SHARP $L^1$–POINCARÉ INEQUALITIES CORRESPOND TO OPTIMAL HYPERSURFACE CUTS

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Abstract. The sharp constant for the $L^1$–Poincaré inequality in $W^{1,1}(\Omega)$ for convex $\Omega \subset \mathbb{R}^n$ and functions with vanishing mean is identical to an independently studied isoperimetric quantity in convex geometry. This simple consequence of the coarea formula seems to have never been stated explicitly (though, implicitly, it is contained in a paper of Cianchi from 1989). As a consequence, these two problems, sharp Poincaré inequalities and bounds on the isoperimetric quantity, have been tackled independently – here we use the correspondence to unify and improve results in both fields.

1. Introduction

Poincaré inequalities embody the principle that in order for a function to be large, it has to grow. The question of how to exclude constants can be handled in different ways: there is the inequality

$$\|u\|_{L^p(\Omega)} \leq C_1(\Omega, p) \|\nabla u\|_{L^p(\Omega)}$$

for functions $u$ satisfying Dirichlet conditions. Another way to exclude constants is to enforce vanishing mean on the function and to consider inequalities of the type

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u(z)dz \right\|_{L^p(\Omega)} \leq C_2(\Omega, p) \|\nabla u\|_{L^p(\Omega)}.$$

There is a large body of work on which properties $\Omega$ needs to satisfy for such an inequality to hold as well as on the behavior of constants and various interdependences thereof.

The case $p = 1$ is much simpler than the general case: by linearity, one can rearrange the gradient along level lines as desired using the coarea formula. It is then natural to concentrate all growth on a level line that is relatively short while still somehow managing a balanced cut.

Theorem (Cianchi, [5]). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and connected. Then we have the sharp inequality

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u(z)dz \right\|_{L^1(\Omega)} \leq \left( \sup_{E \subset \Omega} \frac{2}{|E|}[S|\Omega \setminus S|/|\Omega|] \right) \|\nabla u\|_{L^1(\Omega)},$$

where $E \subset \Omega$ ranges over all surfaces which divide $\Omega$ into two connected subsets $S$ and $\Omega \setminus S$.

It is easy to see that the constant cannot be improved: take a hypersurface cut realizing the infimum, take a function to be constant on both parts (with the constants chosen such that the arising function has mean zero) and mollify. We will describe a 'geometry-free' version of this result in the last section of the paper.

We should emphasize that the proof is quite simple and requires merely the coarea formula. There is a paper of S.-T. Yau [21], where the problem is being discussed in the case of manifolds without boundary, however, the geometric expression in [21, Theorem 1] is incorrect. It also possible to simply adapt the argument of Lefton & Wei [15] for the case of functions vanishing on the boundary (see also Kawohl & Fridman [14]): they prove that for $f \in H_0^1(\Omega)$

$$\|u\|_{L^1} \leq \left( \sup_{D \subset \Omega} \frac{|D|}{|\partial D|} \right) \|\nabla u\|_{L^1},$$

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where \( D \) ranges over all subsets, where \( \partial D \cap \partial \Omega = \emptyset \). The underlying philosophical insight into the structure of these statements dates back at least to the pioneering work of Cheeger [4].

Remark. The author of this short note originally thought he had discovered the statement himself and was at a loss: here is an easy but seemingly unknown consequence of the coarea formula, which at the same time, when combined with other statements, has a number of unknown strong consequences. The author then decided to upload a preprint on the arxiv and send it to several experts. Bernd Kawohl was aware of similar work of Cianchi and Cianchi himself was then able to point out that the statement appeared as a special case of a Lemma in a 1989 paper of his [5, Lemma 1, special case \( q = 1 \)]. Cianchi also used this quantity in a much more recent paper [6]. Hence, while the statement itself is not new, some of its implications have been overlooked. The purpose of this paper is to point them out and to try to condense the underlying insight in the form of sharp weighted \( L^1 \)-Poincaré inequalities for the unit interval.

2. Existing results

2.1. Poincaré inequalities. We will henceforth restrict ourselves to the case of convex \( \Omega \subset \mathbb{R}^n \). We first mention a sharp inequality due to Acosta & Durán [1]. By now, the inequality is widely used in the study of finite element approximations.

**Theorem** (Acosta & Durán). Let \( \Omega \subset \mathbb{R}^n \) be a convex domain of diameter \( d \). Then

\[
\left\| u - \frac{1}{|\Omega|} \int_{\Omega} u(z) \, dz \right\|_{L^1(\Omega)} \leq \frac{d}{2} \| \nabla u \|_{L^1(\Omega)}
\]

and the constant \( 1/2 \) cannot be improved.

The inequality is easily seen to be sharp by considering the unit interval or, by extension, long, thin cylinders in arbitrary space dimension. It can be regarded the \( L^1 \) analogue of the famous Payne-Weinberger inequality [19], which states that the optimal bound for \( p = 2 \) is given by \( d/\pi \).

Recently, Ferone, Nitsch & Trambonetti [9] gave the optimal constants for \( 1 < p < 2 \).

2.2. Hypersurface cuts. Given any subset \( S \) of a convex body \( \Omega \), it is intuitively clear that the boundary \( \partial S \) cannot be too small unless \( S \) itself is either very small or almost containing all of \( \Omega \) (in which case \( \Omega \setminus S \) is small). The problem was formulated in 1989 by Dyer, Frieze & Kannan [11], who also conjectured a lower bound. A first result states that for convex \( \Omega \) and any subset \( S \subset \Omega \)

\[
\mathcal{H}^{n-1} (\partial S \cap \Omega) \geq \frac{1}{\text{diam}(\Omega)} \min (|S|, |\Omega \setminus S|)
\]

and was, using different methods, independently shown by Lovász & Simonovits [16] and Karzanov & Khachiyan [13]. This statement precludes the existence of bottlenecks – it is of utmost importance in proving that random-walk based approximation algorithms (which could be trapped in some regions if a bottleneck was present) can compute the volume of convex bodies in polynomial time; for this seminal work, Dyer, Frieze & Kannan were awarded the Fulkerson prize in 1991. The sharp constant in the statement was given by Dyer & Frieze [7].

**Theorem** (Dyer & Frieze). Let \( \Omega \subset \mathbb{R}^n \) be a convex domain of diameter \( d \) and \( S \subset \Omega \). Then

\[
\mathcal{H}^{n-1} (\partial S \cap \Omega) \geq \frac{2}{d} \min (|S|, |\Omega \setminus S|)
\]

and the constant 2 is optimal.

The statement is again seen to be sharp by considering long, thin cylinders, where it is optimal to take \( S \) to be one half of the cylinder. The constant is sharp but, as is indicated below, these results are only really sharp for long, thin cylinders – much sharper general bounds are given in a recent preprint of Milman [18].
2.3. Convex geometry. There is a third line of research: while the statements on hypersurface cuts were formulated in terms of merely $n$- and $(n-1)$-dimensional measure, one could also consider other geometric quantities such as the diameter. The direct study of the isoperimetric coefficient
\[ \sup_E \frac{|S||\Omega \setminus S|}{H^{n-1}(E)}, \]
where $E$ ranges over all $C^\infty$-surfaces dividing $\Omega$ into two disjoint parts $S, \Omega \setminus S$ seems to have been initiated by Bokowski & Sperner [2]. Bokowski [3] proved, for example,
\[ \sup_E \frac{|S||\Omega \setminus S|}{H^{n-1}(E)} \leq \left(1 - \frac{1}{2n}\right) \frac{(n-1)}{n(n+1)} \omega_{n-1} \text{diam}(\Omega)^{n+1}, \]
where $\omega_{n-1}$ is the volume of the unit $(n-1)$-ball. Further results are due to Santaló [20], Gysin [10] and Mao [17].

3. Improved inequalities

This section collects new results. The first observation is that while the Acosta-Durán inequality is sharp, the only known examples are one-dimensional in the sense that long, thin cylinders can be regarded as a dummy-variable extension of an interval. Indeed, for basically any other shape the inequalities can be improved.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Then
\[ \|u - \frac{1}{|\Omega|} \int_{\Omega} u(z) dz\|_{L^1(\Omega)} \leq \frac{2}{\log 2} M(\Omega) \|\nabla u\|_{L^1(\Omega)}, \]
where $M(\Omega)$ is the average distance between a point in $\Omega$ and the center of gravity of $\Omega$.

*Proof.* This statement is immediately implied by Cianchi’s theorem and a seminal inequality of Kannan, Lovász & Simonovits [12] stating that
\[ \sup_E \frac{|S||\Omega \setminus S|}{H^{n-1}(E)|\Omega|} \leq \frac{M(\Omega)}{\log 2}. \]

The average distance between a point and the center of gravity is trivially bounded by the diameter. However, for ‘generic’ convex bodies it is typically smaller as they tend to concentrate mass tightly around their center. This effect becomes stronger as the number of dimensions increases; for example, the regular $n$-dimensional simplex of diameter 1 satisfies $M(\text{Simplex}) \sim 1/\sqrt{n}$. In light of recent breakthroughs in the understanding of sharp Poincaré inequalities, it is natural to ask whether it is possible to prove a sharp constant for the Kannan, Lovász & Simonovits inequality (or, equivalently, the sharp constant in Theorem 1.)

There is a rather direct improvement of the Dyer-Frieze inequality using the inequality of Acosta & Durán. This is made possible by advantageously changing the algebraic structure of the bound. Equality is again attained for cylinders but thanks to the different algebraic structure we gain up to a factor of 2 as soon as $|S|/|\Omega|$ is close to either 0 or 1.

**Theorem 2.** Let $\Omega$ be convex and $S \subseteq \Omega$. Then
\[ \mathcal{H}^{n-1}(\partial S \cap \Omega) \geq \frac{4}{\text{diam}(\Omega)} \frac{|S||\Omega \setminus S|}{|\Omega|}, \]
and the constant 4 is optimal.

*Proof.* We identify the boundary $\partial S \cap \Omega$ with a hypersurface separating $\Omega$ in two parts $S$ and $\Omega \setminus S$. Then, using the results of Acosta-Durán and Cianchi, we have
\[ \frac{d}{2} \geq \sup \frac{\|f\|_{L^2}}{\|\nabla f\|_{L^2}} \geq \frac{2}{\sup_{E \subset \Omega} \mathcal{H}^{n-1}(E)} \mathcal{H}^{n-1}(\partial S \cap \Omega) \geq \frac{2}{\mathcal{H}^{n-1}(\partial S \cap \Omega)} \frac{|S||\Omega \setminus S|}{|\Omega|}. \]

Rearranging the terms yields the statement.  \[ \square \]
Note that we can rewrite the right-hand side as
\[
\frac{4}{\text{diam}(\Omega)} \frac{|S|}{|\Omega \setminus S|} = \frac{2}{\text{diam}(\Omega)} \min (|S|, |\Omega \setminus S|) \cdot \frac{2 \max (|S|, |\Omega \setminus S|)}{|\Omega|}.
\]

Finally, we are able to complement existing results on hypersurface cuts in a sharp form: the extremal example is again given by long, thin cylinders. This result is generally weaker than the inequality of Kannan, Lovász & Simonovits, we note it here merely because its constant is sharp and it improves statements from the third line of research (i.e. Bokowski & Sperner [2], ...).

**Theorem 3.** Let \( \Omega \subset \mathbb{R}^n \) be convex. Then we have
\[
\sup_{E} \frac{|S|}{\mathcal{H}^{n-1}(E)} \leq \frac{\text{diam}(\Omega)}{4} |\Omega|,
\]
where the constant 1/4 cannot be improved.

4. **Weighted \( L^1 \)-Poincaré inequalities on the unit interval**

In the previous sections we dealt with a domain \( \Omega \subset \mathbb{R}^n \) and functions \( u : \Omega \to \mathbb{R} \). The level sets are thus at least \((n-1)\)-dimensional and induce a one-parameter partition of \( \Omega \). Ultimately the problem is thus merely one-dimensional in nature. The purpose of this section is to state the corresponding results in one dimension. We are not aware of these results being in the literature but considering the overwhelming amount of papers on classical inequalities it seems extremely likely that they are.

Let \( \nu : [0, 1] \to \mathbb{R}_+ \) be a nonvanishing weight \( \nu(x) > 0 \) on the closed unit interval. We are interested in sharp Poincaré constants for the inequality
\[
\int_0^1 |f(x)| \nu(x) dx \leq C \int_0^1 |\nabla f(x)| \nu(x) dx.
\]
We again need to exclude constant functions and, as before, we can do so prescribing either Dirichlet conditions or vanishing mean value.

**Theorem 4.** Let \( \nu : [0, 1] \to \mathbb{R}_+ \) be a nonvanishing weight \( \nu(x) > 0 \) on the closed unit interval. Let \( f : [0, 1] \to \mathbb{R} \) be once differentiable and satisfy \( f(0) = 0 \). Then
\[
\int_0^1 |f(x)| \nu(x) dx \leq \left( \max_{0 \leq z \leq 1} \frac{1}{\nu(x)} \int_x^1 \nu(z) dz \right) \int_0^1 |\nabla f(x)| \nu(x) dx
\]
and the constant is sharp.

**Proof of Theorem 4.** We choose to write the coarea formula in a more elementary variational form. Without loss of generality \( f \) can be taken to be monotonically increasing. Fix some \( 0 < z < 1 \) and let
\[
g_z(x) = \begin{cases} 
0 & \text{if } x \leq z \\
1 & \text{if } x > z.
\end{cases}
\]
We write \( g_{z, \varepsilon} \) for \( g_z(x) \) mollified in the usual way. Then
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_0^1 (f(x) + \varepsilon g_{z, \varepsilon}(x)) \nu(x) dx - \int_0^1 f(x) \nu(x) dx \right) = \int_z^1 \nu(x) dx
\]
as well as
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_0^1 (\nabla f(x) + \varepsilon g_{z, \varepsilon}(x)) \nu(x) dx - \int_0^1 \nabla f(x) \nu(x) dx \right) = \nu(z).
\]
Now, since \( f(0) = 0 \), it can be written as a linear superposition of
\[
f(x) = \int_0^1 f'(z) g_z(x) dz
\]
and linearity implies the result. \( \square \)
The proof actually shows more: it shows that ‘near-extremizers’ can be characterized: they can only increase at places, where the isoperimetric ratio is ‘large’ i.e. close to the set 

\[ A = \left\{ z \in [0,1] : \frac{1}{v(z)} \int_z^1 v(y)dy = \max_{0 \leq x \leq 1} \frac{1}{v(x)} \int_x^1 v(z)dz \right\}. \]

The proof implies the following: for every \( \varepsilon > 0 \) and \( f : [0,1] \to \mathbb{R}_+ \) satisfying \( f(0) = 0 \) and

\[
\int_0^1 |f(x)|v(x)dx \geq \left( \max_{0 \leq x \leq 1} \frac{1}{v(x)} \int_x^1 v(z)dz \right) \int_0^1 |\nabla f(x)|\nu(x)dx - \varepsilon,
\]

there exists a \( \delta > 0 \) depending only on \( \nu \) and \( \varepsilon \) such that \( \delta \to 0 \) as \( \varepsilon \to 0 \) with the following property: if we define

\[ B = \left\{ z \in [0,1] : \inf_{a \in A} |z - a| \geq \delta \right\}, \]

then we have that

\[
\int_B |f'(x)|dx \leq \delta \int_0^1 |f'(x)|dx
\]

and

\[
\int_{[0,1]\setminus B} f'(x)dx \geq (1 - \delta) \int_0^1 |f'(x)|dx.
\]

In other words the gradient \( |f'| \) is only large on the set \([0,1]\setminus B\) and \( f' \) is mostly positive on that interval. Note that \( A \) might contain more than one point (extremizers need not be unique). Of course, this characterization of ‘near-extremizers’ translates easily into the geometric setting.

**Theorem 5.** Let \( \nu : [0,1] \to \mathbb{R}_+ \) be a nonvanishing weight \( \nu(x) > 0 \) on the closed unit interval. Let \( f : [0,1] \to \mathbb{R} \) be once differentiable and have vanishing weighted mean value

\[
\int_0^1 f(x)\nu(x)dx = 0.
\]

Then

\[
\int_0^1 |f(x)|\nu(x)dx \leq \left( \max_{0 \leq x \leq 1} \frac{2}{\nu(x)} \left( \int_0^x \nu(z)dz \right) \left( \int_x^1 \nu(z)dz \right) \int_0^1 |\nabla f(x)|\nu(x)dx \right)
\]

and the constant is sharp.

This statement could be proven in a similar way as Theorem 4, however, there is a more geometric way to deduce it from Cianchi’s statement. The inequality stays clearly unchanged when multiplying the weight \( \nu \to c \cdot \nu \) for some \( c > 0 \). Let us now consider the planar domain

\[
\Omega = \{(x,y) : 0 \leq x \leq 1 \land 0 \leq y \leq c \cdot \nu(x)\} \subset \mathbb{R}^2.
\]

We now use Cianchi’s theorem to compute the sharp \( L^1 \)-Poincare inequality for functions of mean 0 on \( \Omega \) for \( c \to 0 \). It is easy to show that the optimal separating curve \( E \subset \Omega \) converges to a straight line as \( c \to 0 \) and this implies the statement.

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References

[1] G. Acosta, R. Durán, An optimal Poincaré inequality in $L^1$ for convex domains, Proc. Amer. Math. Soc. 132 (2004), 195–202.
[2] J. Bokowski and E. Sperner, Zerlegung konvexer Körper durch minimale Trennflächen (German), J. Reine Angew. Math. 311/312 (1979), 80–100.
[3] J. Bokowski, Ungleichungen für den Inhalt von Trennflächen (German), Arch. Math. (Basel) 34 (1980), no. 1, 84–89.
[4] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, Problems in analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, 195–199.
[5] A. Cianchi. A sharp form of Poincaré inequalities on balls and spheres. Z. Angew. Math. Phys. 40, 558 - 569, 1989.
[6] A. Cianchi, A sharp trace inequality for functions of bounded variation in the ball. Proc. Roy. Soc. Edinburgh Sect. A 142 (2012), no. 6, 1179–1191.
[7] M. Dyer, A. Frieze, Computing the volume of convex bodies: a case where randomness provably helps, Proc. Sympos. Appl. Math., 44, Amer. Math. Soc., Providence, RI, 1991.
[8] H. Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften 153, Springer, 1969.
[9] V. Ferone, C. Nitsch and C. Trombetti, A remark on optimal weighted Poincaré inequalities for convex domains. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 23 (2012), no. 4, 467–475.
[10] L. Gysin, Inequalities for the product of the volumes of a partition determined in a convex body by a surface, Rend. Circ. Mat. Palermo (2) 35 (1986), no. 3, 420–428.
[11] M. Dyer, A. Frieze, R. Kannan, A Random Polynomial Time Algorithm for Approximating the Volume of a Convex Body, Proc. of the 21st ACM Symposium on Theory of Computing, 375–381.
[12] R. Kannan, L. Lovász, M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, Discrete Comput. Geom. 13 (1995), no. 3-4, 541–559.
[13] A. Karzanov, L. Khachiyan, On the conductance of order Markov chains, Order 8 (1991), no. 1, 7–15.
[14] B. Kawohl, V. Fridman, Isoperimetric estimates for the first eigenvalue of the $p$-Laplace operator and the Cheeger constant, Comment. Math. Univ. Carolin. 44 (2003), no. 4, 659–667.
[15] L. Lefton and D. Wei, Numerical approximation of the first eigenpair of the $p$-Laplacian using finite elements and the penalty method, Numer. Funct. Anal. Optim. 18 (1997), no. 3–4, 389–399.
[16] L. Lovász, M. Simonovits, Random walks in a convex body and an improved volume algorithm, Random Structures Algorithms 4 (1993), no. 4, 359–412.
[17] Q. Mao, On an inequality related to the splitting of a convex body by a plane, Geom. Dedicata 47 (1993), no. 2, 237–239.
[18] E. Milman, Sharp isoperimetric inequalities and model spaces of curvature-dimension-diameter condition, [http://arxiv.org/abs/1108.4609v1](http://arxiv.org/abs/1108.4609v1)
[19] L. Payne, H. Weinberger, An optimal Poincaré inequality for convex domains, Arch. Rational Mech. Anal. 5 (1960), 286–292.
[20] L. Santaló, An inequality between the parts into which a convex body is divided by a plane section, Rend. Circ. Mat. Palermo (2) 32 (1983), no. 1, 124–130.
[21] S. T. Yau, Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold, Ann. Sci. École Norm. Sup. (4) 8 (1975), no. 4, 487–507.

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