\textbf{D^*D_\rho \text{ and } B^*B_\rho \text{ strong couplings in light-cone sum rules}}

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Abstract

We present an improved calculation of the strong coupling constants $g_{D^*D_\rho}$ and $g_{B^*B_\rho}$ in light-cone sum rules including the one-loop QCD corrections of leading power with $\rho$ meson distribution amplitudes. We further compute the subleading-power corrections from two-particle and three-particle higher-twist contributions at leading order up to twist-4 accuracy. The next-to leading order corrections to leading power contribution offset the subleading-power corrections to some extend numerically, and our numerical results are consistent with previous works from sum rules. The comparisons between our results and the existing model-dependent estimations are also made.

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I. INTRODUCTION

This paper aims to give a more precise determination of $D^* D \rho$ coupling ($g_{D^* D \rho}$) and $B^* B \rho$ coupling ($g_{B^* B \rho}$). The couplings, describing the low-energy interaction among heavy mesons and light meson, are of great importance to understand the QCD long-distance dynamics. In the first place, the coupling is a fundamental parameter of the effective Lagrangian of heavy meson chiral perturbative theory (HM\(\chi\)PT) [1–4]. Phenomenologically, it describes the strength of the final state interactions [5] which are important in the generation of the strong phase within B decays [6, 7]. Secondly, the coupling relates the pole residue of $D(B)$ to $\rho$ form factors at large momentum transfer by the dispersion relation. Furthermore, although the couplings have been extensively studied in the literature, the theoretical predictions exhibit a widespread of values.

Various theoretical approaches to determine the coupling have been suggested. With the $D(B)$ to $\rho$ form factors obtained at certain region of the momentum transfer from light-cone sum rules (LCSR) [8–10] and the lattice QCD [11], the corresponding pole residue which relates the coupling can be extracted with appropriate extrapolation for the form factors. Phenomenologically, the simplest way is the vector meson dominance (VMD) hypothesis which neglects the continuum spectral. Other modified parameterizations for the form factors have been proposed [12–15]. The strong couplings are estimated from HM\(\chi\)PT with VMD approximation [16–19]. However, these estimations have potential theoretical uncertainties.

Another strategy to obtain the strong coupling is calculation from first principles of QCD. We study the strong coupling constants $g_{D^* D \rho}$ and $g_{B^* B \rho}$ in LCSR by using double dispersion relation. The LCSR was proposed in [20–22] based on the light-cone operator-product-expansion (OPE) relative to the conventional QCD sum rules (QCDSR) method. There have been several works regarding the couplings $g_{D^* D \rho}$ and $g_{B^* B \rho}$, starting from [23] with including two-particle $\rho$ DAs corrections up to twist-3 at leading order (LO). Years later, [24–27] improved the formal calculation by considering the two-particle twist-4 corrections [28] at LO. Meanwhile, $g_{D^* D \rho}$ is also calculated with three-point QCDSR by taking into account the dimension-5 quark-gluon condensate corrections under flavor $SU(3)$ symmetry [29]. With respect to previous works, we give a calculation including $O(\alpha_s)$ corrections to leading power contribution with the resummation of large logarithm to next-to leading
logarithmic accuracy (NLL). For the subleading-power corrections, our results also include
the three-particle twist-4 corrections at LO.

The paper is organized as follows: in section II we calculate the leading power contributions up to NLO. Following the procedure in [33, 34], the hard-collinear factorization is achieved in OPE region with the aid of evanescent operator [35, 36] in the frame work of soft-collinear effective theory (SCET) [37–39] and strategy of regions [40]. Moreover, procedures for analytic continuation and continuum subtraction are similar to [41, 42]. Subleading-power corrections including two-particle and three-particle corrections up to twist-4 at LO are calculated in section III. Section IV provides our numerical results and the phenomenological discussion. We will summarize this work in the last section.

II. THE LEADING POWER CONTRIBUTIONS

A. Hard-collinear factorization at LO in QCD

The strong coupling constant \( g_{H^*H^0} \) is defined by

\[
\langle \rho(p, \eta^*) H(q) \mid H^*(p + q, \varepsilon) \rangle = -g_{H^*H^0} \varepsilon_{pq\rho},
\]

with \( \varepsilon_{pq\rho} = \epsilon^{\mu\nu\alpha\beta} p^\mu q^\nu \eta^{*\alpha} \varepsilon^\beta \). Here we choose \( H^* \) and \( H \) stand for \( D^{*-}(B^{*0}) \) meson and \( \bar{D}^0(B^+) \) meson respectively, and \( \rho \) is \( \rho^- \). \( \varepsilon_\mu \) and \( \eta_\mu \) are the polarization vectors of the \( H^* \) and \( \rho \) mesons respectively. We use the conventions \( \epsilon_{0123} = -1 \) and \( D_\mu = \partial_\mu - ig s T^a A_\mu^a \).

The couplings of different charge states are related by isospin symmetry, for instance,

\[
g_{D^*D^0} \equiv g_{D^*-\bar{D}^0\rho^-} = -\sqrt{2} g_{D^*D^-\rho^0}.
\]

We construct the following correlation function at the starting point

\[
\Pi_\mu(p, q) = \int d^4 x e^{-i (p + q) x} \langle \rho^-(p, \eta^*) \mid T \{ \bar{d}(x) \gamma_\mu \perp Q(x), \bar{Q}(0) \gamma_5 u(0) \} \mid 0 \rangle,
\]

where \( \gamma_\mu \perp = \gamma_\mu - \not{\bar{n}}/2 n_\mu - \not{\bar{\eta}}/2 n_\mu \), \( Q \) is the heavy quark field. The power counting scheme that we take is as follows

\[
|(p + q)^2 - m_Q^2| \sim \mathcal{O}(m_Q^2), \quad |q^2 - m_Q^2| \sim \mathcal{O}(m_Q^2),
\]

\[
n \cdot p \sim \mathcal{O}(m_Q), \quad \bar{n} \cdot p = m_\rho^2/n \cdot p \sim \mathcal{O}(\Lambda_{QCD}^2/m_Q),
\]

(4)
where we have introduced two light-cone vectors \( n_\mu \) and \( \bar{n}_\mu \) with \( n \cdot n = \bar{n} \cdot \bar{n} = 0 \), \( n \cdot \bar{n} = 2 \).

On the hadronic level, taking advantage of the following definitions for decay constants

\[
\langle H^*(p + q, \varepsilon^*) | \bar{d} \gamma^\mu Q | 0 \rangle = f_{H^*} m_{H^*} \varepsilon^\mu, \quad \langle 0 | \bar{Q} \gamma_5 u | H(p) \rangle = -i f_H \frac{m_H^2}{m_Q},
\]

the correlation function (3) can be written as

\[
\Pi_{\mu}^{\text{had}} (p, q) = \frac{g_{H^* H^*} f_{H^*} f_H}{[m_{H^*}^2 - (p + q)^2 - i 0]} \frac{m_H^2 m_{H^*}}{m_Q} \epsilon_{\mu pq n} + \int_{\Sigma} \rho^h (s, s') ds ds' + \cdots,
\]

where the second term counts the contributions from higher resonances and continuum states. The ellipses denote the terms that vanish after double Borel transformation.

After double Borel transformation we get the hadronic representation of the correlation function

\[
\Pi_{\mu}^{\text{had}} (p, q) = \frac{f_H f_{H^*}^2 m_{H^*} m_H}{m_Q} \rho_{H^* H^*} e^{\frac{-m_H^2}{M^2}} \epsilon_{\mu pq n} + \int_{\Sigma} ds ds' e^{\frac{s + s'}{M^2}} \rho^h (s, s').
\]

For the boundary of the integral \( \Sigma \), we take \( s + s' = 2 s_0 \) with \( s_0 \) as the threshold of excited and continuum states. The Borel parameters associated with \((p + q)^2\) and \(q^2\) are quite similar in magnitude, so we set the same value \( M^2 \).

On the quark level, the leading-twist tree diagram is displayed in Fig. 1. The correlation function reads

\[
\Pi_{\mu}^{\text{LT},(0)} (p, q) = - \frac{i}{2 u q^2 + \bar{u} (p + q)^2} \frac{\bar{n} \cdot q}{u \bar{u} m_\rho^2 - m_Q^2} \bar{q}(u p) \gamma_\mu \gamma_5 q(\bar{u} p)
\]

FIG. 1. Diagrammatical representation of the leading-order (LO) contribution.
\begin{equation}
\frac{i}{2} \frac{\bar{n} \cdot q}{u q^2 + \bar{u} (p + q)^2} \frac{\bar{n} \cdot (p + q)^2 - m_Q^2}{m_Q} \bar{q}(u p) \gamma_{\mu \perp} \gamma_5 q(\bar{u} p) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{m_Q^2}\right). \tag{8}
\end{equation}

The spinor structure in (8) is the same as the axial-vector part of the correlation function in \( B \to \gamma l \nu \) in [33]. Following [33] we match the QCD results (8) to SCET. At leading power, we obtain

\begin{equation}
\Pi_{\mu}^{LT,(0)}(p, q) = -\frac{i}{2} \frac{\bar{n} \cdot q}{u' q^2 + \bar{u}' (p + q)^2 - m_Q^2} \langle O_{A,\mu}(u, u') \rangle^{(0)} , \tag{9}
\end{equation}

where the tree level SCET operator matrix elements reads

\begin{equation}
\langle O_{A,\mu}(u, u') \rangle^{(0)} = \bar{\xi}(u p) \gamma_{\mu \perp} \gamma_5 \xi(\bar{u} p) \delta(u - u') , \tag{10}
\end{equation}

with \( u \) is the momentum fraction carried by quark. To establish hard-collinear factorization, we decomposed the SCET operator \( O_{A,\mu} \) in (9) into the light-ray operators

\begin{equation}
O_{A,\mu} = O_{1,\mu} + O_{E,\mu} , \tag{11}
\end{equation}

where

\begin{equation}
O_{j,\mu}(u') = \frac{n \cdot p}{2\pi} \int d\tau e^{-i u' \tau n \cdot p} \bar{\xi}(\tau n) W_c(\tau n, 0) \Gamma_j \xi(0) , \\
\Gamma_1 = \frac{n^\nu}{2} \epsilon_{\mu \nu \alpha \beta} \sigma^{\alpha \beta} , \\
\Gamma_E = \gamma_{\mu \perp} \gamma_5 - \frac{n^\nu}{2} \epsilon_{\mu \nu \alpha \beta} \sigma^{\alpha \beta} . \tag{12}
\end{equation}

The matching equation including the evanescent operator \( O_{E,\mu} \) reads

\begin{equation}
\Pi_{\mu}(p, q) = C_1(u', (p + q)^2, q^2) \langle O_{1,\mu}(u, u') \rangle + C_E(u', (p + q)^2, q^2) \langle O_{E,\mu}(u, u') \rangle , \tag{13}
\end{equation}

at LO the coefficients is

\begin{equation}
C_1^{(0)} = C_E^{(0)} = -\frac{i}{2} \frac{\bar{n} \cdot q}{u' q^2 + \bar{u}' (p + q)^2 - m_Q^2} . \tag{14}
\end{equation}

Using the definition of the leading-twist \( \rho \) DA [28] in appendix A, we get the leading twist tree level factorization formula

\begin{equation}
\Pi_{\mu}^{LT,(0)}(p, q) = -f_\rho^T(\mu) \epsilon_{\mu \rho \perp} \int_0^1 du \phi_\perp(u, \mu) \frac{1}{u q^2 + \bar{u} (p + q)^2 - m_Q^2} . \tag{15}
\end{equation}

The result (15) can be written in the form of double dispersion relation

\begin{equation}
\Pi_{\mu}^{LT,(0)}(p, q) = -f_\rho^T(\mu) \epsilon_{\mu \rho \perp} \int \int ds ds' \frac{\rho_{LT,(0)}^{LT}(s, s')}{[s - q^2][s' - (p + q)^2]} , \tag{16}
\end{equation}

\text{5}
where the double dispersion density $\rho^{LT,(0)}$ is defined as

$$\rho^{LT,(0)}(s, s') = \frac{1}{\pi^2} \text{Im}_y \text{Im}_s \int_0^1 du \frac{\phi_{\perp}(u, \mu)}{u \, s + \bar{u} \, s' - m_Q^2}. \quad (17)$$

Equating (6) and (16) and applying double Borel transformation, then subtracting the continuum states by using quark-hadronic duality, we obtain the master formula as follows

$$g^{LT,(0)} = -\frac{m_Q}{f_H \, f_{H^*} \, m_H^2 \, m_{H^*}} \int_{-\infty}^{+\infty} ds \int_{-\infty}^{2s_0} ds' e^{-\frac{s + s'}{M^2}} \rho^{LT,(0)}(s, s')$$

$$= -\frac{m_Q}{f_H \, f_{H^*} \, m_H^2 \, m_{H^*}} \int_{-\infty}^{2s_0} dt \int_{-\infty}^{+\infty} dv e^{-\frac{t + v}{M^2}} \rho^{LT,(0)}(t, v), \quad (18)$$

where we have chosen the triangle integral region, $s + s' < 2s_0$, for performing continuum subtraction [42], and the variables $t$ and $v$ are defined as

$$t = s + s', \quad v = \frac{s}{s + s'}. \quad (19)$$

The expression of wave function $\phi_{\perp}(u, \mu)$ in terms of Gegenbauer polynomials can be written as

$$\phi_{\perp}(u, \mu) = 6 \, u \, \bar{u} \sum_{n=0}^{\infty} a_n^\perp(\mu) C_n^{3/2}(2u - 1) = \sum_{k=1}^{\infty} b_k \, u^k, \quad (20)$$

where $b_k$ is the function of Gegenbauer moments $a(\mu)$. The spectral density can be obtained as follows [41]

$$\rho^{LT,(0)}(t, v) = \sum_{k=1}^{\infty} b_k \, \frac{1}{\pi^2} \text{Im}_y \text{Im}_s \int_0^1 du \frac{u^k}{u \, s + \bar{u} \, s' - m_Q^2}$$

$$= \sum_{k=1}^{\infty} b_k \, (-1)^{k+1} \frac{1}{k!} \frac{1}{2^{k+1} t} \left( \frac{m_Q^2}{t} - \bar{v} \right)^k \delta^{(k)}(v - \frac{1}{2}) \theta(v \, t - m_Q^2). \quad (21)$$

Compute the corresponding integral of (18) we obtain the leading-twist strong coupling constant at LO

$$g^{LT,(0)} = \frac{m_Q}{f_H \, f_{H^*} \, m_H^2 \, m_{H^*}} \int_{-\infty}^{+\infty} ds \int_{-\infty}^{2s_0} ds' e^{-\frac{s + s'}{M^2}} \rho^{LT,(0)}(s, s') \left[ e^{\frac{m_H^2 + m_{H^*}^2 - 2m_Q^2}{M^2}} - e^{\frac{m_H^2 + m_{H^*}^2 - 2m_0^2}{M^2}} \right]. \quad (22)$$

B. Hard-collinear factorization at NLO in QCD

The one-loop diagrams are shown in Fig. 2. Only the hard region can generate a nonzero contribution since the collinear region corresponding the scaleless integral in dimensional
FIG. 2. NLO QCD corrections of leading-twist contributions.

regularization. With the NDR scheme of the Dirac matrix $\gamma_5$ the results of the one-loop diagrams are

$$\Pi^{(1),h}_{\mu,H^*} = \frac{\alpha_s C_F}{4\pi} \left\{ \left[ \frac{2(1-r_2)}{r_2-r_1} \ln \frac{1-r_1}{1-r_2} - 1 \right] \left[ \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_Q^2} - \ln[(1-r_1)(1-r_2)] \right] 
- \ln[(1-r_1)(1-r_2)] + \frac{1}{r_1-r_2} \left[ 2(1-r_2)(\text{Li}_2(r_1) - \text{Li}_2(r_2)) \right] 
+ \frac{(1-r_1)(1-r_2-2r_2)}{r_1} \ln(1-r_1) - \frac{(1-r_2)(1-3r_2)}{r_2} \ln(1-r_2) \right\} \Pi^{(0)}_{\mu},$$

$$\Pi^{(1),h}_{\mu,H^*} = -\frac{\alpha_s C_F}{4\pi} \left\{ 2 \left[ \frac{1-r_3}{r_1-r_3} \ln \frac{1-r_1}{1-r_3} - 1 \right] \left[ \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_Q^2} - \ln[(1-r_1)(1-r_3)] \right] 
- 2 \ln[(1-r_1)(1-r_3)] - \frac{2}{r_1-r_3} \left[ (1-r_3)(\text{Li}_2(r_1) - \text{Li}_2(r_3)) \right] 
- \frac{(1-r_1)(1-r_1+r_3)}{r_1} \ln(1-r_1) + \frac{1-r_3}{r_3} \ln(1-r_3) \right\} \Pi^{(0)}_{\mu},$$

$$\Pi^{(1)}_{\mu,wfc} = -\frac{\alpha_s C_F}{4\pi} \left\{ \frac{7-r_1}{1-r_1} \left[ \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_Q^2} - \ln(1-r_1) + 1 \right] - \frac{1}{1-r_1} \left[ \frac{1-7r_1}{r_1^2} \ln(1-r_1) \right] + \frac{1}{r_1-3} \right\} \Pi^{(0)}_{\mu}, \quad (23)$$

where $r_1 = (\bar{u} p + q)^2/m_Q^2$, $r_2 = (p + q)^2/m_Q^2$ and $r_3 = q^2/m_Q^2$ and the tree-level $\Pi^{(0)}_{\mu}$ is given in (9). The box diagram is order $\mathcal{O}(\epsilon)$ in dimensional regularization, so has no contribution, which is also confirmed in [33, 34]. Adding up the one-loop results, we get the NLO hard amplitude

$$T^{(1)} |_{\text{NDR}}(r_2, r_3, \mu)$$

$$= \frac{\alpha_s C_F}{4\pi} \left\{ (-2) \left[ \frac{1-r_2}{r_1-r_2} \ln \frac{1-r_1}{1-r_2} + \frac{1-r_3}{r_1-r_3} \ln \frac{1-r_1}{1-r_3} + \frac{3}{1-r_1} \right] \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_Q^2} \right) \right\}.$$
+ 2 \left[ \left( \frac{1 - r_2}{r_1 - r_2} + \frac{1 - r_3}{r_1 - r_3} \right) \text{Li}_2(r_1) - \frac{1 - r_2}{r_1 - r_2} \text{Li}_2(r_2) - \frac{1 - r_3}{r_1 - r_3} \text{Li}_2(r_3) \right] \\
+ 2 \left[ \left( \frac{1 - r_2}{r_1 - r_2} + \frac{1 - r_3}{r_1 - r_3} \right) \ln^2(1 - r_1) - \frac{1 - r_2}{r_1 - r_2} \ln^2(1 - r_2) - \frac{1 - r_3}{r_1 - r_3} \ln^2(1 - r_3) \right] \\
+ \left[ \frac{1 - r_1}{r_1} \left( \frac{1 - r_1 - 2 r_2}{r_1 - r_2} - 2 \left( 1 - r_1 + r_3 \right) \right) + \frac{1 - 6 r_1}{r_1^2} + 1 \right] \ln(1 - r_1) + \frac{1 - 9 r_1}{r_1(1 - r_1)} \\
- \frac{(1 - r_2)(1 - 3 r_2)}{r_2(r_1 - r_2)} \ln(1 - r_2) + \frac{2(1 - r_3)}{r_3(r_1 - r_3)} \ln(1 - r_3) - 3 + 3 \ln \frac{\nu^2}{\mu^2} \right\} C_i^{(0)}, \quad (24)

where \( C_i^{(0)} \) is given by (14). Here we have distinguished the renormalization scale \( \nu \) from factorization scale \( \mu \) because of the non-conservation of the pseudoscalar current in QCD.

It has been proven that the matrix element of the evanescent SCET operator \( \langle O_{E,\mu} \rangle^{(1)} \) vanishes to \( \mathcal{O}(\alpha_s) \) order [33] with the NDR scheme of \( \gamma_5 \) in hard-collinear factorization. So the one-loop hard matching coefficient

\[
C_1^{(1)} = T_1^{(1),\text{reg}}. \quad (25)
\]

The twist-two factorization formula at NLO reads

\[
\Pi_\mu^{LT,(1)}(p, q) = - f_\rho^T(\mu) \epsilon_{\mu pq} \int_0^1 du \frac{\phi_{\perp}(u, \mu)}{u q^2 + \bar{u} (p + q)^2 - m^2_Q} C_i^{(1)} + \mathcal{O}(\alpha_s^2). \quad (26)
\]

Adding to the LO result (15), we obtain

\[
\Pi_\mu^{LT}(p, q) = - f_\rho^T(\mu) \epsilon_{\mu pq} \int_0^1 du \frac{\phi_{\perp}(u, \mu)}{u q^2 + \bar{u} (p + q)^2 - m^2_Q} \left[ 1 + \frac{C_1^{(1)}}{C_1^{(0)}} \right] + \mathcal{O}(\alpha_s^2). \quad (27)
\]

After subtracting the continuum states we obtain the formula

\[
g_{LT,(1)} = - \frac{m_Q}{f_H f_{H^*} m^2_H m_{H^*}} e^{\frac{m^2_H + m^2_{H^*}}{M^2}} f_\rho^T(\mu) m^4_Q \int_{s_0}^{2s_0 - 2} d\sigma \int_{-\infty}^{+\infty} dr \frac{\sigma}{(r + 1)^2} e^{-\frac{\pi^2}{\sqrt{\pi^2}} \rho^{LT,(1)}(r, \sigma)},
\]

(28)

where \( s_0 = s_0/m^2_Q, \hat{M}^2 = M^2/m^2_Q, \) and

\[
\rho^{LT,(1)}(r, \sigma) = \frac{1}{\pi^2} \text{Im}s_0 \text{Im}_s \int_0^1 du \frac{\phi_{\perp}(u)}{u q^2 + \bar{u} (p + q)^2 - m^2_Q} \frac{C_i^{(1)}}{C_i^{(0)}}. \quad (29)
\]

We have used the following Jacobian transformation to obtain (28)

\[
r = \frac{r_2 - 1}{r_3 - 1}, \quad \sigma = r_2 + r_3 - 2. \quad (30)
\]
To get the integral over $r$ in (28) analytically, we take the asymptotic form $\phi_\perp(u) = 6 \, u \, \bar{u}$ for simplify. The NLO spectral density can be obtained as follows [42, 43]

$$
\rho^{\text{LT.}(1)}(r, \sigma) = \frac{1}{\pi^2} \text{Im}_s \int_0^1 du \, \frac{\phi_\perp(u)}{m_Q^2 (\rho - 1)} \frac{C_1^{(1)}}{C_1^{(0)}}
$$

$$
= \frac{1}{\pi^2} \text{Im}_{r_2} \text{Im}_{r_3} \int_0^1 du \, \phi_\perp(u) \, \tilde{T}(r_2, r_3, u)
$$

$$
= \frac{1}{\pi} \text{Im}_{r_2} \int_{r_2}^{r_3} \frac{d\rho}{r_3 - r_2} \phi_\perp(\rho) \frac{1}{\pi} \text{Im}_\rho \tilde{T}(r_2, r_3, \rho)
$$

$$
= \frac{1}{\pi} \text{Im}_{r_2} f(r_2, r_3) = \frac{1}{\pi} \text{Im}_r f(r, \sigma),
$$

with the definitions $\rho = \bar{u} \, r_2 + u \, r_3$ and

$$
\tilde{T}(r_3, r_2, \rho) = \frac{1}{m_Q^2 (\rho - 1)} \frac{C_1^{(1)}}{C_1^{(0)}}, \quad f(r_2, r_3) = \int_{r_2}^{r_3} \frac{d\rho}{r_3 - r_2} \phi_\perp(\rho) \frac{1}{\pi} \text{Im}_\rho \tilde{T}(r_2, r_3, \rho).
$$

The final result reads

$$
\rho^{\text{LT.}(1)}(r, \sigma) = -\frac{1}{2} \frac{1}{m_Q^2} \frac{\alpha_s}{4 \pi} \left\{ \left[ f_1(r, \sigma) + f_2(r, \sigma) \ln r + f_3(r, \sigma) (\ln^2 r - \pi^2) \right] \delta^{(2)}(r - 1)
$$

$$
+ \left[ f_2(r, \sigma) + 2 \, f_3(r, \sigma) \ln r \right] \left[ \frac{d^3}{dr^3} \ln |1 - r| \right] \theta(\sigma) \theta(r) \right\},
$$

where

$$
f_1(r, \sigma) = \tilde{f}_1(r, \sigma) + \Delta f_1(r, \sigma),
$$

$$
\tilde{f}_1(r, \sigma) = -\frac{2 \, (r + 1)}{\sigma^3}
$$

$$
\times \left\{ 6 \, r \, \sigma^2 \left[ 3 \, \text{Li}_2 \left( \frac{-\sigma}{r + 1} \right) + \text{Li}_2 \left( \frac{-r \, \sigma}{r + 1} \right) + \ln \frac{\sigma}{r + 1} \left( \ln \frac{\sigma + r + 1}{r + 1} + \ln \frac{r \, \sigma + r + 1}{r + 1} \right) \right]
$$

$$
+ 3 \, (r + 1) \left[ 2 \, r \, (\sigma + 1) + 5 \, \sigma + 2 \right] \ln \frac{\sigma + r + 1}{r + 1} + 12 \, r \, \sigma^2 \, \ln \frac{r^2}{m_Q^2} + 9 \, r \, \sigma^2 \, \ln \frac{\nu^2}{\mu^2}
$$

$$
+ \frac{\sigma}{r + \sigma + 1} \left[ (3 + 2 \, \pi^2 \, r + 6) \, \sigma^2 + (9 + 2 \, \pi^2 \, r - 6 \, (r + 1) \sigma - 6 \, (r + 1)^2) \right]
$$

$$
- \frac{3 \, r \, \sigma^2}{(r + \sigma + 1) \, (r \, \sigma + r + 1)} \left[ 9 \, r \, \sigma^2 + (7 \, r + 10) \, (r + 1) \sigma + 8 \, (r + 1)^2 \right] \ln \frac{\sigma}{r + 1} \right\},
$$

$$
\Delta f_1(r, \sigma) = -12 \left( 4 + 3 \, \ln \frac{\mu^2}{m_Q^2} \right) m_Q^4 \frac{d}{dm_Q^2} \frac{1}{\sigma} \frac{r \, (r + 1)}{r \, \sigma + r + 1}
$$

$$
f_2(r, \sigma) = -6 \, r \, (r + 1) \left[ \ln \frac{\sigma + r + 1}{r \, \sigma + r + 1} + \frac{2}{r + \sigma + 1} - \frac{2}{\sigma} \ln \frac{\sigma + r + 1}{r \, \sigma + r + 1} \right].
$$
\[ f_3(r, \sigma) = \frac{12r(r+1)}{\sigma}. \]  

(34)

For the tree level density \( \rho^{LT,(0)} \) in (21) which vanishes except \( s = s' \), the integral of \( \rho^{LT,(0)} \) is independent the shape of the integral region. On the other hand, the integral of the NLO density \( \rho^{LT,(1)} \) depends on the duality region we choose. Choosing the triangle region mentioned before, the final results of (28) are as follows

\[ g^{LT,(1)} = -\frac{m_Q}{f_H f_H^* m_H^2 m_{H^*}} e^{\frac{m_H^2 + m_{H^*}^2}{M^2}} f^T_\rho (\mu) \mathcal{F}^{LT,(1)}, \]  

(35)

where

\[ \mathcal{F}^{LT,(1)} = m_Q^4 \int_{-\infty}^{2s_0 - 2} d\sigma \int_{-\infty}^{+\infty} dr \frac{\sigma}{(r+1)^2} e^{-\frac{2s_0^2}{M^2}} \rho^{LT,(1)}(r, \sigma) \]

\[ = \frac{\alpha_s C_F}{4\pi} \left\{ m_Q^2 \int_0^{2s_0 - 2} d\sigma e^{-\frac{2s_0^2}{M^2}} g(\sigma) + \Delta g(\hat{M}^2, m_Q^2) \right\}, \]  

(36)

and

\[ g(\sigma) = 3 \text{Li}_2(-\sigma) + 3 \text{Li}_2(-\sigma - 1) - 6 \text{Li}_2(-\frac{\sigma}{2}) - 3 \ln \frac{\sigma + 2}{2} + 3 \ln(\sigma + 1) \ln(\sigma + 2) \]

\[- \frac{6(\sigma + 1)^2}{(\sigma + 2)^3} \ln(\sigma + 1) + \frac{3(7\sigma^3 + 50\sigma^2 + 100\sigma + 64)}{4(\sigma + 2)^3} \ln \frac{\sigma + 2}{2} + \frac{3}{2} \ln \frac{\sigma + 2}{2} \]

\[ + \frac{3(11\sigma^2 + 28\sigma + 24)}{8(\sigma + 2)^2} - 3 \ln \frac{\mu^2}{m_Q^2} - \frac{9}{4} \ln \frac{\mu^2}{\mu^2} - \frac{\pi^2}{4}, \]

\( \Delta g(\hat{M}^2, m_Q^2) = 3 \left( 4 + 3 \ln \frac{\mu^2}{m_T^2} \right) m_Q^2 e^{-\frac{2m_Q^2}{M^2}}. \]  

(37)

Collecting (22) and (35), we obtain the leading-twist sum rules up to \( \mathcal{O}(\alpha_s) \)

\[ g^{LT} = -\frac{m_Q}{f_H f_H^* m_H^2 m_{H^*}} e^{\frac{m_H^2 + m_{H^*}^2}{M^2}} f^T_\rho (\mu) \left[ -\frac{M^2}{2} \phi_\perp \left( \frac{1}{2}, \mu \right) \left( e^{-\frac{2m_Q^2}{M^2}} - e^{-\frac{2s_0^2}{M^2}} \right) + \mathcal{F}^{LT,(1)} \right], \]  

(38)

Using

\[ \frac{d}{d \ln \mu} m_Q(\mu) = -6 \frac{\alpha_s C_F}{4\pi} m_Q(\mu), \quad \frac{d}{d \ln \mu} f^T_\rho (\mu) = -2 \frac{\alpha_s C_F}{4\pi} f^T_\rho (\mu). \]  

(39)

We find

\[ \frac{d}{d \ln \mu} g^{LT} = 0 + \mathcal{O}(\alpha_s^2), \]  

(40)

so our result is independent of the factorization scale \( \mu \) at one-loop level.
III. THE SUBLEADING-POWER CORRECTIONS AT LO

In this section, we are going to perform the subleading power corrections to coupling constants, which are involved with two-particle and three-particle corrections up to twist-4. The $Q$-quark propagator up to twist-4 in the background field is adopted \[44\]:

$$
\langle 0 \mid T\{\bar{Q}(x), Q(0)\} \mid 0 \rangle \supset \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot x} \frac{i (k + m_Q)}{k^2 - m_Q^2} + i g_s \int \frac{d^4 k}{(2\pi)^4} e^{-i k \cdot x} \int_0^1 du \left[ \frac{u x^\mu}{k^2 - m_Q^2} G^{\mu\nu}(u x) \gamma_\nu - \frac{k + m_Q}{2 (k^2 - m_Q^2)^2} G^{\mu\nu}(u x) \sigma_{\mu\nu} \right]. \tag{41}
$$

A. The two-particle subleading-power corrections

Insert the first term of (41) into correlation function (3), the correlation function in Eq.(3) can be written as

$$
\Pi_\mu(p, q) \supset i \int d^4 x \int \frac{d^4 k}{(2\pi)^4} e^{-i (p + q + k) \cdot x} \frac{1}{k^2 - m_Q^2} \times \langle \rho(p, \eta^*) \mid \bar{d}(x) \left[ k^\nu (g_{\mu\nu} \gamma_5 + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \sigma^{\alpha\beta}) + m_Q \gamma_\mu \gamma_5 \right] u(0) \mid 0 \rangle, \tag{42}
$$

where we used $\gamma_\mu \gamma_\nu = g_{\mu\nu} - i \sigma_{\mu\nu}$ and $\sigma_{\mu\nu} \gamma_5 = i 2 \epsilon_{\mu\nu\alpha\beta} \sigma^{\alpha\beta}$. Substituting the DAs into (42), we gain the higher twist two-particle corrections

$$
\Pi^{2P,HT}_\mu(p, q) = -\frac{1}{4} \epsilon_{\mu p q q} \int_0^1 du \left\{ \frac{-2 m_Q f_\rho m_\rho g_\perp^{(a)}(u) + f_\rho^T(\mu) m_\rho^2 A_T(u)}{|u q^2 + \bar{u} (p + q)^2 - m_Q^2|^2} + \frac{-2 m_Q^2 f_\rho^T(\mu) m_\rho^2 A_T(u)}{|u q^2 + \bar{u} (p + q)^2 - m_Q^2|^3} \right\}. \tag{43}
$$

The twist-3 $g_\perp^{(a)}$ corrections are suppressed by $O(\Lambda_{QCD}/m_Q)$ and twist-4 $A_T$ corrections are suppressed by $O(\Lambda_{QCD}^2/m_Q^2)$ comparing with the leading-twist corrections (27). To keep consistent, we should also consider the subleading-power corrections of leading-twist. With respect to (9) the correlation function reads

$$
\Pi^{LT,NLF}_\mu(p, q) = -\frac{i}{2} \bar{n} \cdot q \frac{u \bar{u} m_\rho^2}{|u q^2 + \bar{u} (p + q)^2 - m_Q^2|^2} q(u p) \gamma_{\mu \perp} \gamma_5 q(\bar{u} p)
\begin{align*}
&= -f_\rho^T \epsilon_{\mu p q q} \int_0^1 du \phi_\perp(u, \mu) \frac{u \bar{u} m_\rho^2}{|u q^2 + \bar{u} (p + q)^2 - m_Q^2|^2}. \tag{44}
\end{align*}
$$
Similar to the leading-twist LO case, after applying double Borel transformations and subtracting the continuum states by using quark-hadronic duality, we obtain the strong coupling constant for two-particle subleading-power corrections at LO

\[ g_{2P} = - \frac{m_Q}{f_H f_{H^*} m_H^2} e^{\frac{m_Q^2 + m_{H^*}^2 - 2m_{Q}^2}{M^2}} \left[ \frac{m_p}{2} f_p^T(\mu) \phi_{\perp} \left( \frac{1}{2}, \mu \right) \right. \]

\[ - m_Q f_p g_{\perp}^{(a)} \left( \frac{1}{2}, \mu \right) + m_p f_p^T(\mu) \left( \frac{1}{2} + \frac{m_Q^2}{M^2} \right) A_T \left( \frac{1}{2}, \mu \right) \]  

(45)

The derivation of the spectral density and the corresponding integral are listed in appendix C.

B. The three-particle subleading-power corrections

At tree level, the correlation function of three-particle \( q\bar{q}g \) corrections can be written as

\[ \Pi_{3P}^{\mu}(p, q) = i g_s \int d^4x \int d^4k (2\pi)^4 \left( \frac{k}{2} \right)^2 e^{-i(k+p+q)\cdot x} \int_0^1 du \left( \rho^{-}(p, \eta) \right) | \overline{d}(x) \gamma_{\perp} \left[ \frac{u x_\alpha}{k^2 - m_Q^2} \gamma_\beta \right. \]

\[ - \frac{k + m_Q}{2(k^2 - m_Q^2)^2} \sigma_{\alpha\beta} \] \( G^{\alpha\beta}(ux) \gamma_5 q(0) | 0 \) \ .

(46)

Employing the definitions of three-particle \( \rho \) DAs, we obtain

\[ \Pi_{3P}^{\mu}(p, q) = - f_p^T(\mu) m_p^2 \epsilon_{uqq} \int_0^1 du \int [da] \frac{\varrho_{3P}^{\mu}(a, u)}{[a_\alpha q^2 + \bar{a}_\alpha (p + q)^2 - m_Q^2]^2} , \] 

(47)

where

\[ \varrho_{3P}^{\mu} = - S(\alpha, \mu) - T_1^{(4)}(\alpha, \mu) + T_2^{(4)}(\alpha, \mu) + (1 - 2u) \left[ \tilde{S}(\alpha, \mu) - T_3^{(4)}(\alpha, \mu) + T_4^{(4)}(\alpha, \mu) \right] . \] 

(48)

Here we used the following definition

\[ \alpha_u = \alpha_q + u \alpha_g , \quad \bar{a}_u = 1 - \alpha_u , \quad \int [da] = \int_0^1 da_q \int_0^1 da_\overline{g} \int_0^1 da_q \delta(1 - \alpha_q - \bar{a}_g - a_g) . \] 

(49)

Comparing with the leading-twist corrections (27), the three-particle twist-4 corrections are suppressed by \( \mathcal{O}(\Lambda_{QCD}^2/m_Q^2) \). The LCSR results for three-particle corrections at LO are as follows

\[ g_{3P} = - \frac{m_Q}{f_H f_{H^*} m_H^2} e^{\frac{m_Q^2 + m_{H^*}^2}{M^2}} f_p^T(\mu) m_p^2 \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dv t e^{-\frac{t}{M^2}} \varrho_{3P}^{\mu}(t, v) \]
\[\begin{align*}
= - \frac{m_Q}{f_H f_H^* m_H^2 m_{H^*}} e^{-\frac{m_Q^2}{M^2}} \left\{ \begin{array}{c}
f^{T}_{\rho}(\mu) m_{\rho}^2 \left\{ - \tilde{S}(\frac{1}{2}, 0) - \tilde{T}_{1}^{(4)}(\frac{1}{2}, 0) + \tilde{T}_{2}^{(4)}(\frac{1}{2}, 0) \\
+ \tilde{S}(\frac{1}{2}, 0) - \tilde{T}_{3}^{(4)}(\frac{1}{2}, 0) + \tilde{T}_{4}^{(4)}(\frac{1}{2}, 0) - 2 \left[ \tilde{S}(\frac{1}{2}, 1) - \tilde{T}_{3}^{(4)}(\frac{1}{2}, 1) + \tilde{T}_{4}^{(4)}(\frac{1}{2}, 1) \right]\end{array} \right\},
\end{align*}\]

where the hat functions of the DAs \( \Phi_{3P} \) are defined as

\[
\hat{\Phi}_{3P}(\alpha_u, l) = \int_{0}^{\alpha_u} d\alpha_q \int_{\alpha_u - \alpha_q}^{\alpha_u} d\alpha_g \frac{(\alpha_u - \alpha_q)^l}{\alpha_g^{l+1}} \Phi_{3P}(\alpha_q, \alpha_g, 1 - \alpha_q - \alpha_g).
\]

Employing the conformal expansion of the three-particle DAs in appendix B, we obtain

\[
g_{3P}^{\rho} = - \frac{m_Q}{f_H f_H^* m_H^2 m_{H^*}} e^{-\frac{m_Q^2}{M^2}} \left\{ f^{T}_{\rho}(\mu) m_{\rho}^2 \left[ \frac{21}{8} - 4 \ln 2 \right] \tilde{t}_{10}(\mu) - \frac{15}{8} s_{00}(\mu) \right\}.
\]

Collecting (38) (45) and (52), the final LCSR reads

\[
g_{H^* H^* \rho} = - \frac{m_Q}{f_H f_H^* m_H^2 m_{H^*}} e^{-\frac{2m_Q^2}{M^2}} \left\{ f^{T}_{\rho}(\mu) \left[ - \frac{M^2}{2} \left( 1 - e^{-\frac{2\alpha_0 - m_Q^2}{M^2}} \right) + \frac{m_{\rho}^2}{4} \right] \phi_{\perp}(\frac{1}{2}, \mu) \right. \\
+ f^{T}_{\rho}(\mu) \mathcal{F}_{LT,(1)} - \frac{m_Q}{2} m_{\rho} \int_{\perp}^{(a)} \left( \frac{1}{2}, \mu \right) \\
+ m_{\rho}^2 f^{T}_{\rho}(\mu) \left[ \frac{1}{2} \left( \frac{1}{2} + \frac{m_Q^2}{M^2} \right) A_{T}(\frac{1}{2}, \mu) + \left( \frac{21}{8} - 4 \ln 2 \right) \tilde{t}_{10}(\mu) - \frac{15}{8} s_{00}(\mu) \right] \right\},
\]

where \( \mathcal{F}_{LT,(1)} \) is defined in (36).

IV. NUMERICAL ANALYSIS

A. Input parameters

The masses of quarks in \( \overline{\text{MS}} \) scheme and the values of decay constants are listed in Table I, and the the values of the parameters in \( \rho \) DAs are listed in Table II [9, 28]. The solution to the two-loop evolution of the Gegenbauer moment \( a^\perp_n(\mu) \) is

\[
f^{T}_{\rho}(\mu) a^\perp_n(\mu) = \left[ E^{NLO}_{T,n}(\mu, \mu_0) + \frac{\alpha_s(\mu)}{4\pi} \sum_{k=0}^{n-2} E^{LO}_{T,n}(\mu, \mu_0) d_{T,n}^k(\mu, \mu_0) \right] f^{T}_{\rho}(\mu_0) a^\perp_n(\mu_0),
\]

where the explicit expressions of \( E_{T,n} \) and \( d_{T,n} \) can be found in [34]. The scale evolution of other nonperturbative parameters to leading logarithmic accuracy is listed in appendix B.
TABLE I. Values of heavy quark masses [45, 46] and decay constants [9, 47–49], with the scale dependent quantity \( f_T^\rho \) given at \( \mu_0 = 1.0 \) GeV.

| \( m_c(m_c) \) (GeV) | \( m_b(m_b) \) (GeV) | \( f_\rho \) (MeV) | \( f_T^\rho (\mu_0) \) (MeV) |
|----------------------|----------------------|-------------------|------------------------|
| 1.288 ± 0.02         | 4.193\(^{+0.022}_{-0.035}\) | 213 ± 5           | 160 ± 7                |
| \( f_D \) (MeV)     | \( f_{D^*} \) (MeV)  | \( f_B \) (MeV)   | \( f_{B^*} \) (MeV)   |
| 209.0 ± 2.4          | 225.3 ± 8.0        | 192.0 ± 4.3       | 182.4 ± 6.2            |

TABLE II. Values of the nonperturbative parameters in the DAs at the scale \( \mu_0 = 1.0 \) GeV.

| \( a_2^\parallel(\mu_0) \) | \( a_2^\perp(\mu_0) \) | \( \zeta_3(\mu_0) \) | \( \omega_3^A(\mu_0) \) | \( \omega_3^V(\mu_0) \) | \( \omega_3^T(\mu_0) \) |
|--------------------------|--------------------------|-----------------------|------------------------|------------------------|------------------------|
| 0.17 ± 0.07              | 0.14 ± 0.06              | 0.032 ± 0.010         | −2.1 ± 1.0             | 3.8 ± 1.8              | 7.0 ± 7.0              |
| \( \zeta_4^T(\mu_0) \) | \( \tilde{\zeta}_4^T(\mu_0) \) | \( \langle\langle Q^{(1)}\rangle\rangle(\mu_0) \) | \( \langle\langle Q^{(3)}\rangle\rangle(\mu_0) \) | \( \langle\langle Q^{(5)}\rangle\rangle(\mu_0) \) |
| 0.10 ± 0.05             | −0.10 ± 0.05            | −0.15 ± 0.15          | 0                      | 0                      |                        |

The factorization scales are taken as \( \mu_c \in [1, 2] \) GeV around the default choice \( m_c \) and \( \mu_b = \frac{m_b + m_b}{2} \) for radiative \( D^* \) and \( B^* \) decays, respectively. As for the Borel mass \( M^2 \) and the threshold parameter \( s_0 \), we take the following intervals [33, 50]

\[
\begin{align*}
    \begin{cases}
        s_0 = 6.0 \pm 0.5 \text{ GeV}^2, & \text{for } D^* \ D \rho; \\
        M^2 = 4.5 \pm 1.0 \text{ GeV}^2, & \text{for } B^* \ B \rho.
    \end{cases}
\end{align*}
\]

Which satisfies the standard criterions [51].

B. Theory predictions

In Table III we present our results of the strong coupling constants with the uncertainties are estimated by varying separate input parameters. The individual uncertainties are presented in Table IV. We add them in quadrature to arrive the final results.

Now we analysis the scaling behavior of different terms in Table III. Based on the following
The results of $g_{H^+H^0}$ (GeV$^{-1}$). Here $g_{LT,LL}^{g}$ corresponds to the tree-level leading power contributions with resummation at LL accuracy; $g_{LT,NLL}^{g}$ corresponds to the leading power contributions up to NLO with resummation at NLL accuracy; $g_{2P,LL}^{g}$ is from sub-leading power two-particle corrections at LO and $g_{3P,LL}^{g}$ is from three-particle corrections at LO.

|       | $g_{LT,LL}^{D^*D_\rho}$ | $g_{LT,NLL}^{D^*D_\rho}$ | $g_{2P,LL}^{D^*D_\rho}$ | $g_{3P,LL}^{D^*D_\rho}$ | Total   |
|-------|--------------------------|---------------------------|--------------------------|--------------------------|---------|
| $D^*D_\rho$ | 3.61$^{+0.54}_{-0.47}$ | 3.18$^{+0.53}_{-0.43}$ | 0.47$^{+0.17}_{-0.16}$ | 0.14$^{+0.09}_{-0.08}$ | 3.80$^{+0.59}_{-0.45}$ |
| $B^*B_\rho$ | 3.59$^{+0.55}_{-0.43}$ | 2.91$^{+0.38}_{-0.42}$ | 0.96$^{+0.23}_{-0.15}$ | 0.022$^{+0.013}_{-0.012}$ | 3.89$^{+0.52}_{-0.48}$ |

The central value and individual uncertainty of $g_{H^+H^0}$ (GeV$^{-1}$) due to the variation of input parameters. We only show the numerically significant uncertainties here.

|       | $g_{D^*D_\rho}^{D^*D_\rho}$ | $g_{B^*B_\rho}^{D^*D_\rho}$ | $\Delta f_{H^+H^0}$ | $\Delta m_Q$ | $\Delta M^2$ | $\Delta \mu$ | $\Delta f_{T_{D_\rho}}$ | $\Delta a_2^\perp$ |
|-------|--------------------------|---------------------------|-----------------|------------|------------|---------|----------------|----------------|
| $g_{D^*D_\rho}^{D^*D_\rho}$ | 3.80$^{+0.50}_{-0.45}$  | +0.14 +0.00 +0.38 +0.20 +0.11 +0.32 |
| $g_{B^*B_\rho}^{D^*D_\rho}$ | 3.89$^{+0.52}_{-0.48}$  | +0.14 +0.12 +0.36 +0.14 +0.12 +0.24 |

where the parameter $\chi$ is of order 1 GeV. We find the scaling of strong couplings from leading-twist contributions in (38)

$$g_{LT}^{D^*D_\rho} \sim \frac{\chi}{\Lambda_{QCD}^2},$$

which are independent of $m_Q$, so $D^*$ and $B^*$ channels have similar values.

The two-particle subleading-power corrections $g_{2P}$ in (45) can be divided into three parts, the leading-twist corrections at NLP $g_{2P,LT}^{2P}$, the twist-3 DA $g_{2P,a}^{(a)}$, the twist-4 DA $A_T$ corrections $g_{2P,A}^{2P}$. They have the following scaling behavior

$$g_{2P,LT}^{2P} \sim \frac{1}{m_Q} \sim \frac{\Lambda_{QCD}^2}{m_Q \chi}, \quad g_{2P,a}^{(a)} \sim \frac{1}{\Lambda_{QCD}} \sim \frac{\Lambda_{QCD}}{\chi}, \quad g_{2P,A}^{2P} \sim \frac{1}{\chi} \sim \frac{\Lambda_{QCD}^2}{\chi^2},$$

where the parameter $\chi$ is of order 1 GeV. We find the scaling of strong couplings from leading-twist contributions in (38)

$$g_{LT}^{D^*D_\rho} \sim \frac{\chi}{\Lambda_{QCD}^2},$$

which are independent of $m_Q$, so $D^*$ and $B^*$ channels have similar values.
TABLE V. The coupling constants $g_{H^*H\rho}$ (GeV$^{-1}$) obtained via sum rules.

|       | this work  | LCSR [24] | LCSR [26] | QCDSR [29] |
|-------|------------|-----------|-----------|------------|
| $g_{D^*D\rho}$ | $3.80^{+0.50}_{-0.45}$ | $4.17 \pm 1.04$ | $3.56 \pm 0.60$ | $4.07 \pm 0.71$ |
| $g_{B^*B\rho}$ | $3.89^{+0.52}_{-0.48}$ | $5.70 \pm 1.43$ | – | – |

the corrections $g^{2P}$ are dominant by $g^{2P,a}$. The suppressed terms $g^{2P,LT}$ and $g^{2P,A}$ have opposite sign with $g^{2P,a}$ and $g^{2P,LT} \sim 1/m_Q$, so $g^{2P}$ for $B^*$ channel is larger than $D^*$ channel.

Similarly, for the three-particles corrections, from (52) the scaling is

$$g^{3P} \sim \frac{1}{m_Q} \sim g^{LT} \frac{\Lambda_{QCD}^2}{\chi m_Q}.$$ (59)

It’s obvious that $g_{B^*B\rho}$ from three-particle corrections is small.

C. Comparison with other approaches

In Table V, we compare our results with other LCSR and QCDSR calculations. Since the NLO effects of leading power cancel with the subleading-power corrections numerically, our results are close to the previous sum rules calculation within errors.

Next we extract the couplings from factors. The $B \rightarrow \rho$ form factors $V(q^2)$ and $T_1(q^2)$ which relate $g_{B^*B\rho}$ are defined as

$$\langle \rho^- (p, \eta^*) | \bar{d} \gamma_\mu b | B^- (p + q) \rangle = \frac{2}{m_B + m_\rho} V(q^2) \epsilon_{\mu pq},$$

$$\langle \rho^- (p, \eta^*) | \bar{d} i \sigma_{\mu\nu} q^\nu b | B^- (p + q) \rangle = -2 T_1(q^2) \epsilon_{\mu pq}. \quad (60)$$

From the dispersion relation of form factor $F_i(q^2)$

$$F_i(q^2) = \frac{r_i}{1 - q^2/m_{B_i}^2} + \int_{(m_B + m_\rho)^2}^{\infty} \frac{\rho(s)}{s - q^2 - i\epsilon}, \quad (61)$$

the strong coupling relates to the pole of the form factors at the unphysical point $q^2 = m_{B_i}^2$.

Then we have the relation

$$r_i^V = \lim_{q^2 \rightarrow m_{B_i}^2} (1 - q^2/m_{B_i}^2) V(q^2) = \frac{m_B + m_\rho}{2 m_B^*} f_{B^*} g_{B^*B\rho}.$$
TABLE VI. The coupling $g_{B^*B\rho}$ (GeV$^{-1}$) from the residue of $B \to \rho$ form factor $V$ and $T_1$ in LCSR fit and compared to our result.

|          | $V(q^2)$ [9] | $T_1(q^2)$ [9] | $V(q^2)$ [10] | $T_1(q^2)$ [10] |
|----------|--------------|----------------|--------------|----------------|
| this work | $3.89^{+0.52}_{-0.48}$ | $8.50 \pm 1.73$ | $8.02 \pm 1.59$ | $6.04^{+1.30}_{-2.34}$ |

$$r_1^{T_1} = \lim_{q^2 \to m_{B^*}^2} (1 - q^2/m_{B^*}^2) T_1(q^2) = \frac{1}{2} f_{B^*}^T g_{B^*B\rho},$$

(62)

where $f_{B^*}^T$ is the tensor coupling of the $B^*$ meson and defined as

$$\langle 0 | \bar{b} \sigma_{\mu\nu} q | B^*(q, \epsilon) \rangle = i f_{B^*}^T (\epsilon_\mu q_\nu - \epsilon_\nu q_\mu).$$

(63)

We choose the size $f_{B^*}^T = f_{B^*}$. Using (62) we extract the coupling $g_{B^*B\rho}$ from the recent LCSR works [9, 10] in table VI. As it shows, our central value is smaller than the extrapolation of LCSR form factors. When considering the errors, our result is compatible with [10].

The HMχPT effective Lagrangian to parametrize the $H^*HV$ coupling can be written as [1]

$$\mathcal{L}_V = i \lambda \text{Tr}[\mathcal{H}_b \sigma^{\mu\nu} F_{\mu\nu}(\rho)_{ba} \hat{H}_a],$$

(64)

where

$$F_{\mu\nu}(\rho) = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu + [\rho_\mu, \rho_\nu], \quad \rho_\mu = i \frac{g_V}{\sqrt{2}} \hat{\rho}_\mu,$$

(65)

where $\hat{\rho}$ is $3 \times 3$ matrices for light meson nonet, and the heavy $H$ and $H^*$ mesons are represented by the doublet field $\mathcal{H}_a$ with the conventional normalization. The parameter $g_V = m_\rho/f_\pi$. In the chiral and heavy quark limits, we have the following relation for the coupling $\lambda$

$$\lambda = \frac{\sqrt{2}}{4} \frac{1}{g_V} g_{H^*H\rho}.$$  

(66)

From the value of $g_{B^*B\rho}$, we find at leading power $\lambda = 0.23 \pm 0.03$ GeV$^{-1}$.

In table VII we compare this value with other estimations. Similar to the estimations from form factors our result is smaller than model predictions. One possibility for this discrepancies may be that the model predictions have a potential large error. Considering a numerical derivation of about 50%, their results are compatible with ours.
TABLE VII. The central value of coupling $\lambda$ (GeV$^{-1}$) from model estimations compared to our result.

|          | VMD [19] | CQM [30] | CQM [31] | QM+VMD [32] |
|----------|----------|----------|----------|-------------|
| this work| 0.23     | 0.56     | 0.60     | 0.47        | 0.33        |

V. CONCLUSION

We compute the $D^* D\rho$ and $B^* B\rho$ strong couplings to subleading-power in LCSR. The large-distance dynamics are incorporated in the $\rho$ DAs. We calculated the $\mathcal{O}(\alpha_s)$ corrections to leading power of the sum rules. For the $\gamma_5$ ambiguity, we take NDR scheme including evanescent SCET operator. Moreover, the evolution of Gegenbauer moments is at two-loop accuracy. The subleading-power corrections are calculated at LO by accounting the two-particle and three-particle wave functions up to twist-4. The analytical results of double spectral density are obtained, and with respect to the previous work, we also performed a continuum subtraction for the higher-twist corrections. The LO results are independent of the choice of duality region since the special form of the double spectral density.

Numerically, the NLO corrections decrease the tree-level results by 10% and 20% for $D^* D\rho$ and $B^* B\rho$ respectively. Contrarily, the subleading-power corrections at LO can give rise to the values about 20% and 30% for $D^* D\rho$ and $B^* B\rho$ respectively. Summing up all the contributions, our values are consistent with the predictions from previous sum rules works. Moreover, we also predict the coupling $\lambda$ in HM$\chi$PT at leading power. The central value of our result is smaller than the existing model-dependent estimations. A better understanding for this discrepancy is beneficial to shed light on the long-distance QCD dynamics.

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Appendix A: The $\rho$ meson DAs

Here we collect the $\rho$ meson DAs up to twist-4 which are defined in [28]

$$
\langle \rho^- (p, \eta^*) | \bar{d}(x) \gamma_\mu u(0) | 0 \rangle 
= f_\rho m_\rho \left\{ \frac{\eta^* \cdot x}{p \cdot x} p_\mu \int_0^1 du e^{i u p \cdot x} \left[ \phi_{\parallel}(u) + \frac{m_\rho^2 x^2}{16} A_{\parallel}(u) \right] 
+ \eta^*_{\perp \mu} \int_0^1 du e^{i u p \cdot x} g_{\perp}(u) - \frac{1}{2} x_\mu \frac{\eta^* \cdot x}{(p \cdot x)^2} m_\rho^2 \int_0^1 du e^{i u p \cdot x} g_3(u) \right\}, \tag{A1}
$$

$$
\langle \rho^- (p, \eta^*) | \bar{d}(x) \gamma_5 u(0) | 0 \rangle = \frac{1}{4} f_\rho m_\rho \epsilon_{\mu \nu p x} \int_0^1 du e^{i u p \cdot x} g_{\perp}^{(a)}(u), \tag{A2}
$$

$$
\langle \rho^- (p, \eta^*) | \bar{d}(x) \sigma_{\mu \nu} u(0) | 0 \rangle = -i f_\rho^T \left\{ \left( \eta^*_{\perp \mu} p_\nu - \eta^*_{\perp \nu} p_\mu \right) \int_0^1 du e^{i u p \cdot x} \left[ \phi_{\perp}(u) + \frac{m_\rho^2 x^2}{16} A_{\perp T}(u) \right] 
+ \left( p_\mu x_\nu - p_\nu x_\mu \right) \frac{\eta^* \cdot x}{(p \cdot x)^2} m_\rho^2 \int_0^1 du e^{i u p \cdot x} h^{(t)}(u) 
+ \frac{1}{2} \frac{\eta^*_{\perp \mu} x_\nu - \eta^*_{\perp \nu} x_\mu}{p \cdot x} m_\rho^2 \int_0^1 du e^{i u p \cdot x} h_3(u) \right\}, \tag{A3}
$$

$$
\langle \rho^- (p, \eta^*) | \bar{d}(x) u(0) | 0 \rangle = -\frac{i}{2} f_\rho^T (\eta^* \cdot x) m_\rho^2 \int_0^1 du e^{i u p \cdot x} h^{(s)}(u), \tag{A4}
$$

$$
\langle \rho^- (p, \eta^*) | \bar{d}(x) g_s \tilde{G}_{\mu \nu}(v x) \gamma_\alpha \gamma_5 u(0) | 0 \rangle 
= -f_\rho m_\rho p_\alpha \left[ p_\nu \eta^*_{\perp \mu} - p_\mu \eta^*_{\perp \nu} \right] \int [D\alpha] e^{i (\alpha_q + v \alpha_g) p \cdot x} A(\alpha) 
- f_\rho m_\rho^3 \frac{\eta^* \cdot x}{p \cdot x} \left[ p_\mu g_{\alpha \nu}^{\perp} - p_\nu g_{\alpha \mu}^{\perp} \right] \int [D\alpha] e^{i (\alpha_q + v \alpha_g) p \cdot x} \tilde{\Phi}(\alpha) 
- f_\rho m_\rho^3 \frac{\eta^* \cdot x}{(p \cdot x)^2} p_\alpha \left[ p_\mu x_\nu - p_\nu x_\mu \right] \int [D\alpha] e^{i (\alpha_q + v \alpha_g) p \cdot x} \tilde{\Psi}(\alpha), \tag{A5}
$$

$$
\langle \rho^- (p, \eta^*) | \bar{d}(x) g_s \tilde{G}_{\mu \nu}(v x) i \gamma_\alpha u(0) | 0 \rangle 
= -f_\rho m_\rho p_\alpha \left[ p_\nu \eta^*_{\perp \mu} - p_\mu \eta^*_{\perp \nu} \right] \int [D\alpha] e^{i (\alpha_q + v \alpha_g) p \cdot x} V(\alpha) 
- f_\rho m_\rho^3 \frac{\eta^* \cdot x}{p \cdot x} \left[ p_\mu g_{\alpha \nu}^{\perp} - p_\nu g_{\alpha \mu}^{\perp} \right] \int [D\alpha] e^{i (\alpha_q + v \alpha_g) p \cdot x} \Phi(\alpha) 
- f_\rho m_\rho^3 \frac{\eta^* \cdot x}{(p \cdot x)^2} p_\alpha \left[ p_\mu x_\nu - p_\nu x_\mu \right] \int [D\alpha] e^{i (\alpha_q + v \alpha_g) p \cdot x} \Psi(\alpha), \tag{A6}
$$
\[ \langle \rho^-(p, \eta^*) | \bar{d}(x) g_s G_{\mu\nu}(vx) \sigma_{\alpha\beta} u(0) | 0 \rangle \]

\[ = f_p^T m_p^2 \frac{\eta^* \cdot x}{2p \cdot x} \left[ p_\alpha p_\mu g_{\beta 0}^\perp - p_\beta p_\mu g_{0\alpha}^\perp \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x \mathcal{T}(\alpha) \]

\[ + f_p^T m_p^2 \left[ p_\alpha \eta_{\perp \mu} g_{\beta 0}^\perp - p_\beta \eta_{\perp \mu} g_{0\alpha}^\perp \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x \mathcal{T}_1^{(4)}(\alpha) \]

\[ + f_p^T m_p^2 \left[ -p_\mu \eta_{\perp \alpha} g_{\beta 0}^\perp - p_\mu \eta_{\perp \alpha} g_{0\nu}^\perp \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x \mathcal{T}_2^{(4)}(\alpha) \]

\[ + f_p^T m_p^2 \left[ -p_\mu \eta_{\perp \alpha} g_{\beta 0}^\perp - p_\mu \eta_{\perp \alpha} g_{0\nu}^\perp \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x \mathcal{T}_3^{(4)}(\alpha) \]

\[ + f_p^T m_p^2 \left[ p_\mu x_\nu - p_\nu x_\mu \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x \mathcal{T}_4^{(4)}(\alpha), \quad \text{(A7)} \]

\[ \langle \rho^-(p, \eta^*) | \bar{d}(x) g_s G_{\mu\nu}(vx) u(0) | 0 \rangle \]

\[ = -i f_p^T m_p^2 \left[ \eta_{\perp \mu}^* p_\nu - \eta_{\perp \nu}^* p_\mu \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x S(\alpha), \quad \text{(A8)} \]

\[ \langle \rho^-(p, \eta^*) | \bar{d}(x) i g_s \tilde{G}_{\mu\nu}(vx) \gamma_5 u(0) | 0 \rangle \]

\[ = i f_p^T m_p^2 \left[ \eta_{\perp \mu}^* p_\nu - \eta_{\perp \nu}^* p_\mu \right] \int [D\alpha] \epsilon^{(\alpha_q + \nu \alpha_g)} p_x \tilde{S}(\alpha), \quad \text{(A9)} \]

where

\[ \tilde{G}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\rho\tau} G^{\rho\tau}. \quad \text{(A10)} \]

**Appendix B: The conformal expansion of \( \rho \) DAs**

Here we list the conformal expansion for these higher twist DAs involved in our calculation [28].

The chiral-even twist-three DA \( g_3^{(a)} \) reads

\[ g_3^{(a)}(u) = 6 u \bar{u} \left\{ 1 + \left[ \frac{1}{4} a_2^2 + \frac{5}{3} \zeta_3 \left( 1 - \frac{3}{16} \omega_3^A + \frac{9}{16} \omega_3^V \right) \right] (5 \xi^2 - 1) \right\}, \quad \text{(B1)} \]

with \( \xi = 2 u - 1 \). For the chiral-odd twist-four DA \( A_T \) we have

\[ A_T(u) = 30 u^2 \bar{u}^2 \left[ \frac{2}{5} \left( 1 + \frac{2}{3} a_2^2 + \frac{10}{3} \zeta_4 - \frac{20}{3} \zeta_4^T \right) + \left( \frac{3}{35} a_2^2 + \frac{1}{40} \zeta_3 \omega_3^T \right) C_2^{5/2}(\xi) \right] \]

\[ - \left[ \frac{18}{11} a_2^2 - \frac{3}{2} \zeta_3 \omega_3^T + \frac{126}{55} \langle \langle Q^{(1)} \rangle \rangle + \frac{70}{11} \langle \langle Q^{(3)} \rangle \rangle \right] [u \bar{u} (2 + 13 u \bar{u}) \]

\[ + 2 u^3 (10 - 15 u + 6 u^2) \ln u + 2 \bar{u}^3 (10 - 15 \bar{u} + 6 \bar{u}^2) \ln \bar{u}] \]. \quad \text{(B2)}
As for the twist-4 three-particle DAs
\[
S(\alpha_i) = 30 \alpha_g^2 \left\{ s_{00} (1 - \alpha_g) + s_{10} \left[ \alpha_g (1 - \alpha_g) - \frac{3}{2} (\alpha_q^2 + \alpha_g^2) \right] + s_{01} \left[ \alpha_g (1 - \alpha_g) - 6 \alpha_q \alpha_g \right] \right\},
\]
\[
\tilde{S}(\alpha_i) = 30 \alpha_g^2 \left\{ \tilde{s}_{00} (1 - \alpha_g) + \tilde{s}_{10} \left[ \alpha_g (1 - \alpha_g) - \frac{3}{2} (\alpha_q^2 + \alpha_g^2) \right] + \tilde{s}_{01} \left[ \alpha_g (1 - \alpha_g) - 6 \alpha_q \alpha_g \right] \right\},
\]
\[
T_1^{(4)}(\alpha_i) = 120 t_{10} (\alpha_q - \alpha_g) \alpha_q \alpha_g \alpha_g ,
\]
\[
T_2^{(4)}(\alpha_i) = -30 \alpha_g^2 (\alpha_q - \alpha_g) \left[ \tilde{s}_{00} + \frac{1}{2} \tilde{s}_{10} (5 \alpha_g - 3) + \tilde{s}_{01} \alpha_g \right] ,
\]
\[
T_3^{(4)}(\alpha_i) = -120 \tilde{t}_{10} (\alpha_q - \alpha_g) \alpha_q \alpha_g \alpha_g ,
\]
\[
T_4^{(4)}(\alpha_i) = 30 \alpha_g^2 (\alpha_q - \alpha_g) \left[ s_{00} + \frac{1}{2} s_{10} (5 \alpha_g - 3) + s_{01} \alpha_g \right] .
\]

The eight parameters in three-particle DAs can be written as
\[
s_{00} = \zeta_4^T, \quad \tilde{s}_{00} = \tilde{\zeta}_4^T ,
\]
\[
s_{10} = -\frac{3}{22} a_2^\perp - \frac{1}{8} \zeta_3 \omega_3^T + \frac{28}{55} \langle \langle Q^{(1)} \rangle \rangle + \frac{7}{11} \langle \langle Q^{(3)} \rangle \rangle + \frac{14}{3} \langle \langle Q^{(5)} \rangle \rangle ,
\]
\[
\tilde{s}_{10} = \frac{3}{22} a_2^\perp - \frac{1}{8} \zeta_3 \omega_3^T - \frac{28}{55} \langle \langle Q^{(1)} \rangle \rangle - \frac{7}{11} \langle \langle Q^{(3)} \rangle \rangle + \frac{14}{3} \langle \langle Q^{(5)} \rangle \rangle ,
\]
\[
s_{01} = \frac{3}{44} a_2^\perp + \frac{1}{8} \zeta_3 \omega_3^T + \frac{49}{110} \langle \langle Q^{(1)} \rangle \rangle - \frac{7}{22} \langle \langle Q^{(3)} \rangle \rangle + \frac{7}{3} \langle \langle Q^{(5)} \rangle \rangle ,
\]
\[
\tilde{s}_{01} = -\frac{3}{44} a_2^\perp + \frac{1}{8} \zeta_3 \omega_3^T - \frac{49}{110} \langle \langle Q^{(1)} \rangle \rangle + \frac{7}{22} \langle \langle Q^{(3)} \rangle \rangle + \frac{7}{3} \langle \langle Q^{(5)} \rangle \rangle ,
\]
\[
t_{10} = -\frac{9}{44} a_2^\perp - \frac{3}{16} \zeta_3 \omega_3^T - \frac{63}{220} \langle \langle Q^{(1)} \rangle \rangle + \frac{119}{44} \langle \langle Q^{(3)} \rangle \rangle ,
\]
\[
\tilde{t}_{10} = \frac{9}{44} a_2^\perp - \frac{3}{16} \zeta_3 \omega_3^T + \frac{63}{220} \langle \langle Q^{(1)} \rangle \rangle + \frac{35}{44} \langle \langle Q^{(3)} \rangle \rangle .
\]

The scale evolution of the nonperturbative parameters to leading logarithmic accuracy is as follows
\[
a_2^\parallel(\mu) = L^{\frac{25}{6} C_F/\beta_0} a_2^\parallel(\mu_0) , \quad \zeta_3(\mu) = L^{(-\frac{1}{2} C_F + 3 C_A)/\beta_0} \zeta_3(\mu_0) ,
\]
\[
\omega_3^T(\mu) = L^{\frac{25}{6} C_F - 2 C_A)/\beta_0} \omega_3^T(\mu_0) , \quad (\zeta_4^T + \tilde{\zeta}_4^T)(\mu) = L^{(3 C_A - \frac{5}{3} C_F)/\beta_0} (\zeta_4^T + \tilde{\zeta}_4^T)(\mu_0) ,
\]
\[
(\zeta_4^T - \tilde{\zeta}_4^T)(\mu) = L^{(11 C_A - 4 C_F)/\beta_0} (\zeta_4^T - \tilde{\zeta}_4^T)(\mu_0) , \quad \langle \langle Q^{(1)} \rangle \rangle(\mu) = L^{(-11 C_F + \frac{11}{3} C_A)/\beta_0} \langle \langle Q^{(1)} \rangle \rangle(\mu_0) ,
\]
⟨⟨Q(3)⟩⟩(µ) = L^{10/3} C_F/β_0 ⟨⟨Q(3)⟩⟩(µ_0), \quad ⟨⟨Q(5)⟩⟩(µ) = L^{-2/3} C_F+5 C_A/β_0 ⟨⟨Q(5)⟩⟩(µ_0),

(B5)

where \( L = \alpha_s(µ)/\alpha_s(µ_0) \) and \( β_0 = 11 - 2 n_f/3 \), \( n_f \) being the number of flavors involved. The scale-dependence of \( ω^V(3)(µ) \)

\[
\begin{pmatrix}
ω^V(3)(µ) - ω^A(3)(µ) \\
ω^V(3)(µ) + ω^A(3)(µ)
\end{pmatrix}
= L^{α_s(µ)/β_0} \begin{pmatrix}
ω^V(3)(µ_0) - ω^A(3)(µ_0) \\
ω^V(3)(µ_0) + ω^A(3)(µ_0)
\end{pmatrix},
\]

(B6)

where \( Γ_ω \) is given by

\[
Γ_ω = \begin{pmatrix}
3 C_F - \frac{3}{3} C_A & \frac{2}{3} C_F - \frac{2}{3} C_A \\
\frac{5}{3} C_F - \frac{4}{3} C_A & \frac{1}{2} C_F + C_A
\end{pmatrix}.
\]

(B7)

Appendix C: The spectral density and integral for higher twist corrections at LO

The wave function \( φ(u) \) can be expressed as a power series

\[
φ(u) = \sum_k c_k u^k. \quad (C1)
\]

For higher-twist contributions, the spectral density of the general term have the form

\[
ρ^{2P}_n(t, v) = \frac{1}{π^2} \text{Im}_s' \text{Im}_s \int_0^1 du \frac{φ(u)}{(u s + \bar{u} s' - m_Q^2)^n}
= \frac{1}{(n-1)!} \frac{d^{n-1}}{(d m_Q^2)^{n-1}} \sum_k c_k \frac{1}{π^2} \text{Im}_s' \text{Im}_s \int_0^1 du \frac{u^k}{(u s + \bar{u} s' - m_Q^2)}
= \frac{1}{(n-1)!} \frac{d^{n-1}}{(d m_Q^2)^{n-1}} \sum_k c_k \frac{-1}{k!} (m_Q^2 - s')^k δ(k)(s' - s) θ(s - m_Q^2)
= \frac{1}{(n-1)!} \frac{d^{n-1}}{(d m_Q^2)^{n-1}} \sum_k c_k \frac{(-1)^{k+1}}{k!} M^2(1/2)^k (v - 1/2) θ(v t - m_Q^2),
\]

(C2)

with \( n \geq 2 \). After continuum subtraction we have the integral

\[
I^{2P}_n = \int_{-∞}^{2s_0} dt \int_{-∞}^{+∞} dv t e^{-\frac{t}{M^2}} ρ^{2P}_n(t, v)
= \frac{1}{(n-1)!} \frac{d^{n-1}}{(d m_Q^2)^{n-1}} \sum_k c_k \frac{-1}{2} M^2 u^k \bigg|_{u=1/2} \left(e^{-\frac{2m_Q^2}{M^2}} - e^{-\frac{2m_Q}{M^2}}\right)
\]

22
\[ = \frac{(-1)^n}{(n-1)!} \left( \frac{2}{M^2} \right)^{n-2} \phi \left( \frac{1}{2} \right) e^{-2m^2 Q/m^2}. \] (C3)

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