The status of quantum geometry in the dynamical sector of loop quantum cosmology

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Abstract
This paper is motivated by the recent papers by Dittrich and Thiemann and, respectively, Rovelli discussing the status of quantum geometry in the dynamical sector of quantum geometry. Since the papers consider model examples, we also study the issue in the case of an example, namely on the loop quantum cosmology model of a space-isotropic universe. We derive the Rovelli–Thiemann–Dittrich partial observables corresponding to the quantum geometry operators of LQC in both Hilbert spaces: the kinematical one and the physical Hilbert space of solutions to the quantum constraints. We find that quantum geometry can be used to characterize the physical solutions, and the operators of quantum geometry preserve many of their kinematical properties.

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1. Introduction

One of the issues of the canonical gravity is the lack of explicit formulae for the Dirac observables. Consequently, the role the kinematic quantization of the gravitational field plays after implementing the quantum Einstein constraints is not known. In loop quantum gravity [1], the operators representing the intrinsic 3-geometry of a given Cauchy surface as well as the extrinsic curvature are known [2]. Their properties, spectra, eigenvalues and eigenfunctions were studied in [3]. But what is their meaning in the dynamical theory? The bottom line is that the kinematical operators are used to define the quantum constraint operators [4]. Therefore, their relevance is unquestionable. However, the open question is which properties of the kinematical geometry operators and other structures of the kinematical quantum theory are preserved by the passage to the Dirac observables. New insights were given recently by work of Dittrich and Thiemann [5]. They study various toy examples of the explicit construction of the Dirac observables by using the so-called partial observables method [6]. That method allows one to construct a Dirac observable from any kinematical observable. It is shown
in [5], however, that the discreteness of the kinematical operators does not imply the discreteness of the corresponding quantum Dirac observables and vice versa. The procedure of turning a kinematical observable into the dynamical one can wash out all the properties and replace them with others. A few days after Dittrich and Thiemann’s paper appeared in the archives, Rovelli sent his response [7]. According to Rovelli, the examples of [5] are too distant from the loop quantum gravity.

These recent works motivated us to check the status of the issue of the quantum Dirac observables in the model of LQG called loop quantum cosmology [8, 9]. We consider the work on the best-understood, ‘improved’ LQC model constructed from the family of the space-isotropic gravitational fields (the Friedmann–Robertson–Walker spacetimes) coupled to a space-isotropic massless scalar fields [10]. Our results also apply to the recent SQLC version of that model [14]. This model has two advantages: on the one hand, it has a lot of the properties of LQG, and is understood as a toy model of LQG. Therefore, it is predicted that many results concerning LQC should admit generalizations to LQG. On the other hand, the model is simple enough to be quite well understood. In particular, the quantum observables of this model can be derived explicitly. This is what we do in this paper. The specific question we focus on is the role of the quantum geometry of LQC in the space of the solutions to that theory. A discussion of the technical subtleties related to our form of the scalar constraint which was adapted here to the question we are studying is contained in the last section.

2. The APS model, positive frequencies

The kinematical Hilbert space $\mathcal{H}_{gr}$ of the gravitational degrees of freedom in the FRW–LQC model is spanned on the basis of orthonormal vectors $|v\rangle$, labeled by all the possible real values of $v \in \mathbb{R}$,

$$(v|v') = \delta_{v,v'} = \begin{cases} 1, & \text{if } v = v' \\ 0, & \text{otherwise}. \end{cases}$$

That is, the Hilbert space and the scalar product $\langle \cdot | \cdot \rangle_{gr}$ are

$$\mathcal{H}_{gr} = \left\{ \sum_{i=1}^{\infty} a_i |v_i\rangle : a_i \in \mathbb{C}, \sum_{i=1}^{\infty} |a_i|^2 < \infty \right\}$$

$$\left( \sum_{i=1}^{\infty} a_i |v_i\rangle \right| \left( \sum_{j=1}^{\infty} b_j |v_j\rangle \right)_{gr} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j \delta_{v_i,v_j}.$$ (2)

The kinematical observables are the quantum volume operator

$$\hat{V} |v\rangle = v |v\rangle$$ (3)

and the ‘improved’ [10] quantum holonomy operator

$$\hat{h}_{\lambda} |v\rangle = |v + \lambda\rangle, \quad \lambda \in \mathbb{R}.$$ (4)

The volume operator $\hat{V}$ represents the 3-volume of an isotropic spacelike section of the universe in the closed FRW case, or some fixed box in an isotropic spacelike section of the universe in the open FRW case. The quantum holonomy operator represents a kinematical observable involving the extrinsic curvature of an isotropic section (we skip some constants and details that can be found in [10]).

The kinematical Hilbert space of the scalar field is the space of the square integrable functions on $\mathbb{R}$ endowed with the Lebesgue measure,

$$\mathcal{H}_{sc} = L^2(\mathbb{R}).$$ (5)
The scalar field operator is just the multiplication,
\[ (\hat{\Phi}\psi)(\phi) = \phi\psi(\phi). \] (6)

The scalar field momentum operator \( \hat{\Pi} \) is
\[ \hat{\Pi}\psi = \frac{1}{i}\frac{d}{d\phi}. \] (7)

Finally, the kinematical Hilbert space of the isotropic gravitational field coupled to the isotropic scalar field is
\[ H_{\text{kin}} = H_{\text{sc}} \otimes H_{\text{gr}}, \] (8)
and the kinematical observables are the following operators
\[ 1 \otimes \hat{V}, \ 1 \otimes \hat{h}_\lambda, \ \hat{\Phi} \otimes 1, \ \hat{\Pi} \otimes 1, \] (9)
where \( \lambda \in \mathbb{R} \) runs through all the set \( \mathbb{R} \). There are only 2 degrees of freedom; hence, the declared set of the ‘momentum’ observables \( \hat{h}_\lambda \) is overcomplete. The reason is that the holonomy operators are unitary. If instead of the operator \( \hat{\Pi} \) we were using an operator \( e^{-i\alpha\hat{\Pi}} \), we would also admit all the values of \( \alpha \).

We now turn to the dynamics. The dynamics of the theory is given by the scalar constraint operator \( \hat{C} \). The massless scalar field we consider here is the best-understood case. Let us start with the simplest from the point of view of our work formulation of the constraint. The scalar constraint used in [10] can be written as
\[ \hat{C}^- = \hat{\Pi} \otimes 1 - 1 \otimes H, \] (10)
where \( H \) is an operator defined in the kinematical Hilbert space \( H_{\text{gr}} \) of the gravitational degree of freedom, it does not involve any of the operators acting in \( H_{\text{sc}} \). The operator \( H \) is not diagonal, i.e. it does not commute with \( \hat{V} \),
\[ [\hat{V}, H] \neq 0. \]

For the sake of completeness, let us consider here the case of a constraint operator of the following form
\[ \hat{C}^\pm = \hat{\Pi} \otimes 1 \pm 1 \otimes H, \] (11)
where we fix either + or −.

A strong Dirac observable is an operator commuting with the constraint \( \hat{C} \) in (a sufficiently large domain in) \( H_{\text{kin}} \). Certainly
\[ [\hat{\Pi}, \hat{C}^\pm] = 0, \] (12)
hence the scalar field quantum momentum is a Dirac observable.

We are particularly interested in those observables which involve the operators acting in the kinematical Hilbert space of the gravitational degrees of freedom which define the quantum geometry: the quantum volume operator \( \hat{V} \) and the quantum holonomy operators \( \hat{h}_\lambda \). None of them is an observable,
\[ [\hat{V}, \hat{C}^\pm] \neq 0 \neq [\hat{h}_\lambda, \hat{C}^\pm]. \]

This is a model version of the outstanding problem in LQG: the quantum geometry has been defined on the kinematical level. The operators of the quantum geometry do not commute with the constraint. However, in the case of either of the constraint \( \hat{C}^- \) or \( \hat{C}^+ \), it is easy to assign a Dirac observable \( \hat{O}^\pm \) to any given operator \( \hat{O} \) in \( H_{\text{gr}} \). Indeed, fix any number \( \phi_0 \in \mathbb{R} \) and define
\[ \hat{O}_{\pm, \phi_0} := e^{\pm i(\hat{\Phi} - \phi_0) \otimes H} \ 1 \otimes \hat{O} e^{\pm i(\hat{\Phi} - \phi_0) \otimes H}. \] (13)
It is easy to check that the result is an operator in $\mathcal{H}_{\text{kin}}$ which satisfies

$$[O_\pm, \hat{\mathcal{C}}_\pm] = i e^{\mp i(\hat{\phi} - \phi_0) \otimes H} (\mp 1 \otimes [O_\pm, H]) e^{\pm i(\hat{\phi} - \phi_0) \otimes H} = 0. \quad (14)$$

The form (13) of a Dirac observable is not a surprise, because the APS constraint $\hat{\mathcal{C}}_\pm$ has exactly the same form as the Rovelli–Schrödinger constraint [7, 13] with the Rovelli time operator $\hat{t}$ replaced with the scalar field operator $\hat{\phi}$. We could just choose $\phi_0 = 0$ in the definition, but the dependence of the operator on $\phi_0$ has the natural meaning of the evolution determined by the dynamics. The Hilbert space $\mathcal{H}_{\text{phys}}$ of ‘solutions’ to the quantum constraint defined by one of the operators $\hat{\mathcal{C}}_\pm$ (11) is identified [10] with the space of $\mathcal{H}_{\text{gr}}$-valued functions defined on the spectrum $\mathbb{R}$ of the scalar field operator,

$$\mathbb{R} \ni \phi \mapsto \psi(\phi) \in \mathcal{H}_{\text{gr}}, \quad (15)$$

which satisfy the equation

$$\frac{-i}{\partial \phi} \psi = \mp H \psi. \quad (16)$$

The scalar product between two solutions is defined by the scalar product in $\mathcal{H}_{\text{gr}}$ calculated at any value of $\phi$ due to the identity

$$(\psi(\phi) | \psi'(\phi'))_{\text{gr}} = (\psi(\phi') | \psi'(\phi))_{\text{gr}}. \quad (17)$$

Now, it is easy to see a general operator acting in $\mathcal{H}_{\text{gr}}$ does not admit a unique action on a solution (15), unless it commutes with $H$. The action can be defined only at a fixed value of $\phi = \phi_0$: given a solution (15) and an operator $\mathcal{O}$, consider another solution

$$\mathbb{R} \ni \phi \mapsto \psi''(\phi) \in \mathcal{H}_{\text{gr}}, \quad (18)$$

such that

$$\psi''(\phi_0) = \mathcal{O} \psi(\phi_0). \quad (19)$$

On the other hand, the action of the corresponding Dirac observable $\hat{O}_\pm, \phi_0$ is well defined on each solution and is

$$\hat{O}_\pm, \phi_0 \psi(\phi) = e^{i(\phi - \phi_0)H} \mathcal{O} e^{i(\phi - \phi_0)H} \psi(\phi). \quad (20)$$

The two actions (19) and (20) coincide. In particular

$$H_\pm, \phi_0 = H.$$ 

In the space of solutions, the action of the scalar momentum operator $\hat{\Pi}$ becomes

$$\hat{\Pi} = \mp H.$$ 

At this point, we are in a position to address the main issue of the paper, the issue of the role and the status of the kinematical Hilbert space of the gravitational degrees of freedom and quantum geometry for the physical space of solutions to the quantum constraints. We observe the following.

- The kinematical Hilbert space $\mathcal{H}_{\text{gr}}$ of the gravitational degrees of freedom is unitarily equivalent to the space of the physical solutions. The dynamics takes the appearance of a generalized Schrödinger equation defined in $\mathcal{H}_{\text{gr}}$ by the Hamiltonian operator $H$. Therefore, the quantum geometry operators have just the same status as the position and the momentum operators in the Schrödinger quantum mechanics.

- Every 1-parameter family of the Dirac observables $\hat{O}_{\pm, \phi_0}$ constructed from a given operator $\mathcal{O}$ in $\mathcal{H}_{\text{gr}}$ is mathematically the same as the Heisenberg picture of the operator $\mathcal{O}$ defined by equation (16) understood as the Schrödinger equation. Hence, it is unitarily equivalent to the original operator $\mathcal{O}$. 

3. An equivalent, systematic construction

The construction of the space $\mathcal{H}_{\text{phys}}$ of the solutions (15) and (16) to the constraint (11) can be performed in a systematic way by using the scheme proposed by Thiemann in the context of the master constraint operator [11]. One begins with the spectral decomposition of the kinematical Hilbert space defined by the spectral decompositions of the Hilbert spaces $\mathcal{H}_{\text{sc}}$ and $\mathcal{H}_g$ corresponding to the operators $\hat{\Pi}$ and $H$ respectively. That is, an element of $\mathcal{H}_{\text{kin}}$ is identified with an assignment

$$R \times R \ni (\pi, E) \mapsto \psi(\pi, E) \in \mathcal{H}_{\hat{\Pi}}^\pi \otimes \mathcal{H}_E^E,$$

where $\mathcal{H}_{\hat{\Pi}}^\pi$ and $\mathcal{H}_E^E$ are some Hilbert spaces assigned to the numbers $\pi$ and $E$, respectively.

The kinematical scalar product in $\mathcal{H}_{\text{kin}}$ reads

$$(\psi|\psi')_{\text{kin}} = \int d\pi dE (\psi(\pi, E)|\psi'(\pi, E))_{\pi,E},$$

where $d\pi$ and $dE$ are some measures and $(\cdot|\cdot)_{\pi,E}$ is the scalar product in $\mathcal{H}_{\hat{\Pi}}^\pi \otimes \mathcal{H}_E^E$. Finally, the action of the operators $\hat{\Pi}$ and, respectively, $H$ in this representation reads

$$(\hat{\Pi}\psi)(\pi, E) = \pi \psi(\pi, E), \quad (H\psi)(\pi, E) = E \psi(\pi, E).$$

(22)

In fact $\mathcal{H}_{\hat{\Pi}}^\pi = \mathbb{C}$, and $d\pi$ is the Lebesgue measure. The measure $dE$ is also known in a large class of cases [12].

In this formulation, a solution to the constraint (11) is just the restriction of the definition (21) to the subset of $R \times R$ such that

$$\pi = \mp E,$$

(either $+$ or $-$, depending on the sign in (11)), i.e., the solution is a map

$$R \ni E \mapsto \psi(\mp E, E) \in \mathcal{H}_{\hat{\Pi}}^\pi \otimes \mathcal{H}_E^E.$$  

(23)

The solutions set a vector space. The scalar product between two solutions is defined to be

$$(\psi|\psi')_{\text{phys}} = \int dE (\psi(\mp E, E)|\psi'(\mp E, E))_{\mp E,E}. $$

The solutions and the scalar product define the physical Hilbert space $\mathcal{H}_{\text{phys}}$. We would like to induce in $\mathcal{H}_{\text{phys}}$ an action of some of the operators introduced in $\mathcal{H}_{\text{kin}}$. To do so, we identify every solution (23) with a linear functional $\langle \psi |$ (defined in some domain $\mathcal{D}_\pm$ in $\mathcal{H}_{\text{kin}}$—we do not bother the reader with the domains of the operators in this paper, but we have to mention this at that point) by

$$D_{\pm} \ni \psi' \mapsto \langle \psi |\psi' \rangle = \int dE (\psi(\pm E, E)|\psi'(\mp E, E))_{\mp E,E}. $$

(24)

Every operator in $\mathcal{H}_{\text{kin}}$ whose range contains $D_\pm$, by the duality, maps each solution (23) into another linear functional in $\mathcal{H}_{\text{kin}}$. If this map preserves the form (23), then the operator induces an operator in $\mathcal{H}_{\text{phys}}$. In particular, every operator commuting with the constraint $C_\pm$ preserves the space of solutions (23). Therefore again, the Dirac observables defined in the previous section pass to $\mathcal{H}_{\pm}$.

This description of $\mathcal{H}_{\text{phys}}$ and the action of the Dirac observables is unitarily equivalent to that used in the previous section.
4. The APS model, all the frequencies

Now, we can turn to the full scalar constraint of the LQC-FRW case. The full scalar constraint operator reads

\[ \hat{C} = \hat{\Pi}^2 \otimes 1 - 1 \otimes \hat{H}^2. \]  (25)

We can write

\[ \hat{C} = (\hat{\Pi} \otimes 1 + 1 \otimes \hat{H})(\hat{\Pi} \otimes 1 - 1 \otimes \hat{H}) = \hat{C}_+ \hat{C}_-, \]  (26)

where the factors commute. One can conjecture that the space of solutions of the constraint (26) consists of the solutions to the constraint \( \hat{C}_- \) and the solutions to the constraint \( \hat{C}_+ \).

Indeed, the conjecture is true in the following sense. Consider again the representation of the elements of \( \mathcal{H}_{\text{kin}} \) by the spectral decomposition (21). Now, a solution to the quantum constraint defined by the constraint operator (26) corresponds to the restriction of the definition (21) to the subset

\[ \{ (\pi, E) \in \mathbb{R} \times \mathbb{R} : \pi^2 = E^2 \}, \]

that is an assignment

\[ \{ (\pi, E) \in \mathbb{R} \times \mathbb{R} : \pi^2 = E^2 \} \ni (\pi, E) \mapsto \psi(\pi, E) \in \mathcal{H}_{\pi} \otimes \mathcal{H}_E. \]  (27)

Certainly, each of the solutions (15) is a solution in the sense of (27). Moreover, every solution (27) is a sum of a pair of solutions (15), one to the constraint \( \hat{C}_- \), and the other one to \( \hat{C}_+ \). That decomposition of (27) is unique, orthogonal, and the components are independent, provided we assume that the subset \( \{0\} \subset \mathbb{R} \) is of measure 0 according to \( dE \). In that case, the physical Hilbert space is

\[ \mathcal{H}_{\text{phys}} = \mathcal{H}_{\text{phys},-} \oplus \mathcal{H}_{\text{phys},+}, \]  (28)

where \( \mathcal{H}_{\text{phys},\pm} \) stands for the Hilbert space of solutions to the constraint \( \hat{C}_\pm \).

In the case of the constraint (26), defining explicitly a large class of (quantum) Dirac observables in \( \mathcal{H}_{\text{kin}} \) is not as easy as before. Still the operator \( \Pi \) defined in \( \mathcal{H}_{\text{kin}} \) passes to \( \mathcal{H}_{\text{phys}} \) and induces therein the following operator

\[ \Pi_{\text{phys}} = H_- - H_+, \]

where \( H_\pm \) annihilates the term \( \mathcal{H}_{\text{phys},\mp} \) in (28) whereas

\[ H_\pm|_{\mathcal{H}_{\text{phys},\mp}} = H. \]

Given a quantum geometry operator \( \mathcal{O} \) defined in \( \mathcal{H}_{\text{gr}} \), a counterpart of the Dirac observable (13) derived along the Rovelli–Thiemann–Dittrich method would heuristically look as

\[ \mathcal{O}_{\text{hs}} = e^{i(\tilde{\phi} - \phi_0)\hat{\Pi}^{-1} \otimes \hat{H}^2} 1 \otimes e^{-i(\tilde{\phi} - \phi_0)\hat{\Pi}^{-1} \otimes \hat{H}^2}. \]  (29)

But completing this definition is not easy.

However, in the previous section we have seen a more general condition on an operator defined in \( \mathcal{H}_{\text{kin}} \) which ensures that the operator naturally induces an operator in \( \mathcal{H}_{\text{phys}} \). To formulate it in the current case, we turn each solution (27) into a linear functional \( \langle \psi | : D \to \mathbb{C} \) (defined in some domain \( D \subset \mathcal{H}_{\text{kin}} \)), such that

\[ \langle \psi | \psi' \rangle = \int dE(\psi(E, E)|\psi'(E, E))_{E,E} + \int dE(\psi(-E, E)|\psi'(-E, E))_{-E,E}. \]  (30)

An operator in \( \mathcal{H}_{\text{kin}} \) whose domain contains \( D \) maps acts by the duality,

\[ \langle \mathcal{O} | \psi \rangle = \langle \psi | \mathcal{O}. \]

The condition is that the right-hand side again be of the solution form (27).
Those who enjoy exploring heuristic formulae can proceed as follows. Assume for this paragraph that it makes sense to make the following replacement in (29):

$$\hat{\Pi}^{-1} \text{ replaced by } \pm H^{-1}.$$ 

Then, the heuristic formula ‘restricted’ to (27) becomes

$$O_{\phi_0} |_{H_{\text{phys}} - \oplus H_{\text{phys}}^+} = e^{\pm i(\hat{\phi} - \phi_0) \otimes H} 1 \otimes O e^{\mp i(\hat{\phi} - \phi_0) \otimes H},$$  

(31)

where the upper/lower sign corresponds to \(H_{\text{phys}}^-\).

The exact form of that conclusion is that given an operator \(1 \otimes O\) in \(H_{\text{kin}} = H_{\text{uc}} \otimes H_{\text{gr}}\), the corresponding Dirac observable operator \(O_{\phi_0}\) can be defined in \(H_{\text{kin}}\) by the assumption that in the spectral decomposition (21) representation, in a neighborhood of the lines

$$\pi = E, \quad \text{or} \quad \pi = -E,$$

it equals (31), and otherwise it is arbitrary.

The resulting operator in \(H_{\text{phys}} = H_{\text{phys}}^- \oplus H_{\text{phys}}^+\) has the expected form,

$$O_{\phi_0} = O_{\phi_0}^- + O_{\phi_0}^+.$$ 

If we relax the assumption that the subset \([0]\) of the spectrum of the operator \(H\) is of the measure 0 with respect to the measure \(dE\), then the resulting Hilbert space \(H_{\text{phys}}\) is still spanned by the spaces \(H_{\text{phys}}^-\) and \(H_{\text{phys}}^+\); however, now

$$H_{\text{phys}}^- \cap H_{\text{phys}}^+ = H_{\text{phys}}^0,$$

where \(H_{\text{phys}}^0 \subset H_{\text{gr}}\) is the space of the vectors (normalizable) annihilated by the operator \(H\). Then, given an operator \(O\) in \(H_{\text{gr}}\) and the ‘instant of time’ \(\phi_0\), in every \(H_{\text{phys}}\) the Dirac observable operator \(O_{\phi_0}\) should coincide with \(O_{\phi_0,\pm}\) on the orthogonal completion of \(H_{\text{phys}}^0\) (on the nonzero modes), whereas on the zero a natural definition is

$$O_{\phi_0} |_{H_{\text{phys}}^0} = O_{\phi_0}^- + O_{\phi_0}^-.$$ 

Those conditions define uniquely an operator in \(H_{\text{phys}}\).

5. Conclusions and outlook

At the end of the first section, we itemized our conclusions concerning the eventual role of the kinematic quantum geometry operators in the case of the first example. The ‘physical’ Hilbert space \(H_{\text{phys}}\) is the space of the negative/positive frequency solutions of the APS model quantum scalar constraint (appropriately adapted, see below). Each quantum solution can be thought of as an evolving state of the (kinematical) quantum geometry. Consequently, the physical Hilbert space \(H_{\text{phys}}\) (let us fix any of the signs) is unitarily equivalent with the kinematical Hilbert space \(H_{\text{gr}}\) of the quantum excitations of the space time geometry, that is

$$H_{\text{phys}} \cong H_{\text{gr}}.$$ 

However, an isometry

$$U_{\phi_0} : H_{\text{phys}} \rightarrow H_{\text{gr}}$$ 

is not unique. It depends on a value \(\phi_0\) of the scalar field. One can interpret that the dependence as ‘evolution’ and think of the parameter \(\phi_0\) as time emerging from LQC. In a consequence, the extension to \(H_{\text{phys}}\) of an operator \(O\) of quantum geometry defined in \(H_{\text{gr}}\) is well defined only when an ‘instant’ \(\phi_0\) of that evolution is given. Hence we may denote it by \(O_{\phi_0}\). The resulting 1-dimensional family of operators, in terms of the QM analogy, is the Heisenberg picture of \(O\). On the other hand, we calculate the Rovelli–Dittrich–Thiemann quantum observable operator
in $\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}} \otimes \mathcal{H}_{\text{gr}}$ assigned to $\mathcal{O}$ upon the choice of the scalar field $\phi$ as a time, and $\phi_0$ as the instant of time. The result is that the RTD observable extended to $\mathcal{H}_{\text{phys}}$ just coincides with $\mathcal{O}_{\phi_0}$. From the mathematical point of view, that interpretation is just equivalent to the Rovelli formulation of quantum mechanics [13]. What is important for us in the current paper, is that the equivalence applies directly to quantum geometry in this LQC model.

We also consider the version of the APS model which admits both, positive and negative frequencies. In the simplest case, if there are no (normalizable) eigenvectors of the $H$ operator, the physical Hilbert space is just the orthogonal sum

$$\mathcal{H}_{\text{phys}} = \mathcal{H}_{\text{phys}}^- \oplus \mathcal{H}_{\text{phys}}^+ \cong \mathcal{H}_{\text{gr}} \oplus \mathcal{H}_{\text{gr}},$$

where the first isometry is natural, and the second again depends on a value of the parameter $\phi_0$.

The action of each quantum geometry operator $\mathcal{O}$ can be extended to $\mathcal{H}_{\text{phys}}$ in the diagonal manner correspondingly to the decomposition above, by using the identification in every given instant $\phi_0$,

$$\mathcal{O}_{\phi_0} = \mathcal{O}_{\phi_0^-} + \mathcal{O}_{\phi_0^+} + \mathcal{O}.$$ 

In this case, the constraint operator is no longer linear in the scalar field momentum operator, hence in $\mathcal{H}_{\text{kin}}$, there is no unique, Rovelli–Dittrich–Thiemann quantum observable operator assigned to the operator $\mathcal{O}$. However, we proposed a more general definition of the quantum observable and applied it to a heuristic quantization of a classical RDT quantum observable. The result again coincides with $\mathcal{O}_{\phi_0}$. The case when there exist in $\mathcal{H}_{\text{gr}}$ vectors annihilated by $H$ is also discussed in section 4.

We have postponed until the end of this paper remarks concerning some transformations we made, to simplify the result. The original gravitational scalar constraint operator in the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ (8) has the following form

$$\hat{C} = \hat{\Pi}^2 \otimes \hat{v}^{-1} + 1 \otimes \hat{C}_{\text{gr}},$$ (32)

where $\hat{v}^{-1}$ is a quantum inverse volume operator. Whereas the self-adjointness of the operator $\hat{C}$ can be proven quite generally [12], the first term does not commute with the second one. For this technical reason, it is reasonable to consider instead the operator

$$\sqrt{\hat{v}^{-1}} \circ \hat{C} \circ \sqrt{\hat{v}^{-1}} = \hat{\Pi}^2 \otimes 1 + 1 \otimes \sqrt{\hat{v}^{-1}} \hat{C}_{\text{gr}} \sqrt{\hat{v}^{-1}}.$$ (33)

On the other hand, in the original APS model another formulation of the constraint operator is considered, namely

$$\hat{v}^{-1} \circ \hat{C} = \hat{\Pi}^2 \otimes 1 + 1 \otimes \hat{v}^{-1} \hat{C}_{\text{gr}},$$ (34)

and the operator is symmetric in a space $\mathcal{H}_{\text{kin}}'$ defined by replacing the kinematical scalar product $(\cdot | \cdot)_{\text{kin}}$ with $(\cdot | v^{-1} \cdot)_{\text{kin}}$. There is a unitary transformation $\mathcal{H}'_{\text{kin}} \rightarrow \mathcal{H}_{\text{kin}}$ which maps (34) into (33), preserves the form of the volume operators and modifies the holonomy operators appropriately.

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