A SPECIALIZATION INEQUALITY FOR TROPICAL COMPLEXES

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Abstract. We prove a specialization inequality relating the dimension of the complete linear series on a variety to the tropical complex of a regular semistable degeneration. Our result extends Baker’s specialization inequality to arbitrary dimension.

1. Introduction

The specialization inequality for curves [Bak08] gives a bound for the dimension of a linear system on a curve in terms of an analogous combinatorial invariant on the dual graph of a degeneration. For higher-dimensional varieties, understanding linear equivalence on the dual complex of a semistable degeneration requires additional information beyond the dual complex, which can be encoded in a tropical complex, as introduced in [Car13]. In this paper, we generalize the specialization inequality to varieties of arbitrary dimension using tropical complexes.

Similar to the case of curves, our specialization inequality applies to a regular, strictly semistable degeneration $\mathcal{X}$ over a discrete valuation ring, meaning that the special fiber of $\mathcal{X}$ is a reduced union of smooth varieties, with simple normal crossings. From $\mathcal{X}$, we can construct the dual complex $\Delta$, which is a regular $\Delta$-complex recording how the components of the special fiber intersect. In addition, the tropical complex $\Delta$ records certain intersection numbers from $\mathcal{X}$, the details of which will be recalled in Section 2. In addition, [Car13] introduced both a specialization map $\rho$ from divisors on the general fiber of $\mathcal{X}$ to the tropical complex $\Delta$ as well as a compatible notion of linear equivalence for divisors on $\Delta$. We define $h^0$ of a divisor on $\Delta$ as the fewest number of rational points such that no linearly effective divisor contains all of these points (see Def. 3.1 for details), and then prove a specialization inequality:

Theorem 1.1. Let $\mathcal{X}$ be a regular strictly semistable degeneration of relative dimension $n$ over a discrete valuation ring. Suppose that the locally closed strata of dimension at most $n - 2$ in $\mathcal{X}$ are affine and that $\mathcal{X}$ is robust in dimensions $n - 1$ and $n$. If $D$ is any divisor on the general fiber $X$ of $\mathcal{X}$, and $\Delta$ is the tropical complex of $\mathcal{X}$, then we have the inequality:

$$\dim H^0(X, O(D)) \leq h^0(\Delta, \rho(D)).$$
The second sentence of Theorem 1.1 places additional requirements on the degeneration beyond the strict semistability, whose definitions we now explain. A closed stratum of dimension \( n - k \) in \( \mathcal{X} \) is a connected component of the intersection of any \( k + 1 \) components of the special fiber. Each closed stratum contains a corresponding locally closed stratum, which is formed by removing all lower-dimensional closed strata, and so Theorem 1.1 requires these differences to be affine varieties, when their dimension is \( n - 2 \) or less. For the closed strata of dimension \( n - 1 \) and \( n \), Theorem 1.1 puts a weaker condition of robustness, which means that the union of the lower dimensional strata form a big divisor, in the sense of birational geometry. See Definition 2.2 for details on both of these definitions.

Although Theorem 1.1 does not apply directly to all regular semistable degenerations because of these hypotheses, it does apply after first modifying by a sequence of blow-ups, at least for projective degenerations:

**Proposition 1.2.** Suppose that \( \mathcal{X} \) is regular, strictly semistable degeneration, and all the components of the special fiber \( \mathcal{X}_0 \) are projective. Then, there exists a series of blow-ups, with centers contained in the special fiber, which results in a regular, strictly semistable degeneration \( \mathcal{X}' \) whose locally closed strata are all affine.

We now outline the structure of the proof of Theorem 1.1, which helps to explain the necessity of the robustness and affine hypotheses. Following the proof of the specialization inequality for curves, our specialization inequality essentially follows from our definition of \( h^0 \) given a specialization map which preserves linear equivalence, effectivity of divisors, and point containment. Preservation of linear equivalence was proved in [Car13]. However, for an effective divisor \( D \), the specialization, \( \rho(D) \), is not always effective, so we introduce a refined specialization in Section 5 which is effective and is linearly equivalent to \( \rho(D) \). Note that, unlike \( \rho(D) \), the refined specialization is not necessarily supported on the \((n - 1)\)-dimensional simplices of \( \Delta \), and so our specialization theorem necessarily requires a framework that includes divisors which intersect the interior of the \( n \)-dimensional simplices, as explained in Section 2.

Although our proof does not use this technology explicitly, the refined specialization of a divisor \( D \), and its relationship to point containment, can be understood in terms of the projection of \( D \) to the skeleton of the Berkovich analytification. More specifically, by results going back to Berkovich [Ber90], the dual complex \( \Delta \) embeds into, and is a strong deformation retract of, the analytification of the general fiber of \( \mathcal{X} \). Restricting to effective divisors in order to avoid cancellation, we have:

**Proposition 1.3.** If \( D \) is an effective divisor in the general fiber of \( \mathcal{X} \), then the projection of the analytification of \( D \) to the skeleton defined by \( \mathcal{X} \) is the union of the refined specialization of \( D \) together with a finite set of polyhedra, all of dimensions at most \( n - 2 \).
While the projection of the analytification always preserves point containment, Proposition 1.3 shows that the same can fail for the refined specialization whenever the projection has components of codimension 2 or greater. The purpose of the robustness and affine hypotheses in Theorem 1.1 is to guarantee that we can lift points from the tropical complex to the algebraic variety in such a way that containing divisors project to a set of codimension 1 on $\Delta$. In the case of curves, the set of codimension 2 in Proposition 1.3 is necessarily empty, which is why the specialization inequality for curves required no hypotheses beyond a strictly semistable degeneration.

As an application of Theorem 1.1, we give an example of a tropical complex which does not lift to any algebraic variety.

**Theorem 1.4.** There exists a 2-dimensional tropical complex $\Delta$ which is not the tropical complex of any regular semistable degeneration.

The underlying $\Delta$-complex of the example in Theorem 1.4 is a triangulation of the product of a cycle and an interval. Therefore, the dual complex is realizable from a degeneration of an algebraic surface, such as the product of an elliptic curve with a projective line. However, in Theorem 1.4 the structure constants of the tropical complex $\Delta$ are chosen so that for any divisor $D$, $h^0(\Delta, nD)$ grows at most linearly in $n$, which combined with the existence of an ample divisor on any smooth, proper surface, shows that $\Delta$ can't be the tropical complex of a projective surface.

An alternative approach to a specialization inequality has been developed in unpublished work of Eric Katz and June Huh. They work not with degenerations, but with tropicalizations of surfaces embedded in $\mathbb{G}_m^N$ over a field with trivial valuation. Moreover, their proof involves choosing linearly equivalent divisors passing through the lowest dimensional toric strata, as opposed to the proof of Theorem 1.1 in which points are lifted from the $n$-dimensional locally closed strata. Therefore, the two approaches should give distinct and possibly complementary bounds on the dimension of linear series.

The rest of this paper is organized as follows. Section 2 recalls the definition of tropical complexes from [Car13] as well as their main properties. In Section 3 we define the invariant $h^0$ from Theorem 1.1 and look at some examples and applications. Section 4 proves compatibility of tropical complexes under ramified base changes followed by toroidal resolutions of singularities, which are the tools used to give a refined specialization in Section 5. Section 6 has the proof of the specialization inequality.

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2. Tropical complexes

Let $X$ be a regular, strictly semistable degeneration over a discrete valuation ring $R$. Specifically, we mean that $X$ is a regular scheme, flat and proper over $\text{Spec } R$, and such that the fiber over the closed point of $\text{Spec } R$ is a reduced simple normal crossing divisor. The closed fiber is called the \textit{special fiber} and denoted $X_0$. We also assume that the residue field of $R$ is algebraically closed. Under these assumptions, we will call $X$ a \textit{degeneration} and we let $n$ denote the dimension of the general fiber of $X$, which is also the dimension of each component of the special fiber. The \textit{general fiber} of $X$, denoted $X$, is the fiber over the generic point of $\text{Spec } R$.

We first recall the construction of the dual complex of $X$. The dual complex is a regular $\Delta$-complex (in the sense of \cite[Sec. 2.1]{Hat02}) that consists of a vertex for each irreducible component of the special fiber $X_0$. In addition, for each set of $k + 1$ of the irreducible components of $X_0$, their intersection is a disjoint union of smooth $(n - k)$-dimensional varieties, and we have a $k$-dimensional simplex for each of these components. We write $C_s$ for the smooth variety corresponding to a $(k + 1)$-dimensional variety, and we have that the simplices are attached such that $C_{s'}$ is a subvariety of $C_s$ if and only if $s'$ is contained in $s$. See the beginning of Section 2 of \cite{Car13} for full details.

The \textit{weak tropical complex} of $X$ is the pair of the dual complex and a function $\alpha$ from the pairs of a vertex $v$ and a ridge $r$ which contains $v$. Here, by \textit{ridge}, we mean an $(n - 1)$-dimensional simplex of the dual complex of $X$. Thus, the corresponding component $C_r$ is a curve which is contained in $C_{s'}$, which is a component of $X_0$ and a divisor on $X$. We define $\alpha(v, r) = -\deg C_v \cdot C_r$, where $\deg C_v \cdot C_r$ denotes the degree of the intersection product on $X$. Note that the intersection is necessarily non-transverse, because $v$ is a vertex of $r$. In the transverse case, the intersection number would be redundant because it number of points of $C_v \cap C_r$, which are in bijection with the $n$-dimensional simplices of the dual complex containing both $v$ and $r$. The weak tropical complex of a degeneration satisfies the following definition by \cite[Prop. 2.6]{Car13}.

\begin{definition}[Def. 2.5 in \cite{Car13}]
An \textit{$n$-dimensional weak tropical complex} $\Delta$ is a pair $(S, \alpha)$, where $S$ a finite, connected, regular $\Delta$-complex, whose simplices all have dimension at most $n$, and $\alpha$ is a function from pairs of a vertex $v$ of $S$ and a ridge $r$ containing $v$ such that for every ridge $r$,
\begin{equation}
\sum_{v \in r_0} \alpha(v, r) = \deg r,
\end{equation}
where $r_0$ denotes the vertices of $r$ and $\deg r$ is the number of $n$-dimensional simplices containing $r$. The values $\alpha(v, r)$ are sometimes referred to as the
\end{definition}
structure constants of the weak tropical complex. We refer to the 0-, 1-, (n − 1)-, and n-dimensional simplices of S as the vertices, edges, ridges, and facets, respectively, of ∆.

Definition 2.1 is more specialized than the one in [Car13] in that we assume that the underlying ∆-complex is regular, meaning that distinct faces of a single simplex are not identified with each other. The dual complex of a simple normal crossing divisor, and thus of a degeneration, is always a regular ∆-complex, and thus regular ∆-complexes are sufficient for the weak tropical complexes which appear in this paper. Assuming that the ∆-complex is regular simplifies the exposition at several points.

We now recall the definitions of locally closed strata and robustness, which appear in the statement of Theorem 1.1. Recall that a divisor D on a smooth variety is called big if the complete linear series on some multiple of D defines a birational map onto its image in projective space [Laz04, Sec. 2.2].

Definition 2.2. Let X be a degeneration and let ∆ be its weak tropical complex. For any (n − k)-dimensional simplex s of ∆, let D_s denote the union \( \bigcup \ s' \), where s' ranges over (n − k + 1)-dimensional simplices containing s'. We define \( C_s \setminus D_s \) to be the locally closed stratum of dimension k, corresponding to s. We also consider D_s as a reduced divisor on C_s, and we say that X is robust in dimension k if, for any (n − k)-dimensional simplex s, D_s is a big divisor on C_s.

Proposition 2.3. Let X be a degeneration, and ∆ its dual complex. Then, the following are equivalent:

1. X is robust in dimension 1.
2. The locally closed strata of dimension 1 in X are affine.
3. Every (n − 1)-dimensional simplex of ∆ is contained in some n-dimensional simplex.

Proof. Let r be an (n − 1)-dimensional simplex of ∆, and thus a ridge. Then, C_r is a curve, and D_r is the union of points corresponding to facets containing r, and thus is non-trivial if and only if there are any such facets. If the divisor D_r is non-empty, then it is ample, and so it is big and also C_r \ D_r is affine. On the other hand, if D_r is the trivial divisor, then it is clearly not big, and C_r \ D_r is a projective curve and so not affine. \( \square \)

For k ≥ 2, if the k-dimensional locally closed strata are affine, then X is robust in dimension k, but not conversely, and neither condition is determined by the dual complex. However, robustness in dimension 2 is determined by its weak tropical complex, for which we need the following definition.

Definition 2.4 (Def. 2.7 in [Car13]). If q is an (n − 2)-dimensional simplex, the local intersection matrix of ∆ at q is the symmetric matrix \( M_q \) whose rows and columns are indexed by the ridges r containing q, and such that
the entry in row \( r \) and column \( r' \) is:

\[
(M_q)_{r,r'} = \begin{cases} 
\# \{ \text{facets containing } r \text{ and } r' \} & \text{if } r \neq r' \\
-\alpha(v,r) & \text{if } r = r',
\end{cases}
\]

where \( v \) refers to the vertex of \( r \) not contained in \( q \). A tropical complex is a weak tropical complex \( \Delta \) such that the local intersection matrix \( M_q \) has exactly one positive eigenvalue for each \((n-2)\)-dimensional simplex \( q \) of \( \Delta \).

**Proposition 2.5** (Prop. 2.9 in [Car13]). Let \( \mathcal{X} \) be a degeneration and \( \Delta \) its weak tropical complex. Then \( \mathcal{X} \) is robust in dimension 2 if and only if \( \Delta \) is a tropical complex.

We now prove Proposition 1.2, showing that any projective degeneration can be modified to one whose locally closed strata are affine, and thus is robust in all dimensions.

**Proof of Prop. 1.2.** We fix a component \( C_v \) of the special fiber \( \mathcal{X}_0 \). By assumption, \( C_v \) is projective, so, by Bertini’s theorem, we can choose smooth and irreducible elements \( H_1, \ldots, H_n \) from the linear system of a very ample divisor on \( \mathcal{X}_0 \), such that the \( H_i \) intersect each other and \( D_v \) transversely.

We now blow-up the points of the intersection \( H_1 \cap \cdots \cap H_n \). Then, we blow-up, for each integer \( k \) from 1 to \( n-1 \), the strict transforms of the \( k \)-dimensional varieties in \( H_1 \cap \cdots \cap H_{n-k} \) for all indices \( 1 \leq i_1 < \cdots < i_{n-k} \leq n \). For each \( k \), these strict transforms are disjoint and thus the order of the blow-ups within a fixed dimension doesn’t matter.

We then repeat the above blow-ups at the strict transform of each component of \( \mathcal{X} \) to obtain the degeneration \( \mathcal{X}' \) from the statement. Since \( \mathcal{X}' \) is the iterated blow-up of a regular degeneration at smooth subvarieties, it is also regular. The centers of the blow-ups intersect the singular locus of the special fiber transversally, so the special fiber \( \mathcal{X}'_0 \) is also reduced and simple normal crossing. Thus, \( \mathcal{X}' \) is a strict degeneration.

In order to show that the locally closed strata of \( \mathcal{X}' \) are all affine, we first consider the case of the case of a stratum \( C'_s \) of \( \mathcal{X}'_0 \) which maps birationally onto its image in \( \mathcal{X} \). We let \( C_s \) be its image in \( \mathcal{X} \), and then \( C'_s \) is formed by blowing up \( C_s \) at the restrictions of intersections of very ample divisors from a component \( C_v \) for each vertex \( v \) in \( C_s \). Each of these blow-ups produces a new component for \( \mathcal{X}' \) and thus the difference \( C'_s \setminus D'_s \) is an open subset of \( C_s \) minus the very ample divisors. This containment may be proper because of the intersection of \( C_s \) with other components of \( \mathcal{X}_0 \). However, the complement of a very ample divisor is affine, and the complement of a Cartier divisor in an affine variety is affine, so \( C'_s \setminus D'_s \) is affine, as desired.

Now, we consider the components introduced by the blow-ups. The center for such a blow-up is a variety \( Y \) within a single component \( C_v \). Since we’ve blown up the intersection of \( Y \) with very ample divisors in the previous step, the complement \( Y \setminus (Y \cap D_v) \) is affine, as in the previous paragraph. The blow-up of \( Y \) is a projective bundle over \( Y \) which intersects \( C_v \) along a section.
Let $C_w$ denote this blow-up and then $C_w \setminus D_w$ is an affine space bundle over an affine variety $Y \setminus (Y \cap D_v)$, and thus it is affine. Further blow-ups only intersect $C_w$ along $D_w$ and therefore do not affect the difference $C_w \setminus D_w$, which remains affine in $X'$. We conclude that the locally closed strata of $X'$ are all affine. □

We now come to divisors on $X$ and their relationship to the combinatorics of the weak tropical complex $\Delta$. Let $D$ be any divisor on $X$, and let $\overline{D}$ denote its closure in $X$. Since $X$ is regular, $\overline{D}$ is a Cartier divisor on $X$, and we can define the following formal sum of ridges of $\Delta$:

$$\rho(D) = \sum_{r \in \Delta_{n-1}} (\deg \overline{D} \cdot C_r)[r],$$

which we call the coarse specialization of $D$. In addition, there is a refined specialization, which will be construction in Section 4. We now describe the machinery of linear and piecewise linear functions on $\Delta$, which will be used to define divisors and linear equivalence.

We first need to generalize beyond just formal sums of the ridges to formal sums of $(n-1)$-dimensional polyhedra, possibly contained in the interior of a facet. To be precise, we first identify a single facet $f$ of $\Delta$ with the standard simplex in $\mathbb{R}^n$, which is the convex hull of the origin together with the $n$ coordinate vectors. This identification is not unique, but a permutation of the vertices induces an affine linear automorphism of $f$, with integral coefficients, and so it does not affect the places where these coordinates are used. We define an $(n-1)$-dimensional polyhedron in $\Delta$ to be an $(n-1)$-dimensional polyhedron of a single facet $f$ of $\Delta$, which is defined by linear inequalities with rational coefficients and real constants, under the above identification.

By a formal sum of $(n-1)$-dimensional polyhedra, we will mean a finite, integral sum of $(n-1)$-dimensional polyhedra. From now on, we will drop the dimension from our terminology, because the only formal sums of polyhedra in this paper will be $(n-1)$-dimensional. Two formal sums of polyhedra are equivalent if they differ by an element of the subgroup generated by the set of formal sums $[P] - [Q_1] - \cdots - [Q_r]$, where $P$ is any $(n-1)$-dimensional polyhedron, and $Q_1, \cdots, Q_r$ are polyhedra such that $P = Q_1 \cup \cdots \cup Q_r$ and $Q_i \cap Q_j$ is either empty or a proper face of both $Q_i$ and $Q_j$.

One can show that any formal sum of polyhedra is equivalent to one where the terms form the maximal cells of a polyhedral complex. While we will not use that result in this paper, it in particular means that, up to equivalence, we can assume that the intersection between any two terms in a formal sum of polyhedra is a face, and thus has dimension less than $n-1$, as in part (1) of the following lemma.

**Lemma 2.6.** Let $Z = \sum a_i [T_i]$ and $Z' = \sum a'_i [T'_i]$ be formal sums of polyhedra which are equivalent to each other on a weak tropical complex $\Delta$. 
(1) If the intersection $T_i \cap T_j$ of any two terms of $Z$ has dimension less than $n - 1$, and the coefficients $a'_i$ of $Z'$ are positive, then the coefficients $a_i$ of $Z$ are non-negative.

(2) If both sets of coefficients $a_i$ and $a'_i$ are positive, then $\bigcup P_i = \bigcup P'_i$.

Proof. We first assume that the any two terms of $Z$ intersect in dimension less than $n - 1$ and that the coefficients of $Z'$ are positive. Let $p$ be a point of an arbitrary term $T_j$ of $Z$. By definition, difference $Z - Z'$ is a finite sum of expressions of the form $[P] - [Q_1] - \cdots - [Q_r]$, where $P = Q_1 \cup \ldots \cup Q_r$, and the intersections $Q_i \cap Q_j$ are proper faces of each. We can assume that $p$ is not contained in any facet of the polyhedra appearing in these expressions or facet of the $T_i$. Therefore, if $p$ is in the polyhedron $P$ of one of these expressions, then $p$ is in exactly one of the corresponding $Q_i$'s. Therefore, equivalence between $Z$ and $Z'$ doesn’t change the sum of the coefficients of the polyhedra containing $p$, meaning that:

\[
\sum_{P_i \ni p} a_i = \sum_{P'_i \ni p} a'_i.
\]

Since the coefficients $a'_i$ are positive, the right hand side of (3) is non-negative. By our assumption on $Z$, the left hand side consists of just the coefficient $a_j$, which is therefore non-negative.

Second, we assume that the terms of $Z$ and $Z'$ are arbitrary, but the coefficients $a_i$ and $a'_i$ are all positive. Again, let $p$ be a point in one term $T_j$ of $Z$, such that $p$ is not contained in any of the facets of the polyhedra appearing in the equivalence between $Z$ and $Z'$, and we again have the equality (3). The left hand side of (3) is positive since $p \in T_j$, so $p$ must also be contained in some term $T'_k$ of $Z'$. Since we were able to choose $p$ from a dense open subset of $\bigcup P_i$, and the union $\bigcup P'_i$ is closed, we have an inclusion $\bigcup P_i \subset \bigcup P'_i$. By symmetry, we have the reverse inclusion, and thus the desired equality. □

Definition 2.7. An equivalence class of formal sums of polyhedra is called effective if it contains a formal sum with all coefficients positive.

The purpose of Lemma 2.6(1) is that it gives a criterion to check whether or not a give formal sum of polyhedra is effective, by using a representative such that the intersections between the terms all have dimension less than $n - 1$. Lemma 2.6(2) shows that the following definition is independent of the representative chosen.

Definition 2.8. The support of an effective equivalence class of formal sums of polyhedra is the union of polyhedra in a representative with positive coefficients.

For example, if we take $P$ to be an $(n - 1)$-dimensional polyhedron, and $Q$ any $(n - 1)$-dimensional polyhedron properly contained in $P$, then $[P] - [Q]$ is effective, even though it is not written with positive coefficients, and its support is $P \setminus Q^o$, where $Q^o$ denotes the relative interior of $Q$. We can find a
subdivision \( P = Q \cup Q_2 \cup \cdots \cup Q_r \), as in the equivalence relation, and then \([P] - [Q]\) is equivalent to \([Q_2] + \cdots + [Q_r]\), which has positive coefficients, thus showing that \([P] - [Q]\) is effective, and its support is \( P \setminus Q \), as claimed.

We now define piecewise linear and linear functions. First, these will allow us to define divisors as equivalence classes of formal sums of polyhedra satisfying a balancing condition. Second, piecewise linear functions will be used to define linear equivalence between divisors, and thus complete linear series.

**Definition 2.9** (Def. 3.1 in [Car13]). A piecewise linear function or \( PL \) function on a weak tropical complex \( \Delta \) is a continuous function \( \phi \) such that, restricted to each simplex of \( \Delta \), \( \phi \) is a piecewise linear function with integral slopes, under the identification of the simplex with a standard simplex in \( \mathbb{R}^k \).

**Definition 2.10** (Const. 3.2 in [Car13]). Let \( \Delta \) be an \( n \)-dimensional weak tropical complex. For any ridge \( r \) of \( \Delta \), we define \( N_r \) to be the simplicial complex which consists of a central \((n - 1)\)-dimensional simplex, together with one \( n \)-dimensional simplex for each facet containing \( r \), and we let \( d \) denote the number of such facets. Thus, if \( N_0 \) is the union of the interiors of the \( n \)-dimensional simplices and the central \((n - 1)\)-dimensional simplices, then \( N_0 \) is homeomorphic to a neighborhood of the interior of \( r \).

We can embed \( N_r \) in the real vector space \( \mathbb{R}^{d+n} \) by sending \( w_i \) to the \( i \)-th coordinate vector of \( \mathbb{R}^{d+n} \) and \( v_i \) to the \((d + i)\)-th coordinate vector, where \( v_1, \ldots, v_n \) are the vertices of the central \((n - 1)\)-dimensional simplex, and \( w_1, \ldots, w_d \) are the other vertices of \( N_r \). Then, we define \( L_r \) to be the quotient of \( \mathbb{R}^{d+n} \) by the one-dimensional vector space generated by \((1, \ldots, 1, -\alpha(v_1, r), \ldots, -\alpha(v_n, r))\). We define \( \phi_r : N_r \to L_r \) to be the composition of the inclusion followed by the projection map.

A linear function \( \phi \) on an open subset \( U \) of a weak tropical complex \( \Delta \) is a PL function such that:

1. For any facet \( f \) of \( \Delta \), the restriction \( \phi|_{U \cap f} \) is the restriction of an affine linear function on \( \mathbb{R}^d \), under the identification of \( f \) with the standard unit simplex in \( \mathbb{R}^d \).
2. For any ridge \( r \) of \( \Delta \), the restriction \( \phi|_{U \cap N_0} \) is a composition \( \ell \circ \phi_r \), where \( \ell : L_r \to \mathbb{R} \) is a linear function.

**Definition 2.11** (Const. 3.5 and Def. 3.7 in [Car13]). Let \( Z \) be a formal sum of \((n - 1)\)-dimensional polyhedra. Let \( Q \) be an \((n - 2)\)-dimensional face of a term of \( Z \) and assume that every for term \( P_i \) of \( Z \) which intersects \( Z \) contains it as a face. Now let \( \phi \) be a linear function on a neighborhood of the interior of \( Q \), which is constant on \( Q \).

For each term \( P_i \), use the coordinates in \( \mathbb{R}^n \) coming from the identification of the facet \( f \subset \Delta \) containing \( P_i \) with the standard unit simplex. We let \( w_i \in \mathbb{Z}^n \) be a vector such that \( w_i \cdot x \) is constant for all \( x \) in \( P_i \). We assume that the entries of \( w_i \) are relatively prime, which uniquely determines \( w_i \) up to a sign. Then let \( v_i \in \mathbb{Z} \) be the normal vector of a supporting hyperplane.
of \(Q\), meaning that for some constant \(c_i \in \mathbb{R}\), \(v_i \cdot x \geq c_i\) for all \(x \in P_i\), and \(Q\) is the set of \(x\) such that \(v_i \cdot x = c_i\). In addition, we can assume that for each integer \(k\), the entries of \(v_i + kw_i\) are relatively prime, which determines \(v_i\) up to a multiple of \(w_i\). Since \(\phi\) is constant on \(Q\), \(\phi|_{P_i}\) can be written as \(\phi|_{P_i}(x) = a_i v_i \cdot x + t\), with \(t \in \mathbb{R}\) and \(a_i\) an integer by the assumption on \(v_i\) and the fact that \(\phi\) has linear slopes. We refer \(a_i\) as the slope of \(\phi\) along \(P_i\).

If \(m_i\) is the multiplicity of \(P_i\) in \(\mathbb{Z}\), then we define the multiplicity of \(\phi\) along \(Q\) of \(\mathbb{Z}\) to be:

\[
\text{mult}_{\mathbb{Z},Q}(\phi) = \sum a_i m_i.
\]

We say that a formal sum of polyhedra \(Z\) is balanced if each \((n-2)\)-dimensional face \(Q\) of a term of \(Z\) is a face of all the terms which contain it, and \(\text{mult}_{\mathbb{Z},Q}(\phi) = 0\) for any linear function \(\phi\) on a neighborhood of the interior of any \((n-2)\)-dimensional face \(Q\) of a term of \(Z\), which is constant on \(Q\).

**Lemma 2.12** (Lem. 3.8 in [Car13]). Let \(Z\) and \(Z'\) be equivalent formal sums of \((n-1)\)-dimensional polyhedra such that any \((n-2)\)-dimensional face of a term of \(Z\) is a face of all terms which contain it, and similarly with \(Z'\). Then, \(Z\) is balanced if and only if \(Z'\) is balanced.

**Definition 2.13** (Def. 3.9 in [Car13]). A (Weil) divisor on a weak tropical complex is an equivalence class of formal sums of \((n-1)\)-dimensional polyhedra such that some representative is balanced.

In [Car13], Cartier divisors on weak tropical complexes are also defined, which are the appropriate tool for intersection theory. With a exception of the Riemann-Roch conjecture at the end of Section 3, only Weil divisors are relevant in this paper, which we usually refer to as divisors. In particular, divisors arise from the specialization of divisors:

**Proposition 2.14** (Prop. 3.16 in [Car13]). Let \(X\) be a degeneration which is robust in dimension 2 and let \(\Delta\) be its tropical complex. If \(D\) is a divisor on \(X\), then \(\rho(D)\) is a divisor on \(\Delta\).

Linear equivalence between divisors on a weak tropical complex is analogous to linear equivalence on an algebraic variety, with PL functions taking the place of rational functions. The following proposition characterizes the divisor associated to a PL function, which closely links them to the linear functions from Definition 2.10.

**Proposition 2.15** (Prop. 4.1 in [Car13]). Let \(\Delta\) be a weak tropical complex and assume that every ridge is contained in a facet. There is a unique function \(\text{div}\) from PL functions on an open subset \(U \subset \Delta\) to equivalence classes of formal sums of polyhedra on \(U\) such that:

(i) For any PL functions \(\phi\) and \(\phi'\) on \(U\), \(\text{div}(\phi + \phi') = \text{div}(\phi) + \text{div}(\phi')\).

(ii) If \(V \subset U\) are open sets, and \(\phi\) is a PL function on \(U\), then \(\text{div}(\phi|_V) = \text{div}(\phi)|_V\).
(iii) A function $\phi$ is linear if and only if $\text{div}(\phi)$ is trivial.
(iv) If $\phi$ is identically zero outside of a single facet $f$ of $\Delta$, and $\phi$ is defined as:
$$\phi(x) = \max\{\lambda \cdot x, 0\},$$
where $x$ is a coordinate vector, using the identification of $f$ with a standard unit simplex in $\mathbb{R}^n$, and $\lambda$ is an integral vector whose entries have no common divisor, then $\text{div}(\phi) = [H]$, where $H = \{x \in f \cap U \mid \lambda \cdot x = 0\}$, using the same coordinates.

In [Car13], Proposition 4.1 also has an additional normalization assumption in order to apply to ridges that aren’t contained in a facet. However, the degenerations of interest in this paper are robust in dimension 1, and thus, by Proposition 2.3, every ridge is contained in a facet.

**Definition 2.16** (Def. 5.1 in [Car13]). Two divisors $D$ and $D'$ on a weak tropical complex $\Delta$ are linearly equivalent if $D - D' = \text{div}(\phi)$ for some PL function $\phi$ on $\Delta$.

**Proposition 2.17** (Prop. 5.2 in [Car13]). Let $X$ be a degeneration which is robust in dimension 2, with $\Delta$ as its tropical complex. If $D$ and $D'$ are linearly equivalent divisors on $X$, then $\rho(D)$ is linearly equivalent to $\rho(D')$.

3. **Linear series**

A complete linear series of a divisor $D$ is the set of all effective divisors linearly equivalent to $D$. The invariant $h^0(\Delta, D)$ appearing in Theorem 1.1 is a measure of how large the complete linear series of $D$ is, analogous to the dimension of the global sections of $\mathcal{O}(D)$ on an algebraic variety. In this section, we define this invariant and give some examples and applications.

**Definition 3.1.** We say that a point $p \in \Delta$ is rational if its coordinates are rational when we identify a $k$-simplex containing $p$ with a standard unit simplex in $\mathbb{R}^k$. We define $h^0(\Delta, D)$ to be the cardinality $m$ of the smallest set of rational points $p_1, \ldots, p_m$ such that there is no effective divisor $D'$ linearly equivalent to $D$ such that $D'$ contains $p_1, \ldots, p_m$. If there is no such finite set of points, then $h^0(\Delta, D)$ is defined to be $\infty$.

The rationality condition on the points in Definition 3.1 is needed for technical reasons in the proof of Theorem 1.1 and we expect that it can be dropped without changing the definition.

**Conjecture 3.2.** The definition of $h^0$ is equivalent to the analogous definition where the $p_i$ are allowed to be arbitrary, not necessarily rational, points.

In the case of a 1-dimensional tropical complex $\Gamma$, the quantity $h^0(\Gamma, D)$ is essentially the same as the rank of the divisor as introduced by Baker and Norine [BN07], and extended to metric graphs in [GK08] and [MZ08], with the exception that our convention differs from theirs by 1. While our
Figure 1. The tropical complex $\Delta$ used in Theorem 1.4 is formed by identifying the edges on the left and right from the triangulation on the left to form the cylinder shown on the right. For the edges forming the top and bottom circles, the structure constants are indicated by the numbers adjacent to each endpoint. The other edges all have degree 2 and all structure constants on these edges are taken to be 1.

The tropical complex $\Delta$ used in Theorem 1.4 is formed by identifying the edges on the left and right from the triangulation on the left to form the cylinder shown on the right. For the edges forming the top and bottom circles, the structure constants are indicated by the numbers adjacent to each endpoint. The other edges all have degree 2 and all structure constants on these edges are taken to be 1.

Definition requires the points to be distinct and rational, these restrictions don’t affect the definition, by the following case of Conjecture 3.2:

**Proposition 3.3.** If $\Gamma$ is a 1-dimensional tropical complex with at least one edge and $r(D)$ is the rank of a divisor $D$ as in [BN07], then $h^0(D) = r(D) + 1$.

**Proof.** Recall that the rank of $D$ is the largest number $r$ such that for any $r$ points $p_1, \ldots, p_r$ in $\Gamma$, the difference $D - [p_1] - \ldots - [p_r]$ is linearly equivalent to an effective divisor $D'$. Thus, $D' + [p_1] + \ldots + [p_r]$ is an effective divisor linearly equivalent to $D$ and it clearly contains the $p_i$, so $h^0(D) \geq r(D) + 1$. To show the reverse inequality, we assume that there exist points $p_1, \ldots, p_{r(D)+1}$ such that $D$ is not linearly equivalent to $D' + [p_1] + \ldots + [p_{r(D)+1}]$ for any effective divisor $D'$. In other words, if we write $|D|$ for the subset of $\Gamma^d$ which are effective divisors linearly equivalent to $D$, then $|D|$ does not intersect the subset

$$\{p_1\} \times \cdots \times \{p_{r(D)+1}\} \times \Gamma \times \cdots \times \Gamma \subset \Gamma^d.$$  

However, $|D|$ is a closed subset [MZ08, Thm. 6.2]. Therefore, since $\Gamma$ is not a point, we can perturb the points $p_i$ slightly and the intersection with $|D|$ will still be empty. In particular, we can make the $p_i$ distinct and rational. Thus, $h^0(D) \leq r(D) + 1$, so we’ve proved the proposition.  

We now give some applications of Theorem 1.1 for 2-dimensional tropical complexes, beginning with the proof of Theorem 1.4, that there exists a 2-dimensional tropical complex which doesn’t lift. For our example, we take the cylinder depicted in Figure 1, which is a variant of Example 4.6 in [Car15], with the circumference of the cylinder increased to 2 so that the underlying complex is a regular $\Delta$-complex. This tropical complex is a tropical analogue of Hopf surface, which are non-algebraic complex surfaces. Tropical Hopf manifolds are explored at greater length, and in a different framework, in [RS16].
Lemma 3.4. Let $\Delta$ be the tropical complex in Figure 1 and let $D$ be the sum of the top two edges. Then any divisor linearly equivalent to $mD$ is the sum of $m$ copies of parallel circles, and thus $h^0(\Delta, mD) = m + 1$.

Proof. Let $\phi$ be a PL function on $\Delta$ such that $\text{div}(\phi) + mD$ is effective. Let $C$ be a horizontal circle on the cylinder $\Delta$ parallel to $D$, but not equal to $D$ or the bottom of the cylinder. Let $p$ be a point of $C$ at which the restriction $\phi|_C$ achieves its maximum. The key point is that the linear functions on a neighborhood $U$ of $p$ embed $U$ in an open subset of $\mathbb{R}^2$ and $C$ remains a straight line in this embedding. Since the divisor of $\phi$ is effective in $U$, $\phi|_U$ is a convex function of $\mathbb{R}^2$. Thus, the restriction to $C$, which is a line segment, is also convex, but $\phi|_C$ achieves its maximum at $p$ and so $\phi|_{C \cap U}$ must be constant. Since $\phi$ is constant in any neighborhood of a point in $C$ where $\phi|_C$ achieves its maximum, $\phi$ is constant on $C$.

Thus, $\phi$ is a function solely of the vertical coordinate on $\Delta$, and $\phi$ defines a linear equivalence between $mD$ and $m$ horizontal circles, possibly not distinct. For any $m$ points, these circles can be chosen to contain them, but for $m + 1$ points chosen to have $m + 1$ distinct heights, no sum of $m$ circles will contain all of the points. Therefore, $h^0(\Delta, mD) = m + 1$, as claimed. □

Proof of Theorem 1.4. Suppose that $X$ is a degeneration over a discrete valuation ring whose weak tropical complex is $\Delta$. Since all the maximal simplices of $\Delta$ are 2-dimensional, and it can be checked that $\Delta$ is a tropical complex, $X$ must be robust in dimensions 1 and 2 by Propositions 2.3 and 2.5 respectively. The general fiber $X$ is a smooth proper surface and therefore projective. We let $A$ be an ample divisor on $X$. By Proposition 2.14, $\rho(A)$ is a divisor, and from the definition of the specialization (2), we know that $\rho(A)$ is a linear combination of the ridges of $\Delta$.

We now examine the possibilities for $\rho(A)$. The set of all formal sums of the 8 ridges of $\Delta$ forms a free Abelian group of rank 8, and those satisfying the balancing condition are a subgroup. In a neighborhood of each vertex $v$ of $\Delta$, the space of linear functions can be computed by considering functions which are linear on the interior of each 2-dimensional simplex, and satisfy the condition in Definition 2.10 along each ridge containing $v$. In each case, the space of linear functions have only one parameter for the slopes. For example, let $v$ be the vertex indicated in Figure 1, and $e_1$, $e_2$, $e_3$, and $e_4$ be the edges incident to $v$, beginning with $e_1$ to the left and continuing counterclockwise. Then, the linear functions are constant along the diagonal edge $e_2$, and affine linear under the embedding shown. Therefore, the slopes along $e_3$ and $e_4$ are equal, and are the negative of the slope along $e_1$. Therefore, if $c_1$, $c_3$, and $c_4$ are the coefficients of these three edges in a divisor, then $c_1 - c_3 - c_4 = 0$. Similarly, the other 3 vertices of $\Delta$ also give linear conditions on the coefficients of a divisor, and one can check that these are independent, so the group of formal sums of the ridges which are divisors is a free Abelian group of rank 4.
Next, one can compute the subgroup of divisors which are linearly equivalent to the trivial one by considering PL functions which are linear on the interior of each 2-dimensional simplex. Using the properties in Proposition 2.15, one can then check that the group of divisor classes which are linearly equivalent to a formal sum of ridges is isomorphic to \( \mathbb{Z} \), with the divisor \( D \) from Lemma 3.4 representing twice a generator.

Therefore, \( \rho(2A) = 2\rho(A) \) is linearly equivalent to \( lD \) for some integer \( l \). Thus, by Theorem 1.1 and Lemma 3.4 we have the inequality

\[
h^0(X, \mathcal{O}(2mA)) \leq h^0(\Delta, mD) = ml + 1.
\]

In particular, \( h^0(X, \mathcal{O}(2mA)) \) grows at most linearly in \( m \), which contradicts the fact that \( A \) is ample divisor on a surface, and so its global sections grow quadratically. Therefore, we conclude that no degeneration \( X \) with weak tropical complex \( \Delta \) can exist. \( \square \)

The following example shows one case where the inequality in Theorem 1.1 can be sharp for a divisor and all of its multiples.

**Example 3.5.** We consider a degeneration of a quartic surface which will yield a tetrahedron as its dual complex. We start with the variety \( \tilde{X} \subset \mathbb{P}^3_R \), where \( R = \mathbb{C}[[t]] \), which is defined by the determinant:

\[
\begin{vmatrix}
xy & (3x^2 + 2y^2 + z^2 - w^2)t \\
3x^2 + y^2 + 2z^2 + w^2 & zw + t(x^2 + y^2 + z^2 + w^2)
\end{vmatrix}
\]

The reason for this determinantal form is that we can immediately see that the general fiber \( X \) contains the scheme defined by the equations in the bottom row of the matrix (4), and one can check that this is a smooth complete intersection and therefore an elliptic curve \( E \). The curve \( E \) is part of a pencil interpolating between \( E \) and the curve defined by the equations in the top row of (4). Thus, the complete linear series of \( E \) defines a map to \( \mathbb{P}^1_K \) with connected fibers, and so \( h^0(\tilde{X}_K, \mathcal{O}(mE)) = m + 1 \) for all \( m \geq 0 \).

Now we want to show how Theorem 1.1 gives a sharp bound for this value of \( h^0(\tilde{X}_K, \mathcal{O}(mE)) \). We can rewrite (4) as \( xyzw + tf \), where

\[
f = xy(x^2 + y^2 + z^2 + w^2) - (3x^2 + 2y^2 + z^2 - w^2)(3x^2 + y^2 + 2z^2 + w^2),
\]
to see that the special fiber of \( \tilde{X} \) is the union of the four coordinate planes. However, \( \tilde{X} \) is not a regular degeneration, but has 24 ordinary double point singularities at the common intersection of the quartic \( f \) and the 6 coordinate lines in \( \mathbb{P}^3 \). We wish to resolve each singularity without introducing any new components in the special fibers, which can be done by blowing up one of the two planes containing the singularity.

As in [Car13, Ex. 2.10], we can obtain a symmetric tropical complex by blowing up one plane at 2 of the 4 singularities along each line and blowing up the other plane at the other 2 singularities, but we pay special attention to the four lines defined by the intersection of one of the planes \( x = 0 \) or...
y = 0 with either z = 0 or w = 0. Note that when either x or y vanishes, f factors as the product of two quadrics, which we write as as
\[ g = 3x^2 + 2y^2 + z^2 - w^2 \quad \text{and} \quad h = 3x^2 + y^2 + 2z^2 + w^2. \]

At each point of intersection of one of these four lines with \( g = 0 \), we blow up either the \( x = 0 \) or \( y = 0 \), as appropriate, and at each point of intersection with \( h = 0 \), we blow up either the \( z = 0 \) or \( w = 0 \) plane. Along the other 2 coordinate lines, we blow up each containing plane at 2 of the singularities, but chosen arbitrarily.

After these blow ups, the resulting scheme \( X \) is a regular semistable degeneration. The special fiber has four components, each of which is the blow up of \( \mathbb{P}^2 \) at 6 points, 2 on each coordinate line. The 1-dimensional strata in each component are the union of the strict transforms of the coordinate lines, and one can check that the union of these lines is a big divisor, so \( X \) is robust and thus we can apply the specialization inequality from Theorem 1.1. The tropical complex \( \Delta \) is a tetrahedron with all structure constants \( \alpha(v,e) \) equal to 1, using the fact that the strict transform of each coordinate line has self-intersection \(-1\).

We first calculate the specialization \( \rho([E]) \), where \( E \) is the elliptic curve on \( X \) defined above. In the original singular degeneration \( \tilde{X} \), the closure \( \overline{E} \) of \( E \) degenerates to the union of two conics, in the \( z = 0 \) and \( w = 0 \) planes, respectively, each defined by the restriction of the polynomial \( h \). However, we chose to blow up the \( z = 0 \) and \( w = 0 \) planes at the intersections of \( h \) with the \( x = 0 \) and \( y = 0 \) lines, which removes the intersections between \( E \) and those lines. Therefore, the only 1-dimensional stratum of \( \tilde{X} \) that \( E \) intersects is the \( x = y = 0 \) line, which it does with multiplicity 2. Thus, \( D = \rho(E) \) is twice one edge of the tetrahedron \( \Delta \).

Finally, to justify our claim that the specialization inequality is sharp, we need to show that \( h^0(\Delta, mD) = m + 1 \) for all non-negative integers \( m \). For this, we start with a PL function \( \phi \) which is constant on the support of \( D \) and increases with slope 1 on each of the two containing facets, which establishes a linear equivalence between \( D \) and any cycle on the tetrahedron parallel to \( D \). Moreover, if we ignore both the edge supporting \( D \) and the edge opposite it in \( \Delta \), we are left with an open cylinder, which has the same theory of linear equivalence as if we removed the top and bottom circles from the cylinder in Figure 1, used in the proof of Theorem 1.4. Therefore, Lemma 3.4 shows any divisor in the complete linear series of \( mD \) consists of \( kD + k'D' \) plus \( l \) cycles parallel to \( D \), where where \( D' \) is twice the edge opposite \( D \) and \( k + k' + l = m \). As in the proof of Lemma 3.4, such divisors can be chosen to contain any \( m \) points, but not \( m + 1 \) points at different distances from \( D \), and so \( h^0(\Delta, mD) = m + 1 \).

The sharpness of the inequality in Example 3.5 depended on our choice of blow-ups in constructing the resolution \( \tilde{X} \). For surfaces and higher-dimensional varieties, unlike the case of curves, there is no single minimal
regular semistable degeneration, and the choice of the model can affect the bounds from Theorem 1.1.

We close this section by stating a conjectural form of Riemann-Roch for 2-dimensional tropical complexes. Although we have no definition for higher cohomology, we can assume Serre duality to justify taking $h^0(\Delta, K_\Delta - D)$ as a replacement for the top cohomology. Here, we take $K_\Delta$ of any tropical complex $\Delta$ to be the formal sum of polyhedra

$$K_\Delta = \sum_r (\deg r - 2)[r],$$

where $r$ ranges over the ridges $r$ of $\Delta$, generalizing the definition for curves. In dimension 1, this approach yielded the Riemann-Roch theorems for graphs and tropical curves [BN07, GK08, MZ08].

In addition, we need two concepts on tropical complexes which we were not defined in Section 2 and are not needed in the rest of the paper. First, [Car13, Def. 4.2] defines Cartier divisors on a weak tropical complex, and every Cartier divisor on a 2-dimensional weak tropical complex is a Weil divisor [Car15, Prop. 2.8]. Second, 2-dimensional tropical complexes have a symmetric, bilinear pairing on divisors, which is integral on Cartier divisors [Car13, Prop. 4.7] and [Car15, Prop. 2.9]. With these ingredients, we conjecture the following, which is a formal analogue to the Riemann-Roch theorem for algebraic surfaces, as it was stated before the introduction of cohomology:

**Conjecture 3.6 (Riemann-Roch).** Let $D$ be a Cartier divisor on a 2-dimensional tropical complex $\Delta$ and assume that $K_\Delta$ is also a Cartier divisor. Then

$$(5) \quad h^0(\Delta, D) + h^0(\Delta, K_\Delta - D) \geq \frac{\deg D \cdot (D - K_\Delta)}{2} + \chi(\Delta),$$

where $\chi(\Delta)$ is the Euler characteristic of underlying topological space of $\Delta$.

**Example 3.7.** We verify Conjecture 3.6 for the tetrahedron $\Delta$ from Example 3.5 and for any multiple $mD$, where $D$ is twice an edge of the tetrahedron. Since every edge is contained in exactly two facets, $K_\Delta$ is trivial. Moreover, Example 3.5 showed that $D$ is linearly equivalent to a parallel circle, which is disjoint from $D$, and so $\deg D^2 = 0$. Therefore, the right hand side of (5) is $\chi(\Delta) = 2$ in this example.

Example 3.5 computed that $h^0(\Delta, mD) = m + 1$ for $m \geq 0$. The same method as in that example shows that if $m > 0$, then $h^0(\Delta, K_\Delta - mD) = h^0(\Delta, -mD) = 0$. (That the negative of an effective divisor is non-effective also holds for a broad class of 2-dimensional tropical complexes, as a consequence of [Car15, Cor. 2.12].) Thus, for any integer $m$, the left hand side of (5) is:

$$h^0(\Delta, mD) + h^0(\Delta, -mD) = \begin{cases} 
m + 1 & \text{if } m \geq 1 \\
2 & \text{if } m = 0 \\
-m + 1 & \text{if } m \leq -1, \end{cases}$$
and so the Riemann-Roch inequality is satisfied. Note that $\frac{1}{2}D$ still has integral coefficients and so is a Weil divisor. Moreover, one can check that $h^0(\Delta, \frac{1}{2}D) = 1$ and $h^0(\Delta, -\frac{1}{2}D) = 0$ and thus the Riemann-Roch inequality is not satisfied for $D$, which doesn’t contradict Conjecture 3.6 because $\frac{1}{2}D$ is not Cartier. □

4. Subdivisions

In this section, we study subdivisions of a weak tropical complex, which correspond to ramified extensions of the discrete valuation ring $R$ followed by toroidal resolution of singularities. Here, the key technology is the relationship between toroidal maps and polyhedral subdivisions as in [KKMSD73]. Combinatorially, passing to a degree $m$ totally ramified extension of the DVR $R$ corresponds to scaling the simplices of $\Delta$ by a factor of $m$, and the toroidal resolution is given by a unimodular subdivision of the scaled simplices. The importance of subdivisions for us is as a tool when we define the refined specialization in the next section.

Construction 4.1 (Subdivisions). By an order-$m$ subdivision of a weak tropical complex $\Delta$, we mean a $\Delta$-complex which is formed by replacing each simplex of $\Delta$ by a unimodular integral subdivision of the standard simplex scaled by $m$, together with the structure constants obtained as follows.

First suppose that $r'$ is a ridge of $\Delta'$ meeting the interior of a facet of $\Delta$. In particular, if $v'_1, \ldots, v'_n$ are the vertices of $r'$, and $w_1$ and $w_2$ contained in the two facets containing $r'$, then the midpoint $(w_1 + w_2)/2$ is contained in the plane spanned by $r'$. Thus, we can write:

$$(w_1 + w_2)/2 = c_1 v'_1 + \cdots + c_n v'_n$$

for some coefficients $c_1, \ldots c_n$ with $c_1 + \cdots + c_n = 1$. We set $\alpha(v'_i, r') = 2c_i$.

On the other hand, suppose that $r'$ is a ridge of $\Delta'$ which is contained in a ridge $r$ of $\Delta$, and $r$ has degree $d$. We represent the points of each facet of $\Delta$ which contains $r$ by $(n + 1)$ non-negative coordinates with total sum equal to $m$. In the $i$th such facet, the unique facet of the subdivision $\Delta'$ containing $r'$ is the convex hull of $r'$ and a unique point, whose coordinate is $(x_{i,1}, \ldots, x_{i,n}, 1)$, by unimodularity. Likewise, the points of $r$ can be given coordinates consisting of $n$ non-negative real numbers, also summing to $m$. In such coordinates, we represent the $i$th vertex $v_i$ of $r'$ by the vector $(y_{i,1}, \ldots, y_{i,n})$. Finally, if $v_1, \ldots, v_n$ are the vertices of $r$, we determine the structure constants of $r'$ by the equation:

$$(\begin{array}{c}
\alpha(v'_1, r') \\
\vdots \\
\alpha(v'_n, r')
\end{array}) = (\begin{array}{ccc}y_{1,1} & \cdots & y_{1,n} \\
\vdots & \ddots & \vdots \\
y_{n,1} & \cdots & y_{n,n}
\end{array})^{-1} (\begin{array}{c}\alpha(v_1, r) + x_{1,1} + \cdots + x_{d,1} \\
\vdots \\
\alpha(v_n, r) + x_{1,n} + \cdots + x_{d,n}
\end{array}).$$

That these structure constants form a weak tropical complex is verified in Proposition 4.3 after the following example. □
Example 4.2. Consider the order 2 subdivision of the complex $\Delta$ shown in Figure 2. Let $e'$ be the upper segment of the central edge, with $v'_1$ the top vertex and $v'_2$ the middle vertex of the edge. If $v_1$ is also the top vertex and $v_2$ is the bottom vertex, then $(y_{1,1}, y_{1,2}) = (2, 0)$ and $(y_{2,1}, y_{2,2}) = (1, 1)$. The vertices adjacent to $e'$ both have coordinate $(1, 0, 1)$. Thus, applying (6), we get:

$$\left(\alpha(v'_1, e'), \alpha(v'_2, e')\right) = \left(\begin{array}{c} 2 \\ 1 \\ 1 \end{array}\right) - \left(\begin{array}{c} 1 + 1 + 1 \\ 1 + 0 + 0 \\ 1 + 0 + 0 \end{array}\right) \left(\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{array}\right) \left(\begin{array}{c} 3 \\ 1 \\ 1 \end{array}\right) = \left(\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right)$$

More generally, we have $\alpha(v, e) = 1$ for every internal edge in the subdivided complex $\Delta'$.

Proposition 4.3. The subdivision in Construction 4.1 results in a weak tropical complex.

Proof. We need to show that the $\alpha(v', r')$ from Construction 4.1 are integral and that they satisfy the required identity (1) for each ridge. We retain the same notation as in Construction 4.1 and we first consider a ridge $r'$ meeting the interior of a facet of $\Delta$. Then the coordinates of $(w_1 + w_2)/2$ will be half-integers, and so the $c_i$ are also half-integers, and $\alpha(v'_i, r') = 2c_i$ is an integer. Moreover, the sum of the $\alpha(v'_i, r')$ will equal 2, which is the degree of $r'$ in $\Delta'$.

Now consider a ridge $r'$ of the subdivision $\Delta'$ contained in a ridge $r$ of $\Delta$. The fact that the vectors $v'_1, \ldots, v'_n$ form the vertices of one simplex of a unimodular triangulation imply that the differences between pairs of the vectors $(y_{i,1}, \ldots, y_{i,n})$ span the integral vectors $\mathbb{Z}^n$ whose sum is 0. Moreover, each vector satisfies the affine linear equation $y_{i,1} + \cdots + y_{i,n} = m$, so the span of these vectors includes all vectors in $\mathbb{Z}^n$ whose sum is a multiple of $m$. Thus, to check that the coefficients defined in (6) are integral, it is sufficient to check that the sum of the entries of the vector from the right hand side of (6) is a multiple of $m$. Indeed, we know that $\alpha(v_1, r) + \cdots + \alpha(v_n, r) = d$ and that $x_{i,1} + \cdots + x_{i,n} = m - 1$, so the total sum is $dm$. Moreover, this
implies that $\alpha(v'_1, r') + \cdots + \alpha(v'_n, r') = d$, which completes the proof that the subdivision is a weak tropical complex.

It is immediate from the definition that a subdivision of a weak tropical complex is homeomorphic to the original complex. In addition, we now show that the subdivision behaves almost identically with the properties of weak tropical complexes introduced in Section 2, namely linear functions and divisors.

**Lemma 4.4.** Let $\Delta'$ be a order $m$ subdivision of a weak tropical complex $\Delta$, and we identify the realizations of $\Delta'$ and $\Delta$ via the natural homeomorphism. Then a function $\phi$ on $\Delta$ is linear (respectively PL) if and only if $m\phi$ is linear (respectively PL) on $\Delta'$.

**Proof.** The agreement of PL functions is clear, so long as the scaled function $m\phi$ on $\Delta'$ has integral slopes on its domains of linearity. However, each facet $f'$ of $\Delta'$ is identified with a $(1/m)$-scaled copy of the standard unit simplex. If we instead identify $f'$ with the standard unit simplex, then the slopes of $\phi$ will be scaled by $1/m$, and so the slopes of $m\phi$ will all be integers.

For linear functions, we let $r'$ be a ridge of of $\Delta'$ and we need to show that linear functions on a neighborhood of $r'$ in $\Delta$, scaled by $m$, agree with linear functions defined by the map $\phi_{r'}: N_{r'} \to L_{r'} \cong R^{d+n-1}$ from Definition 2.10. First suppose that $r'$ intersects the interior of a facet $f$ of $\Delta$. Then, $N_{r'}$ is the union of the two facets containing $r'$, and naturally embeds in $R^{2+n}$ by sending each vertex to a coordinate vector, as in Definition 2.10. Therefore, the inclusion of $N_{r'}$ into $f \subseteq R^n$ factors through a linear map from $R^{2+n}$ to $R^n$, whose kernel contains $(1, 1, -\alpha(v'_1, r'), \ldots, -\alpha(v'_n, r'))$ by the choice of those structure constants in Construction 4.1. Thus, the inclusion of $N_{r'}$ into $f$ factors through the map $\phi_{r'}: N_{r'} \to L_{r'}$, defining linear functions on the open subset $N_{r'}$, and so the linear functions on $\Delta$ and $\Delta'$ agree.

Now suppose that $r'$ is contained in a ridge $r$ of $\Delta$, and we let $v_i, v'_i, x_{i,j}$ and $y_{i,j}$ be as in Construction 4.1 and $\phi_{r}$ as in Definition 2.10. Then, using the fact that $\phi_{r'}$ is affine linear on each facet of $\Delta$, we can evaluate:

$$
\sum_{i=1}^{d} \phi_{r'}(w'_i) = \sum_{i=1}^{d} \frac{1}{m} \left( \phi_{r}(w_i) + \sum_{j=0}^{n} x_{i,j} \phi_{r}(v_j) \right)
$$

$$
= \frac{1}{m} \sum_{j=0}^{n} \alpha(v_j, r) \phi_{r}(v_j) + \frac{1}{m} \sum_{i=1}^{d} \sum_{j=0}^{n} x_{i,j} \phi_{r}(v_j)
$$

$$
= \frac{1}{m} \sum_{j=0}^{n} \left( \alpha(v_j, r) + \sum_{i=1}^{d} x_{i,j} \right) \phi_{r}(v_j)
$$

$$
= \sum_{j=0}^{n} \alpha(v'_j, r') \phi_{r'}(v'_j),
$$
where the last step is by the change of variables in (6), together with the linearity of $\phi_r$ on $r$. Thus, $\phi_r$ satisfies the defining linear relation of $\phi_{r'}$, and so $\phi_{r'}$ is isomorphic to the restriction of $\phi_r$, as in the previous paragraph.

If $\Delta'$ is a subdivision of a weak tropical complex $\Delta$, then any formal sum of polyhedra on $\Delta'$ is also a formal sum of polyhedra on $\Delta$. Conversely, for any formal sum of polyhedra on $\Delta$, we can intersect with all the facets of $\Delta'$ to obtain an equivalent formal sum of polyhedra, all of whose terms are contained in a single facet of $\Delta'$, and thus can be considered as a formal sum of polyhedra on $\Delta'$. Moreover, this correspondence passes to equivalence class of formal sums of polyhedra, so we can identify the equivalence classes on $\Delta$ with those on $\Delta'$, and by the following lemma, we can also identify divisors on $\Delta$ with those on $\Delta'$.

**Lemma 4.5.** Let $\Delta'$ be an order $m$ subdivision of a weak tropical complex $\Delta$. Then a formal sum of polyhedra is a divisor on $\Delta$ if and only if it is a divisor on $\Delta'$.

**Proof.** By Lemma 2.12 passing to equivalent formal sums of polyhedra doesn’t change whether or not it is balanced, so we can assume that the terms of the formal sum of polyhedra are each contained in the facets of $\Delta'$. Then, Lemma 4.4 shows the linear functions on $\Delta$ and $\Delta'$ are the same, up to multiplication by $m$, so we just need to check that the multiplicity $\text{mult}_{Z,Q}(\phi)$ of a linear function along $Q$ of a formal sum of polyhedra $Z$ vanishes in $\Delta$ if and only if it vanishes in $\Delta'$.

Here, the key the difference is between two different embeddings of a single facet $f'$ of $\Delta'$ into $\mathbb{R}^n$. First, we can identify $f'$ with a standard unit simplex in $\mathbb{R}^n$. Second, we identify $f'$ with a simplex contained in a facet $f$ of $\Delta$, and $f$ is identified with a standard unit simplex in $\mathbb{R}^n$. Because the subdivision of $\Delta$ is assumed to be unimodular in Construction 4.1, in the latter identification, $f'$ is a standard unit simplex scaled by $1/m$. Thus, if we transform the vectors $w_i$ and $v_i$ used to compute the multiplicity in $\Delta$ into the coordinates where $f$ is identified with a standard unit simplex, then the transformed vectors have a common multiple of $m$ in their coordinates. Scaling these vectors by $1/m$ gives suitable vectors for computing the multiplicity in $\Delta'$, and so the multiplicity of a given linear function along $Q$ of $Z$ in $\Delta'$ is $1/m$ times the multiplicity in $\Delta$, which completes the proof. □

**Lemma 4.6.** Let $\Delta'$ be an order $m$ subdivision of a weak tropical complex $\Delta$, and further assume that every ridge of $\Delta$ is contained in a facet. Then, divisors on $\Delta$ are linearly equivalent if and only if they are linearly equivalent on $\Delta'$.

**Proof.** To show that linear equivalence on $\Delta$ agrees with that on $\Delta'$, we need to show that, if $\phi$ is a PL function on $\Delta$, then $\text{div}(\phi) = \text{div}'(m\phi)$, where $\text{div}$ and $\text{div}'$ are the divisor functions from Proposition 2.15 on $\Delta$ and $\Delta'$, respectively. Since Proposition 2.15 uniquely characterize $\text{div}$ and $\text{div}'$ in terms of four properties, we need to check that those properties coincide for
and $\Delta'$. The first two properties are that the divisor is linear in the PL function and local, which are independent of the subdivision. Moreover, the third property is that linear functions are characterized by having trivial divisor, and by Lemma 4.4, the linear functions on $\Delta$ and $\Delta'$ agree, up to scaling. The fourth property normalizes the divisor function via a PL function supported on a single facet, which also agrees, because when passing to $\Delta'$ we scale both the coordinates on the complex and the PL function by the same factor of $m$.

On the other hand, the subdivisions of Construction 4.1 mirror what happens for toroidal resolutions of ramified base changes.

**Lemma 4.7.** Let $\mathbb{X}$ be a regular semistable degeneration with dual complex $\Delta$. If $\Delta'$ is an order $m$ subdivision of $\Delta$ and $R'$ is a totally ramified degree $m$ extension of $R$, then the corresponding resolution $\mathbb{X}'$ of $\mathbb{X} \times_{\text{Spec } R} \text{Spec } R'$ has $\Delta'$ as its weak tropical complex.

**Proof.** We first describe the pullback of a divisor $C_v$ from $\mathbb{X}$ to $\mathbb{X}'$. If $v'$ is a vertex of $\Delta'$, contained in a $k$-dimensional simplex $s$ of $\Delta$, then we can express the coordinate of $v'$ by a vector $(x_0, \ldots, x_k)$ with $x_0 + \cdots + x_k = m$. If the vertices of $s$ are $v_0, \ldots, v_k$, then we claim that the coefficient of $C_v$ in the pullback of $C_{v_i}$ is $x_i$. The reason is that in local, toroidal coordinates around $C_s$, the divisor $C_{v_i}$ is defined by a monomial with exponent vector $(0, \ldots, 1, \ldots, 0)$. The valuation of the divisor $C_v'$ in $\mathbb{X}'$ corresponds to the vector $(x_0, \ldots, x_k)$ in the lattice dual to these monomials, meaning that the valuation of the monomial locally defining $C_{v_i}$ is given by the dot product of these two vectors, which is $x_i$.

We now adopt the notation of Construction 4.1. For any facet $f'$ containing $v'$, we compute the intersection number $\pi^{-1}(C_v) \cdot C_{f'}$ in two different ways.

First, using the discussion in the previous paragraph, we can represent $\pi^{-1}(C_v)$ as a linear combination of divisors in $\mathbb{X}'$. The ones which intersect $C_{f'}$ correspond to the vertices of $f'$, together with the vertices in neighboring simplices. All together, these give:

$$x_{1,i} + \cdots + x_{d,i} - \alpha(v'_0, f')y_{0,i} - \cdots - \alpha(v'_n, f')y_{n,i}$$

On the other hand, using the projection formula, $\pi^{-1}(C_v) \cdot C_{f'} = C_v \cdot \pi(C_{f'})$, and we know that $\pi_*(C_{f'})$ is $C_f$, so the intersection number is $-\alpha(v_i, f)$. Putting these equations together for all $i$, we get:

$$
\begin{pmatrix}
y_{0,0} & \cdots & y_{n,0} \\
\vdots & \ddots & \vdots \\
y_{0,n} & \cdots & y_{n,n}
\end{pmatrix}
\begin{pmatrix}
-\alpha(v'_0, F') \\
\vdots \\
-\alpha(v'_n, F')
\end{pmatrix}
+
\begin{pmatrix}
x_{1,0} + \cdots + x_{d,0} \\
\vdots \\
x_{1,n} + \cdots + x_{d,n}
\end{pmatrix}
=
\begin{pmatrix}
-\alpha(v_0, F) \\
\vdots \\
-\alpha(v_n, F)
\end{pmatrix}.
$$

We can solve for the $\alpha(v'_i, F')$ and we get (6). \qed


Figure 3. The black lines give a subdivision of the plane giving a semistable family whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. The red line is the tropicalization of a line in one of the rulings whose specialization to the tropical complex consists the sum of the outside edges of the square minus the diagonal edge. In particular, the specialization is not effective.

5. Refined specialization

In this section, we define a refined specialization map from divisors on the general fiber of $\mathcal{X}$ to divisors on the weak tropical complex $\Delta$. Given a divisor $D$, we first choose a ramified extension of the discrete valuation ring $R$, followed by a toroidal resolution of singularities $\mathcal{X}'$ such that if $D'$ is the pullback of $D$ to the general fiber of $\mathcal{X}'$, the closure $\overline{D'}$ does not contain any strata of the special fiber of $\mathcal{X}'$. Then, the refined specialization of $D$ is $\rho(D')$, which can be considered as a divisor either on the subdivision $\Delta'$ or on $\Delta$ by the results in the previous section. The refined specialization is the higher-dimensional analogue of the map $\tau_*$ from [Bak08, Sec. 2C]. As stated in the introduction, the purpose of the refined specialization is that the refined specialization of an effective divisor is always effective, which is not necessarily true for the coarse specialization $\rho$.

Example 5.1. We construct our example as a toric degeneration. In particular, we take the 3-dimensional fan whose support is $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ and intersection with the plane $\mathbb{R}^2 \times \{1\}$ is the configuration in Figure 3. Projection to the last factor defines a toric morphism from a 3-dimensional toric variety $\mathcal{X}$ to $\mathbb{A}^1$, whose fibers are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, except over the origin. The fiber of the origin consists of 4 components, corresponding to the 4 rays generated by the 4 vertices in the Figure 3. The intersections between these components correspond to the cones they have in common, such that the dual complex of $\mathcal{X}$ is the triangulation of a square, making up the bounded cells in Figure 3.

By Proposition 2.3, the locally closed strata of dimension 1 in $\mathcal{X}$ are affine and we claim that the same is true in dimension 2. Each of the four components is isomorphic to the toric variety corresponding to the star of the corresponding vertex $v$, with the divisor $D_v$ equal to the union of the rays corresponding to the bounded edges. Therefore, in each case, the locally closed stratum $C_v \setminus D_v$ is isomorphic to $\mathbb{A}^2$. 

Now consider a line in one of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ whose tropicalization is the red line in Figure 3. The specialization of this line to $\Delta$ is the sum of four outside edges of the square minus the inner diagonal. In particular, this specialization is not effective and it does not agree with the tropicalization. However, the specialization is linearly equivalent to an effective divisor, namely the intersection of the red tropicalization with $\Delta$. □

In Example 5.1 it was easy to guess the refined tropicalization. In general, we need to show that there exists a subdivision of the weak tropical complex $\Delta$ which will give the refined specialization according to the definition at the beginning of this section, which is the content of the following lemma. The construction is essentially a toroidal version of the partial resolution of a hypersurface singularity coming from its Newton polyhedron.

Lemma 5.2. Let $D$ be an effective divisor on the general fiber $X$ of a degeneration $\mathcal{X}$. Then, there exists a finite ramified extension $R'$ of $R$ and a toroidal resolution of singularities $\mathcal{X}'$ of $\mathcal{X} \otimes_\mathbb{R} R'$ such that $\overline{D'}$ doesn’t contain any of the strata of $\mathcal{X}'$, where $\overline{D'}$ is the closure in $\mathcal{X}'$ of the pullback of $D$ to the general fiber $X'$.

Proof. We let $s$ be any $k$-dimensional simplex of $\Delta$. Let $A$ be the local ring of $\mathcal{X}$ at the generic point of $C_s$. By assumption, $A$ is a regular ring with maximal ideal generated by $x_0, \ldots, x_k$, each of which is the local equation of one of the components of $\mathcal{X}_0$ containing $C_s$. The completion $\widehat{A}$ of $A$ is isomorphic to $S[[x_0, \ldots, x_k]]/\langle x_0 \cdots x_k - \pi \rangle$, where $S$ is a complete, local extension of $R$ and $\pi$ is a uniformizer in $S$. We let $f \in \widehat{A}$ be the local equation of $D$ in the completed local ring $\widehat{A}$. We construct the local Newton polyhedron from $f$ as follows. For each term $s_a x_0^{a_0} \cdots x_k^{a_k}$ in the summation representation of $f$, we translate the positive orthant in $\mathbb{R}^{k+1}$ by $(a_0 + \text{val}(s_a), \ldots, a_k + \text{val}(s_a))$. Then, the local Newton polyhedron of $f$ is the convex hull of the translated orthants coming from all terms of $f$. If $f$ is not a unit, this convex hull will not contain the origin. In this case, we take the dual polyhedron of the local Newton polyhedron and project its bounded faces to the unit $k$-dimensional simplex along lines through the origin. This yields a rational polyhedral subdivision of $s$.

We now claim that these subdivisions are compatible so that taken together they form a rational polyhedral subdivision of $\Delta$. It is sufficient to show compatibility between the subdivision of $s$ and of one of its faces $s'$, which we can also take to be the face indexed by $k$. Thus, in the local ring $A$, the component $C_{s'}$ is defined by the ideal $\langle x_0, \ldots, x_{k-1} \rangle$, and so its local ring is the localization of $A$ at this ideal. We consider the image of $f$, the local equation of $D$ in the tensor product $\widehat{A} \otimes_A A_{(x_0, \ldots, x_{k-1})}$. We can now regroup the terms of $f$ as:

$$\sum_{a_0, \ldots, a_{k-1}} \left( \sum_{a_k} s_a x_k^{a_k} \right) x_0^{a_0} \cdots x_{k-1}^{a_{k-1}}.$$
This expression also holds in the completion of the local ring of $C_s'$, so the local Newton polyhedron of $f$ along $C_s'$ is the projection of the polyhedron along $C_s$. Thus the dual of the Newton polyhedron is the restriction of the dual, which shows the desired compatibility.

Now we have a polyhedral subdivision of $\Delta$. If $m$ is the least common multiple of the denominators of the vertices of the subdivision, then it corresponds to a ramified degree $m$ extension $R'' \supset R$ followed by a toroidal map $x'' \to x \times_R R''$. Although $x''$ is not semistable, the closure $\overline{D''}$ of the pullback of $D$ to $x''$ does not contain any of the strata. By [KKMSD73, Thm. III.4.1], there exists a further extension and toroidal map $x' \to x'' \times_{R''} R'$ such that $x'$ is semistable. Since each stratum of $x'$ maps to a stratum of $x''$, the pullback of $\overline{D''}$ to $x'$ doesn’t contain any strata of the special fiber either. This implies that the pullback of $\overline{D'}$ is equal to the closure of the pullback of $D$. □

After the subdivision given by Lemma 5.2, the specialization of $D$ will be effective, because all of the intersections between $\overline{D}$ and curves $C_r$ will be proper, and thus the coefficients making up the definition of the specialization will all be non-negative. In addition to the above construction, the refined specialization has a more intrinsic interpretation, at least set-theoretically, as the projection of the Berkovich analytification of the divisor to the skeleton defined by the degeneration. Here, we are using the deformation retract from the Berkovich analytification $X^\text{an}$ to the dual complex of $X$. Such a retract is defined in a more general setting in [Ber99] and in the specific case of regular semistable degenerations in [Nic11, Sec. 3.3].

Proof of Proposition 1.3. By Lemma 5.2 we can assume that our degeneration $X$ is chosen such that $\overline{D}$ doesn’t contain any of the closed strata of $\overline{X}$. Because projection of $X^\text{an}$ to the skeleton is compatible with the specialization map, we know that the image of $D^\text{an}$ is contained in those simplices $s$ such that $\overline{D}$ intersects $C_s$. By hypothesis, $\overline{D}$ doesn’t intersect any 0-dimensional strata $C_f$ and whenever it intersects a curve $C_r$ for some ridge $r$, it does so properly and thus $r$ has positive coefficient in the specialization of $D$. Therefore, the projection of the analytification is contained in the union of the support of the refined specialization of $D$, together with some simplices of dimension $0 \leq n - 2$.

It now only remains to show that the projection map is surjective onto the ridges $r$ in the support of the specialization. For this we suppose that $r$ is a ridge such that $\overline{D}$ intersects $C_r$ and let $x$ be a point in the intersection. In a neighborhood of $x$, we can take local defining equations for the divisors containing $C_r$ and use them to define a map to Spec $S$ where $S = R[x_1, \ldots, x_n]/(x_1 \cdots x_n - \pi)$. As in [Nic11, Sec. 3.3, Case 1], the $(n - 1)$-dimensional simplex, which we identify with $r$ embeds in the analytification of Spec $S$ as the skeleton. Since the fiber over $x_1 = \cdots = x_n = 0$ is finite, the map from $\overline{D}$ to Spec $S$ is dominant and so by Proposition 3.4.6(7) of [Ber90],
we can lift any norm corresponding to a point in this skeleton to a point in the analytification of $D$. Since the projection onto the skeleton of $X$ is defined in terms of the norms of the $x_i$, we’ve produced a norm in the analytification of $\overline{D}$ which projects onto any point of $r$. □

The image of the projection in Proposition 1.3 can be larger than the refined specialization whenever $D$ meets a component $C_v$, but doesn’t meet any curves $C_r$ contained in $C_v$.

Example 5.3. Suppose $\mathfrak{X}$ is any degeneration of dimension at least 2 and $x$ is a point of $X$ whose closure in $\mathfrak{X}$ is contained in $C_v \setminus D_v$ for some vertex $v$. Then, the blow-up $\mathfrak{X}'$ of $\mathfrak{X}$ at the closure of $x$ is a regular semistable degeneration and the specialization of the exceptional divisor $E$ of the blow-up is trivial because its closure doesn’t intersect any curve $C_r$ in $X$. However, the analytification of $E$ is not empty and by the compatibility with specialization noted in the proof of Proposition 1.3, its projection to $\Delta$ must be the single point $v$, which has codimension $n \geq 2$. □

6. Proof of the specialization inequality

In this section, we finish the proof of Theorem 1.1. The main remaining ingredient is to show that the refined specialization always preserves point containment, at least for appropriately chosen points. Example 5.3 shows that even for robust degenerations, the specialization of a non-trivial divisor can be trivial, which the importance of the choice of a point. However, we have the following result with the locally closed strata are affine.

Proposition 6.1. Suppose that $\mathfrak{X}$ is a degeneration whose locally closed strata of dimension $m$ are affine for all $2 \leq k \leq m$. If $E$ is a divisor in $X$ and its closure $\overline{E}$ intersects some stratum $C_s$ of the special fiber of $\mathfrak{X}$ and $C_s$ has dimension $m$, then $\overline{E}$ intersects the curve $C_r$ for some ridge $r$ containing $s$.

Proof. The proof is by induction the dimension $m$. If $m = 1$, then $s$ is a ridge and we’re done. Otherwise, by assumption, $C_s \setminus D_s$ is affine, and affine varieties contain no complete subvarieties of positive dimension. However, $\overline{E} \cap C_s$ is a divisor in $C_s$ and so of dimension $m - 1 \geq 1$. Thus, $\overline{E}$ must intersect $D_s$ and thus $C_{s'}$ for some $(n - m + 1)$-simplex $s'$ containing $s$. Applying the inductive hypothesis, we get that $\overline{E}$ intersects $C_r$ for some ridge $r$. □

Proposition 6.1 gives us a tool for proving that the refined specialization contains a given vertex of the dual complex, and together with Lemma 6.5 below, it can be used to give a proof of Theorem 1.1 for degenerations whose locally closed strata are affine. However, Theorem 1.1 applies not only to degenerations where the locally closed strata are affine, but also to those which are only robust in the top two dimensions. In such cases, the conclusion of Proposition 6.1 doesn’t necessarily hold, as shown by Example 5.3. However,
we can still ensure that the refined specialization preserves point containment, so long as the point is chosen sufficiently generically.

**Lemma 6.2.** Assume that $R$ is complete, with fraction field $K$, and that $X$ is robust in dimension $n - 1$ and $n$ and that the locally closed strata of dimension at most $n - 2$ are affine. Then for any vertex $v$ in $\Delta$, there exists a $K$-point $x$ in $X$ such that for any effective divisor $E \subset X$ containing $x$, the refined specialization of $E$ contains $v$.

Since the crux of this lemma is using the robustness hypothesis, we first prove a couple of results about big divisors. We begin with an application of Bertini’s theorem to the connectedness of big divisors.

**Lemma 6.3.** Any big Cartier divisor has a unique connected component which is itself a big Cartier divisor.

*Proof.* Suppose that $E$ is a big divisor on a variety $X$. Then we let $\tilde{X}$ be the blow-up of the base locus and let $O_{\tilde{X}}(1)$ be the relative ample divisor. Let $\tilde{E}$ be the pullback of $E$ and then $\tilde{E}(-1)$ defines a regular morphism from $\tilde{X}$ to $\mathbb{P}^N$. By Bertini’s theorem [Jou83, Thm. 7.1], the support of $\tilde{E}(-1)$ is connected. Therefore, the base locus of $E$ contains all but one connected component of $E$. By taking just that connected component, we’ve only removed fixed divisors, and so we still have a big divisor, which is the desired statement. $\square$

**Lemma 6.4.** Suppose that $E$ is a big Cartier divisor on a variety $X$. If $\phi$ is any morphism from $X$ to $\mathbb{P}^N$, then either $\phi(E) = \phi(X)$ or $\phi$ is generically finite on some irreducible component of $E$.

*Proof.* Suppose, for contradiction, that $E$ is a big divisor on $X$ and $\phi: X \to \mathbb{P}^N$ is a morphism which is not generically finite on any component of $E$ and such that $\phi(X) \neq \phi(E)$. We first consider the case that $\phi$ is generically finite on $X$. Then, $\dim(X) = \dim(\phi(X)) \geq \dim(\phi(E)) + 2$, so for any point in $\phi(X) \setminus \phi(E)$, we can take an intersection with a general linear space to obtain a curve through $\phi(X)$ which doesn’t intersect $\phi(E)$. By taking the preimage, we get a curve passing through any sufficiently general point in $X$, which does not intersect $E$. Since $E$ is big, such curves can’t exist, and so we get a contradiction.

Second, we suppose that $\phi$ is not generically finite on $X$, but still that $\phi(X) \neq \phi(E)$. Then for any point in $\phi(X) \setminus \phi(E)$, the fiber of $\phi$ is at least 1-dimensional and we can choose a curve in this fiber. Again, we’ve constructed curves in $X$ which pass through sufficiently general points, and which do not intersect $E$, so we’ve again contradicted our assumption that $E$ is big. $\square$

*Proof of Lemma 6.2.* Because $R/\mathfrak{m}$ is algebraically closed, we can choose a general point $x_0$ in $C_v \setminus D_v$. By general, we mean that it lies outside of finitely many closed subvarieties, which don’t depend on $E$ and will be
made explicit in the course of the proof. By Hensel’s lemma, we can lift the point \( x_0 \) to an \( R \)-point of \( \mathfrak{X} \), which will give the desired point \( x \).

Now let \( E \) be any divisor of \( X \) containing \( x \) and we first want to show that its closure \( \overline{E} \) in \( \mathfrak{X} \) intersects a 1-dimensional stratum \( C_r \) for some ridge \( r \) containing \( v \). If the dimension \( n \) is 1, then this is immediate, so we assume that \( n \geq 2 \). By assumption, some multiple of \( D_v \) defines a rational map \( \phi_v : C_v \rightarrow \mathbb{P}^N \) which is birational onto its image. We can assume that our choice of \( x_0 \) was in the locus where \( \phi_v \) defines an isomorphism onto its image. Since \( \overline{E} \) contains \( x_0 \), the restriction of \( \phi_v \) to \( E_0 = E \cap C_v \) must also be birational onto its image. Since this restriction is defined by the pullback of the same multiple of \( D_v \), we see that \( D_v \cap E_0 \) is a big divisor on \( E_0 \).

By Lemma 6.3, we can take \( B \) to be a connected component of \( D_v \cap E_0 \) which is a big divisor on \( E_0 \). Since \( B \) is connected, it is either contained in a single stratum \( C_e \) for some edge \( e \) or it meets some \( C_t \), where \( t \) is a 2-simplex of \( \Delta \). If \( n = 2 \), then \( C_e \) is a 1-dimensional stratum intersecting \( \overline{E} \), which is what we wanted to show. So, we assume that \( n \geq 3 \) and, for the moment, we also assume that \( B \) is contained in a single \( C_e \).

By assumption, \( D_e \) is a big divisor in \( C_e \), so some multiple of \( D_e \) defines a map \( \phi_e : C_e \rightarrow \mathbb{P}^M \), birational onto its image. Since this morphism is just defined by a collection of rational functions on \( C_e \), we can extend it to a rational map from \( C_v \) to \( \mathbb{P}^M \), also denoted by \( \phi_v \). The image \( \phi_e(C_v) \) contains the image of \( C_e \), so it is at least \((n-1)\)-dimensional. Therefore, a general fiber of \( \phi_e \) is at most 1-dimensional, so we can assume that \( \phi_e^{-1}(\phi_e(x_0)) \), the fiber containing \( x_0 \), has dimension at most 1. Thus, \( \phi_e(E_0) \) must be at least \((n-2)\)-dimensional. We apply Lemma 6.4 to the restriction \( \phi_e|_{E_0} \), and in either of that lemma’s two cases, \( \phi_e(B) \) is also at least \((n-2)\)-dimensional. Since \( \phi_e(B) \) is positive dimensional, it intersects any hyperplane in \( \mathbb{P}^M \), which shows that \( B \) must intersect \( D_e \), and thus \( C_t \) for some 2-simplex \( t \).

At this point, we’ve shown that if \( n \leq 3 \), then \( \overline{E} \) intersects \( C_r \) for some ridge \( r \) containing \( v \). If \( n > 3 \), then \( \overline{E} \) at least intersects \( C_t \) for some 2-simplex \( t \) containing \( v \), but we can then apply Proposition 6.1 to show that \( \overline{E} \) intersects some \( C_r \) for some ridge containing \( v \) in this case as well.

We now consider the resolution of the base change \( \mathfrak{X}' \) produced by Lemma 5.2. There are two cases, depending on whether or not there exists a facet \( f \) containing \( r \) such that \( \overline{E} \) contains the point \( C_f \). If there does, then we consider the component \( C_f' \) in \( \mathfrak{X}' \) corresponding to \( v \), which maps birationally onto \( C_v \). By the properness of the resolution, there must be a point in \( C_f' \cap \overline{E}' \) mapping to the point \( C_f \), where \( \overline{E}' \) is the closure of \( E \) in \( \mathfrak{X}' \). However, the fiber of \( C_f \) in the resolution \( \mathfrak{X}' \) is a union of toric varieties intersecting along their toric boundaries and so by Proposition 6.1, \( \overline{E}' \) must intersect one of the 1-dimensional strata in this fiber. By Lemma 5.2, this intersection is proper, so it gives a positive coefficient to some ridge containing \( v \).

On the other hand, suppose that \( \overline{E} \) does not intersect \( C_f \) for any facet \( f \) containing \( r \). We’ve assumed that \( D_r \) is a big divisor on \( C_r \), so it is a
of a base change of \( X \) also robust in dimensions \( k \) for all \( k \leq m \). Then the image of \( C'_{s'} \) in \( X' \) is the stratum \( C_s \), where \( s \) is the minimal simplex of \( \Delta \) which contains \( s' \). Therefore, \( k \), the dimension of \( C_s \) satisfies \( k \leq k' \leq m \), and so \( C_s \setminus D_s \) is affine by assumption. From the construction of the toroidal resolution, we know that \( C'_{s'} \) will be a toric variety bundle over \( C_s \) and \( C'_{s'} \setminus D_{s'} \) will be a \( \mathbb{C}^{k-k'} \)-bundle over \( C_s \setminus D_s \). Therefore, the map from \( C'_{s'} \setminus D_{s'} \) to \( C_s \setminus D_s \) is affine and so \( C'_{s'} \setminus D_{s'} \) is affine as well. We conclude that the \( k \)-dimensional locally closed strata of \( X' \) are affine for \( k \leq m \).

We now consider the case that \( X \) is robust in dimensions \( k \) for all \( k \leq m \). Let \( C'_{s'} \) and \( C_s \) be strata of \( X' \) and \( X \) of dimensions \( k' \) and \( k \) as in the previous paragraph. By assumption, some multiple of the divisor \( D_s \) defines a birational map \( \phi \) from an open subset of \( C_s \) to \( \mathbb{P}^N \). We let \( \bar{\phi}(C_s) \) denote the closure of the image of this map and let \( Y \) denote the image of \( C_s \setminus D_s \).
Since $Y$ is the complement of a hyperplane section in a projective variety, $Y$ is affine.

Now, we consider $C_{s'}$, which, as above, is a toric variety bundle over $C_s$, and the toric variety is described by the star of $s'$ in $s$. We can choose some positive multiple of the boundary of this toric variety which contains a spanning set of the characters of the torus, i.e. defines a birational map to projective space. Let $L$ be the corresponding line bundle on $C_{s'}$. Explicitly, $L$ is the line bundle associated to a multiple of the sum of the divisors corresponding to the simplices of $\Delta'$ containing $s'$, which are also contained in $s$. We consider the coherent sheaf on $Y$, $(\phi \circ \pi)_*(\mathcal{L})$, where $\pi: C_{s'} \to C_s$ is the restriction of the map from $X'$ to $\mathcal{X}$.

We can find an open set $U$ of $Y$ such that $\phi$ is an isomorphism on $Y$ and $(\phi \circ \pi)^{-1}(U)$ is the trivial toric variety bundle over $\phi^{-1}(U) \cong U$. Furthermore, we can assume that $U$ is the complement of the variety defined by some element $f$ of the global sections of $Y$, since $Y$ is affine. On $U$, the pushforward $(\phi \circ \pi)_*(\mathcal{L})$ is a free sheaf whose generators can be identified with the sections on the toric variety. In particular, these sections define a rational map from $(\phi \circ \pi)^{-1}(U)$ to projective space which is birational on each fiber of $\pi$. Since $Y$ is affine, we lift these sections from $U$ to $Y$ after multiplying by a sufficiently large power of $f$. If we regard $f$ as a section of $\mathcal{O}(\ell D_s)$ for sufficiently large $\ell$, then what we’ve found are sections of $L \otimes \pi^{-1}(\mathcal{O}(\ell D_s))$ which define an embedding of the torus for a generic fiber of $\pi$. Combining these with the pullbacks of the sections defining $\phi$, we get a birational map from $C_{s'}$, and therefore $D_{s'}$ is a big divisor and $\mathcal{X}'$ is robust in dimensions $k \leq m$.

Proof of Theorem 1.1. By base changing, we can assume that $R$ is complete without changing $h^0(X,D)$ or $\Delta$. Let $r$ be an integer less than $\dim H^0(X,\mathcal{O}(D))$ and let $p_1,\ldots,p_r$ be any $r$ rational points in $\Delta$. We want to show that there exists an effective divisor linearly equivalent to $\rho(D)$ containing these points.

We first make a subdivision of $\Delta$ as follows. Let $m$ be the least common denominator of all the coordinates of the points $p_i$. We rescale the simplices by $m$ and subdivide them using the “regular subdivision” of [KKMSD73, Thm. III.2.22], and now the points $p_i$ are vertices. We let $\mathcal{X}'$ be the corresponding toroidal resolution of a ramified extension of $R$. By Lemma 6.5, the locally closed strata of $\mathcal{X}'$ of dimension at most $n-2$ are affine, and it is robust in dimensions $n$ and $n-1$. By Lemma 6.7, the tropical complex of $\mathcal{X}'$ is the subdivision of $\Delta$ as in Construction 4.1. Finally, by Lemmas 4.5 and 4.6 if we can find a divisor linearly equivalent to $\rho(D)$ on $\Delta'$, then it will also be a divisor linearly equivalent to $\rho(D)$ on $\Delta$. Therefore, we replace $\mathcal{X}$ and $\Delta$ with $\mathcal{X}'$ and $\Delta'$ for the rest of the proof.

Using Lemma 6.2, we can choose points $x_1,\ldots,x_r$ in $X$ corresponding to $p_1,\ldots,p_r$ respectively. Since each $x_i$ is rational over the fraction field of $R$, vanishing on $x_i$ imposes one linear condition on the sections $H^0(X,\mathcal{O}(D))$. 

We’ve assumed that \( r \) is less than the dimension of this vector space, so there exists a non-zero section of \( H^0(X, \mathcal{O}(D)) \) defining a divisor \( D' \) which contains all of the \( x_i \). Let \( D'' \) be the refined specialization of \( D' \). Then \( D'' \) is effective by Lemma \[5.2\] and contains the points \( p_1, \ldots, p_r \) by Lemma \[6.2\] By \[Car13\] Prop. 5.2, \( D'' \) is linearly equivalent to \( \rho(D) \). Therefore, \( h^0(\Delta, \rho(D)) \) is greater than \( r \), which establishes the desired inequality. \( \square \)

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