Membrane Sigma-Models
and Quantization of Non-Geometric Flux Backgrounds

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Abstract

We develop quantization techniques for describing the nonassociative geometry probed by closed strings in flat non-geometric $R$-flux backgrounds $M$. Starting from a suitable Courant sigma-model on an open membrane with target space $M$, regarded as a topological sector of closed string dynamics in $R$-space, we derive a twisted Poisson sigma-model on the boundary of the membrane whose target space is the cotangent bundle $T^*M$ and whose quasi-Poisson structure coincides with those previously proposed. We argue that from the membrane perspective the path integral over multivalued closed string fields in $Q$-space is equivalent to integrating over open strings in $R$-space. The corresponding boundary correlation functions reproduce Kontsevich’s deformation quantization formula for the twisted Poisson manifolds. For constant $R$-flux, we derive closed formulas for the corresponding nonassociative star product and its associator, and compare them with previous proposals for a 3-product of fields on $R$-space. We develop various versions of the Seiberg–Witten map which relate our nonassociative star products to associative ones and add fluctuations to the $R$-flux background. We show that the Kontsevich formula coincides with the star product obtained by quantizing the dual of a Lie 2-algebra via convolution in an integrating Lie 2-group associated to the T-dual doubled geometry, and hence clarify the relation to the twisted convolution products for topological nonassociative torus bundles. We further demonstrate how our approach leads to a consistent quantization of Nambu–Poisson 3-brackets.

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1 Introduction and summary

Background and context

String vacua with $p$-form field fluxes along the extra dimensions are called flux compactifications and have been intensively studied in recent years because of their ability to cure some of the problems suffered by the more conventional Calabi–Yau compactifications [31, 27, 10]. They have also provided broader notions of string geometry, and compactifications on non-geometric spaces have emerged as consistent string vacua. Non-geometric backgrounds can arise from taking T-
duality transformations of conventional geometric backgrounds, and their non-geometric nature is reflected in the fact that the transition functions between local patches typically involve string duality transformations.

A prototypical example of a non-geometric space is obtained by T-dualising a three-torus $T^3$ with non-vanishing three-form $H$-flux \([44, 80, 23]\). T-dualising along one cycle gives rise to a twisted torus with geometric $f$-flux related to a metric connection on the tangent bundle of the nilmanifold. However, T-dualising along two cycles gives rise to a space with non-geometric $Q$-flux whereby one of the cycles is only periodic up to T-duality, which mixes momentum and winding modes; the resulting geometry is thus only well-defined locally and is called a T-fold. An approach to describing these backgrounds mathematically in the context of open string theory was put forward in \([66, 28, 32, 13]\) using the language of noncommutative geometry: The T-fold can be regarded as a fibration of noncommutative two-tori over the base circle. A more radical possibility comes about when one T-dualises along all three circles, which takes a geometric torus $T^3$ with uniform $H$-flux to a space with non-geometric $R$-flux; it provides a realisation of the expectations that there should exist a generalization of T-duality that applies to torus bundles with non-isometric torus action. The precise geometric meaning of $R$-spaces has remained somewhat elusive. In the context of open string theory it is described in \([14, 28]\) using nonassociative twisted convolution algebras, which are defined in \([15]\) as objects internal to a tensor category with a weak monoidal structure: The $R$-space can be regarded generally as a bundle of noncommutative and nonassociative tori.

The open string picture of the geometry of T-folds is a result of the $B$-field experienced by the fibre directions, which depends on the coordinate of the base circle. It can be understood from the way in which noncommutative geometry arises in open string theory as a result of a constant two-form $B$-field on D-branes \([78]\). Canonical quantization of the open string sigma-model results in commutation relations for the string endpoints given by

$$\left[ X^i(\tau, \sigma), X^j(\tau, \sigma') \right] \bigg|_{\sigma = \sigma' = 0, 2\pi} = i \theta^{ij}$$

where $\theta = -2\pi \alpha'(1 + F^2)^{-1} F$, $F = B - F$ with $F$ the gauge field strength two-form on the D-brane. The D-brane worldvolume becomes a noncommutative space with a low-energy effective field theory that is given by a noncommutative gauge theory in a double-scaling limit \(\alpha' \to 0, B \to \infty\) which decouples the open strings from closed strings, i.e. from gravity \([78]\). The resulting twisted gauge symmetries suggest that these noncommutative field theories are models for general relativity which contain emergent and noncommutative gravity (see e.g. \([83]\) for a review).

However, it is more natural to seek noncommutative gravity structures emerging in the sector of closed strings. A similar computation for closed strings reveals that the equal-time equal-position commutator is a function of the worldsheet coordinates, and therefore not a well-defined target space object. However, the authors of \([8, 11]\) define a 3-bracket as the Jacobiator (which they call the “cyclic double commutator”)

$$[X^i, X^j, X^k] := \lim_{\sigma_1 \to \sigma} \left[ \left[ X^i(\tau, \sigma_1), X^j(\tau, \sigma_2) \right], X^k(\tau, \sigma_3) \right] + \text{cyclic}.$$  

Calculating this for the SU(2) WZW model and for a linearized conformal field theory in flat space with $H$-flux they obtain the non-trivial result

$$[X^i, X^j, X^k] = i \alpha \theta^{ijk},$$
where $\theta^{ijk}$ is proportional to the background flux, and $\alpha = 0$ for the $H$-flux background while $\alpha = 1$ after an odd number of T-duality transformations.

In [63] the same type of nonassociativity emerges in a somewhat different context, via T-duality on a three-torus $M = T^3$ with non-vanishing $H$-flux. The basic mechanism is that while on geometric spaces in the presence of 3-flux (either NS–NS $H$-flux or metric $f$-flux) the closed string sigma-model fields commute, $[X^i(\tau, \sigma), X^j(\tau, \sigma)] = 0$, on non-geometric spaces (T-folds or $R$-spaces) they no longer commute, $[X^i(\tau, \sigma), X^j(\tau, \sigma)] \neq 0$. Via T-duality, this maps back to the original geometric space as a noncommutativity relation

$$[X^i(\tau, \sigma), X^*_j(\tau, \sigma)] \neq 0$$

(1.4)

between string coordinates $X^i \in M$ and the dual coordinates $X^*_i \in M^*$. This suggests that T-duality and doubled geometry is the natural framework to investigate closed string noncommutative geometry. This point was noticed already some time ago for tori with constant $B$-fields in [60, 61], while noncommutative correspondence spaces associated to T-folds are described in [16, 13] in the context of open strings. In [63] this was demonstrated by applying T-duality along one direction of a twisted three-torus with non-vanishing geometric flux, which reveals that there is a non-trivial commutation relation between the coordinates in the doubled geometry that is determined by the winding number in the base direction. This led to the conjecture that after three T-dualities on the original $T^3$ with 3-flux the coordinates satisfy a nonassociative phase space coordinate algebra whose Jacobiator reproduces the 3-bracket (1.3), confirming that the source of the nonassociativity structure is the non-trivial $H$-flux [63], as in [14]; in [64] it was pointed out that these phase space relations define a twisted Poisson structure. In this setting, double field theory is the natural framework for investigating the effects of T-duality in the context of double geometry (see e.g. [43] for a review); the doubled geometry in this case is the double torus $M \times M^*$ including the momentum coordinates together with the dual winding coordinates. It provides a means for formulating effective string actions which are invariant under T-duality transformations. Noncommutative and nonassociative phase space structures were also found on geometric twisted tori in [20] from a different point of view as solutions of matrix theory compactification conditions.

In [63] it was further argued that a framework for understanding closed string nonassociative geometry in the context of double field theory is provided by an analogy with open string noncommutativity. Using T-duality, a noncommutative field theory on D-branes may be mapped to an ordinary field theory on D-branes intersecting at angles. A closed string on the doubled $T^3$ geometry has boundary conditions satisfied by momentum states on the double twisted torus as

$$X(\tau, \sigma + 2\pi) = e^{2\pi i \theta} X(\tau, \sigma) \quad \text{and} \quad X^*(\tau + 2\pi, \sigma) = e^{2\pi i \theta} X^*(\tau, \sigma),$$

(1.5)

where $\theta = -n H$ is determined by the dual momentum $n \in \mathbb{Z}$ in the direction along which T-duality acts, while the directions are exchanged for the dual $H$-flux background ($X \leftrightarrow X^*$). This situation resembles the open string case: T-duality switches between the directions of commutative and noncommutative boundary conditions in a manner dictated by the dual momentum. Recall that noncommutativity arises in open string theory as a result of a $B$-field background and a field strength $F$ in the gauge theory on a D-brane. By means of the Seiberg–Witten map this noncommutative gauge theory may be mapped to an ordinary gauge theory [78]. This is similar to the role that T-duality assumes here, and an interesting question is whether there is
a map in closed string theory that exchanges a nonassociative field theory with an associative field theory.

**Summary of results**

In this paper we shall investigate the origins of noncommutative and nonassociative geometry for closed strings in $R$-space, and relate it with open string noncommutativity. We will develop two equivalent nonassociative quantizations of constant $R$-flux backgrounds and connect them to the previous studies reviewed above. Our starting point is the observation that the open string sigma-model with a closed $B$-field background in the limit that decouples it from gravity is given by the Poisson sigma-model, whose boundary correlation functions naturally lead to Kontsevich’s star product for the deformation quantization of fields along the associated Poisson structure [17]. We will argue that the suitable analog for closed strings in flux backgrounds is a higher version of the Poisson sigma-model called the Courant sigma-model; this is a sigma-model on an open three-dimensional membrane, with the boundaries regarded as the closed strings, whose target space is $M$ and whose field content is valued in a Courant algebroid. In the present case the pertinent Courant algebroid over $M$ is the standard Courant algebroid $C = TM \oplus T^*M$. This setting is natural from the point of view of double field theory alluded to above: If $Y$ denotes the double twisted torus which is $T$-dual to $M \times M^*$, then the structure algebra of $C$ given by the twisted Courant–Dorfman bracket coincides with the usual Lie bracket on vector fields in the tangent Lie algebroid $TY$ [45]. The relevance of Courant algebroids in non-geometric flux compactifications and to gauge symmetries in double field theory has also been noted in [34, 46, 9]. We will show that with the structure functions of $C$ appropriate to the pure constant $R$-flux background, the membrane sigma-model reduces to a twisted Poisson sigma-model on the boundary whose target space is the cotangent bundle $T^*M$ of the original target space $M$. The twist is given by a non-flat $U(1)$-gerbe on *momentum space*, and the resulting linear twisted Poisson structure coincides exactly with that proposed in [63, 64]. Our membrane sigma-model thus gives a straightforward dynamical explanation of these nonassociative phase space relations and also a geometric interpretation for the effective target space geometry seen by closed strings in the $R$-flux compactification; it appears to conform with the general expectation that the background fields in non-geometric spaces have non-trivial dependence on the extra dual coordinates in the doubled geometry representation [23].

With the twisted closed string boundary conditions considered in [63, 21], we will then argue that the closed string path integral is equivalent to that of an open string twisted Poisson sigma-model on a disk. This sort of open/closed string duality suggests that considering strings in non-geometric spaces as membranes in geometric spaces appears to rectify the problem of the non-decoupling of gravity, observed in [37] for open strings in $H$-space and in [28] for open strings in $R$-space. As mentioned above, the resulting boundary correlation functions naturally define a quantization of the $R$-flux background, and in fact one can reproduce the entire setting of Kontsevich’s global deformation quantization for twisted Poisson structures. We develop this formalism for arbitrary (not necessarily constant) $R$-flux and describe the resulting nonassociative star product, the corresponding associator, as well as their various derivation properties through the formality maps; for constant $R$-flux we derive explicit closed formulas which resemble the Moyal–Weyl formula. We shall also see that our formalism appears to have the right features to define a proper nonassociative quantization of Nambu–Poisson 3-brackets,
at least for constant trivectors, which are relevant for the quantum geometry of M-branes (see e.g. [25, 74, 75]). We will further demonstrate how the 3-product of fields proposed in [11] (see also [84]) arises in special subsectors of our general formalism.

Using this approach we are also able to clarify the meaning of Seiberg–Witten maps in this setting. The deformation quantization of twisted Poisson structures leads to noncommutative gerbes in the sense of [4]. The nonassociative star product can be “untwisted” to a family of associative star products that are all related by Seiberg–Witten maps. This formulation has a large gauge symmetry given by star commutators with gauge parameters that live on phase space. In the particular setting that we study in this paper, we can also work directly with the nonassociative star product by restricting the class of admissible gauge fields: Seiberg–Witten maps can then be used to describe fluctuations at the boundary of the membrane (or at the endpoints of the open strings) and are closely related to quantized general coordinate transformations. We find two particularly interesting examples. Firstly, a dynamical Seiberg–Witten map from the associative canonical star product on phase space to the nonassociative \( R \)-twisted star product; this map can be computed explicitly to all orders in closed form and may be the first example of its kind. Secondly, Nambu–Poisson maps that can be used to add fluctuations to the (constant) \( R \)-flux background.

Our second approach to the deformation quantization of the \( R \)-flux background introduces the appropriate mathematical language to deal with the target space nonassociativity which should prove helpful in the development of nonassociative deformations of gravity, and ultimately of double field theory. Our twisted Poisson structure is linear and we show that it has the structure of a Lie 2-algebra, which is a categorified version of an ordinary Lie algebra in which the Jacobi identity is weakened to a natural transformation. The pertinent Lie 2-algebra that we use is in fact related to the noncommutative \( Q \)-space background where closed string noncommutativity originally appears, and it can be regarded as a reduction of the structure algebra of the Courant algebroid \( C \) over a point. This Lie 2-algebra can be integrated to a Lie 2-group \( \mathcal{G} \) that is a categorification of the Heisenberg group which defines the double twisted torus \( Y \). By using the nonassociative convolution product induced by horizontal multiplication in \( \mathcal{G} \), we induce a nonassociative star product on the algebra of functions on phase space by embedding it as an algebra object in the category \( \mathcal{G} \), in the spirit of [15]; this mapping can be regarded as a higher version of the Weyl–Wigner quantization map which is familiar from conventional approaches to noncommutative field theory [82]. We demonstrate that this star product is identical to the nonassociative Kontsevich star product; this alternative derivation lends further credibility to our membrane perspective of the closed string dynamics in \( R \)-space. In this manner, we can precisely relate the closed string noncommutative and nonassociative torus bundles that were proposed by [14] to capture the effective geometry of strings in \( R \)-flux compactifications.

Outline

The outline of the remainder of this paper is as follows. In Section 2 we develop the Courant sigma-model which we propose as a description of a sector of the closed string dynamics in \( R \)-space; we reduce the membrane path integral to a string path integral corresponding to a twisted Poisson sigma-model, and argue that it can be quantized as an open string theory with worldsheet a disk. In Section 3 the corresponding boundary correlation functions are considered.
which develop Kontsevich’s global deformation quantization of fields along the twisted Poisson structure. In Section 4 we construct our alternative deformation quantization of the $R$-flux background via convolution in a suitable Lie 2-group and demonstrate that it coincides with the approach based on Kontsevich’s formula. Two appendices at the end of the paper are delegated to some of the more technical aspects of our analysis. In Appendix A we review in some detail all notions regarding the higher algebraic and geometric structures that are employed in the main text. In Appendix B we present some technical details of the explicit computation of Kontsevich’s formula.

2 AKSZ sigma-models in $R$-space

In this section we propose sigma-models for closed strings in $R$-flux backgrounds. The Poisson sigma-model with target space $M$ describes the topological sector of string theory in two-form $B$-field backgrounds. To incorporate non-trivial three-form fluxes, one instead needs a coupling to membranes, which motivates the need for using higher mathematical structures for the twistings that arise in these instances. The effective dynamics in three-form flux backgrounds is thus provided by suitable Courant sigma-models with target space $M$ which describe topological sectors of membrane theories. We will first review how the sigma-model appropriate to $H$-space can be reduced on the boundary of an open membrane to a twisted Poisson sigma-model with target space $M$ [68, 41, 42, 12]. Then we will show that for constant $R$-flux the appropriate Courant sigma-model reduces to a string theory with target space the cotangent bundle of $M$ which coincides with that found in [63, 64]. This geometric interpretation of the $R$-flux background is related to the doubled geometry description of non-geometric flux compactifications [44, 23], and also to the description of the twisted Poisson structure on $T^*M$ as an ordinary Poisson structure on the loop space of $M$ [75].

2.1 Poisson and Courant sigma-models

AKSZ sigma-models whose target spaces comprise a symplectic Lie $n$-algebroid $E$ over a manifold $M$ may be constructed using higher Chern–Simons action functionals [59, 29] (see Appendix A.5 for the relevant details concerning algebroids). A simple case is the cotangent Lie algebroid $E = T^*M$ over a Poisson manifold $M$ with Poisson bivector $\Theta = \frac{1}{2} \Theta^{ij}(x) \partial_i \wedge \partial_j$ where $x = (x^i) \in M$ are local coordinates with $\partial_i := \frac{\partial}{\partial x^i}$; this is a symplectic Lie 1-algebroid with the canonical symplectic structure on the cotangent bundle $T^*M$. Let $\Sigma_2$ be a two-dimensional string worldsheet. The AKSZ construction defines a topological field theory on $C^\infty(T\Sigma_2, T^*M)$ (regarded as a space of Lie algebroid morphisms). A Poisson Lie algebroid-valued differential form on $\Sigma_2$ is given by the smooth embedding $X = (X^i) : \Sigma_2 \rightarrow M$ of the string worldsheet in target space, and an auxiliary one-form field on the worldsheet $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$. The corresponding AKSZ action is

$$S_{\text{AKSZ}}^{(1)} = \int_{\Sigma_2} \left( \xi_i \wedge dX^i + \frac{1}{2} \Theta^{ij}(X) \xi_i \wedge \xi_j \right),$$

which coincides with the action of the Poisson sigma-model [47, 77, 17, 18]. The Poisson sigma-model is the most general two-dimensional topological field theory that can be obtained from the AKSZ construction.
Note that although on-shell the bivector field $\Theta$ is required to have vanishing Schouten–Nijenhuis bracket with itself (in particular so that it defines a differential $d_\Theta$ on the algebra of multivector fields, see Appendix A.3), the perturbative expansion of [17] still makes sense when $\Theta$ is a twisted Poisson bivector and reproduces the Kontsevich formality maps for nonassociative star products [19]; the topological nature of the Poisson sigma-model allows for it to be perturbatively expanded around a non-vacuum solution.

A Courant structure is the first higher analog of a Poisson structure. The corresponding AKSZ sigma-model has target space comprising a symplectic Lie 2-algebroid with a “degree 2 symplectic form”, which is the same thing as a Courant algebroid $E$ over a manifold $M$ [71]. In [71] it is shown that Courant algebroids $E \to M$ are in a canonical bijective correspondence with AKSZ sigma-models on a three-dimensional membrane worldvolume $\Sigma_3$. A Courant algebroid-valued differential form on $\Sigma_3$ is given by the smooth embedding of the membrane worldvolume $X = (X^i) : \Sigma_3 \to M$ in target space, a one-form $\alpha = (\alpha^I) \in \Omega^1(\Sigma_3, X^*E)$, and an auxiliary two-form field on the worldvolume $\phi = (\phi_i) \in \Omega^2(\Sigma_3, X^*T^*M)$. The structure functions of the Lie 2-algebroid are specified by choosing a local basis of sections $\{\psi_I\}$ of $E \to M$ such that the fibre metric $h_{IJ} := \langle \psi_I, \psi_J \rangle$ is constant. We define the anchor matrix $P_I^j$ by $\rho(\psi_I) = P_I^j(x) \partial_j$, and the three-form $T_{IJK}(x) := [\psi_I, \psi_J, \psi_K]_E$. Then the canonical three-dimensional topological field theory associated to the Courant algebroid $E \to M$ is described by the AKSZ action

$$S^{(2)}_{\text{AKSZ}} = \int_{\Sigma_3} \left( \phi_i \wedge dX^i + \frac{1}{2} h_{IJ} \phi^I \wedge d\phi^J - P_I^j(X) \phi_i \wedge \alpha^I + \frac{1}{6} T_{IJK}(X) \alpha^I \wedge \alpha^J \wedge \alpha^K \right), \quad (2.2)$$

which is the action of the Courant sigma-model [48, 41, 72].

### 2.2 $H$-space sigma-models

The Courant algebroid of exclusive interest in geometric flux compactifications of string theory is the standard Courant algebroid $C = TM \oplus T^*M$ twisted by a closed NS–NS three-form flux $H = \frac{1}{6} H_{ijk}(x) \, dx^i \wedge dx^j \wedge dx^k$. The structure maps of $C$ comprise the skew-symmetrization of the $H$-twisted Courant–Dorfman bracket given by [79]

$$[\langle Y_1, \alpha_1 \rangle, \langle Y_2, \alpha_2 \rangle]_H := \langle [Y_1, Y_2]_{TM}, \mathcal{L}_{Y_1} \alpha_2 - \mathcal{L}_{Y_2} \alpha_1 - \frac{1}{2} d(\alpha_2(Y_1) - \alpha_1(Y_2)) + H(Y_1, Y_2, -) \rangle \quad (2.3)$$

for vector fields $Y_1, Y_2 \in C^\infty(M, TM)$ and one-form fields $\alpha_1, \alpha_2 \in \Omega^1(M)$, the metric is the natural dual pairing between $TM$ and $T^*M$,

$$\langle \langle Y_1, \alpha_1 \rangle, \langle Y_2, \alpha_2 \rangle \rangle = \alpha_2(Y_1) + \alpha_1(Y_2), \quad (2.4)$$

and the anchor map is the trivial projection $\rho : C \to TM$ onto the first factor; the map $d : C^\infty(M) \to C^\infty(M, C)$ is given by $df = \frac{1}{2} df$. This is an exact Courant algebroid, i.e. it fits into the short exact sequence

$$0 \longrightarrow T^*M \overset{\rho^*}{\longrightarrow} C \overset{\rho}{\longrightarrow} TM \longrightarrow 0, \quad (2.5)$$

where $\rho^* : T^*M \to C^*$ is the transpose of the anchor map $\rho$ followed by the identification $C^* \cong C$ induced by the pairing on the Courant algebroid. Every exact Courant algebroid on
\(M\) is isomorphic to one of the form \(C = TM \oplus T^*M\) with the structure maps given as above; the isomorphism classes are parametrized by elements \([H] \in H^3(M, \mathbb{R})\) of the degree 3 real cohomology of the target space.

To determine the structure maps of the exact Courant algebroid in a convenient basis, we suppose henceforth that the tangent bundle \(TM \cong M \times \mathbb{R}^d\) is trivial, where \(d = \dim(M)\); this assumption will avoid the appearance of geometric \(f\)-fluxes and other fluxes, as eventually we will want to apply triple T-duality to take us directly into the pure \(R\)-flux background. Then in local coordinates \(x = (x^i)\) for \(M\), a natural frame for \(TM \oplus T^*M\) is given by

\[\eta_i = \partial_i \quad \text{and} \quad \chi^i = dx^i\]  

for \(i = 1, \ldots, d\). Writing \(\eta_i\) for \((\eta_i, 0)\) and \(\chi^i\) for \((0, \chi^i)\) for simplicity, the metric is given by

\[\langle \eta_i, \chi^j \rangle = \delta^j_i.\]  

The corresponding twisted Courant–Dorfman algebra is isomorphic to the algebra with the sole non-trivial brackets

\[[\eta_i, \eta_j]_H = H_{ijk} \chi^k.\]  

The non-vanishing ternary brackets are given by (see Appendix A.5)

\[[\eta_i, \eta_j, \eta_k]_H = H_{ijk}.\]  

As reviewed in [64], the brackets (2.8) and (2.9) for constant \(H\)-flux mimic the phase space quasi-Poisson algebra of a charged particle in the background field of a magnetic monopole [49].

We now write

\[(\alpha^I) = (\alpha^1, \ldots, \alpha^{2d}) := (\alpha^1, \ldots, \alpha^d, \xi_1, \ldots, \xi_d)\]  

where \((\alpha^i) \in \Omega^1(\Sigma_3, X^*TM)\) and \((\xi_i) \in \Omega^1(\Sigma_3, X^*T^*M)\); throughout, upper case indices \(I, J, \cdots \in \{1, \ldots, 2d\}\) run over directions of the doubled geometry, while lower case indices \(i, j, \cdots \in \{1, \ldots, d\}\) run over directions of the original configuration space. Then the action (2.2) becomes

\[S^{(2)}_{WZ} = \int_{\Sigma_3} \left( \phi_i \wedge dX^i + \alpha^i \wedge d\xi_i - \phi_i \wedge \alpha^i + \frac{1}{6} H_{ijk}(X) \alpha^i \wedge \alpha^j \wedge \alpha^k \right).\]  

When \(\Sigma_2 := \partial \Sigma_3 \neq \emptyset\), this is the action of the canonical open topological membrane theory [68]; in this case we can take the consistent Dirichlet boundary conditions \(\alpha^i = \phi_i = 0\) on \(\Sigma_2\) (we could also take \(X^i = \xi_i = 0\) and hybrids thereof; see [41] for a discussion of the resulting modifications). One can also modify the action by adding a boundary term of the form

\[S^0_{WZ} = \int_{\Sigma_2} \left( \xi_i \wedge dX^i + \frac{1}{2} \Theta^{ij}(X) \xi_i \wedge \xi_j + \Gamma^i_{ij}(X) \xi_i \wedge \alpha^j + \frac{1}{2} \Xi^{ij}(X) \alpha^i \wedge \alpha^j \right).\]  

In [41, 42] only the \(\Theta\)-deformation is kept, corresponding to a canonical transformation on the Courant algebroid which gives the boundary/bulk open topological membrane action

\[\tilde{S}^{(2)}_{WZ} = \int_{\Sigma_3} \left( \phi_i \wedge (dX^i - \alpha^i) + \alpha^i \wedge d\xi_i + \frac{1}{6} H_{ijk}(X) \alpha^i \wedge \alpha^j \wedge \alpha^k \right) \frac{1}{2} \Theta^{ij}(X) \xi_i \wedge \xi_j.\]  

(2.13)
In this case the consistent boundary conditions require that \( \Theta = \frac{1}{2} \Theta^{ij}(x) \partial_i \wedge \partial_j \) is an \( H \)-twisted Poisson bivector on \( M \), i.e. its Schouten–Nijenhuis bracket with itself is given by
\[
[\Theta, \Theta]_S = \bigwedge^3 \Theta^2(H), \tag{2.14}
\]
and the Jacobi identity for the corresponding bracket is violated (see Appendix A.3); here \( \bigwedge^3 \Theta^2(H) \) denotes the natural way to turn the three-form \( H \) into a three-vector by using \( \Theta \) to “raise the indices”. After integrating out the two-form fields \( H \) which is the action of the satisfying a suitable integrability condition. The bracket on \( \text{Poisson bivector on } M \) \( R \)-space in \( R^2 \).

The relevance of the topological twisted Poisson sigma-model (2.15) in the effective theory of strings in \( R \)-flux backgrounds was noted in [28, 33]. Here we shall start with the general Courant sigma-model (2.2) and the argument of [34] that the appropriate theory in \( R \)-space is described by a non-topological membrane sigma-model, not a string theory; the membrane action in this case is not generally equivalent to the action of a string theory on the boundary of a membrane. This would also corroborate the observation of [28] that the \( R \)-space geometry does not seem to exist as a low-energy effective description of string theory, in the sense that open strings in \( R \)-space cannot be consistently decoupled from gravity; the absence of a topological limit and the non-decoupling of gravity for open strings in \( H \)-space was also observed in [37]. In a sense to be elucidated below, the membrane theory geometrizes the non-geometric \( R \)-flux background, in a way reminescent of the manner in which M-theory geometrizes string dualities. In [2, 3] potential target space effective actions for non-geometric \( Q \)-flux and \( R \)-flux backgrounds were constructed in the context of double field theory, which provides a geometrical role to the non-geometric fluxes related to gauge transformations (diffeomorphisms); it would be interesting to derive these effective descriptions from membrane sigma-models of the sort described here.

Although the \( R \)-space is not even locally geometric as a Riemannian manifold [80], in this paper we work only at tree-level in the low-energy effective field theory on target space where we can treat the \( R \)-space locally as the original \( d \)-dimensional manifold \( M \).

The Courant algebroid pertinent to the \( R \)-flux background is again the standard Courant algebroid \( C = TM \oplus T^*M \), but now twisted by a trivector flux \( R = \frac{1}{6} R^{ijk}(x) \partial_i \wedge \partial_j \wedge \partial_k \) satisfying a suitable integrability condition. The bracket on \( C \) is the skew-symmetrization of Roytenberg’s \( R \)-twisting of the Courant–Dorfman bracket given by [70, 34, 9]
\[
[(Y_1, \alpha_1), (Y_2, \alpha_2)]_R := (\{Y_1, Y_2\}_{TM} + R(\alpha_1, \alpha_2, -), \tag{2.16}
\mathcal{L}_{Y_1} \alpha_2 - \mathcal{L}_{Y_2} \alpha_1 - \frac{1}{2} d(\alpha_2(Y_1) - \alpha_1(Y_2)) \),
\]
while the remaining structure maps are identical to those of Section 2.2.

Writing the generators of the natural frame for \( TM \oplus T^*M \) as \( \varphi_i \) and \( \chi^i \) as before, the corresponding Roytenberg algebra is isomorphic to the algebra with the non-trivial brackets
\[
[\chi^i, \chi^j]_R = R^{ijk} \varphi_k \tag{2.17}
\]
and the metric (2.7). When $R$ is a constant flux this is the $d$-dimensional Heisenberg algebra; this mimicks the commutation relations for closed string fields which are obtained by applying three T-duality transformations to the $H$-space $M = T^3$ [63, 64], with the remaining non-trivial structure map

$$[\chi^i, \chi^j, \chi^k]_R = R^{ijk}.$$  

(2.18)

Below we will recover the commutation relations of [63, 64] dynamically from an associated twisted Poisson sigma-model.

With the same splitting (2.10), the action (2.2) in the pure $R$-flux background becomes

$$S_R^{(2)} = \int_{\Sigma_3} \left( \phi_i \wedge (dX^i - \alpha^i) + \alpha^i \wedge d\xi_i + \frac{1}{6} R^{ijk}(X) \xi_i \wedge \xi_j \wedge \xi_k \right) + \frac{1}{2} \int_{\Sigma_2} g^{ij}(X) \xi_i \wedge \star \xi_j ,$$

(2.19)

where $g^{-1} = \frac{1}{2} g^{ij}(x) \partial_i \otimes \partial_j$ is the inverse of a chosen metric tensor on target space $M$, and $\star$ is the Hodge duality operator with respect to a chosen metric on the worldsheet $\Sigma_2 = \partial \Sigma_3$; we have again chosen Dirichlet boundary conditions $\alpha^i = \phi_i = 0$ on $\Sigma_2$. As in [34], we have added a metric-dependent term on the boundary $\Sigma_2$ of the membrane, which breaks the topological symmetry of the Courant sigma-model, in order to ensure that the choice $R^{ijk} \neq 0$ is consistent with the equations of motion and also with the gauge symmetries of the field theory [41]. Note that only $g^{-1}$ appears, not the metric $g$ itself; it will play the role of a metric on momentum space later on. Integrating out the two-form fields $\phi_i$ leads to the action

$$S_R^{(2)} = \oint_{\Sigma_2} \xi_i \wedge dX^i + \int_{\Sigma_3} \frac{1}{6} R^{ijk}(X) \xi_i \wedge \xi_j \wedge \xi_k + \oint_{\Sigma_2} \frac{1}{2} g^{ij}(X) \xi_i \wedge \star \xi_j .$$

(2.20)

We will now specialize to the case where both the $R$-flux and the target space metric are constant; this is the situation relevant to the considerations of [8, 63, 11, 64]. On the boundary of the membrane, the equations of motion for $X^i$ then force $\xi_i = dP_i$ to be an exact form (modulo harmonic forms on $\Sigma_2$), where $P_i \in C^\infty(\Sigma_3, X^* T^* M)$ is a section of the cotangent bundle of $M$ restricted to $\Sigma_3$; this solution is also consistent with the equations of motion in the bulk and henceforth we restrict the configuration space for the path integral to this domain of fields. Then the action (2.20) reduces to a pure boundary action of the form

$$S_R^{(2)} = \oint_{\Sigma_2} \left( dP_i \wedge dX^i + \frac{1}{2} R^{ijk} P_i dP_j \wedge dP_k \right) + \oint_{\Sigma_2} \frac{1}{2} g^{ij} dP_i \wedge \star dP_j .$$

(2.21)

This action can be recast in the form

$$S_R^{(2)} = \oint_{\Sigma_2} -\frac{1}{2} \Theta^{-1}_{IJ}(X) dX^I \wedge dX^J + \oint_{\Sigma_2} \frac{1}{2} g_{IJ} dX^I \wedge \star dX^J ,$$

(2.22)

where the fields

$$X = (X^I) = (X^1, \ldots, X^{2d}) := (X^1, \ldots, X^d, P_1, \ldots, P_d)$$

(2.23)

embed the string worldsheet $\Sigma_2$ in the cotangent bundle of $M$, i.e. the effective target space is now phase space, and we have introduced the block matrix on $T^* M$ given by

$$\Theta = (\Theta^{IJ}) = \begin{pmatrix} R^{ijk} & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix}$$

(2.24)
with local phase space coordinates

\[ x = (x^I) = (x^1, \ldots, x^{2d}) := (x^1, \ldots, x^d, p_1, \ldots, p_d) . \]  

(2.25)

The “closed string metric”

\[ (g_{IJ}) = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} \]  

(2.26)

acts on momentum space but not on configuration space. The matrix \( \Theta \) is always invertible and its inverse is given by

\[ \Theta^{-1} = (\Theta^{-1}_{IJ}) = \begin{pmatrix} 0 & -\delta_i^j \\ \delta^i_j & R^{ijk} p_k \end{pmatrix} . \]  

(2.27)

We can linearize the action (2.22) in the embedding fields \( X = (X^I) : \Sigma_2 \to T^* M \) by introducing auxiliary fields \( \eta_I \in \Omega^1(\Sigma_2, X^* T^* (T^* M)) \) to write

\[ S_R^{(2)} = \oint_{\Sigma_2} \left( \eta_I \wedge dX^I + \frac{1}{2} \Theta^{IJ}(X) \eta_I \wedge \eta_J \right) + \oint_{\Sigma_2} \frac{1}{2} G^{IJ} \eta_I \wedge * \eta_J , \]  

(2.28)

where the “open string metric”

\[ (G^{IJ}) = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix} \]  

(2.29)

is related to (2.26) by the usual closed-open string relations [78] that involve \( \Theta \) and the “B-field” \( \Theta^{-1} \) (note that \( (g_{IJ}) \) is not the inverse of \( (G^{IJ}) \)). This is the action of the non-topological generalized Poisson sigma-model for the embedding of the string worldsheet \( \Sigma_2 \) into the cotangent bundle \( T^* M \) of the manifold \( M \) with bivector field

\[ \Theta = \frac{1}{2} \Theta^{IJ}(x) \partial_I \wedge \partial_J , \]  

(2.30)

whose coefficient matrix \( \Theta^{IJ} \) is given by (2.24) and \( \partial_I := \frac{\partial}{\partial x^I} \); below we will write phase space derivatives as \( \partial_i := \frac{\partial}{\partial x^i} \) and \( \tilde{\partial}^i := \frac{\partial}{\partial p_i} \). For completeness, we express the action (2.28) more explicitly in phase space component form by decomposing the one-form fields

\[ (\eta_I) = (\eta_1, \ldots, \eta_{2d}) := (\eta_1, \ldots, \eta_d, p_1, \ldots, p_d) \]  

(2.31)

and writing

\[ S_R^{(2)} = \oint_{\Sigma_2} \left( \eta_i \wedge dX^i + p_i \wedge dP_i + \frac{1}{2} R^{ijk} P_k \eta_i \wedge \eta_j + \eta_i \wedge \pi^i \right) + \oint_{\Sigma_2} \frac{1}{2} g^{ij} \eta_i \wedge * \eta_j . \]  

(2.32)

The first order action (2.32) is equivalent to the string sigma-model (2.21). Note that only the momentum space components \( P_i \) of the strings have propagating degrees of freedom in \( T^* M \); in this sense the generalized Poisson sigma-model is still topological in the original configuration space \( M \). Moreover, the bivector field \( \Theta \) defines a twisted Poisson structure on the cotangent bundle, with twisting provided by a (trivial) non-flat \( U(1) \)-gerbe in momentum space: Computing its Schouten–Nijenhuis bracket with itself yields

\[ [\Theta, \Theta]_S = \wedge^3 \Theta^i(H) , \]  

(2.33)
where
\[ H = \frac{1}{6} R^{ijk} dp_i \wedge dp_j \wedge dp_k \] (2.34)
is a closed three-form \( H \)-flux on the cotangent bundle \( T^*M \); a 2-connection on this gerbe is given by the \( B \)-field
\[ B = \frac{1}{6} R^{ijk} p_k dp_i \wedge dp_j \] (2.35)
with \( H = dB \), which is gauge equivalent to the topological part of the string sigma-model (2.21). In this way our membrane sigma-model (2.19) provides a geometric interpretation of the \( R \)-flux background; this gerbe description will be exploited in Section 3.4.

The antisymmetric brackets at linear order
\[ \{x^i, x^j\}_\Theta = \Theta^{IJ} (x) \] (2.36)
are given explicitly by
\[ \{x^i, x^j\}_\Theta = R^{ijk} p_k , \quad \{x^i, p_j\}_\Theta = \delta^i_j \quad \text{and} \quad \{p_i, p_j\}_\Theta = 0 . \] (2.37)
The corresponding Jacobiator is
\[ \{x^i, x^j, x^K\}_\Theta := [\Theta, \Theta]_S (x^I, x^J, x^K) = \Pi^{IJK} , \] (2.38)
where
\[ \Pi^{IJK} = \frac{1}{3} (\Theta^{KL} \partial_L \Theta^{IJ} + \Theta^{IL} \partial_L \Theta^{JK} + \Theta^{JL} \partial_L \Theta^{KI}) . \] (2.39)
The only non-vanishing components of this trivector field are
\[ \{x^i, x^j, x^K\}_\Theta = R^{ijk} . \] (2.40)
The expressions (2.37) and (2.40) are precisely the nonassociative phase space commutation relations for quantized closed string coordinates which were derived in [63, 64].

Although we are mostly interested in the case of constant \( R^{ijk} \), we can speculate on how to extend our discussion to non-constant \( R \)-flux. By local orthogonal transformations the 3-vector \( R \) can be brought into canonical form wherein its only non-vanishing components are \( R^{ijk} (x) = |R(x)|^{1/3} \varepsilon^{ijk} \) for \( i, j, k = 1, 2, 3 \), where \( \varepsilon^{ijk} \) is the totally antisymmetric tensor and \( |R(x)| \) is the determinant of the matrix \( R^{ij}(x) \), \( J = (jk) \). By a suitable coordinate transformation, \( R^{ijk} \) can thus be taken to be the constant tensor \( \varepsilon^{ijk} \). Depending on how the remaining structure functions of the Courant algebroid \( C \to M \) transform, this may then yield a reduction of the membrane sigma-model (2.19) on \( \Sigma_3 \) to a string sigma-model on the boundary \( \Sigma_2 \) as before.

In any case, the sigma-model (2.32) and its associated brackets also make sense when \( R^{ijk} \) is a general function of \( x \in M \), i.e. a generic trivector field on configuration space. Quantizing these brackets thus provides a means for quantizing generic Nambu–Poisson structures on \( M \) with 3-bracket determined by the trivector \( R \). It provides a geometric way of incorporating nonassociativity into quantized Nambu–Poisson manifolds, extending the (limited) techniques of [25, 75] which relied on associative algebras. Moreover, quantization of the membrane sigma-model provides a dynamical realization of the nonassociative geometry which is an alternative to the reduced target space membrane models of [26]. This quantization is explored in detail below.
2.4 Boundary conditions and correlation functions

It is natural to expect that the path integral for the $R$-twisted Courant sigma-model provides a universal quantization formula for closed strings in $R$-space, regarded as the boundaries of the membranes. For the $H$-space open membrane sigma-model of Section 2.2, it is argued in [41, 42] that the path integral defines a formal quantization for the corresponding twisted Poisson structure, and an explicit prescription is given for quantizing appropriate current algebra and $L_\infty$ brackets of the boundary strings from correlation functions of the open topological membrane theory; for more general deformations of the exact Courant algebroid $C = TM \oplus T^*M$, the path integral is argued to provide a universal quantization formula for generic quasi-Lie bialgebras. The formulas for the worldsheet Poisson algebra nicely resemble those which arise from transgressing the higher bracket structures to loop space [75]. Unfortunately, the complicated nature of the BV formalism which is necessary to quantize the open topological membrane theory obstructs a complete quantization. In particular, the Courant sigma-model with $R$-flux involves very complicated 2-algebroid gauge symmetries; for the general Courant sigma-model the full gauge-fixed action can be found in [72], and it involves both ghost fields and ghosts-for-ghosts.

Here we would like to develop a quantization framework that is based on the induced twisted Poisson sigma-model (2.32), which involves only Lie algebroid gauge symmetries, and whose quantization on the disk is described in [17, 18]. For this, we will interpret the membrane theory as an effective theory of open strings with suitable boundary conditions imposed on the string embedding fields. In [41] (see also [12]) it is proposed that the boundary $\Sigma_2 = \partial \Sigma_3$ can be taken to be an open string worldsheet in the open topological membrane theory by regarding the membrane worldvolume $\Sigma_3$ as a manifold with corners (see e.g. [50]), and allowing for different boundary conditions on the various components of the boundary. In the following we will take another approach that is directly related to the way in which the twisted Poisson structure originates in closed string theory on the $R$-flux background [63, 21] (see [64] for a review). We shall argue that the corners of the membrane worldvolume can be mimicked via branch cuts on a closed surface which give the multivalued string maps responsible for the target space noncommutativity. In this way the membrane serves to provide a sort of open/closed string duality; the analogy between closed strings in non-geometric flux backgrounds and open strings was also pointed out in [63].

The setting of [63, 21] is that of closed strings on the $Q$-space duality frame obtained by applying two T-duality transformations to the three-torus $M = \mathbb{T}^3$ with constant NS–NS three-form flux $H = h \, dx^1 \wedge dx^2 \wedge dx^3$. Locally, this space is a fibration of a two-torus $\mathbb{T}^2$ over a circle $S^1$; globally it is not well-defined as a Riemannian manifold and is the simplest example of a $T$-fold [44]. A representative class of twisted torus fibrations are provided by elliptic T-folds where the monodromies act on the fibre coordinates as rotations. The closed string worldsheet is the cylinder $C = \mathbb{R} \times S^1$ with coordinates $(\sigma^0, \sigma^1)$. The embedding field corresponding to the base direction is denoted $X^3$, while for the fibre directions we use complex fields denoted $Z, \bar{Z} = \frac{1}{\sqrt{2}}(X^1 \pm i X^2)$. As an extended closed string wraps $\tilde{p}^3$ times around the base of the fibration, the fibre directions need only close up to a monodromy corresponding to an $SL(2,\mathbb{Z})$ automorphism of the $\mathbb{T}^2$-fibre. One thus arrives at the twisted boundary conditions

$$Z(\sigma^0, \sigma^1 + 2\pi) = e^{2\pi i \theta} Z(\sigma^0, \sigma^1) \quad \text{and} \quad X^3(\sigma^0, \sigma^1 + 2\pi) = X^3(\sigma^0, \sigma^1) + 2\pi \tilde{p}^3 \quad (2.41)$$
where $\theta = -\hbar \tilde{p}^3$; more precisely, one should impose asymmetric boundary conditions for the left- and right-moving fields in the fibre directions. To linear order in the flux, one can solve the equations of motion of the closed string worldsheet sigma-model in the usual way via oscillator mode expansions for the fibre coordinate fields subject to the twisted boundary conditions (2.41). Standard canonical quantization then shows that the fibre directions acquire a noncommutative deformation determined by the $H$-flux and the winding number (or T-dual Kaluza–Klein momentum) $\tilde{p}^3$ in the $S^1$-direction, in exactly the same way in which open string boundaries are deformed in the presence of a $B$-field. Written in terms of a real parametrization, we may express this closed string noncommutativity generally in the $Q$-flux background via the Poisson brackets

$$\{x^i, x^j\}_Q = Q^{ij}_k \tilde{p}^k \quad \text{and} \quad \{x^i, \tilde{p}^j\}_Q = 0 = \{\tilde{p}^i, \tilde{p}^j\}_Q ,$$

with constant flux $Q^{ij}_k = -2\pi \hbar \varepsilon^{ijk} k$. These brackets define a bona fide Poisson structure, since they are just the relations of a Heisenberg algebra, as in the defining 2-brackets of the corresponding Courant algebroid. A T-duality transformation to the $R$-flux background sends $Q^{ij}_k \mapsto R^{ijk}$ and $\tilde{p}^k \mapsto p_k$, and maps the Poisson brackets (2.42) to the twisted Poisson structure (2.37). This change of duality frame will be useful for some of our later considerations.

This description of the $Q$-space is consistent with its description from [14] as a fibration of stabilized noncommutative two-tori $T^2_\theta$, with Poisson bivector $\theta$ varying over the base $S^1$; low-energy effective open string constructions are given in [28, 32], where the noncommutative fibration is regarded as the algebra of open string field theory in this background, and also in [13] within a $C^*$-algebra framework which describes the T-fold as a topological approximation to a $T^2$-equivariant gerbe with 2-connection on $T^3$. The interplay between open and closed string interpretations of the noncommutative and nonassociative flux backgrounds was noted in [14, 28]; a step towards understanding the pertinent picture was recently carried out in the context of matrix theory compactifications on twisted tori in [20], which constructs solutions with noncommutative and nonassociative cotangent bundles. To explicitly relate the closed and open string pictures, we use the fact that the boundary conditions (2.41) define a twisted sector of an orbifold conformal field theory on the quotient of $M = T^3$ by the free action of a discrete abelian monodromy group; they describe closed strings on the orbifold, which can be regarded as open strings on the covering space $M$. When computing conformal field theory correlation functions, the monodromy can be implemented by inserting a suitable twist field at a point $\sigma^{i'} \in S^1$ which creates a branch cut along the temporal direction $\mathbb{R}$ for the multivalued closed string fields. We now extend the worldsheet $C = \mathbb{R} \times S^1$ to the membrane worldvolume $\Sigma_3 = \mathbb{R} \times (S^1 \times \mathbb{R})$ with coordinates $(\sigma^0, \sigma^1, \sigma^2)$ such that the branch point at $\sigma^{i'} \in S^1$ is blown up to a branch cut $I = \{\sigma^{i'}\} \times \mathbb{R} \subset S^1 \times \mathbb{R}$ extended along the $\sigma^2$-direction, i.e. the branch cut on the closed string worldsheet is blown up to a “branch surface” on a closed membrane worldvolume. The membrane fields are also taken to be multivalued and non-differentiable across the branch cut $I$; hence Stokes’ theorem on $\Sigma_3$ receives contributions from the multivalued fields across the cut whenever integration by parts is used to reduce worldvolume integrals, as we did in Section 2.3. This effectively reduces the membrane to an “open string” with worldsheet $\Sigma_2 := \partial \Sigma_3 = \mathbb{R} \times I$ and coordinates $(\sigma^0, \sigma^2)$; classically, the mapping $\Sigma_3 \rightarrow \Sigma_2$ is a simple application of Stokes’ theorem on the equations of motion $\xi_i = dP_i$. In this way the branch cut $I$ plays the role of a “corner” separating $\Sigma_3$ into regions [50]. The more general orbifolds of [21] can be treated in an analogous way.
We can depict these two reductions from the membrane theory to the closed and open string theories via the following schematic diagram:

![Diagram showing the reduction from membrane theory to closed and open string theories.]

We do not display the temporal direction as it plays no role. The mapping $\Sigma_3 \to C$ is obtained by restriction of the domain of the membrane path integral to fields which are independent of $\sigma^2$; this gives a dimensional reduction of the membrane fields to closed string fields which is reminiscent of the Kaluza–Klein reduction of M-theory to Type IIA string theory. The mapping $\Sigma_3 \to \Sigma_2$ is a restriction of field variables in the membrane path integral to the cut $I$ of the spatial membrane cylinder; via reparametrization of the membrane worldvolume, it also defines a map to the disk $\Sigma_2$ viewed as the complex upper half-plane with boundary the real line $\mathbb{R}$, where the endpoints of the cut at $\pm \infty$ are mapped to finite values. These two restrictions of the field domain in the membrane path integral define the open/closed string duality that we were after; in a certain sense it represents a sort of transmutation between D-branes and fluxes. It is somewhat in line with the recent analysis of [24] which demonstrates how non-geometric doubled space coordinates arise as solutions to Neumann boundary conditions in open string theory on flux backgrounds. Note that in order to ensure independence of the specific location of the branch cut $I$, it is important to assume that the $R$-flux is constant; however, this restriction is no longer needed after we take the 2+1-dimensional Courant sigma-model as the fundamental model for closed strings in the $R$-flux background.

Considering that the endpoints are at $\pm \infty$, it is natural to choose the boundary conditions for the open string on the cut $I$ to coincide with those of [17]. In this sense, the twisted boundary conditions (2.41) on $Q$-space can be made compatible with the Cattaneo–Felder boundary conditions for the open twisted Poisson sigma-model. In the following we will take the topological limit of (2.32) where $g \ll R$; this essentially decouples the open string modes from the closed string modes. Then the propagator of the topological sigma model is given by

$$\langle X^I(w) \eta_J(z) \rangle = \frac{ih}{2\pi} \delta^I_J \frac{dz}{2\pi} \phi^h(z, w), \tag{2.43}$$

where $h$ is a formal expansion parameter, the harmonic angle function

$$\phi^h(z, w) := \frac{1}{2i} \log \frac{(z - w)(z - \overline{w})}{(\overline{z} - w)(\overline{z} - \overline{w})}, \tag{2.44}$$

for $z, w \in \mathbb{C}$ is the Green’s function for the Laplacian on the disk with Neumann boundary conditions, and $dz := dz \frac{\partial}{\partial z} + d\overline{z} \frac{\partial}{\partial \overline{z}}$. In this case, the Feynman diagram expansion of suitable observables in the sigma-model reproduces Kontsevich’s graphical expansion for global deformation quantization of our twisted Poisson structure [58, 17, 18, 19], which we will take as our
proposal for the quantization of the R-flux background. In Section 3 we shall compute the following schematic functional integrals, whose precise meaning will be explained later on and whose precise definitions can be found in [17, 18, 19]. For \( x \in T^* M \), functions \( f_i \in C^\infty(T^* M) \), and a collection of \( n \geq 1 \) multivector fields \( \mathcal{X}_r = \frac{1}{h^r} \mathcal{X}_r^{I_1 \cdots I_r}(x) \partial_{I_1} \cdots \partial_{I_r} \in C^\infty(T^* M, \bigwedge^{k_r} T(T^* M)) \) of degree \( k_r \), define

\[
U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)(f_1, \ldots, f_m)(x) = \int e^{\frac{i}{\hbar} S^{(2)}_{\mathcal{X}}(f_1)} S_{\mathcal{X}_1} \cdots S_{\mathcal{X}_n} \mathcal{O}_x(f_1, \ldots, f_m),
\]

where \( m = 2 - 2n + \sum_{r} k_r \), \( S_{\mathcal{X}_r} = \int_{\Sigma_{2}} \frac{1}{k_r} \mathcal{X}_r^{I_1 \cdots I_r}(X) \partial_{I_1} \cdots \partial_{I_r} \) and \( \mathcal{O}_x(f_1, \ldots, f_m) \) are the boundary observables

\[
\mathcal{O}_x(f_1, \ldots, f_m) = \int_{X(\infty) = x} \left[ f_1(X(q_1)) \cdots f_m(X(q_m)) \right]^{(m-2)}
\]

with \( 1 = q_1 > q_2 > \cdots > q_m = 0 \) and \( \infty \) distinct points on the boundary of the disk \( \partial \Sigma_{2} \); the path integrals are weighted with the full gauge-fixed action and the integrations taken over all fields including ghosts. In particular, for functions \( f, g \in C^\infty(T^* M) \) one may define a star product by the functional integral

\[
(f \ast g)(x) = \int_{X(\infty) = x} f(X(1)) g(X(0)) \ e^{\frac{i}{\hbar} S^{(2)}_{\mathcal{X}}},
\]

whose properties will be thoroughly investigated in what follows.

### 2.5 Twisted and Higher Poisson Structures

We close this section with some general remarks about the twisted Poisson structures we have derived, which will serve to help understand some of the higher structures that will arise in our discussions about quantization. Consider the algebra \( \mathcal{V}^2 = C^\infty(T^* M, \bigwedge^2 T(T^* M)) \) of multivector fields on the cotangent bundle of the target space \( M \). Let \( H = \frac{1}{\hbar} H_{IJK}(x) \ dx^{I} \wedge dx^{J} \wedge dx^{K} \) be the closed three-form (2.34) on \( T^* M \); it extends by the Leibniz rule to give a ternary bracket \([-,-,-]_H\) on \( \mathcal{V}^2 \) of degree 1. Together with the Schouten–Nijenhuis bracket \([-,-,-]_S\), it defines an \( L_\infty \)-structure on \( \mathcal{V}^2 \) with zero differential, generalizing the canonical differential graded Lie algebra structure in the case of vanishing R-flux (see Appendix A for the relevant definitions and background material). On the subspace \( C^\infty(T^* M) \) of smooth functions on \( T^* M \), the \( H \)-twisted Poisson structure (2.30) naturally defines a 2-term \( L_\infty \)-algebra \((V_1 \overset{d}{\rightarrow} V_0)\) where \( V_1 = C^\infty(T^* M) \), \( V_0 \) is the space of vector fields \( \mathcal{X} \in C^\infty(T^* M, T(T^* M)) \) which preserve \( \Theta \) in the sense that \( \mathcal{L}_{\mathcal{X}} \Theta = 0 \) where \( \mathcal{L}_{\mathcal{X}} \) is the Lie derivative along \( \mathcal{X} \), and \( \partial = d_\Theta = -[-,-,\Theta]_S \) is the Lichnerowicz differential which sends a function \( f \in C^\infty(T^* M) \) to its Hamiltonian vector field \( \mathcal{X}_f = \Theta(df, -) \) [70]. The derived bracket (A.13) on \( V_1 \) is just the quasi-Poisson bracket on \( C^\infty(T^* M) \) determined by \( \Theta \) as

\[
\{ f, g \}_\Theta := [df, dg]_S = \Theta(df, dg).
\]

The associated Jacobiator (A.14) can be written as

\[
\{ f, g, h \}_\Theta = H(\mathcal{X}_f, \mathcal{X}_g, \mathcal{X}_h).
\]

Note that here the differential \( d \) is not nilpotent, and the right-hand side of (2.49) can be expressed in terms of \( d^2 \neq 0 \); this is reminiscent of a covariant derivative that does not square to zero when the curvature is non-zero.
The corresponding commutation relations in the associated semistrict Lie 2-algebra $\mathcal{V}$ are (see Appendix A.1)
\[
[X, Y]_{\mathcal{V}} = [X, Y]_{T(T^*M)},
\]
\[
([X, f], [Y, g])_{\mathcal{V}} = ([X, Y]_{T(T^*M)}, X(g) - Y(f) + \{f, g\}_{\Theta}),
\]
while the Jacobiator is
\[
[X, Y]_{\mathcal{V}} = ([X, Y]_{T(T^*M)}, X(g) - Y(f) + \{f, g\}_{\Theta}),
\]
for $X, Y, Z \in C^\infty(T^*M, T(T^*M))$ and $f, g \in C^\infty(T^*M)$. At linear order, denoting the generators $(\partial_I, 0)$ and $(0, x^I)$ by $p_I$ and $x^I$ for simplicity, we have
\[
[p_I, p_J]_{\mathcal{V}} = 0,
\]
\[
[p_I, x^J]_{\mathcal{V}} = \delta_I^J
\]
and
\[
[x^I, x^J]_{\mathcal{V}} = \Theta^{IJ},
\]
with all other brackets vanishing at linear order in $x^i$ and $\partial_i$. These higher Poisson brackets define a 2-term $L_\infty$-algebra structure on $V^\sharp$.

3 Formal deformation quantization

As discussed in Section 2.4, a suitable perturbation expansion of the membrane/string sigma-model of Section 2 motivates an approach to the quantum geometry of the $R$-flux background based on deformation quantization. In [58], Kontsevich constructs a deformation quantization of an arbitrary Poisson structure, based on a graphical calculus which is reproduced by the Feynman diagram expansion of the open Poisson sigma-model on a disk [17]. In this section we shall follow this prescription to derive a nonassociative star product deformation of the usual pointwise product of functions on $T^*M$ along the direction of a generic twisted Poisson bivector $\Theta$, and describe its derivation properties. We then restrict to the case of constant $R$-flux where we derive an explicit closed formula for the star product and its associator, giving a quantization of the 2-brackets (2.37) and the 3-brackets (2.40) respectively. We apply this formalism to derive Seiberg–Witten maps relating nonassociative and associative deformations, and also add fluctuations to the $R$-flux background. We further explain how the 3-product proposed in [11] fits into our formalism.
3.1 Star product

Kontsevich’s formalism relies on the construction of the formality map. The formality map is a sequence of $L_\infty$-morphisms $U_n$, $n \in \mathbb{Z}_{\geq 0}$ that map tensor products of $n$ multivector fields to $m$-differential operators on the manifold $T^*M$; it defines an $L_\infty$-quasi-isomorphism between the differential graded Lie algebra of multivector fields equipped with zero differential and the Schouten–Nijenhuis bracket (see Appendix A.3), and the differential graded Lie algebra of multidifferential operators equipped with the Hochschild differential and the Gerstenhaber bracket (see Appendix A.2). Consider a collection of multivectors $\mathcal{X}_i$ of degree $k_i$ for $i = 1, \ldots, n$. Then $U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ is a multidifferential operator whose degree $m$ is determined by the relation

$$m = 2 - 2n + \sum_{i=1}^{n} k_i.$$  \hfill (3.1)

In particular, $U_0$ yields the usual pointwise product of functions while $U_1$ is the Hochschild–Kostant–Rosenberg map which takes a $k$-vector field to a $k$-differential operator defined by

$$U_1(\mathcal{X}^{I_1 \cdots I_k} \partial_{I_1} \wedge \cdots \wedge \partial_{I_k})(f_1, \ldots, f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \mathcal{X}^{I_{\sigma(1)} \cdots I_{\sigma(k)}} \partial_{I_{\sigma(1)}} f_1 \cdots \partial_{I_{\sigma(k)}} f_k$$  \hfill (3.2)

for $f_i \in C^\infty(T^*M)$. When the multivector fields $\mathcal{X}_i$ are all set equal to the bivector $\Theta$, the star product of functions $f, g \in C^\infty(T^*M)$ is given by the formal power series

$$f \star g := \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_n(\Theta, \ldots, \Theta)(f, g) \equiv \Phi(\Theta)(f, g),$$  \hfill (3.3)

where $\hbar$ is a formal deformation parameter and $U_n(\Theta, \ldots, \Theta)$ is a bidifferential operator by (3.1).

Kontsevich introduced a convenient diagrammatic representation on the upper hyperbolic half-plane $\mathbb{H}$ that provides all possible (admissible) differential operators to each order of the expansion (3.3), and thus determines the formality map $U_n$. Kontsevich diagrams encode the rules for contracting indices and positioning partial derivatives. Each diagram $\Gamma$ consists of:

1. Edges $e$ that are geodesics in $\mathbb{H}$ and represent partial derivatives;
2. A set $q_1, \ldots, q_m \in \mathbb{R}$ of grounded vertices that represent functions; and
3. A set $p_1, \ldots, p_n \in \mathbb{H} \setminus \mathbb{R}$ of aerial vertices that represent the $k_i$-vector fields $\mathcal{X}_i$, and thus $k_i$ edges may emanate from them.

Here the real line $\mathbb{R}$ is the boundary of $\mathbb{H}$. An edge emanating from a given point $p_i$ is labelled as $e_i^{k_i}$. Edges that start from a vertex $v$ can land on any other vertex apart from $v$, while the condition $2n + m - 2 \geq 0$ must be satisfied. The multidifferential operator

$$U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) := \sum_{\Gamma \in G_n} w_\Gamma D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n)$$  \hfill (3.4)

is calculated by summing over operators $D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ in the class $G_n$ of all $n$-th order admissible diagrams $\Gamma$, each contributing with weight $w_\Gamma$ given by the integral [56]

$$w_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \int_{\mathbb{H}_n} \prod_{i=1}^{n} \left( d\phi_{e_i}^h \wedge \cdots \wedge d\phi_{e_i}^h \right),$$  \hfill (3.5)
where $\mathbb{H}_n$ is the space of pairwise distinct points $p_i \in \mathbb{H}$ and the role of the harmonic angles $\phi_{\epsilon_{i_k}}^h$ is explained in Appendix B.

In this setting, the diagrams for the bivector (quasi-Poisson bracket) $\Theta(f, g) = \frac{1}{2} \Theta^{IJ} \partial_I f \partial_J g$ and the trivector (Jacobiator) $\Pi(f, g, h) = \frac{1}{3} \Pi^{IJK} \partial_I f \partial_J g \partial_K h$ contributions are

\[ \begin{array}{c}
\Theta \\
\partial_I \\ f \\
\partial_J \\ g \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\Pi \\
\partial_I \\ f \\
\partial_J \\ g \\
\partial_K \\ h \\
\end{array} \]

which we will call the wedge and triple wedge respectively. The geodesics here have been drawn as straight lines for the sake of clarity. Computing (3.3) then provides the nonassociative star product deformation along the quasi-Poisson structure $\Theta$.

Kontsevich’s construction allows for one or more multivectors to be inserted in $U_n(\Theta, \ldots, \Theta)$. In our case, inserting the trivector $\Pi = [\Theta, \Theta]$ that we acquired from the Schouten–Nijenhuis bracket is of particular interest since it encodes the nonassociativity of our star product. Then on functions $f, g, h \in C^\infty(T^*M)$ the series (3.3) is replaced with

\[ [f, g, h]_* := \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_{n+1}(\Pi, \Theta, \ldots, \Theta)(f, g, h) \equiv \Phi(\Pi)(f, g, h) \tag{3.6} \]

which, as we show in Section 3.2, is the associator for the star product (3.3). The condition (3.1) now implies that $U_{n+1}(\Pi, \Theta, \ldots, \Theta)$ is a tridifferential operator. The map $U_{n+1}$ is calculated as in (3.4); this time though the integrations for the diagrams that give the associated weights (3.5) are much more involved since the edges of the triple wedge can land on any other wedge. Restricting to constant $R$-flux $R^{ijk}$ cures this problem, making the derivation of an explicit expression possible; this will be analysed in Section 3.3.

### 3.2 Derivation properties and associator

In order to define $L_\infty$-morphisms, the maps $U_n$ must satisfy for $n \geq 1$ the formality conditions [58, 65, 54, 83]

\[
d_{\mu_2} U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) + \frac{1}{2} \sum_{\mathcal{I} \cup \mathcal{J} = (1, \ldots, n)} \varepsilon_{\mathcal{X}}(\mathcal{I}, \mathcal{J}) \left[ U_{|\mathcal{I}|}(\mathcal{X}_\mathcal{I}), U_{|\mathcal{J}|}(\mathcal{X}_\mathcal{J}) \right]_G \\
= \sum_{i<j} (-1)^{\alpha_{ij}} U_{n-1}(\mathcal{X}_1, \ldots, \mathcal{X}_i, \mathcal{X}_j, \ldots, \mathcal{X}_n) \tag{3.7}\]

where $d_{\mu_2} U_n := -[U_n, \mu_2]_G$ with $\mu_n : C^\infty(T^*M)^{\otimes n} \to C^\infty(T^*M)$ the usual commutative and associative pointwise product of $n$ functions, $[-,-]_G$ denotes the Gerstenhaber bracket defined in Appendix A.2, and for a multi-index $\mathcal{I} = (i_1, \ldots, i_k)$ we denote $\mathcal{X}_\mathcal{I} := \mathcal{X}_{i_1} \wedge \cdots \wedge \mathcal{X}_{i_k}$ and $|\mathcal{I}| := k$; the sign factor $\varepsilon_{\mathcal{X}}(\mathcal{I}, \mathcal{J})$ is the “Quillen sign” associated with the partition $(\mathcal{I}, \mathcal{J})$ of the integer $n$, $(-1)^{\alpha_{ij}}$ is a prescribed sign rule arising from the $L_\infty$-structure (see Appendix A.1), and the hats denote omitted multivectors. Formality follows from the Ward–Takahashi identities
for the Lie algebroid gauge symmetry of the Poisson sigma-model in the BV formalism. In our case of interest, the conditions (3.7) reduce to

\[ d_\# \Phi(\Theta) = i \hbar \Phi(d_\Theta \Theta), \quad (3.8) \]

where the coboundary operators are \( d_\# = -[-, \#]_G \) and \( d_\Theta = -[-, \Theta]_S \) with \( d_\Theta \Theta = \Pi \). Using (3.8) and (A.15) we derive a formula for the associator (3.6) given by

\[ [f, g, h]_\# = \frac{2i}{\hbar} ((f \ast g) \ast h - f \ast (g \ast h)), \quad (3.9) \]

which is non-zero since the product \( \ast \) is not associative. This formula provides an exact formal expression which can be calculated up to any order in the deformation parameter \( \hbar \) using Kontsevich diagrams.

The formality conditions give rise to derivation properties. Using (3.6) we can define a new function \( \underline{f} \) for every function \( f \) by [54, 83]

\[ \underline{f} = f + \frac{(i \hbar)^2}{2} U_3(f, \Theta, \Theta) + \sum_{n=3}^{\infty} \frac{(i \hbar)^n}{n!} U_{n+1}(f, \Theta, \ldots, \Theta). \quad (3.10) \]

The formality condition is then \( d_\# \underline{f} = i \hbar \Phi(d_\Theta f) \), which tells us that the Hamiltonian vector field \( d_\Theta f \) is mapped to the inner derivation

\[ d_\# \underline{f} = \frac{i}{\hbar} [f, -]_\#, \quad (3.11) \]

where \( [f, g]_\# := f \ast g - g \ast f \) is the star commutator of functions \( f, g \in C^\infty(T^* M) \). Similarly, a new vector field \( \underline{\mathcal{X}} \) for any vector field \( \mathcal{X} \) is defined by

\[ \underline{\mathcal{X}} = \mathcal{X} + \frac{(i \hbar)^2}{2} U_3(\mathcal{X}, \Theta, \Theta) + \sum_{n=3}^{\infty} \frac{(i \hbar)^n}{n!} U_{n+1}(\mathcal{X}, \Theta, \ldots, \Theta). \quad (3.12) \]

The formality condition is now \( d_\# \underline{\mathcal{X}} = i \hbar \Phi(d_\Theta \mathcal{X}) \). \( d_\Theta \)-closed vector fields \( \mathcal{X} \) preserve the twisted Poisson structure, i.e. \( d_\Theta \mathcal{X} = 0 \). The formality condition then implies the derivation property

\[ \underline{\mathcal{X}}(f \ast g) = \underline{\mathcal{X}}(f) \ast g + f \ast \underline{\mathcal{X}}(g) \quad (3.13) \]

for \( f, g \in C^\infty(T^* M) \).

Finally, we consider the formality condition

\[ d_\# \Phi(\Pi) = i \hbar \Phi(d_\Theta \Pi). \quad (3.14) \]

Using (3.6), we can express the left-hand side of this expression as

\[ d_\# \Phi(\Pi)(f, g, h, k) \]

\[ = f \ast [g, h, k]_\# - [f \ast g, h, k]_\# + [f, g \ast h, k]_\# - [f, g, h \ast k]_\# + [f, g, h]_\# \ast k, \quad (3.15) \]

while the Schouten–Nijenhuis bracket on the right-hand side is

\[ d_\Theta \Pi := [\Pi, \Theta]_S = \frac{1}{24} (\Theta^{LM} \partial_M \Pi^{JK} - \Theta^{JM} \partial_M \Pi^{JK} + \Theta^{JM} \partial_M \Pi^{KL} - \Theta^{KM} \partial_M \Pi^{LJ} + \Pi^{LM} \partial_M \Theta^{KL} - \Pi^{LM} \partial_M \Theta^{IJ} + \Pi^{KM} \partial_M \Theta^{LJ} - \Pi^{LM} \partial_M \Theta^{JK} - \Pi^{LM} \partial_M \Theta^{KL} + \Pi^{LM} \partial_M \Theta^{KL} ) \partial_I \wedge \partial_J \wedge \partial_K \wedge \partial_L. \quad (3.16) \]

Then (3.14) relates these two expressions and gives the derivation property for \( f, g, h, k \in C^\infty(T^* M) \).
3.3 Constant $R$-flux

We now turn our attention to the case of constant $R^{ijk}$ considered in Section 2 and calculate the products we found in Section 3.1 explicitly. Let us begin by computing (3.3). The zeroth order diagram is the usual pointwise multiplication. There is only one admissible first order diagram (the wedge) whose weight is found to be $\frac{1}{2}$ in Appendix B, hence $U_1(\Theta)(f, g) = \frac{1}{2} \Theta^{IJ} \partial_I f \partial_J g$. The admissible second order diagrams are

\[ \quad \text{and} \quad \]

The first one represents $\Theta^{KL} \partial_L \Theta^{IJ} \partial_I f \partial_K \partial_J g = R^{ijk} \partial_i f \partial_j \partial_k g = 0$ due to antisymmetry of $R^{ijk}$. The second diagram also vanishes for the same reason. Consequently, all higher order diagrams that contain these two subdiagrams are equal to zero. The third diagram is simply the product of two wedges, therefore its weight is $\frac{1}{4}$. Hence $U_2(\Theta, \Theta)(f, g) = \frac{1}{4} \Theta^{IJ} \Theta^{KL} \partial_I \partial_K f \partial_J \partial_L g$. Since wedges that land on wedges do not contribute to $U_n(\Theta, \ldots, \Theta)$, there is only one admissible diagram to each order of the form

\[ \quad \]

Hence the star product for the constant $R$-flux background is given by the Moyal type formula

\[ f \star g = \mu_2 \left( \exp \left( \frac{i}{\hbar} \Theta^{IJ} \partial_I \otimes \partial_J \right) (f \otimes g) \right), \quad (3.17) \]

where as before $\mu_2$ is the pointwise multiplication map of functions.

The associator (3.6) for constant $R^{ijk}$ can be computed either by calculating Kontsevich diagrams and summing the series or by using the star product (3.17) to compute the left-hand side of (3.9). Here we will follow the second approach, but before doing so it is instructive to calculate diagrams up to third order. A method for calculating Kontsevich diagrams involving two functions for linear Poisson structures was developed in [56]; however we have found this setting unsuitable for calculations involving more than two grounded vertices and so we calculate diagrams in the usual manner. The lowest order admissible diagram is the triple wedge, whose weight is $\frac{1}{6}$ (see Appendix B), thus $U_1(\Pi) = \frac{1}{6} \Pi^{IJK} \partial_I f \partial_J g \partial_K h$. In $U_2(\Pi, \Theta)$ a wedge is added, but since $R^{ijk}$ is constant, diagrams where the wedge lands on the trivector $\Pi$ are zero; thus all non-zero diagrams have weight $\frac{1}{12}$. Third order is more interesting as we now have two wedges that may land on each other. These diagrams are non-zero since the remaining edges all land on different functions. Calculating their weights (see Appendix B) we find that they combine to a trivector diagram according to the formula

\[ \quad \text{and} \quad \]

which when written out explicitly reproduces the formula (2.39) for the Schouten–Nijenhuis bracket $[\Theta, \Theta]_S$ with constant $R^{ijk}$.

To calculate the associator (3.9) explicitly to all orders, we first observe that due to antisymmetry of $R^{ijk}$ the star product (3.17) factorizes as

\[ f \star g = \mu_2 \left( \exp \left( \frac{i}{\hbar} R^{ijk} p_k \partial_i \otimes \partial_j \right) \exp \left[ \frac{i}{\hbar} \left( \partial_i \otimes \partial^i - \partial^i \otimes \partial_i \right) \right] (f \otimes g) \right) =: f \star_p g, \quad (3.18) \]
where as before we write $\partial_i = \frac{\partial}{\partial q_i}$ and $\tilde{\partial}^i = \frac{\partial}{\partial p_i}$. Here we denote the nonassociative product $\star := \star_p$ where $p$ is the dynamical momentum variable. By replacing the dynamical variable $p$ with a constant $\tilde{p}$ we obtain the associative Moyal star product $\star := \star_{\tilde{p}}$. Nonassociativity arises because $\star$ acts non-trivially on the $p$-dependence of $\star := \star_p$ in the associator. Applying this on $(f \star g) \star h$ and $f \star (g \star h)$ using antisymmetry of $R^{ijk}$ we find

\begin{equation}
(f \star g) \star h := (f \star_p g) \star_p h = \left[ \star \left( \exp \left( \frac{\hbar}{4} R^{ijk} \partial_i \otimes \partial_j \otimes \partial_k \right) (f \otimes g \otimes h) \right) \right]_{\tilde{p} \to p}, \tag{3.19}
\end{equation}

\begin{equation}
f \star (g \star h) := f \star_p (g \star_p h) = \left[ \star \left( \exp \left( - \frac{\hbar}{4} R^{ijk} \partial_i \otimes \partial_j \otimes \partial_k \right) (f \otimes g \otimes h) \right) \right]_{\tilde{p} \to p}, \tag{3.20}
\end{equation}

where no ordering is required on the right-hand sides due to associativity of the Moyal product and the operation $[-]_{\tilde{p} \to p}$ reinstates the dynamical momentum dependence. Using (3.9) we therefore find

\begin{equation}
[f, g, h]_{\star} = \frac{4i}{\hbar} \left[ \star \left( \sinh \left( \frac{\hbar^2}{4} R^{ijk} \partial_i \otimes \partial_j \otimes \partial_k \right) (f \otimes g \otimes h) \right) \right]_{\tilde{p} \to p}. \tag{3.21}
\end{equation}

This manner of regarding our nonassociative products is consistent with the observation of Section 2.3 that only the momentum directions in the membrane sigma-model are dynamical on $T^* M$. From (3.21) it follows that our nonassociative star product is cyclic, i.e. the associator is a total derivative and for Schwartz functions $f, g, h \in C^\infty(T^* M)$ we have

\begin{equation}
\int_{T^* M} d^{2d}x \ [f, g, h]_{\star}(x) = 0. \tag{3.22}
\end{equation}

The cyclic property (3.21) also holds for the nonassociative star products derived from open string amplitudes in curved backgrounds [37, 38] (see also [28]).

We conclude by writing out the derivation property (3.15) explicitly for constant $R$-flux. The Schouten–Nijenhuis bracket (3.16) now vanishes, and therefore (3.15) reduces to

\begin{equation}
[f \star g, h, k]_{\star} - [f, g \star h, k]_{\star} + [f, g, h \star k]_{\star} = f \star [g, h, k]_{\star} + [f, g, h]_{\star} \star k \tag{3.23}
\end{equation}

for four functions $f, g, h,$ and $k$, while the remaining derivation properties we found in Section 3.2 remain unaffected. We can interpret (3.22) in the following way. Just as the star commutator provides a quantization of the twisted Poisson structure defined by the antisymmetric 2-bracket $\{f, g\}_\Theta := \Theta(df, dg)$, in the sense that $[f, g]_{\star} = 2i \hbar \{f, g\}_\Theta + \mathcal{O}(\hbar^2)$, the associator (3.20) defines a quantization of the Nambu–Poisson structure defined by the completely antisymmetric 3-bracket

\begin{equation}
\{f, g, h\}_\Theta := \Pi(df, dg, dh), \tag{3.24}
\end{equation}

in the sense that

\begin{equation}
[f, g, h]_{\star} = 6i \hbar \{f, g, h\}_\Theta + \mathcal{O}(\hbar^2). \tag{3.25}
\end{equation}

To this order, the star derivation property (3.22) is just a consequence of the usual Leibniz rule

\begin{equation}
\{f g, h, k\}_\Theta = f \{g, h, k\}_\Theta + \{f, h, k\}_\Theta g \tag{3.26}
\end{equation}

for the Nambu–Poisson bracket (3.23). Likewise, higher derivation properties, when expanded in powers of $\hbar$, should encode the fundamental identity

\begin{equation}
\{\{f_1, f_2, g\}_\Theta, h, k\}_\Theta + \{g, \{f_1, f_2, h\}_\Theta, k\}_\Theta + \{g, h, \{f_1, f_2, k\}_\Theta\}_\Theta \tag{3.27}
\end{equation}

\begin{equation}
= \{f_1, f_2, \{g, h, k\}_\Theta\}_\Theta
\end{equation}
for $f, g, h, k \in C^\infty(T^*M)$; we return to this issue in Section 4.2 where we will see that the equation (3.22) can also be interpreted as the star product version of the pentagon identity (A.36) for the Lie 2-group that we encounter there.

### 3.4 Seiberg–Witten maps

We will now apply the formalism of this section to analyze the effect of adding fluctuations to the membrane boundary and the open string endpoints. To start, we shall recall a few relevant facts of the open string case with an ordinary Poisson structure, and then show how this generalizes to the case of $H$-twisted Poisson structures and ultimately to the membrane setting with $R$-flux.

By studying open strings in a closed string background, Seiberg and Witten [78] found equivalent effective descriptions in terms of ordinary as well as noncommutative gauge theories. Realizing this, they proposed the existence of maps $\hat{A}(a)$ and $\hat{A}(\lambda, a)$ from an ordinary gauge potential $a_\mu$ and gauge parameter $\lambda$ to their noncommutative cousins $\hat{A}_\mu$ and $\hat{A}$, such that an ordinary infinitesimal gauge transformation $\delta g a_\mu = \partial_\mu \lambda$ induces its noncommutative analog $\delta_\lambda \hat{A}_\mu = \partial_\mu \hat{A} + i \hat{A} \star \hat{A}_\mu - i \hat{A}_\mu \star \hat{A}$. Further analysis has revealed [51, 53] that the Seiberg–Witten map can be interpreted as a special generalized change of coordinates induced by an invertible linear operator $D$, which is a non-linear functional of the gauge potential $a$ and maps ordinary spacetime coordinates $x^\mu$ to covariant coordinates $\hat{x}^\mu = D(x^\mu) = x^\mu + \theta^{\mu\nu} \hat{A}_\nu(x)$. The covariantizing map transforms by a star commutator with $\hat{A}$ under gauge transformations, implying the appropriate noncommutative gauge transformation for $\hat{A}_\mu$. For simplicity, we have written here equations for abelian gauge fields and have focused on the case of constant Poisson structure $\theta$. We will continue to focus on the abelian case, but drop all other simplifying assumptions in the following.

A semi-classical version $\rho$ of the covariantizing map is given by the flow generated by the vector field $a_\theta = \theta(a, -) = \theta^{\mu\nu} a_\nu \partial_\mu$, which is obtained by contracting the Poisson bivector field $\theta = \frac{1}{2} \theta^{\mu\nu} \partial_\mu \land \partial_\nu$ with the gauge potential one-form $a = a_\mu(x) dx^\mu$. Under a gauge transformation, $\rho$ transforms via the Poisson bracket with a semi-classical gauge parameter $\hat{\lambda}$. The Poisson bivector $\theta$ is “twisted” by the two-form field strength $f = da$ to a new Poisson bivector $\theta' = \theta(1 + h f \theta)^{-1}$ such that $\rho(f, g) = \{\rho(f), \rho(g)\}_\theta$. The quantum covariantizing map $D$ is similarly obtained as the “flow” of the differential operator $a_\theta$, which is the deformation quantization (3.12) of $a_\theta$. Let $\star$ and $\star'$ likewise be the deformation quantizations along $\theta$ and $\theta'$ respectively according to (3.3). The covariantizing map $D$ relates these star products via $D(f \star' g) = Df \star Dg$, i.e. it is an isomorphism of associative algebras and noncommutative gauge transformations are realised as inner automorphisms. For the technical details of the construction, we refer to [54, 55]. In the following, we will describe several covariantizing maps and the corresponding gauge symmetries relevant for strings in the $R$-flux background.

The presence of a non-trivial closed three-form background $H$ leads to a twisted Poisson structure: The bivector $\Theta$ fails to fulfill the Jacobi identity and its Schouten–Nijenhuis bracket is consequently non-zero: $[\Theta, \Theta]_S = h \land^3 \Theta^f(H)$, where we have introduced a factor $h$ to ensure formal convergence of all expressions in the ensuing construction. (Such a factor is understood to be implicitly included in $\Theta$ in the rest of this article.) From the point of view of background fields and fluctuations, the structure we are dealing with is a gerbe: Given a suitable covering of the target space manifold by contractible open patches (labelled by Greek indices $\alpha, \beta, \ldots$), we can
write $H$ in terms of local two-form fields $B_\alpha$ as $H = dB_\alpha$ on each patch. On the overlap of two patches, the difference $B_\beta - B_\alpha = F_{\alpha\beta}$ is closed, hence exact and can be expressed in terms of one-form fields $a_{\alpha\beta}$ as $F_{\alpha\beta} = da_{\alpha\beta}$. On triple overlaps we then encounter local gauge parameters $\lambda_{\alpha\beta\gamma}$ that satisfy a suitable integrability condition. This hierarchical description in terms of forms has a dual description in terms of multivector fields that is suitable for deformation quantization and leads to noncommutative gerbes in the sense described in [4]: The twisted Poisson bivector $\Theta$ can be locally untwisted by the two-form fields $B$ and leads to noncommutative gerbes in the sense described in [4]: The twisted Poisson bivector $\Theta$ can be locally untwisted by the two-form fields $B$, leading to bonafide Poisson bivectors $\Theta_\alpha = \Theta (1 - h B_\alpha \Theta)^{-1}$. These local Poisson tensors $\Theta_\alpha$ and the corresponding associative star products $\star_\alpha$ are related by covariantizing maps computed from $a_{\alpha\beta}$.

As mentioned in Section 2.3, the relevant geometric structure in $R$-space is a gerbe in momentum space, with curvature (2.34) and 2-connection (2.35). In the present paper, we are dealing with a topologically trivial setting, so the forms and multivector fields are all globally defined. Nevertheless, the constructions of twisted noncommutative gauge theory and Seiberg–Witten maps are non-trivial and interesting. On $T^*M$ the patch index $\alpha$ is replaced by a constant momentum vector $\tilde{p}$ that parameterizes a degree of freedom in the choice of Poisson structure $\Theta_\tilde{p}$ and two-form background field $B_\tilde{p}$. In matrix form, the pertinent bivector and two-form fields are

$$\Theta = \begin{pmatrix} h R^{ijk} p_k & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix}, \quad \Theta_\tilde{p} = \begin{pmatrix} h R^{ijk} \tilde{p}_k & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix} \quad \text{and} \quad B_\tilde{p} = \begin{pmatrix} 0 & R^{ijk} (p_k - \tilde{p}_k) \\ 0 & 0 \end{pmatrix}. \quad (3.27)$$

They satisfy

$$H = dB_\tilde{p}, \quad [\Theta, \Theta]_S = h \Lambda^3 \Theta^2 (H) \quad \text{and} \quad [\Theta_\tilde{p}, \Theta_\tilde{p}]_S = 0, \quad (3.28)$$

together with

$$\Theta = \Theta_\tilde{p} (1 + h B_\tilde{p} \Theta_\tilde{p})^{-1} \quad \text{and} \quad \Theta_\tilde{p} = \Theta (1 - h B_\tilde{p} \Theta)^{-1}. \quad (3.29)$$

The corresponding 1-connection is given by $a_{\tilde{p}, \tilde{p}'} = R^{ijk} p_i (\tilde{p}_k - \tilde{p}'_k) dp_j$. Note that we cannot choose $h B$ to be equal to $\Theta^{-1}$ as that would add terms of order $h^{-1}$ to $H = dB$, which is incompatible with (3.28), and it would lead to convergence problems for the geometric series in (3.29). The deformation quantizations along $\Theta$ and $\Theta_{\tilde{p}}$ yield the nonassociative star product $\star$ and the associative star product $\star_{\tilde{p}}$ respectively. For the special case $\tilde{p} = 0$, $\Theta_0$ and $\star_0$ are respectively the canonical Poisson structure and associative star product on phase space. For fixed three-form $H$, choices for $B$ can differ by any closed (and hence exact) two-form $F = dA$. The corresponding choices of Poisson structures and star products are related by covariantizing maps constructed from gauge potentials

$$A = A_I (x) \, dx^I = a_i (x, p) \, dx^i + \tilde{a}^i (x, p) \, dp_i. \quad (3.30)$$

Associated to these covariantizing maps are Seiberg–Witten maps as explained before. Gauge transformations $\delta_\lambda A = d\lambda$, where $\lambda = \lambda (x, p)$, induce a change of the covariantizing maps by a star commutator with $\hat{\Lambda} (\lambda, a)$.

So far we have discussed ordinary Seiberg–Witten maps for bonafide Poisson bivectors. It would appear to be more natural to find also a construction based directly on the twisted Poisson bivector $\Theta$. Terms involving the non-zero Jacobiator (Schouten–Nijenhuis bracket) usually spoil
such a construction. In the present case it turns out, however, that for the class of gauge potentials of the form \( A = (A_I) = (0, \tilde{a}^i(x,p)) \) (i.e. with \( a_i(x,p) = 0 \)) the unwanted terms drop out, because \( \left[ \Theta, \Theta \right]_{S}^{IJK} A_K \) is proportional to \( R^{ijk} a_k = 0 \). The restriction thus imposed on the class of admissible gauge potentials leads to a corresponding restriction on the class of covariantizing maps. The admissible class of maps is, however, still very large and actually quite interesting: Evaluating \( \Theta(A, -) = \Theta^{IJK} A_J \partial_I = \delta^{i}_{j} \tilde{a}^i(x,p) \partial_i \) shows that any map generated by a vector field of the form \( \tilde{a}^i(x,p) \partial_i \), which acts on configuration space and may even depend on the momentum variables, is admissible. The associated class of ordinary and noncommutative gauge transformations is more restricted: The gauge parameters \( \lambda \) and \( \Lambda \) may only depend on the momenta \( p \).

An interesting subclass of the covariantizing maps just described are generated by gauge potentials of the form \( A = R(a_2, -) \), where \( a_2 \) is a two-form on configuration space and we have used the natural map \( \bigwedge^2 T^*M \to TM \) induced by the three-vector \( R \); in components \( \tilde{a}^i(x) = R^{ijk}(a_2)_{jk}(x) \). The semi-classical version of the resulting maps has been discussed in the context of Nambu–Poisson structures on \( p \)-branes in [52], where \( a_2 \) plays the role of a two-form gauge potential, and (noncommutative) gauge transformations are computed using Nambu–Poisson brackets and \( x \)-dependent one-form gauge parameters. (The restriction to gauge parameters that depend only on momenta is not needed here.) The present article provides a quantization of these Nambu–Poisson maps for membranes \( (p = 2) \) with a constant Nambu–Poisson trivector \( R \).

Another interesting special example concerns the relationship between \( \star_0 \) and \( \star_p \). The corresponding covariantizing map \( D_{\tilde{p}} \) is constructed from \( F_{\tilde{p}} = dA_{\tilde{p}} = B_0 - B_{\tilde{p}} \), with a gauge potential defined by

\[
A_{\tilde{p}} = \tilde{a}^i(p) dp_j = \frac{1}{2} R^{ijk} p_i \tilde{p}_k dp_j \tag{3.31}
\]

up to gauge transformations \( \delta \tilde{a}^i(p) = \tilde{\eta}^j \tilde{\lambda}(p) \). It satisfies \( f \star_p g = D_{\tilde{p}}^{-1}(D_{\tilde{p}} f \star_0 D_{\tilde{p}} g) \). Formally replacing the constant \( \tilde{p} \) by the dynamical momentum variable \( p \) in this equation gives a Seiberg–Witten map from the associative canonical star product \( \star_0 \) to the nonassociative star product \( \star := \star_p \) as

\[
f \star g = \left[ D_{\tilde{p}}^{-1}(D_{\tilde{p}} f \star_0 D_{\tilde{p}} g) \right]_{\tilde{p} \to p} . \tag{3.32}
\]

In view of the foregoing discussion, this can also be written as

\[
f \star g = \left[ D_{\tilde{p}}(f \star g) \right]_{\tilde{p} \to p} = \left[ D_{\tilde{p}} f \star_0 D_{\tilde{p}} g \right]_{\tilde{p} \to p} , \tag{3.33}
\]

since the underlying vector field \( \Theta(A_{\tilde{p}}, -) \) from (3.31) vanishes when \( \tilde{p} \to p \) and the covariantizing map becomes trivial. In the given gauge, the vector field \( \Theta(A_{\tilde{p}}, -) \) receives no quantum corrections from deformation quantization and the Seiberg–Witten map can thus be computed explicitly in closed form. This is one of the very rare cases where this is possible.

From the point of view of noncommutative gauge theory as well as noncommutative string geometry, gauge transformations preserve star products. Expressed in terms of gauge fields, gauge transformations correspond to different choices of one-form potentials \( A \) that preserve the curvature two-form \( F = dA \). From the membrane point of view, however, the three-form \( H \) is the fundamental global quantity and gauge transformations correspond to different choices of two-form potentials \( B \) that preserve the gerbe curvature \( H = dB \). The role of the gauge parameter is taken by a one-form gauge potential \( A \). As we have discussed, such one-forms
generate covariantizing maps $D$. These maps preserve associativity (as well as nonassociativity). The collection of these maps describes the gauge degrees of freedom of our system. Concretely, our construction for the twisted Poisson structure $\Theta$ yielded covariantizing maps $D_\xi$ for all vector fields $\xi$ on configuration space. This evidently generates a huge gauge degree of freedom generated by quantized general coordinate transformations. This point adds some credibility to the terminology “nonassociative gravity” that was coined in [8] to describe the quantum geometry of closed strings in non-geometric flux backgrounds.

3.5 Closed string vertex operators and 3-product

We close this section by comparing our associator with the ternary product for closed strings propagating in a background with constant $R$-flux which was proposed in [11]. Here the authors perform a linearized conformal field theory analysis of the three-point function of tachyon vertex operators in a flat background with constant $H$-flux. After applying three T-dualities they arrive at a nonassociative algebra of closed string vertex operators in the $R$-flux background, from which they propose a deformation of the pointwise product of functions via a 3-product of the form

$$f \bullet g \bullet h = f g h + R^{ijk} \partial_i f \partial_j g \partial_k h + O(R^2) .$$

(3.34)

In the light of the above analysis, we are able to explain this result analytically and relate it to our expressions. Expanding either of the two bracketings (3.19) to linear order we have

$$f * g * h = f g h + R' \cdot [\partial_i f \hat{*} \partial_j g \hat{*} \partial_k h]_{\hat{p} \to p} + O(R'^2) ,$$

(3.35)

from which we conclude that (3.34) agrees with (3.35) to first order in $R' := \pm \frac{\hbar^2}{4} R$, but without the Moyal star product between the derivatives of $f$, $g$ and $h$, and without the dependence on the dynamical variable $p$. For functions that are independent of $p$, the two formulas agree at linear order. The main difference between the two formulas stems from our consideration of the cotangent bundle of $M$ as the effective target space geometry of closed strings in the $R$-flux compactification.

This also explains the proposal of [11] that the binary product of functions is the usual pointwise multiplication, as for closed strings only three and higher point correlation functions experience the effect of the flux background. Setting $p = \hat{p} = 0$ corresponds to the sector of zero winding number in the T-dual $Q$-space frame. It truncates phase space to the original configuration space $M$ and recovers the usual commutative pointwise product $f \hat{*} g = f g = f * g$, consistent with the fact that only extended closed strings with non-trivial winding number (dual momentum) are sensitive to the noncommutative deformation in the $Q$-flux background (see Section 2.4); nevertheless, this sector still retains a non-trivial associator (3.20) of fields in the nonassociative $R$-flux background as in [8, 11]. Moreover, as in [11], higher order associators are not simply related to successive applications of (3.20). Together with the cyclic property (3.21), we see therefore that in this sector our deformation quantization approach agrees with the 3-product of [11]. Similarly to [84], the authors of [11] conjecture an all orders ternary product obtained by exponentiation of the trivector $R$ as a straightforward generalization of the Moyal–Weyl formula. Our results confirm this conjecture insofar that the exponential of $R$ is indeed part of the correct all order expressions (3.19).

As we have discussed, the twisted Poisson structures we have found naturally give rise to an $L_\infty$-structure on the algebra $C^\infty(T^*M)$. In [22] it is shown that correlators of open string
vertex operators in a non-constant $H$-flux background endow the Kontsevich deformation of the algebra of functions on $M$ with the structure of an $A_\infty$-algebra (see Appendix A.1), or more precisely an $A_\infty$-space, which are the natural algebras that appear in generic open-closed string field theories; in particular, the corresponding star commutator algebra is an $L_\infty$-algebra. The correlators of closed string vertex operators computed in [11] also exhibit dilogarithmic singularities analogous to those found in [22] (see also [37]), and it would be interesting to see if they lead to an analogous $A_\infty$-structure; indeed in [1] it is shown that the reflection identity for the Rogers dilogarithm relates four-point correlation functions to two-point correlators in a manner reminiscent of an associator, while the pentagonal identity is related to a factorization property of five-point functions which is reminiscent of the higher coherence relation for the associator. The similarity between open and closed string correlators is also noted in [7]. In Section 4 we will see such structures emerging rather directly in the full quantized algebra of functions.

4 Strict deformation quantization

In [56], Kathotia compares the two canonical deformation quantizations of the linear Kirillov–Poisson structure on the vector space $W = g^*$, where $g$ is a finite-dimensional Lie algebra; these quantizations are provided by the Kontsevich formalism and the associated Lie group convolution algebra. Let us briefly recall how this latter quantization scheme proceeds [69]. We first Fourier transform functions on $W$ to obtain elements in $C^\infty(g)$. We then identify $g$ with its integrating Lie group $G$ in a neighbourhood of the identity element via the exponential mapping. On $G$, we can use the convolution product between functions induced by the group multiplication and the Baker–Campbell–Hausdorff formula. We then perform the inverse operations to pullback the result and obtain a star product on $C^\infty(W)$. For nilpotent Lie algebras, the exponential map between $g$ and $G$ is a global diffeomorphism. In this case, the above construction is equivalent to both Kontsevich’s deformation quantization and quantization via the universal enveloping algebra of $g$ [56]; see also [81] for a comparison directly at the level of formality maps. Since our twisted Poisson structure (2.37) is linear for constant $R$-flux, it is natural to ask if there is an analogous approach which would provide an alternative quantization framework to the combinatorial approach we took in Section 3. In this section we shall develop such an approach based on integrating a suitable Lie 2-algebra to a Lie 2-group which will define a convolution algebra object in a braided monoidal category (see Appendices A.1 and A.6 for the precise definitions), and demonstrate that it is equivalent to the quantization of Section 3 which was based on our proposed membrane sigma model. Here we focus for definiteness on the case of configuration space $M = \mathbb{T}^d$ which is a $d$-dimensional torus with constant $R$-flux. This approach will then further clarify how the $R$-space nonassociativity is realized by a 3-cocycle associated to a nonassociative representation of the translation group, as arises in the presence of a magnetic monopole [49], and its relation to the topological nonassociative tori studied in [14].
4.1 $R$-space and $Q$-space Lie 2-algebras

Let $V \cong \mathbb{R}^{2d}$ be a vector space of dimension $2d$ with a fixed choice of basis elements which we denote by

\[(x^I) = (x^1, \ldots, x^{2d}) = (\hat{x}^1, \ldots, \hat{x}^d, \hat{p}_1, \ldots, \hat{p}_d) ,\]  

(4.1)

where throughout this section we use hats to distinguish abstract vector space and categorical elements from the concrete coordinate functions we used in previous sections. We define a bracket $[-,-]_R : V \wedge V \to V$ by the relations

\[ [\hat{x}^i, \hat{x}^j]_R = i R^{ijk} \hat{p}_k , \quad [\hat{p}_i, \hat{p}_j]_R = 0 \quad \text{and} \quad [\hat{x}^i, \hat{p}_j]_R = i \hbar \delta^i_j = -[\hat{p}_j, \hat{x}^i]_R , \]  

(4.2)

which is just an abstract presentation of the twisted Poisson brackets (2.37). This bracket defines a pre-Lie algebra structure on $V$, i.e. it is antisymmetric but does not satisfy the Jacobi identity; it leads to the non-vanishing Jacobiator

\[ [\hat{x}^i, \hat{x}^j, \hat{x}^k]_R := \frac{1}{2} \left( [\hat{x}^i, [\hat{x}^j, \hat{x}^k]_R] + [\hat{x}^k, [\hat{x}^j, \hat{x}^i]_R] + [\hat{x}^j, [\hat{x}^k, \hat{x}^i]_R] \right) = \hbar R^{ijk} , \]  

(4.3)

and all other Jacobiators vanish. Hence the bracket naturally defines a Lie 2-algebra $\mathcal{V}$ (see Appendix A.1). For this, we set $V_0 = V$, $V_1 = V$, and let $\eta : V_1 \to V_0$ be the identity map $\text{id}_V$. Let $[-,-] : V_0 \wedge V_0 \to V_0$ and $[-,-] : V_0 \otimes V_1 \to V_1$ be the bracket (4.2) of $V$, and let $[-,-,-] : V_0 \wedge V_0 \wedge V_0 \to V_1$ be the Jacobiator (4.3) of $V$. Then $(V_1 \xrightarrow{\eta} V_0)$ is a 2-term $L_\infty$-algebra canonically associated to the pre-Lie algebra $V$.

We can identify the twisted Poisson structure (2.37) on the algebra of functions $C^\infty(T^*M)$ with the natural twisted Poisson structure on the dual $V^*$ of the pre-Lie algebra $V$ as follows. We first identify linear functions on $V^*$ with elements of $V$, and define $\{\hat{v}_1, \hat{v}_2\}_R(x) := \langle x, [\hat{v}_1, \hat{v}_2]_R \rangle$, where $\hat{v}_1, \hat{v}_2 \in V$, $x \in V^*$ and $\langle \cdot, \cdot \rangle : V^* \otimes V \to \mathbb{R}$ denotes the dual pairing. By imposing the Leibniz identity, this defines a quasi-Poisson bracket that extends to polynomial functions on $V^*$, which in turn are dense in $C^\infty(V^*)$.

As we discuss in Appendix A.6, there is no general construction of Lie 2-groups from Lie 2-algebras, but we can build a suitable integration map with some intuition provided from our considerations in Section 2. For this, we will write down an equivalent Lie 2-algebra for which a corresponding Lie 2-group can be “guessed”. We start by replacing the pre-Lie algebra $V$ with a quadratic Lie algebra $\mathfrak{g}$ whose generators $\hat{x}^i, \hat{p}_j$, $i,j = 1, \ldots, d$ have the Lie brackets

\[ [\hat{x}^i, \hat{x}^j]_Q = i R^{ijk} \hat{p}_k \quad \text{and} \quad [\hat{x}^i, \hat{p}_j]_Q = 0 = [\hat{p}_i, \hat{p}_j]_Q , \]  

(4.4)

together with the nondegenerate inner product defined by

\[ \langle \hat{x}^i, \hat{p}_j \rangle = \delta^i_j \quad \text{and} \quad \langle \hat{x}^i, \hat{x}^j \rangle = 0 = \langle \hat{p}_i, \hat{p}_j \rangle \]  

(4.5)

which is invariant under the adjoint action and is of split signature. There are two ways to think about this Lie algebra. Firstly, it is the reduction of the Courant algebroid of Section 2.3 over a point; we may regard $\mathfrak{g} \cong \mathbb{R}^d \oplus (\mathbb{R}^d)^*$ as the cotangent bundle $T^*\mathbb{R}^d$ with its canonical symplectic structure. Secondly, it is an abstract version of the $Q$-space Poisson brackets (2.42), and in particular it coincides with the $\hbar = 0$ limit of the brackets given by (4.2) and (4.3); in this way we will mimick the dynamical quantization of Section 3.3 by first integrating the $d$-dimensional Heisenberg algebra (4.4) involving the “non-dynamical momenta” $\hat{p}_i$, and
then making the momenta “dynamical” \( \hat{p}_i \rightarrow \hat{p}_i \) to recover the T-dual pre-Lie algebra (4.2) appropriate to the \( R \)-space frame with the non-trivial Jacobiator (4.3).

Associated to the quadratic Lie algebra \( g \) is a Lie 2-algebra \( \tilde{\mathcal{V}} \) corresponding to the 2-term \( L_\infty \)-algebra

\[
\tilde{\mathcal{V}} = (\tilde{V}_1 = \mathbb{R} \overset{\mathcal{d}}{\twoheadrightarrow} \tilde{V}_0 = g)
\]  

(4.6)

which is skeletal, i.e. \( \mathcal{d} = 0 \), with brackets \([-,-]: \tilde{V}_0 \wedge \tilde{V}_0 \rightarrow \tilde{V}_0 \) given by the Lie bracket (4.4) of \( g \) and \([-,-]: \tilde{V}_0 \otimes \tilde{V}_1 \rightarrow \tilde{V}_1 \) given by \([\hat{v},c] = 0 \) for \( \hat{v} \in g \), \( c \in \mathbb{R} \), and Jacobiator \([-,-,-]: \tilde{V}_0 \wedge \tilde{V}_0 \wedge \tilde{V}_0 \rightarrow \tilde{V}_1 \) given by \([\hat{v}_1, \hat{v}_2, \hat{v}_3] = \langle [\hat{v}_1, \hat{v}_2]q, \hat{v}_3 \rangle \) for \( \hat{v}_i \in g \); this is just the reduction over a point of the Lie 2-algebra structure (A.34) canonically associated to the exact Courant algebroid \( C \rightarrow M \) of Section 2.3. The corresponding classifying triple is \((g, \mathbb{R}, j)\) where \( \mathbb{R} \) is the trivial representation of \( g \) and the 3-cocycle \( j: g \wedge g \wedge g \rightarrow \mathbb{R} \) is given by

\[
 j(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \langle [\hat{v}_1, \hat{v}_2]q, \hat{v}_3 \rangle .
\]  

(4.7)

The cocycle condition (or equivalently the pentagonal coherence relation (A.7)) follows from adjoint-invariance of the inner product and since \( g \) acts trivially on \( \mathbb{R} \); note that its only non-trivial values on generators are given by

\[
 j(x^i, x^j, x^k) = R^{ijk}
\]  

(4.8)

as in (4.3). The cohomology of the Heisenberg Lie algebra (4.4) is described in [76]; in particular for degree 3 one has

\[
 \dim H^3(g, \mathbb{R}) = D := \frac{1}{6} d(d - 1)(d - 2) - d
\]  

(4.9)

and the space of 3-cocycles

\[
 Z^3(g, \mathbb{R}) = \bigwedge^3 (\hat{x}_1^*, \ldots, \hat{x}_d^*)
\]  

(4.10)

is the vector space of homogeneous elements of degree 3 of the Grassmann algebra over the dual basis to \( \hat{x}_1, \ldots, \hat{x}_d \). It follows that the Jacobiator (4.7) gives rise to a generator \( [j] \) of \( H^3(g, \mathbb{Z}) = \mathbb{Z}^D \), and all generators are obtained via a choice of basis for the space of totally antisymmetric 3-vectors as in (4.8) (modulo linear redefinitions of the central elements \( \hat{p}_1, \ldots, \hat{p}_d \)).

### 4.2 Integrating Lie 2-groups

The classifying data \((g, \mathbb{R}, j)\) of the Lie 2-algebra \( \tilde{\mathcal{V}} \), with \( \mathbb{R} = u(1) \) regarded as the one-dimensional abelian Lie algebra, can be straightforwardly exponentiated to a triple \((G, U(1), \varphi)\) corresponding to a special Lie 2-group \( \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) \) (see Appendix A.6), modulo one subtlety. The universal 2-step nilpotent Lie algebra \( g \) of rank \( d \) integrates to the non-compact simply connected \( d \)-dimensional Heisenberg group \( G \), the associated free 2-step nilpotent Lie group. In order to exponentiate the generator \([j] \in H^3(g, \mathbb{R})\) induced by the Jacobiator (4.7) of \( \tilde{\mathcal{V}} \) to a compact element \([\varphi] \in H^3(G, U(1))\), it is necessary to restrict the space of 3-cocycles (4.10) to a lattice \( \Lambda \cong \mathbb{Z}^d \) of maximal rank in the linear span of the generators \( \hat{x}_1, \ldots, \hat{x}_d \). This lattice injects into a cocompact lattice \( \Gamma \) in \( G \); the resulting quotient \( G/\Gamma \) is a Heisenberg nilmanifold or “double twisted torus”, familiar in \( d = 3 \) dimensions as the doubled space of the geometric T-dual to the three-torus with \( H \)-flux [45]. We assume that the lattice is equipped with a
nondegenerate inner product which is given in a suitable basis by
\[ \eta = (\eta_{ab}) : \Lambda \otimes \Lambda \rightarrow \mathbb{R}, \]
a, b = 1, \ldots, d, with inverse \( \eta^{-1} = (\eta^{ab}) : \Lambda^* \otimes \Lambda^* \rightarrow \mathbb{R} \), and a nondegenerate dual pairing
\[ \Sigma = (\Sigma^a_i) : \Lambda \otimes_{\mathbb{R}} (\mathbb{R}^d)^* \rightarrow \mathbb{R} \]
which is a vielbein for the inner product, i.e. \( \Sigma_a^i \delta_{ij} \Sigma_b^j = \eta_{ab} \).

With these restrictions understood, the Lie 2-algebra \( \tilde{V} \) given by (4.6) integrates to the Lie 2-group
\[ \mathcal{G}_1 = G \times U(1) \xrightarrow{\mathcal{P}} \mathcal{G}_0 = G \]
having \( U(1) \) as the group of automorphisms of its unit object 1 in \( G \), in which the source and target maps \( s, t \) are both projections onto the first factor, vertical multiplication is given by
\( (g, \zeta) \circ (g', \zeta') = (g, \zeta \ast \zeta') \) for \( g \in G \) and \( \zeta, \zeta' \in U(1) \), and horizontal multiplication \( \otimes \) given by group multiplication. The associator
\[ \mathcal{P}_{g,h,k} : (g \otimes h) \otimes k \rightarrow g \otimes (h \otimes k) \]
is the automorphism given by
\[ \mathcal{P}_{g,h,k} = (ghk, \varphi(g, h, k)) , \]
where we have integrated the Lie algebra 3-cocycle (4.7) to the smooth normalised Lie group 3-cocycle \( \varphi : G \times G \times G \rightarrow U(1) \) with
\[ j(\hat{v}_1, \hat{v}_2, \hat{v}_3) = \left. \frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3} \right|_{t_i=0} \varphi(\exp t_1 \hat{v}_1, \exp t_2 \hat{v}_2, \exp t_3 \hat{v}_3) \]
for all \( \hat{v}_i \in \Lambda \). All other structure maps of the Lie 2-group \( \mathcal{G} \) are identity isomorphisms. Finally, to make the transformation to “dynamical” momentum variables \( \hat{p}_i \rightarrow \hat{p}_i \), and hence integrate our original Lie 2-algebra \( \tilde{V} \) with brackets (4.2) and (4.3), we endow \( \mathcal{G} \) with a braiding
\[ \mathcal{B}_{g,h} : g \otimes h \rightarrow h \otimes g \]
which is the automorphism given by
\[ \mathcal{B}_{g,h} = (gh, \beta(g, h)) , \]
where we have integrated the inner product (4.5) to the smooth normalised map \( \beta : G \times G \rightarrow U(1) \) with
\[ \langle \hat{v}_1, \hat{v}_2 \rangle = \left. \frac{\partial^2}{\partial t_1 \partial t_2} \right|_{t_i=0} \beta(\exp t_1 \hat{v}_1, \exp t_2 \hat{v}_2) \]
for all \( \hat{v}_i \in \Lambda \). The braided monoidal category \( \mathcal{G} \) is then the Lie 2-group that integrates the Lie 2-algebra \( \tilde{V} \).

We can make this construction somewhat more concrete and explicit in a way that will be suitable to our ensuing constructions. For this, we formally exponentiate the Lie 2-algebra generators to define
\[ \hat{Z}^a = \exp \left( 2\pi i (\Sigma^{-1})_i^a \hat{x}^i \right) \quad \text{and} \quad \hat{P}_\xi = \exp \left( i \xi^i \hat{p}_i \right) \]
for \( a = 1, \ldots, d \) and \( \xi = (\xi^i) \in \mathbb{R}^d \). We may compute exterior products \( \otimes : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \) of the elements (4.18) in the Lie 2-group \( \mathcal{G} \) by formally applying the Baker–Campbell–Hausdorff
formula using the brackets (4.2) and (4.3); since the bracket functor in this case is nilpotent, the Hausdorff series is still applicable to the sole finite non-vanishing order that we require it without any need of the Jacobi identity. The commutation relations are then given by

\[
\hat{Z}^a \otimes \hat{Z}^b = \hat{P}_{ab} \otimes \hat{Z}^b \otimes \hat{Z}^a ,
\]

(4.19)

\[
\hat{Z}^a \otimes \hat{P}_\xi = e^{2\pi i h (\Sigma^{-1})_{a} \xi} \hat{P}_\xi \otimes \hat{Z}^a ,
\]

(4.20)

\[
\hat{P}_\xi \otimes \hat{P}_\zeta = \hat{P}_\xi \otimes \hat{P}_\zeta ,
\]

(4.21)

where \(\xi_{ab} \in \mathbb{R}^d\) is given by

\[
(\xi_{ab})^i = -4\pi^2 (\Sigma^{-1})^{a}_{j} R^{ijk} (\Sigma^{-1})^{b}_{k} .
\]

(4.22)

In (4.20) we recognize the non-trivial braiding isomorphism \(\mathcal{B}_{\hat{Z}^a,\hat{P}_\xi}\) on 2-group objects given by the map \(\beta : \mathbb{R}^d \times \Lambda^* \to U(1)\) whose only non-trivial values are

\[
\beta(\xi, m) = e^{2\pi i h \xi^i (\Sigma^{-1})_{a} m_a}
\]

(4.23)

for \(\xi = (\xi^i) \in \mathbb{R}^d\) and \(m = (m_a) \in \Lambda^* \cong \mathbb{Z}^d\), while the remaining commutation relations in (4.19)–(4.21) are those of the rank \(d\) Heisenberg group \(G\). The non-trivial associators follow by applying the Baker–Campbell–Hausdorff formula once more to find

\[
(\hat{Z}^a \otimes \hat{Z}^b) \otimes \hat{Z}^c = e^{-2\pi i h R^{abc}} \hat{Z}^a \otimes (\hat{Z}^b \otimes \hat{Z}^c) ,
\]

(4.24)

where

\[
R^{abc} = 2\pi^2 R^{ijk} (\Sigma^{-1})^{a}_{i} (\Sigma^{-1})^{b}_{j} (\Sigma^{-1})^{c}_{k}
\]

(4.25)

are the dimensionless nonassociativity \(R\)-flux parameters. This expression is the Lie 2-group version of the “cyclic double commutator” that was calculated in [8], which we recognise as the action of the non-trivial associator isomorphism \(\mathcal{P}_{\hat{Z}^a,\hat{Z}^b,\hat{Z}^c}\) on 2-group objects. The corresponding 3-cocycle can be regarded as a group homomorphism or tricharacter \(\varphi : \Lambda^* \times \Lambda^* \times \Lambda^* \to U(1)\) defined by

\[
\varphi(m, n, q) = e^{-2\pi i h R^{abc} m_a n_b q_c} .
\]

(4.26)

This map is normalised, i.e. \(\varphi(m, n, q) = 1\) if either of \(m, n,\) or \(q\) is 0; this implies that the two obvious maps from \(\hat{Z}^a \otimes (1 \otimes \hat{Z}^b) = \mathcal{P}((\hat{Z}^a \otimes 1) \otimes \hat{Z}^b)\) to \(\hat{Z}^a \otimes \hat{Z}^b\) are consistent. It is also skew-symmetric, i.e. \(\varphi(m, n, q) = \varphi(n, m, q)^{-1} = \varphi(q, n, m)^{-1} = \varphi(q, n, m)^{-1}\), and it obeys the required pentagonal cocycle identity

\[
\varphi(m, n, q) \varphi(m, n + q, r) \varphi(n, q, r) = \varphi(m + n, q, r) \varphi(m, n, q + r)
\]

(4.27)

for \(m, n, q, r \in \Lambda^*\), which is equivalent to the pentagon identity (A.36) of the category \(\mathcal{B}\). The pentagon identity can also be derived explicitly by iterating the above calculations to find the non-trivial higher nonassociativity relations

\[
\hat{Z}^a \otimes (\hat{Z}^b \otimes (\hat{Z}^c \otimes \hat{Z}^d)) = e^{2\pi i h R^{abc}} \hat{Z}^a \otimes ((\hat{Z}^b \otimes \hat{Z}^c) \otimes \hat{Z}^d)
\]

\[
= e^{2\pi i h (R^{abc} + R^{bde})} (\hat{Z}^a \otimes \hat{Z}^b) \otimes (\hat{Z}^c \otimes \hat{Z}^d)
\]

\[
= e^{2\pi i h (R^{ace} + R^{bde} + R^{bce})} (\hat{Z}^a \otimes (\hat{Z}^b \otimes \hat{Z}^c) \otimes \hat{Z}^d)
\]

\[
= e^{2\pi i h (R^{abc} + R^{acd} + R^{bde} + R^{bce})} ((\hat{Z}^a \otimes \hat{Z}^b) \otimes \hat{Z}^c) \otimes \hat{Z}^d .
\]

(4.28)
As discussed in Appendix A.6, MacLane’s coherence theorem implies that these relations automatically imply all higher associativity relations in the category \( \mathcal{G} \). This is particularly interesting from the perspective of the quantization of Nambu–Poisson structures that we discussed in Section 3.3: As the fundamental identity (3.26) should be encoded in the coherence relations involving five objects, our categorical approach automatically encodes its quantization. This should therefore help to alleviate at least some of the difficulties that arise in implementing the fundamental identity for Nambu–Poisson brackets at the quantum level (see e.g. [25] for a discussion).

4.3 Convolution algebra objects

We will now apply this categorical formalism to the deformation quantization of the algebra of functions \( C^\infty(T^*M) \) on \( T^*M = \mathbb{T}^d \times (\mathbb{R}^d)^* \), regarded as the algebra \( C^\infty(V^*) \) as explained before. Here \( \mathbb{T}^d = \mathbb{R}^d/\Lambda \), and the \( d \times d \) invertible matrix \( \Sigma = (\Sigma_{ai}) \) defines the periods of the directions of the \( d \)-torus \( M = \mathbb{T}^d \), i.e. \( x^i \sim x^i + \Sigma a^i \), with \( a = 1, \ldots, d \); in particular, the (inverse) metric of \( \mathbb{T}^d \) is given by \( \Sigma_{ai} \delta_{ab} \Sigma_{bj} = g^{ij} \). We embed \( C^\infty(T^*M) \) as an algebra object \( A \) of the Lie 2-group \( \mathcal{G} \) via a categorification of the Weyl quantization map, see e.g. [82]; it is defined as the linear isomorphism on \( C^\infty(T^*M) \) given on the dense set of plane waves by

\[
\mathcal{W}(e^{ikI x^I}) = \hat{W}(m, \xi) := \exp \left( i k_I \hat{x}^I \right),
\]

and extended by linearity; here

\[
(k_I) = (k_1, \ldots, k_{2d}) = (k_1, \ldots, k_d, \xi_1, \ldots, \xi_d)
\]

with

\[
k_i = 2\pi (\Sigma^{-1})_i^a m_a, \quad m = (m_a) \in \Lambda^*,
\]

the quantized Fourier momenta appropriate to smooth single-valued functions on \( \mathbb{T}^d \). We regard (4.29) as an object in a suitable enrichment of the Lie 2-group \( \mathcal{G} \) to a linear category over \( \mathbb{C} \), which we think of as an analog of a convolution group algebra generated by the operators (4.18). This map can be applied to an arbitrary Schwartz function \( f \) on \( \mathbb{T}^d \times (\mathbb{R}^d)^* \) by expanding \( f \) in its Fourier transformation

\[
f(x, p) = \sum_{m \in \Lambda^*} e^{2\pi i (\Sigma^{-1})_i^a m_a x^i} \int_{\mathbb{R}^d} \frac{d^d \xi}{(2\pi)^d} f_m(\xi) e^{i \xi^i p_i},
\]

where the inverse Fourier transform is given by

\[
f_m(\xi) = \frac{1}{|\det \Sigma|} \int_{\mathbb{T}^d} d^d x \ e^{-2\pi i (\Sigma^{-1})_i^a m_a x^i} \int_{(\mathbb{R}^d)^*} d^d p \ e^{-i \xi^i p_i} f(x, p).
\]

We then set

\[
\mathcal{W}(f) := \sum_{m \in \Lambda^*} \int_{\mathbb{R}^d} \frac{d^d \xi}{(2\pi)^d} f_m(\xi) \hat{W}(m, \xi).
\]
The convolution product $\circledast$ of two functions $f, g \in C^\infty(T^*M)$ is defined via the horizontal product of two quantized functions as

$$\mathcal{W}(f \circledast g) := \mathcal{W}(f) \otimes \mathcal{W}(g)$$

in the 2-group $\mathcal{G}$ and the inverse map $\mathcal{W}^{-1}$ from (4.29). Another straightforward application of the Baker–Campbell–Hausdorff formula as in (4.19)–(4.21) yields the 2-group multiplication law

$$\hat{W}(m, \xi) \otimes \hat{W}(n, \lambda) = e^{\pi i h (\Sigma^{-1})^a (m_a \lambda^i - n_a \xi^i)} \hat{W}(m + n, \xi + \lambda - R^{abc} m_a n_b \Sigma_c) ,$$

and we obtain

$$\hat{W}^2 (f \circledast g)(x, p) = \sum_{m, n \in \Lambda^*} \int_{\mathbb{R}^d} \frac{d^d \xi}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{d^d \lambda}{(2\pi)^d} f_n(\lambda) g_{m-n}(\xi - \lambda) e^{2\pi i m_a (\Sigma^{-1})^a x^i + i \xi^i p_i} \times e^{-i (\Sigma^{-1})^a (h(m_a \lambda^i - n_a \xi^i) - 2\pi (\Sigma^{-1})^b_{ab} m_a n_b R^{ijk} p_k)} .$$

After introducing a factor of $h$ as in (3.28), this formula is identical to the star product (3.18) that we found by formal deformation quantization along the twisted Poisson structure $\Theta$, and hence the two quantizations are equivalent in this particular case. This result is a Lie 2-algebra version of Kathotia’s theorem [56, Section 5] which asserts the equivalence between Kontsevich’s deformation quantization and the group convolution algebra quantization of the dual of a nilpotent Lie algebra. The crux of this theorem does not rely on the Jacobi identity, and is easily applied to our pre-Lie algebra: By a trivial relabelling of the generators, the commutation relations (4.2) satisfy the conclusions of [56, Theorem 5.2.1]. It is tempting to conjecture that the Lie 2-group convolution algebra quantization that we have developed in this section is equivalent to Kontsevich’s deformation quantization along the linear twisted Poisson bivector field on the dual of any nilpotent pre-Lie algebra. It would be interesting to similarly characterise the nonassociative quantizations of generic semistrict nilpotent Lie 2-algebras, but these questions lie beyond the scope of the present paper.

We conclude by establishing that the algebra of functions $\mathcal{A} = C^\infty(T^d \times (\mathbb{R}^d)^*)$ endowed with the nonassociative product $\circledast$ is really an algebra object of the Lie 2-group $\mathcal{G}$, i.e. it satisfies the associativity relation (A.38) of the category. Using the multiplication law (4.36) of the 2-group $\mathcal{G}$ we compute triple products of the operators (4.29) to get

$$(\hat{W}(m, \xi) \otimes \hat{W}(n, \lambda)) \otimes \hat{W}(q, \eta) = e^{\pi i h R^{abc} m_a n_a q_c} e^{\pi i h (\Sigma^{-1})^a (m_a \lambda^i - n_a \xi^i + (m+n)_a \eta^i - q_a (\xi + \lambda)^i)} \hat{W}(m + n + q, \xi + \lambda + \eta - R^{abc} m_a n_b \Sigma_c - R^{abc} (m + n)_a q_b \Sigma_c) .$$

A completely analogous calculation for the other ordering shows that

$$\hat{W}(m, \xi) \otimes (\hat{W}(n, \lambda) \otimes \hat{W}(q, \eta)) = \varphi(m, n, q) (\hat{W}(m, \xi) \otimes \hat{W}(n, \lambda)) \otimes \hat{W}(q, \eta) = \mathcal{P}[(\hat{W}(m, \xi) \otimes \hat{W}(n, \lambda)) \otimes \hat{W}(q, \eta)] ,$$

where $\varphi$ is the 3-cocycle (4.26) and we have used (4.24) to identify the application of the associator isomorphism $\mathcal{P}$ to 2-group objects (4.29); this formula is extended to operators
(4.34) in the usual way using linearity. Using (4.38) and the quantization map (4.29), (4.35) we now compute the triple convolution product of functions $f, g, h \in C^\infty(T^*M)$ to get
\[
(f \circledast (g \circledast h))(x,p) = \sum_{m,n,q \in \Lambda^*} e^{-\pi i h R^{abc} m_a n_b q_c} e^{2\pi i (\Sigma^{-1})^i a m_a x_i} \int_{\mathbb{R}^d} \frac{d^d \xi}{(2\pi)^d} e^{i \xi^i p_i} 
\]
\[
\times \int_{\mathbb{R}^d} \frac{d\lambda}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{d\eta}{(2\pi)^d} f_{m-n-q}(\xi - \lambda - \eta) g_a(\lambda) h_q(\eta) 
\]
\[
\times e^{\pi i h (\Sigma^{-1})^i a ((m-q)_a \lambda^i - n_a (\xi - \eta)^i + m_a \eta^i - q_a \xi^i)} 
\]
\[
\times e^{i R^{abc} ((m-q)_a n_b + m_a q_b) \Sigma^c i_p_i},
\]
which agrees with the corresponding formula of (3.19). From (4.39) it follows that
\[
(f \circledast (g \circledast h))(x,p) = \mathcal{P}((f \circledast g) \circledast h)(x,p)
\]
as required, where here $\mathcal{P}((f \circledast g) \circledast h)$ is short-hand notation for the composition of morphisms on the right-hand side of (A.38) applied to $(f \circledast g) \circledast h$.

### 4.4 Monopole backgrounds and topological nonassociative tori

We conclude this section by comparing our noncommutative and nonassociative deformation of the cotangent bundle $T^*M = \mathbb{T}^d \times (\mathbb{R}^d)^*$ with some other appearances of nonassociativity in the literature. The relations (4.36) (or (4.19)) are reminiscent of those obeyed by the gauge invariant operators which generate a projective representation of the translation group in the background field of a Dirac monopole [49] (see also [67]), where the projective phase is a 2-cochain determined by the magnetic flux through a 2-simplex; in our case this flux is proportional to $\xi_R(m, n) \in \mathbb{R}^d$ where
\[
\xi_R(m, n)^i = -R^{abc} m_a n_b \Sigma^c i,
\]
and it arises as the gerbe 2-holonomy of the $B$-field (2.35) through the triangle at $p$ formed by the lattice vectors $m, n \in \Lambda^* \subset (\mathbb{R}^d)^*$ in momentum space. The triple product relation (4.39) (or (4.24)) is reminiscent of the nonassociativity relation which arises from the 3-cocycle proportional to the flux through the 3-simplex enclosing the monopole; in our case the 3-cocycle (4.26) is determined by the gerbe $H$-flux (2.34) through the tetrahedron at $p$ formed by the lattice vectors $m, n$ and $q$ in momentum space. See [28] for an open string realization of the monopole background in terms of D0-branes in $H$-space, or equivalently D3-branes in $R$-space.

Let us now compare our construction with the nonassociative tori discussed in [14, 15, 35]. For $n, q \in \Lambda^*$, we use the 3-cocycle (4.26) to define unitary operators $\hat{U}_{n,q}$ on the Hilbert space $L^2(\Lambda^*)$ of square-summable sequences $f_m$ on the momentum lattice $\Lambda^*$ of $M = \mathbb{T}^d$ by
\[
(\hat{U}_{n,q} f)_m = \varphi(m, n, q) f_m.
\]
These operators obey the composition law
\[
\varphi(m, n, q) \hat{U}_{m,n} \hat{U}_{m+n,q} = \alpha_m(\hat{U}_{n,q}) \hat{U}_{m,n+q},
\]
where $\alpha_m$ is the adjoint action by the regular representation $f_n \mapsto f_{n+m}$ of lattice translations by $m \in \Lambda^*$. One then defines the twisted convolution product
\[
(f \circledast \varphi g)_m = \sum_{n \in \Lambda^*} f_n \alpha_n(g_{m-n}) \hat{U}_{n,m-n}
\]
(4.45)
on the algebra $\mathcal{C}^\infty(\Lambda^*, K)$, where $K = \mathcal{K}(\ell^2(\Lambda^*))$ is the algebra of compact operators on $\ell^2(\Lambda^*)$. This defines a nonassociative twisted crossed-product algebra $\mathcal{K}(\ell^2(\Lambda^*)) \rtimes_\varphi \Lambda^*$ which is identified with the algebra of functions on the nonassociative torus. When $\varphi = 1$ ($R = 0$), the operators $\hat{U}_{n,q}$ all act as the identity operator on $\ell^2(\Lambda^*)$ and $\alpha_m$ can be taken to be the identity; then $\otimes_{\varphi=1}$ is just the usual convolution product on the algebra $\mathcal{C}^\infty(\mathbb{T}^d) \otimes K$ of stabilized functions on the torus $\mathbb{T}^d$, which is Morita equivalent to the usual commutative algebra $\mathcal{C}^\infty(\mathbb{T}^d)$. In the general case, by [14, Proposition 3.1] the twisted convolution product $\ast_\varphi$ satisfies (A.38) and hence makes $K(\ell^2(\Lambda^*)) \rtimes_\varphi \Lambda^*$ an algebra object of the tensor category $\mathcal{G}$; in [35] it is shown that this defines a strict (i.e. non-formal) nonassociative deformation quantization.

We can identify a covariant representation of $(\Lambda^*, K(\ell^2(\Lambda^*)))$ by using the commutation relations (4.20) to identify the generators of translations in the lattice $\Lambda^*$ as the operators $\hat{W}(m, 0)$ for $m \in \Lambda^*$. From (4.36) we may then identify operators through the 2-group multiplication law

$$\hat{W}(m, 0) \otimes \hat{W}(n, 0) := \hat{U}_{m,n} \otimes \hat{W}(m + n, 0),$$

where

$$\hat{U}_{m,n} = \hat{P}_{\xi R(m,n)} \tag{4.47}$$

and we have used the Baker–Campbell–Hausdorff formula together with antisymmetry of $R^{abc}$. By (4.39), it follows from [14, Section 3] that these operators coincide with the ones introduced in (4.43). This correspondence is completely analogous to that found in [28, Section 5.2] via an open string analysis of D3-branes in $R$-space; in particular, our representation of the operators $\hat{U}_{m,n}$ is determined by the surface holonomy (4.42) of the pertinent $B$-field as in [28]. However, in our picture, the meaning of the stabilization by the algebra of compact operators $K$ is clear: It represents precisely the additional cotangent degrees of freedom through the unitary momentum operators $\hat{P}_{\xi} \in K$ from (4.18).

We close by commenting on how our nonassociative algebras may be related to associative ones which can be represented as operator algebras on separable Hilbert spaces, hence justifying some of the constructions above. In the context of open strings in non-trivial $H$-flux backgrounds, it was shown in [40, 39] how to map the nonassociative algebra of functions equipped with the Kontsevich star product to an associative algebra by enlarging the deformed configuration space to a deformed phase space; the resulting algebra is interpreted as an algebra of pseudo-differential operators as now both coordinates $x^I$ and derivatives $\partial_I$ appear. This mapping is the analog of the Bopp shift which maps the Heisenberg commutation relations onto trivial commuting variables when viewed as a subalgebra of extended canonical phase space commutation relations. In [20] such Bopp shifts are used to map noncommutative twisted tori onto commutative tori with the same phase space nonassociativity. In our case, the resulting algebra should be compared with the Lie 2-algebra constructed in Section 2.5 which has underlying associative coordinate algebra. The construction of [40, 39] is simply a physical implementation of MacLane's coherence theorem, which states that any monoidal category is equivalent to a strict monoidal category in which the associativity isomorphism (4.12) is simply the obvious identification by rebracketing $(g \otimes h) \otimes k \mapsto g \otimes (h \otimes k)$. In the present case, it is shown in [15] that the equivalence functor is obtained by applying $\mathcal{P}^{-1}$ to (4.45) and it takes an algebra object $\mathcal{A}$ to the associative crossed product algebra $\mathcal{A} \rtimes \Lambda^*$; this augmented algebra is in a sense the “exponentiation” of the extended algebras of [40, 39].
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A Higher Lie algebra structures

In this appendix we collect the pertinent mathematical material on higher structures which are used extensively in the main text.

A.1 Lie 2-algebras

Homotopy Lie algebras. An $L_{\infty}$-algebra or strong homotopy Lie algebra is a graded vector space $V$ together with a collection of totally (graded) antisymmetric $n$-brackets $[-,\ldots, -] : \wedge^n V \to V$, $n \geq 1$ of degree $n - 2$ satisfying the higher or homotopy Jacobi identities

$$\sum_{i=1}^{n} \sum_{\sigma \in \text{Sh}(i,n-i)} (-1)^{\alpha(\sigma)} [[v_{\sigma(1)}, \ldots, v_{\sigma(i)}], v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}] = 0$$

for each $n \geq 1$. Here $(-1)^{\alpha(\sigma)}$ is a prescribed sign rule for permuting homogeneous elements $v_1, \ldots, v_n \in V$, while $\text{Sh}(i,n-i)$ is the set of permutations $\sigma \in S_n$ which preserve the orders of the first $i$ elements and of the last $n-i$ elements, i.e. $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$ for $i = 1,\ldots,n$.

Denote the 1-bracket by $d := [-]$. It has degree $-1$ and the generalized Jacobi identity (A.1) for $n = 1$ reads

$$d^2 = 0,$$

which implies that $d : V \to V$ is a differential making $V$ into a chain complex. For $n = 2$ one has

$$d[v, w] = [dv, w] + (-1)^{|v|} [v, dw],$$

which implies that $d$ is a derivation with respect to the antisymmetric 2-bracket $[-, -] : V \wedge V \to V$. The bracket $[-, -]$ satisfies the usual Jacobi identity only up to a homotopy correction; from (A.1) with $n = 3$ we obtain

$$(-1)^{|v||w|} [[v, w], u] + (-1)^{|w||w|} [[u, v], w] + (-1)^{|v||w|} [[w, u], v]$$

$$= (-1)^{|v||w|+1} (d[v, w, u] + [dv, w, u] + (-1)^{|v|} [v, dw, u] + (-1)^{|v|+|w|} [v, w, du]),$$

which implies that the Jacobiator $[-, -, -] : V \wedge V \wedge V \to V$ is a chain homotopy map. For $n > 3$, the identities (A.1) impose extra coherence relations on this homotopy and all higher homotopies.

If $V$ has trivial grading, then an $L_{\infty}$-algebra is simply an ordinary Lie algebra. More generally, an $L_{\infty}$-algebra with vanishing $n$-brackets for all $n \geq 3$ is a differential graded Lie algebra.
A 2-term $L_\infty$-algebra is a strong homotopy Lie algebra with underlying graded vector space $V = V_0 \oplus V_1$ concentrated in degrees 0 and 1; it has vanishing $n$-brackets for $n > 3$ and the only non-trivial identities in (A.1) occur for $n = 1, 2, 3, 4$. It may be regarded as a 2-term chain complex $V = (V_1 \xrightarrow{d} V_0)$ whose bracket $[-, -] : V_i \otimes V_j \rightarrow V_{i+j}$, $i + j = 0, 1$, is a chain map and whose Jacobiator $[-, -, -] : V_0 \otimes V_0 \otimes V_0 \rightarrow V_1$ is a chain homotopy from the chain map

$$V_0 \otimes V_0 \otimes V_0 \rightarrow V_1, \quad v \wedge w \wedge u \mapsto [v, [w, u]]$$

(A.5)

to the chain map

$$V_0 \otimes V_0 \otimes V_0 \rightarrow V_1, \quad v \wedge w \wedge u \mapsto [[v, w], u] + [w, [v, u]]$$

(A.6)
satisfying the coherence condition

$$[v, [w, u, s]] + [v, [w, u], s] + [v, u, [w, s]] + [[v, w, u], s] + [v, [w, u], s] + [w, [v, u], s] + [w, u, [v, s]].$$

(A.7)

This higher Jacobi identity relates the two ways of using the Jacobiator to rebracket the expression $[[s, v], w], u$.

A related notion is that of an $A_\infty$-algebra, or homotopy associative algebra, which is a graded vector space $A$ endowed with a family of $n$-multiplication operations $\mu_n : A^\otimes n \rightarrow A$ of degree $n - 2$, $n \geq 1$ obeying the higher or homotopy associativity relations

$$\sum_{j+k+l = n} (-1)^{\sigma} \mu_n \circ (\text{id}_{A^\otimes j} \otimes \mu_k \otimes \text{id}_{A^\otimes l}) = 0.$$  

(A.8)

The first two relations

$$d^2 = 0 \quad \text{and} \quad d\mu_2(a, b) = \mu_2(da, b) + (-1)^{|a|} \mu_2(a, db)$$

(A.9)

for $a, b \in A$ make $A$ into a chain complex with differential $d := \mu_1$ which is a graded derivation of the binary product $\mu_2$. The third relation states that the product $\mu_2$ is associative up to the homotopy $\mu_3$, and so on. From an $A_\infty$-algebra structure on $A$ one constructs an $L_\infty$-algebra structure through the antisymmetric $n$-brackets

$$[a_1, \ldots, a_n] := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \mu_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

(A.10)

for $a_1, \ldots, a_n \in A$. However, in general there is no converse enveloping algebra type procedure to construct an $A_\infty$-structure from an $L_\infty$-structure.

**Lie 2-algebras.** 2-term $L_\infty$-algebras are the same things as Lie 2-algebras [6, Theorem 36], which are categorified versions of Lie algebras in which the Jacobi identity is replaced by a Jacobiator isomorphism. For this, recall that a 2-vector space is a linear category $\mathcal{V} = (\mathcal{V}_0, \mathcal{V}_1)$ consisting of a vector space of objects $\mathcal{V}_0$ and a vector space of morphisms $\mathcal{V}_1$, together with source and target maps $s, t : \mathcal{V}_1 \rightarrow \mathcal{V}_0$ sending a morphism to its domain and range, and an inclusion map $1 : \mathcal{V}_0 \rightarrow \mathcal{V}_1$, $v \mapsto 1_v$, sending an object to its identity morphism; the set of composable morphisms is $\mathcal{V}_1 \times_{\mathcal{V}_0} \mathcal{V}_1 = \{(v_1, w_1) \in \mathcal{V}_1 \times \mathcal{V}_1 \mid s(w_1) = t(v_1)\}$. These maps are all linear and compatible in the usual sense with the composition $\circ : \mathcal{V}_1 \times_{\mathcal{V}_0} \mathcal{V}_1 \rightarrow \mathcal{V}_1$ in the category.
A Lie 2-algebra is a 2-vector space $\mathcal{V}$ together with an antisymmetric bilinear bracket functor $[-,-]_{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and a natural antisymmetric trilinear Jacobiator isomorphism on objects satisfying a higher Jacobi identity. A Lie 2-algebra $\mathcal{V}$ is strict if its Jacobiator is the identity isomorphism; in that case both $\mathcal{V}_0$ and $\mathcal{V}_1$ are Lie algebras, and each operation on the category is a homomorphism of Lie algebras. Otherwise $\mathcal{V}$ is semistrict; this is the case of relevance to this paper.

Given a 2-term $L_\infty$-algebra $V = (V_1 \xrightarrow{d} V_0)$, we construct a 2-vector space $\mathcal{V}$ with vector spaces of objects and morphisms given by $\mathcal{V}_0 = V_0$ and $\mathcal{V}_1 = V_0 \oplus V_1$. A morphism $f = (v_0, v_1)$ in $\mathcal{V}_1$ with $v_0 \in V_0$ and $v_1 \in V_1$ has source and target given by $s(v_0, v_1) = v_0$ and $t(v_0, v_1) = v_0 + dv_1$, while the object inclusion is $1_v = (v, 0)$. The composition of two morphisms $f = (v_0, v_1)$ and $f' = (v_0 + dv_1, v_1')$ in $\mathcal{V}_1$ is $f \circ f' := (v_0, v_1 + v_1')$. The bracket functor $[-,-]_{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is defined on objects $v, v' \in \mathcal{V}_0$ by $[v, v']_{\mathcal{V}} = [v, v']$, where $[-,-]$ denotes the bracket in the $L_\infty$-algebra $V$. The bracket of morphisms $f = (v_0, v_1)$ and $f' = (v_0', v_1')$ in $\mathcal{V}_1$ is given by

$$[f, f']_{\mathcal{V}} = ([v_0, v_0'], [v_1, v_0'] + [v_0 + dv_1, v_1']) = ([v_0, v_0'], [v_0, v_1'] + [v_1, v_0' + dv_1']) .$$  \hspace{1cm} (A.11)

Finally, the Jacobiator for $\mathcal{V}$ is defined on $v, w, u \in \mathcal{V}_0$ by

$$[v, w, u]_{\mathcal{V}} := ([v, w], u) ,$$  \hspace{1cm} (A.12)

with source $s([v, w, u]_{\mathcal{V}}) = [[v, w], u]$ and target $t([v, w, u]_{\mathcal{V}}) = [v, [w, u]] + [[v, u], w]$ by (A.4).

The skew-symmetric bracket

$$[v_1, v_1'] = [dv_1, v_1'] = [v_1, dv_1'] ,$$  \hspace{1cm} (A.13)

defined on elements $v_1, v_1' \in V_1$ which figures in the formula (A.11), is called the derived bracket. It satisfies the Jacobiator identity

$$[v_1, [v_1', v_1'']] - [[v_1, v_1'], v_1''] - [v_1', [v_1, v_1'']] = [dv_1, dv_1', dv_1''] .$$  \hspace{1cm} (A.14)

**Classification of Lie 2-algebras.** There is a bijective correspondence between semistrict Lie 2-algebras and certain classifying “Postnikov data” [6], analogous to the Faulkner construction of 3-Lie algebras. The data in question are triples $(g, W, j)$ consisting of a Lie algebra $g$, a representation of $g$ on a vector space $W$, and a 3-cocycle $j$ on $g$ with values in $W$; the isomorphism classes are parametrized by elements $[j] \in H^3(g, W)$ of the degree 3 Lie algebra cohomology.

For a Lie 2-algebra $\mathcal{V}$ obtained from a 2-term $L_\infty$-algebra $V = (V_1 \xrightarrow{d} V_0)$, the corresponding triple $(g, W, j)$ is constructed by firstly setting $g = \ker(d) \subseteq V_1$; since $d = 0$ on $g$ the 2-bracket of the $L_\infty$-structure satisfies the Jacobi identity exactly and makes $g$ into a Lie algebra. Now let $W = \coker(d) \subseteq V_0$, and use the 2-bracket to define an action $g \otimes W \to W$ by $g \triangleright w = [g, w]$ for $g \in g$, $w \in W$; in this correspondence $W$ is regarded as the abelian Lie algebra of endomorphisms of the zero object of $\mathcal{V}$. Finally, the Jacobiator of the $L_\infty$-structure gives a map $[-,-,-] : g \wedge g \wedge g \to W$ which is a Chevalley–Eilenberg 3-cocycle $j$ whose cohomology class $[j] \in H^3(g, W)$ is the obstruction to $\mathcal{V}$ being functorially equivalent to a strict Lie 2-algebra, or equivalently to a differential $\mathbb{Z}_2$-graded Lie algebra.
A.2 Gerstenhaber brackets

Consider the Hochschild complex $H^n(A, A) = \text{Hom}_C(A^\otimes n, A)$ of an algebra $A$ with product $\star \in H^2(A, A)$. The space of $n$-cochains $C^n(A, A) = \text{Hom}_C(\wedge^n A, A)$ is constructed by antisymmetrization, and the Hochschild coboundary operator $d_\star : C^n(A, A) \to C^{n+1}(A, A)$ is defined by

$$d_\star C(f_1, \ldots, f_{n+1}) = f_1 \star C(f_2, \ldots, f_{n+1}) + \sum_{i=1}^{n} (-1)^i C(f_1, \ldots, f_i \star f_{i+1}, \ldots, f_{n+1})$$

$$+ (-1)^{n+1} C(f_1, \ldots, f_n) \star f_{n+1}$$

for $C \in C^n(A, A)$ and $f_i \in A$. From the product on $A$ we construct a cup product $\star : H^n(A, A) \otimes H^n(A, A) \to H^{n+2}(A, A)$ by

$$\langle C_1 \star C_2 \rangle(f_1, \ldots, f_{n_1+n_2}) := C_1(f_1, \ldots, f_{n_1}) \star C_2(f_{n_1+1}, \ldots, f_{n_1+n_2})$$

for $C_1 \in C^{n_1}(A, A)$ and $C_2 \in C^{n_2}(A, A)$.

The Gerstenhaber bracket of $C_1 \in C^{n_1}(A, A)$ and $C_2 \in C^{n_2}(A, A)$ is defined by

$$[C_1, C_2]_G = C_1 \circ C_2 - (-1)^{(n_1+1)(n_2+1)} C_2 \circ C_1$$

in $C^{n_1+n_2-1}(A, A)$, where the composition product is defined as

$$\langle C_1 \circ C_2 \rangle(f_1, \ldots, f_{n_1+n_2-1})$$

$$= C_1(C_2(f_1, \ldots, f_{n_2}), f_{n_2+1}, \ldots, f_{n_1+n_2-1})$$

$$+ \sum_{i=1}^{n_1-2} (-1)^{i n_2} C_1(f_1, \ldots, f_i, C_2(f_{i+1}, \ldots, f_{i+n_2}), f_{i+n_2+1}, \ldots, f_{n_1+n_2-1})$$

$$+ (-1)^{(n_1+1)(n_2+1)} C_1(f_1, \ldots, f_{n_1-1}, C_2(f_{n_1}, \ldots, f_{n_1+n_2-1}))$$

for $f_i \in A$. The coboundary operator is then given by

$$d_\star C = -[C, \star]_G.$$ 

The associativity of the product $\star \in C^2(A, A)$ may be expressed by using

$$[\star, \star]_G(f, g, h) = 2 ((f \star g) \star h - f \star (g \star h)).$$

Associativity is thus equivalent to $d_\star \star = [\star, \star]_G = 0$ or $d_\star^2 = 0$; in that case, the Gerstenhaber algebra $(C^\ast(A, A), d_\star, [-, -]_G)$ is a differential graded Lie algebra.

A.3 Schouten–Nijenhuis brackets

Let $V^\ast = C^\infty(\mathcal{M}, \wedge^\ast T\mathcal{M})$ be the graded-commutative algebra of multivector fields on a smooth manifold $\mathcal{M}$; notice that $V^\ast$ contains the associative algebra $V^0 = C^\infty(\mathcal{M})$ of smooth complex functions on $\mathcal{M}$. The usual Lie bracket of vector fields $[-, -]_{T\mathcal{M}}$ extends to the canonical Schouten–Nijenhuis bracket $[-, -]_S$ on $V^\ast$. It gives $V^\ast$ the structure of a differential graded Gerstenhaber algebra with vanishing differential, i.e. $[-, -]_S$ is a graded Lie bracket of degree $-1$ satisfying the graded Leibniz rule with respect to the associative (graded-commutative)
exterior product. Given homogeneous multivector fields $\mathcal{X} = X^{I_1 \cdots I_{|\mathcal{X}|}} \partial_{I_1} \wedge \cdots \wedge \partial_{I_{|\mathcal{X}|}}$ and $\mathcal{Y} = Y^{I_1 \cdots I_{|\mathcal{Y}|}} \partial_{I_1} \wedge \cdots \wedge \partial_{I_{|\mathcal{Y}|}}$, it is defined by

$$\mathcal{X} \mathcal{Y} = (-1)^{|\mathcal{X}|-1} \mathcal{X} \circ \mathcal{Y} - (-1)^{|\mathcal{X}||\mathcal{Y}|-1} \mathcal{Y} \circ \mathcal{X}$$

(A.21)

in $\mathcal{V}^{|\mathcal{X}|+|\mathcal{Y}|-1}$, where

$$\mathcal{X} \circ \mathcal{Y} := \sum_{l=1}^{|\mathcal{X}|} (-1)^{l-1} X^{I_1 \cdots I_l} Y^{I_{l+1} \cdots I_{|\mathcal{X}|}} \partial_{I_1} \wedge \cdots \wedge \hat{\partial}_{I_l} \wedge \cdots \wedge \partial_{I_{|\mathcal{X}|}} \wedge \partial_{I_{|\mathcal{Y}|}}$$

(A.22)

and the hat indicates an omitted derivative.

The condition for a bivector $\Theta = \frac{1}{2} \Theta^{IJ} \partial_I \wedge \partial_J$ to define a Poisson structure on $C^\infty(M)$ can be expressed through

$$[\Theta, \Theta]_{S} = \frac{1}{3} \left( \Theta^{IK} \partial_I \Theta^{JK} + \Theta^{JK} \partial_J \Theta^{IK} - \Theta^{IL} \partial_L \Theta^{IK} - \Theta^{IK} \partial_K \Theta^{IJ} \right) \partial_I \wedge \partial_J \wedge \partial_K$$

(A.23)

The corresponding antisymmetric bracket $\{f,g\}_\Theta := \Theta(df, dg)$ for $f, g \in C^\infty(M)$ satisfies the Jacobi identity on $C^\infty(M)$ if and only if $[\Theta, \Theta]_{S} = 0$, and thus defines a Poisson bracket. In terms of the Lichnerowicz coboundary operator $d_\Theta : \mathcal{V}^n \rightarrow \mathcal{V}^{n+1}$ defined by

$$d_\Theta = -[-\cdot, \Theta]_{S}$$

(A.24)

the Poisson condition can be expressed as $d_\Theta \Theta = 0$ or $d_\Theta^2 = 0$. The Poisson bracket extends to the cotangent bundle $T^*M$ where it encodes the Schouten–Nijenhuis bracket of multivector fields.

### A.4 Higher derived brackets

Let $\Pi \in \mathcal{V}^\sharp = C^\infty(M, \wedge^\sharp TM)$ be a multivector field satisfying $[\Pi, \Pi]_{S} = 0$. Following [85], we define the $n$-th derived bracket of $\Pi$ as

$$\{X_1, \ldots, X_n\}_\Pi := [\cdots [[\Pi, X_1]_S, X_2]_S, \ldots, X_n]_S$$

(A.25)

for $X_i \in \mathcal{V}^\sharp$ and $n \geq 1$. Then the sequence of brackets $\{-, \ldots, -\}_\Pi$ defines a higher Poisson structure on $\mathcal{V}^\sharp$. Each derived bracket strictly obeys a generalized Leibniz rule with respect to the exterior product on $\mathcal{V}^\sharp$, i.e. $\{-, \ldots, -\}_\Pi$ is a derivation in each argument. By [85, Corollary 1], this sequence of higher Poisson brackets gives $\mathcal{V}^\sharp$ the structure of an $L_\infty$-algebra; the full countable tower of homotopy Jacobi identities is equivalent to the requirement $[\Pi, \Pi]_{S} = 0$.

In this correspondence we use a parity $\mathbb{Z}_2$-grading defined as the multivector degree modulo 2, and then apply the parity reversion functor. Hence we introduce the total $\mathbb{Z}_2$-grading $\mathcal{V}^\sharp = \mathcal{V}_0 \oplus \mathcal{V}_1$ where $\mathcal{V}_0 = C^\infty(M, \wedge^{\text{odd}} TM)$ and $\mathcal{V}_1 = C^\infty(M, \wedge^{\text{even}} TM)$. Owing to the generalized Leibniz rule, in examples it suffices to display the bracket at linear order in the generators of $\mathcal{V}^\sharp$, with $|1| = 1 = |x^I|$ and $|\partial_I| = 0$.

### A.5 Courant algebroids

**Lie algebroids.** A **Lie algebroid** over a smooth manifold $\mathcal{M}$ is a vector bundle $E \rightarrow \mathcal{M}$ endowed with a Lie bracket $[-, -]_E$ on smooth sections of $E$ and a bundle morphism $\rho : E \rightarrow
$T\mathcal{M}$, called the anchor map, which is compatible with the Lie bracket on sections, i.e. the tangent map to $\rho$ is a Lie algebra homomorphism,

$$d\rho_{[\psi_1,\psi_2]} = [d\rho_{\psi_1}, d\rho_{\psi_2}]_{T\mathcal{M}}, \quad \psi_1, \psi_2 \in C^\infty(\mathcal{M}, E),$$

(A.26)

and a Leibniz rule is satisfied when multiplying sections of $E$ by smooth functions on $\mathcal{M}$,

$$[\psi_1, f \psi_2]_E = f [\psi_1, \psi_2]_E + \rho_{\psi_2} (f) \psi_1, \quad \psi_1, \psi_2 \in C^\infty(\mathcal{M}, E), \quad f \in C^\infty(\mathcal{M}).$$

(A.27)

Equivalently, a Lie algebroid is a vector bundle $E \to \mathcal{M}$ endowed with a differential $d_E$ of degree +1 on the free graded-commutative algebra $\bigwedge_{\leq 0}^\infty C^\infty(\mathcal{M}, E)^*$ over $C^\infty(\mathcal{M})$. For $\omega \in \bigwedge_{\leq 0}^{n-1} C^\infty(\mathcal{M}, E)^*$ and $\psi_i \in C^\infty(\mathcal{M}, E)$, the differential $d_E$ is given here by

$$d_E \omega(\psi_1, \ldots, \psi_n) = \sum_{\sigma \in S_n} \left( \rho_{\psi_{\sigma(1)}} (\omega(\psi_{\sigma(2)}, \ldots, \psi_{\sigma(n)})) + \omega([\psi_{\sigma(1)}, \psi_{\sigma(2)}]_E, \psi_{\sigma(3)}, \ldots, \psi_{\sigma(n)}) \right).$$

(A.28)

This defines a differential graded algebra

$$CE(E) = \left( \bigwedge_{\leq 0}^{\leq 1} C^\infty(\mathcal{M}, E)^*, d_E \right)$$

(A.29)

which dually has the structure of a Gerstenhaber algebra with the Lie bracket on $C^\infty(\mathcal{M}, E)$ extended as a biderivation with $[\psi, f]_E = \psi(d_E f)$ for $\psi \in C^\infty(\mathcal{M}, E)$ and $f \in C^\infty(\mathcal{M})$; this bracket generalizes the Schouten–Nijenhuis bracket of multivector fields. The pair (A.29) is called the Chevalley–Eilenberg algebra of the Lie algebroid. It is the complex which computes Lie algebroid cohomology.

A Lie algebroid over a point is just a Lie algebra (with trivial anchor map), and (A.29) is the usual Chevalley–Eilenberg algebra which computes Lie algebra cohomology. More generally, Lie algebra bundles provide natural examples of Lie algebroids.

The tangent Lie algebroid over a manifold $\mathcal{M}$ is $E = T\mathcal{M}$ with the identity anchor map $\rho = \text{id}_{T\mathcal{M}}$ and the usual Lie bracket on vector fields. In this case $CE(T\mathcal{M}) = (\Omega^1(\mathcal{M}), d)$ is the usual de Rham complex.

Any bivector field $\Theta$ on $\mathcal{M}$ induces a map $\Theta^2 : T^* \mathcal{M} \to T\mathcal{M}$ via contraction together with a bracket on $C^\infty(\mathcal{M}, T^* \mathcal{M}) = \Omega^1(\mathcal{M})$ called the Koszul bracket

$$[\alpha, \beta]_\Theta := \mathcal{L}_{\Theta^\sharp(\alpha)} \beta - \mathcal{L}_{\Theta^\sharp(\beta)} \alpha - d\Theta(\alpha, \beta)$$

(A.30)

for $\alpha, \beta \in \Omega^1(\mathcal{M})$, where $\mathcal{L}$ denotes the Lie derivative. Then $E = T^* \mathcal{M}$, $\rho = \Theta^2$, and $[-, -]_E = [-, -]_\Theta$ defines a Lie algebroid on $\mathcal{M}$ if and only if the Schouten–Nijenhuis bracket of $\Theta$ vanishes, i.e. $\Theta$ defines a Poisson structure on $\mathcal{M}$. In this case $d_{T^* \mathcal{M}} = d_\Theta = [\Theta, -, ]_S$ is the Lichnerowicz differential and the Chevalley–Eilenberg algebra (A.29) computes the Poisson cohomology of $\mathcal{M}$.

Courant algebroids. The higher structures which arise in this paper, such as twisted Poisson structures, require a higher extension of the notion of Lie algebroid. For this, consider a vector bundle $E \to \mathcal{M}$ over a smooth manifold $\mathcal{M}$ equipped with a metric $\langle -, - \rangle$ and an antisymmetric bracket $[-, -]_E : C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \to C^\infty(\mathcal{M}, E)$, together with an anchor map $\rho : E \to T\mathcal{M}$. We define the Jacobiator $J : C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \wedge C^\infty(\mathcal{M}, E) \to C^\infty(\mathcal{M}, E)$ by

$$J(\psi_1, \psi_2, \psi_3) = [[\psi_1, \psi_2]_E, \psi_3]_E + [[\psi_2, \psi_3]_E, \psi_1]_E + [[\psi_3, \psi_1]_E, \psi_2]_E,$$

(A.31)
a ternary map \([-,-,-]\)_E : \(C^\infty(M, E) \wedge C^\infty(M, E) \wedge C^\infty(M, E) \to C^\infty(M)\) by
\[
[\psi_1, \psi_2, \psi_3]_E = \frac{1}{3!} \left(\left[\left[\psi_1, \psi_2\right]_E, \psi_3\right] + \left[\left[\psi_2, \psi_3\right]_E, \psi_1\right] + \left[\left[\psi_3, \psi_1\right]_E, \psi_2\right]\right),
\]
and the pullback \(d : C^\infty(M) \to C^\infty(M, E)\) of the exterior derivative \(d\) via the adjoint map \(\rho^*\) by
\[
\langle df, \psi \rangle = \rho_\psi(f),
\]
where \(f \in C^\infty(M)\) and \(\psi, \psi_i \in C^\infty(M, E)\); this map defines a flat connection, \(d^2 = 0\).

Such a vector bundle is called a Courant algebroid [62] if the following conditions are satisfied:

(i) The Jacobi identity holds up to an exact expression: \(J(\psi_1, \psi_2, \psi_3) = d[\psi_1, \psi_2, \psi_3]_E\);

(ii) The anchor map \(\rho\) is compatible with the bracket: \(\rho[\psi_1, \psi_2]_E = [\rho\psi_1, \rho\psi_2]_{TM}\);

(iii) There is a Leibniz rule: \([\psi_1, f \psi_2]_E = f[\psi_1, \psi_2]_E + \rho\psi_1(f) \psi_2 - \frac{1}{2} \langle \psi_1, \psi_2 \rangle df\);

(iv) \(\langle df, dg \rangle = 0\);

(v) \(\rho_\psi(\langle \psi_1, \psi_2 \rangle) = \langle [\psi_1, \psi_1]_E + \frac{1}{2} d(\langle \psi_1, \psi_1 \rangle), \psi_2 \rangle + \langle \psi_1, [\psi_1, \psi_2]_E + \frac{1}{2} d(\langle \psi_1, \psi_2 \rangle)\rangle\);

where \(\psi, \psi_i \in C^\infty(M, E)\) and \(f, g \in C^\infty(M)\).

The graded differential Lie algebra (A.29) is now generalized to a Lie 2-algebra: The structure maps \(d, [-,-]_E, [-,-,-]_E\) of the Courant algebroid \(E \to M\) on the complex
\[
C^\infty(M) \xrightarrow{d} C^\infty(M, E),
\]
extended as \([\psi, f]_E := \frac{1}{2} \langle df, \psi \rangle\) for \(\psi \in C^\infty(M, E)\) and \(f \in C^\infty(M)\), define a 2-term \(L_\infty\)-algebra [73].

A.6 Lie 2-groups

A group is a monoid in which every element has an inverse; 2-groups are categorifications of groups. For this, recall that a tensor or monoidal category is a category \(\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)\) equipped with an exterior product \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\) together with an identity object \(1 \in \mathcal{C}_0\) and three natural functorial isomorphisms: The unity isomorphisms \(1_X := 1 \otimes X \cong X \cong X \otimes 1\) in \(\mathcal{C}_1\) for all objects \(X \in \mathcal{C}_0\), and the associator isomorphisms
\[
\mathcal{P} = \mathcal{P}_{X,Y,Z} : (X \otimes Y) \otimes Z \overset{\sim}{\longrightarrow} X \otimes (Y \otimes Z)
\]
for all objects \(X, Y, Z \in \mathcal{C}_0\). They satisfy the pentagon identities
\[
(1_X \otimes \mathcal{P}_{Y,Z,W}) \circ \mathcal{P}_{X,Y \otimes Z,W} \circ \mathcal{P}_{X,Y,Z \otimes 1_W} = \mathcal{P}_{X,Y,Z \otimes W} \circ \mathcal{P}_{X \otimes Y,Z,W}
\]
which state that the five ways of bracketing four objects commutes, and also the triangle identities which state that the associator isomorphism with \(Y = 1\) is compatible with the unity isomorphisms. For morphisms \(\mathcal{F} : X \to Y\) and \(\mathcal{F}' : X' \to Y'\), their exterior product is the morphism \(\mathcal{F} \otimes \mathcal{F}' : X \otimes X' \to Y \otimes Y'\) in \(\mathcal{C}_1\). By MacLane’s coherence theorem, these identities ensure that all higher associators are consistent.
We call $\mathcal{C}$ braided when there are natural functorial isomorphisms

$$\mathcal{B} = \mathcal{B}_{X,Y} : X \otimes Y \to Y \otimes X$$ (A.37)

for any pair of objects $X, Y \in \mathcal{C}_0$, called commutativity relations. The braiding $\mathcal{B}_{X,Y}$ satisfies two conditions, one expressing $\mathcal{B}_{X \otimes Y, Z}$ in terms of associativity relations $\text{id}_X \otimes \mathcal{B}_{Y,Z}$ and $\mathcal{B}_{Z,X} \otimes \text{id}_Y$, and a similar one for $\mathcal{B}_{X,Y \otimes Z}$.

An object $\mathcal{A} \in \mathcal{C}_0$ in a tensor category $\mathcal{C}$ is an algebra or monoid object if there is a “multiplication” morphism $\otimes : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, written $a \otimes b \mapsto a \otimes b$, which is associative in the category, i.e. it satisfies the associativity condition

$$\otimes \circ (\otimes \otimes \text{id}_A) = \otimes \circ (\text{id}_A \otimes \otimes) \circ \mathcal{P}_{A,A,A}$$ (A.38)

as maps $(\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \to \mathcal{A}$. By MacLane’s coherence theorem, we can deal with nonassociative algebras in this way by expressing usual algebraic operations as compositions of maps and doing the same in the monoidal category with the relevant associator $\mathcal{P}$ inserted between any three objects as needed in order to make sense of expressions. If in addition $\mathcal{C}$ is braided, then $\mathcal{A}$ is commutative if its product morphism obeys

$$\otimes \circ \mathcal{B}_{A,A,A} = \otimes$$ (A.39)

as maps $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. A group object is a monoid object $\mathcal{A}$ together with a “unit” morphism $1_{\mathcal{A}} : 1 \to \mathcal{A}$ satisfying the unit condition

$$\otimes \circ (1_{\mathcal{A}} \otimes \text{id}_A) = \text{id}_A = \otimes \circ (\text{id}_A \otimes 1_{\mathcal{A}})$$, (A.40)

such that every element of $\mathcal{A}$ has an inverse with respect to the product morphism $\otimes$ and the identity $1_{\mathcal{A}}$.

A 2-group is a monoidal category in which every object and morphism has an inverse. A Lie 2-group is a pair $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ of objects in the category of smooth manifolds and smooth maps, with source and target maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$, and a vertical multiplication $\circ : \mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_1$ of morphisms. In addition there is a horizontal multiplication functor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ on objects and morphisms, an identity object 1, and a contravariant inversion functor $(-)^{-1} : \mathcal{G} \to \mathcal{G}$ together with natural isomorphisms provided by the associator $\mathcal{P}_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$, the left and right units $1 \otimes g \cong g \cong g \otimes 1$, and the units and counits $g \otimes g^{-1} \cong 1 \cong g^{-1} \otimes g$ obeying pentagon, triangle and zig-zag identities; see [5, Section 7] for details. If the structure morphisms are all identity isomorphisms, the Lie 2-group $\mathcal{G}$ is called strict; otherwise $\mathcal{G}$ is semistrict.

A Lie 2-group $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is special if its source and target morphisms $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are equal, and the units and counits are all identity isomorphisms. There is a bijective correspondence between special Lie 2-groups and triples $(G, H, \varphi)$ consisting of a Lie group $G$, an action of $G$ as automorphisms of an abelian group $H$, and a normalized smooth 3-cocycle $\varphi : G \times G \times G \to H$; the isomorphism classes are parameterized by elements $[\varphi] \in H^3(G, H)$ in the degree 3 group cohomology with smooth cocycles. Given a triple $(G, H, \varphi)$, the corresponding semistrict Lie 2-group $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ has the Lie group $\mathcal{G}_0 = G$ as the space of objects, the semi-direct product Lie group $\mathcal{G}_1 = G \ltimes H$ as the space of morphisms, and the associator $\mathcal{P}$ is given by the action of $\varphi$; the source and target maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are both projection onto the first factor of $G \times H$, while the cocycle condition on $\varphi$ is equivalent to the pentagon
identities (A.36). In this correspondence the abelian group $H$ is the group of automorphisms of the identity object $1$ in the monoidal category $\mathcal{G}$.

The exponential map takes an ordinary Lie algebra to its integrating simply connected Lie group, while the tangent space at the identity of an ordinary Lie group is the corresponding infinitesimal Lie algebra. In marked contrast, there are no general constructions relating Lie 2-algebras and Lie 2-groups. Integration/differentiation between strict Lie 2-algebras and strict Lie 2-groups is described in [5, 6]; a general procedure for integrating $L_\infty$-algebras is described in [30, 36]. In the semistrict cases of interest to us in this paper, given a triple $(G, H, \varphi)$ representing a special Lie 2-group $G$ (with $H$ an abelian Lie group), by differentiation we obtain a triple $(\mathfrak{g}, W, j)$ representing a 2-term $L_\infty$-algebra $V$ (with $W$ regarded as an abelian Lie algebra); in this case we call the Lie 2-group $G$ an integration of the Lie 2-algebra $V$ corresponding to $V$.

B Weights of Kontsevich diagrams

In this appendix we explain in some detail how to calculate the weights (3.5) of the diagrams that enter into Kontsevich’s formula (3.3) and present some representative examples of the computations.

The edges of a generic diagram $\Gamma$ between two vertices $p, q \in \mathbb{H}$ lie on semicircular geodesics $\ell(p, q)$ in the hyperbolic upper half-plane $\mathbb{H}$. The harmonic angle $\phi^h = \phi^h(p, q)$ is defined to be the angle between an edge $\ell(p, q)$ and the directed geodesic $\ell(p, \infty)$ at $p$; it may be integrated to provide the weight $w_\Gamma$ with which each multidifferential operator contributes to the star product (3.3). This is depicted in the following diagram:

Angles in $\mathbb{H}$ are defined in the usual manner; thus $\phi^h, \phi'^h \in [0, \pi]$ as points $p$ and $p'$ run along the semicircle from the real axis $\mathbb{R}$ (the boundary of $\mathbb{H}$) to $q$ in $\mathbb{H}$. It is important to note that the harmonic angle is measured counterclockwise. This means that $\phi^h \in [0, 2\pi]$ as we cross $q$ to integrate over $\mathbb{H}$ along the semicircle.

**Bivector diagrams.** As an example, let us calculate the weight of the wedge which corresponds to the twisted Poisson bracket $\Theta^{IJ} \partial_I f \partial_J g$; here we denote $\phi^{h_1}$ by $\theta_1$ and $\phi^{h_2}$ by $\psi_1$:
Integrating the two-form $d\theta_1 \wedge d\psi_1$ over $\mathbb{H}$, keeping in mind that $\psi_1 > \theta_1$, is straightforward and gives the weight

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} d\psi_1 \int_0^{\psi_1} d\theta_1 = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\psi_1 \psi_1 = \frac{1}{2}.$$  \hspace{1cm} (B.1)

It is important to note here that changing the order of integration produces a minus sign since $d\psi_1 \wedge d\theta_1 = -d\theta_1 \wedge d\psi_1$. This means that the topologically equivalent tractable wedge has weight equal to $-\frac{1}{2}$. A tractable diagram is one that has the derivatives assigned to its edges reversed, i.e. the tractable wedge corresponds to $\Theta^{IJ} \partial_J f \partial_I g$.

**Triple product diagrams.** Let us now calculate the weights of the following three diagrams which appear at order $\hbar^2$ when we star multiply three functions:
For the first two diagrams we have \( \psi_2 > \theta_2 \) and \( \psi_2 > \psi_1 > \theta_1 \), and thus we get the weights

\[
 w_1 = \frac{1}{(2\pi)^4} \int_0^{2\pi} d\psi_1 \int_0^{\psi_1} d\theta_1 \int_0^{2\pi} d\psi_2 \int_0^{\psi_2} d\theta_2 = \frac{1}{8} \tag{B.2}
\]

for the first diagram and

\[
 w_2 = \frac{1}{(2\pi)^4} \int_0^{2\pi} d\psi_1 \int_0^{\psi_1} d\theta_1 \int_0^{2\pi} d\psi_2 \int_0^{\psi_2} d\theta_2 = -\frac{1}{8} \tag{B.3}
\]

for the second diagram (which is tractable). The third diagram has \( \psi_2 > \theta_2 > \theta_1 \) and \( \psi_2 > \psi_1 > \theta_1 \) which gives the weight

\[
 w_3 = \frac{1}{(2\pi)^4} \int_0^{2\pi} d\psi_1 \int_0^{\psi_1} d\theta_1 \int_0^{2\pi} d\psi_2 \int_0^{\psi_2} d\theta_2 = \frac{1}{12} \tag{B.4}
\]

**Trivector diagrams.** Finally, we calculate the weight of the following trivector diagram that enters the associator \( \Phi(\Pi) \):

Here \( \psi > \theta > \phi \) and the formula (3.5) for the diagram weight gives [58]

\[
 w = \frac{1}{(2\pi)^3} \int_{H_3} d\phi \wedge d\theta \wedge d\psi \ H(\psi - \theta) \ H(\theta - \phi) \ H(\psi - \phi) \\
 = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\psi \int_0^{\psi} d\theta \int_0^{\theta} d\phi = \frac{1}{6} , \tag{B.5}
\]

where \( H \) denotes the Heaviside step function.

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