Supergravity Tensor Calculus in 5D from 6D

Taichiro KUGO* and Keisuke OHASHI**

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

Supergravity tensor calculus in five spacetime dimensions is derived by dimensional reduction from the $d = 6$ superconformal tensor calculus. In particular, we obtain an off-shell hypermultiplet in 5D from the on-shell hypermultiplet in 6D. Our tensor calculus retains the dilatation gauge symmetry, so that it is a trivial gauge fixing to make the Einstein term canonical in a general matter-Yang-Mills-supergravity coupled system.

* E-mail: kugo@gauge.scphys.kyoto-u.ac.jp
** E-mail: keisuke@gauge.scphys.kyoto-u.ac.jp
§1. Introduction

Many physicists have recently been studying seriously the revolutionary idea that our four-dimensional world may be a ‘3-brane’ embedded in a higher dimensional spacetime. This idea should provide us with many new ideas and demands reconsideration of various problems in unification theories; e.g., gauge hierarchy, supersymmetry breaking, hidden sectors, fermion mass hierarchy, cosmology and astrophysics.

This idea originally came from string theory, which now has become recognized as not merely the theory of strings, but the theory describing the totality of various dimensional branes. Hořava and Witten introduced this type of idea for the first time in their investigation of the strongly-coupled heterotic string theory. They argued that the low-energy limit is described by eleven-dimensional supergravity compactified on $S^1/Z_2$, an interval bounded by orientifold planes, and that a ten-dimensional $E_8$ super Yang-Mills theory appears on each plane. Regarding one of the planes as ‘our world’ and the other as ‘hidden’, they offered an interesting proposal for resolving the discrepancy between the GUT scale and the gravity scale.

As a toy model for more realistic phenomenology, in which 6 of the transverse 10 dimensions should be compactified, Mirabelli and Peskin considered a five-dimensional super Yang-Mills theory compactified on $S^1/Z_2$ and clarified how supersymmetry breaking occurring on one boundary is communicated to another. They presented a simple algorithm for coupling the bulk super Yang-Mills theory to the boundary fields with the help of off-shell formulations. It is clearly important to generalize their work to more realistic models in which the bulk is described by five-dimensional supergravity.

One (presumably, the main) obstacle to this investigation has been the lack of an off-shell formulation of five dimensional supergravity. It is quite recent that such an off-shell formulation was first given by Zucker. In a series of two papers, he presented an almost complete supergravity tensor calculus, including the 40+40 minimal Weyl multiplet, Yang-Mills multiplet, linear and nonlinear multiplets, and an invariant action formula applicable to the linear multiplet. Unfortunately and strangely, however, the most important matter multiplet in five dimensions, called a hypermultiplet or scalar multiplet, was missing there. A general system of five-dimensional supergravity should necessarily contain hypermultiplets. Actually, an explicit form of the action for the general matter-Yang-Mills system coupled to supergravity would be very important and useful for phenomenological work, as we know from the example of the work of Cremmer et al. on $N = 1$, $d = 4$ supergravity theory.

In this paper, we present a more complete tensor calculus for 5D supergravity including, in particular, the hypermultiplet. We derive it using dimensional reduction from the known
$d = 6$ superconformal tensor calculus, which was fully described in an excellent and elaborate paper by Bergshoeff, Sezgin and Van Proeyen, referred to as BSVP henceforth. The advantages of this dimensional reduction are two fold: One is obviously that we need not repeat the trial-and-error method to find the multiplet members and their transformation laws. Everything is in principle straightforward, and we can use all the formulas known in 6D. Secondly, the supergravity structure, like supersymmetry transformation laws, is actually simpler in 6D than in 5D, because of its larger symmetry. Knowing the relations between covariant derivatives and curvatures in the 5D and 6D theories, we can obtain deeper understanding of the group structure of 5D supergravity. (Even in applications other than constructing 5D tensor calculus, such knowledge may become useful.) Moreover, we can inherit the advantage of superconformal symmetry of the original 6D theory. Indeed, we retain the dilatation gauge symmetry also in our 5D tensor calculus, and this makes it so that a trivial gauge-fixing renders the Einstein-Rarita-Schwinger term canonical.

The rest of this paper is organized as follows. In §2, we explain in some detail the dimensional reduction procedure to obtain our 5D supergravity tensor calculus from the 6D superconformal one of BSVP. We also present there some general discussion on the covariant derivatives and curvatures, which applies both to supergravity and superconformal theories and turns out to be very useful. Based on this, the transformation law is derived for the 40+40 minimal Weyl multiplet in 5D. Then the transformation laws of Yang-Mills, linear and nonlinear multiplets can be found straightforwardly, as given in §3. However, we need a method to obtain an off-shell hypermultiplet in 5D from the 6D one which exists only as an on-shell multiplet. As explained in §4, we make it off-shell in our reduction procedure using a method similar to that known in $d = 4$ case. The invariant action formulas for the kinetic terms as well as mass terms for such off-shell hypermultiplets are also derived there. Other invariant action formulas and some embedding formulas are given in §5. The final section is devoted to summary and discussion.

We here note that we adopt the metric $\eta_{ab} = \text{diag}(+, -, -, \cdots, -)$, which is different from that of BSVP, but we think it is more familiar to phenomenologists. Our notation and conventions are given in Appendix A, where some useful formulas are also given. The results of the 6D superconformal tensor calculus of BSVP that we need in this paper are briefly summarized in Appendix B.

§2. Dimensional reduction and 5D Weyl multiplet
2.1. Dimensional reduction and gauge-fixing

Our starting point is the superconformal tensor calculus in $d = 6$ dimensions given by BSVP$^{[14]}$ and summarized in Appendix B. The superconformal group in six dimensions is $OSp(6, 2|1) \simeq OSp(4|1; H)$ which has the generators

$$P_{\underline{a}}, M_{\underline{ab}}, U_{ij}, D, K_{\underline{a}}, Q_{\alpha i}, S_{\alpha i},$$

where $\underline{a}, \underline{b}, \cdots$ are tangent vector indices, $\alpha$ is a spinor index, and $i, j, \cdots = 1, 2$ are $SU(2)$ indices. $P_{\underline{a}}$ and $M_{\underline{ab}}$ represent the usual Poincaré generators, $U_{ij}$ represents the $SU(2)$ generators, $D$ is the dilatation, $K_{\underline{a}}$ represents the special conformal boosts, $Q_{\alpha i}$ represents the supersymmetry and $S_{\alpha i}$ represents the conformal supersymmetry, both of which are $SU(2)$-Majorana-Weyl spinors. The gauge fields corresponding to these generators are

$$e_{\mu}^{\underline{a}}, \omega_{\underline{a} \underline{b}}, \chi_{\underline{a} \underline{b}}, \bar{\psi}_{\mu}^{i}, \bar{\phi}_{\mu}^{i},$$

respectively, where the spinor indices of the gauge fields $\bar{\psi}_{\mu}^{i}$ and $\bar{\phi}_{\mu}^{i}$ are suppressed.

The local superconformal algebra in $d = 6$ can only be realized by adding a suitable 'matter' multiplet to these gauge fields, and the $M, K$ and $S$ gauge fields, $\omega_{\underline{a} \underline{b}}, \bar{f}_{\underline{a} \underline{b}}, \phi_{\underline{a} \underline{b}}$, become dependent fields through a set of constraints, (B.1), imposed on the curvatures. Then, the algebra is realized on the set of 40 Bose plus 40 Fermi fields

$$e_{\mu}^{\underline{a}}, \chi_{\underline{a} \underline{b}}, \bar{\psi}_{\mu}^{i}, \bar{\phi}_{\mu}^{i}, T_{\underline{a} \underline{b} \underline{c}}, \chi^{i}, D,$$

called minimal Weyl multiplet, where the last three fields, an anti-self-dual tensor $T_{\underline{a} \underline{b} \underline{c}}$, an $SU(2)$-Majorana-Weyl spinor $\chi^{i}$, and a real scalar $D$, are the added 'matter multiplet'.

We generally add underlines to the quantities in 6 dimensions when necessary to distinguish them from those in 5 dimensions. In particular, the Lorentz index is $\underline{a} = (a, 5)$, and the world vector index is $\underline{\mu} = (\mu, z)$; we use $z$ to denote both the fifth spatial direction and the coordinate itself, $x^\underline{\mu} = (x^{\mu}, x^z = z)$, since we must distinguish the curved index fifth spatial component $v_z \equiv v_{\mu=5}$ from the flat index one $v_5 \equiv v_{a=5}$.

We now perform a 'trivial' dimensional reduction from 6 to 5 dimensions, by simply letting all the fields and local transformation parameters be independent of the fifth spatial coordinate $z$. As in the usual procedure,$^{[14]}$ we fix the off-diagonal local Lorentz $M_{a5}$ gauge by setting $e_{\mu}^{\underline{a}} = 0$, and then the sechsbein $e_{\underline{a}}^{\underline{b}}$ and its inverse take the forms

$$e_{\underline{a}}^{\underline{b}} = \begin{pmatrix} e_{\mu}^{\underline{a}} & e_{\mu}^{5} = \alpha^{-1}A_{\mu} \\ e_{z}^{\underline{a}} = 0 & e_{5}^{\underline{a}} = \alpha^{-1} \end{pmatrix}, \quad e_{\underline{a}}^{\underline{b}} = \begin{pmatrix} e_{\mu}^{\underline{a}} & e_{5}^{\underline{a}} = -e_{\mu}^{\underline{a}}A_{\mu} \\ e_{z}^{\underline{a}} = 0 & e_{5}^{\underline{a}} = \alpha \end{pmatrix}. \quad (2.4)$$

As is well-known since the work of Kaluza and Klein, the field $A_{\mu}$ appearing in the off-diagonal element $e_{\mu}^{5}$ becomes a $U(1)$ gauge field which we call a 'gravi-photon'. Under the
general coordinate (GC) transformation $\delta_{\text{GC}}(\xi^\mu)$ in 6D with the transformation parameter $\xi^\mu = (\xi^\mu, \xi^z)$ taken to be $z$-independent, any field with a world vector index (e.g., gauge fields) $\underline{h}_\mu = (h_\mu, h_z)$ transforms as

$$
\delta_{\text{GC}}(\xi^\nu)h_\mu = \partial_\mu \xi^\nu \cdot h_\mu + \xi^\nu \partial_\mu h_\mu = \delta_{\text{GC}}(\xi^\nu)h_\mu + \partial_\mu \xi^z \cdot h_z,
$$

(2.5)

$$
\delta_{\text{GC}}(\xi^\nu)h_z = \xi^\nu \partial_\nu h_z = \delta_{\text{GC}}(\xi^\nu)h_z,
$$

(2.6)

where $\delta_{\text{GC}}(\xi^\nu)$ is the GC transformation in 5D. For the gravi-photon $A_\mu = e_5^\mu e_5^z$, in particular, we have

$$
\delta_{\text{GC}}(\xi^\nu)A_\mu = \delta_{\text{GC}}(\xi^\nu)A_\mu + \partial_\mu \xi^z,
$$

(2.7)

showing that $A_\mu$ actually transforms as the $U(1)$ gauge field for the transformation parameter $\xi^z$. The discrepancy between the 6D and 5D GC transformations on the other vector fields $\underline{h}_\mu$ [the second term $\partial_\mu \xi^z \cdot h_z$ in Eq. (2.5)] can thus be eliminated by redefining the field as

$$
h_\mu \equiv h_\mu - A_\mu h_z,
$$

(2.8)

which we, therefore, identify with the vector (or gauge) field in 5D. But we note that this identification rule can be rephrased into a simpler rule:

Any field with flat (local Lorentz) indices alone is the same in 6D and 5D. Indeed, if converted into flat indices, we have $h_a = e_\mu^a h_\mu + e_5^a h_z = e_a^\mu (h_\mu - A_\mu h_z) = e_a^\mu h_\mu = h_a$. We hence adopt this rule throughout in our dimensional reduction. Thus we can omit the underlines for the fields with flat indices alone. The $z$-component $h_z$ is just a scalar in 5D and may be denoted $h_z$ without an underline. For the fields with upper curved vector indices, we have $h_\mu = h^\mu, h_z = \alpha h_5 - A_a h^a$.

Now, the 6D GC transformation $\delta_{\text{GC}}(\xi^\mu)$, which appears in the supersymmetry transformation commutator $[\delta_{\text{Q}}(\epsilon_1), \delta_{\text{Q}}(\epsilon_2)]$ with $\xi^\mu = 2i\xi_1^a \gamma_5 \xi_2^a$, reduces to the 5D one, $\delta_{\text{GC}}(\xi^\nu)$, plus the $U(1)$ gauge transformation $\delta_Z(\xi^z)$, which acts only on the gravi-photon field $A_\mu$ among the gauge fields as

$$
\delta_Z(\theta) A_\mu = \partial_\mu \theta.
$$

(2.9)

Below, we identify this transformation $\delta_Z(\theta)$ with the central charge transformation acting on the hypermultiplet fields, which is originally the fifth spatial derivative $\partial_z$ in their supersymmetry transformation law in 6D.

The spinor fields in 6D superconformal theory are all $SU(2)$-Majorana-Weyl spinors $\psi^i_\pm$ satisfying simultaneously the $SU(2)$-Majorana condition

$$
\bar{\psi}^i_\pm \equiv (\psi_\pm)_i \gamma^0 = (\psi^i_\pm)^T C,
$$

(2.10)

and the Weyl condition (of positive or negative chirality)

$$
\gamma_7 \psi^i_\pm = \pm \psi^i_\pm,
$$

(2.11)
where $\gamma^a$ and $C$ are $8 \times 8$ Dirac gamma and charge conjugation matrices in 6D, and $\gamma_7 \equiv -\gamma^0\gamma^1 \cdots \gamma^5$. To make contact with the 4-component spinors in 5D, we can use the following representation of the 6D gamma matrices $\gamma^a$ given in terms of the 5D $4 \times 4$ ones $\gamma^a$ satisfying $\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = 1$:

\[
\begin{cases}
\gamma^a = \gamma^a \otimes \sigma_1 & \text{for } a = 0, 1, 2, 3, 4, \\
\gamma^5 = 1_4 \otimes i\sigma_2, \\
\gamma_7 = -\gamma^0\gamma^1 \cdots \gamma^5 = 1_4 \otimes \sigma_3.
\end{cases}
\tag{2.12}
\]

The charge conjugation matrix $C$ in 6D is given by the 5D one $C$ as

\[
C = C \otimes i\sigma_2.
\tag{2.13}
\]

Then the $SU(2)$-Majorana-Weyl spinors $\psi^\pm_i$ in 6D reduce to the forms

\[
\psi^+_i = \begin{pmatrix} \psi^i \\ 0 \end{pmatrix}, \quad \psi^-_i = \begin{pmatrix} 0 \\ i\psi^i \end{pmatrix},
\tag{2.14}
\]

and both $\psi^i$ are now the 4-component $SU(2)$-Majorana spinors in 5D satisfying

\[
\bar{\psi}^i \equiv (\psi^i)\gamma^0 = (\psi^i)^T C.
\tag{2.15}
\]

Thus the spinors appearing in 5D are all of the $SU(2)$-Majorana type, and we generally use the same symbols to denote the spinors in 5D as those used by BSVP, although they are actually related by Eq. (2.14).

In reducing to 5 dimensions by setting $e^a_a = 0$, it turns out to be simpler to also fix the conformal $S$ and $K$ gauge symmetries by using the following gauge-fixing conditions:

\[
\begin{align*}
M_{a5} : \quad e^a_z &= 0, \\
S^i : \quad \psi^i_5 &= 0, \\
K_a : \quad b_\mu - \alpha^{-1}\partial_\mu \alpha &= 0, \\
K_5 : \quad b_5 &= 0.
\end{align*}
\tag{2.16}
\]

The $S$ gauge is chosen to satisfy $\psi^i_5 = 0$ (implying also $\psi_z = e^5_z \psi_5 = 0$), so as to make the condition $e^a_z = 0$ invariant under the supersymmetry transformation, $\delta Q(\bar{\epsilon}) e^a_z = -2i\bar{\epsilon}\gamma^a \psi_z$.

Note that this gauge also implies the $Q$-invariance of the ‘dilaton’ field $\alpha \equiv (e^5_z)^{-1} = e^z_5$.

Moreover, the $K_a$ gauge $b_\mu = \alpha^{-1}\partial_\mu \alpha$ is chosen so as to make $\alpha$ also covariantly constant $\hat{\partial}_\mu \alpha = 0$ in the reduced 5D theory, as we see below. Thanks to these two properties, the field $\alpha$ carrying Weyl weight $w = 1$ can be treated as if it were a constant and is freely used to adjust the Weyl weights of any quantities to arbitrary desired values.

We, however, keep the dilatation gauge symmetry unfixed, since it becomes useful later when we change the Einstein-Hilbert and Rarita-Schwinger terms in the action into canonical form.

We here note that the extraneous components $\omega^{a5}_\mu$ and $\omega^{ab}_5$ of the spin connection $\omega^{ab}_\mu$ given by Eq. (B.2) now take the following simple form under these gauge-fixing conditions:

\[
e^{\mu a} \omega^{b5}_\mu = \omega^{ab}_5 = -\frac{1}{2\alpha} F^{ab}(A), \quad \omega^{a5}_5 = 0.
\tag{2.17}
\]
Table I. Weyl multiplet in 5D.

| field      | type     | restrictions | $SU(2)$ | Weyl-weight |
|------------|----------|--------------|---------|-------------|
| $e^a_\mu$  | boson    | fünfbein     | 1       | $-1$        |
| $\psi^i_\mu$ | fermion  | $SU(2)$-Majorana | 2       | $\frac{1}{2}$ |
| $V^{ij}_\mu$ | boson    | $V^{ij}_\mu = V^{ji}_\mu = -V^{*}_{ij}$ | 3       | $0$         |
| $A_\mu$    | boson    | gravi-photon $A_\mu = e^{5}_\mu e^5_\mu$ | 1       | $0$         |
| $\alpha$   | boson    | 'dilaton' $\alpha = e^5_\alpha$ | 1       | $1$         |
| $t^{ij}$   | boson    | $t^{ij} = t^{ji} = -t^{*}_{ij}$ ( $=-V^{ij}_5$) | 3       | $1$         |
| $v_{ab}$   | boson    | $v_{ab} = -T_{ab5} + \frac{1}{4\alpha} \hat{F}_{ab}(A)$ | 1       | $1$         |
| $\tilde{\chi}^i$ | fermion  | $SU(2)$-Majorana | 2       | $\frac{3}{2}$ |
| $C$        | boson    | real         | 1       | $2$         |

dependent gauge fields

| $b_\mu$       | boson    | $D$ gauge field $b_\mu = \alpha^{-1} \partial_\mu \alpha$ | 1       | $0$         |
| $\omega^{ab}_\mu$ | boson    | spin connection | 1       | $0$         |

Here $\hat{F}_{\mu\nu}(A)$ is the supercovariantized field strength of the gravi-photon:

$$\hat{F}_{\mu\nu}(A) = F_{\mu\nu}(A) - 2i\alpha \bar{\psi}_\mu \psi_\nu, \quad F_{\mu\nu}(A) = 2\partial_\mu A_\nu.$$  \hspace{1cm} (2.18)

Here a summary of the 5D field contents and their notation may be in order. The fields we are now treating all come from the 40+40 Weyl multiplet (2.3) in 6D, and we also call them a 5D Weyl multiplet. Since $\psi_5$ and $b_\mu$ are eliminated by the above gauge fixing, the remaining fields are now

$$e^a_\mu \rightarrow \left\{ \begin{array}{c} e^a_\mu \\ A_\mu \\ \alpha \end{array} \right\}, \quad V^{ij}_\mu \rightarrow \left\{ \begin{array}{c} V^{ij}_\mu \\ t^{ij} \\ \psi^i_\mu \end{array} \right\}, \quad \psi^i_\mu \rightarrow \psi^i_\mu,$$

$$T_{abc} \rightarrow v_{ab} \equiv -T_{ab5} + \frac{1}{4\alpha} \hat{F}_{ab}(A), \quad \chi^i \rightarrow \chi^i, \quad D \rightarrow D.$$  \hspace{1cm} (2.19)

The quantities on the right-hand sides of the arrows here, $e^a_\mu, A_\mu, \alpha, V^{ij}_\mu, t^{ij}, \psi^i_\mu, v_{ab}, \chi^i$ and $D$, denote our 5D fields. The fields $t^{ij}$ and $v_{ab}$ are defined to be particular combinations of the fields in order to simplify the expressions of the supersymmetry transformation in 5D.

In the same sense, it turns out to be convenient to use the following spinor field $\tilde{\chi}^i$ and scalar field $C$ in place of $\chi^i$ and $D$, respectively:

$$\tilde{\chi}^i \equiv \frac{1}{15}(\chi^i + \frac{3}{4}v_{ab} \hat{R}_{ab}^i(Q)) = \frac{1}{12} \chi^i + 2 \phi_5^i,$$

$$C \equiv \frac{1}{15} \left(D - \frac{3}{4} \hat{R}(M) + 15v \cdot v - \frac{9}{8\alpha^2} \hat{F}(A) \cdot \hat{F}(A) - 30t_j^i t^i_j \right).$$  \hspace{1cm} (2.20)
Here $\hat{R}_{ab}(Q)$ and $\hat{R}(M)$ are the curvatures in 5D theory discussed in detail below, and $\phi_5$ is the fifth spatial component of the 6D $S$-gauge field $\phi_a$. The members of the 5D Weyl multiplet are listed in Table I.

The supersymmetry transformation in 5D is found as follows. The $S$ and $K_a$ gauge-fixing conditions in (2.16) are not invariant under the original $Q$ transformation $\delta Q(\varepsilon)$ in 6D. The deviations of $\hat{\Delta}_Q(\varepsilon)\psi_i^j$ and $\hat{\Delta}_Q(\varepsilon)(b_\mu - \alpha^{-1}\partial_\mu \alpha)$ from zero of course can be transformed back to zero by suitable $S$ and $K_a$ gauge transformations. Thus the following combined transformation of $Q$, $S$ and $K$ is found to leave these conditions intact, and can be defined to give the supersymmetry transformation $\delta Q(\varepsilon)$ in 5D:

$$\delta Q(\varepsilon) = \hat{\Delta}_Q(\varepsilon) + \delta S(\eta(\varepsilon)) + \delta K(\xi^a(\varepsilon)),$$

with $\eta(\varepsilon)^i = -\frac{1}{16\alpha} \gamma \cdot \Gamma(A)\varepsilon^i - \frac{1}{4} \gamma \cdot v \varepsilon^i - t^j \varepsilon^j$,

$$\xi^a(\varepsilon) = -i\varepsilon(\phi^a - \eta(\psi^a) + \frac{1}{24} \gamma^a \chi),$$

$$\xi^5(\varepsilon) = i\varepsilon(\phi_5 - \frac{1}{24} \chi) = \frac{1}{2} i\varepsilon \tilde{\chi}. \quad (2.21)$$

Here, $\gamma \cdot T$ for any tensor $T_{a_1\cdots a_n}$ generally denotes the contraction $\gamma^{a_1\cdots a_n}T_{a_1\cdots a_n}$. Note that the spinors in these expressions and the following already stand for the 5D 4-component spinors defined on the right-hand side of Eq. (2.14).

Now it is straightforward to obtain the supersymmetry transformation laws of the Weyl multiplet in 5D from the superconformal transformation laws (B.3) in 6D by using Eq. (2.21).

Before doing this, however, it is better to define the covariant derivatives and to derive the relations between curvatures in 6D and 5D, since they appear in the supersymmetry transformation laws.

2.2. Covariant derivative and curvatures

We now give somewhat general discussion on the curvatures in supergravity. (Also see the discussion in Refs. 13 and 19.) Let $\{X_A\}$ denote a set of local transformation operators acting on the fields $\phi$, $\varepsilon \hat{A} X_A \phi = \delta A(\varepsilon) \phi$, and satisfy the graded algebra

$$[X_A, X_B] = f_{AB}^C X_C. \quad (2.22)$$

$f_{AB}^C$ here in general depends on the fields and we call it “structure function”. In the (Poincaré or conformal) supergravity theory, the set $\{X_A\}$ contains the ‘translation’ $P_a$, which plays a special role. The transformation operators other than $P_a$ are denoted $X_A$ with no bar over the index $A$. Let us now introduce two kinds of covariant derivatives, excluding and including $P_a$ covariantization:

$$\hat{D}_\mu \phi \equiv \partial_\mu \phi - h^A_{\mu} X_A \phi, \quad \nabla_\mu \phi \equiv \partial_\mu \phi - h^A_{\mu} X_A \phi = \hat{D}_\mu \phi - \epsilon^a_\mu P_a \phi. \quad (2.23)$$
Here, $h_{\mu}^{\dot{A}}$ are gauge fields, and sums over the repeated indices $A$ and $\dot{A}$ are implied. The operator $\nabla_{\mu}$ is the 'usual' covariant derivative, fully covariant with respect to all the gauge transformations, while $\hat{D}_{\mu}$ is the covariant derivative adopted in supergravity. Then, as in Yang-Mills theory, imposing the covariance of the 'usual' one $\nabla_{\mu}$ [i.e. $X_{\dot{A}}(\nabla_{\mu}\phi) = \nabla_{\mu}(X_{\dot{A}}\phi)$] determines the transformation law of the gauge fields as

$$\varepsilon^{\dot{A}}X_{\dot{A}}h_{\mu}^{\dot{A}} \equiv \delta(\varepsilon)h_{\mu}^{\dot{A}} = \partial_{\mu}\varepsilon^{\dot{A}} + \varepsilon^{C}h_{\mu}^{B}f_{BC}^{\dot{A}}; \quad (2.24)$$

and the commutator of $\nabla_{\mu}$ defines the curvature tensors $R_{\mu\nu}^{\dot{A}}$ in the form

$$[\nabla_{\mu}, \nabla_{\nu}] = -R_{\mu\nu}^{\dot{A}}X_{\dot{A}} \rightarrow R_{\mu\nu}^{\dot{A}} \equiv 2\partial_{[\mu}h_{\nu]}^{\dot{A}} - h_{\mu}^{C}h_{\nu}^{B}f_{BC}^{\dot{A}}. \quad (2.25)$$

Now, the particular feature of supergravity is the stipulation that the 'usual' covariant derivative $\nabla_{\mu}$ vanish on any matter field $\phi$ carrying flat indices alone:

$$\nabla_{\mu}\phi = 0 \quad \rightarrow \quad \hat{D}_{\mu}\phi = e^{a}_{\mu}P_{a}\phi. \quad (2.26)$$

In supergravity, thus, the only meaningful covariant derivative is $\hat{D}_{\mu}$, whose flat index version, $\hat{D}_{a} = e^{a}_{\mu}\hat{D}_{\mu}$, gives meaning to the 'translation' transformation $P_{a}$. This stipulation embodied by (2.26) can be imposed if and only if

$$[\nabla_{\mu}, \nabla_{\nu}] = 0 \quad \rightarrow \quad R_{\mu\nu}^{\dot{A}} = 0 \quad (2.27)$$

is satisfied. The curvature in supergravity, $\hat{R}_{ab}^{\dot{A}}$, is defined via the flat $\hat{D}_{a}$ by

$$[\hat{D}_{a}, \hat{D}_{b}] = -\hat{R}_{ab}^{\dot{A}}X_{\dot{A}} \equiv -\hat{R}_{ab}. \quad (2.28)$$

Without carrying out cumbersome computations going back to the original definition of $\hat{D}_{a}$, we can immediately find the following simple expression for this curvature,

$$\hat{R}_{\mu\nu}^{\dot{A}} = e^{b}_{\mu}e^{a}_{\nu}f_{ab}^{\dot{A}} = 2\partial_{[\mu}h_{\nu]}^{\dot{A}} - h_{\mu}^{C}h_{\nu}^{B}f_{BC}^{\dot{A}}, \quad (2.29)$$

where the prime on the structure function indicates that the $[P_{a}, P_{b}]$ commutator parts $f_{ab}^{\dot{A}}$ are excluded from the sum. The first equality follows from the relation $\hat{D}_{a} = P_{a}$, holding on any flat-indexed fields $\phi$, and

$$-\hat{R}_{ab}^{\dot{A}} = [\hat{D}_{a}, \hat{D}_{b}]^{\dot{A}} = [P_{a}, P_{b}]^{\dot{A}} = f_{ab}^{\dot{A}}, \quad (2.30)$$

and the second equality follows from $R_{\mu\nu}^{\dot{A}} = 0$ and Eq. (2.29). Conversely, if $\hat{R}_{\mu\nu}^{\dot{A}}$ is defined by Eq. (2.29), then Eq. (2.28) follows, of course, and it can be used as a convenient formula.
Another convenient formula also follows immediately from \([X_A, \hat{D}_a] = [X_A, P_a] = f_{Aa} \hat{B} X_B\) for the transformation \(\delta(\varepsilon) \equiv \varepsilon^A X_A\) not including \(P_a\):

\[
\delta(\varepsilon) \hat{D}_a \phi = \varepsilon^A \hat{D}_a (X_A \phi) + \varepsilon^A f_{Aa} \hat{B} X_B \phi.
\]  

(2.31)

Using the Jacobi identity \([X, [\hat{D}_a, \hat{D}_b]] + \text{(permutations)} = 0\) and the additional information that \(f_{ab}{}^c = -\hat{R}_{ab}{}^c(P) = 0\) and \(f_{ab}{}^c = \text{const.}\), holding generally in supergravity, we can obtain the following Bianchi identities: For \(X = P_c\), we have

\[
\hat{D}_{[a} \hat{R}_{bc]} A = -\hat{R}_{[ab} f_{c]} B A, \tag{2.32}
\]

and for \(X = X_A\) and \(\delta(\varepsilon) \equiv \varepsilon^A X_A\), we obtain

\[
\delta(\varepsilon) \hat{R}_{[ab}{}^A = 2\varepsilon^B \hat{D}_{[ab} f_{b]} B A + 2\varepsilon^B f_{B[a} C f_{b]} C A - \hat{R}_{ab} C \varepsilon^B f_{BC} A, \tag{2.33}
\]

\[
\hat{R}_{ab}^B \varepsilon^A f_{AB}{}^c = 2\varepsilon^A f_{A[a} B f_{b]} B c. \tag{2.34}
\]

Finally in this general discussion, we add a remark on the meaning of the \(P\) transformation \(\delta_P(\xi) = \xi^a P_a\) on the gauge fields \(h^A_\mu\). The GC transformation of \(h^A_\mu\) given in Eq. (2.5) can be rewritten by using \(R_{\mu\nu}{}^A\) in the form

\[
\delta_{\text{GC}}(\xi^\nu) h^A_\mu = \xi^\nu h^B_\mu X_B h^A_\mu + \xi^\nu R_{\mu\nu}{}^A. \tag{2.35}
\]

Using \(R_{\mu\nu}{}^A = 0\) and extracting the \(P_a\) term from \(X_B\), we find

\[
\delta_P(\xi) h^A_\mu = \left[ \delta_{\text{GC}}(\xi^\nu = \epsilon^\nu \varepsilon^a) - \xi^a h^B_\mu X_B \right] h^A_\mu. \tag{2.36}
\]

Comparing this with \(\delta_P(\xi) = \xi^a \hat{D}_a \phi = (\xi^\mu \partial_\mu - \xi^a h^B_\mu X_B) \phi\) for the flat quantity \(\phi\), the simple derivative term \(\xi^\mu \partial_\mu\) is replaced by the GC transformation \(\delta_{\text{GC}}(\xi^\mu)\) here. Thus the replacement \(\xi^\mu \partial_\mu \rightarrow \delta_{\text{GC}}(\xi^\mu)\) should be generally understood in \(\xi^a \hat{D}_a\) if acting on quantities with curved indices. This is a general rule, because the vielbein \(e^\mu_a\) obeys it, and the conversion of flat indices into curved ones is performed using the vielbein.

The discussion up to this point is general and applies, in particular, both to the present 6D superconformal theory and 5D supergravity, which we obtain from it. Again, to distinguish them, we write the covariant derivative and the curvatures in 6D with underlines as \(\hat{D}_a\) and \(\hat{R}_{ab}{}^A\), while those in 5D as \(\hat{D}_a\) and \(\hat{R}_{ab}{}^A\).

To find relations between \(\hat{R}_{ab}{}^A\) and \(\hat{R}_{ab}{}^A\) by using the formula (2.28), we first need the relation between the covariant derivatives \(\hat{D}_a\) and \(\hat{D}_a\). Note that, in the 5D reduced theory, the transformations \(\{X_A\}\) are only

\[
P_a, \ M_{ab}, \ U_{ij}, \ D, \ Q_{a_i}, \ Z_i; \tag{2.37}
\]
that is, \( M_{a5}, \ S_{ai} \) and \( K_a \) have disappeared in reducing from the 6D superconformal theory, whose generators are given in Eq. (2.1). Using the definition (2.23) of the covariant derivative and noting also the relation (2.21) of the \( Q \) transformations in 5D and 6D, we find

\[
\hat{\mathcal{D}}_a = \mathcal{D}_a - \delta M_{a5}(\omega_{a}^{5b}) - \delta S(\phi_a^i - \eta(\psi_a)) - \delta K_a(f_a^{5b} - \xi^b_5(\psi_a)),
\]

\[
\hat{\mathcal{D}}_5 = \delta Z(\alpha) - \delta M(\omega_5^{ab}) + \delta U(t^{ij}) - \delta S(\phi_5^i) - \delta K_a(f_5^{ab} - \xi_a - G(W_5)).
\]

Note here that the fifth spatial derivative \( \partial_z \) in \( \hat{\mathcal{D}}_b \) has been identified with the central charge transformation \( Z \), so that \( \hat{\mathcal{D}}_5 \) contains \( e_5^a \partial_z \rightarrow \alpha Z = \delta Z(\alpha) \) and \( \hat{\mathcal{D}}_a \) also contains \( e_5^a \partial_\mu + e_5^a \partial_z = \partial_a - A_a Z \). The identification \( \partial_z = Z \) holds exactly on the hypermultiplet, which we discuss below, and, moreover, it is also consistent with the previous central charge term \( \delta Z(\xi^z)A_\mu = \partial_\mu \xi^z \), which appeared as a part of 6D GC transformation of the gravi-photon \( A_\mu \) in Eq. (2.7). Actually, as remarked above, the derivative term \( \xi^z \partial_z \) in the covariant derivative \( \xi^z \hat{\mathcal{D}}_z \) should be understood as the GC transformation \( \delta_{GC}(\xi^z) \) on the gauge fields and, on \( A_\mu \), it indeed yields \( \delta_{GC}(\xi^z)A_\mu = \partial_\mu \xi^z \). This explains the reason that the gravi-photon is identified with the gauge field corresponding to the central charge of the hypermultiplet.

Note also that if Yang-Mills fields \( W_\mu \) with gauge group \( G \) are coupled to the system we should also include \( G \)-covariantization in \( \hat{\mathcal{D}}_a \), so that \( \hat{\mathcal{D}}_a \) contains \( -\delta_G(W_a) \) implicitly and \( \hat{\mathcal{D}}_5 \) contains \( -\delta_G(W_5) \) as written explicitly in the last term.

Using the relations in (2.38) and the formula (2.28) we obtain equations of the form

\[
\hat{\mathcal{R}}_{ab} = [\hat{\mathcal{D}}_b, \hat{\mathcal{D}}_a] = [\hat{\mathcal{D}}_b + \cdots, \hat{\mathcal{D}}_a + \cdots] = \hat{\mathcal{R}}_{ab} + \cdots,
\]

\[
0 = [\hat{\mathcal{D}}_a, \delta Z(\alpha)] = [\hat{\mathcal{D}}_a + \cdots, \hat{\mathcal{D}}_5 + \cdots] = \hat{\mathcal{R}}_{5a} + \cdots.
\]

The commutativity of the central charge with \( \hat{\mathcal{D}}_a \) in the second equation implies the \( U_{(1)} \)-covariance of the latter and should hold as a result of the identification \( \partial_z = Z \) on the hypermultiplet. (Confirming this directly on the gauge fields is also straightforward.) We can obtain various relations from these equations (2.39) by comparing the coefficients of each generator \( X_A \) on both sides. For example, the terms proportional to the central charge \( Z \) in the first equation yield

\[
\hat{F}_{ab}(A) = -2\alpha \omega_{ab}^5,
\]

where \( \hat{\mathcal{R}}_{\mu\nu}(Z) \) here is denoted by the more common notation \( \hat{F}_{\mu\nu}(A) \). The terms proportional to \( P_a \) and \( Z \) in the second equation give, respectively,

\[
\omega_{a5}^b = \omega_{5a}^b, \quad \omega_{5a}^{a5} = 0.
\]
Similarly the following relations can be found:

\[
\begin{align*}
\hat{R}_{5a}^i(Q) &= \frac{1}{16\alpha^2} \gamma_5 \hat{F}(A)\psi_a^i + t^i_j \psi_a^j + \frac{1}{4} \gamma_5 \psi_a^i + \phi_a^i + \gamma_a \phi_5^i \\
\hat{R}_{ab}^i(Q) &= \hat{R}_{ab}^i(Q) - 2\gamma_a \{ \phi_b - \eta^j(\psi_b) \}, \\
\hat{R}_{ab}^{cd}(M) &= \hat{R}_{ab}^{cd}(M) - \frac{1}{2\alpha^2} \hat{F}_{ab}(A) \hat{F}^{cd}(A) - \frac{1}{2\alpha^2} \hat{F}_{[a}^{[c} \hat{F}_{b]d]}(A) \\
&\quad + 8\{ f_{[a}^c - \xi_K^c(\psi_a) \} e_{b]}^d, \\
\hat{R}_{5a}^{5b}(M) &= \frac{1}{4\alpha^2} \hat{F}_{ac}(A) \hat{F}^{bc}(A) + 2\{ f_a^b - \xi_K^a(\psi_b) \} + 2f_5^5 e_a^b, \\
\hat{R}_{5a}^{ij}(U) &= \hat{D}_a t^{ij}, \\
\hat{R}_{ab}^{ij}(U) &= \hat{R}_{ab}^{ij}(U) - \frac{1}{\alpha} \hat{F}_{ab}(A) t^{ij}. 
\end{align*}
\]

Putting these into the constraints (B.1) on the 6D curvatures \( \hat{R}_{\mu\nu}^{\; ab}(M) \) and \( \hat{R}_{\mu\nu}^i(Q) \), we obtain

\[
\begin{align*}
\phi^j_\mu - \eta^j(\psi_\mu) &= \frac{1}{16}(\gamma^{ab} \gamma_\mu - \frac{3}{2} \gamma_\mu \gamma^{ab}) \hat{R}_{ab}^i(Q) - \frac{1}{12\alpha^2} \gamma_\mu \chi^i, \\
\phi^5_\mu &= \frac{1}{40} \gamma^{ab} \hat{R}_{ab}^i(Q) - \frac{1}{12\alpha^2} \chi^i, \\
f_a^a - \xi_K^a(\psi_a) + f_5 = \frac{1}{20} \hat{R}(M) + \frac{1}{80\alpha^2} \hat{F}(A)^2 - \frac{1}{40} D, \\
\hat{R}_{ab}(M) &= \hat{R}_{ac} \gamma_b(M), \quad \hat{R}(M) = \hat{R}_{a}^a(M). 
\end{align*}
\]

2.3. Supersymmetry transformation law of the Weyl multiplet

The commutation relation for the 5D supersymmetry transformation \( \delta_Q(\varepsilon) \) given in Eq. (2.21) is now easily found from the superconformal algebra (B.6) in 6D with the help of the relations (2.38) of covariant derivatives:

\[
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \xi^a \hat{D}_a + \delta_2 (\alpha \xi^5) + \delta_M (\frac{1}{2\alpha^2} \xi^5 \hat{F}(A))_{ab} + \xi_{abcd} t^{cd} + 2\xi_{ij} t^{ij} \\
+ \delta_U (-3\xi^5 t^{ij} - \frac{1}{2\alpha^2} \xi_{ij} \hat{F}(A)^{ab} - 2\xi_{ij} t^{ab}), 
\]

where

\[
\xi^a = 2i \varepsilon_1 \gamma^a \varepsilon_2, \quad \xi^5 = -2i \varepsilon_1 \varepsilon_2, \quad \xi_{abcd} = 2i \varepsilon_1 \gamma_{abcd} \varepsilon_2, \quad \xi_{ij} = 2i \varepsilon_1 \gamma_{ij} \varepsilon_2. 
\]

If we use the relations (2.38) and (2.42), it is now straightforward (although tedious) to obtain, from (B.3) and (B.5), the following supersymmetry transformation law for our Weyl multiplet in 5D. With \( \delta = \delta_Q(\varepsilon) \),

\[
\delta \psi_\mu^a = -2i \varepsilon \gamma^a \psi_\mu, 
\]
\[
\begin{align*}
\delta \psi^i & = D_{\mu} \varepsilon^i + \frac{1}{2} \gamma_{\mu ab} \varepsilon^{i} v^{ab} + \frac{1}{2 \alpha} \gamma^{a} \varepsilon^{i} \dot{F}(A)_{\mu a} + \gamma_{\mu \varepsilon^{i} t^{b} j}, \\
\delta \alpha & = 0, \quad (D_{\mu} \varepsilon^i \equiv \partial_{\mu} \varepsilon^i + \frac{1}{2} b_{\mu} \varepsilon^i - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \varepsilon^i - V_{\mu}^{i} \varepsilon^j), \\
\delta A_{\mu} & = 2 i \alpha \varepsilon \psi_{\mu}, \\
\delta V_{\mu}^{ij} & = -4 i \varepsilon^{i} (\gamma_{\mu} \bar{\kappa}^{j}) - 2 i \varepsilon^{i} (\gamma_{\mu} \bar{\kappa}^{j}) (Q) + \frac{1}{\alpha} \varepsilon^{i} (\gamma_{\mu} \bar{\kappa}^{j}), \\
\delta t^{ij} & = 4 i \varepsilon^{i} (\bar{\kappa}^{j}), \\
\delta v_{ab} & = \frac{1}{4} \varepsilon_{abcd} \bar{R}_{cd} (Q) - 2 i \varepsilon \gamma_{ab} \bar{\kappa}, \\
\delta \bar{\kappa}^{i} & = -\frac{1}{2} \gamma^{a} \varepsilon^{i} \dot{D}^{b} v_{ab} + \frac{1}{2} \dot{D}_{j} v^{j} - \gamma_{\mu} v_{\alpha} \varepsilon^{j} - \frac{1}{4 \alpha} \gamma_{\mu} \bar{\kappa} (A) t^{i} j \varepsilon^{j} \\
& \quad + \frac{1}{2} C \varepsilon^{i} - \frac{1}{4 \alpha^{2}} \gamma_{abcd} \varepsilon^{i} \dot{F}_{ab} (A) \dot{F}_{cd} (A), \\
\delta C & = -2 i \varepsilon \dot{D} \bar{\kappa} + 22 i \varepsilon \bar{\kappa} t_{ij} - 3 i \varepsilon \gamma_{\mu} v \bar{\kappa} - i \varepsilon^{i} \gamma_{\mu} \bar{R}_{ij} (Q) t_{ij}, \\
\delta \omega_{\mu}^{ab} & = -2 i \varepsilon \gamma_{\mu} \bar{R}_{ij} (Q) - i \varepsilon \gamma_{\mu} \bar{R}_{ij} (Q) \\
& \quad + \frac{2 i}{\alpha} \varepsilon \psi_{\mu} \bar{R}_{ab} (A) - 2 i \varepsilon \gamma_{abcd} \psi_{cd} - 4 i \varepsilon^{i} \gamma \psi_{\mu} (A) t_{ij}. \tag{2.47}
\end{align*}
\]

Here $\bar{\kappa}^{i}$ and $C$ are the redefined fields from $\chi^{i}$ and $D$ in Eq. (2.20), and we have also written the transformation law of the spin connection, although it is a dependent field given by (from Eq. (B.2))

\[
\omega_{\mu}^{ab} = \omega_{\mu}^{ab} + i (2 \gamma_{\alpha} [a \psi_{\mu}^{b}] + \psi_{\mu}^{a} \gamma_{\alpha} \psi_{\mu}^{b}) - 2 \epsilon_{\mu}^{[a} \alpha^{-1} \delta^{b]} \alpha. \tag{2.48}
\]

The $[\delta Q, \delta Q]$ commutation relation (2.45) can also be read directly from the structure functions appearing in Eq. (2.47): comparing Eqs. (2.23) and (2.24), we see that, for instance, the last three terms in $\delta \omega_{\mu}^{ab}$ and $\delta V_{\mu}^{ij}$ just give the $\delta \dot{M}$ and $\delta U$ terms in Eq. (2.45), respectively.

When deriving these transformation laws (2.47), we need the following transformation laws of curvature tensors, which follow from the formula (2.34):

\[
\begin{align*}
\delta \dot{F}_{ab} (A) & = 2 i \alpha \varepsilon \bar{R}_{ab} (Q), \\
\delta \dot{R} (M) & = 4 i \varepsilon \dot{D}^{a} \gamma^{b} \bar{R}_{cd} (Q) + 4 i \varepsilon \gamma_{abcd} \bar{R}_{cd} (Q) v^{c} + 4 i \varepsilon \gamma_{ab} \bar{R}_{bc} (Q) v^{c} \\
& \quad + \frac{2}{\alpha} \varepsilon \gamma^{a} \bar{R}_{bc} (Q) \dot{F}_{c} (A) - \frac{4 i}{\alpha} \varepsilon \bar{R}_{ab} (Q) \dot{F}_{c} (A) + 8 i \varepsilon \gamma \cdot \dot{R}^{i} (Q) t_{ij}, \\
\delta \gamma \cdot \dot{R}^{i} (Q) & = 2 (\dot{D}_{\alpha} \gamma_{\mu} v) \gamma^{a} \varepsilon^{i} - 10 \gamma^{a} \varepsilon^{i} \dot{D}^{b} v_{ab} - \frac{1}{\alpha} \gamma^{a} \varepsilon^{i} \dot{D}^{b} \dot{F}_{ab} (A) + 8 \dot{D}_{b} v^{i} \varepsilon^{j} + \\
& \quad + \gamma_{abcd} \varepsilon^{i} \left(2 v_{ab} v_{cd} - \frac{1}{\alpha} v_{ab} \dot{F}_{cd} (A) - \frac{1}{2 \alpha^{2}} \dot{F}_{ab} (A) \dot{F}_{cd} (A) \right) \\
& \quad + \frac{4}{\alpha} \gamma^{a} \varepsilon^{i} \dot{F}_{ae} (A) v_{b} - 12 \gamma_{\mu} v_{t} \varepsilon^{j} \varepsilon^{j} - \frac{6}{\alpha} \gamma \cdot \dot{F} (A) \dot{t}^{j} \varepsilon^{j} - \gamma \cdot \dot{R}^{j} (U) \varepsilon^{j} \\
& \quad - \left(\frac{1}{2} \dot{R} (M) - 6 v \cdot v + \frac{1}{2 \alpha^{2}} \dot{F} (A) \cdot \dot{F} (A) + 20 t_{k} k_{j} \varepsilon^{j} \right) \varepsilon^{i}. \tag{2.49}
\end{align*}
\]

We also need the Bianchi identities following from Eq. (2.32):

\[
\begin{align*}
\bar{D}_{a} \bar{R}_{bc}^{i} (Q) & = -\frac{1}{2} \gamma_{de[a} \bar{R}_{bc]}^{i} (Q) v^{de} - \frac{1}{2 \alpha} \gamma^{d} \bar{R}_{[ab]}^{i} (Q) \bar{F}_{c} (A) - \gamma_{a} \bar{R}_{bc}^{j} (Q) t_{j}^{i}, \\
\bar{D}_{a} \bar{F}_{bc} (A) & = 0. \tag{2.50}
\end{align*}
\]
We will not need explicit expressions for our 5D curvatures in this paper, but we list them for the reader’s convenience:

\[
\begin{align*}
\hat{F}_{\mu\nu}(A) &= 2\partial_{[\mu}A_{\nu]} - 2i\alpha\bar{\psi}_\mu\psi_\nu, \\
\hat{\mathcal{R}}_{\mu\nu}^{ij}(Q) &= 2D_{[\mu}\psi_{\nu]}^{ij} + \gamma_{ab[\mu}\psi^{ij}_{\nu]}v^{ab} - \frac{1}{\alpha} \gamma^a_{\mu}\bar{\psi}^{ij}_{[\mu}\hat{F}_{\nu]}a(A) + 2\gamma_{[\mu}\psi^{ij}_{\nu]}t^{ij}_{\nu]}, \\
\hat{R}_{\mu\nu}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^{[ac}\omega_{\nu]}^{b]} + 4i\bar{\psi}_{[\mu}\gamma^{[a}\hat{R}_{\nu]}^{b]}(Q) + 2i\bar{\psi}_{[\mu}\gamma_{\nu]}\hat{R}^{ab}(Q) \\
&\quad - \frac{2i}{\alpha} \bar{\psi}_{\mu}\psi_{\nu}\hat{F}_{ab}(A) + 2i\bar{\psi}_{\mu}\gamma^{abcd}\psi_{\nu}v_{cd} + 4i\bar{\psi}_{\mu}\gamma^{ab}\psi^{ij}_{\nu}t^{ij}_{\nu],} \\
\hat{R}_{\mu\nu}^{ij}(U) &= 2\partial_{[\mu}V_{\nu]}^{ij} + 2V_{[\mu}^{(i}V_{\nu]}^{j)k} + 8i\bar{\psi}_{[\mu}\gamma_{\nu]}\chi^{ij} + 4i\bar{\psi}_{[\mu}\gamma^{a}\hat{R}_{\nu]}^{ij}(Q) \\
&\quad - \frac{i}{\alpha} \bar{\psi}_{\mu}\gamma_i \hat{F}(A)\psi^{ij}_\nu - 4i\bar{\psi}_{\mu}\gamma_i \psi^{ij}_\nu + 6i\bar{\psi}_{\mu}\psi_{\nu}t^{ij}. \quad (2.51)
\end{align*}
\]

In the following, a quantity \(C\) appears. This is related to \(C\) (or \(D\)) and the scalar curvature \(\hat{R}(M)\) as

\[
\begin{align*}
C &= \frac{1}{6}D + 4(f_a^a - \xi_a^a(\psi_a^a) + f_5^5) \\
&= C + \frac{1}{4}\hat{R}(M) + \frac{1}{8\alpha^2}\hat{F}(A)^2 - v\cdot v + 2t^{ij}_i. \quad (2.52)
\end{align*}
\]

§3. Transformation laws of matter multiplets in 5D

Now it is straightforward to derive the supersymmetry transformation rules for the matter multiplets in 5 dimensions from the superconformal rules in 6D given by BSVP, which are summarized in Appendix B. We can simply apply the formulas (2.21) and (2.38) for our supersymmetry transformation \(\delta_Q(\varepsilon)\) and covariant derivative \(\hat{D}_\mu\).

In the following we omit explicit expressions for the covariant derivatives \(\hat{D}_\mu\phi\) on various fields \(\phi\) for conciseness. In our 5D calculus we have

\[
\begin{align*}
\hat{D}_\mu\phi &= \mathcal{D}_\mu\phi - \delta_Q(\psi_\mu)\phi, \\
\mathcal{D}_\mu\phi &= \left(\partial_\mu - \delta_M(\omega_\mu^{ab}) - \delta_U(V^{ij}_\mu) - \delta_D(\alpha^{-1}\partial_\mu\alpha) - \delta_Z(A_\mu) - \delta_G(W_\mu)\right)\phi, \quad (3.1)
\end{align*}
\]

where \(\mathcal{D}_\mu\) is covariant only with respect to homogeneous transformations \(M_{ab}, U_{ij}, D, Z\) and \(G\) (Yang-Mills gauge transformation). The transformation rules under such homogeneous transformations are obvious, and are the same as those given in (B-4) for the 6D case. The supersymmetry transformation rules are explicitly given below for all the fields, so that the covariant derivatives \(\hat{D}_\mu\phi\) will be clear.

3.1. Vector multiplet

The vector multiplet in 5D derived from the 6D one consists of the fields given in Table [\textsuperscript{II}], where the real scalar \(M\) comes from the fifth spatial component of a 6D vector, \(M \equiv -W_5 = \)
Table II. Matter multiplets in 5D.

| field     | type   | restrictions         | SU(2) | Weyl-weight |
|-----------|--------|----------------------|-------|-------------|
|           | Vector multiplet |                     |       |             |
| $W_\mu$  | boson  | real gauge field     | 1     | 0           |
| $M$      | boson  | real, $M = -W_5$    | 1     | 1           |
| $O^i$    | fermion | $SU(2)$-Majorana       | 2     | $\frac{3}{2}$ |
| $Y_{ij}$ | boson  | $Y^{ij} = Y^{ji} = -Y^{ij*}$ | 3     | 2           |
|           | Linear multiplet |                   |       |             |
| $L^{ij}$ | boson  | $L^{ij} = L^{ji} = -(L_{ij})^*$ | 3     | 3           |
| $\phi^i$ | fermion | $SU(2)$-Majorana | 2     | $\frac{7}{2}$ |
| $E_a$    | boson  | real, constrained by (3.6) | 1     | 4           |
| $N$      | boson  | real, $N = -E_5$   | 1     | 4           |
|           | Nonlinear multiplet |               |       |             |
| $\phi^\alpha_i$ | boson  | $(\phi^\alpha_i)^* = -\phi^\alpha_i$ | 2     | 0           |
| $\lambda^i$ | fermion | $SU(2)$-Majorana | 2     | $\frac{1}{2}$ |
| $V_a$    | boson  | real                 | 1     | 1           |
| $V_5$    | boson  | real                 | 1     | 1           |
|           | Hypermultiplet |                        |       |             |
| $A^\alpha_i$ | boson  | $A^\alpha_i = \gamma^{ij} A^\beta_j \rho_{\beta \alpha} = -(A^\alpha_i)^*$ | 2     | $\frac{3}{2}$ |
| $\zeta^\alpha$ | fermion | $\zeta^\alpha \equiv (\zeta_\alpha)^1 \gamma_0 = \zeta^\alpha T$ | 1     | 2           |
| $F^\alpha_i$ | boson  | $F^\alpha_i = -(F^\alpha_i)^*$ | 2     | $\frac{5}{2}$ |

$-\alpha W_z$. Note that it carries a Weyl weight $w = 1$, since $w(\alpha) = 1$ and $w(W_{\mu}) = 0$. As in 6D, all the component fields are Lie-algebra valued, e.g., $M$ is a matrix $M^{\alpha \beta} = M^A(t_A)^{\alpha \beta}$, where the $t_A$ are (anti-hermitian) generators of the gauge group $G$. The $Q$ transformation rules are found from (B.7) to be

$$
\begin{align*}
\delta W_\mu &= -2i\bar{\epsilon}\gamma_\mu \Omega + 2i\bar{\epsilon}\gamma_\mu M, \\
\delta M &= 2i\bar{\epsilon} \Omega, \\
\delta \Omega^i &= -\frac{1}{4} \gamma \cdot \hat{G}(W) \bar{\epsilon}^i - \frac{1}{2} \gamma^\alpha \bar{\epsilon}^i \hat{D}_\alpha M - Y^{ij} \bar{\epsilon}_j, \\
\delta Y^{ij} &= 2i\bar{\epsilon}^{(i} \tilde{\hat{F}}_{ab} \Omega^{j)} - i\bar{\epsilon}^{(i} \gamma \cdot v \Omega^{j)} + 2i\bar{\epsilon}^{(i} t^j_k \Omega^k + 4i\bar{\epsilon} \Omega t^{ij} - 2ig\bar{\epsilon}^{(i} [M, \Omega^{j)]}, \\
\end{align*}
$$

(3.2)

where $g$ is a gauge coupling constant and $\hat{G}_{ab}(W)$ is the following combination of the super-covariant field strength $\hat{F}_{ab}(W)$ of $W_\mu$ and that of the gravi-photon $A_\mu$:

$$
\hat{G}_{ab}(W) \equiv \hat{F}_{ab}(W) - \frac{1}{\alpha} M \hat{F}_{ab}(A),
$$

(3.3)
which is actually the field strength $\hat{E}_{ab}(W)$ in 6D. The gauge group $G$ can be regarded as a sub-group of the super group, and the above transformation law of the gauge field $W_\mu$ provides us with the additional structure functions, $f_{PQ}^G$ and $f_{QQ}^G$. For example, the transformation law of this mixed field strength can be obtained easily from (2.34):

$$\delta \hat{G}_{ab}(W) = 4i\bar{\varepsilon}_{\gamma [a} \hat{D}_{b]} \Omega - 2i \bar{\varepsilon}_{\gamma a b c d} \Omega v^c d - 4i \bar{\varepsilon}_{\gamma [c a] \Omega v_{b]}^c} - \frac{2i}{\alpha \bar{\varepsilon}_{\gamma c [a} \hat{F}_{b]}^c \Omega t_{ij}}. \quad (3.4)$$

### 3.2. Linear multiplet

The linear multiplet consists of the components listed in Table II and may generally carry a non-Abelian charge of the gauge group $G$. A point which should be noted in the reduction in this case is that, for later convenience in constructing actions, we have lowered the Weyl weight of this multiplet in 5D by one from that in 6D by multiplying each component field by $\alpha^{-1}$.

The $Q$ transformation rules following from (B.8) read

$$\delta L^{ij} = 2i\bar{\varepsilon}^{(i}(\phi^{j)},$$

$$\delta \phi^i = -\hat{D} L^{ij} \varepsilon_j - 4t^i_k L^k \varepsilon^i - 6t^i(k L^j) \varepsilon_j + gM L^{ij} \varepsilon_j + \frac{1}{2} \gamma^a \varepsilon^i E_a + \frac{1}{2} \varepsilon^i N + \frac{1}{2} \gamma^b \hat{F}(A) \varepsilon_j L^{ij} + 2 \gamma \cdot v \varepsilon_j L^{ij},$$

$$\delta E_a = 2i \bar{\varepsilon}_{\gamma a b} \hat{D}^b \varphi - \frac{i}{\alpha} \bar{\varepsilon}_{\gamma a b c} \hat{F}^{b c} \varphi$$

$$- 8i \bar{\varepsilon}^{\gamma a b} \hat{D}^b \varphi = 2i \bar{\varepsilon}_{\gamma a b c} \hat{D}^{b c} (A) - 2i \bar{\varepsilon}_{\gamma a b c} \varphi v^{b c} + 6i \bar{\varepsilon}^{b} \varphi v_{ab}$$

$$- 8i \bar{\varepsilon}^{\gamma a b} \hat{D}^b \varphi t_{ij} + 2i \bar{\varepsilon}^{\gamma a b c} \hat{F}^{b c j} (Q) L_{ij} + 2i g \bar{\varepsilon}_{\gamma a} M \varphi - 4i g \bar{\varepsilon}^{\gamma a} \Omega^i L_{ij},$$

$$\delta N = -2i \hat{D} \varphi - 3i \bar{\varepsilon} \cdot v \varphi - 2i \bar{\varepsilon}^{\gamma} \hat{R}^{\gamma i j} (Q) L_{ij} - 10i \bar{\varepsilon}^{\gamma} \varphi^i t_{ij} + 4i g \bar{\varepsilon} \Omega^i L_{ij}. \quad (3.5)$$

This multiplet apparently contains nine Bose and eight Fermi fields. Thus closure of the algebra requires the following $Q$-invariant constraint, which also follows directly from that in 6D, (B.3):

$$\hat{D}^a E_a + i \hat{\varphi} \gamma \cdot \hat{R} (Q) + g M N + 4i g \hat{\Omega} \varphi + 2g Y_{ij} L_{ij} = 0. \quad (3.6)$$

When the linear multiplet carries no gauge-group charges (i.e., $g = 0$ in the above transformation law), the constraint (3.6) can be solved with respect to $E^a$. Indeed, the constraint can be rewritten in the form $e^{-1} \partial_\lambda (e V^\lambda) = 0$ with

$$V^\lambda \equiv e^a E^a + 2i \bar{\psi}_\rho \gamma^{\lambda \rho} \varphi + 2i \psi_\mu \gamma^{\lambda \mu} \psi_i^j L_{ij}. \quad (3.7)$$

Hence there exists an antisymmetric tensor gauge field $E^{\mu \nu}$, which is a tensor-density and possesses an additional gauge symmetry $\delta E^{\mu \nu} = \partial_\mu (A^{\mu \nu})$:

$$e V^\lambda = -\partial_\mu E^{\lambda \mu} \rightarrow E^a = -e^{-1} e^a \hat{D}_\mu E^{\mu \nu}, \quad (3.8)$$
where \( \tilde{D}_\nu E^{\mu\nu} \) is defined by
\[
\tilde{D}_\nu E^{\mu\nu} = \partial_\nu E^{\mu\nu} + 2ie\bar{\psi}_\nu \gamma^{\mu\nu} \varphi + 2ie\bar{\psi}_\nu \gamma^{\mu\nu} \psi^i L_{ij}.
\]

Because of the covariance of \( E^a \), the \( Q \) transformation law of \( E^{\mu\nu} \) must take the following form, which is also in accordance with a direct calculation:
\[
\delta E^{\mu\nu} = -2ie\bar{\varepsilon} \gamma^{\mu\nu} \varphi - 4ie\bar{\varepsilon} \gamma^{\mu\nu} \psi^i L_{ij}.
\]

### 3.3. Nonlinear multiplet

The nonlinear multiplet is a multiplet whose component fields are transformed nonlinearly. The component fields are also shown in Table II. The first component, \( \Phi^i_\alpha \), carries an additional gauge-group \( SU(2) \) index \( \alpha \) (\( = 1, 2 \)), as well as the superconformal \( SU(2) \) index \( i \). The index \( \alpha \) is also raised and lowered by using the invariant tensors \( \epsilon^{\alpha\beta} \) and \( \epsilon_{\alpha\beta} \), \( \epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \frac{i}{2} \sigma_2 \alpha\beta \):
\[
\Phi^i_\alpha = \Phi^i_\beta \epsilon_{\beta\alpha}, \quad \epsilon^{\gamma\alpha} \epsilon_{\gamma\beta} = \delta^{\alpha}_\beta.
\]
The field \( \Phi^i_\alpha \) takes values in \( SU(2) \) and hence satisfies
\[
\Phi^i_\alpha \Phi^i_\beta = \delta^i_\beta.
\]
The \( Q \) transformation laws of the nonlinear multiplet, following from (B.10), are
\[
\delta \Phi^i_\alpha = 2i\bar{\varepsilon}(i\lambda^j) \Phi^i_\alpha, \quad \delta \lambda^i = -\Phi^i_\alpha \Phi^a_\beta \bar{\varepsilon}^a - \frac{1}{2} \gamma^a V_\alpha \varepsilon^i - \frac{1}{2} V_\alpha \lambda^i - \frac{i}{2} \gamma^a \bar{\varepsilon}_j \gamma_a \lambda^j - \frac{i}{2} \varepsilon_j \lambda^i \lambda^j - \frac{i}{8} \gamma^{ab} \varepsilon_i \lambda_\alpha \lambda^a + \frac{1}{4} \gamma^a \hat{\gamma} \hat{\gamma}(A) \varepsilon^i + \gamma \cdot \gamma \varepsilon^i + 3t_{ij} \varepsilon^i + g M^a_\alpha \Phi^a_\beta \varepsilon^i,
\]
\[
\delta V_\alpha = 2i\bar{\varepsilon} \gamma_\alpha \tilde{D}^b \lambda - \frac{1}{2a} \bar{\varepsilon}_\alpha \gamma_\beta \tilde{D}^b (A) + 2i\bar{\varepsilon} \gamma^b \nu_{ab} \lambda
\]
\[
- i\bar{\varepsilon}_\gamma \gamma \cdot V \lambda - i\bar{\varepsilon}_\gamma \nu_{ab} \lambda - 2i\bar{\varepsilon}_\gamma \nu_\alpha \Phi^a_\beta \Phi^a_\alpha \lambda^j - 4i \bar{\varepsilon}_\gamma \gamma_\alpha \Omega^j_\beta \Phi^i_\beta \Phi^j_\alpha \Phi^i_\beta
\]
\[
+ 4i \bar{\varepsilon}_\gamma \gamma \omega \hat{R}_{ab}(Q) - 2i\bar{\varepsilon}_\gamma \gamma \lambda^j \varepsilon^i - 2i \bar{\varepsilon}_\gamma \gamma_\alpha \lambda^j \Phi^i_\beta \Phi^j_\alpha \Phi^i_\beta,
\]
\[
\delta V_5 = 2i\bar{\varepsilon} \gamma \cdot V \lambda - \frac{1}{2a} \bar{\varepsilon}_\alpha \gamma \cdot \hat{\gamma} \hat{\gamma}(A) \lambda - i\bar{\varepsilon} \gamma^{ab} \hat{R}_{ab} (Q) + 4i \bar{\varepsilon} \hat{\gamma}
\]
\[
- i\bar{\varepsilon} \gamma \cdot V \lambda - i\bar{\varepsilon} \nu_\gamma \lambda - 2i\bar{\varepsilon}_\gamma \Phi^a_\beta \Phi^a_\alpha \lambda^j
\]
\[
- 2i \bar{\varepsilon}_\gamma \lambda^t \lambda^j M^a_\beta \Phi^i_\alpha \Phi^i_\beta - 4ig \bar{\varepsilon} \gamma_\beta \Omega^j_\beta \Phi^i_\beta \Phi^j_\alpha \Phi^i_\beta.
\]
As in the linear multiplet case, the nonlinear multiplet also needs the following \( Q \)-invariant constraint for the closure of the algebra:
\[
2\mathcal{C} + \hat{D}_a V^a + 2i\hat{\lambda} \hat{D} \lambda + \hat{D}^a \Phi^i_\alpha \hat{D} \Phi^a_\alpha - \frac{1}{2} V^a V_a + \frac{1}{2} V_5^2
\]
\[
+ 8i \bar{\lambda} \hat{\gamma} - i\bar{\lambda} \gamma^{ab} \hat{R}_{ab} (Q) - \frac{1}{2a} \bar{\lambda} \gamma \cdot \hat{\gamma} \hat{\gamma}(A) \lambda + i\bar{\lambda} \gamma \cdot \nu \lambda + 2i \bar{\lambda} \Phi^a_\alpha \hat{D} \Phi^a_\alpha \lambda^j
\]
\[
+ 2g Y^{\alpha \beta}_i \Phi^i_\alpha \Phi^i_\beta - 8ig \bar{\lambda} \lambda_\beta \Phi^i_\alpha \Phi^i_\alpha + 2i g \bar{\lambda} \lambda^j M^a_\beta \Phi^i_\beta
\]
\[
+ t^i_j t^i_j - 2g M^a_\beta t^i_j \Phi^i_\alpha \Phi^i_\beta + g^2 M^a_\beta M^a_\beta = 0.
\]
§4. Hypermultiplet

4.1. Off-shell hypermultiplet

The hypermultiplet consisting of a boson $A^\alpha_i$ and a fermion $\zeta^\alpha$ in 6D is an on-shell multiplet, and thus the supersymmetry algebra closes only when it satisfies the equations of motion. When we go down to 5D, we can make it an off-shell multiplet. The procedure is essentially identical to that known for the 4D case. The only difference is that the necessary $U_Z(1)$ gauge multiplet is not added by hand but is automatically included in our case as a multiplet containing the gravi-photon field $A_\mu$.

Explicitly we proceed as follows. In the original supersymmetry transformation law (B.15) of the hypermultiplet in 6D, we first make all the Weyl multiplet and transformation parameters $z$ independent, while keeping only the hypermultiplet members $z$ dependent. Even at this stage, clearly, we still have the same form for the supersymmetry algebra as the original 6D form (B.16), in which $\Gamma^\alpha$ is the non-closure function on $\zeta^\alpha$ given in (B.17). However, as shown by Eq. (B.18), $\Gamma^\alpha$ still closes under the superconformal transformations, together with $C^\alpha_i$, also defined in (B.17). We note that in deriving these transformation laws, the only property required for $\partial z$ is that it commutes with all the other transformations $X_A$. Therefore, even if the fifth spatial derivative $\partial_z$ is replaced by the central charge transformation $Z$ everywhere in these, exactly the same form for the transformation laws will hold, provided that $Z$ is actually a central charge. At this stage, the 6D supersymmetry transformation becomes

$$\delta Q^\epsilon(\varepsilon) = \delta_Q^{\text{org}}(\varepsilon)|_{\partial_z \to Z} . \quad (\varepsilon = \varepsilon(x) : z\text{-independent}) \quad (4.1)$$

However, then, the conditions $\Gamma^\alpha|_{\partial_z \to Z} = 0$ and $C^\alpha_i|_{\partial_z \to Z} = 0$ are no longer the on-shell conditions but become merely defining equations for the central-charge transformations $Z\zeta^\alpha$ and $Z(ZA^\alpha_i)$ in terms of the other fields. Since the ‘on-shell condition’ $C^\alpha_i = 0$ for the boson $A^\alpha_i$ is second order in $\partial_z$, there appears no constraint on the first central charge transformation $ZA^\alpha_i$, so that it defines an auxiliary field

$$F^\alpha_i \equiv \alpha ZA^\alpha_i, \quad (4.2)$$

which is necessary for closing the algebra off-shell and balancing the numbers of boson and fermion degrees of freedom. The factor of the ‘dilaton’ $\alpha$ is included to adjust the Weyl weight of $F^\alpha_i$ so as to have $w(F^\alpha_i) - w(A^\alpha_i) = 1$, for convenience.

* The other way around, it is also well-known to treat the central charge by introducing an extra coordinate $z$. 
The superconformal transformation $X_\alpha$ of $\mathcal{F}_i^\alpha$ can be found from the requirement that $Z$ commutes with all $X_\alpha$ as $[X_\alpha, \mathcal{F}_i^\alpha] = Z(X_\alpha \mathcal{A}_i^\alpha)$. This guarantees the central charge property $[Z, X_\alpha] = 0$ on $\mathcal{A}_i^\alpha$. This property holds also on $\zeta^\alpha$ and $\mathcal{F}_i^\alpha$ if their central-charge transformations $Z\zeta^\alpha$ and $Z\mathcal{F}_i^\alpha = \alpha Z(\mathcal{Z}\mathcal{A}_i^\alpha)$ are defined by $\Gamma^\alpha|_{\partial_z \rightarrow Z} = 0$ and $C_i^\alpha|_{\partial_z \rightarrow Z} = 0$, as above. This is the case because the set of conditions $\Gamma^\alpha = 0$ and $C_i^\alpha = 0$ is invariant under the superconformal transformations. For later convenience, we note that $\Gamma^\alpha$ contains two $\partial_z$ terms, $-i\gamma^z \partial_z \zeta^\alpha = -i\gamma^z (\partial_z \mathcal{A}_j^\alpha) \psi_{\alpha \beta}^j$, so that $\partial_z \zeta$ determined by $\Gamma^\alpha|_{\partial_z \rightarrow Z} = 0$ can be written using $\gamma^z \gamma^z = g^{zz}$ in the form

$$\delta_Z \zeta^\alpha = -i(g^{zz})^{-1} \gamma^z \left( \Gamma^\alpha - \gamma^z (\alpha^{-1} \mathcal{F}_j^\alpha - \partial_z \mathcal{A}_j^\alpha) \psi_{\alpha \beta}^j \right) + \partial_z \zeta^\alpha.$$  \hspace{1cm} (4.3)

Next we fix the gauges of $M_{a_5}$, $S^i$ and $K_{a_5}$ as done above, and then the supersymmetry transformation $\delta_Q(\varepsilon)$ in 5D is given by Eq. (2.21), where $\delta_Q(\varepsilon)$ is the 6D supersymmetry transformation with $\partial_z$ replaced by $Z$, given in (4.1). The relations between the covariant derivatives in 6D and 5D are given exactly by Eq. (2.38). Here also, it is convenient to lower the Weyl weight of the hypermultiplet in 5D by 1/2 from that in 6D by multiplying each component by $\alpha^{-1/2}$. Doing this, the Weyl weights of the hypermultiplet members become those given in Table I.

The supersymmetry transformation law for the 5D hypermultiplet determined this way is given by

$$\delta \mathcal{A}_i^\alpha = 2i\varepsilon^i \zeta^\alpha,$$
$$\delta \zeta^\alpha = -\bar{\mathcal{D}} \zeta^\alpha \varepsilon_i + \mathcal{F}_i^\alpha \varepsilon_i + 3t_{ij} \varepsilon^i \mathcal{A}_j^\alpha + gM_{a \beta} \varepsilon^i \mathcal{A}_{ij}^\beta + \frac{1}{4\alpha} \gamma^i \mathcal{D}(A) \varepsilon_i \mathcal{A}_j^\alpha + \gamma^i \varepsilon_i \mathcal{A}_j^\alpha,$$
$$\delta \mathcal{F}_i^\alpha = -2i\varepsilon^i \left( \bar{\mathcal{D}} \zeta^\alpha - \frac{1}{4\alpha} \gamma^i \mathcal{D}(A) \zeta^\alpha + \frac{1}{2} \gamma^i \varepsilon \zeta^\alpha - \frac{1}{2} \mathcal{A}_i^\alpha \gamma^{ab} \mathcal{R}_{ab} (Q) \right)$$
$$+ 2\mathcal{A}_i^\alpha \mathcal{R}_i \mathcal{A}_j^\alpha + gM_{a \beta} \zeta^\beta + 2g\Omega_{a \beta} \zeta^\alpha \right). \hspace{1cm} (4.4)$$

The central charge transformation law is

$$\alpha \delta_Z(\theta) \mathcal{A}_i^\alpha = \theta \mathcal{F}_i^\alpha,$$
$$\alpha \delta_Z(\theta) \zeta^\alpha = -\theta \left( \bar{\mathcal{D}} \zeta^\alpha - \frac{1}{4\alpha} \gamma^i \mathcal{D}(A) \zeta^\alpha + \frac{1}{2} \gamma^i \varepsilon \zeta^\alpha - \frac{1}{2} \mathcal{A}_i^\alpha \gamma^{ab} \mathcal{R}_{ab} (Q) \right)$$
$$+ 2\mathcal{A}_i^\alpha \mathcal{R}_i \mathcal{A}_j^\alpha + gM_{a \beta} \zeta^\beta + 2g\Omega_{a \beta} \zeta^\alpha \right), \hspace{1cm} (4.5)$$

One should note that the covariant derivatives $\mathcal{D}_\alpha$ here as well as in (4.4) contain also the covariantization term $-\delta_Z(A_\alpha)$ with respect to the central charge, so that these definitions
of $\delta_Z(\theta)\zeta_\alpha$ and $\delta_Z(\theta)F^i_\alpha$ by the second and third equations of (4.5) are recursive. However, they can easily be solved algebraically; for instance, the second equation gives

$$
\delta_Z(\theta)\zeta_\alpha = -\theta^{\alpha} + \frac{1}{\alpha^2 - A^2} \left( \tilde{D}'_\alpha \zeta_\alpha - \frac{1}{4\alpha} \gamma_i \tilde{F}(A) \zeta_\alpha + \frac{1}{2} \gamma_i v \zeta_\alpha - \frac{1}{2} A_\alpha^i \gamma^{ab} \tilde{R}_{ab}(Q) + 2 A_\alpha^i \bar{\chi}_i - g M_\alpha^\beta \zeta_\beta + 2 g \Omega_\alpha^\beta \Lambda_\beta \right),
$$

(4.6)

where $\tilde{D}'_\alpha$ denotes a covariant derivative with the $-\delta_Z(A_a)$ term omitted.

4.2. Hypermultiplet action

Note that we did not actually have to throw away the $z$ dependence of the hypermultiplet in the above 5D supersymmetry transformation. If $z$ is retained, it merely represents a continuous label of an infinite number of copies of the 5D hypermultiplets which all transform in the same way.

In 6D, the action $S_0$ for the hypermultiplet was constructed in the form

$$
S_0 = \int d^5x \int dz e_6 \left\{ A_i^\alpha d_\alpha^\beta C_i^\beta + 2(\bar{\zeta}^\alpha - i \bar{\psi}^\beta \gamma^\mu A_i^\alpha) d_\alpha^\beta \Gamma_\beta \right\},
$$

(4.7)

so that the equations of motion give the desired ‘on-shell’ conditions $\Gamma_\alpha^\beta = C_i^\alpha = 0$, where $e_6$ is the determinant of the sechsbein and $d_\alpha^\beta$ is a $G$-invariant tensor. This action is fully invariant under the original superconformal transformation in 6D.

Here, in the action, let us take all the Weyl multiplet (and transformation parameters) to be $z$ independent, while keeping the hypermultiplet $z$ dependent and using the original $C_i^\alpha$ and $\Gamma_\alpha$, in which $\partial_z$ are not replaced by $Z$. Then, the action represents an action for the infinite copies of the 5D hypermultiplets labeled by $z$, but is, of course, not invariant under the above 6D supersymmetry transformation $Q_{\alpha}(\varepsilon)$ with $\partial_z$ replaced by $Z$, since, for instance, it contains no auxiliary field $F^i_\alpha$. However, we can make it invariant with a small modification as follows.

From our knowledge of global supersymmetric theory, we expect that the quadratic term $\partial_z A_\alpha^i d_\alpha^\beta \partial_z A_\beta^i$ does not appear in the action and that the auxiliary field $F^i_\alpha$ appears as a replacement of $\alpha \partial_z A_\alpha^i$. Thus we are led to trying the following action (before doing the overall rescaling of the hypermultiplet by the factor $\alpha^{-1/2}$):

$$
S = S_0 + \int d^5x \int dz \left\{ -e_6 g^{zz} (\alpha^{-1} F^i_\alpha - \partial_z A_\alpha^i) d_\alpha^\beta (\alpha^{-1} F^i_\beta - \partial_z A_\beta^i) \right\},
$$

(4.8)

Indeed, then the added quadratic term in $\alpha^{-1} F^i_\alpha - \partial_z A_\alpha^i$ exactly cancels the quadratic term $+e_6 g^{zz} \partial_z A_\alpha^i d_\alpha^\beta \partial_z A_\beta^i$, which is contained in the first term $e_6 A_\alpha^i d_\alpha^\beta C_\beta^i$ in $S_0$ after a partial integration. To show that this action $S$ is indeed invariant under the the above
supersymmetry transformation \( \tilde{\delta}_Q(\varepsilon) \) (4.3), we note that \( \tilde{\delta}_Q(\varepsilon) \) transformations of \( \tilde{F}^\alpha_i \) are given by (in 6D spinor notation)

\[
\tilde{\delta}_Q(\varepsilon) \tilde{F}^\alpha_i = 2\varepsilon_i(\delta_Z - \partial_z)\zeta^\alpha = -2i(\varepsilon_i(\Gamma^\alpha - \gamma^2\gamma^z\psi^\alpha_\mu)F_j^\alpha), \\
\tilde{\delta}_Q(\varepsilon)(e_6g^{zz}) = -2ie_6g^{zz}\varepsilon_i\psi^\alpha_2 + 4ie_6\varepsilon^\gamma\gamma^2\psi^\alpha_2 = 2ie_6\varepsilon^\gamma\gamma^2\psi^\alpha_2.
\]

(4.9)

and that the difference between the above \( \tilde{\delta}_Q(\varepsilon) \) transformation and the original 6D supersymmetry transformation \( \tilde{\delta}^\text{org}_Q(\varepsilon) \) exists only in the \( \zeta^\alpha \) field:

\[
\tilde{\delta}_Q(\varepsilon) = \tilde{\delta}^\text{org}_Q(\varepsilon) + \delta'(\varepsilon), \quad \delta'(\varepsilon) = i\gamma^z\varepsilon_i\tilde{F}^\alpha_i, \quad \delta'(\varepsilon)A^\alpha_i = 0, \quad \delta'(\varepsilon)(\text{Weyl multiplet}) = 0.
\]

(4.10)

If we use these equations together with the original action invariance \( \tilde{\delta}^\text{org}_Q(\varepsilon)S_0 = 0 \) and \( \delta S_0/\delta\zeta^\alpha = 4e_6d^\alpha_\beta\Gamma_\beta \), we can immediately confirm the invariance of the action (4.8) under \( \tilde{\delta}_Q(\varepsilon) \). Use has also been made of the relation \( \tilde{F}^\alpha_i = 2\tilde{\delta}^\text{org}_Q(\varepsilon)F^\alpha_i \).

Since the action (4.8) is invariant under \( \tilde{\delta}_Q(\varepsilon) \), as well as all the other 6D superconformal transformations with \( z \)-independent parameters, it gives, after fixing the gauges of \( M_{a5} \), \( S^i \) and \( K_\alpha \), an action that is invariant under the 5D supersymmetry transformation \( \delta_Q(\varepsilon) \) given by Eq. (2.21). Invariance under the central charge transformation \( \delta_Z \) also follows from the \( [\delta_Q, \delta_Q] \) algebra (2.45) and invariance under all transformations other than \( \delta_Z \).

The action (4.8) contains an infinite number of copies of the hypermultiplets. However, we can compactify the \( z \)-direction into a torus of radius \( R \) and Fourier expand the hypermultiplet fields \( \phi^\alpha = (A^\alpha, \zeta^\alpha, F^\alpha) \) into cosine and sine modes as \( \phi^\alpha(x, z) = \sum_n(\phi^\alpha_n(x)\cos(nz/R) + \phi^\alpha_{n}^{(s)}(x)\sin(nz/R)) \). Then, clearly, each set of components, \( \phi^{(c)}_n \) and \( \phi^{(s)}_n \), with the label \( n \) separately gives a 5D hypermultiplet closed under the 5D supersymmetry transformation, and moreover, the multiplets with different labels \( n \) are also separated in the action (4.8), which is (homogeneously) quadratic in \( \phi \), as a result of the conservation of momentum \( p_z = n/R \). The cosine and sine modes with the same label \( n \) mix with each other in the terms containing \( \partial_z \phi \) in the action. Therefore we can retain only the modes with an arbitrary single label \( n \) to be consistent with invariance. The terms containing no \( \partial_z \) give the same forms of kinetic terms for the cosine, \( \phi^{(c)}_n \), and sine, \( \phi^{(s)}_n \), modes (actually, also independent of \( n \)), and the terms containing \( \partial_z \phi \) gives mass terms between \( \phi^{(c)}_n \) and \( \phi^{(s)}_n \). Since the mass square is \( m^2 = (n/R)^2 \) and \( R \) is a free parameter, the kinetic terms and the mass terms give separately invariant actions. Thus, taking also account of the overall rescaling of the hypermultiplet by the factor \( \alpha^{-1/2} \), we first find the action formula for the kinetic terms of the hypermultiplet, which takes the same form as the \( z \)-independent \( n = 0 \) mode,

\[
S_{\text{kin}} = \int d^5x e \left( A^\alpha_i d^\alpha_\beta C^{i\alpha}_\beta + 2(\bar{\zeta}^\alpha - i\bar{\psi}^i_\alpha \gamma^i \psi^\alpha_2) d^\alpha_\beta \Gamma^\alpha_\beta \right)
\]
\[-g^{zz}(\alpha^{-1}F^\alpha_i - \partial_z A^\alpha_i)d_\alpha^\beta(\alpha^{-1}F^\beta_i - \partial_z A^\beta_i)\}, \tag{4.11}\]

where the prime on $C'_\beta$ and $\Gamma'_\beta$ is a reminder that the $\partial_z$ terms in them are omitted, although they vanish automatically, because the fields are now $z$-independent 5D fields. The mass terms, as they stand, are the transition mass terms between the two hypermultiplets $\phi^\alpha_c$ and $\phi^\alpha_s$ (suppressing the label $n$). However, if there is a symmetric $G$-invariant tensor $\eta_{\alpha\beta} = \eta_{\beta\alpha}$, then we can reduce them to a single hypermultiplet by imposing the constraint

\[\phi^\alpha_s = (d^{-1})^{\alpha\beta} \eta_{\beta\gamma} \phi^\gamma_c. \tag{4.12}\]

This constraint is consistent with $G$-invariance, and hence with supersymmetry. The terms containing $\partial_z$ in the action (4.8) have the form $\phi \partial_z \phi$, which is rewritten, after substituting $\phi^\alpha(x,z) = \phi^\alpha_c(x) \cos(nz/R) + \phi^\alpha_s(x) \sin(nz/R)$, performing the $z$ integration, and imposing the constraint (4.12), as

\[\phi \partial_z \phi \rightarrow (n/R) \left( \phi^\alpha_c d_{\alpha\beta} \phi^\beta_s - \phi^\alpha_s d_{\alpha\beta} \phi^\beta_c \right) = 2(n/R) \left( \phi^\alpha_c \eta_{\alpha\beta} \phi^\beta_c \right). \tag{4.13}\]

Using this rule and collecting the $\phi \partial_z \phi$ terms in the action (4.8), we find the action formula for the hypermultiplet mass term to be

\[
\begin{align*}
S_{\text{mass}} = \int d^5x \, e m \eta^{\alpha\beta} \left\{-A^\mu(D_\mu A_{\alpha i})A^i_\beta - i\bar{\zeta}_\alpha \gamma^\gamma \zeta_\beta + 2A_{\alpha i} \bar{\psi}_i^{\gamma} \gamma^\gamma \gamma^\mu \zeta_\beta \\
- i\bar{\psi}_i^{\gamma} \gamma^{\mu \nu} \gamma^\mu \psi^j \gamma^\nu A_{\alpha j} + \alpha^{-1} g^{zz} F^\alpha_{\alpha i} A^i_\beta \right\},
\end{align*}
\tag{4.14}\]

where $D_\mu$ is the covariant derivative with respect to the homogeneous transformations $M_{\alpha\beta}$, $D$, $U^{ij}$ and $G$. The action formulas (4.11) and (4.14) are written using the 6D notation (for, in particular, the spinors and covariant derivatives), and are generally valid independently of the choice of the $M_{\alpha5}$, $S^i$ and $K_a$ gauge-fixing conditions. More explicit expressions for them, valid in the present gauge-fixing and completely written in 5D notation, will be given in the forthcoming paper.\[\square\]

§5. Embedding and invariant action formulas

5.1. Embedding formulas

We now give some embedding formulas that give a (known type of) multiplet using a (set of) multiplet(s). First, however, we discuss the important point that there is a vector submultiplet in our 5D Weyl multiplet.

When going down to 5D from 6D, there appeared a new set of fields $A_\mu$, $\psi_5$ and $\alpha$ from the fifth spatial components of the vielbein and Rarita-Schwinger fields in 6D. It is natural
to wonder if they might give a submultiplet in the 5D Weyl multiplet. Indeed this is the case, and we can easily check that the gravi-photon $A_\mu$ and the dilaton $\alpha = e^\varphi$ have the same transformation law with the usual matter $U(1)$ vector multiplet with the identification

$$(W_\mu, \ M, \ \Omega^i, \ Y^{ij}) = (A_\mu, \ \alpha, \ 0, \ 0). \quad (5.1)$$

We suspect that $\psi_5$ would have appeared as the $\Omega$ component of this multiplet if $\psi_5$ were not set to zero as the $S$ gauge-fixing. We refer to this vector multiplet as a ‘central charge vector multiplet’.

Now we give an embedding formula of the vector multiplets $V^A$ into a linear multiplet. The index $A$ labels the generators $\{t_A\}$ of the gauge group $G$, which is generally non-simple. This formula exists for arbitrary polynomials $f(M)$ of the first components $M^A$ of $V^A$, as long as the degree of $f$ is two or less:

$$f(M) = f_0 + f_{0A}M^A + \frac{1}{2}f_{0AB}M^AM^B. \quad (5.2)$$

For such a polynomial $f$, we can identify the following linear multiplet:

$$L^{ij} = -2f_t^{ij} + f_AY^{Aij} - if_{AB}\tilde{\Omega}^{Ai}\Omega^{Bj},$$

$$\varphi^i = -4f\bar{\chi}^i + f_A\left((\bar{D} - \frac{1}{2}\gamma^i v)\Omega^A + t_j^i \Omega^{Aj} - g[M, \Omega^i]^A\right)$$

$$- f_{AB}\left((\frac{1}{4}\gamma^i \tilde{G}(W)^A - \frac{1}{2}\bar{D}M^A)\Omega^{Bi} + Y^{Ai}_j \Omega^{Bj}\right),$$

$$E_a = \bar{D}^b(4f_{ab} + f_A\tilde{G}_{ab}(W)^A + if_{AB}\tilde{\Omega}^{A\gamma}_{ab}\Omega^B)$$

$$- if_A\tilde{\Omega}^{A\gamma}_{abcde}\tilde{R}^{bc}(Q) + \frac{1}{4\alpha^2}f_0\epsilon_{abcde}\tilde{F}^{bc}(A)\tilde{F}^{de}(A)$$

$$+ \frac{1}{4\alpha^4}\epsilon_{abcde}f_0\tilde{F}^{bc}(W)^A\tilde{F}^{de}(A) + \frac{1}{8}\epsilon_{abcde}f_{0AB}\tilde{F}^{bc}(W)^A\tilde{F}^{de}(W)^B$$

$$- 2igf_A[\tilde{\Omega}, \gamma_0^a]^A - 2igf_{AB}\tilde{\Omega}^{A\gamma}_{a0}[M, \Omega]^B + gf_A[M, \tilde{D}_aM]^A,$$

$$N = -4f(C + 4t^i_j t^j_i) - \bar{D}^a\bar{D}_af$$

$$+ f_A\left(-2\tilde{G}_{ab}(W)^A\gamma^{ab} + \frac{1}{2\alpha}\tilde{G}_{ab}(W)^A\tilde{F}^{ab}(A) + 4t^i_j Y^{Aij}\right)$$

$$+ i\tilde{\Omega}^{A\gamma}_{ab}\tilde{R}_{ab}(Q) - 16i\tilde{\Omega}^{A\gamma}_{a0}\tilde{\chi} + 2ig[\tilde{\Omega}, \gamma_0^a]^A$$

$$+ f_{AB}\left(-\frac{1}{4}\tilde{G}(W)^A\cdot\tilde{G}(W)^B + \frac{1}{2}\bar{D}^aM^A\tilde{D}_aM^B - Y^{Aij}_j Y^{Bj}_i + 2i\tilde{\Omega}^{A\gamma} \tilde{\varphi} \Omega^B$$

$$+ \frac{i}{2\alpha}\tilde{\Omega}^{A\gamma}_{a0}\tilde{R}(A)\Omega^B - i\tilde{\Omega}^{A\gamma}_{a0}\tilde{\chi} + 2i\tilde{\Omega}^{A\gamma} \tilde{\varphi} \Omega^B$$

$$+ \frac{1}{2\alpha}\epsilon_{abcde}\tilde{F}^{bc}(W)^A\tilde{F}^{de}(W)^B + 2i\tilde{\Omega}^{A\gamma} \tilde{\varphi} \Omega^B + 2i\tilde{\Omega}^{A\gamma} \tilde{\varphi} \Omega^B + 2i\tilde{\Omega}^{A\gamma} \tilde{\varphi} \Omega^B, \quad (5.3)$$

where the commutator $[X, Y]^A$ represents $[X, Y]^A t_A \equiv X^B Y^C [t_B, \ t_C]$, and

$$f \equiv f(M), \quad f_A \equiv \frac{\partial f}{\partial M^A} = f_{0A} + f_{0AB}M^B, \quad f_{AB} \equiv \frac{\partial^2 f}{\partial M^A\partial M^B} = f_{0AB}. \quad (5.4)$$

When the transformation of the lowest component $L^{ij}$ of a linear multiplet takes the form $2i\tilde{\varepsilon}^i \varphi^j$, then the supersymmetry algebra (2.45) demands that all the other higher components must uniquely transform in the form given in Eq. (3.5) and that the constraint (3.6)
should hold. Therefore, in order to identify all the above components of the linear multiplet, we have only to examine the transformation law up to the second component, \( \varphi^i \), since the supersymmetry algebra is guaranteed to hold for any function of the vector multiplet fields. The transformation laws of the remaining components \( E^a \) and \( N \), as well as the constraint, are automatic and need not be checked.

For a more general function \( f(M) \), we cannot satisfy the first component transformation form \( \delta L^{ij} = 2i\varepsilon^{(i} \varphi^{j)} \). Therefore, this embedding is impossible for functions other than the polynomial \( f(M) \) of degree two.

In the above derivation we have assumed that the coefficients \( f_0, f_{0A} \) and \( f_{0AB} \) are constants. But actually \( f(M) \) should carry Weyl weight \( w = 2 \). Thus \( f_0, f_{0A} \) and \( f_{0AB} \) in fact each takes the form \( \text{const} \times (\alpha^2, \alpha, 1) \) (and hence \( f(M) \) is actually a homogeneous quadratic polynomial in \( \alpha \) and \( M \)). This is consistent since \( \alpha \) is covariantly constant and \( Q \) invariant.

If \( f(M) \) is \( G \)-invariant, the above linear multiplet does not carry any charge. Then, it must be possible to rewrite the embedded \( E^a \) component into the form (3.8) in terms of an antisymmetric tensor gauge field \( E^{\mu \nu} \). (Note that the last three terms in \( E_a \) of Eq. (5.3) cancel and vanish for \( G \)-invariant \( f(M) \)). In this case, we find

\[
E^{\mu \nu} = -e(4f\psi^{\alpha \beta} - f_A\tilde{G}^{\mu \nu}(W)^A + i f_{AB} \sigma_{AB} + i f_\rho \psi \gamma^{\mu \nu \sigma \rho} \psi_\sigma - 2i f_A \psi \gamma^{\mu \nu \lambda} \Omega^A)
- \frac{1}{2}f_0 e^{\alpha \beta} A_{\lambda} F_{\rho \sigma} \Omega^A - \frac{1}{2} f_{0A} e^{\mu \lambda \rho \sigma} A_{\lambda} F_{\rho \sigma} (W)^A
- \frac{1}{2} \frac{1}{g} W^A \partial_{\rho} W^B - \frac{1}{3} g W^A \Omega^B.
\]

We next consider construction of a linear multiplet from the product of two hypermultiplets \( (A^i, \zeta^i, F^i_a) \) and \( (A^a, \zeta^a, F^a_i) \). This possibility is suggested by \( N = 2, d = 4 \) superconformal tensor calculus. Actually, we find almost the same form of formula to hold:

\[
L^{ij} = \eta^{\alpha \beta} A^i_{(a} A^{ij} \),
\varphi^i = \eta^{\alpha \beta} \left( \zeta_i A^i_{(a} + A^i_{a} \zeta^j \right),
E_a = \eta^{\alpha \beta} \left( A^i_{(a} \tilde{D}_a A^{ij} - (D_a A^i_{a}) A^{ij} - 2i \zeta_i \gamma_{a} \zeta^j \right),
N = \eta^{\alpha \beta} \left( (gM \ast A^i_{a}) A^{ij} + (gM \ast A^i_{a}) A_{a} A^{ij} - 2t_{ij} A^{ij} A^i_{a} - 2i \zeta_i \zeta^j \right),
\]

\[
\left( (gM \ast A^i_{a}) \right)^{\alpha} \equiv (\delta \gamma_g(M) + \delta \gamma_z(\alpha)) A^i_a = gM^a \frac{\alpha}{\beta} A^i_{a} + F^a_i \right)
\]

where \( \eta^{\alpha \beta} \) is an arbitrary \( G \)-covariant tensor. For instance, if \( \eta^{\alpha \beta} \) is proportional to the generator matrices \( (t_A)^{\alpha \beta} = \rho^{\gamma}(t_A)^{\alpha \gamma} \), then this linear multiplet belongs to the adjoint representation of \( G \). Note that even in the case that \( \eta^{\alpha \beta} \) is a \( G \)-invariant tensor, this linear multiplet still carries a \( U(1) \) charge, i.e., it is not invariant under the \( Z \) transformation;
e.g., $\delta_Z(\alpha)L^{ij} = \eta^{\alpha\beta}(F^{ij}_a A^j_\beta + A^{ij}_a F^j_\beta)$. For this linear multiplet, therefore, the ‘group action terms’ like $gML^{ij}$ appearing in the supersymmetry transformation law (3.5) should be understood to contain not only the usual gauge group $G$ action but also the central charge $Z$ action; that is, noting that the vector multiplet associated with the central charge is the multiplet (5.4), the $gM$ action should be understood to be $gM \ast \equiv \delta_G(M) + \delta_Z(\alpha)$, the same action as on $A^\alpha_i$ in the above.

5.2. Invariant action formulas

An invariant action formula exists for the product of a vector multiplet and a linear multiplet in 6D, as given in Eq. (B.20). This leads directly to the following invariant action formula in 5D:

$$e^{-1}L = Y^{ij}L_{ij} + 2i\tilde{\Omega} \varphi + 2i\tilde{\psi}_a^i \gamma^a \Omega^i L_{ij} + \frac{1}{2}M(N - 2i\tilde{\psi}_b^b \gamma^b \varphi - 2i\tilde{\psi}_a^i \gamma^{ab} \tilde{\psi}_b^j L_{ij}) - \frac{1}{2}W_a\varphi + 2i\tilde{\psi}_b^b \gamma^{ab} \psi^j L_{ij}). \quad (5.7)$$

As in 6D, this 5D action is invariant if the vector multiplet is abelian and the linear multiplet carries no gauge group charges or is charged only under the abelian group of this vector multiplet. When the linear multiplet carries no charges at all, the constrained vector field $E^a$ can be replaced by the unconstrained anti-symmetric tensor gauge field $E^\mu\nu$, as shown in (3.8). Using this, the second line of the above action (5.7) can be rewritten, up to a total derivative, as

$$-\frac{1}{2}eW_a\varphi + 2i\tilde{\psi}_b^b \gamma^{ab} \psi^j L_{ij}) \quad \rightarrow \quad +\frac{1}{4}F_{\mu\nu}(W)E^\mu\nu. \quad (5.8)$$

In five dimensions, this formula also leads to a simpler invariant action formula. That is, we have a special vector multiplet (5.1) in 5D which we call the central charge vector multiplet. We can apply (5.7) to this vector multiplet. We then obtain

$$(\alpha e)^{-1}L = N - 2i\tilde{\psi}_b^b \gamma^b \varphi - 2i\tilde{\psi}_a^i \gamma^{ab} \tilde{\psi}_b^j L_{ij} - \frac{1}{\alpha}A_a\varphi + 2i\tilde{\psi}_b^b \gamma^{ab} \psi^j L_{ij}). \quad (5.9)$$

We may call this a linear multiplet action formula, and essentially the same formula was found by Zucker. Again, when the linear multiplet carries no charge at all, the above rewriting (5.8) is of course possible, and the formula becomes extremely simple:

$$L = e\alpha(N - 2i\tilde{\psi}_b^b \gamma^b \varphi - 2i\tilde{\psi}_a^i \gamma^{ab} \tilde{\psi}_b^j L_{ij}) + \frac{1}{2}F_{\mu\nu}(A)E^\mu\nu. \quad (5.10)$$

These action formulas can be used to write the action for a general matter-Yang-Mills system coupled to supergravity. If we use the above embedding formula (5.3) of vector multiplets
into a linear multiplet and apply the last linear multiplet action formula (5.10), then we obtain a general Yang-Mills-supergravity action. If we use the ‘hypermultiplet × hypermultiplet → linear multiplet’ formula (5.6) and apply the linear multiplet action formula (5.9), then we obtain the action for a general hypermultiplet matter system. The kinetic term for the hypermultiplet \((A^i_\alpha, \zeta^\alpha, F^i_\alpha)\) can be obtained if, when using the formula (5.6), we take the central-charge transformed hypermultiplet \(Z(A^i_\alpha, \zeta^\alpha, F^i_\alpha)\) as \((A'^i_\alpha, \zeta'^\alpha, F'^i_\alpha)\). The mass term can be obtained by choosing \((A'^i_\alpha, \zeta'^\alpha, F'^i_\alpha) = (A^i_\alpha, \zeta^\alpha, F^i_\alpha)\) and a symmetric tensor \(\eta^{\alpha\beta}\).

It is interesting to see that these formulas give the same hypermultiplet actions as those which we have independently derived from the 6D action in §4.

§6. Summary and discussion

In this paper we have derived supergravity tensor calculus in five dimensions using dimensional reduction from the known superconformal tensor calculus in six dimensions. Our 5D supergravity tensor calculus results from that in 6D by fixing the gauges of the \(M_{a5}, S^i\) and \(K_a\) symmetries, and so it retains the supersymmetry \(Q^i\), the local Lorentz symmetry \(M_{ab}\), the dilatation symmetry \(D\) and the gauge symmetries of \(SU(2) U_{ij}\), central charge \(Z\) and a group \(G\) transformations. We have derived supersymmetry transformation laws for the vector multiplet, linear multiplet, nonlinear multiplet and hypermultiplet. In particular, we have made the hypermultiplet off-shell, while it existed only as an on-shell multiplet in 6D.

Moreover, we have obtained invariant action formulas and multiplet-embedding formulas, which will become useful when constructing a general matter-gauge field system coupled to supergravity. Some of these formulas are derived directly from the 6D formulas through dimensional reduction, but others are particular to the present five dimensions.

The presence of the dilatation \(D\) in our calculus is extremely useful, because it makes obtaining the canonical form of the Einstein and Rarita-Schwinger terms trivial. Usually, when we write a general system of matter and gauge fields coupled to supergravity, the Einstein and Rarita-Schwinger terms in the resultant action have a non-trivial function of fields as their common coefficient. But if we have dilatation symmetry, the coefficient function can be set equal to 1 simply as a gauge fixing condition of the \(D\) symmetry.

In this respect, it might have been better not to gauge-fix the \(S^i\) supersymmetry either, since the \(S^i\) symmetry could be used to eliminate the mixing of the Rarita-Schwinger and matter fermions. However, we have chosen to fix \(S^i\) by \(\psi_z = 0\) in this paper to avoid

---

* Actually, these two actions for vector multiplets and hypermultiplets do not separately give a consistent supergravity action, but they do when combined. This is discussed in a forthcoming paper.
complications in the dimensional reduction. The price we pay for doing this, however, is that we need to make some field redefinitions to eliminate the fermion mixing. It is also interesting to note that it is actually possible to avoid fixing the gauge altogether (even the $M_{55}$ gauge) in the course of the dimensional reduction (i.e., making all the fields $z$ independent). Then, the resultant 5D theory would have full 6D superconformal symmetry aside from the fact that the 6D GC transformation reduces to the 5D GC transformation plus the central charge transformation. This is interesting because there seems to be no (global) superconformal group in 5D.

By using the formulas given in this paper, it is now easy to write down such a general matter-gauge field system coupled to supergravity. To obtain such supergravity actions in the superconformal framework, we generally need compensating multiplets in addition to the 40+40 supergravity Weyl multiplet. Our five dimensional theory does not have full superconformal symmetries, but its Weyl multiplet is the same size, and it shares common properties with the superconformal theory. A different choice of the compensating multiplets leads to a different off-shell formulation of supergravities. In the $N = 2$, $d = 4$ superconformal case, there are three possibilities for the compensator. The first is to use a nonlinear multiplet, the second a hypermultiplet, and the third a linear multiplet. In the $d = 6$ case, only the third possibility is possible, as described in BSVP. On the other hand, here in our 5D case, all three possibilities are possible. The first possibility was investigated by Zucker in his tensor calculus framework. However, our experience with $N = 1$ and $N = 2$ in $d = 4$ leads us to believe that the second possibility is the most useful choice for constructing the most general matter (hypermultiplet) system as well as for studying the symmetries. We shall carry out these tasks explicitly in a forthcoming paper.

Acknowledgements

The authors would like to thank Antoine Van Proeyen for providing them with precious information on six-dimensional superconformal tensor calculus. They also appreciate the Summer Institute 99 held at Fuji-Yoshida, the discussions at which motivated this work. T. K. is supported in part by the Grant-in-Aid for Scientific Research No. 10640261 from Japan Society for the Promotion of Science and the Grant-in-Aid for Scientific Research on Priority Areas No. 12047214 from the Ministry of Education, Science, Sports and Culture, Japan.
Appendix A

Conventions and Useful Identities

Throughout this paper we use $\eta_{ab} = \text{diag}(+,-,-,-,-)$ as the Lorentz metric, which is different from that of BSVP but is more familiar to phenomenologists. With this metric, the $4 \times 4$ Dirac $\gamma$-matrices $\gamma^a$ in 5D satisfy, as usual,

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}, \quad (\gamma^a)^\dagger = \eta_{ab} \gamma^b.$$  \hspace{1cm} (A.1)

We use the multi-index $\gamma$-matrices defined by

$$\gamma^{(n)} \equiv \gamma^{a_1\cdots a_n} \equiv \gamma^{[a_1} \gamma^{a_2} \cdots \gamma^{a_n]} = \frac{1}{n!} \sum_{p=\text{perm's}} (-1)^p \gamma^{a_1} \gamma^{a_2} \cdots \gamma^{a_n},$$  \hspace{1cm} (A.2)

where the square bracket $[\cdots]$ attached to the indices implies complete antisymmetrization with weight 1. Similarly, $(\cdots)$ is used for complete symmetrization with weight 1. We choose the Dirac matrices to satisfy

$$\gamma^{a_1\cdots a_5} = \epsilon^{a_1\cdots a_5}, \quad \epsilon^{01234} = 1,$$  \hspace{1cm} (A.3)

where $\epsilon^{a_1\cdots a_5}$ is a totally antisymmetric tensor. With this choice, the duality relation reads

$$\gamma^{a_1\cdots a_n} = \frac{S_n}{(5-n)!} \epsilon^{a_1\cdots a_5b_1\cdots b_{5-n}} \gamma^{b_1\cdots b_{5-n}}, \quad S_n = \begin{cases} +1 & \text{for } n = 0, 1, 4, 5 \\ -1 & \text{for } n = 2, 3 \end{cases}.$$  \hspace{1cm} (A.4)

The charge conjugation matrix $C$ in 5D has the properties

$$C^T = -C, \quad C^\dagger C = 1, \quad C\gamma_a C^{-1} = \gamma_a^T.$$  \hspace{1cm} (A.5)

Our spinors carry the $SU(2)$ index $i$ ($i = 1, 2$) of $U_{ij}$. This index is generally raised or lowered by using the $\epsilon$ symbol $\epsilon^{ij} = \epsilon_{ij} = i(\sigma_2)_{ij}$:

$$\psi^i = \epsilon^{ij} \psi_j, \quad \psi_i = \psi^j \epsilon_{ji}.$$  \hspace{1cm} (A.6)

[Note that contraction of the $SU(2)$ indices is always performed according to the northwest-to-southeast convention.] All our 5D spinors satisfy the $SU(2)$-Majorana condition,

$$\bar{\psi}^i \equiv (\psi_i)^\dagger \gamma^0 = (\psi^i)^T C.$$  \hspace{1cm} (A.7)

When $SU(2)$ indices are suppressed in bilinear terms of spinors, the northwest-to-southeast contraction of the $SU(2)$ indices is implied:

$$\bar{\psi}^{(n)} \lambda \equiv \bar{\psi}^i \gamma^{(n)} \lambda_i.$$  \hspace{1cm} (A.8)
The following symmetry of spinor bilinear terms is important:

\[
\bar{\lambda} \gamma^{(n)} \psi = t_n \bar{\psi} \gamma^{(n)} \lambda, \quad t_n = \begin{cases} 
+1 & \text{for } n = 2, 3 \\
-1 & \text{for } n = 0, 1, 4, 5 
\end{cases}.
\]  

(A-9)

If the SU(2) indices are not contracted, the sign becomes opposite.

We often use the Fierz identity, which in 5D reads

\[
\psi^i \bar{\lambda}^j = -\frac{1}{4} (\bar{\lambda}^j \psi^i) - \frac{1}{4} (\bar{\lambda}^j \gamma^a \psi^i) \gamma_a + \frac{1}{8} (\bar{\lambda}^j \gamma^{ab} \psi^i) \gamma_{ab}.
\]  

(A-10)

An important identity can be proved by using this Fierz identity: the antisymmetric trilinear in spinors $\psi_1$, $\psi_2$, $\psi_3$ satisfies

\[
\gamma^a \psi^i_1 (\bar{\psi}^j_2 \gamma_a \psi^k_3) - \psi^i_1 (\bar{\psi}^j_2 \psi^k_3) = 0.
\]  

(A-11)

From this identity and with the help of the Fierz identity again, when necessary, various identities can be derived and used when proving, e.g., the invariance of the action. Two examples are

\[
2 \psi^i_{(a} (\bar{\psi}^j \gamma_{abc} \psi^k_{c)}) - 3 \gamma^{ab} \psi^i_{[a} (\bar{\psi}^j \gamma^c \psi^k_{c}) = 0,
\]

\[
\bar{\psi}^i_{[a} \gamma^d \psi^j_{b} \bar{\psi}^k_{c} \gamma_{abc} \psi^l_{d]} = 0.
\]  

(A-12)

(Six-dimensional analogues to these can actually be found more easily, from which one could derive these by dimensional reduction.)

A useful formula in treating the SU(2) indices is

\[
A^{ij} = -\frac{1}{2} \epsilon^{ij} A + A^{(ij)}, \quad A'_j = +\frac{1}{2} \delta^i_j A + A^i_{,j},
\]  

(A-13)

where $A \equiv A^i_{,i}$ and $A^i_{,j} \equiv A^{(ik)} \epsilon_{kj}$. Often used is the following formula for $\gamma^{\mu_1 \mu_2 \cdots \mu_n}$, which is valid for any number of spacetime dimensions $d$:

\[
\gamma^{\nu_1 \nu_2 \cdots \nu_m} \gamma^{\mu_1 \mu_2 \cdots \mu_n} \gamma_{\nu_1 \nu_2 \cdots \nu_m} = C_{n,m}^d \gamma^{\mu_1 \mu_2 \cdots \mu_n},
\]

\[
C_{n,m}^d = (-1)^{[d/2]+nm} m! \times \{\text{Coefficient of } x^m \text{ in } (1 + x)^{d-n}(1 - x)^n\}.
\]  

(A-14)

More explicit values for $C_{n,m}^d$ in $d = 5$ and 6 cases are given in Table III.

Finally, we mention the formulas holding for $d = 6$ for the (anti-)self-dual tensor $T_{abc}^\pm = \pm (1/6) \epsilon_{abcdef} T_{abcdef}^\pm$, which are useful in the dimensional reduction:

\[
\gamma \cdot T^\pm = 6 T_{abc}^\pm \gamma^{ab} \mathcal{P}^\pm, \quad \left(\mathcal{P}^\pm = \frac{1 \pm \gamma_7}{2}\right),
\]

\[
\gamma_a \gamma \cdot T^\pm = 6 T_{abc}^\pm \gamma^b \mathcal{P}^\pm, \quad \gamma_{[a} \gamma \cdot T^{\pm} \gamma_{b]} = -12 T_{abc}^\pm \gamma^c \mathcal{P}^\mp.
\]  

(A-15)
Table III. The coefficients $C_{n,m}^d$ in the formula Eq. (A.14) for $d = 5$ and 6 cases.

| $d = 5$ | $m = 1$ | $m = 2$ | $d = 6$ | $m = 1$ | $m = 2$ | $m = 3$ |
|---------|---------|---------|---------|---------|---------|---------|
| $n = 0$ | 5       | -20     | $n = 0$ | 6       | -30     | -120    |
| $n = 1$ | -3      | -4      | $n = 1$ | -4      | -10     | 0       |
| $n = 2$ | 1       | 4       | $n = 2$ | 2       | 2       | 24      |
| $n = 3$ | 1       | 4       | $n = 3$ | 0       | 6       | 0       |

Table IV. Correspondence between the notation of BSVP and that used here.

| BSVP         | Ours                        |
|--------------|-----------------------------|
| $x_\mu$      | $x^\mu$ or $-x_\mu$        |
| $\partial_\mu$ | $\partial_\mu$ or $-\partial^\mu$ |
| $\delta_{ab}$ | $-\eta_{ab}$              |
| $\gamma_a$   | $-i\gamma^a$ or $i\gamma_a$ |
| $\gamma^a\partial_a \equiv \phi$ | $-i\gamma^a \partial_a = -i\phi$ |
| $\gamma_7 = i\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$ | $\gamma_7 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$ |
| $\epsilon_{abcde} (\epsilon_{123456} = +1)$ | $-i\epsilon_{abcde}$ or $-i\epsilon_{abcde}$ $(\epsilon_{012345} = +1)$ |

Superconformal generators $X_A$

| BSVP         | Ours                        |
|--------------|-----------------------------|
| $P_a$, $Q$, $D$, $K_a$ | $P_a$, $(1/2)Q$, $D$, $-K_a$ |
| $M_{ab}$, $U$, $S$   | $M_{ab}$, $(1/2)U$, $(1/2)S$ |

Gauge fields $h^A_\mu$ (the same rules for $R^A_{\mu\nu}$ and transformation parameters $\varepsilon^A$)

| BSVP         | Ours                        |
|--------------|-----------------------------|
| $e^a_\mu$, $\psi_\mu$, $b_\mu$, $f^a_\mu$ | $e^a_\mu$, $2\psi_\mu$, $b_\mu$, $-f^a_\mu$ |
| $\omega^{ab}_\mu$, $V^{ij}_\mu$, $\phi_\mu$ | $\omega^{ab}_\mu$, $2V^{ij}_\mu$, $2\phi_\mu$ |
| $T_{abc}$    | $T_{abc}$ or $-T_{-abc}$ |

Appendix B

6D Superconformal Tensor Calculus and Correspondence

Since our notation and, in particular, metric convention are different from BSVP, we here summarize the results of BSVP for 6D superconformal tensor calculus for the parts we need in this paper using our notation. For reader’s convenience, we also give the correspondence between the BSVP notation and our notation in Table IV. In this appendix, we omit the underlines that indicate six dimensional quantities.

The local superconformal tensor calculus is constructed generally by deforming the Yang-Mills theory based on the superconformal group. In 6D, the superconformal
group is $OSp(6,2|1)$, whose generators are given in Eq. (2.1), and the algebra is deformed by imposing the following three constraints on the curvatures:

\[
\hat{R}_{\mu\nu a}(P) = 0 ,
\hat{R}_{\mu\nu b}a(M)e^\nu_b - T_{\mu bc}^- T^{abc} - \frac{1}{12} e_\mu^a D = 0 ,
\gamma^\mu \hat{R}_{\mu\nu i}(Q) = \frac{1}{12} \gamma^\nu \chi^i .
\]

(B.1)

Note that the $\hat{R}_{\mu\nu}(X)$ are the covariant curvatures of the resultant algebra, but not of the original superconformal group.

B.1. Weyl multiplet

As mentioned in Eq. (2.3), the $d = 6$ Weyl multiplet consists of 40 Bose plus 40 Fermi fields, whose properties are summarized in Table V. The $M$, $K$ and $S$ gauge fields $\omega_{\mu}^{ab}$, $f_{\mu}^a$ and $\phi_{\mu}^i$ become dependent fields by the constraints (B.1). For example, the $\omega_{\mu}^{ab}$ is given as follows by solving the first constraint $\hat{R}_{\mu\nu a}(P) = 0$:

\[
\omega_{\mu}^{ab} = \omega_0^{ab} + i(2\bar{\psi}_\mu\gamma^a\psi^b + \bar{\psi}_a\gamma^a\psi^b) - 2e_\mu^a [a, b] ,
\omega_0^{ab} \equiv -2e^{\nu}[a, b]_{e_{\mu}\epsilon_\nu} + e_\mu^a e^b\sigma e_\mu^c \partial_\nu e_{\sigma c} .
\]

(B.2)

The gauge fields of the Weyl multiplet transform under the $Q$, $S$ and $K$ transformation $\delta \equiv \bar{\epsilon}^iQ_i + \bar{\eta}^iS_i + \xi_K^a K_a \equiv \delta_Q(\bar{\epsilon}) + \delta_S(\bar{\eta}) + \delta_K(\xi_K)$ as

\[
\delta e_{\mu}^a = -2i\bar{\epsilon}\gamma^a\psi_{\mu} .
\]
\[\delta \psi^i = D_\mu \psi^i - \frac{1}{24} \gamma^i \cdot T^\gamma \gamma_\mu \psi^i + i \gamma_\mu \eta^i,\]

\[\delta \omega^a_{\mu} = +2 \varepsilon^a_{\gamma} \phi_\mu - 2i \varepsilon_{\gamma} \hat{R}^\gamma_{\mu} (Q) - i \bar{\psi} \gamma_\mu \hat{R}^{ab} (Q) + \frac{i}{6} \varepsilon_{\gamma} \bar{\psi} R^b \eta^i,\]

\[\delta b_\mu = -2 \bar{\psi} \phi_\mu - \frac{i}{12} \bar{\psi} \chi + 2 \bar{\eta} \psi_\mu - 2 \xi^a_{\kappa} \eta^a,\]

\[\delta V^{ij}_\mu = -8 \varepsilon^{(i} \phi_{\mu)} - \frac{i}{3} \bar{\psi} \gamma^{(i} \chi^{j)} - 8 \bar{\eta}^{(i} \phi_j),\]

\[\delta \phi^i_\mu = i f^a_{\mu} \eta_{\alpha} \bar{\psi}^i - \frac{i}{16} (\gamma^{ab} \gamma_\mu - \frac{1}{2} \gamma_\mu \gamma^{ab}) \hat{R}^{ab}_\mu (U) \bar{\psi}^j - \frac{i}{96} \bar{\psi} \gamma^i \bar{\psi}^j + \frac{1}{12} \bar{\phi} \eta \bar{\psi}^i + D_\mu \eta^i - i \xi^a_{\kappa} \gamma_a \psi^i,\] (B.3)

where the derivative \(D_\mu\) is covariant only with respect to the homogeneous transformations \(M_{ab}\), \(D\) and \(U_{ij}\) (and \(G\) for non-singlet fields under the Yang-Mills group \(G\)). These homogeneous (i.e., Weyl weight \(w = 0\)) transformations \(M_{ab}\), \(D\) and \(U_{ij}\) are self-evident from the index structure carried by the fields; our conventions are

\[\delta M(\lambda) \phi^i = \lambda^a_{\mu} \phi^i, \quad \delta M(\lambda) \psi = \frac{1}{4} \lambda_{ab} \gamma_{ab} \psi \quad (\psi : \text{spinor}),\]

\[\delta U(\theta) \phi^i = \theta^j \phi^i, \quad \delta U(\theta) \psi = \theta_{ij} \phi^j = -\theta_i \phi_j,\]

\[\delta G(t) \phi^i = t^{\alpha} \phi^i, \quad \delta D(\rho) \phi = \nu \rho \phi \quad (w : \text{Weyl weight of } \phi).\] (B.4)

The \(Q, S, K\) transformation law of the ‘matter multiplet’ is

\[\delta T^-_{\mu \nu} = - \frac{i}{8} \bar{\varepsilon} \gamma^d \gamma_{abc} \hat{R}_{d e} (Q) + \frac{7}{48} i \bar{\varepsilon} \gamma_{abc} \chi,\]

\[\delta \chi^i = \frac{1}{4} (\bar{\mathcal{D}}_\mu \gamma_\mu \hat{T}^-) \gamma^i \bar{\psi} + \frac{3}{4} \bar{\psi} \hat{R}_\mu (U) \bar{\psi}^j + \frac{1}{2} D e^i + i \gamma \cdot T \eta^i,\]

\[\delta D = -2 \bar{\varepsilon} \bar{\chi} - 4 \bar{\eta} \chi.\] (B.5)

The algebra (2.22) of these transformations is given explicitly by

\[[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \xi^a_{\mu} \bar{\mathcal{D}}_\mu + \delta_M (\xi \cdot T^{-abc}) + \delta_S \left( \frac{i}{24} \xi^d \gamma_{abc} \chi \right) - \delta_K \left( \frac{1}{4} \xi^a \hat{D} e^{T-abc} + \frac{1}{48} \xi^a D \right), \quad (\xi^a \equiv 2i (\bar{\varepsilon}_1 \gamma^a \varepsilon_2)),\]

\[[\delta_S(\eta_1), \delta_Q(\varepsilon)] = \delta_D (-2 \bar{\varepsilon} \eta) + \delta_M (2 \bar{\varepsilon} \gamma^a \eta) + \delta_U (-8 \bar{\varepsilon} \eta^j),\]

\[[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K (2i \bar{\eta} \gamma^a \eta_2).\] (B.6)

This algebra can be read directly from the structure functions which appear in the transformation laws (B.3) of the Weyl multiplet gauge fields (cf. Eq. (2.24)).

B.2. Vector multiplet

The vector multiplet components are summarized in Table [VI] and they are all Lie-algebra valued; e.g., \(W_\mu\) represents the matrix \(W^{\alpha}_\mu = W^A_\mu(t_A)^{\alpha}_\beta\), where the \(t_A\) are the
Table VI. Matter multiplets.

| fields   | type                    | restrictions                              | SU(2) | Weyl-weight |
|----------|-------------------------|------------------------------------------|-------|-------------|
|          |                         |                                          |       |             |
|          | Vector multiplet        |                                          |       |             |
| $W_\mu$  | boson                  | Yang-Mills gauge boson                   | 1     | 0           |
| $\Omega^i$ | fermion               | $\gamma_7 \Omega^i = + \Omega^i$       | 2     | $\frac{3}{2}$ |
| $Y_{ij}$ | boson                  | $Y^{ij} = Y^{ji} = - Y^{*}_{ij}$         | 3     | 2           |
|          | Linear multiplet        |                                          |       |             |
| $L^{ij}$ | boson                  | $L^{ij} = L^{ji} = -(L_{ij})^*$          | 3     | 4           |
| $\varphi^i$ | fermion               | $\gamma_7 \varphi^i = - \varphi^i$     | 2     | $\frac{9}{2}$ |
| $E_a$    | boson                  | real                                     | 1     | 5           |
|          | Nonlinear multiplet     |                                          |       |             |
| $\Phi^i_\alpha$ | boson                | $(\Phi^\alpha_i)^* = - \Phi^\alpha_i$, $\Phi^\alpha_i \Phi^{\alpha_i} = \delta^\alpha_i$, $\Phi^\alpha_i \Phi^{\beta_i} = \delta^\alpha_\beta$ | 2     | 0           |
| $\xi^i$  | fermion                | $\gamma_7 \xi^i = - \xi^i$              | 2     | 1           |
| $V_a$    | boson                  | real                                     | 1     | 1           |
|          | Hypermultiplet          |                                          |       |             |
| $A^i_\alpha$ | boson               | $A^i_\alpha = \epsilon^i j A_{\beta j} = -(A^\alpha_i)^*$ | 2     | 2           |
| $\zeta^i_\alpha$ | fermion         | $\gamma_7 \zeta^i_\alpha = - \zeta^i_\alpha$ | 1     | $\frac{5}{2}$ |

(anti-hermitian) generators of the gauge group $G$. The $Q$, $S$, $K$ gauge transformation law is

$$
\delta W_\mu = -2i \bar{\varepsilon} \gamma_\mu \Omega,
\delta \Omega^i = -\frac{1}{4} \gamma \cdot \hat{F}(W) \varepsilon^i - Y^{ij} \varepsilon_j,
\delta Y^{ij} = 2i \bar{\varepsilon} (i \hat{D} \Omega^j) + 4 \bar{\eta}(i \Omega^j),
$$

where the covariant field strength is $\hat{F}_{\mu\nu}(W) \equiv F_{\mu\nu}(W) + 4i \bar{\psi}_{[\mu} \gamma_{\nu]} \Omega$, with $F_{\mu\nu}(W) \equiv 2 \partial_{[\mu} W_{\nu]} - g[W_{\mu}, W_{\nu}]$, and $\hat{D}_\mu$ is covariant also with respect to $G$.

B.3. Linear multiplet

The linear multiplet consists of the components given in Table VI, and they may generally carry non-Abelian charge. Their $Q$, $S$, $K$ transformation rules are

$$
\delta L^{ij} = 2 \bar{\varepsilon} (i \varphi^i),
\delta \varphi^i = -i \hat{D} L^{ij} \varepsilon_j + \frac{i}{2} \gamma^a \varepsilon^i E_a - 8 L^{ij} \eta_j,
\delta E_a = 2 \bar{\varepsilon} \gamma_{ab} \hat{D}^b \varphi - \frac{1}{12} \bar{\varepsilon} \gamma_a \gamma^i T^i \varphi - \frac{2}{3} i \bar{\varepsilon} \gamma_a \chi^j L_{ij}
- 4ig \bar{\varepsilon} \gamma_a \Omega_j L^{ij} - 10i \bar{\eta} \gamma_a \varphi .
$$

(B.8)
Closure of the algebra requires the following $Q$- and $S$-invariant constraint:

$$\mathcal{D}^a E_a + \frac{1}{2} \bar{\varphi} \chi + 4 g \bar{\Omega} \varphi + 2 g Y^{ij} L_{ij} = 0. \quad (B.9)$$

B.4. Nonlinear multiplet

The nonlinear multiplet consists of the components listed in Table [IV]. Their $Q$, $S$, $K$ transformation rules are

$$\delta \Phi^i = 2 \varepsilon (i) \lambda^j \Phi^i_j, \quad \delta \lambda^i = -i \Phi^i_j \bar{\Phi}^{j \alpha} \varepsilon^i + \frac{i}{2} \gamma^a V_a \varepsilon^i + \frac{1}{2} \gamma_a \varepsilon_j \bar{\lambda}^i \gamma^a \lambda^j$$

$$+ \frac{1}{48} \gamma_{abc} \bar{\varepsilon}^i \bar{\lambda}^j \gamma^{ab} \lambda^j + 4 \eta^i,$$

$$\delta V_a = 2 \bar{\varepsilon} \gamma_{ab} \bar{\Phi}^{b \alpha} + \frac{i}{3} \bar{\varepsilon} \gamma_{a} \chi - \bar{\varepsilon} b_{\gamma} \gamma^b \lambda V_6 - \frac{1}{12} \bar{\varepsilon} \gamma_{a} \gamma^b T^- \lambda$$

$$- 2 \bar{\varepsilon} \gamma_{a} \Phi^i \bar{\Phi}^{j \alpha} \lambda^j - 4 i g \bar{\varepsilon} \gamma_{a} \Omega_{j \beta}^{a} \Phi^j \lambda^j - 2 i \bar{\eta} \gamma_{a} \lambda - 8 \xi_{K a}. \quad (B.10)$$

(We stick to the group transformation convention in (B.4), so that the sign of $g$ here is opposite to that of BSVP.) Closure of the algebra requires the following $Q$-, $S$-, $K$-invariant constraint:

$$\mathcal{D}^a V_a + \frac{1}{3} D - \frac{1}{2} V^a V_a + \mathcal{D}^a \Phi^i \mathcal{D}_a \Phi^i + 2 i \bar{k} \Phi^i \lambda^i - \frac{5}{6} \lambda \gamma \cdot T^- \lambda$$

$$+ 2 i \bar{k} \Phi^i \bar{\Phi}^{j \alpha} \lambda^j + 2 g Y^{ij} \Omega_{j \beta}^{a} \Phi^j \lambda^j + 8 g \lambda \bar{k} \Omega^{a} \Phi^a \lambda^a = 0. \quad (B.11)$$

B.5. Hypermultiplet

The hypermultiplet in 6D is an on-shell multiplet consisting of scalars $A_{\alpha}^i$ and negative chiral spinors $\zeta_{\alpha}$. They carry the index $\alpha = (1, 2, \cdots, 2r)$ of the gauge group $G$, which is raised or lowered by using a $G$-invariant tensor $\rho_{\alpha \beta}$ (and $\rho^{\alpha \beta}$ with $\rho^{\alpha \beta} \rho_{\gamma \beta} = \delta_{\gamma}^0$) like $A_{\alpha}^i = A_{\beta}^i \rho_{\alpha \beta}$. The tensor $\rho_{\alpha \beta}$ can generally be brought into the standard form

$$\rho_{\alpha \beta} = \left( \begin{array} { c c c c } { \epsilon } & { \epsilon } & { \cdots } \\ { \epsilon } & { -1 } & { 0 } \\ { \cdots } & { 0 } & { 1 } \end{array} \right) = \rho^{\alpha \beta}, \quad \epsilon = \left( \begin{array} { c c c c } { 0 } & { 1 } \\ { -1 } & { 0 } \end{array} \right). \quad (B.12)$$

The scalar field $A_{\alpha}^i$ also carries the superconformal $SU(2)$ index $i$ and satisfies the reality condition

$$A_{\alpha}^i = \varepsilon^{ij} A_{\beta j} \rho_{\beta \alpha} = -(A_{\alpha}^i)^*, \quad A_{\alpha i} = (A_{\alpha i})^*, \quad (B.13)$$

so that $A_{\alpha}^i$ can be identified with $r$ quaternions $q_l \equiv q_l^0 + i q_l^1 + j q_l^2 + k q_l^3$ ($l = 1, \cdots, r$); indeed, by the above condition, the $l$-th $2 \times 2$ matrix ($A_{\alpha = 2l-1}^i, A_{\alpha = 2l}^i$) can be written in the
form \( q^1 \mathbf{1}_2 + iq^1 \sigma_1 + iq^2 \sigma_2 + iq^3 \sigma_3 \) with Pauli matrices \( \sigma_1, \sigma_2, \) and \( \sigma_3. \) Thus the group \( G \) acting on the hypermultiplet should be a subgroup of \( GL(r; \mathbf{H}) \):

\[
\delta_C(t) A_i^\alpha = g t^\alpha_\beta A_i^\beta, \quad \delta_C(t) A_i^\alpha = g(t^\alpha_\beta)^* A_i^\beta = -g t_\alpha^\beta A_i^\beta,
\]

\[
t_\alpha^\beta \equiv \rho_{\alpha} t^\gamma_\delta \rho_\delta^\beta = -(t^\alpha_\beta)^*.
\]

The \( Q \) and \( S \) transformations of the hypermultiplet are given by

\[
\delta A_i^\alpha = 2 \bar{\epsilon}^i \zeta_\alpha, \quad \delta \zeta_\alpha = i \bar{\Phi} A_i^\alpha \bar{\epsilon}^j + 4 A_i^\alpha \eta^j.
\]

With these rules, the superconformal algebra (B.6) is not realized on \( \zeta_\alpha \):

\[
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] \zeta_\alpha = [\text{RHS of Eq. (B.6)}] \zeta_\alpha - \gamma_\alpha \Gamma_\alpha (\varepsilon_1 \gamma^\alpha \varepsilon_2).
\]

Therefore, \( \Gamma_\alpha = 0 \) should be an equation of motion for \( \zeta_\alpha \) for the algebra to close on-shell. This fermionic non-closure function \( \Gamma_\alpha \) closes under the superconformal transformations together with its bosonic partner \( \Gamma_\alpha \):

\[
\Gamma_\alpha \equiv -i \bar{\Phi} \zeta_\alpha - \frac{1}{3} A_i^\alpha \chi^i + \frac{i}{12} \gamma^i T^\alpha_\alpha - 2g \tilde{\Omega}^\alpha_\beta A_i^\beta,
\]

\[
C_i^\alpha \equiv (-\bar{D}^\alpha \bar{D}_\alpha + \frac{1}{6} D) A_i^\alpha + \frac{1}{6} \bar{\varsigma}^\alpha \chi_i + 4g \bar{\Omega}_{ij}^\alpha \zeta^\beta - 2g Y^\alpha_\beta A_i^\beta.
\]

\[
\delta \Gamma_\alpha = -C_i^\alpha \epsilon^i + i \gamma^\mu \gamma_\alpha \Gamma_\alpha (\bar{\epsilon}_\mu \psi^\mu),
\]

\[
\delta C_i^\alpha = -2i \bar{\epsilon}^i \bar{\Phi} \Gamma_\alpha - 2(\bar{\psi}_\mu \gamma^\alpha \psi^\mu)(\bar{\epsilon}^3 \gamma_\alpha \Gamma_\alpha) + 4 \bar{\eta} \Gamma_\alpha.
\]

Here, since the algebra does not close on \( \Gamma_\alpha \) either, the covariant derivative \( \bar{\Phi}_\mu \) acting on \( \Gamma_\alpha \) is defined, slightly differently from the usual definition (2.23), to be

\[
\bar{\Phi}_\mu \Gamma_\alpha = \bar{D}_\mu \Gamma_\alpha + C_i^\alpha \psi^i - \frac{1}{2} i \gamma^\nu \gamma_\alpha \Gamma_\alpha (\bar{\psi}_\mu \gamma^\nu \psi^\nu),
\]

so that \( \bar{\Phi}_\mu \Gamma_\alpha \) becomes covariant.

B.6. Invariant action formulas

The action for the product of a vector multiplet and a linear multiplet,

\[
e^{-1} \mathcal{L}_{VL} = \mathcal{L}^i_{VL} L_{ij} + 2 \bar{\Omega} \varphi + 2i \bar{\psi}_i^\alpha \gamma_\alpha \Omega_j L_{ij} - \frac{1}{2} W_a (E^a - 2 \bar{\psi}_b \gamma_\alpha \varphi + 2i \bar{\psi}_b^\gamma \gamma^{abc} \psi^c \bar{\psi}_e L_{ij}),
\]

is superconformal invariant if the vector multiplet is abelian and the linear multiplet carries no gauge group charges or is charged only under the abelian group of this vector multiplet.

The invariant action for the hypermultiplet, which gives as the equations of motion the required on-shell closure condition \( \Gamma_\alpha = 0 \) as well as its bosonic partner \( \Gamma_\alpha = 0 \), is given by

\[
e^{-1} \mathcal{L} = A_i^\alpha d_\alpha^\beta C^i_\beta + 2(\tilde{\zeta}^\alpha - i \bar{\psi}_\mu \gamma_\alpha A_i^\alpha) d_\alpha^\beta \Gamma_\beta.
\]
Here $d_{\alpha \beta}$ is a $G$-invariant hermitian tensor satisfying
\[
(d_{\alpha \beta})^* = d_{\beta \alpha}, \quad d_{\alpha \beta} = -d_{\beta \alpha}, \quad (d_{\alpha \beta} \equiv d_{\alpha \gamma} \rho_{\gamma \beta})
\]
\[
t^\gamma_{\alpha} d_{\gamma \beta} + d_{\alpha \gamma} (t^\gamma_{\beta})^* = 0.
\] (B.22)

It is shown in Ref. 17) that field redefinitions can simultaneously bring $\rho_{\alpha \beta}$ into the standard form (B.12) and $d_{\alpha \beta}$ into the form
\[
d_{\alpha \beta} = \begin{pmatrix} 1_p & \cr & -1_q \end{pmatrix}, \quad (p, q : \text{even})
\] (B.23)

This property and the condition (B.14) imply that the gauge group $G$ should be a subgroup of $USp(p, q)$.

References

[1] I. Antoniadis, Phys. Lett. 246B (1990), 377.
J. D. Lykken, Phys. Rev. D54 (1996), 3693.
N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. 429B (1998), 263.
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. 436B (1998), 257.

[2] I. Antoniadis, C. Muñoz and M. Quirós, Nucl. Phys. B397 (1993), 515.
I. Antoniadis, S. Dimopoulos, A. Pomarol and M. Quiros, Nucl. Phys. B544 (1999), 503.

[3] E. A. Mirabelli and M. E. Peskin, Phys. Rev. D58 (1998), 065002. hep-th/9712214

[4] L. Randall and R. Sundrum, Nucl. Phys. B557 (1999), 79.
M. A. Luty and R. Sundrum, hep-th/9910202

[5] K. R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. 436B (1998), 55; Nucl. Phys. B537 (1999), 47.
S. Abel and S. King, Phys. Rev. D59 (1999), 095010.
N. Arkani-Hamed and S. Dimopoulos, hep-ph/9811353
N. Arkani-Hamed and M. Schmaltz, Phys. Rev. D61 (2000), 033005, hep-ph/9903417
H.-C. Cheng, Phys. Rev. D60 (1999), 075015.
K. Yoshioka, Mod. Phys. Lett. A15 (2000), 29, hep-ph/9904433.

[6] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. D59 (1999), 086004.
M. Maggiore and A. Riotto, Nucl. Phys. B548 (1999), 427.
N. Kaloper and A. Linde, Phys. Rev. D59 (1999), 101303.
N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and J. March-Russell, Nucl. Phys. B567 (2000), 189, hep-ph/9903224; hep-ph/9903239.
A. Riotto, Phys. Rev. D61 (2000), 123506, hep-ph/9904485.

[7] J. Polchinski, String Theory, vol. 1, 2 (Cambridge Univ. Press, Cambridge, 1998).
[8] P. Hořava and E. Witten, Nucl. Phys. B460 (1996), 506, hep-th/9510209. Nucl. Phys. B475 (1996), 94, hep-th/9603142.
[9] M. Zucker, Nucl. Phys. B570 (2000), 267, hep-th/9907082; hep-th/9909144.
[10] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Phys. Lett. 116B (1982), 231; Nucl. Phys. B212 (1983), 413.
[11] E. Bergshoeff, E. Sezgin and A. Van Proeyen, Nucl. Phys. B264 (1986), 653.
[12] T. Kugo and S. Uehara, Nucl. Phys. B222 (1983), 125.
[13] B. de Wit, J. W. van Holten and A. Van Proeyen, Phys. Lett. 95B (1980), 51, P. Breitenlohner and M. F. Sohnius, Nucl. Phys. B187 (1981), 409.
[14] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 76B (1978), 409.
E. Cremmer and B. Julia, Nucl. Phys. B159 (1979), 141.
[15] M. F. Sohnius, Z. Phys. C18 (1983), 229.
[16] A. Van Proeyen, Lecture in the Proceedings of the Winter School in Karpacz, 1983, ed. B. Milewski (World Scientific Pub. Co.).
[17] B. de Wit, P. G. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985), 569.
[18] M. F. Sohnius, Nucl. Phys. B138 (1978), 109.
A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984), 447.
[19] T. Kugo and K. Ohashi, in preparation.
[20] T. Kugo and S. Uehara, Nucl. Phys. B226 (1983), 49; Prog. Theor. Phys. 73 (1985), 235.
[21] B. de Wit, J. W. van Holten and A. Van Proeyen, Nucl. Phys. B167 (1980), 186; B172 (1980), 543 (E); B184 (1981), 77; B222 (1983), 516 (E).
M. de Roo, J. W. van Holten, B. de Wit and A. Van Proeyen, Nucl. Phys. B173 (1980), 175.
E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B182 (1981), 173.
[22] B. de Wit, R. Philippe and A. Van Proeyen, Nucl. Phys. B219 (1983), 143.
[23] S. Ferrara, L. Girardello, T. Kugo and A. Van Proeyen, Nucl. Phys. B223 (1983), 191.
[24] M. Kaku, P. K. Townsend and P. van Niewenhuizen, Phys. Rev. D17 (1978), 3179.
M. Kaku and P. K. Townsend, Phys. Lett. 76B (1978), 54.
M. Kaku, P. K. Townsend and P. van Niewenhuizen, Phys. Rev. D19 (1979), 3166.
[25] P. van Niewenhuizen, Phys. Rep. 68 (1981), 189.