New peculiarities in angular distribution of Cherenkov radiation from relativistic heavy ions caused by their stopping in radiator: numerical and theoretical research

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Abstract. A new method and theoretical base for calculation of the Cherenkov Radiation (ChR) angular distribution from relativistic heavy ions (RHI) taking into account stopping in radiator is suggested. Our method is based on the thin radiator approximation and the Bethe-Bloch formula for ionization energy loss (stopping) and provides fast calculations without using any special software. The simple formula for estimation of the ChR angular distribution width in vicinity of the Cherenkov angle taking into account RHI stopping in radiator is obtained. New peculiarities of ChR - dependence on the RHI charge and mass (isotopic effect), emission wavelength (index of refraction) and radiator length - are studied. These new features of ChR from RHI allow for their possible applications to simultaneously reconstruction of both velocity and charge of RHI penetrating through radiator and even to measure the masses of isotopes or radiator stopping power.

1. Introduction
The Cherenkov radiation (ChR) occurs when a charged particle moves in a medium with a velocity greater than the velocity of light in that medium. ChR is emitted into a cone making the Cherenkov angle with particle velocity vector (particle moves through the radiator with a constant velocity \( v \)). The detectors based on ChR are widely used to measure the ion velocity. It seems that the theory of ChR is developed in detail and confirmed by experiments. It is true, with respect to ChR from relativistic elementary particles. Nowadays new accelerators of RHI (for example, CERN SPS, GSI Darmstadt, FAIR Darmstadt) allow for the new experiments on ChR. In connection with this, we may ask the question – does the angular distribution of ChR from RHI contain any additional information, for example - on ion charge? To answer this question one needs to reconsider the ChR theory for the case of RHI.

The first theory of ChR was developed by Tamm and Frank (TF) in [1]. TF theory describes the ChR of a charged particle that moves rectilinearly in a medium with a constant velocity. It ignores both bending of particle trajectory due to multiple scattering (MS) and decreasing of its velocity due to ionization energy loss (slowing-down or stopping effect). Dedrick [2] was the first who took into account the influence of MS on the angular distribution of ChR, but he neglected the velocity decreasing due to ionization (and radiation) energy loss. In [3] Kuzmin and Tarasov (KT) considered another limiting case: they calculated the angular distribution of ChR from RHI taking into account their velocity decreasing due to stopping (ionization energy loss) in radiator, but neglecting RHI MS. KT model predicts the complex Fresnel diffraction-like structure of ChR angular distribution, but it is...
valid only for thin radiators. To avoid this approximation, the authors of [4] and [5] established another approach which uses computer code SRIM [6] to calculate energy loss and velocity depending on RHI penetration depth into a radiator. The obtained velocity is substituted into the formula of classical electrodynamics for the spectral-angular distribution of radiation from a charged particle that moves in a medium along a trajectory r(t):

\[ \frac{dI}{d\omega d\Omega} = \frac{z^2 e^2 \omega^2}{4\pi^2 c^3} \sqrt{E} \left[ \tilde{n} \times [\tilde{n} \times \tilde{v}] \exp \left\{ i\omega \left[ I - \sqrt{E(\tilde{n} \cdot \tilde{r}(t))c^{-1}} \right] \right\} \right]\, \mathrm{d}t. \quad (1) \]

2. Estimation of the ChR angular distribution width

As is well known, the ChR is emitted forward into a cone making an angle with the ion velocity direction (ion moves through the radiator with a constant velocity $v$). The formula for the Cherenkov angle is:

\[ \cos \theta = c \left( \frac{n}{v} \right). \]

Here $c$ – is the speed of light in vacuum, $n$ – the radiator refractive index.

In the TF theory the width $\Delta \theta_{TF}$ of the ChR angular distribution in the vicinity of the Cherenkov angle is determined by the radiation wave length $\lambda$ and the radiator thickness $L$:

\[ \Delta \theta_{TF} \approx \lambda L^{-1}. \]

It is easy to obtain the estimation of the ChR angular distribution width in the case of the thin radiator. As in the KT model [3], the inverse ion velocity depends on its penetration depth into a radiator $x$ and stopping (ionization energy loss):

\[ \frac{1}{v(x)} = \frac{1}{v_0} - \frac{1}{v_0} \frac{d}{dx} v_0 x, \quad v_0 = \left. \frac{d}{dx} v(x) \right|_{x=0} < 0, \quad (2) \]

\[ v_0 = \frac{dE}{dx} = \frac{c^2}{v} \left( 1 - \frac{\gamma^2}{\varepsilon^2} \right) \frac{dE}{dx} \left|_{x=0} = -\frac{c^2}{E_0 \gamma^2 v_0} S(E_0) \right. \quad (3) \]

Here $E$ – is RHI total energy; $E_0$, $\gamma_0$, $v_0$ – are incident energy, relativistic factor and velocity of RHI; $-\frac{dE}{dx} = S(E_0)$ – radiator’s stopping power (RHI ionization energy loss) which depends on RHI initial velocity. If the thickness of radiator is equal to $L$ ($x = L$) and the approximation (2) is valid, we may easily obtain the RHI velocity $v_1$ at $x = L$. The ChR angles corresponding to initial $v_0$ and final $v_1$ velocities are: $\theta_{0,1} = \arccos \left( \frac{1}{n \beta_{0,1}} \right)$, $\beta_{0,1} = \frac{v_0}{c}$; since the RHI velocity decreases due to stopping in radiator ($v_1 < v_0$), the ChR angle also decreases ($\theta_1 < \theta_0$). Now we may estimate the ChR angular distribution width $\Delta \theta_S$ taking into account stopping in a radiator in the following way:

\[ \Delta \theta_S = \theta_0 - \theta_1 = \arccos \left( \frac{1}{n \beta_0} - \arccos \left( \frac{1}{n \beta_0} \left( 1 - \frac{v_0 L}{v_0} \right) \right) \right). \]

If we consider that ChR is located in a narrow angular interval $\theta_1 \leq \theta \leq \theta_0$, $\theta_0 - \theta_1 \ll 1$, \arccos \left( \frac{1}{n \beta_0} \left( 1 - \frac{v_0 L}{v_0} \right) \right) can be expanded into the Taylor series near $\frac{1}{n \beta_0}$. The expansion reads: \arccos \left( \frac{1}{n \beta_0} \left( 1 - \frac{v_0 L}{v_0} \right) \right) = \arccos \left( \frac{1}{n \beta_0} \right) + \frac{v_0 L}{v_0} \cot \theta_0. After some algebra we obtain a very simple formula for $\Delta \theta$:
\[ \Delta \theta_s \approx \frac{|v_0| L}{v_0} \cot \theta_0 = \frac{c^2}{E_0 \gamma_0^2 v_0} S(E_0) L \cot \theta_0, \quad (4a) \]

\[ \Delta \theta_s = \frac{c^2}{E_0 \gamma_0^2 v_0} S(E_0) \frac{L}{\sqrt{(n\lambda \beta_0^2)^2 - 1}}, \quad (4b) \]

If the thin radiator approximation (2) is not valid, the RHI velocity at \( x = L \) can be calculated with the help of the special computer code, e.g. SRIM [6] or ATIMA [7], and the width of ChR angular distribution again can be estimated as \( \Delta \theta_s = \theta_0 - \theta_1 \approx \arccos \frac{1}{n \beta_0} - \arccos \frac{1}{n \beta_1}. \)

Expressions (4a) and (4b) demonstrate linear dependence of the ChR angular distribution width \( \Delta \theta_s \) on the radiator thickness \( L \), in contrary to the TF theory [1], with angular distribution width \( \Delta \theta_{TF} \approx \lambda L^{-1} \) being inversely proportional to \( L \). Thus, in a very thin radiator (slowing down is insignificant) the ChR angular distribution width is inversely proportional to radiator thickness \( L \) - \( \Delta \theta_{TF} \approx \lambda L^{-1} \), but if \( L \) becomes larger, this dependence transforms to the linear one, i.e. \( \Delta \theta_s \approx SL \). It is clear that if stopping effect is taken into account, the expression for ChR angular distribution radically changes and becomes more complex. Also in (4) we can see an appearance of radiator stopping power \( S(E_0) \) (depends on the RHI charge) and wave length dependence, because \( n = n(\lambda) \). Moreover, (4) contains an isotopic effect [5]: really, slowing down \( S(E_0) \) is proportional to ion charge and is equal for isotopes, but the multiplier \( E_0 = \gamma_0 M c^2 \) contains the dependence on the isotope mass.

### 3. Theoretical analysis

According to TF theory [1], the ChR angular distribution in a finite thickness radiator can be written in the following form:

\[ \frac{dI}{d\omega d\Omega} = \omega L \left( \frac{z e \sin \theta}{c} \right)^2 f_{TF} (\theta, \omega), \quad f_{TF} (\theta, \omega) = \frac{1}{\Delta \theta_{TF}} \left( \frac{\sin x}{x} \right)^2, \quad x = \frac{\pi}{\Delta \theta_{TF}} \left( \cos \theta - \frac{1}{\beta n} \right) \quad (5) \]

Here, \( \Delta \theta_{TF} = \lambda (n L)^{-1} \) is the ChR angular distribution width in the vicinity of the Cherenkov angle, \( z e \) – RHI charge; \( \lambda = 2\pi c \omega^{-1} \) – radiation wave length; \( \omega \) – frequency of the Cherenkov photon. The angular distribution (5) has diffraction-like structure and \( \Delta \theta_{TF} \) is proportional to \( \lambda \cdot L^{-1} \). In [3] the following expression based on (1) was used to analyze the ChR angular distribution and the ChR angular distribution width:

\[ \frac{dW}{d\omega d(\cos \theta)} = \frac{(z e \sin \theta)^2}{2 \pi c^2} k_0 \left| \int_0^L e^{i\Phi(x)} dx \right|^2, \quad (6) \]

Here, \( \Phi(x) = k x \cos \theta - \omega \int_0^x \frac{dx'}{v(x')} \); \( k = \frac{\omega n}{c} \) is the wave number of the Cherenkov photon. Using (2), the authors of [3] obtained analytic expressions for ChR angular distribution of RHI in a thin radiator taking into account their stopping:

\[ \frac{dI}{d\omega d\Omega} = \omega L \left( \frac{z e \sin \theta}{c} \right)^2 f_{KT} (\theta, \omega), \quad (7) \]

Here, the following function appears:
\[ f_{KT}(\theta, \omega) = \frac{1}{2 \Delta \theta_{KT} \sin \theta_0} \left\{ \left( C(u_1) - C(u_0) \right)^2 + \left[ S(u_1) - S(u_0) \right]^2 \right\} \]  \hspace{1cm} (8)

In (8) \( C(u) = \frac{2}{\pi} \int_0^u \cos \tau^2 \, dt \) and \( S(u) = \frac{2}{\pi} \int_0^u \sin \tau^2 \, dt \) - are the Fresnel integrals;

\[ u_{0,1} = -\frac{k \cos \theta - \omega \Delta u_0}{a} ; \quad a = \sqrt{\frac{2 \omega v_0}{v_0^2}} ; \quad \Delta \theta_{KT} = \theta_0 - \theta_1 \approx \left| \frac{v_0'}{v_0} \right| L \cot \theta_0 \]  - ChR angular distribution

width, \( \theta_{0,1} = \arccos \left( \frac{c}{n v_{0,1}} \right) \); \( 1 \approx \frac{1}{v_1} - \frac{1}{v_0} v_L \) - ion velocity on leaving the radiator.

To obtain the expressions for ChR angular distribution we start with the formula of classical electrodynamics (1) and a thin radiator approximation (2) and after some algebra obtained almost the same expressions as in KT model [3]. The difference is in the limits of integration for the Fresnel integrals: we believe that we use the correct expressions for these limits, while the expressions of [3] probably contain misprints. The authors of [3] did not mention, which method was used to calculate the velocity gradient. We did it using the well known Bethe-Bloch formula:

\[-\frac{dE}{dx} = \rho \frac{2 \pi}{\beta^2} \frac{Z}{A} \frac{m_e c^2}{1 + \frac{1}{\gamma} \left( \frac{m_e c^2}{M c^2} \right)^{1/2} + \left( \frac{m_e c^2}{M c^2} \right)^1} Z \left( \frac{2 m_e c^2 \beta^2 T_{\text{max}}}{I^2 (1 - \beta^2)} \right) - 2 \beta^2 \]  \hspace{1cm} (9)

Here, \( T_{\text{max}} = \frac{2 \beta^2 \gamma^2 m_e c^2}{1 + 2 \gamma m_e c^2 (M c^2)^{1/4} + (m_e M c^2)^{1/2}} \) is maximal energy that may be transferred to an electron in a radiator, \( N_A = 6.022 \cdot 10^{23} \text{[mol}^{-1}] \) – Avogadro number; \( z \) – ion charge in terms of elementary charge; \( Z \) and \( A \) – atomic number and atomic weight of the radiator matter; \( m_e c^2 \) – electron rest energy; \( \rho [\text{g cm}^{-3}] \) – radiator density; \( r_e^2 = 2.818 \cdot 10^{-13} \text{[cm]} \) – classical electron radius; \( I = 16 \cdot Z^{0.9} \text{[eV]} \) – average ionization potential.

Thus, TF theory [1] and KT model [3] differ in functions \( f_{TF}(\theta, \omega) \) and \( f_{KT}(\theta, \omega) \). The \( f_{TF}(\theta, \omega) \) contains dependence on the radiation wave length, refractive index and thickness of the radiator. In addition, in \( f_{KT}(\theta, \omega) \) initial ion velocity and radiator stopping power \( dE/dx \) dependencies appear. The radiator length \( L \) dependence is very different: in the first case we see \( L^{-1} \) – inverse dependence, and in the second – linear dependence – \( L \).

The schematics of the ChR angular distributions according to TF theory (left) and taking into account stopping in radiator (right) are presented in figure 1.
The additional broadening of ChR angular distribution width in the right part accompanied with more complex structure appears (only two Cherenkov cones are presented – for initial ($\theta_0$) and final ($\theta_1$) ChR angles).

The main advantages of the new ChR calculation method based on the Bethe-Bloch formula and a thin radiator approximation are: 1) fast calculation speed; 2) no need to use any special software (like SRIM [6] or ATIMA [7]); 3) no limits in choosing radiators, ions, and energies. The main disadvantage: this method may be used only when the thin radiator approximation (2) is valid.

4. New peculiarities in angular distribution of Cherenkov radiation

Now we will discuss the results of our calculations demonstrating the new features of ChR caused by stopping in radiator. The analytical formula for $\Delta\theta_S$ (4) clearly shows the following dependencies of the ChR angular distribution width $\Delta\theta_S$: 1) $L$ - dependence; 2) $\lambda$ – dependence; 3) stopping-power $S(E_0)$ dependence; 4) ion charge $z$ – dependence; 5) mass dependence ($E_0 = \gamma_0 M c^2$). To prove it, we applied equation (8) and performed numerical calculations for different targets and ions. Equation (8) connects together ChR angular distribution structure with radiator parameters (length, refractive index, stopping power) and ion beam parameters (charge, mass). Figure 2 shows the thickness dependence ($L$ – dependence) of ChR angular distribution in vicinity of the Cherenkov angle (corresponding zero in the figure 2) for Au ion in LiF radiator (radiator thickness $L = 0.5$ cm; radiation wave length $\lambda = 390$ nm; refractive index $n = 1.4$; initial energy of ion beam 1000 MeV/u).

And figure 3 shows $z$ – dependence of ChR angular distribution on RHI charge in LiF radiator (radiator thickness $L = 0.5$ cm; radiation wave length $\lambda = 390$ nm; refractive index $n = 1.4$; initial energy of ion beam 1000 MeV/u). Two methods were used for calculations: the first one – suggested in section 3 – based on the Bethe-Bloch formula and thin radiator approximation (2) (left part of figures 2 and 3), and the second one – based on SRIM [6] (right part of figures 2 and 3).

![Figure 2. $L$ – dependence of ChR angular distribution.](image)

It is obvious that ChR width is proportional to the radiator thickness and is almost the same in both methods. Also both methods show that the thicker the radiator is the more diffraction maxima appear. Finally, the computer code SRIM allows obtaining more exact values of RHI velocity depending on
penetration depth into radiator. The difference in results from two methods is that the graphs obtained using SRIM are not symmetric. Obviously, it is the sequence of the thin radiator approximation used in our suggested method of calculation.

Increasing of isotopic charge $z$ leads to additional broadening of the ChR angular distribution width and also to appearance of diffraction maxima.

The newest discovery is that ChR angular distribution width depends on the isotope mass. That’s why isotopic [5] effect appears. One may say that 2 isotopes with different masses $M_1$ and $M_2$ will lose an equal amount of energy, because the Bethe-Bloch formula for stopping power contains very weak dependence on the mass. But the width and fine structure of the ChR angular distribution in vicinity of the Cherenkov angle are remarkable different for isotopes with different masses, at equal initial relativistic factor (velocity) of isotopes. Indeed, let the isotopes with masses $M_1$ and $M_2$ have equal initial relativistic factors $\gamma_{10} = \gamma_{20} = \gamma_0 = \frac{E_1(0)}{M_1 c^2} = \frac{E_2(0)}{M_2 c^2}$, velocities $v_{10} = v_{20} = v_0$ and initial energies $E_1(0)$ and $E_2(0)$. After a thin layer of radiator $\Delta x$ both isotopes will lose an equal amount of energy $\Delta E_1 = \Delta E_2 = -dE/dx = S(\gamma_0, v_0)$, according to the Bethe-Bloch formula (9). For isotopes this values are equal at initial layer of radiator $\Delta x$. The next layer of radiator they will enter with different relativistic factors, etc.:

$$E_1(\Delta x) = E_1(0) - \Delta E_1 = E_1(0) - S(\gamma_0, v_0) \Delta x$$

$$E_2(\Delta x) = E_2(0) - \Delta E_2 = E_2(0) - S(\gamma_0, v_0) \Delta x$$

$$E_1(0) \neq E_2(0) \Rightarrow E_1(\Delta x) \neq E_2(\Delta x) \Rightarrow \gamma_1(\Delta x) \neq \gamma_2(\Delta x)$$

The change in relativistic factor leads to a change in velocity: $\Delta \beta = \Delta \gamma / \gamma^3$. This, for two isotopes the ratio of changes in velocities is:

$$\frac{\Delta \beta_1}{\Delta \beta_2} = \frac{M_2}{M_1}$$

The calculated ChR angular distributions for different isotopes are shown in figure 4:

![Figure 3. $z$ – dependence of ChR angular distribution.](image-url)
Figure 4. Angular distribution of ChR from isotopes: a) Li (4, 7, 8, 12), b) Be (7, 8, 9, 10) c) Ne (20, 21, 22, 23), d) Fe (54, 56, 57, 58); radiator – LiF; radiator thickness = 0.25 cm; radiation wave length $\lambda = 390$ nm; refractive index $n = 1.4$; initial energy of isotopes 1000 MeV/u.

Figure 4 clearly shows that the ChR angular distributions for different isotopes of one element are shifted towards each other. This is because the smaller the isotopic mass is the greater changes in velocity occur in radiator; so the final velocity when leaving the radiator for more heavy ions is greater than for smaller ones; and the Cherenkov angle is also greater, since $\theta_i = \arccos \frac{c}{n \lambda V}$. That is why the isotopic effect is more noticeable for Li isotopes.

5. Conclusions

In this work a new method for calculation of the ChR angular distribution from (RHI) taking into account stopping in radiator is suggested. Our method is based on the thin radiator approximation [3] and the Bethe-Bloch formula for ionization energy loss. Also the simple formula to estimate the ChR angular distribution width from RHI taking into account energy loss in radiator is obtained. New peculiarities if the ChR angular distribution caused by stopping in radiator were analysed and the results were compared with those calculated using computer code SRIM, which gives more correct values of penetration-dependent velocity. The results of numerical calculations based on equation (8), Bethe-Bloch formula for stopping and thin radiator approximation are very close to those based using SRIM-08.

Now we can answer the question formulated in section 1 of this paper: does the width and structure of the Cherenkov ring contain additional information, e.g. on the ion charge or RHI mass? The answer is – yes, and this statement is proved by numerical calculations.

The experimental studies of predicted effects are possible at existing (GSI) and future (FAIR) accelerators, e.g. using RICH detectors. The predicted new features of ChR allow for: possible applications of Cherenkov detectors (e.g. RICH) to simultaneous reconstruction of the velocity and charge of RHI penetrating through radiator; possible application to measure the masses of isotopes; possible application to measure the stopping power of radiator. For these studies, which should be
based on measurements of differential characteristics of ChR, the special optical spectrometer should be designed.

To conclude: only several experiments have been performed until now to observe the ChR from RHI, see [8-10]. An attempt to explain the experimental result was reported in [11], but without detailed analysis. The detailed analysis will be performed by our group and published in separate paper.

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