Generalized symmetries as homotopy Lie algebras

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Abstract

Homotopy Lie algebras are a generalization of differential graded Lie algebras encoding both the kinematics and dynamics of a given field theory. Focusing on kinematics, we show that these algebras provide a natural framework for the description of generalized gauge symmetries using two specific examples. The first example deals with the non-commutative gauge symmetry obtained using Drinfel’d twist of the symmetry Hopf algebra. The homotopy Lie algebra compatible with the twisted gauge symmetry turns out to be the recently proposed braided $L_\infty$-algebra. In the second example we focus on the generalized gauge symmetry of the double field theory. The symmetry includes both diffeomorphisms and gauge transformation and can consistently be defined using a curved $L_\infty$-algebra.
1 Introduction

Symmetries are indispensable in the construction of physical theories and they have proven important both in the description and for the understanding of the phenomena observed in nature. The successful implementation of the concept of gauge symmetries for the development of the Standard Model of particle physics motivated additional research focused on two important questions. The first question is related to finding a precise mathematical description of quantization and renormalization procedures in quantum field theory. The main difficulty here is the lack of rigorous formulation of renormalization of infinities inherent in the standard quantum field theory calculations. The second question is how to apply the lessons from standard model physics and quantum field theory to more general settings including gravity. The issue we are facing in these settings is a need to implement a more general notion of symmetries, like higher gauge symmetries, non-commutative gauge symmetries and dualities, to name a few. A possible systematic approach to both problems is based on the Batalin-Vilkovisky (BV) formalism developed for quantization of field theory [1, 2, 3, 4]. The formalism itself is intimately tied with homotopy Lie algebras [5, 6, 7], as shown by Zwiebach in his seminal work on closed string field theory [6].

Homotopy Lie algebra or $L_\infty$-algebra is a generalization of a differential graded Lie algebra in which the Jacobi identity holds only up to homotopy. It is defined on a graded vector space $V = \bigoplus_{d \in \mathbb{Z}} V_d$, where each vector space $V_d$ contains a physical quantity of assigned grading degree $d$. If we focus on gauge symmetry alone, we define a space of gauge parameters and a space containing gauge fields. Then we define maps on and between these spaces, satisfying a number of consistency conditions that define a gauge algebra in question. Importantly, one can extend this construction to include all data of a (classical, perturbative) field theory, by adding appropriate spaces for equations of motions, Noether identities, potential anomalies and even generators of global symmetries, see Refs.[8, 9, 10] for motivating reviews and additional references. Furthermore, building on the relation to the BV formalism, one can formulate the quantum homotopy algebra relevant for calculation of loop amplitudes [11].

In this review we discuss two examples of generalized gauge symmetries formulated in terms of homotopy Lie algebras. For both of these examples we stress what are the new insights obtained and comment on open questions and further developments. The review is focused on the relevant symmetry structures and it does not include examples of field theory realization and related quantization and renormalization issues.

The first example we shall discuss is relevant for the construction of consistent field theory with non-commutative gauge symmetries. Despite the long history of the topic, see Ref.[12] for an early review, the quantization of non-commutative gauge theories is still not fully understood. However, recent applications of homotopy (Lie) algebras for the constructions and the understanding of consistent non-commutative deformations in field theory might offer new clues.

Similarly to the BV formalism, the $L_\infty$-algebra can be used to construct consistent deformations of a given gauge-invariant, perturbative theory. This idea was applied in ”bootstrapping” non-commutative gauge-invariant theories starting from the star-deformation of the usual gauge algebra [13]. This is a powerful method for perturbative (in orders of
deformation parameter) construction of non-commutative field theories, but it is rather difficult to obtain explicit all order expressions. Thus, it has been suggested that one could use semi-classical approximation based on symplectic embeddings of almost Poisson structures, where one could obtain closed expressions \[14, 15\].

There exists another type of non-commutative deformation of gauge symmetry, based on the Drinfel’d twist of the symmetry Hopf algebra \[16\]. This approach offered well-defined geometric structure and compatible differential calculus, but the physical interpretation of the twisted symmetry remained unclear, see e.g. \[17\]. Recent attempts to understand twisted non-commutative field theories in terms of $L_\infty$-algebra structures resulted in the construction of braided $L_\infty$-algebras \[18, 19\] with appropriately adjusted BV formalism \[20\]. In Sect.2 we discuss some aspects of twisted gauge symmetry in the framework of braided $L_\infty$-algebra. We show that the $L_\infty$-algebra can be rather naturally extended to a Hopf algebra of symmetric graded tensor space \[21\]. Then we twist this extended $L_\infty$-algebra with a Drinfel’d twist, simultaneously twisting its modules. Taking the $L_\infty$-algebra as its own (Hopf) module, we obtain the braided $L_\infty$-algebra constructed in Ref.\[18\].

The second example originates from the gauge symmetry of double field theory. Double field theory (DFT) arose as a proposal of a field theory realizing the T-duality symmetry of strings \[22, 23, 24, 25\]. It is a field theory defined on a doubled $2d$-dimensional configuration space that enjoys global $O(d,d)$ symmetry. The fields of the theory, for the purpose of this review\[2\], are the $d$-dimensional bein $e$ and the 2-form Kalb-Ramond field $B$. The theory also possesses local, gauge symmetry combining standard diffeomorphisms and gauge transformations of $B$, provided that the physical fields satisfy a set of constraints known as the strong constraint. In particular, the algebra of gauge transformation defined by the C-bracket, closes only for fields and gauge parameters obeying the strong constraint. After imposing the strong constraint, the C-bracket reduces to the Courant bracket, the properties of which are captured by Courant algebroids \[27, 28, 29\].

However, in order to describe the properties of the original C-bracket of the theory one needs a more general structure. The first proposal was given in Ref.\[30\] using the geometric structure of a pre-$NQ$ manifold. This structure is defined on non-negatively graded manifolds with a degree 1 vector field which does not square to zero, with the obstruction controlled by the strong constraint. The proposal we shall review here is the DFT algebroid \[31\], which admits a cohomological vector of degree 1 on a graded manifold with general $\mathbb{Z}$-grading \[32\]. In Sect.3 we discuss the construction of the relevant $Q$-structure based on the appropriately defined curved $L_\infty$-algebra corresponding to the DFT algebroid \[32, 33\]. This construction represents the first step toward formulation of the relevant world-volume action as proposed in \[33\].

## 2 Hopf algebra & homotopy Lie algebras

An $L_\infty$-algebra can be defined in several equivalent ways, and here we follow Ref.\[34\] to define it as a $\mathbb{Z}$-graded vector space $X = \bigoplus_{d \in \mathbb{Z}} X_d$ with multilinear graded symmetric maps $b_i : X^\otimes i \to X$ of degree 1 such that the coderivation $D = \sum_{i=0} b_i$ is nilpotent. The

\[2\] See e.g. \[26\] for the geometric description of the dilaton field.
condition of $D^2 = 0$ generates the homotopy relations for the maps $b_i$ of the corresponding $L_\infty$-algebra. As an example we write the first few homotopy relations $\forall x \in X$:

\begin{align}
    b_1(b_0) &= 0 , \\
    b_2(b_0, x) + b_1^2(x) &= 0 , \\
    b_3(b_0, x_1, x_2) + b_2(b_1(x_1), x_2) + (-1)^{|x_1||x_2|} b_2(b_1(x_2), x_1) + b_1(b_2(x_1, x_2)) &= 0 .
\end{align}

When $b_0 = 0$ the $L_\infty$-algebra is called flat and $b_1$ is differential. In that case there exist cochain complex underlying the flat $L_\infty$-algebra,

\[ \cdots \xrightarrow{b_1} X_i \xrightarrow{b_1} X_{i+1} \xrightarrow{b_1} \cdots \]

and for $b_0 \neq 0$ the $L_\infty$-algebra is called curved, constant element $b_0 \in X_1$ is called curvature.

It is important to note that the structure maps $b_i$ are defined on the whole graded symmetric tensor algebra $S(X) := \bigoplus_{n=0}^\infty S^n X$ where $X$ is a $\mathbb{Z}$-graded vector space over the field $K = S^0 X$. We denote the degree of a homogeneous element $x_i \in X$ as $|x_i|$, and the graded symmetric tensor product as $\vee$. The maps $b_i : S^j X \rightarrow S^{i-j+1} X$ act on the full tensor algebra as a coderivation:

\[ b_i(x_1 \vee \cdots \vee x_j) = \sum_{\sigma \in \text{Sh}(i,j-i)} \epsilon(\sigma;x) b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \cdots \vee x_{\sigma(j)}, \quad j \geq i , \]

where $\epsilon(\sigma;x)$ is the Koszul sign,

\[ x_1 \vee \cdots \vee x_k = \epsilon(\sigma;x) x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k)}, \quad x_i \in X , \]

and $\text{Sh}(p, m - p) \in S_m$ denotes those permutations ordered as $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \cdots < \sigma(m)$. We use the conventions that $\text{Sh}(n,0) = \text{Sh}(0,n) = \text{id} \in S_n$. Introducing the permutation map $\tau^\sigma : X^{\otimes i} \rightarrow X^{\otimes i}$ where the $\tau^\sigma$ denotes the action of permutations $\sigma$ including the Koszul sign, e.g. the non-identity permutation of two elements is:

\[ \tau^\sigma(x_1 \vee x_2) = (-1)^{|x_1||x_2|} x_2 \vee x_1 , \]

we can rewrite the coderivation maps (2.2)

\[ b_i \circ \text{id}^{\vee j} = \sum_{\sigma \in \text{Sh}(i,j-i)} (b_i \vee \text{id}^{\vee(j-i)}) \circ \tau^\sigma , \quad j \geq i . \]

Using this notation, the homotopy relations, one for every $i \geq 0$, can be written in the closed form

\[ \sum_{j=0}^i \sum_{\sigma \in \text{Sh}(j,i)} b_{i-j+1}(b_j \vee \text{id}^{\vee i}) \circ \tau^\sigma = 0 . \]

The coderivation $D = \sum_{i=0}^\infty b_i$ satisfies the co-Leibniz property,

\[ \Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta , \]
with the coproduct map $\Delta : S(X) \to S(X) \otimes S(X)$

$$\Delta \circ \text{id}^\lor = \sum_{m=0}^{\infty} \sum_{p=0}^{m} (\text{id}^\lor \circ \text{id}^\lor_{m-p}) \circ \tau^\sigma, \quad p, m \geq 0,$$

defining the coalgebra structure on $S(X)$. Thus, the $L_\infty$-algebra can be defined as a coalgebra with coderivation and counit $\varepsilon : S(X) \to K$, where $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$, $x \in X$.

Moreover, the graded symmetric tensor algebra $S(X)$ has an algebra structure given by the graded symmetric tensor product $\lor$ and a unit map $\eta : K \to S(X)$, where $\eta(1) = 1$. The algebra and coalgebra structure on $S(X)$ are compatible and make up a bialgebra, which furthermore admits a graded antipode map $S$.

Thus it is possible to extend the homotopy Lie algebra defined by the coalgebra structure on the graded symmetric tensor space $S(X)$ to a cocommutative and coassociative Hopf algebra with compatible coderivation [21].

This observation was used in Ref.[21] to introduce a non-(co)commutative deformation of a Hopf/L$\infty$-algebra in the Drinfel’d twist approach. We twist a Hopf algebra $H$ using a twist element $\mathcal{F} \in H \otimes H$, which is invertible and satisfies:

$$(\mathcal{F} \otimes 1)(\Delta \otimes \text{id})\mathcal{F} = (1 \otimes \mathcal{F})(\text{id} \otimes \Delta)\mathcal{F},$$

$$(\varepsilon \otimes \text{id})\mathcal{F} = 1 \otimes 1 = (\text{id} \otimes \varepsilon)\mathcal{F}.$$

It was shown that an Abelian twist $\mathcal{F}$ of a Hopf algebra $H$ results in a new Hopf algebra where only the coproduct gets deformed [35, 36]:

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad h \in H.$$

Thus using the Drinfel’d twist we obtain a twisted $L_\infty$-algebra with deformed coalgebra sector. In the spirit of deformation quantization we simultaneously twist the Hopf algebra modules. Taking the Hopf algebra $^{3}(L_\infty, \lor, \Delta, \varepsilon, S)$ as a module itself, one obtains another Hopf algebra $(L_\infty, \lor, \Delta^*, \varepsilon, S^*)$ with:

$$x_1 \lor x_2 = \tilde{f}^\alpha(x_1) \lor \tilde{f}_\alpha(x_2),$$

$$\Delta^*(x) = x \otimes 1 + \tilde{R}^\alpha \otimes \tilde{R}_\alpha(x),$$

$$S^*(x) = -\tilde{R}^\alpha(x)\tilde{R}_\alpha.$$

Here we used Sweedler’s summation notation to write the twist element and its inverse

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \quad \mathcal{F}^{-1} = \tilde{f}^\alpha \otimes \tilde{f}_\alpha.$$

The invertible $R$-matrix $\mathcal{R} \in S(X) \otimes S(X)$ induced by the twist,

$$\mathcal{R} = f_\alpha \tilde{f}^\beta \otimes f^\alpha \tilde{f}_\beta =: R^\alpha \otimes R_\alpha, \quad \mathcal{R}^{-1} = \tilde{R}^\alpha \otimes \tilde{R}_\alpha$$

\footnote{In lieu with standard notation for Hopf algebra we shall denote this algebra as $(L_\infty, \lor, \Delta, \varepsilon, S)$, where $L_\infty$ denotes algebra as a vector space, $\lor$ and $\Delta$ are product and coproduct respectively, $\varepsilon$ is counit and $S$ antipode.}
controls the non-commutativity of the $\lor\star$-product and provides a representation of the permutation group \[17\]. In particular, the action of a non-identity permutation of two elements is:
\[
\tau_R^\sigma(x_1 \lor x_2) = (-1)^{|x_1||x_2|} \tilde{R}^\sigma(x_2) \lor \tilde{R}_\alpha(x_1)
\]
and it squares to the identity for triangular $R$-matrix. Using the braided permutation map (2.4) we can write the coproduct on the whole tensor algebra:
\[
\Delta \circ \text{id}^{\lor \star m} = \sum_{\sigma \in \text{Sh}(p,m-p)} (\text{id}^{\lor \star p} \otimes \text{id}^{\lor \star (m-p)}) \circ \tau_R^\sigma, \quad p, m \geq 0 .
\]
The coderivation $D_\star = \sum_{i=0}^\infty b_i^\star$ is defined in terms of braided graded symmetric maps $b_i^\star$:
\[
b_i^\star \circ \text{id}^{\lor \star j} = \sum_{\sigma \in \text{Sh}(i,j-i)} (b_i^\star \lor \text{id}^{\lor \star (j-i)}) \circ \tau_R^\sigma, \quad j \geq i ,
\]
\[
b_i^\star(x_1, \ldots, x_m, x_{m+1}, \ldots) = (-1)^{|x_m||x_{m+1}|} b_i^\star(x_{m+1}), \tilde{R}_\alpha(x_m), \ldots ,
\]
with the condition $D_\star^2 = 0$ reproducing the deformed homotopy relations:
\[
\sum_{j=0}^i \sum_{\sigma \in \text{Sh}(j,i)} b_{i-j+1}^\star (b_j^\star \lor \text{id}^{\lor \star i}) \circ \tau_R^\sigma = 0 .
\]
assuming that the maps $b_i^\star$ commute with the action of the twist generators. The braided coproduct (2.5) and the compatible coderivation (2.6) equivariant under the action of the degree 0 twist element reproduce, in the coalgebra picture, the braided $L_\infty$-algebra constructed in \[18\].

Going beyond just the symmetry structure, the braided gauge symmetry was successfully applied in the construction of field theories, including the braided version of general relativity \[18\], a braided version of BF theory in arbitrary dimensions and a new braided version of non-commutative Yang–Mills theory for arbitrary gauge algebras \[19\]. The quantization of the theories with non-commutative braided symmetry is being developed and some preliminary results can be found in Refs.\[20, 37\].

### 3 DFT algebroid & homotopy Lie algebras

The second example of generalized gauge symmetry we discuss originates in the string-inspired double field theory as described in the Introduction. The symmetry algebra of the theory is governed by the C-bracket, whose properties are axiomatically organized in the notion of a DFT algebroid \[31\], in much the same way as the properties of the Courant bracket are collected in the notion of a Courant algebroid. A DFT algebroid is defined as a rank $2d$ vector bundle $E$ over $2d$-dimensional manifold $\mathcal{M}$ corresponding to doubled configuration space of DFT. The skew-symmetric bracket on its sections corresponds to a C-bracket, while the twist of the bracket corresponds to background fluxes of DFT. The

\[4\] Abelian twist induces triangular $R$-matrix.
pairing on the bundle corresponds to the $O(d, d)$ metric which induces the constant $O(d, d)$ metric on the doubled configuration space. An anchor map from the sections of the bundle to the tangent bundle of the base manifold is not a homomorphism of bundles and, more interestingly, it is invertible [32]. That is why the anchor can be related to the generalized bein of DFT [38], where we packaged the field content of the theory, i.e., the d-dimensional bein and Kalb-Ramond field.

In order to define a DFT algebroid in terms of curved $L_\infty$-algebra we start with the following graded vector spaces [32, 33]:

$$X_{-2} \oplus X_{-1} \oplus X_{0} \oplus X_{1}$$

$f \in C^\infty(M)$, $e \in \Gamma(E)$, $h \in \mathcal{X}(M)$, $\eta$

Here we present what was called extended $L_\infty$-algebra in Refs. [32, 33], effectively including the base space in the definition. One should think of the basis of $X_0$ as the one induced by coordinates $x^A$ of a coordinate patch $U \subset M$ that contains point $p$ such that $x^A|_p = 0$. The vector space $X_1$ is spanned by the constant curvature element $\eta$ of degree 1 given by the induced metric on the target. As a next step we define the degree 1 maps $b_i$ such that the homotopy relations (2.3) reproduce the defining properties of the DFT algebroid. For simplicity, in the following we present just a part of the list of maps, the rest can be found in [32, 33]:

$$b_0 = \eta, \quad b_1(f) = Df, \quad b_1(e) = \rho(e), \quad b_2(\eta, f) = -\frac{1}{2}\eta^{-1}(df),$$

where $D : C^\infty(M) \to \Gamma(E)$ is the derivative defined through $\langle Df, e \rangle = \frac{1}{2}\rho(e)f$, using the $O(d, d)$ pairing on the sections of the bundle and the anchor map $\rho : E \to TM$. Applying the second homotopy relation in (2.1) to an element in $X_{-2}$, we obtain that it is satisfied provided

$$\langle (\rho \circ D)f = \frac{1}{2}\eta^{-1}(df), \tag{3.1}$$

which is one of the axioms of the DFT algebroid [32].

Once we are given the $L_\infty$ structure, we can always construct a cohomological vector $Q$, i.e., we can define the corresponding $Q$-manifold locally [39]. First we determine the structure constants of the algebra by evaluating the maps $b_i$ on the basis of the graded vector spaces $\{\tau_\alpha\}$

$$C^\alpha_{\beta_1...\beta_i}\tau_\alpha = b_i(\tau_{\beta_1}, \ldots, \tau_{\beta_i}).$$

Then we construct the cohomological vector $Q$

$$Q = \sum_{i=0}^{\infty} \frac{1}{i!}C^\beta_{\alpha_1...\alpha_i}z^{\alpha_1} \ldots z^{\alpha_i} \frac{\partial}{\partial z^\beta},$$

on the basis $\{z^\alpha\}$ of the dual graded vector space $X^*$. In the case of the DFT algebroid we locally write the $Q$-manifold as $\mathbb{R}[-1] \oplus \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}[1] \oplus \mathbb{R}^{2d}[2]$ with the coordinates\footnote{Here latin indices from the beginning of alphabet $\{A, B, \ldots\}$ denote target indices, while the ones from the middle of alphabet $\{I, J, \ldots\}$ denote bundle indices, just indicating the origin of coordinates on the graded manifold.}...
\( \{ s^{AB}, x^A, e^I, f_A \} \). Note that the coordinate \( s^{AB} \) spans a 1-dimensional vector space. While the full expression for the \( Q \) vector is given in Ref.\[33\], here we set the possible twist of the bracket to zero and anchor map to identity and obtain:

\[
Q = \eta^{AB} \frac{\partial}{\partial s^{AB}} + \left( \delta_i^A e^I - \frac{1}{2} s^{AB} f_B \right) \frac{\partial}{\partial x^A} + \frac{1}{2} \hat{\eta}^{IJS} \delta^A_J \frac{\partial}{\partial e^I},
\]

The constant \( O(d,d) \) metric on the bundle is given in the local coordinates as \( \hat{\eta}_{IJ} \),

\[
\hat{\eta}_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix}, \quad i, j = 1, \ldots, d.
\]

and similarly for the induced constant \( O(d,d) \) metric on the base denoted as \( \eta_{AB} \). In this simplified example, the only condition stemming from the nilpotency of \( Q \) is

\[
\eta^{AB} = \hat{\eta}^{IJS} \delta^A_J \delta^B_I,
\]

which is local coordinate expression of the axiom \[3.1\]. In comparison with the construction of pre-\( NQ \) manifold of Ref.\[30\], we included the induced \( O(d,d) \) metric in the DFT algebroid structure as a curvature element. Thus we obtained an additional coordinate on the \( Q \)-manifold of negative degree, \( s^{AB} \), and bona fide \( Q \)-vector that is nilpotent.

The cohomological vector \( Q \) is related to the BRST operator in the Batalin-Vilkovisky formalism where we replace \( z^\alpha \) with physical fields, and even at the level of the classical action it represents an important geometrical structure. Given a graded \( Q \)-manifold with compatible symplectic structure, known as a \( QP \)-manifold, one can define a field theory over this graded target space. In terms of \( L_\infty \)-algebra, the symplectic structure corresponds to a cyclic pairing compatible with the graded structure making the algebra cyclic \( L_\infty \)-algebra. Namely, the \( L_\infty \) symmetry algebra can always be extended to include equations of motion, but to write the corresponding action one needs additional structure in the form of compatible cyclic pairing. The relation between \( QP \)-manifolds and \( L_\infty \)-structure is particularly nice in the context of the topological field theories, where one can directly write the solution of the classical master equation using the \( QP \)-structure of the target space \[39\]. However, there are few recently analysed examples where one does not have compatible symplectic structure at hand, but can, nevertheless construct the solution of a classical master equation. These examples belong to a class of topological sigma models with Wess-Zumino term, in particular, twisted Poisson sigma model \[40\], Dirac sigma model \[41\] and more generally, twisted R-Poisson sigma models \[42, 43\].

In the context of the DFT algebroid, the relevant question is how to construct the classical field theory action functional compatible with the given symmetry structure. Using the \( L_\infty \)-algebra structure, one can construct equations of motion and propose corresponding world-volume action \[33\]. On the other hand, in the case of odd-dimensional manifolds one can use a contact structure to construct the analog of symplectic potential known as the contact form, and the hamiltonian vector field (function). Therefore, the question would be if the graded \( Q \)-manifold underlying the DFT algebroid admits graded contact structure \[44, 45, 46\] which might be pulled back on the world-volume.
4 Outlook

In this short review we argued that homotopy algebras provide useful framework for describing generalized gauge symmetries. In particular, we showed how to relate an $L_\infty$-algebra with a Hopf algebra, and used this relation to (re)derive the braided $L_\infty$-algebra structure. In short, the braided $L_\infty$-algebra is shown to be a twisted Hopf algebra module \cite{18,21}. Furthermore, we have shown that the generalized gauge symmetry of double field theory can be described using curved $L_\infty$-algebra, a local structure corresponding to the DFT algebroid. In this context it would be interesting to analyze the relevant global symmetries and their potential \textquoteleft t \textquoteleft Hooft anomalies. One question is if the infinitesimal gauge transformations could be integrated to corresponding finite ones. In the standard case of Lie algebras, it is known due to Lie's third theorem that the integrated structure corresponds to Lie groups. For Lie algebroids in general it was shown that there are obstructions to integration to higher structures like Lie groupoids, see review \cite{17} for a categorical approach. For homotopy algebroid structures or $L_\infty$-algebroid the analysis was done in \cite{18} using the graded manifold framework. From physical point of view, the global symmetries are important for understanding of the anomaly structure of the underlying theories. In particular, the investigation of global symmetries through mixed anomaly lead to identification of 2-group structure underlying some standard Abelian and non-Abelian gauge field theories in four dimensions \cite{19}. Important ingredient in this identification is the existence of a higher (1-form, in nomenclature of \cite{19}) global symmetry with conserved 2-form current. The coupling of this current to a 2-form background gauge field modifies the gauge transformation of the gauge fields by inducing a shift proportional to the field strength of the 1-form gauge field. In general, a $q$-form global symmetry is a global symmetry for which the charged operators are of space-time dimension $q$ (like Wilson lines) and the charged excitations have $q$ spatial dimensions (like membranes) \cite{50}.

Finally, we would like to end this contribution by noting that homotopy algebras can provide additional insight even in the case of standard field theories. One important aspect we haven't commented so far is related to the equivalence relations between homotopy algebras, in particular the $L_\infty$-quasi-isomorphisms. These are shown to relate equivalent field theories, thus providing an explanation for a number of phenomenologically established recursion relations and dualities relevant in calculations of scattering amplitudes, see e.g. \cite{9} for a short review. In the special case of MHV amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory the structure of (loop) amplitudes simplifies sufficiently that can be described algebraically using Hopf algebras \cite{51}. Recently, it has been observed that there exist relations between different amplitudes of that type given by the antipode map of the underlying Hopf algebra module structure \cite{52,53}. It would be interesting to understand these relations from the point of view of homotopy algebras and their morphisms.

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