Characteristic classes of star products on Marsden-Weinstein reduced symplectic manifolds

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Abstract

In this note we consider a quantum reduction scheme in deformation quantization on symplectic manifolds proposed by Bordemann, Herbig and Waldmann based on BRST cohomology. We explicitly construct the induced map on equivalence classes of star products which will turn out to be an analogue to the Kirwan map in the Cartan model of equivariant cohomology. As a byproduct we shall see that every star product on a (suitable) reduced manifold is equivalent to a reduced star product.

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1 Introduction

Since ancient times symmetries have played a pivotal role in both physics and mathematics. And so has the art of symmetry reduction, i.e. getting rid of excess degrees of freedom. One prominent example would, of course, be Marsden-Weinstein reduction on symplectic manifolds [27], which will be the main focus of this note. Given a classical system, conceived as a symplectic manifold \((M, \omega)\), with symmetry given by a Hamiltonian (i.e. there is an \(\text{ad}^*\)-equivariant momentum map \(J\)) action of a Lie group \(G\) by symplectomorphisms, Marsden-Weinstein reduction is a two-step-process. First,
take the level set $C := J^{-1}(\{0\})$ (0 should be a value and regular value of $J$) of the momentum map, whereupon we assume the induced group action to be free and proper. This allows one, in the second step, to build the quotient $M_{\text{red}} = C/G$ which turns out to be again a symplectic manifold. The whole situation can be summarized in the diagram

\[ M \xleftarrow{\iota} C \xrightarrow{\pi} M_{\text{red}}. \]

With the advent of Quantum Mechanics, there have been numerous proposals of how to implement symmetry reduction in any given Quantum Theory, starting with Dirac [11]. However, just as with the multitude of quantization schemes developed over time, even in one such scheme, there is typically no “universal” reduction process. Here we will investigate only one quantum reduction scheme in the context of deformation quantization [2] proposed by Bordemann, Herbig and Waldmann in [5] and further developed in [20] by Gutt and Waldmann, which is based on BRST cohomology. We will provide a brief recap to the extent needed later on in Section 2. One of the central ingredients of this reduction scheme is the notion of quantum momentum maps, a direct generalization of the concept of momentum maps on symplectic manifolds (see [35], we will mostly follow conventions of this reduction scheme is the notion of quantum momentum maps, a direct generalization of the

\[ \mathcal{L}_{\xi} = -\frac{1}{\nu} \text{ad}_* (J(\xi)) \] and \[ [J(\xi), J(\eta)]_* = \nu J(\xi, \eta) \]

hold (where we denoted by $X_\xi$ the fundamental vector field of $\xi$). The pair $(\ast, J)$ is then called an equivariant star product and the equivalence classes of equivariant star products where recently shown to be characterized by the second equivariant cohomology (in the Cartan model, see [7][18][22]) $H^2_{\text{red}}(M)$ with respect to $G$ in [32]. Star products on the Marsden-Weinstein reduced symplectic manifold, which we will throughout denote by $M_{\text{red}}$, on the other hand, are classified by the second de Rham cohomology $H^2_{\text{dR}}(M_{\text{red}})$, see [3][10][14][19][30][34] for the symplectic and [25] for the more general Poisson case.

The main question we will be answering is the following: given any equivariant star product $(\ast, J)$ on $M$ with characteristic class $c_\ast (\ast, J) \in \frac{1}{\nu} H^2_{\text{red}}(M)[\nu]$ and the corresponding reduced star product $\ast_{\text{red}}$ with characteristic class $c(\ast_{\text{red}}) \in \frac{1}{\nu} H^2_{\text{dR}}(M_{\text{red}})[\nu]$, what exactly is the relation between these classes? A previous result by Bordemann [4] already gives a partial answer. Using the representation of equivariant differential forms as equivariant maps $\mathfrak{g} \rightarrow \Omega(M)$ (see Section 3) gives maps

\[ H^2_{\text{gr}}(M) \xrightarrow{\text{ev}_0} H^2_{\text{gr}}(M) \xrightarrow{i} H^2_{\text{dR}}(M) \]

where $H^2_{\text{gr}}(M)$ denotes the second invariant de Rham cohomology of $M$ with respect to the action of $G$ (note that, for noncompact $G$, this is different from the invariant part of the de Rham cohomology), $\text{ev}_0$ is induced by the evaluation at $0 \in \mathfrak{g}$ and $i$ is induced by the inclusion of invariant differential forms into differential forms. Both are compatible with taking (equivariant, invariant) characteristic classes of star products, that is the following diagram commutes [32]
where the top map is the inclusion of equivariant star products into star products on \( M \). One can then compare the characteristic classes of \( \star \) and \( \star_{\text{red}} \) on the momentum level set \( C \) used in the classical Marsden-Weinstein reduction via pullbacks

\[
\text{H}_{\text{dr}}(M) \xrightarrow{\iota^*} \text{H}_{\text{dr}}(C) \xleftarrow{\pi^*} \text{H}_{\text{dr}}(M_{\text{red}})
\]

and one finds that \( \iota^* c(\star) = \pi^* c(\star_{\text{red}}) \) \([4]\). However, even for nontrivial \( H^2_{\text{dr}}(M) \) or \( H^2_{\text{dr}}(M_{\text{red}}) \), there are cases where \( H^2_{\text{dr}}(C) = 0 \) and thus this equation does not provide any insights. One such example is given by the Hopf-fibration

\[
\mathbb{C}^{n+1} \backslash \{0\} \xleftarrow{\iota} S^{2n+1} \longrightarrow \mathbb{CP}^n.
\]

To alleviate this problem, we will throughout Section 3 construct a map \( K: \text{H}_{\text{g}}(M) \longrightarrow \text{H}_{\text{dr}}(M_{\text{red}}) \) which circumvents the projection \( \text{H}_{\text{g}}(M) \longrightarrow \text{H}_{\text{dr}}(M) \) and enables us to prove the main theorem in Section 4.

**Theorem (Main theorem)** Let \( M \) be a symplectic manifold equipped with a smooth and proper Hamiltonian \( G \)-action for a finite dimensional, connected Lie group \( G \) and let \( J: M \longrightarrow \mathfrak{g}^* \) be the corresponding \( \text{Ad}^* \)-equivariant momentum map. Assume furthermore that the induced action of \( G \) on \( J^{-1}(\{0\}) \) is free. Given any equivariant star product \( (\star, J) \) on \( M \) and the corresponding reduced star product \( \star_{\text{red}} \) on \( M_{\text{red}} \), we then have

\[
K(c(\star, J)) = c(\star_{\text{red}}).
\]

Furthermore, \( K \) is surjective.

The map \( K \) will turn out to be the Cartan model analogue of the Kirwan map \([23]\), which is defined for the topological, or Borel, model otherwise known as the homotopy quotient. The critical remark here is that we will not restrict ourselves to compact Lie groups and hence the cohomologies of the Cartan and Borel model typically do not agree. The reason we are using the Cartan model at all is of course the fact, that it classifies (also in the noncompact case) equivariant star products on symplectic manifolds \([32]\).

During the construction of \( K \) one pivotal result will be that for any \( G \)-principal bundle \( P \xrightarrow{\pi} B \) the equivariant cohomology (in the Cartan model) of the total space is related to the de Rham cohomology of the base by \( \pi^*: \text{H}_{\text{dr}}(B) \cong \text{H}_{\text{g}}(P) \) \([7,18]\). Since in the context of Marsden-Weinstein reduction the action of \( G \) on the momentum level set \( C \) is proper and free, \( C \) can be viewed as a principal bundle over \( M_{\text{red}} \) \([12]\), which enables us to write \( K \) concisely as

\[
K = (\pi^*)^{-1} \circ \iota^*: \text{H}_{\text{g}}(M) \xrightarrow{\iota^*} \text{H}_{\text{g}}(C) \xrightarrow{(\pi^*)^{-1}} \text{H}_{\text{dr}}(M_{\text{red}})
\]

Finally, the surjectivity of \( K \) shows that any star product on \( M_{\text{red}} \) is equivalent to one obtained by quantum reduction from \( M \).

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### 2 Reduction of Star Products

Throughout this exposition, let \((M, \omega)\) be a connected symplectic manifold equipped with a smooth and proper Hamiltonian \( G \)-action for a finite dimensional, connected Lie group \( G \) and let \( J: M \longrightarrow \mathfrak{g}^* \)
be the corresponding \(\text{Ad}^*\)-equivariant momentum map. We will furthermore require that 0 is a value and regular value of \(J\) and that the \(G\)-action on \(J^{-1}\{0\}\) is free and proper. In this setting we can apply Marsden-Weinstein reduction to obtain the reduced symplectic manifold \(M_{\text{red}} := J^{-1}\{0\}/G\). We then have the following maps

\[
M \xleftarrow{\iota} C := J^{-1}\{0\} \xrightarrow{\pi} M_{\text{red}}
\]

where \(\iota\) is an inclusion of a closed submanifold and \(\pi\) a surjective submersion. The symplectic two-form \(\omega_{\text{red}}\) on \(M_{\text{red}}\) is uniquely determined by \(\iota^*\omega = \pi^*\omega_{\text{red}}\). We will frequently summarize the above situation by stating that \(M_{\text{red}}\) is Marsden-Weinstein reduced \([27]\) from \(M\) via \(C\) (for details see e.g. \([33]\)).

For the convenience of the reader we will briefly recall a construction from \([20]\) to obtain star products on \(M_{\text{red}}\) from star products on \(M\) (see also \([4,5,8,9,15,26]\)). First, since the action of \(G\) is proper, there exists an open neighbourhood \(M_{\text{nice}} \subseteq M\) of \(C\) together with a \(G\)-equivariant diffeomorphism

\[
\Phi: M_{\text{nice}} \rightarrow U_{\text{nice}} \subseteq C \times g^* \quad \text{with} \quad \text{pr}_1 \circ \Phi \circ \iota = \text{id}_C
\]

onto an open neighbourhood \(U_{\text{nice}}\) of \(C \times \{0\}\), where the \(G\)-action on \(C \times g^*\) is the product action of the one on \(C\) and \(\text{Ad}^*\), such that for each \(p \in C\) the subset \(U_{\text{nice}} \cap \{p\} \times g^*\) is star shaped around \(\{p\} \times \{0\}\) and the momentum map \(J\) is given by the projection onto the second factor, i.e. \(J|_{M_{\text{nice}}} = \text{pr}_2 \circ \Phi\). We can use \(\Phi\) to define the following prolongation map:

\[
\text{prol}: \mathcal{C}^\infty(C) \rightarrow \mathcal{C}^\infty(M_{\text{nice}}): \phi \mapsto (\text{pr}_1 \circ \Phi)^* \phi \tag{2.1}
\]

Clearly we have \(\iota^* \circ \text{prol} = \text{id}_{\mathcal{C}^\infty(C)}\). Next consider the (classical) Koszul complex, given by

\[
\mathcal{C}^\infty(M, \Lambda^* g^*) = \mathcal{C}^\infty(M) \otimes \Lambda^* g^* \quad \text{with} \quad \delta = i(J).
\]

\(U_{\text{nice}}\) being star shaped allows to define

\[
(h_k x)(p) = e_a \wedge \int_0^1 t^k \frac{\partial(x \circ \Phi^{-1})}{\partial \mu_a} (c, t \mu) dt
\]

for \(x \in \mathcal{C}^\infty(M, \Lambda^k g^*)\) where we chose a basis \(\{e_a\}\) of \(g^*\) and denoted \(\Phi(p) = (c, \mu)\). The following proposition \([20]\) Prop. 2.1 summarizes some properties of \(h_k\):

**Proposition 2.1** The Koszul complex \((\mathcal{C}^\infty(M_{\text{nice}}, \Lambda^* g^*), \delta)\) is acyclic with explicit homotopy \(h\) and homology \(\mathcal{C}^\infty(C)\) in degree 0. In detail, we have

\[
h_{k-1} \delta_k + \delta_{k+1} h_k = \text{id}_{\mathcal{C}^\infty(M_{\text{nice}}, \Lambda g^*)}
\]

for \(k \geq 0\) and

\[
\text{prol} \iota^* + \delta_1 h_0 = \text{id}_{\mathcal{C}^\infty(M_{\text{nice}})}
\]

as well as \(\iota^* \delta_1 = 0\). Thus the Koszul complex is a free resolution of \(\mathcal{C}^\infty(C)\) as \(\mathcal{C}^\infty(M_{\text{nice}})\)-modules. We have

\[
h_0 \text{prol} = 0
\]

and all the homotopies \(h_k\) are \(G\)-equivariant.

Turning towards quantum reduction, we will exclusively be interested in equivariant (formal) star products on \(M\), so let us give a quick definition (compare \([21, 29, 35]\)):
Definition 2.2 A (formal) star product on $(M, \omega)$ is a bilinear map

\[ \star : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[\nu] : (f, g) \mapsto f \star g = \sum_{k=0}^\infty \nu^k C_k(f, g), \]

such that its $\nu$-linear extension to $\mathcal{C}^\infty(M)[[\nu]]$ is an associative product, all $C_k$ are bidifferential operators, $C_0(f, g) = fg$ and $C_1(f, g) = \{f, g\}$ for all $f, g \in \mathcal{C}^\infty(M)$. An equivariant star product is a pair $(\star, J)$ of a star product $\star$ together with a linear map $J : \mathfrak{g} \to \mathcal{C}^\infty(M)[[\nu]]$ such that

\[ \mathcal{L}_\xi = -\frac{1}{\nu} \text{ad}_\star(J(\xi)) \quad \text{and} \quad J([\xi, \eta]) = [J(\xi), J(\eta)]_\star \]

where we denoted by $\mathcal{L}_\xi$ the Lie derivative with respect to the fundamental vector field $X_\xi$ of $\xi$.

The definition of an equivariant star product immediately implies that $\mathcal{L}_\xi$ is a derivation of $\star$ and, since $G$ is assumed to be connected, that $G$ acts by $\star$-automorphisms. Here we are using the convention $\text{ad}(f)(g) := [f, g]_\star$ with $[ , ]_\star$ being the commutator with respect to $\star$.

We will start by introducing the quantized Koszul operator \cite{20}:

Definition 2.3 (Quantized Koszul operator) Let $\kappa \in \mathbb{C}[\nu]$. The quantized Koszul operator

\[ \partial^{(\kappa)} : \mathcal{C}^\infty(M, \Lambda^*_G \mathfrak{g})[[\nu]] \to \mathcal{C}^\infty(M, \Lambda^{*-1}_G \mathfrak{g})[[\nu]] \]

is defined by

\[ \partial^{(\kappa)} x = i(e_a) x \star J_a + \frac{\nu}{2} C^c_{ab} e_c \wedge i(e_a) i(e_b)x + \nu \kappa i(\Delta)x \]

where $C^c_{ab} = e^c([e_a, e_b])$ are the structure constants of $\mathfrak{g}$ and

\[ \Delta(\xi) = \text{tr} \text{ad}(\xi) \quad \text{for} \quad \xi \in \mathfrak{g} \quad (2.2) \]

is the modular one-form $\Delta \in \mathfrak{g}^*$ of $\mathfrak{g}$.

Here $\{e_a\}$ is assumed to be any basis of $\mathfrak{g}$, $J_a := J(e_a)$ and $i(\xi)x$ denotes the insertion of any $\xi \in \mathfrak{g}$ into the first argument of $x \in \Lambda^{*-1}_G \mathfrak{g}$. We will from now on fix $\kappa$ and omit any explicit mention in all subsequent formulae. Some properties of $\partial$ are collected in \cite{20} Lemma 3.4:

Lemma 2.4 Let $(\star, J)$ be an equivariant star product and $\kappa \in \mathbb{C}[\nu]$. Then one has

i) $\partial$ is left $\star$-linear.

ii) The classical limit of $\partial$ is $\delta$.

iii) $\partial$ is $G$-equivariant.

iv) $\partial \circ \partial = 0$.

Following \cite{2} one can introduce a deformation of the classical restriction map $\iota^*$ by

\[ I^* = \iota^* (\text{id} + (\partial_1 - \delta_1) h_0)^{-1} : \mathcal{C}^\infty(M)[[\nu]] \to \mathcal{C}^\infty(J^{-1}(\{0\}))[[\nu]] \]

where $h$ is a homotopy of the classical Koszul complex. Furthermore, one can find a homotopy $H$ with $H_{-1} = \text{prol}$ such that the augmented complex with $\partial_0 = I^*$ has trivial homology:

\[ H_{k-1} \partial_k + \partial_{k+1} H_k = \text{id}_{\mathcal{C}^\infty(M, \Lambda^*_G \mathfrak{g})[[\nu]]} \]
for $k \geq 0$ and $I^* \text{prol} = \text{id}_{\mathcal{C}^\infty(M)[[\nu]]}$ for $k = -1$. Moreover the maps $I^*$ and $H_k$ are $G$-equivariant. To finally arrive at the reduced star product, one defines a left $*$-ideal $\mathcal{J}_C$ and its normalizer $\mathcal{B}_C$

$$
\mathcal{J}_C := \text{im}(\partial_1) \subseteq \mathcal{C}^\infty(M) [[\nu]]
$$

$$
\mathcal{B}_C := \{ f \in \mathcal{C}^\infty(M) [[\nu]] \mid [f, \mathcal{J}_C]_* \subseteq \mathcal{J}_C \}
$$

to obtain the mutually inverse maps

$$
\frac{\mathcal{B}_C}{\mathcal{J}_C} \rightarrow \pi^* \mathcal{C}^\infty(\mathcal{M}_{\text{red}}) [[\nu]] : [f] \mapsto I^* f
$$

$$
\mathcal{C}^\infty(\mathcal{M}_{\text{red}}) [[\nu]] \rightarrow \frac{\mathcal{B}_C}{\mathcal{J}_C} : u \mapsto [\text{prol}(\pi^* u)]
$$

(2.3)

which enable us to define

$$
\pi^*(u \star_{\text{red}} v) := I^* (\text{prol}(\pi^* u) \star \text{prol}(\pi^* v))
$$

for all $u, v \in \mathcal{C}^\infty(\mathcal{M}_{\text{red}}) [[\nu]]$.

Since we are interested mainly in classifying equivariant star products and their corresponding reduced star products, the first critical property to check is whether equivariantly equivalent star products on $M$ reduce to equivalent star products on $\mathcal{M}_{\text{red}}$.

**Lemma 2.5** Let $T : (\star^1, \mathcal{J}^1) \mapsto (\star^2, \mathcal{J}^2)$ be an equivariant equivalence, then

$$
T_{\text{red}} := ((\pi^*)^{-1} \circ I^*) \circ T \circ (\text{prol} \circ \pi^*)
$$

is an equivalence $T_{\text{red}} : \star^1_{\text{red}} \mapsto \star^2_{\text{red}}$.

**Proof:** First of all, let us check that $T$ induces a map $\mathcal{B}^1_C/\mathcal{J}^1_C \mapsto \mathcal{B}^2_C/\mathcal{J}^2_C$. By extending $T$ onto $\mathcal{C}^\infty(M, \Lambda^* \mathfrak{g}) \cong \mathcal{C}^\infty(M) \otimes \Lambda^* \mathfrak{g}$ as the identity on the second factor, we can calculate for any $x \in \mathcal{C}^\infty(M, \Lambda^* \mathfrak{g})[[\nu]]$

$$
T \partial_1 x = T(i(e^a)x \star^1 J^1_a + \frac{\nu}{2} C^c_{ab} e_c \wedge i(e^a) i(e^b) x + \nu \kappa i(\Delta) x)
$$

$$
= i(e^a) Tx \star^2 T J^1_a + \frac{\nu}{2} C^c_{ab} e_c \wedge i(e^a) i(e^b) Tx + \nu \kappa i(\Delta) Tx
$$

$$
= \partial_2 (Tx)
$$

since $T J^1 = J^2$ and $T$ commutes with all insertions and wedge products of Lie algebra elements. This shows in particular, that $T$ is a chain map between the two quantized Koszul complexes

$$
T : (\mathcal{C}^\infty(M, \Lambda^* \mathfrak{g})[[\nu]], \partial^1) \mapsto (\mathcal{C}^\infty(M, \Lambda^* \mathfrak{g})[[\nu]], \partial^2)
$$

Thus for any $f = \partial^1 x$ we know that $T f = \partial^2 T x$ and hence $T f \in \mathcal{J}^2_C$. Even more, since $T$ is invertible, $\mathcal{J}^1_C \cong \mathcal{J}^2_C$ (as sets) holds. Take then any $j_2 \in \mathcal{J}^2_C$, any $f \in \mathcal{B}^1_C$, define $j_1 := T^{-1} j_2$, and calculate

$$
[T f, j_2]_{\star^2} = [T f, T j_1]_{\star^2} = T [f, j_1]_{\star^1} \in T \mathcal{J}^1_C = \mathcal{J}^2_C,
$$

hence we have $\mathcal{B}^1_C \cong \mathcal{B}^2_C$ (as sets). Furthermore, by (2.3) and the fact that $T$ is an equivalence and thus starts with $\text{id}_{\mathcal{C}^\infty(M)[[\nu]]}$ in 0th order, we know that $T_{\text{red}}$ also has $\text{id}_{\mathcal{C}^\infty(\mathcal{M}_{\text{red}})[[\nu]]}$ in 0th order. The only thing left to check is then that the higher orders of $T_{\text{red}}$ are differential operators on $\mathcal{M}_{\text{red}}$. This however is a direct consequence from the fact that $I^*$ can be decomposed into $\text{id} + \sum_{k=1}^{\infty} \nu^k S_k$

$$
I^* = \epsilon^* \circ \left( \text{id} + \sum_{k=1}^{\infty} \nu^k S_k \right)
$$

with differential operators $S_k$.  \(\blacksquare\)
3 Equivariant Cohomology on Principal Fibre Bundles

As seen in [32], equivariant star products on symplectic manifolds are classified by the second equivariant cohomology (or, to be more precise, by the cohomology of the Cartan complex of equivariant differential forms). In the context of Marsden-Weinstein reduction [Section 2], we will be interested most in the equivariant cohomology of principal bundles, more specifically, the principal bundle \( \pi: C = J^{-1}\{0\} \rightarrow M_{\text{red}} \) (which is a principal bundle since the action on \( C \) is free and proper, see [12]).

To start off, we will first recall the necessary basic definitions of equivariant cohomology (for a detailed exposition consult e.g. [7, 18]). Let \( M \) be a manifold equipped with a smooth \( G \)-action for any finite dimensional, connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \) and consider the complex of equivariant differential forms

\[
\Omega^k_{\mathfrak{g}}(M) := \bigoplus_{2i+j=k} [S^i(\mathfrak{g}^*) \otimes \Omega^j(M)]^G, \quad d_{\mathfrak{g}} = d + i_*
\]

where \( S \) denotes the symmetric tensor algebra, \( \Omega \) the de Rham complex, \( d \) the de Rham differential, \( i_* \), the insertion of fundamental vector fields of the action into the differential form part and invariants are taken with respect to the tensor product of the coadjoint action \( \text{Ad}^* \) of \( G \) on \( S(\mathfrak{g}^*) \) and the pullback on \( \Omega(M) \)

\[
\triangleright: G \times (S(\mathfrak{g}^*) \otimes \Omega(M)) \rightarrow S(\mathfrak{g}^*) \otimes \Omega(M): (g, p \otimes \alpha) \mapsto \text{Ad}^*(g)p \otimes (g^{-1})^*\alpha.
\]

We can view elements of \( \alpha \in [S(\mathfrak{g}^*) \otimes \Omega(M)]^G \) as polynomial maps \( \alpha: \mathfrak{g} \rightarrow \Omega(M) \) such that

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\alpha} & \Omega(M) \\
\text{Ad}(g) & \downarrow & \downarrow (g^{-1})^* \\
\mathfrak{g} & \xrightarrow{\alpha} & \Omega(M)
\end{array}
\]

commutes. Occasionally, for any \( \alpha \in [S^i(\mathfrak{g}^*) \otimes \Omega^j(M)]^G \), we will refer to \( k = 2i + j \) as the total, \( i \) the symmetric and \( j \) the exterior degree of \( \alpha \). Finally, we will frequently make use of the pullback by smooth functions on the complex of equivariant differential forms and the equivariant cohomology, so let us give a brief recap. For any two manifolds \( M \) and \( N \) with \( G \) actions and any equivariant smooth map \( f: M \rightarrow N \) we define for \( p \otimes \alpha \in [S^i(\mathfrak{g}^*) \otimes \Omega^j(N)]^G \) the pullback

\[
f^*(p \otimes \alpha) = p \otimes f^*\alpha.
\]

Clearly, \( f^*(p \otimes \alpha) \in [S^i(\mathfrak{g}^*) \otimes \Omega^j(M)]^G \) since \( f \) is equivariant. Also note that \( \Omega_\mathfrak{g} \) is a contravariant functor, since it can be expressed as

\[
\Omega_\mathfrak{g} = \left[ (-)^G \circ (S(\mathfrak{g}^*) \otimes -) \circ \Omega \right].
\]

Lastly, we will denote by \( H_\mathfrak{g}(M) \) the cohomology of \( \Omega_\mathfrak{g}(M) \) and note that the map \( [p \otimes \alpha]_\mathfrak{g} \mapsto [f^*(p \otimes \alpha)]_\mathfrak{g} \) on cohomology is well defined, since \( i_{f^*}f^* = f^*i_* \) and \( df^* = f^*d \).

**Remark 3.1** \( \Omega_\mathfrak{g}(M) \) in general only computes the equivariant cohomology of \( M \) under special circumstances (e.g. if \( G \) is compact or if the action of \( G \) on \( M \) is free and proper, see [Corollary 3.5]) and hence is, in general, not a model of equivariant cohomology. Thus it is important to note that subsequently we will always refer to \( H_\mathfrak{g}(M) \) by equivariant cohomology.

We will be needing one central result from equivariant cohomology, due to to Cartan [7].
Theorem 3.2 Let $C$ be a $G$-principal bundle. Then

$$H_g(C) \cong H(\Omega_{bas}(C), d)$$

For a (more general) proof consult e.g. [18, 22, 31]. Here however, we will for the convenience of the reader, showcase a shorter, more elementary proof. To this end we will need a well known result about basic differential forms on fibre bundles. Recall that, for a surjective submersion $\pi: M \rightarrow N$, a differential form $\mu$ is called basic if $\imath_Y \mu = 0$ and $\mathcal{L}_Y \mu = 0$ for all $Y \in \ker(T\pi)$. We will denote the complex of basic differential forms on $M$ by $\Omega_{bas}(M)$. The following lemma is widely known:

Lemma 3.3 Let $\pi: M \rightarrow N$ be a surjective submersion such that $\pi^{-1}(y)$ is a connected submanifold of $M$ for all $y \in N$. Then a differential form $\mu \in \Omega(M)$ is basic if and only if there exists a $\nu \in \Omega(N)$ such that

$$\mu = \pi^* \nu$$

Returning to the equivariant cohomology of $C$, there is one additional result needed which involves principal connections on principal bundles. The suitable definition of principal connections on a $G$-principal bundle for our purposes is that of a $g$-valued 1-form $\omega \in \Omega^1(P) \otimes g$ with

$$\text{Ad}_g((g^{-1})^* \omega) = \omega \quad \text{and} \quad \omega(X_\xi) = \xi$$

for all $g \in G$ and $\xi \in g$ (again, $X_\xi$ denotes the fundamental vector field of $\xi$). Let us from now on fix an arbitrary principal connection $\omega \in \Omega^1(C) \otimes g$ (existence is guaranteed e.g. by [1] or [24]). We can then evaluate any $p \in S^1(g)$ on the second tensor factor of $\omega$, for which we will write $p(\omega) \in \Omega^1(C)$. We will now use $\omega$ to define for $k \geq 1$ the following map

$$h_\omega: S^k(g^*) \otimes \Omega(C) \rightarrow S^k(g^*) \otimes \Omega(C): \prod_{i=1}^k p_i \otimes \alpha \mapsto \sum_{j=1}^k \prod_{i \neq j}^k p_i \otimes p_j(\omega) \wedge \alpha \quad (3.1)$$

and $h_\omega = 0$ on $S^0(g^*) \otimes \Omega(C)$.

Lemma 3.4 $h_\omega$ is a contraction of the chain complex $C^k = [S^k(g^*) \otimes \Omega^{n-k}(C)]^G$ with differential $i_*$. (using the convention $\Omega^0(C) = 0$ for $n < 0$):

$$i_* h_\omega + h_\omega i_* = \text{id}$$

Proof: The proof has two parts. First, we have to show that $h$ is a $G$-equivariant map and secondly, that $i_\mu h_\omega + h_\omega i_\mu = \text{id}$ holds. To avoid notational clutter, we will perform calculations only for $k = 1$. All other cases are straightforward generalizations thereof. So let $g \in G$ and $p \otimes \alpha \in S^1(g^*) \otimes \Omega(C)$. Then we know by the equivariance property of $\omega$ that

$$h_\omega(g \triangleright p \otimes \alpha) = h_\omega(\text{Ad}_g^* p \otimes (g^{-1})^* \alpha) = (\text{Ad}_g^* p)(\omega) \wedge (g^{-1})^* \alpha$$

$$= p(\text{Ad}_{g^{-1}} \omega) \wedge (g^{-1})^* \alpha = (g^{-1})^* (p(\omega) \wedge \alpha)$$

$$= g \triangleright h_\omega(p \otimes \alpha)$$

Finally, we can compute (using $\omega(X_\xi) = \xi$ and therefore $i_\xi p(\omega) = p(\xi)$ for all $\xi \in g$)

$$i_* h_\omega(p \otimes \alpha) = i_\mu(p(\omega) \wedge \alpha) = i_\mu(p(\omega) \wedge \alpha - p(\omega) \wedge i_* \alpha) = p \otimes \alpha - h_\omega(i_* p \otimes \alpha) \quad \square$$

The significance of the previous lemma becomes clear, once we view the complex of equivariant differential forms as a double complex $\Omega^i_j(C) = [S^i(g^*) \otimes \Omega^j(C)]^G$, with vertical differential $i_*$ and horizontal differential $d$:
Lemma 3.4 then shows that the columns of $\Omega^{*,*}(C)$ are exact and hence, by a general argument about double complexes with exact columns (see e.g. [6]), we know that the total cohomology, which is precisely the equivariant cohomology, is given by the horizontal cohomology of the kernel of the vertical differential in the bottom row.

Corollary 3.5 Let $G$ be a connected Lie group and $C$ a $G$-principal bundle. Then

$$H_g(C) \cong H_{\text{int}}(C/G)$$

Proof: Let $\alpha \in \Omega^k_g(C)$ be $d_g$-closed and let $\alpha_l$ be the component of $\alpha$ with maximal symmetric degree $l$. Then, since $\alpha$ is closed we must have $i_* \alpha_l = 0$ and therefore, by Lemma 3.4, there must be a $\beta_l$ with $i_* \beta_l = \alpha_l$. By subtracting $d_g \beta_l$ from $\alpha$ its cohomology class stays the same, however $\alpha - d_g \beta_l$ has maximal symmetric degree $l - 1$ or less. Repeating this process, one can find in every cohomology class a representative of symmetric degree zero.

Now, the bottom row complex of $\Omega_g(C)$ is just the complex of invariant differential forms $(\Omega(C)^G, d)$. Consequently all $d_g$-closed forms in the bottom row complex are those that are invariant, $d$-closed and $i_*$-closed, which is equivalent to being basic and $d$-closed. Since the bundle projection $\pi: C \to C/G$ is a surjective submersion, $\pi^*: \Omega_{\text{bas}}(C) \to \Omega(C/G)$ is a chain isomorphism, hence

$$H_g(C) \cong H_{\text{dr}}(C/G).$$

Remark 3.6 Since the rows of $\Omega_g(P)$ are not only exact, but exact by a given homotopy $h_\omega$, we can consider the following map (denote by $Z_g$ ($Z_{\text{bas}}$) closed equivariant (basic) differential forms)

$$\phi: Z_g(C) \to Z_g(C): \alpha \mapsto \alpha - d_g h_\omega \alpha.$$ 

Obviously, $\phi$ induces $\text{id}_{H_g(C)}$ on cohomology. However, on representatives, $\phi$ reduces the maximal symmetric degree of $\alpha$ by at least one and therefore implements the algorithm used for Corollary 3.5 to reduce $\alpha$ to a basic form on $P$ (additionally $\phi$ alters the lower degrees, too. This however is not important here). We can now use $\phi$ to define

$$\Phi := \prod_{k=1}^{\infty} \phi: Z_g(P) \to Z_g(P).$$

Since $\Phi$ stabilizes on $Z^{k,*}_g(C)$ after at most $k$ applications of $\phi$, there are no convergence problems present. Of course, $\Phi$ also induces the identity on cohomology. The important part however is that $\text{im} \Phi \subseteq Z_{\text{bas}}(C)$.

Remark 3.7 From Corollary 3.5 it is clear that $\pi^*: \Omega(C/G) \to \Omega_g(C)$ is a quasi-isomorphism of differential graded associative algebras.
Corollary 3.8 Let $M_{\text{red}}$ be Marsden-Weinstein reduced from $M$ via $C$ by the action of a finite-dimensional, connected Lie group $G$. Then the map

$$K : Z_0(M) \to Z(M_{\text{red}}) : K = (\pi^*)^{-1} \circ \Phi \circ \iota^*$$

is well-defined and induces

$$K : H^2_g(M) \to H^2_{\text{dR}}(M_{\text{red}}) : K = (\pi^*)^{-1} \circ \iota^*$$
on cohomology.

Proof: Apply Corollary 3.5, Remark 3.6 and Remark 3.7 to the case $C/G \cong M_{\text{red}}$.

Remark 3.9 The map $K$ from Corollary 3.8 can be seen as the Cartan-model analogue of the Kirwan map [23]. Again, we emphasize that we cannot use the original Kirwan map since we are working with not necessarily compact Lie groups.

The intriguing question here is of course what we can say about the image of $K$, which is completely determined by the image of $\iota^*$.

Corollary 3.10 Let $M_{\text{red}}$ be Marsden-Weinstein reduced from $M$. Then $K : H^2_g(M) \to H^2_{\text{dR}}(M_{\text{red}})$ from Corollary 3.8 is surjective.

Proof: The very definition of $\text{prol}$ [2.1] extends to the de Rham-complexes of $M$ and $C$:

$$\text{prol} : \Omega(C) \to \Omega(M) : \text{prol} = (\text{pr}_1 \circ \Phi)^*$$

and we clearly have $\iota^* \circ \text{prol} = \text{id}_{\Omega(C)}$. Furthermore, by functoriality of $\text{S}(g^*) \otimes \cdot, \cdot^G$ and cohomology, this equation holds on equivariant cohomology. Thus $\iota^*$ has a right-inverse and hence must be surjective.

4 Characteristic Classes of reduced Star Products

Having the results of the previous sections at hand, we can proceed to prove the main theorem of this paper. It relies heavily on Corollary 3.5, the classification of (equivariant) star products on symplectic manifolds and a result from [4] which relates the characteristic class of a star product with the characteristic class of its reduction. Let us begin by recalling the relevant classification results. On one hand, the set of equivalence classes of star products on symplectic manifolds up to equivalences of star products $\text{Def}(M, \omega)$ is isomorphic to formal power series in the second de Rham cohomology of the manifold (see [3], [10], [19])

$$c : \text{Def}(M, \omega) \overset{\sim}{\to} \frac{\omega}{\nu} + H^2_{\text{dR}}(M)[[\nu]]$$

while on the other hand, the set of equivalence classes of equivariant star products deforming a momentum map $J$ up to equivariant equivalences $\text{Def}(M, \omega, J)$ is isomorphic to power series in the second equivariant cohomology, see [32]

$$c_g : \text{Def}(M, \omega, J) \overset{\sim}{\to} \frac{\omega - J}{\nu} + H^2_g(M)[[\nu]].$$

Both $c$ and $c_g$ are bijections. One can even give explicit expressions of both characteristic classes for the case of (equivariant) Fedosov star products (see [13]), which are essentially all (equivariant) star products (by [3], [32]). Strictly speaking, the Fedosov construction maps pairs of a torsion-free,
symplectic connection $\nabla$ and a formal series of closed two-forms $\Omega \in \nu Z^2(M)[[\nu]]$ to star products. We will however fix once and for all torsion-free, symplectic (and invariant, if applicable) connections on all manifolds involved and henceforth drop all references to them. Instead, we will denote Fedosov star products constructed from $\Omega$ by $F(\Omega)$ for which then the following equations hold:

$$c(F(\Omega)) = \frac{1}{\nu}[\omega + \Omega], \quad c_g(F(\Omega), J) = \frac{1}{\nu}[\omega + \Omega - J]_g.$$  

Finally, from [4] we have the lemma

**Lemma 4.1** Let $M_{\text{red}}$ be Marsden-Weinstein reduced from $M$ via $C$ with inclusion $\iota: C \rightarrow M$ and principal bundle projection $\pi: C \rightarrow M_{\text{red}}$. Additionally, let $(\ast, J)$ be an equivariant star product on $M$ and let $\ast_{\text{red}}$ be the corresponding reduced star product on $M_{\text{red}}$. Then we have

$$\iota^*c(\ast) = \pi^*c(\ast_{\text{red}}).$$

Using all those results, we obtain

**Theorem 4.2** The characteristic class $c(\ast_{\text{red}})$ of $\ast_{\text{red}}$ is given by

$$c(\ast_{\text{red}}) = K(c_g(\ast, J')).$$

**Proof:** For the $(-1)$-th order in $\nu$ this follows directly from the Marsden-Weinstein reduction since $(J\big|_{\nu=0})_{\nu=0} = 0$ and $\iota^*\omega = \pi^*\omega_{\text{red}}$. Thus $\iota^*(\omega - J\big|_{\nu=0}) = \iota^*\omega$ is basic and $\omega_{\text{red}}$ is the unique form on $M_{\text{red}}$ with $\pi^*\omega_{\text{red}} = \iota^*\omega$. For the higher orders, let $F(\Omega_{\text{red}})$ be a Fedosov star product equivalent to $\ast_{\text{red}}$ (for its existence see [3]) and let $(F(\Omega), J)$ be a Fedosov star product equivariantly equivalent to $(\ast, J')$ (which exists due to [32]). Now observe that

$$\iota^*c(F(\Omega), J)_+ := \iota^*\left(c_g(F(\Omega), J) - \left[\frac{\omega - J\big|_{\nu=0}}{\nu}\right]_g\right) = \frac{1}{\nu}[\iota^*(\Omega - J_+)]_g$$

$$\pi^*c(F(\Omega_{\text{red}}))_+ := \pi^*\left(c(F(\Omega_{\text{red}})) - \left[\nu_{\text{red}}\big|_{\nu}\right]_g\right) = \frac{1}{\nu}[\pi^*\Omega_{\text{red}}]_g$$

where we denoted by $J_+$ the terms of order strictly greater $0$ in $\nu$. Using $\tilde{\Omega} = K(c_g(F(\Omega), J)_+)$ as a shorthand notation, we know from the definition of $K$ (see Corollary 3.8) that $\iota^*c_g(F(\Omega), J)_+ = [\pi^*\tilde{\Omega}]_g$. Hence we have

$$[\iota^*(\Omega - J)]_g = [\pi^*\tilde{\Omega}]_g$$

which is equivalent (by using the definition of $d_g$) to

$$\iota^*\Omega - \pi^*\tilde{\Omega} = d\theta \quad \text{and} \quad \iota^*\theta = -J_+$$

for some $\theta \in \Omega^1(C)[[\nu]]$. Additionally, we know from Lemma 4.1 [4] that $\iota^*\Omega - \pi^*\Omega_{\text{red}} = d\mu$ for some $\mu \in \Omega^1(C)[[\nu]]$. Combining those two yields

$$\pi^*\tilde{\Omega} - \pi^*\Omega_{\text{red}} = d(\mu - \theta).$$

Here the left hand side tells us that the form is basic, while from the right hand side, we see that it is exact. Hence we can infer the existence of $\chi \in \Omega^1(M_{\text{red}})[[\nu]]$ such that $\pi^*(\tilde{\Omega} - \Omega_{\text{red}}) = d\pi^*\chi$. But this immediately shows that $[\pi^*\tilde{\Omega}]_g = [\pi^*\Omega_{\text{red}}]_g$ and in turn

$$\iota^*c_g(F(\Omega), J)_+ = \frac{1}{\nu}[\iota^*(\Omega - J)]_g = \frac{1}{\nu}[\pi^*\tilde{\Omega}]_g = \frac{1}{\nu}[\pi^*\Omega_{\text{red}}]_g = \pi^*c(F(\Omega_{\text{red}}))_+.$$  

Finally, remember that $K$ is precisely $(\pi^*)^{-1} \circ \iota^*$ to conclude the proof.  


With Corollary 3.8 and Theorem 4.2 we can deduce the following corollaries:

**Corollary 4.3** Let $M_{\text{red}}$ be Marsden-Weinstein reduced from $M$ by $G$. Then for any star product $*$ on $M_{\text{red}}$, there exists a $G$-equivariant star product $(\tilde{\ast}, J)$ on $M$ such that $*$ and $(\tilde{\ast}, J)_{\text{red}}$ are equivalent.

**Proof:** $K$ is surjective and two star products on $M_{\text{red}}$ are equivalent if and only if their characteristic classes coincide [3].

**Corollary 4.4** If $M_{\text{red}}$ can be Marsden-Weinstein reduced from $M$ by $G$ and the second invariant de Rham cohomology $H^{G,2}_{dR}(M)$ vanishes, then for any star product $*$ on $M_{\text{red}}$ there exists a quantum momentum map $J$ of $F(0)$ such that $(F(0), J)_{\text{red}}$ is equivalent to $\ast$.

**Proof:** First, $F(0)$ is invariant, see [29]. Since $H^{G,2}_{dR}(M) = 0$ any two invariant star products are invariantly equivalent and hence every invariant star product is invariantly equivalent to $F(0)$. But then every equivariant star product $(\ast, J')$ is equivariantly equivalent to $(\ast, M, T')$ whenever $T$ is an invariant equivalence between $\ast$ and $\ast, M$.

Especially the second corollary should be reminiscent of [17] wherein it is shown that $\mathbb{R}^n$ is (up to a cohomological condition) universal with respect to reduction, that is almost every symplectic manifold $M_{\text{red}}$ can be obtained as a reduction of $\mathbb{R}^n$. Here, every star product on a symplectic manifold $M_{\text{red}}$ that has been Marsden-Weinstein reduced from $M$, can be obtained as a reduction of $F(0)$ as long as $H^{G,2}_{dR}(M)$ vanishes. The main difference (and drawback) with Corollary 4.4 is of course that it is restricted to Marsden-Weinstein reduction only whereas in [17] reduction with respect to coisotropic submanifolds is used and it is, to the authors knowledge, not clear which symplectic manifolds arise as Marsden-Weinstein reductions from $\mathbb{R}^n$. On the other hand, reduction of star products by coisotropic manifolds seems to be difficult and only partial results are known (compare [18,9,16]). Also, one can easily see that the condition $H^{G,2}_{dR}(M)$ in Corollary 4.4 poses a real obstruction as seen in the example of $\mathbb{R}^n$ acting on $\mathbb{R}^n$ by translations.

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