Deformed Chern-Simons interaction for nonrelativistic point particles

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Abstract

We deform the interaction between nonrelativistic point particles on a plane and a Chern-Simons field to obtain an action invariant with respect to time-dependent area-preserving diffeomorphisms. The deformed and undeformed Lagrangians are connected by a point transformation leading to a classical Seiberg-Witten map between the corresponding gauge fields. The Schroedinger equation derived by means of Moyal-Weyl quantization from the effective two-particle interaction exhibits

– a singular metric, leading to a splitting of the plane into an interior (bag-) and an exterior region,
– a singular potential (quantum correction) with singularities located at the origin and at the edge of the bag.

We list some properties of the solutions of the radial Schroedinger equation.

1 Introduction

Two-dimensional incompressible fluids, in particular quantum-hall fluids (QHF’s) are well known to be invariant with respect to time-independent area-preserving diffeomorphisms (cp.[1]). In a particle picture a QHF is usually described by neglecting the kinetic energy compared to the magnetic field term leading to noncommutative geometry and a reduced phase space (cp.[2]). In the present paper we generalize this picture by allowing

— the particles to move in full phase space,
— the area-preserving diffeomorphisms to become time-dependent.

To do this we consider a deformation of the interaction of nonrelativistic charged point particles on a plane coupled to a Chern-Simons (CS) field such that

— the deformed action is invariant with respect to time-dependent area-preserving diffeomorphisms \( \nu_{2,t} \),
— the deformed and undeformed particle Lagrangians are connected by a point transformation leading to the classical analogue of a Seiberg-Witten (SW) map between the deformed and undeformed gauge fields.

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There is a second reason for studying such a model. A theory, unifying translational- and $U(1)$-gauge invariance in 2d contains nine independent gauge fields: six dreibein components and three $U(1)$-gauge fields [3]. Our model shows that the restriction of the group of local translations to its subgroup $\nu_{2,t}$ reduces the number of independent gauge fields to three as the dreibeins are built from the deformed U(1)-gauge fields.

We begin by constructing the deformed Lagrangian, consider the equations of motion (EOM) and discuss the properties of the deformed gauge fields. After gauge fixing we solve the nonlinear Gauss-constraint for the two-particle problem and quantize it by means of the Moyal-Weyl prescription. We discuss the structure of the resulting radial Schroedinger equation and list some properties of its solutions. We conclude with some final remarks.

2 Deformed Lagrangian

Infinitesimal elements of $\nu_{2,t}$ are defined by

$$\delta x_i = -\theta \epsilon_{ij} \partial_j \Lambda(x, t), \quad i, j = 1, 2$$

(1)

at fixed time $t$, where $\Lambda$ is an infinitesimal gauge function and $\theta$ is a finite deformation parameter. The corresponding change of a field $f(x, t)$ is defined by

$$\delta_0 f(x, t) := f'(x, t) - f(x, t)$$

(2)

or, if we include the coordinate change, we define

$$\delta f(x, t) := f'(x', t) - f(x, t)$$

(3)

Now, we want to deform

$$L_{\text{part}} := \frac{1}{2} \dot{x}_i^2 + e(A_i(x, t) \dot{x}_i + A_0(x, t))$$

(4)

in such a way that $\hat{L}_{\text{part}}$ becomes quasi-invariant with respect to (1) where $\hat{A}_\mu$ transform as

$$\delta \hat{A}_\mu(x, t) = \partial_\mu \Lambda(x, t)$$

(5)

or, equivalently

$$\delta_0 \hat{A}_\mu(x, t) = \partial_\mu \Lambda(x, t) + \theta \epsilon_{ij} \partial_i \hat{A}_\mu \partial_j \Lambda$$

(6)

ie. the gauge transformations of the $\hat{A}_\mu$ fields mix local $U(1)$-transformations with space transformations (1) (cp. [5])

In order to deform the first term in (4) we define invariant coordinates $\eta_i$ (cp. [8])

$$\eta_i(x, t) := x_i + \theta \epsilon_{ij} \hat{A}_j(x, t).$$

(7)

Obviously, we have

$$\delta \eta_i = 0.$$  

(8)

\footnote{Such transformations are used also in [4].}

\footnote{Deformed quantities are marked by a hat.}

\footnote{The idea of mixing coordinate transformations with gauge transformations has been developed in [6] and extended to noncommutative space in [7].}
Then, by means of the invariant velocity \( \xi_i \)

\[
\xi_i := \frac{d}{dt} \eta_i(x, t),
\]

we define \( \hat{E}_{\text{kin}} \)

\[
\hat{E}_{\text{kin}} := \frac{1}{2} \xi_i^2 .
\]

The deformation of the interaction term in (4) is more involved. It is given by

\[
e(A_i \dot{x}_i + A_0) \rightarrow e(\hat{A}_i \dot{x}_i + \hat{A}_0) + \frac{e\theta}{2} \epsilon_{ij} \hat{A}_i \frac{d}{dt} \hat{A}_j =: \hat{L}_{\text{int}},
\]

where an additional CS-term is needed because \( \delta \dot{x}_i \neq 0 \) and because we want \( \hat{L}_{\text{int}} \) to be quasi-invariant:

\[
\delta \hat{L}_{\text{int}} = \epsilon \frac{d}{dt}(\Lambda - \frac{\theta}{2} \epsilon_{ij} \hat{A}_i \partial_j \Lambda).
\]

To do this it is advantageous to replace \( \hat{L}_{\text{part}} \) by its first-order form

\[
\hat{L}_{\text{part}} = \dot{x}_k(\xi_k + e \hat{A}_k + \theta \epsilon_{ij} (\xi_i + \frac{e}{2} \hat{A}_i) \partial_k \hat{A}_j) - \frac{1}{2} \xi_i \xi_i \\
+ \theta \epsilon_{ij} (\partial_i \hat{A}_j)(\xi_i + \frac{e}{2} \hat{A}_i) + e \hat{A}_0.
\]

By varying the action \( \hat{S}_{\text{part}} \) with respect to \( \xi_i \) and \( x_i \) we get (9) as a constraint

\[
\xi_i = \dot{x}_i + \theta \epsilon_{ij} \frac{d}{dt} \hat{A}_j
\]

and the invariant nonlinear Lorentz-force equation

\[
\dot{\xi}_k = \frac{e}{1 + \theta \hat{F}}(\epsilon_{kl} \xi_l \hat{F} + \hat{F}_{ko})
\]

with the invariant field strength \( \hat{F}_{\mu\nu} \)

\[
\hat{F}_{\mu\nu} := \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \theta \epsilon_{ik} (\partial_i \hat{A}_\mu) \partial_k \hat{A}_\nu
\]

and

\[
\hat{F}_{ij} = \epsilon_{ij} \hat{F} .
\]

We note that in the particular case of a constant external magnetic field \( B \), i.e. for \( \hat{F} = B \), (14) and (15) are equivalent to the EOM given by Duval and Horvathy [9] for a particle which possesses a nonvanishing second central charge \( k \) of the planar Galilei group [10] with \( ek = -\theta \).

Finally, the CS-interaction of the \( \hat{A}_\mu \) field invariant with respect to the gauge transformations (5), is given by [5]

\[
\hat{L}_{\text{CS}} = \frac{\kappa}{2} \int d^2 x \epsilon^{\mu\nu\rho} \hat{A}_\mu (\partial_\nu \hat{A}_\rho + \frac{\theta}{3} \epsilon_{ik} (\partial_i \hat{A}_\nu)(\partial_k \hat{A}_\rho)).
\]

3
3 Deformed gauge fields

Usually the invariant velocity $\xi_i$ is defined in terms of nonrelativistic dreibein fields $E^\nu_\mu$ ($\mu, \nu = 0, 1, 2$) [8]

$$\xi_i = E^i_\mu \dot{x}_\mu + E^0_\mu .$$

Comparing (19) with (14) and considering time as fixed leads to dreibeins expressed in terms of gauge fields $\hat{A}_k$

$$E^\nu_\mu := \theta \epsilon_{\nu k} \partial_\mu \hat{A}_k + \delta_{\nu \mu}$$

$E^0_0 = 1$ and $E^0_k = 0$ ,

which, due to (5), transform covariantly with respect to $\nu_{2,t}$. Note that in the case of arbitrary local translations as an invariance group, dreibeins and the $\hat{A}_\mu$ are independent of each other [3]. Only the restriction to the subgroup $\nu_{2,t}$ allows the relation (20).

From the transformation law (6) we infer that the $\hat{A}_\mu$ are the gauge fields of the classical limit of a noncommutative $U(1)$-gauge theory. This raises the question of a possible Seiberg-Witten (SW) map [9] between the deformed and undeformed gauge fields $\hat{A}_\mu$ and $A_{\mu}$. For this we consider the point transformation

$$x_i \rightarrow \eta_i(x, t)$$

and redefine the gauge fields

$$\hat{A}_\mu(x, t) \rightarrow A_{\mu}(\eta, t)$$

so that

$$\hat{L}_{\text{part}}(\hat{A}_\mu(x, t), \dot{x}_i) = L_{\text{part}}(A_{\mu}(\eta, t), \dot{\eta}_i)$$

(21)-(23) defines the classical analogue of an inverse SW-map between gauge fields on a commutative space [1].

This inverse SW-map may be given explicitly in terms of inverse dreibeins $\{e^\nu_\mu\}$

$$A_{\mu}(\eta(x, t), t) = \frac{1}{2} \hat{A}_{\mu}(x, t)(\delta_{\nu}^\mu + e^\mu_\nu(x, t))$$

with, according to (20),

$$e^i_j = \frac{\delta_{ij} - \theta \epsilon_{ik} \partial_k \hat{A}_j}{1 + \theta F},$$

$$e^i_0 = -\theta \epsilon_{ik} \partial_i \hat{A}_k,$$

$e^0_0 = 1$ and $e^0_i = 0$ .

Solving (24) for $\hat{A}_\mu$ to leading order in $\theta$ we obtain

$$\hat{A}_\mu = A_{\mu} - \frac{\theta}{2} \epsilon_{ik} A_i (\partial_k A_{\mu} + F_{k\mu}) + 0(\theta^2)$$

in agreement with [12].

\footnote{The space indices can be taken equivalently as lower or upper indices.}

\footnote{A different definition of such a classical SW-map has been given in [13].}
4 Gauge fixing and residual symmetry

\( \hat{A}_0 \) is a Lagrange multiplier whose variation in the total action

\[
\hat{S} = \hat{S}_{\text{part}} + \hat{S}_{\text{field}}
\]

leads to the Gauss-constraint

\[
\hat{F}(x, t) = -\frac{1}{\kappa} \sum_{\alpha=1}^{N} e_{\alpha} \delta(x - x_{\alpha}).
\]  

Eq. (28) can be integrated once

\[
\hat{A}_k + \frac{\theta}{2} \epsilon_{\ell S} \hat{A}_\ell \partial_{\kappa} \hat{A}_S = -\frac{1}{2\pi \kappa} \sum_{\alpha} e_{\alpha} \partial_{k} \phi(x - x_{\alpha}) + \partial_{k} \lambda(x, t)
\]

where

\[
\phi(x) := \arctan \frac{x_2}{x_1}
\]

is a singular gauge function which has to be regularized (cp. [11]) and \( \lambda \) is an arbitrary gauge function to be determined by fixing the asymptotic behaviour of \( \hat{A}_\mu \). For that we follow closely the procedure described in [11]:

i) We decompose

\[
\hat{A}_\mu = \bar{A}_\mu + \hat{A}_\mu^{as}
\]

with

\[
\bar{A}_\mu \xrightarrow{r \to \infty} 0(r^{-1}).
\]

In order to fulfill (32) the \( \bar{A}_i \) should be chosen as solutions of (29)\( _{\lambda=0} \).

ii) We require the asymptotically Euclidean metric leading to

\[
\hat{A}_j^{as} = -\frac{1}{\theta} \epsilon_{jk} a_k(t)
\]

ie. the \( \hat{A}_j^{as} \) transform covariantly with respect to translations, local in time (residual symmetry). This requirement fixes also \( \hat{A}_0^{as} \). Thus our procedure fixes \( \lambda \) (gauge fixing).

iii) We redefine the Lagrangian in terms of the new variables \( \{\bar{A}_\mu, a_i(t)\} \) such that the solutions of the Euler-Lagrange equations minimize the new action.

5 The classical two-particle problem

We consider two identical particles of charge \( e \). Applying the Legendre transformation to our Lagrangian and using the Gauss-constraint (28) we obtain

\[
H = \frac{1}{2} \sum_{\alpha=1}^{2} \xi_{i,\alpha}^2
\]

with the constraint

\[
\sum_{\alpha=1}^{2} \xi_{i,\alpha} = 0
\]
arising from the variation of $\dot{a}_i(t)$ in the redefined action. In order to express the $\xi_{i,\alpha}$ in terms of canonical variable $\{x_\alpha, p_\alpha\}$ we need the $\tilde{A}_k$ at the particle positions $x = x_1$ to be solutions of

$$\tilde{A}_k + \frac{\theta}{2} \epsilon_{\ell S} \tilde{A}_\ell \partial_k \tilde{A}_S = \frac{e}{2\pi \kappa} \epsilon_{k\ell} \sum_{\alpha=1}^2 \left( \frac{(x - x_\alpha)_\ell}{|x - x_\alpha|^2} \right)_{\text{reg}}$$

(36)

We find in obvious notation

$$\tilde{A}_{k;2} = \pm \epsilon_{k\ell} (x_1 - x_2) \epsilon_{\ell \chi} (|x_1 - x_2|)$$

(37)

with

$$\chi(r) := \frac{1}{\theta} \left( 1 - \left( 1 - \frac{\tilde{\theta}}{r^2} \right)^{1/2} \right)$$

(38)

where

$$\tilde{\theta} := \frac{e\theta}{\pi \kappa}.$$  

(39)

With (37) and the position and momentum variables for the relative motion

$$x := x_1 - x_2, \quad p = \frac{1}{2}(p_1 - p_2)$$

(40)

we obtain by means of a straightforward computation the two-particle Hamiltonian $H$ in terms of canonical variables

$$H = \left( p_k - \frac{e^2}{2\pi \kappa} \epsilon_{k\ell} x_\ell \right) \left( p_{k'} - \frac{e^2}{2\pi \kappa} \epsilon_{k'\ell'} x_{\ell'} \right) g^{kk'}$$

(41)

with the inverse metric tensor $g^{kk'}$ given by

$$g^{kk'} := (1 - \tilde{\theta}/r^2)^{-1} \delta_{kk'} - \frac{\tilde{\theta}(2 - \tilde{\theta}/r^2)}{r^2 - \tilde{\theta}} x_k x_{k'}.$$  

(42)

In plane spherical coordinates (41) reads

$$H = p_r^2 (1 - \tilde{\theta}/r^2) + \left( \frac{\ell + \frac{e^2}{2\pi \kappa}}{r^2 - \tilde{\theta}} \right)^2$$  

(43)

where $\ell$ is the canonical angular momentum

$$\ell := \epsilon_{ik} x_i p_k.$$  

(44)

From (43), respectively (41,42), we conclude that $H$ is singular at $r^2 = \tilde{\theta} =: r_0^2$ if $\tilde{\theta} > 0$ and so that $E \lesssim 0$ for $r \lesssim r_0$. Thus we conclude that we have no communication between the interior ($r < r_0$) and the exterior ($r > r_0$) space regions. We have a geometric bag determined by the singularity of our dynamically generated metric.

6 The two-particle Schrödinger equation

By applying the Moyal-Weyl quantization procedure (cp. [14] eq. (3.7)) to $H$ given in terms of Cartesian variables (eq. (41)) we obtain the radial Schrödinger equation

$$\left( -\frac{\hbar^2}{r} \partial_r (1 - \tilde{\theta}/r^2) \partial_r + \frac{\bar{m}^2}{r^2 - \tilde{\theta}} + V(r) - E \right) \varphi_{\bar{\pi}}(r) = 0$$

(45)
with a singular potential $V(r)$

$$V(r) := -\frac{\hbar^2 \tilde{\theta}}{2} \left( \frac{1}{r^2 - \tilde{\theta}} - \frac{1}{r^4} \right)$$

(46)

and a fractional (anyonic) angular momentum

$$\overline{m} := m + \frac{e^2}{2\pi \kappa}, \quad m \in \mathbb{Z}.$$  

(47)

Let us list some properties of the solutions of (45):

i) For $r \to r_0$ we obtain a behaviour which is typical for the singularity there (cp. [15])

$$\varphi_{\overline{m}}(r) \simeq \begin{cases} 
C(r_0 - r^2)^{1/4} \exp \left( \frac{-r^2}{(r_0 - r^2)^{1/2}} \right) & \text{if } r < r_0 \\
(r^2 - \tilde{\theta})^{1/4} \left( A \cos \left( \frac{r^2}{(r^2 - \tilde{\theta})^{1/2}} \right) + B \sin \left( \frac{r^2}{(r^2 - \tilde{\theta})^{1/2}} \right) \right) & \text{if } r > r_0 
\end{cases}$$

(48)

With the required continuity of the partial radial current $j_{\overline{m}}$ at $r = r_0$ we infer from (48) $r < r_0$ that

$$j_{\overline{m}}(r_0) = 0.$$  

(49)

Therefore $A/B$ in (48) must be a real number.

Due to (49) we have no communication between the interior (bag-) and the exterior region also in the quantum case. In particular, a scattering wave arriving from the exterior region is totally reflected at the edge of the bag. Thus the bag acts like a white hole.

ii) For $r \to 0$ we obtain

$$\varphi_{\overline{m}}(r) \simeq (r^2)^{S_\pm}, \quad S_\pm := \frac{1}{2} \left( 1 \pm \frac{1}{\sqrt{2}} \right)$$

(50)

ie. all solutions are regular at the origin (cp. [15]).

iii) From (48) and (50) we infer that all solutions are square integrable within the bag region. As we have no additional boundary condition at hand which would determine a discrete spectrum, we expect the spectrum to be continuous. Such a situation is characteristic for singular potentials (cp. [15]).

7 Final remarks

We have shown that a deformed interaction between charged point particles and a CS-field, made invariant with respect to time-dependent area-preserving diffeomorphisms, leads to a two-particle Schrödinger equation of a highly singular nature. We have obtained some properties of its solutions but a complete discussion of its solution structure is still lacking. Work on the continuum generalization and the inclusion of an external magnetic field is in progress.

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8 References

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