Singularities in Space-time Foam Algebras

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Abstract

In this paper we consider the structure of the singularity sets associated with generalized functions in certain space-time foam algebras of generalized functions. In particular, we consider the algebra that is defined in terms of an asymptotic vanishing condition outside sets of first Baire category. It is shown that this algebra is in fact isomorphic to the earlier closed nowhere dense algebras. We also discuss general questions regarding singularities that can be handled in space-time foam algebras.

Keywords: Generalized Functions; Space-time Foam Algebras; Singularities

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1 Introduction

It is well known that the linear theory of distributions is not suited to a comprehensive and systematic treatment of nonlinear problems. This inability of $\mathcal{D}'$ distributions to accommodate nonlinear phenomena is exemplified by the so called Schwarz Impossibility [21]. One formulation of Schwarz’s result is as follows: There is no symmetric bilinear mapping

$$\star : \mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega) \ni (S, T) \rightarrow S \star T \in \mathcal{D}'(\Omega)$$

so that, $S \star T$ is the usual pointwise product of continuous functions, when $S, T \in \mathcal{C}^0(\Omega)$. That is, $\mathcal{D}'(\Omega)$ is not closed under any multiplication that extends the multiplication of continuous functions.

One way in which one may overcome the mentioned impossibility is to embed $\mathcal{D}'(\Omega)$ as a vector subspace in $\mathcal{A}(\Omega)$, where $\mathcal{A}(\Omega)$ is a quotient algebra

$$\mathcal{A}(\Omega) = S/I,$$  \hspace{1cm} (1)

with $S$ a subalgebra in $\mathcal{C}^\infty(\Omega)^\Lambda$, for some index set $\Lambda$, and $I$ an ideal in $S$. This approach was initiated by Rosinger [12] [13] and developed further in [14] [15] [16] [17]. A important particular case of the theory was introduced independently by Colombeau [18] [19]. This version of the theory has seen rapid development and a variety of applications over the past three decades, see for instance [1] [2] [3] [4] [5]. One of the major advantages of the (full) Colombeau algebra $\mathcal{G}(\Omega)$ is the rather natural close connection with distributions. In particular, $\mathcal{G}(\Omega)$ allows a canonical linear embedding of the space of distributions. Furthermore, the partial derivatives in $\mathcal{G}(\Omega)$ coincide with the distributional derivatives, when restricted to $\mathcal{D}'(\Omega)$.

Despite the mentioned utility of $\mathcal{G}(\Omega)$, several deep results obtained within the more general version of the theory appear to have no counterpart in the Colombeau setting. In this regard, we may recall the global version of the Cauchy-Kovalevskaia Theorem [16] [17], obtained in the so called closed nowhere dense algebras. In contradistinction with the closed nowhere dense algebra,
in $\mathcal{G}(\Omega)$ one cannot formulate, let alone prove, such a global existence result [20, Section 3.1]. Indeed, due to the polynomial type growth conditions imposed on generalized functions in $\mathcal{G}(\Omega)$ near singularities, the class of nonlinear operations that can be defined on $\mathcal{G}(\Omega)$, and the types of singularities that generalized functions in $\mathcal{G}(\Omega)$ can handle, are severely limited. In particular, $\mathcal{G}(\Omega)$ fails to be a flabby sheaf.

As mentioned, due to the failure of $\mathcal{G}(\Omega)$ to be a flabby sheaf, the class of singularities that this algebra can deal with is rather limited. Furthermore, the definition of Colombeau algebras on manifolds are severely complicated by this failure, see for instance [8].

Recently, Rosinger [18, 19, 20] introduced a general class of differential algebras, namely, the space-time foam (STF) algebras, which include the earlier nowhere dense algebras as a particular case. Each such algebra admits a linear embedding of $\mathcal{D}'(\Omega)$, and contains $C^\infty(\Omega)$ as a subalgebra. However, the embedding of distributions into STF algebras is in general not canonical. In fact, for any such algebra, there may be infinitely many linear embeddings of $\mathcal{D}'(\Omega)$, none of which are to be preferred above the others. Thus, as far as the embedding of distributions in to differential algebras are concerned, the Colombeau algebra appears a more natural setting than the STF algebras.

Here we may note that a large class of algebras admitting a canonical linear embedding of distributions, which commutes with distributional derivatives, were recently introduced by Vernaeve [23]. These algebras also admits a global version of the Cauchy-Kovalevskaia Theorem.

On the other hand, the STF algebras are able to handle a far larger class of singularities than the Colombeau algebras, and for that matter, any other differential algebra introduced so far. Indeed, STF algebras can deal with singularities that occur on any set $\Sigma$, provided only that $\Omega \setminus \Sigma$ is dense. Also, STF algebras form a fine and flabby sheaf. The mentioned flabby sheaf structure of STF algebras results in a simple construction of generalized functions on manifolds [19], although this fact is often not fully appreciated.

The utility of the STF algebras is further demonstrated by the Global version of the Cauchy-Kovalevskaia Theorem, which holds in suitable STF algebras [20], which is a strengthening of the previously known global existence result for analytic PDEs [17]. Besides this global existence result for large classes of nonlinear PDEs, STF algebras have seen applications in a variety of fields, including abstract differential geometry, de Rham cohomology and quantum gravity, see for instance [9].

In this paper we investigate the structure of singularities in certain STF algebras. In particular, the Baire I algebras $B_{L,B-I}(\Omega)$, where $L$ is an appropriate index set, are investigated, and compared with the nowhere dense algebra. In this regard, we show that these two algebras are, in many cases, identical. We further investigate the extent to which large classes of singularities may be accommodated in STF algebras.

The paper is organized as follows. In Section 2 we consider the relation between the Baire I algebras and chains of closed nowhere dense algebras. In particular, Section 2.1 recalls the basic construction of nowhere dense algebras, while Section 2.2 is concerned with the more general STF algebras. Finally, it is shown that the Baire I algebra is in fact, in many cases, identical with the nowhere dense algebra. A more general discussion of singularities in STF algebras is presented in Section 2.5 and 2.6.

## 2 Baire I Algebras vs Closed Nowhere Dense Algebras

In this section we discuss certain STF algebras, recently introduced by Rosinger [18, 19, 20]. In particular, we consider the algebra $B_{L,B-I}(\Omega)$, which is defined in terms of a dense vanishing condition off sets of first Baire category. These algebras, as part of the larger family of STF algebras, appear to have a rather clear and natural structure. Indeed, as mentioned, STF algebras form flabby sheaves, hence the following two useful and remarkable properties of such algebras:

(a) STF algebras can handle large classes of singularities
STF algebras are defined on general smooth manifolds in a straightforward and simple way. Furthermore, certain STF algebras, including as a particular case \( B_{L,B-I}(\Omega) \), admit a global version of the Cauchy-Kovalevskaia Theorem.

We show that, for a large class of index sets \( L \), the algebra \( B_{L,B-I}(\Omega) \) is in fact isomorphic to the nowhere dense algebra \( A_{L,nd}^\infty(\Omega) \), see [15, 16, 17]. This follows from a generalization of a suitable Baire category argument [17, Chapter 3, Appendix 1].

2.1 Nowhere Dense Algebras

The closed nowhere dense algebras were first introduced by Rosinger [15], and are designed in such a way as to accommodate certain singular generalized solutions of PDEs, such as shock waves, as actual solutions of the respective equations in a differential-algebraic sense. Here we briefly recall the construction of these algebras, and some of their basic features.

In this regard, let \( \Omega \subseteq \mathbb{R}^n \) be a non void, open set. Let \( L = (\Lambda, \leq) \) be an infinite right directed index set. That is, \( \forall \lambda, \mu \in \Lambda : \exists \nu \in \Lambda : \lambda, \mu \leq \nu \).

With respect to the usual componentwise operations, \( C^\infty(\Omega)^\Lambda \) is a unital and commutative algebra, and the set

\[
I_{L,nd}^\infty(\Omega) = \left\{ w = (w_{\lambda})_{\lambda \in \Lambda} \mid \exists \Gamma \subset \Omega \text{ closed nowhere dense : } \forall x \in \Omega \setminus \Gamma : \exists \lambda \in \Lambda : \forall \mu \in \Lambda, \mu \geq \lambda : w_{\mu}(x) = 0 \right\}
\]

is an ideal in \( C^\infty(\Omega)^\Lambda \). Based on a Baire category argument in \( \mathbb{R}^n \), see [17, Chapter 2, Appendix 1] the condition in the definition of \( I_{L,nd}^\infty(\Omega) \) is equivalent to

\[
\exists \Gamma \subset \Omega \text{ closed nowhere dense : } \forall x \in \Omega \setminus \Gamma : \exists \lambda \in \Lambda : \forall y \in V, \mu \in \Lambda, \mu \geq \lambda : D^p w_{\mu}(y) = 0
\]

which is easily seen to be equivalent to

\[
\exists \Gamma \subset \Omega \text{ closed nowhere dense : } \forall x \in \Omega \setminus \Gamma : \exists \lambda \in \mathbb{N}, V \subseteq \Omega \setminus \Gamma \text{ a neighborhood of } x : ,
\]

\[\forall y \in V, p \in \mathbb{N}^n, \mu \in \Lambda, \mu \geq \lambda : D^p w_{\mu}(y) = 0,\] (4)

In view of (4), it follows immediately that the differential operators

\[D^p : C^\infty(\Omega)^\Lambda \ni w = (w_{\lambda}) \mapsto D^p w = (D^p w_{\lambda}) \in C^\infty(\Omega)^\Lambda\]

satisfy the inclusion

\[D^p (I_{L,nd}^\infty(\Omega)) \subseteq I_{L,nd}^\infty(\Omega), \ p \in \mathbb{N}^n.\]

Thus, the usual partial derivative operators on \( C^\infty(\Omega) \) extend to mappings

\[D^p : A_{L,nd}^\infty(\Omega) \ni w + I_{L,nd}^\infty(\Omega) \mapsto D^p w + I_{L,nd}^\infty(\Omega) \in A_{L,nd}^\infty(\Omega), \ p \in \mathbb{N}^n,\]

(6)
where \( \mathcal{A}^\infty_{\text{L,nd}}(\Omega) \) is the quotient algebra
\[
\mathcal{A}^\infty_{\text{L,nd}}(\Omega) = C^\infty(\Omega)^\Lambda / \mathcal{I}^\infty_{\text{L,nd}}(\Omega).
\]

In view of (3) it is clear that the ideal \( \mathcal{I}^\infty_{\text{L,nd}}(\Omega) \) satisfies the \textit{off diagonal or neutrix condition}
\[
\mathcal{I}^\infty_{\text{L,nd}}(\Omega) \cap U^\Lambda(\Omega) = \{0\},
\]
where \( U^\Lambda(\Omega) = \{u(\psi) = (\psi_\lambda)_{\lambda \in \Lambda} \mid \psi_n = \psi, \lambda \in \Lambda\} \) is the diagonal in \( C^\infty(\Omega)^\Lambda \). As such, see [17, Chapter 6], the algebras \( \mathcal{A}^\infty_{\text{nd}}(\Omega) \) contains \( C^\infty(\Omega) \) as a subalgebra. That is, there exists a canonical, injective algebra homomorphism
\[
C^\infty(\Omega) \hookrightarrow \mathcal{A}^\infty_{\text{L,nd}}(\Omega)
\]
In particular, the embedding (8) is an \textit{embedding of differential algebras}. That is, for all \( p \in \mathbb{N}^n \), the diagram
\[
\begin{array}{ccc}
C^\infty(\Omega) & \xrightarrow{D^p} & C^\infty(\Omega) \\
\downarrow & & \downarrow \\
\mathcal{A}^\infty_{\text{L,nd}}(\Omega) & \xrightarrow{D^p} & \mathcal{A}^\infty_{\text{L,nd}}(\Omega)
\end{array}
\]
commutes. The embedding (8) preserves not only the algebraic structure of \( C^\infty(\Omega) \), but also its differential structure.

The neutrix condition (7) implies also the existence of an injective, linear mapping
\[
\mathcal{D}'(\Omega) \hookrightarrow \mathcal{A}^\infty_{\text{L,nd}}(\Omega).
\]
That is, the differential algebra \( \mathcal{A}^\infty_{\text{L,nd}}(\Omega) \) contains the distributions as a linear subspace, see [17, pp. 234–244] where those algebras that admit linear embeddings of distributions are characterized in terms such off diagonal conditions. However, in contradistinction with (8), the embedding (10) does not commute with partial derivatives. In other words, the differential operators on \( \mathcal{A}^\infty_{\text{L,nd}}(\Omega) \) do not, in general, coincide with distributional derivatives, when restricted to \( \mathcal{D}'(\Omega) \).

### 2.2 STF Algebras

Proceeding from the particular to the general, let us now recall the construction of the large class of differential algebras, namely, the STF algebras, first introduced in [18, 19, 20]. In this regard, let \( S \) be a collection of subsets of \( \Omega \) that satisfies the conditions
\[
\forall \Sigma \in S : \quad \Omega \setminus \Sigma \text{ is dense in } \Omega
\]
and
\[
\forall \Sigma, \Sigma' \in S : \quad \exists \Sigma'' \in S : \quad \Sigma \cup \Sigma' \subseteq \Sigma''
\]
For each singularity set \( \Sigma \in S \) we denote by \( \mathcal{J}_{L,\Sigma}(\Omega) \) the ideal in \( C^\infty(\Omega)^\Lambda \) of all \( \Lambda \)-sequences \( w = (w_\lambda)_{\lambda \in \Lambda} \) that, outside the singularity set \( \Sigma \) will satisfy the \textit{asymptotic vanishing} condition
\[
\forall x \in \Omega \setminus \Sigma : \quad \exists \lambda \in \Lambda : \quad \forall p \in \mathbb{N}^n, \mu \in \Lambda, \mu \geq \lambda : \quad D^p w_\mu(x) = 0
\]
It is easily seen that
\[\Sigma \subseteq \Sigma' \Rightarrow J_{L,S}(\Omega) \subseteq J_{L,S'}(\Omega)\]
so that it follows by (12) that
\[J_{L,S}(\Omega) = \bigcup_{\Sigma \in S} J_{L,\Sigma}(\Omega)\] (14)
is also an ideal in \(C^\infty(\Omega)^{\Lambda}\). The space-time foam algebra associated with \(S\) and \(L\) is now defined as
\[B_{L,S}(\Omega) = C^\infty(\Omega)^{\Lambda}/J_{L,S}(\Omega)\] (15)
It is clear that the ideal \(J_{L,S}(\Omega)\) satisfies the off diagonal condition
\[J_{L,S}(\Omega) \cap U^\infty_\Lambda(\Omega) = \{0\}\]
as well as the inclusion
\[D^p(J_{L,S}(\Omega)) \subseteq J_{L,S}(\Omega), \ p \in \mathbb{N}.\]
Consequently, \(B_{L,S}(\Omega)\) contains \(C^\infty(\Omega)^{\Lambda}\) as a subalgebra, and the differential operators on \(C^\infty(\Omega)^{\Lambda}\) extend to \(B_{L,S}(\Omega)\). Furthermore, as in the case of the nowhere dense algebras, \(B_{L,S}(\Omega)\) contains \(D'\) as a linear subspace.
The nowhere dense algebras \(A_{L,nd}^\infty(\Omega)\) is clearly but a particular case of the STF algebras. Indeed, if the family of singularity sets \(S\) is chosen to be the collection of all closed nowhere dense subsets of \(\Omega\), the resulting STF algebra is precisely the algebra \(A_{L,nd}^\infty(\Omega)\).
The nowhere dense algebras \(A_{L,nd}^\infty(\Omega)\) allow singularities which may occur on arbitrary closed nowhere dense sets. As such, these singularity sets are far larger than those that can be dealt with, say, though distributions. In particular, such a singularity set may have arbitrary large positive Lebesgue measure \([11]\). However, general STF algebras are able to handle singularities on even bigger sets. Indeed, due to the requirement \([11]\), any set \(\Sigma \subset \Omega\) may act as the singularity set of a generalized function, provided only that its complement \(\Omega \setminus \Sigma\) is dense in \(\Omega\). In this way, the cardinality of the singularity set of a generalized function may turn out to be greater than that of the set of its regular points.

### 2.3 Baire I Algebras

Another important particular case of the SFT construction, given in Section 2.2, is the so called Baire I algebras \(B_{L, B^{-1}}(\Omega)\). In this case, the family \(S_{B^{-1}}\) of singularity sets is chosen as
\[S_{B^{-1}} = \{\Sigma \subset \Omega : \Sigma \text{ is first Baire category}\}\]
Then the ideal \(J_{L,S_{B^{-1}}}(\Omega) = J_{L,B^{-1}}(\Omega)\) consists of those \(\Lambda\)-sequences of smooth functions \(w = (w_{\lambda})_{\lambda \in \Lambda}\) that satisfy
\[\exists \ \Sigma \subset \Omega \text{ of first Baire category} : \ \
\forall \ x \in \Omega \setminus \Sigma : \ \
\exists \ \lambda \in \Lambda : \ \
\forall \ p \in \mathbb{N}^n, \mu \in \Lambda, \mu \geq \lambda : \ \
w_{\mu}(x) = 0\]
(16)
Since each closed nowhere dense set \(\Gamma \subset \Omega\) is of first Baire category, it is clear that the inclusion
\[T_{L,nd}^\infty(\Omega) \subseteq J_{L,B^{-1}}(\Omega)\] (17)
holds. Consequently, the differential algebras \(B_{L, B^{-1}}(\Omega)\) and \(A_{L,nd}^\infty(\Omega)\) are related through the existence of a canonical, surjective algebra homomorphism
\[A_{L,nd}^\infty(\Omega) \rightarrow B_{L,B^{-1}}(\Omega)\] (18)
In view of the existence of such an algebra homomorphism, the generalized functions in \(B_{L,B^{-1}}(\Omega)\) may be considered to be more regular than those in \(A_{L,nd}^\infty(\Omega)\), see [20, Sect. 1.7] and [22].
2.4 Baire I Algebras are Nowhere Dense Algebras

As mentioned, the Baire I algebras are related to the nowhere dense algebras through the existence of a surjective algebra homomorphism \([18]\). We now proceed to show that, for a large class of index sets \(L\), this algebra homomorphism is in fact an isomorphism.

**Lemma 1** Suppose that \(X\) is a Baire space, and \(Y\) any topological space. Assume that the sequence \((w_\lambda) \subset C(X,Y)\) and the continuous function \(w : X \rightarrow Y\) satisfy

\[
\forall x \in X : \\
\exists \lambda \in \mathbb{N} : \\
\forall \mu \in \mathbb{N}, \mu \geq \lambda : \\
w_\mu(x) = w(x)
\]

Then for every non void, open subset \(A\) of \(X\), there exists a non void, relatively open subset \(U\) of the closure \(\overline{A}\) of \(A\), and \(\nu \in \mathbb{N}\) such that

\[
\forall x \in U : \\
\forall \lambda \in \mathbb{N}, \lambda \geq \nu : \\
w_\lambda(x) = w(x)
\]

**Proof.** We claim that \(\overline{A}\) is a Baire space. In this regard, for each \(\lambda \in \mathbb{N}\), let \(D_\lambda \subseteq \overline{A}\) be relatively open and dense. We may assume, without loss of generality, that \(D_\lambda \subseteq A\). Then, for each \(\lambda \in \mathbb{N}\), the set \(D_\lambda' = (X \setminus \overline{A}) \cup D_\lambda\) is open and dense in \(X\). Since \(X\) is a Baire space, it follows that the set

\[
\bigcap_{\lambda \in \mathbb{N}} D_\lambda'
\]

is dense in \(X\). Since \(A\) is a non void and open, it follows that

\[
\bigcap_{\lambda \in \mathbb{N}} D_\lambda' = \overline{A} \bigcap \left( \bigcap_{\lambda \in \mathbb{N}} D_\lambda' \right) \neq \emptyset.
\]

Thus \(\overline{A}\) is a Baire space.

For every \(\mu \in \mathbb{N}\), let us denote by \(I_\lambda\) the set

\[
A_\mu = \left\{ x \in \overline{A} \mid \forall \lambda \in \mathbb{N}, \lambda \geq \mu, w_\lambda(x) = w(x) \right\}
\]

We clearly have \(\overline{A} = \bigcup_{\mu \in \mathbb{N}} A_\mu\). Since \(w_\lambda\), with \(\lambda \in \mathbb{N}\), and \(w\) are all continuous functions, it follows that each of the sets \(A_\mu\) is a closed set in \(\overline{A}\). Since \(\overline{A}\) is a Baire space, it follows that at least one of the sets \(A_\mu\) must have nonempty interior, relative to \(\overline{A}\). This completes the proof. \(\blacksquare\)

We call the right directed set \(L = (\Lambda, \leq)\) countably co-final if there exists a countable set \(\{\lambda_i : i \in \mathbb{N}\} \subseteq \Lambda\) such that the mapping

\[
\mathbb{N} \ni i \mapsto \lambda_i \in \Lambda
\]

is an order isomorphic embedding, that is,

\[
i \leq j \iff \lambda_i \leq \lambda_j,
\]

and

\[
\forall \lambda \in \Lambda : \\
\exists i \in \mathbb{N} : \\
\lambda \leq \lambda_i
\]
**Theorem 2** Suppose that $L$ is countably co-final. Then $J_{L,B-1} (\Omega) = I_{L,\text{nd}}^{\infty} (\Omega)$.

**Proof.** We only give a proof in case $L = \mathbb{N}$. The extension to general countably co-finite index sets is obvious.

Since we already have the inclusion (17), it is sufficient to show that

$$J_{\mathbb{N},B-1} (\Omega) \subseteq I_{\mathbb{N},\text{nd}}^{\infty} (\Omega).$$

(19)

In this regard, consider any $w = (w_\lambda) \in J_{\mathbb{N},B-1} (\Omega)$. Let $\Sigma \subseteq \Omega$ be the set of first Baire category associated with $w$ through (16). Since $\Omega$ is a Baire space, and $\Sigma$ is of first Baire category, it follows that $E = \Omega \setminus \Sigma$ is also a Baire space. Consequently, we may apply Lemma 1 to the sequence $(w_n)$, with the limiting function $w$ identically 0, to obtain the following:

$$\forall \ y \in V \cap (\Omega \setminus \Sigma), \mu \in \mathbb{N}, \mu \geq \lambda : w_\mu (y) = 0$$

(20)

Since $\Omega \setminus \Sigma$ is dense in $\Omega$, by virtue of $\Sigma$ being of a set of first Baire category, it follows from (20) and the continuity of each $w_\lambda$ that

$$\forall \ y \in V : \forall \ \mu \in \mathbb{N}, \mu \geq \lambda : w_\mu (y) = 0$$

It again follows from the denseness of $\Omega \setminus \Sigma$ that the sequence $w$ satisfies (3). Therefore $w \in I_{\mathbb{N},\text{nd}}^{\infty} (\Omega)$, which verifies (19). This completes the proof. $\blacksquare$

**Corollary 3** If $L$ is countably co-final, then $B_{L,B-1} (\Omega) = A_{L,\text{nd}}^{\infty} (\Omega)$.

Theorem 2 and Corollary 3 may be interpreted as follows: Within the setting of STF algebras of generalized functions, a singularity that occurs on a set of first Baire category will inevitably collapse to a, possibly smaller, closed nowhere dense set. Thus, closed nowhere dense singularity sets are maximal within all singularity sets of first Baire category.

However, one of the main reasons for the use of STF algebras is their ability to handle singularities on large, in particular dense sets. Therefore Corollary 3 raises two important questions: Firstly, are there convenient STF algebras that can in fact handle singularities, even on dense sets? Secondly, what are the possible limitations on the size of singularity sets that the STF algebras can handle? In the following two sections we address these two issues. The former of these two questions is answered in full in Section 2.6. The latter is more subtle, and we give only a brief discussion of the issues involved.

### 2.5 Limitations on the Size of Singularity Sets

It is a rather straightforward matter to construct STF algebras that can handle singularities on large singularity sets, provided that the family of admissible singularity sets is small. In particular, one may choose the family $\mathcal{S}$ in the construction of STF algebras presented in Section 2.2 to consist of a single set $\Sigma \subseteq \Omega$ whose complement $\Omega \setminus \Sigma$ is countable and dense. In this case the singularity set of a generalized function in $B_{L,S} (\Omega)$ has larger cardinality than the set of singular points. In particular, the singularity set $\Sigma$ is uncountable, while the set of singular points $\Omega \setminus \Sigma$ is countable.

However, the algebra $B_{L,S} (\Omega)$ can only handle singularities that occur on one single set, namely, the set $\Sigma$. Hence this algebra is rather limited.

On the other hand, the use of large, and in particular infinite, families of singularity sets appears to place some limitations on the possible size of the singularity sets involved. In this regard, consider any family of singularity sets $\mathcal{S}$ that contains $\mathcal{S}_{B-1} (\Omega)$. Each singularity set $\Sigma \in \mathcal{S}$ satisfies the following smallness condition:

$$\forall \ U \subseteq \Omega \text{ open} : U \cap \Sigma \text{ is not residual in } U$$

7
Indeed, if $\Sigma \cap U$ is residual in $U$ for some $U \subseteq \Omega$, then $\Sigma' = U \setminus (U \cap \Sigma)$ is of first Baire category in $\Omega$ so that $\Sigma' \in \mathcal{S}$. Then, owing to (12), there is some $\Sigma'' \in \mathcal{S}$ such that

$$\Sigma \cup \Sigma' \subseteq \Sigma''.$$ 

However, $U \subseteq \Sigma \cup \Sigma'$, so that $\Sigma''$ does not have dense complement in $\Omega$, contrary to (11).

In this way, we come to realize that singularity sets which contain as particular cases the sets of first Baire category cannot be large, even locally, in the topological sense of being residual. However, we should note that such sets, may still be rather larger with respect to Lebesgue measure. Indeed, a closed nowhere dense set in $\mathbb{R}^n$ may have arbitrary large positive Lebesgue measure, see \[11\].

### 2.6 STF Algebras with Dense Singularities

As mentioned, a key feature of the SFT algebras is the ability of such algebras to deal with large classes of singularities of generalized functions. In particular, and in connection with so called space-time foam structures in general relativity, see \[9\], where the set of singular points in the space-time manifold is dense, there is an interest in singularities that may occur on dense sets.

The STF algebra $B_{L,B-I}(\Omega)$ may, at first, appear to provide precisely such an algebra with dense singularity. Indeed, the typical singularity set associated with a generalized function in $B_{L,B-I}(\Omega)$, which is of first Baire category, is dense in $\Omega$. However, as shown in Section 2.4, singularities that occur on sets of first Baire category inevitably collapse to closed nowhere dense sets, so that $A_{L,nd}(\Omega)$ appears to have a maximal position among those algebras that can handle at least all singularities on closed nowhere dense sets.

In this section we propose an alternative $B_{L,M_0}(\Omega)$ to the algebra $B_{L,B-I}(\Omega)$, with the following two features:

(a) Generalized functions in $B_{L,M_0}(\Omega)$ may exhibit singularities on dense sets.

(b) The algebra $B_{L,M_0}(\Omega)$ is convenient, from the point of view of existence of generalized solutions of large classes of PDEs.

**Example 4** Let $S_{L,M_0}(\Omega)$ denote the collection of all subsets of $\Omega$ with zero Lebesgue measure. Recall that sets of measure 0 may be dense in $\Omega$. Indeed, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$, but being the union of countable many singleton sets, it also has measure 0. Furthermore, sets of measure 0 may not be of first Baire category \[11\]. Hence the generalized functions in the STF algebra $B_{L,M_0}(\Omega) = C^\infty(\Omega) \setminus J_{L,M_0}(\Omega)$ can have singularities on dense sets. Furthermore, since the singularity set may not be of first Baire category, Theorem 4 does not apply, so that the singularity may not collapse to a closed nowhere dense set.

It is easy to show that the algebra $B_{L,M_0}(\Omega)$ admits a global version of the Cauchy-Kovalevskaya Theorem \[20\]. Indeed, one may construct generalized solutions of arbitrary analytic nonlinear PDEs in $A_{L,nd}(\Omega)$, which are analytic everywhere except on a closed nowhere dense set. Furthermore, the closed nowhere dense singularity set may be chosen to have zero Lebesgue measure. It is not hard to show that this generalized solution belongs to $B_{L,M_0}(\Omega)$.

### 3 Concluding Remarks

STF algebras of generalized functions were introduced as a convenient and natural setting in which one may deal with large classes of singularities. In particular, and motivated by space-time foam structures in general relativity, algebras with dense singularity sets were introduced, notably the algebra $B_{L,B-I}(\Omega)$. In this paper, we showed that $B_{L,B-I}(\Omega)$ coincides with the earlier closed nowhere dense algebra $A_{L,nd}(\Omega)$, which admit only singularities that occur on closed nowhere dense sets. From the point of view of generalized solutions of nonlinear PDEs, this is of great interest,
since it shows that singularities in the solutions of a PDE that occur on large sets, namely, dense
sets of first category, inevitable collapse to closed nowhere dense sets.

On the other hand, from the point of view of foam structures in general relativity, and other highly
singular phenomena, it necessary to deal with singularities that occur on dense sets. In this regard,
we constructed an algebra $B_{L,M_0}(\Omega)$ that can handle such singularities. Furthermore, this algebra
is easily seen to be convenient, from the point of view of solutions of large classes of PDEs.

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