PARTIAL DATA INVERSE PROBLEMS FOR THE TIME-HARMONIC MAXWELL EQUATIONS

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ABSTRACT. In this paper we consider an inverse boundary value problem in electromagnetism. We prove that the electromagnetic material parameters of the medium can be uniquely recovered by measuring electric boundary data on a certain part of the boundary and measuring magnetic boundary data roughly on the rest of the boundary. This is an analog of the corresponding result for the partial data Calderón’s inverse conductivity problem due to Kenig, Sjöstrand and Uhlmann.

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1. INTRODUCTION

In the current paper we consider an inverse boundary value problem in electromagnetism. The problem is to recover the electromagnetic material parameters of the medium by making electromagnetic measurements on the boundary. We are interested in the case of the time-harmonic Maxwell equations. We are also considering the case when the measurements are made only on certain parts of the boundary. Let us state more precise mathematical formulation of the problem.

Let $(M, g)$ be a compact 3-dimensional Riemannian manifold with smooth boundary. By $d$ and $*$ we denote the exterior derivative and the Hodge star operator on
We refer the reader to Section 1.1 with the topology defined by the norm $H^m$ pairs $\epsilon$ data inverse problem is to determine where ($\omega > 0$) is a fixed frequency. The complex functions $\mu$ and $\epsilon$ represent the material parameters (permittivity and permeability, respectively). We assume that $\epsilon \in C^3(M)$ and $\mu \in C^3(M)$ have positive real parts in $M$.

Let $i : \partial M \rightarrow M$ be the canonical inclusion. Then we introduce tangential trace of $m$-forms by

$$t : C^\infty \Omega^m(M) \rightarrow C^\infty \Omega^m(\partial M), \quad t(w) = i^*(w), \quad w \in C^\infty \Omega^m(M).$$

We work with following Hilbert space which is the largest domain of $d$ acting on $m$-forms:

$$H_d \Omega^m(M) := \{ w \in L^2 \Omega^m(M) : dw \in L^2 \Omega^{m+1}(M) \}$$

endowed with the inner product

$$(w_1, w_2)_{H_d \Omega^m(M)} := (w_1, w_2)_{L^2 \Omega^m(M)} + (dw_1, dw_2)_{L^2 \Omega^{m+1}(M)}$$

and the corresponding norm $||w||^2_{H_d \Omega^m(M)} := (w, w)_{H_d \Omega^m(M)}$. Then the tangential trace operator has its extensions to bounded operators $t : H_d \Omega^m(M) \rightarrow H^{-1/2} \Omega^m(\partial M)$ and $t : H^1 \Omega^m(M) \rightarrow H^{1/2} \Omega^m(\partial M)$. In fact, $t$ is bounded from $H_d \Omega^m(M)$ into

$$TH_d \Omega^m(\partial M) := \{ t(w) : w \in H_d \Omega^m(M) \}$$

with the topology defined by the norm

$$||f||_{TH_d \Omega^m(\partial M)} := \inf\{ ||w||_{H_d \Omega^m(M)} : t(w) = f, \ w \in H_d \Omega^m(M) \}.$$

We refer the reader to Section 3 for more details.

For open subsets $\Gamma_1, \Gamma_2$ of $\partial M$, we define the Cauchy data set $C^\epsilon,\mu_{\Gamma_1, \Gamma_2}$ to consist of pairs

$$(t(E)|_{\Gamma_1}, t(H)|_{\Gamma_2}) \in TH_d \Omega^1(\partial M)|_{\Gamma_1} \times TH_d \Omega^1(\partial M)|_{\Gamma_2}$$

where $(E, H) \in H_d \Omega^1(M) \times H_d \Omega^1(M)$ solves (1.1) with $\text{supp} \ t(E) \subset \Gamma_1$. The partial data inverse problem is to determine $\epsilon$ and $\mu$ from the knowledge of the Cauchy data set $C^\epsilon,\mu_{\Gamma_1, \Gamma_2}$.

In Appendix A, we show that there is a discrete set $\Sigma$ of frequencies such that for all $\omega \notin \Sigma$, the knowledge of $C^\epsilon,\mu_{\Gamma_1, \Gamma_2}$ is equivalent to the knowledge of the partial admittance map

$$\Lambda^\epsilon,\mu_{\Gamma_1, \Gamma_2} : f \mapsto t(H)|_{\Gamma_2}, \quad f \in TH_d \Omega^1(\partial M), \quad \text{supp} \ f \subset \Gamma_1,$$

where $(E, H) \in H_d \Omega^1(M) \times H_d \Omega^1(M)$ is the unique solution of the system (1.1) with $t(E) = f$.

It was shown by Lassas [19] that this problem can be regarded as a generalization of Calderón’s inverse conductivity problem [3]. More precisely, the latter can be regarded as low-frequency limit of the time-harmonic inverse electromagnetic problem.

Let us now describe previous results. We start with Euclidean setting.
A standard approach to solve this problem is to adopt the method of construction of exponentially growing solution, also known as complex geometrical optics solutions, following the celebrated paper of Sylvester and Uhlmann [29] where they solve Calderón’s inverse conductivity problem. One of the main challenges in adopting the method of [29] is the fact that the system of Maxwell’s equations is not elliptic. In the full data case, Somersalo, Isaacson and Cheney [27] prove uniqueness for the linearized problem at constant material parameters. For the nonlinearized problem, uniqueness was given by Sun and Uhlmann [28] when the coefficients of the Maxwell equations are close to constants. In this paper, to get ellipticity, the Maxwell’s system was reduced to a system with principal part being the Hodge-Laplacian. However, this reduction gives first order terms. For material parameters that are nearly constant, they were able to deal with the first order terms and produce complex geometrical solutions for the Maxwell’s system.

The first global uniqueness result was proven by Ola, Päivärinta and Somersalo [24]. This proof was later simplified by Ola and Somersalo [23]. The important point in the simplified proof is the connection of the Maxwell’s system with a Hodge-Schrödinger equation via certain Hodge Dirac operator, which allowed them to avoid first order terms and construct complex geometrical optics solution for Maxwell’s system. This technique became very popular in subsequent works on various aspects of inverse electromagnetic problem. For $C^1$ coefficients, uniqueness result was given by Caro and Zhou [6].

Much less is known when the boundary data is incomplete. In the work of Caro, Ola and Salo [5], uniqueness result is given when $M$ is a bounded domain in $\mathbb{R}^3$ and $\Gamma_1 = \Gamma_2 = \Gamma \subset \partial M$ provided that the inaccessible part of the boundary for measurements is either part of a hyperplane or part of a sphere. The work is based on reflection approach, following Isakov [15]. There is also a recent uniqueness result with local data due to Brown, Marletta and Reyes [1] when the material parameters are assumed to be known near the boundary.

Now, we describe previous results for non-Euclidean geometries. For this, let us introduce the notion of admissible manifolds.

**Definition.** A compact Riemannian manifold $(M, g)$ with smooth boundary of dimension $n \geq 3$, is said to be admissible if $(M, g) \subset \subset \mathbb{R} \times (M_0, g_0)$, $g = c(e \oplus g_0)$ where $c > 0$ smooth function on $M$, $e$ is the Euclidean metric and $(M_0, g_0)$ is a simple $(n - 1)$-dimensional manifold. We say that a compact manifold $(M_0, g_0)$ with boundary is simple, if $\partial M_0$ is strictly convex, and for any point $x \in M_0$ the exponential map exp$_c$ is a diffeomorphism from its maximal domain in $T_x M_0$ onto $M_0$.

Compact submanifolds of Euclidean space, the sphere minus a point and of hyperbolic space are all examples of admissible manifolds.

The notion of admissible manifolds were introduced by Dos Santos Ferreira, Kenig, Salo and Uhlmann [13] as a class of manifolds admitting the existence of limiting Carleman weights. In fact, the construction of complex geometrical optics solutions
are possible on such manifolds via Carleman estimates approach based on the existence of limiting Carleman weights. Such an approach was introduced by Bukhgeim and Uhlmann [2] and Kenig, Sjöstrand and Uhlmann [17] in the setting of partial data Calderón’s inverse conductivity problem in $\mathbb{R}^n$.

If $(M,g)$ is admissible, points of $M$ can be written as $x = (x_1, x')$, where $x_1$ is the Euclidean coordinate. For the purpose of the paper it is enough to note that the function $\varphi(x) = \pm x_1$ is a natural limiting Carleman weight in $(M,g)$; see [13] for this fact and for the precise definition and for properties of limiting Carleman weights on manifolds.

On admissible manifolds, the uniqueness result for the full-data inverse electromagnetic problem was given by Kenig, Salo and Uhlmann [16]. The results of [16, 23] were extended by Chung, Ola, Salo and Tzou [10] to the case of partial data when $\Gamma_1 = \partial M$ and $\Gamma_2 \subset \partial M$ is a certain open set. They generalize Carleman estimate approach of [2, 17] to Maxwell’s system by reducing the latter to Hodge-Schrödinger-type equation as in [23]. However, this reduction has certain negative parts in the partial data setting. Roughly speaking, appropriate complex geometrical optics solutions for a Hodge-Schrödinger equation were produced using Carleman estimates with boundary terms for the Hodge-Laplacian derived in [9]. Then, in order to relate these solutions to Maxwell’s system, certain Hodge-Dirac operators are applied. This is exactly the step in [10] which causes certain technical difficulties. More precisely, such obtained solutions have only $H^{-1}$ regularity. In order to use these solutions in a certain integral identity, relating boundary measurements and solutions, one needs to integrate them against complex geometrical optics solutions with $H^1$ regularity. The method used in [10] to get complex geometrical optics solutions with $H^1$ regularity does not allow to control boundary behavior of these solutions on $\partial M \setminus \Gamma_1$. Therefore, the authors of [10] have to work with the case when $\Gamma_1$ is the whole boundary.

To state the main result of the paper, let us introduce some notations. For the limiting Carleman weight $\varphi(x) = \pm x_1$, we define

$$
\partial M_{+ \varphi} = \{ x \in \partial M : \partial_\nu \varphi(x) \geq 0 \}, \quad \partial M_{- \varphi} = \{ x \in \partial M : \partial_\nu \varphi(x) \leq 0 \},
$$

where $\nu$ is the unit outer normal to $\partial M$. If $\varphi(x) = x_1$, then we simply write $\partial M_+$ and $\partial M_-.$

In the current paper we improve the result of [10] by assuming that $\Gamma_1$ and $\Gamma_2$ are open neighborhoods of $\partial M_-$ and $\partial M_+$, respectively, in $\partial M$ and the overlap $\Gamma_1 \cap \Gamma_2$ can be arbitrary small. The following theorem is the main result of this paper.

**Theorem 1.1.** Let $(M,g)$ be an admissible manifold of dimension 3 and let $\varphi$ be the limiting Carleman weight $\varphi(x) = x_1$ on $M$. Assume $(\varepsilon_j, \mu_j) \in C^3(M) \times C^2(M)$, $j = 1,2$ are complex valued such that $\text{Re}(\varepsilon_j), \text{Re}(\mu_j) > 0$ in $M$. Suppose that $C_{\Gamma_+, \Gamma_-} = C_{\Gamma_+^0, \Gamma_-^0}$ at fixed frequency $\omega > 0$, for some open neighborhoods $\Gamma_\pm$ of $\partial M_\pm$ in $\partial M$. Then $\varepsilon_1 = \varepsilon_2$ and $\mu_1 = \mu_2$ in $M$.

Throughout the paper we also use the notation $\Gamma_{+ \varphi}$ to denote a neighborhood of $\partial M_{+ \varphi}$ in $\partial M$ mentioned in Theorem 1.1. In other words, we have $\Gamma_{+ \varphi} = \Gamma_+$ if $\varphi(x) = x_1$ and $\Gamma_{+ \varphi} = \Gamma_-$ if $\varphi(x) = -x_1$. 

We also state the following particular case of Theorem 1.1 in $\mathbb{R}^3$. By $\text{ch}(\Omega)$ we denote the convex hull of $\Omega$.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^\infty$ boundary and let $\omega > 0$. Assume $(\varepsilon_j, \mu_j) \in C^3(\Omega) \times C^2(\Omega)$, $j = 1, 2$ are complex valued such that $\Re(\varepsilon_j), \Re(\mu_j) > 0$ in $\Omega$. For a given $x_0 \notin \text{ch}(\Omega)$, define

$$B(x_0) = \{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) \geq 0 \}, \quad F(x_0) = \{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) \leq 0 \}.$$

Suppose that $C^{\varepsilon_1, \mu_1}_{B,F} = C^{\varepsilon_2, \mu_2}_{B,F}$ for some open neighborhoods $B$ and $F$ of $B(x_0)$ and $F(x_0)$, respectively, in $\partial \Omega$. Then $\varepsilon_1 = \varepsilon_2$ and $\mu_1 = \mu_2$ in $\Omega$.

This is an analog of the corresponding result for the partial data Calderón’s inverse conductivity problem of Kenig, Sjöstrand and Uhlmann [17]. Theorem 1.2 can be obtained from Theorem 1.1 using a logarithmic limiting Carleman weight and appropriate change of coordinates as in [10].

In the current work, instead of reducing to a Hodge-Schrödinger equation, following [28], we reduce the Maxwell equations to a system with principal part being the Hodge-Laplacian. Then complex geometrical optics solutions for the reduced system are essentially solutions for the Maxwell’s system. Moreover, using this reduction gives an integral identity whose relation to Maxwell’s equation as well as to its reduced system is more transparent, in contrast to [10]. The latter relation is important in avoiding the loss of regularity of constructed complex geometrical optics solutions.

To construct suitable complex geometrical optics solutions, one needs to derive different Carleman estimate than the one used in [10]. We adopt the idea of Chung [7] to get a Carleman estimate for the Hodge-Laplacian controlling value of 1-forms on an appropriate subset of the boundary. This estimate is also useful in dealing with first order terms in the above mentioned reduced system by getting solutions with sufficient regularity as in the case of magnetic Schrödinger operators; see [7, 13].

The paper is organized as follows. In Section 2 we briefly present basic facts on differential forms and trace operators. Then the trace operators are extended to $H_\delta \Omega^m(M)$ and to the closely related space $H_\delta \Omega^m(M)$. This is discussed in Section 3 where we also study some other important properties of $H_\delta \Omega^m(M)$ and $H_\delta \Omega^m(M)$. Section 4 contains the reduction of the Maxwell equations to a system whose principal part is the Hodge-Laplacian. In Section 5, we derive a local Carleman estimate for Laplace-Beltrami operator, acting on functions, which allows us to control the information about the behavior of the solutions on the boundary. Then in Section 6, we use partition of unity to glue these local Carleman estimate for functions to get a global Carleman estimate for 1-forms. Then we give the construction of the complex geometrical optics solutions for the system to which the Maxwell equations were reduced in Section 4 and then relate these solutions to the Maxwell equations. This is the context of Section 7. We use these solutions to prove Theorem 1.1 in Section 8. Appendix A is devoted to the well-posedness of the Maxwell equations and the corresponding admittance map. In this section, we also solve the eigenvalue
problem for the homogeneous Maxwell equations with homogeneous boundary condition. Finally, Appendix B and Appendix C contain the proofs of some technical results used in Section 5.

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2. Preliminaries

In this section we briefly present basic facts on differential forms and trace operators. For more detailed exposition we refer the reader to the manuscript of Schwarz [26]. Let \((M, g)\) be a compact oriented \(n\)-dimensional Riemannian manifold with smooth boundary. The inner product of tangent vectors with respect to the metric \(g\) is denoted by \(\langle \cdot, \cdot \rangle_g\), and \(| \cdot |_g\) is the notation for the corresponding norm. By \(|g|\) we denote the determinant of \(g = (g_{ij})\) and \((g^{ij})\) is the inverse matrix of \((g_{ij})\). Finally, there is the induced metric \(\iota^*g\) on \(\partial M\) which gives a rise to the inner product \(\langle \cdot, \cdot \rangle_{\iota^*g}\) of vectors tangent to \(\partial M\).

2.1. Basic notations for differential forms. In what follows, for \(F\) some function space \((C^k, L^p, H^k, \text{etc.})\), we denote by \(F\Omega^m(M)\) the corresponding space of \(m\)-forms. In particular, the space of smooth \(m\)-forms is denoted by \(C^\infty\Omega^m(M)\). Let \(* : C^\infty\Omega^m(M) \to C^\infty\Omega^{n-m}(M)\) be the Hodge star operator. For real valued \(\eta, \zeta \in C^\infty\Omega^m(M)\), the inner product with respect to \(g\) is defined in local coordinates as

\[
\langle \eta, \zeta \rangle_g = (\eta \wedge \ast \zeta) = g^{i_1j_1} \cdots g^{i_mj_m} \eta_{i_1 \cdots i_m} \zeta_{j_1 \cdots j_m}.
\]

This can be extended as a bilinear form on complex valued forms. We also write \(|\eta|^2_g = \langle \eta, \overline{\eta} \rangle_g\). The inner product on \(L^2\Omega^m(M)\) is defined as

\[
\langle \eta, \zeta \rangle_{L^2\Omega^m(M)} = \int_M \langle \eta, \zeta \rangle_g \, d\text{Vol}_g = \int_M \eta \wedge \ast \overline{\zeta}, \quad \eta, \zeta \in L^2\Omega^m(M),
\]

where \(d\text{Vol}_g = \ast 1 = |g|^{1/2} dx^1 \wedge \cdots \wedge dx^n\) is the volume form. The corresponding norm is \(|| \cdot ||_{L^2\Omega^m(M)} = (\cdot, \cdot)_{L^2\Omega^m(M)}\). Using the definition of the Hodge star operator \(*\), it is not difficult to check that

\[
\langle \eta, \zeta \rangle_{L^2\Omega^m(M)} = (\ast \eta, \ast \zeta)_{L^2\Omega^{n-m}(M)}.
\]

Let \(d : C^\infty\Omega^m(M) \to C^\infty\Omega^{m+1}(M)\) be the external differential. Then the codifferential \(\delta : C^\infty\Omega^m(M) \to C^\infty\Omega^{m-1}(M)\) is defined as

\[
\langle d\eta, \zeta \rangle_{L^2\Omega^m(M)} = \langle \eta, \delta \zeta \rangle_{L^2\Omega^{m-1}(M)}
\]

for all \(\eta \in C^\infty_0\Omega^{m-1}(M)\), \(\zeta \in C^\infty\Omega^m(M)\). The Hodge star operator \(*\) and the codifferential \(\delta\) have the following properties when acting on \(C^\infty\Omega^m(M)\):

\[
*^2 = (-1)^{m(n-m)}, \quad \delta = (-1)^{m(n-m)-n+m-1} \ast (d \ast)\,.
\]

\[\text{Similarly, } \ast^2 \, \ast = (-1)^{m(n-m)-n+m-1} \ast (d \ast \ast)\,.
\]

\[\ast (\ast \delta \eta) = \delta (\ast \eta)\,.
\]

\[\ast (d \delta \eta) = \delta (d \ast \eta)\,.
\]
For a given $\xi \in C^\infty\Omega^1(M)$, the interior product $i_\xi : C^\infty\Omega^m(M) \to C^\infty\Omega^{m-1}(M)$ is the contraction of differential forms by $\xi$. In local coordinates,

$$i_\xi \eta = g^{ij} \xi_i \eta_{j;i...i_{m-1}}, \quad \eta \in C^\infty\Omega^m(M).$$

The interior product acts on exterior products in the following way

$$i_\xi (\eta \wedge \zeta) = i_\xi \eta \wedge \zeta + (-1)^m \eta \wedge i_\xi \zeta, \quad \eta, \zeta \in C^\infty\Omega^m(M), \quad \zeta \in C^\infty\Omega^k(M). \tag{2.3}$$

It is the formal adjoint of $\xi$, in the inner product $\langle \cdot, \cdot \rangle_g$ on real valued forms, and has the following expression

$$i_\xi \eta = (-1)^{n(m-1)} * (\xi \wedge * \eta), \quad \eta \in C^\infty\Omega^m(M). \tag{2.4}$$

Using this, one can also show that

$$\delta (f \omega) = f \delta \omega - i_\xi df \omega, \quad f \in C^\infty(M), \quad \omega \in C^\infty\Omega^m(M). \tag{2.5}$$

The Hodge Laplacian acting on $\Omega^m(M)$ is defined by $-\Delta = d\delta + \delta d$. Finally, the inner product on $L^2\Omega^m(\partial M)$ is given by

$$(u|v)_{L^2\Omega^m(\partial M)} = \int_{\partial M} (u, v)_{\ast \ast \ast} \, d\sigma_{\partial M}, \quad u, v \in L^2\Omega^m(\partial M),$$

where $(\cdot, \cdot)_{\ast \ast \ast}$ is extended as a bilinear form on complex forms on $\partial M$, and $d\sigma_{\partial M} = v^\ast (i_{\nu} d\text{Vol}_g)$ is the volume form on $\partial M$ induced by $d\text{Vol}_g$.

2.2. The normal and parallel parts of differential forms. The outward unit normal $\nu$ to $\partial M$ can be extended to a vector field near $\partial M$ by parallel transport along normal geodesics (initiating from $\partial M$ in the direction of $-\nu$), and then to a vector field on $M$ via a cutoff function. For $w \in C^\infty\Omega^m(M)$, we introduce

$$\eta_\perp = \nu \wedge i_\nu \eta, \quad \eta_\parallel = \eta - \eta_\perp.$$

Using (2.3), one can see that $i_\nu \eta_\perp = i_\nu \eta$, so $i_\nu \eta_\parallel = 0$. Since $t(\nu) = 0$, we also have $t(\eta_\perp) = 0$, so $t(\eta) = t(\eta_\parallel)$. It is clear that $\nu \wedge \eta_\perp = 0$.

2.3. Integration by parts. Let us first prove the following simple result which will be used in formulating integration by parts formula in appropriate way.

Lemma 2.1. If $\xi \in C^\infty\Omega^m(M)$ and $\zeta \in C^\infty\Omega^{m+1}(M)$, then for an open subset $\Gamma \subset \partial M$ the following holds

$$(t(\eta) | t(i_\nu \zeta))_{L^2\Omega^m(\Gamma)} = \int_{\Gamma} t(\eta \wedge * \zeta).$$

Proof. First, we show that $(\langle \eta, i_\nu \zeta \rangle_g)_{d\sigma_{\partial M}} = t(\eta \wedge * \zeta)$. Since $(\nu \wedge \eta, \zeta)_g = \langle \eta, i_\nu \zeta \rangle_g$, we have

$$(\langle \eta, i_\nu \zeta \rangle_g)_{d\sigma_{\partial M}} = (\nu \wedge \eta, \zeta)_g d\sigma_{\partial M} = (\nu \wedge \eta, \zeta)_g t(i_\nu d\text{Vol}_g) = t(i_\nu((\nu \wedge \eta) \wedge * \zeta)).$$

Using (2.3) and $t(\nu) = 0$, this gives

$$(\langle \eta, i_\nu \zeta \rangle_g)_{d\sigma_{\partial M}} = t(\eta \wedge * \zeta) - t(\nu) \wedge t(i_\nu(\eta \wedge * \zeta)) = t(\eta \wedge * \zeta).$$

Next, we show that $(\eta, i_\nu \zeta)_g = (t(\eta), t(i_\nu \zeta))_{\ast \ast \ast}$ on $\partial M$. Indeed, observe that $(i_\nu \zeta)_\perp = 0$. Therefore, $i_\nu \zeta = (i_\nu \zeta)_\parallel$ and hence on $\partial M$ we get

$$(\eta, i_\nu \zeta)_g = (\eta, (i_\nu \zeta)_\parallel)_g = (\eta_\parallel, (i_\nu \zeta)_\parallel)_g = (t(\eta), t(i_\nu \zeta))_{\ast \ast \ast} = (t(\eta), t(i_\nu \zeta))_{\ast \ast \ast}.$$
Collecting all these, we get \((t(\eta), t(\nu, \zeta))|_{\nu, \zeta} \, \sigma_{\partial M} = t(\eta \wedge \nu, \zeta)\). Finally, integrating over \(\Gamma \subset \partial M\) we get the result.

For \(\eta \in C^\infty \Omega^m(M)\) and \(\zeta \in C^\infty \Omega^{m+1}(M)\), using Stokes’ theorem, Lemma 2.1 (with \(\Gamma = \partial M\) and (2.2), we have the following integration by parts formula for \(d\) and \(\delta\)

\[
(t(\eta)|t(\nu, \zeta)\rangle_{L^2 \Omega^m(\partial M)} = |(d\eta|\zeta\rangle_{L^2 \Omega^{m+1}(M)} - (\eta|\delta\zeta\rangle_{L^2 \Omega^m(M)}.
\]

(2.6)

2.4. Extensions of trace operators. The tangential trace operator \(t\) has an extension to a bounded operator from \(H^1 \Omega^m(M)\) to \(H^{1/2} \Omega^m(\partial M)\). Moreover, for every \(f \in H^{1/2} \Omega^m(\partial M)\), there is \(u \in H^1 \Omega^m(M)\) such that \(t(u) = f\) and

\[
\|u\|_{H^1 \Omega^m(M)} \leq C\|f\|_{H^{1/2} \Omega^m(\partial M)};
\]

see [26, Theorem 1.3.7] and comments.

Next, the operator \(t(\nu, \cdot)\) is bounded from \(H^1 \Omega^m(M)\) to \(H^{1/2} \Omega^{m-1}(\partial M)\). Moreover, for every \(h \in H^{1/2} \Omega^{m-1}(\partial M)\), there is \(\zeta \in H^1 \Omega^m(M)\) such that \(t(\nu, \zeta) = h\) and

\[
\|\zeta\|_{H^1 \Omega^m(M)} \leq C\|h\|_{H^{1/2} \Omega^{m-1}(\partial M)}.
\]

In fact, we can take \(\zeta = \nu \wedge w\), where \(w \in H^1 \Omega^{m-1}(M)\) such that \(t(w) = h\) and

\[
\|w\|_{H^1 \Omega^{m-1}(M)} \leq C\|h\|_{H^{1/2} \Omega^{m-1}(\partial M)}.
\]

Finally, if \(f \in H^{1/2} \Omega^m(\partial M)\) and \(h \in H^{1/2} \Omega^{m-1}(\partial M)\), there is \(\xi \in H^1 \Omega^m(M)\) such that \(t(\xi) = f\), \(t(\nu, \xi) = h\) and

\[
\|\xi\|_{H^1 \Omega^m(M)} \leq C\|f\|_{H^{1/2} \Omega^m(\partial M)} + C\|h\|_{H^{1/2} \Omega^{m-1}(\partial M)}.
\]

This time, we can take \(\xi = u_\parallel + \zeta_\perp\), where \(u \in H^1 \Omega^m(M)\) such that \(t(u) = f\) and \(\|u\|_{H^1 \Omega^m(M)} \leq C\|f\|_{H^{1/2} \Omega^m(\partial M)}\) and \(\zeta \in H^1 \Omega^m(M)\) such that \(t(\nu, \zeta) = h\) and \(\|\zeta\|_{H^1 \Omega^m(M)} \leq C\|h\|_{H^{1/2} \Omega^{m-1}(\partial M)}\).

3. Properties of \(H_\delta \Omega^m(M)\) and \(H_\delta \Omega^m(M)\) spaces

Let \((M, g)\) be a compact oriented \(n\)-dimensional Riemannian manifold with smooth boundary. In this paper we work with the Hilbert spaces \(H_\delta \Omega^m(M)\) and \(H_\delta \Omega^m(M)\) which are the largest domains of \(d\) and \(\delta\), respectively, acting on \(m\)-forms:

\[
H_\delta \Omega^m(M) := \{w \in L^2 \Omega^m(M) : dw \in L^2 \Omega^{m+1}(M)\},
\]

\[
H_\delta \Omega^m(M) := \{u \in L^2 \Omega^m(M) : \delta u \in L^2 \Omega^{m-1}(M)\}
\]

endowed with the inner products

\[
(w_1, w_2)_{H_\delta \Omega^m(M)} := (w_1, w_2)_{L^2 \Omega^m(M)} + (dw_1, dw_2)_{L^2 \Omega^{m+1}(M)},
\]

\[
(u_1, u_2)_{H_\delta \Omega^m(M)} := (u_1, u_2)_{L^2 \Omega^m(M)} + (\delta u_1, \delta u_2)_{L^2 \Omega^{m-1}(M)}
\]

and the corresponding norms

\[
\|w\|^2_{H_\delta \Omega^m(M)} := (w, w)_{H_\delta \Omega^m(M)}, \quad \|u\|^2_{H_\delta \Omega^m(M)} := (u, u)_{H_\delta \Omega^m(M)}.
\]

In the present section we prove some important properties of these spaces.
3.1. **Trace operators.** In this subsection we show that there are bounded extensions \( t : H_0\Omega^m(M) \to H^{-1/2}\Omega^m(\partial M) \) and \( t(i, \cdot) : H_0\Omega^{m+1}(M) \to H^{-1/2}\Omega^m(\partial M) \).

Let \( (\cdot | \cdot)_{\partial M} \) be the distributional duality on \( \partial M \) naturally extending \( (| \cdot )_{L^2\Omega^m(\partial M)} \)

**Proposition 3.1.** (a) The operator \( t : H_0\Omega^m(M) \to H^{1/2}\Omega^m(\partial M) \) has its extension to a bounded operator \( t : H_0\Omega^m(M) \to H^{-1/2}\Omega^m(\partial M) \) and the following integration by parts formula holds

\[
(t(\eta) | t(i, \zeta))_{\partial M} = (d\eta | \zeta)_{L^2\Omega^{m+1}(M)} - (\eta | \delta \zeta)_{L^2\Omega^m(M)}
\]

for all \( \eta \in H_0\Omega^m(M) \) and \( \zeta \in H^1\Omega^{m+1}(M) \).

(b) The operator \( t(i, \cdot) : H_0\Omega^{m+1}(M) \to H^{1/2}\Omega^m(\partial M) \) has its extension to a bounded operator \( t(i, \cdot) : H_0\Omega^{m+1}(M) \to H^{-1/2}\Omega^m(\partial M) \) and the following integration by parts formula holds

\[
(t(i, \zeta) | t(\eta))_{\partial M} = (\zeta | d\eta)_{L^2\Omega^{m+1}(M)} - (\delta \zeta | \eta)_{L^2\Omega^m(M)}
\]

for all \( \zeta \in H_0\Omega^{m+1}(M) \) and \( \eta \in H^1\Omega^m(M) \).

Now we introduce the following space on the boundary \( \partial M \)

\[
TH_0\Omega^m(\partial M) := \{ t(w) : w \in H_0\Omega^m(M) \},
\]

\[
TH_0\Omega^m(\partial M) := \{ t(i, u) : u \in H_0\Omega^m(M) \}
\]

endowed with the norms

\[
\| f \|_{TH_0\Omega^m(\partial M)} := \inf \{ \| w \|_{H_0\Omega^m(M)} : t(w) = f, w \in H_0\Omega^m(M) \},
\]

\[
\| h \|_{TH_0\Omega^m(\partial M)} := \inf \{ \| u \|_{H_0\Omega^m(M)} : t(u) = h, u \in H_0\Omega^m(M) \}.
\]

Then Proposition 3.1 implies that the operators \( t : H_0\Omega^m(M) \to TH_0\Omega^m(\partial M) \) and \( t(i, \cdot) : H_0\Omega^{m+1}(M) \to TH_0\Omega^m(\partial M) \) are bounded under these topologies.

**Proof of Proposition 3.1.** Let us first prove part (a). Let \( w \in C^\infty \Omega^m(M) \) and \( f \in H^{1/2}\Omega^m(\partial M) \). Then using integration parts formula (2.6), we have

\[
(t(w) | f)_{L^2\Omega^m(\partial M)} = (t(w) | t(i, \zeta))_{L^2\Omega^m(\partial M)}
\]

\[
= (dw | \zeta)_{L^2\Omega^{m+1}(M)} - (w | \delta \zeta)_{L^2\Omega^m(M)},
\]

where \( \zeta \in H^1\Omega^{m+1}(M) \) such that \( t(i, \zeta) = f \) and \( \| \zeta \|_{H^1\Omega^{m+1}(M)} \leq C \| f \|_{H^{1/2}\Omega^m(\partial M)} \).

Then

\[
| (t(w) | f)_{L^2\Omega^m(\partial M)} | \leq C \| w \|_{H_0\Omega^m(M)} \| \zeta \|_{H^1\Omega^{m+1}(M)} \leq C \| w \|_{H_0\Omega^m(M)} \| f \|_{H^{1/2}\Omega^m(\partial M)}.
\]

Therefore, \( t \) can be extended to a bounded operator \( H_0\Omega^m(M) \to H^{-1/2}\Omega^m(\partial M) \).

In fact, if \( \eta \in H_0\Omega^m(M) \), then we define \( t(\eta) \) as

\[
(t(\eta) | t(i, \zeta))_{\partial M} = (d\eta | \zeta)_{L^2\Omega^{m+1}(M)} - (\eta | \delta \zeta)_{L^2\Omega^m(M)},
\]

where \( \zeta \in H^1\Omega^{m+1}(M) \).

Now we prove part (b). Let \( w \in C^\infty \Omega^{m+1}(M) \) and \( f \in H^{1/2}\Omega^m(\partial M) \). Then using integration parts formula (2.6), we have

\[
(t(i, w) | f)_{L^2\Omega^m(\partial M)} = (t(i, w) | t(u))_{L^2\Omega^m(\partial M)}
\]

\[
= (wu | \zeta)_{L^2\Omega^{m+1}(M)} - (\delta w | u)_{L^2\Omega^m(M)},
\]
where \( u \in H^1 \Omega^m(M) \) such that \( \mathbf{t}(u) = f \) and \( \|u\|_{H^1 \Omega^m(M)} \leq C \|f\|_{H^{1/2} \Omega^m(\partial M)}. \)

Therefore, we can estimate
\[
|\langle \mathbf{t}(i_\nu w), f \rangle_{L^2 \Omega^m(\partial M)}| \leq C \|w\|_{H^1 \Omega^{m-1}(M)} \|\zeta\|_{H^1 \Omega^m(M)} \leq C \|w\|_{H^1 \Omega^{m+1}(M)} \|f\|_{H^{1/2} \Omega^m(\partial M)}.
\]

Thus, \( \mathbf{t}(i_\nu \cdot) \) can be extended to a bounded operator \( H_\Omega \Omega^{m+1}(M) \rightarrow H^{-1/2} \Omega^m(\partial M). \)

In fact, if \( \zeta \in H_\Omega \Omega^{m+1}(M) \) we define \( \mathbf{t}(i_\nu \zeta) \) as
\[
(\mathbf{t}(i_\nu \zeta) \mathbf{t}(\eta))_{\partial M} = (\zeta \mathbf{d}\eta)_{L^2 \Omega^{m+1}(M)} - (\delta \zeta | \eta)_{L^2 \Omega^m(M)},
\]
where \( \eta \in H^1 \Omega^m(M). \)

### 3.2. Embedding Property

We wish to use Lemma 3.2, and hence we need to show that \( f \), \( v \) and \( h \) satisfy the hypothesis of Lemma 3.3. Obviously, we have \( dw = 0 \) and \( \delta v = 0 \). Integrating by parts and using that \( \mathbf{t}(u) = \mathbf{t}(h) \), we can show that for all \( \chi \in H^1_\Omega \Omega^{m+1}(M) \)
\[
(\mathbf{t}(i_\nu \chi) \mathbf{t}(\eta))_{\partial M} = (u|\chi)_{L^2 \Omega^{m+1}(M)} = (\mathbf{t}(h)|\mathbf{t}(i_\nu \chi))_{L^2 \Omega^m(\partial M)}.
\]
Similary for all $\lambda \in H^1_D(\Omega)$, using the integration by parts formula in part (b) of Proposition 3.1, we can show that
\[(v|\lambda)_{L^2(\Omega)} = (\delta u|\lambda)_{L^2(\Omega)} = -(t(u)v)|_{\partial M} = 0.\]
Next, we show that $t(w) = t(dh)$. For arbitrary $\varphi \in H^{1/2}(\Omega)$, as discussed in Section 2.4, there is $\zeta \in H^1(\Omega)$ such that $t(\varphi \zeta) = \varphi$. Then, using integration by parts formulas in Proposition 3.1, we get
\[(t(w)|\varphi)_{\partial M} = (t(du)|t(\nu \zeta))_{\partial M} = (du|\delta \zeta)_{L^2(\Omega)} = -(t(u)|t(\nu \delta \zeta))_{\partial M}.\]
Since $t(u) = t(h)$, using integration by parts formulas in Proposition 3.1, gives
\[(t(w)|\varphi)_{\partial M} = -(t(h)|t(\nu \delta \zeta))_{\partial M} = -(dh|\delta \zeta)_{L^2(\Omega)} = (t(dh)|\varphi)_{\partial M},\]
which implies $t(w) = t(dh)$. Therefore, applying Lemma 3.3 we find $\psi \in H^1(\Omega)$ such that $d\psi = w$, $\delta \psi = v$ and $t(\psi) = t(h) = t(u)$ and satisfying
\[\|\psi\|_{H^1(\Omega)} \leq C\left(\|w\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right) + C\left(\|t(u)\|_{H^{1/2}(\partial M)} + \|t(h)\|_{H^{1/2}(\partial M)}\right).\]
Using boundedness of $t : H^{1/2}(\Omega) \rightarrow H^{1/2}(\partial M)$ and (3.1),
\[\|t(h)\|_{H^{1/2}(\partial M)} \leq C\|h\|_{H^1(\Omega)} \leq C\|t(u)\|_{H^{1/2}(\partial M)}.\]
Therefore, $\psi$ satisfies the estimate
\[\|\psi\|_{H^1(\Omega)} \leq C\left(\|w\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + \|t(u)\|_{H^{1/2}(\partial M)}\right).\]
Write $\rho = u - \psi$, then $d\rho = 0$ and $\delta \rho = 0$. Therefore, $\rho$ solves $-\Delta \rho = 0$ with $t(\rho) = 0$, $t(\delta \rho) = 0$. By [26, Theorem 2.2.4], it follows that $\rho = 0$. This clearly implies the result.

3.3. Density properties. In this subsection we prove the following two results regarding the density of $C^\infty(\Omega)$ in both $H^1_D(\Omega)$ and $H^1_D(\Omega)$.

Proposition 3.4. The space $C^\infty(\Omega)$ is dense in $H^1_D(\Omega)$.

Proof. The statement is equivalent to showing that if $u \in H^1_D(\Omega)$ is orthogonal to $C^\infty(\Omega)$ in $H^1_D(\Omega)$-inner product, then $u = 0$. Suppose that
\[(u|\phi)_{H^1_D(\Omega)} = (u|\phi)_{L^2(\Omega)} + (\delta u|\delta \phi)_{L^2(\Omega)} = 0, \quad \phi \in C^\infty(\Omega).\]

Let $M$ be a compact manifold with smooth boundary such that $M \subset \subset \tilde{M}^{\text{int}}$ and let by $g$ on $M$ we denote a smooth extension of $g$ from $M$ to $\tilde{M}$. Let $\tilde{u}$ and $\delta \tilde{u}$ denote the extensions of $u$ and $\delta u$ to $\tilde{M}$ by zero. It is clear that $\tilde{u} \in L^2(\tilde{M})$ and $\delta \tilde{u} \in L^2(\tilde{M})$. By (3.2), $\tilde{u}$ and $\delta \tilde{u}$ satisfy
\[(\tilde{u}|\phi)_{L^2(\tilde{M})} + (\delta \tilde{u}|\delta \phi)_{L^2(\tilde{M})} = 0, \quad \phi \in C^\infty(\tilde{M}^{\text{int}}).\]

This in particular implies that $\tilde{u} = -\delta \tilde{u}u$. Since $\tilde{u} \in L^2(\tilde{M})$, we have $\delta \tilde{u} \in H^1_D(\tilde{M})$, therefore $\delta \tilde{u} \in H^1_D(\Omega)$ and $H^1_D(\Omega) \cap H^1_D(\Omega)$. Since $\delta \tilde{u} = 0$ in $\tilde{M} \setminus M$, we have $\delta \tilde{u} = 0$ on $\partial M$. Then by Proposition 3.2, $\delta u \in H^1_D(\Omega)$ and $\delta u \in C^\infty(\Omega)$. There is a sequence $\{\phi_k\}_{k=1}^\infty \subset C^\infty(\Omega)$ such that $\|\delta u -
\( \phi_k \big|_{H^1 \Omega^m - 1(M)} \to 0 \) as \( k \to \infty \). Note also that, in particular, (3.2) gives \( u = d\delta u \).

Using all these facts, we can show that

\[
(u|u)_{L^2 \Omega^m(M)} + (\delta u|\delta u)_{L^2 \Omega^{m-1}(M)} = \lim_{k \to \infty} \left[ (u|d\phi_k)_{L^2 \Omega^m(M)} + (\delta u|\phi_k)_{L^2 \Omega^{m-1}(M)} \right] = \lim_{k \to \infty} \left[ (d\delta u|d\phi_k)_{L^2 \Omega^m(M)} + (\delta u|\phi_k)_{L^2 \Omega^{m-1}(M)} \right].
\]

Integrating by parts and using (3.2), we get

\[
(u|u)_{L^2 \Omega^m(M)} + (\delta u|\delta u)_{L^2 \Omega^{m-1}(M)} = \lim_{k \to \infty} \left[ (\delta u|d\phi_k)_{L^2 \Omega^m(M)} + (u|d\phi_k)_{L^2 \Omega^{m-1}(M)} \right] = 0.
\]

This implies \( u = 0 \) as desired. \( \square \)

**Proposition 3.5.** The space \( C^\infty \Omega^m(M) \) is dense in \( H_d \Omega^m(M) \).

**Proof.** This follows from Proposition 3.4 using the fact that the Hodge star operator \( * \) is an isometry between \( H_d \Omega^m(M) \) and \( H_d \Omega^{n-m}(M) \). \( \square \)

4. **Reduction to system with the Hodge-Laplacian principal part**

In this section we describe the reduction of the Maxwell equations to a system whose principal part is the Hodge-Laplacian. We follow the arguments in \[28\], although we use different notations. As in the papers \[2, 17\], we work with the following Hilbert space

\[ H^1_\Delta \Omega^m(M) := \{ w \in H^1 \Omega^m(M) : \Delta w \in L^2 \Omega^m(M) \}. \]

**Proposition 4.1.** Let \( \omega > 0 \) be a fixed frequency and let \( \varepsilon, \mu \in C^2(M) \) are complex valued with positive real parts in \( M \). If \( (E, H) \in H^1_d \Omega^1(M) \times H^1_d \Omega^1(M) \) satisfies (1.1), then \( E \) satisfies

\[ L_{\varepsilon, \mu} E = (-\Delta - d \circ i_{d \log \varepsilon} + i_{d \log \mu} \circ d - \omega^2 \varepsilon \mu) E = 0, \quad \delta(\varepsilon E) = 0 \quad \text{in } M, \]

in the sense of distributions.

Here and in what follows, we take the principal branch of \( \log \).

**Proof.** The Maxwell equations (1.1) can be rewritten as

\[
\begin{cases}
    dE = i\omega \mu \ast H, \\
    \delta(\ast H) = -i\omega \varepsilon E.
\end{cases}
\]

Taking divergence of the both equations and using (2.4), we obtain

\[
\delta dE = i\omega \mu \delta(\ast H) + i\omega \ast (d\mu \wedge \ast (\ast H)) = \omega^2 \varepsilon \mu E - i\omega i_{d \mu} \ast H = \omega^2 \varepsilon \mu E - i_{d \log \mu} dE \quad (4.1)
\]

and \( \delta(\varepsilon E) = 0 \). Using (2.5), the latter implies

\[
0 = d(\varepsilon^{-1} \delta(\varepsilon E)) = d\delta E - d(i_{d \log \varepsilon} E), \quad (4.2)
\]

in the sense of distributions. Combining this together with (4.1), we finish the proof. \( \square \)

In the following result, we show that converse of Proposition 4.1 is also true.
Proposition 4.2. Let \( \omega > 0 \) be a fixed frequency and let \( \varepsilon, \mu \in C^2(M) \) are complex valued. If \( E \in H^1_\varepsilon \Omega^1(M) \) satisfies
\[
\mathcal{L}_{\varepsilon, \mu} E = (\Delta - d \circ i d \log \varepsilon + i d \log \mu \circ d - \omega^2 \varepsilon \mu) E = 0, \quad \delta(\varepsilon E) = 0 \quad \text{in} \quad M,
\]
then \( H := (i \omega \mu)^{-1} * dE \) is in \( H^1_\varepsilon \Omega^1(M) \) and \( (E, H) \) satisfies (1.1).

Proof. Obviously, \( (E, H) \) satisfy the first equation in the Maxwell system (1.1). It was shown in Proposition 4.1 that \( \delta(\varepsilon E) = 0 \) implies (4.2). Using (4.2) in \( \mathcal{L}_{\varepsilon, \mu} E = 0 \), we obtain (4.1). Hence, using (2.2) and (2.4), we show
\[
* dH = *((i \omega \mu)^{-1} * dE) = (i \omega \mu)^{-1} i d \log \mu dE + (i \omega \mu)^{-1} \delta dE = -i \omega \varepsilon E.
\]
Finally, we want to have that \( H \in H^1_\varepsilon \Omega^1(M) \). But this is clear, since \( H \in L^2_\varepsilon \Omega^1(M) \) and, according to (4.3), \( dH \in L^2_\varepsilon \Omega^2(M) \).

\[\square\]

5. Local Carleman estimates acting on functions

Let \( (M, g) \) be a compact Riemannian manifold of dimension \( n \geq 3 \) with boundary such that
\[
(M, g) \subset \subset \mathbb{R} \times (M_0, g_0), \quad g = e \oplus g_0,
\]
where \( e \) is the Euclidean metric and \( (M_0, g_0) \) is a compact \( (n-1) \)-dimensional manifold Riemannian manifold with boundary.

The purpose of this section is to prove the local Carleman estimate for the operator
\[
\mathcal{L}_{\varphi, \varepsilon} = h^2 e^{\varphi/h} (-\Delta_g) e^{-\varphi/h},
\]
where \( \varphi = \varphi + h^2 \varepsilon / 2 \) and \( \varphi \) is the limiting Carleman weight \( \varphi(x) = \pm x_1 \).

Recall that we write \( \Gamma_{+, \varphi} = \Gamma_+ \) if \( \varphi(x) = x_1 \) and \( \Gamma_{+, \varphi} = \Gamma_- \) if \( \varphi(x) = -x_1 \). We also use the notation \( \Gamma_{+, \varphi} \) for \( \partial M \setminus \Gamma_{+, \varphi} \).

In what follows, for a submanifold \( U \subseteq M \) we shall use the semiclassical norms
\[
\|u\|_{H^{scl}_\varepsilon(U)} = \|u\|_{L^2(U)} + \|h \nabla u\|_{L^2(U)}, \quad \|v\|_{H^{-scl}_\varepsilon(U)} = \sup_{0 \neq \phi \in C_0^\infty(U; \mathbb{R})} \frac{|\langle v, \phi\rangle_U|}{\|\phi\|_{H^{-scl}_\varepsilon(U)}}.
\]

Also, when dealing with estimates in semiclassical norms, the notation \( A \lesssim B \) means \( A \leq CB \) where \( C > 0 \) is a constant independent of \( h \) and \( A, B \). If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \).

Let us denote the projection of \( \mathbb{R} \times M_0 \) onto \( M_0 \) by \( \pi \). The main result of this section is the following Carleman estimate. We mostly follow [7, 8, 9], adopting the approach for the setting under consideration here.

Proposition 5.1. Let \( (M, g) \) be as described above and let \( \varphi \) be the limiting Carleman weight \( \varphi(x) = \pm x_1 \). For \( p \in \Gamma_{+, \varphi} \), let \( U \) be a precompact neighborhood of \( p \) in \( \mathbb{R} \times M_0 \) such that \( M \cap U \) has a smooth boundary. Suppose that there is a smooth \( f : M_0 \rightarrow \mathbb{R} \) such that \( M \cap U \) lies in the set \( A_{f, \varphi} \), which is defined as \( \{ x_1 \geq f(x') \} \) if \( \varphi(x) = x_1 \) and \( \{ x_1 \leq f(x') \} \) if \( \varphi(x) = -x_1 \), and \( \Gamma_{+, \varphi} \cap U \subset \{ x_1 \leq f(x') \} \). Suppose that there is a choice of local coordinates on \( \pi(U) \) such that there are constants \( \delta > 0 \) and a constant vector field \( V \) on \( \pi(M \cap U) \) for which
\[
|g_0 - \text{Id}| \leq \delta, \quad |\nabla_{g_0} f - V|_{g_0} \leq \delta
\]
on $\pi(M \cap U)$. Then for $0 < h \ll \varepsilon \ll 1$ we have
\[
\frac{h}{\varepsilon^{1/2}} \|u\|_{L^2(M \cap U)} \lesssim \|L_{\varphi,\varepsilon} u\|_{H^{-1}_{scl}}(A_f, \varphi),
\]
$u \in C_0^\infty(M^{\text{int}} \cap U)$. We give the proof only when $\varphi(x) = x_1$. Making the change of variables $(x_1, x') \mapsto (-x_1, x')$, we can reduce $\varphi(x) = -x_1$ to the case $\varphi(x) = x_1$. Since we are considering $\varphi(x) = x_1$, throughout this section, we omit $\varphi$ in the notations $A_f, \varphi$ and $\Gamma_{+, \varphi}$.

5.1. Flattening and decomposing into small and large frequency parts. Take $U_1 \subset A_f$ open and precompact such that $U \subset U_1$ and $\Gamma^+_\varphi \subset \partial(M \cap U_1)$. We can choose $U_1$ so close to $U$ so that
\[
|g_0 - 1d| \leq 2\delta, \quad |\nabla g_0 f - V|_{g_0} \leq 2\delta \tag{5.1}
\]
on $\pi(M \cap U_1)$. For convenience, we use the notations $U_M$ and $U_{1,M}$ to denote the intersections $M \cap U$ and $M \cap U_1$, respectively.

Now, let us make the change of variables $\sigma : (x_1, x') \mapsto (x_1 - f(x'), x')$. Under this change of variables, $A_f$ is mapped to $[0, \infty) \times M_0$ and $\Gamma^+_{\varphi}$ is mapped to a subset of $\{0\} \times M_0$. In new coordinate, we have $\varphi(x) = x_1 + f(x')$.

**Proposition 5.2.** Let $\varphi$ be the limiting Carleman weight $\varphi(x) = x_1$. Then for $0 < h \ll \varepsilon \ll 1$ we have
\[
\frac{h}{\varepsilon^{1/2}} \|u\|_{H^1_{scl}(\sigma(M_{U_1}))} \lesssim \|L_{\varphi,\varepsilon} u\|_{L^2(\sigma(M_{U_1}))},
\]
for all $u \in C_0^\infty(\sigma(U_{1,M}^{\text{int}}))$, with
\[
L_{\varphi,\varepsilon,\sigma} = -(1 + |\nabla g_0 f(x')|^2_{g_0}) h \varphi_1^2 + 2\alpha + \langle \nabla g_0 f(x'), h \nabla g_0 \rangle_{g_0} h \varphi_1 - \alpha^2 - h^2 \Delta_{g_0},
\]
where $\alpha = 1 + (h/\varepsilon)(x_1 + f(x'))$ and $\langle , \rangle_{g_0}$ denotes the inner product with respect to $g_0$.

**Proof.** Let $v \in C_0^\infty(U_{1,M}^{\text{int}})$, and let use the notation $v_\sigma(x_1, x') = v(x_1 + f(x'), x')$. Then $v_\sigma \in C_0^\infty(\sigma(U_{1,M}^{\text{int}}))$. Then by the change of variables, we get
\[
\|v_\sigma\|_{L^2(\sigma(U_{1,M}^{\text{int}}))} \approx \|v\|_{L^2(U_{1,M}^{\text{int}})}, \quad \|v_\sigma\|_{H^1_{scl}(\sigma(U_{1,M}^{\text{int}}))} \approx \|v\|_{H^1_{scl}(U_{1,M}^{\text{int}})}
\]
with implicit constants depending on $f$.

Since $L_{\varphi,\varepsilon} v \in C_0^\infty(U_{1,M}^{\text{int}})$, we have $(L_{\varphi,\varepsilon} v)_\sigma \in C_0^\infty(\sigma(U_{1,M}^{\text{int}}))$, and
\[
\|(L_{\varphi,\varepsilon} v)_\sigma\|_{L^2(\sigma(U_{1,M}^{\text{int}}))} \approx \|(L_{\varphi,\varepsilon} v)\|_{L^2(U_{1,M}^{\text{int}})}.
\]
Therefore, by the estimate (4.5) in [13],
\[
\frac{h}{\varepsilon^{1/2}} \|v_\sigma\|_{H^1_{scl}(\sigma(U_{1,M}^{\text{int}}))} \lesssim \|(L_{\varphi,\varepsilon} v)_\sigma\|_{L^2(\sigma(U_{1,M}^{\text{int}}))}.
\]
Using the chain rule, a straightforward calculation gives
\[
(L_{\varphi,\varepsilon} v)_\sigma(x_1, x') = L_{\varphi,\varepsilon} v_\sigma(x_1, x') + hE_1 v_\sigma(x_1, x'),
\]
where $E_1$ is a semiclassical first-order differential operator. Hence, by the change of variables, we have
\[
\|(L_{\varphi,\varepsilon} v)_\sigma\|_{L^2(\sigma(U_{1,M}^{\text{int}}))} \lesssim \|(L_{\varphi,\varepsilon} v)\|_{L^2(\sigma(U_{1,M}^{\text{int}}))} + h\|v_\sigma\|_{H^1_{scl}(\sigma(U_{1,M}^{\text{int}}))}.
\]
Combining this with the previous estimate, for \( \varepsilon > 0 \) sufficiently small we obtain

\[
\frac{h}{\varepsilon^{1/2}} \| v_\sigma \|_{H^1_{\text{cl}}(\sigma(U_M^{\text{int}}))} \lesssim \| \mathcal{L}_{f,\varepsilon} v_\sigma \|_{L^2(\sigma(U_M^{\text{int}}))}.
\]

Now for any \( u \in C_0^\infty(\sigma(U_M^{\text{int}})) \) we take \( v(x_1, x') = w(x_1 - f(x'), x') \).

Now, we do a second change of variables, mapping \( \pi(\sigma(U_1, M)) \) to a subset of \( \mathbb{R}^{n-1} \), then \( \sigma(U_1, M) \) is mapped to a subset of \( \mathbb{R}^n \), the set of points \((x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n \) with \( x_1 > 0 \), and \( \sigma(\Gamma_+) \) is mapped into the hyperplane \( \{0\} \times \mathbb{R}^{n-1} \), that is when \( x_1 = 0 \). We denote the images of \( \sigma(U_M), \sigma(U_1, M) \) and \( \sigma(\Gamma_+) \) under this change of coordinates by \( \tilde{U}, \tilde{U}_1 \) and \( \tilde{\Gamma}_+ \), respectively.

The following proposition can be obtained in a similar way as in Proposition 5.2 via change of coordinates.

**Proposition 5.3.** Let \( \varphi \) be the limiting Carleman weight \( \varphi(x) = x_1 \). Then for \( 0 < h \ll \varepsilon \ll 1 \) we have

\[
\frac{h}{\varepsilon^{1/2}} \| u \|_{H^1_{\text{cl}}(\tilde{U}_1)} \lesssim \| \tilde{L}_{f,\varepsilon} u \|_{L^2(\tilde{U}_1)},
\]

for all \( u \in C_0^\infty(\tilde{U}_1^{\text{int}}) \), with

\[
\tilde{L}_{f,\varepsilon} = -(1 + |\gamma_f|^2) h^2 \partial_1^2 + 2(\sigma_f, h \nabla \tilde{g}_0 ') \partial_1 - \alpha^2 - h^2 \mathcal{L},
\]

where \( \gamma_f, \beta_f \) and \( \tilde{g}_0 \) are the expressions for \( |\nabla g_0, f(x')|g_0, \nabla g_0 f(x') \) and \( \tilde{g}_0 \) in new coordinates on \( \tilde{U}_1 \), and \( \mathcal{L} = g_0^{ij}\partial_{ij} \) with \( i, j = 2, \ldots, n \).

Note that by (5.1), there is a constant \( C_4 > 0 \) with \( C_5 \to 0 \) as \( \delta \to 0 \) such that

\[
|\gamma_f - |\tilde{V}|\tilde{g}_0| \leq C_4, \quad |\beta_f - \tilde{V}|g_0 \leq C_5, \quad |\tilde{g}_0 - \text{Id}| \leq C_6 \quad \text{on} \quad \tilde{U}_1,
\]

where \( \tilde{V} \) is the expression for \( V \) in new local coordinates on \( \tilde{U}_1 \). The constant \( C_5 \) may depend on \( V \), but the latter is fixed.

We also assume that \( \gamma_f, \beta_f \) and \( \tilde{g}_0 \) are extended to the whole \( \mathbb{R}^n \) by keeping the above conditions. Note that \( \tilde{g}_0 \) is very close to the Euclidean metric, so \( |\cdot|g_0 \approx |\cdot| \). Let us denote by \( S(\mathbb{R}^n) \) the set of Schwartz functions on \( \mathbb{R}^n \) restricted to \( \mathbb{R}^n_+ \). Clearly, the functions in \( C_0^\infty(\tilde{U}_1^{\text{int}}) \) are also in \( S(\mathbb{R}^n_+) \).

In what follows, for \( u \in S(\mathbb{R}^n_+) \), by \( \tilde{u}_{\text{sc}}(x_1, \xi) \) we mean the semiclassical Fourier transform of \( u(x_1, x') \) in the \( x' \)-variable,

\[
\tilde{u}_{\text{sc}}(x_1, \xi) = F'_{\text{sc}} u(x_1, x') = \int_{\mathbb{R}^{n-1}} e^{-i\xi \cdot \tilde{x}/h} u(x_1, x') \, dx'.
\]

Choose constants \( \rho_1, \rho_2 > 0 \) such that

\[
\frac{|\tilde{V}|}{1 + |\tilde{V}|^2} < \rho_1 < \rho_2 \leq \frac{1}{2} + \frac{|\tilde{V}|^2}{2(1 + |\tilde{V}|^2)} < 1.
\]

Take also constants \( \delta_1, \delta_2 \) such that \( \delta_2 > \delta_1 > 0 \). Then we consider a cutoff function \( \rho \in C^\infty(\mathbb{R}^{n-1}) \) such that \( \rho(\xi) = 0 \) if \( |\xi|^2 > \rho_2 \) or \( |\xi \cdot \tilde{V}| > \delta_2 \), and \( \rho(\xi) = 1 \) if \( |\xi|^2 \leq \rho_1 \) or \( |\xi \cdot \tilde{V}| \leq \delta_1 \). The choice of the constants \( \rho_1, \rho_2, \delta_1, \delta_2 \) will depend only on \( V \). In particular for \( \delta_2 \), we will need \( 2\delta_2 < \tilde{V} \).
Given \( u \in C^\infty_0(\tilde{U}^{\text{int}}) \), we express it in terms of small frequency and large frequency parts \( u = u_s + u_t \), where
\[
(\tilde{u}_s)_{\text{scl}}(x_1, \xi) = \rho(\xi)\hat{u}_{\text{scl}}, \quad (\tilde{u}_t)_{\text{scl}}(x_1, \xi) = (1 - \rho(\xi))\hat{u}_{\text{scl}}.
\]
We will prove the Carleman estimate for each part separately in Section 5.3 and Section 5.4, and then combine them in Section 5.5.

5.2. The operators. In this subsection we introduce certain operators that will be used in proving Proposition 5.1. Such operators were considered in [7, 8, 9]. Suppose that \( F : \mathbb{R}^{n-1} \to \mathbb{C} \) is smooth such that \( \text{Re} \ F(\xi), |F(\xi)| \approx 1 + |\xi|, \xi \in \mathbb{R}^{n-1} \). We assume that for all multi-indices \( \alpha \) there is \( C_\alpha > 0 \) such that
\[
|\partial_\xi^\alpha F(\xi)| \leq C_\alpha (1 + |\xi|)^{1-|\alpha|}, \quad \xi \in \mathbb{R}^{n-1}.
\]
(5.3)
For \( u \in \mathcal{S}(\mathbb{R}^n_+) \), we define the operator \( J \) by
\[
(\tilde{J}u)_{\text{scl}}(x_1, \xi) = (F(\xi) + h\partial_1)\hat{u}_{\text{scl}}(x_1, \xi).
\]
The adjoint operator \( J^* \) of \( J \) is
\[
(\tilde{J}^*u)_{\text{scl}}(x_1, \xi) = (F(\xi) - h\partial_1)\hat{u}_{\text{scl}}(x_1, \xi).
\]
The right inverses \( J^{-1}, J^{*-1} \) are
\[
(\tilde{J}^{-1}u)_{\text{scl}}(x_1, \xi) = \frac{1}{h} \int_0^{x_1} \hat{u}_{\text{scl}}(s, \xi)e^{\frac{h}{s-x_1}s}F(\xi) ds,
\]
\[
(\tilde{J}^{*-1}u)_{\text{scl}}(x_1, \xi) = \frac{1}{h} \int_{x_1}^{\infty} \hat{u}_{\text{scl}}(s, \xi)e^{\frac{h}{x_1-s}1(s-x_1)F(\xi) ds.
\]
We have the following result on boundedness of these operators, which was shown in [7, 8], although we state it in a different way.

**Lemma 5.4.** The operators \( J, J^{-1}, J^*, J^{*-1} \), defined on \( \mathcal{S}(\mathbb{R}^n_+) \), can be extended to bounded operators
\[
J, J^* : H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n_+), \quad J^{-1}, J^{*-1} : L^2(\mathbb{R}^n_+) \to H^1(\mathbb{R}^n_+),
\]
and the following estimates hold:
\[
\|Ju\|_{L^2(\mathbb{R}^n_+)} \lesssim \|u\|_{H^1_{\text{scl}}(\mathbb{R}^n_+)} , \quad \|J^* u\|_{L^2(\mathbb{R}^n_+)} \lesssim \|u\|_{H^1_{\text{scl}}(\mathbb{R}^n_+)} ,
\]
\[
\|J^{-1}u\|_{H^1_{\text{scl}}(\mathbb{R}^n_+)} \lesssim \|u\|_{L^2(\mathbb{R}^n_+)} , \quad \|J^{*-1} u\|_{H^1_{\text{scl}}(\mathbb{R}^n_+)} \lesssim \|u\|_{L^2(\mathbb{R}^n_+)}.
\]
Moreover, these extensions for \( J^* \) and \( J^{*-1} \) are isomorphisms.

By \( H^1_0(\mathbb{R}^n_+) \) we denote the space of functions in \( H^1(\mathbb{R}^n) \) with zero trace on the boundary, i.e. on the hyperplane \( x_1 = 0 \). The dual space to \( H^1_0(\mathbb{R}^n_+) \) will be denoted by \( H^{-1}(\mathbb{R}^n_+) \).

Using the similar arguments as in [7], we prove the following properties of \( J \), which will be used later.
Lemma 5.5. Assume that \( u, v \in S(\mathbb{R}^n_+) \) and that \( Q \) is a second-order semiclassical differential operator with bounded coefficients in \( C^\infty(\mathbb{R}^n_+) \). Then for sufficiently small \( 0 < h \ll 1 \), the following estimates hold
\[
\|JuJ^{-1}u\|_{L^2(\mathbb{R}^n_+)} \gtrsim \|vu\|_{L^2(\mathbb{R}^n_+)} - h\|u\|_{L^2(\mathbb{R}^n_+)}
\]
and
\[
\|(JQ - QJ)u\|_{H_{sc}^{-1}(\mathbb{R}^n_+)} \lesssim h\|u\|_{H_{sc}^1(\mathbb{R}^n_+)},
\]
with implicit constants depending on the derivatives of \( F \).

Lemma 5.6. For a given \( v \in S(\mathbb{R}^n_+) \), consider \( g \) defined by
\[
\hat{g}_{sc}(x_1, \xi) = 2 \operatorname{Re} \frac{F(\xi)}{h} \int_0^\infty \hat{v}_{sc}(s, \xi)e^{-s\xi_1 + \frac{F(\xi)}{h}s} ds.
\]
Then
\[
\|g\|_{L^2(\mathbb{R}^n_+)} \leq \|v\|_{L^2(\mathbb{R}^n_+)}. \tag{5.4}
\]
Moreover, we have
\[
\|Ju\|_{H_{sc}^{-1}(\mathbb{R}^n_+)} \approx \|v - g\|_{L^2(\mathbb{R}^n_+)}. \tag{5.5}
\]

For the proofs of these results we refer the reader to Appendix C.

5.3. The case of small frequencies. In this subsection is we prove the Carleman estimate for the small frequency part.

Proposition 5.7. There is \( \delta_0 > 0 \) and there are \( \rho_1, \rho_2, \delta_1, \delta_2 \) such that if \( (5.2) \) holds for some \( \delta \leq \delta_0 \), then
\[
\frac{h}{e^{1/2}}\|u_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|Lu_{sc}\|_{H_{sc}^{-1}(\mathbb{R}^n_+)} + \frac{h^2}{e^{1/2}}\|u\|_{L^2(\tilde{U})}
\]
for all \( u \in C^\infty_0(\tilde{U}_{int}) \).

Following [7, Section 6], we start with defining a function \( \Phi : \mathbb{R}^{n-1} \rightarrow \mathbb{C} \) by
\[
\Phi(\xi) = \frac{1}{1 + |\tilde{V}|^2} (1 + i\tilde{V} \cdot \xi + \sqrt{2i\tilde{V} \cdot \xi - |\tilde{V} \cdot \xi|^2 + (1 + |\tilde{V}|^2)|\xi|^2 - |\tilde{V}|^2}), \quad \xi \in \mathbb{R}^{n-1},
\]
where we take the branch of the square root with non-negative imaginary part. This function could play a role of \( F \) in the definitions of the operators in Section 5.2. However, \( \Phi \) is non-smooth and therefore our aim is to approximate it with a certain smooth function \( F_s \), on the support of \( (u_s)_{sc} \), that satisfy all the required conditions for \( F \) in Section 5.2.

Observe that \( \Phi \) is smooth away from the set of those \( \xi \in \mathbb{R}^{n-1} \) when
\[
\tau(\xi) = 2i\tilde{V} \cdot \xi - |\tilde{V} \cdot \xi|^2 + (1 + |\tilde{V}|^2)|\xi|^2 - |\tilde{V}|^2
\]
is real-valued and non-negative, that is when \( \tilde{V} \cdot \xi = 0 \) and \( |\xi|^2 \geq (1 + |\tilde{V}|^2)^{-1}|\tilde{V}|^2 \). This is exactly when \( \tau(\xi) \) is on the branch cut of the above mentioned branch of the square root. Therefore, singular points of \( \Phi \) are those points \( \xi \in \mathbb{R}^{n-1} \) where
\[
\sqrt{\tau(\xi)} \text{ has a discontinuity as a jump of size } 2\sqrt{(1 + |\tilde{V}|^2)|\xi|^2 - |\tilde{V}|^2}.
\]
Since $|\xi|^2 \leq \rho_2$ on the support of $\rho$, we can choose $\rho_2$ sufficiently close to $(1 + |\hat{V}|^2)^{-1} |\hat{V}|^2$ to make the size of the jump sufficiently small.

Therefore, choosing $\rho_2 > 0$ sufficiently close to $(1 + |\hat{V}|^2)^{-1} |\hat{V}|^2$, for arbitrary small $\epsilon > 0$ we can define $F_\ast$ to be a smooth function on the support of $(u_s)_s$ so that

$$|\Phi(\xi) - F_\ast(\xi)| \leq \epsilon.$$ 

Taking $\rho_2$ sufficiently close to $(1 + |\hat{V}|^2)^{-1} |\hat{V}|^2$, in the support of $\rho$, we can show that

$$1 - (1 + |\hat{V}|^2)(1 - |\xi|^2) \leq \delta_2.$$ 

Therefore, in $\text{supp}(u_s)_s$, we have

$$-2\delta_2 \leq \text{Im}(\tau) \leq 2\delta_2$$ 

and

$$-\delta_2 - |\hat{V}|^2 \leq \text{Re}(\tau) \leq \delta_2.$$ 

Hence, we get $|\sqrt{\tau(\xi)}| \leq \delta \sqrt{5}$ in the support of $\rho$.

Taking $\delta_2$ and $\epsilon$ small enough and using the well known inequality

$$- |z| \leq \text{Re}(z) \leq |z|, \quad z \in \mathbb{C},$$

one can show that on $\text{supp}(\rho)$

$$\text{Re} F_\ast(\xi) \geq \text{Re} \Phi(\xi) - \epsilon \geq \frac{1 - \delta_2 \sqrt{5}}{1 + |\hat{V}|^2} - \epsilon \geq \frac{1 - 4\delta_2 - \epsilon(1 + |\hat{V}|^2)}{1 + |\hat{V}|^2} \geq \frac{1}{2(1 + |\hat{V}|^2)},$$

and

$$|F_\ast(\xi)| \geq \text{Re} F_\ast(\xi) > \frac{1}{2(1 + |\hat{V}|^2)}.$$ 

We now fix all the constants $\rho_1, \rho_2, \delta_1, \delta_2, \epsilon$. Then we can extend $F_\ast$ smoothly outside of $\text{supp}(\rho)$ so that $\text{Re} F_\ast(\xi), |F_\ast(\xi)| \approx 1 + |\xi|$ and $\text{Re} F_\ast, |F_\ast| > 1/2(1 + |\hat{V}|^2)$ on $\mathbb{R}^n$.

Thus, we obtain smooth $F_\ast : \mathbb{R}^{n-1} \to \mathbb{C}$ with $\text{Re} F_\ast(\xi), |F_\ast(\xi)| \approx 1 + |\xi|$ on $\mathbb{R}^{n-1}$ and satisfying (5.3). Then by $J_s, J_s^+, J_s^{-1}, J_s^{-1}$ we denote the operators defined as in Section 5.2 with $F$ replaced by $F_\ast$.

Next we give the proof of Proposition 5.7. For this, consider $\chi \in C^\infty(\mathbb{R}_+^n)$ with $0 \leq \chi \leq 1$ such that $\chi = 1$ on $\tilde{U}$ and $\text{supp} \chi \subset \tilde{U}_1$. For a given $u \in C^\infty_0(\tilde{U}^\text{int})$, we have $u_s \in S(\mathbb{R}_+^n)$ and support of $u$ is away from the hyperplane $x_1 = 0$. Then $\chi J_s^{-1} u_s \in C^\infty(\tilde{U}_1)$, and hence by Proposition 5.3 we have

$$\frac{h}{\epsilon^{1/2}} \|\chi J_s^{-1} u_s\|_{H^1_s(\tilde{U}_1)} \lesssim \|\tilde{L}_{\varphi, \epsilon} \chi J_s^{-1} u_s\|_{L^2(\tilde{U}_1)}.$$ 

Since $\chi J_s^{-1} u_s \in C^\infty(\tilde{U}_1)$, this is same as

$$\frac{h}{\epsilon^{1/2}} \|\chi J_s^{-1} u_s\|_{H^1_s(\mathbb{R}_+^n)} \lesssim \|\tilde{L}_{\varphi, \epsilon} \chi J_s^{-1} u_s\|_{L^2(\tilde{U}_1)}.$$ 

Applying Lemma 5.4, we get

$$\frac{h}{\epsilon^{1/2}} \|J_s \chi J_s^{-1} u_s\|_{L^2(\mathbb{R}_+^n)} \lesssim \|\tilde{L}_{\varphi, \epsilon} \chi J_s^{-1} u_s\|_{L^2(\tilde{U}_1)}.$$ 

Using the first estimate in Lemma 5.5 for the left hand-side, we obtain

$$\frac{h}{\epsilon^{1/2}} \|\chi u_s\|_{L^2(\mathbb{R}_+^n)} \lesssim \|\tilde{L}_{\varphi, \epsilon} \chi J_s^{-1} u_s\|_{L^2(\tilde{U}_1)} + \frac{h^2}{\epsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}_+^n)},$$

Since $|\xi|^2 \leq \rho_2$ on the support of $\rho$, we can choose $\rho_2$ sufficiently close to $(1 + |\hat{V}|^2)^{-1} |\hat{V}|^2$ to make the size of the jump sufficiently small.
where the implicit constant depends on the derivatives of $F_s$. The latter depends on $\delta$, and $\delta$ is independent of $h$ and $\varepsilon$.

Let $P$ be the semiclassical pseudodifferential operator of order 0 on $\mathbb{R}^{n-1}$ with symbol $\rho(\xi)$, so $u_s = Pu$. Since $\textsuperscript{supp}u \subset \overline{U}^{\text{int}}$ and $\chi = 1$ on $\overline{U}^{\text{int}}$,

\[ \chi u_s = \chi P u = P \chi u + hE_0 u = Pu + hE_0 u = u_s + hE_0 u \]

for some semiclassical pseudodifferential operator $E_0$ of order 0 on $\mathbb{R}^{n-1}$. Therefore,

\[ \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} \gtrsim \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} - \frac{h^2}{\varepsilon^{1/2}} \|u\|_{L^2(\mathbb{R}^n_+)} \]

and hence,

\[ \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\tilde{L}_{\phi,\varepsilon, \chi} J^{-1}_s u_s\|_{L^2(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} \|u\|_{L^2(\mathbb{R}^n_+)} \]

Taking $h > 0$ sufficiently small, the second term on the right-hand-side can be absorbed into the left-hand-side

\[ \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\tilde{L}_{\phi,\varepsilon, \chi} J^{-1}_s u_s\|_{L^2(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} \|u\|_{L^2(\mathbb{R}^n_+)} \]

Since the commutator $[\tilde{L}_{\phi,\varepsilon, \chi} = hE_1$ for some semiclassical first-order differential operator $E_1$, we obtain

\[ \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\tilde{L}_{\phi,\varepsilon, \chi} J^{-1}_s u_s\|_{L^2(\mathbb{R}^n_+)} + \frac{h}{\varepsilon^{1/2}} \|u\|_{L^2(\mathbb{R}^n_+)} \]

Using Lemma 5.4 and the properties of $\chi$, this implies

\[ \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\tilde{L}_{\phi,\varepsilon, \chi} J^{-1}_s u_s\|_{L^2(\mathbb{R}^n_+)} + h \|u_s\|_{L^2(\mathbb{R}^n_+)} \]

Taking $\varepsilon > 0$ sufficiently small, the second term on the right-hand-side can be absorbed into the left-hand-side and give

\[ \frac{h}{\varepsilon^{1/2}} \|u_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\tilde{L}_{\phi,\varepsilon, \chi} J^{-1}_s u_s\|_{L^2(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} \|u\|_{L^2(\mathbb{R}^n_+)} \] \hspace{1cm} (5.6)

Our next step is to show that

\[ \|v - g\|_{L^2(\mathbb{R}^n_+)} \geq \frac{1}{2} \|v\|_{L^2(\mathbb{R}^n_+)} \]

$\|v\|_{L^2(\mathbb{R}^n_+)}$ and using the expression for $\tilde{L}_{\phi,\varepsilon}$ in the statement of Proposition 5.3, we can write

\begin{align*}
\tilde{g}_{sc} = & \frac{2 \text{Re} F_s}{h} \int_0^\infty \mathcal{F}_{\text{sc}}'(\tilde{L}_{\phi,\varepsilon}w)e^{-\frac{\text{Re} s + \text{Im } s}{h}} ds \\
= & -\frac{2 \text{Re} F_s}{h} \int_0^\infty \mathcal{F}_{\text{sc}}'((1 + |s|^2)h^2 \partial_s^2 w)e^{-\frac{\text{Re} s + \text{Im } s}{h}} ds \\
+ & \frac{2 \text{Re} F_s}{h} \int_0^\infty \mathcal{F}_{\text{sc}}((2 \alpha + \langle \beta f, h \nabla g_0 \cdot g_0 \rangle h \partial_s w)e^{-\frac{\text{Re} s + \text{Im } s}{h}} ds \\
+ & \frac{2 \text{Re} F_s}{h} \int_0^\infty \mathcal{F}_{\text{sc}}((-\alpha^2 - h^2 \mathcal{L})w)e^{-\frac{\text{Re} s + \text{Im } s}{h}} ds.
\end{align*}
Using (5.2) and the fact that \(|1 - \alpha| \lesssim h \varepsilon^{-1}\), this can be rewritten as

\[
\hat{g}_{\text{scl}} = -\frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}}(1 + |\tilde{V}|^2) h^2 \partial_s^2 w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
+ \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} (2 (1 + \tilde{V} \cdot h \nabla') h \partial_s w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
+ \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} ((-1 - h^2 \Delta') w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
+ C_\delta \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} (E_2 w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds,
\]

where \(\nabla'\) and \(\Delta'\) are gradient and Laplacian operators in \(x'\)-variable in \(\mathbb{R}^{n-1}\), and \(E_2\) is a semiclassical second-order differential operator in \(\mathbb{R}^{n-1}_{\varepsilon}\). We apply integration by parts twice for the first term on the right-hand side to get

\[
-\frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} (1 + |\tilde{V}|^2) h^2 \partial_s^2 w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
= -\frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} (\mathcal{F}_{s})^2 (1 + |\tilde{V}|^2) \hat{w}_{\text{scl}} e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds,
\]

with no boundary terms since \(\text{Re} F_s > 0\) and \(v\) is supported away from \(x_1 = 0\), and hence so are both \(\hat{u}_{\text{scl}}\) and \(v = J_{s+1} u\). Similarly, for the second term, we get

\[
\frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} (2 (1 + \tilde{V} \cdot \nabla') h \partial_s w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
= \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} 2 \mathcal{F}_{\text{scl}} (1 + i \tilde{V} \cdot \xi) \hat{w}_{\text{scl}} e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds.
\]

Therefore,

\[
\hat{g}_{\text{scl}} = -\frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} (\mathcal{F}_{s})^2 (1 + |\tilde{V}|^2) \hat{w}_{\text{scl}} e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
+ \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} 2 \mathcal{F}_{\text{scl}} (1 + i \tilde{V} \cdot \xi) \hat{w}_{\text{scl}} e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
+ \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} ((-1 - |\xi|^2) \hat{w}_{\text{scl}} e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds
+ C_\delta \frac{2 \text{Re} F_s}{h} \int_{0}^{\infty} \mathcal{F}_{\text{scl}} (E_2 w) e^{-\frac{F_{s+1} + \overline{F}_{\text{scl}}}{h}} ds.
\]

Observe that \(\Phi(\xi)\) is a solution for the equation

\[
(1 + |\tilde{V}|^2) X^2 - 2(1 + i \tilde{V} \cdot \xi) X + 1 - |\xi|^2 = 0.
\]

Since \(|\Phi(\xi) - F_s(\xi)| \leq \varepsilon\), this implies that on \(\text{supp} \hat{u}_{\text{scl}}\) (thus also on \(\text{supp} \hat{v}_{\text{scl}}\)) we have

\[
|(1 + |\tilde{V}|^2) (F_s(\xi))^2 - 2(1 + i \tilde{V} \cdot \xi) F_s(\xi) + 1 - |\xi|^2| \lesssim \varepsilon |F_s(\xi)|.
\]
where the implicit constant depends only on $\tilde{V}$. Therefore,

$$
\tilde{g}_{\text{sc}} = \frac{2 \text{Re} F_s(\xi)}{h} \int_0^\infty R \tilde{w}_{\text{sc}} e^{-\frac{p_s(t) + \xi^2}{h}} ds + C_\delta \frac{2 \text{Re} F_s}{h} \int_0^\infty F'_{\text{sc}} (E_2 w) e^{-\frac{p_s(t) + \xi^2}{h}} ds
$$

for some $R(\xi)$ such that $|R(\xi)| \lesssim |F_s(\xi)| \lesssim 1 + |\xi|$. Then, using the same reasonings as in the proof of (5.4), we show

$$
\|\tilde{g}_{\text{sc}}\|_{L^2(\mathbb{R}^n_+)} \lesssim \epsilon \|R \tilde{w}_{\text{sc}}\|_{L^2(\mathbb{R}^n_+)} + C_\delta \|E_2 w\|_{L^2(\mathbb{R}^n_+)}.
$$

Using the semiclassical Plancherel’s theorem, this implies

$$
\|g\|_{L^2(\mathbb{R}^n_+)} \lesssim (\epsilon + C_\delta) \|w\|_{H^2_{\text{sc}}(\mathbb{R}^n_+)}.
$$

Using the expression for $\tilde{L}_{\rho,\varepsilon}$ in the statement of Proposition 5.3 together with (5.2) and the fact that $|1 - \alpha| \lesssim h \varepsilon^{-1}$, we have

$$
\|\tilde{L}_{\rho,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)}^2 \gtrsim \left\| \left( (1 + |\tilde{V}|^2) h^2 \partial_1^2 + 2(1 + \tilde{V} \cdot h \nabla') h \partial_1 - (1 + h^2 \Delta') \right) w \right\|_{L^2(\mathbb{R}^n_+)}^2 - C^2_\delta \|w\|_{H^2_{\text{sc}}(\mathbb{R}^n_+)}^2.
$$

By semiclassical Plancherel’s theorem, the first term on the right hand-side can expressed as

$$
\frac{1}{h^{n-1}} \left\| (1 + |\tilde{V}|^2) h^2 \partial_1^2 + 2(1 + i \tilde{V} \cdot \xi) h \partial_1 - (1 - |\xi|^2) \right\|_{L^2(\mathbb{R}^n_+)}^2.
$$

By definition of $w$, for each $x_1$, supp $\tilde{w}_{\text{sc}}(x_1, \cdot) \subset \{ \xi \in \mathbb{R}^{n-1} : |\xi|^2 \leq \rho_2 \}$, where as before

$$
\rho_2 \leq \frac{1}{2} + \frac{|\tilde{V}|^2}{2(1 + |\tilde{V}|^2)} < 1.
$$

Observe that, for each fixed $\xi$ such that $\tilde{w}_{\text{sc}}$ is non-zero, the operator

$$
P_{\xi, \tilde{V}} = -(1 + |\tilde{V}|^2) h^2 \partial_1^2 + 2(1 + i \tilde{V} \cdot \xi) h \partial_1 - (1 - |\xi|^2)
$$

is an elliptic semiclassical second-order differential operator in $x_1$-variable with semiclassical symbol

$$
p_{\xi, \tilde{V}}(s) = (1 + |\tilde{V}|^2)s^2 + 2(1 + i \tilde{V} \cdot \xi) is - (1 - |\xi|^2),
$$

where $s \in \mathbb{R}$ is a variable on a semiclassical Fourier transform side in $x_1$-variable. One can show that the following holds with the implicit constant depending only on $\tilde{V}$

$$
|p_{\xi, \tilde{V}}(s)| \gtrsim 1 + s^2 + |\xi|^2, \quad s \in \mathbb{R}, \quad |\xi|^2 \leq \rho_2 < 1, \quad |\tilde{V} \cdot \xi| \leq \delta_2.
$$

Observe also that, for each $x'$, $u(x_1, x')$ is zero for all $x_1$ outside of $(t_0, t_1) \subset (0, \infty)$, for some $t_0, t_1 > 0$ depending on $\tilde{U}^{\text{int}}$. In particular, $u(\cdot, x') \in H^1_0((t_0, t_1))$ and hence we can extend $u(\cdot, x')$ to the rest of $(-\infty, 0)$ by zero such that $u(\cdot, x') \in H^1(\mathbb{R})$ with supp $u(\cdot, x') \subset [t_0, t_1]$. This implies that, for each $\xi$, $\tilde{w}_{\text{sc}}(\cdot, \xi) \in H^1(\mathbb{R})$ with
supp \( \tilde{w}_{\text{scl}}(\cdot, \xi) \subset [t_0, t_1] \). Therefore, using the semiclassical Plancherel’s theorem in \( x_1 \)-variable, the first term on the right-hand-side of (5.9) is equal to

\[
\frac{1}{h^{n-1}} \| P_{x_1} \tilde{v} \tilde{w}_{\text{scl}} \|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{h^n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |p_{x_1} \tilde{v}(s)|^2 |\mathcal{F}_{x_1}^1(\tilde{w}_{\text{scl}})(s, \xi)|^2 \, ds \, d\xi,
\]

where \( \mathcal{F}_{x_1}^1 \) denotes the semiclassical Fourier transform in \( x_1 \)-variable. Applying (5.10), this gives

\[
\| \tilde{L}_{\varphi, \varepsilon} u \|_{L^2(\mathbb{R}^n)}^2 \gtrsim \| u \|_{H^2_{\text{scl}}(\mathbb{R}^n)} - C^2 \| u \|_{H^2_{\text{scl}}(\mathbb{R}^n)}.
\]

Taking \( \delta > 0 \) sufficiently small, this implies that

\[
\| \tilde{v} \|_{L^2(\mathbb{R}^n)} = \| \tilde{L}_{\varphi, \varepsilon} u \|_{L^2(\mathbb{R}^n)} \gtrsim \| u \|_{H^2_{\text{scl}}(\mathbb{R}^n)}.
\]

Combining this with (5.8) and recalling that \( u = J_{\delta}^{-1} u_s \), we obtain

\[
\| g \|_{L^2(\mathbb{R}^n)} \lesssim (\varepsilon + C_0) \| v \|_{L^2(\mathbb{R}^n)} \quad \Longrightarrow \quad \| g \|_{L^2(\mathbb{R}^n)} \leq \frac{1}{2} \| v \|_{L^2(\mathbb{R}^n)},
\]

taking sufficiently small \( \varepsilon \) and \( \delta \). Then this clearly implies (5.12).

Using (5.12) and Lemma 5.6 in (5.6) gives

\[
\frac{1}{\varepsilon^{1/2}} \| u_s \|_{L^2(\mathbb{R}^n)} \lesssim \| J_{\delta} \tilde{L}_{\varphi, \varepsilon} J_{\delta}^{-1} u_s \|_{H^{-1}_{\text{scl}}(\mathbb{R}^n)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n)}.
\]

Now, applying Lemma 5.5, this implies

\[
\frac{1}{\varepsilon^{1/2}} \| u_s \|_{L^2(\mathbb{R}^n)} \lesssim \| \tilde{L}_{\varphi, \varepsilon} u_s \|_{H^{-1}_{\text{scl}}(\mathbb{R}^n)} + \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n)}.
\]

Taking sufficiently small \( \varepsilon \) and using Lemma 5.4, the second term on the right hand-side can be absorbed into the left hand side

\[
\frac{1}{\varepsilon^{1/2}} \| u_s \|_{L^2(\mathbb{R}^n)} \lesssim \| \tilde{L}_{\varphi, \varepsilon} u_s \|_{H^{-1}_{\text{scl}}(\mathbb{R}^n)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n)}.
\]

The proof of Proposition 5.7 is thus complete.

5.4. The case of large frequency. In this subsection we prove the Carleman estimate for the large frequency case.

**Proposition 5.8.** There is \( \delta_0 > 0 \) and there are \( \rho_1, \rho_2, \delta_1, \delta_2 \) such that if (5.2) holds for some \( \delta \leq \delta_0 \), then

\[
\frac{1}{\varepsilon^{1/2}} \| u_{\epsilon} \|_{L^2(\mathbb{R}^n)} \lesssim \| \tilde{L}_{\varphi, \varepsilon} u_{\epsilon} \|_{H^{-1}_{\text{scl}}(\mathbb{R}^n)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n)}
\]

for all \( u \in C^\infty_0 (\tilde{U}^\text{int}) \).

As in Section 5.3, we consider a function \( \Phi : \mathbb{R}^{n-1} \to \mathbb{C} \) defined as

\[
\Phi(\xi) = \frac{1}{1 + |\tilde{V}|^2} \left( 1 + i\tilde{V} \cdot \xi + \sqrt{2i \tilde{V} \cdot \xi - |\tilde{V} \cdot \xi|^2 + (1 + |\tilde{V}|^2)|\xi|^2 - |\tilde{V}|^2} \right), \quad \xi \in \mathbb{R}^{n-1},
\]

but this time the branch of the square root has non-negative real part. Observe that \( \Phi \) is smooth away from the set of those \( \xi \in \mathbb{R}^{n-1} \) when

\[
\tau(\xi) = 2i \tilde{V} \cdot \xi - |\tilde{V} \cdot \xi|^2 + (1 + |\tilde{V}|^2)|\xi|^2 - |\tilde{V}|^2
\]
is real-valued and non-positive, that is when $\tilde{V} \cdot \xi = 0$ and $|\xi|^2 \leq (1 + |\tilde{V}|^2)^{-1}|\tilde{V}|^2$. Since $(1 + |\tilde{V}|^2)^{-1}|\tilde{V}|^2 < \rho_1$, the function $(u_\ell)_{\text{sc}}$ vanishes in the set of such $\xi$'s. Therefore, $\Phi$ is smooth in $\text{supp}(u_\ell)_{\text{sc}}$.

Since $\Re \sqrt{\tau(\xi)} \geq 0$, it is easy to see that $\Re \Phi(\xi) \geq (1 + |\tilde{V}|^2)^{-1}$, and hence by (5.5), $|\Phi(\xi)| \geq (1 + |\tilde{V}|^2)^{-1}$. Therefore, taking constants $\rho_0, \delta_0$ such that

$$\frac{|\tilde{V}|^2}{1 + |\tilde{V}|^2} < \rho_0 < \rho_1, \quad 0 < \delta_0 < \delta_1,$$

we can take a smooth function $F_\ell$ such that $F_\ell(\xi) = \Phi(\xi)$ for $|\xi| \geq \rho_0$ or $|\tilde{V} \cdot \xi| \geq \delta_0$ and $\Re F_\ell(\xi), |F_\ell(\xi)| \geq (1 + |\tilde{V}|^2)^{-1}$ for all $\xi \in \mathbb{R}^{n-1}$.

For large $\xi \in \mathbb{R}^{n-1}$, one can also see that $\Re \Phi(\xi) \geq 1 + |\xi|$ and $|\Phi(\xi)| \leq 1 + |\xi|$, where the implicit constant depends only on $\tilde{V}$. Therefore, according to (5.5), we have

$$\Re F_\ell(\xi), |F_\ell(\xi)| \approx 1 + |\xi|, \quad \xi \in \mathbb{R}^{n-1}.$$

Thus, we obtain smooth $F_\ell : \mathbb{R}^{n-1} \to \mathbb{C}$ that satisfy all the required conditions for $F$ in Section 5.2. Then by $J_\ell, J^*_\ell, J^{-1}_\ell, J^*-1_\ell$ we denote the operators defined as in Section 5.2 with $F$ replaced by $F_\ell$.

Using the similar approach as in the proof of (5.6), one can obtain

$$\frac{h}{\varepsilon^{1/2}} ||u_{\ell}||_{L^2(\mathbb{R}^n_+)} \lesssim ||\tilde{L}_{\varphi, \varepsilon} J^{-1}_\ell u_{\ell}||_{L^2(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} ||u||_{L^2(\mathbb{R}^n_+)}.$$  (5.11)

Our next step is to show that

$$\|v - g\|_{L^2(\mathbb{R}^n_+)} + h \|u_{\ell}\|_{L^2(\mathbb{R}^n_+)} \geq \frac{1}{2} \|v\|_{L^2(\mathbb{R}^n_+)}, \quad v = \tilde{L}_{\varphi, \varepsilon} J^{-1}_\ell u_{\ell},$$  (5.12)

where $g$ is defined as in Lemma 5.6. Unfortunately, the arguments following (5.6) in Section 5.3 can not be carried out, since in the case of large frequency the operator $\tilde{L}_{\varphi, \varepsilon}$ is not elliptic on $\text{supp} u_{\ell}$.

Consider a cutoff function $\kappa \in C^\infty(\mathbb{R}^{n-1})$ such that $\kappa(\xi) = 1$ if $|\xi|^2 \geq \rho_1$ or $|\tilde{V} \cdot \xi| \geq \delta_1$, and $\kappa(\xi) = 0$ if $|\xi|^2 \leq \rho_0$ or $|\tilde{V} \cdot \xi| \leq \delta_0$. Then define

$$G_\lambda(\xi) = (1 - \kappa(\xi)) F_\ell(\xi)$$

and

$$G^f_\pm(\xi) = \kappa(\xi) \frac{\alpha + i(\beta_f, \xi)_{\tilde{g}_0} + \sqrt{\tau_f(x_1, x', \xi)}}{1 + |\gamma_f|^2} + G_\lambda(\xi),$$

where the branch of the square root has non-negative real part and

$$\tau_f(\xi) = 2i\alpha(\beta_f, \xi)_{\tilde{g}_0} - (\beta_f, \xi)_{\tilde{g}_0}^2 + (1 + |\gamma_f|^2) \tilde{g}_0^{ij} \xi_i \xi_j - |\gamma_f|^2 \alpha^2.$$

Observe that $G^f_\pm$ are smooth away from the set of those $\xi \in \mathbb{R}^{n-1}$ when $\tau_f(\xi)$ is real-valued and non-positive, that is when $\langle \beta_f, \xi \rangle_{\tilde{g}_0} = 0$ and

$$\tilde{g}_0^{ij} \xi_i \xi_j \leq \frac{|\gamma_f|^2 \alpha^2}{1 + |\gamma_f|^2}. $$
According to (5.2),
\[ \hat{g}_0^j \xi_j \xi_j \geq (1 - C_\delta)\|\xi\|^2, \]
where \( C_\delta \to 0 \) as \( \delta \to 0 \). On \( \text{supp} \, \kappa \), we have \( |\xi|^2 \geq \rho_0 > (1 + |\vec{V}|^2)^{-1}|\vec{V}|^2 \) and hence, taking small enough \( \delta \), we get
\[ \hat{g}_0^j \xi_j \xi_j > \frac{|\vec{V}|^2}{1 + |\vec{V}|^2}, \quad \xi \in \text{supp} \, \kappa. \]

Since \( |\alpha - 1| \lesssim \hbar \varepsilon^{-1} \), using (5.2) and taking small enough \( \delta \) and \( \hbar \), this implies that
\[ \hat{g}_0^j \xi_j \xi_j > \frac{|\gamma_f|^2 \alpha^2}{1 + |\gamma_f|^2}, \quad \xi \in \text{supp} \, \kappa. \]

Therefore, \( G^j_x \) are smooth in \( \text{supp} \, \kappa \). It is not difficult to check that \( G^j_x \) are symbols.

Now, let \( T_\kappa \) denote the operator which corresponds to the symbol \( a \). Then
\[ (h \partial_1 - T_{G^j_x})(1 + |\gamma_f|^2)(h \partial_1 - T_{G^j_x}) \]
\[ = (1 + |\gamma_f|^2)h^2 \partial_1^2 - 2\alpha + (\beta_f, h \nabla \psi_0 \cdot \hat{g}_0)h \partial_1 \kappa + (\alpha^2 + h^2 \mathcal{L}) \kappa, \]
\[ - 2(1 + |\gamma_f|^2) T_G + (1 + |\gamma_f|^2)(T_{G^j_x} - T_{G^j_x} + T_{G^j_x} - T_{G^j_x} + T_{G^j_x} + T_{G^j_x} + T_{G^j_x} + T_{G^j_x} + h E_1, \]
where \( E_1(x_1, \cdot) \), for each \( x_1 \), is a first-order semiclassical pseudodifferential operator on \( \mathbb{R}^{n-1} \) with bounds being uniform in \( x_1 \).

Write \( w = J^{-1}_\ell u \). Then \( T_\kappa w = w, \) \( T_\kappa w = w \) and \( T_{G^j_x} w = 0 \) since \( \kappa = 1 \) on \( \text{supp}(1 - \rho) \). Therefore,
\[ (h \partial_1 - T_{G^j_x})(1 + |\gamma_f|^2)(h \partial_1 - T_{G^j_x})w \]
\[ = (1 + |\gamma_f|^2)h^2 \partial_1^2 w - 2\alpha + (\beta_f, h \nabla \psi_0 \cdot \hat{g}_0)h \partial_1 w + (\alpha^2 + h^2 \mathcal{L}) w + h E_1 w. \]

This can be rewritten as
\[ (h \partial_1 - T_{G^j_x})(1 + |\gamma_f|^2)(h \partial_1 - T_{G^j_x})w = \tilde{E}_{\varphi, \varepsilon} w + h E_1 w, \]
where \( E_1 \) is modified but with the same properties as before. Recall that \( v = \tilde{E}_{\varphi, \varepsilon} w \). Then, writing \( b = (1 + |\gamma_f|^2)(h \partial_1 - T_{G^j_x})w \),
\[ \hat{g}_{sc} = \frac{2 \text{Re} F_\ell}{h} \int_0^\infty \mathcal{F}_{\text{sc}}((\hat{E}_{\varphi, \varepsilon} w)e^{-\frac{F_{\xi_1} + \bar{F}_{\xi_1}}{h}}) ds \]
\[ = \frac{2 \text{Re} F_\ell}{h} \int_0^\infty \mathcal{F}_{\text{sc}}((h \partial_1 - T_{G^j_x})b)e^{-\frac{F_{\xi_1} + \bar{F}_{\xi_1}}{h}} ds \]
\[ - \frac{2 \text{Re} F_\ell}{h} \int_0^\infty \mathcal{F}_{\text{sc}}(h E_1 w)e^{-\frac{F_{\xi_1} + \bar{F}_{\xi_1}}{h}} ds. \]

Recall that \( u \) is supported away from \( x_1 = 0 \), and hence so are \( w \) and \( b \). Therefore, integrating by parts, we obtain
\[ \hat{g}_{sc} = \frac{2 \text{Re} F_\ell}{h} \int_0^\infty \mathcal{F}_{\text{sc}}((T_{\bar{F}_\ell} - T_{G^j_x})b)e^{-\frac{F_{\xi_1} + \bar{F}_{\xi_1}}{h}} ds \]
\[ - \frac{2 \text{Re} F_\ell}{h} \int_0^\infty \mathcal{F}_{\text{sc}}(h E_1 w)e^{-\frac{F_{\xi_1} + \bar{F}_{\xi_1}}{h}} ds. \]
Using (5.4) for each term on the right hand side, we come to
\[\|g\|_{L^2(\mathbb{R}^3)} \leq \|(T_{F_\ell} - T_{G_\ell})b\|_{L^2(\mathbb{R}^3)} + h\|E_1 w\|_{L^2(\mathbb{R}^3)}.\]
To find an appropriate estimate for \(\|(T_{F_\ell} - T_{G_\ell})b\|_{L^2(\mathbb{R}^3)}\), consider the symbol of \(T_{F_\ell} - T_{G_\ell}\) on \(\text{supp } \kappa\). Since \(F_\ell = \Phi\) on \(\text{supp } \kappa\),
\[\kappa^{-1}(F_\ell - G_\ell) = \left(1 + \frac{i \tilde{V} \cdot \xi}{1 + |\tilde{V}|^2} - \frac{\alpha + i(\beta_f, \xi) \delta}{1 + |\gamma|^2}\right) + \left(\frac{\sqrt{\tau(\xi)}}{1 + |\tilde{V}|^2} - \frac{\sqrt{\tau_i(\xi)}}{1 + |\gamma|^2}\right) = I + II.

These two terms can be rewritten as
\[I = \frac{(1 + i \tilde{V} \cdot \xi)(|\gamma|^2 - |\tilde{V}|^2) - ((1 - \alpha) + i(\tilde{V} \cdot \xi - (\beta_f, \xi) \delta))}{(1 + |\gamma|^2)}\]
and
\[II = \frac{(1 + |\gamma|^2)(\tau - \tau_f) + ((1 + |\gamma|^2)^2 - (1 + |\tilde{V}|^2)^2)\tau}{(1 + |\gamma|^2)((1 + |\gamma|^2)\sqrt{\tau} + (1 + |\tilde{V}|^2)\sqrt{\tau_f})}.

Each of these terms are first-order symbols multiplied by functions bounded by \(\lesssim C_\delta\). Therefore, we obtain
\[\|(T_{F_\ell} - T_{G_\ell})b\|_{L^2(\mathbb{R}^3)} \lesssim C_\delta \|b\|_{H^1_{\text{aci}}(\mathbb{R}^3)},\]
and hence
\[\|g\|_{L^2(\mathbb{R}^3)} \lesssim C_\delta \|b\|_{H^1_{\text{aci}}(\mathbb{R}^3)} + h\|w\|_{H^2_{\text{aci}}(\mathbb{R}^3)}.
\]
Since \((h \partial_1 - T_{G_\ell})b = \tilde{L}_{\varphi, \varepsilon} w + h E_1 w\) and \(J_\ell' = -h \partial_1 + T_{F_\ell}^\perp\),
\[\|b\|_{H^1_{\text{aci}}(\mathbb{R}^3)} \approx \|J_\ell'^* b\|_{L^2(\mathbb{R}^3)} \lesssim \|(T_{F_\ell}^\perp - T_{G_\ell}^\perp) b\|_{L^2(\mathbb{R}^3)} + \|(h \partial_1 - T_{G_\ell}^\perp) b\|_{L^2(\mathbb{R}^3)} \lesssim C_\delta \|b\|_{H^1_{\text{aci}}(\mathbb{R}^3)} + \|\tilde{L}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}^3)} + h\|w\|_{H^2_{\text{aci}}(\mathbb{R}^3)},\]
where we have used the second part of Lemma 5.4. Taking \(\delta\) sufficiently small, we ensure that \(C_\delta\) is small enough, and hence the first term on the last line can be absorbed to the left side to get
\[\|b\|_{H^1_{\text{aci}}(\mathbb{R}^3)} \lesssim \|\tilde{L}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}^3)} + h\|w\|_{H^2_{\text{aci}}(\mathbb{R}^3)}.
\]
Therefore, since \(w = J_\ell^{-1} u_\ell\), using Lemma 5.4, we come to
\[\|g\|_{L^2(\mathbb{R}^3)} \lesssim C_\delta \|\tilde{L}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}^3)} + h\|w\|_{H^2_{\text{aci}}(\mathbb{R}^3)} \lesssim C_\delta \|\tilde{L}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}^3)} + h\|J_\ell^{-1} u_\ell\|_{H^1_{\text{aci}}(\mathbb{R}^3)} \lesssim C_\delta \|\tilde{L}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}^3)} + h\|u_\ell\|_{L^2(\mathbb{R}^3)}.
\]
Then for small enough \(\delta\), we have
\[\|g\|_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{2}\|v\|_{L^2(\mathbb{R}^3)} + h\|u_\ell\|_{L^2(\mathbb{R}^3)}.
\]
This clearly implies (5.12). Combining (5.12) with (5.11), and using Lemma 5.6, we obtain
\[ \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| J_{\theta} \tilde{L}_{\varphi,\varepsilon} J_{\theta}^{-1} u \|_{H^{-1}_c(\mathbb{R}^n_+)} + h \| u \|_{L^2(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \].

Using the second part of Lemma 5.5, this implies
\[ \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| \tilde{L}_{\varphi,\varepsilon} J_{\theta} J_{\theta}^{-1} u \|_{H^{-1}_c(\mathbb{R}^n_+)} + h \| J_{\theta}^{-1} u \|_{H^{-1}_c(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \].

Applying Lemma 5.4 and taking sufficiently small \( \varepsilon \), the second and third terms on the right-hand side can be absorbed to the left, finishing the proof of Proposition 5.8.

5.5. Proof of Proposition 5.1. Using Proposition 5.7 and Proposition 5.8 together with \( u = u_s + u_{\varepsilon} \), we get
\[ \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| \tilde{L}_{\varphi,\varepsilon} u_s \|_{H^{-1}_c(\mathbb{R}^n_+)} + \| \tilde{L}_{\varphi,\varepsilon} u_{\varepsilon} \|_{H^{-1}_c(\mathbb{R}^n_+)} + \frac{h^2}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \].

Taking \( \varepsilon \) sufficiently small, the last term on the right-hand side can be absorbed into the left hand-side and give
\[ \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| \tilde{L}_{\varphi,\varepsilon} u_s \|_{H^{-1}_c(\mathbb{R}^n_+)} + \| \tilde{L}_{\varphi,\varepsilon} u_{\varepsilon} \|_{H^{-1}_c(\mathbb{R}^n_+)} \].

Observe that \( (1 + |\gamma|^2) > 1 + \| \vec{V} \|^2 - C_\delta \) for small \( \delta \). This implies that
\[ \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| (1 + |\gamma|^2)^{-1} \tilde{L}_{\varphi,\varepsilon} u_s \|_{H^{-1}_c(\mathbb{R}^n_+)} + \| (1 + |\gamma|^2)^{-1} \tilde{L}_{\varphi,\varepsilon} u_{\varepsilon} \|_{H^{-1}_c(\mathbb{R}^n_+)} \].

Recall that \( u_s = Pu \), where \( P \) is a zeroth-order semiclassical pseudodifferential operator on \( \mathbb{R}^{n-1} \) with symbol \( \rho \). Since \( P \) commutes with \( \partial_1 \) and since the coefficient of \( h^2 \partial_1^2 \) term in \( (1 + |\gamma|^2)^{-1} \tilde{L}_{\varphi,\varepsilon} \) is 1, we obtain
\[ \| (1 + |\gamma|^2)^{-1} \tilde{L}_{\varphi,\varepsilon} u_s \|_{H^{-1}_c(\mathbb{R}^n_+)} \lesssim \| P(u_s) \|_{H^{-1}_c(\mathbb{R}^n_+)} \lesssim \| Pu_s \|_{H^{-1}_c(\mathbb{R}^n_+)} + h \| hE_0 \partial_1 u + E_1 u \|_{H^{-1}_c(\mathbb{R}^n_+)} \],

where \( E_0(x_1, \cdot) \) and \( E_1(x_1, \cdot) \), for each fixed \( x_1 \in (0, \infty) \), are semiclassical pseudodifferential operators on \( \mathbb{R}^{n-1} \) of order 0 and 1, respectively. Applying Proposition B.1 for each term on the right-hand side, we get
\[ \| (1 + |\gamma|^2)^{-1} \tilde{L}_{\varphi,\varepsilon} u_s \|_{H^{-1}_c(\mathbb{R}^n_+)} \lesssim \| \tilde{L}_{\varphi,\varepsilon} u \|_{H^{-1}_c(\mathbb{R}^n_+)} + h \| u \|_{L^2(\mathbb{R}^n_+)} \].

In a similar way we can obtain the estimate
\[ \| (1 + |\gamma|^2)^{-1} \tilde{L}_{\varphi,\varepsilon} u_{\varepsilon} \|_{H^{-1}_c(\mathbb{R}^n_+)} \lesssim \| \tilde{L}_{\varphi,\varepsilon} u \|_{H^{-1}_c(\mathbb{R}^n_+)} + h \| u \|_{L^2(\mathbb{R}^n_+)} \],

and hence
\[ \frac{h}{\varepsilon^{1/2}} \| u \|_{L^2(\mathbb{R}^n_+)} \lesssim \| \tilde{L}_{\varphi,\varepsilon} u \|_{H^{-1}_c(\mathbb{R}^n_+)} + h \| u \|_{L^2(\mathbb{R}^n_+)} \].
Taking sufficiently small $\varepsilon$ the last term can be absorbed into the left hand-side, giving us
\[ \frac{h}{\varepsilon^{1/2}}\|u\|_{L^2(\Omega)} \lesssim \|\tilde{L}_{\phi,\varepsilon} u\|_{H^{-1}_{\text{car}}(\mathbb{R}^n_+)} . \]
Making the change of the variables back to the original one, we finish the proof of Proposition 5.1.

6. Carleman estimates acting on 1-forms

In this section we prove a global Carleman estimate for 1-forms. Roughly speaking, for the proof we use partition of unity to glue the local Carleman estimate for functions proved in the previous section. We make the same assumptions on $(M,g)$ as in Section 5.

In what follows we use the semiclassical Sobolev norms of differential forms. One can defined those norms, for example, as
\[ \|\eta\|_{H^{m}_{\text{car}}(M)} = \|\eta\|_{L^2(M)} + \sum_{j=1}^n \|h \nabla_{\mathbf{e}^j} (\phi_\alpha \eta)\|_{L^2(M \cap U_\alpha)} , \quad \eta \in H^{1}_{\text{car}}(\mathbb{R}^n_+) , \]
\[ \|\zeta\|_{H^{-1}_{\text{car}}(M)} = \sup \left\{ \frac{\langle \zeta, w \rangle_M}{\|w\|_{H^{-1}_{\text{car}}(M)}} : w \in H^{1}_{\text{car}}(M), w \neq 0 \right\} , \quad \zeta \in H^{-1}_{\text{car}}(\mathbb{R}^n_+) , \]
where $\langle \cdot, \cdot \rangle_M$ denotes the distributional duality on $M$, $\{U_\alpha\}_\alpha$ is an open cover of $M$, and $\{\phi_\alpha\}_\alpha$ is a subordinate partition of unit, and $\{\mathbf{e}^1_\alpha, \ldots, \mathbf{e}^n_\alpha\}$ is a local orthonormal frame on $TU_\alpha$.

Recall that for the limiting Carleman weight $\phi(x) = \pm x_1$, we use the notation $\Gamma_{+,\phi}$ to denote a neighborhood of $\partial M_{+,\phi}$ in $\partial M$ mentioned in Theorem 1.1.

**Theorem 6.1.** Let $(M,g)$ be as described above and let $\phi$ be the limiting Carleman weight $\phi(x) = \pm x_1$. Suppose that $M_1$ is a smooth compact manifold with boundary such that $M \subset M_1$ and $\partial M \cap \partial M_1 = \Gamma_{+,\phi}$, where $\Gamma_{+,\phi}$ is a neighborhood of $\partial M_{+,\phi}$. Then there is $0 < \varepsilon_0 \ll 1$ such that for $0 < h \ll \varepsilon < \varepsilon_0$ we have
\[ \frac{h}{\varepsilon^{1/2}}\|u\|_{L^2(M)} \lesssim \|h^2 e^{\phi_+ / h} (-\Delta)(e^{-\phi_+ / h} u)\|_{H^{-1}_{\text{car}}(M_1)} , \quad u \in C^\infty_0 \Omega^1(M^\text{int}) . \]

Let us first prove the following important consequence of Theorem 6.1 when the Hodge-Laplacian is perturbed by a first-ordered linear differential operator.

**Corollary 6.2.** Let $(M,g)$ be as described above and let $\phi$ be the limiting Carleman weight $\phi(x) = \pm x_1$. Suppose that $M_1$ is a smooth compact manifold with boundary such that $M \subset M_1$ and $\partial M \cap \partial M_1 = \Gamma_{+,\phi}$, where $\Gamma_{+,\phi}$ is a neighborhood of $\partial M_{+,\phi}$. Assume that $W : H^{1}_{\text{car}}(M_1) \to L^2_{\text{car}}(M_1)$ is a first-order linear differential operator whose purely first-order part has $C^1$ coefficients and zeroth-order linear part has continuous coefficients. Then there is $0 < h_0 \ll 1$ such that for $0 < h \leq h_0$ we have
\[ h\|u\|_{L^2(M)} \lesssim \|h^2 e^{\phi_+ / h} (-\Delta + W)(e^{-\phi_+ / h} u)\|_{H^{-1}_{\text{car}}(M_1)} , \quad u \in C^\infty_0 \Omega^1(M^\text{int}) . \]
Proposition 6.3. If we denote by Proposition 6.3. If we denote by

$$W : H^1 \Omega^1(M_1) \to L^2 \Omega^1(M_1)$$

is a semiclassical first-order linear differential operator and $Q : L^2 \Omega^1(M_1) \to L^2 \Omega^1(M_1)$ is a zeroth-order linear differential operator whose coefficients are uniformly bounded with respect to $h$ and $\varepsilon$. Therefore, by Theorem 6.1

$$\frac{h}{\varepsilon^{1/2}} \|u\|_{L^2 \Omega^1(M)} \lesssim \|h^2 e^{\varphi_\varepsilon/h}(-\Delta + W)(e^{-\varphi_\varepsilon/h}u)\|_{H^{-1}_\mathrm{sc}\Omega^1(M_1)} + h \|u\|_{L^2 \Omega^1(M_1)}$$

for all $u \in C^\infty_0 \Omega^1(M^\text{int})$. Taking $\varepsilon > 0$ sufficiently small, the last term can be absorbed into the left side, and give

$$\frac{h}{\varepsilon^{1/2}} \|u\|_{L^2 \Omega^1(M)} \lesssim \|h^2 e^{\varphi_\varepsilon/h}(-\Delta + W)(e^{-\varphi_\varepsilon/h}u)\|_{H^{-1}_\mathrm{sc}\Omega^1(M_1)}$$

for all $u \in C^\infty_0 \Omega^1(M^\text{int})$. Since $e^{h\varphi_\varepsilon/2\varepsilon}$ is smooth and bounded on $M$, we come to the desired estimate completing the proof.

To prove Theorem 6.1, we need the following two local Carleman estimates for the Hodge-Laplacian $\Delta$ acting on 1-forms. These are obtained from local Carleman estimates for the Laplace-Beltrami operator $\Delta_g$ (acting on functions) in Proposition 5.1 by observing that, locally, the principal part of $\Delta$ is $\Delta_g$.

Proposition 6.3. For $p \in \Gamma_+^+$, let $U$ be a sufficiently small neighborhood of $p$ in $\mathbb{R} \times M_0$ such that $M \cap U$ has a smooth boundary. Suppose that there is a smooth $f : M_0 \to \mathbb{R}$ such that $M \cap U$ lies in the set $A_f := \{x_1 \geq f(x')\}$, and $\Gamma_+^+ \cap U \subset \{x_1 = f(x')\}$. Suppose that there is a choice of local coordinates on $\pi(U)$ such that there are a constant $\delta > 0$ and a constant vector field $K$ on $\pi(M \cap U)$ for which

$$|\nabla g_0 - \text{Id}| \leq \delta, \quad |\nabla g_0 f - K| |g_0| \leq \delta$$

on $\pi(M \cap U)$. Then there is $0 < \varepsilon_0 \ll 1$ such that for $0 < h \ll \varepsilon \ll \varepsilon_0$ we have

$$\frac{h}{\varepsilon^{1/2}} \|u\|_{L^2 \Omega^1(M \cap U)} \lesssim \|h^2 e^{\varphi_\varepsilon/h}(-\Delta)(e^{-\varphi_\varepsilon/h}u)\|_{H^{-1}_\mathrm{sc}\Omega^1(A_f)}, \quad u \in C^\infty_0 \Omega^1(M^\text{int} \cap U).$$

Proof. If we denote by $x'$ the above mentioned local coordinates in $\pi(U)$, then $x = (x^1, \ldots, x^n) = (x_1, x')$ will be local coordinates in $U$. Suppose that the expression of $u \in C^\infty_0 \Omega^1(M^\text{int} \cap U)$ in these local coordinates is $u = u_j(x) \, dx^j$. Then we have

$$(\Delta u)_j(x) \, dx^j = (\Delta_g u_j)(x) \, dx^j + (Eu)_j(x) \, dx^j,$$

where $E$ is a first-order linear differential operator with smooth coefficients; see [30, page 183]. This allows us to write

$$h^2 e^{\varphi_\varepsilon/h}(-\Delta)(e^{-\varphi_\varepsilon/h}u)_j(x) \, dx^j = (L_{\varphi_\varepsilon} u_j)(x) \, dx^j + h(E_{\varphi_\varepsilon} u_j)(x) \, dx^j,$$

where $E_{\varphi_\varepsilon}$ is a semiclassical first-order linear differential operator with smooth coefficients uniformly bounded by $\lesssim (1 + h/2\varepsilon)$. Therefore,

$$\|h^2 e^{\varphi_\varepsilon/h}(\Delta)(e^{-\varphi_\varepsilon/h}u)\|_{H^{-1}_\mathrm{sc}\Omega^1(A_f)} \lesssim \sum_{j=1}^n \|L_{\varphi_\varepsilon} u_j\|_{H^{-1}_\mathrm{sc}\Omega^1(A_f)} - h\|E_{\varphi_\varepsilon} u_j\|_{H^{-1}_\mathrm{sc}\Omega^1(A_f)}.$$
Using Proposition 5.1, this implies
\[ \|h^2 e^{\varphi/h} \Delta (e^{-\varphi/h} u)\|_{H^{-1}_a \Omega^1(A_j)} \gtrsim \sum_{j=1}^{n} \frac{h}{\varepsilon^{1/2}} \|\phi_j\|_{L^2(M \cup U)} - h(1 + \frac{h}{2\varepsilon}) \|u\|_{L^2 \Omega^1(A_j)} \gtrsim \frac{h}{\varepsilon^{1/2}} \|u\|_{L^2 \Omega^1(M \cup U)} - h(1 + \frac{h}{2\varepsilon}) \|u\|_{L^2 \Omega^1(M \cup U)}. \]

Taking sufficiently small \( \varepsilon > 0 \), we can finish the proof as in Proposition 6.3.

**Proposition 6.4.** Let \( U \) be sufficiently small open set in \( \mathbb{R} \times M_0 \). Then there is \( 0 < \varepsilon_0 \ll 1 \) such that for \( 0 < h < \varepsilon < \varepsilon_0 \) we have
\[ \frac{h}{\varepsilon^{1/2}} \|u\|_{L^2 \Omega^1(U)} \lesssim \|h^2 e^{\varphi/h}(-\Delta)(e^{-\varphi/h} u)\|_{H^{-1}_a \Omega^1(U)}, \quad u \in C_0^\infty \Omega^1(U). \]

**Proof.** The proof is the same as in the previous proposition, except instead of Proposition 5.1 one needs to use the estimate (4.7) in [13, Section 4].

Now we are ready to prove Theorem 6.1 by gluing the local Carleman estimates in Proposition 6.3 and Proposition 6.4 via partition of unity.

**Proof of Theorem 6.1.** Let \( U_1, \ldots, U_m \) be finite cover of \( M \) such that each \( M \cap U_j \) has smooth boundary. Suppose that each \( \Gamma_{\epsilon, \varphi} \cap U_j \) is either empty (in this case we assume that such \( U_j \) is sufficiently small) or represented as a graph of the form \( x_1 = f_j(x') \), for some smooth \( f_j : M_0 \to \mathbb{R} \), with \( M \cap U_j \subset A_{f_j} = \{ x_1 \geq f_j(x') \} \) and there is a choice of coordinates in \( \pi(U_j) \) such that \( |g_0 - \text{Id}| \leq \delta_j \) and \( |\nabla g_0 f - K_j|g_0| \leq \delta_j \) for some constant \( \delta_j > 0 \) and for some constant vector field \( K_j \) on \( \pi(M \cap U_j) \).

Consider the partition of unity \( \chi_1, \ldots, \chi_m \) subordinate to \( U_1, \ldots, U_m \). For \( u \in C_0^\infty \Omega^1(M^{\text{int}}) \) define \( \tilde{u}_j = \chi_j u \in C_0^\infty \Omega^1(M^{\text{int}} \cap U_j) \). If \( \Gamma_{\epsilon, \varphi} \cap U_j \neq \emptyset \), by Proposition 6.3
\[ \frac{h}{\varepsilon^{1/2}} \|\tilde{u}_j\|_{L^2 \Omega^1(M \cup U_j)} \lesssim \|h^2 e^{\varphi/h}(-\Delta)(e^{-\varphi/h} \tilde{u}_j)\|_{H^{-1}_a \Omega^1(A_j)}. \]

Otherwise, by Proposition 6.4
\[ \frac{h}{\varepsilon^{1/2}} \|\tilde{u}_j\|_{L^2 \Omega^1(M \cup U_j)} \lesssim \|h^2 e^{\varphi/h}(-\Delta)(e^{-\varphi/h} \tilde{u}_j)\|_{H^{-1}_a \Omega^1(M \cup U_j)}. \]

Suppose that \( \Gamma_{\epsilon, \varphi} \cap U_j \neq \emptyset \). Since \( \partial M_1 \cap \partial M = \Gamma_{\epsilon, \varphi} \) and since \( \partial \varphi < 0 \) on \( \Gamma_{\epsilon, \varphi} \), \( \partial M_1 \) must be represented as a graph of the form \( x_1 = f_j(x') \) near \( \Gamma_{\epsilon, \varphi} \), for some smooth \( f_j : M_0 \to \mathbb{R} \) as above. Therefore, we can assume that each \( A_{f_j} \) coincides with \( M_1 \) near each \( U_j \). Then there is \( \phi_j \in C_0^\infty(\mathbb{R} \times M_0) \) such that \( \phi_j \equiv 1 \) on \( U_j \) and \( \phi_j \equiv 0 \) on the complements of \( A_{f_j} \) and \( M_1^{\text{int}} \). Multiplication by this function is a bounded map from \( H^1_0 \Omega^1(A_{f_j}) \) to \( H^0_0 \Omega^1(M_1) \) and vice versa. This implies that
\[ \|w\|_{H^{-1}_a \Omega^1(M_1)} \approx \|w\|_{H^{-1}_a \Omega^1(A_{f_j})}, \quad w \in C_0^\infty \Omega^1(M^{\text{int}} \cap U_j). \]

Therefore, for the case \( \Gamma_{\epsilon, \varphi} \cap U_j \neq \emptyset \) we have
\[ \frac{h}{\varepsilon^{1/2}} \|\tilde{u}_j\|_{L^2 \Omega^1(M \cup U_j)} \lesssim \|h^2 e^{\varphi/h}(-\Delta)(e^{-\varphi/h} \tilde{u}_j)\|_{H^{-1}_a \Omega^1(M_1)}. \]
Now suppose that $\Gamma_{\pm,\varphi} \cap U_j = \emptyset$. We can assume that $U_j \subset M_1$. Therefore, for the case $\Gamma_{\pm,\varphi} \cap U_j = \emptyset$ we have

\[
\frac{h}{\varepsilon^{1/2}} \| \tilde{u}_j \|_{L^2(\Omega^1(M \cap U_j))} \lesssim \| h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h} \|_{H^{-1}_{\text{ext}}(\Omega^1(M_1))}, \quad (6.2)
\]

Gluing the estimates of the form (6.1) and (6.2) together, we finish the proof. \qed

We finish this section with the following solvability results.

**Proposition 6.5.** Let $(M, g)$ be as described above and let $\varphi$ be the limiting Carleman weight $\varphi(x) = \pm x_1$. Suppose that $M_1$ is a smooth compact manifold with boundary such that $M \subset M_1$ and $\partial M \cap \partial M_1 = \Gamma_{\pm,\varphi}$, where $\Gamma_{\pm,\varphi}$ is a neighborhood of $\partial M_{\pm,\varphi}$. Assume that $W : H^1_0(M_1) \to L^2(M_1)$ is a first-order linear differential operator whose purely first-order part has $C^1$ coefficients and zeroth-order linear part has continuous coefficients. Then there is $0 < h_0 \ll 1$ such that for $0 < h \leq h_0$ and for given $v \in L^2(M)$ there is $u \in H^1_0(M)$ satisfying

\[
h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h} = v \text{ in } M,
\]

and

\[
\| u \|_{H^1_0(M_1)} \lesssim \frac{1}{R} \| v \|_{L^2(M)}.
\]

**Proof.** Define a linear functional $L$ on

\[
(h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h}^* C_0^\infty(M^\text{int}) \subset H^{-1}_0(M_1)
\]

by

\[
L((h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h} w) = (v|w)_{L^2(M)}, \quad w \in C_0^\infty(M^\text{int}).
\]

Then we have

\[
\left| L((h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h} w) \right| \leq \| v \|_{L^2(M)} \| w \|_{L^2(M)}
\]

\[
\lesssim \frac{1}{R} \| v \|_{L^2(M)} \| (h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h}^* w) \|_{H^{-1}_0(M_1)},
\]

where in the last step we have used the Carleman estimate in Corollary 6.2. By the Hahn-Banach theorem, we may extend $L$ to a linear continuous functional $\overline{L}$ on $H^{-1}_0(M_1)$. By the Riesz representation theorem, there exists $u \in H^1_0(M_1)$ such that

\[
\overline{L}(f) = (u|f)_{L^2(M)}, \quad f \in L^2(M).
\]

In particular,

\[
(u|(h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h}^* w)_{L^2(M)} = \overline{L}((h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h} w)
\]

\[
= (v|w)_{L^2(M)},
\]

for all $w \in C_0^\infty(M^\text{int})$. Therefore, we obtain $h^2 e^{\varphi^h/\varepsilon}(\varepsilon - \varphi^{h})^{\varepsilon h} u = v$ and

\[
\| u \|_{H^1_0(M_1)} \lesssim \frac{1}{R} \| v \|_{L^2(M)}.
\]

Finally, $u \in H^1_0(M_1)$ implies that $u|_{\Gamma_{\pm,\varphi}} = 0$. \qed

The following is a consequence of [16, Proposition 4.1].
Suppose that \( E_\rho \) with \( \text{Re} \rho \) concentrated near geodesics on the transversal simple manifold \((M, g_0)\) such that for all \( 0 < h < \hbar \) and for every \( f \in L^2(M) \), there is a unique \( u \in H^2(M) \) solving

\[
e^{\varphi/h} (-\Delta_g) e^{-\varphi/h} u = f
\]

and satisfying the estimate \( \|u\|_{L^2(M)} \lesssim h \|f\|_{L^2(M)} \).

7. Complex geometrical optics solutions

In this section, combining the ideas of [7, 17] and [28], we give the construction of the complex geometrical optics solutions for the system to which the Maxwell equations were reduced in Section 4 and then relate these solutions to the Maxwell equations. More precisely, we construct complex geometrical optics solutions for the system

\[
\mathcal{L}_{\varepsilon, \mu} E = 0, \quad \delta(E) = 0, \quad E|_{\Gamma^c_{+, \varphi}} = 0,
\]

where \( \varphi \) is the limiting Carleman weight \( \varphi(x) = \pm x_1 \). Then using Proposition 4.2, we obtain the appropriate solutions for the Maxwell equations.

Let \((M, g)\) be a 3-dimensional admissible manifold. Throughout the section, we assume that \( M \subset \mathbb{R} \times M^0_{\text{int}} \) and that the metric has the form \( g = e \oplus g_0 \), where \( e \) is Euclidean metric on \( \mathbb{R} \) and \((M_0, g_0)\) is a simple 2-dimensional manifold.

The solutions that we want to construct are of the form

\[
E = e^{-(\varphi+i\psi)/h}(A + R - e^{-\rho/h} B),
\]

where \( \psi \) is a certain real-valued phase to be chosen, \( A \in C^2 \Omega^1(M) \) is specific and concentrated near geodesics on the transversal simple manifold \((M_0, g_0)\), \( \rho \in C^2(M) \) with \( \text{Re} \rho \approx \text{dist}(\cdot, \Gamma^c_{+, \varphi}) \) in a neighborhood of \( \Gamma_{+, \varphi}^c \), \( B \in C^2 \Omega^1(M) \) supported near \( \Gamma_{+, \varphi}^c \) and satisfy \( B|\Gamma_{+, \varphi}^c = A|\Gamma_{+, \varphi}^c \), and \( R \in H^1_{\Delta} \Omega^1(M) \) is the correction term.

Suppose that \( E \) is of the form (7.1). Writing \( \zeta = \varphi + i\psi + \rho \) we can write \( E \) as

\[
E = e^{-\varphi i/h} (A + R) - e^{-\zeta i/h} B.
\]

Then the equation \( \mathcal{L}_{\varepsilon, \mu} E = 0 \) is equivalent to

\[
e^{\varphi+i\psi/h} h^2 \mathcal{L}_{\varepsilon, \mu} e^{-(\varphi+i\psi)/h} R = F_1 + F_2,
\]

where \( F_1 := -e^{\varphi+i\psi/h} h^2 \mathcal{L}_{\varepsilon, \mu} e^{-(\varphi+i\psi)/h} A \) and \( F_2 := e^{\varphi+i\psi/h} h^2 \mathcal{L}_{\varepsilon, \mu} e^{-\zeta i/h} B \).

To choose \( \psi \), recall that the transversal manifold \((M_0, g_0)\) is assumed to be simple. Choose another simple manifold \((\widehat{M}_0, g_0)\) such that \( M_0 \subset \subset \widehat{M}_0 \) and choose \( p \in \widehat{M}_0 \setminus M_0 \). Simplicity of \((\widehat{M}_0, g_0)\) implies that there are globally defined polar coordinates \((r, \theta)\) centered at \( p \). In these coordinates, the metric \( g \) has the form

\[
g = e \oplus \begin{pmatrix} 1 & 0 \\ 0 & m(r, \theta) \end{pmatrix},
\]

where \( m \) is a smooth positive function. Then following [13, Section 5.1], we take \( \psi(x) = r \). In these coordinates it is not difficult to show that

\[
\langle d(\varphi + i\psi), d(\varphi + i\psi) \rangle_g = 0.
\]
Here the Riemannian inner product $\langle \cdot, \cdot \rangle_g$ was extended as a complex bilinear form acting on complex valued 1-forms.

7.1. Transport equation. Using [16, Lemma 6.2], one can show that

$$
F_1 = -e^{(\varphi+i\psi)}/h^2 L_{\varepsilon,\mu} e^{-(\varphi+i\psi)}/h A
$$

$$
= -\langle d(\varphi + iy), d(\varphi + iy) \rangle_g A
+ h(2\nabla \varphi + i\psi) A + \Delta_g (\varphi + i\psi) A + (i d \log \varepsilon A) d(\varphi + iy) - (i d \log \mu d(\varphi + iy)) A
+ h^2 L_{\varepsilon,\mu} A
$$

$$
= h(2\nabla \varphi + i\psi) A + \Delta_g (\varphi + i\psi) A + (i d \log \varepsilon A) d(\varphi + iy) - (i d \log \mu d(\varphi + iy)) A
+ h^2 L_{\varepsilon,\mu} A.
$$

In order to get $\|F_1\|_{L^2\Omega^1(M)} \lesssim h^2$, we should construct $A$ satisfying the following transport equation

$$
2\nabla \varphi + i\psi) A + \Delta_g (\varphi + i\psi) A + (i d \log \varepsilon A) d(\varphi + iy) - (i d \log \mu d(\varphi + iy)) A = 0. \quad (7.3)
$$

Consider the operators

$$
\partial = \frac{1}{2} (\partial_1 - i \partial_r), \quad \overline{\partial} = \frac{1}{2} (\partial_1 + i \partial_r).
$$

Proposition 7.1. Let $\varphi$ be the limiting Carleman weight $\varphi(x) = \pm x_1$ and let $\psi$ be the phase function $\psi(x) = r$. For any $b \in C^\infty(S^1)$ and arbitrary $\lambda, s_0 \in \mathbb{R}$, transport equation $(7.3)$ has a solution in $C^2 \Omega^1(M)$ of the following form:

(a) if $\varphi(x) = x_1$, then

$$
A = e^{i\lambda(x_1 + iy)} b(\theta)^2 \{ |g|^{-1/4} \varepsilon^{-1/2} \Psi e^{i\Phi}(dx_1 - idr) + s_0 |g|^{1/4} d\theta \},
$$

where $\Phi, \Psi \in C^2(M)$ are solutions for

$$
2\overline{\partial} \Phi = \partial_1 \log \varepsilon, \quad 4\overline{\partial} \Psi = -s_0 |g|^{-1/2} \varepsilon^{1/2} e^{-i\Phi} \partial_0 \log \varepsilon \quad \text{in} \quad M.
$$

(b) if $\varphi(x) = x_1$, then

$$
A = e^{i\lambda(x_1 + iy)} b(\theta)^2 \{ |g|^{-1/4} \varepsilon^{-1/2} \Psi (dx_1 + idr) + s_0 |g|^{1/4} d\theta \},
$$

where $\Psi \in C^2(M)$ is a solution for

$$
4\partial \Psi = -s_0 |g|^{-1/2} \varepsilon^{1/2} \partial_0 \log \varepsilon \quad \text{in} \quad M.
$$

(c) if $\varphi(x) = -x_1$, then

$$
A = e^{i\lambda(x_1 - iy)} b(\theta)^2 \{ |g|^{-1/4} \varepsilon^{-1/2} \Psi e^{i\Phi}(dx_1 + idr) + s_0 |g|^{1/4} d\theta \},
$$

where $\Phi, \Psi \in C^2(M)$ are solutions for

$$
2\partial \Phi = -\partial_1 \log \varepsilon, \quad 4\partial \Psi = -s_0 |g|^{-1/2} \varepsilon^{1/2} e^{-i\Phi} \partial_0 \log \varepsilon \quad \text{in} \quad M.
$$

(d) if $\varphi(x) = -x_1$, then

$$
A = e^{i\lambda(x_1 - iy)} b(\theta)^2 \{ |g|^{-1/4} \varepsilon^{-1/2} \Psi (dx_1 - idr) + s_0 |g|^{1/4} d\theta \},
$$

where $\Psi \in C^2(M)$ is a solution for

$$
4\partial \Psi = -s_0 |g|^{-1/2} \varepsilon^{1/2} \partial_0 \log \varepsilon \quad \text{in} \quad M.
$$
Here and in what follows, we take the principal branch of the square root.

**Proof.** We first give the proof for parts (a) and (b), i.e. for the case $\varphi(x) = x_1$. Then the transport equation can be rewritten as

$$2\nabla_{(x_1 + ir)} A + \Delta_g(x_1 + ir) A + (i_d \log \varepsilon) A (x_1 + ir) - (i_d \log \mu) A (x_1 + ir)) A = 0. \quad (7.4)$$

Since the metric $g$ has the form \( \gamma = \gamma_0 \), one can show that

$$\nabla (x_1 + ir) = 2\gamma_0, \quad \Delta_g (x_1 + ir) = \gamma_0 \log |g|$$

and

$$\nabla \partial_1 dx_1 = \nabla \partial_1 dr = \nabla \partial_r dx_1 = \nabla \partial_\theta d\theta = 0, \quad \nabla \partial_\theta d\theta = i(\gamma_0 \log |g|) d\theta.$$  

Then looking for an ansatz of the form $A = A_1 dx_1 + A_r dr + A_\theta d\theta$, we compute

$$2\nabla_{(x_1 + ir)} A = (4\gamma A_1) dx_1 + (4\gamma A_r) dr + (4\gamma A_\theta - 2(\gamma_0 \log |g|)) A_\theta) d\theta,$$

$$(\Delta_g (x_1 + ir)) A = (\gamma_0 \log |g|)(A_1 dx_1 + A_r dr + A_\theta d\theta),$$

$$(i_d \log \varepsilon) A (x_1 + ir) = [(\partial_\theta \log \varepsilon) A_1 + (\partial_\theta \log \varepsilon) A_r] (x_1 + ir),$$

$$(i_d \log \mu) A (x_1 + ir) = (\partial_\theta \log \mu + i\partial_\theta \log \mu)(A_1 dx_1 + A_r dr + A_\theta d\theta).$$

Substituting these expressions in (7.4), the transport equation becomes

$$0 = \left\{ 4\gamma A_1 + (\gamma_0 \log |g|) A_1 + (\partial_\theta \log \varepsilon - \partial_\theta \log \mu - i\partial_\theta \log \mu) A_1 \right\} dx_1$$

$$+ (\partial_r \log \varepsilon) A_r + |g|^{-1}(\partial_\theta \log \varepsilon) A_\theta \right\} dx_1$$

$$+ \left\{ 4\gamma A_r + (\gamma_0 \log |g|) A_r + (i\partial_\theta \log \varepsilon - \partial_\theta \log \mu - i\partial_\theta \log \mu) A_r \right\} dr$$

$$+ \left\{ 4\gamma A_\theta - (\gamma_0 \log |g|) A_\theta - (\partial_\theta \log \mu) A_\theta - i(\partial_\theta \log \mu) A_\theta \right\} d\theta.$$  

Multiplying the above equation by $|g|^{1/4}$ and setting $a_1 = |g|^{1/4} A_1$, $a_r = |g|^{1/4} A_r$ and $a_\theta = |g|^{-1/4} A_\theta$, we obtain

$$0 = \left\{ 4\gamma a_1 + (\partial_\theta \log \varepsilon - \partial_\theta \log \mu - i\partial_\theta \log \mu) a_1 \right\} dx_1$$

$$+ (\partial_r \log \varepsilon) a_r + |g|^{-1/2}(\partial_\theta \log \varepsilon) a_\theta \right\} dx_1$$

$$+ \left\{ 4\gamma a_r + (i\partial_\theta \log \varepsilon - \partial_\theta \log \mu - i\partial_\theta \log \mu) a_r \right\} dr$$

$$+ \left\{ 4\gamma a_\theta - (\partial_\theta \log \mu + i\partial_\theta \log \mu) a_\theta \right\} d\theta. \quad (7.5)$$

To solve this equation we take

$$a_1 = e^{i(\lambda + ir)x_1} b(\theta) \mu^{1/2} \varepsilon^{-1/2} \Psi e^{i\Phi}, \quad a_r = -ia_1, \quad a_\theta = e^{i(\lambda + ir)x_1} b(\theta) \mu^{1/2} s_0,$$

where $\lambda \in \mathbb{R}, \ b \in C^\infty(S^1)$ and $\Phi, \Psi$ are solutions for $2\gamma \Phi = \partial_\theta \log \varepsilon$ and $4\gamma \Psi = - s_0 |g|^{-1/2} \varepsilon^{-1/2} e^{i\Phi} \partial_\theta \log \varepsilon$. Then $\Phi, \Psi$ are in $C^2(M)$ since $\partial_\theta \log \varepsilon \in C^2(M)$. It is
easy to check directly that such chosen \( a_1 \) and \( a_r \) satisfy the equation (7.5), proving part (a).

To prove part (b), we make the choice

\[
a_1 = e^{i\lambda(x_1 + ir)}b(\theta)\varepsilon^{-1/2}\mu^{1/2}s_0, \quad a_r = ia_1, \quad a_\theta = e^{i\lambda(x_1 + ir)}b(\theta)\mu^{1/2}s_0,
\]

where \( \lambda \in \mathbb{R}, \ b \in C^\infty(S^1) \) and \( \Psi \in C^2(M) \) solves

\[
4\partial\Psi = -s_0|g|^{-1/2}\varepsilon^{1/2}\partial_\theta \log \varepsilon.
\]

Part (c) can be treated in the following way. According to part (a), there is a solution \( \overline{A} \) for

\[
2\nabla \nabla(x_1 + ir)\overline{A} + \Delta_g(x_1 + ir)\overline{A} + (i_d \log \varepsilon)\overline{d}(x_1 + ir) - (i_d \log \varepsilon)\overline{d}(x_1 + ir)\overline{A} = 0 \tag{7.6}
\]

of the form

\[
\overline{A} = e^{i(-\lambda)(x_1 + ir)}(-b(\theta))^{1/2}\{ |g|^{-1/4}\varepsilon^{-1/2}\overline{\Psi} e^{i(-\overline{\Phi})}(dx_1 + idr) + s_0|g|^{1/4}d\theta \}
\]

where \( (-\overline{\Phi}), \overline{\Psi} \in C^2(M) \) are solutions for

\[
2\overline{\partial}(-\overline{\Psi}) = \partial_r \log \varepsilon, \quad 4\overline{\partial}(\overline{\Psi}) = -s_0|g|^{-1/2}\varepsilon^{1/2}\partial_\theta \log \varepsilon.
\]

Since (7.6) is linear in \( \overline{A} \) and linear in \( x_1 + ir \), taking its complex conjugate, one can show that

\[
A = e^{i\lambda(x_1 - ir)}b(\theta)\mu^{1/2}\{ |g|^{-1/4}\varepsilon^{-1/2}\Psi e^{\Phi}(dx_1 + idr) + s_0|g|^{1/4}d\theta \}
\]

solves (7.3) with \( \varphi(x) = -x_1 \) and \( \psi(x) = r \), and \( \Phi, \Psi \in C^2(M) \) are solutions for

\[
2\partial_\nu = -\partial_\nu \log \varepsilon \text{ and } 4\partial(\overline{\Psi}) = -s_0|g|^{-1/2}\varepsilon^{1/2}\partial_\theta \log \varepsilon.
\]

Finally, one can prove part (d) following the similar reasonings as in the proof of part (c).

\[
\square
\]

7.2. Approximate solutions for eikonal and transport equations. One can also show that

\[
F_2 = e^{(\varphi + \psi)/h}h^2 L_{\varepsilon, \mu} e^{-\zeta/h} B
\]

\[
e^{(\varphi + \psi)/h} e^{-\zeta/h} \langle d\zeta, d\zeta \rangle \| B \|
\]

\[
-h e^{(\varphi + \psi)/h} e^{-\zeta/h} \{ 2\nabla \nabla \zeta B + (\Delta_g \zeta) B + (i_d \log \varepsilon) B \}
\]

\[
-h^2 e^{(\varphi + \psi)/h} e^{-\zeta/h} L_{\varepsilon, \mu} B.
\]

In order to get \( \|F_2\|_{L^2(\Omega)} \lesssim h^2 \), we want to construct \( \zeta \) and \( B \) satisfying

\[
|e^{(\varphi + \psi)/h} e^{-\zeta/h} \langle d\zeta, d\zeta \rangle| \lesssim h^2 \tag{7.7}
\]

and

\[
|e^{(\varphi + \psi)/h} e^{-\zeta/h} \{ 2\nabla \nabla \zeta B + (\Delta_g \zeta) B + (i_d \log \varepsilon) B \}| \lesssim h, \tag{7.8}
\]

respectively.

**Proposition 7.2.** There is \( \zeta \in C^2(M) \) satisfying (7.7) such that \( \Re \zeta - \varphi \approx \text{dist}(\cdot, \Gamma^*_+; \varphi) \) in a neighborhood of \( \Gamma^*_+; \varphi \).
Let \((x, y) = (x^1, x^2, y)\) be the boundary normal coordinates near \(\Gamma^c_{+,\varphi}\) such that \(\{y < 0\} \subset M^{\text{int}}\). In these coordinates, \(\{y = 0\}\) corresponds to \(\partial M, \nu = \partial_y\) and the metric has the form

\[
g = g_{\alpha\beta} \, dx^\alpha \otimes dx^\beta + dy \otimes dy.
\]

Here and in what follows, we use the convention that Greek indices run from 1 to 2.

Note also that \(x = (x^1, x^2)\) is a local coordinate on \(\Gamma^c_{+,\varphi}\) and that \(|y| \approx \text{dist}(\cdot, \Gamma^c_{+,\varphi})\) near \(\Gamma^c_{+,\varphi}\).

To construct a desired function \(\zeta\), we require

\[
\zeta\big|_{\Gamma^c_{+,\varphi}} = (\varphi + i\psi)|_{\Gamma^c_{+,\varphi}}.
\]

In order to ensure that \(\zeta\) will be different from \(\varphi + i\psi\), we also require

\[
\partial_\nu \zeta|_{\Gamma^c_{+,\varphi}} = -\partial_\nu (\varphi + i\psi)|_{\Gamma^c_{+,\varphi}}.
\]

We will look for \(\zeta\) in the form

\[
\zeta(x, y) = \sum_{j=0}^{4} \zeta_j(x) y^j.
\]

Boundary conditions determine \(\zeta_0\) and \(\zeta_1\):

\[
\zeta_0 = (\varphi + i\psi)|_{\Gamma^c_{+,\varphi}}, \quad \zeta_1 = -\partial_\nu (\varphi + i\psi)|_{\Gamma^c_{+,\varphi}}.
\]

Then

\[
d\zeta = \left( \sum_{j=0}^{4} \partial_\alpha \zeta_j(x) y^j \right) \, dx^\alpha + \left( \sum_{j=0}^{4} j \zeta_j(x) y^{j-1} \right) \, dy,
\]

and hence

\[
\langle d\zeta, d\zeta \rangle_g = (g^{\alpha\beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_0 + \zeta_1^2)
\]

\[
+ y (2g^{\alpha\beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_1 + 4\zeta_1^2)
\]

\[
+ y^2 (2g^{\alpha\beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_2 + 2g^{\alpha\beta} \partial_\alpha \zeta_1 \partial_\beta \zeta_1 + 4\zeta_2^2 + 6\zeta_1^3)
\]

\[
+ y^3 (2g^{\alpha\beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_3 + 2g^{\alpha\beta} \partial_\alpha \zeta_1 \partial_\beta \zeta_2 + 12\zeta_2 \zeta_3 + 8\zeta_1 \zeta_4)
\]

\[
+ y^4 r(\zeta, y),
\]

where \(r\) is uniformly bounded in \(\zeta\) and \(y\), for sufficiently small \(|y| > 0\). First, we show that

\[
|\langle d\zeta, d\zeta \rangle_g| \lesssim \text{dist}(\cdot, \Gamma^c_{+,\varphi})^4
\]

near \(\Gamma^c_{+,\varphi}\). Since \(|y| \approx \text{dist}(\cdot, \Gamma^c_{+,\varphi})\) near \(\Gamma^c_{+,\varphi}\), this is equivalent to showing that

\[
|\langle d\zeta, d\zeta \rangle_g| \lesssim |y|^4.
\]
To ensure this, we would like \( \zeta_0, \zeta_1, \zeta_2, \zeta_3 \) and \( \zeta_4 \) to satisfy the following equations
\[
\begin{align*}
g^{\alpha \beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_0 + \zeta_1^2 &= 0, \quad (7.10) \\
2g^{\alpha \beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_1 + 4\zeta_1 \zeta_2 &= 0, \quad (7.11) \\
2g^{\alpha \beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_2 + 2g^{\alpha \beta} \partial_\alpha \zeta_1 \partial_\beta \zeta_1 + 4\zeta_2^2 + 6\zeta_1 \zeta_3 &= 0, \quad (7.12) \\
2g^{\alpha \beta} \partial_\alpha \zeta_0 \partial_\beta \zeta_3 + 2g^{\alpha \beta} \partial_\alpha \zeta_1 \partial_\beta \zeta_2 + 12\zeta_2 \zeta_3 + 8\zeta_1 \zeta_4 &= 0. \quad (7.13)
\end{align*}
\]
Since \( \zeta_0 \) and \( -\zeta_1 \) are equal to zeroth and first coefficients, respectively, of the Taylor series expansion of \( \varphi + i\psi \) in \( y \) near \( \Gamma^+_c,\varphi \) and since \( \langle d(\varphi + i\psi), d(\varphi + i\psi) \rangle_g = 0 \), the equation (7.10) is satisfied.

Since \( \zeta_1 = -\partial_y(\varphi + i\psi)|_{\Gamma^+_c,\varphi} \), there is a constant \( \varepsilon_0 > 0 \) such that \( |\zeta_1| > \varepsilon_0 \) on \( \Gamma^+_c,\varphi \).

Therefore, the division by \( \zeta_1 \) is possible, and hence we can recursively solve (7.11), (7.12) and (7.13) for \( \zeta_2, \zeta_3 \) and \( \zeta_4 \), respectively.

Thus, we have constructed \( \zeta \in C^2(M) \) satisfying the estimate (7.9) and such that in a neighborhood of \( \Gamma^+_c,\varphi \),
\[
\partial_y \operatorname{Re} \zeta|_{\Gamma^+_c,\varphi} = -\partial_y \varphi|_{\Gamma^+_c,\varphi} > \varepsilon_0, \quad \operatorname{Re} \zeta|_{\Gamma^+_c,\varphi} = \varphi|_{\Gamma^+_c,\varphi}.
\]

Therefore, in a neighborhood of \( \Gamma^+_c,\varphi \), we have \( \operatorname{Re} \zeta - \varphi \approx \operatorname{dist}(\cdot, \Gamma^+_c,\varphi) \). In order to prove (7.7), note that
\[
|e^{(\varphi + i\psi)/h} e^{-\zeta/h} \langle d\zeta, d\xi \rangle_g| \lesssim e^{-(\operatorname{Re} \zeta - \varphi)/h} \operatorname{dist}(\cdot, \Gamma^+_c,\varphi)^4.
\]
If \( \operatorname{dist}(\cdot, \Gamma^+_c,\varphi) \leq h^{1/2} \), then we get (7.7), because \( \operatorname{Re} \zeta - \varphi \approx \operatorname{dist}(\cdot, \Gamma^+_c,\varphi) \) in a neighborhood of \( \Gamma^+_c,\varphi \). If \( \operatorname{dist}(\cdot, \Gamma^+_c,\varphi) \geq h^{1/2} \), then we also get (7.7), because for some constant \( C > 0 \) we have
\[
e^{-(\operatorname{Re} \zeta - \varphi)/h} \leq e^{-C \operatorname{dist}(\cdot, \Gamma^+_c,\varphi)/h} \leq e^{-C/h^{1/2}} \lesssim h^2.
\]
in a neighborhood of \( \Gamma^+_c,\varphi \).

Next, we give the construction of \( B \).

**Proposition 7.3.** There is \( B \in C^2 \Omega^1(M) \) supported near \( \Gamma^+_c,\varphi \) and satisfying (7.8), for \( \zeta \) constructed in Proposition 7.2, and such that \( B|_{\Gamma^+_c,\varphi} = A|_{\Gamma^+_c,\varphi} \) with \( A \) being as in Proposition 7.1.

**Proof.** We work in the same boundary normal coordinates \((x, y) = (x^1, x^2, y)\) near \( \Gamma^+_c,\varphi \) used in Proposition 7.2. We look for \( B \) of the form
\[
B(x, y) = a(x) + yb(x) + b^2c(x),
\]
where
\[
\begin{align*}
a(x) &= a_\alpha(x) \, dx^\alpha + a_y(x) \, dy, \\
b(x) &= b_\alpha(x) \, dx^\alpha + b_y(x) \, dy, \\
c(x) &= c_\alpha(x) \, dx^\alpha + c_y(x) \, dy.
\end{align*}
\]
Boundary condition \( B|_{\Gamma^+_c,\varphi} = A|_{\Gamma^+_c,\varphi} \) determine
\[
a = a_\alpha \, dx^\alpha + a_y \, dy = A|_{\Gamma^+_c,\varphi}.
\]
In boundary normal coordinates \((x^1, x^2, y)\), the following is true for Christoffel symbols
\[
\Gamma^y_{1y} = \Gamma^y_{2y} = \Gamma^y_{yy} = \Gamma^1_{yy} = \Gamma^2_{yy} = 0.
\]
Using this, straightforward but tedious calculation gives
\[
2\nabla_y \zeta B + (\Delta_y \zeta) B + (i d \log \epsilon) B d \zeta - (i d \log \mu d \zeta) B = \left\{(L_a(a) + 2\partial_y \zeta b_a)dx^a + (L_y(a) + 2\partial_y \zeta b_y)dy\right\} + y\left\{(L_a(b) + 4\partial_y \zeta c_a)dx^a + (L_y(b) + 4\partial_y \zeta c_y)dy\right\} + y^2\left\{L_c(c)dx^c + L_y(c)dy\right\}
\]
where \(L_a\) and \(L_y\) are defined for \(f = f_a dx^a + f_y dy\) as
\[
L_a(f) = 2g^{\alpha\beta} \partial_\alpha \zeta (\partial_\beta f_a - \Gamma^\alpha_{\beta\alpha} f_y) - 2\partial_y \zeta \Gamma^\alpha_{\beta\gamma} f_\gamma + (\Delta_y \zeta) f_a,
\]
\[
L_y(f) = 2g^{\alpha\beta} \partial_\alpha \zeta (\partial_\beta f_y - \Gamma^\gamma_{\beta\gamma} f_\gamma),
\]
where \(\partial_y \zeta \approx y_\epsilon(\zeta_1 + yr(\zeta, y))\) for some \(r\) uniformly bounded in \(\zeta\) and \(y\), for sufficiently small \(|y| > 0\). It was also shown that the division by \(\zeta_1\) is possible. Therefore, taking \(|y| > 0\) sufficiently small (that is working sufficiently close to \(\Gamma^c_{+,\varphi}\)), we can ensure that the division by \(\partial_y \zeta\) is possible as well. Hence, we can recursively solve the above equations for \(b\) and \(c\).

Thus, we have constructed \(B \in C^2 \Omega^1(M)\) satisfying (7.14) and \(B|_{\Gamma^c_{+,\varphi}} = A|_{\Gamma^c_{+,\varphi}}\). Recall that \(\text{Re} \zeta - \varphi \approx \text{dist}(\cdot, \Gamma^c_{+,\varphi})\) in a neighborhood of \(\Gamma^c_{+,\varphi}\). In order to prove (7.8), note that
\[
|e^{(\varphi + \psi)/h} e^{-\zeta/h} (2\nabla_y \zeta B + (\Delta_y \zeta) B + (i d \log \epsilon) B d \zeta - (i d \log \mu d \zeta) B)| \leq e^{-((\text{Re} \zeta - \varphi)/h)} \text{dist}(\cdot, \Gamma^c_{+,\varphi})^2.
\]
If \(\text{dist}(\cdot, \Gamma^c_{+,\varphi}) \leq h^{1/2}\), then we get (7.8), because \(\text{Re} \zeta - \varphi \approx \text{dist}(\cdot, \Gamma^c_{+,\varphi})\) in a neighborhood of \(\Gamma^c_{+,\varphi}\). If \(\text{dist}(\cdot, \Gamma^c_{+,\varphi}) \geq h^{1/2}\), then we also get (7.8), because as it was shown in Proposition 7.2, we have \(e^{-((\text{Re} \zeta - \varphi)/h)} \lesssim h^2\) in a neighborhood of \(\Gamma^c_{+,\varphi}\) for some constant \(C > 0\).
Finally, multiplication $B$ by a smooth cut-off function will remain these properties and ensure that $B$ is supported in a neighborhood of $\Gamma_{\varepsilon,\varphi}^\circ$.

7.3. Construction of complex geometrical optics solutions. Now we are ready to construct complex geometrical optics solutions for the equation $\mathcal{L}_{\varepsilon,\mu} E = 0$ which is equivalent to

$$e^{(\varphi+i\psi)/h} h^2 \mathcal{L}_{\varepsilon,\mu} e^{-(\varphi+i\psi)/h} R = F_1 + F_2, \quad (7.15)$$

where $F_1 := -e^{(\varphi+i\psi)/h} h^2 \mathcal{L}_{\varepsilon,\mu} e^{-(\varphi+i\psi)/h} A$ and $F_2 := -e^{(\varphi+i\psi)/h} h^2 \mathcal{L}_{\varepsilon,\mu} e^{-\zeta/h} B$.

According to the discussions of Section 7.1 and Section 7.2, we have

$$\|F_1 + F_2\|_{L^2(\Omega(M))} \lesssim h^2.$$

Then by Proposition 6.5, there is a solution $R \in H^1_\Delta \Omega^1(M)$ of (7.15) such that $R|_{\Gamma_{\varepsilon,\varphi}^\circ} = 0$ and

$$\|R\|_{H^1_\epsilon \Omega^1(M)} \lesssim h.$$

Since $\zeta|_{\Gamma_{\varepsilon,\varphi}^\circ} = (\varphi + i\psi)|_{\Gamma_{\varepsilon,\varphi}^\circ}$ and $B|_{\Gamma_{\varepsilon,\varphi}^\circ} = A|_{\Gamma_{\varepsilon,\varphi}^\circ}$, setting $\rho = \zeta - (\varphi + i\psi)$, we obtain the complex geometrical optics solution $E = e^{-(\varphi+i\psi)/h}(A + R - e^{-\rho/h} B)$ to $\mathcal{L}_{\varepsilon,\mu} E = 0$ such that $E|_{\Gamma_{\varepsilon,\varphi}^\circ} = 0$. Note also that $\text{Re} \rho \approx \text{dist}(\cdot, \Gamma_{\varepsilon,\varphi}^\circ)$ in a neighborhood of $\Gamma_{\varepsilon,\varphi}^\circ$.

Now we want to show that $E$ satisfies the divergence equation $\delta(\varepsilon E) = 0$. Recall from the proofs of Proposition 4.1 and Proposition 4.2 that $\mathcal{L}_{\varepsilon,\mu} E = 0$ is equivalent to

$$\delta(\varepsilon^{-1} dE) + \mu^{-1} d(\varepsilon^{-1} \delta(\varepsilon E)) - \omega^2 \varepsilon E = 0.$$

Taking the divergence, we get

$$\delta(\varepsilon^{-1} d(\varepsilon^{-1} \delta(\varepsilon E))) - \omega^2 \delta(\varepsilon E) = 0.$$

Setting $p = \varepsilon^{-1} \delta(\varepsilon E)$, this is equivalent to

$$\delta(\mu^{-1} dp) - \omega^2 \varepsilon p = 0.$$

Now if we set $q = \mu^{-1/2} p$, then the latter equation can be rewritten as

$$-\Delta_g q + Q q = 0, \quad Q = \mu^{-1/2} (-\Delta_g) \mu^{-1/2} - \omega^2 \varepsilon \mu.$$

Since $E = e^{-\varphi/h} E_1$ with $E_1 = e^{-\varphi/h}(A + R - e^{-\rho/h} B)$, a straightforward computation gives

$$q = e^{-\varphi/h} a,$$

where $a = \mu^{-1/2} \varepsilon^{-1} \delta(\varepsilon E_1) + \mu^{-1/2} (h\varepsilon)^{-1} i_{\delta\varphi} \varepsilon E_1$. Hence, $a \in L^2(M)$ satisfies

$$e^{\varphi/h} (-\Delta_g) e^{-\varphi/h} a = Q a.$$

Then according to Proposition 6.6, there is $0 < h_0 \ll 1$ such that $a = 0$ for all $0 < h \ll h_0$. Therefore, we have shown that $\delta(\varepsilon E) = 0$ for all $0 < h \ll h_0$.

Thus, for any $b \in C^\infty(S^1)$ and for arbitrary $\lambda \in \mathbb{R}$, we have constructed a solution $E \in H^1_\Delta \Omega^1(M)$ for the problem

$$\mathcal{L}_{\varepsilon,\mu} E = 0, \quad \delta(\varepsilon E) = 0, \quad E|_{\Gamma_{\varepsilon,\varphi}^\circ} = 0,$$

of the form

$$E = e^{-(\varphi+i\psi)/h}(A + R - e^{-\rho/h} B)$$
where \( A \in C^2\Omega^1(M) \) is a solution of (7.3) as in Proposition 7.1, \( R \in H^1_{ \Delta}\Omega^1(M) \) is such that \( \| R \|_{ H^1_{ \Delta} \Omega^1(M)} \lesssim h \), \( \rho \in C^2(M) \) satisfies \( \text{Re} \rho \approx \text{dist}(\cdot, \Gamma^c_{+,\varphi}) \) in a neighborhood of \( \Gamma^c_{+,\varphi} \), and \( B \in C^2\Omega^1(M) \) is supported near \( \Gamma^c_{+,\varphi} \) and satisfy \( B|_{\Gamma^c_{+,\varphi}} = A|_{\Gamma^c_{+,\varphi}} \).

Now, by Proposition 4.2, we obtain the following proposition.

**Proposition 7.4.** Let \((M, g)\) be an admissible manifold of dimension \( n = 3 \) with \( g = e \oplus g_0 \). Assume \((\varepsilon, \mu) \in C^3(M) \times C^2(M)\) are complex valued such that \( \text{Re}(\varepsilon), \text{Re}(\mu) > 0 \) in \( M \). Let \( \varphi \) be the limiting Carleman weight \( \varphi(x) = \pm x_1 \) and \( \psi(x) = r \) is the phase function. There is a small \( 0 < h_0 \ll 1 \) such that for all \( 0 < h \ll h_0 \), for any \( b \in C^\infty(S^1) \) and for arbitrary \( \lambda \in \mathbb{R} \), the problem

\[
\begin{cases}
* dE = i\omega \mu H, \\
* dH = -i\omega e E,
\end{cases}
\]

has a solution \((E, H) \in H^1_{ \Delta} \Omega^1(M) \times H^1_\Delta \Omega^1(M) \) of the form

\[ E = e^{-i(\varphi + i\psi)/h}(A + e^{-\rho/h}B), \]

where \( A \in C^2\Omega^1(M) \) is a solution of (7.3) as in Proposition 7.1, \( \rho \in C^2(M) \) such that \( \text{Re} \rho \approx \text{dist}(\cdot, \Gamma^c_{+,\varphi}) \) is a solution of \( \Gamma^c_{+,\varphi} \), and \( B \in C^2\Omega^1(M) \) is supported near \( \Gamma^c_{+,\varphi} \) and satisfy \( B|_{\Gamma^c_{+,\varphi}} = A|_{\Gamma^c_{+,\varphi}} \). Finally, \( R \in H^1_{ \Delta} \Omega^1(M) \) is such that \( \| R \|_{ H^1_{ \Delta} \Omega^1(M)} \lesssim h \).

8. PROOF OF MAIN RESULT

In this section we show that the material parameters of the time-harmonic Maxwell equations can be uniquely determined from the partial boundary measurements.

Let \((M, g)\) be a 3-dimensional admissible manifold, that is \((M, g) \subset \subset \mathbb{R} \times (M_0, g_0)\) with \( g = c(e \oplus g_0) \), where \( c > 0 \) is a smooth function on \( M \) and \((M_0, g_0)\) is a simple manifold of dimension two.

The first ingredient in the proof of Theorem 1.1 is the reduction to the case \( c = 1 \). This was shown in [16, Lemma 7.1]; see also [10, Lemma 3.1].

**Lemma 8.1.** Let \((M, g)\) be a compact Riemannian 3-dimensional manifold with boundary, and let \( \varepsilon, \mu \in C^\infty(M) \) with positive real parts in \( M \). Let \( c > 0 \) be a smooth function on \( M \), and let \( C^{\varepsilon, \mu}_{\Gamma_1, \Gamma_2} \) represent the partial Cauchy data set for \( \varepsilon, \mu \) with respect to the metric \( g \). Then \( C^{\varepsilon, \mu}_{\Gamma_1, \Gamma_2} = C^{c^{1/2} \varepsilon, c^{1/2} \mu}_{\Gamma_1, \Gamma_2} \).

Therefore, it is enough to prove Theorem 1.1 in the case \( c = 1 \). Thus, in the rest of this section we assume that \((M, g) \subset \subset \mathbb{R} \times (M_0, g_0)\) with \( g = e \oplus g_0 \), where \((M_0, g_0)\) is a simple manifold of dimension two.

Next ingredient is the derivation of the main integral identity under the assumption \( C^{\varepsilon_1, \mu_1}_{\Gamma_+, \Gamma_-} = C^{\varepsilon_2, \mu_2}_{\Gamma_+, \Gamma_-} \). For the proof we follow [28, Lemma 0.6].
Proposition 8.2. Suppose \((\varepsilon_j, \mu_j) \in C^3(M) \times C^2(M), j = 1, 2\) are complex valued such that \(\text{Re}(\varepsilon_j), \text{Re}(\mu_j) > 0\) in \(M\). Suppose also that \(C_{T_1, T_2}^\varepsilon, \mu_1^\varepsilon = C_{T_1, T_2}^{\varepsilon_2, \mu_2}^\varepsilon\), at fixed frequency \(\omega > 0\). Assume that \((E_2, H_2) \in H_d\Omega^1(M) \times H_d\Omega^1(M)\) solve 
\[
\begin{align*}
*dE_2 &= i\omega \mu_2 H_2, \\
*dH_2 &= -i\omega \varepsilon_2 E_2
\end{align*}
\]
with \(\text{supp} t(E_2) \subset \Gamma_+\). Assume further that \((\tilde{E}, \tilde{H}) \in H_d\Omega^1(M) \times H_d\Omega^1(M)\) solve 
\[
\begin{align*}
*d\tilde{E} &= i\omega \mu_1 \tilde{H}, \\
*d\tilde{H} &= -i\omega \varepsilon_1 \tilde{E}
\end{align*}
\]
with \(\text{supp} t(\tilde{E}) \subset \Gamma_-\). Then
\[
(i\omega(\varepsilon_1 - \varepsilon_2) E_2(\tilde{E})_{L^2\Omega^1(M)} + (i\omega(\mu_1 - \mu_2) H_2 \tilde{H})_{L^2\Omega^1(M)} = 0.
\]

Proof. According to the hypothesis on the Cauchy data sets, there is \((E_1, H_1) \in H_d\Omega^1(M) \times H_d\Omega^1(M)\) solving 
\[
\begin{align*}
*dE_1 &= i\omega \mu_1 H_1, \\
*dH_1 &= -i\omega \varepsilon_1 E_1
\end{align*}
\]
with \(\text{supp} t(E_1) \subset \Gamma_+\) and such that \(t(E_1)|\Gamma_+ = t(E_2)|\Gamma_+\) and \(t(H_1)|\Gamma_- = t(H_2)|\Gamma_-\). Then, we prove
\[
(*d(H_2 - H_1)|\tilde{E})_{L^2\Omega^1(M)} + (i\omega \mu_1 (H_2 - H_1)|\tilde{H})_{L^2\Omega^1(M)} = 0. \tag{8.1}
\]
By direct calculations, we can show 
\[
(*d(H_2 - H_1)|\tilde{E})_{L^2\Omega^1(M)} + (i\omega \mu_1 (H_2 - H_1)|\tilde{H})_{L^2\Omega^1(M)}
\]
\[
= (*d(H_2 - H_1)|\tilde{E})_{L^2\Omega^1(M)} - (H_2 - H_1)(i\omega \mu_1 \tilde{H})_{L^2\Omega^1(M)}
\]
\[
= (*d(H_2 - H_1)|\tilde{E})_{L^2\Omega^1(M)} - (H_2 - H_1) * d\tilde{E}_{L^2\Omega^1(M)}
\]
\[
= (*d(H_2 - H_1)|\tilde{E})_{L^2\Omega^1(M)} - (H_2 - H_1) * \delta(*\tilde{E})_{L^2\Omega^1(M)}. \tag{8.2}
\]
In the last step we used (2.1) and (2.2). We use Proposition 3.5 and choose a sequence \(\{E_k\}_{k=1}^\infty \subset H_d\Omega^1(M)\) such that \(E_k \rightarrow \tilde{E}\) in \(H_d\Omega^1(M)\). Then \(*E_k \rightarrow *\tilde{E}\) in \(H_d\Omega^1(M)\) and
\[
(*d(H_2 - H_1)|\tilde{E})_{L^2\Omega^1(M)} + (i\omega \mu_1 (H_2 - H_1)|\tilde{H})_{L^2\Omega^1(M)}
\]
\[
= \lim_{k \rightarrow \infty} \left[(*d(H_2 - H_1)|\ast E_k)_{L^2\Omega^1(M)} - (H_2 - H_1) \delta(*E_k)_{L^2\Omega^1(M)}\right]
\]
\[
= \lim_{k \rightarrow \infty} (t(H_2 - H_1)|t(i_\nu \ast E_k))_{\partial M}
\]
\[
= \lim_{k \rightarrow \infty} (t(H_2 - H_1)|t(i_\nu \ast E_k))_{\Gamma^-}. \tag{8.1}
\]
In the last two steps we used integration by parts formula from part (a) of Proposition 3.1 and the fact that \(\text{supp} t(H_2 - H_1) \subset \Gamma_-^c\). We use Proposition 3.5 again and
choose a sequence \( \{H_l\}_{l=1}^{\infty} \subset H_d\Omega^1(M) \) such that \( H_l \to (H_2 - H_1) \) in \( H_d\Omega^1(M) \). Then

\[
(*d(H_2 - H_1)|\vec{E})_{L^2\Omega^1(M)} + (i\omega\mu_1(H_2 - H_1)|\vec{H})_{L^2\Omega^1(M)} = \lim_{l \to \infty} \lim_{k \to \infty} \int_{\Gamma_-} t(H_l \wedge \vec{E}_k)
\]

\[
= \lim_{l \to \infty} \lim_{k \to \infty} (t(i\nu * H_l)|t(\vec{E}))_{L^2\Omega^1(\Gamma_-)}
\]

where we used Lemma 2.1 in the last two lines. Therefore,

\[
(*d(H_2 - H_1)|\vec{E})_{L^2\Omega^1(M)} + (i\omega\mu_1(H_2 - H_1)|\vec{H})_{L^2\Omega^1(M)} = \lim_{l \to \infty} (t(i\nu * H_l)|t(\vec{E}))_{L^2\Omega^1(\Gamma_-)} = 0,
\]

since \( \text{supp} \ t(\vec{E}) \subset \Gamma_- \). Thus, we come to (8.1).

Next, we prove

\[
(*d(E_2 - E_1)|\vec{E})_{L^2\Omega^1(M)} - (i\omega\varepsilon_1(E_2 - E_1)|\vec{E})_{L^2\Omega^1(M)} = 0. \tag{8.3}
\]

For this, observe that \( t(E_2 - E_1) = 0 \). This is because, by hypothesis, \( t(E_1)|\Gamma_+ = t(E_2)|\Gamma_+ \) and \( \text{supp} \ t(E_1) \subset \Gamma_+ \), \( \text{supp} \ t(E_2) \subset \Gamma_+ \). Then using similar arguments as in (8.2),

\[
(*d(E_2 - E_1)|\vec{H})_{L^2\Omega^1(M)} - (i\omega\varepsilon_1(E_2 - E_1)|\vec{H})_{L^2\Omega^1(M)} = (d(E_2 - E_1)|\vec{H})_{L^2\Omega^1(M)} - (E_2 - E_1|\delta(\vec{H}))_{L^2\Omega^1(M)}.
\]

We use Proposition 3.5 and choose a sequence \( \{H_k\}_{k=1}^{\infty} \subset H_d\Omega^1(M) \) such that \( H_k \to \vec{H} \) in \( H_d\Omega^1(M) \). Then \( *H_k \to *\vec{H} \) in \( H_d\Omega^1(M) \) and

\[
(*d(E_2 - E_1)|\vec{H})_{L^2\Omega^1(M)} - (i\omega\varepsilon_1(E_2 - E_1)|\vec{H})_{L^2\Omega^1(M)} = \lim_{k \to \infty} \left( (d(E_2 - E_1)|*\vec{H}_k)_{L^2\Omega^1(M)} - (E_2 - E_1|\delta(\vec{H}_k))_{L^2\Omega^1(M)} \right)
\]

\[
= \lim_{k \to \infty} (t(E_2 - E_1)|t(i\nu * \vec{H}_k))_{\partial\Omega} = 0.
\]

In the last step we used integration by parts formula from part (a) of Proposition 3.1 and the fact that \( t(E_2 - E_1) = 0 \). Thus, we proved (8.3).

Finally, subtracting (8.3) from (8.1), we come to

\[
(*d(H_2 - H_1) + i\omega\varepsilon_1(E_2 - E_1)|\vec{E})_{L^2\Omega^1(M)} - (*d(E_2 - E_1) - i\omega\mu_1(H_2 - H_1)|\vec{H})_{L^2\Omega^1(M)} = 0. \tag{8.4}
\]

Substituting

\[ *d(E_2 - E_1) - i\omega\mu_1(H_2 - H_1) = i\omega(\mu_2 - \mu_1)H_2 \]

and

\[ *d(H_2 - H_1) + i\omega\varepsilon_1(E_2 - E_1) = i\omega(\varepsilon_1 - \varepsilon_2)E_2 \]

in (8.4), we obtain the desired identity. \( \square \)
First, let us consider the case $\varphi(x) = x_1$. Then we can take $\Gamma_{+}\varphi = \Gamma_{+}$. For sufficiently small $h > 0$, for any $b \in C^\infty(S^1)$ and $\lambda \in \mathbb{R}$, by Proposition 7.4, there is $(E_2, H_2) \in H^1_\Delta \Omega^1(M) \times H^1_0\Omega^1(M)$ solving

$$
\begin{cases}
* d E_2 = i \omega \mu_2 H_2, \\
* d H_2 = -i \omega \varepsilon E_2,
\end{cases}
E_2|_{\Gamma_{+}} = 0,
$$

of the form

$$E_2 = e^{-(x_1 + ir)/h}(A_2 + R_2 - B'_2), \quad B'_2 = e^{-\rho_2/h} B_2,$$

where $A_2 \in C^2 \Omega^1(M)$ solves (7.3), $R_2 \in H^1_\Delta \Omega^1(M)$ satisfy

$$\|R_2\|_{H^1_\Delta \Omega^1(M)} \lesssim h, \quad (8.5)$$

$\rho_2 \in C^2(M)$ is such that $\text{Re} \rho_2 \approx \text{dist}(\cdot, \Gamma_{+\varphi}')$ in a neighborhood of $\Gamma_{+\varphi}$, and $B_2 \in C^2 \Omega^1(M)$ supported near $\Gamma_{+\varphi}$. Next, we consider $\varphi(x) = -x_1$. In this case $\Gamma_{+\varphi}$ can be taken as $\Gamma_{-}$. Applying Proposition 7.4, we can also construct $(\widetilde{E}, \widetilde{H}) \in H^1_\Delta \Omega^1(M) \times H^1_0\Omega^1(M)$ solving

$$
\begin{cases}
* d \widetilde{E} = i \omega \mu_1 \widetilde{H}, \\
* d \widetilde{H} = -i \omega \varepsilon \widetilde{E},
\end{cases}
\widetilde{E}|_{\Gamma_{-}} = 0,
$$

of the form

$$\widetilde{E} = e^{(x_1 - ir)/h}(\widetilde{A} + \widetilde{R} - \widetilde{B}'), \quad \widetilde{B}' = e^{-\nu/h} \widetilde{B},$$

where $\widetilde{A} \in C^2 \Omega^1(M)$ solves (7.3), $\widetilde{R} \in H^1_\Delta \Omega^1(M)$ is such that

$$\|\widetilde{R}\|_{H^1_\Delta \Omega^1(M)} \lesssim h, \quad (8.6)$$

$\nu \in C^2(M)$ satisfies $\text{Re} \nu \approx \text{dist}(\cdot, \Gamma_{-})$ in a neighborhood of $\Gamma_{-}$, and $\widetilde{B} \in C^2 \Omega^1(M)$ supported near $\Gamma_{-}$.

Since $\text{supp}(E_2) \subset \Gamma_{+}$ and $\text{supp}(\widetilde{E}) \subset \Gamma_{-}$, we substitute $(E_2, H_2)$ and $(\widetilde{E}, \widetilde{H})$ into the integral identity in Proposition 8.2, and get

$$
((\varepsilon_1 - \varepsilon_2)|E_2|_{L^2 \Omega^1(M)} + (Q * d E_2) * d \widetilde{E})_{L^2 \Omega^1(M)} = 0,
$$

where

$$Q = \omega^{-2}(\mu_1 - \mu_2)\mu_1^{-1} \mu_2^{-1}.$$

Using (2.1), this implies that

$$
((\varepsilon_1 - \varepsilon_2)|E_2|_{L^2 \Omega^1(M)} + (Q d E_2) d \widetilde{E})_{L^2 \Omega^1(M)} = 0.
$$

More precisely, if we write $z = x_1 + ir$, we have

$$\begin{align*}
0 &= ((\varepsilon_1 - \varepsilon_2)(A_2 + R_2 - B'_2) | (\widetilde{A} + \widetilde{R} - \widetilde{B}'))_{L^2 \Omega^1(M)} \\
&+ h^{-2} (Q d z \wedge (A_2 + R_2 - B'_2) | d \mu \wedge (\widetilde{A} + \widetilde{R} - \widetilde{B}'))_{L^2 \Omega^2(M)} \\
&+ h^{-1} (Q d z \wedge (A_2 + R_2 - B'_2) | d \mu \wedge (\widetilde{A} + \widetilde{R} - \widetilde{B}'))_{L^2 \Omega^1(M)} \\
&+ h^{-1} (Q d z \wedge (A_2 + R_2 - B'_2) | d \mu \wedge (\widetilde{A} + \widetilde{R} - \widetilde{B}')_{L^2 \Omega^1(M)} \\
&+ (Q d (A_2 + R_2 - B'_2) | d \mu \wedge (\widetilde{A} + \widetilde{R} - \widetilde{B}'))_{L^2 \Omega^1(M)}
\end{align*}
$$

(8.7)
We start with proving that $\mu_1 = \mu_2$. For this, we need to show that
\[
(Q \, dz \wedge A_2 \mid d\bar{\sigma} \wedge \bar{A})_{L^2(M)} = 0. \tag{8.8}
\]
Let us label the terms in (8.7) in the following way
\[
0 = T_1 + T_2 + T_3 + T_4 + T_5.
\]
Since $\varepsilon_1 - \varepsilon_2$ is bounded, using (8.5), (8.6) and Lemma 8.3 below, it follows that
\[
|T_1| \lesssim (\|A_2\|_{L^2(M)} + \|R_2\|_{L^2(M)} + \|B'_2\|_{L^2(M)}) \times (\|A\|_{L^2(M)} + \|R\|_{L^2(M)} + \|\bar{B}'\|_{L^2(M)}) \lesssim 1.
\]
Similarly, using that $Q$ is bounded, we can show the following estimate
\[
|T_2 - h^{-2}(Q \, dz \wedge A_2 \mid d\bar{\sigma} \wedge \bar{A})_{L^2(M)}| \\lesssim h^{-2}(\|A_2\|_{L^2(M)} + \|R_2\|_{L^2(M)} + \|B'_2\|_{L^2(M)})
\]
\[
+ h^{-2}(\|R_2\|_{L^2(M)} + \|B'_2\|_{L^2(M)}) \|A\|_{L^2(M)}
\]
\[
+ h^{-2}(\|R_2\|_{L^2(M)} + \|B'_2\|_{L^2(M)}) \|\bar{A}\|_{L^2(M)} \|\bar{B}'\|_{L^2(M)}
\]
\[
\lesssim h^{-2+1/2} + h^{-2+1/2} + h^{-1}.
\]
Following the same approach, we estimate the $T_3, T_4$ and $T_5$ terms
\[
|T_3| \lesssim h^{-1}(\|A_2\|_{L^2(M)} + \|R_2\|_{L^2(M)} + \|B'_2\|_{L^2(M)})
\]
\[
\times (\|d\bar{A}\|_{L^2(M)} + \|d\bar{R}\|_{L^2(M)} + \|e^{-\bar{\rho}/h}(d\bar{B} + h^{-1}d\bar{\rho} \wedge \bar{B})\|_{L^2(M)})
\]
\[
\lesssim h^{-2+1/2},
\]
\[
|T_4| \lesssim h^{-1}(\|\bar{A}\|_{L^2(M)} + \|\bar{R}\|_{L^2(M)} + \|\bar{B}'\|_{L^2(M)})
\]
\[
\times (\|d\bar{A}_2\|_{L^2(M)} + \|d\bar{R}_2\|_{L^2(M)} + \|e^{-\rho_2/h}(dB_2 + h^{-1}d\rho_2 \wedge B_2)\|_{L^2(M)})
\]
\[
\lesssim h^{-2+1/2}
\]
and
\[
|T_5| \lesssim (\|d\bar{A}\|_{L^2(M)} + \|d\bar{R}\|_{L^2(M)} + \|e^{-\bar{\rho}/h}(d\bar{B} + h^{-1}d\bar{\rho} \wedge \bar{B})\|_{L^2(M)})
\]
\[
\times (\|dA_2\|_{L^2(M)} + \|dR_2\|_{L^2(M)} + \|e^{-\rho_2/h}(dB_2 + h^{-1}d\rho_2 \wedge B_2)\|_{L^2(M)})
\]
\[
\lesssim h^{-1}.
\]
According to all these estimates, multiplying (8.7) by $h^2$ and letting $h \to 0$, we can establish (8.8).

Using parts (a) and (c) in Proposition 7.1, we take
\[
A_2 = e^{-i\lambda(z_1 + ir)} |g|^{-1/4} b(\theta) e^{-1/2} \mu_1^{1/2} e^{i\Phi} d\sigma, \quad \bar{A} = |g|^{-1/4} \bar{\mu}_1^{-1/2} \bar{\mu}_1^{1/2} e^{i\Psi} dz,
\]
where $\lambda \in \mathbb{R}$, $b \in C^\infty(M)$ and $\Phi, \Psi \in C^2(M)$ are solutions solution for
\[
\bar{\sigma} \Phi = \frac{1}{2} \partial_r \log \varepsilon_2, \quad \partial \Psi = -\frac{1}{2} \partial_r \log \varepsilon_1 \quad \text{in} \quad M,
\]
respectively. Substituting these into (8.8) and using that \( g \) has the special form (7.2), we obtain that
\[
\int_M q e^{-i\lambda(x_1 + ir)} |g|^{-1/2} b(\theta) \, d\text{Vol}_g = 0,
\]
where
\[
q = (\mu_1 - \mu_2) \varepsilon_1^{-1/2} \varepsilon_2^{-1/2} \mu_1^{-1/2} \mu_2^{-1/2} e^{i(\Phi - \Psi)}.
\]
Now we extend \( q \) as zero to \( \mathbb{R} \times M_0 \). Since \( d\text{Vol}_g = |g|^{1/2} \, dx_1 \, d\theta \), we have
\[
\int_{S^1} b(\theta) \int_0^\infty e^{i\lambda x_1} \left( \int_{-\infty}^{\infty} q e^{-i\lambda x_1} \, dx_1 \right) \, dr \, d\theta = 0.
\]
Varying \( b \in C^\infty(M) \) and noting that the term in the brackets is the one-dimensional Fourier transform (classical) of \( q \) with respect to the \( x_1 \)-variable, which we denote by \( \mathcal{F}_1 q \), we get
\[
\int_0^\infty e^{i\lambda r} \mathcal{F}_1 q(\lambda, r, \theta) \, dr = 0, \quad \theta \in S^1.
\]
Recall that \((r, \theta)\) are polar coordinates in \( M_0 \). Therefore, \( r \mapsto (r, \theta) \) is a geodesic in \( M_0 \) and the integral above is the attenuated geodesic ray transform of \( \mathcal{F}_1 q \) on \( M_0 \) with constant attenuation \( \lambda \). Then injectivity of this transform on simple manifolds of dimension two [25, Theorem 1.1] implies that \( \mathcal{F}_1 q(\lambda, \cdot) = 0 \) in \( M_0 \) for all \( \lambda \in \mathbb{R} \). Now, using the uniqueness result for the Fourier transform, we show that \( q = 0 \) and hence \( \mu_1 = \mu_2 \) in \( M \).

To show that \( \varepsilon_1 = \varepsilon_2 \), use \( \mu_1 = \mu_2 \), and consider the integral identity (8.7) with \( Q = 0 \):
\[
0 = ((\varepsilon_1 - \varepsilon_2)(A_2 + R_2 - B_2')((\tilde{A} + \tilde{R} - \tilde{B}')))_{L^2\Omega^1(M)}.
\]
Letting \( h \to 0 \), implies that
\[
((\varepsilon_1 - \varepsilon_2)A_2|\tilde{A})_{L^2\Omega^1(M)} = 0.
\]
Here, we have used similar estimate approach as in the proof of (8.8).

Using parts (b) and (c) in Proposition 7.1, we take
\[
A_2 = e^{-i\lambda(x_1 + ir)} |g|^{-1/4} b(\theta) \varepsilon_2^{-1/2} \mu_2^{-1/2} \, dz, \quad \tilde{A} = |g|^{-1/4} \varepsilon_1^{-1/2} \mu_1^{-1/2} e^{i\Phi} \, dz,
\]
where \( \lambda \in \mathbb{R} \), \( b \in C^\infty(M) \) and \( \Phi \in C^2(M) \) is a solution for
\[
\partial \Phi = -\frac{1}{2} \partial_r \log \varepsilon_1 \quad \text{in} \quad M.
\]
Substitution of these into (8.9) implies that
\[
\int_M f e^{-i\lambda(x_1 + ir)} |g|^{-1/2} b(\theta) \, d\text{Vol}_g = 0,
\]
where
\[
f = (\varepsilon_1 - \varepsilon_2) \varepsilon_1^{-1/2} \varepsilon_2^{-1/2} \mu_1^{-1/2} \mu_2^{-1/2} e^{i\Phi}.
\]
Now, we can proceed similarly as in the proof of \( \mu_1 = \mu_2 \), and get \( \varepsilon_1 = \varepsilon_2 \), finishing the proof of Theorem 1.1.
Finally, let us give the proof of the following result which was used in the proof of Theorem 1.1.

**Lemma 8.3.** Let \( \varphi(x) = \pm x_1 \) be the limiting Carleman weight and let \( \rho \in C^2(M) \) satisfy \( \text{Re} \rho \approx \text{dist}(\cdot, \Gamma^c_{+, \varphi}) \) in a neighborhood of \( \Gamma^c_{+, \varphi} \). Suppose that \( \alpha \in \mathcal{C} \mathcal{R}^m(M) \) is supported in a sufficiently small neighborhood of \( \Gamma^c_{+, \varphi} \). Then for sufficiently small \( 0 < h \ll 1 \),

\[
\| e^{-\rho/h} \alpha \|_{L^2_\Omega^m(M)} \lesssim h^{1/2},
\]

where the implicit constant depends on \( \alpha \).

**Proof.** Let \( U \subset M \) be an open set (in a subset topology of \( M \)) such that \( \text{supp} \alpha \subset \subset U \). We assume that \( U \) is sufficiently close to \( \text{supp} \alpha \) so that \( \text{Re} \rho \approx \text{dist}(\cdot, \Gamma^c_{+, \varphi}) \) in \( U \). For \( 0 < h \ll 1 \) small, let us decompose \( U \) as \( U = U_{h, \leq} \cup U_{h, \geq} \), where

\[
U_{h, \leq} = \{ x \in U : \text{dist}(x, \Gamma^c_{+, \varphi}) \leq h^{1/2} \}, \quad U_{h, \geq} = \{ x \in U : \text{dist}(x, \Gamma^c_{+, \varphi}) \geq h^{1/2} \}.
\]

Then

\[
\| e^{-\rho/h} \alpha \|_{L^2_\Omega^m(M)} = \| e^{-\rho/h} \alpha \|_{L^2_\Omega^m(U_{h, \leq})} + \| e^{-\rho/h} \alpha \|_{L^2_\Omega^m(U_{h, \geq})}.
\]

For the first term, note that \( \| e^{-\rho/h} \|_{L^\infty(U_{h, \leq})} \leq 1 \) and \( \text{Vol}_g(U_{h, \leq}) \lesssim h^{1/2} \). Therefore, using [14, Proposition 6.10], we get the estimate

\[
\| e^{-\rho/h} \alpha \|_{L^2_\Omega^m(U_{h, \leq})} \lesssim h^{1/2}.
\]

For the second term, recall that \( \text{Re} \rho \geq C \text{dist}(\cdot, \Gamma^c_{+, \varphi}) \geq Ch^{2/3} \) on \( U_{h, \geq} \) for some \( C > 0 \). This implies that

\[
\| e^{-\rho/h} \|_{L^\infty(U_{h, \geq})} \leq e^{-C \text{dist}(\cdot, \Gamma^c_{+, \varphi})/h} \lesssim e^{-Ch^{-1/2}} \lesssim o(h^2).
\]

Hence, we can obtain the estimate

\[
\| e^{-\rho/h} \alpha \|_{L^2_\Omega^m(U_{h, \geq})} \lesssim h^2.
\]

Combining these estimates for the two terms, we complete the proof. \( \square \)

**Appendix A. Direct problem and the admittance map**

This section contains well-posedness results of the boundary value problem for the time-harmonic Maxwell equations. These results are well known in Euclidean space. But since we could not find a proper reference, proofs are included here in the setting of Riemannian manifolds.

Let \( (M, g) \) be a compact 3-dimensional Riemannian manifold with smooth boundary. For a given 1-form \( f \) on \( \partial M \), we consider the time-harmonic Maxwell equations

\[
\begin{cases}
\ast dE = i\omega \mu H, \\
\ast dH = -i\omega \varepsilon E,
\end{cases}
\]

(A.1)

with the tangential boundary condition \( t(E) = f \), where \( \omega \) is a complex number. The complex functions \( \mu \) and \( \varepsilon \), which are assumed to be in \( C^2(M) \) with positive real parts in \( M \), represent the material parameters (permittivity and permeability, respectively).
Theorem A.1. Let \((M, g)\) be a compact 3-dimensional Riemannian manifold with smooth boundary. Let \(\varepsilon, \mu \in C^2(M)\) be complex functions with positive real parts. There is a discrete subset \(\Sigma\) of \(\mathbb{C}\) such that for all \(\omega \notin \Sigma\) and for a given \(f \in \mathcal{H}_d\Omega^1(\partial M)\) the Maxwell equation (A.1) with \(t(E) = f\) has a unique solution \((E, H) \in H_d\Omega^1(M) \times H_d\Omega^1(M)\) satisfying
\[
\|E\|_{H_d\Omega^1(M)} + \|H\|_{H_d\Omega^1(M)} \leq C\|f\|_{\mathcal{H}_d\Omega^1(\partial M)}
\]
for some constant \(C > 0\) independent of \(f\).

For \(\omega > 0\) with \(\omega \notin \Sigma\), we define the admittance map \(\Lambda_{\omega}^{\varepsilon, \mu}\) as
\[
\Lambda_{\omega}^{\varepsilon, \mu}(f) = t(H), \quad f \in \mathcal{H}_d\Omega^1(\partial M),
\]
where \((E, H) \in H_d\Omega^1(M) \times H_d\Omega^1(M)\) is the unique solution of the system (A.1) with \(t(E) = f\), guaranteed by Theorem A.1. Moreover, the estimate provided in Theorem A.1 implies that the admittance map is a well-defined and bounded operator \(\Lambda_{\omega}^{\varepsilon, \mu} : \mathcal{H}_d\Omega^1(\partial M) \to \mathcal{H}_d\Omega^1(\partial M)\).

To prove Theorem A.1, we consider the following non-homogeneous problem. Let \(J_e\) and \(J_m\) be 1-forms on \(M\) representing current sources. We consider the non-homogenous time-harmonic Maxwell equations
\[
\begin{cases}
* dE = i\omega \mu H + J_m, \\
* dH = -i\omega \varepsilon E + J_e
\end{cases}
\tag{A.2}
\]
We also work with the space of differential forms in \(H_d\Omega^1(M)\) having zero tangential trace
\[
H_{d,0}\Omega^1(M) := \{ w \in H_d\Omega^1(M) : t(w) = 0 \}.
\]

Theorem A.2. Let \((M, g)\) be a compact 3-dimensional Riemannian manifold with smooth boundary. Let \(\varepsilon, \mu \in C^2(M)\) be complex functions with positive real parts and let \(J_e, J_m \in L^2\Omega^1(M)\). There is a discrete subset \(\Sigma\) of \(\mathbb{C}\) such that for all \(\omega \notin \Sigma\) the Maxwell’s system (A.2) has a unique solution \((E, H) \in H_{d,0}\Omega^1(M) \times H_d\Omega^1(M)\) satisfying
\[
\|E\|_{H_d\Omega^1(M)} + \|H\|_{H_{d,0}\Omega^1(M)} \leq C(\|J_e\|_{L^2\Omega^1(M)} + \|J_m\|_{L^2\Omega^1(M)})
\]
for some constant \(C > 0\) independent of \(J_e\) and \(J_m\).

We first prove Theorem A.2 and then show that this can be used to prove Theorem A.1.

Finally, in Section A.5, we also consider the eigenvalue problem for the Maxwell equation (A.1) with \(t(E) = 0\) under the additional assumption that both \(\varepsilon\) and \(\mu\) are real-valued. Our main result is stated in Theorem A.9.

A.1. Helmholtz decompositions of \(H_d\Omega^1(M)\), \(H_{d,0}\Omega^1(M)\) and \(L^2\Omega^1(M)\). For the proof of Theorem A.2, we will use Helmholtz type decomposition of \(H_{d,0}\Omega^1(M)\) and \(L^2\Omega^1(M)\) suitable for Maxwell’s equations. For the proofs we closely follow [22], see also [18].
For a given \( \alpha \in C^2(M) \) with positive real part, define the spaces
\[
L^2\Omega^1(0,\alpha) = \{ w \in L^2\Omega^1(M) : (\alpha w|dh)_{L^2\Omega^1(M)} = 0, \ h \in H^1_0(M) \}, \\
H_d\Omega^1(0,\alpha) = \{ w \in H_d\Omega^1(M) : (\alpha w|\varphi)_{L^2\Omega^1(M)} = 0, \ \varphi \in H_d(0,\Omega^1(M)) \}, \\
H_{d,0}\Omega^1(0,\alpha) = \{ w \in H_{d,0}\Omega^1(M) : (\alpha w|dh)_{L^2\Omega^1(M)} = 0, \ h \in H^1_0(M) \}.
\]

**Proposition A.3.** The space \( dH^1_0(M) = \{ dh \in L^2\Omega^1(M) : h \in H^1_0(M) \} \) is closed in \( L^2\Omega^1(M) \) and in \( H_{d,0}\Omega^1(M) \), and the following orthogonal decompositions hold
\[
L^2\Omega^1(M) = L^2\Omega^1(0,\alpha) \oplus dH^1_0(M), \quad (A.3) \\
H_d\Omega^1(M) = H_d\Omega^1(0,\alpha) \oplus dH^1_0(M), \quad (A.4)
\]
where all of the projection operators are bounded. Moreover, the projection of \( H_{d,0}\Omega^1(M) \) onto \( H_{d,0}\Omega^1(0,\alpha) \) is the restriction of the projection of \( L^2\Omega^1(M) \) onto \( L^2\Omega^1(0,\alpha) \).

**Proof.** To prove closedness of \( dH^1_0(M) \) in \( L^2\Omega^1(M) \), consider a sequence \( \{h_k\}_{k=1}^\infty \subset H^1_0(M) \) such that \( \|dh_k - u\|_{L^2\Omega^1(M)} \to 0 \) as \( k \to \infty \) for some \( u \in L^2\Omega^1(M) \). In particular, \( \{dh_k\}_{k=1}^\infty \) is a Cauchy sequence in \( L^2\Omega^1(M) \). Then by Poincaré inequality, \( \{h_k\}_{k=1}^\infty \) is a Cauchy sequence in \( L^2(M) \). Hence, \( \{h_k\}_{k=1}^\infty \) is a Cauchy sequence in \( H^1(M) \). Therefore, \( u = dh \) for some \( h \in H^1(M) \). Finally, by closedness of \( H^1_0(M) \) in \( H^1(M) \), we have \( h \in H^1_0(M) \).

Next, to prove closedness of \( dH^1_0(M) \) in \( H_{d,0}\Omega^1(M) \), consider a sequence \( \{h_k\}_{k=1}^\infty \subset H^1_0(M) \) such that \( \|dh_k - u\|_{L^2\Omega^1(M)} \to 0 \) as \( k \to \infty \) for some \( u \in H_{d,0}\Omega^1(M) \). In particular, \( \{dh_k\}_{k=1}^\infty \) is a Cauchy sequence in \( L^2\Omega^1(M) \). Since \( dh_k = 0 \) for all \( k \geq 1 \), \( \{dh_k\} \) is a Cauchy sequence in \( L^2(M) \). Therefore, \( u = dh \) for some \( h \in H^1(M) \). Finally, by closedness of \( dH^1_0(M) \) in \( L^2\Omega^1(M) \), we have \( h \in H^1_0(M) \).

To prove (A.3 – A.4), consider the sesquilinear form \( A \) on \( dH^1_0(M) \) defined as
\[
A(dh,dh') = (\alpha dh|dh')_{L^2\Omega^1(M)}, \quad h,h' \in H^1_0(M).
\]
It is clear that
\[
|A(dh,dh')| \leq C \|dh\|_{L^2\Omega^1(M)} \|dh'\|_{L^2\Omega^1(M)}
\]
and that
\[
\text{Re} A(dh,dh) = (\text{Re}(\alpha)|dh|dh)_{L^2\Omega^1(M)} \geq c \|dh\|_{L^2\Omega^1(M)}^2.
\]
Thus, the form \( A \) is strictly coercive on \( dH^1_0(M) \). For a given \( e \in L^2\Omega^1(M) \), consider the bounded linear functional \( \ell_e : dH^1_0(M) \to \mathbb{C} \) defined as
\[
\ell_e(dh') = (\alpha e|dh')_{L^2\Omega^1(M)}.
\]
Applying the Lax-Milgram’s lemma (see e.g. [22, Lemma 2.21]), we obtain a bounded linear operator \( G : L^2\Omega^1(M) \to H^1_0(M) \) such that
\[
\ell_e(dh') = A(Ge,dh'), \quad e \in L^2\Omega^1(M), \quad h' \in H^1_0(M).
\]
This implies that
\[
(\alpha(e - dGe)|dh')_{L^2\Omega^1(M)} = 0, \quad h' \in H^1_0(M), \quad (A.5)
\]
and hence \( e - dGe \in L^2\Omega^1(M)_{0,\alpha} \).
Thus, we can claim that every \( e \in L^2\Omega^1(M) \) can be uniquely decomposed as \( e = e_0 + dh \) where \( e_0 = (e - dGe) \in L^2\Omega^1(M)_{0,\alpha} \) and \( h = Ge \in H^1_0(M) \). Hence, we have shown (A.3).

If \( e \in H_{d,0}\Omega^1(M) \), then \( e_0 = e - dGe \in H_{d,0}\Omega^1(M) \) since
\[
t(e_0) = t(e) - t(dGe) = -d\partial_M(Ge)|_{\partial M} = 0.
\]
From (A.5) we also can see that \( e_0 \in H_{d,0}\Omega^1(M)_{0,\alpha} \). This gives the decomposition (A.4). \( \square \)

It is easy to see that closedness of \( dH^1_0(M) \) in \( L^2\Omega^1(M) \) imply closedness of the former in \( H_{d}\Omega^1(M) \). Moreover, the sesquilinear form \( A \) in the proof of Proposition A.3 can be defined on \( H_{d}(0,\Omega^1(M)) \); see the definition of the latter space below. The same is true for the linear functional \( \ell_e \). Furthermore, the latter makes sense even for \( e \in L^2\Omega^1(M) \). Therefore, the similar arguments, but \( L^2\Omega^1(M) \) replaced by \( H_{d,0}\Omega^1(M) \) and \( dH^1_0(M) \) replaced by \( H_{d}(0,\Omega^1(M)) \), imply the following result.

**Proposition A.4.** The space \( dH^1_0(M) \) is closed in \( H_{d}\Omega^1(M) \) and the following orthogonal decomposition holds
\[
H_{d}\Omega^1(M) = H_{d}\Omega^1(M)_{\alpha} \oplus H_{d}(0,\Omega^1(M)), \tag{A.6}
\]
where
\[
H_{d}(0,\Omega^1(M)) = \{ \varphi \in H_{d}\Omega^1(M) : d\varphi = 0 \}
\]
and all of the projection operators are bounded.

A.2. **Compact embedding results.** We will also need the following results on compact embedding of \( H_{d,0}\Omega^1(M) \cap H_{d}\Omega^1(M) \) and \( H_{d,0}\Omega^1(M)_{0,\alpha} \) into \( L^2\Omega^1(M) \).

**Proposition A.5.** The inclusion \( H_{d,0}\Omega^1(M) \cap H_{d}\Omega^1(M) \hookrightarrow L^2\Omega^1(M) \) is compact

**Proof.** Follows from Proposition 3.2 and the compactness of the embedding
\[
H^1\Omega^1(M) \hookrightarrow L^2\Omega^1(M),
\]
see e.g. [26, Theorem 1.3.6]. \( \square \)

The following compact embedding result is originally due to Weber [31] in Euclidean case.

**Proposition A.6.** The inclusion \( H_{d,0}\Omega^1(M)_{0,\alpha} \hookrightarrow L^2\Omega^1(M) \) is compact.

**Proof.** We prove this result following [4, Proposition 2.28]. Consider a bounded sequence \( \{u_k\}^\infty_{k=1} \subset H_{d,0}\Omega^1(M)_{0,\alpha} \). Using the Helmholtz decomposition in (A.4) for \( \alpha = 1 \), we can write each \( u_k \) uniquely as \( u_k = u^1_{0,k} + dh^1_k \), where \( u^1_{0,k} \in H_{d,0}\Omega^1(M)_{0,1} \) and \( h^1_k \in H^1_0(M) \). Since \( \{u_k\}|_k^1L^2\Omega^1(M) = (dh^1_k)_{L^2\Omega^1(M)} \), we have \( \|u_k\|_{H_{d}\Omega^1(M)} \leq \|u^1_{0,k}\|_{H_{d,0}\Omega^1(M)} \) and hence
\[
\|u^1_{0,k}\|_{H_{d,0}\Omega^1(M)} \leq C\|u_k\|_{H_{d}\Omega^1(M)}.
\]
Thus, the sequence \( \{u^1_{0,k}\}^\infty_{k=1} \subset H_{d,0}\Omega^1(M)_{0,1} \) is bounded. Since \( H_{d,0}\Omega^1(M)_{0,1} \subset H_{d,0}\Omega^1(M) \cap H_{d}\Omega^1(M) \), Proposition A.5 implies that there is \( u \in L^2\Omega^1(M) \) and a subsequence \( \{u^1_{0,k'}\}^\infty_{k'=1} \) such that
\[
\|u - u^1_{0,k'}\|_{L^2\Omega^1(M)} \to 0 \quad \text{as} \quad k' \to \infty. \tag{A.7}
\]
Now, using the Helmholtz decomposition in (A.3), we can write \( u \) uniquely as \( u = u^α + dh^α \), where \( u^α \in L^2Ω^1(M)_{0,α} \) and \( h^α \in H^0_0(M) \). Then
\[
(α(u^α - u_k^α))(u^α - u_k^α)_{L^2Ω^1(M)} = (α(u^α - u_k^α))(u^α + dh^α - u_k^α + dh_k^α)_{L^2Ω^1(M)} = (α(u^α - u_k^α))(u^α - u_k^α)_{L^2Ω^1(M)}.
\]
Together with (A.7) this gives that
\[
∥u^α - u_k^α∥_{L^2Ω^1(M)} ≤ C∥u - u^α_k∥_{L^2Ω^1(M)} \to 0 \quad \text{as} \quad k' \to ∞.
\]
Thus, the subsequence \( \{u^α_{k'}\}_{k'=1}^∞ \) converges to \( u^α \) in \( L^2Ω^1(M) \). The proof is complete.

A.3. Proof of Theorem A.2. Now, we are ready to give the proof. For this, we follow the standard variational-methods used in [12, 16, 18, 20, 22]. Substituting the second equation of (A.2) into the first equation of (A.2), we obtain the following second-order equation
\[
δ(μ^{-1}dE) - ω^2εE = iωJ_e + *d(μ^{-1}J_m).
\]
If we find a unique solution \( E \in H_{d,0}Ω^1(M) \) of this equation satisfying
\[
∥E∥_{H_{d,0}Ω^1(M)} ≤ C(∥J_e∥_{L^2Ω^1(M)} + ∥J_m∥_{L^2Ω^1(M)}),
\]
defining \( H = -iω^{-1}μ^{-1}(*dE - J_m) \) we obtain a unique \( (E,H) \in H_{d,0}Ω^1(M) × H_dΩ^1(M) \) solving the Maxwell equations (A.2) and hence satisfying
\[
∥E∥_{H_{d,0}Ω^1(M)} + ∥H∥_{H_{d,0}Ω^1(M)} ≤ C(∥J_e∥_{L^2Ω^1(M)} + ∥J_m∥_{L^2Ω^1(M)}).
\]
Therefore, the problem is reduced to finding a unique \( E \in H_{d,0}Ω^1(M) \) such that
\[
(μ^{-1}dE|e')_{L^2Ω^1(M)} - (ω^2εE|e')_{L^2Ω^1(M)} = (iωJ_e|e')_{L^2Ω^1(M)} + (μ^{-1}J_m|e')_{L^2Ω^1(M)}
\]
for all \( e' \in H_{d,0}Ω^1(M) \).

Using (A.4), we can decompose \( E \) uniquely as \( E = E_0 + dh \), where \( E_0 \in H_{d,0}Ω^1(M)_{0,ε} \) and \( h \in H^0_0(M) \). Since \( iωε^{-1}J_e \in L^2Ω^1(M) \), this can be uniquely decomposed as \( iωε^{-1}J_e = J_{e,0} + dj_e \), where \( J_{e,0} \in L^2Ω^1(M)_{0,ε} \) and \( j_e \in H^0_0(M) \). We note here that
\[
∥j_e∥_{H^1(M)} ≤ C∥J_e∥_{L^2Ω^1(M)}.
\]
Using these decompositions, (A.9) can be rewritten as
\[
(μ^{-1}dE_0|e')_{L^2Ω^1(M)} - (ω^2εE_0|e')_{L^2Ω^1(M)} - (ω^2εdh|e')_{L^2Ω^1(M)} = (εJ_{e,0}|e')_{L^2Ω^1(M)} + (εdj_e|e')_{L^2Ω^1(M)} + (μ^{-1}J_m|e')_{L^2Ω^1(M)}
\]
for all \( e' \in H_{d,0}Ω^1(M) \).

Our first step is to extract \( h \) from (A.11). For this, use \( e' = dh' \) for arbitrary \( h' \in H^0_0(M) \) in (A.11). Since \( E_0 \in H_{d,0}Ω^1(M)_{0,ε} \) and \( J_{e,0} \in L^2Ω^1(M)_{0,ε} \), we obtain
\[
-(ω^2εdh|h')_{L^2Ω^1(M)} = (εdj_e|h')_{L^2Ω^1(M)}
\]
for all \( h' \in H^0_0(M) \). We rewrite this as
\[
(εd(ω^2h + j_e)|h')_{L^2Ω^1(M)} = 0
\]
and take \( h' = \omega^2 h + j \). Then we obtain \( h' = 0 \), which implies that \( h = -\omega^2 j \).

Now, we use \( h = -\omega^2 j \) in (A.11) and get
\[
(\mu^{-1} d E_0 | d e')_{L^2(\Omega^2)} - (\omega^2 \varepsilon E_0 | e')_{L^2(\Omega^2)} = (\varepsilon J_{e,0} | e')_{L^2(\Omega^2)} + (\mu^{-1} \ast J_m | d e')_{L^2(\Omega^2)}
\]
for all \( e' \in H_{d,0} \Omega^1(M) \). Thus, our next step is to find a unique \( E_0 \in H_{d,0} \Omega^1(M)_{0,\varepsilon} \) satisfying
\[
\delta(\mu^{-1} d E_0) - \omega^2 \varepsilon E_0 = \varepsilon J_{e,0} + \delta(\mu^{-1} \ast J_m).
\]
To solve this equation, we need the following result on existence of a solution operator

**Proposition A.7.** There are a constant \( \lambda > 0 \) and a bounded linear map \( T_\lambda : (H_{d,0} \Omega^1(M))^\prime \to H_{d,0} \Omega^1(M) \) such that
\[
\delta(\mu^{-1} d T_\lambda u) + \lambda \varepsilon T_\lambda u = u, \quad u \in (H_{d,0} \Omega^1(M))^\prime
\]
and
\[
T_\lambda(\delta(\mu^{-1} d e) + \lambda \varepsilon e) = e, \quad e \in H_{d,0} \Omega^1(M).
\]
Further, if \( \langle u, dh' \rangle_M = 0 \) for all \( h' \in H^1_0(M) \), then \( T_\lambda u \in H_{d,0} \Omega^1(M)_{0,\varepsilon} \). Moreover, if \( \varepsilon \) and \( \mu \) are positive, then \( T_\lambda \) is self-adjoint with respect to the \( L^2(\Omega^1(M)) \)-inner product.

Here and in what follows, \( \langle \cdot, \cdot \rangle_M \) is the duality between \((H_{d,0} \Omega^1(M))^\prime \) and \( H_{d,0} \Omega^1(M) \) naturally extending the \( L^2(\Omega^1(M)) \)-inner product.

**Proof.** Consider the bilinear form on \( H_{d,0} \Omega^1(M) \)
\[
B(e, e') := (\mu^{-1} d e | d e')_{L^2(\Omega^2)}, \quad e, e' \in H_{d,0} \Omega^1(M).
\]
Then
\[
|B(e, e')| \leq C ||e||_{H_{d,0} \Omega^1(M)} ||e'||_{H_{d,0} \Omega^1(M)}.
\]
It is also easy to see that
\[
\text{Re} B(e, e) \geq C_0 ||d e||_{L^2(\Omega^2)}^2 \geq c_0 ||e||_{H_{d,0} \Omega^1(M)}^2 - C_0 ||e||_{L^2(\Omega^1)}^2
\]
for some constants \( c_0, C_0 > 0 \) independent of \( e \). Thus, there is constant \( \lambda > 0 \) such that the form \( B(e, e') + (\lambda \varepsilon e | e')_{L^2(\Omega^1)} \) is strictly coercive on \( H_{d,0} \Omega^1(M) \). In fact, we can take \( \lambda > 0 \) satisfying \( \lambda \geq C_0 / \min_M \text{Re} (\varepsilon) \). Applying the Lax-Milgram’s lemma, we obtain a bounded linear operator \( T_\lambda : (H_{d,0} \Omega^1(M))^\prime \to H_{d,0} \Omega^1(M) \) such that
\[
(\mu^{-1} d T_\lambda u | d e')_{L^2(\Omega^2)} + (\lambda \varepsilon T_\lambda u | e')_{L^2(\Omega^2)} = \langle u, e' \rangle_M
\]
for all \( u \in (H_{d,0} \Omega^1(M))^\prime \) and \( e' \in H_{d,0} \Omega^1(M) \), where \( \langle \cdot, \cdot \rangle_M \) is the duality between \((H_{d,0} \Omega^1(M))^\prime \) and \( H_{d,0} \Omega^1(M) \). Thus, \( T_\lambda \) is the operator which maps \( u \in (H_{d,0} \Omega^1(M))^\prime \) to the unique solution \( e \in H_{d,0} \Omega^1(M) \) of \( \delta(\mu^{-1} d e) + \lambda \varepsilon e = u \).

In particular, if \( \langle u, dh' \rangle_M = 0 \) for all \( h' \in H^1_0(M) \), setting \( e' = dh' \) in (A.14) we get
\[
(\varepsilon T_\lambda u | dh')_{L^2(\Omega^1)} = 0
\]
and hence \( T_\lambda u \in H_{d,0} \Omega^1(M)_{0,\varepsilon} \).
To prove that \( T_\lambda \) is self-adjoint, suppose \( \varphi, \varphi' \in L^2\Omega^1(M) \). Then
\[
(T_\lambda \varphi|\varphi')_{L^2\Omega^1(M)} = (T_\lambda \varphi|\delta(\mu^{-1}dT_\lambda \varphi') + \lambda \varepsilon T_\lambda \varphi'_{L^2\Omega^1(M)}) = (\mu^{-1}dT_\lambda \varphi|dT_\lambda \varphi')_{L^2\Omega^1(M)} + (\lambda \varepsilon T_\lambda \varphi|T_\lambda \varphi'_{L^2\Omega^1(M)}) = (\delta(\mu^{-1}dT_\lambda \varphi) + \lambda \varepsilon T_\lambda \varphi|T_\lambda \varphi'_{L^2\Omega^1(M)}) = (\varphi|T_\lambda \varphi')_{L^2\Omega^1(M)}.
\]
Thus, \( T_\lambda^* = T_\lambda \). \( \square \)

Then \( E_0 \in H_{d,0}\Omega^1(M)_{0,\varepsilon} \) solves (A.12) if and only if
\[
E_0 - (\omega^2 + \lambda)\tilde{T}_\lambda E_0 = T_\lambda \big( \varepsilon J_{r,0} + \delta(\mu^{-1} \ast J_m) \big) \tag{A.15}
\]
where \( \tilde{T}_\lambda = T_\lambda \circ m_\varepsilon \circ P_\varepsilon \), \( m_\varepsilon \) is multiplication by \( \varepsilon \), and \( P_\varepsilon \) is the bounded orthogonal projection of \( L^2\Omega^1(M) \) onto \( L^2\Omega^1(M)_{0,\varepsilon} \) constructed in Proposition A.3. Note that for all \( h' \in H^1_0(M) \) we have
\[
\langle \varepsilon J_{r,0} + \delta(\mu^{-1} \ast J_m), dh' \rangle_M = \langle \varepsilon J_{r,0} | dh' \rangle_{L^2\Omega^1(M)} + (\mu^{-1} \ast J_m | dh')_{L^2\Omega^1(M)} = 0,
\]
since \( J_{r,0} \in L^2\Omega^1(M)_{0,\varepsilon} \). Therefore, by the second part of Proposition A.7, this implies that \( T_\lambda \big( \varepsilon J_{r,0} + \delta(\mu^{-1} \ast J_m) \big) \in H_{d,0}\Omega^1(M)_{0,\varepsilon} \).

Second part of Proposition A.7 implies also that \( \tilde{T}_\lambda \) can be considered as a bounded linear operator
\[
\tilde{T}_\lambda : L^2\Omega^1(M)_{0,\varepsilon} \xrightarrow{m_\varepsilon} L^2\Omega^1(M)_{0,1} \xrightarrow{T} H_{d,0}\Omega^1(M)_{0,\varepsilon} \hookrightarrow L^2\Omega^1(M) \xrightarrow{P_\varepsilon} L^2\Omega^1(M)_{0,\varepsilon}
\]
and
\[
\tilde{T}_\lambda : L^2\Omega^1(M)_{0,\varepsilon} \xrightarrow{m_\varepsilon} L^2\Omega^1(M)_{0,1} \xrightarrow{T} H_{d,0}\Omega^1(M)_{0,\varepsilon} \tag{A.16}
\]
The equation (A.15) has a unique solution \( E_0 \) if and only if either \( \omega^2 = -\lambda \) or \( (\omega^2 + \lambda)^{-1} \notin \text{Spec}(\tilde{T}_\lambda) \). By Proposition A.6, the inclusion \( H_{d,0}\Omega^1(M)_{0,\varepsilon} \hookrightarrow L^2\Omega^1(M) \) is compact. This implies that \( \tilde{T}_\lambda \) is compact as an operator from \( L^2\Omega^1(M)_{0,\varepsilon} \) to itself.

According to Fredholm’s alternative (see e.g. [14, Theorem 0.38]), this implies that \( 0 \notin \text{Spec}(\tilde{T}_\lambda) \) and \( \text{Spec}(\tilde{T}_\lambda) \) is discrete. Therefore, (A.15) has a unique solution \( E_0 \) for any \( \omega \notin \Sigma \), where
\[
\Sigma = \{ \omega \in \mathbb{C} \setminus \{ \pm i\lambda^{1/2} \} : (\omega^2 + \lambda)^{-1} \notin \text{Spec}(\tilde{T}_\lambda) \}
\]
which is discrete. Since \( \text{Id} - (\omega^2 + \lambda)\tilde{T}_\lambda : H_{d,0}\Omega^1(M)_{0,\varepsilon} \to H_{d,0}\Omega^1(M)_{0,\varepsilon} \), for all \( \omega \notin \Sigma \) we have \( \text{Id} - (\lambda + \omega^2)\tilde{T}_\lambda \) is invertible, and since the right-hand-side of (A.15) is in \( H_{d,0}\Omega^1(M)_{0,\varepsilon} \), this implies that the solution \( E_0 \) belongs to \( H_{d,0}\Omega^1(M)_{0,\varepsilon} \). Finally, setting \( E = E_0 - \omega^{-2}d_J \), we obtain a unique \( H_{d,0}\Omega^1(M) \) solution for (A.8) such that
\[
\| E \|_{H_{d,0}\Omega^1(M)} \leq C(\| J_{r} \|_{L^2\Omega^1(M)} + \| J_{m} \|_{L^2\Omega^1(M)}),
\]
since \( \| j_{r} \|_{H^1(M)} \leq C(\| J_{r} \|_{L^2\Omega^1(M)} + \| J_{m} \|_{L^2\Omega^1(M)}) \). Therefore, (A.10) shows that the proof of Theorem A.2 is thus complete.
A.4. **Proof of Theorem A.1.** For a fixed $\omega \in \mathbb{C}$, consider the following space

$$\mathcal{M}_{\epsilon,\mu,\omega} = \{(E,H) \in H_{d}\Omega^{1}(M) \times H_{d}\Omega^{1}(M) : (E,H) \text{ is a solution of (A.1)}\}.$$ 

The topology on this space is the subspace topology in $H_{d}\Omega^{1}(M) \times H_{d}\Omega^{1}(M)$. It is not difficult to check that $\mathcal{M}_{\epsilon,\mu,\omega}$ is closed in $H_{d}\Omega^{1}(M) \times H_{d}\Omega^{1}(M)$.

For a given $(E,H) \in \mathcal{M}_{\epsilon,\mu,\omega}$ define $t_{E}(E,H) := t(E) \in TH_{d}\Omega^{1}(\partial M)$. Since the inclusion $\mathcal{M}_{\epsilon,\mu,\omega} \rightarrow H_{d}\Omega^{1}(M) \times H_{d}\Omega^{1}(M)$ is bounded, it is clear that $t_{E} : \mathcal{M}_{\epsilon,\mu,\omega} \rightarrow TH_{d}\Omega^{1}(\partial M)$ is bounded.

We now prove the following proposition which clearly implies Theorem A.1.

**Proposition A.8.** There is a discrete set $\Sigma \subset \mathbb{C}$ such that for all $\omega \notin \Sigma$ the operator $t_{E} : \mathcal{M}_{\epsilon,\mu,\omega} \rightarrow TH_{d}\Omega^{1}(\partial M)$ is a homeomorphism.

**Proof.** Let $\Sigma$ be as in Theorem A.2 and let us take any $\omega \notin \Sigma$. If we show that the bounded operator $t_{E} : \mathcal{M}_{\epsilon,\mu,\omega} \rightarrow TH_{d}\Omega^{1}(\partial M)$ is one-to-one and onto, the result follows from Open Mapping Theorem.

First, we prove injectivity of $t_{E}$. Suppose that $(E_{1},H_{1}),(E_{2},H_{2}) \in \mathcal{M}_{\epsilon,\mu,\omega}$ satisfy $t_{E}(E_{1},H_{1}) = t_{E}(E_{2},H_{2})$. Then $(E,H) \in \mathcal{M}_{\epsilon,\mu}$ and $t(E) = 0$, where $E := E_{1} - E_{2}$ and $H := H_{1} - H_{2}$. Uniqueness part of Theorem A.2 (with $J_{c} = J_{m} = 0$) gives that $E = 0$ and $H = 0$.

Now, we prove surjectivity of $t_{E}$. For a given $f \in TH_{d}\Omega^{1}(\partial M)$, by definition of $TH_{d}\Omega^{1}(\partial M)$, there is $E' \in H_{d}\Omega^{1}(M)$ such that $t(E') = f$. Applying Theorem A.2 with $J_{c} = i\omega E'$ and $J_{m} = *dE'$, we obtain a unique $(E_{0},H_{0}) \in H_{d,0}\Omega^{1}(M) \times H_{d}\Omega^{1}(M)$ solving

$$\begin{cases} *dE_{0} = i\omega \mu H_{0} + *dE', \\
*dh_{0} = -i\omega \epsilon E_{0} + i\omega \epsilon E'. \end{cases}$$

Then $(E,H) \in \mathcal{M}_{\epsilon,\mu}$ with $t_{E}(E,H) = t(E) = f$, where $E := E_{0} + E'$ and $H := H_{0}$. The proof is complete. \( \square \)

A.5. **Spectral problem for homogeneous Maxwell equations.** In this section we consider the eigenvalue problem for the boundary value problem

$$\begin{cases} *dE = i\omega \mu H, \\
*dh = -i\omega \epsilon E, \\
t(E) = 0 \tag{A.17} \end{cases}$$

under the additional assumption that both $\epsilon$ and $\mu$ are real-valued. We study this problem since the result of this section will be used in future works. Our main result is the following theorem.

**Theorem A.9.** Let $(M,g)$ be a compact 3-dimensional Riemannian manifold with smooth boundary and let $\epsilon,\mu \in C^{2}(M)$ be positive functions. There is a sequence of positive $\{\omega_{k}\}_{k=1}^{\infty} \subset \mathbb{R}$ and the corresponding sequence $\{(\epsilon_{k},h_{k})\}_{k=1}^{\infty} \in H_{d,0}\Omega^{1}(M)_{\epsilon,\mu} \times H_{d}\Omega^{1}(M)_{\mu}$ satisfying

$$\begin{cases} *d\epsilon_{k} = i\omega_{k} \mu h_{k}, \\
*dh_{k} = -i\omega_{k} \epsilon \epsilon_{k}. \tag{A.18} \end{cases}$$

The eigenvalues $\omega_{k} > 0$ have finite multiplicity, $0 < \omega_{1} \leq \omega_{2} \leq \cdots \rightarrow \infty$ as $k \rightarrow \infty$. The set $\{\epsilon_{k}\}_{k=1}^{\infty}$ forms an orthonormal basis in $H_{d,0}\Omega^{1}(M)_{\epsilon,\mu}$ with respect to the inner product $(\cdot,\cdot)_{L^{2}\Omega^{1}(M)} := (\epsilon \cdot | \cdot)_{L^{2}\Omega^{1}(M)}$ and the set $\{h_{k}\}_{k=1}^{\infty}$ forms a basis in
integrating by parts we show that
\[
\int_{\Omega} \omega E \cdot \nabla \phi + \mu \phi \varepsilon e \cdot \nabla E = 0, 
\]
and
\[
\int_{\Omega} \omega E \cdot \nabla \phi = 0. 
\]
For the proof, observe that the boundary value problem (A.17) can be written as
\[
\delta \mu^{-1} dE - \omega^2 \varepsilon E = 0, \quad t(E) = 0. 
\]
We first consider the case \( \omega \neq 0 \). Then the latter boundary value problem has a solution \( E \) in \( H_{d,0,\Omega^1}(M)_{0,\varepsilon} \) if and only if
\[
E - (\omega^2 + \lambda) \tilde{T}_\lambda E = 0, 
\]
where \( \tilde{T}_\lambda \) is defined as in the proof of Theorem A.2. We first show that this operator is in fact a self-adjoint operator with respect to certain inner product.

**Lemma A.10.** If both \( \varepsilon \) and \( \mu \) are strictly positive on \( M \), then the restriction of \( \tilde{T}_\lambda \) onto \( L^2\Omega^1(M)_{0,\varepsilon} \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{L^2\Omega^1(M)} \).

**Proof.** For \( \varphi, \varphi' \in L^2\Omega^1(M)_{0,\varepsilon} \) we have \( \tilde{T}_\lambda \varphi = T_\lambda (\varepsilon \varphi) \) and \( \tilde{T}_\lambda \varphi' = T_\lambda (\varepsilon \varphi') \). Therefore, using integration by parts,
\[
\langle \tilde{T}_\lambda \varphi, \varphi' \rangle_{L^2\Omega^1(M)} = \langle \varepsilon T_\lambda (\varphi) | \varphi' \rangle_{L^2\Omega^1(M)} = \langle T_\lambda (\varepsilon \varphi) | \varepsilon \varphi' \rangle_{L^2\Omega^1(M)}. 
\]
According to the hypotheses and Proposition A.7, \( T_\lambda \) is self-adjoint with respect to the \( L^2\Omega^1(M) \)-inner product. Therefore,
\[
\langle \tilde{T}_\lambda \varphi, \varphi' \rangle_{L^2\Omega^1(M)} = \langle \varepsilon \varphi | T_\lambda (\varepsilon \varphi') \rangle_{L^2\Omega^1(M)} = \langle \varphi, \tilde{T}_\lambda \varphi' \rangle_{L^2\Omega^1(M)}. 
\]
This finishes the proof. \( \square \)

It was shown in the previous section that the operator \( \tilde{T}_\lambda \) is bounded and compact from \( L^2\Omega^1(M)_{0,\varepsilon} \) to itself. Moreover, by Lemma A.10, the assumptions that \( \varepsilon \) and \( \mu \) are strictly positive imply that the operator \( \tilde{T}_\lambda \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{L^2\Omega^1(M)} \). Then by Fredholm’s alternative and Spectral theorem (see e.g. Proposition 6.6 in [30, Appendix A]) there is a sequence \( \{\kappa_k\}_{k=1}^\infty \subset \mathbb{R} \) consisting of eigenvalues of finite multiplicity such that \( \kappa_k \searrow 0 \) as \( k \to \infty \). Associated to the eigenvalues \( \kappa_k \) we have the eigenfunctions \( e_k \in L^2\Omega^1(M)_{\varepsilon,\mu} \), forming an orthonormal basis in \( L^2\Omega^1(M)_{0,\varepsilon} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{L^2\Omega^1(M)} \) and satisfying \( \tilde{T}_\lambda e_k = \kappa_k e_k \). Moreover, each \( e_k \) is in \( H_{d,0,\Omega^1(M)_{0,\varepsilon}} \) (by (A.16)) and solves
\[
e_k - (\omega_k^2 + \lambda) \tilde{T}_\lambda e_k = 0 \]
if \( \omega_k^2 = \kappa_k^{-1} - \lambda \). Then \( e_k \) also solves \( \delta (\mu^{-1} d e_k) - \omega_k^2 \varepsilon e_k = 0 \). Using this and integrating by parts we show that
\[
(\mu^{-1} d e_k | d e_k)_{L^2\Omega^1(M)} - \omega_k^2 (\varepsilon e_k | e_k)_{L^2\Omega^1(M)} = (\delta (\mu^{-1} d e_k) - \omega_k^2 \varepsilon e_k | e_k)_{L^2\Omega^1(M)} = 0. 
\]
Since \( \varepsilon \) and \( \mu \) are assumed to be strictly positive, this implies that
\[
\omega_k^2 = \frac{(\mu^{-1} d e_k | d e_k)_{L^2\Omega^1(M)}}{(\varepsilon e_k | e_k)_{L^2\Omega^1(M)}} > 0. 
\]
We may choose $\omega_k > 0$, and hence $\omega_k = (\kappa_k^{-1} - \lambda)^{1/2}$. Since $\kappa_k \searrow 0$ as $k \to \infty$, we have $\omega_k \to \infty$ as $k \to \infty$.

Next, we define the sequence $\{h_k\}_{k=1}^\infty \subset L^2\Omega^1(M)$ as $\ast d e_k = i \omega_k \mu h_k$. Then, by direct calculations, it is not difficult to see that each $(e_k, h_k)$ satisfy \((A.18)\) and hence also $h_k \in H_d\Omega^1(M)$. Moreover, $h_k \in H_d\Omega^1(M)_\mu$, since for all $\varphi \in H_d(0, \Omega^1(M))$, integrating by parts, we have

$$(h_k, \varphi)_{L^2_\mu\Omega^1(M)} = (\mu h_k | \varphi)_{L^2\Omega^1(M)} = (i\omega_k)^{-1}(* d e_k | \varphi)_{L^2\Omega^1(M)} = (i\omega_k)^{-1}(e_k | d \varphi)_{L^2\Omega^1(M)} = 0.$$

Further, using \((A.18)\)

$$(h_k, h_l)_{L^2_\mu\Omega^1(M)} = (\mu h_k | h_l)_{L^2\Omega^1(M)} = (\omega_k \omega_l)^{-1}(\mu^{-1} * d e_k | d e_l)_{L^2\Omega^1(M)} = (\omega_k \omega_l)^{-1}(\delta | e_l)_{L^2\Omega^1(M)} = \frac{\omega_k}{\omega_l}(e_k | e_l)_{L^2\Omega^1(M)}.$$

Therefore,

$$(h_k, h_l)_{L^2_\mu\Omega^1(M)} = \frac{\omega_k}{\omega_l}(e_k | e_l)_{L^2\Omega^1(M)} = \delta_{kl},$$

i.e. $\{h_k\}_{k=1}^\infty$ forms an orthonormal set with respect to $(\cdot, \cdot)_{L^2_\mu\Omega^1(M)}$.

To show that $\{h_k\}_{k=1}^\infty$ is complete in $H_d\Omega^1(M)_\mu$, with respect to $(\cdot, \cdot)_{L^2_\mu\Omega^1(M)}$, take $\psi \in H_d\Omega^1(M)_\mu$ such that $(h_k, \psi)_{L^2_\mu\Omega^1(M)} = 0$ for all $k \geq 1$ integer. Then

$$0 = i \omega_k (\mu h_k | \psi)_{L^2\Omega^1(M)} = (\ast d e_k | \psi)_{L^2\Omega^1(M)} = (e_k | \delta \ast \psi)_{L^2\Omega^1(M)} = (e_k | d \psi)_{L^2\Omega^1(M)}.$$

Setting $\phi = \varepsilon^{-1} * d \psi \in L^2\Omega^1(M)$, this implies that $(e_k, \phi)_{L^2\Omega^1(M)} = 0$ for all $k \geq 1$ integer. Suppose that $\phi \in L^2\Omega^1(M)_{0, \omega}$. Then by completeness of $\{e_k\}_{k=1}^\infty$ in $L^2\Omega^1(M)_{0, \omega}$ with respect to the inner product $(\cdot, \cdot)_{L^2\Omega^1(M)}$, we get $\phi = 0$ and hence $\psi \in H_d(0, \Omega^1(M))$. Then $\psi = 0$ according to the Helmholtz decomposition \((A.6)\).

Now, we show that $\phi \in L^2\Omega^1(M)_{0, \omega}$. For this, we need to show that $(\varepsilon \phi | d \varphi)_{L^2\Omega^1(M)} = 0$ for all $\varphi \in H^0_0(M)$. By density, it is enough to consider the case when $\varphi \in C^0_0(M_m)$. Then, integrating by parts,

$$(\varepsilon \phi | d \varphi)_{L^2\Omega^1(M)} = (\varepsilon d \psi | d \varphi)_{L^2\Omega^1(M)} = (d \psi | * d \varphi)_{L^2\Omega^1(M)} = (t(\psi) | t(\varphi))_{\partial M}.$$

Since $t(\varphi) = d_{\partial M}(\varphi | \partial M) = 0$, by Lemma 2.1, we obtain

$$(t(u) | t(i_n * d \varphi))_{L^2\Omega^1(\partial M)} = \int_{\partial M} t(u) \wedge t(d \varphi) = 0$$

for all $u \in C^\infty\Omega^1(M)$. Therefore, $t(i_n * d \varphi) = 0$ and hence $(\varepsilon \phi | d \varphi)_{L^2\Omega^1(M)} = 0$. This proves the completeness.

Finally, we mention that $\omega = 0$ is also an eigenvalue of \((A.17)\) with infinite dimensional eigenspace $H_{d,0}(0, \Omega^1(M)) \times H_d(0, \Omega^1(M))$. 
APPENDIX B. TRANSVERSAL SEMICLASSICAL PSEUDODIFFERENTIAL OPERATORS

Suppose \( m \geq 0 \) is an integer and \( a \in C^{\infty}(\mathbb{R}^n_+ \times \mathbb{R}^{n-1}) \) is such that for all multi-indices \( \alpha = (\alpha_1, \alpha') \) and \( \beta \)

\[
|\partial_{x_1}^{\alpha_1} \partial_{x'}^{\alpha'} \partial_{\xi}^{\beta} a(x_1, x', \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\beta|},
\]

for some constant \( C_{\alpha, \beta} > 0 \). Thus, for each \( x_1 \in (0, \infty) \) and \( \alpha_1 \geq 0 \) integer, \( (\partial_{x_1}^{\alpha_1} a)(x_1, x', \xi) \) is a symbol on \( \mathbb{R}^{n-1} \) of order \( m \) with bounds being uniform in \( x_1 \).

Then we consider an operator \( A \) defined for \( u(x_1, x') \in S(\mathbb{R}^n_+) \) as a semiclassical pseudodifferential operator on \( \mathbb{R}^{n-1} \) acting on \( x' \) variable, for each fixed \( x_1 \in (0, \infty) \), with symbol \( a(x_1, x', \xi) \) via standard quantization, i.e.

\[
Au(x_1, x') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi} a(x_1, x', h \xi) \tilde{u}(x_1, \xi) \, d\xi,
\]

where \( h > 0 \) is a small semiclassical parameter.

**Proposition B.1.** Suppose that \( s \in \mathbb{R} \) and \( A \) is an operator on \( S(\mathbb{R}^n_+) \) defined as above. Then \( A \) has an extension to a bounded operator \( H^s(\mathbb{R}^n_+) \to H^{s-m}(\mathbb{R}^n_+) \) satisfying the estimate

\[
\|Au\|_{H^{s-m}(\mathbb{R}^n_+)} \lesssim \|u\|_{H^s(\mathbb{R}^n_+)} , \quad u \in H^s(\mathbb{R}^n_+),
\]

where the implicit constant depends only on \( s \) and \( m \).

**Proof.** The case when \( s \) is an integer such that \( s \geq m \) was established in [7, Lemma 4.3]. We use this result to prove the proposition for the case when \( s \) is arbitrary real number using duality and interpolation.

First we consider the case when \( s \leq 0 \) is an integer. We first observe that \( A^* \) is an operator defined in a similar way as \( A \) but with symbol \( b(x_1, x', \xi) = a(x_1, -\xi, x') \).

Since \(-s + m \geq m\), this implies that \( A^*: H^{-s+m}(\mathbb{R}^n_+) \to H^{-s}(\mathbb{R}^n_+) \) is bounded and

\[
\|A^* u\|_{H^{-s}(\mathbb{R}^n_+)} \lesssim \|u\|_{H^{-s+m}(\mathbb{R}^n_+)} , \quad u \in H^{-s+m}(\mathbb{R}^n_+).
\]

In particular, \( A^* \) is bounded from \( H^{-s+m}_0(\mathbb{R}^n_+) \) into \( H^{-s}_0(\mathbb{R}^n_+) \). Then by duality, \( A \) is bounded from \( H^s(\mathbb{R}^n_+) \) into \( H^{s-m}(\mathbb{R}^n) \) and

\[
\|Au\|_{H^{s-m}(\mathbb{R}^n)} \leq \sup_{\psi \in H^{s-m}_0(\mathbb{R}^n_+)} \frac{\langle Au, \psi \rangle_{\mathbb{R}^n_+}}{\|\psi\|_{H^{s-m}_0(\mathbb{R}^n_+)}} \leq \sup_{\psi \in H^{s-m}_0(\mathbb{R}^n_+)} \frac{\langle u, A^* \psi \rangle_{\mathbb{R}^n_+}}{\|\psi\|_{H^{s-m}_0(\mathbb{R}^n_+)}} \lesssim \sup_{\psi \in H^{s-m}_0(\mathbb{R}^n_+)} \frac{\langle u, A^* \psi \rangle_{\mathbb{R}^n_+}}{\|A^* \psi\|_{H^{-s}_0(\mathbb{R}^n_+)}} \lesssim \|u\|_{H^s(\mathbb{R}^n_+)}.
\]

Now, suppose that \( s \) is any real number in \( (0, m) \). According to what we have proven so far, we know that \( A \) is bounded from \( L^2(\mathbb{R}^n_+) \) into \( H^{-m} \) and from \( H^m(\mathbb{R}^n_+) \) into \( L^2(\mathbb{R}^n_+) \), and the following estimates hold

\[
\|Au\|_{H^{s-m}(\mathbb{R}^n_+)} \lesssim \|u\|_{L^2(\mathbb{R}^n_+)}, \quad \|Av\|_{L^2(\mathbb{R}^n_+)} \lesssim \|v\|_{H^{s-m}(\mathbb{R}^n_+)}, \quad u \in L^2(\mathbb{R}^n_+), \, v \in H^m(\mathbb{R}^n_+).
\]
Proposition written as Observe that first-order semiclassical pseudodifferential operator on have are semiclassical pseudodifferential operators of orders 0, 1 and 2. Therefore, we \( \partial \)

Since \( J \)

Now, we prove the second estimate. Since \( E \)

where \( E_0(x_1,\cdot) \), for each fixed \( x_1 \in (0,\infty) \), is a zeroth-order semiclassical pseudodifferential operator on \( \mathbb{R}^{n-1} \) with bounds being uniform in \( x_1 \)-variable. By Proposition B.1, the operator \( E_0 \) is bounded from \( \mathcal{L}^2(\mathbb{R}^n_+) \) to \( \mathcal{L}^2(\mathbb{R}^n_+) \). Hence, by Lemma 5.4, this implies

\[
\| JvJ^{-1}u \|_{\mathcal{L}^2(\mathbb{R}^n_+)} \geq \| vJ^{-1}u \|_{\mathcal{L}^2(\mathbb{R}^n_+)} - h\| hJ^{-1}u \|_{\mathcal{L}^2(\mathbb{R}^n_+)} \\
\geq \| vu \|_{\mathcal{L}^2(\mathbb{R}^n_+)} - h\| u \|_{\mathcal{L}^2(\mathbb{R}^n_+)}.
\]

Finally, these arguments can be used to deal with the case when \( s \in \mathbb{R} \) is non-integer such that \( s < 0 \) or \( s > m \). This finishes the proof.

**Appendix C. Proofs of Technical Results**

*Proof of Lemma 5.5.* We begin with the proof of the first estimate. Let \( T \) be the first-order semiclassical pseudodifferential operator on \( \mathbb{R}^{n-1} \) with symbol \( F(\xi) \), acting on functions on \( \mathbb{R}^n_+ \) on the \( x' \)-variable. Then we show

\[
\| JvJ^{-1}u \|_{L^2(\mathbb{R}^n_+)} = \| (T + h\partial_1)vJ^{-1}u \|_{L^2(\mathbb{R}^n_+)} \\
\geq \| v(T + h\partial_1)J^{-1}u \|_{L^2(\mathbb{R}^n_+)} - \| hE_0J^{-1}u \|_{L^2(\mathbb{R}^n_+)},
\]

where \( E_0(x_1,\cdot) \), for each fixed \( x_1 \in (0,\infty) \), is a zeroth-order semiclassical pseudodifferential operator on \( \mathbb{R}^{n-1} \) with bounds being uniform in \( x_1 \)-variable. By Proposition B.1, the operator \( E_0 \) is bounded from \( \mathcal{L}^2(\mathbb{R}^n_+) \) to \( \mathcal{L}^2(\mathbb{R}^n_+) \). Hence, by Lemma 5.4, this implies

\[
\| JvJ^{-1}u \|_{L^2(\mathbb{R}^n_+)} \geq \| vJ^{-1}u \|_{L^2(\mathbb{R}^n_+)} - h\| hJ^{-1}u \|_{L^2(\mathbb{R}^n_+)} \\
\geq \| vu \|_{L^2(\mathbb{R}^n_+)} - h\| u \|_{L^2(\mathbb{R}^n_+)}. 
\]

Now, we prove the second estimate. Since \( J = T + h\partial_1 \), the operator \( Q \) can be written as

\[
Q = A_0h^2\partial_1^2 + A_1h\partial_1 + A_2,
\]

where \( A_0(x_1,\cdot) \), \( A_1(x_1,\cdot) \) and \( A_2(x_1,\cdot) \), for each fixed \( x_1 \in (0,\infty) \), are semiclassical differential operators of orders 0, 1 and 2 on \( \mathbb{R}^{n-1} \) with bounds being uniform in \( x_1 \)-variable. Since \( u \in \mathcal{S}(\mathbb{R}^n_+) \), we have \( Qu \in \mathcal{S}(\mathbb{R}^n_+) \), and hence

\[
\| (JQ - QJ)u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} = \| [h\partial_1 + T, A_0h^2\partial_1^2 + A_1h\partial_1 + A_2]u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)}. 
\]

Since \( \partial_1T = T\partial_1 \), we obtain

\[
\| (JQ - QJ)u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} \leq \| [h\partial_1, Q]u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} + \| [T, A_0]h^2\partial_1^2u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} \\
+ \| [T, A_1]h\partial_1u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} + \| [T, A_2]u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)}.
\]

Observe that

\[
[h\partial_1, Q] = hD_2, \quad [T, A_0] = hP_0, \quad [T, A_1] = hP_1, \quad [T, A_2] = hP_2,
\]

where \( D_2 \) is a second-order semiclassical differential operator, and \( P_0 \), \( P_1 \) and \( P_2 \) are semiclassical pseudodifferential operators of orders 0, 1 and 2. Therefore, we have

\[
\| (JQ - QJ)u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} \leq h\| D_2u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} + h\| P_0h^2\partial_1^2u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} \\
+ h\| P_1h\partial_1u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)} + h\| P_2u \|_{H^{-\frac{n}{2}}(\mathbb{R}^n_+)}. 
\]
Applying Lemma B.1 to the terms on the right-hand side of the above estimate, we finish the proof.

**Proof of Lemma 5.6.** Assuming \( g \in L^2(\mathbb{R}_+^n) \), let us prove the lemma. Observe that

\[
(Jg)_{sc1}(x_1, \xi) = (F(\xi) + h\partial_1)\hat{g}_{sc1}(x_1, \xi) = 0.
\]

This implies that

\[
\|Jv\|_{H^{-1}_{sc1}(\mathbb{R}_+^n)} = \sup_{0 \neq w \in H^1_{0}(\mathbb{R}_+^n)} \frac{|(Jv, w)_{\mathbb{R}_+^n}|}{\|w\|_{H^1_{sc1}(\mathbb{R}_+^n)}} = \sup_{0 \neq w \in H^1_{0}(\mathbb{R}_+^n)} \frac{|(v - g, J^*w)_{\mathbb{R}_+^n}|}{\|w\|_{H^1_{sc1}(\mathbb{R}_+^n)}}.
\]

Since by Lemma 5.4, the operator \( J^* \) is an isomorphism from \( H^1(\mathbb{R}_+^n) \) with semi-classical norm to \( L^2(\mathbb{R}_+^n) \), we have

\[
\|Jv\|_{H^{-1}_{sc1}(\mathbb{R}_+^n)} \lesssim \|v - g\|_{L^2(\mathbb{R}_+^n)}.
\]

For the opposite side estimate, write \( v - g = J^*J^{-1}(v - g) \). By Lemma 5.4, we have \( J^{-1}(v - g) \in H^1(\mathbb{R}_+^n) \). Also, by the definition of \( g \)

\[
(J^{-1}g)_{sc1}(0, \xi) = \frac{2}{h} \int_0^\infty e^{-\frac{F(\xi)}{h}} \frac{1}{h} \int_0^\infty \hat{v}_{sc1}(t, \xi)e^{-\frac{F(\xi) s + F(\xi) t}{h}} dt ds
\]

\[
= \frac{2}{h} \int_0^\infty e^{-\frac{2\sqrt{F(\xi)}}{h}} \frac{1}{h} \int_0^\infty \hat{v}_{sc1}(t, \xi)e^{-\frac{F(\xi) t}{h}} dt ds
\]

\[
= (J^{-1}v)_{sc1}(0, \xi).
\]

Therefore, \( J^{-1}(v - g)(0, x') = 0 \) and hence \( J^{-1}(v - g) \in H^1_0(\mathbb{R}_+^n) \). If \( v - g = 0 \), then we are done by (C.1). If not, we make the choice \( w = J^{-1}(v - g) \) in (C.1) and get the desired estimate.

Finally, let us show that \( g \in L^2(\mathbb{R}_+^n) \). Using the definition of \( \hat{g}_{sc1} \), the direct calculations give

\[
\int_0^\infty |\hat{g}_{sc1}(x_1, \xi)|^2 dx_1 = \frac{2}{h} \int_0^\infty \hat{v}_{sc1}(s, \xi)e^{-\frac{F(\xi)}{h}} ds.
\]
Applying Hölder’s inequality, we get
\[
\int_0^\infty |\hat{g}_{\text{scl}}(x_1, \xi)|^2 \, dx_1 \leq \frac{2 \text{Re} F(\xi)}{h} \left( \int_0^\infty |\tilde{v}_{\text{scl}}(x_1, \xi)|^2 \, dx_1 \right) \left( \int_0^\infty e^{-\frac{2 \text{Re} F(\xi)}{h} s} \, ds \right) = \int_0^\infty |\tilde{v}_{\text{scl}}(x_1, \xi)|^2 \, dx_1.
\]
Integrating over $\mathbb{R}^{n-1}$ with respect to $\xi$-variable and using the semiclassical Plancherel’s theorem, we get
\[
\|g\|_{L^2(\mathbb{R}^n_+)} \leq \|v\|_{L^2(\mathbb{R}^n_+)}.
\]
completing the proof.

REFERENCES

[1] M. Brown, M. Marletta, J.M. Reyes, Uniqueness for an inverse problem in electromagnetism with partial data, J. Differen. Eq. 260 (2016), 6525–6546.
[2] A. Bukhgeim, G. Uhlmann, Recovering a potential from partial Cauchy data, Comm. Partial Diff. Eq. 27 (2002), no. 3-4, 653–668.
[3] A. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)
[4] S. Caorsi, P. Fernandes, M. Raffetto, On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems, SIAM J. Numer. Anal., 38, 580–607.
[5] P. Caro, P. Ola, M. Salo, Inverse boundary value problem for Maxwell equations with local data, Comm. PDE 34 (2009), 1425–1464.
[6] P. Caro, T. Zhou, On Global Uniqueness for an IBVP for the Time-harmonic Maxwell’s Equations, Analysis and PDE, 7 (2014), No. 2, 375–405.
[7] F. J. Chung, A partial data result for the magnetic Schrödinger inverse problem, Analysis and PDE, 7 (2014), 117–157.
[8] F. J. Chung, Partial data for the Neumann-Dirichlet map, Journal of Fourier Analysis and Applications, 21 (2015), 628–665.
[9] F. J. Chung, M. Salo, L. Tzou, Partial data inverse problems for the Hodge Laplacian, to appear in Analysis and PDE.
[10] F. J. Chung, P. Ola, M. Salo, L. Tzou, Partial data inverse problems for Maxwell equations via Carleman estimates, preprint, arXiv:1502.01618
[11] M. Costabel, A remark on the regularity of solutions of Maxwell’s equations in Lipschitz domains, Math. Meth. Appl. Sci., 12, 365–368.
[12] M. Costabel, A coercive bilinear form for Maxwell’s equations, J. Math. Anal. Appl. 157 (1991) 527–541.
[13] D. Dos Santos Ferreira, C. Kenig, M. Salo, G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, Invent. Math. 178 (2009), no. 1, 119–171.
[14] G. B. Folland, Introduction to Partial Differential Equations, 2nd edition, Princeton University Press, 1995.
[15] V. Isakov, On uniqueness in the inverse conductivity problem with local data, Inverse Probl. Imaging, 1 (2007), 95–105.
[16] C. Kenig, M. Salo, G. Uhlmann, Inverse problems for the anisotropic Maxwell equations, Duke Math. J. 157 (2011), no. 2, 369–419.
[17] C. Kenig, J. Sjöstrand, G. Uhlmann, The Calderón problem with partial data, Ann. of Math. (2) 165 (2007), no. 2, 567–591.
[18] A. Kirsch, F. Hettlich, The Mathematical Theory of Time-Harmonic Maxwell’s Equations, Applied Mathematical Sciences 190, Springer, 2015
[19] M. Lassas, Impedance imaging problem as a low frequency limit, Inverse Problems, 13 (1997), 1503–1518.
[20] R. Leis, Zur theorie elektromagnetischer schwingungen in anisotropen inhomogenen medien, Math. Z., 106 (1968), pp. 213–224.

[21] W. McLean, Strongly elliptic systems and boundary integral equations, Cambridge University Press, 2000.

[22] P. Monk, Finite Element Methods for Maxwell’s Equations, Numer. Math. Sci. Comput., Oxford University Press, New York, 2003.

[23] P. Ola, E. Somersalo, Electromagnetic inverse problems and generalized Sommerfeld potentials, SIAM J. Appl. Math. 56 (1996), 1129–1145.

[24] P. Ola, L. Päivärinta, E. Somersalo, An inverse boundary value problem in electromagnetics, Duke Math. J. 70 (1993), 617–655.

[25] M. Salo, G. Uhlmann, The attenuated ray transform on simple surfaces J. Diff. Geom. 88 (2011), no. 1, 161–187.

[26] G. Schwarz, Hodge decomposition – a method for solving boundary value problems, Lecture notes in mathematics 1607, Springer, 1995.

[27] E. Somersalo, D. Isaacson, M. Cheney, A linearized inverse boundary value problem for Maxwell’s equations, J. Comp. Appl. Math. 42 (1992), 123–136.

[28] Z. Sun, G. Uhlmann, An Inverse Boundary Value Problem for Maxwell’s Equations, Arch. Rational Mech. Anal., 119 (1992), 71–93.

[29] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. (2) 125 (1987), no. 1, 153–169.

[30] M. E. Taylor, Partial differential equations I: Basic theory, Springer, 1999.

[31] C. Weber, A local compactness theorem for Maxwell’s equations, Math. Meth. Appl. Sci., 2 12–25.

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