Right-truncated Archimedean and related copulas

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Abstract

The copulas of random vectors with standard uniform univariate margins truncated from the right are considered and a general formula for such right-truncated conditional copulas is derived. This formula is analytical for copulas that can be inverted analytically as functions of each single argument. This is the case, for example, for Archimedean and related copulas. The resulting right-truncated Archimedean copulas are not only analytically tractable but can also be characterized as tilted Archimedean copulas. This finding allows one, for example, to more easily derive analytical properties such as the coefficients of tail dependence or sampling procedures of right-truncated Archimedean copulas. As another result, one can easily obtain a limiting Clayton copula for a general vector of truncation points converging to zero; this is an important property for (re)insurance and a fact already known in the special case of equal truncation points, but harder to prove without aforementioned characterization. Furthermore, right-truncated Archimax copulas with logistic stable tail dependence functions are characterized as tilted outer power Archimedean copulas and an analytical form of right-truncated nested Archimedean copulas is also derived.

Keywords

Right truncation, conditional copulas, Archimedean copulas, Archimax and nested Archimedean copulas, tilted and outer power transformations.

1 Introduction and motivation

Juri and Wüthrich (2002) and Charpentier and Segers (2007) studied the practically relevant problem of determining the copula \( C_t \) and its properties of a bivariate random vector \((U_1, U_2)\) distributed according to some copula \( C \) given that, componentwise, \((U_1, U_2) \leq (t, t)\) for some truncation point \( t \in (0, 1) \). The copula \( C_t \) is the copula of \((U_1, U_2) \mid (U_1, U_2) \leq (t, t)\), that is the copula of the conditional distribution of \((U_1, U_2)\) given \((U_1, U_2) \leq (t, t)\), a right-truncated \((U_1, U_2)\). In the first reference, \( C_t \) is called “extreme tail dependence copula relative to \( C \) at the level \( t \)” as the limiting copula for \( t \downarrow 0 \) is of interest, and in the second reference \( C_t \) is called “lower tail dependence copula relative to \( C \) at level \( t \)”; note that \( u \) instead of \( t \) is used as single truncation point. Both references show that if \( C \) is Archimedean, then \( C_t \) is also Archimedean, and that if \( C \) is a Clayton copula, then \( C_t \) equals \( C \) – a fact relevant

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for (re)insurance. Larsson and Nešlehová (2011) address the $d$-dimensional case, but also focus on the limit for the truncation point converging to zero; the corresponding copulas are referred to as “limiting lower threshold copulas”.

In comparison to the aforementioned publications, our contributions are as follows:

1) We consider a fixed $d$-dimensional vector $t = (t_1, \ldots, t_d) \in (0, 1]^d$ as right truncation point. In particular, the thresholds do not have to be the same for each component and we do not focus on the limiting case $t \downarrow 0$ alone.

2) We derive a formula for the copula $C т$ of $U | U \leq t$ in terms of the copula $C$ of $U = (U_1, \ldots, U_d)$; see Proposition 2.5. The formula is analytical if $C$ is componentwise analytically invertible.

3) We consider the case where $U$ follows an Archimedean copula and show that the family of copulas of $U | U \leq t$ is not only analytically available but actually also known, namely tilted Archimedean; see Theorem 3.1.

4) We consider the case of $U$ following outer power Archimedean copulas or Archimax copulas with logistic stable tail dependence function and show that the family of copulas of $U | U \leq t$ is tilted outer power Archimedean; see Sections 4.1 and 4.2.

5) We consider $U$ following nested Archimedean copulas and derive the corresponding right-truncated copulas; see Theorem 4.1.

Various examples are given and further properties discussed in the appendix.

The operation of right-truncation is important for (re)insurance as the copula $C т$ allows one to study the dependence between the components of a truncated loss random vector $L | L \leq w$ for any $w \in \mathbb{R}^d$ such that $\mathbb{P}(L \leq w) > 0$, where $L \sim F_L$ with continuous margins $F_{L_1}, \ldots, F_{L_d}$ and copula $C$. To see this, note that the distribution function of $L | L \leq w$ can be written as $\mathbb{P}(L \leq x | L \leq w) = \mathbb{P}(U \leq u | U \leq t)$ for $U = (F_{L_1}(L_1), \ldots, F_{L_d}(L_d)) \sim C$, $u = (F_{L_1}(x_1), \ldots, F_{L_d}(x_d))$ and $t = (F_{L_1}(w_1), \ldots, F_{L_d}(w_d))$, so in terms of the distribution function of $X = (U \leq u | U \leq t)$ whose copula $C т$ for a fixed truncation point $t \in (0, 1]^d$ is the main objective in this work; occasionally, we also address the case $t \downarrow 0$. As the copula of a right-truncated distribution function, we simply refer to the copula $C т$ of $U | U \leq t$ as right-truncated copula in what follows.

Our findings also apply to the survival copula $\hat{C}$ of a left-truncated $U$, that is $U | U \geq t$. To see this let $\hat{u} = 1 - u$, $\hat{t} = 1 - t$ and $\hat{U} = 1 - U$, and note that, in distribution, $(U \geq u | U \geq t) = (1 - U \leq 1 - u | 1 - U \leq 1 - t) = (\hat{U} \leq \hat{u} | \hat{U} \leq \hat{t})$, so the survival copula of a left-truncated $U$ (that is the copula of the survival distribution of $U \sim C$ given $U \geq t$) equals the copula of $\hat{U}$ (that is the survival copula $\hat{C}$ corresponding to $C$) right-truncated at $\hat{t}$. This fits in our framework if one considers the copulas we work with (Archimedean, Archimax, etc.) as survival copulas $\hat{C}$. 
2 Right-truncated copulas

We start with the general form of the distribution function \( F_t \) (and its margins \( F_{t,1}, \ldots, F_{t,d} \)) of the right-truncated random vector \( U \mid U \leq t \) for \( U \) following a \( d \)-dimensional copula \( C \), that is the distribution function \( F_t \) of the conditional distribution of \( U \sim C \) given \( U \leq t \).

**Lemma 2.1 (Right-truncated distribution function and its margins)**

Let \( U \sim C \) for a \( d \)-dimensional copula \( C \) and let \( t \in (0, 1]^d \) such that \( C(t) > 0 \). Furthermore, let \( \min\{x, t\} = (\min\{x_1, t_1\}, \ldots, \min\{x_d, t_d\}) \). Then the distribution function \( F_t \) of \( X = (U \mid U \leq t) \) is given by

\[
F_t(x) = \frac{C(\min\{x, t\})}{C(t)}, \quad x \in [0, t],
\]

with margins \( F_{t,j}(x_j) = \frac{C(x_j; t_{-j})}{C(t)}, x_j \in [0, t_j], j = 1, \ldots, d \), where, for all \( j = 1, \ldots, d \), \( t_{-j} = (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_d) \) and

\[
C(x_j; t_{-j}) = C(t_1, \ldots, t_{j-1}, x_j, t_{j+1}, \ldots, t_d), \quad x_j \in [0, t_j].
\]

**Proof.** The distribution function \( F_t \) of \( U \mid U \leq t \) is given by

\[
F_t(x) = \mathbb{P}(U \leq x \mid U \leq t) = \frac{\mathbb{P}(U \leq x, U \leq t)}{\mathbb{P}(U \leq t)} = \frac{\mathbb{P}(\min\{x, t\})}{\mathbb{P}(U \leq t)}
\]

which equals \( \frac{\mathbb{P}(\min\{x, t\})}{\mathbb{P}(U \leq t)} = \frac{C(x)}{C(t)} \) for \( x \in [0, t] \). The \( j \)th margin is obtained by letting \( x_j = t_j \) for all \( j \neq j \).

It is immediate from Lemma 2.1 that if \( C \) has density \( c \), then \( F_t \) has density \( f_t(x) = c(x)/C(t), x \in (0, t) \).

The right-truncated distribution function \( F_t \) has a unique copula which we denote by \( C_t \) and call **right-truncated copula**.

**Definition 2.2 (Right-truncated copula)**

Let \( U \sim C \) for a \( d \)-dimensional copula \( C \) and let \( t \in (0, 1]^d \) such that \( C(t) > 0 \). The copula \( C_t \) of the distribution function \( F_t \) of \( U \mid U \leq t \) is called **right-truncated copula at \( t \)** or the copula \( C \) **right-truncated at \( t \)**.

A straightforward sampling procedure for \( C_t \) is the following.

**Algorithm 2.3 (Rejection sampling)**

1) For \( i = 1, \ldots, n \), do: Repeat sampling \( U \sim C \) until \( U \leq t \), then set \( X_i = U \).
2) Return \( (F_{t,1}(X_{i1}), \ldots, F_{t,d}(X_{id})), i = 1, \ldots, n \). Alternatively, for sufficiently large \( n \), return the pseudo-observations of \( X_1, \ldots, X_n \); see Genest et al. (1995).

The following example shows pseudo-samples from right-truncated Marshall–Olkin copulas.
Example 2.4 (Right-truncated bivariate Marshall–Olkin copulas)

Figure 1 shows 5000 pseudo-observations from bivariate right-truncated Marshall–Olkin copulas \( C(u_1, u_2) = \min\{u_1^{1-\alpha_1}u_2, \ u_1u_2^{1-\alpha_2}\} \) with parameters \( \alpha_1 = 0.2, \ \alpha_2 = 0.7 \) and truncation points as indicated; for \( t = (1, 1) \) in the top left plot, no truncation takes place and thus a sample from \( C \) is shown. We see how right-truncation allows one to change the shape of Marshall–Olkin samples. In particular, right-truncation allows one to cut out a lower-left region of the copula samples (appropriately scaled to again be copula samples after truncation) and thus to shift the top right end point of the singular component to

**Figure 1** \( n = 5000 \) pseudo-observations from a Marshall–Olkin copula with parameters \( \alpha_1 = 0.2, \ \alpha_2 = 0.7 \) right-truncated at the indicated points \( t = (t_1, t_2) \).
points other than \((1,1)\). Although we consider Archimedean and related copulas in what follows, this neither symmetric nor radially symmetric example nicely demonstrates the operation of right-truncation.

We now derive a general formula for the right-truncated copula \(C_t\) of a \(d\)-dimensional copula \(C\). To this end and for later, it will be convenient to write \(C(\{u_j\}_j)\) for \(C(u_1, \ldots, u_d)\) and to let \(y_j \mapsto C^{-1}[y_j; t_{-j}]\) denote the generalized inverse of the increasing (that is non-decreasing) function \(x_j \mapsto C(x_j; t_{-j})\); see Embrechts and Hofert (2013) for the notion of generalized inverses.

**Proposition 2.5 (Right-truncated copulas)**

Let \(C\) be a \(d\)-dimensional copula and let \(t \in (0,1]^d\) such that \(C(t) > 0\). Then the right-truncated copula at \(t\) is given by

\[
C_t(u) = \frac{C(\{C^{-1}[C(t)u_j; t_{-j}]\}_j)}{C(t)}, \quad u \in [0,1]^d.
\]

**Proof.** By Lemma 2.1, the quantile function of \(F_{t,j}\) is \(F_{t,j}^{-1}(u_j) = C^{-1}[C(t)u_j; t_{-j}]\). By Sklar’s Theorem, the right-truncated copula can thus be obtained from \(F_t\) via

\[
C_t(u) = F_t(F_{t,1}^{-1}(u_1), \ldots, F_{t,d}^{-1}(u_d)) = \frac{C(F_{t,1}^{-1}(u_1), \ldots, F_{t,d}^{-1}(u_d))}{C(t)} = \frac{C(\{C^{-1}[C(t)u_j; t_{-j}]\}_j)}{C(t)}.
\]

**Remark 2.6**

If \(U \sim U(0,1)\) and \(t \in (0,1]\), then \(X = U \mid U \leq t\) has distribution function \(F_t(x) = P(U \leq x \mid U \leq t) = x/t, \ x \in [0,t]\), which equals the distribution function of the random variable \(Y = tU\). As such, one might be tempted to believe that in the multivariate case for \(U \sim C\), the random vector \(U \mid U \leq t\) is in distribution equal to (and thus can be sampled as) \(tU = (t_1U_1, \ldots, t_dU_d)\). However, note that \(tU\) is a simple componentwise scaled version of \(U \sim C\) and thus, by the invariance principle, the copula of \(tU\) is also \(C\).

**Example 2.7 (Independence, independent blocks and comonotonicity copulas)**

If \(C\) is \textit{componentwise analytically invertible}, that is componentwise invertible in analytic form, we see from Proposition 2.5 that the right-truncated copula \(C_t\) is given analytically. The following are immediate examples.

1) For the independence copula \(C(u) = \prod_{j=1}^d u_j\), we have \(C(x_j; t_{-j}) = x_j \prod_{j \neq j} t_j, \ x_j \in [0, t_j]\), so that \(C^{-1}[y_j; t_{-j}] = y_j / \prod_{j \neq j} t_j, \ y_j \in [0, C(t_j; t_{-j}) = C(t)]\), and thus

\[
C_t(u) = \frac{C(\{C^{-1}[C(t)u_j; t_{-j}]\}_j)}{C(t)} = \prod_{j=1}^d \frac{C(t)u_j}{\prod_{j \neq j} t_j} = C(t)^{d-1} \prod_{j=1}^d \frac{u_j}{\prod_{j \neq j} t_j} = C(t)^{d-1} \frac{C(u)}{C(t)^{d-1}} = C(u),
\]
which confirms the intuition that right-truncating independent random variables (the independence copula) leads to independent random variables (the independence copula).

2) For a hierarchical copula of the form \( C(u) = \prod_{s=1}^{S} C_s(u_s) \) (independent dependent blocks) with \( u = (u_1, \ldots, u_S) \), copulas \( C_s, s = 1, \ldots, S \), and a truncation point \( t = (t_1, \ldots, t_S) \) we have \( C(x_{sj}; t_{-sj}) = C_s(x_{sj}; t_{s(-j)}) \prod_{s \neq s} C_s(t_s) \), where \( t_{-sj} \) denotes \( t \) without the \( j \)th component in block \( s \) and \( t_{s(-j)} \) denotes \( t_s \) without the \( j \)th component. Therefore \( C^{-1}[y_{sj}; t_{-sj}] = C^{-1}_s(y_{sj}/ \prod_{s \neq s} C_s(t_s); t_{s(-j)}) \) and thus

\[
C_t(u) = \frac{C\left(\left\{ C_s^{-1}(C(t)u_{sj}/ \prod_{s \neq s} C_s(t_s); t_{s(-j)})\right\}_{s,j}\right)}{C(t)} = \frac{C\left(\{C_s^{-1}(C_s(t_s)u_{sj}; t_{s(-j)})\}_{s,j}\right)}{C(t)} = \prod_{s=1}^{S} C_s(t_s),
\]

that is the product of the copulas \( C_1, \ldots, C_S \) right-truncated at \( t_1, \ldots, t_S \), respectively. This confirms the intuition that right-truncating independent random vectors (their copulas) leads to independent right-truncated random vectors (their copulas).

3) For the comonotonicity copula \( C(u) = \min\{u_1, \ldots, u_d\} \), we have

\[
C(x_j; t_{-j}) = \min\{t_1, \ldots, t_{j-1}, x_j, t_{j+1}, \ldots, t_d\} = \begin{cases} x_j, & x_j \leq \min\{t_{-j}\}, \\ \min\{t_{-j}\}, & \min\{t_{-j}\} < x_j \leq t_j, \end{cases}
\]

where the second case is void if \( t_j \leq \min\{t_{-j}\} \). Therefore, \( C^{-1}[y_j; t_{-j}] = y_j, y_j \in [0, C(t_j; t_{-j}) = C(t)], \) and thus

\[
C_t(u) = \frac{C\left(\{C^{-1}(C(t)u_j; t_{-j})\}_{j}\right)}{C(t)} = \frac{\min\{\{C(t)u_j\}_{j}\}}{C(t)} = \frac{C(t)\min\{\{u_j\}_{j}\}}{C(t)} = C(u),
\]

which confirms the intuition that right-truncating comonotone random variables (the comonotonicity copula) leads to comonotone random variables (the comonotonicity copula).

**Example 2.8 (Bivariate right-truncated Marshall–Olkin copulas)**

Another example of a componentwise analytically invertible copula is the bivariate Marshall–Olkin copula \( C(u_1, u_2) = \min\{u_1^{1-\alpha_1}u_2, u_1u_2^{1-\alpha_2}\} \) for \( \alpha_1, \alpha_2 \in (0, 1) \). Note that \( u_1^{1-\alpha_1}u_2^{1-\alpha_2} \leq u_1u_2^{1-\alpha_2} \) if and only if \( u_2^{\alpha_2} \leq u_1^{\alpha_1} \) if and only if \( u_2 \leq u_1^{\alpha_1/\alpha_2} \) if and only if \( u_1 \geq u_2^{\alpha_2/\alpha_1} \). Let \( \alpha_{-j} = \alpha_1 \) if \( j = 2 \) and \( \alpha_{-j} = \alpha_2 \) if \( j = 1 \). Then

\[
C(x_j; t_{-j}) = \begin{cases} t_j^{1-\alpha_{-j}}x_j, & 0 \leq x_j \leq t_j^{\alpha_{-j}/\alpha_j}, \\ t_jx_j^{1-\alpha_j}, & t_j^{\alpha_{-j}/\alpha_j} \leq x_j \leq 1, \end{cases}
\]
which is continuous and strictly increasing, and equal to $t_{-j}$ for $x_j = 1$. Therefore,
\[
C^{-1}[y_j; t_{-j}] = \begin{cases} 
\frac{y_j}{t_{-j}^{1-\alpha_j}}, & 0 \leq y_j \leq t_{-j}^{1-\alpha_j+\alpha_j/\alpha_j}, \\
\left(y_j/t_{-j}\right)^{1-\alpha_j}, & t_{-j}^{1-\alpha_j+\alpha_j/\alpha_j} < y_j \leq t_{-j};
\end{cases}
\]

note that since $\alpha_j \leq 1$, we have $1 - \alpha_j + \alpha_j/\alpha_j \geq 1$ so that $t_{-j}^{1-\alpha_j+\alpha_j/\alpha_j} \leq t_{-j}$.

**Case 1.** If $t_2^{\alpha_2} \leq t_1^{\alpha_1}$, we have $C(t) = t_1^{\alpha_1}t_2$ and a calculation shows that
\[
C^{-1}[C(t)u_1; t_2] = \begin{cases} 
t_1^{1-\alpha_1}t_2^{\alpha_2}u_1, & 0 \leq u_1 \leq \left(t_2^{\alpha_2}/t_1^{\alpha_1}\right)^{(1-\alpha_1)/\alpha_1}, \\
t_1^{\alpha_1}u_1^{1-\alpha_2}, & \left(t_2^{\alpha_2}/t_1^{\alpha_1}\right)^{(1-\alpha_1)/\alpha_1} < u_1 \leq 1,
\end{cases}
\]

and that $C^{-1}[C(t)u_2; t_1] = t_2u_2$, $0 \leq u_2 \leq 1$. We thus obtain that
\[
C_t(u_1, u_2) = \min\{C^{-1}[C(t)u_1; t_2]^{1-\alpha_1}C^{-1}[C(t)u_2; t_1], C^{-1}[C(t)u_2; t_2]C^{-1}[C(t)u_2; t_1]^{1-\alpha_2}\}
\]

Furthermore, the singular component of $C_t(u_1, u_2)$ is given by all $(u_1, u_2) \in [0, 1]^2$ such that $C^{-1}[C(t)u_1; t_2]^{\alpha_1} = C^{-1}[C(t)u_2; t_1]^{\alpha_2}$, which are all $u_2$ such that
\[u_2^{\alpha_2} = \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2}u_1^{1-\alpha_1}, \quad 0 \leq u_2 \leq \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_1)/\alpha_1}.
\]

**Case 2.** If $t_2^{\alpha_2} > t_1^{\alpha_1}$, we have $C(t) = t_1t_2^{1-\alpha_2}$ and a calculation shows that $C^{-1}[C(t)u_1; t_2] = t_1u_1$, $0 \leq u_1 \leq 1$, and that
\[
C^{-1}[C(t)u_2; t_1] = \begin{cases} 
t_1^{\alpha_1}t_2^{1-\alpha_2}u_2, & 0 \leq u_2 \leq \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2}, \\
t_2u_2^{1-\alpha_2}, & \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2} < u_2 \leq 1.
\end{cases}
\]

We thus obtain that
\[
C_t(u_1, u_2) = \begin{cases} 
\min\{u_1^{1-\alpha_1}u_2, \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2}u_1u_2^{1-\alpha_2}\}, & 0 \leq u_2 \leq \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2}, \\
\min\left\{\left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)u_1^{1-\alpha_1}u_2^{1-\alpha_2}, u_1u_2\right\}, & \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2} < u_2 \leq 1,
\end{cases}
\]

which can also be obtained from the case $t_2^{\alpha_2} \leq t_1^{\alpha_1}$ by interchanging $t_1, t_2$ and $\alpha_1, \alpha_2$ and $u_1, u_2$. Furthermore, the singular component is given by all $u_2$ such that
\[u_2^{\alpha_2} = \left(t_1^{\alpha_1}/t_2^{\alpha_2}\right)^{(1-\alpha_2)/\alpha_2}u_1^{1-\alpha_1}, \quad 0 \leq u_1 \leq 1.
\]

Figure 2 shows the bivariate right-truncated Marshall–Olkin copulas corresponding to the samples displayed in Figure 1 with singular components depicted on the graphs of $C_t$.

One can also use above formulas for $C_t$ to verify that if $t_1 = t_2 = t$, then $\lim_{t \to 0} C_{(t,t)}(u_1, u_2) = u_1u_2$ if $\alpha_1 \neq \alpha_2$ and $\lim_{t \to 0} C_{(t,t)}(u_1, u_2) = \min\{u_1^{1-\alpha_1}u_2, u_1u_2^{1-\alpha_2}\}$ if $\alpha_1 = \alpha_2 = \alpha$, so limiting copulas of right-truncated Marshall–Olkin copulas are the independence copula or Cuadras–Augé copulas.
Figure 2 Bivariate Marshall–Olkin copulas with parameters $\alpha_1 = 0.2$, $\alpha_2 = 0.7$ right-truncated at the indicated points $t$. 
The following result provides a scaling property of right-truncated extreme value copulas. It implies that unless no truncation takes place (so unless \( t = 1 \)), right-truncated extreme value copulas are not extreme value copulas anymore. In particular, the bivariate right-truncated Marshall–Olkin copulas from Example 2.3 are not extreme value anymore.

**Proposition 2.9 (Scaling of extreme value copulas)**

If \( C \) is an extreme value copula, then \( C_t(u^\alpha) = C_{t^{1/\alpha}}^\alpha(u), \ u \in [0,1]^d \), for all \( \alpha > 0 \).

*Proof.* Note that \( C^{-1}[y_j^\alpha; t_{-j}] = v \) if and only if \( C(t_1, \ldots, t_{j-1}, v, t_{j+1}, \ldots, t_d) = y_j^\alpha \) if and only if \( C^{1/\alpha}(t_1, \ldots, t_{j-1}, v, t_{j+1}, \ldots, t_d) = y_j \) if and only if \( C(t_1^{1/\alpha}, \ldots, t_{j-1}^{1/\alpha}, v^{1/\alpha}, t_{j+1}^{1/\alpha}, \ldots, t_d^{1/\alpha}) = y_j \) if and only if \( C^{-1}[y_j; t_{-j}^{1/\alpha}] = v^{1/\alpha} \) if and only if \( C^{-1}[y_j; t_{-j}^{1/\alpha}]^\alpha = v \). Therefore, \( C^{-1}[y_j^\alpha; t_{-j}] = C^{-1}[y_j; t_{-j}^{1/\alpha}]^\alpha = C_t(u^\alpha) \) and thus

\[
C_t(u^\alpha) = \frac{C(\{C^{-1}[C(t)u_j^\alpha; t_{-j}]\})}{C(t)} = \frac{C(\{C^{-1}[C(t^{1/\alpha})u_j; t_{-j}^{1/\alpha}]\})}{C(t)} \cdot \frac{C(\{C^{-1}[C(t^{1/\alpha})u_j; t_{-j}^{1/\alpha}]\})}{C(t)} = \left( \frac{C(\{C^{-1}[C(t^{1/\alpha})u_j; t_{-j}^{1/\alpha}]\})}{C(t)} \right)^\alpha = C_{t^{1/\alpha}}^\alpha(u), \quad u \in [0,1]^d.
\]

Selected further properties of general right-truncated copulas are derived in Appendix A.

### 3 Right-truncated Archimedean copulas

In this section we characterize right-truncated Archimedean copulas as tilted Archimedean copulas, address their properties and consider examples of right-truncated Archimedean copulas.

#### 3.1 Characterization and properties

Archimedean copulas are widely used in finance, insurance and risk management. A copula \( C \) is an Archimedean copula if it admits the form

\[
C(u) = \psi(\psi^{-1}[u_1] + \cdots + \psi^{-1}[u_d]), \quad u = (u_1, \ldots, u_d) \in [0,1]^d,
\]

where \( \psi : [0,\infty) \to [0,1] \) is known as (Archimedean) generator. Let \( \Psi_d \) denote the set of all Archimedean generators which generate a \( d \)-dimensional Archimedean copula; see, for example, McNeil and Nešlehová (2009).

It is immediate from (2) that Archimedean copulas \( C \) are componentwise analytically invertible and thus lead to analytical right-truncated copulas \( C_t \). The following result not only shows that right-truncated Archimedean copulas are Archimedean again (known in the bivariate case and for equal truncation points; see Juri and Wüthrich (2002) and
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Charpentier and Segers (2007), but also that we know their type, even in the general $d$-dimensional case and for a general truncation point $t$. As it turns out, right-truncated Archimedean copulas are so-called tilted Archimedean copulas, that is Archimedean copulas with a tilted generator $\tilde{\psi}$ of the form

$$\tilde{\psi}(t) = \frac{\psi(t + h)}{\psi(h)}, \quad t \in [0, \infty),$$

for some tilt $h \geq 0$. Also note that in what follows we interpret a sum of the form $\sum_{i=1}^{n} a_i + b$ as $(\sum_{i=1}^{n} a_i) + b$.

**Theorem 3.1 (Right-truncated Archimedean copulas)**

Let $C$ be a $d$-dimensional Archimedean copula with generator $\psi \in \Psi_d$. For $t \in (0, 1]^d$ such that $C(t) > 0$, let

$$\tilde{\psi}(t) = \frac{\psi(t + \psi^{-1}[C(t)])}{C(t)}, \quad t \in [0, \infty),$$

with corresponding $\tilde{\psi}^{-1}[u] = \psi^{-1}[C(t)u] - \psi^{-1}[C(t)]$, $u \in [0, 1]$. Then the right-truncated copula at $t$ is given by

$$C_t(u) = \frac{\psi(\sum_{j=1}^{d} \psi^{-1}[C(t)u_j] - (d - 1)\psi^{-1}[C(t)])}{\tilde{\psi}\left(\sum_{j=1}^{d} \psi^{-1}[u_j]\right)}, \quad u \in [0, 1]^d,$$

that is $C_t$ is Archimedean with tilted generator $\tilde{\psi}(t) = \psi(t + h)/\psi(h)$ and tilt $h = \psi^{-1}[C(t)]$.

**Proof.** For $j \in \{1, \ldots, d\}$, $C(x_j; t_{-j}) = \psi(\psi^{-1}[x_j] + \sum_{j \neq j} \psi^{-1}[t_j])$ and thus $C^{-1}[y_j; t_{-j}] = \psi(\psi^{-1}[y_j] - \sum_{j \neq j} \psi^{-1}[t_j])$. By Proposition 2.5, we thus have that for all $u \in [0, 1]^d$,

$$C_t(u) = \frac{C(\{C^{-1}[C(t)u_j; t_{-j}]\}_{j=1}^{d})}{\psi(\sum_{j=1}^{d} \psi^{-1}[C(t)u_j] - \sum_{j \neq j} \psi^{-1}[t_j]) = \frac{\psi(\sum_{j=1}^{d} \psi^{-1}[C(t)u_j] - \sum_{j \neq j} \psi^{-1}[t_j])}{\psi(\sum_{j=1}^{d} \psi^{-1}[C(t)u_j] - \sum_{j \neq j} \psi^{-1}[t_j])}}.$$
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Remark 3.2

1) Tilted Archimedean copulas were introduced and studied in Hofert (2010, Section 4.2.1) under the assumption of \( \psi \in \Psi_\infty \). By Bernstein’s Theorem, see Feller (1971, pp. 439), it is well known that \( \psi \in \Psi_\infty \) if and only if \( \psi \) is the Laplace–Stieltjes transform of a distribution function \( F \) on the positive real line known as frailty distribution (in short: \( \psi = LS[F] \) or \( F = LS^{-1}[\psi] \)); such \( \psi \) are completely monotone (that is \( (-1)^k \psi^{(k)}(t) \geq 0, t \in (0, \infty) \), for all \( k \in \mathbb{N}_0 \)) and many Archimedean generators fulfill this property (see below for examples). If \( \psi \in \Psi_\infty \) then

\[
\tilde{\psi}(t) = \frac{\psi(t + h)}{\psi(h)} = \int_0^\infty \exp(-tv) \frac{\exp(-hv)}{\psi(h)} dF(v) = \int_0^\infty \exp(-tv) d\tilde{F}(v),
\]

so that \( \tilde{F} = LS^{-1}[\tilde{\psi}] \). If \( F \) has density \( f \) (probability mass function \( (p_k)_{k \in \mathbb{N}} \)), then \( \tilde{F} \) has the exponentially tilted density \( \tilde{f} \) (probability mass function \( (\tilde{p}_k)_{k \in \mathbb{N}} \)) given by

\[
\tilde{f}(v) = \frac{\exp(-hv)}{\psi(h)} f(v), \quad v \in (0, \infty), \quad \left( \tilde{p}_k = \frac{\exp(-hk)}{\psi(h)} p_k, \quad k \in \mathbb{N} \right). \tag{4}
\]

This explains the “tilted” in the name of tilted Archimedean copulas. Note that for right-truncated Archimedean copulas, \( \psi(h) = C(t) \), in particular, neither the order of the truncation points nor whether they are all equal (as long as \( C(t) \) remains constant) affects \( h \) and thus how much \( \psi \) is tilted.

2) Knowing a sampling algorithm for the frailty distribution \( F \) is crucial for efficiently sampling Archimedean copulas with the so-called Marshall–Olkin algorithm; see Marshall and Olkin (1988). This is the case for many well-known Archimedean families; see, for example, Hofert (2010, Section 2.4). For sampling the corresponding tilted Archimedean copulas, one can in principle use rejection based on \( F \), however, the resulting rejection constant is \( 1/\psi(h) \), which can be large. Under the assumption that \( \psi^{1/m} \in \Psi_\infty \) for all \( m \in \mathbb{N} \) and that there is a sampling algorithm for \( \psi^{1/m} \), a fast rejection algorithm with rejection constant \( \log(1/\psi(h)) \) is derived in Hofert (2010, Section 4.2.1); see also Hofert (2011) and Hofert (2012). The algorithm is available in the \texttt{R} package \texttt{copula} for the implemented Archimedean families. This allows one to more efficiently sample from right-truncated Archimedean copulas in comparison to Algorithm 2.3 which becomes slow especially in large dimensions if at least one component of \( t \) is small.

3) As a consequence of Theorem 3.1 right-truncated Archimedean copulas – as Archimedean copulas – are exchangeable, independently of the choice of truncation point \( t \in (0,1]^d \). This is rather unexpected in the light of Proposition 2.5 because \( C(\cdot;t_{-,j}), \quad j = 1, \ldots, d \), are not all equal (and so aren’t \( C^{-1}[\cdot;t_{-,j}], \quad j = 1, \ldots, d \) unless all truncation points are equal). The reason why right-truncated Archimedean copulas are exchangeable follows from Step 3 in the proof of Theorem 3.1 since the summation over all \( j \) makes the sums \( \sum_{j \neq j} \tilde{\psi}^{-1}[t_j] \) (which can differ for different \( j \)'s) equal.

4) Another advantage of knowing that right-truncated Archimedean copulas are tilted Archimedean copulas is that we can easily obtain the limit for \( t \downarrow \mathbf{0} \) (that is, \( t_j \downarrow 0 \) for
at least one \( j \in \{1, \ldots, d\} \) under the assumption that \( \psi \in \text{RV}_{-1/\theta}^{\infty} \) for \( \theta > 0 \) (that is \( \psi \) is regularly varying at infinity with index \(-1/\theta\)). This result is known since Larsson and Nešlehová (2011) (Juri and Wüthrich (2002) and Charpentier and Segers (2007) considered the bivariate case and assumed equal truncation points). To see it through the lens of tilted Archimedean copulas, let \( \psi \in \text{RV}_{-1/\theta}^{\infty} \) for \( \theta > 0 \), let \( \tilde{\psi} \) be as in Theorem 3.1 and, without loss of generality, let \( h = \psi^{-1}[C(t)] > 0 \). Most importantly, we can now use that \( \tilde{\psi} \) and \( \tilde{\psi}(ht) \) generate the same copula and that

\[
\lim_{t \downarrow 0} \tilde{\psi}(ht) = \lim_{h \to \infty} \frac{\psi(ht + h)}{\psi(h)} = \lim_{h \to \infty} \frac{\psi(h(1 + t))}{\psi(h)} = (1 + t)^{-1/\theta},
\]

which is a generator of a Clayton copula with parameter \( \theta \). Therefore, \( \lim_{t \downarrow 0} C_t \) equals a Clayton copula with parameter \( \theta \) in this case. Note that this result is only of limited use as several well-known Archimedean generators are not regularly varying with negative index (see later) and thus a limiting Clayton copula is not an adequate model in these situations.

5) For a formula for Kendall’s tau for tilted Archimedean copulas, see Hofert (2010, Section 4.2.1).

The following corollary shows that if an Archimedean copula \( C \) admits a density (which is the case for many well-known examples), then so does its right-truncated version \( C_t \). Moreover, it also reveals that the density of \( C_t \) is numerically as tractable as that of \( C \), an important property for (log-)likelihood-based inference of right-truncated Archimedean (and thus Archimedean) copulas.

**Corollary 3.3 (Densities of right-truncated Archimedean copulas)**

Let \( C \) be a \( d \)-dimensional Archimedean copula with generator \( \psi \in \Psi_d \) that is \( d \) times continuously differentiable. Then the corresponding right-truncated copula \( C_t \) admits the density

\[
c_t(u) = \frac{\tilde{\psi}(d)(\sum_{j=1}^d \tilde{\psi}^{-1}[u_j])}{\prod_{j=1}^d \psi'(\tilde{\psi}^{-1}[u_j])}, \quad u \in (0, 1)^d,
\]

where \( \tilde{\psi}(d)(t) = \psi(d)(t + h)/\psi(h) \). Numerically stable proper logarithms of \( \psi(d) \) are available in many cases; see Hofert, Mächler, et al. (2012) or the \texttt{R} package \texttt{copula} for more details.

Archimedean copulas are especially of interest due to their ability to capture tail dependence. The following lemma provides formulas for the coefficients of tail dependence for right-truncated Archimedean copulas.

**Lemma 3.4 (Tail dependence for right-truncated Archimedean copulas)**

Let \( C \) be a bivariate Archimedean copula with generator \( \psi \in \Psi_2 \) and let \( t = (t_1, t_2) \in (0, 1)^2 \). Assuming the limits to exist, the coefficients of lower and upper tail dependence of the
right-truncated Archimedean copulas \( C_t \) are given by
\[
\lambda^C_{t_1} = \lim_{t \uparrow 1} \frac{\psi(2t + h)}{\psi(t + h)} \quad \text{and} \quad \lambda^C_{t_2} = 2 - \lim_{t \downarrow 0} \frac{1 - \psi(2t + h)/\psi(h)}{1 - \psi(t + h)/\psi(h)}.
\]

If \( \psi \) is differentiable and the limits exist, then
\[
\lambda^C_{t_1} = 2 \lim_{t \uparrow 1} \frac{\psi'(2t + h)}{\psi'(t + h)} \quad \text{and} \quad \lambda^C_{t_2} = 2 - 2 \lim_{t \downarrow 0} \frac{\psi'(2t + h)}{\psi'(t + h)}.
\]

**Proof.** For the coefficient of lower tail dependence, we have
\[
\lambda^C_{t_1} = \lim_{u \uparrow 0} \frac{C_t(u, u)}{u} = \lim_{u \uparrow 0} \frac{\bar{\psi}(2\bar{\psi}^{-1}[u])}{u} = \lim_{t \uparrow 1} \frac{\bar{\psi}(2t)}{\bar{\psi}(t)} = \lim_{t \uparrow 1} \frac{\psi(2t + h)}{\psi(t + h)}
\]
and, if \( \psi \) is differentiable, an application of l'Hôpital’s rule leads to
\[
\lambda^C_{t_1} = 2 \lim_{t \uparrow 1} \frac{\psi'(2t + h)}{\psi'(t + h)}.
\]

For the coefficient of upper tail dependence, we have
\[
\lambda^C_{t_2} = \lim_{u \uparrow 1} \frac{1 - 2u + C_t(u, u)}{1 - u} = \lim_{u \uparrow 1} \frac{1 - 2u + \bar{\psi}(2\bar{\psi}^{-1}(u))}{1 - u} = 2 - \lim_{u \uparrow 1} \frac{1 - \psi(2\psi^{-1}(u))}{1 - u}
\]
and, if \( \psi \) is differentiable, an application of l'Hôpital’s rule leads to
\[
\lambda^C_{t_2} = 2 - 2 \lim_{t \downarrow 0} \frac{\psi'(2t + h)}{\psi'(t + h)}.
\]

The formulas in Lemma 3.4 generalize those known for Archimedean copulas (take \( h = 0 \)). They allow us to derive the coefficients of tail dependence for any truncation point, regardless of whether the truncation takes place in a symmetric manner \((t_1 = t_2)\) or not. This simply comes from the fact that tilted Archimedean copulas are Archimedean copulas themselves and thus exchangeable, the truncation point only enters the generator as a parameter through the tilt \( h = \psi^{-1}[C(t)] \); compare with Example A.2.

In line with intuition, the following result shows that a majority of right-truncated Archimedean copulas satisfy \( \lambda^C_{t_1} = \lambda^C \) and \( \lambda^C_{t_2} = 0 \).

**Proposition 3.5 (Preserving lower tail dependence, cutting out the upper one)**
Let \( C \) be a bivariate Archimedean copula with generator \( \psi \in \Psi_2 \) and let \( t = (t_1, t_2) \in (0, 1]^2 \).

1) If \( C \) has coefficient of lower tail dependence \( \lambda^C \) and \( \lim_{t \uparrow 1} \frac{\psi(t)}{\psi(t + h/2)} \) exists, then \( \lambda^C_{t_1} \geq \lambda^C \).

Furthermore, if \( \psi \in RV_{-\alpha} \) for \( \alpha > 0 \), then \( \lambda^C_{t_1} = \lambda^C \).

2) If \( \psi \) is differentiable and \( t \neq (1, 1) \), then \( \lambda^C_{t_2} = 0 \).

**Proof.** 1) By Lemma 3.4,
\[
\lambda^C_{t_1} = \lim_{t \uparrow 1} \frac{\psi(2t + h)}{\psi(t + h)} = \lim_{t \uparrow 1} \frac{\psi(2(t + h/2))}{\psi(t + h/2)} \lim_{t \uparrow 1} \frac{\psi(t + h/2)}{\psi(t + h/2 + h/2)} = \lambda^C \lim_{t \uparrow 1} \frac{\psi(t)}{\psi(t + h/2)}.
\]
3 Right-truncated Archimedean copulas

In particular, since \( \psi(t + h/2) \leq \psi(t) \), we have \( \lambda_i^{Ct} \geq \lambda_i^C \). For the remaining part, let \( \varepsilon > 0 \) and note that for all \( t \) sufficiently large, \( \psi((1+\varepsilon)t) \leq \psi(t + h/2) \leq \psi(t) \). If \( \psi \in \text{RV}_- \) for \( \alpha > 0 \), we obtain that

\[
1 \leq \lim_{t \to \infty} \frac{\psi(t)}{\psi(t + h/2)} \leq \lim_{t \to \infty} \frac{\psi((1+\varepsilon)t)}{\psi((1+\varepsilon)t)} = \lim_{t \uparrow \infty} \frac{\psi((1+\varepsilon)t)}{\psi(t)} = \frac{1}{(1 + \varepsilon)^\alpha} = (1 + \varepsilon)^\alpha
\]

which converges to 1 for \( \varepsilon \downarrow 0 \) and thus \( \lambda_i^{Ct} = \lambda_i^C \).

2) If \( \psi \) is differentiable and \( t \neq (1, 1) \), then \( h = \psi^{-1}[C(t)] > 0 \) and thus \( \psi'(h) \in (-\infty, 0) \) which, by Lemma 3.4 implies that \( \lambda_i^{Ct} = 2 - 2\psi'(h)/\psi'(h) = 0 \). \( \square \)

Many properties of right-truncated Archimedean copulas follow immediately from the fact that they are tilted and thus Archimedean copulas. For example, the Kendall distribution \( K_{Ct}(u) = \mathbb{P}(C_t(U_t) \leq u) \) (for \( U_t \sim C_t \)) of a right-truncated Archimedean copula \( C \) with tilt \( h = \psi^{-1}[C(t)] \) is given by

\[
K_{Ct}(u) = \sum_{k=0}^{d-1} \left( \psi^{-1}[C(h)u] - h \right)^k \frac{(-1)^k \psi^{(k)}(\psi^{-1}[C(h)u])}{k! \psi(h)}
\]

\[
= \sum_{k=0}^{d-1} \left( \psi^{-1}[C(t)u] - \psi^{-1}[C(t)] \right)^k \frac{(-1)^k \psi^{(k)}(\psi^{-1}[C(t)u])}{k! C(t)}, \quad u \in [0, 1];
\]

the formulas of Barbe et al. (1996) and McNeil and Nešlehová (2009) are obtained for \( t = 1 \). Note that the appearing generator derivatives are known for several well-known Archimedean families, see Hofert, Mächler, et al. (2012), and so \( K_{Ct} \) can be computed for such families.

3.2 Examples

Let us now turn to specific examples of right-truncated Archimedean copulas.

Example 3.6 (Right-truncated Clayton copulas)

The generator \( \psi(t) = (1 + t)^{-\theta} \), \( \theta > 0 \), of a Clayton copula has inverse \( \psi^{-1}[u] = u^{-\theta} - 1 \). We thus obtain that

\[
\tilde{\psi}(t) = \frac{\psi(t + \psi^{-1}[C(t)])}{C(t)} = \frac{1 + t + \psi^{-1}[C(t)]^{-1/\theta}}{C(t)} = \frac{(t + C(t)^{-\theta})^{-1/\theta}}{C(t)}
\]

\[
= (1 + t/C(t)^{-\theta})^{-1/\theta} = \psi(t/C(t)^{-\theta}).
\]

Therefore, \( \tilde{\psi} \) is of the form \( \psi(ct) \) for some \( c > 0 \) and thus generates the same Archimedean copula as \( \psi \), namely a Clayton copula with parameter \( \theta \). In other words, right-truncated Clayton copulas are again Clayton copulas with the same parameter. It is then no surprise that the coefficients of tail dependence of \( C_t \) are those of the Clayton copula \( C \) in this case, which we can also easily verify from Proposition 3.3 since \( \psi \in \text{RV}_{-1/\theta} \) and \( \psi \) is differentiable.
Remark 3.7
The fact that right-truncated Clayton copulas are again Clayton copulas (even with the same parameter) is one reason for the popularity of Clayton copulas in insurance applications, although, as we showed in Theorem 3.1, all right-truncated Archimedean copulas are tilted Archimedean and thus enjoy the corresponding tractability. Furthermore, all generators in the remaining examples below are not regularly varying in the sense of Remark 3.2 and thus a limiting Clayton model is neither adequate, nor necessary to consider. For example, the following two examples will show that both right-truncated Ali–Mikhail–Haq and right-truncated Frank copulas are again Ali–Mikhail–Haq and Frank copulas, respectively (albeit with different parameters) and so one can work with an exact (instead of a limiting) model in these cases. Also right-truncated Gumbel and Joe copulas are tractable as we will see.

Example 3.8 (Right-truncated Ali–Mikhail–Haq copulas)
The generator \( \psi(t) = (1 - \theta)/(\exp(t) - \theta), \theta \in [0, 1) \), of an Ali–Mikhail–Haq copula has \( \tilde{\psi}(t) = \psi(t + h)/\psi(h) = (1 - e^{-h\theta})/\exp(t - e^{-h\theta}), h \geq 0 \), which is of the same form as \( \psi \) if \( \theta \) is replaced by \( e^{-h\theta} \). So unlike in the Clayton case, right-truncated Ali–Mikhail–Haq copulas are not the same Ali–Mikhail–Haq copulas, however, they are Ali-Mikhail-Haq copulas again, with parameter tilted by \( e^{-h} \). The frailty distribution for Ali–Mikhail–Haq copulas is the geometric distribution Geo(1 - \( \theta \)) on \( \mathbb{N} \). The frailty distribution for right-truncated Ali–Mikhail–Haq copulas is thus Geo(1 - \( e^{-h\theta} \)) on \( \mathbb{N} \), which could have also been derived from \( \Psi \); we demonstrate this in the next example for Frank copulas. Note that for \( h = \psi^{-1}[C(t)], e^{-h} = e^{-\psi^{-1}[C(t)]} = \frac{C(t)}{1 - \theta(1 - C(t))} \), so that right-truncated Ali–Mikhail–Haq copulas with truncation point \( t \) have frailty distribution Geo\( \frac{1 - \theta}{1 - \theta(1 - C(t))} \) on \( \mathbb{N} \) and thus can easily be sampled.

Example 3.9 (Right-truncated Frank copulas)
For \( p = 1 - e^{-\theta} \), the generator \( \psi(t) = -\log(1 - p \exp(-t))/\theta, \theta \in (0, \infty) \), of a Frank copula has corresponding logarithmic frailty distribution Log(\( p \)) with probability mass function \( p_k = p_k\theta/(\log(1 - p))k \) at \( k \in \mathbb{N} \); for a sampling algorithm, see the algorithm “LK” of Kemp (1981). Using that \( e^{-h} = e^{-\psi^{-1}[C(t)]} = (1 - e^{-\theta C(t)})/p \), it follows from (4) that the frailty distribution of a right-truncated Frank copula has probability mass function

\[
\tilde{p}_k = \frac{(pe^{-\theta})^k}{k \log(1 - p)C(t)} = \frac{(1 - e^{-\theta C(t)})^k}{k \theta C(t)} = \frac{\tilde{p}_k}{-k \log(1 - \tilde{p})} \quad \text{for} \quad \tilde{p} = 1 - e^{-\theta C(t)}.
\]

This is the probability mass function of a Log(\( \tilde{p} \)) distribution. We see that right-truncated Frank copulas are again Frank copulas, with parameter \( \theta \) replaced by \( \theta C(t) \), and thus can easily be sampled.

Example 3.10 (Right-truncated Gumbel copulas)
The generator \( \psi(t) = \exp(-t^{1/\theta}), \theta \geq 1 \), of a Gumbel copula has inverse \( \psi^{-1}[u] = (-\log u)^{\theta} \).
With \( h = \psi^{-1}[C(t)] \), we obtain that

\[
\tilde{\psi}(t) = \frac{\psi(t + \psi^{-1}[C(t)])}{C(t)} = \frac{\psi(t + h)}{\psi(h)} = \frac{\exp(-(t + h)^{1/\theta})}{\exp(-h^{1/\theta})} = \exp(-(t + h)^{1/\theta} - h^{1/\theta}),
\]

which is the Laplace–Stieltjes transform of the exponentially tilted stable distribution \( \tilde{S}(1/\theta, 1, \cos^\theta(\pi/(2\theta)), 1_{\{\theta = 1\}}, h 1_{\{\theta \neq 1\}}; 1 \); see Hofert (2010, Section 4.2.2), Hofert (2011), and the R package copula for more details about this distribution. In particular, this distribution and thus the corresponding right-truncated Gumbel copulas can be easily be sampled.

Gumbel copulas \( C \) are the only Archimedean extreme value copulas, so this example also shows that right-truncated Gumbel copulas \( C_t \) cannot be extreme value (unless \( t = 1 \); this is in line with Proposition 2.9). Also, by Lemma 3.4 and Proposition 3.5 C_t has no lower or upper tail dependence.

**Example 3.11 (Right-truncated Joe copulas)**

The generator \( \psi(t) = 1 - (1 - \exp(-t))^{1/\theta}, \theta \in [1, \infty) \), of a Joe copula has a Sibuya (Sib(1/\theta)) frailty distribution with probability mass function \( p_k = \binom{1/\theta}{k}(-1)^{k-1} \) at \( k \in \mathbb{N} \). An efficient sampling algorithm for Sib(1/\theta) distributions was introduced in Hofert (2011) Proposition 3.2; use \( \theta_0 = 1 \) there. By (4), the probability mass function of the frailty distribution of the corresponding right-truncated Joe copula is an exponentially tilted Sib(1/\theta) distribution with probability mass function \( \tilde{p}_k = \binom{1/\theta}{k}(-1)^{k-1}(1 - (1 - C(t))^{\theta})^k/C(t) \) at \( k \in \mathbb{N} \), which is not a Sibuya distribution anymore, so right-truncated Joe copulas are Archimedean but not Joe copulas anymore. However, an efficient sampling algorithm for the frailty distribution corresponding to \( (\tilde{p}_k)_{k \in \mathbb{N}} \) can be given; see the following algorithm (choose \( \alpha = 1/\theta \) and \( p = 1 - (1 - C(t))^{\theta} \) there).

**Algorithm 3.12 (Sampling exponentially tilted Sibuya distributions)**

Let \( \alpha \in (0, 1] \) and \( p_k^{\text{Sib}} = \binom{\alpha}{k}(-1)^{k-1}, k \in \mathbb{N} \), be the probability mass function of a Sib(\( \alpha \)) distribution with Laplace–Stieltjes transform \( \psi(t) = 1 - (1 - \exp(-t))^{\alpha} \). Let \( h > 0, p = e^{-h} \in (0, 1) \) and let \( \tilde{p}_k^{\log(p)} = p_k^{\log(p)}\exp(-\log(1 - p)k), k \in \mathbb{N} \), be the probability mass function of Log(p). Furthermore, let \( \tilde{F} \) be the distribution function with exponentially tilted Sib(\( \alpha \)) probability mass function \( \tilde{p}_k = \frac{(e^{-h})^k}{\psi(h)p_k} p_k^{\text{Sib}^{\alpha}} = \frac{p_k^\alpha}{1 - (1 - p)^\alpha} p_k^{\text{Sib}^{\alpha}}, k \in \mathbb{N} \), and Laplace–Stieltjes transform \( \tilde{\psi}(t) = \psi(t + h)/\psi(h) = \psi(t - \log(p))/\psi(-\log(p)) \). Then \( \tilde{V} \sim \tilde{F} \) can be sampled as follows.

1) If \( p \leq -\log((1 - p)^\alpha) \), independently sample \( V \sim \text{Sib}(\alpha), U \sim U(0, 1) \) until \( U \leq p^{V - 1} \) and then return \( \tilde{V} = V \).

2) And if \( p > -\log((1 - p)^\alpha) \), independently sample \( V \sim \text{Log}(p), U \sim U(0, 1) \) until \( Vp^{\text{Sib}^{\alpha}}/\alpha \) (see the proof for equivalent conditions) and then return \( \tilde{V} = V \).
4 Right-truncated copulas related to Archimedean copulas

The rejection constant of this algorithm overall is bounded above by $1/(1 - 1/e) \approx 1.5820$.

**Proof.** On the one hand,

$$p_k = \frac{p^k}{1 - (1 - p)^\alpha} p_k^{\text{Sib}(\alpha)} \leq \frac{p^k}{1 - (1 - p)^\alpha} \tilde{p}_k^{\text{Sib}(\alpha)}, \quad k \in \mathbb{N},$$

which implies that we can sample $\tilde{F}$ by rejection from a $\text{Sib}(\alpha)$ distribution with rejection constant $c^{\text{Sib}(\alpha)} = \frac{p^k}{1 - (1 - p)^\alpha} \geq 1$ (decreasing as a function of $p$, $1/\alpha$ for $p = 0$, 1 for $p = 1$) and acceptance condition $U c^{\text{Sib}(\alpha)} \tilde{p}_V^{\text{Sib}(\alpha)} \leq \tilde{p}_V$, equivalent to $U \leq p V^{-1}$. On the other hand, since

$$k p_k^{\text{Sib}(\alpha)} = k \left(\frac{\alpha}{k}\right) (-1)^{k-1} = k \alpha \prod_{j=1}^{k-1} (1 - \alpha/j) \leq \alpha, \quad k \in \mathbb{N},$$

we also have that

$$\tilde{p}_k = \frac{p^k}{1 - (1 - p)^\alpha} p_k^{\text{Sib}(\alpha)} \leq \frac{p^k}{1 - (1 - p)^\alpha} \tilde{p}_k^{\text{Sib}(\alpha)} = \frac{-\log((1 - p)^\alpha)}{1 - (1 - p)^\alpha} k p_k^{\text{Sib}(\alpha)} \leq \frac{-\log((1 - p)^\alpha)}{1 - (1 - p)^\alpha} \tilde{p}_k^{\text{Sib}(\alpha)}, \quad k \in \mathbb{N},$$

and so we can also sample $\tilde{F}$ by rejection from a $\text{Log}(p)$ distribution with rejection constant $c^{\text{Log}(p)} = \frac{-\log((1 - p)^\alpha)}{1 - (1 - p)^\alpha} \geq 1$ (maximal value $1/(1 - 1/e)$ is attained for $p = 1 - e^{-1/\alpha}$) and acceptance condition $U c^{\text{Log}(p)} \tilde{p}_V^{\text{Log}(p)} \leq \tilde{p}_V$, equivalent to $U \leq V p_V^{\text{Sib}(\alpha)} / \alpha = (\frac{\alpha}{\alpha-1})(-1)^V = (\frac{\alpha}{\alpha-1}) = \prod_{j=1}^{V-1} (1 - \alpha/j)$. Finally, note that $c^{\text{Sib}(\alpha)} \leq c^{\text{Log}(p)}$ if and only if $p \leq -\log((1 - p)^\alpha)$ in which case also $c^{\text{Sib}(\alpha)}$ is bounded above by $1/(1 - 1/e)$. \qed

4 Right-truncated copulas related to Archimedean copulas

In this section we cover the right-truncated copulas of outer power Archimedean copulas, Archimax copulas with logistic stable tail dependence function and nested Archimedean copulas.

4.1 Right-truncated outer power Archimedean copulas

If $\psi \in \Psi$, then $\psi(t) = \psi(t^\alpha) \in \Psi$ for all $\alpha \in (0,1]$. The Archimedean copulas generated by $\psi$ are known as outer power Archimedean copulas; see Nelsen (2006, Section 45.1) and Hofert (2010, Section 2.4). Outer power generators $\psi$ have become popular in applications due to the added flexibility through the additional parameter $\alpha$, which can be beneficial for modeling purposes; see Hofert and Scherer (2011) or Górecki et al. (2020). If the base generator $\psi$ is from Clayton’s family, the resulting outer power Clayton copula can reach any lower and upper tail dependence of interest.
4 Right-truncated copulas related to Archimedean copulas

By Theorem 3.1, the right-truncated copula $C_t$ of an outer power Archimedean copula with generator $\psi(t) = \psi(t^\alpha), \alpha \in (0,1]$, is (tilted outer power) Archimedean with generator

$$\tilde{\psi}(t) = \frac{\tilde{\psi}(t + \tilde{\psi}^{-1}[\tilde{C}(t)])}{\tilde{C}(t)},$$

where $\tilde{C}$ is the outer power Archimedean copula generated by $\tilde{\psi}$. If $F = \mathcal{L}S^{-1}[\psi]$ denotes the frailty distribution corresponding to $\psi$ and $V \sim F$, then $\tilde{F} = \mathcal{L}S^{-1}[\tilde{\psi}]$ has stochastic representation

$$\tilde{V} = S V^{1/\alpha},$$

where $S \sim S(\alpha, 1, \cos^{1/\alpha}(\alpha \pi / 2), 1\{\alpha = 1\}; 1)$ denotes the stable distribution with Laplace–Stieltjes transform $\psi_S(t) = \exp(-t^\alpha)$; see Hofert (2010, Theorem 4.2.6).

Under the assumptions stated in Remark 3.2 4), it is also straightforward to verify that $\lim_{t \downarrow 0} C_t$ for a right-truncated outer power Archimedean copula with parameter $\alpha \in (0,1]$ is a Clayton copula with parameter $\theta/\alpha$.

4.2 Right-truncated Archimax copulas with logistic stable tail dependence function

Archimax copulas, see Capéraà et al. (2000) and Charpentier, Fougères, et al. (2014), are copulas of the form

$$C(u) = \psi(\ell(\psi^{-1}[u_1], \ldots, \psi^{-1}[u_d])), \quad u \in [0,1]^d,$$

where $\psi \in \Psi_d$ and $\ell: [0, \infty)^d \rightarrow [0, \infty)$ is a stable tail dependence function; see Ressel (2013) and Charpentier, Fougères, et al. (2014) for a characterization of stable tail dependence functions. A popular family of stable tail dependence functions are logistic stable tail dependence functions $\ell(x) = \ell_\alpha(x) = \left(\sum_{j=1}^d x_j^{1/\alpha}\right)^\alpha, x \in [0, \infty)^d, \alpha \in (0,1]$; logistic stable tail dependence functions are the stable tail dependence functions of logistic copulas (extreme value copulas) also known as Gumbel copulas we already encountered.

The class of outer power Archimedean copulas is equivalent to the class of Archimax copulas with logistic stable tail dependence functions. By the above considerations we thus also know that right-truncated Archimax copulas with logistic stable tail dependence functions are tilted outer power Archimedean and thus Archimedean copulas again.

For Archimax copulas with other stable tail dependence functions, the corresponding right-truncated copulas may not be available analytically anymore as stable tail dependence functions are typically not componentwise analytically invertible unless in the (hierarchical) logistic case; see Example 4.3 3) below for the hierarchical case.
4 Right-truncated copulas related to Archimedean copulas

4.3 Right-truncated nested Archimedean copulas

In this section we consider nested Archimedean copulas of the form

$$C(u) = C_0(C_1(u_1), \ldots, C_S(u_S)) = \psi_0\left(\sum_{s=1}^{S} \psi_0^{-1}\left[\psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}[u_{sj}]\right)\right]\right), \quad (5)$$

where \(u = (u_1, \ldots, u_S) \in [0, 1]^d\) with \(u_s = (u_{s1}, \ldots, u_{sd_s})\), \(s \in \{1, \ldots, S\}\), and \(C_0, C_1, \ldots, C_S\) are Archimedean copulas of dimensions \(S, d_1, \ldots, d_S\) (such that \(\sum_{s=1}^{S} d_s = d\)) with generators \(\psi_0, \psi_1, \ldots, \psi_S \in \Psi_{\infty}\), respectively. We assume the sufficient nesting condition of McNeil (2008) to hold \(((\psi_0^{-1} \circ \psi_s)'\) being completely monotone for all \(s = 1, \ldots, S\)), so that \(C\) in (5) is guaranteed to be a proper copula. Furthermore, for ease of notation, let

\[t = (t_1, \ldots, t_S) = (t_{11}, \ldots, t_{1d_1}, \ldots, t_{S1}, \ldots, t_{Sd_S})\]

and, similarly to (1),

\[x_s \mapsto C(x_s; t_{-s}) = C(t_1, \ldots, t_{s-1}, x_s, t_{s+1}, \ldots, t_S).\]

The following result provides the right-truncated copulas of nested Archimedean copulas of form (5).

Theorem 4.1 (Right-truncated nested Archimedean copulas)

Let \(C\) be a \(d\)-dimensional nested Archimedean copula of form (5). For \(t \in (0, 1]^d\) such that \(C(t) > 0\), the right-truncated copula at \(t\), that is \(C_t(u)\), is given by

$$C_t(u) = \frac{\psi_0\left(\sum_{s=1}^{S} \psi_0^{-1}\left[\psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}[C(t)_{t_{sj}}] - \psi_0^{-1}[C(x_s; t_{-s})]\right)\right]\right)}{\psi_0\left(\sum_{s=1}^{S} \psi_0^{-1}\left[\psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}[C(t)_{t_{sj}}] - \psi_0^{-1}[C(x_s; t_{-s})]\right)\right]\right)}, \quad (6)$$

where \(\psi_0^{-1}[C(1; t_{-s})] = \psi_0^{-1}[C(t)] - \psi_0^{-1}[C_t(x_s)].\)

Proof. We have that

$$C(x_{sj}; t_{-sj}) = \psi_0\left(\sum_{s \neq s}^{S} \psi_0^{-1}[C_s(t_{s})] + \psi_0^{-1}\left[\psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}[t_{sj}] + \psi_0^{-1}[x_{sj}]\right)\right]\right)$$

and thus

$$C^{-1}[y_{sj}; t_{-sj}] = \psi_s\left(\psi_0^{-1}\left[\psi_0^{-1}[y_{sj}] - \psi_0^{-1}[C_s(t_{s})]\right]\right) - \sum_{j=1}^{d_s} \psi_0^{-1}[t_{sj}]$$

and

$$\begin{align*}
C^{-1}[y_{sj}; t_{-sj}] &= \psi_s\left(\psi_0^{-1}\left[\psi_0^{-1}[y_{sj}] - \psi_0^{-1}[C(1; t_{-s})]\right]\right) - \sum_{j=1}^{d_s} \psi_0^{-1}[t_{sj}]
\end{align*}.$$
where we used that $\sum_{s=1}^{S} \phi_{s}^{-1}(C(t_{s})) = \phi_{0}^{-1}(C_{t}(1; t_{s}))$. Using $y_{sj} = C(t)u_{sj}$, $s = 1, \ldots, S$, $j = 1, \ldots, d_{s}$, we obtain from Proposition 2.5 that $C_{t}(u)$ equals

$$
\frac{C(\{C^{-1}(C(t)u_{sj}; t_{-s})\})_{s,j}}{C(t)}
\psi_{0}\left(\sum_{s=1}^{S} \phi_{s}^{-1}\left[\left(\sum_{j=1}^{d_{s}} (\phi_{s}^{-1}(C(t)u_{s}) - \phi_{0}^{-1}(C_{t}(t_{-s}))\right) - \sum_{j=1}^{d_{s}} \phi_{s}^{-1}(t_{s,j})\right)\right]
= \frac{C(t)}{C(t)}
$$

Expanding the sum over $j$ and using the fact that

$$\sum_{j=1}^{d_{s}} \phi_{s}^{-1}(t_{s,j}) = \sum_{j=1}^{d_{s}} (d_{s} - 1) \phi_{s}^{-1}(t_{s,j}) = (d_{s} - 1) \phi_{s}^{-1}(C_{t}(t_{s})),$$

we obtain the form of $C_{t}$ as claimed. To see the representation for $\phi_{0}^{-1}(C_{t}(1; t_{-s}))$, note that $\phi_{0}^{-1}(C_{t}(1; t_{-s})) = \sum_{s=1}^{S} \phi_{s}^{-1}(C_{t}(t_{s})) = \sum_{s=1}^{S} \phi_{s}^{-1}(C_{t}(t_{s})) = \phi_{0}^{-1}(C_{t}) - \phi_{s}^{-1}(C_{t}(t_{s})).$

It is readily checked that $C_{t}$ in (6) is grounded. To verify that it has standard uniform univariate margins, consider the $s$th margin and let $u_{s} = 1$ for all $s \neq \tilde{s}$ or $j \neq \tilde{j}$. This implies that

$$\psi_{0}(\phi_{0}^{-1}(C_{t}(u_{s})) - \phi_{0}^{-1}(C_{t}(t_{-s}))
= \psi_{0}(\phi_{0}^{-1}(C_{t}(u_{s})) - \phi_{0}^{-1}(C_{t}) + \phi_{0}^{-1}(C_{t}(t_{s})))
= \begin{cases} 
\phi_{0}(\phi_{0}^{-1}(C_{t}(u_{s})) - \phi_{0}^{-1}(C_{t}) + \phi_{0}^{-1}(C_{t}(t_{s}))), & s = \tilde{s} \text{ and } j = \tilde{j}, \\
C_{t}(u_{s}), & s \neq \tilde{s} \text{ or } j \neq \tilde{j}.
\end{cases}$$

(7)

It then follows from (6) that

$$C_{t}(1, u_{\tilde{s}}), C(t)
= \psi_{0}\left(\sum_{s=1}^{S} \phi_{s}^{-1}\left[\psi_{s}\left(\sum_{j=1}^{d_{s}} (\phi_{s}^{-1}(C(t)u_{s}) - \phi_{0}^{-1}(C_{t}(t_{-s}))\right) + \phi_{0}^{-1}(C_{t}(t_{s}))\right)\right]
- (d_{s} - 1) \phi_{s}^{-1}(C_{t}(t_{s}))\right]\right)
= \psi_{0}\left(\sum_{s=1}^{S} (\phi_{0}^{-1}(C_{t}(t_{s})) + \phi_{0}^{-1}(C_{t}(t_{s}))\right)
+ \phi_{s}^{-1}(C(t)u_{\tilde{s}}) - \phi_{0}^{-1}(C_{t}(t_{s}))\right]\right)
+ \phi_{0}^{-1}(C_{t}(t_{s}))\right]\right)$$

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4 Right-truncated copulas related to Archimedean copulas

\[ - (d_\delta - 1) \psi_\delta^{-1} [C_\delta(t_\delta)] \]

\[ = \psi_0 \left( \sum_{s=1}^{S} \psi_0^{-1} [C_s(t_s)] + \psi_0^{-1} [\psi_0^{-1} [\psi_0^{-1} [\psi_0^{-1} \left( \psi_0^{-1} [C(t) u_{s_j}] - \psi_0^{-1} [C(t)] + \psi_0^{-1} [C_s(t_s)] \right)] \right] \right) \]

\[ = \psi_0 \left( \sum_{s=1}^{S} \psi_0^{-1} [C_s(t_s)] + \psi_0^{-1} [C(t) u_{s_j}] - \psi_0^{-1} [C(t)] + \psi_0^{-1} [C_s(t_s)] \right) \]

\[ = \psi_0 \left( \sum_{s=1}^{S} \psi_0^{-1} [C_s(t_s)] + \psi_0^{-1} [C(t) u_{s_j}] - \psi_0^{-1} [C(t)] \right) \]

\[ = \psi_0 \left( \psi_0^{-1} [C(t)] + \psi_0^{-1} [C(t) u_{s_j}] - \psi_0^{-1} [C(t)] \right) = \psi_0 \left( \psi_0^{-1} [C(t) u_{s_j}] \right) = C(t) u_{s_j}, \]

from which we correctly obtain that \( C_t(1, u_{s_j}, 1) = u_{s_j} \).

Based on (7), a similar but more tedious calculation can be done to derive the bivariate margins of (6).

**Corollary 4.2 (Bivariate margins of a right-truncated nested Archimedean copula)**

The copula \( C_t \) in (6) has bivariate margin \( C_t(1, u_{s_1}, 1, u_{s_2}, 1) \) given by

\[
\begin{align*}
\psi_0 \left( \tilde{h} + \psi_0^{-1} \left[ \psi_s \left( \sum_{s=1}^{S} \psi_0^{-1} \left[ \psi_0^{-1} [C(t) u_{s_j}] - h \right] \right) - \psi_0^{-1} [C_s(t_s)] \right] \right), & \quad s_1 = s_2 = s, \\
\tilde{\psi}_0 \left( \tilde{h} + \psi_0^{-1} [u_{s_1 j}] + \psi_0^{-1} [u_{s_2 j}], \right) & \quad s_1 \neq s_2,
\end{align*}
\]

where \( \tilde{h} = \psi_0^{-1} [C(t)] - \psi_0^{-1} [C_s(t_s)] \) if \( s_1 = s_2 = s \) and where \( \tilde{\psi}_0(t) = \psi_0(t + h) / \psi_0(h) \) with \( h = \psi_0^{-1} [C(t)] \) if \( s_1 \neq s_2 \). In particular, bivariate margins of right-truncated nested Archimedean copulas are tilted Archimedean copulas if the corresponding indices belong to different sectors. And the fact that this does not hold in general if they belong to the same sectors implies that a pairwise margin of a truncated nested Archimedean copula not necessarily equals the truncated corresponding pairwise margin of a nested Archimedean copula.

**Example 4.3 (Independent Archimedean copulas, exchangeable nested Archimedean copulas, hierarchical logistic stable tail dependence function)**

1) If \( C_0 \) is the independence copula (for example if \( \psi_0(t) = \exp(-t) \)), then

\[
\psi_0 \left( \psi_0^{-1} [C(t) u_{s_j}] - \psi_0^{-1} [C(1; \pi t)] \right) = \psi_0 \left( \psi_0^{-1} [C(t) u_{s_j}] - \psi_0^{-1} [C(t)] + \psi_0^{-1} [C_s(t_s)] \right) = C(t) u_{s_j} \frac{1}{C(t)} C_s(t_s) = C_s(t_s) u_{s_j}.
\]
so that
\[
C_t(u) = \frac{\psi_0\left(\sum_{s=1}^{S} \psi_0^{-1}\left[\psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}\left[\psi_s\left(C_s(t_s)u_{sj}\right]\right) - (d_s - 1)\psi_s^{-1}[C_s(t_s)]\right]\right)\right]}{C(t)},
\]
\[
= \frac{\prod_{s=1}^{S} \psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}\left[C_s(t_s)u_{sj}\right] - (d_s - 1)\psi_s^{-1}[C_s(t_s)]\right)}{C(t)},
\]
\[
= \frac{\prod_{s=1}^{S} \psi_s\left(\sum_{j=1}^{d_s} \psi_s^{-1}\left[C_s(t_s)u_{sj}\right] - (d_s - 1)\psi_s^{-1}[C_s(t_s)]\right)}{C_s(t_s)} = \prod_{s=1}^{S} C_s(t_s)(u_s),
\]

where \(C_s(t_s)\) denotes \(C_s\) truncated at \(t_s\); compare with Example 2.7.2. 2) If \(\psi_0 = \psi_1 = \ldots = \psi_S = \psi\), we obtain from (6) by canceling out compositions \(\psi^{-1} \circ \psi\) that \(C_t(u)\) equals
\[
= \frac{\psi\left(\sum_{s=1}^{S} \left(\sum_{j=1}^{d_s} \psi^{-1}[C(t)u_{sj}] - \psi^{-1}[C(t)] + \psi^{-1}[C_s(t_s)]\right) - (d_s - 1)\psi^{-1}[C_s(t_s)]\right)}{C(t)}
\]
\[
= \frac{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - d_s\psi^{-1}[C(t)] + \psi^{-1}[C_s(t_s)]\right)}{C(t)}
\]
\[
= \frac{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - \psi^{-1}[C(t)] + \psi^{-1}[C_s(t_s)]\right)}{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - d_s\psi^{-1}[C(t)] + \psi^{-1}[C_s(t_s)]\right)}
\]
\[
= \frac{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - \psi^{-1}[C(t)]\right)}{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - d_s\psi^{-1}[C(t)]\right)}
\]
\[
= \frac{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - \psi^{-1}[C(t)]\right)}{\psi\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}] - \psi^{-1}[C(t)]\right)}.
\]

This is a right-truncated copula as in Theorem 3.1, that is a tilted Archimedean copula, which is intuitive since taking equal generators in (5) results in an Archimedean copula.

3) If \(\psi_s(t) = \psi_s(t) = \psi(t^{\alpha_s})\), \(s = 0, 1, \ldots, S\), are outer power Archimedean generators with parameters \(\alpha_s \in (0, 1)\), \(s = 0, 1, \ldots, S\), satisfying \(\alpha_0 \geq \max\{\alpha_1, \ldots, \alpha_S\}\) (sufficient nesting condition), then, by (6), \(C_t(u)\) equals
\[
= \frac{\psi\left(\left(\sum_{s=1}^{S} \psi^{-1}[C(t)u_{sj}]\right)^{\frac{\alpha_0}{\alpha}} - \psi^{-1}[C(1; t_{-s})]\right)^{\frac{\alpha_0}{\alpha}} - (d_s - 1)\psi^{-1}[C_s(t_s)]^{\frac{\alpha_0}{\alpha}}}{C(t)}
\]
\[
\]
This is also the form of a right-truncated Archimax copula with hierarchical stable tail dependence function
\[
\ell(x) = \ell_{\alpha_0}(\ell_{\alpha_1}(x_1), \ldots, \ell_{\alpha_S}(x_S)) = \left(\sum_{s=1}^{S} \left(\sum_{j=1}^{d_s} \frac{1}{\alpha_s}\right)\right)^{\alpha_0}, \quad x = (x_1, \ldots, x_S) \in [0, \infty)^d,
\]
5 Conclusion

since the choice of generators $\psi_0, \psi_1, \ldots, \psi_S$ implies that

$$C(u) = \psi\left(\left(\sum_{s=1}^S \left(\sum_{j=1}^{d_s} \psi^{-1}[u_{sj}]^{\frac{1}{\alpha_s}}\right)_{\alpha_s}^0\right)_{\alpha_0}^0\right),$$

which is an Archimax copula with generator $\psi$ and hierarchical stable tail dependence function $\ell$.

**Example 4.4 (Right-truncated nested Clayton and Gumbel copulas)**

Figure 3 shows 5000 samples from truncated nested Clayton (left column) and truncated nested Gumbel (right column) copulas. The nested copulas are of the form $C(u) = C_0(u_{11}, C_1(u_{21}, u_{22}))$, where $C_0, C_1$ are from the respective Archimedean family with parameters $\theta_0, \theta_1$ chosen such that the corresponding Kendall’s taus are 0.5, 0.75, respectively. The truncation points are $t = (1, 1, 1)$ (no truncation; top row), $t = (0.2, 0.5, 0.5)$ and $t = (0.9, 0.9, 0.9)$ (middle row), and $t = (0.2, 0.1, 0.9)$ and $t = (0.5, 0.5, 0.5)$ (bottom row). As is clear from Corollary 4.2 and Example 3.6, for the right-truncated Clayton copulas, the bivariate margins of pairs with indices belonging to different sectors (so $(U_1, U_2)$ and $(U_1, U_3)$) do not change when changing the truncation point. We can also see that the within-sector margin (so $(U_2, U_3)$) is not much affected by a change of the truncation point. For the right-truncated Gumbel copulas, already moderate right-truncation will lead to weaker dependence as right-truncation especially affects the upper-right tail, in line with Proposition 3.5.

5 Conclusion

For $U$ following a copula $C$, we considered the copulas $C_t$ of $U \mid U \leq t$, termed right-truncated copulas with truncation point $t = (t_1, \ldots, t_d)$. In comparison to the existing literature, we focused on the case of a fixed truncation point $t \in (0, 1)^d$, so neither the case of equal truncation points $t = (t, \ldots, t)$ nor the limit for $t \downarrow 0$; some results for the former case can be found in the appendix. In this setup, we derived a formula for $C_t$ which is analytically tractable if $C$ is componentwise analytically invertible; see Proposition 2.5.

We then considered the case where $U$ follows an Archimedean copula and show that the family of copulas of $U \mid U \leq t$ can be characterized as tilted Archimedean; see Theorem 3.1. For various well-known Archimedean copulas (where a limiting Clayton model is not be adequate; see Remark 3.7), we were thus able to identify the corresponding right-truncated copulas; in particular, right-truncated Ali–Mikhail–Haq or Frank copulas, for example, are again Ali–Mikhail–Haq and Frank copulas, respectively. We also considered outer power Archimedean copulas $C$ (or Archimax copulas with logistic stable tail dependence function) and showed that their right-truncated copulas $C_t$ are tilted outer power Archimedean; see Sections 4.1 and 4.2. Furthermore, we derived the right-truncated copulas of nested Archimedean copulas; see Theorem 4.1.

One open problem that remains concerns exchangeability of bivariate margins of right-truncated copulas if not all truncation points are equal; see Example A.2.
Figure 3 \( n = 5000 \) pseudo-observations from nested Clayton (left column) and nested Gumbel (right column) copulas (of the form \( C(u) = C_0(u_{11}, C_1(u_{21}, u_{22})) \) with parameters \( \theta_0, \theta_1 \) of \( C_0, C_1 \) chosen such that the corresponding Kendall’s taus are 0.5, 0.75, respectively) right-truncated at the indicated points \( t \).
A Properties of right-truncated copulas

In this section we gather selected properties of right-truncated copulas, mainly in the case of equal truncation points, so $t = (t, \ldots, t)$ for some $t \in (0, 1]$.

The following result provides a sufficient condition for $C_t$ to be exchangeable.

**Corollary A.1 (Exchangeability)**

If $C$ is exchangeable, that is permutation symmetric in its arguments, and $t = (t, \ldots, t)$ for some $t \in (0, 1]$, then $F_t$ and thus $C_t$ are exchangeable.

**Proof.** If $\pi$ denotes a permutation of $\{1, \ldots, d\}$ and $x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(d)})$, then Lemma 2.1 implies that $F_t(x_\pi) = C(\min\{x_{\pi(1)}, t\}, \ldots, \min\{x_{\pi(d)}, t\}) = C_t(\min\{x_{\pi(1)}, t\}, \ldots, \min\{x_{\pi(d)}, t\})$ so that if $t = (t, \ldots, t)$ for some $t \in (0, 1]$, $F_t$ is exchangeable. By Hofert, Kojadinovic, et al. (2018, Proposition 2.5.5), $C_t$ is exchangeable. \qed

As we have already seen in Theorem 3.1 for Archimedean copulas (whose right-truncated copulas are tilted Archimedean copulas and thus exchangeable), exchangeable copulas $C$ can lead to exchangeable right-truncated copulas $C_t$ even for non-equal truncation points. The following example describes an open problem concerning exchangeability.

**Example A.2 (Right-truncated bivariate survival Gumbel copulas)**

Figure 4 shows 5000 pseudo-observations from bivariate survival Gumbel copulas $C(u_1, u_2) = -1 + u_1 + u_2 + \psi(\psi^{-1}[1 - u_1] + \psi^{-1}[1 - u_2])$ for $\psi(t) = \exp(-t^{1/\theta})$ with parameter $\theta = 2$ (Kendall’s tau equals 0.5), right-truncated at the truncation points as indicated. As we can see from the bottom row of Figure 4, it seems that such bivariate copulas are exchangeable. It remains an open problem to show this property mathematically. In three or more dimensions, non-equal truncation points of survival Gumbel copulas lead to non-exchangeable copulas, but each of their bivariate margins again seem to be exchangeable; see Figure 5.

The following result addresses the coefficients of tail dependence of right-truncated exchangeable copulas.

**Proposition A.3 (Tail dependence of exchangeable copulas right-truncated at equal truncation points)**

Let $C$ be a bivariate exchangeable copula with existing coefficient of lower and upper tail dependence $\lambda_l^C$ and $\lambda_u^C$, respectively. Furthermore, let $t = (t, t)$ for some $t \in (0, 1]$ such that $C(t) > 0$. Assuming the limits to exist, the coefficients of lower and upper tail dependence of the truncated copula $C_t$ at $t$ are given by

$$\lambda_l^{C_t} = \lim_{u \downarrow 0} \frac{C(u, u)}{C(u, t)} = \lambda_l^C \lim_{u \downarrow 0} \frac{u}{C(u, t)} \quad \text{and} \quad \lambda_u^{C_t} = 2 - \lim_{u \uparrow t} \frac{C(t, t) - C(u, u)}{C(t, t) - C(u, t)}.$$

In particular, $\lambda_l^{C_t} \geq \lambda_l^C$. 

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A Properties of right-truncated copulas

Figure 4 $n = 5000$ pseudo-observations from a survival Gumbel copula (with parameter such that Kendall’s tau equals 0.5) right-truncated at the indicated points $t$. 
A Properties of right-truncated copulas

Figure 5 \( n = 5000 \) pseudo-observations from a trivariate survival Gumbel copula (with parameters such that all pairwise Kendall’s tau equal 0.5) right-truncated at \( t = (0.05, 0.95, 0.4) \).

Furthermore, assuming the limits to exist,

\[
\lambda^C_{t l} = \frac{\lambda^C_l}{D_1 C(0, t)} \quad \text{and} \quad \lambda^C_{u t} = 2 - \frac{\delta^C_C(t)}{D_1 C(t, t)},
\]

where \( D_1 C(s, t) = \frac{\partial}{\partial u} C(u, t) \big|_{u=s} \) for \( s \in [0, t] \) and \( \delta^C_C(t) = \frac{\partial}{\partial u} \delta^C_C(u) \big|_{u=t} \) for \( \delta^C_C(u) = C(u, u) \).

**Proof.** Assuming the limits to exist, we have

\[
\lambda^C_{t l} = \lim_{u \downarrow 0} \frac{C_t(u, u)}{u} = \lim_{u \downarrow 0} \frac{C(C^{-1}[C(t)u; t], C^{-1}[C(t)u; t])}{C(t)u} = \lim_{v \downarrow 0} \frac{C(C^{-1}[v; t], C^{-1}[v; t])}{v} = \lim_{u \downarrow 0} \frac{C(u, u)}{C(u, t)} = \lim_{u \downarrow 0} \frac{C(u, u)}{C(u, t)}.
\]

If the coefficient of lower tail dependence \( \lambda^C_1 \) of \( C \) and the limit \( \lim_{u \downarrow 0} \frac{u}{C(u, t)} \) exist, then

\[
\lambda^C_{t l} = \lim_{u \downarrow 0} \frac{C(u, u)}{u} \lim_{u \downarrow 0} \frac{u}{C(u, t)} = \lambda^C_1 \lim_{u \downarrow 0} \frac{u}{C(u, t)}.
\]

To see that \( \lambda^C_{t l} \geq \lambda^C_1 \), note that \( C(u, t) \leq \min\{u, t\} = u \) for \( u \leq t \).
Now consider upper tail dependence. Similar as before, we obtain for $C(t) > 0$ that

$$
\lambda_u^{C_t} = \lim_{u \uparrow 1} \frac{1 - 2u + C_t(u, u)}{1 - u} = 2 - \lim_{u \uparrow 1} \frac{1 - C_t(u, u)}{1 - u}
$$

$$
= 2 - \lim_{u \uparrow 1} \frac{1 - C(C^{-1}[C(t)u]; t), C^{-1}[C(t)u])}{1 - u}
$$

$$
= 2 - \lim_{u \uparrow 1} \frac{C(t) - C(C^{-1}[v; t], C^{-1}[v; t])}{C(t) - v} = 2 - \lim_{u \uparrow 1} \frac{C(t, t) - C(u, u)}{C(t; t) - C(u, t)}.
$$

The final formulas for $\lambda_u^{C_t}$ and $\lambda_u^{L_t}$ follow from an application of l'Hôpital's rule. □

**Corollary A.4 (Tail dependence for truncated survival Archimedean copulas)**

Let $C$ be the survival copula of an Archimedean copula $C_\psi$ with generator $\psi$ satisfying $\psi'(0) = -\infty$; this is the case for all Archimedean copulas with completely monotone generators with upper tail dependence; see Embrechts and Hofert (2011). If $t = (t, t)$ for $t \in (0, 1]$, then $\lambda_t^{C_t} = \lambda_u^{C_\psi}$ and $\lambda_u^{C_t} = 0$.

**Proof.** We have $C(u, t) = -1 + u + t + \psi(\psi^{-1}[1 - u] + \psi^{-1}[1 - t])$, so that

$$
D_1 C(u, t) = 1 - \frac{\psi'(\psi^{-1}[1 - u] + \psi^{-1}[1 - t])}{\psi'(\psi^{-1}[1 - u])}
$$

and thus $D_1 C(0, t) = 1 - \psi'(0 + \psi^{-1}[1 - t]) / \psi'(0) = 1$. By Proposition A.3, $\lambda_t^{C_t} = \lambda_t^{C} = \lambda_u^{C_\psi}$.

Furthermore, $D_1 C(t, t) = 1 - \psi'(2\psi^{-1}[1 - t]) / \psi'(\psi^{-1}[1 - t])$ and $\lambda_t^{C_t} = 2 - 2\psi'(2\psi^{-1}[1 - t]) / \psi'(\psi^{-1}[1 - t]) = 2 D_1 C(t, t)$ so that, by Proposition A.3, $\lambda_t^{C_t} = 2 - 2 = 0$. □

For example, for the survival Gumbel copula $C$ of Example A.2 we obtain from Corollary A.4 that the corresponding right-truncated copula $C_t$ with equal truncation points has coefficients of lower and upper tail dependence given by $\lambda_u^{C_t} = \lambda_u^{C_\psi} = 2 - 2^{1/\theta} = 2 - \sqrt{2} \approx 0.5858$ and $\lambda_u^{C_t} = 0$, respectively; compare with the top row of Figure 4.

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