Some Open Problems in Optimal AdaBoost and Decision Stumps

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Abstract

The significance of the study of the theoretical and practical properties of AdaBoost is unquestionable, given its simplicity, wide practical use, and effectiveness on real-world datasets. Here we present a few open problems regarding the behavior of “Optimal AdaBoost,” a term coined by Rudin, Daubechies, and Schapire in 2004 to label the simple version of the standard AdaBoost algorithm in which the weak learner that AdaBoost uses always outputs the weak classifier with lowest weighted error among the respective hypothesis class of weak classifiers implicit in the weak learner. We concentrate on the standard, “vanilla” version of Optimal AdaBoost for binary classification that results from using an exponential-loss upper bound on the misclassification training error. We present two types of open problems. One deals with general weak hypotheses. The other deals with the particular case of decision stumps, as often and commonly used in practice. Answers to the open problems can have immediate significant impact to (1) cementing previously established results on asymptotic convergence properties of Optimal AdaBoost, for finite datasets, which in turn can be the start to any convergence-rate analysis; (2) understanding the weak-hypotheses class of effective decision stumps generated from data, which we have empirically observed to be significantly smaller than the typically obtained class, as well as the effect on the weak learner’s running time and previously established improved bounds on the generalization performance of Optimal AdaBoost classifiers; and (3) shedding some light on the “self control” that AdaBoost tends to exhibit in practice.
1 Introduction

Due to space constraints, we concentrate on stating the open problems and conjectures without entering into the details. We refer the reader to our manuscript on the convergence properties of Optimal AdaBoost for additional details [Belanich and Ortiz 2012], recently updated for presentation and clarification purposes. We also refer the reader to that manuscript for further discussion of the important implications, briefly listed in the Abstract, that answers to the open problems and conjectures stated here would have.

We note that our main interest is not highly synthetic, “low-dimensional” examples that contradict the conjectures unless, of course, such examples are the simple start of more sophisticated constructions of non-trivial and realistic counterexamples.

Technical Preliminaries and Notation. Let $\mathcal{X}$ denote the feature space (i.e., the set of all inputs) and $\{-1,+1\}$ be the set of (binary) output labels. To simplify notation, let $D \equiv \mathcal{X} \times \{-1,+1\}$ be the set of possible input-output pairs. In typical AdaBoost, we want to learn from a given, fixed dataset of $m$ training examples $D \equiv \{(x^{(1)},y^{(1)}),(x^{(2)},y^{(2)}),\ldots,(x^{(m)},y^{(m)})\}$, where each input-output pair $(x^{(l)},y^{(l)}) \in D$, for all examples $l = 1,\ldots,m$. We make the standard assumption that each example in $D$ comes i.i.d. from a probability space $(\mathcal{D},\Sigma,P)$, where $\mathcal{D}$ is the outcome space, $\Sigma$ is the (sigma-algebra) set of possible events with respect to $\mathcal{D}$ (i.e., subsets of $\mathcal{D}$), and $P : \Sigma \rightarrow \mathbb{R}$ is the probability measure.

We denote the set of hypotheses that the weak learner that AdaBoost uses, or simply the weak-hypothesis class, by $\mathcal{H}$. We say that $\mathcal{H}$ is AdaBoost-natural with respect to $D$ if (1) the hypothesis $h$, such that, for all $x \in \mathcal{X}$, $h(x) = 1$, is in $\mathcal{H}$; (2) if $h \in \mathcal{H}$, then $-h \in \mathcal{H}$; and (3) for all $h \in \mathcal{H}$, there exists an $(x,y) \in D$ such that $h(x) \neq y$.

In our work, we study Optimal AdaBoost as a dynamical system of the weights $w$ over the examples in $D$; we also refer to such $w$ as the example or sample weights, in a way similar to previous work [Rudin et al. 2004]. In particular, we take a dynamical system view to the Optimal AdaBoost update rule of the example weights $w$ on $D$. That is, each $w$ is a probability distribution over the $m$ examples in $D$. The set of all such $w$’s, denoted by $\Delta_m$, corresponds to the state space of the AdaBoost-induced dynamical system. Denote by $(w_1,w_2,\ldots)$ the infinite sequence of examples weights that AdaBoost would generate if it were run infinitely (i.e., the total number of rounds $T \rightarrow \infty$). We deviate slightly in the initialization of $w_1$, which is often uniform over the set of examples: i.e., $w_1(l) = \frac{1}{m}$ for $l = 1,\ldots,m$. Instead, we let $w_1 \sim \text{Uniform}(\Delta_m)$.

We also assume that the weak learner has a deterministic tie-breaking rule; i.e., denoting $\text{err}(h;w,D) \equiv \sum_{l=1}^{m} w(l) \mathbb{I}[h(x^{(l)}) \neq y^{(l)}]$ and $\mathcal{H}(w,D) \equiv \arg\min_{h \in \mathcal{H}} \text{err}(h;w,D)$, for every example weight $w$ that AdaBoost could generate, the weak learner always outputs the same weak hypothesis $h^* \in \mathcal{H}(w,D)$. We call $h^*$ the (weak-learner’s) representative hypothesis of the set $\mathcal{H}(w,D)$. In addition, we assume that if the set $\mathcal{H}(w,D) = \mathcal{H}(w',D)$ for any other $w' \neq w$, then the representative hypothesis of $\mathcal{H}(w',D)$ is also $h^*$.
2 The No-Ties Conjecture

The following conjecture essentially states that Optimal AdaBoost eventually has no ties in the selection of the best weak-classifier at each round. We use this no-ties condition to establish the convergence of the AdaBoost classifier, its generalization error, and in fact, the time/per-round average of any $L_1$ measurable function of the $w_t$’s generated by Optimal AdaBoost, which include the output classifier’s margin, the example margins, as well as the weighted error $\epsilon_t$’s and the weak-hypothesis weight $\alpha_t$’s of the selected hypothesis $h_t$’s at each round $t$.

We denote by $\mu$ both the Borel and the countable measure, as appropriate and clear from context. In the statement below, we assume that the characterization of the set of all $\mu$-probability spaces, and all $\mu$-measurable spaces over $H$, each depend on their own different set of parameters with Borel or counting measurable spaces, as appropriate for the corresponding $\sigma$-algebras.

Conjecture 1 (No-Ties Conjecture) For $\mu$-almost all probability spaces $(D, \Sigma, P)$ and any dataset $D \sim P$, and $\mu$-almost all $H$ that are AdaBoost-natural with respect to $D$, there exist $m', T' > 0$, such that if $m > m'$ is the size of $D$, then, $P$-almost surely, either (1), for all $t > T'$ rounds of Optimal AdaBoost, either (1.a) $|H(w_t, D)| = 1$; or (1.b) for all pairs $h_t, h_t' \in H(w_t, D)$, $h_t(x^{(l)}) = h_t'(x^{(l)})$ for all $l = 1, \ldots, m$; or (2) $\lim_{t \to \infty} \sum_{t=1}^{m} w_t(l) 1[h_t(x^{(l)}) \neq h_t'(x^{(l)})] = 0$, where $h_t, h_t' \in H(w_t, D)$.

Conjecture 2 (Measure-Zero-Decision-Boundary Conjecture) For $\mu$-almost all probability spaces $(D, \Sigma, P)$ and any dataset $D \sim P$, and $\mu$-almost all $H$ that are AdaBoost-natural with respect to $D$, there exist $m', T' > 0$, such that if $m > m'$, and $T > T'$ is the total number of rounds of Optimal AdaBoost, then the decision boundary of the binary classifier that Optimal AdaBoost outputs after $T$ rounds when given dataset $D$ as input has $P$-measure zero.

In our work we employ tools from ergodic theory to establish our convergence results. We provide a non-constructive proof of the existence of a measure for which the Optimal-AdaBoost update is measure-preserving.

Conjecture 3 (Constructive-Proof Conjecture) For $\mu$-almost all probability spaces $(D, \Sigma, P)$ and any dataset $D \sim P$, and $\mu$-almost all $H$ that are AdaBoost-natural with respect to $D$, there exists $m' > 0$, such that if $m > m'$, then there exists a constructive proof of existence of a measure for which the Optimal AdaBoost weight update is measure-preserving, $P$-almost surely.

3 AdaBoosting Decision Stumps

For simplicity, we concentrate on the feature space $X = [0, 1]^n$, the $n$-dimensional hypercube, so that $D = [0, 1]^n \times \{-1, +1\}$. Denote by $H$ the finite set of decision stumps on the finite dataset $D$ induced by using the so-called midpoint rule. In this rule, we project $D$ along each feature dimension $i$ and create a decision stump $h$ based on the midpoint between any pair of distinct consecutive examples $x_i^{(l)} < x_i^{(l+1)}$.
with different labels \(y^{(l_j)} \neq y^{(l_{j+1})}\), such that, denoting the corresponding midpoint by \(x'_i = \frac{x^{(l_j)}_i + x^{(l_{j+1})}_i}{2}\), we can define the decision stump as \(h(x) = \text{sign}(x_i - x'_i)\).

We eliminate from \(\hat{H}\) any dominated hypothesis; that is, we do not need to consider any \(h \in \hat{H}\), such that there exists another \(h' \in \hat{H}\), with the property that \(h'(x^{(l)}) \neq y^{(l)} \Rightarrow h(x^{(l)}) \neq y^{(l)}\) for all \(l = 1, \ldots, m\). Denote the resulting effective set by \(\hat{E}\).

Further, denote by \(\hat{U}_T \equiv \bigcup_{t=1}^{T} \{h_t\}\) the set of unique decision stumps actually selected by Optimal AdaBoost from \(\hat{E}\) after \(T\) rounds.

**Problem 1 (Bounding the Number of Effective Stumps)** Given a measurable space \((D, \Sigma, P)\), a dataset \(D \sim P\), and the number of rounds of Optimal AdaBoost \(T\). Provide non-trivial upper and lower bounds on \(|\hat{H}|\), \(|\hat{E}|\), and \(|\hat{U}_T|\).

Trivial upper and lower bounds are \(1 < |\hat{U}_T| \leq |\hat{E}| \leq |\hat{H}| \leq 2(n(m - 1) + 1)\).

**Conjecture 4 (Logarithmic Growth on Unique Stumps)** For \(\mu\)-almost all probability spaces \((D, \Sigma, P)\) and any dataset \(D \sim P\), there exist \(m', T' > 0\), such that if \(m > m'\) and \(T > T'\), we have \(\mathbb{E} \left[ |\hat{U}_T| \right] \leq (\log T + 1)^c\), for some \(c \in [1, 2)\), \(P\)-almost surely.

**References**

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