical thermodynamics as a limit of quantum statistical mechanics. This has furthermore led to a significant contribution in the study of the classical limit of ground states [12, 13, 14, 18]. More precisely, in such works a mathematically rigorous description of the limit of ground states $\omega_N$ of $H_{cw,N}$ in the regime of large particles $N \to \infty$ is given. In particular, a classical counterpart $\omega_\infty$ (i.e. a probability measure) of the quantum ground state $\omega_N \in S(B_N^\pi)$ is constructed with the property that $\omega_\infty(a) := \lim_{N \to \infty} \omega_N(Q_N(a))$ for all $a \in C(S(B))$. Additionally, this algebraic approach has revealed the existence of several physical emergent phenomena, see [19] for an overview. These results are consistent with the point of view of [11] — which is also the one considered in this paper — for which a quantum theory is pre-existing and the classical limit is computed in a second step, not vice versa.

As characteristic for mean-field models, the Curie-Weiss Hamiltonian describes the energy of a system of $N$ spin sites under the assumption that the interaction is non-local, namely that every spin site interacts with all other spin site. This leads to interesting results, but it is ultimately an approximation as one would rather expect each spin site to interact with finitely many neighbouring spin sites. An exemplary model based on such a local interaction is the
celebrated quantum Heisenberg Hamiltonian (for a spin chain) \[6\ §6.2\]

\[
H_{\text{He},N} := - \sum_{j=0}^{N-1} I^{N-2-j} \otimes \sum_{p,q=1}^{3} J^{pq} \sigma_p \otimes \sigma_q \otimes I^j - \sum_{j=0}^{N-1} I^{N-1-j} \otimes \sum_{p=1}^{3} h^p \sigma_p \otimes I^j , \tag{3}
\]

where \(J^{pq}\) is the symmetric matrix describing the spin interaction while \(h^p\) are the components of an external magnetic field —here for \(j = N - 1\) the contribution in the first sum reads \(\sum_{p,q=1}^{3} J^{pq} \sigma_q \otimes I^{N-2} \otimes \sigma_p\). For this model the interaction is restricted to two neighbouring sites.

Similarly to what happens with mean-field models one may wonder whether there exists a strict deformation quantization of a suitable \(C^*\)-algebra such that

\[
H_{\text{He},N}/N = Q_N(h_{\text{He}}) + O(1/N) .
\]

The purpose of this paper is to prove that this is in fact the case, cf. Theorem 24. In [15] a different (though similar in spirit) point of view is taken, and a strict deformation is considered such that \(H_{\text{He},N} = Q_\kappa(h_{\text{He},N}) + O(1/\kappa)\) where the semi-classical parameter \(\kappa\) corresponds to the increasing dimension of the single site algebra \(B = \mathcal{M}_\kappa(\mathbb{C})\) for a fixed number \(N\) of lattice sites. In contrast, this paper deals with an arbitrary but fixed dimension \(\kappa \in \mathbb{N}\) considering instead the increasing number \(N\) of spin sites as the semi-classical parameter.

Our result is particularly relevant because it provides an excellent basis for studying the classical limit of local quantum spin systems. Similarly to the case of mean-field theories [12, 13, 18, 19], one may now consider a rigorous \(C^*\)-algebraic formalization of the limit of ground states or Gibbs states [8, 9]. The latter can be used for the study of spontaneous symmetry breaking and phase transitions in realistic models such as the Heisenberg model.

From a technical point of view, the methods of this paper profit of those of [12, 16] for mean-field models. Nevertheless the results obtained therein do not apply straight away to our case. As a matter of fact the strict deformation quantization for mean-field models (like the Curie-Weiss Hamiltonian) profits of:

1. A large symmetry group, that is, mean-field models are symmetric under the permutation group \(\mathfrak{S}_N\) of all \(N\) spin sites. This leads to a high symmetry property which can be exploited in several steps of the construction, cf. [12, 16].

2. A fairly explicit description of the classical algebra \([B]_\pi^\infty = C(S(B))\). One may define \([B]_\gamma^\infty\) in terms of equivalence classes of “symmetric sequences” —cf. Remark 3— but the description in terms of \(C(S(B))\) simplifies the discussion, e.g. it allows to identify a Poisson structure in a rather direct way.

Contrary to this case, local quantum spin Hamiltonians (e.g. the Heisenberg model defined in (3)) are invariant under the strictly smaller subgroup generated by a fixed cyclic permutation of \(N\) objects. This spoils the possibility of applying the arguments of [12, 16]. The latter have to be reconsidered to take into account the smaller symmetry group. Moreover, the classical algebra \([B]_\gamma^\infty\) for such models does not have a “simple” explicit description. As a matter of fact, \([B]_\gamma^\infty\) is
defined as the C*-algebra generated by (equivalence classes of) “γ-sequences” —cf. Definition 5. Nevertheless it is still possible to prove all properties of $[B]_{\gamma}^\infty$ relevant for the discussion of its strict deformation quantization.

The paper is structured as follows. In Section 2 we introduce the notion of “γ-sequences” —cf. Definition 2— and discuss their properties. The main result in this section is the proof that the C*-algebra $[B]_{\gamma}^\infty$ generated by (equivalence classes of) γ-sequences is a commutative C*-algebra. The latter will play the role of the classical algebra $\mathcal{A}_\infty$ for which we will present a strict deformation quantization.

In Section 3 we state and prove the main theorem of this paper, which provides a strict deformation quantization of the commutative C*-algebra $[B]_{\gamma}^\infty$. To this avail, Section 3.1 is devoted to prove Proposition 12 which provides the continuous bundle of C*-algebras $[B]_{\gamma}^N$ needed in the formulation of Theorem 24. The main technical hurdle of this section is to prove that, given a γ-sequence $(a_N)_N$, the sequence of the norms $(\|a_N\|)_N$ is convergent. While this is straightforward for symmetric sequences (i.e. those used when dealing with mean-field models) for γ-sequences this is non-trivial and has to be discussed carefully. Sections 3.2-3.3 discuss further relevant properties of (equivalence classes of) γ-sequences as well as the Poisson structure on the C*-algebra $[B]_{\gamma}^\infty$. Eventually Theorem 24 is proved by recollecting all results from the previous sections.

For the sake of clarity the following theorem recollects in a concise fashion the content of the main Theorem 24 together with the other relevant results of the paper.

**Theorem 1 (main results):** The algebra $[B]_{\gamma}^\infty := [\mathcal{B}]_{\gamma}^\infty$ of equivalence classes of γ-sequences —cf. Definitions 2-5— is a commutative C*-algebra which is also endowed with a Poisson structure $\{\cdot,\cdot\}_{\gamma}$ —cf. Propositions 6-22.

Moreover, the data $[B]_{\gamma}^\infty$ and $B_{\gamma}^N := \mathcal{A}_N(B^N)$ —cf. Equation (7)— define a continuous bundle $[B]_{\gamma}^N$ of C*-algebras —cf. Proposition 12.

Finally, there exists a family of quantization maps $Q_N: [\mathcal{B}]_{\gamma}^\infty \to B_{\gamma}^N$, $N \in \mathbb{N}$ such that the data $[B]_{\gamma}^\infty, [B]_{\gamma}^N, \{Q_N\}_{N \in \mathbb{N}}$ define a strict deformation quantization —cf. Theorem 24. ♦

**Acknowledgements.** We are indebted with V. Moretti for countless helpful discussions on this project and to K. Landsman for his precious feedback. N.D. is grateful to M. Dippell for a brief, yet effective, discussion on the proof of Lemma 9. C. J. F. van de Ven is supported by a postdoctoral fellowship granted by the Alexander von Humboldt Foundation (Germany).

**Data availability statement.** Data sharing is not applicable to this article as no new data were created or analysed in this study.

**Conflict of interest statement.** The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter discussed in this manuscript.
2 The algebra of \(\gamma\)-sequences

2.1 Definition of \(\gamma\)-sequences

In this section we will introduce \(\gamma\)-sequences and discuss their properties.

To fix some notations, let \(\kappa \in \mathbb{N}\) and set \(B := M_\kappa(\mathbb{C})\). For the sake of simplicity we shall denote by \(B^N := B^\otimes N\), where \(N \in \mathbb{N}\), with the convention that \(B^0 = \mathbb{C}\). The state space over \(B^N\) will be denoted by \(S(B^N)\); Given \(\eta \in S(B)\) we set \(\eta^N := \eta^\otimes N \in S(B^N)\). Whenever needed we will denote \(\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}\).

Following [12] we denote by \(I, b_1, \ldots, b_{\kappa^2-1}, I \in B\) being the identity matrix, a basis of \(B\) (as a \(\mathbb{R}\)-vector space) abiding by the requirements

\[
\text{tr}(b_j) = 0, \quad b^*_j = b_j, \quad [b_j, b_\ell] = ic_{j_\ell}^m b_m, \quad \forall j, \ell = 1, \ldots, \kappa^2 - 1.
\]

where \(c_{j_\ell}^m\) denotes the structure constants of \(su(\kappa)\). In the particular case \(\kappa = 2\) we may choose \(b_j = \sigma_j / 2\) while \(c_{j_\ell}^m = \varepsilon_\ell s \delta^{sm}, \varepsilon\) being the Levi-Civita symbol. We will denote by \(\tilde{B}\) the vector space generated by \(\{b_j\}_{j=1}^{\kappa^2-1}\). The latter corresponds to the ker \(\tau\), being \(\tau: B \to \mathbb{C}\) the normalized trace defined by \(\tau(a) := \text{tr}(a) / \kappa\).

We then consider the linear operator (left-shift operator) \(\gamma_N: B^N \to B^N\) uniquely defined by continuous and linear extension of the following map defined on elementary tensors

\[
\gamma_N(a_1 \otimes \ldots \otimes a_N) := a_2 \otimes \ldots \otimes a_N \otimes a_1 \quad a_1, \ldots, a_N \in B.
\]

(5)

The operator \(\gamma_N\) is an algebra endomorphism, moreover, \(\gamma_N^2 = \text{Id}_B\). \(\text{Id}_B: B \to B\) being the identity operator. We denote by \(\overline{\gamma}_N: B^N \to B^N\) the averaged \(\gamma_N\) operator, defined by

\[
\overline{\gamma}_N := \frac{1}{N} \sum_{j=0}^{N-1} \gamma_N^j.
\]

(6)

Clearly \(\gamma_N \circ \overline{\gamma}_N = \overline{\gamma}_N \circ \gamma_N\): We denote by

\[
B^N_{\gamma} := \overline{\gamma}_N(B^N),
\]

(7)

the \(C^*\)-subalgebra of \(B^N\) made by \(\gamma_N\)-invariant elements.

Through this paper we will mostly consider sequences \((a_N)_{N \in \mathbb{N}} = (a_N)_{N \in \mathbb{N}}\) with \(a_N \in B^N\) for all \(N \in \mathbb{N}\). A sequence \((a_N)_{N \geq K}\) will be implicitly extended to \((a'_N)_{N \in \mathbb{N}}\) where \(a'_N = a_N\) for \(N \geq K\) and \(a'_N = 0\) for \(N < K\).

\[\text{In the forthcoming discussion we will use the notation } a_N \text{ to denote an element } a_N \in B^N. \text{ When we will need to use a subindex without necessarily stating the degree of the element we will use the notation } a_{(k)} \text{ so that } a_{(k)} \in B^{M(k)}, M(k) \in \mathbb{N}.\]
Definition 2: A sequence \((a_N)_N\) is called \(\gamma\)-sequence if there exists \(M \in \mathbb{N}\) and \(a_M \in B^M\) such that
\[
a_N = \tau_N^M a_M := \begin{cases} \tau_N(I^{N-M} \otimes a_M) & N \geq M \\ 0 & N < M \end{cases},
\]
where \(I \in B\) denotes the identity of \(B\) and \(I^N = I^\otimes N\).

Remark 3:

(i) For fixed \(N,M \in \mathbb{N}, N \geq M\), \(\tau_N^M : B^M \rightarrow B^N\) is a linear operator with operator norm smaller than 1. This implies that, \((\tau_N^M a_M)_N\) is bounded with
\[
\|\tau_N^M a_M\|_\infty := \sup_{N \in \mathbb{N}} \|\tau_N^M a_M\|_N \leq \|a_M\|_M,
\]
where \(\|\|_M\) denotes the norm on \(B^M\).

(ii) It is worth comparing our construction with the one presented in the literature \[11, 12, 16\], based on symmetric sequences. We stress that the latter are exploited to deal with the Curie-Weiss Hamiltonian —or more generally with mean-field theories \[11, \S 10\]— which prescribe a non-local interaction between spin sites. On the other hand we are interested in models compatible with Hamiltonian describing a local interaction between spin sites —e.g. the Heisenberg Hamiltonian, cf. Remark 4. To describe the non-local interaction algebraically one considers the symmetrization operator \(S_N : B^N \rightarrow B^N\) defined by continuous and linear extension of
\[
S_N(a_1 \otimes \cdots \otimes a_N) := \frac{1}{N!} \sum_{\kappa \in \mathfrak{S}_N} a_{\kappa(1)} \otimes \cdots \otimes a_{\kappa(N)} \quad a_1, \ldots, a_N \in B,
\]
where \(\mathfrak{S}_N\) is the set of permutation of \(N\) objects \[11, 16\]. Considering the \(C^*\)-subalgebra \(B^N_\pi := S_N B^N \subset B^N\) one then defines a symmetric-sequence (shortly, \(\pi\)-sequence) to be a sequence \((a_N)_N\) such that there exists \(M \in \mathbb{N}\) and \(a_M \in B^M_\pi\) fulfilling
\[
(a_N)_N = (\pi_N^M a_M)_N := (S_N(I^{N-M} \otimes a_M))_{N \geq M}.
\]
One immediately sees the relation with Definition 2. Actually a \(\gamma\)-sequence is defined in a way similar to \(\pi\)-sequences but averaging over a strictly smaller subgroup of \(\mathfrak{S}_N\). In fact \(\gamma\)-sequences and \(\pi\)-sequences share many similar properties, although \(\pi\)-sequences are generally speaking better behaved.

\(\diamondsuit\)
2.2 Asymptotic properties of $\gamma$-sequences

In what follows we will be mainly interested in the asymptotic behaviour as $N \to \infty$ of the sequences under investigations. For this reason, following [16], we introduce the $\sim$-equivalence relation

$$(a_N)_N \sim (b_N)_N \iff \lim_{N \to \infty} \|a_N - b_N\|_N = 0.$$  

(9)

For a given sequence $(a_N)_N$ we will denoted by $[a_N]_N := [(a_N)_N]$ the corresponding equivalence class with respect to (9). The $\sim$-equivalence relation (9) has a nice interplay with the full $C^*$-product $\prod_{N \in \mathbb{N}} B^N$ defined by

$$\prod_{N \in \mathbb{N}} B^N := \{(a_N)_N | \|a_N\|_N \in \ell^\infty(\mathbb{N})\}.$$  

(10)

As it is well-known [11] $\prod_{N \in \mathbb{N}} B^N$ is a $C^*$-algebra with respect to sup norm $\|(a_N)_N\|_{\infty} := \sup_{N \in \mathbb{N}} \|a_N\|_N$. Moreover, the direct $C^*$-sum

$$\bigoplus_{N \in \mathbb{N}} B^N := \{(a_N)_N \in \prod_{N \in \mathbb{N}} B^N | \lim_{N \to \infty} \|a_N\|_N = 0\},$$  

(11)

is a closed two-sided ideal in $\prod_{N \in \mathbb{N}} B^N$ and thus we may consider the quotient

$$[B]_\sim := \prod_{N \in \mathbb{N}} B^N / \bigoplus_{N \in \mathbb{N}} B^N,$$  

(12)

which is nothing but the space of $\sim$-equivalence classes $[a_N]_N$ for bounded sequences $(a_N)_N$. Importantly, $[B]_\sim$ is a $C^*$-algebra with norm

$$\|[a_N]_N\|_{[B]_\sim} = \limsup_{N \to \infty} \|a_N\|_N.$$  

(13)

Remark 4:

(i) Since both $\gamma$- and $\pi$-sequences are bounded —cf. Remark 3-i— they lead to well-defined elements $[\gamma^M_N a_M]_N, [\pi^M_N a_M]_N \in [B]_\sim$. One may wonder whether $[\gamma^M_N a_M]_N = [0]_N$ for a non-zero $a_M \in B^M$. This is in fact possible, but we postpone this discussion to Section 3.2 where we will prove that, for a given equivalence class $[\gamma^M_N a_M]_N$ it is possible to extract a “canonical representative” —cf. Definition 16— with the property that $[\gamma^M_N a_M]_N = [0]_N$ if and only if the canonical representative vanishes —cf. Proposition 18.

(ii) With reference to Equation (3) we have (considering $\kappa = 2$)

$$\frac{1}{N} H_{He,N} = \gamma^2_N \left( \sum_{p,q=1}^3 J^{pq} \sigma_p \otimes \sigma_q \right) + \gamma^1_N \left( \sum_{p=1}^3 h^p \sigma_p \right),$$  

8
showing the relation between $\gamma$-sequences and the Heisenberg Hamiltonian. Similarly, as discussed in \cite{12}, Equation (2) leads to

$$\frac{1}{N} H_{cw,N} = -\pi^2_N(J\sigma_3 \otimes \sigma_3) + \pi^1_N(h\sigma_1) + O(1/N),$$

showing that $(H_{cw,N}/N)_{N \geq 1}$ is equivalent to a $\pi$-sequence.

At this stage it is worth observing that $\gamma$-sequences model an arbitrary Hamiltonian with local spin interaction. We say that $H_N \in B^N$ is a (translation invariant) Hamiltonian with local spin interaction if and only if

$$H_N = \sum_{|i-j| \leq \ell, p \leq 1} J_{pq} \sigma_p(i) \sigma_q(j) + \sum_{p=1}^N h_p \sigma_p(i), \quad (14)$$

were $\sigma_p(i) \sigma_q(j)$ is a short notation for $I_{N-i+1} \otimes \sigma_p \otimes I_{1-M} \otimes \sigma_q \otimes I_{N-j}$ and similarly $\sigma_p(i) = I_{N-i+1} \otimes \sigma_p \otimes I_{N-j}$. The parameter $\ell \in \mathbb{N}$ determines the number of spin sites which interact with a fixed spin site $i$—e.g. for the Heisenberg Hamiltonian $\ell = 1$. The strength of the interaction and of the external magnetic field is determined by $J_{pq}, h_p$. Notice that the latter do not depend on the spin site: This entails that we are considering translation invariant local spin interactions.

Any Hamiltonian $H_N$ as per Equation (14) leads to a $\gamma$-sequences as per Definition (2). Indeed, we have

$$H_N/N = \sum_{m=0}^{\ell-1} \gamma^m_N \left( \sum_{p,q=1}^3 J_{pq} \sigma_p(i) \sigma_q(j) + \sum_{p=1}^N h_p \sigma_p(i) \right),$$

(iii) The $\sim$-equivalence relation (9) provides a first example showing the different behaviour of $\gamma$-sequences with respect to $\pi$-sequences. To this avail, let $a_M \in B^M$ and let us consider the $\pi$-sequence $(\pi^N_N a_M)_{N \geq 1}$. By direct inspection one immediately sees that, for all $N' \geq N \geq M$

$$\pi^N_N \pi^M_M a_M = S_N \left[ I^{N-N'} \otimes S_N(I^{N-M} \otimes a_M) \right] = \pi^M_M a_M,$$

which shows that the family of maps $\pi^M_M : B^M \rightarrow B^N$ is “consistent”, namely $\pi^N_N \circ \pi^M_M = \pi^M_M$. The same property does not apply for $\gamma$-sequences, but it holds only asymptotically. Indeed for $a_M \in B^M$ one has, for $N \geq M$,

$$\gamma^M_N a_M = \frac{1}{N} \sum_{j=0}^{N-M} \gamma^j_N (I^{N-M} \otimes a_M) + R_N$$

$$= \frac{1}{N} \sum_{j=0}^{N-M} I^{N-M-j} \otimes a_M \otimes I^j + R_N, \quad \|R_N\|_N \leq \frac{M-1}{N} \|a_M\|_M.$$
where we used the $\gamma$-invariance while

$$\|R_N\|_{N'} = \left\| \frac{N - 1}{N} \sum_{j=0}^{N-M} I^{N-M-j} \otimes a_M + R_N \right\|_{N'} \leq \| R_N \|_N + \frac{M - 1}{N} \| a_M \|_M = O(1/N).$$

This shows that, although $\bar{\gamma}_N \circ \bar{\gamma}_N \neq \bar{\gamma}_N$ one still has

$$\lim_{N \to \infty} \left\| \frac{N - 1}{N} \sum_{j=0}^{N-M} I^{N-M-j} \otimes a_M + R_N \right\|_{N'} - \left\| \frac{N - 1}{N} a_M \right\|_{B_{N'}} = 0. \quad (15)$$

As we shall see, naively speaking most the results obtained for $\pi$-sequences holds true also for $\gamma$-sequences but only asymptotically — in the sense of relation (9). \hfill \Box

### 2.3 The algebra generated by $\gamma$-sequences

In what follows we will consider the $*$-algebra $\bar{B}_\gamma^\infty \subset \prod_{N \in \mathbb{N}} B_N$ generated by $\gamma$-sequences together with its projection $[\bar{B}_\gamma^\infty] \subset [B]_\infty$. As we will see, the latter algebra enjoys remarkable properties, in particular, it can be completed to a commutative $C^*$-algebra $[B]_\gamma^\infty$.

**Definition 5:** Let $\bar{B}_\gamma^\infty$ be the $*$-algebra generated by $\gamma$-sequences — cf. Definition 2. We denote by $[B]_\gamma^\infty \subset [B]_\infty$ the projection of $\bar{B}_\gamma^\infty$ in $[B]_\infty$, that is, $[B]_\gamma^\infty$ is the $*$-algebra generated by equivalence classes of $\gamma$-sequences. Thus, $[a_N]_N \in [B]_\gamma^\infty$ if and only if

$$[a_N]_N = \sum_{\ell,k_1,\ldots,k_\ell} c^{k_1\ldots k_\ell} \frac{\gamma^{M(k_1)}(a_{(k_1)}) \cdots \gamma^{M(k_\ell)}(a_{(k_\ell)})}{\gamma} \in [B]_\gamma^\infty,$$

where $a_{(k_j)} \in B^{M(k_j)}$ while the sum over $\ell, k_1, \ldots, k_\ell$ is finite. We denote by $[B]_\gamma^\infty := \overline{[B]_\gamma^\infty}$ the closure of $[B]_\gamma^\infty$ in $[B]_\infty$, that is, the $C^*$-algebra generated by equivalence classes of $\gamma$-sequences. To wit, an equivalence class $[a_N]_N$ belongs to $[B]_\gamma^\infty$ if and only if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ and $[a'_N]_N \in [B]_\gamma^\infty$ such that $\|a_N - a'_N\|_N < \varepsilon$ for all $N \geq N_\varepsilon$. \hfill \Box

**Proposition 6:** Let $a_{M_1}, \ldots, a_{M_\ell} \in \mathbb{N}$, be such that $a_{M_j} \in B^{M_j}$, $j = 1, \ldots, \ell$. Then:

$$[\gamma_{M_1}(a_{M_1}) \cdots \gamma_{M_\ell}(a_{M_\ell})]_N = \gamma_N \left( \frac{1}{N^{\ell-1}} \sum_{|j| = \ell} I^{j_1} \otimes a_{M_1} \otimes \cdots \otimes I^{j_\ell} \otimes a_{M_\ell} \right)_N, \quad (16)$$

where $|j| := j_1 + \ldots + j_\ell$, $|M|_\ell = M_1 + \ldots + M_\ell$ is a short notation while

$$I^{j_1} \otimes a_{M_1} \otimes \cdots \otimes I^{j_\ell} \otimes a_{M_\ell} := \frac{1}{\ell} \sum_{\varepsilon \in \mathbb{E}} I^{j_1} \otimes a_{M_{(1)}} \otimes \cdots \otimes I^{j_\ell} \otimes a_{M_{(\ell)}}, \quad (17)$$

10
denotes “total weighted symmetrization” over the factor \(a_{M_1}, \ldots, a_{M_\ell}\) \(^2\) In particular \([B]_\gamma^\infty\) is a commutative C\(^*\)-subalgebra of \([B]_\sim\).

Proof. Let \(a_{M_1} \in B^{M_1}, \ldots, a_{M_\ell} \in B^{M_\ell}, \ell \in \mathbb{N}\). We will prove that, for \(N\) large enough,

\[
\gamma_N^{M_1}(a_{M_1}) \cdots \gamma_N^{M_\ell}(a_{M_\ell}) = \gamma_N \left( \frac{1}{N^{\ell-1}} \sum_{|j| = N-|M_\ell|} I^{j_1} \otimes a_{M_1} \otimes \cdots \otimes I^{j_\ell} \otimes a_{M_\ell} \right) + R_N, \tag{18}
\]

where \(R_N \in B^N\) is such that \(\|R_N\|_N = O(1/N)\). On account of 9 this implies Equation 10.

We proceed by induction over \(\ell \in \mathbb{N}\). For \(\ell = 1\) the right-hand side of Equation (18) reduces to \(\gamma_N^{M_1} a_{M_1} + R_N\) so that we may choose \(R_N = 0\). For \(\ell = 2\) we find, for \(N\) large enough (say, \(N \geq 2(M_1 + M_2)\)),

\[
\gamma_N^{M_1}(a_{M_1}) \gamma_N^{M_2}(a_{M_2}) = \gamma_N \left( \left( I^{N-M_1} \otimes a_{M_1} \right) \gamma_N \left( I^{N-M_2} \otimes a_{M_2} \right) \right)
\]

\[
= \gamma_N \left( \left( I^{N-M_1} \otimes a_{M_1} \right) \frac{1}{N} \sum_{j=0}^{N-1} \gamma_N^j \left( I^{N-M_2} \otimes a_{M_2} \right) \right)
\]

\[
= \gamma_N \left( \left( I^{N-M_1} \otimes a_{M_1} \right) \frac{1}{N} \sum_{j=1}^{N-M_2-1} \gamma_N^j \left( I^{N-M_2} \otimes a_{M_2} \right) \right) + R_N
\]

\[
= \gamma_N \left( \frac{1}{N} \sum_{j_1+j_2=N-M_1-M_2} I^{j_1} \otimes a_{M_1} \otimes I^{j_2} \otimes a_{M_2} \right) + R_N,
\]

where in the last line we used the symmetry of \(j_1, j_2\) as well as the \(\gamma_N\)-invariance of the whole term. The remainder \(R_N\) coincides with

\[
\|R_N\|_N = \left\| \gamma_N \left( \left( I^{N-M_1} \otimes a_{M_1} \right) \frac{1}{N} \sum_{j \in \{0, \ldots, M_1-1\} \cup \{N-M_2, \ldots, N-1\}} \gamma_N^j \left( I^{N-M_2} \otimes a_{M_2} \right) \right) \right\|_N \leq \frac{C_{M_1,M_2}}{N},
\]

where \(C_{M_1,M_2} > 0\) is a constant depending on \(a_{M_1}, a_{M_2}\). Roughly speaking, we removed the values of \(j\) for which \(a_{M_1}\) and \(a_{M_2}\) have “overlapping positions”. This happens in \(M_1 + M_2\) cases, whose fraction vanishes as \(N \to \infty\).

This proves Equation (18) for \(\ell = 2\). Proceeding by induction on \(\ell\), we now assume that

\(^2\)For example \(I^{j_1} \otimes a_{M_1} \otimes I^{j_2} \otimes a_{M_2} = (I^{j_1} \otimes a_{M_1} \otimes I^{j_2} \otimes a_{M_2} + I^{j_1} \otimes a_{M_2} \otimes I^{j_2} \otimes a_{M_1})/2\).
Equation (18) holds for all $\ell' < \ell$ and prove it for $\ell$. To this avail we consider, for $N \geq 2|M|_{\ell}$,

$$
\gamma_{N}^{M_{1}}(a_{M_{1}}) \cdot \gamma_{N}^{M_{\ell}}(a_{M_{\ell}})
= \gamma_{N} \left( \frac{1}{N^{\ell-2}} \sum_{|j|_{\ell-1}=N-|M|_{\ell-1}} P_{j_{1}} \otimes a_{M_{1}} \otimes \ldots \otimes P_{j_{\ell-1}} \otimes a_{M_{\ell-1}} \right) \gamma_{N}^{M_{\ell}}(a_{M_{\ell}}) + R_{N} \gamma_{N}^{M_{\ell}}(a_{M_{\ell}})
$$

where $\|R_{N}'\|_{N} \leq \|R_{N}\|_{N}\|a_{M_{\ell}}\| = O(1/N)$. Thus, we focus on

$$
\gamma_{N} \left( \frac{1}{N^{\ell-1}} (P_{j_{1}} \otimes a_{M_{1}} \otimes \ldots \otimes P_{j_{\ell-1}} \otimes a_{M_{\ell-1}}) \gamma_{N}^{M_{\ell}}(a_{M_{\ell}}) \right)
$$

where $j_{1}, \ldots, j_{\ell-1}$ are such that $|j|_{\ell-1} = N - |M|_{\ell-1}$. We now proceed as in the case $\ell = 2$ by considering only those values $j_{\ell}$ for which the position of $a_{M_{\ell}}$ “overlaps” with the ones of $P_{j_{1}}, \ldots, P_{j_{\ell}}$ and not with those of $a_{M_{1}}, \ldots, a_{M_{\ell-1}}$. Notice that, in focusing only on these $j_{\ell}$’s we are neglecting a contribution $R_{N}'$ with $\|R_{N}'\|_{N} = O(1/N)$. We obtain

$$
\gamma_{N} \left( \frac{1}{N^{\ell-1}} (P_{j_{1}} \otimes a_{M_{1}} \otimes \ldots \otimes P_{j_{\ell-1}} \otimes a_{M_{\ell-1}}) \sum_{j_{\ell}=0}^{N-1} \gamma_{N}^{j_{\ell}}(I^{N-M_{\ell}} \otimes a_{M_{\ell}}) \right)
$$

$$
= \gamma_{N} \left( \frac{1}{N^{\ell-1}} \sum_{h_{1}=0}^{j_{\ell}-M_{\ell}} P_{j_{1}} \otimes a_{M_{1}} \otimes P_{j_{1}-M_{\ell}-h_{1}} \otimes a_{M_{1}} \otimes \ldots \otimes P_{j_{\ell-1}} \otimes a_{M_{\ell-1}} \right)
$$

$$
+ \ldots + \gamma_{N} \left( \frac{1}{N^{\ell-1}} P_{j_{1}} \otimes a_{M_{1}} \otimes \ldots \otimes \sum_{0 \leq h_{\ell-1} \leq M_{\ell}} P_{j_{\ell-1}-M_{\ell}-h_{\ell-1}} \otimes a_{M_{\ell-1}} \right) + R_{N}''
$$

(19)

where $\|R_{N}''\|_{N} = O(1/N)$ while the sum over the $h_{p}$ is empty if $j_{p} < M_{\ell}$ —notice that at least one of these sums is not empty if $N$ is large enough. Notice that each of the $\ell - 1$ sets of $\ell$ indexes

$$
\{h_{1}, j_{1} - M_{\ell} - h_{1}, j_{2}, \ldots, j_{\ell-1}\}, \{j_{1}, h_{2}, j_{2} - M_{\ell} - h_{2}, j_{3}, \ldots, j_{\ell-1}\}, \ldots \{j_{1}, \ldots, j_{\ell-2}, h_{\ell-1}, j_{\ell-1} + M_{\ell} - h_{\ell-1}\}.
$$

is such that its elements sum to $N - |M|_{\ell}$. Considering now the summation over $j_{1}, \ldots, j_{\ell-1}$ and using the $\gamma_{N}$-invariance each subset of indexes provides the same contribution. We are lead
to

\[
\mathcal{T}_N \left( \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell-1} = N-|M|_{\ell-1}} (I^{j_1} \otimes a_{M_1} \otimes \ldots \otimes I^{j_{\ell-1}} \otimes a_{M_{\ell-1}}) \mathcal{T}_N^{M_\ell}(a_{M_\ell}) \right)
\]

\[
= \mathcal{T}_N \left( \frac{\ell - 1}{N^{\ell-1}} \sum_{|j|_{\ell} = N-|M|_{\ell}} I^{j_1} \otimes a_{M_1} \otimes \ldots \otimes I^{j_\ell} \otimes a_{M_\ell} \right) + R_N^m,
\]

where in the last line we used that for all \( \zeta \in \Theta_\ell \) there are \( \ell \) permutations which are equivalent to \( \zeta \) up to a cyclic permutation. Indeed, for any permutation of \( a_{M_1}, \ldots, a_{M_\ell} \) we may use the \( \gamma_N \)-invariance to write the corresponding contribution fixing the position of the factor \( a_{M_\ell} \). This boils down to a permutation of \( a_{M_1}, \ldots, a_{M_{\ell-1}} \) which is repeated \( \ell \) times.

By induction this proves Equation (18) for all \( \ell \in \mathbb{N} \) and thus Equation (16). \( \square \)

**Remark 7:**

(i) The appearance of the total weighted symmetrization \([17]\) ensures that, when \( a_{M_j} = I^{M_j} \) for all \( j \in \{1, \ldots, \ell\} \), the right-hand side of Equation (16) coincides with \([I^N]_N\). This is related to the fact that \( \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N-|M|_{\ell}} = (\ell - 1)! + O(1/N) \).

(ii) A closer inspection to the remainder term \( R_N \) of Equation (18) reveals that

\[
\|[R_N, \mathcal{T}_N^{M_1}(a_{M_1}) \ldots \mathcal{T}_N^{M_\ell}(a_{M_\ell})]\|_N = O(1/N^2),
\]

for all \( \ell, M_1, \ldots, M_\ell \in \mathbb{N} \), and \( a_M \in B^M \). Roughly speaking, the reason for this is due to the fact that both \( (R_N)_N \) and

\[
(\mathcal{T}_N^{M_1} a_{M_1} \ldots \mathcal{T}_N^{M_\ell} a_{M_\ell})_N
\]

are sequences with “an increasing number of identities”. In more details, Equation (15) implies

\[
\mathcal{T}_N^{M_1}(a_{M_1}) \ldots \mathcal{T}_N^{M_\ell}(a_{M_\ell}) = \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N-|M|_{\ell}} \mathcal{T}_N(I^{j_1} \otimes a_{M_1} \otimes \ldots \otimes I^{j_\ell} \otimes a_{M_\ell}) + R_N',
\]

where \( \|R_N'\|_N = O(1/N) \). This implies

\[
[R_N, \mathcal{T}_N^{M_1}(a_{M_1}) \ldots \mathcal{T}_N^{M_\ell}(a_{M_\ell})] = [R_N, R_N']
\]

\[
+ \left[ R_N, \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N-|M|_{\ell}} \mathcal{T}_N(I^{j_1} \otimes a_{M_1} \otimes \ldots \otimes I^{j_\ell} \otimes a_{M_\ell}) \right].
\]
The first contribution is estimated by \(\| R_N, R'_N \|_N = O(1/N^2)\) while for the second contribution we have

\[
\left\| R_N, \frac{1}{N^p-1} \sum_{|j|_e=N-|M'|_e} \gamma_N(I^1 \otimes a_{M'_1} \otimes \ldots \otimes I^p \otimes a_{M'_p}) \right\| \\
\leq \frac{1}{N} \sum_{p=0}^{N-1} \left\| R_N, \frac{1}{N^p-1} \sum_{|j|_e=N-|M'|_e} \gamma^p_N(I^1 \otimes a_{M'_1} \otimes \ldots \otimes I^p \otimes a_{M'_p}) \right\| \\
\leq \frac{L}{N^2} C_{M_1,\ldots,M_p},
\]

where \(C_{M_1,\ldots,M_p} > 0\) does not depend on \(N\). In the last inequality we used the estimate \(\| R_N \|_N = O(1/N)\) and that, on account of the structure of \(R_N\) — cf. the proof of Proposition 4 — and of \(I^1 \otimes a_{M'_1} \otimes \ldots \otimes I^p \otimes a_{M'_p}\), the sum over \(p\) is non-vanishing for finitely many values, say \(L\), where \(L\) is \(N\)-independent. This proves Equation (20).

(iii) In complete analogy with Definition 5 one may introduce the \(C^*\)-algebra \([B]_\pi^\infty \subset [B]_\sim\) generated by equivalence classes of \(\pi\)-sequences [16, Def. II.1]. Moreover, as shown in [11, 12, 16], for any \(a_M \in B^M_\pi\) and \(a_{M'} \in B^{M'}_\pi\) one finds

\[
[\pi^M_N(a_M)\pi^M_{N'}(a_{M'})]_N = [\pi^{M+M'}_{N}(S_{M+M'}(a_M \otimes a_{M'}))]_N,
\]

which shows that also \([B]_\pi^\infty\) is a commutative \(C^*\)-algebra. In fact, the product of \(\pi\)-sequences is (asymptotically as \(N \to \infty\)) a \(\pi\)-sequence. Additionally, one may prove that the system \(\{B^N_\pi\}_{N \in \mathbb{N}}, \{\pi^N_M\}_{N \geq M}\) is a generalized inductive system [4, 2]. This streamlines the identification of a bundle of \(C^*\)-algebras \(\prod_{N \in \mathbb{N}} B^N_\pi\) out of which a strict deformation quantization can be constructed [12, 16].

The situation for \(\gamma\)-sequences is slightly different. Indeed, Equation (16) shows that the product of \(\gamma\)-sequences is not a \(\gamma\)-sequence, even if its \(\sim\)-equivalence class is of \(\pi\). Nevertheless, Equation (16) shows that the product of equivalence classes of \(\gamma\)-sequences is commutative. As we shall see in Section 3.1 this will be enough to identify a continuous bundle of \(C^*\)-algebras \(\prod_{N \in \mathbb{N}} B^N_\gamma\) out of which a strict deformation quantization is obtained.

Finally it is worth observing that, for all \(a \in B\), one finds \(\gamma^N_N(a) = \pi^N_N(a)\) so that, given the results of [11, 16], \([B]_\pi^\infty \subset [B]_\gamma^\infty\).

(iv) By standard Gelfand duality [4, § II.2] we find

\([B]_\gamma^\infty \simeq C(K([B]_\gamma^\infty))\),

where \(K([B]_\gamma^\infty)\) denotes the character space over \([B]_\gamma^\infty\). An element \(\varphi \in K([B]_\gamma^\infty)\) is completely characterized by

\[
\varphi_M(a_M) := \varphi(\pi^M_N(a_M)N), \quad a_M \in B^M,
\]

14
which identifies a sequence \( \{ \varphi_M \}_{M \in \mathbb{N}} \) of states with \( \varphi_M \in S(B^M) \). These states are “asymptotically equivalent” because of Equation (15). Indeed considering \( \varpi_N : B^M \to B^N \), 
\( \varpi_N a_M = \varpi_N (I^{N-M} \otimes a_M) \), we find
\[
\lim_{N \to \infty} (\varphi_N \circ \varpi_N^{-1})(a_M) = \lim_{N \to \infty} \varphi_N (\varpi_N \varpi_N^{-1} a_M) = \varphi_N (\varpi_N^{-1} a_M) = \varphi_M(a_M) .
\]

A similar argument goes for \( [B]_{\infty}^\gamma \), where \( K([B]_{\infty}^\gamma) \) can be explicitly characterized. In particular \( K([B]_{\infty}^\gamma) \simeq S(B) \) [16, Lem. IV.4]. This identification may also be seen as a consequence of the prominent quantum De Finetti Theorem [11, Thm. 8.9]. As shown in [12], \( S(B) \) is a stratified manifold which carries a Poisson structure.

\[\diamondsuit\]

3 **Strict deformation quantization of \( \gamma \)-sequences**

The goal of this section is to construct a strict deformation quantization of the commutative algebra \([B]_{\infty}^\gamma\). To this avail in Section 3.1 we will identify a suitable continuous bundle of \( C^* \)-algebras \([B]_\gamma\) by means of a standard construction [14]. In Section 3.2 we will introduce the notion of “canonical representative” for an element \( \{ \alpha_N \}_{N \in \mathbb{N}} \in \gamma \) — cf. Definitions 16, 19.

Eventually, in Section 3.3 we will show that \( [B]^\gamma_{\infty} \) carries a Poisson structure and we will prove Theorem 24 which provides the strict deformation quantization of \([B]^\gamma_{\infty}\).

3.1 The continuous bundle of \( C^* \)-algebras \([B]^\gamma_{\infty}\) associated with \([B]^\gamma_{\infty}\)

In this section we will define a continuous bundle of \( C^* \)-algebras \([B]^\gamma_{\infty}\) over \( \mathbb{N} = \mathbb{N} \cup \{ \infty \} \) whose fibers are \([B]^\gamma_{N} := B^\gamma_{N}\) for \( N \in \mathbb{N} \) and \([B]^\gamma_{\infty}\), defined as per Definition 5, for \( N = \infty \).

To this avail we briefly recall the main definitions and results we need — cf. [11, App. C.19], [13, §IV.1.6]. We denote by \( C(\mathbb{N}) \) the space of \( \mathbb{C} \)-valued sequences \( \{ \alpha_N \}_{N \in \mathbb{N}} \) such that \( \alpha_\infty := \lim_{N \to \infty} \alpha_N \in \mathbb{C} \) exists. A continuous bundle (or field) of \( C^* \)-algebras over \( \mathbb{N} \) is a triple \( \mathcal{A}, \{ \mathcal{A}_N \}_{N \in \mathbb{N}} ; \{ \psi_N \}_{N \in \mathbb{N}} \) made by \( C^* \)-algebras \( \mathcal{A}, \mathcal{A}_N, \ N \in \mathbb{N} \), and surjective homomorphisms \( \psi_N : \mathcal{A} \to \mathcal{A}_N \) such that:

(i) The norm of \( \mathcal{A} \) is given by \( ||a||_\mathcal{A} := \sup_{N \in \mathbb{N}} ||\psi_N(a)||_{\mathcal{A}_N} \);

(ii) For all \( \alpha = (\alpha_N)_{N \in \mathbb{N}} \in C(\mathbb{N}) \) and \( a \in \mathcal{A} \) there exists a \( \alpha a \in \mathcal{A} \) with the property that \( \psi_N(\alpha a) = \alpha_N \psi_N(a) \).

(iii) For all \( a \in \mathcal{A} \), \( ||\psi_N(a)||_{\mathcal{A}_N} \) \( N \in \mathbb{N} \in C(\mathbb{N}) \).

A continuous section of \( \mathcal{A} \) is an element \( a \in \prod_{N \in \mathbb{N}} A_N \) such that there exists \( a' \in \mathcal{A} \) fulfilling \( a_N = \psi_N(a') \) for all \( N \in \mathbb{N} \). Clearly \( \mathcal{A} \) can be identified with its continuous sections, therefore, in the forthcoming discussion we shall always regard \( a \in \mathcal{A} \) as an element \( a \in \prod_{N \in \mathbb{N}} A_N \). For this reason from now on we will implicitly identify \( \psi_N, N \in \mathbb{N} \), with the projection \( \prod_{N \in \mathbb{N}} A_N \to A_N \).
Since this holds true for all convergent subsequences we conclude that 

\[ C_{\text{smallest continuous bundle of}} \]

Then defining 

\[ A \]

Remark 8: In applications, it is often difficult to identify a continuous bundle of \( C^* \)-algebras by assigning the triple \( \mathcal{A}, \{A_N\}_{N \in \mathbb{N}}, \{\psi_N\}_{N \in \mathbb{N}} \) directly. However, a useful result—cf. [10, Prop. 1.2.3], [11, Prop. C.124]—shows that it is in fact sufficient to identify a dense set of (a posteriori) continuous sections of \( \mathcal{A} \). Actually, let \( \tilde{A} \subseteq \prod_{N \in \mathbb{N}} A_N \) be such that:

1. For all \( N \in \mathbb{N} \) the set \( \{a_N \mid a \in \tilde{A}\} \) is dense in \( A_N \);
2. \( \tilde{A} \) is a \( * \)-algebra;
3. For all \( \tilde{a} \in \tilde{A} \), it holds
   \[ \lim_{N \to \infty} \|\tilde{a}_N\|_{A_N} = \|\tilde{a}_\infty\|_{\tilde{A}_\infty} \text{, i.e. } (\|\tilde{a}_N\|_{A_N})_{N \in \mathbb{N}} \in C(\mathbb{N}). \]

Then defining \( A \) by

\[ A := \left\{ a \in \prod_{N \in \mathbb{N}} A_N \mid \forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N}, \exists a' \in \tilde{A}: \|a_N - a'\|_{A_N} < \varepsilon \forall N \geq N_\varepsilon \right\}, \tag{22} \]

one may prove that \( A \) is a continuous bundle of \( C^* \)-algebras over \( \mathbb{N} \) [10, 11]. In fact, \( A \) is the smallest continuous bundle of \( C^* \)-algebras over \( \mathbb{N} \) which contains \( \tilde{A} \). \( \diamond \)

We will now prove that \( \prod_{N \in \mathbb{N}} [B]^N_\gamma \) identifies a continuous bundle of \( C^* \)-algebras where \( [B]^N_\gamma := B^N_\gamma \) for \( N \in \mathbb{N} \) while \( [B]^\infty_\gamma \) denotes the \( C^* \)-algebra introduced in Definition 5. To this avail we will identity a subset \( \bar{A} \subseteq \prod_{N \in \mathbb{N}} [B]^N_\gamma \) fulfilling conditions 1-2 of Remark 8. From a technical point of view, condition 3 will require to prove that, for all \( [a_N]_N \in [B]^\infty_\gamma \), the sequence \( (\|a_N\|_N)_N \) has a limit as \( N \to \infty \): This is proved in Proposition 10. To this avail, the following Lemma comes in handy.

Lemma 9: Let \( (\alpha_N)_{N \in \mathbb{N}} \) be a bounded sequence of real numbers such that

\[ \exists C_1, C_2 \in \mathbb{R}, \exists N_0 \in \mathbb{N} : \alpha_N \geq \alpha_K + \alpha_K + C_1 \frac{1}{K} + C_2 \frac{K}{N} \forall N \geq K \geq N_0. \tag{23} \]

Then \( (\alpha_N)_{N \in \mathbb{N}} \in C(\mathbb{N}) \), i.e. \( \alpha_\infty := \lim_{N \to \infty} \alpha_N \in \mathbb{R} \) exists. \( \diamond \)

Proof. Let \( (\alpha_{N_j})_{j \in \mathbb{N}} \) be a convergent subsequence of \( (\alpha_N)_{N \in \mathbb{N}} \). Then for all \( K \geq N_0 \) we find

\[ \lim_{j \to \infty} \alpha_{N_j} \geq \lim_{j \to \infty} \left( \alpha_K + C_1 \frac{1}{K} + C_2 \frac{K}{N_j} \right) = \alpha_K + \frac{C_1}{K}. \]

Since this holds true for all convergent subsequences we conclude that

\[ \liminf_{N \to \infty} \alpha_N \geq \alpha_K + \frac{C_1}{K} \quad \forall K \geq N_0. \]

Considering again a convergent subsequence \( (\alpha_{K_j})_{j \in \mathbb{N}} \) of \( (\alpha_N)_{N \in \mathbb{N}} \) the above inequality implies

\[ \lim_{j \to \infty} \alpha_{K_j} \leq \lim_{j \to \infty} \left( \liminf_{N \to \infty} \alpha_N - \frac{C_1}{K_j} \right) = \liminf_{N \to \infty} \alpha_N. \]
Since this holds for all convergent subsequences we conclude that
\[
\limsup_{N \to \infty} \alpha_N \leq \liminf_{N \to \infty} \alpha_N ,
\]
therefore, \( \lim_{N \to \infty} \alpha_N = \liminf_{N \to \infty} \alpha_N = \limsup_{N \to \infty} \alpha_N \) exists and it is finite. \( \square \)

**Proposition 10:** For all \( [a_N]_N \in [B]^\infty \_\gamma \) the sequence \( (\|a_N\|_N)_N \) is in \( C(\mathbb{N}) \). In particular we have
\[
\| [a_N]_N \|_{[B]^\infty \_\gamma} (= \| [a_N]_N \|_{[B]^\infty}) = \lim_{N \to \infty} \| a_N \|_N .
\]

**Proof.** To begin with we prove the claim for \( [a_N]_N = [\gamma^M_N a_M]_N \). We will then move to \( [a_N]_N \in [\dot{B}]^\infty \_\gamma \) eventually discussing \( [a_N]_N \in [B]^\infty \_\gamma \).

Let \( a_M \in B^M, \ M \in \mathbb{N}, \) and let us consider \( [\gamma^M_N a_M]_N \). Let \( N, K \in \mathbb{N}, \ N \geq K \geq M \) and consider \( \omega_K \in S(B^K) \). We decompose \( \omega_K \) in a finite convex combination of product states
\[
\omega_K = \sum_{p_1, \ldots, p_K} \omega^{p_1 \ldots p_K}_K \eta_{p_1} \otimes \ldots \otimes \eta_{p_K},
\]
where \( \eta_{p_\ell} \in S(B) \) for all \( \ell = 1, \ldots, K \). Then we consider
\[
\omega_{K,N} := \sum_{p_1, \ldots, p_K} \omega^{p_1 \ldots p_K}_K \tau^r \otimes (\eta_{p_1} \otimes \ldots \otimes \eta_{p_K})^q \in S(B^N),
\]
where \( N = r + qK, \ q \in \mathbb{N} \) and \( r \in \{0, \ldots, K - 1\} \) while \( \tau \in S(B) \) is normalized the trace state. We consider
\[
\omega_{K,N}(\gamma^M_N a_M) = \omega_{K,N} \left( \frac{1}{N} \sum_{j=0}^{N-1} I^{N-M-j} \otimes a_M \otimes I^j \right). \]

By direct inspection we have that, for all \( \ell \in \{0, \ldots, K - 1\},
\[
\frac{1}{N} \left[ \sum_{p_1, \ldots, p_K} \omega^{p_1 \ldots p_K}_K \tau^r \otimes (\eta_{p_1} \otimes \ldots \otimes \eta_{p_K})^q \right] \left( I^{N-M-\ell} \otimes a_M \otimes I^{\ell} \right)
\]
\[
= \frac{1}{N} \omega_K (\gamma^K_M (I^{K-M} \otimes a_M)).
\]
The same contribution arises if \( j \leq N - r - M = qK - M \) and \( j = \ell \mod K \). The number of such \( j \)'s is roughly
\[
q - M/K = N/K - r/K - M/K = N/K + O(1),
\]

17
where the \( O(1) \) contribution is bounded both in \( N \) and in \( K \). The net result is

\[
\omega_{K,N}(\gamma^M_N a_M) = \sum_{\ell=0}^{K-1} \left( \frac{N}{K} + O(1) \right) \frac{1}{N} \omega_K(\gamma^j_K (I^{K-M} \otimes a_M)) \\
+ \frac{1}{N} \sum_{j=N-r-M+1}^{N-1} \omega_{K,N}(\gamma^j_N (I^{N-M} \otimes a_M)) \\
= \omega_K(\gamma^M_K a_M) + O(K/N),
\]

where we observed that the sum over \( j \in [N - r - M + 1, N - 1] \) contains at most \( r + M - 1 = O(K) \) terms each of which is bounded by \( \|a_M\|_M \). Overall we find

\[
\|\gamma^M_N a_M\|_N \geq |\omega_{K,N}(\gamma^M_N a_M)| = \|\omega_K(\gamma^M_K a_M) + C\frac{K}{N}\| \geq |\omega_K(\gamma^M_K a_M)| - C\frac{K}{N},
\]

where \( C > 0 \) depends on \( M \) but not on \( N \) or \( K \). The arbitrariness of \( \omega_K \in S(B^K) \) leads to

\[
\|\gamma^M_N a_M\|_N \geq \|\gamma^M_K a_M\|_K - C\frac{K}{N}.
\]

Thus, Lemma 9 applies to the sequence \( \{\|\gamma^M_N a_M\|_N\} \), proving that \( \lim_{N \to \infty} \|\gamma^M_N a_M\|_N \) exists.

We now consider an arbitrary element \([a_N]_N\). Although our proof works for an arbitrary element of \([a_N]_N \in \bar{B}^\infty_\gamma\), for the sake of (notational) simplicity we restrict ourself to the case

\[
[a_N]_N = \left[ \sum_{k_1,k_2} c^{k_1 k_2 M(k_1)} \gamma^M_N (a(k_1)) \gamma^M_N (a(k_2)) \right]_N,
\]

where the sum over \( k_1, k_2 \) is finite. To prove that \( \{\|a_N\|_N\} \) has a limit as \( N \to \infty \) we rely on Equation (16) together with an argument similar in spirit to the one used for the case of a single \( \gamma \)-sequence. In fact, Proposition 8 implies that

\[
\left\| \sum_{k_1,k_2} c^{k_1 k_2 M(k_1)} \gamma^M_N (a(k_1)) \gamma^M_N (a(k_2)) \right\|_N \\
= \left\| \sum_{k_1,k_2} \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \gamma_N (I^{j_1} \otimes a(k_1) \otimes I^{j_2} \otimes a(k_2)) \right\|_N + O(1/N),
\]

so that we may restrict to the first factor on the right-hand side.

As for the case of single \( \gamma \)-sequence let \( N, K \in \mathbb{N} \) be such that \( N \geq K \geq \max\{2(M(k_1) + M(k_2))\} \) where the maximum is taken over all pairs \( k_1, k_2 \in \mathbb{N} \) appearing in the sum.
where $N = r + qK$, $q \in \mathbb{N}$ and $r \in \{0, \ldots, K - 1\}$ while $\omega_K = \sum_{\omega_{p_1 \ldots p_K}} \omega_{p_1 \ldots p_K} \eta_{p_1} \otimes \ldots \otimes \eta_{p_K}$ is an arbitrary finite convex decomposition of $\omega_K$ into product states. We then evaluate

$$\omega_{K, N} \left( \sum_{k_1, k_2} c_{k_1, k_2}^1 \frac{1}{N} \sum_{j_1 + j_2 = N - M(k_1) - M(k_2)} \tau_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right)$$

$$= \sum_{k_1, k_2} c_{k_1, k_2}^1 \frac{1}{N} \sum_{j_1 + j_2 = N - M(k_1) - M(k_2)} \omega_{K, N} \left( \tau_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right).$$

To this avail we fix $k_1, k_2$ and split the sum over $j_2$ in two cases:

(a) Let consider the sum for $0 \leq j_2 \leq N - M(k_1) - M(k_2) - r$. For $0 \leq \ell \leq K - M(k_1) - M(k_2)$ we find, with the same argument used for a single $\gamma$-sequence,

$$\omega_{K, N} \left( \tau_N \left( I^{N - M(k_1) - M(k_2) - \ell} \otimes a_{(k_1)} \otimes I^{\ell} \otimes a_{(k_2)} \right) \right)$$

$$= \omega_K \left( \tau_K \left( I^{K - M(k_1) - M(k_2) - \ell} \otimes a_{(k_1)} \otimes I^{\ell} \otimes a_{(k_2)} \right) \right) + O(K/N),$$

The number of $j_2$’s such that $0 \leq j_2 \leq N - M(k_1) - M(k_2) - r$ and $j_2 = \ell \mod K$ is roughly $q = N/K + O(1)$, therefore, summing over such $j_2$’s leads to a contribution of

$$\omega_{K, N} \left( \frac{1}{N} \sum_{j_1 + j_2 = N - M(k_1) - M(k_2)} \tau_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right)$$

$$= \omega_K \left( \frac{1}{K} \sum_{j_1 + j_2 = K - M(k_1) - M(k_2)} \tau_K \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right) + O(K/N).$$

It remains to discuss the sum over $0 \leq j_2 \leq N - M(k_1) - M(k_2) - r$ with $j_2 \in \mathbb{N}$.
In this case we find

\[
\left| \omega_{K,N} \left( \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \Pi_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right) \right| \\
\leq \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \|a_{(k_1)}\| \|M(k_1)\| \|a_{(k_2)}\| \|M(k_2)\| \\
\leq \frac{1}{N} \left( \frac{N}{K} + O(1) \right) (M(k_1) + M(k_2) - 1) \|a_{(k_1)}\| \|M(k_1)\| \|a_{(k_2)}\| \|M(k_2)\| = O(1/K) ,
\]

where we observed that, for each of the \(M(k_1)+M(k_2)-1\) values of \(\ell \in [K-M(k_1)-M(k_2), K-1]\), there are \(q = N/K + O(1)\) values of \(j_2 \leq N - M(k_1) - M(k_2) - r\) such that \(a_{(k_2)}\) overlaps with the (translated) position of \(a_{(k_1)}\). This does not allow to reconstruct \(\omega_K\), therefore, these cases are estimated by \(O(1/K)\).

(b) If \(j_2 \in [N-M(k_1)-M(k_2)-r+1, N-M(k_1)-M(k_2)]\) —which is empty if \(r = 0\)— we have

\[
\omega_{K,N} \left( \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \Pi_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right) = O(K/N) .
\]

Recollecting our result we have

\[
\left| \omega_{K,N} \left( \sum_{k_1,k_2} c_{k_1,k_2} \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \Pi_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right) \right| \\
= \left| \omega_K \left( \sum_{k_1,k_2} c_{k_1,k_2} \frac{1}{K} \sum_{j_1+j_2=K-M(k_1)-M(k_2)} \Pi_K \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right) \right| + O(1/K) + O(K/N) \\
= \left| \omega_K \left( \sum_{k_1,k_2} c_{k_1,k_2} \frac{1}{K} \sum_{j_1+j_2=K-M(k_1)-M(k_2)} \Pi_K \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right) \right| - \frac{C_1}{K} - \frac{C_2 K}{N} 
\]

de where \(C_1, C_2 > 0\) do not depend neither on \(N\) nor on \(K\). The arbitrariness of \(\omega_K \in S(B^K)\) leads to

\[
\left\| \sum_{k_1,k_2} \frac{1}{N} \sum_{j_1+j_2=N-M(k_1)-M(k_2)} \Pi_N \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right\|_N \\
\geq \left\| \sum_{k_1,k_2} \frac{1}{K} \sum_{j_1+j_2=K-M(k_1)-M(k_2)} \Pi_K \left( I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)} \right) \right\|_K - \frac{C_1}{K} - \frac{C_2 K}{N} .
\]
Thus, Lemma [11] applies and the limit
\[
\lim_{N \to \infty} \|\pi_N^{M(k_1)}(a_{(k_1)})\pi_N^{M(k_2)}(a_{(k_2)})\|_N,
\]
exists and it is finite.

Finally, let \([a_N]_N \in [B]_\gamma^\infty\). Then, for all \(\varepsilon > 0\) there exists \(N_\varepsilon \in \mathbb{N}\) and \([a'_N]_N \in [\hat{B}]_\gamma^\infty\) such that
\[
\|a_N - a'_N\|_N < \varepsilon \quad \forall N \geq N_\varepsilon.
\]
Moreover, since \((\|a'_N\|_N)_N\) is convergent, there exists \(N'_\varepsilon \in \mathbb{N}\) such that
\[
\|a'_N\|_N - \|a'_M\|_M < \varepsilon \quad \forall N, M \geq N'_\varepsilon.
\]
For \(N, M \geq \max\{N_\varepsilon, N'_\varepsilon\}\) we then have
\[
\|a_N\|_N - \|a_M\|_M \leq \|a_N\|_N - \|a'_N\|_N + \|a'_N\|_N - \|a'_M\|_M + \|a'_M\|_M - \|a_M\|_M
\]
\[
\leq \|a_N - a'_N\|_N + \|a'_N\|_N - \|a'_M\|_M + \|a'_M - a_M\|_M \leq 3\varepsilon,
\]
proving that \((\|a_N\|_N)_N\) is a Cauchy sequence.

\[\square\]

**Remark 11:** The result of Proposition [11] applies also for \(\pi\)-sequences. For this latter case the proof streamlines because
\[
\|\pi_N^M a_M\|_N = \|\pi_K^M a_M\|_N \leq \|\pi_K^M a_M\|_K,
\]
so that \((\|\pi_N^M a_M\|)_N\) is decreasing. The difficulties in moving from \([B]_\gamma^\infty\) to \([B]_\gamma^\infty\) is twofold. On the one hand, for \(\gamma\)-sequences \(\|\pi_N^M a_M\|_N\) is not decreasing, although it fulfils a similar properties asymptotically. On the other hand, the product of \(\gamma\)-sequences is not a \(\gamma\)-sequence, even when equivalence classes are considered. This requires a different strategy to ensure the existence of the limit \(\lim_{N \to \infty} \|a_N\|_N\) for \([a_N]_N \in [B]_\gamma^\infty\).

The following proposition proves the existence of the continuous bundle of \(C^*\)-algebras of interest.

**Proposition 12:** Let \(\{B^N_\gamma\}_{N \in \mathbb{N}}\) be the family of \(C^*\)-algebras introduced in Equation (11). Let \(\{[B]_\gamma^N\}_{N \in \mathbb{N}}\) be defined by \([B]_\gamma^N := B^N_\gamma\) for \(N \in \mathbb{N}\) while \([B]_\gamma^\infty\) is the \(C^*\)-algebra generated by equivalence classes of \(\gamma\)-sequences, cf. Definition 11. Let \([\hat{B}]_\gamma \subset \prod_{N \in \mathbb{N}} [B]_\gamma^N\) be the subset defined by

\[
[\hat{B}]_\gamma := \left\{ (A_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} [B]_\gamma^N \mid \exists (a_N)_N \in [\hat{B}]_\gamma^\infty : A_N = \begin{cases} a_N & N \in \mathbb{N} \\ [a_N]_N & N = \infty \end{cases} \right\}.
\] (24)
Then \([\hat{B}]_\gamma\) fulfills conditions 1-2-3 and thus it leads to a continuous bundle of \(C^*\)-algebras

\[
\begin{align*}
[B]_\gamma := \left\{(A_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} [B]_\gamma^N \mid \forall \varepsilon > 0 \exists N \in \mathbb{N}, \exists A' \in [\hat{B}]_\gamma : \|A_N - A'_N\|_N < \varepsilon \forall N \geq N_\varepsilon \right\}.
\end{align*}
\]

(25)

\[\Box\]

Proof. We will prove conditions 1-2-3 of Remark 8. The space \([\hat{B}]_\gamma\) is a \(*\)-algebra, therefore, condition 2 is fulfilled. Concerning condition 1, we have to prove that

\[
Z_M := \{A_M \in [B]^M_\gamma \mid (A_N)_{N \in \mathbb{N}} \in [\hat{B}]_\gamma \subseteq [B]^M_\gamma \},
\]

is dense in \([B]^M_\gamma\) for all \(M \in \mathbb{N}\). For \(M \in \mathbb{N}\) it is enough to observe that, for all \(a_M \in [B]^M_\gamma = B^M_{\gamma}\), we may consider \((A_N)_{N \in \mathbb{N}} \in [\hat{B}]_\gamma\) defined by

\[
A_N = \begin{cases} \gamma^M_M a_M & N \in \mathbb{N} \\ \gamma^M_N a_M & N = \infty \end{cases},
\]

which leads to \(A_M = \gamma^M_M a_M = a_M\), i.e. \(Z_M = [B]^M_\gamma\). If \(M = \infty\) we have \(Z_{\infty} = [\hat{B}]_{\gamma}\) whose closure is per definition \([B]_{\gamma}^\infty\) — cf. Definition 5.

Finally condition 3 is equivalent to

\[
\lim_{N \to \infty} \|A_N\|_N = \lim_{N \to \infty} \|a_N\|_N = \|[a_N]_N\|_{[B]^\infty_\gamma} = ||A_{\infty}\|[B]_{\gamma}^\infty \quad \forall (A_N)_{N \in \mathbb{N}} \in [\hat{B}]_\gamma,
\]

where the existence of \(\lim_{N \to \infty} \|a_N\|_N\) is ensured by Proposition 10. \(\Box\)

3.2 Canonical representative of \([a_N]_N \in [\hat{B}]_{\gamma}^\infty\)

To proceed further in the construction of the deformation quantization of \([B]^\infty_\gamma\) we have to discuss the possibility of identifying a canonical representative of an element \([a_N]_N \in [\hat{B}]_{\gamma}^\infty\). This is required for both endowing \([B]_{\gamma}^\infty\) with a Poisson structure as well as for defining the quantization maps \(Q_N : [B]_{\gamma}^\infty \to [B]_{\gamma}^N\) — cf. Theorem 24.

To begin with, we address the following problem: Given \([\gamma^M_N a_M]_N \in [\hat{B}]_{\gamma}^\infty\) does it hold

\[
[\gamma^M_N a_M]_N = [0]_N \iff a_M = 0 ?
\]

A positive answer in this direction would imply that, given an equivalence class \([\gamma^M_N a_M]_N\), one is able to determine uniquely the \(\gamma\)-sequence \((\gamma^M_N a_M)_N\). Unfortunately, the answer to this question is negative because

\[
[\gamma^M_N a_M]_N = [\gamma^M_N (I^K \otimes a_M)]_N = [\gamma^M_N (a_M \otimes I^K)]_N,
\]

which leads to

\[
\lim_{N \to \infty} \|a_N\|_N = \lim_{N \to \infty} ||[a_N]_N\|_{[B]^\infty_\gamma} = ||A_{\infty}\|[B]_{\gamma}^\infty \quad \forall (A_N)_{N \in \mathbb{N}} \in [\hat{B}]_\gamma,
\]

where the existence of \(\lim_{N \to \infty} \|a_N\|_N\) is ensured by Proposition 10.
although the associated sequences are not the same. Indeed
\[ \gamma^M_M + (I^K \otimes a_M) = 0 \neq \gamma^M_M a_M = \gamma^M_M a_M. \]
This counterexample suggests to focus on the \( C^*\)-subalgebra \( \tilde{B}^M \) where \( \tilde{B} = \ker \tau \), \( \tau \in S(B) \) being the trace state — cf. Section 2. In fact, therein the situation is slightly better as shown by the following Lemma.

**Lemma 13:** Let \( \tilde{a}_M \in \tilde{B}^M \) be such that \( [\gamma^M_N \tilde{a}_M]_N = [0]_N \). Then \( \tilde{a}_M = 0 \).

**Proof.** Per definition \( [\gamma^M_N \tilde{a}_M]_N = [0]_N \) if and only if \( \lim_{N \to \infty} \| \gamma^M_N \tilde{a}_M \|_N = 0 \). Let \( \omega_M \in S(B^M) \) and let \( \tau \in S(B) \) be the normalized trace state \( \tau(a) := \text{tr}(a)/\kappa \). Let \( N \geq M + 1, q \in \mathbb{N} \) and \( r \in \{0, \ldots, M\} \) be such that \( N = r + q(M + 1) \). We consider the state
\[ \omega_{M,N} := \tau^r \otimes (\tau \otimes \omega_M)^q \in S(B^N). \]

By direct inspection we find that
\[ \omega_{M,N}([\gamma^M_N \tilde{a}_M]_N) = [\tau^r \otimes (\tau \otimes \omega_M)^q] \left( \frac{1}{N} \sum_{j=0}^{N-1} \gamma^j_N (I^{N-M} \otimes \tilde{a}_M) \right) = \frac{1}{M + 1} \omega_M(\tilde{a}_M) + O(1/N), \]

(26)

Indeed, for \( j = 0 \) the resulting contribution is \( \omega_M(\tilde{a}_M)/N \). The same contribution appears when \( j = 0 \mod M + 1 \): Since \( j \in \{0, \ldots, N-1\} \) this happens \( q \) times, moreover, \( q = N/(M+1) + O(1) \) leading to the right-hand side of Equation (26). Whenever \( j \neq 0 \mod M + 1 \) the resulting contribution is 0, on account of the fact that \( \tau \) vanishes on \( \tilde{B} \).

Equation (26) implies that, for all \( \omega_M \in S(B^M) \),
\[ 0 = \lim_{N \to \infty} \| \gamma^M_N \tilde{a}_M \| \geq \frac{1}{M + 1} \| \omega_M(\tilde{a}_M) \|. \]

The arbitrariness of \( \omega_M \) leads to \( \| \tilde{a}_M \|_M = 0 \), that is, \( \tilde{a}_M = 0 \).

Thus, although the equivalence class \( [\gamma^M_N a_M]_N \) does not identify a unique sequence \( (\gamma^M_N a_M)_N \), Lemma 13 suggests that a (a posteriori unique) canonical representative may be extracted by working with the “\( \tilde{B} \)-irreducible components” of the \( \gamma \)-sequence. To this avail, we introduce the notion of \( \tilde{B} \)-irreducibility. This identifies those elements in \( \tilde{B}^M \) which cannot be written as \( I \otimes a_{M-1} \) or \( a_{M-1} \otimes I \) for some \( a_{M-1} \in B^{M-1} \).

**Definition 14:** An element \( a_M \in B^M \) is called **\( \tilde{B} \)-irreducible**, and we write \( a_M \in B^{M}_{\text{irr}} \), if either \( M = 0 \) or
\[ (\tau \otimes \omega_{M-1})(a_M) = (\omega_{M-1} \otimes \tau)(a_M) = 0, \]

(27)

for all \( \omega_{M-1} \in S(B^{M-1}) \).
We stress that some of the $a$'s may vanish in the process. However, it is important to observe that, moving from $a_M$ to $a'_M$, the $\gamma$-sequence (and thus its equivalence class) does not change. Notice that, if we replace $a_M$ with $I^K \otimes a_M$ or $a_M \otimes I^K$, the $B$-irreducible elements $\{a'_M\}_{M=0}$ do not change.

**Definition 16:** Let $(\gamma^M a_M)_N$ be a $\gamma$-sequence and let $\sum_{j=0}^{M} I^{M-j} \otimes a'_j$ be the element defined as per Equation (29), where $a'_j \in B^j_{\text{IRR}}$ for all $j \in \{0, \ldots, M\}$. The sequence

$$\sum_{j=0}^{M} (\gamma^M a'_j)_N \in \hat{B}^\infty_\gamma,$$

is called the **canonical representative** of $(\gamma^M a_M)_N$. \hfill \blacklozenge
Remark 17:

(i) It is worth pointing out that, while $(\tau^M_Na_M)_N = (\tau^M_Na'_M)_N$ for $a'_M = \sum_{j=0}^M t^{M-j} \otimes a'_j$, for the canonical representative we only have equality of equivalence classes, i.e. $[\tau^M_Na_M]_N = \sum_{j=0}^M [\tau^j_Na'_j]_N$. In particular we have

$$(\tau^M_Na_M)_N = (\tau^M_Na'_M)_N = \sum_{j=0}^M (\tau^j_Na'_j)_N + R_N,$$  \hspace{1cm} (30)

where $\|R_N\| = O(1/N^\infty)$. For example if $a_M = a_0 I^M$ then the canonical representative is the constant sequence $a_N = a_0 I^N$, $N \in \mathbb{N}$, which coincides with $(\tau^M_Na_M)_N$ only for $N \geq M$.

(ii) On account of the previous discussion we observe that the algebra generated by $\gamma$-sequences of the form $(\tau^M_Na_M)_N$ for $a_M \in B^M_{\mathbb{H}rr}$, $M \in \mathbb{N}$, exhaust the whole space $B^\infty_{\mathbb{H}}$.

\diamond

The following proposition shows that the canonical representative introduced in Definition 15 is unique.

Proposition 18: Let $M \in \mathbb{N}$ and $a_j \in B^M_{\mathbb{H}rr}$ for all $j = 0, \ldots, M$. Then

$$\lim_{N \to \infty} \left\| \sum_{j=0}^M \tau^j_Na_j \right\|_N = 0 \iff a_0 = 0, \ldots, a_M = 0. \hspace{1cm} (31)$$

\diamond

Proof. The proof is similar to the one of Lemma 13. By direct inspection we have

$$0 = \lim_{N \to \infty} \left\| \sum_{j=0}^M \tau^j_Na_j \right\|_N \geq \lim_{N \to \infty} \tau^N \left( \sum_{j=0}^M \tau^j_Na_j \right) = |\tau^0_N a_0|. \hspace{1cm} (30)$$

Let now $\eta \in S(B)$ and let $\omega_{\eta,N} := \tau^r \otimes (\tau^{2M-1} \otimes \eta)^q \in S(B^N)$, where $N = r + 2Mq$, $q \in \mathbb{N}$ and $r \in \{0, \ldots, 2M-1\}$. We have

$$0 = \lim_{N \to \infty} \left\| \sum_{j=0}^M \tau^j_Na_j \right\|_N \geq \lim_{N \to \infty} \left| \omega_{\eta,N} \left( \sum_{j=1}^M \tau^j_Na_j \right) \right| = \left| \frac{1}{2M} \eta(a_1) \right|,$$

which implies $a_1 = 0$ because of the arbitrariness of $\eta \in S(B)$. Notice that $\omega_{\eta,N}(\tau^I_Na_j) = 0$ for all $j \geq 2$ on account of the assumption $a_j \in B^j_{\mathbb{H}rr}$.

Proceeding by induction we may assume that $a_1 = \ldots = a_{\ell-1} = 0$ and prove that $a_{\ell} = 0$. To this avail let $\eta_1, \ldots, \eta_{\ell} \in S(B)$ and set

$$\omega_{\eta_1,\ldots,\eta_{\ell},N} := \tau^r \otimes (\tau^{2M-\ell} \otimes \eta_1 \otimes \ldots \otimes \eta_{\ell})^q \in S(B^N),$$

25
where \( N = r + 2Mq, \ q \in \mathbb{N} \) and \( r \in \{0, \ldots, 2M-1\} \). Using the inductive hypothesis we find

\[
0 = \lim_{N \to \infty} \left\| \sum_{j=\ell}^{M} \mathbb{P}_{N}^{j} a_{j} \right\|_{N} \geq \lim_{N \to \infty} \left| \omega_{\eta_{\ell}, \ldots, \eta_{N}} \left( \sum_{j=\ell}^{M} \mathbb{P}_{N}^{j} a_{j} \right) \right| = \frac{1}{2M} \left| (\eta_{1} \otimes \ldots \otimes \eta_{\ell})(a_{\ell}) \right| ,
\]

where, with the same argument as above, the contributions arising from \( a_{j}, \ j \geq \ell + 1 \), vanish. The arbitrariness of \( \eta_{1}, \ldots, \eta_{\ell} \in S(B) \) implies \( a_{\ell} = 0 \).

Summing up, every equivalence class \( [\gamma_{N}^{M} a_{M}]_{N} \in [\hat{B}]_{\gamma}^{\infty} \) has a unique canonical representative obtained by decomposing \( a_{M} \) into its \( \hat{B} \)-irreducible components.

We shall now discuss the notion of canonical representative for a generic element \( [a_{N}]_{N} \in [\hat{B}]_{\gamma}^{\infty} \). Proposition 6 and Remark 17-ii lead to the following definition.

**Definition 19:** Let \( [a_{N}]_{N} \in [\hat{B}]_{\gamma}^{\infty} \) be such that

\[
[a_{N}]_{N} = \sum_{\ell, k_{1}, \ldots, k_{\ell}} c^{k_{1} \ldots k_{\ell}} \left[ \mathbb{P}_{N}^{M(k_{1})} (a_{(k_{1})}) \ldots \mathbb{P}_{N}^{M(k_{\ell})} (a_{(k_{\ell})}) \right]_{N}
\]

\[
= \sum_{\ell, k_{1}, \ldots, k_{\ell}} c^{k_{1} \ldots k_{\ell}} \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N - |M(k)|_{\ell}} \mathbb{P}_{N} (I^{j_{1}} \otimes a_{(k_{1})} \otimes \ldots \otimes I^{j_{\ell}} \otimes a_{(k_{\ell})}) \right]_{N}.
\]

where \( a_{k_{j}} \in B_{M(k_{j})}^{M(k_{j})} \) for all \( k_{j} \), while the sum over \( \ell, k_{1}, \ldots, k_{\ell} \) is finite and \( |M(k)|_{\ell} := M(k_{1}) + \ldots + M(k_{\ell}) \). The sequence

\[
\sum_{\ell, k_{1}, \ldots, k_{\ell}} c^{k_{1} \ldots k_{\ell}} \left( \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N - |M(k)|_{\ell}} \mathbb{P}_{N} (I^{j_{1}} \otimes a_{(k_{1})} \otimes \ldots \otimes I^{j_{\ell}} \otimes a_{(k_{\ell})}) \right)_{N} \geq |M(k)|_{\ell}, \tag{32}
\]

is called the **canonical representative** of \( [a_{N}]_{N} \).

Similarly to Proposition 18 we have the following result, showing that the canonical representative introduced in Definition 19 is unique.

**Proposition 20:** It holds

\[
\lim_{N \to \infty} \left\| \sum_{\ell, k_{1}, \ldots, k_{\ell}} c^{k_{1} \ldots k_{\ell}} \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N - |M(k)|_{\ell}} \mathbb{P}_{N} (I^{j_{1}} \otimes a_{(k_{1})} \otimes \ldots \otimes I^{j_{\ell}} \otimes a_{(k_{\ell})}) \right\|_{N} = 0 \]

\[
\iff \sum_{\ell, k_{1}, \ldots, k_{\ell}} c^{k_{1} \ldots k_{\ell}} \frac{1}{N^{\ell-1}} \sum_{|j|_{\ell} = N - |M(k)|_{\ell}} \mathbb{P}_{N} (I^{j_{1}} \otimes a_{(k_{1})} \otimes \ldots \otimes I^{j_{\ell}} \otimes a_{(k_{\ell})}) = 0 \quad \forall N \in \mathbb{N} , \tag{33}
\]

where the sum over \( \ell, k_{1}, \ldots, k_{\ell} \in \mathbb{N} \) is finite and \( a_{k_{j}} \in B_{M(k_{j})}^{M(k_{j})} \) for all \( k_{j} \).

**Proof.** For the sake of clarity, we will discuss the proof for \( \ell \leq 2 \). This simplifies the construction without affecting the validity of the argument. We thus consider the sequence

\[
a_{N} := \sum_{k_{1}, k_{2}} c^{k_{1} k_{2}} \frac{1}{N} \sum_{|j| = N - |M(k)|_{2}} \mathbb{P}_{N} (I^{j_{1}} \otimes a_{(k_{1})} \otimes I^{j_{2}} \otimes a_{(k_{2})}) , \tag{34}
\]
where the sum over $k_1, k_2$ is finite and $a_{(k)} \in \mathcal{B}^{M(k)}_{\text{irr}}$ for all $k$. Notice that, whenever $M(k_1) = 0$ or $M(k_2) = 0$ the corresponding contribution reduces to a single $\gamma$-sequence up to a remainder $O(1/N)$. We have to prove that $\lim_{N \to \infty} \|a_N\|_N = 0$ implies $a_N = 0$ for all $N \in \mathbb{N}$.

We observe that
\[
0 = \lim_{N \to \infty} \|a_N\|_N \geq |\tau^N(a_N)| = \sum_{k_1: M(k_1)=0 \atop k_2: M(k_2)=0} e^{k_1 k_2} a_{(k_1)} a_{(k_2)},
\]
so that we may assume $(M(k_1), M(k_2)) \neq (0, 0)$ in (34).

We now analyse (34) with the help of the following parameters:
\[
\bar{M} := \max_{k_1, k_2} \{M(k_1), M(k_2)\}, \quad M_1 := \min_{k_1, k_2} \max\{M(k_1), M(k_2)\}, \quad M_2 := \min_{k_1: M(k_1) \leq M_1 \atop k_2: M(k_2) \leq M_1} \min\{M(k_1), M(k_2)\}. \quad (35)
\]

Roughly speaking $\bar{M}$ is the maximal degree of the $a_{(k)}$’s appearing in (34). The parameter $M_1 \leq \bar{M}$ is the minimal “bigger length” among all pairs $(k_1, k_2)$ appearing in (34). Notice that $M_1 > 0$ on account of the hypothesis $(M(k_1), M(k_2)) \neq (0, 0)$. Finally $M_2 \leq M_1$ is the minimal length of the $a_{(k)}$’s appearing when considering only those pairs $(k_1, k_2)$ for which $\max\{M(k_1), M(k_2)\} \leq M_1$ —notice that this implies $M(k) = M_1$ for at least one between $k \in \{k_1, k_2\}$.

Let $\omega_{M_1} \in S(B^M)$, $\omega_{M_2} \in S(B^{M_2})$ and let $\omega_{M_1, M_2, N} \in S(B^N)$ be defined by
\[
\omega_{M_1, M_2, N} := \tau^M \otimes \omega_{M_1} \otimes \tau^{M_2} \otimes \omega_{M_2}, \quad (36)
\]
where $N = r + (2\bar{M} + M_1 + M_2)q$, $q \in \mathbb{N}$, $r \in \{0, \ldots, 2\bar{M} + M_1 + M_2 - 1\}$. We consider
\[
\omega_{M_1, M_2, N}(a_N) = \sum_{k_1, k_2} e^{k_1 k_2} \frac{1}{N} \sum_{[j] = N - |M(k)|} \omega_{M_1, M_2, N} \left[ \gamma_N(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}) \right] \\
= \sum_{k_1: M(k_1) \leq M_1 \atop k_2: M(k_2) \leq M_1} \frac{1}{N} \sum_{[j] = N - |M(k)|} \omega_{M_1, M_2, N} \left[ \gamma_N(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}) \right],
\]
where in the second line we observed that
\[
\omega_{M_1, M_2, N} \left[ \gamma_N(I^{j_1} \otimes a_{(k_1)} \otimes I^{j_2} \otimes a_{(k_2)}) \right] = 0 \quad \text{if} \quad \min\{M(k_1), M(k_2)\} > M_1;
\]
This follows from the fact that, if $M_1 < M_2 \leq \bar{M}$, for all $a_{M_2} \in \mathcal{B}^{M_2}_{\text{irr}}$ we have
\[
\omega_{M_1, M_2, N} \left( \gamma_N(a_{N-M_2} \otimes a_{M_2}) \right) = 0,
\]
27
no matter the choice of $a_{N-M_2} \in B^{N-M_2}$. In fact, for all $j \in \{0, \ldots, N-1\}$ one finds that
\[
\omega_{M_1, M_2, N} \left[ \gamma_N^j(a_{N-M_2} \otimes a_{M_2}) \right] = \tau^\mu \otimes (\tau^\mu \otimes \omega_{M_1} \otimes \tau^\mu \otimes \omega_{M_2})^\eta \left[ \gamma_N^j(a_{N-M_2} \otimes a_{M_2}) \right],
\]
is non vanishing only if the position of $a_{M_2}$ “overlaps completely” with either $\omega_{M_1}$ or with $\omega_{M_2}$, however, this is not possible because $M_2 > M_1 \geq M_2$.

Notice that overlapping with both states is impossible since each pair $\omega_{M_1}, \omega_{M_2}$ is separated by $\tau^\mu$ and $M_2 \leq M$. We now analyse the remaining contributions (37) of $\omega_{M_1, M_2, N}(a_N)$. Notice that the condition $M(k_1) \leq M_1$ and $M(k_2) \leq M_1$ implies $M(k) = M_1$ for at least one between $k_1, k_2$. In fact, we also have $M(k_1), M(k_2) \geq M_2$ which implies $M(k_2) = M_2$ or $M(k_1) = M_2$ for at least one pair $(k_1, k_2)$. Moreover, by direct inspection:

(a) If $j_2 = \overline{M} \mod 2M + M_1 + M_2$ then
\[
\omega_{M_1, M_2, N} \left[ \gamma_N^j(I^1 \otimes a_{(k_1)} \otimes I^2 \otimes a_{(k_2)}) \right] = \frac{1}{2M + M_1 + M_2} \left[ \omega_{M_1}(I^{M_1-M(k_1)} \otimes a_{(k_1)}) \omega_{M_2}(I^{M_2-M(k_2)} \otimes a_{(k_2)}) \right.
\]
\[
+ \omega_{M_1}(I^{M_2-M(k_1)} \otimes a_{(k_1)}) \omega_{M_2}(I^{M_1-M(k_2)} \otimes a_{(k_2)}) \left] + O(1/N), \quad (38)
\]
with the convention that the contribution vanishes if, say, $M_2 < M(k_1)$ —this may happen if $M_2 < M_1$ and $M(k_1) = M_1$. This restricts the non-vanishing contributions to those pairs $(k_1, k_2)$ such that $\{M(k_1), M(k_2)\} = \{M_1, M_2\}$. Notice that there exists at least one such pair on account of the definition of $M_2$ —cf. Equation (37).

To prove (38) it suffices to observe that for all $\ell \in \{0, \ldots, N-1\}$ we have
\[
\omega_{M_1, M_2, N} \left[ \gamma_N^\ell(I^1 \otimes a_{(k_1)} \otimes I^2 \otimes a_{(k_2)}) \right] = \begin{cases} 
\omega_{M_1}(I^{M_1-M(k_1)} \otimes a_{(k_1)}) \omega_{M_2}(I^{M_2-M(k_2)} \otimes a_{(k_2)}) & \text{if } \ell = 0 \mod M_1 + M_2 + 2M \\
\omega_{M_2}(I^{M_2-M(k_1)} \otimes a_{(k_1)}) \omega_{M_2}(I^{M_2-M(k_2)} \otimes a_{(k_2)}) & \text{if } \ell = M_1 + \overline{M} \mod M_1 + M_2 + 2M \\
0 & \text{otherwise}
\end{cases}
\]
Since the number of $\ell \in \{0, \ldots, N-1\}$ such that $\ell = 0 \mod 2M + M_1 + M_2$ (resp. $\ell = M_1 + \overline{M} \mod 2M + M_1 + M_2$) is roughly $N/(2M + M_1 + M_2) + O(1)$ the formula for $\omega_{M_1, M_2, N} \left[ \gamma_N(I^1 \otimes a_{(k_1)} \otimes I^2 \otimes a_{(k_2)}) \right]$ follows.
(b) Similarly, if \( j_2 = 2M + M_1 \mod 2M + M_1 + M_2 \) then

\[
\omega_{M_1, M_2, N}(I_{j_1} \otimes a_{(k_1)} \otimes I_{j_2} \otimes a_{(k_2)})
= \frac{1}{2M + M_1 + M_2} \left[ \omega_{M_1}(I_{M_1 - M(k_1)} \otimes a_{(k_1)}) \omega_{M_2}(I_{M_2 - M(k_2)} \otimes a_{(k_2)}) \\
+ \omega_{M_2}(I_{M_2 - M(k_1)} \otimes a_{(k_1)}) \omega_{M_1}(I_{M_1 - M(k_2)} \otimes a_{(k_2)}) \right] + O(1/\mathcal{N}),
\]

where again the contribution is non-vanishing if and only if \( \{M(k_1), M(k_2)\} = \{M_1, M_2\} \).

(c) In all other cases the contribution vanishes.

The number of \( j_2 \in \mathbb{N} \) such that \( j_2 \leq N - |M(k)|_2 \) and \( j_2 = M_1 \mod 2M + M_1 + M_2 \) (resp. \( j_2 = 2M + M_1 \mod 2M + M_1 + M_2 \)) is roughly \( N/(2M + M_1 + M_2) + O(1) \). Moreover, we have

\[
\omega_{M_1}(I_{M_1 - M(k)} \otimes a_{(k)}) = (\omega_{M_1 - M_2} \otimes \omega_{M_2})(I_{M_1 - M(k)} \otimes a_{(k)}),
\]

where \( \omega_{M_1 - M_2} \in \mathcal{S}(B_{M_1 - M_2}) \) is arbitrarily chosen.

Thus, combining cases (a)-(b) we find

\[
\omega_{M_1, M_2, N}(a_N)
= \sum_{(k_1, k_2): \{M(k_1), M(k_2)\} = \{M_1, M_2\}} c_{k_1 k_2}^{k_1 k_2} \frac{1}{(2M + M_1 + M_2)^2} \left[ \frac{1}{2} \omega_{M_1} + \frac{1}{2} \omega_{M_2 - M_1} \otimes \omega_{M_2} \right]^2 \left[ I_{M_1 - M(k_1)} \otimes a_{(k_1)} \otimes I_{M_2 - M(k_2)} \otimes a_{(k_2)} \right] + O(1/\mathcal{N}).
\]

Since \( \|a_N\|_N \geq |\omega_{M_1, M_2, N}(a_N)| \) and \( \|a\|_N \to 0 \) we find

\[
\sum_{(k_1, k_2): \{M(k_1), M(k_2)\} = \{M_1, M_2\}} c_{k_1 k_2}^{k_1 k_2} \left[ \frac{1}{2} \omega_{M_1} + \frac{1}{2} \omega_{M_2 - M_1} \otimes \omega_{M_2} \right]^2 \left[ I_{M_1 - M(k_1)} \otimes a_{(k_1)} \otimes I_{M_2 - M(k_2)} \otimes a_{(k_2)} \right] = 0,
\]

(39)

for all \( \omega_{M_1} \in \mathcal{S}(B_{M_1}) \), \( \omega_{M_2} \in \mathcal{S}(B_{M_2}) \) and \( \omega_{M_2 - M_1} \in \mathcal{S}(B_{M_2 - M_1}) \). Choosing

\[
\omega_{M_1} = \omega_{M_1 - M_2} \otimes \omega_{M_2},
\]

we have

\[
\sum_{(k_1, k_2): \{M(k_1), M(k_2)\} = \{M_1, M_2\}} c_{k_1 k_2}^{k_1 k_2} (\omega_{M_1 - M_2} \otimes \omega_{M_2})^2 \left[ I_{M_1 - M(k_1)} \otimes a_{(k_1)} \otimes I_{M_2 - M(k_2)} \otimes a_{(k_2)} \right] = 0,
\]

(40)

29
for all \( \omega_{M_i} \in S(B_{M_i}') \) and \( \omega_{M_i-M_j} \in S(B_{M_i-M_j}') \). This implies that in the general case, unfolding \( (\omega_{M_i} + \omega_{M_i-M_j} \otimes \omega_{M_j})^2 \) and using Equation (40), we have

\[
\sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_j}) \left[ I_{M_i-M(k_1)} \otimes a_{(k_1)} \otimes I_{M_i-M(k_2)} \otimes a_{(k_2)} \right] 
+ \sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_i-M_j} \otimes \omega_{M_j}) \left[ (I_{M_i-M(k_1)} \otimes a_{(k_1)}) \otimes (I_{M_i-M(k_2)} \otimes a_{(k_2)}) \right] = 0,
\]

for all \( \omega_{M_i} \in S(B_{M_i}') \), \( \omega_{M_j} \in S(B_{M_j}') \) and \( \omega_{M_i-M_j} \in S(B_{M_i-M_j}') \) while \( a \otimes a' := a \otimes a' + a' \otimes a \). Equation (41) is now linear in \( \omega_{M_i-M_j} \otimes \omega_{M_j} \). Since convex combinations of states in \( S(B_{M_i-M_j}') \otimes S(B_{M_j}') \) generate \( S(B_{M_i}') \) we find that

\[
\sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_j}) \left[ I_{M_i-M(k_1)} \otimes a_{(k_1)} \otimes I_{M_i-M(k_2)} \otimes a_{(k_2)} \right] 
+ \sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_i-M_j}) \left[ (I_{M_i-M(k_1)} \otimes a_{(k_1)}) \otimes (I_{M_i-M(k_2)} \otimes a_{(k_2)}) \right] = 0,
\]

for all \( \omega_{M_i}, \omega_{M_i}' \in S(B_{M_i}') \). Choosing \( \omega_{M_i} = \omega_{M_i}' \), we find

\[
\sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_i}') \left[ I_{M_i-M(k_1)} \otimes a_{(k_1)} \otimes I_{M_i-M(k_2)} \otimes a_{(k_2)} \right] = 0,
\]

which implies that in the general case, unfolding \( (\omega_{M_i} + \omega_{M_i-M_j} \otimes \omega_{M_j})^2 \) and using Equation (40), we have

\[
\sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_i}) \left[ I_{M_i-M(k_1)} \otimes a_{(k_1)} \otimes I_{M_i-M(k_2)} \otimes a_{(k_2)} \right] 
+ \sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (\omega_{M_i} \otimes \omega_{M_i-M_j} \otimes \omega_{M_j}) \left[ (I_{M_i-M(k_1)} \otimes a_{(k_1)}) \otimes (I_{M_i-M(k_2)} \otimes a_{(k_2)}) \right] = 0,
\]

for all \( \omega_{M_i}, \omega_{M_i}' \in S(B_{M_i}') \).

The arbitrariness of \( \omega_{M_i}, \omega_{M_i}' \in S(B_{M_i}') \) and the fact that any \( \omega_{2M_i} \in S(B_{2M_i}) \) can be written as a convex combination of product states in \( \omega_{M_i} \otimes \omega_{M_i}' \) lead to

\[
\sum_{\{M(k_1),M(k_2)\} = \{M_i,M_j\}} c^{k_1k_2} (I_{M_i-M(k_1)} \otimes a_{(k_1)}) \otimes (I_{M_i-M(k_2)} \otimes a_{(k_2)}) = 0.
\]
On account of the symmetry in \( k_1, k_2 \) in the sum, we may assume that \( M(k_1) = \overline{M}_1 \) and \( M(k_2) = \overline{M}_2 \) for all pairs \((k_1, k_2)\). Equation (46) reduces to
\[
\sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} a(k_1) \otimes \pi (\mathcal{I}^{M_1 - \overline{M}_2} \otimes a(k_2)) = 0. \tag{45}
\]

Let \( \omega_{\overline{M}_1} \in S(B^{\overline{M}_1}) \) and \( \omega_{\overline{M}_2} \in S(B^{\overline{M}_2}) \) and let
\[
\omega_{M_1, \overline{M}_2} := \frac{1}{2} \omega_{M_1} \otimes \tau^{M_1 - \overline{M}_2} \otimes \omega_{\overline{M}_2} \quad \text{and} \quad \omega_{\overline{M}_1, M_2} := \frac{1}{2} \tau^{M_1 - \overline{M}_2} \otimes \omega_{M_1} \otimes \omega_{\overline{M}_2}.
\]

Then Equation (45) leads to
\[
0 = \omega_{M_1, \overline{M}_2} \left( \sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} a(k_1) \otimes \pi (\mathcal{I}^{M_1 - \overline{M}_2} \otimes a(k_2)) \right)
= \left( \omega_{M_1} \otimes \omega_{\overline{M}_2} \right) \left( \sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} a(k_1) \otimes \pi a(k_2) \right),
\]
with the convention that \( \omega_{M} a(k) = 0 \) if \( M(k) \neq M \). This shows that Equation (45) implies
\[
\sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} a(k_1) \otimes \pi a(k_2) = 0. \tag{46}
\]

We now observe that Equation (46) is equivalent to
\[
\sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} \frac{1}{N} \sum_{j_1 + j_2 = N - M_1 - M_2} \tau_N (I^{j_1} \otimes a(k_1) \otimes I^{j_2} \otimes a(k_2)) = 0, \quad \forall N \geq M_1 - M_2. \tag{47}
\]

Indeed, by direct inspection Equation (46) implies that
\[
\sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} I^{j_1} \otimes a(k_1) \otimes I^{j_2} \otimes a(k_2) = 0, \quad \forall j_1, j_2 \in \mathbb{N}. \tag{48}
\]

and thus it implies Equation (47). Conversely, if Equation (47) holds true then evaluation on the state \( \tau^{f_1} \otimes \omega_{\overline{M}_1} \otimes \tau^{f_2} \otimes \omega_{\overline{M}_2} \) leads to
\[
0 = (\tau^{f_1} \otimes \omega_{\overline{M}_1} \otimes \tau^{f_2} \otimes \omega_{\overline{M}_2}) \left( \sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} \frac{1}{N} \sum_{j_1 + j_2 = N - M_1 - M_2} \tau_N (I^{j_1} \otimes a(k_1) \otimes I^{j_2} \otimes a(k_2)) \right)
= \frac{1}{N} (\omega_{\overline{M}_1} \otimes \omega_{\overline{M}_2}) \left( \sum_{k_1: M(k_1) = \overline{M}_1, \; k_2: M(k_2) = \overline{M}_2} c_{k_1 k_2} \frac{1}{N} a(k_1) \otimes \pi a(k_2) \right),
\]

31
where \( \ell_1, \ell_2 \) are such that \( \ell_1 + \ell_2 = N - M_1 - M_2 \) while \( \omega_{M_1} \in S(B^{M_1}) \) and \( \omega_{M_2} \in S(B^{M_2}) \) are arbitrary states. This implies Equation (45).

By comparison with (34) we conclude that Equation (47) is nothing but the sum of the terms in \( a_N \) whose pairs \( k_1, k_2 \) fulfills \( \{ M(k_1), M(k_2) \} = \{ M_1, M_2 \} \).

At this stage we may either argue that this is in contradiction with the definition of \( M_1, M_2 \) —unless \( a_N = 0 \) — because \( \min_{k_1: M(k_1) \leq M_1, k_2: M(k_2) \leq M_2} \min \{ M(k_1), M(k_2) \} > M_2 \). Alternatively we may consider the remaining contribution to \( a_N \) and argue again as above identifying new values \( M_1, M_2, M \). In either case we have \( a_N = 0 \) for all \( N \in \mathbb{N} \) as claimed.

\[ \text{Remark 21: The notion of canonical representative applies also for symmetric sequences. Indeed, let } [\pi_N^M a_M]_N \in [B]^\infty \text{ where } a_M \in B^M. \text{ As for } \gamma \text{-sequences one has } [\pi_N^M a_M]_N = [\pi_N^{M+K}(I^K \otimes a_M)]_N \text{ so that the } \pi \text{-sequence generating } [\pi_N^M a_M]_N \text{ is not uniquely determined. Nevertheless, since } a_M \in B^M, \text{ one obtain the following unique decomposition:} \]

\[ a_M = S_M(\tilde{a}_0 I^M + I^{M-1} \otimes \tilde{a}_1 + \ldots + \tilde{a}_M) = S_M \left( \sum_{j=0}^M I^{M-j} \otimes \tilde{a}_j \right) \in S_M \left( \bigoplus_{j=0}^M I^{M-j} \otimes \tilde{B}_\pi^j \right). \]

With this decomposition at hand the canonical representative of \( [\pi_N^M a_M]_N \) is defined by

\[ \left( \sum_{j=0}^M \pi_N^j \tilde{a}_j \right)_N. \]

This point of view is equivalent to the one adopted in [12].

\[ \diamond \]

### 3.3 The Poisson structure of \([B]^\infty_\gamma\)

In this section we will endow \([B]^\infty_\gamma\) with a Poisson structure defined on \([\dot{B}]^\infty_\gamma\). Eventually we will discuss the deformation quantization of \([B]^\infty_\gamma\).

We recall that a Poisson structure over a C*-algebra \( A \) is given by a bilinear map \( \{ \ , \ \} : A_0 \times A_0 \to A_0 \) defined on a dense *-subalgebra \( A_0 \subset A \) which fulfills:

\[ \{ a, a' \} = -\{ a', a \}, \quad \{ a, a' \}^* = \{ a^*, a'^* \} \]

\[ \{ a, a^* a'' \} = \{ a, a' \} a'' + a' \{ a, a'' \}, \]

\[ \{ a, \{ a', a'' \} \} = \{ a, a' \} a'' + \{ a', \{ a, a'' \} \}, \]

for all \( a, a', a'' \in A_0 \).

### Proposition 22:

Let \( \{ \ , \ \}_\gamma : [\dot{B}]^\infty_\gamma \times [\dot{B}]^\infty_\gamma \to [\dot{B}]^\infty_\gamma \) be the bilinear map defined by

\[ \{ [a_N]_N, [a'_N]_N \}_\gamma := [i N [a^N_C A_N, a'^N_C A_N]]_N, \]

where \( [a^N_C A_N]_N \) denotes the canonical representative of \( [a_N]_N \) —cf. Definitions 16-19. Then \( \{ \ , \ \}_\gamma \) is a Poisson structure on \([B]^\infty_\gamma\).

\[ \diamond \]

32
Proof. Notice that \( \{ \cdot \},_\gamma \) fulfills condition (19) because so does the pointwise commutator \( i[\cdot,\cdot] \).

The non-trivial part of the proof is to prove that \( \{ \cdot \},_\gamma \) is well-defined, namely that \( \{a_N|N, a'_N|N\}_\gamma \) is a well-defined element of \( [\hat{B}]_\gamma^\infty \). Moreover, we also have to prove conditions (50) - (51): The latter do not follow from the properties of the commutator because Equation (52) uses the canonical representative and in general \( [a_N^{\text{can}}, a_N^{\text{can}}] \neq [a_N^{\text{can}}, a_N^{\text{can}}]^{\text{can}}. \) For these reasons we proceed in several steps:

As a first step, we prove that \( \{a_N|N, a'_N|N\}_\gamma \in [\hat{B}]_\gamma^\infty \) for the case of two equivalence classes of \( \gamma \)-sequences. Since the commutator is linear, on account of Remark 11 we may reduce to the case \( a_N|N = [\gamma_N^M a_M]|N, a'_N|N = [\gamma_N^{M'} a_{M'}]|N \) for \( a_M \in B_{\text{irr}}^M \) and \( a_{M'} \in B_{\text{irr}}^{M'} \).

In this latter case we find, for large enough \( N \), say \( N \geq 2(M + M') \),

\[
i_N[\gamma_N^M a_M, \gamma_N^{M'} a_{M'}] = i_N[\gamma_N^M a_M, \gamma_N^{M'} a_{M'}] - i_N[\gamma_N^M a_M, \gamma_N^{M} a_{M'}] + i_N[\gamma_N^{M'} a_{M'}, \gamma_N^{M} a_{M'}]
\]

which implies

\[
\{[\gamma_N^M a_M]|N, [\gamma_N^{M'} a_{M'}]|N\}_\gamma = \{[\gamma_N^M a_M, \gamma_N^{M'} a_{M'}]|N = [\gamma_N^{M+M'} a_M a_{M'}]|N \in [\hat{B}]_\gamma^\infty .
\]

This proves that the Poisson bracket between \( [\gamma_N^M a_M]|N \) and \( [\gamma_N^{M'} a_{M'}]|N \) is an element of \( [\hat{B}]_\gamma^\infty \).
At this stage we observe that

$$[iN\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]_N \in [\hat{B}]^\infty_N.$$  

Indeed, this is due to the identity

$$[a_N, a'_N a''_N] = [a_N, a'_N a''_N + a'_N [a_N, a''_N]],$$

together with the fact that the result holds true for $\ell = \ell' = 1$. Moreover, we have

$$||N[R_N, \gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]||_N = O(1/N)$$

where the latter estimate is due to Remark 7-ii. Overall we have shown that

$$iN[a_{\text{CAN}}^N, a_{\text{CAN}}^N] = iN[\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}] + R_{\text{CN}},$$

where $||R_{\text{CN}}||_N = O(1/N)$ and by direct inspection fulfills Equation (20). This implies in particular that

$$[iN[a_{\text{CAN}}^N, a_{\text{CAN}}^N]]_N = [iN[\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]]_N \in [\hat{B}]^\infty_N,$$

so that $\{,\}$ is well-defined.

By proceeding in a completely analogous way one also proves conditions (50)- (51).

Indeed, considering without loss of generality

$$[a_N]_N = [\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}]_N, \quad [a'_N]_N = [\gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]_N, \quad [a''_N]_N = [\gamma_i N a_{M_1''} \cdots \gamma_i N a_{M_{\ell''}}]_N,$$

we find

$$(iN)^2[a_{\text{CAN}}^N, a_{\text{CAN}}^N] = iN[a_{\text{CAN}}^N, iN[\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]] + iN[a_{\text{CAN}}^N, R_N]$$

$$= iN[\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, iN[\gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}, \gamma_i N a_{M_1''} \cdots \gamma_i N a_{M_{\ell''}}]]$$

$$+ iN[R'_N, iN[\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]] + iN[a_{\text{CAN}}^N, R_N].$$

The first contribution fulfills (51). With an argument similar in spirit to Remark 7-ii the second contribution can be estimated by

$$||[R'_N, iN[\gamma_i N a_{M_1} \cdots \gamma_i N a_{M_\ell}, \gamma_i N a_{M_1'} \cdots \gamma_i N a_{M_{\ell'}}]]||_N = O(1/N^2).$$

34
Finally \( \|N[a_{N}^{\text{CAN}}, R_{N}]\|_{N} = O(1/N) \) because of Remark 13 so that

\[
(i N)^{2}[a_{N}^{\text{CAN}}, [a_{N}^{\text{CAN}}, a_{N}^{\text{rCAN}}]] = i N[M_{N}^{a_{M_{1}}} \cdots M_{N}^{a_{M_{r}}}, i N[M_{N}^{M_{r}^{0}}, \cdots M_{N}^{M_{r}^{0}}, a_{M_{r}^{0}}] + R_{N}^{m},
\]

with \( \|R_{N}^{m}\|_{N} = O(1/N) \). This proves condition (51) for \( \{ , \gamma \} \).

By proceeding in a similar fashion we also have

\[
i N[a_{N}^{\text{rCAN}}, a_{N}^{\text{CAN}}, a_{N}^{\text{rCAN}}] = i N[M_{N}^{a_{M_{1}}} \cdots M_{N}^{a_{M_{r}}}, i N[M_{N}^{M_{r}^{0}}, \cdots M_{N}^{M_{r}^{0}}, a_{M_{r}^{0}}] + R_{N}^{m},
\]

where \( \|R_{N}^{m}\|_{N} = O(1/N) \) while the first contribution fulfils (50). This proves condition (50) for \( \{ , \gamma \} \).

\[\square\]

**Remark 23:** The proof of Proposition 22 shows that, if \( a_{M} \in B_{irr}^{M} \) and \( a_{M'} \in B_{irr}^{M'} \) then

\[\{ [M_{N}^{a_{M_{1}}} \cdots M_{N}^{a_{M_{r}}}, [M_{N}^{M_{r}^{0}}, \cdots M_{N}^{M_{r}^{0}}, a_{M_{r}^{0}}] \} \gamma = [M_{N}^{M+2M'} a_{M+2M'}],\]

where \( a_{M+2M'} \in B_{irr}^{M+2M'} \) is not \( \tilde{B} \)-irreducible in general.

At last, we can finally state and prove the main theorem of this paper.

**Theorem 24:** Let \( [B]_{\gamma} \subset \prod_{N \in \mathbb{N}}[B]_{\gamma}^{N} \) be the continuous bundle of \( C^{*} \)-algebras defined as per Proposition 12. For \( K \in \mathbb{N} \) let \( Q_{K} : [B]_{\gamma}^{\infty} \to [B]_{\gamma}^{K} \) be the linear map defined by

\[
Q_{K}([a_{N}]_{N}) := \begin{cases} a_{K}^{\text{CAN}} & K \in \mathbb{N} \\ [a_{N}]_{N} & K = \infty \end{cases}
\]

(53)

where \( (a_{K}^{\text{CAN}})_{N} \) is the canonical representative of \( [a_{N}]_{N} \) as per Definitions 16-19. Then the family of maps \( \{Q_{N}\}_{N \in \mathbb{N}} \) defines a strict deformation quantization of \( [B]_{\gamma}^{\infty} \).

\[\square\]

**Proof.** Notice that \( Q_{N} \) is well-defined for all \( N \in \mathbb{N} \) on account of the uniqueness of the canonical representative — cf. Propositions 18,20.

With reference to Section 11 we have

\[
[B]_{\gamma} \leftrightarrow \coprod_{N \in \mathbb{N}} A_{N}, \quad [B]_{\gamma}^{N} \leftrightarrow A_{N}, \quad [\hat{B}]_{\gamma}^{\infty} \leftrightarrow \hat{A}_{\infty}, \quad [B]_{\gamma}^{\infty} \leftrightarrow A_{\infty}.
\]

We will now prove conditions 3a,3b,3c.

\[\square\] Per definition we have \( Q_{\infty} := \text{Id}_{[B]_{\gamma}^{\infty}} \) as well as \( Q_{N}([a_{N}]_{N})^{*} = Q_{N}([a_{N}^{*}]_{N}) \) for all \( [a_{N}]_{N} \in [B]_{\gamma}^{\infty} \). Moreover, Equation (53) defines an element in the space \( [\hat{B}]_{\gamma} \) — cf. Equation (24) — and thus a continuous section of \( [B]_{\gamma} \) as per Proposition 12.

35
By direct inspection one has
\[ [iK(Q_K([a_N]_N), Q_K([a'_N]_N))]_K = [iK[a^CAN_K, a'^CAN_K]]_K = \{[a_N]_N, [a'_N]_N\}_\gamma, \]
which implies Equation (1). Notice that, on account of Remark 23 in general
\[ Q_K([a_N]_N, [a'_N]_\gamma) \neq iK[a^CAN_K, a'^CAN_K], \]
despite the fact that the equivalence classes of the associated sequences are equal.

By direct inspection one finds that \( Q_M([\tilde{B}]_\gamma) = B^M_\gamma \) for all \( M \in \mathbb{N} \). Indeed, by proceeding as in the proof of Proposition 12, let \( a_M \in B^M_\gamma \), for \( M \in \mathbb{N} \). Then we have \( a_M = \gamma_M(a_M) = \sum_{j=0}^M \gamma^j_M a_j \) for \( a_j \in B^j_{\text{irr}} \) — cf. Equation (29). This implies that
\[ a_M = \gamma_M(a_M) = \sum_{j=0}^M \gamma^j_M a_j = Q_M\left(\left[\sum_{j=0}^M \gamma^j_M a_j\right]_N\right), \]
thus proving that \( Q_M([\tilde{B}]_\gamma) = B^M_\gamma \).

\( \square \)

### A Characterization of \( \tilde{B} \)-irreducible elements

This section is devoted to characterize the set \( B^M_{\text{irr}} \) of \( \tilde{B} \)-irreducible elements in \( B^M \) — cf. Definition 14. To this avail, we introduce the following convenient family of linear maps.

**Definition 25:** Let \( M, \ell \in \mathbb{N}, \ell \geq 2, \) and \( j_1, \ldots, j_{\ell-1} \in \mathbb{N} \) such that \( j_1 + \ldots + j_{\ell-1} = M - \ell \). We denote by \( \iota^{j_1 \ldots j_{\ell-1}}_M : \tilde{B}^\ell \to B^M \) the linear map defined by
\[ \iota^{j_1 \ldots j_{\ell-1}}_M(\tilde{a}_\ell) := \sum_{k_1, \ldots, k_\ell} c^{k_1 \ldots k_\ell} b_{k_1} \otimes I^{j_1} \otimes \ldots \otimes I^{j_{\ell-1}} \otimes b_{k_\ell}, \] (54)
where \( I, b_1, \ldots, b_{n-1} \) is a basis of \( B \) fulfilling (4) and \( \tilde{a}_\ell = \sum_{k_1, \ldots, k_\ell} c^{k_1 \ldots k_\ell} b_{k_1} \otimes \ldots \otimes b_{k_\ell} \), the sum over \( k_1, \ldots, k_\ell \) being finite.

**Remark 26:**

(i) If \( \ell = M \) one has \( \iota^{j_1 \ldots j_{\ell-1}}_M(\tilde{a}_M) = \tilde{a}_M \). Moreover, by direct inspection \( \iota^{j_1 \ldots j_{\ell-1}}_M \) does not depend on the chosen basis \( I, b_1, \ldots, b_{n-1} \).

(ii) On account of Definition 14 we have \( \iota^{j_1 \ldots j_{\ell-1}}_M(\tilde{B}^\ell) \subseteq B^M_{\text{irr}} \). Moreover, \( \iota^{j_1 \ldots j_{\ell-1}}_M \) is injective. Indeed, if \( \iota^{j_1 \ldots j_{\ell-1}}_M(\tilde{a}_\ell) = 0 \) then for all \( \eta_1, \ldots, \eta_\ell \in S(B) \) we have
\[ 0 = [\eta_1 \otimes \tau^{j_1} \otimes \ldots \otimes \tau^{j_{\ell-1}} \otimes \eta_\ell](\iota^{j_1 \ldots j_{\ell-1}}_M(\tilde{a}_\ell)) = (\eta_1 \otimes \ldots \otimes \eta_\ell)(\tilde{a}_\ell). \]
The arbitrariness of \( \eta_1, \ldots, \eta_\ell \) entails \( \omega_\ell(\tilde{a}_\ell) = 0 \) for all \( \omega_\ell \in S(B^\ell) \), therefore, \( \tilde{a}_\ell = 0 \).

36
Let \( I, b_1, \ldots, b_{n-1} \) be a basis of \( B \) fulfilling (44) and let \( a_M \in B^{M}_{\text{irr}}, M \geq 2 \). By considering Equation (28) for \( a_M \in B^{M}_{\text{irr}} \) we find

\[
a_M = \varepsilon^{M-2}(\tilde{a}_2) + \sum_{j_1+j_2=M-3} \varepsilon^{j_1j_2}(\tilde{a}_{3|j_1j_2}) + \ldots + \varepsilon^{j_1\ldots j_{\ell-1}}(\tilde{a}_{\ell|j_1\ldots j_{\ell-1}}) + \ldots + \tilde{a}_M, \quad (55)
\]

where \( \tilde{a}_{\ell|j_1\ldots j_{\ell-1}} \in \tilde{B}^{\ell} \) for all \( \ell, j_1, \ldots, j_{\ell-1} \).

Equation (55) provides a description of \( B^{M}_{\text{irr}} \) in terms of “\( \tilde{B} \)-components”. To this avail we consider the vector space

\[
\tilde{B}^M := \tilde{B}^2 \oplus \bigotimes_{j_1+j_2=M-3} \tilde{B}^3 \oplus \ldots \oplus \bigotimes_{j_1+\ldots+j_{\ell-1}=M-\ell} \tilde{B}^{\ell} \oplus \ldots \oplus \tilde{B}^M, \quad (56)
\]

where \( \tilde{B}^0 = \mathbb{C} \) and \( \tilde{B}^1 = \tilde{B} \). We then define the linear map

\[
\Phi_M : \tilde{B}^M \to B^M \quad \Phi_M(\tilde{a}_M) := \begin{cases} a_0 & M = 0 \\ \tilde{a}_1 & M = 1 \\ \varepsilon^{M-2}(\tilde{a}_2) + \ldots + \varepsilon^{j_1\ldots j_{\ell-1}}(\tilde{a}_{\ell|j_1\ldots j_{\ell-1}}) + \ldots + \tilde{a}_M & M \geq 2 \end{cases}, \quad (57)
\]

where \( \tilde{a}_M \in \tilde{B}^M \) is given by

\[
\tilde{a}_M = \begin{cases} a_0 & M = 0 \\ \tilde{a}_1 & M = 1 \\ \tilde{a}_2 \oplus \bigotimes_{j_1+j_2=M-3} \tilde{a}_{3|j_1j_2} \oplus \ldots \oplus \bigotimes_{j_1+\ldots+j_{\ell-1}=M-\ell} \tilde{a}_{\ell|j_1\ldots j_{\ell-1}} \oplus \ldots \oplus \tilde{a}_M & M \geq 2 \end{cases}. \quad (58)
\]

Equation (55) can be rephrased by saying that for all \( a_M \in B^{M}_{\text{irr}} \) there exists \( \tilde{a}_M \in \tilde{B}^M \) such that \( \Phi_M(\tilde{a}_M) = a_M \). The following lemma shows that \( \Phi_M \) is in fact an isomorphism, proving that \( B^{M}_{\text{irr}} \cong \tilde{B}^M \).

**Lemma 27:** For all \( M \in \mathbb{N} \), the map \( \Phi_M : \tilde{B}^M \to B^M_{\text{irr}} \) is an isomorphism. \( \diamond \)

**Proof.** There is nothing to prove for \( M \in \{0, 1\} \), therefore, we assume \( M \geq 2 \).

From equations (55), (57), we have that \( \Phi_M \) is linear and surjective: Thus, it remains to prove that \( \Phi(\tilde{a}_M) = 0 \) implies \( \tilde{a}_M = 0 \). We now prove that, if \( \Phi(\tilde{a}_M) = 0 \), then all components of \( \tilde{a}_M \) appearing in Equation (58) vanish.

37
To this avail let $\eta_1, \eta_2 \in S(B)$. By direct inspection we have

$$0 = (\eta_1 \otimes \tau^{M-2} \otimes \eta_2)[\Phi(\tilde{a}_M)] = (\eta_1 \otimes \eta_2)(\tilde{a}_2).$$

Notice that no other term from $\Phi(\tilde{a}_M)$ provides a non-vanishing contribution because $\tau(\tilde{B}) = \{0\}$. The arbitrariness of $\eta_1, \eta_2 \in S(B)$ leads to $\omega_2(\tilde{a}_2) = 0$ for all $\omega_2 \in S(B^2)$ and thus $\tilde{a}_2 = 0$.

We now proceed by proving that $\tilde{a}_3|_{j_1 j_2} = 0$ for all $j_1 + j_2 = M - 3$. To this avail let $j_1, j_2$ be such that $j_1 + j_2 = M - 3$ and let $\eta_1, \eta_2, \eta_3 \in S(B)$. Since we already proved that $\tilde{a}_2 = 0$ it follows that

$$0 = (\eta_1 \otimes \tau^{j_1} \otimes \eta_2 \otimes \tau^{j_2} \otimes \eta_3)[\Phi(\tilde{a}_M)] = (\eta_1 \otimes \eta_2 \otimes \eta_3)(\tilde{a}_{3|j_1 j_2}).$$

Once again, the arbitrariness of $\eta_1, \eta_2, \eta_3 \in S(B)$ (as well as the one of $j_1, j_2$) leads to $\tilde{a}_{3|j_1 j_2} = 0$ for all $j_1 + j_2 = M - 3$. Proceeding by induction we find $\tilde{a}_{\ell|j_1 \ldots j_{\ell-1}} = 0$ for all $j_1 + \ldots + j_{\ell-1} = M - \ell$. Thus $\tilde{a}_M = 0$.

References

[1] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D., Deformation Theory and Quantization. 1. Deformations of Symplectic Structures, Annals Phys. 111 (1978) 61

[2] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D., Deformation Theory and Quantization. 2. Physical Applications, Annals Phys. 111 61

[3] Berezin F.A., General concept of quantization, Commun. Math. Phys. 40, 153-174

[4] Blackadar B., Operator algebras - Theory of $C^*$-algebras and von Neumann algebras, Springer-Verlag Berlin Heidelberg

[5] Blackadar B., Kirchberg E., Generalized inductive limits of finite-dimensional $C^*$-algebras, Math. Ann. 307, 343-380

[6] Bratteli O., Robinson D.W., Operator algebras and quantum statistical mechanics II Springer-Verlag Berlin Heidelberg

[7] Dixmier J., $C^*$-Algebras, North-Holland, (1977).

[8] Drago, N., van de Ven, C.J.F., DLR-KMS correspondence on lattice spin systems, Letters in Mathematical Physics 113, 88

[9] Friedli S., Velenik Y., Statistical mechanics of lattice systems - A concrete mathematical introduction, Cambridge University Press

[10] Landsman K., Mathematical Topics Between Classical and Quantum Mechanics, Springer New York, NY
[11] Landsman K., Foundations of Quantum Theory: from classical concepts to operators algebras, Springer Cham (2017).
[12] Landsman K., Moretti V., van de Ven C. J. F., Strict deformation quantization of the state space of $M_k(\mathbb{C})$ with application to the Curie-Weiss model, Rev. Math. Phys. Vol. 32, No. 10, 2050031 (2020).
[13] Moretti V., van de Ven C. J. F., Bulk-boundary asymptotic equivalence of two strict deformation quantizations, Letters in Mathematical Physics 110(11), 2941-2963 (2020).
[14] Moretti V., van de Ven C. J. F., The classical limit of Schrödinger operators in the framework of Berezin quantization and spontaneous symmetry breaking as emergent phenomenon, International Journal of Geometric Methods in Modern Physics (2022).
[15] Murro S., van de Ven C. J. F. Injective tensor products in strict deformation quantization, Math Phys Anal Geom 25, 2 (2022).
[16] Raggio G. A., Werner R. F., Quantum statistical mechanics of general mean field systems, Helv. Phys. Acta 62 980 (1989).
[17] Rieffel M. A., Quantization and $C^*$-algebras, Contemp. Math. 167 67-97 (1994).
[18] van de Ven C. J. F., The classical limit of mean-field quantum spin systems, J. Math. Phys. 61, 121901 (2020).
[19] van de Ven C. J. F., The classical limit and spontaneous symmetry breaking in algebraic quantum theory, Expo. Math. 40, 3 (2022).