The spectral sequence of the canonical foliation of a Vaisman manifold

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Abstract
In this paper we investigate the spectral sequence associated to a Riemannian foliation which arises naturally on a Vaisman manifold. Using the Betti numbers of the underlying manifold we establish a lower bound for the dimension of some terms of this cohomological object. This way we obtain cohomological obstructions for two-dimensional foliations to be induced from a Vaisman structure. We show that if the foliation is quasi-regular the lower bound is realized. In the final part of the paper we discuss two examples.

Keywords: locally conformally Kähler, canonical foliation, Vaisman manifold, spectral sequence.

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1. Introduction

A Hermitian manifold \((M, J, g)\) (\(\dim_{\mathbb{R}} M \geq 2\)) is Vaisman if its fundamental two-form \(\omega(X, Y) := g(X, JY)\) satisfies \(d\omega = \theta \wedge \omega\) for a non-zero one-form \(\theta\) (called the Lee form) which is parallel with respect to the Levi-Civita connection of the metric. In particular, a Vaisman metric is locally conformally Kähler (LCK), see e.g. [Dr-O].

A Vaisman metric is a Gauduchon metric \((\bar{\partial} \omega^{n-1} = 0)\) and hence, on compact \(M\), it is unique, up to homothety, in a given conformal class. But
not every conformal class of LCK metrics contains a Vaisman one: e.g. the Inoue surfaces do not admit Vaisman metrics, [Be].

Most of the compact complex surfaces are LCK, [Br], and among them, many are Vaisman: Hopf surfaces of rank 1, Kodaira surfaces etc., see [Be]. Higher dimensional examples of Vaisman manifolds are the diagonal Hopf manifolds, see [O-Ve3].

Throughout this paper we shall assume $\theta$ is not exact, that is, the manifold is not globally conformally Kähler. In particular, $b_1(M) \geq 1$.

On a Vaisman manifold $(M, J, g)$, the vector fields metrically equivalent with $\theta$ and respectively $-J\theta$ (called the anti-Lee form) define a real 2-dimensional distribution $\mathcal{F}$ which is integrable and generate a Riemannian, totally geodesic and holomorphic foliation [V], usually called the canonical foliation. It was widely studied, especially in the compact case, e.g. in [Ts], where it is used to show that a direct product of compact Vaisman manifolds cannot carry LCK metrics. Geometric properties of this foliation were discussed in [Ch-Pi], while in [P] (see also the last section of this paper) a comprehensive description of the canonical foliation is presented for some Hopf surfaces, addressing the transverse structure of the foliation, a classification of leaves and in some particular cases even a characterization of the space of leaves.

The purpose of this paper is to investigate the spectral sequence $(E_k, d_k)$ of the canonical foliation of a compact Vaisman manifold (for definition see [A-K, To]). The dimensions of the spectral terms $E_k$ with $k \geq 2$ are known to be invariant with respect to a leafwise homeomorphism (i.e. a homeomorphism which sends leaves to leaves), [A-M]. As in the bundle case, the spectral sequence converges to the de Rham cohomology of $M$.

For other spectral sequences defined on a foliated manifolds we refer to [VT] for symplectic foliations and more recently to [W] for a spectral sequence related to the transverse geometry. Also, for the particular case when the transverse distribution is integrable, see [Po].

Note that the page $E_2$ (which is the most relevant part of the spectral sequence) contains as a subset the groups of the basic cohomology (the spectral terms with 0 leafwise degree). The basic de Rham complex has been intensively studied and in fact can be regarded as the counterpart of the de Rham complex for the space of leaves.

The transverse geometry of a foliation is encoded in its basic cohomology (for closed Kähler manifolds see e.g. [Go], for Sasakian and 3-Sasakian manifolds see [Bo-Gal2], [Bo-Gal1] etc.). The formulae which relate the basic Betti numbers with respect to the canonical foliation of a Vaisman manifold to the usual Betti numbers of the manifold where given in [V]. Moreover, using the minimality of the canonical foliation and the Poincaré duality defined in [Mas] it is possible to compute the dimension of the spectral terms with top leafwise degree 2.

In this paper we go further and compute the dimension of the remaining terms of order 2 in terms of the basic Betti numbers. We rely to the fact that
the remaining terms are related to the Lee and anti-Lee one-forms which are defined in the leafwise direction. Together, these two steps give informations about all relevant spectral terms.

Our main tool is a Hodge theoretic approach, devised in this particular setting by Álvarez-López and Kordyukov [A-K]. We stress that this approach is different from the previous cohomological method used for computing the spectral sequence (see e.g. [Do]).

On compact Vaisman manifolds, some spectral terms are nontrivially related to the basic cohomology groups and we obtain a lower bound for their dimension in terms of basic Betti numbers. For quasi-regular foliations (i.e. foliations with all leaves compact) we show that the lower bound is attained and for a class of examples we determine the dimension of all spectral terms $E_2$. Note that a Vaisman structure can always be deformed to one with quasi-regular canonical foliation, see [O-Ve2].

Our result can be viewed as a cohomological obstruction for a given foliated structure on a complex manifold to be determined by some Vaisman structure.

The paper is organized as follows. In Section 2 we present the basic properties of the Vaisman manifolds and of the spectral sequence associated to a Riemannian foliation. In Section 3 we establish a lower bound for the dimension of some spectral terms for the canonical foliation of a compact Vaisman manifold. In Section 4 we prove that the lower bound is attained if the foliation is quasi-regular. In Section 5 we present two examples: one (the diagonal Hopf manifold) which satisfies the obstructions, and another one on which the obstruction is effective.

2. Preliminaries

2.1. Vaisman manifolds. We present the needed background for Vaisman manifolds. For details, proofs and examples, please see [Dr-O] and more recent papers by Ornea and Verbitsky.

Let $(M, J)$ be a complex manifold of real dimension $2n + 2$, with $n \geq 1$ (tacitly assumed to be connected, of class $C^\infty$).

**Definition 2.1:** A Vaisman metric on $(M, J)$ is a Hermitian metric $g$ such that its fundamental two-form $\omega(X, Y) := g(X, JY)$ satisfies the integrability condition:

$$d\omega = \theta \wedge \omega,$$

for some one-form $\theta$, called Lee form, which is parallel with respect to the Levi-Civita connection of $g$.

Then $g$ gives a Vaisman structure on $(M, J)$, and $(M, J, g)$ is called a Vaisman manifold.

**Remark 2.2:** A Vaisman manifold is locally conformally Kähler (LCK), as $\nabla \theta = 0$ implies $d\theta = 0$. 
As $\theta$ is parallel, it can be considered of norm 1. Let $\theta^c := -J\theta$ be the anti-Lee form ($\theta^c(X) = \theta(JX)$). The vector fields metrically equivalent with $\theta$ and $\theta^c$ will be denoted by $U$ and $V$, and they are unitary:

$$\theta(U) = 1, \quad \theta^c(V) = 1, \quad \theta(V) = 0, \quad \theta^c(U) = 0.$$  

As the universal Riemannian cover of a Vaisman manifold is a metric cone of a Sasaki manifold (see e.g. [O-Ve1]), the local structure of a Vaisman manifold implies the existence of a local Sasaki structure transverse to the flow generated by $U$. Thus we may assume the existence of a local orthonormal frame $\{f_1, \ldots, f_n, f_{n+1}, \ldots, f_{2n}, U, V = \xi\}$, such that $\xi$ is the Reeb vector field and

$$J(e_i) = e_{n+i}, \quad J(e_{n+i}) = -e_i.$$  

The dual frame will be denoted $\{f^\flat_1, \ldots, f^\flat_n, f^\flat_{n+1}, \ldots, f^\flat_{2n}, \theta, \theta^c\}$.

The parallelism of $\theta$ immediately implies:

**Lemma 2.3:** ([V]) The following equations hold on a Vaisman manifold:

\begin{align*}
\nabla_{e_i} U &= 0, \\
\nabla_{e_i} V &= J(e_i), \\
\nabla U U &= \nabla V U = 0, \\
\nabla U V &= \nabla V V = 0.
\end{align*}  

In particular, $U$ and $V$ generate a foliation $\mathcal{F}$ with leafwise dimension 2 (called the canonical foliation [Ch-Pi, P, Ts]), the leaves of which are minimal submanifolds.

Moreover, it can be shown, [V], that $g$ is a bundle-like metric, which means the foliation is locally a Riemannian submersion. Such spaces are called Riemannian foliations [To, Mo].

The metric $g$ induces a splitting of the tangent bundle of the underlying manifold $M$,

$$TM = Q \oplus T\mathcal{F},$$  

with $T\mathcal{F}$ being the leafwise tangent bundle and $Q$ the orthogonal complement. If $\pi_Q, \pi_{T\mathcal{F}}$ are the canonical projection operators and

$$\nabla^Q := \pi_Q \circ \nabla, \quad \nabla^{T\mathcal{F}} := \pi_{T\mathcal{F}} \circ \nabla,$$

then the Levi-Civita connection also splits

$$\nabla = \nabla^Q + \nabla^{T\mathcal{F}}.$$
The following result collects several useful computational facts:

**Lemma 2.4:** The relations below hold on a Vaisman manifold:

\[
\begin{align*}
\nabla^{TF} U^b &= 0, \\
\nabla^{TF} V^b &= 0, \\
\n\nabla^{TF} e_i^b &= \nabla^{TV} e_i^b = 0, \\
\n\nabla_U U^b &= \nabla_V V^b = 0, \\
\n\nabla_U V^b &= \nabla_V U^b = 0.
\end{align*}
\]

\[ (2.3) \]

**Proof.** The first two, and the last two relations are direct consequences of the first equation (2.1). For the remaining one, just note that

\[
\nabla_U e_i^b(U) = -e_i^b(\nabla_U U) = 0.
\]

\[ 2.2. \] The spectral sequence associated to a Riemannian foliation.

We now introduce the spectral sequence of a Riemannian foliation. It encodes many properties concerning the homotopy of the foliation, [A-K, To].

Let \( M \) be a closed (i.e. compact and without boundary) manifold, and let \( F \) be a foliation of dimension \( p \) and codimension \( q \) (hence \( n = p + q \)).

\[ (2.4) \quad \Omega_k = \{ \omega \in \Omega^r \mid \iota_X \omega = 0, X = X_1 \wedge \cdots \wedge X_{r-k+1}, X_i \in \Gamma(TF) \}, \]

where \( \iota \) denotes the interior product. This way, the set of differential forms \( \Omega \) on \( M \) becomes a filtered complex:

\[
\Omega = \Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_q \supset \Omega_{q+1} = 0.
\]

We follow [A-K, Sec. 2], [To] for the definition of the spectral sequence associated to \( F \). Set \( E^u,v_k := \Omega^u,v_k/\Omega^u,v_{k+1} \), for \( 0 \leq u \leq q, \ 0 \leq v \leq p \), and construct the “page” of order \( k + 1 \) inductively as \( E_{k+1} := H(E_k, d_k) \), where \( d_k \) is canonically induced by \( d \). More precisely:

\[
E^u,v_{k+1} = \frac{\ker (d_k : E^u-k,v+k-1_k \to E^u,v_k)}{\im (d_k : E^u,v_k \to E^u,v_{k-1})}
\]

An explicit description of the above terms is possible:

\[
E^u,v_k = \frac{Z^u,v_k}{Z^u,v_{k-1} + B^u,v_k},
\]

where the spaces \( Z^u,v_k, B^u,v_k \) are:

\[
Z^u,v_k := \Omega^u,v_k \cap d^{-1}(\Omega^u,v_{k+1}),
\]

\[
B^u,v_k := \Omega^u,v_k \cap d(\Omega^u,v_{k-1}).
\]

\[ (2.5) \]
Definition 2.5: The family of cohomological complexes \((E_k, d_k)_{k \geq 0}\) is called the spectral sequence of the foliation \(\mathcal{F}\).

Remark 2.6: The spectral terms \(\{E^{u,0}_1\}_{0 \leq u \leq q}\) can be identified with the spaces of basic forms (with respect to \(\mathcal{F}\)):

\[ \Omega^u_b := \{ \alpha \in \Omega^u \mid \iota_X \alpha = 0, \iota_X d\alpha = 0 \ (\forall) X \in \Gamma(T\mathcal{F}) \}, \]

while \(\{E^{u,0}_2\}_{0 \leq u \leq q}\) can be identified with the groups \(\{H^u_b\}_{0 \leq u \leq q}\) of the basic de Rham cohomology of the foliation.

The following result can be seen as an extension of the topological invariance of the groups of basic cohomology stated in [E-N1]:

Theorem 2.7: ([A-M]) On a compact Riemannian foliation the dimension of the spectral terms \(E^{u,v}_k\) is invariant with respect to foliated homeomorphisms (i.e. homeomorphisms which send leaves to leaves) for \(k \geq 2, 0 \leq u \leq q, 0 \leq v \leq p\).

Let now \((M, J, g)\) be a closed Vaisman manifolds of dimension \(2n + 2\) and let \(\mathcal{F}\) be the canonical foliation. We denote by \(\{b_i\}_{0 \leq i \leq 2n+2}\) the Betti numbers of \(M\) and by \(\{e_i\}_{0 \leq i \leq 2n}\) the basic Betti numbers. Then we have:

Theorem 2.8: [V, Theorem 4.2] On a compact Vaisman manifold the numbers \(\{e_i\}\) are uniquely determined by \(\{b_i\}\) and, conversely, starting with Betti numbers \(\{b_i\}\) we can compute the basic Betti numbers \(\{e_i\}\).

As a consequence, if we describe the dimension of all spectral terms of order 2 using the numbers \(\{e_i\}\), then it would be possible to express them using only the de Rham complex of \(M\). We have

\[ \dim E^{u,0}_2 = \dim H^u_b := e_u. \]

On the other hand, using the Poincaré duality for the basic de Rham complex of a minimal foliation (see e.g. [To, Chapter 7]), the duality stated in [Mas] gives

\[ \dim E^{u,0}_2 = \dim E^{u,2}_2. \]

Then, the difficulty consists in studying the terms \(E^{u,1}_2\), for \(0 \leq u \leq 2n\). To overcome it, we use a Hodge-type theory for the terms of the spectral sequence, [A-K], that we further describe.

2.2.1. A Hodge-type theory. Let \(\mathcal{F}\) be a closed Riemannian foliation. The splitting (2.2) induces a corresponding bigrading of \(\Omega\) such that

\[ \Omega_k = \bigoplus_{u \geq k} \Omega^u. \]
Therefore \((2.4)\) can be regarded as the set of differential forms with transverse degree at least \(k\). Then:

\[
\Omega_{u,v} = \mathcal{C}^\infty \left( \bigwedge^u Q^* \oplus \bigwedge^v T^*F^* \right), \quad u, v \in \mathbb{Z}.
\]

Let \(\pi_{u,v} : \Omega \to \Omega_{u,v}^k\) be the canonical projections determined by the bigrading of \(\Omega\). Define the topological vector spaces:

\[
z_{k}^{u,v} = \pi_{u,v}(Z_{k}^{u,v}), \quad b_{k}^{u,v} = \pi_{u,v}(B_{k}^{u,v}), \quad e_{k}^{u,v} = z_{k}^{u,v}/b_{k-1}^{u,v}.
\]

One can see that

\[
Z_{k}^{u,v} \cap \text{Ker} \pi_{u,v} = Z_{k-1}^{u+1,v-1},
\]

and this induces the continuous linear isomorphism

\[
E_{k}^{u,v} \simeq e_{k}^{u,v}.
\]

Let now \(d_{k}\) be the operator introduced above. In [A-K Section 5.1] a sequence of Laplace type operators is inductively constructed: \(\Delta_0, \Delta_1, \ldots\), as well as a sequence of corresponding kernel spaces \(H_1 \supseteq H_2 \supseteq \cdots\), such that:

\[
\Omega = H_1 \oplus \text{Im} d_0 \oplus \text{Ker} \delta_0, \quad H_1 = H_2 \oplus \text{Im} d_1 \oplus \text{Ker} \delta_1, \quad \vdots
\]

where the overline denotes closure with respect to \(\mathcal{C}^\infty\) topology. These decompositions are the analogues of the Hodge theory in our setting. If \(H_{k}^{u,v} := \pi_{u,v}(H_{k})\) then these vector spaces are related by the following isomorphism [A-K Section 5.1] (for \(k = 2\) see also [A-K Theorem 2.2(iv)]):

\[
e_{k}^{u,v} \simeq H_{k}^{u,v}
\]

so together with \((2.7)\) we obtain

\[
H_{k}^{u,v} \simeq E_{k}^{u,v}, \quad \text{for } k \geq 2.
\]

The de Rham differential and codifferential decompose with respect to the bigrading \((2.6)\) as \((13)\):

\[
d = d_{0,1} + d_{1,0} + d_{2,-1}, \quad \delta = \delta_{0,-1} + \delta_{1,0} + \delta_{-2,1},
\]

with

\[
d_{i,j} : \Omega^{u,v} \to \Omega^{u+i,v+j}.
\]

Note that \(\delta_{i,j}\) is the adjoint operator of \(d_{i,j}\). Also, \(d_0 \equiv d_{0,1}\).

**Remark 2.9**: We can define the basic de Rham operators \(d_b, \delta_b\) restricting \(d_{1,0}\) and \(\delta_{-1,0}\) to \(\Omega_b\). If the foliation has vanishing mean curvature (for instance in the case of canonical foliations on Vaisman manifolds), then these operators coincide with the usual de Rham operators on some local transverse submanifold. Consequently, the **basic Laplace operator** \(\Delta_b := d_b \delta_b + \delta_b d_b\) coincides with the corresponding transverse operator (see e.g. -7-
Similarly, using the 1-st order operators \(d_{0,1}, \delta_{0,-1}\), we construct \(\Delta_0 := d_{0,1}\delta_{0,-1} + \delta_{0,-1}d_{0,1}\). The leafwise Laplace operator \(\Delta_F\) can be constructed using the restrictions \(d_F, \delta_F\) of the first order operators \(d_{0,1}, \delta_{0,-1}\) to \(\Omega^0\).

Finally, we notice that \(d_{0,1}, \delta_{0,-1}\) vanish on basic forms.

We investigate the kernel space \(H_2\) using the adiabatic limit of the foliation.

The metric tensor can be written according to (2.2):

\[
g = g_Q \oplus g_{TF}.
\]

Introducing a parameter \(h > 0\), we define the family of metrics:

\[
g_h = h^{-2}g_Q \oplus g_{TF}.
\]

The pair represented by the manifold \(M\) and the “limit” of the Riemannian manifolds \((M, g_h)\) when \(h \downarrow 0\) is called the “adiabatic limit” of the foliation \((M, F)\). This concept was introduced for the first time by Witten, being a necessary tool for the study of the “eta” invariant of the Dirac operator; it was also used and extended by [Bi-F] and [Maz-Me].

Let \(\Theta_h : (\Omega, g_h) \to (\Omega, g)\) be ([Maz-Me]):

\[
\Theta_h \omega = h^u \omega, \quad \forall \omega \in \Omega^u,v, \quad u, v \in \mathbb{N}.
\]

We define the differential and codifferential obtained by the “rescaling” procedure:

\[
d_h = \Theta_h d \Theta_h^{-1}, \\
\delta_h = \Theta_h \delta g_h \Theta_h^{-1}.
\]

Then (2.9) implies:

\[
\begin{align*}
d_h &= d_{0,1} + h d_{1,0} + h^2 d_{2,-1}, \\
\delta_h &= \delta_{0,-1} + h \delta_{1,0} + h^2 \delta_{2,1}.
\end{align*}
\]

The corresponding Laplace and Dirac operators are:

\[
\begin{align*}
\Delta_h := \Theta_h \Delta_{g_h} \Theta_h^{-1} &= d_h \delta_h + \delta_h d_h, \\
D_h := d_h + \delta_h.
\end{align*}
\]

It can be shown that \(D_h\) is (formally) self-adjoint and \(D_h^2 = \Delta_h\).

We then have:

**Theorem 2.10**: [A-K] Section 1] Let \(\{\alpha_i\}\) be a sequence of differential forms in \(\Omega^r\), with \(\|\alpha\| = 1\), and let \(h_i\) be a sequence of real numbers such that \(h_i \downarrow 0\). If

\[
\langle \Delta_h \alpha_i, \alpha_i \rangle \in o(h_i^2),
\]

then there exists a subsequence of \(\alpha_i\) which converges in \(H_2^r\).

**Theorem 2.11**: [A-K] Section 5.1] The spaces \(H_2^{u,v}\) are uniquely determined by \(z_2^{u,v}, b_1^{u,v}\) and \(b_0^{u,v}\) as follows:

\[
z_2^{u,v} + b_0^{u,v} = H_2^{u,v} \oplus (b_1^{u,v} + b_0^{u,v}).
\]
where the closure of $b^0$ is considered in the $C^\infty$ topology.

**Remark 2.12:** In [A-K], the above are proven for general $H_k$, not only for $k = 2$.

Two direct consequences will be of interest for us:

**Lemma 2.13:** If $\alpha \in \Omega^u$ verifies
\begin{align}
&d_{0,1}\alpha = 0, \quad \delta_{0,-1}\alpha = 0, \\
&d_{1,0}\alpha = 0, \quad \delta_{-1,0}\alpha = 0,
\end{align}
then $\alpha \in H^u_2$.

**Proof.** We fix $\alpha_i = \alpha$. Clearly we can assume, without restricting the generality, that $\|\alpha\| = 1$. Then, for any $h_i \downarrow 0$, from (2.10) and (2.11) we obtain:
\[
\langle \Delta h_i \alpha, \alpha \rangle = \|D h_i \alpha\|^2 = h_i^4 \|(d_{2,-1} + \delta_{-2,1})\alpha\|^2 \in o(h_i^2).
\]
The result now follows from Theorem 2.10.

**Lemma 2.14:** If $\alpha \in H^u_2$, then there exist $\beta_i \in \Omega^{u+1,v-1}$ and $\gamma_i \in \Omega^{u-1,v+1}$ such that
\begin{enumerate}[label=(i),ref=(i)]
\item $d_{0,1}\alpha = 0, \quad \delta_{0,-1}\alpha = 0$, \\
\item $d_{1,0}\alpha + d_{0,1}\beta_i \rightarrow 0, \quad \delta_{-1,0}\alpha + \delta_{0,-1}\gamma_i \rightarrow 0$ in the $L^2$ norm.
\end{enumerate}

**Proof.** (i) is just [Remark 2.9](#) taking into account that $H^2_2 \subset H^1_1$, and hence $\Delta_{0}\alpha = 0$.

To prove (ii), note that $\alpha \in H^u_2$ implies (via Theorem 2.11) $\alpha \in z_2^u + b_0^u$. Then $\alpha = \alpha' + \alpha''$, with $\alpha' \in z_2^u$ and $\alpha'' \in b_0^u$. If $\alpha' \in z_2^u$, then from the definition [(2.5)] of the spaces $Z^u_k$ there exists a differential form $\beta \in \Omega^{u+1,v-1}$ such that
\[
\begin{align}
&d_{1,0}\alpha' + d_{0,1}\beta = 0, \\
&d_{1,0}\beta \in d_{0,1}(\Omega^{u+1,v-1}),
\end{align}
\]
and thus $d_{1,0}\beta' \in d_{0,1}(\Omega^{u+1,v-1})$.

As $d^2 = 0$, one has ([A-K]):
\[
\begin{align}
&d_{1,0}d_{0,1} + d_{0,1}d_{1,0} = 0.
\end{align}
\]

Let $P$ be the projection of $\Omega$ to $d_{0,1}(\Omega)$. Then from [2.13](#) we derive ([A-K] Lemma 2.3):
\[
\begin{align}
&d_{1,0}P = P d_{1,0}P, \\
&\text{and hence } d_{1,0}\alpha'' \in d_{0,1}(\Omega^{u+1,v-1}).
\end{align}
\]

Then $d_{1,0}\alpha \in d_{0,1}(\Omega^{u+1,v-1})$, and clearly there is a sequence $\beta_i$ such that
\[
\begin{align}
&d_{1,0}\alpha + d_{0,1}\beta_i \rightarrow 0
\end{align}
\]
in the $L^2$ norm.

For the last part we use the idea in [A-K Corollary 5.15](#). If $M$ is not orientable, take its two-sheeted oriented cover. Denoting by $\ast$ the Hodge star...
operator, we obtain the above result for $*\alpha$, with a corresponding sequence $\beta_i$. As

$$*d_{1,0} = (-1)^{r+1}\delta_{-1,0}^*, \quad *d_{0,1} = (-1)^{r+1}\delta_{0,-1}^*,$$

when the above operators are applied to a differential form of degree $r$, we obtain the necessary sequence $\gamma_i$ from $\beta_i$. ■

3. A LOWER BOUND FOR $E_2^{u,1}$

In this section, $(M,J,g)$ is a closed Vaisman manifold.

We start with a technical result about the operators defined in formula (2.9):

**Lemma 3.1:** On a closed Vaisman manifold the 1-st order differential operators $d_{0,1}$, $d_{1,0}$, and their adjoints vanish on $\theta$ and $\theta^c$:

$$d_{1,0}\theta = 0, \quad \delta_{-1,0}\theta = 0, \quad d_{1,0}\theta^c = 0, \quad \delta_{-1,0}\theta^c = 0,$$
$$d_{0,1}\theta = 0, \quad \delta_{0,-1}\theta = 0, \quad d_{0,1}\theta^c = 0, \quad \delta_{0,-1}\theta^c = 0.$$

**Proof.** From $d\theta = 0$, we obtain

$$d_{0,1}\theta = 0, \quad d_{1,0}\theta = 0.$$ 

On the other hand, using the transverse complex coordinates $\{z^a\}$ and the metric coefficients $g_{\overline{a}\overline{b}}$ with respect to these coordinates, we have (V):

$$d\theta^c = -ig_{\overline{a}\overline{b}}dz^a \wedge d\overline{z}^b,$$

and hence

$$d_{0,1}\theta^c = 0, \quad d_{1,0}\theta^c = 0.$$

Along the leaves of $\mathcal{F}$, the deRham codifferential coincide with the codifferential operator on the leaves (considered as immersed submanifolds). Applying (2.3), we get

$$\delta_{0,-1}\theta = -\iota_U \nabla_U^{\mathcal{F}} \theta - \iota_V \nabla_V^{\mathcal{F}} \theta = 0.$$ 

Similarly, $\delta_{0,-1}\theta^c = 0$. As $\theta, \theta^c \in \Omega^{0,1}$, the result follows considering the bigrading. ■

We now give the lower bound estimate. Recall that $e_i$ are the basic Betti numbers with respect to $\mathcal{F}$.

**Theorem 3.2:** If $(M,\mathcal{F})$ is the canonical foliation of a closed Vaisman manifold, then:

$$\dim E_2^{u,1} \geq 2e_u,$$

for $0 \leq u \leq 2n.$
Proof. We prove that if $\eta \in \Omega^u_\delta$ is a basic harmonic differential form, then $\eta \wedge \theta$ and $\eta \wedge \theta^c \in \mathcal{H}^2_{2,1}$.

Using [Remark 2.9] we get

\[ d_{0,1}(\eta \wedge \theta) = (-1)^u \eta \wedge d_{0,1} \theta, \]
\[ \delta_{0,-1}(\eta \wedge \theta) = (-1)^u \eta \wedge \delta_{0,-1} \theta. \]

From [Lemma 3.1] we then have

\[ d_{0,1}(\eta \wedge \theta) = 0, \quad \delta_{0,-1}(\eta \wedge \theta) = 0. \]

We show now that $d_{1,0}(\eta \wedge \theta) = 0$ and $\delta_{-1,0}(\eta \wedge \theta) = 0$. Because $\eta$ is basic and harmonic with respect to the basic Laplace operator $\Delta_\delta$, using [Remark 2.9] we get

\[ d_{1,0} \eta = 0, \quad \delta_{-1,0} \eta = 0. \]

From the hypothesis, [Lemma 3.1] and (2.3) we obtain

\[ d_{1,0}(\eta \wedge \theta) = d_{1,0} \eta \wedge \theta + (-1)^u \eta \wedge d_{1,0} \theta^c = 0. \]

Then, again using (2.3), we have:

\[ \delta_{-1,0}(\eta \wedge \theta) = \left( - \sum_i t_{e_i} \nabla_{e_i} - t_U \nabla_U - t_V \nabla_V \right)_{-1,0}^{\eta \wedge \theta} \]
\[ = \delta_{-1,0} \eta \wedge \theta + (-1)^u \eta \wedge \delta_{-1,0} \theta \]
\[ + \sum_i t_{e_i} \eta \wedge \nabla_{e_i}^{T,F} \theta + (-1)^u \nabla_{e_i}^{T,F} \eta \wedge \nabla_{U} \theta \]
\[ + (-1)^u \nabla_{V}^{T,F} \eta \wedge \nabla_{V} \theta = 0. \]

The conclusion comes from [Lemma 2.13] and equation (2.8). As for $\theta^c$, the proof is similar. 

**Corollary 3.3:** Using [V] Theorem 4.2, we can also express the lower bound using the Betti numbers of the underlying manifold:

\[ \dim E^u_{2,1,2} \geq 2 (-1)^u \sum_{i=0}^{\lfloor u/2 \rfloor} \left( \begin{array}{c} \frac{u}{2} \\ i \end{array} \right) (b_{2i} - b_{2i+(-1)^u}), \quad \text{for} \ 0 \leq u \leq n. \]

For $n + 1 \leq u \leq 2n$ we can use the Poincaré duality.

**Remark 3.4:** In particular, $\dim E^0_{2,1,2} \geq 2$.

This can be used to obtain the following obstruction for a 2-dimensional foliation on a compact complex manifold to be associated to a Vaisman structure:

**Proposition 3.5:** Let $(M,J)$ be a closed, complex, foliated manifold with a 2-dimensional foliation which admits a bundle-like metric. If $e_{2n} = \dim E^{2n,0}_2$
= 0 or \( \dim E_2^{0,1} < 2 \), then the foliation does not come from a Vaisman structure.

**Remark 3.6:** The top dimensional basic cohomology group \( H_b^{2n} \) is related to the *tautness* of the foliation (see e.g. [Ca, Mas]), but the geometrical meaning of the spectral term \( E_2^{0,1} \) is still not understood. It seems that it could be related to the existence of a minimal sub-flow on the underlying manifold. We notice here that on a Vaisman manifold there are two such flows, generated by the vector fields \( A \) and \( B \).

### 4. Quasi-regular foliations

In this section we show that if the canonical foliation has compact leaves, then the inequalities in [Theorem 3.2] become equalities.

Recall that a foliated map is a pair \((U, \varphi)\) such that \( \varphi(U) \cong O_1 \times O_2 \), with \( O_1 \in \mathbb{R}^q, O_2 \in \mathbb{R}^p \), and \( \varphi^{-1}(x_1 \times O_2) \) is on the same leaf, for any \( x_1 \in O_1 \).

If around any point \( x \) there exists a foliated map with the property that any leaf \( L \) intersects a transversal through \( x \) at most a finite number of times \( N(x) \), then the foliation is said to be *quasi-regular*. If, furthermore \( N(x) = 1 \) for any \( p \), then it is *regular*. The quasi-regularity is known to be equivalent to the compactness of all leaves [Bo-Gal1].

**Theorem 4.1:** If the canonical foliation of a closed Vaisman manifold is quasi-regular, then \( \dim E_2^{u,1} = 2 e_u \).

**Proof.** We prove that if \( \alpha \in H^{u,1} \) then it is possible to decompose it as

\[
\alpha = \eta_1 \wedge \theta + \eta_2 \wedge \theta^c,
\]

with \( \eta_1, \eta_2 \) basic harmonic forms with respect to the basic Laplace operator \( \Delta_b \).

The proof is divided in two steps: first we obtain \( \eta_i \in \Omega^{u}_b, 1 \leq i \leq 2 \), then we show that \( \eta_i \) are also harmonic.

**Step 1: \( \eta_i \) are basic.** Let \( \alpha \in H^{u,1}_2 \) be fixed arbitrarily. From [Lemma 2.14 (ii)] we have

\[
d_{0,1} \alpha = 0, \quad \delta_{0,-1} \alpha = 0.
\]

We then decompose \( \alpha \) as

\[
\alpha = f_1 \tilde{\eta}_1 \wedge \theta + f_2 \tilde{\eta}_2 \wedge \theta^c,
\]

with \( \tilde{\eta}_i \in \Omega^{u}_b \) basic forms for \( i = 1, 2 \), and \( f_i \in C^{\infty}(M) \). As the foliation is quasi-regular, all leaves will be compact, diffeomorphic with the real 2-torus \( T^2 \).

**The proof for \( u = 0 \).** We have

\[
\alpha = f_1 \theta + f_2 \theta^c.
\]

Note that the above functions \( f_i \) are constant on the leaves of \( \mathcal{F} \), and hence they are basic. Indeed, \( d_{0,1} \alpha = 0, \delta_{0,-1} \alpha = 0 \) imply \( \Delta_{\mathcal{F}} \alpha = 0 \), as \( \Delta_{\mathcal{F}} \) is the Laplace operator on the torus \( T^2 \) endowed with flat Euclidean metric.
If we fix a leaf $L$, then $\alpha_L := \alpha|_L$ will be a harmonic 1-form and as $\theta|_L$, $\theta^c|_L$ are also harmonic (as $\theta$ is parallel), then $f_1|_L, f_2|_L$ are constant, as $\dim \ker \Delta_{\mathcal{F}|_L} = 2$.

Moreover, we have
\begin{align}
V(f_1) &= U(f_2), \\
U(f_1) &= -V(f_2),
\end{align}
which are equivalent with the conditions
\begin{align*}
d_{0,1} \alpha &= (\theta \wedge \nabla^T_u \theta^c + \theta^c \wedge \nabla^T_v \theta^c) \alpha = 0, \\
\delta_{0,-1} \alpha &= (-\iota_u \nabla^T_u \theta^c - \iota_v \nabla^T_v \theta^c) \alpha = 0.
\end{align*}

**The proof for $u \geq 1$.** Write now
\begin{align*}
d_{0,1} \alpha &= \tilde{\eta}_1 \wedge V(f_1) \theta^c \wedge \theta + \tilde{\eta}_2 \wedge U(f_2) \theta \wedge \theta^c \\
&= (U(f_2) \tilde{\eta}_2 - V(f_1) \tilde{\eta}_1) \wedge \theta \wedge \theta^c,
\end{align*}
and
\[\eta_{0,-1} \alpha = -U(f_1) \tilde{\eta}_1 - V(f_2) \tilde{\eta}_2.\]

We show that the differential forms $f_i \tilde{\eta}_i$ are basic. As above, we fix a leaf $L$ and we suppose $\tilde{\eta}_i|_L \neq 0$. As $\tilde{\eta}_i \in \Omega^{u,0}_b$, from (4.2) we have
\[\tilde{\eta}_2 = a \tilde{\eta}_1|_L,
\]
with $a \in \mathbb{R}$, and we obtain (4.3) for $f_1|_L, a f_2|_L$. Then $f_i$ are basic, as well as $\eta_i$.

**Step 2:** $\eta_i$ are harmonic with respect to the basic Laplacian. Let $\beta \in \Omega^{u+1,0}$. We study the convergence
\[d_{1,0} \alpha + d_{0,1} \beta_i \longrightarrow 0
\]
in $L^2$ around a regular point $x \in M$. As the subset of regular points is open and dense (see e.g. [Ma] Chapter 3), taking a transversal $T$ small enough we can consider a foliated map $(\mathcal{U}, \varphi)$, $x \in \mathcal{U}$, such that $\mathcal{U} \simeq T \times T^2$, all leaves in $\mathcal{U}$ being regular, diffeomorphic to $T^2$.

We introduce transverse coordinates $y = (y^1, \ldots, y^{2n})$ and leafwise coordinate $(t, s)$, such that $\varphi(\mathcal{U}) = \mathcal{O} \times (0, 2\pi) \times (0, 2\pi)$, with $\mathcal{O} \subseteq \mathbb{R}^{2n}$ and such that
\begin{align}
U &= c_1 \frac{\partial}{\partial t}, \\
V &= c_2 \frac{\partial}{\partial s},
\end{align}
with $c_1, c_2$ real constants.

As the leaves have trivial holonomy and the metric is bundle-like, sliding along leaves we can construct on $\mathcal{U}$ a transverse orthonormal basis $\{e_i\}_{1 \leq i \leq q}$ such that $\{e_i^\perp\} \in \Omega^1_b$, with $\{e_i^\perp\}$ basic forms. As above, for local computation we consider the dual basis $\{e^1_1, \ldots, e^1_{2n}, \theta, \theta^c\}$.
Locally, we can write
\[ \beta = g^{i_1 \ldots i_{u+1}} e_{i_1}^b \wedge \ldots \wedge e_{i_{u+1}}^b = g_I e_I^b, \]
for \( 1 \leq i_1, \ldots, i_{u+1} \leq q \), and the multi-index \( I := (i_1, \ldots, i_{u+1}) \), with \( e_I^b := e_{i_1}^b \wedge \ldots \wedge e_{i_{u+1}}^b \).

Using (4.4), we obtain
\[
d_{0,1} \beta = c_1 \frac{\partial g_I}{\partial t} \theta \wedge e_I^b + c_2 \frac{\partial g_I}{\partial s} \theta^c \wedge e_I^b \]
\[
= \frac{\partial g_I}{\partial t} e_I^b \wedge \theta + \frac{\partial g_I}{\partial s} e_I^b \wedge \theta^c, \tag{4.5} \]
with \( g_I^i := (-1)^{u+1} c_i \cdot g_I^i, \ i = 1, 2. \)

From (4.1) and (2.3) we get
\[
d_{1,0} \alpha = h_I^1 e_I^b \wedge \theta + h_I^2 e_I^b \wedge \theta^c, \tag{4.6} \]
As \( d_{1,0} \delta_i, e_I^b \) are basic differential forms, \( h_I^1 \) are basic functions, \( i = 1, 2. \)

In the sequel we check the \( L^2 \) convergence stated in Lemma 2.14 (ii) on the local chart \( \mathcal{U} \). From (4.5) and (4.6), we have
\[
\mathcal{I} := \int_{\mathcal{U}} \|d_{1,0} \alpha + d_{0,1} \beta\|^2 d\mu_g \\
= \sum_I \int_{\varphi(\mathcal{U})} \left( h_I^1 + \frac{\partial g_I^1}{\partial t} \right)^2 + \left( h_I^2 + \frac{\partial g_I^2}{\partial s} \right)^2 \sqrt{\det G} \ ds \ dt \ dy, \]
where \( G \) is the matrix of the metric \( g \) with respect to the local chart, \( d\mu_g \)
is the volume form canonically associated to the metric \( g \), \( dy \) corresponding
to the transverse coordinates.

We fix now \( I \). Clearly the Riemannian metric \( g \) is non-degenerate and \( \mathcal{U} \)
can be chosen relatively compact. Then we can find a constant \( c \) such that
\[
\sqrt{\det G} > c > 0
\]
on \( \mathcal{U} \). Then, as \( h_I^1 \) is basic (so it does not depend on \( t \) and \( s \)), using the Fubini formula, we have
\[
\int_{\varphi(\mathcal{U})} \left( h_I^1 + \frac{\partial g_I^1}{\partial t} \right)^2 \sqrt{\det G} \ ds \ dt \ dy > c4\pi^2 \int_0^{2\pi} (h_I^1)^2 \ dy \\
+ 2c \int_0^{2\pi} \int_0^t \left( \frac{\partial g_I^1}{\partial t} \right) ds \ dy \\
+ c \int_{\varphi(\mathcal{U})} \left( \frac{\partial g_I^1}{\partial t} \right)^2 \ ds \ dt \ dy.
\]
Because on the torus $g_1^I(0) = g_1^I(2\pi)$, the second term vanishes and the third is positive, then arguing in a similar manner, for arbitrary $I$, we can write

$$\|d_{1,0} \alpha + d_{0,1} \beta\|_{L^2}^2 \geq \mathcal{I} > 4\pi^2 c \sum_I ((h_1^I)^2 + (h_2^I)^2)dy.$$ 

As the last expression depends only on $\alpha$ and on $\mathcal{U}$, we can have a sequence $\beta_i \in \Omega^{u+1,0}$ such that

$$d_{1,0} \alpha + d_{0,1} \beta_i \rightarrow 0 \text{ in } L^2$$

if and only if $h_i^I = 0$ on $\mathcal{U}$, for any $I$ and $1 \leq i \leq 2$. All mathematical objects that we use are of type $C^\infty$, $x$ is a regular point arbitrarily chosen and the set of regular points is dense. Consequently we obtain the desired relation

(4.7) \hspace{1cm} d_{1,0} \delta_i = 0

on $M$.

We still have to prove $\delta_{-1,0} \delta_i = 0$. We proceed in a similar way as above. Notice that

$$\delta_{-1,0} \alpha = \delta_{-1,0} \eta_1 \wedge \theta + \delta_{-1,0} \eta_2 \wedge \theta^c$$

$$= h_1^I e_I^\beta \wedge \theta + h_2^I e_I^\beta \wedge \theta^c,$$

with $I := (i_1, \ldots, i_{u-1})$, $1 \leq i_1, \ldots, i_{u-1} \leq 2n$ and $h_i^I$ being basic functions.

We consider $\gamma \in \Omega^{u-1,2}$; on $\mathcal{U}$ we write

$$\gamma = g^I e_I^\beta \wedge \theta \wedge \theta^c.$$ 

Then

$$\delta_{0,-1} \gamma = (-1)^{u-1} (c_1 U (g^I) e_I^\beta \wedge \theta^c - c_2 V (g^I) e_I^\beta \wedge \theta))$$

$$= \frac{\partial g_1^I}{\partial s} e_I^\beta \wedge \theta + \frac{\partial g_2^I}{\partial t} e_I^\beta \wedge \theta^c.$$

and, consequently

$$\mathcal{J} := \int_{\mathcal{U}} \|\delta_{-1,0} \alpha + \delta_{0,-1} \gamma\|^2 d\mu_g$$

$$= \sum_I \int_{\varphi(\mathcal{U})} \left( h_1^I + \frac{\partial g_1^I}{\partial s} \right)^2 + \left( h_2^I + \frac{\partial g_2^I}{\partial t} \right)^2 \sqrt{G} ds dt dy.$$ 

As before, we get

$$\|\delta_{-1,0} \alpha + \delta_{0,-1} \gamma\|_{L^2}^2 \geq \mathcal{J} > 0,$$

and we obtain $h_i^I = 0$ on $\mathcal{U}$ for any $I$. From here, arguing as above,

(4.8) \hspace{1cm} \delta_{-1,0} \delta_i = 0.
From (4.7) and (4.8) it results that $\delta_1$, $\delta_2$ are basic harmonic forms with respect to the basic operator $\Delta_b$. Using (2.8)
\[ \dim E^{u,1}_2 = \dim H^{u,1}_2 = 2 e_u, \]
and the theorem is proved. ■

5. Examples

In this final section we present two foliated manifolds. The first one is the canonical foliation of a diagonal Hopf surface, for which we apply Theorem 3.2 and Theorem 4.1 to explicitly compute the dimension of all spectral terms of order 2. The second example is a class of foliations of arbitrary large transverse dimension which does not come from a Vaisman structure, in accordance with Proposition 3.5.

5.1. The diagonal Hopf surface. Let $W := C^2 \setminus \{0\}$ and $a, b \in C \setminus \{0\}$ and let $\gamma : W \to W$ be given by
\[ \gamma : (z_1, z_2) \mapsto \left( az_1, bz_2 \right). \]
The quotient
\[ H_{a,b} := W/\gamma, \]
is a Hopf surface. \cite{Gau-O} \cite{P}. For any $(z_1, z_2) \in W$ there is an unique real number $\varphi$ such that
\[ |z_1|^2 |a|^{-2\varphi} + |z_2|^2 |b|^{-2\varphi} = 1. \]
The map $\tilde{\psi}$ defined by
\[ \tilde{\psi} : (z_1, z_2) \mapsto (\varphi \mod Z, z_1 a^{-\varphi}, z_2 b^{-\varphi}) \]
is compatible with $\gamma$ and the quotient map $\psi := \tilde{\psi}/\gamma$ establishes a homeomorphism between $H_{a,b}$ and the product of spheres $S^1 \times S^3$, \cite{Gau-O}. The Hermitian form
\[ \Omega := -\sqrt{-1} \frac{1}{4} \frac{\partial \bar{\partial} \Phi}{\Phi}, \quad \text{with} \quad \log \Phi := (\log |a| + \log |b|)\varphi, \]
produces a Vaisman metric $g$ on $H_{a,b}$, with $g(\cdot, \cdot) := -\Omega(J\cdot, \cdot)$. The topological properties of the leaves of the canonical foliation are investigated in \cite{P}, where the author proves that all leaves are compact (and consequently the foliation is quasi-regular) if and only if there are positive integers $c, d$ such that $a^c = b^d$.

We now compute the dimensions of the terms of order 2 of the spectral sequence. The Betti numbers of the Hopf surface are
\[ b_0 = 1, \quad b_1 = 1, \quad b_2 = 0, \quad b_3 = 1, \quad b_4 = 1. \]
From \cite{V}, the basic Betti numbers are:
\[ e_0 = 1, \quad e_1 = 0, \quad e_2 = 1. \]
By our lower estimate of \( \dim E_{2}^{u,1} \) and the Poincaré duality, we obtain:

**Proposition 5.1:** The dimension of the terms of the second order of the canonical Vaisman foliation associated to \( H_{\alpha,\beta} \) satisfy:

\[
\begin{align*}
\dim E_{2}^{u,0} &= \dim E_{2}^{u,2} = \frac{1 + (-1)^{u}}{2}, \\
\dim E_{2}^{u,1} &\geq 1 + (-1)^{u}.
\end{align*}
\]

(5.1)

**Remark 5.2:** If \( a^{c} = b^{d} \) for \( c, d \in \mathbb{N} \setminus \{0\} \), then the foliation is quasi-regular and we get equality in the second line of (5.1).

5.2. A suspension of an odd-dimensional torus. We use an idea presented in [E-N2] to construct foliations of arbitrary large transverse dimension.

Consider the matrix

\[
A := \begin{pmatrix}
d_{1} & 1 & 1 & \cdots & 1 \\
1 & d_{2} & 0 & \cdots & 0 \\
1 & 0 & d_{3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & d_{2n+1}
\end{pmatrix}
\]

where \( d_{1} = 1 \) and \( d_{k} = 1 + d_{1} \cdot d_{2} \cdot \cdots \cdot d_{k-1} \), for \( 2 \geq k \geq 2n + 1 \). Then:

1. \( A \in \text{SL}(2n+1, \mathbb{Z}) \).
2. All its eigenvalues \( \lambda_{i} \) have multiplicity 1 and are uniformly distributed in the intervals \((d_{1}, d_{3}), (d_{3}, d_{4}), \ldots, (d_{2n+1}, \infty)\).
3. The coordinates \( \{v_{1}^{i}, \ldots, v_{2n+1}^{i}\} \) of each eigenvector \( v_{i} \) are linearly independent over the field \( \mathbb{Q} \).

As in [Ca] for dimension 2 and [Do] for dimension 3, we take the semidirect product \( G := \mathbb{R} \times \mathbb{R}^{2n+1} \) and use \( A \) to induce a Lie group structure on \( G \) by the multiplication

\[
(t_{1}, z_{1}) \cdot (t_{2}, z_{2}) := (t_{1} + t_{2}, A t_{1} z_{2} + z_{1}).
\]

The eigenvectors \( v_{i} \) become tangent vectors at the origin of this Lie group, and we denote by \( \{V_{i}\}_{1 \leq i \leq 2n+1} \) the left invariant vector fields induced by \( \{v_{i}\} \). Consider the frame \( \{V_{0} := \partial/\partial t, V_{i}\} \). The corresponding coframe determined by the canonical left invariant metric is denoted by \( \{\alpha_{0} := dt, \alpha_{i}\}, \)

\( 1 \leq i \leq 2n + 1 \).

As \( A \in \text{SL}(2n+1, \mathbb{Z}) \), the subgroup \( \Gamma := \mathbb{Z} \times \mathbb{Z}^{2n+1} \) is cocompact. The quotient manifold \( T_{A}^{2n+2} := \Gamma \backslash G \) is a suspension of the torus \( T_{A}^{2n+1} := \mathbb{R}^{2n+1} / \mathbb{Z}^{2n+1} \) with respect to the automorphism canonically induced on the torus by the matrix \( A \). [E-N1, Ca]. For the construction of a suspension we indicate [Mo, Chapter I].

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Notice that the metric, the left invariant frame and coframe can be projected on $T_{2n+2}^A$; for convenience we use the same symbols to denote the projected objects.

We construct a foliation which is different from the classical foliation associated to a suspension. Precisely, the fields $V_{2n}, V_{2n+1}$ generate two flows which induce a foliation of leafwise dimension 2 and transverse dimension $2n$ on $T_{2n+2}^A$. As in [Ca, Do], the above metric is bundle-like.

**Proposition 5.3:** For the above foliation the spectral terms $E_{2}^{2n}$ and $E_{2}^{0,1}$ vanish,

$$E_{2}^{2n} = 0, \quad E_{2}^{0,1} = 0.$$

**Proof.** To compute $E_{2}^{2n}$ and $E_{2}^{0,1}$ we use an argument similar to the one in [Do] (see also [Ca]).

We first look at $E_{1}^{u,v}$, $0 \leq u \leq 2n$, $0 \leq v \leq 2$. As $d_{0,1} \alpha_{i} = 0$, these terms decompose as

$$E_{1}^{u,v} = \bigwedge^{u}(\alpha_{0}, \ldots, \alpha_{2n-1}) \otimes E_{1}^{0,v}.$$  

As $v_{i}^{j}$ are independent over $\mathbb{Q}$, there exist some constants $C_{i}$ and $\delta_{i}$ such that the following Diophantine condition is satisfied [S, II, 4.1].

$$|\langle m, v_{i} \rangle| \geq \frac{C_{i}}{||m||^{\delta_{i}}},$$

for any $m \in \mathbb{Z}^{2n+1}\setminus\{0\}$ and any $1 \leq i \leq 2n+1$. Then, for any function

$$h : \mathbb{R} \times \mathbb{R}^{2n+1} \to \mathbb{R} \quad \text{with} \quad h(t, z) = h(t + 1, A(z))$$

there exists a smooth function $f$ with the same properties such that

$$V_{i}(f) = h - h_{0},$$

with $h_{0} \equiv h_{0}(t) = h_{0}(t + 1)$. As in [Do], using (5.2) we obtain

$$E_{1}^{u,v} = \bigwedge^{u}(\alpha_{0}, \ldots, \alpha_{2n-1}) \otimes \bigwedge^{v}(\alpha_{2n}, \alpha_{2n+1}) \otimes \Omega^{0}(S^{1}).$$

We use $E_{2} = H(E_{1}, d_{1})$ to compute the dimension of $E_{2}^{2n,0}$. Let $\alpha \in E_{1}^{2n,0}$. From (5.3) we can write

$$\alpha = h(t) \alpha_{0} \land \cdots \land \alpha_{2n-1},$$

with $h(t) = h(t + 1)$. As in [Ca, Proposition 2], we construct a differential form $\alpha' \in E_{1}^{2n-1,0}$ such that $d_{1} \alpha' = \alpha$, $d_{1}$ being the basic de Rham operator. We choose

$$\alpha' = f(t) \alpha_{1} \land \cdots \land \alpha_{2n-1}.$$  

Then we compute:

$$d_{1} \alpha' = (f'(t) - (\log \lambda_{1} + \cdots + \log \lambda_{2n-1})f(t))\alpha_{0} \land \cdots \land \alpha_{2n-1}$$

$$= (f'(t) + f(\lambda f(t)))\alpha_{0} \land \cdots \land \alpha_{2n-1},$$
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where \( \lambda := \lambda_{2n} \cdot \lambda_{2n+1} \). By the distribution of the eigenvalues of \( A \) on the real axis, \( \lambda \neq 1 \), and the differential equation

\[
 f'(t) + \log \lambda f(t) = h(t)
\]

has the solution

\[
 f(t) = \lambda^{-t}(k + \int_0^t \lambda^x h(t)dx).
\]

Choosing

\[
 k = \frac{1}{\lambda - 1} \int_0^1 \lambda^x h(t)dx,
\]

leads to \( f(t) = f(t+1) \). Then \( E^{2n,0}_2 \equiv 0 \), and the foliation is taut.

Finally, we determine the kernel of the operator \( d_1 : E^{0,1}_2 \to E^{1,1}_2 \). If \( \alpha \in E^{0,1}_1 \), then

\[
 \alpha = f_1 \alpha_{2n} + f_2 \alpha_{2n+1},
\]

and \( f_i = f_i(t) = f_i(t+1) \). As above,

\[
 d_1 \alpha = (f'_1(t) - \log \lambda_{2n} f_1(t))\alpha_{2n} + (f'_2(t) - \log \lambda_{2n+1} f_2(t))\alpha_{2n+1}.
\]

From \( f'_i(t) - \log \lambda_{2n-1+i} f_i(t) = 0 \), as the functions are periodic, we get \( f_i \equiv 0 \), \( 1 \leq i \leq 2 \). Then \( E^{0,1}_2 \equiv 0 \). \( \blacksquare \)

Using Proposition 3.5 we obtain:

Corollary 5.4: The above foliation on \( T^{2n+2}_A \) does not come from a Vaisman structure.

Remark 5.5: Any two vector fields from the set \( V_1, \ldots, V_{2n+1} \) can be used to construct the foliation. However, if \( \frac{\partial}{\partial t}, J\left(\frac{\partial}{\partial t}\right) \in TF \), where \( J \) is the complex structure, then the existence of a Vaisman structure on the above suspension of the torus would imply the existence of a Sasaki structure on the odd dimensional torus which is the fibre of the suspension. But it is known that odd-dimensional tori do not admit Sasaki structures (in fact, not even K-contact structures, see [Bo-Gal2], [I]). This means that our obstruction becomes important only when \( \frac{\partial}{\partial t}, J\left(\frac{\partial}{\partial t}\right) \) are transverse to the foliation.

Remark 5.6: On the other hand \( T^{2n+2}_A \) is orientable, and the differential one-form \( \omega_0 \equiv dt \) is closed but not exact, so \( b_1 > 0 \). Then, the existence of a Vaisman structure on our manifold cannot be precluded using this Betti number (see the Introduction). Consequently, in some instances, the terms of the spectral sequence may be a useful obstruction in deciding whether a 2 dimensional foliated structure on a closed manifold is generated by a Vaisman structure or not.

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