Nonstandard Treatment of Two Dimensional Taylor Series with Reminder Formulas

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ABSTRACT
The aim of this paper is to establish some new two dimensional Taylor series formulas using some concepts of nonstandard analysis given by Robinson and axiomatized by Nelson

Keyword: nonstandard analysis, infinitely near, Taylor series.

1- Introduction:
Let $f$ be a continuous function defined on a domain $D$ and posses its derivatives up to order $n$ in $D$, then the Taylor development of $f(x)$ about $x_0$ with remainder form is given by:
\[ f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_o)}{k!} (x - x_o)^k + R_{n-1}(x), \]

where \( x_o \in D \) and \( R_{n-1}(x) \) is the remainder, which takes one of the following forms:

\[
R_{n-1}(x) = \sum_{k=n}^{\infty} \frac{f^{(k)}(x_o)}{k!} (x - x_o)^k
\]

\[
R_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!} (x - x_o)^n, \quad \text{for} \quad \xi \in [x_o, x]
\]

\[
R_{n-1}(x) = \frac{1}{(n-1)!} \int_{x_o}^{x} (x - t)^{n-1} f^{(n)}(t) \, dt
\]

Through this paper we need the following nonstandard concepts:

Every set or element defined in a classical mathematics is called standard [1].

**Definition 1.1**

A real number \( X \) is called limited if there exists a positive standard real number \( r \) such that \(|x| \leq r\), otherwise it is called unlimited. The set of all unlimited real numbers is denoted by \( \overline{R} \) [1].

**Definition 1.2**

A real number \( X \) is called infinitesimal if \(|x| \leq r\), for all positive standard real numbers \( r \) [1].

**Definition 1.3**

Two real numbers, \( X \) and \( Y \) are infinitely close if \( X - Y \) is infinitesimal, and is denoted by \( X \cong Y \) [1].

**Definition 1.4**

A function \( f \) is differentiable at \( x_o \), denoted by \( f'(x_o) \), if there exists a standard number \( \lambda \) such that:

\[
f'(x_o) = \lambda \equiv \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}. [3]
\]

2- Higher Order Differentiation

In [2] and [6] a brief introduction of higher order differentiation is given. Suppose that \( Z = f(x, y) \) is a function of two variables with continuous partial derivatives of first order, then the differentiation of \( Z \), denoted by \( dZ \), is defined by:
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\[ dz = df(x,y) = f_x(x,y)dx + f_y(x,y)dy \]
since \( dz \) is also a function of \( x \) and \( y \), so if the second order partial derivatives of \( f \) exists then differentiation of \( dz \) exists, and it is called second order differentiation, which is denoted by \( d^2z \).

It is important to emphasize that the quantities \( dx \) and \( dy \) are assumed to be constants. Therefore we have:

\[ d^2z = d^2f(x,y) = d( df(x,y)) \]
\[ = (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \]
\[ = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \]
\[ = (D_xdx + D_ydy)^2 f(x,y), \text{ where } D_x = \frac{\partial}{\partial x} \]

that is

\[ d^2f(x,y) = (D_xdx + D_ydy)^2 f(x,y). \] ...

...(2.1)

In general

\[ d^n f(x,y) = (D_xdx + D_ydy)^n f(x,y) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} D_x^{n-k} D_y^k dx^{n-k} dy^k f(x,y) \]

...(2.2)

Consider now \( z = f(x,y) \) such that:

\( x = u(t) \) and \( y = v(t) \) then \( df(x,y) \) and \( d^2f(x,y), \ldots \) are given as follows:

\[ df(x,y) = f_x(x,y)dx + f_y(x,y)dy \]

where \( dx \) and \( dy \) are differentials of other functions not still constant, therefore

\[ d^2f(x,y) = (D_xdx + D_ydy)^2 f(x,y) + (D_xd^2x + D_yd^2y)f(x,y) \]

and

\[ d^3f(x,y) = (D_xdx + D_ydy)^3 f(x,y) + f_x d^3x + 2f_{xx} d^2x^2 + f_y d^3y + 2f_{xy} d^2x d^2y + 3f_{xx} d^2xdy + 3f_{xy} d^2y. \]

Therefore

\[ d^4f(x,y) = (D_xdx + D_ydy)^4 f(x,y) + \sum_{i=0}^{4} \binom{4}{i} (f_{x,i} d^{4-i}x^i + f_{y,i} d^{4-i}y^i) + g(D_x, D_y, dx, dy), \]

where \( g(D_x, D_y, dx, dy) = 3f_{xy} d^2xdy + 3f_{yx} dx d^2y. \)
The following lemma gives a general form of any compound function \( f(x, y) \)

**Lemma 2.1**

Let \( f(x, y) \) be a continuous function of two variables \( x \) and \( y \) such that \( x = u(t) \) and \( y = v(t) \) where \( a \leq t \leq b \) for \( a, b \in \mathbb{R} \), then the \( n^{th} \) order differentiation of \( f(x, y) \) is given by:

\[
\frac{d^n f(x, y)}{dt^n} = \left( \frac{D_x d^n f}{dt^n} + D_y d^n f \right) + \sum_{i=1}^{n-1} \binom{n-1}{i} \left( f_{x,i} d^{n-1-i} x^i + f_{y,i} d^{n-1-i} y^i \right) + \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} \sum_{k=1}^{n-1-i} \alpha_i \binom{n}{i} \left( f_{x,y} d^{i-j-1} x^i y^{j} d^{n-i-k-1} y^k \right)
\]

where \( \alpha_i \) are real constants

**Proof:**

Use mathematical induction to get the result.

3- **Taylor Expansion of \( f(x,y) \)**

Let \( f \) be a real valued function defined on a domain \( D \), then

\[
\Delta f(x_o) = f(x) - f(x_o) = f(x_o + \Delta x) - f(x_o), \quad \ldots(3.1)
\]

where \( \Delta x = x - x_o \) (later we shall use \( h = x - x_o \)).

Therefore

\[
\Delta f(x_o) = \sum_{k=1}^{\infty} \frac{f^{(k)}(x_o)}{k!} \Delta x^k \Delta^k x + R_{n+1}(x_o) \quad \ldots(3.2)
\]

where \( R_{n+1}(x_o) = \frac{f^{(n)}(\xi)}{n!} \) for some \( \xi \in [x_o, x] \) [3].

Now by using Definition (1.4) we get that \( \Delta y \equiv f'(x_o) \Delta x \), and then

\[
dy \equiv \Delta y \Rightarrow dy \equiv f'(x_o) \Delta x, \quad \ldots(3.3)
\]

therefore

\[
\Delta f(x_o) \equiv \sum_{k=1}^{\infty} \frac{d^k f(x_o)}{k!}
\]

thus
The formulas (3.4) and (3.5) represent differential formulas of a Taylor series expansion with remainder. Similarly with a necessary modification we can define a Taylor series expansion of multiple variable functions [2], [5]. Let \( z = f(x, y) \) be a function of two variables defined in a rectangular region \( D \) such that its \( n \)-partial derivatives are defined and continuous in \( D \). By using (3.1) and (3.4) we find that:

\[
f(x, y) = \sum_{k=0}^{n-1} \frac{d^k f(x_o, y_o)}{k!} + R_{n-1}(x_o, y_o)
\]

with the assumption that

\[
f(x_o, y_o) = d^0 f(x_o, y_o) \quad \text{and} \quad R_{n-1}(x_o, y_o) = \frac{1}{n!} d^n f(\xi, \lambda) \quad \text{for some} \quad \xi \in [a, x] \quad \text{and} \quad \lambda \in [c, y] \quad \text{in} \quad D = \{(x, y): a \leq x \leq b, c \leq y \leq d\}.
\]

Now putting \( h_x = x - x_o \) and \( h_y = y - y_o \), and then applying (2.2) and (3.6) we get:

\[
f(x, y) = \sum_{k=0}^{n-1} \frac{1}{k!} \sum_{s=0}^{\min(s, k)} \binom{s}{k} D_x^{s-k} D_y^k h_x^{s-k} h_y^k f(x_o, y_o) + R_{n-1}(x_o, y_o)
\]

where

\[
R_{n-1}(x_o, y_o) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} D_x^{n-k} D_y^k h_x^{n-k} h_y^k f(\xi, \lambda) \quad \text{for some} \quad \xi \in [a, x], \lambda \in [c, y] \quad [6].
\]

Consequently, with the first formula of (2.2) we can write the exponential Taylor expansion formula of a function of two variables as:

\[
f(x, y) \approx f(x_o, y_o) + \sum_{s=0}^{n-1} \frac{1}{s!} (D_x h_x + D_y h_y)^s f(x_o, y_o) \approx e^{(D_x h_x + D_y h_y)} f(x_o, y_o) \quad \text{for} \quad \text{unlimited} \quad n.
\]

In the next section we try to deduce new formulas of Taylor series with different forms of remainders.

4- Integral Formula of Taylor Series with Remainders
The integral formula of Taylor series of a function of two variables is based on the line integral on a curve. Let \( z = f(x, y) \) be a two variables function whose partial derivatives \( f_x \) and \( f_y \) are defined and continuous in an open rectangle region \( D \) and its differentiation is given by:

\[
df(x, y) = f_x \, dx + f_y \, dy = P \, dx + Q \, dy,
\]
provided that \( f(x, y) \) posses its integral line \( \int_C \df(x, y) \) where \( C \) is a curve in \( D \). Let \( A(x_o, y_o) \) be the initial point of \( C \) and \( B(x, y) \) be the terminal point of \( C \), then

\[
\int_C \df(x, y) = \int_{A(x_o, y_o)}^{B(x,y)} \df(x, y), \quad \text{...(4.2)}
\]

Therefore

\[
\int_C (P \, dx + Q \, dy) = f(x, y) - f(x_o, y_o), \quad \text{...(4.3)}
\]

provided that the differentiation is not exact whenever we used it, since the line integral of exact differentiation will vanish.

**Theorem 4.1**

Let \( z = f(x, y) \) be a function of two variables whose \( n \) partial derivatives in \( x \) and \( y \) are continuous in an open rectangular region \( D \) such that \( f(x, y) \) has a total differential of any order over a sectionally smooth curve \( C \) contained completely in \( D \) with initial point \((x_o, y_o)\) and terminal point \((x, y)\). Then the Taylor series of \( f(x, y) \) whose integral form of the remainder is given by:

\[
f(x, y) = f(x_o, y_o) + \sum_{k=1}^{n-1} \left( \int_C \right)^k \, d^k f(x_o, y_o) + R_{n-1}(x_o, y_o), \quad \text{where}
\]

\[
R_{n-1}(x_o, y_o) = \frac{1}{n!} \int_C \cdots \int_C \, d^n f(s, u) \quad \text{for some } s \in [a, x] \quad \text{and} \quad u \in [c, y] \quad \text{in} \quad D = \{(x, y) : a \leq x \leq b, \quad c \leq y \leq d\}.
\]

**Proof:**

Since

\[
\int_C \df(s, u) = \int_{A(x_o, y_o)}^{B(x,y)} \df(s, u) = f(x, y) - f(x_o, y_o),
\]
then by using (2.1) we get
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\[ f(x, y) = f(x_0, y_0) + \int df(s, u) \]

\[ = f(x_0, y_0) + \left[ \int df(x_0, y_0) + \frac{1}{2} \int d^2f(s, u) \right] \]

where \( df(x_0, y_0) = df(x, y) \bigg|_{x=x_0, y=y_0} \).

In general we obtain:

\[ f(x, y) = f(x_0, y_0) + \sum_{k=1}^{\infty} \left( \int \frac{1}{k!} d^k f(x_0, y_0) + \int \frac{1}{k!} d^k f(s, u) \bigg|_{x=x_0, y=y_0} \right) \]

\[ = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \int \frac{1}{k!} d^k f(s, u) \right) \text{for some } s \in [a, x] \text{ and } u \in [c, y]. \]

**Corollary 4.2**

Let \( z = f(x, y) \) be a two variables function satisfying the conditions of Theorem 4.1, then:

\[ R_{n-1}(x_0, y_0) = \sum_{k=1}^{\infty} \left( \int \frac{1}{k!} d^k f(x_0, y_0) \right) \]

\[ = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \int \frac{1}{k!} d^k f(s, u) \bigg|_{x=x_0, y=y_0} \right) \]

**Proof:**

For finding its Taylor expansion, expand \( f \) in a Taylor series and use formula (2.2).

**Theorem 4.3**

Let \( f(x, y) \) be a function whose \( n \) partial derivatives in \( x \) and \( y \) are continuous in an open rectangular region \( D \) such that \( f(x, y) \) has a total differential of any order over a sectionally smooth curve \( C \) where \( C \) is a curve from \( A(0, x) \) to \( B(0, y) \). Then the Taylor series of \( f(x, y) \) with integral form of the remainder is given by:

\[ f(x, y) = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{(x^k)(y^r)}{k!r!} D_x^k D_y^r f(x, y) \bigg|_{x=x_0, y=y_0} + R_{n-1}(x_0, y_0), \]

where

\[ R_{n-1} = \frac{1}{2^n(n-1)!} \left( \int_0^x (x-u)^{n-1} f_x(u, t) du + \int_0^y (y-v)^{n-1} f_y(v, s) dv \right) + \frac{1}{2^n} \sum_{k=0}^{m-2} \frac{(m-2)_k}{(m-k)!} \times 1, \]

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and \[ I = \int_0^x \int_0^y (x - u)^{m-k}(y - v)^k f_{g^{-1}_k}(u,v)\,du\,dv \]

Proof:
Put \( f_o = f(X_o) = f(x_o, y_o) \), then using theorem (4.1) we get
\[
f = f_o + \sum_{k=1}^{n-1} \frac{1}{k!} \left( \int_C \right)^k d^k f_o, \]
\[
= f_o + \frac{1}{2} \int_{X_o} (x)\,dx + f_y(X_o)\,dy \]
\[
+ \frac{1}{4} \int_{X_o} \int_{X_o} \left\{ f_{xx}(X_o)\,dx^2 + 2f_{xy}(X_o)\,dx\,dy + f_{x}(X_o)\,dx \right\} + \sum_{k=3}^{n-1} \frac{1}{k!} \left( \int_C \right)^k d^k f_o \]
\[
= f(x_o) + \frac{1}{2} [xf_x(x_o) + yf_y(x_o)] \]
\[
+ \frac{1}{4} \left[ \frac{x^2}{2} f_{xx}(x_o) + 2xyf_{xy}(x_o) + \frac{y^2}{2} f_{yy}(x_o) \right] + \sum_{k=3}^{n-1} \frac{1}{k!} \left( \int_C \right)^k d^k f_o \]

In general applying formula (2.2) to expand each \( d^n \) and integrate the result term by term we obtain:
\[
f = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{k}{i} \left( xD_x \right)^{k-i} \left( yD_y \right)^i f(x,y) \bigg|_{x=x_o}^{y=y_o}, \]

for determination of \( R_{n-1} \), follows from Theorem 4.1, thus
\[
R_{n-1} = \frac{1}{2^n} \int_C \cdots \int_C D^n s f(s,t) \quad \text{for some } s \in [0,x] \text{ and } t \in [0,y]. \]

Therefore
\[
R_{n-1} = \frac{1}{2^n} \int_C \cdots \int_C \sum_{i=0}^{n} \binom{n}{i} D_i s f(s,t)ds^{n-s}dt^i \]
for the last formula (4.4) we use integration by part to get the first and final terms of $R_{n-1}$ then using the result obtained by calculating the values of between in terms of $R_{n-1}$ to get the final result of $R_{n-1}$ as follows:

$$R_{n-1} = \frac{1}{2^n(n-1)!} \int_{y_0}^{y} \left( (x-u)^{n-1} f_x(u,t)du \right) + \frac{1}{2^n} \sum_{k=0}^{m+n-2} \frac{\binom{m+2}{k+1}}{(m-k)!k!} x^I$$

where

$$I = \int_{x_0}^{x} \int_{y_0}^{y} (x-u)^{m-k} (y-v)^k f_{x^{m-k}y^{k+1}}(u,v)dudv$$

**Theorem 4.4**

Let $z = f(x,y)$ be a function whose $n$ partial derivatives in $x$ and $y$ are continuous in an open rectangular region $D$ such that $f(x,y)$ has a total differential of any order over a sectionally smooth curve $C$ where $C$ is a curve whose parametric equations are given by $x = h(t)$, $y = g(t)$ $\alpha \leq t \leq \beta$ $\alpha, \beta \in \mathbb{R}$, where the initial point is $A(x_0,y_0) = (h(\alpha),g(\alpha))$ and the terminal point is $B(x,y) = (h(\beta),g(\beta))$ for some $t \in [\alpha, \beta]$. Therefore the Taylor series of $f$ whose remainder is given by:

$$f(x(t),y(t)) = f(x(\alpha),y(\alpha)) + f'(\alpha)(t-\alpha) + \frac{f''(\alpha)}{2}(t-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1} + R_{n-1}(x_0,y_0),$$

where

$$R_{n-1}(x_0,y_0) = \frac{1}{(n-1)!} \int_{\alpha}^{t} (t-u)s^{(n)}(u)du$$

And $s(u)$ is the integral of the quantity

$$P(x(u),y(u))x'(u) + Q(x(u),y(u))y'(u)$$

**Proof:**
We have
\[\int_C \left( Pdx + Qdy \right) = \int_{(x_0,y_0)}^{(x,y)} Pdx + Qdy\]
\[= \int_a^t \left[ P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t) \right] dt,\]
where \(\alpha < t \leq \beta\).

Now using equation (4.3) to get
\[f(x(t), y(t)) = f(x(\alpha), y(\alpha)) + \int_a^t s'(u)du\]
Then applying integration by part on the last equation \(n\)-times we get:
\[f(x(t), y(t)) = f(x(\alpha), y(\alpha)) + \sum_{k=1}^{n-1} \frac{s^{(k)}(\alpha)}{k!} + R_{n-1},\]
where
\[R_{n-1}(x_0, y_0) = \frac{1}{(n-1)!} \int_a^t (t-u)s^{(n)}(u)du\]
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