Nonlinear QED in an ultrastrong rotating electric field: Signatures of the momentum-dependent effective mass

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The specific features of nonlinear pair production and radiation processes in an ultrastrong rotating electric field are investigated, taking into account that this field models the antinodes of counterpropagating laser beams. It is shown that a particle in a rotating electric field acquires an effective mass which depends on its momentum absolute value as well as on its direction with respect to the field plane. This phenomenon has an impact on the nonlinear Breit-Wheeler and nonlinear Compton processes. The spectra of the produced pairs in the first case, and the emitted photon in the second case, are shown to bear signatures of the effective mass. In the first case, the threshold for pair production by a γ-photon in the presence of this field varies according to the photon propagation direction. In the second case, varying the energy of the incoming electron allows for the measurement of the momentum dependence of the effective mass. Two corresponding experimental setups are suggested.

A strong field may modify the mass of the particles with which it interacts. This phenomenon, originally introduced in the context of particle physics (the Higgs mechanism [1]), may be also found in condensed matter [2], plasma [3], and strong field QED [4-6]. In the latter, the effective mass significantly deviates from the vacuum mass for large values of the classical strong field parameter \( \xi \equiv ea/m^2 \), where \( e \) is the amplitude of the laser vector potential \( A_\mu \), and \( -e \) and \( m \) are the electron charge and mass, respectively; relativistic units \( \hbar = c = 1 \) are used. Contemporary optical lasers [8, 9] may reach \( \xi \sim 100 \) and a significant increase is expected in the next generation laser facilities [10, 11]. Consequently, the effective mass is expected to play a significant role in the interaction of such intense beams with matter.

In the realm of the strong field QED, the perturbation treatment is developed based on solutions of the Dirac equation in the presence of the external field [12]. The fundamental quantity of this theory is the quantum strong field parameter \( \chi \equiv e \sqrt{(F_{\mu \nu}P_\nu)/m^3} \), where \( P_\mu = (E, \mathbf{P}) \) is the kinematic four-momentum, a bold letter stands for a 3-vector and \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field tensor. In the limit \( \chi \rightarrow 0 \) the classical electrodynamics is recovered. The lowest order processes described by this theory are [2, 13] non-linear Compton (NLC), where an electron absorbs s-laser photons to emit an energetic photon, nonlinear Breit-Wheeler (NLBW), where an electron positron pair is created following the absorption of a γ-photon and s-laser photons, and the Schwinger mechanism [14], where the strong field induces a pair creation from the vacuum. Due to the dressing of the electron mass by a strong laser field, the kinematic associated with these processes is modified with respect to the weak field case. In particular, one may show [7] that the quantity appearing in the energy-momentum conservation is the cycle-averaged momentum \( \overline{P}_\mu \).

The NLC and NLBW processes are crucial parts in the physics of the electron-positron pairs in high-intensity laser fields. Due to the dressing of the electron mass by a strong field QED were reported [24, 27], bringing closer the perspective to measure the effective mass.

Theoretical investigation of the effective mass requires a solution for the dynamics of the particle in the presence of the field. It is well known that for a plane wave field (PWF) the Dirac equation admits an analytical solution [28]. Owing to its high relevance to laser matter experiment, most of the existing literature concerning the effective mass relies on this solution. It was shown to depend on the laser polarization [7] and the shape of the laser pulse [29], and leaves signatures in the radiation spectrum [30]. The definition of the effective mass becomes more elusive for non-periodic fields such as few cycles pulses [31, 35]. Another field configuration bearing significance to laser matter interaction is the oscillating electric field. This field models the antinode of a standing wave, created by two counterpropagating laser beams. Note that the electrons are expected to be trapped in the antinodes of the standing laser wave in the anomalous radiative trapping regime [36]. Several approximations to the corresponding wave function were discussed in the context of various strong field processes [37, 48]. The effective mass and its consequences, however, were explicitly considered only for the limiting case of a vanishing particle momentum [49].

In this letter, the role of the effective mass for nonlinear QED in a strong rotating electric field (REF) is investigated. We derive the analytic expression for the effective mass of a particle in the presence of REF and show that it depends not only on the field parameters but also on the particle momentum. Namely, two particles propagating in different direction or velocity in the same field acquire a different mass. The effect of the dressed mass on the probabilities of NLC and NLBW processes, and on the spectra of photons or created electron-positron pairs are explored by analytical and numerical means. Furthermore, two experimental scenarios are suggested to detect a measurable signature of this phenomenon.

A possible realization of REF in laboratory may be achieved using counterpropagating circularly polarized laser beams, as illustrated schematically in Fig. 1. In the antinodes of the standing wave created by the beams the magnetic components of the two beams cancel each other and the field can be approximated as REF.
Accordingly, the minimal value of \( m_\perp \) leads to
\( \mu = \frac{\sqrt{m_\perp^2}}{\sqrt{2}} \).
The vector potential of REF is defined
\( a \) standing wave and the
\( \text{PWF value} m_\perp^0 \) as a function of \( \theta \) and \( p/m \).
One may observe that for \( \theta = 0 \) or \( p \ll m_\xi \) the normalized
value of the effective mass tends to 1, in agreement with the
analytical result. The values for \( p \gg m_\xi \) coincides to a very
good approximation with Eq. (1). Fig. 2(b) presents the same
quantity as a function of \( p/m \) and \( \xi \) for \( \theta = \pi/2 \).
The limits of \( p \) much higher / lower than \( m_\xi \) hold here as in Fig. 2(a).
Notice that the minimal value of the normalized effective mass
is 1/\( \sqrt{2} \approx 0.71 \) and that a significant decrease appears for
\( p \gg m_\xi, \xi \sim 1 \), in accordance with Eq. (1).

Since the effective mass is embedded in the kinematics associated
with the NLC and NBW processes, its fingerprint may be
found in the corresponding spectra. In a previous work [48]
we have examined in details the NLC probability for this field
configuration. It was found that as long as \( \varepsilon \gg m_\xi \), the rate
coincides to an excellent approximation with the one obtained
with the semiclassical formula introduced by Baier and Katkov
[51, 52]. Due to the crossing symmetry between the matrix
elements of the Compton and Breit-Wheeler processes [53],
this conclusion holds for the NBW process as well. For this
reason, we calculate here the rate according to the semiclassical
expression. In this case, the probability to emit a photon with a
four-momentum \( k' = (\omega', k'_z) \) reads
\[
d\mathcal{P} = \frac{\alpha}{(2\pi)^2\omega'} |M|^2 d^3k',
\] (2)
where \( \alpha \approx 1/137 \) is the fine structure constant,
\[
|M|^2 \equiv -\frac{e^2}{2} |\mathcal{T}_\mu|^2 + \frac{m^2\omega^2}{2e^2g^2}\mathcal{I},
\] (3)

and \( \varepsilon' = \varepsilon - \omega' \). The integrals \( \mathcal{I} \) and \( \mathcal{T}_\mu \) are defined as follows
\[
\mathcal{I} \equiv \int_{-\infty}^{\infty} dt e^{i\psi}, \quad \mathcal{T}_\mu \equiv \int_{-\infty}^{\infty} dt v_\mu e^{i\psi},
\] (4)
where the phase reads \( \psi \equiv \frac{\xi}{\sqrt{2}} \cdot \mathbf{k}' \cdot \mathbf{x}(t) \), the velocity is \( v_\mu = P_\mu/\mathcal{E} \)
and \( \mathbf{x}(t) \) designates the classical trajectory. The probability
associated with the NBW takes analogous form where \( d^3k' \)
is replaced by the momentum of the outgoing electron \( d^3p' \) and
\( \varepsilon' = \omega' - \varepsilon \). It follows from Eqs. (2)-(4) that the probability
is determined according to the trajectory of the electron in the presence of the field. It provides an explanation to the fact that for \( \theta = 0 \) the effective mass coincides with that of the PWF, as seen from Eq. (1). In this case, the particle is simply moving on a circle in the \((x,y)\)-plane while drifting along the \(z\)-axis, which is identical to the particle motion in a PWF.

In the applied scheme of Fig. 1, a particle would experience REF rather than a standing wave only if it propagates along the antinode plane (perpendicular to the beams axis), namely with \( \theta = \pi/2 \). On the other hand, we wish to detect the angle dependence of the effective mass. According to Fig. 2, this dependence is slow and monotonous. Thus, finding another configuration corresponding to \( \theta = 0 \) may be sufficient. As explained above, the latter case is theoretically equivalent to a particle in the presence of a PWF. Hence, our reference configuration would be a \( \gamma \)-photon interacting with a circularly polarized PWF with the same \( \xi \) value. Since for the PWF the effective mass depends solely on \( \xi \), the angle between the \( \gamma \)-photon and the laser may be chosen according to convenience. In the following we assumed that this angle would be \( \theta = \pi/2 \). Namely, the reference configuration is identical to the one presented in Fig. 1, where only a single laser beam is active.

We start with the NLBW scattering. For this process to take place, the center of mass energy, \( E_s = \sqrt{(sk + k')^2} = \sqrt{2s(k \cdot k')} \) should exceed \( 2m_e \), where \( s \) is the number of absorbed field photons and their wavevector reads \( k = (\omega, 0, 0) \). This threshold suggests a simple way to measure the effective mass. Since for the set up illustrated in Fig. 1 we have \( k \cdot k' = \omega \omega' \), the threshold energy for the incoming \( \gamma \)-photon is

\[
\omega'_s = \frac{2m_e^2}{s\omega}.
\]

Accordingly, increasing \( \omega' \) for fixed laser parameters leads to a discrete change in the number of allowed channels in the vicinity of \( \omega'_s \), leading to an abrupt jump in the total probability. In order to detect this discontinuity two requirements should be fulfilled. First, the laser normalized amplitude should lay in the perturbative regime (i.e. \( \xi \leq 1 \)), so that high harmonics are inhibited and the main contribution originates from the \( s \)th channel under consideration. Second, the threshold \( \omega'_s \) should be remote from the sequential one \( \omega'_{s+1} \), so that the influence of the \( s \)th channel would be distinguishable. Therefore, as Eq. (5) implies, low harmonics are preferable. The total probability of pair production in dependence of the incoming \( \gamma \)-photon energy is shown in Fig. 3(a). Since high \( \omega' \) \( \gamma \)-photon energies are difficult to achieve, we propose to increase \( \omega \) by using harmonics of the laser radiation, and consider the following laser parameters: \( \xi = 0.4, \omega = 4.65 \text{ eV} \), corresponding to the 3rd harmonic of Ti:S laser with intensity of \( 6 \times 10^{18} \text{ W/cm}^2 \). As mentioned above, observing effective mass effects requires multi-cycle pulse. A 10 cycle pulse with the desired intensity focused on a spot with diameter of 10 wavelengths corresponds to 4 mJ, which is realizable with the present laser technique [54]. The \( \gamma \)-energies lie in the same GeV-range as those achieved in the E-144 experiment [22, 23]. One may observe that the thresholds are \( \omega'_s = 65.2 \text{ GeV} \) for \( \theta = 0 \), and \( 60.5 \text{ GeV} \) for \( \theta = \pi/2 \), which using Eq. (5) correspond to \( m_e(\theta = 0) = m_e^0 \) and \( m_e(\theta = \pi/2) = 0.96m_e^0 \), in accordance with Eq. (1). Notice that for \( \theta = 0 \) the quantum parameter is \( \chi = \xi\omega\omega'/m^2 \) whereas for \( \theta = \pi/2 \) it reads \( \xi\omega\omega'/m^2\sin(\omega t) \). Accordingly, the average value of \( \chi \) is lower in the second case and so is the corresponding rate.

Another indication to the effective mass may be observed in the spectrum of the created pair, as follows. From the energy momentum conservation \( s_0 + P_\mu = P'_\mu + k' \), a restriction on the incoming particles energy arises [50]. For a given number of absorbed photons \( s \), one may show that

\[
\left| \epsilon - \omega' \right| < \frac{\Delta_s}{2}, \quad \Delta_s = \omega' \sqrt{1 - \frac{s_0}{s}},
\]

where \( s_0 = 2m_e^2/(\omega \omega') \). As an example, the spectral probability associated with the created pair is depicted in Fig. 3(b). The \( \gamma \)-photon energy is 50 GeV and the laser parameters as described above. The widths of the 3rd harmonic are \( \Delta_3 = 0.34\omega', 0.43\omega' \) for \( \theta = 0, \pi/2 \) respectively. Employing Eq. (6) one obtained the same effective mass values written above.

Furthermore, the effective mass is manifested in the NLC spectra (the PWF case was discussed in [30]). A straightforward kinematical calculation [50], shows that for a given \( s \), the emitted photon has a cutoff energy, known as “edge”

\[
\omega'_s = \frac{s\omega\epsilon}{\sqrt{m_e^2 + \epsilon^2}}, \quad \tilde{\nu} = \frac{p}{\sqrt{m_e^2 + p^2}},
\]

where \( \tilde{\nu} \) is the absolute value of the cycle-averaged velocity. As a result, the effective mass affects the edge location. In principle, since the effective mass is momentum-dependent (as shown in Fig. 2), it may differ for the incoming and outgoing particles. We study the process in the classical regime, \( \chi \ll 1 \), because the regime where both \( \xi \sim 1 \) and \( \chi \sim 1 \) are fulfilled would require very high frequency colliding laser beams (with photon energies of MeV range, which is beyond contemporary
The edge locations corresponding to $\theta = \pi/2$ deduced using Eq. (7), which is in accordance with the function summarized in Fig. 4(b). From the latter the effective mass $m_e = m'\xi_p$ for $\theta = \pi/2$, normalized by $\omega_e$, as a function of $p/m$ (red line). As a reference, the prediction of Eq. (7) for the limiting cases $m_e(p = 0)$ (dash-dotted blue line) and $m_e(p = m\xi_p)$ (dashed green line).

Figure 4(a) presents the NLC spectrum for $\theta = 0$ (thin blue line) and $\theta = \pi/2$ (green line). Simulation parameters: $\omega = 1.55 \text{ eV}, \xi = 2$, $p/m = 20$. The vertical dash-dotted red (dashed black) line shows the edge energy of the first harmonic $\theta = 0 (\theta = \pi/2)$. (b) The edge energy for $\theta = \pi/2$, normalized by $\omega_e$, as a function of $p/m$ (red line). As a reference, the prediction of Eq. (7) for the limiting cases $m_e(p = 0)$ (dash-dotted blue line) and $m_e(p = m\xi_p)$ (dashed green line).

FIG. 4. (a) NLC emission spectrum: $\theta = 0$ (thin blue line) and $\theta = \pi/2$ (green line). Simulation parameters: $\omega = 1.55 \text{ eV}, \xi = 2$, $p/m = 20$. The vertical dash-dotted red (dashed black) line shows the edge location of the first harmonic $\theta = 0 (\theta = \pi/2)$. From the latter the edge location. The shift of the edge from the calculated spectra is expected, the curve obtained from the edge location shift (solid red line) interpolates continuously between the two other ones.

Concluding, the emergence of a momentum-dependent effective mass in the presence of a strong REF has been demonstrated. As a result, the pair production threshold by a $\gamma$-photon and the harmonic edges in the pair spectrum, depend on its angle with respect of the field plane. Moreover, from the edges of the photon emission spectrum of an electron in the presence of this field, the momentum-dependence of the effective mass could be measured. These predictions may be put to the test with present day facilities.

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Supplemental Materials
to the paper “Nonlinear QED in an ultrastrong rotating electric field: Signatures of
the momentum-dependent effective mass”

I. TRAJECTORY

In the semiclassical formalism, employed in this paper, the
classical trajectory of the particle is the cornerstone of the rate
calculation. Let us calculate explicitly the trajectory of an
electron moving in a rotating electric field (REF), described
by a vector potential \( \mathbf{A}(t) = a(\cos \omega t, \sin \omega t, 0) \). The particle
location is given by

\[
\mathbf{x}(t) = \int_0^t dt' \mathbf{u}(t') = \int_0^t dt' \frac{\mathbf{P}(t')}{\mathcal{E}(t')},
\]

(1)

Due to the canonical momentum conservation, the kinetic
momentum is straightforwardly derived

\[
\mathbf{P}(t) = \mathbf{p} + e \mathbf{A}(t),
\]

(2)

and using the dispersion relationship \( \mathcal{E}(t) = \sqrt{\mathbf{P}^2(t) + m^2} \), one
arrives at

\[
\mathcal{E}(t) = \sqrt{\omega^2 + (ea)^2} \sin \theta \cos(\omega t - \nu),
\]

(3)

where \( \tan \nu = p_y / p_x \). It may be represented as

\[
\mathcal{E} = G \sqrt{1 - \mu \sin^2 \left( \frac{\omega t - \nu}{2} \right)},
\]

(4)

where the following quantities are introduced

\[
G \equiv \sqrt{m^2(1 + \xi^2)p^2 + 2m\xi p \sin \theta},
\]

(5)

and

\[
\mu \equiv \frac{4m\xi p \sin \theta}{G^2}.
\]

(6)

Substituting the explicit expression for the energy into Eq. (1),
one obtains the particle coordinate. Its \( x \) component reads

\[
x(t) = \frac{1}{G} \int_0^t dt' \left[ \frac{p_x}{\sqrt{1 - \mu \sin^2 \left( \frac{\omega t' - \nu}{2} \right)}} + \frac{ea \cos(\omega t')}{\sqrt{1 - \mu \sin^2 \left( \frac{\omega t' - \nu}{2} \right)}} \right].
\]

(7)

A variable change \( \phi \equiv (\omega t - \nu)/2 \) yields

\[
x(t) = \frac{2}{\omega G} \int_{\phi_0}^\phi d\phi' \left[ \frac{p_x}{\sqrt{1 - \mu \sin^2 \phi'}} + \frac{ea \cos(2\phi' + \nu)}{\sqrt{1 - \mu \sin^2 \phi'}} \right].
\]

(8)

The latter takes the form

\[
x(t) = \frac{2}{\omega G} \left[ p_x \mathcal{J}_1 + m\xi (\cos \nu \mathcal{J}_2 - \sin \nu \mathcal{J}_3) \right],
\]

(9)

where the following integrals are defined:

\[
\mathcal{J}_1(x|\mu) = \int_0^\infty dx' \frac{1}{\sqrt{1 - \mu \sin^2 x'}},
\]

(10)

\[
\mathcal{J}_2(x|\mu) = \int_0^\infty dx' \frac{\cos(2x')}{\sqrt{1 - \mu \sin^2 x'}},
\]

(11)

\[
\mathcal{J}_3(x|\mu) = \int_0^\infty dx' \frac{\sin(2x')}{\sqrt{1 - \mu \sin^2 x'}},
\]

(12)

and \( x = \omega t/2 \). These integrals admit analytical solution

\[
\mathcal{J}_1(x|\mu) = E_1(x|\mu),
\]

(13)

\[
\mathcal{J}_2(x|\mu) = \frac{(\mu - 2)E_1(x|\mu) + 2E_2(x|\mu)}{\mu},
\]

(14)

\[
\mathcal{J}_3(x|\mu) = \frac{2}{\mu} \left[ 1 - \sqrt{1 - \mu \sin^2 x} \right],
\]

(15)

where \( E_1(x|\mu), E_2(x|\mu) \) are the incomplete elliptic integral of
first and second kind, respectively

\[
E_1(x|\mu) = \int_0^x \frac{1}{\sqrt{1 - \mu \sin^2 x'}} dx',
\]

(16)

\[
E_2(x|\mu) = \int_0^x \frac{\sin(2x')}{\sqrt{1 - \mu \sin^2 x'}} dx'.
\]

(17)

Analogously, for the \( y \) component of the coordinate one
obtains

\[
y(t) = \frac{2}{\omega G} \int_{\phi_0}^\phi d\phi' \left[ \frac{p_y}{\sqrt{1 - \mu \sin^2 \phi'}} + \frac{ea \sin(2\phi' + \nu)}{\sqrt{1 - \mu \sin^2 \phi'}} \right],
\]

(18)

which reads

\[
y(t) = \frac{2}{\omega G} \left[ p_y \mathcal{J}_1 + m\xi (\sin \nu \mathcal{J}_2 + \cos \nu \mathcal{J}_3) \right].
\]

(19)

The vector potential has no \( z \) component, so for \( z(t) \) we simply have

\[
z(t) = \frac{2}{\omega G} p_z \mathcal{J}_1.
\]

(20)

In addition to the trajectory, the average velocity is required
as well for the purpose of rate calculation. Applying the definition
\( \bar{v}_x \equiv \left[ x(T) - x(0) \right]/T \), where \( T = 2\pi/\omega \), we obtain

\[
\bar{v}_x = \frac{1}{\pi G} \left[ p_x \mathcal{J}_1 + m\xi (\cos \nu \mathcal{J}_2 - \sin \nu \mathcal{J}_3) \right]_{\phi_0}^{\pi}.
\]

(21)
One may easily find
\[
\begin{align*}
J_1 (x|\mu) \bigg|_{0}^{\pi} &= 2E_1(\mu), \\
J_2 (x|\mu) \bigg|_{0}^{\pi} &= \frac{2(\mu - 2)E_1(\mu) + 4E_2(\mu)}{\mu}, \\
J_3 (x|\mu) \bigg|_{0}^{\pi} &= 0,
\end{align*}
\]
where the complete elliptic integrals are given by \(E_1(\mu) = E_1(\frac{x}{2} \mu), E_2(\mu) = E_2(\frac{1}{2} \mu)\). Accordingly, the \(x\) component of the average velocity takes the form
\[
\bar{v}_x = \frac{2}{\pi G} \left[ p_x E_1(\mu) + m_\xi \cos \left( \frac{2E_1(\mu) + (\mu - 2)E_1(\mu)}{\mu} \right) \right].
\]
Similarly, the other components are given by
\[
\begin{align*}
\bar{v}_y &= \frac{2}{\pi G} \left[ p_y E_1(\mu) + m_\xi \sin \left( \frac{2E_1(\mu) + (\mu - 2)E_1(\mu)}{\mu} \right) \right], \\
\bar{v}_z &= \frac{2}{\pi G} p_z E_1(\mu).
\end{align*}
\]

**II. APPROXIMATED EFFECTIVE MASS**

As explained in the main text, the effective mass is defined as \(\sqrt{\vec{p}^2}\). Since \(\vec{P} = p\) we have
\[
m_\varepsilon = \sqrt{\vec{E}^2 - p^2}.
\]
The cycle-averaged energy is given by
\[
\bar{E} = \frac{2}{\pi} GE_2(\mu).
\]
In the following we would like to Taylor expand the effective mass for \(p \gg m_\varepsilon\). As we show below, the first order vanishes and, therefore, we evaluate it up to second order. We introduce the following quantities
\[
\delta \equiv \frac{4m_\varepsilon |p| \sin \theta}{R^2}, \quad R \equiv \sqrt{m^2 \xi^2 + m^2 + p^2}.
\]
\(G, \mu\) defined above read in terms of these variables:
\[
G = R \sqrt{1 + \frac{\delta}{2}}, \\
\mu = \frac{\delta}{1 + \delta/2}.
\]
Substituting Eqs. (32) and (31) into Eq. (29), one finds
\[
\bar{E} = \frac{2R}{\pi} \sqrt{1 + \frac{\delta}{2}} E_2 \left( \frac{\delta}{1 + \delta/2} \right).
\]
One may notice that up to third order
\[
\delta \approx 4|\sin \theta| \left( \frac{m_\varepsilon}{p} \right) + O \left( \frac{m_\varepsilon^3}{p^3} \right).
\]
As a result, up to second order, we may expand with respect to \(\delta\) instead of \(m_\varepsilon^3/p\). Employing the following Taylor expansions
\[
E_2(x) \approx \frac{\pi}{2} \left( 1 - \frac{x}{4} - \frac{3x^2}{64} \right),
\]
\[
\sqrt{1 + x} \approx 1 + \frac{x}{2} - \frac{x^2}{8},
\]
and substituting Eq. (33) into Eq. (28), one obtains
\[
m_\varepsilon^2 = (m_\varepsilon^2)^2 - \frac{\delta^2 R^2}{32},
\]
where \(m_\varepsilon^2 = m \sqrt{1 + \xi^2}\). Since
\[
\delta^2 R^2 = \frac{16m^2 p^2 \xi^2 \sin^2 \theta}{R^2} = 16m^2 \xi^2 \sin^2 \theta,
\]
one may see that Eq. (37) becomes
\[
\frac{m_\varepsilon}{m_\varepsilon^2} = \sqrt{1 - \frac{\xi^2}{2(1 + \xi^2)} \sin^2 \theta}.
\]

**III. NONLINEAR COMPTON SCATTERING**

**A. Probability**

As demonstrated in [1], under the condition \(\epsilon \gg m_\varepsilon\), the quantum and semiclassical [2, 3] approaches coincide. Since the latter allows for simpler calculation, it will be used in this work. According to this approximation, the probability of a Dirac particle to emit a photon with a four-momentum \(k'\) is given by
\[
d\mathcal{P} = \frac{\alpha}{(2\pi)^2} d^3 k' \\
\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 N_2 \exp \left[ i \epsilon' \cdot [\mathbf{x}(t_1) - \mathbf{x}(t_2)] \right].
\]

where \(\rho_\varepsilon = [t, \mathbf{x}(t)]\) and \(\mathbf{x}(t)\) is the particle classical trajectory found above. The prefactor is given by
\[
N_2 \equiv \frac{1}{2e^2} \left( \epsilon_\varepsilon^2 + \epsilon_\varepsilon^2 \right) [\mathbf{u}(t_1) \cdot \mathbf{u}(t_2) - 1] + \frac{\omega^2 m^2}{\varepsilon^2}.
\]
It should be mentioned that this expression already contains averaging over the incoming electron spin and summing over the outgoing electron (photon) spin (polarization), respectively. Since \(\mathbf{u}(t_1) \mathbf{u}(t_2) = -\mathbf{u}(t_1) \mathbf{u}(t_2)\), Eq. (40) can be cast in the following form
\[
d\mathcal{P} = \frac{\alpha}{(2\pi)^2} |\mathcal{K}|^2 d^3 k',
\]
where
\[
|\mathcal{K}|^2 \equiv - \frac{(\varepsilon^2 + \varepsilon^2)}{2e^2} \left[ \bar{F}_\mu^2 + \frac{m_\varepsilon^2 \omega^2}{2e^2} |I|^2 \right].
\]
with
\[ I \equiv \int_{-\infty}^{\infty} dt e^{i\psi}, \quad T_\mu \equiv \int_{-\infty}^{\infty} dt v_\mu(t)e^{i\psi}, \quad (44) \]
and
\[ \psi = \frac{E}{E'} k' \cdot x(t) = \frac{E\omega'}{E'} (1 - x \cdot n'), \quad (45) \]
where \( k' = \omega'(1, n') \). Since we are dealing with a periodic motion, the phase may be decomposed to a periodic and non-periodic parts, \( \psi = \psi_p + \psi_{np}\epsilon\omega \) with
\[ \psi_p = \frac{E}{E'} n' \cdot x_p, \quad \psi_{np} = \frac{E\omega'}{E'} (1 - \bar{\nu} \cdot n'), \quad (46) \]
where the periodic part of the trajectory is given by
\[ x_p(t) = x(t) - \bar{\nu} t. \quad (47) \]
Since we assume that the incoming electron propagates along the x-axis, \( p_x = (\epsilon, p_x, 0, 0) \), the emitted photon parametrization is defined accordingly
\[ n' = (\cos \theta_e, \sin \theta_e \sin \varphi_e, \sin \theta_e \cos \varphi_e), \quad (48) \]
where \( \theta_e, \varphi_e \) are the polar and azimuthal angles with respect to the x-axis, respectively. Replacing the periodic part of the integrands by their Fourier series, the integrals are solved
\[ T^\mu = 2\pi \sum_s T^\mu_s(\Omega_s), \quad I = 2\pi \sum_s I_s(\Omega_s), \quad (49) \]
where the argument of the delta function reads
\[ \Omega_s \equiv \psi_{np} - s\omega, \quad (50) \]
and the Fourier coefficients are
\[ T^\mu_s = \frac{1}{T} \int_0^T dt v_\mu(t)e^{i(\omega - \omega')s}, \quad (51) \]
\[ I_s = \frac{1}{T} \int_0^T dt e^{i\omega' s}. \quad (52) \]
With the aid of the condition \( \Omega_s = 0 \), forced by the delta functions, the angle \( \theta_e \) is found
\[ \cos \theta_e = \frac{1}{\bar{\nu}} \left( 1 - \frac{s\omega}{\omega'} \right). \quad (53) \]
Using Eq. (48), the periodic part of the phase takes the form
\[ \psi_p = \frac{E\omega'}{E - \omega'} \left[ \cos \theta_e x_p(t) + \sin \theta_e \sin \varphi_e y_p(t) \right]. \quad (54) \]
Substituting Eq. (49) into Eq. (43), and using the identity
\[ \delta^2(\Omega_s) = \frac{1}{2\pi} \delta(\Omega_s), \]
with the interaction time \( \tau \), one obtains
\[ |K|^2 = 2\pi \sum_s |\mathcal{K}_s|^2(\Omega_s)\tau, \quad (55) \]
where
\[ |\mathcal{K}_s|^2 = \left( \frac{\omega^2 + \epsilon^2}{2\epsilon^2} \right)|T^\mu_s|^2 + \frac{m^2\omega^2}{2\epsilon^2}\bar{\nu}^2|I_s|^2. \quad (56) \]
Using \( d^3K' = \omega'^2 d(\cos \theta_e)d\varphi_e \) and integrating Eq. (42) over \( \cos \theta_e \) yields
\[ \frac{dI}{d\omega'd\varphi_e} = \frac{1}{(2\pi)^2} \sum_s |\mathcal{K}_s|^2 \frac{d\Omega_s}{d(\cos \theta_e)}, \quad (57) \]
where the relation \( dI = \omega' dP/\tau \) between the probability and the radiation intensity was employed. Since \( \bar{\nu} \cdot n' = \bar{\nu} \cos \theta_e \), from Eq.(50) it follows that
\[ \left| \frac{d\Omega_s}{d(\cos \theta_e)} \right| = \frac{\epsilon\omega'}{\bar{\nu}}. \quad (58) \]
Hence, the final expression takes the form
\[ \frac{dI}{d\omega'd\varphi_e} = \frac{a\omega'}{4\pi^3\epsilon'} \sum_s \left[ -\epsilon^2 \left( \epsilon^2 + m^2 \right) |T^\mu_s|^2 + m^2(\omega')^2|I_s|^2 \right]. \quad (59) \]

**B. Kinematics**

The highest possible value of \( \omega' \) associated with a given harmonics \( s \) may be derived from kinematic considerations. Using \( \epsilon' = \epsilon - \omega' \), the emitted photon energy stems from the kinematic relation Eq. (53)
\[ \omega' = \frac{s\omega E'}{E(1 - \bar{\nu} \cos \theta_e) + s\omega}. \quad (60) \]
The maximal value of \( \omega' \) corresponds to \( \cos \theta_e = 1 \). This result may be derived by an alternative kinematic approach. The energy momentum conservation of this process reads
\[ P_\mu + s\kappa_\mu = P'_\mu + k'_\mu, \quad (61) \]
where \( k_\mu = (\omega, 0, 0, 0) \). Therefore, the spatial momentum conservation yields
\[ p'_\parallel + k'_\parallel = p, \quad p'_\perp = -k'_\perp, \quad (62) \]
where \( \parallel \) and \( \perp \) designate the parallel and transverse components of the momenta with respect to the incoming particle direction, respectively. Then the total outgoing momentum reads
\[ p' = \sqrt{m^2 + p'_\parallel^2 + p'_\perp^2} = \sqrt{\omega'^2 + p'^2 - 2s\omega' \cos \theta_e}, \quad (63) \]
where \( k'_\parallel = \omega' \cos \theta_e \) and \( k'_\perp = \omega' \sin \theta_e \). Substituting \( p' \) into the energy conservation equation
\[ \bar{E} + s\omega = \omega' + \sqrt{m^2 + p'^2}. \quad (64) \]
one obtains the energy of the emitted photon
\[ \omega' = \frac{2s\omega \bar{E} + \epsilon' \omega^2}{2(\bar{E} - p \cos \theta_e + s\omega)}. \quad (65) \]
Recalling that \( \bar{P} = p \), the absolute value of the average velocity is given by \( \bar{\nu} = p/\bar{E} \). As a result we have
\[ \omega' = \frac{s\omega \bar{E}}{\bar{E}(1 - \bar{\nu} \cos \theta_e) + s\omega}. \quad (66) \]
where \( s\omega \ll \bar{E} \) was assumed. Approximating \( \bar{E} \approx \epsilon \) one returns to Eq. (60) given above.
IV. NONLINEAR BREIT-WHEELER PROCESS

A. Probability

Owing to the crossing symmetry relating the NLC and NLBW processes [3, 4], the semiclassical probability associated with the latter takes the form

\[ dP = \frac{\alpha}{(2\pi)^2 \omega'} |K|^2 d^3p. \]  

(67)

The difference with respect to the photon emission expression of Eq. (42) is the outgoing particle phase space, namely \( d^3k' \rightarrow d^3p \). Therefore, the final result Eq. (59) should be only multiplied by a factor \( \varepsilon^2 / \omega'^2 \). Moreover, in this case we are interested in the emission rate rather than intensity, leading to additional \( 1 / \omega' \) factor. Finally, one obtains

\[ dW / d\varepsilon d\phi_e = \frac{\alpha}{4\pi \varepsilon \omega'^2} \sum_s [ -\varepsilon^2 (\varepsilon^2 + \varepsilon'^2) |T_s^\mu|^2 + m^2 \omega'^2 |I_s|^2], \]  

(68)

where \( T_s^\mu, I_s \) are given by Eqs. (51) and (52). Another modification with respect to NLC scattering lies in the periodic part of the phase. Since the incoming photon propagates along the \( x \)-axis, one may write \( \mathbf{n}' = (1, 0, 0) \). Therefore, the outgoing electron four-momentum is parameterized as

\[ p_{\mu} = (\varepsilon, p \cos \theta_e, p \sin \theta_e \sin \varphi_e, p \sin \theta_e \cos \varphi_e). \]  

(69)

Accordingly, \( \psi_p \) may be written as

\[ \psi_p = \frac{\varepsilon \omega'}{\omega' - \varepsilon} \cos \theta_e x_p(t), \]  

(70)

where the trajectory is given by Eq. (9), and the relation \( \varepsilon' = \omega' - \varepsilon \) is used.

B. Kinematics

As in the NLC case, the effective mass may be inferred from the maximal value of the outgoing particle energy for a given harmonic \( s \). The kinematic relation Eq. (53), together with \( \varepsilon' = \omega' - \varepsilon \), yields

\[ \cos \theta_e = \frac{1}{\bar{\nu}} \left[ 1 - \frac{s \omega(\omega' - \varepsilon)}{\omega' \varepsilon} \right]. \]  

(71)

Since \( \cos \theta_e \leq 1 \) and employing

\[ 1 - \bar{\nu} = \frac{\varepsilon - \sqrt{\varepsilon^2 - m^2}}{\varepsilon} \approx \frac{m^2}{2 \varepsilon^2}, \]  

(72)

one obtains

\[ \frac{s \omega(\omega' - \varepsilon)}{\omega' \varepsilon} \geq \frac{m^2}{2 \varepsilon^2}. \]  

(73)

Solving the quadratic inequality for \( \varepsilon \) one arrives at

\[ |\varepsilon - \omega'| / 2 \leq \sqrt{1 - s_0 / s}, \]  

(74)

where \( s_0 = 2m^2 / (\omega \omega') \).

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