Existence and Construction of Resonances for
Atoms Coupled to the Quantized Radiation
Field

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Abstract
For a nonrelativistic atom, which is minimally coupled to the quantized radiation field, resonances emerging from excited atomic eigenstates are constructed by an iteration scheme inspired by [1] and [4]. This scheme successively removes an infrared cut off in momentum space and yields a convergent algorithm enabling us to calculate the resonance eigenvalues and eigenstates, to arbitrary order in the fine-structure constant $\alpha \sim 1/137$, and is thus an alternative method of proof of a similar result obtained in [2].

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1 Introduction and Main Result

1.1 Resonances, Time-Decay, and Metastable States

The general framework of quantum mechanics is given by a Hilbert space $\mathcal{H}_S$, whose normalized elements $\psi_S(t) \in \mathcal{H}_S$ - called wave functions - represent the state of the system at any time $t \in \mathbb{R}$, and a self-adjoint Hamiltonian $H_S = H^*_S$ acting on (a dense domain contained in) $\mathcal{H}_S$. The subscript $S$ stands for system. The dynamics of the system evolving from a state $\psi_S(0) \in \mathcal{H}_S$ is determined by Schrödinger’s equation

$$\forall t > 0 : \dot{\psi}_S(t) = -iH_S\psi_S(t),$$

(1.1)

which is solved by $\psi_S(t) = e^{-iH_st}\psi_S(0)$, showing that the dynamical information of (1.1) is fully contained in the spectral decomposition of $H_S$.

In case that $\psi_S(0)$ is an eigenvector of $H_S$ corresponding to an eigenvalue $E_S \in \mathbb{R}$, the dynamics (1.1) leaves $\psi_S(0)$ unchanged, up to multiplication by a phase factor, i.e., $\psi_S(t) = e^{-iE_st}\psi_S(0)$. For many physical systems, most of the eigenvectors of $H_S$ are unstable in the sense that a generic perturbation of $H_S$ - no matter how small it may be - lets the eigenvalue disappear and turns it into a resonance.

There are several definitions of a resonance (see, e.g., [3]), but with varying degree of mathematical rigor and no clear logical hierarchy of stronger assumptions implying weaker ones. In this paper we define resonances as eigenvectors and corresponding eigenvalues of a complex deformation $H_S(\theta)$ of the Hamiltonian $H_S \equiv H_S(\theta = 0)$, following [4, 20]. Here, $\theta \in \mathbb{C}$ is a deformation parameter, and $\theta \mapsto H_S(\theta)$ defines a type-A analytic family in a neighbourhood of $\theta = 0$, i.e., $D_r \ni \theta \mapsto H_S(\theta)[H_S + i]^{-1} \in \mathcal{B}[\mathcal{H}_S]$ is an analytic $\mathcal{B}[\mathcal{H}_S]$-valued map, for some $r > 0$. The usual choice of complex deformation, which we also make here, is defined by $H_S(\theta) := U(\theta)H_SU^{-1}(\theta)$, where $U(\theta) := e^{i\theta M}$ and $M = M^*$ is a self-adjoint operator on $\mathcal{H}_S$ such that $H$ is sufficiently $M$-regular.

The relation of (1.1) to $H_S(\theta)$ is given through an application of the Laplace transform and analytic continuation in $\theta$,

$$\langle \varphi | e^{-iH_st}\psi \rangle = \frac{-1}{2\pi i} \int_{\mathbb{R}+i\epsilon} e^{-izt} \left\langle \varphi \mid (H_S - z)^{-1} \psi \right\rangle dz$$

(1.2)

$$= \frac{-1}{2\pi i} \int_{\mathbb{R}+i\epsilon} e^{-izt} \left\langle \varphi_{\theta} \mid (H_S(\theta) - z)^{-1} \psi_{\theta} \right\rangle dz,$$

where $\varphi$ and $\psi$ belong to a suitable dense domain (e.g., analytic vectors...
of $M$ in $\mathcal{D}_S$ and $\theta$ varies in a range for which the spectrum of $H_S(\theta)$ doesn’t intersect $\mathbb{C}^+$. 

Now suppose that the Hamiltonian is of the perturbative form $H_{S,\alpha} = H_{S,0} + W_{S,\alpha}$, where $W_{S,\alpha} \to 0$, as $\alpha \to 0$, and that $E_{S,0} \in \mathbb{R}$ is an eigenvalue of $H_{S,0}$. Then, $E_{S,0}$ is also an eigenvalue of $H_{S,0}(\theta)$, for all $\theta \in D_r \cap \mathbb{R}$, because $H_{S,0}(\theta)$ is unitary equivalent to $H_{S,0}$. The analyticity of $\theta \mapsto H_{S,0}(\theta)$ then implies that $E_{S,0}$ is, in fact, an eigenvalue of $H_{S,0}(\theta)$, for all $\theta \in D_r$. This implies that $z \mapsto (H_{S,0}(\theta) - z)^{-1}$ is singular, as $z \to E_{S,0}$, and hence, close to $E_{S,0}$, the integration contour in (1.2) cannot be deformed from the upper half-plane into the lower half-plane. If $\alpha \neq 0$, however, then the eigenvalue $E_{S,0}$ often turns into an eigenvalue $E_{S,\alpha} \in \mathbb{C}^-$ of $H_{S,\alpha}(\theta)$ in the lower half-plane, and the real axis about $E_{S,0}$ is free of spectrum of $H_{S,\alpha}(\theta)$. This, in turn, implies that, for $\alpha \neq 0$, the integration contour in (1.2) can, indeed, be deformed from the upper half-plane into the lower half-plane, yielding a local exponential decay in time. The eigenvalue $E_{S,\alpha} \in \mathbb{C}^-$ of $H_{S,\alpha}(\theta)$ is then called a resonance and the corresponding eigenvector of $H_{S,\alpha}(\theta)$ a metastable state.

We remark that complex deformations leave isolated eigenvalues of finite or infinite multiplicity unchanged. In particular, $\sigma_{\text{disc}}[H_S(\theta)] \supset \sigma_{\text{disc}}[H_S]$, for all $\theta \in D_r$. Furthermore, the generator $M$ of the deformation group is necessarily unbounded in order to have the deforming effect on the spectrum of $H_S$ described above.

The relation of the spectral information on $H_S(\theta)$ to the time-decay properties of $e^{-iH_S t}$ is rather indirect. Exponential decay of the form $|\langle \varphi | e^{-iH_S t} \psi \rangle| \leq e^{-\kappa t}$, for some $\kappa > 0$, for instance, necessarily requires that $\sigma(H_S) = \mathbb{R}$, for otherwise a contradiction to the Paley-Wiener theorem would emerge. Therefore, for a semibounded Hamiltonian $H_S$, one cannot expect better decay estimates than

$$|\langle \varphi | e^{-iH_S t} \psi \rangle| \leq C_1 e^{-\kappa t} + C_2(t),$$

(1.3)

with $|C_1| \gg |C_2(0)|$, but $|C_1| e^{-\kappa t} \ll |C_2(t)|$, for $t \gg 1$. Under mild further assumptions involving functions of Gevrey type [5], rather than analytic functions, the error term obeys the estimate $|C_2(t)| \leq \bar{C}_\beta \exp[-\bar{\kappa}_\beta |t|^\beta]$, where $\beta < 1$ may be chosen arbitrarily close to 1. Accepting that an estimate of the form (1.3) is optimal, the question arises in which sense there is a natural choice for the decay rate $\kappa$. For many models which allow for a perturbative formulation as $H_{S,\alpha}$ above, this natural choice for $\kappa$ is to leading order in $\alpha$ determined by second-order perturbation theory - leading to the weak-coupling limit of the dynamics. Many physical processes show a varying decay rate on different time scales, however, and the adequate mathematical description of these phenomena remains a challenge.
1.2 Excited Eigenvalues turning into Resonances

In the present paper, we study a model for a one-electron atom in interaction with the quantized radiation field. The Hilbert space of this system is

$$\mathcal{H}^0 = \mathcal{H}_{at} \otimes \mathcal{F}^0,$$

(1.4)

where $$\mathcal{H}_{at} = L^2(\mathbb{R}^3)$$ is the Hilbert space of one spinless particle, and $$\mathcal{F}^0 \equiv \mathfrak{F}_b[\mathfrak{h}^0]$$ is the boson Fock space of photons, with $$\mathfrak{h}^0 := L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$$ being the corresponding one-photon Hilbert space. The dynamics is generated by the Hamiltonian

$$H = \left(\frac{1}{i}\nabla + A^0(x)\right)^2 \mathcal{H}^0 + V + \tilde{H}^0,$$

(1.5)

where the potential $$V \geq 0$$ is a multiplication operator in the electro position $$x \in \mathbb{R}^3$$ which is assumed to be an infinitesimal perturbation of $$-\Delta$$ and to decay to zero at infinity. The field energy is represented by

$$\tilde{H}^0 = \int \omega(k) a^*(k) a(k) dk,$$

(1.6)

with $$k = (\vec{k}, \mu) \in \mathbb{R}^3 \times \mathbb{Z}_2$$, $$\int f(k) dk := \sum_{\mu \in \mathbb{Z}_2} \int f(\vec{k}, \mu) d^3k$$, and $$\omega(k) := |\vec{k}|$$. The fine structure constant $$\alpha > 0$$ is assumed to be a sufficiently small coupling constant, ignoring that its physical value is about 1/137. The vector potential of the quantized magnetic field in Coulomb gauge is

$$A^0(x) := \frac{\alpha^{3/2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3 \times \mathbb{Z}_2} \frac{\kappa(k)}{(2\omega(k))^{1/2}} \left\{ \vec{e}(k) e^{-i\alpha \vec{k} \cdot x} a^*(k) + \vec{e}(k)^* e^{i\alpha \vec{k} \cdot x} a(k) \right\} dk,$$

(1.7)

where the $$a^*(k)$$ and $$a(k)$$ are the creation and annihilation operators representing the canonical commutation relations on $$\mathcal{F}^0$$ as

$$[a(k), a^*(k')] = \delta(k-k') \cdot 1, \quad [a(k), a(k')] = [a^*(k), a^*(k')] = 0, \quad a(k) \Omega = 0,$$

(1.8)

for all $$k, k' \in \mathbb{R}^3 \times \mathbb{Z}_2$$, in the sense of operator-valued distributions, i.e., it is understood that (1.8) is integrated against sufficiently regular test functions. The ultraviolet cut-off $$\kappa$$ is (the restriction to the real line of) an entire function and of sufficiently rapid decay. Analyticity of $$\kappa$$ is important because we use the method of complex deformations, and the characteristic function of a set would not be an admissible choice. A natural choice would be $$\kappa(k) := \exp[-k^2]$$. Furthermore, the transversal polarization vectors $$\vec{e}(\vec{k}, \mu) \equiv \vec{e}(\vec{k}/|\vec{k}|, \mu)$$ are measurable maps on $$\mathbb{S}_2$$ which, together with $$\vec{e}(\vec{k}/|\vec{k}|)$$, constitute an orthonormal basis (Dreibein) in $$\mathbb{R}^3$$, for (almost) all $$\vec{k} \neq 0$$. Hence, $$\vec{e}(r\vec{k}, \mu) = \vec{e}(\vec{k}, \mu)$$, for all $$r > 0$$. 
The atomic potential $V$ is assumed to be dilation analytic and compact relatively to the kinetic energy $-\Delta$ of the electron. Consequently, the essential spectrum $\sigma_{\text{ess}}(H_{\text{at}}) = \mathbb{R}_{+}^+$ of $H_{\text{at}} := -\Delta + V$ is the positive half-axis, and its discrete spectrum $\sigma_{\text{disc}}(H_{\text{at}}) = \{e_0, e_1, e_2, \ldots, e_M\} \subseteq \mathbb{R}^-$ lies on the negative axis, where $M \in \mathbb{N}$ or $M = \infty$ and $e_0 < e_1 < \cdots < e_M$. We assume in this paper that $M \geq 2$, i.e., that $H_{\text{at}}$ has at least two isolated eigenvalues of finite multiplicity.

Under these assumptions it is easy to see that the spectrum $\sigma(H_0) = [e_0, \infty)$ of $H_0 = H_{\text{el}} \otimes 1 + 1 \otimes \tilde{H}^0$ covers the half-axis above $e_0$ and is entirely essential, $\sigma(H_0) = \sigma_{\text{ess}}(H_0) = [e_0, \infty)$. The atomic eigenvalues $\{e_0, e_1, e_2, \ldots, e_M\}$ are eigenvalues of $H_0$, as well, but now embedded in a continuum. Fixing $j \in \{1, 2, \ldots, M\}$, it is expected that $e_j$ is unstable under perturbations and that $H$ has no (real) eigenvalue in the vicinity of $e_j$ - no matter how small $\alpha > 0$ may be.

In the past two decades, a lot of research has been carried out on the model defined by (1.4)-(1.8) to establish this picture. The first basic step is to establish the existence of the model, as it is defined by an unbounded operator. After several works of increasing generality, it was established in [6, 13] that $H$ is a semibounded, self-adjoint operator with domain $\text{dom}(H_0)$, for all $\alpha > 0$. Its ground state energy

$$E_{\text{gs}}(\alpha) = \inf \{\sigma(H_\alpha)\}$$

is a simple eigenvalue, which was first shown for small $\alpha > 0$ in [15] and for all $\alpha > 0$ in [8]. Its simplicity is a consequence of the assumption that the electron is spinless. In reality, electrons are spin-$\frac{1}{2}$ fermions, and the ground state is two-fold degenerate [9].

The half-line above $E_{\text{gs}}(\alpha)$ is filled with essential spectrum of $H$. In particular, $H$ has no isolated eigenvalues. Absence of eigenvalues, positive commutator estimates, and limiting absorption principles in the interval $(E_{\text{gs}}(\alpha), \Sigma(\alpha))$ have been established, where

$$\Sigma(\alpha) := \inf \left\{\sigma\left((-i\nabla - A^0)^2 + \tilde{H}^0\right)\right\},$$

which, in particular, proves the instability of $e_j$ for $j \geq 1$ [7].

Our main result is to further establish resonances for $j = 1$ (or any other $j \geq 2$) and show that the eigenvalue $e_j$ is not only absent from the real axis but pushed into the lower half-plane.

The coupling function (often called “form factor” which, however, is a misnomer) $\frac{1}{(2\omega(k))^{1/2}}$ (see (1.7)) yields major difficulties for the study of resonances, indeed, in this case the perturbation is (superficially) marginal in
the renormalization group sense. On the contrary, we can control the renormalization scheme in presence of any regularization of the form factor, i.e., $1/\sqrt{|k|^{1/2-\mu}}$, as $k \to 0$, with $\mu > 0$. We cannot handle $H$ directly but instead we can handle another Hamiltonian which is unitarily equivalent (and therefore physically equivalent). This new Hamiltonian is obtained by a change of gauge which is called the Pauli-Fierz transformation. In the new Hamiltonian the interaction with the photons is modeled by a less singular form factor, but the long-distance behavior in the electron variable is more difficult to handle. The relation between small momenta and long distances is manifest in the oscillating phase factors $(e^{-i\alpha \vec{k} \cdot \vec{x}})$ that appear in Eq. (1.7). Roughly speaking, we trade small momenta for long distances by applying the Pauli-Fierz transformation. The long-distance behavior in the electron variable can be controlled by exploiting the exponential localization of the eigenfunctions. The use of a Pauli-Fierz transformation similar to the one we use was first done by Sigal in [2].

Now we describe the Pauli-Fierz transformation and the Hamiltonian that we analyze. We select a function $\eta \in C_0^\infty(\mathbb{R})$ that is identically 1 in a neighborhood of zero. We define the self-adjoint operator

$$
\chi_{PF}^0 := \frac{\alpha^{3/2}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3 \times \mathbb{R}_2} \left\{ \bar{\varepsilon}(k) \cdot \eta\left(||x||k\right) x a^*(k) + \varepsilon(k)^* \cdot \eta\left(||x||k\right) x a(k) \right\} dk.
$$

(1.11)

The Pauli-Fierz transformation is the unitary operator $e^{-i\lambda_0 PF}$ and the Hamiltonian that we study is the conjugation of $H$ with respect to this unitary operator:

$$
H^0 := e^{-i\lambda_0 PF} H e^{i\lambda_0 PF}.
$$

(1.14)

The operator $H^0$ is actually the energy operator in a different representation, which is described by the Pauli-Fierz transformation. The Pauli-Fierz transformation can be regarded as a change of gauge, as formula (1.14) suggests.

Thanks to our analyticity assumptions on $V$ and $\kappa$, we can trace the eigenvalue and associate to the Hamiltonian a natural analytic family $\{H^0(\theta)\}$ of operators which are complex deformations of $H^0$. If $\text{Im} \theta$ is strictly positive (and small enough), and $\alpha$ is small, we prove that $H^0(\theta)$ possess a simple eigenvalue $E_{\infty}$ in a neighbourhood of $e_1$ (the same proof holds for $e_j$ with $j > 1$). We prove furthermore that the imaginary part of $E_{\infty}$ is strictly negative and that it is, therefore, a resonance. We prove additionally that locally, in a neighbourhood of $E_{\infty}$ and $e_1$, there is no point of the spectrum
of \( H^0(\theta) \) with imaginary part larger than \( \text{Im} \ E_\infty \) and we estimate the norm of the resolvent \( \| \frac{1}{H^0(\theta) - z} \| \) for such points \( z \) in the resolvent set. We prove also that the eigenvector associated to \( E_\infty \) is exponentially localized in the electron variable by a rather simple method.

The precise formulation of our main result is given in Section 6.4. This result has already been established in [2] by methods different from those used in the present paper. In [2] the renormalization group based on the Feshbach-Schur map is used to prove Section 6.4 while our approach borrows ideas from [14] and is based on a successive reduction of momentum slices known as Pizzo’s method. We include new ingredients in the method taking advantage of the maximum modulus principle for analytic functions and using the Feshbach-Schur map as a tool to restrict the domain of our Hamiltonian to functions that are exponentially localized in the electron variable. As we do not use renormalization based on the Feshbach-Schur map, we do not deal with the usual combinatoric problems of the theory of renormalization.

1.3 Strategy of the Proof and Description of the Content of the Paper

We describe the strategy of our proof. We restrict the momentum of the photon \( |k| \) to be larger than a positive number \( s \) and we implement this restriction in all operators and Hilbert spaces. The parameter \( s \) is actually an infrared cut-off (when strictly positive). We take a decreasing sequence of positive numbers \( \{ \sigma_n \}_{n \in \mathbb{N} \cup \{0\}} \) in which each \( \sigma_n \) represents an infrared cut-off and \( \sigma_n \to 0 \), for \( n \to \infty \). We denote by \( \overset{n}{H} \) the Hamiltonian with the restriction \( |k| \geq \sigma_n \). We can construct (see Theorem 5.7) an eigenvalue \( E_0 \) of \( \overset{n}{H} \) in a neighbourhood of \( e_1 \). This eigenvalue is actually isolated (by the infrared cut-off), which makes it simple to analyze. We can prove the existence of \( E_0 \), for sufficiently small \( \alpha \). We can not proceed in this way to construct eigenvalues of \( \overset{n}{H} \) because the maximal magnitude of \( \alpha \), that we could still admit, would tend to zero, as \( n \to \infty \). To construct \( E_n \), we rather proceed inductively. Assuming to have already constructed the eigenvalues \( E_m \), for \( m \leq n \), we construct the eigenvalue \( E_{n+1} \) of \( \overset{n+1}{H} \). After completing the induction step (see Section 6.3.3) we conclude that, for any \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), there is an eigenvalue \( E_n \) of \( \overset{n}{H} \). We conclude by proving that the limit

\[
E_\infty := \lim_{n \to \infty} E_n
\]

exists and is an eigenvalue of \( \overset{\infty}{H} := H^0 \) (see Section 6.4).
We now briefly describe the structure of our paper. In Section 2 we describe the mathematical objects that we use and the physical model. In Section 3 we compile the results that we need concerning the atom Hamiltonian. The proofs of the statements in this section are given in Section 7. In Section 4 we state basic bounds on the interaction with respect to the non-interacting Hamiltonian and use this results to prove some basic estimates. The proofs of the statements in this section are given in Section 8. In Section 5 we prove the induction basis of our induction scheme. The main results of this section are summarized in Theorem 5.7. We prove that $E_0$ is a simple eigenvalue of $H$, we estimate its imaginary part, we construct the projection onto the corresponding eigenspace, and we derive some estimates for the resolvent operator and the resolvent set. The proofs of these assertions are carried out in Section 9, there we use similar estimates as in [15]. In our case the analysis is much simpler because our eigenvalues are isolated.

Section 6 is the heart of the paper. In that Section we state and prove all our results. In Section 6.1 we introduce some mathematical tools that we need and prove exponential decay of the eigenfunctions (see Theorem 6.5). In Section 6.2 we define the sequence of Hamiltonians and fix some notations and assumptions. Section 6.3 is the core of our proof. There we establish the inductive scheme and prove the induction step. We inductively construct $E_n$ and prove that it is a simple eigenvalue of $H$, we construct the projection onto the corresponding eigen-space and we derive some estimates for the resolvent operator and the resolvent set. In the proof of the induction step we make use of the exponential decay of the eigenfunctions by restricting the domain of our operators to a space of functions which decay exponentially in the electron variable. The mathematical object that permits us to justify this restriction is the Feshbach-Schur map. Here we use it just as a mathematical tool that allows us to use the exponential decay of the eigenfunctions, and not as a fundamental object on which a renormalization scheme is based. In this paper we do not use renormalization group techniques. In Section 6.4 we state and prove the main results using Section 6.3. We take the limit $n \to \infty$ and prove that $\tilde{H}$ has a simple eigenvalue ($E_\infty$) in a neighbourhood of $e_1$. We furthermore prove that the imaginary part of $E_\infty$ is strictly negative and it is, therefore, a resonance. We additionally prove that locally, in a neighbourhood of $E_\infty$ and $e_1$, there is no point of the spectrum of $\tilde{H}$ with imaginary part larger than $\text{Im} E_\infty$ and we estimate the norm of the resolvent $\| \frac{1}{\tilde{H}(\theta) - z} \|$ for such points $z$ in the resolvent set.
2 Definition of the Model

2.1 The Atom Hamiltonian

2.1.1 The Group of Dilation Operators

For every $\theta \in \mathbb{R}$, we denote by $u(\theta) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ the group of dilation operators. $u(\theta)$ is defined by the formula,

$$(u(\theta)\phi)(x) := e^{3\theta/2} \phi(e^\theta x), \forall \phi \in L^2(\mathbb{R}^3).$$

(2.15)

2.1.2 The Electric Potential

The electric potential is a complex valued function defined in $\mathbb{R}^3$. It depends on the parameters $\alpha, \beta$ and $\theta$. $\alpha$ is a positive real number less than one and $\theta$ and $\beta$ are complex numbers with $|\theta|, |\beta| < \frac{1}{2}$. We denote the electric potential by

$$V(\theta) : \mathbb{R}^3 \to \mathbb{C}$$

and, in general, in the notation we omit the dependence on $\alpha$ and $\beta$

$$V(\theta, \alpha, \beta) := V(\theta).$$

(2.16)

The electric potential is the sum of 3 operators,

$$V(\theta) := V(\theta) + \alpha^3 V_{PF}(\theta) + \tilde{V}(\theta, \beta).$$

(2.17)

The function $V(0)$ is the physical electric potential and $V(\theta)$ is an analytic continuation of $V(0)$ (see Section 2.1.2.1). The term $V_{PF}(\theta)$ is an effective potential that comes from the Pauli-Fierz transformation (see Sections 2.1.2.2 and 2.3.3.3) and the term $\tilde{V}(\theta, \beta)$ is a potential yielded by the conjugation with the exponential $e^{\beta(x)}$, see (2.24) and (2.23). We use this conjugation to prove exponential decay for the eigenfunctions of the Hamiltonians (see Sections 2.1.2.3, 3.1 and Theorem 6.5) and to control the long distance behavior for the electron (see (6.125) for example). Adding the potential $\tilde{V}(\theta, \beta)$ allows us to collocate an exponential decreasing factor $e^{-\beta(x)}$ in a convenient place using the exponential decay of terms that we can control to compensate the term $e^{\beta(x)}$ (for real and positive $\beta$, see (6.125)).

2.1.2.1 The Physical Electric Potential

We assume that the physical electric potential satisfies the following properties:
• For any $\theta = \mu + i\nu$

$$\lim_{|x| \to \infty} \sup_{|\theta| < 1/2} |V(\theta)(x)| = 0, \quad V(\theta)^* = V(\bar{\theta}), \quad V(\theta) = u(\mu)V(iv)u(\mu)^{-1}. \quad (2.18)$$

• The function

$$\theta \to V(\theta)\frac{1}{-\Delta + 1}, \quad |\theta| < \frac{1}{2}, \quad (2.19)$$

is an operator valued analytic function, where $\Delta$ is the Laplace operator. We suppose furthermore that

$$\lim_{r \to \infty} \sup_{|\theta| < 1/2} \| V(\theta)\frac{1}{-\Delta + r} \| = 0, \quad (2.20)$$

where the norm $\| \cdot \|$ is the operator norm.

### 2.1.2.2 The Effective Potential $V_{PF}(\theta)$

This potential comes from the Pauli-Fierz transformation (see Section 2.3.3.3). In particular, it appears in Eq. (2.23).

Let $\eta \in C_0^\infty([0, \infty))$ be a decreasing function such that

$$\eta(r) = \begin{cases} 1, & \text{if } r \leq 1, \\ 0, & \text{if } r \geq 2. \end{cases} \quad (2.24)$$

We denote by $\bar{\varepsilon}$, a fixed function $\bar{\varepsilon} := (\varepsilon_1, \varepsilon_2, \varepsilon_3) : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{C}^3$ that satisfies

$$\bar{\varepsilon}(\vec{k}, \lambda)^* \cdot \bar{\varepsilon}(\vec{k}, \mu) = \delta_{\lambda, \mu}, \quad \vec{k} \cdot \bar{\varepsilon}(\vec{k}, \lambda) = 0, \quad \bar{\varepsilon}(\vec{k}, \lambda) = \bar{\varepsilon}(\vec{k}, \lambda), \quad \bar{\varepsilon}(r\vec{k}, \lambda) = \bar{\varepsilon}(\vec{k}, \lambda), \quad r > 0, \quad (2.25)$$

where $\bar{\varepsilon}(\vec{k}, \lambda)$ is the vector whose entries are the complex-conjugate of the entries of $\bar{\varepsilon}(\vec{k}, \lambda)$ and $\delta_{\lambda, \mu}$ is the Kronecker symbol.

The effective potential is defined by the following:

$$V_{PF}(\theta)(x) := e^{-\theta^2 \sum_{\lambda=1}^2 \frac{1}{2(2\pi)^3}} \int_{\mathbb{R}^3} \left[ \exp(-2e^{-2\theta}|\vec{k}|^2) \right] \cdot |\eta(|x||\vec{k}|)\bar{\varepsilon}(\vec{k}, \lambda) \cdot x|d\vec{k}, \quad (2.26)$$

for any $\theta \in \mathbb{C}$ with $|\theta| < 1/2.$
2.1.2.3 The Potential $\tilde{V}(\theta, \beta)$

We denote by
$$\langle x \rangle := (1 + |x|^2)^{1/2}.$$  (2.23)

We define
$$\tilde{V}(\theta, \beta)(x) := -e^{-2\theta}e^{-\beta(x)} \Delta e^{\beta(x)} + e^{-2\theta} \Delta, \theta, \beta \in \mathbb{C}, |\theta|, |\beta| < \frac{1}{2},$$  (2.24)

where $\Delta$ is the Laplace operator.

2.1.3 The Atom Hamiltonian

The atom Hamiltonian is a closed operator with domain contained in the atom Hilbert space
$$\mathcal{H}_{at} := L^2(\mathbb{R}^3).$$

It is defined by the formula
$$H_{at}(\theta) := -e^{-2\theta} \Delta + V(\theta).$$

We do not write explicitly the dependence of $H_{at}(\theta)$ on $\alpha, \beta$ (see 2.16). If it is required we also write
$$H_{at}(\theta, \alpha, \beta) := H_{at}(\theta), \theta, \beta \in \mathbb{C}, |\theta|, |\beta| < \frac{1}{2}, \alpha \in [0, 1].$$

Hypothesis 2.1 (The Energy of the First Excited State of $H_{at}$). We denote by
$$e_0 := \inf(\sigma(H_{at}(0, 0, 0))), e_1 := \inf(\sigma(H_{at}(0, 0, 0)) \setminus \{e_0\}),$$

where $\sigma(O)$ is the spectrum of the operator $O$.

We assume that
$$e_0 < e_1 < 0,$$

and that $e_1$ is a non-degenerate eigenvalue.

Definition 2.1. We denote by
$$\delta_{at} := \text{dist}(e_1, \sigma(H_{at}(0, 0, 0)) \setminus \{e_1\}).$$  (2.25)
2.2 The Photon Hamiltonian

2.2.1 The Photon Hilbert Space

The Hilbert space of one photon restricted to energies between $s$ and $t$ (with $0 \leq s < t \leq \infty$) is denoted by

$$\mathcal{h}^{s,t} := L^2(\mathcal{K}^{s,t}),$$

(2.26)

where

$$\mathcal{K}^{s,t} := \{(\vec{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2 \mid s \leq |\vec{k}| < t\}.$$  

(2.27)

A pair $(\vec{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$ is denoted by

$$k := (\vec{k}, \lambda).$$

(2.28)

The modulus of an element $k = (\vec{k}, \lambda)$ is

$$|k| := |\vec{k}|.$$  

(2.29)

The integral of a function $f : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathbb{C}$ is defined by

$$\int f(k)dk = \int f(\vec{k},0)d\vec{k}_1d\vec{k}_2d\vec{k}_3 + \int f(\vec{k},1)d\vec{k}_1d\vec{k}_2d\vec{k}_3,$$

(2.30)

where $\mathbb{Z}_2 = \{0, 1\}$.

The Hilbert space of $N$ photons with energies between $s$ and $t$ is

$$\mathcal{F}_N^{s,t} := \mathcal{S}_N \otimes \mathcal{h}^{s,t}, \quad \mathcal{F}_0^{s,t} := \mathbb{C},$$

(2.31)

where $\mathcal{S}_N$ is the projection onto the space of totally symmetric functions. The photon Fock space is the direct sum

$$\mathcal{F}^{s,t} := \bigotimes_{N=0}^{\infty} \mathcal{F}_N^{s,t}.$$  

If $t = \infty$ we omit the variable $t$ and write

$$\mathcal{F}^s := \mathcal{F}^{s,\infty}.$$  

The vacuum vector is the element

$$\Omega^{s,t} := (1, 0, 0, \cdots) \in \mathcal{F}^{s,t}.$$  

Remark 2.2. We use frequently the superscript $s, t$ in our formulas throughout the paper. We will use the convention that if the variable $t$ is omitted then it is $\infty$. 


2.2.2 The Photon Hamiltonian

The photon Hamiltonian is the operator that takes an element $\phi = (\phi_j)_{j \in \mathbb{N} \cup \{0\}} \in \mathcal{F}^{s,t}$ to the element $\hat{H}^{s,t}(\theta)\phi$ with

$$ (\hat{H}^{s,t}(\theta)\phi)(k_1, \cdots, k_j) := e^{-\theta(|k_1| + \cdots + |k_j|)}\phi_j(k_1, \cdots, k_j), \quad j \geq 1, $$

$$ (\hat{H}^{s,t}(\theta)\phi)_0 = 0, $$

for any $\theta \in \mathbb{C}$, with $|\theta| < \frac{1}{2}$. If $t = \infty$ we omit the variable $t$:

$$ \hat{H}^{s,t}(\theta) := \hat{H}(\theta). $$

2.3 The Atom-Photon Hamiltonian

2.3.1 The Atom-Photon Hilbert Space

The atom-photon Hilbert space is the tensor product

$$ \mathcal{H}^{s,t} := \mathcal{H}_{\text{at}} \otimes \mathcal{F}^{s,t} $$

(If $t = \infty$ we omit it in our notations).

2.3.2 The Free Atom-Photon Hamiltonian

The free atom-photon Hamiltonian is the operator

$$ H_0^{s,t}(\theta) := H_{\text{at}}(\theta) \otimes 1_{\mathcal{F}^{s,t}} + 1_{\mathcal{H}_{\text{at}}} \otimes \hat{H}^{s,t}(\theta), \quad (2.32) $$

where 1 denotes the identity operator. $H_0^{s,t}(\theta)$ depends also on the variables $\alpha$ and $\beta$. If it is necessary we write this dependence explicitly

$$ H_0^{s,t}(\theta, \alpha, \beta) := H_0^{s,t}(\theta), \quad \theta, \beta \in \mathbb{C}, |\theta|, |\beta| < \frac{1}{2}, \quad \alpha \in [0, 1]. \quad (2.33) $$

(If $t = \infty$ we omit it in our notations).

2.3.3 The Atom-Photon Hamiltonian

The atom-photon Hamiltonian is a Schrödinger operator with a second quantized magnetic potential (see Section 2.3.3.2). The singularity at $k=0$ of the form factor that models the interaction between the electron and the photons does not allow to study these resonances directly. To overcome this difficulties we transform it to another one that is unitarily equivalent via the Pauli-Fierz transformation (see Section 2.3.3.3), which is a gauge transformation. The new Hamiltonian has to be analytically continued in order to
study the resonances (see Section 2.3.3.4). The long distance behavior in the electron variable becomes worse after the Pauli-Fierz transformation, this is the price to pay for a less singular form factor, that in fact will turn out to be regular at $k=0$. To overcome this problem we include a new parameter ($\beta$) that is technically convenient to analyze the long distance regime for the electron (see Section 2.3.3.5). The parameter $\beta$ is also useful to prove exponential decay of the eigenfunctions (see Section 3.1 and Theorem 6.5).

Finally we introduce an infrared cut off into the Hamiltonian that will be eventually removed (see Section 2.3.3.6). In Section 2.3.3.6 we present the final Hamiltonian that we study in the rest of this article, and we give a precise description of the steps mentioned above.

### 2.3.3.1 Creation and Annihilation Operators

For any function $f \in \mathfrak{h}^{s,t}$, the creation operator $a^\ast(f)$ is the operator that takes an element $\phi \in \mathcal{F}^s_t N$ to the vector

$$a^\ast(f)(\phi) := \sqrt{N+1} S_{N+1} f \otimes \phi \in \mathcal{F}_N^{s,t+1},$$

see (2.31). We extend $a^\ast(f)$ below by linearity and take the closure to define the creation operator $a^\ast(f)$ in $\mathcal{F}^{s,t}$. The annihilation operator $a(f)$ is the adjoint of $a^\ast(f)$.

The creation and annihilation operators satisfy the canonical commutation relations:

$$[a^\ast(f) , a^\ast(g)] = [a(f) , a(g)] = 0, \ [a(f) , a^\ast(g)] = \langle f | g \rangle,$$

$$a(f) \Omega^{s,t} = 0,$$

where $[\cdot , \cdot]$ and $\langle \cdot | \cdot \rangle$ denote the commutator and the scalar product, respectively.

For any vector valued function $\vec{f} \in (\mathfrak{h}^{s,t})^3$ we define $a^\ast(\vec{f})$ and $a(\vec{f})$ component-wise. We use the isomorphism

$$\mathcal{H}^{s,t} = \mathcal{H}_{s,t} \otimes \mathcal{F}^{s,t} \cong L^2(\mathbb{R}^3; \mathcal{F}^{s,t})$$

to extend the definition of creation and annihilation operators to $\mathcal{H}^{s,t}$. Let $f : \mathbb{R}^3 \times K^{s,t} \rightarrow \mathbb{C}$ be such that for all $x \in \mathbb{R}^3$, the function $f_x(\cdot) := f(x, \cdot)$ belongs to $\mathfrak{h}^{s,t}$. We define

$$a(f)^\ast \phi(x) := a^\ast(f_x)\phi(x), \ \phi \in L^2(\mathbb{R}^3; \mathcal{F}^{s,t}).$$
The definition of $a(f)$ is similar. Both definitions can be extended to vector valued functions component-wise. For any operator $O$ defined in $\mathcal{H}_{at}$ and any function $f : \mathbb{R}^3 \times \mathcal{K}^{s,t} \to \mathbb{C}$ as before we identify
\begin{equation}
    a^*(Of) := O \otimes 1_{\mathcal{F}^{s,t}} a^*(f), \quad a(Of) := a(f) O^* \otimes 1_{\mathcal{F}^{s,t}},
\end{equation}
and accordingly we define $a^*(fO)$, $a(fO)$.

It is convenient to define the creation and annihilation operators pointwise ((X.74) and (X.75)). They are operator-valued tempered distributions on the Fock space $\mathcal{F}^{s,t}$ obeying the canonical commutation relations,
\begin{align}
    [a^\#(\vec{k}_1, \lambda), a^\#(\vec{k}_2, \mu)] &= 0, \quad [a(\vec{k}_1, \lambda), a^*(\vec{k}_2, \mu)] = \delta_{\lambda,\mu} \delta(\vec{k}_1 - \vec{k}_2),
    \label{eq:commutation_relations}
\end{align}
where $a^\# = a$ or $a^*$. For any $O$ and $f$ are as in (2.35) we write
\begin{align}
    a^*(Of) := \int_{\mathcal{K}^{s,t}} Of(x,k) a^*(k) dk, \quad a(Of) := \int_{\mathcal{K}^{s,t}} f(x,k) O^* a(k) dk,
    \label{eq:quantized_operators}
\end{align}
and accordingly we write $a^*(fO)$ and $a(fO)$.

2.3.3.2 The Atom-Photon Hamiltonian: The Second-Quantized Magnetic Schrödinger Equation

For any $\theta$ in $\mathbb{C}$ with $|\theta| < \frac{1}{2}$ we define the function $G^{s,t}(\theta) : \mathbb{R}^3 \times \mathcal{K}^{s,t} \to \mathbb{C}^3$ by
\begin{equation}
    G^{s,t}(\theta)(x,k) := \frac{\alpha^{3/2}}{(2\pi)^{3/2}} e^{-\theta} \exp(e^{-2\theta} |k|^2) e^{-i\alpha \vec{k} \cdot \vec{\varepsilon}(k)}, \quad \label{eq:atom_photon_function}
\end{equation}
where the polarization vector $\vec{\varepsilon}(k)$ is defined in Section 2.1.2.2. The constant $\alpha$ is the fine structure constant but it will be assumed to be sufficiently small. The exponential function $\exp(e^{-2\theta} |k|^2)$ plays the role of an ultraviolet cut off. The parameter $s$ is an infrared cut off if it is strictly positive.

The second quantized magnetic potential is the operator
\begin{equation}
    A^{s,t}(\theta) := a^*(G^{s,t}(\theta)) + a(G^{s,t}(\overline{\theta})). \quad \label{eq:quantized_magnetic_potential}
\end{equation}
In the formula above we omit the parameter $t$ when it is infinity.

$A^0(\theta)$ is an analytic continuation of the physical second quantized magnetic potential $A^0(0)$. The electron-photon Hamiltonian is the operator
\begin{equation}
    H := (-i\nabla \otimes 1_{\mathcal{F}} - A^0(0))^2 + V(0,0,0) \otimes 1_{\mathcal{F}} + 1_{\mathcal{H}_{el}} \otimes \hat{H}^0 \quad \label{eq:atom_photon_hamiltonian}
\end{equation}
(see (2.16)), which is a second quantized version of the magnetic Schrödinger operator.
2.3.3.3 The Pauli-Fierz Transformation

The form factor \( \frac{1}{\sqrt{|k|}} \) (see (2.38)) yields major difficulties in the study of resonances, indeed, in this case the perturbation is (superficially) *marginal*.

On the contrary, we can control the renormalization scheme in presence of any regularization of the form factor, i.e., \( 1/\sqrt{|k|}^{1/2-\mu} \) as \( k \to 0 \), with \( \mu > 0 \). We cannot handle directly \( H \) but instead we can handle another Hamiltonian which is unitarily equivalent (and therefore physically equivalent). This new Hamiltonian is obtained by a change of gauge which is called the Pauli-Fierz transformation. In the new Hamiltonian the interaction of the electron with the photons is modeled by a less singular form factor, but the long distance behavior in the electron variable is more difficult to handle. The relation between small momenta and long distances can be read from the oscillatory functions \( (e^{-i\alpha \vec{k} \cdot \vec{x}}) \). In some sense we can say that we trade small momenta for long distances via the Pauli-Fierz transformation. The long distance behavior in the electron variable can be controlled by exploiting the exponential localization of the eigenfunctions.

We define the operator

\[
\lambda_{PF}^{s,t}(\theta) := a^\dagger(G^{s,t}(\theta)(0, k) \cdot \eta(|x||k|)e^{\theta}x) + a(G^{s,t}(\theta)(0, k) \cdot \eta(|x||k|)e^{\theta}x) \quad (2.41)
\]

(see (2.19)). We omit the variable \( t \) if it is infinity and we do not write the dependence on \( \alpha \).

The Pauli-Fierz transformed Hamiltonian is the operator

\[
H^0(0, \alpha, 0) := e^{-i\lambda_{PF}^{0}(0)} H e^{i\lambda_{PF}^{0}(0)}. \quad (2.42)
\]

Explicitly we have (see (2.16, 2.17) and appendix B)

\[
H^0(0, \alpha, 0) = (-i \nabla \otimes 1^{F_0} - A^0(0))^2 + b^0(0) \quad (2.43)
+ V(0, \alpha, 0) \otimes 1^{F_0} + 1_{\text{int}} \otimes \check{H}^0,
\]

where

\[
A^{s,t}(\theta) := A^{s,t}(\theta) - (e^{-\theta} \nabla \otimes 1^{F_{-s,t}}) \lambda_{PF}^{s,t}(\theta), \quad (2.44)
\]

\[
b^{s,t}(\theta) := a^\dagger(ie^{-\theta}|k|G^{s,t}(\theta)(0, k) \cdot \eta(|x||k|)x)
+ a(ie^{-\theta}|k|G^{s,t}(\theta)(0, k) \cdot \eta(|x||k|)x)
\]

and we do not write \( t \) if it is infinity.
2.3.3.4 Analytic Continuation

The Hamiltonian $H^0(0)$ is the object of our study. It represents the energy of the electron-photon system in the representation that follows from the change of gauge in formula (2.42). We are interested on the resonances of the Hamiltonian $H^0(0)$. To study the resonances we need an analytic continuation of $H^0(0)$. It is given by (2.17) and (2.44) as follows

$$H^0(\theta, \alpha, 0) = (-ie^{-\theta} \nabla \otimes 1_{F_0} - A^0(\theta))^2 + b^0(\theta)$$

$$+ V(\theta, \alpha, 0) \otimes 1_{F_0} + e^{-\theta} 1_{H_{\text{el}}} \otimes \tilde{H}^0.$$  \hfill (2.45)

For real $\theta$ we define the unitary operator

$$U(\theta) := u(\theta) \otimes (1_{\mathbb{C}} \oplus \bigoplus_{j=1}^{\infty} u(-\theta)^{\otimes j})$$

(see (2.15)) acting on the Hilbert space $\mathcal{H}^0$. It is clear from the definitions that

$$H^0(\theta, \alpha, 0) := U(\theta) H^0(0, \alpha, 0) U(\theta)^{-1}. \quad \hfill (2.46)$$

In Section 4 we prove that $H^0(\theta, \alpha, 0)$ is closed and self-adjoint for real $\theta$. We prove furthermore that it is an analytic family of operators (see Theorem 4.4).

2.3.3.5 Controlling the Long-Distance Behavior for the Electron Variable: The parameter $\beta$

The Pauli-Fierz transformation yields a form factor that is regular at $k = 0$, but as a consequence the long distance behavior of the electronic part is worse. To control the electron at log distances we make use of the exponential decay of the eigenfunctions. This is done by introducing a new parameter $\beta \in \mathbb{C}$, $|\beta| < \frac{1}{2}$, in the Hamiltonian. We define

$$H^0(\theta, \alpha, \beta) := e^{-\beta(x)} H^0(\theta, \alpha, 0) e^{\beta(x)}$$

$$= (-ie^{-\theta} \nabla \otimes 1_{F_0} - ie^{-\theta} \beta(\nabla \langle x \rangle) \otimes 1_{F_0} - A^0(\theta))^2 + b^0(\theta)$$

$$+ V(\theta, \alpha, 0) \otimes 1_{F_0} + e^{-\theta} 1_{H_{\text{el}}} \otimes \tilde{H}^0,$$  \hfill (2.47)

(see (2.23)). The parameter $\beta$ is also useful to prove exponential decay of the eigenfunctions (see Section 3.1 and Theorem 6.5).
2.3.3.6 The Infrared cut off: The Final Hamiltonian

We conclude our construction of the Hamiltonian by including an infrared cut off (that will be eventually removed). We restrict the momentum of the photons to be larger or equal to $s$ and smaller than $t$. We point out that if $s = 0$ there is no infrared cut off. Finally, we define the Hamiltonian

$$H_{s,t}^{s,t}(\theta, \alpha, \beta) := (-ie^{-\theta} \nabla \otimes 1_{x_{s,t}} - A_{s,t}(\theta))^2$$

$$+ ie^{-\theta} \beta(\nabla \langle x \rangle) \cdot A_{s,t}(\theta) + A_{s,t}(\theta) \cdot ie^{-\theta} \beta(\nabla \langle x \rangle)$$

$$+ b_{s,t}(\theta) + V(\theta, \alpha, \beta) \otimes 1_{x_{s,t}} + e^{-\theta} 1_{\mathcal{H}_{x_{s,t}}} \otimes \tilde{H}_{s,t}$$

$$= H_{0}^{s,t}(\theta, \alpha, \beta) + W_{s,t}^{s,t}(\theta, \alpha, \beta),$$

(2.48)

where

$$W_{s,t}^{s,t}(\theta, \alpha, \beta) := H_{s,t}^{s,t}(\theta, \alpha, \beta) - H_{0}^{s,t}(\theta, \alpha, \beta)$$

(2.49)

is the interaction (see the definition of $H_{0}^{s,t}(\theta, \alpha, \beta)$ in (2.32)).

Whenever we do not use explicitly the parameters $\alpha$ and $\beta$ we simply write

$$H_{s,t}^{s,t}(\theta) := H_{s,t}^{s,t}(\theta, \alpha, \beta), \ W_{s,t}^{s,t}(\theta) := W_{s,t}^{s,t}(\theta, \alpha, \beta).$$

(2.50)

(If $t = \infty$ we omit it in our notation).

Using (2.36) and (2.37) we can write the interaction in the form

$$W_{s,t}^{s,t}(\theta, \alpha, \beta) := W_{1,0}^{s,t}(\theta, \alpha, \beta) + W_{0,1}^{s,t}(\theta, \alpha, \beta) + W_{2,0}^{s,t}(\theta, \alpha, \beta)$$

$$+ W_{0,2}^{s,t}(\theta, \alpha, \beta) + W_{1,1}^{s,t}(\theta, \alpha, \beta) + W_{0,0}^{s,t}(\theta, \alpha, \beta).$$

(2.51)

where

$$W_{m,n}^{s,t}(\theta, \alpha, \beta) := \int_{(K_{s,t})_{m+n}} dk_1 \cdots dk_m d\tilde{k}_1 \cdots d\tilde{k}_n$$

$$\left[ w_{m,n}^{s,t}(\theta, \alpha, \beta)(k_1, \ldots, k_m, \tilde{k}_1 \cdots \tilde{k}_n) a^*(k_1) \cdots a^*(k_m) a(\tilde{k}_1) \cdots a(\tilde{k}_n) \right].$$

(2.52)

The functions $w_{m,n}^{s,t}(\theta, \alpha, \beta)$ can be derived from (2.49). We report them in (9.260)-(9.265). The operator on the r-h-s of Eq. (2.52) is understood in the sense of quadratic forms (see Theorem X.44 and (X.74, X.75) of [10]).

3 Estimates for the Atom Hamiltonian

In this section we state some results for the electron Hamiltonian that are used later on. The proofs are deferred to Section 7.
Lemma 3.1. For every $\epsilon > 0$ there exists a constant $b_{\epsilon}$ such that, $\forall|\theta| < \frac{1}{2}$:

$$\|V(\theta, \alpha, \beta)\|_{L^2(\mathbb{R}^3)} \leq \epsilon\| - \Delta \phi\|_{L^2(\mathbb{R}^3)} + b_{\epsilon}\|\phi\|_{L^2(\mathbb{R}^3)} \quad (3.53)$$

for any $\phi \in H^2(\mathbb{R}^3)$, where $H^2(\mathbb{R}^3)$ is the Sobolev space of functions in $L^2(\mathbb{R}^3)$ with distributional derivatives up to order 2 in $L^2(\mathbb{R}^3)$.

Proof: Eq. (3.53) follows from (2.18), (2.21) and (2.24). \qed

Definition 3.2. The following constants are repeatedly used through the paper:

$$a := a(\frac{1}{19}(\frac{32}{\Delta_{a_t}})^{-1})^{1/3}, \quad (3.54)$$

$$b := \frac{1}{19}(50 \cdot (4 + (2|e_0| + 2b_{1/2} + 1)\frac{32}{\Delta_{a_t}}))^{-1}. \quad (3.55)$$

$$C_{3.55} := \sup_{\alpha \leq 1, |\beta| \leq 1} \max_{|\theta| = \frac{1}{120}} \|V(\theta, \alpha, \beta)\|_{L^2(\mathbb{R}^3)} \quad (3.56)$$

$$C_{3.56} := \left(\frac{1}{2(1/60 - 1/120)^2}\right)^{3.55} + 2e^{1/15} \cdot \frac{4}{3} \left(1 + 2|e_0| + \frac{\Delta_{a_t}}{16} + b_{1/4} + 1 \right). \quad (3.57)$$

Theorem 3.3. We suppose that $\alpha \leq a$, $|\beta| \leq b$ and that

$$|\theta| \leq \min\left((32C_{3.56})^{-1}, \frac{1}{120}\right). \quad (3.58)$$

The following holds true:

- There are only two points $\{e_0(\theta, \alpha, \beta), e_1(\theta, \alpha, \beta)\}$ in the spectrum of $H_{a_t}(\theta, \alpha, \beta)$ with real part less than $e_1 + \frac{16}{16}\delta_{a_t}$ (see Definition 2.7). They are simple eigenvalues and they do not depend on $\beta$ and $\theta$ (they are therefore real). They satisfy

$$|e_j(\theta, \alpha, \beta) - e_j| \leq \frac{\delta_{a_t}}{16}, \quad j \in \{0, 1\}. \quad (3.59)$$

It follows that

$$e_j(\theta, \alpha, \beta) = e_j(0, \alpha, 0) := e_j(\alpha), \quad j \in \{0, 1\}. \quad (3.60)$$

We omit writing the dependence on $\alpha$ when it is not strictly necessary.
&bullet; Let

\[ P_{at,j}(\theta, \alpha, \beta), \ j \in \{0, 1\} \] (3.60)

be the projection onto the eigen-space corresponding to \( e_j, \ j \in \{0, 1\} \) respectively. It follows that

\[
\| P_{at,j}(0, \alpha, \beta) - P_{at,j}(0, 0, 0) \| \leq \frac{1}{8},
\]

\[
\| P_{at,j}(\theta, \alpha, \beta) - P_{at,j}(0, \alpha, \beta) \| \leq \frac{1}{8}. \tag{3.61}
\]

**Definition 3.4.** We denote by

\[ e'_i := e_i - \frac{\delta_{at}}{16}, \ i \in \{1, 2\}. \tag{3.62} \]

and

\[ \delta = \frac{7\delta_{at}}{8}. \tag{3.63} \]

We notice that

\[ \delta \leq e_1 - e_0, \ e'_i \leq e_i, \ i \in \{0, 1\}, \tag{3.64} \]

for any \( \alpha \leq a \).

We define the projection operator \( P_{disc}(\theta) \) by

\[ P_{disc}(\theta, \alpha, \beta) := P_{at,0}(\theta, \alpha, \beta) + P_{at,1}(\theta, \alpha, \beta). \tag{3.65} \]

For any projection \( P \) we define

\[ \overline{P} := 1 - P. \tag{3.66} \]

**Remark 3.5.** From the proof of Theorem 3.3 (more precisely from Eq. (7.206)) it follows that

\[
\| (1 - \Delta) \frac{1}{H_{at}(0, \alpha, \beta) - z} \| \leq C_{3.68}^2,
\]

where

\[
C_{3.68}^2 := \frac{4}{3} \left( 1 + 2 |e_0| + \delta_{at} / 16 + b_{1/4} + 1 \right), \tag{3.68}
\]

for any \( z \in \mathbb{C} \) with \( \text{Re}(z) \leq 0 \) and \( \text{dist}(z, \sigma(H_{at}(0, 0, 0))) \geq \frac{\delta_{at}}{16} \).
Theorem 3.6. Suppose that \( \theta \) satisfies (3.57). Suppose furthermore that \( \alpha \leq a \) and \( |\beta| \leq b \). Let \( z \in \mathbb{C} \) with \( \text{Re}(z) < e_1 + \frac{7}{8} \delta_{at} \). Then,

\[
\| (H_{at}(\theta, \alpha, \beta) - z)^{-1} \overline{P}_{disc}(\theta, \alpha, \beta) \| \leq 54 \frac{1}{|z - e_0|}.
\]

(3.69)

where

\[
e_{3.70} := e_1 + \frac{15}{16} \delta_{at},
\]

(3.70)

and

\[
C_{3.71} := 972 \frac{|e_0|}{\delta_{at}}.
\]

(3.71)

Corollary 3.7. Suppose that \( \theta \) satisfies (3.57). Suppose furthermore that \( \alpha \leq a \) and \( |\beta| \leq b \). Let \( z \in \mathbb{C} \setminus \{e_0, e_1\} \) with \( \text{Re}(z) < e_1 + \frac{7}{8} \delta_{at} \). The following estimates hold true,

\[
\| (H_{at}(\theta, \alpha, \beta) - z)^{-1} \| \leq 4 C_{3.71} \left( \frac{1}{|z - e_0|} + \frac{1}{|z - e_1|} \right).
\]

(3.72)

\[
\| (H_{at}(\theta, \alpha, \beta) - z)^{-1} \overline{P}_{at,1}(\theta, \alpha, \beta) \| \leq 4 C_{3.71} \frac{1}{|z - e_0|}.
\]

(3.73)

Corollary 3.8. Suppose that \( \theta \) satisfies (3.57). Suppose furthermore that \( \alpha \leq a \) and \( |\beta| \leq b \). Let \( z, \mu \in \mathbb{B}_{\delta_{at}/6}(e_1) \), it follows that

\[
\| (H_{at}(\theta, \alpha, \beta) - z) \overline{P}_{at,1}(\theta, \alpha, \beta) (H_{at}(\theta, \alpha, \beta) - \mu) \| \leq 4 + 4 C_{3.71}.
\]

(3.74)

3.1 Exponentially Boundedness of the Eigenfunctions

It is well known that the eigenfunctions of the electron Hamiltonian are exponentially bounded (see [22]). But for our proposes we need uniform bounds in \( \theta \) and \( \alpha \). We provide below some explicit bounds for the exponentially boundedness of the eigenvalues as well as explicit uniform estimates for the exponential decay rate.

Theorem 3.9. Suppose that \( \theta \) satisfies (3.57). Suppose furthermore that \( \alpha \leq a \) and \( |\beta| \leq b \). Then the range of \( P_{at,1}(\theta, \alpha, 0) \) is contained in the domain of \( e^{\beta(x)} \) (see (2.23)) and

\[
\| e^{\beta(x)} P_{at,1}(\theta, \alpha, 0) \| \leq 8 \| e^{\beta(x)} \phi \|_{H_{at}}.
\]

(3.75)
In particular the following uniform bounds (in $\theta$ and $\alpha$) hold true

$$\|e^{\beta(x)}P_{at,1}(\theta, \alpha, 0)\| \leq C_{3.78},$$  

and

$$\|(1 + |x|^2)P_{at,1}(\theta, \alpha, 0)\| \leq C_{3.79},$$

where

$$C_{3.78} = 8\|e^{b(x)}\phi\|_{H_{at}},$$

$$C_{3.79} = 8\|e^{b(x)}\phi\|_{H_{at}}(1 + e^{-2}(1 + 4/b^2)),$$

and $\phi$ is any unit eigenvector of $P_{at,1,0,0}(0)$.

### 4 Relative Boundedness of the Atom-Photon Hamiltonian and Analyticity.

In this section we establish some basic properties of the atom-photon Hamiltonian as well as some useful estimates. The proofs of this sections are deferred to Section 8.

First we recall a basic result the proof of which can be found for example in Lemma 1 of $[13]$.

**Lemma 4.1.** Let $g_i : \mathbb{R}^3 \to h^{s,t}$, $i \in \{1, 2\}$, be uniformly bounded functions. We suppose furthermore that the functions $x \to |k|^{-1/2}g_i(x)$ are uniformly bounded with values in $h^{s,t}$ (see (2.22)-(2.27)).

For any $a^\#, \tilde{a}^\# \in \{a, a^*\}$, it follows that

$$\text{dom}(1_{H_{at}} \otimes (\tilde{H}^{s,t})^{1/2}) \subset \text{dom}(a^\#(g_i))$$

and

$$a^\#(g_i)[\text{dom}(1_{H_{at}} \otimes \tilde{H}^{s,t})] \subset \text{dom}(a^\#(g_j)).$$

It follows furthermore that for every $\rho > 0$ and every $\phi \in \text{dom}(1_{H_{at}} \otimes (\tilde{H}^{s,t})^{1/2})$, $\psi \in \text{dom}(1_{H_{at}} \otimes \tilde{H}^{s,t})$,

$$\|a^\#(g_i)\phi\|_{\tilde{H}^{s,t}} \leq \|g_i\|_\rho \|1_{H_{at}} \otimes (\tilde{H}^{s,t} + \rho)^{1/2}\phi\|_{\tilde{H}^{s,t}},$$

$$\|\tilde{a}^\#(g_2) a^\#(g_1)\psi\|_{\tilde{H}^{s,t}} \leq \|g_2\|_\rho \|g_1\|_\rho \|1_{H_{at}} \otimes (\tilde{H}^{s,t} + \rho)\psi\|_{\tilde{H}^{s,t}},$$

where

$$\|g_i\|_\rho := \sup_{x \in \mathbb{R}^3} \frac{1}{\rho^{1/2}} \|g_i(x)\|_{h^{s,t}} + \sup_{x \in \mathbb{R}^3} \|k|^{-1/2}g_i(x)\|_{h^{s,t}}.$$
In the following remark we fix some notations that will be used later.

**Remark 4.2.** Let \( s' \leq s < t \) be positive real or infinite numbers.

The union
\[
K^{s',t} = K^{s',s} \cup K^{s,t}
\]
gives rise to the isomorphisms
\[
\mathcal{F}^{s',t} \cong \mathcal{F}^{s,t} \otimes \mathcal{F}^{s',s},
\]
\[
\mathcal{H}^{s',t} \cong \mathcal{H}^{s,t} \otimes \mathcal{F}^{s',s},
\]
\[
\tilde{\mathcal{H}}^{s',t} = \tilde{\mathcal{H}}^{s,t} \otimes 1_{\mathcal{F}^{s',s}} + 1_{\mathcal{F}^{s',s}} \otimes \tilde{\mathcal{H}}^{s',s}.
\]

Given an operator \( O \) on \( \mathcal{H}^{s,t} \), we use the same symbol to denote the operator
\[
O := O \otimes 1_{\mathcal{F}(s',s)}.
\]

The next lemma establishes the relative boundedness of the interaction with respect to the free atom-photon Hamiltonian (see \((2.48)-(2.50))\), which implies that the atom-photon Hamiltonian is closed (for small \( \alpha \) and \( |\beta| \)) and furthermore that it is self-adjoint for \( \theta \in \mathbb{R} \) and \( \Re \beta = 0 \) (see Theorem 1.1 page 190 of \[11\] and the Kato Rellich Theorem (Theorem X.12 of \[10\]). For the proofs see Section 8.

**Lemma 4.3.** Let \( s' \leq s < t \) be positive real numbers, possibly \( t \) is infinity.

Suppose that \( \alpha \leq a \) and \( |\beta| \leq b \) (see \((3.54))\).

For every \( \rho > 0 \) and every \( \theta, \eta \in \mathbb{C} \) with \( |\theta| \leq \frac{1}{120} \) and \( |\theta + \eta| \leq \frac{1}{120} \) there exist constants \( C_{8.241} \) and \( C_{8.246} \) such that the following estimates hold true
\[
\| W^{s,t}(\theta) \phi \|_{\mathcal{H}^{s',t}} \leq C_{8.241} (1 + \frac{1}{\rho^{1/2}})^{3/2} (1 + \frac{1}{\rho^{1/2}}^2 \alpha^3).
\]
\[
(\| (\tilde{\mathcal{H}}^{s',t}(\theta) + \rho) \phi \|_{\mathcal{H}^{s',t}} + \| \phi \|_{\mathcal{H}^{s',t}}),
\]

moreover, if \( -\rho \) belongs to the resolvent set of \( H_0^{s,t}(0, \alpha, \beta) \),
\[
\| (W^{s,t}(\theta + \eta) - W^{s,t}(\theta)) \|_{\mathcal{H}^{s',t}(0) + \rho} \leq |\eta| C_{8.246} (1 + \frac{1}{\rho^{1/2}})^{3/2} (1 + \| (\tilde{\mathcal{H}}^{s',t}(0) + \rho)^{-1} \|).
\]

and the operator-valued function \( \theta \to W^{s,t}(\theta) \) is analytic for \( |\theta| < \frac{1}{120} \).

Explicit values for the constants are given in \((8.241)\) and \((8.246)\).
In the next theorem we establish the analyticity of the Hamiltonian.

**Theorem 4.4.** Suppose that $\alpha \leq a$ and $|\beta| \leq b$ (see 3.54). For every $\rho > 0$ such that $-\rho$ belongs to the resolvent set of $H_{0}^{s,t}(0)$, the function $\theta \rightarrow H^{s,t}(\theta)\frac{1}{H_{0}^{s,t}(0)+\rho}$ is an operator-valued analytic function for $|\theta| < \frac{1}{120}$.

Moreover, for every $\theta, h \in \mathbb{C}$ such that $|\theta| < \frac{1}{120}, |\theta + h| < \frac{1}{120}$ the following estimate holds

$$\|(H^{s,t}(\theta + h) - H^{s,t}(\theta))\frac{1}{H_{0}^{s,t}(0)+\rho}\| \leq C_{4.91}|h|(1 + \frac{1}{H_{0}^{s,t}(0)+\rho})$$

(4.90)

where

$$C_{4.91}(\alpha, \rho) := C_{4.91} := \frac{1}{128}(1 + \frac{1}{\rho^{1/2}})\alpha^{3/2} + (1 + \frac{1}{\rho^{1/2}})^{2}\alpha^{3}$$

(4.91)

$$+ \frac{4}{3}(1 + b_{1/4})(\frac{C_{3.55}}{2(1/60 - 1/120)^2} + 3e^{1/15})$$

Using Lemma 4.3 and Theorem 4.4 we can prove the following basic estimates that are useful for the next chapters.

**Corollary 4.5.** Suppose that $\alpha \leq a$ and $|\beta| \leq b$ (see 3.54). Suppose furthermore that $\alpha^{3/2} \leq \frac{1}{128}$ and that $z \in \mathbb{C}$ is such that $\Re(z) \leq \frac{3}{2}e_{0} - \frac{3}{4}$.

Then $H^{s,t}(0, \alpha, \beta) - z$ is invertible and

$$\frac{1}{H^{s,t}(0, \alpha, \beta) - z} \leq \left(\frac{4}{3}\right)^{2}\frac{1}{|z - e_{0}|}$$

(4.92)

$$\|(H_{0}^{s,t}(0, \alpha, \beta) - z)\frac{1}{H^{s,t}(0, \alpha, \beta) - z}\| \leq 2.$$  

(4.93)

Suppose furthermore that $\theta, h \in \mathbb{C}$ are such that $|\theta| < \frac{1}{120}$ and $|\theta + h| < \frac{1}{120}$, then

$$\|(H^{s,t}(\theta + h, \alpha, \beta) - H^{s,t}(\theta, \alpha, \beta))\frac{1}{H^{s,t}(0, \alpha, \beta) - z}\| \leq |h| \cdot 2\left(1 + \frac{4}{3}\right)$$

$$\cdot \left(\frac{1}{16} + \frac{4}{3}(1 + b_{1/4})(\frac{C_{3.55}}{2(1/60 - 1/120)^2} + 3e^{1/15})\right).$$

(4.94)
In particular if
\[ |h| \leq \frac{1}{4} \left( 2 + \left( \frac{4}{3} \right)^2 \right) \cdot \left( \frac{1}{16} + \frac{4}{3} \left( 1 + b_{1/4} \right) \left( \frac{C_{3.55}}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \right)^{-1}, \]
it follows that
\[ \left\| \frac{1}{H^{s,t}(\theta, \alpha, \beta) - z} \right\| \leq \left( \frac{4}{3} \right)^3 \cdot \frac{1}{|z - e_0|}. \]  

**Corollary 4.6.** Suppose that \( \alpha \leq a \) and \( |\beta| \leq b \) (see (3.54)). Suppose furthermore that \( \alpha^{3/2} \leq \frac{1}{128 \cdot 8.246} \). Then for every \( \theta \in B_{1/120}^C(0) \), \( h \in B_{1/120}^C(0) \cap (i\mathbb{R}) \) with
\[ |\theta| < \frac{1}{16} \left( 2 + \left( \frac{4}{3} \right)^2 \right) \cdot \left( \frac{1}{16} + \frac{4}{3} \left( 1 + b_{1/4} \right) \left( \frac{C_{3.55}}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \right)^{-1} \]
the following estimates hold
\[ \left\| (H^{s,t}_0(0, \alpha, \beta) - \rho) \frac{1}{H^{s,t}(\theta, \alpha, \beta) - \rho} \right\| \leq 2, \]
\[ \left\| (H^{s,t}_0(0, \alpha, \beta) - \rho) \frac{1}{H^{s,t}(\theta, \alpha, \beta) - e^h(\rho + r)} \right\| \leq 8, \]
where \( r \in \mathbb{R} \) is any negative number and \( \rho \leq 2e_0 - 1. \)

## 5 Resonances for the Infrared-Regularized Hamiltonian

In this section we construct an eigenvalue of the infrared regularized Hamiltonian in a neighbourhood of \( e_1 \). We can estimate the imaginary part of it and give conditions for it to be strictly negative (which implies that the eigenvalue is actually a resonance). The main results are collected in Theorem 5.7. Theorem 5.7 is the starting point of an inductive argument that takes the infrared cut off to zero and proves the existence of resonances for the non-regularized Hamiltonian. The latter is the main result of this paper and it is proved in the next section (Section 6).

In this section we fix the parameter \( \theta \),
\[ \theta = i\nu \]  
with
\[ \nu \in \mathbb{R} \setminus \{0\}. \]
We suppose that it satisfies (3.57) and (4.96).
We recall that the parameter \( s \) in the Hamiltonian denotes an infrared cut off. In this section we suppose that
\[
s \leq \frac{1}{2} \delta_{at}
\]  
(5.101)
(see (2.25)).

We start by setting some notations.
We define the following subsets of the complex plane
\[
\mathcal{E}^{s,t}(\theta) = \mathcal{E}^s(\theta) := \left[ e_1 - s/4, e_1 + s/4 \right] + i\left[ -\left( |\sin(\nu)|/2 \right)s, (|\sin(\nu)|/2)s \right] - e^{-i\nu}[0, \infty),
\]  
(5.102)
\[
\mathcal{A}^{s,t}(\theta) := \mathcal{A}^s(\theta) := \mathcal{E}^s(\theta) \setminus B_{\left( |\sin(\nu)|/2 \right)s/4}(e_1), \tag{5.103}
\]
where \( B_{\rho}(z) \) is the ball of radius \( \rho \) and center \( z \) in the complex plane. The factor \( |\sin(\nu)|/2 \) ensures that \( e_0 \) is not contained in \( \mathcal{A}^s(\theta) \) and that
\[
\mathcal{A}^s(\theta) - re^{-i\nu} \subset \mathcal{A}^s(\theta), \tag{5.104}
\]
for every \( r \in \{0\} \cup [s, \infty) \).

For every \( z \) in the resolvent set of \( H_{0}^{s,t}(\theta, \alpha, \beta) \) we denote by \( R_{0}^{s,t}(\theta, \alpha, \beta)(z) \) the resolvent of \( H_{0}^{s,t}(\theta, \alpha, \beta) \) at \( z \):
\[
R_{0}^{s,t}(\theta, \alpha, \beta)(z) := \frac{1}{H_{0}^{s,t}(\theta, \alpha, \beta) - z}. \tag{5.105}
\]

We identify (see 3.60)
\[
P_{at}(\theta, \alpha, \beta) \equiv P_{at}(\theta, \alpha, \beta) \otimes 1_{\mathfrak{F}^{s,t}}, \quad \overline{P}_{at}(\theta, \alpha, \beta) := \overline{P}_{at}(\theta, \alpha, \beta) \otimes 1_{\mathfrak{F}^{s,t}}. \tag{5.106}
\]

We will prove that the set \( \mathcal{A}^s(\theta) \) is contained in the resolvent set of the Hamiltonian \( H^{s,t}(\theta, \alpha, \beta) \). We prove furthermore that \( \mathcal{E}^s(\theta) \) contains only one point of the spectrum of \( H^{s,t}(\theta, \alpha, \beta) \). This point is a simple eigenvalue.

**Lemma 5.1.** Suppose that \( \theta = \pm i\nu \), where \( \nu \in \mathbb{R} \) is different from zero. Suppose furthermore that it satisfies (3.57) and (4.96) and that \( \alpha \leq a \) and \( |\beta| \leq b \). Then, for every \( \mu \geq 0 \) and every \( z \in \mathcal{E}^s(\theta) \setminus \{e_1\} \),
\[
\|R_{0}^{s,t}(\theta, \alpha, \beta)(z)\| \leq \frac{50|e_0| |\sin(\nu)|}{\delta |\sin(\nu)|} \frac{1}{|z - e_1|}, \tag{5.107}
\]
\[
\||R_{0}^{s,t}(\theta, \alpha, \beta)(z)\| (\mathcal{H}^{s,t} + \mu)\| \leq \frac{50|e_0| |\sin(\nu)|}{\delta |\sin(\nu)|} \left( 1 + \frac{\mu}{|z - e_1|} \right), \tag{5.108}
\]
\[ \parallel R_{s}^{t}(\theta, \alpha, \beta)(z) |(H_{at} \pm i\delta)\parallel \leq \frac{C_{5.111}}{|\sin(\nu)|} \left( 1 + \frac{1}{|z - e_{1}|} \right). \quad (5.109) \]

For any \( z \in E^{(1/2s_{at})} \),
\[ \parallel |P_{at,1}(\theta, \alpha, \beta)R_{0}^{s_{at}}(\theta, \alpha, \beta)(z)\parallel \leq \min \left( \frac{16C_{3.71}}{|\sin(\nu)|\delta}, \frac{20|e_{0}|C_{3.71}}{|\sin(\nu)|\delta |z - e_{0}|} \right), \quad (5.110) \]

where
\[ C_{5.111} := 2 \left( 1 + 8C_{3.71} \right) + 8C_{3.71} \left( 1 + \frac{|e_{0}|}{\delta} \right) + \frac{8C_{3.71}}{|\sin(\nu)| \delta + |e_{1}|}. \quad (5.111) \]

In the next Lemma we prove the basic estimates on the interaction that are necessary to study the spectral properties of our Hamiltonian.

For the next Lemma recall (2.51), (2.52), (4.83)-(4.86), and (9.26)-(9.265).

**Lemma 5.2.** Suppose that \( s' \leq s < t \) and that \( \theta = \pm i\nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \). Suppose furthermore that \( \theta \) satisfies (3.57) and (4.96) and that \( \alpha \leq a \) and \( |\beta| \leq b \). For every \( z \in E^{s}(\theta) \setminus \{e_{1}\} \) there is a constant \( C_{9.288} \) such that
\[ \parallel |R_{0}^{s_{at}}(\theta, \alpha, \beta)(z)\parallel \leq \frac{8C_{3.71}}{|\sin(\nu)|} \left( 1 + \frac{|e_{0}|}{\delta} \right) + \frac{8C_{3.71}}{|\sin(\nu)| \delta + |e_{1}|} \]. (5.112)

For any \( z \in E^{1/2s_{at}} \),
\[ \parallel |W^{s_{at}}(\theta, \alpha, \beta)|P_{at}(\theta, \alpha, \beta)R_{0}^{s_{at}}(\theta, \alpha, \beta)(z)\parallel \leq C_{9.289} \alpha^{3/2} \]. (5.115)

The explicit values of the constants \( C_{9.288} \) and \( C_{9.289} \) are given in (9.288) and (9.289).
For any $z$ in the resolvent set of $H^{s,t}(\theta, \alpha, \beta)$, we define by

$$R^{s,t}(\theta, \alpha, \beta)(z) := \frac{1}{H^{s,t}(\theta, \alpha, \beta) - z}.$$  \hspace{1cm} (5.116)

Lemma 5.2 provides the key ingredient to construct the resolvent of $H^{s,t}(\theta, \alpha, \beta)$ for any $z \in A^s(\theta)$ (see (5.103)) in terms of a Neumann series.

**Theorem 5.3.** Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha$ is such that

$$C_9 \cdot 288 \cdot |\sin(\nu)|^3/2 \cdot \left(1 + \left(\frac{|\sin(\nu)|/2}{s/4}\right)^{1/2}\right) \leq \delta |\sin(\nu)| \alpha^3 \left(1 + \left(\frac{|\sin(\nu)|}{\delta}\right)^{1/2}\right),$$

and that $\alpha \leq a$ and $|\beta| \leq b$.

Then $A^s(\theta)$ is contained in the resolvent set of $H^{s,t}(\theta, \alpha, \beta)$ and for every $z \in A^s(\theta)$

$$\|R^{s,t}(\theta, \alpha, \beta)(z)\| \leq \frac{60|e_0|C_3}{\delta|\sin(\nu)|} \cdot \frac{1}{|z - e_1|}.$$  \hspace{1cm} (5.118)

**Proof.**

We construct $R^{s,t}(\theta, \alpha, \beta)(z)$ by a norm-convergent Neumann series

$$R^{s,t}(\theta, \alpha, \beta)(z) = \sum_{n=0}^{\infty} R_0^{s,t}(\theta, \alpha, \beta)(z) \left[-W^{s,t}(\theta, \alpha, \beta)R_0^{s,t}(\theta, \alpha, \beta)(z)\right]^n.$$  \hspace{1cm} (5.119)

We estimate the $n^{th}$ order term using Lemma 5.2

$$\|R_0^{s,t}(\theta, \alpha, \beta)(z) \left[-W^{s,t}(\theta, \alpha, \beta)R_0^{s,t}(\theta, \alpha, \beta)(z)\right]^n\|

\leq \|R_0^{s,t}(\theta, \alpha, \beta)(z)^{-1}R_0^{s,t}(\theta, \alpha, \beta)(z)\|^{n+1} \cdot \|R_0^{s,t}(\theta, \alpha, \beta)(z)\| \cdot \|R_0^{s,t}(\theta, \alpha, \beta)(z)\|^{1/2}W^s_{\theta,\alpha,\beta}|R_0^{s,t}(\theta, \alpha, \beta)(z)|^{1/2}\|n

\leq \|R_0^{s,t}(\theta, \alpha, \beta)(z)\| \left(\frac{60|e_0|C_3}{\delta|\sin(\nu)|}\right)^n.$$  \hspace{1cm} (5.120)

This proves the convergence of the Neumann series and, by Lemma 5.1, the bound in (5.118).
Once we know that \( R^{s,t}(\theta, \alpha, \beta)(z) \) exists for \( z \in A^s(\theta) \), we can define the projections

\[
P^{s,t}(\theta, \alpha, \beta) := \frac{i}{2\pi} \int_{\gamma_{\theta}^{s,t}} R^{s,t}(\theta, \alpha, \beta)(z) \, dz,
\]

\[
P^{s,t}_0(\theta, \alpha, \beta) := \frac{i}{2\pi} \int_{\gamma_{\theta}^{s,t}} R^{s,t}_0(\theta, \alpha, \beta)(z) \, dz,
\]

where \( \gamma_{\theta}^{s,t} : [0, 2\pi) \to \mathbb{C} \) is the curve given by

\[
\gamma_{\theta}^{s,t}(t) := e_1 + (|\sin(\nu)|/2) se^{-it}.
\]

(5.121)

Lemma 5.1 and Eqs. (5.119) and (5.120) imply that

\[
\| R^{s,t}(\theta, \alpha, \beta)(z) - R^{s,t}_0(\theta, \alpha, \beta)(z) \| \leq \frac{|\sin(\nu)|}{4} \frac{1}{|z - e_1|}.
\]

(5.123)

Integrating over \( \gamma_{\theta}^{s,t} \) we obtain

\[
\| P^{s,t}(\theta, \alpha, \beta) - P^{s,t}_0(\theta, \alpha, \beta) \| \leq \frac{|\sin(\nu)|}{4}.
\]

(5.124)

As

\[
P^{s,t}_0(\theta, \alpha, \beta) = P_{at,1}(\theta, \alpha, \beta) \otimes \Omega^{s,t},
\]

(5.125)

(3.61) implies

\[
1 - \frac{2}{8} \leq \| P^{s,t}_0(\theta, \alpha, \beta) \| \leq 1 + \frac{2}{8}.
\]

(5.126)

Hence, using (5.124) and that \( |\nu| < \frac{1}{120} \), we have that

\[
1 - \frac{2}{8} - \frac{1}{120} \leq \| P^{s,t}(\theta, \alpha, \beta) \| \leq 1 + \frac{2}{8} + \frac{1}{120}.
\]

(5.127)

For \( \beta = 0 \), \( P_{at}(0, \alpha, 0) \) is an orthogonal projection and its norm is, therefore, 1. Using (3.61) we can conclude that

\[
1 - \frac{1}{8} - \frac{1}{120} \leq \| P^{s,t}(\theta, \alpha, 0) \| \leq 1 + \frac{1}{8} + \frac{1}{120}.
\]

(5.128)

**Remark 5.4.** By Theorem 3.3 and (5.125), \( e_1 \) is the only point in the spectrum of \( H_{0}^{s,t}(\theta, \alpha, \beta) \) that belongs to \( E^s(\theta) \) (see (5.102)) and it is a simple eigenvalue.

It follows from (5.124) that the range of the projection \( P^{s,t}(\theta, \alpha, \beta) \) is one-dimensional. Thus, by (5.118), there exists a unique point in the spectrum of \( H^{s,t}(\theta, \alpha, \beta) \) that belongs to \( E^s(\theta) \) and it is a simple eigenvalue, we denote it by \( E^{s,t}(\theta) \). Clearly, by Theorem 3.3, \( E^{s,t}(\theta) \) is contained in the disk \( B^C_{(|\sin(\nu)|/2)s/4}(e_1) \).
Corollary 5.5. Suppose that \( \theta = \pm i \nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \), and that \( \theta \) satisfies (3.57) and (4.90). Suppose furthermore that \( \alpha \) satisfies (5.117) and that \( \alpha \leq a \) and \( |\beta| \leq b \). Then for every \( z \in \mathcal{E}^s(\theta) \),

\[
\| R^{s,t}(\theta, \alpha, \beta)(z) \overline{P}^{s,t}(\theta, \alpha, \beta) \| \leq \frac{100|e_0| C^{3.71}}{\delta |\sin(\nu)| \left( |\sin(\nu)|/2 \right)^{s}} ,
\]

(5.129)

where \( \overline{P}^{s,t}(\theta, \alpha, \beta) := 1 - P^{s,t}(\theta, \alpha, \beta) \).

Proof.

Let \( \psi, \tilde{\psi} \) belong to \( \mathcal{H}^{s,t} \). The function

\[
f(z) := \langle \psi | R^{s,t}(\theta, \alpha, \beta)(z) \overline{P}^{s,t}(\theta, \alpha, \beta) \tilde{\psi} \rangle
\]

is analytic in \( \mathcal{E}^s(\theta) \). Hence, by the maximum modulus principle and Theorem 5.3, we have that

\[
\max_{z \in B_{3.71}[\sin(\nu)]/2(0)} |f(z)| \leq \frac{60|e_0| C^{3.71}}{\delta |\sin(\nu)| \left( |\sin(\nu)|/2 \right)^{s}} \|\psi\|_{\mathcal{H}^{s,t}} \|\overline{P}^{s,t}(\theta, \alpha, \beta) \tilde{\psi}\|_{\mathcal{H}^{s,t}},
\]

which proves (5.129) for \( z \in B_{3.71}[\sin(\nu)]/2(0) \) (we use (5.127)). The rest follows from Theorem 5.3. \( \square \)

In Remark 5.4 we establish the existence of an eigenvalue \( E^{s,t}(\theta) \) of the infrared-regularized Hamiltonian \( H^{s,t}(\theta, \alpha, 0) \). We claim that this eigenvalue is a resonance, i.e., that its imaginary part is strictly negative. This is the content of the next Theorem which is proven in Section 9.2.

Theorem 5.6. Suppose that \( \theta = \pm i \nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \), and that \( \theta \) satisfies (3.57) and (4.90) and that \( \beta = 0 \). Suppose furthermore that \( \alpha \leq a \) satisfies (5.117) and that \( |\sin(\nu)|/2s \leq 1 \) and \( s = \alpha^\nu \) for some \( \nu \in (0, 2) \).

Let (see (9.260))

\[
\tilde{E}_I := -\pi \int_{S^2} dS \| P_{a,0}(0, \alpha, 0)|e_1 - e_0| \cdot w_{1,0}(0, \alpha, 0)(x, \frac{k}{|k|}|e_1 - e_0|)\psi_0 \|^2,
\]

where \( S^2 \) is the sphere and \( \psi_0 \) is a unit eigenvector of \( P_{a,1}(0, \alpha, 0) \).

There is a constant \( C^{9.352} \) such that

\[
|\tilde{E}_I - \text{Im}(E^{s,t}(\theta))| \leq 4C^{9.352} \alpha^{3(\alpha^\nu) [\log(\alpha^\nu)] + \alpha^\nu + \alpha^{2\nu} + \alpha^{(3-\nu)/2}(1 + \alpha^{(3-\nu)/2})^3}
\]

(5.131)

The explicit value of the constant \( C^{9.352} \) is written in (9.352).
We conclude this section by collecting the main results. They establish the existence of resonances for the infrared-regularized Hamiltonian as well as some estimates on the resolvent. In the next section we remove the infrared cut off. We do it step by step using an inductive argument. The induction basis is the content of the next theorem, where we make use of the previous results.

**Theorem 5.7.** Suppose that $\theta = \pm iv$, with $v \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha$ satisfies (5.117) that $s$ satisfies (5.101) and that $\alpha \leq a$ and $|\beta| \leq b$.

We assume also that $(|\sin(\nu)|/2)s \leq 1$ and $s = \alpha^\nu$ for some $\nu \in (0, 2)$. Then there is only one point $(E^{s,t}(\theta))$ in the spectrum of $H^{s,t}(\theta, \alpha, \beta)$ included in the set $E^s(\theta)$ (see (5.102)). It is a simple eigenvalue and for $\beta = 0$ its imaginary part is estimated by (5.131). The distance of $E^{s,t}(\theta)$ to $e_1$ is estimated by

$$|E^{s,t}(\theta) - e_1| \leq \frac{(|\sin(\nu)|/2)s}{4},$$

(5.132)

see (5.105).

The Riesz projection (see (5.121)) of the Hamiltonian corresponding to the eigenvalue $E^{s,t}(\theta)$ satisfies the Bounds (5.124) (5.127) and (5.128).

For any $z \in E^s(\theta)$ the resolvent operator $(5.116)$ satisfies the following bound:

$$\|R^{s,t}(\theta, \alpha, 0)(z)\bar{P}^{s,t}(\theta, \alpha, 0)\| \leq \frac{1000|e_0|C_{3.71}}{\delta|\sin(\nu)|} \left(\frac{1}{(|\sin(\nu)|/2)s + |z - E^{s,t}(\theta)|}\right),$$

(5.133)

where $\bar{P}^{s,t}(\theta, \alpha, 0) := 1 - P^{s,t}(\theta, \alpha, 0)$.

**Proof:** The only results not proven yet are Eqs. (5.132) and (5.133). Eq. (5.132) follows from (5.103) and (5.118). Eq. (5.133) follows from Theorem 5.3 (5.128) and (5.132) for $|z - e_1| \geq (|\sin(\nu)|/2)s$, and from Corollary 5.5 and (5.132) for $|z - e_1| \leq (|\sin(\nu)|/2)s$. ⊓⊔

### 6 Resonances for the Non-Regularized Hamiltonian: The Infrared Limit

This is the main section of the paper. Here we prove the existence of resonances (see Section 6.4.1), we provide an explicit formula for the imaginary part of the resonant eigenvalue up to order $\alpha^3$ (see Section 6.4.2). We
prove also that the resonant eigenvalue is non-degenerate (see Section 6.4.3),
we give estimates for the resolvent operator and the resolvent set (see Sec-
tion 6.4.4) and prove exponential decay of eigenfunctions (See The orem 6.5).

Most of the conclusive results of this section are stated and proved in
Sections 6.4.1-6.4.4 but the core of our proofs relies on the results of Sec-
tion 6.3.3 which is the main technical ingredient of our paper.

To accomplish our proofs we use an inductive argument. The induction
basis is Theorem 5.7. We take the infrared cut off to zero step by step
inductively (Section 6.3.3).

6.1 The Feshbach Map

In all statements of this section we suppose that \( \alpha \leq a \) and \( |\beta| \leq b \).

We present here an appropriate Feshbach map that is a useful too l for
our proofs. The proof that it is well defined as well as some basic estimates
are done in a similar way as in Section 6.2 where complete detailed proofs
are included. Some of the proofs of the basic results are based on the proofs
of Section 6.2. In these cases we refer to Section 6.2 to follow the ar gument.

We fix some parameters \( s' \), \( s \) and \( t \), with \( s' \leq s < t \leq \infty \). We define the
following projection on \( H_{s,t} \) (see (3.60))

\[
\mathcal{P} := \mathcal{P}_\beta := \mathcal{P}(\theta) := P_{\alpha,1}(\theta, \alpha, \beta) \otimes 1_{F_{s,t}} ,
\]

\[
\overline{\mathcal{P}} := \overline{\mathcal{P}_\beta} := \overline{\mathcal{P}(\theta)} := 1 - \mathcal{P}(\theta) .
\]

We denote by

\[
\dot{H} := \dot{H}_\beta := H_{s,t}(\theta) \otimes 1_{F_{s',s}} + 1_{F_{s,t}} \otimes \dot{H}_{s',s}
\]

and by

\[
\dot{H}_{\overline{\theta}} := \dot{H}_{\overline{\theta}_\beta} := \overline{\mathcal{P}} \dot{H} \overline{\mathcal{P}} .
\]

We define (formally) the Feshbach map corresponding to the projection (6.1)

\[
\mathcal{F}_\mathcal{P} := \mathcal{F}_{P,\beta} := \mathcal{F}_\mathcal{P}(\dot{H} - z) := \mathcal{P}(\dot{H} - z)\mathcal{P}
\]

\[
-\mathcal{P} \dot{H} \overline{\mathcal{P}} (\dot{H}_{\overline{\theta}} - z)^{-1} \overline{\mathcal{P}} \dot{H} \mathcal{P} .
\]

If \( \alpha \) is sufficiently small, \( \mathcal{F}_\mathcal{P} \) defines a closed operator. Its importance lies
in the facts that \( z \in \sigma(H_{s,t}(\theta)) \) if and only if \( 0 \in \sigma(\mathcal{F}_\mathcal{P}) \) and that there are
explicit formulae for the resolvent of each one of those operators in terms of
the other as well as for the eigenvectors. We will state precisely and prove
this facts in the remaining of this sub-section.
Lemma 6.1. Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha^{3/2} \leq \frac{1}{2^{9.289}}$. Then $\mathcal{E}_{\text{sat}/2}(\theta)$ is contained in the resolvent set of $\dot{H}_{\mathcal{F}}$ and for every $\mu \in \mathcal{E}_{\text{sat}/2}(\theta)$,

$$
\| (\dot{H}_{\mathcal{F}} - \mu)^{-1}\mathcal{P} \| \leq \min \left( \frac{32C^{3.71}}{|\sin(\nu)|\delta}, \frac{40|e_0|C^{3.71}}{|\sin(\nu)|\delta} \right) \frac{1}{|z - e_0|}. \tag{6.5}
$$

Proof: The proof is similar to the one of Lemma 9.2, here we use (5.110) and (5.113).

Lemma 6.2. Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha^{3/2} \leq \frac{1}{2^{9.289}}$. Then for every $z \in \mathcal{E}_{\text{sat}/2}(\theta)$,

$$
\| \mathcal{P}(\dot{H}_{\mathcal{F}} - z)^{-1}\mathcal{P}\dot{H}_{\mathcal{F}}\mathcal{P} \| < 4\alpha^{3/2}C^{9.289}, \tag{6.6}
$$

$$
\| \mathcal{P}\dot{H}_{\mathcal{F}}(\dot{H}_{\mathcal{F}} - z)^{-1}\mathcal{P} \| < 4\alpha^{3/2}C^{9.289},
$$

Proof: The proof of (6.6) is similar to the one of (6.5), here we use Lemma 5.2 instead of Lemma 5.1.

Using (3.61), (4.87) and (6.6) we obtain for $\rho = 1$

$$
\| (\mathcal{P}W_{s,t}(\theta,\alpha,\beta)\mathcal{P} - \mathcal{P}W_{s,t}(\theta,\alpha,\beta)\mathcal{P}(\dot{H}_{\mathcal{F}} - z)^{-1}W_{s,t}(\theta,\alpha,\beta)\mathcal{P})\phi \|_{H^{s,t}} \leq (2 + 4\alpha^{3/2}C^{9.289})6C^{8.241}\alpha^{3/2}
$$

$$
\cdot \left( \| \mathcal{P}(H_{0}^{s,t}(\theta,\alpha,\beta) + 1)\mathcal{P}\phi \|_{H^{s,t}} + \| \mathcal{P}\phi \|_{H^{s,t}} \right). \tag{6.7}
$$

As $\mathcal{P}(H_{0}^{s,t}(\theta,\alpha,\beta) + 1)\mathcal{P}$ is closed, we conclude (see Theorem 1.1 page 190 of [11]) that $\mathcal{F}_{\mathcal{P}}$ is closed for any $\alpha$ such that $(2 + 4\alpha^{3/2}C^{9.289})6C^{8.241}\alpha^{3/2} < 1$. Actually the domain of $\mathcal{F}_{\mathcal{P}}$ is the same as the domain of $\mathcal{P}H_{0}^{s,t}(\theta,\alpha,\beta)\mathcal{P}$.

Lemma 6.3. Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha^{3/2} \leq \frac{1}{2^{9.289}}$. Then for every
\[ z \in \mathcal{E}^{\delta_{\nu}/2}(\theta) \] and every \( \rho \in \mathbb{C} \),

\[
\|(H_0^{s,t}(\theta, \alpha, \beta) + \rho)\overline{P}(\hat{H}_{\overline{\tau}} - z)^{-1}\overline{P}\|
\leq 6 \min \left( (1 + |\rho| + |\epsilon_0|)^{32\left(\frac{3.71}{\sin(\nu)|\delta|}\right)} + 1 + (|\rho| + |\epsilon_0|)^{32\left(\frac{3.71}{\sin(\nu)|\delta|}\right)} + \frac{40|\epsilon_0|\left(\frac{3.71}{\sin(\nu)|\delta|}\right)}{\sin(\nu)|\delta|} \right).
\]  

(6.8)

**Proof:** The result follows from Eq. (5.115), the Neumann expansion (9.295) with \( \mathcal{P} \) instead of \( \overline{\mathcal{P}} \) and (6.5).

\[ \square \]

**Lemma 6.4.** Suppose that \( \theta = \pm i \nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \), and that \( \theta \) satisfies (3.57) and (4.96). Suppose furthermore that \( \alpha^{3/2} \leq 1 / 2^{289} \) and that \( (2 + 4\alpha^{3/2}) < 1 \). Then for any \( z \in \mathcal{E}^{\delta_{\nu}/2}(\theta) \), the operator \( \mathcal{F}_P(\hat{H} - z) \) (see 6.4) is closed and the following holds true:

(i) \( \mathcal{F}_P(\hat{H} - z) \) is invertible on \( \mathcal{P}\mathcal{H}^{s,t} \) if and only if \( H^{s,t}(\theta) - z \) is invertible on \( \mathcal{H}^{s,t} \) and the following formulas hold

\[
(\mathcal{F}_P)^{-1} := \mathcal{P}(\hat{H} - z)^{-1}\mathcal{P}
\]  

(6.9)

\[
(\hat{H} - z)^{-1} = \left[ \mathcal{P} - \overline{P}(\hat{H}_{\overline{\tau}} - z)^{-1}W^{s,t}(\theta, \alpha, \beta)\overline{P} \right].
\]  

(6.10)

(ii) If \( \hat{H}\psi = z\psi \) for some eigenvector \( 0 \neq \psi \in \mathcal{H}^{s,t} \) and \( z \in \mathcal{E}^{\delta_{\nu}/2}(\theta) \), then \( 0 \neq \mathcal{P}\psi \in \mathcal{P}\mathcal{H}^{s,t} \) solves \( \mathcal{F}_P\mathcal{P}\psi = 0 \) and

\[
\|\mathcal{P}\psi\| \geq \frac{\|\psi\|}{1 + \|\overline{P}(\hat{H}_{\overline{\tau}} - z)^{-1}\overline{P}\mathcal{H}\|}.
\]  

(6.11)

(iii) If \( \mathcal{F}_P\phi = 0 \) for some eigenvector \( 0 \neq \phi = \mathcal{P}\phi \in \mathcal{P}\mathcal{H}^{s,t} \), then the vector \( 0 \neq \psi \in \mathcal{H}^{s,t} \), defined by \( \psi := [\mathcal{P} - (\hat{H}_{\overline{\tau}} - z)^{-1}\overline{P}W^{s,t}(\theta, \alpha, \beta)\overline{P}]\phi \), solves \( \hat{H}\psi = z\psi \).

(iv)

\[ \dim \ker(\hat{H} - z) = \dim \ker \mathcal{F}_P. \]  

(6.12)
Proof: See the proof of Theorem II.1 [17].

\[ \text{Theorem 6.5 (Exponential Boundedness of the Eigenvalues). Suppose that } \theta = \pm i \nu, \text{ with } \nu \in \mathbb{R} \setminus \{0\}, \text{ and that } \theta \text{ satisfies } (3.57) \text{ and } (4.96). \text{ Suppose furthermore that } \alpha^{3/2} \leq \frac{1}{16(3.289)}(2 + 4\alpha^{3/2}C_{9.289}6C_{8.241})^{3/2} < 1, \text{ that } \alpha \leq a \text{ and } \beta = 0 \text{ (see 3.54). If } z \in E_{\delta/2}(\theta) \text{ is an eigenvalue of } \hat{H}_0 \text{ and } \psi \neq 0 \text{ a corresponding eigenfunction, then}
\]
\[
\begin{align*}
\|e^{\beta(x)}\psi\| &\leq 12C_{3.78} \\
\|\psi\| &\leq 3C_{3.78}
\end{align*}
\]

The constant $C_{3.78}$ defined in (3.78) does not depend on $\alpha$, $\beta$, and $\theta$, nor on $s'$ and $s$ and $t$ (see the beginning of this section).

\[ \text{Proof:} \]

By Lemma 6.4, the eigenvectors on the kernel of $\hat{H}_0 - z$ are of the form $\psi = [P_0 - (\hat{H}_{\mathcal{P}_0,0} - z)^{-1}P_0W^{s,t}(\theta, \alpha, 0)P_0]\phi$, where $\mathcal{F}_P\phi = 0$ (see (6.1)-(6.4)). We have that
\[
\begin{align*}
\|e^{\beta(x)}\psi\| &= \|e^{\beta(x)}[P_0 - (\hat{H}_{\mathcal{P}_0,0} - z)^{-1}P_0W^{s,t}(\theta, \alpha, 0)P_0]e^{-\beta(x)}e^{\beta(x)}\phi]\nonumber \\
&= \|[P_{-\beta} - (\hat{H}_{\mathcal{P}_{-\beta},0} - z)^{-1}P_{-\beta}W^{s,t}(\theta, \alpha, -\beta)P_{-\beta}]e^{\beta(x)}P_0\phi]\nonumber \\
&\leq \|[P_{-\beta} - (\hat{H}_{\mathcal{P}_{-\beta},0} - z)^{-1}P_{-\beta}W^{s,t}(\theta, \alpha, -\beta)P_{-\beta}]\| \cdot \|e^{\beta(x)}P_0\| \cdot \|\phi\| \nonumber \\
&\leq 3C_{3.78}\|\phi\| 
\end{align*}
\]

where we used (3.61), (3.76) and (6.6). Notice that $\phi \in \mathcal{P}_0\mathcal{H}^{s',t}$ implies that $\phi = \mathcal{P}_0\phi$. By (6.6) we have that
\[
\|\psi\| \geq (1 - \frac{1}{4})\|\phi\| \geq \frac{1}{4}\|\phi\|. 
\]

(6.13) is a direct consequence of (6.14) and (6.15).
6.2 The Sequence of Infrared-Regularized Hamiltonians (Notation, Definitions and Assumptions)

Assumptions 6.1. In this section we suppose that the parameters $\theta$, $\alpha$ and $\beta$ satisfy the following:

- $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$ and $\theta$ satisfies (3.57) and (4.90).
- $\alpha \leq a$ (see 3.54) and it satisfies (5.117) with $\sigma_0$ instead of $s$ (see $\sigma_0$ below),

$$\alpha^{3/2} \leq \frac{1}{16} \frac{1}{3.289} + \frac{1}{128} \frac{1}{8.246} + \frac{1}{64(1+8(3.71^{1/4})C.7)} \frac{1}{A.7}$$

$$+ \frac{1}{24(0.241(1+8(3.71^{1/4})))^{6}} \frac{1}{8.241}$$

- $|\beta| \leq b$ (see 3.54).

Next we define an important constant that in some sense includes all the constants appearing in the error bounds in our estimations. This constant plays a fundamental role in our inductive procedure. It depends only on the parameters of the physical atom Hamiltonian (without field) and $\theta$. It does not depend on $\alpha$ and $\beta$ satisfying Assumptions 6.1.

Definition 6.6. We denote by $C_{6.16}$ the following constant

$$C_{6.16} := \frac{1000|e_0(3.71^{1/4})C.7)}{\delta |\sin(\nu)|} + C_{6.110} + C_{6.139} + C_{6.150}$$

$$+ C_{6.155} + C_{6.197} + C_{6.121} + C_{6.112} + 1. \quad (6.16)$$

Explicit values of the constants in the right hand side of Eq. (6.16) are written in (3.71) and (6.110), (6.139), (6.150), (6.155) and (6.197) below. They depend only on the physical atom Hamiltonian (with $\alpha = \beta = \theta = 0$) and $\theta$, for $\alpha$ and $\beta$ satisfying Assumptions 6.1.

In the following definition we introduce a sequence of numbers that represent the infrared infrared-cut off. Taking the sequence index to infinity corresponds to removing the cut off in our induction scheme.

Definition 6.7. We fix two constants $(\sigma_0, B)$ satisfying the following properties:

$$C_{6.16}^1 \sigma_0^{1/2} \leq \frac{1}{100}, \quad C_{6.16}^2 \sigma_0^{1/2} \leq \frac{1}{100}, \quad \frac{\alpha^{3/2}}{8} \leq 1,$$

$$C_{6.16}^2 \sigma_0 \leq \frac{1}{10}(|\sin(\nu)|/2), \quad \sigma_0 = \alpha^\nu, \quad \sigma_0 \leq \frac{1}{2}\delta_{at}, \quad (6.17)$$
for some $v \in (0, 2)$. Notice that $\sigma_0 < 1$, $B < 1$ are consequences of \[6.17\].

We define a decreasing sequence $(\sigma_n)_{n=0}^{\infty}$ by setting

$$\sigma_n := B^n \sigma_0, \quad \sigma_\infty = 0.$$  \eqref{6.18}

### 6.2.1 Scale of Hamiltonians

In this section we take $\beta = 0$ whenever we do not write $\beta$ explicitly.

For any $\sigma_m < \sigma_n$, the disjoint union

$$K^\sigma_m = K^\sigma_m \cup K^\sigma_n.$$  \eqref{6.19}

Gives rise to the isomorphisms

$$\mathcal{F}^\sigma_m \cong \mathcal{F}^\sigma_n \otimes \mathcal{F}^\sigma_m, \sigma_n,$$  \eqref{6.20}

$$\mathcal{H}^\sigma_m \cong \mathcal{H}^\sigma_n \otimes \mathcal{F}^\sigma_m, \sigma_n.$$  \eqref{6.21}

We simplify our notation using

$$\bar{H}^n := \bar{H}(\theta) := H^\sigma_n(\theta, \alpha, 0), \quad R(z) = \frac{1}{\bar{H} - z},$$

$$\bar{H}_0^n := \bar{H}_0^n(\theta) := H_0^\sigma_n(\theta, \alpha, 0), \quad R_0(z) = \frac{1}{\bar{H}_0 - z},$$  \eqref{6.22}

To compare the Hamiltonians $\bar{H}^n$ and $\bar{H}^{n+1}$ at successive energy scales we introduce the Hamiltonians,

$$\tilde{H}^n := H \otimes 1_{\mathcal{F}^\sigma_{n+1, \sigma_n}} + e^{-\theta} 1_{\mathcal{H}^\sigma_n} \otimes \tilde{H}^\sigma_{n+1, \sigma_n}, \quad \tilde{R}^n(z) = \frac{1}{\tilde{H}^\sigma - z},$$

$$\tilde{H}^\infty := H \otimes 1_{\mathcal{F}^\sigma_{\infty, \sigma_n}} + e^{-\theta} 1_{\mathcal{H}^\sigma_n} \otimes \tilde{H}^\sigma_{\infty, \sigma_n}, \quad \tilde{R}^\infty(z) = \frac{1}{\tilde{H}^\sigma_{\infty} - z}.$$  \eqref{6.23}

We introduce the velocity operator acting on $\mathcal{H}^\sigma_n$ by the formula

$$\bar{v}^\sigma_n(\theta) := \bar{v}^\sigma_n := -ie^{-\theta} \nabla - A^\sigma_n(\theta),$$  \eqref{6.24}

see \[2.44\], and the operator\(\bar{v}^\sigma_n\)(we omit the variable $t$ if it is infinity) Then we have (see \[2.48\]),

$$\bar{H}^n = (\bar{v}^\sigma_n)^2 + b^\sigma_n + e^{-\theta} \bar{H}^\sigma_n + V(\theta, \alpha, 0).$$  \eqref{6.25}
We identify $\vec{\nu}(\theta)^{\sigma_n}$ with $\vec{\nu}(\theta)^{\sigma_n} \otimes 1_{\sigma_m,\sigma_n}$, which is acting on $\mathcal{H}^{\sigma_m}$ for any $m \geq n + 1$. Then we have,

$$\vec{\nu}(\theta)^{\sigma_m} = \vec{\nu}(\theta)^{\sigma_n} - A^{\sigma_m,\sigma_n}(\theta). \quad (6.26)$$

We define the operator (for $m > n$)

$$W_{m,n}(\theta) := W_{m,n} := (\vec{\nu}(\theta)^{\sigma_m})^2 - (\vec{\nu}(\theta)^{\sigma_n})^2 + b^{\sigma_m} - b^{\sigma_n} \cdot A^{\sigma_m,\sigma_n}(\theta) + (A(\theta)^{\sigma_m,\sigma_n})^2 \quad (6.27)$$

With this notation we have that,

$$H_{n+1} = \tilde{H}^n + W_{n+1}. \quad (6.28)$$

Now we define

$$G^{s,t}_\theta := G^{s,t}_\theta(x) := (G^{s,t}_{\theta,1}(x), G^{s,t}_{\theta,2}(x), G^{s,t}_{\theta,3}(x))$$

$$:= G^{s,t}(\theta)(x, \cdot) - e^{-\theta}Q^{s,t}(x, \cdot), \quad (6.29)$$

where the central dot ($\cdot$) represents the variable $k$ and (see also 8.232)

$$Q^{s,t}(\theta)(x, k) := G^{s,t}(\theta)(0, k) \cdot \left(\eta(|x||k|)e^{\theta|x}\right). \quad (6.30)$$

Then we have that

$$A^{s,t}(\theta) = a^*(G^{s,t}_\theta) + a(G^{s,t}_\theta). \quad (6.31)$$

As the variable $\theta$ is fixed, we will omit it in our notations unless we need to write it explicitly.

### 6.3 Inductive Scheme

We remember that the energy $E^{s,t}(\theta)$ is defined in Remark 5.4 and the sets $\mathcal{E}^{s,t}(\theta)$ are defined in (5.102). For every $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $\sigma_n$ is defined in (6.18).

Our selection of $\alpha$ and $\theta$ assure that there exist only one point on the spectrum of $\tilde{H}$ in the set $\mathcal{E}^{\sigma_n}(\theta)$ and that it is a simple eigenvalue (see Remark 5.4). This is possible only for $\sigma_0$ because the selection of the alpha that fulfills the hypothesis of Lemma 5.3 goes to zero as $s$ goes to infinity (see 5.117), thus only $\alpha = 0$ would assure the existence of a unique simple
eigenvalue of $H$ in $E^\sigma_n(\theta)$.

We wish to find simple eigenvalues of $H$ for every $n \in \mathbb{N}$ and then take the limit when $n$ goes to infinity in order to find an eigenvalue for $\overline{H}$.

We find the eigenvalues recursively and prove by induction that they satisfy certain properties.

We construct inductively (and simultaneously) a sequence of numbers \( \{E_j\}_{j \in \mathbb{N} \cup \{0\}} \), a sequence of subsets of the complex plane \( \{\mathcal{E}_j\}_{j \in \mathbb{N} \cup \{0\}} \) and a sequence of operators \( \{P_j\}_{j \in \mathbb{N} \cup \{0\}} \) that satisfy certain properties that we specify below.

### 6.3.1 Induction Basis

Here we define the number $E_0$, the set $\mathcal{E}_0$ and the operator $P_0$. They satisfy some properties that we list below (see Theorem 5.7).

1. \[
E_0 = E^\sigma_0(\theta) .
\]

$E_0$ is a simple eigenvalue of $H$. Remember that the parameter $t$ does not appear if it is infinity.

2. \[
\mathcal{E}_0 := \begin{cases} 
E^\sigma_0(\theta) \setminus (E_0 + \mathbb{R} - i((|\sin(\nu)|/2)\sigma_0/2, \infty)), & \text{if } \text{Im}(\theta) > 0 , \\
E^\sigma_0(\theta) \setminus (E_0 + \mathbb{R} + i((|\sin(\nu)|/2)\sigma_0/2, \infty)), & \text{if } \text{Im}(\theta) < 0 .
\end{cases}
\]

By Theorem 5.7, $E_0$ is the only point in the spectrum of $H$.

3. \[
P_0 := P^\sigma_0(\theta, \alpha, 0) .
\]

4. Theorem 5.7 and (6.16) imply that
\[
\| R(z) \overline{P}_0 \| \leq C_{6.16} \frac{1}{|\sin(\nu)/2|\sigma_0 + |z - E_0|} ,
\]

where $\overline{P}_0 = 1 - P_0$. 

Remark 6.8. Theorem 5.7 implies that $P_0$ satisfies

$$
\| P_{at,1}(\theta, \alpha, 0) - P_0 \| \leq \frac{|\sin(\nu)|}{4},
$$

$$
1 - \frac{1}{8} - \frac{1}{120} \leq \| P_0 \| \leq 1 + \frac{1}{8} + \frac{1}{120}.
$$

(6.36)

6.3.2 Induction Hypothesis

We suppose that we have already defined the numbers $E_m$, the sets $\mathcal{E}_m$ and the operators $P_m$ for every $m \leq n$. We suppose furthermore that they satisfy the properties listed below.

1a. $E_m$ is a simple eigenvalue of $H$.

1b.

$$
|E_m - E_{m-1}| < C_{6.16}^m \frac{3/2}{6.16} (\sigma_{m-1})^2.
$$

(6.37)

2.

$$
\mathcal{E}_m := \begin{cases}
\mathcal{E}^e_0(\theta) \setminus (E_m + \mathbb{R} - i((|\sin(\nu)|/2)\sigma_m/2, \infty)), & \text{if } \text{Im}(\theta) > 0, \\
\mathcal{E}^o_0(\theta) \setminus (E_m + \mathbb{R} + i((|\sin(\nu)|/2)\sigma_m/2, \infty)), & \text{if } \text{Im}(\theta) < 0.
\end{cases}
$$

(6.38)

$E_m$ is the only point in the spectrum of $H$ in $\mathcal{E}_m$.

3a.

$$
P_m := \frac{i}{2\pi} \int_{\gamma_m} R(z)dz,
$$

(6.39)

where $\gamma_m : [0, 2\pi] \to \mathbb{C}$ is the curve given by

$$
\gamma_m(t) := E_m + (|\sin(\nu)|\sigma_m)e^{-it}.
$$

(6.40)

3b.

$$
\| P_m - P_{m-1} \otimes P_{\Omega^{(\sigma_m, \sigma_{m-1})}} \| \leq C_{6.16}^{m+2} \sigma_{m-1}^{1/2},
$$

(6.41)

where $P_{\Omega^{(\sigma_m, \sigma_{m-1})}}$ is the projection on the vacuum state $\Omega^{(\sigma_m, \sigma_{m-1})}$.

4.

$$
\| R(z)P_m \| \leq C_{6.16}^{m+1} \frac{1}{(|\sin(\nu)|/2)\sigma_m + |z - E_m|},
$$

(6.42)

where

$$
P_m := 1 - P_m.
$$

(6.43)
Remark 6.9. From (6.17), (6.36), and (6.41) it follows that

\[ 1 - \frac{1}{8} - \frac{1}{120} - \frac{1}{100} \sum_{i=0}^{m-1} \left( \frac{1}{100} \right)^i \leq \|P_m\| \]

\[ \|P_m\| \leq 1 + \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{m-1} \left( \frac{1}{100} \right)^i. \]  

(6.44)

Remarks.

Items 1b. and 3b. do not appear in the induction basis, if we wish to have them then we could define 

\[ E_{-1} = E_0, \quad P_{-1} = P_0 \quad \text{and} \quad \sigma_{-1} = \sigma_0. \]

Once we have defined the value \( E_m \), we can construct the sets \( \mathcal{E}_m \). If we know that the eigenvalue \( E_m \) is the only spectral point of \( H \) in \( \mathcal{E}_m \), then the projection \( P_m \) is well defined. To accomplish the induction step, we have to construct \( E_{n+1} \), to prove that it is simple and the only eigenvalue in \( \mathcal{E}_{n+1} \) and to verify that (6.37), (6.41) and (6.42) are valid.

The following equation, that is a consequence of the induction basis and the induction hypothesis, is used below

\[ \|P_m - P_{n+1}(0,0,0) \otimes P_{\Omega(\sigma_m,\infty)}\| \leq \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{m-1} \left( \frac{1}{100} \right)^i. \]  

(6.45)

6.3.3 Induction Step

The main result in this section is the Theorem below. The proof of this Theorem is the content of this subsection. It is based in Theorems 6.31, 6.34 and 6.36.

**Theorem 6.10.** Suppose that for any \( m \in \mathbb{N} \cup \{0\}, m \leq n \), we have defined the number \( E_m \), the set \( \mathcal{E}_m \) and the operator \( P_m \) that satisfy the properties stated in the induction hypothesis. Then there exists a number \( E_{n+1} \) a set \( \mathcal{E}_{n+1} \) and the operator \( P_{n+1} \) satisfying the same properties.

Proof: The proof is a consequence of Remarks 6.32, 6.35 and 6.37 (which describe Theorems 6.31, 6.34 and 6.36).

Once Theorem 6.10 is proved, then we conclude that the numbers \( E_n \) the sets \( \mathcal{E}_n \) and an operators \( P_n \) are well defined for any \( n \in \mathbb{N} \). Parallel to the sets \( \mathcal{E}_m \) we define

\[ \mathcal{E}_{(m,\infty)} := \begin{cases} \mathcal{E}_0^0(\theta) \setminus (E_m + \mathbb{R} + i(\sigma_m, \infty)), & \text{if } \text{Im}(\theta) > 0, \\ \mathcal{E}_0^0(\theta) \setminus (E_m + \mathbb{R} - i(\sigma_m, \infty)), & \text{if } \text{Im}(\theta) < 0, \end{cases} \]  

(6.46)
\[ \tilde{E}_m = \mathcal{E}^{\sigma_0}(\theta) \setminus (E_m + \mathbb{R} + i\frac{1}{2}\sin(-\nu)(\sigma_{m+1}, \infty)). \] (6.47)

**Corollary 6.11** (from the proof of Theorem 6.10). Suppose that the induction hypothesis are valid. It follows that for every \( z \in \mathcal{E}_{(n, \infty)} \)

\[
\| (1 + |x|^2)^{-1} W_n^\infty \tilde{R}_\infty^n (z) \| \leq C_n^{n+26.16} \sigma_n^{n+1},
\]

\[
\| W_n^\infty \tilde{R}_\infty^n (z) \| \leq C_n^{n+26.16}.
\] (6.48)

Furthermore, \( z \) belongs to the resolvent set of \( \infty H \) and

\[
\| (\infty H - z)^{-1} \| \leq C_n^{n+26.16} \frac{1}{(|\sin(\nu)|/2)\sigma_{n+1} + |z - E_n|}.
\] (6.49)

**Proof:** It follows from Theorem 6.24 and Theorem 6.29, see also (6.197).

We remark that since Theorem 6.10 closes the inductive scheme, then the conclusions of Corollary are valid for any \( n \in \mathbb{N} \).

### 6.3.3.1 Proof of Theorem 6.10

In this paragraph we prove Theorem 6.10. As in the case of the infrared-regularized Hamiltonian, the estimates for the interaction are very important. In the regularized case Theorem 5.2 is the key ingredient for the proof of Theorem 5.7, which establish the existence of resonances. In the first sub paragraph of this paragraph we do estimates for the interaction, which are stated in Theorem 6.24.

In Theorem 5.2, we estimate the interaction with respect to the free (non-interacting) resolvent \( (R_0^{s.t}(\theta, \alpha, \beta)) \). This makes the analysis simple but it has the disadvantage that \( \alpha \) depends on the infrared cut off parameter \( s \) (see 5.117) and that it goes to zero as \( s \) goes to zero, which implies that only the non-interacting case (\( \alpha = 0 \)) is analyzable if we remove the infrared cut-off.

Here we estimate the interaction (6.27) with respect to the resolvent (5.23). This analysis is more delicate than the one of Theorem 5.2 because the resolvent (6.23) already has a piece of interaction. The resolvent (6.23) is very singular and we cannot handle it as in the case of the free resolvent. We need extra (physical) information to control it. We have to make use of the exponential decay (in the atom variable) of the eigenfunctions. To do this,
we restrict our operators to a space that is generated (in the atomic part) by an eigenvector corresponding to the first excited eigenvalue of the atom Hamiltonian (see (6.1)). This is done using the Feshbach map (See (6.4)).

In the second subparagraph we estimate the Feshbach map of our Hamiltonians. The key ingredient is Theorem 6.27 which is the analogous result of Theorem 5.2 and Theorem 6.24. Theorem 6.27 (and Neumann expansions) imply the invertibility of the Feshbach map applied to the Hamiltonian. In the third subparagraph we study the resolvent of the original Hamiltonian using (6.10) and prove (6.41) for the induction step as well as the existence of $E_{n+1}$ and the fact that it is non-degenerate (see Remark 6.32). In the forth subparagraph we prove (6.37) (see Remark 6.35) and conclude the induction step by proving (6.42) in Theorem 6.36, which finish the induction scheme (see Remark 6.37).

Estimates for the Interaction

As in the case of the infrared-regularized Hamiltonian, the estimates for the interaction are very important. In the regularized case Theorem 5.2 is the key ingredient for the proof of Theorem 5.7 which establish the existence of resonances. In this sub paragraph we do estimates for the piece of the interaction with photon energies between $\sigma_{n+1}$ and $\sigma_n$ (see (6.27)). The main result is stated in Theorem 6.24.

To study the operator $W_{n+1}$ we analyze many terms. To simplify our notation we organize some of these terms in different sets which are introduced in the following definition together with some other notations.

**Definition 6.12.** We define the following sets:

$$W_2(\iota, m) = \left\{ (1 + |x|^2)^{-\iota/2} (A_{j}^{\sigma_m,\sigma_n})^* A_{q}^{\sigma_m,\sigma_n}, 1 + |x|^2)^{-\iota/2} (A_{j}^{\sigma_m,\sigma_n})^* A_{q}^{\sigma_m,\sigma_n}, (1 + |x|^2)^{-\iota/2} (A_{j}^{\sigma_m,\sigma_n})^* A_{q}^{\sigma_m,\sigma_n}, (1 + |x|^2)^{-\iota/2} (A_{j}^{\sigma_m,\sigma_n})^* A_{q}^{\sigma_m,\sigma_n} i, j, q \in \{1, 2, 3\} \right\}. \quad (6.50)$$

$$W_1(\iota, m) = \left\{ (1 + |x|^2)^{-\iota/2} b_{\sigma_m,\sigma_n}, (1 + |x|^2)^{-\iota/2} A_{j}^{\sigma_m,\sigma_n} \right\}. \quad (6.51)$$
The constant
\[ C_{6.52} := 40C_i^2(1 + C_{3.79}), \] (6.52)
and the function
\[ \mathcal{G}_\alpha(\sigma, \rho) := \alpha^{3/2}(\sigma^2\rho^{-1/2} + \sigma^{3/2}) \] (6.53)
is repeatedly used in this section.

**Remark 6.13.** Eq. (6.23) implies that \( E_n \) is a simple eigenvalue of \( \tilde{H}^n \). We denote by
\[ \tilde{P}_n = P_n \otimes P_{\Omega^\sigma_{n+1}} \] (6.54)
the projection onto the corresponding eigenvector. It is given by the integral
\[ \tilde{P}_n := \frac{i}{2\pi} \int_{\tilde{\gamma}_n} \tilde{R}^n(z)dz, \] (6.55)
where \( \tilde{\gamma}_n : [0, 2\pi] \to \mathbb{C} \) is the curve
\[ \tilde{\gamma}_n(t) := E_n + \left( |\sin(\nu)|/2 \right) \sigma_{n+1} e^{-it}. \] (6.56)

**Lemma 6.14.** For any \( z \in \tilde{E}_n \)
\[ \| \tilde{R}^n(z) P_n \| \leq \frac{2}{(|\sin(\nu)|/2)} \left( C_{6.16}^{n+1} + 2 \right) \frac{1}{(|\sin(\nu)|/2)\sigma_{n+1} + |z - E_n|}, \] (6.57)
for any \( z \in E_{(n, \infty)} \)
\[ \| \tilde{R}^n(z) \| \leq \frac{2}{(|\sin(\nu)|/2)} \left( C_{6.16}^{n+1} + 2 \right) \frac{1}{(|\sin(\nu)|/2)\sigma_{n+1} + |z - E_n|}, \] (6.58)

**Proof.**
We prove (6.57) and suppose that \( \text{Im}(\theta) > 0 \). The other cases are similar. By functional calculus,
\[ \| \tilde{R}^n(z) P_n \| \leq \| \tilde{R}^n(z) P_n \otimes 1_{H^\sigma_{n+1}} \| + \| \tilde{R}^n(z) P_n \otimes P_{\Omega^\sigma_{n+1}} \| \]
\[ \leq \sup_{r \in \sigma(\tilde{R}^n_{\sigma_{n+1}})} \| \tilde{R}(z - e^{-\theta} r) P_n \| + \sup_{r \in [\sigma_{n+1}, \infty)} \left\| \frac{P_n}{z - E_n + r e^{-\theta}} \right\| \]
\[ \leq \sup_{r \in \sigma(\tilde{R}^n_{\sigma_{n+1}})} \left( C_{6.16}^{n+1} \right) \frac{1}{(|\sin(\nu)|/2)\sigma_{n+1} + |z - E_n|} \]
\[ + \sup_{r \in [\sigma_{n+1}, \infty)} \frac{1}{|z - E_n - r e^{-\theta}|}, \]
where we used (6.42) and (6.44).

It is clear from the definition of \( \tilde{E}_n \) that for \( z \) in this set
\[ (|\sin(\nu)|/2)\sigma_{n+1} \leq |z - E_n - r e^{-\theta}|, \quad \forall r \in [\sigma_{n+1}, \infty). \] (6.60)
We denote by $C_n$ the set
\[ C_n := \{ z = (z_1, z_2) \in \tilde{E}_n : z_1 > \text{Re}(E_n) \} \]
and define the following sets:
\[ L_{n,d} := \{ E_n + de^{i\pi/2 - \theta} - re^{-\theta} : t \in \mathbb{R} \}, \quad (6.61) \]
\[ L_n := \bigcup_{d \geq 0} L_{n,d} \cap C_n. \quad (6.62) \]

By construction
\[ \text{dist}(L_{n,d}, E_n) = d. \quad (6.63) \]

and
\[ |z - E_n| \leq |z - E_n - re^{-\theta}|, \forall z \in \tilde{E}_n \setminus L_n, \forall r \in [\sigma_{n+1}, \infty). \quad (6.64) \]

Let $Z_{1,d}$ be the intersection of $L_{n,d}$ with the line $E_n - \frac{d}{\sin(\nu)}\sigma_{n+1} + \mathbb{R}$ and let $Z_{2,d} = L_{n,d} \cap (E_n + \mathbb{R})$. We define furthermore $Z_{3,d} = E_n + de^{i\pi/2 - \theta}$. It follows that
\[ \sup_{Z \in L_{n,d} \cap C_n} (|Z - E_n|) = |Z_{1,d} - E_n|^2 = d^2 + |Z_{3,d} - Z_{1,d}| \]
\[ = d^2 + (|Z_{3,d} - Z_{2,d}| + |Z_{2,d} - Z_{1,d}|)^2 = d^2 + \left( \frac{d}{\tan(\nu)} + \frac{1}{2}\sigma_{n+1} \right)^2 \quad (6.65) \]

Eqs. (6.60), (6.63) and (6.65) imply that
\[ \frac{|z - E_n|}{|z - E_n - re^{-\theta}|} \leq \left( \left( \frac{\cos(\nu)}{\sin(\nu)} + \frac{1}{\sin(\nu)} \right)^2 + 1 \right)^{1/2} \leq \frac{2}{\sin(\nu)}, \quad (6.66) \]
for $r \in [\sigma_{n+1})$ and $z \in L_n$, which together with (6.64) implies that
\[ \frac{1}{|z - E_n - re^{-\theta}|} \leq \frac{2}{\sin(\nu)} \frac{1}{|z - E_n|}, \quad (6.67) \]
for every $r \in [\sigma_{n+1})$ and $z \in \tilde{E}_n$.

Eqs. (6.60) and (6.67) imply that
\[ \frac{1}{|z - E_n - re^{-\theta}|} \leq \frac{2}{|\sin(\nu)|} \frac{1}{|z - E_n|}, \forall z \in \tilde{E}_n, \forall r \in [\sigma_{n+1}, \infty). \quad (6.68) \]

Finally Eq. (6.57) follows from (6.59) and (6.68).

In the next Lemma we use (2.44), (6.24) and (6.31). Remember that we omit the variable $\theta$. \qed
Lemma 6.15. Denote by $P^0 = 1$ and $P^1 = P$. For every $t \in \{0, 1\}$, $\rho > 0$, $m > n$ (including $\infty$) and any $w \in W_j(t, m)$.

$$\|w(\hat{H}^{\sigma_m, \sigma_n} + \rho)^{-j/2}P^{1-t}\| \leq C_{0, \frac{3}{2}, \rho}(\sigma_n, \rho)^j$$  \hspace{1cm} (6.69)

**Proof.**

We consider first the term $(A^{\sigma_m, \sigma_n})^2$ and $t = 0$. By (6.31),

$$(A^{\sigma_m, \sigma_n})^2 = a^*(G_{\theta}^{\sigma_m, \sigma_n}) \cdot a^*(G_{\theta}^{\sigma_m, \sigma_n})$$

$$+ a(G_{\theta}^{\sigma_m, \sigma_n}) \cdot a(G_{\theta}^{\sigma_m, \sigma_n}) + a^*(G_{\theta}^{\sigma_m, \sigma_n}) \cdot a(G_{\theta}^{\sigma_m, \sigma_n}) \cdot a^*(G_{\theta}^{\sigma_m, \sigma_n})$$  \hspace{1cm} (6.70)

We analyze the term $a(G_{\theta}^{\sigma_m, \sigma_n}) \cdot a(G_{\theta}^{\sigma_m, \sigma_n})$, the other terms can be treated analogously.

By Lemma 4.1 and Remark A.3 we have

$$\|a(G_{\theta}^{\sigma_m, \sigma_n}) \cdot a(G_{\theta}^{\sigma_m, \sigma_n})\| \leq \|a(G_{\theta}^{\sigma_m, \sigma_n} \cdot a(G_{\theta}^{\sigma_m, \sigma_n})\|$$

$$\leq \|a(G_{\theta}^{\sigma_m, \sigma_n}(1 + |x|^2)^{-1/2}) \cdot a(G_{\theta}^{\sigma_m, \sigma_n}(1 + |x|^2)^{-1/2})\|$$

$$\cdot (1 + |x|^2)P$$  \hspace{1cm} (6.71)

$$\leq \|G_{\theta}^{\sigma_m, \sigma_n}(1 + |x|^2)^{-1/2}\|\|G_{\theta}^{\sigma_m, \sigma_n}(1 + |x|^2)^{-1/2}\|$$

$$\cdot (1 + |x|^2)P \leq C_{0, \frac{3}{2}, \rho}((\sigma_n)^2 \rho^{-1/2} + (\sigma_n)^3/2) .$$

The desired result for this term follows from (6.70)-(6.71) and similar estimates.

For the terms involving the commutator $[A_j^{\sigma_m, \sigma_n}, \tilde{v}_{\theta, q}]$, we compute

$$[A_j^{\sigma_m, \sigma_n}, \tilde{v}_{\theta, q}] = ie^{-\theta}a^*(\frac{\partial}{\partial x_q}G_{\theta}^{\sigma_m, \sigma_n}) + ie^{-\theta}a(\frac{\partial}{\partial x_q}G_{\theta}^{\sigma_m, \sigma_n})$$  \hspace{1cm} (6.72)

and estimate as before. The rest of the terms are analyzed similarly.

For $t = 0$ we proceed in a similar way using the following:

$$[(1 + |x|^2)^{-1/2}A_j^{\sigma_m, \sigma_n}, \tilde{v}_{\theta, q}] = ie^{-\theta}a^*(\frac{\partial}{\partial x_q}(1 + |x|^2)^{-1/2}G_{\theta}^{\sigma_m, \sigma_n})$$

$$+ ie^{-\theta}a(\frac{\partial}{\partial x_q}(1 + |x|^2)^{-1/2}G_{\theta}^{\sigma_m, \sigma_n})$$  \hspace{1cm} (6.73)
Lemma 6.16. Let $h \in \mathbb{C}$ satisfy the properties for $\theta$ in Assumptions 6.1. Suppose furthermore that $\rho \leq e_0 - 1$ and $m > n$. For every $i \in \{0, 1\}$ and any $w \in \mathcal{W}_j(i, m)$,

\[ \|w - \frac{1}{H_0(h) - \rho}\| \leq 40C(6.52) \left(1 + b_{1/4}\left(\frac{G_n}{\sigma_n}\right)^j\right) \]  

(6.74)

Proof. First we notice that by (3.53),

\[ \frac{3}{4}\|H^{\sigma_m} - \rho - \Delta\|_{\mathcal{H}^{\sigma_m, \sigma_n}} \leq \]  

(6.75)

and that

\[ \|H^{\sigma_m} - e^{-h}H^{\sigma_m} - \Delta + e^{-2h}\Delta\|_{\mathcal{H}^{\sigma_m}} \leq (|1 - e^{h}| + |1 - e^{-2h}|)\|H^{\sigma_m} - \rho - \Delta\|_{\mathcal{H}^{\sigma_m}}. \]  

(6.76)

Eqs. (3.72), (6.75) and (6.76) imply

\[ \|(-\Delta + H^{\sigma_m} - \rho)\frac{1}{H_0(h) - \rho}\| \leq 2(1 + 8b_{1/4}C(3.71)). \]  

(6.77)

Using functional calculus we prove that

\[ \|\left(-\Delta + H^{\sigma_m} - \rho\right)\frac{1}{-\Delta + H^{\sigma_m} - \rho}\| \leq 1. \]  

(6.78)

We conclude using (6.77), (6.78) and the proof of Lemma 6.15.

Lemma 6.17. Suppose that $m > n$ and either $z \in \mathcal{E}_n \setminus \{E_n\}$ (see 6.44) for $m = n + 1$ or $z \in \mathcal{E}_{(n, \infty)}$ (see 6.46) for $m > n + 1$. Suppose furthermore that $\rho > 0$ is such that $\rho \leq |z - E_n|$, then

\[ \left\|\frac{H^{\sigma_m, \sigma_n} + \rho}{e^{-\theta}H^{\sigma_m, \sigma_n} - (z - E_n)}\right\| \leq \frac{10}{|\sin(\nu)|}. \]  

(6.79)
Proof:
We take $m = n + 1$, the other cases are similar.
First we notice that
\[ \| \| e^{-\theta \hat{H}^{\sigma_n+\rho}} - (z-E_n) \| \| \leq 1 + \sup_{r \in [\sigma_n, \infty]} \| (z-E_n) \|^{j/2-1} , \]
then, by functional calculus,
\[ \| \| e^{-\theta \hat{H}^{\sigma_n+\rho}} - (z-E_n) \| \| \leq 1 + \sup_{r \in [\sigma_n, \infty]} \| (z-E_n) \|^{j/2-1} , \]
where in the last inequality we used (6.67).

\[ \| \| e^{-\theta \hat{H}^{\sigma_n+\rho}} - (z-E_n) \| \| \leq 1 + \frac{4}{(\| \sin(\nu) \|/2)} , \]
where we used Lemmata 6.15 and 6.17.

By functional calculus
\[ \| \| e^{-\theta \hat{H}^{\sigma_n+\rho}} - (z-E_n) \| \| \leq \sup_{r \in [\sigma_n+1, \infty]} \| (E_n - (z-e^{-\theta}r)) \|^{n} \]
where we used Lemmata 6.15 and 6.17.

Lemma 6.18. Suppose that $z \in \tilde{E}_n \setminus \{ E_n \}$ (see 6.47). Let $P^0 = 1$ and $P^1 = P$. Then for every $\iota \in \{0, 1\}$ and any $w \in W_{j}(\iota, n + 1)$
\[ \| wP^{1-\iota} \tilde{R}^{n}(z) \| \leq C(3.52) \mathcal{G}_{\alpha}(\sigma_n, |z-E_n|)j^{j/2-1} \]
The same estimates holds if we substitute $\tilde{R}^{n}(z)$ by $\tilde{R}^{n}_{\infty}(z)$ (see (6.28)), $E^{n}$ by $E^{(n, \infty)}$ (see (6.40)) and $n + 1$ for $\infty$.

Proof.
We prove (6.82) for $\iota = 1$ and $w = (A^{\sigma_{n+1}, \sigma_{n}})^2$, the other cases are similar.
Take $\rho = |z-E_n|$ , we have that
\[ \| (A^{\sigma_{n+1}, \sigma_{n}})^2 P \tilde{R}^{n}(z) \| \leq \| (A^{\sigma_{n+1}, \sigma_{n}})^2 (\hat{H}^{\sigma_{n+1}, \sigma_{n}} + \rho)^{-1} P \| \]
\[ \cdot \| e^{-\theta \hat{H}^{\sigma_{n+1}, \sigma_{n}} - (z-E_n)} \| \| (e^{-\theta \hat{H}^{\sigma_{n+1}, \sigma_{n}} - (z-E_n)}) \tilde{R}^{n}(z) \| \]
\[ \leq C(3.52) \mathcal{G}_{\alpha}(\sigma_n, |z-E_n|)^{j} \]
\[ \cdot \| (e^{-\theta \hat{H}^{\sigma_{n+1}, \sigma_{n}} - (z-E_n)}) \tilde{R}^{n}(z) \| , \]
where we used Lemmata 6.15 and 6.17.
By functional calculus
\[ \| (e^{-\theta \hat{H}^{\sigma_{n+1}, \sigma_{n}} - (z-E_n)}) \tilde{R}^{n}(z) \| \]
\[ \leq \sup_{r \in [0, \sigma_{n+1}, \infty]} \| (E_n - (z-e^{-\theta}r)) \|^{n} \]
\[ \| (E_n - (z-e^{-\theta}r)) \|^{n} \]
Next we choose a fixed number \(r \in \{0\} \cup [\sigma_{n+1}, \infty)\). Eqs. (6.42) and (6.44) imply
\[
\| (E_n - (z - e^{-\theta}r))^n \bar{R}(z - e^{-\theta}r) \| \leq \| P_n \| \tag{6.85}
\]
\[
+ \| (E_n - (z - e^{-\theta}r))^n \bar{R}(z - e^{-\theta}r) P_n \| \leq 2 + C_{n+1}^{6.16} \leq 3C_{n+1}^{6.16}.
\]
Eq. (6.82) follows from (6.83)-(6.85).

For the case \(j = 1\) we use
\[
\| (\tilde{H}_{\sigma_{n+1}, \sigma_n} + \rho)^{1/2} \tilde{R}_n(z) \| \leq \| (\tilde{H}_{\sigma_{n+1}, \sigma_n} + \rho)^{1/2} \tilde{R}_n(z) \|^{1/2} \| \tilde{R}_n(z) \|^{1/2}.
\]

Remember that we defined the velocity operator \(v_{\sigma_n}^\sigma\) in (6.24).

**Lemma 6.19.** Suppose that \(h \in \mathbb{C}\) satisfies the properties of \(\theta\) in Assumptions 6.1.
For any \(\rho \in (-\infty, e_0 - 1]\), any \(m \geq n\) and any \(j, q \in \{1, 2, 3\}\)
\[
\| (v_j^\sigma)^* v_q^\sigma \frac{1}{H_0(h) - \rho} \| \leq 20C_{3.71} (b_{1/4} + 1). \tag{6.86}
\]

**Proof:**
We use here that \(\alpha^{3/2} \leq \frac{1}{64(1+4\xi)} \left( \frac{1}{8.24} \right)^{1/4} \left( \frac{1}{A_{71}} \right) \) and \(\alpha^{3/2} \leq \frac{1}{24(1+8\xi)} \left( \frac{1}{8.24} \right)^{1/4} \left( \frac{1}{A_{71}} \right)\)
(see Assumptions 6.1).
We take \(m = n\), the other cases follow in the same way.

We denote by
\[
p := (p_1, p_2, p_3) := -i \nabla \tag{6.87}
\]
the momentum operator.
We calculate (remember that \(\theta = i\nu, \nu \in \mathbb{R}\),
\[
(v_j^\sigma)^* v_q^\sigma = p_j p_q - e^{-\theta} (a^\ast (p_j G_{\theta,q}^\sigma) - a(p_j G_{\theta,q}^\sigma)) \tag{6.88}
\]
\[
- e^{-\theta} A_j^\ast (\theta) p_j - A_j^\ast (\bar{\theta}) e^{-\theta} p_q + A_j^\ast (\theta) A_q^\ast (\theta).
\]

We estimate separately the terms appearing in (6.88). First we notice that
\[
\| p_j p_q \frac{1}{H^\sigma - \rho - \Delta} \| \leq 1. \tag{6.89}
\]
Then we have as in (6.77) that,
\[ \| p_j p_n - \frac{1}{H_0(h) - \rho} \| \leq 2(1 + 8C_{3.71} b_1/4) . \] (6.90)
Following the proof of (4.87), using (3.72) functional calculus and our selection of \( \rho \) we get for \( \alpha^{3/2} \leq \frac{1}{64(1+8C_{3.71} b_1/4)} \)
\[ \| (e^{-\theta} A_q^\sigma(\theta) p_j + A_j^\sigma(\theta) e^{-\theta} p_q + A_j^\sigma(\theta) A_q^\sigma(\theta)) \frac{1}{H_0(0) - \rho} \| \leq \frac{1}{4} . \] (6.91)
Finally by Lemma 4.1 and Remark A.3
\[ \| (a^*(p_j G^\sigma_{\theta,q}) + a(p_j G^\sigma_{\theta,q})) \frac{1}{H_0 - \rho} \| \leq 4\alpha^{3/2} C_{A.7} . \] (6.92)
Thus, using (6.77), (6.78) and that \( \alpha^{3/2} \leq \frac{1}{64(1+8C_{3.71} b_1/4)} \), we get
\[ \| (a^*(p_j G^\sigma_{\theta,q}) + a(p_j G^\sigma_{\theta,q})) \frac{1}{H_0(0) - \rho} \| \leq \frac{1}{4} . \] (6.93)
Eq. (6.86) follows from (6.88), (6.90), (6.91), (6.93) and the fact that \( C_{3.71} > 1 \).

\[ \square \]

**Lemma 6.20.** Suppose that \( h \in \mathbb{C} \) satisfies the properties of \( \theta \) in Assumptions 6.1.
For any \( \rho \in (-\infty, e_0 - 1] \), any \( m \geq n \) and any \( j \in \{1, 2, 3\} \)
\[ \| v_j^\sigma \frac{1}{H_0(h) - \rho} \| \leq 20C_{3.71} (b_1/4 + 1) . \] (6.94)

**Proof:**
We take \( m = n \), the other cases are similar.
By Lemma 6.19,
\[ \| |v_j^\sigma| \frac{1}{H_0(h) - \rho} \| \leq 20C_{3.71} (b_1/4 + 1) . \] (6.95)
As \( |H_0(h) - \rho| \) is positive (see Theorem X.8 of [10]),
\[ |v_j^\sigma| \leq 20C_{3.71} (b_1/4 + 1)|H_0(h) - \rho| \] (6.96)
and then it follows that for any \( \phi \in \text{dom} H_0(h) - \rho \)

\[
\| v^n_j \phi \|_{H^n} = \| v^n_j \phi \|_{H^n} \\
\leq (20C^{3.71}(b_1/4 + 1))^{1/2} \| H_0(h) - \rho \|^{1/2} \| \phi \|_{H^n}.
\]

Eq. (6.94) follows from (6.97) and the fact that \( \| | H_0(h) - \rho |^{-1/2} \| \leq 8^{1/2}(C^{3.71})^{1/2} \), which is a consequence of (3.72) and

\[
\langle \frac{1}{| H_0(h) - \rho |^{-1/2}} \phi, \frac{1}{| H_0(h) - \rho |^{-1/2}} \phi \rangle \\
= \langle \frac{1}{| H_0(h) - \rho |^{-1}} \phi, \phi \rangle \\
\leq \| | H_0(h) - \rho |^{-1} \phi \| \| \phi \| \\
= \| (H_0(h) - \rho)^{-1} \phi \| \| \phi \|.
\]

\[\square\]

**Lemma 6.21.** Suppose that \( z \in \tilde{\mathcal{E}}_n \setminus \{ E_n \} \)

Take \( \rho = 2e_0 - 1 \), then the following estimate holds true,

\[
\| (\tilde{H}_0(0) - \rho)\tilde{R}_n(z) \| \leq \frac{700}{| \sin(\nu) |} C^{n+1} \cdot (1 + |e_0|)(1 + \frac{1}{|z-E_n|}) \cdot \| \tilde{H}_n(\theta) - e^{-\theta}(\rho)\tilde{R}_n(z) \|. \]

The same estimate is valid for \( z \in \mathcal{E}_{n,\infty} \) replacing \( \tilde{R}_n(z) \) by \( \tilde{R}^\infty_n(z) \) and \( \tilde{H}_0(0) \) by \( \tilde{H}_0(0) \).

**Proof:**

First we notice that

\[
\| (\tilde{H}_0(0) - \rho)\tilde{R}_n(z) \| \leq \| (\tilde{H}_0(0) - \rho)\tilde{R}_n(e^{-\theta}) \|.
\]

Next we use functional calculus and Corollary 4.6 to estimate,

\[
\| (\tilde{H}_0(0) - \rho)\tilde{R}_n(e^{-\theta}) \| \\
= \sup_{r \in \{0\} \cup [\sigma_{n+1},\infty)} \| (\tilde{H}_0(0) - \rho)\frac{1}{\tilde{H}(\theta)-e^{-\theta}(\rho-r)} \| \leq 8.
\]
By (6.44), (6.54) and Lemma 6.14
\[ \left\| \left( \tilde{H}^n(\theta) - z + (z - E_n) - (e^{-\theta} \rho - E_n) \right) \tilde{R}^n(z) \right\| \leq 1 + \frac{2}{(\sin(\nu)/2)}(C_{n+1}^{6.16} + 2) + 2 + \frac{2(2\varepsilon_0 - |E_n|)^2(C_{n+1}^{6.16} + 2)}{|z - E_n|}
\] (6.102)

Remark 5.4, 6.17 and Eq. (6.37) imply that
\[ |E_n - E_0| \leq (|\sin(\nu)|/2)\sigma_0, \quad |E_0 - e_1| \leq (|\sin(\nu)|/2)\sigma_0 \] (6.103)
and therefore we have
\[ |E_n| \leq |\sin(\nu)|\sigma_0 + |\epsilon_0| \] (6.104)

Eq. (6.99) follows from (6.100) - (6.102) and (6.104) (notice that \( \sigma_0 \leq 1 \) and \( C_{6.16} \geq 1 \)).

\[ \Box \]

**Lemma 6.22.** Suppose that \( z \in \tilde{E}_n \setminus \{E_n\} \).
Denote by \( P^0 = 1 \) and \( P^1 = \mathcal{P} \). For every \( \iota \in \{0, 1\} \),
\[ \left\| (1 + |x|^2)^{-\iota/2} A^{\sigma_{n+1}, \sigma_n} \cdot \tilde{\nu}^{\sigma_n} P^{1-\iota} \tilde{R}^n(z) \right\| \leq 700C_{n+1}^{6.16} \frac{30}{\sin(\nu)^2} \cdot \mathcal{G}_\alpha(\sigma_n, |z - E_n|)^2 \frac{b_{1/4} + 1}{C_{3.71}^{3.90} (b_{1/4} + 1)} \] (6.105)
\[ \cdot \left( 1 + |\epsilon_0| \right) \left( 1 + \frac{1}{|z - E_n|} \right)^{1/2} . \]
\[ \left\| (1 + |x|^2)^{-\iota/2} \tilde{\nu}^{\sigma_n} P^{1-\iota} \tilde{R}^n(z) \right\| \leq C_{n+1}^{6.16} \frac{30}{\sin(\nu)^2} \cdot \mathcal{G}_\alpha(\sigma_n, |z - E_n|)^2 \frac{b_{1/4} + 1}{C_{3.71}^{3.90} (b_{1/4} + 1)} \] (6.106)
\[ \cdot \left( 1 + |\epsilon_0| \right) \left( 1 + \frac{1}{|z - E_n|} \right)^{1/2} + \| (1 + |x|^2)^{-\iota/2} A^{\sigma_{n+1}, \sigma_n} \cdot \tilde{\nu}^{\sigma_n} P^{1-\iota} \tilde{R}^n(z) \| . \]

The estimates are also valid for \( z \in \mathcal{E}_{(n, \infty)} \) if we replace \( \tilde{R}^n(z) \) for \( \tilde{R}^n_{\infty}(z) \) and \( n + 1 \) for \( \infty \).
Proof:

Take \( \psi = P \tilde{R}^n(z) \phi \), for some \( \phi \in \mathcal{H}^{\sigma_n+1} \). A simple calculation leads us to

\[
\langle A^{\sigma_n+1, \sigma_n} \cdot \tilde{v}_{\sigma_n}^n \psi \mid A^{\sigma_n+1, \sigma_n} \cdot \tilde{v}_{\sigma_n}^n \psi \rangle_{\mathcal{H}^{\sigma_n+1}}
\]

\[
= \sum_{j, q \in \{1, 2, 3\}} \langle (v_{q}^{\sigma_n})^* v_{j}^{\sigma_n} \psi \mid (A_{j}^{\sigma_n+1, \sigma_n})^* A_{q}^{\sigma_n+1, \sigma_n} \psi \rangle_{\mathcal{H}^{\sigma_n+1}}
\]

\[
+ \sum_{j, q \in \{1, 2, 3\}} \langle v_{j}^{\sigma_n} \psi \mid (A_{j}^{\sigma_n+1, \sigma_n})^*[A_{q}^{\sigma_n+1, \sigma_n}, v_{q}^{\sigma_n}] \psi \rangle_{\mathcal{H}^{\sigma_n+1}}
\]

\[
+ \sum_{j, q \in \{1, 2, 3\}} \langle v_{j}^{\sigma_n} \psi \mid [(A_{j}^{\sigma_n+1, \sigma_n})^*, v_{q}^{\sigma_n}] A_{q}^{\sigma_n+1, \sigma_n} \psi \rangle_{\mathcal{H}^{\sigma_n+1}}.
\]

Eq. (6.105) follows from Lemmata 6.18, 6.19, 6.20, 6.21 and Eq. (6.107). For (6.106) we take the commutator and use Lemma 6.18.

\[\square\]

Lemma 6.23. Suppose that \( z \in \bar{\mathcal{E}}_n \setminus \{E_n\} \), \( \iota \in \{0, 1\} \). Denote by \( P^0 = 1 \) and \( P^1 = \mathcal{P} \). For every \( \rho \leq e_0 - 1 \) and every \( h \in \mathbb{C} \) satisfying the properties of \( \theta \) in Assumptions 6.1.

\[
\| (1 + |x|^2)^{-\iota} A^{\sigma_n+1, \sigma_n} \cdot \tilde{v}_{\sigma_n}^n \|_{H_0^{\nu(h) - \rho}} \leq 200 C_{6.52} \|3.71 \| (1 + b_{1/4}) G_{\alpha}(\sigma_n, 1), \]

\[
\| (1 + |x|^2)^{-\iota} \tilde{w}_{\sigma_n}^n \cdot A^{\sigma_{n+1}, \sigma_n} \|_{H_0^{\nu(h) - \rho}} \leq 240 C_{6.52} \|3.71 \| (1 + b_{1/4}) G_{\alpha}(\sigma_n, 1).
\]

The estimates are also valid for \( z \in \mathcal{E}_{(n, \infty)} \) if we replace \( n + 1 \) for \( \infty \).

Proof:

The proof is similar as the one of Lemma 6.22, here we use Lemma 6.16 (we used also that \( C_{6.52} \geq 1 \)).

\[\square\]

Theorem 6.24. Denote by \( P^0 = 1 \) and \( P^1 = \mathcal{P} \). Suppose that \( z \in \bar{\mathcal{E}}_n \setminus \{E_n\} \) and that \( |z - E_n| \geq \frac{(|\sin(\nu)|/2)^{\sigma_{n+1}}}{10} \).

Then for every \( \iota \in \{0, 1\} \)

\[
\| (1 + |x|^2)^{-\iota} W_{n+1} P^{1-\iota} \tilde{R}^n(z) \| \leq C_{6.112} \|3.12 \| C_{n+1, \sigma_n}, \]

(6.110)
\[ \|W_{n+1} P R(z)\| \leq C_{6.112} \frac{\alpha^{3/2}}{8} C_n^{n+1}, \quad (6.111) \]

where \(C_{6.112}\) is a constant independent of \(n\) and \(B\), given by the formula
\[ C_{6.112} := 10^6 \frac{6.52 \left( b_{1/4} + 1 \right)}{\sin(\nu)^2} \left( b_{1/4} + 1 \right) \left( 1 + |\epsilon_0| \right) \] \( (6.112) \)

The estimates are also valid for \(z \in E_{(n, \infty)}\) if we replace \(\tilde{R}_n(z)\) for \(\tilde{R}_\infty(z)\) and \(n + 1\) for \(\infty\).

**Proof:**
The proof of (6.110) follows from (6.27) and Lemmata 6.18 and 6.22. We used also that the constants \(C_{6.112}\) and \(C_{3.71}\) are larger than 1 and that \(\sigma_0 \leq \frac{1}{2}\) (see also (6.53)).
The proof of (6.111) is similar.

---

**Estimates for the Feshbach Map**

In this subparagraph we estimate the Feshbach map of our Hamiltonians. The key ingredient is Theorem 6.27 which is the analogous result of Theorem 5.2 and Theorem 6.24. Theorem 6.27 (and Neumann expansions) imply the invertibility of the Feshbach map applied to the Hamiltonian.

**Proposition 6.25.** For any \(z \in \tilde{E}_n\),
\[ \|PW_{n+1}^{\dagger} H^\dagger P_{\theta} - z\| \leq C_{6.114} \frac{\alpha^{3/2}}{8} \sigma_n^{3/2}, \quad (6.113) \]
\[ \|PW_{n+1} H^\dagger P_{\theta} - z\| \leq C_{6.114} \frac{\alpha^{3/2}}{8} \sigma_n^{3/2}, \]

where \(C_{6.114}\)
\[ C_{6.114} := 10^6 \frac{6.52 \left( b_{1/4} + 1 \right)}{\sin(\nu)^2} \left( b_{1/4} + 1 \right) \left( 1 + |\epsilon_0| \right) \] \( (6.114) \)

The estimates are also valid for \(z \in E_{(n, \infty)}\) if we replace \(n + 1\) for \(\infty\) in the first equation in (6.113).

**Proof:**
We prove the first inequality in (6.113), the other can be proved in the same way.
The proof follows from (6.27), Lemmata 6.3, 6.16, 6.23 and the following estimate,

\[ \|P W_{n+1}^{-1} \| \leq \left( \|P (1 + |x|^2)\| + 1 \right) \]

\[ \cdot \| (1 + |x|^2)^{-1} W_{n+1}^{-1} P^{-1} \|_{H \theta} \]

\[ \cdot \| (n+1) P_{n+1} H \theta (\rho) - (e_0 - 1) \|_{P^{-1} P \rho z} \].

(6.115)

Notice that the term \( \|P (1 + |x|^2)\| + 1 \) is already considered in the constant \( C_{6.52} \).

Lemma 6.26. Suppose that \( z \in \tilde{E}_n \setminus \{E_n\} \) and that \( |z - E_n| \geq \frac{(|\sin(\nu)|/2)\sigma_{n+1}}{10} \).

Then the following holds true.

\[ \| W^{\sigma_{n+1}} P \tilde{R}^n (z) \| \leq C_{6.117} \frac{\alpha^{3/2}}{B} C_{6.10}^{n+1} \frac{1}{\sigma_n} \]  

(6.116)

where

\[ C_{6.117} := 384 C_{8.241} \left( 1 + 2 + \frac{6}{(|\sin(\nu)|/2)} \right) \]

\[ \cdot \left( 1 + 10(|\sin(\nu)| + 3|e_0' - 1|) / (|\sin(\nu)|/2) \right) \]  

(6.117)

The estimate is also valid for \( z \in E_{(n, \infty)} \), if we replace \( \tilde{R}^n(z) \) for \( \tilde{R}_\infty^n(z) \) and \( n + 1 \) for \( \infty \).

Proof:

Take \( \rho = 2e_0 - 1 \) and use (4.37) to get (remember that \( H_{0}^{\sigma_{n+1}}(0) \) is self-adjoint)

\[ \| W^{\sigma_{n+1}} P \frac{1}{H_{0}^{\sigma_{n+1}}} \| \leq 24 C_{8.241}^{3/2}. \]  

(6.118)

As in the proof of (4.93) we have (notice that \( \tilde{H}^n(0) \) is self-adjoint and therefore we can use functional calculus)

\[ \| (H_{0}^{n+1} - \rho) \frac{1}{H_{0}^{n+1} - \rho} \| \leq 2 \]  

(6.119)
It follows from the proof of Corollary 4.6 that
\[ \| (H_{n+1} - \rho) \frac{1}{H_n(\theta) - \rho} \| \leq 8. \tag{6.120} \]

We obtain finally
\[ \| W_{\sigma_n+1} \frac{1}{H_n(\theta) - \rho} \| \leq 384 C_{6.241}^{3/2}. \tag{6.121} \]

Using (6.44), Lemma 6.14, (6.104) and that \( \sigma_{n+1} = B \sigma_n \) we obtain
\[ \| W_{\sigma_n+1} \frac{1}{H_n(\theta) - z} \| \leq 384 C_{6.241}^{3/2}(1 + (2 + 2(1 + 10(\frac{|\sin(\nu)|/2)}{|\sin(\nu)|/2}B \sigma_n)) \tag{6.122} \]

Finally (6.116) follows from (6.122).

\[ \square \]

**Theorem 6.27.** Suppose that \( z \in \tilde{E}_n \setminus \{E_n\} \) and that \( |z - E_n| \geq \frac{(|\sin(\nu)|/2)\sigma_{n+1}}{10} \). Then the following holds true.

\[ \| (\mathcal{F}_p(H_{n+1} - z) - \mathcal{F}_p(\tilde{H}_n - z)) \frac{1}{\mathcal{F}_p(H_n - z)} \| \leq C_{6.123}^{1/2} \sigma_n^{3/2} C_{6.116}^{n+1} \tag{6.123} \]

where
\[ C_{6.123} := 4C_{6.112}^{6} + 2C_{6.114}^{6} C_{6.117}^{6} + 8(1 + 4e^{-2}/b^2)C_{3.78}^{9} C_{9.289}^{9} C_{3.113}^{6} C_{6.117}^{6} + 4C_{3.289}^{7} C_{6.112}^{6}. \tag{6.124} \]

The estimate is also valid for \( z \in E_{n,\infty} \), if we replace \( \tilde{H}_n(z) \) for \( \tilde{H}_n(z) \) and \( n + 1 \) for \( \infty \).
Proof:
First we notice that by (6.4), \( \frac{1}{\mathcal{F}_p(H^n-z)} = \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \).
We compute
\[
\| (\mathcal{F}_p(\tilde{H}^{n+1} - z)) - \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 \\
\leq \| \mathcal{P} W_{n+1} \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 + \| \mathcal{P} W_{n+1} \mathcal{P} (\frac{1}{H^n-z} \mathcal{P} - z)^{-1} \|_2 \\
\cdot \| W_{n+1} \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 \\
\cdot \| e^{\beta(x)} \| \cdot \| e^{-\beta(x)} W_{n+1} \mathcal{P} (\mathcal{P} H \mathcal{P} - z)^{-1} e^{\beta(x)} \|_2 \\
\cdot \| W_{n+1} \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 + \| \mathcal{P} W_{n+1} \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 \\
\cdot \| W_{n+1} \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 \\
\leq 4C_{6.112}^{3/2} C_{6.116}^{n+1} \sigma_{n+1}^{3/2} \sigma_n \sigma_{6.117}^{3/2} C_{6.110}^{n+1} \frac{1}{\mathcal{P}} \sigma_n \\
+ \| \mathcal{P} e^{\beta(x)} \|_2 \\
\cdot \| e^{-\beta(x)} W_{n+1} \mathcal{P} (\mathcal{P} H \mathcal{P} - z)^{-1} e^{\beta(x)} \|_2 \\
\cdot \| e^{-\beta(x)} (1 + |x|^2) \| \cdot \| (1 + |x|^2)^{-1} W_{n+1} \mathcal{P} \frac{1}{H^n-z} \mathcal{P} \|_2 \\
\cdot 2C_{6.117}^{3/2} C_{6.112}^{n+1} \frac{1}{\mathcal{P}} \sigma_n \\
+ 4 \alpha^{3/2} C_{6.289}^{3/2} C_{6.112}^{n+1} \frac{1}{\mathcal{P}} \sigma_n \sigma_{6.116}^{3/2} C_{6.113}^{n+1} \sigma_n ,
\]
where we used (6.6), (6.110), (6.113) and (6.116).
We obtain (6.123) from the proof of (6.113) and noting that (see (2.23))
\[
\| e^{-\beta(x)} (1 + |x|^2) \| \leq \| e^{-|x|} (1 + |x|^2) \| \leq 1 + 4 e^{-2}/b^2 .
\]

(6.126)
and that
\[\|e^{-\beta(x)}W^\sigma(\theta, \alpha, 0)\bar{P}(\bar{P}^{n+1}H(\theta, \alpha, 0)\bar{P} - z)^{-1}e^{\beta(x)}\|\]
\[= \|W^\sigma(\theta, \alpha, \beta)(\bar{P}_{at,1}(\theta, \alpha, \beta)H^{n+1}(\theta, \alpha, \beta)\bar{P}_{at,1}(\theta, \alpha, \beta) - z)\|\quad (6.127)\]
\[\leq 4\alpha^{3/2}C^{9.289}\]
see the proof of (6.6). Finally we take \(\beta = b\) and use (3.79).

**Corollary 6.28.** Suppose that \(z \in \tilde{E}_n \setminus \{E_n\}\) and that \(|z - E_n| \geq \frac{|\sin(\nu)|/2^{\sigma_{n+1}}}{10}\). Then
\[\|\mathcal{F}_P(\bar{H}^{n+1} - z)^{-1}\| \leq \frac{64}{(|\sin(\nu)|/2)^2}C^{n+1} \frac{1}{6.16(|\sin(\nu)|/2)^{\sigma_{n+1}}+|z-E_n|} \quad (6.128)\]
This estimate is also valid for \(z \in E_{(n, \infty)}\), if we replace \(n + 1\) for \(\infty\).

**Proof:**
From (6.16) and (6.17) it follows that \(C^{n+1} \frac{1}{6.16(|\sin(\nu)|/2)^{\sigma_{n+1}}+|z-E_n|} < \frac{1}{2}\). We use Theorem 6.27 and Neumann Series to obtain
\[\|\mathcal{F}_P(\bar{H}^{n+1} - z)^{-1}\| \leq 2\|\mathcal{F}_P(\bar{H}^{n} - z)^{-1}\| = 2\|\mathcal{F}(\bar{H}^{n} - z)^{-1}\|\]
\[\leq \frac{16}{(|\sin(\nu)|/2)^2} \left(\frac{C^{n+1}}{6.16} + 2 \frac{1}{|\sin(\nu)|/2^{\sigma_{n+1}}+|z-E_n|}\right)\]
\[\leq \frac{16}{(|\sin(\nu)|/2)^2} \left(\frac{C^{n+1}}{6.16} + 2 \frac{1}{|\sin(\nu)|/2^{\sigma_{n+1}}+|z-E_n|}\right)\]
\[+320 \frac{1}{(|\sin(\nu)|/2)^{\sigma_{n+1}}+|z-E_n|},\]
where we used (3.61), Lemma 6.4, (6.44), (6.57) and that \(|\nu| \leq \frac{1}{120}\).

Estimates for the Resolvent
Theorem 6.29. Suppose that \( z \in \hat{E}_n \setminus \{E_n\} \) and that \( |z - E_n| \geq \frac{(\sin(\nu)/2)\sigma_{n+1}}{10} \). Then

\[
\| (H - z)^{-1} \| \leq \min \left\{ (2 + 4\alpha^{3/2}C(6.289))^2 \left( \frac{64}{|\sin(\nu)/2|^2} \right)^{\frac{n+1}{6.16}} \frac{1}{\sigma_{n+1}} + 6 \left( \frac{3.71}{|\sin(\nu)|^2} \right), \right. \\
\left. \left( 2 + 4\alpha^{3/2}C(6.289) \right)^2 \left( \frac{64}{|\sin(\nu)/2|^2} \right)^{\frac{n+1}{6.16}} \frac{1}{|\sin(\nu)|^2} \sigma_{n+1 + |z - E_n|} \right. \\
\left. + 0 \right. \left( \frac{40|e_0|}{|\sin(\nu)|^2} \right)^{\frac{16|e_0|}{\sin(\nu)} \delta \sin(\nu)} \left( \frac{20}{|\sin(\nu)/2| \sigma_{n+1 + |z - E_n|}} \right) \right.
\]

(6.130)

This estimate is also valid for \( z \in E_{(n,\infty)} \) if we replace \( n + 1 \) for \( \infty \).

Proof: We use (6.5), (6.6), (6.10), Corollary 6.28 and the following estimates.

Given two points \( z_1 \) and \( z_2 \) in the complex plane and a real number \( \ell > 1 \), the set of points \( z \) such that \( |z - z_1| = \frac{1}{\ell} |z - z_2| \), is a circle with center at \( z_1 + \frac{1}{\ell^2} (z_1 - z_2) \) and radius \( \frac{1}{\ell^2} |z_1 - z_2| \). It follows from (3.58), (5.102), (6.17), (6.38), (6.39) and (6.47) that dist\((\epsilon_0, E_n) \geq \delta |\sin(\nu)|/4\), which implies that if \((\ell^2 - 1) |e_0| \leq \delta |\sin(\nu)|/4\) then \(|z - E_n| \leq \ell |z - e_0|\), for \( z \in \hat{E}_n \).

We take \( \ell = \frac{16|e_0|}{\delta |\sin(\nu)|} \). We have

\[
\frac{1}{|z - e_0|} \leq \frac{16|e_0|}{\delta |\sin(\nu)|} \frac{1}{|z - E_n|} \leq \frac{16|e_0|}{\delta |\sin(\nu)|} \left( \frac{20}{|\sin(\nu)/2| \sigma_{n+1 + |z - E_n|}} \right).
\]

(6.131)

Lemma 6.30. Suppose that \( z \in \hat{E}_n \) and that \( |z - E_n| \geq \frac{(\sin(\nu)/2)\sigma_{n+1}}{10} \). Then

\[
\| (H^n - z)^{-1} \|
\]

(6.132)

\[
\leq C(6.133) \alpha^{3/2} B^{2 \sigma_{n+2}} C^{2\alpha^{1/2}} + 6 \left( \frac{3.71}{|\sin(\nu)|^2} \right),
\]

where

\[
C(6.133) = 2 \left( C(3.78) (1 + 4e^{-2/b^2}) 4C(6.289) + 1 \right) C(6.114) \left( 2 + 4C(6.289) \right) \cdot \frac{64}{|\sin(\nu)/2|^2}
\]

(6.133)

\[
+(2 + 4C(6.289)) \frac{64}{|\sin(\nu)/2|^2} C(6.124).
\]
Proof: First we notice that

$$\|PW^{\sigma+1}(\mathcal{P}H^m\mathcal{P} - z)^{-1} - PW^{\sigma}(\mathcal{P}H^m\mathcal{P} - z)^{-1}\|$$

$$\leq \|PW^{n+1}_{n+1}(\mathcal{P}H^m\mathcal{P} - z)^{-1}\|$$

$$+\|Pe^{\beta(z)}\| : \|e^{-\beta(z)}W^{\sigma+1}\mathcal{P}(\mathcal{P}H^m\mathcal{P} - z)^{-1}e^{\beta(z)}\|$$

$$\cdot \|e^{-\beta(z)}W^{n+1}_{n+1}\mathcal{P}H^m\mathcal{P} - z\|$$

$$\leq \left(\|Pe^{\beta(z)}\|(1 + 4e^{-2}/b^2)4\alpha^{3/2}C^{0.289} \right)^3 \sigma_n^{3/2}$$

Where we used (6.113) (and its proof, see (6.115)), (6.126) and (6.127).

Similarly we have

$$\|W^{\sigma+1}_nP - (\mathcal{P}H^m\mathcal{P} - z)^{-1}W^{\sigma}_n\mathcal{P}\|$$

$$\leq \left(\|Pe^{\beta(z)}\|(1 + 4e^{-2}/b^2)4\alpha^{3/2}C^{0.289} \right)^3 \sigma_n^{3/2}$$

Using (6.5), (6.6), (6.10), Corollary 6.28 and (6.134-6.135) we get

$$\|((H - z)^{-1} - (\mathcal{P}H^m\mathcal{P} - z)^{-1}\|$$

$$\leq \|P - (\mathcal{P}H^m\mathcal{P} - z)^{-1}W^{\sigma+1}_n\mathcal{P}\|$$

$$+ 2\left(\|Pe^{\beta(z)}\|(1 + 4e^{-2}/b^2)4\alpha^{3/2}C^{0.289} \right)^3 \sigma_n^{3/2}$$

$$\cdot (2 + 4\alpha^{3/2}C^{0.289}) \cdot \frac{64}{\|\sin(\nu)/2\|^2} \sum_{n+1}^{\infty} \frac{1}{6.16^{n+1}} \sigma_n^{3/2}$$

$$+ 6.14 \cdot \frac{3.7}{|\sin(\nu)/2|} \cdot \sum_{n+1}^{\infty} \frac{1}{6.16^{n+1}} \sigma_n^{3/2}$$

(6.136)
Eq. (6.6), Theorem 6.27, Corollary 6.28 and the second resolvent equation imply that
\[
\| [P - (\overline{P} H^n P - z)^{-1} W^{\sigma_n} P] \cdot (\mathcal{F}_p (H - z)^{-1} - \mathcal{F}_p (\overline{H}^n - z)^{-1})
\cdot [P - (\overline{P} H P - z)^{-1} W^{\sigma_{n+1}} P] \| 
\leq (2 + 4 \alpha^{3/2} C_{6}^{2/3} \frac{64}{|\sin(\nu)|/2} C_{n+1}^{2/3} \frac{1}{6.16^{n+1}})
\]
\[
C_{6.138} \leq \left( \frac{|\sin(\nu)|}{2} \right) \left( C_{6.139} + 64 C_{6.133}^{2/3} \frac{64}{|\sin(\nu)|/2} \right)
\]

Remark 6.32. As \( C_{6.139} \leq C_{6.16} \), Theorem 6.31 establish the induction step for (6.41). It also follows the existence of \( E_{n+1} \) and its non degeneracy (see (6.17)). Together with Theorem 6.28, Theorem 6.31 implies that \( E_{n+1} \) is the only eigenvalue in \( \overline{E}_n \).
Theorem 6.33. For every \( z \in \tilde{E}_n \)

\[
\| R(z) P_{n+1} \| \leq C_{6.142} \frac{1}{10} \| \sin(\nu)/2\| \sigma_{n+1} |z - E_n|,
\]

(6.141)

where

\[
C_{6.142} := 18 \left( 2 + \frac{4\alpha^3}{2} C_9.289 \right)^2 \frac{64}{\| \sin(\nu)/2 \|} C_{6.16}
\]

(6.142)

Proof: By (6.138), there is only one eigenvalue of \( n+1 H \) in \( \tilde{E}_n \). Then we have that \( R(z) P_{n+1} \) is analytic in \( \tilde{E}_n \). For \( |z - E_n| \geq \frac{\| \sin(\nu)/2\| \sigma_{n+1}}{10} \), the result follows from (6.130) (Remark 6.32 imply that \( ||P_{n+1}|| \leq 2 \)). For \( |z - E_n| \leq \frac{\| \sin(\nu)/2\| \sigma_{n+1}}{10} \) it follows from the maximum modulus principle.

\( \square \)

Estimates for the Energy. Proof of (6.37) and (6.42) for the induction step. Completion of the Induction Step

For any vector \( \phi \in \mathcal{H}^{\sigma_k} \) and every \( k \leq \tilde{k} \leq \infty \) we identify

\[
\phi := \phi \otimes \Omega^{\sigma_k, \sigma_k}.
\]

(6.143)

We take a unit eigenvector \( (\phi_{at}) \) of \( P_{at,1}(0, \alpha, 0) \). We define for every \( m \leq n+1 \)

\[
\psi_m(\theta) := P_m \phi_{at}.
\]

(6.144)

By (3.61), (6.17), (6.36) and (6.41) we have that for \( m \leq n+1 \)

\[
\| \psi_m(\theta) - \phi_{at} \| \leq \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{n-1} \left( \frac{1}{100} \right)^i,
\]

(6.145)

which implies that

\[
1 - \frac{1}{8} - \frac{1}{120} - \frac{1}{100} \sum_{i=0}^{n-1} \left( \frac{1}{100} \right)^i \leq \| \psi_n(\theta) \|
\]

\[
\leq 1 + \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{n-1} \left( \frac{1}{100} \right)^i,
\]

(6.146)
and this in turn implies
\[
|\langle P_{n+1} \psi_n(\theta) \rangle \phi_{at} | \geq - |\langle (P_{n+1} - P_n) \psi_n(\theta) \rangle \phi_{at} | + |\langle \psi_n(\theta) \rangle \phi_{at} | \geq 1 - \left( \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{m-1} \left( \frac{1}{100} \right)^i \right) - \frac{2}{100}
\] (6.147)

and
\[
|\langle \psi_n(\theta) \rangle P_{n+1} \psi_n(\theta) | \geq |\langle \phi_{at} \rangle P_{n+1} \psi_n(\theta) | - |\langle \psi_n(\theta) \rangle - \phi_{at} | P_{n+1} \psi_n(\theta) | \\
\geq 1 - \left( \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{m-1} \left( \frac{1}{100} \right)^i \right) - \frac{2}{100} - (1 + \frac{1}{100}) \left( 1 + \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{n-1} \left( \frac{1}{100} \right)^i \right)
\] (6.148)

\[
\cdot \left( \frac{1}{8} + \frac{1}{120} + \frac{1}{100} \sum_{i=0}^{n-1} \left( \frac{1}{100} \right)^i \right).
\]

**Theorem 6.34.** The following inequality holds true:
\[
|E_{n+1} - E_n| \leq C \frac{6.150}{6.16} n^2 C_{n+1} \alpha^{3/2}
\] (6.149)

\[
C_{6.150} := C_{6.112} \frac{\left| \sin(\nu) \right| / 2}{10} \cdot 2(1 + 4e^{-2/b^2}) \cdot 4 \cdot 24\zeta_{3.78}.
\] (6.150)

**Proof:** We have that
\[
E_{n+1} = \frac{\langle \psi_n(\theta) \rangle P_{n+1} \psi_n(\theta) \rangle}{\langle \psi_n(\theta) \rangle P_{n+1} \psi_n(\theta) \rangle} \\
= \frac{\langle (\tilde{H}^{n+1})^* + (W_{n+1}(\theta))^* \psi_n(\theta) \rangle P_{n+1} \psi_n(\theta) \rangle}{\langle \psi_n(\theta) \rangle P_{n+1} \psi_n(\theta) \rangle} \\
= E_n + \frac{\langle e^{-\beta(x)} W_{n+1}(\theta) \psi_n(\theta) \rangle e^{\beta(x)} P_{n+1} \psi_n(\theta) \rangle}{\langle \psi_n(\theta) \rangle P_{n+1} \psi_n(\theta) \rangle}.
\] (6.151)

now we estimate for any z such that \( |z - E_n(\bar{\theta})| = \frac{|\sin(\nu)| / 2 \sigma_{n+1}}{10} \)
\[
\| e^{-\beta(x)} W_{n+1}(\theta) \psi_n(\theta) \|
\]
\[
= |z - E_n(\bar{\theta})| \| e^{-\beta(x)} W_{n+1}(\theta) \| \frac{1}{|H_n(\theta) - z|} \| \psi_n(\theta) \|
\]
\[
\leq \frac{|\sin(\nu)| / 2 \sigma_{n+1}}{10} \| e^{-\beta(x)} (1 + |x|^2) \| \| \psi_n(\theta) \|
\]
\[
\leq \sigma_{n+1} \frac{C_{n+1}}{6.16} \frac{3/2}{6.112} \cdot 2(1 + 4e^{-2/b^2}).
\] (6.152)
where we used (6.110), (6.126) and (6.146).

(6.13), (6.146), (6.147), (6.151) and (6.152) imply

$$|E_{n+1} - E_n| \leq \sigma_n^2 C_{n+1}^{3/2} \cdot \frac{\sin(\nu)/2}{10} \cdot 2(1 + 4e^{-2/b^2}) \cdot 4 \cdot 24C_{3.78}. \quad (6.153)$$

Eq. (6.149) follows from (6.153).

Remark 6.35. As $C_{6.150} < C_{6.16}$, Theorem 6.34 establish the induction step for (6.37).

Theorem 6.36. For every $z \in \tilde{E}_n$

$$\| P_{n+1}(z) \| \leq C_{6.155} C_{n+1}^{3/2} \left( \frac{1}{\sin(\nu)/2} \right) \sigma_{n+1} + |z - E_{n+1}|, \quad (6.154)$$

where,

$$C_{6.155} := \frac{10}{9} C_{6.142}. \quad (6.155)$$

Proof: By (6.17) and (6.149) we have that

$$|E_{n+1} - E_n| \leq \frac{1}{10} (|\sin(\nu)|/2) \sigma_n, \quad (6.156)$$

thus

$$\frac{1}{|\sin(\nu)|/2} \sigma_{n+1} + |z - E_n| \leq \frac{10}{9} \left( \frac{1}{|\sin(\nu)|/2} \right) \sigma_{n+1} + |z - E_{n+1}|. \quad (6.157)$$

(6.154) follows from (6.141) and (6.157).

Remark 6.37. Notice that (6.17) implies that $\sigma_n^2 C_{n+2}^{3/2} \leq \frac{1}{10} \sin(\nu) \sigma_{n+1}$ and this in turn, together with (6.34), implies that $E_{n+1} \subset \tilde{E}_n$ (see (6.38) and (6.47)).

As $C_{6.155} < C_{6.16}$, Theorem 6.36 establish the induction step for (6.42). This Remark together with Remark 6.32 and Remark 6.35 complete the proof of the induction step (see the Remarks on Section 6.3).
6.4 Existence of Resonances for the non-Regularized Hamiltonian

In this section we suppose that Assumptions 6.1 are valid and that Definition 6.7 is satisfied. The resonances are eigenvalues of \(H^0(\theta)\) (see 2.50) with imaginary part strictly negative. Here we prove the existence of such eigenvalues.

We use notation (6.22) and identify
\[
H^0(\theta) = H^{0,\infty}(\theta) = \tilde{H}.
\]

It is obvious from (6.17) and (6.37) that the sequence of eigenvalues \(\{E_n\}_{n \in \mathbb{N}}\) converges. We define
\[
E_\infty := \lim_{n \to \infty} E_n.
\]

Let \(\phi_{at}\) be a unit eigenvector of the atom Hamiltonian \(H_{at}(0, \alpha, 0)\) corresponding to the first excited state \(e_1\). We define the sequence of vectors
\[
\psi_n := P_n \phi_{at}.
\]

In the previous equation we identify vectors in \(\mathcal{H}_{at}\) and vectors in \(\mathcal{H}^{\sigma_n}\) with vectors in \(\tilde{\mathcal{H}}^{\infty}\) by applying a tensor product with the corresponding vacuum state.

In this section we prove that the limit
\[
\psi_\infty := \lim_{n \to \infty} \psi_n
\]
exists that it is an eigenvector of \(\tilde{H}\) corresponding to the eigenvalue \(E_\infty\).

We prove furthermore that \(E_\infty\) is non-degenerate and that its imaginary part is strictly negative and that it is, thus, a resonance.

6.4.1 Existence a Resonant Eigenvalue \(E_\infty\)

Theorem 6.38. The complex number \(E_\infty\) is an eigenvalue of \(\tilde{H}\) with corresponding eigenvector \(\psi_\infty\).

Proof:

We first prove that \(E_\infty\) is ein eigenvalue and that \(\psi_\infty\) is a corresponding eigenvector. By (6.17) and (6.41)
\[
\|\psi_{k+1} - \psi_k\| \leq \frac{1}{100} \left(\frac{1}{100}\right)^k.
\]

(6.161)
and therefore the series \( \{\psi_k\}_{k \in \mathbb{N}} \) converges.

From (6.143)-(6.146), it follows that
\[
3/4 \leq \|\psi_\infty\| \leq \frac{5}{4} \tag{6.162}
\]
and
\[
|\langle \psi_\infty | \phi_{at} \rangle| \geq 3/4 \tag{6.163}
\]
We select some \( z_n \in \mathcal{E}_{(n,\infty)} \) with \( |z_n - E_n| = \sigma_n \) (see (6.46)) and compute
\[
\hat{H}\psi_n = (\hat{H}_\infty^n + W^n_\infty)\psi_n
= E_n\psi_n + (z_n - E_n)W^n_\infty \hat{R}_\infty^n(z)\psi_n. \tag{6.164}
\]
Thus we have that (See Corollary 6.11)
\[
\|H\psi_n - E_n\psi_n\| \leq C_6^{n+2}\sigma_n, \tag{6.165}
\]
which implies by (6.17) that
\[
\lim_{n \to \infty} \hat{H}\psi_n = E_\infty\psi_\infty. \tag{6.166}
\]
As also
\[
\lim_{n \to \infty} \psi_n = \psi_\infty \tag{6.167}
\]
and \( \hat{H} \) is closed, we conclude that \( \psi_\infty \) belongs to the domain of \( \hat{H} \) and that
\[
\hat{H}\psi_\infty = E_\infty\psi_\infty, \tag{6.168}
\]
which proves the statement.

\[\square\]

### 6.4.2 Approximations For the Imaginary Part of \( E_\infty \)

Here we give an explicit expression of the imaginary part of \( E_\infty \) up to order \( \alpha^3 \). We prove that, under certain conditions, \( E_\infty \) has a strictly negative imaginary part and it is, therefore, a resonance. The main result of this subsection is Theorem 6.41.
Definition 6.39. We utilize the symbols of Landau. Let $Y$ be a normed space. Suppose that $F(\alpha, \cdot)$ is a $Y$-valued function that depends on $\alpha$ and another parameter. We say that

$$F = O(\alpha^\mu)$$

(6.169)

if there is a constant $C$ (independent of $\alpha$ and the other parameters) such that

$$\|F(\alpha, \cdot)\| \leq C\alpha^\mu$$

(6.170)

for sufficiently small $\alpha$.

The imaginary part of $E_\infty$ is given by $\tilde{E}_I$ (see (5.130)) up to order $\alpha^3$ (see (6.186)) below. But unfortunately $\tilde{E}_I$ depends on $\alpha$. In the next lemma we extract an $\alpha$-independent quantity that equals $\tilde{E}_I$ up to order $\alpha^3$.

Lemma 6.40. Let $\phi_{at}$ we a unit eigenvector of $H_{at}(0,0,0)$ corresponding to the eigenvalue $e_1$ (see Hypothesis 2.1). We define (see Hypothesis 2.1, (3.59), (3.60) and (9.260))

$$E_{Im} := \pi \int_{S^2} dS \| P_{at,0}(0,0,0)|e_1 - e_0| \cdot \tilde{w}_{1,0}(0,0,0)(x, |k|e_1 - e_0)|^{1/2},$$

(6.171)

where

$$\tilde{w}_{1,0}(0,0,0) = \frac{1}{\alpha^{3/2}} w_{1,0}(0,0,0)$$

(6.172)

(see (9.260)).

It follows that

$$|E_{Im} + \frac{1}{\alpha^3} \tilde{E}_I| = O(\alpha).$$

(6.173)

proof:

We select a $\beta > 0$ be satisfying Assumptions 6.1, we denote by

$$\hat{k} := \frac{k}{|k|}$$

. We notice that (see (2.38), (5.130) and (9.260)) for $\psi_0$ as in (5.131)

$$\|(\tilde{w}(0,0,0)(x, |e_1 - e_0|\hat{k}) - \tilde{w}(0,0,0)(x, |e_1 - e_0|\hat{k}))\psi_0\|_{H_{at}}$$

$$= \|(2G^0(0)(x, |e_1 - e_0|\hat{k}) - 2G^0(0)(0, |e_1 - e_0|\hat{k})) \cdot \nabla \psi_0\|_{H_{at}}$$

$$= \|(2G^0(0)(x, |e_1 - e_0|\hat{k}) - 2G^0(0)(0, |e_1 - e_0|\hat{k}))e^{-\beta(x)} \cdot (e^{\beta(x)}$$

$$\nabla e^{-\beta(x)})(2e_0 - 1) - e_1 e^{\beta(x)}\|_{H_{at}} = O(\alpha),$$

(6.174)
where we used (3.67) and (3.75). It follows from (2.16) and (2.17) and standard perturbative arguments with Neumann series that
\[
\|P_{at,0}(0, \alpha, 0) - P_{at,0}(0, 0, 0)\| = \mathcal{O}(\alpha^3),
\] (6.175)
which implies that for \(i \in \{0, 1\}\)
\[
\|e_i - e_i\| = \mathcal{O}(\alpha^3). \tag{6.176}
\]
Using (6.176) we prove as in (6.174) that
\[
\|\tilde{w}(0, 0, 0)(x, |e_1 - e_0| \mathbf{k}) - \tilde{w}(0, 0, 0)(x, |e_1 - e_0| \mathbf{k})\|_{\mathcal{H}_{at}} = \mathcal{O}(\alpha), \tag{6.177}
\]
We conclude from (6.174)-(6.177) that
\[
|\frac{1}{\alpha^3} \hat{E}_I + \pi \int_{S^2} dS \|P_{at,0}(0, 0, 0)|e_1 - e_0|\| P_{at,1}(0, \alpha, 0)\phi_{at} = \mathcal{O}(\alpha) \tag{6.178}
\]
Now we take
\[
\psi_0 = \frac{1}{\|P_{at,1}(0, \alpha, 0)\phi_{at}\|} P_{at,1}(0, \alpha, 0)\phi_{at} \tag{6.179}
\]
and notice that by (6.175)
\[
\|\psi_0 - \phi_{at}\| = \mathcal{O}(\alpha^3). \tag{6.180}
\]
For any \(h \in \mathcal{H}_{at}\) we have that
\[
|\langle h | P_{at,0}(0, 0, 0)|e_1 - e_0| \cdot \tilde{w}_{1,0}(0, 0, 0)(x, |e_1 - e_0| \mathbf{k})\rangle(\psi_0 - \phi_{at})\rangle| = |\langle |e_1 - e_0| \cdot (\tilde{w}_{1,0}(0, 0, 0)(x, |e_1 - e_0| \mathbf{k}))\rangle(\psi_0 - \phi_{at})\rangle|^* \tag{6.181}
\]
\[
P_{at,0}(0, 0, 0)h |(\psi_0 - \phi_{at}))| \leq C \|h\|_{\mathcal{H}_{at}} \alpha^3,
\]
for some constant \(C\). Eq. (6.181) implies that
\[
\|P_{at,0}(0, 0, 0)|e_1 - e_0| \cdot \tilde{w}_{1,0}(0, 0, 0)(x, |e_1 - e_0| \mathbf{k})\|(\psi_0 - \phi_{at})\| = \mathcal{O}(\alpha^3) \tag{6.182}
\]
which together with (6.178) implies (6.173).
Theorem 6.41.
The imaginary part of $E_\infty$ is given by $-\alpha^3 E_{\text{Im}}$ up to order $\alpha^3$, i.e.

$$|\text{Im} E_\infty - (-\alpha^3 E_{\text{Im}})| = O(\alpha^9/10).$$  \hspace{1cm} (6.183)

In particular if we assume that $E_{\text{Im}}$ is non-zero then the imaginary part of $E_\infty$ is strictly negative for small $\alpha$ and it is therefore a resonance. Notice that $E_{\text{Im}}$ does not depend on $\alpha$.

Proof:
It follows from (6.17) and (6.37) that

$$|E_\infty - E_0| \leq \frac{100}{99} C^2 \frac{3/2}{\sigma_0} \left(1 - \frac{3/2}{\sigma_0} \right),$$  \hspace{1cm} (6.184)

which implies together with (5.131) that

$$|\text{Im} E_\infty - \tilde{E}_I| \leq \frac{100}{99} C^2 \frac{3/2}{\sigma_0} \left(1 - \frac{3/2}{\sigma_0} \right) + 4C \alpha^3 |\log(\alpha^3)| + \alpha^v + \alpha^{2v} + \alpha^{(3-v)/2} (1 + \alpha^{-v/2})^3.$$  \hspace{1cm} (6.185)

Taking $v = \frac{6}{5}$ we get

$$|\text{Im} E_\infty - \tilde{E}_I| = O(\alpha^9/10).$$  \hspace{1cm} (6.186)

Eq. (6.183) follows from (6.173) and (6.186).

\[ \square \]

6.4.3 Non Degeneracy of $E_\infty$

Lemma 6.42. For every $\psi \in H^{(0,\infty)}$

$$\lim_{r \to \infty} 1_{H_{\text{x}} \otimes F^{(r,\infty)}} \otimes (1 - P_{\Omega^{(0,\infty)}}) \psi = 0$$  \hspace{1cm} (6.187)

Proof. Let $\psi = (\psi_n)_{n=0}^\infty$, where for any $n \geq 1$, $\psi_n \in L^2(\mathbb{R}^3 \times (\mathcal{K}^{(0,\infty)})^n)$ is symmetric on the variable belonging to $(\mathcal{K}^{(0,\infty)})^n$, and $\psi_0 \in L^2(\mathbb{R}^3)$.

We identify the function $\psi_n$ with the element in $H^{(0,\infty)}$ such that its $n$-component is equal to $\psi_n$ and the others are equal to zero.

As $\psi \in H^{(0,\infty)}$ and $1_{H_{\text{x}} \otimes F^{(r,\infty)}} \otimes (1 - P_{\Omega^{(0,\infty)}})$ is an orthogonal projection, for a given $\epsilon > 0$ we can choose $N$ such that

$$\sum_{n=N}^\infty \left\|1_{H_{\text{x}} \otimes F^{(r,\infty)}} \otimes (1 - P_{\Omega^{(0,\infty)}}) \psi_n \right\|^2$$

$$\leq \sum_{n=N}^\infty \left\|\psi_n \right\|^2 < \epsilon.$$  \hspace{1cm} (6.188)
uniformly in \( r \). As \( \sigma_r \) goes to zero, we have that there exist \( R \in \mathbb{N} \) such that for \( r > R \)
\[
\sum_{n=0}^{N} \| 1_{H_{\text{at}} \otimes F^{(\sigma_r, \infty)}} \otimes (1 - P_{\Omega(0, \sigma_r)}) \psi_n \|^2 < \epsilon
\] (6.189)
(6.188) and (6.189) imply (6.187).

\[ \square \]

**Theorem 6.43.** The eigenvalue \( E_{\infty} \) is non degenerate.

**Proof:**

Suppose that \( \psi \) is such that \( H = E_{\infty} \psi \), we take \( \gamma_n \) as in (6.40) and \( u \in \mathcal{E}_{(n, \infty)} \) with \( |u - E_n| = \sigma_n \) (see 6.46). We use Corollary 6.11

\[
P_n 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \psi
\]
\[= \frac{i}{2\pi} \int_{\gamma_n} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \psi dz
\]
\[= \frac{i}{2\pi} \int_{\gamma_n} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\] (6.190)
\[= \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\]
\[= \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\]
\[+ \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\]
\[= \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\]
\[+ \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\]
\[= \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{n}{1} \overline{R(z)} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \frac{(\overline{R(z)} - z)}{E_{\infty} - z} \psi dz
\]

\[
\left( 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} \psi \right) + \frac{i}{2\pi} \int_{\gamma_n} 1_{H_{\text{at}} \otimes F^{(\sigma_n, \infty)}} \otimes P_{\Omega(0, \sigma_n)} ((z - u) \overline{R(z)} + 1) \overline{R(z)} \frac{W_n}{E_{\infty} - z} e^{\beta(x)} \psi
\]
where $\beta > 0$ satisfies Assumptions 6.1. Using Corollary 6.11 we conclude that
\[
\lim_{n \to \infty} \frac{i}{2\pi} \int_{\gamma_n} 1_{\mathcal{H}_n} \otimes F_{(\sigma_n, \infty)} \otimes P_{\Omega(0, \sigma_n)} \cdot ((z - u)R(z) + 1) \tilde{R}_n(u) W_{n}^{\sigma_n} e^{-\beta(x)} E_{\infty} e^{-\beta(x)} \psi = 0,
\]
(6.191)
Thus by Lemma 6.42
\[
\lim_{n \to \infty} P_n 1_{\mathcal{H}_n} \otimes F_{(\sigma_n, \infty)} \otimes P_{\Omega(0, \sigma_n)} \psi = P_{\infty} \psi =
\]
(6.192)
and we conclude that
\[
P_{\infty} \psi = \psi,
\]
(6.193)
which implies that $E_{\infty}$ is non degenerate.

6.4.4 Estimates for the Resolvent

Theorem 6.44. For any $z \in \mathcal{E}_{(n, \infty)}$, $z$ belongs to the resolvent set of $\tilde{H}$ and
\[
\|(\tilde{H} - z)^{-1}\| \leq C_{6.194}^{n+2} \frac{1}{(|\sin(\nu)|/2)\sigma_{n+1} + |z - E_n|}.
\]
(6.194)
In particular the set
\[
\mathcal{E}_{\infty} := \begin{cases} 
\mathcal{E}_{\sigma_0}(\theta) \setminus (E_{\infty} + \mathbb{R} - i[0, \infty)), & \text{if } \text{Im}(\theta) > 0, \\
\mathcal{E}_{\sigma_0}(\theta) \setminus (E_{\infty} + \mathbb{R} + i[0, \infty)), & \text{if } \text{Im}(\theta) < 0.
\end{cases}
\]
(6.195)
is contained in the resolvent set of $\tilde{H}$.

Proof: It follows from Theorem 6.29 that
\[
\|(\tilde{H} - z)^{-1}\| \leq C_{6.197} \frac{C_{6.16}^{n+1} 1}{(|\sin(\nu)|/2)\sigma_{n+1} + |z - E_n|},
\]
(6.196)
where
\[
C_{6.197} := (2 + 4C_{6.289})^2 \frac{64}{(|\sin(\nu)|/2)} + 6 \frac{40|\phi|_{4}(3, 71)}{|\sin(\nu)|} \frac{16|\phi|_{4}}{8|\sin(\nu)|} 20.
\]
(6.197)
Eq. (6.194) follows from (6.16).
7 Proofs of Section 3

7.1 Proof of Theorem 3.3

Theorem 3.3. We suppose that \( \alpha \leq a, |\beta| \leq b \) and that

\[
|\theta| \leq \min \left( \frac{32C_{3.56}}{3.56}^{-1}, \frac{1}{120} \right),
\]

the following holds true:

- There are only two points \( \{e_0(\theta, \alpha, \beta), e_1(\theta, \alpha, \beta)\} \) in the spectrum of \( H_{at}(\theta, \alpha, \beta) \) with real part less than \( e_1 + \frac{15}{76} \delta_{at} \) (see (2.1)). They are simple eigenvalues and they do not depend on \( \beta \) and \( \theta \) (they are therefore real). They satisfy

\[
|e_j(\theta, \alpha, \beta) - e_j| \leq \frac{\delta_{at}}{16}, j \in \{0, 1\}. \tag{3.58}
\]

It follows that

\[
e_j(\theta, \alpha, \beta) = e_j(0, \alpha, 0) := e_j(\alpha), j \in \{0, 1\}. \tag{3.59}
\]

We omit writing the dependence on \( \alpha \) it is not required.

- Let

\[
P_{at,j}(\theta, \alpha, \beta), j \in \{0, 1\} \tag{3.60}
\]

be the projection into the eigen-space corresponding to \( e_j, j \in \{0, 1\} \) respectively. It follows that

\[
\|P_{at,j}(0, \alpha, \beta) - P_{at,j}(0, 0, 0)\| \leq \frac{1}{8},
\]

\[
\|P_{at,j}(\theta, \alpha, \beta) - P_{at,j}(0, \alpha, \beta)\| \leq \frac{1}{8}. \tag{3.61}
\]

We prove the assertions in several steps.

**Step 1:**

We prove that for any \( \alpha \leq a \), any \( |\beta| \leq b \) and any \( z \in \mathbb{C} \) such that \( \text{dist}(z, \sigma(H_{at}(0, 0, 0))) \geq \frac{\delta_{at}}{32} \) and \( \text{Re} z \leq |e_0| \),

\[
\|\alpha^3 V_{P-F}(\theta) + \hat{V}(\theta, \beta) \frac{1}{H_{at}(0, 0, 0) - z}\| \leq \frac{1}{9}, \forall \theta \in \mathbb{C}, |\theta| < \frac{1}{2}. \tag{7.198}
\]

It follows from (3.53) that

\[
\|(1 - \Delta)_{H_{at}(0, 0, 0) - z} \| \leq 2(1 + |z|) \frac{1}{H_{at}(0, 0, 0) - z} \|
\]

\[
+(b_{1/2} + 1/2) \frac{1}{H_{at}(0, 0, 0) - z} \|. \tag{7.199}
\]
Eqs. (2.21) and (2.24) imply that
\[ |V_{PF}(\theta)(x)| \leq \min( |x|^2, \frac{1}{|x|} ), \quad \forall x \in \mathbb{R}^3, \forall \theta, |\theta| < \frac{1}{2}, \] (7.200)
\[ \| \tilde{V}(\theta, \beta) \frac{1}{1 - \Delta} \| \leq 50 \beta, \quad \forall \theta, |\theta| < \frac{1}{2}, \] (7.201)

Eq. (7.198) follows from (7.199-7.201) and our assumptions on $\alpha$ and $\beta$.

**Step 2:**
We prove (3.58) and (3.61) for $\theta = 0$.

Using Neumann series and (7.198) we prove that for any $z \in \mathbb{C}$ such that $\text{dist}(z, \sigma(H_{at}(0, 0, 0))) \geq \frac{\delta_{at}}{32}$ and $\text{Re} z \leq |e_0|$, 
\[ \| \frac{1}{H_{at}(0, \alpha, \beta) - z} \| \leq \frac{9}{8} \text{dist}(z, \sigma(H_{at}(0, 0, 0))). \] (7.202)

By theorem XIII.46 [12] the ground state $e_0$ defined in Hypothesis 2.1 is non-degenerate and by hypothesis $e_1$ is non-generate. The rank-one projections $P_{at,j}(0, 0, 0) \ (j \in \{0, 1\})$ are the integrals
\[ P_{at,j}(0, 0, 0) = \frac{i}{2\pi} \int_{\partial D_{at}/2(e_j)} \frac{1}{H_{at}(0, \alpha, \beta) - z} dz. \] (7.203)

Using again Neumann series and (7.198) we obtain that
\[ \| \frac{i}{2\pi} \int_{\partial D_{at}/2(e_j)} \frac{1}{H_{at}(0, \alpha, \beta) - z} dz - P_{at,j}(0, 0, 0) \| \leq \frac{1}{8}, \] (7.204)
which implies that for any $j \in \{0, 1\}$ there is only one eigenvalue $-e_j(0, \alpha, \beta)$ of $H_{at}(0, \alpha, \beta)$ with $|e_j(0, \alpha, \beta) - e_j| < \delta_{at}/2$. Eq. (3.58) follows from (7.202).

The first half of (3.61) is a consequence of (7.204), since
\[ P_{at,j}(0, \alpha, \beta) = \frac{i}{2\pi} \int_{\partial D_{at}/2(e_j)} \frac{1}{H_{at}(0, \alpha, \beta) - z} dz. \] (7.205)

**Step 3:** We prove (3.61).

Using (3.58) and (7.202) we obtain that for any $z \in \mathbb{C}$ such that $\text{dist}(z, \sigma(H_{at}(0, 0, 0))) \geq \frac{\delta_{at}}{32}$ and $\text{Re} z \leq |e_0|$, 
\[ \| (1 - \Delta) \frac{1}{H_{at}(0, \alpha, \beta) - z} \| \leq \frac{4}{3} \left( 1 + 2 \frac{|z| + b_{1/4} + 1 \text{dist}(z, \sigma(H_{at}(0, 0, 0)))}{\text{dist}(z, \sigma(H_{at}(0, 0, 0)))} \right), \] (7.206)
Lemma A.3 implies that for $\theta \in \mathbb{C}$ with $|\theta| \leq \frac{1}{120}$,
\[ \| (H_{at}(\theta, \alpha, \beta) - H_{at}(0, \alpha, \beta)) \frac{1}{1 - \Delta} \| \leq |\theta| \left( \frac{1}{2(1/60 - 1/120)^2} C_{3.58} + 2e^{1/15} \right). \] (7.207)
Then we have that for any \( z \in \mathbb{C} \) with \( \text{Re}(z) \leq |e_0| \) and \( \text{dist}(z, \sigma(\mathcal{H}_{at})) \geq \frac{\delta_{at}}{16} \)

\[
\|(\mathcal{H}_{at}(\theta, \alpha, \beta) - \mathcal{H}_{at}(0, \alpha, \beta))\frac{1}{\mathcal{H}_{at}(0, \alpha, \beta) - z}\| \leq C_{3.56}|\theta|,
\]

(7.208)

It follows from (3.57), (7.202), (7.208) and the Neumann series that for every \( j \in \{0, 1\} \), we can construct the projection \( P_{at,j}(\theta, \alpha, \beta) \) using the Dunford integral,

\[
P_{at,j}(\theta, \alpha, \beta) := \frac{i}{2\pi} \int_{|z-e_j| = \delta_{at}/2} \frac{dz}{\mathcal{H}_{at}(\theta, \alpha, \beta) - z}
\]

(7.209)

and that

\[
\|P_{at,j}(\theta, \alpha, \beta) - P_{at,j}(0, \alpha, \beta)\| \leq \frac{1}{8}.
\]

(7.210)

**Step 4:** We prove (3.59). First we notice that

\[ e_j(\theta, \alpha, 0) = e_j(0, \alpha, 0) \]

follows from Theorem III.36 [12].

Suppose that \( \phi \) is ein eigenvector of \( \mathcal{H}_{at}(\theta, \alpha, 0) \) corresponding to \( e_j(\theta, \alpha, 0) \). It is easy to see that \( e^{-\beta(x)}\phi \in \text{dom}(\mathcal{H}_{at}(\theta, \alpha, \beta)) \) and that

\[ \mathcal{H}_{at}(\theta, \alpha, \beta)e^{-\beta(x)}\phi = e_j(0, \alpha, 0)e^{-\beta(x)}\phi, \]

which implies that \( e_j(0, \alpha, 0) \) is an eigenvalue of \( \mathcal{H}_{at}(\theta, \alpha, \beta) \). Eq. (3.61) implies that the range \( P_{at,j}(\theta, \alpha, \beta) \) is unidimensional. The only eigenvalue that can be associated to this projection is \( e_j(0, \alpha, 0) \).

\[ \square \]

### 7.2 Proof of Theorem 3.6

**Theorem 3.6.** Suppose that \( \theta \) satisfies (3.57). Suppose furthermore that \( \alpha \leq a \) and \( |\beta| \leq b \). Let \( z \in \mathbb{C} \) with \( \text{Re}(z) < e_1 + \frac{5}{8}\delta_{at} \). Then,

\[
\|(\mathcal{H}_{at}(\theta, \alpha, \beta) - z)^{-1}\mathcal{P}_{disc}(\theta, \alpha, \beta)\| \leq \frac{54}{|z-e_0|} \leq C_{3.71} \frac{1}{|z-e_0|},
\]

(3.69)

where

\[ e_{3.71} := e_1 + \frac{15}{16}\delta_{at}, \]

(3.70)

and

\[ C_{3.71} := 972 \frac{|e_0|}{\delta_{at}}. \]

(3.71)
Let $\mathfrak{A}$ be the set $\{ \mu \in \mathbb{C} : \text{Re}(\mu) \leq 18 \}$ and $\mathfrak{F}$ the function defined by the rule

$$\mathfrak{F}(\mu) := (H_{at}(\theta, \alpha, \beta) - \mu)^{-1}P_{\text{disc}}(\theta, \alpha, \beta).$$

(7.211)

$\mathfrak{F}$ is analytic.

Given two points $z_1$ and $z_2$ in the complex plane and a real number $\ell > 1$, the set of points $z$ such that $|z - z_1| = \frac{1}{\ell}|z - z_2|$, is a circle with center at $z_1 + \frac{1}{\ell^2 - 1}(z_1 - z_2)$ and radius $\frac{\ell}{\ell^2 - 1}|z_1 - z_2|$. We take $\ell = 4$ and we define $D := D_{4/15, 0} - \mathfrak{F}(e_1 + \frac{1}{15}(e_1 - e_{3.70})) \cup D_{4/15, 0} - \mathfrak{F}(e_0 + \frac{1}{15}(e_0 - e_{3.70}))$.

For any $z \in \mathbb{C} \setminus D$, $|z - e_{3.70}| \leq 4|z - e_0|$, $|z - e_{3.70}| \leq 4|z - e_1|$.

(7.212)

For any $z \in D_{4/15, 0} - \mathfrak{F}(e_1 + \frac{1}{15}(e_1 - e_{3.70}))$ and any $\mu \in \partial D_{4/15, 0} - \mathfrak{F}(e_1 + \frac{1}{15}(e_1 - e_{3.70}))$, $i \in \{0, 1\}$

$$|z - e_{3.70}| \leq \frac{1 + \frac{1}{\ell^2 - 1} + \frac{1}{\ell^2 - 1}}{1 + \frac{1}{\ell^2 - 1} + \frac{1}{\ell^2 - 1}}|\mu - e_{3.70}| \leq 1.67|\mu - e_{3.70}|.$$  

(7.213)

Eqs. (3.57), (7.202), (7.208) and the Neumann series imply

$$\frac{1}{H_{at}(\theta, \alpha, \beta) - z} \leq 2\frac{1}{\text{dist}(z, \sigma(H_{at}(0, 0, 0)))}.$$  

(7.214)

From (3.61) follows

$$1 - \frac{2}{8} \leq \|P_{at,j}(\theta, \alpha, \beta)\| \leq 1 + \frac{2}{8}.  \quad (7.215)$$

By (7.212), (7.214) and (7.215)

$$\|\mathfrak{F}(z)\| \leq 32\frac{1}{|z - e_{3.70}|}, \quad z \in \mathbb{C} \setminus D, \quad \text{Re}(z) \leq e_{3.70}.  \quad (7.216)$$

If $z \in D$, it follows from the maximum modulus principle, (7.213) and (7.216) that

$$\|\mathfrak{F}(z)\| \leq 54\frac{1}{|z - e_{3.70}|}.  \quad (7.217)$$

Now we take $z_1 = e_{3.70}, z_2 = e_0$ and $\ell = 18\frac{|e_0|}{\delta_{at}}$. We have that $D_{\ell/(\ell^2-1), \mu}(e_{3.70} + \frac{1}{\ell^2 - 1}(e_{3.70} - e_0))$ is contained in the set $\{z \in \mathbb{C} : \text{Re}(z) \geq e_1 + \frac{7}{8}\delta_{at}\}$ and $|z - e_0| \leq \ell|z - e_{3.70}|$, $z \in \mathbb{C}, \text{Re}(z) \leq e_1 + \frac{7}{8}\delta_{at}$.  

(7.218)
7.3 Proof of Corollary 3.7

Corollary 3.7. Suppose that $\theta$ satisfies (3.57). Suppose furthermore that $\alpha \leq a$ and $|\beta| \leq b$. Let $z \in \mathbb{C} \setminus \{e_0, e_1\}$ with $\text{Re}(z) < e_1 + \frac{7}{8} \delta_{at}$. The following estimates hold true,

\[
\|(H_{at}(\theta, \alpha, \beta) - z)^{-1}\| \leq 4C_{3.71} \left( \frac{1}{|z - e_0|} + \frac{1}{|z - e_1|} \right). \tag{3.72}
\]

\[
\|(H_{at}(\theta, \alpha, \beta) - z)^{-1}P_{at,1}(\theta, \alpha, \beta)\| \leq 4C_{3.71} \frac{1}{|z - e_0|}. \tag{3.73}
\]

We have that

\[
(H_{at}(\theta, \alpha, \beta) - z)^{-1}P_{disc}(\theta, \alpha, \beta)
= (e_0 - z)^{-1}P_{at,0}(\theta, \alpha, \beta) + (e_1 - z)^{-1}P_{at,1}(\theta, \alpha, \beta),
\] (7.219)

thus

\[
\|(H_{at}(\theta, \alpha, \beta) - z)^{-1}P_{disc}(\theta, \alpha, \beta)\|
\leq |e_0 - z|^{-1}\|P_{at,0}(\theta, \alpha, \beta)\| + |e_1 - z|^{-1}\|P_{at,1}(\theta, \alpha, \beta)\|. \tag{7.220}
\]

Eqs. (7.215) and (7.220) imply

\[
\|(H_{at}(\theta, \alpha, \beta) - z)^{-1}P_{disc}(\theta, \alpha, \beta)\| \leq 2|e_0 - z|^{-1} + 2|e_1 - z|^{-1}. \tag{7.221}
\]

Eq. (3.72) follows from (7.221) and Theorem 3.6. Analogously we can probe (3.73).

7.4 Proof of Corollary 3.8

Corollary 3.8. Suppose that $\theta$ satisfies (3.57). Suppose furthermore that $\alpha \leq a$ and $|\beta| \leq b$. Let $z, \mu \in B_{\delta_{at}/6}(e_1)$, it follows that

\[
\|(H_{at}(\theta, \alpha, \beta) - z)^{-1}P_{at,1}(\theta, \alpha, \beta)\| \leq 4 + 4C_{3.71}\tag{3.74}
\]

\[
\frac{\|P_{at,1}(\theta, \alpha, \beta)\|}{\|H_{at}(\theta, \alpha, \beta) - \mu\|} \leq 4 + 4C_{3.71}. \tag{3.74}
\]
by Corollary 3.7 and (7.215) we have that,
\[
\frac{\mathcal{P}_{\text{at},1}(\theta,\alpha,\beta)}{(H_{\text{at}}(\theta,\alpha,\beta) - z)} = \mathcal{P}_{\text{at},1}(\theta,\alpha,\beta)(1 + \frac{\mu - z}{(H_{\text{at}}(\theta,\alpha,\beta) - \mu)}) \leq 4 + 4C^{3.71}.
\]
(7.222)

\[\Box\]

### 7.5 Proof of Theorem 3.9

**Theorem 3.9** Suppose that \(\theta\) satisfies (3.57). Suppose furthermore that \(\alpha \leq a\) and \(|\beta| \leq b\). Then the range of \(P_{\text{at},1}(\theta,\alpha,0)\) is contained in the domain of \(e^{\beta(x)}\) (see (2.23)) and

\[
\|e^{\beta(x)}P_{\text{at},1}(\theta,\alpha,0)\| \leq 8\|e^{\beta(x)}\phi\|_{\mathcal{H}_{\text{at}}},
\]
(3.75)

In particular the following uniform bounds hold true

\[
\|e^{\beta(x)}P_{\text{at},1}(\theta,\alpha,0)\| \leq C^{3.78},
\]
(3.76)

and

\[
\|(1 + |x|^2)P_{\text{at},1}(\theta,\alpha,0)\| \leq C^{3.79},
\]
(3.77)

where

\[
C^{3.78} = 8\|e^{b(x)}\phi\|_{\mathcal{H}_{\text{at}}},
\]
(3.78)

\[
C^{3.79} = 8\|e^{b(x)}\phi\|_{\mathcal{H}_{\text{at}}}(1 + e^{-2}(1 + 4/b^2)),
\]
(3.79)

and \(\phi\) is any unit eigenvector of \(P_{\text{at},1,0,0}(0)\).

By (2.23), \(P_{\text{at},1}(\theta,\alpha,\beta)\) is an analytic operator valued function with respect to \(\beta\). For purely imaginary \(\beta\), the operator \(e^{\beta(x)}\) is unitary and therefore for such \(\beta\)

\[
P_{\text{at},1}(\theta,\alpha,\beta) = e^{-\beta(x)}P_{\text{at},1}(\theta,\alpha,0)e^{\beta(x)}, \quad \text{Re} \beta = 0.
\]

By analyticity it follows that for any \(\phi, \psi \in C_{0}^{\infty}(\mathbb{R}^3)\),

\[
\langle \psi | P_{\text{at},1}(\theta,\alpha,\beta)\phi \rangle_{\mathcal{H}_{\text{at}}} = \langle e^{-\beta(x)\psi} | P_{\text{at},1}(\theta,\alpha,0)e^{\beta(x)}\phi \rangle_{\mathcal{H}_{\text{at}}}, \quad |\beta| < \frac{1}{2}.
\]

As \(C_{0}^{\infty}(\mathbb{R}^3)\) is dense in \(\mathcal{H}_{\text{at}}\) and the range of \(P_{\text{at},1}(\theta,\alpha,0)\) is one dimensional, \(P_{\text{at},1}(\theta,\alpha,0)[C_{0}^{\infty}(\mathbb{R}^3)]\) covers the range of \(P_{\text{at},1}(\theta,\alpha,0)\) and therefore for any \(\phi\) in the range of \(P_{\text{at},1}(\theta,\alpha,0)\) there is a constant \(C\) such that,

\[
\langle e^{-\beta(x)\psi} \phi \rangle_{\mathcal{H}_{\text{at}}} \leq C\|\psi\|_{\mathcal{H}_{\text{at}}}, \quad \forall \psi \in C_{0}^{\infty}(\mathbb{R}^3),
\]
which implies that

\[ \|e^{1/4(x)} \phi\|_{H_{at}} \leq C^{1/2} \|\phi\|_{H_{at}}^{1/2}, \]

thus, the range of \( P_{at,1}(\theta, \alpha, 0) \) is contained in the domain of \( e^{1/4(x)} \). Related results can be found in Lemma, page 196, [12].

Let \( \phi_{0,0} \in H_{at}(0, 0, 0) \) be such that \( P_{at,1}(0, 0, 0) \phi_{0,0} = \phi_{0,0} \) and \( \|\phi_{0,0}\|_{H_{at}} = 1 \), and take \( \phi_{\alpha, \theta} := P_{at,1}(\theta, \alpha, 0) \phi_{0,0} \). By (7.210) and (7.215) the following estimates hold true

\[
\|P_{at,1}(\bar{\theta}, \alpha, 0)(1 - P_{at,1}(\theta, \alpha, 0))\| = 1 - \frac{2}{8} \leq \|\phi_{\alpha, \theta}\|_{H_{at}} \tag{7.223}
\]

\[
\|(P_{at,1}(\bar{\theta}, \alpha, 0) - P_{at,1}(\theta, \alpha, 0))(1 - P_{at,1}(\theta, \alpha, 0))\| \leq \frac{3}{4} ,
\]

\[
= \|(P_{at,1}(\theta, \alpha, 0) - P_{at,1}(0, 0, 0))\phi_{0,0} + \phi_{0,0}\| \leq 1 + \frac{2}{8} . \tag{7.224}
\]

Now we prove that if \( \phi_{\alpha, \theta} \neq 0 \) is such that \( P_{at,1}(\theta, \alpha, 0)\phi_{\alpha, \theta} = \phi_{\alpha, \theta} \) then

\[
\|e^{\beta(x)} P_{at,1}(\theta, \alpha, 0)\|
\leq \frac{\|e^{\beta(x)} \phi_{\alpha, \theta}\|_{H_{at}}}{\|\phi_{\alpha, \theta}\|_{H_{at}}(1 - \|P_{at,1}(\theta, \alpha, 0)\|^{2})^{1/2}} . \tag{7.225}
\]

Suppose that \( \phi \neq 0 \), as the range of \( P_{at,1}(\theta, \alpha, 0) \) is one dimensional we can write

\[
\phi = \lambda_{\phi}\phi_{\alpha, \theta} + \bar{\phi} , \tag{7.266}
\]

where \( (1 - P_{at,1}(\theta, \alpha, 0))\bar{\phi} = \bar{\phi} \).

A simple computation yields

\[
\|\phi\|_{H_{at}} \geq \lambda_{\phi}^{2}\|\phi_{\alpha, \theta}\|_{H_{at}}^{2} + \|\bar{\phi}\|_{H_{at}}^{2} - 2\lambda_{\phi}\|(P_{at,1}(\theta, \alpha, 0)\phi_{\alpha, \theta})(1 - P_{at,1}(\theta, \alpha, 0))\bar{\phi})| \geq \lambda_{\phi}^{2}\|\phi_{\alpha, \theta}\|_{H_{at}}^{2} + \|\bar{\phi}\|_{H_{at}}^{2} - 2\lambda_{\phi}\|\phi_{\alpha, \theta}\|_{H_{at}}\|\bar{\phi}\|_{H_{at}}\|P_{at,1}(\bar{\theta}, \alpha, 0)(1 - P_{at,1}(\theta, \alpha, 0))\| \geq \lambda_{\phi}^{2}\|\phi_{\alpha, \theta}\|_{H_{at}}^{2}(1 - \|P_{at,1}(\bar{\theta}, \alpha, 0)(1 - P_{at,1}(\theta, \alpha, 0))\|^{2}) . \tag{7.227}
\]

Then we have that

\[
\|e^{\beta(x)} P_{at,1}(\theta, \alpha, 0)\| = \sup_{\phi \neq 0, \lambda_{\phi} \neq 0} \frac{\|e^{\beta(x)} P_{at,1}(\theta, \alpha, 0)\|}{\|\phi\|} \leq \sup_{\phi \neq 0, \lambda_{\phi} \neq 0} \frac{\|\lambda_{\phi} e^{\beta(x)} \phi_{\alpha, \theta}\|_{H_{at}}}{\lambda_{\phi}\|\phi_{\alpha, \theta}\|_{H_{at}}(1 - \|P_{at,1}(\theta, \alpha, 0)(1 - P_{at,1}(\theta, \alpha, 0))\|^{2})^{1/2}} , \tag{7.228}
\]
which implies (7.225).

We calculate

\[ \|e^{\beta(x)} \phi_{\alpha,\theta}\|_{\mathcal{H}_{at}} = \|e^{\beta(x)} P_{at,1}(\theta, \alpha, 0) \phi_{0,0}\|_{\mathcal{H}_{at}} \]

\[ = \|P_{at,1}(\theta, \alpha, -\beta) e^{\beta(x)} \phi_{0,0}\|_{\mathcal{H}_{at}} , \tag{7.229} \]

Using (7.215), (7.223)-(7.225) and (7.229) we obtain

\[ \|e^{\beta(x)} P_{at,1}(\theta, \alpha, 0)\| \leq \|P_{at,1}(\theta, \alpha, -\beta)\| e^{\beta(x)} \phi_{0,0}\|_{\mathcal{H}_{at}} \]

\[ \leq 8 \|e^{\beta(x)} \phi_{0,0}\|_{\mathcal{H}_{at}} . \tag{7.230} \]

To prove (3.77) we notice that

\[ \|e^{-b|x|}(1 + |x|^2)\| \leq \|e^{-b|x|}(1 + |x|^2)\| \]

\[ \leq (1 + e^{-2}(1 + 4/b^2)) . \tag{7.231} \]

\[ \square \]

8 Proofs of Section 4

8.1 Proof of Lemma 4.3

Lemma 4.3. Let \( s' \leq s \leq t \) be positive real or infinite numbers. Suppose that \( \alpha \leq a \) and \( |\beta| \leq b \) (see 3.54).

For every \( \rho > 0 \) and every \( \theta, \eta \in \mathbb{C} \) with \( |\theta| \leq \frac{1}{120} \) and \( |\theta + \eta| \leq \frac{1}{120} \) there exist constants \( C_{8.241} \) and \( C_{8.246} \) such that the following estimates hold true.

\[ \|W^{s,t}(\theta)\phi\|_{\mathcal{H}^s,t} \leq C_{8.241} \left(1 + \frac{1}{\rho^{1/2}}\right) \alpha^{3/2} + \left(1 + \frac{1}{\rho^{1/2}}\right)^2 \alpha^3, \tag{8.187} \]

\[ \left(\|H_0^{s,t}(\theta + \rho)\phi\|_{\mathcal{H}^s,t} + \|\phi\|_{\mathcal{H}^s,t}\right), \]

more over, if \( -\rho \) belongs to the resolvent set of \( H_0^{s,t}(0, \alpha, \beta) \),

\[ \left\|\left(W^{s,t}(\theta + h) - W^{s,t}(\theta)\right)\frac{1}{H_{2,1}(0) + \rho}\right\| \leq |h| C_{8.246} \left(1 + \frac{1}{\rho^{1/2}}\right) \alpha^{3/2} + \left(1 + \frac{1}{\rho^{1/2}}\right)^2 \alpha^3 \right) \left(1 + \||H_0^{s,t}(0) + \rho^{-1}\|\right), \tag{8.188} \]
and the operator-valued function $\theta \to W^{s,t}(\theta)_{H^{s,t}(0)+\rho}$ is analytic for $|\theta| < \frac{1}{120}$.

Explicit values for the constants are given in (8.241) and (8.246).

We introduce the functions

$$Q^{s,t}(\theta)(x,k) := G^{s,t}(\theta)(0,k) \cdot \left( \eta \left( \frac{|x||k|}{\rho} \right) e^{\theta x} \right),$$

$$Q^{s,t}(\theta)(x,k) := 2(G^{s,t}(\theta)(x,k) - e^{-\theta} \nabla Q^{s,t}(\theta)(x,k)),$$

$$(ie^{-\theta} \frac{x}{(1+|x|^2)^{1/2}} + i e^{-\theta} |k| Q^{s,t}(\theta)(x,k)) .$$

and the operator

$$\tilde{b}^{s,t}(\theta) := a^*(Q^{s,t}(\theta)) + a(Q^{s,t}(\theta)) .$$

From Eqs. (2.48)-(2.50) we can write

$$W^{s,t}(\theta) := e^{-\theta} A^{s,t}(\theta) \cdot (i \nabla) \otimes \mathbf{1}_{\mathcal{H}^{s,t}} + e^{-\theta} (i \nabla) \otimes \mathbf{1}_{\mathcal{H}^{s,t}} \cdot A^{s,t}(\theta)$$

$$+ A^{s,t}(\theta) \cdot A^{s,t}(\theta) + \tilde{b}^{s,t}(\theta) .$$

Below we do not write the operator $\mathbf{1}_{\mathcal{H}^{s,t}}$, in the understanding that we identify $i \nabla$ with $(i \nabla) \otimes \mathbf{1}_{\mathcal{H}^{s,t}}$.

By Lemma 4.1 (see also (2.44)), for every $\phi \in \text{dom}(\tilde{H}^{s,t}) \cap \text{dom}(-\Delta)$

$$\|e^{-\theta} A^{s,t}(\theta) \cdot \nabla \phi\|_{H^{s,t}} \leq 2e^{1/30}$$

$$\sum_{j=1}^{3} \sup_{|\theta| \leq \frac{1}{30}} \|G^{s,t}(\theta)_{j} + e^{-\theta} \partial_{j} Q^{s,t}(\theta)\|_{p}(\tilde{H}^{s,t} + \rho)^{1/2}(-i \partial_{j})\phi\|_{H^{s,t}}$$

$$\leq e^{1/30}(\sum_{j=1}^{3} \sup_{|\theta| \leq \frac{1}{30}} \|G^{s,t}(\theta)_{j} + e^{-\theta} \partial_{j} Q^{s,t}(\theta)\|_{p})$$

$$\|(H^{s,t} + \rho - \Delta)\phi\|_{H^{s,t}},$$

$$\|A^{s,t}(\theta) \cdot A^{s,t}(\theta)\|_{H^{s,t}}$$

$$\leq 4(\sum_{j=1}^{3} \sup_{|\theta| \leq \frac{1}{30}} \|G^{s,t}(\theta)_{j} + e^{-\theta} \partial_{j} Q^{s,t}(\theta)\|_{p})^{2})$$

$$\|(H^{s,t} + \rho - \Delta)\phi\|_{H^{s,t}} .$$

(8.235)
By (3.53),
\[(1 - \epsilon)\| (\tilde{H}^{s,t} + \rho - \Delta)\phi \|_{\mathcal{H}^{\nu,t}} \leq \| (\tilde{H}^{s,t} + \rho - \Delta + V(\theta))\phi \|_{\mathcal{H}^{\nu,t}} + b_k\| \phi \|_{\mathcal{H}^{\nu,t}}.\] (8.236)

We have also that,
\[
\| (\tilde{H}^{s,t} - e^{-\theta} \tilde{H}^{s,t} - \Delta + e^{-2\theta} \Delta)\phi \|_{\mathcal{H}^{\nu,t}} 
\leq (|1 - e^\theta| + |1 - e^{-2\theta}|)\| (\tilde{H}^{s,t} + \rho - \Delta)\phi \|_{\mathcal{H}^{\nu,t}}.\] (8.237)

Now we use Remark A.2 to estimate
\[
\| G^{s,t}(\theta) \|_{\mathcal{H}^{\nu,t}} + e^{-\theta} \partial_\theta Q^{s,t}(\theta) \|_{\mathcal{H}^{\nu,t}} 
\leq (1 + \frac{1}{\rho^{1/2}}) C_{A,5} (1 + (1 + 6\| \eta'' \|_{\infty}))^{3/2},\] (8.238)

where we used (2.19), (8.232). Notice that
\[
\eta(|x||k|) = 0 \text{ if } |x||k| \geq 2 \text{ and thus we can bound } |x| \text{ by } \frac{2}{|k|} \text{ when we estimate the integrals.}
\]

Analogously we get
\[
\| e^{-2\theta} Q^{s,t}(\theta) \|_{\mathcal{H}^{\nu,t}} \leq 3e^{1/15} (1 + \frac{1}{\rho^{1/2}}) C_{A,5} (4\| \eta'' \|_{\infty} + 2\| \eta''' \|_{\infty})^{3/2},
\]
\[
\| Q^{s,t}(\theta) \|_{\mathcal{H}^{\nu,t}} \leq 6e^{1/15} (1 + \frac{1}{\rho^{1/2}}) C_{A,5}^{3/2}
+ 12\beta (1 + \frac{1}{\rho^{1/2}}) C_{A,5}^{3/2} (1 + 2\| \eta'' \|_{\infty})\] (8.239)

Using (8.235)-(8.239), that
\[
\| (\tilde{H}^{s,t} + \rho)^{1/2}\phi \|_{\mathcal{H}^{\nu,t}} \leq \frac{1}{2} (\| (\tilde{H}^{s,t} + \rho)\phi \|_{\mathcal{H}^{\nu,t}} + \| \phi \|_{\mathcal{H}^{\nu,t}}),\] (8.240)

the fact that \(|1 - e^\theta| + |1 - e^{-2\theta}| \leq 1/4 \text{ for } |\theta| \leq \frac{1}{30} \text{ and similar computations, we deduce Eq. (4.87) with}
\]
\[
C_{8.241} := 30 (1 + b_{1/4}) \left( (3 + 6\| \eta'' \|_{\infty} + 2\| \eta''' \|_{\infty})^{2} (C_{A,5}^{4} + C_{A,5}^{2}) + 24bC_{A,5}^{4} (1 + 2\| \eta'' \|_{\infty}) \right).\] (8.241)

The function \(G^{s,t}(\theta)(x,k)\) in Eq. (2.33) is analytic in \(\theta\) when we leave \(x\) and \(k\) fixed, we denote by \(G^{s,t}(\theta)(x,k)\) its derivative with respect to \(\theta\). We denote by \(A^{s,t}(\theta)'\) the operator that we obtain after writing \(G^{s,t}(\theta)(x,k)\) instead of \(G^{s,t}(\theta)(x,k)\) in (2.39), (2.41) and (2.44). In the same way we define
the formal derivative with respect to $\theta$ of the operator $\tilde{b}_{s,t}(\theta)$ introduced in Eq. (8.233) and we denote it by $\tilde{b}_{s,t}(\theta)'$. The formal derivative of the interaction is the operator

$$W^{s,t}(\theta)' := e^{-\theta} A^{s,t}(\theta)' \cdot (i\nabla) + e^{-\theta}(i\nabla) \cdot A^{s,t}(\theta)' + e^{-\theta} A^{s,t}(\theta) \cdot (-i\nabla) + e^{-\theta}(-i\nabla) \cdot A^{s,t}(\theta) + A^{s,t}(\theta) \cdot (A^{s,t}(\theta))' + (A^{s,t}(\theta))' \cdot A^{s,t}(\theta) + \tilde{b}^{s,t}(\theta)' .$$

(8.242)

By Lemma 4.1, we obtain,

$$\|e^{-\theta} (A^{s,t}(\theta + h) - A^{s,t}(\theta)) - (A^{s,t}(\theta))' \cdot \nabla \phi\|_{H^{s',t}} \leq 2e^{1/30} \left( \sum_{j=1}^{3} \left\| G^{s,t}(\theta + h) - G^{s,t}(\theta) \right\| + (G^{s,t}(\theta))' \right)$$

$$+ e^{-\theta} b_{s,t} \cdot (\tilde{b}^{s,t}(\theta) - e^{-\theta} \partial_{j} Q^{s,t}(\theta)) - (e^{-\theta} \partial_{j} Q^{s,t}(\theta))' \|_{\rho} \cdot$$

$$\|\tilde{H}^{s,t} + \rho - \Delta) \phi\|_{H^{s',t}} \leq 4 \left( \sum_{j=1}^{3} 2 \left\| G^{s,t}(\theta + h) - G^{s,t}(\theta) \right\| + (G^{s,t}(\theta))' \right)$$

$$+ e^{-\theta} b_{s,\partial} \cdot (\tilde{b}^{s,t}(\theta) - e^{-\theta} \partial_{j} Q^{s,t}(\theta)) - (e^{-\theta} \partial_{j} Q^{s,t}(\theta))' \|_{\rho} +$$

$$\|G_{\Theta,j}^{s,t} + e^{-\theta} \partial_{j} Q^{s,t}(\theta)\|_{\rho} +$$

$$\|G^{s,t}(\theta + h) - G^{s,t}(\theta)\|_{\rho} + e^{-\theta} b_{s,j} \cdot (\tilde{b}^{s,t}(\theta + h) - e^{-\theta} \partial_{j} Q^{s,t}(\theta))\|_{\rho}$$

$$\|G_{\Theta}^{s,t}(\theta + h) - G_{\Theta}^{s,t}(\theta)\|_{\rho} + e^{-\theta} b_{s,j} \cdot (\tilde{b}^{s,t}(\theta + h) - e^{-\theta} \partial_{j} Q^{s,t}(\theta)\|_{\rho})$$

$$\|\tilde{H}^{s,t} + \rho - \Delta) \phi\|_{H^{s',t}} \leq 81$$

(8.243)

Using (A.12), it follows from (8.236) (8.241), (8.243), (8.244) and similar
estimates that
\[
\| (W_{s,t}(\theta + h) - W_{s,t}(\theta)) \phi \|_{\mathcal{H}_{t_{0}}} \leq \left( \frac{1}{(1/60 - 1/120)^3} + \frac{1}{4(1/60 - 1/120)^4} \right) |h| C_{8.241}
\]
and therefore \( W_{s,t}(\theta) \frac{1}{H_{0}\_t(0)+\rho} \) is analytic in for \(|\theta| < \frac{1}{120}\), whenever \( \rho \) belongs to the resolvent set of \( H_{0}\_t(0) \).

Finally (4.88) can be proved as in (8.245) with
\[
C_{8.246} := \left( \frac{1}{(1/60 - 1/120)^3} + \frac{1}{4(1/60 - 1/120)^4} \right) C_{8.241}
\]
(8.246)

8.2 Proof of Theorem 4.4

Theorem 4.4. For every \( \rho > 0 \) such that \(-\rho\) belongs to the resolvent set of \( H_{0}\_t(0) \), the function \( \theta \rightarrow H_{s,t}(\theta) \frac{1}{H_{0}\_t(0)+\rho} \) is an operator-valued analytic function for \(|\theta| < \frac{1}{120}\). Moreover, for every \( \theta, h \in \mathbb{C} \) such that \(|\theta| < \frac{1}{120}, |\theta + h| < \frac{1}{120}\) the following estimate holds
\[
\| (H_{s,t}(\theta + h) - H_{s,t}(\theta)) \frac{1}{H_{0}\_t(0)+\rho} \| \leq C_{4.91} |h| (1 + \| \frac{1}{H_{0}\_t(0)+\rho} \|), \tag{4.90}
\]
where
\[
C_{4.91} := C_{4.91}(\alpha, \rho) := \left( C_{8.241} \left( \frac{1}{\rho^{3/2}} \right) \alpha^{3/2}
\right)
\]
\[+(1 + \frac{1}{\rho^{1/2}})^2 \alpha^3) + \frac{1}{3} \left( 1 + b_{1/4} \right) \left( \frac{C_{3.55}}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \tag{4.91}
\]
By (8.236) we have that, \( \| \frac{1}{H_{0}\_t(0)+\rho} \| \leq \frac{1}{1/4} (1 + (b_{1/4} + 1) \| \frac{1}{H_{0}\_t(0)+\rho} \|). \)

Then (4.90) follows from (4.88), (A.12), functional calculus and the following,
\[ \| (V(\theta + h, \alpha, \beta) - V(\theta, \alpha, \beta) + e^{-2\theta - 2h} - e^{-2\theta} + e^{-\theta} H_{s,t} - e^{-\theta} \hat{H}_{s,t}) \|_{H_{0,t}^s(0)+\rho} \]
\[ \leq (\| (V(\theta + h, \alpha, \beta) - V(\theta, \alpha, \beta)) - \Delta \|_{1-\Delta} + \| e^{-2(\theta + h)} - e^{-2\theta} \|_{1-\Delta} + \| e^{-\theta} - e^{-\theta} \|_{1-\Delta} + \| \hat{H}_{s,t} \|_{H_{0,t}^s(0)+\rho}) \]
\[ \| H_{s,t}(0, \alpha, \beta) - z \| \leq \left( \frac{4}{3} \right)^2 \left( \frac{1}{|z - e^0|} \right) \cdot \left( \frac{1}{16} + \frac{4}{3} (1 + b_1/4) \left( \frac{C_3.55}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \right) \]
\[ \| H_{s,t}(\theta, \alpha, \beta) - z \| \leq \left( \frac{4}{3} \right)^3 \left( \frac{1}{|z - e^0|} \right) \cdot \left( \frac{1}{16} + \frac{4}{3} (1 + b_1/4) \left( \frac{C_3.55}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \right) \]

The proof for analyticity can be done analogously.

\[ (8.247) \]

### 8.3 Proof of Corollary 4.5

**Corollary 4.5.** Suppose that \( \alpha \leq a \) and \( |\beta| \leq b \) (see 3.54). Suppose furthermore that \( \alpha^{3/2} \leq \frac{1}{128} \) and \( z \in \mathbb{C} \) is such that \( \text{Re}(z) \leq \frac{3}{2} e^0 - \frac{3}{4} \). Then \( H_{s,t}(0, \alpha, \beta) - z \) is invertible and

\[ \| (H_{s,t}^s(0, \alpha, \beta) - z) \| \leq 2 \]  
(4.92)

Suppose furthermore that \( \theta, h \in \mathbb{C} \) are such that \( |\theta| < \frac{1}{120} \) and \( |\theta + h| < \frac{1}{120} \), then

\[ \| (H_{s,t}^s(0, \alpha, \beta) - z) \| \leq |h| \cdot 2 \left( 1 + \left( \frac{4}{3} \right)^2 \right) \left( \frac{1}{16} + \frac{4}{3} (1 + b_1/4) \left( \frac{C_3.55}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \right) \]
(4.93)

In particular if

\[ |h| \leq \frac{1}{4} \left( 1 + \left( \frac{4}{3} \right)^2 \right) \left( \frac{1}{16} + \frac{4}{3} (1 + b_1/4) \left( \frac{C_3.55}{2(1/60 - 1/120)^2} + 3e^{1/15} \right) \right)^{-1} \]

it follows that

\[ \| H_{s,t}(\theta, \alpha, \beta) - z \| \leq \left( \frac{4}{3} \right)^3 \left( \frac{1}{|z - e^0|} \right) \]
(4.95)
Proof of (4.92):
By (7.198), for any $z \in \mathbb{C}$ with $\operatorname{Re}(z) \leq e_0 - \frac{\delta}{16}$,
\[ \| \frac{1}{H_{at}(0, \alpha, \beta) - z} \| \leq \frac{4}{3} \left| \frac{1}{z - e_0} \right|. \] (8.248)

By functional calculus,
\[ \| \frac{1}{H^{s,t}_0(0, \alpha, \beta) - z} \| \leq \max_{r > 0} \| \frac{1}{H_{at}(0, \alpha, \beta) - z + r} \| \leq \frac{4}{3} \left| \frac{1}{z - e_0} \right|. \] (8.249)

Then we use (4.87) to show that for our choice of $\alpha$ (notice that $C^{(8.246)} \geq 1$)
\[ \| W^{s,t}(0) \frac{1}{H^{s,t}_0(0, \alpha, \beta) - \operatorname{Re}(z)} \| \leq \frac{1}{16}, \] (8.250)
\[ \| W^{s,t}(0) \frac{1}{H^{s,t}_0(0, \alpha, \beta) - z} \| \leq \frac{1}{16} \| \frac{H^{s,t}_0(0, \alpha, \beta) - \operatorname{Re}(z)}{H^{s,t}_0(0, \alpha, \beta) - z} \| \leq \frac{1}{16} (1 + \frac{4}{3}) \leq \frac{1}{4}. \]

Now we use the Neumann series to compute $\frac{1}{H^{s,t}_0(0, \alpha, \beta) - z}$:
\[ \frac{1}{H^{s,t}_0(0, \alpha, \beta) - z} = \frac{1}{H^{s,t}_0(0, \alpha, \beta) - z} \sum_{j=0}^{\infty} \left[ - W^{s,t}(0) \frac{1}{H^{s,t}_0(0, \alpha, \beta) - z} \right]^j. \] (8.251)

Eq. (8.250) assures the convergence of the Neumann series and the bound (4.92).

Proof of (4.93):
Using (4.87) and taking on account our selection of $\alpha$ we obtain,
\[ \| W^{s,t}(0) \phi \|_{\mathcal{H}^{s,t}} \leq \frac{1}{32} (\| (H^{s,t}_0(0, \alpha, \beta) - z) \phi \|_{\mathcal{H}^{s,t}} + (1 + \| \operatorname{Im}(z) \|) \| \phi \|_{\mathcal{H}^{s,t}}). \] (8.252)

Then we have
\[ (1 - \frac{1}{32}) \| (H^{s,t}_0(0, \alpha, \beta) - z) \phi \|_{\mathcal{H}^{s,t}} \leq \| (H^{s,t}_0(0, \alpha, \beta) - z) \phi \|_{\mathcal{H}^{s,t}} + \frac{1}{32} (1 + \| \operatorname{Im}(z) \|) \| \phi \|_{\mathcal{H}^{s,t}}. \] (8.253)

Using (4.92) we get,
\[ \| (H^{s,t}_0(0, \alpha, \beta) - z) \frac{1}{H^{s,t}_0(0, \alpha, \beta) - z} \| \leq \frac{32}{31} (1 + \frac{1}{16} \left( \frac{4}{3} \right)^2). \] (8.254)

Proof of (4.94)
Eq. (4.94) follows from Theorem 4.4 and (4.92)-(4.93).
Proof of 4.95

We use (4.92), (4.94) and the Neumann series.

8.4 Proof of Corollary 4.6

Corollary 4.6. Suppose that $\alpha \leq a$ and $|\beta| \leq b$ (see 3.54). Suppose furthermore that $\alpha^{3/2} \leq \frac{1}{128} \frac{1}{8.246}$. Then for every $\theta \in D_{1/120}(0)$, $h \in D_{1/120}(0) \cap (i\mathbb{R})$ with

$$|\theta| < \frac{1}{16}\left(2\left(1 + \left(\frac{1}{4}\right)^2\right) \cdot \left(\frac{1}{16} \cdot \frac{4}{3} \cdot \frac{3.55}{2(1/80-1/120)^2} + 3e^{1/15}\right)\right)^{-1},$$

(4.96)

the following estimate holds

$$\| (H_{s,t}^0(0,\alpha,\beta) - \rho) \frac{1}{H_{s,t}(\theta,\alpha,\beta) - \rho} \| \leq 2,$$

(4.97)

$$\| (H_{s,t}^0(0,\alpha,\beta) - \rho) \frac{1}{H_{s,t}(\theta,\alpha,\beta) - e^h(\rho + r)} \| \leq 8,$$

(4.98)

where $r \in \mathbb{R}$ is any negative number and $\rho \leq 2e_0 - 1$.

Let $z = x + iy$ with $x \leq \rho$. From (4.87) and its proof we obtain that

$$\| W_{s,t}^z(\theta) \phi \|_{\mathcal{H}^{s,t}} \leq \frac{1}{128}(\| (H_{s,t}^0(0) - z) \phi \|_{\mathcal{H}^{s,t}} + (1 + |y|)\| \phi \|_{\mathcal{H}^{s,t}})$$

(8.255)

where we used that $C_{8.246} \geq 60C_{8.241} \geq 1$.

From the proof of Theorem 4.4, in particular from (8.247) it follows that

$$\| (H_{0,s,t}(\theta) - H_{0,s,t}(0)) \frac{1}{H_{0,s,t}(0) - z} \|_{\mathcal{H}^{s,t}} \leq \frac{1}{16},$$

(8.256)

where we used the assumptions on $\alpha$ and $\theta$ (see the proof of (1.94)).

Using (8.255) and (8.256) we conclude that

$$\begin{align*}
(1 - \frac{1}{128} - \frac{1}{16})\| (H_{0,s,t}(\theta) - z) \phi \|_{\mathcal{H}^{s,t}} & \leq \| (H_{s,t}^0(0) - z) \phi \|_{\mathcal{H}^{s,t}} + \frac{1}{128}(1 + |y|)\| \phi \|_{\mathcal{H}^{s,t}}
\end{align*}$$

(8.257)
We obtain finally from (4.95)
\[
\| (H_{s,t}^0(0) - z) - z \| \leq \frac{1}{(1 - \frac{1}{128} - \frac{1}{16})} (1 + \frac{2}{128} (4/3)^3),
\]
which proves (4.97).

Now we prove (4.98). We compute
\[
\| (H_{s,t}^0(0, \alpha, \beta) - \rho)_{H^{s,t}(\theta) - e^\theta (\rho + r)} \| \leq \frac{1}{(1 - \frac{1}{128} - \frac{1}{16})} (1 + \frac{2}{128} (4/3)^3)
\]
\[
+\| (e^\theta (\rho + r) - \rho)_{H^{s,t}(\theta) - e^\theta (\rho + r)} \| \leq \frac{1}{(1 - \frac{1}{128} - \frac{1}{16})} (1 + \frac{2}{128} (4/3)^3)
\]
\[
+(\frac{4}{3})^3 + (\frac{4}{3})^3 \frac{|\rho_0 - \rho|}{|\rho_0 - e^\theta (\rho + r)|} \leq 8,
\]
where we used (4.95). This last equation proves (4.98).

\[
\square
\]

9 Proofs of Section 5

9.1 Proof of Lemmata 5.1 and 5.2

Before starting our estimates we write down, for the convenience of the reader, the explicit expression of the functions \( w_{s,t}^{m,n}(\theta, \alpha, \beta) \) (see (2.51) and (2.52)). We use notations (8.232)-(8.234).

\[
w_{s,t}^{1,0}(\theta, \alpha, \beta)(x, k) := w_{s,t}^{1,0}(\theta, \alpha, \beta)(k)
\]
\[
:= 2 (G^{s,t}(\theta)(x, k) - e^{-\theta} (\nabla Q^{s,t}(\theta)(x, k))) \cdot (-ie^{-\theta} \nabla)
\]
\[
+ie^{-2\theta} \Delta Q^{s,t}(\theta)(x, k) + Q^{s,t}(\theta)(x, k).
\]

\[
w_{s,t}^{0,1}(\theta, \alpha, \beta)(x, k) := w_{s,t}^{0,1}(\theta, \alpha, \beta)(k)
\]
\[
:= 2 (G^{s,t}(\theta)(x, k) - e^{-\theta} (\nabla Q^{s,t}(\theta)(x, k))) \cdot (-ie^{-\theta} \nabla)
\]
\[
+ie^{-2\theta} \Delta Q^{s,t}(\theta)(x, k) + Q^{s,t}(\theta)(x, k).
\]
\[ w_{s,t}^{k,k'}(\theta, \alpha, \beta) := \frac{G_{s,t}(\theta)(x,k) - e^{-\theta}(\nabla Q_{s,t}(\theta)(x,k))}{\sin(\nu)} \]

\[ w_{s,t}^{k,k'}(\theta, \alpha, \beta) := \frac{G_{s,t}(\theta)(x,k) - e^{-\theta}(\nabla Q_{s,t}(\theta)(x,k))}{\sin(\nu)} \]

9.1.1 Proof of Lemma 5.1

**Lemma 5.1.** Suppose that \( \theta = \pm i\nu \), where \( \nu \in \mathbb{R} \) is different from zero and that \( \alpha \leq a \), \( |\beta| \leq b \). Suppose furthermore that it satisfies (3.57) and (4.96). Then, for every \( \mu \geq 0 \) and every \( z \in E(\theta) \cap \{e_1\} \),

\[ \|R_{s,t}^{k,k'}(\theta, \alpha, \beta)(z)\| \leq \frac{50|e_0|C_{3.71}}{\delta|\sin(\nu)|} \frac{1}{|z - e_1|} \]

\[ \|H_{s,t}^{k,k'}(\theta, \alpha, \beta)(z)\| \leq \frac{50|e_0|C_{3.71}}{\delta|\sin(\nu)|} \frac{1}{|z - e_1|} \]

\[ \|H_{s,t}^{k,k'}(\theta, \alpha, \beta)(z)\| \leq \frac{50|e_0|C_{3.71}}{\delta|\sin(\nu)|} \frac{1}{|z - e_1|} \]

For any \( z \in E(1/2\delta t, \infty) \),

\[ \|P_{s,t}(\theta, \alpha, \beta)(z)\| \leq \min \left( \frac{16C_{3.71}}{|\sin(\nu)|\delta}, \frac{20|e_0|C_{3.71}}{|\sin(\nu)|\delta} \frac{1}{|z - e_0|} \right) \]
where
\[ C_{5.111} := 2 \left( 1 + 8C_{3.71} \right) + \frac{8G_{3.71}}{\sin(\nu)} \left( 1 + \frac{|e'_0|}{\delta} \right) + \frac{8G_{3.71}}{\sin \nu} \left( \delta + |e'_1| \right). \] (5.111)

First we stress the following: Suppose that \( O_1, O_2 \) are closed operators, densely defined on a Hilbert space. Suppose furthermore that \( O_1 \) and \( O_1O_2 \) are bounded and that \( O_1O_2 = O_2O_1 \), then
\[ \|O_1O_2\| = \|O_1\|O_2\| = \|O_1\||O_2\|. \] (9.266)

We prove now (5.108). By the functional calculus and Corollary 3.7,
\[ \|P_0^{s,t}(\theta, \alpha, \beta)(\hat{H}^{s,t} + \mu)\| \]
\[ \leq \sup_{r \in \{0\} \cup [s, \infty)} \left( \| (H_{at}(\theta, \alpha, \beta) - (z - e^{-i\nu}r))^{-1} \| (r + \mu) \right) \] (9.267)
\[ \leq 4C_{3.71} \sup_{r \in \{0\} \cup [s, \infty)} \left( \frac{r + \mu}{|z - e^{-i\nu}r - e_0|} + \frac{r + \mu}{|z - e^{-i\nu}r - e_1|} \right). \]

It is geometrically clear that for \( r \geq s \),
\[ |z - e^{-i\nu}r| \geq s|\sin(\nu)|/2. \]

It is also easy to prove that for \( r \geq s \) and for
\[ z \in C_\nu := \{ z = z_1 + iz_2 \in \mathcal{E}^{s,t}(\theta) : z_1 > e_1 \}, \]
\[ |z - e_1| \leq s \left( |\tan(\nu)/4| + |\sin(\nu)/2| + \sqrt{(1/4)^2 + (\sin(\nu)/2)^2} \right). \] (9.268)

Therefore for \( z \in C_\nu \) and \( r \geq s \)
\[ |z - e_1| \leq \frac{\tan(\nu)/4 + |\sin(\nu)/2| + \sqrt{(1/4)^2 + (\sin(\nu)/2)^2}}{|\sin(\nu)|/2}. \] (9.269)

As \( |\nu| \leq \frac{1}{120} \) and
\[ |z - e^{-i\nu}r - e_1| \geq |z - e_1| \]
for \( z \in \mathcal{E}^{s,t}(\theta) \setminus C_\nu \), we have that
\[ \frac{|z - e_1|}{|z - e^{-i\nu}r - e_1|} \leq \frac{1}{|\sin(\nu)|}, \quad \forall z \in \mathcal{E}^{s,t}(\theta) \setminus \{e_1\}, \forall r \in \{0\} \cup [s, \infty). \] (9.270)

Similarly we obtain that
\[ \frac{1}{|z - e^{-i\nu}r - e_0|} \leq \min \left( \frac{1}{\delta (3/4 - \cos(\nu)/2)|\sin(\nu)|}, \frac{5|e_0|}{|\sin(\nu)| |z - e_0|}, \frac{1}{|\sin(\nu)| |z - e_0|} \right). \] (9.271)
for all $z \in E^{s,t}(\theta)$.

Now we denote by

$$C_{\nu,1} := \{ z = z_1 + iz_2 \in E^{s,t}(\theta) : z_1 > e_0 \}.$$ 

For every $z \in C_{\nu,1}$,

$$|z - e_1| \leq |e_0|(|\tan(\nu)|5/4 + |\sin(\nu)|/2 + (5/4)^2)^{1/2} \leq 2|e_0|$$

so, by (9.271),

$$\frac{1}{|z - e^{-i\nu}r - e_0|} \leq \frac{8|e_0|}{\delta|z - e_1||\sin(\nu)|^1},$$

for $z \in C_{\nu,1}$.

For $z = z_1 + iz_2 \in E^{s,t}(\theta) \setminus C_{\nu,1}$ we have that

$$|z - e^{-i\nu}r - e_0| \geq |z_2| \geq (1/5)^{1/2}|\sin(\nu)||z - e_1|.$$ 

Then we have,

$$\frac{1}{|z - e^{-i\nu}r - e_0|} \leq \frac{8|e_0|}{\delta|z - e_1||\sin(\nu)|^1} \forall z \in E^{s,t}(\theta) \setminus \{e_1\}. \quad (9.272)$$

Eq. (5.108) follows from (9.267) - (9.272).

To estimate (5.109), we proceed as in the proof of Lemma 3.12 of [15]. By the functional calculus,

$$\|R_0^{s,t}(\theta, \alpha, \beta)(z)(H_{at}(0, \alpha, \beta) \pm i\delta)\| \leq \sup_{r \in \{0\} \cup [s, \infty)} \|Y_{\pm}\|, \quad (9.273)$$

where,

$$Y_{\pm} := (H_{at}(\theta, \alpha, \beta) - z + e^{-i\nu}r)^{-1}(H_{at}(0, \alpha, \beta) \pm i\delta). \quad (9.274)$$

The identity,

$$Y_{\pm} = 1 - Y_{\pm}(H_{at}(0, \alpha, \beta) \pm i\delta)^{-1}(H_{at}(\theta, \alpha, \beta) - H_{at}(0, \alpha, \beta)) +$$

$$(\pm i\delta + z - e^{-i\nu}r)(H_{at}(\theta, \alpha, \beta) - z + e^{-i\nu}r)^{-1}$$

yields

$$\sup_{r \in \{0\} \cup [t, \infty)} \|Y_{\pm}\| \leq \frac{1}{1 - \frac{2}{3.56}}(1 + 8C_{3.71} + 8C_{3.71} |\sin(\nu)|) \left(1 + \frac{|e_0|}{\delta}\right) +$$

$$\frac{8C_{3.71}}{|\sin(\nu)|} \left(\delta + |e_1|\right) \frac{1}{|z - e_1|}.$$
where we used (7.208), Corollary 3.7, (9.270) and (9.271). Eq. (5.109) follows from (9.276).

Finally (5.107) follows from the functional calculus, Corollary 3.7 and (9.270, 9.272). (5.110) is proved in the same way.

\[ \Box \]

9.1.2 Proof of Lemma 5.2

Lemma 5.2. Suppose that \( s' \leq s < t \) and that \( \theta = \pm i\nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \). Suppose furthermore that \( \theta \) satisfies (3.57) and (4.96) and that \( \alpha \leq a, |\beta| \leq b \). For every \( z \in \mathcal{E}^{s,t}(\theta) \setminus \{e_1\} \) there is a constant \( C_{9.288} \) such that

\[
\begin{align*}
\| R^{s,t}_{0}(\theta, \alpha, \beta)(z) \|^2 W^{s,t}_{m,n}(\theta, \alpha, \beta) R^{s,t}_{0}(\theta, \alpha, \beta)(z) \|^{1/2} & \leq \frac{C_{9.288}}{|\sin(\nu)|^2} \alpha^3/m + n/2 \left(1 + \frac{1}{|z-e_1|}\right)^{1/2}, m + n \geq 1 , \\
\| R^{s,t}_{0}(\theta, \alpha, \beta)(z) \|^2 W^{s,t}_{m,n}(\theta, \alpha, \beta) R^{s,t}_{0}(\theta, \alpha, \beta)(z) \|^{1/2} & \leq \frac{C_{9.288}}{|\sin(\nu)|} \alpha^3(1 + \frac{1}{|z-e_1|}) , \\
\| R^{s,t}_{0}(\theta, \alpha, \beta)(z) \|^2 W^{s,t}_{m,n}(\theta, \alpha, \beta) R^{s,t}_{0}(\theta, \alpha, \beta)(z) \|^{1/2} & \leq \frac{C_{9.288}}{|\sin(\nu)|^2} \alpha^3/2 \left(1 + \frac{1}{|z-e_1|}\right)^{1/2} + \frac{C_{9.288}}{|\sin(\nu)|} \alpha^3 \left(1 + \frac{1}{|z-e_1|}\right) , \\
\end{align*}
\]

For any \( z \in \mathcal{E}^{(1/2)\text{sat}, \infty} \),

\[
\| W^{s,t}(\theta, \alpha, \beta) P_{\text{at}, 1}(\theta, \alpha, \beta) R^{s,t}_{0}(\theta, \alpha, \beta)(z) \| \leq C_{9.288} \alpha^3/2 .
\]

The explicit value of the constants \( C_{9.288} \) and \( C_{9.289} \) are given in (9.288) and (9.289).

Lemma 9.1. Suppose that \( \theta = \pm i\nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \). Suppose furthermore that \( \theta \) satisfies (3.57) and (4.96) and that \( \alpha \leq a, |\beta| \leq b \). Let \( \mu \in B_{(|\sin(\nu)|/2)s}(e_1) \) and \( z \in \mathbb{C} \) be such that \( |z-e_1| = (|\sin(\nu)|/2)s \) then

\[
\|(H^{s,t}_{0}(\theta, \alpha, \beta) - z) P_{\text{at}, 1}(\theta, \alpha, \beta) + P_{\Omega^{s,t}} H^{s,t}_{0}(\theta, \alpha, \beta) - \mu \| \leq 100(1 + C_{3.71}) , \quad (9.277)
\]

where \( P_{\Omega^{s,t}} : 1 - P_{\Omega^{s,t}}, P_{\Omega^{s,t}} \) being the projection over the vacuum \( \Omega^{s,t} \).
Proof:
By the functional calculus
\[
\| (H_0^{s.t}(\theta, \alpha, \beta) - z) \mathbf{T}_{at,1}(\theta, \alpha, \beta) + \mathbf{T}_{\nu,t} \| \\
\leq \sup_{r \in (0, \infty)} \| (H_{at}(\theta, \alpha, \beta) - (z - r e^{-\nu r})) \mathbf{T}_{at,1}(\theta, \alpha, \beta) - (\mu - r e^{-\nu r}) \| \\
+ \sup_{r \in [s, \infty)} \| (H_{at}(\theta, \alpha, \beta) - (z - r e^{-\nu r})) \mathbf{T}_{at,1}(\theta, \alpha, \beta) - (\mu - r e^{-\nu r}) \| \\
\leq \| (H_{at}(\theta, \alpha, \beta) - z) \mathbf{T}_{at,1}(\theta, \alpha, \beta) \| \\
+ \sup_{r \in [s, \infty)} \| (H_{at}(\theta, \alpha, \beta) - (z - r e^{-\nu r})) \mathbf{T}_{at,1}(\theta, \alpha, \beta) - (\mu - r e^{-\nu r}) \| \\
\cdot (1 + \| \mathbf{T}_{at,1}(\theta, \alpha, \beta) \|). \\
\]  
(9.278)

As in (9.270) we prove that
\[
\frac{|z - e_1|}{|\mu - e^{-\nu s} - e_1|} \leq 1, \ r \geq s. \\
\]  
(9.279)

using Corollary 3.7 and (9.271) we obtain (remember that \( s \leq \frac{1}{2} \delta_{at} \leq \delta \))
\[
\| (H_{at}(\theta, \alpha, \beta) - (z - r e^{-\nu r})) \mathbf{T}_{at,1}(\theta, \alpha, \beta) - (\mu - r e^{-\nu r}) \| \\
\leq 1 + \| \frac{|\mu - z|}{H_{at}(\theta, \alpha, \beta) - (\mu - r e^{-\nu r})} \| \leq 1 + 4C_{3.71}(2 + \frac{8(|\sin(\nu)|/2)^{12}}{\delta|\sin(\nu)|}) \\
\leq 1 + 30C_{3.71}. \\
\]  
(9.280)

Eq. (9.277) follows from (7.215), Corollary 3.8, (9.278) and (9.280).

\[\Box\]

**Proof of Lemma 5.2**

First remember that \( \omega(k) = |k| \).

The proof follows from the proof of Lemma 3.13 of [15], using Lemma 5.1 instead of Lemma 3.12 of [15]. For the reader’s convenience we sketch the proof.

We use the expression (2.51), (2.52) for the interaction. First we bound the term \( \| R_0^{s.t}(\theta, \alpha, \beta)(z) \|^{1/2} W_{0,1}^{s.t}(\theta, \alpha, \beta) \| R_0^{s.t}(\theta, \alpha, \beta)(z) \|^{1/2} \). By Eq. (3.118)
By (9.270), for $\omega(k) \leq 2|z - e_1|$, we have that (1 + $\frac{\omega(k)}{|z-e_1|}$) $\leq \frac{3}{\sin(\nu)}(1 + \omega(k))(1 + \frac{1}{|z-e_1|})$. For $\omega(k) > 2|z-e_1|$ we have that (1 + $\frac{\omega(k)}{|z-e_1|}$)$\leq 4(1 + \omega(k))(1 + \frac{1}{|z-e_1|})$. Using (8.238), (8.239) (with $\rho = 1$) and the triangle inequality we obtain,

$$
\|R^{s,t}_0(\theta, \alpha, \beta)(z)\|^{1/2} \leq \left( \int_{K^{s,t}} \frac{dk}{\omega(k)} \right) \|R^{s,t}_0(\theta, \alpha, \beta)(z)\|^{1/2} \left( 12C^{3.68}C^{4.5}(1 + (1 + 6\|\eta'\|_\infty))\alpha^{3/2}
+ 6\epsilon^{1/15}C^{A.5}(4\|\eta'\|_\infty + 2\|\eta''\|_\infty)\alpha^{3/2} + 12\epsilon^{1/15}C^{A.5}\alpha^{3/2}
+ 24\beta C^{A.5}\alpha^{3/2}(1 + 2\|\eta'\|_\infty)(1 + \frac{1}{|z-e_1|})^{1/2}. \right)
$$

Now we analyze the term containing $W_{0,1}(\theta, \alpha, \beta)$. By (3.121) and (3.122) of [15],
\[
\| R_{0}^{s,t}(\theta, \alpha, \beta)(z) \|_{0,2}^{1/2} W_{0,2}^{s,t}(\theta, \alpha, \beta) R_{0}^{s,t}(\theta, \alpha, \beta)(z) \|_{0,2}^{1/2} \|
\leq \int_{(K^{s,t})^{2}} \frac{dkdk'}{\omega(k)\omega(k')} \| R_{0}^{s,t}(\theta, \alpha, \beta)(z) \|_{0,2}^{1/2} (\tilde{H}_{0}^{s,t} + \omega(k) + \omega(k'))^{1/2} \|
\cdot \| R_{0}^{s,t}(\theta, \alpha, \beta)(z - (\omega(k) + \omega(k'))e^{-i\nu}) \|_{0,2}^{1/2} (\tilde{H}_{0}^{s,t} + \omega(k) + \omega(k'))^{1/2} \|
\cdot |w_{0,2}^{s,t}(\theta, \alpha, \beta)(k, k'; \theta)|^{2}
\leq \frac{(50|e_{0}|^{3/2})}{|\sin(\nu)|^{2}} \int_{(K^{s,t})^{2}} \frac{dkdk'}{\omega(k)\omega(k')} (1 + \frac{\omega(k) + \omega(k')}{|z - e_{1}|})(1 + \frac{\omega(k')}{|z - e_{1} - (\omega(k) + \omega(k'))e^{-i\nu}|})
\cdot |w_{0,2}^{s,t}(\theta, \alpha, \beta)(k, k'; \theta)|^{2},
\] (9.283)
where we used Lemma 5.1.

By (9.270), for \( r \in [s, \infty) \),
\[
\frac{r}{|z - e_{1} - re^{-i\nu}|} \leq \frac{|z - e_{1} - re^{-i\nu}|}{|z - e_{1} - re^{-i\nu}|} + \frac{|z - e_{1}|}{|z - e_{1} - re^{-i\nu}|}
\leq 1 + \frac{1}{|\sin(\nu)|} .
\] (9.284)

By (8.238), (9.283) and (9.284),
\[
\| R_{0}^{s,t}(\theta, \alpha, \beta)(z) \|_{0,2}^{1/2} W_{0,2}^{s,t}(\theta, \alpha, \beta) R_{0}^{s,t}(\theta, \alpha, \beta)(z) \|_{0,2}^{1/2} \|
\leq \frac{1000|e_{0}|}{|\sin(\nu)|^{1/2}} \frac{3.71}{A.5} C^{2} (1 + (1 + 6\|\eta'\|_{\infty})^{2})^{2} \alpha^{3} (1 + \frac{1}{|z - e_{1}|})^{1/2},
\] (9.285)
where we used that
\[
\frac{1}{\omega(k)} + \frac{1}{\omega(k')|z - e_{1}|} + \frac{1}{\omega(k')|z - e_{1}|} \leq (\frac{1}{\omega(k)} + \frac{1}{\omega(k)}) (1 + \frac{1}{|z - e_{1}|}) .
\]
Similarly we can analyze the term $W_{s,t}^{s,t}(\theta, \alpha, \beta)$ and obtain the bound
\[
\| |R_{s,t}^{s,t}(\theta, \alpha, \beta)(z)|^{1/2}W_{s,t}^{s,t}(\theta, \alpha, \beta)|R_{0}^{s,t}(\theta, \alpha, \beta)(z)|^{1/2}\| \\
\leq \left( \int_{(K, \nu)^2} \frac{dkdk'}{\omega(k)\omega(k')} \| R_{0}^{s,t}(\theta, \alpha, \beta)(z - \omega(k)e^{-i\nu})^{1/2}(\tilde{H}_{s,t}^{s,t} + \omega(k))^{1/2}\|^{2} \cdot \| w_{1,1}(\theta, \alpha, \beta)(k, k'; \theta) \|^{2}\right)^{1/2} \\
\leq \left( \frac{50|\nu|}{\delta \sin(\nu)}^{2} \int_{(K, \nu)^2} \frac{dkdk'}{\omega(k)\omega(k')} (1 + \frac{\omega(k)}{z - e_{1} - \omega(k)e^{-i\nu}})(1 + \frac{\omega(k')}{z - e_{1} - \omega(k)e^{-i\nu}}) \right)^{1/2} \cdot \| w_{1,1}(\theta, \alpha, \beta)(k, k'; \theta) \|^{2}\right)^{1/2} \\
\leq \frac{1000|\nu|}{\delta \sin(\nu)^2} C^{2} A.5 \left(1 + (1 + 6\| \eta' \|_{\infty})^{2}\alpha^{3}\right),
\] (9.286)

We also have
\[
\| |R_{0}^{s,t}(\theta, \alpha, \beta)(z)|^{1/2}W_{s,t}^{s,t}(\theta, \alpha, \beta)|R_{0}^{s,t}(\theta, \alpha, \beta)(z)|^{1/2}\| \\
\leq \frac{50|\nu|}{\delta \sin(\nu)}^{2} C^{2} A.5 \left(1 + (1 + 6\| \eta' \|_{\infty})^{2}\alpha^{3}(1 + \frac{1}{|z - e_{1}|})\right).
\] (9.287)

Now we define
\[
C_{0.288} := 5^{\frac{|\nu|}{\delta}} \left( 1000 C^{2} A.5 \left(1 + (1 + 6\| \eta' \|_{\infty})^{2}\right) + 2\left(50 C^{2} A.5 \right)^{1/2} (1, 1 + 6\| \eta' \|_{\infty}) \right) + 6e^{1/15} C^{4} A.5 \left(4\| \eta' \|_{\infty} + 2\| \eta'' \|_{\infty} \right) + 12e^{1/15} C^{4} A.5 \\
+ 24b C^{4} A.5 \left(1 + 2\| \eta' \|_{\infty}\right)
\] (9.288)

Finally other terms in (2.51) can treated as in (9.282) and (9.285).

To prove (5.115) we use (4.87) with $\rho = 1 - e_{0}$, (5.110), (7.215) and
\[
C_{0.289} := 6 C^{3} A.5 + 16(2 + 2|\nu'_{0}|) C^{3} A.5 \left(\frac{1}{\sin(\nu)\delta}\right) + \frac{20|\nu|}{\sin(\nu)\delta} C^{3} A.5.
\] (9.289)
9.2 Proof of Theorem 5.6

9.2.1 The Feshbach Map

We define the projection operator on $\mathcal{H}_{s,t}$

$$P_{s,t}(\theta) := \mathbf{P} := P_{at.1}(\theta, \alpha, \beta) \otimes P_{\Omega_{s,t}}, \quad \mathbf{P}^{*t}(\theta) := \mathbf{P} := 1 - \mathbf{P}, \quad (9.290)$$

where $P_{at.1}(\theta, \alpha, \beta)$ is defined in (3.60) and $P_{\Omega_{s,t}}$ is the projection on the vacuum vector of $\mathcal{F}_{s,t}$.

We denote by

$$\mathcal{F}_{s,t}(\theta) := \mathcal{P}_{H_{s,t}(\theta, \alpha, \beta)} = \mathcal{F}_{\mathcal{P}}(\theta, \alpha, \beta), \quad (9.291)$$

We prove below that for any $z \in \mathcal{E}_{s}(\theta)$ (see (5.102)) $H_{s,t}(\theta) - z$ is invertible on the range of $\mathbf{P}$ and that

$$\|\mathbf{P}(H_{s,t}(\theta) - z)^{-1}\mathbf{P}H_{s,t}(\theta)\| < \infty, \quad (9.292)$$

We define the Feshbach map, $f_{s,t}(\theta) - z$, by

$$F_{s,t}(\theta) := F_{s,t}(\theta, \alpha, \beta) = \mathbf{P}(H_{s,t}(\theta) - z)\mathbf{P}, \quad (9.293)$$

on the range of $\mathbf{P}$.

The Feshbach map is discussed in detail in [16] and [17], in this text we use it to estimate the value of $\mathcal{E}_{s,t}(\theta)$. The remaining of this section is devoted to prove the invertibility of $H_{s,t}(\theta) - z$ and (9.292).

**Lemma 9.2.** Suppose that $\theta = \pm \nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha$ satisfies (5.117) and that $\alpha \leq a, \beta \leq b$. Then $\mathcal{E}_{s}(\theta)$ is contained in the resolvent set of $H_{s,t}(\theta)$ and for every $\mu \in \mathcal{E}_{s}(\theta)$,

$$\|\mathbf{P}(H_{s,t}(\theta) - \mu)^{-1}\mathbf{P}\| \leq 150(1 + C_{3.71} \frac{50|e_{0}|C_{3.71}}{\delta \sin(\nu)} \frac{1}{(\sin(\nu)/2)^{s}}. \quad (9.294)$$

**Proof.**

Let $\mu \in \mathcal{E}_{s,t}(\theta) \setminus \{e_{1}\}$. We construct $(H_{s,t}(\theta) - \mu)^{-1}\mathbf{P}$ by a norm-convergent Neumann series

$$(H_{s,t}(\theta) - \mu)^{-1}\mathbf{P} = \sum_{n=0}^{\infty} \frac{\mathbf{P}}{H_{0}^{s,t}(\theta, \alpha, \beta) - \mu} \left[ - W_{s,t}(\theta, \alpha, \beta) \frac{\mathbf{P}}{H_{0}^{s,t}(\theta, \alpha, \beta) - \mu} \right]^{n} \quad (9.295)$$
Next we estimate the $n^{th}$ order term. We take $z = \mu$ if $|\mu - e_1| \geq (|\sin(\nu)|/2)s$, and $z = z_0$ with $|z_0 - e_1| = (|\sin(\nu)|/2)s$ if $|\mu - e_1| < (|\sin(\nu)|/2)s$.

Using the proof of Lemma 9.1 and Lemma 5.2 we obtain,

\[
\|H_{s,t}^0(\theta, \alpha, \beta) - z\|_{\mathcal{F}} \leq \|H_{s,t}^0(\theta, \alpha, \beta) - \mu\|_{\mathcal{F}}^{n+1} - \|H_{s,t}^0(\theta, \alpha, \beta) - z\|_{\mathcal{F}}^{n+1}
\]

This proves the convergence of the Neumann series and, by Lemma 5.1, the bound (9.294).

If $\mu = e_1$, the same estimate follows because the right hand side of Eq. (9.295) is analytic in $E_s(\theta)$ \{e_1\} and bounded in a neighbourhood of $e_1$, so it is analytic in $E_s(\theta)$.

\[\Box\]

**Lemma 9.3.** Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha$ satisfies (5.117) and that $|\alpha| \leq a$, $|\beta| \leq b$. Then for every $\mu \in E_s(\theta)$ and every $z$ with $|z - e_1| = (|\sin(\nu)|/2)s$,

\[
\|H_s^0(\theta, \alpha, \beta) - z\|^{1/2}(H_p^0(\theta) - \mu)^{-1}P[H_s^0(\theta, \alpha, \beta) - z]^{1/2}\| \leq 150(1 + C^{3.71})
\]

(9.297)

**Proof.**

It can be proved using similar estimates as in the proof of Lemma 9.2.

\[\Box\]

**Lemma 9.4.** Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha$ satisfies (5.117) and that
\[ \alpha \leq a, \ |\beta| \leq b. \] Then
\[ \| \overline{P}(H_{\theta}^{s,t}(\theta) - z)^{-1}\overline{P} H_{\theta}^{s,t}(\theta, \alpha, \beta)P \| \leq 150(1 + C_{3.71}) \]
\[ \cdot \left( \frac{0.288}{|\sin(\nu)|^2} \alpha^{3/2} \left( 1 + \frac{1}{(|\sin(\nu)|/2)^{s/4}} \right)^{1/2} + \frac{0.288}{|\sin(\nu)|^2} \alpha^3 \left( 1 + \frac{1}{(|\sin(\nu)|/2)^{s/4}} \right) \right) \cdot 2 \left( \frac{50|e_0|}{\delta|\sin(\nu)|} \right)^{1/2}, \]
\[ \| P H_{\theta}^{s,t}(\theta, \alpha, \beta) \overline{P}(H_{\theta}^{s,t}(\theta) - z)^{-1}\overline{P} \| \leq 150(1 + C_{3.71}) \]
\[ \cdot \left( \frac{0.288}{|\sin(\nu)|^2} \alpha^{3/2} \left( 1 + \frac{1}{(|\sin(\nu)|/2)^{s/4}} \right)^{1/2} + \frac{0.288}{|\sin(\nu)|^2} \alpha^3 \left( 1 + \frac{1}{(|\sin(\nu)|/2)^{s/4}} \right) \right) \cdot 2 \left( \frac{50|e_0|}{\delta|\sin(\nu)|} \right)^{1/2}. \]

(9.298)

**Proof.**
The proof of (9.298) is similar to the one of (9.294) (see also (9.307) below), here we use Lemmata 5.1 and 5.2 instead of Lemma 5.1 itself. We make use of (7.215) also.

\[ (H_{\theta}^{s,t}(\theta, \alpha, \beta) - z)^{-1} = \]
\[ [P - \overline{P}(H_{\theta}^{s,t}(\theta) - z)^{-1} W_{s,t}(\theta, \alpha, \beta)P] \cdot \overline{F}_{\theta}^{-1} [P - PW_{s,t}(\theta, \alpha, \beta)\overline{P}(H_{\theta}^{s,t}(\theta) - z)^{-1}] \]
\[ + \overline{P}(H_{\theta}^{s,t}(\theta) - z)^{-1}\overline{P}, \]

for every \( z \in E^{s,t}(\theta) \setminus \{ E^{s,t}(\theta) \} \).

### 9.2.2 Estimations for \( E^{s,t}(\theta) \)

As the annihilation operator applied to the vacuum is zero and thus the creation operator followed by the projection on the vacuum is zero, the following is clear.

**Remark 9.5.** For any \( m \) and \( n \) belonging to \( \{0, 1, 2\} \)
\[ PW_{m,n}^{s,t}(\theta, \alpha, \beta)P = 0, \ 1 \leq m + n \leq 2. \] (9.300)

Furthermore
\[ W_{0,2}^{s,t}(\theta, \alpha, \beta)P = 0 = PW_{2,0}^{s,t}(\theta, \alpha, \beta) \] (9.301)
and

\[ W^{s,t}_{0,1}(\theta, \alpha, \beta) = 0 = PW^{s,t}_{0,1}(\theta, \alpha, \beta). \]  \tag{9.302} \]

In the next Lemma we should remember (5.102).

**Lemma 9.6.** Suppose that \( \theta = \pm i \nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \), and that \( \theta \) satisfies (3.57) and (4.9). Suppose furthermore that \( \alpha \leq a, |\beta| \leq b \).

For any \( z \in E^{s,t}(\theta) \setminus \{e_1\} \),

\[ \| |R^{s,t}_{0}(\theta, \alpha, \beta)(z)|^{1/2} W^{s,t}_{m,n}(\theta, \alpha, \beta)P \| \]

\[ \leq 2^{0.288} |\sin(\nu)|^{3} |z - e_1|^{1/2} (1 + |z - e_1|)^{1/2}, \quad m + n \geq 1 \]  \tag{9.303} \]

\[ \| |PW^{s,t}_{m,n}(\theta, \alpha, \beta)|R^{s,t}_{0}(\theta, \alpha, \beta)(z)|^{1/2} \| \]

\[ \leq 2^{0.288} |\sin(\nu)|^{3} |z - e_1|^{1/2} (1 + |z - e_1|)^{1/2}, \quad m + n \geq 1 \]  \tag{9.304} \]

\[ \| |R^{s,t}_{0}(\theta, \alpha, \beta)(z)|^{1/2} W^{s,t}_{0,0}(\theta, \alpha, \beta)P \| \]

\[ \leq 2^{0.288} |\sin(\nu)|^{3} (|z - e_1|^{1/2} + |z - e_1|^{-1/2}), \]  \tag{9.305} \]

\[ \| |PW_{0,0}(\theta, \alpha, \beta)|R^{s,t}_{0}(\theta, \alpha, \beta)(z)|^{1/2} \| \]

\[ \leq 2^{0.288} |\sin(\nu)|^{3} (|z - e_1|^{1/2} + |z - e_1|^{-1/2}). \]  \tag{9.306} \]

**Proof.**

We use (7.215), Lemma 5.2 and the bound

\[ \| |R^{s,t}_{0}(\theta, \alpha, \beta)(z)|^{1/2}P \| \leq 2|z - e_1|^{1/2}. \]  \tag{9.307} \]

\( \square \)

**Lemma 9.7.** Suppose that \( \theta = \pm i \nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \), and that \( \theta \) satisfies (3.57) and (4.9). Suppose furthermore that \( \alpha \leq a, |\beta| \leq b \). Then

\[ P(W^{s,t}_{0,1}(\theta, \alpha, \beta) + W^{s,t}_{1,0}(\theta, \alpha, \beta)) \frac{P}{H^{s,t}_{0,0}(\theta, \alpha, \beta)} W^{s,t}_{0,1}(\theta, \alpha, \beta) \]

\[ + W^{s,t}_{1,0}(\theta, \alpha, \beta))P = \int_{K^{s,t}} P_{at,1}(\theta, \alpha, \beta) w_{0,1}(\theta, \alpha, \beta)(k) \]

\[ + \frac{1}{H^{s,t}_{at}(\theta, \alpha, \beta) - (z - e^{-i\nu}|k|)} w_{1,0}(\theta, \alpha, \beta)(k) P_{at,1}(\theta, \alpha, \beta) P_{\Omega^{s,t}}. \]  \tag{9.308}
Proof.

By Remark 9.5:

\[
P(W_{0,1}^{s,t}(\theta, \alpha, \beta) + W_{1,0}^{s,t}(\theta, \alpha, \beta) \frac{P}{H_0^{s,t}(\theta, \alpha, \beta) - z}(W_{0,1}^{s,t}(\theta, \alpha, \beta)) + W_{1,0}^{s,t}(\theta, \alpha, \beta)P = PW_{0,1}^{s,t}(\theta, \alpha, \beta)
\]

(9.309)

\[
\frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - z} W_{1,0}^{s,t}(\theta, \alpha, \beta)P.
\]

As \( P \) projects on the vacuum photon space, it is enough to apply \( P W_{0,1}^{s,t}(\theta, \alpha, \beta) \frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - z} W_{1,0}^{s,t}(\theta, \alpha, \beta) \) to a function \( \psi : \mathbb{R}^3 \to \mathcal{F}^{s,t} = \mathbb{C} \Omega^{s,t} \). The function

\[
\psi_1 := W_{1,0}^{s,t}(\theta) \psi
\]

is a function form \( \mathbb{R}^3 \) to \( \mathcal{F}^{s,t} \):

\[
\psi_1(x)(k) = w_{1,0}(\theta, \alpha, \beta)(x, k)\psi(x).
\]

Therefore

\[
\frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - z} \psi_1 = \frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - \left(z - e^{-\theta|k|}\right)} \psi_1.
\]

(9.312)

We obtain that

\[
(W_{0,1}^{s,t}(\theta, \alpha, \beta) \frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - z} \psi_1)(x)
= \int_{k,s,t} dk \ w_{0,1}(\theta, \alpha, \beta)(x, k) \frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - (z - e^{-\theta|k|})} \psi_1(x, k).
\]

(9.313)

Eqs. (9.311) and (9.313) imply

\[
P(W_{0,1}^{s,t}(\theta, \alpha, \beta) + W_{1,0}^{s,t}(\theta, \alpha, \beta) \frac{P}{H_0^{s,t}(\theta, \alpha, \beta) - z}(W_{0,1}^{s,t}(\theta, \alpha, \beta)) + W_{1,0}^{s,t}(\theta, \alpha, \beta)P = \int_{k,s,t} P w_{0,1}(\theta, \alpha, \beta)(k)
\]

(9.314)

\[
\frac{1}{H_0^{s,t}(\theta, \alpha, \beta) - (z - e^{-\theta|k|})} w_{1,0}(\theta, \alpha, \beta)(k)P.
\]

To obtain (9.308) we notice (see (9.290)) that \( w_{1,0}(\theta, \alpha, \beta)(k)P = w_{1,0}(\theta, \alpha, \beta)(k) (P_{at,1}(\theta, \alpha, \beta) \otimes 1) (1 \otimes P_{\Omega^{s,t}}) = P_{\Omega^{s,t}} w_{1,0}(\theta, \alpha, \beta)(k) P_{at,1}(\theta, \alpha, \beta) \).

On the next lemma we should remember the definition of the Feshbach map (9.293).
Lemma 9.8. Suppose that $\theta = \pm iv$, with $v \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (5.57) and (4.90). Suppose furthermore that $\alpha$ satisfies (5.117) and that $\alpha \leq a$, $|\beta| \leq b$. We assume additionally that $\left(\frac{|\sin(v)|}{2}\right)s \leq 1$. Then for every $z \in \mathbb{C}$ with $|z - e_1| \leq \left(\frac{|\sin(v)|}{2}\right)s$,

$$\mathcal{F}_P = (e_1 - z)P - \int_{K_{s,t}} P_{at,1}(\theta, \alpha, \beta)w_{0,1}(\theta, \alpha, \beta)(k)$$

$$\frac{1}{H_{at}^{s,t}(\theta, \alpha, \beta) - (z - e^{-iu}|k|)}w_{1,0}^{s,t}(\theta, \alpha, \beta)(k)P_{at}(\theta, \alpha, \beta)P_{\Omega_{s,t}}$$

$$(9.315)$$

$$+PW_{0,0}(\theta, \alpha, \beta)P + Rem_0 + Rem_1,$$

where

$$\|Rem_0\| \leq C_{Rem_0} \frac{\alpha^{9/2}}{((|\sin(v)|/2)s)^{1/2}}(5 + \frac{\alpha^{3/2}}{((|\sin(v)|/2)s)^{1/2}})^3,$$  

$$\|Rem_1\| \leq C_{Rem_1} \frac{4\alpha^3}{((|\sin(v)|/2)s)^{1/2}}(2\alpha^{3/2} + \frac{4\alpha^3}{((|\sin(v)|/2)s)^{1/2}}),$$  

and

$$C_{Rem_0} := 4(150(1 + C_{3.71}))^2\left(\frac{4C_{0.288}}{|\sin(v)|^2}\right)^3$$  

$$C_{Rem_1} := 8(150(1 + C_{3.71}))\left(\frac{4C_{0.288}}{|\sin(v)|^2}\right)^2$$  

$$\text{(9.316)} \quad \text{(9.317)}$$

Proof.

By (9.290), (9.293) and (9.300),

$$\mathcal{F}_P = (e_1 - z)P + PW_{0,0}(\theta, \alpha, \beta)P$$

$$-PW^{s,t}(\theta, \alpha, \beta)\overline{P}(H^{s,t}_P(\theta) - z)^{-1}\overline{P}W^{s,t}(\theta, \alpha, \beta)P.$$  

$$\text{(9.320)}$$

Next we use Lemma 9.7 to obtain,

$$PW^{s,t}(\theta, \alpha, \beta)\overline{P}(H^{s,t}_P(\theta) - z)^{-1}\overline{P}W^{s,t}(\theta, \alpha, \beta)P$$

$$= \int_{K_{s,t}} Pw_{0,1}(\theta, \alpha, \beta)(k) \frac{1}{H_{at}^{s,t}(\theta, \alpha, \beta) - (z - e^{-iu}|k|)}w_{1,0}(\theta, \alpha, \beta)(k)P$$

$$+Rem_0 + Rem_1,$$

where

$$Rem_0 := PW^{s,t}(\theta, \alpha, \beta)\overline{P}(H^{s,t}_P(\theta) - z)^{-1}$$

$$-(\overline{P}H^{s,t}_0(\theta, \alpha, \beta)\overline{P} - z)^{-1})W^{s,t}(\theta, \alpha, \beta)P.$$  

$$\text{(9.321)} \quad \text{(9.322)}$$
and

\[
\text{Rem}_1 := PW^{s,t}(\theta, \alpha, \beta)\overline{P}(H^{s,t}_0(\theta, \alpha, \beta) - z)^{-1}\overline{P}W^{s,t}(\theta, \alpha, \beta)P
\]

\[
- P(W^{s,t}_{1,0}(\theta, \alpha, \beta) + W_{0,1}(\theta, \alpha, \beta))\overline{P}(H^{s,t}_0(\theta, \alpha, \beta) - z)^{-1}
\]

\[
\cdot \overline{P}(W^{s,t}_{1,0}(\theta, \alpha, \beta) + W_{0,1}(\theta, \alpha, \beta))P.
\]

To estimate \( \text{Rem}_0 \) we use the second resolvent equation,

\[
\text{Rem}_0 = - PW^{s,t}(\theta, \alpha, \beta)\overline{P}(H^{s,t}_0(\theta) - z)^{-1}\overline{P}W^{s,t}(\theta, \alpha, \beta)P
\]

\[
\cdot (PH^{s,t}_0(\theta, \alpha, \beta)P - z)^{-1}\overline{P}W^{s,t}(\theta, \alpha, \beta)P
\]

\[
= - PW^{s,t}(\theta, \alpha, \beta)H^{s,t}_0(\theta, \alpha, \beta) - \tilde{z}^{1/2}
\]

\[
\cdot |H^{s,t}_0(\theta, \alpha, \beta) - \tilde{z}|^{1/2}(H^{s,t}_0(\theta) - z)^{-1}\overline{P}|H^{s,t}_0(\theta, \alpha, \beta) - \tilde{z}|^{1/2}
\]

\[
\cdot \overline{P}|H^{s,t}_0(\theta, \alpha, \beta) - \tilde{z}|^{-1/2}W^{s,t}(\theta, \alpha, \beta)|H^{s,t}_0(\theta, \alpha, \beta) - \tilde{z}|^{-1/2}
\]

\[
\cdot \overline{P}|H^{s,t}_0(\theta, \alpha, \beta) - \tilde{z}|^{-1/2}W^{s,t}(\theta, \alpha, \beta)P.
\]

where \( \tilde{z} \in \mathbb{C} \) is such that \( |\tilde{z} - e_1| = (|\sin(\nu)|/2)s \). We used that \( \overline{P} \) commutes with \( H^{s,t}_0(\theta, \alpha, \beta) \), that the range of \( (H^{s,t}_0(\theta) - z)^{-1} \) is contained in the range of \( \overline{P} \).

Eq. (9.316) follows from (9.277), Lemma 5.2, (9.297), Lemma 9.6 and (9.324).

Eq. (9.317) is deduced similarly, there we only use Lemma 9.6.

\[ \square \]

**Theorem 9.9.** Suppose that \( \theta = \pm i\nu \), with \( \nu \in \mathbb{R} \setminus \{0\} \), and that \( \theta \) satisfies (3.57) and (4.96). Suppose furthermore that \( \alpha \) satisfies (5.117) and that \( \alpha \leq a, |\beta| \leq b \). We assume additionally that \( (|\sin(\nu)|/2)s \leq 1 \). Then for every \( z \in \mathbb{C} \) with \( |z - e_1| \leq (|\sin(\nu)|/2)s \),

\[
\mathcal{F}_P = (e_1 - z)P - \alpha^3 Z^{s,t}_d(\theta) - \alpha^3 Z^{s,t}_c(\theta)
\]

\[
+ \text{Rem}_0 + \text{Rem}_1 + \text{Rem}_2 + PW_{0,0}(\theta, \alpha, \beta)P,
\]

(9.325)
where,
\[ Z_{od}^{\pm,t}(\theta) := \frac{1}{\alpha} \int_{K_{x,t}} dk P_{at,t}(\theta, \alpha, \beta) w_{0,1}(\theta, \alpha, \beta)(k) P_{\Omega,t} \]  
(9.326)
\[ \cdot \frac{1}{H_{at}(\theta, \alpha, \beta) - e_1 - \omega |k|} w_{1,0}(\theta, \alpha, \beta)(k) P_{at,t}(\theta) P_{\Omega,t} , \]
\[ Z_{d}^{\pm,t}(\theta) := \frac{1}{\alpha} \int_{K_{x,t}} \frac{dk}{e^{-w |k|}} P_{at,t}(\theta) w_{0,1}(\theta, \alpha, \beta)(k) P_{at,t}(\theta) \]
(9.327)
\[ \cdot w_{1,0}(\theta, \alpha, \beta)(k) P_{at,t}(\theta) P_{\Omega,t} , \]
Rem₂ ≤ \( C_{\text{9.328}} \) \( s^3 (1 + \log(s) + 4)^3 \)
(9.328)

\[ C_{\text{9.328}} := 4C^2_{\text{4.5}} \sqrt{24(|e_1'| + \delta + 1)^{1/2} C_{\text{9.328}}} (12 + 6b + 42\|\eta'\|_{\infty} + 18\|\eta''\|_{\infty})^2 (1 + \frac{3}{2} (4C_{\text{9.71}})^2 (\frac{4}{\delta |\sin \nu|} + 1)^2) \]

and Rem₀ and Rem₁ are defined in Lemma 9.8 and b is defined in (3.34).

Proof.
We use Lemmata 9.7 and 9.8 to obtain,
\[ \mathcal{F}_{P} = (e_1 - z) P - \alpha^3 Z_{od}^{\pm,t}(\theta) - \int_{K_{x,t}} dk P_{at,t}(\theta, \alpha, \beta) \]
\[ \cdot w_{0,1}(\theta, \alpha, \beta)(k) P_{at,t}(\theta, \alpha, \beta) \cdot \frac{1}{e_1 - z + \omega |k|} w_{1,0}(\theta, \alpha, \beta)(k) \]
\[ \cdot P_{at,t}(\theta, \alpha, \beta) P_{\Omega,t} \]
(9.329)
\[ - \int_{K_{x,t}} dk P_{at,t}(\theta, \alpha, \beta) w_{0,1}(\theta, \alpha, \beta)(k) \frac{1}{H_{at}(\theta, \alpha, \beta) - e_1 - \omega |k|} w_{1,0}(\theta, \alpha, \beta)(k) \]
\[ \cdot P_{at,t}(\theta, \alpha, \beta) P_{\Omega,t} + PW_{0,0}(\theta, \alpha, \beta) P + Rem₀ + Rem₁ , \]

where we used that 1 = \( P_{at,t}(\theta, \alpha, \beta) + \mathcal{F}_{at,t}(\theta, \alpha, \beta) \). Next note that for \(|k| ≥ s \) and \(|e_1 - z| ≤ \frac{s}{\omega} \),
\[ \left| \frac{1}{e_1 - z + \omega |k|} - \frac{1}{e^{-\omega |k|}} \right| ≤ \frac{s}{|k|^2} . \]
(9.330)
Using the second resolvent equation and (3.73), (9.271) we get
\[ \mathcal{F}_{at,t}(\theta, \alpha, \beta) \left( \frac{1}{H_{at}(\theta, \alpha, \beta) - e_1 - \omega |k|} - \frac{1}{H_{at}(\theta, \alpha, \beta) - e_1 + \omega |k|} \right) \]
\[ ≤ \frac{3}{2} (4C_{\text{3.71}})^2 \left( \frac{4}{\delta |\sin \nu|} \right)^2 s \]
(9.331)
For fixed $k$, the operator $w_{1,0}(\theta, \alpha, \beta)(k)$ is an operator on $H_{at}$, we have by (9.260) and (3.67) that
\[
\|w_{1,0}(\theta, \alpha, \beta)(k)P_{at,1}(\theta, \alpha, \beta)\| \leq 12\left(|e_1| + \delta\right)^{1/2}C3.68 \left( G^{s,t} + \nabla Q^{s,t}(\theta) \right) + 2|e^{s,t}(\theta)| \leq 24\left(|e_1| + \delta + 1\right)^{1/2}C3.68 \left( 12 + 6|b| + 42\|\eta\|_\infty \right) + 18\|\eta\|_\infty |\beta| + 18\|\eta''\|_\infty \left( 1 + |k| \right) \max_{j \in \{1,2,3\}} |G_{\theta,j}|.
\]
and a similar bound holds for $w_{0,1}(\theta, \alpha, \beta)(k)P_{at,1}(\theta, \alpha, \beta)$. Then we obtain (9.325) by (7.215), (9.329-9.332) and Remark A.2.

9.2.3 Proof of Theorem 5.6

Theorem 9.10. Suppose that $\theta = \pm i\nu$, with $\nu \in \mathbb{R} \setminus \{0\}$, and that $\theta$ satisfies (3.57) and (4.96). Suppose furthermore that $\alpha \leq a$ satisfies (5.117) and that $|\sin(\nu)/2|s \leq 1$ and $s = \alpha^v$ for some $v \in (0,2)$. We assume that $\beta = 0$.

Let (see (9.260))
\[
\tilde{E}_I := -\pi \int_{S^2} dS\|P_{at,0}(0, \alpha, 0)|e_1 - e_0| \cdot w_{1,0}(0, \alpha, 0)(x, \frac{1}{|x|}|e_1 - e_0|)\psi_0\|^2,
\]
where $S^2$ is the sphere and $\psi_0$ is a unit eigenvector of $P_{at,1}(0, \alpha, 0)$ (see (9.333)).

There is a constant $C_{9.352}$ such that
\[
|\tilde{E}_I - \text{Im}(E^{s,t}(\theta))| \leq 4C_{9.352} \alpha^{v/3}(\alpha^v|\log(\alpha^v)| + \alpha^v + \alpha^{2v}) + \alpha^{(3-v)/2}(1 + \alpha^{(3-v)/2})^3.
\]

The explicit value of the constant $C_{9.352}$ is written in (9.352).

The dilation operator (2.15) is a one parameter group of unitary operators when $\theta$ is real. The set of analytic vectors for the generator of the group is dense in $L^2(\mathbb{R}^3)$. We select an analytic vector $\psi$ such that
\[
\psi_0 := P_{at,1}(0, \alpha, 0)\psi \neq 0, \quad \|\psi_0\|_{H_{at}} = 1.
\]
We define
\[ \psi_\theta := P_{at,1}(\theta, \alpha, 0)u(\theta)\psi. \] (9.334)

It follows that
\[ \langle \psi_\theta | \psi_\theta \rangle_{H_{at}} \] (9.335)
is analytic in \( \theta \) for any theta satisfying (3.57) and (4.96). As \( u(\theta) \) is unitary for real \( \theta \), (9.335) is constant for real theta, thus
\[ \langle \psi_\theta | \psi_\theta \rangle_{H_{at}} = 1. \] (9.336)

Eq. (9.336) implies that the operator
\[ |\psi_\theta \rangle \langle \psi_\theta| \]
is a projection in \( L^2(\mathbb{R}^3) \). The range of \( |\psi_\theta \rangle \langle \psi_\theta| \) equals the range of \( P_{at,1}(\theta, \alpha, 0) \).

As the null space of \( P_{at,1}(\theta, \alpha, 0) \) is the orthogonal complement of the range of \( P_{at,1}(\theta, \alpha, 0)^* = P_{at,1}(\theta, \alpha, 0) \) we conclude that the null space of both projections coincide and therefore
\[ P_{at,1}(\theta, \alpha, 0) = |\psi_\theta \rangle \langle \psi_\theta|. \] (9.337)

Eq. (9.337) implies that
\[ Z_s,\infty,od(\theta) = z_{s,\infty,od}(\theta)P_{at,1}(\theta, \alpha, 0) \otimes P_{\Omega,\infty}, \]
\[ Z_s,\infty,d(\theta) = z_{s,\infty,d}(\theta)P_{at,1}(\theta, \alpha, 0) \otimes P_{\Omega,\infty}, \] (9.338)
\[ PW_{0,0}(\theta, \varsigma, \beta) = w_{s,\infty}(\theta)P_{at,1}(\theta, \alpha, 0) \otimes P_{\Omega,\infty}, \]
where
\[ z_{s,\infty,od}(\theta) = \langle \psi_\theta \otimes P_{\Omega,\infty} | Z_{s,\infty,od}(\theta)\psi_\theta \otimes P_{\Omega,\infty} \rangle_{H_{\infty,\infty}}, \]
\[ z_{s,\infty,d}(\theta) = \langle \psi_\theta \otimes P_{\Omega,\infty} | Z_{s,\infty,d}(\theta)\psi_\theta \otimes P_{\Omega,\infty} \rangle_{H_{\infty,\infty}}, \] (9.339)
\[ w_{s,\infty}(\theta) = \langle \psi_\theta \otimes P_{\Omega,\infty} | W_{0,0}(\theta, \alpha, \beta)\psi_\theta \otimes P_{\Omega,\infty} \rangle_{H_{\infty,\infty}}. \]

It follows from (3.265), (9.326) and (9.327) that
\[ Z_{od,\infty}(\theta) = z_{od,\infty}(\theta), \]
\[ z_{od,\infty}(\theta) = z_{od,\infty}(\theta), \]
\[ w_{0,\infty}(\theta) = w_{s,\infty}(\theta). \] (9.340)
Using (3.61), (3.73), (9.271), (9.332), (9.334) and remark A.2 we obtain
\[ |z_{0,s}(\theta)| \leq 4C_{\delta}^{3.68} \left( 24(|e_1| + \delta + 1)^{1/2}C_{\delta}^{3.68}(12 + 6|b|) + 42\|\eta^\prime\|_\infty + 18\|\eta^\prime\|_\infty|b| + 18\|\eta^\prime\|_\infty \right)^2 \]
(9.341)

Similarly we get
\[ |z_{d,0,s}(\theta)| \leq 2\left( 24(|e_1| + \delta + 1)^{1/2}C_{\delta}^{3.68}(12 + 6|b|) + 42\|\eta^\prime\|_\infty + 18\|\eta^\prime\|_\infty|b| + 18\|\eta^\prime\|_\infty \right)^2 \cdot \sup_{|\theta| \leq \frac{1}{120}} 16\|u(\theta)\|_2^2C_{A.5}^{2.82} \]
(9.342)

and
\[ |w_{0,s}(\theta)| \leq 4\left( 24C_{\delta}^{3.68}(12 + 42\|\eta^\prime\|_\infty + 18\|\eta^\prime\|_\infty) \right)^2 \cdot \sup_{|\theta| \leq \frac{1}{120}} 16\|u(\theta)\|_2^2C_{A.5}^{2.82} \]
(9.343)

The numbers \( z_{0,\infty}^{0,s}(\theta) \), \( z_{d,\infty}^{0,s}(\theta) \) and \( w_{0,\infty}(\theta) \) do not depend on \( \theta \). We show this for the case of \( z_{d,\infty}^{0,s}(\theta) \) to present the argument: We define the functions \( \theta \rightarrow f_\theta \), \( \theta \rightarrow g_\theta \) by
\[
 f_\theta(x, k) := \frac{1}{e^{-|k|^2}}P_{ad}(\theta, \alpha, 0)w_{1,0}(\theta, \alpha, 0)\psi_\theta,
 g_\theta(x, k) := (w_{1,0}(\theta, \alpha, 0))^*\psi_\theta
\]
(9.344)

with values in \( L^2(\mathbb{R}^3 \times K^{0,\infty}) \).

It is clear that
\[ z_{d,\infty}^{0,s}(\theta) = \langle g_\theta | f_\theta \rangle_{L^2(\mathbb{R}^3 \times K^{0,\infty})}. \]
(9.345)

For real theta
\[ \langle g_\theta | f_\theta \rangle_{L^2(\mathbb{R}^3 \times K^{0,\infty})} \]
\[ = \langle u(\theta) \otimes u(-\theta)g_0 | u(\theta) \otimes u(-\theta)f_0 \rangle_{L^2(\mathbb{R}^3 \times K^{0,\infty})} \]
(9.346)

\[ = \langle g_0 | f_0 \rangle_{L^2(\mathbb{R}^3 \times K^{0,\infty})}, \]
since \( u(\theta) \) is unitary (see 2.13). As \( z_{d}^{0,\infty}(\theta) \) is analytic, we conclude that it is constant. Similarly we prove that \( z_{d}^{0,\infty}(\theta) \) and \( w_{\infty,0}(\theta) \) are constant. We denote by \( z_{d}^{0,\infty}(0) \) the limit \( \theta \to 0 \) of \( z_{d}^{0,\infty}(\theta) \). From (9.327) taking \( \theta \) to zero we conclude that \( z_{d}^{0,\infty}(\theta) \) is real and similarly we conclude that \( w_{\infty,0}(\theta) \) is real. We compute now the imaginary part of \( z_{d}^{0,\infty}(\theta) \).

\[
\text{Im} \ z_{d}^{0,\infty}(\theta) = \lim_{\vartheta \to 0} \frac{1}{2\pi i} \int_{K^{0,\infty}} dk \langle \psi_{i\vartheta} | w_{0,1}(i\vartheta, \alpha, 0)(k) \rangle 
\]

\[
\cdot \frac{P_{at,0}(i\vartheta, \alpha, 0) + (1 - P_{at,0}(i\vartheta, \alpha, 0) - P_{at,1}(i\vartheta, \alpha, 0))}{P_{at}(i\vartheta, \alpha, 0) - (e_{1} - e^{i\vartheta}|k|)} w_{1,0}(i\vartheta, \alpha, 0)(k) \psi_{i\vartheta} \rangle \mathcal{H}_{at}
\]

\[
- \frac{1}{2\pi i} \int_{K^{0,\infty}} dk \langle \psi_{i\vartheta} | w_{0,1}(i\vartheta, \alpha, 0)(k) \rangle 
\]

\[
\cdot \frac{P_{at,0}(i\vartheta, \alpha, 0) + (1 - P_{at,0}(i\vartheta, \alpha, 0) - P_{at,1}(i\vartheta, \alpha, 0))}{P_{at}(i\vartheta, \alpha, 0) - (e_{1} - e^{i\vartheta}|k|)} w_{1,0}(i\vartheta, \alpha, 0)(k) \psi_{i\vartheta} \rangle \mathcal{H}_{at}
\]

\[
= \lim_{\vartheta \to 0} \frac{1}{2\pi i} \int_{K^{0,\infty}} dk \langle \psi_{0} | w_{0,1}(0, \alpha, 0)(k) \rangle \left( \frac{P_{at,0}(0, \alpha, 0)}{e_{0} - (e_{1} - e^{i\vartheta}|k|)} \right) w_{1,0}(0, \alpha, 0)(k) \psi_{0} \rangle \mathcal{H}_{at}
\]

\[
= \lim_{R \to \infty} \lim_{\vartheta \to 0} \frac{1}{2\pi i} \int_{S^{2}} dS \int_{0}^{R} dr |e_{0} - e_{1}|^{2}
\]

\[
\langle \psi_{0} | w_{0,1}(0, \alpha, 0)(|e_{0} - e_{1}|^{2}) \left( e^{-i\vartheta} \frac{P_{at,0}(0, \alpha, 0)}{e_{0} - e_{1} + e^{i\vartheta}|k|} \right) w_{1,0}(0, \alpha, 0)(|e_{0} - e_{1}|^{2}) \psi_{0} \rangle \mathcal{H}_{at},
\]

where in the last step we use the exponential decay of \( w_{1,0}(0, \alpha, 0) \), \( S^{2} \) is the 2-sphere, \( dS \) is the volume element in \( S^{2} \) and \( \frac{k}{|k|} \in S \).

To compute the radial integral we use the analyticity of the function \( \tilde{z} \to \frac{1}{e_{0} - e_{1} + k} \), then we have for large \( R \)

\[
\lim_{\vartheta \to 0} \frac{1}{2\pi i} \int_{0}^{R} \left( \frac{e^{-i\vartheta}}{e_{0} - e_{1} + e^{i\vartheta}|k|} - \frac{e^{i\vartheta}}{e_{0} - e_{1} + e^{i\vartheta}|k|} \right)
\]

\[
= \frac{1}{2\pi i} \int_{S^{2}} \frac{1}{e_{0} - e_{1} + e^{i\vartheta}|k|} d\tilde{z} = 1 ,
\]

where \( S^{2} \) is the 2-sphere, \( dS \) is the volume element in \( S^{2} \) and \( \frac{k}{|k|} \in S \).
It follows from Lemma 9.8, Theorem 9.9 and Eqs. (9.341) to (9.343) that
\[ \mathcal{F}_P = (e_1 - z + w^{0,\infty}(0) - \alpha^3 z_{od}^{0,\infty}(0) - \alpha^3 z_d^{0,\infty}(0))P + Rem, \] (9.350)
where
\[ \|Rem\| \leq C_{9.352} \alpha^3 (s \log|s| + s + s^2 + \frac{\alpha^{3/2}}{s^{1/2}} (1 + \frac{\alpha^{3/2}}{s^{1/2}})^3) \] (9.351)
and
\[ C_{9.352} := 5^3 C_{Rem_0} + C_{Rem_1} (|\sin(\nu)|/2)^2 + 10C_{9.328} (1 + \sup_{|\theta| \leq 1/2} ||u(\theta)|/2||)^2. \] (9.352)

Now we take
\[ s = \alpha^v, \quad v \in (0,2). \] (9.353)
It follows from the fact that \( e_1, w^{0,\infty}(0) \) and \( z_d^{0,\infty}(0) \) are real that if
\[ \frac{2C_{9.352} \alpha^3 (\alpha^v \log(\alpha^v)) + \alpha^v + \alpha^{2v} \alpha^{(3-v)/2} (1 + \alpha^{(3-v)/2})^3}{|\text{Im}(z + \alpha^3 z_{od}^{0,\infty}(0))|} < \frac{1}{2}, \] (9.354)
then \( \mathcal{F}_P \) is invertible and by (9.299) \( H^{s,t}(\theta,\alpha,0) \) is also invertible. We conclude by remark 5.4 that
\[ |\text{Im}(E^{s,t}(\theta) + \alpha^3 z_{od}^{0,\infty}(0))| \leq 4C_{9.352} \alpha^3 (\alpha^v \log(\alpha^v)) + \alpha^v + \alpha^{2v} \]
\[ + \alpha^{(3-v)/2} (1 + \alpha^{(3-v)/2})^3, \] (9.355)
which proves (5.131).

A Relevant Integrals for the Interaction Terms

In this appendix we estimate some integrals and define some constants that are used frequently in most parts of the text.

Lemma A.1. For any positive real numbers \( s, t \) with \( s > t \), the following
estimates hold (see (2.34) and (2.38)),

\[
\begin{align*}
\sup_{\theta \leq 1/30, \vec{x} \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{s \geq |\vec{k}| \geq t} |G(\theta)_j(x,k)|^2 dk & \leq \alpha^3 \min(C_1^2, C_2^2 s(s-t)) , \\
\sup_{\theta \leq 1/30, \vec{x} \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{s \geq |\vec{k}| \geq t} \frac{|G(\theta)_j(x,k)|^2}{|\vec{k}|^2} dk & \leq \alpha^3 \min(C_{1,\omega}^2, C_2^2(s-t)) , \\
\sup_{\theta \leq 1/30, \vec{x} \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{|\vec{k}| \geq s} \frac{|G(\theta)_j(x,k)|^2}{|\vec{k}|^2} dk & \leq \alpha^3 C_2^2(|\log(s)| + 1) , \\
\sup_{\theta \leq 1/30, \vec{x} \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{|\vec{k}| \geq s} \frac{||k||G(\theta)_j(x,k)||^2}{|\vec{k}|^2} dk & \leq \alpha^3 \min(C_1^2, C_2^2 s^2(s-t)) , \\
\sup_{\theta \leq 1/2, \vec{x} \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{s \geq |\vec{k}| \geq t} \frac{||k||G(\theta)_j(x,k)||^2}{|\vec{k}|^2} dk & \leq \alpha^3 \min(C_1^2, C_2^2 s^2(s-t)) ,
\end{align*}
\]

where

\[
\begin{align*}
C_1 & := 2 \frac{e^{3/4}}{2(2\pi)^{1/2}} , \quad C_2 := 2^{1/2} \frac{e^{1/4}}{2\pi} , \\
C_{1,\omega} & := 2^{1/2} \frac{e^{1/2}}{2^{1/4}(2\pi)^{1/4}} \frac{e^{1/4}}{2^{1/4}} .
\end{align*}
\]

Proof.

By (2.38), using spherical coordinates we get,

\[
\begin{align*}
\int_{s \geq |\vec{k}| \geq t} |G(\theta)_j(x,k)|^2 dk & \leq 2 \frac{e^{3/2}}{2(2\pi)^{1/2}} \frac{e^{1/4}}{2\pi} \int_0^s \exp(-2e^{-1}r^2) r \, dr \\
& \leq \min(2 \frac{e^{3/2}}{2(2\pi)^{1/2}} \frac{e^{1/4}}{2\pi} , 2 \frac{e^{3/2}}{4(2\pi)^{1/2}} e^{-3/2}) .
\end{align*}
\]

The second inequality in (A.1) is estimated similarly.

For the third inequality we use,

\[
\begin{align*}
\int_0^\infty \exp(-2e^{-1}r^2) \frac{1}{r} & \leq |\log(s)| \exp(-2e^{-1}s^2) + \int_1^\infty dt \log(t^2) \exp(-2e^{-1}t^2) 4e^{-1}t \\
& \leq |\log(s)| + 2e^{-1} \int_1^\infty dt \log(t) \exp(-2e^{-1}t) \leq |\log(s)| + \int_1^\infty dt \frac{1}{t} \exp(-2e^{-1}t) \\
& \leq |\log(s)| + \int_1^\infty dt \exp(-2e^{-1}t) \leq |\log(s)| + 1 ,
\end{align*}
\]

where in the first and third inequalities we used integration by parts.

The forth integral is calculated similarly, here we either compute the integral \(\int_0^\infty \exp(-2e^{-1}r^2)r^3\) or use the bound \(\int_0^s \exp(-2e^{-1}r^2)r^3 \leq s^2 \int_0^s \exp(-2e^{-1}r^2)r\). The fifth integral is estimated similarly, additionally we have to compute the variance of the Gaussian \(\exp(-2e^{-1}r^2)\).
Remark A.2. We define
\[ C_{A.5} := \sqrt{\frac{e}{2}} \max(C_1, C_{1,\omega}, C_2). \] (A.5)
From the previous lemma it follows that
\[ \sup_{|\theta| \leq 1/30, x \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{s \geq |k| \geq t} |k|^m |G^{s,t}(\theta)_j(x, k)|^2 dk \]
\[ \leq \alpha^3 \min(C_{A.5}^2, C_{A.5}^{m+2}), \quad m \in \{-1, \cdots 2\}, \]
\[ \sup_{|\theta| \leq 1/30, x \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{|k| \geq t} |k|^{-2} |G^{s,t}(\theta)_j(x, k)|^2 dk \]
\[ \leq \alpha^3 C_{A.5}^2 \left( |\log(t)| + 1 \right). \] (A.6)
Remember that the functions \( G_{s,t}^{\theta,j} \) are defined in (6.29).

Remark A.3. We define
\[ C_{A.7} := 3^{1/2} C_{A.5} \left( 4 + 4 \|\eta'\|_\infty + 2 \|\eta''\|_\infty \right). \] (A.7)
From the previous lemma it follows that
\[ \sup_{|\theta| \leq 1/30, x \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{s \geq |k| \geq t} |k|^m |G_{s,t}^{\theta,j}(x, k)(1 + |x|^2)^{-1/2}|^2 dk \]
\[ \leq \alpha^3 C_{A.7}^{m+4}, \quad m \in \{-1, \cdots 2\}, \]
\[ \sup_{|\theta| \leq 1/30, x \in \mathbb{R}^3, j \in \{1,2,3\}} \int_{s \geq |k| \geq t} |k|^m \left| \frac{\partial}{\partial x} G_{s,t}^{\theta,j}(x, k) \right|^2 dk \]
\[ \leq \alpha^3 C_{A.7}^{m+4}, \quad m \in \{-1, \cdots 2\}. \] (A.8)
Proving: The proof of (A.8) follows from the proof of Lemma A.1, Remark A.2 and the following. By (2.38), (6.29) and (8.232) we have that
\[ |G_{s,t}^{\theta,j}(x, k)(1 + |x|^2)^{-1/2}| \leq \alpha |k||G^{s,t}(\theta)_j(0, k)| \left| \frac{|x|}{(1+|x|^2)^{1/2}} \right|^2 \]
\[ + |G^{s,t}(\theta)_j(0, k)| \cdot |1 - \eta(|x||k|)| \left| \frac{|x|}{(1+|x|^2)^{1/2}} \right|^2 \]
\[ + |k||G^{s,t}(\theta)(0, k) \cdot x(1 + |x|^2)^{-1/2}||\eta'||_\infty \]
\[ \leq |k||G^{s,t}(\theta)(0, k)|(2 + \|\eta'\|_\infty), \]
where we used that \(|1 - \eta(|x||k|)| = 0\) for \(|x||k| \leq 1\) and therefore \(|1 - \eta(|x||k|)| \leq |1 - \eta(|x||k|)| \cdot |x||k| \).
To prove (A.9) we notice that
\[
|\frac{\partial}{\partial x} G_{s,t}^{s,t}(x,k) | \leq |G^{s,t}(0,k)| \cdot |k| (\alpha + 4\|\eta\|_\infty + |x||k| \cdot |\eta''(|x||k|)|) \]
and that \(|x||k||\eta''(|x||k|)| \leq 2|\eta''(|x||k|)|
\]
(A.11)

Lemma A.4. Let \( r \in (0, \frac{1}{2}) \) and \( \iota \) be a vector-valued analytic function defined on a neighbourhood the closed ball \( D_r^c(0) := \{ \theta \in \mathbb{C} : |\theta| \leq r \} \) and let \( \gamma := \{ re^{it} : t \in [0, 2\pi] \} \).
For any \( s < r \), every \( \theta \in D_s(0) \), every \( h \in \mathbb{C} \) such that \( \theta + h \in D_s(0) \) the following estimations hold.
\[
\| \iota(\theta + h) - \iota(\theta) - h \iota' \| \leq \frac{|h|}{2(r-s)^2} \max_{z \in \gamma} \| \iota(z) \| ,
\]
(A.12)
\[
\| \iota(\theta + h) - \iota(\theta) \| \leq \frac{|h|}{2(r-s)^2} \max_{z \in \gamma} \| \iota(z) \| .
\]
In particular for every \( j \in \{1, 2, 3\} \) and every \( \theta \in D_{1/120}(0) \), every \( h \in \mathbb{C} \) such that \( \theta + h \in D_{1/120}(0) \),
\[
\| G^{s,t}(\theta + h)_{j} - G^{s,t}(\theta)_{j} \|_{\rho} \leq \alpha^{3/2} \frac{|h|}{2(1/60-1/120)^2} C_{A.5} (1 + \frac{1}{\rho^{1/2}}) ,
\]
(A.13)
where \( C_{A.5} \) is defined in (A.5).
Proof:
We estimate the first inequality in (A.12), the second is estimated similarly.

We use the Cauchy’s integral formula the contour \( \gamma := \{ re^{it} : t \in [0, 2\pi] \} \) to get,
\[
|\frac{\iota(\theta + h) - \iota(\theta)}{h} - \iota'(\theta)| = \frac{|h|}{2\pi} \left| \int_{\gamma} \frac{\iota(z)}{(z-\theta)(z-(\theta+h))} dz \right| \leq \frac{|h|}{2(r-s)^3} \sup_{z \in \gamma} |\iota(z)| .
\]
(A.14)
Now we choose \( \iota \) to be given by
\[
\iota(\theta) := G^{s,t}(\theta)_{j}(x,k) .
\]
(A.15)
Eqs. (A.13) follows from (A.12) and Remark A.2
The Pauli-Fierz Transformation

The operator \( \lambda^0_{PF}(0) \) defined in (2.41) given by (see (6.30))

\[
\lambda^0_{PF}(0) = a^*(Q^0(0)) + a(Q^0(0))
\]

is self-adjoint in \( \mathcal{H}^0 \) and therefore the operator

\[
e^{-i\lambda^0_{PF}(0)}
\]

is unitary.

Assumptions (2.20) imply that the components of the magnetic potential (see (2.39)) commute between each other:

\[
[A^0(x)_\nu, A^0(y)_\mu] = 0, \ \nu, \mu \in \{1, 2, 3\}.
\]

(B.17)

By (B.17),

\[
e^{-i\lambda^0_{PF}(0)} A^0 e^{i\lambda^0_{PF}(0)} = A^0.
\]

(B.18)

We have that,

\[
\frac{\partial}{\partial t} e^{-i\lambda^0_{PF}(0)} (i\nabla) e^{i\lambda^0_{PF}(0)} = (\nabla \lambda^0_{PF}(0)).
\]

(B.19)

Since this last expression does not depend on \( t \), we can integrate with respect to \( t \) to obtain:

\[
e^{-i\lambda^0_{PF}(0)} (-i\nabla) e^{-i\lambda^0_{PF}(0)} = -i\nabla + (\nabla \lambda^0_{PF}(0)).
\]

(B.20)

It is easy to prove that

\[
[\hat{H}^0, a^*(|k\rangle Q^0(0))] = a^*(|k\rangle Q^0(0)).
\]

(B.21)

Taking adjoints we get

\[
[\hat{H}^0, a(Q^0(0))] = -a(|k\rangle Q^0(0)).
\]

(B.22)

Using (B.21) and (B.22) we get (see also (2.39)),

\[
\frac{\partial}{\partial t} e^{-it\lambda^0_{PF}(0)} \hat{H}^0 e^{it\lambda^0_{PF}(0)}
\]

\[
= e^{-it\lambda^0_{PF}(0)} i(a^*(|k\rangle Q^0(0)) - a(|k\rangle Q^0(0))) e^{it\lambda^0_{PF}(0)}.
\]

(B.23)

By the commutation relations (2.34) we have for \( \nu, \mu \in \{1, 2, 3\} \),

\[
\frac{\partial}{\partial t} e^{-it\lambda^0_{PF}(0)} i(a^*(|k\rangle Q^0(0)) - a(|k\rangle Q^0(0))) e^{it\lambda^0_{PF}(0)}
\]

\[
= \langle Q^0(0)|[k|Q^0(0)] + \langle |k\rangle Q^0(0)| Q^0(0). 
\]

(B.24)
Since this last quantity does not depend on $t$, we can integrate and obtain
\[
e^{-it\lambda_0 P_F(0)} (a^* (|k|Q^0(0)) - a (|k|Q^0(0))) e^{it\lambda_0 P_F(0)}
= i (a^* (|k|Q^0(0)) - a (|k|Q^0(0))) + 2t \langle Q^0(0) | |k|Q^0(0) \rangle.
\] (B.25)

Integrating (B.23) we get,
\[
e^{-i\lambda_0 P_F(0)} \hat{H}^0 e^{i\lambda_0 P_F(0)} = \hat{H}^0 + i (a^* (|k|Q^0(0)) - a (|k|Q^0(0)))
+ \langle Q^0(0) | |k|Q^0(0) \rangle.
\] (B.26)

Eqs. (2.21), (B.20) and (B.26) imply (2.43).

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