Viète’s fractal distributions and their momenta

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Abstract

Solutions of Schröder-Poincaré’s polynomial equations \( f(az) = P(f(z)) \) usually do not admit a simple closed form representation in terms of known standard functions. We show that there is a one-to-one correspondence between zeros of \( f \) and a set of discrete functions stable at infinity. The corresponding Viète-type infinite products for zeros of \( f \) are also provided. This allows us to obtain a special kind of closed-form representation for \( f \) based on the Weierstrass-Hadamard factorization. From this representation, it is possible to derive explicit momenta formulas for zeros. Obtaining explicit closed-form expressions is the main motivation for this work. Finally, all the branches of multivalued function \( f^{-1} \) are computed explicitly.

Keywords: Poincaré’s equation, Schröder’s equation, Viète’s formula, Weierstrass-Hadamard factorization, polynomial dynamics

1. Introduction and main results

The classical Viète’s formula

\[
\frac{2}{\pi} = \sqrt{\frac{2}{2}} \cdot \sqrt{\frac{2 + \sqrt{2}}{2}} \cdot \sqrt{\frac{2 + \sqrt{2 + \sqrt{2}}}{2}} \ldots
\]

uses nested square radicals to represent the constant \( \pi \). Wiki says ”By now many formulas similar to Viète’s involving either nested radicals or infinite products of trigonometric functions are known for \( \pi \), as well as for other constants such as the golden ratio”, see, e.g., [1] [2] [4] [3]. In this note, we derive formulas for zeros of functions satisfying Schröder-Poincaré’s polynomial equations. In general, the formulas for zeros involve various nested-radicals products similar to Viète’s. These formulas can be used in Weierstrass-Hadamard factorization to obtain various closed-form expressions.

Finally, looking through ”A chronology of continued square roots and other continued compositions” [11], I found paper [12], where a detailed analysis of real roots of \( f \), satisfying \( f(az) = f(z)^2 + c \), is provided. Many interesting facts are presented in [11], e.g., an interesting story of the famous formula

\[
\varepsilon_0 \sqrt{2 + \varepsilon_1 \sqrt{2 + \varepsilon_2 \sqrt{2} + \ldots}} = 2 \sin \left( \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{\varepsilon_0 \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n}{2^n} \right),
\]
where $\varepsilon_i = -1, 0, 1$.

We assume facts about existence of entire solutions of SP-equation to be known, see, e.g., [6, 9]. Let $P$ be some polynomial of degree $\geq 2$. Let $b$ be its repelling point $P(b) = b$, $|a| := |P'(b)| > 1$. Consider the entire solution $f$ of SP-equation $f(az) = P(f(z))$ satisfying $f(0) = b$, $f'(0) = 1$. Under broad assumptions, such solution exists. For simplicity, let us assume $b \neq 0$. This is not a restriction, since $\tilde{f} := f + c$, $c \in \mathbb{C}$, also satisfies some polynomial SP-equation.

Let $g := f^{-1}$ be the inverse function. It is analytic in some neighborhood of $b$, and it satisfies $g(b) = 0$, $g'(b) = 1$. Let us define the notation: $P^n(z) = P \circ \ldots \circ P(z)$ is the composition; if a composition inverse exists then it is denoted by $P^{-1}(z)$, and $P^{-n}(z) = P^{-1} \circ \ldots \circ P^{-1}(z)$. Using

\[
f(z) = P^n(f(a^{-n}z))
\]

we obtain

\[
g(w) = a^n g(P_0^{-n}(w)),
\]

where $P_0^{-1}$ is the branch of $P^{-1}$ analytic in a neighborhood of $b$ and satisfying $P_0^{-1}(b) = b$. Since $b$ is the attraction point for $P_0^{-1}$, we deduce $P_0^{-n}(w) \to b$. Hence, by (2), we conclude

\[
g(w) = \lim_{n \to \infty} g_n(w), \quad g_n(w) := a^n(P_0^{-n}(w) - b),
\]

since $g(b) = 0$, $g'(b) = 1$. Introduce

\[
Q(z) := \frac{P(z) - P(b)}{z - b} = \frac{P(z) - b}{z - b}.
\]

Using (3), (4), we obtain

\[
g_n(w) = \frac{a^n(P_0^{-n}(w) - b)Q(P_0^{-n}(w))}{Q(P_0^{-n}(w))} = g_{n-1}(w) \frac{a}{Q(P_0^{-n}(w))}.
\]

Thus, (5) and (3) give us

\[
g(w) = (w - b) \prod_{n=1}^{\infty} \frac{a}{Q(P_0^{-n}(w))}.
\]

This is a closed-form expression for $g$ involving algebraic functions only. Differentiating (3), we obtain also a closed-form expression for the derivative

\[
g'(w) = \prod_{n=1}^{\infty} \frac{a}{P'(P_0^{-n}(w))},
\]

but we do not use it in this note.

Product (6) converges exponentially fast. Indeed, using (3) and $Q(b) = P'(b) = a$, we conclude that

\[
Q(P_0^{-n}(w)) = Q(b + O(a^{-n})) = a + O(a^{-n}).
\]
Hence, the terms in (6) are of the order $1 + O(a^{-n-1})$. Product (7) converges exponentially fast as well.

Suppose that $P$ is a polynomial of power $d+1$. Let $P^{-1}_1, ..., P^{-1}_d$ be the branches of $P^{-1}$ apart from $P^{-1}_0$ already defined. Suppose that $z_0$ is a zero of $f$. Then, by (7), we have that $f(a^{-n}z_0)$ is a zero of the polynomial $P^n(w)$. Thus

$$f(a^{-n}z_0) = P^{-1}_{j_n} \circ \cdots \circ P^{-1}_{j_1}(0)$$

(8)

for some $(j_i) \in \{0, ..., d\}^n$. If $n$ is sufficiently large then we can apply $g$ to (8). Then, using (6), we obtain

$$z_0 = a^n(P^{-1}_{j_n} \circ \cdots \circ P^{-1}_{j_1}(0) - b)\prod_{n=1}^{\infty} \frac{a}{Q(P^{-n}_{0}(P^{-1}_{j_n} \circ \cdots \circ P^{-1}_{j_1}(0)))}.$$  

(9)

The first term can be expanded by the same way as in (5):

$$a^n(P^{-1}_{j_n} \circ \cdots \circ P^{-1}_{j_1}(0) - b) = -b\prod_{i=1}^{n} \frac{a}{Q(P^{-1}_{j_i} \circ \cdots \circ P^{-1}_{j_1}(0))}.$$  

(10)

Introduce

$$\Sigma = \{ \sigma : \mathbb{N} \to \{0, ..., d\}, \lim_{n \to \infty} \sigma_n = 0 \}.$$  

(11)

Then, combining (9) and (10), we obtain

$$z_0 = -b\prod_{n=1}^{\infty} \frac{a}{Q(P^{-n}_{\sigma_n} \circ \cdots \circ P^{-1}_{\sigma_1}(0))},$$

where $\sigma = (j_1, ..., j_n, 0, 0, 0, ...) \in \Sigma$. Under sufficiently broad assumptions, all the statements given above are invertible and we can state:

**Theorem 1.1.** The set of zeros of $f$ coincides with $\{z(\sigma)\}_{\sigma \in \Sigma}$, where

$$z(\sigma) = -b\prod_{n=1}^{\infty} \frac{a}{Q(P^{-n}_{\sigma_n} \circ \cdots \circ P^{-1}_{\sigma_1}(0))}.$$  

(12)

Each zero is counted according to its multiplicity. In other words, the multiplicity of $z_0$ as zero of $f$ is $\#\{\sigma \in \Sigma : z(\sigma) = z_0\}$.

I set a theorem to highlight this statement, for ease of reading. In fact, we can use other values $w \in \mathbb{C}$ apart from $w = 0$ in (8). Thus, we obtain all the branches of super multi-valued function $g = f^{-1}$:

$$g_\sigma(w) = (w - b)\prod_{n=1}^{\infty} \frac{a}{Q(P^{-n}_{\sigma_n} \circ \cdots \circ P^{-1}_{\sigma_1}(w))}, \quad \sigma \in \Sigma.$$  

(13)
Remark. In general, the order of $f$ is $\rho = \ln(d + 1)/\ln a$, which can be easily obtained by substituting $e^{A|z|^\rho}$ into the functional equation $f(az) = P(f(z))$ and equating the leading terms. The condition $\rho < 1$ guarantees the existence of zeros of $f$. Moreover, there is the Weierstrass-Hadamard factorization

$$f(z) = b \prod_{\sigma \in \Sigma} \left( 1 + \frac{z}{b} \prod_{n=1}^{\infty} \frac{Q(P_{\sigma_n}^{-1} \circ \cdots \circ P_{\sigma_1}^{-1}(0))}{a} \right)$$

(14)

without an exponent-of-polynomial leading term. This is a special closed form representation for a complex function $f$. Note that $Q$, $P$ are algebraic functions, and the product does not contain implicit terms. There is also the momentum formula for zeros

$$\sum_{\sigma \in \Sigma} \prod_{n=1}^{\infty} \frac{Q(P_{\sigma_n}^{-1} \circ \cdots \circ P_{\sigma_1}^{-1}(0))}{a} = f'(0) = 1.$$  

(15)

Let us note how to compute explicitly other momenta of zeros. First, differentiating $f(az) = P(f(z))$ at $z = 0$ and using $f(0) = b$, $f'(0) = 1$, $P'(b) = a$, we obtain recurrent formulas for determining all the derivatives:

$$f''(0) = (a^2 - a)^{-1} P''(b),$$

(16)

$$f^{(m)}(0) = (a^m - a)^{-1} \sum_{j=2}^{m} P^{(j)}(b) B_{m,j}(f'(0), \ldots, f^{(m-j+1)}(0)), \quad m \geq 2,$$

(17)

where $B_{m,j}$ are Bell polynomials. They are given by

$$B_{m,j}(x_1, \ldots, x_{m-j+1}) = \sum_{k_1, \ldots, k_{m-j+1}} \frac{m!}{k_1! \cdots k_{m-j+1}!} \frac{x_1^{k_1}}{1!} \cdots \frac{x_{m-j+1}^{k_{m-j+1}}}{(m-j+1)!},$$

(18)

where the sum is taken over all sequences $k_1, k_2, \ldots, k_{m-j+1}$ of non-negative integers such that the two conditions are satisfied:

$$\sum_{i=1}^{m-j+1} k_i = j, \quad \sum_{i=1}^{m-j+1} ik_i = m,$$

(19)

see more about Faà di Bruno’s formula for high order derivatives of compositions in, e.g., wiki. Now, differentiating $\ln f(z)$ at $z = 0$ and using (14), we obtain the momenta formulas of high orders $m \geq 2$:

$$\sum_{\sigma \in \Sigma} \prod_{n=1}^{\infty} \frac{Q(P_{\sigma_n}^{-1} \circ \cdots \circ P_{\sigma_1}^{-1}(0))^2}{a^2} = f'(0)^2 - bf''(0) = 1 - \frac{bP''(b)}{a^2 - a}$$

(20)

$$\sum_{\sigma \in \Sigma} \prod_{n=1}^{\infty} \frac{Q(P_{\sigma_n}^{-1} \circ \cdots \circ P_{\sigma_1}^{-1}(0))^m}{a^m} = \sum_{j=1}^{m} \frac{(-b)^{m-j}(j-1)!}{(m-1)!} B_{m,j}(f'(0), \ldots, f^{(m-j+1)}(0)).$$

(21)
Finally, it is easy to use the above arguments to derive
\[ f(z) = w + (b - w) \prod_{\sigma \in \Sigma} \left( 1 - \frac{z}{g_{\sigma}(w)} \right), \]  
(22)
where \( g_{\sigma} \) are branches of \( f^{-1} \), see (13). Thus, another type of Vieta formulas are
\[ \sum_{\sigma \in \Sigma} w - b g_{\sigma}(w) = f'(0) = 1, \quad \sum_{\sigma \neq \tau} (w - b)^2 g_{\sigma}(w) g_{\tau}(w) = \frac{f''(0)}{2} = \frac{P''(b)}{2(a^2 - a)} \]  
(23)
and so on. The momenta formulas for (22) can be obtained in the same way as (20), (21).

We do not discuss here the Mellin transform
\[ \int_{0}^{+\infty} x^{m-1} \ln(f(x) - w)dx \]  
which can express
momenta for non-integer \( m \), defining a Riemann-like function.

2. Examples

1. Consider the case \( P(z) = 2z^2 - 1 \). SP-equation is \( f(az) = 2f(z)^2 - 1 \). We take \( f(0) = b = 1, \ f'(0) = 1 \). Then \( a = (2z^2 - 1)|_{z=b} = 4 \). Polynomial (4) is
\[ Q(z) = \frac{2z^2 - 1 - 1}{z-1} = 2z + 2. \]
There are two branches of \( P^{-1} \):
\[ P_{1}^{-1}(w) = \sqrt{\frac{1+w}{2}}, \quad P_{-1}^{-1}(w) = -\sqrt{\frac{1+w}{2}}. \]
To parameterize zeros of \( f \), we should use the set \( \Sigma = \{ \sigma: \mathbb{N} \rightarrow \{\pm 1\}, \lim_{n \to \infty} \sigma_{n} = 1 \} \).
Then zeros of \( f \) have form (12)
\[ z(\sigma) = -\prod_{n=1}^{\infty} \frac{4}{2 + 2\sigma_{n} \sqrt{\frac{1}{2} + \ldots + \frac{\sigma_{1}}{2} \sqrt{\frac{1}{2}}}} = -\prod_{n=1}^{\infty} \frac{1}{\frac{1}{2} + \frac{\sigma_{n}}{2} \sqrt{\frac{1}{2} + \ldots + \frac{\sigma_{1}}{2} \sqrt{\frac{1}{2}}}}. \]
Computations show
\[ z(1, 1, 1, ...) = -\frac{\pi^2}{8}, \quad z(-1, 1, 1, ...) = -\frac{9\pi^2}{8}, \quad z(-1, -1, 1, ...) = -\frac{25\pi^2}{8}, \quad z(1, -1, 1, ...) = -\frac{49\pi^2}{8} \]
and so on. This is in full agreement with expected values, since \( f(z) = \cos \sqrt{-2z} \). In this case, the formulas for zeros are, in fact, modified Viète’s formulas, see also [11, 21]. The order of entire function \( f \) is \( 1/2 \). WH-factorization is
\[ \cos \sqrt{-2z} = \prod_{n=1}^{\infty} \left( 1 + \frac{8z}{(2n-1)^2 \pi^2} \right) = \prod_{\sigma \in \Sigma} \left( 1 + z \prod_{n=1}^{\infty} \left( \frac{1}{2} + \frac{\sigma_{n}}{2} \sqrt{\frac{1}{2} + \ldots + \frac{\sigma_{1}}{2} \sqrt{\frac{1}{2}}} \right) \right). \]
2. Consider the case $P(z) = z^2 - 1$. The SP-equation is $f(az) = f(z)^2 - 1$. We take $f(0) = b = \frac{\sqrt{5} + 1}{2}$, $f'(0) = 1$. Then $a = (z^2 - 1)\big|_{z=b} = 2b$. Polynomial (4) is

$$Q(z) = \frac{z^2 - 1 - b}{z - b} = z + b.$$  

There are two branches of $P^{-1}$:

$$P_1^{-1}(w) = \sqrt{1 + w}, \quad P_{-1}^{-1}(w) = -\sqrt{1 + w}.$$  

To parameterize zeros of $f$, we should use the same set as in the previous example

$$\Sigma = \{\sigma : \mathbb{N} \to \{\pm 1\}, \lim_{n \to \infty} \sigma_n = 1\}.$$  

Then zeros of $f$ have the form

$$z(\sigma) = -b \prod_{n=1}^{\infty} \frac{2b}{b + \sigma_n \sqrt{1 + \ldots + \sigma_1 \sqrt{1}}}.$$  

The first negative zero $z(1, 1, 1, \ldots) = -2C$ relates to the so-called Paris constant $C$ appearing in the approximation of golden ratio by nested square radicals, see [4, 5, 10]. Zeros of $f$ are also related to the polynomial dynamics generated by $P = z^2 - 1$ and, hence, approximate the corresponding Julia set growing up to infinity, see more in [7, 8, 9]. The zeros form impressive fractal structures, see Fig. 1. The order of entire function $f$ is $\ln 2 / \ln a < 1$. Hence, there is WH-factorization

$$f(z) = b \prod_{\sigma \in \Sigma} \left(1 + \frac{z}{b} \prod_{n=1}^{\infty} \frac{b + \sigma_n \sqrt{1 + \ldots + \sigma_1 \sqrt{1}}}{2b}\right).$$
This is the closed-form representation for the very complex function $f$. There are infinitely many complex zeros of multiplicities $2^n$ for any $n \geq 0$, see \cite{10}. All the multiplicities are taken into account in WH-factorization mentioned above. The first, second and third moment formulas for zeros, see \cite{15}, \cite{20} and \cite{21}, are

$$
\sum_{\sigma \in \Sigma} \prod_{n=1}^{\infty} \left( b + \sigma_n \sqrt{1 + \ldots + \sigma_1 \sqrt{1}} \right)^m = \begin{cases} 
1, & m = 1, \\
1 - \frac{1}{\sqrt{5}}, & m = 2, \\
\frac{2}{5}, & m = 3.
\end{cases}
$$

3. Let us consider the cubic SP-equation $f(az) = f(z)^3 - 6$, $f(0) = b = 2$, $f'(0) = 1$. Then $a = 3b^2 = 12$. The order of the entire function $f(z)$ is $\ln 3 / \ln 12 < 1$. Hence, we can use \cite{15}, \cite{20} to obtain explicit momentum formulas

$$
\sum_{k_n \in \{0,1,2\}; \lim k_n = 0} \prod_{n=1}^{\infty} \left( \frac{e^{\frac{2\pi i k_n}{3}} \sqrt[3]{6 + \ldots + e^{\frac{2\pi i k_1}{3}} \sqrt[3]{6}}}{12} + 2e^{\frac{2\pi i k_n}{3}} \sqrt[3]{6 + \ldots + e^{\frac{2\pi i k_1}{3}} \sqrt[3]{6}} + 4 \right)^2 = 1,
$$

and so on. In these formulas, the distribution of Viète’s products in $\mathbb{C}$ for different $k_n = 0, 1, 2$ is sufficiently complex, but all of their momenta are rational numbers.

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