An Indirect Spectral Collocation Method Based on Shifted Jacobi Functions for Solving Some Class of Fractional Optimal Control Problems

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Abstract. A new approximation formula of the Riemann-Liouville fractional derivatives is derived based on shifted classical Jacobi polynomial in spectral approximations. This formula is presented to approximate indirect solution of fractional optimal control problems (FOCPs) with a fractional differential equation as the dynamic constrain. The properties of new formula allows us to use spectral collocation method to reduce FOCPs by indirect method to a system of liner/nonlinear algebraic equations. Four test examples are presented to examine the applicability and validity of a newly purposed method.

Keywords. Spectral method, collocation method, fractional optimal control problems, Jacobi functions, frictional Riemann-Liouville derivatives.

1. Introduction
Fractional optimal control problems can be found in several scientific and engineering applications. It has recently become a vivid and successful research area, [25, 10]. Generally, the given FOCPs can have two types of solutions: Analytical and numerical solutions. Analytical solution is very difficult or almost impossible to obtain. Because of this, most researchers have used numerical methods to obtain the solution of a given FOCPs. Usually, the numerical methods to solve FOCPs can be divided into direct and indirect methods. The first step in the indirect method is to obtain the optimality conditions of optimal control problems. This leads to boundary-value problems (BVPs). The BVPs can then be solved numerically to obtain the extremals, then the optimal solution can be obtained [19, 8].
The first work that implemented the indirect method to solve FOCPs by driving necessary and sufficient optimality conditions began in a conference paper by Pooseh et al [19]. They used expansion formula given in [3] to approximate the FOCP solution, and the necessary and sufficient optimality conditions was derived for FOCPs in the -Liouville fractional sense. The readers interested in the indirect method to solve FOCPs is referred to [7, 22, 24, 12, 26, 4, 21] and references therein.

This paper is dedicated to present the collocation scheme use to solve linear/ nonlinear FOCPs by the indirect method. Therefore, the researcher’s aim, is to solve the necessary optimality conditions in a form of linear/nonlinear fractional two-po

int of BVP. A formula based on spectral modified Jacobi functions is also defined depending on shifted classical Jacobi polynomial, then it is used to solve a system of fractional differential equation by the collocation methods. The paper is organized as follows: In the second section, definitions are given with some notations for Jacobi Gauss quadrature rule and fractional integrals and derivatives. Section 3 includes FOCP model and optimality necessary conditions. In Section 4, the basic formulation of the proposed approximate formulas of left and right Riemann-Liouville fractional derivatives has been obtained. In Section 5, four examples were included to demonstrate the validity and applicability of the proposed technique. In the last section, a brief conclusion and some remarks were given.

2. Preliminaries and notations
The relative definitions and notations used in this paper are introduced Li and Zeag (2015).

Let \(0 < \alpha \leq 1\) is a real number and \(g: [a, b] \to \mathbb{R}\) is continuous function then:

\[
\frac{\partial^\alpha}{\partial \tau^\alpha} g(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau (\tau - t)^{\alpha-1} g(t) dt, \quad (1)
\]

\[
\frac{\partial^\alpha}{\partial \bar{\tau}^\alpha} g(\tau) = \frac{1}{\Gamma(\alpha)} \int_b^\tau (\tau - t)^{\alpha-1} g(t) dt, \quad (2)
\]

is the left and right Riemann-Liouville fractional integrals of order \(\alpha\), respectively. And the left and right Riemann-Liouville fractional derivative of order \(\alpha\), are defined as:

\[
D^\alpha_{a,\tau} g(\tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^\tau (\tau - t)^{-\alpha} g(t) dt, \quad (3)
\]

\[
D^\alpha_{\tau,b} g(\tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_\tau^b (t - \tau)^{-\alpha} g(t) dt, \quad (4)
\]

where \(n-1 \leq \alpha < n\) and \(n\) is an integer number then the left and right Riemann-Liouville fractional derivative are defined as:

\[
D^\alpha_{a,\tau} g(\tau) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^\tau (\tau - t)^{n-\alpha-1} g(t) dt, \quad (5)
\]

\[
D^\alpha_{\tau,b} g(\tau) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_\tau^b (t - \tau)^{n-\alpha-1} g(t) dt. \quad (6)
\]

The left and right Caputo fractional derivative are defined as:

\[
^{c}D^\alpha_{a,\tau} g(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_a^\tau (\tau - t)^{n-\alpha-1} g^{(n)}(t) dt, \quad (7)
\]

\[
^{c}D^\alpha_{\tau,b} g(\tau) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_\tau^b (t - \tau)^{n-\alpha-1} g^{(n)}(t) dt. \quad (8)
\]

There are many properties of the Riemann-Liouville and Caputo fractional derivatives, for further interest, see [18, 20, 10, 17, 9, 14, 13, 25].
3. Jacobi Gausse quadrature formula

The Jacobi polynomials are applied in a wide range of engineering disciplines as well as system analysis, optimal control, numerical analysis, and signal analysis for representation solution of problems [14, 19]. The Jacobi Gausse quadrature formula:

\[
\int_{-1}^{1} g(\tau)w^{a,b}(\tau)d\tau = \sum_{i=0}^{n-1} w_i g(\tau_i),
\]

is exact for any polynomials \( g \in P_{2n-1} = \text{span}\{1, \tau, \ldots, \tau^{2n-1}\} \), where

\[
w_i = \frac{2^{a+b+1}\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(n+a+b+1)(1-\tau_i^a)\Gamma(p^{(a,b)}_n(\tau_i))},
\]

and \( \tau_0, \tau_1, \ldots, \tau_{n-1} \) are the Jacobi Gauss nodes (i.e., zeros of \( p^{(a,b)}_n(\tau) \)) see [20].

According to the definition of Jacobi polynomials, various types of functions with fractional order have been constructed based on them. In the following, important theorem and some properties of the classical Jacobi polynomials are presented which are used later to establish our method.

Theorem 1 Let \( r - \alpha > -1 \) and \( q + \alpha > -1 \). Then, for \( t \in [0,1] \), we have:

\[
D^{\alpha}_{0,t} \left[t^r p_k^{(q,r)}(2t-1)\right] = \frac{\Gamma(k+r+1)}{\Gamma(k+r-\alpha+1)} t^{-\alpha} p_k^{(q+r-a)}(2t-1)
\]

Proof. To proof of this theorem see [11, 30, 6]. Some applications of the above theorem can be found in [31, 29, 38, 32, 33, 28, 2].

Some properties of the classical Jacobi polynomials are summarized and used the present work:

1. \( p_j^{(a,b)}(1) = (j+a)^{\frac{1}{2}} \), \( p_j^{(a,b)}(-1) = (-1)^j (j+a)^{\frac{1}{2}} \), and

\[
p_j^{(a,b)}(-t) = (-1)^j p_j^{(b,a)}(t).
\]

2. \( \frac{d^n}{dt^n} p_j^{(a,b)}(t) = \frac{j^n}{(j+a)^n} p_j^{(a^n+b^n)}(t) \), \( j \geq n, n \in \mathbb{N} \),

\[
\frac{d^n}{dt^n} p_j^{(a,b)}(t) = \frac{j^n}{j(j+a)^n+1} \Gamma(j+a+b+1) p_j^{(a^n+b^n)}(t).
\]

In particular, one has

\[
\frac{d}{dt} p_j^{(a,b)}(t) = \frac{1}{2} (j+a+b+1) p_{j-1}^{(a+1,b+1)}(t), \quad n \geq 1.
\]

3. The Jacobi polynomials also satisfy the following properties:

\[
p_k^{(a,b)}(2t-1) = \frac{k+a+b}{2k+a+b} p_k^{(a,b)}(2t-1) + \frac{k+a}{2k+a+b} p_{k-1}^{(a,b)}(2t-1),
\]

and

\[
p_{k-1}^{(a,b)}(2t-1) = \frac{k+a+b}{2k+a+b} p_k^{(a,b)}(2t-1) - \frac{k+b}{2k+a+b} p_{k-1}^{(a,b)}(2t-1).
\]

4. Problem statement

In the present work, the following FOCPs in the Riemann-Liouville derivative sense are examined [5, 19, 23]. The aim is to find control \( u(t) \in \mathbb{R}^{n_u} \) and state \( x(t) \in \mathbb{R}^{n_s} \) functions that minimize the functional

\[
j(u) = \int_a^t F(t,x(t),u(t))dt
\]

subject to the dynamic system constraints

\[
A \dot{x}(t) + B \int_a^t D^a_{0,t} x(t) = G(t,x(t),u(t)),
\]

and the conditions

\[
x(a) = x_0, \quad x(t_f) = x_f, \quad t \in [a, t_f].
\]
The system of fractional differential equations (12)–(13) for the modified Jacobi functions is defined depending on $\alpha$, which simplifies the problem. Shifted shiftedes can be obtained in a closed form (22), as a basis of $(\alpha+2)−\alpha^2$. Further, the left and right fractional differentiations of $f$ with respect to the weight function $w(t)$ are defined as follows:

\[ f^L(t) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+2)−\alpha^2} f(t) \]

where $\Gamma(\alpha+2)$ is the Gamma function. Then, the system (20)-(21) contains necessary and sufficient condition for optimal solutions $\mathbf{u}^*(t)$ and $x^*(t)$.

5. Numerical approximation

In this section, a formula based on spectral modified Jacobi functions is defined depending on shifted classical Jacobi polynomial, then it is used to solve the system of fractional differential equations (12)-(13) by collocation methods. Let us present the set $\{t(1−t) f_k^{(a,b)}(2t−1), k = 0, 1, \cdots\}$, as a basis of modified Jacobi functions, where $a, b > −1$, $t ∈ [0,1]$. And let as use the special case $a = b = 1$ then the modified Jacobi functions are orthogonal with respect to the weight function $w(t) = t^{−1}(1−t)^{−1}$ as follows:

\[ \int_0^1 t(1−t) f_j^{(1,1)}(2t−1).t(1−t) f_i^{(1,1)}(2t−1) w(t) dt = \frac{\Gamma(j+2)}{\Gamma(2j+3)} \delta_{i,j} \]  

where $\delta_{i,j}$ is the Kronecker delta function. Let as use the special case of modified Jacobi functions to derive the left and right fractional differentiation of $\varphi_k(t)$ and

\[ \varphi_k(t) = \frac{(k+2)(k+3)}{k+1} t(1−t) f_k^{(1,1)}(2t−1) \]  

Consider $\{\varphi_k(t), k = 0, 1, \cdots, n\}$ as the basis of functions. A good feature of these basis functions is that, the left and right fractional derivatives can be obtained in a closed form, which this fact simplifies the discretization stage.

To derive the left and right fractional derivatives of $\varphi_k(t)$, let

\[ D^\alpha_0 \varphi_k(t) = \psi_k(t) \]

It’s easy to evaluate $\psi_0(t)$, such that:

\[ \psi_0(t) = \frac{6\Gamma(2)}{\Gamma(2−\alpha)} t^{1−\alpha} − \frac{6\Gamma(3)}{\Gamma(2−\alpha)} t^{2−\alpha} \]

To evaluate $\psi_k(t)$ for $k ≥ 1$, we begin from the fact that:
\[
\varphi_k(t) = \frac{(k+2)(2k+3)}{k+1} t P_k^{(1,1)}(2t - 1) - \frac{t^2 P_k^{(1,1)}}{k+1}(2t - 1),
\] (27)

because of the fact that the Riemann-Liouville and Caputo fractional derivative of order \(0 < \alpha < 1\) of \(\varphi_k(t)\) are equivalent, we can using linearity property to obtain

\[
D_{0+}^\alpha \varphi_k(t) = \frac{(k+2)(2k+3)}{k+1} \left( D_{0+}^\alpha t P_k^{(1,1)}(2t - 1) \right) - \frac{D_{0+}^\alpha t^2 P_k^{(1,1)}}{k+1}(2t - 1)).
\] (28)

Now using Theorem 1 to compute the left-sided fractional derivative of the first term of the above equation as:

\[
D_{0-}^\alpha \left[ \frac{(k+2)(2k+3)}{k+1} t P_k^{(1,1)}(2t - 1) \right] = \frac{(2k+3)\Gamma(k+3)}{(k+1)\Gamma(k+2-\alpha)} \left( \frac{(k+1)}{(k+2)} \right) t^{1-\alpha} P_k^{(1+\alpha,1-\alpha)}(2t - 1).
\] (29)

The second term of the equation (28) can be obtained by using the Jacobi polynomials property (10), for values \(\alpha = 1\) and \(b = 2\):

\[
P_k^{(1,1)}(2t - 1) = \frac{(k+3)}{(2k+3)} P_k^{(1,2)}(2t - 1) + \frac{(k+1)}{(2k+3)} P_k^{(1,2)}(2t - 1)).
\] (30)

Replacing \(P_k^{(1,1)}(2t - 1)\) in the second term of (28) by (30), we have left- sided fractional derivative as:

\[
D_{0+}^\alpha \left[ \frac{(k+2)(2k+3)}{k+1} t^2 P_k^{(1,1)}(2t - 1) \right] = \frac{(k+2)(2k+3)}{k+1} D_{0+}^\alpha \left[ \frac{(k+2)}{(k+3)} P_k^{(1,2)}(2t - 1) + \frac{(k+1)}{(2k+3)} P_k^{(1,2)}(2t - 1)) \right].
\] (31)

Again by using Theorem 1, we have

\[
D_{0+}^\alpha \left[ \frac{(k+2)(2k+3)}{k+1} t^2 P_k^{(1,1)}(2t - 1) \right] = \frac{(k+2)(k+4)}{(k+1)\Gamma(k+3-\alpha)} \left( \frac{(k+1)}{(k+2)} \right) t^{2-\alpha} P_k^{(1+\alpha,2-\alpha)}(2t - 1) + \frac{\Gamma(k+3)}{\Gamma(k+2-\alpha)} t^{2-\alpha} P_k^{(1+\alpha,2-\alpha)}(2t - 1).
\] (32)

By substituting equations (29) and (32) into the relation (28), therefore we can evaluate \(\psi_k(t)\) and \(\psi_k(t) = t^{1-\alpha} - \frac{(k+2)(k+4)}{(k+1)\Gamma(k+3-\alpha)} \left( \frac{(k+1)}{(k+2)} \right) t^{2-\alpha} P_k^{(1+\alpha,2-\alpha)}(2t - 1) - \frac{\Gamma(k+3)}{\Gamma(k+2-\alpha)} t^{2-\alpha} P_k^{(1+\alpha,2-\alpha)}(2t - 1).
\] (33)

Now we establish to obtain a closed form of the right side fractional integral and derivatives of modified Jacobi functions where \(n - 1 \leq \alpha < n\).

\[
I_{1+}^\alpha t(1-t) \left( P_j^{(1,1)}(2t - 1) \right) = \frac{1}{\Gamma(a)} \int_{u=1}^{u=1} (u-t)^{a-1}, u(1-u) P_j^{(1,1)}(2u - 1) du
\]

by using the change of variable \(u = 1 - s\) and \(t,s,u \in [0,1]\), together with the first property of the Jacobi polynomials (9) we obtain:

\[
I_{1+}^\alpha t(1-t) \left( P_j^{(1,1)}(2t - 1) \right) = \frac{1}{\Gamma(a)} \left[ (1-s) \right] \left( (1-s)^{a-1}, (1-s)s P_j^{(1,1)}(2s - 1) ds
\]

And by using properties of fractional derivative and integral with the (33) then we have:

\[
D_{1-}^\alpha t(1-t) \left( P_j^{(1,1)}(2t - 1) \right) = (-1)^k I_{1+}^\alpha t(1-t) \left( P_j^{(1,1)}(2t - 1) \right) \left( 1 - (1-t) \right)
\] (34)

The equation (35) is a closed form of right side Riemann- Liouville fractional derivatives of modified Jacobi functions. Depending on this fact and equation (33) the right fractional derivative of \(\varphi_k(t)\) can be evaluated,

\[
D_{1-}^\alpha \varphi_k(t) = (-1)^k \psi_k(1-t).
\] (36)
6. Error upper bound of the approximate fractional derivatives

To achieve this objective, we need the following theorem.

Theorem 3 Suppose that \( Y \) is a closed subspace of a Hilbert space \( H \), such that \( y_1, y_2, \ldots, y_n \) is a basis for \( Y \) of dimension \( n \). Let \( x \) be an arbitrary element in a Hilbert space \( H \) and \( y_0 \) be the unique best approximation to \( x \) out of \( Y \). Then

\[
\| x - y_0 \|_2^2 = \frac{G(x, y_1, y_2, \ldots, y_n)}{G(y_1, y_2, \ldots, y_n)}
\]

where

\[
G(x, y_1, y_2, \ldots, y_n) = \begin{vmatrix}
\langle x, x \rangle & \langle x, y_1 \rangle & \cdots & \langle x, y_n \rangle \\
\langle y_1, x \rangle & \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle y_n, x \rangle & \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle 
\end{vmatrix}
\]

The proof of this theorem can be found in [15].

Now to evaluate error upper bound for the approximate fractional derivatives let as consider \( \{\varphi_k(t), k = 0, 1, \ldots, n\} \) as the basis of functions and \( \varphi_k(t) \) is defined in equation (24). A good feature of these basis functions is that, the left and right fractional derivatives can be obtained in a closed form (see equations (25) and (36)), which is the error upper bound for fractional derivatives of \( \varphi_k(t) \) equal to zero. On the other hand, by using the above theorem 3, it is easy to evaluate the error upper bound of \( x(t) \) such that \( x(t) \) can be approximated by \( x_n(t) \) as:

\[
x(t) \approx x_n(t) = \sum_{k=0}^{n} \varphi_k(t) a_k
\]

\[
\| x(t) - \sum_{k=0}^{n} \varphi_k(t) a_k \|_2 = \left( \frac{G(x(t), \varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t))}{G(\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t))} \right)^{\frac{1}{2}},
\]

where \( a_k \in \mathbb{R}^n \) are unknown vector.

Remark. Where \( \alpha = 1 \), then: \( \dot{\psi}_0(t) = 12 - 18t \), and for \( k \geq 1 \)

\[
\dot{\psi}_k(t) = \frac{(2(k+2)\Gamma(k+2))}{(k+1)\Gamma(k+1)} p_k^{(2,0)} (2t - 1) - \frac{(k+2)\Gamma(k+4)}{(k+1)\Gamma(k+2)} t P_k^{(2,1)} (2t - 1) - \frac{\Gamma(k+3)}{\Gamma(k+1)} t P_{k-1}^{(2,1)} (2t - 1).
\]

Remark. Our goal is to approximate the unknown function \( x(t), u(t), l(t) \) in problem (20)–(21) via spectral collocation method based on the Gauss Jacobi nodes. We begin in the first step by transforming the interval \([a, t_f]\) to interval \([0, 1]\). Where \( x(r) \) is a real-valued function defined on the interval \([a, t_f]\), and \( r_i, i = 0, 1, \ldots, m \) is the collocation points on the reference interval \([-1, 1]\), then \( t = \frac{4r-a-3t_f}{t_f-a}, \quad t \in [0, 1] \).

To illustrate our numerical methods, suppose that the state function \( x(t) \) can be approximated by \( x_n(t) \) as:

\[
x(t) \approx x_n(t) = x_0 + (x_\ell - x_0) t + \sum_{k=0}^{n} \varphi_k(t) a_k
\]

such that \( x_0 \) and \( x_\ell \) are lower and upper boundary conditions for the stat function \( x(t) \), and we add \( x_0 + (x_\ell - x_0) t \) to satisfy the boundary conditions of the problem (20)-(21). And \( \varphi_k(t) \) is defined in equation (24). Now, from equation (38) we conclude

\[
x(t) \approx \dot{x}_n(t) = (x_\ell - x_0) + \sum_{k=0}^{n} \dot{\psi}_k(t) a_k.
\]

\[
\frac{c D_0^\alpha x(t)}{\Gamma(2-\alpha)} \approx \frac{\Gamma(2)}{\Gamma(2-\alpha)} (x_\ell - x_0) t^{1-\alpha} + \sum_{k=0}^{n} \psi_k(t) a_k,
\]

(40)
such that \( \psi_k(t) \) is defined in equation (25). Also can be approximated Lagrange multiplier function \( \lambda(t) \) by \( \lambda_n(t) \):

\[
\lambda(t) \approx \lambda_n(t) = \lambda_0 + (\lambda_0 - \lambda_T)(1 - t) + \sum_{k=0}^n \psi_k(t)b_k, \quad (41)
\]

such that \( \lambda_T \) and \( \lambda_0 \) are upper and lower boundary conditions for the Lagrange multiplier function \( \lambda(t) \), and we add \( \lambda_T + (\lambda_0 - \lambda_T)(1 - t) \) to satisfy the boundary conditions of the problem (20)-(21). Now, from equation (41) we conclude

\[
\lambda(t) \approx \lambda_n(t) = -(\lambda_0 - \lambda_T) + \sum_{k=0}^n \psi_k(t)b_k, \quad (42)
\]

\[
D^{(1-t)^{-\alpha}}_{\alpha, \alpha} \lambda_T + \frac{\Gamma(2)}{\Gamma(2-\alpha)} (\lambda_0 - \lambda_T)(1 - t)^{1-\alpha} + \sum_{k=0}^n (-1)^k \psi_k(t)(1 - t)b_k, \quad t \in [0, 1]. \quad (43)
\]

Where \( a_k, b_k \in \mathbb{R}^{n_x} \) are unknown vectors, which should be obtained. Note that the approximation of \( x(t) \) satisfies the boundary conditions (22). By substituting the above approximations functions in the fractional system equation (20)-(21), we obtain:

\[
c(t) + \sum_{k=0}^n \{ A\phi_k(t)a_k + B\psi_k(t)b_k \} = M(t, x_n(t), \lambda_n(t)), \quad (44)
\]

\[
d(t) + \sum_{k=0}^n \{ A\phi_k(t)b_k - (-1)^k \psi_k(1 - t)b_k \} = -N(t, x_n(t)), \quad (45)
\]

\[
x(0) = x_0, \quad x(t_f) = x_f, \quad \lambda(0) = \lambda_0, \quad \lambda(t_f) = \lambda_T, \quad (46)
\]

where

\[
c(t) = (x_T - x_0)(A + \frac{\Gamma(2)}{\Gamma(2-\alpha)} Bt^{1-\alpha})
\]

and

\[
d(t) = (\lambda_T - \lambda_0)(A + \frac{\Gamma(2)}{\Gamma(2-\alpha)} B(1 - t)^{1-\alpha} - B \frac{(1-t)^{-\alpha}}{\Gamma(1-\alpha)} \lambda_T, \quad (47)
\]

The above system (44)-(45) of linear/non linear algebraic equations, we can approach it to obtain the reminder by collocation strategy. One of the best advantages of this strategy is that it is easy to drive operational matrices of differentiation and we can implement this method in any mathematical software.

Let us define the Gauss Jacobi nodes where weight function \( w^{1,1}(t) \) on interval the \([0, 1] \) as: \( \tau_i^{(1,1)}, \ i = 0, \ldots, n \) be the zeros of \( w^{(1,1)}(2t - 1) \). By collocation method, with nodes \( \tau_i^{(1,1)}, \ i = 0, \ldots, n, \) the system of fractional problem (44)-(45) will be:

\[
c(\tau_i^{(1,1)}) + \sum_{k=0}^n \{ A\phi_k(\tau_i^{(1,1)})a_k + B\psi_k(\tau_i^{(1,1)})b_k \} = M(\tau_i^{(1,1)}, x_n(\tau_i^{(1,1)}), \lambda_n(\tau_i^{(1,1)})),
\]

\[
d(\tau_i^{(1,1)}) + \sum_{k=0}^n \{ A\phi_k(\tau_i^{(1,1)})b_k - (-1)^k \psi_k(1 - \tau_i^{(1,1)})b_k \} = -N(\tau_i^{(1,1)}, x_n(\tau_i^{(1,1)}), \lambda_n(\tau_i^{(1,1)}))
\]

The above system of equations can be expressed in the following matrix form:

\[
c^T + A\tilde{P}a^T + B\mathbb{D}_L a^T = M, \quad (48)
\]

\[
d^T + A\tilde{P}b^T - B\mathbb{D}_R b^T = -N, \quad (49)
\]

where

\[
a = [a_0, a_1, \ldots, a_n]^T, \quad b^T = [b_0, b_1, \ldots, b_n]^T,
\]

\[
c = [c(\tau_0^{(1,1)}), c(\tau_1^{(1,1)}), \ldots, c(\tau_n^{(1,1)})]^T,
\]

\[
d = [d(\tau_0^{(1,1)}), d(\tau_1^{(1,1)}), \ldots, d(\tau_n^{(1,1)})]^T,
\]

and

\[
\tilde{P} = \begin{bmatrix}
\phi_0(\tau_0^{(1,1)}) & \phi_0(\tau_1^{(1,1)}) & \phi_0(\tau_n^{(1,1)}) \\
\phi_1(\tau_0^{(1,1)}) & \phi_1(\tau_1^{(1,1)}) & \phi_1(\tau_n^{(1,1)}) \\
\vdots & \vdots & \vdots \\
\phi_n(\tau_0^{(1,1)}) & \phi_n(\tau_1^{(1,1)}) & \phi_n(\tau_n^{(1,1)})
\end{bmatrix}
\]
The proposed method is implemented with MATLAB.

Example 1  Consider FOCPs [22, 19]:

\[
\begin{align*}
\frac{d^\alpha x(t)}{dt^\alpha} + \frac{2t^{\alpha+2}}{\Gamma(3+\alpha)} u(t) - t^2 = 0, \quad x(0) = 0, \quad x(1) = \frac{2}{\Gamma(3+\alpha)},
\end{align*}
\]

The exact solution: \( x^*(t) = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} \), \( u^*(t) = 2t^{\alpha+1} \). Hamiltonian equation defined as:

\[
H(t,x,u,\lambda) = [tx(t) - (\alpha + 2)x(t)]^2 + \lambda(t).[u(t) + t^2].
\]

By using Theorem 2 we get fractional differential equations:

\[
\begin{align*}
\dot{x}(t) + \frac{\alpha}{2t^\alpha} \frac{d^\alpha x(t)}{dt^\alpha} + \frac{\lambda(t)}{2t^2} - \frac{(\alpha+2)x(t)}{t} - t^2 &= 0, \\
\lambda(t) - D_{t,c}^\alpha \lambda(t) + \frac{(\alpha+2)}{\Gamma(3+\alpha)} \lambda(t) &= 0,
\end{align*}
\]

To approximate the solution of FOCPs with boundary value conditions apply the algorithm of collocation strategy method. The Figures 1 represents the obtained control and state functions with error function by
the presented method, with values of $n = 30$, and $\alpha = 0.8$. The CPU time for the first and second figures is 0.383580 seconds. In Table 1 th maximum errors in the state $x$ and in the control $u$ for different values of $n$ are compared with the results of [22]. This table shows that our method batter than the method used in [22].

**Example 2** Consider FOCPs [22]:

$$
\min J = \int_0^1 (x(t) - u(t))^2 dt,
$$

with the system of fractional equation

$$
\dot{x}(t) + ^CD_{0+}^\alpha x(t) = u(t) - x(t) + \frac{6t^{\alpha+2}}{\Gamma(\alpha+3)} + t^3,
$$

$$
\dot{\lambda}(t) - D_{0+}^\alpha \lambda(t) = -2(u(t) - x(t)) - \lambda(t), \quad \lambda(1) = 0.
$$

We can define Hamiltonian equation and than by using Theorem (2) we get fractional differential equations:

$$
\dot{x}(t) + ^CD_{0+}^\alpha x(t) = u(t) - x(t) + \frac{6t^{\alpha+2}}{\Gamma(\alpha+3)} + t^3,
$$

$$
\dot{\lambda}(t) - D_{0+}^\alpha \lambda(t) = -2(u(t) - x(t)) - \lambda(t), \quad \lambda(1) = 0.
$$

With the optimal control law $u(t) = x(t)$. The exact solution $x^*(t) = \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}$ and $u^*(t) = \frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}$. To approximate the solution of FOCP with boundary value conditions apply the algorithm of collocation strategy method. Figure 2, shows the obtained control and state functions with error function by the presented method, with values of $n = 60$, and $\alpha = 0.6$. The CPU time for the first and second figures is 0.415150 seconds. Figure 4 shows that the control function for different values of $\alpha$. Figure 3 of the semilog plot of $E^2_\alpha(u)$. And $E^2_\alpha(u)$ are the $\ell_2$ norm of error in the control function, i.e.

$$
E^2_\alpha(u) := \left(\frac{1}{n}\sum_{i=1}^{n} (u(t_i) - u_n(t_i))^2\right)^{1/2},
$$

where $u_n(t_i)$ is the obtained control function by presenting method with $n$ nodes.

**Example 3** Consider FOCPs:

$$
\min J = \int_0^1 [(x_1(t) - t^{3.3})^2 + (x_2(t) - (1 + t)^2)^2 + (u(t) - x_1(t))^2] dt,
$$

with the system of fractional equation

$$
\dot{x}_1(t) + ^CD_{0+}^\alpha x_1(t) = 3.3t^{-1}x_1(t) + \frac{\Gamma(4.2)}{\Gamma(4.3-\alpha)} t^{3.3-\alpha},
$$

$$
\dot{x}_2(t) - ^CD_{0+}^\alpha x_2(t) = 2(t + 1) - \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{4}{\Gamma(2-\alpha)} t^{1-\alpha},
$$

the analytical solution of the problem is:

$$
x_1^*(t) = t^{3.3}, \quad x_2^*(t) = (t + 1)^2, \quad u^*(t) = t^{3.3}.
$$

To approximate the solution of FOCP with boundary value conditions apply the algorithm of collocation strategy method. We can define Hamiltonian equation
as: 

\[ H(t, x, u, \lambda) = F(t, x, u) + \lambda G(t, x, u). \]

\[ H = (x_1(t) - t^{3.3})^2 + (x_2(t) - (t + 1)^2)^2 + (u(t) - x_1(t))^2 + \lambda_1(t)[3.3 t^{-1} x_1(t) + \frac{\Gamma(4.3)}{\Gamma(3-a)} t^{3.3-a}] + \lambda_2(t)[2(t + 1) - \frac{\Gamma(3)}{\Gamma(2-a)} t^2 - a - \frac{4}{\Gamma(2-a)} t^{1-a}]. \]

By using Theorem (2) we get fractional differential equations:

\[
\begin{align*}
\dot{x}_1(t) + c D^\alpha_0,t x_1(t) &= 3.3 t^{-1} x_1(t) + \frac{\Gamma(4.3)}{\Gamma(3-a)} t^{3.3-a}, \\
\dot{x}_2(t) - c D^\alpha_0,t x_2(t) &= 2 t + 1 - \frac{\Gamma(3)}{\Gamma(2-a)} t^{2-a} - \frac{4}{\Gamma(2-a)} t^{1-a}, \\
\dot{\lambda}_1(t) + D^\alpha_0,t \lambda_1(t) &= -2[x_1(t) - t^{3.3}] + 2[u(t) - x_1(t)] + 3.3 t^{-1} \lambda_1(t), \\
\dot{\lambda}_2(t) - D^\alpha_0,t \lambda_2(t) &= -2[x_2(t) - (t + 1)^2], \\
x_1(0) &= 0, \quad x_1(1) = 1, \quad x_2(0) = 1, \\
x_2(1) &= 4, \quad \lambda_1(1) = 0, \quad \lambda_2(1) = 0,
\end{align*}
\]

with optimal control low \( \frac{\partial H}{\partial u} = 0 \), and then \( u(t) = x_1(t) \). To approximate the solution of FOCP with boundary value conditions apply the algorithm of collocation method. Figures 5- represents the obtained control and state functions \( x_1(t), x_2(t), u(t) \) with error function by the presented method, with values of \( n = 30 \), and \( \alpha = 0.3 \).

**Example 4** Consider FOCPs [7, 27]:

\[
\min J = \frac{1}{2} \int_0^1 [x_1^2(t) + x_2^2(t) + u^2(t)] dt,
\]

with the system of fractional equation

\[
\begin{align*}
D^\alpha_0,t x_1(t) &= -2x_1(t) + x_2(t) + u(t), \\
D^\alpha_0,t x_2(t) &= -2x_2(t), \\
x_1(0) &= x_2(0) = 1.
\end{align*}
\]

Where \( \alpha = 1 \) the exact solution of this problem: \( x_1^*(t) = x_2^*(t) = e^{-2t}, u^*(t) = -e^{-2t} \), with conditions \( x_1(1) = x_2(1) = e^{-2}, \dot{x}_1(0) = -2, \dot{x}_2(1) = -2e^{-2} \).

Hamiltonian equation define as:

\[
H(t, x, u, \lambda) = \frac{1}{2} (x_1^2(t) + x_2^2(t) + u^2(t)) + \lambda_1(t)(-2x_1(t) + x_2(t) + u(t)) - 2\lambda_2(t)x_2(t).
\]

By using theorem (2) we get fractional differential equations:

\[
\begin{align*}
D^\alpha_0,t x_1(t) + 2x_1(t) - x_2(t) + \lambda_1(t) &= 0, \\
D^\alpha_0,t x_2(t) + 2x_2(t) &= 0, \\
D^\alpha_0,t \lambda_1(t) + \lambda_1(t) - x_1(t) &= 0, \\
D^\alpha_0,t \lambda_2(t) + 2\lambda_2(t) - \lambda_1(t) - x_2(t) &= 0, \\
-\lambda_1(t) &= u(t), \\
x_1(0) &= x_2(0) = 1, \lambda_1(1) = \lambda_2(1) = 0, \\
and \quad x_1(1) &= x_2(1) = e^{-2}.
\end{align*}
\]

To approximate the solution of FOCP with boundary value conditions apply the algorithm of collocation strategy method. Figures 6a and Figures 6b, represents the obtained control and state functions with error function by the presented method, with values of \( n = 8 \), and \( \alpha = 1 \). In Table 2 the maximum errors in the state and in the control functions for different values of \( n \) are compared with the results of [27]. This table
shows that the current method is better than the method used in [27]. The figure 7 represents the approximated state and control functions where $\alpha = 0.8, 0.9, 0.95, 1$ and $n = 8$.

Table 2: Maximum absolute errors in the state and in the control functions for different values of $n$, $\alpha = 1$ together with the results of [27]. (Example 4).

| $n$  | $\text{max error in } x_1$ and $x_2$ | $\text{max error in } u$ | $\text{max error in } x_1$ and $x_2$ | $\text{max error in } u$ |
|------|-----------------------------------|-----------------|-----------------------------------|-----------------|
| 3    | $0.1221e^{-2}$                     | $0.3764e^{-2}$  | $0.1221e^{-2}$                     | $0.1411e^{-2}$  |
| 4    | $0.1426e^{-3}$                     | $0.5105e^{-3}$  | $0.1426e^{-3}$                     | $0.1350e^{-3}$  |
| 6    | $0.8020e^{-8}$                     | $0.6947e^{-7}$  | $0.8020e^{-8}$                     | $0.9161e^{-7}$  |

8. Conclusion

A new approximation of the left Riemann-Liouville fractional derivatives is derived based on spectral modified Jacobi functions with collocation of Jacobi Guss nodes. It is presented for approximations solution of FOCPs. The properties spectral modified Jacobi functions are used to reduce FOCPs to solve a system of liner/nonlinear algebraic equations. Global approximations of functions are defined and constructed on the interval $[0,1]$. There are some basic challenges in solving BVP (20) - (21) numerically. The first challenge is that the problem contains both left and right fractional derivatives. The second challenge is that the fractional derivatives are nonlocal and singular operators. These main challenges are tackled in the current study by deriving the left and right fractional differentiation matrices and using the collocation spectral method. The method used in this work is efficient and provides accurate results, whereas a small number of collocation points are used and a low CPU time is consumed. Moreover, our approach can be easily implemented to cover FOCPs with boundary conditions. Four numerical examples are integrated to show the validity and applicability of the new technique.

Figure 1: The obtained control and error functions by the presented method, with $n = 30$ and $\alpha = 0.8$. The CPU time is 0.383580 seconds (Example 1).
Figure 2: The obtained control and error function obtained by the presented method, with $n = 60$ and $\alpha = 0.6$. The CPU time is 0.415150 seconds (Example 2).

Figure 3: The figure represents norm-2 of error for the different values of $n = 1, 2, 3, \ldots, 150$, $\alpha = 0.5$ (Example 2).

Figure 4: Approximated solution for $\alpha = 0.3, 0.4, 0.5, 0.6, 0.7$ with $n = 8$ nodes (Example 2).
Figure 5: The obtained control function $u(t)$ and error function by the presented method, with $n = 30$ and $\alpha = 0.3$. The CPU time is 0.450341 seconds (Example 3).

Figure 6. a: The obtained state function, and absolute error by the presented method, with $n = 8$ and $\alpha = 1$. The CPU time is 0.34354 seconds (Example 4).

Figure 6. b: The obtained control function, and absolute error by the presented method, with $n = 8$ and $\alpha = 1$. The CPU time is 0.34354 seconds (Example 4).
Figure 7: Approximated solution for state and control functions where $\alpha = 0.8, 0.9, 0.95, 1$ and $n = 8$ nodes (Example 4).

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