Decentralized Stochastic Variance Reduced Extragradient Method

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Abstract

This paper studies decentralized convex-concave minimax optimization problems of the form

$$\min_x \max_y f(x,y) \triangleq \frac{1}{m} \sum_{i=1}^m f_i(x,y),$$

where $m$ is the number of agents and each local function can be written as $f_i(x,y) = \frac{1}{n} \sum_{j=1}^n f_{i,j}(x,y)$. We propose a novel decentralized optimization algorithm, called multi-consensus stochastic variance reduced extragradient, which achieves the best known stochastic first-order oracle (SFO) complexity for this problem. Specifically, each agent requires $O\left( (n + \kappa \sqrt{n}) \log(1/\varepsilon) \right)$ SFO calls for strongly-convex-strongly-concave problem and $O\left( (n + \sqrt{nL}/\varepsilon) \log(1/\varepsilon) \right)$ SFO calls for general convex-concave problem to achieve $\varepsilon$-accurate solution in expectation, where $\kappa$ is the condition number and $L$ is the smoothness parameter.

The numerical experiments show the proposed method performs better than baselines.

1 Introduction

The minimax optimization problem has received increasing attention recently because of it contains lot of popular applications such as empirical risk minimization [41, 53], robust optimization [4, 9, 11, 37], AUC maximization [12, 49], fairness-aware machine learning [50], policy evaluation [8], PID control [13], game theory [3, 40] and etc.

We focus on decentralized first-order methods for convex-concave minimax problem where agents are connected by an undirected network. Decentralized optimization algorithm is more communication efficient than centralized one because of each agent is only allowed to access its neighbors [21], which reduces the communication cost on the busiest agent. For large-scale machine learning model, the number of training samples could be very large and we are interested in designing stochastic first-order (SFO) algorithms to reduce the computational cost. Such ideas have been widely used for solving minimization problem [14, 15, 20, 21, 31, 38, 43, 47]. Recently, researchers studied decentralized first-order algorithms for minimax optimization. Mukherjee and Chakraborty [28] proposed GT-ExtraGradient (GT-EG) algorithm for decentralized minimax problem, which is based on the classical extragradient method [18, 39] and gradient tracking technique [29, 32, 34]. GT-EG has linear convergence rate for unconstrained strongly-convex-strongly-concave minimax problem, but its iterations depend on expensive full gradient oracles. Beznosikov et al. [5] considered more general convex-concave minimax problem and proposed decentralized extra step method (DESM), which iterates with stochastic gradient oracle and achieves sub-linear convergence rate under bounded variance assumption. Additionally, Liu et al. [23], Xian et al. [42] studied minimax problem without convex-concave assumption.

In this paper, we propose a novel stochastic optimization algorithm, called multi-consensus stochastic variance reduced extragradient (MC-SVRE), which is designed by integrating the ideas of extragradient method [18, 39], gradient tracking [29, 33, 34], variance reduction [1, 2, 6, 7, 10, 16, 17, 19, 24, 26, 30, 36, 42, 47, 52] and multi-consensus [46, 48]. MC-SVRE achieves the best known SFO complexity for decentralized convex-concave minimax optimization problem. It is the first stochastic decentralized algorithm that enjoys linear convergence for strongly-convex-strongly-concave minimax problem and also works for general convex-concave case. Furthermore, we can derive a deterministic decentralized algorithm, called multi-consensus extragradient (MC-EG), by replacing the stochastic variance reduced gradient in MC-SVRE by full gradient. MC-EG requires more computational cost than MC-SVRE, while it (nearly) matches the communication and computational lower bound of deterministic first-order algorithms. We also conduct the numerical experiments on several machine learning applications to show the outperformance of proposed algorithms.

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Paper Organization In Section 2 we give the problem setting and preliminaries. In Section 3 we survey the previous work for decentralized minimax optimization. In Section 4 we propose MC-SVRE method and provide the complexity analysis. convex-concave assumption. In Section 5 we derive the deterministic algorithm MC-EG from MC-SVRE and provide the related theoretical results. In Section 6 we provide the empirical studies to show the effectiveness of proposed algorithms. Finally, we conclude this work in Section 7. All proofs are deferred to the appendix.

2 Problem Setting and Preliminaries

We study the following decentralized minimax optimization problem of the form

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) \overset{\triangle}{=} \frac{1}{m} \sum_{i=1}^{m} f_i(x,y),$$

where $m$ is the number of agents and $f_i(x, y)$ is the local function on $i$-th agent with $n$ components as follows

$$f_i(x, y) = \frac{1}{n} \sum_{j=1}^{n} f_{i,j}(x,y).$$

Our task is finding the saddle point of $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ of $f(x,y)$ which satisfies $$f(x^*, y) \leq f(x^*, y^*)$$

for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. For the ease of presentation, we also denote $z = [x; y] \in \mathbb{R}^d$ and $z^* = [x^*; y^*] \in \mathbb{R}^d$, where $d = d_x + d_y$.

We consider the following assumptions for problem (1).

Assumption 2.1. We suppose each $f_{i,j}(x,y)$ is $L$-smooth for $L > 0$, that is, we have

$$\|\nabla f_{i,j}(x,y) - \nabla f_{i,j}(x',y')\|^2 \leq L^2(\|x - x'\|^2 + \|y - y'\|^2).$$

for any $x, x' \in \mathbb{R}^{d_x}$ and $y, y' \in \mathbb{R}^{d_y}$.

Assumption 2.2. We suppose $f(x,y)$ is convex-concave, that is, we have

$$f(x',y) \geq f(x,y) + \langle \nabla_x f(x,y), x' - x \rangle$$

and

$$f(x,y') \leq f(x,y) + \langle \nabla_y f(x,y), y' - y \rangle$$

for any $x, x' \in \mathbb{R}^{d_x}$ and $y, y' \in \mathbb{R}^{d_y}$.

Assumption 2.3. We suppose $f(x,y)$ is $\mu$-strongly-convex-$\mu$-strongly-concave for $\mu > 0$, that is, the function defined as $f(x,y) - \frac{\mu}{2} \|x\|^2 + \frac{\mu}{2} \|y\|^2$ is convex-concave.

Assumption 2.4. We suppose $\mathcal{X} = \mathbb{R}^{d_x}$ and $\mathcal{Y} = \mathbb{R}^{d_y}$.

Assumption 2.5. We suppose $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ and $\mathcal{Y} \subseteq \mathbb{R}^{d_y}$ are convex and compact with diameter $D$, that is, we have $\|x - x'\|^2 + \|y - y'\|^2 \leq D^2$ for any $x, x' \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$.

Based on above assumptions, we will study decentralized minimax optimization for the following three cases:

(a) The objective function $f(x,y)$ is $\mu$-strongly-convex-$\mu$-strongly-concave and the problem is unconstrained (Assumption 2.1, 2.3 and 2.4).
We also introduce the aggregate gradient operator as

where $\bar{x}$ and $\bar{y}$ are local variables of the $i$-th agent. We also denote $z = [x, y] \in \mathbb{R}^{m \times d}$, where $z(i) = [x(i); y(i)] \in \mathbb{R}^{d}$ and $d = d_x + d_y$.

We define the gradient operator of $f(x, y)$ as

$$g(x, y) = [\nabla_x f(x, y); \nabla_y f(x, y)] \in \mathbb{R}^{d}.$$  

We also introduce the aggregate gradient operator as

$$g(z) = [g_1(z(1)), \cdots, g_m(z(m))]^\top \in \mathbb{R}^{m \times d},$$

where $g_i(z(i)) = [\nabla_x f_i(x(i), y(i)); -\nabla_y f_i(x(i), y(i))].$

We use the lower case with a bar to represents the mean vector for the rows of corresponding aggregate variable, such as $\bar{z} = m^{-1}1^\top z \in \mathbb{R}^{1 \times d}$. We denote $\bar{z}^*$ and $\bar{g}(\bar{z})$ as the transpose of $z^*$ and $g(\bar{z}^\top)$ respectively.

In decentralized optimization, the communication step is typically written as the matrix multiplication

$$z^{\text{new}} = W z^{\text{old}},$$

where $W \in \mathbb{R}^{m \times m}$ is the gossip matrix associate with $m$ agents and it holds the following assumption.

**Assumption 2.6.** We suppose the gossip matrix $W$ satisfies:

(a) $W \in \mathbb{R}^{m \times m}$ is symmetric with $W_{i,j} \neq 0$ if and only if $i$ and $j$ are connected or $i \neq j$;

(b) $0 \preceq W \preceq I$, $W1 = 1$ and $\text{null}(I - W) = \text{span}(1)$;

where $I$ is identity matrix and $1 = [1, \cdots, 1]^\top \in \mathbb{R}^m$.

We denote $\lambda_2(W)$ as the second largest eigenvalue of $W$. Note that Assumption 2.6 means $\lambda_2(W)$ is strictly less than one. Hence, we define $\chi \triangleq 1/(1 - \lambda_2(W))$.

Liu and Morse [22] proposed an efficient way to achieve average of local variables described in Algorithm 1 and it follows the convergence result.

**Lemma 2.1** (Liu and Morse [22, Lemma 2]). Under Assumption 2.6, Algorithm 1 holds

$$\frac{1}{m}1^\top z^{(K)} = \bar{z}^{(0)}$$

and

$$\|z^{(K)} - 1\bar{u}\| \leq \left(1 - \sqrt{1 - \lambda_2(W)}\right)^K \|z^{(0)} - 1\bar{z}^{(0)}\|,$$

where $\bar{z}^{(0)} = \frac{1}{m}1^\top z^{(0)}$ and $\|\cdot\|$ is the Frobenius norm.
We also denote \( Z = \{ z = [x; y] : x \in \mathcal{X} \text{ and } y \in \mathcal{Y} \} \) and define the projection from \( \mathbb{R}^{d_x+d_y} \) to \( Z \) as
\[
P_Z(z) = \arg \min_{z' \in Z} \| z - z' \|.
\]
Similarly, the corresponding projection operator for aggregate variable is defined as
\[
P_Z(z) = [P_Z(z(1)), \cdots, P_Z(z(m))]^\top.
\]

3 Related Work

The extragradient (EG) method \cite{korpelevich1976extragradient, Nazin2017} is the optimal \cite{zhang2019solving} batch gradient algorithm for convex-concave minimax problem on single machine, whose iteration is based on
\[
\begin{cases}
z_{t+1/2} = P_Z(z_t - \eta_1 g(z_t)), \\
z_{t+1} = P_Z(z_t - \eta_1 g(z_{t+1/2})), 
\end{cases}
\]
where \( \eta_1 = \Theta(1/L) \) is the stepsize.

Mukherjee and Chakraborty \cite{mukherjee2019extragradient} combined the idea of EG gradient with gradient tracking and proposed GT-Extragradient (GT-EG) algorithm for unconstrained decentralized minimax optimization problem \cite{chavdarova2020decentralized}. The update rule of GE-EG has the following compact form
\[
\begin{aligned}
        & z_{t+1/2} = z_t - \eta_2 s_t, \\
        & s_{t+1/2} = s_t + g(z_{t+1/2}) - g(z_t), \\
        & z_{t+1} = W z_t - \eta_2 s_{t+1/2}, \\
        & s_{t+1} = W s_t + g(z_{t+1}) - g(z_t),
\end{aligned}
\]
where \( \eta_2 = \Theta((\frac{1}{\kappa \sqrt{\lambda}}) \lor \frac{1}{\kappa \sqrt{\chi \sqrt{\lambda}}}) \lor \frac{1}{\kappa \sqrt{\chi \lambda}} ) \). The analysis shows GT-EG has linear convergence rate under Assumption 2.3 and 2.4 but it converges slower than EG on single machine since \( \eta_2 \) could be much smaller than \( \eta_1 \) when \( \kappa \) or \( \chi \) is large. Additionally, the iteration of GT-EG requires each agent to compute the full gradient of local function, which is very expensive for large-scale problems. Beznosikov et al. \cite{bezdosikov2021decentralized} proposed a stochastic algorithm for decentralized minimax optimization, called decentralized extra step method (DESM), which iterates with stochastic gradient to reduce the computational cost. DESM has sub-linear convergence rate under bounded variance assumption\cite{jiang2021desm}.

Variance reduction is a popular way to improve the convergence of stochastic optimization algorithms and it has been successfully used in large-scale minimax optimization \cite{chavdarova2020decentralized, bertsimas2020faster, chavdarova2020decentralized, chavdarova2020asynchronous, alacaoglu2021solving, alacaoglu2021decentralized}. Chavdarova et al. \cite{chavdarova2020decentralized} first incorporated variance reduction technique with EG algorithm. Later, Alacaoglu and Malitsky \cite{alacaoglu2021solving} proposed another variance reduced algorithm based on the framework of loopless SVRG \cite{chavdarova2020decentralized, bertsimas2020faster}, which achieves the optimal SFO complexity for single machine algorithms \cite{chavdarova2020decentralized}.

4 Multi-Consensus Stochastic Variance Reduced Extragradient

We present our multi-consensus stochastic variance reduced extragradient (MC-SVRE) method in Algorithm 2. Each agent iterate with local variance reduced gradient estimator
\[
v_{t+1/2}(i) = g_i(w_t(i)) + g_{i,j}(z_{t+1/2}(i)) - g_{i,j}(w_t(i))
\]
where \( w_t \) is the most recent iteration with \( g_{i,j}(\cdot) \) being exactly evaluated. We use multi-consensus (Algorithm 1) to achieve gradient estimators \( s_t \) and \( s_{t+1/2} \) for updating \( z_{t+1/2} \) and \( z_{t+1} \) respectively, resulting the following lemma.

\footnote{The settings for the analysis of DESM \cite{jiang2021desm} is different from ours. It depends on the assumption that we can access the stochastic gradient operator \( g_i(z; \xi) \) such that \( \mathbb{E}_\xi \| g_i(z) - g_i(z; \xi) \|^2 \leq \sigma^2 \), while this paper suppose each \( f_i \) has the finite-sum form of \cite{jiang2021desm}.}
Algorithm 2: Multi-Consensus Stochastic Variance Reduced Extragradient (MC-SVRE)

1: Initialize: \( \mathbf{w}_0 = \mathbf{z}_0 = [z_0^1; \ldots; z_0^d] \) with \( z_0 \in \mathcal{X} \times \mathcal{Y} \), \( \mathbf{v}_0 = \mathbf{s}_0 = \mathbf{0} \), \( p = 1/(2n) \), \( \alpha = 1 - p \).
2: \( \mathbf{v}_0 = \mathbf{s}_0 = \text{FastMix}(\mathbf{g}(\mathbf{z}_0), K) \)
3: for \( t = 0, 1, \ldots, T - 1 \) do
4: \( \mathbf{z}_t' = \alpha \mathbf{z}_t + (1 - \alpha)\mathbf{w}_t \)
5: \( \mathbf{s}_t = \text{FastMix}(\mathbf{s}_{t-1} + \mathbf{v}_t - \mathbf{v}_{t-1}, K) \)
6: \( \mathbf{z}_{t+1/2} = \text{FastMix}(\mathbf{P}_\mathcal{Z}(\mathbf{z}_t' - \eta \mathbf{s}_t), K) \)
7: parallel for \( i = 1, \ldots, m \) do
8: \( \mathbf{v}_{t+1/2}(i) = \mathbf{g}_i(\mathbf{w}_t(i)) + g_{i,j}(\mathbf{z}_{t+1/2}(i)) - g_{i,j}(\mathbf{w}_t(i)) \)
9: end parallel for
10: \( \mathbf{v}_{t+1/2} = \text{FastMix}(\mathbf{s}_t + \mathbf{v}_{t+1/2} - \mathbf{v}_t, K) \)
11: \( \mathbf{z}_{t+1} = \text{FastMix}(\mathbf{P}_\mathcal{Z}(\mathbf{z}_t' - \eta \mathbf{s}_{t+1/2}), K) \)
12: \( \begin{cases} \mathbf{w}_{t+1} = \mathbf{z}_{t+1}, & \mathbf{v}_{t+1} = \mathbf{g}(\mathbf{z}_{t+1}), & \mathbf{s}_{t+1} = \text{FastMix}(\mathbf{s}_t + \mathbf{v}_{t+1} - \mathbf{v}_t, K) \quad \text{with probability } p, \\ \mathbf{w}_{t+1} = \mathbf{w}_t, & \mathbf{v}_{t+1} = \mathbf{v}_t, & \mathbf{s}_{t+1} = \mathbf{s}_t \quad \text{with probability } 1 - p. \end{cases} \)
14: end for

Lemma 4.1. We suppose Assumption 2.1 holds. For Algorithm 2, it holds that \( \| \tilde{\mathbf{s}}_t - \tilde{\mathbf{g}}(\tilde{\mathbf{w}}_t) \| \leq \frac{1}{\sqrt{m}} \| \mathbf{w}_t - \mathbf{1}\tilde{\mathbf{w}}_t \| \) and \( \| \mathbb{E}[\tilde{\mathbf{z}}_{t+1/2}] - \tilde{\mathbf{g}}(\tilde{\mathbf{z}}_{t+1/2}) \| \leq \frac{1}{\sqrt{m}} \| \mathbf{z}_{t+1/2} - \mathbf{1}\tilde{\mathbf{z}}_{t+1/2} \| \) .

Lemma 4.1 indicates \( \tilde{\mathbf{s}}_t \) and \( \tilde{\mathbf{s}}_{t+1/2} \) are good gradient estimators at \( \tilde{\mathbf{z}}_t \) and \( \tilde{\mathbf{w}}_t \) respectively when the consensus error \( \| \mathbf{w}_t - \mathbf{1}\tilde{\mathbf{w}}_t \| \) and \( \| \mathbf{z}_{t+1/2} - \mathbf{1}\tilde{\mathbf{z}}_{t+1/2} \| \) are small. Hence, we can characterize the convergence of Algorithm 2 by showing how the mean variables \( \tilde{\mathbf{z}}_t \) and \( \tilde{\mathbf{w}}_t \) converge to \( \tilde{\mathbf{z}}^* \).

We also introduce the vector

\( \nu_t^2 = \left[ \| \mathbf{z}_t - \mathbf{1}\tilde{\mathbf{z}}_t \|^2, \| \mathbf{w}_t - \mathbf{1}\tilde{\mathbf{w}}_t \|^2, \eta^2 \| \mathbf{s}_t - \mathbf{1}\tilde{\mathbf{s}}_t \|^2 \right]^\top. \)

for analyzing the consensus error. We always denote that \( \rho \doteq (1 - \sqrt{1 - \lambda_2(W)})^K < 1 \), where \( K \) is the number of iterations in FastMix (Algorithm 1). For MC-SVRE (Algorithm 2), we use \( \mathbb{E}_t[\cdot] \) to present the expectation by fixing \( \mathbf{z}_0, \ldots, \mathbf{z}_t, \mathbf{w}_0, \ldots, \mathbf{w}_t \).

The remainder of this section provides the convergence analysis for proposed MC-SVRE (Algorithm 2) under different kinds of assumptions. Table 2 presents the detailed comparison of proposed methods with existing algorithms for decentralized minimax optimization.

4.1 Unconstrained Case

We first consider the unconstrained minimax problem with \( \mu \)-strongly-convex-\( \mu \)-strongly-concave assumption. In such case, projection steps in Algorithm 2 are unnecessary and Lemma 2.1 implies the update of mean variables can be written as

\[
\begin{aligned}
\mathbf{Z}_t' &:= \alpha \tilde{\mathbf{z}}_t + (1 - \alpha)\tilde{\mathbf{w}}_t, \\
\tilde{\mathbf{z}}_{t+1/2} &:= \mathbf{Z}_t' - \eta \tilde{\mathbf{s}}_t, \\
\tilde{\mathbf{z}}_{t+1} &:= \mathbf{Z}_t' - \eta \tilde{\mathbf{s}}_{t+1/2}.
\end{aligned}
\]

By imitating the analysis of centralized algorithms and considering the consensus error from decentralized setting, we obtain the following lemma.
Lemma 4.4. Under the settings of Lemma 4.3, define and establish the linear convergence of the weighed sum of \( \| \rho \) connecting Lemma 4.2 and 4.3 with sufficient small each iteration. Hence, MC-SVRE has \( O(K) \) by taking where \( \sqrt{\chi} \).

Suppose Assumption 2.1, 2.3 and 2.4 hold. For Algorithm 2 with \( C \) where \( \kappa \).

We also have the recursion of consensus error as follows.

**Lemma 4.3.** Under the settings of Lemma 4.2 and suppose \( \rho \leq 1/\sqrt{291} \). Then for Algorithm 2 holds that

\[
E_t \left[ V_{t+1} \right] \leq \left( 1 - \frac{1}{3n} \right) 1^{\top} r_t^2 + 4 \rho^2 m \delta_t.
\]

Note that the term related to \( \delta_t \) in (4) is typically ignored in the study of single machine convex-concave minimax optimization [27], while it is useful to the analysis of decentralized algorithms. Specifically, connecting Lemma 4.2 and 4.3 with sufficient small \( \rho \) and appropriate \( c_1 \), we can cancel the last term of (5) and establish the linear convergence of the weighed sum of \( \| \bar{z}_t - \bar{z}^* \|^2 \), \( \| \bar{w}_t - \bar{z}^* \|^2 \) and \( 1^{\top} r_t \).

**Lemma 4.4.** Under the settings of Lemma 4.3, define

\[
V_t = \| \bar{z}_t - \bar{z}^* \|^2 + \frac{c_1 \| \bar{w}_t - \bar{z}^* \|^2}{1 + \frac{\eta\mu}{c_1 p}} + \frac{c_2 1^{\top} r_t}{1 + \frac{\eta\mu}{c_1 p}} \]

where \( C = \frac{8L\eta(2\kappa + 3L\eta)}{m} \), \( c_1 = \frac{2\eta\mu + 4\eta}{\eta\mu + 4p} \) and \( c_2 = \frac{3}{p} \). Then Algorithm 2 holds that

\[
E_t \left[ V_{t+1} \right] \leq \left( 1 - \frac{1}{6(n + 4\kappa \sqrt{n})} \right) V_t.
\]

by taking \( K = \left[ \sqrt{\chi} \log(\max \{ \sqrt{12mnc_2}C, \sqrt{291} \}) \right] \).

The update rule of \( v_{t+1} \) indicates the full gradient \( g(z_{t+1}) \) is computed with probability \( p = 1/2n \) at each iteration. Hence, MC-SVRE has \( O(Tnp) = O(T) \) SFO complexity totally in expectation. Note that Algorithm 2 runs FastMix with \( K_0 \) iterations on \( g(z_0) \) in line 2, which is used to upper bound the term \( 1^{\top} r_0 \) in \( V_0 \). Finally, we obtain our main result for the unconstrained and strongly-convex-strongly-concave case.

| Algorithm | SFO Complexity | Communication Complexity | Reference |
|-----------|----------------|--------------------------|-----------|
| MC-SVRE   | \( O((n + \kappa \sqrt{n}) \log \left( \frac{1}{\rho} \right)) \) | \( O((n + \kappa \sqrt{n}) \sqrt{\log(\kappa) \log \left( \frac{1}{\rho} \right)}) \) | Algorithm 2 (Theorem 4.1) |
| MC-EG     | \( O(\kappa \log \left( \frac{1}{\rho} \right)) \) | \( O(\kappa \sqrt{\log \left( \frac{1}{\rho} \right)} \log \left( \frac{1}{\rho} \right)) \) | Algorithm 3 (Theorem 4.2) |
| GTEG      | \( O((\kappa^{4/3} \chi^{4/3} + \kappa \chi^2) \log \left( \frac{1}{\rho} \right)) \) | \( O((\kappa^{4/3} \chi^{4/3} + \kappa \chi^2) \log \left( \frac{1}{\rho} \right)) \) | Mukherjee and Chakraborty [28] |

Table 1: Complexities to obtain \( \| \bar{z} - \bar{z}^* \|^2 \leq \varepsilon \) for unconstrained and strongly-convex-strongly-concave case.

| Algorithm | SFO Complexity | Communication Complexity | Reference |
|-----------|----------------|--------------------------|-----------|
| MC-SVRE   | \( O((n + \kappa \sqrt{n}) \log \left( \frac{1}{\rho} \right)) \) | \( O((n + \kappa \sqrt{n}) \sqrt{\log(\kappa) \log \left( \frac{1}{\rho} \right)}) \) | Algorithm 2 (Theorem 4.2) |
| MC-EG     | \( O(\kappa \log \left( \frac{1}{\rho} \right)) \) | \( O(\kappa \sqrt{\log \left( \frac{1}{\rho} \right)} \log \left( \frac{1}{\rho} \right)) \) | Algorithm 3 (Theorem 5.2) |
| DESM      | \( O\left( \frac{\kappa^2}{\mu^2 \tau^2} \right) \) | \( O(\kappa \sqrt{\log(\tau) \log \left( \frac{1}{\rho} \right)}) \) | Beznosikov et al. [5] |

Table 2: Complexities to obtain \( \| \bar{z} - \bar{z}^* \|^2 \leq \varepsilon \) for constrained and strongly-convex-strongly-concave case.
Table 3: Complexities to obtain $\mathbb{E}[f(\hat{x}, y^*) - f(x^*, \hat{y})] \leq \varepsilon$ for constrained and convex-concave case

| Algorithm   | SFO Complexity            | Communication Complexity | Reference |
|-------------|---------------------------|--------------------------|-----------|
| MC-SVRE     | $O\left(n + \sqrt{nL}\right)$ | $O\left(\sqrt{nL} \log \left(\frac{n}{\varepsilon}\right)\right)$ | Algorithm 2 (Theorem 4.3) |
| MC-EG       | $O\left(\frac{n}{\varepsilon}\right)$ | $O\left(\frac{L^2}{\varepsilon} \log \left(\frac{1}{\varepsilon}\right)\right)$ | Algorithm 3 (Theorem 5.3) |
| DESM        | $O\left(\frac{L^2}{\varepsilon} \log \left(\frac{1}{\varepsilon}\right)\right)$ | $O\left(\frac{L^2}{\varepsilon}\right)$ | Beznosikov et al. [5] |

**Theorem 4.1.** Under the settings of Lemma 4.4, let

$$T = \left[6(n + 4\kappa \sqrt{n}) \log \left(\frac{6\|z_0 - z^*\|^2 + \varepsilon}{\varepsilon}\right)\right], \quad K_0 = \left[\sqrt{\gamma} \log \left(\frac{10\kappa}{3m\sqrt{nL}\varepsilon}\right) \left\|g(z_0) - \frac{1}{m}11^\top g(z_0)\right\|^2\right]$$

and suppose $n \geq 2$ for Algorithm 2. Then it requires at most $O\left((n + \kappa \sqrt{n}) \log \left(\frac{1}{\varepsilon}\right)\right)$ SFO calls on each agent in expectation and $O\left((n + \kappa \sqrt{n}) \sqrt{\log(kn) \log \left(\frac{1}{\varepsilon}\right)}\right)$ communication rounds to obtain $\mathbb{E}_T \|\hat{z}_T - \tilde{z}^*\|^2 \leq \varepsilon$.

### 4.2 The Constrained Case

For the constrained minimax problem, the updates rule of mean vector can be written as

$$\begin{cases} z_t' = \alpha \tilde{z}_t + (1 - \alpha)\bar{w}_t, \\ \tilde{z}_{t+1/2} = P(z_t' - \eta \bar{s}_t) + \Delta_t, \\ \tilde{z}_{t+1} = P(z_t' - \eta \bar{s}_{t+1/2}) + \Delta_{t+1/2}. \end{cases}$$

where $\Delta_{t+1/2} = \frac{1}{m}1^\top P(z_t' - \eta \bar{s}_{t+1/2}) - P(z_t' - \eta \bar{s}_t) + \Delta_t = \frac{1}{m}1^\top P(z_t' - \eta \bar{s}_t) - P(z_t' - \eta \bar{s}_t)$. Compared with update rule 11 for unconstrained case, we need to deal with the additional term $\Delta_t$ and $\Delta_{t+1/2}$.

For the strongly-convex-strongly-concave case, we establish the recursion relationship of $\tilde{z}_t$ and $\bar{w}_t$ as follows.

**Lemma 4.5.** Suppose Assumptions 2.1, 2.3 and 2.5 hold. For Algorithm 2 with $\eta = 1/(6\sqrt{nL})$, we have

$$\begin{align*} &\left(1 + \frac{\eta t}{2} - c_1p\right) \mathbb{E}_t \|\tilde{z}_{t+1} - z^*\|^2 + c_1 \mathbb{E}_t \|\bar{w}_{t+1} - \tilde{z}\|^2 \\ &\leq \left(1 - p\right) \|\tilde{z}_t - z^*\|^2 + (p + c_1(1 - p)) \|\bar{w}_t - \tilde{z}\|^2 + \zeta_t \end{align*}$$

where

$$\zeta_t = \frac{2L\eta(2\gamma + 3L\eta)}{m} \left(\mathbb{E}_t \|\tilde{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \|\bar{w}_t - 1\bar{w}_t\|^2\right)$$

$$\quad + \mathbb{E}_t \|\tilde{z}_{t+1} - z^*\|^2 + \mathbb{E}_t (\|\tilde{z}_{t+1/2} - z^*\|^2 + \|\Delta_{t+1/2}\|) + \mathbb{E}_t (\|\tilde{z}_{t+1/2} - z^*\|^2 + \|\Delta_{t+1/2}\| + \|\Delta_t\|).$$

Using the bounded assumption on $X \times Y$ and Lemma 2.1, we bound the term $\zeta_t$ as follows.

**Lemma 4.6.** Under the settings of Lemma 4.4, for Algorithm 2 with $K \geq \sqrt{\gamma} \log (2(\sqrt{m}LD + \delta')/\delta')$ and $\|s_0 - 1\bar{s}_0\| \leq \delta'$ for some $\delta' > 0$, it holds that

$$\zeta_t \leq \frac{4L\eta(2\gamma + 3L\eta)\delta'^2}{m} + \frac{6(C_1 + C_1/2)\delta}{\sqrt{m}}$$

where $C_1 = 2D + \frac{1}{\sqrt{m}L}2L^2D^2 + \frac{1}{m} \sum_{i=1}^m \|g_i(z^*)\|^2$ and $C_1/2 = D + \frac{1}{\sqrt{m}L}6L^2D^2 + \frac{1}{m} \sum_{i=1}^m \|g_i(z^*)\|^2$.

By appropriate setting of $K_0$, we can make $\delta'$ and $\zeta_t$ be sufficient small and derive the complexity of Algorithm 2 for constrained strongly-convex-strongly-concave case.
Algorithm 3 Multi-Consensus Extragradient (MC-EG)

1: Initialize: $z_0 = [z_0^1; \ldots; z_0^n]$ with $z_0 \in \mathcal{X} \times \mathcal{Y}$
2: $s_0 = \text{FastMix}(g(z_0), K)$
3: for $t = 0, 1, \ldots, T - 1$ do
4: $z_{t+1/2} = \text{FastMix}(P_{z_t} (z_t - \eta s_t), K)$
5: $s_{t+1/2} = \text{FastMix}(s_t + g(z_{t+1/2}) - g(z_t), K)$
6: $z_{t+1} = \text{FastMix}(P_z (z_t - \eta s_{t+1/2}), K)$
7: $s_{t+1} = \text{FastMix}(s_t + g(z_{t+1}) - g(z_t), K)$
8: end for

Theorem 4.2. Under the settings of Lemma 4.6, let

$$T = \left\lceil 6 \left( n + 4 \kappa \sqrt{n} \right) \log \left( \frac{10 \|z_0 - z^*\|^2}{\varepsilon} \right) \right\rceil$$

and $K_0 = \left\lceil \sqrt{n} \log \left( \frac{1}{\delta'} \left\| g(z_0) - \frac{1}{m} \nabla g(z_0) \right\| \right) \right\rceil$.

and suppose $n \geq 2$ for Algorithm 2. Then it requires at most $O \left( (n + \kappa \sqrt{n}) \log(1/\varepsilon) \right)$ SFO calls on each agent in expectation and $O \left( (n + \kappa \sqrt{n}) \sqrt{\log(n \kappa / \varepsilon)} \log(1/\varepsilon) \right)$ communication rounds to obtain $E_T \left\| \bar{z}_T - \bar{z}^* \right\| ^2 \leq \varepsilon$, where $\delta' = \min \left\{ \frac{\sqrt{1 + 16 L^2 n}}{4 m \kappa} \right\}$.

For the convex-concave case, we can upper bound the consensus error in the similar way to Lemma 4.6 and obtain the following result.

Theorem 4.3. Suppose Assumption 2.1, 2.2 and 2.5 hold. We let $\eta = 1 / (6 \sqrt{n} L)$, $T = \left\lceil 12 \sqrt{n} L \|z_0 - z^*\|^2 / \varepsilon \right\rceil$, $K = \left\lceil \sqrt{n} \log \left( 2 \left( \sqrt{n} L D + \delta' / \delta'' \right) \right) \right\rceil$ and

$$K_0 = \left\lceil \sqrt{n} \log \left( \frac{1}{\delta'} \left\| g(z_0) - \frac{1}{m} \nabla g(z_0) \right\| \right) \right\rceil$$

for Algorithm 2 where

$$\delta' = \min \left\{ \frac{m \eta \sqrt{\kappa}}{4 \sqrt{2 n L^2 + 16 \kappa \beta n L}}, \sqrt{\frac{m \eta \kappa}{4 \beta}} \right\}, \beta = \frac{2 D^2}{\varepsilon}$$

and $C_1, C_2$ follow definitions in Lemma 4.6. We let

$$(\hat{x}, \hat{y}) = \left( \frac{1}{T} \sum_{t=0}^{T-1} \hat{x}_{t+1/2}, \frac{1}{T} \sum_{t=0}^{T-1} \hat{y}_{t+1/2} \right)$$

as the output. Then it requires at most $O \left( (n + \sqrt{n} L / \varepsilon) \right)$ SFO complexity on each agent in expectation and $O \left( (n \sqrt{n} / \varepsilon) \log(n L / \varepsilon) \right)$ communication rounds to obtain $E_T \left[ f(\hat{x}, y^*) - f(x^*, \hat{y}) \right] \leq \varepsilon$.

5 The Deterministic Algorithm

By eliminating all randomness, MC-SVRE (Algorithm 2) reduces to the deterministic method shown in Algorithm 3. We name it as multi-consensus extragradient (MC-EG).

The analysis of MG-EG can directly follows the one of MC-SVRE with $n = 1$. Since the variable $w_t$ is unnecessary, we only need to analyze the consensus by the two-dimensional vector

$$\hat{r}_t^2 = \left[ \|z_t - 1 \bar{z}_t\|^2, \eta^2 \|s_t - 1 \bar{s}_t\|^2 \right] \top.$$

Both of the computation and communication complexities of MC-EG (nearly) match the lower bound for decentralized convex-concave minimax optimization [3]. The reminder of this section formally present the theoretical results for MC-EG in difference cases.
5.1 Unconstrained Case

Following the ideas in Section 4.1, we first establish the recursive relationship for $\|z_{t+1} - \bar{z}\|^2$ and $1^\top r_t$ as follows.

Lemma 5.1. Suppose Assumptions 2.1, 2.3, and 2.4 hold. For Algorithm 3 with $\eta = 1/(6L)$, we have

$$\|z_{t+1} - \bar{z}\|^2 \leq \left(1 - \frac{\mu - \frac{\mu L}{2}}{2}\right)\|z_{t} - \bar{z}\|^2 - \frac{5}{6} \|\bar{z}_{t+1/2} - \bar{z}_{t}\|^2 - \frac{1}{2} \|\bar{z}_{t+1/2} - \bar{z}_{t+1}\|^2 + \frac{4L\eta(3L\eta + 2\kappa)}{m}1^\top r_t.$$

Lemma 5.2. Under the settings of Lemma 5.1, we have

$$1^\top r_{t+1}^2 \leq 52\rho^21^\top r_t^2 + 2\rho^2m\delta_t,$$

where $\delta_t = \|\bar{z}_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{z}_t\|^2$.

Connecting Lemma 5.1 and Lemma 5.2, we have the following convergence result.

Lemma 5.3. Under the settings of Lemma 5.2, we let $\eta = 1/(6L)$ and $K = \left[\sqrt{\frac{\chi}{\delta}}\right]$ for Algorithm 3 where

$$\delta = \min \left\{\frac{1}{2}, \sqrt{\frac{1}{2}} \left[\frac{5}{6} - \frac{5\eta}{2}\right], \frac{3(2 - \eta)}{2(4\kappa + 157)}\right\}.$$

Then it holds that

$$\|z_{t+1} - \bar{z}\|^2 + \frac{1^\top r_{t+1}}{m} \leq \left(1 - \frac{1}{12\kappa}\right)\left(\|z_{t} - \bar{z}\|^2 + \frac{1^\top r_{t}^2}{m}\right).$$

By appropriate setting of $K_0$, we obtain the main result for MC-EG (Algorithm 3) in unconstrained strongly-convex-strongly-concave case, which indicates the algorithm has lower communication cost than MC-SVRE.

Theorem 5.1. Under the settings of Lemma 5.3, Algorithm 3 with

$$K_0 = \left[\frac{\sqrt{\chi}}{2} \log \left(\frac{1}{m\varepsilon} \left\|g(z_0) - \frac{1}{m} \text{11}^\top g(z_0)\right\|^2\right)\right]$$

and $T = \left[12\kappa \log \left(\frac{\|\bar{z}_0 - \bar{z}\|^2}{\varepsilon} + 1\right)\right]$ requires at most $O(\kappa n \log (1/\varepsilon))$ SFO complexity on each agent and $O(\kappa \sqrt{\chi} \log \kappa \log (1/\varepsilon))$ communication rounds to obtain $\|\bar{z}_T - \bar{z}\|^2 \leq \varepsilon$.

5.2 Constrained Case

The analysis of constrained case follows the one in Section 4.2. We present the results for the strongly-convex-strongly-concave case and the convex-concave case respectively in Theorem 5.2 and 5.3 respectively.

Theorem 5.2. Suppose Assumptions 2.1, 2.3, and 2.4 hold. We let $\eta = 1/(6L)$, $T = \left[12\kappa \log \left(2\|\bar{z}_0 - \bar{z}\|^2 / \varepsilon\right)\right]$, $K_0 = \left[\frac{\sqrt{\chi}}{2} \log \left(\left\|g(z_0) - \frac{1}{m} \text{11}^\top g(z_0)\right\|^2 / \delta'\right)\right]$ and $K = \left[\frac{\sqrt{\chi}}{2} \log \left(2(\sqrt{\kappa} + \delta') / \delta'\right)\right]$ for Algorithm 3 where

$$\delta' = \min \left\{\frac{m\varepsilon}{(6\kappa + 1)}, \frac{\sqrt{m\varepsilon}}{(384\kappa C_1)}\right\}$$

then we require at most $O(\kappa n \log (1/\varepsilon))$ SFO complexity on each agent and $O(\kappa \sqrt{\chi} \log \log (\kappa/\varepsilon)/(1/\varepsilon))$ communication rounds to obtain $\|\bar{z}_T - \bar{z}\|^2 \leq \varepsilon$.

Theorem 5.3. Suppose Assumptions 2.1, 2.3, and 2.4 hold. We let $\eta = 1/(6L)$, $K = \left[\frac{\sqrt{\chi}}{2} \log \left(2(\sqrt{\kappa} + \delta') / \delta'\right)\right]$, $K_0 = \left[\frac{\sqrt{\chi}}{2} \log \left(\left\|g(z_0) - \frac{1}{m} \text{11}^\top g(z_0)\right\|^2 / \delta'\right)\right]$ and $T = \left[12L \|\bar{z}_0 - \bar{z}\|^2 / \varepsilon\right]$ where

$$\delta' = \min \left\{\frac{mn\varepsilon/48}{\eta^2 + \beta\eta L^2 + \bar{C}_1 \sqrt{m\varepsilon}}, \frac{2D^2}{\varepsilon}\right\}, \beta = \frac{2D^2}{\varepsilon}$$

and $\bar{C}_1 = 2D + \frac{1}{12\kappa} \sqrt{\frac{2L^2 D^2 + \frac{2}{m} \sum_{i=1}^{m} \|g_i(z^*)\|^2}{\varepsilon}}$. Then we require $O(nL/\varepsilon)$ SFO complexity on each agent and $O((L/\sqrt{\kappa} / \varepsilon) \log (LD/\varepsilon))$ communication rounds to obtain $f(\hat{x}, y^*) - f(x^*, \hat{y}) \leq \varepsilon$, where $\hat{x} = \frac{1}{T} \sum_{t=0}^{T-1} \bar{x}_{t+1/2}$ and $\hat{y} = \frac{1}{T} \sum_{t=0}^{T-1} \bar{y}_{t+1/2}$.
Remark 5.1. Based on results summarized in Table 1–3 we observe that MC-EG is more communication efficient than MC-SVRE, but it has more computational cost. This comparison implies we can introduce a trade-off between computation and communication for MC-SVRE in practice. Specifically, we can replace $\nu_{t+1/2}(i)$ by the mini-batch variance reduced estimator as follows

$$g_i(\mathbf{w}_t(i)) + \frac{1}{l} \sum_{k=1}^{l} (g_{i,j_{i,k}}(\mathbf{z}_{t+1/2}(i)) - g_{i,j_{i,k}}(\mathbf{w}_t(i))) ,$$

where $l$ is the mini-batch size and each $j_{i,k}$ is sampled from $\{1, \ldots, n\}$ uniformly. The probability of computing the full gradient should be scaled into $p = 1/(2n)$ for such strategy.

6 Numerical Experiments

In this section, we provide numerical experiments on the applications of AUC maximization and distributionally robust optimization (DRO). We evaluate the performance by the norm of gradient operator $g(z)$ and the norm of gradient mapping $h(z) = \|z - P_{\mathcal{Z}}(z - \gamma g(z))\|/\tau$ for unconstrained and constrained problems respectively, where we set $\tau = 0.5$. All experiments are conducted in a random graph of ten nodes and the $10 \times 10$ gossip matrix related to this graph satisfies Assumption 2.6. Furthermore, our experiments of AUC maximization and distributionally robust optimization are conducted on two datasets: “a9a” and “w8a”. The samples of dataset are uniformly distributed into the ten agents. We compare the proposed MC-SVRE (Algorithm 2) and MC-EG (Algorithm 3) with existing algorithms GTEG [28] and DESM [3]. To achieve the fair comparison, we tune the parameters for all algorithms properly.

6.1 AUC Maximization

AUC maximization [12, 49] is targeted to find the binary classifier $\theta \in \mathbb{R}^d$ on training set $\{(a_{i,j}, b_{i,j})\}_{i,j}$ with $mn$ samples, where $a_{i,j} \in \mathbb{R}^d$, $b_{i,j} \in \{+1, -1\}$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We denote $N^+$ be the numbers of positive and negative instances and let $q = N^+/(mn)$. The unconstrained minimax formulation for this model is

$$\min_{x \in \mathbb{R}^{d+2}} \max_{y \in \mathbb{R}} f(x, y) \triangleq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x, y; a_{i,j}, b_{i,j}, \lambda),$$

where $x = [\theta; u; v] \in \mathbb{R}^{d+2}$, $\lambda$ is the regularization parameter and each component function $f_{i,j}(x, y; a_{i,j}, b_{i,j}, \lambda)$ is defined as

$$f_{i,j}(x, y; a_{i,j}, b_{i,j}, \lambda) \triangleq \frac{\lambda}{2} \|x\|_2^2 - q(1 - q)y^2 + (1 - q) ((\theta^T a_{i,j} - u)^2 - 2(1 + y)\theta^T a_{i,j}) I_{\{b_{i,j}=1\}}$$

$$+ q ((\theta^T a_{i,j} - v)^2 + 2(1 + y)\theta^T a_{i,j}) I_{\{b_{i,j}=-1\}}.$$

We set $\lambda = 0.01$ for our experiment.
6.2 Distributionally Robust Optimization

We consider the distributionally robust optimization with logistic loss and \( \ell_1 \)-ball constraint \([9, 44]\). Given a training set \( \{(a_{i,j}, b_{i,j})\}_{i,j} \) with \( mn \) samples where \( a_{i,j} \in \mathbb{R}^d \), \( b_{i,j} \in \{1, -1\} \), \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). The constrained minimax formulation for this model is

\[
\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) \triangleq \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j}(x, y),
\]

where \( f_{i,j}(x, y) = y_{i,j}l_{i,j}(x) + \frac{\lambda_2}{2} \|x\|^2 - V(y) \), \( l_{i,j}(x) = \log(1 + \exp(-b_{i,j}a_{i,j}^\top x)) \) and \( V(y) = \frac{\lambda_3}{2} \|mny - 1\|^2 \).

The constrained sets are \( \mathcal{X} = \{x \in \mathbb{R}^d : \|x\|_1 \leq \lambda_1\} \) and \( \mathcal{Y} = \{y \in \mathbb{R}^{mn} : 0 \leq y_i \leq 1, \sum_{i=1}^{mn} y_i = 1\} \). We set \( \lambda_1 = 1 \), \( \lambda_2 = 0.1 \) and \( \lambda_3 = 1/n^2 \) for our experiment.

6.3 Experimental Results and Discussion

We report our experimental results in Figure 1 and 2. We observe that MC-EG outperforms GTEG since GTEG lacks the multi-consensus step and each agent only communicates with its neighbors only once for each update of the variable. At the same time, the results show that MC-EG is both more communication efficient and more computation efficient than DESM. This is because of DESM does not include the gradient-tracking step, leading to the algorithm requires more communication rounds when it tries to obtain a high precision solution. On the other hand, the comparisons show that MC-SVRE achieves the lowest computation cost among all algorithms, but it commonly requires more communication cost than MC-EG, which implies MC-SVRE is suitable for computation sensitive cases. All of these empirical results validates our theoretical analysis.

7 Conclusion

In this paper, we have proposed a variance reduced extragradient method for decentralized convex-concave minimax optimization. We prove the algorithm achieves best known SFO complexity in theoretical. We also provide a deterministic variant for this method, which is more communication efficient. The numerical experiments on machine learning applications of AUC maximization and distributionally robust optimization validate the effectiveness of proposed algorithms. It would be interesting to extend our algorithms to more general case, such as nonconvex-concave or nonconvex-nonconcave minimax optimization problems.

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Appendices are organized as follows. In Section A, we provide some useful lemmas for our proofs. In Section B, we give the detailed proofs for Section 4.1. In Section C, we give the detailed proofs for Section 4.2. In Section D, we give the detailed proofs for Section 5. We always use $T(\cdot)$ to present the procedure of FastMix (Algorithm 1), that is

$$T(z) = \text{FastMix}(z, K).$$

Recall that we have defined

$$\rho = \left(1 - \sqrt{1 - \lambda_2(W)}\right)^K,$$

where $K$ is the number of iterations in Algorithm 1.

## A Some Useful Lemmas

We first provide some useful lemmas will be used in the analysis of MC-SVRE and MC-EG.

### Lemma A.1

For any $a_1, \ldots, a_m \in \mathbb{R}^d$, we have

$$\left\| \frac{1}{m} \sum_{i=1}^{m} a_i \right\|^2 \leq \frac{1}{m} \sum_{i=1}^{m} \|a_i\|^2.$$

**Proof.** We have

$$\frac{1}{m} \sum_{i=1}^{m} \|a_i\|^2 - \left( \frac{1}{m} \sum_{i=1}^{m} a_i \right)^2 = \frac{1}{m^2} \left( m \sum_{i=1}^{m} \|a_i\|^2 - \sum_{i=1}^{m} \sum_{j=1}^{m} \langle a_i, a_j \rangle \right)$$

$$= \frac{1}{m^2} \left( (m - 1) \sum_{i=1}^{m} \|a_i\|^2 - 2 \sum_{i \neq j} \langle a_i, a_j \rangle \right)$$

$$= \frac{1}{m^2} \left( (m - 1) \sum_{i=1}^{m} \|a_i\|^2 - 2 \sum_{i \neq j} \langle a_i, a_j \rangle \right)$$

$$= \frac{1}{m^2} \sum_{i \neq j} \|a_i - a_j\|^2 \geq 0.$$

### Lemma A.2 (Ye et al. [46, Lemma 3])

For any matrix $z \in \mathbb{R}^{m \times d}$, we have $\|z - 1\bar{z}\| \leq \|z\|$ where $\bar{z} = \frac{1}{m} 1^\top z$.

### Lemma A.3

Under Assumption 2.1, we have $\|g(z) - g(z')\| \leq L \|z - z'\|$ for any $z, z' \in \mathbb{R}^{m \times d}$.

**Proof.** Assumption 2.1 means $g_i$ is $L$-Lipschitz continuous. Then we have

$$\|g(z) - g(z')\|$$

$$= \left\| \begin{bmatrix} g_1(z(1))^\top \\ \vdots \\ g_m(z(m))^\top \end{bmatrix} - \begin{bmatrix} g_1(z'(1))^\top \\ \vdots \\ g_m(z'(m))^\top \end{bmatrix} \right\|^2$$

$$= \sum_{i=1}^{m} \|g_i(z(i)) - g_i(z'(i))\|^2$$
\[ L^2 \sum_{i=1}^{m} \| z(i) - z'(i) \|^2 = L^2 \| z - z' \|^2. \]

**Lemma A.4** (Rockafellar [35, Theorem 1]). For any \( z, z' \in \mathbb{R}^d \), we have
\[ \langle g(z) - g(z'), z - z' \rangle \geq 0 \]
under Assumption 2.2 and
\[ \langle g(z) - g(z'), z - z' \rangle \geq \mu \| z - z' \| \]
under Assumption 2.3.

**Lemma A.5.** For Algorithm 2, we have \( \bar{s}_t = \frac{1}{m} \mathbf{1}^\top g(w_t) \) and \( \mathbb{E}_t[\bar{s}_{t+1/2}] = \frac{1}{m} \mathbf{1}^\top g(z_{t+1/2}). \)

**Proof.** We prove this lemma by induction. For \( t = 0 \), we have
\[ \bar{s}_0 = \frac{1}{m} \mathbf{1}^\top s_0 = \frac{1}{m} \mathbf{1}^\top g(z_0) = \frac{1}{m} \mathbf{1}^\top g(w_0) \]
and
\[ \mathbb{E}_0[\bar{s}_{1/2}] = \frac{1}{m} \mathbf{1}^\top \mathbb{E}_0[s_{1/2}] = \frac{1}{m} \mathbf{1}^\top \mathbb{E}_t[s_0 + v_{1/2} - v_0] = \frac{1}{m} \mathbf{1}^\top \mathbb{E}_t[v_{1/2}] = \frac{1}{m} \mathbf{1}^\top g(z_{1/2}). \]
Suppose the statement holds for \( t \leq t' \). Then for \( t = t' + 1 \), Lemma 2.1 and induction base means
\[ \bar{s}_{t+1} = \bar{s}_t + \bar{v}_{t+1} - \bar{v}_t \]
\[ = \frac{1}{m} \mathbf{1}^\top g(w_t) + \frac{1}{m} \mathbf{1}^\top g(w_{t+1}) - \frac{1}{m} \mathbf{1}^\top g(w_t) \]
\[ = \frac{1}{m} \mathbf{1}^\top g(w_{t+1}), \]
and
\[ \mathbb{E}_t[\bar{s}_{t+1/2}] = \mathbb{E}_t[\bar{s}_t + \bar{v}_{t+1/2} - \bar{v}_t] \]
\[ = \mathbb{E}_t \left[ \frac{1}{m} \mathbf{1}^\top g(w_t) + \bar{v}_{t+1/2} - \frac{1}{m} \mathbf{1}^\top g(w_t) \right] \]
\[ = \mathbb{E}_t \left[ \bar{v}_{t+1/2} \right] \]
\[ = \frac{1}{m} \mathbf{1}^\top \mathbb{E}_t \left[ g_1(w_t(1))^T + g_{1,j_1}(z_{t+1/2}(1))^T - g_{1,j_1}(w_t(1))^T \right. \]
\[ \vdots \]
\[ \left. g_m(w_t(m))^T + g_{m,j_m}(z_{t+1/2}(m))^T - g_{m,j_m}(w_t(m))^T \right] \]
\[ = \frac{1}{m} \mathbf{1}^\top g(z_{t+1/2}). \]

**Lemma A.6.** For Algorithm 2, we have
\[ \| \mathbf{1}^\top z' - \bar{z} \| \leq \alpha \| z_t - 1\bar{z}_t \| + (1 - \alpha) \| w_t - 1\bar{w}_t \|. \]

**Proof.** The update rule of \( z'_t = \alpha z_t + (1 - \alpha) w_t \) means
\[ \| z'_t - 1\bar{z}_t \| = \| \alpha z_t + (1 - \alpha) w_t - 1(\alpha \bar{z}_t + (1 - \alpha) \bar{w}_t) \| \leq \alpha \| z_t - 1\bar{z}_t \| + (1 - \alpha) \| w_t - 1\bar{w}_t \|. \]
B  The Proof Details for Section 4.1

This section provide the detailed proofs for theoretical results of MC-SVRE in unconstrained case.

B.1  The Proof of Lemma 4.1

Proof. Assumption 2.1 and Lemma A.5 means

\[ \| \bar{s}_t - \bar{g}(\bar{w}_t) \|^2 = \left\| \frac{1}{m} \sum_{i=1}^{m} (g_i(w_t(i)) - g_i(w_t^T)) \right\|^2 \leq \frac{1}{m} \sum_{i=1}^{m} \| g_i(w_t(i)) - g_i(w_t^T) \|^2 \leq \frac{L^2}{m} \sum_{i=1}^{m} \| w_t(i) - w_t^T \|^2 = \frac{L^2}{m} \| w_t - 1w_t \|^2. \]

Similarly, we can also prove \( \mathbb{E}_t[\bar{v}_{t+1/2}] = \frac{1}{m} 1^T g(z_{t+1/2}). \) \( \square \)

B.2  The Proof of Lemma 4.2

Proof. Lemma 2.31 means

\[ \bar{z}_{t+1/2} = \frac{1}{m} 1^T (z_t' - \eta s_t) = \bar{z}_t' - \eta \bar{s}_t \quad \text{and} \quad \bar{z}_{t+1} = \frac{1}{m} 1^T (z_t' - \eta s_{t+1/2}) = \bar{z}_t' - \eta \bar{s}_{t+1/2}. \]

which implies

\[ 2 \langle \bar{z}_{t+1} - \bar{z}_t', \bar{z}^* - \bar{z}_{t+1} \rangle + 2 \langle \bar{z}_{t+1/2} - \bar{z}_t', \bar{z}_{t+1} - \bar{z}_{t+1/2} \rangle + 2\eta \langle \bar{s}_{t+1/2}, \bar{z}^* - \bar{z}_{t+1/2} \rangle + 2\eta \langle \bar{s}_{t+1/2} - \bar{s}_t, \bar{z}_{t+1/2} - \bar{z}_{t+1} \rangle = 0. \]  

(6)

Using the fact \( 2 \langle a, b \rangle = \| a + b \|^2 - \| a \|^2 - \| b \|^2, \) we have

\[ 2 \langle \bar{z}_{t+1} - \bar{z}_t', \bar{z}^* - \bar{z}_{t+1} \rangle = 2 \langle \bar{z}_{t+1} - \alpha \bar{z}_t - (1-\alpha)\bar{w}_t, \bar{z}^* - \bar{z}_{k+1} \rangle \]
\[ = 2\alpha \langle \bar{z}_{k+1} - \bar{z}_t, \bar{z}^* - \bar{z}_{k+1} \rangle + 2(1-\alpha) \langle \bar{z}_{t+1} - \bar{w}_t, \bar{z}^* - \bar{z}_{t+1} \rangle \]
\[ = \alpha \left( \| \bar{z}_t - \bar{z}^* \|^2 - \| \bar{z}_{t+1} - \bar{z}^* \|^2 - \| \bar{z}_{t+1} - \bar{z}_t \|^2 \right) + (1-\alpha) \left( \| \bar{w}_t - \bar{z}^* \|^2 - \| \bar{z}_{t+1} - \bar{z}^* \|^2 - \| \bar{z}_{t+1} - \bar{w}_t \|^2 \right). \]  

(7)

and

\[ 2 \langle \bar{z}_{t+1/2} - \bar{z}_t', \bar{z}_{t+1} - \bar{z}_{t+1/2} \rangle = \alpha \| \bar{z}_t - \bar{z}_{t+1} \|^2 + (1-\alpha) \| \bar{w}_t - \bar{z}_{t+1} \|^2 - \| \bar{z}_{t+1/2} - \bar{z}_t \|^2 + (1-\alpha) \| \bar{z}_{t+1} - \bar{z}_{t+1/2} \|^2 - \alpha \| \bar{z}_{t+1} - \bar{w}_t \|^2. \]

(8)
Additionally, we have
\[
2\mathbb{E}_t \left[ (\bar{s}_{t+1/2}, \bar{z}^* - \bar{z}_{t+1/2}) \right] \\
= 2\langle g(\bar{z}_{t+1/2}), \bar{z}^* - \bar{z}_{t+1/2} \rangle + 2\mathbb{E}_t [\bar{s}_{t+1/2}] - g(\bar{z}_{t+1/2}), \bar{z}^* - \bar{z}_{t+1/2} \\
\leq 2\langle g(\bar{z}^*), \bar{z}^* - \bar{z}_{t+1/2} \rangle - 2\mu \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\|^2 + 2\mathbb{E}_t [\bar{s}_{t+1/2}] - g(\bar{z}_{t+1/2}), \bar{z}^* - \bar{z}_{t+1/2} \\
\leq -\mu \mathbb{E}_t \left\| \bar{z}_{t+1} - \bar{z}^* \right\|^2 + 2\mu \mathbb{E}_t \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\|^2 + \frac{4}{\mu} \left\| \mathbb{E}_t [\bar{s}_{t+1/2}] - g(\bar{z}_{t+1/2}) \right\|^2 + \frac{\mu}{4} \left\| \bar{z}_{t+1/2} - \bar{z}^* \right\|^2 \\
\leq -\mu \mathbb{E}_t \left\| \bar{z}_{t+1} - \bar{z}^* \right\|^2 + 2\mu \mathbb{E}_t \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\|^2 + \frac{4L^2}{m\mu} \left\| \bar{z}_{t+1/2} - \bar{1}\bar{z}_{t+1/2} \right\|^2 + \frac{\mu}{2} \left\| \bar{z}_{t+1} - \bar{z}^* \right\|^2 \\ 
(9)
\]
where the first inequality follows Lemma \[A.1\] the second inequality is according to Lemma \[C.2\] and Young’s inequality; and the last inequality is based on Lemma \[4.1\].

Moreover, the update rule \( s_{t+1} = T(s_t + g(w_{t+1}) - g(w_t)), v_t = g(w_t) \) and Lemma \[2.1\] implies
\[
\mathbb{E}_t \left\| \bar{s}_{t+1/2} - \bar{s}_t \right\|^2 \\
= \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^T s_{t+1/2} - \frac{1}{m} \bar{1}^T s_t \right\|^2 \\
= \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^T \left( s_t + v_{t+1/2} - v_t \right) - \frac{1}{m} \bar{1}^T s_t \right\|^2 \\
= \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^T \left( v_{t+1/2} - v_t \right) \right\|^2 \\
= \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^T \left[ g_1(w_t(1))^T + g_1,j_1(z_{t+1/2}(1))^T - g_1,j_1(w_t(1))^T - g_1(w_t(1))^T \right] \right\|^2 \\
\leq \mathbb{E}_t \left\| \frac{1}{m} \sum_{i=1}^m \left\| g_{i,j_1}(z_{t+1/2}(i)) - g_{i,j_1}(w_t(i)) \right\|^2 \\
\leq \mathbb{E}_t \left\| \frac{L^2}{m} \sum_{i=1}^m \left\| z_{t+1/2}(i) - w_t(i) \right\|^2 \right\| \\
= \frac{L^2}{m} \mathbb{E}_t \left\| z_{t+1/2} - w_t \right\|^2 \\
\leq 3\frac{L^2}{m} \mathbb{E}_t \left[ \left\| z_{t+1/2} - \bar{1}\bar{z}_{t+1/2} \right\|^2 + \left\| 1\bar{z}_{t+1/2} - 1\bar{w}_t \right\|^2 + \left\| w_t - 1\bar{w}_t \right\|^2 \right] \\
= 3\frac{L^2}{m} \mathbb{E}_t \left\| \bar{z}_{t+1/2} - \bar{w}_t \right\|^2 + \frac{3L^2}{m} \mathbb{E}_t \left\| z_{t+1/2} - 1\bar{z}_{t+1/2} \right\|^2 + \frac{3L^2}{m} \mathbb{E}_t \left\| w_t - 1\bar{w}_t \right\|^2 
\]
where the first inequality use Lemma \[A.1\] the second inequality use \[A.3\] and the last inequality is due to the fact \( \left\| a + b + c \right\|^2 \leq 3 \left( \left\| a \right\|^2 + \left\| b \right\|^2 + \left\| c \right\|^2 \right) \).

By Young’s inequality, there also holds
\[
\mathbb{E}_t \left[ 2\eta \left\langle \bar{s}_{t+1/2} - \bar{s}_t, \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\rangle \right] \\
\leq 2\eta^2 \mathbb{E}_t \left\| \bar{s}_{t+1/2} - \bar{s}_t \right\|^2 + \frac{1}{2} \mathbb{E}_t \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\|^2 \\
\leq 6\eta^2 L^2 \mathbb{E}_t \left\| \bar{z}_{t+1/2} - \bar{w}_t \right\|^2 + \frac{6\eta^2 L^2}{m} \mathbb{E}_t \left\| z_{t+1/2} - 1\bar{z}_{t+1/2} \right\|^2 + \frac{6\eta^2 L^2}{m} \mathbb{E}_t \left\| w_t - 1\bar{w}_t \right\|^2 \\
+ \frac{1}{2} \mathbb{E}_t \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\|^2 \\ 
(10)
\]
Plugging results of (7), (8), (9) and (10) into inequality (6), we obtain that
\[\alpha \| \bar{z}_t - \bar{z}^* \|^2 + (1 - \alpha) \| \bar{w}_t - \bar{z}^* \|^2 - \mathbb{E}_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 - \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 \\
- \alpha \| \bar{z}_{t+1/2} - \bar{z}_t \|^2 - (1 - \alpha) \| \bar{z}_{t+1/2} - \bar{w}_t \|^2 \\
- \eta \mu \mathbb{E}_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 + \frac{5\eta \mu}{2} \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 + \frac{4L^2 \eta}{m \mu} \| \bar{z}_{t+1/2} - \bar{1} \bar{z}_{t+1/2} \|^2 + \frac{\eta \mu}{2} \| \bar{z}_{t+1} - \bar{z}^* \|^2 \\
+ 6\eta^2 L^2 \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{w}_t \|^2 + \frac{1}{2} \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 \\
+ \frac{6\eta^2 L^2}{m} \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{1} \bar{z}_{t+1/2} \|^2 + \frac{6\eta^2 L^2}{m} \mathbb{E}_t \| \bar{w}_t - \bar{1} \bar{w}_t \|^2 \geq 0.
\]
that is
\[\left(1 + \frac{\eta \mu}{2}\right) \| \bar{z}_{t+1} - \bar{z}^* \|^2 \leq \alpha \| \bar{z}_t - \bar{z}^* \|^2 + (1 - \alpha) \| \bar{w}_t - \bar{z}^* \|^2 - \left(\frac{1}{2} + \frac{5\eta \mu}{2}\right) \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 \\
- \alpha \| \bar{z}_{t+1/2} - \bar{z}_t \|^2 - (1 - \alpha - 6\eta^2 L^2) \| \bar{z}_{t+1/2} - \bar{w}_t \|^2 \\
+ \frac{2L\eta(2\kappa + 3L\eta)}{m} \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{1} \bar{z}_{t+1/2} \|^2 + \frac{6\eta^2 L^2}{m} \mathbb{E}_t \| \bar{w}_t - \bar{1} \bar{w}_t \|^2.
\]
On the other hand, by update rule of \( \bar{w}_{t+1} \) means
\[\mathbb{E}_t \| \bar{w}_{t+1} - \bar{z}^* \|^2 \]
\[= \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^\top (\bar{z}_{t+1} - \bar{z}^*) \right\|^2 \\
= (1 - p) \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^\top (\bar{z}_{t+1} - \bar{z}^*) \right\|^2 + (1 - p) \mathbb{E}_t \left\| \frac{1}{m} \bar{1}^\top \bar{w}_t - \bar{z}^* \right\|^2 \\
= (1 - p) \mathbb{E}_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 + (1 - p) \mathbb{E}_t \| \bar{w}_t - \bar{z}^* \|^2,
\]
where \( \bar{z}^* = [z^*, \ldots, z^*]^\top \in \mathbb{R}^{m \times d} \).
Combining the results of (11), (12) and the setting \( \alpha = 1 - p \), we have
\[\left(1 + \frac{\eta \mu}{2}\right) \mathbb{E}_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 + \mathbb{E}_t \| \bar{w}_{t+1} - \bar{z}^* \|^2 \\
\leq (1 - p) \| \bar{z}_t - \bar{z}^* \|^2 + p \| \bar{w}_t - \bar{z}^* \|^2 - \left(\frac{1}{2} + \frac{5\eta \mu}{2}\right) \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 \\
- (1 - p) \| \bar{z}_{t+1/2} - \bar{z}_t \|^2 - (p - 6\eta^2 L^2) \| \bar{z}_{t+1/2} - \bar{w}_t \|^2 \\
+ \mathbb{E}_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 + \mathbb{E}_t \| \bar{w}_t - \bar{z}^* \|^2 \\
+ \frac{2L\eta(2\kappa + 3L\eta)}{m} \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{1} \bar{z}_{t+1/2} \|^2 + \frac{6\eta^2 L^2}{m} \mathbb{E}_t \| \bar{w}_t - \bar{1} \bar{w}_t \|^2,
\]
which implies
\[
\begin{align*}
&\left(1 + \frac{\eta \mu}{2} - c_1 p \right) \mathbb{E}_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 + c_1 \mathbb{E}_t \| \bar{w}_{t+1} - \bar{z}^* \|^2 \\
\leq & (1 - p) \| \bar{z}_t - \bar{z}^* \|^2 + (p + c_1 (1 - p)) \| \bar{w}_t - \bar{z}^* \|^2 - \left( \frac{1}{2} - \frac{5 \eta \mu}{2} \right) \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 \\
&- (1 - p) \| \bar{z}_{t+1/2} - \bar{z}_t \|^2 - (p - 6 \eta^2 L^2) \| \bar{z}_{t+1/2} - \bar{w}_t \|^2 \\
&+ \frac{2 L \eta (2 \kappa + 3 L \eta)}{m} \mathbb{E}_t \| \bar{z}_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 + \frac{6 \eta^2 L}{m} \mathbb{E}_t \| \bar{w}_t - 1 \bar{w}_t \|^2 \\
\leq & (1 - p) \| \bar{z}_t - \bar{z}^* \|^2 + (p + c_1 (1 - p)) \| \bar{w}_t - \bar{z}^* \|^2 - \left( \frac{1}{2} - \frac{5 \eta \mu}{2} \right) \mathbb{E}_t \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 \\
&- (1 - p) \| \bar{z}_{t+1/2} - \bar{z}_t \|^2 - (p - 6 \eta^2 L^2) \| \bar{z}_{t+1/2} - \bar{w}_t \|^2 \\
&+ \frac{2 L \eta (2 \kappa + 3 L \eta)}{m} \mathbb{E}_t \| \bar{z}_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 + \mathbb{E}_t \| \bar{w}_t - 1 \bar{w}_t \|^2 \\
\leq & (1 - p) \| \bar{z}_t - \bar{z}^* \|^2 + (p + c_1 (1 - p)) \| \bar{w}_t - \bar{z}^* \|^2 - \frac{1}{3 \eta \delta_t} \\
&+ \frac{8 L \eta (2 \kappa + 3 L \eta)}{m} \left( \| \bar{z}_t - 1 \bar{z}_t \|^2 + \| \bar{w}_t - 1 \bar{w}_t \|^2 + \eta \| \bar{s}_t - 1 \bar{s}_t \|^2 \right),
\end{align*}
\]
\]

where the last step use \( \eta = 1/(6 \sqrt{n} L) \) and Lemma B.1 which leads to
\[
\begin{align*}
\| \bar{z}_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 + \| \bar{w}_t - 1 \bar{w}_t \|^2 \\
\leq & 3 \rho^2 \left( \| \bar{z}_t - 1 \bar{z}_t \|^2 + \| \bar{w}_t - 1 \bar{w}_t \|^2 + \eta^2 \| \bar{s}_t - 1 \bar{s}_t \|^2 \right) + \| \bar{w}_t - 1 \bar{w}_t \|^2 \\
\leq & 4 \left( \| \bar{z}_t - 1 \bar{z}_t \|^2 + \| \bar{w}_t - 1 \bar{w}_t \|^2 + \eta^2 \| \bar{s}_t - 1 \bar{s}_t \|^2 \right).
\end{align*}
\]

\]

### B.3 The Proof of Lemma B.3

We first introduce some lemmas for the consensus error of each variables.

**Lemma B.1.** Suppose Assumption \( \ref{as:1} \) holds. For Algorithm \( \ref{alg:1} \) we have
\[
\| \bar{z}_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 \leq \rho \left( \| \bar{z}_t - 1 \bar{z}_t \| + \| \bar{w}_t - 1 \bar{w}_t \| + \eta \| \bar{s}_t - 1 \bar{s}_t \| \right).
\]

**Proof.** Using Lemma B.1 and the update rule \( \bar{z}_t' = \alpha \bar{z}_t + (1 - \alpha) \bar{w}_t \), we have
\[
\| \bar{z}_{t+1/2} - 1 \bar{z}_{t+1/2} \| \\
= \left\| \mathbb{T}(\bar{z}_t' - \eta \bar{s}_t) - \frac{1}{m} 11^\top \mathbb{T}(\bar{z}_t' - \eta \bar{s}_t) \right\| \\
\leq \rho \left\| (\bar{z}_t' - \eta \bar{s}_t) - \frac{1}{m} 11^\top (\bar{z}_t' - \eta \bar{s}_t) \right\| \\
= \rho \left\| (\bar{z}_t' - \eta \bar{s}_t) - (1(\bar{z}_t' - \eta \bar{s}_t)) \| \\
\leq \rho \| \bar{z}_t' - 1 \bar{z}_t ' \| + \rho \eta \| \bar{s}_t - 1 \bar{s}_t \| \\
= \rho \left( \| \bar{z}_t - 1 \bar{z}_t \| + (1 - \alpha) \| \bar{w}_t - 1 \bar{w}_t \| + \eta \| \bar{s}_t - 1 \bar{s}_t \| \right) \\
\leq \rho \left( \| \bar{z}_t - 1 \bar{z}_t \| + \| \bar{w}_t - 1 \bar{w}_t \| + \eta \| \bar{s}_t - 1 \bar{s}_t \| \right).
\]

**Lemma B.2.** Suppose Assumption \( \ref{as:1} \) holds. For Algorithm \( \ref{alg:1} \) we have
\[
\| \bar{s}_{t+1/2} - 1 \bar{s}_{t+1/2} \| \leq \rho \left( \| \bar{s}_t - 1 \bar{s}_t \| + L \| \bar{z}_{t+1/2} - 1 \bar{z}_{t+1/2} \| + L \sqrt{m} \| \bar{w}_t - 1 \bar{w}_t \| + L \| \bar{w}_t - 1 \bar{w}_t \| \right).
\]

\[
\]

\]
Proof. We have
\[
\|s_{t+1/2} - 1 s_{t+1/2}\|
\]
\[
= \left\| T(s_t + v_{t+1/2} - v_t) - \frac{1}{m} 11^T T(s_t + v_{t+1/2} - v_t) \right\|
\]
\[
\leq \rho \| s_t + v_{t+1/2} - v_t - \frac{1}{m} 11^T (s_t + v_{t+1/2} - v_t) \|
\]
\[
\leq \rho \left( \| s_t - 1 s_t \| + \| v_{t+1/2} - v_t \| \right),
\]
where the first inequality uses Lemma 2.1 and the last one uses Lemma A.2.

Then we bound the last term as follows
\[
\| v_{t+1/2} - v_t \|^2
\]
\[
= \left\| \begin{bmatrix} g_1(w_t(1))^T + g_{1,j_1}(z_{t+1/2}(1))^T - g_1(w_t(1))^T \\
g_m(w_t(m))^T + g_{m,j_m}(z_{t+1/2}(m))^T - g_m(w_t(m))^T \\
\end{bmatrix} \right\|^2
\]
\[
= \sum_{i=1}^m \| g_{i,j_i}(z_{t+1/2}(i)) - g_{i,j_i}(w_t(i)) \|^2
\]
\[
\leq L^2 \sum_{i=1}^m \left( \| z_{t+1/2}(i) - z^* \|^2 + \| w_t(i) - z^* \|^2 \right)
\]
\[
= 2L^2 \left( \| z_{t+1/2} - 1 z^* \|^2 + \| w_t - 1 z^* \|^2 \right)
\]
\[
\leq L \left( \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \| + \| 1 \bar{z}_{t+1/2} - 1 \bar{w}_t \| + \| w_t - 1 \bar{w}_t \| \right)
\]
\[
= L \left( \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \| + \sqrt{m} \| \bar{z}_{t+1/2} - \bar{w}_t \| + \| w_t - 1 \bar{w}_t \| \right),
\]
where the first inequality use the Lipschitz continuity of $g_{ij}$ and the second inequality use Young’s inequality.

Combining the results of (14) and (15), we have
\[
\| s_{t+1/2} - 1 \bar{s}_{t+1/2} \| \leq \rho \left( \| s_t - 1 \bar{s}_t \| + L \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \| + L \sqrt{m} \| \bar{z}_{t+1/2} - \bar{w}_t \| + L \| w_t - 1 \bar{w}_t \| \right).
\]

Lemma B.3. Suppose Assumption 2.4 holds. For Algorithm 2, we have
\[
\| z_{t+1} - 1 \bar{z}_{t+1} \| \leq \rho \left( \| z_t - 1 \bar{z}_t \| + \| w_t - 1 \bar{w}_t \| + \| s_{t+1/2} - 1 \bar{s}_{t+1/2} \| \right).
\]
Proof. Using Lemma 2.1, Lemma A.2 and the update rule of $z'_t$, we have
\[
\| z_{t+1} - 1 \bar{z}_{t+1} \|
\]
\[
= \left\| T(z'_t - \eta s_{t+1/2}) - \frac{1}{m} 11^T T(z'_t - \eta s_{t+1/2}) \right\|
\]
\[
\leq \rho \left( \| z'_t - \eta s_{t+1/2} - \frac{1}{m} 11^T (z'_t - \eta s_{t+1/2}) \| \right)
\]
\[
\leq \rho \left( \| z'_t - \eta s_{t+1/2} - 1 (z'_t - \eta \bar{s}_{t+1/2}) \| \right)
\]
\[
\leq \rho \left( \| z'_t - 1 \bar{z}_{t} \| + \| s_{t+1/2} - 1 \bar{s}_{t+1/2} \| \right)
\]
\[
\leq \rho \left( \alpha \| z_t - 1 \bar{z}_t \| + (1 - \alpha) \| w_t - 1 \bar{w}_t \| + \| s_{t+1/2} - 1 \bar{s}_{t+1/2} \| \right).
\]
\[
\leq \rho \left( \left\| \mathbf{s}_t - \mathbf{1} \bar{z}_t \right\| + \left\| \mathbf{w}_t - \mathbf{1} \bar{w}_t \right\| + \eta \left\| \mathbf{s}_{t+1/2} - \mathbf{1} \bar{s}_{t+1/2} \right\| \right).
\]

Lemma B.4. For Algorithm 2, we have

\[
\mathbb{E}_t \left\| \mathbf{w}_{t+1} - \mathbf{1} \bar{w}_{t+1} \right\|_2^2 = \frac{1}{2n} \mathbb{E}_t \left\| \mathbf{z}_{t+1} - \mathbf{1} \bar{z}_{t+1} \right\|_2^2 + \left( 1 - \frac{1}{2n} \right) \mathbb{E}_t \left\| \mathbf{w}_t - \mathbf{1} \bar{w}_t \right\|_2^2.
\]

Proof. Using the update rule of \(\mathbf{z}_{t+1}\) and \(p = 1/(2n)\), we have

\[
\mathbb{E}_t \left\| \mathbf{w}_{t+1} - \mathbf{1} \bar{w}_{t+1} \right\|_2^2 = \frac{p \mathbb{E}_t \left\| \mathbf{z}_{t+1} - \mathbf{1} \bar{z}_{t+1} \right\|_2^2 + (1 - p) \mathbb{E}_t \left\| \mathbf{w}_t - \mathbf{1} \bar{w}_t \right\|_2^2}{2n}.
\]

Lemma B.5. Suppose Assumption 2.4 holds. For Algorithm 2, we have

\[
\mathbb{E}_t \left\| \mathbf{s}_{t+1} - \mathbf{1} \bar{s}_{t+1} \right\| \leq \rho \mathbb{E}_t \left\| \mathbf{s}_t - \mathbf{1} \bar{s}_t \right\| + \rho \mathbb{E}_t \left[ \left\| \mathbf{z}_{t+1} - \mathbf{1} \bar{z}_{t+1} \right\| + 2 \left\| \mathbf{z}_{t+1/2} - \mathbf{1} \bar{z}_{t+1/2} \right\| + \left\| \mathbf{w}_t - \mathbf{1} \bar{w}_t \right\| \right] + \rho L \mathbb{E}_t \left[ \left\| \mathbf{s}_{t+1} - \mathbf{1} \bar{s}_{t+1} \right\| + \left\| \mathbf{z}_{t+1/2} - \mathbf{1} \bar{z}_{t+1/2} \right\| + \left\| \mathbf{w}_t - \mathbf{1} \bar{w}_t \right\| \right].
\]

Proof. Using Lemma 2.1 and the update rule of \(\mathbf{s}'_t\) and \(\mathbf{w}_{t+1}\), we have

\[
\mathbb{E}_t \left\| \mathbf{s}_{t+1} - \mathbf{1} \bar{s}_{t+1} \right\| = \mathbb{E}_t \left\| \mathbf{s}_t + \mathbf{g}(\mathbf{w}_{t+1}) - \mathbf{g}(\mathbf{w}_t) \right\| + \frac{1}{m} \mathbb{E}_t \left\| \mathbf{s}_t + \mathbf{g}(\mathbf{w}_{t+1}) - \mathbf{g}(\mathbf{w}_t) \right\| \leq \rho \mathbb{E}_t \left\| \mathbf{s}_t - \mathbf{1} \bar{s}_t \right\| + \rho \mathbb{E}_t \left\| \mathbf{g}(\mathbf{w}_{t+1}) - \mathbf{g}(\mathbf{w}_t) \right\| + \rho \mathbb{E}_t \left\| \mathbf{w}_{t+1} - \mathbf{w}_t \right\|.
\]
We use the notations of
\[ r_t = \left\| \mathbf{z}_t - \mathbf{1} \tilde{z}_t \right\|_2 \quad \text{and} \quad r_t^2 = \left\| \mathbf{z}_t - \mathbf{1} \tilde{z}_t \right\|_2^2. \]

Then we provide the proof of Lemma 4.3.

**Proof.** Using Lemma B.1 we have
\[ \left\| \mathbf{z}_{t+1/2} - \mathbf{1} \tilde{z}_{t+1/2} \right\| \leq \rho \begin{bmatrix} 1 & 1 \end{bmatrix} r_t. \]

Using Lemma B.2 we have
\[
\begin{align*}
\left\| \mathbf{s}_{t+1/2} - \mathbf{1} \tilde{s}_{t+1/2} \right\| &\leq \rho \left( \begin{bmatrix} 0 & L & 1/\eta \end{bmatrix} r_t + L \left\| \mathbf{z}_{t+1/2} - \mathbf{1} \tilde{z}_{t+1/2} \right\| + L \sqrt{m} \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\| \right) \\
&= \rho \left( \begin{bmatrix} 0 & L & 1/\eta \end{bmatrix} r_t + \rho L \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} r_t + L \sqrt{m} \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\| \right) \\
&\leq \rho \left[ L \begin{bmatrix} 2L & L + 1/\eta \end{bmatrix} r_t + \rho L \sqrt{m} \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\| \right. \\
&\leq \rho \left[ L \begin{bmatrix} 2L & 3/(2\eta) \end{bmatrix} r_t + \rho L \sqrt{m} \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\| \right].
\end{align*}
\]

Note that \( \eta \leq 1/(4L) \). Using Lemma B.3 we have
\[
\begin{align*}
\left\| \mathbf{z}_{t+1} - \mathbf{1} \tilde{z}_{t+1} \right\| &\leq 4 \rho^2 \begin{bmatrix} 6.25 & 9 & 9 \end{bmatrix} r_t^2 + \rho^2 m \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\|^2 \\
&\leq \rho^2 \begin{bmatrix} 25 & 36 & 36 \end{bmatrix} r_t^2 + \rho^2 m \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\|^2.
\end{align*}
\]

Using Lemma B.4 we have
\[
\begin{align*}
\mathbb{E}_t \left\| \mathbf{w}_{t+1} - \mathbf{1} \tilde{w}_{t+1} \right\|_2^2 &\leq \rho \mathbb{E}_t \left\| \mathbf{z}_{t+1} - \mathbf{1} \tilde{z}_{t+1} \right\|_2^2 + (1 - \rho) \left\| \mathbf{w}_t - \mathbf{1} \tilde{w}_t \right\|_2^2 \\
&\leq \frac{1}{2n} \left( \rho^2 \begin{bmatrix} 25 & 36 & 36 \end{bmatrix} r_t^2 + \rho^2 m \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\|^2 \right) + \left( 1 - \frac{1}{2n} \right) \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} r_t^2 \\
&\leq \frac{25 \rho^2}{2n} \begin{bmatrix} 1 & 1 - 36 \rho^2 & 18 \rho^2 \end{bmatrix} \begin{bmatrix} 2n \end{bmatrix} r_t^2 + \frac{\rho^2 m}{2n} \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\|^2.
\end{align*}
\]

Using Lemma B.5 we have
\[
\begin{align*}
\mathbb{E}_t \left\| \mathbf{s}_{t+1} - \mathbf{1} \tilde{s}_{t+1} \right\| &\leq \rho \begin{bmatrix} 0 & L & 1/\eta \end{bmatrix} r_t + \rho L \begin{bmatrix} 2.5 & 3 & 3 \end{bmatrix} r_t + \rho \left( 1 - \frac{36 \rho^2}{2n} \right) \frac{18 \rho^2}{2n} r_t^2 \\
&+ \rho \begin{bmatrix} 2L & 4L & 1/\eta + 3L \end{bmatrix} r_t + \rho \left( 2L \begin{bmatrix} 2L & 0 \end{bmatrix} + \rho \left( 1 - \frac{36 \rho^2}{2n} \right) \frac{18 \rho^2}{2n} r_t^2 \right) \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\| \\
&+ \rho \begin{bmatrix} 2L & 2L \end{bmatrix} r_t + \rho \begin{bmatrix} 2L & 2L \end{bmatrix} \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\| + \left\| \tilde{z}_{t+1/2} - \tilde{w}_t \right\|.
\end{align*}
\]
\[ \leq \rho \left[ 4.5L \ 6L \ 1/\eta + 7L \right] r_t + 1.5\rho L \sqrt{mE_t} \left[ \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{w}_t\| \right], \]

that is

\[ \eta E_t \|s_{t+1} - 1s_{t+1}\| \leq \rho \left[ 9/8 \ 3/2 \ 11/4 \right] r_t + \frac{3}{8} \rho \sqrt{mE_t} \left[ \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{w}_t\| \right], \]

which implies

\[ \eta^2 E_t \|s_{t+1} - 1s_{t+1}\| \leq 4\rho^2 \left[ (9/8)^2 \ (3/2)^2 \ (11/4)^2 \right] r_t + \frac{36}{64} \rho^2 mE_t \left[ \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{w}_t\| \right]^2 \]

\[ \leq \rho^2 \left[ \frac{25}{16} \frac{36}{16} \frac{121}{4} \right] r_t + \frac{9}{8} \rho^2 mE_t \left[ \|z_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{w}_t\|^2 \right]. \]

Combing all above results, we have

\[ E_t \left[ r_{t+1}^2 \right] \leq \left[ \frac{25\rho^2}{2n} \frac{36\rho^2}{2n} \frac{18\rho^2}{n} \right] r_t^2 + \left[ \frac{\rho^2 m}{2n} \left| z_{t+1/2} - \bar{w}_t \right|^2 \right]. \]

Consider that \( \rho \leq 1/\sqrt{291} \), we have

\[ E_t \left[ 1^T r_{t+1}^2 \right] \leq \left( 1 \ - \ \frac{1}{3n} \right) 1^T r_t^2 + 4\rho^2 mE_t \left[ \|z_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{w}_t\|^2 \right]. \]

\[ \square \]

B.4 The Proof of Lemma 4.4

Proof. Using Lemma 4.2 and Lemma 4.3, we have

\[ \left( 1 + \frac{\eta \mu}{2} - c_1 \rho \right) E_t \|z_{t+1} - \bar{z}^*\|^2 + c_1 E_t \|\bar{w}_{t+1} - \bar{z}^*\|^2 \leq (1 - p) \|\bar{z}_t - \bar{z}^*\|^2 + (p + c_1 (1 - p)) \|\bar{w}_t - \bar{z}^*\|^2 - \frac{1}{3n} \delta_t + C1^T r_t^2 \]

and

\[ 1^T r_{t+1}^2 \leq \left( 1 - \frac{1}{3n} \right) 1^T r_t^2 + 4\rho^2 m\delta_t = \left( 1 - \frac{2p}{3} \right) 1^T r_t^2 + 4\rho^2 m\delta_t. \]

Hence, we combining above results, we obtain

\[ \left( 1 + \frac{\eta \mu}{2} - c_1 \rho \right) E_t \|z_{t+1} - \bar{z}^*\|^2 + c_1 E_t \|\bar{w}_{t+1} - \bar{z}^*\|^2 + c_2 C E_t \left[ 1^T r_{t+1} \right] \leq (1 - p) \|\bar{z}_t - \bar{z}^*\|^2 + (p + c_1 (1 - p)) \|\bar{w}_t - \bar{z}^*\|^2 - \frac{1}{3n} \delta_t + C1^T r_t^2 + c_2 C \left( 1 - \frac{2p}{3} \right) 1^T r_t^2 + 4c_2 C \rho^2 m\delta_t \]

\[ = (1 - p) \|\bar{z}_t - \bar{z}^*\|^2 + (p + c_1 (1 - p)) \|\bar{w}_t - \bar{z}^*\|^2 + \left( 1 + c_2 \left( 1 - \frac{2p}{3} \right) \right) C1^T r_t^2 - \left( \frac{1}{3n} - 4c_2 C \rho^2 m \right) \delta_t \leq (1 - p) \|\bar{z}_t - \bar{z}^*\|^2 + (p + c_1 (1 - p)) \|\bar{w}_t - \bar{z}^*\|^2 + \left( 1 + c_2 \left( 1 - \frac{2p}{3} \right) \right) C1^T r_t^2 \]

where the last step is due to the choice of \( K \) leads to \( \rho \leq 1/\sqrt{12mn\delta_2} \) and \( 1/(3n) \geq 4c_2 C \rho^2 m \).
We rewrite above inequality as
\[
\mathbb{E}_t \| \tilde{z}_{t+1} - \bar{z}^* \|^2 + \frac{c_1}{1 + \frac{\eta}{2} - c_1 p} \mathbb{E}_t \| \tilde{w}_{t+1} - \bar{z}^* \|^2 + \frac{c_2}{1 + \frac{\eta}{2} - c_1 p} \mathbb{E}_t \left[ \mathbf{1}^\top r_{t+1} \right] \leq \frac{1 - p}{1 + \frac{\eta}{2} - c_1 p} \| \tilde{z}_t - \bar{z}^* \|^2 + \frac{p + c_1 (1 - p)}{1 + \frac{\eta}{2} - c_1 p} \| \tilde{w}_t - \bar{z}^* \|^2 + \frac{1 + c_2 (1 - \frac{2p}{3})}{1 + \frac{\eta}{2} - c_1 p} \mathbf{C} \mathbf{1}^\top r_t^2. \tag{16}
\]

The settings \( c_1 = \frac{2n + 4p}{\eta + 4p} \) and \( \eta = \frac{1}{\sqrt{n}} \) mean
\[
\frac{1 - p}{1 + \frac{\eta}{2} - c_1 p} = 1 - \frac{\eta}{2} - p(c_1 - 1) = 1 - \frac{\eta}{2} - \frac{2n + 4p}{(\eta + 4p)(2n + 4p)} = 1 - \frac{2n + 4p}{(1 + \frac{\eta}{2})(\eta + 4p) - p(2\eta + 4p)} = 1 - \frac{(2 + \eta(\eta + 4p))}{2\eta + 4p} \leq 1 - \frac{\eta^2 \mu^2 + 2\eta \mu p}{2\eta + 4p} \leq 1 - \frac{\eta^2 \mu^2 + 2\eta \mu p}{3\eta \mu + 8p}
\]

and
\[
\frac{p + c_1 (1 - p)}{c_1} = 1 - \frac{p(c_1 - 1)}{c_1} = 1 - \frac{(2n + 4p)}{2n + 4p} = 1 - \frac{\eta \mu}{\eta + 4p} = 1 - \frac{1}{2} \frac{1 + \frac{\eta}{2}}{\eta + 4p} = 1 - \frac{1}{4n + 24k\sqrt{n}}.
\]

Additionally, the value of \( c_2 = \frac{3p}{6n} \) means
\[
1 + \frac{c_2 (1 - \frac{2p}{3})}{c_2} = 1 + \frac{3p}{6n} \left( 1 - \frac{2p}{3} \right) = 1 - \frac{p + \frac{1}{3}}{3} = 1 - \frac{1}{6n}.
\]

Hence, the inequality \( 16 \) implies
\[
\mathbb{E}_t [V_{t+1}] \leq \max \left\{ 1 - \frac{1}{24k\sqrt{n}}, 1 - \frac{1}{n + 24k\sqrt{n}}, 1 - \frac{1}{6n} \right\} V_t \leq \left( 1 - \frac{1}{6(n + 4\sqrt{n})} \right) V_t.
\]

\[
\Box
\]

**B.5 The Proof of Theorem 4.1**

**Proof.** Note that
\[
1 + \frac{\eta \mu}{\eta \mu + 4p} \leq c_1 = \frac{2\eta \mu + 4p}{\eta \mu + 4p} \leq \frac{2\eta \mu + 8p}{\eta \mu + 4p} = 2
\]

which means
\[
1 + \frac{\eta \mu}{\eta \mu + 4p} \leq \frac{2}{1 + \frac{\eta}{2} - 2p} = \frac{2}{1 + \frac{1}{12\sqrt{n}} - \frac{1}{n}} \leq 4
\]

We also have
\[
C = \frac{8q (2\kappa + 3L\eta)}{m} = \frac{8\sqrt{n} (2\kappa + \frac{3L}{\sqrt{n}})}{m} = \frac{2(4\kappa\sqrt{n} + 1)}{3mnL}
\]

and
\[
\frac{c_2 C}{1 + \frac{\eta}{2} - c_1 p} \leq \frac{12(4\kappa\sqrt{n} + 1)}{m} \leq \frac{24(4\kappa\sqrt{n} + 1)}{m} \leq \frac{120\kappa\sqrt{n}}{m}
\]

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Using above inequalities, we have
\[ V_t = \| \bar{z}_t - \bar{z}^* \|^2 + \frac{c_1}{1 + \frac{\eta}{2} - c_1 p} \| \bar{w}_t - \bar{z}^* \|^2 + \frac{c_2}{1 + \frac{\eta}{2} - c_1 p} C 1^\top r_t^2. \]
Since \( z_0 = w_0 = 1 \bar{z}_0 = 1 \bar{w}_0 \), we have
\[ 1^\top r_0^2 = \| z_0 - 1 \bar{z}_0 \|^2_2 + \| w_0 - 1 \bar{w}_0 \|^2_2 + \frac{\eta \| s_0 - 1 \bar{s}_0 \|_2^2}{36 nL^2} = \frac{1}{36 nL^2} \| s_0 - 1 \bar{s}_0 \|^2_2 \]
and
\[ \frac{c_2}{1 + \frac{\eta}{2} - c_1 p} C 1^\top r_0^2 \leq \frac{120 \kappa \sqrt{n}}{36 mnL^2} \| s_0 - 1 \bar{s}_0 \|^2_2 = \frac{10 \kappa}{3m \sqrt{nL^2}} \| s_0 - 1 \bar{s}_0 \|^2_2 \]
which implies
\[ V_0 = \| \bar{z}_0 - \bar{z}^* \|^2 + \frac{c_1}{1 + \frac{\eta}{2} - c_1 p} \| \bar{w}_0 - \bar{z}^* \|^2 + \frac{c_2}{1 + \frac{\eta}{2} - c_1 p} C 1^\top r_0^2 \]
\[ \leq 5 \| \bar{z}_0 - \bar{z}^* \|^2 + \frac{10 \kappa}{3m \sqrt{nL^2}} \| s_0 - 1 \bar{s}_0 \|^2_2 \]
\[ = 5 \| \bar{z}_0 - \bar{z}^* \|^2 + \frac{10 \kappa}{3m \sqrt{nL^2}} \| g(z_0) - \frac{1}{m} 11^\top g(z_0) \|^2 \]
\[ \leq 5 \| \bar{z}_0 - \bar{z}^* \|^2 + \frac{10 \kappa \rho}{3m \sqrt{nL^2}} \| g(z_0) - \frac{1}{m} 11^\top g(z_0) \|^2. \]
Then, by setting
\[ K_0 \geq \sqrt{\chi} \log \left( \frac{10 \kappa}{3m \sqrt{nL^2}} \| g(z_0) - \frac{1}{m} 11^\top g(z_0) \|^2 \right) = O \left( \sqrt{\chi} \log \left( \frac{n}{mnL^2 \epsilon} \right) \right) \]
we have
\[ V_0 \leq \| \bar{z}_0 - \bar{z}^* \|^2 + 5 \| \bar{w}_0 - \bar{z}^* \|^2 + \frac{\epsilon}{\sqrt{\chi}} \leq 6 \| \bar{z}_0 - \bar{z}^* \|^2 + \epsilon. \]
Hence, we require
\[ T = 6(n + 4 \kappa \sqrt{n}) \log \left( \frac{6 \| \bar{z}_0 - \bar{z}^* \|^2 + \epsilon}{\epsilon} \right) = O \left( (n + \kappa \sqrt{n}) \log \left( \frac{1}{\epsilon} \right) \right) \]
to obtain
\[ \mathbb{E}_t \left[ \| \bar{z}_T - \bar{z}^* \|^2 \right] \leq \mathbb{E}_t \left[ V_T \right] \]
\[ \leq \left( 1 - \frac{1}{6(n + 4 \kappa \sqrt{n})} \right)^T V_0 \]
\[ \leq \left( 1 - \frac{1}{6(n + 4 \kappa \sqrt{n})} \right)^T \left( 6 \| \bar{z}_0 - \bar{z}^* \|^2 + \epsilon \right) \]
\[ \leq \epsilon. \]
Since each iteration requires \( 1 + np = O(1) \) SFO calls in expectation, the total complexity is
\[ O((1 + np)T) = O \left( (n + \kappa \sqrt{n}) \log \left( \frac{1}{\epsilon} \right) \right) \]
in expectation. The number of communication rounds is
\[ KT + K_0 = O \left( (n + \kappa \sqrt{n}) \sqrt{\log(n)} \log \left( \frac{1}{\epsilon} \right) \right). \]
The Proof Details for Section 4.2

We first give two lemmas to address the constraint in the problem.

**Lemma C.1** (Luo et al. [26, Lemma 6]). For any \( u \in \mathbb{R}^d \) and \( v \in X \times Y \), we have \( \langle \mathcal{P}_u(u) - u, \mathcal{P}_v(u) - v \rangle \leq 0 \).

**Lemma C.2.** Under Assumption 2.1 and 2.2, we have \( \langle g(z^*), z - z^* \rangle \geq 0 \) for any \( z \in X \times Y \).

**Proof.** Since the objective function is convex-concave, for any \( x \in X \) and \( y \in Y \), we have

\[
\langle \nabla_x f(x^*, y^*), x - x^* \rangle \geq 0 \quad \text{and} \quad \langle -\nabla_y f(x^*, y^*), y - y^* \rangle \geq 0
\]

which means \( \langle g(z^*), z - z^* \rangle \geq 0 \). \( \Box \)

**Lemma C.3** (Ye et al. [48, Lemma 11]). For any \( u \in \mathbb{R}^{m \times d} \), it holds that

\[
\left\| \mathcal{P}_u \left( \frac{1}{m} \mathbf{1}^T u \right) - \frac{1}{m} \mathbf{1}^T \mathcal{P}_u (u) \right\| \leq \| u - \mathbf{1} \bar{u} \|.
\]

Then we provide the detailed proofs for theoretical results of MC-SVRE in constrained case.

### C.1 The Proof of Lemma 4.5

**Proof.** Lemma 4.1 means

\[
\bar{z}_{t+1/2} = \frac{1}{m} \mathbf{1}^T \mathcal{P}(\mathbf{z}_t' - \eta s_t) = \frac{1}{m} \mathbf{1}^T \mathcal{P}(\mathbf{z}_t' - \eta s_t + \Delta_t)
\]

and

\[
\bar{z}_{t+1} = \frac{1}{m} \mathbf{1}^T \mathcal{P}(\mathbf{z}_t' - \eta s_{t+1/2}) = \frac{1}{m} \mathbf{1}^T \mathcal{P}(\mathbf{z}_t' - \eta s_{t+1/2} + \Delta_{t+1/2})
\]

which implies

\[
\mathcal{P}(\mathbf{z}_t' - \eta \bar{g}_t) = \bar{z}_{t+1/2} - \Delta_t \quad \text{and} \quad \mathcal{P}(\mathbf{z}_t' - \eta \bar{g}_{t+1/2}) = \bar{z}_{t+1} - \Delta_{t+1/2}.
\]

Using Lemma C.4 with \( u = \mathbf{z}_t' - \eta \bar{g}_{t+1/2} \) and \( v = \bar{z}^* \), we have

\[
\langle \bar{z}_{t+1} - \Delta_{t+1/2} - \bar{z}_t', + \eta s_{t+1/2}, \bar{z}_{t+1} - \Delta_{t+1/2} - \bar{z}^* \rangle \leq 0.
\]

Using Lemma C.4 with \( u = \mathbf{z}_t' - \eta \bar{g}_t \) and \( v = \bar{z}_{t+1} \), we have

\[
\langle \bar{z}_{t+1/2} - \Delta_t - \bar{z}_t', + \eta s_t, \bar{z}_{t+1/2} - \Delta_t - \bar{z}_{t+1} \rangle \leq 0.
\]

Summing over above inequalities, we have

\[
\langle \bar{z}_{t+1} - \bar{z}_t', + \eta s_{t+1/2}, \bar{z}^* - \bar{z}_{t+1} \rangle + \langle \bar{z}_{t+1} - \bar{z}_t', + \eta s_{t+1/2} + \bar{z}_{t+1} - \bar{z}^*, \Delta_{t+1/2} \rangle - \| \Delta_{t+1/2} \|^2 \\
+ \langle \bar{z}_{t+1/2} - \bar{z}_t', + \eta \bar{g}_t, \bar{z}_{t+1} - \bar{z}_{t+1/2} \rangle + \langle \bar{z}_{t+1/2} - \bar{z}_t', + \eta \bar{g}_t + \bar{z}_{t+1/2} - \bar{z}_{t+1}, \Delta_t \rangle - \| \Delta_t \|^2 \geq 0.
\]
which means
\[
\langle \tilde{z}_{t+1} - \tilde{z}'_t, \tilde{z}^* - \tilde{z}_{t+1} \rangle + \langle \tilde{z}_{t+1/2} - \tilde{z}'_t, \tilde{z}_{t+1} - \tilde{z}_{t+1/2} \rangle + \eta \langle \tilde{s}_{t+1/2}, \tilde{z}^* - \tilde{z}_{t+1/2} \rangle + \eta \langle \tilde{s}_{t+1/2} - \tilde{s}_t, \tilde{z}_{t+1/2} - \tilde{z}_{t+1} \rangle
\]
\[
+ \langle \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_{t+1/2} + \tilde{z}_{t+1} - \tilde{z}^*, \Delta_{t+1/2} \rangle
\]
\[
+ \langle \tilde{z}_{t+1/2} - \tilde{z}'_t + \eta \tilde{s}_t + \tilde{z}_{t+1/2} - \tilde{z}_{t+1}, \Delta_t \rangle \geq 0.
\]

In constrained case. It is easy to verify the results of (7), (8) and (10) also hold. Additionally, we also have (9) due to Lemma C.2.

Plugging results of (7), (8), (9) and (10) into (17), we obtain that
\[
\alpha \| \tilde{z}_t - \tilde{z}^* \|^2 + (1 - \alpha) \| \tilde{w}_t - \tilde{z}^* \|^2 - \mathbb{E}_t \| \tilde{z}_{t+1} - \tilde{z}^* \|^2 - \mathbb{E}_t \| \tilde{z}_{t+1/2} - \tilde{z}_{t+1} \|^2
\]
\[
- \alpha \| \tilde{z}_{t+1/2} - \tilde{z}_{t+1} \|^2 - (1 - \alpha) \| \tilde{z}_{t+1/2} - \tilde{w}_t \|^2
\]
\[
- \eta \mathbb{E}_t \| \tilde{z}_{t+1} - \tilde{z}^* \|^2 + \frac{5 \eta \mu}{2} \mathbb{E}_t \| \tilde{z}_{t+1} - \tilde{z}_{t+1/2} \|^2 + \frac{4 L^2 \eta}{\mu} \mathbb{E}_t \| \tilde{z}_{t+1} - 1 \tilde{z}_{t+1/2} \|^2 + \frac{\eta \mu}{2} \| \tilde{z}_{t+1} - \tilde{z}^* \|^2
\]
\[
+ 6 \eta^2 L^2 \mathbb{E}_t \| \tilde{z}_{t+1/2} - \tilde{w}_t \|^2 + \frac{1}{2} \mathbb{E}_t \| \tilde{z}_{t+1/2} - \tilde{z}_{t+1} \|^2
\]
\[
+ \frac{6 \eta^2 L^2}{m} \mathbb{E}_t \| \tilde{z}_{t+1} - 1 \tilde{z}_{t+1/2} \|^2 + \frac{6 \eta^2 L^2}{m} \| \tilde{w}_t - \tilde{w}_t \|^2
\]
\[
+ \mathbb{E}_t \left[ \langle \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_{t+1} + \tilde{z}_{t+1} - \tilde{z}^*, \Delta_{t+1/2} \rangle \right]
\]
\[
+ \mathbb{E}_t \left[ \langle \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_t + \tilde{z}_{t+1} - \tilde{z}_{t+1}, \Delta_t \rangle \right] \geq 0.
\]

Combining inequality (18) with
\[
\langle \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_{t+1} + \tilde{z}_{t+1} - \tilde{z}^*, \Delta_{t+1/2} \rangle \leq \| \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_{t+1} + \tilde{z}_{t+1} - \tilde{z}^* \| \| \Delta_{t+1/2} \|
\]
and
\[
\langle \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_t + \tilde{z}_{t+1} - \tilde{z}_{t+1}, \Delta_t \rangle \leq \| \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_t + \tilde{z}_{t+1} - \tilde{z}_{t+1} \| \| \Delta_t \|
\]
we have
\[
\left( 1 + \frac{\eta \mu}{2} - c_1 p \right) \mathbb{E}_t \| \tilde{z}_{t+1} - \tilde{z}^* \|^2 + c_1 \mathbb{E}_t \| \tilde{w}_t - \tilde{z}^* \|^2
\]
\[
\leq (1 - p) \| \tilde{z}_t - \tilde{z}^* \|^2 + (p + c_1 (1 - p)) \| \tilde{w}_t - \tilde{z}^* \|^2
\]
\[
+ 2 \eta \frac{2 \eta (2 \delta + 3 L \eta)}{m} \left( \mathbb{E}_t \| \tilde{z}_{t+1} - 1 \tilde{z}_{t+1/2} \|^2 + \| \tilde{w}_t - \tilde{w}_t \|^2 \right)
\]
\[
+ \mathbb{E}_t \left[ \| \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_{t+1} + \tilde{z}_{t+1} - \tilde{z}^* \| \| \Delta_{t+1/2} \| \right] + \mathbb{E}_t \left[ \| \tilde{z}_{t+1} - \tilde{z}'_t + \eta \tilde{s}_t + \tilde{z}_{t+1} - \tilde{z}_{t+1} \| \| \Delta_t \| \right]
\]
\[
= (1 - p) \| \tilde{z}_t - \tilde{z}^* \|^2 + (p + c_1 (1 - p)) \| \tilde{w}_t - \tilde{z}^* \|^2 + \zeta.
\]

\[\square\]

C.2 The Proof of Lemma 4.6

We first give a lemma for consensus error.

**Lemma C.4.** Suppose Assumption 2.1 and 2.5 hold and we have \( \| s_0 - 1 \tilde{s}_0 \| \leq \delta' \) for some \( \delta' > 0 \). Then Algorithm 2 with \( \eta = 1/(6 \sqrt{m} L) \) and
\[
K \geq \sqrt{\chi} \log \left( \frac{2 (\sqrt{m} L D + \delta')}{\delta'} \right),
\]
holds that \( \| z_t - 1 \tilde{z}_t \| \leq \delta' \), \( \| z_{t+1/2} - 1 \tilde{z}_{t+1/2} \| \leq \delta' \), \( \| w_t - 1 \tilde{w}_t \| \leq \delta' \), \( \| s_t - 1 \tilde{s}_t \| \leq \delta'/\eta \) and \( \| s_{t+1/2} - 1 \tilde{s}_{t+1/2} \| \leq \delta'/\eta \) for any \( t \geq 0 \).
Proof. We prove this lemma by induction. It is obvious the statement holds for $t = 0$. Suppose we have $\|z_t - 1 \tilde{z}_t\| \leq \delta'$, $\|w_t - 1 \tilde{w}_t\| \leq \delta'$, $\|s_t - 1 \tilde{s}_t\| \leq \delta'/\eta$ for $t \geq 1$, then Lemma [2.1] and Lemma [B.2] means

\[
\|s_{t+1/2} - 1 \tilde{s}_{t+1/2}\| \\
\leq \rho \left( \|s_t - 1 \tilde{s}_t\| + L \|z_{t+1/2} - 1 \tilde{z}_{t+1/2}\| + L \sqrt{m} \| \tilde{z}_{t+1/2} - \tilde{w}_t\| + L \|w_t - 1 \tilde{w}_t\| \right) \\
\leq \rho \left( \|s_t - 1 \tilde{s}_t\| + L \sqrt{mD} + L \sqrt{mD} + L \sqrt{mD} \right) \\
\leq \rho \left( \delta'/\eta + 3 \sqrt{mLD} \right) \leq \delta'/\eta.
\]

We also have

\[
\|s_{t+1} - 1 \tilde{s}_{t+1}\| \\
= \|T(s_t + g(w_{t+1}) - g(w_t)) - \frac{1}{m} 11^T T(s_t + g(w_{t+1}) - g(w_t))\| \\
\leq \rho \left\| s_t + g(w_{t+1}) - g(w_t) - \frac{1}{m} 11^T (s_t + g(w_{t+1}) - g(w_t)) \right\| \\
\leq \rho \| s_t - 1 \tilde{s}_t \| + \rho \left\| g(w_{t+1}) - g(w_t) - \frac{1}{m} 11^T (g(w_{t+1}) - g(w_t)) \right\| \\
\leq \rho \| s_t - 1 \tilde{s}_t \| + \rho \| g(w_{t+1}) - g(w_t) \| \\
\leq \rho \| s_t - 1 \tilde{s}_t \| + \rho \| w_{t+1} - w_t \| \\
\leq \rho (\delta'/\eta + \sqrt{mLD}) \leq \delta'/\eta.
\]

We modify the proof of Lemma [B.1] as follows

\[
\|z_{t+1/2} - 1 \tilde{z}_{t+1/2}\| \\
= \|T(P_Z(z'_t - \eta s_t)) - \frac{1}{m} 11^T T(P_Z(z'_t - \eta s_t))\| \\
\leq \rho \| P_Z(z'_t - \eta s_t) - \frac{1}{m} 11^T P_Z (z'_t - \eta s_t) \| \\
\leq \rho \| P_Z(z'_t - \eta s_t) - P_Z (1(\tilde{z}'_t - \eta \tilde{s}_t))\| + \rho \left\| P_Z (1(\tilde{z}'_t - \eta \tilde{s}_t)) - \frac{1}{m} 11^T P_Z (\tilde{z}'_t - \eta \tilde{s}_t) \right\| \\
\leq \rho \| z'_t - \eta s_t - 1(\tilde{z}'_t - \eta \tilde{s}_t)\| + \rho \| (z'_t - \eta s_t) - 1(\tilde{z}'_t - \eta \tilde{s}_t)\| \\
\leq 2\rho \| z'_t - 1 \tilde{z}_t \| + 2\rho \| s_t - 1 \tilde{s}_t \| \\
\leq \rho \left( 2 \sqrt{mD} + 2 \delta' \right) \leq \delta',
\]

where the first inequality use Lemma [2.1] and third inequality use the non-expansiveness of projection and Lemma [C.3] with $\tilde{a} = z_t - \eta w_t$.

Similarly, we modify the proof of Lemma [B.3] as follows

\[
\|z_{t+1} - 1 \tilde{z}_{t+1}\| \\
= \|T(P_Z(z'_t - \eta s_{t+1/2})) - \frac{1}{m} 11^T T(P_Z(z'_t - \eta s_{t+1/2}))\| \\
\leq \rho \| P_Z(z'_t - \eta s_{t+1/2}) - \frac{1}{m} 11^T P_Z (z'_t - \eta s_{t+1/2}) \| \\
\leq \rho \| P_Z(z'_t - \eta s_{t+1/2}) - P_Z (1(\tilde{z}'_t - \eta \tilde{s}_{t+1/2}))\| + \rho \left\| P_Z (1(\tilde{z}'_t - \eta \tilde{s}_{t+1/2})) - \frac{1}{m} 11^T P_Z (\tilde{z}'_t - \eta \tilde{s}_{t+1/2}) \right\| \\
\leq \rho \| z'_t - \eta s_{t+1/2} - 1(\tilde{z}'_t - \eta \tilde{s}_{t+1/2})\| + \rho \| (z'_t - \eta s_{t+1/2}) - 1(\tilde{z}'_t - \eta \tilde{s}_{t+1/2})\| \\
\leq 2\rho \| z'_t - 1 \tilde{z}_t \| + \eta \| s_{t+1/2} - 1 \tilde{s}_{t+1/2} \| \\
\leq \rho \left( 2 \sqrt{mD} + 2 \delta' \right) \leq \delta'.
\]
Finally, we have
\[ \| \mathbf{w}_{t+1} - \mathbf{1} \tilde{w}_{t+1} \| \leq \max \{ \| \mathbf{z}_{t+1} - \mathbf{1} \tilde{z}_{t+1} \|, \| \mathbf{w}_{t} - \mathbf{1} \tilde{w}_{t} \| \} \leq \delta'. \]

Then we provide the proof of Lemma 4.6.

**Proof.** Using the non-expansiveness of projection, the update rule \( z'_i = \alpha z_i + (1 - \alpha) \mathbf{w}_t \) and Lemma C.4, we have

\[
\| \Delta_t \| = \left\| \frac{1}{m} \mathbf{1}^T \mathcal{P}_\mathcal{Z} (z'_t - \eta s_t) - \mathcal{P}_\mathcal{Z} (\bar{z}_t - \eta \tilde{s}_t) \right\|
\]

\[
= \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left\| \mathcal{P}_\mathcal{Z} (z'_t(i) - \eta s_t(i)) - \mathcal{P}_\mathcal{Z} (\bar{z}_t^T - \eta \tilde{s}_t^T) \right\|^2}
\]

\[
\leq \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left\| (z'_t(i) - \eta s_t(i)) - (\bar{z}_t^T - \eta \tilde{s}_t^T) \right\|^2}
\]

\[
\leq \sqrt{\frac{1}{m} \sum_{i=1}^{m} 2 \left( \| z'_t(i) - \bar{z}_t^T \|^2 + \eta^2 \| s_t(i) - \tilde{s}_t \|^2 \right)}
\]

\[
\leq \sqrt{\frac{1}{m} \left( \| z'_t - \mathbf{1} \tilde{z}_t \|^2 + \eta^2 \| s_t - \mathbf{1} \tilde{s}_t \|^2 \right)}
\]

\[
\leq \sqrt{\frac{6 \delta}{\sqrt{m}}}
\]

Similarly, we obtain

\[ \| \Delta_{t+1/2} \| \leq \frac{6 \delta'}{\sqrt{m}}. \]

We also have

\[
\| \bar{z}_{t+1/2} - z'_t + \eta \tilde{s}_t + \bar{z}_{t+1/2} - \tilde{z}_{t+1} \|
\]

\[
\leq \| \bar{z}_{t+1/2} - z'_t \| + \| \bar{z}_{t+1/2} - \tilde{z}_{t+1} \| + \eta \| \tilde{s}_t \|
\]

\[
\leq 2D + \eta \left\| \frac{1}{m} \mathbf{1}^T g(\mathbf{w}_t) \right\|
\]

\[
= 2D + \eta \frac{1}{m} \sum_{i=1}^{m} g_i(\mathbf{w}_t(i))
\]

\[
\leq 2D + \eta \sqrt{\frac{1}{m} \sum_{i=1}^{m} \| g_i(\mathbf{w}_t(i)) \|^2}
\]

\[
\leq 2D + \eta \sqrt{\frac{2}{m} \sum_{i=1}^{m} \left( \| g_i(\mathbf{w}_t(i)) - g_i(\mathbf{z}^*) \|^2 + \| g_i(\mathbf{z}^*) \|^2 \right)}
\]

\[
\leq 2D + \eta \sqrt{\frac{2}{m} \sum_{i=1}^{m} \left( L^2 \| \mathbf{w}_t(i) - \mathbf{z}^* \|^2 + \| g_i(\mathbf{z}^*) \|^2 \right)}
\]
Similarly, we have
\[
\|\hat{z}_{t+1} - \hat{z}' + \eta \hat{s}_{t+1/2} + \bar{z}_{t+1} - \bar{z}^*\|
\leq \|\hat{z}_{t+1} - \hat{z}'\| + \|\bar{z}_{t+1} - \bar{z}^*\| + \eta \|\hat{s}_{t+1/2}\|
\leq 2D + \eta \|\hat{s}_{t+1/2}\|
\leq 2D + \eta \left\| \frac{1}{m} \sum_{i=1}^{m} \left( g_i(w_t(i)) + g_{i,j}(z_{t+1/2}(i)) - g_{i,j}(w_t(i)) \right) \right\|
\leq 2D + \eta \left\| \frac{1}{m} \sum_{i=1}^{m} \left[ g_i(w_t(i)) + g_{i,j}(z_{t+1/2}(i)) - g_{i,j}(w_t(i)) \right] \right\|^2
= 2D + \eta \left\| \frac{3}{m} \sum_{i=1}^{m} \left[ g_i(w_t(i)) - g_i(z^*) \right]^2 + \|g_{i,j}(z_{t+1/2}(i)) - g_{i,j}(w_t(i))\|^2 + \|g_i(z^*)\|^2 \right\|
\leq 2D + \eta \left\| \frac{3}{m} \sum_{i=1}^{m} \left[ L^2\|w_t(i) - z^*\|^2 + L^2\|z_{t+1/2}(i) - w_t(i)\|^2 + \|g_i(z^*)\|^2 \right] \right\|
\leq 2D + \frac{1}{6\sqrt{nL}} \sqrt{2L^2D^2 + \frac{3}{m} \sum_{i=1}^{m} \|g_i(z^*)\|^2}
\triangleq C_{1/2}.
\]
Plugging all above results into the expression of \(\zeta_t\), we have
\[
\zeta_t \leq \frac{4L\eta (2\kappa + 3L\eta) \delta'^2}{m} + \frac{6(C_1 + C_{1/2})\delta'}{\sqrt{m}}.
\]

C.3 The Proof of Theorem 4.2

Proof. The setting of \(K_0\) means \(\|s_0 - 1\|_\infty \leq \delta'\). Then Lemma 4.5 and Lemma 4.6 imply
\[
\|\hat{z}_{t+1} - \bar{z}^*\|^2 + \frac{c_1}{1 + \frac{m\eta}{2} - c_1p} \|\bar{w}_{t+1} - \bar{z}^*\|^2
\leq \frac{1 - p}{1 + \frac{m\eta}{2} - c_1p} \|\hat{z}_t - \bar{z}^*\|^2 + \frac{p + c_1(1 - p)}{1 + \frac{m\eta}{2} - c_1p} \|\bar{w}_t - \bar{z}^*\|^2 + \frac{\zeta_t}{1 + \frac{m\eta}{2} - c_1p}
\leq \left( 1 - \frac{1}{6(n + 4\kappa\sqrt{n})} \right) \|\hat{z}_t - \bar{z}^*\|^2 + \frac{c_1}{1 + \frac{m\eta}{2} - c_1p} \|\bar{w}_t - \bar{z}^*\|^2
+ \frac{4L\eta (2\kappa + 3L\eta) \delta'^2}{m} + \frac{6(C_1 + C_{1/2})\delta'}{\sqrt{m}},
\]
where the last step follows the proof of Lemma 4.4. Recall that
\[
c_1 = \frac{2\eta\mu + 4p}{\eta\mu + 4p} \leq 2 \quad \text{and} \quad \frac{c_1}{1 + \frac{m\eta}{2} - c_1p} \leq \frac{2}{1 + \frac{m\eta}{2} - 2p} = \frac{2}{1 + \frac{2\kappa\sqrt{n}}{\eta} - \frac{\delta'}{2}} \leq 4.
\]
Then we have
\[
\|\tilde{z}_T - \tilde{z}^*\|^2 + \frac{c_1}{1 + \frac{m}{2} - c_1 p} \|\tilde{w}_T - \tilde{z}^*\|^2 \\
 \leq \left( 1 - \frac{1}{6(n + 4\kappa\sqrt{n})} \right)^T \left( \|\tilde{z}_0 - \tilde{z}^*\|^2 + \frac{c_1}{1 + \frac{m}{2} - c_1 p} \|\tilde{w}_0 - \tilde{z}^*\|^2 \right) \\
 + \frac{6(n + 4\kappa\sqrt{n})}{1 + \frac{m}{2} - c_1 p} \left( \frac{4L\eta(2\kappa + 3L\eta)\delta'^2}{m} + \frac{6(C_1 + C_{1/2})\delta'}{\sqrt{m}} \right) \\
 \leq 5 \left( 1 - \frac{1}{6(n + 4\kappa\sqrt{n})} \right)^T \|\tilde{z}_0 - \tilde{z}^*\|^2 + 12(n + 4\kappa\sqrt{n}) \left( \frac{4L\eta(2\kappa + 3L\eta)\delta'^2}{m} + \frac{C_1 + C_{1/2}}{\sqrt{m}} \right).
\]

The value of $T$ and $\delta'$ means we have
\[
5 \left( 1 - \frac{1}{6(n + 4\kappa\sqrt{n})} \right)^T \|\tilde{z}_0 - \tilde{z}^*\|^2 \leq \frac{\varepsilon}{2}
\]
and
\[
12(n + 4\kappa\sqrt{n}) \left( \frac{4L\eta(2\kappa + 3L\eta)\delta'^2}{m} + \frac{C_1 + C_{1/2}}{\sqrt{m}} \right) \leq \frac{\varepsilon}{2},
\]
which implies
\[
E_t \|\tilde{z}_T - \tilde{z}^*\|^2 \leq E_t \|\tilde{z}_T - \tilde{z}^*\|^2 + \frac{c_1}{1 + \frac{m}{2} - c_1 p} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Since each iteration requires $1 + np = \mathcal{O}(1)$ SFO calls in expectation, the total complexity is
\[
\mathcal{O}((1 + np)T) = \mathcal{O} \left( (n + \kappa\sqrt{n}) \log \left( \frac{1}{\varepsilon} \right) \right)
\]
in expectation. The number of communication rounds is
\[
KT + K_0 = \mathcal{O} \left( (n + \kappa\sqrt{n}) \sqrt{\chi} \log \left( \frac{KN}{\varepsilon} \right) \right).
\]

C.4 The Proof of Lemma 4.3

Proof. Since the objective function is non-strongly-convex and non-strongly-concave, we first modify (9) as follows
\[
2E_t \left[ \langle \tilde{z}_{t+1/2}, \tilde{z}^* - \tilde{z}_{t+1/2} \rangle \right] \\
= 2\langle \tilde{g}(\tilde{z}_{t+1/2}), \tilde{z}^* - \tilde{z}_{t+1/2} \rangle + 2\langle E_t[\tilde{s}_{t+1/2}] - \tilde{g}(\tilde{z}_{t+1/2}), \tilde{z}^* - \tilde{z}_{t+1/2} \rangle \\
\leq 2\langle \tilde{g}(\tilde{z}^*), \tilde{z}^* - \tilde{z}_{t+1/2} \rangle + 2\langle E_t[\tilde{s}_{t+1/2}] - \tilde{g}(\tilde{z}_{t+1/2}), \tilde{z}^* - \tilde{z}_{t+1/2} \rangle \\
\leq \beta \left\| E_t[\tilde{s}_{t+1/2}] - \tilde{g}(\tilde{z}_{t+1/2}) \right\| \| \tilde{z}_{t+1/2} - \tilde{z}^* \|^2 + \frac{1}{\beta} \| \tilde{z}_{t+1/2} - \tilde{z}^* \|^2
\]
where we use Lemma A.4 and C.2.

Plugging results of (7), (8), (19) and (10) into (17), we obtain that
\[
\alpha \|\tilde{z}_t - \tilde{z}^*\|^2 + (1 - \alpha) \|\tilde{w}_t - \tilde{z}^*\|^2 - E_t \|\tilde{z}_{t+1} - \tilde{z}^*\|^2 - E_t \|\tilde{z}_{t+1/2} - \tilde{z}_{t+1}\|^2 \\
- \alpha \|\tilde{z}_{t+1/2} - \tilde{z}_t\|^2 - (1 - \alpha) \|\tilde{z}_{t+1/2} - \tilde{w}_t\|^2 + \beta \eta \|E_t[\tilde{s}_{t+1/2}] - \tilde{g}(\tilde{z}_{t+1/2})\|^2 + \frac{\eta}{\beta} \|\tilde{z}_{t+1/2} - \tilde{z}^*\|^2 \\
+ 6\eta^2 L^2 E_t \|\tilde{z}_{t+1/2} - \tilde{w}_t\|^2 + \frac{1}{2} E_t \|\tilde{z}_{t+1/2} - \tilde{z}_{t+1}\|^2 \\
+ \frac{6\eta^2 L^2}{m} E_t \|\tilde{z}_{t+1/2} - \tilde{z}_{t+1/2}\|^2 + \frac{6\eta^2 L^2}{m} E_t \|\tilde{w}_t - \tilde{z}_{t+1}\|^2 \\
+ E_t \left[ \langle \tilde{z}_{t+1} - \tilde{z}_t + \eta \tilde{z}_{t+1/2} + \tilde{z}_{t+1} - \tilde{z}^* + \Delta_{t+1/2} \rangle + \langle \tilde{z}_{t+1/2} - \tilde{z}_t + \eta \tilde{z}_{t+1} + \tilde{z}_{t+1/2} - \tilde{z}_{t+1} + \Delta_{t+1/2} \rangle \right] \\
\geq 2\eta E_t \left[ \langle \tilde{g}(\tilde{z}_{t+1/2}), \tilde{z}_{t+1/2} - \tilde{z}^* \rangle \right],
\]
Using the update rule of $w_t$, we have
\[
E_t \| \bar{w}_{t+1} - \bar{z}^* \|^2 = (1-p)E_t \| \bar{w}_t - \bar{z}^* \|^2 + pE_t \| \bar{z}_{t+1} - \bar{z}^* \|^2.
\]

Connecting the inequalities (20) and (21) implies
\[
2\eta E_t \langle \bar{g}(\bar{z}_{t+1/2}), \bar{z}_{t+1/2} - \bar{z}^* \rangle + E_t \| \bar{w}_{t+1} - \bar{z}^* \|
\leq (1-p) \| \bar{z}_t - \bar{z}^* \|^2 + p \| \bar{w}_t - \bar{z}^* \|^2 + \frac{\eta}{\beta} \| \bar{z}_{t+1/2} - \bar{z}^* \|^2 + \eta \beta \| E_t[\bar{s}_{t+1/2}] - g(\bar{z}_{t+1/2}) \|^2
+ \frac{6\eta^2 L^2}{m} E_t \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 + \frac{6\eta^2 L^2}{m} E_t \| w_t - 1 \bar{w}_t \|^2
+ E_t \left[ \langle \bar{z}_{t+1} - \bar{z}'_t + \eta \bar{s}_{t+1/2} + \bar{z}_t - \bar{z}^*, \Delta_t \rangle + \langle \bar{z}_{t+1/2} - \bar{z}'_t + \eta \bar{s}_t + \bar{z}_{t+1/2} - \bar{z}_{t+1}, \Delta_t \rangle + p \| \bar{z}_{t+1} - \bar{z}^* \|^2 + (1-p) \| \bar{w}_t - \bar{z}^* \|^2 \right]
\leq (1-p) \| \bar{z}_t - \bar{z}^* \|^2 - E_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 + \frac{\eta D^2}{\beta} + \frac{\beta L^2 \eta}{m} \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2
+ \frac{6\eta^2 L^2}{m} E_t \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 + \frac{6\eta^2 L^2}{m} E_t \| w_t - 1 \bar{w}_t \|^2
+ E_t \left[ \| \bar{z}_{t+1} - \bar{z}'_t + \eta \bar{s}_{t+1/2} + \bar{z}_t - \bar{z}^* \|^2 \Delta_t + \| \bar{z}_{t+1/2} - \bar{z}'_t + \eta \bar{s}_t + \bar{z}_{t+1/2} - \bar{z}_{t+1} \| \Delta_t \right]
+ p \| \bar{z}_{t+1} - \bar{z}^* \|^2 + (1-p) \| \bar{w}_t - \bar{z}^* \|^2,
\]

that is
\[
2\eta E_t \langle \bar{g}(\bar{z}_{t+1/2}), \bar{z}_{t+1/2} - \bar{z}^* \rangle
\leq (1-p) \left( \| \bar{z}_t - \bar{z}^* \|^2 - E_t \| \bar{z}_{t+1} - \bar{z}^* \|^2 \right) + \left( \| \bar{w}_t - \bar{z}^* \|^2 - E_t \| \bar{w}_{t+1} - \bar{z}^* \|^2 \right) + \zeta'_t
\]

where
\[
\zeta'_t = \frac{\eta D^2}{\beta} + \frac{6(\eta^2 + \beta \eta) L^2}{m} E_t \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \|^2 + \frac{6\eta^2 L^2}{m} E_t \| w_t - 1 \bar{w}_t \|^2
+ E_t \left[ \| \bar{z}_{t+1} - \bar{z}'_t + \eta \bar{s}_{t+1/2} + \bar{z}_t - \bar{z}^* \|^2 \Delta_t + \| \bar{z}_{t+1/2} - \bar{z}'_t + \eta \bar{s}_t + \bar{z}_{t+1/2} - \bar{z}_{t+1} \| \Delta_t \right]
\leq \frac{\eta D^2}{\beta} + \frac{6(2\eta^2 + \beta \eta) L^2 \delta'^2}{m} + \frac{6(C_1 + C_{1/2}) \delta'}{\sqrt{m}}.
\]

The upper bound of $\zeta'_t$ follows the proof of Lemma 4.4.

Summing over (22) with $t = 0, \ldots, T-1$, we obtain
\[
E_t \left[ f(\bar{x}, y^*) - f(x^*, y) \right]
\leq \frac{1}{T} \sum_{t=0}^{T-1} E_t \left[ f(\bar{x}_{t+1/2}, y^*) - f(x^*, \bar{y}_{t+1/2}) \right]
\leq \frac{1}{T} \sum_{t=0}^{T-1} E_t \langle g(\bar{z}_{t+1/2}), \bar{z}_{t+1/2} - \bar{z}^* \rangle
\leq (1-p) \left( \| z_0 - z^* \|^2 - E_T \| z_T - z^* \|^2 \right) + \| w_0 - z^* \|^2 - E_T \| w_T - z^* \|^2 + \sum_{t=0}^{T-1} \zeta'_t
\leq \frac{\| z_0 - z^* \|^2}{\eta T} + \frac{1}{2\eta} \left( \frac{\eta D^2}{\beta} + \frac{6(2\eta^2 + \beta \eta) L^2 \delta'^2}{m} + \frac{6(C_1 + C_{1/2}) \delta'}{\sqrt{m}} \right)
\leq \frac{\varepsilon}{2} + \varepsilon + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon.
\]

where the first inequality use Jensen’s inequality; the second inequality use the objective function is convex-concave; the third inequality use the upper bound of $\zeta'_t$; and the last one is based on the value of parameter settings.
Since each iteration requires $1 + np = \mathcal{O}(1)$ SFO calls in expectation and we the algorithm needs to compute the full gradient at first, the total SFO complexity is

$$\mathcal{O}(n + (1 + np)T) = \mathcal{O}\left(n + \frac{\sqrt{n}L}{\varepsilon}\right)$$

in expectation. The number of communication rounds is

$$KT + K_0 = \mathcal{O}\left(\frac{\sqrt{n}L}{\varepsilon} \log \left(\frac{nL}{\varepsilon}\right)\right).$$

\[\square\]

## D The Proof Details for Section 5

This section provide the detailed proofs for theoretical results of MC-EG.

### D.1 The Proof of Lemma 5.1

**Proof.** Using the fact $2\langle a, b \rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2$, we have

$$2(\bar{z}_t - \bar{z}_{t+1}, \bar{z}_{t+1} - \bar{z}^*) = \|\bar{z}_t - \bar{z}^*\|^2 - \|\bar{z}_t - \bar{z}_{t+1}\|^2 - \|\bar{z}_{t+1} - \bar{z}^*\|^2,$$

and

$$2(\bar{z}_t - \bar{z}_{t+1/2}, \bar{z}_{t+1/2} - \bar{z}_{t+1}) = \|\bar{z}_t - \bar{z}_{t+1}\|^2 - \|\bar{z}_t - \bar{z}_{t+1/2}\|^2 - \|\bar{z}_{t+1/2} - \bar{z}_{t+1}\|^2. \quad (23)$$

Hence, we have

$$2\eta(\bar{s}_{t+1/2}, \bar{z}_{t+1/2} - \bar{z}^*)$$

$$= 2\eta(\bar{s}_{t+1/2}, \bar{z}_{t+1} - \bar{z}^*) + 2\eta(\bar{s}_{t+1/2}, \bar{z}_{t+1/2} - \bar{z}_{t+1})$$

$$= 2\eta(\bar{z}_{t} - \bar{z}_{t+1}, \bar{z}_{t+1} - \bar{z}^*) + 2\eta(\bar{z}_{t} - \bar{z}_{t+1}, \bar{z}_{t+1/2} - \bar{z}_{t+1}) + 2\eta(\bar{z}_{t+1/2} - \bar{z}_{t+1}, \bar{z}_{t+1/2} - \bar{z}_{t+1})$$

$$= \|\bar{z}_t - \bar{z}^*\|^2 - \|\bar{z}_t - \bar{z}_{t+1}\|^2 - \|\bar{z}_{t+1} - \bar{z}^*\|^2 + \|\bar{z}_t - \bar{z}_{t+1}\|^2 - \|\bar{z}_t - \bar{z}_{t+1/2}\|^2 - \|\bar{z}_{t+1/2} - \bar{z}_{t+1}\|^2$$

$$+ 2\eta(\bar{z}_{t+1/2} - \bar{s}_t, \bar{z}_{t+1/2} - \bar{z}_{t+1})$$

$$\leq \|\bar{z}_t - \bar{z}^*\|^2 - \|\bar{z}_t - \bar{z}_{t+1}\|^2 - \|\bar{z}_{t+1} - \bar{z}^*\|^2 - \|\bar{z}_t - \bar{z}_{t+1/2}\|^2 - \|\bar{z}_t - \bar{z}_{t+1/2}\|^2 - \|\bar{z}_{t+1/2} - \bar{z}_{t+1}\|^2$$

$$+ 2\eta^2 \|\bar{z}_{t+1/2} - \bar{s}_t\|^2 + \frac{\mu}{2} \|\bar{z}_{t+1/2} - \bar{z}_{t+1}\|^2$$

$$\leq \|\bar{z}_t - \bar{z}^*\|^2 - \|\bar{z}_t - \bar{z}_{t+1}\|^2 - \|\bar{z}_t - \bar{z}_{t+1}/2\|^2 - \|\bar{z}_t - \bar{z}_{t+1/2}\|^2 - (1 - 6\eta L^2) \|\bar{z}_{t+1/2} - \bar{z}_t\|^2$$

$$+ \frac{6L^2\eta^2}{m} \left(\|\bar{z}_{t+1/2} - \bar{z}_t\|^2 + \|\bar{z}_t - \bar{z}_t\|^2\right),$$

where the second equality is based on the update rule; the third one is based on (23), (24) and Lemma 2.1; the inequality is due to $2\langle a, b \rangle \leq \|a\|^2 + \|b\|^2$; the last step is because of

$$\|\bar{s}_{t+1/2} - \bar{s}_t\|^2$$

$$\leq 3 \left(\|\bar{s}_{t+1/2} - \bar{g}(\bar{z}_{t+1/2})\|^2 + 3 \|\bar{g}(\bar{z}_{t+1/2}) - \bar{g}(\bar{z}_t)\|^2 + 3 \|\bar{g}(\bar{z}_t) - \bar{s}_t\|^2\right) \leq 3 \left(\frac{L^2}{m} \|\bar{z}_{t+1/2} - \bar{z}_t\|^2 + L^2 \|\bar{z}_{t+1/2} - \bar{z}_t\|^2 + \frac{L^2}{m} \|\bar{z}_t - \bar{z}_t\|^2\right). \quad (26)$$

We also have

$$2\eta(\bar{s}_{t+1/2}, \bar{z}_{t+1/2} - \bar{z}^*)$$

$$= 2\eta(\bar{g}(\bar{z}_{t+1/2}), \bar{z}_{t+1/2} - \bar{z}^*) + 2\eta(\bar{g}(\bar{z}_{t+1/2}), \bar{z}_{t+1/2} - \bar{z}^*)$$

$$\geq 2\eta \|\bar{g}(\bar{z}^*)\| \|\bar{z}_{t+1/2} - \bar{z}^*\|^2 + 2\eta \|\bar{z}_{t+1/2} - \bar{z}^*\|^2 - 4\eta \mu \|\bar{s}_{t+1/2} - \bar{g}(\bar{z}_{t+1/2})\|^2 - \frac{\eta \mu}{4} \|\bar{s}_{t+1/2} - \bar{g}(\bar{z}_{t+1/2})\|^2$$

$$\geq \eta \mu \|\bar{z}_t - \bar{z}^*\|^2 - 2\eta \mu \|\bar{z}_t - \bar{z}_{t+1/2}\|^2 - 4\eta \mu \|\bar{s}_{t+1/2} - \bar{g}(\bar{z}_{t+1/2})\|^2 - \frac{\eta \mu}{4} \|\bar{s}_{t+1/2} - \bar{z}^*\|^2, \quad (27)$$

$$34$$
where the first inequality follows Lemma A.4 and Young’s inequality; the second inequality is according to Lemma D.1.

Connecting inequalities (25) and (27), we have

\[
\eta\mu \left\| z_t - z^* \right\|^2 - 2\eta\mu \left\| \bar{z}_t - \bar{z}_{t+1/2} \right\|^2 \\
\leq 2\eta \langle \delta_{t+1/2}, \bar{z}_{t+1/2} - z^* \rangle + \frac{4\eta}{\mu} \left\| \bar{z}_{t+1/2} - \bar{g}(\bar{z}_{t+1/2}) \right\|^2 + \frac{\mu\eta}{4} \left\| \bar{z}_{t+1/2} - \bar{z}^* \right\|^2 \\
\leq \left\| \bar{z}_t - z^* \right\|^2 - \left\| \bar{z}_{t+1} - z^* \right\|^2 - \frac{1}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\| - (1 - 6\eta^2 L^2) \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 \\
+ \frac{6L^2\eta^2}{m} \left( \left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\|^2 + \left\| z_t - 1 \bar{z}_t \right\|^2 \right) \\
+ \frac{4\eta}{\mu} \left\| \bar{z}_{t+1/2} - g(\bar{z}_{t+1/2}) \right\|^2 + \frac{\mu\eta}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 + \frac{\mu\eta}{2} \left\| \bar{z}_t - z^* \right\|^2 \\
\leq \left\| \bar{z}_t - z^* \right\|^2 - \left\| \bar{z}_{t+1} - z^* \right\|^2 - \frac{1}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\| - (1 - 6\eta^2 L^2) \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 \\
+ \frac{6L^2\eta^2}{m} \left( \left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\|^2 + \left\| z_t - 1 \bar{z}_t \right\|^2 \right) \\
+ \frac{4\eta L^2}{m\mu} \left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\|^2 + \frac{\mu\eta}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 + \frac{\mu\eta}{2} \left\| \bar{z}_t - z^* \right\|^2 ,
\]

where the last step follows the proof of Lemma D.1. By rearranging above result, we have

\[
\left\| \bar{z}_{t+1} - z^* \right\|^2 \\
\leq \left( 1 - \frac{\mu\eta}{2} \right) \left\| \bar{z}_t - z^* \right\|^2 - \frac{1}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\| - (1 - 6\eta^2 L^2) \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 \\
+ \frac{6L^2\eta^2}{m} \left( \left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\|^2 + \left\| z_t - 1 \bar{z}_t \right\|^2 \right) + \frac{4\eta L^2}{m\mu} \left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\|^2 \\
+ \frac{\mu\eta}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\| + 2\eta\mu \left\| \bar{z}_t - \bar{z}_{t+1/2} \right\|^2 \\
= \left( 1 - \frac{\mu\eta}{2} \right) \left\| \bar{z}_t - z^* \right\|^2 - \frac{1}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\| - \left( 1 - \frac{5\mu\eta}{2} - 6\eta^2 L^2 \right) \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 \\
+ \frac{2L\eta (3\eta L + 2\kappa)}{m} \left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\|^2 + \frac{6L^2\eta^2}{m} \left\| z_t - 1 \bar{z}_t \right\|^2 \\
\leq \left( 1 - \frac{\mu\eta}{2} \right) \left\| \bar{z}_t - z^* \right\|^2 - \frac{1}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\| - \left( 1 - \frac{5\mu\eta}{2} - 6\eta^2 L^2 \right) \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 \\
+ \frac{2L\eta (3\eta L + 2\kappa)}{m} (\rho \left\| z_t - 1 \bar{z}_t \right\| + \rho \eta \left\| s_t - 1 \bar{s}_t \right\|) + \frac{6L^2\eta^2}{m} \left\| z_t - 1 \bar{z}_t \right\|^2 \\
\leq \left( 1 - \frac{\mu\eta}{2} \right) \left\| \bar{z}_t - z^* \right\|^2 - \frac{1}{2} \left\| \bar{z}_{t+1/2} - \bar{z}_{t+1} \right\| - \left( 1 - \frac{5\mu\eta}{2} - 6\eta^2 L^2 \right) \left\| \bar{z}_{t+1/2} - \bar{z}_t \right\|^2 \\
+ \frac{4L\eta (3\eta L + 2\kappa) \rho}{m} \left\| \bar{z}_t - z^* \right\|^2 ,
\]

where the second inequality use Lemma D.1.

\[\square\]

**D.2 The Proof of Lemma 5.2**

We first introduce some lemmas for the consensus error of each variables. We use the notations of

\[
\hat{r}_t = \begin{bmatrix} \| z_t - 1 \bar{z}_t \|_2 \\ \eta \| s_t - 1 \bar{s}_t \|_2 \end{bmatrix} \quad \text{and} \quad \bar{r}_t^2 = \begin{bmatrix} \| z_t - 1 \bar{z}_t \|_2^2 \\ \eta^2 \| s_t - 1 \bar{s}_t \|_2^2 \end{bmatrix}.
\]

**Lemma D.1.** Suppose Assumption 2.2 holds. For Algorithm 3, we have

\[
\left\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \right\| \leq \rho \left( \| z_t - 1 \bar{z}_t \| + \eta \| s_t - 1 \bar{s}_t \| \right).
\]
Proof. Similar to the proof of Lemma B.3, we have
\[
\begin{align*}
\|z_{t+1/2} - 1\tilde{z}_{t+1/2}\| = & \left\| T(z'_t - \eta s_t) - \frac{1}{m}11^\top T(z'_t - \eta s_t) \right\| \\
\leq & \rho \left\| (z'_t - \eta s_t) - \frac{1}{m}11^\top (z'_t - \eta s_t) \right\| \\
= & \rho \| (z'_t - \eta s_t) - 1(\tilde{z}'_t - \eta \tilde{s}_t) \| \\
\leq & \rho \| z'_t - 1\tilde{z}'_t \| + \rho \eta \| s_t - 1\tilde{s}_t \|.
\end{align*}
\]

Lemma D.2. Suppose Assumption 2.4 holds. For Algorithm B, we have
\[
\|s_{t+1/2} - 1\tilde{s}_{t+1/2}\| \leq \rho \left( L \| z_t - 1\tilde{z}_t \| + L \| z_{t+1/2} - 1\tilde{z}_{t+1/2} \| + \| s_t - 1\tilde{s}_t \| + L\sqrt{m} \| \tilde{z}_{t+1/2} - \tilde{z}_t \| \right).
\]
Proof. Similar to the proof of Lemma B.2, we have
\[
\begin{align*}
\|s_{t+1/2} - 1\tilde{s}_{t+1/2}\| = & \left\| T(s_t + g(z_{t+1/2}) - g(z_t)) - \frac{1}{m}11^\top T(s_t + g(z_{t+1/2}) - g(z_t)) \right\| \\
\leq & \rho \left\| s_t + g(z_{t+1/2}) - g(z_t) - \frac{1}{m}11^\top (s_t + g(z_{t+1/2}) - g(z_t)) \right\| \\
\leq & \rho \left( \| s_t - 1\tilde{s}_t \| + \| g(z_{t+1/2}) - g(z_t) \| \right) \\
\leq & \rho \left( \| s_t - 1\tilde{s}_t \| + L \| z_{t+1/2} - z_t \| \right) \\
\leq & \rho \left( \| s_t - 1\tilde{s}_t \| + L \| z_{t+1/2} - 1\tilde{z}_{t+1/2} \| + L\sqrt{m} \| \tilde{z}_{t+1/2} - \tilde{z}_t \| + L \| z_t - 1\tilde{z}_t \| \right).
\end{align*}
\]

Lemma D.3. Suppose Assumption 2.4 holds. For Algorithm B, we have
\[
\|z_{t+1} - 1\tilde{z}_{t+1}\| \leq \rho \left( \| z_t - 1\tilde{z}_t \| + \eta \| s_{t+1/2} - 1\tilde{s}_{t+1/2} \| \right).
\]
Proof. Similar to the proof of Lemma B.3, we have
\[
\begin{align*}
\|z_{t+1} - 1\tilde{z}_{t+1}\| = & \left\| T(z_t - \eta s_{t+1/2}) - \frac{1}{m}11^\top T(z_t - \eta s_{t+1/2}) \right\| \\
\leq & \rho \left\| (z_t - \eta s_{t+1/2}) - \frac{1}{m}11^\top (z_t - \eta s_{t+1/2}) \right\| \\
\leq & \rho \| z_t - \eta s_{t+1/2} - 1(\tilde{z}_t - \eta \tilde{s}_{t+1/2}) \| \\
\leq & \rho \left( \| z_t - 1\tilde{z}_t \| + \eta \| s_{t+1/2} - 1\tilde{s}_{t+1/2} \| \right)
\end{align*}
\]

Lemma D.4. Suppose Assumption 2.4 holds. For Algorithm B, we have
\[
\|s_{t+1} - 1\tilde{s}_{t+1}\| \leq \rho \| s_t - 1\tilde{s}_t \| + \rho L \left( \| z_{t+1} - 1\tilde{z}_{t+1} \| + 2 \| z_{t+1/2} - 1\tilde{z}_{t+1/2} \| + \| z_t - 1\tilde{z}_t \| \right)
\]
\[
+ \rho L \left( \sqrt{m} \| \tilde{z}_{t+1} - \tilde{z}_{t+1/2} \| + \sqrt{m} \| \tilde{z}_{t+1/2} - \tilde{z}_t \| \right).
\]
Proof. Similar to the proof of Lemma \[B.5\] we have

\[
\|s_{t+1} - 1s_{t+1}\|
\]

\[
\leq \rho \|s_{t} + g(z_{t+1}) - g(z_{t}) - \frac{1}{m} \mathbf{1}\mathbf{1}^\top(s_{t} + g(z_{t+1}) - g(z_{t}))
\]

\[
\leq \rho \|s_{t} + g(z_{t+1}) - g(z_{t}) - \frac{1}{m} \mathbf{1}\mathbf{1}^\top(s_{t} + g(z_{t+1}) - g(z_{t}))
\]

\[
\leq \rho \|s_{t} - 1s_{t}\| + \rho \|g(z_{t+1}) - g(z_{t}) - \frac{1}{m} \mathbf{1}\mathbf{1}^\top(g(z_{t+1}) - g(z_{t}))
\]

Then we give the proof of Lemma \[B.2\]

Proof. The analysis is similar to the proof of Lemma \[B.3\]. Using Lemma \[D.1\] we have

\[
\|z_{t+1/2} - 1\tilde{z}_{t+1/2}\| \leq \rho \begin{bmatrix} 1 & 1 \end{bmatrix} \hat{r}_t.
\]

Using Lemma \[D.2\] we have

\[
\|s_{t+1/2} - 1\tilde{s}_{t+1/2}\|
\]

\[
\leq \rho \begin{bmatrix} L & 1/\eta \end{bmatrix} \hat{r}_t + \rho L \|z_{t+1/2} - 1\tilde{z}_{t+1/2}\| + L\sqrt{m} \|\tilde{z}_{t+1/2} - \tilde{z}_t\|
\]

\[
\leq \rho \begin{bmatrix} L & 1/\eta \end{bmatrix} \hat{r}_t + L\sqrt{m} \|\tilde{z}_{t+1/2} - \tilde{z}_t\|.
\]

Using Lemma \[D.1\] we have

\[
\|z_{t+1} - 1\tilde{z}_{t+1}\|
\]

\[
= 2\rho \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{r}_t + \eta \|s_{t+1/2} - 1\tilde{s}_{t+1/2}\|
\]

\[
\leq 2\rho \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{r}_t + \eta \rho \begin{bmatrix} L & 3/(2\eta) \end{bmatrix} \hat{r}_t + \eta \rho L\sqrt{m} \|\tilde{z}_{t+1/2} - \tilde{z}_t\|
\]

\[
\leq 2\rho \hat{r}_t + \eta \rho L\sqrt{m} \|\tilde{z}_{t+1/2} - \tilde{z}_t\|
\]

\[
\leq 2\rho \begin{bmatrix} 2.5 & 3 \end{bmatrix} \hat{r}_t + 0.5\rho \sqrt{m} \|\tilde{z}_{t+1/2} - \tilde{z}_t\|
\]

and

\[
\|z_{t+1} - 1\tilde{z}_{t+1}\| \leq 4\rho^2 \begin{bmatrix} 6.25 & 9 \end{bmatrix} \hat{r}_t^2 + \rho^2 m \|\tilde{z}_{t+1/2} - \tilde{z}_t\|^2
\]

\[
\leq \rho^2 \begin{bmatrix} 25 & 36 \end{bmatrix} \hat{r}_t^2 + \rho^2 m \|\tilde{z}_{t+1/2} - \tilde{z}_t\|^2.
\]
Note that $\eta \leq 1/4L$. Using Lemma [D.3] we have
\[
\|s_{t+1} - 1s_{t+1}\|
\leq \rho \left[ L/\eta \right] \hat{r}_t + \rho L \left( \rho \left[ 2.5 \ 3 \right] \hat{r}_t + 0.5\rho\sqrt{m}\|z_{t+1/2} - \bar{z}_t\| \right)
+ 2\rho^2 L \left[ 1 \ 1 \right] \hat{r}_t + \rho L\sqrt{m} \left[ \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{z}_t\| \right)
\leq \rho \left[ 3.5L \ 1/\eta \right] + 3L \left[ 1 \ 1 \right] \hat{r}_t + 0.5\rho^3 L\sqrt{m} \left[ z_{t+1} - \bar{z}_{t+1/2}\right]
+ \rho \left[ 2L \ 2L \right] \hat{r}_t + \rho L\sqrt{m} \left[ \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{z}_t\| \right)
\leq \rho \left[ 5.5L \ 1/\eta + 5L \right] \hat{r}_t + 1.5\rho^2 L\sqrt{m} \left[ \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{z}_t\| \right]
\]
that is
\[
\eta \|s_{t+1} - 1s_{t+1}\| \leq \rho \left[ 11/8 \ 9/4 \right] \hat{r}_t + \frac{3}{8}\rho\sqrt{m} \left( \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{z}_t\| \right),
\]
which implies
\[
\eta^2 \|s_{t+1} - 1s_{t+1}\|^2
\leq 3\rho^2 \left[ (11/8)^2 \ (9/4)^2 \right] \hat{r}_t^2 + \frac{27}{64}\rho^2 m \left( \|z_{t+1} - \bar{z}_{t+1/2}\| + \|\bar{z}_{t+1/2} - \bar{z}_t\| \right)^2
\leq \rho^2 \left[ \frac{363}{64} \frac{243}{16} \right] \hat{r}_t^2 + \frac{27}{32}\rho^2 m \left( \|z_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{z}_t\|^2 \right).
\]
In summary, we have
\[
\hat{r}_t^2_{t+1} \leq \left[ \frac{25\rho^2}{363\rho^2} \frac{36\rho^2}{243\rho^2} \frac{1}{16} \right] \hat{r}_t^2 + \left[ \frac{\rho^2 m \|z_{t+1} - \bar{z}_t\|^2}{\|z_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{z}_t\|^2} \right].
\]
Hence, we have
\[
1^T \hat{r}_t^2_{t+1} \leq 52\rho^2 1^T \hat{r}_t^2 + 2\rho^2 m \left( \|z_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{z}_t\|^2 \right).
\]

D.3 The Proof of Lemma [5.3]

Proof. Using Lemma [5.3] and Lemma [5.2] we have
\[
\|\bar{z}_{t+1} - z^*\|^2 + \frac{1}{m}1^T \hat{r}_t^2 + \frac{1}{m}1^T \hat{r}_t^2
\leq \left( 1 - \frac{\mu m}{2} \right) \|\bar{z}_t - z^*\|^2 + \frac{1}{m} \left( \|\bar{z}_{t+1} - \bar{z}_t\|^2 - \left( 1 - \frac{5\mu m}{2} - 6\eta^2 L^2 \right) \|\bar{z}_{t+1} - \bar{z}_t\|^2 \right)
+ \frac{4L\eta(3L\eta + 2\kappa) - \rho^2}{m} \left[ \frac{1}{m} - \frac{5\mu m}{2} \right] \hat{r}_t^2 + \frac{52\rho^2}{m} \left[ \frac{1}{m} + 2\rho^2 \right] \left( \|\bar{z}_{t+1} - \bar{z}_{t+1/2}\|^2 + \|\bar{z}_{t+1/2} - \bar{z}_t\|^2 \right)
- \left( \frac{1}{2} - 2\rho^2 \right) \|\bar{z}_{t+1} - \bar{z}_t\|^2 - \left( 1 - \frac{5\mu m}{2} - 6\eta^2 L^2 - 2\rho^2 \right) \|\bar{z}_{t+1} - \bar{z}_t\|^2
\leq \left( 1 - \frac{\mu m}{2} \right) \left( \|\bar{z}_t - \bar{z}^*\|^2 + \frac{1}{m} \hat{r}_t^2 \right)
= \left( 1 - \frac{12\kappa}{12\kappa} \right) \left( \|\bar{z}_t - \bar{z}^*\|^2 + \frac{1}{m} \hat{r}_t^2 \right)
\]
where the last inequality is due to the value of $K$ implies
\[
\rho \leq \min \left\{ \frac{1}{2}, \sqrt{\frac{1}{2} \left( 1 - \frac{5\mu m}{2} - 6\eta^2 L^2 \right)}, \frac{1 - \frac{\mu m}{2}}{4L\eta(3L\eta + 2\kappa) + 52} \right\} = \min \left\{ \frac{1}{2}, \sqrt{\frac{1}{2} \left( \frac{5 - 5\mu m}{2} \right)}, \frac{3(2 - \mu\eta)}{2(4\kappa + 157)} \right\}.
\]

D.4 The Proof of Theorem 5.1

Proof. The value of $K_0$ implies $1^\top r_0^2 = \|s_0 - 1\tilde{s}_0\|^2 \leq m\epsilon$. Then Lemma 5.3 implies

$$
\|\tilde{z}_T - \tilde{z}^*\|^2 \\
\leq \|\tilde{z}_T - \tilde{z}^*\|^2 + \frac{1}{m} 1^\top r_T^2 \\
\leq \left(1 - \frac{1}{12\kappa}\right)^T (\|\tilde{z}_0 - \tilde{z}^*\|^2 + \frac{1}{m} 1^\top r_0^2) \\
\leq \left(1 - \frac{1}{12\kappa}\right)^T (\|\tilde{z}_0 - \tilde{z}^*\|^2 + \epsilon).
$$

Hence, the value of $T$ means $\|\tilde{z}_T - \tilde{z}^*\|^2 \leq \epsilon$.

Since each iteration requires $O(n)$ SFO calls, the total complexity is

$$
O(nT) = O\left(\kappa n \log \left(\frac{1}{\epsilon}\right)\right)
$$

and the number of communication round is

$$
KT + K_0 = O\left(\kappa \sqrt{\chi} \log \kappa \log \left(\frac{1}{\epsilon}\right)\right).
$$

D.5 The Proof of Theorem 5.2

We first provide some lemmas for our proof.

Lemma D.5. Suppose Assumption 2.1 and 2.3 hold. For Algorithm 3 with $\eta = 1/(6L)$, we have, we have

$$
\|\tilde{z}_{t+1} - \tilde{z}^*\|^2 \leq \left(1 - \frac{\mu\eta}{2}\right)\|\tilde{z}_t - \tilde{z}^*\|^2 + \hat{\zeta}_t
$$

where

$$
\hat{\zeta}_t = 4L\eta (3L\eta + 2\kappa) \left(\|z_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \|z_t - 1\tilde{z}_t\|^2\right) \\
+ \left\|\tilde{z}_{t+1} - \tilde{z}_t + \eta\tilde{s}_{t+1/2} + \tilde{z}_{t+1} - \tilde{z}^*\right\| \left\|\tilde{\Delta}_{t+1/2}\right\| \\
+ \left\|\tilde{z}_{t+1} - \tilde{z}_t + \eta\tilde{s}_{t+1/2} + \tilde{z}_{t+1} - \tilde{z}^*\right\| \left\|\tilde{\Delta}_{t+1/2}\right\|.
$$

Proof. Lemma 2.1 means

$$
\tilde{z}_{t+1/2} = \frac{1}{m} 1^\top (P(z_t - \eta s_t)) = \frac{1}{m} 1^\top P(z_t - \eta s_t) \\
= P(\tilde{z}_t - \eta\tilde{s}_t) + \frac{1}{m} 1^\top P(z_t - \eta s_t) - P(\tilde{z}_t - \eta\tilde{s}_t) \\
= P(\tilde{z}_t - \eta\tilde{s}_t) + \tilde{\Delta}_t
$$

and

$$
\tilde{z}_{t+1} = \frac{1}{m} 1^\top (P(z_t - \eta v_{t+1/2})) = \frac{1}{m} 1^\top P(z_t - \eta v_{t+1/2}) \\
= P(\tilde{z}_t - \eta\tilde{v}_{t+1/2}) + \frac{1}{m} 1^\top P(z_t - \eta s_{t+1/2}) - P(\tilde{z}_t - \eta\tilde{s}_{t+1/2}) \\
= P(\tilde{z}_t - \eta\tilde{v}_{t+1/2}) + \tilde{\Delta}_{t+1/2}.
$$

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which implies

\[ \mathcal{P}(\bar{z}_t - \eta \tilde{s}_t + 1/2) + \hat{\Delta}_{t+1/2}. \]

Using Lemma 0.4 with \( u = \bar{z}_t - \eta \tilde{g}_{t+1/2} \) and \( v = \tilde{z}^* \), we have

\[ \langle \bar{z}_{t+1} - \hat{\Delta}_{t+1/2} - \bar{z}_t + \eta \tilde{s}_{t+1/2}, \bar{z}_{t+1} - \hat{\Delta}_{t+1/2} - \tilde{z}^* \rangle \leq 0. \]

Using Lemma 0.4 with \( u = \bar{z}_t - \eta \tilde{g}_t \) and \( v = \bar{z}_{t+1} \), we have

\[ \langle \bar{z}_{t+1/2} - \hat{\Delta}_t - \bar{z}_t + \eta \tilde{s}_t, \bar{z}_{t+1/2} - \hat{\Delta}_t - \bar{z}_{t+1} \rangle \leq 0. \]

Summing over above inequalities, we have

\[
\begin{align*}
&\langle \bar{z}_{t+1} - \bar{z}_t + \eta \tilde{s}_{t+1/2}, \tilde{z}^* - \bar{z}_{t+1} \rangle + \langle \bar{z}_{t+1} - \bar{z}_t + \eta \tilde{s}_{t+1/2} + \bar{z}_{t+1} - \tilde{z}^*, \hat{\Delta}_{t+1/2} \rangle - \left\| \hat{\Delta}_{t+1/2} \right\|^2 \\
&+ \langle \bar{z}_{t+1/2} - \bar{z}_t + \eta \tilde{s}_{t+1/2}, \bar{z}_{t+1} - \tilde{z}_t + \eta \tilde{s}_{t+1/2} + \bar{z}_{t+1} - \tilde{z}_t + \hat{\Delta}_t \rangle - \left\| \hat{\Delta}_t \right\|^2 \geq 0,
\end{align*}
\]

which means

\[ \langle \bar{z}_{t+1} - \bar{z}_t, \tilde{z}^* - \bar{z}_{t+1} \rangle + \langle \bar{z}_{t+1} - \bar{z}_t, \bar{z}_{t+1} - \tilde{z}_{t+1/2} \rangle + \langle \bar{z}_{t+1/2} - \bar{z}_t + \eta \tilde{s}_{t+1/2} \rangle + \langle \bar{z}_{t+1/2} - \bar{z}_t + \eta \tilde{s}_{t+1/2} + \bar{z}_{t+1} - \tilde{z}_t + \hat{\Delta}_t \rangle \geq 0. \tag{28} \]

In constrained case, it is easy to verify the results of (29), (24), (27) and (20) also hold. Hence, we have

\[
\begin{align*}
2\eta \langle \bar{z}_{t+1} - \bar{z}_t, \tilde{z}^* - \bar{z}_{t+1} \rangle &= 2\eta \langle \bar{z}_{t+1} - \bar{z}_t, \tilde{z}^* - \bar{z}_{t+1} \rangle \\
&= 2\langle \bar{z}_t - \bar{z}_{t+1}, \tilde{z}^* - \bar{z}_t - \bar{z}_{t+1} \rangle + 2\langle \bar{z}_t - \bar{z}_{t+1}, \tilde{z}^* - \bar{z}_{t+1} \rangle + 2\langle \bar{z}_t - \bar{z}_{t+1}, \tilde{z}^* - \bar{z}_{t+1} \rangle \\
&= \| \bar{z}_t - \tilde{z}^* \|^2 - \| \bar{z}_t - \bar{z}_{t+1} \|^2 - \| \bar{z}_t - \tilde{z}_{t+1/2} \|^2 \geq 0. \tag{29} \\
&+ 2\eta \langle \bar{z}_{t+1} - \bar{z}_t, \tilde{z}_t + \eta \tilde{s}_{t+1/2} \rangle + \langle \bar{z}_{t+1} - \bar{z}_t + \eta \tilde{s}_{t+1/2} + \bar{z}_{t+1} - \tilde{z}_t, \hat{\Delta}_{t+1/2} \rangle \\
&+ \langle \bar{z}_{t+1/2} - \bar{z}_t + \eta \tilde{s}_t + \bar{z}_{t+1} - \tilde{z}_t, \hat{\Delta}_t \rangle \\
&\leq \| \bar{z}_t - \tilde{z}^* \|^2 - \| \bar{z}_t - \bar{z}_{t+1} \|^2 - \| \bar{z}_t - \tilde{z}_{t+1/2} \|^2 - \| \bar{z}_t - \bar{z}_{t+1} \|^2 \geq 0. \tag{29}
\end{align*}
\]

Combining (29) with (27), we have

\[
\begin{align*}
\| \bar{z}_{t+1} - \tilde{z}_t \|^2 &\leq \left( 1 - \frac{m \eta}{2} \right) \| \bar{z}_t - \tilde{z}^* \|^2 - \frac{1}{2} \| \bar{z}_{t+1/2} - \bar{z}_{t+1} \|^2 - \left( 1 - \frac{5m \eta}{2} - 6\eta^2 L^2 \right) \| \bar{z}_{t+1/2} - \tilde{z}_t \|^2 \\
&+ \frac{4L^2 \eta (3L \eta + 2\kappa)}{m} \left( \| \bar{z}_{t+1/2} - \bar{z}_{t+1/2} \|^2 + \| \bar{z}_t - \bar{z}_t \|^2 \right).
\end{align*}
\]
Using Lemma D.2 we have
\[ t \leq \left( 1 - \frac{\ln \eta}{2} \right) \| \bar{z}_t - z^* \|^2 + \hat{c}_t. \]

**Lemma D.6.** Suppose Assumption 2.1 and 2.5 and we have \( \| s_0 - 1 \bar{s}_0 \| \leq \delta' \). For Algorithm 3 with \( \eta = 1/(6L) \) and
\[ K \geq \sqrt{\chi} \log \left( \frac{2 (\sqrt{m}LD + \delta')}{\delta'} \right), \]
holds that \( \| z_t - 1 \bar{z}_t \| \leq \delta' \), \( \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \| \leq \delta' \), \( \| s_t - 1 \bar{s}_t \| \leq \delta'/\eta \) and \( \| s_{t+1/2} - 1 \bar{s}_{t+1/2} \| \leq \delta'/\eta \) for any \( t \geq 0 \).

**Proof.** The analysis is similar to the proof of Lemma [C.3]. It is obvious the statement holds for \( t = 0 \). Suppose we have that \( \| z_t - 1 \bar{z}_t \| \leq \delta' \), \( \| w_t - 1 \bar{w}_t \| \leq \delta' \), \( \| s_t - 1 \bar{s}_t \| \leq \delta'/\eta \) for \( t \geq 1 \), then Lemma [D.1] means
\[
\begin{align*}
\| z_{t+1/2} - 1 \bar{z}_{t+1/2} \| \\
&= \| \mathbb{T}(\mathcal{P}_z(z_t - \eta s_t)) - \frac{1}{m} 11^T \mathbb{T}(\mathcal{P}_z(z_t - \eta s_t)) \| \\
&\leq \rho \| \mathcal{P}_z(z_t - \eta s_t) - \frac{1}{m} 11^T \mathcal{P}_z(z_t - \eta s_t) \| \\
&\leq \rho \| \mathcal{P}_z(z_t - \eta s_t) - \mathcal{P}_Z(1(\bar{z}_t - \eta \bar{s}_t)) + \rho \| \mathcal{P}_Z(1(\bar{z}_t - \eta \bar{s}_t)) - \frac{1}{m} 11^T \mathcal{P}_Z(z_t - \eta s_t) \| \\
&\leq 2 \rho \| z_t - \eta s_t - 1(\bar{z}_t - \eta \bar{s}_t) \| + 2 \rho \| z_t - \eta s_t \| \leq (2 \sqrt{m}D + 2 \delta') \leq \delta'.
\end{align*}
\]
Using Lemma [D.2] we have
\[
\begin{align*}
\| s_{t+1/2} - 1 \bar{s}_{t+1/2} \| \\
&\leq \rho \left( L \| z_t - 1 \bar{z}_t \| + L \| z_{t+1/2} - 1 \bar{z}_{t+1/2} \| + \| s_t - 1 \bar{s}_t \| + L \sqrt{m} \| \bar{z}_{t+1/2} - \bar{z}_t \| \right) \\
&\leq \rho \left( 3L \sqrt{m}D + \delta'/\eta \right) \leq \delta'/\eta
\end{align*}
\]
Using Lemma [D.4] we have
\[
\begin{align*}
\| z_{t+1} - 1 \bar{z}_{t+1} \| \\
&= \| \mathbb{T}(\mathcal{P}_z(z_t - \eta s_{t+1/2})) - \frac{1}{m} 11^T \mathbb{T}(\mathcal{P}_z(z_t - \eta s_{t+1/2})) \| \\
&\leq \rho \| \mathcal{P}_z(z_t - \eta s_{t+1/2}) - \frac{1}{m} 11^T \mathcal{P}_z(z_t - \eta s_{t+1/2}) \| \\
&\leq \rho \| \mathcal{P}_z(z_t - \eta s_{t+1/2}) - \mathcal{P}_Z(1(\bar{z}_t - \eta \bar{s}_{t+1/2})) + \rho \| \mathcal{P}_Z(1(\bar{z}_t - \eta \bar{s}_{t+1/2})) - \frac{1}{m} 11^T \mathcal{P}_Z(z_t - \eta s_{t+1/2}) \| \\
&\leq 2 \rho \| z_t - \eta s_{t+1/2} - 1(\bar{z}_t - \eta \bar{s}_{t+1/2}) \| + 2 \rho \| z_t - \eta s_{t+1/2} \| \leq (2 \sqrt{m}D + 2 \delta') \leq \delta'.
\end{align*}
\]
Using Lemma [D.3] we have
\[
\| s_{t+1} - 1 \bar{s}_{t+1} \|
\]
Similarly, we obtain
\[ \|\hat{\Delta}_{t+1/2}\| \leq \frac{4\tilde{\delta}^t}{\sqrt{m}}. \]

We also have
\[
\begin{align*}
\|\tilde{s}_{t+1/2} - \tilde{s}_t + \eta \tilde{s}_t + \tilde{s}_{t+1/2} - \tilde{s}_{t+1}\| \\
= \|\tilde{s}_{t+1/2} - \tilde{s}_t\| + \|\tilde{s}_{t+1/2} - \tilde{s}_{t+1}\| + \eta \|\tilde{s}_t\| \\
\leq 2D + \eta \left\| \frac{1}{m} \mathbf{1}^\top \mathbf{g}(\mathbf{z}_t) \right\| \\
= 2D + \eta \left\| \frac{1}{m} \sum_{i=1}^m g_i(\mathbf{z}_t(i)) \right\|.
\end{align*}
\]

\[\square\]

**Lemma D.7.** Under the settings of Lemma D.5 and Lemma D.6, we have
\[ \hat{\zeta} \leq \frac{4Ln(2\kappa + 3Ln)\tilde{\delta}^2}{m} + \frac{8\hat{C}_t\tilde{\delta}^t}{\sqrt{m}} \]

**Proof.** The analysis is similar to the proof of Lemma 4.6. Using the non-expansiveness of projection, the update rule \( z'_t = \alpha z_t + (1 - \alpha) w_t \), and Lemma D.6, we have
\[
\|\hat{\Delta}_t\| = \left\| \frac{1}{m} \mathbf{1}^\top \mathcal{P}_Z(z_t - \eta s_t) - \mathcal{P}_Z(\tilde{z}_t - \eta \tilde{s}_t) \right\|
\]
\[
= \left\| \frac{1}{m} \sum_{i=1}^m \left[ \mathcal{P}_Z(z_t(i) - \eta s_t(i)) - \mathcal{P}_Z(\tilde{z}_t(i) - \eta \tilde{s}_t(i)) \right] \right\|
\]
\[
\leq \left\| \frac{1}{m} \sum_{i=1}^m \left[ (z_t(i) - \eta s_t(i)) - (\tilde{z}_t(i) - \eta \tilde{s}_t(i)) \right] \right\|
\]
\[
\leq \left\| \frac{1}{m} \sum_{i=1}^m \left[ (z_t(i) - \tilde{z}_t(i)) + \eta s_t(i) - \eta \tilde{s}_t(i) \right] \right\|
\]
\[
\leq \left\| \frac{1}{m} \sum_{i=1}^m \left[ (z_t(i) - \tilde{z}_t(i)) + \eta s_t(i) - \eta \tilde{s}_t(i) \right] \right\|
\]
\[
\leq \frac{4\tilde{\delta}^t}{\sqrt{m}}.
\]

Similarly, we obtain
\[
\|\hat{\Delta}_{t+1/2}\| \leq \frac{4\tilde{\delta}^t}{\sqrt{m}}.
\]
\[ \leq 2D + \eta \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left\| g_i(z_t(i)) \right\|^2} \]

\[ \leq 2D + \eta \sqrt{\frac{2}{m} \sum_{i=1}^{m} \left( \left\| g_i(z_t(i)) - g_i(z^*) \right\|^2 + \left\| g_i(z^*) \right\|^2 \right)} \]

\[ \leq 2D + \eta \sqrt{\frac{2}{m} \sum_{i=1}^{m} \left( L^2 \left\| z_t(i) - z^* \right\|^2 + \left\| g_i(z^*) \right\|^2 \right)} \]

\[ \leq 2D + \frac{1}{6L} \left( 2L^2 D^2 + \frac{2}{m} \sum_{i=1}^{m} \left\| g_i(z^*) \right\|^2 \right) \]

\[ \hat{\delta} \triangleq \hat{C}_1 \]

and

\[ \left\| \tilde{z}_{t+1} - \hat{\delta}_t + \eta \hat{\delta}_{t+1/2} + \tilde{z}_{t+1} - z^* \right\| \leq \hat{C}_1. \]

Combing above results, we have

\[ \hat{\delta}_t = \frac{2L\eta (2\kappa + 3L\eta)}{m} \left( \left\| z_{t+1/2} - 1z_{t+1/2} \right\|^2 + \left\| z_t - 1\tilde{z}_t \right\|^2 \right) \]

\[ + \left\| \tilde{z}_{t+1} - \hat{\delta}_t + \eta \hat{\delta}_{t+1/2} + \tilde{z}_{t+1} - z^* \right\| \left\| \Delta_{t+1/2} \right\| + \left\| \tilde{z}_{t+1/2} - \hat{\delta}_t + \eta \hat{\delta}_t + \tilde{z}_{t+1/2} - \tilde{z}_{t+1} \right\| \left\| \Delta_t \right\| \]

\[ \leq \frac{4L\eta (2\kappa + 3L\eta) \hat{\delta}^2}{m} + \frac{8\hat{C}_1 \hat{\delta}'^2}{\sqrt{m}}. \]

Then we provide the proof of Theorem \ref{thm:main}.\hfill \Box

**Proof.** The setting of $K_0$ means $\|s_0 - 1\bar{s}_0\| \leq \hat{\delta}'$. Then Lemma \ref{lem:grad} and Lemma \ref{lem:grad2} imply

\[ \|\tilde{z}_{t+1} - \hat{\delta}_t\|^2 \leq \left( 1 - \frac{1}{12\kappa} \right) \|\tilde{z}_t - \hat{\delta}_t\|^2 + \frac{4L\eta (2\kappa + 3L\eta) \hat{\delta}^2}{m} + \frac{8\hat{C}_1 \hat{\delta}'^2}{\sqrt{m}}. \]

Then we have

\[ \|\tilde{z}_T - \hat{\delta}_T\|^2 \leq \left( 1 - \frac{1}{12\kappa} \right)^T \|\tilde{z}_t - \hat{\delta}_t\|^2 + \frac{4L\eta (2\kappa + 3L\eta) \hat{\delta}^2}{m} + \frac{8\hat{C}_1 \hat{\delta}'^2}{\sqrt{m}} \]

\[ = \left( 1 - \frac{1}{12\kappa} \right)^T \|\tilde{z}_t - \hat{\delta}_t\|^2 + \frac{4\kappa (4\kappa + 1) \hat{\delta}^2}{m} + \frac{96\kappa \hat{C}_1 \hat{\delta}'^2}{\sqrt{m}} \]

\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \]

where the last step is due to the value of $\eta$, $T$ and $\hat{\delta}'$.

Since each iteration requires $O(n)$ SFO calls in expectation and we the algorithm needs to compute the full gradient at first, the total complexity is

\[ O(nT) = O \left( \kappa n \log \left( \frac{1}{\varepsilon} \right) \right). \]

The number of communication rounds is

\[ O(nT) = O \left( \kappa \sqrt{n} \log \left( \frac{\kappa}{\varepsilon} \right) \log \left( \frac{1}{\varepsilon} \right) \right). \]

\hfill \Box

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D.6 The Proof of Theorem 5.3

Proof. Since the objective function is non-strongly-convex and non-strongly-concave, we first modify (27) as follows

\[
2\eta(\tilde{s}_{t+1/2}, \tilde{z}_{t+1/2} - \bar{z}^*) \\
= 2\eta(g(\tilde{z}_{t+1/2}), \tilde{z}_{t+1/2} - \bar{z}^*) + 2\eta(\tilde{s}_{t+1/2} - g(\tilde{z}_{t+1/2}), \tilde{z}_{t+1/2} - \bar{z}^*) \\
\geq 2\eta(g(\bar{z}^*), \tilde{z}_{t+1/2} - \bar{z}^*) - \beta \eta \|\tilde{s}_{t+1/2} - g(\tilde{z}_{t+1/2})\|^2 - \frac{\eta}{\beta} \|\tilde{z}_{t+1/2} - \bar{z}^*\|^2 
\]

(30)

where we use Lemma A.4, C.2 and \( \beta = 2D^2/\varepsilon \).

It is easy to verify (25) and (26) still holds. Connecting the with and (30), we have

\[
2\eta(\tilde{g}(\tilde{z}_{t+1/2}), \tilde{z}_{t+1/2} - \bar{z}^*) - \beta \eta \|\tilde{s}_{t+1/2} - \tilde{g}(\tilde{z}_{t+1/2})\|^2 - \frac{\eta}{\beta} \|\tilde{z}_{t+1/2} - \bar{z}^*\|^2 \\
\leq 2\eta(\tilde{s}_{t+1/2}, \tilde{z}_{t+1/2} - \bar{z}^*) \\
\leq \|\tilde{z}_t - \bar{z}^*\|^2 - \|\tilde{z}_{t+1} - \bar{z}^*\|^2 - \frac{1}{2} \|\tilde{z}_{t+1/2} - \tilde{z}_{t+1}\|^2 - (1 - 6\eta^2L^2) \|\tilde{z}_{t+1/2} - \bar{z}_t\|^2 \\
+ \frac{6L^2\eta^2}{m} \left( \|\mathbf{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \|\mathbf{z}_t - 1\tilde{z}_t\|^2 \right) \\
+ \|\tilde{z}_{t+1} - \tilde{z}_t + \eta\tilde{s}_{t+1} + \tilde{z}_{t+1} - \bar{z}^*\| \|\Delta_{t+1/2}\| + \|\tilde{z}_{t+1/2} - \bar{z}_t + \eta\tilde{s}_t + \tilde{z}_{t+1/2} - \tilde{z}_{t+1}\| \|\hat{\Delta}_t\|, 
\]

which implies

\[
2\eta(\tilde{g}(\tilde{z}_{t+1/2}), \tilde{z}_{t+1/2} - \bar{z}^*) \\
\leq \|\tilde{z}_t - \bar{z}^*\|^2 - \|\tilde{z}_{t+1} - \bar{z}^*\|^2 + \beta \eta \|\tilde{s}_{t+1/2} - \tilde{g}(\tilde{z}_{t+1/2})\|^2 + \frac{\eta}{\beta} \|\tilde{z}_{t+1/2} - \bar{z}^*\|^2 \\
+ \frac{6L^2\eta^2}{m} \left( \|\mathbf{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \|\mathbf{z}_t - 1\tilde{z}_t\|^2 \right) \\
+ \|\tilde{z}_{t+1} - \tilde{z}_t + \eta\tilde{s}_{t+1} + \tilde{z}_{t+1} - \bar{z}^*\| \|\Delta_{t+1/2}\| + \|\tilde{z}_{t+1/2} - \bar{z}_t + \eta\tilde{s}_t + \tilde{z}_{t+1/2} - \tilde{z}_{t+1}\| \|\hat{\Delta}_t\| \\
\leq \|\tilde{z}_t - \bar{z}^*\|^2 - \|\tilde{z}_{t+1} - \bar{z}^*\|^2 + \frac{3L^2\eta^2}{m} \|\mathbf{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \frac{\eta D^2}{\beta} \\
+ \frac{6L^2\eta^2}{m} \left( \|\mathbf{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \|\mathbf{z}_t - 1\tilde{z}_t\|^2 \right) \\
+ \|\tilde{z}_{t+1} - \tilde{z}_t + \eta\tilde{s}_{t+1} + \tilde{z}_{t+1} - \bar{z}^*\| \|\Delta_{t+1/2}\| + \|\tilde{z}_{t+1/2} - \bar{z}_t + \eta\tilde{s}_t + \tilde{z}_{t+1/2} - \tilde{z}_{t+1}\| \|\hat{\Delta}_t\| \\
= \|\tilde{z}_t - \bar{z}^*\|^2 - \|\tilde{z}_{t+1} - \bar{z}^*\|^2 + \frac{3L^2\eta^2}{m} \|\mathbf{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \bar{\zeta}_t, 
\]

where

\[
\bar{\zeta}_t = \frac{\eta D^2}{\beta} + \frac{6(\eta^2 + \beta \eta)L^2}{m} \left( \|\mathbf{z}_{t+1/2} - 1\tilde{z}_{t+1/2}\|^2 + \|\mathbf{z}_t - 1\tilde{z}_t\|^2 \right) \\
+ \|\tilde{z}_{t+1} - \tilde{z}_t + \eta\tilde{s}_{t+1} + \tilde{z}_{t+1} - \bar{z}^*\| \|\Delta_{t+1/2}\| + \|\tilde{z}_{t+1/2} - \bar{z}_t + \eta\tilde{s}_t + \tilde{z}_{t+1/2} - \tilde{z}_{t+1}\| \|\hat{\Delta}_t\| 
\]

and the second inequality is based on the fact

\[
\|\tilde{s}_{t+1/2} - g(\tilde{z}_{t+1/2})\|^2 \\
= \left( \frac{1}{m} \sum_{i=1}^{m} \left( g_i(\mathbf{z}_{t+1/2}(i)) - g_i(\tilde{z}_{t+1/2}) \right) \right)^2 \\
\leq \frac{1}{m} \sum_{i=1}^{m} \left( g_i(\mathbf{z}_{t+1/2}(i)) - g_i(\tilde{z}_{t+1/2}) \right)^2 
\]

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\[
\frac{L^2}{m} \sum_{i=1}^{m} \left\| z_{t+1/2}(i) - \bar{z}_{t+1/2}^T \right\|^2 \\
= \frac{L^2}{m} \left\| z_{t+1/2} - \mathbf{1}\bar{z}_{t+1/2} \right\|^2 .
\]

Following the proof of Lemma \[ \text{D.7} \] we have

\[
\hat{\zeta}'_t \leq \frac{\eta D^2}{\beta} + \frac{12(\eta^2 + \beta \eta)L^2 \hat{\delta}'^2}{m} + \frac{8C_1 \hat{\delta}'}{\sqrt{m}}.
\]

Summing over (31) with \( t = 0, \ldots, T - 1 \), we obtain

\[
f(\tilde{x}, y^*) - f(x^*, \tilde{y}) \\
\leq \frac{1}{T} \sum_{t=0}^{T-1} \left( f(\tilde{x}_{t+1/2}, y^*) - f(x^*, \tilde{y}_{t+1/2}) \right) \\
\leq \frac{1}{T} \sum_{t=0}^{T-1} \langle g(\tilde{z}_{t+1/2}), \tilde{z}_{t+1/2} - z^* \rangle \\
\leq \frac{1}{T} \left( \| \bar{z}_0 - z^* \|^2 - \| \bar{z}_T - z^* \|^2 \right) + \sum_{t=0}^{T-1} \hat{\zeta}'_t \\
\leq \frac{\| \bar{z}_0 - z^* \|^2}{2\eta T} + \frac{1}{2\eta} \left( \frac{\eta D^2}{\beta} + \frac{12(\eta^2 + \beta \eta)L^2 \hat{\delta}'^2}{m} + \frac{8C_1 \hat{\delta}'}{\sqrt{m}} \right) \\
\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \frac{\epsilon}{8} = \epsilon.
\]

where the first inequality use Jensen’s inequality; the second inequality use the objective function is convex-concave; the third inequality use the upper bound of \( \hat{\zeta}'_t \); and the last one is based on the value of parameter settings.

Since each iteration requires \( O(n) \) SFO calls in expectation and we the algorithm needs to compute the full gradient at first, the total complexity is

\[
O(nT) = O \left( \frac{nL}{\epsilon} \right).
\]

The number of communication round is

\[
KT + K_0 = O \left( \frac{L \sqrt{N}}{\epsilon} \log \left( \frac{L}{\epsilon} \right) \right).
\]