THE SECOND VANISHING THEOREM FOR LOCAL COHOMOLOGY MODULES

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Abstract. We prove the Second Vanishing Theorem for local cohomology modules of an unramified regular local ring in its full generality. As an application of our vanishing theorem for unramified regular local rings, we extend our topological characterization of the highest Lyubeznik number of an equal-characteristic local ring to the setting of mixed characteristic. Some observations and open questions are also presented along the way.

1. Introduction

The study of vanishing of local cohomology modules has a long and rich history. In [Har67, p. 79], Grothendieck stated the following problem.

Problem 1.1 (Grothendieck). Let $R$ be a commutative noetherian local ring, $a$ be an ideal of $R$, and $n$ be an integer. Find conditions under which $H^i_a(M) = 0$ for all $i > n$ and all $R$-modules $M$.

Grothendieck proved that $H^i_a(M) = 0$ for all $i > \dim(R)$ and all $R$-modules $M$ ([Har67]), which solved Problem 1.1 for $n = \dim(R)$. A solution to Problem 1.1 in the case when $n = \dim(R) - 1$ was found in [Har68, Theorem 3.1] and has been referred as the Hartshorne-Lichtenbaum Vanishing Theorem. To explain a solution to Problem 1.1 in the case when $n = \dim(R) - 2$, we consider the following definition:

Definition 1.2. Let $(R, m)$ be a $d$-dimensional noetherian local ring and let $\tilde{R}$ denote the completion of the strict henselization of the completion of $R$. We say that the Second Vanishing Theorem holds for $R$ if, for each ideal $a$ in $R$, the following conditions are equivalent:

1. $H^j_a(M) = 0$ for all $j > d - 2$ and all $R$-modules $M$;
2. $\dim(R/a) \geq 2$ and the punctured spectrum of $\tilde{R}/a\tilde{R}$ is connected.

If $R$ is not regular, then the Second Vanishing Theorem may not hold for $R$, cf. [HZ18, 7.7]. When $R$ is regular, some positive results are known. When $R$ is a polynomial ring over a field and $a$ is a homogeneous ideal, then the Second Vanishing Theorem holds, as proved by Hartshorne ([Har68, 7.5]) who also coined the name ‘Second Vanishing Theorem’ and proposed the following problem in [Har68, p. 445]:

Problem 1.3 (Hartshorne). Prove that the Second Vanishing Theorem holds for all regular local rings.

It is clear that a solution to Problem 1.3 produces a solution to Grothendieck’s original Problem 1.1 for regular local rings when $n = \dim(R) - 2$. Subsequently, Ogus ([Ogu73, Corollary 2.11]) proved that the Second Vanishing Theorem holds for regular local rings of equal-characteristic 0 and Peskine-Szpiro showed in [PS73, III 5.5] that the Second Vanishing Theorem holds for regular local rings of equal-characteristic $p$. [HL90] provided a unified proof that the Second Vanishing Theorem holds for regular local rings of equal-characteristic. Extending the Second Vanishing

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Theorem to rings that do not contain a field has been a major open problem in the study of local cohomology. In §2 we resolve Problem 1.3 for unramified regular local rings of mixed characteristic as follows:

**Theorem 1.4.** If $R$ is an unramified regular local ring of mixed characteristic, then the Second Vanishing Theorem holds for $R$.

A special case of Theorem 1.4 when $\dim(R/\mathfrak{a}) \geq 3$ and $R/\mathfrak{a}$ is equidimensional, can be found in [HunnBPW18].

In prime characteristic $p > 0$, we produce a new proof of the Second Vanishing Theorem using the action of Frobenius and some equivalent formulations of the Second Vanishing Theorem in characteristic $p > 0$, which can be found in §4.

As an application of Theorem 1.4 we extend our results in [Zha07] to local rings of mixed characteristic. Before stating our extension, we recall the definition of the Hochster-Huneke graph of a local ring. Let $A$ be a noetherian local ring. The Hochster-Huneke graph $\Gamma_A$ of $A$ is defined as follows. Its vertices are the top-dimensional minimal prime ideals of $A$, and two distinct vertices $P$ and $Q$ are joined by an edge if and only if $h_P(P + Q) = 1$.

The main result in §5 is the following:

**Theorem 1.5.** Let $(A, m, k)$ be a $d$-dimensional noetherian local ring. Assume that $A = R/I$ where $R$ is an $n$-dimensional unramified regular local ring $(R, m)$ of mixed characteristic $(0, p)$. Then $\dim_k(\text{Hom}_R(k, H^d_I H^{n-d}_I(R)))$ is the number of connected components of the Hochster-Huneke graph $\Gamma_{\tilde{A}}$ of $\tilde{A}$, where $\tilde{A}$ is the completion of the strict Henselization of the completion of $A$.

The paper is organized as follows: In §2 we prove Theorem 1.4. In §3 we reduce the Second Vanishing Theorem to prime ideals of dimension 2. In §4 we consider some equivalent characterizations of the Second Vanishing Theorem in prime characteristic $p$. In §5 we prove Theorem 1.5. In our last section §6 we present some observations along with some open questions.

2. **Proof of Theorem 1.4**

To prove Theorem 1.4 we need the following result from [PS73].

**Theorem 2.1** (Théorème III.5.1 in [PS73]). Let $(R, m)$ be a $d$-dimensional complete regular local ring with a separably closed residue field and let $\mathfrak{a}$ be an ideal of $R$. Assume that $\text{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}$ is connected and $\dim(R/\mathfrak{a}) \geq 2$. Then the following are equivalent.

1. $H^i_\mathfrak{a}(R)$ is artinian for all $i \geq d - 1$;
2. $H^i_\mathfrak{a}(R) = 0$ for all $i \geq d - 1$.

**Remark 2.2.** There are many characterizations of artinianness; one such characterization (cf. [HK91] Remark 1.3) asserts that, if $(A, m, k)$ is complete local ring, then an $A$-module $M$ is artinian if and only if that $\text{Supp}(M) = \{m\}$ and $M$ has finite dimensional socle, i.e. $\text{Hom}_R(k, M)$ is a finite dimensional $k$-space.

We now prove Theorem 1.4.

**Proof of Theorem 1.4** Since $H^d_\mathfrak{a}(M) \otimes_R \tilde{R} \cong H^d_\mathfrak{a}(M \otimes_R \tilde{R})$ by flat base change and $\tilde{R}$ is faithfully flat over $R$, we may assume that $R$ is complete with a separably closed residue field by replacing $R$ with $\tilde{R}$.

(1)$\Rightarrow$(2). If $\dim(R/\mathfrak{a}) \leq 1$, then either $H^d_\mathfrak{a}^{-1}(R) \neq 0$ (when $\dim(R/\mathfrak{a}) = 1$) or $H^d_\mathfrak{a}(R) \neq 0$ (when $\dim(R/\mathfrak{a}) = 0$). Hence $\dim(R/\mathfrak{a}) \geq 2$. If the punctured spectrum of $R/\mathfrak{a}$ were disconnected, then there would be two ideals $I, J$ of height at most $d - 1$ such that $I \cap J = \mathfrak{a}$ and $\sqrt{I + J} = \mathfrak{m}$. The Mayer-Vietoris sequence says

$$0 = H^d_\mathfrak{a}^{-1}(R) \to H^d_{I+J}(R) = H^d_\mathfrak{m}(R) \to H^d_I(R) \oplus H^d_J(R).$$
Since $ht(I), ht(J) \leq d - 1$, the Hartshorne-Lichtenbaum Vanishing Theorem ([Har68, Theorem 3.1]) implies $H^d_I(R) = H^d_J(R) = 0$ which would imply that $H^d_m(R) = 0$, a contradiction. Hence the punctured spectrum of $R/a$ must be connected.

(2) $\Rightarrow$ (1). It was observed in [Har68] that $H^t_I(M) = 0$ for all $j > t$ and all $R$-modules $M$ if and only if $H^t_I(R) = 0$ for all $j > t$. Hence it suffices to show $H^{d-1}_a(R) = H^d_a(R) = 0$ (since $H^2_d(R) = 0$ by Grothendieck Vanishing). Combining Theorem 2.1 and Remark 2.2, it suffices to show that both of $H^d_a(R)$ and $H^{d-1}_a(R)$ are supported only at the maximal ideal and have finite dimensional socle. It follows from the Hartshorne-Lichtenbaum vanishing theorem ([Har68, Theorem 3.1]) that $H^d_a(R) = 0$ and $\text{Supp}(H^{d-1}_a(R)) = \{m\}$. On the other hand, [Lyu00, Theorem 1] (or [NnB13, Theorem 1.2]) shows that $H^{d-1}_a(R)$ has finite dimensional socle since $R$ is an unramified complete regular local ring. This finishes the proof.

**Remark 2.3.** The analogue of Theorem 1.4 in the ramified case remains open in general.

Our approach in the proof of Theorem 1.4 provides a unified approach to the Second Vanishing Theorem: the same proof also works for equal-characteristic regular local rings. It follows from our proof of Theorem 1.4 that: let $(R, m, k)$ be a catenary noetherian local ring and $a$ be an ideal such that $\text{Spec}(R/a) \setminus \{m\}$ is connected and $\dim(R/a) \geq 2$, if $\dim_0 \text{Hom}_R(k, H^{a-1}_a(R)) < \infty$, then $H^{a-1}_a(R) = 0$.

Note that $\dim_0 \text{Hom}_R(k, H^{a-1}_a(R))$ is one of the Bass numbers of the local cohomology module $H^{a-1}_a(R)$. One ought to remark that the finiteness of Bass numbers of local cohomology modules of a ramified regular local ring of mixed characteristic was first conjectured in [Hum92] and has been a long standing open problem. This is one of the reasons we consider a reduction and some characterizations of the Second Vanishing Theorem in characteristic $p$ in the subsequent sections.

Section 3 contains some open questions which may be viewed as different approaches to the Second Vanishing Theorem in mixed characteristic.

### 3. Reduction to Dimension 2

In this section, we show that the Second Vanishing Theorem can be reduced to the case when the ideal $a$ is a prime ideal of dimension 2, i.e. $\dim(R/a) = 2$, and discuss a related approach to proving the Second Vanishing Theorem in general.

We begin with the following result which was implicitly contained in the proof of [III90, Theorem 2.9].

**Proposition 3.1.** Let $(A, m)$ be a catenary noetherian local ring and $a$ be an ideal of $A$. Let $\{p_1, \ldots, p_t\}$ be the set of minimal primes of $a$. Assume that $\text{Spec}(A/a) \setminus \{m\}$ is connected. Then there exists $p_i$ such that $\text{Spec}(A/b_i) \setminus \{m\}$ is connected where $b_i = \bigcap_{j \neq i} p_j$.

**Proof.** The proof follows the same of line of reasoning as in line 7-15 from the bottom on page 79 in [III90]. Since [III90, Theorem 2.9] only treats rings of equal-characteristic, we opt to reproduce the proof here for the sake of completeness and clarity.

When $t = 1$, there is nothing to prove. Assume that $t \geq 2$. Then $\dim(A/p_j) \geq 2$ for each $1 \leq j \leq t$ (otherwise $p_j \notin \bigcap_{i \neq j} p_i$ would be $m$-primary and hence $\text{Spec}(A/a) \setminus \{m\}$ would be disconnected). Consider the graph $\mathbb{G}$ with vertices $1, \ldots, t$ in which $i$ and $j$ are joined by an edge if $p_i + p_j$ is not $m$-primary. One can see that $\text{Spec}(A/a) \setminus \{m\}$ is connected if and only if so is this graph $\mathbb{G}$. Since $\mathbb{G}$ is a finite connected graph, it admits a spanning tree by [Bol79, Corollary 5, p. 7]. The number of edges in a tree is less than the number of vertices, it follows from [Bol79, Corollary 7, p. 8] that this tree has a vertex from which only one edge emanates. The graph $\mathbb{G}$ minus this particular vertex is connected. Assume this particular vertex is $i$. Then, $\text{Spec}(A/b_i) \setminus \{m\}$ is connected where $b_i = \bigcap_{j \neq i} p_j$. \hfill $\Box$

Our main result is the following:
Theorem 3.2. Let \((R, \mathfrak{m}, k)\) be a \(d\)-dimensional complete regular local ring with a separably closed residue field. If \(H^d_p(R) = H^{d-1}_p(R) = 0\) for all prime ideals \(p\) with height \(d - 2\), then \(H^d_a(R) = H^{d-1}_a(R) = 0\) for all ideals \(I\) such that \(\text{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}\) is connected and that \(\dim(R/\mathfrak{a}) \geq 2\).

Proof. We will use reverse induction on the height of \(\mathfrak{a}\). Since the conclusion only depends on the radical of \(\mathfrak{a}\), we may assume that \(\mathfrak{a}\) is radical. First, assume \(\text{ht}(\mathfrak{a}) = d - 2\) and write \(\mathfrak{a} = \cap_{i=1}^t \mathfrak{p}_i\). Since \(\text{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}\) is connected, each prime \(\mathfrak{p}_i\) must have height \(d - 2\). The Hartshorne-Lichtenbaum Vanishing Theorem implies that \(H^d_a(R) = 0\) We will use induction on \(t\) to show that \(H^d_a(R) = 0\). When \(t = 1\), this is precisely our assumption. Assume that \(t \geq 2\). By Proposition 3.1 and renumbering the minimal primes if necessary, we may assume that \(\text{Spec}(R/\cap_{j \geq 2} \mathfrak{p}_j) \setminus \{\mathfrak{m}\}\) is connected. We consider the exact sequence

\[
\cdots \to H^{d-1}_{\mathfrak{p}_1}(R) \oplus H^{d-1}_{\mathfrak{p}_1 \cap \cap_{i=2}^t \mathfrak{p}_i}(R) \to H^{d-1}_a(R) \to H^d_{\mathfrak{p}_1 \cap \cap_{i=2}^t \mathfrak{p}_i}(R).
\]

Since \(\text{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}\) is connected, \(\mathfrak{p}_1 + \cap_{i=2}^t \mathfrak{p}_i\) is not \(\mathfrak{m}\)-primary; consequently it follows from Hartshorne-Lichtenbaum Vanishing Theorem that \(H^d_{\mathfrak{p}_1 + \cap_{i=2}^t \mathfrak{p}_i}(R) = 0\). The induction hypothesis asserts that \(H^{d-1}_{\mathfrak{p}_1 \cap \cap_{i=2}^t \mathfrak{p}_i}(R) = 0\). Therefore, \(H^d_a(R) = 0\).

Now assume that \(\text{ht}(\mathfrak{a}) \leq d - 3\). First, we treat the case when \(\mathfrak{a}\) is a prime ideal. In this case pick \(r \in \mathfrak{m} \setminus \mathfrak{a}\), then by Faltings’ connectedness theorem \(\text{Spec}(R/(\mathfrak{a}, r)) \setminus \{\mathfrak{m}\}\) is connected since the number of generators of \((\mathfrak{r})\) in \(R/\mathfrak{a}\) is at most \(\dim(R/\mathfrak{a}) - 2\). As \(\text{ht}((\mathfrak{a}, r)) = \text{ht}(\mathfrak{a}) + 1\), induction hypotheses implies that \(H^{d-1}_{(\mathfrak{a}, r)}(R) = H^{d-1}_a(R) = 0\). Since \(\text{Supp}(H^d_a(R)) \subseteq \{\mathfrak{m}\}\), we have \(H^{d-1}_a(R) = 0\). Now the exact sequence \(H^{j-1}_{(\mathfrak{a}, r)}(R) \to H^d_{\mathfrak{a}}(R) \to H^{d-1}_a(r)\) shows that \(H^d_a(R) = H^{d-1}_a(R) = 0\). Next assume that all minimal prime ideals of \(\mathfrak{a}\) have the same height. Then induction on the number of minimal primes of \(\mathfrak{a}\) shows that \(H^d_a(R) = H^{d-1}_a(R) = 0\). Finally, in the general case, write \(\mathfrak{a} = I \cap J\) where all minimal primes of \(I\) have height \(\text{ht}(\mathfrak{a})\) and \(\text{ht}(J) \geq \text{ht}(\mathfrak{a}) + 1\). Then induction hypothesis and the Mayer-Veritoris sequence \(H^{j-1}_{I+J}(R) \to H^{j-1}_I(R) \oplus H^{j-1}_J(R) \to H^{j-1}_a(R) \to H^{j-1}_{I+J}(R)\) to prove that \(H^d_a(R) = H^{d-1}_a(R) = 0\). \(\square\)

4. Second Vanishing Theorem in prime characteristic \(p\), Revisited

In this section we consider the Second Vanishing Theorem in characteristic \(p\) from a different perspective, including some equivalent characterizations.

We recall the following result due to Lyubeznik (Theorem 1.1 in [Lyu06b]).

Theorem 4.1. Let \((R, \mathfrak{m})\) be a regular local ring of dimension \(n\) of prime characteristic \(p\) and \(A\) be a homomorphic image of \(R\). Let \(I\) be the kernel of \(R \to A\). Then \(H^i_{I^{-1}}(R) = 0\) if and only if the Frobenius action on \(H^i_{\mathfrak{m}}(A)\) is nilpotent.

Remark 4.2. In the statement of the Second Vanishing Theorem, the residue field of \(\bar{R}\) is separably closed. By gonflement [Bou06] chapitre IX], one can find a flat local extension \((\bar{R}, \bar{\mathfrak{m}}) \to (R', \mathfrak{m}')\) such that such that \(\mathfrak{m}' = \mathfrak{m}R'\) and \(R'/\mathfrak{m}'\) is an algebraic closure of \(R/\bar{R}\). Since \(R'\) is faithfully flat over \(\bar{R}\), one has \(H^d_{\mathfrak{a}}(\bar{R}) = 0 \Leftrightarrow H^d_{\mathfrak{a}R'}(R') = 0\). Furthermore, since \(\bar{R}/\bar{\mathfrak{m}}\) is separably closed, the punctured spectrum of \(\bar{R}/\mathfrak{a}\) is connected if and only if the same hold for \(R'/\mathfrak{a}R'\). Hence when proving the Second Vanishing Theorem, one may assume that the residue field is algebraically closed.

Proposition 4.3. Let \((A, \mathfrak{m})\) be a noetherian local ring which is a homomorphic image of a Gorenstein local ring. If \(A\) satisfies Serre’s condition \((S_i)\) for some \(i < \dim(A)\). Then \(H^i_{\mathfrak{m}}(A)\) has finite length.
Remark 4.5. The following statements are equivalent:

- For every prime ideal \( \mathfrak{p} \) in \( B \), every element in \( \mathcal{H}_\mathfrak{p}(A, B) \) is \( \mathfrak{p} \)-nilpotent.
- Every \( \mathfrak{p} \)-module is \( \mathfrak{p} \)-finite.
- For every \( \mathfrak{p} \)-module \( M \), the \( \mathfrak{p} \)-module \( \mathfrak{p}M \) is \( \mathfrak{p} \)-finite.

This finishes the proof. \( \square \)

An alternative proof of Second Vanishing Theorem in characteristic \( p \). Let \( (R, \mathfrak{m}) \) be a noetherian regular local ring of prime characteristic \( p \). Set \( n = \text{dim}(R) \) and \( I \) be an ideal of height at most \( n - 2 \). Combining Remark 4.2 and Proposition 3.2, we may assume that \( R \) is a complete noetherian regular local ring of prime characteristic \( p \) with an algebraically closed residue field and \( I \) is a height-(\( d - 2 \)) prime ideal.

The proof of the implication that the vanishing \( H_I^n(R) = H_I^{n-1}(R) = 0 \) implies the connectedness of \( \text{Spec}(R/I) \setminus \{\mathfrak{m}\} \) is the same as in the proof of Theorem 4.4. We will focus on the other implication.

Assume that \( \text{Spec}(R/I) \setminus \{\mathfrak{m}\} \) is connected and we wish to show \( H_I^n(R) = H_I^{n-1}(R) = 0 \). The vanishing \( H_I^n(R) = 0 \) follows from Hartshorne-Lichtenbaum vanishing. It remains to show \( H_I^{n-1}(R) = 0 \). According to Theorem 4.1, this is equivalent to the nilpotence of the Frobenius action on \( H_I^1(A) \).

Since \( A \) is a local integral domain and hence satisfies Serre’s condition \( (S_1) \), by Proposition 4.3, \( H_I^1(A) \) has finite length. Let \( f \) denote the Frobenius action on \( H_I^1(A) \). Since \( H_I^1(A) \) has finite length, every element in \( \mathfrak{m}H_I^1(A) \) is \( f \)-nilpotent. It follows that \( H_I^1(A) \) is \( f \)-nilpotent if and only if \( H_I^1(A) \) is \( f \)-nilpotent if and only if \( \bigcup_{i} f^i(H_I^1(A)) \) is 0.

Set \( U = \text{Spec}(R/I) \setminus \{\mathfrak{m}\} \). Then [HS77, 3.1] asserts that there is an exact sequence

\[
0 \to H_I^0(A) \to A_s \to H^0(U, \mathcal{O}_U) \to H_I^1(A) \to 0.
\]

Since \( U \) is connected and \( A \) is a domain (whose residue field is algebraically closed), the map in the middle \( A_s \to H^0(U, \mathcal{O}_U) \) is the isomorphism \( k \to k \) where \( k \) is the residue field of \( A \). Hence \( H_I^1(A) = 0 \). This proves that \( H_I^1(A) \) is \( f \)-nilpotent and hence \( H_I^{n-1}(R) = 0 \). \( \square \)

Next we will consider some equivalent formulations of the Second Vanishing Theorems in characteristic \( p \). To this end, we recall some basic facts regarding \( S_2 \)-ification from [HH94].

Remark 4.4. Let \( (A, \mathfrak{m}) \) be a complete local domain with a canonical module \( \omega \), then

1. \( \text{Hom}_A(\omega, \omega) \) is a commutative complete local ring and the natural map \( A \to \text{Hom}_A(\omega, \omega) \) is an injective module-finite ring homomorphism;
2. \( \text{Hom}_A(\omega, \omega) \) satisfies Serre’s \( (S_2) \)-condition as both an \( a \)-module and as a ring on its own.

Theorem 4.5. The following statements are equivalent:

1. The Second Vanishing Theorem holds for regular local rings of characteristic \( p \).
2. Let \( (A, \mathfrak{m}) \) be a 2-dimensional complete local domain of prime characteristic \( p > 0 \) with a canonical module \( \omega \). Assume \( A/\mathfrak{m} \) is separably closed. For each element \( \phi \in \text{Hom}_A(\omega, \omega) \) there is an integer \( e \) such that \( \phi^{\ast e} \in A \), i.e. there is an element \( a \in A \) such that \( \phi^{\ast e} \) is the multiplication by \( a \) on \( \omega \).
Let $A$ be a 2-dimensional complete local domain of characteristic $p$ with a separably closed residue field. Then there exists a positive integer $n$ such that, for all systems of parameters $x, y$, one has $\langle x : y \rangle^{[p^n]} = (x^{p^n})$ for all $n \geq n$.

**Proof.** First we prove that $(1) \Rightarrow (2)$. Assume that the Second Vanishing Theorem holds in characteristic $p$. Let $(A, m)$ be as in $(2)$. Set $S := \text{Hom}_A(\omega, \omega)$. Consider the short exact sequence $0 \to A \to S \to \mathbb{C} \to 0$. $(2)$ is equivalent to proving that the Frobenius on $C$ is nilpotent.

The short exact sequence induces a long exact sequence on local cohomology

$$0 = H^0_m(S) \to H^0_m(C) \to H^1_m(A) \to H^1_m(S) = 0$$

where $H^0_m(S) = H^1_m(S) = 0$ since $S$ satisfies $(S_2)$-condition. Write $A = R/I$ where $R$ is an $n$-dimensional complete regular local ring. By the Second Vanishing Theorem, $H^n_i(R) = 0$. By Theorem 4.1, the Frobenius on $H^0_m(A)$ must be nilpotent. Hence so is the Frobenius on $H^0_m(C)$. We claim that $H^0_m(C) = C$ and we reason as follows. It suffices to show that $C$ is supported in the maximal ideal only. Let $p$ be any non-maximal prime ideal. Since $A_p$ is Cohen-Macaulay (note that $\dim(A) = 2$), we have $S_p \cong A_p$ and hence $C_p = 0$. This shows that $C$ is supported in the maximal ideal only. It is clear that $C$ is a finite $A$-module, thus $H^0_m(C) = C$. So, the Frobenius on $C$ is nilpotent and, equivalently, $(2)$ holds.

Next, we prove $(2) \Rightarrow (1)$. Assume now $(2)$ holds, and we wish to prove that Second Vanishing Theorem. To this end, let $(R, m)$ be an $n$-dimensional prime-characteristic complete regular local ring with a separably closed residual field. By Proposition 5.1, it suffices to prove that $H^1_p(R) = 0$ for all prime ideals $P$ of height $n - 2$. Set $A = R/P$. Then $A$ satisfies hypotheses in $(2)$. By the argument in previous paragraph, we see that the Frobenius on $H^1_p(A)$ is nilpotent. According to Theorem 4.1 we have $H^1_p(R) = 0$. This completes the proof of $(2) \Rightarrow (1)$ and hence $(1) \Leftrightarrow (2)$.

Next we prove that $(1) \Leftrightarrow (3)$. We have seen that $(1)$ is equivalent to $H^1_m(A)$ being $f$-nilpotent where $(A, m)$ is a 2-dimensional complete local domain of characteristic $p$ with a separably closed residue field. Given an arbitrary system of parameters $x, y$ in $A$, each element in $H^1_m(A)$ can be written as $[\frac{a}{x}, \frac{b}{y}]$ such that $ax = by$. Since $H^1_m(A)$ is artinian, it is $f$-nilpotent if and only if there is an integer $n$ such that $f^n(H^1_m(A)) = 0$ (and consequently $f^n(H^1_m(A)) = 0$ for all $n' \geq n$). This holds if and only if $[\frac{a}{x}, \frac{b}{y}]^{p^n} = 0$ for all $a, b, x, y$ such that $ax = by$. Note that $[\frac{a}{x}, \frac{b}{y}]^{p^n} = 0$ if and only if $a^{p^n} \in (x^{p^n})$ and $ax = by$ if and only if $a \in (x : y)$. This completes the proof of $(1) \Leftrightarrow (3)$. 

5. **The highest Lyubeznik number of a local ring of mixed characteristic**

In [Lyu93] §4, Lyubeznik introduced a set of integers attached to an equal-characteristic complete local ring which have been referred to as Lyubeznik numbers ever since. Because of the connections with topology of algebraic varieties (cf. GLS98, Wal01, BB05, Zha07, Zha11, Swi15, RSW21), studying Lyubeznik numbers has become an active research area. In [Zha07], the author proved a topological characterization of the highest Lyubeznik number for all local rings of equal-characteristic. The goal of this section is to extend the main theorem in [Zha07] to mixed characteristic, using our Theorem 4.1.

Let $(A, m, k)$ be a complete local ring of mixed characteristic. By Cohen’s structure theorem, $A$ admits a surjection $\pi: R \to A$ from a complete unramified regular local ring $(R, m, k)$. Let $I$ be the kernel of $\pi$ and $n$ denote $\dim(R)$. We have the following.

**Proposition 5.1.** Let $A, R, I, n$ be as above. Then

$$\dim_k \text{Hom}_R(k, H^i_m H^{n-j}_i(R))$$

depends only on $A, i, j$, but not on the choices of $R$ or $\pi$. 

Proof. The proof follows the same line of ideas as in [Lyu93, 4.1] and [NnBW13, 3.4], hence we will provide a sketch only.

Let \( V \) be a coefficient ring of \( A \) (whose existence is guaranteed by Cohen’s structure theorem). Then one can reduce the proof to proving the following:

\[
\dim_k \text{Hom}_R(k, H^i_m H^{n-j}_I(R)) = \dim_k \text{Hom}_R(k, H^i_{(m,x)} H^{n+1-j}_{(I,x)}(R[[x]]))
\]

where \( x \) is an indeterminate over \( R \). Set \( S = R[[x]] \). For each \( R \)-module \( M \), define \( G(M) := M \otimes_R H^1_x(S) \) (\( G \) was introduced in [Lyu93, Proof of 4.3] and further studied in [NnBW14, §3]). Then by [NnBW14, 3.10], \( G(H^i_m H^{n-j}_I(R)) = H^i_{(m,x)} H^{n+1-j}_{(I,x)}(R[[x]]) \). Now (5.1.1) follows from [NnBW14, 3.12] which asserts that \( \text{Hom}_R(k, M) = \text{Hom}_S(k, G(M)) \) for all \( R \)-modules \( M \).

\( \square \)

Definition 5.2. Let \((A, m, k)\) be a noetherian local ring of mixed characteristic and let \( \hat{A} \) denote its completion. Let \( \pi : R \to \hat{A} \) be a surjection from an \( n \)-dimensional complete unramified regular local ring \((R, m, k)\) of mixed characteristic. Define

\[
\lambda_{i,j}(A) := \dim_k \text{Hom}_R(k, H^i_m H^{n-j}_I(R)).
\]

Remark 5.3. By Cohen’s Structure Theorem of complete local rings, if \( A \) is a complete local ring of mixed characteristic, then \( A \) admits a surjection \( \pi : R \to \hat{A} \) from a complete unramified regular local ring \( R \).

If \( A \) (not necessarily complete) itself admits a surjection \( R' \to A \) from an \( n' \)-dimensional complete unramified regular local ring \((R', m', k)\) with kernel \( I' \), then one can check that

\[
\lambda_{i,j}(A) = \dim_k \text{Hom}_{R'}(k, H^i_{m'} H^{n-j}_{I'}(R')).
\]

Since \( R \) is an unramified regular local ring of mixed characteristic, it follows from [Lyu00] that \( \lambda_{i,j}(A) \) are finite.

Remark 5.4. If \((A, m, k)\) is a noetherian local ring containing a field and admits presentation \( A = R/I \) where \((R, m, k)\) is an \( n \)-dimensional regular local ring containing the same field, then it follows from [Lyu93, 1.4, 4.1] that

\[
\dim_k \text{Hom}_R(k, H^i_m H^{n-j}_I(R)) = \dim_k \text{Ext}^1_R(k, H^{n-j}_I(R)).
\]

However, when \( A \) does not contain a field, (5.4.1) may no longer hold.

Let \( R = \mathbb{Z}_2[[x_0, \ldots, x_5]] \) and let \( I \) be the ideal of \( R \) generated by the 10 monomials

\[
\{x_0 x_1 x_2, x_0 x_1 x_3, x_0 x_2 x_4, x_0 x_3 x_5, x_0 x_4 x_5, x_1 x_2 x_5, x_1 x_3 x_4, x_1 x_4 x_5, x_2 x_3 x_4, x_2 x_3 x_5\}.
\]

Let \( m \) denote \((2, x_0, \ldots, x_5)\) and set \( k = R/m, A = R/I \). Then it is proved in [DSZ, 5.5] that

\[
H^4_I(R) \cong H^6_{(x_0, \ldots, x_5)}(R/(2)).
\]

Since \( H^6_{(x_0, \ldots, x_5)}(R/(2)) \) admits an injective resolution (as an \( R \)-module)

\[
0 \to H^6_{(x_0, \ldots, x_5)}(R/(2)) \to E_R(R/m) \xrightarrow{2} E_R(R/m) \to 0,
\]

one can check that

\[
\dim_k \text{Hom}_R(k, H^6_{(x_0, \ldots, x_5)}(R/(2)) = 0 \neq 1 = \dim_k \text{Ext}^1_R(k, H^4_I(R)).
\]

Theorem 1.5 is one reason why we use \( \dim_k \text{Hom}_R(k, H^6_{(x_0, \ldots, x_5)}(R/(2)) \) as our definition of \( \lambda_{i,j}(A) \).

Next, we focus on \( \lambda_{d,d}(A) \) where \( d = \dim(A) \) and prove our Theorem 1.5 which extends our results in [Zha07] to local rings of mixed characteristic.
Proof of Theorem 1.5. Since both completion and strict Henselization are faithfully flat, we may assume that both $A$ and $R$ are complete with separably closed residue fields. Assume that $\Gamma_1, \ldots, \Gamma_t$ are the connected components of $\Gamma_A$. For $1 \leq j \leq t$, let $I_j$ be the intersection of the minimal primes of $R$ that are vertices of $\Gamma_j$. Similar to the proof of [Lyu06a Proposition 2.1], using the Mayer-Vietoris sequence of local cohomology, one can prove that

$$H_{I_j}^{n-d}(R) = \oplus_{j=1}^t H_{I_j}^{n-d}(R).$$

Hence

$$\dim_k \text{Hom}_R(k, H_{m}^d H_{I_j}^{n-d}(R)) = \sum_{j=1}^t \dim_k \text{Hom}_R(k, H_{m}^d H_{I_j}^{n-d}(R)).$$

We are reduced to proving that $\dim_k \text{Hom}_R(k, H_{m}^d H_{I_j}^{n-d}(R)) = 1$ when $\Gamma_A = \Gamma_{R/I}$ is connected and $A$ is equidimensional. The rest of the proof follows the same strategy as in [Zha07].

We will use induction on $\dim(A)$. First, assume that $\dim(A) = \dim(R/I) = 2$. Since $\Gamma_A$ is connected and $A$ is equidimensional, Spec$(A) \setminus \{m\}$ is also connected. Our Theorem 1.4 now implies that $H_{I_j}^2(R) = H_{I_j}^{n-1}(R) = 0$. Hence the following is an injective resolution of $H_{I_j}^{n-2}(R)$:

$$0 \to H_{I_j}^{n-2}(R) \to \bigoplus_{I \subseteq p; \text{ht}(p)=n-2} E(R/p) \to \bigoplus_{I \subseteq q; \text{ht}(q)=n-1} E(R/q) \to E(R/m) \to 0.$$

Therefore, we have $H_{m}^2 H_{I_j}^{n-2}(R) = E(R/m)$ and $H_{m}^2 H_{I_j}^{n-2}(R) = 0$ for $j \neq 2$. This proves that case when $\dim(A) = 2$.

Assume now $\dim(A) \geq 3$. We will pick an element $r \in m$ as follows (same as in the proof of [Zha07 Theorem 1.4]). If Supp$(H_{I_j}^{n-d+1}(R)) \neq \{m\}$, then by prime avoidance we pick $r$ that is not in any minimal prime of $I$ nor in any minimal element of Supp$(H_{I_j}^{n-d+1}(R))$. If Supp$(H_{I_j}^{n-d+1}(R)) = \{m\}$, then we pick $r \in m$ that is not in any minimal prime of $I$. Then $\dim(R/I + (r)) = \dim(A) - 1$ and $R/I + (r)$ is also equidimensional. Our theorem now follows from the following statements:

1. $H_{m}^d H_{I_j}^{n-d}(R) \cong H_{m}^{d-1} H_{I_j + (r)}^{n-d+1}(R)$, and
2. $\Gamma_{R/I + (r)}$ is connected.

These two statements appeared as Proposition 2.1 and Proposition 2.2, respectively, in [Zha07]. The proofs of these two statements in [Zha07] do not require the ring to contain a field. This completes the proof our theorem.

\[\square\]

Corollary 5.5. If a $d$-dimensional noetherian local ring $A$ satisfies the Serre’s $(S_2)$ condition, then $\lambda_{d,d}(A) = 1$.

Proof. When $A$ contains a field, this is known (cf. [NnBWZ16 Theorem 4.6]). Assume $A$ doesn’t contain a field. According to our Theorem 1.5 it suffices to show that that Hochster-Huneke graph $\Gamma_A$ of $\tilde{A} = \tilde{A}^\text{sh}$ is connected. Since $A$ is $S_2$, so is $\tilde{A}$. Then [Har62 Remark 2.4.1] implies that $\tilde{A}$ is equidimensional. Therefore, $\Gamma_A$ must be connected by [HH94 Theorem 3.6].

Remark 5.6. Let $(A, m, k)$ be a complete local ring of prime characteristic $p$. Then, by Cohen Structure Theorem, one can write $A = R/I$ where $R = k[[x_1, \ldots, x_n]]$ is a formal power series ring over $k$. Denote the maximal ideal of $R$ by $n$. One may consider

$$\dim_k \text{Hom}_R(k, H_{n}^i H_{I}^{n-j}(R))$$

which agrees with the Lyubeznik number $\lambda_{i,j}(A)$ (cf. [Lyu93 1.4, 4.1]). On the other hand, $A$ can be also written as $R'/I'$ where $R'$ is a complete unramified regular local ring of mixed characteristic.
Denote the maximal ideal of \( R' \) by \( n' \) and \( \dim(R') \) by \( n' \). Following Definition 5.2 one may consider

\[
(5.6.2) \quad \dim_k \Hom_{R'}(k, H^n_{R'}(R')).
\]

A natural question is that whether (5.6.1) agrees with (5.6.2) (for fixed \( i \) and \( j \)).

When \( i = \dim(A) \) and \( j = \dim(A) \), it follows immediately from our Theorem 1.5 and the main theorem in [Zha07] that the numbers (5.6.2) and (5.6.1) coincide, both of which agree with the number of the connected components of the Hochster-Huneke graph of \( A \).

Remark 5.7. The proof of Theorem 1.5 is an example of applying our Theorem 1.4 to extend results, previously only known in equal-characteristic, to mixed characteristic. One may also apply Theorem 1.4 to extend other results (for instance, some results in [NnBSW19]) to mixed characteristic, which we will leave to another project.

6. Remarks and Questions

In this section, we observe some natural mixed characteristic extensions of the links between vanishing of local cohomology and nilpotence of Frobenius action, and raise awareness of some open questions.

6.1. Connections with Frobenius. Recall that Lyubeznik’s vanishing theorem links the vanishing of \( H^{n-i}_a(R) \) and the action of Frobenius on \( H^i_m(R/a) \) where \((R, m)\) is a regular local ring of prime characteristic \( p \) and \( a \) is an ideal of \( R \). In this section, we consider some (partial) extensions to mixed characteristic.

Theorem 6.1. Let \((R, m)\) be an \( n \)-dimensional unramified regular local ring of mixed characteristic \((0, p)\) and \( a \) be an ideal of \( R \). Assume that \( p \in a \) (hence \( R/a \) contained a field of characteristic \( p \)). If \( H^i_m(R/a) \) is Frobenius nilpotent, then

\[
H^{n-i}_a(R) = 0.
\]

Proof. Set \( \overline{R} = R/(p) \) and \( \overline{a} = a/(p) \). Then \( R/a = \overline{R}/\overline{a} \). Since \( R \) is unramified, \( \overline{R} \) is an \((n - 1)\)-dimensional regular local ring of characteristic \( p \). By our assumption on \( H^i_m(R/a) \) and Theorem 4.1, we have \( H^{(n-1)-i}_\overline{a}(\overline{R}) = 0 \). The short exact sequence \( 0 \to R \to \overline{R} \to \overline{R} \to 0 \) induces an exact sequence of local cohomology modules:

\[
0 = H^{(n-1)-i}_\overline{a}(\overline{R}) = H^{(n-1)-i}_a(R) = H^i_m(R/a) = H^{n-i}_a(R).
\]

Since \( p \in a \), this forces \( H^{n-i}_a(R) = 0 \).

We ought to remark that Theorem 6.1 only extends one implication in Theorem 4.1. As to the other implication, we ask:

Question 6.2. Does the converse to Theorem 6.1 hold?

When \( i = 1 \), the answer is affirmative and it follows from the Second Vanishing Theorems in characteristic \( p \) and in unramified mixed characteristic (i.e. our Theorem 1.4).

Theorem 6.3. Let \((R, m)\) be an \( n \)-dimensional unramified regular local ring of mixed characteristic \((0, p)\) and \( a \) be an ideal of \( R \). Assume that \( p \in a \) (hence \( R/a \) contained a field of characteristic \( p \)). Then \( H^{n-1}_a(R) = 0 \) if and only if \( H^i_m(R/a) \) is Frobenius nilpotent.

Proof. \( \Leftarrow \) is a special case of Theorem 6.1 it remains to prove \( \Rightarrow \).

Set \( \overline{R} = R/(p) \) and \( \overline{a} = a/(p) \). Then \( R/a = \overline{R}/\overline{a} \). Note that, since \( R \) is regular, \( H^0_a(R) = 0 \) by the Hartshorne-Lichtenbaum vanishing theorem. By Theorem 1.4, the punctured spectrum of \( \overline{R}a\overline{R} \) is connected (\( \overline{R} \) is the completion of the strict henselization of the completion of \( R \)) and
dim(Ra) \geq 2. Since \( p \in a \), the same will hold for \( R \) and hence \( H^p_{\pi} = H^{p-1}_{\pi} = 0 \) by the Second Vanishing Theorem in characteristic \( p \). It now follows from Theorem \ref{thm:vanishing} that \( H^1_m(R/\pi) = H^1_m(R/a) \) is Frobenius nilpotent.

Question \ref{ques:vanishing} concerns a specific cohomological degree. It turns out that one can draw a weaker conclusion if one considers all vanishing above a specific cohomological degree.

**Proposition 6.4.** Let \((R, m)\) be an \( n \)-dimensional unramified regular local ring of mixed characteristic \((0, p)\) and \( a \) be an ideal of \( R \). Assume that \( p \in a \) (hence \( R/a \) contained a field of characteristic \( p \)). If \( H^{n-j}_a(R) = 0 \) for all \( j \leq t \) for a fixed integer \( t \), then \( H^{j-1}_m(R/a) \) is Frobenius nilpotent for all \( j \leq t \).

**Proof.** Set \( \overline{R} = R/(p) \) and \( \pi = a/(p) \). Then \( R/a = \overline{R}/\pi \). Consider the long exact sequence of local cohomology induced by \( 0 \to R^p \to R \to \overline{R} \to 0 \). Since \( H^{n-j}_a(R) = 0 \) for all \( j \leq t \), one has \( H^{n-j}_a(\overline{R}) = H^{n-j}_a(\overline{R}) = 0 \) for all \( j \leq t \). Theorem \ref{thm:vanishing} shows that

\[
H^{n-j}_m(R/a) = H^{j-1}_m(\overline{R}/\pi) = H^{j-1}_m(\overline{R}/\pi)
\]

is Frobenius nilpotent for \( j \leq t \).

When \( R \) is a ramified, the situation seems to be much more mysterious since \( R/(p) \) is no longer a regular ring. We are only able to obtain a weaker version of Theorem \ref{thm:vanishing} as follows.

**Theorem 6.5.** Let \((R, m)\) be an \( n \)-dimensional ramified regular local ring of mixed characteristic \((0, p)\) and \( a \) be an ideal of \( R \). Assume that \( p \in a \) (hence \( R/a \) contained a field of characteristic \( p \)). Assume that \( H^1_m(R/a) \) is Frobenius nilpotent for \( j \leq t \) for a fixed integer \( t \), then

\[
H^{n+1-j}_a(R) = 0 \quad \text{for} \quad j \leq t.
\]

**Proof.** Without loss of generality, we may assume that \( R \) is complete. By Cohen’s Structure Theorem, \( R \cong V[[x_1, \ldots, x_n]]/(p - f) \) where \( f \in m^n \). Set \( A = (V/pV)[[x_1, \ldots, x_n]] \) (an \( n \)-dimensional regular local ring of characteristic \( p \)). We will denote the image of \( f \) in \( A \) by \( f \) again. Then

\[
R/(p) \cong A/(f).
\]

Set \( \overline{R} = R/(p) \) and \( \overline{a} = a/(p) \). We may view \( \overline{a} \) as an ideal in \( A/(f) \). Let \( b \) be the ideal in \( A \) such that \( b/(f) = \pi \). It is clear that

\[
R/a \cong A/b.
\]

And hence \( H^{j}_m(A/b) \) is Frobenius nilpotent for \( j \leq t \). Theorem \ref{thm:vanishing} asserts that \( H^{n-j}_b(A) = 0 \) for \( j \leq t \). The exact sequence of local cohomology induced by \( 0 \to A^p \to A \to A/(f) \to 0 \) shows that

\[
H^{n-j}_b(A/(f)) = 0
\]

for \( j \leq t \). (This is where we need to assume vanishing above a cohomological degree instead of vanishing at a single degree.) Consequently \( H^{n-j}_a(R/(p)) = H^{n-j}_a(R/(p)) = 0 \) for \( j \leq t \).

Consider the exact sequence of local cohomology induced by the exact sequence \( 0 \to R^p \to R \to \overline{R} \to 0: \)

\[
H^{n-j}_a(R/(p)) \to H^{n+1-j}_a(R) \to H^{n+1-j}_a(R)
\]

for \( j \leq t \). Since \( p \in a \), this forces \( H^{n+1-j}_a(R) = 0 \) for \( j \leq t \).

Recall that the local cohomological dimension of an ideal \( I \) in a noetherian ring \( A \), denoted by \( l_{\text{ld}}(I) \), is \( \max\{j \mid H^j_I(A) \neq 0\} \).
For a noetherian local ring \((A, \mathfrak{m})\) of prime characteristic \(p\), its local cohomology modules \(H^i_{\mathfrak{m}}(A)\) are equipped with an action of Frobenius \(f : H^i_{\mathfrak{m}}(A) \rightarrow H^i_{\mathfrak{m}}(A)\) induced by the Frobenius endomorphism on \(A\). The \(F\)-depth of \(A\) is defined as 

\[
F\text{-depth}(A) := \min\{j \mid H^j_{\mathfrak{m}}(A) \text{ is not nilpotent under } f\}.
\]

We are in position to prove:

**Theorem 6.6.** Let \((R, \mathfrak{m})\) be an \(n\)-dimensional regular local ring of mixed characteristic \((0, p)\) and let \(a\) be an ideal that contains \(p\). Then

1. \(\text{lcd}_R(a) \leq n - F\text{-depth}(R/a)\), when \(R\) is unramified; and
2. \(\text{lcd}_R(a) \leq n + 1 - F\text{-depth}(R/a)\), when \(R\) is ramified.

**Proof.** Since \(R\) is regular and \(p \in a\), it follows from Theorem 6.1 (unramified case), for each \(t \leq F\text{-depth}(R/a)\),

\[
H^i_a(R) = 0 \text{ for } i \geq n - t
\]

Therefore,

\[
\text{lcd}_R(a) \leq n - F\text{-depth}(R/a).
\]

The ramified case follows similarly from Theorem 6.5.

\[\square\]

### 6.2. Some Open questions.

In [HL90, Theorem 2.5], Huneke and Lyubeznik proved an ‘induction theorem’ which enables them to provide a unified proof of the Second Vanishing Theorem in equal-characteristic. Their induction theorem is a refinement of a theorem due to Faltings [Fal80, Satz 1].

Before we can recall the theorem due to Huneke-Lyubeznik, we need to reproduce some definitions. For a local ring \(A\), set

\[
\text{mdim}(A) := \min\{\dim(A/Q) \mid Q \text{ is a minimal prime of } A\},
\]

and

\[
c(I) := \text{embdim}(A) = \text{mdim}(A/I)
\]

for every ideal \(I\) of \(A\), where \(\text{embdim}(A)\) denotes the embedding dimension of \(A\).

**Theorem 6.7** (Faltings). Let \(A\) be a complete local ring containing its separably closed residue field. Let \(I\) be an ideal of \(A\) and let \(n > c(I)\) be an integer and \(M\) be a finitely generated \(A\)-module. Assume that, for every integer \(s\) with \(0 < s < c(I)\) and for every prime ideal \(p \subseteq A\) with \(\dim(A/p) > s\), \(H^q_{I, A_p}(M_p) = 0\) for all \(q \geq n - s\). Then

\[
H^q_I(M) = 0, \quad \forall q \geq n.
\]

**Theorem 6.8** (Huneke-Lyubeznik). Let \((A, \mathfrak{m})\) be a complete local ring containing its separably closed residue field. Let \(I\) be an ideal of \(A\) and let \(n > c(I)\) be an integer and \(M\) be a finitely generated \(A\)-module. Assume that, for every integer \(s\) with \(0 < s < c(I)\) and for all \(q \geq n - s\), the following hold

1. \(H^q_{I, A_p}(M_p) = 0\) for all \(p \in \text{Spec}(A)\) such that \(I \subseteq p\) and \(\dim(A/p) > s + 1\)
2. \(H^q_{I, A_p}(M_p) = 0\) for all \(p \in \text{Spec}(A)\) such that \(I \subseteq p\), \(\dim(A/p) = s + 1\), and \(p + q\) is \(\mathfrak{m}\)-primary for some minimal prime \(q\) of \(I\).

Then

\[
H^q_I(M) = 0, \quad \forall q \geq n.
\]

These two theorems have produced a family of results on local cohomological dimension, in equal-characteristic. More specifically, Theorem 6.8 implies that the Second Vanishing Theorem holds for all regular local rings of equal-characteristic.

**Question 6.9.** Do Theorems 6.7 and 6.8 hold in mixed characteristic?
A positive answer to Question 6.9 will produce a new family of results on local cohomological dimension in mixed characteristic. More specifically, extending Theorem 6.8 to local rings of mixed characteristic is another approach to proving the Second Vanishing Theorem in mixed characteristic. To the best of our knowledge, Question 6.9 is wide open.

One may also consider extending the Second Vanishing Theorem to non-regular rings. In [Lyu02], Lyubeznik asked the following.

Question 6.10 (Lyubeznik). Let \((A, \mathfrak{m})\) be a complete local domain of dimension \(d\) whose residue field is separably closed.

1. Find necessary and sufficient condition on \(I\) under which \(H^j_I(A) = 0\) for all \(j > d - 2\).
2. Let \(I\) be a prime ideal. Assume that \(I + p\) is not \(\mathfrak{m}\)-primary for every height-1 prime ideal \(p\). Is it true that \(H^j_I(A) = 0\) for all \(j > d - 2\)?

As shown in [HZ18 7.7], Question 6.10(b) has a negative answer in general as stated. However, Question 6.10(a) remains open. More specifically, we ask

Question 6.11. (1) Let \(R\) be an equal-characteristic regular local ring whose residue field is separably closed and let \(G\) be a linearly reductive group acting on \(R\). Does the Second Vanishing Theorem hold for the invariant subring \(R^G\)?
(2) Analogously, assume \(R\) is a polynomial ring over a separably closed field and let \(G\) be a linearly reductive group acting on \(R\). Does the Second Vanishing Theorem hold for homogeneous ideals in \(R^G\)?

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