UNITARILY INVARIANT NORM INEQUALITIES INVOLVING $G_1$ OPERATORS

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Abstract. In this paper, we present some upper bounds for unitarily invariant norms inequalities. Among other inequalities, we show some upper bounds for the Hilbert-Schmidt norm. In particular, we prove

$$\|f(A)Xg(B) \pm g(B)Xf(A)\|_2 \leq \left\| \frac{(I + |A|)X(I + |B|) + (I + |B|)X(I + |A|)}{d_A d_B} \right\|_2,$$

where $A, B, X \in \mathbb{M}_n$ such that $A, B$ are Hermitian with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f, g$ are analytic on the complex unit disk $\mathbb{D}$, $g(0) = f(0) = 1$, $\text{Re}(f) > 0$ and $\text{Re}(g) > 0$.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ with the identity $I$. In the case when $\text{dim} \mathcal{H} = n$, we determine $\mathbb{B}(\mathcal{H})$ by the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having associated with entries in the complex field. If $z \in \mathbb{C}$, then we write $z$ instead of $zI$. For any operator $A$ in the algebra $\mathbb{K}(\mathcal{H})$ of all compact operators, we denote by $\{s_j(A)\}$ the sequence of singular values of $A$, i.e. the eigenvalues $\lambda_j(|A|)$, where $|A| = (A^* A)^{1/2}$, enumerated as $s_1(A) \geq s_2(A) \geq \cdots$ in decreasing order and repeated according to multiplicity. If the rank $A$ is $n$, we put $s_k(A) = 0$ for any $k > n$. Note that $s_j(X) = s_j(X^*) = s_j(|X|)$ and $s_j(AXB) \leq \|A\| \|B\| s_j(X) (j = 1, 2, \cdots)$ for all $A, B \in \mathbb{B}(\mathcal{H})$ and all $X \in \mathbb{K}(\mathcal{H})$.

A unitarily invariant norm is a map $\|\| \cdot \|\| : \mathbb{K}(\mathcal{H}) \rightarrow [0, \infty]$ given by $\|\|A\|\| = g(s_1(A), s_2(A), \cdots)$, where $g$ is a symmetric norming function. The set $\mathcal{C}_{\|\| \cdot \|\|}$ including $\{A \in \mathbb{K}(\mathcal{H}) : \|\|A\|\| < \infty\}$ is a closed self-adjoint ideal $\mathcal{J}$ of $\mathbb{B}(\mathcal{H})$ containing finite rank operators. It enjoys the property [6]:

$$\|\|AXB\|\| \leq \|\|A\|\| \|\|B\|\| \|\|X\|\|$$

for $A, B \in \mathbb{B}(\mathcal{H})$ and $X \in \mathcal{J}$. Inequality (1.1) implies that $\|\|UXV\|\| = \|\|X\|\|$, where $U$ and $V$ are arbitrary unitaries in $\mathbb{B}(\mathcal{H})$ and $X \in \mathcal{J}$. In addition, employing the polar decomposition of $X = W|X|$ with $W$ a partial isometry and (1.1),

2010 Mathematics Subject Classification. Primary 15A60, Secondary 30E20, 47A30, 47B10, 47B15.

Key words and phrases. $G_1$ operator; unitarily invariant norm; commutator operator; the Hilbert-Schmidt; analytic function.

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we have \( |||X||| = |||X||| \). An operator \( A \in K(\mathcal{H}) \) is called Hilbert-Schmidt if \( \|A\|_2 = \left( \sum_{j=1}^{\infty} s_j^2(A) \right)^{1/2} < \infty \). The Hilbert-Schmidt norm is a unitarily invariant norm. For \( A = [a_{ij}] \in M_n \), it holds that \( \|A\|_2 = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2} \). We use the notation \( A \oplus B \) for the diagonal block matrix \( \text{diag}(A, B) \). Its singular values are \( s_1(A), s_1(B), s_2(A), s_2(B), \ldots \). It is evident that

\[
||A \oplus B|| = \max\{\|A\|, \|B\|\} \quad \text{and} \quad ||A \oplus B||_2 = \left( \|A\|_2^2 + \|B\|_2^2 \right)^{1/2}.
\]

The inequalities involving unitarily invariant norms have been of special interest; see e.g., [3, 4, 5, 10] and references therein.

An operator \( A \in B(\mathcal{H}) \) is called \( G_1 \) operator if the growth condition

\[
\|(z - A)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(A))}
\]

holds for all \( z \) not in the spectrum \( \sigma(A) \) of \( A \), where \( \text{dist}(z, \sigma(A)) \) denotes the distance between \( z \) and \( \sigma(A) \). It is known that normal (more generally, hyponormal) operators are \( G_1 \) operators (see e.g., [17]). Let \( A \in B(\mathcal{H}) \) and \( f \) be a function which is analytic on an open neighborhood \( \Omega \) of \( \sigma(A) \) in the complex plane. Then \( f(A) \) denotes the operator defined on \( \mathcal{H} \) by the Riesz-Dunford integral as

\[
f(A) = \frac{1}{2\pi i} \int_C f(z)(z - A)^{-1}dz,
\]

where \( C \) is a positively oriented simple closed rectifiable contour surrounding \( \sigma(A) \) in \( \Omega \) (see e.g., [9, p. 568]). The spectral mapping theorem asserts that \( \sigma(f(A)) = f(\sigma(A)) \).

Throughout this note, \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denotes the unit disk, \( \partial \mathbb{D} \) stands for the boundary of \( \mathbb{D} \) and \( d_A = \text{dist}(\partial \mathbb{D}, \sigma(A)) \). In addition, we denote

\[
\mathfrak{A} = \{ f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, } \text{Re}(f) > 0 \text{ and } f(0) = 1 \}.
\]

The Sylvester type equations \( AXB \pm X = C \) have been investigated in matrix theory; see [7]. Several perturbation bounds for the norm of sum or difference of operators have been presented in the literature by employing some integral representations of certain functions; see [12, 15, 18] and references therein.
In the recent paper [15], Kittaneh showed that the following inequality involving \( f \in A \)
\[
\|\| f(A)X - Xf(B)\|\| \leq \frac{2}{d_A d_B} \|\| AX - XB\|\|
\]
where \( A, B, X \in \mathcal{B}(\mathcal{H}) \) such that \( A \) and \( B \) are \( G_1 \) operators with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \).

In [16], the authors extended this inequality for two functions \( f, g \in A \) as follows
\[
\|\| f(A)X - Xg(B)\|\| \leq \frac{2\sqrt{2}}{d_A d_B} \|\| AX| + |XB|\|\| \quad (1.2)
\]
and
\[
\|\| f(A)X + Xg(B)\|\| \leq \frac{2\sqrt{2}}{d_A d_B} \|\| AXB| + |X|\|\| , \quad (1.3)
\]
in which \( A, B, X \in \mathcal{B}(\mathcal{H}) \) such that \( A \) and \( B \) are \( G_1 \) operators with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \).

They also showed that
\[
\|\| f(A)X - Xg(B)\|\| \leq \frac{2\sqrt{2}}{d_A d_B} \|\| AX| + |XB|\|\| \quad (1.4)
\]
and
\[
\|\| f(A)Xg(B) + X\|\| \leq \frac{2\sqrt{2}}{d_A d_B} \|\| AXB| + |X|\|\| , \quad (1.5)
\]
where \( A, B, X \in \mathcal{B}(\mathcal{H}) \) such that \( A \) and \( B \) are \( G_1 \) operators with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \).

In this paper, by using some ideas from [15, 16] we present some upper bounds for unitarily invariant norms of the forms \( \|\| f(A)X + Xf(A)\|\| \) and \( \|\| f(A)X - Xf(A)\|\| \) involving \( G_1 \) operator and \( f \in A \). We also present the Hilbert-Schmidt norm inequality of the form
\[
\| f(A)Xg(B) \pm g(B)Xf(A) \|_2 \leq \left\| \frac{(I + |A|)X(I + |B|) + (I + |B|)X(I + |A|)}{d_AD_B} \right\|_2 ,
\]
where \( A, B, X \in \mathbb{M}_n \) such that \( A \) and \( B \) are Hermitian matrices with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \) and \( f, g \in \mathfrak{A} \).

## 2. Main Results

Our first result is some upper bounds for the Hilbert-Schmidt norm inequalities.

**Theorem 2.1.** Let \( A, B \in \mathbb{M}_n \) be Hermitian matrices with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \) and \( f, g \in \mathfrak{A} \). Then
\[
\| f(A)X + Xg(B) \pm f(A)Xg(B) \|_2 \leq \left\| \frac{X + |A|X}{d_A} + \frac{X + X|B|}{d_B} + \frac{(I + |A|)X(I + |B|)}{d_AD_B} \right\|_2
\]
and

\[ \| f(A)Xg(B) \pm g(B)Xf(A) \|_2 \leq \left\| \frac{(I + |A|)X(I + |B|) + (I + |B|)X(I + |A|)}{d_AD_B} \right\|_2, \]

where \( X \in \mathbb{M}_n \).

**Proof.** Let \( A = U\Lambda U^* \) and \( B = V\Gamma V^* \) be the spectral decomposition of \( A \) and \( B \) such that \( \Lambda = \text{diag} (\lambda_1, \ldots , \lambda_n) \), \( \Gamma = \text{diag} (\gamma_1, \ldots , \gamma_n) \) and let \( U^*XV := [y_{jk}] \). It follows from \( |e^{i\alpha} - \lambda_j| \geq d_A \) and \( |e^{i\beta} - \gamma_k| \geq d_B \) that

\[ \| f(A) + Xg(B) \|_2 \]

\[ = \sum_{j,k} |f(\lambda_j)y_{j,k} + y_{j,k}g(\gamma_k) \pm f(\lambda_j)y_{j,k}g(\gamma_k)|^2 \]

\[ = \sum_{j,k} |f(\lambda_j) + f(\lambda_j)g(\gamma_k) + g(\gamma_k)|^2|y_{j,k}|^2 \]

\[ = \sum_{j,k} \left| \int_0^{2\pi} \int_0^{2\pi} e^{i\alpha} + \lambda_j \frac{e^{i\alpha} - \lambda_j}{e^{i\alpha} - \lambda_j} + \frac{e^{i\beta} + \gamma_k}{e^{i\beta} - \gamma_k} - \left( e^{i\alpha} + \lambda_j \frac{e^{i\alpha} + \lambda_j}{e^{i\alpha} - \lambda_j} - \frac{e^{i\beta} + \gamma_k}{e^{i\beta} - \gamma_k} \right) du(\alpha)du(\beta) \right|^2 |y_{j,k}|^2 \]

\[ \leq \sum_{j,k} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{|e^{i\alpha} + \lambda_j|}{|e^{i\alpha} - \lambda_j|} + \frac{|e^{i\beta} + \gamma_k|}{|e^{i\beta} - \gamma_k|} + \frac{|e^{i\alpha} + \lambda_j||e^{i\beta} + \gamma_k|}{|e^{i\alpha} - \lambda_j||e^{i\beta} - \gamma_k|} du(\alpha)du(\beta) \right)^2 |y_{j,k}|^2 \]

\[ \leq \sum_{j,k} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{|e^{i\alpha} + \lambda_j|}{d_A} + \frac{(1 + |\lambda_j|)(1 + |\gamma_k|)}{d_A d_B} + \frac{1 + |\gamma_k|}{d_B} du(\alpha)du(\beta) \right)^2 |y_{j,k}|^2 \]

\[ \leq \sum_{j,k} \left( \frac{1 + |\lambda_j|}{d_A} + \frac{1 + |\gamma_k|}{d_B} + \frac{(1 + |\lambda_j|)(1 + |\gamma_k|)}{d_A d_B} \right)^2 |y_{j,k}|^2 \]

\[ = \left\| \frac{X + |A|X}{d_A} + \frac{X + |B|X}{d_B} + \frac{(I + |A|)X (I + |B|)}{d_Ad_B} \right\|_2. \]
Then we get the first inequality. Similarly,
\[
\|f(A)Xg(B) \pm g(B)X f(A)\|_2^2 \\
= \sum_{j,k} \left| f(\lambda_j) y_{j,k} g(\gamma_k) \pm g(\gamma_j) y_{j,k} f(\lambda_k) \right|^2 \\
= \sum_{j,k} \left| f(\lambda_j) g(\gamma_k) \pm g(\gamma_j) f(\lambda_k) \right|^2 |y_{j,k}|^2 \\
= \sum_{j,k} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{(e^{i\alpha} + \lambda_j)(e^{i\beta} + \gamma_k)}{(e^{i\alpha} - \lambda_j)(e^{i\beta} - \gamma_k)} \pm \frac{(e^{i\beta} + \gamma_j)(e^{i\alpha} + \lambda_k)}{(e^{i\beta} - \gamma_j)(e^{i\alpha} - \lambda_k)} d\mu(\alpha)d\mu(\beta) \right)^2 |y_{j,k}|^2 \\
\leq \sum_{j,k} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{|e^{i\alpha} + \lambda_j| |e^{i\beta} + \gamma_k|}{|e^{i\alpha} - \lambda_j| |e^{i\beta} - \gamma_k|} \pm \frac{|e^{i\beta} + \gamma_j| |e^{i\alpha} + \lambda_k|}{|e^{i\beta} - \gamma_j| |e^{i\alpha} - \lambda_k|} d\mu(\alpha)d\mu(\beta) \right)^2 |y_{j,k}|^2 \\
\leq \sum_{j,k} \left( \frac{(1 + |\lambda_j|)(1 + |\gamma_k|)}{d_A d_B} + \frac{(1 + |\gamma_j|)(1 + |\lambda_k|)}{d_A d_B} \right)^2 |y_{j,k}|^2 \\
\leq \sum_{j,k} \left( \frac{(1 + |\lambda_j|)|y_{j,k}(1 + |\gamma_k|)}{d_A d_B} + \frac{(1 + |\gamma_j|)y_{j,k}(1 + |\lambda_k|)}{d_A d_B} \right)^2 \\
= \frac{\left\| (I + |A|)X(I + |B|) + (I + |B|)X(I + |A|) \right\|}{d_A d_B}.
\]
\[\square\]

Now, if we put \( X = I \) in Theorem 2.1, then we get the next result.

**Corollary 2.2.** Let \( A, B \in \mathbb{M}_n \) be Hermitian matrices with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \) and \( f, g \in A \). Then
\[
\| f(A) + g(B) \pm f(A)g(B) \|_2 \leq \left\| \frac{I + |A|}{d_A} + \frac{I + |B|}{d_B} + \frac{(I + |A|)(I + |B|)}{d_A d_B} \right\|_2
\]
and
\[
\| f(A)g(B) \pm g(B)f(A) \|_2 \leq \left\| \frac{(I + |A|)(I + |B|) + (I + |B|)(I + |A|)}{d_A d_B} \right\|_2.
\]

To prove the next results, the following lemma is required.

**Lemma 2.3.** Let \( A, B, X, Y \in \mathbb{B}(\mathcal{H}) \) such that \( X \) and \( Y \) are compact. Then
\[
(a) \ s_j(AX \pm YB) \leq 2\sqrt{\|A\|\|B\|} s_j(X \pm Y) \ (j = 1, 2, \cdots);
\]
(b) \[ ||(AX \pm YB) \oplus 0|| \leq 2\sqrt{\|A\|\|B\| ||X \oplus Y||}. \]

**Proof.** Using [12, Theorem 2.2] we have

\[ s_j(AX \pm YB) \leq (\|A\| + \|B\|)s_j(X \oplus Y) \quad (j = 1, 2, \ldots). \]

If we replace \( A, B, X \) and \( Y \) by \( tA, \frac{B}{t}, \frac{X}{t} \) and \( tY \), respectively, then we get

\[ s_j(AX \pm YB) \leq \left( t\|A\| + \frac{\|B\|}{t} \right)s_j(X \oplus Y) \quad (j = 1, 2, \ldots). \]

It follows from \( \min_{t>0}(t\|A\| + \frac{\|B\|}{t}) = 2\sqrt{\|A\|\|B\|} \) that we reach the first inequality. The second inequality can be proven by the first inequality and the Ky Fan dominance theorem [6, Théorème IV.2.2]; see also [2]. \( \square \)

Now, by applying Lemma 2.3 we obtain the following result.

**Theorem 2.4.** Let \( A, B, X, Y \in B(\mathcal{H}) \) and \( f, g \in \mathfrak{A} \). Then

\[ ||((f(A) - g(B))X \pm Y(f(B) - g(A))) \oplus 0|| \leq \frac{4\sqrt{2}}{d_A d_B} \|A\| + \|B\| ||X \oplus Y|| \]

and

\[ ||((f(A) + g(B))X \pm Y(f(B) + g(A))) \oplus 0|| \leq \frac{4\sqrt{2}}{d_A d_B} \|I + |AB|\| ||X \oplus Y||, \]

where \( X, Y \) are compact and \( A, B \) are \( G_1 \) operators with \( \sigma(A) \cup \sigma(B) \subset \mathbb{D} \).

**Proof.** Using Lemma 2.3 and inequalities (1.2) and (1.3) we have

\[ \begin{align*}
||((f(A) - g(B))X \pm Y(f(B) - g(A))) \oplus 0|| &
\leq 2\|f(A) - g(B)\| \frac{\sqrt{2}}{d_A} \|f(B) - g(A)\| \frac{\sqrt{2}}{d_B} ||X \oplus Y|| \quad (\text{by Lemma 2.3}) \\
&\leq 2 \sqrt{\frac{2\sqrt{2}}{d_A d_B} \|A\| + \|B\|} \sqrt{\frac{2\sqrt{2}}{d_A d_B} \|B\| + \|A\|} ||X \oplus Y|| \\
&\quad \text{(by inequality (1.2))} \\
&= \frac{4\sqrt{2}}{d_A d_B} \|A\| + \|B\| ||X \oplus Y||.
\end{align*} \]
Similarly,
\[
\left\| (f(A) + g(B))X \pm Y(f(B) + g(A)) \right\| \leq 2\|f(A) + g(B)\|^{1/2}\|f(B) + g(A)\|^{1/2}\|X \pm Y\| \quad (by \ Lemma \ 2.3)
\]
\[
\leq 2\sqrt{2\|I\|} \sqrt{2\|I\|} \left\| AX \pm Y \right\| \quad (by \ inequalities \ (1.3))
\]
\[
= \frac{4\sqrt{2}}{d_A d_B} \|I\| \left\| X \oplus Y \right\|.
\]

\[\square\]

**Theorem 2.5.** Let \(A, B \in \mathcal{B}(\mathcal{H})\) be \(G_1\) operators with \(\sigma(A) \cup \sigma(B) \subset \mathbb{D}\) and \(f \in \mathfrak{A}\). Then for every \(X \in \mathcal{B}(\mathcal{H})\)

\[
\left\| f(A)X + Xf(B) \right\| \leq \frac{2}{d_A d_B} \left\| X - AXB^* \right\| \quad (2.1)
\]

and

\[
\left\| f(A)X - Xf(B) \right\| \leq \frac{2\sqrt{2}}{d_A d_B} \left\| AX + XB^* \right\|, \quad (2.2)
\]

**Proof.** Using the Herglotz representation theorem (see e.g., [8, p. 21]) we have

\[
f(z) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) + i \text{Im} f(0) = \int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha),
\]

where \(\mu\) is a positive Borel measure on the interval \([0, 2\pi]\) with finite total mass \(\int_0^{2\pi} d\mu(\alpha) = f(0) = 1\). Hence

\[
f(\bar{z}) = \int_0^{2\pi} \frac{e^{i\alpha} + \bar{z}}{e^{i\alpha} - \bar{z}} d\mu(\alpha) = \int_0^{2\pi} \frac{e^{-i\alpha} + \bar{z}}{e^{-i\alpha} - \bar{z}} d\mu(\alpha),
\]

where \(\bar{f}\) is the conjugate function of \(f\) (i.e., \(\bar{f} f = |f|^2\)). So

\[
f(A)X + Xf(B)
\]

\[
= \int_0^{2\pi} (e^{i\alpha} + A) (e^{i\alpha} - A)^{-1} X + X (e^{-i\alpha} + B^*) (e^{-i\alpha} - B^*)^{-1} d\mu(\alpha)
\]

\[
= \int_0^{2\pi} (e^{i\alpha} - A)^{-1} \left[ (e^{i\alpha} + A) X (e^{-i\alpha} - B^*)
\right.
\]

\[
+ (e^{i\alpha} - A) X (e^{-i\alpha} + B^*) \left. \right] (e^{-i\alpha} - B^*)^{-1} d\mu(\alpha)
\]

\[
= 2 \int_0^{2\pi} (e^{i\alpha} - A)^{-1} (X - AXB^*) (e^{-i\alpha} - B^*)^{-1} d\mu(\alpha).
\]
Hence

\[ \left\| f(A)X + X\tilde{f}(B) \right\| = \left\| \int_{0}^{2\pi} (e^{i\alpha} + A)(e^{i\alpha} - A)^{-1} X + X(e^{-i\alpha} + B^*) (e^{-i\alpha} - B^*)^{-1} d\mu(\alpha) \right\| \]

\[ = 2 \left\| \int_{0}^{2\pi} (e^{i\alpha} - A)^{-1} (X - AXB^*) (e^{-i\alpha} - B^*)^{-1} d\mu(\alpha) \right\| \]

\[ \leq 2 \int_{0}^{2\pi} \left\| (e^{i\alpha} - A)^{-1} (X - AXB^*) (e^{-i\alpha} - B^*)^{-1} \right\| d\mu(\alpha) \]

\[ \leq 2 \int_{0}^{2\pi} \| (e^{i\alpha} - A)^{-1} \| \| (e^{i\alpha} - B)^{-1} \| \| X - AXB^* \| d\mu(\alpha) \]

(by inequality (1.1)).

Since $A$ and $B$ are $G_1$ operators, it follows from $\| (e^{i\alpha} - A)^{-1} \| = \frac{1}{\text{dist}(\partial E, \sigma(A))} \leq \frac{1}{d_A}$ and $\| (e^{i\alpha} - B)^{-1} \| \leq \frac{1}{d_B}$ that

\[ \left\| f(A)X + X\tilde{f}(B) \right\| \leq \left( \frac{2}{d_A d_B} \int_{0}^{2\pi} d\mu(\alpha) \right) \| X - AXB^* \|
\]

\[ = \left( \frac{2}{d_A d_B} f(0) \right) \| X - AXB^* \|
\]

\[ = \frac{2}{d_A d_B} \| X - AXB^* \|. \]

Then we have the first inequality. Using the inequality

\[ \left\| e^{-i\alpha}AX + e^{i\alpha}XB^* \right\| = \left\| \begin{bmatrix} e^{-i\alpha}AX + e^{i\alpha}XB^* & 0 \\ 0 & 0 \end{bmatrix} \right\|
\]

\[ = \left\| \begin{bmatrix} e^{-i\alpha} & e^{i\alpha} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} AX & 0 \\ XB^* & 0 \end{bmatrix} \right\|
\]

\[ \leq \left\| \begin{bmatrix} e^{-i\alpha} & e^{i\alpha} \\ 0 & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} AX & 0 \\ XB^* & 0 \end{bmatrix} \right\| \quad \text{(by inequality (1.1))}
\]

\[ = \sqrt{2} \left\| \begin{bmatrix} AX & 0 \\ XB^* & 0 \end{bmatrix} \right\|
\]

\[ = \sqrt{2} \left\| (|AX|^2 + |XB^*|^2)^{\frac{1}{2}} \oplus 0 \right\|
\]

\[ \leq \sqrt{2} \left\| (|AX| + |XB^*|) \oplus 0 \right\|
\]

(applying [1, p. 775] to the function $h(t) = t^{\frac{1}{2}}$).
the Ky Fan dominance theorem we have
\[
||| e^{-i\beta}AX + e^{i\alpha}XB^* ||| \leq \sqrt{2} \left( |||AX||| + |||XB^*||| \right).
\] (2.3)

It follows from (2.3) and the same argument of the proof of the first inequality that we have the second inequality and this completes the proof. \(\square\)

Remark 2.6. Let \(f(x + yi) = u(x, y) + v(x, y)i\), where \(u, v\) are real and imaginary parts of \(f\), respectively. If \(f, \bar{f} \in A\), then the Cauchy-Riemann equations for complex analytic functions (i.e., \(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\) and \(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\)) implies that \(v(x, y) = k\) for some \(k \in \mathbb{C}\). The condition \(f(0) = 1\) conclude that \(v(x, y) = 0\). Hence, \(f\) is a real valued function. So, for arbitrary functions \(f, g \in A\), we can not replace \(g\) by \(\bar{f}\) in inequalities (1.2) and (1.3). Thus, in Theorem 2.5 we have been established some upper bounds for \(|||f(A)X + X\bar{f}(B)|||\) and \(|||f(A)X - X\bar{f}(B)|||\) in terms of \(|||X - AXB^*|||\) and \(|||AX| + |XB^*|||\), respectively, that can not be derived from inequality (1.2) and (1.3) for an arbitrary function \(f \in A\).

Remark 2.7. If \(A, B \in \mathbb{B}(\mathcal{H})\) are \(G_1\) operators with \(\sigma(A) \cup \sigma(B) \subseteq \mathbb{D}\) and \(f \in A\), then with a similar argument in the proof of Theorem 2.5 we get the following inequalities
\[
|||\bar{f}(A)X + Xf(B)||| \leq \frac{2}{d_A d_B} |||AX - A^*XB|||
\] (2.4)
and
\[
|||\bar{f}(A)X - Xf(B)||| \leq \frac{2\sqrt{2}}{d_A d_B} |||A^*X| + |XB|||,
\]
where \(X \in \mathbb{B}(\mathcal{H})\).

Remark 2.8. For an arbitrary operator \(A \in \mathbb{B}(\mathcal{H})\), the numerical range is definition by \(W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, ||x|| = 1\}\). It is well-known that \(W(A)\) is a bounded convex subset of the complex plane \(\mathbb{C}\). Its closure \(\overline{W(A)}\) contains \(\sigma(A)\) and is contained in \(\{z \in \mathbb{C} : |z| \leq ||A||\}\). In [11], it is shown
\[
\frac{1}{\text{dist}(z, \sigma(A))} \leq ||(z - A)^{-1}|| \quad (z \notin \sigma(A))
\]
and
\[
||(z - A)^{-1}|| \leq \frac{1}{\text{dist}(z, \overline{W(A)})} \quad (z \notin \overline{W(A)}).
\]

Now, if we replace the hypophysis \(G_1\) operators by the conditions \(\overline{W(A)} \cup \overline{W(B)} \subseteq \mathbb{D}\) in Theorem 2.5, then in inequalities (1.2)-(1.5), the constants \(d_A\) and \(d_B\) interchange.
to $D_A$ and $D_B$, respectively, where $D_A = \text{dist}(\partial \mathbb{D}, W(A))$, $D_B = \text{dist}(\partial \mathbb{D}, W(A))$. Also inequalities (2.1) and (2.2) appear of the forms

$$|||f(A)X + X\bar{f}(B)||| \leq \frac{2}{D_AD_B} |||X - AXB^*|||$$

and

$$|||f(A)X - X\bar{f}(B)||| \leq \frac{2\sqrt{2}}{D_AD_B} |||AX| + |XB^*|||.$$

where $f \in \mathfrak{A}$. For example, for every contraction operator $A$ (i.e., $A^*A \leq I$) and $0 < \epsilon < 1$, the operator $\epsilon A$ has the property $W(\epsilon A) \subseteq \mathbb{D}$.

If we take $X = I$ in Theorem 2.5, then we get the following result.

**Corollary 2.9.** Let $A, B \in \mathbb{B}(\mathcal{H})$ be normal operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f \in \mathfrak{A}$. Then for every $X \in \mathbb{B}(\mathcal{H})$

$$|||f(A) + \bar{f}(B)||| \leq \frac{2}{d_A d_B} |||I - AB^*|||.$$

In particular, for $B = A$ we have

$$|||\text{Re}(f(A))||| \leq \frac{1}{d_A^2} |||I - AA^*|||.$$

For the next result we need the following lemma (see also [14]).

**Lemma 2.10.** If $A, B, X \in \mathbb{B}(\mathcal{H})$ such that $A$ and $B$ are self-adjoint and $0 < mI \leq X$ for some positive real number $m$, then

$$m |||A - B||| \leq |||AX + XB|||.$$

**Proof.**

$$m |||A - B||| \leq \frac{1}{2} |||(A - B)X + X(A - B)||| \quad \text{(by [19, Lemma 3.1])}$$

$$= \frac{1}{2} |||AX - XB + (XA - BX)|||$$

$$\leq \frac{1}{2} (|||AX - XB||| + |||XA - BX|||)$$

$$= |||AX - XB||| \quad \text{(since $||A|| = ||A^*||$).}$$

\[\square\]

**Proposition 2.11.** Let $A, B \in \mathbb{B}(\mathcal{H})$ be $G_1$ operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$, let $X \in \mathbb{B}(\mathcal{H})$ such that $0 < mI \leq X$ for some positive real number $m$ and $f \in \mathfrak{A}$. Then

$$m |||\text{Re}(f(A)) - \text{Re}(f(B))||| \leq \frac{1}{d_A d_B} (|||X - AXB^*||| + |||X - A^*XB|||), \quad (2.5)$$
In particular, if $A$ and $B$ are unitary operators, then

$$m \|\| \Re(f(A)) - 
\Re(f(B)) \|\| \leq \frac{2}{d_A d_B} \|\| X - AXB^* \|\|$$

Proof.

$$m \|\| \Re(f(A)) - \Re(f(B)) \|\| \leq \|\| \Re(f(A)) X + X \Re(f(B)) \|\|$$

(by Lemma 2.10)

$$= \frac{1}{2} \|\| f(A) X + X \bar{f}(B) + \bar{f}(A) X + X f(B) \|\|$$

$$\leq \frac{1}{2} \left( \|\| f(A) X + X \bar{f}(B) \|\| + \|\| \bar{f}(A) X + X f(B) \|\| \right)$$

$$\leq \frac{1}{d_A d_B} \left( \|\| X - AXB^* \|\| + \|\| X - A^* XB \|\| \right)$$

(by inequalities (2.1) and (2.4)).

Hence we get the first inequality. Especially, it follows from inequality (2.5) and equation

$$\|\| X - AXB^* \|\| = \|\| A^* XB - X \|\| = \|\| A^* XB - X \|\|. \quad \Box$$

Remark 2.12. Using Lemma 2.3 we have

$$\|\| (f(A) + \bar{f}(B)) X - Y (f(B) + \bar{f}(A)) \oplus 0 \|\| \leq 2 \|\| f(A) + \bar{f}(B) \|\| \|\| f(B) + \bar{f}(A) \|\| \|\| X \oplus Y \|\|$$

$$= 2 \|\| f(A) + \bar{f}(B) \|\| \|\| X \oplus Y \|\|.$$ 

Now, If we apply inequality (2.1), then we reach

$$\|\| f(A) + \bar{f}(B) \|\| \|\| X \oplus Y \|\| \leq \frac{2}{d_A d_B} \|\| I - AB^* \|\| \|\| X \oplus Y \|\|,$$

whence

$$\|\| (f(A) + \bar{f}(B)) X - Y (f(B) + \bar{f}(A)) \oplus 0 \|\| \leq \frac{4}{d_A d_B} \|\| I - AB^* \|\| \|\| X \oplus Y \|\|.$$

Hence, if we put $B = A$, then we get

$$\|\| \Re(f(A)) X - Y \Re(f(A)) \oplus 0 \|\| \leq \frac{2}{d_A^2} \|\| I - AA^* \|\| \|\| X \oplus Y \|\|.$$
REFERENCES

1. T. Ando and X. Zhan, *Norm inequalities related to operator monotone functions*, Math. Ann. 315 (1999), no. 4, 771–780.
2. R. Alizadeh and M.B. Asadi, *An extension of Ky Fan’s dominance theorem*, Banach J. Math. Anal. 6 (2012), no. 1, 139–146.
3. M. Bakherad, M. Krnic and M.S. Moslehian, *Reverses of the Young inequality for matrices and operators*, Rocky Mountain J. Math. 46 (2016), no. 4, 1089–1105.
4. M. Bakherad and M.S. Moslehian, *Reverses and variations of Heinz inequality*, Linear Multilinear Algebra 63 (2015), no. 10, 1972–1980.
5. M. Bakherad and F. Kittaneh, *Numerical Radius Inequalities Involving Commutators of G1 Operators*, Complex Anal. Oper. Theory (to appear) DOI 10.1007/s11785-017-0726-9.
6. R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
7. L. Bao, Y. Lin and Y. Wei, *Krylov subspace methods for the generalized Sylvester equation*, Appl. Math. Comput. 175 (2006), no. 1, 557–573.
8. W.F. Donoghue, *Monotone Matrix Functions and Analytic Continuation*. Springer, New York (1974).
9. N. Dunford and J. Schwartz, *Linear Operators I*, Interscience, New York, 1958.
10. J.I. Fujii, M. Fujii, T. Furuta and M. Nakamoto, *Norm inequalities equivalent to Heinz inequality*, Proc. Amer. Math. Soc. 118 (1993), 827–830.
11. S. Hildebrandt, *Über den numerischen Wertebereich eines operators*, Math. Ann. 163 (1966) 230–247.
12. O. Hirzallah and F. Kittaneh, *Singular values, norms, and commutators*, Linear Algebra Appl. 432 (2010), no. 5, 1322–1336.
13. F. Kittaneh, *On some operator inequalities*, Linear Algebra Appl. 208/209 (1994), 1928.
14. F. Kittaneh, M.S. Moslehian and T. Yamazaki, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. 471 (2015), 46–53.
15. F. Kittaneh, *Norm inequalities for commutators of G1 operators*, Complex Anal. Oper. Theory 10 (2016), no. 1, 109–114.
16. F. Kittaneh, M.S. Moslehian and M. Sababheh, *Unitarily invariant norm inequalities for elementary operators involving G1 operators*, Linear Algebra Appl. 513 (2017) 84–95.
17. C.R. Putnam, *Operators satisfying a G1 condition*, Pacific J. Math. 84 (1979), 413–426.
18. A. Seddik, *Rank one operators and norm of elementary operators*, Linear Algebra Appl. 424 (2007), no. 1, 177–183.
19. J.L. van Hemmen and T. Ando, *An inequality for trace ideals*, Comm. Math. Phys. 76(2) (1980) 143–148.

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