New Fundamental Equation for Classical Waves and Its Physical Applications

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Abstract. The irreducible representations of the extended Galilean group are used to derive the symmetric and asymmetric, in time and space derivatives, wave equations. It is shown that among these equations only a new asymmetric wave equation is fundamental. By being fundamental the equation gives the most complete description of propagating waves as it accounts for the Doppler effect, forward and backward waves, and makes the wave speed to be the same in all inertial frames. To demonstrate these properties, the equation is applied to acoustic waves propagation in an isothermal atmosphere. The derived fundamental wave equation plays the same role for classical waves as Newton’s law of inertia plays for classical particles.

1. Introduction

In modern physics, a dynamical equation is called fundamental if it is local, has its Lagrangian and remains invariant with respect to the spatio-temporal transformations that form a group of the metric, and internal symmetries that form a gauge group [1]; the latter is related to interactions, which do not affect the description of free particles and waves. The law of inertia for classical particles [2-4], and the Schrödinger [5] and Levy-Leblond [6] equations for quantum particles are examples of nonrelativistic fundamental equations. Moreover, all basic equations of relativistic classical and quantum physics are fundamental [1].

The wave equation is one of the best known equations in classical physics and it describes the propagation of waves in a given background medium [7-9]; however, the wave equation is not fundamental. Therefore, the main purpose of this paper is to find a new fundamental wave equation for freely propagating classical waves, and demonstrate its properties by applying it to acoustic wave propagation in an isothermal atmosphere.

In nonrelativistic physics, space and time are Galilean and their metrics are $ds_1^2 = dx^2 + dy^2 + dz^2$ and $ds_2^2 = dt^2$, respectively, with $x$, $y$ and $z$ being the spatial coordinates and $t$ being time. All transformations that leave the Galilean metrics unchanged form the Galilean group of the metric [9,10], which means that the metrics preserve their forms in all inertial frames, and observers associated with these frames are called *Galilean observers*. 
For free particles in Classical Mechanics (CM), the law of inertia is fundamental as it is local with its well-known Lagrangian [2-4], and it is also invariant with respect to the Galilean group of the metric [11], which means that for all Galilean observers the form of the equation describing this law is the same. On the other hand, the second law of dynamics may or may not be Galilean invariant depending on the form of its force [2-4].

The Galilean group of the metric can be extended to make its structure similar to that of the Poincaré group [11,12]. Let \( G_e \) be the extended Galilean group [9-12] with its mathematical structure \( G_e = [O(3) \otimes s B(3)] \otimes s [T(3+1) \otimes U(1)] \), where \( O(3) \) and \( B(3) \) are subgroups of rotations and boosts, respectively. In addition, \( T(3+1) \) is an invariant Abelian subgroup of combined translations in space and time, and \( U(1) \) is a one-parameter unitary subgroup. The subgroup \( T(3+1) \) plays an important role in \( G_e \) because its irreducible representations (irreps) are well-known [13,14] and they provide labels for all the irreps of \( G_e \) [13-15].

The Schrödinger equation of Quantum Mechanics (QM) is Galilean invariant since its form remains the same when all transformations of \( G_e \) are applied to it; however, the invariance requires that the phase factor is introduced [5,16-18]. By being Galilean invariant, local and with its Lagrangian known, the Schrödinger equation is the fundamental equation of QM as its form remains the same for all Galilean observers. Moreover, the scalar wavefunction of the equation transforms as one of the irreps of \( G_e \), which guarantees that all Galilean observers identify the same physical object represented by the function.

The Levy-Leblond equation [6] whose spinor wavefunction describes elementary particles with spin in nonrelativistic QM is linear, has Lagrangian, and is also Galilean invariant, which means that it is fundamental. Thus, the Levy-Leblond and Schrödinger equations are two fundamental wave equations of nonrelativistic QM, and they describe particles with and without spin, respectively.

Theories of classical waves are based on the wave equation that is not fundamental because the characteristic wave speed is a frame-dependent quantity; Galilean invariance requires the wave speed to be the same in all inertial frames of references. However, for classical waves the Galilean law of adding speeds or velocities in moving frames must be used, thus, violating Galilean invariance. It must be noted that Schrödinger equations were used in theories of classical waves [19-23] but those equations were not Galilean invariant.

In relativistic classical physics, the wave equation for electromagnetic waves is fundamental since the speed of light remains the same in all inertial frames. However, there is not similar equation in nonrelativistic physics, thus, the main aim of this paper is to derive such an equation by following the previous work [24] in which a new asymmetric wave equation was found and used to formulate a theory of cold dark matter [25]. Here, the conditions for the new asymmetric wave equation to become a fundamental wave equation for classical waves are established and discussed. To compare the wave description given by the non-fundamental and fundamental wave equations,
both equations are used to describe the propagation of acoustic waves in an isothermal atmosphere and to determine Lamb’s cutoff frequency.

The paper is organized as follows: in Section 2, the basic equations are derived and discussed; wave equations and their Lagrangians are obtained in Section 3; Galilean invariance of the wave equations is investigated in Section 4; applications of the obtained results to acoustic wave propagation are presented in Section 5; and conclusions are given in Section 6.

2. Derivation of symmetric and asymmetric equations

The invariant Abelian subgroup \( T(3 + 1) \) of combined translations in space and time plays an important role in \( \mathcal{G}_e \) because its irreducible representations (irreps) are well-known [12-15] and they provide labels for all the irreps of \( \mathcal{G}_e \) [13,14]. The conditions that the scalar wavefunction \( \phi(t, x) \) transforms as one of the irreps of \( \mathcal{G}_e \) are given by the following eigenvalue equations [17,18] (see the Appendix A for their derivation):

\[
\frac{i}{\partial t} \phi(t, x) = \omega \phi(t, x),
\]

and

\[
-i \nabla \phi(t, x) = k \phi(t, x),
\]

where \( \phi(t, x) \) is an eigenfunction of the generators of \( T(3 + 1) \), and the eigenvalues \( \omega \) and \( k \) are real constants that label the irreps. The generator of translation in time is \( \hat{E} = i \partial / \partial t \), and the generator of translations in space is \( \hat{P} = -i \nabla \), with \( [\hat{E}, \hat{P}] = 0 \). The group also has the generator of boosts given by \( \hat{V} = t \hat{P} \), which means that the eigenvalues for these two operators must be the same [17,18]. The fact that \( \phi(t, x) \) obeys Eqs (1) and (2) and transforms as one of the irreps of \( \mathcal{G}_e \) means that all Galilean observers identify the same object, which is a wave under consideration, and their description of this wave is identical.

The obtained eigenvalue equations can be used to derive all wave equations of physics for scalar wavefunctions that are allowed to exist in the Galilean space and time. In general, the derived dynamical equations can be divided into two separate families, namely, the symmetric equations, with the same order of space and time derivatives, and the asymmetric equations, with different orders of space and time derivatives [17]. Moreover, the equations can be of any order [18] but in this paper only the second-order equations are considered.

The only second-order symmetric equation that can be derived from the eigenvalue equations is given by

\[
\left[ \frac{\partial^2}{\partial t^2} - C_1 \nabla^2 \right] \phi(t, x) = 0,
\]

where \( C_1 = \omega^2/k^2 \), with \( k^2 = (k \cdot k) \). Since \( C_1 \) is a real constant coefficient of arbitrary value, there is an infinite set of these second-order equations, and they are called the wave-like equations [24].
Two different asymmetric second-order equations resulting from Eqs. (1) and (2) can also be obtained [24]

\[
\left[ i \frac{\partial}{\partial t} + C_2 \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
\]

(4)

and

\[
\left[ \frac{\partial^2}{\partial t^2} - i C_3 \mathbf{k} \cdot \nabla \right] \phi(t, \mathbf{x}) = 0 ,
\]

(5)

where \( C_2 = \omega/k^2 \) and \( C_3 = \omega^2/k^2 = C_1 \) are arbitrary constants. This means that there are two infinite sets of the second-order asymmetric equations.

The form of the asymmetric equation given by Eq. (4) is the same as that of the Schrödinger equation [5], except the coefficient \( C_2 \). Therefore, all equations of the same form as Eq. (4) are called Schrödinger-like equations. However, the equations given by Eq. (5) with different coefficients \( C_3 \) are called new asymmetric equations [24]. It must also be noted that the constants \( C_1, C_2 \) and \( C_3 \) are expressed in terms of the eigenvalues, which label the irreps of \( G_e \).

In the previous work [24], it was shown that by using the de Broglie relationship [5], the coefficient \( C_2 \) expressed in terms of the labels of the irreps \( \omega \) and \( k \) can be evaluated, and it becomes the same as the coefficient in the Schrödinger equation of QM [5]. The obtained Schrödinger equation does not include any potentials, which means that it describes free quantum particles of ordinary matter. Moreover, the coefficient \( C_3 \) of the new asymmetric equation was also evaluated and the resulting equation was used to describe a quantum structure of dark matter particles [25]. In the following, the coefficients \( C_1, C_2 \) and \( C_3 \) are evaluated in such a way that the resulting equations describe classical waves.

3. Wave equations for classical waves

By identifying the labels of the irreps \( \omega \) and \( k \) as the wave frequency and wave number, respectively, the coefficients \( C_1, C_2 \) and \( C_3 \) can be evaluated and expressed in terms of the characteristic wave speed, \( c_w = \omega/k \). Then, Eq. (3) becomes

\[
\left[ \frac{\partial^2}{\partial t^2} - c_w^2 \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
\]

(6)

which is the well-known standard wave equation SWE [7-9]. Moreover, the Schrödinger-like and new asymmetric wave equations for classical waves can be written as

\[
\left[ i \frac{\partial}{\partial t} + \frac{c_w^2}{\omega} \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
\]

(7)

and

\[
\left[ \frac{\partial^2}{\partial t^2} - ic_w^2 \mathbf{k} \cdot \nabla \right] \phi(t, \mathbf{x}) = 0 .
\]

(8)

It is seen that the obtained standard, Schrödinger-like, and new asymmetric wave equations are of different forms and yet they can be used to describe free propagation...
of classical waves. By making the Fourier transforms in space and time, the same dispersion relation \( \omega = kc_w \) is obtained for each one of the above wave equations.

The derived wave equations are second-order, thus, they are local, which is one of the requirements for them to be fundamental. Since the wave equations describe freely propagating waves, the requirement of gauge invariance [1] does not have to be considered. However, it still remains to determine whether these equations have Lagrangians and whether they are Galilean invariant or not.

4. Lagrangians for wave equations

The Lagrangian formalism requires prior knowledge of a Lagrangian, and then it shows how to obtain the resulting dynamical equation from this Lagrangian. Typically, the Lagrangians are presented without explaining their origin because there no methods to derive them from first principles. Historically, most equations of physics were established first and only then their Lagrangians were found, often by guessing. Once the Lagrangians are known, the process of finding the resulting equations is straightforward and it requires substitution of these Lagrangians into the Euler-Lagrange (E-L) equation. Despite some progress in deriving Lagrangians for physical systems described by ordinary differential equations (ODEs) (e.g., [26-31]), similar work for partial differential equations (PDEs) has only limited applications (e.g., [9,32,33]).

Let \( L(\phi, \partial_t \phi, \nabla \phi) \), where \( \partial_t = \partial/\partial t \), be a Lagrangian that satisfies the E-L equation

\[
\frac{\partial L}{\partial \phi} - \partial_t \left( \frac{\partial L}{\partial (\partial_t \phi)} \right) - \nabla \cdot \left( \frac{\partial L}{\partial (\nabla \phi)} \right) = 0 .
\]

(9)

Substituting \( L(\phi, \partial_t \phi, \nabla \phi) \) into Eq. (9) gives the required dynamical equation if, and only if, the Lagrangian is a priori known. In case the equation is given first, its Lagrangian must be constructed in such a way that when substituted into Eq. (9) the desired dynamical equation is obtained; this is the Lagrangian formalism.

Since the SWE given by Eq. (6) is hyperbolic, its Lagrangian can be constructed [9,32] and the result is

\[
L_{\text{swe}}(\partial_t \phi, \nabla \phi) = \frac{1}{2} \left[ c_w^{-2} (\partial_t \phi)^2 - (\nabla \phi)^2 \right] .
\]

(10)

It is easy to verify that substitution of the Lagrangian \( L = L_s(\partial_t \phi, \nabla \phi) \) into Eq. (9) gives the required symmetric wave-like equation.

The Schrödinger-like wave equation given by Eq. (7) is parabolic, thus, its Lagrangian must be of a special form that involves both \( \phi \) and its complex conjugate \( \phi^* \) [32]. The form of this Lagrangian is

\[
L_{\text{Sch}}(\phi, \phi^*, \partial_t \phi, \partial_t \phi^*, \nabla \phi, \nabla \phi^*) = \frac{i}{2} (\phi^* \partial_t \phi - \phi \partial_t \phi^*) - \frac{c_w^2}{\omega}(\nabla \phi^* ) \cdot (\nabla \phi) .
\]

(11)

This Lagrangian gives the Schrödinger-like wave equation when substituted to the E-L equation for the variations in \( \phi^* \). On the other hand, the variations in \( \phi \) lead to the complex conjugate Schrödinger-like wave equation, which becomes important when the
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probability density $|\phi|^2$ is required to be calculated. However, in theories of classical waves $|\phi|^2$ does not play significant role as it does in QM [5].

To find the Lagrangian for the new asymmetric wave equation given by Eq. (8), the Lagrangian for the Schrödinger-like wave equation must be modified as

$$L_{asy}(\phi, \phi^*, \partial_t \phi, \partial_t \phi^*, \nabla \phi, \nabla \phi^*) = (\partial_t \phi^*)(\partial_t \phi) - \frac{i}{2} c^2 \left[ \phi (k \cdot \nabla \phi^*) - \phi^* (k \cdot \nabla \phi) \right].$$ (12)

Then, this Lagrangian is substituted to the E-L equation

$$\frac{\partial L_{asy}}{\partial \phi^*} - \partial_t \left( \frac{\partial L_{asy}}{\partial (\partial_t \phi^*)} \right) - (k \cdot \nabla) \cdot \left( \frac{\partial L_{asy}}{\partial (k \cdot \nabla \phi^*)} \right) = 0,$$ (13)

and the new asymmetric wave equation (see Eq. 8) is obtained.

The presented results demonstrate that Lagrangians exist for the symmetric and asymmetric wave equations, and that these equations are local. Therefore, the last requirement for a wave equation to be called fundamental is its Galilean invariance, which is now investigated.

5. Galilean invariance of wave equations

5.1. Known fundamental equations of nonrelativistic physics

Let $S$ and $S'$ be two intertial frames moving with respect to each other with the velocity $\mathbf{v} = \text{const}$; this allows to write a boost as $\mathbf{x} = \mathbf{x}' + \mathbf{v}t'$ with $t' = t$. Then the Galilean metric in space is $ds^2 = d\mathbf{x} \cdot d\mathbf{x} = dx^2 + dy^2 + dz^2$ with $ds^2 = ds'^2$, and in time $dt^2 = dt'^2$.

By performing the Galilean transformations (translations in space and time, rotations and boosts) that form the Galilean group of the metric [11], or the extended Galilean group $\mathcal{G}_e$ [10], Galilean invariance of the metrics can be verified. The invariance means that the forms of the metrics remain the same for all Galilean observers.

Similarly, for a dynamical equation to be Galilean invariant, it is required that the form of the equation remains the same in all inertial frames; this means that the coefficients of this equation must also be the same in all inertial frames. With the equation retaining its form, solutions of this equation are also the same for all Galilean observers. The simplest example is the second-order ODE describing the law of inertia, whose invariance with respect to all transformations that form the Galilean group of the metrics is well-known [3,11]. It is also known that the Lagrangian of the law of inertia is not Galilean invariant [2,3,11]; however, it was recently shown that Galilean invariance of the Lagrangian can be restored by using the so-called null Lagrangians [34]. Thus, the law of inertia is the fundamental equation of classical mechanics.

As shown above, space and time in Galilean relativity are separated and obey different metrics. Therefore, for dynamical equations to be Galilean invariant they must be asymmetric in space and time derivatives. The ODE describing the law of inertia is asymmetric as it does not have any space derivative. However, among the wave equations obtained in this paper and given by Eqs. (6), (7) and (8), the SWE is symmetric and the two other wave equations are asymmetric. As a result, the SWE
is not Galilean invariant and, thus, it is not fundamental. In other words, for classical particles, the law of inertia is the fundamental equation but there is no corresponding fundamental equation for classical waves; the main objective of this paper is find such an equation and apply it to wave theories.

The Schrödinger equation of QM is asymmetric and its Galilean invariance is well-known and it requires a phase factor, whose form is frame-dependent [5,10,16-18]. The existence of this phase factor makes the wavefunction to be different for each Galilean observer, which may imply that the equation is not Galilean invariance. However, the presence of the phase factor in the solutions does not violate Galilean invariance because in QM only the square of the absolute value of the wavefunction is the measureable quantity, and this quantity remains the same for all Galilean observers. Thus, the Schrödinger equation for free quantum particles is the fundamental equation of QM. Similarly, the Levy-Leblond equation for its spinor wavefunction is Galilean invariant and fundamental equation of QM [6,10]. It must also be pointed out that the Lagrangians for the Schrödinger and Levy-Leblond equations are Galilean invariant.

Having demonstrated that the law of inertia and the Schrödinger and Levy-Leblond equations are the fundamental equations of nonrelativistic physics, and that the SWE cannot be a fundamental equation for classical waves, it remains now to determine whether Eq. (7) and Eq. (8) are fundamental.

5.2. Schrödinger-like wave equation

Applying the Galilean transformations to Eq. (7), the transformed Schrödinger-like wave equation can be written as

\[ i \frac{\partial}{\partial t'} + c_w^2 \frac{\omega'}{\omega} \nabla'^2 \phi'(t', \mathbf{x}') = 0, \]  

(14)

where the original and transformed wavefunctions are related by

\[ \phi(t, \mathbf{x}) = \phi(t', \mathbf{x}' + \mathbf{v}t') = \phi'(t', \mathbf{x}') e^{i\eta(t', \mathbf{x}')}, \]  

(15)

with the phase factor being given by

\[ \eta(t', \mathbf{x}') = \frac{\omega'}{2c_w^2} \left( \mathbf{v} \cdot \mathbf{x}' + \frac{v^2 t'^2}{2} \right). \]  

(16)

For the obtained transformed Schrödinger-like wave equation to be Galilean invariant, it is also required that \( c_w^2 / \omega = c_w^2 / \omega' \). This condition is satisfied when

\[ \mathbf{k}' = \mathbf{k} - \frac{\omega}{2c_w^2} \mathbf{v}, \]  

(17)

and

\[ \omega' = \omega \left( 1 + \frac{v^2}{4c_w^2} \right) - \mathbf{k} \cdot \mathbf{v}. \]  

(18)

The above results demonstrate that the Schrödinger-like wave equation preserves its form in all inertial frames if, and only if, the wavefunction transforms according to Eq. (15) and Eqs. (17) and (18) are satisfied. The existence of the phase factor given by
Eq. (16), which is a frame-dependent quantity, is well-known and its presence does not violate Galilean invariance of the Schrödinger equation in quantum mechanics because of its requirement that only $|\phi(t, x)|^2 = |\phi'(t', x')|^2$ must be valid for all Galilean observers [5,10,16-18].

For classical waves, the wavefunction $\phi(t, x)$ represents one of the physical variables describing a wave; thus, to get the same wave description by all Galilean observers, the solutions for $\phi(t, x)$ and $\phi'(t', x')$ must be the same in all inertial frames. However, they are not because of the presence of the phase factor (see Eq. 15) that is different in different inertial frames. As a result, Galilean observes describe waves differently in their inertial frames, which means that the Schrödinger-like equation for classiacal waves is not Galilean invariant and therefore it is not fundamental.

5.3. New asymmetric wave equation

Let $\phi(t, x)$ be the wavefunction of Eq. (8) and $\phi(x', t')$ be the transformed wavefunction. After performing the Galilean transformations, Eq. (8) becomes

$$\left[ \frac{d^2}{dt'^2} - ic'_w k \cdot \nabla \right] \phi(t', x') = \left[ 2(v \cdot \nabla') \frac{\partial}{\partial t'} - (v \cdot \nabla')^2 \right] \phi(t', x') \, .$$

Comparison of this equation to Eq. (8) shows that its LHS is of the same form as the new asymmetric wave equation if, and only if, the RHS is zero. Let $\phi'(x', t') = \phi(r')$, where $r' = x' + vt'/2$. Then, with $\phi(x', t') = \phi(r')$, the LHS of Eq. (19) can be written as

$$\left[ \frac{d^2}{dr'^2} - i \left( \frac{2c'_w}{v} \right)^2 k' \cdot \frac{d}{dr'} \right] \phi(r') = 0 \, .$$

Using the Galilean transformations, the variable $r'$ is transformed to $r' = x - vt/2 \equiv r$. With $r' = r$ and $\phi(r') = \phi(r)$, Eq. (8) becomes

$$\left[ \frac{d^2}{dr^2} - i \left( \frac{2c'_w}{v} \right)^2 k \cdot \frac{d}{dr} \right] \phi(r) = 0 \, ,$$

which is of the same form as Eq. (20) if, and only if, $c'_w = c_w$ and $k' = k$. To show that these conditions are valid, let the phase of a wave in the inertial frame $S'$ be given by

$$k' \cdot r' = k' \cdot x' + \frac{1}{2} (k' \cdot v) t' = k' \cdot x' + \frac{1}{2} \left( \frac{\omega'}{c'_w} \right) (\hat{k}' \cdot \hat{v}) vt' \, ,$$

where $\hat{k}$ and $\hat{v}$ are unit vectors corresponding to $k$ and $v$, respectively. It must also noted that the dispersion relation $\omega' = k'c'_w$ was also used to obtain the wave phase.

After the Galilean transformations, Eq. (22) becomes

$$k' \cdot r' = k' \cdot x - \frac{1}{2} \left( \frac{\omega'}{c'_w} \right) (\hat{k}' \cdot \hat{v}) vt \, .$$

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On the other hand, the wave phase in the inertial frame $S$ is

$$\mathbf{k} \cdot \mathbf{r} = \mathbf{k} \cdot \mathbf{x} - \frac{1}{2} \left( \frac{\omega}{c_w} \right) (\mathbf{k} \cdot \dot{\mathbf{v}}) vt.$$  \hspace{1cm} (24)

The requirement of Galilean invariance is that the phases are the same, $\mathbf{k'} \cdot \mathbf{r'} = \mathbf{k} \cdot \mathbf{r}$, thus, one obtains

$$\mathbf{k}' = \mathbf{k} \text{ and } \frac{\omega'}{c_w'} = \frac{\omega}{c_w}.$$  \hspace{1cm} (25)

With $\mathbf{k}' = \mathbf{k} = \text{const}$ (see Eq. [2]), Eqs (19) and (20) can be written in the following form

$$\frac{d}{d(\mathbf{k}' \cdot \mathbf{r'})} \left[ \frac{d}{d(\mathbf{k} \cdot \mathbf{r})} - i \left( \frac{2c_w'}{v} \right)^2 \right] \phi(\mathbf{k}' \cdot \mathbf{r'}) = 0.$$  \hspace{1cm} (26)

and

$$\frac{d}{d(\mathbf{k} \cdot \mathbf{r})} \left[ \frac{d}{d(\mathbf{k} \cdot \mathbf{r})} - i \left( \frac{2c_w}{v} \right)^2 \right] \phi(\mathbf{k} \cdot \mathbf{r}) = 0.$$  \hspace{1cm} (27)

Since $\mathbf{k}' \cdot \mathbf{r'} = \mathbf{k} \cdot \mathbf{r}$, $\phi(\mathbf{k}' \cdot \mathbf{r'}) = \phi(\mathbf{k} \cdot \mathbf{r})$ and $c_w' = c_w$, the above equations are of the same form and they are Galilean invariant. However, the form of Eq. (27) is very different than that of the original new asymmetric equation given by Eq. (8), which means that in order for this equation to be fundamental, the existence of its Lagrangian must still be established.

5.4. New fundamental equation for classical waves

The Galilean invariant equation (Eq. (27)) is an ordinary differential equation, whose Lagrangian can be found by one of the methods previously developed for ODEs (e.g., [26-31]). The Lagrangian for Eq. (27) can be written as

$$L_{\text{new}}(d_{kr} \phi, \mathbf{k} \cdot \mathbf{r}) = \frac{1}{2} [d_{kr} \phi(\mathbf{k} \cdot \mathbf{r})]^2 e^{-4i(\mathbf{k} \cdot \mathbf{r})c_w^2/v^2},$$  \hspace{1cm} (28)

where $d_{kr} = d/d(\mathbf{k} \cdot \mathbf{r})$. The derived Lagrangian depends on the wave phase $\mathbf{k} \cdot \mathbf{r}$ that involves both $\mathbf{x}$ and $t$.

In CM, the dependence of Lagrangians on $t$ implies that the total energy of a dynamical system is not conserved and, as a result, the energy function must be calculated [3,4]. For physical systems with their Lagrangians explicitly time-dependent, the exponentially decaying or increasing terms are present, like in the well-known Caldirola-Kanai Lagrangian [35,36], originally written for the Bateman oscillator [37,38]. However, the Lagrangian given by Eq. (28) is of a different form as its exponential term is periodic in $\mathbf{k} \cdot \mathbf{r}$ instead. Since the first term on the RHS in Eq. (28) represents the wave kinetic energy, the exponential term shows that this energy is required to oscillate in time and space in the Lagrangian, so that the correct wave equation is obtained. This is a new phenomenon in classical waves and, thus, $L_{\text{new}}(d_{kr} \phi, \mathbf{k} \cdot \mathbf{r})$ forms a separate class among all Lagrangians known in physics.
To demonstrate that $L_{aw}(d_z \phi, z)$ is Galilean invariant, the Galilean transformations are applied and the following transformed Lagrangian is found

$$L'_{new}(d'_{kr} \phi', k' \cdot r') = \frac{1}{2} \left[ d'_{kr} \phi'(k' \cdot r') \right]^2 e^{-4i(k' \cdot r')c_w^2/v^2},$$

(29)

where $d'_{kr} = d/d(k' \cdot r')$. The transformed Lagrangian is of the same form as the original one given by Eq. (28) because $k \cdot r = k' \cdot r'$. $\phi(k \cdot r) = \phi'(r' \cdot k')$ and $c_w^2/v^2 = c'_w/v^2$. Therefore, the Lagrangian $L_{new}(d_{kr} \phi, k \cdot r)$ is Galilean invariant.

After substituting the Lagrangian given by Eq. (28) into the E-L equation

$$\frac{d}{d(k \cdot r)} \left( \frac{dL_{as}}{d(d_{kr} \phi)} \right) - \frac{dL_{as}}{d\phi} = 0,$$

(30)

the following equation is obtained

$$\left[ \frac{d^2 \phi}{d(k \cdot r)^2} - i \left( \frac{2c_w}{v} \right)^2 \frac{d\phi}{d(k \cdot r)} \right] e^{-4i(k \cdot r)c_w^2/v^2} = 0.$$

(31)

Since $e^{-4i(k \cdot r)c_w^2/v^2} \neq 0$, the terms in the square brackets must be zero, which gives Eq. (27). This shows that in addition to be local and Galilean invariant, Eq. (27) can also be derived from the Lagrangian given by Eq. (28). With its Lagrangian known and Galilean invariance of the Lagrangian verified, Eq. (27) is the new fundamental equation for classical waves.

By being fundamental, the new asymmetric wave equation gives the most comprehensive description of free classical waves as it accounts for the Doppler effect, the forward and backward waves, and makes the wave speed to be the same in all inertial frames. The equation plays the same role for classical waves as the law of inertia plays for classical particles. The mathematical forms of both equations are similar as they are second-order ODEs but the new fundamental wave equation has one additional term that allows for periodic solutions, as shown in Sec. 6a.

The wave speed $c_w$ is constant for all Galilean observers, since $v = \text{const}$, the coefficient $4c_w^2/v^2 = \text{const}$. This is an interesting result. It shows that this coefficient plays similar role for classical waves in Galilean relativity as the speed of light $c$ plays in Special Theory of Relativity (STR) for electromagnetic waves. However, while $c = \text{const}$ is the basic principle of Nature and the foundation of STR, the coefficient is the necessary condition for Galilean invariance, and its validity is guaranteed by selecting the wave phase as a variable describing the waves.

6. Applications to acoustic wave propagation

6.1. Freely propagating acoustic waves

Acoustic waves propagate freely in uniform media; however, the presence of such gradients does affect the wave propagation as it is shown below. It is interesting to compare solutions to the SWE, which is not fundamental, to those found for the new asymmetric wave equation, which is fundamental.
The solutions of the SWE are the plane waves, $\phi(t, x) = \phi_o \ e^{-i\omega t - k \cdot x}$, whose frequency $\omega$ and the wave vector $k$ are frame-dependent (the Doppler effect); this means that the Galilean observers see the plane waves with their frequencies and wave vectors being different in their respective inertial frames moving with the constant velocity $v$.

Finding the solutions to the fundamental wave equation given by Eq. (27) is straightforward. After two integrations, it yields

$$\phi(k \cdot r) = c_1 e^{i(2cs/v)^2(k \cdot r)} + c_2 ,$$

where $c_1$ and $c_2$ are integration constants, $c_s$ is the speed of sound, and the explicit form of the phase of acoustic wave is

$$\theta_s \equiv \left(\frac{2cs}{v}\right)^2 (k \cdot r) = \left(\frac{2cs}{v}\right)^2 \left[ k \cdot x - \frac{1}{2} \left( \frac{v}{2cs} \right) (\hat{k} \cdot \hat{v}) \omega t \right] .$$

The solution for $\phi(k \cdot r)$ represents the forward propagating acoustic wave. This is seen as the backward propagating acoustic wave in an inertial frame $S'$ because of the condition $k \cdot r = k' \cdot r'$, with the latter defined by Eq. (24). The above condition and $(2cs/v)^2 = (2cs'/v)^2$ guarantee that the solution is the same in all inertial frames. Thus, there is a full symmetry between the forward and backward acoustic waves. The solution given by Eqs (32) and (33) also accounts for the Doppler effect. The above solution shows that the new asymmetric wave equation is fundamental and its description of freely propagating acoustic waves is much more comprehensive than that given by the SWE.

In the next section, the assumption of uniform media is removed and gradients in physical parameters describing the media are allowed to exist.

6.2. Lamb's cutoff frequency

In his original work, Lamb [39-41] considered acoustic waves propagating in the $z$-direction in the background medium with gravity $\vec{g} = -g\hat{z}$ and density gradient $\rho_0(z) = \rho_{00} \exp(-z/H)$, where $\rho_{00}$ is the gas density at the height $z = 0$, and $H = c_s^2/\gamma g$ is the density scale height, with $\gamma$ denoting the ratio of specific heats. In his model, the background gas pressure $p_0$ and gas density $\rho_0$ vary with height $z$; however, the temperature $T_0$ remains constant. As a result, $H = \text{const}$ and $c_s = \text{const}$.

This stratified but otherwise isothermal medium is often referred to as an isothermal atmosphere, and acoustic waves in this atmosphere are described by the following variables: velocity $u(t, z)$, pressure $p(t, z)$ and density $\rho(t, z)$ perturbations. The resulting acoustic wave equation (AWE) is derived for the transformed wave variables $u_1(t, z) = u(t, z)\rho_0^{1/2}$, $p_1(t, z) = p(t, z)\rho_0^{-1/2}$ and $\rho_1(t, z) = \rho(t, z)\rho_0^{1/2}$ using the hydrodynamic equations [40-42,46]. The resulting wave equation can be written as

$$\left[ \frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial z^2} + \Omega_{ac}^2 \right] \left[ u_1(t, z), p_1(t, z), \rho_1(t, z) \right] = 0 .$$

where the speed of sound is $c_s = [\gamma p_0(z)/\rho_0(z)]^{1/2} = [\gamma RT_0/\mu]^{1/2}$, while the acoustic cutoff frequency $\Omega_{ac} = c_s/2H = \gamma g/2c_s$ remains constant in the entire isothermal
atmosphere [39-41,42,46]. The Lamb cutoff frequency describes the effects of the atmospheric density gradient on the acoustic wave propagation, and it is used to determine the wave propagation conditions (see Sec. 6d). Note also that the form of the wave equation is the same for each wave variable in an isothermal atmosphere.

The fact that the form of the derived AWE remains the same at every height in an isothermal atmosphere is well-known and first shown by Lamb [39-41]. However, different inertial observers see the waves differently, namely, with their different characteristic speeds, frequencies, and wave vectors. Different waves seen in different inertial frames means that the theory of waves based on the AWE is not fundamental because it is not the same for all Galilean observers.

In numerous studies of propagation of acoustic waves that followed Lamb’s work, different aspects of the wave propagation were investigated by using methods based on either global and local dispersion relations, or the WKB approximation, or finding analytical solutions to acoustic wave equations for special cases [43-45]. A method to determine the cutoff frequency for linear and adiabatic acoustic waves propagating in non-isothermal media without gravity was also developed [46] with a method based on transformations of wave variables that lead to standard wave equations, and using oscillation theorem to determine the turning point frequencies. Then, physical arguments are used to select the largest of these frequencies as the Lamb cutoff frequency. In this paper, the Lamb cutoff frequency is obtained for the new fundamental wave equation.

6.3. New fundamental wave equation and Lamb’s cutoff frequency

The acoustic wave equation given by Eq. (34) is obtained from the hydrodynamic equations. It is easy to show that neither the Schrödinger-like wave equation nor the new asymmetric wave equation can be derived using only the hydrodynamic equations. However, both wave equations can be derived from the hydrodynamic equations if, and only if, these equations are supplemented by the eigenvalue equations. Specifically, the Schrödinger-like wave equation is obtained when the eigenvalue equation given by Eq. (1) is applied to Eq. (34). Since the results presented in Sec. 5b showed that the Schrödinger-like wave equation is not fundamental, the equation will not be further considered here.

Instead, the new fundamental wave equation is considered. By applying the eigenvalue equation given by Eq. (2) to Eq. (34), the following new asymmetric wave equation is obtained

\[
\left[ \frac{\partial^2}{\partial t^2} - i k c_s^2 \frac{\partial}{\partial z} + \Omega_{ac}^2 \right] \phi(t, z) = 0 .
\]

For the considered acoustic wave propagation along the z-axis, the label of the irreps of \( G_e \) is \( k = k \cdot \hat{z} \), and the wavefunction \( \phi(t, z) \) represents one of the acoustic wave variables in Eq. (34).
The results presented in Sec. 5c demonstrated that the new asymmetric wave equation can be converted into the form that is Galilean invariant (see Eq. 27). Then, the resulting wave equation is

\[ \frac{d^2}{d\chi^2} - i \left( \frac{2c_s}{v} \right)^2 \frac{d}{d\chi} + \left( \frac{2\Omega_{ac}}{kv} \right)^2 \phi(\chi) = 0, \]

where \( \chi = k \cdot r = k(z - vt/2) \). Since \( \Omega_{ac} = \gamma g/2c_s \), with \( c_s = c'_s \), \( k = k' \), and with \( \gamma \) and \( g \) being the same in all inertial frames, Eq. (36) and its solutions are the same for all Galilean observers; this means that the derived equation is fundamental. The obtained fundamental wave equation describes the effects of an isothermal atmosphere on the acoustic wave propagation. Thus, Eq. (36) generalizes Eq. (27), which describes only freely propagating acoustic waves in a medium without any gradients.

As a result of the Galilean transformations, the term that represents Lamb’s cutoff frequency is now modified by the factor \( 2/kv \), which describes the effects of moving inertial frames on the cutoff; these effects are more prominent for smaller velocities \( v \) and wavevectors \( k \). Since the considered acoustic wave propagation is restricted to one-dimension (the \( z \)-axis), it is easy to generalize the obtained results to three-dimensions by requiring the factor to be \( 2/(k \cdot v) \).

### 6.4. Conditions for acoustic wave propagation

As originally demonstrated by Lamb [39-41], the frequency \( \Omega_{ac} \) uniquely determines whether acoustic waves in an isothermal atmosphere are propagating or evanescent. Since \( \Omega_{ac} = \text{const} \) in the isothermal atmosphere, after making the Fourier transforms in time and space, the AWE (Eq. 34) gives the global dispersion relation: \( (\omega^2 - \Omega_{ac}^2) = k^2c_s^2 \), where \( \omega \) is the wave frequency and \( k = k_z \) is the wave vector. The obtained dispersion relation is valid in one selected inertial frame, which is called stationary. In this frame, the waves are propagating when \( \omega > \Omega_{ac} \) and \( k \) is real, and they are non-propagating (evanescent) when either \( \omega = \Omega_{ac} \) with \( k = 0 \) or \( \omega < \Omega_{ac} \) with \( k \) being imaginary.

When a Galilean observer moves with velocity \( v \) with respect to the stationary frame, then the wave frequency (the Doppler effect), wave vector and characteristic wave speed change, which means that \( (\omega^2 - \Omega_{ac}^2) = k'^2c_s'^2 \); the dispersion relation preserves its form but the values of the wave parameters are changed from one inertial frame to another. With \( \Omega_{ac} \neq \Omega'_{ac} \), the acoustic cutoff frequency is different in different inertial frame.

The same conditions for wave propagation are obtained when the Fourier transforms in time and space are performed in the new asymmetric wave equation (Eq. 35), and the result is \( (\omega^2 - \Omega_{ac}^2) = k^2c_s^2 \), which is the same as the dispersion relation obtained for the AWE. Thus, the conditions for the wave propagation are also the same. However, neither the AWE given by Eq. (34) nor the new asymmetric wave equation given by Eq. (35) is fundamental. The only fundamental wave equation is given by Eq. (36). The conditions for wave propagation resulting from this equation are now determined and discussed.
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To find the conditions for the acoustic wave propagation in an isothermal atmosphere, the fundamental wave equation given by Eq. (36) must be solved. The obtained solutions $\phi_1(\chi)$ and $\phi_2(\chi)$ are

$$\phi_{1,2}(\chi) = \exp \left[ \frac{i}{2} \left( \frac{2c_s}{v} \right)^2 \left( 1 \pm \sqrt{1 + \left( \frac{v}{2c_s} \right)^2 \frac{\Omega^2_{ac}}{\omega^2 - \Omega^2_{ac}} \chi} \right) \right],$$  \hspace{1cm} (37)

and their superposition gives the general solution $\phi(\chi) = C_1 \phi_1(\chi) + C_2 \phi_2(\chi)$; note that the dispersion relation $k^2 c_s^2 = (\omega^2 - \Omega^2_{ac})$ was used to derive Eq. (37). Using the dispersion relation, $\chi = k(z - vt/2)$ can be written as

$$\chi = \left( \frac{z}{c_s} - \frac{1}{2} \frac{v}{c_s} t \right) \sqrt{\omega^2 - \Omega^2_{ac}},$$  \hspace{1cm} (38)

which allows writing the wave phases of the solutions given by Eq. (37) in the following form

$$\theta_{1,2}(t,z) = \frac{c_s}{v} \left[ \sqrt{\omega^2 - \Omega^2_{ac}} \pm \sqrt{\left( \omega^2 - \Omega^2_{ac} \right) + \left( \frac{v}{2c_s} \right)^2 \Omega^2_{ac}} \left( 2\frac{z}{v} - t \right) \right],$$  \hspace{1cm} (39)

and the general solution becomes

$$\phi(z,t) = C_1 e^{i\theta_1(z,t)} + C_2 e^{i\theta_2(z,t)}.$$  \hspace{1cm} (40)

This solution and its wave phases are now used to determine the conditions for the propagation of acoustic waves in an isothermal atmosphere.

The general solution given by Eq. (37) shows that any real $\theta_1(z,t)$ and $\theta_2(z,t)$ describe the propagating waves. On the other hand, imaginary wave phases make the general solution exponentially decaying, which corresponds to non-propagating (or evanescent) waves. There are several cases of interest that are now considered. If $\omega \gg \Omega_{ac}$, then the wave phases are $\theta_1(z,t) = (c_s/v)(2z/v - t)\omega$ and $\theta_2(z,t) = 0$, with the first phase representing a freely propagating acoustic wave along the $z$-axis, and the second phase is a trivial (constant) solution that show no acoustic wave; these results are consistent with a more general (3-dimensional) solution given by Eqs (32) and (34). The obtained results show that the propagation of very high frequency acoustic waves is not affected by stratification of the isothermal atmosphere.

The effects of medium stratification on the acoustic wave propagation become important when $\omega \gtrsim \Omega_{ac}$; in this case, the wave phase is given by Eq. (39) and both solutions contribute to $\phi(z,t)$ given by Eq. (40). The most interesting case is when $\omega = \Omega_{ac}$, which gives $\phi_{1,2}(z,t) = \pm (z/v - t/2)\Omega_{ac}$, showing that the propagating acoustic waves cease to exist as they are replaced by oscillations of the atmosphere with its natural frequency $\Omega_{ac}$. The existence of oscillations in the planetary, solar and stellar atmospheres is well known [47-49]. The origin of oscillations in the solar atmosphere is attributed to either acoustic waves trapped in the solar interior (the solar 5-min oscillations, e.g., [48]) or acoustic waves propagating in the solar atmosphere (the solar 3-min oscillations, e.g., [50]). The above results demonstrate that at $\omega = \Omega_{ac}$, when the propagation of acoustic waves is terminated, the atmospheric oscillations originate and
their characteristic frequency is the atmosphere natural frequency $\Omega_{ac}$, which is also the cutoff frequency for acoustic waves as it was first shown by Lamb [39-41].

Having demonstrated that the acoustic wave propagation is terminated in the limit when $\omega \to \Omega_{ac}$, which means that $\Omega_{ac}$ is the Lamb (acoustic) cutoff frequency, it must be now verified that the wave phases given by Eq. (39) become imaginary for any $\omega < \Omega_{ac}$, which physically means that the solutions $\phi_{1,2}(z, t)$ are exponentially decaying and the waves are evanescent. In case of $\omega \lesssim \Omega_{ac}$, the wave phases are given by the imaginary equation

$$
\theta_{1,2}(t, z) = i \frac{c_s}{v} \left[ \sqrt{\Omega_{ac}^2 - \omega^2} \pm \sqrt{\left(1 - \frac{v^2}{4c_s^2}\right) \Omega_{ac}^2 - \omega^2} \right] \left(2 \frac{z}{v} - t\right).
$$

(41)

In general, the term $[(1 - v^2/4c_s^2)\Omega_{ac}^2 - \omega^2] > 0$ but it may also become negative if $v > 2c_s$, which means that the second term of these phases becomes imaginary, then this term would give oscillatory solutions. However, this does not affect the general solution as the exponential decay caused by the first term takes over and makes the entire solution evanescent. Similarly, when $\omega << \Omega_{ac}$, the wave phases become

$$
\theta_{1,2}(t, z) = i \frac{c_s}{v} \left[ \Omega_{ac} \pm \sqrt{\left(1 - \frac{v^2}{4c_s^2}\right) \Omega_{ac}} \right] \left(2 \frac{z}{v} - t\right),
$$

(42)

which shows that the solutions are exponentially decaying. Based on the above discussion, the obtained results are valid in both cases when $v \leq 2c_s$ as well as when $v > 2c_s$. Thus, acoustic waves of all frequencies lower than $\Omega_{ac}$ are always evanescent.

The presented results show that the fundamental AWE can be derived from the hydrodynamic equations after using the eigenvalue equation given by Eq. (2). As a result, the fundamental AWE directly displays the characteristic atmospheric frequency $\Omega_{ac}$ similar as the SWE does. By solving the fundamental AWE, it is demonstrated that $\Omega_{ac}$ is the Lamb (acoustic) cutoff frequency that uniquely determines the conditions for the acoustic wave propagation, which is consistent with the original results presented by Lamb in 1910 [39]. However, there are main differences between the results presented in this paper and those obtained by Lamb [39-41], namely, Lamb’s results are valid in only one stationary inertial frame $S$, which is selected to describe the waves, while the presented results are the same for all inertial observers in the Galilean space and time. In other words, for all Galilean observers, the waves have the same wave speed, frequency and wavenumber, and their propagation conditions remain also frame-independent, which shows that the formulated theory of acoustic waves is fundamental.

7. Conclusions

A method based on the irreps of the extended Galilean group is used to derive infinite sets of symmetric and asymmetric second-order partial differential equations with constant coefficients of arbitrary real values. The obtained results demonstrate that among these equations only one asymmetric equation is a new fundamental wave equation. This
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equation gives the most complete description of propagating waves as it accounts for the Doppler effect, forward and backward waves, and makes the wave speed to be the same in all inertial frames. It appears as if the fundamental wave equation plays the same role for classical waves as Newton’s law of inertia plays for classical particles.

The fundamental wave equation is applied to the propagation of acoustic waves in an isothermal atmosphere. The analysis shows that the wave propagation conditions are uniquely determined by the existence of the atmospheric natural frequency, which is identified with the acoustic cutoff frequency originally introduced by Lamb [39]. However, while Lamb’s wave description and its cutoff frequency are frame-dependent, the wave description given by the new fundamental wave equation (Eq. 36) and its acoustic cutoff remain the same for all Galilean observers in their inertial frames.

The presented theory of waves based on the fundamental wave equation also predicts the existence of atmospheric oscillations with the natural atmospheric frequency, that are driven by the process of the propagating waves becoming evanescent when their frequencies become equal to the Lamb frequency.

[1] L. Susskind, A. Friedman, Special Relativity and Classical Field Theory, Basic Books, New York, 2017
[2] L.D. Landau and E.M. Lifschitz, Mechanics, Pergamon Press, Oxford, 1969
[3] H. Goldstein, C.P. Poole, J.L. Safko, Classical Mechanics, 3rd Edition, Addison-Wesley, San Francisco, CA, 2002
[4] J.V. José, E.J. Saletan, Classical Dynamics, A Contemporary Approach, Cambridge Univ. Press, Cambridge, 2002
[5] E. Merzbacher, Quantum Mechanics, Wiley & Sons, Inc., New York, 1998
[6] J.-M. Levy-Leblond, Nonrelativistic particles and wave equations, Comm. Math. Phys. 6 (1967) 286
[7] W.C. Elmore, M.A. Heald, Physics of Waves, Dover Publ., Inc., New York, 1969
[8] I.G. Main, Vibrations and Waves in Physics, Cambridge Uni. Press, New York, 1993
[9] G.B. Whitman, Linear and Nonlinear Waves, John Wiley & Sons, Inc, New York, 1999
[10] J.-M. Levy-Leblond, Galilean group and nonrelativistic quantum mechanics, J. Math. Phys. 4 (1963) 776
[11] J.-M. Levy-Leblond, Group theoretical foundations of classical mechanics: the Lagrange gauge problem, Comm. Math. Phys. 12 (1969) 64
[12] Y.S. Kim and M.E. Noz, Theory and Applications of the Poincaré Group, Reidel, Dordrecht, 1986
[13] E. Inönü and E.P. Wigner, Representations of the Galilei group, Nuovo Cimento 9 (1952) 705.
[14] V. Bargmann, On unitary ray representations of continuous groups, Ann. Math. 59 (1954) 1
[15] S. Sternberg, Group Theory and Physics, Cambridge Uni. Press, Cambridge, 1994.
[16] A.B. van Oosten, Covariance of the Schrödinger equation under low velocity boosts, Apeiron 13 (2006) 449
[17] Z.E. Musielak and J.L. Fry, Physical theories in Galilean space-time and the origin of Schrödinger-like equations, Ann. Phys. 324 (2009) 296
[18] Z.E. Musielak and J.L. Fry, General dynamical equations for free particles and their Galilean invariance, Int. J. Theor. Phys. 48 (2009) 1194
[19] S.M. Flatté, The Schrödinger equation in classical physics, Am. J. Phys. 54 (1986) 1088
[20] S.A. Hilbert, H. Batelaan, Acoustic analog to quantum mechanical level splitting, Am. J. Phys. 75 (2007) 1003
[21] T. Aktosun, R. Weber, Inverse spectra-scattering problem with two sets of discrete spectra for the radial Schrödinger equation, Inverse Problems 22 (2006) 89
New Fundamental Equation for Classical Waves and Its Physical Applications

[22] H. White, P. Bailey, J. Lawrence, J. George and J. Vera, J., A discussion on a dynamic vacuum model: Derivation of Helmholtz equation from Schrödinger equation, Phys. Open 1 (2019) 100009
[23] N.M. Kocherginsky, Interpretation of Schrödinger equation based on classical mechanics and spin, Quantum Stud.: Math. Found. 8 (2021) 217
[24] Z.E. Musielak, New Equation of Nonrelativistic Physics and Theory of Dark Matter, Int. J. Mod. Phys. A, 28 (2021) 2150042 (12pp)
[25] Z.E. Musielak, New Nonrelativistic Quantum Theory of Cold Dark Matter, Int. J. Mod. Phys. A, 37 (2022) 2250137 (10pp)
[26] R.A. El-Nabulsi, Fiz., A14, 289, 2005
[27] M.C. Nucci and P.G.L. Leach, Lagrangians galore, J. Math. Phys., 48 (2007) 123510
[28] Z.E. Musielak, Standard and non-standard Lagrangians for dissipative dynamical systems with variable coefficients, J. Phys. A Math. Theor., 41 (2008) 055205
[29] J.L. Cieśliński and T. Nikiciuk, A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients, J. Phys. A Math. Theor., 43 (2010) 175205
[30] J. Lopuszanski, The Inverse Variational Problems in Classical Mechanics, World Scientific, Singapore, 1999
[31] Z.E. Musielak, General conditions for the existence of non-standard Lagrangians for dissipative dynamical systems, Chaos, Solitons Fractals, 42 (2009) 2640
[32] N.A. Daughtly, Lagrangian Interactions, Addison-Wesley Publ. Comp., Inc., Sydney, 1990
[33] Z.E. Musielak, Standard Lagrangians for wave, Laplace and Tricomi-like equations with variable coefficients, J. Phys. A Math. Theor. 43 (2010) 425205
[34] Z. E. Musielak and T. B. Watson, Gauge functions and Galilean invariance of Lagrangians, Phys. Let. A 384 (2020) 126642
[35] P. Caldirola, Forze non conservative nella meccanica quantista, Nuovo Cim. 18 (1941) 393
[36] E. Kanai, On the quantization of the dissipative systems, Prog. Theor. Phys. 3 (1948) 44
[37] H. Bateman, On dissipative systems and related variational principles, Phys. Rev. 38 (1931) 815
[38] L.C. Vestal, Z.E. Musielak, Bateman Oscillators: Caldirola-Kanai and null Lagrangians and gauge functions, Physics, 3 (2021) 449
[39] H. Lamb, On atmospheric oscillations, Proc. R. Soc. London A 34 (1910) 551.
[40] H. Lamb, Hydrodynamics, Dover Publ., Inc, New York, 1945
[41] H. Lamb, The Dynamical Theory of Sound, Dover Publ., Inc, Mineola, NY, 1960
[42] L.M.B.C. Campos, On waves in gases. Part 2: Interaction of sound with magnetic and internal modes, Rev. Mod. Phys. 59 (1987) 363.
[43] N. Suda, K. Nawa and Y. Fukao, Earth’s background oscillations, Science 279 (1998) 2089.
[44] E.M. Solomons, R. Blumrich and D. Heimann, Eulerian time-dependent model for sound propagation over a finite-impedance ground surface. Comparison with frequency-domain model, Acta Acoustica with Acoustica 88 (2002) 483.
[45] J. Rhie and B. Romanowicz, Excitation of Earth’s continuous free oscillations by atmospheric-ocean-seafloor coupling, Nature 431 (2004) 552.
[46] Z.E. Musielak, D.E. Musielak and H. Mobashi, Method to determine acoustic cutoff frequencies in nonisothermal atmosphere, Phys. Rev. E 73 (2006) 036612-1.
[47] U. Lee, Acoustic oscillations of Jupiter, Astrophys. J. 405 (1993) 359.
[48] S. Basu, Global seismology of the Sun, Living Reviews in Solar Physics 13 (2016) 2.
[49] R. Garcia and J. Ballot, Asteroseismology of solar-type stars, Living Reviews in Solar Physics 16 (2019) 4.
[50] B. Fleck and F. Schmitz, The 3-min oscillations of the solar chromosphere, Astron. Astrophys. 250 (1991) 235.
Appendix A. Derivation of the eigenvalue equations

Let us consider a set of \( N \) functions that forms a basis of an \( N \)-dimensional representation given by a set of \( N \times N \) matrices \( A \) for each irrep, and for each element of the group

\[
\hat{a}_l^{(k)} = \sum_m A_{ml}(\hat{a}) f_m^{(k)}, \quad (A.1)
\]

where \( \alpha \) is one of the elements of the group, \( k \) labels the irreps and \( l \) is one of the members of the set of \( N \) functions satisfying Eq. (1). In addition, the sum on \( m \) is over the \( N \) members of the set, and the matrices \( A \) are unitary.

Writing Eq. \( (A.1) \) for space translations \( \vec{a} \), the result is

\[
\hat{T}_a \psi(t, x) \equiv \psi(t, x + a) = e^{i\vec{k} \cdot \vec{a}} \psi(t, x) . \quad (A.2)
\]

Making the Taylor series expansion of \( \phi(\vec{r} + \vec{a}) \), one gets

\[
\phi(t, x + a) = \exp[i(-i \vec{a} \cdot \nabla)] \phi(t, x) . \quad (A.3)
\]

Comparing Eq. \( (A.3) \) to Eq. \( (A.2) \), the following eigenvalue equation is obtained

\[
-i \nabla \phi(t, x) = k \phi(t, x) , \quad (A.4)
\]

which is one of the eigenvalue equations given by Eq. \( \{1\} \).

For the time translation \( t_0 \), one obtains

\[
\hat{T}_{t_0} \psi(t, x) \equiv \psi(t + t_0, x) = e^{-i\omega t_0} \psi(t, x) . \quad (A.5)
\]

Comparision of this equation to the Taylor series expansion

\[
\phi(t + t_0, x) = \exp[i(-it_0 \partial/\partial t)] \phi(t, x) . \quad (A.6)
\]

gives

\[
i \frac{\partial}{\partial t} \phi(t, x) = \omega \phi(t, x) , \quad (A.7)
\]

which is the eigenvalue equation given by Eq. \( \{1\} \).

The derived eigenvalue equations represent the necessary conditions that \( \phi(t, x) \) transforms as one of the irreps of \( T(3+1) \); [17]. The above results also show that the unitary irreps of the group \( T(3+1) \) are labeled by the real vector \( k \) and the real scalar \( \omega \), and there are no other restrictions on these quantities. It must also be mentioned that these labels are preserved in the irreps of the entire \( G_\epsilon \) because \( T(3+1) \) is its invariant subgroup [10].