Thermal Non-equilibrium Double Diffusive Convection in a Maxwell Fluid with Internal Heat Source

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Abstract. In this study, double-diffusive natural convection in a Maxwell fluid saturated porous layer with internal heat under local thermal non-equilibrium model (LTNM) is studied analytically. The study has been carried out based on using linear stability analysis. The modified Darcy model for the viscoelastic fluid of the Maxwell type is used to govern the momentum equation. A two-field model for energy equation has been applied for fluid and solid phases. The onset criterion for stationary convection has been derived analytically. The effects of inter-phase heat transfer coefficient and internal heat parameter on the stability of the system are investigated.

1. Introduction

In recent years, the problem of double-diffusive natural convection in porous layers and cavities has received considerable attention and this is because of its wide applications in industries and natural systems. For example, the effect of pollutants on underground water and lakes, chemical processes, energy storage, processing of materials, atmospheric pollution, and food industries, to name a few. The difference of temperature and the difference of concentration in the fluid layer are the two main factors that affect the buoyancy force in double diffusive natural convection process. The books by Nield and Bejan [1], Ingham and Pop [2] and Vafai [3] gave a detailed and wide discussion on double-diffusion natural convection. The onset of thermal instability of a fluid saturated horizontal porous layer was firstly investigated extensively by Horton and Rojers [4] and Lapwood [5].

The local thermal equilibrium (LTE) model has been considered in the modeling of many works of studying fluids saturated porous medium. LTE governs situations when the temperature gradient at any location between the solid and fluid phases is considered to be negligible. If the heat transfer between the two phases is significant, then the local thermal equilibrium model is no longer valid. In this case, the local thermal non-equilibrium model (LTNM) must be considered. Flows with high-speed and a big difference in temperature between the fluid and solid phases which appear in many applications are leads to use LTNM, since the LTE is inadequate to
govern such situations [6]. LTNM plays an important role in many applications. Freezing of foods, drying, microwave heating, and computer cooling are some of these important and common applications [7]. Malashetty et al. [6] studied the effect of LTNM on the onset of double-diffusive natural convection in a porous layer. Their results showed that the onset of instability as stationary or oscillatory modes is affected by the processes of thermal and solutal diffusion. In addition, they found that the concentration gradients has a stabilizing effect on the process of double-diffusive convection. Thermosolutal instability in a horizontal porous layer has been studied by Chen et al. [8]. Their study is based on stability analysis using LTNM. Linear and nonlinear double-diffusive natural convection in a viscoelastic fluid saturated a porous layer under LTNM is investigated by Kumar and Bhadauria [9].

Different models and applications are taking to account the heat generation source. Where in many situations the porous media has its own source of heat and the convection is driven by the internal heat. Due to the several applications such as reactor safety analysis, fire and combustion studies, geophysics, and storage of radioactive materials, the internal heat parameter is necessary to be considered in the mathematical model. Bhadauria [10] was carried out the effect of internal heat on the instability and heat and mass transport in double-diffusive convection in an anisotropic porous layer. Altawallbeh et al. [11] investigated the effect of internal heat and Soret parameters on the onset of convection in a saturated anisotropic porous layer. Also, they studied the effect of these parameters on the steady heat and mass transport. A destabilizing effect was observed by increasing the internal heat parameter.

As an increasing of importance of non-Newtonian fluids in industry and modern technology, the studies concerning such fluids become desirable. The non-Newtonian fluids are more realistic for modeling some natural phenomena than Newtonian fluids. There are many applications for non-Newtonian fluids in industry, for example, lubricants and suspension solutions, the extrusion of polymer fluids and solidification of liquid crystals. Different studies on double-diffusive convection in a porous medium saturated by non-Newtonian fluids can be found in the literature, but have not given much attention. Awad et al. [12] investigated the linear stability analysis of thermosolutal convection in a porous layer saturated with a Maxwell fluid based on the modified Darcy-Brinkman model. The study of double-diffusive convection in a porous medium saturated with a Maxwell fluid with Soret effect has carried out by Shaowei and Wenchang [13]. They found that the system is destabilized by the effect of Soret parameter and the instability set in as an oscillatory convection. The relaxation time parameter also enhances the instability of the system. Malashetty and Biradar [14] investigated the onset of instability in a porous layer saturated with Maxwell fluid taking to account the Soret and Duffour effects. The effect of internal heat on the onset of double-diffusive natural convection in a Maxwell fluid saturated porous layer was studied by Zhao et al. [15].

Even though there are some studies on double-diffusive natural convection in a porous medium saturated with non-Newtonian fluids, the studies concerning Maxwell fluids and LTNM have not given enough attention. So, the aim of this work is to study the effect internal heat driven double-diffusion natural convection in a Maxwell fluid saturated isotropic porous layer taking to account local thermal non-equilibrium model.

2. Mathematical formulation
In the recent physical model, an infinite horizontal isotropic porous layer is considered. The layer is filled by a viscoelastic fluid taking to account the effect of internal heat generation. The porous layer is confined between two parallel horizontal planes at \( z = 0 \) and \( z = d \) with a distance \( d \) apart. A Cartesian frame of reference is chosen in such a way that the origin lies on the lower plane and the \( z \)-axis as vertical upward, where gravity force \( g \) is acting vertically downward. Adverse temperature and concentration gradients are applied across the porous layer. The lower plane is kept at temperature \( T_0 + \Delta T \) and concentration at \( S_0 + \Delta S \), while, the upper plane is
kept at temperature $T_0$ and concentration at $S_0$, where $\Delta T$ is the temperature difference, while $\Delta S$ is concentration difference between walls. The Oberbeck-Boussinesq approximation has been used to account the effect of density variations.

For the modeling of energy equation, the fluid and solid structures are assumed to be in thermal non-equilibrium. In general, there are two approaches to model the energy equation. The first one, the fluid and solid structures are considered in LTE. This assumption is appropriate for small-pore media. For example, fibrous insulations, geothermal reservoirs and small temperature differences between the two phases. The second one, the fluid and solid are considered in LTNM. In deed, this model is suitable for different applications involving flows with high-speed or big temperature difference between the fluid and solid phases. A two-field model that represents the fluid and solid phases separately is employed for the energy equation that represents the fluid and solid phases separately is employed for the energy equation with high-speed or big temperature difference between the fluid and solid phases. A two-field model that represents the fluid and solid phases separately is employed for the energy equation.

These assumptions, the governing equations can be written as follows (see Kumar and Bhadauria for this study, which means that the model of thermal non-equilibrium is considered. Under these assumptions, the governing equations can be written as follows (see Kumar and Bhadauria [9] and Malashetty and Biradar [14]):

$$\nabla \cdot \mathbf{q} = 0$$  \hspace{1cm} (1)

$$\left( \frac{\lambda}{\partial t} + 1 \right) \left( \frac{\rho_0}{\epsilon} \nabla P - \rho g \right) + \frac{\mu}{K} \mathbf{q} = 0$$ \hspace{1cm} (2)

$$\epsilon (\rho_0 c)_f \frac{\partial T_f}{\partial t} + (\rho_0 c)_f \mathbf{q} \cdot \nabla T_f = \epsilon (\kappa_f \nabla^2 T_f) + h(T_s - T_f) + Q(T_f - T_0)$$ \hspace{1cm} (3)

$$(1 - \epsilon) (\rho_0 c)_s \frac{\partial T_s}{\partial t} = (1 - \epsilon) (\kappa_s \nabla^2 T_s) - h(T_s - T_f)$$ \hspace{1cm} (4)

$$\frac{\partial C}{\partial t} + \frac{1}{\epsilon} (\mathbf{q} \cdot \nabla C) = D \nabla^2 C$$ \hspace{1cm} (5)

$$\rho = \rho_0 [1 - \beta_T (T_f - T_0) + \beta_s (C - C_0)]$$ \hspace{1cm} (6)

with the following boundary conditions:

$$T = T_0 + \Delta T, \hspace{1cm} C = C_0 + \Delta C \hspace{1cm} \text{at} \hspace{1cm} z = 0$$

$$T = T_0, \hspace{1cm} C = C_0 \hspace{1cm} \text{at} \hspace{1cm} z = d.$$ \hspace{1cm} (7)

Where $\mu$ is the fluid viscosity, $\lambda$ is the relaxation time, $\mathbf{q}$ is velocity $(u, v, w)$, $K$ is the permeability, $T_f$ and $T_s$ are the temperature for fluid and solid phases, respectively. $\beta_T$ is the thermal expansion coefficient, $\beta_s$ is the concentration expansion coefficient, $\kappa_f$ and $\kappa_s$ are the thermal conductivities for fluid and solid phases, $D$ is the concentration diffusivity, $h$ is the inter-phase heat transfer coefficient, $\epsilon$ is the porosity, $\rho$ is the density, $g = (0, 0, -g)$ is gravitational acceleration, $Q$ is the internal heat source, while $\rho_0$ is the reference density.

2.1. Basic state and perturbed state

The basic state of the fluid is assumed to be quiescent. So,

$$q_b = (0, 0, 0), T_f = T_{fb}(z), T_s = T_{sb}(z), C = C_b(z),$$

$$P = P_b(z), \rho = \rho_b(z).$$ \hspace{1cm} (8)

Substitute Eq. (8) into Eqs. (1)–(5), to obtain

$$\kappa_f \frac{d^2 T_{fb}}{dz^2} + Q(T_{fb} - T_0) = 0, \hspace{1cm} \frac{d^2 T_{sb}}{dz^2} = 0, \hspace{1cm} \frac{d^2 C_b}{dz^2} = 0.$$ \hspace{1cm} (9)

By solving the obtained equations in (9), we find $T_{fb}(z)$, $T_{sb}(z)$ and $C_b(z)$.
On the basic state we superimpose perturbations by an infinitesimal perturbation and study the stability of the system, so that

\[ \mathbf{q} = \mathbf{q}_b + \mathbf{q}_t, T_f = T_{fb} + T_f, T_s = T_{sb} + T_s, C = C_b + C, \rho = \rho_b + \rho. \] (10)

Now, substitute Eq. (10) into Eqs. (1)–(6), and using basic state equations in (9) we obtain

\[ \nabla \cdot \mathbf{q}' = 0 \] (11)

\[ \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left( \frac{1}{V_a} \frac{\partial}{\partial t} \nabla^2 \psi + Ra_T \frac{\partial T_f}{\partial x} - Ra_s \frac{\partial S}{\partial x} \right) + \nabla^2 \psi = 0 \] (12)

\[ \left( \frac{\partial}{\partial t} - \nabla^2 - R_i \right) T_f - \frac{\partial (\psi, T_f)}{\partial (x, z)} + \frac{\partial \psi}{\partial x} = H(T_s - T_f) \] (13)

\[ \left( \frac{\tau}{Le} - \nabla^2 \right) S = \gamma H(T_f - T_s) \] (14)

\[ \left( \frac{\partial}{\partial t} - \frac{1}{Le} \nabla^2 \right) S - \frac{\partial (\psi, S)}{\partial (x, z)} + \frac{\partial \psi}{\partial x} = 0. \] (15)

Eliminate pressure from Eq. (12) by operating curl twice on it, and use the following transformation for dimensionless variables:

\[ (x^*, z^*) = (\frac{x}{d^*}, \frac{z}{d^*}), t = (\frac{\rho_0 c_f d^2}{\kappa_f}) t^*, (u', v', w') = (\frac{\kappa_f}{\rho_0 c_f d^*} (u^*, v^*, w^*), T_f = (\Delta T) T_f^*, T_s = (\Delta T) T_s^*, C' = (\Delta C) S^*. \] (16)

Substitute Eq. (16) into Eqs. (12)– (15) and use the stream function, which is defined by \((u, w) = \left( \frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x} \right)\). The resulted dimensionless system can be written as follows (asterisks removes for simplicity)

\[ \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left( \frac{1}{V_a} \frac{\partial}{\partial t} \nabla^2 \psi + Ra_T \frac{\partial T_f}{\partial x} - Ra_s \frac{\partial S}{\partial x} \right) + \nabla^2 \psi = 0 \] (17)

\[ \left( \frac{\partial}{\partial t} - \nabla^2 - R_i \right) T_f - \frac{\partial (\psi, T_f)}{\partial (x, z)} + \frac{\partial \psi}{\partial x} = H(T_s - T_f) \] (18)

\[ \left( \frac{\tau}{Le} - \nabla^2 \right) S = \gamma H(T_f - T_s) \] (19)

\[ \left( \frac{\partial}{\partial t} - \frac{1}{Le} \nabla^2 \right) S - \frac{\partial (\psi, S)}{\partial (x, z)} + \frac{\partial \psi}{\partial x} = 0. \] (20)

Where \( Ra_T = \frac{\beta_T q K \Delta T d}{\nu \kappa d} \) the thermal Rayleigh number, \( Ra_s = \frac{\beta_s q K \Delta C d}{\nu \kappa s} \) the concentration Rayleigh number, \( \nu = \mu/\rho_0 \) the kinematic viscosity, \( Le = \frac{d^*}{\kappa_s} \) the Lewis number, \( H = \frac{h d^2}{\kappa_f} \) the interphase heat transfer coefficient, \( \gamma = \frac{\kappa_f}{1 - \epsilon} \kappa_s \) porosity modified conductivity ratio. \( R_i = \frac{Q d^2}{\kappa_f} \) is the internal heat parameter.

Now, Eqs. (17)–(20) are solved with the following boundary conditions

\[ W = T_f = T_s = S = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \] (21)

For next steps the asterisks are removed for simplicity.
3. Linear stability theory
To make the linear stability study, we neglect the Jacobian in Eqs. (17)–(20), and assume the solutions to be periodic waves of the form

\[
\begin{bmatrix}
w \\
T_f \\
T_s \\
S 
\end{bmatrix} = e^{\sigma t} \begin{bmatrix}
W \sin \alpha x \\
\Theta \cos \alpha x \\
\Phi \cos \alpha x \\
\Omega \cos \alpha x 
\end{bmatrix} \sin \pi z \tag{22}
\]

where \((W, \Theta, \Phi, \Omega)\) are the amplitudes of \((w, T_f, T_s, S)\), \(\omega = \omega_r + i\omega_i\) is a growth rate and \(\alpha\) is a horizontal wave number.

Now, substitute Eq. (22) in the linearized form of Eqs. (17)–(20) to obtain the following system

\[
\begin{bmatrix}
A & \alpha(1 + \lambda\sigma) Ra_T & 0 & -\alpha(1 + \lambda\sigma) Ra_s \\
2\alpha F & \sigma + \delta^2 - R_i + H & -H & 0 \\
0 & -\gamma H & \tau\sigma + \delta^2 + \gamma H & 0 \\
-1 & 0 & 0 & \sigma + \frac{\delta^2}{Le} 
\end{bmatrix} \begin{bmatrix}
\Psi \\
\Theta \\
\Phi \\
\Omega 
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 
\end{bmatrix}
\]

where \(\delta^2 = \pi^2 + \alpha^2, A = \left(1 + \lambda\sigma\right) \frac{\sigma}{V a} + 1 \right) \delta^2\) and \(F = \int_0^1 f(z) \sin^2(\pi z) dz\). For the nontrivial solution of the above system, the determinant of coefficient matrix must vanish.

Now, the occurrence of convection depends on the value of sigma. For the onset of stationary convection, \(\sigma\) must be zero, which means that the real and imaginary part of \(\sigma\) are equal to zero. So, the stationary Rayleigh number equal to

\[
Ra_T^{st} = -\frac{(4\pi^2 - R_i)(\delta^4 + Le Ra_s \alpha)(-H\delta^2 \gamma - \delta^4 + HR_i \gamma - \delta^2 + R_i \delta^2)}{4(\alpha^2 \pi^2 \delta^2 (H\gamma + \delta^2))}. \tag{23}
\]

4. Results and Discussion
In this section, the results given in Eq. (23) have been illustrated graphically. The figures include the neutral stability curves. The variation of the thermal Rayleigh number \(Ra_T\) with respect to wave number \(\alpha\) for stationary convection is depicted. The results are simulated for the following fixed values of \(Le = 2.5, Ra_s = 100, \gamma = 0.5\), with variation of \(H\) and \(R_i\) in each case, as shown in the following figures. The criteria of stability which is based on the linear stability theory is expressed in terms of the critical thermal Rayleigh number \(Ra_T^{st}\), below which the system is stable, while above which it is unstable.

The effect of interphase heat transfer parameter \(H\) is shown in Fig. 1. The simulated curves are studied with different values of \(H\) with \(\{1, 10, 50, 100\}\) and fix the other parameters with \(Le = 2.5, \gamma = 0.5, R_i = 2\). It is observed that increasing the values of \(H\) increases the critical values of \(Ra_T^{st}\), hence stabilize the system. Fig. 2 illustrates the effect of internal heat source \(R_i\) on the system stability. The curves are simulated for different values of \(R_i\) as shown in the figure and fix the other parameters with values \(Le = 2.5, \gamma = 0.5, H = 10\). The figure shows that increasing the values of \(R_i\) decreases the values of thermal Rayleigh number \(Ra_T^{st}\), hence destabilize the system.
5. Conclusion
Double-diffusive natural convection in a Maxwell fluid saturated porous layer with internal heat under local thermal non-equilibrium model (LTNM) is studied analytically. Linear stability theory was used to derive the critical Rayleigh number and wave number for stationary mode. The effect of the inter-phase heat transfer coefficient $H$, and internal heat parameter $R_i$ on the stationary convection have been investigated and were given graphically. The effect of increasing the values of inter-phase heat transfer parameter $H$ is to stabilize the system. On the other hand, a destabilizing effect was observed when the internal heat parameter is increased.

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