Twisted Rindler space-times

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Abstract

The (linearized) noncommutative Rindler space-times associated with canonical, Lie-algebraic and quadratic twist-deformed Minkowski spaces are provided. The corresponding deformed Hawking spectra detected by Rindler observers are derived as well.
1 Introduction

One of the most interesting discovery of modern theoretical physics deals with the deep relation between horizons of black hole and thermodynamics. At the beginning of 1970s there was observed by Bekenstein (see [1]), that laws of black hole dynamics can be provided with thermodynamical interpretation, if one identifies entropy with the area of black hole horizon and temperature with its "surface gravity". This observation has been confirmed by Hawking in his seminal articles [2], [3], in which, it was predicted that a black hole should radiate with a temperature

\[ T_{\text{Black Hole}} = \frac{\hbar g}{2\pi k c} , \]

where \( g \) denotes the gravitational acceleration at the surface of the black hole, \( k \) is Boltzman’s constant, and \( c \) is the speed of light. Subsequently, it was shown separately by Davies [4] and Unruh [5], that uniformly accelerated observer in vacuum detects a radiation (a thermal field) with the same temperature as \( T_{\text{Black Hole}} \)

\[ T_{\text{Vacuum}} = \frac{\hbar a}{2\pi k c} , \]

but with inserted acceleration of the detector \( a \). Formally, such an observer "lives" in so-called Rindler space-time [6], which can be obtained by the following transformation from Minkowski space with coordinates \((x_0, x_1, x_2, x_3)\)

\[ x_0 = N(z_1) \sinh(a z_0) , \]
\[ x_1 = N(z_1) \cosh(a z_0) , \]
\[ x_2 = z_2 , \]
\[ x_3 = z_3 , \]

where \( N \) is a positive function of the coordinate. The Minkowski metric \( ds^2 = -dx_0^2 + \sum_{i=1}^{3} dx_i^2 \) transforms to

\[ ds^2 = -a N^2(z_1) dz_0^2 + (N')^2(z_1) dz_1^2 + dz_2^2 + dz_3^2 . \]

Recently, in [7], there was proposed the noncommutative counterpart of Rindler space, so-called (linearized) \( \kappa \)-Rindler space

\[ [ z_0, z_{2,3} ] = \frac{i}{a z_1 \kappa} z_{2,3} \cosh(a z_0) , \]

\[ [ z_1, z_{2,3} ] = \frac{i}{a \kappa} z_{2,3} \sinh(a z_0) , \]
\[ [ z_{2,3}, z_{2,3} ] = 0 , \]

associated with the well-known \( \kappa \)-deformed Minkowski space-time [8], [9]

\[ [ x_0, x_i ] = \frac{i}{\kappa} x_i , \]
\[ [ x_i, x_j ] = 0 ; \quad i, j = 1, 2, 3 , \]

\( ^1 c = 1. \)
equipped with the mass-like parameter \( \kappa \). Moreover, following the content of the papers \([1]-[5]\) (see also \([10], [11]\)), there has been found the leading \( 1/\kappa \) correction to the Hawking thermal spectrum, detected by noncommutative and uniformly accelerated (\( \kappa \)-Rindler) observer.

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in \([12]\). Recently, there were also found formal arguments based mainly on Quantum Gravity \([13], [14]\) and String Theory models \([15], [16]\), indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature.

Presently, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries \([17], [18]\), one can distinguish three basic types of space-time noncommutativity:

1) The canonical (soft) deformation

\[
[ x_\mu, x_\nu ] = i \theta_{\mu \nu},
\]

(11)

with constant and antisymmetric tensor \( \theta_{\mu \nu} \). The explicit form of corresponding Poincare Hopf algebra has been provided in \([19], [20]\), while its nonrelativistic limit has been proposed in \([21]\).

2) The Lie-algebraic case

\[
[ x_\mu, x_\nu ] = i \theta^\rho_{\mu \nu} x_\rho,
\]

(12)

with particularly chosen constant coefficients \( \theta^\rho_{\mu \nu} \). Particular kind of such space-time modification has been obtained as representations of \( \kappa \)-Poincare \([22], [23]\) and \( \kappa \)-Galilei \([24]\) Hopf algebras. Besides, the Lie-algebraic twist deformations of relativistic and nonrelativistic symmetries have been provided in \([25], [26]\) and \([21]\), respectively.

3) The quadratic deformation

\[
[ x_\mu, x_\nu ] = i \theta^\rho_\mu \theta^\tau_\nu x_\rho x_\tau,
\]

(13)

with constant coefficients \( \theta^\rho_\mu \theta^\tau_\nu \). Its Hopf-algebraic realization was proposed in \([27], [28]\) and \([26]\) in the case of relativistic symmetry, and in \([29]\) for its nonrelativistic counterpart.

In this article, following the scheme proposed in \([7]\), we provide the noncommutative counterparts of Rindler space-time, associated with canonical (1)), Lie-algebraic (2)), and quadratic (3)) twisted Poincare Hopf algebras, respectively. Further, we investigate the gravito-thermodynamical radiation detected by such twist-deformed Rindler observers in the vacuum, i.e. we find the thermal (Hawking) spectra for considered classes of twisted space-times. However, it should be noted, that the most detailed calculation of such a spectrum has been performed in the case of canonical deformation 1).

The paper is organized as follows. In first section we recall the basic facts concerning the twisted Poincare Hopf algebras and the corresponding quantum space-times. The

\[2 \theta_{\mu \nu} = - \theta_{\nu \mu}.\]
second section is devoted to the canonically, Lie-algebraically and quadratically twisted Rindler spaces, obtained from their noncommutative Minkowski counterparts. The deformed Hawking radiation spectra detected by twisted Rindler observers are derived in section three. The final remarks are discussed in the last section.

2 Twisted relativistic symmetries and the corresponding quantum space-times

2.1 Twisted Poincare Hopf algebras

In this subsection we recall basic facts related with the twist-deformed relativistic symmetries provided in [20], [25] and [26].

In accordance with the general twist procedure [30]-[33], the algebraic sectors for all discussed below Hopf algebra structures remain undeformed, i.e. $(\eta_{\mu\nu} = (-, +, +, +))$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i (\eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma}) ,$$

$$[M_{\mu\nu}, P_{\rho}] = i (\eta_{\nu\rho} P_{\mu} - \eta_{\mu\rho} P_{\nu}) , \ [P_{\mu}, P_{\nu}] = 0 , \ (14)$$

while the coproducts and antipodes transform as follows

$$\Delta_0(a) \rightarrow \Delta(a) = F \circ \Delta_0(a) \circ F^{-1} , \ S(a) = u. S_0(a) u^{-1} , \ (15)$$

with $\Delta_0(a) = a \otimes 1 + 1 \otimes a$, $S_0(a) = -a$ and $u = \sum f_{(1)} S_0(f_{(2)})$ (we use Sweedler’s notation $F = \sum f_{(1)} \otimes f_{(2)}$). The twist element $F \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$, present in the above formula, satisfies the classical cocycle condition

$$F_{12} \cdot (\Delta_0 \otimes 1) \ F = F_{23} \cdot (1 \otimes \Delta_0) \ F , \ (16)$$

and the normalization condition

$$(\epsilon \otimes 1) \ F = (1 \otimes \epsilon) \ F = 1 , \ (17)$$

with $F_{12} = F \otimes 1$ and $F_{23} = 1 \otimes F$.

Let us start with the first, canonically twisted Poincare Hopf algebra $\mathcal{U}_0(\mathcal{P})$ provided in [20]. Its algebraic part remains classical (see formula (14)), while the corresponding twist factor and coproducts take the forms

$$F_0 = \exp i (\theta^{\mu\nu} P_\mu \wedge P_\nu) ; \ a \wedge b = a \otimes b - b \otimes a , \ (18)$$

and

$$\Delta_{\theta^{\mu\nu}}(P_{\rho}) = \Delta_0(P_{\rho}) , \ (19)$$

$$\Delta_{\theta^{kl}}(M_{\mu\nu}) = \Delta_0(M_{\mu\nu}) - \theta^{kl}[(\eta_{k\mu} P_\nu - \eta_{k\nu} P_\mu) \otimes P_l + P_k \otimes (\eta_{l\mu} P_\nu - \eta_{l\nu} P_\mu)] + \theta^{kl}[(\eta_{k\mu} P_\nu - \eta_{k\nu} P_\mu) \otimes P_l + P_k \otimes (\eta_{l\mu} P_\nu - \eta_{l\nu} P_\mu)] . \ (20)$$

\(^3\)For their nonrelativistic counterparts see [21].

\(^4\)In this article we consider only Abelian twist deformations of Poincare Hopf structures (see [30]-[33]).
Of course, for deformation parameter $\theta_{\mu\nu}$ approaching zero we get the undeformed (classical) Poincare Hopf structure $\mathcal{U}_0(\mathcal{P})$.

The second twist-deformed Poincare Hopf algebra $\mathcal{U}_\kappa(\mathcal{P})$ has been provided in [25] and [26]. Its algebraic part remains classical while coproducts take the form

$$\Delta_\kappa(P_\mu) = \Delta_0(P_\mu) + (-i)^\gamma \sinh \left( \frac{i\gamma}{2\kappa} \zeta^\lambda P_\lambda \right) \wedge (\eta_{\mu\alpha} P_\beta - \eta_{\beta\mu} P_\alpha)$$

$$\Delta_\kappa(M_{\mu\nu}) = \Delta_0(M_{\mu\nu}) + M_{\alpha\beta} \wedge \frac{1}{2\kappa} \zeta^\lambda (\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu)$$

$$+ i [M_{\mu\nu}, M_{\alpha\beta}] \wedge (-i)^\gamma \sinh \left( \frac{i\gamma}{2\kappa} \zeta^\lambda P_\lambda \right)$$

$$+ [M_{\mu\nu}, M_{\alpha\beta}] \wedge 1 \sinh \left( \frac{i\gamma}{2\kappa} \zeta^\lambda P_\lambda \right)$$

$$+ M_{\alpha\beta}(-i)^\gamma \sinh \left( \frac{i\gamma}{2\kappa} \zeta^\lambda P_\lambda \right) \wedge \frac{1}{2\kappa} \zeta^\lambda (\psi_\lambda P_\alpha - \chi_\lambda P_\beta)$$

$$+ \frac{1}{2\kappa} \zeta^\lambda (\psi_\lambda \eta_{\alpha\alpha} P_\beta + \chi_\lambda \eta_{\beta\beta} P_\alpha) \wedge M_{\alpha\beta}(-1)^{1+\gamma} \sinh \left( \frac{i\gamma}{2\kappa} \zeta^\lambda P_\lambda \right) - 1) ,$$

where $a \perp b = a \otimes b + b \otimes a$, $\psi_\gamma = \eta_{\gamma \gamma} \eta_{ki} - \eta_{i\gamma} \eta_{kj}$, $\chi_\gamma = \eta_{j\gamma} \eta_{ki} - \eta_{i\gamma} \eta_{kj}$ and $\zeta^\lambda$ - arbitrary fourvector. The corresponding twist factor looks as follows

$$F_\kappa = \exp \frac{i}{2\kappa} (\zeta^\lambda P_\lambda \wedge M_{\alpha\beta}) ; \; \lambda \neq \alpha, \beta .$$

Obviously, for deformation parameter $\kappa$ running to infinity the above Hopf structure becomes classical.

The last twist deformation of relativistic symmetries, as we shall see below, generates the quadratic space-time noncommutativity, and is associated with the following twist factor

$$F_\xi = \exp \frac{i}{2\xi} (M_{\alpha\beta} \wedge M_{\gamma\delta}) ; \; \alpha, \beta, \gamma, \delta - \text{different and fixed} .$$

This type of deformation has been proposed in [26] but, unfortunately, due to the nontrivial technical problems, the explicit form of its coalgebraic sector has not been provided in explicit form.

### 2.2 Twisted Minkowski space-times

In this subsection we introduce the generalized relativistic space-times corresponding to the Poincare Hopf algebras provided in the pervious section. They are defined as quantum representation spaces (Hopf modules) for quantum Poincare algebras, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules [34], [35], [20].
The action of Poincare algebra on its Hopf module of functions depending on space-time coordinates $x_\mu$ is given by

$$P_\mu \triangleright f(x) = i \partial_\mu f(x), \quad M_{\mu\nu} \triangleright f(x) = i (x_\mu \partial_\nu - x_\nu \partial_\mu) f(x),$$

while the $\star$-multiplication of arbitrary two functions is defined as follows

$$f(x) \star g(x) := \omega \circ (\mathcal{F}^{-1} \triangleright f(x) \otimes g(x)).$$

In the above formula $\mathcal{F}$ denotes twist factor corresponding to the proper Poincare algebra and $\omega \circ (a \otimes b) = a \cdot b$.

Hence, we get the following twisted Minkowski space-times:

i) Canonical deformation of relativistic space

$$[x_\mu, x_\nu]_{\star_\theta} = i \theta_{\mu\nu},$$

ii) Lie-algebraically deformed Minkowski space-time

$$[x_\mu, x_\nu]_{\star_\kappa} = i C^\rho_{\mu\nu} x_\rho,$$

with coefficients

$$C^\rho_{\mu\nu} = \frac{1}{\kappa} \zeta^\mu \left( \eta_{\beta\nu} \delta^\rho_\alpha - \eta_{\alpha\nu} \delta^\rho_\beta \right) + \frac{1}{\kappa} \zeta^\nu \left( \eta_{\alpha\mu} \delta^\rho_\beta - \eta_{\beta\mu} \delta^\rho_\alpha \right),$$

and

iii) Quadratic modification of relativistic space

$$[x_\mu, x_\nu]_{\star_\xi} = i C^\rho_{\mu\nu} x_\rho x_\tau,$$

where $C^\rho_{\mu\nu} = C^\rho_{\mu\nu}(\xi)$ denotes the proper function of deformation parameter $\xi$, such that

$$[x_\mu, x_\nu]_{\star_\xi} = \text{itanh} \left( \frac{\xi}{2} \right) \left( \eta_{\alpha\mu} \eta_{\gamma\nu} \{ x_\beta, x_\delta \}_\star^\xi - \eta_{\alpha\mu} \eta_{\delta\nu} \{ x_\beta, x_\gamma \}_\star^\xi + \eta_{\beta\mu} \eta_{\gamma\nu} \{ x_\alpha, x_\delta \}_\star^\xi + \eta_{\beta\mu} \eta_{\delta\nu} \{ x_\alpha, x_\gamma \}_\star^\xi \right).$$

Of course, if parameters $\theta_{\mu\nu}$ and $\xi$ goes to zero and parameter $\kappa$ approaches infinity, the above space-times become classical Minkowski space.

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$^5[a, b]_\star := a \star b - b \star a.$

$^6\{a, b\}_\star^\xi := a \star \xi b + b \star \xi a.$
3 Twisted Rindler space-times

Let us now introduce the new objects - the twisted Rindler spaces - corresponding to the twisted Minkowski space-times described in previous section. Following the scheme proposed in the case of $\kappa$-Poincare deformation \[7\], we define such space-times as the quantum spaces with noncommutativity given by the proper $\star$-multiplications. Similarly to the twisted Minkowski space-time the new $\star$-multiplications are defined by the new $Z$-factors, which play the role of counterparts of factors (18), (23) and (24). Hence, first of all, we recall the standard transformation rules from commutative Minkowski space (described by $x_\mu$ variables) to the accelerated and commutative as well (Rindler) space-time ($z_\mu$) \[6\]

\[
\begin{align*}
x_0 &= z_1 \sinh(az_0), \\
x_1 &= z_1 \cosh(az_0), \\
x_2 &= z_2, \\
x_3 &= z_3,
\end{align*}
\] (32-35)

where $a$ denotes the acceleration parameter, i.e. we have chosen function $N(z_1) = z_1$ in formulas (3)-(6). Next, we rewrite the Minkowski twist factors (18), (23) and (24) (depending on commutative $x_\mu$ variables and defining the $\star$-multiplication (26)) in terms of $z_\mu$ variables. In such a way, we get the Rindler $Z$-factors and, consequently, we have:

i) Canonically deformed Rindler space-time.

In such a case, due to the transformation rules (32)-(35)\[7\], the wanted $\theta$-product takes the form

\[
(f(z) \star \theta g(z)) := \omega \circ (Z^{-1} \triangleright f(z) \otimes g(z)),
\] (36)

where

\[
Z^{-1}_\theta = \exp(-2i(\theta^{01} f_0(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1}) + \sum_{a=2}^{3} \theta^{0a} f_0(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge \partial_{z_a} - \theta^{23} \partial_{z_2} \wedge \partial_{z_3} + \sum_{a=2}^{3} \theta^{1a} f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1}) \wedge \partial_{z_a} \right),
\] (37)

and

\[
\begin{align*}
f_0(z_0, z_1, \partial_{z_0}, \partial_{z_1}) &= -\sinh(az_0)i\partial_{z_1} + (\cosh(az_0)/az_1)i\partial_{z_0}, \\
f_1(z_0, z_1, \partial_{z_0}, \partial_{z_1}) &= \cosh(az_0)i\partial_{z_1} - (\sinh(az_0)/az_1)i\partial_{z_0}.
\end{align*}
\] (38-39)

However, to simplify, we consider the following differential operator

\[
(Z^{-1}_\theta)_{\text{Lin}} = 1 + O_\theta(z, \partial_z),
\] (40)

7By straightforward but tedious calculations, we find $\partial_{z_0} = (-\sinh(az_0)\partial_{z_1} + (\cosh(az_0)/az_1)\partial_{z_0})$ and $\partial_{z_1} = (\cosh(az_0)\partial_{z_1} - (\sinh(az_0)/az_1)\partial_{z_0})$. 

7
which contains only the terms linear in deformation parameter $\theta^{\mu}_{\nu}$8. Hence, the linearized $^\ast$-Rindler multiplication is given by the formula (40), but with differential operator (41) instead of the complete factor $Z^{-1}_\theta$. Consequently, for $f(z) = z_\mu$ and $g(z) = z_\nu$, we get

$$ [ z_\mu, z_\nu ]_i = \theta^{\mu\sigma} \left[ (f_\rho(z, \partial_z)z_\mu)(f_\tau(z, \partial_z)z_\nu) - (f_\tau(z, \partial_z)z_\mu)(f_\rho(z, \partial_z)z_\nu) \right],$$

(41)

with $f_2(z, \partial_z) = \partial_z z$, $f_3(z, \partial_z) = \partial_z z$. The above commutation relations define the canonically twisted Rindler space-time associated with canonical Minkowski space (27).

ii) Lie-algebraically deformed Rindler space.

Here, due to the rules (32)-(35), the $^\ast_{\kappa}$-multiplication look as follows

$$ f(z) \ast_{\kappa} g(z) = \omega \circ \left( Z^{-1}_{\kappa} \circ f(z) \otimes g(z) \right),$$

(42)

where $(\delta_{ab/c} = \delta_{ab} \text{ or } \delta_{ac})$.

$$ Z^{-1}_{\kappa} = \exp \frac{i}{2\kappa} \zeta^\lambda \left( (\delta_{\lambda 2/3} z_\alpha + \delta_{\alpha 0/1} f_\lambda (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right) \wedge \left( (\delta_{\alpha 2/3} z_\alpha + \delta_{\alpha 0/1} g_\alpha (z_0, z_1)) (\delta_{2/3} z_\beta + \delta_{0/1} f_\beta (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

(43)

$$ - \left( (\delta_{2/3} z_\beta + \delta_{0/1} g_\beta (z_0, z_1)) (\delta_{\alpha 2/3} z_\alpha + \delta_{\alpha 0/1} f_\alpha (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right) \wedge \left( (\delta_{\gamma 2/3} z_\gamma + \delta_{\gamma 0/1} g_\gamma (z_0, z_1)) (\delta_{2/3} z_\delta + \delta_{0/1} f_\delta (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

$$ + \left( (\delta_{\gamma 2/3} z_\gamma + \delta_{\gamma 0/1} g_\gamma (z_0, z_1)) (\delta_{\gamma 2/3} z_\delta + \delta_{\gamma 0/1} g_\delta (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

(44)

and

$$ g_0(z_0, z_1) = z_1 \sinh(a z_0), \quad g_1(z_0, z_1) = z_1 \cosh(a z_0).$$

Consequently, the corresponding (linearized) Rindler space-time takes the form

$$ [ z_\mu, z_\nu ]_i = \frac{i}{\kappa} \zeta^\lambda \left[ (\mathcal{A}_\lambda (z, \partial_z) z_\mu) (\mathcal{A}_{\alpha \beta} (z, \partial_z) z_\nu) - (\mathcal{A}_{\alpha \beta} (z, \partial_z) z_\mu) (\mathcal{A}_\lambda (z, \partial_z) z_\nu) \right],$$

(45)

where we use the linearized approximation to $Z^{-1}_{\kappa}$ (see (40)).

iii) Quadratic deformation of Rindler space.

In such a case, the $^\ast_\xi$-multiplication takes the form

$$ f(z) \ast_\xi g(z) = \omega \circ \left( Z^{-1}_{\xi} \circ f(z) \otimes g(z) \right),$$

(46)

with factor

$$ Z^{-1}_{\xi} = \exp \frac{i}{2} \xi \left( (a_{2/3} z_\alpha + a_{0/1} g_\alpha (z_0, z_1)) (b_{2/3} z_\beta + b_{0/1} f_\beta (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

(47)

$$ - (b_{2/3} z_\beta + b_{0/1} g_\beta (z_0, z_1)) (a_{2/3} z_\alpha + a_{0/1} f_\alpha (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \wedge (b_{2/3} z_\beta + b_{0/1} g_\beta (z_0, z_1)) (a_{2/3} z_\alpha + a_{0/1} f_\alpha (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

$$ + (b_{2/3} z_\beta + b_{0/1} g_\beta (z_0, z_1)) (b_{2/3} z_\delta + b_{0/1} g_\delta (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

(48)

$$ - (b_{2/3} z_\beta + b_{0/1} g_\beta (z_0, z_1)) (b_{2/3} z_\delta + b_{0/1} g_\delta (z_0, z_1, \partial_{z_0}, \partial_{z_1})) \right)$$

(49)

$$ = \exp \frac{i}{2} \xi \left( (\mathcal{A}_{\alpha \beta} (z, \partial_z)) \wedge (\mathcal{A}_{\gamma \delta} (z, \partial_z)) \right) = \exp \mathcal{O}_\xi (z, \partial_z).$$

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8 As we shall see in the next section, we look for the corrections to Hawking radiation linear in deformation parameter.
Then, the (linearized) quadratically deformed Rindler space looks as follows
\[ [z_\mu, z_\nu]_{\hat{\xi}} = -i\xi \left( \left[ A_{\alpha\beta}(z, \partial_z) z_\mu \right] \left( A_{\gamma\delta}(z, \partial_z) z_\nu \right) - \left( A_{\gamma\delta}(z, \partial_z) z_\mu \right) \left( A_{\alpha\beta}(z, \partial_z) z_\nu \right) \right), \] (48)
with \( \hat{\xi} \)-multiplication defined by the linear approximation to (47) (see (40)).

Obviously, for both deformation parameters \( \theta^{\mu\nu} \) and \( \xi \) approaching zero, and parameter \( \kappa \) running to infinity, the above (twisted) Rindler space-times become classical.

### 4 Hawking thermal spectra for twisted Rindler space-times

In this section we find the corrections to the gravito-thermodynamical process, which occur in twisted (noncommutative) space-times. It should be noted, however, that the more detailed calculations of a proper spectrum have been performed only in the case of canonical deformation 1).

As it was mentioned in Introduction, such effects as Hawking radiation [2], can be observed in vacuum by uniformly accelerated observer [4], [5]. First of all, following [7], we recall the calculations performed for gravito-thermodynamical process in commutative relativistic space-time [10], [11]. Firstly, we consider the on-shell plane wave corresponding to the massless mode with positive frequency \( \hat{\omega} \) moving in \( x_1 = x \) direction of Minkowski space \( (x_0 = t) \)
\[ \phi(x, t) = \exp(\hat{\omega} x - \hat{\omega} t). \] (49)

In terms of Rindler variables this plane wave takes the form \( (z_0 = \tau, z_1 = z) \)
\[ \phi(x(z, \tau), t(z, \tau)) \equiv \phi(z, \tau) = \exp(i\hat{\omega} z e^{-\alpha \tau}), \] (50)
i.e. it becomes nonmonochromatic and instead has the frequency spectrum \( f(\omega) \), given by Fourier transform
\[ \phi(z, \tau) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} f(\omega) e^{-i\omega \tau}. \] (51)
The corresponding power spectrum is given by \( P(\omega) = |f(\omega)|^2 \) and the function \( f(\omega) \) can be obtained by inverse Fourier transform
\[ f(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\hat{\omega} z e^{-\alpha \tau}} e^{i\omega \tau} = \left( -\frac{1}{a} \right) (\hat{\omega} z)^{i\omega/a} \Gamma \left( -\frac{i\omega}{a} \right) e^{\pi\omega/2a}, \] (52)
where \( \Gamma(x) \) denotes the gamma function [36]. Then, since
\[ \left| \Gamma \left( \frac{i\omega}{a} \right) \right|^2 = \frac{\pi}{(\omega/a) \sinh(\pi\omega/a)}, \] (53)
we get the following power spectrum at negative frequency
\[ \omega P(-\omega) = \omega |f(-\omega)|^2 = \frac{2\pi/a}{e^{2\pi\omega/a} - 1}, \] (54)
which corresponds to the Planck factor \((e^{\hbar\omega/kT} - 1)\) associated with temperature \(T = \hbar a/2\pi k c\) (the temperature of radiation seen by Rindler observer (see formula (2))).

Let us now turn to the case of twisted (noncommutative) space-times provided in pervious section. In order to find the power spectra for such deformed Rindler spaces, we start with the (fundamental) formula (50) for scalar field, equipped with the twisted (linearized) \(\hat{*}\)-multiplications

\[
\phi_{\text{Twisted}}(z, \tau) = \exp(\hat{i}\hat{\omega}z \hat{*} e^{-a\tau}) .
\] (55)

Then

\[
f_{\text{Twisted}}(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\hat{\omega}z \hat{*} e^{-a\tau} \hat{*} e^{i\omega \tau}} ,
\] (56)

and, in accordance with the pervious considerations, we get

\[
f_{\text{Twisted}}(\omega) = f(\omega) + \int_{-\infty}^{+\infty} d\tau \omega \circ (\mathcal{O}(\tau, z, \partial_{\tau}, \partial_{z}) \triangleright e^{i\hat{\omega}z \hat{*} e^{-a\tau} \otimes e^{i\omega \tau}})
\]

\[
+ \int_{-\infty}^{+\infty} d\tau e^{i\hat{\omega}z \hat{*} e^{-a\tau} \hat{*} e^{i\omega \tau}} \circ (\mathcal{O}(\tau, z, \partial_{\tau}, \partial_{z}) \triangleright i\hat{\omega}z \otimes e^{-a\tau}) = f(\omega) + f(\omega) .
\] (57)

Consequently, the power spectrum at negative frequency takes the form (see e.g. [10], [11])

\[
\omega P_{\text{Twisted}}(-\omega) = \omega |f_{\text{Twisted}}(-\omega)|^2 = \omega P(-\omega) + \omega P(-\omega) = \frac{2\pi}{e^{2\pi\omega/a} - 1} + \omega P(-\omega) ,
\] (58)

with the corrections given by

\[
P(-\omega) = |f_{\text{Twisted}}(-\omega)|^2 + f_{\text{Twisted}}(-\omega) \cdot f_{\text{Twisted}}(-\omega) + f_{\text{Twisted}}(-\omega) \overline{f_{\text{Twisted}}(-\omega)} .
\] (59)

As an example, let us consider the most simple case of space-time noncommutativity - the canonical deformation of classical space 1). In such a case the operator \(\mathcal{O}(\tau, z, \partial_{\tau}, \partial_{z}) = \mathcal{O}_0(\tau, z, \partial_{\tau}, \partial_{z})\) is given by the formula (37) and, we get

\[
f_{\mathcal{O}_0}(\omega) = \frac{2i\theta^{01} \omega \hat{\omega}}{az} \int_{-\infty}^{+\infty} d\tau e^{i\hat{\omega}z \hat{*} e^{-a\tau} \hat{*} e^{i\omega \tau} - \frac{2\theta^{01} \hat{\omega}}{z} \int_{-\infty}^{+\infty} d\tau e^{i\hat{\omega}z \hat{*} e^{-a\tau} \hat{*} e^{i\omega \tau}} .
\] (60)

Consequently, in accordance with (57) one obtains

\[
f_{\mathcal{O}_0}^{\text{Twisted}}(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\hat{\omega}z \hat{*} e^{-a\tau} \hat{*} e^{i\omega \tau}} \left(1 + \frac{2\theta^{01} \hat{\omega}}{z} e^{-a\tau} \left[\frac{i\omega}{a} - 1\right]\right) .
\] (61)
The above integral can be evaluated with the help of standard identity for Gamma function \( \Gamma(y + 1) = y \Gamma(y) \); one gets

\[
f_{\theta}^{\text{Twisted}}(\omega) = \left( -\frac{1}{a} \right) (\omega z)^{i\omega/a} \Gamma\left( \frac{-i\omega}{a} \right) e^{\pi \omega/2a} \left( 1 + \frac{2\theta_{01}\omega}{az^2} \left[ \frac{i\omega}{a} - 1 \right] \right),
\]

Hence, it is easy to deduce the thermal power spectrum at negative frequency, which looks as follows

\[
\omega P_{\theta}^{\text{Twisted}}(-\omega) = \frac{1}{T} \frac{1}{e^{\omega/T} - 1} \left( 1 - \frac{2\theta_{01}\omega}{\pi T z^2} \right) + \mathcal{O}(\theta_{01}^2),
\]

with Hawking temperature \( T = a/2\pi \) associated with the acceleration of twisted observer.

5 Final remarks

In this article we provide three (linearized) Rindler spaces corresponding to the canonically, Lie-algebraically and quadratically twist-deformed Minkowski space-times [20], [26]. Further, we demonstrated that in the case of canonical deformation, there appear corrections to the Hawking thermal radiation which are linear in parameter \( \theta_{\mu\nu} \).

It should be noted, that the above results can be extended in different ways. First of all, the complete form of Rindler space-times can be find with the use of complete twist differential operators

\[
\mathcal{Z}^{-1} = \exp \mathcal{O}(z, \partial_z),
\]

which appear respectively in the formulas (36), (42) and (46). However, due to the complicated form of operators \( \mathcal{O}(z, \partial_z) \) such a problem seems to be quite difficult to solve from technical point of view. Besides, one can provide the quantum Rindler spaces associated with the so-called generalized Minkowski space-times

\[
[ x_\mu, x_\nu ] = i\theta_{\mu\nu} + i\theta_{\mu\rho} x_\rho,
\]

investigated recently in the series of papers [37]-[39]. The studies in these directions are already in progress.

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