Anomalous dispersion of density waves in the early universe with positive cosmological constant

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Abstract

Density perturbations in the flat ($K = 0$) Robertson-Walker universe with radiation ($p = \varepsilon/3$) and positive cosmological constant ($\Lambda > 0$) are investigated. The phenomenon of anomalous dispersion of acoustic waves on $\Lambda$ is discussed.
1 Introduction

The image the perturbations create on the last scattering surface depends on how they propagate in the radiational era. In the flat radiation dominated universe with vanishing cosmological constant the density perturbations form sound waves and propagate with the constant sound velocity $v = 1/\sqrt{3}$. They obey linear dispersion relation $\omega = k/\sqrt{3}$, therefore, their phase and group velocities are identical. Their propagation does not depend on the wave number, therefore, no critical scale appear and no gravitationally bound structures form. The above is the subject of theorem proved by Sachs and Wolfe ([1] sec. II), confirmed independently by Lukash [2], Chibisov and Mukhanov [3] on the basis of Hamilton formalism in Field-Shepley variables [4]. The result holds in the gauge-invariant descriptions [5] as well as in the original Lifshitz formalism [6].

The situation changes significantly in the universe with the negative space curvature ($K = -1$), where the acoustic wave propagates as a scalar field with the mass $m = -K$. The dispersion relation is nonlinear and, in consequence, some critical frequency $\omega_{cr}$ appear. Below $\omega_{c}$ the wave propagation is forbidden. The dispersion of the acoustic wave on the curvature affects the microwave background fluctuations and in principle can be used to measure the cosmological density parameter $\Omega$ [5].

In this letter we show that also the cosmological constant $\Lambda$ is a source of dispersive phenomena. The interest in $\Lambda$-cosmology grows, owing to speculations that cosmological constant may take substantial values in the present epoch [8]. We investigate the flat ($K = 0$) universe with radiation ($p = \epsilon/3$) and positive cosmological constant ($\Lambda > 0$). We find exact solutions for gauge-invariant perturbation equations [9]–[16] and discuss their properties to show, that in the presence of positive $\Lambda$ the dispersion of the acoustic field has an anomalous character.

2 Homogeneous background

In the flat radiation-dominated universe with a positive cosmological constant, Friedman equations are satisfied by the scale factor $a(t)$:

$$a(t) = \left(\frac{\mathcal{M}}{\Lambda}\right)^{1/4} \sqrt{\sinh(2\sqrt{\Lambda/3}t)} = \left(\frac{\mathcal{M}}{\Lambda}\right)^{1/4} \sqrt{\sinh \tau}$$  \hspace{1cm} (1)

where $\mathcal{M}$ is the constant of motion $\mathcal{M} = \epsilon a^4$, $\epsilon$ means the energy density and the dimensionless time parameter $\tau$ is related to the metric time as $\tau = 2\sqrt{\Lambda/3}t$. To simplify notation in the forthcoming formulae we also introduce the constant $\beta$ defined as $\beta = \sqrt{3/[16\mathcal{M}\Lambda]^{1/4}}$. The conformal time $\eta$

$$\eta = F\left(\arccos\left[\frac{1 - \sinh \tau}{1 + \sinh \tau}\right], \frac{1}{\sqrt{2}}\right) \beta$$  \hspace{1cm} (2)

is finite $\eta \in [0, 2K(2^{-1/2})\beta] = [0, \frac{1}{2}\pi^{-1/2}\Gamma(\frac{1}{4})^2\beta]$, where $F(\varphi, m)$ stands for the elliptic integral of the first kind and $K(m)$ is the complete elliptic integral of the first kind [17]. The scale factor $a(\eta)$ expresses by the Jacobi elliptic functions $sn(u, m)$, $cn(u, m)$ of the ratio $\eta/\beta$.

$$a(\eta) = 2\beta \sqrt{\mathcal{M}/3} \frac{\text{sn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}{1 + \text{cn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}.$$  \hspace{1cm} (3)

The scale factor $a(\eta)$ grows monotonically from 0 to infinity. In the neighborhood of the initial singularity the radiation is dynamically dominant and the effect of the cosmological constant can be omitted. The opposite boundary of the conformal time interval ($\eta = \frac{1}{2}\pi^{-1/2}\Gamma(\frac{1}{4})^2\beta$) correspond to the de Sitter epoch where $\Lambda$ dominates and the radiation plays a marginal role.

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1Gauge-invariant perturbation equation takes the form of eq. (5.4) of [4] with the minimal coupling ($\xi = 0$). The same is also the form of eq. (21) of [4] and eq. (4.6) of [4].
3 The perturbation equation

By varying Raychaudhuri and the continuity equations one obtains the propagation equation for the density contrast \( \delta \epsilon / \epsilon \). While \( \delta \epsilon / \epsilon \) is measured on the flow-orthogonal hypersurfaces \([12]\) the equation reads

\[
- \frac{1}{2} \coth^2(\tau) X(\tau, x) + \frac{1}{2} \coth(\tau) X'(\tau, x) + X''(\tau) = v^2 \beta^2 \csc(\tau) \nabla^2 X(\tau, x). \quad (4)
\]

\( \nabla^2 \) denotes the Laplace operator in the conformal space, consequently its eigenvalue \(-k^2\) is related to the time invariant “commoving” wave vector \( k \) with \( k = |k| \). The same equation can be obtained in other gauge-invariant theories \([9]–[11], [13]–[16]\) under necessary redefinition of the perturbation variables \([5]\). This is convenient to express the equation (4) in terms of the conformal time

\[
\frac{1}{\beta^2} \left( 1 - \frac{2}{\sin^2(\beta, 1/\sqrt{2})} \right) X(\eta, x) - v^2 \nabla^2 X(\eta, x) + X''(\eta, x) = 0 \quad (5)
\]

and search for its solutions in the form of the Fourier integral

\[
X(\eta, x) = \int A_k u_k(\eta, x) \, d^3k + c.c. \quad (6)
\]

Prime stands for the conformal time derivative, Fourier coefficients \( A_k \) are arbitrary functions of the wave number \( k \). Modes \( u_k(\eta, x) \) satisfy both, the Helmholtz equation

\[
\nabla^2 u_k(\eta, x) = -k^2 u_k(\eta, x) \quad (7)
\]

and the time equation (5), and divide into two classes distinguished by the critical value \( k = \frac{1}{\sqrt{2/3 \beta}} = (4 \Lambda \Lambda)^{1/4}. \)

Below, we employ the dimensionless wave vector \( |\kappa| = \kappa = \sqrt{2} v \beta k = \sqrt{2/3} \beta k \). In this notation both classes of solutions are defined by inequalities \( \kappa > 1 \) and \( \kappa \leq 1 \). They will be discussed separately.

4 Subcritical perturbations: solutions for \( \kappa > 1 \)

The short-scale solutions \( (\kappa > 1) \) propagate as waves of variable amplitude and variable frequency. Elementary solutions \( u_\kappa(\eta, x) \) take the form

\[
uu(\eta, x) = \frac{\sin(2 \sqrt{\beta} \beta^{-1} [\Gamma(1/4)]^2, 1/\sqrt{2}) \sqrt{2 + (\kappa^2 - 1) \sin^2(\eta, 1/\sqrt{2})}}{\sqrt{2 + (\kappa^2 - 1) \sin^2([2 \sqrt{\beta} \beta^{-1} [\Gamma(1/4)]^2, 1/\sqrt{2}) \sin^2(\eta, 1/\sqrt{2})}} \exp[i \Theta(\eta, x)] \quad (8)
\]

where the phase \( \Theta(\eta, x) \) is given by

\[
\Theta(\eta, x) = \frac{\kappa \cdot x}{\sqrt{2} \beta v} - \frac{\kappa \sqrt{\kappa^2 - 1}}{\sqrt{2} (\kappa^2 - 1)} \left[ \frac{\eta}{\beta} - \Pi \left( \frac{\eta \beta^{-1}}{1/\sqrt{2}}, -\frac{1}{2} (\kappa^2 - 1), \frac{1}{\sqrt{2}} \right) \right]. \quad (9)
\]

Function \( \Pi(\varphi, \alpha^2, m) \) is the elliptic integral of the third kind and \( \text{am}(u, m) \) is the amplitude elliptic function \([17]\). The solution is singular at both ends of the conformal time interval. In the vicinity of the initial singularity

\( ^2 \text{Although we restrict ourselves to } v = 1/\sqrt{3} \text{ we explicitly keep } v \text{ in perturbation equations to make clear their structure and correspondence to other formalisms.} \)
The phase velocity
\[ \omega(\eta, \kappa) = \frac{\partial \Theta(\eta, \kappa)}{\partial \eta} = \frac{\kappa \sqrt{k^4 - 1} \sin^2\left(\frac{\eta}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \beta (2 + (\kappa^2 - 1) \sin^2\left(\frac{\eta}{\sqrt{2}}\right))}, \] (10)
is a time-dependent quantity. The interval \( \eta \in [\eta_1, \eta_2] \) contained between two extrema of the second derivative \( \frac{\partial^2 \omega(\eta, \kappa)}{\partial \eta^2} \)
\[ \eta_1 = F\left(\arcsin \sqrt{2}, \frac{1}{\sqrt{2}}\right) \beta, \quad \eta_2 = \frac{1}{2} \sqrt{\pi}^{-1/2} [\Gamma(\frac{1}{4})]^2 \beta - F\left(\arcsin \sqrt{2}, \frac{1}{\sqrt{2}}\right) \beta \]
form the oscillatory stage, where the perturbation propagates as the acoustic wave. \( \omega(\eta, \kappa) \) maintains an approximately constant value there. The quantity
\[ \omega_{\text{loc}}(\kappa) = \frac{\kappa \sqrt{k^4 - 1}}{\sqrt{2} \beta (k^2 + 1)}, \] (11)
may be understood as the basic frequency, while its the modulation \( \omega_{\text{mod}}(\eta, \kappa) \) is given by
\[ \omega_{\text{mod}}(\eta, \kappa) = \frac{\sqrt{2} \kappa \sqrt{k^4 - 1} (-1 + \sin^2\left(\frac{\eta}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right))}{\beta (1 + \kappa^2)(2 + (\kappa^2 - 1) \sin^2\left(\frac{\eta}{\sqrt{2}}\right))}. \] (12)

Beyond this interval \([\eta_1, \eta_2] \) the division into basic frequency and the modulation loses its physical meaning. While \( \eta \to 0 \) or \( \eta \to \frac{\beta}{2 \sqrt{\pi}} [\Gamma(\frac{1}{4})]^2 \) the frequency formally defined in (11) tends to zero and the amplitude grows indefinitely. In fact, the concept of oscillation, breaks down in these limits.

Formula (10) plays the role of time-dependent dispersion relation, and (11) is its analog for the basic frequency. Both relations are nonlinear. The number \( \kappa = 1 \) \( (k = \sqrt{2/3} \beta = [4MA]^1/4) \) forms the critical value of the wave number, and naturally defines the critical wave-length \( \lambda_{\text{cr}} = \frac{\sqrt{2 \pi}}{[MA]^{1/4}} \). Below this scale the wave propagation is forbidden. While \( \kappa \to 1 \), the frequency tends to zero, therefore, perturbations longer than the critical size do not form travelling waves. In contrast to dispersion on the space curvature, the dispersion on \( \Lambda \) manifests its anomalous character: critical behaviour concerns the scale of the perturbation (the wave number \( k \)), not the frequency \( \omega \). The anomaly is also clear from the phase and the group velocities behaviour
\[ v_\ell(\kappa, \beta) = \frac{\omega(\kappa, \beta)}{k} = \frac{\sqrt{k^4 - 1} \sin^2\left(\frac{\eta}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \beta (2 + (\kappa^2 - 1) \sin^2\left(\frac{\eta}{\sqrt{2}}\right))}, \] (13)
\[ v_g(\kappa, \beta) = \frac{\partial \omega(\kappa, \beta)}{\partial k} = \frac{[6k^4 - 2 + (\kappa^2 - 1)(k^4 - 2\kappa^2 - 1) \sin^2\left(\frac{\eta}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)] \sin^2\left(\frac{\eta}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}{\sqrt{3} \kappa^4 - 1 [2 + (\kappa^2 - 1) \sin^2\left(\frac{\eta}{\sqrt{2}}\right)]^2}. \] (14)
The phase velocity \( v_\ell(\kappa, \beta) \) decreases with increasing wave-length \( [4] \) and, in consequence, the group velocity \( v_g(\kappa, \beta) \) is always greater than the phase velocity \( v_\ell(\kappa, \beta) \). In the \( \kappa \to 1 \) limit the phase velocity tends to zero, while the group velocity formally expressed as \([3] \) grows indefinitely. In this regime the concept of the wave packet loses its sense\([4] \) .

\( \text{see} \[18] \)

\( \text{The space curvature defines the minimal frequency like the plasma frequency.} \)

\( \text{Taylor series for } \omega \text{ cannot be cut at linear terms. Higher order terms are responsible for the diffusion of the wave packets} \[20] \) .
5 Supercritical perturbations: solutions for $\kappa \leq 1$

The large-scale solutions ($\kappa \leq 1$) do not propagate as travelling waves. The space of solutions consists of the combinations of the two modes $u^{(1)}_\kappa \exp(i \kappa \cdot x)$ and $u^{(2)}_\kappa \exp(i \kappa \cdot x)$, where both $u^{(1)}_\kappa$ and $u^{(2)}_\kappa$

$$u^{(1)}_\kappa(\eta) = \frac{\text{sn}([2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}}])}{\sqrt{2} - \text{sn}^2([2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}}])} \sqrt{2 - \text{sn}^2(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})} \text{sn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}$$

$$u^{(2)}_\kappa(\eta) = u^{(1)}_\kappa(\eta) \int_0^\eta \frac{1}{|u^{(1)}_\kappa(\eta')|^2} d\eta'.$$  \hspace{1cm} (15)

are real functions. In the vicinity of the initial singularity $u^{(1)}_\kappa$ behave as the decaying mode, while $u^{(2)}_\kappa$ plays the role of the growing one. In this particular sense the behaviour of the large-scale perturbations mimics that of modes in the pressureless matter. Later on the solutions evolve differently. Both modes keep approximately constant amplitude in the epoch when the $\kappa > 1$ modes oscillate. Finally, in the de Sitter regime both modes $u^{(1)}_\kappa$ and $u^{(2)}_\kappa$ blow up exponentially.

The two limit cases $\kappa = 0$ and $\kappa = 1$ express in simple formulae

$$u^{(1)}_\kappa(\eta) = \frac{\text{sn}(2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}}) \sqrt{2} - \text{sn}^2(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}{\sqrt{2} - \text{sn}^2(2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}})} \text{sn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}.$$  \hspace{1cm} (17)

$$u^{(2)}_\kappa(\eta) = \frac{\text{sn}(2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}}) \sqrt{2} - \text{sn}^2(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}{\sqrt{2} - \text{sn}^2(2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}})}$$

$$\left[ \frac{2E\left(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}} \right) - \text{cn}\left(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}} \right) - \text{am}\left(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}} \right) \right] \right] \frac{1}{\sqrt{2} \text{sn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}.$$  \hspace{1cm} (18)

for $\kappa = 0$ and

$$u^{(1)}_\kappa(\eta) = \text{sn}\left(2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}} \right) \frac{1}{\text{sn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})}.$$  \hspace{1cm} (19)

$$u^{(2)}_\kappa(\eta) = \text{sn}\left(2\sqrt{\pi} \beta^{-1} [\Gamma(\frac{1}{4})]^2, \frac{1}{\sqrt{2}} \right) \frac{1}{\text{sn}(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}})} \left[ \text{am}\left(\frac{\eta}{\beta}, \frac{1}{\sqrt{2}} \right) - \frac{\eta}{\beta} \right] \left( \frac{1}{\sqrt{2}} \right)$$  \hspace{1cm} (20)

for $\kappa = 1$.

The dynamics of large-scale perturbations does not confirm their exceptional role in the structure formation process: their amplitudes are of the same range as those of oscillating modes. However, since these perturbations do not propagate, the density and expansion excesses are correlated to each other throughout the space for decaying modes, and anti- correlated for the growing ones. Present observations of the microwave background,
which are limited to the single physical quantity (temperature), are not complete enough to distinguish between traveling waves and the perturbations of other types. Yet it is possible that we will get more complete data from the last scattering surface in future. Therefore, some estimations of linear diameters and appropriate angular scales on the sky are worth mentioning.

We assume that the universe after decoupling evolves as filled with pressureless matter. Then the lapse of conformal time between the emission and the observation moments is given by

$$\Delta \eta(z) = \sqrt{3/\Lambda} \left[ \frac{\epsilon_0}{m} \right]^{1/3} \left( (1 + z)^2 \right) F_1 \left[ \frac{1}{3} \frac{1}{2} \frac{4}{3} ; -(1 + z)^3 \frac{\epsilon_0}{\Lambda} \right] - 2 F_1 \left[ \frac{1}{3} \frac{1}{2} \frac{4}{3} \frac{\epsilon_0}{\Lambda} \right] \right)$$

where $z$ means the redshift of the last scattering surface, $m = \epsilon a^3$ is the constant of motion for the late universe, subscript 0 refers to the present time, and $2 F_1 [a; b; c; x]$ stands for the hypergeometric function [22]. Sky equator covers $n$ regions of the critical length diameter where

$$n \left( \frac{\epsilon_0}{A} , z \right) = \frac{2 \pi \Delta \eta}{\lambda_{cr}} = 6 \left[ \frac{\epsilon_0}{A(1 + z)} \right]^{1/4} \left( (1 + z)^2 F_1 \left[ \frac{1}{3} \frac{1}{2} \frac{4}{3} ; -(1 + z)^3 \frac{\epsilon_0}{\Lambda} \right] - 2 F_1 \left[ \frac{1}{3} \frac{1}{2} \frac{4}{3} \frac{\epsilon_0}{\Lambda} \right] \right) .$$

For $\epsilon_0/A = 0.2$ and $z = 100$ the number $n$ is about $n = 0.8$, therefore, supercritical perturbations may contribute to the dipole anisotropy of the MBR. If the radiation plays its dynamical role after decoupling the number $n$ can be greater, but always less than $n = 2$.

6 Conclusions

Like the space curvature, the cosmological constant causes dispersion of the acoustic field in the early universe. In both cases the dispersive phenomena are induced by the space-time geometry and depend neither on the initial conditions for acoustic field nor on the character of cosmological epochs prior to the radiational era. Dispersion on the cosmological constant is a measurable effect in the sense that appropriate critical length scale is comparable with the region of the last scattering surface we observe today. Nevertheless, to detect dispersion one needs more complete data from the last scattering epoch, in particular, the second dynamically independent quantity should be known (for instance the velocity field). In contrast to dispersion on the space curvature, the dispersion on $\Lambda$ is anomalous. The two phenomena differ enough to uniquely distinguish their provenience. Both might be employed to measure values of $\Lambda$ and $\Omega$ respectively.

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