INTERPOLATION THEORY
FOR THE HK-FOURIER TRANSFORM

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Abstract. We use the Henstock–Kurzweil integral and interpolation theory to extend the Fourier cosine transform operator, broadening some classical properties such as the Riemann–Lebesgue lemma. Furthermore, we show that a qualitative difference between the cosine and sine transform is preserved on differentiable functions.

1. Introduction

We shall deal with real Banach spaces denoted by $X$ and with their complexification given by $X + iX$. Also, given two Banach spaces $X$ and $Y$, we denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators $T : X \to Y$ with the operator norm given by $\|T\|_{\mathcal{L}(X, Y)} = \sup \{\|T(x)\|_Y : \|x\|_X \leq 1\}$. For any $T \in \mathcal{L}(X, Y)$ we define

$$\tilde{T}(x + iy) := T(x) + iT(y) \quad (x, y \in X).$$

It follows that $\|T\|_{\mathcal{L}(X, Y)} = \|\tilde{T}\|_{\mathcal{L}(X+iX,Y+iY)}$. This procedure has been used by several authors [24, 2, 17].

We recall that for any $p \in [1, \infty)$ and $X \subset \mathbb{R}$, the symbol $L^1(X)$ denotes the space of all Lebesgue measurable functions $f : X \to \mathbb{R}$ with

$$\|f\|_{L^p} := \left(\int_X |f(x)|^p \, dx\right)^{1/p} < \infty.$$ 

Moreover, we denote by $W_p = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = 0 \text{ a.e.}\}$ the subspace of $L^p(X)$ on which $\|\cdot\|_{L^p}$ vanishes. It is known that $\|\cdot\|_{L^p}$ is a seminorm for all $p \in [1, \infty)$ and induces a norm on the quotient space $L^p(X)/W_p$, under which it is complete. We will denote this space with respect to its norm by $L^p(X)$, [27]. Similarly, for $p \in [1, \infty)$ we define $L^p(X, \mathbb{C})$ and $L^p(X, \mathbb{C})$ by considering functions $f : X \to \mathbb{C}$.

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For $p = \infty$ and $f : X \to \mathbb{R}$, we define $\|f\|_\infty$ as the essential supremum of $|f|$, and $L^\infty(X)$ is the vector space of all Lebesgue measurable functions $f$ for which $\|f\|_\infty < \infty$. Similarly, we define $L^\infty(X, \mathbb{C})$ and $L^\infty(X, C)$. If $A \subseteq X$ is a Lebesgue measurable set and $m$ denotes the Lebesgue measure, then given a Lebesgue measurable function $f$ defined on $A$ such that $m(X \setminus A) = 0$ we will denote by the same symbol $f$ the trivial extension of $f$ to a (measurable) function on $X$. Furthermore, for a function $f \in L^p(X)$ or $f \in L^p(X, \mathbb{C})$, we will call by the same symbol $f$ the (unique) element that defines this function in $L^p(X)$ or in $L^p(X, \mathbb{C})$, respectively. Also, the characteristic function of a set $E$ is given by $\chi_E(x) = 1$ if $x \in E$ and zero otherwise.

If $f$ belongs to $L^1(\mathbb{R}) \setminus L^p(\mathbb{R})$, the Fourier transform is defined for every real number $s$ as

$$
\mathcal{F}_p(f)(s) := \int_{\mathbb{R}} e^{-isx} f(x) \, dx = \int_{\mathbb{R}} \cos(sx) f(x) \, dx - i \int_{\mathbb{R}} \sin(sx) f(x) \, dx
$$

with

$$
\|\mathcal{F}_p(f)\|_p \leq \gamma_p \|f\|_q,
$$

where the integral is taken in the Lebesgue sense. $\mathcal{F}_p^c$ and $\mathcal{F}_p^s$ are called Fourier cosine and Fourier sine transforms, respectively. Furthermore, by interpolation theory, the operator $\mathcal{F}_p(f)$ is extended to $L^p(\mathbb{R})$ for $p \in [1, 2]$ as a bounded operator

$$
\mathcal{F}_p : L^p(\mathbb{R}) \to L^q(\mathbb{R})
$$

where $1/p + 1/q = 1$ and

$$
\gamma_p = \begin{cases} 
1 & \text{if } p = 1, \\
(2\pi)^{\frac{1}{2}} \left( \frac{p-1}{p} \right)^{\frac{1}{p}} \left( \frac{p+1}{2p} \right)^{\frac{1}{2p}} & \text{if } 1 < p \leq 2.
\end{cases}
$$

The value of $\gamma_p$ is given by the Hausdorff–Young inequality [25], the sharp Hausdorff–Young inequality [15, Theorem 5.7] and [3].

For any unbounded subset $X \subset \mathbb{R}$, the space $C_\infty(X)$ denotes the complex valued continuous functions on $X$ vanishing at infinity [25]. We denote the space of bounded variation functions by $BV(\mathbb{R})$ and by $BV_0(\mathbb{R})$ the subspace of functions vanishing at infinity, [12, 4, 31]. Also $BV_0(\mathbb{R}, \mathbb{C})$ is the corresponding complexification of $BV_0(\mathbb{R})$.

In [30] the Henstock–Kurzweil integral was employed to study the Fourier transform. In [20, 22] it was proved that (1.1) makes sense as a Henstock–Kurzweil integral on $BV_0(\mathbb{R})$. In fact, we have the following statement in [23].
Definition 1.1. The HK-Fourier transform exists for every \( s \neq 0 \), and is defined by

\[
\mathcal{F}_{HK} : L^1(\mathbb{R}) + BV_0(\mathbb{R}) \to C_\infty(\mathbb{R}\setminus\{0\}),
\]

\[
\mathcal{F}_{HK}(f)(s) := \int_{-\infty}^{\infty} e^{-isx} f(x) \, dx,
\]

where the integral is in the Henstock–Kurzweil sense. Analogously, we define the HK-Fourier cosine transform \( \mathcal{F}_c^{HK} \) and the HK-sine Fourier transform \( \mathcal{F}_s^{HK} \) as in (1.1).

We say “HK-Fourier transform” in order to emphasize the use of the Henstock–Kurzweil integral \([30]\). Moreover, \( \mathcal{F}_{HK}(f)(s) \) is pointwise defined and is continuous except at zero; see example 3(d) in \([30]\). Note that \( \mathcal{F}_{HK} \) is well defined because the Henstock–Kurzweil integral contains the Lebesgue integral, \([14, 19]\). \( \mathcal{F}_1 \) can be seen as an extension of the HK-Fourier transform restricted to \( BV_0(\mathbb{R}) \),

\[
\mathcal{F}_{HK} : BV_0(\mathbb{R}) \to C_\infty(\mathbb{R}\setminus\{0\}).
\]

Moreover, \( \mathcal{F}_p \) is an extension of \( \mathcal{F}_1 \), so that \( \mathcal{F}_p \) is an extension of \( \mathcal{F}_{HK} \).

The relation between \( \mathcal{F}_p \) and \( \mathcal{F}_{HK} \) was first studied in \([23]\), while the operator \( \mathcal{F}_c^{HK} \) was studied in \([3]\). This work builds on these references.

2. Henstock–Kurzweil Fourier transform

The space of Henstock–Kurzweil integrable functions defined on an interval \( I \) is denoted by \( \mathcal{H}K(I) \). This space is a seminormed space with the Alexiewicz seminorm, defined as

\[
\|f\|_{\mathcal{H}K} = \sup \left\{ \left| \int_c^d f(x) \, dx \right| : [c, d] \subset I \right\}.
\]

The quotient space \( \mathcal{H}K/W(I) \) will be denoted by \( HK(I) \), where \( W(I) \) is the subspace of \( HK(I) \) for which the Alexiewicz seminorm vanishes \([7]\). The completion will be denoted by \( \widehat{HK}(I) \) and its complexification will be written as \( \widehat{HK}(\mathbb{R}, \mathbb{C}) \).

We study the HK-Fourier cosine transform defined by

\[
\mathcal{F}_{c}^{HK}(f)(s) = \int_{-\infty}^{\infty} \cos(sx) f(x) \, dx \quad (s \neq 0).
\]

Notice that for \( s = 0 \) and \( f \in BV_0(\mathbb{R}) \), \( \mathcal{F}_{c}^{HK}(f)(0) \) might not be defined. Also, we have that

\[
\mathcal{F}_{c}^{HK}(f)(s) = \mathcal{F}_{c}^{HK}(f)(s)
\]

(2.1)

for all \( f \in L^1(\mathbb{R}) \cap BV_0(\mathbb{R}) \) and \( s \in \mathbb{R} \). However, a partial result about the question of continuity at \( s = 0 \) was proved in \([3, Theorem 1]\). In fact, \( \mathcal{F}_{c}^{HK} \) is bounded while \( \mathcal{F}_{s}^{HK} \) is not. Actually, Theorem 1 and Proposition 3 in \([3]\) imply the following statement.
Theorem 2.1.

(i) The HK-Fourier cosine transform is a bounded linear operator from $BV_0(\mathbb{R})$ into $HK(\mathbb{R})$.

(ii) The Fourier transform is a densely defined closed operator from $L^2(\mathbb{R})$ into $HK(\mathbb{R})$.

We shall show that differences and similitudes between the Fourier cosine and Fourier sine transforms also hold on the classical Sobolev space $W^{1,1}(\mathbb{R})$. It is expected that these transforms are bounded operators with the same domain and codomain for functions with enough smoothness, for example as in the Schwartz space [25]. See also [16].

3. Interpolation theory

We consider a couple $(X,Y)$ of complex Banach spaces such that $X$ and $Y$ are continuously embedded in a Hausdorff topological vector space $V$, i.e., $X \subset V$ and $Y \subset V$ with continuous inclusion. This couple is called a complex interpolation couple. In this case the intersection $X \cap Y$ is a linear subspace of $V$, and it is a Banach space under the norm

$$\|v\|_{X \cap Y} = \max\{\|v\|_X, \|v\|_Y\}.$$ 

The sum $X + Y = \{x + y : x \in X, y \in Y\}$ is a linear subspace of $V$ and it is endowed with the norm

$$\|v\|_{X + Y} = \inf\{\|x\|_X + \|y\|_Y : x \in X, y \in Y, x + y = v\}.$$ 

Remark 3.1. It follows from [18] that the space $X + Y$ is isometric to the quotient space $(X \times Y)/D$, where $D = \{(d, -d) : d \in X \times Y\}$. Since $V$ is a Hausdorff space, $D$ is closed, so $X + Y$ is a Banach space. Moreover, $X$ and $Y$ are continuously embedded in $X + Y$.

Throughout this section we shall consider $S = \{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1\}$ and we shall use the complex space $X + Y$ and the space $\mathcal{F}(X, Y)$ of functions $f : S \to X + Y$ holomorphic on the interior of the strip $S$ and continuous up to its boundary, such that the maps $t \mapsto f(it)$ and $t \mapsto f(1 + it)$ are continuous from the real line into $X$ and $Y$, respectively. Therefore, $\mathcal{F}(X, Y)$ is a Banach space with the norm given by

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_X, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_Y \right\} < \infty.$$ 

These facts can be consulted in [18] Ch. 2, [6] Ch. 4, [9] Ex. 2.6.6, [28] Ch. 2, [13] Ch. 4 and [10] 1–4.

Definition 3.2. For every $\theta \in (0, 1)$, the space $[X,Y]_\theta$ consists of all $a \in X + Y$ such that $a = f(\theta)$ for some $f \in \mathcal{F}(X, Y)$ and the norm on $[X,Y]_\theta$ is

$$\|a\|_{[X,Y]_\theta} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = a, f \in \mathcal{F}(X, Y)\}.$$ 

Remark 3.3. The space $X \cap Y$ is dense in $[X,Y]_\theta$ and $[X,Y]_\theta$ is isomorphic to the quotient space $\mathcal{F}(X, Y)/\mathcal{N}_\theta$, where $\mathcal{N}_\theta$ is the subset of $\mathcal{F}(X, Y)$ consisting of the functions vanishing at $z = \theta$. Moreover, $\mathcal{N}_\theta$ is closed (see [6] [13]).
Theorem 3.4. The space $[X,Y]_{\theta}$ is a Banach space and an intermediate space with respect to $(X,Y)$, i.e.,

$$X \cap Y \subset [X,Y]_{\theta} \subset X + Y$$

with continuous inclusion.

Remark 3.5. It follows from [18] Corollary 2.8, Proposition 2.10 that for each $\theta \in (0,1)$,

$$(X,Y)_{\theta,1} \subset [X,Y]_{\theta} \subset (X,Y)_{\theta,\infty},$$

where the spaces $(X,Y)_{\theta,p}$ are defined by the real method of interpolation. See also [6] Theorem 4.7.1.

Theorem 3.6. Let $(X_1,Y_1),(X_2,Y_2)$ be complex interpolation couples. If $T$ belongs to $\mathcal{L}(X_1,X_2) \cap \mathcal{L}(Y_1,Y_2)$, then the restriction of $T$ to $[X_1,Y_1]_{\theta}$ belongs to $\mathcal{L}([X_1,Y_1]_{\theta},[X_2,Y_2]_{\theta})$ for every $\theta \in (0,1)$. Moreover,

$$\|T\|_{\mathcal{L}([X_1,Y_1]_{\theta},[X_2,Y_2]_{\theta})} \leq \|T\|_{\mathcal{L}(X_1,X_2)}^{1-\theta} \|T\|_{\mathcal{L}(Y_1,Y_2)}^\theta.$$

In order to construct the interpolation space of $L^1(\mathbb{R})$ and $BV_0(\mathbb{R})$ we consider the space $\mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})$ with given norm $\|\cdot\|_{\mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})} := \max\{\|\cdot\|_{L^1}, \|\cdot\|_{BV}\}$.

Lemma 3.7. $\mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})$ is a Banach space with the given norm.

Proof. Since $BV_0(\mathbb{R})$ is a Banach space, then given a Cauchy sequence $(f_n)_{n \geq 1}$ on $\mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})$ there is $f \in BV_0(\mathbb{R})$ such that

$$\|f_n - f\|_{BV} \to 0 \quad (n \to \infty).$$

This yields uniform convergence of the sequence to $f$. Similarly, there exists $[\tilde{f}] \in L^1(\mathbb{R})$ such that

$$\|f_n - \tilde{f}\|_{L^1} \to 0 \quad (n \to \infty).$$

It follows that there exists a subsequence $(f_{n_k})_{k \geq 1}$ of $(f_n)_{n \geq 1}$ converging pointwise a.e. to $f$; see [27] [8]. From the fact that $(f_n)_{n \geq 1}$ converges uniformly to $f$, we get that $f(x) = \tilde{f}(x)$ a.e., yielding $f \in \mathcal{L}^1(\mathbb{R})$ and

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f_n(x) - f(x)| \, dx = 0.$$ 

On the product space $\mathcal{L}^1(\mathbb{R}) \times BV_0(\mathbb{R})$ with given norm $\|(f,g)\|_{\mathcal{L}^1(\mathbb{R}) \times BV_0} := \|f\|_{L^1} + \|g\|_{BV}$, we consider the quotient space $(\mathcal{L}^1(\mathbb{R}) \times BV_0(\mathbb{R}))/D$ where $D := \{(f,-f) \in \mathcal{L}^1(\mathbb{R}) \times BV_0(\mathbb{R}) : f \in \mathcal{L}^1(\mathbb{R}) \cap BV_0(\mathbb{R})\}$. So, we set

$$\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}) := (\mathcal{L}^1(\mathbb{R}) \times BV_0(\mathbb{R}))/D.$$

Therefore, if $a \in \mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})$, then it is an equivalence class given by $a = (f,g) + D$. Nevertheless, we shall write $a = f + g$ to simplify notation. Also, we define

$$\|a\|_{\mathcal{L}^1 + BV_0} := \inf_{(h,-h) \in D} \|f - h\|_{L^1} + \|g + h\|_{BV}.$$

This is a norm, by standard arguments. Then we consider the completion of the space $\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})$, denoted by $\overline{\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})}$. In addition, on the product
space \( L^1(\mathbb{R}) \times BV_0(\mathbb{R}) \) with the usual norm \( \|([f], g)\|_{L^1 \times BV_0} = \|f\|_{L^1} + \|g\|_{BV_0} \), we make

\[
D' = \{ ([f], -f) \in L^1(\mathbb{R}) \times BV_0(\mathbb{R}) : f \in L^1(\mathbb{R}) \cap BV_0(\mathbb{R}) \}. 
\]

Due to \( L^1(\mathbb{R}) \cap BV_0(\mathbb{R}) \) being complete, \( D' \) is closed in \( L^1(\mathbb{R}) \times BV_0(\mathbb{R}) \). We define

\[
L^1(\mathbb{R}) + BV_0(\mathbb{R}) = (L^1(\mathbb{R}) \times BV_0(\mathbb{R}))/D'.
\]

Thus the sum space \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \) is a Banach space with the quotient norm \([26]\).

Its elements are equivalence classes of the form \( \bar{a} = [f + g] = ([f], g) + D' \); however, we will just write \( \bar{a} = f + g \). We have the following characterization.

**Proposition 3.8.** The space \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \) is isometric to \( \widehat{L^1(\mathbb{R}) + BV_0(\mathbb{R})} \).

**Proof.** \( f \in L^1(\mathbb{R}) \) yields \( [f] \in L^1(\mathbb{R}) \) with \( \|[f]\|_{L^1} = \|f\|_{L^1} \). Conversely, if \([f]\) belongs to \( L^1(\mathbb{R}) \) then there is \( \tilde{f} \in L^1(\mathbb{R}) \) such that \( f = \tilde{f} \) a.e. Then, for each \( a = (f, g) + D \in L^1(\mathbb{R}) + BV_0(\mathbb{R}) \), we define \( \bar{a} = ([f], g) + D' \) in \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \). We get

\[
\|(f, g) + D\|_{L^1 + BV_0} = \inf_{(h, -h) \in D} \|f - h\|_{L^1} + \|g + h\|_{BV} = \inf_{(h, -h) \in D'} \|[f - h]\|_{L^1} + \|g + h\|_{BV} = \|([f], g) + D'\|_{L^1 + BV_0}.
\]

Therefore, the map \( a \mapsto \bar{a} \) from \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \) into \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \) has dense range, due to \( L^1(\mathbb{R}) \) being dense in \( L^1(\mathbb{R}) \). The map extends to an isometry from the completion \( \widehat{L^1(\mathbb{R}) + BV_0(\mathbb{R})} \) onto \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \), implying the Proposition. \( \square \)

Therefore, we have characterized the real space \( L^1(\mathbb{R}) + BV_0(\mathbb{R}) \). The complexification space of this space is given by

\[
L^1(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C}) := (L^1(\mathbb{R}) + BV_0(\mathbb{R})) + i(L^1(\mathbb{R}) + BV_0(\mathbb{R})).
\]

Similarly, we define the real space \( L^\infty(\mathbb{R}) + \overline{HK}(\mathbb{R}) \) and its complexification \( L^\infty(\mathbb{R}, \mathbb{C}) + \overline{HK}(\mathbb{R}, \mathbb{C}) \). We will consider complex spaces and omit the symbol \( (\mathbb{R}, \mathbb{C}) \) to simplify notation. Furthermore, for the complex interpolation couples

\[
(L^1, BV_0) \quad \text{and} \quad (L^\infty, \overline{HK}) \tag{3.1}
\]

we say that \( T \) is a bounded linear operator from \( (L^1, BV_0) \) to \( (L^\infty, \overline{HK}) \) if and only if \( T \in \mathcal{L}(L^1 + BV_0, L^\infty + \overline{HK}) \) such that \( T \in \mathcal{L}(L^1, L^\infty) \) and \( T \in \mathcal{L}(BV_0, \overline{HK}) \).

We say that the complex spaces \( \mathfrak{A} \) and \( \mathfrak{B} \) are intermediate spaces between the couples in (3.1) if and only if

\[
L^1 \cap BV_0 \subset \mathfrak{A} \subset L^1 + BV_0 \quad \text{and} \quad L^\infty \cap \overline{HK} \subset \mathfrak{B} \subset L^\infty + \overline{HK}.
\]

\( \mathfrak{A} \) and \( \mathfrak{B} \) are called interpolation spaces with respect to the couples in (3.1) if and only if \( \mathfrak{A} \) and \( \mathfrak{B} \) are intermediate spaces with the following property: \( T \in \mathcal{L}(L^1 + BV_0, L^\infty + \overline{HK}) \) implies that the restriction of \( T \) to \( \mathfrak{A} \) belongs to \( \mathcal{L}(\mathfrak{A}, \mathfrak{B}) \).

From Theorem 3.4 we have the interpolation spaces \( [L^1, BV_0]_\theta \) and \( [L^\infty, \overline{HK}]_\theta \) with respect to \( (L^1, BV_0) \) and \( (L^\infty, \overline{HK}) \) for each \( \theta \in (0, 1) \). We deal with the
operators $\mathcal{F}_1^c$ and $\mathcal{F}^c_{\text{HK}}$ given in (1.1) and Definition 1.1 and their extensions to the complexification of the spaces. We use the same symbols for the extended operators. Then we define the operator

$$
\mathfrak{F}_1^c : L^1(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C}) \rightarrow L^\infty(\mathbb{R}, \mathbb{C}) + \hat{\mathbb{H}}k(\mathbb{R}, \mathbb{C})
$$

(3.2)

$$
\mathfrak{F}_1^c(f + g)(s) := \mathcal{F}_1^c(f)(s) + \mathcal{F}^c_{\text{HK}}(g)(s).
$$

Formula (3.2) is well defined on $L^1(\mathbb{R}) + BV_0(\mathbb{R})$. By interpolation theory, $\mathfrak{F}_1^c$ is extended to $L^1(\mathbb{R}) + BV_0(\mathbb{R})$ for each $s \neq 0$. Thus, from Theorem 2.1 and Theorem 3.6 we conclude that

$$
\mathfrak{F}_1^c \in \mathcal{L}([L^1, BV_0]_\theta, [L^\infty, \hat{\mathbb{H}}k]_\theta).
$$

The following estimate for its norm is valid:

$$
\|\mathfrak{F}_1^c\|_{\mathcal{L}(L^1, BV_0}_\theta, [L^\infty, \hat{\mathbb{H}}k]_\theta) \leq \|\mathcal{F}_1^c\|_{\mathcal{L}(L^1, L^\infty)}^{1-\theta} \|\mathcal{F}_{\text{HK}}\\|_{\mathcal{L}(BV_0, HK)}^\theta \leq \mathfrak{C}^\theta,
$$

for every $\theta \in (0, 1)$, where

$$
\mathfrak{C} = 4\pi \text{Si}(\pi) \quad \text{and} \quad \text{Si}(x) := \frac{2}{\pi} \int_0^x \frac{\sin(y)}{y} dy.
$$

Furthermore, from Remark 3.5 we have that

$$
(L^1, BV_0)_{\theta, 1} \subset [L^1, BV_0]_\theta \subset (L^1, BV_0)_{\theta, \infty}
$$

holds for each $\theta \in (0, 1)$.

**Proposition 3.9.** For $f \in [L^1, BV_0]_\theta$, the formula

$$
\mathfrak{F}_1^c(f)(s) = \int_{-\infty}^{\infty} \cos(sx)f(x) \, dx
$$

holds true pointwise almost everywhere and the Riemann–Lebesgue lemma is satisfied: $\mathfrak{F}_1^c(f)(s) \to 0$ as $|s| \to \infty$.

**Proof.** If $f = f_1 + f_0 = \tilde{f}_1 + \tilde{f}_0$ belongs to $L^1(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C})$, then

$$
(f_1 - \tilde{f}_1, f_0 - \tilde{f}_0) \in D'.
$$

This yields $f_1 - \tilde{f}_1 = \tilde{f}_0 - f_0$ with $f_1 - \tilde{f}_1 \in L^1(\mathbb{R}, \mathbb{C})$ and $\tilde{f}_0 - f_0 \in BV_0(\mathbb{R}, \mathbb{C})$.

Since $\mathcal{F}_1^c$ and $\mathcal{F}_{\text{HK}}$ coincide on $L^1(\mathbb{R}, \mathbb{C}) \cap BV_0(\mathbb{R}, \mathbb{C})$ due to (2.1), we conclude that

$$
\mathcal{F}_1^c(f_1) + \mathcal{F}_{\text{HK}}(f_0) = \mathcal{F}_1^c(\tilde{f}_1) + \mathcal{F}_{\text{HK}}(\tilde{f}_0).
$$

As a consequence, the value of $\mathfrak{F}_1^c(f)(s)$ does not depend on the representation of $f \in [L^1, BV_0]_\theta$, for each $\theta \in (0, 1)$. From Theorem 3.6, for every $f \in [L^1, BV_0]_\theta$, there exist $f_1 \in L^1(\mathbb{R}, \mathbb{C})$ and $f_0 \in BV_0(\mathbb{R}, \mathbb{C})$ such that $f = f_1 + f_0$ and for each
\[ s \neq 0, \]
\[ \mathcal{F}_1^c(f)(s) = \mathcal{F}^c(f_1 + f_0)(s) \]
\[ = \mathcal{F}_1^c(f_1)(s) + \mathcal{F}_1^c(f_0)(s) \]
\[ = \int_{-\infty}^{\infty} \cos(sx)f_1(x)\,dx + \int_{-\infty}^{\infty} \cos(sx)f_0(x)\,dx \]
\[ = \int_{-\infty}^{\infty} \cos(sx)(f_1(x) + f_0(x))\,dx \]
\[ = \int_{-\infty}^{\infty} \cos(sx)f(x)\,dx. \]  

(3.4)

Then, (3.4) establishes that the HK-Fourier cosine transform on \([L^1, BV_0]\) has an integral representation. From the Riemann–Lebesgue lemma [21, Lemma 2] we conclude that \(F_1^c(f)(s) \to 0\) as \(|s| \to \infty\).

In [1, 8], the Sobolev spaces \(W^{1,p}(\mathbb{R})\) are defined and for their complexification \(W^{1,p}(\mathbb{R}, \mathbb{C}) := W^{1,p}(\mathbb{R}) + iW^{1,p}(\mathbb{R})\) we have the next statement.

**Lemma 3.10.** For each \(\theta \in (0, 1)\),
\[ W^{1,1}(\mathbb{R}, \mathbb{C}) \subset [L^1(\mathbb{R}, \mathbb{C}), BV_0(\mathbb{R}, \mathbb{C})]_{\theta} \]
with continuous inclusion.

**Proof.** First we recall that \(W^{1,1}(\mathbb{R}) \subset L^1(\mathbb{R}) \cap BV_0(\mathbb{R})\). From [4, Theorem 7.5] we have
\[ \|u\|_{L^1 \cap BV_0} := \max\{\|u\|_{L^1}, \|u\|_{BV}\} \leq \|u\|_{L^1} + \|u\|_{BV} \]
\[ = \|u\|_{L^1} + \|u'\|_{L^1} \]
\[ = \|u\|_{W^{1,1}}. \]

If \(u\) belongs to \(W^{1,1}(\mathbb{R}, \mathbb{C})\), from [18, Proposition 2.4] we get
\[ \|u\|_{[\theta]} \leq \max\{\|u\|_{L^1}, \|u\|_{BV}\} \leq \|u\|_{W^{1,1}(\mathbb{R}, \mathbb{C})}, \]
for every \(\theta \in (0, 1)\). \(\square\)

**Corollary 3.11.** For \(u \in W^{1,1}(\mathbb{R}, \mathbb{C})\), \(\mathcal{F}_1^c(u)\) belongs to \(HK(\mathbb{R}, \mathbb{C})\).

The proof of Corollary 3.11 follows from the fact that \(W^{1,1}(\mathbb{R}) \subset BV_0(\mathbb{R})\), and then by Theorem 2.1
\[ \mathcal{F}_{HK}^c(W^{1,1}(\mathbb{R})) \subset HK(\mathbb{R}). \]

Therefore, the range of the Sobolev space \(W^{1,1}(\mathbb{R}, \mathbb{C})\) under the HK-Fourier cosine transform is contained in \(HK(\mathbb{R}, \mathbb{C})\). Explicitly,
\[ \mathcal{F}_1^c(W^{1,1}(\mathbb{R}, \mathbb{C})) \subset HK(\mathbb{R}, \mathbb{C}). \]

(3.5)

The Fourier cosine and sine transforms are continuous operators on \(L^2(\mathbb{R}, \mathbb{C})\), while their qualitative differences appear even on the space of functions \(W^{1,1}(\mathbb{R}, \mathbb{C})\) that have a degree of regularity. In the following example we show this difference.
Example 3.12. Let us define
\[ h(x) := \begin{cases} \frac{1}{2 - \log(x)} & \text{if } x \in (0, 1], \\ 0 & \text{if } x > 1. \end{cases} \]
For each \( x \in (0, 1) \), we have \( h'(x) = \frac{1}{x[2 - \log(x)]^2} \).

We extend \( h \) over \( \mathbb{R} \) as an odd map. Also, we consider an even function \( \varphi \in C_c^\infty(\mathbb{R}) \) such that \( 0 \leq \varphi(x) \leq 1 \), with \( \varphi(x) = 1 \) for \( |x| \leq 1/2 \) and vanishing for \( |x| \geq 1 \). We define \( f(x) := h(x)\varphi(x) \), for all \( x \in \mathbb{R} \). Thus, \( f \) is an odd map belonging to \( W^{1,1}(\mathbb{R}) \subset L^1(\mathbb{R}) \setminus BV_0(\mathbb{R}) \) and
\[ F_{HK}^s(f)(s) = 2 \int_0^\infty \sin(s x) f(x) \, dx, \quad \text{for all } s \geq 0. \]

We analyze the convergence of the Henstock–Kurzweil integral:
\[ \int_0^\infty F_{HK}^s(f)(s) \, ds. \quad (3.6) \]

For any fixed \( s > 0 \), the map \( x \mapsto \sin(s x) \) belongs to \( HK[0, M] \) for \( 0 < M < \infty \), and we have
\[ \| \sin(s \cdot) \|_{HK[0,M]} = \sup_{0<u<v<M} \left| \int_u^v \sin(st) \, dt \right| \leq \frac{2}{s}. \]
Thus, for \( 0 < b < \infty \), we get from Lebesgue’s dominated convergence theorem, Fubini’s theorem and Hake’s theorem [4]:
\[ \int_0^b \int_0^{\infty} \sin(s x) f(x) \, dx \, ds = \lim_{M \to \infty} \int_0^M \frac{1 - \cos(b x)}{x} f(x) \, dx. \]

In fact,
\[ \int_0^{M} \frac{1 - \cos(b x)}{x} f(x) \, dx = \int_0^{b} \frac{1 - \cos(y)}{y} f(y/b) \, dy. \]

Now, for \( \delta = 1/4 \),
\[ \int_0^{b} \frac{1 - \cos(y)}{y} f(y/b) \, dy = \int_0^{\delta} \frac{1 - \cos(y)}{y} f(y/b) \, dy \quad + \int_{\delta}^{b} \frac{1 - \cos(y)}{y} f(y/b) \, dy. \quad (3.7) \]

Since \( f(y/b) \to 0 \) as \( b \to \infty \), we have that
\[ \lim_{b \to \infty} \int_0^{\delta} \frac{1 - \cos(y)}{y} f(y/b) \, dy = 0. \]

For the second integral on the right side of (3.7) we have
\[ \int_{\delta}^{b} \frac{1 - \cos(y)}{y} f(y/b) \, dy = \int_{\delta}^{b} \frac{f(y/b)}{y} \, dy + \int_{\delta}^{b} \frac{-\cos(y)}{y} f(y/b) \, dy = I_1 + I_2. \]
Integrating by parts,
\[
\lim_{b \to \infty} I_2 = \lim_{b \to \infty} \int_{\delta}^{b} \sin(y) \left( \frac{b^{-1}y f'(y/b) - f(y/b)}{y^2} \right) dy
= \lim_{b \to \infty} \int_{\delta/b}^{1} \frac{\sin(bt)}{bt} f'(t) dt - \int_{\delta}^{b} \frac{\sin(y)f(y/b)}{y^2} dy.
\]

By Lebesgue’s dominated convergence theorem we conclude that
\[
\lim_{b \to \infty} \int_{\delta}^{b} \frac{\sin(y)f'(y/b)}{by} dy = \lim_{b \to \infty} \int_{\delta}^{b} \frac{\sin(y)}{y^2} f(y/b) dy = 0.
\]

Therefore the limit of \(I_2\) is zero. Writing explicitly the integrand of the integral \(I_1\) we get
\[
I_1 = \int_{\delta/b}^{1} \frac{1}{u[2 - \log(u)]} + \int_{1/2}^{1} \frac{\varphi(u)}{u[2 - \log(u)]} du
\]
and
\[
\lim_{b \to \infty} \int_{\delta/b}^{1/2} \frac{1}{u[2 - \log(u)]} du = \infty.
\]

Therefore, the integral in (3.6) does not exist and \(\mathcal{F}_{\text{HK}}^c(f)\) does not belong to \(HK(\mathbb{R})\). In conclusion,
\[
\mathcal{F}_{\text{HK}}^c(W^{1,1}) = \mathcal{F}_{\text{HK}}^c(W^{1,1}) = \mathcal{F}_{\text{HK}}^c(W^{1,1}(\mathbb{R})) \not\subset HK(\mathbb{R}). \tag{3.8}
\]

Then, the HK-Fourier sine transform remains unbounded on \(W^{1,1}(\mathbb{R})\), in contrast with relation (3.5). Also, in [3, Example 1] it was established that
\[
\mathcal{F}_{\text{HK}}(BV_0(\mathbb{R}) \setminus W^{1,1}(\mathbb{R})) \not\subset HK(\mathbb{R}).
\]

The function given in Example 3.12 is a slight variation of one considered in [11].

We can proceed in the same way to define the space \(L^2(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C})\). Therefore, we have the continuous inclusions
\[
L^2(\mathbb{R}, \mathbb{C}) \cap BV_0(\mathbb{R}, \mathbb{C}) \subset [L^2(\mathbb{R}, \mathbb{C}), BV_0(\mathbb{R}, \mathbb{C})]_{\theta} \subset L^2(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C}),
\]
for \(0 < \theta < 1\). Now for the extended operators \(\mathcal{F}_{\text{HK}}^c\) and \(\mathcal{F}_{\text{HK}}^c\) on \(L^2(\mathbb{R}, \mathbb{C})\) and on \(BV_0(\mathbb{R}, \mathbb{C})\) respectively, we define the map
\[
\mathcal{F}_{\text{HK}}^c : L^2(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C}) \longrightarrow L^2(\mathbb{R}, \mathbb{C}) + \hat{HK}(\mathbb{R}, \mathbb{C})
\]
\[
\mathcal{F}_{\text{HK}}^c(f + g) := \mathcal{F}_{\text{HK}}^c(f) + \mathcal{F}_{\text{HK}}^c(g).
\]

So, by Theorem 3.6 we have
\[
\mathcal{F}_{\text{HK}}^c \in \mathcal{L}([L^2, BV_0]_\theta, [L^2, \hat{HK}]_\theta),
\]
with the following estimate for its norm:
\[
\|\mathcal{F}_{\text{HK}}^c\|_{\mathcal{L}([L^2, BV_0]_\theta, [L^2, \hat{HK}]_\theta)} \leq \|\mathcal{F}_{\text{HK}}^c\|_{\mathcal{L}(L^2, L^2)^{1-\theta}} \|\mathcal{F}_{\text{HK}}^c\|_{\mathcal{L}(BV_0, \hat{HK})} \leq (2\pi)^{\frac{1-\theta}{2}} e^\theta.
\]
for every $\theta \in (0, 1)$ and $c$ given by (3.3).

Similarly, for the operators $F^c_p$ and $F^c_{HK}$ on $L^p(\mathbb{R}, \mathbb{C})$ and on $BV_0(\mathbb{R}, \mathbb{C})$ respectively, we define

$$\mathfrak{F}^c_p : L^p(\mathbb{R}, \mathbb{C}) + BV_0(\mathbb{R}, \mathbb{C}) \rightarrow L^q(\mathbb{R}, \mathbb{C}) + \mathfrak{H}K(\mathbb{R}, \mathbb{C})$$

$$\mathfrak{F}^c_p(f) := F^c_p(f_p) + F^c_{HK}(g),$$

where $f = f_p + g$ with $1/p + 1/q = 1$. This operator is a generalization of the map considered in [23, Corollary 1]. For the couples given by $X_1 = L^p(\mathbb{R}, \mathbb{C})$ and $X_2 = L^q(\mathbb{R}, \mathbb{C})$, with $1 \leq p \leq 2$, and $Y_1 = BV_0(\mathbb{R}, \mathbb{C})$ and $Y_2 = \mathfrak{H}K(\mathbb{R}, \mathbb{C})$ we have from Theorem 3.6 that

$$\mathfrak{F}^c_p \in \mathcal{L}([L^p, BV_0], [L^q, \mathfrak{H}K])$$

for every $\theta \in (0, 1)$, and the following estimate for the norm:

$$\|\mathfrak{F}^c_p\|_{\mathcal{L}(L^p, BV_0; L^q, \mathfrak{H}K)} \leq \|F^c_p\|_{\mathcal{L}(L^p, L^q)}^{1-\theta} \|F^c_{HK}\|_{\mathcal{L}(BV_0, \mathfrak{H}K)}^{\theta} \leq \gamma^{1-\theta} C^\theta,$$

where $C$ is given in (3.3).

For $1 < p < 2$, the relation between $\mathfrak{F}^c_1$, $\mathfrak{F}^c_2$ and $\mathfrak{F}^c_3$ is given by the decomposition of $L^p(\mathbb{R})$ in [23], which implies that for each $f_p + g \in L^p(\mathbb{R}) + BV_0(\mathbb{R})$ there exist $f_1 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ such that

$$f_p + g = (f_1 + g) + (f_2 + f).$$

**Corollary 3.13.** For $1 < p < 2$, $\mathfrak{F}^c_p(f_p + g) = \mathfrak{F}^c_1(f_1 + g) + \mathfrak{F}^c_2(f_2)$.

**Proof.** This follows by taking a sequence $(f_n)_{n \geq 1}$ on $L^p(\mathbb{R})$ such that $f_n \rightarrow f$ with $f = f_1 + f_2 \in L^p(\mathbb{R})$, and using that the sequence $f_n = f_{1,n} + f_{2,n}$ has the property $f_{1,n} \rightarrow f_1$ in the norm of $L^1(\mathbb{R})$, $i = 1, 2$; see [23].

**Proposition 3.14.** For $f \in W^{1,p}(\mathbb{R})$ with $1 < p \leq 2$, the formula

$$F_p(f')(s) = is F_p(f)(s)$$

holds true pointwise almost everywhere.

**Proof.** If $f \in W^{1,p}(\mathbb{R})$ then $f, f'$ belong to $L^p(\mathbb{R})$ with $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$; see [8, Corollary 8.9]. Next, for each $n \geq 1$, we let $\varphi_n(x) := \chi_{[-n,n]}(x)f(x)$ and $\gamma_n(x) := \chi_{[-n,n]}f'(x)$. So $\|\varphi_n - f\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$, and there exists a subsequence $(\varphi_{n_k})_{k \geq 1}$ such that $F_p(\varphi_{n_k})(s) \rightarrow F_p(f)(s)$ a.e. as $k \rightarrow \infty$. Therefore,

$$F_p(f)(s) = \lim_{k \rightarrow \infty} F_p(\varphi_{n_k})(s) = \lim_{k \rightarrow \infty} \int_{-n_k}^{n_k} e^{-isx} f(x) \, dx$$

almost everywhere. Integrating by parts [8, Corollary 8.10], for each $k \geq 1$,

$$\int_{-n_k}^{n_k} e^{-isx} f'(x) \, dx = e^{-isx} f(x) \bigg|_{-n_k}^{n_k} - \int_{-n_k}^{n_k} ise^{-isx} f(x) \, dx.$$
Thus,

\[
\mathcal{F}_p(f')(s) = \lim_{k \to \infty} \mathcal{F}_p(\gamma_{nk})(s) = \lim_{k \to \infty} \int_{-nk}^{nk} e^{-isx} f'(x) \, dx
\]

\[
= is \lim_{k \to \infty} \int_{-nk}^{nk} e^{-isx} f(x) \, dx = is \mathcal{F}_p(f)(s)
\]

almost everywhere. □

**Proposition 3.15.** If \( f \in W^{1,p}(\mathbb{R}) \) with \( 1 < p \leq 2 \), then \( \mathcal{F}_p(f) \in L^1(\mathbb{R}) \).

**Proof.** If \( f \in W^{1,p}(\mathbb{R}) \) then \( \mathcal{F}_p(f) \) belongs to \( L^q(\mathbb{R}) \) with \( 1/p + 1/q = 1 \) and for \( A = \{s \in \mathbb{R} : |s| \geq 1\} \) we get, by Proposition 3.14 and Hölder’s inequality,

\[
\int_A |\mathcal{F}_p(f)(s)| \, ds = \int_A \left| \frac{1}{s} \mathcal{F}_p(f')(s) \right| ds \leq \|1/(\cdot)\|_{L^p(A)} \|\mathcal{F}_p(f')\|_{L^q(A)} < \infty,
\]

with \( 1/p + 1/q = 1 \). Therefore, \( \mathcal{F}_p(f) \in L^1(\mathbb{R}) \). □

As a consequence of Proposition 3.15, the range of \( W^{1,p}(\mathbb{R}) \), for \( 1 < p \leq 2 \), under the action of the \( L^p \)-Fourier transform operator is contained in \( L^1(\mathbb{R}) \). Explicitly,

\[
\mathcal{F}_p(W^{1,p}(\mathbb{R})) \subset L^1(\mathbb{R}) \subsetneq HK(\mathbb{R}).
\]

This relation contrasts with (3.8).

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