Removing Spurious Features can Hurt Accuracy and Affect Groups Disproportionately

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Abstract

The presence of spurious features interferes with the goal of obtaining robust models that perform well across many groups within the population. A natural remedy is to remove spurious features from the model. However, in this work we show that removal of spurious features can decrease accuracy due to the inductive biases of overparameterized models. We completely characterize how the removal of spurious features affects accuracy across different groups (more generally, test distributions) in noiseless overparameterized linear regression. In addition, we show that removal of spurious feature can decrease the accuracy even in balanced dataset—each target co-occurs equally with each spurious feature; and it can inadvertently make the model more susceptible to other spurious features. Finally, we show that robust self-training can remove spurious features without affecting the overall accuracy. Experiments on the Toxic-Comment-Detectoin and CelebA datasets show that our results hold in non-linear models.

1. Introduction

Machine learning models are vulnerable to fitting spurious features. For example, models for toxic comment detection assign different toxicity scores to the same sentence with different identity terms (“I’m gay” and “I’m straight”) (Dixon et al., 2018), and models for object recognition make different predictions on the same object against different backgrounds (Xiao et al., 2020; Ribeiro et al., 2016). A common strategy to make models robust against spurious features is to remove such features, e.g., removing identity terms from a comment (Garg et al., 2019), removing background of an image (Elhabian et al., 2008), or learning a new representation from which it is impossible to predict the spurious feature (Zemel et al., 2013; Louizos et al., 2015; Beutel et al., 2017). However, removing spurious features can lower accuracy (Zemel et al., 2013; Wang et al., 2019), and moreover, this drop varies widely across groups within the population (Yurochkin and Sun, 2020).

Previous work has identified two reasons for this accuracy drop: (i) Core (non-spurious) features are noisy or not expressive enough (Khani and Liang, 2020; Kleinberg and Mullainathan, 2019), so spurious features are needed even by the optimal model to achieve the best accuracy. (ii) Removing spurious features corrupts core features, especially when learning a new representation uncorrelated with the spurious features (Zhao and Gordon, 2019).

In this work, we show that even in the absence of the aforementioned two reasons, removing spurious features can still lead to a drop in accuracy due to the dominant effects of inductive bias in overparametrized models. For our theoretical analysis, we consider noiseless linear regression: We have \( d \) core features \( z \) which determine the prediction target \( y = \theta^\top z \) and the spurious feature \( s = \beta^\top z \). See Figure 1 for the causal graph. Importantly, (i) \( s \) adds no information about \( y \) beyond what already exists in \( z \), and (ii) removing \( s \) does not corrupt \( z \) (since we are not required to remove the correlated features to \( s \)). We consider two models: the core model \( M^{-s} \), which only uses \( z \) to predict \( y \); and the full model \( M^{+s} \), which also uses the spurious feature \( s \). In this simple setting, one might conjecture that removing the spurious feature should only help accuracy. However, in this work we show the contrary. The main question we wish to answer is: For which groups (or more generally, test distributions) does removing the spurious feature \( s \) help or hurt accuracy?

In the overparametrized regime, since number of training examples is less than the number of features, there are some
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| True Parameters | Training Example | Model estimated parameters | Model prediction |
|-----------------|------------------|---------------------------|------------------|
| $\theta^* = [2, 2]^T$ | $y = [2, 2]z$ | Full model $\hat{\theta} = [1, 0]^T$, $w = 1$ | Core model $\hat{y} = [2, 0]z$ |
| $\beta^* = [1, \alpha]^T$ | spurious $s = [1, \alpha]z$ | Core model $\hat{y} = [2, 0]z$ |
| $\alpha$ | $z$ | Full model $\hat{y} = [1, 0]z + s = [2, \alpha]z$ |
| $\beta_i$ | $s$ | $y$ | $2$ |
| $z$ | $1$ | $w$ | $1$ |

Table 1. A simple linear regression example that shows when removing a spurious feature increases the error. (a) There are 2 core features, $z = [z_1, z_2]$, one spurious feature $s = \beta^* z$ and the target is $y = \theta^* z$. (b) There is only one training example with no information about $z_2$. (c) Core model’s estimate for the parameters is $[2, 0]$; however, the full model interpolates data with a smaller norm by putting weight $w = 1$ for the spurious feature $s$. (d) For the prediction, we can replace $s$ by $\beta^* z = [1, \alpha] z$, which results in the full model implicitly assigning weight of $\alpha$ for the second feature; while the core model assigns weight of 0 to the second feature. As a result, if $|\alpha - 2| < |0 - 2|$ then removing $s$ increases the error.

directions of data variation that are not observed in the training data. For instance, Table 1 shows a simple example with only one training data and two core features ($z_1$ and $z_2$). In this example, we do not observe any information about the second feature ($z_2$). The core model assigns weight 0 to the unseen directions (weight 0 for the second feature in Table 1(c)). On the other hand, in the presence of a spurious feature, the full model can fit the training data perfectly with a smaller norm by assigning a non-zero weight for the spurious feature (weight 1 for the feature $s$ in Table 1(c)). This non-zero weight for the spurious feature leads to a different inductive bias for the full model. In particular, the full model does not assign weight 0 to the unseen directions. In Table 1(d), the full model implicitly assigns weight $\alpha$ to the second feature (unseen direction at training), while the core model assigns weight 0. As a result, if $|\alpha - 2|$ is small, then the full model which uses $s$ has lower error, and if $|\alpha - 2|$ is large then core model which does not use $s$ has lower error.

Intuitively, by using the spurious feature, the full model incorporates $\beta^*$ into its estimate. The true target parameter ($\theta^*$) and the true spurious feature parameters ($\beta^*$) agree on some of the unseen directions and do not agree on the others. Thus, depending on which unseen directions are weighted heavily in the test time, removing $s$ can lower or higher the error. We formalize the conditions under which the removal of the spurious feature ($s$) increases the error in Proposition 1. In particular, Proposition 1 states that in addition to the true parameters ($\theta^*, \beta^*$), the distribution of core-features in train and test data are also critical on the impact of removing $s$ on error. Consequently, conditions only on the true parameters (e.g., the target and spurious feature are determined by disjoint features, $\theta_i^* \beta_i^* = 0$, for all $i$), or only on the spurious feature and target (e.g., balanced dataset, $y \perp s$ in the train and test data) are not sufficient, and removing the spurious feature may aggravate the error even under these favorable conditions.

We then study multiple spurious features and show that removing one spurious feature inadvertently makes the model more susceptible to the rest of the spurious features, in line with the recent empirical results (Yin et al., 2019). Finally, we show how to leverage unlabeled data and the recently introduced Robust Self-Training (RST) (Raghu

2. Setup

Let $z \in \mathbb{R}^d$ denote the core features which determine the prediction target, $y = f(z)$. Let $s = g(z)$ denote a spurious feature. We study the overparameterized regime, where we observe $n < d$ triples $(z_i, s_i, y_i)$ as training data. For an arbitrary loss function $\ell$, the standard error for a model $M : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is:

$$\text{Error}(M) = \mathbb{E}[\ell(y, M(z, s))],$$

where the expectation is over test points $(z, s, y)$. We are also interested in robustness of a model against spurious feature. Let $S$ denote the set of different values for the
spurious feature. We define the robust error as follows:

\[
\text{RobustError}(M) = \mathbb{E}[\max_{s' \in S} \ell(y, M(z, s'))]
\]

(2)

RobustError measures the worst case error for each data point \(z\) with respect to change in \(s\). Ideally we want our model prediction to be robust with respect to change in the spurious feature (i.e., \(\text{Error}(M) = \text{RobustError}(M)\)).

Let \(C\) denote a function that measures the complexity of a model (e.g., the \(L2\)-norm of its parameters), and \(\lambda\) be a parameter to tune the trade-off between complexity of a model and its empirical risk. We are interested in the performance of the following two models:

- **Full model** (\(M^s\)): which uses the core features \(z\) and the spurious \(s\) to predict \(y\).

\[
M^s = \arg\min_M \sum_i \ell(M(z_i, s_i), y_i) + \lambda C(M)
\]

(3)

- **Core model** (\(M^\cdot\)): which only uses the core features \(z\) to predict \(y\).

\[
M^\cdot = \arg\min_M \sum_i \ell(M(z_i, 0), y_i) + \lambda C(M)
\]

(4)

The core model does not use the spurious feature, therefore its RobustError is equal to its Error. However, there might be a large gap between Error and RobustError for the full model.

3. When does the Full Model Outperform the Core Model?

For the theory section of the paper, we consider the noiseless linear regression setup. We assume there are true parameters \(\theta^*, \beta^* \in \mathbb{R}^d\) such that the prediction target, \(y = \theta^\top z\), and the spurious feature \(s = \beta^\top z\) (in Section 3.3 we study multiple spurious features and their interactions). Motivated by recent work in deep learning, which shows gradient descent converges to the minimum-norm solution that fit training data perfectly (Gunasekar et al., 2017), we consider the minimum-norm solution (i.e., the complexity function \(C\) in (3) and (4) returns the \(L2\)-norm of the parameters with regularization strength tending to 0). We consider squared-error for the loss function.

Let \(Z \in \mathbb{R}^{n \times d}\) be the matrix of observed core features, \(Y \in \mathbb{R}^n\) be the targets, and \(S \in \mathbb{R}^n\) be the spurious features.

We first analyze the minimum-norm estimate for the core model. Let \(\hat{\theta}^s\) denote the estimated parameters of the core model which can be obtained through solving the following optimization problem.

\[
\hat{\theta}^s = \arg\min_{\theta} \|\theta\|_2^2
\]

\[
s.t. \quad Z\theta = Y.
\]

(5)

Let \(\Pi = Z^\top (ZZ^\top)^{-1}Z\) be the projection matrix to columns of \(Z\) then \(\hat{\theta}^s = \Pi \theta^*\).

The full model uses the spurious feature in addition to the core features. Let \(\hat{\theta}^s\) and \(\hat{\theta}\) denote the estimated parameters for the full model which can be obtained through solving the following optimization problem.

\[
\hat{\theta}^s, \hat{\theta} = \arg\min_{\theta, w} \|\theta\|_2^2 + w^2
\]

\[
s.t. \quad Z\theta + Sw = Y,
\]

(6)

where \(w\) denote the weight associate with the spurious feature. Note that in (6) we can always set \(\hat{w} = 0\) to obtain \(\hat{\theta}^s\), so the full model only achieves smaller norm by optimizing over \(w\). In particular, instead of having norm of \(\|\Pi \theta^*\|_2\), the full model can use \(s\) with weight \(\hat{w}\) and have norm \(\|\Pi \theta^* - \hat{w} \beta^* \|_2^2 + \hat{w}^2\) instead. As a result, \(\hat{w}\) is larger if \(\theta^*\) and \(\beta^*\) are correlated in column space of the training data (\(\Pi\)). The optimum value for \(\hat{w}\) which minimizes the norm is:

\[
\hat{w} = \frac{\theta^\top \Pi \beta^*}{1 + \beta^\top \Pi \beta^*}
\]

(7)

See Appendix A for details. The estimated parameters for the full and core models are shown in Table 5 (first and second rows).

To understand the difference between inductive biases of the core model and full model and how they affect different groups, we extend the example in the introduction to contain two unseen directions, see Table 2. In this example, there are \(d = 3\) core features and only one training example with no information about the second and the third features. Let \(\theta^* = [2, 2, 2]^\top\) which results in \(y = 2\). Let \(\beta^* = [1, 2, -2]_\top\) which results in \(s = 1\). Without the spurious feature, the estimated parameter is \(\hat{\theta}^s = [2, 0, 0]^\top\) however, by using \(s\) the full model fits the training data perfectly with a smaller norm by setting \(\hat{w} = 1\). By substituting \(s\) with \(\beta^\top z\), the full model implicitly assigns weights of \(2\) and \(-2\) to the second and third features, respectively. Therefore, in this example, removing \(s\) decreases the error for the groups with high variance on the second feature, and removing \(s\) increases the error for the groups with high variance on the third feature.
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| True parameters | Training example | Model estimated parameters | Model prediction |
|-----------------|------------------|---------------------------|------------------|
| $\theta^* = [2, 2, 2]^T$ | $z$ $s$ $y$ | Core model $\theta^* = [2, 0, 0]^T$ | Core model $\hat{y} = [2, 0, 0]z$ |
| $\beta^* = [1, 2, -2]^T$ | $[1, 0, 0]^T$ 1 2 | Full Model $\hat{\theta}^* = [1, 0, 0]^T, w = 1$ | Full model $\hat{y} = [1, 0, 0]z + s = [2, 2, -2]z$ |

Table 2. There is only one training example with no information about $z_2$ and $z_3$. The core model assigns weight of 0 to these two features. The full model, uses the spurious feature $(s)$, which results in a good estimation for real weight of $z_2$ but not $z_3$.

The following proposition, which is our main result, provides general conditions that characterize precisely when the core model (removing the spurious feature) increases the error over the full model.

**Proposition 1.** Let $\Sigma = \mathbb{E}[zz^T]$ denote the covariance matrix of the test data (or covariance matrix for any group), and let $\Pi$ denote the column space of training data. $\text{Error}(M^{**}) < \text{Error}(M^*)$ if:

$$\text{sign}(\beta^T \Pi \theta^*) = \text{sign}(\beta^T (I - \Pi) \Sigma (I - \Pi) \theta^*),$$

and

$$\left| \frac{\beta^T \Pi \theta^*}{1 + \beta^T \Pi \beta^*} \right| < \frac{2 \beta^T (I - \Pi) \Sigma (I - \Pi) \theta^*}{\beta^* (I - \Pi) \Sigma (I - \Pi) \beta^*}.$$  

Intuitively, this proposition states that removing the spurious feature increases the error if the correlation between $\beta^*$ and $\theta^*$ in column space of training data (seen directions) is similar to the correlation of $\beta^*$ and $\theta^*$ in the null-space of training data (unseen directions) scaled by the covariance matrix. Applying (9) to the example in Table 2, we see that removing $s$ reduces the error at the test time with covariance matrix $\Sigma$ if $2\Sigma_{33} + 3\Sigma_{22} \leq \Sigma_{33}$. This is in line with our intuition since the full model recovers $\theta^*_3$ exactly, however, its estimation for $\theta^*_3$ is worse than the core model’s estimate.

### 3.1. Disjoint Parameters and Balanced Dataset are not Enough

Proposition 1 characterizes when removal of the spurious feature decreases the overall error. Now we investigate if removing spurious feature always decreases the error for some special settings. For example, if the features that determine the spurious feature $(s)$ and the target $(y)$ respectively are disjoint (i.e., for all $i$ we have: $\beta_s^* \theta^*_i = 0$, or when spurious features and the target are independent (i.e., $s \perp y$ in the train and test empirical distribution)).

The following corollary states that there is no condition on the true parameters that guarantees error reduction by removing the spurious feature.

**Corollary 1.** (Disjoint parameters are not enough) For $d \geq 4$, consider any non-zero $\theta^*, \beta^* \in \mathbb{R}^d$, such that there is no scalar $c$ where $\beta^* = c\theta^*$. For any $n < d - 1$, we can construct $Z \in \mathbb{R}^{n \times d}$ as training and $Z', Z'' \in \mathbb{R}^{n \times d}$ as test data such that if we train $M^{**}$ and $M^*$ using $Z$ as training data, then $\text{Error}(M^{**}) < \text{Error}(M^*)$ on $Z'$, and $\text{Error}(M^{**}) > \text{Error}(M^*)$ on $Z''$. On the contrary, if $\beta^* = c\theta^*$, then training both models on any $Z$, results in $\text{Error}(M^{**}) \leq \text{Error}(M^*)$ on any $Z'$.

For any $\beta^*$ and $\theta^*$, we can choose the training data $Z$, such that the parameters have positive correlation in the column space of training data. We can then select the test data in a way that the correlation between $\theta^*$ and $\beta^*$ in the null space has the same or opposite sign. See Appendix A for the full proof. Note that even in the special case of disjoint parameters ($\theta^*_i \beta^*_i = 0$, for all $i$), removing $s$ can increase the error depending on the core features distribution.

Can we guarantee error reduction by conditioning on the vector of observed spurious features and the targets in train and test data? In fact, one of the proposed ways to reduce the sensitivity of the model against the spurious features is having a balanced dataset. (i.e., collecting the dataset such that $y$ and $s$ are independent in train and test data, $\mathbb{P}[y | s] = \mathbb{P}[y]$). For example, in comment toxicity detection, Dixon et al. (2018) suggest adding new examples in training data to equalize the number of toxic comments and non-toxic comments for each identity term. Although they show mitigation with this method, Wang et al. (2019) demonstrate that a balanced dataset is not enough, and the model can still be sensitive to the spurious features.

We now show that no measure depending on the spurious features and the target in train and test can guarantee that removing a spurious feature decreases the error.

**Corollary 2.** (Balanced dataset is not enough) Consider any $d \geq 4$, $n < d$ and $S, Y \in \mathbb{R}^n$, where there is no scalar $c$ such that $Y = cS$. We can construct $Z, Z', Z'' \in \mathbb{R}^{n \times d}$ such that if we use $(Z, S, Y)$ as training set to train $M^{**}$ and $M^*$, then the $\text{Error}(M^{**}) < \text{Error}(M^*)$ on $(Z', S, Y)$ and $\text{Error}(M^{**}) > \text{Error}(M^*)$ on $(Z'', S, Y)$. On the contrary, if $Y = cS$ then training both models on any $(Z, S, Y)$, results in $\text{Error}(M^{**}) \leq \text{Error}(M^*)$ on any $(Z', S, Y)$.

See Appendix A for the proof. At a high level, for proving this corollary, we first rewrite the formulation of $\hat{w}$ in terms
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Table 3. An example demonstrating that $s$ and $y$ can be exactly the same in train and test time (blue filled cells), but the full model (which uses $s$) has a worse performance in comparison to the core model (which does not use $s$).

| Training examples | True parameters | Estimated parameters |
|-------------------|-----------------|----------------------|
| $z$               | $\theta^* = [1, 0, 1, 0]^T$ | $\hat{\theta}^* = [1, 0, 0, 0]^T$ |
| $s$               | $\beta^* = [1, 0, 1, -1]^T$  | $\hat{\beta}^* = [2/3, -1/3, 0, 0]^T$, $w = 1/3$ |

Table 4. An example with two spurious features ($s_1$ and $s_2$). There are two models: one model which only uses $s_1$ and another model which uses $s_1$ and $s_2$. Removing $s_2$ increases the weight for $s_1$ (weight 1 in comparison to 2/3, blue filled cells). As a result, the model that only uses $s_1$ is more susceptible to change in $s_1$ (increase the RobustError with respect to $s_1$) and it performs worse on groups with high variance on $z_2$ (weight of −3 instead of −2 while the true weight is 2, red filled cells).

| True parameters | Training example | Estimated parameters |
|-----------------|-----------------|----------------------|
| $\theta^* = [2, -2, 2]^T$ | $z$, $s_1$, $s_2$, $y$ | $\theta$ with $s_1$ and $s_2$ |
| $\beta_1^* = [1, -3, 0]^T$  | $[1, 0, 0]$ | $[2/3, 0, 0]^T$, $2/3$ |
| $\beta_2^* = [1, 0, -3]^T$  | $1$ | $[2, -2, -2]^T$, $0/3$ |

We then construct $Z$ such that the dot product of $S$ and $Y$ projected on $(ZZ^T)^{-1}$ is non-zero. We then construct $Z''$, $Z''^*$, $\theta^*$, $\beta^*$ such that $Y = Z\theta^* = Z''\theta^* = Z''^*\theta^*$, and $S = Z\beta^* = Z''\beta^* = Z''^*\beta^*$. Furthermore, we select $Z'$ to have high variance on the unseen directions that $\theta^*$ and $\beta^*$ have negative correlation, while choosing $Z''$ to have high variance on the unseen directions with positive correlation between $\theta^*$ and $\beta^*$. See Appendix A for the proof.

$\hat{w} = \frac{\theta^*\Pi\beta^*}{1 + \beta^*\Pi\beta^*} = \frac{S^T(ZZ^T)^{-1}Y}{1 + S^T(ZZ^T)^{-1}S}$ \hspace{1cm} (10)$

There are two main messages in Corollary 2, (i) The core model can have lower error than the full model even when there is no shift in the distribution of the spurious feature and target (i.e., $S$ and $Y$ are exactly the same in train and test). See Table 3 for a simple example. (ii) We can have independent targets and spurious features at train and the test, and still, the full model (which uses $s$) has lower error than the core model.

Corollary 1 and 2 together state that we cannot compute the sensitivity of a model to the spurious feature, or the effect of removing spurious feature by only observing the relation between $s$ and $y$ or knowing the relationship between $\theta^*$ and $\beta^*$. Therefore, naively collecting a balanced dataset is not enough, and we need to consider other features as well.

3.2. RobustError analysis

Proposition 1 compares the error of the full model and core model, and show each model can outperform the other on some conditions. In this section, we show that RobustError of the full model is always larger than the RobustError of the core model (recall that for the core model RobustError($M^*$) = Error($M^*$)). Intuitively for any $(z, s)$ data point we can perturb $s$ such that the full model makes the same prediction as the core model, therefore, the full model’s RobustError is always lower than the core model error.

Proposition 2. If $\|z\| \leq \gamma$ and consequently the set of different values for the spurious feature is $S = \{ -\gamma \beta^*\|s\|, \gamma \beta^*\|s\| \}$ then:

$RobustError(M^*) \leq RobustError(M^{**})$ \hspace{1cm} (11)$

Note that without any bounds on the spurious feature the RobustError can be infinity. By bounding the norm of $z$, we can bound the perturbation set of the spurious feature. Here $\|\cdot\|$ indicates the dual norm, see Appendix A for the proof.
3.3. Multiple spurious features

We now extend our framework to \( k \) spurious features, \( s_1, \ldots, s_k \), where \( s_i = \beta^{i\star} \). We characterize the effect of removing one spurious feature on the other spurious features. In particular, we show that removing one spurious feature can make the model more susceptible to other spurious features.

Extending the notation from Section 3, let \( Z \in \mathbb{R}^{n \times d} \) be the core features, \( Y \in \mathbb{R}^n \) be the target, and \( S \in \mathbb{R}^{n \times k} \) denote the spurious features. Similar to previous section, we obtain the minimum-norm estimate by solving the following optimization problem:

\[
\hat{\theta}^s, \hat{w} = \arg \min_{\theta,w} \|\theta\|_2^2 + \|w\|_2^2 \\
\text{s.t. } Z\theta + Sw = Y,
\]

where \( \hat{w} \in \mathbb{R}^k \) denote the optimal weights on spurious features.

**Proposition 3.** The weight of the \( i \)th spurious feature of the minimum norm estimator is

\[
\hat{w}_i = \frac{\theta^{i\star\top} \Pi \beta^{i\star} - \sum_{j \neq i} \hat{w}_j \beta^{j\star\top} \Pi \beta^{i\star}}{1 + \beta^{i\star\top} \Pi \beta^{i\star}}.
\]

Consider the special case of \( k = 2 \) with \( s_1 \) and \( s_2 \) as spurious features, where \( \beta^{1\star} \) and \( \beta^{2\star} \) are positively correlated in the column space of training data. Table 4 shows a simple example of this setup. As shown in (13), removing \( s_2 \) increases the weight for \( s_1 \), which makes the model more sensitive against changes in \( s_1 \). For instance, in Table 4, after removing \( s_2 \) the weight of \( s_1 \) changed from 2/3 to 1 (green cells).

In addition, recall that using \( s_1 \) causes high error for groups with high variance on the unseen directions where \( \beta^{1\star} \) differs from \( \theta^\star \). Removing \( s_2 \) exacerbates the problem even more in these groups. In our example, using \( s_1 \) and \( s_2 \) the model’s estimate for \( \theta^\star \) is \( -2 \) which is different from the true value of \( \theta^\star = 2 \); therefore, groups with high variance on the second feature incur high error. Removing \( s_2 \) changes the estimate of \( \theta^\star \) from \(-2\) to \(-3\), which exacerbates the error on the groups with high variance on \( s_2 \). This is in line with the recent empirical results suggesting that making a model robust against one type of noise might make it more vulnerable against other types of noise (Yin et al., 2019).

4. Self-Training

In the previous examples shown in Table 1 and Table 2, we showed that the full model which uses the spurious feature (\( M^\star \)) has an equivalent form which has weight 0 on the spurious feature. This equivalent form is useful when feature \( s \) is not available, or when we test our model in a new domain with different \( (z, s) \) distribution. We now explain how to recover the equivalent \( s \)-oblivious form with finite labeled data and access to \( m > d \) independent unlabeled examples.

In order to recover the \( s \)-oblivious equivalent form of the full model, we use Robust Self Training (RST). RST introduced by Carmon et al. (2019); Raghunathan et al. (2020); Najafi et al. (2019); Uesato et al. (2019) leverage unlabeled data to make the model robust to adversarial perturbations (small perturbation on the model’s input) while guaranteeing no decrease in average accuracy. In particular, RST first pseudo label all unlabeled data, with a non-robust model (with low error). It then trains a robust model on labeled data and pseudo labeled data to recover a model with both low error and low RobustError.

We now explain how to use RST in our setup. Assume in addition to \( n \) labeled examples \((z_i, s_i, y_i)\), we have access to \( m \) unlabeled examples \((z_i^\nu, s_i^\nu)\). We are interested in a model that 1) it is robust against \( s \), and 2) it has the same prediction as full model \( M^\star \) on unlabeled data. We recover \( M^\star_{\text{RST}} \) as follows:

\[
M^\star_{\text{RST}} = \arg \min_M \left( \sum_i \ell(M(z_i,0),y_i) + \eta \sum_i \ell(M(z_i^\nu,0),M^\star(z_i^\nu,s_i^\nu)) + \lambda C(M) \right),
\]

where \( \lambda, \eta \) are parameters to tune the trade-off between the true labeled data, pseudo-labeled data generated by the full model, and complexity of the model.

We now show that in our linear regression setup, \( M^\star_{\text{RST}} \) is a \( s \)-robust equivalent form of \( M^\star \). Let \( Z^\nu \in \mathbb{R}^{m \times d} \) and \( S^\nu \in \mathbb{R}^{m} \) denote the unlabeled data. Let \( \theta^\star_{\text{RST}} \) denote the estimated parameters of RST model obtained by solving the following optimization problem.

\[
\theta^\star_{\text{RST}} = \arg \min_{\theta} \|\theta\|_2^2 \\
\text{s.t. } Z^\nu \theta = Y \\
Z^\nu \theta = Z^\nu \hat{\theta}^\star + S^\nu w
\]

The following proposition shows the optimum parameters for (15) and prove that for any data point \( M^\star_{\text{RST}} \) has the same prediction as \( M^\star \).

**Proposition 4.** The optimum parameters for (15) are:

\[
\hat{w} = \frac{\beta^{\star\top} \Pi \theta^\star}{1 + \beta^{\star\top} \Pi \beta^\star}
\]

\[
\hat{\theta}_{\text{RST}}^\star = \Pi \theta^\star + w(I - \Pi)\beta^\star
\]

and for any data point \((z, s)\), we have:

\[
M^\star_{\text{RST}}(z,0) = M^\star(z,s)
\]
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| model | weight for s | weights for z | Error |
|-------|--------------|---------------|-------|
| M^a   | 0            | Πθ^*          | E[(y - z^T Πθ^*)^2] |
| M^b   | w            | Πθ^* - wβ^*   | E[(y - z^T Πθ^*)^2] + E[w^2(s - z^T Πβ^*)^2] - E[2w(y - z^T Πθ^*)(s - z^T Πβ^*)] |
| M^RST | 0            | Πθ^* + w(1 - Π)β^* | E[(y - z^T Πθ^*)^2] + E[w^2(z^T β^* - z^T Πβ^*)^2] - E[2w(y - z^T Πθ^*)(z^T β^* - z^T Πβ^*)] |

Table 5. Estimated parameters and error for different models. w = β^* Πθ^* / (1 + β^* Πθ^*)

Figure 2. The identity terms used as spurious features. The Red line indicates the ratio of positive comments containing the identity terms over the negative comments containing the identity term. The difference between the ratio of TPR/TNR of the core and full model can be small (straight, young) or large (male, Chinese) independent of #positive/#negative ratio for the identity term.

See Appendix A for details. Note that M^RST simply learn β^* from unlabeled data and replace w/s with w/β^* z. See Table 5 for the estimated parameters of the three introduced models.

5. Experiments

We now investigate the effects of removing spurious features in non-linear models trained on real-world datasets. Although our theory assumptions do not hold anymore, we find similar results as Section 3. In particular, we show that removal of a spurious feature lowers the average accuracy, has disproportionate effects on different groups, and makes the model less robust to other spurious features. We then show core+RST model can achieve higher accuracy than the core model while not relying on the spurious feature.

5.1. Datasets and Setup

Double-MNIST The MNIST dataset (LeCun et al., 1998) consists of 60K images of handwritten digits between 0 to 9. We synthetically construct a new dataset from the MNIST dataset, which we call Double-MNIST. We concatenate each image in MNIST with another random image from the same class with probability 0.9 and a random image from other classes with probability 0.1. The original image’s label is the target (y) and the concatenated image’s label is the spurious feature (s). Note that the features that determine the target (the first image) are completely disjoint from the feature that determine the spurious feature (the second image). We train a two-layer neural network with 128 hidden units on this dataset. Using 50K for the training data, the model achieves 98.3% accuracy. However, for our experiments (where we need unlabeled data), we used 1K labeled examples, 50K unlabeled examples, and 10K for test data.

CelebA The CelebA dataset (Liu et al., 2015) contains photos of celebrities along with 40 different attributes. We choose wearing lipstick which indicates if a celebrity is wearing lipstick as the target and wearing earrings as the spurious feature. We train a two-layer neural network with 128 hidden units on this dataset. For our purposes in this work, we use 1K labeled examples, 50K unlabeled examples, and 20K for test data.

Toxic-Comment-Detection. The Toxic comment dataset is a public Kaggle dataset containing 160K Wikipedia comments. Each comment is labeled by human raters as toxic or non-toxic, where toxicity is defined by Dixon et al. (2018) as “rude, disrespectful, or unreasonable comment that is likely to make you leave a discussion.”. We cleaned the data by re-

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1https://www.kaggle.com/c/jigsaw-toxic-comment-classification-challenge
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| name                  | core features (z) | target (y) | spurious feature (s) | Example: z       | y    | s   |
|-----------------------|-------------------|------------|----------------------|------------------|------|-----|
| Double-MNIST          | two MNIST images  | label of the left image | label of the right image | ![Image](57) | 5   | 7   |
| CelebA                | celebrities photo | wearing lipstick | wearing earrings | ![Image](CelebA) | True | True |
| Toxic-Comment-Detection | comment (w/o identity terms) | toxic or not identity terms | cuz i shouldn’t be blocked non-toxic black just for being UNK |

Table 6. A summary of the three datasets that we used in this work. Double-MNIST is a synthetically generated dataset where each data point is a concatenation of two images from the MNIST dataset.

|                      | Double-MNIST | CelebA | Toxic-Comments |
|----------------------|--------------|--------|---------------|
|                      | all          | same labels | different labels | all          | no lipstick | only earrings | only lipstick | both lipstick and earrings | all          |
| group size (in percentage) | 100          | 90      | 10             | 100          | 52.0        | 3.4          | 28.8          | 15.8          | 100          |
| full model accuracy   | 92.7±0.06    | 96.6±0.05 | 53.4±0.4       | 83.5±0.1     | 85.4±0.4    | 33.3±0.9     | 79.6±0.6     | 95.0±0.3      | 90.1±0.1     |
| core model accuracy   | 92.1±0.06    | 94.8±0.06 | 64.1±0.3       | 82.5±0.1     | 82.0±0.5    | 59.1±1.0     | 85.2±0.6     | 84.1±0.7      | 89.0±0.1     |
| full model robust accuracy | 47.3±0.92    | 49.6±0.96 | 24.4±0.7       | 66.7±0.4     | 58.7±0.9    | 33.3±0.9     | 79.6±0.6     | 76.4±0.8      | 74.9±0.5     |
| core+RST accuracy     | 92.7±0.06    | 96.2±0.05 | 56.7±0.5       | 83.2±0.1     | 83.4±0.5    | 58.7±1.1     | 84.8±0.7     | 85.2±0.7      | 89.6±0.1     |

Table 7. Accuracy declines as we remove spurious feature (the core model vs. the full model), however, note that this drop varies widely among different group. The core+RST achieves similar average accuracy as the full model while having high robust accuracy and lower gap among different groups accuracy.

Placing abbreviations (e.g., changing “we’ve” to “we have”). We used term frequency-inverse document frequency (tf-idf) to extract features and used a logistic regression model to train on the extracted features. Splitting data to the 80-20 train-test, we achieve a similar AUC (95.8) as reported in Garg et al. (2019). Dixon et al. (2018) provide 50 identity terms that a model should be robust against them. We choose 17 of these identity terms that exist in positive and negative class at least 30 times (see Figure 2 for the list). We use these terms as the spurious feature and replace them with a special token for the core model. For the full model we did not remove these identity terms. We choose the subset of the dataset that contains these 17 adjectives, consisting of 11K examples. We did 20 – 80 train-test split, use 500 examples as labeled data, and the rest of examples as unlabeled data.

See Table 6 for a summary of datasets. We run each experiment 50 times and report the average accuracy and the standard deviation.

5.2. Results

Core model vs. Full model. Removing the spurious feature decreases the overall accuracy in all three datasets, see the first two rows of Table 7. This is in particular surprising for the Double-MNIST dataset where the target and the spurious feature get determined by disjoint set of features. Note that the change in accuracy varies widely among different groups in the CelebA dataset removing the spurious wearing-earrings feature increases the accuracy for celebrities who only wear either lipstick or earrings, and in the Double-MNIST dataset removing the second image label increases the accuracy for data points where the concatenated images have the different labels).

For the Toxic-Comment-Detection dataset, for each identity term u, we construct two groups: one group consist of toxic comments that have u in the comment text, and another group consists of non-toxic comments that have u in the comment text, resulting in total of 34 groups. The maximum gap among accuracy of the 17 groups containing toxic comments (max-TPR-gap) drop from 70.1 to 28.4, and the maximum gap among accuracy of groups that are non-toxic (max-TNR-gap) drops from 20.5 to 1.8. Toxic-gay and non-toxic-gay groups incurred the maximum change in accuracy (~40 and +19 respectively) due to removal of identity terms. Figure 2 shows the change in true positive rate, true negative rate ratio for different identity terms.

Robust accuracy. Recall that robust accuracy is the worst-case accuracy for each data point with respect to change in the spurious feature. First, note that the robust accuracy of

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1. https://github.com/conversationai/unintended-ml-bias-analysis/blob/master/unintended_ml_bias/bias_madlibs_data/adjectives_people.txt
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|                        | all           | no lipstick or earrings | only earrings | only lipstick | both lipstick and earrings |
|------------------------|---------------|-------------------------|---------------|---------------|---------------------------|
| With hair color and necklace | accuracy 83.9 ± 0.1 | 85.0 ± 0.5             | 37.0 ± 1.1    | 81.8 ± 0.7    | 94.2 ± 0.3               |
|                        | robust accuracy 68.5 ± 0.4 | 60.9 ± 1.0             | 37.0 ± 1.1    | 81.8 ± 0.7    | 76.8 ± 0.8               |
| With only necklace     | accuracy 83.5±0.1 | 85.4±0.4               | 33.3±0.9      | 79.6±0.6      | 95.0±0.3                 |
|                        | robust accuracy 66.7±0.4 | 58.7±0.9               | 33.3±0.9      | 79.6±0.6      | 76.4±0.8                 |

Table 8. Two models trained on the CelebA dataset. Top model uses hair color and lipstick (a binary feature indicating if a person is wearing lipstick or not), bottom model only uses hair color. 1) the robust accuracy against hair color drop more rapidly for the bottom model. 2) The group that has lowest accuracy for the bottom model (only earrings) have a better accuracy when hair color is also used as an extra spurious feature.

the full model (third row) is much lower than its accuracy (first row), which shows the full model’s reliance on the spurious feature. Furthermore, this reliance helps some groups (e.g., celebrities who wear earrings and lipstick) while it does not have any effect on other groups (e.g., celebrities who wear earrings but not lipstick). Recall that the robust accuracy of the core model is exactly equal to its accuracy since it does not use the spurious feature. Finally, in line with our theory in Proposition 4, the robust accuracy of the full model is always lower than the robust accuracy of the core model.

Robust Self-Training. The forth row of the Table 7 shows the accuracy core+RST model as explained in Section 4. The core+RST has a better accuracy than the core model. Recall that core+RST does not use the spurious feature; therefore, its robust accuracy (unlike the full model) is exactly equal to its accuracy. In Section 4, we prove that the full model and the core+RST should have the same predictions; however, unlike our theory, we observe that the large gap among the accuracy of different groups in the full model is mitigated by the core+RST. We have not observed a large accuracy boost using unlabeled data in the Toxic-Comment-Detection dataset. After some error analysis we observed that as mentioned in Garg et al. (2019), there are some examples that can be toxic with respect to some identity terms but not the others. Furthermore, we observed some biases in annotations in this dataset (e.g., ’username is gay’ has labeled as toxic). As a result some accuracy drop of the core model is inherent and cannot be mitigated by unlabeled data.

Effect of training data size. Figure 3 shows that as we increase the training data size, the gap between the accuracy of the core model and full model is reduced. In particular, it shows the error on two different groups in the Double-MNIST dataset: the majority (different labels) and minority (same labels). When we train a model on more training examples and consequently, observing more directions in the training data, we reduce the effect of inductive bias enforced to the model by the spurious feature. As a result, we observe a decrease in the gap between the full and the core models.

Multiple spurious features. Table 4 shows the accuracy of two models in the CelebA dataset: A model which uses hair color and wearing earrings as the spurious features and a model that only uses wearing earrings as the spurious feature. In line with our theory in Section 3.3, we observe that: 1) Robust accuracy against wearing-earrings has a bigger gap from standard accuracy when we remove hair color. 2) The accuracy of people who wear lipstick but not earrings (the group that is more vulnerable to using earrings as a spurious feature and had the lowest accuracy) drops as we remove the hair color attribute.

6. Related work and discussion

This work is motivated by work in fairness in machine learning that aims to construct a model that is robust against changes in sensitive features. The techniques used in this work is similar to work in robust machine learning that tries to understand the tradeoff between the robust accuracy against perturbed inputs and the standard accuracy. In the following, we discuss related work in these two fields.

Fairness in Machine Learning. There are mainly two common flavors of fairness notions in machine learning concerning a sensitive feature (e.g., nationality). (i) The statistical notions which measure how much a model loss is different among groups according to the sensitive features (Hardt et al., 2016; Khani et al., 2019; Agarwal et al., 2018; Woodworth et al., 2017). These notions operate at the group level, and it does not provide any guarantees at the individual level. (ii) Counterfactual notion of fairness which measure how much two “similar” individuals are incurred different losses because of their sensitive feature (Kusner et al., 2017; Chiappa, 2019; Loftus et al., 2018; Khani and Liang, 2020; Kilbertus et al., 2017; Garg et al., 2019).

This work is related to the counterfactual notion of fairness since we study the models that are robust against sensitive (spurious) features. Note that there are many concerns and critics regarding counterfactual reasoning when sensitive features are immutable (Holland, 2003; Freedman, 2004). However, there are works that try to learn a new representation entirely uncorrelated to the sensitive feature in
We focused on a specific spurious feature (with a known worse error) with the cost of a drop in accuracy. There have been many attempts on achieving a robust model and practitioners should consider other features as well. The closest work to us is Raghunathan et al. (2020), where they show augmenting the training data with adversarially input perturbation comes with the cost of a drop in accuracy. There have been many explanations regarding this drop in accuracy (Tsipras et al., 2019; Zhang et al., 2019; Fawzi et al., 2018; Nakkiran et al., 2019). The recent line of work on “double descent” (Bartlett et al., 2019; Gan et al., 2017; Nakkiran et al., 2019; Belkin et al., 2019; Mei and Montanari, 2019) has shed some lights on this trade-off and show that unlike conventional under-parameterized regime in the new over-parameterized regime, more data and fewer parameters might result in a worse error.

The closest work to us is Raghunathan et al. (2020), where they show augmenting the training data with adversarially perturbed inputs changes the inductive bias and consequently can hurt the accuracy. In contrast to their work, we focused on a specific spurious feature (with a known correlation with other features) instead of an arbitrary data perturbation set for data augmentation. As a result, we were enabled to analyze the drop in accuracy in a more interpretable way and show how removing the spurious feature affects different groups. There is also another difference between our work and work on robustness to input perturbation. In our work, by removing s, we have a robust model independent of the data distribution. However, robustness to input perturbation always depends on the distribution.

7. Conclusion

In this work, we first showed that overparameterized models are incentivized to use spurious features in order to fit the training data with a smaller norm. Then we demonstrated how removing these spurious features altered the model’s inductive bias. Theoretically and empirically, we showed that this change in inductive bias could hurt the overall accuracy and affect groups disproportionately. We then proved that robustness against spurious features (error reduction by removing the spurious features) cannot be guaranteed under any condition of the target and spurious feature. Consequently, balanced datasets do not guarantee a robust model and practitioners should consider other features as well. Studying the effect of removing noisy spurious features is an interesting future direction.

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A. Missing proofs

**Proposition 1.** Let $\Sigma = \mathbb{E}[zz^\top]$ denote the covariance matrix of the test data (or covariance matrix for any group), and let $\Pi$ denote the column space of training data. $\text{Error}(M^+s) < \text{Error}(M^-s)$ iff:

\[
\text{sign} \left( \beta^* \Pi \theta^* \right) = \text{sign} \left( \beta^* (I - \Pi) \Sigma (I - \Pi) \theta^* \right),
\]

and

\[
\frac{\beta^* \Pi \theta^*}{1 + \beta^* \Pi \beta^*} < \frac{2 \beta^* (I - \Pi) \Sigma (I - \Pi) \theta^*}{\beta^* (I - \Pi) \Sigma (I - \Pi) \beta^*}.
\]

**Proof.** Recall, the parameters of $M^-$ can be obtained by solving the following optimization problem.

\[
\hat{\theta}^-s = \arg \min_{\theta} \|\theta\|_2^2 \quad \text{s.t.} \quad Z\theta = y,
\]

and the parameters of $M^{+s}$ is obtained by the following optimization problem.

\[
\hat{\theta}^{+s}, \hat{\omega} = \arg \min_{\theta, \omega} \|\theta\|_2^2 + \omega^2 \quad \text{s.t.} \quad Z\theta + s\omega = y,
\]

where $\hat{\omega}$ denotes the weight associate with the feature $s$.

For the first scenario (5) we have:

\[
\theta^-s = Z^T (ZZ^T)^{-1} y = \Pi \theta^*
\]

In the second scenario, when we use feature $s$ we have:

\[
\theta^{+s} = Z^T (ZZ^T)^{-1} (y - s\omega) = \Pi \theta^* - \Pi \beta^* \hat{\omega}
\]

Writing the L2-norm of the estimate and taking the derivative with respect with $\hat{\omega}$, we have:

\[
\frac{\partial \|\hat{\theta}^{+s}\| + \partial \hat{\omega}^2}{\partial \hat{\omega}} = \frac{\partial (y - s\omega)^T (ZZ^T)^{-1} Z^T (ZZ^T)^{-1} (y - s\omega)}{\partial \hat{\omega}} + 2\omega
\]

\[
= 2s \omega^T (ZZ^T)^{-1} s - 2y^T (ZZ^T)^{-1} s + 2\omega
\]
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therefore, the optimum \( \hat{w} \) is:

\[
\hat{w} = \frac{y^\top (ZZ^\top)^{-1} s}{1 + s^\top (ZZ^\top)^{-1} s} = \frac{\theta^\top \Pi \beta^*}{1 + \beta^* \Pi \beta^*}
\]  

We can now compute the error for each model:

\[
\text{Error}(\mathcal{M}^*) = E[\theta^*^\top z - E[\theta^{-*}]^\top z]^2 = \theta^*^\top (I - \Pi) \Sigma(I - \Pi) \theta^* 
\]  

\[
\text{Error}(\mathcal{M}^{**}) = E \left[ \theta^{**}^\top z - \theta^{**+s}^\top z - w s \right]^2 
\]  

\[
= E \left[ \theta^{**}^\top z - (\Pi \theta^*)^\top z + \hat{w}(\Pi \beta^*)^\top z - \hat{w} \beta^*^\top z \right]^2 
\]  

\[
= E \left[ \theta^{**}^\top (I - \Pi) z - \hat{w} \beta^* (I - \Pi) z \right]^2 
\]  

\[
= \theta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \theta^* 
\]  

\[
+ \hat{w}^2 \beta^*^\top (I - \Pi) \Sigma(I - \Pi) \beta^* - 2 \hat{w} \theta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \beta^* 
\]  

\[
\text{Error}(\mathcal{M}^{**}) - \text{Error}(\mathcal{M}^*) = \hat{w}^2 \beta^*^\top (I - \Pi) \Sigma(I - \Pi) \beta^* - 2 \hat{w} \theta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \beta^* 
\]  

In order for (35) to be negative \( \hat{w} \) should be non zero and \( \hat{w} \) and \( \theta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \beta^* \) should have the same sign. Therefore, \( \text{Error}(\mathcal{M}^{**}) < \text{Error}(\mathcal{M}^*) \), iff:

\[
\text{sign}(\beta^*^\top \Pi \theta^*) = \text{sign}(\theta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \beta^*) 
\]  

\[
\left| \frac{\beta^*^\top \Pi \theta^*}{1 + \beta^*^\top \Pi \beta^*} \right| < \left| \frac{2 \beta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \theta^*}{\beta^{**}^\top (I - \Pi) \Sigma(I - \Pi) \beta^*} \right| 
\]  

\[
\square
\]

Corollary 1. (Disjoint parameters are not enough) For \( d \geq 4 \), consider any non-zero \( \theta^* \), \( \beta^* \in \mathbb{R}^d \), such that there is no scalar \( c \) where \( \beta^* = c \theta^* \). For any \( n < d - 1 \), we can construct \( Z \in \mathbb{R}^{n \times d} \) as training and \( Z', Z'' \in \mathbb{R}^{n \times d} \) as test data such that if we train \( \mathcal{M}^{**} \) and \( \mathcal{M}^* \) using \( Z \) as training data, then \( \text{Error}(\mathcal{M}^{**}) < \text{Error}(\mathcal{M}^*) \) on \( Z' \), and \( \text{Error}(\mathcal{M}^{**}) > \text{Error}(\mathcal{M}^*) \) on \( Z'' \). On the contrary, if \( \beta^* = c \theta^* \), then training both models on any \( Z \), results in \( \text{Error}(\mathcal{M}^{**}) \leq \text{Error}(\mathcal{M}^*) \) on any \( Z' \).

Proof. We find three directions \((a_1, a_2, a_3)\) in the feature space where the dot product between \( \beta^* \) and \( \theta^* \) is non-zero in the first direction, positive in the second direction, and negative in the third direction. We then construct a training dataset in the first direction, and construct two test datasets in the other two directions. We then use Proposition 1 to prove the corollary.
Let $b \in \mathbb{R}^d$ be a vector that is orthogonal to both $\beta^*$ and $\theta^*$. We define $a_2 = (\frac{\theta^*}{\|\theta^*\|} + \frac{\beta^*}{\|\beta^*\|} + 2b)$ and $a_3 = (\frac{\theta^*}{\|\theta^*\|} - \frac{\beta^*}{\|\beta^*\|})$.

We now find $a_1$ such that it is orthogonal to $a_2$ and $a_3$ (i.e., $a_1^\top a_2 = 0$ and $a_1^\top a_3 = 0$) but not orthogonal to $\theta^*$ and $\beta^*$ (i.e., $a_1^\top \theta^* \neq 0$ and $a_1^\top \beta^* \neq 0$).

\begin{align*}
a_1^\top \left( \frac{\theta^*}{\|\theta^*\|} + \frac{\beta^*}{\|\beta^*\|} + 2b \right) &= 0 \quad (38) \\
a_1^\top \left( \frac{\theta^*}{\|\theta^*\|} - \frac{\beta^*}{\|\beta^*\|} \right) &= 0 \quad (39) \\
a_1^\top \theta^* &\neq 0 \quad (40) \\
a_1^\top \beta^* &\neq 0 \quad (41)
\end{align*}

Fixing $b$ and for a $x > 0$ we can simplifying the above equations and finding a solution by solving:

\begin{align*}
a_1^\top b &= -x \quad (42) \\
a_1^\top \frac{\theta^*}{\|\theta^*\|} &= x \quad (43) \\
a_1^\top \frac{\beta^*}{\|\beta^*\|} &= x \quad (44)
\end{align*}

Since $\theta^*, \beta^*, b$ are not parallel the above problem has a solution for any $x$.

We now construct $Z$ to consist of $n$ examples of $a_1$, and construct $Z'$ and $Z''$ to have $n$ examples of $\frac{1}{n} a_2$ and $\frac{1}{n} a_3$ respectively. Note that instead of copying the same examples to both train and test from we can add examples from directions that are orthogonal to $a_1, a_2, a_3,$ and $\theta^*, \beta^*$.

Note that in this case $\Pi = \frac{a_1 a_1^\top}{\|a_1\|^2}$. As a result $\text{sign}(\theta^* \Pi \beta^*) = \text{sign}(x^2)$ is positive. However, the dot product of $\theta^*$ and $\beta^*$ is negative when projected on $a_3$. 

We now show that the dot product of \( \theta^*\top (I - a_1a_1^\top) (a_3a_3^\top) (I - a_1a_1^\top) \beta^* = \theta^*\top (a_3^\top a_3) \beta^* \) (45)

\[
= \theta^*\top (\frac{\theta^*}{\|\theta^*\|} - \frac{\beta^*}{\|\beta^*\|}) (\frac{\theta^*}{\|\theta^*\|} - \frac{\beta^*}{\|\beta^*\|})^\top \beta^*
\]

\[
= (\|\theta^*\| - \frac{\theta^*\top \beta^*}{\|\beta^*\|}) (\frac{\theta^*\top \beta^*}{\|\beta^*\|} - \|\beta^*\|)
\]

\[
= - \|\theta^*\|\|\beta^*\| \left(1 - \frac{\theta^*\top \beta^*}{\|\theta^*\|\|\beta^*\|}\right)^2
\]

<0 (49)

According to (45) and (8), Error(M***) > Error(M*) on \( Z'' \).

We now show that the dot product of \( \theta^*, \beta^* \) projected on \( a_2 \) is positive.

\[
\theta^*\top (I - a_1a_1^\top) (a_2a_2^\top) (I - a_1a_1^\top) \beta^* = \theta^*\top (a_2^\top a_2) \beta^* \]

\[
= \theta^*\top (\frac{\theta^*}{\|\theta^*\|} + \frac{\beta^*}{\|\beta^*\|} + \frac{\theta^*\top \beta^*}{\|\beta^*\|} + \frac{\theta^*\top \beta^*}{\|\beta^*\|})^\top \beta^*
\]

\[
= (\|\theta^*\| + \frac{\theta^*\top \beta^*}{\|\beta^*\|}) (\|\beta^*\| + \frac{\theta^*\top \beta^*}{\|\beta^*\|})
\]

\[
= \|\theta^*\|\|\beta^*\| \left(1 + \frac{\theta^*\top \beta^*}{\|\theta^*\|\|\beta^*\|}\right)^2
\]

>0 (54)

In (50) we showed that (8) holds for \( Z' \), in addition we should also show that (9) also holds.

\[
\left| \frac{\beta^*\top \Pi \theta^*}{1 + \beta^*\top \Pi \beta^*} \right| \leq \left| \frac{2\beta^*\top (I - \Pi) \Sigma (I - \Pi) \theta^*}{\beta^*\top (I - \Pi) \Sigma (I - \Pi) \beta^*} \right|
\]

\[
\left| \frac{\beta^*\top (a_1a_1^\top) \theta^*}{1 + \beta^*\top (a_1a_1^\top) \beta^*} \right| \leq \left| \frac{2\beta^*\top (a_2a_2^\top) \theta^*}{\beta^*\top (a_2a_2^\top) \beta^*} \right|
\]

\[
\frac{x^2}{a^\top a + x^2} \leq \frac{2\|\theta^*\| + \|\theta^*\|\|\theta^*\|}{\|\theta^*\| + \|\theta^*\|}
\]

\[
\frac{x^2}{a^\top a + x^2} \leq \frac{2\|\theta^*\|}{\|\beta^*\|}
\]

since we assumed \( d \geq 4 \) we can choose a small \( x \) and increase the norm of \( a \) on other directions to satisfy this inequality.

Now we show that if \( \theta^* = c\beta^* \) then Error(M***) \leq Error(M*). First note that if \( \tilde{w} = 0 \) then Error(M***) = Error(M*). Furthermore, if \( \theta^*\top (I - \Pi) \Sigma (I - \Pi) \beta^* = 0 \) then \( \beta^*\top (I - \Pi) \Sigma (I - \Pi) \beta^* = 0 \) which implies Error(M***) = Error(M*).
We now assume $\hat{w} \neq 0$, and $\beta^* (I - \Pi) \Sigma(I - \Pi) \beta^* \neq 0$, and we show that (8) and (9) always hold.

\[
\begin{align*}
\text{sign}(\theta^* \Pi \beta^*) &= \text{sign}(c \beta^* \Pi \beta^*) = \text{sign}(c) \\
\text{sign}(\theta^* (I - \Pi) \Sigma(I - \Pi) \beta^*) &= \text{sign}(c \beta^* (I - \Pi) \Sigma(I - \Pi) \beta^*) = \text{sign}(c)
\end{align*}
\]  

(59)\hspace{6cm} (60)

\[
\begin{align*}
\left| \frac{\beta^* \Pi \theta^*}{1 + \beta^* \Pi \beta^*} \right| &< \frac{2 \beta^* (I - \Pi) \Sigma(I - \Pi) \theta^*}{\beta^* (I - \Pi) \Sigma(I - \Pi) \beta^*} \\
\frac{\beta^* \Pi \beta^*}{1 + \beta^* \Pi \beta^*} |c| &< 2 |c|
\end{align*}
\]  

(61)\hspace{6cm} (62)

\[\square\]

**Corollary 2.** *(Balanced dataset is not enough)* Consider any $d \geq 4$, $n < d$ and $S, Y \in \mathbb{R}^n$, where there is no scalar $c$ such that $Y = cS$. We can construct $Z, Z', Z'' \in \mathbb{R}^{n \times d}$, such that if we use $(Z, S, Y)$ as training set to train $M^*$ and $M^i$, then the $\text{Error}(M^i) < \text{Error}(M^*)$ on $(Z', S, Y)$ and $\text{Error}(M^i) < \text{Error}(M^*)$ on $(Z'', S, Y)$. On the contrary, if $Y = cS$ then training both models on any $(Z, S, Y)$, results in $\text{Error}(M^i) \leq \text{Error}(M^*)$ on any $(Z', S, Y)$.

**Proof.** For any vector $u \in \mathbb{R}^{k \times d}$ let $\bar{u} \in \mathbb{R}^{k \times 2}$ denote its first 2 columns. Similarly for $v \in \mathbb{R}^d$, let $\bar{v} \in \mathbb{R}^2$ denote its first 2 elements.

Define $\bar{\theta}^* = [1, 1]$ and $\bar{\beta}^* = [1, 0]$. We construct $\bar{Z} \in \mathbb{R}^{n \times 2}$ such that $Z \theta^* = Y$ and $Z \beta^* = S$. In particular, $\bar{Z} = [S, Y - S]$. Assign the rest of elements in $Z$ to be zero, $Z = [\bar{Z}, 0]$. As before, let $\Pi$ be the column space of training data, $\Pi = Z^\dagger Z$.

Since we assumed $S$ and $Y$ are not parallel, $\Pi$ is a zero matrix with a $2 \times 2$ identity matrix and the top left (If all the rows of $Z$ are parallel then it implies that $S$ and $Y - S$ are parallel which means $S$ and $Y$ are parallel).

Now we find $a, a' \in \mathbb{R}^d$ such that (i) $a \theta^* = 0$ and $a' \theta^* = 0$, (ii) $a \beta^* = 0$ and $a' \beta^* = 0$, and (iii) conditions on Proposition 1 holds for $Z + A$ but not for $Z + A'$, where $A, A' \in \mathbb{R}^{n \times d}$ are $n$ copy of $\frac{1}{n} a, \frac{1}{n} a'$ respectively.

The first two conditions ensure that adding $a, a'$ to the rows of $Z$ does not change the targets or spurious features. Define $Z' = Z + A$ and $Z'' = Z + A'$.

Since we assumed $a \theta^* = 0$ we can write:

\[
0 = a \theta^* = a(I - \Pi) \theta^* + a \Pi \theta^* \implies a(I - \Pi) \theta^* = -a \Pi \theta^*
\]  

(63)
Also note that \((I - \Pi)Z = 0\). Now we rewrite the RHS of (8).

\[
\begin{align*}
\theta^*^\top (I - \Pi)(Z + A)^\top (Z + A)(I - \Pi)\beta^* 
&= \theta^*^\top (I - \Pi)A^\top A(I - \Pi)\beta^* \\
&= \theta^*^\top (I - \Pi)\alpha^\top a(I - \Pi)\beta^* \\
&= \theta^*^\top \Pi a^\top \Pi \beta^* \\
&= \theta^* a^\top \alpha^\top \beta^*,
\end{align*}
\]

where the last inequality holds since \(\Pi\) is a zero matrix with a \(2 \times 2\) identity matrix at the top left corner.

Now define \(\bar{a} = (\bar{d}\theta^* + \bar{\beta}\beta^*)^\top\) and \(\bar{a}' = (\bar{d}\theta^* - \bar{\beta}\beta^*)^\top\). Let \(\bar{0} \in \mathbb{R}^{d-4}\) denote an all zero vector (when \(d = 4\) this is empty). Define \(\theta^* = [\bar{d}\theta, \bar{0}, 1, 0]\) and \(\beta^* = [\bar{\beta}\beta, \bar{0}, 0, 1]\). We now show that \(a = [\bar{a}, \bar{0}, -\bar{a}\theta^*, -\bar{a}\beta^*]\) and \(a' = [\bar{a}', \bar{0}, -\bar{a}'\theta^*, -\bar{a}'\beta^*]\) satisfy all three conditions. For the first two conditions we have: \(a\theta^* = [\bar{a}, \bar{0}, -\bar{a}\theta^*, -\bar{a}\beta^*][\theta^*, \bar{0}, 1, 0]^\top = \bar{a}\theta^* - \bar{a}\theta^* = 0\), similarly, \(a\beta^* = a'\theta^* = a'\beta^* = 0\). Similar to (50) and (45) we can show the (8) holds for \(Z'\) but not \(Z''\). Now we only need to show:

\[
\frac{\|\beta^*^\top \Pi \theta^*\|}{1 + \beta^*^\top \Pi \beta^*} \leq \frac{2\beta^*^\top (I - \Pi)\Sigma(I - \Pi)\theta^*}{\beta^*^\top (I - \Pi)\Sigma(I - \Pi)\beta^*}
\]

(68)

\[
\frac{\beta^*^\top \theta^*}{1 + \beta^*^\top \beta^*} \leq \frac{2\beta^*^\top \bar{a}\alpha^\top \theta^*}{\beta^*^\top \bar{a}\alpha^\top \beta^*}
\]

(69)

\[
\frac{\bar{\beta}^*^\top \bar{\theta}^*}{1 + \bar{\beta}^*^\top \bar{\beta}^*} \leq \frac{2\bar{\beta}^*^\top (\bar{d}\theta^* + \bar{\beta}\beta^*)^\top (\bar{d}\theta^* + \bar{\beta}\beta^*)^\top \bar{\theta}^*}{\bar{\beta}^*^\top (\bar{d}\theta^* + \bar{\beta}\beta^*)^\top (\bar{d}\theta^* + \bar{\beta}\beta^*)^\top \bar{\beta}^*}
\]

(70)

\[
\frac{1}{2} \leq \frac{2\|\theta^*\|}{\|\beta^*\|} = 2\sqrt{2}
\]

(71)

We now show if \(Y = cS\) then full model always outperforms the core model. Let \(Z \in \mathbb{R}^{n \times d}\) be the training data and \(Z' \in \mathbb{R}^{n \times d}\) be the test data.

\[
Z\theta^* - cZ\beta^* = Y - cS = 0 \implies Z(\theta^* - c\beta^*) = 0 \implies Z^\top Z(\theta^* - c\beta^*) = 0 \implies \Pi\theta^* = c\Pi\beta^*
\]

(72)

Let \(A = Z' - Z\). Due to our assumption we have \(A\theta^* = A\beta^* = 0\), therefore,

\[
0 = A\theta^* = A(I - \Pi)\theta^* + A\Pi\theta^* \implies A(I - \Pi)\theta^* = -A\Pi\theta^*
\]

(73)
Removing Spurious Features can Hurt Accuracy and Affect Groups Disproportionately

We can now show:

\[ \theta^* (I - \Pi)(Z + A)^T (Z + A)(I - \Pi) \beta^* = \theta^* (I - \Pi) A^T A(I - \Pi) \beta^* \]
\[ = \theta^* \Pi A^T A \Pi \beta^* \]

(74)

(75)

We can write conditions on Proposition 1 as follows:

\[ \text{sign}(c\beta^* \Pi \beta^*) = \text{sign}(c\beta^* \Pi A A^T \beta^*) \]

(76)

which is always true, we can write the second condition:

\[ \frac{\beta^* \Pi \theta^*}{1 + \beta^* \Pi \beta^*} \leq \frac{2\beta^* (I - \Pi) \Sigma (I - \Pi) \theta^*}{\beta^* (I - \Pi) \Sigma (I - \Pi) \beta^*} \]

(77)

\[ \frac{c\beta^* \Pi \beta^*}{1 + \beta^* \Pi \beta^*} \leq \frac{2c\beta^* \Pi A A^T \beta^*}{\beta^* \Pi A A^T \Pi \beta^*} \]

(78)

\[ \frac{\beta^* \Pi \beta^*}{1 + \beta^* \Pi \beta^*} \leq 2 \]

(79)

(80)

Which is again always true. Therefore, for any choice of \( Z \) and \( Z' \) full model always have lower or equal error to the core model on \( (Z, S, Y) \).

As a special case, consider the setup where features that determine \( s \) and \( y \) are disjoint, (i.e., \( \beta^*_i \neq 0 \implies \theta^*_i = 0 \) and \( \theta^*_i \neq 0 \implies \beta^*_i = 0 \)). First of all, note that it is still possible that \( \theta^* \) and \( \beta^* \) have a large correlation on the observed direction (i.e., projected on \( \Pi \)), which results in \( w \neq 0 \). In this case, if we assume the covariance matrix at the test time is identity, then removing \( s \) always helps.

**Corollary 3.** If true features that determine \( s \) and \( y \) are disjoint, \( \beta^*_i \neq 0 \implies \theta^*_i = 0 \) and \( \theta^*_i \neq 0 \implies \beta^*_i = 0 \), and the test time covariance matrix \( \Sigma = I \) then removing \( s \) always reduce the error.

**Proof.**

\[ \beta^* \Pi \theta^* = \beta^* \Pi \theta^* + \beta^* (I - \Pi) \theta^* = 0 \]

(81)

WLOG, we can assume \( \beta^* \Pi \theta^* > 0 \). Using (??) and the assumption that \( \Sigma = I \), we know that removing \( s \) increases the error if \( \beta^* (I - \Pi) \theta^* > 0 \) which is impossible since we assume \( \beta^* \Pi \theta^* = 0 \).
Proposition 2. If $\|z\| \leq \gamma$ and consequently the set of different values for the spurious feature is $S = [-\gamma \|\beta^\star\|_\star, \gamma \|\beta^\star\|_\star]$ then:

$$\text{RobustError}(M^\star) \leq \text{RobustError}(M'^\star)$$  \hfill (11)

Proof. We will show that for each data point $(z, s)$ we can shift $s$ in its perturbation set to $s'$ such that $M'^\star(z, s') = M^\star(z, s)$, which implies $\text{RobustError}(M'^\star) \geq \text{Error}(M^\star) = \text{RobustError}(M^\star)$. Define $s' = (I - \Pi) \beta^\star \top z$

$$M'^\star(z, s') = (\Pi \theta^\star) \top z - \hat{w}((I - \Pi) \beta^\star) \top z + ws'$$  \hfill (82)

$$= (\Pi \theta^\star) \top z - \hat{w}((I - \Pi) \beta^\star) \top z + \hat{w}((I - \Pi) \beta^\star) \top z$$  \hfill (83)

$$= (\Pi \theta^\star) \top z$$  \hfill (84)

$$= M^\star(z, s)$$  \hfill (85)

Now it suffices to prove $s' \in S$.

$$|s'| = |((I - \Pi) \beta^\star) \top z|$$  \hfill (86)

$$= |\beta^\star \top (I - \Pi) z|$$  \hfill (87)

$$\leq \gamma \|\beta^\star\|_\star$$  \hfill (88)

where in the last inequality we used the fact that $(I - \Pi)$ is a projection matrix, and multiplying a vector with a projection matrix results in a vector with a smaller norm.  \hfill \Box

Proposition 3. The weight of the $i^{th}$ spurious feature of the minimum norm estimator is

$$\hat{w}_i = \frac{\theta^\star \top \Pi \beta^i - \sum_{j \neq i} \hat{w}_j \beta^i \top \Pi \beta^j}{1 + \beta^i \top \Pi \beta^i} \top z$$  \hfill (13)

Proof. Similar to (24) we have:

$$\hat{\theta}^\star = \Pi \theta^\star - \Pi \beta^\star \hat{w}$$  \hfill (89)

Similar to (27) we can compute the optimum value for $\hat{w}$ as follows:

$$\frac{\partial \|\hat{\theta}^\star\| + \partial \sum_j w_j^2}{\partial w_i} = \frac{\partial}{\partial w_i} \left( \sum_{j \neq i} w_i w_j \beta^i \top \Pi \beta^j - 2 w_i \theta^\star \top \Pi \beta^i \right) + 2 w_i$$  \hfill (90)

$$= \sum_{j \neq i} 2 w_j \beta^i \top \Pi \beta^j + 2 w_i \beta^i \top \Pi \beta^i - 2 \theta^\star \top \Pi \beta^i + 2 w_i$$  \hfill (91)
Therefore, the optimum value for \( w_i \) is:

\[
w_i = \frac{\theta^* \beta_i^* - \sum_{j \neq i} w_j \beta_j^* \Pi \beta_i^*}{1 + \beta_i^* \beta_i^*}
\]  

(92)

**Proposition 4.** The optimum parameters for (15) are:

\[
\hat{w} = \frac{\beta^* \Pi \theta^*}{1 + \beta^* \Pi \beta^*}
\]  

(16)

\[
\hat{\theta}^*_{RST} = \Pi \theta^* + \hat{w} (I - \Pi) \beta^*
\]  

(17)

and for any data point \((z, s)\), we have:

\[
M^s_{RST}(z, 0) = M^s(z, s)
\]  

(18)

**Proof.** We show how to derive the estimated parameters for the core model + RST as introduced in Section 4. Recall that we are interested in the following optimization problem

\[
\hat{\theta}^*_{RST} = \arg \min_{\theta} \|\theta\|_2^2
\]  

(93)

s.t

\[
Z \theta = Y
\]  

(94)

\[
Z_u \theta = Z_u \hat{\theta}^* + S_u \hat{w}
\]  

(95)

Substituting \( \hat{\theta}^* \), \( Y \), and \( S_u \) in terms of \( \beta^* \) and \( \theta^* \) we have:

\[
\hat{\theta}^*_{RST} = \arg \min_{\theta} \|\theta\|_2^2
\]  

(96)

s.t

\[
Z \theta = Z \theta^*
\]  

(97)

\[
Z_u \theta = Z_u (\Pi \theta^* - \hat{w} \beta^*) + Z_u \beta^* \hat{w} = Z_u (\Pi \theta^* - \hat{w} (I - \Pi) \beta^*)
\]  

(98)

as explained in (27), we have: \( \hat{w} = \frac{\beta^* \Pi \theta^*}{1 + \beta^* \Pi \beta^*} \). Since we assumed we have \( m > d \) unlabeled examples then solving (98) results in

\[
\hat{\theta}^*_{RST} = \Pi \theta^* + \hat{w} (I - \Pi) \beta^*
\]  

(99)
We now prove $M_{\text{RST}}^3$ makes the same predictions as $M^*$. 

$$M_{\text{RST}}^3(z, 0) = (\Pi \theta^* + \hat{\omega}(I - \Pi)\beta^*)^\top z = (\Pi \theta^* - \hat{\omega} \Pi \beta^*)^\top z + \hat{\omega} \beta^*^\top z = (\Pi \theta^* - \hat{\omega} \Pi \beta^*)^\top z + w s = M^*(z, s)$$ (100)