General slow-roll spectrum for gravitational waves

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Abstract
We derive the power spectrum $P_\psi (k)$ of the gravitational waves produced during general classes of inflation with second-order corrections. Using this result, we also derive the spectrum and the spectral index in the standard slow-roll approximation with new higher order corrections.

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1. Introduction
Inflation [1], among many of its features, is believed to have created the scalar perturbations from primordial quantum fluctuations in the inflaton field $\phi$. However, not only scalar curvature perturbations are produced during inflation but tensor fluctuations associated with the metric are also generated along with the scalar perturbations [2]. The tensor fluctuations, which emerge as gravitational waves, are not accompanied by the density perturbations responsible for the structure formation in the observed universe. This makes them not directly accessible to observations. They are currently believed to only influence the cosmic microwave background [3] on large angular scales and the polarization [4], which is one of the major aims for the future cosmic microwave background observations. Gravitational wave detectors such as the Laser Interferometer Space Antenna (LISA) [5] and the Laser Interferometric Gravitational Wave Observatory (LIGO) [6] are also currently in progress.

Gravitational waves, although not observed yet, possess several potential uses. One noteworthy feature is that they are directly associated with the energy scale of inflation. Once detected, they should lead us to determine the inflationary energy scale without any obstacle. Moreover, for the case of models of inflation with a single degree of freedom for $\phi$, the tilt of the tensor perturbations is related to the tensor-to-scalar amplitude ratio, known as the consistency relation. If this relation turns out to be false, we should abandon the single field inflationary models and consider models with multiple degrees of freedom for $\phi$ [7, 13] or models in generalized gravity [8]. Also, it is interesting that different models for the generation of the universe, such as ekpyrotic [9], cyclic [10] and pre-big bang models [11], generally predict the
power spectrum of gravitational waves strongly tilted to the blue. Therefore detection of the
tensor perturbations would be a powerful discriminator for these models, as well as inflation.

In this paper, we follow Green’s function method [12, 13] within the generalized slow-roll
approximation [14, 15] and present the power spectrum for the tensor perturbations, \( P_\psi(k) \),
with second-order corrections. Throughout this paper, we set \( c = \hbar = 8\pi G = 1 \).

2. Power spectrum for gravitational waves

In this section, we derive the power spectrum for gravitational waves produced during inflation.
First, we present the basic principles, and then derive the formulae for \( P_\psi(k) \) in the general
slow-roll scheme.

2.1. Preliminaries

The linear tensor perturbations in general can be written as [16]
\[
dx^2 = a^2(\eta) [d\eta^2 - (\delta_{ij} + 2h_{ij}) dx^i dx^j],
\]
where we assume that \( |h_{ij}| \ll 1 \). Although the tensor \( h_{ij} \) has six degrees of freedom, imposing
traceless and transverse conditions, we can remove four degrees of freedom and are left with
two physical degrees of freedom, or polarizations. Thus, we can write the tensor \( h_{ij} \) in Fourier
components as
\[
h_{ij} = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\lambda=1}^2 \psi_{k,\lambda}(\eta) e_{ij}(k, \lambda) e^{ikx},
\]
where the polarization tensors \( e_{ij} \) satisfy the relations
\[
e_{ij} = e_{ji}, \quad e_i^i = 0, \quad k^i e_{ij} = 0,
e_{ij}^\mu(k, \mu) e^{ij}(k, \nu) = \delta_{\mu\nu} \quad \text{and} \quad e_{ij}(-k, \lambda) = e_{ij}^\ast(k, \lambda),
\]
and the power spectrum for the tensor perturbations, or gravitational waves, is defined as
\[
\langle \psi_{k,\lambda} \psi_{l,\lambda}^\ast \rangle \equiv \frac{2\pi^2}{k^3} P_\psi(k) \delta^{(3)}(k - l).
\]

2.2. General slow-roll formulae for \( P_\psi(k) \)

To apply the general slow-roll scheme, we begin with the action for the tensor perturbations
[17]
\[
S = \int \frac{1}{2} a^2 \left[ \left( \frac{\partial h_{ij}}{\partial \eta} \right)^2 - (\nabla h_{ij})^2 \right] d\eta d^3x
= \int \frac{1}{2} \sum_{\lambda=1}^2 \left[ \frac{\partial v_{k,\lambda}}{\partial \eta} \right]^2 - \left( k^2 - \frac{1}{a} \frac{d^2a}{d\eta^2} \right) |v_{k,\lambda}|^2 \right] d\eta d^3k,
\]
where we define
\[
v_{k,\lambda} \equiv a \psi_{k,\lambda}.
\]
Then, the equation of motion for the Fourier modes is given as
\[
\frac{d^2 v_k}{d\eta^2} + \left( k^2 - \frac{1}{a} \frac{d^2a}{d\eta^2} \right) v_k = 0,
\]
where the solution \( v_k \) satisfies the boundary conditions
\[
v_k \rightarrow \begin{cases} \sqrt{2} e^{-ik\eta} & \text{as } -k\eta \to \infty \\ A_k a & \text{as } -k\eta \to 0. \end{cases}
\]  
(8)

Now, defining \( \nu = \sqrt{2} v_k \), \( x = -k\eta \) and
\[
p(\ln x) = \frac{2\pi}{k} x a.
\]  
(9)

Equation (7) becomes
\[
\frac{d^2 \nu}{dx^2} + \left(1 - \frac{2}{x^2}\right) \nu = \frac{1}{x^2} q \nu,
\]  
(10)

where
\[
q = \frac{p'' - 3p'}{p}
\]  
(11)

and \( p' = dp/d\ln x \). Then, we can write the solution of equation (10) using Green’s function as
\[
\nu(x) = \nu_0(x) + \frac{i}{2} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} du' q(u)\nu_0(x)\nu_0(u)\nu_0(x) - \nu_0(x)\nu_0(u)\nu_0(x).
\]  
(12)

where
\[
\nu_0(x) = \left(1 + \frac{i}{x}\right)e^{ix}
\]  
(13)
is the desired homogeneous solution of equation (10). The power spectrum for the tensor perturbations, \( P_\psi(k) \), can be written, using equations (4) and (9), as
\[
P_\psi(k) = \frac{k^3}{2\pi^2} \lim_{-k\eta \to 0} \frac{v_k}{a^2} = \lim_{x \to 0} \frac{x^2}{p} \nu_0^2.
\]  
(14)

Here, we note that equation (7) has exactly the same structure as that for the scalar perturbations, which was the subject of [15]. The only difference is the definition of the function \( p(\ln x) \) in equation (9). So, we can just follow the same steps as those used in [15] to write \( P_\psi(k) \). We assume that \( \nu(x) \) is dominated by the scale invariant, homogeneous solution \( \nu_0(x) \), that is, \( q \) is small. Then, for the spectrum with second-order corrections, we iterate equation (12) twice,
\[
\nu(x) \simeq \nu_0(x) + L(x, u)\nu_0(u) + L(x, u)v_0(v),
\]  
(15)

and substituting into equation (14), after some calculations, we can obtain \( P_\psi(k) \) as
\[
\ln P_\psi(k) = \ln \left(\frac{1}{p^2}\right) - 2 \int_0^{\infty} \frac{du}{u} w(p, u) \nu_0^2 \nu_0^2 + 2 \left[ \int_0^{\infty} \frac{du}{u} \chi(u) \nu_0^2 \right]^2 - 4 \int_0^{\infty} \frac{du}{u} \nu_0^2 \chi(u) \nu_0^2 \nu_0^2 \int u \frac{p'}{v^2} \nu_0^2,
\]  
(16)

where the subscript \( * \) means some convenient point of evaluation around horizon crossing, and the window functions are given by
\[
w(x) = \frac{\sin(2x)}{x} - \cos(2x), \quad \chi(x) = \frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x),
\]  
(17)
and
\[ w_{\theta}(x, x) = w(x) - \theta(x_\ast - x). \]  
(18)

We can see that the integrals in equation (16) are well defined as \( x \to 0 \), since \( w(x) \) and \( \chi(x) \) behave asymptotically as
\[ \lim_{x \to 0} w(x) = 1 + O(x^2) \quad \text{and} \quad \lim_{x \to 0} \chi(x) = \frac{2}{3} x^3 + O(x^5). \]  
(19)

It is crucial to realize that equation (16) is independent of the evaluation point \( \ast \), i.e. we have the same spectrum irrespective of when we evaluate. It is manifest from equation (16) that the \( \ast \) dependence cancels out because of the step function. Note that although the form of equation (16) is the same as that for the scalar perturbations [15], differences appear as we specify the function \( p(\ln x) = \frac{2\pi}{k a} x^{\alpha} \), compared with \( f(\ln x) = \frac{2\pi}{k a} \dot{\phi} H \) for the scalar case. Some examples of different \( p \) will be presented in the following sections.

3. Applications

In this section, using our result, equation (16), we present \( P_\psi(k) \) for several physically interesting cases. The utility of this section is twofold: first, it gives specific examples of our result, so that we can see the broad applicability of equation (16). Also, we can verify that it successfully reproduces the familiar, well-known result for the spectrum [2]
\[ P_\psi(k) = \left( \frac{H}{2\pi} \right)^2 \]  
(20)
and the additional corrections which make \( P_\psi(k) \) more accurate.

3.1. Standard slow-roll approximation

It is very illuminating to use equation (16) to derive \( P_\psi(k) \) in the standard slow-roll approximation with one higher order, i.e. third-order corrections, since we have all the necessary information with only one undetermined coefficient [15]. Previously, \( P_\psi(k) \) was known up to second-order corrections in the standard slow-roll approximation [18, 19] and third-order corrections under some special conditions [19]. Here we give the third-order corrections in the standard slow-roll approximation.

From equation (16), we can remove the logarithm and expand \( p'/p \) in terms of \( \ln(x/x_\ast) \), which implies we are applying the standard slow-roll approximation, to obtain the result
\[ P_\psi(k) = \frac{1}{p_\ast^2} \left\{ 1 - 2\alpha_\ast \frac{p'}{p_\ast} + \left( -\alpha_\ast^2 + \frac{\pi^2}{12} \right) \frac{p''}{p_\ast} + \left( 3\alpha_\ast^2 - 4 + \frac{5\pi^2}{12} \right) \left( \frac{p'_\ast}{p_\ast} \right)^2 \right. 
\]
\[ + \left[ \frac{1}{3} \alpha_\ast^3 + \frac{\pi^2}{12} \alpha_\ast - \frac{4}{3} + \frac{2}{3} \xi(3) \right] \frac{p'''}{p_\ast} \right. 
\]
\[ + \left[ 3\alpha_\ast^3 - 8\alpha_\ast + \frac{7}{12} \pi^2 \alpha_\ast + 4 - 2\xi(3) \right] \frac{p'_\ast p''}{p_\ast^2} 
\]
\[ + \left[ -4\alpha_\ast^3 + 16\alpha_\ast - \frac{5}{3} \pi^2 \alpha_\ast - 8 + 6\xi(3) \right] \left( \frac{p'_\ast}{p_\ast} \right)^3 + \cdots \} , \]  
(21)

where
\[ \alpha_\ast \equiv \alpha - \ln x_\ast, \quad \alpha \equiv 2 - \ln 2 - \gamma \simeq 0.729 \, 637, \]  
(22)
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\( \gamma \approx 0.577216 \) is the Euler–Mascheroni constant, \( \xi \) is the Riemann zeta function and we have used the results of [15], where we determined the last coefficient using some exactly known solutions.

Now, we write the slow-roll parameters in the standard slow-roll approximation

\[
\epsilon_1 = -\frac{\dot{H}}{H^2} = O(\xi), \quad \text{and} \quad \epsilon_{n+1} = \frac{1}{H^n} \left( \frac{d}{dt} \right)^n \epsilon_1 = O(\xi^{n+1}),
\]

where \( \xi \) is some small parameter and \( n \geq 1 \) is some integer. Using these parameters, we can write the function \( p \) in terms of these slow-roll parameters as

\[
\frac{1}{p^3} = \left( \frac{H}{2\pi} \right)^2 \left[ 1 - 2\epsilon_1 + \epsilon_1^2 - 2\epsilon_2 \epsilon_1 - 2\epsilon_3 - O(\xi^4) \right],
\]

\[
\frac{p'}{p} = -\epsilon_1 - \epsilon_1^2 - \epsilon_2 - \epsilon_3 - 4\epsilon_1 \epsilon_2 - 4\epsilon_3 + O(\xi^4),
\]

\[
\frac{p''}{p} = \epsilon_1^2 + 2\epsilon_1^3 + 6\epsilon_1 \epsilon_2 + 3\epsilon_3 + O(\xi^4),
\]

\[
\frac{p'''}{p} = -\epsilon_1^3 - 4\epsilon_1 \epsilon_2 - 3 + O(\xi).\]

Substituting these into equation (21), we obtain the power spectrum as

\[
\mathcal{P}_\psi(k) = \left( \frac{H_*}{2\pi} \right)^2 \left\{ 1 + (2\alpha_* - 2) \epsilon_1, + \left( 2 \alpha_*^2 - 2 \alpha_* - 3 + \frac{\pi^2}{2} \right) \epsilon_1^2, + \left( -\alpha_*^2 + 2 \alpha_* - 2 + \frac{\pi^2}{12} \right) \epsilon_2, + \left[ \frac{4}{3} \alpha_*^3 - 8 \alpha_*^2 + \pi^2 \alpha_3 + 16 \frac{3}{3} - 4 \zeta(3) \right] \epsilon_1^3, + \left[ -\frac{5}{3} \alpha_*^3 + 2 \alpha_*^2 + 8 \alpha_* - \frac{11 \pi^2}{12} - \frac{2}{3} \cdot \frac{6}{3} + \frac{2}{3} - 8 \zeta(3) \right] \epsilon_1 \epsilon_2, + \left[ \frac{1}{3} \alpha_*^3 - 2 \alpha_*^2 + 2 \alpha_* - \frac{\pi^2}{12} \alpha_3 - \frac{2}{3} + \frac{\pi^2}{12} - \frac{2}{3} \zeta(3) \right] \epsilon_3 \right\}.
\]

In addition, we can calculate the spectral index

\[
n_\psi(k) = \frac{d \ln \mathcal{P}_\psi(k)}{d \ln k}
\]

easily from \( \mathcal{P}_\psi(k) \) as

\[
n_\psi(k) = -2\epsilon_1, -2\epsilon_1^2 + (2\alpha_* - 2) \epsilon_2, + \left( -\alpha_*^2 + 2 \alpha_* - 2 + \frac{\pi^2}{12} \right) \epsilon_3, - 2\epsilon_1^4, + \left[ -\frac{5}{3} \alpha_*^3 + 2 \alpha_*^2 + 20 \alpha_* - \frac{11 \pi^2}{12} - \alpha_* - \frac{92}{3} + \frac{31 \pi^2}{6} - \frac{44}{3} \zeta(3) \right] \epsilon_1 \epsilon_2, + \left[ \alpha_*^3 - 7 \alpha_*^2 + 22 \alpha_* - \frac{5 \pi^2}{4} - \alpha_* - 16 + \frac{19 \pi^2}{12} - 2 \zeta(3) \right] \epsilon_1 \epsilon_2, + \left[ \alpha_*^3 - 4 \alpha_*^2 + 16 \alpha_* - \frac{13 \pi^2}{12} - \alpha_* - \frac{38}{3} + \frac{4 \pi^2}{3} - \frac{2}{3} \zeta(3) \right] \epsilon_2, + \left[ \frac{1}{3} \alpha_*^3 - 2 \alpha_*^2 + 2 \alpha_* - \frac{\pi^2}{12} \alpha_3 - \frac{2}{3} + \frac{\pi^2}{12} - \frac{2}{3} \zeta(3) \right] \epsilon_4.
\]

1 Note that these parameters are defined differently from in [18, 19], where \( \epsilon_0 = -\frac{\dot{H}}{H^2} \) and \( \epsilon_{n+1} = \frac{d \ln \epsilon_1}{d \ln n} \).
Note that the slow-roll parameters, equations (23), are given as functions of $H$ only, so these results are applicable to models with multiple degrees of freedom for inflaton field $\phi$ [7, 13], as well as single field cases [12, 14, 15].

3.2. de Sitter background

For a perfect de Sitter space, where $H$ is constant, we have

$$x = \frac{k}{aH} \quad \text{and} \quad p = \frac{2\pi}{H},$$

(28)

from which it follows that

$$\frac{p'}{p} = 0 \quad \text{and} \quad q = \frac{p'' - 3p'}{p} = 0.$$  

(29)

Therefore, from equation (16), only the leading term survives and we obtain the simple result

$$\mathcal{P}_\psi(k) = \left(\frac{H}{2\pi}\right)^2,$$

(30)

and the spectral index is of course exactly flat, that is,

$$n_\psi(k) = 0.$$    

(31)

This is because $\mathcal{P}_\psi(k)$ depends only on $H$ during inflation, not the detailed dynamics of the inflaton field $\phi$ due to the potential $V(\phi)$. That is, the gravitational waves depend only on the energy scale associated with the inflaton potential, which is one characteristic feature of the tensor perturbations. So, even if $V(\phi)$ has some features, e.g. a linear potential with a slope change [15], as long as $V(\phi)$ is very close to flat, so that de Sitter space is a good approximation, we obtain a featureless, nearly flat $\mathcal{P}_\psi(k)$, but nontrivial $\mathcal{P}_\zeta(k)$.

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