UNWEIGHTED DONALDSON–THOMAS THEORY OF THE BANANA 3-FOLD WITH SECTION CLASSES

by OLIVER LEIGH†

(School of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia, Department of Mathematics, The University of British Columbia, Vancouver, British Columbia V6T 1Z2, Canada and Matematiska institutionen, Stockholms universitet, 106 91 Stockholm, Sweden)

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Abstract

We further the study of the Donaldson–Thomas theory of the banana 3-folds which were recently discovered and studied by Bryan [3]. These are smooth proper Calabi–Yau 3-folds which are fibred by Abelian surfaces such that the singular locus of a singular fibre is a non-normal toric curve known as a ‘banana configuration’. In [3], the Donaldson–Thomas partition function for the rank 3 sub-lattice generated by the banana configurations is calculated. In this article, we provide calculations with a view towards the rank 4 sub-lattice generated by a section and the banana configurations. We relate the findings to the Pandharipande–Thomas theory for a rational elliptic surface and present new Gopakumar–Vafa invariants for the banana 3-fold.

1. Introduction

1.1. Donaldson–Thomas partition functions

Donaldson–Thomas theory provides a virtual count of curves on a 3-fold. It gives us valuable information about the structure of the 3-fold and has strong links to high-energy physics.

For a non-singular Calabi–Yau 3-fold $Y$ over $\mathbb{C}$, we let

$$\text{Hilb}^{\beta,n}(Y) = \left\{ Z \subset Y \mid [Z] = \beta \in H_2(Y), n = \chi(\mathcal{O}_Z) \right\}$$

be the Hilbert scheme of one-dimensional proper subschemes with fixed homology class and holomorphic Euler characteristic. We can define the $(\beta,n)$ Donaldson–Thomas invariant of $Y$ by

$$\text{DT}_{\beta,n}(Y) = 1 \cap [\text{Hilb}^{\beta,n}(Y)]^{\text{vir}}.$$ 

Behrend proved the surprising result in [1] that the Donaldson–Thomas invariants are actually weighted Euler characteristics of the Hilbert scheme:

$$\text{DT}_{\beta,n}(Y) = e(\text{Hilb}^{\beta,n}(Y), v) := \sum_{k \in \mathbb{Z}} k \cdot e(v^{-1}(k)).$$

†Corresponding author: E-mail: oleigh@math.su.se

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Here \( v : \text{Hilb}^{n,n}(Y) \to \mathbb{Z} \) is a constructible function called the Behrend function and its values depend formally locally on the scheme structure of \( \text{Hilb}^{n,n}(Y) \) [8]. We also define the unweighted Donaldson–Thomas invariants to be

\[
\hat{\text{DT}}_{\beta,n}(Y) = e(\text{Hilb}^{n,n}(Y)).
\]

These are often closely related to Donaldson–Thomas invariants and their calculation provides insight to the structure of the 3-fold. Moreover, many important properties of Donaldson–Thomas invariants such as the PT/DT correspondence and the flop formula also hold for the unweighted case [19, 20].

The depth of Donaldson–Thomas theory is often not clear until one assembles the invariants into a partition function. Let \( \{C_1, \ldots, C_N\} \) be a basis for \( H_2(Y, \mathbb{Z}) \), chosen so that if \( \beta \in H_2(Y, \mathbb{Z}) \) is effective then \( \beta = d_1C_1 + \cdots + d_NC_N \) with each \( d_i \geq 0 \). The Donaldson–Thomas partition function of \( Y \) is

\[
Z(Y) := \sum_{\beta \in H_2(Y, \mathbb{Z})} \sum_{n \in \mathbb{Z}} \text{DT}_{\beta,n}(Y) Q^\beta p^n := \sum_{d_1, \ldots, d_N \geq 0} \sum_{n \in \mathbb{Z}} \text{DT}_{(\sum_i d_iC_i),n}(Y) Q_1^{d_1} \cdots Q_N^{d_N} p^n.
\]

We also define the analogous partition function \( \hat{Z} \) for the unweighted Donaldson–Thomas invariants.

**Remark 1.1.1** This choice of variable is not necessarily the most canonical as shown in [3] where the variable \( p \) is substituted for \( -p \). However, in this article, we will be focusing on the unweighted Donaldson–Thomas invariants where this choice makes the most sense.

The Donaldson–Thomas partition function is very hard to compute. Indeed, for proper Calabi–Yau 3-folds, the only known examples of a complete calculation are in computationally trivial cases. However, when we restrict our attention to subsets of \( H_2(Y, \mathbb{Z}) \), there are many remarkable results. Two interesting cases which are related to the computations in this article are the Schoen (Calabi–Yau) 3-fold of [18] and the banana (Calabi–Yau) 3-fold of [3].

We will employ computational techniques developed in [5] for studying Donaldson–Thomas theory of local elliptic surfaces.

### 1.2. Donaldson–Thomas theory of banana 3-folds

The banana 3-fold is of primary interest to us and is defined as follows. Let \( \pi : S \to \mathbb{P}^1 \) be a generic rational elliptic surface with a section \( \zeta : \mathbb{P}^1 \to S \). We will take \( S \) to be \( \mathbb{P}^2 \) blown up at 9 points and \( \pi \) given by a generic pencil of cubics. This gives rise to 9 natural choices for \( \zeta \) and we choose one. The associated banana 3-fold is the blow-up

\[
X := \text{Bl}_\Delta (S \times_{\mathbb{P}^1} S),
\]

where \( \Delta \) is the diagonal divisor in \( S \times_{\mathbb{P}^1} S \). The surface \( S \) is smooth, but the morphism \( \pi : S \to \mathbb{P}^1 \) is not. It is singular at 12 points of \( S \) which are the nodes of the nodal fibres of \( \pi \). This gives rise to
12 conifold singularities of $S \times \mathbb{P}^1 S$ that all lie on the divisor $\Delta$. It also makes $X$ a conifold resolution of $S \times \mathbb{P}^1 S$. $X$ is a non-singular simply connected proper Calabi–Yau 3-fold [3, Proposition 28].

There is a natural projection $pr : X \to \mathbb{P}^1$ and a unique section $\sigma : \mathbb{P}^1 \to X$ arising canonically from $\zeta$. The generic fibres of the map $pr : X \to \mathbb{P}^1$ are Abelian surfaces of the form $E \times E$, where $E = \pi^{-1}(x)$ is the elliptic curve given by the fibre of a point $x \in \mathbb{P}^1$. The projection map $pr$ also has 12 singular fibres which are non-normal toric surfaces. They are each a compactification of $\mathbb{C}^* \times \mathbb{C}^*$ by a reducible singular curve called a banana configuration [c.f. Definition 1.2.1]. Furthermore, the normalization of a singular $pr^{-1}(x)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at 2 points [3, Proposition 24].

The rational elliptic surface $\pi : S \to \mathbb{P}^1$, together with the section $\zeta : \mathbb{P}^1 \to S$, is a Weierstrass fibration. This means that there is a consistent way of choosing Weierstrass coordinates for each fibre (see [11, III.1.4]). Thus, we have an involution $\iota : S \to S$ which gives rise to a canonical group law on each fibre, where the identity is defined by $\zeta$ and the inverse defined by $\iota$.

We will fix four natural divisors of $X$ for the remainder of this article. The first two arise from considering the natural projections $pr_i : X \to S$ and the sections $S_i : S \to X$ arising from $\zeta$. We denote the corresponding divisors by $S_1$ and $S_2$.

The third and fourth natural divisors of $X$ arise by considering the diagonal $\Delta$ and anti-diagonal $\Delta_{op}$ (the graph of $\iota$) of $S \times \mathbb{P}^1 S$. The anti-diagonal intersects the diagonal in a curve on $\Delta_{op}$, so it is unaffected by the blow up. We denote the anti-diagonal divisor in $X$ by $S_{op}$ and the proper transform
of the diagonal by $S_\Delta$. The latter is a rational elliptic surface blown up at the 12 nodal points of the fibres.

**Definition 1.2.1** A banana configuration is a union of three curves $C_1 \cup C_2 \cup C_3$, where $C_i \cong \mathbb{P}^1$ with $N_{C_i/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{z_1, z_2\}$ where $z_1, z_2 \in X$ are distinct points. Also, there exist formal neighbourhoods of $z_1$ and $z_2$ such that the curves $C_i$ become the coordinate axes in those coordinates. We label these curves by their intersection with the natural surfaces in $X$. That is $C_1$ is the unique banana curve that intersects $S_1$ at one point. Similarly, $C_2$ intersects $S_2$ and $C_3$ intersects $S_{op}$.

The banana 3-fold contains 12 copies of the banana configuration. We label the individual banana curves by $C^{(j)}_i$ (and simply $C_i$ when there is no confusion or distinction to be made). We have that $C^{(j_1)}_i \sim C^{(j_2)}_i$ in $H_2(X, \mathbb{Z})$ for each choice of $i, j_1, j_2$. The banana curves $C_1, C_2$ and $C_3$ generate a sub-lattice $\Gamma_0 \subset H_2(X, \mathbb{Z})$ and we can consider the partition function restricted to these classes

$$Z_{\Gamma_0} := \sum_{\beta \in \Gamma_0} \sum_{n \in \mathbb{Z}} \text{DT}_{\beta, n}(X) Q^n p^n.$$  

In [3, Theorem 4], this rank three partition function is computed to be

$$Z_{\Gamma_0} = \prod_{d_1, d_2, d_3 \geq 0} \prod_k (1 - Q_1^{d_1} Q_2^{d_2} Q_3^{d_3} (-p)^k)^{-12c(\|d\|, k)},$$

where $d = (d_1, d_2, d_3)$ and the second product is over $k \in \mathbb{Z}$ unless $d = (0, 0, 0)$ in which case $k > 0$. (Note the change in variables from [3].) The powers $c(\|d\|, k)$ are defined by

$$\sum_{a=-1}^{\infty} \sum_{k \in \mathbb{Z}} c(a, k) Q^a y^k := \frac{\sum_{k \in \mathbb{Z}} Q^{k^2} (-y)^k}{\left( \sum_{k \in \mathbb{Z} + \frac{1}{2}} Q^{2k^2} (-y)^k \right)^2} = \frac{\vartheta_4(2\tau, z)}{\vartheta_1(4\tau, z)^2}$$

such that $\|d\| := 2d_1d_2 + 2d_2d_3 + 2d_3d_1 - d_1^2 - d_2^2 - d_3^2$, while $\vartheta_1$ and $\vartheta_4$ are Jacobi theta functions with change of variables $Q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$. 

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**Figure 3.** On the left is a depiction of the banana configuration. On the right is the normalization of the singular fibre $F_{\text{ban}} = \text{pr}^{-1}(x)$ with the restrictions of the surfaces $S_1$, $S_2$ and $S_{op}$. 

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\[ \text{Downloaded from https://academic.oup.com/qjmath/advance-article-abstract/doi/10.1093/qmathj/haaa007/5851528 by guest on 09 June 2020} \]
Remark 1.2.2 The calculation of (2) uses a motivic method where the values of the Behrend function are explicitly calculated at the contributing points \([3, Proposition 23]\). By removing these weights, we can calculate the unweighted partition function \(\hat{Z}_{\Gamma_0}\) directly. In this case, removing the weights corresponds to the change of variables \(Q_i \mapsto -Q_i\) and \(p \mapsto -p\) in the Donaldson–Thomas partition function.

We can include the class of the section \(\sigma\) to generate a larger sub-lattice \(\Gamma \subset H_2(X, \mathbb{Z})\). The partition function of this sub-lattice is currently unknown. The purpose of this article is to make progress towards understanding this partition function. We will be calculating the unweighted Donaldson–Thomas theory in the classes

\[ \beta = \sigma + (0, d_2, d_3) := \sigma + 0C_1 + d_2 C_2 + d_3 C_3, \]

by computing the following partition function

\[ \hat{Z}_{\sigma + (0, \bullet, \bullet)} := \sum_{d_2, d_3 \geq 0} \sum_{n \in \mathbb{Z}} \delta_{\beta, n}(Y) Q_2^{d_2} Q_3^{d_3} p^n, \]

which we give in terms of the MacMahon functions \(M(p, Q) = \prod_{m > 0} (1 - p^m Q)^{-m}\) and their simpler version \(M(p) = M(p, 1)\).

Theorem A. The above unweighted Donaldson–Thomas functions are

1. \(\hat{Z}_{\sigma + (0, \bullet, \bullet)}\) is

\[ \hat{Z}_{\sigma + (0, \bullet, \bullet)} = \left(1 - p\right)^2 \prod_{m > 0} \frac{1}{(1 - Q_2^m Q_3^m)^8 (1 - p Q_2^m Q_3^m)^2 (1 - p^{-1} Q_2^m Q_3^m)^2}, \]

where \(\hat{Z}_{(0, \bullet, \bullet)}\) is the \(Q_1^0\) part of the unweighted version of the \(\Gamma_0\) partition function (2) and is given by

\[ M(p)^{24} \prod_{d > 0} \frac{M(p, Q_2^d Q_3^d)^{24}}{(1 - Q_2^d Q_3^d)^{12} M(p, -Q_2^{d-1} Q_3^{d-1})^{12} M(p, -Q_2^d Q_3^d)^{12}}. \]

In the following corollary, the connected unweighted Pandharipande–Thomas version of the above formula is identified as the connected version of the Pandharipande–Thomas theory for a rational elliptic surface \([5, Corollary 2]\).

Corollary B. The connected unweighted Pandharipande–Thomas partition function is

\[ \hat{Z}^{\text{PT,Con}}_{\sigma + (0, \bullet, \bullet)} := \log \left( \frac{\hat{Z}_{\sigma + (0, \bullet, \bullet)}}{\hat{Z}_{(0, \bullet, \bullet)} | Q_i = 0} \right) \]

\[ = \frac{-p}{(1 - p)^2} \prod_{m > 0} \frac{1}{(1 - Q_2^m Q_3^m)^8 (1 - p Q_2^m Q_3^m)^2 (1 - p^{-1} Q_2^m Q_3^m)^2}. \]
We will also be computing the unweighted Donaldson–Thomas theory in the classes

\[ \beta = b \sigma + (0, 0, d_3), \quad \beta = b \sigma + (0, 1, d_3) \quad \text{and} \quad \beta = b \sigma + (1, 1, d_3) \]

and the permutations involving \( C_1 \) and \( C_2 \). So for \( i, j \in \{0, 1\} \) we define

\[ \hat{Z}_{\bullet \sigma + (i, j, \bullet)} := \sum_{b, d_3 \geq 0} \sum_{n \in \mathbb{Z}} \hat{\text{DT}}_{\beta, n}(Y) Q_\sigma^b Q_3^{d_3} p^n. \]

The formulas will be given in terms of the functions which are defined for \( g \in \mathbb{Z} \):

\[ \psi_g = \psi_g(p) := \left( p^{\frac{1}{2}} - p^{-\frac{1}{2}} \right)^{2g-2} = \left( \frac{p}{(1-p)^2} \right)^{1-g}. \]

**Theorem C.** The above unweighted Donaldson–Thomas functions are

1. \( \hat{Z}_{\bullet \sigma + (0, 0, \bullet)} \) is

\[ M(p)^{24} \prod_{m \geq 0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m}. \]

2. \( \hat{Z}_{\bullet \sigma + (0, 1, \bullet)} = \hat{Z}_{\bullet \sigma + (1, 0, \bullet)} \) is

\[ \hat{Z}_{\bullet \sigma + (0, 0, \bullet)} \cdot \left( (12 \psi_0 + Q_3(24 \psi_0 + 12 \psi_1) + Q_3^2(12 \psi_0)) + Q_\sigma Q_3(12 \psi_0 + 2 \psi_1) \right). \]

3. \( \hat{Z}_{\bullet \sigma + (1, 1, \bullet)} \) is

\[ \hat{Z}_{\bullet \sigma + (0, 0, \bullet)} \cdot \left( (144 \psi_{-1} + 24 \psi_0 + 12 \psi_1) + Q_3(576 \psi_{-1} + 384 \psi_0 + 72 \psi_1 + 12 \psi_2) \ight. \\
+ Q_3^2(864 \psi_{-1} + 720 \psi_0 + 264 \psi_1 + 24 \psi_2) \\
+ Q_3^3(576 \psi_{-1} + 384 \psi_0 + 72 \psi_1 + 12 \psi_2) + Q_3^4(144 \psi_{-1} + 24 \psi_0 + 12 \psi_1) \bigg) \\
+ Q_\sigma \left( (12 \psi_0 + 2 \psi_1) + Q_3(288 \psi_{-1} + 96 \psi_0 + 44 \psi_1) \\
+ Q_3^2(576 \psi_{-1} + 600 \psi_0 + 156 \psi_1 + 24 \psi_2) \\
+ Q_3^3(288 \psi_{-1} + 96 \psi_0 + 44 \psi_1) + Q_3^4(12 \psi_0 + 2 \psi_1) \right) \\
+ Q_\sigma^2 Q_3^2(144 \psi_{-1} + 48 \psi_0 + 4) \bigg). \]

The connected unweighted Pandharipande–Thomas versions of the formulae in Theorem C contain the same information, but are given in a much more compact form. In fact, we can present the
Table 1. The non-zero $\hat{n}_\beta^g$ for $\beta = \sigma + (i,j,d_3)$ where $i,j \in \{0,1\}$ and $d_3 \geq 0$.

| $(d_1, d_2, d_3)$ | (0, 0, 0) | (0, 1, 1) | (1, 0, 1) | (1, 1, 0) | (1, 1, 1) | (1, 1, 2) | (1, 1, 3) | (1, 1, 4) |
|-------------------|----------|----------|----------|----------|----------|----------|----------|----------|
| $g = 0$           | 1        | 12       | 12       | 48       | 216      | 48       | 12       |          |
| $g = 1$           | 0        | 2        | 2        | 2        | 44       | 108      | 44       | 2        |
| $g = 2$           | 0        | 0        | 0        | 0        | 24       | 0        | 0        | 0        |

same information in an even more compact form using the unweighted Gopakumar–Vafa invariants $\hat{n}_\beta^g$ via the expansion

$$Z_{\text{PT,Con}}^g(X) := \sum_{\beta \in \Gamma \setminus \{0\}} \sum_{g \geq 0} \sum_{m > 0} \hat{n}_\beta^g \psi_g (p^m) (-Q)^{m\beta} := \sum_{b, d_1, d_2, d_3 \geq 0} \sum_{g \geq 0} \sum_{m > 0} \hat{n}_{(b,d_1,d_2,d_3)}^g \psi_g (p^m) (-Q_\sigma)^{mb} (-Q_1)^{md_1} (-Q_2)^{md_2} (-Q_3)^{md_3}. $$

As noted before, these express the same information as the previous generating functions. For $\beta = (d_1, d_2, d_3)$, these invariants are given in [3, §A.5]. We present the new invariants for $\beta = b\sigma + (i,j,d_3)$ where $b > 0$.

**Corollary D.** Let $i,j \in \{0,1\}$, $b > 0$ and $\beta = b\sigma + (i,j,d_3)$. The unweighted Gopakumar–Vafa invariants $\hat{n}_\beta^g$ are given by

1. If $b > 1$, then we have $\hat{n}_\beta^g = 0$.
2. If $b = 1$, then the non-zero invariants are given above in Table 1.

**Remark 1.2.3** We note that the values given only depend on the quadratic form $\|d\| := 2d_1d_2 + 2d_1d_3 + 2d_2d_3 - d_1^2 - d_2^2 - d_3^2$ appearing in the rank 3 Donaldson–Thomas partition function of [3, Theorem 4]. However, there is no immediate geometric explanation for this fact.

Corollaries B and D will be proved in Section 6.1.

1.3. **Notation**

The main notations for this article have been defined above in Section 1.2. In particular, $X$ will always denote the banana 3-fold as defined in equation (1).

1.4. **Future**

The calculation here is for the unweighted Donaldson–Thomas partition function. However, the method of [5] also provides a route (up to a conjecture [5, Conjecture 21]) of computing the
Donaldson–Thomas partition function. The following are needed in order to convert the given calculation:

1. A proof showing the invariance of the Behrend function under the \((\mathbb{C}^*)^2\)-action used on the strata.

2. A computation of the dimensions of the Zariski tangent spaces for the various strata.

A comparison of the unweighted and weighted partition functions of the rank 3 lattice of [3] reveals the likely differences:

In the variables chosen in this article, one can pass from the unweighted to the weighted partition functions by the change of variables \(Q_i \mapsto -Q_i\) and \(p \mapsto -p\).

Moreover, the conifold transition formula reveals further insight by a comparison with the Donaldson–Thomas partition function of the Schoen 3-fold with a single section and all fibre classes. The Donaldson–Thomas theory of the Schoen 3-fold with a section class was shown in [12] (via the reduced theory of the product of a K3 surface with an elliptic curve) to be given by the weight 10 Igusa cusp form.

As we mentioned previously the Donaldson–Thomas partition function is very hard to compute. So much so that for proper Calabi–Yau 3-folds, the only known complete examples are computationally trivial cases. This is even true conjecturally and even a conjecture for the rank 4 partition function is highly desirable. The work here shows underlying structures that a conjectured partition function must have.

2. Overview of the computation

2.1. Overview of the method of calculation

We will closely follow the method of [5] developed for studying the Donaldson–Thomas theory of local elliptic surfaces. However, due to some differences in geometry, a more subtle approach is required in some areas. In particular, the local elliptic surfaces have a global action which reduces the calculation to considering only the so-called partition thickened curves.

Our method is based around the following continuous map:

\[
\text{Cyc} : \text{Hilb}^{\beta,n}(X) \to \text{Chow}^n(X),
\]

which takes a one-dimensional subscheme to its 1-cycle. Here \(\text{Chow}^n(X)\) is the Chow variety parametrizing 1-cycles in the class \(\beta \in H_2(X, \mathbb{Z})\) (as defined in [9, Theorem I 3.20]). The fibres of this map are of particular importance and we denote them by \(\text{Hilb}^{\beta,\text{Cyc}}(X, q)\) where \(q \in \text{Chow}_1(X)\).

Each \(\text{Hilb}^{\beta,\text{Cyc}}(X, q)\) is a closed subset of \(\text{Hilb}^{n,\beta}(X)\) and hence has the natural structure of a reduced subscheme of \(\text{Hilb}^{n,\beta}(X)\) (see [17, Tag 01J3] for more details).

**Remark 2.1.1** No such morphism exists in the algebraic category. In fact, we note from [9, Theorem I 6.3] that there is only a morphism from the semi-normalization \(\text{Hilb}^{\beta,n}(X)^{SN} \to \text{Chow}^n(X)\). However, the semi-normalization \(\text{Hilb}^{\beta,n}(X)^{SN}\) is homeomorphic to \(\text{Hilb}^{\beta,n}(X)\), which gives rise to the above continuous map.
Remark 2.1.2 While no Hilbert–Chow morphism exists for the Chow variety, there is a promising theory of relative cycles developed in [16, Paper IV] which allows for the construction of a morphism of functors. These results were used in [15] to study the Donaldson–Thomas theory of smooth curves.

Broadly, we will be calculating the Euler characteristics $e(\text{Hilb}^n(X))$ using the following method:

1. Push forward the calculation to a Euler characteristic on Chow(r(X), weighted by the constructible function $(\text{Cyc}_r)(q) := e(\text{Hilb}^n_{\text{Cyc}}(X, q))$. This is further described in Sections 2.2 and 2.3.

2. Analyse the image of Cyc and decompose it into combinations of symmetric products where the strata are based on the types of subscheme in the fibres Hilb^n_{\text{Cyc}}(X, q). This is done in Section 3.

3. Compute the Euler characteristic of the fibres $e(\text{Hilb}^n_{\text{Cyc}}(X, q))$ and show that they form a constructible function on the combinations of symmetric products. This is done in Section 5.

4. Use the following lemma to give the Euler characteristic partition function.

Lemma 2.1.3 [5, Lemma 32] Let $Y$ be finite type over $\mathbb{C}$ and let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}((p))$ be any function with $g(0) = 1$. Let $G : \text{Sym}^d(Y) \to \mathbb{Z}((p))$ be the constructible function defined by

$$G(ax) = \prod_i g(a_i),$$

where $ax = \sum_i a_i x_i \in \text{Sym}^d(Y)$ and $x_i \in Y$ are distinct points. Then

$$\sum_{d=0}^{\infty} e(\text{Sym}^d(Y), G)q^d = \left(\sum_{a=0}^{\infty} g(a)q^a\right)^{\text{e}(Y)},$$

where the G-weighted Euler characteristic $e(-, G)$ is defined in Equation (3).

To compute the Euler characteristics of the fibres $(\text{Cyc}_r)(q) := e(\text{Hilb}^n_{\text{Cyc}}(X, q))$, we use the following method made rigorous in Section 4:

1. Denote the open subset consisting entirely of Cohen–Macaulay subschemes by $\text{Hilb}^n_{\text{CM}}(X, q) \subset \text{Hilb}^n_{\text{Cyc}}(X, q)$, and define the notation $\text{Hilb}^{\bullet}_{\text{CM}}(X, q) := \bigsqcup_{m \in \mathbb{Z}} \text{Hilb}^m_{\text{CM}}(X, q)$.

2. Consider the constructible map which takes a subscheme $Z$ to the maximal Cohen–Macaulay subscheme $Z_{\text{CM}} \subset Z$ and denote the constructible map by $\kappa_n : \text{Hilb}^n_{\text{Cyc}}(X, q) \to \text{Hilb}^{\bullet}_{\text{CM}}(X, q)$.

3. Note the equality of the Euler characteristic $e(\text{Hilb}^n_{\text{Cyc}}(X, q))$ and that of the weighted Euler characteristic $e(\text{Hilb}^{\bullet}_{\text{CM}}(X, q), (\kappa_n)_*)$ where $(\kappa_n)_* 1$ is the constructible function $((\kappa_n)_* 1)(p) := e(\kappa_n^{-1}(p))$.

4. Define a $(\mathbb{C}^*)^2$-action on $\text{Hilb}^{\bullet}_{\text{CM}}(X, q)$ and show that $\kappa_1(p) = \kappa_*(\alpha \cdot p)$ meaning $e(\text{Hilb}^n_{\text{Cyc}}(X, q)) = e(\text{Hilb}^{\bullet}_{\text{CM}}(X, q)(\mathbb{C}^*)^2, \kappa_1)$. This technique is discussed in Section 4.2.
5. Identify the \((\mathbb{C}^*)^2\)-fixed points \(\text{Hilb}^\bullet_{\text{CM}}(X, q)(\mathbb{C}^*)^2\) as a discrete subset containing partition thickened curves. These neighbourhoods and this action are given explicitly in Section 4.4.

6. Calculate the Euler characteristics \(e(\text{Hilb}^\bullet_{\text{CM}}(X, q)(\mathbb{C}^*)^2, \kappa \ast 1)\) using the Quot scheme decomposition and topological vertex method of [5]. The concept of this is depicted in Fig. 5 and described below. Further technical details are given in Section 4.5.

The Euler characteristic calculation of \(e(\text{Hilb}^\bullet_{\text{CM}}(X, q)(\mathbb{C}^*)^2, \kappa \ast 1)\) for Theorems A and C follow similar methods, but have different decompositions. The calculations are completed by considering the different types of topological vertex that occur for each fixed point in \(\text{Hilb}^m_{\text{CM}}(X, q)(\mathbb{C}^*)^2\) for \(m \in \mathbb{Z}\).

Since the fixed locus \(\text{Hilb}^m_{\text{CM}}(X, q)(\mathbb{C}^*)^2\) will be a discrete set, we can consider the individual subschemes \(C \in \text{Hilb}^m_{\text{CM}}(X, q)(\mathbb{C}^*)^2\) and their contribution to the Euler characteristic \(e(\text{Hilb}^\bullet_{\text{CM}}(X, q)(\mathbb{C}^*)^2, \kappa \ast 1)\). To compute the contribution from \(C\), we must decompose \(X\) as follows:

1. Take the complement \(W = X \setminus C\).
2. Consider, \(C^\circ\), the set of singularities of the underlying reduced curve.
3. Define \(C^\circ = C^{\text{red}} \setminus C^\circ\) to be its complement.
The curve \( C \) will be partition thickened. So each formal neighbourhood of a point \( x \in C^o \) will give rise to a 3D partition asymptotic to a collection of three partitions (depicted on the right-hand side of Fig. 5). Similarly, points on \( C^o \) and \( W \) will give rise to 3D partitions asymptotic to collections of three partitions. However, for \( C^o \) only one of the three partitions will be non-empty and for \( W \) all three partitions will be empty (depicted respectively on the bottom-left and top-left parts of Figure 5). Using techniques from Section 4.5 the Euler characteristics can then be determined.

This calculation for Theorem A is finalized in Section 5.1. Generalities for the proof of Theorem C are given in Section 5.2 and the individual calculations are given in Sections 5.3, 5.4 and 5.5.

2.2. Review of Euler characteristic

We begin by recalling some facts about the (topological) Euler characteristic. For a scheme \( Y \) over \( \mathbb{C} \), we denote by \( e(Y) \) the topological Euler characteristic in the complex analytic topology on \( Y \). This is independent of any non-reduced structure of \( Y \), is additive under decompositions of \( Y \) into open sets and their complements, and is multiplicative on Cartesian products. In this way, we see that the Euler characteristic defines a ring homomorphism from the Grothendieck ring of varieties to the integers

\[
e : K_0(\text{Var}_\mathbb{C}) \longrightarrow \mathbb{Z}.
\]

If \( Y \) has a \( \mathbb{C}^* \)-action with fixed locus \( Y^{\mathbb{C}^*} \subset Y \), then the Euler characteristic also has the property \( e(Y^{\mathbb{C}^*}) = e(Y) \) [2, Corollary 2].
The interaction of Euler characteristic with constructible functions plays a key role in this article. Recall that a function \( \mu : Y \to \mathbb{Z}(p) \) is constructible if \( \mu(Y) \) is finite and \( \mu^{-1}(c) \) is the union of finitely many locally closed sets for all non-zero \( c \in \mu(Y) \). The \( \mu \)-weight Euler characteristic is a ring homomorphism

\[
e(-, \mu) : K_0(\text{Var}_\mathbb{C}) \to \mathbb{Z}(p)
\]

defined by \( e(Y, \mu) = \sum_{k \in \mathbb{Z}(p)} k \cdot e(\mu^{-1}(k)) \). The constant function 1 and the Behrend function \( \nu \) are two canonical examples of constructible functions with images in \( \mathbb{Z} \subset \mathbb{Z}(p) \). Moreover, the usual Euler characteristic is \( e(Y) = e(Y, 1) \) where 1 is the constant function.

For a scheme \( Z \) over \( \mathbb{C} \), a constructible map \( f : Y \to Z \) is a finite collection of continuous functions \( f_i : Y_i \to Z_i \) where \( Y = \bigsqcup_i Y_i \) is a decomposition into locally closed subsets and \( Z_i \subseteq Z \). A constructible homeomorphism is a constructible map such that each \( f_i \) is a homeomorphism and \( Z = \bigsqcup_i Z_i \) is a decomposition into locally closed subsets. When \( f : Y \to Z \) is a constructible map, we define the constructible function \( f_* \mu : Z \to \mathbb{Z}(p) \) by

\[
(f_* \mu)(x) := e(f^{-1}(x), \mu).
\]

This has the important property \( e(Z, f_* \mu) = e(Y, \mu) \). If \( \omega : Z \to \mathbb{Z}(p) \) is another constructible function, then \( \mu \cdot \omega \) is a constructible function on \( Y \times Z \) and \( e(Y \times Z, \mu \cdot \omega) = e(Y, \mu) \cdot e(Z, \omega) \).

It will be useful to consider the rings of formal power series in \( Q_i \) and formal Laurent series in \( p \) with coefficients in \( K_0(\text{Var}_\mathbb{C}) \). An element \( P \in K_0(\text{Var}_\mathbb{C})[Q_i][p]((p)) \) is an indexed disjoint union of varieties where the indexing is given by monomials. A constructible function \( \mu : \sum_{d_i} \sum_n Q_i^{d_i} p^n Y_{d_i,n} \to \mathbb{Z}(p) \) is an indexed collection of constructible functions \( \mu_{d_i,n} : Y_{d_i,n} \to \mathbb{Z}(p) \). Moreover, we extend Euler characteristic to a ring homomorphism

\[
e(-, \mu) : K_0(\text{Var}_\mathbb{C})[Q_i][p]((p)) \to \mathbb{Z}[Q_i][p]((p))
\]

preserving the indexing \( e(\sum_{d_i} \sum_n Q_i^{d_i} p^n Y_{d_i,n}, \mu) := \sum_{d_i} \sum_n Q_i^{d_i} p^n e(Y_{d_i,n}, \mu_{d_i,n}) \).

Lastly, we extend the definition of constructible map to formal series of varieties in two different ways. First, a constructible map \( f : \sum_n p^n Y_n \to Z \) is an indexed collection of constructible maps \( f_n : Y_n \to Z \), and secondly, a constructible map \( g : \sum_{d_i} \sum_n p^n Y_{d_i,n} \to \sum_{d_i} Z_{d_i} \) is an indexed collection of constructible maps \( g_{d_i} : \sum_n Y_{d_i,n} \to Z_{d_i} \). We can now also define the push-forwards of constructible functions as before.

2.3. Pushing forward to the Chow variety

Recall that the Chow variety \( \text{Chow}^\beta(X) \) is a space parametrizing the one-dimensional cycles of \( X \) in the class \( \beta \in H_2(X, \mathbb{Z}) \). We then have a constructible map

\[
\rho_\beta : \sum_n p^n \text{Hilb}^{\beta,n}(X) \to \text{Chow}^\beta(X).
\]

The strategy for calculating the partition functions is to analyse \( \text{Chow}^\beta(X) \) and the fibres of the map \( \rho_\beta \). These will often involve the symmetric product and where possible we will apply Lemma 2.1.3.
It will be convenient to employ the following \(\bullet\)-notations for the Hilbert schemes

\[
\text{Hilb}^{\sigma+(0,\bullet,\bullet)}(X) := \sum_{d_2,d_3 \geq 0} \sum_{n \in \mathbb{Z}} Q_{d_2} Q_{d_3}^2 Q_{d_3}^3 p^n \text{Hilb}^{\sigma+(0,d_2,d_3),n}(X)
\]

\[
\text{Hilb}^{\bullet+(i,j,\bullet)}(X) := \sum_{b,d_3 \geq 0} \sum_{n \in \mathbb{Z}} Q_{d_3} b Q_{i}^j Q_{d_3}^3 p^n \text{Hilb}^{b+(i,j,d_3),n}(X)
\]

and for the Chow varieties

\[
\text{Chow}^{\sigma+(0,\bullet,\bullet)}(X) := \sum_{d_2,d_3 \geq 0} \sum_{n \in \mathbb{Z}} Q_{d_2} Q_{d_3}^2 Q_{d_3}^3 \text{Chow}^{\sigma+(0,d_2,d_3),n}(X)
\]

\[
\text{Chow}^{\bullet+(i,j,\bullet)}(X) := \sum_{b,d_3 \geq 0} \sum_{n \in \mathbb{Z}} Q_{d_3} b Q_{i}^j Q_{d_3}^3 \text{Chow}^{b+(i,j,d_3),n}(X),
\]

where we have viewed the Hilbert scheme and Chow variety as elements in the Grothendieck ring of varieties and \(i, j \in \{0, 1\}\). The notation \(q \in \text{Chow}^{\sigma+(0,\bullet,\bullet)}(X)\) and \(q \in \text{Chow}^{\bullet+(i,j,\bullet)}(X)\) will denote \(q \in \text{Chow}^{\beta}(X)\) for some \(\beta \in H_2(X, \mathbb{Z})\). This is what we will mean by the ‘points’ of \(\text{Chow}^{\sigma+(0,\bullet,\bullet)}(X)\) and \(\text{Chow}^{\bullet+(i,j,\bullet)}(X)\). \(q\) will often be given without any associated monomial since that is usually implicitly understood. The \(\bullet\)-notation is extended to symmetric products by

\[
\text{Sym}^\bullet_Q(Y) := \sum_{n \in \mathbb{Z}_{>0}} Q^n \text{Sym}^n(Y),
\]

and we use the following notation for elements of the symmetric product

\[
ay := \sum_i a_i y_i \in \text{Sym}^n(Y),
\]

where \(y_i\) are distinct points on \(Y\) and \(a_i \in \mathbb{Z}_{>0}\). We also denote a tuple of partitions \(\alpha\) of a tuple of non-negative integers \(a\) by \(\alpha \vdash a\). As was the case with the Chow variety, we think of \(ay\) as being a point in \(\text{Sym}^\bullet_Q(Y)\).

Using the \(\bullet\)-notation for the maps \(\rho_\beta\), we create the following constructible maps:

\[
\rho_\bullet : \text{Hilb}^{\sigma+(0,\bullet,\bullet)}(X) \rightarrow \text{Chow}^{\sigma+(0,\bullet,\bullet)}(X)
\]

\[
\eta_\bullet^{ij} : \text{Hilb}^{\bullet+(i,j,\bullet)}(X) \rightarrow \text{Chow}^{\bullet+(i,j,\bullet)}(X)
\]

and we also use the notation \(\eta_\bullet = \eta_\bullet^{00} + \eta_\bullet^{01} + \eta_\bullet^{11}\). The fibres of these maps will be formal sums of subsets of the Hilbert schemes parametrizing one-dimensional subschemes with a fixed 1-cycle. Specifically, let \(C \subset X\) be a one-dimensional subscheme in the class \(\beta \in H_2(X)\) with 1-cycle \(\text{Cyc}(C)\). Define \(\text{Hilb}_{\text{Cyc}}^n(X, \text{Cyc}(C)) \subset \text{Hilb}^{\beta,n}(X)\) to be the closed subset

\[
\text{Hilb}_{\text{Cyc}}^{\beta}(X, \text{Cyc}(C)) = \{ Z \in \text{Hilb}^{\beta,n}(X) \mid \text{Cyc}(Z) = \text{Cyc}(C) \}.
\]

The maps \(\rho_\bullet\) and \(\eta_\bullet\) are explicitly described in Lemmas 3.5.1 and 3.5.3, respectively.
3. Parametrizing underlying 1-cycles

3.1. Related linear systems in rational elliptic surfaces

In this section, we consider some basic results about linear systems on a rational elliptic surface. Some of these result can be found in [5, Section A.1].

Recall our notation that \( \pi : S \rightarrow \mathbb{P}^1 \) is a generic rational elliptic surface with a canonical section \( \zeta : \mathbb{P}^1 \rightarrow S \). Consider the following classical results for rational elliptic surfaces from [11, II.3]:

\[
\pi_* \mathcal{O}_S \cong \pi_* \mathcal{O}_S(\zeta) \cong \mathcal{O}_{\mathbb{P}^1}, \quad R^1 \pi_* \mathcal{O}_S \cong \mathcal{O}_{\mathbb{P}^1}(-1) \quad \text{and} \quad R^1 \pi_* \mathcal{O}_S(\zeta) \cong 0.
\]

After applying the projection formula, we have the following:

\[
\pi_* \mathcal{O}_S(dF) \cong \pi_* \mathcal{O}_S(\zeta + dF) \cong \mathcal{O}_{\mathbb{P}^1}(d) \quad \text{(4)}
\]

as well as

\[
R^1 \pi_* \mathcal{O}_S(dF) \cong \mathcal{O}_{\mathbb{P}^1}(d - 1) \quad \text{and} \quad R^1 \pi_* \mathcal{O}_S(\zeta + dF) \cong 0. \quad \text{(5)}
\]

**Lemma 3.1.1** We have the following isomorphisms:

\[
H^1(S, \mathcal{O}_S(dF)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1)) \quad \text{and} \quad H^1(S, \mathcal{O}_S(\zeta + dF)) \cong 0.
\]

**Proof.** The second isomorphism is immediate from the vanishing of \( R^i \pi_* \mathcal{O}_S(\zeta + dF) \) for \( i > 0 \) (see for example [7, III Ex. 8.1]) and \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong 0 \).

To show the first isomorphism, we consider the following exact sequence arising from the Leray spectral sequence:

\[
0 \rightarrow H^0(S, \mathcal{O}_S(dF)) \rightarrow H^1(\mathbb{P}^1, \pi_* \mathcal{O}_S(dF)) \rightarrow H^1(S, \mathcal{O}_S(dF)) \rightarrow H^0(S, \mathcal{O}_S(\zeta + dF)) \rightarrow 0.
\]

We have from (4) that \( H^1(\mathbb{P}^1, \pi_* \mathcal{O}_S(dF)) \cong 0 \) and we have the desired isomorphism after considering (5). \( \square \)

**Lemma 3.1.2** Consider a fibre \( F \) of a point \( z \in \mathbb{P}^1 \) by the map \( S \rightarrow \mathbb{P}^1 \) and the image of a section \( \zeta : \mathbb{P}^1 \rightarrow S \). Then there are isomorphisms of the linear systems

\[
|dF|_S \cong |\zeta + dF|_S \cong |dz|_{\mathbb{P}^1} \quad \text{and} \quad |b\zeta + F|_S \cong |z|_{\mathbb{P}^1}.
\]

**Proof.** The isomorphism \( |\zeta + dF|_S \cong |dz|_{\mathbb{P}^1} \) is immediate from the vanishing of \( R^i \pi_* \mathcal{O}_S(\zeta + dF) \) for \( i > 0 \) and (4) (see for example [7, III Ex. 8.1]).

We continue by showing \( |dF|_S \cong |\zeta + dF|_S \). Consider the long exact sequence arising from the divisor sequence for \( \zeta \) twisted by \( \mathcal{O}_S(\zeta + dF) \):

\[
0 \rightarrow H^0(S, \mathcal{O}_S(dF)) \rightarrow H^0(S, \mathcal{O}_S(\zeta + dF)) \rightarrow H^0(S, \mathcal{O}_{\mathbb{P}^1}(\zeta + dF)) \rightarrow H^1(S, \mathcal{O}_S(\zeta + dF)) \rightarrow 0.
\]
where we have applied the results from Lemma 3.1.1. From intersection theory, we have that
\[ \xi \mathcal{O}_{\mathbb{P}^1}(\xi + dF) \cong \xi \mathcal{O}_{\mathbb{P}^1}(d - 1). \]
So \( g \) and hence \( f \) are isomorphisms.

The isomorphism \(|b \xi + F|_S \cong |z|_2\) will follow inductively from the divisor sequence for \( \xi \) on \( S \):
\[
0 \longrightarrow \mathcal{O}_S(k\xi + F) \longrightarrow \mathcal{O}_S((k + 1)\xi + F) \longrightarrow \mathcal{O}_S((k + 1)\xi + F) \longrightarrow 0.
\]

Intersection theory shows us that \( \mathcal{O}_S((k + 1)\xi + F) \) is a degree \(-k\) line bundle on \( \mathbb{P}^1 \) which shows that its 0th cohomology vanishes. Hence, we have isomorphisms
\[
H^0(S, \mathcal{O}_S(F)) \cong \cdots \cong H^0(S, \mathcal{O}_S(b\xi + F)).
\]

\[ \square \]

3.2. Curve Classes and 1-cycles in the 3-fold

Recall from Definition 1.2.1 that the banana curves \( C_i \) are labelled by their unique intersections with the rational elliptic surfaces \( S_1, S_2 \) and \( S_{op} \).

These are smooth effective divisors on \( X \). Hence, a curve \( C \) in the class \((d_1, d_2, d_3)\) will have the following intersections with these divisors:
\[
C \cdot S_1 = d_1, \quad C \cdot S_2 = d_2 \quad \text{and} \quad C \cdot S_{op} = d_3.
\]

The full lattice \( H_2(X, \mathbb{Z}) \) is generated by
\[
C_1, \ C_2, \ C_3, \ \sigma_{11}, \ \sigma_{12}, \ \ldots, \ \sigma_{19}, \ \sigma_{21}, \ \ldots, \ \sigma_{99},
\]
where the \( \sigma_{ij} \) are the 81 canonical sections of \( pr : X \rightarrow \mathbb{P}^1 \) arising from the 9 canonical sections of \( \pi : S \rightarrow \mathbb{P}^1 \). However, there are 64 relations between the \( \sigma_{ij} \)s giving the lattice rank of 20 (see [3, Proposition 28 and Proposition 29]).

**Lemma 3.2.1** There are no relations in \( H_2(X, \mathbb{Z}) \) of the form
\[
n \cdot \sigma_{ij} + d_1 C_1 + d_2 C_2 + d_3 C_3 = \sum_{(k,l) \neq (i,j)} a_{k,l} \cdot \sigma_{k,l} + d'_1 C_1 + d'_2 C_2 + d'_3 C_3,
\]
where \( n, a_{k,l}, d_i, d'_t \in \mathbb{Z}_{\geq 0} \) for all \( k, l \in \{1, \ldots, 9\} \) and \( t \in \{1, 2, 3\} \).

**Proof.** Any such relation must push forward to relations on \( S \) via the projections \( pr_i : X \rightarrow S_i \). However, \( S \) is isomorphic to \( \mathbb{P}^2 \) blown up at 9 points. The exceptional divisors of these blow ups correspond to the sections \( \zeta_i : \mathbb{P}^1 \rightarrow S \). Hence,
\[
\text{Pic } S \cong \text{Pic } \mathbb{P}^2 \times \zeta_1 \times \cdots \times \zeta_9 \cong \mathbb{Z}^{10}
\]
and there are no relations of this form. \[ \square \]
The next lemma allows us to consider the curves in our desired classes by decomposing them.

**Lemma 3.2.2** Let \(d_1, d_2, d_3, b \in \mathbb{Z}_{\geq 0}\) and \(i, j \in \{0, 1\}\).

1. Let \(C\) be a Cohen–Macaulay curve in the class \((d_1, d_2, d_3)\). Then the support of \(C\) is contained in fibres of the projection map \(pr : X \to \mathbb{P}^1\).
2. A curve \(C\) in the class \(\sigma + (d_1, d_2, d_3)\) is of the form \(C = \sigma \cup C_0\), where \(C_0\) is a curve in the class \((d_1, d_2, d_3)\).
3. A curve in the class \(b \sigma + (i, j, d_3)\) is of the form \(C = C_\sigma \cup C_0\) where \(C_\sigma\) is a curve in the class \(b \sigma\) and \(C_0\) is a curve in the class \((i, j, d_3)\). The same result holds for permutations of \(b \sigma + (i, j, d_3)\).

**Proof.** Consider a curve in one of the given classes and its image under the two projections \(pr_i : X \to S_i\). For (1) these must be in the classes \(|d_1f_1|\) and \(|d_2f_1|\), for (2) the classes \(|\xi + d_1F_1|\) and \(|\xi + d_2F_2|\), and for (3) the classes \(|if_1|\) and \(|jf_1|\). Lemma 3.1.2 now shows that the curve must have the given form. \(\square\)

3.3. **Analysis of 1-cycles in smooth fibres of \(pr\)**

Consider a fibre \(F_x = pr^{-1}(x)\) which is smooth. Then there is an elliptic curve \(E\) such that \(F_x \cong E \times E\). Consider a curve \(C\) with underlying 1-cycle contained in \(E \times E\), then this gives rise to a divisor \(D\) in \(E \times E\). Hence, we must analyse divisors in \(E \times E\) and their classes in \(X\). The class of such a curve is determined uniquely by its intersection with the surfaces \(S_1, S_2\) and \(S_{op}\).

**Lemma 3.3.1** Let \(C \subset X\) correspond to a divisor \(D\) in \(E \times E\).

1. If \(C\) is in the class \((0, d_2, d_3)\), then \(d_2 = d_3\) and \(D\) is the pullback of a degree \(d_2\) divisor on \(E\) via the projection to the second factor.
2. The result in i is true for \((d_1, 0, d_3)\) and projection to the first factor.

**Proof.** If \(C\) is in the class \((0, d_2, d_3)\), then it does not intersect with the surface \(S_1\). When we restrict to \(E \times E\), this is the same condition as not intersecting with a fibre of the projection to the second factor. The only divisors that this is true for are those pulled back from \(E\) via the projection to the second factor. A divisor of this form will have intersection with \(S_2\) of \(d_2\) and intersection with \(S_{op}\) of \(d_2\). Hence, we have that \(d_2 = d_3\). The proof for part (2) is completely analogous. \(\square\)

**Lemma 3.3.2** Let \(C \subset X\) be in the class \((1, 1, d)\) and correspond to a divisor \(D\) in \(E \times E\). Then \(d \in \{0, \ldots, 4\}\) and occurs in the following situations:

1. If \(E\) has \(j(E) \neq 0, 1728\), then
   
   (a) \(d = 0\) occurs when \(D\) is a translation of the graph \(\{(x, -x)\}\).
(b) \( d = 4 \) occurs when \( D \) is a translation of the graph \( \{(x, x)\} \).
(c) \( d = 2 \) occurs when \( D \) is the union of a fibre from the projection to the first factor and a fibre from the projection to the second factor.

2. If \( j(E) = 1728 \) and \( E \cong \mathbb{C}/i \), then we have the cases (a) to (c) as well as
(d) \( d = 2 \) occurs when \( D \) is a translation of the graph \( \{(x, \pm ix)\} \).

3. If \( j(E) = 0 \) and \( E \cong \mathbb{C}/\tau \) with \( \tau = \frac{1}{2}(1 + i\sqrt{3}) \) (j-invariant \( j(E) = 0 \)), then we have the cases (a) to (c) as well as
(e) \( d = 1 \) occurs when \( D \) is a translation of the graph \( \{(x, -\tau x)\} \) or the graph \( \{(x, (\tau - 1)x)\} \).
(f) \( d = 3 \) occurs when \( D \) is a translation of the graph \( \{(x, \tau x)\} \) or the graph \( \{(x, (\tau + 1)x)\} \).

Proof. Denote the projection maps by \( p_i : E \times E \to E \) and let \( C \subset X \) be in the class \((1, 1, d)\) and correspond to a divisor \( D \in E \times E \). Suppose \( D \) is reducible. Then from Lemma 3.3.1 we see that \( D \) must be the union \( p_1^{-1}(x_1) \cup p_2^{-1}(x_2) \) where \( x_1, x_2 \in E \) are generic points. We also have that \( D \) is in the class \((1, 1, 2)\).

Suppose \( D \) is irreducible. The surfaces \( S_1 \) and \( S_2 \) intersect \( D \) exactly once. So the restrictions \( p|_D : D \to E \) are degree 1 and hence isomorphisms. Thus, \( D \) is the translation of the graph of an automorphism of \( E \).

All elliptic curves have the automorphisms \( x \mapsto \pm x \). Also we have

\begin{itemize}
  \item if \( E \cong \mathbb{C}/i \) (\( j \)-invariant \( j(E) = 1728 \)), then \( E \) also has the automorphisms \( x \mapsto \pm ix \), and
  \item if \( E = \mathbb{C}/\tau \) with \( \tau = \frac{1}{2}(1 + i\sqrt{3}) \) (\( j \)-invariant \( j(E) = 0 \)), then \( E \) also has the automorphisms \( x \mapsto \pm \tau x \) and \( x \mapsto \pm (\tau - 1)x \).
\end{itemize}

So to complete the proof, we have to calculate the intersections \( #(\Gamma_\xi \cap S_{\text{op}}) \) where \( \Gamma_\xi \) is the graph of an automorphism \( \xi \). Also, \( S_{\text{op}}|_E \cong \Gamma_{-1} \) hence we calculate \( #(\Gamma_\xi \cap \Gamma_{-1}) = #\{(x, \xi(x)) = (x, -x)\} \) in the surface \( F_x \). For all the elliptic curves, we have

(a) \( #(\Gamma_1 \cap \Gamma_{-1}) \) is given by the four 2-torsion points \( \{0, \frac{1}{2}, \frac{1}{2}(1 + \tau)\} \).
(b) \( #(\Gamma_{-1} \cap \Gamma_{-1}) = 0 \) since one copy can be translated away from the other.

For \( E \cong \mathbb{C}/i \) (\( j \)-invariant \( j(E) = 1728 \)), we have

(d) \( #(\Gamma_{\pm i} \cap \Gamma_{-1}) \) is given by the two points \( \{0, \frac{1}{2}(1 + \tau)\} \).

For \( E = \mathbb{C}/\tau \) with \( \tau = \frac{1}{2}(1 + i\sqrt{3}) \) (\( j \)-invariant \( j(E) = 0 \)), we have

(e) \( #(\Gamma_\tau \cap \Gamma_{-1}) \) and \#(\( \Gamma_{(1-\tau)i} \cap \Gamma_{-1} \)) are both determined by the three points \( \{0, \frac{1}{2}(1 + \tau)\}, \frac{1}{2}(1 + \tau) \} \).
(f) \#(\( \Gamma_{-\tau} \cap \Gamma_{-1} \)) and \#(\( \Gamma_{(1-\tau)i} \cap \Gamma_{-1} \)) are both given by the single point \( \{0\} \).

\[ \square \]

3.4. Analysis of 1-cycles in singular fibres of \( pr \)

We denote the fibres of the projection \( pr \) by \( F_i := pr^{-1}(x) \). The singular fibres are all isomorphic so we denote a singular fibre by \( F_{\text{ban}} \) and its normalization by \( v : F_{\text{ban}} \to F_{\text{ban}} \). From [3, Proposition 24] we have that \( F_{\text{ban}} \cong Bl_2(\mathbb{P} \times \mathbb{P}) \) and if we choose the coordinates on each \( \mathbb{P} \) so that the 0 and \( \infty \) map to a nodal singularity, then the two points blown up are \( z_1 = (0, \infty) \) and \( z_2 = (\infty, 0) \):

\[
\begin{array}{ccc}
Bl_{z_1, z_2}(\mathbb{P} \times \mathbb{P}) & \xrightarrow{\nu} & F_{\text{ban}} \\
\downarrow \text{bl} & & \\
\mathbb{P} \times \mathbb{P} & \xrightarrow{\rho} & F_{\text{ban}}
\end{array}
\]
We also let $N := \pi^{-1}(x)$ be a nodal elliptic fibre in $S$, and denote the natural projections by $q_i : F_{\text{ban}} \to N$ (these are the morphisms $\text{pr}_i : X \to S$ with restricted domain and codomain).

3.4.1 Denote the divisors in $\tilde{F}_{\text{ban}}$ corresponding to the banana curve $C_i$ by $\tilde{C}_i$ and $\tilde{C}_i'$. They are identified in $F_{\text{ban}}$ by $\nu(\tilde{C}_i) = \nu(\tilde{C}_i') = C_i$.

For $i = 1, 2$ we also denote $\hat{C}_i = \text{bl}(\tilde{C}_i)$ and $\hat{C}_i' = \text{bl}(\tilde{C}_i')$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. The curve classes in $\tilde{F}_{\text{ban}}$ are generated by the collection of $\tilde{C}_i$ and $\tilde{C}_i'$s with the relations

$$\tilde{C}_1 + \tilde{C}_3 \sim \tilde{C}_1' + \tilde{C}_3' \quad \text{and} \quad \tilde{C}_2 + \tilde{C}_3 \sim \tilde{C}_2' + \tilde{C}_3'.$$

3.4.2 Let $f_1$ and $f_2$ be fibres of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ not equal to any $\hat{C}_i$ or $\hat{C}_i'$ and let $\tilde{f}_1$ and $\tilde{f}_2$ be their proper transforms. Then we also have the relations

$$\tilde{f}_1 \sim \tilde{C}_1 + \tilde{C}_3 \quad \text{and} \quad \tilde{f}_2 \sim \tilde{C}_2 + \tilde{C}_3.$$

Moreover, if $\tilde{D}$ is a divisor in $\tilde{F}_{\text{ban}}$ such that $\nu(\tilde{D})$ is in the class $(d_1, d_2, d_3)$, then $D$ is in a class

$$a_1 \tilde{C}_1 + a_1' \tilde{C}_1' + a_2 \tilde{C}_2 + a_2' \tilde{C}_2' + a_3 \tilde{C}_3 + a_3' \tilde{C}_3',$$

where $d_i + d_i' = d_i$.

**Lemma 3.4.3** Let $C \subset X$ correspond to a divisor $D$ in $F_{\text{ban}}$.

1. $C$ is in the class $(0, 0, d_3)$ if and only if $D$ has 1-cycle $d_3 C_3$.

2. $C$ is in the class $(0, d_2, d_3)$ if and only if $D$ has 1-cycle $\tilde{D} + a_2 C_2^{(j)} + a_3 C_3^{(j)}$ where $\tilde{D}$ is the pullback of a degree $a_2$ divisor from the smooth part of $N$ via the projection $q_i : F_{\text{ban}} \to N$ such that $a_j + a_2 = D_2$ and $a_j + a_3 = D_3$. Moreover, $\tilde{D}$ is in the class $(0, a_j, a_j)$.
Proof. Let $C \subset X$ be a curve in the class $(0, d_2, d_3)$ and correspond to a divisor $D$ in $F_{\text{ban}}$. There exists a divisor $\tilde{D}$ in $\tilde{F}_{\text{ban}} \cong \text{Bl}_{c_1,c_2}(\mathbb{P}^1 \times \mathbb{P}^1)$ with $\nu(\tilde{D}) = D$.

From the discussion in 3.4.2, we have that $\text{bl}(\tilde{D})$ is in the class of $d_2f_2$ and is hence in its corresponding linear system. So, $\tilde{D}$ is the union of the proper transform of $\text{bl}(\tilde{D})$ and curves supported at $\tilde{C}_3$ and $\tilde{C}_3'$. The result now follows. $\square$

Lemma 3.4.4 Let $C \subset X$ be an irreducible curve in the class $(1, 1, d)$ and correspond to a divisor $D$ in $F_{\text{ban}}$. Then $D$ is the image under $\nu$ of the proper transform under $\text{bl}$ of a smooth divisor in $[f_1 + f_2]$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, the value of $d$ is determined the intersection of $D$ with points in $\mathbb{P} = \{(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)\}$. That is, if $D$ intersects

1. $(0, 0)$ and $(\infty, \infty)$ only, then $d = 2$.
2. $(0, \infty)$ and $(\infty, 0)$ only, then $d = 0$.
3. $(0, 0)$ only or $(\infty, \infty)$ only, then $d = 2$.
4. $(0, \infty)$ only or $(\infty, 0)$ only, then $d = 1$.
5. no points of $\mathbb{P}$, then $d = 2$.

Moreover, there are no smooth divisors in $[f_1 + f_2]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ that intersect other combinations of these points.

Proof. Let $C \subset X$ be an irreducible curve in the class $(1, 1, d)$ and correspond to a divisor $D$ in $F_{\text{ban}}$. There exists an irreducible divisor $\tilde{D}$ in $\tilde{F}_{\text{ban}} \cong \text{Bl}_{c_1,c_2}(\mathbb{P}^1 \times \mathbb{P}^1)$ with $\nu(\tilde{D}) = D$. $\tilde{D}$ does not contain either of the exceptional divisor $\tilde{C}_3$ and $\tilde{C}_3'$. Hence, it must be the proper transform of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

From the discussion in 3.4.2, we have that $\text{bl}(\tilde{D})$ is in the class of $f_1 + f_2$ and is hence in its corresponding linear system. The only irreducible divisors in $[f_1 + f_2]$ are smooth and can only pass through the combinations of points in $\mathbb{P}$ that are given. We refer to the Appendix 6.2.3 for the proof of this. The total transform in any divisor in $[f_1 + f_2]$ will correspond to a curve in the class $C_1 + C_2 + 2C_2$. Hence, the classes of the proper transforms depend on the number of intersections with the set $\{(0, \infty), (\infty, 0)\}$. The values are immediately calculated to be those given. $\square$

3.5. Parametrizing 1-cycles

For $i \in \{1, 2, \text{op}\}$, we use the notation:

1. $B_i = \{b_i^1, \ldots, b_i^{12}\}$ is the set of the 12 points in $S_i$ that correspond to nodes in the fibres of the projection $\pi_i := \text{pr}|_{S_i} : S_i \to \mathbb{P}^1$.
2. $S_i^0 = S_i \setminus B_i$ is the complement of $B_i$ in $S_i$.

Lemma 3.5.1 In the case $\beta = \sigma + (0, d_2, d_3) \in H_2(X, \mathbb{Z})$, there is the following constructible homeomorphism in $K_0(\text{Var}_{\mathbb{C}})[[Q_2, Q_3]]$:

$$\text{Chow}^{\sigma + (0, \bullet, \bullet)} \cong Q_\sigma \times \text{Sym}^*_{Q_2, Q_3}((S_2^0) \times \text{Sym}^*_{Q_2, Q_3}(B_2) \times \text{Sym}^*_{Q_3, Q_3}(B_{\text{op}})).$$
Moreover, if the points of \( \text{Chow}^{\sigma + (0, \bullet, \bullet)}(X) \) are identified using this constructible homeomorphism, then for \( x = (ay, mb_2, nb_\text{op}) \in \text{Chow}^{\sigma + (0, \bullet, \bullet)}(X) \) the fibre of the cycle map is \( \rho^{-1}_\bullet(x) = \text{Hilb}^{\bullet}_\text{Cyc}(X, q) \) where

\[
q = \sigma + \sum_i a_i \text{pr}_2^{-1}(y_i) + \sum_i m_i C_2^{(i)} + \sum_i n_i C_3^{(i)}.
\]

**Proof.** From Lemma 3.2.2 part 2 it is enough to consider curves in the class \( (0, d_2, d_3) \). Also from 3.2.2 part 1 we know that the curves are supported on fibres of the map \( pr : X \to \mathbb{P}^1 \). From Lemma 3.3.1 part 1, we know that the curves supported on smooth fibres of \( pr \) must be thicken fibres of the projection \( \text{pr}_2 : X \to S \). Similarly, we know from Lemma 3.4.3 part 2 that the curves supported on singular fibres of \( pr \) must be the union of thicken fibres of \( \text{pr}_2 \) and curves supported on the \( C_2 \) and \( C_3 \) banana curves. The result now follows. \( \square \)

We also use the notation:

1. \( N_j \subset S_j \) are the 12 nodal fibres of \( \pi_j : S_j \to \mathbb{P}^1 \) with the nodes removed and
   
   \[
   N_j = N_j^\sigma \sqcup N_j^\theta \text{ where } N_j^\sigma := N_j \cap \sigma \text{ and } N_j^\theta := N_j \setminus \sigma.
   \]
2. \( \text{Sm}_j = S_j^\circ \setminus N_j \) is the complement of \( N_j \) in \( S_j^\circ \) and
   
   \[
   \text{Sm}_j = \text{Sm}_j^\sigma \sqcup \text{Sm}_j^\theta \text{ where } \text{Sm}_j^\sigma := \text{Sm}_j \cap \sigma \text{ and } \text{Sm}_j^\theta := \text{Sm}_j \setminus \sigma.
   \]
3. \( J^0 \) and \( J^{1728} \) to be the subsets of points \( x \in \mathbb{P}^1 \) such that \( \pi^{-1}(x) \) has \( j \)-invariant 0 or 1728, respectively, and \( J = J^0 \sqcup J^{1728} \).
4. \( L \) to be the linear system \( |f_1 + f_2| \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with the singular divisors removed where \( f_1 \) and \( f_2 \) are fibres of the two projection maps.
5. \( \overline{\text{Aut}}(E) := \text{Aut}(E) \setminus \{ \pm 1 \} \).

**Remark 3.5.2** The following lemma should be parsed in the following way. For \( i, j \in \{0, 1\} \) and \( b, d_3 \in \mathbb{Z}_{\geq 0} \), a subscheme in the class \( \beta = b\sigma + (i, j, d_3) \in H_2(X, \mathbb{Z}) \) will have 1-cycle of the following form:

\[
q = b\sigma + D + \sum_i n_i C_3^{(i)},
\]

where \( D \) is reduced and does not contain \( \sigma \) or \( C_3^{(i)} \). Then \( D \) is in the class \( (i, j, n) \in H_2(X, \mathbb{Z}) \) for some \( n \in \mathbb{Z}_{\geq 0} \).

The Chow groups parameterize the different possible 1-cycles that \( D \) can have. Moreover, these possibilities depend on \( i \) and \( j \):

- If \( i = j = 0 \), then \( D \) is the empty curve.
- If \( i = 0 \) and \( j = 1 \), then \( D \) can be either a fibre of the projection \( \text{pr}_2 \) or \( C_2^{(i)} \).
- If \( i = j = i \), then \( D \) can be reducible or irreducible. If \( D \) is reducible, then it is some combination of fibres and banana curves. We call the collection where \( D \) is irreducible the **diagonals**.
Lemma 3.5.3  In the cases \( \beta = d\sigma + (i,j,d_3) \in H_2(X,\mathbb{Z}) \), there is the following constructible homeomorphism in \( K_0(\text{Var}_\mathbb{C})[[Q_\sigma,Q_3]] \):

\[
\text{Chow}^{\sigma+(i,j,\bullet)}(X) \cong \sum_{b \in \mathbb{Z}_{\geq 0}} Q_b^{\bullet} \text{Chow}^{(i,j,\bullet)}(X).
\]

Moreover, using the identification \( Q_b^{\bullet} \text{Chow}^{(i,j,\bullet)}(X) = \{b\} \times \text{Chow}^{(i,j,\bullet)}(X) \) the points of \( (b,x) \in \text{Chow}^{\sigma+(i,j,\bullet)}(X) \) give the fibres \( (\eta_{ij}^{\bullet})^{-1}(b,x) = \text{Hilb}_\text{Cyc}(X,q) \) with \( q = b\sigma + q' \) where \( q' \in \text{Chow}^{(i,j,\bullet)}(X) \).

We also have the following decompositions of \( \text{Chow}^{(i,j,\bullet)}(X) \), by constructible homeomorphisms in \( K_0(\text{Var}_\mathbb{C})[[Q_\sigma,Q_3]] \):

1. For \( i = j = 0 \), we have the decomposition of \( \text{Chow}^{(0,0,\bullet)}(X) \) with parts:
   a. \( \text{Sym}_{Q_3}^\bullet(B_{\text{op}}) \)
      The corresponding fibres are then \( (\eta_{00}^{\bullet})^{-1}(x) = \text{Hilb}_\text{Cyc}(X,q) \) where:
   a. If \( x = nb_{\text{op}} \), then \( q = \sum_i n_i C_3^{(i)} \).

2. For \( i = 0 \) and \( j = 1 \), we have a decomposition of \( \text{Chow}^{(0,1,\bullet)}(X) \) with parts:
   a. \( Q_2 S_2^\infty \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}}) \)
   b. \( Q_2 \bigcup_{k=1}^{12} \text{Sym}_{Q_3}^\bullet([b_{\text{op}}]) \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}} \setminus \{b_{\text{op}}\}) \)
      The corresponding fibres are then \( (\eta_{01}^{\bullet})^{-1}(x) = \text{Hilb}_\text{Cyc}(X,q) \) where:
   a. If \( x = (y, nb_{\text{op}}) \), then \( q = \text{pr}_2^{-1}(y) + \sum_i n_i C_3^{(i)} \).
   b. If \( x = (a_k b_{\text{op}}, nb_{\text{op}}) \), then \( q = C_2^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)} \).

3. For \( i = j = 1 \), we have a decomposition of \( \text{Chow}^{(1,1,\bullet)}(X) \) with parts:
   a. \( Q_2 Q_3 S_1^\infty \times S_2^\infty \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}}) \)
   b. \( Q_2 Q_3 \bigcup_{k=1}^{12} S_1^\infty \times \text{Sym}_{Q_3}^\bullet([b_{\text{op}}]) \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}} \setminus \{b_{\text{op}}\}) \)
   c. \( Q_2 Q_3 \bigcup_{k=1}^{12} S_2^\infty \times \text{Sym}_{Q_3}^\bullet([b_{\text{op}}]) \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}} \setminus \{b_{\text{op}}\}) \)
   d. \( Q_2 Q_3 \bigcup_{k,l=1}^{12} \text{Sym}_{Q_3}^\bullet([b_{\text{op}}]) \times \text{Sym}_{Q_3}^\bullet([b_{\text{op}}]) \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}} \setminus \{b_{\text{op}}, b_{\text{op}}\}) \)
   e. \( Q_2 Q_3 \bigcup_{k=1}^{12} \text{Sym}_{Q_3}^\bullet([b_{\text{op}}]) \times \text{Sym}_{Q_3}^\bullet(B_{\text{op}} \setminus \{b_{\text{op}}\}) \)
   f. \( Q_2 Q_3 \text{Diag}^\bullet \)

where \( \text{Diag}^\bullet \) will be defined by a further decomposition. The corresponding fibres of (a) – (e) are \( (\eta_{11}^{\bullet})^{-1}(x) = \text{Hilb}_\text{Cyc}(X,q) \) where
(a) If \( x = (y_1, y_2, nb_{op}) \), then \( q = pr_1^{-1}(y_1) + pr_2^{-1}(y_2) + \sum_i n_i C_3^{(i)} \).

(b) If \( x = (y_1, a_k b_{op}^k, nb_{op}) \), then \( q = pr_1^{-1}(y_1) + C_2^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)} \).

(c) If \( x = (y_2, a_k b_{op}^k, nb_{op}) \), then \( q = pr_2^{-1}(y_2) + C_1^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)} \).

(d) If \( x = (a_k b_{op}^k, a_i b_{op}^i, nb_{op}) \), then \( q = C_1^{(k)} + C_2^{(i)} + a_k C_3^{(k)} + a_i C_3^{(i)} + \sum_i n_i C_3^{(i)} \).

(e) If \( x = (a_k b_{op}^k, nb_{op}) \), then \( q = C_1^{(k)} + C_2^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)} \).

For part (f), \( \text{Diag}^* \) is defined by the further decomposition:

\[
\begin{align*}
&\text{(g) } \mathbb{S}m_1 \times \text{Sym}_{Q_3}^*(B_{op}) \\
&\text{(h) } \mathbb{S}m_2 \times \text{Sym}_{Q_3}^*(B_{op}) \\
&\text{(i) } \bigcup_{y \in \mathbb{E}} \mathbb{E}_{\pi(y)} \times \text{Aut}(\mathbb{E}_{\pi(y)}) \times \text{Sym}_{Q_3}^*(B_{op}) \\
&\text{(j) } \bigcup_{k=1}^{12} \mathbb{1} \times \text{Sym}_{Q_3}^*(b_{op}^k) \times \text{Sym}_{Q_3}^*(B_{op} \setminus \{b_{op}^k\}).
\end{align*}
\]

The corresponding fibres of (g) – (j) are \((\mathcal{H}_\ast)^{-1}(x) = \text{Hilb}_{\text{Cyc}}(X, q)\) where

\[
\begin{align*}
&\text{(g) } x = (y, nb_{op}), \text{ then } q = D_y + \sum_i n_i C_3^{(i)} \text{ where } D_y \text{ is the graph of the map } f(z) = z + y|_{E_{\pi(y)}} \text{ in the fibre } F_{\pi(y)} = E_{\pi(y)} \times E_{\pi(y)}. \\
&\text{(h) } x = (y, nb_{op}), \text{ then } q = D_y + \sum_i n_i C_3^{(i)} \text{ where } D_y \text{ is the graph of the map } f(z) = -z + y|_{E_{\pi(y)}} \text{ in the fibre } F_{\pi(y)} = E_{\pi(y)} \times E_{\pi(y)}. \\
&\text{(i) } x = (y, nb_{op}), \text{ then } q = D_y + \sum_i n_i C_3^{(i)} \text{ where } D_y \text{ is the graph of the map } f(z) = A(z) + y|_{E_{\pi(y)}} \text{ for some } A \in \text{Aut}(E_{\pi(y)}) \setminus \{\pm 1\}. \\
&\text{(j) } x = (z, a_k b_{op}^k, nb_{op}), \text{ then } q = \nu(\tilde{L}_z) + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)} \text{ where } \tilde{L}_z \text{ is the proper transform of the divisor } L_z \text{ in } \mathbb{P}^1 \times \mathbb{P}^1 \text{ and } \nu \text{ is the normalization of the } k\text{th singular fibre.}
\end{align*}
\]

**Proof.** The decomposition \( \text{Chow}_{\ast + (i, j, \bullet)}^\ast(X) \cong \bigoplus_{b \in \mathbb{Z}_{\geq 0}} Q_b^\ast \text{Chow}_{\ast + (i, j, \bullet)}(X) \) is immediate from Lemma 3.2.2 part 3. Hence, it is enough to parametrize the curves in the class \( \beta = (i, j, \bullet) \). Also from 3.2.2 part 1 we know that the curves are supported on fibres of the map \( pr : X \to \mathbb{P}^1 \). We must have that

\[
\text{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}
\]

for some minimal reduces curve \( D \) in the class \((1, 1, n)\) for \( n \geq 0 \) minimal. The possible \( D \) curves are described in Lemmas 3.3.1, 3.3.2, 3.4.3 and 3.4.4. The result now follows. \( \square \)

**Remark 3.5.4** Using Lemma 3.5.3 and the identification \( Q_b^\ast \text{Chow}_{\ast + (i, j, \bullet)}(X) = \{b\} \times \text{Chow}_{\ast + (i, j, \bullet)}(X) \), we make the following identification for notational convenience in discussing the points in Section 5.1:

\[
\text{Chow}_{\ast + (i, j, \bullet)}(X) \cong \mathbb{Z}_{\geq 0} \times \text{Chow}_{\ast + (i, j, \bullet)}(X).
\]
4. Techniques for calculating Euler characteristic

4.1. Quot schemes and their decomposition

This section is a summary of required results from [5]. First, we consider the following subset of the Hilbert scheme.

**Definition 4.1.1** Let $C \subset X$ be a Cohen–Macaulay subscheme of dimension 1. Consider the Hilbert scheme parameterizing one-dimensional subschemes $Z \subset X$ with class $[Z] = [C] \in H_2(X, \mathbb{Z})$ and $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_C) + n$ for some $n \in \mathbb{Z}_{\geq 0}$. This contains the following closed subset:

$$\text{Hilb}^n(X, C) := \{Z \subset X \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n\}.$$

It is convenient to replace the Hilbert scheme here with a Quot scheme. Recall the Quot scheme $\text{Quot}_n^X(\mathcal{F})$ parametrizing quotients $\mathcal{F} \twoheadrightarrow \mathcal{G}$ on $X$, where $\mathcal{G}$ is zero-dimensional of length $n$. It is related to the above Hilbert scheme in the following way.

**Lemma 4.1.2** [5, Lemma 5], [14, Lemma 5.1]. The following equality holds in $K_0(\text{Var}_C)(p)$:

$$\text{Hilb}^\bullet(X, C) := \sum_{n \in \mathbb{Z}_{\geq 0}} \text{Hilb}^n(X, C) = \sum_{n \in \mathbb{Z}_{\geq 0}} \text{Quot}_X^n(\mathcal{O}_C).$$

We also consider the following subscheme of these Quot schemes.

**Definition 4.1.3** [5, Definition 12] Let $\mathcal{F}$ be a coherent sheaf on $X$ and $S \subset X$ a locally closed subset. We define the locally closed subset of $\text{Quot}_n^X(\mathcal{F})$

$$\text{Quot}_X^n(\mathcal{F}, S) := \{[\mathcal{F} \twoheadrightarrow \mathcal{G}] \in \text{Quot}_X^n(\mathcal{F}) \mid \text{Supp}_{\text{red}}(\mathcal{G}) \subset S\}.$$  

This allows us to decompose the Quot schemes in the following way.

**Lemma 4.1.4** [5, Proposition 13] Let $\mathcal{F}$ be a coherent sheaf on $X$, $S \subset X$ a locally closed subset and $Z \subset X$ a closed subset. Then if $Z \subset S$ and $n \in \mathbb{Z}_{\geq 0}$, then there is a geometrically bijective constructible map:

$$\text{Quot}_X^n(\mathcal{F}, S) \longrightarrow \coprod_{n_1 + n_2 = n} \text{Quot}_X^{n_1}(\mathcal{F}, S \setminus Z) \times \text{Quot}_X^{n_2}(\mathcal{F}, Z).$$

4.2. An action on the formal neighbourhoods

Let $C \subset X$ be a one-dimensional subscheme in the class $\beta \in H_2(X)$ with 1-cycle $q = \text{Cyc}(C)$. We recall the notation defining $\text{Hilb}^\beta_{\text{Cyc}}(X, q) \subset \text{Hilb}^{\beta,n}(X)$ to be the following reduced subscheme

$$\text{Hilb}^\beta_{\text{Cyc}}(X, q) := \{[Z] \in \text{Hilb}^{\beta,n}(X) \mid \text{Cyc}(Z) = q\}.$$

Furthermore, we define

$$\text{Hilb}^\beta_{\text{CM}}(X, q) \subset \text{Hilb}^\beta_{\text{Cyc}}(X, q)$$

to be the open subset containing Cohen–Macaulay subschemes of $Z$. 


Lemma 4.2.1 Suppose \( Z \subset X \) is a one-dimensional Cohen–Macaulay subscheme such that

1. \( Z \) has the decomposition \( Z = C \cup \bigcup_i Z_i \) where \( C \) is reduced, \( Z_i \cap Z_j = \emptyset \) for \( i \neq j \) and for each \( i \) we have \( C \cap Z_i \) is finite.

2. There are formal neighbourhoods \( V_i \) of \( Z_i \) in \( X \) such that \((\mathbb{C}^*)^2 \) acts on each and fixes \( Z_i^{\text{red}} \).

3. \( C \cap (\bigcup_i V_i) \) is invariant under the \((\mathbb{C}^*)^2\)-action on \( V_i \).

Then there is a \((\mathbb{C}^*)^2\)-action on \( \text{Hilb}_{CM}^n (X, \text{Cyc}(Z)) \) such that if \( \alpha \in (\mathbb{C}^*)^2 \) and \( Y \in \text{Hilb}_{CM}^n (X, \text{Cyc}(Z)) \), then

\[
\alpha \cdot Y = \tilde{C} \cup \alpha \cdot (Y|_{\bigcup_i V_i}).
\]

Proof. Let \( \beta = [Z] \in H_2(X, \mathbb{Z}) \) and use the simplifying notation \( \mathcal{H} = \text{Hilb}_{CM}^n (X, \text{Cyc}(Z)) \) and \( H := \text{Hilb}_{CM}^n (X, \text{Cyc}(Z)) \). The composition \( H \hookrightarrow H_{\text{Cyc}(X, \text{Cyc}(Z))} \hookrightarrow \mathcal{H} \) defines an immersion \( H \hookrightarrow \mathcal{H} \) expressing \( H \) as a locally closed (reduced) subscheme of \( \mathcal{H} \). Moreover, the immersion also defines the following flat family over \( H \): \[ Z \mathcal{H} \xrightarrow{p_2} X \times H \]

We consider \( Z \cap (\bigcup_i Z_i^{\text{red}} \times H) \) and define \( C := \overline{Z \setminus (Z \cap (\bigcup_i Z_i^{\text{red}} \times H))} \), the scheme-theoretic closure of the scheme-theoretic complement. Also, we denote by \( Z^{\dagger} := \widehat{Z \setminus C} \) the formal completion of \( Z \setminus C \) in \( Z \). This gives a decomposition of \( Z \) by

\[ Z = C \cup Z^{\dagger}. \]

For all closed points \( x \in H \), the fibres of the composition \( C \hookrightarrow Z \hookrightarrow H \) have property \((C)_x = C \) as subschemes of \( X \). Hence, \( C \cap (C \times H) \) contains all of the closed points of \( C \times H \). Thus, \( C \times H = C \cap (C \times H) \) and we have the following immersion over \( H \)

\[
C \times H \xrightarrow{\alpha} C \xrightarrow{f_2} H \]

where both of \( f_1 \) and \( f_2 \) are proper with \( f_1 \) being flat. Also, since \( H \) is Noetherian, [17, Tag 01TX, Tag 05XD] shows that there is an open set \( U \subset H \) containing all the closed points such that \( \alpha_U \) is an isomorphism. Hence, \( C \times H = C \) as subschemes of \( X \times H \).

Now, using a similar argument to the previous paragraph, we have that the underlying reduced schemes of \( Z^{\dagger} \) and \( \bigcup_i Z_i^{\text{red}} \times H \) are equal as subschemes in \( X \times H \). This means that they both have the same formal completion in \( X \times H \).

The formal completion of \( \bigcup_i Z_i^{\text{red}} \times H \) in \( X \times H \) is given by \( \bigcup_i V_i \times H \), so we have inclusion \( Z^{\dagger} \hookrightarrow \bigcup_i V_i \times H \). Furthermore, we have an inclusion

\[
Z \hookrightarrow C \cup \bigcup_i (V_i \times H) = \left(C \cup \bigcup_i V_i\right) \times H.
\]
Now, by letting $G := (\mathbb{C}^*)^2$ and $W := C \cup \bigcup_i V_i$, we have the following diagram:

\[
\begin{array}{c}
G \times Z \xrightarrow{id \oplus \iota} G \times (W \times H) \xrightarrow{\beta} G \times (W \times H) \xrightarrow{id \oplus (j \circ \iota)} G \times (X \times H) \\
| \quad | \quad | \quad | \quad | \\
G \times H \xrightarrow{id \oplus \iota} G \times H \xrightarrow{id \oplus q_2} G \times H \xrightarrow{id \oplus p_2} G \times H
\end{array}
\]

where $\beta$ is defined by $(g, (x, y)) \mapsto (g, (g \cdot x, y))$, $j : W \hookrightarrow X$ is the natural inclusion and $q_2 : W \times H \to H$ is the projection onto the second factor. Taking the composition of the top row defines the following flat family in $\mathcal{H}$ over $G \times H$:

\[
\zeta := \begin{pmatrix}
G \times Z \\
G \times H
\end{pmatrix} \longrightarrow 
\begin{pmatrix}
X \times (G \times H)
\end{pmatrix}
\]

The flat family $\zeta$ defines a morphism $\Phi : G \times H \to \mathcal{H}$.

We have that $G \times H$ is reduced, so the scheme-theoretic image $\text{Sch.Im.}(\Phi)$ is also reduced. Moreover, every closed point of the $\text{Sch.Im.}(\Phi)$ is contained in $H$. Hence, we have $\text{Sch.Im.}(\Phi) \subseteq H$. Thus, $\Phi$ defines a morphism $G \times H \to H$. It is now straightforward to show that this morphism satisfies the identity and compatibility axioms of a group action. $\blacksquare$

**Remark 4.2.2** In the case where $Z$ is smooth, an analysis similar to that in the proof of Lemma 4.2.1 was carried out in [15]. However, the analysis there is scheme theoretic. Moreover, the equality $\kappa^{-1}(z) = \text{Quot}_X^*(I_Z)$, which will appear in the proof of the next lemma (Lemma 4.2.3), was proven scheme theoretically in the case of $Z$ begin smooth.

Define $\text{Hilb}^\bullet_{\text{CM}}(X, q) := \bigsqcup_{m \in \mathbb{Z}} \text{Hilb}^m_{\text{CM}}(X, q)$ and consider the constructible map

\[
\kappa : \text{Hilb}^\bullet_{\text{Cyc}}(X, q) \longrightarrow \text{Hilb}^\bullet_{\text{CM}}(X, q),
\]

where $Z \subset X$ is mapped to the maximal Cohen–Macaulay subscheme $Z_{\text{CM}} \subset Z$ (also forgetting the indexing variable $p$). Then for $z \in \text{Hilb}^\bullet_{\text{CM}}(X, q)$ corresponding to $Z \subset X$ we have

\[
\kappa^{-1}(x) = \sum_{m \in \mathbb{Z}} p^m \text{Hilb}^{m-x(O_Z)}(X, Z) = p^x(O_Z) \text{Hilb}^\bullet(X, Z).
\]

Moreover, we have

\[
e\left(\text{Hilb}^\bullet_{\text{Cyc}}(X, q)\right) = e\left(\text{Hilb}^\bullet_{\text{CM}}(X, q), \kappa_* 1\right)
= e\left(\text{Hilb}^\bullet_{\text{CM}}(X, q)(\mathbb{C}^*)^2, \kappa_* 1\right),
\]

where $(\kappa_* 1)(z) := e(\kappa^{-1}(z))$ and the last line comes from the following lemma.
Lemma 4.2.3 The constructible function $\kappa_\ast 1$ is invariant under the $(\mathbb{C}^*)^2$-action. That is if $\alpha \in (\mathbb{C}^*)^2$ and $z \in \text{Hilb}^n_{CM}(X, q)$ then $(\kappa_\ast 1)(z) = (\kappa_\ast 1)(\alpha \cdot z)$.

Proof. Let $\alpha \in (\mathbb{C}^*)^2$ and $z \in \text{Hilb}^n_{CM}(X, q)$ correspond to $Z \subset X$. Also let $Z_i$ and $V_i$ be as in Lemma 4.2.1 with $\tilde{Z} := \bigcup_i Z_i$ and $\tilde{V} := \bigcup_i V_i$. Then the fibre $\kappa^{-1}(z)$ is

$$
\kappa^{-1}(z) = p^{x(O_Z)} \text{Hilb}^\bullet(X, Z) = p^{x(O_Z)} \text{Quot}^\bullet_X(I_Z),
$$

where the last equality is in $K_0(\text{Var}_C)((p))$ from Lemma 4.1.2. Also from Lemma 4.1.4 we have a geometrically bijective constructible map:

$$
\begin{align*}
\text{Quot}^n_X(I_Z) \to \coprod_{n_1 + n_2 = n} \text{Quot}^{n_1}_X(I_Z, X \setminus \tilde{V}) \times \text{Quot}^{n_2}_\tilde{X}(I_Z, \tilde{V}).
\end{align*}
$$

We have $I_{\alpha \cdot Z}|_{X \setminus \tilde{V}} = I_Z|_{X \setminus \tilde{V}}$ so $\text{Quot}^{n_1}_X(I_Z, X \setminus \tilde{V}) \cong \text{Quot}^{n_1}_X(I_{\alpha \cdot Z}, X \setminus \tilde{V})$. Moreover, we have isomorphisms

$$
\text{Quot}^{n_2}_\tilde{X}(I_Z, \tilde{V}) \cong \text{Quot}^{n_2}_\tilde{V}(I_Z|_\tilde{V})
$$

and $Z_{\tilde{V}} \cong \alpha \cdot Z_{\tilde{V}}$ so we have an isomorphism

$$
\text{Quot}^{n_2}_\tilde{X}(I_Z, \tilde{V}) \cong \text{Quot}^{n_2}_\tilde{V}(I_{\alpha \cdot Z}, \tilde{V}).
$$

Taking Euler characteristic now shows that $e(\kappa^{-1}(z)) = e(\kappa^{-1}(\alpha \cdot z))$.

4.2.4 We will now consider a useful tool in calculating Euler characteristics of the form given in (6). First, let $z \in \text{Hilb}^n_{CM}(X, q)$ correspond to $Z \subset X$ such that $Z$ is locally monomial. In other words, for every geometric point $z \in Z$, the restriction of $Z$ to the formal neighbourhood of $z$ in $X$ is of the form $\text{Spec} \mathbb{C}[x, y, z]/I_z$, where $I_z$ is an ideal generated by monomials in $\mathbb{C}[x, y, z]$. Then the fibre $\kappa^{-1}(z)$ is

$$
\kappa^{-1}(z) = p^{x(O_Z)} \text{Hilb}^\bullet(X, Z) = p^{x(O_Z)} \text{Quot}^\bullet_X(I_Z),
$$

where the last equality is in $K_0(\text{Var}_C)((p))$ from Lemma 4.1.2. To compute this fibre, we employ the following method:

1. Decompose $X$ by $X = Z \amalg W$ where $W := X \setminus Z$.
2. Let $Z^\circ$ be set of singularities of $Z^{\text{red}}$.
3. Let $\coprod_i Z_i = Z \setminus Z^\circ$ be a decomposition into irreducible components.

Then applying Euler characteristic to Lemma 4.1.4 we have

$$
e(\text{Quot}^\bullet_X(I_Z)) = e(\text{Quot}^\bullet_X(I_Z, W)) \prod_{z \in Z^\circ} e(\text{Quot}^\bullet_X(I_Z, \{z\})) \prod_i e(\text{Quot}^\bullet_X(I_Z, Z_i)).$$
4.3. Partitions and the topological vertex

We recall the terminology of 2D partitions, 3D partitions and the topological vertex from \([4, 13]\). A 2D partition \(\lambda\) is an infinite sequence of weakly decreasing integers that are zero except for a finite number of terms. The size of a 2D partition \(|\lambda|\) is the sum of the elements in the sequence and the length \(l(\lambda)\) is the number of non-zero elements. We will also think of a 2D partition as a subset of \((\mathbb{Z}_{\geq 0})^2\) in the following way:

\[
\lambda \leftrightarrow \{(i,j) \in (\mathbb{Z}_{\geq 0})^2 \mid \lambda_i \geq j \geq 0 \text{ or } i = 0\}.
\]

A 3D partition is a subset \(\eta \subset (\mathbb{Z}_{\geq 0})^3\) satisfying the following condition:

1. \((i,j,k) \in \eta\) if and only if one of \(i, j\) or \(k\) is zero or one of \((i-1,j,k), (i,j-1,k)\) or \((i,j,k-1)\) is also in \(\eta\).

Given a triple of 2D partitions \((\lambda, \mu, \nu)\), we also define a 3D partition asymptotic to \((\lambda, \mu, \nu)\) is a 3D partition \(\eta\) that also satisfies the conditions:

1. \((j,k) \in \lambda\) if and only if \((i,j,k) \in \eta\) for all \(i \gg 0\).
2. \((k,i) \in \mu\) if and only if \((i,j,k) \in \eta\) for all \(j \gg 0\).
3. \((i,j) \in \nu\) if and only if \((i,j,k) \in \eta\) for all \(k \gg 0\).

The leg of \(\eta\) in the \(i\)th direction is the subset \(\{(i,j,k) \in \eta \mid (j,k) \in \lambda\}\). We analogously define the legs of \(\eta\) in the \(j\) and \(k\) directions. The weight of a point in \(\eta\) is defined to be

\[
\xi_\eta(i,j,k) := 1 - \# \{\text{legs of } \eta \text{ containing } (i,j,k)\}.
\]

Using this we define the renormalized volume of \(\eta\) by

\[
|\eta| := \sum_{(i,j,k) \in \eta} \xi_\eta(i,j,k). \tag{7}
\]
The topological vertex is the formal Laurent series:

\[ V_{\lambda,\mu,\nu} := \sum_{\eta} p^{|\eta|}, \]

where the sum is over all 3D partitions asymptotic to \((\lambda, \mu, \nu)\). An explicit formula for \(V_{\lambda,\mu,\nu}\) is derived in [13, Equation 3.18] to be

\[ V_{\lambda,\mu,\nu} = M(p) p^{-\frac{1}{2}(\|\lambda\|^2 + \|\mu\|^2 + \|\nu\|^2)} \sum_{\eta} S_{\lambda,\eta}(p^{\rho}) S_{\mu,\eta}(p^{\rho}) S_{\nu,\eta}(p^{\rho}). \]

4.4. **Partition thickened section, fibre and banana curves**

In this subsection, we consider the non-reduced structure of curves in our desired classes. The partition thickened structure will be the fixed points of a \((\mathbb{C}^*)^2\)-action.

4.4.1 Recall that the section \(\zeta \in S\) is the blow up of a point in \(z \in \mathbb{P}^2\). Choose once and for all a formal neighbourhood \(\text{Spec} \mathbb{C}\llbracket s, t \rrbracket\) of \(z \in \mathbb{P}^2\). The blow up gives the formal neighbourhood of \(\zeta \in S\) with 2 coordinate charts:

\[ \mathbb{C}\llbracket s, t \rrbracket \llbracket u \rrbracket / (t - su) \cong \mathbb{C}\llbracket s \rrbracket \llbracket u \rrbracket \quad \text{and} \quad \mathbb{C}\llbracket s, t \rrbracket \llbracket v \rrbracket / (s - tv) \cong \mathbb{C}\llbracket t \rrbracket \llbracket v \rrbracket \]

with change of coordinates \(s \mapsto tv\) and \(u \mapsto v^{-1}\). This gives the formal neighbourhood of \(\sigma \in X\) with 2 coordinate charts:

\[ \mathbb{C}\llbracket s_1, s_2 \rrbracket \llbracket u \rrbracket \quad \text{and} \quad \mathbb{C}\llbracket t_1, t_2 \rrbracket \llbracket v \rrbracket \]

with change of coordinates \(s_i \mapsto t_i v\) and \(u \mapsto v^{-1}\). We call these coordinates the canonical formal coordinates around \(\sigma \in X\).

4.4.2 Now consider a reduced curve \(D\) in \(X\) that intersects \(\sigma\) transversely with length 1. When \(D\) is restricted to the formal neighbourhood of \(\sigma\), it is given by

\[ \mathbb{C}\llbracket s_1, s_2 \rrbracket \llbracket u \rrbracket / (a_0 u - a_1, b_0 s_1 - b_1 s_2) \quad \text{and} \quad \mathbb{C}\llbracket t_1, t_2 \rrbracket \llbracket v \rrbracket / (a_0 - a_1 v, b_0 t_1 - b_1 t_2) \]

for some \([a_0 : a_1], [b_0 : b_1] \in \mathbb{P}^1\). We use this to define the change of coordinates:

\[ \tilde{s}_1 \mapsto b_0 s_1 - b_1 s_2 \quad \text{and} \quad \tilde{s}_2 \mapsto b_1 s_1 + b_0 s_2 \]

\[ \tilde{t}_1 \mapsto b_0 t_1 - b_1 t_2 \quad \text{and} \quad \tilde{t}_2 \mapsto b_1 t_1 + b_0 t_2. \]

We call these coordinates the canonical formal coordinates relative to \(D\).

**Definition 4.4.3** Let \(\mathbb{C}\llbracket s_1, s_2 \rrbracket \llbracket u \rrbracket\) and \(\mathbb{C}\llbracket t_1, t_2 \rrbracket \llbracket v \rrbracket\) be either the formal canonical coordinates of 4.4.1 or those of 4.4.2. Then we define
Figure 8. Depiction of the subscheme in \( \mathbb{C}^2 \) given by the monomial ideal \((y^3, y^2x, y^1x^2, y^1x^3, x^4)\) associated to the partition \((3,2,1,1,0,\ldots)\).

1. The canonical \((\mathbb{C}^*)^2\)-action on these coordinates by \((s_1, s_2) \mapsto (g_1s_1, g_2s_2)\) and \((t_1, t_2) \mapsto (g_1t_1, g_2t_2)\).

2. Let \( \lambda = (\lambda_1, \ldots, \lambda_j, 0, \ldots) \) be a 2D partition. The \(\lambda\)-thickened section denoted by \(\lambda\sigma\) is the subscheme of \(X\) defined by the ideal given in the coordinates by

\[ (s_1^{\lambda_1}, \ldots, s_1^{\lambda_j}, s_2^{l_1}, s_2^{l_2}) \quad \text{and} \quad (t_1^{\lambda_1}, \ldots, t_1^{l_1}, t_2^{l_2}). \]

4.4.4 We now consider a canonical formal neighbourhood of the banana curve \(C_3\). We follow much of the reasoning from [3, Section 5.2]. Let \(x \in S\) correspond to a point where \(\pi : S \to \mathbb{P}^1\) is singular. Let formal neighbourhoods in the two isomorphic copies of \(S\) be given by

\[ \text{Spec } \mathbb{C}[s_1, t_1] \quad \text{and} \quad \text{Spec } \mathbb{C}[s_2, t_2] \]

and the map \(S \to \mathbb{P}^1\) be given by \(r \mapsto s_1t_1\). Then the formal neighbourhood of a conifold singularity in \(X\) is given by

\[ \text{Spec } \mathbb{C}[s_1, t_1, s_2, t_2]/(s_1t_1 - s_2t_2), \]

and the restriction to a fibre of the projection \(S \times_{\mathbb{P}^1} S \to \mathbb{P}^1\) is

\[ \text{Spec } \mathbb{C}[s_1, t_1, s_2, t_2]/(s_1t_1 - s_2t_2). \]

Now, blowing up along \([s_1 = t_2 = 0]\) (which is canonically equivalent to blowing up along \([s_1 - t_1 = s_2 - t_2 = 0]\)), we have the two coordinate charts:

\[ \mathbb{C}[s_1, t_2, s_2, t_2][u]/(s_1 - ut_2, s_2 - ut_1) \cong \mathbb{C}[t_1, t_2][u], \quad \text{and} \]

\[ \mathbb{C}[s_1, t_2, s_2, t_2][v]/(t_1 - vs_2, t_2 - vs_1) \cong \mathbb{C}[s_1, s_2][v], \]

where the change of coordinates is given by \(t_1 \mapsto vs_2, t_2 \mapsto vs_1\) and \(u \mapsto v^{-1}\). We call these coordinates the canonical formal coordinates around the banana curve \(C_3\).
4.4.5 With these coordinates, we have

1. The restriction to the fibre of \(\pi r : X \to \mathbb{P}^1\) is

\[
\mathbb{C}[t_1, t_2][u]/(t_1 t_2 u) \quad \text{and} \quad \mathbb{C}[s_1, s_2][v]/(s_1 s_2 v).
\]

2. The banana curve \(C_3\) is given by

\[
\mathbb{C}[t_1, t_2][u]/(t_1, t_2) \quad \text{and} \quad \mathbb{C}[s_1, s_2][v]/(s_1, s_2).
\]

4.4.6 Similar to 4.4.2 we also consider canonical relative coordinates for a \(C_3\) banana curve. Recall 3.4.4 and let \(D\) be the image under \(v : \tilde{F}_{\text{ban}} \to F_{\text{ban}}\) of the proper transform under \(bl : \text{Bl}(0, \infty) \times (\infty, 0) (\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1 \times \mathbb{P}^1\) of a smooth divisor in \(|f_1 + f_2|\) on \(\mathbb{P}^1 \times \mathbb{P}^1\).

If \(D\) intersects \((0, 0)\), then the restriction of \(D\) to the formal neighbourhood of \(C_3\) is given by

\[
\mathbb{C}[s_1, s_2][v]/(s_1 - as_2, v)
\]

for some \(a \in \mathbb{C}^*\). In this case, we define canonical formal coordinates relative to \(D\) around a \(C_3\) banana by the following change of coordinates:

\[
\begin{align*}
\tilde{s}_1 & \mapsto s_1 - as_2 & \text{and} & \quad \tilde{s}_2 & \mapsto s_1 + as_2 \\
\tilde{t}_1 & \mapsto at_1 + t_2 & \text{and} & \quad \tilde{t}_2 & \mapsto -at_1 + t_2.
\end{align*}
\]

We similarly define the same relative coordinates if for \(D\) intersects \((\infty, \infty)\) in the ideal \((-at_1 + t_2, u)\). Note that these coordinates are compatible if \(D\) intersects both \((0, 0)\) and \((\infty, \infty)\).

**Definition 4.4.7** Let \(\mathbb{C}[s_1, s_2][u]\) and \(\mathbb{C}[t_1, t_2][v]\) be either the canonical coordinates or relative coordinates.

1. The canonical \((\mathbb{C}^*)^2\)-action on these coordinates is defined by

\[
(s_1, s_2, v) \mapsto (g_1 s_1, g_2 s_2, v) \quad \text{and} \quad (t_1, t_2, u) \mapsto (g_2 t_1, g_1 t_2, u).
\]

2. Let \(\lambda = (\lambda_1, \ldots, \lambda_3, 0, \ldots)\) be a 2D partition. The \(\lambda\) thickened banana curve \(C_3\) denoted by \(\lambda C_3\) is the subscheme of \(X\) defined by the ideal given in the coordinates by

\[
(s_1^{\lambda_1}, \ldots, s_1^{d-1} s_2^{\lambda_2}, s_1) \quad \text{and} \quad (t_1^{\lambda_1}, \ldots, t_1^{d-1} t_2)^l.
\]

(Note the change in coordinates compared to Definition 4.4.3.)

**Remark 4.4.8** If \(D\) intersects both \((0, 0)\) and \((\infty, \infty)\) and \(\lambda C_3\) is partition thickened in the coordinates relative to \(D\), then ideals for \(D \cup \lambda C_3\) at the points \((0, 0)\) and \((\infty, \infty)\) are

\[
(s_1^{\lambda_1}, \ldots, s_1^{d-1} s_2^{\lambda_2}, s_1) \cap (s_1, v) \quad \text{and} \quad (t_1^{\lambda_1}, \ldots, t_1^{d-1} t_2)^l \cap (t_2, u),
\]

respectively. These both give 3D partitions asymptotic to \((\lambda, \emptyset, \square)\).
Lemma 4.4.9 Let $D$ be as described in the first paragraph of 4.4.6. Let $V$ be the formal neighbourhood of $C_3$ in $X$. If $D$ intersects $(0,0)$ and/or $(\infty,\infty)$, then use the relative coordinates of 4.4.6, otherwise use the canonical coordinates of 4.4.4. Then $D \cap V$ is invariant under the $(\C^*)^2$-action.

Proof. We have $D \cap V \neq \emptyset$ if and only if it intersects at least one of $(0,0)$, $(0,\infty)$, $(\infty,0)$, $(\infty,\infty)$. The possible combinations are

1. $(0,0)$ and/or $(\infty,\infty)$: this is by construction of the relative coordinates.
2. Exactly one of $(0,\infty)$ or $(\infty,0)$: then $D$ is given by the ideal $(v-a,s_1)$ or $(v-a,s_2)$ for some $a \in \C^*$, which are $(\C^*)^2$-invariant.
3. $(0,\infty)$ and $(\infty,0)$: then $D$ is given by the ideal $(v-a,s_1s_2)$ for some $a \in \C^*$ which is $(\C^*)^2$-invariant. \hfill \Box

4.4.10 It is also shown in [3, Section 5.2] that there are the following formal coordinates on $C_2$ compatible with the canonical formal coordinates around $C_3$:

$$
\C[[s_1,v]][s_2] \quad \text{and} \quad \C[[t_1,u]][t_2],
$$

where the change on coordinates is given by $s_2 \mapsto t_2, s_1 \mapsto t_1t_2$ and $v \mapsto t_2u$. We can define partition thickenings and a compatible $(\C^*)^2$-action in these coordinates.

Definition 4.4.11 Let $\C[[s_1,v]][s_2]$ and $\C[[t_1,u]][t_2]$ be the above canonical coordinates.

1. The canonical $(\C^*)^2$-action on these coordinates is defined by

$$(s_1,v,s_2) \mapsto (g_1s_1,v,g_2s_2) \quad \text{and} \quad (t_1,u,t_2) \mapsto (g_2t_1,u,g_1t_2).$$

2. Let $\mu = (\mu_1,\ldots,\mu_k,0,\ldots)$ be a 2D partition. The $\mu$-thickened banana curve $C_2$ denoted by $\mu C_2$ is the subscheme of $X$ defined by the ideal given in the coordinates by

$$(s_1^{\mu_1},\ldots,s_2^{\mu_k},v^{\mu_1},\ldots,v^{\mu_k}) \quad \text{and} \quad (t_1^{\mu_1},\ldots,t_2^{\mu_k},u^{\mu_1},\ldots,u^{\mu_k}).$$

(Note the change in coordinates compared to Definition 4.4.7.)

3. Let $\lambda = (\lambda_1,\ldots,\lambda_l,0,\ldots)$ be another 2D partition. The $(\mu,\lambda)$-thickened banana curve denoted is the union $\mu C_2 + \lambda C_3$.

Remark 4.4.12 The $C_2$ and $C_3$ banana curves meet in exactly 2 points. At these two points, a $(\mu,\lambda)$-thickened banana curve will define two 3D partitions. One will be asymptotic to $(\mu,\lambda,0)$ the other will be asymptotic to $(\mu',\lambda',\emptyset)$ (or equivalently $(\lambda,\mu,\emptyset)$).

We will now consider fibres of the projection map $\text{pr}_2 : X \to S$.

Definition 4.4.13 Recall the definition of $\text{pr}_2 : X \to S$ from Section 1.2. Let $x \in S$ be such that $f_x := \text{pr}_2^{-1}(x)$ is smooth. Then we define

1. Canonical coordinates on a formal neighbourhood $V_x$ of $f_x$ are formal coordinates $\C[[s,t]]$ of $S$ at $x$ such that $V_x := f_x \times \text{Spec} \C[[s,t]]$ and for $\{x\} \cap \sigma \neq \emptyset$ we have that $\sigma$ restricted to $\C[[s,t]]$ is given by the ideal $(s)$. 

2. The canonical $(\mathbb{C}^*)^2$-action on these coordinates by $(s,t) \mapsto (g_1 s, g_2 t)$.

3. Let $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots)$ be a 2D partition. The $\lambda$-thickened smooth fibre at $x$ denoted by $\lambda f_x$ is the subscheme of $X$ given by the ideal:

$$ (l^{\lambda_1}, \ldots, s^{l-1} l^{\lambda_l}, s^l). $$

4.4.14 Let $N$ be a nodal fibre of $\pi : S \to \mathbb{P}^1$, $x \in N \setminus \{ \text{node} \}$ and $C = \text{pr}_2^{-1}(x)$. Let $F_{\text{ban}} = \text{pr}_2^{-1}(N)$ and $\widehat{F}_{\text{ban}}$ be its formal completion in $X$. The formal completion of $C$ in $\widehat{F}_{\text{ban}}$ is the same as the formal completion of $C$ in $\widehat{F}_{\text{ban}}$. Let $\mathbb{P}^1 \times \mathbb{P}^1$ be the formal neighbourhood of $\mathbb{P}^1 \times \mathbb{P}^1$ in the total space of its canonical bundle. It is shown in [3, Proposition 25] that there is a natural étale map $\tau : \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1) \to \widehat{F}_{\text{ban}}$ whose restriction to the underlying reduced subschemes is the normalization map for $F_{\text{ban}}$.

Using this and the results of [3, Section 5.2] we have charts for the formal completion of the normalization of $C$ in $\text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$ given by

$$ \mathbb{C}[y_0, z_0][x_0] \quad \text{and} \quad \mathbb{C}[y_1, z_1][x_1] $$

with an isomorphism on the complements of $(x_0)$ and $(x_1)$ given by

$$ x_0 \mapsto \frac{1}{x_1} \quad \text{and} \quad y_0 \mapsto y_1 x_2 \quad \text{and} \quad z_0 \mapsto z_1. $$

The identification at the node of $C$ is given by the restriction of the morphism $\tau : \text{Bl}(\mathbb{P}^1 \times \mathbb{P}^1) \to \widehat{F}_{\text{ban}}$ is

$$ z_0 = z_1 \quad \text{and} \quad x_0 = y_1 \quad \text{and} \quad y_0 = x_1. $$

Moreover, these coordinates can be chosen such that $\sigma$ restricted to $\mathbb{C}[y_0, z_0][x_0]$ is given by the ideal $(x_0 - 1, z_0)$.

**Definition 4.4.15** Using 4.4.14, we define

1. Canonical coordinates on a formal neighbourhood $V_C$ of $C$ are formal coordinates given in 4.4.14.

2. The canonical $(\mathbb{C}^*)^2$-action on given respectively on $\mathbb{C}[y_0, z_0][x_0]$ and $\mathbb{C}[y_1, z_1][x_1]$ by

$$ (x_0, y_0, z_0) \mapsto \left( g_1 x_0, \frac{1}{g_1} y_0, g_2 z_0 \right) \quad \text{and} \quad (x_1, y_1, z_1) \mapsto \left( \frac{1}{g_1} x_1, g_1 y_1, g_2 z_1 \right). $$

3. Let $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots)$ be a 2D partition. The $\lambda$-thickened fibre with one node at $x$ denoted by $\lambda C$ is the subscheme of $X$ given by the ideal which restricted to $\mathbb{C}[y_0, z_0][x_0]$ is

$$ (z_0^{\lambda_1}, \ldots, x_0^{l-1} z_0, x_0^l) \cap (z_0^{\lambda_1}, \ldots, y_0^{l-1} z_0, y_0^l). $$
and when restricted to $\mathbb{C}[y_1, z_1][x_1]$ is

$$(z_1^{\lambda_1}, \ldots, y_1^{l-1}z_1^{\lambda_l}, y_1^l) \cap (z_1^{\lambda_1}, \ldots, x_1^{l-1}z_1^{\lambda_1}, x_1^l).$$

**Remark 4.4.16** The partition thickened curves described in this section are easily shown to be the only Cohen–Macaulay subschemes supported in these neighbourhoods that are invariant under the $(\mathbb{C}^*)^2$-action. This is because the invariant Cohen–Macaulay subschemes must be generated by monomial ideals.

**Lemma 4.4.17** Let $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots)$ and $\mu = (\mu_1, \ldots, \mu_k, 0, \ldots)$ be 2D partitions, and let $x \in S$ such that $f_x := pr_2^{-1}(x)$ contains no banana curves. Then we have the holomorphic Euler characteristics:

1. $\chi(\mathcal{O}_{C_\lambda}) = \frac{1}{2}(\|\lambda\|^2 + \|\lambda^{\prime}\|^2)$,
2. $\chi(\mathcal{O}_{\lambda_{f_x}}) = 0$,
3. $\chi(\mathcal{O}_{\sigma\cap\lambda_{f_x}}) = \lambda_1$,
4. $\chi(\mathcal{O}_{\mu_{C_2} \cup \lambda_{C_3}}) = |\eta_1| + |\eta_2| + \frac{1}{2}(\|\mu\|^2 + \|\mu^{\prime}\|^2 + \|\lambda\|^2 + \|\lambda^{\prime}\|^2)$ where $|\eta_i|$ are the renormalized volumes of the minimal 3D partitions associated to $(\mu, \lambda, 0)$ and $(\mu^{\prime}, \lambda^{\prime}, 0)$.

**Proof.** (2) and (3) are proved in [5, Lemma 11], the rest are from [3, Proposition 23].

### 4.5. Relation between Quot schemes on $\mathbb{C}^3$ and the topological vertex

This section is predominately a summary of required results from [5]. For 2D partitions $\lambda$, $\mu$ and $v$, we define the following subscheme of $\mathbb{C}^3$:

$$C_{\lambda, \mu, v} = C_{\lambda, 0, 0} \cup C_{0, \mu, 0} \cup C_{0, 0, v} \subset \text{Spec} \mathbb{C}[r, s, t],$$

where $C_{\lambda, 0, 0}$ is defined by the ideal $I_{\lambda, 0, 0} := (r^{\lambda_1}, \ldots, s^{l-1}r^{\lambda_l}, s^l)$, with $C_{0, \mu, 0}$ and $C_{0, 0, v}$ being cyclic permutations of this. In general, we define the ideal $I_{\lambda, \mu, v} = I_{\lambda, 0, 0} \cap I_{0, \mu, 0} \cap I_{0, 0, v}$.

Now we consider the Quot scheme of length $n$ quotients of $I_{\lambda, \mu, v}$ that are set-theoretically supported at the origin and we employ the following simplifying notation:

$$\text{Quot}^n(\lambda, \mu, v) := \text{Quot}^n_{C^3}(I_{\lambda, \mu, v}, \{0\}).$$

A quotient parametrized here will have kernel that is the ideal sheaf of a one-dimensional scheme $Z$ with underlying Cohen–Macaulay (formal) curve $C_{\lambda, \mu, v}$. The embedded points of this scheme are all supported at the origin, but $Z$ does not have to be locally monomial. We use the following variation of the notation for the topological vertex:

$$\tilde{V}_{\lambda, \mu, v} := e(\text{Quot}^*(\lambda, \mu, v)) \in \mathbb{Z}[p].$$

**Lemma 4.5.1** Let $C$ be a partition thickened section, fibre or $C_3$-banana curve thickened by $\lambda$. Then

1. If $x \in C$ is a smooth point, then $e(\text{Quot}^n_C(I_C, \{x\})) = \tilde{V}_{\lambda, 0, 0}$. 


2. If $C$ is a thickened nodal fibre, then $e\left(\text{Quot}^a_X(I_C, \{x\})\right) = \tilde{\mathcal{V}}_{\lambda;\emptyset}$. Let $C'$ be a reduced curve intersecting $C$ at $y \in C$ such that $I_{C'} \cap I_C$ is locally monomial, and there are formal local coordinates $\mathbb{C}[r, s, t]_y$ at $y$ such that

1. $I_{C'} \cap I_C = \left(r^{\lambda_1}, \ldots, s^{l-1}r^{\lambda_l}, s^l\right) \cap (r, s)$, then $e\left(\text{Quot}^a_X(I_C, \{x\})\right) = \tilde{\mathcal{V}}_{\lambda\emptyset\emptyset}$.
2. $I_{C'} \cap I_C = \left(r^{\lambda_1}, \ldots, s^{l-1}r^{\lambda_l}, s^l\right) \cap (r, s) \cap (r, t)$, then $e\left(\text{Quot}^a_X(I_C, \{x\})\right) = \tilde{\mathcal{V}}_{\lambda\emptyset\square}$.

Proof. The proof is the same as [5, Lemma 15]. □

Lemma 4.5.2 Let $D$ be a one-dimensional Cohen–Macaulay subscheme of $X$.

1. We have

$$e\left(\text{Quot}^a_X(I_D, X \setminus D)\right) = \left(\tilde{\mathcal{V}}_{\emptyset\emptyset\emptyset}\right)^{e(X) - e(C)}.$$

2. Let $\lambda$ be a 2D partition and $\lambda C \subset D$ be either a partition thickened section, fibre or $C^3$ banana and let $T$ be finite set of points on $C$ such that $C \setminus T$ is smooth. Then

$$e\left(\text{Quot}^a_X(I_D, C \setminus T)\right) = \left(\tilde{\mathcal{V}}_{\lambda\emptyset\emptyset}\right)^{e(C) - e(T)}.$$

Proof. The argument is the same as that given for [5, equation (9)]. □

The standard $(\mathbb{C}^*)^3$-action on $\mathbb{C}^3$ induces an action on the Quot schemes. The invariant ideals $I \subset \mathbb{C}[r, s, t]$ are precisely those generated by monomials. Also, since there is a bijection between locally monomial ideals and 3D partitions we see that

$$\tilde{\mathcal{V}}_{\lambda;\mu\nu} = e\left(\text{Quot}^*(\lambda, \mu, \nu)_{(\mathbb{C}^*)^3}\right)$$

$$= \sum_{\eta} p^{n(\eta)},$$

where we are summing over 3D partitions asymptotic to $(\lambda, \mu, \nu)$ and $n(\eta)$ is the number of boxes not contained in any legs. Note that the lowest order term in $\tilde{\mathcal{V}}_{\lambda;\mu\nu}$ is one, which is not true about $\mathcal{V}_{\lambda;\mu\nu}$ in general. In fact, we have the relationship

$$\mathcal{V}_{\lambda;\mu\nu} = p^{||\eta_{\text{min}}||} \tilde{\mathcal{V}}_{\lambda;\mu\nu},$$

where $\eta_{\text{min}}$ is the 3D partition associated to $C_{\lambda;\mu\nu}$, and $|| \cdot ||$ is the renormalised volume defined in equation (7).

Lemma 4.5.3 If $\lambda$ is a 2D partition, then we have the following equalities:

1. $\mathcal{V}_{\lambda;\emptyset\emptyset} = \tilde{\mathcal{V}}_{\lambda;\emptyset\emptyset}$
2. $\mathcal{V}_{\lambda;\emptyset\emptyset} = p^{-\lambda_1} \tilde{\mathcal{V}}_{\lambda;\emptyset\emptyset}$
3. $\mathcal{V}_{\lambda;\emptyset\square} = p^{-\lambda_1 - \lambda_1} \tilde{\mathcal{V}}_{\lambda;\emptyset\square}$
4. $\mathcal{V}_{\lambda;\lambda;\emptyset} = p^{-||\lambda||^2} \tilde{\mathcal{V}}_{\lambda;\lambda;\emptyset}$. 

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Figure 9. Depiction of the decomposition of the Chow subscheme that parametrizes the vertical fibres of \( \text{pr}_2 \). The red dots indicate when the fibres do not intersect the section i.e. \( \text{Sm}_2^\emptyset \) and \( N_2^\emptyset \). The white dots indicate when the fibres do intersect the section i.e. \( \text{Sm}_1^\sigma \) and \( N_1^\sigma \).

**Proof.** Parts (1), (2) and (4) are directly from [5, Lemma 17]. For part (3), there are \( \lambda_1 \) boxes that are in the \( \lambda \)-leg and one of the \( \square \)-legs. There are \( \lambda_1^j \) boxes that are in the \( \lambda \)-leg and the other \( \square \)-leg. There is one box that is contained in all three so the renormalized volume is calculated to be

\[
(\lambda_i - 1)(1 - 2) + (\lambda_1^j - 1)(1 - 2) + (1)(1 - 3) = -\lambda_i - \lambda_1^j.
\]

\( \square \)

5. Calculating the Euler characteristic from the fibres of the Chow map

5.1. Calculation for the class \( \sigma + (0, \bullet, \bullet) \)

We now recall some previously introduced notation:

1. \( B_i = \{b_1^i, \ldots, b_1^{i2}\} \) is the set of the 12 points in \( S_i \) that correspond to nodes in the fibres of the projection \( \pi_i := \text{pr}|_{S_i} : S_i \to \mathbb{P}^1 \).

2. \( S_i^\emptyset = S_i \setminus B_i \) is the complement of \( B_i \) in \( S_i \).

3. \( N_i \subset S_i \) are the 12 nodal fibres of \( \pi_i : S_i \to \mathbb{P}^1 \) with the nodes removed and

\[
N_i = N_i^\sigma \sqcup N_i^\emptyset \text{ where } N_i^\sigma := N_i \cap \sigma \text{ and } N_i^\emptyset := N_i \setminus \sigma.
\]

4. \( \text{Sm}_i = S_i^\emptyset \setminus N_i \) is the complement of \( N_i \) in \( S_i^\emptyset \) and

\[
\text{Sm}_i = \text{Sm}_i^\sigma \sqcup \text{Sm}_i^\emptyset \text{ where } \text{Sm}_i^\sigma := \text{Sm}_i \cap \sigma \text{ and } \text{Sm}_i^\emptyset := \text{Sm}_i \setminus \sigma.
\]

Now from Lemma 3.5.1 we can further decompose \( \text{Chow}^{\sigma + (0, \bullet, \bullet)}(X) \) as

\[
\text{Sym}^\bullet_{Q_2 Q_3}(\text{Sm}_2^\emptyset) \times \text{Sym}^\bullet_{Q_2 Q_3}(N_2^\sigma) \times \text{Sym}^\bullet_{Q_2 Q_3}(\text{Sm}_2^\emptyset) \times \text{Sym}^\bullet_{Q_2 Q_3}(N_2^\emptyset) \\
\times \text{Sym}^\bullet_{Q_2}(B_2) \times \text{Sym}^\bullet_{Q_3}(B_{\text{op}}).
\]
Moreover, if \( q = (ax, cy, dz, lw, mb_2, nb_{op}) \in \text{Chow}^{\sigma + (0, \ast)}(X) \) then the fibre is given by 
\[
\rho^{-1}(q) \cong \text{Hilb}_{\text{Cyc}}(X, q)
\]
where
\[
q = \sigma + \sum_i a_i pr_2^{-1}(x_i) + \sum_i c_i pr_2^{-1}(y_i) \\
+ \sum_i d_i pr_2^{-1}(z_i) + \sum_i l_i pr_2^{-1}(w_i) + \sum_i m_i C_2^{(i)} + \sum_i n_i C_3^{(i)}.
\]

5.1.1 Suppose \( C \) is Cohen–Macaulay with the cycle given above. Note that \( C \) can be decomposed into a part supported on \( C_2 \cup C_3 \) and a part supported away from the banana configuration. This gives the following formal neighbourhoods and \((\mathbb{C}^*)^2\)-actions:

1. Let \( U_i \) be the formal neighbourhood of \( C_2^{(i)} \cup C_3^{(i)} \) in \( X \). These have a canonical \((\mathbb{C}^*)^2\)-action described in 4.4.7 and 4.4.11.

2. Let \( V_i \) be the formal neighbourhood of \( pr_2^{-1}(y_i) \) in \( X \). These have a canonical \((\mathbb{C}^*)^2\)-action described in Definition 4.4.13 and \( \sigma \cap V_i \) is either empty or invariant under this action.

Hence the conditions of Lemma 4.2.1 are satisfied and there is a \((\mathbb{C}^*)^2\)-action defined on \( \text{Hilb}_{\text{CM}}^\bullet(X, q) \). Using the partition thickened notation introduced in Section 4.4, we introduce the subschemes:

\[
C_{\alpha, \gamma, \delta, \lambda, \mu, \nu} := \sigma \cup \bigcup_i (\alpha^{(i)} f_{x_i}) \cup \bigcup_i (\gamma^{(i)} f_{y_i}) \cup \bigcup_i (\delta^{(i)} f_{z_i}) \\
\cup \bigcup_i (\lambda^{(i)} f_{w_i}) \cup \bigcup_i (\mu^{(i)} C_2^{(i)}) \cup \bigcup_i (\nu^{(i)} C_3^{(i)})
\]

and their ideals \( I_{\alpha, \gamma, \delta, \lambda, \mu, \nu} \) in \( X \) where \( \alpha, \gamma, \delta, \lambda, \mu \) and \( \nu \) are tuples of partitions of \( a, c, d, l, m \) and \( n \), respectively. Then using this notation we can identify the fixed points of the action as the following formal sum of discrete sets:

\[
\text{Hilb}_{\text{CM}}^\bullet(X, q)^{\mathbb{C}^*} = \prod_{\alpha + a, \gamma + c, \delta + d, \lambda + l, \mu + m, \nu + n} \left\{ C_{\alpha, \gamma, \delta, \lambda, \mu, \nu} \right\}.
\]
Using the result of 4.2.3, we have
\[ e\left(\text{Hilb}_{\text{CYC}}(X, q)\right) = e\left(\text{Hilb}_{\text{CM}}(X, q)(\mathbb{C}^*)^2, \kappa_{s1}\right) = \sum_{\sigma \subseteq \alpha, \gamma \subseteq \beta, \lambda \subseteq \mu, v \subseteq \nu} e\left((\text{Hilb}^*(X, C_{\alpha, y, \delta, \lambda, \mu, v})\right) \rho^X(C_{\alpha, y, \delta, \lambda, \mu, v}) \]
\[ = \sum_{\sigma \subseteq \alpha, \gamma \subseteq \beta, \lambda \subseteq \mu, v \subseteq \nu} e\left((\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v})\right) \rho^X(C_{\alpha, y, \delta, \lambda, \mu, v}).\]

5.1.2 Using the decomposition method of 4.2.4 following method:

1. Decompose \( X \) by \( X = W \sqcup C_{\alpha, y, \delta, \lambda, \mu, v} \) where \( W := X \setminus C_{\alpha, y, \delta, \lambda, \mu, v} \).
2. Let \( C_{\alpha, y, \delta, \lambda, \mu, v} \) be set points given by the following disjoint sets:
   (a) \( \sigma^\circ := \sigma \cap C_{\alpha}^{\text{red}} \)
   (b) \( \sigma^\circ_{\gamma} := \sigma \cap C_{\gamma}^{\text{red}} \)
   (c) \( C_{\gamma}^\circ \) the set of nodes of \( C_{\gamma} \)
   (d) \( C_{\lambda}^\circ \) the set of nodes of \( C_{\lambda} \)
   (e) \( B^\circ = \bigcup_{i}(C_2^{(i)} \cap C_3^{(i)}) \).
3. Denote the components supported on smooth reduced subcurves by:
   (a) \( \sigma^\circ := \sigma \setminus \sigma^\circ \)
   (b) \( C_{\alpha}^\circ := C_{\alpha} \setminus C_{\alpha}^\circ \)
   (c) \( C_{\gamma}^\circ := C_{\gamma} \setminus (\sigma^\circ_{\gamma} \cup C_{\lambda}^\circ) \)
   (d) \( C_{\lambda}^\circ := C_{\lambda} \setminus C_{\lambda}^\circ \)
   (e) \( C_{\mu}^\circ := C_{\mu} \setminus B^\circ \)
   (f) \( C_{\nu}^\circ := C_{\nu} \setminus B^\circ \).

5.1.3 Then applying Euler characteristic to Lemma 4.1.4 we have
\[ e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v})\right) = e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, W)\right)e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, \sigma^\circ)\right)\]
\[ e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, \sigma^\circ_{\alpha})\right)e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\alpha}^\circ)\right)\]
\[ e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\gamma})\right)e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\gamma}^\circ)\right)\]
\[ e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\lambda})\right)e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\lambda}^\circ)\right)\]
\[ e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, B^\circ)\right)e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\mu}^\circ)\right)e\left(\text{Quot}^*(I_{\alpha, y, \delta, \lambda, \mu, v}, C_{\nu}^\circ)\right).\]
Lemma 5.1.4  The holomorphic Euler characteristic of \( C_{\alpha,\delta,\lambda,\mu,v} \) is

\[
\chi(\mathcal{O}_{C_{\alpha,\delta,\lambda,\mu,v}}) = 1 + \sum_i \chi(\mathcal{O}_{\mu^{(i)}C_2^{(i)} \cup \nu^{(i)}C_3^{(i)}}) - \sum_i \alpha^{(i)} - \sum_i \gamma^{(i)}
\]

and we have

\[
p \chi(\mathcal{O}_{\mu^{(i)}C_2^{(i)} \cup \nu^{(i)}C_3^{(i)}}) \tilde{\nu}_{(\mu^{(i)}\nu^{(i)})\emptyset} \tilde{\nu}_{(\nu^{(i)})\emptyset} = \frac{1}{p^2}(\|\mu^{(i)}\| + \|\nu^{(i)}\|)^2 + \|\nu^{(i)}\|)^2 \mathcal{V}_{(\nu^{(i)})\emptyset} \mathcal{V}_{(\mu^{(i)}\nu^{(i)})\emptyset}.
\]

Proof.  Define the sheaves:

- \( \mathcal{F}^\sigma := \mathcal{O}_\sigma \oplus \left( \bigoplus_i \mathcal{O}_{\mu^{(i)}f_{x_i}} \right) \oplus \left( \bigoplus_i \mathcal{O}_{\mu^{(i)}f_{y_i}} \right) \)
- \( \mathcal{F}^\emptyset := \left( \bigoplus_i \mathcal{O}_{\mu^{(i)}f_{x_i}} \right) \oplus \left( \bigoplus_i \mathcal{O}_{\mu^{(i)}f_{y_i}} \right) \)
- \( \mathcal{F}^\cap := \left( \bigoplus_i \mathcal{O}_{\mu^{(i)}f_{x_i}} \right) \oplus \left( \bigoplus_i \mathcal{O}_{\mu^{(i)}f_{y_i}} \right) \)
- \( \mathcal{B} := \bigoplus_i \mathcal{O}_{\mu^{(i)}C_2^{(i)} \cup \nu^{(i)}C_3^{(i)}} \)

The exact sequence decomposing \( C_{\alpha,\delta,\lambda,\mu,v} \) is then

\[
0 \rightarrow \mathcal{O}_{C_{\alpha,\delta,\lambda,\mu,v}} \rightarrow \mathcal{F}^\sigma \oplus \mathcal{F}^\emptyset \oplus \mathcal{B} \rightarrow \mathcal{F}^\cap \rightarrow 0.
\]

The result now follows from 4.4.17 and the fact that \( \chi(\mathcal{O}_\sigma) = 1 \). \( \square \)

5.1.5  Applying lemmas 5.1.4, 4.5.1 and 4.5.2 we have

\[
e(\text{Quot}^*_X(\mathcal{O}_{C_{\alpha,\delta,\lambda,\mu,v}})) p \chi(\mathcal{O}_{C_{\alpha,\delta,\lambda,\mu,v}})
\]

\[
= p \cdot \left( \tilde{\nu}_{\emptyset\emptyset} \right)^{e(W)} \left( \tilde{\nu}_{\emptyset\emptyset} \right)^{e(\sigma^\cap)}
\]

\[
\cdot \prod_i \left( p^{-\alpha} \tilde{\nu}_{\mu^{(i)}\emptyset} \right) \prod_i \left( \tilde{\nu}_{\mu^{(i)}\emptyset} \right)^{-1}
\]

\[
\cdot \prod_i \left( p^{-\gamma} \tilde{\nu}_{\nu^{(i)}\emptyset} \right) \prod_i \left( \tilde{\nu}_{\nu^{(i)}\emptyset} \right)^{-1}
\]

\[
\cdot \prod_i \left( \tilde{\nu}_{\delta^{(i)}\emptyset} \right)^0
\]

\[
\cdot \prod_i \left( \tilde{\nu}_{\lambda^{(i)}\emptyset} \right)^0 \prod_i \left( \tilde{\nu}_{\lambda^{(i)}\emptyset} \right)^0
\]

\[
\cdot \prod_i \left( p^{\chi(\mathcal{O}_{\mu^{(i)}C_2^{(i)} \cup \nu^{(i)}C_3^{(i)}})} \tilde{\nu}_{(\mu^{(i)}\nu^{(i)})\emptyset} \tilde{\nu}_{(\nu^{(i)})\emptyset} \right) \prod_i \left( \tilde{\nu}_{(\mu^{(i)}\nu^{(i)})\emptyset} \right)^0 \prod_i \left( \tilde{\nu}_{(\nu^{(i)})\emptyset} \right)^0.
\]
We note that \( e(X) = 24 \) and \( e(\sigma) = 2 \), so from Lemma 4.5.3 we now have

\[
e(\text{Quot}_\lambda(I_{\alpha,\gamma,\delta,\lambda,\mu,\nu})) p^\lambda(\mathcal{O}_{\alpha,\gamma,\delta,\lambda,\mu,\nu})
\]

\[
= p \cdot \left( \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} \right)^{24} \prod_i \left( \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} + \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} \right) 
\]

\[
\cdot \prod_i \left( \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} p^{\| \mu \|^2 + \| \nu \|^2} \frac{\mathcal{V}_{\mu,\nu}}{\mathcal{V}_{\emptyset,\emptyset}} \right) \prod_i \left( \frac{\mathcal{V}_{\lambda,\mu}}{\mathcal{V}_{\emptyset,\emptyset}} \right) 
\]

\[
\cdot \prod_i \left( \frac{p^{\| \mu \|^2 + \| \nu \|^2} \mathcal{V}_{\mu,\nu}}{\mathcal{V}_{\emptyset,\emptyset}} \right). 
\]

We now define the functions

1. \( g_{Sm^c} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}(p) \) is defined by \( g_{Sm^c}(a) = \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} \sum_{a=0} \mathcal{V}_{a,\emptyset} \).

2. \( g_{N^c} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}(p) \) is defined by \( g_{N^c}(c) = \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} \sum_{\gamma+c} \mathcal{V}_{\emptyset,\emptyset} \).

3. \( g_{Sm^g} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}(p) \) is defined by \( g_{Sm^g}(d) = \sum_{\delta-d} 1 \).

4. \( g_{N^g} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}(p) \) is defined by \( g_{N^g}(i) = \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} \).

5. \( g_B : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}(p) \) is defined by the equation
   
   \[
g_B(m, n) = \sum_{\mu + m \atop v + n} p^{\| \mu \|^2 + \| \nu \|^2} \frac{\mathcal{V}_{\mu,\nu}}{\mathcal{V}_{\emptyset,\emptyset}}.
\]

So the constructible function \((\rho_\bullet)_1 : \text{Chow}^{\sigma + (0,\bullet,\bullet)}(X) \rightarrow \mathbb{Z}(p)\) is calculated for \( q = (ax, cy, dz, lw, mb_2, nb_{op}) \) by

\[
((\rho_\bullet)_1)(q)
\]

\[
= e(\rho_\bullet^{-1}(q))
\]

\[
= \sum_{\alpha + a, \gamma + c, \delta + \delta, \lambda + \lambda, \mu + m, \nu + n} e(\text{Quot}_\lambda(I_{\alpha,\gamma,\delta,\lambda,\mu,\nu})) p^\lambda(\mathcal{O}_{\alpha,\gamma,\delta,\lambda,\mu,\nu})
\]

\[
= p \left( \frac{\mathcal{V}_{\emptyset,\emptyset}}{\mathcal{V}_{\emptyset,\emptyset}} \right)^{24} \prod_i g_{Sm^c}(a_i) \prod_i g_{N^c}(c_i) \prod_i g_{Sm^g}(d_i) \prod_i g_{N^g}(l_i) \prod_i g_B(m_i, n_i).
\]
So we can now apply Lemma 2.1.3 and reintroduce the formal variables $Q_2$ and $Q_3$ to obtain

\[
e\left(\text{Chow}_{\sigma + (0, \bullet, \bullet)}(X), (\rho_\bullet)_\ast 1\right)
= p \cdot \left(\frac{V_{\emptyset \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}}\right)^{24}\left(\frac{V_{\emptyset \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \sum_\alpha (Q_2 Q_3)^{|\alpha|} \frac{V_{\alpha \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \right)
\times \left(\frac{V_{\emptyset \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \sum_\gamma (Q_2 Q_3)^{|\gamma|} \frac{V_{\gamma \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \right)
\times \left(\frac{V_{\emptyset \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \sum_\delta (Q_2 Q_3)^{|\delta|} \frac{V_{\delta \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \right)
\cdot \left(\frac{V_{\emptyset \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \sum_\lambda (Q_2 Q_3)^{|\lambda|} \frac{V_{\lambda \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \right)
\cdot e\left(\text{Sym}_Q^0 (B_2) \times \text{Sym}_Q^0 (B_{\text{op}}), G_B\right).
\]

where $G_B$ is the constructible function

\[
G_B : \text{Sym}_Q^0 (B_2) \times \text{Sym}_Q^0 (B_{\text{op}}) \to \mathbb{Z}(p)
\]

defined by $G_B(mb_2, nb_{\text{op}}) := \prod_{i=1}^{12} g_B(m_i, n_i)$. However, since $B_2 = \{b_2^1, \ldots, b_2^{12}\}$ and $B_{\text{op}} = \{b_{\text{op}}^1, \ldots, b_{\text{op}}^{12}\}$, we have

\[
\text{Sym}_Q^0 (B_2) \times \text{Sym}_Q^0 (B_{\text{op}}) \equiv \prod_{i=1}^{12} \text{Sym}_Q^0 \left(\{b_2^{(i)}\}\right) \times \text{Sym}_Q^0 \left(\{b_{\text{op}}^{(i)}\}\right).
\]

Defining $G_B^{(i)} := G_B|_{\text{Sym}_Q^0 (\{b_2^{(i)}\}) \times \text{Sym}_Q^0 (\{b_{\text{op}}^{(i)}\})}$ gives us

\[
e\left(\text{Sym}_Q^0 (B_2) \times \text{Sym}_Q^0 (B_{\text{op}}), G_B\right)
= \prod_{i=1}^{12} e\left(\text{Sym}_Q^0 \left(\{b_2^{(i)}\}\right) \times \text{Sym}_Q^0 \left(\{b_{\text{op}}^{(i)}\}\right), G_B^{(i)}\right)
= \left(\sum_{\mu, \nu} Q_2^{(\mu)} Q_3^{(\nu)} p^{\frac{1}{2}(\|\mu\|^2 + \|\mu'\|^2 + \|\nu\|^2 + \|\nu'\|^2)} \frac{V_{\mu \nu \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \frac{V_{\mu' \nu' \emptyset \emptyset}}{V_{\emptyset \emptyset \emptyset}} \right)^{12}.
\]
5.1.6 Applying the vertex formulas of Lemmas 6.3.5 and 6.3.2 and Lemma 6.1, we have

\[ e \left( \text{Chow}^{\sigma + (0, \bullet, \bullet)}(X), (\rho_{\bullet})_1 \right) = M(p)^{24} \frac{p}{(1 - p)^2} \left( \prod_{d > 0} \frac{1 - Q_d^d Q_3^d}{(1 - p Q_2^d Q_3^d)(1 - p^{-1} Q_2^d Q_3^d)} \right)^{-10} \]

\[ \cdot \left( \prod_{d > 0} \frac{M(p, Q_2^d Q_3^d)}{(1 - p Q_2^d Q_3^d)} \right)^{12} \]

\[ \cdot \left( \prod_{d > 0} \frac{1}{(1 - Q_2^d Q_3^d)} \right)^{10} \left( \prod_{d > 0} \frac{M(p, Q_2^d Q_3^d)}{(1 - Q_2^d Q_3^d)} \right)^{-12} \]

\[ \cdot \left( \prod_{d > 0} \frac{M(p, Q_2^d Q_3^d)^2}{(1 - Q_2^d Q_3^d)^8 (1 - p Q_2^d Q_3^d)^2 (1 - p^{-1} Q_2^d Q_3^d)^2} \right) \]

\[ \cdot \left( \prod_{d > 0} \frac{M(p, Q_2^d Q_3^d)^2}{(1 - Q_2^d Q_3^d) M(p, -Q_2^d Q_3^d) M(p, -Q_2^d Q_3^d - 1)} \right)^{12} \]

which completes the proof of Theorem A.

5.2 Preliminaries for classes of the form \( \bullet \sigma + (i, j, \bullet) \)

We recall from Lemma 3.5.3 that there is a decomposition of \( \text{Chow}^{\bullet \sigma + (i, j, \bullet)}(X) \) such that for any point \( q \in \text{Chow}^{\bullet \sigma + (i, j, \bullet)}(X) \) the fibre is

\[ (\eta_{\bullet})^{-1}(q) \cong \text{Hilb}_{\text{Cyc}}^{\bullet}(X, \text{Cyc}(C)) \]

for some one-dimensional subscheme \( C \) of \( X \) with

\[ \text{Cyc}(C) = q = a \sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}, \tag{8} \]

where \( D \) is a one-dimensional reduced subscheme of \( X \). We see from Lemma 3.5.3 that the intersection of \( D \) with \( \sigma \) has length 0, 1 or 2. We consider the following formal neighbourhoods around components of \( C \):
1. Let $U_i$ be the formal neighbourhood of $C_3^{(i)}$ in $X$. These have a canonical $(\mathbb{C}^*)^2$-action described in 4.4.7 and the $(\mathbb{C}^*)^2$-invariance of $D \cap U_i$ is shown in Lemma 4.4.9.

2. Let $V$ be the formal neighbourhood of $\sigma$ in $X$ with the coordinates:
   
   (a) If $\#(D \cap \sigma) = 0, 2$, then let $V$ have the canonical coordinates of 4.4.1 and of $(\mathbb{C}^*)^2$-action described in 4.4.3.
   
   (b) If $\#(D \cap \sigma) = 1$, then let $V$ have the canonical coordinates of 4.4.2 and of $(\mathbb{C}^*)^2$-action described in 4.4.3.

By construction, the restrictions of $D$ to these neighbourhoods are invariant under these actions. Hence, the conditions of Lemma 4.2.1 are satisfied and there is a $(\mathbb{C}^*)^2$-action defined on $\text{Hilb}^n_{CM}(X, \text{Cyc}(C))$. We introduce the notation for subschemes of $X$:

$$C_{\alpha, \mu} = C_{\alpha, \mu^{(1)}, \ldots, \mu^{(12)}} = \alpha \sigma \cup D \cup \bigcup_{i=1}^{12} \mu_i C_3^{(i)}$$

and their ideals $I_{\alpha, \mu}$. Then using this notation we can identify the fixed points of the action as the following discrete set:

$$\text{Hilb}^n_{CM}(X, q)^{(\mathbb{C}^*)^2} = \bigsqcup_{\alpha \vdash a, \mu \vdash m} \{ C_{\alpha, \mu} \}.$$

Using the result of 4.2.3, we have

$$e\left( \text{Hilb}^\bullet_{CM}(X, q) \right)^{(\mathbb{C}^*)^2} = e\left( \text{Hilb}^\bullet_{CM}(X, q)^{(\mathbb{C}^*)^2}, \kappa_1 \right)$$

$$= \sum_{\alpha \vdash a, \mu \vdash m} e\left( (\text{Hilb}^\bullet(X, C_{\alpha, \mu}) \right) p_x^X(O_{C_{\alpha, \mu}})$$

$$= \sum_{\alpha \vdash a, \mu \vdash m} e\left( \text{Quot}^\bullet(X, I_{\alpha, \mu}) \right) p_x^X(O_{C_{\alpha, \mu}})$$

where the holomorphic Euler characteristic $\chi(O_{C_{\alpha, \mu}})$ is given by the following lemma.
Lemma 5.2.1  The holomorphic Euler characteristic of $C_{\alpha, \mu}$ is

$$
\chi(O_{C_{\alpha, \mu}}) = \chi(O_D) + \left( \chi(O_{\alpha \sigma}) - |D \cap \alpha \sigma| \right) 
+ \left( \sum_{i=1}^{12} \chi(O_{\mu^{(i)} C^{(i)}_3}) - \sum_{i=1}^{12} |D \cap \mu^{(i)} C^{(i)}_3| \right).
$$

Proof. This is immediate from the exact sequence decomposing $C_{\alpha, \mu}$ into irreducible components:

$$
0 \rightarrow O_{C_{\alpha, \mu}} \rightarrow O_D \oplus O_{\alpha \sigma} \oplus \bigoplus_i O_{\mu^{(i)} C^{(i)}_3} \rightarrow O_{D \cap \alpha \sigma} \oplus \bigoplus_i O_{D \cap \mu^{(i)} C^{(i)}_3} \rightarrow 0.
$$

□

5.2.2 Using the decomposition method of 4.2.4, we take the following steps:

1. Decompose $X$ by $X = W \sqcup C_{\alpha, \mu}$ where $W := X \setminus C_{\alpha, \mu}$.

2. Let $C_{\alpha, \mu}^{\circ}$ be set points given by the following disjoint sets:
   (a) $D^{\circ}$ is the set of nodes of $D \setminus (\sigma \cup_i C^{(i)}_3)$.
   (b) $D^*$ is the set singularities of $D \setminus (\sigma \cup_i C^{(i)}_3)$ that are locally isomorphic to the coordinate axes in $\mathbb{C}^3$.
   (c) $\sigma^{\circ} := \sigma \cap D$.
   (d) $B_i^{\circ} = (C^{(i)}_3 \cap D)$ for $i \in \{1, \ldots, 12\}$.

Note that $D^{\circ} \cup D^*$ is the set of singularities of $D \setminus (\sigma \cup_i C^{(i)}_3)$.

3. Denote the components supported on smooth reduced sub-curves by
   (a) $D^{\circ} = D \setminus (D^{\circ} \cup D^*)$,
   (b) $\sigma^{\circ} := \sigma \setminus \sigma^{\circ}$,
   (c) $B_i^{\circ} = C^{(i)}_3 \setminus B_i^{\circ}$ for $i \in \{1, \ldots, 12\}$.

5.2.3 Then applying Euler characteristic to Lemma 4.1.4, we have

$$
e^\bigl(\text{Quot}_X(I_{\alpha, \mu})\bigr) p^\chi(O_{C_{\alpha, \mu}})
= e^\bigl(\text{Quot}_X(I_{\alpha, \mu}), W\bigr)
\cdot p^\chi(O_D) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, D^{\circ})\bigr) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, D^*)\bigr) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, D^\circ)\bigr) (9)
\cdot p^\chi(O_{\alpha \sigma}) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, \sigma^{\circ})\bigr) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, \sigma^{\circ})\bigr) (10)
\cdot \prod_{i=1}^{12} p^\chi(O_{\mu^{(i)} C^{(i)}_3}) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, B_i^{\circ})\bigr) e^\bigl(\text{Quot}_X(I_{\alpha, \mu}, B_i^{\circ})\bigr). (12)$$
5.2.4 We have that \( e(X) = 24 \) and \( e(\sigma) = e(C_{3}^{(i)}) = 2 \). So the Euler characteristic of \( W \) is

\[
e(W) = e(X) - e(\sigma) - \sum_{i=1}^{12} e(C_{3}^{(i)}) - e(D^o) - e(D^o) - e(D^o)
\]

\[
= -2 - e(D^o) - e(D^o) - e(D^o).
\]

Hence, we now have from Lemma 4.5.2 that lines (9)–(10) from above will be

\[
\Psi(D) := p^{\mathcal{X}(\mathcal{O}_D)} \left( V_{\mathcal{O}_\alpha} \right) e(W) \left( V_{\mathcal{O}_\alpha} \right) e(D^o) \left( V_{\mathcal{O}_\alpha} \right) e(D^o) \left( V_{\mathcal{O}_\alpha} \right) = p^{\mathcal{X}(\mathcal{O}_D)} \left( V_{\mathcal{O}_\alpha} \right)^{-2} \left( V_{\mathcal{O}_\alpha} \right) e(D^o) \left( V_{\mathcal{O}_\alpha} \right) e(D^o) \left( V_{\mathcal{O}_\alpha} \right) e(D^o) \left( V_{\mathcal{O}_\alpha} \right).
\]

The intersection of \( D \) and \( \alpha \sigma \) will determine line (11) from above. From Lemmas 4.5.2 and 4.5.3, it will be one of

1. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

2. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

3. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

4. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

Similarly, the factors of line (12) from above are determined by the intersections \( D \cap C_{3}^{(i)} \) to be (the fourth comes from 4.4.8):

1. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

2. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

3. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

4. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \)

5. \( p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) = p^{\frac{1}{2}(\|a\|^{2}+\|a\|^{'2})} \left( V_{\mathcal{O}_\alpha} V_{\mathcal{O}_\alpha} \right) \).

5.2.5 We can calculate \( e(\text{Hilb}^*_\text{Cyc}(X, q)) \) using the above results and notation from 5.2.4:

\[
e \left( \text{Hilb}^*_\text{Cyc}(X, q) \right) = \sum_{\alpha \cap \alpha', \mu \cap \mu'} p^{\mathcal{X}(\mathcal{O}_{\alpha, \mu})} e \left( \text{Quot}^*_\text{Cyc}(I_{\alpha, \mu}) \right)
\]

\[
= \Psi(D) \Phi(a) \prod_{i=1}^{12} \Phi(m_i),
\]
where $\Phi_\sigma$ and $\Phi_i$ are determined by the intersections of $\sigma$ and $C_3^{(i)}$, respectively, to be one of the following functions:

1. $\Phi^{\emptyset,\emptyset}(a) := \sum_{\alpha \vdash a} p^\frac{1}{2}(\|\alpha\|^2 + \|\alpha'\|^2)(V_{\emptyset \emptyset \emptyset \emptyset} V_{\emptyset \emptyset \emptyset \emptyset})$

2. $\Phi^{-\emptyset}(a) := \sum_{\alpha \vdash a} p^\frac{1}{2}(\|\alpha\|^2 + \|\alpha'\|^2)(V_{\emptyset \emptyset \emptyset \emptyset})$

3. $\Phi^{-\emptyset}(a) := \sum_{\alpha \vdash a} p^\frac{1}{2}(\|\alpha\|^2 + \|\alpha'\|^2)(V_{\emptyset \emptyset \emptyset \emptyset})$

4. $\Phi^{-\emptyset}(a) := \sum_{\alpha \vdash a} p^\frac{1}{2}(\|\alpha\|^2 + \|\alpha'\|^2)(V_{\emptyset \emptyset \emptyset \emptyset})$

5. $\Phi^{+,\emptyset}(a) := \sum_{\alpha \vdash a} p^\frac{1}{2}(\|\alpha\|^2 + \|\alpha'\|^2 + 1)(V_{\emptyset \emptyset \emptyset \emptyset})$

6. $\Phi^{+,+}(a) := \sum_{\alpha \vdash a} p^\frac{1}{2}(\|\alpha\|^2 + \|\alpha'\|^2 + 2)(V_{\emptyset \emptyset \emptyset \emptyset})$

5.3. Calculation for the class $\bullet \sigma + (0,0,\bullet)$

From Lemma 3.5.3, we have the decomposition of Chow$^{\bullet \sigma + (0,0,\bullet)}(X)$ into

$$Z_{\geq 0} \times \text{Sym}_{Q_3}(B_{\text{op}}).$$

Recall equation (8) from Section 5.2 and the notation:

$$\text{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}.$$

In this class, we have $D = \emptyset$. Hence, we have the following summary of results from 5.2.4 and 5.2.5.

$$\chi(O_D) = 0$$

$$e(\eta_{\bullet}^{-1}(a, m)) = \frac{1}{(V_{\emptyset \emptyset \emptyset})^2} \cdot Q_\sigma^{a} \Phi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$$

Now we have

$$e\left( Z_{\geq 0} \times \text{Sym}_{Q_3}(B_{\text{op}}), (\eta_{\bullet})_{-1} \right) = \frac{1}{(V_{\emptyset \emptyset \emptyset})^2} \left( \sum_{a} Q_\sigma^{a} \Phi^{\emptyset,\emptyset}(a) \right) \left( \sum_{m} Q_3^{m} \Phi^{\emptyset,\emptyset}(m) \right)^{12}$$

$$= M(p)^{24} \prod_{m>0} (1 + p^m Q_\sigma^{m}) (1 + p^m Q_3^{12m}),$$

where the last equality is from 6.1 part 2 and 6.3.2 part 1.
5.4. Calculation for the class $\bullet \sigma + (0, 1, \bullet)$

Recall the previously introduced notation:

1. $B_i = \{b_1^i, \ldots, b_1^{12}\}$ is the set of the 12 points in $S_i$ that correspond to nodes in the fibres of the projection $\pi : S_i \to \mathbb{P}^1$.
2. $S^\sigma_i = S_i \setminus B_i$ is the complement of $B_i$ in $S_i$.
3. $N_i \subset S_i$ are the 12 nodal fibres of $\pi : S_i \to \mathbb{P}^1$ with the nodes removed and $N_i = N_i^\sigma \sqcup N_i^\emptyset$ where $N_i^\sigma := N_i \cap \sigma$ and $N_i^\emptyset := N_i \setminus \sigma$.
4. $S_m_i = S^\sigma_i \setminus N_i$ is the complement of $N_i$ in $S_i$ and $S_m_i = S_m_i^\sigma \sqcup S_m_i^\emptyset$ where $S_m_i^\sigma := S_m_i \cap \sigma$ and $S_m_i^\emptyset := S_m_i \setminus \sigma$.

Now from Lemma 3.5.3 we can further decompose $\text{Chow}^{\bullet \sigma + (0, 1, \bullet)}(X)$ into the five parts:

1. $Z_{\geq 0} \times S_m_i^{\sigma} \times \text{Sym}_{Q_3}^{\bullet}(B_{\text{op}})$
2. $Z_{\geq 0} \times S_m_i^{\emptyset} \times \text{Sym}_{Q_3}^{\bullet}(B_{\text{op}})$
3. $Z_{\geq 0} \times N_i^{\sigma} \times \text{Sym}_{Q_3}^{\bullet}(B_{\text{op}})$
4. $Z_{\geq 0} \times N_i^{\emptyset} \times \text{Sym}_{Q_3}^{\bullet}(B_{\text{op}})$
5. $\bigcup_{k=1}^{12} Z_{\geq 0} \times \text{Sym}_{Q_3}^{\bullet}(\{b_{k_{\text{op}}}^i\}) \times \text{Sym}_{Q_3}^{\bullet}(B_{\text{op}} \setminus \{b_{k_{\text{op}}}^i\})$.

Recall equation (8) from Section 5.2 and the notation

$$\text{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(j)}.$$ 

Each part will be characterized by the type of $D$. We consider parts 1–4 separately to part 5.

5.4.1 Parts 1–4: in parts 1–4 the curve $D$ is a fibre of the projection $pr_2 : X \to S$. The following table is the summary of results from 5.2.4 and 5.2.5 when applied to the particular $D$'s arising in each strata:

$$Z_{\geq 0} \times U \times \text{Sym}_{Q_3}^{\bullet}(B_{\text{op}}).$$
The union of parts 1–4 is $\mathbb{Z}_{\geq 0} \times S^2 \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}})$ so we have

$$e\left( \mathbb{Z}_{\geq 0} \times S^2 \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}), (\eta_\bullet)_* 1 \right)$$

$$= e\left( \mathbb{Z}_{\geq 0} \times \text{Sm}_2^\sigma \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}), (\eta_\bullet)_* 1 \right)$$

$$+ e\left( \mathbb{Z}_{\geq 0} \times \text{Sm}_1^\emptyset \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}), (\eta_\bullet)_* 1 \right)$$

$$+ e\left( \mathbb{Z}_{\geq 0} \times N_2^\sigma \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}), (\eta_\bullet)_* 1 \right)$$

$$+ e\left( \mathbb{Z}_{\geq 0} \times N_2^\emptyset \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}), (\eta_\bullet)_* 1 \right)$$
which becomes

\[
e \left( \mathbb{Z}_{\geq 0} \times S^3 \times \text{Sym}^*_{Q_3}(B_{\text{op}}), (\eta_\bullet)_+, 1 \right)
\]

\[
e (\text{Sm}_2^\sigma) Q_2 Q_3 p \frac{(V_{\Box\Box})}{(V_{\Box\Box})^2} \left( \sum_{a \geq 0} Q_\sigma^a \Phi^{a \sigma, \chi} (a) \right) \left( \sum_{m \geq 0} Q_3^m \Phi^{m, \chi} (m) \right)^{12}
\]

\[
+ e (\text{Sm}_2^\theta) Q_2 Q_3 p \frac{(V_{\Box\Box})}{(V_{\Box\Box})^3} \left( \sum_{a \geq 0} Q_\sigma^a \Phi^{a \sigma, \chi} (a) \right) \left( \sum_{m \geq 0} Q_3^m \Phi^{m, \chi} (m) \right)^{12}
\]

\[
+ e (N_2^\sigma) Q_2 Q_3 \frac{1}{(V_{\Box\Box})^2} \left( \sum_{a \geq 0} Q_\sigma^a \Phi^{a \sigma, \chi} (a) \right) \left( \sum_{m \geq 0} Q_3^m \Phi^{m, \chi} (m) \right)^{12}
\]

\[
+ e (N_2^\theta) Q_2 Q_3 \frac{1}{(V_{\Box\Box})^2} \left( \sum_{a \geq 0} Q_\sigma^a \Phi^{a \sigma, \chi} (a) \right) \left( \sum_{m \geq 0} Q_3^m \Phi^{m, \chi} (m) \right)^{12}
\]

From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. \(V_{\Box\Box\Box} = M(p)\)
2. \(V_{\Box\Box\Box} = M(p) \frac{1}{1-p}\)
3. \(V_{\Box\Box\Box} = M(p) \frac{p^2 - p + 1}{p(1-p)^2}\)
4. \(\sum_{m \geq 0} Q^m \Phi^{m, \chi} (m) = M(p)^2 \prod_{m > 0} (1 + m Q)^m\)
5. \(\sum_{m \geq 0} Q^m \Phi^{m, \chi} (m) = M(p)^2 \frac{1 + Q}{1-p} \prod_{m > 0} (1 + m Q)^m\).

So we have

\[
e \left( \mathbb{Z}_{\geq 0} \times S^3 \times \text{Sym}^*_{Q_3}(B_{\text{op}}), (\eta_\bullet)_+, 1 \right)
\]

\[
= Q_\sigma Q_2 Q_3 M(p)^{24} \left( \prod_{m > 0} (1 + m Q_\sigma)^m (1 + m Q_3)^{12m} \right) \left( 2 + 12 \frac{p}{(1-p)^2} \right)
\]

\[
= Q_\sigma Q_2 Q_3 M(p)^{24} \left( \prod_{m > 0} (1 + m Q_\sigma)^m (1 + m Q_3)^{12m} \right) (2\psi_1 + 12\psi_0)
\]
5.4.2 **Part 5:** we have 12 separate isomorphic strata:

\[ \mathbb{Z}_{\geq 0} \times \text{Sym}^*_{Q_3}([b_{\text{op}}^k]) \times \text{Sym}^*_{Q_3}(B_{\text{op}} \setminus \{b_{\text{op}}^k\}). \]

These parameterize when \( D = C_2^{(k)}. \) The following is the summary of results from 5.2.4 and 5.2.5.

| \( U = \{k\} \) | \( e(U) = 1 \) | \( \chi(\mathcal{O}_D) = 1 \) |
|-----------------|----------------|----------------|
| \( e(\eta^* \langle a, m_k, m \rangle) = \frac{1}{V_{\emptyset, \emptyset}^2} \cdot Q_2^\alpha \Phi_0,0(a) \cdot Q_3^m \Phi^{\cdots, -}(m_k) \cdot \prod_{i=1 \atop i \neq k}^{12} Q_3^m \Phi_0,0(m_i) \) | | |

From Lemmas 6.3.2 and 6.1, we have

1. \( V_{\emptyset, \emptyset} = M(p) \)
2. \( \sum_{m \geq 0} Q^m \Phi^{0,0}(m) = M(p)^2 \prod_{m > 0} (1 + p^m Q)^m \)
3. \( \sum_{m \geq 0} Q^m \Phi^{\cdots, -}(m) = M(p)^2(\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0 Q^2) \prod_{m > 0} (1 + p^m Q)^m. \)

Since the strata are isomorphic, we have

\[
e\left( \bigotimes_{k=1}^{12} \mathbb{Z}_{\geq 0} \times \text{Sym}^*_{Q_3}([b_{\text{op}}^k]) \times \text{Sym}^*_{Q_3}(B_{\text{op}} \setminus \{b_{\text{op}}^k\}) \right) \right)^{11} \right) \right) \right) \right)

\]

5.4.3 Thus, combining parts 1–5 we have that the overall formula is

\[
e\left( \text{Chow}^{*+(0,1 \cdot)}(X), (\eta^*)_*1 \right)

= Q_2 M(p)^{24} \left( \prod_{m > 0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) \left( \psi_0 + (\psi_1 + 2\psi_0)Q_3 + \psi_0 Q_3^2 \right). \]
5.5. Calculation for the class $\bullet \sigma + (1, 1, \bullet)$

We have a decomposition from Lemma 3.5.3 of Chow$^{(1, 1, \bullet)}(X)$ into the parts:

(a) $S^o_1 \times S^o_2 \times \text{Sym}^\bullet_{Q^3}(B_{op})$

(b) $\bigsqcup_{k=1}^{12} S^o_1 \times \text{Sym}^\bullet_{Q^3}([b^k_{op}]) \times \text{Sym}^\bullet_{Q^3}(B_{op} \setminus [b^k_{op}])$

(c) $\bigsqcup_{k=1}^{12} S^o_2 \times \text{Sym}^\bullet_{Q^3}([b^k_{op}]) \times \text{Sym}^\bullet_{Q^3}(B_{op} \setminus [b^k_{op}])$

(d) $\bigsqcup_{k,l=1, k \neq l}^{12} \text{Sym}^\bullet_{Q^3}([b^k_{op}]) \times \text{Sym}^\bullet_{Q^3}([b^l_{op}]) \times \text{Sym}^\bullet_{Q^3}(B_{op} \setminus [b^k_{op}, b^l_{op}])$

(e) $\bigsqcup_{k=1}^{12} \text{Sym}^\bullet_{Q^3}([b^k_{op}]) \times \text{Sym}^\bullet_{Q^3}(B_{op} \setminus [b^k_{op}])$

(f) $\text{Diag}^\bullet$.

We also recall the notation from equation (8) from Section 5.2 and the notation

$$\text{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C^{(i)}_3.$$ 

Each part will be characterized by the type of $D$. We will consider each case a–f separately and will use the following the previously introduced notation throughout:

1. $B_i = \{b^1_i, \ldots, b^{12}_i\}$ is the set of the 12 points in $S_i$ that correspond to nodes in the fibres of the projection $\pi : S_i \to \mathbb{P}^1$.

2. $S^o_i = S_i \setminus B_i$ is the complement of $B_i$ in $S_i$.

3. $N_i \subset S_i$ are the 12 nodal fibres of $\pi : S_i \to \mathbb{P}^1$ with the nodes removed and

\[ N_i = N^\sigma_i \amalg N^\theta_i \text{ where } N^\sigma_i := N_i \cap \sigma \text{ and } N^\theta_i := N_i \setminus \sigma. \]

4. $S^\sigma_i = S^o_i \setminus N_i$ is the complement of $N_i$ in $S^o_i$ and

\[ S^\sigma_i = S^\sigma_i \amalg S^\theta_i \text{ where } S^\sigma_i := S^\sigma_i \cap \sigma \text{ and } S^\theta_i := S^\sigma_i \setminus \sigma. \]

We will also use the new notation

$$D := \{(x, x) \in S^o_1 \times S^o_1\}.$$ 

5.5.1 Part a: we have the following stratification of $S^o_1 \times S^o_1$:

1. $\left(\left(N^\sigma_1 \times N^\sigma_2\right) \cap D \amalg (S^\sigma_1 \times S^\sigma_2) \cap D\right)$
2. \( \bigcup (N_1^\sigma \times N_2^\sigma \setminus D \bigcup N_1^\sigma \times Sm_2^\sigma \bigcup Sm_1^\sigma \times N_2^\sigma \bigcup Sm_1^\sigma \times Sm_2^\sigma \setminus D ) \)

3. \( \bigcup (N_1^\sigma \times N_2^\sigma \setminus D \bigcup (N_1^\sigma \times N_2^\sigma) \cap D \bigcup N_1^\sigma \times Sm_2^\sigma \bigcup Sm_1^\sigma \times N_2^\sigma \bigcup Sm_1^\sigma \times Sm_2^\sigma \setminus D ) \)

4. \( \bigcup (N_1^\sigma \times N_2^\sigma \setminus D \bigcup (N_1^\sigma \times N_2^\sigma) \cap D \bigcup N_1^\sigma \times Sm_2^\sigma \bigcup Sm_1^\sigma \times N_2^\sigma \bigcup Sm_1^\sigma \times Sm_2^\sigma \setminus D ) \)

5. \( \bigcup (N_1^\sigma \times N_2^\sigma \setminus D \bigcup (N_1^\sigma \times N_2^\sigma) \cap D \bigcup Sm_1^\sigma \times N_2^\sigma \bigcup N_1^\sigma \times Sm_2^\sigma \bigcup Sm_1^\sigma \times Sm_2^\sigma \setminus D ) \).

Here we have grouped by the number and type of intersection with \( \sigma \).

Grouping 1: the following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 1:

\[ Z_{\geq 0} \times U \times \text{Sym}_2 \mathcal{Q}(B_{\text{op}}). \]

| \( U = (N_1^\sigma \times N_2^\sigma) \cap D \) | \( e(U) = 12 \) | \( \chi(O_D) = -1 \) |
|---|---|---|
| \( e(\eta^{-1}(a, x, m)) = Q_1 Q_2 Q_3 p \cdot \frac{(V_{\square \square \sigma})^2}{(V_{\square \sigma})^2} \cdot Q_2^{a, \Phi + \Theta}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\sigma, \sigma}(m_i) \) | \( U = (Sm_1^\sigma \times Sm_2^\sigma) \cap D \) | \( e(U) = -10 \) | \( \chi(O_D) = -1 \) |

From Lemmas 6.3.2 and 6.1, we have

1. \( V_{\emptyset \emptyset \emptyset} = M(p) \)
2. \( V_{\square \square \emptyset} = M(p) \frac{1}{1-p} \)
3. \( V_{\square \emptyset \emptyset} = M(p) \frac{p^{2} - p + 1}{p(1-p)^2} \)
4. \( \sum_{m \geq 0} Q^m \Phi^{\emptyset, \emptyset}(m) = M(p)^2 \prod_{m > 0} (1 + p^m Q)^m \)
5. \( \sum_{m \geq 0} Q^m \Phi^{\emptyset, \emptyset}(m) = M(p)^2 (1 + \psi_0 + (\psi_1 + 2\psi_0) Q + \psi_0 Q^2) \prod_{m > 0} (1 + p^m Q)^m. \)
So the contribution is

\[ Q_1 Q_2^2 M(p)^2 \prod_{m > 0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \left( \psi_0 + (\psi_1 + 2\psi_0) Q_\sigma + \psi_0 Q_3^2 \right) \cdot \left( \frac{2(p^4 + 8p^3 - 12p^2 + 8p + 1)}{(p - 1)^2p} \right). \]

**Grouping 2:** the following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 2:

| \( U = N_1^\sigma \times N_2^\sigma \setminus D \) | \( e(U) = 132 \) | \( \chi(\mathcal{O}_D) = 0 \) |
|---|---|---|
| \( e(\eta_1^{-1}(a, x, m)) = Q_1 Q_2 Q_3^2 \frac{(V_{\square \square \square})^2}{(V_{\square \square \square})^2} \cdot Q_\sigma^a \Phi^{-\cdot}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i) \) |
| \( U = N_1^\sigma \times \text{Sym}_2^\ast (B_{op}) \times \text{Sym}_2^\ast (B_{op}) \) | \( e(U) = -120 \) | \( \chi(\mathcal{O}_D) = 0 \) |
| \( e(\eta_2^{-1}(a, x, m)) = Q_1 Q_2 Q_3^2 \frac{(V_{\square \square \square})^2}{(V_{\square \square \square})^2} \cdot Q_\sigma^a \Phi^{-\cdot}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i) \) |
| \( U = N_1^\sigma \times N_2^\sigma \setminus D \) | \( e(U) = 110 \) | \( \chi(\mathcal{O}_D) = 0 \) |
| \( e(\eta_3^{-1}(a, x, m)) = Q_1 Q_2 Q_3^2 \frac{1}{(V_{\square \square \square})^2} \cdot Q_\sigma^a \Phi^{-\cdot}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i) \) |

From Lemmas 6.3.2 and 6.1, we have

1. \( V_{\square \square \square} = M(p) \)
2. \( V_{\square \square \square} = M(p)^{\frac{1}{1-p}} \)
3. \( V_{\square \square \square} = M(p)^{\frac{p^2 - p + 1}{p(1-p)^2}} \)
4. $\sum_{m \geq 0} Q^m \Phi^{0,0}(m) = M(p)^2 \prod_{m > 0} (1 + p^m Q)^m$

5. $\sum_{m \geq 0} Q^m \Phi^{-,-}(m) = M(p)^2 \frac{1}{p} (\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0Q^2) \prod_{m > 0} (1 + p^m Q)^m$.

So the contribution is

$$Q_1Q_2Q_3^2M(p)^{24} \left( \prod_{m > 0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) \frac{1}{p} (\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0Q^2) \cdot \left( \frac{2(p^4 + 8p^3 + 48p^2 + 8p + 1)}{(p - 1)^2} \right).$$

**Grouping 3:** the following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 3:

$$\mathbb{Z}_{\geq 0} \times U \times \operatorname{Sym}^*_{Q_3}(B_{op}).$$

| $U$ | $e(U)$ | $\chi(\mathcal{O}_D)$ |
|-----|--------|-------------------|
| $U = N_1^\sigma \times N_2^\sigma \setminus D$ | $\eta^{-1}(a, x, m) = Q_1Q_2Q_3^2p^2\frac{(V_{\sigma,\theta})^2}{(V_{\theta,\theta})(V_{\varphi,\varphi})^5} \cdot Q_\sigma^3\Phi^{-,-}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i)$ | $-132$ | $0$ |
| $U = (N_1^\sigma \times N_2^\sigma) \cap D$ | $\eta^{-1}(a, x, m) = Q_1Q_2Q_3^2p^2\frac{(V_{\sigma,\theta})^2}{(V_{\theta,\theta})(V_{\varphi,\varphi})^5} \cdot Q_\sigma^3\Phi^{-,-}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i)$ | $-12$ | $-1$ |
| $U = N_1^\sigma \times \operatorname{Sm}_2^\theta$ | $\eta^{-1}(a, x, m) = Q_1Q_2Q_3^2p^2\frac{(V_{\sigma,\theta})^2}{(V_{\theta,\theta})(V_{\varphi,\varphi})^5} \cdot Q_\sigma^3\Phi^{-,-}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i)$ | $120$ | $0$ |
| $U = \operatorname{Sm}_1^\sigma \times N_2^\theta$ | $\eta^{-1}(a, x, m) = Q_1Q_2Q_3^2p^2\frac{(V_{\sigma,\theta})^2}{(V_{\theta,\theta})(V_{\varphi,\varphi})^5} \cdot Q_\sigma^3\Phi^{-,-}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i)$ | $120$ | $0$ |
| $U = \operatorname{Sm}_1^\sigma \times N_2^\theta$ | $\eta^{-1}(a, x, m) = Q_1Q_2Q_3^2p^2\frac{(V_{\sigma,\theta})^2}{(V_{\theta,\theta})(V_{\varphi,\varphi})^5} \cdot Q_\sigma^3\Phi^{-,-}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{0,0}(m_i)$ | $120$ | $0$ |
From Lemmas 6.3.2, 6.3.4, and 6.1, we have

1. \( V_{\emptyset \emptyset \emptyset} = M(p) \)
2. \( V_{\square \emptyset \emptyset} = M(p) \frac{1}{1-p} \)
3. \( V_{\square \square \emptyset} = M(p) \frac{p^2 - p + 1}{p (1-p)^2} \)
4. \( \sum_{m>0} Q^m \Phi^{\beta, \beta}(m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^m \)
5. \( \sum_{m>0} Q^m \Phi^{-\beta}(m) = M(p)^2 \frac{1 + Q}{1-p} \prod_{m>0} (1 + p^m Q)^m. \)

The contribution from grouping 3 is

\[
Q_1 Q_2 Q_3^2 M(p)^2 \left( \prod_{m>0} (1 + p^m Q) (1 + p^m Q_3)^{12m} \right) \left( \frac{1 + Q_\sigma}{1-p} \right) \cdot \left( \frac{2 (p^2 + 10p + 1) (p^4 - 2p^3 + 8p^2 - 2p + 1)}{(p - 1)^2 p} \right).
\]

Grouping 4: the results for grouping 4 are identical to those of grouping 3 under the symmetry of the banana 3-fold.

The contribution from grouping 4 is

\[
Q_1 Q_2 Q_3^2 M(p)^2 \left( \prod_{m>0} (1 + p^m Q) (1 + p^m Q_3)^{12m} \right) \left( \frac{1 + Q_\sigma}{1-p} \right) \cdot \left( \frac{2 (p^2 + 10p + 1) (p^4 - 2p^3 + 8p^2 - 2p + 1)}{(p - 1)^2 p} \right).
\]
Grouping 5: the following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 5:

\[ Z_{\geq 0} \times U \times \text{Sym}^*_Q(R_{\text{op}}). \]

| \( U = N_1^0 \times N_2^0 \setminus D \) | \( e(U) = 132 \) | \( \chi(\mathcal{O}_D) = 0 \) |
|---|---|---|
| \( e(\eta^{-1}_*(a, x, m)) = Q_1 Q_2 Q_3^2 \left( \frac{V_{\theta_0 \theta}}{V_{\theta_0 \theta}} \right)^2 \cdot Q_{\sigma}^a \Phi_0^\theta, \theta(a) \cdot \prod_{i=1}^{12} Q_{m_i}^3 \Phi_0^\theta, \theta(m_i) \) | | |

| \( U = (N_1^0 \times N_2^0) \cap D \) | \( e(U) = 12 \) | \( \chi(\mathcal{O}_D) = -1 \) |
|---|---|---|
| \( e(\eta^{-1}_*(a, x, m)) = Q_1 Q_2 Q_3^2 \left( \frac{V_{\theta_0 \theta}}{V_{\theta_0 \theta}} \right)^2 \cdot Q_{\sigma}^a \Phi_0^\theta, \theta(a) \cdot \prod_{i=1}^{12} Q_{m_i}^3 \Phi_0^\theta, \theta(m_i) \) | | |

| \( U = S_m^0 \times N_2^0 \) | \( e(U) = -120 \) | \( \chi(\mathcal{O}_D) = 0 \) |
|---|---|---|
| \( e(\eta^{-1}_*(a, x, m)) = Q_1 Q_2 Q_3^2 \left( \frac{V_{\theta_0 \theta}}{V_{\theta_0 \theta}} \right)^2 \cdot Q_{\sigma}^a \Phi_0^\theta, \theta(a) \cdot \prod_{i=1}^{12} Q_{m_i}^3 \Phi_0^\theta, \theta(m_i) \) | | |

| \( U = N_1^0 \times S_m^0 \) | \( e(U) = -120 \) | \( \chi(\mathcal{O}_D) = 0 \) |
|---|---|---|
| \( e(\eta^{-1}_*(a, x, m)) = Q_1 Q_2 Q_3^2 \left( \frac{V_{\theta_0 \theta}}{V_{\theta_0 \theta}} \right)^2 \cdot Q_{\sigma}^a \Phi_0^\theta, \theta(a) \cdot \prod_{i=1}^{12} Q_{m_i}^3 \Phi_0^\theta, \theta(m_i) \) | | |

| \( U = S_m^0 \times S_m^0 \setminus D \) | \( e(U) = 110 \) | \( \chi(\mathcal{O}_D) = 0 \) |
|---|---|---|
| \( e(\eta^{-1}_*(a, x, m)) = Q_1 Q_2 Q_3^2 \left( \frac{V_{\theta_0 \theta}}{V_{\theta_0 \theta}} \right)^2 \cdot Q_{\sigma}^a \Phi_0^\theta, \theta(a) \cdot \prod_{i=1}^{12} Q_{m_i}^3 \Phi_0^\theta, \theta(m_i) \) | | |

| \( U = (S_m^0 \times S_m^0) \cap D \) | \( e(U) = -10 \) | \( \chi(\mathcal{O}_D) = -1 \) |
|---|---|---|
| \( e(\eta^{-1}_*(a, x, m)) = Q_1 Q_2 Q_3^2 \left( \frac{V_{\theta_0 \theta}}{V_{\theta_0 \theta}} \right)^2 \cdot Q_{\sigma}^a \Phi_0^\theta, \theta(a) \cdot \prod_{i=1}^{12} Q_{m_i}^3 \Phi_0^\theta, \theta(m_i) \) | | |
From Lemmas 6.3.2 and 6.1, we have

1. \( V_{\emptyset \emptyset \emptyset} = M(p) \)
2. \( V_{\Box \emptyset \emptyset} = M(p) \frac{1}{1-p} \)
3. \( V_{\Box \Box \emptyset} = M(p) \frac{p^2 - p + 1}{p(1-p)^2} \)
4. \( \sum_{m \geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m > 0} (1 + p^m Q)^m \)

Summing the contributions from the above groupings, we arrive at the overall contribution from part a:

\[
e(\mathbb{Z}_{\geq 0} \times S_1^\sigma \times S_2^\sigma \times \text{Sym}^\bullet_{Q_3}(B_{\text{op}}), (\eta_\bullet)_* 1) = Q_1 Q_2 Q_3^2 M(p)^{24} \left( \prod_{m > 0} (1 + p^m Q_3)^m \right)^{12m} \left( \frac{2 (p^2 + 10p + 1) (p^4 - 2p^3 + 8p^2 - 2p + 1)}{(p - 1)^4 p} \right) = Q_1 Q_2 Q_3^2 Q_\sigma M(p)^{24} \left( \prod_{m > 0} (1 + p^m Q_3)^m (1 + p^m Q_3)^{12m} \right) \cdot \left( 120 \psi_0 + Q_\sigma \left( 144 \psi_0^2 + 48 \psi_0 + 4 \right) \right).
\]

5.5.2 Part b–c: by the symmetry of \( X \), we only need to consider part b, with part c being completely analogous. For each \( k \in \{1, \ldots, 12\} \), we begin by decomposing \( S_1^\sigma \) into the following six parts:

\[
S_1^\sigma \cup S_2^\emptyset \cup N_1^{\sigma,(k)} \cup N_1^{\emptyset,c} \cup N_1^{\emptyset,(k)} \cup N_1^{\emptyset,c},
\]

where \( N_1^{\sigma,(k)} \) is the connected component of \( N_1^{\sigma} \) corresponding the \( k \)th banana configuration and \( N_1^{\emptyset,c} \) is its complement in \( N_1^{\emptyset} \). The same definition is true for \( N_1^{\emptyset} \).

We use the above size part decomposition for

\[
\mathbb{Z}_{\geq 0} \times S_1^\emptyset \times \text{Sym}^\bullet_{Q_3}(\{b_{\text{op}}^k\}) \times \text{Sym}^\bullet_{Q_3}(B_{\text{op}} \setminus \{b_{\text{op}}^k\}).
\]

The following table is the summary of results from 5.2.4 and 5.2.5 for this stratification.
From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. \( V_{\emptyset \emptyset \emptyset} = M(p) \)
2. \( V_{\square \emptyset} = M(p) \frac{1}{1-p} \)
3. \( V_{\square \square \emptyset} = M(p) \frac{p^2 - p + 1}{p(1-p)^2} \)
4. \( V_{\square \square \square} = M(p) \frac{p^4 - p^3 + p^2 - p + 1}{p^2(1-p)^3} \)
5. \( \sum_{m \geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^m \)
6. \( \sum_{m \geq 0} Q^m \Phi^{-\emptyset}(m) = M(p)^2 \frac{1+Q}{1-p} \prod_{m>0} (1 + p^m Q)^m \)
5.5.3 Part d–e: parts d and e parametrize the cases when \( D \) is the union of \( C_2^{(k)} \) and \( C_2^{(l)} \). We have the spaces:

1. \( \bigcup_{k \neq l}^{12} \text{Sym}_{\mathcal{O}_3}^\bullet ([b^k_{\text{op}}]) \times \text{Sym}_{\mathcal{O}_3}^\bullet ([b^l_{\text{op}}]) \times \text{Sym}_{\mathcal{O}_3}^\bullet (B_{\text{op}} \setminus \{b^k_{\text{op}}, b^l_{\text{op}}\}) \),

2. \( \bigcup_{k=1}^{12} \text{Sym}_{\mathcal{O}_3}^\bullet ([b^k_{\text{op}}]) \times \text{Sym}_{\mathcal{O}_3}^\bullet (B_{\text{op}} \setminus \{b^k_{\text{op}}\}) \).

The following table is the summary of results from 5.2.4 and 5.2.5 for this stratification.

| \( U = \{(k, l)\}, k \neq l \) | \( e(U) = 1 \) | \( \chi(\mathcal{O}_D) = 2 \) |
|---|---|---|
| \( e(\eta^{-1}_{\bullet}(a, c, d, m)) = Q_1 Q_2 p^2 \frac{1}{(\psi_{\theta\theta})^2} \cdot Q_3^a \Phi^{\theta,0}(a) \cdot Q_3^b \Phi^{-,-}(c) \cdot Q_3^c \Phi^{-,-}(d) \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\theta,0}(m_i) \) | | |

| \( U = \{(k, k)\} \) | \( e(U) = 1 \) | \( \chi(\mathcal{O}_D) = 0 \) |
|---|---|---|
| \( e(\eta^{-1}_{\bullet}(a, m_k, m)) = Q_1 Q_2 p^2 \frac{1}{(\psi_{\theta\theta})^2} \cdot Q_3^a \Phi^{\theta,\delta}(a) \cdot Q_3^{m_k} \Phi^{\theta,\delta}(m_k) \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\theta,0}(m_i) \) | | |

From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. \( \mathcal{V}_{\psi_{\theta\theta}} = M(p) \)
2. \( \sum_{m \geq 0} Q^m \Phi^{\theta,\theta}(m) = M(p)^2 \prod_{m > 0} (1 + p^m Q)^m \)
3. \( \sum_{m \geq 0} Q^m \Phi^{-,-}(m) = M(p)^2 \frac{1}{p} (\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0 Q^2) \prod_{m > 0} (1 + p^m Q)^m \).
There are 136 choices for two distinct fibres. Hence, the contribution from part d is

\[ e\left(\prod_{k,i=1, k \neq i}^{12} \text{Sym}_{Q_3}^\bullet \{b^k_{\text{op}}\} \times \text{Sym}_{Q_3}^\bullet \{b^i_{\text{op}}\} \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}} \setminus \{b^k_{\text{op}}, b^i_{\text{op}}\}), (\eta_\bullet)_* 1\right) \]

\[ = 132Q_1Q_2M(p)^{24} \left( \prod_{m=0} (1 + p^m Q_3)^m (1 + p^m Q_3)^{12m} \right) (\psi_0 + (\psi_1 + 2\psi_0)Q_3 + \psi_0Q_3^2)^2. \]

The 12 singular fibres give the contribution of part e as

\[ e\left(\prod_{k=1}^{12} \text{Sym}_{Q_3}^\bullet \{b^k_{\text{op}}\} \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}} \setminus \{b^k_{\text{op}}\}), (\eta_\bullet)_* 1\right) \]

\[ = 12Q_1Q_2M(p)^{24} \left( \prod_{m=0} (1 + p^m Q_3)^m (1 + p^m Q_3)^{12m} \right) \]

\[ \cdot \left( (Q_3^2\psi_0 + Q_3(2\psi_0 + \psi_1) + \psi_0)^2 + (Q_3^4(2\psi_0 + \psi_1) + Q_3^2(8\psi_0 + 6\psi_1 + \psi_2) \right. \]

\[ + Q_3^2(12\psi_0 + 10\psi_1 + 2\psi_2) \]

\[ + Q_3(8\psi_0 + 6\psi_1 + \psi_2) + (2\psi_0 + \psi_1) \right) \cdot \left(144(Q_3^2\psi_0 + Q_3(2\psi_0 + \psi_1) + \psi_0)^2 + 12(Q_3^4(2\psi_0 + \psi_1) + Q_3^2(8\psi_0 + 6\psi_1 + \psi_2) \right. \]

\[ + Q_3^2(12\psi_0 + 10\psi_1 + 2\psi_2) \]

\[ + Q_3(8\psi_0 + 6\psi_1 + \psi_2) + (2\psi_0 + \psi_1) \right) \).

Summing the contributions of parts d and e, we have

\[ Q_1Q_2M(p)^{24} \left( \prod_{m=0} (1 + p^m Q_3)^m (1 + p^m Q_3)^{12m} \right) \]

\[ \cdot \left(144(Q_3^2\psi_0 + Q_3(2\psi_0 + \psi_1) + \psi_0)^2 + 12(Q_3^4(2\psi_0 + \psi_1) + Q_3^2(8\psi_0 + 6\psi_1 + \psi_2) \right. \]

\[ + Q_3^2(12\psi_0 + 10\psi_1 + 2\psi_2) \]

\[ + Q_3(8\psi_0 + 6\psi_1 + \psi_2) + (2\psi_0 + \psi_1) \right) \).

### Part f:

Recall from Lemma 3.5.3 that part f, Diag\* has the further decomposition:

1. \( Sm_1 \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}) \)
2. \( Sm_2 \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}) \)
3. \( \bigsqcup_{y \in J} E_{\pi(y)} \times \text{Aut}(E_{\pi(y)}) \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}}) \)
4. \( \bigsqcup_{k=1}^{12} L \times \text{Sym}_{Q_3}^\bullet \{b^k_{\text{op}}\} \times \text{Sym}_{Q_3}^\bullet (B_{\text{op}} \setminus \{b^k_{\text{op}}\}) \).
where we have used the notation:

1. \( \mathcal{J}^0 \) and \( \mathcal{J}^{1728} \) to be the subsets of points \( x \in \mathbb{P}^1 \) such that \( \pi^{-1}(x) \) has \( j \)-invariant 0 or 1728, respectively, and \( \mathcal{J} = \mathcal{J}^0 \sqcup \mathcal{J}^{1728} \).

2. \( \mathcal{L} \) to be the linear system \( [f_1 + f_2] \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with the singular divisors removed where \( f_1 \) and \( f_2 \) are fibres of the two projection maps.

3. \( \widetilde{\text{Aut}}(E) := \text{Aut}(E) \setminus \{ \pm 1 \} \).

5.5.5 **Parts g-i:**

The results for parts g–i will all be very similar. The key differences are

1. The overall factor of \( Q_3 \) may be different.

2. The Euler characteristics of the space parametrizing the \( D_s \) may be different.

We define \( U \) to be one of

\[
\begin{align*}
(g) \quad & \text{Sm}_1 \ 	ext{noting that } e(\text{Sm}_1 \cap \{ \sigma \}) = -10 \text{ and } e(\text{Sm}_1 \setminus \{ \sigma \}) = 10. \\
(h) \quad & \text{Sm}_2 \ 	ext{noting that } e(\text{Sm}_2 \cap \{ \sigma \}) = -10 \text{ and } e(\text{Sm}_2 \setminus \{ \sigma \}) = 10. \\
(i) \quad & E_{\pi(y)} \text{ for } y \in \mathcal{J} \ 	ext{noting that } e(E_{\pi(y)} \cap \{ \sigma \}) = 1 \text{ and } e(E_{\pi(y)} \setminus \{ \sigma \}) = -1. 
\end{align*}
\]

| \( U \cap \{ \sigma \} \) | \( \chi(\mathcal{O}_D) = 0 \) |
|--------------------------|-----------------|
| \( e(\eta_{-1}(a, x, m_k, m)) = Q_1 Q_2 Q_3 \frac{1}{(V_{\emptyset\emptyset})^2} \cdot Q_3^a \Phi^{-, \emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^m \Phi^{, \emptyset}(m_i) \) | ![Diagram](image1.png) |

| \( U \setminus \{ \sigma \} \) | \( \chi(\mathcal{O}_D) = 0 \) |
|--------------------------|-----------------|
| \( e(\eta_{-1}(a, x, m_k, m)) = Q_1 Q_2 Q_3 \frac{1}{(V_{\emptyset\emptyset})^2} \cdot Q_3^a \Phi^{, \emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^m \Phi^{, \emptyset}(m_i) \) | ![Diagram](image2.png) |

From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. \( V_{\emptyset\emptyset} = M(p) \)

2. \( V_{\sqsupset\emptyset} = M(p)^{\frac{1}{1-p}} \)
3. \( \sum_{m \geq 0} Q^m \Phi^{0,0} (m) = M(p)^2 \prod_{m > 0} (1 + p^m Q)^m \)

4. \( \sum_{m \geq 0} Q^m \Phi^{-,0} (m) = M(p)^2 \frac{1 + Q}{1 - p} \prod_{m > 0} (1 + p^m Q)^m. \)

The overall factors of \( Q_3^n \) are calculated in 3.3.2 to be

1. \( n = 4 \) for (g) and \( n = 0 \) for (h).
2. If \( j(E) = 1728 \) and \( E \cong \mathbb{C}/i \), then
   - \( n = 2 \) occurs when \( D \) is a translation of the graph \( \{(x, \pm i\alpha)\} \).
3. If \( j(E) = 0 \) and \( E \cong \mathbb{C}/\tau \) with \( \tau = \frac{1}{2} (1 + i\sqrt{3}) \), then
   - \( n = 1 \) occurs when \( D \) is a translation of the graph \( \{(x, -\tau x)\} \) or the graph \( \{(x, (\tau - 1)x)\} \).
   - \( n = 3 \) occurs when \( D \) is a translation of the graph \( \{(x, \tau x)\} \) or the graph \( \{(x, (-\tau + 1)x)\} \).

Lastly, in a generic pencil, we have \( e(\sigma^0) = 4 \) and \( e(\sigma^{1728}) = 6 \).

Hence, the contribution for parts g–i is

\[ Q_1 Q_2 Q_3 M(p)^2 \left( \prod_{m > 0} (1 + p^m Q_3)^m (1 + p^m Q_3)^{12m} \right) \left( -10 + 8Q_3 + 12Q_3^2 + 8Q_3^3 - 10Q_3^4 \right). \]

### 5.5.6 Part j:

In the Appendix 6.2.2, we give the following decomposition for \( L \) into groupings:

1. \( \bigcup L^\sigma_{(0,0),(\infty,\infty)} \cup L^\sigma_{(0,0),(\infty,0)} \cup L^\sigma_{(0,0),(0,\infty)} \cup L^\sigma_{(\infty,\infty),(\infty,0)} \cup L^\sigma_{(\infty,\infty),(0,\infty)} \)
2. \( \bigcup L^\sigma_{(0,0)} \cup L^\sigma_{(0,0)} \cup L^\sigma_{(\infty,\infty)} \cup L^\sigma_{(\infty,\infty)} \)
3. \( \bigcup L^\sigma_{(0,\infty)} \cup L^\sigma_{(0,\infty)} \cup L^\sigma_{(\infty,0)} \cup L^\sigma_{(\infty,0)} \)
4. \( \bigcup L^\sigma_{\emptyset} \cup L^\sigma_{\emptyset} \).

The Euler characteristics of the parts of this decomposition are computed in 6.2.3 and the overall factors of \( Q_1, Q_2 \) and \( Q_3 \) are calculated in Lemma 3.4.4.

**Grouping 1:** the following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 1:

\[ \mathbb{Z}_{\geq 0} \times U \times \text{Sym}_{Q_1}^\bullet (\{b_{op}^k\}) \times \text{Sym}_{Q_3}^\bullet (B_{op} \setminus \{b_{op}^k\}). \]
Note that the vertex is different for $L_{(0,0), (\infty, \infty)}$ as described in 4.4.8.

| $U = L^\sigma_{(0,0), (\infty, \infty)}$ | $e(U) = 1$ | $\chi(\mathcal{O}_D) = 1$ |
|-------------------------------------|-------------|------------------|
| $e(\eta^{-1}(a, x, m_k, m)) = $ | $Q_1Q_2Q_3^2p\frac{1}{(V_{\theta \theta})(V_{\theta \theta})} Q_3^a \Phi^{-,\theta}(a)Q_3^{m_k} \Phi^{-,\theta}(m_k) \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\theta, \theta}(m_i)$ | ![Diagram](image) |

| $U = L^\theta_{(0,0), (\infty, \infty)}$ | $e(U) = -1$ | $\chi(\mathcal{O}_D) = 1$ |
|-------------------------------------|-------------|------------------|
| $e(\eta^{-1}(a, x, m_k, m)) = $ | $Q_1Q_2Q_3^2p\frac{1}{(V_{\theta \theta})(V_{\theta \theta})} Q_3^a \Phi^{+,\theta}(a)Q_3^{m_k} \Phi^{+,\theta}(m_k) \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\theta, \theta}(m_i)$ | ![Diagram](image) |

| $U = L^\sigma_{(0,0), (\infty, 0)}$ | $e(U) = 1$ | $\chi(\mathcal{O}_D) = 0$ |
|-------------------------------------|-------------|------------------|
| $e(\eta^{-1}(a, x, m_k, m)) = $ | $Q_1Q_2\frac{1}{(V_{\theta \theta})^2} Q_3^a \Phi^{-,\theta}(a)Q_3^{m_k} \Phi^{-,\theta}(m_k) \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\theta, \theta}(m_i)$ | ![Diagram](image) |

| $U = L^\theta_{(0,0), (\infty, 0)}$ | $e(U) = -1$ | $\chi(\mathcal{O}_D) = 0$ |
|-------------------------------------|-------------|------------------|
| $e(\eta^{-1}(a, x, m_k, m)) = $ | $Q_1Q_2\frac{1}{(V_{\theta \theta})^2} Q_3^a \Phi^{+,\theta}(a)Q_3^{m_k} \Phi^{+,\theta}(m_k) \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\theta, \theta}(m_i)$ | ![Diagram](image) |

From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. $V_{\emptyset \emptyset} = M(p)$
2. $V_{\emptyset \emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m \geq 0} Q^m \Phi^{\emptyset, \emptyset}(m) = M(p)^2 \prod_{m \geq 0} (1 + p^m Q)^m$
4. $\sum_{m \geq 0} Q^m \Phi^{-, \emptyset}(m) = M(p)^2 \frac{1 + Q}{1-p} \prod_{m \geq 0} (1 + p^m Q)^m$
5. $\sum_{m \geq 0} Q^m \Phi^{+, \emptyset}(m) = M(p)^2 (\psi_0 + (\psi_1 + 2\psi_0) Q + \psi_0 Q^2) \prod_{m \geq 0} (1 + p^m Q)^m$
6. $\sum_{m \geq 0} Q^m \Phi^{-, -}(m) = M(p)^2 (\psi_0 + (2\psi_0 + \psi_1) Q + (\psi_0 + \psi_1) Q^2) \prod_{m \geq 0} (1 + p^m Q)^m$
7. $\sum_{m \geq 0} Q^m \Phi^{+, \emptyset}(m) = M(p)^2 (\psi_1 + \psi_0 + (\psi_1 + 2\psi_0) Q + \psi_0 Q^2) \prod_{m \geq 0} (1 + p^m Q)^m$. 
So after accounting for the 12 singular fibres, we have the contribution from grouping 1 as

\[ Q_1 Q_2 M(p)^{24} \left( \prod_{m>0} (1 - p^m Q_\sigma)^m (1 - p^m Q_3)^{12m} \right) \]

\[ \cdot 12 Q_\sigma Q_3^2 \left( (\psi_0 + \psi_1) + Q_3 (2\psi_0 + \psi_1) + 2Q_3^2 \psi_0 + Q_3^3 (\psi_0 + \psi_1) \right). \]

**Grouping 2:** we compute the results for \( L_{(0,0)} \) with \( L_{(\infty,\infty)} \) being completely analogous. The following table is the summary of results from \S 5.2.4 and \S 5.2.5 for the strata in grouping 2:

\[ \mathbb{Z}_{\geq 0} \times U \times \text{Sym}^\bullet_{Q_3} \{ b^{k}_{\text{op}} \} \times \text{Sym}^\bullet_{Q_3} (B_{\text{op}} \setminus \{ b^{k}_{\text{op}} \}). \]

| \( U = L_{(0,0)}^\sigma \) | \( e(U) = -1 \) | \( \chi(\mathcal{O}_D) = 1 \) |
|-------------------|------------------|------------------|
| \( e(\eta^{-1}(a, x, m_k, m)) = \) | \( Q_1 Q_2 Q_3^2 \left( \frac{1}{\sqrt[12]{(\psi_0 + \psi_1)}} \right) Q_3^m \Phi^{\psi_0, \psi_1} (a) Q_3^m \Phi^{\psi_0, \psi_1} (m_k) \prod_{i=1, i \neq j}^{12} Q_3^m \Phi^{\psi_0, \psi_1} (m_i) \) | \|
| \( U = L_{(0,0)}^\emptyset \) | \( e(U) = 1 \) | \( \chi(\mathcal{O}_D) = 1 \) |
| \( e(\eta^{-1}(a, x, m_k, m)) = \) | \( Q_1 Q_2 Q_3^2 \left( \frac{1}{\sqrt[12]{(\psi_0 + \psi_1)}} \right) Q_3^m \Phi^{\psi_0, \psi_1} (a) Q_3^m \Phi^{\psi_0, \psi_1} (m_k) \prod_{i=1, i \neq j}^{12} Q_3^m \Phi^{\psi_0, \psi_1} (m_i) \) | \|

From Lemmas 6.3.2, 6.1 and 6.3.4, we have:

1. \( V_{\oplus \emptyset \emptyset} = M(p) \)
2. \( V_{\boxtimes \emptyset \emptyset} = M(p) \frac{1}{1-p} \)
3. \( \sum_{m \geq 0} Q^m \Phi^{\psi_0, \psi_1} (m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^m \)
4. \( \sum_{m \geq 0} Q^m \Phi^{\psi_0, \psi_1} (m) = M(p)^2 \frac{1+Q}{1-p} \prod_{m>0} (1 + p^m Q)^m \)

Accounting for both \( L_{(0,0)} \) and \( L_{(\infty,\infty)} \), the contribution for grouping 2 is

\[ Q_1 Q_2 M(p)^{24} \left( \prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) \]

\[ \cdot (-24) Q_\sigma Q_3^2 (\psi_0 + Q_3 \psi_0). \]
Grouping 3: we compute the results for $L_{(0,\infty)}$ with $L_{(\infty,0)}$ being completely analogous. The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 3:

$$Z_{\geq 0} \times U \times \text{Sym}^*_{Q_3}([b^k_{\text{op}}]) \times \text{Sym}^*_{Q_3}(B_{\text{op}} \setminus \{b^k_{\text{op}}\}).$$

| $U = L_{(0,\infty)}^\sigma$ | $e(U) = -1$ | $\chi(\mathcal{O}_D) = 1$ |
|-----------------------------|-------------|-----------------|
| $e(\eta^{-1}(a, x, m_k, m_m)) = Q_1Q_2Q_3p\frac{1}{(V_{\emptyset\emptyset})^2}Q^\emptyset_3^\emptyset_3\Phi^{-\emptyset}(a)Q_{3m_k}^\Phi\Phi^{-\emptyset}(m_k) \prod_{i=1}^{12}Q_{3i}^\Phi\Phi^{-\emptyset}(m_i)$ | $\sigma$ |
| $U = L_{(0,\infty)}^\emptyset$ | $e(U) = 1$ | $\chi(\mathcal{O}_D) = 1$ |
| $e(\eta^{-1}(a, x, m_k, m_m)) = Q_1Q_2Q_3p\frac{(V_{\emptyset\emptyset})^2}{(V_{\emptyset\emptyset})^3}Q^\emptyset_3^\emptyset_3\Phi^{-\emptyset}(a)Q_{3m_k}^\Phi\Phi^{-\emptyset}(m_k) \prod_{i=1}^{12}Q_{3i}^\Phi\Phi^{-\emptyset}(m_i)$ | $\sigma$ |

From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. $\mathcal{V}_{\emptyset\emptyset} = M(p)$
2. $\mathcal{V}_{\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m=0}^{\infty} Q^m \Phi^{\emptyset, \emptyset}(m) = M(p)^2 \prod_{m=0}^{\infty} (1 + p^m Q)^m$
4. $\sum_{m=0}^{\infty} Q^m \Phi^{-\emptyset, \emptyset}(m) = M(p)^2 \frac{1 + Q}{1 - p} \prod_{m=0}^{\infty} (1 + p^m Q)^m$.

Accounting for both $L_{(0,\infty)}$ and $L_{(\infty,0)}$, the contribution for grouping 3 is

$$Q_1Q_2M(p)^{24} \left( \prod_{m=0}^{\infty} (1 + p^m Q_\emptyset)^m (1 + p^m Q_3)^{12m} \right) \cdot (-24)Q_\emptyset Q_3^2 \left( \psi_0 + Q_3 \psi_0 \right).$$

Grouping 4: the following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping 4:

$$Z_{\geq 0} \times U \times \text{Sym}^*_{Q_3}([b^k_{\text{op}}]) \times \text{Sym}^*_{Q_3}(B_{\text{op}} \setminus \{b^k_{\text{op}}\}).$$
From Lemmas 6.3.2, 6.1 and 6.3.4, we have

1. $V_{\emptyset\emptyset} = M(p)$

2. $V_{\square\emptyset} = M(p) \frac{1}{1-p}$

3. $\sum_{m \geq 0} Q^n \Phi^{0,0}(m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^n$

4. $\sum_{m \geq 0} Q^n \Phi^{-,0}(m) = M(p)^2 \frac{1 + Q}{1 - p} \prod_{m>0} (1 + p^m Q)^n$.

So the contribution for grouping 4 is

$$Q_1 Q_2 M(p)^{24} \left( \prod_{m>0} (1 + p^m Q)^n (1 + p^m Q_3)^{12m} \right) \cdot 24 Q_3 Q_3^2 \psi_0.$$ 

Combining groupings 1–4, we have the overall contribution for part j is

$$Q_1 Q_2 M(p)^{24} \left( \prod_{m>0} (1 + p^m Q)^n (1 + p^m Q_3)^{12m} \right) \cdot 12 Q_3 \left( (\psi_0 + \psi_1) + Q_3 \psi_1 + Q_3^3 \psi_1 + Q_3^4 (\psi_0 + \psi_1) \right).$$

6. Appendix

6.1. Connected invariants and their partition functions

For the rank four sub-lattice $\Gamma \subset H_2(X,\mathbb{Z})$ generated by a section and banana curves, we can consider the connected unweighted Pandharipande–Thomas invariants. They are defined formally via the following partition function

$$\hat{Z}_{\Gamma,\text{Con}}^{\text{PT}}(X) := \log \left( \frac{\hat{Z}_\Gamma(X)}{\hat{Z}_{\emptyset\emptyset}(0,\emptyset)|_{Q_i=0}} \right).$$
For the partition function in Theorem A, we consider the first terms of the expansion in $Q_\sigma$ and $Q_1$:

$$\frac{\hat{Z}_\Gamma(X)}{Z_{(0,\star,\star)}|_{Q_i=0}} = \frac{\hat{Z}_{(0,\star,\star)}(0,\star,\star)}{Z_{(0,\star,\star)}|_{Q_i=0}} + Q_\sigma \frac{\hat{Z}_{\sigma+(0,\star,\star)}(0,\star,\star)}{Z_{(0,\star,\star)}|_{Q_i=0}} + \cdots$$

$$= \frac{\hat{Z}_{(0,\star,\star)}(0,\star,\star)}{Z_{(0,\star,\star)}|_{Q_i=0}} \left(1 + Q_\sigma \frac{\hat{Z}_{\sigma+(0,\star,\star)}}{Z_{(0,\star,\star)}} + \cdots\right).$$

So the first terms of the expansion in $Q_\sigma$ and $Q_1$ of the connected partition function are

$$\hat{Z}_{\Gamma,\text{Con}}(X) = \frac{\hat{Z}_{(0,\star,\star)}(0,\star,\star)}{Z_{(0,\star,\star)}|_{Q_i=0}} - Q_\sigma \frac{\hat{Z}_{\sigma+(0,\star,\star)}}{Z_{(0,\star,\star)}} + \cdots.$$

In particular, we have the connected version of $\hat{Z}_{\sigma+(0,\star,\star)}$ as

$$\hat{Z}_{\Gamma,\text{Con}}^{\sigma+(0,\star,\star)} = \frac{-p}{(1-p)^2} \prod_{m>0} \frac{1}{(1 - Q^m_2 Q^m_3)^8 (1 - p Q^m_2 Q^m_3)^2 (1 - p^{-1} Q^m_2 Q^m_3)^2},$$

proving Corollary B. For the partition function in Theorem C, we consider the first terms of the expansion in $Q_1$ and $Q_2$:

$$\frac{\hat{Z}_\Gamma(X)}{Z_{(0,\star,\star)}|_{Q_i=0}} = \frac{\hat{Z}_{(0,\star,\star)}(0,\star,\star)}{Z_{(0,\star,\star)}|_{Q_i=0}} \left(1 + Q_1 \frac{\hat{Z}_{\sigma+(1,0,\star)}(0,\star,\star)}{Z_{\sigma+(0,0,\star)}(0,\star,\star)} + Q_2 \frac{\hat{Z}_{\sigma+(0,1,\star)}(0,\star,\star)}{Z_{\sigma+(0,0,\star)}(0,\star,\star)} + Q_1 Q_2 \frac{\hat{Z}_{\sigma+(1,1,\star)}(0,\star,\star)}{Z_{\sigma+(0,0,\star)}(0,\star,\star)} + \cdots\right).$$

So the first terms of the expansion in $Q_1$ and $Q_2$ of the connected partition function are

$$\hat{Z}_{\Gamma,\text{Con}}^{\text{PT}}(X) = \log \left(\frac{\hat{Z}_{(0,\star,\star)}(0,\star,\star)}{Z_{(0,\star,\star)}|_{Q_i=0}}\right) - Q_1 \frac{\hat{Z}_{\sigma+(1,0,\star)}(0,\star,\star)}{Z_{\sigma+(0,0,\star)}(0,\star,\star)} - Q_2 \frac{\hat{Z}_{\sigma+(0,1,\star)}(0,\star,\star)}{Z_{\sigma+(0,0,\star)}(0,\star,\star)}$$

$$+ Q_1 Q_2 \left(\frac{\hat{Z}_{\sigma+(1,0,\star)}(0,\star,\star) - \hat{Z}_{\sigma+(0,1,\star)}(0,\star,\star)}{(\hat{Z}_{\sigma+(0,0,\star)}(0,\star,\star))^2} - \frac{\hat{Z}_{\sigma+(1,1,\star)}(0,\star,\star)}{\hat{Z}_{\sigma+(0,0,\star)}(0,\star,\star)}\right) + \cdots.$$
In particular, we have the connected version of $\hat{Z}_{\sigma+(0,0,\bullet)}$ as

$$\hat{Z}_{\sigma+(0,0,\bullet)}^{\text{PT,Con}} = \log\left(\frac{\hat{Z}_{\sigma+(0,0,\bullet)}}{\hat{Z}_{(0,\bullet,\bullet)}}|_{Q_i=0}\right)$$

$$= \log\left(\prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m}\right)$$

$$= \sum_{n>0} \frac{p^n}{(1-p^n)^2} \frac{(-Q_\sigma)^n}{n} + \sum_{n>0} 12 \frac{p^n}{(1-p^n)^2} \frac{(-Q_3)^n}{n}$$

and the connected version of $\hat{Z}_{\sigma+(0,1,\bullet)}$ (and also of $\hat{Z}_{\sigma+(1,0,\bullet)}$) given by

$$\hat{Z}_{\sigma+(0,1,\bullet)}^{\text{PT,Con}} = -\left(12 \psi_0 + Q_3(24 \psi_0 + 12 \psi_1) + Q_3^2(12 \psi_0) + Q_\sigma Q_3(\psi_0 + 2 \psi_1)\right)$$

and the connected version of $\hat{Z}_{\sigma+(1,1,\bullet)}$ given by

$$\hat{Z}_{\sigma+(1,1,\bullet)}^{\text{PT,Con}} = \left(12 (Q_3^4(2 \psi_0 + \psi_1) + Q_3^3(8 \psi_0 + 6 \psi_1 + \psi_2) + Q_3^2(12 \psi_0 + 10 \psi_1 + 2 \psi_2)\right)$$

$$+ Q_3(8 \psi_0 + 6 \psi_1 + \psi_2) + (2 \psi_0 + \psi_1))\right)$$

$$+ Q_\sigma \left(12 \psi_0 + 2 \psi_1) + Q_3(48 \psi_0 + 44 \psi_1) + Q_3^2(216 \psi_0 + 108 \psi_1 + 24 \psi_2)\right)$$

$$+ Q_3^3(48 \psi_0 + 44 \psi_1) + Q_3^4(12 \psi_0 + 2 \psi_1)\right).$$

Corollary D now follows immediately.

6.2. Linear system in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we consider a stratification of the following linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ with strata determined by the intersections of the associated divisors with a collection of points.

Consider the fibres of the projection maps $pr_i: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and a fibre from each $f_i$. The linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the sum of a fibre from each map is $|f_1 + f_2| = \mathbb{P}^3$. This is the collection of bi-homogeneous polynomials of degree $(1,1)$:

$$\left\{ ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1 = 0 \mid \{a : b : c : d\} \in \mathbb{P}^3 \right\}.$$
6.2.1 There are five points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are of interest to us:

\[
\sigma = (1:1, 1:1) \quad \text{and} \quad \mathcal{P} := \{(0, 0), (0, \infty), (\infty, 0), (\infty, \infty)\},
\]

where we have used the standard notation $0 = [0:1]$ and $\infty = [1:0]$. We will decompose $|f_1 + f_2|$ into strata based on which points the divisor intersects. Consider a divisor $D \in |f_1 + f_2|$. Then $D$ passes through

1. $(0, 0)$ if and only if $d = 0$;
2. $(0, \infty)$ if and only if $c = 0$;
3. $(\infty, 0)$ if and only if $b = 0$;
4. $(\infty, \infty)$ if and only if $a = 0$.

6.2.2 Define the following convenient notation for $y, x \in \mathcal{P}$:

1. $\text{Sing} \subset |f_1 + f_2|$ is the subset of singular divisors.
2. $L_y \subset (|f_1 + f_2| \setminus \text{Sing})$ is the subset of smooth curves not passing through any points of $\mathcal{P}$.
3. $L_x \subset (|f_1 + f_2| \setminus \text{Sing})$ is the subset of smooth curve passing through $x$ but no other points of $\mathcal{P}$.
4. $L_{x,y} \subset (|f_1 + f_2| \setminus \text{Sing})$ is the subset of smooth curve passing through $x$ and $y$ but no other points of $\mathcal{P}$.
5. Also let $L_y^\sigma, L_x^\sigma$ and $L_{x,y}^\sigma$ be subsets of $L_y, L_x$ and $L_{x,y}$, respectively, with the further condition that the curves pass through $\sigma$.
6. Let $L_y^\emptyset, L_x^\emptyset$ and $L_{x,y}^\emptyset$ be the complements of $L_y^\sigma, L_x^\sigma$ and $L_{x,y}^\sigma$ in $L_y, L_x$ and $L_{x,y}$, respectively.
With this notation, we have the following decomposition of \( |f_1 + f_2| \):

\[
|f_1 + f_2| = \text{Sing} \sqcup L_{(0,0),\infty} \sqcup L_{(0,\infty),\infty} \\
\sqcup L_{(0,0)} \sqcup L_{(0,\infty)} \sqcup L_{(\infty,0)} \sqcup L_{(\infty,\infty)} \\
\sqcup L_{\emptyset}.
\]

6.2.3 We now consider the strata of this collection and their Euler characteristics:

\textbf{ban:} a curve in \( |f_1 + f_2| \) is singular if and only if the equation for the curve factorizes:

\[
ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1 = (\alpha x_0 + \beta x_1)(\gamma y_0 + \delta y_1) = 0.
\]

where \([\alpha : \beta], [\gamma : \delta] \in \mathbb{P}^1 \). Hence, \( \text{Sing} \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and the Euler characteristic is \( e(\text{ban}) = e(|f_1 + f_2|) = 4 \).

\textbf{L}_{x,y}: we consider for \( x = (0,0) \) and \( y = (\infty, \infty) \) with the case \((0, \infty) \) and \((\infty, 0) \) being completely analogous. The points \([a : b : d : c] \in |f_1 + f_2| \) correspond to a curve passing through \( x \) and \( y \) if and only if \( a = d = 0 \). Moreover, this is singular when either \( b = 0 \) or \( c = 0 \). Hence, \( L_{x,y} \cong \mathbb{P}^1 \setminus \{0, \infty\} \) and \( e(L_{x,y}) = 0 \).

\textbf{L}_x: the set \( L_x \) is when \( b+c = 0 \), which is a point in \( \mathbb{P}^1 \). So we have \( e(L_{x,0}) = 1 \) and \( e(L_{x,\infty}) = -1 \).

\textbf{L}_y: we consider the case \( x = (0,0) \) with the other cases being completely analogous. So the subspace of all divisors passing through \( x \) is \([a : b : d : c] \in |f_1 + f_2| \) where \( d = 0 \). This is a \( \mathbb{P}^2 \subset \mathbb{P}^3 \). The subspace where the curve doesn’t pass through one of the other points is where \( a, b, c \neq 0 \) which is given by \( \mathbb{C}^* \times \mathbb{C}^* \cong \mathbb{P}^2 \setminus \{(a = 0) \cup \{(b = 0) \cup \{c = 0\}\). None of the equations for these curves factorize since such a factorization would require either \( b = 0 \) or \( c = 0 \). Hence, \( L_x \cong \mathbb{C}^* \times \mathbb{C}^* \) and \( e(L_x) = 0 \).

The subset \( L_x^\sigma \) is defined by the further condition \( a + b + c = 0 \) which gives

\[
L_x^\sigma = \left\{ [a : b : c] \in \mathbb{P}^2 \mid a, b, c \neq 0 \text{ and } a + b = 1 \right\} \cong \mathbb{C}^* \setminus \text{pt.}
\]

Hence, we have the Euler characteristics \( e(L_x^\sigma) = -1 \) and \( e(L_x^\emptyset) = 1 \).

\textbf{L}_{\emptyset}: the set of curves not passing through any points of \( P \) is given by

\[
\left\{ [a : b : c : d] \in |f_1 + f_2| \mid a, b, c, d \neq 0 \right\} \cong \left\{ b, c, d \in (\mathbb{C}^*)^3 \right\}.
\]

The singular curves are given by the factorization condition:

\[
x_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1 = (x_0 + \beta x_1)(y_0 + \delta y_1)
\]

which is the condition that \( d = bc \). So the subspace of curves which are singular is \((\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^3 \). Hence, \( L_{\emptyset} \cong \{(b, c, d) \in (\mathbb{C}^*)^3 \mid b \neq dc \} \) and \( e(L_{\emptyset}) = 0 \).
\( \mathbb{L}_0^\sigma \) is given by the further condition that \( b + c + d = 0 \), so we have
\[
\mathbb{L}_0^\sigma \cong \left\{ (b, c, d) \in (\mathbb{C}^*)^3 \mid d \neq bc \text{ and } 1 + b + c + d = 0 \right\}
\cong \left\{ (b, c) \in (\mathbb{C}^*)^2 \mid (b + 1)(c + 1) \neq 0 \text{ and } b + c \neq -1 \right\}
\cong \left\{ (b, c) \in (\mathbb{C}^* \setminus \{-1\})^2 \left| b + c \neq -1 \right. \right\}
\cong (\mathbb{C}^* \setminus \{-1\})^2 - (\mathbb{C} \setminus \{2\text{pt}\}).
\]

Hence, we have the Euler characteristics \( e(\mathbb{L}_0^\sigma) = 2 \) and \( e(\mathbb{L}_0^\emptyset) = -2 \).

### 6.3. Topological vertex formulas

In this section of the appendix, we collect some useful formulas for partition functions involving the topological vertex.

Define the ‘MacMahon’ notation:
\[
M(p, Q) = \prod_{m > 0} (1 - p^m Q)^{-m}
\]
and the simpler version \( M(p) = M(p, 1) \).

**Lemma 6.3.1** We have the equality
\[
V_{\lambda \square \square} V_{\lambda \emptyset \emptyset} = \frac{1}{p} V_{\lambda \emptyset \emptyset} V_{\lambda \emptyset \square} + V_{\lambda \square \emptyset} V_{\lambda \emptyset \square}.
\]

**Proof.** We prove the equivalent equation
\[
\frac{V_{\square \square v}}{V_{\emptyset \emptyset v}} = \frac{1}{p} + \frac{V_{\emptyset \emptyset v}}{(V_{\emptyset \emptyset v})^2}.
\]

From the definition, we have
\[
\frac{V_{\square \square v}}{V_{\emptyset \emptyset v}} = \frac{1}{p} \sum_{\eta \subset \square} S_{\square/\eta}(p^{-v - \rho}) S_{\square/\eta}(p^{-v' - \rho})
= \frac{1}{p} \left( S_{\square/\emptyset}(p^{-v - \rho}) S_{\square/\emptyset}(p^{-v' - \rho}) + S_{\square/\square}(p^{-v - \rho}) S_{\square/\square}(p^{-v' - \rho}) \right)
= \frac{V_{\emptyset \emptyset v} V_{\lambda \emptyset \square}}{(V_{\emptyset \emptyset v})^2} + \frac{1}{p}.
\]

\( \square \)
Lemma 6.3.2  We have

1. \( V_{\emptyset\emptyset\emptyset} = M(p) \)
2. \( V_{\Box\emptyset\emptyset} = M(p) \frac{1}{1-p} \)
3. \( V_{\Box\Box\emptyset} = M(p) \frac{p^2-p+1}{p(1-p)^2} \)
4. \( V_{\Box\Box\Box} = M(p) \frac{p^4-p^3+p^2-p+1}{p^2(1-p)^3} \).

Proof. Part 1 is immediate from the definition. For part 2, we have

\[
V_{\Box\emptyset\emptyset} = M(p)p^{-\frac{1}{2}}S_\emptyset(p^{-\rho}) \sum_\eta S_{\Box/\eta}(p^{-\rho})S_{\emptyset/\eta}(p^{-\rho})
\]
\[
= M(p) \frac{1}{1-p}.
\]

For part 3, we have

\[
V_{\Box\Box\emptyset} = M(p)p^{-1}S_\emptyset(p^{-\rho}) \sum_\eta S_{\Box/\eta}(p^{-\rho})S_{\Box/\eta}(p^{-\rho})
\]
\[
= M(p)p^{-1}\left(S_{\Box/\emptyset}(p^{-\rho})S_{\Box/\emptyset}(p^{-\rho}) + S_{\Box/\Box}(p^{-\rho})S_{\Box/\Box}(p^{-\rho})\right)
\]
\[
= M(p)p^{-1}\left(\frac{p}{(1-p)^2} + 1\right)
\]
\[
= M(p)\frac{p^2-p+1}{p(1-p)^2}.
\]

Part 4 follows from parts 2 and 3 and lemma 6.3.1:

\[
V_{\Box\Box\Box} = \frac{1}{p} V_{\Box\emptyset\emptyset} + \frac{V_{\Box\Box\emptyset} V_{\Box\Box\Box}}{V_{\emptyset\emptyset\emptyset}}.
\]

6.3.3 It is shown in \[3, \text{Section 4.3}\] that

\[
\sum_{v,\alpha,\mu} Q_1^{v_1} Q_2^{[\alpha]} Q_3^{[\mu]} p^{\frac{1}{2}(\|v\|^2+\|v\|^2+\|\alpha\|^2+\|\alpha\|^2+\|\mu\|^2+\|\mu\|^2)} (V_{v\mu\alpha} V_{v^\prime\mu^\prime\alpha^\prime}).
\]

\[
= \prod_{d_1, d_2, d_3 \geq 0} \prod_k (1 - (-Q_1)^{d_1} (-Q_2)^{d_2} (-Q_3)^{d_3} p^k)^{-c(k)}.
\]
Lemma 6.3.4 We have the following equalities:
\[
\sum_{\alpha \in \mathcal{A}} \sum_{k \in \mathbb{Z}} c(\alpha, k) Q^\alpha y^k := \frac{\sum_{k \in \mathbb{Z}} Q^{2k} (-y)^k}{\left( \sum_{k \in \mathbb{Z} + 1/2} Q^{2k} (-y)^k \right)^2} = \frac{\vartheta_4(2\tau, z)}{\vartheta_4(4\tau, z)^2}
\]

and \(\|d\| := 2d_1d_2 + 2d_2d_3 + 2d_3d_1 - d_1^2 - d_2^2 - d_3^2\). Also, if we recall the notation that
\[
\psi_{r_k} := \left( \frac{p}{(1-p)^2} \right)^{1-g}
\]
then we have the following corollary.

Lemma 6.3.5 We have
\[
1. \sum_{\alpha, \mu} Q_2^{\mu|\alpha|} Q_3^{\mu|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2 + \|\alpha\'|^2 + \|\mu\|^2 + \|\mu\'|^2)} (V_{\psi_0, (\alpha \mu)} V_{(\alpha \mu)\alpha'}) = M(p)^2 \prod_{m > 0} \left( 1 - Q_2^m Q_3^m \right) M(p - Q_2^m Q_3^m) M(p - Q_2^{m-1} Q_3^m)
\]
\[
2. \sum_{\alpha} Q^{\alpha|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2 + \|\alpha\'|^2)} (V_{\psi_0, (\alpha \mu)} V_{(\alpha \mu)\alpha'}) = M(p)^2 \prod_{m > 0} \left( 1 + Q_2^m Q_3^m \right)
\]
\[
3. \sum_{\alpha} Q^{\alpha|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2 + \|\alpha\'|^2) + 1} (V_{\psi_0, (\alpha \mu)} V_{(\alpha \mu)\alpha'}) = M(p)^2 (\psi_0 + (\psi_1 + 2\psi_0) Q + \psi_0 Q^2) \prod_{m > 0} \left( 1 + p^m Q \right)^m
\]
\[
4. \sum_{\alpha} Q^{\alpha|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2 + \|\alpha\'|^2) + 2} (V_{\psi_0, (\alpha \mu)} V_{(\alpha \mu)\alpha'}) = M(p)^2 \prod_{m > 0} \left( 1 + p^m Q \right)^m Q^3 (8\psi_0 + 6\psi_1 + \psi_2)
\]
\[
+ Q^2 (12\psi_0 + 10\psi_1 + 2\psi_2) + Q(8\psi_0 + 6\psi_1 + \psi_2) + (2\psi_0 + \psi_1).\]

Proof: These are all coefficients of the partition function in 6.3.3. For example part (3) is the coefficient of \(Q_1 Q_0^0\). □

Lemma 6.3.5 We have the following equalities:
\[
1. \sum_{\alpha} Q^{\alpha|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2 + \|\alpha\'|^2)} (V_{\psi_0, (\alpha \mu)} V_{(\alpha \mu)\alpha'}) = M(p)^2 \frac{1 + Q}{1 - p} \prod_{m > 0} \left( 1 + p^m Q \right)^m
\]
\[
2. \sum_{\alpha} Q^{\alpha|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2 + \|\alpha\'|^2) + 1} (V_{\psi_0, (\alpha \mu)} V_{(\alpha \mu)\alpha'}) = M(p)^2 (\psi_0 + \psi_1) Q + \psi_0 Q^2 \prod_{m > 0} \left( 1 + p^m Q \right)^m
\]
3. $\sum_{\alpha} Q^{\alpha} p^{1/2(\|\alpha\|^2 + \|\alpha'\|^2)} + 1 (V_{\emptyset\emptyset \emptyset})^2 = M(p)^2 (\psi_0 + (2\psi_0 + \psi_1)Q + (\psi_0 + \psi_1)Q^2) \prod_{m>0} (1 + p^mQ)^m$.

Proof. Part (1) is given by

$$\sum_{\alpha} Q^{\alpha} p^{1/2(\|\alpha\|^2 + \|\alpha'\|^2)} (V_{\emptyset\emptyset \emptyset})^2 = p^{-1/2} M(p)^2 \sum_{\alpha} Q^{\alpha} \sum_{\eta} S_{\alpha'/\eta}(p^{-\rho})S_{\emptyset'/\eta}(p^{-\rho}) \sum_{\delta} S_{\alpha'/\delta}(p^{-\rho})S_{\emptyset'/\delta}(p^{-\rho})$$

$$= p^{-1/2} M(p)^2 \sum_{\alpha} Q^{\alpha} \left( S_{\alpha'/\emptyset}(p^{-\rho})S_{\emptyset'}(p^{-\rho}) + S_{\alpha'/\emptyset}(p^{-\rho}) \right) S_{\alpha}(p^{-\rho})$$

$$= p^{-1/2} M(p)^2 \left( S_{\emptyset'}(p^{-\rho}) \sum_{\alpha_{\emptyset'\emptyset'}} S_{\emptyset'\emptyset'}(p^{-\rho})S_{\alpha'/\emptyset}(Qp^{-\rho}) + \sum_{\alpha_{\emptyset'\emptyset'}} S_{\emptyset'\emptyset'}(p^{-\rho})S_{\alpha'/\emptyset}(Qp^{-\rho}) \right).$$

After applying [10, Equation 2, p. 96], the equation becomes

$$p^{-1/2} \cdot M(p)^2 \prod_{i,j>0} (1 + p^{i+j}Q)$$

$$\left( S_{\emptyset'}(p^{-\rho}) \sum_{\tau \subset \emptyset} S_{\emptyset'/\tau}(p^{-\rho})S_{\emptyset'/\tau}(Qp^{-\rho}) + \sum_{\tau \subset \emptyset} S_{\emptyset'/\tau}(p^{-\rho})S_{\emptyset'/\tau}(Qp^{-\rho}) \right)$$

$$= p^{-1/2} \cdot M(p)^2 \prod_{m>0} (1 + p^mQ)^m (1 + Q) \frac{p^{1/2}}{1 - p}.$$

Part (2) follows from Lemmas 6.1 and 6.3.1:

$$\sum_{\alpha} Q^{\alpha} p^{1/2(\|\alpha\|^2 + \|\alpha'\|^2)+1} (V_{\emptyset\emptyset \emptyset \emptyset})^2 = \sum_{\alpha} Q^{\alpha} p^{1/2(\|\alpha\|^2 + \|\alpha'\|^2)} (V_{\emptyset\emptyset \emptyset \emptyset})^2 + \sum_{\alpha} Q^{\alpha} p^{1/2(\|\alpha\|^2 + \|\alpha'\|^2)+1} (V_{\emptyset\emptyset \emptyset \emptyset})^2.$$
Part (3) is given by

\[
\sum_{\alpha} Q^{||\alpha||^2+\|\alpha''\|^2} Q^{\|\alpha\|^2+1} (V_{\square \kappa})^2
\]

\[
= \sum_{\alpha} Q^{\|\alpha\|^2+1} (V_{\alpha} \square) \cdot p^{\|\alpha''\|^2+1} (V_{\varphi \kappa})
\]

\[
= M(p)^2 \sum_{\alpha} Q^{\|\alpha\|^2} S_\square (p^{-\rho}) \sum_{\delta} S_{\alpha/\delta} (p^{-\square -\rho}) S_{\varphi/\delta} (p^{-\varphi -\rho})
\]

\[
= M(p)^2 S_\square (p^{-\rho}) \sum_{\alpha} S_\alpha (Q p^{-\square -\rho}) \left( S_{\alpha'} (p^{-\rho}) S_\square (p^{-\rho}) + S_{\alpha' \square} (p^{-\rho}) \right).
\]

After applying [10, Equation 2, p. 96], the equation becomes

\[
M(p)^2 S_\square (p^{-\rho}) (1 + Q) \prod_{m>0} (1 + Q p^m)^m \left( S_\square (p^{-\rho}) + S_\square (p^{-\square -\rho}) \right).
\]

The result follows from a quick computation involving \(V_{\varphi \square} = V_{\square \varphi}\) showing that

\[
S_\square (p^{-\rho}) S_\square (p^{-\square -\rho}) = S_\square (p^{-\rho})^2 + 1.
\]

Lemma 6.3.6  The following are true:

1. \[\sum_{\alpha} Q^{||\alpha||^2} = \prod_{d>0} \frac{1}{1-Q^d}\]

2. \[\sum_{\alpha} Q^{||\alpha||^2} \frac{(V_{\alpha \square})(V_{\alpha} \varphi)}{(V_{\alpha \varphi})(V_{\alpha \varphi})} = \prod_{d>0} \frac{(1-Q^d)}{1-p Q^d (1-p^{-1}Q^d)}\]

3. \[\sum_{\alpha} p^{\|\alpha\|^2} Q^{||\alpha||^2} \frac{(V_{\alpha \alpha' \varphi})(V_{\alpha \varphi})}{(V_{\alpha \varphi})(V_{\alpha \varphi})} = \prod_{d>0} \frac{M(p, Q^d)}{(1-Q^d)}\]

4. \[\sum_{\alpha} p^{\|\alpha\|^2} Q^{||\alpha||^2} \frac{(V_{\alpha \alpha' \varphi})(V_{\alpha \varphi})}{(V_{\alpha \varphi})(V_{\alpha \varphi})} = \prod_{d>0} \frac{M(p, Q^d)}{(1-p Q^d)(1-p^{-1}Q^d)}\]

Proof. The first is a classical result and the other three are the content of [6, Theorem 3].

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