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New features of scattering from a one-dimensional non-Hermitian (complex) potential

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Abstract

For complex one-dimensional potentials, we propose the asymmetry of both reflectivity and transmittivity under time reversal: \( R(-k) \neq R(k) \) and \( T(-k) \neq T(k) \), unless the potentials are real or PT-symmetric. For complex PT-symmetric scattering potentials, we propose that \( R_{\text{left}}(-k) = R_{\text{right}}(k) \) and \( T(-k) = T(k) \). So far, the spectral singularities (SS) of a one-dimensional non-Hermitian scattering potential are witnessed/conjectured to be at most 1. We present a new non-Hermitian parametrization of the Scarf II potential to reveal its four new features. Firstly, it displays the just acclaimed (in)variances. Secondly, it can support two spectral singularities at two pre-assigned real energies \( E^* = \alpha^2, \beta^2 \) either in \( T(k) \) or in \( T(-k) \), when \( \alpha \beta > 0 \). Thirdly, when \( \alpha \beta < 0 \) it possesses one SS in \( T(k) \) and the other in \( T(-k) \). Fourthly, when the potential becomes PT-symmetric \( (\alpha + \beta) = 0 \), we obtain \( T(k) = T(-k) \), it possesses a unique SS at \( E = \alpha^2 \) in both \( T(-k) \) and \( T(k) \). Lastly, for completeness, when \( \alpha = i\gamma \) and \( \beta = i\delta \) there are no SS, instead we get two real energies \(-\gamma^2\) and \(-\delta^2\) of the complex PT-symmetric Scarf II belonging to the two well-known branches of discrete bound-state eigenvalues. We find them as \( E^+_M = -((\gamma - M)^2) \) and \( E^+_N = -((\delta - N)^2) \); \( M(N) = 0, 1, 2, \ldots \) with 0 \( \leq M(N) < \gamma(\delta) \).

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A particle subject to a potential in the Schrödinger equation is what lies beneath the foundation of non-relativistic quantum mechanics. A variety of potentials in various situations keep throwing new results where proofs are elusive (or incomplete) and experimental verification could be challenging. In the non-Hermitian domain, for over a decade, the complex PT-symmetric potentials [1] have attracted a large number of investigations of both theoretical [2] and experimental [3] types. A Hamiltonian which is invariant under the joint transformation of parity \( (P : x \rightarrow -x) \) and time reversal \( (T : i \rightarrow -i) \) is called PT-symmetric. Later these new Hamiltonians have been covered under the more general concept of pseudo-Hermiticity [4]. However, the language of PT-symmetry is physically more appealing.
This work brings out new features of scattering from a one-dimensional complex potential. Earlier, for a general non-Hermitian (complex) scattering potential it has been proved that [5]

\[ R_{\text{left}} \neq R_{\text{right}} \quad \text{and} \quad T_{\text{left}} = T_{\text{right}}. \]  

(1)

The reflectivity turns out to be symmetric: \( R_{\text{left}} = R_{\text{right}} \), if the complex potential is spatially symmetric. For complex PT-symmetric potentials which are essentially spatially asymmetric it is found that if the particle enters from the side where potential is absorptive (\( \text{Im}(V(x)) < 0 \)), then the reflectivity (\( R(E) \)) is normal < 1. But if it enters from the other side, \( R(E) \) would be anomalous (>1) [7] for some domain of energy. This phenomenon is called left/right-handedness of the reflectivity for a complex PT-symmetric scattering potential.

In this paper, we claim that the proof for (1) (see equations (7) and (9) in [5]) subsequently for a complex non-Hermitian potential also yields the asymmetry of both reflectivity and transmittivity under time reversal as

\[ R(-k) \neq R(k), \quad T(-k) \neq T(k). \]  

(2)

The indicated proof of (2) is not sufficient to rule out the exclusive PT-symmetric scattering potentials for which we conjecture

\[ R_{\text{left}}(-k) = R_{\text{right}}(k), \quad T(-k) = T(k). \]  

(3)

We claim that models of complex PT-symmetric scattering potentials discussed so far [6–9] and others indeed conform to these new invariances (3); yet a proof is welcome. A similar lacunae has been faced earlier wherein complex periodic PT-symmetric potentials could admit [10] real energy bands; yet a proof is desired. The unique signature/feature of complex PT-symmetric potentials even in scattering is a desideratum. In this regard, the conjecture (3) along with the left/right-handedness (1) of the reflectivity could be the unique feature/signature of scattering from a complex PT-symmetric potential perspective.

One feature of scattering from a complex potential which is now important [8] though present (see figure 2 in [7]) got overlooked then. This feature is of an anomalous (≫1) peak in both transmission (\( T(E) \)) and (left) reflection (\( R(E) \)) coefficients. Recently, the positive real discrete energy (\( E = E_n > 0 \)) at which both \( R(E) \) and \( T(E) \) become infinite has been well investigated as spectral singularity or zero width resonance [8]. These are real discrete energies where the Jost functions become linearly dependent. It is good to recall that in Hermitian quantum mechanics the physical energy poles of \( T(E) \) and \( R(E) \) if real, represent the bound states (\( E_n < 0 \)) and resonances (meta-stable states) if complex: \( E_n = -i\Gamma_n/2 \), with \( E_n > 0 \).

The spectral singularities of a non-Hermitian one-dimensional potential have also been discussed earlier in connection with the super-symmetric quantum mechanics [9]. However, seeing [8] them as positive energy discrete poles of \( T(E) \) and \( R(E) \) is more transparent and also it connects well to their experimental realization in wave propagation experiments [11]. The role of spectral singularity in the completeness of the bi-orthogonal basis has been well debated [12]. Spectral singularities seem to have been known in the theory of differential equations with complex and variable coefficients (see [8, 9]).

The solvable Scarf II can be expressed as

\[ V(x) = V_1 \text{sech}^2 x + V_2 \text{sech} x \tanh x \]  

(4)

this can be complexified by taking \( V_2 = iU \). This complexification has contributed quite considerably to the complex PT-symmetric quantum mechanics. This is the first exactly solvable model of complex PT-symmetric potential for scattering [7], discrete spectrum (real and complex-conjugate) [13] and the spectral singularity [14]. It has also helped in various
other investigations [15]. An explicit condition involving \( V_1 \) and \( V_2 \) could be derived along with a simple explicit expression for the spectral singularity [14]. These results have been found by analytically continuing the well-known [16, 17] expressions of transmission and reflection amplitudes of the Hermitian Scarf II potential. Later, in an update on the complex PT-symmetric Scarf II potential (4) a part (when \( V_1 < 0 \)) of these results on the spectral singularity [14] have been rederived [18].

So far, for one-dimensional non-Hermitian potentials the spectral singularity has been witnessed/conjectured to be at most 1 [8, 9]. In this paper, we wish to reveal existence of two other investigations [15]. An explicit condition involving

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\( \text{reflection amplitudes of the Hermitian Scarf II potential.} \]

Later, in an update on the complex SS will also occur here as special cases.

For the Scarf II potential

\[ V(x) = (B^2 - A^2 - A) \text{sech}^2 x + B(2A + 1) \text{sech} x \tanh x. \]  

(5)

Let \( 2 \mu = h^2 \) and \( k = \sqrt{E} \), where \( E \) is the energy. Following [16, 17] we can write the transmission amplitude for (5)

\[ t_{A,B}(k) = \frac{\Gamma[-A - ik] \Gamma[1 + A - ik] \Gamma[1/2 - iB - ik] \Gamma[1/2 + iB - ik]}{\Gamma[-ik] \Gamma[1 - ik] \Gamma[2][1/2 - ik]} \]  

\[ r_{A,B}(k) = t_{A,B}(k) \left[ \frac{\cos \pi A \sin \pi B}{\cosh \pi k} + i \frac{\sin \pi A \cos \pi B}{\sinh \pi k} \right]. \]  

(6)

The transmittivity \( T(E) = |t(k)|^2 \) and the reflectivity \( R(E) = |r(k)|^2 \). We have rederived (6) to find that for (5)

\[ t_{\text{left}}(k) = t_{A,B}(k), \quad r_{\text{left}}(k) = r_{A,B}(k) \]

and

\[ t_{\text{right}}(k) = t_{A,-B}(k), \quad r_{\text{right}}(k) = r_{A,-B}(k). \]  

(7)

For the Hermitian case both \( A \) and \( B \) are real and equation (6) displays various invariances: \( R(-k) = R(k), \quad T(-k) = T(k) \) and \( R_{\text{left}} = R_{\text{right}}, \quad T_{\text{left}} = T_{\text{right}} \) as usual. For a general non-Hermitian potential, we have \( A = A_1 + iA_2 \) and \( B = B_1 + iB_2 \); we can check that the acclaimed properties (2) are followed. When \( A \) is real and \( B \) is purely imaginary, we get a complex PT-symmetric potential in equation (5); one can indeed verify the acclaimed (3) PT-symmetry of the reflectivity and transmittivity from (6) by the help of (7).

Now, we parametrize \( A \) and \( B \) in (5) and (6) to bring out two spectral singularities. Let

\[ A = -(m + 1) + i\alpha, \quad m \in \mathbb{I}^+ + \{0\}; \]  

(8)

then the second Gamma function in the numerator of (3) is \( \Gamma[-m + (\alpha - k)i] \) which becomes infinite \( (\Gamma(-m) = \infty) \) when \( k = \alpha \), giving us the first spectral singularity: \( E_{s1} = \alpha^2 \). Next, when we assign

\[ B = \beta + i(n + 1/2), \quad n \in \mathbb{I}^+ + \{0\}, \]  

(9)

time the third Gamma function in the numerator of (3) is \( \Gamma[-n + (\beta - k)i] \) which becomes infinite \( (\Gamma(-n) = \infty) \) when \( k = \beta \), giving us the second spectral singularity: \( E_{s2} = \beta^2 \).

The potential(s) possessing these two spectral singularities is Scarf II as in (4) which by the help of (5) gives a new parametrization of \( V_1 \) and \( V_2 \) as

\[ V_1(\alpha, \beta) = [\alpha^2 + \beta^2 - (m + 1)^2 - (n + 1/2)^2 + (m + 1)] + i[2m + 1] \alpha + (2n + 1) \beta, \]

\[ V_2(\alpha, \beta) = -[(2n + 1) \alpha + (2m + 1) \beta] + i[2 \alpha \beta - (m + 1)(2n + 1) + (n + 1/2)], m, n \in \mathbb{I}^+ + \{0\}. \]  

(10)
Figure 1. For the complex Scarf II potential (with the new parametrization (equation (10))), the transmitivity $T(E)$ is plotted as a function of energy $E$. The dark curve represents the $T(k)$ and the faint curve represents the time-reversed transmitivity $T(-k)$. In (a) $\alpha = \sqrt{2}$ and $\beta = \sqrt{3}$, note two peaks in the dark curve at $E = 2$ and 5 and no peak in the faint one. In (b) $\alpha = -\sqrt{2}$ and $\beta = -\sqrt{3}$, note two peaks in the faint curve at $E = 2$ and 5 and no peak in the dark curve. In (c) $\alpha = -\sqrt{2}$ and $\beta = \sqrt{3}$, note one peak in the faint curve at $E = 2$ and one peak in the dark curve at $E = 5$. In (d) in the case when the potential becomes complex PT-symmetric ($\alpha = -\beta = \sqrt{2}$) both transmitivities coincide (the dark and the faint curves merge together) and there is a single spectral singularity at $E = \frac{1}{2}$. Here we have taken $n = m = 0$ (see equation (10)). The spectral singularities occur at $E = \alpha^2$ and $\beta^2$.

In the following, we discuss SS in terms of transmitivity since the relevant positive energy poles of $R(k)$ and $T(k)$ are common (see equation (6)). In figure 1, the transmitivity is represented by the dark curve and the time-reversed transmitivity by the faint curve. Three cases arise here.

1. **Two spectral singularities (general non-Hermiticity).** When $\alpha \beta > 0$, then there exist SS at $k_{s1} = \alpha$ and $k_{s2} = \beta$ or $k_{s1} = -\alpha$ and $k_{s2} = -\beta$. Alternatively, we can state that both spectral singularities $E_{s1} = \alpha^2$ and $E_{s2} = \beta^2$ will occur either in the transmitivity (see two peaks in the dark curve in figure 1(a)) or in the time-reversed transmitivity (see two peaks in the faint curve figure 1(b)).

2. **Single spectral singularity (non-Hermiticity).** When $\alpha \beta < 0$ ($\alpha + \beta \neq 0$), then there exist SS at $k_\alpha = \alpha$ or at $k_\beta = -\beta$. Alternatively, one of the spectral singularities $E_{s1} = \alpha^2$ or $E_{s2} = \beta^2$ will occur in the transmitivity and the other one in the time-reversed transmitivity. In figure 1(c) see one peak in the dark curve and one in the faint curve. When $\alpha = iy$ ($y > 0$), there exists one bound state eigenvalue at $(E = -y^2)$ (see...
spectrum as function yields the physical poles as where and the bound states when Im \( \beta < 0 \). Let us use \( A = -(m + 1) - \gamma \) (see (8)) in (6) to find that the second Gamma function yields the physical poles as \( k = i(m + \gamma - M) \), where \( M \) is a non-negative integer: \( M = 0, 1, 2, \ldots < m + \gamma \). From this physical pole, we get one branch of the negative discrete spectrum as

\[
E_M^+ = -(\gamma + M - M)^2, \quad M = 0, 1, 2, \ldots < m + \gamma. \tag{11}
\]

Similarly, the third Gamma function in (6) also yields physical poles as \( k = i(\delta + n - N) \), where \( N \) is a non-negative integer such that \( N = 0, 1, 2, \ldots < n + \delta \). We get the second branch of the negative discrete spectrum as

\[
E_N^- = -(\delta + N - N)^2, \quad N = 0, 1, 2, \ldots < n + \delta. \tag{12}
\]

These eigenvalue formulas which are derived here from the physical (Im(\( V_1 \)) > 0) \( k \)-poles of the transmission amplitude (6) can be verified as newly expressed forms of the energy eigenvalue formulae derived earlier [13, 15] for the complex PT-symmetric Scarf II potential.

It is demonstrated that physical poles of \( t(k) \) and \( r(k) \) yield the spectral singularities and the bound states when Im(\( k \)) = 0 and when Im(\( k \)) > 0, respectively [8]. It needs to be emphasized that \( m \) and \( n \) themselves are non-negative integral parameters to be chosen along with \( \gamma \) and \( \delta \) for fixing the potential (5) using (8)–(10). The energies \(-\gamma^2\) and \(-\delta^2\) will

| Hamiltonian                  | Reflectivity        | Transmittivity   |
|-----------------------------|---------------------|------------------|
| [1] Hermitian \( R_{\text{left}} = R_{\text{right}} \) | \( T_{\text{left}} = T_{\text{right}} \) | \( T(k) = T(-k) \) |
| [2] Non-Hermitian \( P \)-symmetric \( R_{\text{left}} = R_{\text{right}} \) | \( T_{\text{left}} = T_{\text{right}} \) | \( T(-k) \neq T(k) \) |
| [3] Non-Hermitian \( R_{\text{left}} \neq R_{\text{right}} \) | \( T_{\text{left}} = T_{\text{right}} \) | \( T(-k) = T(k) \) |
| [4] Non-Hermitian \( P \)-symmetric \( R_{\text{left}} = R_{\text{right}} \) | \( T_{\text{left}} = T_{\text{right}} \) | \( T(-k) = T(k) \) |

Note: The table is a simplified representation of the discussed scenarios and does not include all permutations and combinations as mentioned in the text.
essentially belong to the bound-state eigenvalues $E^+_M$ (11) and $E^-_N$ (12), respectively. So for instance when $m = n = 0$, these will be the ground state eigenvalues of the two branches. It turns out that spectral singularity is not the necessary feature of a complex PT-symmetric scattering potential. In these cases, spectral singularity is more probable when the imaginary part of the potential is sufficiently stronger than that of its real part.

Table 1 displays the update on various (in)variances in the scattering from one-dimensional potentials. The results {1} and {2} [5] are already known. The result {3} is a new proposal (see equation (2)). All three results can be proved readily using the method of [5]. For complex PT-symmetric potentials, for the conjecture in {4} (see equation (3)) a proof is welcome. It, however, along with the left/right handedness of the reflectivity endows the complex PT-symmetric scattering potentials a unique signature. We believe that the present results coming from a new non-Hermitian parametrization of the Scarf II potential could be the features of a general one-dimensional scattering potential ($V(\pm\infty) = 0$). This opens up scope for further investigations. It is also desired to investigate the possibility of more than two (one) spectral singularities in one-dimensional non-Hermitian (complex PT-symmetric) scattering potentials.

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