TABLEAUX POSETS AND THE FAKE DEGREES OF COINVARIANT ALGEBRAS

SARA C. BILLEY, MATJAŽ KONVALINKA, JOSHUA P. SWANSON

ABSTRACT. We introduce two new partial orders on the standard Young tableaux of a given partition shape, in analogy with the strong and weak Bruhat orders on permutations. Both posets are ranked by the major index statistic offset by a fixed shift. The existence of such ranked poset structures allows us to classify the realizable major index statistics on standard tableaux of arbitrary straight shape and certain skew shapes. By a theorem of Lusztig–Stanley, this classification can be interpreted as determining which irreducible representations of the symmetric group exist in which homogeneous components of the corresponding coinvariant algebra, strengthening a recent result of the third author for the modular major index. Our approach is to identify patterns in standard tableaux that allow one to mutate descent sets in a controlled manner. By work of Lusztig and Stembridge, the arguments extend to a classification of all nonzero fake degrees of coinvariant algebras for finite complex reflection groups in the infinite family of Shephard–Todd groups.

1. Introduction

Let SYT(\( \lambda \)) denote the set of all standard Young tableaux of partition shape \( \lambda \). We say \( i \) is a descent in a standard tableau \( T \) if \( i + 1 \) comes before \( i \) in the row reading word of \( T \), read from bottom to top along rows in English notation. Equivalently, \( i \) is a descent in \( T \) if \( i + 1 \) appears in a lower row in \( T \). Let \( \text{maj}(T) \) denote the major index statistic on SYT(\( \lambda \)), which is defined to be the sum of the descents of \( T \). The major index generating function for SYT(\( \lambda \)) is given by

\[
\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.
\]

The polynomial \( \text{SYT}(\lambda)^{\text{maj}}(q) \) has two elegant closed forms, one due to Steinberg [Ste51] based on dimensions of irreducible representations of \( \text{GL}_n(\mathbb{F}_q) \).
and one due to Stanley [Sta79] generalizing the Hook-Length Formula; see Theorem 2.11.

For fixed $\lambda$, consider the fake degree sequence

$$b_{\lambda,k} := \# \{ T \in \text{SYT}(\lambda) : \text{maj}(T) = k \}$$

for $k = 0, 1, 2, \ldots$

The fake degrees have appeared in a variety of algebraic and representation-theoretic contexts including Green’s work on the degree polynomials of unipotent $\text{GL}_n(\mathbb{F}_q)$-representations [Gre55, Lemma 7.4], the irreducible decomposition of type $A$ coinvariant algebras [Sta79, Prop. 4.11], Lusztig’s work on the irreducible representations of classical groups [Lus77], and branching rules between symmetric groups and cyclic subgroups [Ste89, Thm. 3.3]. The term “fake degree” was apparently coined by Lusztig [Car89], perhaps because $\# \text{SYT}(\lambda) = \sum_{k \geq 0} b_{\lambda,k}$ is the degree of the irreducible $S_n$-representation indexed by $\lambda$, so a $q$-analog of this number is not itself a degree but related to the degree.

We consider three natural enumerative questions involving the fake degrees:

(I) which $b_{\lambda,k}$ are zero?
(II) are the fake degree sequences unimodal?
(III) are there efficient asymptotic estimates for $b_{\lambda,k}$?

We completely settle (I) with the following result. Denote by $\lambda'$ the conjugate partition of $\lambda$, and let $b(\lambda) := \sum_{i \geq 1} (i - 1)\lambda_i$.

**Theorem 1.1.** For every partition $\lambda \vdash n \geq 1$ and integer $k$ such that $b(\lambda) \leq k \leq \binom{n}{2} - b(\lambda')$, we have $b_{\lambda,k} > 0$ except in the case when $\lambda$ is a rectangle with at least two rows and columns and $k$ is either $b(\lambda) + 1$ or $\binom{n}{2} - b(\lambda') - 1$. Furthermore, $b_{\lambda,k} = 0$ for $k < b(\lambda)$ or $k > \binom{n}{2} - b(\lambda')$.

As a consequence of the proof of Theorem 1.1, we identify two ranked poset structures on $\text{SYT}(\lambda)$ where the rank function is determined by $\text{maj}$. Furthermore, as a corollary of Theorem 1.1 we have a new proof of a complete classification due to the third author [Swa18, Thm. 1.4] generalizing an earlier result of Klyachko [Kly74] for when the counts

$$a_{\lambda,r} := \{ T \in \text{SYT}(\lambda) : \text{maj}(T) \equiv_n r \}$$

for $\lambda \vdash n$ are nonzero.

The easy answer to question (II) is “no”. The fake degree sequences are not always unimodal. For example, $\text{SYT}(4,2)^{\text{maj}}(q)$ is not unimodal. See Example 2.13. Nonetheless, certain inversion number generating functions $p_n^{(k)}(q)$ which appear in a generalization of $\text{SYT}(\lambda)^{\text{maj}}(q)$ are in fact unimodal; see Definition 7.7 and Corollary 7.10. Furthermore, computational evidence suggests $\text{SYT}(\lambda)^{\text{maj}}(q)$ is typically not far from unimodal.

Questions (II) and (III) will be addressed in a forthcoming article [BKS18]. In particular, we show in that article that the coefficients of $\text{SYT}(\lambda^{(i)})^{\text{maj}}(q)$
are asymptotically normal for any sequence of partitions \(\lambda^{(1)}, \lambda^{(2)}, \ldots\) such that \(\text{aft}(\lambda^{(i)})\) approaches infinity where \(\text{aft}(\lambda)\) is the number of boxes outside the first row or column, whichever is smaller. The aft statistics on partitions is in FindStat as [RS+18, St001214].

We note that there are polynomial expressions for the fake degrees \(b_{\lambda,k}\) in terms of parameters \(H_i\), the number of cells of \(\lambda\) with hook length equal to \(i\). These polynomials are closely related to polynomials that express the number of permutations \(S_n\) of a given inversion number \(k \leq n\) as a function of \(n\) by work of Knuth. See Lemma 3.1 and Corollary 3.3. These polynomials are useful in some cases, however, we find that in practice Stanley’s formula is the most effective way to compute a given fake degree sequence for partitions up to size 200. See Remark 2.12 for more on efficient computation using cyclotomic polynomials.

Symmetric groups are the finite reflection groups of type \(A\). The classification and invariant theory of both finite irreducible real reflection groups and complex reflection groups developed over the past century builds on our understanding of the type \(A\) case [Hum90]. In particular, these groups are classified by Shephard–Todd into an infinite family \(G(m,d,n)\) together with 34 exceptions. Using work of Stembridge on generalized exponents for irreducible representations, the analog of (1) can be phrased for all Shephard–Todd groups as

\[
\begin{align*}
g(\Lambda)^d(q) := \frac{\#(\Lambda)^d}{d} \left[ \prod_{i=1}^{m} \text{SYT}(\lambda^{(i)})_{\text{maj}}(q^m) = \sum b_{\Lambda^d, k} q^k
\end{align*}
\]

where \(\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})\) is a sequence of \(m\) partitions with \(n\) cells total, \(\alpha(\Lambda) = (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|) \equiv n, d \mid m\), and \(\{\Lambda^d\}\) is the orbit of \(\Lambda\) under the group \(C_d\) of \((m/d)\)-fold cyclic rotations; see Corollary 8.2. The polynomials \(\left[ \frac{n}{\alpha(\Lambda)} \right]_{q^d}\) are deformations of the usual \(q\)-multinomial coefficients which we explore in Section 7. The coefficients \(b_{\Lambda^d, k}\) are the fake degrees in this case.

We use (3) and Theorem 1.1 to completely classify all nonzero fake degrees for coinvariant algebras for all Shephard–Todd groups \(G(m,d,n)\), which includes the finite real reflection groups in types \(A\), \(B\), and \(D\). See Corollary 6.4 and Corollary 8.4 for the type \(B\) and \(D\) cases, respectively. See Theorem 6.3 and Theorem 8.3 for the general \(C_m \wr S_n\) and \(G(m,d,n)\) cases, respectively.

The rest of the paper is organized as follows. In Section 2, we give background on tableaux combinatorics, Shephard–Todd groups, and their irreducible representations. Section 3 describes the polynomial formulas for fake degrees in type \(A\). Section 4 presents our combinatorial argument proving Theorem 1.1 and giving poset structures on tableaux of a given shape. Section 5 uses the argument in Section 4 to answer in the affirmative a question of Adin–Elizalde–Roichman about internal zeros of \(\text{SYT}(\lambda)^{\text{des}}(q)\); see Corollary 5.3.
In Section 6, we begin to address the question of characterizing nonzero fake degrees by starting with the wreath products $C_m \wr S_n = G(m,1,n)$; see Theorem 6.3. In Section 7, we define the deformed $q$-multinomials $\left[\begin{array}{c} n \\ \alpha \end{array}\right]_q$ as rational functions and give a summation formula, Theorem 7.6, which shows they are polynomial. Finally, in Section 8, we complete the classification of nonzero fake degrees for $G(m,d,n)$ and spell out how (3) relates to Stembridge’s original generating function for the fake degrees in $G(m,d,n)$; see Theorem 8.3 and Corollary 8.2.

2. Background

In this section, we review some standard terminology and results on combinatorial statistics and tableaux. Many further details in this area can be found in [Sta12, Sta99]. We also review background on the finite complex reflection groups and their irreducible representations. Further details in this area can be found in [Car89, Sag91].

2.1. Word and Tableaux Combinatorics. Here we review standard combinatorial notions related to words and tableaux.

**Definition 2.1.** Given a word $w = w_1w_2\cdots w_n$ with letters $w_i \in \mathbb{Z}_{>1}$, the *content* of $w$ is the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ where $\alpha_i$ is the number of times $i$ appears in $w$. Such a sequence $\alpha$ is called a (weak) composition of $n$, written as $\alpha \models n$. Trailing 0’s are often omitted when writing compositions, so $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ for some $m$. Note, a word of content $(1,1,\ldots,1) \models n$ is a permutation in the symmetric group $S_n$ written in one-line notation. The *inversion number* of $w$ is

$$\text{inv}(w) := \# \{ (i,j) : i < j, w_i > w_j \}.$$  

The *descent set* of $w$ is

$$\text{Des}(w) := \{ 0 < i < n : w_i > w_{i+1} \}$$

and the *major index* of $w$ is

$$\text{maj}(w) := \sum_{i \in \text{Des}(w)} i.$$  

The study of permutation statistics is a classical topic in enumerative combinatorics. The major index statistic on permutations was introduced by Percy MacMahon in his seminal works [Mac13, Mac17]. At first glance, this function on permutations may be unintuitive, but it has inspired hundreds of papers and many generalizations; for example on Macdonald polynomials [HHL05], posets [ER15], quasisymmetric functions [SW10], cyclic sieving [RSW04, AS17], and bijective combinatorics [Foa68, Car75].
Definition 2.2. Given a finite set $W$ and a function $\text{stat}: W \to \mathbb{Z}_{\geq 0}$, write the corresponding ordinary generating function as
$$W^{\text{stat}}(q) := \sum_{w \in W} q^{\text{stat}(w)}.$$

Definition 2.3. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \models n$. We use the following standard $q$-analogues:
\[ [n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}, \quad (q\text{-integer}) \]
\[ [n]_q! := [n]_q[n-1]_q\cdots[1]_q, \quad (q\text{-factorial}) \]
\[ \binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \in \mathbb{Z}_{\geq 0}[q], \quad (q\text{-binomial}) \]
\[ \binom{n}{\alpha}_q := \frac{[n]_q!}{[\alpha_1]_q!\cdots[\alpha_m]_q!} \in \mathbb{Z}_{\geq 0}[q], \quad (q\text{-multinomial}). \]

Example 2.4. The identity statistic on the set $W = \{0, \ldots, n-1\}$ has generating function $[n]_q$. The “sum” statistic on $W = \prod_{j=1}^{n} \{0, \ldots, j-1\}$ has generating function $[n]_q!$. It is straightforward to show that $S_n^{\text{inv}} := \sum_{w \in S_n} q^{\text{inv}(w)} = [n]_q!$ also.

For $\alpha \models n$, let $W_\alpha$ denote the words of content $\alpha$. A classic result of MacMahon is that $\text{maj}$ and $\text{inv}$ have the same distribution on $W_\alpha$ which is determined by the corresponding $q$-multinomial.

Theorem 2.5. [Mac17, §1] For each $\alpha \models n$,
\[ W^{\text{maj}}_\alpha(q) = \binom{n}{\alpha}_q = W^{\text{inv}}_\alpha(q). \] (4)

Definition 2.6. A polynomial $P(q) = \sum_{i=0}^{n} c_i q^i$ is symmetric if $c_i = c_{n-i}$ for $0 \leq i \leq n$. We generally say $P(q)$ is symmetric also if there exists an integer $k$ such that $q^k P(q)$ is symmetric. We say $P(q)$ is unimodal if
$$c_0 \leq c_1 \leq \cdots \leq c_j \geq c_{j+1} \geq \cdots \geq c_n$$
for some $0 \leq j \leq n$. Furthermore, $P(q)$ has no internal zeros provided that $c_j \neq 0$ whenever $c_i, c_k \neq 0$ and $i < j < k$.

From Theorem 2.5 and the definition of the $q$-multinomials, we see that each $W^{\text{maj}}_\alpha(q)$ is a symmetric polynomial with constant and leading coefficient 1. Indeed, these polynomials are unimodal generalizing the well-known case for Gaussian coefficients [Sta80, Thm 3.1] and [Zei89]. It also follows easily from MacMahon’s theorem that $W^{\text{maj}}_\alpha(q)$ has no internal zeros.
2.2. Partitions and Standard Young Tableaux.

Definition 2.7. A composition \( \lambda \vdash n \) such that \( \lambda_1 \geq \lambda_2 \geq \ldots \) is called a partition of \( n \), written as \( \lambda \vdash n \). The size of \( \lambda \) is \( |\lambda| := n \) and the length \( \ell(\lambda) \) of \( \lambda \) is the number of non-zero entries. The Young diagram of \( \lambda \) is the upper-left justified arrangement of unit squares called cells where the \( i \)th row from the top has \( \lambda_i \) cells following the English notation; see Figure 1a. The cells of a tableau are indexed by matrix notation when we refer to their row and column. The hook length of a cell \( c \in \lambda \) is the number of cells in \( \lambda \) in the same row as \( c \) to the right of \( c \) and in the same column as \( c \) and below \( c \), including \( c \) itself; see Figure 1b. A corner of \( \lambda \) is any cell with hook length 1. A notch of \( \lambda \) is any \((i, j)\) not in \( \lambda \) such that both \((i - 1, j)\) and \((i, j - 1)\) are in \( \lambda \). Note that notches cannot be in the first row or column of \( \lambda \). A bijective filling of \( \lambda \) is any labeling of the cells of \( \lambda \) by the numbers \([n] = \{1, 2, \ldots, n\}\). The symmetric group \( S_n \) acts on bijective fillings of \( \lambda \) by acting on the labels.

![Young diagram of \( \lambda \).](image1)

![Hook lengths of \( \lambda \).](image2)

Figure 1. Constructions related to the partition \( \lambda = (6, 3, 3) \vdash 12 \). The partition has corners at positions \((3, 3)\) and \((1, 6)\) and one notch at position \((2, 4)\).

Definition 2.8. A skew partition \( \lambda/\nu \) is a pair of partitions \((\nu, \lambda)\) such that the Young diagram of \( \nu \) is contained in the Young diagram of \( \lambda \). The cells of \( \lambda/\nu \) are the cells in the diagram of \( \lambda \) which are not in the diagram of \( \nu \), written \( c \in \lambda/\nu \). We identify straight partitions \( \lambda \) with skew partitions \( \lambda/\emptyset \) where \( \emptyset = (0, 0, \ldots) \) is the empty partition. The size of \( \lambda/\nu \) is \( |\lambda/\nu| := |\lambda| - |\nu| \). The notions of bijective filling, hook lengths, corners, and notches naturally extend to skew partitions as well.

Definition 2.9. Given a sequence of partitions \( \Lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \), we identify the sequence with the block diagonal skew partition obtained by translating the Young diagrams of the \( \lambda^{(i)} \) so that the rows and columns occupied by these components are disjoint, form a valid skew shape, and they appear in order from top to bottom as depicted in Figure 2.

Definition 2.10. A standard Young tableau of shape \( \lambda/\nu \) is a bijective filling of the cells of \( \lambda/\nu \) such that labels increase to the right in rows and down
Figure 2. Diagram for the skew partition $\lambda/\nu = 76443/4433$, which is also the block diagonal skew shape $\underline{\lambda} = ((3,2),(1,1),(3))$.

columns; see Figure 3. The set of standard Young tableaux of shape $\lambda/\nu$ is denoted $\text{SYT}(\lambda/\nu)$. The descent set of $T \in \text{SYT}(\lambda/\nu)$ is the set $\text{Des}(T)$ of all labels $i$ in $T$ such that $i + 1$ is in a strictly lower row than $i$. The major index of $T$ is

$$\text{maj}(T) := \sum_{i \in \text{Des}(T)} i.$$ 

Figure 3. On the left is a standard Young tableau of straight shape $\lambda = (6,3,3)$ with descent set $\{2,4,7,9,10\}$ and major index 32. On the right is a standard Young tableau of block diagonal skew shape $(7,5,3)/(5,3)$ corresponding to the sequence of partitions $((2),(2),(3))$ with descent set $\{2,6\}$ and major index 8.

The block diagonal skew partitions $\underline{\lambda}$ allow us to simultaneously consider words and tableaux as follows. Let $W_\alpha$ be the set of all words with content $\alpha = (\alpha_1, \ldots, \alpha_k)$. Letting $\underline{\lambda} = ((\alpha_k), \ldots, (\alpha_1))$, we have a bijection

$$\phi: \text{SYT}((\underline{\lambda})) \rightarrow W_\alpha$$

which sends a tableau $T$ to the word whose $i$th letter is the row number in which $i$ appears in $T$, counting from the bottom up rather than top down. For example, using the skew tableau $T$ on the right of Figure 3, we have $\phi(T) = 131231 \in W_{(3,2,2)}$. It is easy to see that $\text{Des}(\phi(T)) = \text{Des}(T)$, so that $\text{maj}(\phi(T)) = \text{maj}(T)$. 


2.3. **Major Index Generating Functions.** Stanley gave the following analogue of Theorem 2.5 for standard Young tableaux of a given shape. It generalizes the famous Frame–Robinson–Thrall Hook-Length Formula [FRT54, Thm. 1] or [Sta99, Cor. 7.21.6] obtained by setting \( q = 1 \).

**Theorem 2.11.** [Sta99, 7.21.5] Let \( \lambda \vdash n \) with \( \lambda = (\lambda_1, \lambda_2, \ldots) \). Then

\[
\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c \in \lambda} [h_c]_q}
\]

where \( b(\lambda) := \sum (i-1)\lambda_i \) and \( h_c \) is the hook length of the cell \( c \).

**Remark 2.12.** Since \( \# \text{SYT}(\lambda) \) typically grows extremely quickly, Stanley’s formula offers a very useful way to compute \( \text{SYT}(\lambda)^{\text{maj}}(q) \) by expressing both the numerator and denominator, up to a \( q \)-shift, as a product of cyclotomic polynomials and canceling all factors from the denominator. We prefer to use cyclotomic factors over linear factors in order to avoid arithmetic in cyclotomic fields.

**Example 2.13.** For \( \lambda = (4,2) \), \( b(\lambda) = 2 \) and the multiset of hook lengths is \( \{1^2, 2^2, 4, 5\} \) so \( |\text{SYT}(\lambda)| = 9 \) by the Hook-Length Formula. The major index generating function is given by

\[
\text{SYT}(4,2)^{\text{maj}}(q) = q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2
\]

\[
= q^2 \frac{[6]_q!}{[5]_q[4]_q[2]_q[2]_q} = q^2 \frac{[6]_q[3]_q}{[2]_q}.
\]

Note, \( \text{SYT}(4,2)^{\text{maj}}(q) \) is symmetric but not unimodal.

For \( \lambda = (4,2,1) \), \( b(\lambda) = 4 \) and the multiset of hook lengths is \( \{1^3, 2, 3, 4, 6\} \) so \( |\text{SYT}(\lambda)| = 35 \) by the Hook-Length Formula. The major index generating function is given by

\[
\text{SYT}(4,2,1)^{\text{maj}}(q) = q^{14} + 2q^{13} + 3q^{12} + 4q^{11} + 5q^{10} + 5q^9 + 5q^8 + 4q^7 + 3q^6 + 2q^5 + q^4
\]

\[
= q^4 \frac{[7]_q!}{[6]_q[4]_q[3]_q[2]_q} = q^4 [7]_q[5]_q.
\]

Note, \( \text{SYT}(4,2,1)^{\text{maj}}(q) \) is symmetric and unimodal.

**Example 2.14.** We recover \( q \)-integers, \( q \)-binomials, and \( q \)-Catalan numbers, up to \( q \)-shifts as special cases of the major index generating function for tableaux as follows:

\[
\text{SYT}(\lambda)^{\text{maj}}(q) = \begin{cases} 
q[n]_q & \text{if } \lambda = (n, 1), \\
q^{\binom{k+1}{2}} \binom{n}{k}_q & \text{if } \lambda = (n-k+1, 1^k), \\
q^n \frac{1}{[n+1]_q} \binom{2n}{n}_q & \text{if } \lambda = (n, n).
\end{cases}
\]
The following strengthening of Stanley’s formula to $\lambda$ is well known (e.g., see [Ste89, (5.6)]), though since it is somewhat difficult to find explicitly in the literature, we include a short proof.

**Theorem 2.15.** Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ where $\lambda^{(i)} \vdash \alpha_i$ and $n = \alpha_1 + \cdots + \alpha_m$. Then

\[
\text{SYT}(\lambda)^{\text{maj}}(q) = \left(\frac{n}{\alpha_1, \ldots, \alpha_m}\right) \cdot \prod_{i=1}^{m} \text{SYT}(\lambda^{(i)})^{\text{maj}}(q).
\]

**Proof.** The stable principal specialization of skew Schur functions is given by

\[
s_{\frac{\lambda}{\nu}}(1, q, q^2, \ldots) = \frac{\text{SYT}(\lambda/\nu)^{\text{maj}}(q)}{\prod_{j=1}^{\lambda/\nu}(1 - q^j)};
\]
see [Ste89, Lemma 3.1] or [Sta99, Prop.7.19.11]. On the other hand, it is easy to see from the definition of a skew Schur function as the content generating function for semistandard tableaux of the given shape that

\[
s_{\frac{\lambda}{\nu}}(x_1, x_2, \ldots) = \prod_{i=1}^{m} s_{\frac{\lambda}{\nu}}^{(i)}(x_1, x_2, \ldots).
\]

The result quickly follows. \qed

**Remark 2.16.** Theorem 2.11 and Theorem 2.15 have several immediate corollaries. First, we recover MacMahon’s result, Theorem 2.5, from Theorem 2.15 when $\lambda = ((\alpha_m), (\alpha_{m-1}), \ldots)$ by using the maj-preserving bijection $\phi$ in (5). Second, each $\text{SYT}(\frac{\lambda}{\nu})^{\text{maj}}(q)$ is symmetric (up to a $q$-shift) with leading coefficient 1. In particular, there is a unique “maj-minimizer” and “maj-maximizer” tableau in each $\text{SYT}(\frac{\lambda}{\nu})$. Moreover,

\[
\text{min maj}(\text{SYT}(\lambda)) = b(\lambda)
\]
and

\[
\text{max maj}(\text{SYT}(\lambda)) = \left(\frac{n}{2}\right) - b(\lambda') = b(\lambda) + \left(\frac{\lambda}{2}\right) - \sum_{c \in \lambda} h_c
\]

where $b(\lambda) := \sum_i b(\lambda^{(i)})$ and $b(\lambda') := \sum_i b(\lambda^{(i)'})$.

For general skew shapes, $\text{SYT}(\frac{\lambda}{\nu})^{\text{maj}}(q)$ does not factor as a product of cyclotomic polynomials times $q$ to a power. A “$q$-Naruse” formula due to Morales–Pak–Panova, [MPP15, (3.4)], gives an analogue of Theorem 2.11 involving a sum over “excited diagrams,” though the resulting sum has a single term precisely for the block diagonal skew partitions $\lambda$. 

2.4. **Complex Reflection Groups.** A complex reflection group is a finite subgroup of $GL(C^n)$ generated by pseudo-reflections, which are elements which pointwise fix a codimension-1 hyperplane. Shephard–Todd, building on work of Coxeter and others, famously classified the complex reflection groups [ST54]. The irreducible representations were constructed by Young, Specht, Lusztig, and others. We now summarize these results and fix some notation.

**Definition 2.17.** A pseudo-permutation matrix is a matrix where each row and column has a single non-zero entry. For positive integers $m, n$, the wreath product $C_m \wr S_n \subset GL(C^n)$ is the group of $n \times n$ pseudo-permutation matrices whose non-zero entries are complex $m$th roots of unity. For $d \mid m$, let $G(m, d, n)$ be the Shephard–Todd group consisting of matrices $x \in C_m \wr S_n$ where the product of the non-zero entries in $x$ is an $(m/d)$th root of unity. In fact, $G(m, d, n)$ is a normal subgroup of $C_m \wr S_n$ of index $d$ with cyclic quotient $(C_m \wr S_n)/G(m, d, n) \cong C_d$ of order $d$.

**Theorem 2.18.** [ST54] Up to isomorphism, the complex reflection groups are precisely the direct products of the groups $G(m, d, n)$, along with 34 exceptional groups.

**Remark 2.19.** Special cases of the Shephard–Todd groups include the following. The Weyl group of type $A_{n-1}$, or equivalently the symmetric group $S_n$, is isomorphic to $G(1, 1, n)$. The Weyl groups of both types $B_n$ and $C_n$ are $G(2, 1, n)$, the group of $n \times n$ signed permutation matrices. The subgroup of the group of signed permutations whose elements have evenly many negative signs is the Weyl group of type $D_n$, or $G(2, 2, n)$ as a Shephard–Todd group. We also have that $G(m, m, 2)$ is the dihedral group of order $2m$, and $G(m, 1, 1)$ is the cyclic group $C_m$ of order $m$.

The complex irreducible representations of $S_n$ were constructed by Young [You77] and are well known to be certain modules $S^\lambda$ canonically indexed by partitions $\lambda \vdash n$. These representations are beautifully described in [Sag91]. Specht extended the construction to irreducibles for $G \wr S_n$ where $G$ is a finite group.

**Theorem 2.20.** [Spe35] The complex inequivalent irreducible representations of $C_m \wr S_n$ are certain modules $S^\Delta$ indexed by the sequences of partitions $\Delta = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ for which $|\Delta| = |\lambda^{(1)}| + \cdots + |\lambda^{(m)}| = n$.

**Remark 2.21.** The version we give of Theorem 2.20 was stated by Stembridge [Ste89, Thm. 4.1]. The $C_m$-irreducibles are naturally though non-canonically indexed by $\mathbb{Z}/m$ up to one of $\phi(m)$ additive automorphisms, where $\phi(m)$ is Euler’s totient function. Correspondingly, one may identify $\mathbb{Z}/m$ with \{1, \ldots, m\} and obtain $\phi(m)$ different indexing schemes for the $C_m \wr S_n$-irreducibles. The
resulting indexing schemes are rearrangements of one another, and our results will be independent of these choices.

Clifford described a method for determining the branching rules of irreducible representations for a normal subgroup of a given finite group [Cl37]. Stembridge combined this method with Specht’s theorem to describe the irreducible representations for all Shephard–Todd groups from the $C_m \wr S_n$-irreducible representations.

We use Stembridge’s terminology where possible. In particular, for $d \mid m$, the $(m/d)$-fold cyclic rotations are the elements in the subgroup isomorphic to $C_d$ of $S_m$ generated by $\sigma_m^{m/d}$, where $\sigma_m = (1, 2, \ldots, m)$ is the long cycle. Let $S_m$ act on block diagonal partitions of the form $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ by permuting the blocks. This action restricts to $C_d = \langle \sigma_m^{m/d} \rangle$ as well. Let $\{\lambda\}^d$ denote the orbit of $\lambda$ under the $(m/d)$-fold cyclic rotations in $C_d$. Note, the number of block diagonal partitions in such a $C_d$-orbit, denoted $\#\{\lambda\}^d$, always divides $d$, but could be less than $d$ if $\lambda$ contains repeated partitions.

For example, take $d = 2$ and $m = 6$. If $\lambda = ((1), (2), (3, 2), (4), (5), (6, 1))$, then $\{\lambda\}^2$ has two elements, $\lambda$ and $((4), (5), (6, 1), (1), (2), (3, 2))$. If $\mu = ((1), (2), (3, 2), (1), (2), (3, 2))$, then $\{\mu\}^2$ only contains the element $\mu$.

**Theorem 2.22.** [Ste89, Remark after Prop. 6.1] The complex inequivalent irreducible representations of $G(m, d, n)$ are certain modules $S^{(\lambda)^d, c}$ indexed by the pairs $(\{\lambda\}^d, c)$ where $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ is a sequence of partitions with $|\lambda| = n$, $\{\lambda\}^d$ is the orbit of $\lambda$ under $(m/d)$-fold cyclic rotations, and $c$ is any positive integer $1 \leq c \leq \#\{\lambda\}^d$.

**Remark 2.23.** As with $C_m \wr S_n$, the indexing scheme is again non-canonical in general up to a choice of orbit representative, though our results relying on this work are independent of these choices. In fact, Stembridge uses $\lambda = (\lambda^{(m-1)}, \ldots, \lambda^{(0)})$, which is the most natural setting for Theorem 2.22 and Theorem 2.39 below. The fake degrees for irreducibles $S\Delta$ of $C_m \wr S_n$ are invariant up to a $g$-shift under all permutations of $\lambda$ in $S_m$, so for our purposes the indexing scheme is largely irrelevant. The fake degrees for irreducibles $S^{(\lambda)^d, c}$ of $G(m, d, n)$, however, are only invariant under the $(m/d)$-fold cyclic rotations of $\lambda$ in general. In this case, strictly speaking our $\lambda^{(i)}$ corresponds to the irreducible cyclic group representation $\chi^{i-1}$ defined by $\chi^{i-1}(\sigma_m) = \omega_m^{i-1}$ where $\omega_m$ is a fixed primitive $m$th root of unity in the sense that

$$S\Delta := \left(\chi^0 \otimes \chi^{(1)} \otimes \cdots \otimes \chi^{m-1} \otimes \chi^{\lambda(m)}\right)_{C_m \wr S_n}$$

see [Ste89, (4.1)]. Since we have no need of these explicit representations, we have used the naive indexing scheme throughout.
Example 2.24. For the type $B_n$ group $G(2,1,n)$, the irreducible representations are indexed by pairs $(\lambda, \mu)$ since $C_1$ is the trivial group and so in each case $c = 1$.

Example 2.25. For the type $D_n$ group $G(2,2,n)$, the irreducible representations can be thought of as being indexed by the sets $\{\lambda, \mu\}$ with $\lambda \neq \mu$ and $|\lambda| + |\mu| = n$ together with the pairs $(\nu, 1)$ and $(\nu, 2)$ where $\nu \vdash n/2$. The orbits alone can be thought of as the 2 element multisets $\{\lambda, \mu\}$ with $|\lambda| + |\mu| = n$ to together with the pairs $(\nu, 1)$ and $(\nu, 2)$ where $\nu \vdash n/2$.

2.5. Coinvariant Algebras. As mentioned in the introduction, Stanley [Sta79] and Lusztig (unpublished) determined the graded irreducible decomposition of the type $A$ coinvariant algebra via the major index generating function on standard Young tableaux. Stembridge was the first to publish a complete proof of this result and extended it to the complex reflection groups $G(m,d,n)$ [Ste89]. We now summarize these results.

Definition 2.26. Any group $G \subset \text{GL}(\mathbb{C}^n)$ acts on the polynomial ring with $n$ variables $\mathbb{C}[x_1,\ldots,x_n]$ by identifying $\mathbb{C}^n$ with $\text{Span}_\mathbb{C}\{x_1,\ldots,x_n\}$ and extending the $G$-action multiplicatively. The coinvariant algebra of $G$ is the quotient of $\mathbb{C}[x_1,\ldots,x_n]$ by the ideal generated by homogeneous $G$-invariant polynomials of positive degree, which is thus a graded $G$-module.

Definition 2.27. Let $R_n$ denote the coinvariant algebra of $S_n$. For $\lambda \vdash n$, let $g^\lambda(q)$ be the fake degree polynomial whose $k$th coefficient is the multiplicity of $S^\lambda$ in the $k$th degree piece of $R_n$.

Theorem 2.28 (Lusztig–Stanley, [Sta79, Prop. 4.11]). For a partition $\lambda$,

$$g^\lambda(q) = \text{SYT}(\lambda)^{\text{maj}}(q).$$

Equivalently, the multiplicity of $S^\lambda$ in the $k$th degree piece of the type $A$ coinvariant algebra $R_n$ is $b_{\lambda,k}$, the number of standard tableaux of shape $\lambda \vdash n$ with major index $k$.

Definition 2.29. Let $R_{m,n}$ denote the coinvariant algebra of $C_m \wr S_n$. Set $b_{\Delta,k} :=$ the multiplicity of $S^\Delta$ in the $k$th degree piece of $R_{m,n}$.

Write the corresponding fake degree polynomial as

$$g^\Delta(q) := \sum_k b_{\Delta,k} q^k.$$

Definition 2.30. Given a sequence of partitions $\underline{\lambda} = (\lambda^{(1)},\ldots,\lambda^{(m)})$, recall

$$b(\alpha(\underline{\lambda})) = \sum_{i=1}^m (i-1)|\lambda^{(i)}|.$$
on the block diagonal skew partition $\lambda$. We will abuse notation and define $b(\alpha(T)) := b(\alpha(\lambda))$ for any $T \in \text{SYT}(\lambda)$, which is not necessary in the next theorem but will be essential for the general Shephard–Todd groups $G(m, d, n)$.

**Theorem 2.31.** [Ste89, Thm. 5.3] For $\lambda = (\lambda(1), \ldots, \lambda(m))$ with $|\lambda| = n$,

\[
g_\lambda(q) = q^{b(\lambda)} \text{SYT}(\lambda)_{\text{maj}}(q^m).
\]

Equivalently, the multiplicity of $S_\lambda$ in the $k$th degree piece of the $C_m \wr S_n$ coinvariant algebra $R_{m,n}$ is the number of standard tableaux $T$ of block diagonal shape $\lambda$ with $k = b(\alpha(T)) + m \cdot \text{maj}(T)$.

**Remark 2.32.** By (7), we have an explicit product formula for $g_\lambda(q)$ also. Furthermore, in [BKS18], we characterize the possible limiting distributions for the coefficients of the polynomials $\text{SYT}(\lambda)_{\text{maj}}(q)$. We show that in most cases, the limiting distribution is the normal distribution. Consequently, that characterization can be interpreted as a statement about the asymptotic distribution of irreducible components in different degrees of the $C_m \wr S_n$ coinvariant algebras.

**Corollary 2.33.** In type $B_n$, the irreducible indexed by $(\lambda, \mu)$ with $|\lambda| = k$ and $|\mu| = n - k$ has fake degree polynomial

\[
g_{(\lambda, \mu)}(q) = q^{|\lambda| + 2b(\lambda) + 2b(\mu)} \binom{n}{k} q^{|\lambda| - k} \prod_{c \in \lambda} [h_c q^2] \prod_{c \in \mu} [h_c q^2].
\]

**Definition 2.34.** Let $R_{m,d,n}$ denote the coinvariant algebra of $G(m, d, n)$ assuming $d \mid m$. For an orbit $\{\lambda\}^d$ of a sequence of $m$ partitions with $n$ total cells under $(m/d)$-fold cyclic rotations, set

\[
b_{(\lambda)}^d,k := \text{the multiplicity of } S_{(\lambda)}^{d,c} \text{ in the } k\text{th degree piece of } R_{m,d,n},
\]

which in fact depends only on the orbit $\{\lambda\}^d$ and not the number $c$ by [Ste89, Prop. 6.3]. Write the corresponding fake degree polynomial as

\[
g_{(\lambda)}^d(q) = \sum_k b_{(\lambda)}^d,k q^k.
\]

**Theorem 2.35.** [Ste89, Cor. 6.4] Let $\{\lambda\}^d$ be the orbit of a sequence of $m$ partitions $\lambda$ with $|\lambda| = n$ under $(m/d)$-fold cyclic rotations. Then

\[
g_{(\lambda)}^d(q) = \frac{(\{\lambda\}^d)^{b(\lambda)}(q)}{[d]_{q^{n/m}}^{\text{maj}}(q^m)} \text{SYT}(\lambda)_{\text{maj}}(q^m)
\]

where

\[
(\{\lambda\}^d)^{b(\lambda)}(q) := \sum_{\mu \in \{\lambda\}^d} q^{b(\mu)}.
\]
Corollary 2.36 ([Lus77, Sect. 2.5], [Ste89, Cor. 6.5]). In type $D_n$, an irreducible indexed by $\{\lambda\}^2$ with $\lambda = (\lambda, \mu)$ and $|\lambda| = k$, $|\mu| = n - k$ has fake degree polynomial
\[
g(\lambda)^2(q) = \kappa_{\lambda\mu} q^{2b(\lambda) + 2b(\mu)} \frac{q^k + q^{n-k}}{1 + q^n} \frac{k!}{\prod_{c \in \lambda} [h_c]_{q^2}} \frac{(n-k)!}{\prod_{c \in \mu} [h_c]_{q^2}},
\]
where $\kappa_{\lambda\mu} = 1$ if $\lambda \neq \mu$ and $\kappa_{\lambda\lambda} = 1/2$.

Observe that Theorem 2.31 gives a direct tableaux interpretation of the coefficients of $g^\lambda(q)$. More generally, Stembridge gave a tableaux interpretation of the coefficients of $g^{\{\lambda\}^d}(q)$ which we next describe.

Definition 2.37. For a given $m, d, n$, let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ be a sequence of $m$ partitions with $|\lambda| = n$. Let $\{\lambda\}^d$ be the orbit of $\lambda$ under $(m/d)$-fold rotations. The cyclic group $C_d = \langle \sigma_m^d \rangle$ acts on the disjoint union $\bigsqcup_{\mu \in \{\lambda\}^d} \text{SYT}(\mu)$ as follows. Given $\mu = (\mu^{(1)}, \ldots, \mu^{(m)}) \in \{\lambda\}^d$, each $T \in \text{SYT}(\mu)$ may be considered as a sequence $\underline{T} = (T^{(1)}, \ldots, T^{(m)})$ of fillings of the shapes $\mu^{(i)}$. The group $C_d$ acts by $(m/d)$-fold rotations of this sequence of fillings. Write the resulting orbit as $\{\underline{T}\}^d$, which necessarily has size $d$. For such $\underline{T}$, the largest entry of $\underline{T}$, namely $n$, appears in some $T^{(k)}$. If among the elements of the orbit $\{\underline{T}\}^d$ of $\underline{T}$ this value $k$ is minimal for $T$ itself, then we call $\underline{T}$ the canonical standard tableau representative for $\{\underline{T}\}^d$. Let
\[
\text{SYT}(\{\lambda\}^d) \subset \bigsqcup_{\mu \in \{\lambda\}^d} \text{SYT}(\mu)
\]
be the set of canonical standard tableaux representatives of orbits $\{\underline{T}\}^d$ for $T \in \text{SYT}(\lambda)$. Recall, $b(\alpha(\underline{T})) := b(\alpha(\mu)) = \sum (i-1)|\mu^{(i)}|$ if $\underline{T} \in \text{SYT}(\mu)$, so $b \circ \alpha$ is not generally constant on $\text{SYT}(\{\lambda\}^d)$.

Remark 2.38. When the parts $\lambda^{(i)}$ are all non-empty, the set $\text{SYT}(\{\lambda\}^d)$ is the set of standard block diagonal skew tableaux of some shape $\mu \in \{\lambda\}^d$ where $n = |\lambda|$ is in the upper-right-most partition possible among the $(m/d)$-fold cyclic rotations of its blocks. Since every orbit $\{\underline{T}\}^d$ has size $d$, we have
\[
\# \text{SYT}(\{\lambda\}^d) = \frac{\# \{\lambda\}^d}{d} \# \text{SYT}(\lambda).
\]

Theorem 2.39. [Ste89, Thm. 6.6] Let $\lambda$ be a sequence of $m$ partitions with $|\lambda| = n$. Let $\{\lambda\}^d$ be the orbit of $\lambda$ under $(m/d)$-fold cyclic rotations. Then
\[
g^{\{\lambda\}^d}(q) = \text{SYT}(\{\lambda\}^d)^{b(\alpha(\underline{T}))+m \cdot \text{maj}(\underline{T})}(q).
\]
Equivalently, the multiplicity of $S^{\{\lambda\}^d,c}$ in the $k$th degree piece of the $G(m, d, n)$ coinvariant algebra $R_{m,d,n}$ is the number of canonical standard tableaux $\underline{T} \in \text{SYT}(\{\lambda\}^d)$ where with $k = b(\alpha(\underline{T})) + m \cdot \text{maj}(\underline{T})$. 

3. Polynomial Formulas For Fake Degrees

In this section, we briefly show how to construct polynomial formulas for the fake degrees $b_{\lambda,k}$ directly from Stanley’s $q$-hook length formula. We will use these polynomials in the next section for small changes from the minimal major index. Our results extend to a formula for counting permutations of a given inversion number.

Given $\lambda$, let

$$H_i(\lambda) = \# \{ c \in \lambda : h_c = i \},$$

(10)

$$m_i(\lambda) = \# \{ k : \lambda_k = i \}.$$  

(11)

If $\lambda$ is understood, we abbreviate $H_i = H_i(\lambda)$. For any nonnegative integer $k$ and polynomial $f(q)$, let $[q^k]f(q)$ be the coefficient of $q^k$.

**Lemma 3.1.** For every $\lambda \vdash n$ and $k = b(\lambda) + d$, we have

$$b_{\lambda,k} = [q^{b(\lambda)+d}]\text{SYT}(\lambda)^{\text{maj}}(q) = \sum_{\mu \vdash n} \prod_{i=1}^{\lfloor \lambda \rfloor} \binom{H_i + m_i(\mu) - 2}{m_i(\mu)}$$

(12)

which is a polynomial in the $H_i$’s for every positive integer $n$.

**Proof.** By Theorem 2.11, we have

$$q^{-b(\lambda)}\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{\lfloor n \rfloor!}{\prod_{c \in \lambda} [h_c]_q} = \prod_{i=1}^{n} (1 - q^i)^{-H_i-1}.$$  

(13)

The result follows using the expansion $(1 - q^i)^{-j} = \sum_{n=0}^{\infty} (j+n-1)q^n$ and multiplication of ordinary generating functions. \hfill $\square$

Note that if $H_i(\lambda) = 0$ and $m_i(\mu) = 1$, then the corresponding binomial coefficient in (12) is $-1$, so it is not obvious from this formula that the coefficients $b_{\lambda,k}$ are all nonnegative, which is clearly true by definition.

**Remark 3.2.** The first few polynomials are given by

$$[q^{b(\lambda)+1}]\text{SYT}(\lambda)^{\text{maj}}(q) = H_1 - 1$$

$$= \# \{ \text{notches of } \lambda \},$$

$$[q^{b(\lambda)+2}]\text{SYT}(\lambda)^{\text{maj}}(q) = \binom{H_1}{2} + H_2 - 1,$$

$$[q^{b(\lambda)+3}]\text{SYT}(\lambda)^{\text{maj}}(q) = \binom{H_1 + 1}{3} + (H_1 - 1)(H_2 - 1) + (H_3 - 1)$$

$$[q^{b(\lambda)+4}]\text{SYT}(\lambda)^{\text{maj}}(q) = \binom{H_1 + 2}{4} + \binom{H_2}{2} + \binom{H_1}{2}(H_2 - 1)$$

$$+ (H_1 - 1)(H_3 - 1) + (H_4 - 1).$$
These exact formulas hold for all $|\lambda| \geq 4$. For smaller size partitions some terms will not appear.

It is interesting to compare these polynomials to the ones described by Knuth for the number of permutations with $k \leq n$ inversions in $S_n$ in [Knu73, p.16]. See also [Sta12, Ex. 1.124] and [OEI18, A008302]. We can extend Knuth’s formulas to all $0 \leq k \leq \binom{n}{2}$ using the same idea.

**Corollary 3.3.** For fixed positive integers $k$ and $n$, we have

$$\#\{w \in S_n : \text{inv}(w) = d\} = \sum_{\mu \vdash d} (-1)^{\#\{\mu_i > 1\}} \binom{n + m_1(\mu) - 2}{m_1(\mu)}$$

where the sum is over all partitions $\mu \vdash d$ such that $\mu_1 \leq n$ and all of the parts of $\mu$ larger than 1 are distinct.

The proof follows in exactly the same way from the formula $\sum_{w \in S_n} q^{\text{inv}(w)} = \prod_{i=1}^{n}[i]^q = \prod_{i=1}^{n}[i]^q/[1]^q$. In essence, this is the case of the $q$-hook length formula when all of the hooks are length 1.

**Remark 3.4.** Let $T(d, n)$ be the number of partitions $\mu \vdash d$ such that $\mu_1 \leq n$ and all of the parts of $\mu$ larger than 1 are distinct. The triangle of numbers $T(d, n)$ for $1 \leq n \leq d$ is [OEI18, A318806].

## 4. Type A Internal Zeros Classification

As a corollary of Stanley’s formula, we know that for every partition $\lambda \vdash n \geq 1$ there is a unique tableau with minimal major index $b(\lambda)$ and a unique tableau with maximal major index $\binom{n}{2} - b(\lambda')$. These two agree for shapes consisting of one row or one column, and otherwise they are distinct. It is easy to identify these two tableaux in SYT$(\lambda)$; see Definition 4.1 below. Then, we classify all of the values $k$ such that $b(\lambda) < k < \binom{n}{2} - b(\lambda')$ and the fake degree $b_{\lambda,k} = 0$. We refer to such $k$ as internal zeros, meaning the location of zeros in the fake degree sequence for $\lambda$ between the known minimal and maximal nonzero locations.

**Definition 4.1.**

1. The max-maj tableau for $\lambda$ is obtained by filling the outermost, maximum length, vertical strip in $\lambda$ with the largest possible numbers $|\lambda|, |\lambda| - 1, \ldots, |\lambda| - \ell(\lambda) + 1$ starting from the bottom row and going up, then filling the rightmost maximum length vertical strip containing cells not previously used with the largest remaining numbers, etc.

2. The min-maj tableau of $\lambda$ is obtained similarly by filling the outermost, maximum length, horizontal strip in $\lambda$ with the largest possible numbers $|\lambda|, |\lambda| - 1, \ldots, |\lambda| - \lambda_1 + 1$ going right to left, then filling the lowest
maximum length horizontal strip containing cells not previously used
with the largest remaining numbers, etc.

See Figure 4 for an example. Note that the max-maj tableau of \( \lambda \) is the
transpose of the min-maj tableau of \( \lambda' \).

\[
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 9 & 13 \\
4 & 6 & 10 & 14 & & \\
7 & 11 & 15 & & & \\
8 & 12 & 16 & & & \\
17 & & & & & \\
\end{array}
\quad \quad
\begin{array}{cccccc}
1 & 3 & 4 & 11 & 16 & 17 \\
2 & 6 & 7 & 15 & & \\
5 & 9 & 10 & & & \\
8 & 13 & 14 & & & \\
12 & & & & & \\
\end{array}
\]

(a) A max-maj tableau and its outermost vertical strip. (b) A min-maj tableau and its outermost horizontal strip.

**Figure 4.** Max-maj and min-maj tableau for \( \lambda = (6, 4, 3, 3, 1) \).

The \( q^{b(\lambda)+1} \) coefficients of \( \text{SYT}(\lambda)_{\text{maj}}(q) \) can be computed as in Lemma 3.1, resulting in the following.

**Corollary 4.2.** We have \( q^{b(\lambda)+1} \text{SYT}(\lambda)_{\text{maj}}(q) = 0 \) if and only if \( \lambda \) is a rectangle. If \( \lambda \) is a rectangle with more than one row and column, then \( q^{b(\lambda)+2} \text{SYT}(\lambda)_{\text{maj}}(q) = 1 \).

A similar statement holds for \( \text{maj}(T) = \binom{n}{2} - b(\lambda') - 1 \) by symmetry. Thus, \( \text{SYT}_{\text{maj}}(q) \) has internal zeros when \( \lambda \) is a rectangle with at least two rows and columns. We will show these are the only internal zeros of type \( A \) fake degrees, proving Theorem 1.1.

**Definition 4.3.** Let \( \mathcal{E}(\lambda) \) denote the set of exceptional tableaux of shape \( \lambda \) consisting of the following elements.

(i) For all \( \lambda \), the max-maj tableau for \( \lambda \).
(ii) If \( \lambda \) is a rectangle, the min-maj tableau for \( \lambda \).
(iii) If \( \lambda \) is a rectangle with at least two rows and columns, the unique tableau in \( \text{SYT}(\lambda) \) with major index equal to \( \binom{n}{2} - b(\lambda') - 2 \). It is obtained from the max-maj tableau of \( \lambda \) by applying the cycle \( (2, 3, \ldots, \ell(\lambda) + 1) \), which reduces the major index by 2.

For example, \( \mathcal{E}(64331) \) consists of just the max-maj tableau for 64331 in Figure 4a, while \( \mathcal{E}(555) \) has the following three elements:
We prove Theorem 1.1 by constructing a map
\[ \varphi: \text{SYT}(\lambda) \setminus \mathcal{E}(\lambda) \rightarrow \text{SYT}(\lambda) \]
with the property
\[ \text{maj}(\varphi(T)) = \text{maj}(T) + 1. \]
For most tableau \( T \), we can find another tableau \( T' \) of the same shape such that \( \text{maj}(T') = \text{maj}(T) + 1 \) by applying some simple cycle to the values in \( T \), meaning a permutation whose cycle notation is either \((i, i+1, \ldots, k-1, k)\) or \((k, k-1, \ldots, i+1, i)\) for some \( i < k \). We will show there are 5 additional rules that must be added to complete the definition.

We note that technically the symmetric group \( S_n \) does not act on \( \text{SYT}(\lambda) \) for \( \lambda \vdash n \) since this action will not generally preserve the row and column strict requirements for standard tableaux. However, \( S_n \) acts on the set of all bijective fillings of \( \lambda \) using the alphabet \( \{1, 2, \ldots, n\} \) by acting on the values.

We will only apply permutations to tableaux after locating all values in some interval \([i, j]\) = \([i, i+1, \ldots, j]\) in \( T \). The reader is encouraged to verify that the specified permutations always maintain the row and column strict properties.

4.1. Rotation Rules. We next describe certain configurations in a tableau which imply that a simple cycle will increase \( \text{maj} \) by 1. Recall, the cells of a tableau are indexed by matrix notation.

**Definition 4.4.** Given \( \lambda \vdash n \) and \( T \in \text{SYT}(\lambda) \), a positive rotation for \( T \) is an interval \([i, k]\) \( \subset [n] \) such that if \( T' := (i, i+1, \ldots, k-1, k) \cdot T \), then \( T' \in \text{SYT}(\lambda) \) and there is some \( j \) for which
\[ \{j\} = \text{Des}(T') - \text{Des}(T) \quad \text{and} \quad \{j-1\} = \text{Des}(T) - \text{Des}(T'). \]
Intuitively, a positive rotation is one for which \( j-1 \in \text{Des}(T) \) becomes \( j \in \text{Des}(T') \) and all other entries remain the same. Consequently, \( \text{maj}(T') = \text{maj}(T) + 1 \). We call \( j \) the moving descent for the positive rotation.

The positive rotations can be characterized explicitly as follows. The proof is omitted since it follows directly from the pictures in Figure 5.

**Lemma 4.5.** An interval \([i, k]\) is a positive rotation for \( T \in \text{SYT}(\lambda) \) if and only if \( i < k \) and there is some necessarily unique moving descent \( j \) with \( 1 \leq i \leq j \leq k \leq n \) such that
(a) \( i, \ldots, j-1 \) forms a horizontal strip, \( j-1, j \) forms a vertical strip, and \( j, j+1, \ldots, k \) forms a horizontal strip;
(b) if \( i < j \), then \( i \) appears strictly northeast of \( k \) and \( i-1 \) is not in the rectangle bounding \( i \) and \( k \);
(c) if \( i = j \), then \( i-1 \) appears in the rectangle bounding \( i \) and \( k \);
(d) if \( j < k \), then \( k \) appears strictly northeast of \( k-1 \) and \( k+1 \) is not in the rectangle bounding \( k \) and \( k-1 \); and
(e) if \( j = k \), then \( k + 1 \) appears in the rectangle bounding \( k \) and \( k - 1 \).

See Figure 5 for diagrams summarizing these conditions.

\[
\begin{array}{c}
\begin{array}{ccc}
& & i \ldots j-1 \\
\hline
k & & \\
\hline
j & \ldots & k-1
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccc}
i+1 & \ldots & j \\
\hline
i & & \\
\hline
j+1 & \ldots & k
\end{array}
\end{array}
\]

(A) Schematic of a positive rotation with \( i < j < k \).

\[
\begin{array}{c}
\begin{array}{ccc}
i & i+1 & \ldots & k-1 \\
\hline
k & & i+1 & \ldots & k-1 & i+1 & \ldots & k+1
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
i+1 & \ldots & k-1 \\
\hline
i & & \\
\hline
k & & k+1
\end{array}
\end{array}
\]

(B) Schematic of a positive rotation with \( i < j = k \).

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
i-1
\hline
i & \ldots & i+1 \\
\hline
k
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
i-1
\hline
i & \ldots & i+2 \\
\hline
k
\end{array}
\end{array}
\end{array}
\]

(C) Schematic of a positive rotation with \( i = j < k \).

\textbf{Figure 5.} Summary diagrams for positive rotations.

In addition to the positive rotations above, we can also apply negative rotations, which are defined exactly as in Definition 4.4 with \((i, i+1, \ldots, k-1, k)\) replaced by \((k, k-1, \ldots, i+1, i)\) and the rest unchanged. Combinatorially, negative rotations can be obtained from positive rotations by applying inverse-transpose moves, that is, by applying negative cycles \((k, k-1, \ldots, i)\) to the transpose of the configurations in Figure 5 and reversing the arrows. Explicitly, we have the following analogue of Lemma 4.5. See Figure 6 for the corresponding diagrams.

\textbf{Lemma 4.6.} An interval \([i, k]\) is a negative rotation for \( T \in \text{SYT}(\lambda) \) if and only if \( i < k \) and there is some necessarily unique moving descent \( j \) with \( 1 \leq i \leq j \leq k \leq n \) such that

1. \( i, \ldots, j \) forms a vertical strip, \( j, j+1 \) forms a horizontal strip, and \( j+1, \ldots, k \) forms a vertical strip;
2. if \( i < j \), then \( i+1 \) appears strictly southwest of \( i \) and \( i-1 \) is not in the rectangle bounding \( i \) and \( i+1 \);
3. if \( i = j \), then \( i-1 \) appears in the rectangle bounding \( i \) and \( i+1 \);
4. if \( j < k \), then \( i \) appears strictly southwest of \( k \) and \( k+1 \) is not in the rectangle bounding \( i \) and \( k \); and
5. if \( j = k \), then \( k+1 \) appears in the rectangle bounding \( i \) and \( k \).
Example 4.7. The tableau
\[
\begin{array}{cccccc}
1 & 2 & 6 & 7 & 9 \\
3 & 4 & 8 & 13 \\
5 & 11 & 12 & 15 \\
10 & 14 \\
\end{array}
\]
allows positive rotation rules with \([i, k] \in \{[5,6], [8,9], [8,10], [8,11], [9,13]\}\), and the tableau
\[
\begin{array}{cccccc}
1 & 3 & 8 & 10 & 15 \\
2 & 4 & 9 & 11 \\
5 & 7 & 13 & 14 \\
6 & 12 \\
\end{array}
\]
allows negative rotation rules with \([i, k] \in \{[4,6], [6,7], [11,12]\}\).

It turns out that for the vast majority of tableaux, some negative rotation rule applies. The positive rotations can be applied in many of the remaining cases. For example, among the 81,081 tableaux in SYT(5442), there are only 24 (i.e., 0.03%) on which we cannot apply any positive or negative rotation rule.
For example, no rotation rules can be applied to the following two tableaux:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 \\
14 & 15 & & \\
\end{array}
\quad \quad \quad \quad 
\begin{array}{cccc}
1 & 2 & 3 & 8 & 12 \\
4 & 6 & 9 & 13 \\
5 & 7 & 10 & 14 \\
11 & 15 & & \\
\end{array}
\]

The following lemma and its corollary give a partial explanation for why negative rotation rules are so common. Given a tableaux \( T \), let \( T|_{\{z\}} \) denote the restriction of \( T \) to those values in \( \{z\} \).

**Lemma 4.8.** Let \( T \in \text{SYT}(\lambda) \setminus \mathcal{E}(\lambda) \). Suppose \( z \) is the largest value such that \( T|_{\{z\}} \) is contained in \( \text{maxmaj}(\mu) \) for some \( \mu \). If \( T|_{\{z+1\}} \) is not of the form
\[
\begin{array}{cccc}
1 & 2 & \cdots & i \\
i+1 & z+1 & & \\
i+2 & & & \\
\vdots & & & \\
z & & & \\
\end{array}
\]
then some negative rotation rule applies to \( T \).

**Proof.** Since \( T \notin \mathcal{E}(\lambda) \), \( T \) is not \( \text{maxmaj}(\lambda) \), so \( \lambda \) is not a one row or column shape. We have \( z \geq 2 \) since both two-cell tableaux are the max-maj tableau of their shape. Since \( \text{maxmaj}(\mu) \) is built from successive, outermost, maximal length, vertical strips as in Figure 4a, the same is true of \( T|_{\{z\}} \).

First, suppose \( z \) is not in the lowest row of \( T|_{\{z\}} \). Let \( i \) be the value in the topmost corner cell in \( T|_{\{z\}} \) which is strictly below \( z \). Let \( j \geq i \) be the bottommost cell in the vertical strip of \( T|_{\{z\}} \) which contains \( i \). See Figure 7a. We verify the conditions of Lemma 4.6, so the negative \([i, z]\)-rotation rule applies with moving descent \( j \). By construction, \( i, \ldots, j \) forms a vertical strip, \( j, j+1 \) forms a horizontal strip, and \( j+1, \ldots, z \) forms a vertical strip. If \( i < j \), then since \( i \) is a corner cell, \( i+1 \) appears strictly southwest of \( i \), and \( i-1 \) is above both \( i \) and \( i+1 \) so \( i-1 \) is not in the rectangle bounding \( i \) and \( i+1 \). If \( i = j \), we see that \( i-1 \) appears in the rectangle bounded by \( i \) and \( i+1 \). We also see that \( i \) appears strictly southwest of \( z \), and \( z+1 \) is not in the rectangle bounding \( i \) and \( z \) since \( i \) is a topmost corner and \( z \) is maximal.

Now suppose \( z \) is in the lowest row of \( T|_{\{z\}} \). In this case, \( T|_{\{z\}} \) is the max-maj tableau of its shape, so that \( z < |\lambda| \) and \( z+1 \) exists in \( T \) since \( T \notin \mathcal{E}(\lambda) \). By maximality of \( z \), \( z+1 \) cannot be in row 1 or below \( z \). Let \( i < z \) be the value in the rightmost cell of \( T|_{\{z\}} \) in the row immediately above \( z+1 \). See Figure 7b. We check that the negative \([i, z]\)-rotation rule applies with moving descent \( j = z \) using the conditions in Lemma 4.6. By construction, \( i, \ldots, z \) forms a vertical strip. Since \( z+1 \) is not below \( z \), we see that \( z, z+1 \) forms a horizontal strip. Since \( z+1 \) is in the row below \( i \), \( i+1 \) appears strictly southwest of \( i \).
For the tableau on the left above, \( i = 8 \) and \( z = 12 \) since \( T[i][12] \) is contained the max-maj tableau of shape 44322, 12 is not in the lowest row, 8 is in the closet corner to 12 in \( T[i][12] \) and below 12. Apply the negative rotation \((12,11,10,9,8)\) to get the tableau on the right, and observe maj has increased by 1. The moving descent is \( j = 10 \).

For the tableau on the left above, \( i = 7 \) and \( z = 10 \) since \( T[i][10] \) is the max-maj tableau of shape 33211, 10 is in the lowest row, 11 is in row 3, and 7 is the largest value in \( T[i][10] \) in row 2. Apply the negative rotation \((10,9,8,7)\) to get the tableau on the right, and observe maj has increased by 1. The moving descent is \( j = z = 10 \).

\[
\begin{array}{cccc}
1 & 3 & 6 & 11 \\
2 & 4 & 7 & 12 \\
5 & 8 & \rightarrow & 5 & 12 \\
9 & 13 & \rightarrow & 8 & 13 \\
10 & & & & 9
\end{array}
\]

(A) We also see that \( z + 1 \) appears in the rectangle bounded by \( i \) and \( z \) by choice of \( i \). It remains to show that \( i - 1 \) is not in the rectangle bounding \( i \) and \( i + 1 \). Suppose to the contrary that \( i - 1 \) is in the rectangle bounding \( i \) and \( i + 1 \). Then \( i \) would have to be in row 1 by choice of \( i < z \). Consequently \( i + 1 \) is in row 2 and strictly west of \( i \), forcing \( i - 1 \) to be in row 1 also. It follows from the choice of \( z \) that \( T[i][i] \) is a single row, the values \( i, i + 1, \ldots, z \) form a vertical strip, and \( T[z+1] \) is of the above forbidden form, giving a contradiction. □

**Corollary 4.9.** If \( T \in \text{SYT}(\lambda) \setminus \mathcal{E}(\lambda) \) and \( 1 \in \text{Des}(T) \), then some negative rotation rule applies to \( T \).

*Proof.* Let \( z \) be as in Lemma 4.8. Clearly \( z \geq 2 \) and \( T[2] \) is a single column, so \( T[z+1] \) cannot possibly be of the forbidden form. □

We also have the following variation on Lemma 4.8. It is based on finding the largest value \( q \) such that \( T[q] \) is contained in an exceptional tableau of type (iii). The proof is again a straightforward verification of the conditions in Lemma 4.6, and is omitted.

**Figure 7.** Examples of the negative rotations obtained from Lemma 4.8.
Lemma 4.10. Let $T \in \text{SYT}(\lambda) \setminus \mathcal{E}(\lambda)$. Suppose the initial values of $T$ are of the form

\[
\begin{array}{ccccccc}
1 & 2 & & & & & \\
3 & p+1 & & & & & \\
4 & & & & & & \\
\vdots & \ddots & & & & & \\
q & & & & & & \\
p & & & & & & \\
\end{array}
\begin{array}{ccccccc}
1 & 2 & \ell+1 & \cdots & p+1 & & \\
3 & z+1 & & & & & \\
4 & & & & & & \\
\vdots & \ddots & & & & & \\
q & & & & & & \\
p & & & & & & \\
\end{array}
\begin{array}{ccccccc}
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\end{array}
\quad \text{or} \quad
\begin{array}{ccccccc}
1 & 2 & \ell & \cdots & q+1 & & \\
3 & z+2 & & & & & \\
4 & & & & & & \\
\vdots & \ddots & & & & & \\
q & & & & & & \\
p & & & & & & \\
\end{array}
\begin{array}{ccccccc}
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\end{array}
\begin{array}{ccccccc}
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\vdots & \ddots & & & & & \\
\end{array}
\]

In either case, the $(p, q)$-negative rotation rule applies to $T$.

4.2. Initial Block Rules. Here we describe a collection of five additional block rules which may apply to a tableau that is not in the exceptional set. In each case, if the rule applies, then we specify a permutation of the entries so that we either add 1 into the descent set and leave the other descents unchanged, or we add 1 into the descent set, increase one existing descent by 1, and decrease one existing descent by 1. Thus, maj will increase by 1 in all cases. While these additional rules are certainly not uniquely determined by these criteria, they are also not arbitrary.

Example 4.11. For a given $T \in \text{SYT}(\lambda)$, one may consider all $T' \in \text{SYT}(\lambda)$ where $\text{maj}(T') = \text{maj}(T) + 1$. If $T' = \sigma \cdot T$ where $\sigma$ is a simple cycle, then one of the rotation rules may apply to $T$. Table 1 summarizes five particular $T$ for which no rotation rules apply. These examples have guided our choices in defining the block rules. In all but one of these examples, there is a unique $T'$ with $\text{maj}(T') = \text{maj}(T) + 1$, though in the third case there are two such $T'$, one of which ends up being easier to generalize.

| Tableau $T$ | Tableaux $T'$ | $\sigma$ | Block rule |
|-------------|---------------|----------|------------|
| 1, 2, 3, 4, 5, 6, 7 | 1, 3, 4, 6, 5, 7, 8 | (2, 3, 4) (6, 7) | B1 |
| 1, 2, 3, 4 | 1, 3, 4, 6, 5, 6, 7 | (2, 3, 4, 7, 6, 5) | B2 |
| 1, 2, 3, 4 | 1, 3, 6, 4, 5, 2, 6, 7 | (2, 3, 6, 4), (2, 4) (3, 5) | B3, — |
| 1, 2, 7, 3, 5, 8, 4, 6, 9 | 1, 4, 8, 2, 5, 9, 6, 3, 10 | (2, 4, 3) (7, 8, 9, 10) | B4 |
| 1, 2, 3, 5, 4, 6, 7 | 1, 5, 2, 6, 3, 4 | (2, 5, 6, 7, 4, 3) | B5 |

Table 1. Some tableaux $T \in \text{SYT}(\lambda)$ together with all $T' = \sigma \cdot T \in \text{SYT}(\lambda)$ where $\text{maj}(T') = \text{maj}(T) + 1$. See Definition 4.13 for an explanation of the final column.
In the remainder of this subsection, we describe the block rules, abbreviated B-rules. Then, we prove that if no rotation rules are possible for a tableau then either it is in the exceptional set or we can apply one of the B-rules. The B-rules cover disjoint cases so no tableau admits more than one block rule. To state the B-rules precisely, assume \( T \in \text{SYT}(\lambda) \setminus \mathcal{E}(\lambda) \) and no rotation rule applies.

**Notation 4.12.** Let \( c \) be the largest possible value such that \( T|_c \) is contained in the min-maj tableau of a rectangle shape with \( a \) columns and \( b \) rows. Consequently, the first \( a \) numbers in row \( i \), \( 1 \leq i \leq b-1 \), of \( T \) are \((i-1)a+1, \ldots, ia\), and row \( b \) begins with \((b-1)a+1, (b-1)a+2, \ldots, c\).

Assuming \( 1 \notin D(T) \) and \( T \notin \mathcal{E}(\lambda) \), we know \( a, b \geq 2 \) and \( c \geq 3 \). If \( c+1 \) is in \( T \), then it must be either in position \((1, a+1)\) or \((b+1, 1)\). If \( c = ab \), then \( c < |\lambda| \) since \( T \notin \mathcal{E}(\lambda) \), otherwise \( c = |\lambda| \) is possible. For example, the tableaux

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 16 \\
6 & 7 & 8 & 9 & 10 & 17 \\
\end{array},
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
\end{array},
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 11 \\
6 & 7 & 8 & 9 & 10 & 12 \\
\end{array}
\]

have \((a, b, c)\) equal to \((5, 3, 15)\), \((5, 3, 13)\), \((4, 2, 5)\), and \((2, 2, 3)\), respectively.

**Definition 4.13.** Using Notation 4.12, we identify the block rules with further required assumptions as follows. See Figure 8 for summary diagrams.

- **Rule B1:** Assume \( c = ab \), \( T_{(1,a+1)} = c+1 \), \( T_{(2,a+1)} = c+2 \), and \( a < c-2 \).
  In this case, we perform the rotations \((2, \ldots, a+1)\) and \((c, c+1)\) which are sufficiently separated by hypothesis. Then, \( 1, a+1 \) and \( c \) become descents, and \( a \) and \( c+1 \) are no longer descents, so the major index is increased by 1. The B1 rule is illustrated here with \( a = 5, b = 3 \):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 16 & \text{Boxed numbers} \\
6 & 7 & 8 & 9 & 10 & 17 & \\
\end{array}
\begin{array}{cccccccc}
1 & 3 & 4 & 5 & 6 & 15 & \text{Boxed numbers} \\
2 & 7 & 8 & 9 & 10 & 17 & \\
\end{array}
\]

The boxed numbers represent descents of the tableau on the left/right that are not descents of the tableau on the right/left. The elements not shown (i.e., \(18, 19, \ldots, |\lambda|\)) can be in any position.

- **Rule B2:** Assume \( c < ab \) and there exists a \( 1 \leq k < a \) such that \( T_{(b,k)} = c \) and \( T_{(b,k+1)} = c+1 \). In this case, we perform the rotation \((2, 3, \ldots, a, 2a, 3a, \ldots, a(b-1), c, c-1, \ldots, c-k+1 = a(b-1)+1, a(b-2)+1, \ldots, 2a+1, a+1)\) around the perimeter of \( T|_c \). Now 1 becomes a descent, and the other descents stay the same so the major index again increases by 1. The B2 rule is illustrated by the following (here \( a = 5, b = 2 \) and \( k = 3 \)):
The crossed out number 14 means that 14 is not in position (3, 4): it can either be in positions (1, 6) or (4, 1), or it can be that $\lambda = 553$. Again, the numbers 15, $\ldots$, $|\lambda|$ can be anywhere in $T$.

- **Rule B3**: Assume $a \geq 3$, $c = a + 1$, and there exists $k \geq 2$ such that $T_{(2,2)} = a + k + 1$, $T_{(3,2)} = a + k + 2$, and for all $i \in \{1, 2, \ldots, k\}$ we have $T_{(i+1,1)} = a + i$. Then we apply the rotation $(2, 3, \ldots, a, a + k + 1, a + 1)$. Now 1 becomes a descent, and the rest of the descent set remains unchanged, so the major index again increases by 1. The B3 rule is illustrated by the following (here $a = 4$, $k = 4$):

```
1 2 3 4 10
5 9
B3: 6 10
7
8
```

- **Rule B4**: Assume that $a = 2$, $c = 3$, and there exists $k \geq 2$ such that \{3, 4, $\ldots$, $k + 1$\} appear in column 1 of $T$, \{2 + k + 3, $\ldots$, 2k\} appear in column 2 in $T$. Further assume that the set \{2k + 1, 2k + 2, $\ldots$, 3k\} appears in column 3, \{3k + 1, 3k + 2, $\ldots$, 4k\} appears in column 4, etc., until column $l$ for some $l > 2$ and $T_{(k+1,1)} = kl + 1$ and $T_{(k+1,2)} = kl + 2$. In this case, we can perform the two rotations $(k + 1, k, \ldots, 3, 2)$ and $(k(l-1) + 1, k(l-1) + 2, \ldots, kl, kl+1)$. Now 1, $k + 1$ and $k(l-1) + 1$ enter the descent set, and $k$ and $k(l-1) + 1$ leave it, so the major index increases by 1. The B4 rule is illustrated by the following (here $k = 3$ and $l = 4$):

```
1 2 7 10
3 5 8 11
4 6 9 12
B4: 1 4 7 11
2 5 8 12
3 6 9 13
```

- **Rule B5**: Assume that $a = 2$, $c = 3$, and there exists $k > 3$ such that \{3, 4, $\ldots$, $k$\} appear in column 1 of $T$, \{k + 1, k + 2, $\ldots$, 2k - 2\} appear in column 2 in $T$. Furthermore, assume $T_{(k,1)} = 2k - 1$ and $T_{(k,2)} = 2k$. Then apply the cycle $(k, k - 1, \ldots, 3, 2, k + 1, k + 2, \ldots, 2k - 1)$ to $T$. Now 1 becomes a descent, and the rest of the descent set remains unchanged, so the major index increases by 1. The B5 rule is illustrated by the
following (here $k = 5$):

\[
\begin{array}{ccc}
1 & 2 & 1 \\
3 & 6 & 2 \\
\end{array}
\]

\begin{array}{ccc}
B5: & 4 & 7 \\
& 3 & 8 \\
& 5 & 9 \\
& 5 & 8 \\
\end{array}

Lemma 4.14. If $T \in \text{SYT}(\lambda)$, $T \not\in \mathcal{E}(\lambda)$, and $1, 2 \not\in D(T)$, then either some rotation rule applies to $T$ or a B1, B2 or B3 rule applies.

Proof. Let $c$ be the largest possible value such that $T|_{[c]}$ is contained in the min-maj tableau of a rectangle shape with $a$ columns and $b$ rows, as in Notation 4.12. Since $1, 2 \not\in D(T)$ and $T \not\in \mathcal{E}(\lambda)$, we know $1, 2, 3$ are in the first row of $T$ so $a \geq 3$, $b \geq 2$, and $a + 2 \leq |\lambda|$. By construction, we have $T_{(2,1)} = a + 1$ and $a + 2$ must appear in position $(1, a + 1)$, $(2, 2)$, or $(3, 1)$ in $T$.

Case 1: $T_{(1,a+1)} = a + 2$. Observe that

\[
T|_{[a+2]} = \begin{array}{ccc}
1 & 2 & 3 \\
& a + 2 \\
\end{array}
\]

and $z \geq a + 2$. Consequently, $T|_{[z]}$ cannot be of the form forbidden by Lemma 4.8, so a negative rotation rule applies.

Case 2: $T_{(2,2)} = a + 2$. First suppose $c = ab$, then $T_{(1,a+1)} = c + 1$ by choice of $c$. Now consider the two subcases, $T_{(2,a+1)} = c + 2$ and $T_{(2,a+1)} = c + 2$. In the former case, as in Figure 8a, the B1 rule applies to $T$. In the latter case, one may check that an $[i, c + 1]$-positive rotation rule applies to $T$ where $i = T_{(b,1)}$. On the other hand, if $c < ab$, then a B2 rule applies to $T$ as in Figure 8b.

Case 3: $T_{(3,1)} = a + 2$. Let $k = \min\{j \geq 2 \mid a + j \not\in D(T)\}$ so $T_{(k+1,1)} = a + k$ and $T_{(k+2,1)} = a + k + 1$. Since $T \not\in \mathcal{E}(\lambda)$, we know $a + k + 1$ exists in $T$ either in position $(1, a + 1)$ or $(2, 2)$, so $T|_{[a+k+1]}$ looks like

\[
\begin{array}{ccc}
1 & 2 & 3 \\
& a + 1 \\
& a + 2 \\
& \vdots \\
& a + k \\
\end{array}
\]

\[
\begin{array}{ccc}
a + k + 1 & a + k + 1 \\
& a + 2 \\
& \vdots \\
& a + k \\
\end{array}
\]

If $T_{(1,a+1)} = a + k + 1$, then Lemma 4.8 shows that a negative rotation rule applies to $T$. On the other hand, if $T_{(2,2)} = a + k + 1$, then observe that either a B3 move applies or the rotation $(a + k, a + k + 1)$ applies to $T$, depending on whether $T_{(3,2)} = a + k + 2$ or not.

Lemma 4.15. If $T \in \text{SYT}(\lambda)$, $T \not\in \mathcal{E}(\lambda)$, $1 \not\in D(T)$, and $2 \in D(T)$, then either some rotation rule applies to $T$ or a B1, B2, B4 or B5 rule applies.
Proof. Let \( k = \min\{ j \geq 3 \mid j \not\in D(T) \} \) so the consecutive sequence \([3, k]\) appears in the first column of \( T \) and \( k + 1 \) does not. By definition of \( k \) and the fact that \( T \not\in \mathcal{E}(\lambda) \), \( T \) must have \( k + 1 \) in position (1,3) or (2,2). If \( T_{(1,3)} = k + 1 \), then a negative rotation rule holds by Lemma 4.8.
Assume \( T_{(2,2)} = k+1 \). Let \( \ell \) be the maximum value such that \([k+1, \ell]\) appears as a consecutive sequence in column 2 of \( T \). If \( \ell < 2(k-1) \), then the negative rotation rule for \((\ell, \ell-1, \ldots, k)\) applies to \( T \) by the first case of Lemma 4.10.

If \( \ell = 2(k-1) \) and \( T_{(1,3)} = \ell + 1 \), let \( m \) be the maximum value such that \([\ell + 1, m]\) appears as a consecutive sequence in column 3 of \( T \). We subdivide on cases for \( m \) again. If \( m < 3(k-1) \), then the negative rotation rule \((m, m-1, \ldots, \ell)\) applies to \( T \) by the second case of Lemma 4.10. If \( m = 3(k-1) \), we consider the maximal sequence of columns containing a consecutive sequence

---

Figure 8. Summary diagrams for block rules.
in rows \([1, k - 1]\) to the right of column 2 until one of two conditions hold

\[
\begin{array}{cccc}
1 & 2 & \ell + 1 & \cdots \\
3 & k + 1 & & \\
& \vdots & \ddots & \\
k & \ell & m & \\
\end{array}
\]

In the first picture, \(T[p]\) is not a rectangle, so we may apply a negative rotation by the second case of Lemma 4.10, so consider the second picture. In the second picture, \(T[p]\) is a rectangle and we know \(p + 1\) exists in \(T\) since \(T[p]\) is an exceptional tableau for a rectangle shape. If \(p + 2\) is in row \(k\), column 2, a \((p, p + 1)\) rotation rule applies. If \(p + 2\) is not in row \(k\), column 2, then a B4-move applies.

Finally, consider the case \(\ell = 2(k - 1)\) and \(T(k, 1) = \ell + 1\). If \(T(k, 2) \neq \ell + 2\) and \(k > 3\), then a B5-move applies. If \(T(k, 2) = \ell + 2\) and \(k > 3\), then the rotation \((\ell, \ell + 1)\) applies to \(T\) since \(\ell - 1\) is above \(\ell\). If \(T(k, 2) = \ell + 2\) and \(k = 3\), then \(\ell = 4 = T(2, 2)\) and \(T(3, 1) = 5\) so \(T\) contains

\[
\begin{array}{cccc}
1 & 2 \\
3 & 4 \\
5 \\
\end{array}
\]

In this case, consider the subcases \(c = ab\) or \(c < ab\) with \(a = 2\). If \(c = ab\), then \(T(1, 3) = c + 1\) since \(T \notin E(\lambda)\). Either a B1-move applies if \(T(2, 3) = c + 2\) and a \((c, c + 1)\) rotation applies otherwise. On the other hand, if \(c < ab\) then a B2-rule applies.

We may now prove Theorem 1.1 from the introduction.

**Proof of Theorem 1.1.** Given any \(T \in \operatorname{SYT}(\lambda) \setminus E(\lambda)\), we define \(\varphi(T)\) with the property \(\operatorname{maj}(\varphi(T)) = \operatorname{maj}(T) + 1\). If \(1 \in D(T)\), define \(\varphi(T) = (z, z - 1, \ldots, i)T\) as in Corollary 4.9. If \(1, 2 \notin D(T)\), then Lemma 4.14 applies, so define \(\varphi(T)\) using the specific B1, B2, B3 or rotation rule identified in the proof of that lemma. If \(1 \notin D(T)\) and \(2 \in D(T)\), then Lemma 4.15 applies, so define \(\varphi(T)\) using the specific B1, B2, B4, B5, or negative rotation rule identified in the proof of that lemma. These rules cover all possible cases.

We may define two poset structures on standard tableaux of a given shape using the preceding combinatorial operations. We call them “strong” and “weak” in analogy with the strong and weak Bruhat orders on permutations. Recall an inverse-transpose block rule is a block rule obtained from transposing the diagrams in Figure 8 and reversing the arrows.
Definition 4.16. As sets, let \( P(\lambda) \) and \( Q(\lambda) \) be either

\[ SYT(\lambda) \setminus \{ \minmaj(\lambda), \maxmaj(\lambda) \} \]

if \( \lambda \) is a rectangle with at least two rows and columns, or \( SYT(\lambda) \) otherwise.

- (Strong SYT Poset) Let \( P(\lambda) \) be the partial order with covering relations given by rotations, block rules, and inverse-transpose block rules increasing maj by 1.
- (Weak SYT Poset) Let \( Q(\lambda) \) be the partial order with covering relations given by \( S < T \) if \( \varphi(S) = T \) or \( \varphi(T') = S' \) where \( S', T' \) are the transpose of \( S, T \), respectively.

Corollary 4.17. As posets, \( P(\lambda) \) and \( Q(\lambda) \) are ranked with a unique minimal and maximal element. If \( \lambda \) is not a rectangle, the rank function is given by \( \text{rk}(T) = \text{maj}(T) - b(\lambda) \). If \( \lambda \) is a rectangle with at least two rows and columns, then the rank function is given by \( \text{rk}(T) = \text{maj}(T) - b(\lambda) - 2 \).

Proof. By Corollary 4.2, \( P(\lambda) \) and \( Q(\lambda) \) have a single element of minimal maj and of maximal maj. Any element \( T \) besides these is covered by \( \varphi(T) \) and covers \( \varphi(T') \), so is not maximal or minimal. By construction maj increases by 1 under covering relations. The result follows. \( \square \)

In Figure 9, we show an example of both the Weak SYT Poset and the Strong SYT poset for \( \lambda = (3, 2, 1) \). More examples of these partial orders are given at https://sites.math.washington.edu/~billey/papers/syt.posets.

Remark 4.18. Observe that both the positive and negative rotation rules apply equally well to any skew shape tableaux in \( SYT(\lambda/\nu) \). The block rules apply to skew shape tableaux as well when \( T_z \) is a straight shape tableau. However, in order to define the analogous posets on \( SYT(\lambda/\mu) \), one must include additional block moves. This is part of an ongoing project.

5. Internal zeros for des on SYT(\( \lambda \))

The results of Section 4 show that \( SYT(\lambda)^{\text{maj}}(q) \) almost never has internal zeros. Adin–Elizalde–Roichman analogously considered the internal zeros of the descent number generating functions \( SYT(\lambda/\nu)^{\text{des}}(q) \) where \( \text{des}(T) = \# \text{Des}(T) \) is the number of descents in a tableau \( T \).

Question 5.1. [AER18, Problem 7.5] Is \( \{ \text{des}(T) : T \in SYT(\lambda/\nu) \} \) an interval consisting of consecutive integers, for any skew shape \( \lambda/\nu \)? That is, does \( SYT(\lambda/\nu)^{\text{des}}(q) \) ever have internal zeros?

The minimum and maximum descent numbers are easily described as follows. The argument involves constructions similar to the sequences of vertical and horizontal strips used in Definition 4.1.
Figure 9. Hasse diagram of the Weak SYT Poset and the Strong SYT Poset of \( \lambda = (3, 2, 1) \). Each tableau is represented by its row reading word in these pictures.

Lemma 5.2. [AER18, Lemma 3.7(1)] Let \( \lambda/\nu \) be a skew shape with \( n \) cells. Let \( c \) be the maximum length of a column and \( r \) be the maximum length of a row. Then

\[
\begin{align*}
\min \text{des}(\text{SYT}(\lambda/\nu)) &= c - 1 \\
\max \text{des}(\text{SYT}(\lambda/\nu)) &= (n - 1) - (r - 1) = n - r.
\end{align*}
\]
Indeed, it is easy to see that minmaj(\(\lambda\)) constructed as in Definition 4.1 has \(\lambda'_1 - 1 = c - 1\) descents, and symmetrically that maxmaj(\(\lambda\)) has \(n - r\) descents. The arguments in Section 4 consequently resolve Question 5.1 affirmatively in the straight-shape case.

**Corollary 5.3.** For \(\lambda \vdash n\), we have
\[
\{\text{des}(T) : T \in \text{SYT}(\lambda)\} = \{\lambda'_1 - 1, \lambda'_1, \ldots, n - \lambda_1 - 1, n - \lambda_1\}.
\]

In particular, \(\text{SYT}(\lambda)_{\text{des}}(q)\) has no internal zeros.

**Proof.** First suppose \(\lambda\) is not a rectangle with at least two rows and columns. Iterating the \(\varphi\) map creates a chain from minmaj(\(\lambda\)) to maxmaj(\(\lambda\)). At each step, \(\varphi\) either applies a rotation rule or a block rule. Rotation rules preserve descent number. Block rules always increase the descent number by exactly 1. Since minmaj(\(\lambda\)) and maxmaj(\(\lambda\)) have the minimum and maximum number of descents possible, the result follows.

If \(\lambda\) is a rectangle with at least two rows and columns, it is easy to see that the unique tableau of major index \(b(\lambda) + 2\) has exactly one more descent than minmaj(\(\lambda\)). The result follows as before by iterating the \(\varphi\) map. \(\square\)

**Remark 5.4.** The same argument shows that \(\text{SYT}(\lambda)_{\text{maj}} - \text{des}(q)\) also has no internal zeros. Indeed, applying a rotation rule increases increases maj – des by 1 while fixing des, and applying a B-rule fixes maj – des and increases des by 1. In this sense, the strong or weak posets \(P(\lambda)\) and \(Q(\lambda)\) have a \(\mathbb{Z} \times \mathbb{Z}\) ranking given by (maj – des, des).

### 6. Internal zeros for fake degrees of \(C_m \wr S_n\)

In this section, we classify which irreducible representations appear in which degrees of the corresponding coinvariant algebras for all finite groups of the form \(C_m \wr S_n\). The goal is to classify when the fake degrees \(b_{\lambda,k} \neq 0\). We will use the following helpful lemma which is straightforward to prove.

**Lemma 6.1.** Suppose that \(f\) and \(g\) are polynomials in \(\mathbb{Z}[q]\) with non-negative coefficients, that \(f\) has no internal zeros and has at least two non-zero coefficients, and that \(g\) has no adjacent internal zeros. Then, \(fg\) has no internal zeros.

**Lemma 6.2.** Let \(\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})\) be a sequence of partitions. The polynomial \(\text{SYT}(\underline{\lambda})_{\text{maj}}(q)\) has no internal zeros except when \(\underline{\lambda}\) has a single non-empty block \(\lambda^{(i)}\) which is a rectangle with at least two rows and columns. In this latter case, the only internal zero up to symmetry occurs at \(k = b(\lambda^{(i)}) + 1\).

**Proof.** If \(\underline{\lambda}\) has only one nonempty partition, then the characterization of internal zeros follows from Theorem 1.1, so assume \(\underline{\lambda}\) has two or more nonempty
partitions. From Theorem 2.15, we have

\[ \text{SYT}(\Lambda)_{\text{maj}}(q) = \left( \left| \lambda^{(1)} \right|, \ldots, \left| \lambda^{(m)} \right| \right) \prod_{q=1}^{m} \text{SYT}(\lambda^{(i)})_{\text{maj}}(q). \]

By MacMahon’s Theorem 2.5, we observe that the \( q \)-multinomial coefficients have no internal zeros. Furthermore, the \( q \)-multinomial in (15) is not constant whenever \( \Lambda \) has two or more non-empty partitions. From Theorem 1.1, we know \( \text{SYT}(\lambda^{(i)})_{\text{maj}}(q) \) has no adjacent internal zeros for any \( 1 \leq i \leq m \). Consequently, by Lemma 6.1, the overall product in (15) has no internal zeros. \( \Box \)

**Theorem 6.3.** Let \( \Lambda \) be a sequence of \( m \) partitions with \( |\Lambda| = n \), and assume \( g^\Lambda(q) = \sum_k b_{\Lambda,k} q^k \). Then for \( k \in \mathbb{Z} \), \( b_{\Lambda,k} \neq 0 \) if and only if

\[ \frac{k - b(\alpha(\Lambda))}{m} - b(\Lambda) \in \left\{ 0, 1, \ldots, \left( \frac{n+1}{2} \right) - \sum_{c\in\Lambda} h_c \right\} \setminus D_\Lambda, \]

where \( D_\Lambda \) is empty unless \( \Lambda \) has a single non-empty partition \( \lambda^{(i)} \) which is a rectangle with at least two rows and columns, in which case

\[ D_\Lambda = \left\{ 1, \left( \frac{n+1}{2} \right) - \sum_{c\in\lambda^{(i)}} h_c - 1 \right\}. \]

**Proof.** By Theorem 2.31,

\[ g^\Lambda(q) = q^{b(\alpha(\Lambda))} \text{SYT}(\Lambda)_{\text{maj}}(q^m) \]

which implies \( b_{\Lambda,k} \neq 0 \) only if \( k - b(\alpha(\Lambda)) \) is a multiple of \( m \). By Lemma 6.2, we know \( \text{SYT}(\Lambda)_{\text{maj}}(q) \) has either no internal zeros or internal zeros at degree \( 1 + b(\Lambda) \) and degree one less than the maximal major index for \( \Lambda \) in the case of a rectangle with at least 2 rows and columns. By (8) and (9), the minimal major index for \( \Lambda \) is \( b(\Lambda) := \sum_i b(\lambda^{(i)}) \), and the maximal major index is \( b(\Lambda) + \left( \frac{n+1}{2} \right) - \sum_{c\in\Lambda} h_c \). Hence, the result follows. \( \Box \)

**Corollary 6.4.** In type \( B_n \), the irreducible representation indexed by \((\lambda, \mu)\) with \(|\lambda| + |\mu| = n \) appears in degree \( k \) of the coinvariant algebra of \( G(2, 1, n) \) if and only if

\[ \frac{k - |\mu|}{2} - b(\lambda) - b(\mu) \in \left\{ 0, 1, \ldots, \left( \frac{n+1}{2} \right) - \sum_{c\in\lambda} h_c - \sum_{c'\in\mu} h_{c'} \right\} \setminus D_{(\lambda, \mu)}. \]

**Example 6.5.** Consider the type \( B_6 \) case, where \( m = 2, d = 1, n = 6 \). For \( \Lambda = ((2), (31)) \), we have \( b(\Lambda) = 1 \) and

\[ g^\Lambda(q) = q^{26} + 2q^{24} + 4q^{22} + 5q^{20} + 7q^{18} + 7q^{16} + 7q^{14} + 5q^{12} + 4q^{10} + 2q^8 + q^6. \]

For \( \mu = (\varnothing, (33)) \), we have \( b(\mu) = 3 \) and

\[ g^\mu(q) = q^{24} + q^{20} + q^{18} + q^{16} + q^{12}. \]
In both cases, the nonzero coefficients are determined by Corollary 6.4.

7. Deformed Gaussian Multinomial Coefficients

We now turn our attention to extending Theorem 6.3 to general Shephard–Todd groups \( G(m,d,n) \). We begin by introducing a deformation of the \( q \)-multinomial coefficients arising from Theorem 2.35 in the special case when \( A = ((\alpha_1), (\alpha_2), \ldots, (\alpha_m)) \) is a sequence of one row partitions. After several lemmas, we give an alternative formulation for these deformed \( q \)-multinomials in terms of inversion generating functions on words with a bounded first letter.

**Definition 7.1.** Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \vdash n \) be a weak composition of \( n \) with \( m \) parts. Recall the long cycle \( \sigma_m = (1, 2, \ldots, m) \in S_m \), so
\[
\sigma_m \cdot \alpha = (\alpha_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}).
\]

Let \( d \mid m \), \( \tau = \sigma_m^{m/d} \), and \( C_d = \langle \tau \rangle = \langle \sigma_m^{m/d} \rangle \) so \( C_d \) acts on length \( m \) compositions by \( (m/d) \)-fold cyclic rotations as in Definition 2.37. Set
\[
\binom{n}{\alpha}_{q,d} = \sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)} \binom{n}{\alpha} q^m
\]
where
\[b(\alpha) = \sum_{i=1}^{m} (i-1)\alpha_i.
\]

Note that when \( q = 1 \), we have \( \binom{n}{\alpha}_{1,d} = \binom{n}{\alpha} \), and when \( d = 1 \), we have \( \binom{n}{\alpha}_{q,1} = q^{b(\alpha)} \binom{n}{\alpha} q^m \), where \( m \) is the number of parts of \( \alpha \). As usual, we also write
\[
\binom{n}{k}_{q,d} = \binom{n}{k, n-k}_{q,d} = \binom{n}{n-k, k}_{q,d},
\]
where \( m = 2 \) in this case. Note that \( \binom{n}{\alpha}_{q,d} \) is invariant under the \( C_d \)-action on \( \alpha \), though this is not typically true of general permutations of \( \alpha \).

**Example 7.2.** Observe that \( \binom{n}{\alpha}_{q,m} \) alone is generally not divisible by \( [d]_{q^{nm/d}} \).

For example, if \( n = 5 \), \( \alpha = (2, 1, 1, 1) \), and \( d = 2 \), we have
\[
\binom{5}{2,1,1,1}_{q^4} = q^{36} + 3q^{32} + 6q^{28} + 9q^{24} + 11q^{20} + 11q^{16} + 9q^{12} + 6q^8 + 3q^4 + 1
\]
which is not divisible by \( [2]_{q^{5/2}} = q^{10} + 1 \). However, \( \sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)} = q^8 + q^6 \) and \( (q^8 + q^6) \binom{5}{2,1,1,1}_{q^4} \) is divisible by \( q^{10} + 1 \) giving
\[
\binom{5}{2,1,1,1}_{q^2} = q^{34} + q^{32} + 3q^{30} + 3q^{28} + 6q^{26} + 5q^{24} + 8q^{22} + 6q^{20} + 8q^{18} + 5q^{16} + 6q^{14} + 3q^{12} + 3q^{10} + q^8 + q^6.
\]

See Figure 10 for a larger example.
Figure 10. A plot of the coefficients for the deformed $q$-multinomial $\left[\begin{array}{c} n \\ \alpha \end{array} \right]_{qd}$ with $\alpha = (2, 1, 3, 1, 4, 5)$ and $d = 3$.

**Lemma 7.3.** Given $\alpha = (\alpha_1, \ldots, \alpha_m) \models n$, we have

$$b(\sigma_m \cdot \alpha) - b(\alpha) = n - m\alpha_m,$$

and

$$b(\tau \cdot \alpha) - b(\alpha) = nm/d - m(\alpha_m + \alpha_{m-1} + \cdots + \alpha_{m-1/d+1}).$$

**Proof.** The second claim follows by iterating the first for $\tau = \sigma_{m/d}$. For the first, we have

$$b(\sigma_m \cdot \alpha) - b(\alpha) = (\alpha_1 + 2\alpha_2 + \cdots + (m-1)\alpha_{m-1})$$

$$- (\alpha_2 + 2\alpha_3 + \cdots + (m-1)\alpha_m)$$

$$= \alpha_1 + \alpha_2 + \cdots + \alpha_{m-1} - (m-1)\alpha_m,$$

which simplifies to $n - m\alpha_m$. \qed

If $\alpha \models n$, let $\downarrow_i \alpha$ be the vector obtained from $\alpha$ by decreasing $\alpha_i$ by 1. Extend the definition of $\left[\begin{array}{c} n \\ \alpha \end{array} \right]_q$ to $m$-tuples of integers by declaring $\left[\begin{array}{c} n \\ \alpha \end{array} \right]_q = 0$ if any $\alpha_i$ is negative. The following lemma is well known but we include a proof for completeness.

**Lemma 7.4.** We have the following recurrence for $q$-multinomial coefficients,

$$\left( \begin{array}{c} n \\ \alpha_1, \ldots, \alpha_m \end{array} \right)_q = \sum_{i=1}^{m} q^{\alpha_1 + \cdots + \alpha_{i-1}} \left( \begin{array}{c} n-1 \\ \downarrow_i \alpha \end{array} \right)_q.$$

**Proof.** By MacMahon's Theorem, the left-hand side is the inversion number generating function on length $n$ words with $\alpha_i$ copies of the letter $i$ for each $i$. If the first letter in such a word is $i$, the number of inversions involving the first letter is $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}$, from which the result quickly follows. \qed

The non-trivial deformation of the $q$-binomial coefficients in Definition 7.1 has the following more explicit form. In particular, these rational functions are
polynomials with non-negative integer coefficients that satisfy a Pascal-type formula.

**Lemma 7.5.** In the case \( d = m = 2 \), we have

\[
\binom{n}{k}_{q,2} = \frac{q^k + q^{n-k}}{1 + q^n} \binom{n}{k} = q^{n-k} \binom{n-1}{k-1}_{q,2} + q^k \binom{n-1}{k}_{q,2} \in \mathbb{Z}_{20}[q].
\]

**Proof.** The first equality is immediate from Definition 7.1. For the second, we use the well-known “\( q \)-Pascal” identities

\[
\binom{n}{k} = q^k \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n-1}{k} + q^{n-k} \binom{n-1}{k-1},
\]

which arise from Lemma 7.4. Thus,

\[
q^k \binom{n}{k}_{q,2} = q^k \binom{n-1}{k}_{q,2} + q^{n-k} \binom{n-1}{k-1}_{q,2}
\]

and

\[
q^{n-k} \binom{n}{k}_{q,2} = q^{n-k} \binom{n-1}{k}_{q,2} + q^{n-k} \binom{n-1}{k-1}_{q,2}.
\]

Hence,

\[
(q^k + q^{n-k}) \binom{n}{k}_{q,2} = (1 + q^n) \left( q^k \binom{n-1}{k}_{q,2} + q^{n-k} \binom{n-1}{k-1}_{q,2} \right)
\]

so the second equality in (17) holds. \( \square \)

We next generalize Lemma 7.5 to all \( \binom{n}{\alpha}_{q,d} \) for any \( \alpha \vdash n \). The proof that follows is independent of Theorem 2.35, which can also be used to prove they are polynomials with non-negative integer coefficients.

**Theorem 7.6.** Let \( \alpha \) be a weak composition of \( n \) with \( m \) parts, and let \( d \mid m \). Then

\[
\binom{n}{\alpha}_{q,d} = \sum_{\sigma \in \mathcal{C}_d} q^{b(\sigma \cdot \alpha)} \frac{m}{d} \binom{m-((\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{v-1})}{v} \binom{n-1}{v \cdot (\sigma \cdot \alpha)}_{q,m}.
\]

In particular, \( \binom{n}{\alpha}_{q,d} \) is a polynomial with non-negative coefficients.

**Proof.** Observe from the definition that \( \binom{n}{\alpha}_{q} = \binom{n}{\sigma \cdot \alpha}_{q} \) for any \( \sigma \in C_d \). Thus, by Lemma 7.4, we can rewrite the numerator of \( \binom{n}{\alpha}_{q,d} \) as

\[
\sum_{\sigma \in \mathcal{C}_d} q^{b(\sigma \cdot \alpha)} \binom{n}{\alpha}_{q,m} = \sum_{j=1}^{d} q^{b(\tau_j \cdot \alpha)} \binom{n}{\tau_j \cdot \alpha}_{q,m} = \sum_{j=1}^{d} \sum_{i=1}^{m} q^{c(i,j,\alpha)} \binom{n-1}{\tau_j \cdot \alpha}_{q,m}
\]
where

\[ \epsilon(i, j, \alpha) := b(\tau^j \cdot \alpha) + \tau^j \cdot \alpha. \]

It is straightforward to check that \( \downarrow_i (\sigma_m \cdot \alpha) = \downarrow_{i-1} \alpha \), so that \( \downarrow_i (\tau^j \cdot \alpha) = \tau^j \cdot \downarrow_{i-m/d} \alpha \), where indices are taken modulo \( m \). Thus,

\[ \sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)} \binom{n}{\alpha} q^m = \sum_{i=1}^m \sum_{j=1}^d q^{\epsilon(i, j, \alpha)} \binom{n-1}{\downarrow_{i-m/d} \alpha} q^m. \]

Group the terms on the right according to the value \( i - jm/d \equiv t \in [m] \). Note that \( j \in [d] \) could be equivalently represented as \( j \in \mathbb{Z}/d \), though \( i \in [m] \) cannot be treated similarly here. One may check that the set of \( (i, j) \in [m] \times \mathbb{Z}/d \) such that \( i - jm/d \equiv t \) can be described as

\[ \{(t + sm/d, s) : s \in [-p_t, d - 1 - p_t]\} \]

where \( t = p_t m/d + v_t \) for some unique \( p_t \in [0, d - 1] \) and \( v_t \in [m/d] \). Consequently,

\[ \sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)} \binom{n}{\alpha} q^m = \sum_{t=1}^{m-d-1} \left( \sum_{s=-p_t}^{d-p_t} q^{\epsilon(t + sm/d, s, \alpha)} \right) \binom{n-1}{\downarrow_t \alpha} q^m. \]

Next, we evaluate the incremental change

\[ \epsilon(t + (s + 1)m/d, s + 1, \alpha) - \epsilon(t + sm/d, s, \alpha) \]

for a given \( s \). Let \( \beta = \tau^s \cdot \alpha \). By Lemma 7.3,

\[ b(\tau^{s+1} \cdot \alpha) - b(\tau^s \cdot \alpha) = b(\tau \cdot \beta) - b(\beta) = nm/d - m \cdot (\beta_m + \cdots + \beta_{m-m/d+1}). \]

We also find

\[ (\tau \cdot \beta)_1 + \cdots + (\tau \cdot \beta)_{t+(s+1)m/d-1} = \beta_{m-m/d+1} + \cdots + \beta_m + \beta_1 + \cdots + \beta_{t+sm/d-1} \]

so

\[ (\tau \cdot \beta)_1 + \cdots + (\tau \cdot \beta)_{t+(s+1)m/d-1} - \beta_1 - \cdots - \beta_{t+sm/d-1} = \beta_{m-m/d+1} + \cdots + \beta_m. \]

Combining these observations,

\[ \epsilon(t + (s + 1)m/d, s + 1, \alpha) - \epsilon(t + sm/d, s, \alpha) = nm/d. \]

It follows that

\[ \sum_{s=-p_t}^{d-p_t} q^{\epsilon(t + sm/d, s, \alpha)} = q^{\epsilon(t-p_t m/d, -p_t, \alpha)}[d]_{q^n m/d} = q^{\epsilon(v_t, -p_t, \alpha)}[d]_{q^n m/d}. \]
Since we have a bijection \([0, d - 1] \times [m/d] \rightarrow [m]\) given by \((p, v) \mapsto pm/d + v\), we have
\[
(19) \quad \sum_{\sigma \in C_d} q^{\ell(\sigma \cdot \alpha)} \binom{n}{\alpha}_{q^m} = [d]_{q^{nm/d}} \sum_{p=0}^{d-1} \sum_{v=1}^{m/d} q^{\ell(v, p, \alpha)} \binom{n-1}{\downarrow \nu + pm/d \alpha}_{q^m},
\]
proving the polynomiality of \([n]_{\alpha}^{m/d}\).

We can further refine (19). From (18), we observe that
\[
\ell(v, -p, \alpha) = \ell(v, 0, \tau^{-p} \cdot \alpha),
\]
and since \(\tau = \sigma_m^{m/d}\), we have
\[
\downarrow v + pm/q \alpha = \tau^{p} \cdot \downarrow v (\tau^{-p} \cdot \alpha).
\]
So,
\[
q^{\ell(v, -p, \alpha)} \binom{n-1}{\downarrow v + pm/d \alpha}_{q^m} = q^{\ell(v, 0, \tau^{-p} \cdot \alpha)} \binom{n-1}{\downarrow v (\tau^{-p} \cdot \alpha)}_{q^m},
\]
which implies
\[
\sum_{\sigma \in C_d} q^{\ell(\sigma \cdot \alpha)} \binom{n}{\alpha}_{q^m} = [d]_{q^{nm/d}} \sum_{\sigma \in C_d} q^{\ell(\sigma \cdot \alpha)} \binom{m/d}{\nu + \sigma \cdot \alpha_{1} \cdots \sigma \cdot \alpha_{v-1}} \binom{n-1}{\downarrow v (\sigma \cdot \alpha)}_{q^m}.
\]
The result follows by dividing by \([d]_{q^{nm/d}}\). \(\square\)

In light of Theorem 7.6, we define the following polynomials.

**Definition 7.7.** Let \(\alpha = (\alpha_1, \ldots, \alpha_m) \vdash n\), and say \(1 \leq k \leq m\). Define the \(\alpha, k\)-partial sum multinomial by
\[
P^{(k)}_{\alpha}(q) = \sum_{i=1}^{k} q^{\alpha_1 + \cdots + \alpha_{i-1}} \binom{n-1}{\downarrow i \cdot \alpha}_{q}.
\]

**Remark 7.8.** By Lemma 7.4, \(p^{(m)}_{\alpha}(n) = \binom{n}{\alpha}_{q}\), and more generally the same argument shows that
\[
P^{(k)}_{\alpha}(q) = \{w \in W_\alpha : w_1 \leq k\}^{\text{inv}}(q)
\]
is an inversion number generating function.

It is very well-known that the multinomial coefficients can be written as a product of binomial coefficients. More generally, \(q\)-multinomial coefficients can be written as a product of \(q\)-binomial coefficients. This holds true even for the \(\alpha, k\)-partial sum multinomials as follows.

**Lemma 7.9.** Let \(\alpha = (\alpha_1, \ldots, \alpha_m) \vdash n\) and \(1 \leq k \leq m\). We have
\[
P^{(k)}_{\alpha}(q) = \prod_{i=1}^{k} \binom{\alpha_1 + \cdots + \alpha_{i}}{\alpha_{i} q} \cdot \prod_{i=k+1}^{m} \binom{\alpha_1 + \cdots + \alpha_{i} - 1}{\alpha_{i} q}.
\]
Proof. Recall that \( p^{(k)}_\alpha(q) = \{ w \in W_\alpha : w_1 \leq k \}^{\text{inv}}(q) \). Partition the set \( \{ w \in W_\alpha : w_1 \leq k \} \) into \( (\alpha_1 + \ldots + \alpha_k - n + 1) \) subsets according to the placement of all \( k + 1, k + 2, \ldots, m \)'s in positions 2, 3, \ldots, \( n \). For each such placement, there are \((\alpha_1 + \ldots + \alpha_k)\) ways to place numbers 1, 2, \ldots, \( k \) in the remaining positions. Since each inversion in a word \( w \in W_\alpha \) is between two letters \( \leq k \); between two letters \( \geq k + 1 \), or between a letter \( \leq k \) and a letter \( \geq k + 1 \), it follows that

\[
\{ w \in W_\alpha : w_1 \leq k \}^{\text{inv}}(q) = \binom{\alpha_1 + \ldots + \alpha_k}{\alpha_1, \ldots, \alpha_k}^{-1}(q) \binom{\alpha_k + \ldots + \alpha_1}{\alpha_k, \ldots, \alpha_1}(q).
\]

by MacMahon’s Theorem 2.5. Factoring each \( q \)-multinomial in (22) into \( q \)-binomials gives (21). □

Corollary 7.10. Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \vdash n \) and \( 1 \leq k \leq m \). The \( \alpha, k \)-partial sum multinomial \( p^{(k)}_\alpha(q) \) is symmetric and unimodal.

Proof. A result of Andrews [And76, Thm. 3.9] states that the product of symmetric, unimodal polynomials with non-negative coefficients is symmetric and unimodal with non-negative coefficients. The \( q \)-binomials are symmetric with non-negative coefficients, and it is a well-known, non-trivial fact that they are also unimodal. See [Zei89] for a combinatorial proof of this fact and further historical references. The result now follows from Lemma 7.9. □

Lemma 7.11. Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \vdash n \) and \( 1 \leq k \leq m \). Then \( p^{(k)}_\alpha(q) \neq 0 \) if and only if \( \alpha_1 + \ldots + \alpha_k > 0 \). In this case, \( p^{(k)}_\alpha(q) \) has constant coefficient \( 1 \), degree \( D_\alpha = \alpha_{k+1} - \ldots - \alpha_m \) where \( D_\alpha = \binom{n}{2} - \binom{\alpha_i}{2} \) is the degree of \( \binom{n}{\alpha}q^\alpha \), and has no internal zeros. Furthermore, \( p^{(k)}_\alpha(q) \) is non-constant except when

- \( \alpha_1 + \ldots + \alpha_k = 0 \), in which case \( p^{(k)}_\alpha(q) = 0 \);
- \( \alpha_1 + \ldots + \alpha_k = 1 \) and \( \alpha_i = n - 1 \) for some \( i > k \), in which case \( p^{(k)}_\alpha(q) = 1 \);
- or
- \( \alpha_i = n \) for some \( i \leq k \), in which case \( p^{(k)}_\alpha(q) = 1 \).

Proof. Each claim follows easily from the fact that \( p^{(k)}_\alpha(q) \) is the inversion generating function for \( \{ w \in W_\alpha : w_1 \leq k \} \). Alternatively, one may use Lemma 7.9. □

We have the following summary statement.

Corollary 7.12. Let \( \alpha \) be weak composition of \( n \) with \( m \) parts, and let \( d \mid m \). Let \( \underline{\alpha} = ((\alpha_1), (\alpha_2), \ldots, (\alpha_m)) \) be the corresponding sequence of one row partitions. Then,

\[
\binom{n}{\underline{\alpha}}_{q^{\text{inv}}} = \binom{n}{\alpha} q^{b(\sigma, \alpha)} \frac{q^m}{d} \prod_{\sigma \in C_d} \binom{\alpha}{q^{m/d}} q^{\sigma} = \sum_{\sigma \in C_d} q^{b(\sigma, \alpha) \sigma(q^m/d)} = \frac{d}{\# \{ \underline{\alpha} \}^d} g^{(\underline{\alpha})^d}(q).
\]
Proof. The first equality is just the definition. The second equality follows from Theorem 7.6. The third equality follows from Theorem 2.35 and the fact that
\[ \sum_{\sigma \in C_d} q^{b(\sigma \cdot \alpha)} \equiv \frac{d}{\#(\alpha)^d} \left( (\alpha)^d \right)^{\text{box}} (q). \]

\[ \square \]

Remark 7.13. We note that since \( [n]_{q,d} = \frac{d}{\#(\alpha)^d} g(\alpha)^d(q) \), we knew from Stembridge’s work that the deformed multinomial coefficients are polynomials in \( q \) even though they are defined as rational functions. Our proof in Theorem 7.6 gives an alternate, direct proof of this fact without going through representation theory. Furthermore, we use the summation formula in Corollary 7.12 to characterize the internal zeros of \( g(\Lambda)^d(q) \) in the next section.

8. Internal zeros for \( G(m, d, n) \)

We now extend the results of Section 6 to all Shephard–Todd groups \( G(m, d, n) \). We thus give a remarkably simple and completely general description for which irreducible representations appear in which degrees of the coinvariant algebras of essentially arbitrary complex reflection groups. Recall the notation established in Section 2.4. Let \( \Lambda \) be the orbit of \( \lambda \) under \((m/d)\)-fold rotations in \( C_d = \langle \sigma_m^{m/d} \rangle \).

Definition 8.1. Given a sequence \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \) with \( |\lambda| = n \), let
\[ \alpha(\Lambda) = (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|) \equiv n. \]

Similarly, let \( \alpha(\Lambda) \) be the length \( m \) sequence of partitions whose \( i \)th partition is the single row partition of size \( |\lambda^{(i)}| \).

The map \( \alpha \) may not be injective on \( \{\lambda\}^d \), though it has constant fiber sizes since \( \alpha \) is \( C_d \)-equivariant. For example, when \( m = 4, d = 4 \), we have
\[
\alpha : ((2), \varnothing, (1^2), \varnothing) \mapsto ((2), \varnothing, (2), \varnothing),
\]
\[
(\varnothing, (2), \varnothing, (1^2)) \mapsto (\varnothing, (2), \varnothing, (2)),
\]
\[
((1^2), \varnothing, (2), \varnothing) \mapsto ((2), \varnothing, (2), \varnothing),
\]
\[
(\varnothing, (1^2), \varnothing, (2)) \mapsto (\varnothing, (2), \varnothing, (2)).
\]

Generalizing Theorem 2.15, we have the following corollary of Stembridge’s Theorem 2.35 and Definition 7.1.

Corollary 8.2. Let \( \Lambda \) be a sequence of \( m \) partitions with \( |\lambda| = n \). Let \( \{\lambda\}^d \) be the orbit of \( \lambda \) under \((m/d)\)-fold cyclic rotations. Then
\[ g(\Lambda)^d(q) = \frac{\#(\lambda)^d}{d} \cdot \left[ \binom{n}{\alpha(\Lambda)} \right]_{q,d} \prod_{i=1}^{m} \text{SYT}(\lambda^{(i)})^{\text{maj}}(q^m). \]
Proof. By Theorem 2.35,

\[ g_{\{\lambda\}}(q) = \frac{\#\{\lambda\}_d^{b_{\alpha}}(q)}{[d]_{q^{m/d}}} \text{SYT}(\lambda)_{\text{maj}}(q^m). \]

We have

\[ (\{\lambda\}_d^{b_{\alpha}}(q) = \frac{\#\{\lambda\}_d}{\#\{\alpha(\lambda)\}_d^d} \cdot (\{\alpha(\lambda)\}_d^{b_{\alpha}}(q) \]

and

\[ (\{\alpha(\lambda)\}_d^d^{b_{\alpha}}(q) = \frac{\#\{\alpha(\lambda)\}_d}{d} \sum_{\sigma \in C_d} q^{b(\sigma(\lambda))}. \]

Consequently, using Theorem 2.15 and Definition 7.1, we have

\[ g_{\{\lambda\}}(q) = \frac{\#\{\lambda\}_d}{d} \cdot \frac{\sum_{\sigma \in C_d} q^{\sigma(\alpha(\lambda))}}{[d]_{q^{m/d}}} \cdot \left( \frac{n}{\alpha(\lambda)} \right) \cdot \prod_{i=1}^{m} \text{SYT}(\lambda(i))_{\text{maj}}(q^m) \]

\[ = \frac{\#\{\lambda\}_d}{d} \cdot \left[ \frac{n}{\alpha(\lambda)} \right] \cdot \prod_{i=1}^{m} \text{SYT}(\lambda(i))_{\text{maj}}(q^m). \]

\[ \square \]

We will now prove the general classification theorem for nonzero fake degrees as mentioned in the introduction. The reader may find it useful to compare the statement to the type A case in Theorem 1.1 and the \( C_m \bowtie S_n \) case in Theorem 6.3.

**Theorem 8.3.** Let \( \lambda \) be a sequence of \( m \) partitions with \( |\lambda| = n \geq 1 \), let \( d \mid m \), and let \( (\lambda)\}_d \) be the orbit of \( \lambda \) under the group \( C_d \) of \( (m/d) \)-fold cyclic rotations. Then \( b_{(\lambda)\}_d} = k \neq 0 \) if and only if for some \( \mu \in \{\lambda\} \) we have \( |\mu| + \cdots + |\mu(m/d)| > 0 \) and

\[ \frac{k - b(\mu)}{m} - b(\mu) \in \left\{ 0, 1, \ldots, |\mu(1)| + \cdots + |\mu(m/d)| + \left( \frac{n}{2} \right) - \sum_{\epsilon \in \mu} h_{\epsilon} \right\} \setminus D_{\mu, d}. \]

Here \( D_{\mu, d} \) is empty unless either

(1) \( \mu \) has a partition \( \mu \) of size \( n \); or

(2) \( \mu \) has a partition \( \mu \) of size \( n - 1 \) and \( |\mu(1)| + \cdots + |\mu(m/d)| = 1 \),

where in both cases \( \mu \) must be a rectangle with at least two rows and columns.

In case (1), we have

\[ D_{\mu, d} := \left\{ 1, \left( \frac{n+1}{2} \right) - \sum_{\epsilon \in \mu} h_{\epsilon} - 1 \right\}, \]

and in case (2) we have

\[ D_{\mu, d} := \left\{ 1, \left( \frac{n}{2} \right) - \sum_{\epsilon \in \mu} h_{\epsilon} \right\}. \]
Proof. Let $\alpha = \alpha(\lambda)$. Using Corollary 8.2 and Corollary 7.12, we have

$$g^{(\lambda)}(q) = \frac{\# \{ \lambda \}}{d} \cdot \sum_{\sigma \in C_d} q^{b(\sigma, \alpha)} p_{\sigma, \alpha}^{(m/d)}(q^m) \cdot \prod_{i=1}^{m} \text{SYT}(\lambda(i))^{\text{maj}}(q^m).$$

Thus, we consider the locations of the nonzero terms in

$$p_{\sigma, \alpha}^{(m/d)}(q) \prod_{i=1}^{m} \text{SYT}(\lambda(i))^{\text{maj}}(q).$$

Recall that $p_{\sigma, \alpha}^{(m/d)}(q) = 0$ whenever $(\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{m/d} = 0$, so assume $(\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{m/d} > 0$. Since $\text{SYT}(\lambda(i))^{\text{maj}}(q) \neq 0$ for all partitions $\lambda(i)$, we can also assume (24) is not zero.

By Lemma 7.11, $p_{\sigma, \alpha}^{(m/d)}(q)$ has no internal zeros, degree $(\alpha_2) - \sum (\alpha_i) - (\sigma \cdot \alpha)_{m/d+1} - \cdots - (\sigma \cdot \alpha)_m$, and constant term $p_{\sigma, \alpha}^{(m/d)}(0) = 1$. Thus the minimal degree term of (24) with nonzero coefficient is $q^{b(\lambda)}$ by (8), and the maximal degree term is $q$ to the power

$$\binom{n}{2} - \sum (\alpha_i) - (\sigma \cdot \alpha)_{m/d+1} - \cdots - (\sigma \cdot \alpha)_m + \sum \deg(\text{SYT}(\lambda(i))^{\text{maj}}(q)).$$

Since $\alpha_i = |\lambda(i)|$, we know by (9) that $\deg(\text{SYT}(\lambda(i))^{\text{maj}}(q)) = (\alpha_2) - b(\lambda(i))$, so (25) simplifies to

$$\binom{n}{2} - (\sigma \cdot \alpha)_{m/d+1} - \cdots - (\sigma \cdot \alpha)_m - b(\lambda')$$

where $b(\lambda') := \sum_i b(\lambda(i))$. From (9), we also know $\binom{n}{2} - b(\lambda') = b(\lambda) + \binom{n+1}{2} - \sum_{\alpha \lambda} h_c$ and $\sigma \cdot \alpha \vdash n$, so we conclude that the maximal degree of (24) is

$$b(\lambda) + \sum (\sigma \cdot \alpha)_{m/d+1} + \cdots + (\sigma \cdot \alpha)_{m/d} + \binom{n}{2} - \sum_{\alpha \lambda} h_c.$$

If $p_{\sigma, \alpha}^{(m/d)}(q) \neq 1$, then the product in (24) also has no internal zeros by Theorem 1.1 and Lemma 6.1. The cases where $p_{\sigma, \alpha}^{(m/d)}(q) = 1$ are listed in Lemma 7.11. By assumption, $(\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{m/d} > 0$ so the remaining cases are when $(\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{m/d} = 1$ and some $\alpha_i = n-1$ for $i > m/d$, or $\alpha_i = n$ for some $i \leq m/d$. In either of the remaining cases, determining the nonzero coefficients of (24) reduces to the case of a single partition described in Theorem 1.1.

To finish the proof, we observe from (23) that $b(\lambda')_{i,k} \neq 0$ if and only if there exists some $\sigma \in C_d$ such that $(\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{m/d} > 0$ and the corresponding product

$$q^{b(\sigma, \alpha)} p_{\sigma, \alpha}^{(m/d)}(q^m) \cdot \prod_{i=1}^{m} \text{SYT}(\lambda(i))^{\text{maj}}(q^m)$$
has nonzero coefficient of $q^k$. Thus, by our analysis of the location of nonzero coefficients in (24), we observe that $b_{(\lambda),k} \neq 0$ if and only if

$$\frac{k - b(\sigma \cdot \alpha)}{m} - b(\lambda) \in \left\{ 0, 1, \ldots, (\sigma \cdot \alpha)_1 + \cdots + (\sigma \cdot \alpha)_{m/d} + \left( \frac{n}{2} \right) - \sum_{c \in \lambda} h_c \right\} \setminus \mathcal{D}_{\mu/d}$$

where $\mu = \sigma \cdot \lambda$. Observing that $b(\lambda) = b(\mu)$, $\sum_{c \in \lambda} h_c = \sum_{c \in \mu} h_c$, and $b(\sigma \cdot \alpha) = b(\alpha(\mu))$ completes the proof of the theorem.

**Corollary 8.4.** In type $D_n$, an irreducible with orbit $\{\lambda, \mu\}$ where $|\lambda| + |\mu| = n$ appears in degree $k$ of the coinvariant algebra of $G(2,2,n)$ if and only if either $\lambda \neq \varnothing$ and

$$\frac{k - |\mu|}{2} - b(\lambda) - b(\mu) \in \left\{ 0, 1, \ldots, |\lambda| + \left( \frac{n}{2} \right) - \sum_{c \in \lambda} h_c - \sum_{c \in \mu} h_c' \right\} \setminus \mathcal{D}_{(\lambda,\mu);2}$$

or $\mu \neq \varnothing$ and

$$\frac{k - |\lambda|}{2} - b(\lambda) - b(\mu) \in \left\{ 0, 1, \ldots, |\mu| + \left( \frac{n}{2} \right) - \sum_{c \in \lambda} h_c - \sum_{c \in \mu} h_c' \right\} \setminus \mathcal{D}_{(\mu,\lambda);2}.$$

**Example 8.5.** Consider the type $D_6$ case, where $m = 2, d = 2, n = 6$. For $\lambda = ((2), (31))$, we have $b(\lambda) = 1$ and

$$g^{(\lambda)^2}(q) = q^{20} + 3q^{18} + 6q^{16} + 8q^{14} + 9q^{12} + 8q^{10} + 6q^8 + 3q^6 + q^4.$$  

On the other hand, if $\underline{\mu} = (\varnothing, (33))$ then $b(\mu) = 3$ and

$$g^{(\mu)^2}(q) = q^{18} + q^{14} + q^{12} + q^{10} + q^6.$$

In both cases, the nonzero coefficients are determined by Corollary 8.4. One may notice that $g^{(\nu)^2}(q) \neq g^{(\mu)^1}(q)$, which appeared in Example 6.5. However, for $\nu = ((33), \varnothing)$, one can check $\{\underline{\mu}\}^2 = \{\underline{\nu}\}^2$ and $g^{(\nu)^2}(q) = g^{(\nu)^2}(q) = g^{(\nu)^1}(q)$.

**Acknowledgments**

We would like to thank Christian Stump, William McGovern, Andrew Ohana, Alejandro Morales, Greta Panova, Mihael Perman, Martin Raïc, Victor Reiner, Sheila Sundaram, Vasu Tewari, Lauren Williams and Alex Woo for helpful discussions related to this work.

**References**

[AER18] Ron Adin, Serge Elizalde, and Yuval Roichman. Cyclic descents for near-hook and two-row shapes, 2018. arXiv:1801.00044.

[And76] George E. Andrews. *The Theory of Partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.
[AS17] Connor Ahlbach and Joshua P. Swanson. Refined cyclic sieving. Sém. Lothar. Combin., 78B:Art. 48, 12, 2017.
[BKS18] Sara C. Billey, Matjaž Konvalinka, and Joshua P. Swanson. Asymptotic normality of the major index on tableaux and limit laws for cyclotomic generating functions. In Preparation, 2018.
[Car75] L. Carlitz. A combinatorial property of $q$-Eulerian numbers. Amer. Math. Monthly, 82:51–54, 1975.
[Car89] Roger W. Carter. Simple Groups of Lie Type. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.
[Cli37] Alfred H. Clifford. Representations induced in an invariant subgroup. Ann. of Math. (2), 38(3):533–550, 1937.
[ER15] Richard Ehrenborg and Margaret Readdy. A poset view of the major index. Adv. in Appl. Math., 62:1–14, 2015.
[Foa68] D. Foata. On the Netto inversion number of a sequence. Proc. Amer. Math. Soc., 19:236–2479–1130, 1968.
[FRT54] J. S. Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric groups. Canadian J. Math., 6:316–324, 1954.
[Gre55] J. A. Green. The characters of the finite general linear groups. Trans. Amer. Math. Soc., 80:402–447, 1955.
[HHL05] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. J. Amer. Math. Soc., 18(3):735–761, 2005.
[Hum90] James E. Humphreys. Reflection Groups and Coxeter Groups, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
[Kly74] A. A. Klyachko. Lie elements in the tensor algebra. Siberian Mathematical Journal, 15(6):914–920, 1974.
[Knu73] D. E. Knuth. The Art of Computer Programming, volume 3. Addison–Wesley, Reading, MA, 1973.
[Lus77] George Lusztig. Irreducible representations of finite classical groups. Invent. Math., 43(2):125–175, 1977.
[Mac13] P. A. MacMahon. The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects. Amer. J. Math., 35(3):281–322, 1913.
[Mac17] P. A. MacMahon. Two applications of general theorems in combinatorial analysis: (1) to the theory of inversions of permutations; (2) to the ascertainmment of the numbers of terms in the development of a determinant which has amongst its elements an arbitrary number of zeros. Proc. London Math. Soc., S2-15(1):314, 1917.
[MPP15] Alejandro Morales, Igor Pak, and Greta Panova. Hook formulas for skew shapes I. $q$-analogues and bijections, 2015. arXiv:1512.08348.
[OEI18] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2018. Online. http://oeis.org.
[RS*18] Martin Rubey, Christian Stump, et al. FindStat - The combinatorial statistics database. http://www.FindStat.org, 2018.
[RSW04] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. J. Combin. Theory Ser. A, 108(1):17–50, 2004.
[Sag91] B. Sagan. The Symmetric Group. Wadsworth, Inc., Belmont, CA, 1991.
W. Specht. Die irreduziblen darstellungen der symmetrischen gruppe. *Mathematische Zeitschrift*, 39:696–711, 1935.

G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

Richard P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc. (N.S.)*, 1(3):475–511, 1979.

Richard P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Alg. Disc. Meth.*, 1(2):168–184, 1980.

R. P. Stanley. *Enumerative Combinatorics. volume 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

R. P. Stanley. *Enumerative Combinatorics. volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

R. Steinberg. A geometric approach to the representations of the full linear group over a Galois field. *Trans. Amer. Math. Soc.*, 71:274–282, 1951.

John R. Stembridge. On the eigenvalues of representations of reflection groups and wreath products. *Pacific J. Math.*, 140(2):353–396, 1989.

John Shareshian and Michelle L. Wachs. Eulerian quasisymmetric functions. *Adv. Math.*, 225(6):2921–2966, 2010.

Joshua P. Swanson. On the existence of tableaux with given modular major index. *Algebraic Combinatorics*, 1(1):3–21, 2018.

Alfred Young. *The Collected Papers of Alfred Young (1873–1940)*. University of Toronto Press, Toronto, Ont., Buffalo, N. Y., 1977.

Doron Zeilberger. Kathy O’Hara’s constructive proof of the unimodality of the Gaussian polynomials. *Amer. Math. Monthly*, 96(7):590–602, 1989.

**BILLEY:** DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA

**KONVALINKA:** FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 21, LJUBLJANA, SLOVENIA, AND INSTITUTE FOR MATHEMATICS, PHYSICS AND MECHANICS, JADRANSKA 19, LJUBLJANA, SLOVENIA

**SWANSON:** DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO (UCSD), LA JOLLA, CA 92093-0112