Effects of the Tsallis distribution in the linear sigma model

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Abstract. The effects of the Tsallis distribution which has a parameter $q$ on physical quantities are studied using the linear sigma model in chiral phase transitions. The temperature dependences of the condensate and mass for various $q$ are shown, where the Tsallis distribution approaches the Boltzmann-Gibbs distribution as $q$ approaches 1. The critical temperature and energy density are described with digamma function, and the $q$ dependences of these quantities and the extension of Stefan-Boltzmann limit of the energy density are shown. The following facts are clarified. The chiral symmetry restoration for $q > 1$ occurs at low temperature, compared with the restoration for $q = 1$. The sigma mass and pion mass reflect the restoration. The critical temperature decreases monotonically as $q$ increases. The small deviation from the Boltzmann-Gibbs distribution results in the large deviations of physical quantities, especially the energy density. It is displayed from the energetic point of view that the small deviation from the Boltzmann-Gibbs distribution is realized.

1 Introduction

A power-like distribution appears in various branches of science and has been investigated. One of them is called Tsallis distribution, and it has been studied in the last few decades. The distribution is one parameter extension of the Boltzmann-Gibbs distribution. This distribution has been used to analyze various phenomena [1]. An example is the momentum distribution in high energy collisions.

A problem in high energy collisions is to obtain the momentum distribution of emitted particles. The distribution shows power-like behavior, and has been analyzed with the Tsallis distribution. The momentum distribution is fitted with a Tsallis distribution which has a parameter $q$, where the Tsallis distribution with $q = 1$ is the Boltzmann-Gibbs distribution. It was shown that the momentum distribution is fitted well by the Tsallis distribution with the parameter $q$ which is close to 1 [2,3,4,5,6].

The phase transition is an important topic in high energy heavy ion collisions. A topic is the study of the equation of state in the nonextensive statistics [7,8]. The NJL model is often used to study the phase transition. This model was used to study the effects of the nonextensive statistics [9]. The linear sigma model is also used to study the phase transition at high energies. The model is the extension of $x^4$ model, and the effects of the Tsallis statistics was studied in the $x^4$ model in quantum mechanics [10,11]. The phase transition should be investigated when the distribution is described with the Tsallis distribution.

There are several ways to derive the Tsallis distribution. One of them is to extremize the Tsallis entropy [12,13]. The entropy is not only the origin of the distribution, but some origins of the distribution also exist: temperature fluctuation [6,14,15], multiplicative noise [15,16,17], etc [15,17]. Therefore, the definition of the expectation value of physical quantities is not always obvious, as discussed in the study of the Tsallis nonextensive statistics.

The purpose of this paper is to clarify the effects of the Tsallis distribution on physical quantities in chiral phase transitions. We employ the linear sigma model and study the $q$ dependences of the physical quantities, where the parameter $q$ is introduced in the Tsallis distribution. The effects of the distribution on the following quantities are investigated under the massless free particle approximation: the condensate, the mass, the critical temperature, and the energy density.

The expressions of the physical quantities are obtained when the momentum distribution is the Tsallis distribution. The following things are shown. The chiral symmetry restoration for $q > 1$ occurs at low temperature, compared with the restoration for $q = 1$. The sigma mass and pion mass also changes, reflecting the chiral symmetry restoration. It is shown that the critical temperature decreases as $q$ increases. The extension of the Stefan-Boltzmann limit of the energy density are shown: The energy density changes remarkably as a function of $q$. This fact indicates that the deviation of $q$ from $q = 1$ is small.

The outline of this paper is as follows: in sect. 2 the expressions of physical quantities are calculated in the linear sigma model, when the momentum distribution is the Tsallis distribution. In sect. 3 the numerical values of the quantities are shown for various $q$. Section 4 is assigned for discussion and conclusion.
2 Expressions of some physical quantities

2.1 Condensate and mass

Scalar fields \( \phi = (\phi_0, \phi_1, \ldots, \phi_{N-1}) \) are used and the Hamiltonian density of the linear sigma model is given by

\[
\mathcal{H} = \frac{1}{2} (\partial^2 \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \lambda \left( \phi^2 - v^2 \right)^2 - H \phi_0, \tag{1}
\]

where \((\partial^2 \phi)^2 \equiv \sum_{i=0}^{N-1} (\partial^0 \phi_i)^2\), \((\nabla \phi)^2 \equiv \sum_{i=0}^{N-1} (\nabla \phi_i)^2\), and \(\phi^2 \equiv \sum_{i=0}^{N-1} (\phi_i)^2\). The Hamiltonian is rewritten by inserting the decomposition \(\phi_i = \phi_{ic} + \phi_{ih}\) into eq. (1), where \(\phi_{ic}\) represents the condensate and \(\phi_{ih}\) is the remaining part.

The expectation value of the Hamiltonian is calculated. The expectation value \(\langle (\phi_{ic})^{2s+1} \rangle\) is zero under the free particle approximation, where \(s\) is a non-negative integer and \(\langle \rangle\) represents the expectation value of the physical quantity \(\mathcal{O}\). The expectation value of Hamiltonian under the approximation is given by

\[
\langle \mathcal{H}(\phi) \rangle = \mathcal{H}(. \phi_{ic}) + \frac{1}{2} \left( (\partial^0 \phi_{ih})^2 + \frac{1}{2} (\nabla \phi_{ih})^2 \right) + \frac{\lambda}{2} \left( \phi_{ic}^2 - v^2 \right)^2 \langle \phi_{ih}^2 \rangle + \frac{\lambda}{4} \langle \phi_{ih}^4 \rangle \tag{2}
\]

where \(\phi_{ic}^2 \equiv \sum_{i=0}^{N-1} (\phi_{ic})^2\) and \(\phi_{ih}^2 \equiv \sum_{i=0}^{N-1} (\phi_{ih})^2\).

The mass squared \(m_i^2\) is defined by

\[
m_i^2 := \frac{\partial^2 \langle \mathcal{H}(\phi) \rangle_{\text{MFPA}}}{\partial \phi_{ic}^2} \equiv \lambda \left[ \phi_{ic}^2 + (N + 2) K_q(T) - v^2 \right] \phi_{ic} - H \delta_{i0} = 0. \tag{3}
\]

where \(K_q(T)\) is applied to the calculate the approximate value of the various physical quantities. The quantity \(\langle (\phi_{ih})^2 \rangle\) under the approximation is given by

\[
\langle (\phi_{ih})^2 \rangle = \int \frac{d^2k}{(2\pi)^2} \left( a_{j,k}^\dagger a_{j,k} \right)_{\text{MFPA}} \tag{4}
\]

where \(a_{j,k}\) is the annihilation operator for the field \(\phi_{ih}\). The second term is discarded in the present calculation, because this term is the contribution of the vacuum. The expectation value of \((\phi_{ih})^2\) under MFPA is represented as

\[
\langle (\phi_{ih})^2 \rangle_{\text{MFPA}} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k} f_q(k), \tag{5}
\]

where \(f_q(k)\) is the distribution function.

The Tsallis distribution is useful to describe the momentum distribution in the analysis of the experiments of high energy heavy ion collisions. Therefore, the Tsallis distribution is assumed as \(f_q(k)\), where \(q\) is a parameter introduced in the Tsallis distribution. The Tsallis distribution for boson in MFPA is given by

\[
f_q(k, T) = \frac{1}{1 + (q - 1) \left( \frac{T}{\hbar} \right)^{1-q}} - 1, \tag{6}
\]

where \([x]_q = x\) for \(x \geq 0\) and \([x]_q = 0\) for \(x < 0\). We define \(K_q(T)\) as \(K_q(T) := \langle (\phi_{ih})^2 \rangle_{\text{MFPA}} \). The expectation value of Hamiltonian density \(\langle H(\phi) \rangle_{\text{MFPA}}\) is given by

\[
\langle H(\phi) \rangle_{\text{MFPA}} = \mathcal{H}(\phi_{ic}) + \frac{\lambda}{2} \left( (N + 2) \phi_{ic}^2 - N v^2 \right) K_q(T) \tag{7}
\]

where \(\phi_{ic}^2 \equiv \sum_{i=0}^{N-1} (\phi_{ic})^2\), \(\phi_{ih}^2 \equiv \sum_{i=0}^{N-1} (\phi_{ih})^2\), and

\[
The last three terms in the right-hand side of eq. (6) are independent of \(\phi_{ic}\).

The potential in eq. (11) is tilted to the \(\phi_0\) direction when \(H \neq 0\). Therefore, the value of the condensate \(\phi_{ic}\) for \(j \neq 0\) is zero for \(H \neq 0\). Equation (11) for \(\phi_{ic}\) is reduced to the following equation:

\[
\phi_{ic}^2 + [(N + 2) K_q(T) - v^2] \phi_{ic} - H / \lambda = 0. \tag{8}
\]

The mass squared \(m_i^2\) is given by

\[
m_i^2 = \lambda \left[ (1 + 2\delta_{i0}) (\phi_{ic})^2 + (N + 2) \phi_{ih}^2 \right] \tag{9}
\]

Once \(K_q(T)\) is given, the condensate \(\phi_{ic}\) and mass \(m_i\) are obtained. The quantity \(K_q(T)\) is expressed with digamma function \(\psi(x)\):

\[
K_q(T) = \frac{I(1, -1)}{2\pi^2} = \frac{T}{2\pi^2(q-1)} \{ \psi(2-q) - \psi(3-2q) \} \quad (q < 3/2), \tag{10}
\]

where \(I(1, -1)\) is given by eq. (23a) in appendix A. The restriction of \(q\) comes from the condition that the integral converges. The quantity \(K_q(T)\) approaches \(T^2/12\) as \(q\) approaches 1, as is expected [18, 19].
2.2 Critical temperature and energy density

As is well-known, the critical temperature $T_c$ cannot be defined definitely when $H \neq 0$. In the present study, the critical temperature is the temperature at which a local minimum of the potential vanishes: a local minimum and local maximum merge. This critical temperature $T_c(q)$ is given by

$$T_c(q) = \frac{2\pi^2}{(N+2)} \frac{q - 1}{\psi(2-q) - \psi(3-2q)} \left[ \sqrt{v^2 - 3\left(\frac{4H}{\lambda}\right)^{2/3}} \right] \quad (q < 3/2). \quad (12)$$

Therefore, the ratio $T_c(q)/T_c(q = 1)$ is

$$\frac{T_c(q)}{T_c(q = 1)} = \frac{\pi^2}{6} \frac{q - 1}{\psi(2-q) - \psi(3-2q)} \quad (q < 3/2). \quad (13)$$

This ratio is independent of the number of fields $N$.

The mass of the field $\phi_0$ has a minimum as a function of the temperature. The temperature at the minimum, $T_0(q)$, is calculated by differentiating $m_0^2$. The ratio $T_0^+(q)/T_c(q)$ is

$$\frac{T_0^+(q)}{T_c(q)} = \frac{\sqrt{v^2 - 3\left(\frac{4H}{\lambda}\right)^{2/3}}}{\sqrt{v^2 - \frac{3}{4}\left(\frac{4H}{\lambda}\right)^{2/3}}}. \quad (14)$$

This ratio is independent of the parameter $q$ and the number of fields $N$. This value is determined only by the parameters of the linear sigma model. The mass $m_0$ and condensate $\phi_0$ at $T_0$ are given by

$$m_0(T_0) = \sqrt{6\lambda} \left(\frac{H}{4\lambda}\right)^{1/3}, \quad (15a)$$

$$\phi_0(T_0) = \frac{H}{4\lambda} \quad (15b)$$

The mass of the field $\phi_j$ ($j \neq 0$) does not have extremum at $T \neq 0$.

The energy density $\varepsilon$ of free particles for one field ($N = 1$) and pressure density $p$ are given by

$$\varepsilon = \int \frac{dk}{(2\pi)^3} \omega(k, T) f_0(k, T), \quad (16a)$$

$$p = \int \frac{dk}{(2\pi)^3} \frac{k^2}{3\omega(k, T)} f_0(k, T), \quad (16b)$$

where $\omega(k, T)$ is the energy of a particle. The relation $\varepsilon = 3p$ is also hold in the massless case. The energy density divided by $T^4$ for $N$ fields is calculated with eq. (16a) in appendix A

$$\frac{\varepsilon(q)}{T^4} = \frac{N}{2\pi^2(q-1)^3} \left\{ \psi(2-q) - \psi(5-4q) \right\} \quad (q < 5/4). \quad (17)$$

The right-hand side is independent of the temperature. This is the extension of the Stefan-Boltzmann limit of the energy density. The restriction of $q$ also comes from the condition that the integral converges. The ratio $\varepsilon(q)/\varepsilon(q = 1)$ is given by

$$\frac{\varepsilon(q)}{\varepsilon(q = 1)} = \frac{15}{2\pi^4(q-1)^3} \left\{ \psi(2-q) - \psi(5-4q) \right\} - 3 \left[ \psi(3-2q) - \psi(4-3q) \right] \quad (q < 5/4). \quad (18)$$

The numerical values of the above quantities are shown in the next section.

3 Numerical results

In this section, the values of the physical quantities are calculated numerically under the approximations discussed in the previous section. The number of the fields $N$ is set to 4. Therefore, $\phi_0$ and $\phi_j$ are interpreted as the sigma field and pion fields, respectively. The values of the parameters of the linear sigma model are set to $\lambda = 20$, $\nu = 87.4$ MeV, and $H = (119$ MeV$)^3$. At $T = 0$, the parameters generate $m_0 = 600$ MeV, $m_j = 135$ MeV ($j = 1, 2, 3$), and pion decay constant $f_\pi = 92.5$ MeV.

Firstly, the temperature dependences of the condensate $\phi_0$, sigma mass $m_0$, and pion mass $m_j$ ($j = 1, 2, 3$) are shown for $q = 0.9, 1.0, 1.1$. These temperature dependences for $q = 1$ are well-known. Figure (1a) shows the temperature dependences of the condensate for $q = 0.9, 1.0, 1.1$. It is shown that the temperature dependence at $q \neq 1$ is similar to that at $q = 1$. High temperature is required to restore the symmetry for small $q$. This $q$ dependence is easily explained by the fact that the ratio of the number of the particles with large momentum at $q > 1$ is larger than that at $q = 1$. The difference between different $q$ in the condensate can be seen at $T \sim T_c$. Figure (1b) shows the temperature dependence of the sigma mass for $q = 0.9, 1.0, 1.1$. The sigma mass has a minimum and the mass at the minimum is independent of $q$. The mass at the minimum is 302.5 MeV from eq. (15a) for the present values of the parameters, $\lambda, \nu$, and $H$. This value is also obtained from the numerical calculation of $m_0$ which is given in fig. (1b). Figure (1c) shows the temperature dependence of the pion mass for $q = 0.9, 1.0, 1.1$. Contrarily, the pion mass does not have the extremum at $T \neq 0$. The difference of the mass between different $q$ can be seen at high temperature at which the symmetry is (partially) restored.

Secondly, the behavior of the critical temperature $T_c$ as a function of $q$ is shown. The ratio $T_c(q)/T_c(q = 1)$ is shown in fig. (2). The parameter $q$ is less than 3/2 as discussed in the previous section. The critical temperature decreases as $q$ increases in the figure. The critical temperature at $T_c(q = 1)$ in the present case is approximately 90 MeV [20]. Therefore, the critical temperature at $q = 1.1$ is approximately 10 MeV lower than $T_c(q = 1)$. The ratio
of the ratio of the temperature $T_0^c(q)$ to the critical temperature $T_c(q)$ is approximately 1.574 in the present case. The $q$ dependence of $T_0^c(q)$ is the same as that of $T_c(q)$. Therefore, the behavior of the ratio $T_0^c(q)/T_c(q = 1)$ is the same, like $T_c(q)/T_c(q = 1)$.

Finally, the ratio $\varepsilon(q)/\varepsilon(q = 1)$ is shown in fig. 3. The energy density increases markedly as $q$ increases, as shown in fig. 3(a). The increase of the energy density around $q = 1$ is shown in fig. 3(b). The energy density at $q = 1.1$ is approximately three times larger than that at $q = 1$. Much energy is stored for $q > 1$ even when the Tsallis distribution is close to the Boltzmann-Gibbs distribution. As shown in fig. 3(a), the energy density diverges at $q = 5/4$. Therefore, the temperature $T(q)$ is restricted, because $q$ is restricted from energetic point of view. We note the values of the ratio $T_c(q)/T_c(q = 1)$.

The ratio $T_c(q)/T_c(q = 1)$ for $q = 3/2, 5/4$, and 0 are 0, 0.685, and 1.814 respectively. The restriction caused by the energy density, $q < 5/4$, indicates that the ratio $T_c(q)/T_c(q = 1)$ is larger than $T(q = 5/4)/T(q = 1)$ from fig. 2.

4 Discussion and Conclusion

We studied the effects of the Tsallis distribution in the linear sigma model. The Tsallis distribution was assumed as the distribution function, because the Tsallis distribution describes many phenomena well, such as momentum distribution. The Tsallis distribution has a parameter $q$, and the distribution at $q = 1$ is the Boltzmann-Gibbs distribution. The effects of the distribution on physical quantities, such as condensate, were investigated. We note that the expectation value in the present study is not the $q$-expectation value used in the Tsallis nonextensive statistics.

At the same temperature, the condensate for $q > 1$ is smaller than that for $q = 1$ and the condensate for $q < 1$ is larger than that for $q = 1$. The behavior depends on the expectation value of the square of a field, $(\phi_{ih})^2$, which increases as $q$ increases. This increase comes from the fact that the tail of the distribution is long for $q > 1$. The parameter $q$ is restricted in the region of $q < 3/2$ in the calculation of the expectation value of $(\phi_{ih})^2$.

The increase of the expectation value of $(\phi_{ih})^2$ affects the mass. The pion mass $m_j$ ($j = 1, 2, 3$) for $q > 1$ is heavier than that for $q = 1$ and the pion mass for $q < 1$ is lighter than that for $q = 1$. At high temperature, the sigma mass for $q > 1$ is heavier than that for $q = 1$. However, at low temperature, the sigma mass for $q > 1$ is lighter than that for $q = 1$. This behavior at low temperature comes from the behavior of the condensate. The quantity $(\phi_{ih})^2$ for $q > 1$ is large, compared with the value for $q = 1$, and the value of the condensate is small for $q > 1$ at low temperature, compared with that for $q = 1$. The sigma mass for $q > 1$ decreases rapidly at low temperature, reflecting the value of the condensate.

The critical temperature $T_c(q)$ decreases as $q$ increases. This behavior depends on the expectation value of the square of a field, $(\phi_{ih})^2$, as the condensate does. The critical temperature changes even when $q$ changes slightly. The
The energy density $\varepsilon(q)$ is proportional to $T^4$. The ratio $\varepsilon(q)/(NT^3)$ depends only on the parameter $q$ under the massless free particle approximation, where $N$ is the number of fields. The value $\varepsilon(q)/T^4$ is the extension of the Stefan-Boltzmann limit of the energy density. The parameter $q$ is restricted in the region of $q < 5/4$ in the calculation of $\varepsilon(q)$, though $q$ is restricted in the region of $q < 3/2$ in the calculation of the expectation value of $(\phi_B)^2$. Physically, the deviation of $q$ from $q = 1$ should be small, because the energy density increases remarkably as $q$ increases. The critical temperature $T(q)$ is also restricted from the restriction of $q$: the ratio $T(q)/T(q = 1)$ is larger than $T(q = 5/4)/T(q = 1)$.

In summary, the expressions of physical quantities such as critical temperature, are obtained when the momentum distribution is the Tsallis distribution. The chiral symmetry restoration for $q > 1$ occurs at lower temperature, compared with the symmetry restoration for $q = 1$, and the changes of the sigma mass and pion mass reflect the chiral symmetry restoration. The effect of the Tsallis distribution on the condensate is shown at $T \sim T_c$ and that on the mass is shown at $T > T_c$. The critical temperature decreases monotonically as $q$ increases. The small deviation of $q$ from $q = 1$ results in the large deviations of physical quantities, especially the energy density. The extension of the Stefan-Boltzmann limit of the energy density is shown, and it is expected from energetic point of view that the small deviation of $q$ from $q = 1$ is realized.

The results will be useful to study the phenomena in high energy collisions, and the further works related to the Tsallis distribution will be performed in the near future.

![Fig. 3. Ratio of the energy density $\varepsilon(q)$ to $\varepsilon(q = 1)$ for (a) $0 < q < 5/4$ and (b) $0.9 < q < 1.1$.]({"alt":"Fig. 3. Ratio of the energy density $\varepsilon(q)$ to $\varepsilon(q = 1)$ for (a) $0 < q < 5/4$ and (b) $0.9 < q < 1.1.$"})

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A Integrals in the calculations

The following integral appears in the calculations:

\[
I(\mu; \xi) := \int_0^\infty dk \frac{k^\mu}{[1 + (q - 1)\beta k]^1/(q - 1) + \xi} \quad (\xi = -1, 0, 1),
\]

where the parameter \( \mu \) is a non-negative integer, and the notation \([x]_+\) is defined as follows:

\[
[x]_+ = \begin{cases} 
  x & (x \geq 0) \\
  0 & (x < 0) 
\end{cases}.
\]

The parameter \( \xi \) is \(-1\) for boson, 0 for classical particle, and 1 for fermion.

This integral is represented by changing of variables:

\[
I(\mu; \xi) = \frac{1}{\beta^{\mu+1}(q-1)^\mu} \int_0^1 dy \left( \frac{y^{1-q}[y^{1-q}-1]^{\mu/2}}{1+\xi y} \right).
\]

The integral can be represented with digamma function \( \psi(x) \) [21,22] which is given by

\[
\psi(x) = \frac{1}{\Gamma(x)} \left( \frac{d\Gamma(x)}{dx} \right).
\]

The integrals \( I(1, \xi) \) for \( q < 3/2 \) are calculated:

\[
\begin{align*}
I(1, -1) &= \frac{1}{\beta^2(q-1)} \left\{ \psi(2-q) - \psi(3-2q) \right\}, \\
I(1, 0) &= \frac{1}{\beta^2(q-1)} \left\{ \frac{1}{3-2q} - \frac{1}{2-q} \right\}, \\
I(1, 1) &= \frac{1}{\beta^2(q-1)} \left\{ \frac{1}{2} \left[ \psi(2-q) - \psi\left( \frac{3-2q}{2} \right) \right] \\
&\quad - \frac{1}{2} \left[ \psi\left( \frac{3-q}{2} \right) - \psi\left( \frac{2-q}{2} \right) \right] \right\}.
\end{align*}
\]

The integrals \( I(3, \xi) \) for \( q < 5/4 \) are calculated:

\[
\begin{align*}
I(3, -1) &= \frac{1}{\beta^3(q-1)^3} \left\{ \psi(2-q) - \psi(5-4q) \right. \\
&\quad - 3 \left[ \psi(3-2q) - \psi(4-3q) \right] \left\} \right., \\
I(3, 0) &= \frac{1}{\beta^3(q-1)^3} \left\{ \frac{1}{5-4q} - \frac{1}{2-q} \right. \\
&\quad - 3 \left[ \frac{1}{4-3q} - \frac{1}{3-2q} \right] \left\} \right., \\
I(3, 1) &= \frac{1}{\beta^3(q-1)^3} \left\{ [J_3 - J_0] - 3 [J_2 - J_1] \right\},
\end{align*}
\]

where \( J_p \) is given by

\[
J_p = \frac{1}{2} \left\{ \psi \left( \frac{(3+p)-(1+p)q}{2} \right) - \psi \left( \frac{(2+p)-(1+p)q}{2} \right) \right\}, \quad \left( q < \frac{2+p}{1+p} \right).
\]

Some quantities are represented with these expressions.