On the normalized arithmetic Hilbert function

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Tuesday 20th October, 2015, 00:37

Abstract
Let \(X \subset \mathbb{P}^N\) be a subvariety of dimension \(n\), and \(H_{\text{norm}}(X; \cdot)\) the normalized arithmetic Hilbert function of \(X\) introduced by Philippon and Sombra. We show that this function admits the following asymptotic expansion

\[
H_{\text{norm}}(X; D) = \frac{\hat{h}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad \forall D \gg 1,
\]

where \(\hat{h}(X)\) is the normalized height of \(X\). This gives a positive answer to a question raised by Philippon and Sombra.

Keywords: Arithmetic Hilbert function, Height.

MSC: 11G40, 11G50, 11G35.

1 Introduction
In [5], Philippon and Sombra introduce an arithmetic Hilbert function defined for any subvariety in \(\mathbb{P}^N\), the projective space of dimension \(N\) over \(\mathbb{Q}\). This function measures the binary complexity of the subvariety. In the case of toric subvarieties, a result of Philippon and Sombra shows that the asymptotic behaviour of the associated normalized arithmetic Hilbert function is related to the normalized height of the subvariety considered, see [5, Proposition 0.4]. This result is an important step toward the proof of the main theorem of [5], that is an explicit formula for the normalized height of projective translated toric varieties, see [5, Théorème 0.1].

In [5, Question 2.2], the authors ask if the normalized arithmetic Hilbert function admits an asymptotic expansion similar to the toric case. More precisely, given \(X\) a subvariety of dimension \(n\) in \(\mathbb{P}^N\) the projective space of dimension \(N\) over \(\mathbb{Q}\), can we find a real \(c(X) \geq 0\) such that

\[
H_{\text{norm}}(X; D) = \frac{c(X)}{(n+1)!} D^{n+1} + o(D^{n+1})?
\]

If this is the case, do we have \(c(X) = \hat{h}(X)\)? where \(\hat{h}(X)\) is the normalized height of \(X\).

In this article, we give an affirmative answer to this question. We prove the following theorem

**Theorem 1.1.** [Theorem 2.5] Let \(X \subset \mathbb{P}^N\) be a subvariety of dimension \(n\) in \(\mathbb{P}^N\). Then the normalized arithmetic Hilbert function associated to \(X\) admits the following asymptotic expansion

\[
H_{\text{norm}}(X; D) = \frac{\hat{h}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}), \quad \forall D \gg 1.
\]
The notion of normalized height plays an important role in the diophantine approximation on tori, in particular in Bogomolov’s and generalized Lehmer’s problems, see [3], [2]. A result of Zhang shows that a subvariety $X$ with a vanishing normalized height is necessarily a union of toric subvarieties, see [8].

Gillet and Soulé proved an arithmetic Hilbert-Samuel formula as a consequence of the arithmetic Riemann-Roch theorem, see [4]. Roughly speaking, this formula describes the asymptotic behaviour of the arithmetic degree of a hermitian module defined by the global sections of the tensorial power of a positive hermitian line bundle on an arithmetic variety. Moreover, the leading term is given by the arithmetic degree of the hermitian line bundle. Later Abbès and Bouche gave a new proof for this result without using the arithmetic Riemann-Roch theorem, see [1]. Randriambololona extends the result Gillet and Soulé to the case of coherent sheaf provided as a subquotient of a metrized vector bundle on an arithmetic variety, see [7].

1.1 Notations

Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Z}$ the ring of integers, $K$ a number field and $\mathcal{O}_K$ its ring of integers. For $N$ and $D$ two integers in $\mathbb{N}$ we set $\mathcal{N}^N_{D+1} := \{a \in \mathbb{N}^{N+1}|a_0 + \cdots + a_N = D\}$. $\mathbb{C}[x_0, \ldots, x_N]_D$ (resp. $K[x_0, \ldots, x_N]_D$) denotes the complex vector space (resp. $K$-vector space) of homogeneous polynomials of degree $D$ in $\mathbb{C}[x_0, \ldots, x_N]$ (resp. in $K[x_0, \ldots, x_N]$).

For any prime number $p$ we denote by $| \cdot |_p$ the $p$-adic absolute value on $\mathbb{Q}$ such that $|p|_p = p^{-1}$ and by $| \cdot |_{\infty}$ or simply $| \cdot |$ the standard absolute value. Let $M_{\mathbb{Q}}$ be the set of these absolute values. We denote by $M_K$ the set of absolute values of $K$ extending the absolute values of $M_{\mathbb{Q}}$, and by $M_K^\infty$ the subset in $M_K$ of archimedean absolute values.

We denote by $\mathbb{P}^N$ the projective space over $\overline{\mathbb{Q}}$ of dimension $N$. A variety is assumed reduced and irreducible.

1.1.1 Acknowledgements

I am very grateful to Martín Sombra for his helpful conversations and encouragement during the preparation of this paper. I would like to thank Vincent Maillot for his useful discussions.

2 The proof of Theorem (1.1)

We keep the same notations as in [3]. Let $\omega$ be the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$. For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we denote by $h_k$ the hermitian metric on $\mathcal{O}(1)$ given as follows

$$h_k(\cdot, \cdot) = \frac{| \cdot |^2}{(|x_0|^{2k} + \cdots + |x_N|^{2k})^\frac{1}{2k}}, \forall k \in \mathbb{N}_{\geq 1} \quad \text{and} \quad h_\infty(\cdot, \cdot) = \frac{| \cdot |^2}{\max(|x_0|, \ldots, |x_N|)^2},$$

and we let $\mathcal{O}(1)_k := (\mathcal{O}(1), h_k)$ and $\omega_k := c_1(\mathcal{O}(1), h_k)$ for any $k \in \mathbb{N} \cup \{\infty\}$. Note that $\omega_k = \frac{1}{k}[k]^{*}\omega$, where $[k] : \mathbb{P}^N(\mathbb{C}) \rightarrow \mathbb{P}^N(\mathbb{C})$, $[x_0 : \ldots : x_N] \mapsto [x_0^k : \ldots : x_N^k]$. Observe that the sequence $(\omega_k)_{k \in \mathbb{N}_{\geq 1}}$ converges weakly to the current $\omega_\infty$. We consider the following normalized volume form

$$\Omega_k := \omega_k^{\wedge N} \quad \forall k \in \mathbb{N}_{\geq 1} \cup \{\infty\}.$$
For any \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \), the metric of \( O(1)_k \) and \( \Omega_k \) define a scalar product \( \mathbb{C}[x_0, \ldots, x_N]_D \) denoted by \( \langle \cdot, \cdot \rangle_k \) given as follows

\[
\langle f, g \rangle_k := \int_{\mathbb{P}^N(\mathbb{C})} h_k^D(f, g) \Omega_k,
\]

for any \( f = \sum_a f_a x^a \), \( g = \sum_a g_a x^a \) in \( \mathbb{C}[x_0, \ldots, x_N]_D \) with \( f_a, g_a \in \mathbb{C} \). We denote by \( \| \cdot \|_k \) the associated norm for any \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \). Note that, \( \langle f, g \rangle_\infty = \sum_a f_a g_a \) and \( \| x^a \|_\infty = 1 \) for any \( a \in \mathbb{N}^{N+1}_D \) and \( D \in \mathbb{N} \).

Let \( X \subset \mathbb{P}^N \) be a subvariety defined over a number field \( K \). Let \( v \in M_K^\infty \) and \( \sigma_v : K \to \mathbb{C} \) the corresponding embedding. For any \( p_1, \ldots, p_l \in K[x_0, \ldots, x_N]_D \) we set

\[
\| p_1 \wedge \cdots \wedge p_l \|_{k,v} := \| \sigma_v(p_1) \wedge \cdots \wedge \sigma_v(p_l) \|_k \quad \forall k \in \mathbb{N} \cup \{\infty\}.
\]

Let \( O(D) := O(1)^{\otimes D} \). We set \( M := \Gamma(\Sigma, O(D)_{|\Sigma}) \) the \( O_K \)-module of global sections of \( O(D)_{|\Sigma} \), where \( \Sigma \) is the Zariski closure of \( X \) in \( \mathbb{P}^N_K \). For any \( v \in M_K^\infty \), we set \( \Gamma(\Sigma, O(D)_{|\Sigma})_{\sigma_v} := \Gamma(\Sigma, O(D)_{|\Sigma}) \otimes_{O_K} \mathbb{C} \). We consider the following restriction map

\[
\pi : \Gamma(\mathbb{P}^N_K, O(D))_{\sigma_v} \to \Gamma(\Sigma, O(D)_{|\Sigma})_{\sigma_v} \to 0.
\]

The space \( \Gamma(\mathbb{P}^N_K, O(D))_{\sigma_v} \) is identified canonically to \( K[x_0, \ldots, x_N]_D \). For any \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \), this space can be endowed by the scalar product induced by \( \Omega_k \) and \( h_k \), denoted by \( \langle \cdot, \cdot \rangle_{k,v} \):

\[
\langle f, g \rangle_{k,v} = \langle \sigma_v(f), \sigma_v(g) \rangle_k,
\]

for any \( f, g \in \Gamma(\mathbb{P}^N_K, O(D))_{\sigma_v} \). Since \( O(1) \) is ample, then there exists \( D_0 \in \mathbb{N} \) such that for any \( D \geq D_0 \), the restriction map is surjective. Let \( D \geq D_0 \), for any \( k \in \mathbb{N} \cup \{\infty\} \), we denote by \( \| \cdot \|_{k,v,\text{quot}} \) the quotient norm induced by \( \pi \) and \( \| \cdot \|_{k,v} \). Following [5] p.348], we endow \( \Gamma(\Sigma, O(D)_{|\Sigma})_{\sigma_v} \) with \( \| \cdot \|_{k,v,\text{quot}} \), for any \( k \in \mathbb{N}_{\geq 1} \cup \{\infty\} \). By this construction, \( M \) can be equipped with a structure of a hermitian \( O_K \)-module, denoted by \( \mathcal{M}_k \). If \( f_1, \ldots, f_s \in M \), is a \( K \)-basis for \( M \otimes_{O_K} K \), then

\[
\overline{\deg}(M_k) = \overline{\deg}(\Gamma(\Sigma, O(D)_{|\Sigma})_k) := \frac{1}{[K : \mathbb{Q}]} \left( \log \text{Card} \left( \bigwedge M/(f_1 \wedge \cdots \wedge f_s) \right) - \sum_{v : K \to \mathbb{C}} \log \| f_1 \wedge \cdots \wedge f_s \|_{k,v} \right).
\]

## 2.1 The normalized arithmetic Hilbert function

Let \( X \subset \mathbb{P}^N \) be a subvariety defined over a number field \( K \) and \( I := I(X) \subset K[x_0, \ldots, x_N] \) its ideal of definition. We set

\[
\mathcal{H}_{\text{geom}}(X; D) := \dim_K \left( K[x_0, \ldots, x_N]/I \right)_D = \binom{D + N}{N} - \dim_K (I_D).
\]

This function \( \mathcal{H}_{\text{geom}}(X; \cdot) \) is known as the classic geometric Hilbert function. In [5], Philippon and Sombra introduce an arithmetic analogue of this function. Let \( m := \mathcal{H}_{\text{geom}}(X; D), \ l := \dim_K (I_D) \) and

\[
\bigwedge^l K[x_0, \ldots, x_N]_D,
\]

the \( l \)-th exterior power product of \( K[x_0, \ldots, x_N]_D \). For \( f \in \bigwedge^l K[x_0, \ldots, x_N]_D \) and \( v \in M_K \) we denote by \( |f|_v \) the sup-norm of the coefficients of \( f \) at the place \( v \), with respect to the standard basis of \( \bigwedge^l K[x_0, \ldots, x_N]_D \).
**Definition 2.1.** (Définition 2.1) Let \( p_1, \ldots, p_l \) be a \( K \)-basis of \( I_D \), we set
\[
\mathcal{H}_{\text{norm}}(X; D) = \sum_{v \in M_K' } \left[ \frac{K_v}{K} : \mathbb{Q} \right] \log |p_1 \wedge \cdots \wedge p_l|_v.
\]

By the product formula, this definition does not depend on the choice of the basis, also it is invariant by finite extensions of \( K \). \( \mathcal{H}_{\text{norm}}(X; \cdot) \) is called the normalized arithmetic Hilbert function of \( X \).

Following Philippon and Sombra, this arithmetic Hilbert function measures, for any \( D \in \mathbb{N} \), the binary complexity of the \( K \)-vector space of forms of degree \( D \) in \( K[x_0, \ldots, x_N] \) modulo \( I \). As pointed out by Philippon and Sombra, when \( X \) is a toric variety, the asymptotic behaviour of its associated normalized arithmetic Hilbert function is related to \( \hat{h}(X) \), the normalized height of \( X \), see [5, Proposition 0.4]. The authors ask the following question:

Given \( X \) a subvariety in \( \mathbb{P}^N \) of dimension \( n \), can we find a real \( c(X) \geq 0 \) such that
\[
\mathcal{H}_{\text{norm}}(X; D) = \frac{c(X)}{(n+1)!} D^{n+1} + o(D^{n+1})?
\]

If this is the case, do we have \( c(X) = \hat{h}(X) \)?

We recall the following proposition, which gives a dual formulation for \( \mathcal{H}_{\text{norm}} \).

**Proposition 2.2.** Let \( q_1, \ldots, q_m \in K[x_0, \ldots, x_N]_D^\ast \) be a \( K \)-basis of \( \text{Ann}(I_D) \), then
\[
\mathcal{H}_{\text{norm}}(X; D) = \sum_{v \in M_K' } \left[ \frac{K_v}{K} : \mathbb{Q} \right] \log |q_1 \wedge \cdots \wedge q_m|_v.
\]

**Proof.** See [5, Proposition 2.3]. □

For any \( k \in \mathbb{N}_+ \cup \{ \infty \} \), we consider the following arithmetic function,
\[
\mathcal{H}_{\text{arith}}(X; D; k) := \sum_{v \in M_K' } \left[ \frac{K_v}{K} : \mathbb{Q} \right] \log ||p_1 \wedge \cdots \wedge p_l||_{k,v}
+ \sum_{v \in M_K' \setminus M_K'} \left[ \frac{K_v}{K} : \mathbb{Q} \right] \log |p_1 \wedge \cdots \wedge p_l|_v + \frac{1}{2} \log(\gamma(N, D, k)),
\]

where \( p_1, \ldots, p_l \) is a \( K \)-basis of \( I_D \) and
\[
\gamma(N; D, k) := \prod_{a \in \mathbb{N}^{N+1}_+} (\langle a, a \rangle_k)^{-1}.
\]

Notice that for \( k = 1 \), \( \mathcal{H}_{\text{arith}}(X; \cdot, 1) \) corresponds, up to a constant, to the arithmetic function \( \mathcal{H}_{\text{arith}}(X; \cdot) \) considered in [5, p. 346].

Similarly to \( \mathcal{H}_{\text{norm}} \), the function \( \mathcal{H}_{\text{arith}} \) admits a dual formulation. The scalar product \( \langle \cdot, \cdot \rangle_k \) induces the following linear isomorphism
\[
\eta_k : \mathbb{C}[x_0, \ldots, x_N] \rightarrow \mathbb{C}[x_0, \ldots, x_N]^\vee, \quad f \mapsto \langle \cdot, f \rangle_k.
\]
Thus \( \mathbb{C}[x_0, \ldots, x_N]^\vee \) can be endowed with the dual scalar product, given as follows
\[
\langle \eta_k(f), \eta_k(g) \rangle_k := \langle f, g \rangle_k, \quad \forall f, g \in \mathbb{C}[x_0, \ldots, x_N]_D.
\]
We can check easily that, for any $k \in \mathbb{N} \cup \{\infty\}$ we have $\|\theta\|^2_k := \sup_{g \in \mathbb{C}[x_0, \ldots, x_N]} \|\hat{g}(\theta)\| = \|f\|_k$ where $f \in \mathbb{C}[x_0, \ldots, x_N]$ is such that $\theta = \eta_k(f)$. Then, $\|\theta\|^2_k = \langle \theta, \theta \rangle_k$ for any $\theta \in \mathbb{C}[x_0, \ldots, x_N]^*$. It follows that,

$$\langle \theta, \zeta \rangle_k = \sum \langle x^b, x^b \rangle_k^{-1} \theta_b \zeta_b. \quad (4)$$

This product extends to $\wedge^m \left( \mathbb{C}[x_0, \ldots, x_N]^* \right)$ as follows

$$\langle \theta_1 \wedge \cdots \wedge \theta_m, \zeta_1 \wedge \cdots \wedge \zeta_m \rangle_k := \det(\langle \theta_i, \zeta_j \rangle_k)_{1 \leq i, j \leq m}.$$

**Proposition 2.3.** Let $q_1, \ldots, q_m \in K[x_0, \ldots, x_N]^*$ be a $K$-basis of $\text{Ann}(I_D)$, then

$$H_{\text{arith}}(X; D, k) = \sum_{v \in M_K^{\infty}} \left[ K_v : \mathbb{Q}_v \right] \log \|q_1 \wedge \cdots \wedge q_m\|_{v, k} + \sum_{v \in M_K \setminus M_K^{\infty}} \left[ K_v : \mathbb{Q}_v \right] \log |q_1 \wedge \cdots \wedge q_m|_v.$$

**Proof.** The proof is similar to [5, Prop. 2.5]. \qed

**Lemma 2.4.** There exists $D_1$ such that for any $D \geq D_1$ and any $k \in \mathbb{N}$, we have

$$H_{\text{arith}}(X; D, k) = \deg(\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})) - \frac{1}{2} H_{\text{geom}}(X; D) \log \left( \frac{D + N}{N} \right).$$

**Proof.** The proof is similar to [5, lemme 2.6]. Let $\mathcal{I}$ be the ideal sheaf of $\Sigma$ and $\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D))$ the $\mathcal{O}_K$-module of global sections of $\mathcal{I}\mathcal{O}(D)$, endowed with the scalar products induced by the hermitian $\mathcal{O}_K$-modules $\langle \cdot, \cdot \rangle_{\mathcal{O}_K}$. We claim that there exists $D_1$ an integer which does not depend on $k$ such that for any $D \geq D_1$, we have

$$\deg(\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})) = \deg(\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D))) - \deg(\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D))_{|\Sigma})_{k}.$$

Indeed, we can find $D_1 \in \mathbb{N}$ such that $\forall D \geq D_1$, the following sequence is exact

$$0 \to \Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D)_{|\Sigma}) \to \Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{O}(D)) \to \Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma}) \to 0,$$

and then by [5, lemme 2.3.6], the following sequence of hermitian $\mathcal{O}_K$-modules is exact

$$0 \to \Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D)_{|\Sigma})_{k} \to \Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{O}(D))_{k} \to \Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})_{k} \to 0,$$

where the metrics of $\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D)_{|\Sigma})_{k}$ and $\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma})_{k}$ are induced by the metric of $\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{I}\mathcal{O}(D))_{k}$.

We have

$$\tilde{\deg}(\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{O}(D)))_{k} = \frac{1}{2} \log(\gamma(N; D, k)) + \frac{1}{2} \left( \frac{D + N}{N} \right) \log \left( \frac{N + D}{N} \right). \quad (5)$$

As in the proof of [5, Lemme 2.6], and keeping the same notations we have,

$$\tilde{\deg}(\Gamma(\Sigma, \mathcal{O}(D)_{|\Sigma}))_{k} = \frac{1}{2} \log(\gamma(N; D, k)) + \frac{1}{2} H_{\text{geom}}(X; D) \log \left( \frac{N + D}{N} \right)$$

$$+ \sum_{v \in M_K^{\infty}} \left[ K_v : \mathbb{Q}_v \right] \log \|p_1 \wedge \cdots \wedge p_l\|_{v, k} = \frac{1}{[K : \mathbb{Q}]} \log \text{Card}(\hat{\Lambda}(I_{\mathcal{O}_K})/(p_1 \wedge \cdots \wedge p_l)). \quad (6)$$
The last term in (6) does not depend on the metric. It is computed in [5, p. 349]; we have
\[
\frac{1}{|K : \mathbb{Q}|} \log \text{Card} \left( \bigwedge (I_{\mathfrak{o}_K})/(p_1 \wedge \cdots \wedge p_l) \right) = - \sum_{v \in M_K \setminus M_K^\prime} \frac{[K_v : \mathbb{Q}_v]}{|K : \mathbb{Q}|} \log |p_1 \wedge \cdots \wedge p_l|_v.
\]
This ends the proof of the lemma.

By [7, Théorème A], we have
\[
\deg(\Sigma, \mathcal{O}(D)_{|x}) = \frac{h_{\mathcal{O}(1)}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad \forall D \gg 1,
\]
where \( h_{\mathcal{O}(1)}(X) \) denotes the height of the Zariski closure of \( X \) in \( \mathbb{P}^N_{\mathcal{O}_K} \) with respect to \( \mathcal{O}(1)_k \).
Since \( \frac{1}{2} H_{\text{geom}}(X; D) \log \left( \frac{D+N}{D} \right) = o(D^{n+1}) \) for \( D \gg 1 \). Then, by Lemma (2.4), we get
\[
\mathcal{H}_{\text{arith}}(X; D, k) = \frac{h_{\mathcal{O}(1)}(X)}{(n+1)!} D^{n+1} + o(D^{n+1}) \quad \forall D \gg 1.
\]
Let \( q_1, \ldots, q_m \in K[x_0, \ldots, x_N]^\vee \) be a \( K \)-basis of \( \text{Ann}(I_D) \). For any finite subset \( M \) in \( \mathbb{N}^{D+1}_N \) of cardinal \( m \), we set \( q_M := (q_{j,b})_{1 \leq j \leq m, b \in M} \in K^{m \times m} \) where the \( q_{j,b} \) are such that \( q_j = \sum_{b \in \mathbb{N}^{D+1}} q_{j,b}(x_b)^\vee \). For any \( v \in M_K^\prime \), we have
\[
|q_1 \wedge \cdots \wedge q_m|_v = \max \{ |\det(q_M)|_v : M \subset \mathbb{N}^{D+1}, \text{Card}(M) = m \}
\leq \left( \sum_{M, \text{Card}(M) = m} \left( \prod_{b \in M} (b, b)^{-1}_v \right) |\det(q_M)|_v^2 \right)^{\frac{1}{2}},
\]
(We use the following inequality: \( \langle x^a, x^a \rangle_k = \int_{\mathbb{P}^n(\mathbb{C})} h_{\mathcal{O}_k} \mathcal{O}(x^a, x^a) \Omega_k \leq 1 \) for any \( a \in \mathbb{N}^{D+1} \), which follows from \( h_{\mathcal{O}(k)}(x^a, x^a) \leq h_{\mathcal{O}(D)_\infty}(x^a, x^a) \leq 1 \) on \( \mathbb{P}^n(\mathbb{C}) \), and the fact that \( \Omega_k \) is positive on \( \mathbb{P}^n(\mathbb{C}) \) and \( \int_{\mathbb{P}^n(\mathbb{C})} \Omega_k = 1 \).

Then,
\[
|q_1 \wedge \cdots \wedge q_m|_v \leq \|q_1 \wedge \cdots \wedge q_m\|_{k,v} \quad \forall k \in \mathbb{N}.
\]
By Propositions (2.2) and (2.3), we get,
\[
\mathcal{H}_{\text{norm}}(X; D) \leq \mathcal{H}_{\text{arith}}(X; D, k) \quad \forall k \in \mathbb{N}.
\]
By (5), the previous inequality gives
\[
\limsup_{D \to \infty} \frac{(n+1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) \leq \frac{h_{\mathcal{O}(1)}(X)}{(n+1)!} \quad \forall k \in \mathbb{N}.
\]
We know that \( (h_k)_{k \in \mathbb{N}} \) converges uniformly to \( h_\infty \) on \( \mathbb{P}^N(\mathbb{C}) \). Let \( 0 < \varepsilon < 1 \), which will be fixed in the sequel, then there exists \( k_0 \in \mathbb{N} \) such that for any \( k \geq k_0 \), we have
\[
(1 - \varepsilon)^{2D} \leq \frac{\max(|x_0|, \ldots, |x_N|)^{2D}}{|x_0|^2k + \cdots + |x_N|^{2k}} \leq (1 + \varepsilon)^{2D} \quad \forall x \in \mathbb{P}^N(\mathbb{C}), \forall D \in \mathbb{N}.
\]
Thus, for any \( k \geq k_0, D \in \mathbb{N}_{\geq 1} \) and \( a \in \mathbb{N}^{D+1} \) we get
\[
\langle x^a, x^a \rangle_k \geq (1 - \varepsilon)^{2D} \int_{\mathbb{P}^N(\mathbb{C})} h_{\infty}^{\otimes D}(x^a, x^a) \omega_0^N.
\]
\[
\int_{P_N(C)} h_\infty^N(x^a, x^a) \omega_k^N = \int_{\mathbb{C}^N} \frac{|z^{2a}|}{\max(1, |z_1|, \ldots, |z_N|^{2D})} \prod_{i=1}^N \frac{|z_i|^{2(k-1)}}{(1 + \sum_{i=1}^N |z_k|)^{N+1}} \prod_{i=1}^N d\bar{z}_i \\
= 2^N \int_{(\mathbb{R}^+)^N} \frac{k^{N-a+k-1}}{\max(1, r_1, \ldots, r_N)^{2D}} \prod_{i=1}^N dr_i \\
= 2^N \int_{(\mathbb{R}^+)^N} \frac{r^\frac{N-a+k-1}{2}}{\max(1, r_1, \ldots, r_N)^{\frac{N}{2}}} \prod_{i=1}^N dr_i \\
= 2^N \sum_{j=0}^N \int_{E_j} \frac{r^\frac{N-a+k-1}{2}}{\max(1, r_1, \ldots, r_N)^{\frac{N}{2}}} \prod_{i=1}^N dr_i 
\]

where \( E_j := \{ x \in (\mathbb{R}^+)^N \mid x_j \geq 1, x_j \leq x_j \text{ for } l = 1, \ldots, N \} \) for \( j = 1, \ldots, N \) and \( E := \{ x \in (\mathbb{R}^+)^N \mid x_l \leq 1, l = 1, \ldots, N \} \). Using the following application

\[
(\mathbb{R}^+)^N \to (\mathbb{R}^+)^N, \quad x = (x_1, \ldots, x_N) \mapsto \left( \frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{1}{x_j}, \ldots, \frac{x_N}{x_j} \right)
\]

for \( j = 1, \ldots, N \), we can show that there exists \( b^{(j)} = (b^{(j)}_1, \ldots, b^{(j)}_N) \in \mathbb{N}^N \) such that

\[
\int_{E_j} \frac{r^\frac{N-a+k-1}{2}}{\max(1, r_1, \ldots, r_N)^{\frac{N}{2}}} \prod_{i=1}^N dr_i = \int_{E} \frac{r^\frac{N-a+k-1}{2}}{\max(1, r_1, \ldots, r_N)^{\frac{N}{2}}} \prod_{i=1}^N dr_i \quad (14)
\]

We set \( b^{(0)} := a \). Then,

\[
\int_{P_N(C)} h_\infty^N(x^a, x^a) \omega_k^N = 2^N \sum_{j=0}^N \int_{E_j} r^\frac{N-a+k-1}{2} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} \quad (15)
\]

Let \( 0 < \delta < 1 \), and set \( E_\delta := \{ x \in E \mid x_l \geq \delta \text{ for } l = 1, \ldots, N \} \). From (13) and (15), we obtain

\[
(x^a, x^a)_k \geq (1 - \varepsilon)^{2D} 2^N \sum_{j=0}^N \int_{E_\delta} r^\frac{N-a+k-1}{2} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} \geq (1 - \varepsilon)^{2D} 2^N (N+1) \delta^\frac{N}{2} \mu_\delta,
\]

where \( \mu_\delta := \int_{E_\delta} \frac{\prod_{i=1}^N dr_i}{(1 + \sum_{i=1}^N r_i)^{N+1}} \).

Thus,

\[
(x^a, x^a)_k \leq (1 - \varepsilon)^{-2D} \delta^{-\frac{N}{2}} \mu_\delta^{-1} \quad (16)
\]

Then, for any \( k \geq k_0 \) and \( D \geq D_1 \),

\[
\| q_1 \wedge \cdots \wedge q_m \|_{k, v} \leq \left( \sum_{M : \text{Card}(M) = m} \left( \prod_{b \in M} (b, b)^{-1}_{v,k} \right) \right)^\frac{1}{2} |q_1 \wedge \cdots \wedge q_m|_v \\
\leq \text{Card}\{ M \subset \mathbb{N}_D^{N+1} \mid \text{Card}(M) = m \} \left( \prod_{b \in M} (1 - \varepsilon)^{-mD} \delta^{-m \frac{N}{2}} \mu_\delta^{-m} \right) |q_1 \wedge \cdots \wedge q_m|_v \quad \text{by (16)} \\
\leq \text{Card}[\mathbb{N}_D^{N+1}](1 - \varepsilon)^{-mD} \delta^{-m \frac{N}{2}} \mu_\delta^{-m} |q_1 \wedge \cdots \wedge q_m|_v \\
= \left( \frac{N + D}{N} \right)^\frac{1}{2} (1 - \varepsilon)^{-mD} \delta^{-m \frac{N}{2}} \mu_\delta^{-m} |q_1 \wedge \cdots \wedge q_m|_v.
\]
Therefore,
\[
\mathcal{H}_{\text{arith}}(X; D, k) \leq \mathcal{H}_{\text{norm}}(X; D) + \frac{1}{2} \log \left( \frac{N + D}{N} \right) - D \mathcal{H}_{\text{geom}}(X; D) \log(1 - \varepsilon) - \frac{D \mathcal{H}_{\text{geom}}(X; D)}{k} \log \delta - \mathcal{H}_{\text{geom}}(X; D) \log \mu \delta.
\] (18)

By (8), we obtain that
\[
h_{\mathcal{O}(1)}(X) \leq \lim inf_{D \to \infty} \frac{(n + 1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) + O(\varepsilon) + \frac{\log \delta}{k} O(1), \quad \forall k \geq k_0.
\] (19)

Gathering (12) and (19), we conclude that for any \(0 < \varepsilon < 1\), there exists \(k_0 \in \mathbb{N}\) such that
\[
\lim sup_{D \to \infty} \frac{(n + 1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) \leq h_{\mathcal{O}(1)}(X) \leq \lim inf_{D \to \infty} \frac{(n + 1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) + O(\varepsilon) + \frac{\log \delta}{k} O(1), \quad \forall k \geq k_0.
\] (20)

Since \(\lim_{k \to \infty} h_{\mathcal{O}(1)}(X) = h_{\mathcal{O}(1)}(X)\) (see for instance [2]) and \(h_{\mathcal{O}(1)}(X) = \hat{h}(X)\) (see [5, p. 342]) we get
\[
\lim inf_{D \to \infty} \frac{(n + 1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) = \lim sup_{D \to \infty} \frac{(n + 1)!}{D^{n+1}} \mathcal{H}_{\text{norm}}(X; D) = \hat{h}(X).
\] (21)

Thus we proved the following theorem

**Theorem 2.5.** Let \(X \subset \mathbb{P}^N\) be a subvariety of dimension \(n\) in \(\mathbb{P}^N\). Then the normalized arithmetic Hilbert function associated to \(X\) admits the following asymptotic expansion
\[
\mathcal{H}_{\text{norm}}(X; D) = \frac{\hat{h}(X)}{(n + 1)!} D^{n+1} + o(D^{n+1}) \quad D \gg 1.
\]

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