A coloring of the square of the 8-cube with 13 colors

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Abstract

Let $\chi_k(n)$ be the number of colors required to color the $n$-dimensional hypercube such that no two vertices with the same color are at a distance at most $k$. In other words, $\chi_k(n)$ is the minimum number of binary codes with minimum distance at least $k+1$ required to partition the $n$-dimensional Hamming space. By giving an explicit coloring, it is shown that $\chi_2(8) = 13$.

1 Introduction

For any pair $u, v \in \{0, 1\}^n$, the Hamming distance between $u$ and $v$, denoted by $d_H(u, v)$, is the number of positions in which $u$ and $v$ differ. A binary $(n, M, d)$ code $C$ is a subset of $\{0, 1\}^n$ for which $|C| = M$ and the minimum Hamming distance between any two distinct elements of $C$ is $d$. The parameters $n$, $M$, and $d$ are called the length, the size, and the minimum distance of $C$, respectively.

The $n$-dimensional hypercube, also called the $n$-cube, denoted by $Q_n$, is the graph with vertex set $V = \{0, 1\}^n$ such that two vertices are adjacent if and only if their Hamming distance is exactly 1. Given a graph $G$, the $k$th power of $G$, denoted by $G^k$, is the graph obtained from $G$ by adding edges between all pairs of vertices that have distance at most $k$ in $G$. In particular, $G^2$ is called the square of $G$.

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A proper vertex coloring of $Q_k^n$ corresponds to a partition of $\{0, 1\}^n$ into binary codes of minimum distance at least $k + 1$. The chromatic number of $Q_k^n$ is denoted by $\chi_k(n)$. The problem of finding bounds and exact values for $\chi_k(n)$ arises from the problem of scalability of certain optical networks and has attracted wide interest in coding theory and combinatorics; see for example [1-5].

2 Determining $\chi_2(8)$

The size of a binary code of length 8 and minimum distance 3 is at most 20 [6]. Therefore, at least $\left\lceil \frac{2^8}{20} \right\rceil = 13$ colors are needed to color the square of the 8-cube. Colorings with 14 colors were first obtained by Hougardy in 1991 [3] and Royle in 1993 [7, Section 9.7], but it has been an open problem whether 13 colors suffice.

We give a partition of $\{0, 1\}^8$ into 13 codes of minimum distance at least 3 in Table 1 which shows that $\chi_2(8) = 13$. To save space, the elements of $\{0, 1\}^8$ are given as integers from 0 to 255. Twelve of the codes are $(8, 20, 3)$ codes and the remaining code is an $(8, 16, 4)$ code.

The listed coloring is one of many colorings found with a computer-aided approach. The computational techniques will be discussed in detail in a full paper. It will further be checked whether these colorings can be used as substructures to obtain colorings of the square of the 9-cube with 13 colors.

References

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Table 1: A partition of \( \{0,1\}^8 \) into 13 codes of minimum distance at least 3

| \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) | \( C_5 \) | \( C_6 \) | \( C_7 \) | \( C_8 \) | \( C_9 \) | \( C_{10} \) | \( C_{11} \) | \( C_{12} \) | \( C_{13} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 3 | 23 | 8 | 10 | 2 | 14 | 1 | 4 | 6 | 7 | 5 | 0 |
| 18 | 29 | 24 | 20 | 21 | 13 | 25 | 15 | 11 | 19 | 12 | 16 | 30 |
| 37 | 38 | 33 | 31 | 35 | 39 | 36 | 22 | 17 | 28 | 26 | 27 | 45 |
| 56 | 40 | 47 | 46 | 44 | 52 | 55 | 42 | 54 | 32 | 41 | 34 | 51 |
| 63 | 59 | 50 | 49 | 48 | 57 | 58 | 60 | 61 | 43 | 53 | 62 | 75 |
| 71 | 68 | 66 | 65 | 70 | 81 | 67 | 76 | 78 | 69 | 64 | 72 | 85 |
| 92 | 90 | 77 | 82 | 73 | 94 | 84 | 80 | 87 | 74 | 83 | 79 | 102 |
| 96 | 109 | 100 | 103 | 95 | 107 | 104 | 91 | 88 | 89 | 93 | 86 | 120 |
| 110 | 112 | 121 | 123 | 101 | 108 | 111 | 99 | 98 | 116 | 106 | 97 | 135 |
| 115 | 119 | 126 | 124 | 122 | 114 | 113 | 117 | 105 | 127 | 118 | 125 | 153 |
| 134 | 133 | 139 | 131 | 132 | 142 | 128 | 136 | 130 | 137 | 129 | 138 | 170 |
| 149 | 150 | 140 | 154 | 143 | 144 | 141 | 157 | 158 | 159 | 148 | 156 | 180 |
| 155 | 152 | 145 | 160 | 146 | 151 | 147 | 167 | 165 | 174 | 162 | 164 | 204 |
| 163 | 175 | 166 | 173 | 185 | 161 | 171 | 176 | 168 | 178 | 184 | 169 | 210 |
| 172 | 177 | 189 | 182 | 190 | 186 | 188 | 187 | 179 | 181 | 191 | 183 | 225 |
| 202 | 201 | 199 | 196 | 209 | 197 | 198 | 194 | 193 | 192 | 203 | 195 | 255 |
| 205 | 206 | 212 | 207 | 220 | 200 | 216 | 215 | 221 | 214 | 222 | 213 |  |
| 208 | 211 | 218 | 217 | 224 | 219 | 223 | 228 | 239 | 227 | 231 | 238 |  |
| 246 | 226 | 232 | 234 | 235 | 230 | 229 | 233 | 244 | 237 | 236 | 240 |  |
| 249 | 252 | 243 | 245 | 247 | 253 | 242 | 254 | 250 | 248 | 241 | 251 |  |

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