Nonlinear topological Toda quasicrystal

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Topological edge states are known to emerge in certain quasicrystals. We investigate a topological quasicrystal in the presence of nonlinearity by generalizing the Toda lattice to include modulated periodic hoppings, where the period is taken irrational to the original lattice. It is found that topological edge states in a quasicrystal survive against nonlinearity based on the quench dynamics. It is also found that an extended-localization transition is induced by the quasicrystal hopping modulation. The present model is experimentally realizable by a transmission line with variable capacitance diodes, where the inductance is modulated.

I. INTRODUCTION

Quasicrystal is an ordered crystal which has no periodicity. Quasicrystals can be topological [1–9]. One of the simplest models is the Aubry-André-Harper (AAH) model [10–11], which is a one-dimensional (1D) lattice model with a periodic on-site potential whose period is irrational to the original lattice. A 1D quasicrystal is understood in terms of a corresponding 2D ancestor model. For example, the AAH model has the corresponding Harper model [10] as an ancestor, which is a Chern insulator. In a similar way, the off-diagonal AAH model [1–3, 4] has a corresponding Chern insulator as an ancestor. They are characterized by the Hofstadter diagram and the emergence of chiral edge states. Localization occurs in these strongly modulated quasicrystals.

Nonlinear topological physics is an emerging field of topological physics. It is realized in photonic [12–21], mechanical [24–25] and electric circuit [26–28] systems. Although the Toda lattice [29–31] is a nontopological nonlinear system, there is a rich variety of generalizations by introducing inhomogeneous hoppings. For instance, a topological Toda lattice can be constructed [28] by dimerizing the Toda lattice as in the case of the Su-Schrieffer-Heeger model.

Quench dynamics starting from a localized state at an edge site presents a strong signal to detect whether it is topological or trivial even in nonlinear systems. The state almost remains at the initial edge site in the topological phase because of the existence of the localized topological edge states. On the other hand, the state rapidly spreads into the bulk in the trivial phase. This method has proved to be useful in photonic [19–21], mechanical [24–25] and electric circuit [28] systems.

In this paper, we study an effect of nonlinearity in a topological quasicrystal. It introduces a hopping modulation to the Toda lattice, which makes it a nonlinear quasicrystal when the period is irrational to the original lattice. In the linear limit, this model is reduced to the off-diagonal AAH model, which has topological edge states. We study the quench dynamics starting from an edge or bulk site. We find that there are distinct behaviors between them. There remains a finite oscillation at the initial site for the quench dynamics starting from the edge site. However, right and left going propagating waves are dominant for the quench dynamics starting from the bulk site. It evidences the emergence of the topological edge states even in nonlinear systems. The present model is experimentally realizable in an electric circuit with the use of the variable capacitance diodes as in the case of the original Toda model [32–38].

II. GENERALIZED TODA LATTICE

We propose the generalized Toda model in the form of

\[ \frac{1}{\xi} \frac{d^2}{d\tau^2} \log \left( 1 + \xi \psi_n \right) = \left[ M_{nm} - \sum_n M_{nm} \right] \psi_m, \]

(1)

where \( M_{nm} \) represents a hopping from the lattice site \( n \) to \( n \). Nonlinearity is controlled by the parameter \( \xi \), where the linear limit is given by \( \xi \to 0 \). When we choose the hopping matrix as

\[ M = \kappa \sum_n (|n\rangle \langle n+1| + |n+1\rangle \langle n|), \]

(2)

the original Toda model is recovered. All quantities are assumed to be dimensionless.

We investigate the quench dynamics by imposing the initial condition,

\[ \psi_n(0) = \delta_{n,n_0}, \quad \dot{\psi}_n(0) = 0. \]

(3)

Namely, we explore the time evolution of an input starting from the initial site \( n_0 \) at the initial time \( \tau = 0 \).

III. OFF-DIAGONAL AUBRY-ANDRÉ-HARPER MODEL

In the present work, we study the generalized Toda model [1], where the hopping matrix is given by the off-diagonal AAH model [1–4, 10–11],

\[ M_\phi = \sum_n J_n^\phi (|n\rangle \langle n+1| + |n+1\rangle \langle n|), \]

(4)

with

\[ J_n^\phi = \kappa + J \cos (2\pi \alpha n + \phi), \]

(5)

where the hopping amplitude \( J_n^\phi \) is modulated with the period \( J/\alpha \) with \( 0 < \alpha < 1 \), \( 0 \leq J/\kappa \leq 1 \) and \( \phi \) is the phase of the modulation. It is reduced to the original Toda model [29–31] when \( J = 0 \), and to the dimerized Toda model [28] when
\(\alpha = 1/2\). The original Toda system is not topological, while the dimerized Toda system is topological \[28\].

The off-diagonal AAH model is equivalent to the 2D ancestor model given by \[2\]

\[
H^{2D} = \sum_{n,m} (\kappa |n, m\rangle \langle n + 1, m| \\
+ \frac{J}{2} e^{-2\pi i \alpha n} |n, m\rangle \langle n + 1, m + 1| \\
+ \frac{J}{2} e^{2\pi i \alpha n} |n, m\rangle \langle n + 1, m - 1|) + \text{c.c.},
\]

where the modulation phase \(\phi\) has been traded with an extra synthetic dimension. We can confirm the equivalence by introducing the Fourier transformation

\[
|m\rangle = \sum_{\phi} e^{-im\phi} |\phi\rangle
\]

into Eq. (6), and rewrite it as

\[
H^{2D} = \sum_{\phi} M_{\phi} |\phi\rangle \langle \phi|,
\]

where \(M_{\phi}\) is given by Eq. (4). The model (6) describes a Chern insulator \[2\] for any \(\alpha\), and hence, the off-diagonal AAH model (6) also describes a topological insulator. Note that the edge states in the off-diagonal AAH model are mapped to the chiral edge states in the 2D ancestor model.

We confirm this correspondence numerically by considering a finite chain with length \(L\) in the rest of this section.

The eigenspectrum of the matrix \(M\) in Eq. (4) is calculated and shown in Fig. 1(a) as a function of rational number \(\alpha\). The result is a well-known Hofstadter diagram, consisting of edge states marked in red and green in addition to the bulk states marked in lime green. The Hofstadter diagram is peculiar to a Chern insulator in two dimensions.

The system describes a quasicrystal when \(\alpha\) is taken to be an irrational number. For definiteness, we explicitly take \(\alpha\) to be the inverse golden ratio \((\sqrt{5} - 1)/2\). The energy spectrum is shown as a function of \(\phi\) in Fig. 1(b). We find left-localized and right-localized edge states as a function of \(\phi\), as depicted by red and green curves connecting two bulk bands. Note that these edge states are present for all \(\phi\). This energy spectrum is also peculiar to a Chern insulator with the left-moving and right-moving chiral edge states in two dimensions.

The linear limit (\(\xi = 0\)) of Eq. (1) with (4) is the dynamical AAH model, where the above topological property holds as it is. In what follows, we explore how these topological edge states survive as the nonlinearity \(\xi\) increases.

**IV. QUENCH DYNAMICS**

We first investigate the quench dynamics in the original Toda lattice, where \(J = 0\). The time evolution of an input starting from the bulk site \(n_0 = L/2\) is shown in Fig. 2(a1)-(c1). There are only propagating waves and no localized modes. In weak nonlinearity (\(\xi = 0.1\)), there are ripples between the two wave fronts as in Fig. 2(a1). In strong nonlinearity (\(\xi = 10\)), on the other hand, there are no ripples because solitary waves are formed as in Fig. 2(c1). We also show the time evolution at the initial site in Fig. 2(a2)-(c2) and Fig. 2(d1)-(f3).

The time evolution starting from the edge site \(n_0 = 1\) is shown in Fig. 2(d1)-(f1). The time evolution at the initial site is similar between the quench dynamics starting from the bulk and edge sites as shown in Fig. 2(a2)-(f2), which indicates that there are no edge states with \(J = 0\) in consistent with the fact that the system is not topological.

Next, we study the quench dynamics starting from the bulk site in the case of \(J \neq 0\), where the system is topological. There appear right and left going propagating waves in addition to localized modes in the vicinity of the initial site, as shown in Fig. 2(a3)-(c3). The localized mode emerges due to the effect of the quasicrystal hopping modulations. In addition, by comparing Fig. 2(a1)-(c1) and Fig. 2(a3)-(c3), the velocity of the wave propagation becomes smaller in the presence of the quasicrystal modulation. It may be also due to the interference of the hopping modulation.

The time evolution at the edge is shown in Fig. 2(d3)-(f3). The quench dynamics starting from the edge site is quite different from that starting from the bulk site. In contrast to...
FIG. 2. Time evolution of $\psi_n$ starting from the edge site $n_0 = 1$ for the left panel, and from the bulk site $n_0 = L/2$ for the right panel. (a1)∼(f1), (a3)∼(f3) and (a5)∼(f5) Spatial profile of the time evolution of $\psi_n$. (a2)∼(f2), (a4)∼(f4) and (a6)∼(f6) The time evolution of $\psi_{n_0}$ at the initial site. We have taken a finite chain with length $L = 100$. We have set $\phi = 0$ and $T = 30$. (a1)∼(f2) $J = 0$, (a3)∼(f4) $J = 0.75\kappa$ and (a5)∼(f6) $J = \kappa$. There is no localization phenomenon nor edge enhancement in the original Toda model ($J = 0$). The localization phenomenon occurs due to the quasicrystal hopping modulation for $J \neq 0$ but it is small for small $J$. It reaches the maximum at $J = \kappa$. An extended-localization transition point is at $J = 0.76\kappa$, about which we refer to Fig.3(d). The edge enhancement occurs for all $J \neq 0$ due to the emergence of topological edge states.

the quench dynamics starting from the bulk site shown in Fig.2(a3)∼(c3), the intensity of the propagating wave is weak but almost all states are localized at the edge as shown in Fig.2(d3)∼(f3). The amplitude of oscillation is stationary after a certain time as in Fig.2(d4)∼(f4), which is larger than that of the bulk shown in Fig.2(a4)∼(c4). These phenomena indicate the emergence of the topological edge states even in a nonlinear quasicrystal.

Then, we study the quench dynamics in the case of $J = \kappa$. The state is almost localized at the initial site for the quench dynamics starting from both the edge and bulk sites, as shown in Fig.2(a5)∼(f5). Especially, amplitude of the oscillation is quite large as shown in Fig.2(d6)∼(f6). It indicates that the strong localization occurs due to the strong quasicrystal hopping modulation.

We proceed to study systematically the effects due the quasicrystal hopping modulation and the nonlinearity by introducing an index

$$\Psi \equiv \max_{T/2 < \tau < T} (|\psi_{n_0}|),$$

which is the maximum value of $|\psi_{n_0}|$ in the time span $T/2 < \tau < T$, where $T$ is large enough so that the oscillation is stationary.

The index $\Psi$ is shown as a function of the periodic-modulation magnitude $J$ for various nonlinearity $\xi$ in Fig.3(a) and (b), where the quench dynamics starting from the edge and bulk sites are studied. There are some distinguishable features between them. First, $\Psi$ is larger for the quench dynamics starting from the edge site. It indicates the existence of the topological edge states in the nonlinear quasicrystal. Next, $\Psi$ increases as a function of $J$ in both the cases, which indicates that the localization is enhanced in the presence of the quasiperiodic potential.
FIG. 3. Index $\Psi$ as a function of $J$ for various $\xi$ at (a) the edge site $n_0 = 1$ and (b) bulk site $n_0 = L/2$. Black curve indicates $\xi = 0$, green curve indicates $\xi = 0.1$, magenta curve indicates $\xi = 1$ and cyan curve indicates $\xi = 10$. We have set $T = 50$ and $L = 100$. The horizontal axis is $J$ and the vertical axis is $\Psi$. (c) The IPR as a function of $J$ for $P$, and (d) that for $P_b$.

FIG. 4. Index $\Psi$ as a function of $\xi$ for various $J$ at (a) the edge site $n_0 = 1$ and (b) bulk site $n_0 = L/2$. Black curve indicates $J = 0$, orange curve indicates $J = 0.25\kappa$, green curve indicates $J = 0.5\kappa$, magenta curve indicates $J = 0.75\kappa$ and cyan curve indicates $J = \kappa$. We have set $T = 50$ and $L = 100$. The horizontal axis is $\xi$ and the vertical axis is $\Psi$.

In order to understand the localization phenomenon in the quasicrystal, we calculate the inverse-participation ratio (IPR) defined by

$$P \equiv \sum_{n=1}^{L} \left| \tilde{\psi}_n \right|^4,$$

where $\tilde{\psi}_n$ is the normalized eigenstate of $M_n$. The IPR satisfies $0 \leq P \leq 1$. The IPR measures the magnitude of the localization, where the system is localized when $P$ is close to 1, while it is extended when $P$ is close to 0. We show the IPR as a function of $J$ in Fig. 4(c). It linearly increases as a function of $J$, which is consistent with Fig. 4(a).

The IPR defined by Eq. (10) contains a large contribution from the edge, which is irrelevant to the localization phenomenon. Hence, we define the IPR for the bulk by

$$P_{b} \equiv \sum_{n=2}^{L-1} \left| \psi_n \right|^4,$$

which is shown in Fig 3(d). It exhibits a clear transition around $J \approx 0.76\kappa$. It is an extended-localization transition induced by the quasicrystal hopping modulation $J_n^\phi$. The quench dynamics at $J = 0.75\kappa$ in Fig. 3(a)~(f) is in the one near this transition point.

Finally, in order to study the effect of the nonlinearity on the localization due to the quasicrystal hopping modulation $J_n^\phi$, we show $\Psi$ as a function of $\xi$ for various $J$ in Fig. 4. $\Psi$ remains almost unchanged even when nonlinearity increases, which indicates that the topological edge states survive for strong nonlinearity.

V. ELECTRIC-CIRCUIT REALIZATION

The original Toda lattice is realized by a transmission line with the use of variable-capacitance diodes and inductors\[28, 39\]. Here we show how to realize the generalized Toda lattice with the use of variable-capacitance diodes and inductors.

We consider a transmission line as shown in Fig. 5. The Kirchhoff law is given by

$$L_n \frac{dJ_n}{dt} = v_n - v_{n+1},$$

$$\frac{dQ_n}{dt} = J_{n-1} - J_n,$$

where $v_n$ is the voltage, $J_n$ is the current and $Q_n$ is the charge at the node $n$, while $L_n$ is the inductance for the inductor between the nodes $n$ and $n+1$, as illustrated. The Kirchhoff law is summarized in the form of the second-order differential equation\[28, 39]\,

$$\frac{d^2Q_n}{dt^2} = \frac{dJ_{n-1}}{dt} - \frac{dJ_n}{dt} = \frac{1}{L_{n-1}} (V_{n-1} - V_n) - \frac{1}{L_n} (V_n - V_{n+1}),$$

where we have introduced a new variable $V_n$ by $v_n = V_0 + V_n$.

The capacitance is a function of the voltage $V_n$ in the variable-capacitance diode, and it is well given by\[33]\,

$$C_n(V_n) = \frac{Q_n(V_0)}{F_0 + V_n - V_0},$$
where $F_0$ is a constant characteristic to the variable-capacitance diode. Especially, we have

\[ C(V_0) = \frac{Q(V_0)}{F_0}. \] (16)

The charge is given by

\[ Q_n = \int_0^{V_n} C(V) \, dV = Q(V_0) \log \left[ 1 + \frac{V_n}{F_0} \right] + \text{const.} \] (17)

It follows from Eqs. (14) and (17) that

\[ Q(V_0) \frac{d^2}{dt^2} \log \left[ 1 + \frac{V_n}{F_0} \right] = \frac{V_{n+1}}{L_n} - \left( \frac{1}{L_{n-1}} + \frac{1}{L_n} \right) V_n + \frac{V_{n-1}}{L_{n-1}}. \] (18)

This is a closed form of the differential equation for $V_n$.

Eq. (18) has the same form as the generalized Toda equation (1) with

\[ M_{nm} = \sum_n \kappa_n \langle n | (n + 1) + (n + 1) | n \rangle, \] (19)

when we set

\[ \tau = t/\sqrt{L_0 Q(V_0)} V_1^2/F_0, \quad \psi_n = V_n/V_1, \]

\[ \xi = V_1/f_0, \quad \kappa_n = L_0/L_n. \] (20)

where $1/L_0$ is the mean of $1/L_n$. For instance, by choosing the inductance as

\[ 1/L_n = \kappa + J \cos (2\pi n + \phi) \] (21)

Eq. (5) is reproduced.

Finally, the initial condition (3) is rewritten as

\[ V_n = V_{1\delta n} \text{ and } \dot{V}_n = 0 \text{ at } \tau = 0. \] (22)

Hence, the nonlinearity $\xi$ is controlled only by changing the initial voltage $V_1$ without changing samples in experiments. The quench dynamics is experimentally observed by measuring the time evolution of the voltage.

VI. CONCLUSION

We have investigated topological properties of a nonlinear quasicrystal by generalizing Toda lattice. The existence of the topological edge states is well signaled by comparing the quench dynamics starting from an edge site and a bulk site. Stationary oscillation occurs at the edge even in nonlinear regime, which indicates that the system is a nonlinear topological quasicrystal. The present model is experimentally realizable based on electric circuits with variable-capacitance diodes and inductors. It is interesting to study further interplay among, topology, quasicrystal and nonlinearity.

The author is very much grateful to M. Kawamura, S. Kasumoto and N. Nagaosa for helpful discussions on the subject. This work is supported by the Grants-in-Aid for Scientific Research from MEXT KAKENHI (Grants No. JP17K05490 and No. JP18H03676). This work is also supported by CREST, JST (JPMJCR16F1 and JPMJCR20T2).

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