ON EXISTENCE OF THE PRESCRIBING $k$-CURVATURE OF THE EINSTEIN TENSOR

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Abstract. In this paper, we study the problem of conformally deforming a metric on a 3-dimensional manifold $M^3$ such that its $k$-curvature equals to a prescribed function, where the $k$-curvature is defined by the $k$-th elementary symmetric function of the eigenvalues of the Einstein tensor, $1 \leq k \leq 3$. We prove the solvability of the problem and the compactness of the solution sets on manifolds when $k = 2$ and 3, provided the conformal class admits a negative $k$-admissible metric with respect to the Einstein tensor.

1. Introduction

Let $(M^n, g)$ be an $n$-dimensional compact Riemannian manifold with or without boundary, $n \geq 3$. Let $\text{Ric}$ and $R$ be the Ricci tensor and the scalar curvature, respectively. Then the Einstein tensor is defined by

$$E_g = \frac{1}{n-2}(\text{Ric} - \frac{R}{2}g),$$

which can be also viewed as a special case of the following modified Schouten tensor with a parameter $\tau$ that was introduced by Gursky and Viaclovsky [15], and A. Li and Y.-Y. Li [24] independently:

$$A^\tau_g = \frac{1}{n-2}(\text{Ric} - \frac{\tau R}{2(n-1)}g),$$

where $\tau \in \mathbb{R}$ and the Einstein tensor is just the case $\tau = n - 1$.

Einstein tensor plays a key role in general relativity, and was extensively studied by many researchers. In this paper, we focus on its property in the conformal class.

Let $\lambda(A^\tau_g) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of $A^\tau_g$ with respect to $g$. We also define the $k$-curvature of $\lambda(A^\tau_g)$ as

$$\sigma_k(\lambda) = \Sigma_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

the $k$-th elementary symmetric polynomial, and

$$\Gamma_k^+ = \{ \lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0 \text{ for } j = 1, \cdots, k \}$$

the corresponding open, convex cone in $\mathbb{R}^n$.

Define $\Gamma_k^- = \{ \lambda \in \mathbb{R}^n \mid -\lambda \in \Gamma_k^+ \}$. We call $g$ is $k$-admissible if $\lambda(g^{-1}A^\tau_g) \in \Gamma_k^+$ or negative $k$-admissible if $\lambda(g^{-1}A^\tau_g) \in \Gamma_k^-$. 

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With the conformal transformation \( \tilde{g} = u^{\frac{4}{n-k}} g \), the modified Schouten tensor changes by
\[
A_{\tilde{g}}^\gamma = -\frac{2}{n-2} u^{-1} \nabla^2 u - \frac{2(1-\tau)}{(n-2)^2} u^{-1} (\Delta u)g + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{-2} |\nabla u|^2 g + A_g^\gamma.
\]
When \( \tau = n - 1 \), we have
\[
E_{\tilde{g}} = -\frac{2}{n-2} u^{-1} \nabla^2 u + \frac{2}{(n-2)^2} u^{-1} (\Delta u)g + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{-2} |\nabla u|^2 g + E_g.
\]
If \( \tilde{g} = e^{-2u} g \), we have
\[
A_{\tilde{g}}^\gamma = A_g^\gamma + \nabla^2 u + \frac{1}{n-2} \Delta u + du \otimes du - \frac{2-\tau}{2} |\nabla u|^2 g.
\]
In particular, when \( \tau = n - 1 \), we have
\[
E_{\tilde{g}} = E_g + \nabla^2 u - \Delta u + du \otimes du + \frac{n-3}{2} |\nabla u|^2 g.
\]

Similar to the \( k \)-Yamabe problem, the \( k \)-curvature of the Einstein tensor is defined by
\[
\sigma_k(E_g) = \sigma_k(\lambda(g^{-1}E_g)).
\]
It is natural to ask: can we find a \( k \)-admissible (or negative \( k \)-admissible) metric in the conformal class of \( g \) with constant \( k \)-curvature of the Einstein tensor? This problem is equivalent to find a solution to the following equation:
\[
\sigma_k(E_g + \nabla^2 u - \Delta u + du \otimes du + \frac{n-3}{2} |\nabla u|^2 g) = 1.
\]
or
\[
\sigma_k(-\nabla^2 u + \Delta u - du \otimes du - \frac{n-3}{2} |\nabla u|^2 g - E_g) = 1.
\]
The equation (13) is elliptic, which in some sense corresponds to the \( k \)-Yamabe equation in the positive cone, i.e
\[
\sigma_k(A_g^\gamma + \nabla^2 u + \frac{1}{n-2} \Delta u + du \otimes du - \frac{2-\tau}{2} |\nabla u|^2 g) = 1
\]
with \( \tau = 1 \).

The \( k \)-Yamabe problem have been extensively studied in the last decade. When \( k = 1 \), equations (1.4) and (1.3) become to the classical Yamabe equation. The existence of its solution has been solved by Yamabe [9], Trudinger [35], Aubin [1] and Schoen [31]. The answer for the compactness of the solution set is positive when the dimension \( n \leq 24 \) and negative when \( n \geq 25 \) (14, 32). When \( \tau = 1 \), it was initially studied by Viaclovsky [38]. The existence of the solution to the equation was solved for \( k = 2, n \geq 4 \) in [31, 22], for \( k \geq \frac{n}{2} \) by [10, 36, 37], for \( 2 < k < n/2 \) and \((M,g)\) being locally conformally flat case by [12, 23]. The compactness of the solution set was proved for \( k = 2, n = 4 \) by [5], for \( k \geq \frac{n}{2} \) by [10, 36, 37, 28], and for \( 2 < k < n/2 \) and \((M,g)\) being locally conformally flat case by [24].

For general \( \tau \), the case \( \tau \leq 1 \) and \( \tau \geq n-1 \) is somehow meaningful since in these cases the equation is elliptic, for \( \neq 1 \), equation (1.4) usually does not have the variational structure even the manifold is locally conformally flat, which requires some other ways to approach this problem. In [22, 34], the authors studied these cases and they give a positive answer to the problems on existence and
compactness for \( \tau < 1 \) in the negative cone and \( \tau > n - 1 \) in the positive cone using a parabolic flow argument and proved the priori estimates are exponentially decayed. In [34], the authors also give a positive answer to long time existence of the flow for \( \tau < 1 \) in the positive cone and \( \tau > n - 1 \) in the negative cone on locally conformally flat manifolds. But the convergence of the flow is still unknown even on locally conformally flat manifold.

In this paper we mainly study the \( k \)-curvature equation of Einstein tensor \((\tau = n - 1)\) in the negative cone. This corresponds to the \( k \)-curvature of the Schouten tensor being in the positive cone. We only deal with the three dimensional case since the admissibility of the Einstein tensor in the cases of \( k = 2 \) and \( 3 \) imply the non-negativity of Ricci tensor. By use of the idea in [28] and [16] we can get the \( C^0 \) estimate of the solutions. For the higher dimensional case, the non-negativity of Ricci tensor can not be obtained from this point.

Our main result is

**Theorem 1.1.** Let \((M^3, g)\) be a 3-dimensional closed Riemannian manifold and \( k = 2 \), or \( k = 3 \). Assume

1. \( g \) is negative \( k \)-admissible with respect to the Einstein tensor, and
2. \((M^3, g)\) is not conformally equivalent to the standard sphere.

Then for any given smooth positive function \( h \in C^\infty(M) \), there exist a solution \( u \in C^\infty(M) \) of \( \sigma_k(-\lambda(\tilde{g})) = h(x) \), where \( \tilde{g} = e^{-2u}g \), and the set of all such solutions is compact in the \( C^m \)-topology for any \( m > 0 \).

The rest of the paper is organized as follows. In section 2 we introduce the Liouville theorem and a Riemannian version of Hawking’s singularity theorem in relativity. In section 3 we give the deformation of equation (1.3). In section 4 we give some local estimates of the solutions to the deformation equation, and then by contradiction argument, we show that there is at most one blow-up point in the manifold and establish the explicit blow-up speed around the blow-up point. Then using the Bishop-Gromov volume comparison theorem we get the contradiction, and finish the proof of Theorem 1.1.

2. Preliminary

In this section, we first introduce some basic properties of the elementary symmetric functions and Newton transformation.

The \( k \)-th Newton transformation associated with a real symmetric matrix \( A \) is defined as follows:

\[
T_k(A) = \sigma_k(A)I - \sigma_{k-1}(A)A + \cdots + (-1)^k A^k.
\]

and we have

\[
T_k(A)^i_j = \frac{1}{k!} \delta_{j_1 \cdots j_k}^i A_{i_1 j_1} \cdots A_{i_k j_k},
\]
Theorem 2.3. (Theorem 1.1 of \cite{24}) For $n \geq 3$, assume that $u \in C^2(\mathbb{R}^n)$ is a superharmonic solution of $(2.1)$ for some $-\infty < p \leq (n + 2)/(n - 2)$. Then either $u \equiv \text{constant}$ or $p = (n + 2)/(n - 2)$ and
\begin{equation}
(2.2)
    u(x) \equiv \frac{a}{\left(1 + cb^2|x - \bar{x}|^2\right)^{n-2}}, \quad x \in \mathbb{R}^n
\end{equation}
for some fixed $\bar{x} \in \mathbb{R}^n$ and for some positive constants $a$ and $b$ satisfying $\sigma_k(\lambda(2b^2a^{-2}I)) = 1$, where $c = \frac{(n-2)}{(-2n+2+n\pi)}$. 

Proof. A direct calculation gives that $T_1(-E_g) = \text{Ric}_g$, by Proposition 2.1(iv), the Ricci tensor is positive definite.

Now consider the equation
\begin{equation}
(2.1)
    \sigma_k(\lambda(-E_{g_u})) = u^{p-(n+2)/(n-2)}
\end{equation}
in the case $-\infty < p \leq (n + 2)/(n - 2)$ and $n \geq 3$. We have a Liouville type theorem:

Corollary 2.2. Let $(M, g)$ be a 3-dimensional Riemannian manifold and $x \in M$. $-E_g$ is 2-admissible at $x$, then
\[ \text{Ric}_g(x) \geq 0. \]
Modifying some constants in Theorem 1.3 of [24], and noticing that $\sigma_k$ satisfies the condition of $f$ in the theorem, we can prove Theorem 2.3. We omit the proof here.

Next we introduce a Riemannian version of Hawking’s singularity theorem which will play an important role in the blow-up analysis.

**Proposition 2.4.** (see [28] or [18]) Let $(N^n, g)$ be a complete smooth Riemannian manifold with smooth boundary $\partial N$. If $\text{Ric}_g \geq -(n-1)\alpha^2$ for some $\alpha \geq 0$ and if the mean curvature $H$ of $\partial N$ with respect to its inward pointing normal satisfies $H \geq (n-1)c_0 \geq (n-1)\alpha$. Then

\begin{equation}
\sum_{i=1}^n |\lambda_i(M) - \lambda_{\sigma(i)}(\tilde{M})| < C(n)\epsilon.
\end{equation}

The following lemma is about the symmetric matrix, which is needed in our proof of the main theorem. The proof can be found in [28].

**Lemma 2.5.** (28 Lemma A.1) For an $n \times n$ real symmetric matrix $M$, let $\lambda_1(M), \ldots, \lambda_n(M)$ denote its eigenvalues. There exists a constant $C(n) > 0$ such that $\forall \epsilon > 0$ and any two symmetric matrices $M$ and $\tilde{M}$ satisfying $|M - \tilde{M}| < \epsilon$, there holds for some permutation $\sigma = \sigma(M, \tilde{M})$ that

\begin{equation}
\sum_{i=1}^n |\lambda_i(M) - \lambda_{\sigma(i)}(\tilde{M})| < C(n)\epsilon.
\end{equation}

3. **Deformation of the equation**

In this section we deform equation (1.3) to a equation which has a unique solution. Consider

\begin{equation}
\sigma_k^\frac{1}{k} \left( \lambda(g^{-1}|\lambda_k g - \varphi(t)g - \varphi(t)E_g - \nabla^2 u + \Delta u g - du \otimes du + \frac{3-n}{2} |\nabla u|^2 g) \right)
= \varphi(t)h(x)e^{-2u} + (1 - t)\left( \int_M e^{-(n+1)u)\frac{2}{n+1} \right)
\end{equation}

where $\varphi \in C^1[0,1]$ satisfies $0 \leq \varphi(t) \leq 1$, $\varphi(0) = 0$, $\varphi(t) = 1$ for $t \geq \frac{1}{2}$; and $\lambda_k = (C_k^n)^{-\frac{1}{k}}$. This deformation is valid for any $k$ and $n$.

At $t = 1$, (3.1) becomes (1.3); while $t = 0$, it turns to

\begin{equation}
\sigma_k^\frac{1}{k} \left( \lambda(g^{-1}|\lambda_k g - \nabla^2 u + \Delta u g - du \otimes du + \frac{3-n}{2} |\nabla u|^2 g) \right)
= \left( \int_M e^{-(n+1)u)\frac{2}{n+1} \right).
\end{equation}

We can show that this equation has the unique solution $u(x) \equiv 0$. It is easy to see that $u \equiv 0$ is a solution. The uniqueness can be shown as follows: let $x_0$ be the maximum point of $u$ on $M$. At
this point we have $\nabla u|_{x_0} = 0$, and $\nabla^2 u|_{x_0}$ is negative semi-definite and then $-\nabla^2 u + \Delta u g$ is negative semi-definite. Then at $x_0$ we get

$$\lambda_k = \sigma_k^+ (\lambda(g^{-1} \cdot g))$$

$$\geq \sigma_k^+ (\lambda(g^{-1}\lambda_k g - \nabla^2 u + \Delta u g + du \otimes du + \frac{3-n}{2} |\nabla u|^2 g))$$

$$= \left( \int_M e^{-(n+1)u} \right)^{\frac{n}{n+1}}$$

Similarly, at the minimum point of $u$, we have $\lambda_k \leq (\int_M e^{-(n+1)u})^{\frac{n}{n+1}}$. Therefore, $\lambda_k = (\int_M e^{-(n+1)u})^{\frac{n}{n+1}}$.

On the other hand, by Newton-Maclaurin inequality, we have,

$$\lambda_k = \sigma_k^+ (\lambda(g^{-1}|\lambda_k g - \nabla^2 u + \Delta u g - du \otimes du + \frac{3-n}{2} |\nabla u|^2 g))$$

$$\leq \frac{1}{n} \sigma_1 (\lambda(g^{-1}|\lambda_k g - \nabla^2 u + \Delta u g + du \otimes du + \frac{3-n}{2} |\nabla u|^2 g))$$

$$= \frac{1}{n} \left( (n-1)\Delta u - \frac{n(n-3)+2}{2} |\nabla u|^2 + n\lambda_k \right).$$

Then we get

$$\left( \frac{n}{2} - 1 \right) \int_M |\nabla u|^2 \leq \int_M \Delta u = 0$$

thus $u \equiv \text{constant} = 0$.

Now we define the operator as in [16][19]

$$\Psi_t = \sigma_k^+ (\lambda(g^{-1}|\lambda_k (1-\varphi(t))g - \varphi(t)E_g - \nabla^2 u + \Delta u g - du \otimes du + \frac{3-n}{2} |\nabla u|^2 g))$$

$$- \varphi(t)h(x)e^{-2u} - (1-t)\left( \int_M e^{-(n+1)u} \right)^{\frac{n}{n+1}}.$$}

When $t = 0$, $\Psi_0|u = 0$ has unique solution $u \equiv 0$ and the linearization of $\Psi_0$ at $u \equiv 0$ is invertible.

We define the Leray-Schauder degree $\deg(\Psi_t, \mathcal{O}, 0)$ as in [21][17], where $\mathcal{O} = \{ u \in C^{4,\alpha}(M) : u \text{ is } k\text{-admissible}, \text{dist}(\lambda(A_g), \partial \Gamma^+_{k}) > \frac{1}{2} \}$. Then $\deg(\Psi_0, \mathcal{O}, 0) \neq 0$ at $t = 0$. Of course we would like to use the homotopy-invariance of the degree to conclude $\deg(\Psi_t, \mathcal{O}, 0)$ is non-zero for some open set $\mathcal{O} \subset C^{4,\alpha}$. To do this we need to establish a priori estimate for (3.1). By [21], $C^1$ and $C^2$ estimates have been already done. Once we get the $L^\infty$ bound, by the standard Evans-Krylov theory, we have the $C^{2,\alpha}$ estimate and then the higher order estimates can be established.

Here we also note for $t \in [0, 1 - \delta]$ and $\delta$ sufficiently small, the equation (3.1) has $C^0$ estimate by the work of He-Sheng [20]. In fact this can be done by use of the $C^1$ and $C^2$ estimates in [20] and a similar argument in [12]. So we just need to focus on the case $t = 1$.

4. The priori estimates

In this section, we derive the $C^0$ bound by a contradiction argument.
In [19, 20], the authors have already got the point-wise $C^1$ and $C^2$ estimate (4.1) with $g_u = e^{-2u}g$:

\[
sup_M (|\nabla^2 u| + |\nabla u|^2)(x) \leq C (1 + e^{-2\inf_M u}).
\]

Here for convenience we use the form $g_u = u^{\frac{4}{n-2}}g$, substitute it to (4.1) we have

\[
sup_M (|\nabla \log u| + |\nabla \log u|^2)(x) \leq C(1 + \max_M u^{\frac{4}{n-2}}).
\]

Now we only need to derive the $C^0$ bound.

Without loss of generality, we may assume that $\sigma_2(\lambda(-E_{g_{can}})) = 1$ on $S^n$, where $g_{can}$ is the standard metric on $S^n$.

First we’ll show the upper bound of $u$ implies it’s lower bound. Assume at this moment we have established the estimate

$$\max u \leq C.$$  

This implies that

\[
sup_M (|\nabla^2 u| + |\nabla u|^2)(x) \leq C.
\]

Now we’ll show that:

$$\min_M u \geq \frac{1}{C}.$$  

We argue it by contradiction. Suppose there exists a sequence \{u_i\} such that

\[
\min_M u_i \to 0.
\]

By definition, the metrics $g_i = u_i^{-\frac{4}{n-2}}g$ satisfy

\[
\sigma_k(\lambda(E_{g_i})) = 1 - \lambda(E_{g_i}) \in \Gamma_k^+.
\]

Note here $g_i = (\frac{u_i}{u_1})^{\frac{4}{n-2}}g_1$, evaluating \[18\] at the maximum point $\bar{x}_i$ of $\frac{u_i}{u_1}$, since we have the relationship between $E_{g_i}$ and $E_{g_1}$:

\[
E_{g_i} = -\frac{2}{n-2}(\frac{u_i}{u_1})^{-1}\nabla^2(\frac{u_i}{u_1}) + \frac{2}{(n-2)^2}(\frac{u_i}{u_1})^{-1}(\Delta(\frac{u_i}{u_1}))g_1 + \frac{2n}{(n-2)^2}(\frac{u_i}{u_1})^{-2}d(\frac{u_i}{u_1}) \otimes d(\frac{u_i}{u_1})
\]

and $-\frac{2}{n-2}(\frac{u_i}{u_1})^{-1}\nabla^2(\frac{u_i}{u_1}) + \frac{2}{(n-2)^2}(\frac{u_i}{u_1})^{-1}(\Delta(\frac{u_i}{u_1}))g_1$ is positive definite at $\bar{x}_i$ (by $\tau < 1$), we obtain

$$1 \geq (\frac{u_i}{u_1})^{-\frac{n-2}{4}}\sigma_k(\lambda(E_{g_1}(\bar{x}_i))) = (\frac{u_i}{u_1})^{-\frac{n-2}{4}}$$

which implies $\max_M u_i \geq u_1(\bar{x}_i) \geq \min_M u_1$. On the other hand we have the Harneck inequality $\min_M u_i \geq \frac{1}{e} \max_M u_i$ which comes from the $C^1$ estimate. Combining these inequalities, we have $\min_M u_i \geq \frac{1}{e} \min_M u_1$ which contradicts (4.4). Since we have established the $L^\infty$, $C^1$ and $C^2$ estimates, the higher order estimate on $\log u$ follows from the standard Evans-Krylov’s and Schauder’s estimates.
Now it is sufficient to check $\max_M u \leq C$ by the above argument. We also argue it by contradiction. Assume $\exists \{u_i\}$ a sequence of smooth positive functions on $M$ such that $g_i = u_i^{4n-2} g$ satisfy (1.3) but $u_i(x_i) = \max_M u_i \to \infty$ as $i \to \infty$.

Suppose $x_\infty$ is a blow-up point, we have $x_i \to x_\infty$ in the metric topology induced by the initial metric on $M$.

4.1. The unique blow-up point. Now we show the blow-up point is unique, in fact we can control the speed of blow-up of the sequence $u_i$,

\begin{equation}
  u_i \leq Cd_g(x, x_i)^{-\frac{n-2}{2}}, \quad \forall x \in M \setminus \{x_i\},
\end{equation}

where $C$ is a constant independent of $i$.

This property was observed by Y.Y.Li-Nguyen [28]. Our proof follows the work in [28]. The following lemma plays an important role to establish (4.6). It reveals some concentration property of the volume of a small neighborhood of the blow-up sequence $\{x_i\}$.

**Lemma 4.1.** (Lemma 3.1 of [28]) Assume for some $C_1 \geq 1, K_i \to \infty$ and $y_i \in M$, that $u_i \to \infty$; set $D_{K_i} = \{y \in M | d_g(y, y_i) \leq K_i u_i(y_i)^{-\frac{n-2}{2}}\}$, and assume

\begin{equation}
  \sup_{D_{K_i}} u_i \leq C_1 u_i(y_i).
\end{equation}

Then for any $0 < \mu < 1$, there exists $K = K(C_1, \mu)$ such that for $i$ sufficiently large

\[ \text{Vol}_{g_i}(D_K) \geq (1 - \mu) \text{Vol}_{g_i}(M) \]

**Proof.** This argument comes from Lemma 3.1 of [28]. We need a little modification here since the coefficients in the formula of conformal change of Einstein tensor $E_g$ is different to the Schouten tensor $A_g$ in [28]. We state the outline of the proof. Let $p \in \mathbb{R}^n, a > 0$ and $b = \frac{n-2}{2n-2-n\tau}$, define

\begin{equation}
  U_{a,p,b}(x) = \left( \frac{2a}{1 + ba^2|x-p|^2} \right)^{\frac{n-2}{2}}
\end{equation}

$S^n = \{z = (z_1, \cdots, z_n) \in \mathbb{R}^{n+1} | z_1^2 + \cdots + z_n^2 = 1\}$.

Let $(x_1, \cdots, x_n) \in \mathbb{R}^n$ be the stereographic projection coordinates of $S^n$, then

\[ g_{\text{can}} = |dz|^2 = \left( \frac{2}{1 + |x|^2} \right)^2 |dx|^2. \]

Let $x = \sqrt{b}x'$ we then have

\[ g_{\text{can}} = |dz|^2 = \left( \frac{2}{1 + b|x'|^2} \right)^2 b|dx'|^2 = U_{1,0,b}^{\frac{1}{2}}(x') b|dx'|^2 = U_{1,0}^{\frac{1}{2}}(x') g_{\text{flat}} \]

where $g_{\text{flat}} = |dx|^2$ is the standard Euclidean metric on $\mathbb{R}^n$. It follows that

\[ f(\lambda(-E^\tau_{U_{a,p,b} g_{\text{flat}}})) = 1. \]
Now we define a map from the tangent space to the manifold \( \Phi_i : T_y(M, g) \to M \) by
\[
\Phi_i(x) = \exp_y \left( \frac{2x}{u_i(y_i)} \right),
\]
then we can get \( f(\lambda(A^r_{(u_i, \Phi_i)} \frac{1}{\alpha} \Phi_i^* g)) = 1 \) and let
\[
\tilde{u}_i(x) = \frac{2\alpha^2}{u_i(y_i)} u_i \circ \Phi_i(x), x \in \mathbb{R}^n.
\]

By Lemma 3.1 of [28], we know \( \tilde{u}_i \) subconverges to some positive \( \tilde{u}_* \in C^2(\mathbb{R}^n) \) and actually by the Liouville theorem in section 2, \( \tilde{u}_* = U_{a_*, x_*, b} \) for some bounded \( a_* > 0 \) and \( x_* \in \mathbb{R}^n \).

Process as lemma 3.1 of [28], this implies that \( \forall \epsilon > 0, \exists R = R(\epsilon, C_1) > 0 \) such that
\[
|\text{Vol}_{g_i}(\Phi_*(B(0, R))) - \text{Vol}(S^n)| \leq C\epsilon^n
\]
for some \( C \) independent of \( i \) and \( \epsilon \). Let \( H_{g_i} \) be the mean curvature of \( \partial \Phi_i(B(0, R)) \) with respect to \( g_i \), this also implies \( H_{g_i} \geq \epsilon^{-1} \).

By the Hawking’s lemma in section 2 and the above property, we see that
\[
\text{diam}_{g_i}(M \setminus \Phi(B(0, R))) \leq C\epsilon.
\]
this inequality together with Bishop’s comparison theorem imply that
\[
\text{Vol}_{g_i}(M \setminus \Phi(B(0, R))) \leq C\epsilon^n,
\]
then \( \text{Vol}_{g_i}(\Phi(B(0, R))) \geq (1 - \epsilon^n)C \).

Now let \( K = \frac{1}{2}R \) and \( \mu = \epsilon^n \), then \( D_{K_i} = \Phi(B(0, R)) \), the proof is complected. \( \square \)

With this lemma we can immediately get (4.10) by the argument in [28].

Besides, we can also get the similar higher order estimates as in step 2 of [28] :
(4.9) \[
|\nabla^k \log u_i(x)| \leq C d_{g_i}^{-k} \text{ for } x \neq x_i, k = 1, 2.
\]
and the similar convergence behavior as in step 3 of [28] :
\[
\lim_{i \to \infty} u_i = u_\infty
\]
with \( u_\infty(x) \equiv 0 \) in \( M \setminus \{x_\infty\} \). Unfortunately this limit function is difficult to analysis. In order to get some useful result we need to rescale it as follows: fix some point \( p \in M \setminus \{x_\infty\} \) and let
\[
v_i(x) = \frac{u_i(x)}{u_i(p)}
\]
v_i subconverges, for every \( 0 < \alpha < 1 \), in \( C^{1, \alpha}(M \setminus \{x_\infty\}, g) \) to some positive function \( v_\infty \in C^{1,1}(M \setminus \{x_\infty\}, g) \) which satisfies \( v_\infty(p) = 1 \) and
(4.10) \[
|\nabla^k \log v_\infty(x)| \leq C d_{g_\infty}^{-k} \text{ for } x \neq x_i, k = 1, 2.
\]
Furthermore, \( -E_{g_\infty} \) is on the boundary of the Garding cone \( \Gamma_k^+ \) in viscosity sense as \( u_i(p) \) trends to 0.
4.2. The blow-up order. In this section, we show that $v_{\infty}$ has an asymptotic behavior near $x_{\infty}$ of order $n-2$, i.e

\begin{equation}
\lim_{x \to x_{\infty}} v_{\infty}d_g(x, x_{\infty}) = a
\end{equation}

where $a \in (0, +\infty)$ is a constant.

By (4.10), we can immediately get

\begin{equation}
\limsup_{x \to x_{\infty}} v_{\infty}d_g(x, x_{\infty}) = A < +\infty.
\end{equation}

We only need to show

\begin{equation}
\liminf_{x \to x_{\infty}} v_{\infty}d_g(x, x_{\infty}) = a
\end{equation}

and $A = a$.

As in [28], note here $-E_{\nabla v_{\infty}} \in \partial \Gamma_k^+$. Then we have the super-harmonicity of $v_{\infty}$, i.e $\Delta_g v_{\infty} - \frac{(n-2)}{4(n-1)} R_g v_{\infty} \leq 0$. Let $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ be the conformal Laplacian, then $L_g v_{\infty} \leq 0$. We then have the following lemma.

**Lemma 4.2.** (Lemma 3.3 of [28]) Let $\Omega$ be an open neighborhood of a point $p \in M$. If $w$ is a nonnegative lower semi-continuous function in $\Omega \setminus \{p\}$ and satisfies $L_g w \leq 0$ in the viscosity sense in $\Omega \setminus \{p\}$, then

\begin{equation}
\lim_{r \to 0} r^{n-2} \min_{\partial B_r(p, r)} w < +\infty.
\end{equation}

By Lemma 4.2 we immediately know that $a$ is finite.

Now it remains to show $A = a$. We prove it by contradiction also. Assume $A > a$, then we can find a sequence $\{y_i\}$ such that for some $\epsilon > 0$,

\begin{equation}
A + \epsilon \geq d_g(y_i, x_{\infty})^{n-2} v_{\infty}(y_i) \geq a + 2\epsilon,
\end{equation}

where $x_{\infty} = \lim_{i \to \infty} y_i$. Also, by (4.13), we have

\begin{equation}
d_g(y_i, x_{\infty})^{n-2} \min_{d_g(y, x_{\infty}) = d_g(y_i, x_{\infty})} v_{\infty}(y) \leq a + \epsilon.
\end{equation}

Let $R_i = d_g(y_i, x_{\infty})^{-1}$, define the exponential map $E_i : B_{\delta R_i} \subset T_{x_{\infty}} M \to M$ by

\[ \Theta_i(y) = \exp_{x_{\infty}}(R_i^{-1}y), \]

where $\delta$ is sufficiently small, and as in subsection 4.1 we have $R_i^2 \Theta_i^* g$ converges on compact subsets to the standard Euclidean metric. Now set $\hat{v}_i(y) = R_i^{2-n} v_{\infty} \circ E_i(y)$, then $\hat{v}_i \in C^{1,1}_{loc}(B_{\delta R_i} \setminus \{0\})$. By a direct computation we have

\begin{equation}
\begin{cases}
\lambda(-E_{\hat{v}_i(y)} \frac{\nabla}{\nabla^{R_i^2 \Theta_i^* g}}) \in \partial \Gamma_k^+ \text{ in } B_{\delta R_i} \setminus \{0\} \\
\min_{\partial B_1} \hat{v}_i \leq a + \epsilon \text{ and } \max_{\partial B_1} \hat{v}_i \geq a + 2\epsilon.
\end{cases}
\end{equation}
Then by the estimates (4.13) in subsection 4.1, a subsequence of \( \hat{v} \) converges uniformly to a limit \( \hat{v}_* \in C^{1,1}_{loc}(\mathbb{R}^n \setminus \{0\}) \) and satisfies in the viscosity sense

\[
\lambda \left( -E_{\frac{1}{n-2}g_{falt}} \right) \in \partial \Gamma_k^+ \text{ in } \mathbb{R}^n \setminus \{0\} \tag{4.15}
\]

then we get \( v_* \) is radially symmetric by the following lemma.

**Lemma 4.3.** (cf. Theorem 1.18 in [26]) For \( n \geq 3 \), let \( \Gamma_k^+ \) be the Garding cone, and let \( u \) be a positive, locally Lipschitz viscosity solution of

\[
\lambda \left( -E_{\frac{1}{n-2}g_{falt}} \right) \in \partial \Gamma_k^+ \text{ in } \mathbb{R}^n \setminus \{0\} \tag{4.16}
\]

Then \( u \) is radially symmetric about the origin and \( u'(r) \leq 0 \) for almost all \( 0 < r < \infty \).

**Proof.** Note

\[
-E_{\frac{1}{n-2}g_{falt}} = \frac{2}{n-2} u^{-1} \nabla^2 u - \frac{2}{(n-2)} u^{-1}(\Delta u) g - \frac{2n}{(n-2)^2} u^{-2} du \otimes du + \frac{2}{(n-2)^2} u^{-2} |\nabla u|^2 g.
\]

The proof in Theorem 1.18 of [26] mainly depend on the ellipticity of the operator in the formula of \( A_{\frac{1}{n-2}g_{falt}} \) and in our case, the operator \( \frac{2}{n-2} u^{-1} \nabla^2 - \frac{2}{(n-2)} u^{-1} \Delta g \) is elliptic. The proof in Theorem 1.18 of [26] is still hold, we omit it here. \( \square \)

This makes a contradiction to the second line of (4.14), we then get \( A = a \).

Now we only need to show that \( a > 0 \). We do not use the condition \( n = 3 \) and \( k = 2 \) or \( k = 3 \) until this moment, in the next lemma, we'll assume these conditions, and since \( \Gamma_3^- < \Gamma_2^- \), we only need to prove the case \( k = 2 \).

In normal coordinates at \( x_i \), let \( r = |x| \). The following lemma is the key ingredient, which is a special version of Lemma 3.4 in [28]

**Lemma 4.4.** There exists some small \( r_1 > 0 \) depending only on \( (M, g) \) such that for all \( 0 < \delta < \frac{1}{4} \), the function \( v_\delta := r^{-1-2\delta} e^r \) satisfies

\[
\lambda(E_{v_\delta}) = \nabla^2 R_{v_\delta} \in \mathbb{R}^n \setminus \Gamma_2^- \text{ in } \{0 < r < r_1\} \tag{4.17}
\]

**Proof.** Let \( b = 1 - 2\delta \), when \( n = 3 \), the Einstein tensor for \( g_\delta = \frac{1}{v_\delta^{1-2\delta}} g \) reads

\[
E_{g_\delta} = -\frac{2}{n-2} v_{\delta}^{-1} \nabla^2 v_{\delta} + \frac{2}{(n-2)} v_{\delta}^{-1}(\Delta v_\delta) g + \frac{2n}{(n-2)^2} v_{\delta}^{-2} dv_\delta \otimes dv_\delta - \frac{2}{(n-2)^2} v_{\delta}^{-2} |\nabla v_\delta|^2 g + E_g
\]

\[
= -2v_{\delta}^{-1} \nabla^2 v_{\delta} + 2v_{\delta}^{-1}(\Delta v_\delta) g + 6v_{\delta}^{-2} dv_\delta \otimes dv_\delta - 2v_{\delta}^{-2} |\nabla v_\delta|^2 g + E_g
\]

\[
= D_1 I - D_2 \frac{x}{r} \otimes \frac{x}{r} + D_3
\]

where \( I \) is the identity matrix and

\[
D_1 = 2 \frac{v'_{\delta}}{v_\delta} - 2 \frac{(v'_{\delta})^2}{v_\delta^2} + 2 \frac{v''_{\delta}}{v_\delta} = 2 \frac{r-a}{r^2} - 2 \frac{(r-a)^2}{r^2} + 2 \frac{(r-a)^2 + a}{r^2} = \frac{2}{r}
\]
$D_2 = 2v_\delta^{-1}(v_\delta'' - \frac{v_\delta'}{r}) - 6\frac{v_\delta^2}{v_\delta'} = \frac{4(1-a)a + 2(4a-1)r - 4r^2}{r^2}$

$|D_3| \leq C(1 + rv_\delta^{-1}|v_\delta'| + r^2v_\delta^{-2}|v_\delta'|^2) \leq C \leq CrD_1$

The eigenvalue of $D_1I - D_2 I \frac{x}{r} \otimes \frac{x}{r}$ with respect to $I$ are $D_1 - D_2, D_1, D_1$, we can apply Lemma 2.5 to see that the eigenvalues $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of $E_{g_\delta}$ with respect to $g_\delta$ satisfies

$|\lambda_1 - v_\delta^{-4}(D_1 - D_2)| + |\lambda_2 - v_\delta^{-4}D_1| + |\lambda_3 - v_\delta^{-4}D_1| \leq Cv_\delta^{-4} \leq Crv_\delta^{-4}D_1$

We have $\sigma_1(D_1 - D_2, D_1, D_1) = 3D_1 - D_2 = \frac{-4(1-a)(a-2r) + 4r^2}{r^2} < 0$ for $a < 1$ and $r$ is sufficiently small. Then $\sigma_2(D_1 - D_2, D_1, D_1) = D_1(3D_1 - 2D_2) < 0$. This implies $(D_1 - D_2, D_1, D_1)$ lying outside of $\bar{\Gamma}_2$ (also $\bar{\Gamma}_3$ immediately). Thus $\lambda(E_{g_\delta})$ lies outside of $\bar{\Gamma}_2$ (also $\bar{\Gamma}_3$) since $r$ is sufficiently small.

From $v_i(p) = 1$ and \[4.9\], there exists some positive constant $C$ independent of $i$ and $\delta$ such that $v_i \geq \frac{1}{C}v_\delta$, on $\{r = r_1\}$. For some $K = K(\delta) > 0$ large enough, let

$\bar{\gamma} = \sup \left\{ 0 < \gamma < \frac{1}{C} : v_i \geq \gamma v_\delta \text{ in } \{ Ku_i(x_i)^{-\frac{1}{n-2}} < r < r_1 \} \right\}$.

By Lemma 4.4 and the comparison principle, there exist $\hat{x}_i$ with $|\hat{x}_i| = r_1$ such that

$v_i(\hat{x}_i) = \bar{\gamma}v_\delta(\hat{x}_i)$.

Then follow the argument in \[28\], we have $\bar{\gamma} = \frac{1}{C}$. This shows $v_i \geq \frac{1}{C}v_\delta \geq \frac{1}{C\bar{d}_g(x, x_{1_1})^{n-2-2\delta}}$ in $\{ Ku_i(x_i)^{-\frac{1}{n-2}} < r < r_1 \}$ for $i \geq N$ with $N$ sufficiently large. When $i \to \infty$, we have

$v_\infty \geq \frac{1}{C}v_\delta \geq \frac{1}{C\bar{d}_g(x, x_{\infty})^{n-2-2\delta}}$ in $\{0 < r < r_1\}$

for all sufficiently small $\delta > 0$, finally this implies $a > 0$.

Now note we already get the bound $C^{-1}\bar{d}_g^{-n}(x, x_{\infty}) \leq v_\infty(x) \leq C\bar{d}_g^{-n}(x, x_{\infty})$, also note the Ricci curva-ure is semi-positive definite when $n = 3$ and $k = 2$, we can follow the work in \[28\] \[16\] \[36\] to show $(M \setminus x_{\infty}, \frac{1}{\bar{d}_g^{-n}}g)$ is isomorphic to $(\mathbb{R}^n, g_{fial})$ by the Bishop-Gromov comparison theorem. Then by the consequence of Gursky-Viaclovsky\[16\], this implies that $(M, g)$ is conformally equivalent to the standard sphere, which is a contradiction to the initial hypothesis. This finishes our proof of the boundness of $u$.

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