Strong and Electromagnetic Decays of The $D$-wave Heavy Mesons

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We calculate the $\pi$, $\rho$, $\omega$, and $\gamma$ coupling constants between the heavy meson doublets $(1^-, 2^-)$ and $(0^-, 1^-)/(0^+, 1^+)$ within the framework of the light-cone QCD sum rule at the leading order of heavy quark effective theory. Most of the sum rules are stable with the variations of the Borel parameter and the continuum threshold. Then we calculate the strong and electromagnetic decay widths of the $(1^-, 2^-)$ $D$-wave heavy mesons. Their total widths are around several tens of MeV, which is helpful in the future experimental search.

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I. INTRODUCTION

Heavy quark effective theory (HQET) [1] is a framework which is widely used to study the spectra and transition amplitudes of heavy hadrons containing one heavy quark. In HQET, the expansion is performed in terms of $1/m_Q$, where $m_Q$ is the mass of the heavy quark involved. At the leading order of $1/m_Q$, the HQET Lagrangian respects the heavy quark flavor-spin symmetry, therefore heavy hadrons form a series of degenerate doublets. The two members in a doublet share the same quantum number $j$, the angular momentum of the light components. The $j_l = \frac{1}{2}$ $S$-wave doublet $(0^-, 1^-)$ is conventionally denoted as $H$ and the $j_l = \frac{1}{2}/\frac{3}{2}$ $P$-wave doublets $(0^+, 1^+)/(1^+, 2^+)$ are conventionally denoted as $S/T$. We denote the $j_l = \frac{3}{2}/\frac{5}{2}$ $D$-wave doublets $(1^-, 2^-)/(2^-, 3^-)$ as $M/N$.

Shifman-Vainshtein-Zakharov (SVZ) sum rules [2] is a nonperturbative approach used to determine hadronic parameters such as the hadron mass. The vacuum expectation value of the $T$ product of two interpolating currents is considered in this approach. After performing the operator product expansion (OPE), one obtains sum rules which relate the hadronic parameters to expressions containing vacuum condensates parameterizing the QCD nonperturbative effect. In the late 1980s, light-cone QCD sum rules (LCQSR) [3] was developed to calculate various hadronic transition form factors. Now the OPE of the $T$ product of two interpolating currents sandwiched between the vacuum and an hadronic state is performed near the light-cone rather than at a small distance as in the conventional SVZ sum rules.

The $\rho$ coupling constants $g_{B^*B\rho}$ and $g_{D^*D\rho}$ were calculated with LCQSR in full QCD in Ref. [4]. The couplings $g_{H+H^*\rho}$, $\tilde{f}_{H+H^*\rho}$, $g_{H\rho}$, and $\tilde{f}_{H\rho}$ were calculated in full QCD in Ref. [5]. Their values in the limit $m_Q \rightarrow \infty$ are also discussed in this paper. The $\rho$ coupling constants between the three doublets $H$, $S$, $T$ and within the two doublets $H$, $S$ are systematically studied with LCQSR at the leading order of HQET in Ref. [6]. The $\pi$ coupling constants between the $S$-wave and $P$-wave heavy mesons have been studied using QCD sum rules or/and LCQSR in Ref. [7]. The $\pi$ coupling constants between $M/N$ and $H/S/T$ are calculated with LCQSR at the leading order of HQET in Ref. [8, 9]. The radiative decay between $H$, $S$, and $T$ are studied using the light-cone QCD sum rule at the leading order of HQET in Ref. [10]. In Ref. [11], the radiative decays of $D_{sJ}(2317)$ and $D_{sJ}(2460)$ are studied using LCQSR approach.

In this work, we use LCQSR to calculate the $\pi$, $\rho$, $\omega$, and $\gamma$ coupling constants between the doublets $M$ and $H/S$. Because of the covariant derivative in the interpolating currents of the $M$ doublet, the contribution from the 3-particle light-cone distribution amplitudes of $\pi$, $\rho$, $\omega$, and $\gamma$ have to be included. We work in HQET to differentiate the two states with the same $J^P$ value and yet quite different decay widths. The interpolating currents $j_{l,J,P,J_l}$ adopted in

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our work have been properly constructed in Ref. \[12\]. They satisfy
\[
\langle 0|J_{\beta j}^{\alpha_1 \cdots \alpha_j}(0)|\bar{j}', P', j'\rangle = f_{P j', j'} \delta_{j j'} \delta_{P P'} \delta_{\beta j} \eta^{\alpha_1 \cdots \alpha_j},
\]
\[
i(0)T\{J_{\beta j}^{\alpha_1 \cdots \alpha_j}(x)J_{\beta_2 j}^{\alpha_1 \cdots \alpha_j}(0)\}|0\rangle = \delta_{j j'} \delta_{P P'} \delta_{\beta j} (-1)^j S \eta^{\alpha_1 \beta_1} \cdots \eta^{\alpha_j \beta_j} \int dt \delta(x - vt) \Pi_{P j}(x),
\]
in the limit $m_Q \rightarrow \infty$. Here $\eta^{\alpha_1 \cdots \alpha_j}$ is the polarization tensor for the spin $j$ state, $v$ is the velocity of the heavy quark, $g_{l}^{\alpha \beta} = g^{\alpha \beta} - v^{\alpha} v^{\beta}$, $S$ denotes symmetrizing the indices and subtracting the trace terms separately in the sets $(\alpha_1 \cdots \alpha_j)$ and $(\beta_1 \cdots \beta_j)$.

II. SUM RULES FOR THE $\pi$ COUPLING CONSTANTS

We shall perform the calculation at the leading order of HQET. According to Ref. \[12\], the interpolating currents for the doublets $(0^-, 1^-)$, $(0^+, 1^+)$, and $(1^-, 2^-)$ read as
\[
J_{\alpha \rightarrow \mp}^1 = \sqrt{\frac{1}{2}} \bar{h}_v \gamma_5 q,
\]
\[
J_{\alpha \rightarrow \pm}^1 = \sqrt{\frac{1}{2}} \bar{h}_v \gamma_\alpha q,
\]
where $h_v$ is the heavy quark field in HQET, $\gamma_5^\alpha \equiv \gamma^\alpha - v^\alpha v^\nu$, $D_\alpha^\gamma \equiv D^\gamma - (D \cdot v) v^\nu$, $g_1^{\mu \nu} \equiv g^{\mu \nu} - v^\mu v^\nu$, and $v^\nu$ is the velocity of the heavy quark.

We consider the $\pi$ decay of $M_2$ to $H_1$ to illustrate our calculation. Here the subscript of $\mathcal{M}(H)$ indicates the spin of the meson involved. Owing to the conservation of the angular momentum of the light components in the limit $m_Q \rightarrow \infty$, there is only one independent $\pi$ coupling constant between doublets $M$ and $H$. We denote it as $g_{M H \pi}^p$ where $p$ and the number following it indicate the orbital and total angular momentum $(l, j_h)$ of the final $\pi$ meson respectively. $g_{M H \pi}^p$ can be defined in terms of the decay amplitude $\mathcal{M}(M_2 \rightarrow H_1 + \pi)$ as
\[
\mathcal{M}(M_2 \rightarrow H_1 + \pi) = I \eta_{\alpha_1 \alpha_2} [\epsilon^{\star \alpha_1} q_1^{\alpha_2} - \frac{1}{3} g_1^{\alpha_1 \alpha_2} (\epsilon^{\star} \cdot q_1)] g_{M_2 H_1 \pi}^p,
\]
where $\eta$ and $\epsilon^{\star}$ denote the polarization tensors of the initial and final heavy mesons respectively, $q$ is the momentum of the $\pi$ meson. The transversal tensor are defined as $\epsilon^{\star \mu} \equiv \epsilon^\mu - (\epsilon \cdot v) v^\mu$, $\epsilon^{\star \mu} \equiv q^\mu - (q \cdot v) v^\mu$, and $g_1^{\mu \nu} \equiv g^{\mu \nu} - v^\mu v^\nu$.

To obtain the sum rules for the coupling constants $g_{M_2 H_1 \pi}^p$, we consider the correlation function
\[
\int d^4 x e^{-i k \cdot x} \langle \pi(q) | T\{J_{\alpha_1 \cdots \alpha_j}^1(0)J_{\alpha_2 \cdots \alpha_j}^1(x)\}|0\rangle = I \left[ \frac{1}{2} (g_1^{\alpha_1 \alpha_2} q_1^{\alpha_2} + g_1^{\alpha_2 \alpha_1} q_1^{\alpha_1}) - \frac{1}{3} g_1^{\alpha_1 \alpha_2} q_1^{\beta} \right] G_{M_2 H_1 \pi}^p(\omega, \omega'),
\]
where $\omega \equiv 2v \cdot k$, $\omega' \equiv 2v \cdot (k - q)$. At the leading order of HQET, the heavy quark propagator reads as
\[
\langle 0| T\{h_v(0)\bar{h}_v(x)\}|0\rangle = \frac{1 + \hat{v}}{2} \int dt \delta^4(-x - vt).
\]
The correlation function can now be expressed as

\[-\frac{i}{4} \int dx e^{-ikx} \int_0^\infty dt \delta(-x - vt) \text{Tr} \left\{ \frac{\gamma_\mu}{2} + \frac{\gamma_5}{2} \left[ \gamma_\mu \gamma_5 D_\mu^{\alpha2} + \gamma_\mu \gamma_5 D_\mu^{\alpha1} - \frac{2}{3} g_\mu^{\alpha1\alpha2} \hat{D}_\mu \right] \langle \pi(q)|q(x)\bar{q}(0)|0 \rangle \right\}.

\tag{7}

It can be further calculated using the light-cone wave functions of the \(\pi\) meson. To our approximation, we need the 2- and 3-particle light-cone wave functions. Their definitions are collected in Appendix A.

At the hadron level, \(G_{M_2H_1\pi}^{(1)}(\omega, \omega')\) in (3) has the following pole terms

\[
G_{M_2H_1\pi}^{(1)}(\omega, \omega') = \frac{f_{-1/2} f_{-3/2} g_{M_2H_1\pi}^{(1)}}{(2\Lambda_{-1/2} - \omega')(2\Lambda_{-3/2} - \omega')} + \frac{c}{2\Lambda_{-1/2} - \omega'} + \frac{c'}{2\Lambda_{-3/2} - \omega'},
\tag{8}

where \(\Lambda_{-1/2} = m_H - m_Q, \Lambda_{-3/2} = m_M - m_Q, f_{-1/2}, \text{ etc.}\) are the overlap amplitudes of their interpolating currents with the heavy mesons.

\(G_{M_2H_1\rho}^{(1)}(\omega, \omega')\) can now be expressed by the \(\pi\) meson light-cone wave functions. After the Wick rotation and the double Borel transformation with \(\omega\) and \(\omega'\), the single-pole terms in (8) are eliminated. We arrive at

\[
g_{M_2H_1\pi}^{(1)} f_{-1/2} f_{-3/2} e^{\frac{\hat{A}_{-3/2} + \hat{A}_{-1/2}}{2}}
= \frac{1}{48} \pi \left\{ 12 \left[ \phi (\bar{u}_0) - (u\phi)(\bar{u}_0) \right] T^0 f_1 \left( \frac{\omega}{T} \right)
- \frac{4m^2}{m_u + m_d} \left[ T^0 \phi (\bar{u}_0) - 6 \phi (\bar{u}_0) + 6 (u\phi)(\bar{u}_0) \right] T f_0 \left( \frac{\omega}{T} \right)
+ 3m^2 \left[ A (\bar{u}_0) - 2 \frac{A}{3} (\bar{u}_0) + 8 \frac{A}{16} (\bar{u}_0) + 8 (u\bar{B}) \right] \left( \frac{1}{T} \right)
\right\},
\tag{9}

where \(f_n(x) = 1 - e^{-x} \sum_{k=0}^n x^k / k!\) is the continuum subtraction factor, and \(\omega_c\) is the continuum threshold, \(u_0 = T_1/T_1 + T_2, T = T_1T_2/(T_1 + T_2), \text{ and } \bar{u}_0 = 1 - u_0.\) \(T_1\) and \(T_2\) are the two Borel parameters. We have employed the Borel transformation \(B_T e^{\omega} = \delta(\alpha - 1/T)\) to obtain (9). In the above expressions, we have used the functions \(F^{(0)}(\bar{u}_0)\) and \(F^{(1,0)}(\bar{u}_0)\) which are defined in Appendix B.

The \(\pi\) coupling constant between doublets \(M\) and \(S/T/M\) can be defined similarly:

\[
\mathcal{M}(M_2 \to S_1 + \pi) = I \eta_{\alpha_1 \alpha_2} e^{\beta_1 \gamma_1} q_1 q_2 \gamma_1 \gamma_2 g_{M_2S_1\pi}^{(2)},
\mathcal{M}(M_1 \to T_1 + \pi) = I (\eta \cdot \epsilon_1) g_{M_1T_1\pi}^{(1)} + \left[ (\eta \cdot q_1)(\epsilon^* \cdot q_2) - \frac{q_1^2}{3} (\eta \cdot \epsilon_1) \right] g_{M_1T_1\pi}^{(2)},
\mathcal{M}(M_2 \to M_1 + \pi) = 2I \eta_{\alpha_1 \alpha_2} \left[ \epsilon_1^{\alpha_1} q_2 - \frac{1}{3} g_{M_2M_1\pi}^{(1)} \right] \left[ \epsilon_2^\alpha \cdot q_2 \right] g_{M_2M_1\pi}^{(1)} + I \eta_{\alpha_1 \alpha_2} \left( q_1^\alpha q_2^\beta (\epsilon^* \cdot q_2) - \frac{q_1^2}{3} (2 \epsilon_1^{\alpha_1} q_2^\alpha + g_{M_2M_1\pi}^{(1)} (\epsilon^* \cdot q_2)) \right) g_{M_2M_1\pi}^{(3)}.
\tag{10}

Here the vector notations in the Levi-Civita tensor come from an index contraction between the Levi-Civita tensor and the vectors, for example, \(e^{\alpha_1 \beta_1 \gamma_1} \equiv e^{\alpha_1 \gamma_1} q_1 v_3.\) The Levi-Civita tensor is defined as \(e^{0123} = 1.\) The sum rules for the coupling constants in Eq. (10) are

\[
g_{M_2S_1\pi}^{(2)} f_+ f_- e^{\frac{\Lambda_{-3/2} + \Lambda_{-1/2}}{2}}
= -\frac{1}{24} f_\pi \left[ 12 \left[ \phi (\bar{u}_0) - (u\phi)(\bar{u}_0) \right] T f_1 \left( \frac{\omega}{T} \right)
- \frac{4m^2}{m_u + m_d} \left[ T^0 \phi (\bar{u}_0) - 6 \phi (\bar{u}_0) + 6 (u\phi)(\bar{u}_0) \right] T f_0 \left( \frac{\omega}{T} \right)
+ 3m^2 \left[ A (\bar{u}_0) - 2 \frac{A}{3} (\bar{u}_0) + 8 \frac{A}{16} (\bar{u}_0) + 8 (u\bar{B}) \right] \left( \frac{1}{T} \right)
\right],
\]

\[
g_{M_1T_1\pi}^{(2)} f_+ f_- e^{\frac{\Lambda_{-3/2} + \Lambda_{-1/2}}{2}}
= \frac{f_\pi}{48} \left[ u(1 - u) \phi (\bar{u}_0) \right] T^4 f_3 \left( \frac{\omega}{T} \right) + \frac{f_\pi m^2}{24m_u m_d} \left[ m^2 - m_{ud}^2 \phi (\bar{u}_0) + (u(1 - u)\phi)(\bar{u}_0) + (1 - \alpha_2 T)^{3/2} (\bar{u}_0) \right] \left( \frac{1}{T} \right)
- \left( \alpha_3 T \right)^{3/2} (\bar{u}_0) \right] T^3 f_3 \left( \frac{\omega}{T} \right) - \frac{f_\pi m^2}{12} \left[ \frac{3}{8} A (\bar{u}_0) + \frac{1}{16} \right] \left( u(1 - u)\phi (\bar{u}_0) + \frac{1}{2} \bar{B} (\bar{u}_0) \right)
\right].
\[
\frac{1}{2} (u(1-u)B)' (u_0) + (u(1-u)\phi_\pi)' (u_0) + (2A_\parallel + 2A_\perp + V_\parallel + V_\perp)^{[1,0]} (u_0) + ((1-\alpha_2)(A_\parallel + V_\perp))^2_{[0]} (u_0)
\]
\[
-(\alpha_3(A_\parallel + V_\perp))^{[2,1]} (u_0) T^2 f_1 (\frac{\omega_c}{T}) - \frac{m_\pi^2}{2m_\pi} \left[ \left( u(1-u)\phi_\pi \right) (u_0) + \frac{m_\pi^2 - m^2_{ud}}{6m_\pi^2} \phi_\pi (u_0) + \left( (1-\alpha_2)T \right)^{[1,0]} (u_0) \right.
\]
\[
-(\alpha_3T)^{[1,1]} (u_0) \left. T f_0 (\frac{\omega_c}{T}) + \frac{f_\pi m_\pi^2}{3} \left[ \frac{1}{16} (u(1-u)\alpha') (u_0) + (uB - \frac{1}{2}B^2) (u_0) - \frac{1}{2} (u(1-u)B)^{-1} (u_0) \right) \right]
\]
\[
+ (A_\parallel + A_\perp + 2V_\parallel + 2V_\perp)^{-1,0} (u_0) - \left( (1-\alpha_2)(A_\parallel + V_\perp) \right)^{[0,0]} (u_0) + (\alpha_3(A_\parallel + V_\perp))^{[0,1]} (u_0) \right]
\]

\[
g_{M_1 M_1}^{\pi} f_2 f_3 e^{-\frac{\Delta_{3,2} + \Delta_{3,2}}{2}}
\]
\[
= \frac{f_\pi m_\pi^2}{2m_\pi} \left[ \frac{5}{8} \delta (u_0) - \frac{1}{16} (u(1-u)\alpha') (u_0) - \left( (1-\alpha_2)T \right)^{[1,1]} (u_0) \right]
\]
\[
+ 4 \left( (A_\parallel + A_\perp)^{[0,0]} (u_0) + ((1-\alpha_2)(A_\parallel + A_\perp))^{[2,0]} (u_0) + (\alpha_3(A_\parallel + A_\perp))^{[1,1]} (u_0) - 5(\nabla_\parallel + \nabla_\perp)^{[0,0]} (u_0) \right)
\]
\[
T f_0 (\frac{\omega_c}{T}) + \frac{f_\pi m_\pi^2}{2m_\pi} \left[ \frac{1}{16} (u(1-u)\phi_\pi) (u_0) - \left( (1-\alpha_2)(A_\parallel + A_\perp) \right)^{-2,0} (u_0) + (\alpha_3(A_\parallel + A_\perp))^{[-1,1]} (u_0) - ((1-\alpha_2)(A_\parallel + A_\perp))^{[-1,1]} (u_0) \right]
\]
\[
g_{M_2 M_2}^{\pi} f_2 f_3 e^{-\frac{\Delta_{3,2} + \Delta_{3,2}}{2}}
\]
\[
= \frac{\sqrt{6} f_\pi m_\pi^2}{2m_\pi} \left[ \frac{1}{16} (u(1-u)\alpha') (u_0) - \left( (1-\alpha_2)T \right)^{[0,0]} (u_0) \right]
\]
\[
+ (\alpha_3 T)^{[0,1]} (u_0) + \sqrt{6} f_\pi m_\pi^2 \left[ \frac{1}{16} (u(1-u)\phi_\pi) (u_0) - \left( (1-\alpha_2)(A_\parallel + A_\perp) \right)^{-2,0} (u_0) + (\alpha_3(A_\parallel + A_\perp))^{[-1,1]} (u_0) \right]
\]
\[
+ (\alpha_3(A_\parallel + A_\perp))^{[-1,1]} (u_0) \right]
\]
\]
\[
(11)
\]
\\
with \( m_{ud} = m_u + m_d \). The \( \pi \) coupling constants of the other decay channels are defined as

\[
\mathcal{M}(M_1 \rightarrow H_0 + \pi) = I(\eta \cdot q_1) g_{M_1 H_0 \pi}^{p_1}
\]
\[
\mathcal{M}(M_1 \rightarrow H_1 + \pi) = I(\epsilon^* \cdot q_1) g_{M_1 H_1 \pi}^{p_2}
\]
\[
\mathcal{M}(M_1 \rightarrow S_1 + \pi) = I \left[ (\eta \cdot q_1)(\epsilon^* \cdot q_1) - \frac{1}{3} (\eta \cdot \epsilon^*) q_1^2 \right] g_{M_1 S_1 \pi}^{p_1}
\]
\[
\mathcal{M}(M_2 \rightarrow S_0 + \pi) = I(\eta_{\alpha_1 \alpha_2} g_{\alpha_1 \alpha_2}^0 q_1^2 \epsilon^* \cdot q_1) g_{M_2 S_0 \pi}^{p_1}
\]
\[
\mathcal{M}(M_1 \rightarrow T_2 + \pi) = 2I(\alpha_{\beta_1 \beta_2} \epsilon^* \cdot q_1 q_{\alpha_1 \alpha_2} q_{\beta_1 \beta_2} g_{M_1 T_2 \pi}^{p_1}
\]
\[
\mathcal{M}(M_2 \rightarrow T_2 + \pi) = 2I(\alpha_{\beta_1 \beta_2} \epsilon^* \cdot q_1 q_{\alpha_1 \alpha_2} q_{\beta_1 \beta_2} g_{M_2 T_2 \pi}^{p_1}
\]
\[
\mathcal{M}(M_1 \rightarrow T_2 + \pi) = 2I(\alpha_{\beta_1 \beta_2} \epsilon^* \cdot q_1 q_{\alpha_1 \alpha_2} q_{\beta_1 \beta_2} g_{M_1 T_2 \pi}^{p_1}
\]
\[
+ I(\eta_{\alpha_1 \alpha_2} g_{\alpha_1 \alpha_2}^0 q_1^2 \epsilon^* \cdot q_1 q_{\alpha_1 \alpha_2} q_{\beta_1 \beta_2} g_{M_2 T_2 \pi}^{p_1}
\]
We notice that

\[ g_{M_2,π}^0 = \sqrt{6} g_{M_1,π}^0 = \sqrt{6} g_{M_1,π}^0, \]

\[ g_{M_2,π}^2 = -\sqrt{6} g_{M_2,π}^0 = -\sqrt{6} g_{M_2,π}^0, \]

\[ g_{M_1,π}^0 = g_{M_2,π}^0, \]

\[ g_{M_2,π}^1 = -\sqrt{6} g_{M_1,π}^2 = -\sqrt{6} g_{M_1,π}^2, \]

\[ g_{M_1,π}^1 = -\sqrt{6} g_{M_2,π}^0 = -\sqrt{6} g_{M_2,π}^0. \]

These relations are consistent with the HQET expectation at the leading order.

To get the numerical values of the six independent coupling constants, we need the following mass parameters \( \bar{\Lambda} \)’s and \( f \)’s, the overlap amplitudes of these interpolating currents:

| \( \bar{\Lambda} \) | \( (0^-, 1^-) \) [13] | \( (0^+, 1^+) \) [14] | \( (1^+, 2^+) \) [14] | \( (1^-, 2^-) \) [9] |
|-------------------|----------------|----------------|----------------|----------------|
| \( f \) | \( (0.25 \pm 0.04) \) GeV\(^{3/2} \) | \( (0.36 \pm 0.10) \) GeV\(^{3/2} \) | \( (0.26 \pm 0.06) \) GeV\(^{5/2} \) | \( (0.39 \pm 0.03) \) GeV\(^{5/2} \) |

The \( \pi \) decay constant \( f_\pi = 131 \) MeV. \( \mu_\pi = m_\pi/(m_u + m_d) \) is given in Ref. [13]: \( \mu_\pi = (1.573 \pm 0.174) \) GeV. The parameters appear in the \( \pi \) distribution amplitudes are listed below. We use the values at the scale \( \mu = 1 \) GeV in our calculation under the consideration that the heavy quark behaves almost as a spectator of the decay processes in our discussion at the leading order of HQET.

| \( a_2 \) | \( \eta_3 \) | \( \omega_3 \) | \( \eta_4 \) | \( \omega_4 \) | \( h_{00} \) | \( v_{00} \) | \( a_{10} \) | \( v_{10} \) | \( h_{01} \) | \( h_{10} \) |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.25      | 0.015     | -1.5      | 10        | 0.2       | -3.33     | -3.33     | 5.14      | 5.25      | 3.46      | 7.03      |

We will work at the symmetry point, i.e., \( T_1 = T_2 = 2T, u_0 = 1/2 \). This comes from the consideration that every reliable sum rule has a working interval of the Borel parameter \( T \) within which the sum rule is insensitive to the variation of \( T \). So it is reasonable to choose a common point \( T_1 = T_2 \) at the overlap of \( T_1 \) and \( T_2 \). Furthermore, choosing \( T_1 = T_2 \) will enable us to subtract the continuum contribution cleanly, while the asymmetric choice will lead to the very difficult continuum subtraction.

From the convergence requirement of the operator product expansion and the requirement that the pole contribution is larger than 40\%, we get the working interval of the Borel parameter \( T \). The resulting sum rules are plotted with \( \omega_c = 3.2, 3.4, 3.6 \) GeV in Figs. [14]. The numerical values of these coupling constants are collected in Table [14].

#### III. SUM RULES FOR THE \( \rho, \omega \) COUPLING CONSTANTS

The sum rules for the \( \rho \) (\( \omega \)) coupling constants can be obtained using the same approach. Now the tensor structure of the decay amplitudes are a little more complicated than those of the \( \pi \) meson, due to the spin of the final \( \rho \) (\( \omega \))
meson. For example, the decay amplitude $\mathcal{M}(M_2 \to H_1 + \rho)$ can now be written as

$$
\mathcal{M}(M_2 \to H_1 + \rho) = 2I\eta_{\alpha_1 \alpha_2} \left[ -\varepsilon^{\alpha_1 \alpha_2 \sigma_1} \varepsilon^{\sigma_1 \sigma_2} + \frac{1}{3} g_{\alpha_1 \alpha_2} \varepsilon^{\sigma_1 \sigma_2} \right] g_{M_2 H_1 \rho}^1 + 2I\eta_{\alpha_1 \alpha_2} \left[ -\varepsilon^{\alpha_1 \alpha_2 \sigma_1} \varepsilon^{\sigma_1 \sigma_2} + \frac{1}{3} g_{\alpha_1 \alpha_2} \varepsilon^{\sigma_1 \sigma_2} \right] g_{M_2 H_1 \rho}^2 + 2I\eta_{\alpha_1 \alpha_2} \left[ -\varepsilon^{\alpha_1 \alpha_2 \sigma_1} \varepsilon^{\sigma_1 \sigma_2} (\varepsilon^* \cdot q_t) - \frac{3}{5} \left( \varepsilon^{\alpha_1 \alpha_2 \sigma_1} \varepsilon^{\sigma_1 \sigma_2} + \varepsilon^{\alpha_1 \alpha_2 \sigma_1} \varepsilon^{\sigma_1 \sigma_2} \right) \right] g_{M_2 H_1 \rho}^2 \tag{14}
$$

where $\varepsilon^*$ is the polarization vector of the final $\rho$ meson.

| $g_{M_2 H_1 \pi}$ | $g_{M_2 S_1 \pi}$ | $g_{M_1 T_1 \pi}$ | $g_{M_2 H_1 \pi}$ | $g_{M_2 S_1 \pi}$ | $g_{M_1 T_1 \pi}$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| $g_{c}$ | 1.13 | -0.68 | -0.40 | -1.09 | 0.04 | -0.46 |
| $g_{2}$ | 0.78 | 1.65 | -1.07 | -0.47 | -0.83 | -0.15 | -1.67 | -0.75 | 0.03 | 0.05 | -0.62 | -0.34 |

TABLE I: The $\pi$ coupling constants in units of [GeV]$^{-j}$ with $j$ the orbital angular momentum of the final pion. $g_c$ corresponds to the central values of the overlap amplitudes and $g$ with $\omega_c = 3.4$ GeV and $T = 3.5$ GeV where $g_{M_2 H_1 \pi} \equiv g_{M_2 H_1 \pi} f_{-\frac{1}{2}} f_{-\frac{1}{2}} - f_{\frac{1}{2}} f_{\frac{1}{2}}$ etc. The ranges of $g$ are determined according to the uncertainty of the overlap amplitudes and the ranges of $g$ with $3.0 < T < 4.0$ GeV and $3.2 < \omega_c < 3.6$ GeV. We use asterisk to indicate the coupling constants from sum rules without a stable working interval.
We define other $\rho$ coupling constants as

\[
\mathcal{M}(M_1 \to H_0 + \rho) = I e^{\nu^e q_v} g_{M_1 H_0 \rho}^{p_1},
\]

\[
\mathcal{M}(M_1 \to H_1 + \rho) = \left(\eta \cdot e^*_\eta\right) (e^* \cdot q_t) (\eta \cdot q_t) (e^* \cdot e^*_\eta) + \frac{2}{3} (\eta \cdot e^*_\eta) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot e^*_\eta) \right) g_{M_1 H_1 \rho}^{p_2}
\]

\[
\mathcal{M}(M_2 \to H_0 + \rho) = 2 i \eta_{\alpha_1 \alpha_2} \left(\eta \cdot e^*_\eta\right) (e^* \cdot q_t) (\eta \cdot q_t) (e^* \cdot e^*_\eta) + \frac{2}{3} (\eta \cdot e^*_\eta) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot e^*_\eta) \right) g_{M_2 H_0 \rho}^{p_2}
\]

\[
\mathcal{M}(M_1 \to S_0 + \rho) = I (\eta \cdot e^*_\eta) g_{M_1 S_0 \rho}^{p_1} + \frac{1}{3} (\eta \cdot q_t) (e^* \cdot q_t) (\eta \cdot q_t) (e^* \cdot e^*_\eta) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot e^*_\eta) \right) g_{M_1 S_0 \rho}^{p_1}
\]

\[
\mathcal{M}(M_2 \to S_0 + \rho) = 2 i \eta_{\alpha_1 \alpha_2} \left(\eta \cdot e^*_\eta\right) (e^* \cdot q_t) (\eta \cdot q_t) (e^* \cdot e^*_\eta) + \frac{2}{3} (\eta \cdot e^*_\eta) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot e^*_\eta) \right) g_{M_2 S_0 \rho}^{p_1}
\]

\[
\mathcal{M}(M_2 \to S_1 + \rho) = 2 i \eta_{\alpha_1 \alpha_2} \left(\eta \cdot e^*_\eta\right) (e^* \cdot q_t) (\eta \cdot q_t) (e^* \cdot e^*_\eta) + \frac{2}{3} (\eta \cdot e^*_\eta) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot q_t) + \frac{2}{3} (\eta \cdot q_t) (e^* \cdot e^*_\eta) \right) g_{M_2 S_1 \rho}^{p_1}
\]

Because of heavy quark symmetry, there are only six independent coupling constants involving these decay modes. Their sum rules read

\[
G_{M_1 H_0 \rho}^{p_1} = \frac{1}{64} \left(4 f_f^T [u \varphi_1 ](u \varphi_1 ) + 4 f_f m_\rho \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] T f_0 \left[ \frac{\omega}{T} \right] + f_0 m_\rho^2 \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] \right),
\]

\[
G_{M_2 H_0 \rho}^{p_1} = \frac{1}{320} \left(12 f_f^T [u \varphi_1 ](u \varphi_1 ) + 4 f_f m_\rho \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] T f_0 \left[ \frac{\omega}{T} \right] + f_0 m_\rho^2 \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] \right),
\]

\[
G_{M_1 H_0 \rho}^{p_2} = \frac{1}{64} \left(4 f_f^T [u \varphi_1 ](u \varphi_1 ) + 4 f_f m_\rho \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] T f_0 \left[ \frac{\omega}{T} \right] + f_0 m_\rho^2 \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] \right),
\]

\[
G_{M_2 H_0 \rho}^{p_2} = \frac{1}{320} \left(12 f_f^T [u \varphi_1 ](u \varphi_1 ) + 4 f_f m_\rho \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] T f_0 \left[ \frac{\omega}{T} \right] + f_0 m_\rho^2 \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] \right),
\]

\[
G_{M_1 H_0 \rho}^{p_3} = \frac{1}{64} \left(4 f_f^T [u \varphi_1 ](u \varphi_1 ) + 4 f_f m_\rho \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] T f_0 \left[ \frac{\omega}{T} \right] + f_0 m_\rho^2 \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] \right),
\]

\[
G_{M_2 H_0 \rho}^{p_3} = \frac{1}{320} \left(12 f_f^T [u \varphi_1 ](u \varphi_1 ) + 4 f_f m_\rho \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] T f_0 \left[ \frac{\omega}{T} \right] + f_0 m_\rho^2 \left[ A_{[1]}(u \varphi_1 ) + 2 A_{[2]}(u \varphi_1 ) \right] \right),
\]
\[-64(u \varphi_l)^{[2]}(\bar{u}_0) - 64 \varphi^0_{\parallel}(\bar{u}_0) + 4A(\bar{u}_0) - 4(uA)(\bar{u}_0) + 6A^{[1]}(\bar{u}_0) - 64 g^{[e]}_{\parallel}(\bar{u}_0) + 64(u g^{[e]}_{\parallel})^{[2]}(\bar{u}_0) + 64(u g^{[e]}_{\parallel})^{[2]}(\bar{u}_0) + 164 g^{[e]}_{\parallel}(\bar{u}_0) + 16\psi^{[-1,0]}(u_0) \frac{1}{T} + 8 f^T \rho m^4 \left[A^{[1]}(\bar{u}_0) - (u A_T)^{[1]}(\bar{u}_0) + 16 B_T^{[3]}(\bar{u}_0) \right]
\]
\[-16(u B_T)^{[3]}(\bar{u}_0) - 32 B_T^{[4]}(\bar{u}_0) + 8 T^{[-2,0]}(u_0) + 16 T_1^{[-2,0]}(u_0) + 16 T_2^{[-2,0]}(u_0) + 16 T_3^{[-2,0]}(u_0) + 16 T_4^{[-2,0]}(u_0) \]
\[+16 T_4^{[-2,0]}(u_0) \frac{1}{T^2} + 16 f_\rho m^5 \left[A^{[3]}(\bar{u}_0) - A^{[2]}(\bar{u}_0) + (u A)^{[2]}(\bar{u}_0) - 16 \psi^{[-3,0]}(u_0) \right] \frac{1}{T^3} \bigg \}, \tag{17}\]

\[g_{M_3 H \rho f^{-2}} = -\frac{3}{2} e^{-\frac{\Delta_{S/2} + \Delta_{3/2}}{T}} \]
\[g_{M_3 S \rho f} = -\frac{3}{2} e^{-\frac{\Delta_{S/2} + \Delta_{3/2}}{T}} \]
\[g_{M_3 S \rho f} = -\frac{3}{2} e^{-\frac{\Delta_{S/2} + \Delta_{3/2}}{T}} \]
\[g_{M_3 S \rho f} = -\frac{3}{2} e^{-\frac{\Delta_{S/2} + \Delta_{3/2}}{T}} \]

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\begin{align*}
&= \frac{1}{32} \left\{ 4 f^T_\rho \left[ \varphi_\perp(u_0) - (u_\varphi)(u_0) \right] T f_0(\omega_{\perp}) + 2 f_\rho m_\rho \left[ (u g^{(a)}_\perp)(u_0) - g^{(a)}_\perp(u_0) + 4 A^{(0,0)}(u_0) \right] \\
&\quad + f^T_\rho m_\rho^2 \left[ (u A)(u_0) - A(u_0) + 16 T_2^{[-1,0]}(u_0) + 16 T_3^{[-1,0]}(u_0) \right] \frac{1}{T} \right\}. 
\end{align*}
\]

(21)

The other \( \rho \) coupling constants are related to the above ones by the following relations:

\[
\begin{align*}
&g^{p1}_{M2H_1\rho} = \frac{\sqrt{6}}{4} g^{p1}_{M2H_2\rho} = -\frac{\sqrt{6}}{2} g^{p1}_{M2H_1\rho}, \\
&g^{p2}_{M2H_1\rho} = \frac{\sqrt{6}}{6} g^{p2}_{M2H_2\rho} = -\frac{1}{2} g^{p2}_{M2H_1\rho}, \\
&g^{f2}_{M2H_1\rho} = \frac{\sqrt{6}}{6} g^{f2}_{M2H_2\rho} = -\frac{1}{2} g^{f2}_{M2H_1\rho}, \\
&g^{s1}_{M2S1\rho} = \frac{\sqrt{6}}{4} g^{s1}_{M2S_0\rho} = -\frac{6}{2} g^{s1}_{M2S1\rho}, \\
&g^{d1}_{M2S1\rho} = -\frac{\sqrt{6}}{4} g^{d1}_{M2S_0\rho} = \sqrt{6} g^{d1}_{M2S1\rho}, \\
&g^{d2}_{M2S1\rho} = -\frac{\sqrt{6}}{6} g^{d2}_{M2S_0\rho} = -\frac{1}{2} g^{d2}_{M2S1\rho}. 
\end{align*}
\]

(22)

In our numerical analysis, the parameters that appear in the distribution amplitudes of the \( \rho \) meson take the values from Ref. [18].

| \( f_\rho \) [MeV] | \( f^T_\rho \) [MeV] | \( a_2^\parallel \) | \( a_2^\perp \) | \( \zeta_3^\parallel \) | \( \zeta_3^\perp \) | \( \omega_3^\parallel \) | \( \omega_3^\perp \) | \( \zeta_4^\parallel \) | \( \zeta_4^\perp \) | \( \zeta_4^\parallel \) |
|------------------|------------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 216(3) | 165(9) | 0.15(7) | 0.14(6) | 0.030(10) | 0.09(3) | 0.15(5) | 0.55(25) | 0.07(3) | 0.03(1) | 0.03(5) | 0.08(5) |

TABLE II: The \( \rho/\omega \) coupling constants in units of \([\text{GeV}]^{-1}\) with \( j \) the orbital angular momentum of the final \( \rho/\omega \) meson. \( g_e \)'s correspond to the central values of the overlap amplitudes and \( g_s \) with \( \omega_c = 3.4 \text{ GeV} \) and \( T = 2.75 \text{ GeV} \).

The ranges of \( g_s \) are determined according to the uncertainty of the overlap amplitudes and the ranges of \( g_s \) with \( 2.5 < T < 3.0 \text{ GeV} \) and \( 3.2 < \omega_c < 3.6 \text{ GeV} \).

The resulting sum rules are plotted with \( \omega_c = 3.2, 3.4, 3.6 \text{ GeV} \) in Figs. 38. The numerical values of these coupling constants are collected in Table II.

Replacing the \( \rho \) meson parameters by those for the \( \omega \) meson, one obtains the \( \omega \) meson couplings with the heavy mesons, which are also listed in Table II. Here we take the following values for the parameters \( f_\omega, f^T_\omega \) [17], and \( m_\omega \):

- \( f_\omega = 0.195 \text{ GeV} \), \( f^T_\omega = 0.145 \text{ GeV} \), and \( m_\omega = 0.78 \text{ GeV} \).

IV. SUM RULES FOR THE \( \gamma \) COUPLING CONSTANTS

As an example, we consider the decay \( M_1 \to H_1 + \gamma \). The decay amplitude should now be expressed according to the total angular momentum of the final photon, namely \( m_1, e^2 \cdots \):

\[
\mathcal{M}(M_1 \to H_1 + \gamma) = e i \left[ (\eta \cdot e_i^\ast)(e^\ast \cdot q_f) - (\eta \cdot q_f)(e^\ast \cdot e_i^\ast) \right] g^{m_1}_{M_1H_1\gamma}
\]
where we have adopted the Fock-Schwinger gauge $x^\mu A_\mu(x) = 0$ to express the electromagnetic vector potential in terms of the gauge invariant $F_{\mu\nu}$. In our calculation, we take the $u$ quark as the light quark involved and therefore $e_u = \frac{2}{3}$. The second diagram involves the nonperturbative interaction of the photon with the light quark in terms of the photon light-cone distribution amplitudes.

Now the problem can be tackled using the same approach as in Section II and III. We list the final sum rules below:

$$g_{M_1\gamma}^1 f_{-1/2} f_{-3/2}$$
\[-2(u\psi^*)(\bar{u}\bar{0}) + 4a^{[1]}(u\bar{0}) + 2e^{[1]}(u\bar{0}) \Bigl[ T_f_0(\frac{\omega}{T}) + \langle \tilde{q}\bar{q} \rangle \Bigr] (A(\bar{u}\bar{0}) - (uA)(\bar{u}\bar{0}) + 8h_{1}^{[2]}(\bar{u}\bar{0}) - 8h_{1}^{[6]}(\bar{u}\bar{0}) + 8(uh_{1})(\bar{u}\bar{0}) + 32\bar{S}^{[2]}(u\bar{0}) + 16T_{3}^{[0]}(u\bar{0}) - 16T_{4}^{[0]}(u\bar{0}) \Bigr) \bigr) ,
\]

\[g_{M_{1}H_{1}\gamma}^{2}f_{+}+f_{-}+4e^{-\frac{\lambda-3}{2}+\lambda+1/2}\]

\[-\frac{1}{16\sqrt{6}}e^{T_{f}^{2}(\frac{\omega}{T}) - 4g_{M_{1}S_{1}\gamma}^{[4]}(2\frac{u_{0}}{\pi^{2}}T_{f}^{2}(\frac{\omega}{T}) - 4\chi(\tilde{q}\bar{q})[c_{1}(\bar{u}\bar{0}) - (u\phi_{1})(\bar{u}\bar{0})] T_{f_0}(\frac{\omega}{T}) + 2f_{3}\gamma \left[ 4\psi_{1}^{a}(\bar{u}\bar{0}) + 2\phi_{1}^{[1]}(\bar{u}\bar{0}) - (u\phi_{1})(\bar{u}\bar{0}) + A^{[1]}(u\bar{0}) - 8S_{1}^{[2]}(u\bar{0}) + 16T_{4}^{[1]}(u\bar{0}) - 16T_{4}^{[1]}(u\bar{0}) \Bigr) \right] ,
\]

Similarly, the electromagnetic coupling constants between doublets \( M \) and \( S \) can be defined as

\[\mathcal{M}(M_{1} \rightarrow S_{1} + \gamma) = e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{1}S_{1}\gamma}^{[1]} + e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{1}S_{1}\gamma}^{[2]} \]

and the sum rules for \( g_{M_{1}S_{1}\gamma}^{[1]} \) and \( g_{M_{1}S_{1}\gamma}^{[2]} \) read

\[-\frac{\sqrt{6}}{32}e^{(f_{+}+f_{-})+\lambda+1/2}\]

The \( \gamma \) coupling constants of the other channels are defined as

\[\mathcal{M}(M_{1} \rightarrow H_{0} + \gamma) = e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{1}H_{0}\gamma}^{[1]} \]

\[\mathcal{M}(M_{2} \rightarrow H_{0} + \gamma) = e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{2}H_{0}\gamma}^{[1]} \]

\[\mathcal{M}(M_{2} \rightarrow H_{1} + \gamma) = 2e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{2}H_{1}\gamma}^{[1]} \]

\[\mathcal{M}(M_{1} \rightarrow S_{0} + \gamma) = e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{1}S_{0}\gamma}^{[1]} \]

\[\mathcal{M}(M_{2} \rightarrow S_{0} + \gamma) = 2e(\epsilon_{\gamma}^{[1]} - \epsilon_{\gamma}^{[2]}) g_{M_{2}S_{0}\gamma}^{[1]} \]
The renormalization scale is \( \langle \kappa \rangle \approx 1 \) for the decays \( \ell^- \rightarrow \mu^- \). There are simple relations among the coupling constants of a certain decay type \( (e1, m1 \ldots) \) within two doublets:

\[
\begin{align*}
g_{M_1 H_1 \gamma}^{m1} &= -\frac{1}{2} g_{M_1 H_0 \gamma}^{m1} - \frac{\sqrt{6}}{3} g_{M_2 H_1 \gamma}^{m1}, \\
g_{M_1 H_1 \gamma}^{e2} &= -\sqrt{6} g_{M_2 H_0 \gamma}^{e2} = \sqrt{6} g_{M_2 H_1 \gamma}^{e2}, \\
g_{M_1 S_1 \gamma}^{m1} &= -\frac{1}{2} g_{M_1 S_0 \gamma}^{m1} = \frac{\sqrt{6}}{3} g_{M_2 S_1 \gamma}^{m1}, \\
g_{M_2 S_1 \gamma}^{m2} &= \frac{\sqrt{6}}{2} g_{M_2 S_0 \gamma}^{m2} = -\sqrt{6} g_{M_2 S_1 \gamma}^{m2}.
\end{align*}
\]

The resulting sum rules are plotted with \( \omega_c = 3.2, 3.4, 3.6 \text{ GeV} \) in Figs. 9-12. The numerical values of these coupling constants are collected in Table 1111. The quark condensate at the same renormalization scale is \( \langle \bar{q} q \rangle = (0.0245 \text{ GeV})^3(q = u, d) \). We adopt the value \( \chi = (3.15 \pm 0.3) \text{ GeV}^{-2} \) for the magnetic susceptibility of the quark condensate.
\[
\begin{array}{cccc}
g_{M_1 H_1 \gamma} & g_{M_1 H_1 \gamma} & g_{M_1 S_1 \gamma} & g_{M_1 S_1 \gamma} \\
[\text{GeV}^{-1}] & [\text{GeV}^{-3}] & [\text{GeV}^{-2}] & [\text{GeV}^{-2}] \\
g_c & -0.05 & 0.37 & 0.06 & 0.16 \\
g & -0.07 \sim -0.03 & 0.26 \sim 0.53 & 0.04 \sim 0.10 & 0.10 \sim 0.28
\end{array}
\]

| TABLE III: The $\gamma$ coupling constants. $g_c$ correspond to the central values of the overlap amplitudes and $\tilde{g}_s$ with $\omega_c = 3.4$ GeV and $T = 2.75$ GeV for $g_{M_1 H_1 \gamma}$. The ranges of $g_s$ are determined according to the uncertainty of the overlap amplitudes and the ranges of $\tilde{g}_s$ with $T$ in their working intervals and $3.2 < \omega_c < 3.6$ GeV.

V. DECAY WIDTHS

It is straightforward to calculate the decay widths of the $(1^{-}, 2^{-}) D$-wave heavy mesons with the coupling constants extracted in the previous sections. For completeness, we make rough estimates of the decay widths of the single and double-pion channels with the given coupling constants, although some of the coupling constants are extracted from sum rules without a stable working interval, as shown in Section III and IV. The formulas for the single-pion and the radiative decay widths are collected in Appendix C.

To get the numerical values of these decay widths, we need the mass parameters of the heavy mesons concerned. They are collected in Table IV.

| \(J^P\) | 0^- | 1^- | 0^+ | 1^+ | 2^+ | 1^- | 2^- |
|------|------|------|------|------|------|------|------|
| \(m_D\) [GeV] | 1.87 | 2.01 | 2.40 | 2.43 | 2.42 | 2.46 | 2.82 | 2.83 |
| \(m_B\) [GeV] | 5.28 | 5.33 | 5.70 | 5.73 | 5.72 | 5.75 | 6.10 | 6.11 |
| \(m_{D^*}\) [GeV] | 1.97 | 2.11 |
| \(m_{B^*}\) [GeV] | 5.37 | 5.41 |

| TABLE IV: Heavy meson masses used in our calculation. We take the values of \(m_D, m_{D^*}, m_{D^*_1}, m_{D^*_0}, m_{D^*_1}, m_{D^*_2}, m_B, m_{B^*}, m_{B^*_1}, m_{B^*_0}, m_{B^*_1}, m_{B^*_2}, m_{D^*_s},\) and \(m_{B^*_s}\) from Ref. [21]. Masses of the other heavy mesons are from the quark model prediction made in Ref. [22].

The resulting decay widths are presented in Table V. Here we also make rough estimates of the decay widths of \(D + \eta/K^+\) channels assuming the flavor SU(3) symmetry. The column \(\Gamma_{D\rightarrow D+P}\) sums over the widths of all pseudoscalar meson decay channels, including the widths of \(D\pi^0\) channels which are about half of the widths of the corresponding \(D\pi^+\) channels.

The Feynman diagram of the double-pion decays is shown in Fig. 13. We will not present the expressions for

\[
\begin{tikzpicture}
  \draw[->] (0,0) -- (0,2) node[right] {$H/S$};
  \draw[->] (0,0) -- (-2,-2) node[above left] {$M$};
  \draw[->] (0,0) -- (2,-2) node[above right] {$\rho$};
  \draw[->] (0,0) -- (0,-2) node[below left] {$\pi$};
  \draw[->] (0,0) -- (-2,2) node[below right] {$\pi$};
\end{tikzpicture}
\]

FIG. 13: The double-pion decays of \(D\)-wave heavy mesons via a virtual \(\rho\) meson.

The widths of these double-pion decay channels due to the length of these expressions. Their numerical values are collected in Table V together with those of the radiative decay channels. We take \(f_{\rho\pi\pi} = 6.14\) in our calculation.
appear in these light-cone distribution amplitudes. The uncertainty in the light-cone distribution amplitudes of the involved mesons or photon, and the uncertainty in the parameters that define the higher twist terms in the OPE near the light-cone, the variation of the coupling constant with the continuum threshold $\omega_c$. For the other sum rules, we can not find a stable working interval of $T$. The extracted vector meson heavy meson coupling constants may be helpful in the study of the interaction between two $B(D)$ mesons.

Some possible sources of the errors in our calculation include the inherent inaccuracy of LCQSR: the omission of the higher twist terms in the OPE near the light-cone, the variation of the coupling constant with the continuum threshold $\omega_c$ and the Borel parameter $T$ in the working interval, the omission of the higher conformal partial waves in the light-cone distribution amplitudes of the involved mesons or photon, and the uncertainty in the parameters that appear in these light-cone distribution amplitudes. The uncertainty in $f$’s and $\Lambda$’s is another source of errors. It’s also known that the $1/m_Q$ correction may turn out to be quite large for the charm mesons while such a correction is under control for the bottom system.

With these extracted coupling constants, we have made a very rough estimate of the decay widths of the $(1^-, 2^-)$ $D$-wave heavy mesons. Their dominant decay modes are $M \rightarrow T + \pi$ and $M \rightarrow H + \pi$. The former mode is of $S$-wave while the latter decay occurs in the $P$-wave but with a larger phase space. The total width of the $D$-wave heavy

### TABLE V: The widths of the single-pion channels in units of MeV, including their ranges and the values corresponding to the central values of $g_\pi$.

| $D_1^\prime \rightarrow$ | $B_\pi^+ \ B_\pi^{*+} \ B_0^* \ B_1^\prime$ | $B_2^*$ |
|-------------------------|-----------------------------------|------|
| $D_1^\prime \rightarrow$ | $D^\pi^0 \ D^{*\pi^0} \ D_0^{\pi^0} \ D_1^{\prime\pi^0}$ | $2.1 - 9.3$ |
| $B_1^\prime \rightarrow$ | $19.1 - 76.5$ | $217.3 - 1382.1$ |
| $B_2^*$ | $5.8 - 0.2$ | $0.02 - 0.06$ |
| $B_2^*$ | $2.1 - 9.3$ | $0.01 - 0.06$ |
| $198.1 - 1234.1$ | $152.5 - 1008.4$ & $0.02 - 0.12$ | $1.0 - 8.7$ |
| $9.4 - 18.8$ | $9.5 - 22.2$ & $0.5$ | $0.8$ |

### TABLE VI: The widths of the double-pion and radiative decay channels in units of keV.

### VI. CONCLUSION

We have calculated the $\pi$, $\rho$, $\omega$, and $\gamma$ couplings with heavy mesons at the leading order of HQET within the framework of LCQSR. Most of the sum rules obtained are stable with the variations of the Borel parameter $T$ and the continuum threshold $\omega_c$. For the other sum rules, we can not find a stable working interval of $T$. The extracted vector meson heavy meson coupling constants may be helpful in the study of the interaction between two $B(D)$ mesons.

Some possible sources of the errors in our calculation include the inherent inaccuracy of LCQSR: the omission of the higher twist terms in the OPE near the light-cone, the variation of the coupling constant with the continuum threshold $\omega_c$ and the Borel parameter $T$ in the working interval, the omission of the higher conformal partial waves in the light-cone distribution amplitudes of the involved mesons or photon, and the uncertainty in the parameters that appear in these light-cone distribution amplitudes. The uncertainty in $f$’s and $\Lambda$’s is another source of errors. It’s also known that the $1/m_Q$ correction may turn out to be quite large for the charm mesons while such a correction is under control for the bottom system.

With these extracted coupling constants, we have made a very rough estimate of the decay widths of the $(1^-, 2^-)$ $D$-wave heavy mesons. Their dominant decay modes are $M \rightarrow T + \pi$ and $M \rightarrow H + \pi$. The former mode is of $S$-wave while the latter decay occurs in the $P$-wave but with a larger phase space. The total width of the $D$-wave heavy
mesons is roughly several tens of MeV. Therefore the \((1^-, 2^-)\) \(D\)-wave heavy mesons are not expected to be extremely broad, which is helpful in the future experimental search.

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Appendix A: The definitions of the \(\pi\), \(\rho\) meson and the photon light-cone distribution amplitudes

The 2-particle distribution amplitudes of the \(\pi\) meson are defined as [15]

\[
\langle 0 | \bar{u}(z) \gamma_\mu \gamma_5 d(-z) | \pi^- (P) \rangle = i f_\pi p_\mu \int_0^1 du e^{i \xi p_z u} \phi_\pi (u) + i \frac{f_\pi m_\pi^2}{p_z} \int_0^1 du e^{i \xi p_z u} g_\pi (u),
\]

\[
\langle 0 | \bar{u}(z) i \gamma_5 d(-z) | \pi (P) \rangle = \frac{f_\pi m_\pi^2}{m_u + m_d} \int_0^1 du e^{i \xi p_z u} \phi_\rho (u),
\]

\[
\langle 0 | \bar{u}(z) \sigma_{\alpha \beta} \gamma_5 d(-z) | \pi (P) \rangle = -i \frac{f_\pi m_\pi^2}{3 (m_u + m_d)} (p_\alpha z_\beta - p_\beta z_\alpha) \int_0^1 du e^{i \xi p_z u} \phi_\sigma (u),
\]

where \(\xi = 2u - 1\), \(\phi_\pi\) is the leading twist-2 distribution amplitude, \(\phi_{(p, \sigma)}\) are of twist-3. All the above distribution amplitudes \(\phi = \{\phi_\pi, \phi_\rho, \phi_\sigma, g_\pi\}\) are normalized to unity: \(\int_0^1 du \phi (u) = 1\).
There is one 3-particle distribution amplitudes of twist-3, defined as [15]

\[ \langle 0 | \bar{u}(z) \sigma_{\mu\nu} \gamma_5 g G_{\alpha\beta}(vz) d(-z) | \pi^- (P) \rangle = i f_\pi m_\pi^2 \left( p_\alpha p_\mu g^{\perp}_{\mu\beta} - p_\alpha p_\nu g^{\perp}_{\nu\beta} - p_\beta p_\mu g^{\perp}_{\mu\alpha} + p_\beta p_\nu g^{\perp}_{\nu\alpha} \right) T(v, pz), \]  

(A2)

where we used the following notation for the integral defining the 3-particle distribution amplitude:

\[ T(v, pz) = \int D\alpha e^{-i p_z (\alpha_u - \alpha_d + \nu_\alpha)} T(\alpha_d, \alpha_u, \alpha_g). \]  

(A3)

Here \( \alpha \) is the set of three momentum fractions \( \alpha_d, \alpha_u, \) and \( \alpha_g \). The integration measure is

\[ \int D\alpha = \int_0^1 d\alpha_d d\alpha_u d\alpha_g (1 - \alpha_u - \alpha_d - \alpha_g). \]  

(A4)

The 3-particle distribution amplitudes of twist-4 are

\[ \langle 0 | \bar{u}(z) \gamma_\mu \gamma_5 g G_{\alpha\beta}(vz) d(-z) | \pi^- (P) \rangle = p_\mu (p_\alpha z_\beta - p_\beta z_\alpha) \left( \frac{1}{p_z} f_\pi m_\pi^2 A_\parallel (v, pz) + (p_\beta g^{\perp}_{\alpha\mu} - p_\alpha g^{\perp}_{\beta\mu}) f_\pi m_\pi^2 A_\perp (v, pz), \right), \]

(A5)

where \( \tilde{G}_{\alpha\beta} \) is the dual \( G_{\alpha\beta} \equiv \frac{1}{2} \xi_{\alpha\beta\gamma} G^{\gamma\delta} \).

The definitions of the distribution amplitudes of the \( \rho \) meson used in the text read as [17, 18]

\[ \langle 0 | \bar{u}(z) \gamma_\mu d(-z) | \rho^-(P, \lambda) \rangle = f_\rho m_\rho \left[ \frac{e^{(\lambda)} \cdot z}{p \cdot z} \int_0^1 du e^{i \xi p_z} \varphi_\parallel (u, \mu^2) + e^{(\lambda)} \mu^2 \int_0^1 du e^{i \xi p_z} g_\perp(u, \mu^2) \right]^{-1}, \]

\[ \langle 0 | \bar{u}(z) \gamma_\mu d(-z) | \rho^-(P, \lambda) \rangle = \frac{1}{2} f_\rho m_\mu n_\nu \left( \frac{e^{(\lambda)} \cdot \nu}{p \cdot \mu} \int_0^1 du e^{i \xi p_z} g^{(\lambda)} (u, \mu^2) \right), \]

\[ \langle 0 | \bar{u}(z) \gamma_\mu d(-z) | \rho^-(P, \lambda) \rangle = i f_\rho \left( \frac{e^{(\lambda)} \cdot \mu}{p \cdot \mu} \int_0^1 du e^{i \xi p_z} \varphi_\perp (u, \mu^2) \right) \]

\[ + (p_\mu z_\nu - p_\nu z_\mu) \frac{e^{(\lambda)} \cdot \mu}{p \cdot z} \int_0^1 du e^{i \xi p_z} h^{(t)} (u, \mu^2) \]

\[ + \frac{1}{2} (e^{(\lambda)} \cdot \mu z_\nu - e^{(\lambda)} \cdot \nu z_\mu) \int_0^1 du e^{i \xi p_z} h^{(s)} (u, \mu^2) \],

\[ \langle 0 | \bar{u}(z) d(-z) | \rho^-(P, \lambda) \rangle = -i f_\rho \left( e^{(\lambda)} z \int_0^1 du e^{i \xi p_z} h^{(s)} (u, \mu^2) \right). \]  

(A6)

The distribution amplitudes \( \varphi_\parallel \) and \( \varphi_\perp \) are of twist-2, \( g^{(o)}_\perp, g^{(a)}_\perp, h^{(s)}_\parallel \) and \( h^{(t)}_\parallel \) are twist-3 and \( g_3, h_3 \) are twist-4. All functions \( \phi = \{ \varphi_\parallel, \varphi_\perp, g^{(o)}_\perp, g^{(a)}_\perp, h^{(s)}_\parallel, h^{(t)}_\parallel, g_3, h_3 \} \) are normalized to satisfy \( \int_0^1 du \phi (u) = 1 \).

The 3-particle distribution amplitudes of the \( \rho \) meson are defined as [17, 18]

\[ \langle 0 | \bar{u}(z) g G_{\mu\nu} \gamma_5 d(-z) | \rho^-(P, \lambda) \rangle = f_\rho m_\rho p_\alpha [p_\nu e^{(\lambda)} \cdot \perp \mu - p_\mu e^{(\lambda)} \cdot \perp \nu] A (v, pz) \]

\[ + \frac{f_\rho m_\rho^3 e^{(\lambda)} \cdot \perp p_z}{p z} [p_\mu g^{\perp}_{\mu\nu} - p_\nu g^{\perp}_{\nu\mu}] \tilde{\Phi} (v, pz) \]

\[ + \frac{f_\rho m_\rho^3 e^{(\lambda)} \cdot \perp p_z}{p z} [p_\mu z_\nu - p_\nu z_\mu] \tilde{\Psi} (v, pz), \]

\[ \langle 0 | \bar{u}(z) g G_{\mu\nu} i \gamma_5 d(-z) | \rho^-(P) \rangle = f_\rho m_\rho p_\alpha [p_\nu e^{(\lambda)} \cdot \perp \mu - p_\mu e^{(\lambda)} \cdot \perp \nu] \mathcal{V} (v, pz) \]

\[ + \frac{f_\rho m_\rho^3 e^{(\lambda)} \cdot \perp p_z}{p z} [p_\mu g^{\perp}_{\mu\nu} - p_\nu g^{\perp}_{\nu\mu}] \Phi (v, pz) \]
Here the distribution amplitudes \( \langle 0 | \bar{u} (z) \sigma_{\alpha \beta} g G_{\mu \nu} (v z) | \rho^- (P, \lambda) \rangle = \int d^4 \tau \langle 0 | \partial_{\tau} \bar{u} (\lambda \tau) \sigma_{\alpha \beta} g G_{\mu \nu} (v z) | \rho^- (P, \lambda) \rangle \),

\[
\langle 0 | \bar{u} (z) \sigma_{\alpha \beta} g G_{\mu \nu} (v z) d (-z) | \rho^- (P, \lambda) \rangle = f_{\rho} T \rho^2 \frac{e^{i \lambda}}{(p z)} \frac{1}{2} [ \partial_{\mu} g_{\mu \nu} - \partial_{\nu} g_{\mu \nu} ] T (v, p z)
\]

The definitions of the 2-particle distribution amplitudes of the photon read as [19]

\[
\langle 0 | \bar{q} (z) \sigma_{\alpha \beta} q (z) | \gamma^{(\lambda)} (q) \rangle = i e_q e_\gamma \langle q q \rangle \left( \gamma^{(\lambda)}_{\alpha} - \gamma^{(\lambda)}_{\beta} \right) \int_0^1 du e^{i \xi q z} \phi_{\gamma} (u, \mu)
\]

\[
\langle 0 | \bar{q} (z) \gamma_{\lambda} q (z) | \gamma^{(\lambda)} (q) \rangle = e_q e_{\lambda} \int_{-1}^1 d u e^{i \xi q z} \psi^{(\nu)} (u, \mu)
\]

The 3-particle distribution amplitudes of the photon are defined as [19]

\[
\langle 0 | \bar{q} (z) g G_{\mu \nu} (v z) \gamma_{\lambda} \sigma_{\alpha \beta} q (z) | \gamma^{(\lambda)} (q) \rangle = e_q e_{\gamma} \langle q q \rangle \left( \gamma^{(\lambda)}_{\alpha} - \gamma^{(\lambda)}_{\beta} \right) \int_0^1 du e^{i \xi q z} \psi^{(\nu)} (u, \mu)
\]

The definitions of the distribution amplitudes \( A, V \) and \( T \) are of twist-3 and the other 3-particle distribution amplitudes are of twist-4.

Appendix B: The definitions of \( F^{[n]} \) and \( F^{[m, n]} \)

The functions \( F^{[n]} \) and \( F^{[m, n]} \) used in the text are defined as

\[
F^{[n]} (\bar{u} \cdots \bar{v} x_{12} \cdots x_{n}) = \int_0^{\xi_0} \cdots \int_0^{\xi_2} \int_0^{\xi_3} F (x_1) dx_1 dx_2 \cdots dx_n,
\]

Here \( \phi_\gamma \) is the leading twist-2 distribution amplitude, \( \psi^{(\nu)} \), \( \psi^{(\nu)} \), \( A \), and \( V \) are of twist-3, \( h \), \( S \), \( S \), and \( T_{1, 2, 3, 4} \) are of twist-4.
\[
\begin{align*}
\mathcal{F}^{[0,0]}(u_0) & \equiv \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\alpha_3} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[0,1]}(u_0) & \equiv \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\alpha_3^2} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[1,0]}(u_0) & \equiv \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\alpha_3} \, d\alpha_3 \, d\alpha_2 - \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\alpha_3} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[1,1]}(u_0) & \equiv \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\alpha_3^2} \, d\alpha_3 \, d\alpha_2 - \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\alpha_3^2} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[2,0]}(u_0) & \equiv \frac{\mathcal{F}(0, u_0, 1-\alpha_0) + \mathcal{F}(1-\alpha_0 - \alpha_3, u_0, \alpha_3)}{1-\alpha_0} \bigg|_{\alpha_3=0} \\
& \quad + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_3} \bigg|_{\alpha_3=u_0-\alpha_2} \, d\alpha_3 \, d\alpha_2, \\
& \quad - \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_2} \bigg|_{\alpha_2=u_0} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[2,1]}(u_0) & \equiv \frac{\mathcal{F}(1-\alpha_0 - \alpha_3, u_0, \alpha_3)}{\alpha_3} \bigg|_{\alpha_3=0} + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_3} \bigg|_{\alpha_3=u_0-\alpha_2} \, d\alpha_3 \, d\alpha_2, \\
& \quad + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_2} \bigg|_{\alpha_2=u_0} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[3,0]}(u_0) & \equiv \frac{\partial \mathcal{F}(0, 1-\alpha_0, \alpha_3)}{\partial \alpha_0} \bigg|_{\alpha_0=1-\alpha_0} + \frac{\partial \mathcal{F}(u_0 - \alpha_2 - \alpha_3, 1-\alpha_0)}{\partial \alpha_0} \bigg|_{\alpha_2=0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \\
& \quad + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial^2 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0^2} \bigg|_{\alpha_0=1-\alpha_0} \, d\alpha_3 \, d\alpha_2, \\
& \quad - \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial^2 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0^2} \bigg|_{\alpha_0=1-\alpha_0} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{[3,1]}(u_0) & \equiv \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0} \bigg|_{\alpha_0=1-\alpha_0} + \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0} \bigg|_{\alpha_0=1-\alpha_0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \\
& \quad + \frac{\partial \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0} \bigg|_{\alpha_0=1-\alpha_0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \bigg|_{\alpha_2=u_0, \alpha_3=0} \\
& \quad + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial^2 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0^2} \bigg|_{\alpha_0=1-\alpha_0} \, d\alpha_3 \, d\alpha_2, \\
& \quad + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial^2 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0^2} \bigg|_{\alpha_0=1-\alpha_0} \, d\alpha_3 \, d\alpha_2, \\
& \quad - \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \frac{\partial^2 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3)}{\partial \alpha_0^2} \bigg|_{\alpha_0=1-\alpha_0} \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{-1,0}(u_0) & \equiv \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3) \, d\alpha_3 \, d\alpha_2, \\
& \quad + \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3) \, d\alpha_3 \, d\alpha_2, \\
\mathcal{F}^{-1,1}(u_0) & \equiv \frac{1}{2} \left[ \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3) \, d\alpha_3 \, d\alpha_2 ight. \\
& \quad + \frac{1}{2} \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \left( u_0 - \alpha_2 \right) \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3) \, d\alpha_3 \, d\alpha_2, \\
& \quad + \frac{1}{2} \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \left( u_0 - \alpha_2 \right)^2 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3) \, d\alpha_3 \, d\alpha_2, \\& 
& \quad \left. + \frac{1}{2} \int_0^{u_0} \int_{u_0-u_0}^{u_0-\alpha_2} \left( u_0 - \alpha_2 \right)^3 \mathcal{F}(1-\alpha_2 - \alpha_3, \alpha_2, \alpha_3) \, d\alpha_3 \, d\alpha_2 \right].
\end{align*}
\]
The single-pion decay widths can be expressed in terms of the 3-momentum $q$ of the final pion:

$$
\Gamma(M_1 \to H_0\pi^+) = \frac{m_{H_0}}{24\pi m_{M_1}} (g_{H_0H_0\pi}^1)^2 |q|^3, \\
\Gamma(M_1 \to H_1\pi^+) = \frac{m_{H_1}}{12\pi m_{M_1}} (g_{H_1H_1\pi}^1)^2 |q|^3, \\
\Gamma(M_1 \to S_1\pi^+) = \frac{m_{S_1}}{36\pi m_{M_1}} (g_{S_1S_1\pi}^2)^2 |q|^5 + \frac{1}{54\pi m_{M_1} m_{S_1}} (g_{S_1S_1\pi}^0)^2 |q|^7, \\
\Gamma(M_1 \to T_1\pi^+) = \frac{m_{T_1}}{8\pi m_{M_1}} (g_{T_1T_1\pi}^1)^2 |q|^3 + \frac{1}{24\pi m_{M_1} m_{T_1}} (g_{T_1T_1\pi}^0)^2 |q|^3 + \frac{1}{18\pi m_{M_1} m_{T_1}} (g_{T_1T_1\pi}^0)^2 |q|^5, \\
\Gamma(M_1 \to T_2\pi^+) = \frac{m_{T_2}}{6\pi m_{M_1}} (g_{T_2T_2\pi}^1)^2 |q|^5 + \frac{1}{54\pi m_{M_1} m_{T_2}} (g_{T_2T_2\pi}^0)^2 |q|^7, \\
\Gamma(M_2 \to H_1\pi^+) = \frac{m_{H_1}}{24\pi m_{M_2}} (g_{H_1H_1\pi}^1)^2 |q|^3 + \frac{1}{60\pi m_{M_2} m_{H_1}} (g_{H_1H_1\pi}^0)^2 |q|^5, \\
\Gamma(M_2 \to S_1\pi^+) = \frac{m_{S_1}}{40\pi m_{M_2}} (g_{S_1S_1\pi}^2)^2 |q|^5, \\
\Gamma(M_2 \to T_1\pi^+) = \frac{m_{T_1}}{10\pi m_{M_2}} (g_{T_1T_1\pi}^1)^2 |q|^5, \\
\Gamma(M_2 \to T_2\pi^+) = \frac{m_{T_2}}{2\pi m_{M_2}} (g_{T_2T_2\pi}^1)^2 |q|^5 + \frac{1}{45\pi m_{M_2} m_{T_2}} (g_{T_2T_2\pi}^0)^2 |q|^5, \\
\Gamma(M_2 \to T_2\pi^+) = \frac{7}{30\pi} m_{M_2} m_{T_2} |q|^5 + \frac{7}{80\pi m_{M_2} m_{T_2}} (g_{T_2T_2\pi}^0)^2 |q|^5 + \frac{2}{45\pi m_{M_2} m_{T_2}} (g_{T_2T_2\pi}^0)^2 |q|^5.
$$

The above formulas for the decays $M_{1,2}(1) \to H_{(0,1)} + \pi^+$ are also valid for the decays $M_{1,2}(1) \to H_{(0,1)} + K^+/\eta$, except for an extra $1/6$ isospin factor in the case of $\eta$ decays.
Similarly, the radiative decay widths read as

\[ \Gamma(M_1 \to H_0\gamma) = \frac{e^2 m_{H_0}}{12\pi m_{M_1}} (g_{M_1 H_0\gamma}^{m_1})^2 |q|^3, \]

\[ \Gamma(M_1 \to H_1\gamma) = \frac{e^2 m_{H_1}}{6\pi m_{M_1}} (g_{M_1 H_1\gamma}^{m_1})^2 |q|^3 + \frac{e^2}{12\pi m_{M_1} m_{H_1}} (g_{M_1 H_1\gamma}^{m_1})^2 |q|^5 + \frac{e^2}{12\pi m_{M_1} m_{H_1}} g_{M_1 H_1\gamma}^{m_1} g_{M_1 H_1\gamma}^{e_1} |q|^7, \]

\[ \Gamma(M_1 \to S_0\gamma) = \frac{e^2 m_{S_0}}{12\pi m_{M_1}} (g_{M_1 S_0\gamma}^{m_1})^2 |q|^5, \]

\[ \Gamma(M_1 \to S_1\gamma) = \frac{e^2 m_{S_1}}{6\pi m_{M_1}} (g_{M_1 S_1\gamma}^{m_1})^2 |q|^5 + \frac{e^2}{6\pi m_{M_1}} (g_{M_1 S_1\gamma}^{m_1})^2 |q|^7 + \frac{e^2}{12\pi m_{M_1} m_{S_1}} (g_{M_1 S_1\gamma}^{m_1})^2 |q|^9, \]

\[ \Gamma(M_2 \to H_0\gamma) = \frac{e^2 m_{H_0}}{40\pi m_{M_2}} (g_{M_2 H_0\gamma}^{e_1})^2 |q|^7, \]

\[ \Gamma(M_2 \to H_1\gamma) = \frac{e^2 m_{H_1}}{3\pi m_{M_2}} (g_{M_2 H_1\gamma}^{e_1})^2 |q|^3 + \frac{e^2}{10\pi m_{M_2} m_{H_1}} (g_{M_2 H_1\gamma}^{e_1})^2 |q|^5 + \frac{e^2}{10\pi m_{M_2} m_{H_1}} g_{M_2 H_1\gamma}^{e_1} g_{M_2 H_1\gamma}^{e_2} |q|^7, \]

\[ \Gamma(M_2 \to S_0\gamma) = \frac{e^2 m_{S_0}}{10\pi m_{M_2}} (g_{M_2 S_0\gamma}^{m_1})^2 |q|^5, \]

\[ \Gamma(M_2 \to S_1\gamma) = \frac{e^2 m_{S_1}}{3\pi m_{M_2}} (g_{M_2 S_1\gamma}^{m_1})^2 |q|^5 + \frac{3e^2 m_{S_1}}{5\pi m_{M_2}} (g_{M_2 S_1\gamma}^{m_1})^2 |q|^7 + \frac{e^2}{10\pi m_{M_2} m_{S_1}} (g_{M_2 S_1\gamma}^{m_1})^2 |q|^9, \]

\[ -\frac{e^2}{5\pi m_{M_2} m_{S_1}} (g_{M_2 S_1\gamma}^{m_1})^2 |q|^7 + \frac{e^2}{10\pi m_{M_2} m_{S_1}} (g_{M_2 S_1\gamma}^{m_1})^2 |q|^9. \] (C2)