Renormalization Group Study of the Intrinsic Finite Size Effect in 2D Superconductors

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Vortices in a thin-film superconductor interact logarithmically out to a distance on the order of the two-dimensional (2D) magnetic penetration depth $\lambda_\perp$, at which point the interaction approaches a constant. Thus, because of the finite $\lambda_\perp$, the system exhibits what amounts to an intrinsic finite size effect. It is not described by the 2D Coulomb gas but rather by the 2D Yukawa gas (2DYG). To study the critical behavior of the 2DYG, we map the 2DYG to the massive sine-Gordon model and then perform a renormalization group study to derive the recursion relations and to verify that $\lambda_\perp$ is a relevant parameter. We solve the recursion relations to study important physical quantities for this system including the renormalized stiffness constant and the correlation length. We also address the effect of current on this system to explain why finite size effects are not more prevalent in experiments given that the 2D magnetic penetration depth is a relevant parameter.

I. INTRODUCTION

Interest in the critical behavior of vortices in two-dimensional (2D) superconductors can be traced back to at least the studies of Kosterlitz and Thouless \cite{1} and Berezinskii \cite{3} in the early 1970’s. It was stated in Ref. \cite{1} however, that the theory as formulated did not apply to 2D superconductors where there is a coupling to the magnetic field. Because of this coupling, the interaction is logarithmic only up to the length scale $\lambda_\perp = 2\lambda^2/d$, (where $\lambda$ is the London penetration depth and $d$ is the film thickness,) at which point it approaches a constant. \cite{1} The application of the Kosterlitz-Thouless-Berezinskii (KTB) transition to 2D superconducting systems dates only to the late 1970’s and the work of Beasley, Mooij, and Orlando \cite{4} and of Halperin and Nelson \cite{5} who pointed out that $\lambda_\perp$ can be quite large (relative to the film size) in superconducting thin films and that therefore these films should behave as the charged analog of superfluid thin films. As a result, a flurry of experimental papers reported investigations of KTB behavior in thin-film superconductors in the early 1980’s. \cite{6}

Sparked partially by the discovery of the high-temperature superconductors, interest in the critical behavior of 2D superconductors (SC’s) and Josephson Junction arrays (JJA’s) has been renewed in this decade. Sujani, Chattopadhyay, and Shenoy \cite{9} have derived recursion relations for 2D superconductors that incorporate nonzero current drives. Minnhagen and coworkers \cite{10} have been investigating the dynamics of the transition numerically while Ammirata et al. \cite{11,12} have applied dynamic scaling to $I$-$V$ data from 2D superconductors and JJA’s.

The limiting large scale $\lambda_\perp$, a length intrinsic to 2D superconductors, acts as an unavoidable source of finite size effects. Other sources, characterized by other lengths, are extrinsic, arising from the finite size (or extent) of a system and from a current. Some of the recent work on 2D SC’s and JJA’s has focused on the expected finite size effects (of any origin) in these systems. Simkin and Kosterlitz \cite{13} have investigated numerically and analytically the finite size effects in these systems, while Repaci et al. \cite{14} claimed experimental evidence of finite size effects in ultra thin film YBCO. Herbert et al. \cite{15} have extended this work to JJA’s. We will focus here on the influence of a finite $\lambda_\perp$, which we refer to as the intrinsic finite size effect.

Despite all of the work on 2D SC’s and JJA’s, there is still no analytical theory of the influence of intrinsic finite size effect on their critical properties. This is particularly striking since it is generally accepted that the intrinsic finite size effect should be relevant, yet reports of its experimental observation are relatively sparse. Our objective in this paper is to fill in the theoretical gap and to address the small number of experimental reports of finite size effects. We perform a RG study of the vortices in 2D superconductors by modeling them as a 2D Yukawa gas (2DYG) and then mapping this system to the massive sine-Gordon (SG) Hamiltonian. We then perform a momentum-space shell RG study to derive the recursion relations, which we then analyze to study the shift in the critical temperature and the behavior of the system near it. We will also address the effect of the presence of a driving current on the critical behavior of the system.

This paper is organized as follows. In Section \ref{section:2}, the partition function for the 2DYG is given and shown to map to the massive sine-Gordon. In Section \ref{section:3}, the recursion relations are derived. These equations are analyzed in Section \ref{section:4} and their experimental ramifications are explored in Section \ref{section:5}. Finally, the paper is summarized in Section \ref{section:6}.
II. MODEL AND RENORMALIZATION GROUP STUDY

As indicated above, our approach is to model the vortices in two dimensional superconductors as a vortex gas interacting via the 2D Yukawa potential. Therefore, we begin this section by describing the partition function for the 2DYG. We will then map it to the massive sine-Gordon model, and perform a RG study on the latter.

A. Model

The partition function for the neutral vortex gas (N+ vortices and N- = N+ antivortices) in a 2D superconductor is

\[ Z = \sum_N y^{2N} \frac{1}{(N!)^2} \int d^2 r_1 \int d^2 r_2 \ldots \int d^2 r_{2N} \times \exp \left[ -\frac{\beta}{2} \sum_{i \neq j} p_i p_j V(|r_i - r_j|) \right] \]  

(1)

where 2N is the total number of particles (N = N+), ri are the coordinates of the ith charge pi = ±p and y = exp(βμ) is the fugacity, where μ = -E_c and β = 1/k_B T.

V(R) is the interaction of a pair of vortices, the strength of which is p^2 = πn_s^2D h^2/2m (n_s^2D = n_s d) is the areal superfluid density, n_s is the superfluid electron density, and m is the mass of a free electron.) For the 2D Coulomb gas, the interaction is logarithmic V(R) = ln(R/τ) (where τ is the vortex core size) for all lengths. But for a superconductor, the interaction is only logarithmic out to λ⊥, [1] at which point it approaches a constant as 1/R due to the coupling to the magnetic field. We will approximate this behavior for the potential by writing it in terms of a modified Bessel function,

\[ V(R) = \ln(\lambda_{\perp}/\tau) - K_0(R/\lambda_{\perp}) \]  

(2)

which has the limits calculated by Pearl. [1] We will refer to the vortex gas in 2D superconductors as the 2D Yukawa Gas (2DYG) since this potential is the 2D equivalent of the more familiar 3D Yukawa potential.

B. Mapping to the Massive sine-Gordon

A standard way of treating the 2DCG is to map it to the 2D sine-Gordon Hamiltonian. [14] For the case of the 2D vortex gas in a superconductor, as we now review, the correct mapping is to the massive sine-Gordon model: [17]

\[ -\mathcal{I} \int d^2 r \left\{ \langle \nabla \phi(r) \rangle^2 + \left[ m \phi(r) \right]^2 \right\} + 2y \int d^2 r \cos \phi(r) \]  

(3)

where \( \mathcal{I} = 1/[2\pi \beta p^2] \) is the coupling strength or “stiffness,” \( \phi \) is the SG field and m is the mass, which we will relate to \( \lambda_{\perp} \) below. For m = 0, Eq. (3) reduces to the SG Hamiltonian.

In Ref. [18] the mapping of a layered vortex gas with arbitrary interactions to a SG-like Hamiltonian was studied. By examining the 2D limit of this mapping, we can establish the link between the 2DYG and the massive sine-Gordon. Following Ref. [18], one can show that Eq. (3) can be written as

\[ Z = \int D\phi(r) \exp \left\{ -H_o + 2y \int d^2 r \cos(\phi(r)) \right\}, \]  

(4)

where the Hamiltonian \( H_o \) is Gaussian in the field \( \phi \). The connection between the sine-Gordon field and the potential \( V(R) \) is given by:

\[ \beta p^2 V(R) = \langle \phi(R) \rangle_0 > 0 - \langle \phi^2(0) \rangle_0, \]  

(5)

where \( \langle \ldots \rangle_0 \) denotes an average with respect to a Hamiltonian \( H_o \) Gaussian in the field \( \phi \).

Making use of Eq. (3), one can readily show that \( \beta p^2 V(R) = \ln(\lambda_{\perp}/\tau) - K_0(R/\lambda_{\perp}) \) when \( H_o = \mathcal{I} \int d^2 r \left\{ \langle \nabla \phi(r) \rangle^2 + \left[ m \phi(r) \right]^2 \right\}, \beta p^2 = 1/[2\pi \mathcal{I}], \) and m = τ/λ⊥. The calculation of the right hand side of Eq. (3) is straightforward and quite similar to that done for the layered case in Ref. [18]. (See the calculation of \( g_o(R) \) in Appendix B of that reference.) After performing the Gaussian integral over the fields, one is left with

\[ \beta p^2 V(R) = \int \frac{d^2 q}{(2\pi)^2} \frac{1 - \cos(q \cdot R)}{2\mathcal{I}(q^2 + m^2)}, \]  

(6)

where the angle is integrated from 0 to 2π and q is integrated from zero out to its infrared cutoff \( q_o \sim 2\pi/\tau \). Performing the integral over the angle first and then over q, one obtains

\[ \beta p^2 V(R) = \frac{1}{2\pi \mathcal{I}} \left\{ \left[ 1 - J_0(q_o R) \right] \ln R - K_0(mR) - \ln(mR) \right\}, \]  

(7)

where R is now expressed in units of \( \tau \) and q in units of 1/τ. Neglecting the oscillating behavior [20] of the zeroeth order Bessel function \( J_0(q_o R) \) at large R, which arises from the sharp cutoff, this has the desired behavior of Eq. (3).

This completes that mapping between the 2DYG and the massive 2D SG Hamiltonian. We will now proceed to a RG study on the latter.

C. Renormalization Group Study

The advantage of the above mapping is that it is easier to perform a momentum space RG study on Eq. (3).
than a real space RG calculation based on Eq. (1). In this subsection we show how to perform a RG study in momentum space on Eq. (1) to obtain the recursion relations for the three parameters in the system $I$, $y$, and $m$. The recursion relations for this system were previously derived in Ref. [21], but as we will explain below, a technical point was neglected which rendered the results formally incorrect.

To briefly overview the process, the RG study consists of integrating out the large wavevector $q$ components in a thin shell: $[q_0(1 - \epsilon) < q < q_0]$, which correspond to small scale structure. The system is then shrink back so that the cutoff in momentum space is restored to the original. We follow the procedure of Knops and den Ouden [10] as outlined in Ref. [18] for a layered vortex gas. Because the calculation is closely related to the previous work of the present authors, [19] we do not need to show much detail here.

The first step in the RG study is to Fourier analyze the first term in the free energy,

$$F = -\frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \phi(q)\phi(-q)[I(q^2 + m^2)] + 2y \int d^2r \cos[\phi(r)],$$  \hspace{1cm} (8)

it is then convenient to define the Gaussian terms in Eq. (8) as

$$F_0 = \phi(q)\phi(-q)[I(q^2 + m^2)].$$  \hspace{1cm} (9)

There are two differences between the 2DYG and the layered SG model of Ref. [13]: First, $F_0$ for the layered case is not given by Eq. (9), but rather by

$$\phi_k(q)\phi_{-k}(-q)[Iq^2 + 2K(1 - \cos(kz))]$$

where $s$ is the interlayer distance and $k$ is the $z$-axis wave vector. ($K$ also has a different meaning there.) Second, the system here is purely two-dimensional and so the integration over the third dimension $k$ in Ref. [13] does not appear. Keeping these two distinctions in mind and carrying out the RG study in the manner of Ref. [13], one finds

$$\mathcal{I}' = \mathcal{I} + \epsilon A(\mathcal{I}, m),$$  \hspace{1cm} (10)

$$y' = y\{1 + \epsilon[2 - f(R = 0)/2]\},$$  \hspace{1cm} (11)

and

$$m' = m\{1 + \epsilon[1 - y^2 A(\mathcal{I}, m)/\mathcal{I}]\},$$  \hspace{1cm} (12)

where

$$A(\mathcal{I}, m) = \int d^2R(q \cdot R)^2 f(R)e^{-g(R)/2},$$  \hspace{1cm} (13)

$$f_s(R) = \int_0^{2\pi} \int_{q_0(1 - \epsilon)}^{q_0} \frac{d^2q}{(2\pi)^2} \frac{e^{iq \cdot R}}{\mathcal{I}(q^2 + m^2)},$$  \hspace{1cm} (14)

and

$$g_s(R) = \int_0^{2\pi} \int_{q_0}^{q_0} \frac{d^2q}{(2\pi)^2} \frac{1 - \cos(q \cdot R)}{\mathcal{I}(q^2 + m^2)}.$$  \hspace{1cm} (15)

Integrating Eqs. (14) and (15) we have

$$f(R) = \frac{\epsilon}{2\pi(1 + m^2)} J_0(R),$$  \hspace{1cm} (16)

and

$$g(R) = \frac{1}{\pi \mathcal{I}} [(1 - J_0(q_0 R)) \ln R - K_0(m R) - \ln(m R)].$$  \hspace{1cm} (17)

Substituting these into $A(\mathcal{I}, m)$ and expanding to first order in $m$, one finds [19]

$$A(\mathcal{I}, m) = A(\mathcal{I}) - A_m(\mathcal{I}) m \ln m.$$  \hspace{1cm} (18)

The values of $A(\mathcal{I})$ and $A_m(\mathcal{I})$ depend upon the cutoff used and the nature of core used. Here, as in Ref. [19], we will assume a sharp cutoff in momentum space and a hard-core.

The recursion relations can be put in more convenient differential form. Introducing also the stiffness constant, $K = 1/[2\pi \mathcal{I}](\equiv \beta \nu^2)$, we can write the final result in the form

$$\frac{dK}{de} = -\frac{1}{2} K^2 y^2 (1 - 4.5 m \ln m),$$  \hspace{1cm} (19)

$$\frac{dy}{de} = \frac{y}{2} \frac{4 - K}{1 + m^2},$$  \hspace{1cm} (20)

and

$$\frac{dm}{de} = m(1 - y^2 K/4).$$  \hspace{1cm} (21)

Our definition of the parameter $K$ differs from others in the literature by a factor of $2\pi$.

The $m = 0$ limit of these equations correctly reduce to the original Kosterlitz recursion relations for the 2DCG. Furthermore, the $m$ corrections to those relations are of the form that one would expect physically. For example, in the recursion relation for $K$, the screening effect of small pairs on $K$ is increased by $m$ (since $m \ll 1$). This is because the vortex pairs are more weakly bound for a superconductor than for a superfluid and so they are more susceptible to screening. (This is quite consistent with the opposite result found in the case of a layered vortex gas. There, the vortex pairs are more strongly bound than in the 2DCG and the screening effects in the recursion relation for $K$ diminished. In the recursion relation for $y$, the number of vortex pairs increases in the presence of finite size effects because the energy of a vortex pair is reduced. In the recursion relation for $m$, there are two terms. The first enters through the shrinking
scale step of the RG calculation. The second, of $O(y^2)$, enters through the integrating out of small scale structure, which in this case consists of the small vortex pairs. This explains the $y^2$ dependence. This second term arises in a somewhat complicated way from the renormalization of the coefficient of the $(\nabla \phi(r))^2$ term in Eq. (3). This enters into the mass term $m$ because the coefficient of $(m \phi(r))^2$ is $2m^2$ and so $m$ must be renormalized in the second term to compensate the renormalization of $\mathcal{L}$ in the first term.

The recursion relations Eqs. (19)-[21] differ from those of Ref. [21] in the $m$ correction to the equation for $K$. This is because of an approximation made there which was pointed out to be incorrect by Knops and den Ouden. [16] The approximation, common in the literature, [24,25] is to Taylor expand a cosine term (see the first term in Eq. (A8) of Ref. [19]) and then to neglect the higher order terms. As Knops and den Ouden showed, the higher order terms are not negligible and in fact contribute in a significant way. In our case, the contribution of the higher order terms is to give an $m$ correction to the recursion relation for $K$ that is opposite in sign to that obtained by Ichinose and Mukaida. [21] (As discussed in the previous paragraph, the sign found here makes more physical sense.) However, the $m$ correction to the recursion relation that most influences the system and the integration of the recursion relations turns out to be that for $y$, Eq. (20), and this renders the effect of the error discussed above less harmful.

III. ANALYSIS OF THE RECURRENCE RELATIONS

By integrating Eqs. (19)-(21), the analysis of the effects of the finite length scale on the 2DCG is possible. One sees at once that the size effect, or mass, term is relevant. Because $y^2$ is small, the growth of $m$ with $\epsilon$ is nearly exponential, which leads to the intrinsic finite size effects dominating the integration of the recursion relations at large length scales and, in certain temperature regimes, at smaller length scales too. In this section we analyze this behavior and the temperature regimes where these effects are most significant. We will first analyze the RG flows, move on to discuss the correlation length, and then examine the temperature dependences and the behavior of the renormalized stiffness constant.

In Figure 1, we show a plot of the “flows” of the three system parameters as a function of length scale. To facilitate the comparison of these results with those of the 2DCG in the literature, we use the parameter $x = 4/K - 1$. The flows are obtained by integrating the recursion relations for representative initial values of the three parameters: $K_1 = 8$ $(x_1 = -0.5)$, $n_i = 10^{-5}$, and $y_i = 0.56, 0.58, 0.60, 0.62, 0.64, 0.66, 0.68$. (The initial values of each parameter will be denoted below by the subscript $i$.) The recursion relations are integrated until they cease to be valid, which is when $y$ or $m$ become large [$y(\epsilon) \geq 0.7$ or $m(\epsilon) \geq 0.7$] and the approximations that we made in deriving them break down. As one can see, the flows always tend towards large $m$ at the end of the flows. This reflects that $m(\lambda_\perp)$ is a relevant parameter: finite size effects will ultimately influence or determine the critical behavior. This parameter dominates the end of the flows, which corresponds to the largest length scales: finite size effects result in free vortices, and the average distance between these free vortices corresponds to these long lengths. This will become more evident as we examine other aspects of the flow.

![Figure 1](image-url)

FIG. 1. The RG flows for the 2DYG. Because $\lambda_\perp$ is a relevant parameter, all flows will eventually flow out of the $x$-$y$ plane. The direction of the flows in the figures can be inferred from the location of the starting points, as given in the text.

A projection of these flows onto the $x$-$y$ plane is shown (solid lines) in Figure 2 to better illustrate the effect of $m$ on the 2DCG. Also plotted in the figure are the RG flows (dashed lines) for the 2DCG (i.e., the $m = 0$ 2DYG). We first explain the RG flows for the 2DCG. The low temperature ($T < T_{KTB}$) flows all iterate to $y = 0$, which corresponds to zero vortex density since there are no pairs at the largest separations for $T < T_{KTB}$. This follows from the relation between vortex fugacity and vortex density. The high temperature ($T > T_{KTB}$) flows tend toward $y = 0$ initially but ultimately towards $y = \infty$. The initial decrease reflects the vortex pair density decrease, while the upturn and subsequent rise is a result of the increasing number of free vortices above the transition temperature. The isotherm that divides the low temperature flows from the high temperature flows represents the critical temperature. For the linearized recursion relations for the 2DCG, the critical isotherm is the line $y = -x$. 

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Looking now at the RG flows of the 2DYG contrasted to the 2DCG, one sees that it is at the largest values of $\epsilon$ that the largest deviation from the 2DCG flows occurs. What is also visible here and in Figure 1 is that none of the flows end at $m, y = \infty$. In other words, all of the flows have the same qualitative behavior. This confirms the expected behavior that there is no phase transition in the 2DYG since there are free vortices present at all temperatures. Further, the fact that more RG flows move towards the higher temperature limit than for the 2DCG shows that the apparent phase transition temperature does shift towards lower temperature before being wiped out. Along with the tendency of the 2DYG flows to move toward $y = \infty$, we note that the corresponding tendency for the flows to more rapidly approach large $x$ than those of the 2DCG, which is especially evident for the three highest temperatures in Figure 2. This reflects the tendency of the renormalized stiffness constant $K$ to approach zero more rapidly in the 2DYG.

It is instructive to look solely at $y$ as a function of length scale because this quantity represents the density of vortices as a function of length scale. For the same temperatures as in Figures 1 and 2, we plot $y$ as a function of $\epsilon = \ln \tau$ in Figure 3 (solid lines) along with $y(\epsilon)$ for the 2DCG (dashed lines). For temperatures far below $T_{KTB}$, one can see that the density of vortices is dropping rapidly with length scale. As mentioned above this range of $\epsilon$ represents vortex pairs. At large enough length scales, there is always an upturn in $y(\epsilon)$. This is due to free vortices present because of finite size effects. As the temperature is increased, while still remaining below $T_{KTB}$, the upturn in $y$ at large $\epsilon$ moves to lower values of $\epsilon$. At temperatures above $T_{KTB}$, the behavior is similar but more pronounced. This is because the upturn in $y$ at large $\epsilon$ is always present in the 2DCG for $T > T_{KTB}$ since free vortices can be spontaneously created in this temperature range.

Further insights can be gained through an examination of the correlation length, which is obtained as follows. As one integrates the recursion relations, $y$, $K$, and $m$ are parameterized by $\epsilon$, the natural log of the length scale. As mentioned above, the recursion relations are integrated until they break down. The value of $\epsilon$ at that point is denoted by $\epsilon_{max}$. Because there is only one length scale in the system and it is the vortex correlation length $\xi(T)$, one makes the association $\epsilon_{max} = \ln \xi(T)$. The temperature dependence enters through the first integral of the recursion relations of the (non-linearized) 2DCG: $y^i_0 - 2x^i_0 + \ln(1 + c) = c$ where $c \propto T - T_{KTB}$. For small $m$, we assume that this association remains valid.

The meaning of the 2DCG correlation length below $T_{KTB}$, $\xi_-(T)$, deserves mention. Kosterlitz [2] originally defined $\xi_-(T)$ to be infinite because the susceptibility below the transition temperature is infinite. Ambegaokar et al. [23] on the other hand, defined a finite diverging correlation length for $T < T_{KTB}$, based on the critical behavior of the dielectric constant. (Because the two descriptions have different meanings, they do not contradict one another.) Later, Sinakin and Kosterlitz [24] stated that $\xi_-(T)$ is the length scale at which $K(\epsilon)$ is essentially at its asymptotic value [25]. In other words, $\xi_-(T)$ has meaning both above and below $T_{KTB}$ and here, we take it to represent the characteristic size of the largest vortex pairs. $\xi_+(T)$, on the other hand, is defined by where the recursion relations become invalid and can be thought of as the length scale at which the influence of free vortices is significant. The relationship of $\xi_-$ to $\xi_+$ can be seen in Figure 3. $\xi_-$ is the length scale at which the density of paired vortices becomes small, while $\xi_+$ is the length scale at which the density of free vortices is large. In the case of the 2DYG, both $\xi_+$ and $\xi_-$ can be defined at all temperatures.
Defining the correlation length as the quantity $\xi$ introduced by Simkin and Kosterlitz [13] is particularly important in the case of the 2DYG. Because $m$ is a relevant RG parameter, the condition for terminating the integration of the recursion relations will always be determined by $m$ becoming too large. This will result in a featureless $\xi$ for $T < T_{KT B}$. But $\xi$, computed in accordance with Ref. [13] by using the condition that $y$ reaches an arbitrary low value, has a clear physical meaning.

The correlation length, defined as explained above, is plotted in Figure 4, for various initial values of $m$, versus $c \propto T / T_{KT B} - 1$ for $m_i = 10^{-20}$, $10^{-15}$, $10^{-10}$, and $10^{-7}$. The plateau is where the correlation length exceeds $\lambda_{\perp}$.

The second quantity that can be studied, at least qualitatively, is the transition temperature. Before the divergence of the correlation length is cut off, one can see that the effective transition temperature has shifted down. This is as expected since the vortex pairs interact more weakly. The temperature shift should have the same dependence as the quantity in Eq. (22) but this can not be verified here because the critical behavior is rounded off.

We now examine the renormalization of the stiffness constant (superfluid density) $K$. For the 2DCG, this quantity, which is evaluated for $\epsilon = \infty$, first decreases linearly with increasing temperature and then decreases more rapidly near $T_{KT B}$ with the temperature dependence of the well-known square-root cusp. Precisely at $T_{KT B}$, it jumps to zero from its universal value of 4 (in these units).

In the case of the 2DYG, $K(\infty)$ is equal to zero, independent of temperature, because there are free vortices at all temperatures. Hence, one must look at $K$ for length scales $\xi$, as shown in Figure 5. $\xi$ is the pertinent length scale especially for $T \ll T_{KT B}$ where the cutoff determining $\xi$ is $y(\epsilon)$ small, because it characterizes vortex pairs, and it is the vortex pair properties (in the presence of a current) that allow one to measure $K$ via the $I$-$V$ exponent. $K(\xi)$ is plotted in Fig. 5 for various initial values of $m$. One can see for $m_i = 0$, that $K$ has the expected behavior for the 2DCG including the jump from universal value of 4 (for our definition of this parameter) to zero. As the value of $m_i$ is increased, the jump becomes considerably more rounded. The temperature width over which the jump is smoothed is given by $\tau_{fs}$. At the largest value of $m_i$, one can see that even the $T \ll T_{KT B}$ values of $K$ are depressed.

![Figure 4](image4.png)

**FIG. 4.** $\epsilon_{max}$ (the logarithm of the correlation length) versus the integration constant $c \propto T / T_{KT B} - 1$ for $m_i = 10^{-20}$, $10^{-15}$, $10^{-10}$, and $10^{-7}$. The plateau is where the correlation length exceeds $\lambda_{\perp}$.

![Figure 5](image5.png)

**FIG. 5.** The renormalized stiffness constant $K(\xi)$ as a function of temperature for $m_i = 0$, $10^{-20}$, $10^{-15}$, $10^{-10}$, and $10^{-5}$.
be expected to be qualitatively similar and difficult to distinguish experimentally (see Ref. [30] for effects of a finite system), there is an important difference. In both cases there are free vortices below the transition temperature. But the effect of the length $\lambda_\perp$ on the vortex interaction is a little larger than that of the system size. As a result, we expect that the transition temperature will not be depressed as much in the finite extent system. There may be more differences in terms of the dynamics, and this deserves further study.

To close this section we remark on the implications of this work for the 2D massive SG Hamiltonian itself. The recursion relations found here for the 2D massive sine-Gordon differ slightly from those of Ref. [21]. As discussed in Section II C, the sign of the correction to the recursion relation for $K$ found here is opposite to that found by those authors. [21] We nonetheless feel that the behavior claimed by those authors would be affected that found by those authors. [21] We nonetheless feel that the transition temperature will not be depressed as much in the finite extent system. There may be more differences in terms of the dynamics, and this deserves further study.

IV. EXPERIMENTAL RAMIFICATIONS AND THE EFFECT OF CURRENTS

In this paper, we have rigorously verified the well-known notion that the intrinsic finite size effect characterized by the relevant parameter $\lambda_\perp$ should eventually dominate any critical behavior. One should therefore ask why this effect is not more prevalent in experimental studies. To address this question, one must incorporate the effect of an applied current since most experimental studies of KTB behavior in superconductors involve such a current. There one expects to see finite size effects for $T < T_{KTB}$ when the probing length of the current $r_c (\propto 1/I)$ exceeds $\lambda_\perp$. [13,14] We know from the work here that the renormalized value of $\lambda_\perp$ decreases nearly exponentially, [see Eq. (21),] and so one would expect finite size effects to always be seen. However, through a deeper examination of this condition, we will see that this is not the case. To do that, we must know the behavior of $r_c$ under renormalization.

Sujani et al. [1] have derived the recursion relations for the 2DCG in the presence of a current. These can also be obtained by taking the 2D limit of the recursion relations of the layered vortex gas in the presence of a current, as derived in Ref. [23]. (See also the work of Cserti [21] on 2D dislocation systems and the effect of stress.) The latter results include a correction to the recursion relation for $K$ due to the current $I$ not found in the former. Otherwise, the results are in agreement. [21]

For the case of the 2DYG in the presence of a current, the recursion relations would consist of those for the 2DCG along with the first order corrections from both the finite size variable $m$ calculated here and the current calculated in Refs. [1] and Refs. [23]. The relevant recursion relation here is that for $I$: $dI/d\epsilon = I$.

Let us now return to the condition for seeing inherent finite size effects in the presence of a current: $r_c \gg \lambda_\perp$. Even though $\lambda_\perp$ decreases nearly exponentially under renormalization, it will never get smaller than $r_c$ unless as its bare value starts out smaller than the bare value of $r_c$, because $r_c$ is decreasing exponentially under renormalization. This explains the experimental situation.

We now briefly address the expected effect of finite size effects on the current-voltage relationship for the 2DCG. The first order effects have been worked out for the layered case. [32] Taking the 2D limit of Equation (11) in Ref. [13], (i.e., $I_0 \to 0$) one obtains

$$ V = I(a + bI^{2\alpha})^{1/2}, $$

where $\alpha$ is the familiar 2DCG $I$-$V$ exponent and $a$ and $b$ are temperature dependent parameters. As expected, the $I$-$V$ relationship is ohmic at small currents, due to free vortices induced by finite size effects, and then becomes a power-law due to vortex pairs for larger currents. The crossover from ohmic to non-ohmic behavior is determined by the vortex correlation length $\xi_\perp$, thereby setting limits on $a$ and $b$. Eq. (23) is based on approximations assuming weak current. We expect that a more rigorous treatment of the $I$-$V$ relation in the presence of finite size effects would include extensions to finite current, such as a dynamic [13,12] or finite size scaling approach.

V. SUMMARY

In this work, we have derived the recursion relations for the two-dimensional Yukawa gas, relevant to vortices in 2D superconductors. We have examined the renormalized stiffness constant as a function of length scale, the RG flows, and the width of the region around $T_{KTB}$ that is no longer critical because of the finite size effect. We have also examined the condition for observing finite size effect in electrical transport measurements on superconducting films.

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