Equivariant Poisson Cohomology and a Spectral Sequence Associated with a Moment Map

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Abstract

We introduce and study a new spectral sequence associated with a Poisson group action on a Poisson manifold and an equivariant momentum mapping. This spectral sequence is a Poisson analog of the Leray spectral sequence of a fibration. The spectral sequence converges to the Poisson cohomology of the manifold and has the $E_2$-term equal to the tensor product of the cohomology of the Lie algebra and the equivariant Poisson cohomology of the manifold. The latter is defined as the equivariant cohomology of the multi-vector fields made into a $G$-differential complex by means of the momentum mapping. An extensive introduction to equivariant cohomology of $G$-differential complexes is given including some new results and a number of examples and applications are considered.

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1 Introduction

The main result of the present paper is the existence of a spectral sequence associated with an equivariant momentum mapping of a Poisson manifold and converging to the Poisson cohomology of the manifold. When the manifold is symplectic, the spectral sequence is isomorphic to that in the de Rham cohomology for the dual data: the group action rather than the momentum mapping. Namely, let $P$ be a symplectic manifold acted on by a compact Poisson Lie group $G$. Then the spectral sequence in question is isomorphic to the Leray spectral sequence of the principal $G$-bundle $P \times EG \to (P \times EG)/G$. If the momentum mapping is a submersion onto an open subset, and so the action is locally free, one can take the projection $P \to P/G$ as well.

To extend the spectral sequence to Poisson manifolds, one thus needs to dualize the notion of equivariant de Rham cohomology, which leads to what we call equivariant Poisson cohomology. The latter arises as a particular case of the equivariant cohomology of a $G$-differential complex, for an equivariant momentum mapping makes the complex of multi-vector fields into one.

Equivariant de Rham cohomology has proven itself to be a very convenient tool in the study of Hamiltonian group actions because it renders a powerful topological machinery to one’s service. However, no analog of such a cohomology has been known for Poisson manifolds. The introduction of equivariant Poisson cohomology fills in the gap. The cohomology readily lends itself to a variety of applications (e.g., the spectral sequence and calculations of Poisson cohomology) in a way similar to de Rham cohomology, but with algebraic topology replaced by some algebra. Certain aspects of the new techniques have remained unexplored in this paper. For example, one may see that the equivariant Poisson cohomology spaces behave well under symplectic reduction analogously to equivariant de Rham cohomology. However, no rigorous results in this direction have been proved yet. It is also worth noticing that, although the formal algebraic machinery has been developed long time ago, Poisson equivariant cohomology is still one of very few examples, if not the only example, where $G$-differential complexes essentially different from the de Rham complex are employed.

The spectral sequence introduced in the paper can be generalized to virtually any Poisson map. The generalization is particularly transparent for Poisson submersions. This gives an affirmative answer to a question asked by Alan Weinstein whether there exists a spectral sequence associated with a “Poisson fibration”. In general, the $E_1$ and $E_2$-terms tend to have a very complicated structure. However, for a momentum mapping, the spectral sequence is relatively easy to analyze and, as illustrated by a number of examples, use to calculate the Poisson cohomology, to which it converges.
Before the spectral sequence can be defined and the $E_2$ calculated, a rather cumbersome machinery has to be developed. Some work to this end has already been carried out in [Gi2]. The exposition in the present paper naturally evolves around the study of equivariant cohomology of $G$-differential complexes and its Poisson counterpart. The paper is organized as follows.

We begin with an extensive introduction, enlivened with examples, to equivariant cohomology of $G$-differential complexes. This approach, developed originally by H. Cartan [Ca1], is based on the axiomatization within an algebraic framework of the key features of differential forms needed to define the equivariant cohomology of a manifold with real coefficients. Thus a $G$-differential complex is, by definition, a complex $A^\ast$ equipped with a $G$-action and contractions with the elements of $\mathfrak{g}$ satisfying the natural compatibility conditions. A fundamental example is the de Rham complex $A^\ast = \Omega^\ast(M)$, where $M$ is a manifold acted on by a group $G$. The equivariant cohomology $H^\ast_G(A^\ast)$ is then just the cohomology of the complex $(A^\ast \otimes S^\ast \mathfrak{g}^\ast)^G$, which for the de Rham complex gives $H^\ast_G(M, \mathbb{R})$, provided that $G$ is compact.

In the opening subsections of Section 2, we give all the necessary definitions and state first basic theorems including the result that for “locally free $G$-differential algebras” the equivariant cohomology coincides with the basic cohomology. An alternative source for this material is Appendix B of [GLS] or the second part of [DKV].

Then we turn to one of our main examples of $G$-differential complex – the Chevalley–Eilenberg complex $C^\ast(\mathfrak{g}; V)$, where $V$ is a Fréchet $G$-module. This is a complex over a $G$-differential locally free algebra $C^\ast(\mathfrak{g})$, and so $H^\ast_K(\mathfrak{g}; V) = H^\ast(\mathfrak{g}; t; V)$ for any subgroup $K \subset G$. In Section 2.3.2, we recall that $H^\ast(\mathfrak{g}; t; V) = H^\ast(\mathfrak{g}, t) \otimes V^G$, when $G$ is compact, which is rather easy to prove using the so-called van Est spectral sequence (cf., [Gi2]). A proof that does not rely on the spectral sequence and gives a little bit stronger parametric version of the theorem is provided in the appendix. The results of this section play a key role in the explicit calculation of (equivariant) Poisson cohomology of certain Poisson manifolds, e.g., $G^\ast$, and of the $E_2$-term of the spectral sequence. Conceptually, this role is not entirely dissimilar to that of Hodge theory in analysis on manifolds.

In Section 2.4 we study further properties of equivariant cohomology. For example, we analyze the equivariant cohomology in low degrees and introduce a spectral sequence analogous to the standard spectral sequence for equivariant cohomology, and thus having $E_2 = H^\ast(A^\ast) \otimes (S^\ast \mathfrak{g}^\ast)^G$ and converging to $H^\ast_G(A^\ast)$.

Finally, in Section 2.5 we call into action the major figure of this paper – a spectral sequence relating the cohomology of $A^\ast$, the cohomology of $\mathfrak{g}$, and the equivariant cohomology $A^\ast$. As we have mentioned, the spectral sequence is a generalization of the Leray spectral sequence of $M \times EG \to (E \times EG)/G$, where $M$ is a space acted on by $G$. It has $E_2 = H^\ast_G(A^\ast) \otimes H^\ast(\mathfrak{g})$ and converges to the ordinary cohomology $H^\ast(A^\ast)$.

Section 3 is a review of the results from Poisson geometry used in the subsequent sections. Along with the standard material, which can be found elsewhere (see, e.g., [Va] and bibliography therein), we prove some new results and outline a technique developed in [Gi2] to be applied in the present paper.

Section 4 is entirely devoted to equivariant Poisson cohomology. The data sufficient to define this cohomology, $H^\ast_{\pi,G}(P)$, are the Poisson manifold $P$ and an equivariant momentum mapping $\mu: P \to G^\ast$, and so a Poisson action of a Poisson Lie group $G$. We start with a discussion of the general properties of the cohomology and carry out some calculations in low degrees. The “locally free” case where $\mu$ is a submersion onto an open subset on $G^\ast$ is a key to understanding the geometrical meaning of the cohomology. In Section 4.3.
we show that in this case, \( H^\ast_{\pi,G}(P) \) can be identified with the cohomology of \( G \)-invariant multi-vector fields tangent to the \( \mu \)-fibers. (It is instructive to contrast this result with the identification of the equivariant and basic de Rham cohomology for locally free actions.)

The equivariant cohomology of \( G^\ast \) with respect to a subgroup \( K \subset G \) is found in Section 4.4: \( H^\ast_{\pi,K}(G^\ast) = H^\ast(g\bullet t) \otimes C^\infty(G^\ast)^G \), and, in particular, \( H^\ast_{\pi,G}(G^\ast) = C^\infty(G^\ast)^G \). Little proving needs to be done throughout Section 4, for the results proved previously for general \( G \)-differential complexes readily apply here.

Finally, in Section 5 we define the spectral sequence associated with a momentum mapping, calculate its \( E_2 \)-term and analyze a variety of examples.

Throughout the paper the cohomology of a manifold are taken with real coefficients unless specified otherwise. All Lie groups are assumed to be connected.

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2 Equivariant cohomology

In this section we study the equivariant cohomology in the general algebraic context of \( G \)-differential complexes, introduced originally in by H. Cartan in [Ca1]. The material reviewed in the first two subsections is fairly standard and the proofs are omitted. The reader interested in more details should consult, e.g., [DKV] or Appendix B of [GLS]. (Many other relevant references are spread throughout the text.) Two concluding subsections contain some newer results crucial for what follows and their detailed proofs are provided.

2.1 Equivariant cohomology of \( G \)-differential complexes

Let \((A^\ast, d)\) be a complex of Fréchet spaces with the differential \( d \) of degree one. For the sake of simplicity we assume throughout this section that the grading on \( A^\ast \) is positive: \( A^n = 0 \) when \( n < 0 \). Assume that a connected Lie group \( G \) acts smoothly on \( A^\ast \) and that the action commutes with \( d \). Denote by \( L_\xi \) the infinitesimal action on \( A^\ast \) of an element \( \xi \) of the Lie algebra \( \mathfrak{g} \) of \( G \). The complex \( A^\ast \) is said to be a \( G \)-differential complex if for any \( \xi, \zeta \in \mathfrak{g} \) it is equipped with a continuous linear mapping \( i_\xi: A^\ast \to A^\ast \) of degree -1 such that the following conditions hold for all \( \xi \) and \( \zeta \) in \( \mathfrak{g} \) and \( g \in G \):

(i) \( i_\xi i_\zeta + i_\zeta i_\xi = 0 \);

(ii) \( gi_\xi g^{-1} = i_{\text{ad}_g \xi} \);

(iii) \( L_\xi = di_\xi + i_\xi d \) (Cartan’s identity).

If instead of a \( G \)-action, \( A^\ast \) carries only an infinitesimal \( \mathfrak{g} \)-action, as above, commuting with the differential, \( A^\ast \) is said to be a \( \mathfrak{g} \)-differential complex, provided that we have “contractions” \( i_\xi \), (i) and (iii) hold, and (ii) is replaced by its infinitesimal version:

(ii') \( i_{[\xi, \zeta]} = L_\xi i_\zeta - i_\zeta L_\xi \).

Let \( A^\ast \) be a graded-commutative differential algebra, i.e., \( ab = (-1)^{\deg a \deg b} ba \), and let the graded Leibniz identity hold for \( d \) and the contractions, i.e., \( i_\xi(ab) = (i_\xi a)b + (-1)^{\deg a}ai_\xi b \). Then we call \( A^\ast \) a \( G \)- or \( \mathfrak{g} \)-differential algebra.
Remark 2.1 Our definition of a \( g \)-differential complex is redundant in the sense that the action of \( g \) on \( A^* \) can be recovered from \( d \) and the “contractions”. To be more precise, assume that \( A^* \) is a complex and \( i_\xi \) are given so that (i) and (ii') hold. Define the infinitesimal action \( L_\xi \) via Cartan’s identity (iii). The commutation relation \( L_\xi i_\xi = L_\xi i_\xi - L_\xi L_\xi \) is then a consequence of (i) and (ii'). Furthermore, if \( A^* \) is a \( G \)- or \( g \)-differential algebra, the multiplication is automatically \( G \)- or \( g \)-invariant.

Note also that a \( g \)-differential complex may fail to integrate to a \( G \)-differential complex even if \( G \) is compact.

Consider the tensor product \( A^* \otimes S^*g^* \) with the grading \( \deg(A^n \otimes S^mg^*) = n + 2m \). This tensor product can be viewed as the space of \( A^* \)-valued polynomial functions on \( g \). Thus for \( \alpha \in A^* \otimes S^*g^* \) and \( \xi \in g \) we denote by \( \alpha(\xi) \in A^* \) the value of such a polynomial at \( \xi \). Let us define a homomorphism \( d_G : A^* \otimes S^*g^* \to A^* \otimes S^*g^* \) of degree one as follows:

\[
(d_G\alpha)(\xi) = d(\alpha(\xi)) - i_\xi \alpha(\xi).
\]

A routine calculation shows that \( d_G^2 \alpha(\xi) = -L_\xi \alpha(\xi) \), and so \( d_G \) is not a differential on the entire tensor product. Note however that \( A^* \otimes S^*g^* \) is a \( g \)-module with the diagonal action and denote by \( A^*_g \) its subspace \( (A^* \otimes S^*g^*)^g \) of \( g \)-invariant elements. Clearly, \( A^*_g \) is closed under \( d_G \) and \( d_G^2 = 0 \) on \( A^*_g \).

Definition 2.2 The cohomology of the complex \( (A^*_g, d_G) \), denoted \( H^*_G(A^*) \), is called the \textit{equivariant cohomology} of the \( g \)-differential complex \( A^* \). The complex \( (A^*_g, d_G) \) is said to be the \textit{Cartan model} for equivariant cohomology.

Let us emphasize that in what follows we use the notation \( H^*_G(A^*) \) even when \( A^* \) carries only an infinitesimal \( G \)-action, i.e., when \( A^* \) is a \( g \)-module.

Recall that since \( d_G \) is linear over \( (S^*g^*)^g \), the equivariant cohomology is a graded \( (S^*g^*)^g \)-module. If \( A^* \) is a \( g \)-differential algebra, the multiplication descends to \( H^*_G(A) \) making \( H^*_G(A^*) \) into an \( (S^*g^*)^g \)-algebra.

Example 2.3 The standard example of equivariant cohomology is the equivariant de Rham cohomology. Let \( M \) be a manifold acted on by \( G \). We take the de Rham complex of \( M \) with the natural action of \( G \) and natural contractions as the \( G \)-differential algebra \( (\Omega^*(M), d) = (\Omega^*(M), d) \). Then the cohomology \( H^*_G(M) \) is called the equivariant cohomology of \( M \) with real coefficients. To be more precise, let \( H^*_G(M, \mathbb{R}) \) be the ordinary (real) cohomology of the quotient \( (M \times EG)/G \) where \( EG \) is the classifying space of \( G \). Then \( H^*_G(M, \mathbb{R}) = H^*_G(A) \) when \( G \) is compact (the equivariant de Rham theorem).

The \( g \)-differential algebra construction goes through even if \( G \) is not compact or if \( M \) is equipped only with an infinitesimal \( G \)-action. In these cases, the equivariant cohomology of \( A \) will also be called the equivariant cohomology of \( M \) even though the topological construction may then lead to an entirely different result \[AB\].

Likewise, let \( E \rightarrow M \) be a vector bundle (with a Fréchet fiber) endowed with a flat connection. We use horizontal lifts of vector vector fields to obtain a \( G \)-action on \( E \). Then the complex \( A^* = \Omega^*(M, E) \) of \( E \)-valued differential forms turns naturally into a \( G \)-differential complex, and we have the equivariant cohomology \( H^*_G(M; E) \). When \( G \) is compact, this space is just the standard equivariant cohomology of \( M \) with coefficients in the sheaf (a local system) of flat sections of \( E \). Note that now \( A^* \) is not a \( G \)-differential algebra. However, it will be such when the fibers of \( E \) are algebras and multiplication is invariant of holonomy.
The comparison of algebraic and topological constructions of equivariant cohomology will be continued in Example \ref{ex:example}.

\section{The Weil complex and locally free $G$-differential complexes}

In this section we outline a slightly different, though equivalent, approach due to Atiyah and Bott \cite{At} to the definition of equivariant cohomology. This approach relying on the usage of the Weil algebra of $g$ appears to be considerably more convenient for our present purposes. However, it should be mentioned that adapting the methods of \cite{Ca2} one can prove Theorem \ref{thm:main} and Corollary \ref{cor:cor} without ever leaving the realm of complexes $A^*_\mathbb{Z}$.

Let $W^*(g)$ be the Weil algebra of $g$. Recall that, as a graded commutative algebra, $W^*(g) = \wedge^* g^* \otimes S^* g^*$ with the symmetric part given the even grading. The elements of $W^*(g)$ can be thought of either as symmetric (even) functions on $g$ with values in $\wedge^* g^*$ or odd (skew-symmetric) functions on $g$ with values in $S^* g^*$. The second interpretation leads to the $S^* g^*$-linear contractions $i_\xi : W^*(g) \to W^*(g)$ for $\xi \in g$. To be more precise, $i_\xi (\delta \varphi \otimes f) = (i_\xi \varphi) \otimes f$, where $\varphi \in \wedge^* g^*$ and $f \in S^* g^*$.

Furthermore, $W^*(g)$ carries a differential $d_W = d_L - \delta$, Here $d_L$ is the Chevalley–Eilenberg differential of $g$ with coefficients in the $g$-module $E = S^* g^*$ (see (\ref{def:CE}) of the next section) and $\delta \phi (\xi) = i_\xi \phi (\xi)$, where $\phi \in W^*(g)$ is thought of as a function on $g$ with values in $\wedge^* g^*$. For example, when $\lambda \in \wedge^1 g^* = g^*$, we have $d_W \lambda = d_L \lambda - f_\lambda$, where $f_\lambda = \lambda$ is viewed as an element of $S^1 g^*$. The differential $d_W$, the contractions $i_\xi$, and the natural action of $G$ fit together to make $W^*(g)$ into a $G$-differential algebra. (See, e.g., \cite{Fu} or \cite{AB} for more details.)

The Weil complex can be characterized by the following universal property. (See, e.g., \cite{Fu} for the proof.) Namely, given a multiplicative complex $A^*$, any linear mapping $\Phi : g^* \to A^1$ can be extended in a unique way to a multiplicative homomorphism of complexes: $\Phi : W^*(g) \to A^*$. Note that $\Phi (f_\lambda) = \wedge^2 \Phi (d_L \lambda) - d_A \Phi (\lambda)$, where $\lambda$ and $f_\lambda$ are as above. The Weil complex is known to be acyclic. (See, e.g., \cite{Fu}, for an explicit homotopy between $id$ and zero.)

Recall that for a $g$-differential complex $C^*$, the basic subcomplex $C^*_b$ is formed by $g$-invariant elements $c$ such that $i_\xi c = 0$ for all $\xi \in g$. Equivalently, $c \in C^*_b$ if and only if $i_\xi c = 0$ and $i_\xi dc = 0$ for all $\xi$. The cohomology $H^*_b (C^*) = H^* (C^*_b)$ is called the basic cohomology of $C^*$.

The tensor product of two $g$-differential complexes (algebras) is, in a natural way, a $g$-differential complex (algebra). In particular, given a $g$-differential complex (algebra) $A^*$, the product $C^* = A^* \otimes W^*(g)$ is a $g$-differential complex (algebra) and the basic cohomology $H^*_b (A^* \otimes W^*(g))$ is defined.

The inclusion $(S^* g^*)^G \to (W^*(g))_b$ is actually an isomorphism because $i_\xi (\varphi \otimes f) = 0$ for all $\xi$, where $\varphi \in \wedge^* g^*$ and $f \in S^* g^*$, means that $\varphi = 1$. Observe also that $d_W = 0$ on $(S^* g^*)^G$. Therefore, $H^*_b (W^*(g)) = (W^*(g))_b = (S^* g^*)^G$. In particular, this implies that $H^*_b (A^* \otimes W^*(g))$ is an $(S^* g^*)^G$-module.

\begin{theorem}
$H^*_b (A^* \otimes W^*(g)) \simeq H^*_G (A^*)$ as topological $(S^* g^*)^G$-modules (or algebras if $A^*$ is an algebra).
\end{theorem}

The proof of the theorem can be found, for example, in Appendix B of \cite{GLS}. The argument is rather standard: one shows that the very complexes $(A^* \otimes W^*(g))_b$ and $(A^* \otimes$
Infinitesimal actions can be made equivariant by applying the averaging over $G$ of Example 2.3, $\Omega^\star(G)$ generally, the de Rham complex is a locally free $g$-module with the differentials and contractions. This means that $g$-differential complex $A^\star$ is a locally free $g$-module and the multiplication is compatible with the differentials and contractions. This means that $d(ac) = (da)c + (-1)^n a dc$ and $i_\xi(ac) = (i_\xi a)c + (-1)^n a(i_\xi c)$ for $a \in A^n$ and $c \in C^*$. 

**Definition 2.5** A $g$-differential algebra $A^\star$ is said to be locally free if $A^0$ has a unit 1, and so $\mathbb{R} \subseteq A^1$, and there exists a $g$-equivariant linear homomorphism $\Theta: g^\star \to A^1$ such that $\lambda(\xi) = i_\xi \Theta(\lambda)$ for all $\lambda \in g^\star$ and $\xi \in g$.

**Remark 2.6** If $G$ is compact, any $\Theta: g^\star \to A^1$ satisfying the above condition $\lambda(\xi) = i_\xi \Theta(\lambda)$ can be made equivariant by applying the averaging over $G$.

Definition 2.5 dates back to H. Cartan’s original papers [Ca1] and [Ca2]. (Among more recent publications see [DKV] (Definition 16, the existence of a connection) and also [GLS] (Appendix B).)

Consider now a $g$-differential algebra $A^\star$ and a $g$-differential complex $C^\star$. We say that $C^\star$ is a $g$-differential $A^\star$-module if $C^\star$ is a graded $A^\star$-module and the multiplication is compatible with the differentials and contractions. This means that $d(ac) = (da)c + (-1)^n a dc$ and $i_\xi(ac) = (i_\xi a)c + (-1)^n a(i_\xi c)$ for $a \in A^n$ and $c \in C^*$.

**Theorem 2.7** Let $C^\star$ be a module over a locally free $g$-differential algebra. Then $H^\star_G(C^\star) \simeq H^\star_b(C^\star)$ as topological vector spaces.

**Corollary 2.8** Assume that $A^\star$ is locally free. Then $H^\star_G(A^\star) = H^\star_b(A^\star)$ as topological algebras.

The proof of Theorem 2.7 is only a minor modification of that of Corollary 2.8, which is essentially due to H. Cartan. (See [Ca1] and [Ca2].) The proofs can also be found in [DKV] (Theorem 17 of Part I, the ring of polynomials $S^\star g^\star$ being replaced by functions on $g^\star$) and [GLS] (Appendix B). We omit the argument here.

**Example 2.9** (Continuing Example 2.3.) Let $G$ act on $M$ locally free, i.e., so that all stabilizers are discrete. Then $A^\star = \Omega^\star(M)$ is a locally free $G$-differential algebra. More generally, the de Rham complex is a locally free $g$-differential algebra when $M$ is given an infinitesimal $G$-action generated by a monomorphism of $g$ into the space of vector fields $X^1(M)$ on $M$. We shall call such an infinitesimal action locally free as well. In the notation of Example 2.3, $\Omega^\star(M, E)$ is a module over $\Omega^\star(M)$, and the equivariant cohomology is then the same as the cohomology of the basic complex $\Omega^\star(M, E)_G$.

Assume now that $G$ is compact and that $M$ carries a genuine $G$-action. Then, as well known, $H^\star_G(M) = H^\star_b(\Omega(M)) = H^\star(M/G)$. One way to see this is to consider the Leray spectral sequence of the projection $\rho: (M \times EG)/G \to M/G$. The fiber $\rho^{-1}(x)$, where $x \in M/G$, is homotopy equivalent to $BG_x$. The Leray spectral sequence of $\rho$ with real coefficients collapses in the $E_2$-term, since $H^\star_{>0}(BG_x, \mathbb{R}) = 0$. This yields an isomorphism $H^\star_G(M) \simeq H^\star(M/G, \mathbb{R})$. (See, e.g., Lemma 5.3 of [31] and references therein for more details.) Furthermore, the basic subcomplex $\Omega(M)_G$ can be identified with the de Rham

\[S^\star g^\star)^G \text{ are isomorphic.}\] The isomorphism is just the natural inclusion $(A^\star \otimes S^\star g^\star)^G \to (A^\star \otimes W^\star(g))_G$ and its inverse can be written explicitly.

1Given two complexes, the result (e.g., Theorem 2.3) that their cohomology spaces are isomorphic may arguably be called “trivial” when the complexes are themselves isomorphic and “nontrivial” otherwise. This classification does not appear to be completely meaningless.

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complex of the orbifold $M/G$. A similar reasoning applies to the differential forms with values in a vector bundle.

The Weil complex $W^*(\mathfrak{g})$ is only a nominally different example of a locally free algebra. Recall that geometrically $W^*(\mathfrak{g})$ can be thought of as an algebraic model for $\Omega^*(EG)$ and the $G$-action on $EG$ is free. Clearly, $(W^*(\mathfrak{g}))_b = (S^*\mathfrak{g}^*)^G$ with the trivial differential, and so $H^*_b(W^*(\mathfrak{g})) = (S^*\mathfrak{g}^*)^G$.

2.3 The Chevalley–Eilenberg complex as a $K$-differential complex

2.3.1 The Chevalley–Eilenberg complex

Let $K \subset G$ be a connected subgroup of $G$ with the Lie algebra $\mathfrak{t}$ and let $V$ be a locally convex differentiable $G$-module. Consider the complex $C^*(\mathfrak{g}, K; V) = (V \otimes \wedge^*(\mathfrak{g}/\mathfrak{t})^*)^K$ with the differential $d_{Lie}$ defined as

\[ d_{Lie} \phi(\xi_1, \ldots, \xi_{n+1}) = \sum_{1 \leq j < k \leq n+1} (-1)^{j+k-1} \phi([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_j, \ldots, \hat{\xi}_i, \ldots, \xi_{n+1}) \]

where $\xi v$ denotes the action of $\xi \in \mathfrak{g}$ on $v \in V$ and $C^*(\mathfrak{g}, K; V)$ is identified with a subspace of $C^*(\mathfrak{g}; V) = V \otimes \wedge^*\mathfrak{g}^*$. In this way, we do obtain a complex called the Chevalley–Eilenberg complex of $\mathfrak{g}$ relative to $K$ with coefficients in $V$. (See, e.g., [Fu] or [Gu] for more details.) The cohomology of this complex is denoted by $H^*(\mathfrak{g}, K; V)$.

Take, in particular, $(C^*, d) = (C^*(\mathfrak{g}, V), d_{Lie})$. A straightforward calculation shows that the natural $G$-action and contractions make $C^*$ into a $G$-differential complex, and the equivariant cohomology $H^*_K(C^*)$ is defined. Note also that if $V$ is a topological $G$-algebra, i.e., the multiplication $G$-invariant, $C^*(\mathfrak{g}, K; V)$ is a multiplicative complex, and so $C^*$ is a $G$-differential algebra. In any case

**Theorem 2.10** Assume that $\mathfrak{t} \subset \mathfrak{g}$ has a $K$-invariant complement with respect to the adjoint action. Then $H^*_K(C^*) = H^*(\mathfrak{g}, K; V)$ as topological vector spaces. Furthermore, these spaces are isomorphic as topological algebras if $V$ is a $G$-algebra.

**Remark 2.11** A $K$-invariant complement $\mathfrak{t}^\perp$ exists, for example, when $K$ is compact or $\mathfrak{t}$ is semisimple. (But also in some other cases as well.) The existence of $\mathfrak{t}^\perp$ is equivalent to the existence of a $K$-equivariant projection $\theta: \mathfrak{g} \to \mathfrak{t}$, for we can take $\mathfrak{t}^\perp = \ker \theta$.

**Proof.** Observe that $A^* = C^*(\mathfrak{g})$ is a free $\mathfrak{t}$-differential algebra. (The linear map $\Theta = \theta^*: \mathfrak{t}^* \to \mathfrak{g}^*$ is $\mathfrak{t}$-equivariant.) Furthermore, $C^* = C^*(\mathfrak{g}; V)$ is a differential $A^*$-module. Hence by Theorem 2.7, $H^*_K(C^*) = H^*_b(C^*)$. It is easy to see that $C^*(\mathfrak{g}; V)_b = C^*(\mathfrak{g}, K; V)$ as complexes, and $H^*_b(C^*) = H^*(\mathfrak{g}, K; V)$.

\[\square\]

\[\square\] 2According to the classification of our previous footnote, Theorem 2.10 as well as Theorem 2.7 it depends on, is “nontrivial”.

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Recall that for every \( v \in G \) representation on \( \Omega^*(G, V) \) is a differential \( K \)-complex with respect to the right action of \( K \) on \( G \). Since the action is free, we have \( H^*_K(\Omega^*(G; V)) = H^*(G/K; V) \). The isomorphism is induced by the pull-back \( \pi^*: \Omega^*(G/K; V) \to \Omega^*(G; V)_b \) (cf., [Ca2]). The complex \( C^*(\mathfrak{g}, K; V) \) can be identified with the subcomplex of \( \Omega^*(G/K; V) \) formed by \( V \)-valued differential forms invariant under the diagonal action. The averaging yields \( H^*(\mathfrak{g}, K; V) = H^*_R(G/K; V) \), provided that \( G \) is compact. (See the next section for a detailed discussion.) Similarly, \( C^*(\mathfrak{g}; V) \) is a subcomplex of forms invariant under the diagonal action on \( \Omega^*(G; V) \). It is not hard to see that \( C^*(\mathfrak{g}; V) \) and \( \Omega^*(G; V) \) have equal \( K \)-equivariant cohomology when \( G \) is compact. Thus

\[
H^*(\mathfrak{g}, K; V) = H^*(G/K; V) = H^*_K(\Omega^*(G; V)) = H^*_K(C^*),
\]

which gives a topological proof of Theorem 2.10 for a compact \( G \).

2.3.2 Lie algebra cohomology with coefficients in Fréchet modules

We have just seen how the explicit calculation of certain equivariant cohomology spaces can be reduced to finding relative Lie algebra cohomology. In the subsequent sections we shall see that some Poisson cohomology can also be treated as the Lie algebra cohomology. Various Lie algebra or Lie group cohomology spaces with coefficients in other Fréchet modules also arise in studying the rigidity of group actions (e.g., Poisson actions) and the existence of pre-momentum mappings \([G2]\). In this section we recall how to carry out the calculation of cohomology explicitly when the group is compact.

Let \( G \) be a compact Lie group and \( V \) a smooth Fréchet \( G \)-module. Denote by \( \rho \) the representation on \( G \) on \( V \) and by \( V^G \) the (closed) subspace of all \( G \)-invariant vectors in \( V \). Recall that for every \( v \in V \), the map \( g \mapsto \rho(g)v \) of \( G \) to \( V \) is smooth and therefore \( V \) is a \( \mathfrak{g} \)-module. Furthermore, this map is integrable with respect to a bi-invariant Haar measure \( dg \) on \( G \) and the projector \( A: V \to V^G \) given by averaging over \( G \) is continuous. Let \( C^* \) be the Chevalley–Eilenberg complex \( C^*(\mathfrak{g}, K; V) \) of \( \mathfrak{g} \) relative \( K \) with values in \( V \). In this section we use the notation \( d_\rho \) for the differential \( d_{\text{Lie}} \) of (6) to emphasize its dependence of \( \rho \). Denote \( B^n \subset C^n \) the subspace of all exact \( n \)-cochains and by \( Z^n \) the subspace formed by cocycles. The space \( C^n \) is Fréchet as well as \( Z^n \), and \( B^n \), being equipped with the induced topology, turns into a locally convex topological vector space.

Sometimes one also needs to consider a family \( \rho_x \), \( x \in X \), of representations of \( G \) on \( V \) smoothly parameterized by a manifold \( X \). Then the operator \( d_{\rho_x} \) depends on \( x \) but the spaces \( C^n \) do not.

Theorem 2.13

(i) The inclusion \( V^G \to V \) gives rise to a topological isomorphism in the Lie algebra cohomology whose inverse is induced by \( A \). In particular,

\[
H^*(\mathfrak{g}, K; V) = H^*(\mathfrak{g}, K; V^G) = H^*(\mathfrak{g}, K) \otimes V^G.
\]

(ii) For every \( n > 0 \), there exists a continuous linear map

\[
\mathcal{H}: C^n(\mathfrak{g}, K; V) \to C^{n-1}(\mathfrak{g}, K; V)
\]

such that \( d\mathcal{H}\big|_{B^n} = \text{id} \).
Given a smooth family of representations $\rho_x$, the family of operators $H_x$ from (ii) can be chosen to be smooth in $x$.

**Corollary 2.14** In the notation of Theorem 2.10, $H^*_K(C^*) = H^*(g, K) \otimes V^G$, provided that $G$ is compact and $V$ is a Fréchet $G$-module.

Assertion (i), which is sufficient for a majority of applications, can be easily proved by employing the so-called van Est spectral sequence [Es], e.g., as in [Gi2], combined with a theorem by Mostow which implies that $\Omega^*(G/K; V)$ is continuously strongly injective. (See [BW], Proposition 5.4 of Chapter IX.) In the appendix, we outline a direct and relatively elementary proof, yet also using Mostow’s theorem, of all the assertions.

Another immediate consequence of the theorem is the following

**Corollary 2.15** Let a compact group $G$ act on a manifold $M$. Then $H^*(g; C^\infty(M)) = H^*(g) \otimes (C^\infty(M))^G$.

**Remark 2.16** 1. For a finite-dimensional $V$, assertion (i) of Theorem 2.13 is well known (e.g., [Gu], Section II.11) and (ii), (iii) are evident. Moreover, it is a classical result that $H^*(g; V) = 0$, when $g$ is a semisimple real or complex finite-dimensional Lie algebra and $V$ an irreducible finite-dimensional nontrivial $g$-module. (See, e.g., [Gu] or [Fu] and references therein.) Of course, this also follows from (i) by complexifying, in the real case, both $g$ and $V$ and then passing to the representation of the compact form of $g$ on $V_C$.

2. It is essential that the $g$-module structure on $V$ integrates to that of a $G$-module. Theorem 2.13 does not extend to just $g$-modules [GW]. Nor does it hold when $G$ is a noncompact simple group (e.g., $G = SL(2, \mathbb{R})$).

### 2.4 Further properties of equivariant cohomology

In this section, we routinely extend the standard properties of the equivariant de Rham cohomology to the equivariant cohomology of $G$-differential complexes.

Let us start with a calculation of equivariant cohomology in low degrees. The zeroth cohomology is particularly simple: $H^0_G(A^*) = \ker(d: A^0 \to A^1)$, for the elements of the kernel are $g$-invariant.

**Theorem 2.17**

(i) Let $H^1(g) = 0$. Then every $d$-closed $g$-invariant element of $A^1$ is automatically $d_g$-closed, and, as a consequence,

$$H^1_G(A) = \{a \in A^1 \mid da = 0 \text{ and } i_\xi a = 0 \text{ for all } \xi \in g\}/d(A^0)^G.$$ 

(ii) Let $G$ be compact semisimple and let $A^*$ be a $G$-differential complex. Then the forgetful homomorphism $H^*_C(A) \to H^*(A)$ is an isomorphism in degree one, $\ast = 1$, and an epimorphism in degree two, $\ast = 2$.

**Proof.** Denote by $Z^n$ the space of all $d$-closed elements in $A^n$. It is clear that $Z^n$ is a Fréchet $g$- or $G$-submodule of $A^n$. We will need the following observation.
Let $a \in (\mathbb{Z}^n)^\mathfrak{g}$. Then $c: \mathfrak{g} \to \mathbb{Z}^{n-1}$, defined as $c(\xi) = i_\xi a$, is a $\mathfrak{g}$-cocycle.

This fact an immediate consequence of the identity $i_{[\xi,\zeta]} a = L_\xi a - L_\zeta a$ combined with the assumption that $a$ is closed and $\mathfrak{g}$-invariant.

Let us prove (i). We have to show that

$$i_\xi a = 0 \text{ for all } \xi \in \mathfrak{g} \iff L_\xi a = 0 \text{ for all } \xi \in \mathfrak{g},$$

when $a \in \mathbb{Z}^1$. Clearly, $da = 0$ and $i_\xi a = 0$ yield $L_\xi a = 0$. Conversely, assume that the right hand condition holds. Then $c$ defined as above is a cocycle on $\mathfrak{g}$ with coefficients in $\mathbb{Z}^0$. Since $\mathbb{Z}^0$ is a trivial $\mathfrak{g}$-module, $H^1(\mathfrak{g}; \mathbb{Z}^0) = H^1(\mathfrak{g}) \otimes \mathbb{Z}^0 = 0$, and $c$ is exact. An exact one-cocycle with coefficients in a trivial module is identically zero. Hence $i_\xi a = 0$ for all $\xi \in \mathfrak{g}$.

The forgetful homomorphism is injective in degree one because $A^0_\mathfrak{g} = (A^0)^G$. Hence we just need to show that it is onto. Let $a \in A^1$ be $d$-closed. Without loss of generality we may assume that $a$ is $G$-invariant. (Indeed, since $G$ is compact, the mean $\int_G ga dg$ exists, is $G$-invariant and homologous to $a$ due to Cartan’s formula.) Then by (i), $a$ is also $d^G_\mathfrak{g}$-closed.

It remains to show that the forgetful homomorphism is surjective in degree two. Pick $a \in \mathbb{Z}^2$. As before, we may assume that $a$ is $G$-invariant, i.e., $a \in (\mathbb{Z}^2)^G$. Then $c(\xi) = i_\xi a$ is a cocycle on $\mathfrak{g}$ with values in $\mathbb{Z}^1$. Since $\mathfrak{g}$ is semisimple, $H^1(\mathfrak{g}) = 0$. Thus, by Theorem 2.13, $H^1(\mathfrak{g}; \mathbb{Z}^1) = H^1(\mathfrak{g}) \otimes \mathbb{Z}^1 = 0$, and $c$ is exact. In other words, there exists $b \in (\mathbb{Z}^1)^G$ such that $i_\xi a = L_\xi b$. It is clear that the element $a + i_\xi b \in A^2_\mathfrak{g}$ is $d^G_\mathfrak{g}$-closed and projects onto $a$ under the forgetful homomorphism.

Let us now turn to some general properties of equivariant cohomology. The following proposition is evident:

**Proposition 2.18** (The long exact sequence.) Let $0 \to B^* \to A^* \to C^* \to 0$ be an exact sequence of $G$-differential complexes. If $G$ is compact, the induced sequence

$$0 \to B^*_G \to A^*_G \to C^*_G \to 0$$

is also exact, and we have a long exact sequence for equivariant cohomology:

$$\ldots \to H^*_G(B^*) \to H^*_G(A^*) \to H^*_G(C^*) \to \ldots.$$  

**Remark 2.19** Let $A^* = \Omega^*(M)$, where $M$ is a smooth manifold. The splitting map $p: (M \times ET)/T \to (M \times EG)/G$ is homotopy equivalent to a fibration with fiber $G/T$. The map $p$ is known to induce a monomorphism $H^*_G(M) \to H^*_T(M)$ whose image is $H^*_T(M)^W$, where $W$ is the Weil group of $G$. This result can also be generalized to equivariant cohomology of $G$-differential complexes when $G$ is compact (cf., [DKV], p. 155). We omit the details since this fact, although very interesting, is irrelevant to the subject matter of this paper.

To introduce the standard spectral sequence for equivariant cohomology, consider the decreasing filtration

$$F_0 = A_G^* \supset F_1 \supset \ldots \supset F_n \supset F_{n+1} \supset \ldots$$

of the complex $A_G^*$ by its subcomplexes

$$F_n = \bigoplus_{2j \geq n} (A^* \otimes S^j \mathfrak{g}^*)^G.$$  

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Theorem 2.20 The spectral sequence of the filtration $F_p$ converges to $H^*(A^*)$. If $G$ is compact,

$$E_1^{pq} = E_2^{pq} = H^q(A^*) \otimes (S^{p/2}g^*)^G$$

when $p$ is even and $E_1^{pq} = E_2^{pq} = 0$ when $p$ is odd. Furthermore,

$$d_2([a] \otimes \phi)(\xi) = [i_\xi a] \phi(\xi),$$

where $\phi \in S^*g^*$ and $[a]$ is the cohomology class of $a \in A^*$.

Remark 2.21 The $E_2$-term can be sometimes found even when $G$ is not compact. For example, if $G$ is abelian, we have $E_2^{pq} = H^q((A^*)^G) \otimes S^{p/2}g^*$. (Generically zero!) It is clear that in general $E_2 \neq H^*(A^*) \otimes (S^*g^*)^G$.

Theorem 2.17 can be derived from Theorem 2.20. However, a direct elementary proof given above seems more transparent.

When $A^* = \Omega^*(M)$, the spectral sequence turns into the Leray spectral sequence of the fibration $M \to (M \times EG)/G \to BG$.3

Proof. The convergence of $E_r$ is a general fact which holds for any spectral sequence arising from a filtered complex.

To evaluate the $E_2$-term, note that by definition $E_1^{pq}$ is the $(p + q)$-th cohomology of $E_0^{pq} = F_p^*/F_{p+1}^*$ with respect to the induced differential $d_0$. It is clear that $E_0^{pq}$ is just $(A^* \otimes S^{p/2}g^*)^G$ when $p$ is even and zero otherwise. The differential $d_0$ is the restriction of the original differential $d$ of $A^*$ extended to act trivially on $S^{p/2}g^*$. When $G$ is compact, we have

$$H^*((A^* \otimes S^{p/2}g^*)^G) = H^*(A^* \otimes S^{p/2}g^*)^G.$$

By the definition of $d_1$, the right hand side is just $(H^*(A^*) \otimes S^{p/2}g^*)^G$. Furthermore, since $A^*$ is a differential $G$-module, $G$ acts trivially on the cohomology of $A^*$. Thus $E_1^{pq} = H^q(A^*) \otimes (S^{p/2}g^*)^G$. Clearly, $d_1 = 0$. Hence, $E_2 = E_1$ and $d_2$ is simply induced by the differential $d_G$.

2.5 The second spectral sequence for equivariant cohomology

In this section we introduce a spectral sequence relating the equivariant cohomology of a differential $G$-complex $A^*$, the cohomology of $G$, and the cohomology of $A^*$. To be more precise, the spectral sequence converges to $H^*(A^*)$ and has the product $H^*(g) \otimes H^*_G(A^*)$ as its $E_2$-term. This result will be used in Section 5 to construct a spectral sequence in Poisson cohomology associated with an equivariant momentum mapping.

---

3As a matter of tradition, when the $E_2$-term of the Leray spectral sequence of a fibration is expressed as the tensor product, the cohomology of the base is taken first and the cohomology of the fiber second. This convention is systematically broken throughout the paper. To follow the rule, one would have to switch the order in the tensor product, i.e., to start with $(S^*g^* \otimes A^*)^G$, so that $E_2$ would become $(S^*g^*)^G \otimes H^*(A^*)$. Of course, the results obtained for different orders are canonically isomorphic. The same applies to the spectral sequence introduced in the next section.
Example 2.22 Let $M$ be a smooth manifold acted on by a compact group $G$. As in Example 2.3, take $A^* = \Omega^*(M)$. If the action is free, the natural projection $M \times EG \to (M \times EG)/G$ is a principal $G$-bundle. The Leray spectral sequence of this bundle converges to $H^*(M)$ because $EG$ is contractible and has $E_2 = H^*(G) \otimes H^*_G(M)$. The same is true when the action is just locally free.

Theorem 2.23 Assume that $G$ is compact. There exists a spectral sequence converging to $H^*(A)$ and such that $E^{pq}_1 = H^q(g) \otimes (A^* \otimes W^*(g))^p_0$ and $E^{pq}_2 = H^q(g) \otimes H^p_G(A^*)$.

Remark 2.24 As clear from the proof, when $G$ is not compact, one still has $E^{pq}_1 = H^q(g; (A^* \otimes W^*(g))_0)$, where $(A^* \otimes W^*(g))_0$ is the submodule of $A^* \otimes W^*(g)$ formed by $a \otimes w$ which have zero contraction, $i_\xi(a \otimes w) = 0$, with every $\xi \in g$.

Proof. The $G$-differential complex $B^* = A^* \otimes W^*(g)$ carries the following decreasing filtration by subcomplexes:

$$F^{p+q}_p = \{ b \in B^{p+q} \mid i_{\xi_1} i_{\xi_2} \ldots i_{\xi_{q+1}} b = 0 \text{ for all } \xi_1, \ldots, \xi_{q+1} \in g \}.$$

In degree $n$, the smallest nonzero term $F^n_p$ of this filtration is formed by $b \in B^n$ such that $i_\xi b = 0$ for all $\xi \in g$. The spectral sequence $E_r$ in question is just that associated with the filtration. It is clear that $E_r$ converges to $H^*(A)$ because $W^*(g)$ is acyclic.

We claim that $E^{pq}_1 = F^{p+q}_p/F^{q+1}_p$ is naturally isomorphic to $C^q(g; F^p_p) = \text{Hom}(\wedge^q g, F^p_p)$. For $b \in F^{p+q}_p$, let

$$c_b(\xi_1, \ldots, \xi_q) = i_{\xi_1} \ldots i_{\xi_q} b.$$

By the definition of the filtration, $c_b(\xi_1, \ldots, \xi_q) \in F^p_p$. Thus $c_b$ can be viewed as a linear mapping $\wedge^q g \to F^p_p$. Furthermore, $c$ is also linear in $b$ and $c_b \equiv 0$ if and only if $b \in F^{p+q}_p$.

This means that $b \mapsto c_b$ is a monomorphism $E^{pq}_0 \to C^q(g; F^p_p)$. This monomorphism is actually an epimorphism, and so an isomorphism. To see this, let us pick $\phi \in C^q(g; F^p_p)$ and show that $\phi = c_b$ for some $b \in F^{p+q}_p$. In effect, we may just take $b = \sum \phi(e_j) \otimes e_j^*$, where $e_1, \ldots, e_r$ is a basis in $\wedge^q g$ and $e_1^*, \ldots, e_r^*$ is the dual basis.

The next step is to prove that the differential $d_0$ is, up to a sign, the Chevalley–Eilenberg differential $d_{\text{Lie}}$ on $C^*(g; F^p_p)$ with $F^p_p$ equipped with its natural $g$-module structure inherited from $B^*$. (Note in this connection that $F^p_p$ is indeed a submodule of $B^*$ as follows from (ii'): $i_{[\xi, \zeta]} = L_\xi \zeta - \zeta L_\xi$.) This is equivalent to showing that $c_{db} = -d_{\text{Lie}} c_b$, which is easy to verify by a straightforward calculation. Namely, using Cartan’s formula, we obtain

$$c_{db}(\xi_1, \ldots, \xi_{q+1}) = i_{\xi_{q+1}} \ldots i_{\xi_q} db = -i_{\xi_{q+1}} \ldots i_{\xi_2} di_{\xi_1} b + i_{\xi_{q+1}} \ldots L_{\xi_1} db = i_{\xi_{q+1}} \ldots i_{\xi_2} di_{\xi_1} b - i_{\xi_{q+1}} \ldots i_{\xi_2} L_{\xi_1} i_{\xi_1} b + i_{\xi_{q+1}} \ldots L_{\xi_1} db$$

$$= \sum_{1 \leq k \leq q+1} (-1)^{k-1} i_{\xi_{q+1}} \ldots i_{\xi_{k+1}} L_{\xi_k} i_{\xi_{k-1}} \ldots i_{\xi_1} b + (-1)^q d_{\xi_{q+1}} \ldots i_{\xi_1} b$$

$$= \sum_{1 \leq k \leq q+1} (-1)^{k-1} i_{\xi_{q+1}} \ldots i_{\xi_{k+1}} L_{\xi_k} i_{\xi_{k-1}} \ldots i_{\xi_1} b.$$
where the last term vanishes because \( b \in F^p_{p+q} \). Employing (ii'), we can switch \( L_{\xi_k} \) and the contractions so that to have \( L_{\xi_k} \) applied last:

\[
\begin{aligned}
c_{db}(\xi_1, \ldots, \xi_{q+1}) &= - \sum_{1 \leq j < \ell \leq q+1} \sum_{1 \leq k \leq q+1} (-1)^{j-1} \xi_{\ell+1} \cdots \xi_{\xi_{j+1}} \cdot i_{[\xi_j, \xi_{\ell}]} \xi_{\xi_{j-1}} \cdots \xi_j \cdot i_{\xi_1} b \\
&\quad + \sum_{1 \leq k \leq q+1} (-1)^{k-1} L_{\xi_k} \xi_{q+1} \cdots \xi_1 b \\
&= \sum_{1 \leq j < \ell \leq q+1} \sum_{1 \leq k \leq q+1} (-1)^{|j\ell|} \xi_{\ell+1} \cdots \xi_\ell \cdots \xi_j \cdots \xi_1 b \\
&\quad + \sum_{1 \leq k \leq q+1} (-1)^{k-1} L_{\xi_k} \xi_{q+1} \cdots \xi_1 b \\
&= -d_{\text{Lie}} c_b(\xi_1, \ldots, \xi_{q+1}).
\end{aligned}
\]

The \( E_1 \) term is now easy to find: \( E_1^{pq} = H^q(\mathfrak{g}; F^p_{p+q}) \). By Theorem 2.13, this is just \( H^q(\mathfrak{g}) \otimes (F^p_{p+q})^G \) because \( G \) is compact and \( F^p_{p+q} \) is clearly a Fréchet \( G \)-module. Finally, recall that \( F^p_{p+q} \) is formed by \( b \in B^p \) such that \( i_\xi b = 0 \) for any \( \xi \in \mathfrak{g} \). Thus \((F^p_{p+q})^G = B^p_0 \), which yields \( E_1^{pq} = H^q(\mathfrak{g}) \otimes (A^* \otimes W^*(\mathfrak{g}))^G_0 \).

It remains to prove that \( d_1 \) is just the natural differential \( d \) inherited by \( B^*_0 \) from \( B^* \) and acting on the second term only in the tensor product expression for \( E_1^{pq} \). To this end, pick a representative \( \phi \otimes f \) of an element of \( E_1^{pq} \) such that \( \phi \) is a \( q \)-cocycle on \( \mathfrak{g} \) and \( f \in B^p_0 \). We need to show that \( d_1(\phi \otimes f) = \phi \otimes df \). Recall that \( d_1 \) is induced by \( d = d_A \pm d_W \), where \( d_W = d_L - \delta \). (See Section 2.2.) Since \( \phi \) is a cocycle, \( d_L \phi = 0 \), and so \( d(\phi \otimes f) = \pm(\delta \phi) \otimes f + \phi \otimes df \). The first term \((\delta \phi) \otimes f \) belongs to \( F^p_{p+q+1} \), because \( f \) is basic. Thus it makes no contribution into \( d_1 \), which therefore has the desired form. As a result, \( E_2^{pq} = H^q(\mathfrak{g}) \otimes H^p_0(B^*) = H^q(\mathfrak{g}) \otimes H^p_G(A^*) \).

One can also start with a filtration of \( A^* \) instead of the filtration of \( A^* \otimes W^* \) considered above. Namely, let us take

\[
F^p_{p+q} = \{ a \in A^{p+q} \mid i_{\xi_1} i_{\xi_2} \cdots i_{\xi_{q+1}} a = 0 \text{ for all } \xi_1, \ldots, \xi_{q+1} \in \mathfrak{g} \}.
\]

This filtration gives rise to a spectral sequence converging to \( H^*(A^*) \) but having in general an absolutely incomprehensible structure. However, if \( A^* \) is locally free, the spectral sequence is even simpler than that of Theorem 2.23 and essentially equivalent to it.

**Example 2.25** In the notation of Example 2.22, the projection \( M \to M/G \) gives rise to a filtration on \( \Omega^*(M) \) but rather little can be said about the resulting spectral sequence except that it converges to \( H^*(M) \) when no extra assumption on the action is made. However, if the action is locally free, we have \( E_1^{pq} = H^q(G) \otimes \Omega^p_0(M) \) and \( E_2^{pq} = H^q(G) \times H^p(M/G) \). In fact, the spectral sequence of this projection coincides with that from Example 2.22 beginning with the \( E_2 \)-term. When the action is free, the fibration \( M \to M/G \) is homotopy equivalent to \( M \times EG \to (M \times EG)/G \).

The proof of our next theorem is omitted, for it is quite similar to that of Theorem 2.23 (cf., the proof of Theorem 2.13).
Assume that $G$ is compact and $A^*$ is a locally free $G$-differential algebra. The spectral sequence associated with the filtration (3) converges to $H^*(A^*)$ and has $E_1^{pq} = H^q(g) \otimes A^*_b$ and $E_2^{pq} = H^q(g) \otimes H^p_b(A^*)$.

**Remark 2.27** In fact, Theorem 2.26 follows from Theorem 2.23 under a very natural extra hypothesis that $A^*$ is a $G$-differential algebra with unit. Indeed, then $B^* = A^* \otimes W^*(g)$ becomes a locally free $G$-differential algebra, and applying Theorem 2.26 to $B^*$, we obtain Theorem 2.23. Furthermore, it is not hard to extend Theorem 2.26 to complexes over locally free $G$-differential algebras. Then it would yield Theorem 2.23 in its full generality.

### 3 Some results from Poisson geometry

This section is a brief review on the geometry of Poisson manifolds heavily biased toward our present needs. In addition to the standard material, which can be found, for example, in [V], we also prove some new theorems and recall the results of [Gi2] essential for what follows.

#### 3.1 Poisson manifolds and momentum mappings

Let $P$ be a Poisson manifold. Denote by $\mathcal{X}^*(P)$ the algebra of multi-vector fields on $P$. Recall that the space of one-forms $\Omega^1(P)$ on $P$ is a (local) Lie algebra with the bracket

$$\{\alpha, \beta\} = d\langle \alpha \wedge \beta, \pi \rangle + i_{\pi^\#}d\beta - i_{\pi^\#}d\alpha$$

where $\alpha$ and $\beta$ are one-forms, $\langle \cdot, \cdot \rangle$ denotes the pairing between $k$-forms and $k$-vector fields and $\pi^\#: \Omega^1(P) \to \mathcal{X}^1(P)$ is the paring with $\pi$, i.e., $\beta(\pi^\# \alpha) = \langle \beta \wedge \alpha, \pi \rangle$. Note that $\pi^#$ and $d:\mathcal{C}^{\infty}(P) \to \Omega^1(P)$ are Lie algebra homomorphisms and $\pi^#$ extends to an algebra homomorphism $\Omega^*(P) \to \mathcal{X}^*(P)$ by multiplicativity: $\pi^#(\alpha_1 \wedge \alpha_2) = \pi^#(\alpha_1) \wedge \pi^#(\alpha_2)$.

Following Bhaskara and Viswanath [BV], let us define the “Lie derivative” of a multi-vector field $w$ in the direction of a one-form $\alpha$ by Cartan’s identity:

$$L_\alpha w = i_\alpha d_\pi w + d_\pi i_\alpha w \quad (5)$$

where $d_\pi w = -[\pi, w]$, with $[\cdot, \cdot]$ being the Schouten bracket, and $i_\alpha w$ stands for the contraction of $\alpha$ with $w$. To be more precise, $i_\alpha w$ is characterized by the equation $\langle \beta, i_\alpha w \rangle = \langle \alpha \wedge \beta, w \rangle$.

As shown in [BV], the action $L$ makes the space of multi-vector fields $\mathcal{X}^*(P)$ into a module over $\Omega^1(P)$:

$$L_{\{\alpha, \beta\}} = L_\alpha L_\beta - L_\beta L_\alpha$$

and

$$i_{\{\alpha, \beta\}} = L_\alpha i_\beta - i_\beta L_\alpha \quad (6)$$

Furthermore, $L_\alpha$ satisfies the Leibniz identity

$$L_\alpha (w_1 \wedge w_2) = (L_\alpha w_1) \wedge w_2 + w_1 \wedge (L_\alpha w_2) \quad (7)$$

where $w_1$ and $w_2$ are multi-vector fields. In accord with [B] we have

$$L_\alpha f = L_{\pi^#_\alpha} f$$
for \( f \in C^\infty(P) \). The “Lie derivative” \( \mathcal{L}_\alpha \) of a vector field \( v \) is related to the standard Lie derivative along \( \xi = \pi^\# \alpha \) by the following equation verified in [Gi2]:

\[
\mathcal{L}_\alpha v = L_\xi v + \pi^\# i_v d\alpha .
\] (8)

As a consequence,

\[
\mathcal{L}_\alpha w = L_\xi w + \sum_{j=1}^q v_1 \wedge \ldots \wedge v_{j-1} \wedge (\pi^\# i_v j d\alpha) \wedge v_{j+1} \wedge \ldots \wedge v_q ,
\] (9)

where \( v_1, v_2, \ldots, v_q \) are vector fields and \( w = v_1 \wedge v_2 \wedge \ldots \wedge v_q \) [Gi2]. This shows that \( \pi^\# : \Omega^*(P) \to \mathcal{X}^*(P) \) is a homomorphism of \( \Omega^1(P) \)-modules, provided that the domain is given the \( \Omega^1(P) \)-action via the standard Lie derivative \( L \) while the range via \( \mathcal{L} \). In other words,

\[
\pi^\# L_{\pi^\# \alpha} = \mathcal{L}_\alpha \pi^\# \beta ,
\] (10)

where \( \alpha \in \Omega^1(P) \) and \( \beta \in \Omega^*(P) \).

With the forthcoming applications in mind, let us rewrite (9) in a more compact form. For \( w \in \mathcal{X}^q(P) \) consider a \( C^\infty(P) \)-linear homomorphism

\[
\tilde{i}_w : \Omega^q(P) \to \Omega^{q-1}(P) \otimes \mathcal{X}^{1}(P)
\]
defined by the formula

\[
\tilde{i}_w \beta = \sum_{j=1}^q (-1)^{j-1} i_{v_j} \beta \otimes v_1 \wedge \ldots \wedge v_{j-1} \wedge v_{j+1} \wedge \ldots \wedge v_q ,
\] (11)

where as above \( w = v_1 \wedge v_2 \wedge \ldots \wedge v_q \), together with the assumption that \( \tilde{i} \) is also \( C^\infty(P) \)-linear in \( w \). Extend \( \pi^\# : \Omega^1(P) \otimes \mathcal{X}^k(P) \to \mathcal{X}^{1+k}(P) \) so that \( \pi^\# (\beta \otimes w) = (\pi^\# \beta) \wedge w \). Then (9) turns into

\[
\mathcal{L}_\alpha w = L_\xi w + \pi^\# \tilde{i}_w d\alpha ,
\] (12)
an equation similar to (8). In Lemma 4.10 we shall use the following two simple properties of \( \tilde{i} \):

\[
\tilde{i}_w \alpha = i_\alpha w ,
\] (13)

where \( \alpha \in \Omega^1(P) \) and \( w \in \mathcal{X}^*(P) \), and

\[
\tilde{i}_w (\alpha_1 \wedge \alpha_2) = (-1)^{(k-1)} \alpha_2 \wedge \tilde{i}_w \alpha_1 + (-1)^{k} \alpha_1 \wedge \tilde{i}_w \alpha_2 ,
\] (14)

where \( \alpha_1 \in \Omega^1(P) \) and \( \alpha_2 \in \Omega^1(P) \).

In the same vein, one can define the “Lie derivative” \( \mathcal{L}_\alpha \beta \) of a differential from \( \beta \in \Omega^*(P) \) in the direction of a one-form \( \alpha \). (See [BV], [V], and also [Gi2] for a more detailed treatment.) When \( \beta \) is a one-form, the derivative is given by (1): \( \mathcal{L}_\alpha \beta = \{ \alpha, \beta \} \). In general, \( \mathcal{L}_\alpha \) satisfies the Leibniz identity on forms (cf., (7)). Finally, \( \mathcal{L}_\alpha \) of \( w \in \mathcal{X}^k(P) \) and \( \beta \in \Omega^k(P) \) are related by the following formula from [Gi2]

\[
\mathcal{L}_\alpha \langle \beta, w \rangle = \langle \mathcal{L}_\alpha \beta, w \rangle + \langle \beta, \mathcal{L}_\alpha w \rangle .
\] (15)

Let now \( P \) be acted on by a group \( G \). Denote by \( a : g \to \mathcal{X}^1(P) \) the infinitesimal action homomorphism. The action is said to be cotangential if \( a \) can be lifted to \( \Omega^1(P) \) as a linear
map, i.e., there exists a linear map $\tilde{a}: g \to \Omega^1(P)$ such that $a = \pi^\# \circ \tilde{a}$. We call $\tilde{a}$ a cotangent lift or a pre-momentum mapping. A pre-momentum mapping $\tilde{a}$ is said to be an equivariant if $\tilde{a}$ is an anti-homomorphism of Lie algebras.

It is clear that an action with an equivariant pre-momentum mapping is cotangential and a cotangential action is tangential, i.e., tangent to the symplectic foliation of $\pi$. (The converse is not true [Gi2].) Note that here we do not require the action to preserve $\pi$ or even to be Poisson. The problem of existence and uniqueness of an (equivariant) pre-momentum mapping is analyzed in [Gi2] in detail. In particular, it is shown that a cotangential action of a compact group with $H^2(g) = 0$ admits an equivariant pre-momentum mapping.

Let $G$ be a Poisson Lie group and $G^*$ its dual group. Recall that $\pi_G$ vanishes at $e \in G$ and the linearization of $\pi_G$ at $e$ is a linear homomorphism $\delta: g \to g \wedge g$, which is a cocycle with respect to the adjoint action. The Lie algebra of $G^*$ is $g^*$ with the bracket defined as $\delta^*: g^* \wedge g^* \to g^*$ or equivalently via (1). Thus every $\xi \in g$ can be thought of as an element of $(g^*)^* = T_e G^*$. Denote by $\theta_\xi$ the left-invariant one-form on $G^*$ extending $\xi$. Note that the dressing action of $\xi$ on $G^*$ is given by the vector field $- (\pi_G^\#)^\xi \theta_\xi$.

Recall that $G$ is said to act on $P$ is a Poisson way if the action map $G \times P \to P$ is Poisson. The following is a very convenient infinitesimal criterion for an action of a connected group to be Poisson [LM]: An action is Poisson if and only if, for all $\xi \in g$,

$$d_\pi a(\xi) = a \delta(\xi) \; . \quad (16)$$

Here we use the same notation $a$ for the infinitesimal action $g \to \mathcal{X}^1(P)$ and for the induced homomorphisms $\wedge^* a: \wedge^* g \to \mathcal{X}^*(P)$.

As defined by Lu in [Lu1], a momentum mapping for a Poisson action is a map $\mu: P \to G^*$ such that

$$a(\xi) = - \pi^\# \mu^* \theta_\xi \; . \quad (17)$$

A momentum mapping is said to be equivariant if it is such with respect to the dressing action or, equivalently, if it is Poisson [Lu1]. For example, the identity map is an equivariant momentum mapping for the dressing action. An (equivariant) momentum mapping gives rise to an (equivariant) cotangential lift by the formula $\tilde{a}(\xi) = - \mu^* \theta_\xi$. Thus an action with a momentum mapping is necessarily tangential and even cotangential.

The following theorem will play a crucial role in the subsequent analysis. Recall that once a pre-momentum mapping $\tilde{a}$ is fixed, the spaces $\mathcal{X}^*(P)$ and $\Omega^*(P)$ become $g$-modules via $L_{\tilde{a}(-)}$. (Note that the structure of $g$-modules may depend on the choice of $\tilde{a}$.) Assume that $P$ is equipped with a genuine, not only infinitesimal, $G$-action.

**Theorem 3.1**

(i) [Gi2] The $g$-modules $\mathcal{X}^*(P)$ and $\Omega^*(P)$ integrate to representations the universal covering $\tilde{G}$ of $G$. The resulting $\tilde{G}$-action $\mathcal{X}^*(P)$ commutes with $d_\pi$ and induces a trivial $\tilde{G}$-action on $H^*(P)$ (see Section 3.2).

(ii) Assume that $G$ is compact and $\tilde{a}$ is associated with an equivariant momentum mapping. Then the infinitesimal action integrates to representations on $\mathcal{X}^*(P)$ and $\Omega^*(P)$ of a finite covering $\tilde{G}$ of $G$.

**Proof.** Assertion (i) was proved in [Gi2]. By (17), it is sufficient to prove (ii) for $\mathcal{X}^*(P)$. Recall that some finite covering $\tilde{G}$ of $G$ can be decomposed as $T \times K$, where $T$ is a torus and
$K$ is a compact simply connected (semisimple) Lie group. We claim that the infinitesimal action integrates to a representation of $G$.

A straightforward calculation with the cocycle $\delta$ shows that $\pi^G$ vanishes along $T$, and so $T$ is a Poisson subgroup of $G$. (Note that in contrast with $T$, the subgroup $K$ does not, in general, inherit the structure of a Poisson Lie group, and we cannot apply (i) to it.) Furthermore, $t^*$ is an abelian subalgebra in $g^*$ and we have a Poisson projection $G^* \to t^*$ (cf., [LW]). The composition of this projection with the momentum mapping is an equivariant momentum mapping for the induced $T$-action. Clearly, $d\tilde{\alpha}(\xi) = 0$ for $\xi \in t$, which yields, by (12), $L_{\tilde{\alpha}(\xi)} = L_{\tilde{\alpha}(\xi)}$ for $\xi \in t$. This shows that the infinitesimal $t$-action on $X^{\star}(P)$ and $\Omega^{\star}(P)$ integrates to a $T$-action. Now it is a routine to conclude that the $g$-modules integrate to modules over $T \times K$.

\[\blacksquare\]

We will also need the following result (cf., equation (16)).

Lemma 3.2 Let $\tilde{\alpha}$ arise from an equivariant momentum mapping. Then, for every $\xi \in g$,

\[d\tilde{\alpha}(\xi) = \tilde{\alpha}(\delta(\xi)),\] (18)

where as before, the same notation $\tilde{\alpha}$ is used for the pre-momentum mapping and for the induced maps $\wedge g \to \Omega(P)$.

Proof. It is clear that equation (18) holds on $G^*$. (See, e.g., [Lu1].) On $P$, (18) is just the pull-back of the equation on $G^*$.

\[\blacksquare\]

Remark 3.3 Assume that $G$ is compact or semisimple. Then the cocycle $\delta$ is exact: there exists $r \in \wedge^2 g$ such that $\delta(\xi) = [\xi, r]$. As a consequence, $d\tilde{\alpha}(\xi) = L_{\tilde{\alpha}(\xi)}\tilde{\alpha}(r)$, when $\tilde{\alpha}$ arises from an equivariant momentum mapping.

It is well-known that $r$ exists when $g$ is semi-simple: any one-cocycle on a finite-dimensional semisimple Lie algebra is exact due to Whitehead’s lemma. (See, e.g., [Gu], Section II.11.) Assume that $G$ is compact. The function $\Delta: G \to \wedge^2 g$, defined as $\Delta(g) = (R_g^{-1})_*\pi_g$, where $R_g$ stands for the left translation, is a nonhomogeneous cocycle on $G$ with respect to the adjoint action. (See, e.g., [LW] and references therein.) Since $G$ is compact, any cocycle is exact [Gu]. The cocycle $\delta$ is the image of $\Delta$ under the natural homomorphism from the cochains on $G$ to the cochains on $g$. Thus $\delta$ is exact as well. Furthermore, we can even produce an explicit expression for $r$ by applying a homotopy formula from [Gu]:

\[r = \int_G \text{Ad}_{g^{-1}}(R_g^{-1})_*\pi_g dg.\]

Rather little seems to be known on the existence of equivariant momentum mappings when the Poisson structure on $G$ is nontrivial. (See a discussion in [Gi2].) When $P$ is symplectic and simply connected, nonequivariant momentum mappings $P \to G^*$ exist and are parameterized by elements of $G^*$ [Lu1]. If $G$ is semisimple, an equivariant momentum mapping $P \to G^*$ is at most unique [Gi2].
3.2 Poisson cohomology

In this section we recall the definition and some of the properties of the Poisson cohomology spaces, a notion introduced by Lichnerowicz in [Li]. (A general introduction can be found, for example, in [Va]. A detailed discussion more oriented toward our present goals is given in [Gi2].)

Since $[\pi, \pi] = 0$, the operation $d\pi = -[, \pi]: X^{*}(P) \to X^{*+1}(P)$ is a differential: $d^{2}\pi = 0$.

The Poisson cohomology $H^{\star}_{\pi}(P)$ is the cohomology of the complex $(X^{\star}(P), d\pi)$.

Example 3.4 The homomorphism $\pi^{\#}: \Omega^{\star}(P) \to X^{\star}(P)$ commutes with the differentials up to a sing, and so induces a homomorphism $\pi^{\#}: H^{\star}(P) \to H^{\star}_{\pi}(P)$. When $P$ is symplectic, $\pi^{\#}$ is an isomorphism on the level of complexes and in cohomology. On the other hand, $\pi^{\#} = 0$ when $\pi = 0$, and $H^{\star}_{\pi}(P) = X^{\star}(P)$. In general, Poisson cohomology shares the properties of de Rham cohomology and multi-vector fields on $P$. In effect, the Poisson cohomology classes on $P$ can be thought of as multi-vector vector fields on $P$ in the “Poisson category” [GL].

Example 3.5 Interpretations of Poisson cohomology:

(0) $H^{0}_{\pi}$ is the algebra of the so-called Casimir functions, i.e., functions constant on the leaves of the symplectic foliation. Note that $H^{*}_{\pi}(P)$ is an algebra over $H^{0}_{\pi}(P)$.

(1) $H_{\pi}^{1}(P) = \text{Poiss/\text{Ham}}$ is a Lie algebra (over $\mathbb{R}$). It is the quotient of the Lie algebra of $\pi$-preserving vector fields, called Poisson, over the ideal of Hamiltonian vector fields.

(2) $H^{2}_{\pi}(P)$ is the space of infinitesimal deformations of $\pi$ modulo those given by infinitesimal diffeomorphisms.

The following simple result is a very particular case of a general interpretation due to Lu [Lu2] of the Poisson cohomology of Poisson homogeneous spaces via certain Lie algebra cohomology. When $\pi_{G} = 0$, it was obtained by Koszul [Ko].

Proposition 3.6 Let $G$ be a Poisson Lie group and let $U$ be an open $G$-invariant subset of $G^{*}$. Then

$$H^{\star}_{\pi}(U) = H^{\star}(\mathfrak{g}; C^{\infty}(U)),$$

where $C^{\infty}(U)$ is made into a $\mathfrak{g}$-module by means of the dressing action of $\mathfrak{g}$ on $G^{*}$.

In effect, we again have an isomorphism of complexes: $\mathcal{X}^{*}(U) \simeq \wedge^{*}\mathfrak{g}^{*} \otimes C^{\infty}(U)$. When $G$ is compact, the cohomology can be calculated explicitly.

Theorem 3.7 [GW]. Assume that $G$ is compact. Then $H^{\star}_{\pi}(U) = H^{\star}(\mathfrak{g}) \otimes (C^{\infty}(U))^{G}$.

The theorem follows immediately from Corollary 2.15 and Proposition 3.6 we just stated.

4 Equivariant Poisson cohomology

4.1 Basic properties of equivariant Poisson cohomology

Let $(P, \pi)$ be a Poisson manifold with an infinitesimal action of a Lie group $G$. Assume that the action admits an equivariant pre-momentum mapping $\tilde{\alpha}: \mathfrak{g} \to \Omega^{1}(P)$.

As we have shown above, $\mathcal{X}^{*}(P)$ is a $\mathfrak{g}$-module with the $\mathfrak{g}$-action defined via $\mathcal{L}_{\tilde{\alpha}()}$. It is easy to see that $\mathcal{X}^{*}(P)$ equipped with the differential $d_{\pi}$ becomes a $\mathfrak{g}$-differential complex.
if we set \( i_\xi w = i_{\tilde{a}(\xi)}w \) where \( \xi \in \mathfrak{g} \) and \( w \in \mathfrak{X}^*(P) \). We call the \( G \)-equivariant cohomology of this complex the \textit{equivariant Poisson cohomology} of \( P \) and denote it by \( H^*_{\pi,G}(P) \).

When \( G \) is a Poisson Lie group and the action is Poisson and has an equivariant momentum mapping \( \mu: P \to G^* \), the equivariant cohomology taken for \( \tilde{a}(\xi) = -\mu^*\theta_\xi \) is said to be associated with \( \mu \).

**Remark 4.1** We emphasize that \( H^*_{\pi,G}(P) \) is defined regardless of whether we have a genuine or infinitesimal \( G \)-action on \( P \). It is also irrelevant for the definition, but not for calculations, whether the action is Poisson or not. The only data required is \((P, \pi)\), the \( \pi \)-action, and \( \pi \)-equivariant pre-momentum mapping \( \tilde{a} \), which may exist even when the action is not Poisson. For example, \( \tilde{a} \) always exists and is unique when \( P \) is symplectic. The complex \( \mathfrak{X}^*(P)_{\mathfrak{g}} \) employed to define the equivariant cohomology depends on the choice of \( \tilde{a} \) and \( \tilde{a} \) is not unique in general. Furthermore, the very equivariant cohomology spaces appear to depend on \( \tilde{a} \). (Thus the notation \( \mathfrak{X}^*(P)_{\tilde{a}} \) and \( H^*_{\pi,\tilde{a}}(P) \) would be more appropriate.) However, it is not clear whether this still can occur when \( G \) is compact semisimple.

**Remark 4.2** Assume that \( P \) is given a genuine \( G \)-action. By Theorem 3.1(i), the structure of \( \mathfrak{g} \)-module on \( \mathfrak{X}^*(P) \) via \( \mathcal{L}_{\tilde{a}} \) integrates to a \( G \)-module.

Moreover, it may happen that \( \mathfrak{X}^*(P) \) is in effect a \( G \)-module. This is the case, for example, when \( P \) is symplectic or when the \( G \)-action preserves \( \pi \) and \( \tilde{a} \) arises from an equivariant momentum mapping or just the one-forms \( \tilde{a}(\xi) \) are closed. Knowing that \( \mathfrak{X}^*(P) \) is a \( G \)-differential complex may sometimes simplify the calculation of equivariant cohomology. When \( G \) is compact and \( \tilde{a} \) is associated with an equivariant momentum mapping, we can always assume from the very beginning that \( \mathfrak{X}^*(P) \) is a \( G \)-module due to Theorem 3.1(ii). Indeed, the replacement of \( G \) by its finite covering \( \tilde{G} \) has no effect on the equivariant cohomology.

**Example 4.3** As with ordinary Poisson cohomology, \( \pi^\# \) induces a homomorphism

\[
H^*_G(P) \longrightarrow H^*_{\pi,G}(P).
\]

Similarly to Example 3.4, \( \pi^\# \otimes id \) is an isomorphism of the complexes \( \Omega^*(P)_{\mathfrak{g}} \to \mathfrak{X}^*(P)_{\mathfrak{g}} \) when \( P \) is symplectic, and \( H^*_{\pi,G}(P) \simeq H^*_G(P) \).

The geometrical meaning of the equivariant Poisson cohomology will be clarified in the next two sections. Here we just point out that \( H^*_{\pi,G}(P) \) has nothing to do with the Poisson cohomology of \( P/G \) even when the action is free. As shown in Section 4.3, the equivariant Poisson cohomology is related to the cohomology of the \( G \)-invariant multi-vector fields tangent to the fibers of the momentum mapping.

**Example 4.4** Take \( P = \mathbb{C}^n \setminus 0 \) with its standard symplectic structure and the standard \( G = \mathbb{T}^1 \)-action by rotations. Then \( H^*_{\pi,G}(P) \simeq H^*_G(P) \simeq H^*(P/G) = H^*(\mathbb{C}P^{n-1}) \) because \( P \) is symplectic. On the other hand, one can show that \( H^k_\pi(P/G) = C^\infty(\mathbb{R}^+) \) when \( k = 0, 2n - 1 \) and zero otherwise.

All the results of Section 3 apply to equivariant Poisson cohomology. In particular, \( H^*_{\pi,G}(P) \) is an algebra over \( (S^*\mathfrak{g}^*)^G \).
Example 4.5 Torus actions. Let $G = \mathbb{T}^n$ be an $n$-dimensional torus $\mathbb{T}^1 \times \ldots \times \mathbb{T}^1$, acting on $P$ with an equivariant pre-momentum mapping. Denote by $(\mathcal{X}^*(P))^\mathfrak{g}$ the subcomplex of $\mathcal{X}^*(P)$ formed by multi-vector fields $w$ with $\mathcal{L}_{\tilde{\alpha}}(\xi)w = 0$. Let us identify $S^*\mathfrak{t}^*$ with $\mathbb{R}[u_1, \ldots, u_n]$. As follows from the definition, $H^*_{\pi, \mathfrak{t}}(P)$ is the cohomology of the complex $\mathcal{X}^*(P)^\mathfrak{g} \otimes S^*\mathfrak{t}^*$ with the differential $d_\pi = \sum u_j i_{\tilde{\alpha}(\xi_j)}$, where $\xi_j$ is the generator of $t_j$.

Remark 4.6 In a similar way, one can introduce the algebroid equivariant cohomology. The data required are an infinitesimal action of $G$ on the underlying manifold $M$ of an algebroid $\mathcal{A}$ and its lift to an action on $\mathcal{A}$ satisfying the natural axioms. This is sufficient to make the complex $\wedge^* \mathcal{A}^*$ used in the definition of algebroid cohomology into a $\mathfrak{g}$-differential algebra.

Remark 4.7 It is interesting that $\mathfrak{g}$-differential complexes arise in Poisson geometry in other ways as well. For example, as recently pointed out by Jian-Hua Lu, the complex of Poisson homology $H^*_{\pi}(P)$ with the differential induced by the de Rham differential on $P$ is a $\mathfrak{g}$-differential complex over the first Poisson cohomology Lie algebra $\mathfrak{g} = H^1_{\pi}(P)$. More generally, $H^*_{\pi}(P)$ is a $\mathfrak{g}$-differential complex over the super Lie algebra $H^*_{\pi}(P)$. This is similar (and equivalent when $\pi = 0$) to $\Omega^*(P)$ being a $\mathfrak{g}$-differential complex over $\mathfrak{g} = \mathcal{X}^1(P)$ or $\mathfrak{g} = \mathcal{X}^*(P)$. Although the case of $\pi = 0$ does not seem to yield any noteworthy results, the general case may be more interesting.

4.2 Equivariant Poisson cohomology of low degrees

Let us start by explicitly writing down the beginning of $\mathcal{X}^*_\mathfrak{g}$:

$$0 \longrightarrow \mathcal{X}^0(\mathfrak{g}) \xrightarrow{d_{\mathcal{L}_{\tilde{\alpha}}} = d_\pi} \mathcal{X}^1(\mathfrak{g}) \xrightarrow{d_{\mathcal{L}_{\tilde{\alpha}}}} \mathcal{X}^2(\mathfrak{g}) \xrightarrow{d_{\mathcal{L}_{\tilde{\alpha}}}} \ldots$$

It is easy to see that $\mathcal{X}^0(\mathfrak{g})$ is the space of smooth $G$-invariant functions, since $L_{\pi\#\tilde{\alpha}(\xi)} = \mathcal{L}_{\tilde{\alpha}}(\xi)$ on $C^\infty(P)$. Clearly $\text{Cas} \subseteq C^\infty(P)^G$ and, in this degree, $d_G = d_\pi$. Thus

- $H^0_{\pi, G}(P) = H^0_{\pi}(P) = \text{Cas}$ is just the algebra of Casimir functions, and so $H^*_{\pi, G}(P)$ is an algebra over $\text{Cas}$.

Clearly, $\mathcal{X}^1(\mathfrak{g})$ is the space $\mathcal{X}^1(\mathfrak{g})^\mathfrak{g}$ of vector fields invariant with respect to $\mathcal{L}_{\tilde{\alpha}}$. The differential on this space is $d_Gw = d_\pi w + i_{\tilde{\alpha}(\cdot)} w$. Finally, in the same notation, $\mathcal{X}^2(\mathfrak{g}) = \mathcal{X}^2(\mathfrak{g})^\mathfrak{g} \oplus (C^\infty(P) \otimes \mathfrak{g}^*)^G$.

Theorem 4.8

(i) Let $H^1(\mathfrak{g}) = 0$. Then

$$H^1_{\pi, G}(P) = \{ w \in \mathcal{X}^1(P) \mid d_\pi w = 0 \text{ and } i_{\tilde{\alpha}(\xi)} w = 0 \text{ for all } \xi \in \mathfrak{g} \} / d_\pi(C^\infty(P)^G)$$

(ii) Assume that $\tilde{\alpha}$ arises from an equivariant momentum mapping $\mu : P \rightarrow G^*$. Then $H^1_{\pi, G}(P)$ is the quotient of the space of $G$-invariant Poisson vector fields tangent to the $\mu$-fibers over the space of Hamiltonian vector fields with $G$-invariant Hamiltonians.

(iii) Let $G$ be compact semisimple. Then the forgetful homomorphism $H^*_{\pi, G}(P) \rightarrow H^*_{\pi}(P)$ is an isomorphism in degree one and an epimorphism in degree two.
Remark 4.9 In other words, under the hypothesis of (ii), $H^1_{\pi,G}(P)$ is the quotient of vector fields whose (local) flows preserve $\pi$, the action, and $\mu$ over all such Hamiltonian vector fields. Assertion (ii) gives an interpretation of in the spirit of Example 3.5.

Proof. Assertion (i) is an immediate consequence of Theorem 2.17. To prove (iii), recall that by Theorem 3.1 the $\mathfrak{g}$-module structure on $\mathcal{X}^*(P)$ integrates to a $\bar{G}$-module for some finite covering $\bar{G}$ of $G$. According to our definition, $H^*_{\pi,G}(P) = H^*_{\pi,G}(P)$. The covering $\bar{G}$ is compact semisimple because so is $G$, and (iii) follows from the second assertion of Theorem 2.17.

Let us prove (ii). It suffices to show that in $\mathcal{X}^1(P)^g$ every cohomology class can be represented by a $G$-invariant Poisson vector field. Since $L_{\bar{a}(\xi)}w = 0$, this is a consequence of the following observation:

Lemma 4.10 Assume that $\bar{a}$ arises from an equivariant momentum mapping $\mu$. Let $w \in \mathcal{X}^*(P)$ be tangent to the fibers of $\mu$, i.e., $i_{\bar{a}(\xi)}w = 0$ for all $\xi \in \mathfrak{g}$. Then $L_{\bar{a}(\xi)}w = L_{\bar{a}(\xi)}w$ for all $\xi \in \mathfrak{g}$.

Proof. First recall that

$$L_{\bar{a}(\xi)}w - L_{\bar{a}(\xi)}w = \pi^# i_w \bar{d}\bar{a}(\xi)$$

due to (12). Thus we just need to show that the right hand side is zero. Since $\bar{a}$ arises from an equivariant momentum mapping, Lemma 3.2 yields $d\bar{a}(\delta(\xi)) = \bar{a}(\delta(\xi))$ for $\delta(\xi) \in \wedge^2 \mathfrak{g}$, and so

$$\bar{a}(\delta(\xi)) = \bar{a}(\xi_1) \wedge \bar{a}(\xi_1) + \ldots + \bar{a}(\xi_n) \wedge \bar{a}(\xi_n)$$

for some elements $\xi_j$ and $\zeta_j$, $j = 1$, 2, $\ldots$, $n$, of $\mathfrak{g}$. According to equations (13) and (14),

$$\bar{i}_w \bar{a}(\delta(\xi)) = \sum \bar{a}(\xi_j) \otimes i_{\bar{a}(\xi_j)}w - \bar{a}(\xi_j) \otimes i_{\bar{a}(\xi_j)}w.$$

By the hypothesis, $i_{\bar{a}(\xi_j)}w = i_{\bar{a}(\xi_j)}w = 0$, and so $\bar{i}_w \bar{a}(\rho\ell) = 0$, which completes the proof.

Remark 4.11 It appears that the second equivariant Poisson cohomology should have an interpretation along the line of Example 3.5 as well. Namely, one may hope that under certain conditions $H^2_{\pi,G}(P)$ becomes the tangent space to the moduli space of deformations of $\pi$, the momentum mapping, and the action subject to some constraints. (By the moduli space we mean the quotient of the space of deformations by the action of a group of diffeomorphisms.) At the moment, it is not clear to the author how to make such an interpretation rigorous.

Recall that when a Poisson Lie group $G$ acts on $P$ in a Poisson manner, the graded space $\mathcal{X}^*(P)^G$ of $G$-invariant multi-vector fields on $P$ is closed under $d_\pi$ [Lu2]. (Here we take the standard $G$-action on multi-vector fields. The differential $d_\pi$ need not commute with the action, but $\mathcal{X}^*(P)^G$ is still a subcomplex.) The cohomology $H^*(\mathcal{X}^*(P)^G, d_\pi)$ is called the invariant Poisson cohomology of $P$. We will denote it by $H^*_{\pi}(P)_G$. The inclusion of complexes induces a homomorphism $j^*: H^*_{\pi}(P)_G \rightarrow H^*_{\pi}(P)$.

Example 4.12 It is not hard to see that $H^*_\pi(G^*)^G_* = H^*(\mathfrak{g})$. Here, as in a number of similar examples considered before, we have an isomorphism of complexes: $\mathcal{X}^*(G^*)^G_* \simeq C^*(\mathfrak{g})$. 

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Corollary 4.13 Assume that $G$ is compact semisimple and the action of $G$ on $P$ admits an equivariant momentum mapping. Then the induced homomorphism $j^1: H^1_\pi(P)_G \to H^1_\pi(P)$ is surjective.

Proof. We need to show that every cohomology class in $H^1_\pi(P)$ can be represented by a $G$-invariant vector field. According to Theorem 4.8, it can be represented by a basic vector field $w$, i.e., $w$ such that $d_\pi w = 0$ and $i_{\widehat{a}(\xi)} w = 0$ for all $\xi \in \mathfrak{g}$. By Lemma 4.10, $w$ is $G$-invariant in the standard sense. 

Remark 4.14 The corollary speaks in favor of a conjecture [Gi2] that $j^* \ast$ is surjective when the group is compact and the action admits a momentum mapping.

Another conjecture on the homomorphism $j^* \ast$ having interesting applications is that $j^* \ast$ is injective (perhaps under some weak additional assumptions). When the action preserves $\pi$, both conjectures are proved in [Gi2].

4.3 Equivariant Poisson cohomology for locally free actions

In this section we examine the equivariant Poisson cohomology in the case where the momentum mapping is a submersion or, more generally, the pre-momentum mapping is in a certain sense locally free.

Let $P$ be a Poisson manifold acted on in a Poisson fashion by a Poisson Lie group $G$; also let, as usual, $\hat{a}$ be an equivariant pre-momentum mapping. Set

$$\mathcal{X}^\ast(\hat{a}) = \{ w \in \mathcal{X}^\ast(P) \mid i_{\hat{a}(\xi)} w = 0 \text{ and } L_{\hat{a}(\xi)} w = 0 \text{ for all } \xi \in \mathfrak{g} \} .$$

When $\hat{a}$ is associated with an equivariant moment map $\mu$, this is just the space of all $G$-invariant multi-vector fields which are tangent to the $\mu$-fibers. In this case, we will also use the notation $\mathcal{X}^\ast(\mu)$. Recall that by definition the basic subcomplex of $\mathcal{X}^\ast(P)$ is

$$\mathcal{X}^\ast(P)_b = \{ w \in \mathcal{X}^\ast(P) \mid i_{\tilde{a}(\xi)} w = 0 \text{ and } L_{\tilde{a}(\xi)} w = 0 \text{ for all } \xi \in \mathfrak{g} \} .$$

Our first objective is to compare the graded spaces $\mathcal{X}^\ast(\mu)$ and $\mathcal{X}^\ast(P)_b$.

Proposition 4.15 Assume that $\tilde{a}$ arises from an equivariant momentum mapping $\mu$. Then $\mathcal{X}^\ast(\mu) = \mathcal{X}^\ast(P)_b$. In particular, $\mathcal{X}^\ast(\mu)$ is a subcomplex of $\mathcal{X}^\ast(P)$ whose cohomology denoted from now on by $H^\ast_\pi(\mu)$, or $H^\ast(\tilde{a})$, coincides with the basic cohomology $H^\ast_{\pi,b}(P)$.

The corollary is an immediate consequence of Lemma 4.10.

Theorem 4.16 Assume that $G$ is compact and $\tilde{a}$ is associated with a momentum mapping $\mu$ which is a submersion onto its image. Then

$$H^\ast_{\pi,G}(P) = H^\ast_{\pi,b}(P) = H^\ast_\pi(\mu) .$$

According to the theorem, the Poisson equivariant cohomology has a very simple geometrical interpretation when $\mu$ is surjective. Namely, it is just the cohomology of the complex of multi-vector fields which are $G$-invariant and tangent to the $\mu$-fibers.
Before we prove Theorem 4.16, let us state a more general result. An equivariant pre-momentum mapping \(\tilde{a}\) is said to be locally free if it makes \(\mathcal{X}^*(P)\) into a locally free \(G\)-differential algebra in the sense of Definition 2.5. More explicitly, this means that there exists a \(g\)-equivariant linear map \(\Theta : \mathfrak{g}^* \to \mathcal{X}^1(P)\) such that

\[
\langle \tilde{a}(\xi), \Theta(\lambda) \rangle = \lambda(\xi)
\]

for all \(\lambda \in \mathfrak{g}^*\) and \(\xi \in \mathfrak{g}\).

**Example 4.17** When \(P\) is symplectic, \(\tilde{a}\) is locally free if and only if the action is locally free, i.e., the infinitesimal action homomorphism \(a : \mathfrak{g} \to \mathcal{X}^1(P)\) is injective. Then \(\pi^#\) induces an isomorphism between the basic subcomplex \(\Omega^*(P)_b = \Omega^*(P/G)\) of the de Rham complex and \(\mathcal{X}^*(\mu)\). Hence, in particular, \(H^*(P/G) = H^*_G(\mu)\).

In general, it is easy to find an example where \(\tilde{a}\) is locally free, but the \(G\)-action on \(P\) is trivial.

As a particular case of Corollary 2.8, we have

**Theorem 4.18** Let \(\tilde{a}\) be locally free. Then \(H^*_{\pi,G}(P) = H^*_{\pi,b}(P)\).

*Proof of Theorem 4.16.* By Theorem 4.18 it suffices to show that \(\tilde{a}\) is locally free. For \(\lambda \in \mathfrak{g}^* = T_eG^*\), denote by \(\dot{\lambda}\) the left-invariant vector field on \(G^*\) whose value at \(e\) is \(\lambda\). Then \(\theta_\xi(\lambda) = \lambda(\xi)\).

Fix a connection on the fiber bundle \(\mu : P \to G^*\) and let \(\Theta(\lambda)\) be the horizontal lift of \(\dot{\lambda}\). Then \(\langle \tilde{a}(\xi), \Theta(\lambda) \rangle = \theta_\xi(\lambda) = \lambda(\xi)\). Therefore, (19) holds for \(\Theta\). As mentioned in Remark 2.6, we can make \(\Theta\) be \(\mathfrak{g}\)-equivariant, while keeping (19), by averaging \(\Theta\) over the \(G\)-action.

\[\square\]

### 4.4 Calculations of equivariant Poisson cohomology

Let now \(G\) be a Poisson Lie group and \(K \subset G\) its closed connected subgroup. By definition, the dressing action of \(G\) on \(G^*\) has an equivariant momentum mapping \(\mu = id\). Thus the restriction of the action to \(K\) has a natural equivariant pre-momentum mapping, which we denote by \(\tilde{a}\) and use in the equivariant cohomology \(H^*_{\pi,K}(U)\), where \(U\) is a \(G\)-invariant subset of \(G^*\). The following result is an equivariant version of Proposition 3.6.

**Theorem 4.19** \(H^*_{\pi,K}(U) = H^*(\mathfrak{g}, K; C^\infty(U))\).

**Corollary 4.20** \(H^0_{\pi,G}(U) = C^\infty(U)^G\) and \(H^*_{\pi,G} > 0(U) = 0\).

*Proof of Theorem 4.19.* In the case at hands \(\mathcal{X}^*(\mu)\) is just the Chevalley–Eilenberg complex \(C^*(\mathfrak{g}, K; C^\infty)\). Hence, the theorem is an immediate consequence of Theorem 4.16 which applies because \(\mu = id\).

Alternately, one may follow the line of the proof of Proposition 3.6. The complex \((\mathcal{X}^*(U), d_e)\) can be naturally identified with the Chevalley–Eilenberg complex \(C^*(\mathfrak{g}, K; C^\infty(U), d_{Lie})\). Under this identification, the Cartan model for the equivariant Poisson cohomology of \(U\) turns into the Cartan model for the \(K\)-equivariant cohomology of the Chevalley–Eilenberg complex \(C^*\) as in Section 2.3. Hence, \(H^*_{\pi,K}(U) = H^*_K(C^*)\). By Theorem 2.10, \(H^*_K(C^*) = H^*(\mathfrak{g}, K; C^\infty(U))\).
Combining Corollary 2.14 and Theorem 4.19, we obtain

Theorem 4.21 Assume that $G$ is compact. Then $H^*_r(K)(U) = H^*(g, K) \otimes C^\infty(U)^G$.

Finally, Theorem 2.20 implies the following

Proposition 4.22 Let $G$ be compact. Then there exists a spectral sequence of $(S^*g^*)^G$-modules with $E_1^{pq} = E_2^{pq} = H_q^*(P) \otimes (S^p/2g^*)^G$ which converges to $H^*_r(P)$.

5 A spectral sequence associated with a momentum mapping

Now we are ready to introduce and study the main object of this paper – a new spectral sequence associated with an equivariant momentum mapping. The examples are analyzed in Section 5.2.

5.1 The spectral sequence

Let $P$ be a Poisson manifold acted on in a Poisson fashion by a Poisson group $G$ and let the action admit an equivariant momentum mapping $\mu$. The following result is an immediate consequence of Theorems 2.23 and 3.3.

Theorem 5.1 There exists a spectral sequence converging to $H^*_r(P)$ and having $E_1^{pq} = H^q(g) \otimes (X^*(P) \otimes W^*(g))^p$ and $E_2^{pq} = H^q(g) \otimes H^p_{\pi,G}(P)$.

Remark 5.2 For the sake of simplicity, we deliberately chose to state the theorem under rather restrictive assumptions. The hypotheses of the theorem can be relaxed. Clearly, it is sufficient to assume that we are given an equivariant pre-momentum mapping for an action of some compact group $G$ (not necessarily Poisson) on $P$ and that $X^*(P)$ integrates to a $G$-module.

As in Section 2.3, one can also consider a decreasing filtration of $X^*(P)$ arising from the moment map:

\[ X^{p+q}_m(P) = \{ w \in X^{p+q}(P) \mid i_{\alpha_1} i_{\alpha_2} \ldots i_{\alpha_{q+1}} w = 0 \text{ for all } \alpha_1, \ldots, \alpha_{q+1} \} , \quad (20) \]

where $\alpha_1, \ldots, \alpha_{q+1}$ are the pull-backs by $\mu$ of some one-forms on $G^*$. In effect, there exists a similar filtration for any Poisson map $P \to B$ of Poisson manifolds. When $B = G^*$, it is sufficient to take only $\alpha_i$ which are pull-backs of the left-invariant one-forms on $G^*$, i.e., $\alpha_i = \mu^* \theta_{\xi_i}$. In general, there seems to be little chance to tell anything interesting about the spectral sequence for this filtration in addition to the fact that it converges to $H^*_r(P)$. (See, however, Example 5.12.) As in Theorem 2.26, the situation becomes much more favorable when $X^*(P)$ is a locally free $G$-differentiable algebra. As we know from the proof of Theorem 4.18, this is the case when $\mu$ is a submersion onto an open subset of $G^*$, which is then necessarily $G$-invariant.

Theorem 5.3 Assume that $G$ is compact and the momentum mapping $\mu$ is a submersion onto an open subset $U$ of $G^*$. The filtration (20) gives rise to a spectral sequence which converges to $H^*_r(P)$ and has $E_1^{pq} = H^q(g) \otimes X^p(\mu)$ and $E_2^{pq} = H^q(g) \otimes H^p_{\pi}(\mu)$ in the notation of Section 4.3.
**Example 5.4** Let $P$ symplectic. It is easy to see $\pi^#$ induces an isomorphism from the spectral sequence of the principal $G$-bundle $M \times EG \to (M \times EG)/G$ (Example 2.22) to that in Theorem 5.1.

Assume also that $\mu$ is a submersion. Then according to Example 4.17, $\pi^#$ induces an isomorphism between the basic de Rham complex $\Omega^*(P)_b$ (e.g., $\Omega^*(P/G)$ if the action is free) and $\mathcal{X}^*(\mu)$. In particular, $\pi^#$ gives rise to the identification $H^*(P/G) = H^* G(\mu)$. Now we can add to it that $\pi^#$ is an isomorphism between the spectral sequences of $M \to M/G$ (Example 2.23) and the spectral sequence from Theorem 5.3. We will elaborate on this example in the next section.

Finally note that nontrivial systems of local coefficients do not arise in these examples (nor in Theorems 5.1 and 5.3) because $G$ is assumed to be connected.

**Proof of Theorem 5.3.** This result is an immediate consequence of Theorems 2.26 and 3.1 for the hypotheses guarantee that $\mathcal{X}^*(P)$ is a locally free $G$-differential algebra. However, since the proof of Theorem 2.26 was omitted, we briefly outline a direct proof of Theorem 5.3 in order to illuminate the geometrical meaning and the nature of the initial terms and the differentials of the spectral sequence. As mentioned above, the argument follows closely the proof of Theorem 2.23.

First recall that by the second assertion of Theorem 5.1, we may assume, replacing if necessary $G$ by its finite covering, that the $g$-action on $\mathcal{X}^*(P)$ via $\mathcal{L}_g$ integrates to a $G$-action.

Denote, as above, by $\alpha_j$'s the pull-backs of some left-invariant forms on $G^*$. In other words, $\alpha_j = \mu^* \xi_j$ for some $\xi_j \in g$, and so the space of $\alpha_j$'s can be identified with $g$. Observe that for $w \in \mathcal{X}_p^{p+q}(P)$ the contraction

$$c_w(\alpha_1, \ldots, \alpha_q) = i_{\alpha_1} \cdots i_{\alpha_q} w$$

is a $p$-vector field tangential to the $\mu$-fibers, i.e., an element of $\mathcal{X}_p^{p}(P)$. Hence, the correspondence $w \mapsto c_w$ gives rise to a linear homomorphism from $\mathcal{X}_p^{p+q}(P)$ to the space $C^q(g; \mathcal{X}_p^{p}(P))$ of $q$-cochains on $g$ with coefficients in $\mathcal{X}_p^{p}(P)$. The kernel of this homomorphism is clearly $\mathcal{X}_p^{p+q}(P)$. Thus $w \mapsto c_w$ descends to a monomorphism, also denoted by $c$,

$$E_0^{pq} = \mathcal{X}_p^{p+q}(P)/\mathcal{X}_p^{p+q}(P) \longrightarrow C^q(g; \mathcal{X}_p^{p}(P)) \quad .$$  

(21)

It is easy to see that $c$ is onto, and therefore an isomorphism. (One can prove this using a distribution transversal to the $\mu$-fibers, which exists because $\mu$ is a submersion.) From now on we identify $E_0^{pq}$ and $C^q(g; \mathcal{X}_p^{p}(P))$.

To find $d_0$ one uses an expression for $d_\pi$ which is analogous to the Cartan formula for the de Rham differential but with the roles played by vector fields and differential forms interchanged. (See, e.g., [BV] or formula (4.8) in Section 4.2 of [Va].) Then a straightforward calculation identical to that in the proof of Theorem 2.23 shows that $d_0$ is equal to the Lie algebra differential $d_{Lie}$ on $C^*(g; \mathcal{X}_p^{p}(P))$. Here we can take any one of two $g$-module structures, via $\mathcal{L}$ or $L$, because by Lemma 4.10 they coincide on $\mathcal{X}_p^{p}(P)$. Thus we have

$$E_1^{pq} = H^q(g; \mathcal{X}_p^{p}(P))$$

and, since $G$ is compact, $E_1^{pq} = H^q(g) \otimes (\mathcal{X}_p^{p}(P))^G$ due to Theorem 2.13. By definition, $(\mathcal{X}_p^{p}(P))^G = \mathcal{X}^p(\mu)$, which yields

$$E_1^{pq} = H^q(g) \otimes \mathcal{X}^p(\mu) \quad .$$
(Recall that $\mathcal{X}^p(\mu)$ is the space of $G$-invariant $p$-vector fields which are tangent to the fibers of $\mu$; see Section 4.3.)

It remains to show that $d_1$ coincides with the restriction of $d_\pi$ to $\mathcal{X}^*(\mu)$. The rest of the proof is a routine, although lengthy, analysis of $d_1$ serving to this end and based on the assumptions that $G$ is compact and $\mu$ is surjective. (Example 5.4 lends itself as a hint.)

Let $w \in F_p^{q+q}$ represent an element $[w]$ of $E_1^{pq}$. Note that $d_\pi w = 0$ yields $d_\pi w \in F_p^{q+q+1}$, which in turn guarantees that $i_{\alpha_q} \ldots i_{\alpha_1} d_\pi w \in \mathcal{X}^{q+1}(\mu)$. By definition, $d_1[w] \in E_1^{q+q+1}$ is represented by the cocycle

$$(\alpha_1, \ldots, \alpha_q) \rightarrow i_{\alpha_q} \ldots i_{\alpha_1} d_\pi w$$

with values in $\mathcal{X}^{q+1}(\mu)$. To ensure that $d_1$ has the desired form, it suffices to show that for some particular choice of $w$ in $[w]$, the same up to a sign cocycle is obtained when the contractions with $\alpha_1 \ldots \alpha_q$ are followed by $d_\pi$, i.e.,

$$i_{\alpha_q} \ldots i_{\alpha_1} d_\pi w = \pm d_\pi i_{\alpha_q} \ldots i_{\alpha_1} w .$$

(Since $\mathcal{X}^*(\mu)$ is a subcomplex, the right hand side is indeed in $\mathcal{X}^{q+1}(\mu)$.) This amounts to proving that

$$(i_{\alpha_q} \ldots i_{\alpha_1} d_\pi w)(\beta_1, \ldots, \beta_{q+1}) = (-1)^q (d_\pi i_{\alpha_q} \ldots i_{\alpha_1} w)(\beta_1, \ldots, \beta_{q+1})$$  (23)$$

at every point $x \in P$, for any one-forms $\beta_1, \ldots, \beta_{q+1}$ on the tangent space to the $\mu$-fiber through $x$. Thus let $\beta_j$ form a basis in $T_x \mu^{-1}(\mu(x))$. To verify (23), we will extend $\beta_j$’s to global one-forms near $x$ in a particularly convenient way and then apply the Cartan formula for $d_\pi$.

The extension is to be carried out so that to obtain the $G$-invariant one-forms with respect to the $G$-action on $\Omega^1(P)$ arising from $L$. (See Theorem 3.1.) Let us begin by extending $\beta_j$’s to $G$-invariant differential forms on $\mu$-fibers defined in a neighborhood of $x$. To do so we first extend $\beta_j$ along a slice transversal to the $G$-orbit through $x$ and then use the translations by the $G$-action to extend the forms to a neighborhood of $x$. (Note that along the $\mu$-fibers the derivatives $L$ and $L_\mu$ coincide and so do the two lifts of $G$-action to the tangent (and cotangent) bundles to the $\mu$-fibers.) Finally, let us fix a $G$-invariant projection $K$ of $TP$ to the tangent bundle to the $\mu$-fibers. Here the invariance is understood with respect to the action arising from $L$. Such a projection exists because the fibers have constant dimension and $G$ is compact. Taking the composition of this projection with $\beta_j$’s, which have so far been defined along $\mu$-fibers, we obtain genuine one-forms on a neighborhood of $x$. These forms, denoted again by $\beta_j$, are $G$-invariant by (13).

According to the representation of $E_1^{pq}$ as a tensor product, a multi-vector field $w$ in the class $[w]$ can be chosen in the form $w = y_q \wedge v_p$, where $v_p$ is a $p$-vector field tangent to the $\mu$-fibers and $y_q$ is a $q$-vector field which is the horizontal lift of a $G^*$-invariant multi-vector field on $G^*$. The horizontality is understood, of course, with respect to the “connection” ker $K$.

Applying the Cartan formula for $d_\pi$ to evaluate the left hand side of (23), we obtain a sum of the terms involving the derivatives $L_{\beta_j}$ and $L_{\alpha_i}$ and various pairwise brackets of $\beta_j$ and $\alpha_i$. These terms can be divided into three groups,

$$(i_{\alpha_q} \ldots i_{\alpha_1} d_\pi w)(\beta_1, \ldots, \beta_{q+1}) = I + II + III ,$$

as follows.
The first group contains the terms with $L_{\beta_j}$ and the brackets $\{\beta_i, \beta_k\}$. The sum for this group is exactly the right hand side of (23):

$$I = \sum_{1 \leq l < k \leq p+1} (-1)^{2q+k+l-1} w(\{\beta_l, \beta_k\}, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_l, \ldots, \beta_k, \ldots, \beta_{p+1})$$

$$+ \sum_{1 \leq j \leq p+1} (-1)^{q+j} L_{\beta_j} w(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_j, \ldots, \beta_{p+1})$$

$$= (-1)^q (d_\pi i_{\alpha_q} \ldots i_{\alpha_1} w)(\beta_1, \ldots, \beta_{p+1}).$$

The second group is made up of the terms with $L_{\alpha_i}$ and the brackets $\{\alpha_i, \alpha_k\}$. Each of the summands in the group vanishes because $w$ is taken in the form $y_q \wedge v_p$:

$$II = \sum_{1 \leq l < k \leq q} (-1)^{k+l-1} w(\{\alpha_l, \alpha_k\}, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_{p+1})$$

$$+ \sum_{1 \leq j \leq p+1} (-1)^j L_{\alpha_i} w(\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_{p+1})$$

$$= 0.$$

The third group involves the terms with $\{\alpha_i, \beta_j\}$. The brackets vanish, for the forms $\beta_j$ are $G$-invariant, and so $\{\alpha_i, \beta_j\} = L_{\alpha_i} \beta_j = 0$:

$$III = \sum_{i,j} (-1)^{q+i+j-1} w(\{\alpha_i, \beta_j\}, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_j, \ldots, \beta_{p+1}) = 0.$$

This completes the proof of (23) and the proof of the theorem.

\[\square\]

Remark 5.5 The identifications (21) and (23) hold even when $G$ is not compact, but $\mu$ is still a submersion. However, the $E_2$-term is, in this case, much more difficult to calculate.

As clear from the proof, the cohomology $H^*(g)$ can be understood with some extra insight as the invariant Poisson cohomology $H^*_\pi(G^*)_{G^*}$ (Example 4.12) or, when $P$ is symplectic, as $H^*(G)$ (see Example 5.4).

5.2 Examples and applications

Throughout this section $G$ is assumed to be a compact connected (Poisson) Lie group.

Assume first that $P$ is a symplectic manifold. Then, as we noted in Example 4.2, $\pi^\#$ gives rise to an isomorphism between the spectral sequence in the de Rham cohomology and that in the Poisson cohomology making the latter particularly easy to write down. We illustrate this point by two simple examples where the Poisson structure on $G$ is assumed to be trivial.

Example 5.6 Let $G$ be the circle acting on the standard symplectic $\mathbb{R}^{2n} = \mathbb{C}^n$ by rotations with Hamiltonian $\sum |z_j|^2/2$. Since the manifold is (equivariantly) contractible, its (equivariant) cohomology is the same as of a point. The spectral sequence from Theorem 5.1 has $E_2^{pq} = H^q(S^1) \otimes H^p(\mathbb{C}P^\infty)$. It converges to $H^*(pt)$: the differential $d_2$ kills all the terms but $E_2^{00} = \mathbb{R}$. 

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Take now $P = \mathbb{C}^n \setminus 0$ with the same action. The momentum mapping on $P$ is a submersion and we are in a position to apply Theorem 5.3. The spectral sequence converges to $H^*_\pi(P) = H^*(S^{2n-1})$ and has $E^{pq}_2$ which is the tensor product of $H^q(S^1)$ and $H^*_\pi(\mu) = H^*(\mathbb{C}P^{n-1})$. (The easiest way to see the latter isomorphism is via $\pi^\#: \mu$.) Finally, $d_2$ kills all the $E_2$-terms but $E^{00}_2 = \mathbb{R}$ and $E^{2n-11}_2 = \mathbb{R}$.

**Example 5.7** Let $P = T^*G$ with its natural symplectic structure and the (left) $G$-action. The spectral sequence converges to $H^*_\pi(T^*G) = H^*(G)$ by collapsing already in the $E_2$-term. We have $E_2 = H^*(g) \otimes H^*_\pi(\mu)$ with $H^*(g) = H^*(G)$ and $H^*_\pi(\mu) = H^*(T^*G/G) = H^*(pt)$.

Let us now turn to the examples where $P$ is a genuine Poisson manifold.

**Example 5.8** Let $P$ be an open $G$-invariant subset of $G^*$ with $\mu = id; P \hookrightarrow G^*$ and the standard dressing action of $G$. Observe that $H^*_\pi(\mu) = C^\infty(P)^G$ when $q = 0$ and zero otherwise, since $\mu = id$. Then the spectral sequence collapses in the $E_2$-term and $H^*_\pi(P) = H^*(g) \otimes C^\infty(P)^G = E_2$ gives the decomposition of $E_2$ as the tensor product. (Note that we thus obtain Theorem 5.7 as a corollary of Theorem 5.3.)

**Example 5.9** Consider $P = su(2)^* \setminus 0$. Let $\mu$ be a Casimir function without critical points, e.g., the distance to the origin. We have $P = S^2 \times \mathbb{R}_+$ with the Poisson structure $\pi = t\pi_0$, where $\pi_0$ is the standard symplectic Poisson structure on $S^2$ and $\mu = t$ is the coordinate on $\mathbb{R}_+$. Clearly, $\mu$ gives rise to the trivial circle action on $P$, and we can apply Theorem 5.3.

The resulting spectral sequence converges to $H^*_\pi(P) = H^*(su(2)) \otimes \mathcal{A} = \mathcal{A} \oplus [u \wedge \pi_0] \cdot \mathcal{A}$, where $u = \partial_t$ and for the sake of brevity the algebra of Casimir functions $C^\infty(\mathbb{R}_+)$ is denoted by $\mathcal{A}$. Since the level sets of $\mu$ are just the symplectic leaves, $H^*_\pi(\mu)$ can be identified with the **tangential** Poisson cohomology $H^*_{\pi,\text{tan}}(P)$ introduced in [Gi2], i.e., the cohomology of multi-vector fields tangent to the symplectic leaves. It is not hard to see that $H^*_{\pi,\text{tan}}(P) = \mathcal{A} \oplus [\pi_0] \cdot \mathcal{A}$. Then, fixing also the generators of $H^*(S^1)$, we get

\[
E^{pq}_2 \simeq \begin{cases} 
C^\infty(\mathbb{R}_+) & \text{if } p = 0, \ 2 \text{ and } q = 0, \ 1 \\
0 & \text{otherwise}
\end{cases}.
\]

The differential $d_1$ is an isomorphism $E^{01}_2 \to E^{20}_2$ canceling these two terms and leaving $E^{00}_2$ and $E^{21}_2$ intact.

Of course, the Poisson cohomology in this and the next example has been calculated repeatedly by many authors using various methods. See, in particular, [VK], [Xu] and references therein.

**Example 5.10** In the notation of Example 5.9, let us take $P = S^2 \times \mathbb{R}_+$ with $\pi = f(t)\pi_0$, where $f$ is a nonvanishing function and $\mu = t$ the projection onto $\mathbb{R}_+$. (Of course, in what follows $\mathbb{R}_+$ can be replaced by $\mathbb{R}$ or $S^1$.) We will use the spectral sequence of Theorem 5.3 to calculate the Poisson cohomology of $P$ and thus answer a question of Gregg Zuckerman. It is easy to see that the $E_2$-term is independent of $f$ and therefore given by (24). As before, the spectral sequence stabilizes no later than in the $E_3$-term. Moreover, $E^{00}_2$ and $E^{21}_2$ are not touched by $d_2$, which reflects the fact that $H^0_\pi(P)$ and $H^1_\pi(P)$ are independent of $f$. The only effect $f$ has is via $d_2: E^{01}_2 \to E^{20}_2$. (For example, $d_2 = 0$ when $f = \text{const}$ and, as we
have seen, $d_2$ is an isomorphism when $f(t) = t$. One can show that under the identification (24), $d_3 \colon \mathcal{A} \to \mathcal{A}$ is just the multiplication by $f'$. Thus denoting by $I$ the interior of the set $\{f' = 0\}$, we get:

$$H^k_\pi(P) = \begin{cases} 
\mathcal{A} = \text{Cas} & \text{when } k = 0 \\
[u] C^\infty_0(I) & \text{when } k = 1 \\
[\pi_0] (\mathcal{A}/(f' \cdot \mathcal{A})) & \text{when } k = 2 \\
[u \wedge \pi_0] \mathcal{A} & \text{when } k = 3 
\end{cases},$$

where $C^\infty_0(I)$ is the space of functions on $I$ which, being set zero on the complement to $I$, extend smoothly to $\mathbb{R}_+$. The requirement that $f$ is nowhere constant renders $I = \emptyset$, and so $H^1_\pi(P) = 0$. Furthermore, if $f$ is a Morse function with $n$ critical points, $H^2_\pi(P) \simeq \mathcal{A}/(f' \cdot \mathcal{A}) = \mathbb{R}^n$.

**Example 5.11** In the setting of Example 5.10, let us take $P = S^2 \times S^1$ so that $P$ becomes compact and $\mathcal{A} = C^\infty(S^1)$. Also, take $f$ such that $I$ is a proper subset of $S^1$, i.e., $I$ is neither $\emptyset$ nor $S^1$. Then $C^\infty_0(I)$ is not a finitely generated $\mathcal{A}$-module. As a consequence, $H^*_\pi(P)$ is not a finitely generated module over the algebra of Casimir functions even though $P$ is compact and the symplectic foliation is just the direct product $S^2 \times S^1$. Of course, $\pi$ itself fails to be real analytic even though its symplectic foliation is such.

Finally, let us analyze an example where we have only a Poisson map but no group action.

**Example 5.12** Let $P = B \times X$ and let the symplectic leaves of $\pi$ be the fibers $b \times X$, $b \in B$. The natural projection $\mu : P \to B$ is a Poisson submersion, provided that $B$ is equipped with the zero Poisson structure. As mentioned above, we still have the filtration (24) associated with $\mu$ and thus a spectral sequence converging to $H^*_\pi(P)$. One may show that $E^p_2 = \mathcal{X}^q(B) \otimes H^p(X)$. This decomposition, unlike $d_2$, is independent of $\pi$ as long as the symplectic foliation remains fixed.

It appears to be a safe and provable conjecture that $H^*_\pi(P)$ is a finitely generated $C^\infty(B)$-module when $\pi$ is real analytic.

Assume that $B$ is an open subset of $\mathbb{R}^n$. Then along the line of Example 5.10, we may write $E^p_2 = H^q(\mu^0) \otimes H^p_\pi(\mu)$, where $\mu$ is viewed as the momentum mapping of a trivial $T^n$-action. As before we have $H^p_\pi(\mu) = H^p_\pi,\text{tan}(P)$, which can also be identified with the tangential de Rham cohomology of the foliation into $\mu$-fibers.

### 6 Appendix: Proof of Theorem 2.13

Throughout the proof we keep the notation of Section 2.3.2. Thus $C^* = C^*(\mathfrak{g}, K; V)$.

Denote by $I : C^n \to C^n(\mathfrak{g}, K; V^G)$ the averaging over the $G$-action:

$$(I \phi)(\xi_1, \ldots, \xi_n) = \int_G \rho(g) \phi(\xi_1, \ldots, \xi_n) \, dg,$$

where $\xi_1, \ldots, \xi_n$ are elements of $\mathfrak{g}$ and $\phi \in C^n$. A direct calculation shows that $I$ commutes with the differential $d_\rho$, i.e., $d_\rho I = I d_\rho$, even though, in general, the operators $\rho(g)$ do not. (This easily follows from the fact that $I(\rho_\xi(v)) = 0$ for any $\xi \in \mathfrak{g}$ and $v \in V$.) To prove (i), it suffices to show that $I \phi - \phi$ is exact when $\phi$ is closed.
Let us identify $C^*$ with a certain subcomplex $\Omega^*_p$ of the de Rham complex $\Omega^*$ of differential forms on $G/K$ with values in $V$. Denote by $T^*$ the action of $G$ on $\Omega^*$ induced by the left translations. Consider the diagonal action $T^0$ of $G$ on $\Omega^* = \Omega^*(G/K) \otimes V$, i.e., the action via $T^*$ on the first term and via $\rho$ on the second. Let us identify $C^*$ with $\wedge^* T^0_p(G/K)$ as vector spaces. Then the complex $C^*$ is topologically isomorphic to the subcomplex $\Omega^*_p \subset \Omega^*$ of $T^0$-invariant forms. The isomorphism $\Phi: C^* \to \Omega^*_p$ sends a cochain $\phi$ to the form $\phi^0$ which is a unique extension of $\phi$ to a $T^0$-invariant form. In particular, $\phi^0 = \phi$ and the evaluation at $e$ is the inverse to $\Phi$. We do have an isomorphism of complexes: (2) coincides with the Cartan formula for $d_{dR}$. From now on we identify $C^*$ and $\Omega^*_p$ and omit $\Phi$ from our notation. Hence $d_{dR} = d_\rho$ on $\Omega^*_p$.

It is easy to see that the averaging operator $\omega \mapsto \int_G T^*_g \omega \, dg$ preserves $\Omega^*_p$ and its restriction to $\Omega^*_p$ coincides with $I$. Thus we denote this averaging $\Omega^* \to \Omega^*$ by $I$ again. Similarly to the averaging of real-valued forms, $I$ commutes with $d_{dR}$ ([Gu], Appendix E). Furthermore, using Lemma E.1 of [Gu], it is not hard to show that $I$ induces the identity map on the de Rham cohomology, i.e., $I(\omega) - \omega$ is exact when $\omega \in \Omega^*$ is closed.

Consider a continuous projection $P: \Omega^* \to \Omega^*_p$ defined by

$$P(\omega) = \int_G T^*_g \omega \, dg$$

A direct though lengthy calculation shows $P$ is a homomorphism of complexes, i.e., $P$ commutes with $d_{dR}$. The calculation is based on the observation that on $\Omega^0$, i.e., on smooth functions, $P$ commutes with the Lie derivatives along the left-invariant vector fields. (Note that, unlike $I$, the projection $P$ does not induce the identity homomorphism in the cohomology unless the action of $G$ on $V$ is trivial.)

Now we are ready to complete the proof of the theorem. To prove (i), pick $\phi \in Z^n \subset C^n$. Then $I\phi - \phi = d_\rho \beta$ for some form $\beta$. The left hand side of this equation is in the subcomplex $C^* = \Omega^*_p$ of $\Omega^*$. Applying $P$, we get $I\phi - \phi = d_\rho P\beta$, for $P d_{dR} = d_\rho P$.

To prove (ii) and (iii), we need the following property of $\Omega^*$ to get around the standard usage of the Hodge theory, which cannot be applied to $V$-valued forms. Namely, there exists a continuous linear map $H_{dR}: \Omega^n \to \Omega^{n-1}$ such that $H_{dR} |_{\Omega^*_p}$ is a right-inverse to $d_{dR}$, i.e., $d_{dR} H_{dR} |_{\Omega^*_p} = id$. This observation is an immediate consequence of a theorem by Mostow that $\Omega^*$ is a strongly injective. (See [BW], Proposition 5.4 of Chapter IX or [Gu], Section D.1.1 and Lemma E.1.) This means that for every $n$, the inclusion of $\Omega^n_{dR} = \ker d_{dR}$ into $\Omega^n$ and that of $\Omega^n/\Omega^n_{dR}$ into $\Omega^{n+1}$ admit continuous left inverses.

Let us set $H = PH_{dR} |_{C^*}$. $H$ is a composition of continuous linear maps, $H$ is continuous and linear. It is sufficient to show that $d_\rho H(\phi) = \phi$ when $\phi \in B^n$. This is equivalent to showing that $d_{dR} P H_{dR}(\omega) = \omega$ for any $d_{dR}$-exact $\omega \in \Omega^n_{dR}$. Since $P$ and $d_{dR}$ commute, and $H_{dR} |_{\Omega^*_p}$ is a right-inverse $d_{dR}$, we have $d_{dR} P H_{dR}(\omega) = P d_{dR} H_{dR}(\omega) = P \omega = \omega$.

Finally note that $H_{dR}$ is independent of $\rho$ and $P$ and $\Phi$ are both smooth in $\rho$. As a consequence, $H$ depends smoothly on $\rho$ which proves (iii).

\[
\square
\]

Remark 6.1 It is clear from the proof that $B^n$ is a closed subspace of $C^n$ and hence a Fréchet space. Indeed, $B^n = \Phi^{-1}(\Omega^*_p \cap B^n_{dR})$, where as shown above $\Omega^n$ and $B^n_{dR}$ are closed in $\Omega^n$.
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