WEYL FORMULA FOR THE NEGATIVE DISSIPATIVE EIGENVALUES OF MAXWELL’S EQUATIONS

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ABSTRACT. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup generated by Maxwell’s equations in an exterior domain $\Omega \subset \mathbb{R}^3$ with dissipative boundary condition 
$E_{\tan} - \gamma(x)(\nu \wedge B_{\tan}) = 0$, $\gamma(x) > 0, \forall x \in \Gamma = \partial \Omega$. We study the case when $\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$ and $\gamma \neq 1$ is a constant. We establish a Weyl formula for the counting function of the negative real eigenvalues of $G_b$.

1. INTRODUCTION

Let $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ be an open connected domain and let $\Omega = \mathbb{R}^3 \setminus \overline{K}$ be connected domain with $C^\infty$ smooth boundary $\Gamma$. Consider the boundary problem

\begin{equation}
\partial_t E = \nabla \times B, \quad \partial_t B = -\nabla \times E \quad \text{in} \quad \mathbb{R}_+^3 \times \Omega,
\end{equation}

\begin{align*}
E_{\tan} - \gamma(x)(\nu \wedge B_{\tan}) &= 0 \quad \text{on} \quad \mathbb{R}_+^3 \times \Gamma, \quad (1.1) \\
E(0,x) &= E_0(x), \quad B(0,x) = B_0(x).
\end{align*}

with initial data $f = (E_0, B_0) \in (L^2(\Omega))^6 = \mathcal{H}$. Here $\nu(x)$ is the unit outward normal to $\partial \Omega$ at $x \in \Gamma$ pointing into $\Omega$, $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{C}^3$, $u_{\tan} := u - (u, \nu) \nu$, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. The solution of the problem (1.1) is described by a contraction semigroup

\[ (E, B) = V(t)f = e^{tG_b}f, \quad t \geq 0, \]

where the generator $G_b$ has domain $D(G_b)$ which is the closure in the graph norm of functions $u = (v, w) \in (C^\infty(0; \mathbb{R}^3))^3 \times (C^\infty(0; \mathbb{R}^3))^3$ satisfying the boundary condition $u_{\tan} - \gamma(\nu \wedge w_{\tan}) = 0$ on $\Gamma$.

In [1] it was proved that the spectrum of $G_b$ in the open half plan $\{z \in \mathbb{C} : \text{Re } z < 0\}$ is formed by isolated eigenvalues with finite multiplicities. Note that if $G_b f = \lambda f$ with $\text{Re } \lambda < 0$, the solution $u(t,x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they are very important for the inverse scattering problems (see [2]). In particular, the eigenvalues $\lambda$ with $\text{Re } \lambda \to -\infty$ imply a very fast decay of the corresponding solutions. In [2] the existence of eigenvalues of $G_b$ has been studied for the ball $B_3 = \{x \in \mathbb{R}^3, |x| < 1\}$ assuming $\gamma$ constant. It was proved for $\gamma = 1$ there are no eigenvalues in $\{z \in \mathbb{C} : \text{Re } z < 0\}$, while for $\gamma \neq 1$ there is always an infinite number of real eigenvalues $\lambda_m$ and with exception of one they satisfy the estimate

\[ \lambda_m \leq -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}} = -c_0, \quad (1.2) \]

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where \( \gamma_0 = \max\{\gamma, \frac{1}{\gamma}\} \).

In this Note we study the distribution of the negative eigenvalues and our purpose is to obtain a Weyl formula for the counting function
\[
N(r) = \#\{\lambda \in \sigma_p(G_b) \cap \mathbb{R}^- : |\lambda| \leq r\}, \quad r > r_0(\gamma),
\]
where every eigenvalues \( \lambda_m \) is counted with its algebraic multiplicity given by
\[
\text{mult}(\lambda_m) = \frac{\text{rank} 1}{2\pi i} \int_{|\lambda_n - z| = \epsilon} (z - G_b)^{-1} dz,
\]
where \( 0 < \epsilon \ll 1 \). Our main result is the following

**Theorem 1.1.** Let \( \gamma \neq 1 \) be a constant and let \( \gamma_0 = \max\{\gamma, \frac{1}{\gamma}\} \). Then the counting function \( N(r) \) for the ball \( B_3 \) has the asymptotic
\[
N(r) = (\gamma_0^2 - 1)r^2 + O_\gamma(r), \quad r \geq r_0(\gamma) > c_0. \tag{1.3}
\]

The proof of Theorem 1.1 is based on a precise analysis of the roots of the equation (3.1) involving spherical Hankel functions \( h_n^{(1)}(\lambda) \) of first kind. We show in Section 3 that for \( \gamma > 1 \) this equation has only one real root \( \lambda_n < 0 \). Moreover, we have \( \lambda_{n+1} < \lambda_n, \forall n \in \mathbb{N} \), so we have a decreasing sequence of eigenvalues. The geometric multiplicity of \( \lambda_n \) is \( 2n + 1 \). Since \( G_b \) is not a self-adjoint operator the geometric multiplicity could be less than the algebraic one. In our case these multiplicities coincide and the proof is based on a representation of \( (G_b - z)^{-1} \).

To estimate \( \lambda_n \) as \( n \to \infty \), we apply an approximation of the exterior semiclassical Dirichlet to Neumann map for the operator \( (h^2\Delta + z) \) established in [6] (see also [8]) combined with an application of Rouché theorem.

We conjecture that in the general case of strictly convex obstacles and \( \min_{y \in \Gamma} \gamma(y) = \gamma_1 > 1 \) we have the asymptotic
\[
N(r) = \frac{1}{4\pi} \left( \int_{\Gamma} (\gamma^2(y) - 1) dS_y \right) r^2 + O_\gamma(r), \quad r \geq r_0(\gamma_0).
\]

For the ball \( B_3 \) this agrees with (1.3).

2. **Boundary problem for Maxwell system**

Our purpose is to study the eigenvalues of \( G_b \) in case the obstacle is equal to the ball \( B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\} \). Setting \( \lambda = i\mu, \text{Im} \mu > 0 \), an eigenfunction \((E, B) \neq 0\) of \( G_b \) satisfies
\[
\text{curl} E = -i\mu B, \quad \text{curl} B = i\mu E. \tag{2.1}
\]
Replacing \( B \) by \( H = -B \) yields for \((E, H) \in (H^2(|x| \leq 1))^6\),
\[
\begin{aligned}
\text{curl} E &= i\mu H, \quad \text{curl} H = -i\mu E, \quad \text{for} \quad x \in B_3, \\
E_{\text{tan}} + \gamma(\nu \land H_{\text{tan}}) &= 0, \quad \text{for} \quad x \in \mathbb{S}^2.
\end{aligned} \tag{2.2}
\]
Expand \( E(x), H(x) \) in the spherical functions \( Y_n^m(\omega), n = 0, 1, 2, ..., |m| \leq n, \omega \in \mathbb{S}^2 \) and the spherical Hankel functions of first kind
\[
h_n^{(1)}(z) := \frac{H_n^{(1)}(z)}{\sqrt{z}}, n \geq 1
\]
An application of Theorem 2.50 in [3] (in the notation of [3] it is necessary to replace \( \omega \) by \( \mu \in \overline{\mathbb{C}} \setminus \{0\} \)) says that the solution of the system (2.2) for \( x = |x| \omega, r = |x| > 0, \omega = \frac{x}{|x|} \) has the form

\[
E(x) = \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \alpha_n^m \sqrt{n(n+1)} \frac{h_n^{(1)}(\mu r)}{r} Y_n^m(\omega) \right. \\
+ \left. \frac{\alpha_n^m}{r} (\mu r) h_n^{(1)}(\mu r) U_n^m(\omega) \right], \tag{2.3}
\]

\[
H(x) = \frac{1}{i\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \beta_n^m \sqrt{n(n+1)} \frac{h_n^{(1)}(\mu r)}{r} Y_n^m(\omega) \right. \\
+ \left. \frac{\beta_n^m}{r} (\mu r) h_n^{(1)}(\mu r) U_n^m(\omega) \right]. \tag{2.4}
\]

Here \( U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \text{grad} \kappa Y_n^m(\omega) \) and \( V_n^m(\omega) = \kappa U_n^m(\omega) \) for \( n \in \mathbb{N}, -n \leq m \leq n \) form a complete orthonormal basis in \( L^2(S^2) = \{ u(\omega) \in (L^2(S^2))^3 : \langle \omega, u(\omega) \rangle = 0 \text{ on } S^2 \} \).

To find a representation of \( \kappa \cdot H_{tan} \), observe that \( \kappa \cdot (\kappa \cdot U_n^m) = -U_n^m \), so for \( r = 1 \) one has

\[
(\kappa \cdot H_{tan})(\omega) = -\frac{1}{i\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \beta_n^m \left( h_n^{(1)}(\mu) + \frac{d}{d \mu} h_n^{(1)}(\mu r) \right) \right] \bar{V}_n^m(\omega)
\]

and the boundary condition in (2.2) is satisfied if

\[
\alpha_n^m \left[ \frac{d}{d \mu} h_n^{(1)}(\mu) \right] = 0, \forall n \in \mathbb{N}, |m| \leq n, \tag{2.5}
\]

\[
-\frac{\beta_n^m}{i\mu} \left[ \frac{d}{d \mu} h_n^{(1)}(\mu) \right] = 0, \forall n \in \mathbb{N}, |m| \leq n. \tag{2.6}
\]

3. ROOTS OF THE EQUATION \( g_n(\lambda) = 0 \)

To examine the eigenvalues of \( G_b \) it is necessary to find the roots of the equations (2.3) and (2.4). Since \( h_n^{(1)}(\mu) \neq 0 \) for \( \text{Im} \mu > 0 \), the problem is reduced to study the roots \( \lambda \in \mathbb{R}^+ \) of the equation

\[
1 + \frac{d}{d \mu} h_n^{(1)}(-i\lambda) \bigg|_{\mu=1} (h_n^{(1)}(-i\lambda))^{-1} - \lambda \gamma = 0 \tag{3.1}
\]

and the same equation with \( \gamma \) replaced by \( \frac{1}{\gamma} \). Clearly, if \( \mu = -i\lambda \) is such that the expressions in the brackets \([\ldots]\) in (2.3) and (2.4) are non-vanishing for every \( n \geq 1 \), we must have \( \alpha_n^m = \beta_n^m = 0 \) which implies \( E_{tan} = B_{tan} = 0 \). Hence \( (E, B) = 0 \) because the boundary problem with \( \gamma = 0 \) has no eigenvalues in \( \{ z \in \mathbb{C} : \text{Re} z < 0 \} \). In this section we suppose that \( \gamma \neq 1 \) and examine the equation

\[
g_n(\lambda) := \frac{1}{\lambda} + \frac{d}{d \lambda} \left( h_n^{(1)}(-i\lambda) \right) (h_n^{(1)}(-i\lambda))^{-1} - \gamma = 0. \tag{3.2}
\]
It is well known that (see [5])
\[ h_n^{(1)}(-i\lambda) = (-i)^{n+1} \frac{e^{\lambda}}{-i\lambda} R_n \left( \frac{i}{-2i\lambda} \right) = (-i)^n \frac{e^{\lambda}}{\lambda} R_n \left( \frac{-1}{2\lambda} \right) \]
with
\[ R_n(z) := \sum_{m=0}^{n} a_{m,n} z^m, \quad a_{m,n} = \frac{(n+m)!}{m!(n-m)!} > 0. \]
We will prove the following

**Proposition 3.1.** For \( \lambda < 0 \) we have
\[ G_{n,n+1}(\lambda) = \frac{d}{d\lambda} h_{n+1}^{(1)}(-i\lambda) - \frac{d}{d\lambda} h_n^{(1)}(-i\lambda) > 0. \]  \hspace{1cm} (3.3)

**Proof.** The purpose is to show that
\[ \left( h_n^{(1)}(-i\lambda) \frac{d}{d\lambda} h_{n+1}^{(1)}(-i\lambda) - h_n^{(1)}(-i\lambda) \frac{d}{d\lambda} h_{n+1}^{(1)}(-i\lambda) \right) \left( h_n^{(1)}(-i\lambda) h_{n+1}^{(1)}(-i\lambda) \right)^{-1} > 0. \]
Introduce the functions
\[ \xi_n(\lambda) := \frac{e^{\lambda}}{\lambda} R_n \left( \frac{-1}{2\lambda} \right), \quad \eta_n(\lambda) := \lambda \xi_n(\lambda). \]
Then \( h_n^{(1)}(-i\lambda) = (-i)^n \xi_n(\lambda) \) and the above inequality is equivalent to
\[ \left( \xi_n(\lambda) \frac{d}{d\lambda} \xi_{n+1}(\lambda) - \xi_{n+1}(\lambda) \frac{d}{d\lambda} \xi_n(\lambda) \right) \left( \xi_{n+1}(\lambda) \xi_n(\lambda) \right)^{-1} = \left( \eta_n(\lambda) \frac{d}{d\lambda} \eta_{n+1}(\lambda) - \eta_{n+1}(\lambda) \frac{d}{d\lambda} \eta_n(\lambda) \right) \left( \eta_{n+1}(\lambda) \eta_n(\lambda) \right)^{-1} > 0. \]
Since \( \eta_n(\lambda) \eta_{n+1}(\lambda) > 0 \) for \( \lambda < 0 \), it suffices to show that the function
\[ F(\lambda) = \eta_n(\lambda) \frac{d}{d\lambda} \eta_{n+1}(\lambda) - \eta_{n+1}(\lambda) \frac{d}{d\lambda} \eta_n(\lambda) \]
has positive values for \( \lambda \in (-\infty, 0) \). Consider the derivative
\[ F'(\lambda) = \eta_n(\lambda) \frac{d^2}{d\lambda^2} \eta_{n+1}(\lambda) - \eta_{n+1}(\lambda) \frac{d^2}{d\lambda^2} \eta_n(\lambda). \]
We have
\[ \eta_n(\lambda) = i^{n+1} h_n^{(1)}(-i\lambda)(-i\lambda) = i^{n+1} \Xi_n(-i\lambda) = -i^{n-1} \Xi_n(-i\lambda). \]
The function \( \Xi_n(z) = z h_n^{(1)}(z) \) satisfies the equation
\[ \Xi''_n(z) + \left( 1 - \frac{n^2 + n}{z^2} \right) \Xi_n(z) = 0 \]
and
\[ \frac{d^2}{d\lambda^2} \eta_n(\lambda) = i^{n-1} \Xi''_n(-i\lambda) = -i^{n-1} \left( 1 + \frac{n^2 + n}{\lambda^2} \right) \Xi_n(-i\lambda) = \left( 1 + \frac{n^2 + n}{\lambda^2} \right) \eta_n(\lambda). \]
Consequently,
\[ F'(\lambda) = \left[ \frac{(n+1)^2 + n + 1}{\lambda^2} - \frac{n^2 + n}{\lambda^2} \right] \eta_n(\lambda) \eta_{n+1}(\lambda) \]
\[ = 2(n+2) \frac{\eta_n(\lambda) \eta_{n+1}(\lambda)}{\lambda^2} > 0. \]
On the other hand,

\[ F(\lambda) = e^{\lambda}R_n \left( -\frac{1}{2\lambda} \right) \frac{d}{d\lambda} \left( e^{\lambda}R_{n+1} \left( -\frac{1}{2\lambda} \right) \right) - e^{\lambda}R_{n+1} \left( -\frac{1}{2\lambda} \right) \frac{d}{d\lambda} \left( e^{\lambda}R_n \left( -\frac{1}{2\lambda} \right) \right) \]

\[ = \frac{e^{2\lambda}}{2\lambda^2} \left[ R_n \left( -\frac{1}{2\lambda} \right) R'_{n+1} \left( -\frac{1}{2\lambda} \right) - R_{n+1} \left( -\frac{1}{2\lambda} \right) R_n \left( -\frac{1}{2\lambda} \right) \right] \]

and

\[ \lim_{\lambda \to -\infty} F(\lambda) = 0, \quad \lim_{\lambda \to 0} F(\lambda) = +\infty \]

since

\[ \lim_{w \to +\infty} \left[ R_n(w)R'_{n+1}(w) - R_{n+1}(w)R'_n(w) \right] = +\infty. \]

Finally, the function \( F(\lambda) \) in the interval \((-\infty, 0]\) is increasing from 0 to \(+\infty\) and this completes the proof. \( \square \)

Now if \( \lambda_n < 0 \) is a solution the equation

\[ g_n(\lambda) := \frac{1}{\lambda} + \left( \frac{d}{d\lambda} h_n^{(i)}(-i\lambda) \right) (h_n^{(i)}(-i\lambda))^{-1} - \gamma = 0, \]  

(3.4)

one has

\[ g_{n+1}(\lambda_n) = \frac{1}{\lambda_n} + \left( \frac{d}{d\lambda} h_{n+1}^{(i)}(-i\lambda_n) \right) (h_{n+1}^{(i)}(-i\lambda_n))^{-1} - \gamma = G_{n,n+1}(\lambda_n) > 0, \]

so \( \lambda_n \) is not a root of the equation

\[ g_{n+1}(\lambda) = \frac{1}{\lambda} + \left( \frac{d}{d\lambda} h_{n+1}^{(i)}(-i\lambda) \right) (h_{n+1}^{(i)}(-i\lambda))^{-1} - \gamma = 0. \]

In the following we assume that \( \gamma > 1 \). Then for \( \lambda \to -\infty \) we have \( g_{n+1}(\lambda) \to 1 - \gamma < 0 \), and since \( g_{n+1}(\lambda_n) > 0 \) the equation \( g_{n+1}(\lambda) = 0 \) has at least one root \(-\infty < \lambda_{n+1} < \lambda_n. \)

**Lemma 3.1.** Let \( \gamma > 1 \). For every \( n \geq 1 \) the equation \( g_n(\lambda) = 0 \) in the interval \((-\infty, 0]\) has exactly one root \( \lambda_n < 0. \)

**Proof.** Setting \( w = -\frac{1}{\lambda} \), we write the equation (3.2) as \( R_n(w) := w^2R'_n(w) + \alpha R_n(w) = 0 \), where \( \alpha = \frac{1-\gamma}{2} < 0 \). We will show that this equation has exactly one positive root. Since

\[ w^2R'_n(w) = \sum_{k=1}^{n} k a_{k,n} w^{k+1}, \quad R_n(w) = \sum_{k=0}^{n} a_{k,n} w^k, \]

the polynomial \( R_n(w) \) has the representation

\[ R_n(w) = \sum_{k=0}^{n+1} b_{k,n} w^k \]

with

\[
\begin{cases}
    b_{k,n} = (k-1)a_{k-1,n} + \alpha a_{k,n}, & 0 \leq k \leq n, \quad a_{-1,n} = 0, \\
    b_{n+1,n} = \frac{(2n)!}{(n-1)!}.
\end{cases}
\]

Taking into account the form of \( a_{k,n} \), we deduce

\[ b_{k,n} = \frac{(n+k-1)!}{(n-k+1)!k!} \left( (k-1) + \alpha(n+k)(n-k+1) \right), \quad 0 \leq k \leq n+1. \]  

(3.5)
Thus the sign of $b_{k,n}$ depends on the sign of the function
\[ B(k) := (1 - \alpha)k^2 + (\alpha - 1)k + \alpha(n^2 + n) \]
which for $k \geq 1$ is increasing since
\[ B'(k) = 2(1 - \alpha)k + \alpha - 1 \geq 1 - \alpha > 0. \]
Clearly, $b_{0,n} = \alpha < 0$ and $b_{n+1,n} > 0$. There are two cases:

(i) $b_{1,n} \leq 0$. Then there is only one change of sign in the Descartes’ sequence
\[ \{b_{n+1,n}, b_{n,n}, \ldots, b_{1,n}, b_{0,n}\}. \]

(ii) $b_{1,n} > 0$. Then $b_{k,n} > 0$ for $1 \leq k \leq n + 1$ and in the Descartes’ sequence
\[ \{b_{n+1,n}, b_{n,n}, \ldots, b_{1,n}, b_{0,n}\} \] one has again only one change of sign.

Applying the Descartes’ rule of signs, we conclude that the number of the positive roots of $R_n(w) = 0$ is exactly one.

Combining Proposition 3.1 and Lemma 3.1, one obtain the following

**Corollary 3.1.** Let $\gamma > 1$. Then the generator $G_b$ has an infinite sequence of real eigenvalues
\[ -\infty < \ldots < \lambda_n < \ldots < \lambda_2 < \lambda_1 < 0 \]
and $\lambda_n$ has geometric multiplicity $2n + 1$.

The last statement concerns the geometric multiplicity since the functions $\{Y_{m,n}(\omega)\}_{m=-n}^n$ are linearly independent. The algebraic multiplicity of $\lambda_m$ will be discussed in Section 5.

### 4. Estimation of the roots

Throughout this section we assume $\gamma > 1$. Set $\lambda = \frac{i\sqrt{z}}{h}$, $0 < h \ll 1$ with $z = -1 + i\eta$, $0 \leq |\eta| \leq h^{1/2}$, $\eta \in \mathbb{R}$. Consider the Dirichlet problem

\[
\begin{cases}
(h^2\Delta + z)w = 0, & |x| > 1, w \in H^2(|x| > 1), \\
w = f, & |x| = 1
\end{cases}
\]
and note that $\Delta + \frac{i\eta}{h} = \Delta - \lambda^2$. The solution of (4.1) has the form
\[
w(r\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(-i\lambda r)(h_n^{(1)}(-i\lambda))^{-1} \alpha_{n,m} Y_{n,m}(\omega),
\]
where
\[
f(\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n,m} Y_{n,m}(\omega).
\]

The semiclassical Dirichlet-to-Neumann operator $N_{ext}(h, z) = \frac{h}{i \frac{d}{dr}} w|_{r=1}$ related to (4.1) becomes

\[
N_{ext}(h, z) = -i\sqrt{z} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (h_n^{(1)})'(-i\lambda)(h_n^{(1)}(-i\lambda))^{-1} \alpha_{n,m} Y_{n,m}
\]
\[
= \sqrt{z} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{d}{d\lambda} \left(h_n^{(1)}(-i\lambda)(h_n^{(1)}(-i\lambda))^{-1} \alpha_{n,m} Y_{n,m}\right).
\]
By using the approximation of $N_{ext}(h, z)$ established in [8],[6] for $z = -1 + i\eta$, one deduces
\[
\|N_{ext}(h, z)f - Op_{h}(\rho)f\|_{L^{2}(\mathbb{S}^{2})} \leq C\left(\frac{|z|}{|\lambda|}\right)^{\frac{1}{2}}\|f\|_{L^{2}(\mathbb{S}^{2})}, \ 0 < h \leq h_{0}
\]
with $\rho = \sqrt{z - r_{0}(x', \xi')}$ and a constant $C > 0$ independent of $z, \lambda$ and $f$. Here $r_{0}(x', \xi')$ is the principal symbol of the semiclassical Laplace-Beltrami operator $-h^{2}\Delta_{\mathbb{S}^{2}} = \hat{x}^{2}\Delta_{\mathbb{S}^{2}}$. Moreover, $\sqrt{z} = i\sqrt{1 - h^{2}} = i(1 - \frac{w}{2} + O(h^2))$ and
\[
Re \lambda = -\frac{1}{h} + O(1), \ \operatorname{Im} \lambda = O(h^{-1/2}).
\]
Hence, for $0 < h \leq h_{0}$ we get
\[
\lambda \in \Lambda_{0} = \{ z \in \mathbb{C} : |\operatorname{Im} z| \leq c_{0}h^{1/2} |\operatorname{Re} z|, \ \operatorname{Re} \lambda < -\epsilon < 0, |\lambda| \geq \lambda_{0} \}.
\]
On the other hand,
\[
\left\| Op_{h}(\rho) - \sqrt{z}\left(\sqrt{1 - \frac{\Delta_{\mathbb{S}^{2}}}{\lambda^{2}}}\right)\right\|_{L^{2}(\mathbb{S}^{2}) \rightarrow L^{2}(\mathbb{S}^{2})} \leq C_{1}|\lambda|^{-1}, \ \lambda \in \Lambda_{0}.
\]
Applying the spectral theorem, one deduces
\[
\left(\sqrt{1 - \frac{\Delta_{\mathbb{S}^{2}}}{\lambda^{2}}}f\right) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(1 + \frac{n(n+1)}{\lambda^{2}}\right)\alpha_{n,m}Y_{n,m}
\]
and
\[
\left\|N_{ext}(h, -z) - \sqrt{z}\left(\sqrt{1 - \frac{\Delta_{\mathbb{S}^{2}}}{\lambda^{2}}}f\right)\right\|_{L^{2}(\mathbb{S}^{2})}^{2} = |z| \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|\frac{d}{d\lambda} \left(h_{n}(-i\lambda)\right)(h_{n}(-i\lambda))^{-1}ight|^{2} \left|\frac{n(n+1)}{\lambda^{2}}\right|^{2} |a_{n,m}|^{2}.
\]
This implies
\[
\left|\frac{d}{d\lambda} \left(h_{n}^{\omega}(1)(-i\lambda)\right)(h_{n}^{\omega}(1)(-i\lambda))^{-1} - \sqrt{1 + \frac{n(n+1)}{\lambda^{2}}}\right| \leq C_{2}|\lambda|^{-1}, \ \forall n \in \mathbb{N}, \ \lambda \in \Lambda_{0} \ (4.2)
\]
which we write as
\[
\left|\left[\frac{1}{2} + \frac{d}{d\lambda} \left(h_{n}^{\omega}(1)(-i\lambda)\right)(h_{n}^{\omega}(1)(-i\lambda))^{-1} - \gamma\right] - \left[\sqrt{1 + \frac{n(n+1)}{\lambda^{2}}} - \gamma\right]\right| \leq C_{0}|\lambda|^{-1}. \ (4.3)
\]
Remark 4.1. For bounded $1 \leq n \leq N_{0}$ and sufficiently large $|\lambda|$ the estimate (4.2) follows easily from the fact that $\frac{R_{n}(w)}{R_{n}(w)} = n(n+1) + O(|w|)$ as $|w| \to 0$.

Remark 4.2. The estimate (4.2) is similar to that in Proposition 2.1 in [7], where the function $J_{\nu}(\lambda)$ for $\nu \geq 0$ and $0 < C \leq |\operatorname{Im} \lambda| \leq \delta |\operatorname{Re} \lambda|$, $\operatorname{Re} \lambda > C_{1}$ has been approximated. Here $J_{\nu}(z)$ is the Bessel function, while the boundary problem examined in [7] is in the domain $|x| < 1$.

Put $z = \lambda$ and for $z \in \Lambda_{0}$ consider the function
\[
f_{n}(z) := \sqrt{1 + \frac{n(n+1)}{z^{2}}} - \gamma
\]
with zeros
\[
z_{n}^{\pm} = \pm \sqrt{\frac{n^{2} + n}{\gamma^{2} - 1}}.
\]
In the following we set \( z_n = -\sqrt{\frac{n(n+1)}{\gamma^2-1}} \). Clearly,
\[
f_n'(z) = -\frac{1}{z} \frac{n(n+1)}{z^2 + \frac{n(n+1)}{z^2}}
\]
and \( \frac{n(n+1)}{z_n^2} = \gamma^2 - 1 \), \( f_n'(z_n) = -\frac{1}{z_n} \). A calculus yields the second derivative
\[
f_n''(z) = \frac{1}{z^2} \left[ 3n(n+1) \left(1 + \frac{n(n+1)}{z^2} \right) \right] \]
\[
- \frac{n^2(n+1)^2}{z^4} \left(1 + \frac{n(n+1)}{z^2} \right)^{-1/2} \left(1 + \frac{n(n+1)}{z^2} \right)^{-1}.
\]
For \( n \) large enough and \( a > 0 \) to be fixed below introduce the contour
\[
C_n(a) := \{ z = z_n + ae^{i\varphi}, 0 \leq \varphi < 2\pi \} \subset \Lambda_0.
\]
Our purpose is to choose \( a \) so that
\[
|f_n(z)| \geq \frac{C_0}{z}, \quad \forall z \in C_n(a). \tag{4.4}
\]
We have
\[
z^2 = z_n^2 + 2z_n ae^{i\varphi} + a^2 e^{2i\varphi}
\]
and
\[
\frac{n(n+1)}{z^2} = (\gamma^2 - 1) \left(1 + O\left(\frac{1}{n}\right) a + O\left(\frac{1}{n^2}\right) a^2\right)^{-1}, \quad z \in C_n(a). \tag{4.5}
\]
On the other hand,
\[
\sqrt{\frac{n(n+1)}{z^2}} + 1 = \left[ \frac{\gamma^2 + O\left(\frac{1}{n}\right) a + O\left(\frac{1}{n^2}\right) a^2}{1 + O\left(\frac{1}{n}\right) a + O\left(\frac{1}{n^2}\right) a^2} \right]^{1/2}.
\]
Clearly, one has the estimate
\[
|f_n(z)| \geq \frac{\gamma^2 - 1}{\gamma |z_n|} - \frac{a^2}{2} \sup_{z \in C_n(a)} |f_n''(z)|, \quad z \in C_n(a). \tag{4.6}
\]
Set \( C_\gamma = \frac{\gamma^2 - 1}{\gamma} > 0 \) and choose \( a > 0 \) so that \( C_\gamma a > 4C_0 \). We fix \( a \) and obtain
\[
\frac{C_\gamma a}{2|z_n|} > \frac{2C_0}{|z_n|} > \frac{C_0}{|z_n||1 + \frac{ae^{i\varphi}}{|z_n|}|}, \quad 0 \leq \varphi < 2\pi,
\]
taking \( n \) large enough to satisfy the inequality
\[
\frac{1}{1 + \frac{ae^{i\varphi}}{|z_n|}} < 2.
\]
Next we arrange the inequality
\[
\frac{C_\gamma a}{2|z_n|} - \frac{a^2}{2} \sup_{z \in C_n(a)} |f_n''(z)| > 0. \tag{4.7}
\]
It is clear that
\[
f_n''(z) = \frac{1}{z^2} G\left(\frac{n(n+1)}{z^2}\right),
\]
where
\[
G(\zeta) = \left[ 3\zeta \sqrt{\zeta + 1} - \zeta^2 (\zeta + 1)^{-1/2} \right] (\zeta + 1)^{-1}.
\]
Note that for $z \in C_n(a)$ and $n$ large enough according to (4.11), the function $|G(a_{n+1})|$ is bounded by a constant $B_{\gamma,a}$ depending on $\gamma$ and $a$. Thus for large $n$ we get
\[
\sup_{z \in C_n(a)} |f_n''(z)| \leq B_{\gamma,a} \sup_{z \in C_n(a)} \frac{1}{|z|^2} = B_{\gamma,a} \sup_{z \in C_n(a)} \frac{1}{1 + \frac{\text{Re}z}{z_n}} \leq 4B_{\gamma,a} \frac{1}{|z_n|^2}
\]
and the proof of (4.3) is reduced to
\[
C_{10} > 4B_{\gamma,a} \frac{a}{|z_n|}
\]
which is satisfied taking again $n$ large. Finally, we proved the estimate (4.3) and we can apply Rouché theorem for the functions $g_n(z)$ and $f_n(z)$ and conclude that the function $g_n(z)$ has exactly one simple zero $\lambda_n$ in $C_n(a)$. Since $g_n(z)$ has only real zeros (see Appendix in [2]), this implies the following

Lemma 4.1. There exist $n_0(\gamma)$ and $a(\gamma) > 0$ depending on $\gamma$ such that for $n \geq n_0(\gamma)$ the negative root $\lambda_n$ of the equation (3.3) satisfies the estimate
\[
|\lambda_n + \sqrt{n(n + 1)}| \leq a(\gamma).
\]

Remark 4.3. According to Proposition 2.1, $n_0(\gamma)$ must satisfy the inequality
\[
n_0(\gamma) \geq \frac{\sqrt{\gamma^2 - 1}}{\max\{\gamma - 1, \sqrt{\gamma - 1}\}}.
\]

5. WEYL ASYMPTOTICS

We start with the analysis of the multiplicity of $\lambda_n$.

Lemma 5.1. For $n \geq n_0(\gamma)$ we have $\text{mult}(\lambda_n) = 2n + 1$.

Proof. Since the geometric multiplicity of $\lambda_n$ is $2n + 1$, it is sufficient to show that
\[
\text{mult}(\lambda_n) \leq 2n + 1.
\]
Let $\lambda \in \Lambda_n$, where $\Lambda_0$ is the set introduced in the previous section and let $\lambda \notin \sigma(G_b)$. If $0 \neq (f, g) \in \text{Image } G_b \cap L^2(\Omega)$, one has div $f = \text{div } g = 0$ and for $(u, v) = (G_b - \lambda)^{-1}(f, g)$ we get div $u = \text{div } v = 0$. Consider the skew self-adjoint operator
\[
A = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}
\]
with boundary condition $\nu \wedge u = 0$ on $S^2$. Then $\sigma(A) \subset i\mathbb{R}$ and let $(u_0(x; \lambda), v_0(x; \lambda)) = (A - \lambda)^{-1}(f, g)$, that is
\[
\begin{cases}
(A - \lambda) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} & \text{for } |x| > 1, \\
\nu \wedge u_0 = 0 & \text{on } S^2.
\end{cases}
\]
Since $\text{div } u_0 = \text{div } v_0 = 0$, the well known coercive estimates yield $(u_0, v_0) \in H^1(\Omega)$. Moreover the resolvent $(A - \lambda)^{-1}$ is analytic in $\{z \in \mathbb{C} : \text{Re } z < 0\}$ and
For $u(x; \lambda), v_0(x; \lambda)$ depend analytically on $\lambda$. We write $(u, v) = (u_0, v_0) + (u_1, v_1)$, where $(u_1(x; \lambda), v_1(x; \lambda))$ is the solution of the problem
\[
\begin{cases}
(G - \lambda) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0 \\ (u_1)_{tan} - \gamma (\nu \wedge (v_1)_{tan}) = -\gamma (\nu \wedge (v_0)_{tan}(x; z)) \end{cases}
\quad (5.3)
\]
for $|x| > 1$.

To solve (5.3), note that $-\gamma (\nu \wedge (v_0)_{tan}(\omega; z)) = F(\omega; \lambda) L^2(S^2)$ with $F(\omega; \lambda)$ analytical in $\lambda$ for $\lambda \in \Lambda_0$. Thus we may write
\[
F(\omega; \lambda) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{\alpha}^m_n(\lambda) U^m_n(\omega) + \tilde{\beta}^m_n(\lambda) V^m_n(\omega)
\]
with analytical coefficients $\tilde{\alpha}^m_n(\lambda), \tilde{\beta}^m_n(\lambda)$. Now we can solve (2.3), (2.4) with right hand part $(\tilde{\alpha}^m_n(\lambda), \tilde{\beta}^m_n(\lambda))$. Finally, we obtain a representation of the solution of (5.3) with meromorphic coefficients
\[
\alpha^m_n(\lambda) = \frac{\tilde{\alpha}^m_n(\lambda)}{h^{(1)}_n(-i\lambda) \left(1 + \frac{d}{dr}(h^{(1)}_n(-i\lambda))\right)}
\]
\[
\beta^m_n(\lambda) = -\frac{\lambda \tilde{\beta}^m_n(\lambda)}{\gamma h^{(1)}_n(-i\lambda) \left(1 + \frac{d}{dr}(h^{(1)}_n(-i\lambda))\right)}
\]
If $\gamma > 1$ the analysis in the previous section shows that for $\lambda \in \Lambda_0$ the meromorphic function $\alpha^m_n(\lambda)$ has a simple pole at $\lambda_n < 0$, while $\beta^m_n(\lambda)$ is analytic in $\Lambda_0$. For $0 < \gamma < 1$ the function $\alpha^m_n(\lambda)$ is analytic in $\Lambda_0$ and $\beta^m_n(\lambda)$ is meromorphic.

Next we integrate $(u(x; \lambda), v(x; \lambda))$ over the circle $|\lambda_n - \lambda| = \epsilon$, where $\epsilon$ is sufficiently small. The integral of $(u_0(x; \lambda), v_0(x; \lambda))$ vanish, while for the integral of $(u_1(x; \lambda), v_1(x; \lambda))$, taking into account the representation of the solution of (5.3), we will obtain a sum
\[
S_n = \begin{cases}
\sum_{m=-n}^{n} \tilde{\alpha}^m_n(\lambda_n) U^m_n(\omega), c_n \neq 0, \gamma > 1, \\
\sum_{m=-n}^{n} \tilde{\beta}^m_n(\lambda_n) \gamma^{-1} V^m_n(\omega), d_n \neq 0, 0 < \gamma < 1.
\end{cases}
\]

This completes the proof of (5.1).

Passing to the analysis of $N(r)$, consider first the case $\gamma > 1$. The root $\lambda_n$ has algebraic multiplicity $2n + 1$ and to find a lower bound of $N(r)$ we apply the estimate
\[
|\lambda_n| \leq \sqrt{n(n+1)} \frac{\alpha(\gamma)}{\gamma^2 - 1} + a(\gamma) < \frac{n+1}{\sqrt{\gamma^2 - 1}} a(\gamma) \leq r
\]
for $r \geq a(\gamma) + \frac{n\alpha(\gamma)+1}{\sqrt{\gamma^2 - 1}}$. Then
\[
N(r) \geq \sum_{j=n_0(\gamma)}^{[r-a(\gamma)]\sqrt{\gamma^2-1}-1} (2j+1) = (\gamma^2 - 1)r^2 + O(\gamma) + A_\gamma.
\]

To get a upper bound for $N(r)$, we use the estimate
\[
|\lambda_n| \geq \sqrt{n(n+1)} \frac{\alpha(\gamma)}{\gamma^2 - 1} - a(\gamma) \geq \frac{n}{\sqrt{\gamma^2 - 1}} a(\gamma) \geq r
\]
for
\[ n \geq (r + a(\gamma))\sqrt{\gamma^2 - 1} \geq 2a(\gamma)\sqrt{\gamma^2 - 1} + n_0(\gamma) + 1, \]
hence
\[ N(r) \leq \sum_{j=n_0(\gamma)+1}^{[(r+a(\gamma))\sqrt{\gamma^2-1}]+1} (2j + 1) + D_\gamma = (\gamma^2 - 1)r^2 + O_\gamma(r) + A_\gamma'. \]

If \(0 < \gamma < 1\), we have \(\frac{1}{\gamma} > 1\) and one applies our argument to the the equation (2.6). This completes the proof of theorem 1.1

References

[1] F. Colombini, V. Petkov and J. Rauch, Spectral problems for non elliptic symmetric systems with dissipative boundary conditions, J. Funct. Anal. 267 (2014), 1637-1661.
[2] F. Colombini, V. Petkov and J. Rauch, Eigenvalues for Maxwell’s equations with dissipative boundary conditions, Asymptotic Analysis, 99 (1-2) (2016), 105-124.
[3] A. Kirsch and F. Hettlich, The Mathematical Theory of Time-Harmonic Maxwells Equations, vol. 190 of Applied Mathematical Sciences, Springer, Switzerland, 2015.
[4] P. Lax and R. Phillips, Scattering theory for dissipative systems, J. Funct. Anal. 14 (1973), 172-235.
[5] F. Olver, Asymptotics and Special Functions, Academic Press, New York, London, 1974.
[6] V. Petkov, Location of the eigenvalues of the wave equation with dissipative boundary conditions, Inverse Problems and Imaging, 10 (4) (2016), 1111-1139.
[7] V. Petkov and G. Vodev, Localization of the interior transmission eigenvalues for a ball, Inverse Problems and Imaging, 11 (2) (2017), 355-372.
[8] G. Vodev, Transmission eigenvalue-free regions. Commun. Math. Phys. 336 (2015), 1141-1166.

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