Asymptotic Parameter Tracking Performance with Measurement Data of 1-bit Resolution

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Abstract—The problem of signal parameter estimation and tracking with measurement data of low amplitude resolution is considered. In particular the performance-loss of a simplistic receiver with 1-bit resolution in comparison to an ideal receiver with infinite resolution is investigated. For the case where the measurement data is available after a hard-limiting device with 1-bit amplitude resolution, it is well-understood that the performance for low SNR channel parameter estimation degrades moderately by $2/\pi$ ($\approx 1.96$ dB). Here we show that the relative 1-bit quantization-loss can be significantly smaller if additional side information about the temporal evolution of the channel parameters in form of a state-space model is taken into account. Through the analysis of a Bayesian bound for the achievable tracking performance, we attain the result that the quantization-loss in dB is in general by factor two smaller if the channel evolution is slow. For the low SNR regime this is equivalent to a reduced loss of $\sqrt{2/\pi}$ ($\approx 0.98$ dB). By simulation of non-linear filters for a satellite-based ranging application (GPS) and a UWB channel estimation problem, both with low-complexity 1-bit analog-to-digital converter (ADC) at the receiver, we verify that the analytical characterization of the tracking performance is accurate, such that the performance-loss due to observations with low amplitude resolution can in practice be indeed much less pronounced than indicated by classical results. Finally, we discuss the implication of the attained result for medium SNR applications like channel quality estimation in the context of mobile wireless communications.

Index Terms—parameter estimation, tracking, hard-limiter, 1-bit ADC, satellite-based positioning, UWB channel estimation

I. INTRODUCTION

When analyzing parameter estimation methods and algorithms in the context of statistical signal processing it is often assumed that the measurement data is available in digital form with high amplitude resolution. Therefore, quantization effects can be neglected in the underlying system model and an ideal receiver with infinite amplitude resolution is usually assumed for the analytical characterization of the receiver performance. However, as in practice the hardware complexity and the power dissipation of an ADC scales exponentially $O(2^b)$ with the number of resolution bits $b$, such high resolution devices are expensive to build and power consuming during system operation. Further, the speed of the temporal sampling process is limited at high amplitude resolution [1]. A work-around to this unattractive property of high resolution signal processing systems is to adapt the estimation and tracking algorithm by intention to measurement data of low resolution. This enables the use of an ADC with low complexity, production cost and power consumption or to operate the sampling device at ultra fast sampling rates. In the extreme case the conversion from the analog to the digital domain is performed with a symmetric hard-limiter, providing a digital measurement output of 1-bit resolution. For such an ADC the circuit design becomes trivial as it can be realized by a single comparator with zero threshold voltage. Further, this extreme approach has the advantage that low-level digital signal processing operations, which involve the binary receive data, can be carried out hardware-efficiently by the use of 1-bit arithmetics. Nevertheless, due to the strong non-linearity the conceptual simplicity of low-resolution analog-to-digital conversion comes with a significant loss in performance if compared to receive systems working at high ADC resolution. The focus of this work is the characterization of the performance gap between a simplistic 1-bit signal processing system and an ideal receiver with $\infty$-bits ADC resolution in the context of signal parameter estimation and tracking.

A. Related Works

An interesting and long-standing result in statistical signal processing with quantized receive data [2] is that for low SNR applications the performance loss associated with 1-bit hard-limiting is moderate with $2/\pi$ ($\approx 1.96$ dB) [3]. Due to the attractive simplicity of ADCs with 1-bit amplitude resolution, a variety of works [4] [5] [6] [7] have analyzed the loss associated with this non-linear operation in the context of signal parameter estimation. Focusing on the problem of reliable communication over a noisy channel, the work [8] establishes the theoretical limit of the transmission rate with a 1-bit ADC at the receiver. Another line of works studies different methods with respect to the reduction of the 1-bit quantization-loss. In [9] [10] [11] the possibility to increase the temporal sampling rate with 1-bit ADC is discussed in the context of communication theory, while [12] takes into account the optimization of the hard-limiting threshold. In [13] the quantization threshold is adaptively adjusted, whereas [14] [15] consider the method of dithering for signal parameter estimation from quantized data. In contrast [16] analyzes the benefit of dithering strategies with additional feedback. Adding noise prior to the quantization operation and exploiting the effect of stochastic resonance is studied in [17]. [18] proves that a constant quantization threshold maximizes the Fisher information measure and it’s Bayesian version. The
work of [19] reveals that noise correlation can be beneficial for the information flow (Shannon’s information measure) through highly non-linear ADC devices, while by means of an estimation theoretic approach (Fisher’s information measure), [20] and [21] show how to exploit this effect for statistical estimation theoretic approach (Fisher’s information measure), through highly non-linear ADC devices, while by means of an work of [19] reveals that noise correlation can be beneficial for the subject of state estimation with quantized measurements principles, a stochastic model which describes the short-time duration and the channel in general follows basic physical principles, a stochastic model which describes the short-time temporal evolution of the channel parameters can be derived in many situations. Such a model forms an additional source of information which can be exploited within the digital part of the receiver at high internal resolution. In particular for signal processing systems where the measurement data is acquired from a low-complexity (or ultra-fast) sampling device with low amplitude resolution, the embedding of available side information into the formulation of the estimation problem might play an important role. In fact, through an asymptotic performance analysis based on a theoretic Bayesian bound for signal parameter tracking [27] [28] [29] [30], we show that significant performance gains can be achieved for quantized receivers through incorporation of a state-space model into the estimation algorithm. In contrast to preliminary works on the subject of state estimation with quantized measurements [22] [23], we carry out an asymptotic performance analysis under slow channel parameter evolution and obtain an explicit relative loss of $\sqrt{2/\pi}$ ($-0.98$ dB) in the low SNR regime. Interestingly, under the condition of slow channel evolution, for the medium to high SNR regime the relative quantization-loss with state-space model turns out to be in general by factor two smaller if characterized in dB. Analyzing the duration of the transient phase of the tracking process, a similar result is derived for the convergence behavior of the tracking error. With Monte-Carlo simulations of suboptimal particle filters for channel estimation tasks in the context of low SNR satellite-based ranging and UWB communication, we verify that all our theoretic results can be directly translated into signal processing practice. For the completeness of the discussion, in the beginning we briefly review the theory of two classical estimation approaches without a state-space model and discuss the performance of efficient methods for operation with 1-bit measurement data attained within these frameworks.

II. Observation Model

For the discussion an amplified sensor signal

$$y(t) = \gamma s(t; \theta(t)) + \eta(t), \quad (1)$$

$y(t) \in \mathbb{R}$, is assumed. The analog signal $y(t)$ consists of a deterministic transmit signal $s(t; \theta(t)) \in \mathbb{R}$, attenuated by factor $\gamma \in \mathbb{R}$. The signal $s(t; \theta(t))$ is modulated by a parameter $\theta(t) \in \mathbb{R}$, which evolves over time $t \in \mathbb{R}$. White random noise $\eta(t) \in \mathbb{R}$, due to an analog low-noise amplifier behind the receiver, distorts the receive signal in an additive way. The receive signal $y(t)$ is low-pass filtered to a bandwidth of $B$ (one-sided) and sampled with a rate of $f_s = 2B = 1/c$. In the $k$-th processing block of duration $T_o = NT_s$ we combine $N$ subsequent samples to an observation vector

$$y_k = y(s(\theta_k) + \eta_k), \quad (2)$$

$y_k, s(\theta_k), \eta_k \in \mathbb{R}^N$, with the individual vector entries

$$[y_k]_n = y((k-1)NT_s + (n-1)T_s)$$
$$[s(\theta_k)]_n = s((k-1)NT_s + (n-1)T_s; \theta_k)$$
$$[\eta_k]_n = \eta((k-1)NT_s + (n-1)T_s). \quad (3)$$

By following this model we assume that the temporal evolution of the channel parameter $\theta(t)$ is slow compared to the sampling process, such that we can approximate the parameter $\theta_k$ to be constant within the $k$-th block. Note that this forms no general restriction as in practice the sampling rate $f_s$ or the block length $N$ can be chosen such that the assumption of a constant block parameter is fulfilled with sufficiently high accuracy. The temporal evolution over subsequent blocks can then be described in the form of a transition probability density function $p(\theta_k | \theta_{k-1})$ with an initial prior $p(\theta_0)$ modeling the uncertainty about the channel parameter at the beginning of the receive process. The noise samples $\eta_k$ form a multivariate Gaussian random variable with the properties

$$E_{\eta} [\eta_k] = 0, \quad \forall k,$$
$$E_{\eta} [\eta_k \eta_k^T] = I, \quad \forall k, \quad (4)$$

such that the conditional probability density of the receive signal $y_k$ in the $k$-th block can be written

$$p(y_k | \theta_k) = \frac{1}{(2\pi)^{-N/2}} e^{-\frac{1}{2} \left(y_k - \gamma s(\theta_k)\right)^T \left(y_k - \gamma s(\theta_k)\right)}$$
$$= \frac{1}{(2\pi)^{-N/2}} \prod_{n=0}^{N-1} e^{-\frac{1}{2} \left(y_k - \gamma s(\theta_k)\right)_n^2}. \quad (5)$$

In order to take into account an ADC with low-resolution at the receiver, in the following the receive signal is considered to be exclusively available in quantized form

$$r_k = \text{sign}(y_k), \quad (6)$$

where $\text{sign}(x)$ is the element-wise signum function. After this hard-limiting operation the conditional probability of each
binary receive sample \([r_k]_n\) is

\[
p([r_k]_n = +1|\theta_k) = \int_{-\infty}^{\infty} p_{\eta}(\eta_k)[n]d[\eta_k]_n = Q(-\gamma(s[\theta_k])_n) = 1 - Q(\gamma(s[\theta_k])_n)
\]

such that

\[
p([r_k]_n = -1|\theta_k) = \int_{-\infty}^{\infty} p_{\eta}(\eta_k)[n]d[\eta_k]_n = 1 - Q(-\gamma(s[\theta_k])_n)
\]

\[
= Q(\gamma(s[\theta_k])_n),
\]

with \(Q(x)\) being the Q-function. The final task of the receiver is to calculate a block-wise estimate \(\hat{\theta}(r_k)\) from the receive signal \(r_k\). The quality of the estimate \(\hat{\theta}(r_k)\) is judged on the basis of a quadratic error measure

\[
\epsilon_k = (\hat{\theta}(r_k) - \theta_k)^2.
\]

**III. HARD-LIMITING LOSS - FISHER THEORY**

First we discuss the problem under a Fisher theoretic perspective \[31\]. Therefore, the parameter \(\theta_k\) is considered to be a deterministic but unknown entity. Further, each block is processed independently without taking into account the temporal evolution of the channel parameter \(\theta_k\). In this case the optimum block-wise inference procedure is the maximum likelihood estimator (MLE)

\[
\hat{\theta}_{\text{ML}}(r_k) = \arg \max_{\theta_k \in \Theta} p(r_k|\theta_k).
\]

As under the above assumptions the estimator is asymptotically efficient, for a sufficiently large number of samples \(N\), the mean square error (MSE) of the estimator

\[
\text{MSE}(\theta_k) = E_{r_k|\theta_k}[{(\hat{\theta}_{\text{ML}}(r_k) - \theta_k)}^2]
\]

in block \(k\) reaches the theoretical limit, the so-called Cramér-Rao lower bound (CRLB)

\[
\text{MSE}(\theta_k) \geq \frac{1}{F(\theta_k)}.
\]

The Fisher information measure \(F(\theta_k)\) is defined

\[
F(\theta_k) = E_{r_k|\theta_k}[{(\partial \log p(r_k|\theta_k))}^2]
\]

\[
= \sum_{n=1}^{N} E_{r_k|\theta_k}[{(\partial \log p([r_k]_n|\theta_k))}^2] + \sum_{n=1}^{N} p([r]_n = +1|\theta_k) \left( \frac{\partial p([r]_n = +1|\theta_k)}{\partial \theta_k} \right)^2 + \sum_{n=1}^{N} p([r]_n = -1|\theta_k) \left( \frac{\partial p([r]_n = -1|\theta_k)}{\partial \theta_k} \right)^2.
\]

With the derivatives of the conditional probability function

\[
\frac{\partial p([r]_n = +1|\theta_k)}{\partial \theta_k} = +\frac{\gamma}{\sqrt{2\pi}} e^{-\gamma^2(s[\theta_k])^2} \left[ \frac{\partial s(\theta_k)}{\partial \theta_k} \right]_n
\]

\[
\frac{\partial p([r]_n = -1|\theta_k)}{\partial \theta_k} = -\frac{\gamma}{\sqrt{2\pi}} e^{-\gamma^2(s[\theta_k])^2} \left[ \frac{\partial s(\theta_k)}{\partial \theta_k} \right]_n
\]

the information measure is found to be given by

\[
F(\theta_k) = \frac{\gamma^2}{2\pi} \sum_{n=1}^{N} Q(\gamma(s[\theta_k])_n) \frac{\partial s(\theta_k)}{\partial \theta_k} \left[ \frac{\partial s(\theta_k)}{\partial \theta_k} \right]^2.
\]

As a performance reference for the non-linear 1-bit receiver \[5\], we consider an ideal receiver which has access to the high resolution signal \(y_k\). For this kind of receive system the Fisher information measure in the \(k\)-th block is found to be

\[
F_{\infty}(\theta_k) = E_{y_k|\theta_k}[{(\frac{\partial \log p(y_k|\theta_k)}{\partial \theta_k})}^2]
\]

\[
= \gamma^2 \left[ \frac{\partial s(\theta_k)}{\partial \theta_k} \right]_n^T \frac{\partial s(\theta_k)}{\partial \theta_k} + \gamma^2 \sum_{n=1}^{N} \left[ \frac{\partial s(\theta_k)}{\partial \theta_k} \right]_n^2.
\]

In order to compare both receivers, we define the relative performance-loss by the block-wise information ratio

\[
\chi_k = \frac{F(\theta_k)}{F_{\infty}(\theta_k)}.
\]

As

\[
\lim_{\kappa \to 0} \frac{e^{-\kappa^2}}{\kappa} = 4,
\]

the loss for asymptotically small SNR is

\[
\lim_{\kappa \to 0} \chi_k = \frac{2}{\pi}.
\]

**IV. HARD-LIMITING LOSS - BAYESIAN THEORY**

The Bayesian perspective is slightly different. Here the parameter \(\theta_k\) is treated as a random variable which is distributed according to a block-wise prior \(p(\theta_k)\). Still each block is processed independently, but the a priori knowledge \(p(\theta_k)\) is incorporated into the estimation process. In such a situation the optimum algorithm for the inference of the parameter \(\theta_k\) is the conditional mean estimator (CME)

\[
\hat{\theta}_{\text{CM}}(r_k) = E_{\theta_k|r_k}[\theta_k] = \int_{\Theta} \theta_k p(\theta_k|r_k)d\theta_k.
\]

The mean square error (MSE) of this estimator

\[
\text{MSE}_k = E_{\theta_k|r_k}[{(\hat{\theta}_{\text{CM}}(r_k) - \theta_k)}^2]
\]

can be lower bounded by a Bayesian version of the CRLB

\[
\text{MSE}_k \geq \frac{1}{J_k}.
\]
where the block-wise Bayesian Fisher information measure is

\[
J_k = E_{p_k, \theta_k} \left[ \left( \frac{\partial \ln p(r_k, \theta_k)}{\partial \theta_k} \right)^2 \right]
\]

\[
= E_{\theta_k} \left[ E_{p_k|\theta_k} \left[ \left( \frac{\partial \ln p(r_k|\theta_k)}{\partial \theta_k} \right)^2 \right] \right] + E_{\theta_k} \left[ \left( \frac{\partial \ln p(\theta_k)}{\partial \theta_k} \right)^2 \right]
\]

\[
= E_{\theta_k} [F(\theta_k)] + J_{p,k}
\]

\[
= F(\theta_k) + J_{p,k}.
\]

Equivalently, for the ideal reference receiver we have

\[
J_{\infty,k} = E_{p_k, \theta_k} \left[ \left( \frac{\partial \ln p(y_k, \theta_k)}{\partial \theta_k} \right)^2 \right]
\]

\[
= \bar{F}_\infty(\theta_k) + J_{p,k}.
\]

Defining the relative performance gap between both systems

\[
\psi_k = \frac{J_k}{J_{\infty,k}} = \frac{F(\theta_k) + J_{p,k}}{\bar{F}_\infty(\theta_k) + J_{p,k}}
\]

(24)

and assuming that the estimation theoretic quality of the prior is constant for each block, i.e. \(J_{p,k} = J_p\), allows to perform an asymptotic performance analysis in the low SNR regime after increasing the block-length \(N\) to infinity

\[
\lim_{\gamma \to 0} \lim_{N \to \infty} \psi_k = \lim_{\gamma \to 0} \frac{F(\theta_k)}{\bar{F}_\infty(\theta_k)} = \frac{2}{\pi}.
\]

(27)

V. HARD-LIMITING LOSS - TRACKING

Finally we assume that the available stochastic model \(p(\theta_k|\theta_{k-1})\), describing the temporal evolution of the channel parameter from one block to another, is taken into account in an optimal way. This allows to perform parameter estimation with tracking over subsequent blocks and to calculate the current block estimate \(\hat{\theta}_k\) based on the observations of the current block and all preceding blocks. For simplicity we assume that the channel parameter \(\theta_k\) follows a probabilistic linear evolution model of first order (autoregressive model of order one)

\[
\theta_k = \alpha \theta_{k-1} + z_k,
\]

(28)

where \(\alpha \in \mathbb{R}\) and the innovation \(z_k \in \mathbb{R}\) is a random Gaussian variable with the properties

\[
E[z_k] = 0,
\]

\[
E[z_k^2] = \sigma^2,
\]

\[\forall k,
\]

(29)

such that the transition probability function is given by

\[
p(\theta_k|\theta_{k-1}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\theta_k - \alpha \theta_{k-1})^2}{2\sigma^2}}.
\]

(30)

For the first block we assume an initial priori distribution

\[
p(\theta_0) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(\theta_0 - \mu_0)^2}{2\sigma_0^2}}.
\]

(31)

Note that for such a state-space model the mean and the variance of the parameter evolve according to

\[
E_{\theta_k}[\theta_k] = \alpha^k \mu_0
\]

(32)

\[
E_{\theta_k}[(\theta_k - E_{\theta_k}[\theta_k])^2] = \alpha^{2k} \sigma_0^2 + \sum_{i=1}^{k} \alpha^{2(k-i)} \sigma^2.
\]

(33)

In order to avoid divergence of the state-space variance, we restrict \(\alpha\) to the range \(0 < \alpha < 1\) and in the limit attain

\[
\lim_{k \to \infty} E_{\theta_k}[\theta_k] = 0
\]

(34)

\[
\lim_{k \to \infty} E_{\theta_k}[(\theta_k - E_{\theta_k}[\theta_k])^2] = \frac{1}{1 - \alpha^2} \sigma^2.
\]

(35)

The optimum estimator in such a setup is the CME with all past observation blocks

\[
\hat{\theta}_{\text{CM}}(R_k) = E_{\theta_k|R_k}[\theta_k]
\]

(36)

where the observation matrix

\[
R_k = [r_k \ r_{k-1} \ldots \ r_1]
\]

(37)

contains the receive signals of all past blocks up to the \(k\)-th block. The mean square error (MSE) of this estimator

\[
\text{MSE}_k = E_{R_k, \theta_k} \left[ (\hat{\theta}_{\text{CM}}(R_k) - \theta_k)^2 \right]
\]

(38)

can be lower bounded by

\[
\text{MSE}_k \geq \frac{1}{U_k},
\]

(39)

where the tracking information measure \(U_k\) in block \(k\)

\[
U_k = D_{k-1}^{10} - D_{k-1}^{11} (U_{k-1} + D_{k-1}^{11} - D_{k-1}^{12})
\]

(40)

is calculated recursively with

\[
D_{k-1}^{10} = E_{\theta_{k-1}, \theta_k} \left[ \left( \frac{\partial \ln p(\theta_k|\theta_{k-1})}{\partial \theta_{k-1}} \right)^2 \right]
\]

(41)

\[
D_{k-1}^{11} = E_{\theta_{k-1}, \theta_k} \left[ \left( \frac{\partial \ln p(\theta_k|\theta_{k-1})}{\partial \theta_{k-1}} \right) \frac{\partial \ln p(\theta_k|\theta_{k-1})}{\partial \theta_k} \right]
\]

(42)

\[
D_{k-1}^{12} = E_{\theta_{k-1}, \theta_k} \left[ \left( \frac{\partial \ln p(\theta_k|\theta_{k-1})}{\partial \theta_{k-1}} \right)^2 + \left( \frac{\partial \ln p(\theta_k|\theta_{k-1})}{\partial \theta_k} \right)^2 \right]
\]

\[
+ E_{\theta_{k-1}, \theta_k} \left[ \left( \frac{\partial \ln p(\theta_{k-1}|\theta_k)}{\partial \theta_k} \right)^2 \right] + E_{\theta_{k-1}, \theta_k} [F(\theta_k)].
\]
With the exemplary state-space model (28), the required derivatives are
\[
\frac{\partial \ln p(\theta_k | \theta_{k-1})}{\partial \theta_{k-1}} = \frac{(\theta_k - \alpha \theta_{k-1})}{\sigma^2} \\
\frac{\partial \ln p(\theta_k | \theta_{k-1})}{\partial \theta_k} = -\frac{(\theta_k - \alpha \theta_{k-1})}{\sigma^2},
\]
(44)
such that
\[
E_{\theta_{k-1}} \left[ E_{\theta_k | \theta_{k-1}} \left[ \left( \frac{\partial \ln p(\theta_k | \theta_{k-1})}{\partial \theta_{k-1}} \right)^2 \right] \right] = \frac{\alpha^2}{\sigma^2} \\
E_{\theta_k | \theta_{k-1}} \left[ \left( \frac{\partial \ln p(\theta_k | \theta_{k-1})}{\partial \theta_k} \right)^2 \right] = \frac{1}{\sigma^2} \\
E_{\theta_{k-1}} \left[ E_{\theta_k | \theta_{k-1}} \left[ \frac{\partial \ln p(\theta_k | \theta_{k-1})}{\partial \theta_{k-1}} \frac{\partial \ln p(\theta_k | \theta_{k-1})}{\partial \theta_k} \right] \right] = -\frac{\alpha^2}{\sigma^2}.
\]
(45) (46) (47)
Consequently, the recursive rule for the computation of the tracking information measure \( U_k \) is given by
\[
U_k = \frac{1}{\sigma^2} - \frac{\alpha^2}{\sigma^4} (U_{k-1} + \frac{\alpha^2}{\sigma^2})^{-1} + E_{\theta_k} [F(\theta_k)] \\
= \left( \sigma^2 + \frac{\alpha^2}{U_{k-1}} \right)^{-1} + E_{\theta_k} [F(\theta_k)].
\]
(48)
For the ideal receiver the recursion is given accordingly
\[
U_{\infty, k} = \left( \sigma^2 + \frac{\alpha^2}{U_{\infty, k-1}} \right)^{-1} + E_{\theta_k} [F_{\infty}(\theta_k)].
\]
(49)
VI. ASYMPTOTIC TRACKING PERFORMANCE
After an initial transient phase the tracking algorithm reaches a steady-state where the estimation error saturates and
\[
U_k = U_{k-1}.
\]
(50)
We can thus claim that
\[
U = \lim_{k \to \infty} U_k \\
= \frac{1}{2\sigma^2} + \frac{\hat{F}}{2} + \sqrt{\left( \frac{1}{2\sigma^2} + \frac{\hat{F}}{2} \right)^2 + \frac{\alpha^2 F}{\sigma^2}},
\]
(51)
where the expected steady-state Fisher information is
\[
\hat{F} = \lim_{k \to \infty} E_{\theta_k} [F(\theta_k)].
\]
(52)
The situation that the last term \( \frac{\alpha^2 F}{\sigma^2} \) in (51) dominates the tracking information measure \( U \) arises if the two conditions
\[
\left( \frac{1}{2\sigma^2} \right)^2 \ll \frac{\alpha^2 \hat{F}}{\sigma^2} \\
\left( \frac{\hat{F}}{2} \right)^2 \ll \frac{\alpha^2 \hat{F}}{\sigma^2}
\]
are fulfilled. The first condition (53) can be reformulated
\[
(1 - \alpha^2)^2 \ll \alpha^2 \sigma^2 \hat{F}
\]
(55)
and the second condition (54) can be stated as
\[
\hat{F} \ll \frac{\alpha^2}{\sigma^2},
\]
(56)
Substituting (56) into (55), we get
\[
1 - \alpha^2 \ll \alpha^2,
\]
(57)
which is satisfied if we set \( \alpha \) close to one. This means that, if \( \alpha \) is close to one (see eq. (57)) and the informative quality of the state-space model, indicated by \( \frac{\alpha^2}{\sigma^2} \) (see eq. (45)), is much higher than the expected steady-state Fisher information \( \hat{F} \) of the observation model (56), the steady-state tracking information measure \( U \) can be approximated by
\[
U \approx \sqrt{\frac{\alpha^2 \hat{F}}{\sigma^2}}.
\]
(58)
For the comparison between the quantized receiver and the ideal system, we define the 1-bit quantization loss for parameter estimation and tracking in the \( k \)-th block as
\[
\rho_k = \frac{U_k}{U_{\infty, k}},
\]
(59)
such that in steady-state
\[
\rho = \lim_{k \to \infty} \rho_k \\
= \frac{U}{U_{\infty}},
\]
(60)
where the steady-state tracking information measure \( U_{\infty} \) for the ideal reference receiver is
\[
U_{\infty} = \frac{1}{2\sigma^2} + \frac{\hat{F}_{\infty}}{2} + \sqrt{\left( \frac{1}{2\sigma^2} + \frac{\hat{F}_{\infty}}{2} \right)^2 + \frac{\alpha^2 \hat{F}_{\infty}}{\sigma^2}},
\]
(61)
with the expected steady-state Fisher information
\[
\hat{F}_{\infty} = \lim_{k \to \infty} E_{\theta_k} [F_{\infty}(\theta_k)].
\]
(62)
Under the assumption that the state-space model has much higher informative value than the observation model independent of the form of the receiver, i.e.
\[
\hat{F} \ll \frac{\alpha^2}{\sigma^2} \quad \hat{F}_{\infty} \ll \frac{\alpha^2}{\sigma^2},
\]
(63) (64)
it is possible to evaluate the loss for an asymptotic slow temporal evolution of the channel parameter according to
\[
\lim_{\alpha \to 1} \rho \approx \sqrt{\frac{\hat{F}}{\hat{F}_{\infty}}},
\]
(65)
Note that as long as (63) and (64) are fulfilled the result (65) holds in general, independent of the considered SNR regime. This implies that, compared to the Fisher or the Bayesian approach, tracking the parameter with the use of an accurate state-space model leads to a 1-bit quantization-loss which is by factor two smaller if measured in dB. With the result (65) we can make the explicit statement, that for signal parameter...
estimation and tracking in the low SNR regime the relative 1-bit quantization-loss is

$$\lim_{\gamma \to 0} \lim_{\alpha \to 1} \rho \approx \sqrt{\frac{2}{\pi}}. \quad (66)$$

VII. CONVERGENCE AND TRANSIENT PHASE ANALYSIS

In order to further analyze the behavior of the 1-bit quantized system we consider the convergence of the recursive information measure (48). The goal is to determine the amount of blocks which is required to fulfill the steady-state condition (50). To this end, we define a transient phase of quality λ > 1 with duration

$$\bar{k}_\lambda = \inf \left\{ k \geq 1 \left| |U_k - U| \leq 10^{-\lambda} |U| \right. \right\}. \quad (67)$$

The measure \( \bar{k}_\lambda \) characterizes the delay from the start of the tracking procedure to the steady-state entry point. The rate of convergence \( \nu \in \mathbb{R} \) of recursion (48) is found by solving

$$\lim_{k \to \infty} \frac{|U_k - U|}{|U_{k-1} - U|^\nu} = \xi \quad (68)$$

for \( \nu \) with constant \( \xi \in \mathbb{R}, \xi < \infty \). As the derivative

$$\frac{\partial U_k}{\partial U_{k-1}} \bigg|_{U_{k-1} = U} = \alpha^2 (\sigma^2 U + \alpha^2)^{-2} \neq 0 \quad (69)$$

we have \( \nu = 1, \) i.e. the order of convergence is linear and

$$\xi = \alpha^2 (\sigma^2 U + \alpha^2)^{-2}. \quad (70)$$

With \( |U_k - U| \approx \xi^k |U| \) the duration \( \bar{k}_\lambda \) is found to be approximately

$$\bar{k}_\lambda \approx -\frac{\lambda}{\log \xi}. \quad (71)$$

Assuming that the conditions (63) and (64) are satisfied and \( \alpha \) is close to to one

$$\xi = (\sqrt{\sigma^2 F} + \alpha)^{-2} \approx \alpha^{-2}, \quad (72)$$

such that the delay \( \bar{k}_\lambda \) in this case is

$$\bar{k}_\lambda \approx -\frac{\lambda}{2 \log(\alpha)} \approx \frac{\lambda}{2(1 - \alpha)}. \quad (73)$$

Note that in (73) the order of magnitude is \( (1 - \alpha)^{-1} \). Specifying the additional relative delay \( \Delta \) which is introduced with 1-bit quantization by

$$\Delta = \frac{\bar{k}_\lambda}{k_{\infty, \lambda}}, \quad (74)$$

where \( \bar{k}_{\infty, \lambda} \) is the duration of the transient phase for the ideal receive system, we find

$$\Delta = \frac{\log(\sqrt{\sigma^2 F} + \alpha)}{\log(\sqrt{\sigma^2 F})} \approx \frac{F_{\infty}}{F}, \quad (75)$$

independent of the choice of the steady-state accuracy \( \lambda \), similar to the loss result (65). Therefore, in the low SNR regime the delay is \( \Delta \approx 1.25 \), i.e. the transient phase with the 1-bit receiver takes approximately 25% more time than with the ideal system.

VIII. SATELLITE-BASED POSITIONING AT LOW SNR

As an application we consider a satellite-based ranging problem where a transmitter sends a known periodic signal of the form

$$s(t) = \sum_{c = -\infty}^{\infty} [b(1+\text{mod}(c,C))] g(t - cT_c). \quad (76)$$

The vector \( b \) is a binary sequence with \( C \) symbols, each of duration \( T_c \) and \( g(t) \) is a band-limited rectangular transmit pulse. A doppler-compensated receiver observes an attenuated and delayed copy of the transmit signal

$$y(t) = \gamma s(t - \theta(t)) + \eta(t) \quad (77)$$

with additive noise \( \eta(t) \). Band-limiting and sampling the receive signal, the ideal receiver has available the digital receive signal

$$y_k = \gamma s(\theta_k) + \eta_k, \quad (78)$$

while a low-cost version of the receiver operates on the basis of the signal sign

$$r_k = \text{sign} (y_k) = \text{sign} (\gamma s(\theta_k) + \eta_k). \quad (79)$$

The temporal evolution of the time-delay parameter \( \theta_k \) can be approximated by

$$\theta_k = \alpha \theta_{k-1} + z_k. \quad (80)$$

Note that in this radio-based ranging example \( \alpha - 1 \) corresponds to the relative velocity (normalized by the speed of light) between transmitter and receiver. For simplicity we assume that the state-space parameter \( \alpha \) is constant over the considered amount of blocks and known at the receiver. The receivers task is to estimate the distance to the transmitter in each block \( k \) by measuring the time-delay parameter \( \theta_k \).

A. Tracking with a non-linear Filter

As the optimum estimator is difficult to calculate in this situation, for simulations we use a suboptimal non-linear filter called particle filter (29). The filter is based on approximating the posterior filtered density

$$p(\theta_k | R_k) \approx \sum_{l=1}^{L} w^l_k \delta (\theta_k - \theta^l_k) \quad (81)$$

by \( L \) particles \( \theta^l_k \). The particle weights \( w^l_k \geq 0 \) satisfy

$$\sum_{l=1}^{L} w^l_k = 1, \quad (82)$$
such that a block-wise estimate $\hat{\theta}_k$ can be calculated by
\[ \hat{\theta}_k = \sum_{i=1}^{L} u_i^j \theta_k^i. \] (83)

Using the transitional probability function $p(\theta_k|\theta_{k-1})$ as the importance density, the particle weights are updated recursively
\[ \tilde{w}_k^i = w_{k-1}^i p(r_k^i|\theta_k^i) \] (84)
and normalized
\[ w_k^i = \frac{\tilde{w}_k^i}{\sum_{l=1}^{L} \tilde{w}_k^l}. \] (85)

If the effective number of particles
\[ L_{\text{eff}} = \frac{1}{\sum_{l=1}^{L} |w_k^l|^2} \] (86)
falls below a certain threshold $\kappa$, i.e.
\[ L_{\text{eff}} \leq \kappa L, \] (87)
a resampling step is performed by replacing the particles by sampling $L$ times from $\tilde{p}(\theta_k|R_k)$.

### B. Results

For simulations we use the signal of the 5-th GPS satellite with $C = 1023$, $T_e = \frac{1}{1.023 \text{ MHz}}$ and a rectangular transmit pulse $g(t)$. Accordingly to the chip rate, the receiver bandwidth is set to $B = 1.023$ MHz (one-sided). The sampling rate is set to $f_s = 2B$ and each block has duration $T_o = 1$ ms, i.e. a block contains $N = 2046$ samples. The signal-to-noise ratio is set to $\text{SNR} = -15.0$ dB. For the state-space model we choose $\alpha = 1 - 10^{-3}$ and $\sigma = 10^{-3}$ and the initialization setup is $\theta_0 = 398.7342 \cdot T_e$ and $\sigma_0 = 0.1 \cdot T_e$. For $K = 250$ blocks we generate 100 delay processes and run the non-linear filters with $L = 100$ particles for each delay process 1000 times with independent noise realizations while the resampling threshold is set to $\kappa = 0.66$. The results depicted in Fig. 1 show that the block-wise analytic tracking errors $U_k^{-1}$ and $U_{\infty,k}^{-1}$ approach the asymptotic steady-state errors $U^{-1}$ and $U_{\infty}^{-1}$. Further it can be observed, that both non-linear filters perform efficient such that the errors MSE$_k$ and MSE$_{\infty,k}$ reach the theoretic tracking bounds $U_k^{-1}$ and $U_{\infty,k}^{-1}$. Therefore, in Fig. 2 the quantization-loss $\rho_k$ for the ranging problem is visualized based on the analytic bounding expressions. It is observed, that at the beginning the performance gap between both receivers is moderate ($-1.38$ dB at $k = 1$) due to the same initial knowledge of variance $\sigma_0^2$. In the following transient phase, the quantization-loss becomes quite pronounced ($-1.90$ dB at $k = 15$) while after reaching the steady-state phase the loss converges to $-0.93$ dB.

### IX. UWB Channel Estimation at Low SNR

As a second application, we consider the estimation of the channel quality in the context of UWB communication. Similar to the ranging application the receive signal of a synchronized receiver during a pilot phase can be modeled
\[ y_k = \theta_k s_k + \eta_k, \] (88)
where $s_k$ is the time-discrete form of a known pilot signal with an analog structure (76) and $\theta_k$ is the channel coefficient. Note, that in contrast to the ranging problem the parameter $\theta_k$ in the ideal receive model (88) shows up in a linear form. The task of a low-cost UWB receiver
\[ r_k = \text{sign} \left( y_k \right) = \text{sign} \left( \theta_k s_k + \eta_k \right) \] (89)
is to estimate the signal attenuation $\hat{\theta}_k$ for each pilot block, while the channel coefficient follows the temporal evolution model (28). In contrast to the ranging application we assume $B = 528$ MHz, a Nyquist transmit pulse $g(t)$ of bandwidth $B$ and $C = 10$ with $\text{SNR} = -15.0$ dB. The state-space model parameters are $\alpha = 1 - 10^{-4}$ and $\sigma = \sqrt{(1 - \alpha^2)} \text{SNR}$, while the initialization setup is $\theta_0 = \sqrt{\text{SNR}}$ and $\sigma_0 = 0.05$. In Fig. 3 it can be seen, like in the ranging application, that
the non-linear filters, simulated with 1000 channel coefficient processes and 100 independent noise realizations, perform efficient and therefore close to the tracking bounds $U_k^{-1}$ or $U_k^{-\infty}$, which asymptotically equal the analytic steady-state errors $U^{-1}$ and $U^{-\infty}$. In Fig. 4 the performance loss $\rho_k$ is depicted in dB. As in the ranging problem, it is observed, that the loss after the initial transient phase recovers and approaches $-1.02$ dB. Note that for both of the considered applications the asymptotic loss is slightly different from $-0.98$ dB as the low SNR or the slow channel evolution assumption are not fully valid for the chosen simulation setups.

X. ENABLING 1-BIT ESTIMATION AT MEDIUM SNR

As a low-cost radio front-end design might in particular be interesting for mobile communication receivers, we finally investigate the potential tracking performance for signal parameter estimation in the medium SNR regime. As here the 1-bit quantization-loss is much more pronounced, using a low-cost 1-bit ADC might make it impossible to meet the specified technical requirements. However, the use of a state-space model bears the potential to reduce the quantization error, such that low-cost ADCs become a possible system design option. For the considered scenario we assume a mobile communication channel with $B = 2.5$ MHz, a pilot sequence of $C = 10$ symbols and a medium channel quality of SNR $= 6.0$ dB. The task of the receiver is to estimate the channel coefficient $\hat{\theta}_k$ in each pilot block, while the initial knowledge is assumed to be $\theta_0 = \sqrt{\text{SNR}}$ under the uncertainty

$$
\sigma_0 = \left( \sqrt{\mathbb{E}_{\theta_0} [F_\infty(\theta_0)]} \right)^{-1}.
$$

(90)

The process noise is set to $\sigma = \sqrt{(1-\alpha^2)\text{SNR}}$ while we study the tracking loss $\rho$ and $\rho_k$ for variable $\alpha = 1 - \beta$.

In Fig. 5 the steady-state loss $\rho$ is plotted for different $\beta$. In comparison to the quantization-loss $\psi$ without tracking, it can be seen that the quantization-loss becomes small when $\alpha$
XI. Conclusion

We have analyzed the performance gap between two extreme receive systems in the context of parameter estimation and tracking. The reference receiver performs analog-to-digital conversion with infinite amplitude resolution while the low-cost receive system has a simple symmetric hard-limiting ADC with 1-bit output resolution. If consecutive blocks are processed independently we attain the well-established loss of $\frac{2}{\pi} \approx -1.96$ dB for low SNR applications. If in contrast additional side information about the temporal evolution of the channel in form of a state space model is optimally taken into account, the loss can be significantly lower. For asymptotically slow channel evolution we attain $\sqrt{\frac{2}{\pi}} \approx -0.98$ dB in the low SNR regime while for medium to high SNR the loss in dB is in general by factor two smaller compared to the case where the side information is not taken into account. Through simulations of a non-linear filtering algorithm for a satellite-based ranging and a UWB channel estimation application we have verified that this small theoretical loss can be translated into practical applications. In particular for technical applications with medium SNR the result is interesting as here the quantization-loss is pronounced. The embedding of additional information into the estimation and tracking process might allow to suppress the loss due to a non-linear radio front-end and therefore enable new low-cost design options.

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