A Naturally Renormalized Quantum Field Theory

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Abstract

It was shown that quantum metric fluctuations smear out the singularities of Green’s functions on the light cone \([1]\), but it does not remove other ultraviolet divergences of quantum field theory. We have proved that the quantum field theory in Krein space, \textit{i.e.} indefinite metric quantization, removes all divergences of quantum field theory with exception of the light cone singularity \([2, 3]\). In this paper, it is discussed that the combination of quantum field theory in Krein space together with consideration of quantum metric fluctuations, results in quantum field theory without any divergences.

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1 Introduction

One of the greatest challenges of physics today is the achievement of a proper theory of quantized gravitational fields. In other words a theory, which quantizes the gravitational fields without any anomaly, is sought for the past seven decades without a thorough success. The element of time, which has two completely different concepts in quantum mechanics as opposed to general relativity, is the most outstanding problem of this theory. By introduction of background field method this problem has been resolved-although it could not be applied to the very early moments of evaluation of universe (the Planck scale) where the perturbation of metric is in the same order as the background metric itself. The next problem, which appears in the background field method is the non-renormalizability of the theory of quantum gravity. Three divergent opinions have been stated for explanation of this anomaly. The first view sees the problem as an inherent problem of general relativity and the second, as an intrinsic problem of quantum mechanics. The third group, believe that both, quantum mechanics and general relativity are problematic and an alternative theory has to replace them.

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Combination of quantum theory and special relativity results in appearance of singularity in QFT. To cope with this problem regularization and renormalization procedures, which are not related to fundamental concepts of quantum mechanic and/or special relativity, have been successfully utilized to remove divergences for certain problems. This procedure, however, cannot be extended to the quantum general relativity, and the theory remains divergent. We believe the root of this anomaly lies in the QFT, not in general relativity.

The singular behavior of Green’s function at short relative distances (ultraviolet divergence) or in the large relative distances (infrared divergence) leads to divergences of the QFT. The ultraviolet divergence appears in the following terms of Green’s function in the limit $\sigma = 0$:

\[
\frac{1}{\sigma}, \ln \sigma, \text{ and } \delta(\sigma).
\]

It was conjectured long ago [4, 5] that quantum metric fluctuations might smear out the singularities of Green’s functions on the light cone i.e. $\delta(\sigma)$. Along the this line the model described by Ford [1], which is based on the quantum metric fluctuations, does smear out the light cone singularities, but it does not remove other ultraviolet divergences of quantum field theory.

The quantum field theory in Krein space, i.e. indefinite metric quantization, studied previously for other problems [6, 7], was utilized for the covariant quantization of the minimally coupled scalar field in de Sitter space [2]. In this method, the auxiliary negative norm states (negative frequency solution) have been utilized, the modes of which do not interact with the physical states or real physical world. One of the interesting results of this construction is that the quantum field theory in Krein space removes all ultraviolet divergences of quantum field theory with exception of the light cone singularity. In the previous works, we had shown that presence of negative norm states play the role of an automatic renormalization device for certain problems [2, 3, 8, 9, 10, 11, 12].

In the next section, we have shown the quantum field theory in Krein space removes all ultraviolet divergences of quantum field theory with exception of the light cone singularity. In the following section, we briefly recall that quantum metric fluctuations as a tool to remove the singularities of Green’s functions on the light cone [1]. We have established that the combination of quantum field theory in Krein space together with consideration of quantum metric fluctuations, results in quantum field theory without any divergences. Finally, we explicitly calculate for $\lambda \phi^4$ theory, the transition amplitude of the state $|q_1, q_2; in >$ to the state $|p_1, p_2; out >$ for s-channel contribution in the one-loop approximation.

## 2 Krein space quantization

Recently, the existence of a non-zero cosmological constant has been proposed to explain the luminosity observations on the farthest supernovas [13, 14]. If this hypothesis is validated, our ideas on the large-scale universe should to be changed and the de Sitter metric will play a further important role. Thus the quantization of the massless tensor spin-2 field on dS space, i.e. a linear gravitational field, without infrared divergence presents an excellent modality for further research. The linear quantum gravity is an important element in the understanding of quantum cosmology and of quantum gravity. However the graviton propagator in the linear approximation, for largely separated points, either has a pathological behavior (infrared divergence) or
violates dS invariance [15, 16, 17]. Antoniadis, Iliopoulos and Tomaras [18] have shown that
the pathological large-distance behavior of the graviton propagator on a dS background does
not manifest itself in the quadratic part of the effective action in the one-loop approximation.
This means that the pathological behavior of the graviton propagator is gauge dependent and
so should not appear in an effective way as a physical quantity. Recently, de Vega and al. [19]
have also shown that, the infrared divergence does not appear in the physical world. This result
has been also obtained by other authors [20, 21, 22]. The dS linear gravity could indeed be
constructed from minimally coupled scalar field in the ambient space notation, i.e. $k_{\alpha\beta} = D_{\alpha\beta}\phi$
[23, 24, 25]. Thus a large volume of literature has been devoted to the quantization problem
for the minimally-coupled massless field in de Sitter space [26, 27].

It is proven that for the minimally coupled scalar field, in de Sitter space, one can not
construct a covariant quantization with only positive norm states. In addition there appears
an infrared divergence in the two point function [26]. It has been proved that the use of the
two sets of solutions (positive and negative norms states) is an unavoidable feature if one is
resolved to preserve causality (locality), covariance and elimination of the infrared divergence
in quantum field theory for the minimally coupled scalar field in de Sitter space [2]. We
maintain the covariance principle and remove the positivity condition similar to the Gupta-
Bleuler quantization of electrodynamics in Minkowski space.

One of the very interesting results of this construction is that the Green’s function, at
large relative distances, does not diverge. In other words the previous infrared divergence
disappears [2, 11] and the ultraviolet divergence in the stress tensor disappears as well, which
means the quantum free scalar field in this method is automatically renormalized. The effect
of “unphysical” states (negative norm states) appears in the physics as a natural renormalization
procedure. By the use of this method for linear gravity (the traceless rank-2 “massless” tensor
field) a fully covariant quantization in dS space is obtained [25]. Consequently the corresponding
two point function is free of any infrared divergence [10, 23]. It is important to note that
following this method, a natural renormalization of the certain problems, have been already
attained [2, 3, 8, 9, 10, 11].

We briefly recall the Krein space quantization of the minimally coupled scalar field in de
Sitter space [2]. As proved by Allen [26], the covariant canonical quantization procedure with
positive norm states fails in this case. The Allen’s result can be reformulated in the following
way: the Hilbert space generated by a set of modes (named here the positive modes, including
the zero mode) is not de Sitter invariant,

$$\mathcal{H} = \{ \sum_{k\geq 0} \alpha_k \phi_k; \sum_{k\geq 0} |\alpha_k|^2 < \infty \}.$$  

This means that it is not closed under the action of the de Sitter group generators. In order
to resolved this problem, we have to deal with an orthogonal sum of a positive and negative
inner product space, which is closed under an indecomposable representation of the de Sitter
group. The negative values of the inner product are precisely produced by the conjugate modes:
$$\langle \phi_k^*, \phi_k^* \rangle = -1, \ k \geq 0.$$

We do insist on the fact that the space of solution should contain the unphysical states with negative norm. Now, the decomposition of the field operator into positive
and negative norm parts reads

$$\phi(x) = \frac{1}{\sqrt{2}} [\phi_p(x) + \phi_n(x)], \quad (1)$$

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A direct consequence of these formulas is the positivity of the energy standard QFT based on CCR lies in the following on the commutation relations requirement:

\[ a_k|0 >= 0, \quad [a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad b_k|0 >= 0, \quad [b_k, b_{k'}^\dagger] = -\delta_{kk'}. \]  

(3)

A direct consequence of these formulas is the positivity of the energy \textit{i.e.}

\[ \langle \langle \hat{k} | T_{00} | \hat{k} \rangle \rangle \geq 0, \]

for any physical state \( |\hat{k}\rangle \) (those built from repeated action of the \( a_k^\dagger \)'s on the vacuum). This quantity vanishes if and only if \( |\hat{k}\rangle = |0\rangle \). Therefore the “normal ordering” procedure for eliminating the ultraviolet divergence in the vacuum energy, which appears in the usual QFT is not needed [2]. Another consequence of this formula is a covariant two-point function, which is free of any infrared divergence [11].

It is important to note that, similar to the gauge quantum field theory, the auxiliary negative norm states are presented, after the calculation of the Green function, by imposing the condition on the field operator, are eliminated. The negative norm states cannot propagate in the physical world and they only play the rule of an automatic renormalization device for the theory.

Within the framework of our approach, the “Wightman” two-point function is the imaginary part the usual Wightman two-point function, which is built from the positive norm states

\[ W(x, x') =< 0 | \phi(x)\phi(x') | 0 > = \frac{1}{2} [W_p(x, x') + W_n(x, x')] = i\tilde{W}_p(x, x'), \]

(4)

where \( W_n = -W_p^* \). The time-ordered product of fields is defined as

\[ iG_T(x, x') =< 0 | T\phi(x)\phi(x') | 0 >= \theta(t - t')W(x, x') + \theta(t' - t)W(x', x). \]

(5)

In this case we obtain

\[ G_T(x, x') = \frac{1}{2} [G_F^p(x, x') + (G_F^p(x, x'))^*] = \Re G_F^p(x, x'). \]

(6)

For scalar field \( \phi(x) \) in 4-dimensional Minkowski space-time, the positive norm state time-ordered product of fields or Feynman two point function is [28]

\[ G_F^p(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-x')} \tilde{G}_F(k) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik.(x-x')}}{k^2 - m^2 + i\epsilon} \]

\[ = -\frac{1}{8\pi}\delta(\sigma_0) + \frac{m^2}{8\pi}\theta(\sigma_0) \left[ J_1(\sqrt{2m^2\sigma_0}) - iN_1(\sqrt{2m^2\sigma_0}) \right] - \frac{im^2}{4\pi^2}\theta(-\sigma_0) \frac{K_1(\sqrt{2m^2(-\sigma_0)})}{\sqrt{2m^2(-\sigma_0)}}, \]

(7)

where \( J_1, N_1 \) and \( K_1 \) are the Bessel functions. Then the time-ordered product of fields in the Krein space quantization is:

\[ G_T(x, x') = \Re G_F^p(x, x') = -\frac{1}{8\pi}\delta(\sigma_0) + \frac{m^2}{8\pi}\theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \sigma_0 \geq 0. \]

(8)

This function is singular only on the light cone. In the next section, we briefly recall that quantum metric fluctuations as a tool to remove the singularities of Green’s functions on the light cone [1].
3 Quantum metric fluctuation

Reviewing the effective methods of quantum metric fluctuation for removal of light cone singularities, we present Ford’s vivid paper in this section [1]. Consideration of a flat background space time with a linearized perturbation \( h_{\mu\nu} \) propagating upon it, constitute the basic modality of quantum metric fluctuation, i.e.

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h| < |\eta|. \tag{9}
\]

In the unperturbed space time, the square of the geodesic separation of points \( x \) and \( x' \) is

\[
2\sigma_0 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu). \tag{10}
\]

In the general curved space-time, the presence of the perturbation \( h_{\mu\nu} \), the square of the geodesic separation is set to be

\[
2\sigma = \sigma_0 + \sigma_1 + O(h^2),
\]

so \( \sigma_1 \) is the first order shift in \( \sigma \) and it is an operator in the linear quantum gravity.

In flat space time, the retarded Green’s function for a massless scalar field is

\[
G_{\text{ret}}^{(0)}(x - x') = \frac{\theta(t - t')}{4\pi} \delta(\sigma_0), \tag{11}
\]

which has a delta-function singularity on the future light cone and is zero elsewhere. In the presence of a classical metric perturbation, the retarded Green’s function has its delta-function singularity on the perturbed light cone, where \( \sigma = 0 \). In general, it may also become nonzero on the interior of the light cone due to backscattering off of the curvature. However, we are primarily interested in the behavior of the near ”new” light cone, so we replace \( G_{\text{ret}}^{(0)}(x - x') \) by

\[
G_{\text{ret}}(x - x') = \frac{\theta(t - t')}{4\pi} \delta(\sigma). \tag{12}
\]

This may be expressed as

\[
G_{\text{ret}}(x - x') = \frac{\theta(t - t')}{4\pi} \int_{-\infty}^{\infty} da e^{ia\sigma_0} e^{ia\sigma_1}. \tag{13}
\]

We now replace the classical metric perturbations by gravitons in a vacuum state \( |\psi\rangle \). Then \( \sigma_1 \) becomes a quantum operator which is linear in the graviton field operator, \( h_{\mu\nu} \). Because a vacuum state is a state such that \( \sigma_1 \) may be decomposed into positive and negative frequency parts, i.e., we may find \( \sigma_1^+ \) and \( \sigma_1^- \) so that \( \sigma_1^+ |\psi\rangle = 0, \langle \psi | \sigma_1^- = 0, \) and \( \sigma_1 = \sigma_1^+ + \sigma_1^- \). Thus when we average over the metric fluctuations, the retarded Green’s function is replaced by its quantum expectation value:

\[
\langle G_{\text{ret}}(x - x') \rangle = \frac{\theta(t - t')}{4\pi} \int_{-\infty}^{\infty} da e^{ia\sigma_0} e^{ia\langle\sigma_1\rangle}. \tag{14}
\]

This integral converges only if \( \langle \sigma_1^2 \rangle > 0 \), in which case it may be evaluated to yield

\[
\langle G_{\text{ret}}(x - x') \rangle = \frac{\theta(t - t')}{8\pi^2} \sqrt{\frac{\pi}{2\langle \sigma_1^2 \rangle}} \exp\left(-\frac{\sigma_0^2}{2\langle \sigma_1^2 \rangle}\right). \tag{15}
\]
Note that this averaged Green’s function is indeed finite at $\sigma_0 = 0$ provided that $\langle \sigma_1^2 \rangle \neq 0$. Thus the light cone singularity has been removed.

Therefore the quantum field theory in Krein space, including the quantum metric fluctuation, removes all ultraviolet divergences of the theory:

$$\langle G_T(x - x') \rangle = -\frac{1}{8\pi \sqrt{2\langle \sigma_1^2 \rangle}} \exp \left( -\frac{\sigma_0^2}{2\langle \sigma_1^2 \rangle} \right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}.$$  \hfill (16)

In the case of $2\sigma_0 = \eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) = 0$, due to the metric quantum fluctuation, $h_{\mu\nu}$, $\langle \sigma_1^2 \rangle \neq 0$, and we have

$$\langle G_T(0) \rangle = -\frac{1}{8\pi \sqrt{2\langle \sigma_1^2 \rangle}} + \frac{m^2}{8\pi} \frac{1}{2}.$$  \hfill (17)

It should be noted that $\langle \sigma_1^2 \rangle$ is related to the density of gravitons [1].

### 4 $\lambda \phi^4$ theory in Krein space

An immediate consequence of this construction is a finite quantum field theory. A vivid example is the realization of finite $\lambda \phi^4$ theory. To demonstrate this we explicitly calculate the transition amplitude of the state $|q_1, q_2; \text{in}>$ to the state $|p_1, p_2; \text{out}>$ for s-channel contribution in the one-loop approximation. It is given by [30]

$$T \equiv \langle p_1, p_2; \text{out}|q_1, q_2; \text{in}\rangle = \int d^4y_1d^4y_2d^4x_1d^4x_2 e^{ip_1.y_1+i p_2.y_2-i q_1.x_1-i q_2.x_2}$$

$$G_T(y_1 - z_2)G_T(y_2 - z_1)G_T(x_1 - z_2)G_T(x_2 - z_2),$$

where the Feynman Green function $G_F^p$ is replaced by the time-order product Green function $G_T$. We obtain

$$T = \frac{\lambda^2}{2} \int d^4z_1d^4z_2 e^{i(p_1+p_2).z_1-i(q_1+q_2).z_2} [G_T(z_1 - z_2)]^2 - \frac{\lambda^2}{2} (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

$$\int d^4z e^{i(p_1+p_2).z} \left( \frac{1}{8\pi \sqrt{2\langle \sigma_1^2 \rangle}} \exp \left( -\frac{\sigma_0^2}{2\langle \sigma_1^2 \rangle} \right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} \right)^2,$$  \hfill (18)

where $2\sigma_0 = (z_1 - z_2)^2 = z^2$. The second integral for the space-like separated pair $(z_1, z_2)$ is zero. The first integral is gaussian and the second is a fourier transformation of a non-singular function. Therefore the transition amplitude in Krein space quantization, is finite in the one-loop approximation.

### 5 Conclusion

The problem of divergence disappears in QFT, if the principle of the positivity of the norms is overlooked. In the other word, the problem appears when the negative norm states, which are
also solutions of the field equation in the classical level, are eliminated by one of the principles of quantum theory, i.e. the principle of positivity. The direct result of the principle of positivity is appearance of distractive anomalies in QFT, divergences and breaking of the covariance of the minimally coupled scalar field and linear quantum gravity in de Sitter space.

Implementing of this method to the linear gravity (traceless part) on de Sitter space, a covariant two-point function, free of the pathological large-distance behavior was obtained [10]. The quantum field theory in Krein space, which includes quantum metric fluctuations, resolves the ultraviolet divergences of quantum field theory. Finally the theory of quantum gravity in the back ground field method is achieved without any divergencies.

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