A note on a modified Bessel function integral

R. B. PARIS

Division of Computing and Mathematics,
University of Abertay Dundee, Dundee DD1 1HG, UK

Abstract

We investigate the integral
\[ \int_{0}^{\infty} \cosh^{\mu} t K_{\nu}(z \cosh t) \, dt \quad \Re(z) > 0, \]
where \( K \) denotes the modified Bessel function, for non-negative integer values of the parameters \( \mu \) and \( \nu \). When the integers are of different parity, closed-form expressions are obtained in terms of \( z^{-1}e^{-z} \) multiplied by a polynomial in \( z^{-1} \) of degree dependent on the sign of \( \mu - \nu \).

Mathematics Subject Classification: 30E20, 33C10

Keywords: Modified Bessel function, integrals, spherical Bessel function

1. Introduction

An integral arising in neutron scattering [1, 2] is given by
\[ F(\mu, \nu; z) := \int_{0}^{\infty} \cosh^{\mu} t K_{\nu}(z \cosh t) \, dt \quad \Re(z) > 0, \quad (1.1) \]
where \( K_{\nu}(z) \) is the modified Bessel function of the second kind with the parameters \( \mu \geq 0 \) and \( \nu \in C \). The asymptotic expansion of \( F(\mu, \nu; z) \) for \( |z| \to \infty \) in \( \Re(z) > 0 \) has been obtained by Birrell in [1]. The integral (1.1) can also be expressed in terms of a combination of three \( \text{}_1F_2(z^2/4) \) generalised hypergeometric functions. The large-\( z \) asymptotics could therefore be derived by means of the asymptotic theory of integral functions of hypergeometric type; see, for example, the summary presented in [5, Section 2.3].

Birrell [1] placed particular emphasis on the special case when \( \mu \) and \( \nu \) are positive integers of different parity, where \( F(\mu, \nu; z) \) reduces to a finite expansion in inverse powers of \( z \) multiplied by \( z^{-1}e^{-z} \). The approach adopted in [1] was the use of a recurrence relation satisfied by the integral in (1.1) combined with direct evaluation when \( \mu = 0, \nu \text{ odd} \) and \( \mu = 1, \nu \text{ even} \). In this note, we consider the case of integer \( \mu \) and \( \nu \) from a different viewpoint.
2. Evaluation for non-negative integer \( \mu \)

We consider the evaluation of \( F(\mu, \nu; z) \) when \( \mu \) and \( \nu \) are non-negative integers of different parity by using a well-known property of spherical Bessel functions.

2.1 The case of even \( \mu \)

Let us first consider \( \mu = 2n \) and \( \nu = 2m + 1 \), where \( m \) and \( n \) are non-negative integers. Then we have [3, p. 136]

\[
\cosh^{2n} t = \frac{1}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{k} \cosh 2(n-k)t,
\]

where the prime after the summation sign indicates that the term \( k = n \) is multiplied by the factor \( \frac{1}{2} \). We also make use of the result [4, p. 253]

\[
\int_{0}^{\infty} \cosh(a-b)t K_{a+b}(2x \cosh t) \, dt = \frac{1}{2} K_{a}(x) K_{b}(x) \quad (\Re(x) > 0; \ a, b \in \mathbb{C}).
\]

It therefore follows that

\[
F(2n, 2m + 1; z) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n}{k} K_{m+n-k+\frac{1}{2}} \left( \frac{1}{2} z \right) K_{m-n+k+\frac{1}{2}} \left( \frac{1}{2} z \right),
\]

where the Bessel functions are of semi-integer order and so have finite expressions given by [4, p. 264]

\[
K_{s+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k=0}^{s} \frac{(s+k)!}{k!(s-k)!} (2x)^{-k} \quad (s = 0, 1, 2, \ldots).
\]

Then, when \( m \geq n \) we obtain

\[
F(2n, 2m + 1; z) = \frac{\pi e^{-z}}{2^{2n} z} \sum_{k=0}^{n} \binom{2n}{k} \sum_{r=0}^{m-n-k} a_r z^{-r} \sum_{s=0}^{m-n+k} b_s z^{-s}
\]

\[
= \sum_{k=0}^{n} \binom{2n}{k} \sum_{p=0}^{2m} c_p(k) z^{-p},
\]

where

\[
a_r = \frac{(m+n-k+r)!}{r!(m+n-k-r)!}, \quad b_s = \frac{(m-n+k+s)!}{s!(m-n+k-s)!}
\]

and we have defined

\[
c_p(k) \equiv c_p(k; m, n) = \sum_{r+s=p} a_r b_s
\]

\[
= \sum_{r=0}^{p} \frac{(m+n-k+r)!(m-n+k+p-r)!}{r!(p-r)!(m+n-k-r)!(m-n+k-p+r)!}.
\]

We therefore find the result

\[
F(2n, 2m + 1; z) = \frac{\pi e^{-z}}{2^{2n} z} \sum_{p=0}^{2m} C_p(m, n) z^{-p} \quad (m \geq n),
\]
A Bessel function integral

where the coefficients \( C_p(m, n) \) are given by

\[
C_p(m, n) = \sum_{k=0}^{n} \binom{2n}{k} c_p(k) \quad (0 \leq p \leq 2m). \tag{2.5}
\]

In the case \( m < n \), we can proceed in a similar manner but now care has to be taken with the second Bessel function appearing in (2.1). Making use of the fact that \( K_{-\nu}(x) = K_{\nu}(x) \), we write

\[
K_{m-n+k+\frac{1}{2}}(\frac{1}{2}z) = \begin{cases} 
K_{n-m-k-\frac{1}{2}}(\frac{1}{2}z) & \text{for } k \leq n-m-1 \\
K_{m-n+k+\frac{1}{2}}(\frac{1}{2}z) & \text{for } k \geq n-m.
\end{cases} \tag{2.6}
\]

Then the product in (2.2) becomes

\[
\sum_{r=0}^{m+n-k} a_r z^{-r} \sum_{s=0}^{S} b_s z^{-s}, \quad S = \begin{cases} 
n-m-k-1 & \text{for } k \leq n-m-1 \\
m-n+k & \text{for } k \geq n-m
\end{cases}
\]
to yield

\[
\mathcal{F}(2n, 2m+1; z) = \frac{\pi e^{-z}}{2^{2n} z} \sum_{k=0}^{n} \binom{2n}{k} \sum_{p=0}^{2n-1} d_p(k) z^{-p}.
\]

The coefficients \( d_p(k) \equiv d_p(k; m, n) \) are defined by

\[
d_p(k) = \sum_{r+s=p} a_r \hat{b}_s, \quad \hat{b}_s = \begin{cases} 
\frac{(n-m-k-1+s)!}{s!(n-m-k-1-s)!} & \text{for } k \leq n-m-1 \\
\frac{(m-n+k+s)!}{s!(m-n+k-s)!} & \text{for } k \geq n-m,
\end{cases} \tag{2.7}
\]

where the \( a_r \) are as specified in (2.3).

Consequently, we obtain the result

\[
\mathcal{F}(2n, 2m+1; z) = \frac{\pi e^{-z}}{2^{2n} z} \sum_{p=0}^{2n-1} D_p(m, n) z^{-p} \quad (m < n), \tag{2.8}
\]

where the coefficients \( D_p(m, n) \) are given by

\[
D_p(m, n) = \sum_{k=0}^{n} \binom{2n}{k} d_p(k) \quad (0 \leq p \leq 2n-1). \tag{2.9}
\]

2.2 The case of odd \( \mu \)

When \( \mu = 2n + 1 \), we can make use of the expansion [3, p. 136]

\[
cosh^{2n+1}t = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \sinh(2n - 2k + 1)t
\]

to obtain

\[
\mathcal{F}(2n + 1, 2m; z) = \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{2n+1}{k} K_{m+n-k+\frac{1}{2}}(\frac{1}{2}z) K_{m-n+k-\frac{1}{2}}(\frac{1}{2}z). \tag{2.10}
\]
Following the same procedure as in the even $\mu$ case, we obtain when $m > n$

$$\mathcal{F}(2n + 1, 2m; z) = \frac{\pi e^{-z}}{2^{2n+1}z} \sum_{k=0}^{n} \binom{2n+1}{k} \sum_{p=0}^{2m-1} \hat{C}_p(m, n) z^{-p} \quad (m > n), \quad (2.11)$$

where

$$\hat{C}_p(m, n) = \sum_{k=0}^{n} \binom{2n+1}{k} \hat{c}_p(k) \quad (0 \leq p \leq 2m - 1), \quad (2.12)$$

with

$$\hat{c}_p(k) = \sum_{r+s=p} \frac{(m+n-k+r)!(m-n+k-1+s)!}{r!(m+n-k-r)!(m-n+k-1-s)!}.$$  

When $m \leq n$, the second Bessel function in (2.10) is written as in (2.14) with $k$ replaced by $k - 1$ to find

$$\mathcal{F}(2n + 1, 2m; z) = \frac{\pi e^{-z}}{2^{2n+1}z} \sum_{k=0}^{n} \binom{2n+1}{k} \sum_{p=0}^{2n} \hat{D}_p(m, n) z^{-p} \quad (m \leq n), \quad (2.13)$$

where

$$\hat{D}_p(m, n) = \sum_{k=0}^{n} \binom{2n+1}{k} \hat{d}_p(k) \quad (0 \leq p \leq 2n), \quad (2.14)$$

with $\hat{d}_p(k)$ as defined in (2.7) with $k \to k - 1$.

We summarise the results obtained in the following theorem.

**Theorem 1.** Let $m$ and $n$ be non-negative integers and $\Re(z) > 0$. Then

$$\mathcal{F}(2n, 2m + 1; z) = \frac{\pi e^{-z}}{2^{2n}z} P(z), \quad \mathcal{F}(2n + 1, 2m; z) = \frac{\pi e^{-z}}{2^{2n+1}z} \hat{P}(z), \quad (2.15)$$

where $P(z)$ and $\hat{P}(z)$ are polynomials in $z^{-1}$ given by

$$P(z) = \begin{cases} 2m \\ \sum_{p=0}^{2m} C_p(m, n) z^{-p} \quad (m \geq n) \\ 2n-1 \sum_{p=0}^{2n-1} D_p(m, n) z^{-p} \quad (m < n) \end{cases}, \quad \hat{P}(z) = \begin{cases} 2m \\ \sum_{p=0}^{2m} \hat{C}_p(m, n) z^{-p} \quad (m > n) \\ 2n \sum_{p=0}^{2n} \hat{D}_p(m, n) z^{-p} \quad (m \leq n). \end{cases}$$

The coefficients $C_p(m, n), D_p(m, n), \hat{C}_p(m, n)$ and $\hat{D}_p(m, n)$ are defined in (2.5), (2.9), (2.12) and (2.14).

### 3. Concluding remarks

As an example, when $\mu = 4, \nu = 7$ ($m = 3, n = 2$) and $\mu = 4, \nu = 3$ ($m = 1, n = 2$) the values of the coefficients $C_p(m, n)$ are

$$8, \ 208, \ 2520, \ 17880, \ 76800, \ 184320, \ 184320 \quad (m = 3, n = 2; \ 0 \leq p \leq 2m),$$

$$8, \ 48, \ 120, \ 120 \quad (m = 1, n = 2; \ 0 \leq p \leq 2n - 1);$$
whence we find
\[ \mathcal{F}(4, 7; z) = \frac{\pi e^{-z}}{2z} (1 + 26z^{-1} + 315z^{-2} + 2235z^{-3} + 9600z^{-4} + 23040z^{-5} + 23040z^{-6}) \]
and
\[ \mathcal{F}(4, 3; z) = \frac{\pi e^{-z}}{2z} (1 + 6z^{-1} + 15z^{-2} + 15z^{-3}). \]

We note from Theorem 1 that the finite series in \( z^{-1} \) terminate at different \( p \) index. For even \( \mu \), the degree of the polynomial \( P(z) \) is \( 2m \) when \( m \geq n \) and \( 2n - 1 \) when \( m < n \). For odd \( \mu \), the degree of the polynomial \( \hat{P}(z) \) is \( 2m \) when \( m > n \) and \( 2n \) when \( m \leq n \). This was observed in [1], although the case \( m = n \) was not mentioned.

Since it is easily verified that \( c_0(k) = d_0(k) = \hat{c}_0(k) = \hat{d}_0(k) = 1 \), the leading coefficients in the expansions in (2.15) satisfy
\[
C_0(m, n) = D_0(m, n) = \sum_{k=0}^{n} \binom{2n}{k} = 2^{2n} - \frac{n!}{2^{n+1} n} \binom{2n}{n},
\]
\[
\hat{C}_0(m, n) = \hat{D}_0(m, n) = \sum_{k=0}^{n} \binom{2n+1}{k} = 2^{2n},
\]
which are independent of \( m \).

Finally, we mention that the integral
\[ \mathcal{G}(2n, \nu; z) = \int_{0}^{\infty} \sinh^{2n} t K_{\nu}(z \cosh t) \, dt \]
when \( \nu = 2m + 1 \) can be dealt with in the same manner. Since [3, p. 136]
\[
\sinh^{2n} t = \frac{1}{2^{2n-1}} \sum_{k=0}^{n} {\nu \choose k} (-1)^{k} \binom{2n}{k} \cosh 2(n - k)t,
\]
it follows immediately from Section 2.1 that
\[ \mathcal{G}(2n, 2m + 1; z) = \frac{\pi e^{-z}}{2^{2n} z} P(z), \]
where the polynomial \( P(z) \) is as specified in Theorem 1, with the coefficients \( C_p(m, n) \) and \( D_p(m, n) \) now defined by
\[
C_p(m, n) = \sum_{k=0}^{n} (-1)^{k} \binom{2n}{k} c_p(k), \quad D_p(m, n) = \sum_{k=0}^{n} (-1)^{k} \binom{2n}{k} d_p(k).
\]
The leading coefficients are again independent of \( m \) and given by
\[
C_0(m, n) = D_0(m, n) = \sum_{k=0}^{n} (-1)^{k} \binom{2n}{k} = 0;
\]
this means that a \( z^{-1} \) can be factored out leaving the polynomial \( P(z) \) with degree reduced by unity.
References

[1] J. Birrell, Evaluation of a family of Bessel function integrals. (2015) arXiv:1509.06308.

[2] J. Birrell, C.T. Yang and J. Rafelski, Relic neutrino freeze-out: dependence on natural constants. Nucl. Phys. B 890 (2015) 481.

[3] A. Jeffrey and H-H. Dai, Handbook of Mathematical Formulas and Integrals, 4th edition, Academic Press, Oxford, 2008.

[4] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

[5] R.B. Paris and D. Kaminski, Asymptotics and Mellin-Barnes Integrals, Cambridge University Press, Cambridge, 2001.