Gaussian Waves on the Regular Tree

Yehonatan Elon
Department of Physics of Complex Systems,
The Weizmann Institute of Science, 76100 Rehovot, Israel

Abstract. We consider the family of real (generalized) eigenfunctions of the adjacency operator on $T_d$ - the $d$-regular tree. We show the existence of a unique invariant Gaussian process on the ensemble and derive explicitly its covariance operator.

We investigate the typical structure of level sets of the process. In particular we show that the entropic repulsion of the level sets is uniformly bounded and prove the existence of a critical threshold, above which the level sets are all of finite cardinality and below it an infinite component appears almost surely.

1. Introduction and main results

The regular tree $T_d$ (also known as the Bethe-lattice), is a connected, cycle-free, infinite graph where each vertex is connected to $d$ neighbors. In an abuse of notation, we shall use the notation $T_d$ for both the graph and the set of its vertices.

For a function $f : T_d \to \mathbb{R}$ the adjacency operator $A$ acts on the components of $f$ by

$$(Af)(v) = \sum_{|v-v'|=1} f(v')$$

where $v, v' \in T_d$ and $|v - v'|$ is the distance in $T_d$ from $v$ to $v'$.

The spectrum $[\Pi]$ of $A$ is absolutely continuous, supported on the interval

$$\sigma(T_d) = [-2\sqrt{d-1}, 2\sqrt{d-1}],$$

with a spectral density, given by

$$\rho(\lambda) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - \lambda^2}}{d^2 - \lambda^2} \mathbb{I}_{\lambda \in \sigma(T_d)}$$
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where $I$ is the indicator function. For a given $\lambda \in \sigma(T_d)$, a function $\psi : T_d \to \mathbb{C}$ will be referred as a wave (or a generalized eigenfunction) if $(A - \lambda I)\psi = 0$, where $I$ is the identity operator on $T_d$.

In this paper, we investigate the existence and properties of the following Gaussian process on $T_d$:

**Theorem 1.1.** For every $d \geq 3$ and $\lambda \in \sigma(T_d)$, there exists a unique random process $\mathcal{GS}_d(\lambda) = \{\Omega, \mu\}$, associating $\forall \omega \in \Omega$ a function $\psi_\omega : T_d \to \mathbb{R}$ with the following properties:

(i) for almost every $\omega \in \Omega$, $(A - \lambda I)\psi_\omega = 0$ .

(ii) $\mu$ is a Gaussian measure, where $\forall v \in T_d$, the marginal variance $\text{Var}(\psi_\omega(v)) = 1$ .

(iii) for every automorphism $\Phi : T_d \to T_d$ and vertices $v, v' \in T_d$,

$$\mathbb{E}(\psi_\omega(v)\psi_\omega(v')) = \mathbb{E}(\psi_\omega(\Phi(v))\psi_\omega(\Phi(v')))$$

Gaussian processes are frequently used in various branches of physics, such as semiclassical analysis [2], optics [3] or cosmology [4] (to list only a few) - usually as a heuristic model to study systems with random perturbations. In particular, $\mathcal{GS}_d(\lambda)$ was conjectured in [5], as a limiting process for the distribution of eigenvectors of random regular graphs.

Theorem 1.1 will be established in section 2 where we investigate the properties of $\mathcal{GS}_d(\lambda)$, by calculating the covariance operator explicitly. In addition, we prove that the process has a Markov property, in a sense that will be defined in theorem 2.3.

We will also characterize the structure of a typical realization $\psi_\omega$, by considering its level sets:

For a function $f : T_d \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, we define the induced subgraph $T_\alpha(f) \subset T_d$, by keeping only vertices above the threshold $\alpha$:

$$T_\alpha(f) = \{v \in T_d, f(v) > \alpha\}$$

and define the $\alpha$-level sets of $f$, to be the connected components of $T_\alpha(f)$.

One aspect that will be investigated is the entropic repulsion induced by the process,
namely the distribution of $\psi_\omega(v)$, conditioned on the diameter of the $\alpha$-level set, containing $v$. Let

$$V = \{v_j\}_{j=1}^n \subset T_d$$

be a simple path of length $n-1$, so that $\forall 1 < j < n, |v_{j+1} - v_j| = |v_j - v_{j-1}| = 1$ but $v_{j-1} \neq v_{j+1}$.

For a given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, we define the conditional probability

$$\mathbb{P}_{n,\alpha}^\alpha(\cdot) = \mathbb{P}(\cdot | \forall v \in V, \psi_\omega(v) > \alpha)$$

and argue the following:

**Theorem 1.2.** $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, there exist $\psi_0(\lambda, \alpha) < \infty$ and $0 < c_1(\lambda), c_2(\lambda) < \infty$, so that $\forall n \in \mathbb{N}, 1 \leq k \leq n$ and $x > \psi_0$

$$\mathbb{P}_{n,\alpha}^\alpha(\psi_\omega(v_k) \geq x) < c_1 e^{-c_2 x^2}$$

A natural question which arise, when considering the structure of the $\alpha$-level sets of $\mathcal{G}_S_d(\lambda)$, is related to the existence, or the absence, of an infinite level set and the transition between the two regimes. This question is answered by the following theorem:

**Theorem 1.3.** $\forall \lambda \in \sigma(T_d)$, there exists an $\alpha_c \in \mathbb{R}$ so that for almost every realization $\psi_\omega \in \mathcal{G}_S_d(\lambda)$, $T_\alpha(\psi_\omega)$ has an infinite component for $\alpha < \alpha_c$, but only finite components for $\alpha > \alpha_c$.

The theorem is proved in section 4, following a 'quasi-Bernoulli' criterion, introduced by Lyons in [6], for random percolation processes on tree graphs.

1.1. Relations with previous results

Gaussian waves on $\mathbb{R}^n$ were first suggested in [2] as a model for the limiting behavior of eigenfunctions of chaotic systems. While the model is not supported by any rigorous derivation, it was found consistent with some numerical observations, such as [7, 8, 9]. In [3], a modified 'random waves' model was introduced, in order to describe the statistics of adjacency eigenvectors of the ensemble $G(n,d)$, consisted of all $d$–regular
graphs on \( n \) vertices and equipped with the uniform measure.

The ensemble \( G(n,d) \) serves frequently as a convenient model for random expander graphs (for a review, consider \[10\]). Eigenvectors of such graphs are used in various algorithms (e.g. \[11 \ 12 \ 13\]), however not much is known about their characteristics. \( G(n,d) \) graphs have drawn recently a considerable attention in the physical community as a plausible ‘toy-model’ for generic chaotic systems. In \[14\] it was claimed, based on numerical simulations, that in the limit \( n \to \infty \), the local level distribution of such graphs follows the predictions of the GOE random matrices ensemble, as believed to hold for chaotic billiards as well \[15\]. This result was recently strengthened by an analytic derivation \[16\] of the 2-levels form-factor asymptotics. The spectral properties of \( G(n,d) \) were also suggested in \[17\] as a natural finite dimensional model for the regular tree, in the context of Anderson (de-) localization.

The model conjectured in \[5\], has to do with these three aspects: it relates \( G(n,d) \) graphs to an additional universality class associated with chaotic behavior, it predicts that the eigenvectors of such graphs are extended (corresponding to the appearance of an ac spectrum in the corresponding lattice). Lastly, it predicts with a high accuracy variate statistical properties of the eigenvectors, which are of interest - for example, the nodal domains statistics of such graphs, which was measured in \[18\], but have not found any explanation.

In this paper we provide a rigorous construction of the Gaussian waves model on \( T_d \), which is a first step towards the analysis of the relations between the process \( GS_d(\lambda) \) and the eigenvectors of a random \( G(n,d) \) graph.

The statistics of level sets of Gaussian random waves in \( \mathbb{R}^2 \) and specifically their nodal sets was measured and characterized in \[7 \ 19\]. The observed statistics found an intriguing explanation in \[9\], where it was conjectured that the nodal statistics of 2 dimensional Gaussian waves can be approximated by a critical (non correlated) percolation model.

The last section of the current work provides a rigorous proof for the critical behavior of
the Gaussian waves model in \( T_d \). However, the characters of the transition are different then the ones of uncorrelated percolation.

2. Properties of the process \( GS_d(\lambda) \)

As a Gaussian process is characterized by its covariance operator, theorem 1.1 will follow from lemmas 2.1 and 2.2, where we prove the existence of the described process and the uniqueness of its covariance.

We denote by \( \delta_v \in l^2(T_d) \) the indicator function supported at \( v \in T_d \). We define for \( z \in \mathbb{C} \setminus \sigma(T_d) \) the resolvent operator \( R(A, z) : T_d \to T_d \) by

\[
R(A, z) = (A - zI)^{-1}
\]

We will also make use of the Chebyshev Polynomials of the second kind, defined as

\[
U_n(x) = \sin \left( (n + 1) \cos^{-1}(x) \right) / \sin \left( \cos^{-1}(x) \right)
\]

and follow the convention \( U_{-1}(x) = U_1(x) \).

**Lemma 2.1.** For every \( \lambda \in \sigma(T_d) \), the Gaussian process determined by the covariance operator \( \text{Cov}(\psi_\omega(v), \psi_\omega(v')) = \langle \delta_v, C_\lambda \delta_{v'} \rangle \), where

\[
\langle \delta_v, C_\lambda \delta_{v'} \rangle = \lim_{\epsilon \to 0^+} \frac{\text{Im} \langle \delta_v, R(A, \lambda + i\epsilon) \delta_{v'} \rangle}{\text{Im} \langle \delta_v, R(A, \lambda + i\epsilon) \delta_v \rangle}
\]

Is consistent with the requirements of theorem 1.1

**Proof.** The existence of the limit appearing in equation 2.2 can be verified for \( \lambda \in \sigma(T_d) \), by considering the spectral representation of the resolvent and recalling the smoothness of the spectral density (see, for example sections 1.3, 1.4 of [20]). As

\[
\text{Im}(A - (\lambda + i\epsilon)I)^{-1} = \epsilon((A - \lambda I)^2 + \epsilon^2 I)^{-1}
\]

The resolvent is positive definite \( \forall \epsilon > 0 \), therefore equation 2.2 defines an appropriate covariance operator.

Following equation 2.2, \( \text{Var}(\psi_\omega(v)) = 1 \). Moreover, as the resolvent is invariant under any automorphism \( \Phi : T_d \to T_d \) requirements (ii) and (iii) of theorem 1.1 are satisfied.
Lastly, let $\psi_\omega$ be a random realization of the Gaussian process generated by $C_\lambda$. As the law of $\psi_\omega$ is invariant under reflections,

$$\mathbb{E}(\langle \delta_v, (A - \lambda I)\psi_\omega \rangle) = 0.$$ 

In addition, by the invariance of the measure,

$$\mathbb{E}(\langle \delta_v, (A - \lambda I)\psi_\omega \rangle^2) =$$

$$(d + \lambda^2)\mathbb{E}(\psi_\omega(v)^2) - 2d\lambda\mathbb{E}(\psi_\omega(v)\psi_\omega(v'))_{|v - v'|=1} + d(d - 1)\mathbb{E}(\psi_\omega(v)\psi_\omega(v''))_{|v - v'|=2} =$$

$$\mathbb{E}(\langle \delta_v, (A - \lambda I)^2\psi_\omega \rangle \langle \psi_\omega, \delta_v \rangle) =$$

$$\langle \delta_v(A - \lambda I)^2, C_\lambda \delta_v \rangle =$$

$$\lim_{\epsilon \to 0} \frac{\langle \delta_v, (A - \lambda I)^2((A - \lambda I)^2 + \epsilon^2 I)^{-1}\delta_v \rangle}{\langle \delta_v, ((A - \lambda I)^2 + \epsilon^2 I)^{-1}\delta_v \rangle} = 0$$

where in the first and second step we have expanded the bilinear form into elements (bearing in mind that $\mathbb{E}(\psi_\omega(v)\psi_\omega(v'))$ depends only on $|v - v'|$) and recollected it; in the third step we have followed the definition of the covariance operator: $C_\lambda = \mathbb{E}(\psi_\omega\psi_\omega^T)$ and in the fourth, we have followed equations [2.2, 2.3].

As $\langle \delta_v, (A - \lambda I)\psi_\omega \rangle$ is a Gaussian random variable, with zero mean and variance, it equals zero almost surely, establishing by that requirement $(i)$ of theorem [1.1] \(\square\)

Note that the last proof relies only on the smoothness of the spectral density of $A$, and the invariance of the process. Therefore Gaussian wave models can be generated by the covariance operator [2.2] for broader classes of conducting graphs.

In order to find an explicit expression for the covariance of the process $\mathcal{GS}_d(\lambda)$ for a given $\lambda \in \sigma(T_d)$, we would like to introduce the function \(\Pi [21] \phi^{(\lambda)} : \mathbb{N} \to \mathbb{R}\) defined as

$$\phi^{(\lambda)}(n) = (d - 1)^{-n/2} \left( \frac{d - 1}{d} U_n \left( \frac{\lambda}{2\sqrt{d - 1}} \right) - \frac{1}{d} U_{n-2} \left( \frac{\lambda}{2\sqrt{d - 1}} \right) \right)$$

and state the following:

**Lemma 2.2.** Let $\mathcal{GS}_d(\lambda) = \{\Omega, \mu\}$ be a random Gaussian process, consistent with the requirements of theorem [1.1]. Then, the covariance of $\mathcal{GS}_d(\lambda)$ is given by

$$\text{Cov}(\psi_\omega(v), \psi_\omega(v')) = \phi^{(\lambda)}(|v - v'|)$$
Proof. We begin by considering a general property of waves on $T_d$.

Let $f : T_d \to \mathbb{R}$, so that $(A - \lambda I) f = 0$. For a given $v \in T_d$ and $k \in \mathbb{N}$, denote the sphere of radius $k$ around $v$ by

$$\Lambda_k(v) = \{ v' \in T_d, |v' - v| = k \}$$

and define

$$S_k(f, v) = \sum_{v' \in \Lambda_k(v)} f(v')$$

to be the sum of $f$ over the $k$–sphere. As $(A - \lambda I)f = 0$ and since for $k \geq 2$, every vertex in the $(k - 1)^{th}$ sphere has $d - 1$ neighbors in the $k^{th}$ sphere, we get that:

$$S_0 = f(v), \quad S_1 = \lambda f(v)$$

$$\lambda S_k = (d - 1)S_{k-1} + S_{k+1} \quad \text{for } k \geq 2 \quad (2.5)$$

Recalling that Chebyshev polynomials are related by the recursion relation

$$2xU_k(x) = U_{k-1}(x) + U_{k+1}(x)$$

one can verify that

$$S_k(f, v) = |\Lambda_k| \phi^{(\lambda)}(k) \cdot f(v) \quad (2.6)$$

is the (unique) solution to $(2.5)$.

Now, assume that a process $\mathcal{G}S_d(\lambda) = \{\Omega, \mu\}$ follows the requirements made in theorem 1.1. Then,

$$\mathbb{E}(\psi_\omega(v) \cdot \psi_\omega(v')|_{|v - v'| = k} = \mathbb{E}(\psi_\omega(v) \cdot \frac{1}{|\Lambda_k|} S_k(\psi_\omega, v)) \quad (2.7)$$

$$= \phi^{(\lambda)}(k) \mathbb{E}(\psi^2_\omega(v))$$

$$= \phi^{(\lambda)}(k)$$

Where we have followed properties $(iii)$, $(i)$ and $(ii)$ of the process $\mathcal{G}S_d(\lambda)$ respectively. \qed

In the rest of this paper, we will be often interested in the restriction of the process $\mathcal{G}S_d(\lambda)$ to finite subsets of $T_d$. For this reason we would like to introduce the following
notation:

For a set $V = \{v_i\}_{i=1}^n \subset T_d$ we denote

$$\psi_\omega(V) = (\psi_\omega(v_1), \ldots, \psi_\omega(v_n)) \quad d\psi_\omega(V) = \prod_{i=1}^n d\psi_\omega(v_i).$$

The density of the measure on $\mathcal{G}S_d(\lambda)$ will be denoted by

$$d\mu(\psi_\omega(V)) = p(\psi_\omega(V))d\psi_\omega(V)$$

The adjacency operator is local, i.e. it contains only nearest neighbors interactions.

For tree graphs, such as $T_d$, this property has the following consequence:

Consider two adjacent vertices $v_1, v_2 \in T_d$, define the partition of $T_d$ into

$$T^{(1)}(v_1, v_2) = \{v \in T_d, |v - v_1| < |v - v_2|\}$$
$$T^{(2)}(v_1, v_2) = \{v \in T_d, |v - v_1| > |v - v_2|\}$$

(see figure 1) and Let $V_1 \subset T^{(1)} \setminus v_1$ and $V_2 \subset T^{(2)} \setminus v_2$ be finite subsets of the two subgraphs.

For a given $\lambda \in \sigma(T_d)$ and $X \in \mathbb{R}^{|V_1|+2}$, consider the family of waves on $T_d$, where we fix the value of the function on $V_0$ and $V_1$ to $X$:

$$F = \{f : T_d \to \mathbb{R}, (A - \lambda I)f = 0, f(V_0 \cup V_1) = X\}.$$
Due to the constraints which are imposed by the adjacency operator, fixing \( f(V_0) \) might impose constraints on \( f(V_2) \) which must be satisfied \( \forall f \in F \). However, note that by fixing \( f(V_0) \), the adjacency operator does not mix vertices from \( V_1 \) and \( V_2 \). As a result, the constraints on \( f(V_2) \), imposed by fixing \( f(V_0 \cup V_1) \) are identical to the one imposed by fixing \( f(V_0) \) alone.

This property is inherited by the process \( G\Sigma_d(\lambda) \) in the following sense:

**Theorem 2.3.** Let \( V_0 = \{v_1, v_2\} \subset T_d \), so that \( |v_1 - v_2| = 1 \), \( V_1 \subset T^{(1)}(v_1, v_2) \) and \( V_2 \subset T^{(2)}(v_1, v_2) \). Then the distribution of \( \psi_\omega(V_2) \) conditioned on \( \psi_\omega(V_0) \) is independent of \( \psi_\omega(V_1) \):

\[
p(\psi_\omega(V_2)|\psi_\omega(V_0 \cup V_1)) = p(\psi_\omega(V_2)|\psi_\omega(V_0))
\]

**Proof.** As \( \psi_\omega(V_2) \) is a Gaussian random vector, it is enough to show that

\[
\mathbb{E}(\psi_\omega(V_2)|\psi_\omega(V_0 \cup V_1)) = \mathbb{E}(\psi_\omega(V_2)|\psi_\omega(V_0))
\]

and that

\[
C_\lambda(\psi_\omega(V_2)|\psi_\omega(V_0 \cup V_1)) = C_\lambda(\psi_\omega(V_2)|\psi_\omega(V_0))
\]

where \( C_\lambda \) is the conditional covariance operator.

To do so, we set \( n_1 = |V_1|, n_2 = |V_2| \) and denote by \( V = \{v_k\}_{k=1}^{2^n_1+n_2} \) the union of \( V_0, V_1, V_2 \). We consider first the case where the adjacency operator do not impose constraints on \( \psi_\omega(V) \), so that the covariance matrix \( (C_\lambda(V))_{ml} = \phi^{(3)}(|v_m - v_l|) \) is strictly positive. \( C_\lambda(V) \) can be written in the next blocks form:

\[
C_\lambda(V) = \begin{pmatrix} C_{(11)} & C_{(12)} \\ C_{(21)} & C_{(22)} \end{pmatrix}
\]

Where \( C_{(11)} \) is the \( (n_1 + 2) \times (n_1 + 2) \) covariance matrix of \( \psi_\omega(V_0 \cup V_1) \) and \( C_{(22)} \) is the \( n_2 \times n_2 \) covariance matrix of \( \psi_\omega(V_2) \).

We set, in a similar fashion to the proof of lemma 2.2:

\[
\tilde{\Lambda}_k(v_2) = \{v \in T^{(2)}, |v - v_2| = k\}
\]
\[ S_k(\psi, v_2) = \sum_{v \in \tilde{\Lambda}_k(v_2)} \psi(v) \]

As \((A - \lambda I)\psi = 0\), the \(\tilde{S}_k\)'s are determined by the recursion relation

\[ \begin{align*}
\tilde{S}_0 &= \psi(v_2), \\
\tilde{S}_1 &= \lambda \psi(v_2) - \psi(v_1) \\
\lambda \tilde{S}_k &= (d - 1) \tilde{S}_{k-1} + \tilde{S}_{k+1}
\end{align*} \]

Therefore \(\forall k \in \mathbb{N}\), \(\tilde{S}_k\) is determined by \(\psi(V_1)\). By the invariance of the process \(GS_d(\lambda)\), we obtain that \(\forall k \geq 0\) and \(v \in \tilde{\Lambda}_k\),

\[ \mathbb{E}(\psi(v)|\psi(V_0), \psi(V_1)) = \tilde{S}_k/|\tilde{\Lambda}_k| \]

As a result, since \(V_2 \subset \bigcup_k S_k\), we find that

\[ \mathbb{E}(\psi(V_2)|\psi(V_0), \psi(V_1)) = \mathbb{E}(\psi(V_2)|\psi(V_0)) \]

The conditional expectation and covariance operator of \(\psi(V_2)\) are given by the formulae

\[ \mathbb{E}(\psi(V_2)|\psi(V_0), \psi(V_1)) = C_{(21)}C_{(11)}^{-1}\psi(V_0 \cup V_1) \quad (2.8) \]

\[ C_\lambda(V_2)|_{\psi(V_0), \psi(V_1)} = C_{(22)} - C_{(21)}C_{(11)}^{-1}C_{(12)} \]

Therefore, since \(\mathbb{E}(\psi(V_2)|\psi(V_0), \psi(V_1))\) is independent of \(\psi(V_1)\), \((C_{(21)}C_{(11)}^{-1})_{ij}\) must vanish \(\forall j > 2\) and \((C_{(21)}C_{(11)}^{-1})_{ij}\) is independent of the set \(V_i\) for \(i = 1, 2\). As a result, \(C_{(21)}C_{(11)}^{-1}C_{(12)}\) is independent of \(V_1\) as well, therefore

\[ C_\lambda(V_2)|_{\psi(V_0), \psi(V_1)} = C_\lambda(V_2)|_{\psi(V_0)} \]

Establishing by that the suggested independence.

As was noted above, if the adjacency operator does impose constrains on the distribution of \(\psi(V)\), these constraints can be decoupled into separate constrains on \(\psi(V_1)\) and \(\psi(V_2)\). Therefore, there exists a partition of \(V_1, V_2\) into a free and constrained subsets: \(V_1 = V_1^F \cup V_1^C\) and \(V_2 = V_2^F \cup V_2^C\), so that \(C_\lambda(V_0 \cup V_1^F \cup V_2^F)\) is strictly positive, while \(\psi(V_1^C)\) and \(\psi(V_2^C)\) are determined uniquely by \(\psi(V_0 \cup V_1^F)\) and \(\psi(V_0 \cup V_2^F)\) correspondingly.
Therefore, from the proof to the unconstrained case we obtain that
\[ p \left( \psi_\omega(V_2^F) | \psi_\omega(V_0 \cup V_1) \right) = p \left( \psi_\omega(V_2^F) | \psi_\omega(V_0 \cup V_1^F) \right) = p \left( \psi_\omega(V_2^F) | \psi_\omega(V_0) \right) \]

As \( \psi_\omega(V_2^C) \) is uniquely determined by \( \psi_\omega(V_2^F) \), the theorem follows. \( \square \)

3. Distribution of level sets

As was suggested in section 1, the \( \alpha \)-level sets of \( \psi_\omega \in \mathcal{GS}_d(\lambda) \), can be naturally related to a random process \( \{ \Omega, \mathbb{P}_\alpha \} \) on the Bethe lattice, associating \( \forall \omega \in \Omega \), an induced subgraph \( T_\alpha(\psi_\omega) \subset T_d \), according to the rule:
\[ T_\alpha(\psi_\omega) = \{ v \in T_d, \psi_\omega(v) > \alpha \} \]

For \( \alpha \in \mathbb{R} \), \( \psi_\omega \in \mathcal{GS}_d(\lambda) \) and \( v \in T_d \), we set \( C_\omega^\alpha(v) \) to denote the connected component of \( v \) in \( T_\alpha(\psi_\omega) \).

In the following, we will consider the conditional distribution of \( \psi_\omega(v) \), where we condition on the diameter of the \( \alpha \)-level set which contains \( v \).

Let \( V = \{ v_j \}_{j=1}^n \) be a simple path in \( T_d \). For a given \( \alpha \in \mathbb{R} \) we set
\[ \Omega_n^{+\alpha} = \{ \omega \in \Omega, v_1 \in C_\omega^\alpha(v_n) \} \]
as the restriction of the sample space \( \Omega \) to events in which \( V \) is contained in an \( \alpha \)-level set. Similarly, we use the symbols
\[ \mathbb{P}_n^{+\alpha}(\cdot) = \mathbb{P}(\cdot | v_1 \in C_\omega^\alpha(v_n)) \quad \mathbb{E}_n^{+\alpha}(\cdot) = \mathbb{E}(\cdot | v_1 \in C_\omega^\alpha(v_n)) \quad p_n^{+\alpha}(\cdot) = p(\cdot | v_1 \in C_\omega^\alpha(v_n)) \]
to denote probabilities, expectations and densities, conditioned on the event \( v_1 \in C_\omega^\alpha(v_n) \).

The main result of this section is theorem 1.2. The proof of the theorem will follow the next lines:

First, we calculate in lemma 3.1 the probability density \( p_n^+(\psi_\omega(v)|\psi_\omega(V \setminus v)) \) and find that it is concentrated around a linear combination of \( \{ \psi_\omega(v' \in V), |v - v'| \leq 2 \} \), with a bounded variance and Gaussian tails.

Next, in lemma 3.2 we observe that above some finite threshold \( \psi_0(\lambda, \alpha) < \infty \), the
suggested linear combination becomes convex. Therefore, the probability to find that \( \psi_\omega(v) > x \) decays rapidly for \( x > \psi_0 \), unless \( \psi_\omega(v) \) is significantly smaller then the average of its neighbors. Finally, we show that the convexity of the distribution, results in the concentration of \( \psi_\omega(v) \), establishing by that theorem [1.2]

**Lemma 3.1.** Let \( V = \{v_j\}_{j=1}^n \subset T_d \) be a simple path. Then \( \forall \lambda \in \sigma(T_d), \alpha \in \mathbb{R} \) and \( v_k \in V \), there exist constants \( a_{k\pm 1}(\lambda, \alpha), a_{k\pm 2}(\lambda, \alpha) \) and \( \sigma_k^2 < 1 \), so that

\[
p_{n,k}^{+}\alpha(\psi_\omega(v_k) = x|\psi_\omega(V \setminus v_k)) = \frac{1}{Z} \exp \left( \frac{- (x - E_k)^2}{2\sigma_k^2} \right) \mathcal{I}_{x > \alpha}
\]

where \( \mathcal{I} \) is the indicator function, \( Z = \int_\alpha^\infty dy \exp \left( \frac{- (y - E_k)^2}{2\sigma_k^2} \right) \) and

\[
E_k(\omega) = \sum_{|k-k'| \leq 2} a_{k'}(\lambda, \alpha) \psi_\omega(v_{k'})
\]

**Proof.** First, note that

\[
p_{n,k}^{+}\alpha(\psi_\omega(v_k) = x|\psi_\omega(V \setminus v_k)) = \frac{\mathcal{I}_{x > \alpha}}{Z} \cdot p(\psi_\omega(v_k) = x|\psi_\omega(V \setminus v_k))
\]

Therefore, as \( p(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k)) \) is Gaussian, equation 3.1 follows with

\[
E_k(\omega) = \mathbb{E}(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k)), \quad \sigma_k^2 = \text{Var}(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k))
\]

Next, according to theorem [2.3], we get that

\[
p(\psi_\omega(v_k) = x|\psi_\omega(V \setminus v_k)) = p(\psi_\omega(v_k) = x|\psi_\omega(\tilde{V}))
\]

where \( \tilde{V} = (v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}) \). Therefore, following formula [2.8], we find that \( \text{Var}(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k)) < \text{Var}(\psi_\omega(v_k)) = 1 \) and that \( \mathbb{E}(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k)) \) is a linear combination of \( \psi_\omega(\tilde{V}) \), establishing by that the lemma.

By a straight forward calculation (which involves the inversion of a \( 4 \times 4 \) matrix), we obtain that the conditional expectation of \( \psi_\omega(v_k) \) is given by

\[
E_1(\omega) = \frac{\lambda \psi_\omega(v_2) - \psi_\omega(v_3)}{d-1}
\]

\[
E_2(\omega) = \frac{(d-1)\lambda \psi_\omega(v_1) + d\lambda \psi_\omega(v_3) - (d-1)\psi_\omega(v_4)}{\lambda^2 + (d-1)^2}
\]

\[
E_k(\omega) = a_1(\lambda) \frac{\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1})}{2} - a_2(\lambda) \frac{\psi_\omega(v_{k-2}) + \psi_\omega(v_{k+2})}{2}
\]
for $2 < k < n - 1$, where

$$a_1(\lambda) = \frac{2d\lambda}{\lambda^2 + (d-1)^2 + 1}, \quad a_2(\lambda) = \frac{2(d-1)}{\lambda^2 + (d-1)^2 + 1}.\]$$

$E_{n-1}, E_n$ are obtained from $E_1, E_0$ by a reindexation of $V$.

According to the last lemma, $p_n^{+,\alpha}(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k))$ is concentrated with Gaussian tails near its expectation value, which is bounded from above by

$$\mathbb{E}_n^{+,\alpha}(\psi_\omega(v_k)|\psi_\omega(V \setminus v_k)) < \max(E_k(\omega), \alpha) + 1 \tag{3.3}$$

An important observation, which will have a significant role in the proof of theorem 1.2, is the convexity of $E_k(\omega)$. Note that $\forall \lambda \in \sigma(T_d)$ and $1 \leq k \leq n$, the sum of coefficients appearing in equation 3.2 is smaller than one, implying that $\mathbb{E}_n^{+,\alpha}(\psi_\omega(v_k))$ cannot exceed significantly the average of its neighbors. Introducing the notation $\psi_\omega(v_{-1}) \equiv \psi_\omega(v_1), \psi_\omega(v_{n+1}) \equiv \psi_\omega(v_{n-1})$, the next lemma follows:

**Lemma 3.2.** $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, $\exists \psi_0(\lambda, \alpha) < \infty$ and $c_2(\lambda), c_3(\lambda) > 0$, so that $\forall n \in \mathbb{N}, 1 \leq k \leq n$ and $x > \psi_0$

$$\mathbb{P}_n^{+,\alpha}\left(\psi_\omega(v_k) > x \land \psi_\omega(v_k) > \frac{1-c_3}{2}(\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1}))\right) < \frac{1}{2} e^{-c_2x^2}$$

For the sake of brevity and in order to avoid messy calculations, we consider here only the case where $\lambda < d - \sqrt{2(d-1)}$ and $2 < k < n - 1$, where the completion of the proof is postponed to Appendix A. Note that as $\sigma(T_d) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$, the following proof is incomplete only for $d \leq 10$, where $d - \sqrt{2(d-1)} < 2\sqrt{d-1}$.

**Proof. (partial)** According to equation 3.2 and as $\forall \omega \in \Omega^+, \psi_\omega(v_{k\pm2}) > \alpha$, we observe that

$$E_k(\omega) < \frac{a_1(\lambda)}{2}(\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1})) + a_2(\lambda)|\alpha|$$

Note that for $\lambda < d - \sqrt{2(d-1)}$, $a_1(\lambda) < 1$. Therefore, either $E_k(\omega)$ is bounded from above, or $E_k(\omega) < (\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1}))/2$. Setting

$$c_3(\lambda) = \frac{1-a_1(\lambda)}{3}, \quad \psi_0(\lambda, \alpha) = \max\left\{\frac{a_2(\lambda)|\alpha|}{c_3(\lambda)}(1-c_3), 2(|\alpha| + 1)\right\}$$
we obtain that $c_3 > 0$ and
\[ E_k(\omega) < \frac{1 - 3c_3}{2}(\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1})) + \frac{c_3}{1 - c_3} \psi_0, \]
therefore, according to equation 3.3
\[ P_n^{+, \alpha}(\psi_\omega(v_k) > x \wedge \psi_\omega(v_k) > \frac{1 - c_3}{2}(\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1}))) \]
\[ < P_n^{+, \alpha}(\psi_\omega(v_k) - \max(\alpha + 1, E_k(\omega) + 1) > \frac{c_3}{1 - c_3} \chi) \]
\[ < \frac{1}{2} e^{-c_2x^2} \]
where $c_2 = c_3^2/2\sigma_k^2(1 - c_3)^2$ and following lemma 3.1 in the last inequality.

The proof of the lemma to the extreme vertices of $V$ (i.e. $k = 1, 2, n - 1$ and $n$) is similar and do not bare any difficulty. However, for $\lambda \geq d - \sqrt{2(d - 1)}$ the arguments made above are insufficient, as in that case $a_1(\lambda) \geq 1$. This obstacle is removed by considering the role of $a_2(\lambda)$ in equation 3.2 to show that the event
\[ \left\{ \omega \in \Omega_n^{+, \alpha}, \psi_\omega(v_k) > x \wedge \psi_\omega(v_k) > \frac{1 - c_3}{2}(\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1})) \right\} \]
can occur only if for some $v_{k'} \in V$, $\psi_\omega(v_{k'}) - E_{k'}(\omega)$ is proportional to $x$.

As according to the last lemma, the probability to find that $\psi_\omega(v_k) > x$ is small, unless one of its nearest neighbors is considerably larger than $x$, theorem 1.2 follows:

**Proof. of theorem 1.2.** For a given $\lambda$ and $\alpha$, set $\psi_0, c_2, c_3$ as in lemma 3.2.

A first observation we make is that if for some $\omega \in \Omega_n^{+, \alpha}$ and $x > 0$
\[ x \leq \psi_\omega(v_k) \leq \frac{1 - c_3}{2}(\psi_\omega(v_{k-1}) + \psi_\omega(v_{k+1})) \]
then necessarily $\max(\psi_\omega(v_{k-1}), \psi_\omega(v_{k+1})) > (1 - c_3)^{-1}x$.

Now, assume without the loss of generality that $\psi_\omega(v_{k+1}) \geq \psi_\omega(v_{k-1})$. Then, either $\psi_\omega(v_{k+1}) > \frac{1 - c_3}{2}(\psi_\omega(v_k) + \psi_\omega(v_{k+2}))$, or $\psi_\omega(v_{k+2}) > (1 - c_3)^{-2}x$.

By iterating the last step (and keeping in mind the convention $\psi_\omega(v_{-1}) \equiv$
$\psi_\omega(v_1), \psi_\omega(v_{n+1}) \equiv \psi_\omega(v_{n-1})$, we find out that if $\psi_\omega(v_k) > x$ then there must exist $1 \leq j \leq n$, so that $\psi_\omega(v_j) > (1 - c_3)^{|j-k|}x$ and in addition

$$\psi_\omega(v_j) > \frac{1 - c_3}{2} (\psi_\omega(v_{j-1}) + \psi_\omega(v_{j+1}))$$

As a result, according to lemma 3.2, we obtain that

$$P_{n}^{+}\alpha (\psi_\omega(v_k) > x)$$

$$< \sum_{j=0}^{n} P_{n}^{+}\alpha (\psi_\omega(v_j) > (1 - c_3)^{-|j-k|}x \land \psi_\omega(v_j) > \frac{1 - c_3}{2}(\psi_\omega(v_{j-1}) + \psi_\omega(v_{j+1})))$$

$$< \frac{1}{2} \sum_{j=0}^{n} \exp (-c_2(1 - c_3)^{-2|j-k|}x^2)$$

where

$$c_1 = \sum_{j=0}^{\infty} \exp (-c_2[(1 - c_3)^{-2} - 1]x^2) < \infty$$

In the next section we will be interested in the conditional distribution $P_{n}^{\lambda}(\psi_\omega(v_j))$, where in addition we condition on $\psi_\omega(v_1), \psi_\omega(v_2)$. For this purpose we introduce the following variation on theorem 1.2.

**Corollary 3.3.** \(\forall \lambda \in \sigma(T_d) \text{ and } \alpha \in \mathbb{R}, \exists \psi_0(\lambda, \alpha) < \infty \text{ and } 0 < c_1(\lambda), c_2(\lambda) < \infty\), so that \(\forall n \in \mathbb{N}, 3 \leq k \leq n, x_1, x_2 > \alpha \text{ and } x > \max(\psi_0, x_2)\)

$$P_{n}^{\lambda}(\psi_\omega(v_k) \geq x|(\psi_\omega(v_1), \psi_\omega(v_2)) = (x_1, x_2)) < c_1 e^{-c_2x^2}$$

**Proof.** For a given $\lambda$ and $\alpha$, set $\psi_0, c_2$ and $c_3$ as in lemma 3.2. In a similar manner to the proof of theorem 1.2, we notice that if $\psi_\omega(v_k) \leq x$ for some $3 \leq k \leq n$, then there must exist $3 \leq j \leq n$, so that $\psi_\omega(v_j) > (1 - c_3)^{|j-k|}x$ and in addition $\psi_\omega(v_j) > \frac{1 - c_3}{2}(\psi_\omega(v_{j-1}) + \psi_\omega(v_{j+1}))$.

As was shown above, the probability for such an event decays with $x$ in a Gaussian manner. 

\[\square\]
4. Phase Transition of the \( \alpha \)-level sets

In this section we consider, for a given \( \lambda \in \sigma(T_d) \) and \( \alpha \in \mathbb{R} \), the distribution of the large components of the random process \( \{ \Omega, \mathbb{P}_\alpha \} \), or the large \( \alpha \)-level sets in \( GS_d(\lambda) \).

In particular we prove theorem 1.3 and the existence of a critical threshold \( \alpha_c \), so that for \( \alpha > \alpha_c \) the level sets are almost surely all finite, while for \( \alpha < \alpha_c \) a level-set of an infinite cardinality will almost surely appear.

Due to the tree structure of \( T_d \), we can focus our inquiries in the following measures over simple paths:

Let \( V = \{ v_j \}_{j=1}^n \subset T_d \) be a simple path. We denote probability densities along the path by the shorthand notation

\[
p(x_{i_1}, x_{i_2} | x_{j_1}, x_{j_2}) = p((\psi_\omega(v_{i_1}), \psi_\omega(v_{i_2})) = (x_{i_1}, x_{i_2}) | (\psi_\omega(v_{j_1}), \psi_\omega(v_{j_2})) = (x_{j_1}, x_{j_2}))
\]

and Similarly

\[
p_n^{+\alpha}(x_{i_1}, x_{i_2} | x_{j_1}, x_{j_2}) =
\]

\[
p((\psi_\omega(v_{i_1}), \psi_\omega(v_{i_2})) = (x_{i_1}, x_{i_2}) | (\psi_\omega(v_{j_1}), \psi_\omega(v_{j_2})) = (x_{j_1}, x_{j_2}), v_1 \in C^\alpha_\omega(v_n))
\]

For a given \( \lambda \in \sigma(T_d) \) and \( \alpha \in \mathbb{R} \) we define

\[
P^{(n)}_\alpha = \mathbb{P}(v_n \in C^\alpha_\omega(v_n))
\]

as the probability that a given path of length \( n \) is contained in an \( \alpha \)-level set. The probability for the same event, where we condition on \( \psi_\omega(v_1), \psi_\omega(v_2) \) will be denoted by

\[
F^{(n)}_\alpha(x_1, x_2) = \mathbb{P}(v_n \in C^\alpha_\omega(v_n) | (\psi_\omega(v_1), \psi_\omega(v_2)) = (x_1, x_2))
\]

The existence of infinite \( \alpha \)-level sets for small enough \( \alpha \) is proven in [22], where general invariant percolation processes on \( T_d \) are considered. Using the mass-transport method it is shown that if the survival probability of an edge in such a process is larger than \( 2/d \), an infinite cluster will appear in almost every realization of the process. Since for any \( \lambda \in \sigma(T_d) \) the survival probability of an edge in \( GS_d(\lambda) \) is approaching 1 as \( \alpha \to -\infty \), the existence of an infinite component in \( T_\alpha(\psi_\omega) \) is promised \( \forall \alpha \) below some (calculable) threshold.
Gaussian Waves on the Regular Tree

The absence of an infinite component for high values of \( \alpha \) results from the following lemma:

**Lemma 4.1.** \( \forall \lambda \in \sigma(T_d) \) there exist \( \beta(\lambda) > 0 \), so that \( \forall \alpha > 0 \) and \( n \in \mathbb{N} \)

\[
P^{(n)}_\alpha < e^{-\beta n^2}
\]

**Proof.** For every \( \alpha > 0 \), \( P^{(n)}_\alpha \) can be bounded from above by

\[
P^{(n)}_\alpha = \mathbb{P}(\forall 1 \leq j \leq n, \psi_\omega(v_i) > \alpha)
< \mathbb{P}(\Psi_\omega(V) > n\alpha)
\]

where we set \( \Psi_\omega(V) = \sum_{j=1}^{n} \psi_\omega(v_j) \). Note that \( \Psi_\omega \) is a Gaussian random variable, with variance

\[
\text{Var}(\Psi_\omega(V)) = \mathbb{E} \left( \sum_{ij} \psi_\omega(v_i)\psi_\omega(v_j) \right)
= n \left( \phi^{(\lambda)}(0) + 2 \sum_{j=1}^{n-1} \frac{n-j}{n} \phi^{(\lambda)}(j) \right) < n\Phi^{(\lambda)}
\]

where \( \Phi^{(\lambda)} = \phi^{(\lambda)}(0) + 2 \sum_{j=1}^{\infty} |\phi^{(\lambda)}(j)| \) (see equation 2.4 and lemma 2.2). Note that, as \( |\phi^{(\lambda)}(j)| = O((d-1)^{-j/2}) \), \( \Phi^{(\lambda)} \) is finite \( \forall \lambda \in \sigma(T_d) \). As a consequence,

\[
P^{(n)}_\alpha < \frac{1}{\sqrt{2\pi n\Phi^{(\lambda)}}} \int_{n\alpha}^{\infty} \exp \left( -\frac{x^2}{2n\Phi^{(\lambda)}} \right) < e^{-\beta n^2}
\]

where \( \beta = (2\Phi^{(\lambda)})^{-1} \).

Recalling that the volume of a sphere in \( T_d \) is \( |\Lambda_n| = d(d-1)^{n-1} \), we obtain that \( |\Lambda_n|P^{(n)}_\alpha \) decays exponentially for any \( \alpha > \sqrt{(d-1)/\beta} \), implying that almost surely no infinite component will appear.

In order to verify the existence of a critical threshold between the two phases, we would like to present the following classification of random processes on trees, introduced in [6]:

**Definition 4.2.** A random process \( \{\Omega, \mathbb{P}\} \) on a tree graph \( \Gamma \), associating \( \forall \omega \in \Omega \) an induced subgraph \( \Gamma_\omega \subset \Gamma \), is a quasi Bernoulli process, if \( \exists M < \infty \), such that
Figure 2. For a quasi-bernoulli process, the probability to find \( v_0 \in C_\omega^\alpha(v_1) \) (continuous purple line) conditioned that \( v_0 \in C_\omega^\alpha(v_2) \) (dotted green line) is uniformly bounded by the probability that \( v_0 \in C_\omega^\alpha(v_1) \), conditioned that \( v_0 \in C_\omega^\alpha(v_0 \land v_1 \land v_0) \) (dashed blue line).

\[
\forall v_0, v_1, v_2 \in \Gamma:
\frac{\mathbb{P}(v_1 \in C_{\Gamma_\omega}(v_0)|v_2 \in C_{\Gamma_\omega}(v_0))}{\mathbb{P}(v_1 \in C_{\Gamma_\omega}(v_0)|v_0 \land 1 \land 2 \in C_{\Gamma_\omega}(v_0))} \leq M .
\]

(4.1)

where \( v_0 \land 1 \land 2 \) is the intersection of the simple paths in \( \Gamma \) between the three vertices (see figure 2) and \( C_{\Gamma_\omega}(v_0) \) is the connected component of \( v_0 \) in \( \Gamma_\omega \).

Definition 4.2 provides a simple criterion for the existence (or the absence) of an infinite component in \( \Gamma_\omega \). For the sake of clarity, we provide here a partial version of a theorem, derived in [6]:

**Lemma 4.3. (Lyons)** Let \( \{\Omega, \mathbb{P}\} \) be a quasi Bernoulli process on \( T_d \), which is invariant under the automorphism group of \( T_d \) and associates \( \forall \omega \in \Omega \) an induced graph \( T_\omega \subset T_d \). If

\[
\lim_{|v' - v| \to \infty} (\mathbb{P}(v' \in C_{T_\omega}(v)))^{1/|v' - v|} < \frac{1}{d-1}
\]

then, with a high probability, all the connected components of \( T_\omega \) are finite. If

\[
\lim_{|v' - v| \to \infty} (\mathbb{P}(v' \in C_{T_\omega}(v)))^{1/|v' - v|} > \frac{1}{d-1}
\]

\( T_\omega \) will have an infinite component with probability 1. \( \square \)
In order to verify that the level sets of $G\mathcal{S}_d(\lambda)$ are quasi-Bernoulli, we provide the following bound on $P_\alpha^{(n)}$:

**Lemma 4.4.** $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, $\exists 0 < c_1, c_2 < \infty$ so that $\forall n, m \in \mathbb{N}$

$$c_1 P_\alpha^{(n)} P_\alpha^{(m)} < P_\alpha^{(n+m)} < c_2 P_\alpha^{(n)} P_\alpha^{(m)}$$  \hspace{1cm} (4.2)

Note that the right inequality in equation 4.2 is the restriction of equation 4.1 to the case where $v_0$ is along the simple path between $v_1$ to $v_2$.

**Proof.** According to theorem 1.2 and corollary 3.3, $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, there exists a finite threshold $\alpha < \psi_1(\lambda, \alpha) < \infty$, so that $\forall n \in \mathbb{N}$ and $j < n$,

$$\mathbb{P}^+ \alpha_n (\psi_\omega(v_j), \psi_\omega(v_{j+1})) \in (\alpha, \psi_1)^2 > \frac{1}{2}$$ and

$$\mathbb{P}^+ \alpha_{n+2} (\psi_\omega(v_3), \psi_\omega(v_4)) \in (\alpha, \psi_1)^2(\psi_\omega(v_1), \psi_\omega(v_2)) \in (\alpha, \psi_1)^2 > \frac{1}{2}$$

Recalling that

$$\mathbb{P}(v_1 \in C_\omega^{\alpha}(v_n) \wedge (\psi_\omega(v_1), \psi_\omega(v_2)) \in (\alpha, \psi_1)^2) = \int_\alpha^{\psi_1} dy_1 dy_2 p(y_1, y_2) F_\alpha^{(n)}(y_1, y_2)$$

and that

$$\mathbb{P} (v_1 \in C_\omega^{\alpha}(v_{n+2}) \wedge (\psi_\omega(v_3), \psi_\omega(v_4)) \in (\alpha, \psi_1)^2(\psi_\omega(v_1), \psi_\omega(v_2)) = (x_1, x_1))$$

$$= \int_\alpha^{\psi_1} dx_3 dx_4 p(x_3, x_4|x_1, x_2) F_\alpha^{(n)}(x_3, x_4)$$

We obtain by applying bayes’ theorem that $\forall (x_1, x_2) \in (\alpha, \psi_1)^2$,

$$\frac{1}{2} P_\alpha^{(n)} < \int_\alpha^{\psi_1} dy_1 dy_2 p(y_1, y_2) F_\alpha^{(n)}(y_1, y_2) < P_\alpha^{(n)}$$  \hspace{1cm} (4.3)

$$\frac{1}{2} F_\alpha^{(n+2)}(x_1, x_2) < \int_\alpha^{\psi_1} dx_3 dx_4 p(x_3, x_4|x_1, x_2) F_\alpha^{(n)}(x_3, x_4) < F_\alpha^{(n+2)}(x_1, x_2)$$

As $p(y_1, y_2)$ and $p(x_3, x_4|x_1, x_2)$ are strictly positive and bounded, $\exists 0 < \tilde{c}_1(\lambda, \alpha), \tilde{c}_2(\lambda, \alpha) < \infty$ so that for every $\{x_1, x_2, x_3, x_4, y_1, y_2\} \in (\alpha, \psi_1)$:

$$\tilde{c}_1 = \frac{p(x_3, x_4|x_1, x_2)}{p(y_1, y_2)} < \tilde{c}_2$$

Therefore, we get from equation 4.3 that $\forall (x_1, x_2) \in (\alpha, \psi_1)^2$,

$$\frac{\tilde{c}_1}{2} P_\alpha^{(n)} < F_\alpha^{(n+2)}(x_1, x_2) < 2\tilde{c}_2 P_\alpha^{(n)}$$
Since, by the Markov Property of $\mathcal{G}\mathcal{S}_d(\lambda)$,

$$P_{\alpha}^{(n+m)} = P_{\alpha}^{(n)} \int_{\alpha}^{\infty} dx_{n-1} dx_{n} P_{n}^+(x_{n-1}, x_{n}) F_{\alpha}^{(m+2)}(x_{n-1}, x_{n})$$

We get that

$$P_{\alpha}^{(n+m)} < P_{\alpha}^{(n)} \int_{\alpha}^{\psi_1} dx_{n-1} dx_{n} P_{n}^+(x_{n-1}, x_{n}) F_{\alpha}^{(m+2)}(x_{n-1}, x_{n}) < c_2 P_{\alpha}^{(n)} P_{\alpha}^{(m)}$$

where $c_2 = 2\tilde{c}_2 \int_{\alpha}^{\psi_1} dx_{n-1} dx_{n} P_{n}^+(x_{n-1}, x_{n})$. Similarly,

$$P_{\alpha}^{(n+m)} > \frac{1}{2} P_{\alpha}^{(n)} \int_{\alpha}^{\psi_1} dx_{n-1} dx_{n} P_{n}^+(x_{n-1}, x_{n}) F_{\alpha}^{(m+2)}(x_{n-1}, x_{n}) > c_1 P_{\alpha}^{(n)} P_{\alpha}^{(m)}$$

where $c_1 = \tilde{c}_1 \int_{\alpha}^{\psi_1} dx_{n-1} dx_{n} P_{n}^+(x_{n-1}, x_{n})/4$.

Note that according to lemma 4.3, $\lim_{n \to \infty} (P_{\alpha}^{(n)})^{1/n} = \lim_{n \to \infty} (P_{\alpha}^{(n)})^{1/n}$. As a result, $\lim_{n \to \infty} (P_{\alpha}^{(n)})^{1/n}$ exists and

**Corollary 4.5.** $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, $\exists 0 < c_3, c_4, c_5 < \infty$, so that $\forall n \in \mathbb{N}$ and $\epsilon > 0$

$$c_3 e^{-(c_5+\epsilon)n} \leq P_{\alpha}^{(n)} \leq c_4 e^{-(c_5-\epsilon)n}$$

The dependence of $F_{\alpha}^{(n)}(x_1, x_2)$ in its argument, can be bounded in the following manner

**Lemma 4.6.** $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, $\exists 6, c_7 < \infty$ so that $\forall n \in \mathbb{N}$ and $x_1, x_2 > \alpha$

$$F_{\alpha}^{(n)}(x_1, x_2) < c_6 \cdot (1 + |x_2|)^{c_7} P_{\alpha}^{(n)}$$

**Proof.** Since $\forall (x_1, x_2) \in \mathbb{R}^2$, $F_{\alpha}^{(n)}(x_1, x_2) \leq 1$, corollary 4.5 implies that $\forall \epsilon > 0$,

$$F_{\alpha}^{(n)}(x_1, x_2) < \frac{c_6}{c_7} P_{\alpha}^{(n)}$$

Therefore, if $x_2 > (d-1)^{n/6}$, the lemma follows with $c_6 = c_3^{-1}$ and $c_4 = 6c_3 / \log (d-1)$.

Otherwise, set $X_1 = (x_1, x_2)$, $m = \lceil 6 \log_{d-1}(|x_2|) \rceil < n$ (where $\lceil \cdot \rceil$ stands for the integer part). By the Markov property of $\mathcal{G}\mathcal{S}_d(\lambda)$, $F_{\alpha}^{(n)}(x_1, x_2)$ equals

$$F_{\alpha}^{(n)}(x_1, x_2) = F_{\alpha}^{(m)}(x_1, x_2) \int_{\alpha}^{\infty} dx_{m-1} dx_{m} P_{m}^+(x_{m-1}, x_{m}) F_{\alpha}^{(n-m+2)}(x_{m-1}, x_{m})$$

Since according to corollary 3.3, $\exists \psi_2(\lambda, \alpha) < \infty$ so that

$$\mathbb{P}_{n}^{+,\alpha} ((\psi_\omega(v_{m-1}), \psi_\omega(v_m)) \in (\alpha, x_2 + \psi_2)^2 | (\psi_\omega(v_1), \psi_\omega(v_2)) = (x_1, x_2)) > 1/2$$
And as
\[
F^{(m)}_{\alpha}(x_1, x_2)p^{1}_{m}(x_{m-1}, x_{m}|x_1, x_2)
\]
\[
= p(x_{m-1}, x_{m}|x_1, x_2)p(x_{m-1}, x_{m}|v_1 \in C^{\alpha}_{v}(v_m)|x_1, x_2, x_{m-1}, x_{m})
\]
\[
< p(x_{m-1}, x_{m}|x_1, x_2)
\]
We obtain that
\[
F^{(n)}_{\alpha}(x_1, x_2) \leq 2 \int_{\alpha}^{x_1+\psi_2} dx_{m-1} dx_m p(x_{m-1}, x_{m}|x_1, x_2) F^{(n-m+2)}_{\alpha}(x_{m-1}, x_{m})
\]  \hspace{1cm} (4.4)

Next, we would like to evaluate the conditional density \( p(x_{m-1}, x_{m}|x_1, x_2) \) in terms of \( p(x_{m-1}, x_{m}) \). To do so, we set
\[
C_{11} = C_{22} = \begin{pmatrix} \phi^{(\lambda)}(0) & \phi^{(\lambda)}(1) \\ \phi^{(\lambda)}(1) & \phi^{(\lambda)}(0) \end{pmatrix}, \quad C_{12} = \begin{pmatrix} \phi^{(\lambda)}(m-2) & \phi^{(\lambda)}(m-1) \\ \phi^{(\lambda)}(m-2) & \phi^{(\lambda)}(m-1) \end{pmatrix}
\]

and \( X_m = (x_{m-1}, x_{m}) \). According to equation 2.8, the investigated density is given by
\[
p(x_{m-1}, x_{m}|x_1, x_2) = \frac{1}{2\pi \sqrt{\det(C)}} \exp \left( -\frac{1}{2} \langle (X_m - \mu), C^{-1}(X_m - \mu) \rangle \right)
\]  \hspace{1cm} (4.5)

where \( \mu = C_{12}C_{22}^{-1}X_0 \) and \( C = C_{11} - C_{12}C_{22}^{-1}C_{12} \).

Since \( \phi^{(\lambda)}(m) = O((d-1)^{-m/2}) \), we get that \( \|C_{12}\|_2 = O((d-1)^{-m/2}) \) while \( \|C_{11}\|_2 = 1 + |\lambda|/d \). As a result, by considering the Taylor expansion of \( C \) and \( \mu \), we find that
\[
C^{-1} = C_{11}^{-1} + C_p, \quad \text{where} \quad \|C_p\|_2 = O((d-1)^{-m}) \quad \text{and} \quad \|\mu\|_2 = O(\|X_1\|_2(d-1)^{-m/2}).
\]

Recalling that \( m = \log_{d-1}(1 + \|X_1\|_\infty) \), we obtain from equation 4.5 that
\[
\forall x_{m-1}, x_{m} < x_2 + \psi_2,
\]
\[
p(x_{m-1}, x_{m}|x_1, x_2) = \frac{1}{2\pi \sqrt{\det(C)}} \exp \left( -\frac{1}{2} \langle X_m, C_{11}^{-1}X_m \rangle + f(X_1) \right)
\]

where
\[
f(X_1) = -\frac{1}{2} \langle X_m, C_p X_m \rangle - \langle X_m, C^{-1} \mu \rangle \xrightarrow{\|X_1\|_\infty \to 0} 0
\]

is a bounded function of \( X_1 \). Since
\[
p(x_{m-1}, x_{m}) = \frac{1}{2\pi \sqrt{\det(C_{11})}} \exp \left( -\frac{1}{2} \langle X_m, C_{11}^{-1}X_m \rangle \right)
\]
and as $\det(C_{11})/\det(C) = 1 + O((d - 1)^{-m}) < 2$, we find that

$$p(x_{m-1}, x_m|x_1, x_2) < \tilde{c}_1 p(x_{m-1}, x_m)$$

where $\tilde{c}_1(\lambda, \alpha) = 2 \max_{\mathbb{R}_2} f(X_1) < \infty$. Returning to equation 4.4 we find that

$$F_{\alpha}^{(n)}(x_1, x_2) \leq \tilde{c}_1 \int_{\alpha}^{x_1 + \nu_2} dx_{m-1} dx_m p(x_{m-1}, x_m) F_{\alpha}^{(n-m+2)}(x_{m-1}, x_m)$$

$$< \tilde{c}_1 \int_{\alpha}^{\infty} dx_{m-1} dx_m p(x_{m-1}, x_m) F_{\alpha}^{(n-m+2)}(x_{m-1}, x_m)$$

$$= \tilde{c}_1 P_{\alpha}^{(n-m+2)}$$

Finally, as according to corollary 4.5 $\forall \epsilon > 0$, $P_{\alpha}^{(n-m+2)} \leq (c_4/c_3)e^{(c_5+\epsilon)(m-2)} P_{\alpha}^{(n)}$,

setting $c_6 = \tilde{c}_1 c_4/c_3$, $c_7 = 6c_5/\log(d - 1)$, the lemma follows.

\[\square\]

Having lemmas 4.4 and 4.6 in hand, we are ready to prove theorem 1.3.

\textbf{Proof. of theorem 1.3} First, we note that $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, the $\alpha$-level sets of $\mathcal{G}_d(\lambda)$ are quasi-Bernoulli.

Indeed, let $v_0, v_1, v_2 \in T_d$. If $v_1(v_2)$ is on the pass, connecting $v_0$ to $v_2(v_1)$, condition 4.1 is trivially satisfied, with $M = 1$. If $v_0$ connects $v_1$ to $v_2$, then according to lemma 4.4 condition 4.1 is satisfied with $M = c_2(\lambda, \alpha)$.

Otherwise, $v_{0}^{\wedge} \notin \{v_0, v_1, v_2\}$. We denote by $v_0', v_1', v_2'$ the vertices which are adjacent to $v_{0}^{\wedge}$ on the simple path leading to $v_0, v_1, v_2$ correspondingly; We rewrite the LHS of equation 4.1 as:

$$\frac{\mathbb{P}(v_1 \in C^\alpha_{\wedge}(v_0) | v_2 \in C^\alpha_{\wedge}(v_0))}{\mathbb{P}(v_1 \in C^\alpha_{\wedge}(v_0) | v_{0}^{\wedge} \notin C^\alpha_{\wedge}(v_0))} = \frac{\mathbb{P}(\{v_1, v_2\} \subset C^\alpha_{\wedge}(v_0)) \mathbb{P}(v_{0}^{\wedge} \notin C^\alpha_{\wedge}(v_0))}{\mathbb{P}(v_1 \in C^\alpha_{\wedge}(v_0)) \mathbb{P}(v_2 \in C^\alpha_{\wedge}(v_0))}$$

and set $n_j = |v_{0}^{\wedge} - v_j|$ for $j = 0, 1, 2$.

By the Markov property of $\mathcal{G}_d(\lambda)$, we express:

$$\mathbb{P}(\{v_1, v_2\} \subset C^\alpha_{\wedge}(v_0)) = \int_{-\infty}^{\alpha} dx_{0}^{\wedge} dx_0 dx_1 dx_2$$

$$\times p(x_{0}^{\wedge}, x_0', x_1', x_2') \prod_{j=0,1,2} F_{\alpha}^{(n_j)}(x_{0}^{\wedge}, x_j')$$
Following lemma 4.6, the RHS is bounded by

\[ P(\{v_1, v_2\} \subset C_\omega^\alpha(v_0)) \leq c_8(\lambda, \alpha) \prod_{j=0,1,2} P_{\alpha}^{(n_j)} \]

where

\[ c_8(\lambda, \alpha) = c_6^3 \int_{-\infty}^{\alpha} dx_0 dx_1 dx_2 \prod_{j=0,1,2} (1 + |x_j|)^{c_7} \]

Note that as \( c_8(\lambda, \alpha) \) is a moment of a Gaussian distribution, it is finite.

Finally, following lemma 4.4, we get that

\[ P(\{v_1, v_2\} \subset C_\omega^\alpha(v_0)) \leq c_8(\lambda, \alpha) \]

establishing by that the quasi-Bernoulli property of \( \{\Omega, \mathbb{P}_\alpha\} \).

As (following lemma 4.4) for \( \alpha > \sqrt{(d-1)/\beta} \), \( \lim_{n \to \infty} (P_{\alpha}^{(n)})^{1/n} < 1/(d - 1) \), while according to [22], \( \lim_{n \to \infty} (P_{\alpha}^{(n)})^{1/n} > 1/(d - 1) \) for small enough \( \alpha \) and since \( \lim_{n \to \infty} (P_{\alpha}^{(n)})^{1/n} \) is strictly decreasing in \( \alpha \), theorem 1.3 follows, where \( \alpha_c \) is given by the (implicit) expression

\[ \lim_{n \to \infty} (P_{\alpha_c}^{(n)})^{1/n} = \frac{1}{d - 1} \]

\[ \square \]

Appendix A. proof to lemma 3.2

In section 3 we have established lemma 3.2 for vertices in the bulk of the path \((2 < k < n - 1)\) and \( \lambda < d - \sqrt{2(d-1)} \). We begin by proving the lemma \( \forall \lambda \in \sigma(T_d) \) for the case \( k = 1 \) \((k = n)\). The main theme in the proof is the partition of the event

\[ \left\{ \omega \in \Omega_n^{+,\alpha}, \psi_{\omega}(v_k) > x \wedge \psi_{\omega}(v_k) > \frac{1 - c_3}{2}(\psi_{\omega}(v_{k-1}) + \psi_{\omega}(v_{k+1})) \right\} \]

into a finite union of events, so that in every subevent, \( \psi_{\omega}(v_{k'}) \) exceed \( E_{k'}(\omega) \) significantly (see equation 3.2) for some \( k' \).
Given $\psi_0$ and $c_3$, we would like to evaluate the probability of the event

$$A(x) = \{ \omega \in \Omega_n^{+\alpha}, \psi_\omega(v_1) > x \wedge \psi_\omega(v_1) > (1 - c_3)\psi_\omega(v_2) \}$$

for $x > \psi_0$ by decomposing it into

$$A_1(x) = \left\{ \omega \in A, \psi_\omega(v_3) > \lambda \psi_\omega(v_2) - \frac{d - 1}{1 + c_3} \psi_\omega(v_1) \right\}$$

$$A_2(x) = \left\{ \omega \in A \setminus A_1, \psi_\omega(v_4) > \lambda \psi_\omega(v_1) + \frac{d\lambda}{d - 1} \psi_\omega(v_3) - \frac{\lambda^2 + (d - 1)^2}{(d - 1)(1 + c_3)} \psi_\omega(v_2) \right\}$$

$$A_3(x) = \left\{ \omega \in A \setminus \bigcup_{j=1}^3 A_j, \psi_\omega(v_5) > \frac{\lambda d}{d - 1} (\psi_\omega(v_2) + \psi_\omega(v_4)) - \psi_\omega(v_1) - \frac{\lambda^2 + (d - 1)^2 + 1}{(d - 1)(1 + c_3)} \psi_\omega(v_3) \right\}$$

$$A_4(x) = A \setminus \bigcup_{j=1}^3 A_j$$

This partition is chosen, following equation 3.2, so that $\forall 1 \leq j \leq 3$ and $\omega \in A_j(x)$,

$$\psi_\omega(v_j) - E_j(\omega) > \beta_j x$$

where $\beta_j$ are some strictly positive functions of $c_3$. Therefore, according to lemma 3.1 and equation 3.3, we find out that $\exists \psi_0(\lambda, \alpha)$, so that $\forall x > \psi_0$,

$$\mathbb{P}_n^{+\alpha}(\omega \in A_j(x)) < \exp \left( -\frac{\beta_j^2 x^2}{2} \right)$$

As a result, the lemma will follow by showing that $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$, there exists $c_3(\lambda, \alpha) > 0$ and $\psi_0(\lambda, \alpha) < \infty$, so that $\forall x > \psi_0$, $A(x) = \bigcup_{j=1}^3 A_j(x)$.

For $\lambda \leq 0$, as $\forall \omega \in A, \psi_\omega(v_2) > \alpha$, we obtain that

$$A_1(x) \supset \left\{ \omega \in A, \alpha > -|\lambda|\psi_\omega(v_2) - \frac{d - 1}{1 + c_3} \psi_\omega(v_1) \right\}$$

Therefore $\forall c_3 > 0$, setting $\psi_0 = -(1 + |\lambda|)(1 + c_3)\alpha/(d - 1)$, we find that $A(x) \subset A_1(x)$ for every $x > \psi_0$.

Similarly, if $0 < \lambda \leq d - 1$ then

$$A_1(x) \supset \left\{ \omega \in A, \alpha > \frac{\lambda - (d - 1)}{1 + c_3} \psi_\omega(v_1) \right\}$$

Therefore $\forall c_3 > 0$, setting $\psi_0 = (1 + c_3)\alpha/(d - 1 - \lambda)$, we get that $A(x) \subset A_1(x)$ for every $x > \psi_0$. Note that if $d > 5$ then $\max(\sigma(T_d)) = 2\sqrt{d - 1} < d - 1$, and the proof is
done.

If \( d - 1 \leq \lambda < (d - 1 + \sqrt{d^2 + 2d - 3})/2 \), we find that

\[
A_2(x) \supset \left\{ \omega \in A \setminus A_1, \alpha > - \left( 1 + \frac{dc_3}{d-1} \right) \lambda^2 + (d - 1 - (d + c_3)c_3)\lambda + d - 1 \right\}
\]

As in the limit \( \psi_\omega(v_1) \to \infty, c_3 \to 0 \) and \( \forall d - 1 < \lambda < (d - 1 + \sqrt{d^2 + 2d - 3})/2 \)

\[
\alpha > - \left( 1 + \frac{dc_3}{d-1} \right) \lambda^2 + (d - 1 - (d + c_3)c_3)\lambda + d - 1 \frac{\psi_\omega(v_1)}{1 - c_3}
\]

we obtain that for any \( \lambda \) smaller then \( (d - 1 + \sqrt{d^2 + 2d - 3})/2 \), there exist finite and positive \( \psi_0, c_3 \), so that \( \forall x > \psi_0, A(x) = A_1(x) \cup A_2(x) \).

As \( \max(\sigma(T_d)) < (d - 1 + \sqrt{d^2 + 2d - 3})/2 \) for \( d \geq 3 \), we are left with the case \( d = 3 \) and \( 1 + \sqrt{3} < \lambda \leq 2\sqrt{2} \). Preforming a similar calculation, one finds that for an appropriate choice of \( \psi_0 \) and \( c_3 \), \( A_3(x) \supset A(x) \setminus \bigcup_{j=1,2} A_j(x) \), as long as \( \lambda^3 - 2\lambda^2 - 4\lambda + 4 < 0 \). As this is indeed the case for every \( 1 + \sqrt{3} < \lambda \leq 2\sqrt{2} \), the proof is complete.

**the case \( k > 2 \) :**

The proof is similar to the above, but require few more iterations. Using the shorthand notation

\[
\psi_\omega^{(j)}(v_k) = \frac{1}{2}(\psi_\omega(v_{k-j}) + \psi_\omega(v_{k+j}))
\]

we decompose, for a given \( \psi_0 \) and \( c_3 \), the event

\[
A(x) = \left\{ \omega \in \Omega_n^+, \psi_\omega(v_k) > x \land \psi_\omega(v_k) > (1 - c_3)\psi_\omega^{(1)}(v_k) \right\}
\]

into

\[
A_1(x) = \left\{ \omega \in A, \psi_\omega^{(2)}(v_k) > \frac{d\lambda}{d-1} \psi_\omega^{(1)}(v_k) - \frac{\lambda^2 + (d - 1)^2 + 1}{(d - 1)(1 + c_3)} \psi_\omega(v_k) \right\}
\]

\[
A_2(x) = \left\{ \omega \in A \setminus A_1, \frac{d\lambda}{d-1} (\psi_\omega(v_k) + \psi_\omega^{(2)}(v_k)) - \left( \frac{\lambda^2 + (d - 1)^2 + 1}{(d - 1)(1 + c_3)} - 1 \right) \psi_\omega^{(1)}(v_k) \right\}
\]

\[
A_j(x) = \left\{ \omega \in A \setminus \bigcup_{i=1}^{j-1} A_i, \psi_\omega^{(j+1)}(v_k) > \frac{d\lambda}{d-1} (\psi_\omega^{(j-2)}(v_k) + \psi_\omega^{(j)}(v_k)) - \psi_\omega^{(j-3)}(v_k) - \frac{\lambda^2 + (d - 1)^2 + 1}{(d - 1)(1 + c_3)} \psi_\omega^{(j-1)}(v_k) \right\}
\]
where we assume that $x > \psi$ and $j \geq 3$.

As before, the intervals are chosen so that if $\omega \in A_{j+1}$ for some $j \geq 0$, then

$$\psi^{(j)}(v_k) - \frac{1}{2}(E_{k-j}(\omega) + E_{k+j}(\omega)) > \beta_jx$$

for some $\beta_j(c_3) > 0$, implying that for large enough $\psi_0$

$$\forall x > \psi_0, \quad \mathbb{P}_n^{+}(\omega \in A_j(x)) < \exp\left(-\frac{\beta_j^2x^2}{2}\right)$$

As a result, the lemma will follow by verifying that $\forall \lambda \in \sigma(T_d)$ and $\alpha \in \mathbb{R}$ there exist $\psi_0 < \infty$ and $c_3 > 0$ so that $\forall x > \psi_0, A(x) = \bigcup_{j=1}^{p(\lambda)} A_j(x)$ for some finite integer $p(\lambda)$.

As was demonstrated in the (partial) proof at section 3, if $\lambda < d - \sqrt{2(d-1)}$ then for an appropriate choice of $\psi_0$ and $c_3$, we find that $A(x) = A_1(x)$.

If $\lambda \geq d - \sqrt{2(d-1)}$, we find that $A_2(x) = A \setminus A_1(x)$ for small enough $c_3$ and large enough $\psi_0$, as long as

$$-d\lambda^3 + 2(d^2 - d + 1)\lambda^2 - d(d^4 - 4d + 4)\lambda - 2d^3 + 4d^2 - 4d + 2 < 0 \quad (A.1)$$

The last polynomial has a single real root $\lambda_0(\lambda)$, where $\forall d > 5, \lambda_0 > 2\sqrt{d-1}$. Therefore, for these cases, condition $\text{A.1}$ is fulfilled and the lemma follows.

Iterating the process four more times (where each iteration involves the evaluation of the roots of a polynomial of increasing degree), we find for $d = 4, 5$ that $\forall \lambda \in \sigma(T_d)$, $A(x) = \bigcup_{j=1}^{3} A_j(x)$ (for an appropriate choice of $\psi_0$ and $c_3$). For $d = 3$ we get that $A(x) = \bigcup_{j=1}^{6} A_j(x)$, by that establishing the lemma for $2 < k < n - 1$.

The proof for the case $k = 2$ ($k = n - 1$) is identical and therefore will be omitted.

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\[ \lambda_0 = \frac{2(d^2-d+1)}{3d(d^2+6d^3-21d^4+38d^5-39d^6+24d^8-8+3\lambda^2)} \]
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