The number of non-crossing perfect plane matchings
is minimized (almost) only by point sets in convex position

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Abstract

It is well-known that the number of non-crossing perfect matchings of $2k$ points in convex position in the plane is $C_k$, the $k$th Catalan number. García, Noy, and Tejel proved in 2000 that for any set of $2k$ points in general position, the number of such matchings is at least $C_k$. We show that the equality holds only for sets of points in convex position, and for one exceptional configuration of 6 points.

Introduction, notation, result

Let $S$ be a set of $n = 2k$ points in general position (no three points lie on the same line) in the plane. Under a perfect matching of $S$ we understand a geometric perfect matching of the points of $S$ realized by $k$ non-crossing segments. The number of perfect matchings of $S$ will be denoted by $\text{pm}(S)$.

In general, $\text{pm}(S)$ depends on (the order type of) $S$. Only for very special configurations an exact formula is known. The well-known case is that of points in convex position:

**Theorem 1** (Classic/Folklore/Everybody knows). If $S$ is a set of $2k$ points in convex position, then $\text{pm}(S) = C_k = \frac{1}{k+1} \binom{2k}{k}$, the $k$th Catalan number.

There are several results concerning the maximum and minimum possible values of $\text{pm}(S)$ over all sets of size $n$. For the maximum possible value of $\text{pm}(S)$, only asymptotic bounds are known. The best upper bound up to date is due to Sharir and Welzl [3] who proved that for any $S$ of size $n$, we have $\text{pm}(S) = O(10.05^n)$. For the lower bound, García, Noy, and Tejel [2] constructed a family of examples which implies the bound of $\Omega(3^n n^{O(1)})$; it was recently improved by Asinowski and Rote [1] to $\Omega(3.09^n)$.

As for the minimum possible value of $\text{pm}(S)$ for sets of size $n = 2k$, García, Noy, and Tejel [2] showed that it is attained by sets in convex position, and thus, by Theorem 1 it is $C_k = \Omega(2^n / n^{3/2})$:

**Theorem 2** (García, Noy, and Tejel, 2000 [2]). For any set $S$ of $n = 2k$ points in general position in the plane, we have $\text{pm}(S) \geq C_k$.

However, to the best of our knowledge, the question of whether only sets in convex position have exactly $C_k$ perfect matchings, was never studied. In this note we show that this is almost the case: there exists a unique (up to order type) exception shown in Figure 1:

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Figure 1: A set of six points in non-convex position that has five perfect matchings.

**Theorem 3.** Let $S$ be a planar set of $2k$ points in general position. We have $\text{pm}(S) = C_k$ only if $S$ is in convex position, or if $k = 3$ and $S$ is a set with the order type as in Figure 1.

We recall the recursive definition of Catalan numbers: $C_0 = 1$; and for $k \geq 1$,

$$C_k = \sum_{i=0}^{k-1} C_i C_{k-1-i}.$$

For two distinct points $A$ and $B$, the straight line through $A$ and $B$ will be denoted by $\ell(AB)$. We say that segment $AB$ pierces segment $CD$ if the segments do not cross, but the line $\ell(AB)$ crosses $CD$.

**Discussion**

First we recall, for the sake of completeness, the proof of Theorem 2 by García, Noy, and Tejel.

**Proof.** For $k = 0$ and $k = 1$ the claim is trivial/clear.

Let $k \geq 2$. Refer to Figure 2. Let $A_1$ be any point of $S$ that lies on the boundary of $\text{conv}(S)$. Label other points of $S$ by $A_2, A_3, \ldots, A_n$ according to the clockwise polar order with respect to $A_1$ (so that $A_2$ is the immediate successor and $A_n$ is the immediate predecessor of $A_1$ on the boundary of $\text{conv}(S)$). For $i = 0, 1, \ldots, k - 1$ we bound the number of perfect matchings in which $A_1$ is connected to $A_{2i+2}$, as follows. The line $\ell(A_1A_{2i+2})$ splits $S \setminus \{A_1, A_{2i+2}\}$ into two subsets of sizes $2i$ and $n - 2 - 2i$. Therefore, if we start constructing a perfect matching by choosing the segment $A_1A_{2i+2}$ to be its member, we can complete its construction by choosing arbitrary perfect matchings of these subsets. By induction, the numbers of inner perfect matchings of these subsets are (respectively) at least $C_i$ and at least $C_{k-1-i}$. Thus, the number of perfect matchings of $S$ in which $A_1$ is matched to $A_{2i+2}$ is at least $C_i C_{k-1-i}$, and the total number of perfect matchings of $S$ is at least

$$\sum_{i=0}^{k-1} C_i C_{k-1-i} = C_k,$$

as claimed.

For sets of points in convex position this argument essentially proves Theorem 1 as in this case it counts all perfect matchings. However, for general point sets it is quite rough, since it does
not count (1) all the perfect matchings in which $A_1$ is matched to $A_j$ with odd $j$ (for example, in Figure 2 we miss perfect matchings that contain the edge $A_1A_5$), and (2) all the perfect matchings in which $A_1$ is matched to $A_j$ with even $j$, and some edges connect pairs of points separated by the line $\ell(A_1A_j)$ (in Figure 2 we miss perfect matchings that contain the edge $A_1A_4$ and some edges that cross $\ell(A_1A_4)$). Therefore one could expect that we have the equality $\text{pm}(S) = C_k$ for a set of size $2k$ only if it is in convex position and, maybe, for a limited number of exceptional configurations. Figure 1 shows such a configuration: it has exactly 5 ($= C_3$) perfect matchings (the central point can be connected to any other point, and then a perfect matching can be completed in a unique way). Thus, in our Theorem 3 we essentially claim that this is the only exceptional configuration.

**Proof of Theorem 3**

The main tool will be the following observation.

**Observation 4.** Let $S$ be a planar set of $2k$ points in general position. Suppose that $S$ has a perfect matching $M$ in which there are two segments $AB$ and $CD$ such that one of the endpoints of $AB$ lies on the boundary of $\text{conv}(S)$, and $AB$ pierces $CD$ (see Figure 3 for an illustration). Then $\text{pm}(S) > C_k$.

**Proof.** Set $A_1 := A$ an apply the proof of Theorem 2 for this choice. It gives $\text{pm}(S) \geq C_k$, but, as we explained after the proof of Theorem 2, at least the perfect matching $M$ is not counted. Therefore we have $\text{pm}(S) > C_k$. \hfill \Box

The property of perfect matchings as in the assumption of Observation 4 will be called the **piercing property**. We shall show that any set of $2k$ points in general position, except the sets in convex position and those with order type as in the example from Figure 1 has a perfect matching with the piercing property. In Propositions 5 and 6 we prove this under assumption that the interior of $\text{conv}(S)$ contains exactly one or, respectively, several points of $S$. 

Figure 2: Illustration to the proof (by García, Noy and Tejel [2]) of Theorem 2.
Figure 3: The piercing property.

**Proposition 5.** Let $S$ be a set of $n = 2k$ points in general position such that the interior of $\text{conv}(S)$ contains exactly one point of $S$. Then $S$ has a perfect matching with the piercing property, unless $S$ has the order type as in Figure 1.

**Proof.** Let $Q \in S$ be the point that lies in the interior of $\text{conv}(S)$. Label all other points of $S$ by $A_1, A_2, \ldots, A_{n-1}$ according to the clockwise cyclic order in which they appear on the boundary of $\text{conv}(S)$. We apply the standard rotating argument on directed lines that pass through $Q$: we choose one such line and rotate it around $Q$, keeping track of the difference $\delta$ between the number of points of $S$ to the right of the line and the number of points of $S$ to its left. We observe that when the line makes half a turn, $\delta$ changes the sign; that $\delta$ can only change by $\pm 1$ when the line meets or leaves one of the points of $S$; and that $\delta$ is even if and only if the line contains one of the points of $S \setminus \{Q\}$. It follows that for some $j \in \{1, 2, \ldots, n-1\}$ we have $\delta = 0$ for the line $\ell(QA_j)$. Thus $\ell(QA_j)$ halves $S$: there are $k - 1$ points of $S$ in each open half-plane bounded by this line.

**Case 1:** $k$ is even. Refer to Figure 4(a). We assume without loss of generality that $\ell(QA_1)$ halves $S$. Consider the perfect matching $\{A_1Q, A_2A_3, A_4A_5, \ldots, A_{n-2}A_{n-1}\}$. In this matching, $A_1Q$ pierces $A_{k-1}A_{k+1}$. Thus, this matching has the piercing property.

**Case 2:** $k$ is odd. Here we have two subcases.

**Subcase 2a:** The line $\ell(QA_j)$ halves $S$ not for all $j \in \{1, 2, \ldots, n-1\}$. Refer to Figure 4(b). We apply the rotating argument again and conclude that for some $j_0 \in \{1, 2, \ldots, n-1\}$ we have $\delta = \pm 2$ for the line $\ell(QA_{j_0})$ (the two signs corresponding to different ways to orient the line). In other words, the open half-planes bounded by $\ell(QA_{j_0})$ contain $k - 2$ and $k$ points of $S$. We assume without loss of generality that $j_0 = 1$, and that the open half-planes bounded by $\ell(QA_1)$ contain respectively the following sets of points: $\{A_2, A_3, \ldots, A_{k-1}\}$ and $\{A_k, A_{k+1}, \ldots, A_{n-1}\}$. Consider the perfect matching $\{A_1Q, A_2A_3, A_4A_5, \ldots, A_{n-2}A_{n-1}\}$. In this matching, $A_1Q$ pierces $A_{k-1}A_k$. Thus, it has the piercing property.

**Subcase 2b:** The line $\ell(QA_j)$ halves $S$ for all $j \in \{1, 2, \ldots, n-1\}$. Refer to Figure 4(c). Assume $k \geq 5$. The segment $A_{k-1}A_{k+2}$ does not cross the segment $A_1Q$: indeed, the line $\ell(QA_{k-2})$ halves $S$ and thus crosses the segment $A_{n-3}A_{n-2}$; thus the points $A_{k-1}, A_{k+2}$ on one hand and the point $A_1$ on the other hand lie in different open half-planes bounded by $\ell(QA_{k-2})$. Consider the
perfect matching

\{A_1Q, A_{k-1}A_{k+2}, A_kA_{k+1}, A_iA_{i+1} : i \in \{2, 4, \ldots, k-3, k+3, k+5, \ldots, n-4, n-2\}\}

In this matching, \(A_1Q\) pierces \(A_{k-1}A_{k+2}\) (and \(A_kA_{k+1}\)). Thus, it has the piercing property.

For \(k = 3\) the argument above ("the segment \(A_{k-1}A_{k+2}\) does not cross the segment \(A_1Q\") does not apply. It is easy to verify by case distinction that only for the order type from Figure 1 there is no perfect matching with the piercing property.

\(\square\)

\begin{figure}
\centering
\includegraphics{figure4}
\caption{Illustration of the proof of Proposition 5.}
\end{figure}

**Proposition 6.** Let \(S\) be a set of \(n = 2k\) points in general position such that the interior of \(\text{conv}(S)\) contains several points of \(S\). Then \(S\) has a matching with the piercing property.

**Proof.** Label the points of \(S\) that lie on the boundary of \(\text{conv}(S)\) by \(A_1, A_2, \ldots, A_\ell\) according to the clockwise cyclic order in which they appear on the boundary.

Let \(Q\) and \(R\) be two points of \(S\) that lie in the interior of \(\text{conv}(S)\). Assume without loss of generality that \(\ell(QR)\) crosses the segment \(A_1A_2\) so that \(R\) lies on this line between \(Q\) and \(\ell(QR) \cap A_1A_2\).

Consider the triangle \(A_1A_2Q\). The point \(R\) lies in its interior. Let \(R'\) be the point of \(S\) in the interior of triangle \(A_1A_2Q\) for which the angle \(\angle A_1A_2R'\) is the smallest. Denote \(T = \ell(A_1Q) \cap \ell(A_2R')\). Due to the choice of \(R'\), the interior of triangle \(A_1A_2T\) does not contain any point of \(S\).

The line \(\ell(A_1Q)\) crosses the boundary of \(\text{conv}(S)\) twice: in point \(A_1\), and in a point \(U\) that belongs to the interior of some segment \(A_jA_{j+1}\). Let \(S_1\) and \(S_2\) be the sets of points of \(S\) that lie respectively in the open half-planes bounded by \(\ell(A_1Q)\) (so that \(S_1\) contains \(A_2\) and \(A_j\), and \(S_2\) contains \(A_{j+1}\) and \(A_\ell\)). Figure 5 illustrates the introduced notation (the black points belong to \(S\), and the white points are reference points that do not belong to \(S\)).

We start constructing a perfect matching \(M\) by taking the segment \(A_1Q\).

If \(|S_1|\) and \(|S_2|\) are odd, we take the segment \(A_jA_{j+1}\) to be a member of \(M\), and then complete constructing \(M\) by taking arbitrary perfect matchings of \(S_1 \setminus \{A_j\}\) and of \(S_2 \setminus \{A_{j+1}\}\). The obtained perfect matching \(M\) has the piercing property since \(A_1Q\) pierces \(A_jA_{j+1}\).
If $|S_1|$ and $|S_2|$ are even, we complete constructing $M$ by taking an arbitrary perfect matching of $S_2$ and some perfect matching of $S_1$ that contains $A_2R'$ (this is possible since the interior of triangle $A_1A_2T$ does not contain points of $S$). The obtained perfect matching $M$ has the piercing property since $A_2R'$ pierces $A_1Q$.

Notice that our proof applies as well for the special case when $A_2 = A_j$ (and no other coincidence among the points $A_1, A_2, A_j, A_{j+1}$ is possible).

Now we can complete the proof of Theorem 3. In Propositions 5 and 6 we showed that any set of $2k$ points in general position has a matching with the piercing property, unless it is in convex position or has the order type as in the example from Figure 1. From Observation 4 we know that if $S$ has a matching with the piercing property, then $\text{pm}(S) > C_k$.

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