Order Independence in Asynchronous Cellular Automata

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Sequential Dynamical Systems – Definitions

► An SDS is a triple consisting of:

- A graph $Y$ with vertex set $\{1, 2, \ldots, n\}$.
- A vertex function $f_i$ such that $f_i(y_i) = y_{i-1}$.
- A word $w$ of length $m$ over $v[Y] = \{1, 2, \ldots, n\}$.

The SDS map generated by the triple $(Y, (F_i)_1^n, w)$ is

$$[[\delta_Y, w] = F_w(m) \circ F_w(m-1) \circ \cdots \circ F_w(1).$$
Let \( Y = \text{Circ}_n \), the circular graph on \( n \) vertices.

If \( k = a_7 a_6 a_5 a_4 a_3 a_2 a_1 a_0 \) in binary, then \textit{Wolfram rule} \( k \) is defined by \( \text{wolf}^{(k)} : (y_{i-1}, y_i, y_{i+1}) \mapsto z_i \) by the following table.

| \( y_{i-1} y_i y_{i+1} \) | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|-------------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| \( z_i \)              | \( a_7 \) | \( a_6 \) | \( a_5 \) | \( a_4 \) | \( a_3 \) | \( a_2 \) | \( a_1 \) | \( a_0 \) |

Let \( \text{Wolf}^{(k)} : \mathbb{F}_2^n \to \mathbb{F}_2^n \) be the corresponding local function, and \( \mathcal{W}\text{olf}_n^{(k)} = (\text{Wolf}^{(k)}) \) the sequence of local functions of \( \text{Circ}_n \).

The SDS map \( [\mathcal{W}\text{olf}_n^{(k)}, \pi] \), where \( \pi \in S_Y \), is an \textit{asynchronous cellular automata} \textsc{(ACA)}. 
Tags of Wolfram rules

We can arrange the binary digits of $k$ in the following table.

There are 4 possibilities for each pair:

- ‘1’ = \[
\begin{array}{c}
1 \\
1
\end{array}
\]
- ‘0’ = \[
\begin{array}{c}
0 \\
0
\end{array}
\]
- ‘-’ = \[
\begin{array}{c}
1 \\
0
\end{array}
\]
- ‘x’ = \[
\begin{array}{c}
0 \\
1
\end{array}
\]

The tag of Rule $k$ is $p_4p_3p_2p_1$, where each $p_i \in \{0, 1, -, x\}$.

The substring $p_4p_1$ represents the symmetric part of Rule $k$, and $p_3p_2$ represents the antisymmetric part.
$w$-independence

A sequence $\mathcal{F}_Y$ is $\pi$-independent ($w$-independent) if $\text{Per}[\mathcal{F}_Y, w] = \text{Per}[\mathcal{F}_Y, w']$ for all $w$ and $w'$ in $S_Y$ (fair words in $W_Y$).

**Proposition**

$\mathcal{F}_Y$ is $\pi$-independent iff it is $w$-independent.
Main theorem

Theorem (Hansson, Mortveit, Reidys, 2005)

Of the 16 symmetric Wolfram rules, exactly 11 are w-independent for all $n > 3$.

Theorem (Macauley, McCammond, Mortveit, 2007)

Of the 256 Wolfram rules, exactly 104 are w-independent. More precisely, $\text{Wolf}_n^{(k)}$ is w-independent for all $n > 3$ iff $k \in \{0, 1, 4, 5, 8, 9, 12, 13, 28, 29, 32, 40, 51, 54, 57, 60, 64, 65, 68, 69, 70, 71, 72, 73, 76, 77, 78, 79, 92, 93, 94, 95, 96, 99, 102, 105, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 132, 133, 136, 137, 140, 141, 147, 150, 152, 153, 156, 157, 160, 164, 168, 172, 184, 188, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 204, 205, 206, 207, 216, 218, 220, 221, 222, 223, 224, 226, 228, 230, 232, 234, 235, 236, 237, 238, 239, 248, 249, 250, 251, 252, 253, 254, 255\}.

These 104 rules constitute 41 distinct classes up to equivalence (inversion and reflection).
Major classes of $w$-independent Wolfram rules

- The 104 Wolfram rules fall into one of three categories:
  - Invertible rules
  - Rules of the following form:
    
    \[
    \begin{array}{c|cc|c}
    * & 0 & 0 \\
    0 & * & 0 \\
    0 & 1 & 1 \\
    0 & 0 & 0 \\
    \end{array}
    \]
    
    - One of 6 exceptional cases: 32, 40, 152, 184, 28, 29.

The main technique used in the second case was the use of potential functions.
Disregarding the constant states 0 and 1, the following are the only sets of periodic points that arise up to inversion:

\[ P_{n,1} : \{ \text{No '11', '000'} \}, \]
\[ P_{n,2} : \{ \text{No '11', '010'} \}, \]
\[ P_{n,3} : \{ \text{No '11', '101'} \}, \]
\[ P_{n,4} : \{ \text{No '000', '111', '1100'} \}, \]
\[ P_{n,5} : \{ \text{No '000', '111'} \}, \]
\[ P_{n,6} : \{ \text{No '101', '010'} \}, \]
\[ P_{n,7} : \{ \text{No '11'} \}, \]
\[ P_{n,8} : \{ \text{No '101'} \}, \]
\[ P_{n,9} : \{ \text{No '111'} \} \]

Note that \{0\}, \{1\}, and \{0, 1\} also arise as periodic point sets.
Periodic point poset

- The sets of periodic points form the following poset:
**Definitions**

**Proposition**

*If \( \mathcal{F}_Y \) is \( w \)-independent, then each \( F_i \) is bijective on \( P := \text{Per}(\mathcal{F}_Y) \).*

Let \([\mathcal{F}_Y, \omega]^*\) denote the restriction of \([\mathcal{F}_Y, \omega]\) to the set of periodic points.

If \( W' \subseteq W_Y \) then the group

\[
H(\mathcal{F}_Y, W') = \langle [\mathcal{F}_Y, \omega]^* : \omega \in W' \rangle
\]

is called the *dynamics group* of \( \mathcal{F}_Y \) and \( W' \).

- **Full dynamics group:** \( G(\mathcal{F}_Y) := H(\mathcal{F}_Y, W_Y) = \langle F_i^* : F_i \in \mathcal{F}_Y \rangle \),
- **Permutation dynamics group:** \( H(\mathcal{F}_Y) := H(\mathcal{F}_Y, S_Y) = \langle [\mathcal{F}_Y, \pi]^* : \pi \in S_Y \rangle \).
Computation

- The dynamics group is the homomorphic image of a Coxeter group: $|F_i| \leq 2$ and $|F_i F_j| = m_{ij}$ for $m_{ij} \in \{1, 2, 3, 4, 6, 12\}$.

- Of the 41 non-equivalent rules, only 15 of them have a non-trivial dynamics group.

- $SL(n)$ or $AL(n)$: Rules 1, 9, 110, 126.
- $\mathbb{Z}_2^n$: Rules 28, 29, 51.
- $A_n$ or $A_{n-1}$: Rules 54, 57
- $GL(n,2)$: Rule 60.
- Not sure: Rules 73, 105, 108, 150, 156
Flips

For each of the 8 neighborhood state configurations \((y_{i-1}, y_i, y_{i+1})\), Wolfram rule \(k\) can be thought of as either preserving, or “flipping” the value \(y_i\).

| # flips | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---------|----|----|----|----|----|----|----|----|----|
| # w-independent rules | 1  | 8  | 26 | 34 | 26 | 4  | 4  | 0  | 1  |
| # of rules     | 1  | 8  | 28 | 56 | 70 | 56 | 28 | 8  | 1  |
| Percentage     | 100% | 100% | 93% | 61% | 37% | 7% | 14% | 0% | 100% |

All 5 \(w\)-independent rules with more than 5 flips are invertible.

This can be extended to SDSs. Call 0 \(\mapsto\) 1 and \textit{up-flip} and 1 \(\mapsto\) 0 a \textit{down-flip}. Define the \textit{signature} of \(\mathcal{Y}\) to be the number of up-flips minus the number of down-flips.

The signature is an indication of stability, and a good starting point for the study of update-order stochastic SDSs.
## Table of the 104 rules

| $p_4 p_1$ | $p_3$ | \(-\) | \(-\) | 0 | 0 | - | 1 | 1 | - | x | x |
|-----------|-------|---------|---------|---|---|---|---|---|---|---|---|
| --        | 132   | 204     | 196     | 140| 132| 206| 220| 222| 198| 156| 150|
| 0-        | 4     | 76      | 68      | 12 | 4  | 78 | 92  | 94 | 70 | 28 |
| -0        | 128   | 200     | 192     | 136| 128| 202| 216 | 218| 194| 152|
| 1-        | 164   | 236     | 228     | 172| 164| 238| 252 | 254| 230| 188|
| -1        | 133   | 205     | 197     | 141| 133| 207| 221 | 223| 199| 157|
| 10        | 160   | 232     | 224     | 168| 160| 234| 248 | 250| 226| 184|
| 01        | 5     | 77      | 69      | 13 | 5  | 79 | 93  | 95 | 71 | 29 |
| 00        | 0     | 72      | 64      | 8  | 0  |
| x0        | 32    | 96      | 40      | 32 |
| 0x        | 1     | 73      | 65      | 9  | 1  |
| -x        | 129   | 201     | 193     | 137| 129| 195| 153 | 147|
| x-        | 36    | 108     |         | 110| 124| 126| 102 | 60 | 54 |
| x1        | 37    | 109     |         | 111| 125| 127|
| 1x        | 161   |         |         | 235| 249| 251|
| 11        | 165   | 237     |         | 239| 253| 255|
| xx        | 33    | 105     |         | 99 | 57 | 51 |

Table: The 104 $w$-independent rules arranged by symmetric and asymmetric parts of their tags.
Table of the 104 rules, arranged by $m_{ij}$

| $p_3$ | $p_2$ | $x$ | $-x$ | $-x$ | $-0$ | $0$ | $-1$ | $1$ | $1$ |
|-------|-------|-----|------|------|------|-----|------|-----|-----|
| 1     |       | 6   | 6    | 6    | 6    | 6   | 6    | 6   | 6   |
| 37    |       | 6   | 6    | 6    | 6    | 6   | 6    | 6   | 6   |
| 129   | 12    | 6   | 6    | 6    | 6    | 6   | 6    | 6   | 6   |
| 36    | 12    | 6   | 6    | 6    | 6    | 6   | 6    | 6   | 6   |
| 33    | 2     | 6   | 6    | 6    | 6    | 6   | 6    | 6   | 6   |
| 132   | 3     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 128   | 1     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 164   | 1     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 160   | 1     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 133   | 2     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 4     | 2     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 5     | 2     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 0     | 0     | 1   | 1    | 1    | 1    | 1   | 1    | 1   | 1   |
| 165   | 1     |     | 1    | 1    | 1    |     | 1    | 1   | 1   |
| 161   | 1     |     | 1    | 1    |     | 1    |     | 1   | 1   |
| 32    | 1     |     | 1    |     |     | 1    | 1    |     | 1   |

Table: The 104 $w$-independent rules arranged by $m_{ij}$
Future research

- Finish analyzing the dynamics groups.
- Analyze the other 152 rules.
- Extend these ideas and techniques to general SDSs.
- Use these ideas and techniques to study stochastic systems.
- Compare to the dynamics of classical (synchronous) CAs.
SDS – Collaborators, Papers, Info

**Joint work with:** Jon McCammond, Henning Mortveit

**Preprint:** [http://arxiv.org/abs/0707.2360](http://arxiv.org/abs/0707.2360)

**SDS course web page with link to papers:**

```
Web: [http://www.math.vt.edu/people/hmortvei/class_home/4984_15748.html](http://www.math.vt.edu/people/hmortvei/class_home/4984_15748.html)
```

**NDSSL:**

```
Web: [http://ndssl.vbi.vt.edu](http://ndssl.vbi.vt.edu)
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