COBORDISM CATEGORIES AND MODULI SPACES OF ODD DIMENSIONAL MANIFOLDS

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Abstract. We prove that the stable moduli space of \((n - 1)\)-connected, \(n\)-parallelizable, \((2n + 1)\)-dimensional manifolds is homology equivalent to an infinite loopspace for \(n \geq 4, n \neq 7\). The main novel ingredient is a version of the cobordism category incorporating surgery data in the form of Lagrangian subspaces.

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1. Introduction

1.1. History and Motivation. The study of diffeomorphisms of smooth manifolds has been a focus of differential topology from its inception. After initial geometric techniques were developed, the method of choice for attacking such automorphism groups was a combination of surgery theory and Waldhausen’s A-theory, with the former providing input on block diffeomorphisms and the latter on the difference between block and honest diffeomorphisms, see [WeWi 14] for a modern formulation. With Tillmann’s work [Ti 97] on cobordism categories of surfaces and Madsen and Weiss’ proof of the Mumford conjecture [MaWe 07], however, a new method emerged, whose application to high dimensional manifolds was pioneered by Galatius and Randal-Williams in [GaRW 10, GaRW 14, GaRW 18] culminating in [GaRW 17]. Their work focuses on manifolds of even dimension and in the simplest case proceeds roughly as follows: Write \(d\) for \(2n\) or \(2n + 1\) and set \(\theta_d^n: BO(d)(n) \to BO(d)\)
for the classifying map of the universal $d$-dimensional vector bundle on an $n$-connected space. One can then form the monoid (under boundary-connected sum)
\[ \mathcal{M}_d = \coprod_W \text{BDiff}^\partial(W), \]
where the index runs through all $(n - 1)$-connected $n$-parallelizable nullcobordisms of $S^{d-1}$; $n$-parallelizable means that the classifying map for the tangent bundle $W \to BO(d)$ admits a lift along $\theta_d^n: BO(d)/(n) \to BO(d)$, or equivalently that $W$ admits a framing on some $n$-skeleton. In three largely independent steps, Galatius and Randal-Williams now show that

i) The scanning map of Madsen and Tillmann
\[ \mathcal{M}_{2n} \to \Omega^\infty MT_2^n, \]
induces a weak equivalence
\[ \Omega B\mathcal{M}_{2n} \to \Omega^\infty MT_2^n, \]
where the right hand side is a certain Thom spectrum, whose homology is readily computable by the standard toolkit of algebraic topology.

ii) Denoting by $W_{g,1}^{2n}$ the $g$-fold connected sum of $S^n \times S^n$ with a disc removed the inclusions
\[ \text{BDiff}^\partial(W_{g,1}) \to \mathcal{M}_{2n} \]
induce an isomorphism
\[ \colim_{g \to \infty} H_*(\text{BDiff}^\partial(W_{g,1})) \to H_*(\Omega_0 B\mathcal{M}_{2n}), \]
with structure maps extending diffeomorphisms by the identity outside $W_{g,1}^{2n} \subset W_{g+1,1}^{2n}$.

iii) The structure maps
\[ \text{BDiff}^\partial(W_{g,1}) \to \text{BDiff}^\partial(W_{g+1,1}) \]
in the colimit above induce isomorphism on homology in degrees below $\frac{g-3}{2}$.

The results of Galatius and Randal-Williams hold in far greater generality, but a similarly complete description of the homology of diffeomorphism groups in the case of any odd dimensional manifolds is as of yet conjectural at best, even in the highly connected case. Homological stability (i.e. the analogue of the third part) for the diffeomorphism groups of certain odd dimensional manifolds (among them connected sums of $S^n \times S^{n+1}$) was recently established by the second author in \cite{Pe17a} and \cite{Pe18} and part ii) has a well-known (if more complicated) analogue, as we shall explain below. The first statement, however, does not yet have a meaningful replacement. Indeed, due to the vanishing of certain characteristic classes (by work of Ebert \cite{Eb09}) it was previously known that $\Omega_0 B\mathcal{M}_{2n+1}$ is not equivalent to $\Omega^\infty MT_2^n$ or any close variant. Thus it was an open question, whether the homology of stabilized diffeomorphism groups in odd dimensions is governed by a spectrum at all. The main result of the present paper answers this affirmatively:
Main Theorem. For $n \geq 4$, $n \neq 7$ the group completion $\Omega B\mathcal{M}_{2n+1}$ of $\mathcal{M}_{2n+1}$ carries an infinite loop space structure.

Let us remark that the monoid $\mathcal{M}_n$ is well-known to carry an $E_n$-structure but is not expected to carry even an $E_{n+1}$-structure before group completion (we shall, however, not make use of these structures in the remainder and therefore will say nothing further about this point). We will exhibit an explicit infinite loop space structure on $\Omega B\mathcal{M}_{2n+1}$ by modifying the definition of cobordism categories in odd dimensions, see Section 1.4 below. To explain this construction, which is probably more important than the result itself, and to place our results in context, we begin by presenting in Section 1.2 a more detailed sketch of the techniques from [GaRW 14], [GaRW 10] and [GaMaTiWe 09] used to establish the first two statements of the list above, explain in their failure in odd dimensions and our solution in 1.4.

Note that our result makes the situation for odd dimensions analogous to that in the surface case in the 1990’s, where Tillman in [Ti 97] showed the homotopy type of the group completion of $\mathcal{M}_2$ and thus the homology of

$$\colim_{g \to \infty} B\operatorname{Diff}(\Sigma_{g,1})$$

to be that of an infinite loop space, whereas the homotopy type of the underlying spectrum was only identified much later by Madsen and Weiss in [MaWe 07].

1.2. The even dimensional case. For any map $\theta : B \to BO(d)$ with $d = 2n$ or $2n + 1$, recall the cobordism category $\operatorname{Cob}_\theta$ defined in [GaMaTiWe 09]: Objects of $\operatorname{Cob}_\theta$ are given by $(d-1)$-dimensional, closed submanifolds $M \subset \mathbb{R}^\infty$, equipped with a bundle map $\ell_M : TM \oplus \epsilon^1 \to \theta^* \gamma^d$, i.e. a $\theta$-structure. A morphism between objects $M$ and $N$ is given by a $d$-dimensional embedded cobordism $W \subset [0, t] \times \mathbb{R}^\infty$, equipped with a bundle map $\ell_W : TW \to \theta^* \gamma^d$ that restricts to $\ell_M$ and $\ell_N$ over the boundary. The category $\operatorname{Cob}_\theta$ is topologized so that for each $M, N \in \operatorname{Ob} \operatorname{Cob}_\theta$ there is a weak homotopy equivalence

$$\operatorname{Cob}_\theta(M, N) \simeq \bigsqcup_W B\operatorname{Diff}_\theta(W, M \sqcup N),$$

where the union ranges over all diffeomorphism classes of compact manifolds $W$ equipped with a specified identification $\partial W \cong M \sqcup N$. The main theorem from [GaMaTiWe 09] yields a weak equivalence

$$\Omega B\operatorname{Cob}_\theta \xrightarrow{\simeq} \Omega^\infty M\Gamma \theta,$$

where $M\Gamma \theta$ is the Thom spectrum associated to the virtual vector bundle $-\theta^* \gamma^d$ over $B$.

Now for any $(n-1)$-connected nullcobordism $W$ of $S^{d-1}$ the space of extensions of a fixed $\theta^*_d$-structure on $S^{d-1}$ to $W$ is either empty or contractible, hence in the latter case

$$B\operatorname{Diff}^\theta(W) \simeq B\operatorname{Diff}^\theta_{\theta^*_d}(W \cup_D D^d, D^d).$$
Removing two disks from the fixed disk on the right hand side therefore produces a multiplicative map

\[(1.2.2) \quad \mathcal{M}_d \rightarrow \text{Cob}_{\theta_d}^n(S^{d-1}, S^{d-1}),\]

where the right-hand side is the endomorphism monoid on the standard sphere \(S^{d-1} \subset \mathbb{R}^\infty\), equipped with its essentially unique \(\theta_d^n\)-structure compatible with the orientation.

The scanning map of i) above can then be factored as

\[(1.2.3) \quad \Omega B\mathcal{M}_d \rightarrow \Omega^\infty \text{MT}\theta_d^n \rightarrow \Omega B\text{Cob}_{\theta_d}^n(S^{d-1}, S^{d-1}) \rightarrow \Omega B\text{Cob}_{\theta_d}^n.\]

The majority of the technical work in [GaRW 14] is then devoted to establishing that the composite

\[(1.2.4) \quad B\mathcal{M}_{2n} \rightarrow B\text{Cob}_{\theta_d}^n(S^{d-1}, S^{d-1}) \rightarrow B\text{Cob}_{\theta_{2n}}^n\]

is a weak equivalence onto a path component (the term in the middle needs modification to make this true for the individual maps as well). This result is achieved via a sequence of parametrized surgery arguments, making first the morphisms and then the objects ever higher connected.

Part ii) is obtained by an application of the group completion theorem of McDuff and Segal [MDSe 75]. It shows that

\[H_*(\Omega B\mathcal{M}_d) = H_*(\mathcal{M}_d)_W,\]

the localization taken with respect to \(W = \pi_0(\mathcal{M}_d) \subseteq H_0(\mathcal{M}_d)\). By a result of Kreck, the monoid \(W_{2n}\) is generated under saturation by the single element \(s = [W_{2n}^{2n}]\), which implies

\[H_*(\mathcal{M}_{2n})_{W_{2n}} = \text{colim} \left( H_*(\mathcal{M}_{2n}) \xrightarrow{s} H_*(\mathcal{M}_{2n}) \xrightarrow{s} \ldots \right).\]

Denoting by \(B^{2n}\) the path component of \(D^{2n}\) in the homotopy colimit over

\[\mathcal{M}_{2n} \xrightarrow{W_{2n}^{2n}} \mathcal{M}_{2n} \xrightarrow{W_{2n}^{2n}} \ldots\]

we obtain a homology isomorphism

\[B^{2n}_\infty \rightarrow \Omega_0 B\mathcal{M}_{2n},\]

which gives assertion ii). We will refer to the space \(B^{2n}_\infty\) as the stable moduli space of highly-connected manifolds of dimension \(2n\).

As it will play no role in the paper, let us refrain from expounding the proof of part iii).
1.3. The odd-dimensional case. It is tempting to try to carry out a similar program to study the stable moduli spaces of odd-dimensional manifolds. While it is not true that $\mathcal{W}_{2n+1}$ is obtained from a single element under saturation, the localization $H_\ast(\mathcal{M}_{2n+1})_{\mathcal{W}_{2n+1}}$ can still be computed as a colimit: Fixing a system of generators $\{s_i\}_{i \in \mathbb{N}}$ for $\mathcal{W}_{2n+1}$ (it is countable), we have

$$H_\ast(\mathcal{M}_{2n+1})_{\mathcal{W}_{2n+1}} = \text{colim} \left( H_\ast(\mathcal{M}_{2n+1}) \xrightarrow{s_1} H_\ast(\mathcal{M}_{2n+1}) \xrightarrow{s_2^2} H_\ast(\mathcal{M}_{2n+1}) \xrightarrow{s_3^3} \ldots \right).$$

Again form the stable moduli space of manifolds $\mathcal{B}_{2n+1}$ as the path component of the disk $D_{2n+1}$ in the corresponding homotopy colimit over $\mathcal{M}_{2n+1}$. In replacement of statement ii) one obtains a homology isomorphism

$$\mathcal{B}_{2n+1} \to \Omega_0 B\mathcal{M}_{2n+1}.$$ 

By homotopy commutativity of $\mathcal{M}_{2n+1}$, the homotopy type of the limiting space $\mathcal{B}_{2n+1}$ does not depend on the choice of generating set.

However, the arguments from [GaRW 14] cannot be used to prove that the map

$$B\mathcal{M}_{2n+1} \to B\text{Cob}_{2n+1}$$

is the inclusion of a path component, as the techniques employed by Galatius and Randal-Williams meet a surgery obstruction when applied in the middle dimension of even dimensional manifolds (in the step making the objects of $\text{Cob}_{2n+1}$ into homotopy spheres). This failure is not a mere technicality since as mentioned above it had previously been established by Ebert in [Eb 09] that for an odd dimensional manifold the scanning map is never injective in rational cohomology, not even in a range. Thus, in order to complete the picture in odd dimensions, replacements for the cobordism category $\text{Cob}_\theta$ and the spectrum $MT\theta$ are needed.

1.4. Statement of results. Let $n \neq 1, 3, 7$; this restriction implies the existence of the quadratic form $\mu$ used below. The following category is the main object of study:

**Definition.** The topological category $\text{Cob}_{2n+1}^\mathcal{L}$ has as its objects pairs $(M, L)$ that satisfy the following conditions:

(i) $M$ is an object of $\text{Cob}_{2n+1}^{\theta}$, i.e. $M \subset \mathbb{R}^\infty$ is a $2n$-dimensional closed submanifold equipped with a $\theta_{2n+1}^\theta$-structure.

(ii) $L \leq H_n(M)$ is a Lagrangian subspace with respect to the intersection and selfintersection form $(H_n(M), \lambda, \mu)$. By Lagrangian we mean that $L \perp L$ with respect to $\lambda$ and $\mu|_L = 0$.

The morphism space $\text{Cob}_{2n+1}^\mathcal{L}((M, L_M), (N, L_N))$ is the following subspace of $\text{Cob}_{2n+1}^{\theta}(M, N)$. A cobordism $W \subseteq [0, t] \times \mathbb{R}^\infty$ from $M$ to $N$ is a morphism in $\text{Cob}_{2n+1}^\mathcal{L}((M, L_M), (N, L_N))$ if:

(a) The pair $(W, N)$ is $(n - 1)$-connected;

(b) $i^\text{in}(L_M) = i^\text{out}(L_N)$, where $i^\text{in}: H_n(M) \to H_n(W)$ and $i^\text{out}: H_n(N) \to H_n(W)$ are the maps induced by the boundary inclusions.
Since $H_n(S^{2n}) = 0$, we obtain a factorization
\[ \mathcal{M}_{2n+1} \to \text{Cob}_2^{2n+1}(S^{2n}, S^{2n}) \to \text{Cob}_{2n+1}^n(S^{2n}, S^{2n}) \]
and thus a map $\Omega B\mathcal{M}_{2n+1} \to \Omega B\text{Cob}_2^{2n+1}$.

The following theorem is our main technical result:

**Theorem A.** Let $n \geq 4$ be a natural number except 7. Then the map
\[ \Omega B\mathcal{M}_{2n+1} \to \Omega B\text{Cob}_2^{2n+1} \]
just described is a weak homotopy equivalence.

The operation of disjoint union almost makes $\text{Cob}_2^{2n+1}$ into a symmetric monoidal category (almost, due to embedded cobordism having a length). More precisely, we endow $B\text{Cob}_2^{2n+1}$ with the structure of a special $\Gamma$-space. Applying the results of Segal [Se 74], we therefore obtain a (connective) spectrum $\mathbb{M}L_{2n+1}^+$, together with a map $\mathbb{M}L_{2n+1}^+ \to \mathbb{M}\theta_{2n+1}^n$, such that:

**Corollary B.** Let $n \geq 4$ be an integer except 7. Then the scanning map factors through an equivalence $\Omega B\mathcal{M}_{2n+1} \to \Omega^\infty \mathbb{M}L_{2n+1}^+$.

As explained in the previous sections one can now deduce information about stable diffeomorphism groups by applying the group completion theorem.

**Corollary C.** Let $n \geq 4$ be an integer except 7. Then the stable moduli space of manifolds $B_{2n+1}^\infty$ has the homology type of the infinite loop space $\Omega^\infty \mathbb{M}L_{2n+1}^+$.

Finally, we can say a little bit about the cohomology of $\Omega^\infty \mathbb{M}L_{2n+1}^+$: Denoting the scanning map $BDiff(W, D^{2n+1}) \to \Omega_0^\infty \mathbb{M}\theta_{2n+1}^n$ for $W \in \mathcal{W}_{2n+1}$ by $\mathcal{P}_W$ and $F : \Omega^\infty \mathbb{M}L_{2n+1}^+ \to \Omega^\infty \mathbb{M}\theta_{2n+1}^n$ the infinite loop map induced by applying $\Omega$ to the composite
\[ B\text{Cob}_2^{2n+1} \longrightarrow B\text{Cob}_{2n+1}^n \longrightarrow \Omega^\infty \mathbb{M}\theta_{2n+1}^n, \]
we find:

**Corollary D.** Let $n \geq 4$ be an integer except 7. Then the kernel of the homomorphism
\[ F^* : H^*(\Omega_0^\infty \mathbb{M}\theta_{2n+1}^n; \mathbb{Q}) \longrightarrow H^*(\Omega^\infty \mathbb{M}L_{2n+1}^+; \mathbb{Q}) \]
is equal to the common kernel of the collection of the maps
\[ \mathcal{P}_W^* : H^*(\Omega_0^\infty \mathbb{M}\theta_{2n+1}^n; \mathbb{Q}) \longrightarrow H^*(BDiff(W, D^{2n+1}); \mathbb{Q}) \]
for all $W \in \mathcal{W}_{2n+1}$.

It follows from the main result of [Eb 09] that the common kernel of the maps
\[ \mathcal{P}_W^* : H^*(\Omega_0^\infty \mathbb{M}\theta_{2n+1}^n; \mathbb{Q}) \longrightarrow H^*(BDiff(W, D^{2n+1}); \mathbb{Q}) \]
contains the tautological classes associated to Hirzebruch’s $L$-polynomials, and it is a simple calculation that most of these do not vanish in the source.
1.5. Remarks. While our construction of the category $\text{Cob}^L$ works equally well for arbitrary $n$-connected $\theta$ it is currently unclear how to define it in general. The reason is the non-existence of a self-intersection form at the homology level. A possible solution is suggested by the work of Ranicki, i.e. to work directly with the chains on a manifold rather than its homology. At the chain level all relevant structure should be present without connectivity assumptions. Carrying out such a generalization, however, presents serious technical difficulties, although recent work of Steimle and the first author on cobordism categories of Poincaré chain complexes indicates this may come within reach in the near future.

The stabilisation with respect to many manifolds instead of just the $W_{q,1}^{2n}$ is also present in \cite{GaRW14} for the treatment of more general tangential structures and manifolds and is therefore not a new phenomenon in odd dimensions. For even dimensions, however, the need for it was later removed as the main result of \cite{GaRW17} by showing that the homology of $B_\infty^\theta$ no longer is affected by further stabilisation. An analogue of this result is work in progress and could replace stabilization with respect to all elements of $W_{2n+1}$, by stabilization with only $S^n \times S^{n+1}$, for which the second author proved homological stability in \cite{Pe17a}. Such results improving on homological stability are largely independent of the present work though.

Finally, the most pressing question to be addressed in future work is that of the homotopy type of $MT\mathcal{L}_{2n+1}$, in particular whether its cohomology contains entirely new classes not coming from $MT\theta_{2n+1}$. One would expect such an analysis to be simpler than a direct analysis of the monoid $\mathcal{M}_{2n+1}$ just as $B\text{Cob}_\theta$ can be identified with $\Omega^{\infty-1}MT\theta$ using Pontryagin-Thom theory. While there are obvious guesses to be made, we cannot currently offer a concrete conjecture based on more than speculation.

1.6. Organization of the paper. Section 2 is devoted to recollections about spaces of submanifolds and the extension of their definition to include homological data. In Section 3 we repeat the definition of the usual cobordism category, explain those steps in the program of Galatius and Randal-Williams that also work in odd dimension. In Section 4 we define the category $\text{Cob}^L_\theta$ and show how to derive Theorem A and its corollaries from a number of technical results whose proofs make up the rest of the paper: Section 5 constructs the $\Gamma$-space structure on $B\text{Cob}^L_{2n+1}$, Sections 6 and 7 contain preliminary technicalities, which enable us to apply the parametrized surgery techniques of \cite{GaRW14} in the final three sections.

The paper is not meant to be read in isolation: To avoid lengthy repetitions of established material we shall make frequent use of the terminology and results of the two papers \cite{GaRW10} and \cite{GaRW14}. For example in several places we will only indicate which parts of a definition needs to be changed. The reader is therefore advised to have copies of both papers close by.

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2. Preliminaries on Spaces of Manifolds

2.1. Spaces of manifolds. We begin by briefly reviewing some basic constructions from [GaRW 10] and [GaRW 14]. Recall that a tangential structure is a map \( \theta : B \to BO(d) \). A \( \theta \)-structure on an \( m \)-dimensional manifold \( M \) (with \( m \leq d \)) is a bundle map \( TM \oplus \epsilon^{d-m} \to \theta^* \gamma^d \) (i.e. a fiberwise linear isomorphism).

Fix a tangential structure \( \theta : B \to BO(d) \). Recall, for \( U \subseteq \mathbb{R}^n \), the space \( \Psi_{\theta,l}(U) \) from [GaRW 10 Section 2], consisting of pairs \( (M,\ell) \) where \( M \subseteq U \) is an \( l \)-dimensional submanifold without boundary and closed as a subspace of \( U \), while \( \ell \) is a \( \theta \)-structure on \( M \). These are topologized so that the assignment \( U \mapsto \Psi_{\theta,l}(U) \) defines a sheaf on \( \mathbb{R}^n \), valued in topological spaces. We will slightly extend the construction of that topology below in Subsection 2.3. As in [GaRW 10] we will need to consider particular subspaces of \( \Psi_{\theta,l}(\mathbb{R}^n) \) consisting of submanifolds \( M \subset \mathbb{R}^n \) that are open in a fixed number of directions. We repeat [GaRW 10 Definition 3.5]:

**Definition 2.1.1.** For \( k \leq n \), \( \psi_{\theta,l}(n,k) \subset \Psi_{\theta,l}(\mathbb{R}^n) \) is the subspace consisting of those \( \theta \)-manifolds \( (M,\ell) \) such that \( M \subset \mathbb{R}^k \times (-1,1)^{n-k} \). The space \( \psi_{\theta,l}(\infty,k) \) is defined to be the colimit of the \( \psi_{\theta}(n,k) \) taken as \( n \to \infty \).

For \( l = d \), we shall drop the index \( l \). Let \( x_1 : \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R} \), denote the projection onto the first factor. We will often consider \( \psi_{\theta,l}(\infty,k) \) as the space submanifolds \( W \subseteq \mathbb{R} \times \mathbb{R}^\infty \) and for any subset \( K \subseteq \mathbb{R} \), we write

\[
W|_K = W \cap x_1^{-1}(K)
\]

If \( \ell \) is a \( \theta \)-structure on \( W \) and \( W|_K \) a submanifold of \( W \), then we write \( \ell|_K \) for the restriction of \( \ell \) to \( TW|_K \).

2.2. Homological preliminaries. We will need to work with homology groups of elements of \( \Psi_{\theta}(\mathbb{R}^\infty) \), which are in general non-compact manifolds, and it turns out that for our purposes the locally finite/Borel-Moore homology \( H^\lf_k \) as defined in [Sp 93] is a convenient set-up. We shall only really have to consider \( H^\lf_k(M,A) \) for an \( m \)-manifold \( M \) with a closed codimension 0 submanifold \( A \) both of which have cylindrical ends (see Definition 2.3.3 below). In this case

\[
H^\lf_k(M,A) = \lim_{K \subseteq M} H_k(M, A \cup (M \setminus K)),
\]
see e.g. by [Sp 93, Theorem 10.1] for the absolute case and then use the long exact sequences. We will effectively treat the right hand side as a definition. Recall then that locally finite homology is covariantly functorial in proper maps (by restricting to the final system $f^{-1}(K)$ in the limit defining the source) and contravariantly functorial in open embeddings. In the special case above with $A = 0$ the contravariant functoriality for $V \subseteq M$ is given by

$$H_{lf}^k(M) \xrightarrow{\cong} \lim_{K \subseteq M} H_k(M, M \setminus K) \xrightarrow{\cong} \lim_{K \subseteq V} H_k(M, M \setminus K) \xrightarrow{\cong} \lim_{K \subseteq V} H_k(V, V \setminus K) \xrightarrow{\cong} H_k^f(V),$$

with the middle maps the projection to the indicated components of the limit and excision, respectively. Furthermore, stemming from Poincaré duality for non-compact manifolds there are Gysin homomorphisms. We shall only need the following version: Let $j : Z \hookrightarrow M$ be the inclusion of a compact, oriented submanifold of dimension $n$ into $M$, which we also assume oriented and further $U \subseteq M$ a closed tubular neighborhood of $Z$. The homomorphism

$$j_! : H^f_k(M) \rightarrow H_{k+n-m}(Z, \partial Z)$$

is defined to be the composition

$$H^f_k(M) \rightarrow H_k(M, M \setminus \text{Int} U) \xrightarrow{\cong} H_k(U, \partial U) \xrightarrow{\cong} H^{m-k}(U) \xrightarrow{\cong} H^{m-k}(Z) \xrightarrow{\cong} H_{k+n-m}(Z, \partial Z),$$

where the first map is the canonical projection (of the inverse limit onto one of its factors), the second is excision and the third and fifth arrows are given by Lefschetz duality. Since any two tubular neighborhoods of $Z$ are isotopic (see [Hi 76, Theorem 5.3]) it follows that the definition of the map $j_!$ is independent of the choice of tubular neighborhood $U$. We will call both these types of maps restrictions and denote them by

$$x \mapsto x|_Z.$$ 

**Remark 2.2.1.** This leads to very little ambiguity: If the open subset $V \subseteq X$ is given as the interior of some compact codimension 0 submanifold $Z \subseteq X$ we claim that $H^f_k(V)$ and $H_k(Z, \partial Z)$ are canonically isomorphic in a fashion making

$$H^f_k(V) \xrightarrow{\cong} H_k(Z, \partial Z)$$

commutative. To this end choose an open collar $C$ of $\partial Z$ in $Z$. As $V - C$ is a compact subset of $V$ we obtain a map $H^f_k(V) \rightarrow H_k(V, C - \partial Z)$, which we shall momentarily see is an isomorphism (since $V$ has $C - \partial Z$ as its cylindrical ends as defined below). Clearly, $H_k(V, C - \partial Z)$ and $H_k(Z, \partial Z)$ are canonically isomorphic (for example via their inclusions into $H_k(Z, C)$). It is now readily checked that this identification is independent of the chosen collar and the diagram above indeed commutes.

As mentioned we will mainly work with a particularly simple class of non-compact manifolds:
Definition 2.2.2. A manifold $M$ is said to have *cylindrical ends* if there exists some compact codimension-0 submanifold $B \subset X$ (possibly with boundary), such that the complement $M \setminus \text{Int}(B)$ is homeomorphic to the cylinder $\partial B \cup A \times [0, \infty)$, relative to $\partial B$, for some codimension 0 submanifold $A$ of $\partial B$.

In this case we find an isomorphism $H^k_h(M) \cong H_k(B, A)$, since every compact subset $K \subset M$ is contained in a submanifold $B$ as above, so

$$H^k_h(M) \cong \lim_{B \subseteq M} H_k(M \setminus B) \cong \lim_{B \subseteq M} H_k(B, A)$$

by finality and the latter system is evidently constant.

2.3. Spaces of manifolds equipped with homological data. We will need to consider spaces of manifolds equipped with a choice of subspace of its homology group. These spaces (defined below) will enable us to topologize the cobordism category $\text{Cob}^L_\theta$ (discussed in the introduction) and the semi-simplicial spaces introduced in Section 6.

Definition 2.3.1. Fix a tangential structure $\theta : B \to BO(d)$. For an open subset $U \subset \mathbb{R}^m$, let $\Psi^\Delta_{\theta, l}(U)$ denote the set of triples $(M, \ell, V)$ with $(M, \ell) \in \Psi_{\theta, l}(U)$ and $V \leq H^k_h(M)$ a subgroup.

We topologize the set $\Psi^\Delta_{\theta}$ in complete analogy with [GaRW 10, Section 2.1], where a topology on the spaces $\Psi_{\theta}$ is described in three steps. Let us briefly indicate the necessary changes:

Construction 2.3.2. Step 1: Define the *compactly supported topology* on $\Psi^\Delta_{\theta, l}(U)$ for $U \subseteq \mathbb{R}^n$ in the same fashion as [GaRW 10, Step 1, page 1248], using instead of the map $c_M$ the partially defined map

$$\Gamma_{cM} (\nu M) \to \Psi^\Delta_{\theta, l}(U), \quad s \mapsto ((id_M + p \circ s)(M), \ell \circ D(id_M + p \circ s)^{-1}, (id_M + p \circ s)_*(V))$$

for some $(M, \ell, V) \in \Psi^\Delta_{\theta, l}(U)$, where $p : \nu M \to \mathbb{R}^n$ is the projection in the fibre direction. This makes $\Psi^\Delta_{\theta, l}(U)$ into a covering of $\Psi_{\theta}(U)$ with fiber over $(M, \ell)$ the set of subgroups of $H_n(M)$ although we shall not need this.

Step 2: To construct the *$K$-topology*, for some compact $K \subset U$, proceed as in [GaRW 10, Step 2, page 1249], identifying two elements of $\Psi^\Delta_{\theta, l}(U)$ if

$$(M \cap A, \ell|_{M \cap A}, V|_{M \cap A}) = (M' \cap A, \ell'|_{M' \cap A}, V'|_{M' \cap A})$$

for some neighbourhood $A$ of $K$.

Step 3: Finally, give $\Psi^\Delta_{\theta}(U)$ the terminal topology making all identities into the various $K$-topologies continuous, no changes required.

With $\Psi^\Delta_{\theta}(U)$ topologized as above one may proceed as in [GaRW 10, Section 2.2] to obtain direct analogues to the basic properties of $\Psi_{\theta}(U)$ proven in that section. For example we shall need the analogue of [GaRW 10, Theorem 2.7], its proof applies verbatim:
Proposition 2.3.3. Let $U' \subseteq U$ be an open subset. The restriction map
\[ \Psi^\Delta_\theta(U) \to \Psi^\Delta_\theta(U'), \quad (M, \ell, V) \mapsto (M \cap U', \ell_{|M \cap U'}, V_{|M \cap U'}) \]
is continuous.

Definition 2.3.4. For each $k \leq m$ we define $\Psi^\Delta_\theta(m, k) \subset \Psi^\Delta_\theta(\mathbb{R}^m)$ to be the subspace consisting of those $(M, \ell, V)$ such that $(M, \ell) \in \Psi_\theta(m, k)$. We define $\Psi^\Delta_\theta(\mathbb{R}^\infty)$ and $\Psi_\theta(\mathbb{R}^\infty)$ to be the colimits of the above spaces, taken as $m \to \infty$.

Finally, let us give a criterion for lifting the continuity of maps into $\Psi^\Delta_\theta(U)$ from that of their projection to $\Psi_\theta(U)$. For the maps that we will be concerned with later (in Sections 8 and 9) it is satisfied by the discussion in [GaRW 14, Sections 4 and 5].

Definition 2.3.5. A map $\phi: X \to \Psi_\theta(U)$, written $(W_t, \ell_t) \in \Psi_\theta(U)$, is said to be locally generated by vector fields if for every $t_0 \in X$ and compact subset $A \subseteq U$, there exists a neighbourhood $V \subseteq X$ of $t_0$ and a map $s: V \to \Gamma_c(\nu W_{t_0})$ such that
\[ W_t \cap A = (id_M + p \circ s_t)(W_{t_0} \cap A) \quad \text{and} \quad \ell_t\big|_{A \cap W_t} = (\ell_{t_0} \circ D(id_M + p \circ s_t)^{-1})\big|_{A \cap W_t} \]
for all $t \in V$.

Construction 2.3.6. Let $\phi: X \to \Psi_\theta(U)$, written $t \mapsto (W_t, \ell_t)$, be a continuous map and $Q \subseteq U$ be an open subset such that the family $(W_t, \ell_t)$ is constant when restricted to $U \setminus Q$. Let $W'$ denote the (constant) complement $W_t \setminus (W_t \cap Q)$ and
\[ \beta_t: H^0_{\text{lf}}(W') \to H^0_{\text{lf}}(W_t) \]
be the homomorphism induced by inclusion $W' \hookrightarrow W_t$. For a subspace $V \subseteq H^0_{\text{lf}}(W')$ let
\[ V_t := \beta_t(V) \leq H^0_{\text{lf}}(W_t). \]
This gives a lift
\[ \phi_Q: X \to \Psi^\Delta_\theta(U), t \mapsto (W_t, \ell_t, V_t). \]

Proposition 2.3.8. If $\phi: X \to \Psi_\theta(U)$ is locally generated by vector fields, then $\phi_Q: X \to \Psi^\Delta_\theta(U)$ is continuous.

Proof. By definition of the topology it suffices to prove that the families
\[ (W_t, \ell_t, V_t)_K \in \Psi^\Delta_\theta(U), \]
are continuous for the various $K$-topologies. Since $\phi$ is locally generated by vector fields, there exists a neighbourhood $V$ of $t$, such that $(W_t \cap A, \ell_t|_A)$ is given by a family of vector fields $s: V \to \Gamma_c(\nu W_{t_0})$ on $V$. But then each $s_t$ has to vanish on $W' \cap A$, whence
\[ (s_t)_*(V_{t_0})|_{W_t \cap A} = V_t|_{W_t \cap A}. \]
This means that the manifestly continuous family $(s_t(W_{t_0}) \cap A, \ell_{s_t(W_{t_0}) \cap A}, (s_t)_*(V_{t_0})|_{s_t(W_{t_0}) \cap A})$ identifies to the same map as $\phi|_V$ in the quotient defining the $K$-topology.

\[ \square \]
3. Cobordism Categories of Highly Connected Odd Dimensional Manifolds

In this section we collect the relevant parts of \cite{GaRW14} that still hold true in the odd-dimensional setting and briefly explain the failure of the key statement. In particular, we give the relationship between the stabilized diffeomorphism group, the monoid $\mathcal{M}_{2n+1}$ from the introduction and the cobordism category, which are entirely analogous to the even dimensional, non-highly connected situation.

3.1. Some subcategories of $\text{Cob}_\theta$. We start out by repeating \cite{GaRW10} Definition 3.7:

\textbf{Definition 3.1.1.} We let the non-unital topological category $\text{Cob}_\theta$ have object space $\psi_{\theta,d-1}(\infty,0)$. The morphism space is the following subspace of $\mathbb{R} \times \psi_{\theta}(1+\infty,1)$: A pair $(t, (W, \ell))$ is a morphism if there exists an $\varepsilon > 0$ with

$$W|_{(-\infty, \varepsilon)} = (-\infty, \varepsilon) \times W_0 \quad \text{and} \quad W|_{(t-\varepsilon, \infty)} = (t-\varepsilon, \infty) \times W_t$$

as $\theta$-manifolds, where $(-\infty, \varepsilon) \times W_0$ and $(t-\varepsilon, \infty) \times W_t$ are equipped with the product $\theta$-structures induced from $\ell|_0$ and $\ell|_t$ on $W_0$ and $W_t$ using. The source of such a morphism is $W_0$ and the target $W_t$, equipped with their respective restrictions of the $\theta$-structure $\ell$ on $W$.

We need to establish some notation. Let $n \geq 0$ be an integer and fix a map $\theta : B \to BO(2n+1)\langle n \rangle$ giving rise to a tangential structure, which we shall also call $\theta$ (we follow Galatius and Randal-Williams in using $\langle n \rangle$ to indicate the $n$-connected cover of space). Also, fix once and for all a 2n-dimensional disk

$$D \subset \left( -\frac{1}{2}, 0 \right) \times (-1,1)^{\infty-1},$$

which near $\{0\} \times \mathbb{R}^{\infty-1}$ agrees with $(-1,0] \times \partial D$ and a $\theta$-structure $\ell_D : TD \oplus \varepsilon^1 \to \theta^*\gamma^{2n+1}$ on it. Let $\ell_{\mathbb{R} \times D}$ denote the $\theta$-structure on $\mathbb{R} \times D$ induced by $\ell_D$. Recall, finally, the notion of weak once-stability of a tangential structure $\theta : B \to BO(2n+1)$ from \cite{GaRW14} Definition 5.4], which implies that the $\theta$ on a cobordism $(M, \ell_M)$ to $(N, \ell_N)$ can be changed to give a cobordism from $(N, \ell_N)$ to $(M, \ell_M)$, something that is not true in general (\cite{GaRW14} Section 5.2).

\textbf{Definition 3.1.3.} Define a sequence of subcategories of $\text{Cob}_\theta$ as follows:

(i) $\text{Cob}_\theta^m \subset \text{Cob}_\theta$ has the same space of objects, and the morphisms from $(M, \ell_M)$ to $(N, \ell_N)$ are given by those $(t, W, \ell)$ for which the pair $(W|_{[0,t]}, W|_t)$ is $(n-1)$-connected.

(ii) $\text{Cob}_\theta^D \subset \text{Cob}_\theta^m$ has as its objects those $(M, \ell)$ such that

$$M \cap \left( [-1,0] \times (-1,1)^{\infty-1} \right) = D,$$

and that the restriction of $\ell$ to $D$ agrees with $\ell_D$. Similarly, it has as its morphisms those $(W, \ell)$ such that $W \cap \left[ \mathbb{R} \times [-1,0] \times (-1,1)^{\infty-1} \right] = \mathbb{R} \times D$, and the restriction of $\ell$ to $\mathbb{R} \times D$ agrees with $\ell_{\mathbb{R} \times D}$.

(iii) Let $l \in \mathbb{Z}_{\geq -1}$. The topological subcategory $\text{Cob}_\theta^l \subset \text{Cob}_\theta^D$ is the full subcategory on those objects $(M, \ell)$ such that $M$ is $l$-connected.
(iv) Assume $\theta$ weakly once-stable. Then $\text{Cob}_\theta^0 \subset \text{Cob}_\theta$ is the full subcategory on those $\theta$-manifolds $(M, \ell)$ that are $\theta$-nullcobordant, i.e. admit a morphism to (or from) the empty set.

(v) Define $\text{Cob}_\theta^{l,0}$ to be the intersection $\text{Cob}_\theta^0 \cap \text{Cob}_\theta^l$.

Just as in the even dimensional case $\text{Cob}_\theta^n$ relates via a group completion argument to the diffeomorphisms of $n-1$-connected, $n$-parallelizable manifolds. We establish this in the next section. In the last we will discuss the relation between $\text{Cob}_\theta^n$ and $\text{Cob}_\theta$ and the homotopy type of the latter.

3.2. Reduction to a monoid and group completion. We proceed to establish the relation between $\text{Cob}_\theta^n$ and stabilised diffeomorphism groups. Assume that the tangential structure $\theta : B \to BO(2n+1)$ is weakly once-stable. It follows directly from the discussion of weak once-stability (and reversibility) that the subspace $B\text{Cob}_\theta^n \subset B\text{Cob}_\theta$ is a single path component of $B\text{Cob}_\theta$. For $B\text{Cob}_\theta^{l,0}$ the analogous property relies on a connectivity assumption on $\theta$. For $l \leq n-1$ it then follows from Theorem 3.3.2 and for $l = n$ we need the following strengthening:

**Proposition 3.2.1.** If $B$ is $n$-connected, the object space $\text{ObCob}_\theta^n,0$ is a path component of $\text{ObCob}_\theta^n$ and consequently $B\text{Cob}_\theta^n,0$ is a path component of $B\text{Cob}_\theta^n$.

**Proof.** Note first, that $\text{ObCob}_\theta^n,0$ is clearly a union of path components of $\text{ObCob}_\theta^n$, so it will suffice to show that $\text{ObCob}_\theta^n,0$ is path connected. To this end observe that, since $BO(2n+1)(n)$ is $n$-connected, it follows that for any object $(M, \ell) \in \text{ObCob}_\theta^n,0$, the manifold $M$ is diffeomorphic to the standard sphere $S^{2n}$: Indeed, $M$ is $n$-connected and thus is a homotopy sphere. Let $(W, \ell_W)$ be a $\theta$-null-bordism of $(M, \ell)$. By the connectivity assumption the $\theta$-structure provides $W$ with a parallelization over an $n$-skeleton and then by [Wa 62a], we may perform a sequence of surgeries on the interior of $W$ so that the resulting manifold $\tilde{W}$ is contractible. Since $n \geq 4$, it follows from the h-cobordism theorem that $\tilde{W}$ is diffeomorphic to $D^{2n+1}$, and thus $M \cong S^{2n}$. Thus there is a weak homotopy equivalence

$$\text{ObCob}_\theta^n,0 \simeq \text{Bun}_\theta(TS^{2n} \oplus \epsilon^1, \theta^*\gamma^{2n+1}; \ell_D) \parallel \text{Diff}(S^{2n}, D^{2n}),$$

where $\text{Bun}_\theta(TS^{2n} \oplus \epsilon^1, \theta^*\gamma^{2n+1}; \ell_D)$ is the space of $\theta$-structures $\ell$ on $S^{2n}$ that agree with the structure $\ell_D$ when restricted to a fixed disk $D^{2n} \subset S^{2n}$, such that $(S^{2n}, \ell)$ is $\theta$-cobordant to the empty set. We will show that $\text{Bun}_\theta(TS^{2n} \oplus \epsilon^1, \theta^*\gamma^{2n+1}; \ell_D)$ is path connected.

Since the space of $\theta$-structures on $D^{2n+1}$ (fixed on a boundary hemisphere) is contractible, it will suffice to show that every element of $\text{Bun}_\theta(TS^{2n} \oplus \epsilon^1, \theta^*\gamma^{2n+1}; \ell_D)$ is the restriction to the boundary of some $\theta$-structure on $D^{2n+1}$. To see this we need only observe that the surgeries making $W$ into a disk can be chosen compatible with $\theta$. This is ensured by the dicussion in [GaRW 14, Section 4.1].

The addendum stating that $B\text{Cob}_\theta^n,0$ is a path component of $B\text{Cob}_\theta^n$ now follows from the fact that $\text{Cob}_\theta^n,0 \subset \text{Cob}_\theta^n$ is a full subcategory. \qed
To proceed, fix once and for all an object \((S, \ell_S) \in \text{Ob}\, \text{Cob}_\theta^{n,\emptyset}\). We define
\[ \mathcal{M}_\theta \subset \text{Cob}_\theta^{n,\emptyset} \]
to be the endomorphism monoid on the object \((S, \ell_S)\). Since the object space \(\text{Ob}\, \text{Cob}_\theta^{n,\emptyset}\) is path-connected and the combined source-target map is well known to be a fibration (for example it follows from the work of Lima [Li 64], the extension to \(\theta\)-structures is explicitly handled in [RaSt 18, Lemma 4.1]) the homotopy type of this topological monoid \(\mathcal{M}_\theta\) is independent of the choice of object \((S, \ell_S)\).

**Proposition 3.2.2.** For an \(n\)-connected tangential structure \(\theta\), the inclusion \(B\mathcal{M}_\theta \hookrightarrow B\text{Cob}_\theta^{n,\emptyset}\) is a weak homotopy equivalence.

The proof is entirely analogous to [GaRW 14, Section 7.1] using the path-connectivity of \(\text{Ob}\, \text{Cob}_\theta^{n,\emptyset}\). By the contractibility of embedding spaces into infinite euclidean space we find
\begin{equation}
\mathcal{M}_\theta \simeq \bigsqcup_W \text{BDiff}_\theta(W, D^{2n+1}),
\end{equation}
with union ranging over diffeomorphism classes of \((n - 1)\)-connected, \((2n + 1)\)-dimensional, closed \(\theta\)-manifolds \(W\), equipped with an embedding \(D^{2n+1} \hookrightarrow W\) compatible with the \(\theta\)-structure. Finally, we have:

**Proposition 3.2.4.** The monoid \(\mathcal{M}_\theta\) is homotopy commutative.

The proof is just as in [GaRW 10, Proposition 4.27] and so we also omit it. For a particular choice of tangential structure we obtain the monoid \(\mathcal{M}_{2n+1}\) defined in the introduction: Let \(\theta^n: BO(2n+1)\langle n \rangle \rightarrow BO(2n+1)\) denote the projection. Since for any \((n - 1)\)-connected, \((2n + 1)\)-dimensional closed manifold \(W\) that admits a \(\theta^n\)-structure, the space of \(\theta^n\)-structures on \(W\) is weakly contractible (relative to the one chosen on the embedded disk, see [GaRW 14, Lemma 7.16]) it follows that there is a weak homotopy equivalence
\[ \mathcal{M}_{\theta^n} \simeq \mathcal{M}_{2n+1}. \]
We will use these two monoids interchangeably.

**Definition 3.2.5.** A manifold admitting a \(\theta^n\)-structure is called \(n\)-parallelizable. Let \(\mathcal{W}_{2n+1}\) denote the set of diffeomorphism classes of oriented, \((n - 1)\)-connected, \((2n + 1)\)-dimensional, closed, manifolds, that are \(n\)-parallelizable.

Clearly \(\mathcal{W}_{2n+1} \cong \pi_0\mathcal{M}_{2n+1}\), in particular \(\mathcal{W}_{2n+1}\) is a monoid under connected sum. Recall that a monoid \(M\) is said to be finitely saturated if its group completion can be constructed by inverting just finitely many elements of \(M\) (or equivalently a single one).

**Proposition 3.2.6.** The monoid \(\mathcal{W}_{2n+1}\) is countable, but not finitely generated or even finitely saturated.
Proof. Recall that two oriented manifolds $M$ and $M'$ are said to be almost diffeomorphic if $M$ is diffeomorphic to $M' \# \Sigma$ where $\Sigma$ is an oriented homotopy sphere. Since the set of homotopy spheres in a given dimension is finite, it follows that there are only finitely many diffeomorphism types in a given almost-diffeomorphism class. In [Wa 67], Wall shows that the almost diffeomorphism class of any $(n - 1)$-connected, $(2n + 1)$-dimensional manifold $M$, is determined by a finite collection of algebraic invariants, each of which it turns out can take only countably many values. In the case that $n$ is even and $W \in W_{2n+1}$, these (almost) diffeomorphism invariants are given by the linking form $b : \tau H_n(W) \otimes \tau H_n(W) \rightarrow \mathbb{Q}/\mathbb{Z}$, and cohomology classes $\hat{\phi} \in H^{n+1}(W; \mathbb{Z}/2)$ and $\hat{\beta} \in H^{n+1}(W; \pi_n(SO))$ (see [Wa 67] Theorem 7). The linking form is a nonsingular, $(-1)^{n+1}$-symmetric, bilinear pairing, and according to the classification in [Wa 64], there are countably many such objects up to isomorphism. It follows that the set of almost-diffeomorphism classes (and hence the diffeomorphism classes) of elements of $W_{2n+1}$ is countably infinite.

That $W_{2n+1}$ cannot be finitely saturated, follows from the analogous fact the monoid (under direct sum) of isomorphism classes of finite abelian groups, to which $W_{2n+1}$ surjects (via taking $n$-th homology), since by [Wa 67] all non-degenerate pairings on finite abelian groups are realised by linking forms (and e.g. by [Wa 64] there is at least one such on every finite group); the latter monoid clearly is not finitely saturated (as finitely many elements can only ever account for torsion at finitely many primes). Since finite saturation is inherited by quotients the claim follows.

Since $\mathcal{M}_{2n+1}$ is homotopy commutative we may apply the group completion theorem of McDuff and Segal from [MDSe 75] (see [Ni] for a concise modern treatment). The main result of [MDSe 75] implies that the natural map $\mathcal{M}_{2n+1} \rightarrow \Omega B \mathcal{M}_{2n+1}$, induces an isomorphism

$$H_*(\mathcal{M}_{2n+1}) \left[\pi_0(\mathcal{M}_{2n+1})^{-1}\right] \xrightarrow{\cong} H_*(\Omega B \mathcal{M}_{2n+1}).$$

(3.2.7)

Using the language of [RW 13], this isomorphism may also be expressed as a certain map

$$(\mathcal{M}_{2n+1})_\infty \rightarrow \Omega B \mathcal{M}_{2n+1}$$

being a homology equivalence (even acyclic), where $(\mathcal{M}_{2n+1})_\infty$ is the colimit of the direct system

$$\begin{align*}
\mathcal{M}_{2n+1} & \xrightarrow{-W_1} \mathcal{M}_{2n+1} \xrightarrow{-W_1-W_2} \mathcal{M}_{2n+1} \xrightarrow{-W_1-W_2-W_3} \mathcal{M}_{2n+1} \xrightarrow{-W_1-W_2-W_3-W_4} \cdots
\end{align*}$$

(3.2.8)

where the $W_i$ give a generating system of $\pi_0(\mathcal{M})$ under saturation. Restricting to the path component of $\mathcal{M}_{2n+1}$ corresponding to the sphere $S^{2n+1}$ produces the direct system

$$\text{BDiff}(W_1, D^{2n+1}) \rightarrow \text{BDiff}(W_1^{#2} \# W_2, D^{2n+1}) \rightarrow \text{BDiff}(W_1^{#3} \# W_2^{#2} \# W_3, D^{2n+1}) \rightarrow \cdots$$

We denote the colimit of this direct system by $\mathbf{B}_\infty$, dropping the superscripted dimension from the introduction. Let $\Omega_0 B \mathcal{M}_{2n+1} \subseteq \Omega B \mathcal{M}_{2n+1}$ denote the path-component that contains the constant loop. We obtain the desired conclusion:
Proposition 3.2.9. The above construction produces a homology equivalence, in fact an acyclic map,
\[ B_\infty \to \Omega_0BM_{2n+1} \]
and therefore so is the composite
\[ B_\infty \to \Omega_0^\infty\text{Cob}_\theta^n. \]

3.3. The homotopy type of the cobordism category. Let us finally explain the failure of the method from [GaRW 14] in odd dimensions briefly. First of all we recall the main result of [GaMaTiWe 09] in the language of [GaRW 10]: If we denote by $MT\theta$ the Thom spectrum associated to the $-d$-dimensional virtual vector bundle $-\theta^*\gamma^d$ over $B$, Galatius and Randal-Williams construct (zig-zags of) weak equivalences
\[ B\text{Cob}_\theta \simeq \psi_\theta(\infty, 1) \simeq \Omega^{\infty-1}MT\theta, \]
that model the scanning map mentioned from the introduction. Since we will not explicitly make use of further properties of these equivalence let us refrain from spelling them out. Note that this result is completely insensitive to the parity of the manifold dimension. The same is true for the following two results.

Theorem 3.3.1. The inclusions $B\text{Cob}_\theta^D \hookrightarrow B\text{Cob}_\theta^m \hookrightarrow B\text{Cob}_\theta$ are weak homotopy equivalences.

This is immediate from [GaRW 14, Proposition 2.15 & Theorem 3.1] and the next statement follows from [GaRW 14, Theorem 4.1].

Theorem 3.3.2. Let $l \leq n-1$ and suppose that $\theta : B \to BO(2n+1)$ has $B$ $l$-connected and $\pi_{l+1}(B)$ finitely generated. Then the inclusion $B\text{Cob}_\theta^l \hookrightarrow B\text{Cob}_\theta^{l-1}$ is a weak homotopy equivalence.

The alert reader will notice that our assumption are weaker than those in the cited theorem of Galatius and Randal-Williams: Their result requires $B$ to be $(l+1)$-connected, since (in their notation) the map $L \to B$ is assumed an $(l+1)$-equivalence. The stronger assumption is use at only one place in the proof, namely to show that the relative homotopy group $\pi_{l+1}(\ell)$ is finitely generated (where $\ell : M \to B$ denotes a $\theta$-structure). Following Wall, this is known to follow from the weaker assumption that $\ell$ be an $l$-equivalence and $B$ of type $F_{l+1}$, i.e. admitting an $l+1$-equivalence from some finite complex (compare the proof of [CrMaL{{"u}, Lemma 3.81]). This in turn is guaranteed by our assumptions.

Theorems 3.3.1 and 3.3.2 together imply the weak homotopy equivalences
\[ B\text{Cob}_\theta^{n-1} \simeq B\text{Cob}_\theta \simeq \Omega^{\infty-1}MT\theta \]
for $(n-1)$-connected $B$ with $\pi_n(B)$ finitely generated. In contrast to the even dimensional case (see [GaRW 14, Theorem 5.3]), however, 3.3.2 cannot even be extended to yield a weak homotopy equivalence between $B\text{Cob}_\theta^n$ and $B\text{Cob}_\theta^{n-1}$ for $n$-connected $B$! Indeed, the inclusion
\( \Omega B\text{Cob}_\theta^n \rightarrow \Omega B\text{Cob}_\theta^n \) cannot be a weak homotopy equivalence: Consider the homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega_0 B M_{2n+1} & \longrightarrow & \Omega_0^\infty M \theta^n \\
\cong_{\mathbb{H}_*} & & \downarrow \mathbb{P} \\
B_\infty & \longrightarrow & \prod_{i \in \mathbb{N}} \text{BDiff}(W_1^i \# \ldots \# W_i, D^{2n+1})
\end{array}
\]

from the previous section, where the bottom-horizontal map is induced by the inclusions of the terms into their colimit, the top arrow is induced by the composite

\[
B M_{2n+1} \xrightarrow{\cong} B\text{Cob}_\theta^n \longrightarrow B\text{Cob}_\theta^n \xrightarrow{\cong} \Omega^\infty-1 M \theta^n,
\]

and the right-vertical map is the scanning map, whose effect in rational cohomology is well-known to give the tautological or Morita-Miller-Mumford classes of manifold bundles. Since homology preserves colimits, the bottom map is certainly surjective in homology, and thus \( B\text{Cob}_\theta^n \longrightarrow B\text{Cob}_\theta^n \) being a weak equivalence onto a path component would imply surjectivity (in homology) of the right-vertical map. However, the main theorem of [Eb 09] implies that this map has a non-trivial kernel in rational cohomology (consisting at least of the tautological classes of Hirzebruch’s \( \mathcal{L} \)-polynomials, most of which do not vanish in the cohomology of \( \Omega^\infty-1 M \theta^n \)), and thus cannot be surjective in homology.

We therefore see that the failure of Theorem 3.3.2 in the case that \( l = n \) is fundamental, and not merely a technical shortcoming of the methods of [GaRW 14].

4. Cobordism Categories of Manifolds Equipped with Lagrangians

In this section we set out to describe our modification of the replacement for the cobordism category in detail (Definition 4.2.1) and state the main technical theorems we will prove in the paper (Theorems 4.3.3 - 4.3.1). The results from the introduction are immediate consequences and we spell this out at the end of the section. First, however, we need to cover some preliminaries regarding bilinear and quadratic forms and their Lagrangian subspaces.

4.1. Preliminaries on quadratic forms and Lagrangian subspaces. Let \( \varepsilon = \pm 1 \). An \( \varepsilon \)-symmetric bilinear form is a pair \((P, \lambda)\) where \( P \) is a finitely generated \( \mathbb{Z} \)-module and \( \lambda : P \otimes P \rightarrow \mathbb{Z} \) is a bilinear map with the property that \( \lambda(x, y) = \varepsilon \cdot \lambda(y, x) \) for all \( x, y \in P \). An \( \varepsilon \)-symmetric bilinear form \((P, \lambda)\) is said to be non-singular if the map

\[
P \rightarrow \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}), \quad x \mapsto \lambda(x, -)
\]

becomes an isomorphism after modding out torsion. If \( V \leq P \) is a submodule, we let \( V^\perp \) denote the orthogonal complement of \( V \) in \( P \), i.e.

\[
V^\perp = \{ x \in P | \lambda(x, v) = 0 \text{ for all } v \in V \}.
\]
An $\varepsilon$ quadratic form is a triple $(P, \lambda, \mu)$, such that $(P, \lambda)$ is an $\varepsilon$-symmetric form and $\mu \to \mathbb{Z}/(1-\varepsilon)$ is a quadratic refinement of $\lambda$ in the sense that
\[ \mu(kx) = k^2 \mu(x) \quad \text{and} \quad \mu(p + q) = \mu(p) + \mu(q) + [\lambda(p, q)] \]
hold for all $k \in \mathbb{Z}$ and $x, y \in P$.

**Definition 4.1.1.** Let $\varepsilon = \pm 1$, and let $(P, \lambda, \mu)$ be an $\varepsilon$-quadratic form. A submodule $L \leq P$ will be called a Lagrangian if $L = L^\perp$ and $\mu|_L = 0$.

**Remark 4.1.2.** Let us note immediately, that a symmetric form (i.e. Lagrangian) be called a refinement, only if it is even in the sense that $\lambda(p, p)$ is even for every $p \in P$. If that is the case, then there exists a unique refinement, namely $\mu(p) = \lambda(p, p)/2$. In particular the second condition in the definition of Lagrangian is implied by the first in this case and one can wholly disregard the quadratic refinement.

In the case of an anti-symmetric form the situation is quite the opposite: Any torsionfree such form admits a subspace $L$ with $L^\perp = L$, and thus admits a quadratic refinement (since it can then be split apart into standard hyperbolics all of which do admit a quadratic refinement), but such a quadratic refinement is neither unique nor forced to vanish on $L$. Let $M$ be a $2n$-dimensional compact oriented manifold. The main example of a bilinear form that we will consider is the intersection pairing $\lambda: H_n(M) \otimes H_n(M) \to \mathbb{Z}$. Let $D: H_n(M) \to H^n(M, \partial M)$ be a tangential structure. With $\dim(M) = 2n$ the construction of the quadratic refinement breaks down in two separate cases depending on the parity of the integer $n$. Let again $\theta: B \to BO(2n + 1)$ be a tangential structure.

**Construction 4.1.3.** Suppose $n$ even and consider a $2n$-dimensional $\theta$-manifold $M$. It follows immediately that the $n$th Wu class $v_n(TM)$ vanishes. For even $n$, the element $v_n(TM)$ is characteristic for the modulo-2 intersection pairing (i.e. $\lambda(x, x) \equiv \lambda(p_2x, v_n)$ modulo 2 for all $x \in H_n(M)$) by the Wu formula, and thus $v_n(TM) = 0$ forces this pairing to be even. From Remark 4.1.2, we automatically obtain a quadratic refinement of the intersection form for such manifolds.

In the case of odd $n$ one has to work harder to obtain a quadratic refinement for the intersection form of a $2n$-dimensional manifold. In general it was shown by Browder in his work on the Arf-Kervaire invariant, that a Wu-orientation, i.e. a lift of the classifying map $M \to BO$ of the normal
bundle to the fibre of the map \( v_{n+1}: BO \to K(\mathbb{Z}/2, n+1) \), determines such a refinement. Moreover, it follows from a calculation of Stong [St 63], that all \( n \)-parallelized 2n-manifolds are canonically Wu-oriented, unless \( n \) is a Hopf dimension. We start out with the latter claim:

**Lemma 4.1.4.** We have \( 0 = v_{n+1} \in H^{n+1}(BO(n), \mathbb{Z}/2) \), whenever \( n \neq 0, 1, 3, 7 \), and consequently \(-\gamma\) admits a canonical Wu-orientation on \( BO(2n+1)n \).

**Proof.** For \( n = 5 \) (or more generally when \( n + 1 \) is not a power of 2) this is immediate, since then \( v_6 \in H^6(BO, \mathbb{Z}/2) \) (just like all non-two-power degree elements) is decomposable over the Steenrod-algebra. For general \( n \geq 9 \) Stong’s calculations imply that \( v_{n+1} \in H^{n+1}(BO(n-1), \mathbb{Z}/2) \) lies in the image of the Postnikov section \( BO(n-1) \to K(\pi_n BO, n) \), which in turn implies that \( 0 = v_{n+1} \in H^{n+1}(BO(n), \mathbb{Z}/2) \).

For the second claim, note that in \( BO(n) \) we necessarily have \( v_{n+1} = v_{n+1}(-\gamma) \), since the inversion on \( BO(n) \) is an automorphism of \( H^{n+1}(BO(n)) \) and this group is either 0 or \( \mathbb{Z}/2 \). This shows the existence of a Wu-orientation. The uniqueness follows from obstruction theory: Lifts are parametrised by \( H^n(BO(2n+1)n, \mathbb{Z}/2) = 0 \) once the obstruction \( v_{n+1} \) vanishes.

We shall now follow Brown [Br 72], who gave a simple construction of Browder’s quadratic refinement, see [JoRe 78] for another published account. In fact, for \( n \)-parallelized manifolds Brown’s construction may be simplified substantially:

**Construction 4.1.5.** Let \( n \) be odd with either \( n = 5 \) or \( n \geq 9 \) and \( \theta: B \to BO(2n+1)n \). Let \( M \) be a 2\( n \)-dimensional \( \theta \)-manifold. By the discussion above, the stable normal bundle of \( M \) has a canonical Wu-orientation. Given a class \( x \in H_n(M, \mathbb{Z}/2) \) we can represent its Poincaré dual by a map \( (M, \partial M) \to (K(\mathbb{Z}/2, n), pt) \). This map determines an element \( \mu(x) \) in the relative bordism group \( \Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \) of \( n \)-parallelized manifolds (which is represented by the spectrum usually (mis)named \( MO(n) \)). With our assumptions on the integer \( n \), it turns out that this bordism group is isomorphic to \( \mathbb{Z}/2 \), see below.

By identifying \( \Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \cong \mathbb{Z}/2 \), the assignment \( x \mapsto \mu(x) \) yields the desired quadratic refinement of the intersection form (see [Br 72] Corollary 1.11)).

The argument that \( \Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \cong \mathbb{Z}/2 \) proceeds as follows: First observe that the map \( \Sigma^\infty K(\mathbb{Z}/2, n) \to H(\mathbb{Z}/2, n) \) admits a lift into the homotopy fibre \( F \) of

\[
Sq^{n+1}: H(\mathbb{Z}/2, n) \to H(\mathbb{Z}/2, 2n+1).
\]

Any such lift turns out to be a \((2n+1)\)-equivalence by a direct calculation of the cohomology groups involved. In particular, \( \Omega_{2n}^{(n)}(K(\mathbb{Z}/2, n), pt) \cong \Omega_{2n}^{(n)}(F) \). Smashing the fibre sequence defining \( F \) with \( MO(n) \) gives an exact sequence

\[
H_{n+1}(MO(n), \mathbb{Z}/2) \xrightarrow{Sq^{n+1}} H_0(MO(n), \mathbb{Z}/2) \to \Omega_{2n}^{(n)}(F) \to H_n(MO(n), \mathbb{Z}/2).
\]

Clearly the fourth term is 0, the second one \( \mathbb{Z}/2 \) and the first map may be identified with

\[
\chi(Sq^{n+1})*: H^{n+1}(MO(n), \mathbb{Z}/2)* \to H^0(MO(n), \mathbb{Z}/2)*
\]
Since $\chi(S_q^{n+1})(u) = v_{n+1}u$ by definition of the Wu-class ($u \in H^0(MO(n), \mathbb{Z}/2)$ the Thom class), the map vanishes by the previous lemma and we obtain the desired isomorphism.

The two most important facts about this refinement for us are that

i) it is preserved under codimension zero embeddings and

ii) on a closed $(n-1)$-connected, $2n$-dimensional manifold $M$ it precisely obstructs representability of degree $n$ homology classes by embedded spheres with trivial normal bundle.

Upon investing that every class in $H_n(M)$ can be represented by an embedded sphere unique up to regular homotopy by Haefliger’s embedding theorem [Ha 61] and the Smale-Hirsch theorem, the second statement is proven in [Br 72, Corollary 1.13]. In fact, the standard machinery of surgery in the middle dimension then gives:

**Theorem 4.1.6.** Let $n \geq 4, n \neq 7$ and $\theta : B \to BO(2n+1)$ be weakly once stable with $B$ $n$-connected. Let $(M, \ell)$ be an $(n-1)$-connected, $2n$-dimensional, closed, $\theta$-manifold. Let $L \leq H_n(M)$ be a Lagrangian subspace for the selfintersection pairing. Then there exists a finite set $\Sigma$ and an embedding $f : \Sigma \times S^n \times D^n \to M$ that satisfies the following conditions:

(a) The $\theta$-structure on $\Sigma \times S^n \times D^n$ given by the composition,

$$T(\Sigma \times S^n \times D^n) \oplus e^1 \xrightarrow{Df \oplus Id} TM \oplus e^1 \xrightarrow{\ell} \theta^* \gamma^{2n+1},$$

extends to a $\theta$-structure on $\Sigma \times D^{n+1} \times D^n$.

(b) The homology classes, $[f|_{\sigma} \times S^n \times \{0\}] \in H_n(M), \sigma \in \Sigma$, yield a basis for the Lagrangian subspace $L \leq H_n(M)$.

Note that the assumptions on $M$ automatically force $H_n(M)$ to be torsion-free and thus make the Lagrangian a free module as well, so that (b) indeed makes sense. Condition (b) in particular implies that the manifold $\tilde{M}$ obtained by performing surgery on the embedding $f$ is $n$-connected, i.e. a homotopy sphere.

For a proof see e.g. [Ra 80, Proposition 5.2] or [CrMaLü, Proposition 4.13] with two comments: First, the cited references use Wall’s intersection pairing defined on the group of regular homotopy classes of immersions $S_n(M)$ (see [Wa 70, Theorem 5.2] or [Wa 62b] for a definition). To compare this to Browder’s, recall that a tangential structure map $l : M \to B$ gives a map $\pi_{n+1}(l) \to S_n(M)$ (see [CrMaLü, Lemma 4.60] for a pleasant exposition). For $n$-connected $B$ we then obtain a surjection $\pi_{n+1}(l) \to H_n(M)$ from the exact sequence of $l$ and it follows that the two compositions,

$$\pi_{n+1}(l) \to H_n(M) \to \mathbb{Z}/2 \quad \text{and} \quad \pi_{n+1}(l) \to S_n(M) \to \mathbb{Z}/2,$$

agree, since both precisely obstruct the desired representability. Secondly, the references work with stable bundles over Poincaré complexes, but the Poincaré condition does not enter into the special case above, and the construction of the required bundle data really only uses weak once-stability (see the discussion in [GaRW 14, Sections 4.1 & 5.1]).

For the reader’s convenience we supply a proof of the first fact:
**Proposition 4.1.7.** For an embedding \( i : W \to M \) for a closed 2n-manifold \( M \) and a compact 2n-manifold \( W \), the map \( i_* : H_n(W) \to H_n(M) \) preserves both the intersection and the selfintersection pairing.

**Proof.** That the intersection pairing is preserved is a simple calculation:

\[
\lambda_M(i_*(x), i_*(y)) = \langle i_*(x), D^M i_*(y) \rangle_M
\]

\[
= \langle i_*(x), i^! D(W, \partial W) y \rangle_M
\]

\[
= \langle i^! i_*(x), D(W, \partial W) y \rangle(W, \partial W)
\]

\[
= \langle incl_*(x), D(W, \partial W) y \rangle(W, \partial W)
\]

\[
= \lambda_W(x, y)
\]

Where the shriek maps are induced by \((M, \emptyset) \to (M, M - \text{Int} W) \leftarrow (W, \partial W)\) (the right arrow inducing an isomorphism by excision), \(incl\) denotes the inclusion \((W, \emptyset) \to W, \partial W\) and \(D^M i_*(y) = i^! D(W, \partial W) y\) follows from the naturality of cap products \(H_*(X, A \cup B) \times H^*(X, A) \to H_*(X, B)\) applied to \((M, \emptyset, \emptyset) \to (M, M - \text{Int} W, \emptyset)\) using the fact that \(i_!([M]) = [W, \partial W]\).

The preservation of the selfintersection pairing now follows since also in \(\Omega^{(n)}(W, \partial W)\) we have \(i_!([M]) = [W, \partial W]\) for the fundamental classes represented by the identity maps (since this can be checked locally around some point, by the definition of fundamental classes), so

\[
\mu_M(i_*(y)) = [D^M i_*(y) : (M, \emptyset) \to (K(Z, n), pt)]
\]

\[
= [i^! D(W, \partial W) y : (M, \emptyset) \to (K(Z, n), pt)]
\]

\[
= (D(W, \partial W) y)\iota([M])
\]

\[
= (D(W, \partial W) y)_\iota([W, \partial W])
\]

\[
= [D(W, \partial W) y : (W, \partial W) \to (K(Z, n), pt)]
\]

\[
= \mu(W, \partial W)(y)
\]

\[\Box\]

### 4.2. Cobordism categories of manifolds equipped with Lagrangian subspaces.

Fix a tangential structure \( \theta : B \to B\Omega(2n + 1)\langle n \rangle \) and suppose \( n \geq 4, n \neq 7 \).

**Definition 4.2.1.** The non-unital topological category \( \text{Cob}_\theta^n \) has as its object space the subspace of \( \psi_{\theta, 2n}^\Delta, n(\infty, 0) \) given by those tuples \((M, \ell, L)\) for which \( L \leq H_n(M) \) is a Lagrangian subspace with respect to the intersection form \((H_n(M), \lambda, \mu)\). The space of morphisms is given by the following subspace of the product \( \mathbb{R} \times \psi_{\theta, 2n+1}^\Delta(1 + \infty, 1) \times \psi_{\theta, 2n}^\Delta(\infty, 0) \times \psi_{\theta, 2n}^\Delta(\infty, 0)\): A tuple

\[
(t, (W, \ell), (M, \ell_M, L_M), (N, \ell_N, L_N))
\]

is a morphism (from \((M, \ell_M, L_M)\) to \((N, \ell_N, L_N)\)) if

(i) \( (t, W, \ell) \in \text{Cob}_\theta^n((M, \ell_M), (N, \ell_N)) \)
Here, \( \eta_{\text{in}} : H_n(M) \to H_n(W_{[0,\ell]}) \) and \( \eta_{\text{out}} : H_n(N) \to H_n(W_{[0,\ell]}) \) are the homomorphisms induced by the boundary inclusions \( M = W|_0 \twoheadrightarrow W_{[0,\ell]} \twoheadrightarrow W|_{\ell} = N \).

**Remark 4.2.2.** We note that the forgetful functor \( \text{Cob}_0^\ell \to \text{Cob}_\theta \) is faithful. It is, however, the increase in extra structure on objects that makes the definition of the entire morphism space of \( \text{Cob}_0^\ell \) more complicated than that of \( \text{Cob}_\theta \) as a cobordism no longer determines its source or target.

When denoting a morphism in \( \text{Cob}_0^\ell \) we will (nevertheless) usually drop the source and target from the notation and just write \((t,W,\ell)\) for \((t,(W,\ell),(M,\ell_M,L_M),(N,\ell_N,L_N))\). We proceed to filter the cobordism category \( \text{Cob}_0^\ell \) by subcategories analogous to those from Definition 3.1.3. Let \( D \subset (-\frac{1}{2},0] \times (-1,1)^{\infty-1} \) be the \( 2n \)-dimensional disk from \((3.1.2)\). Let \( \ell_D \) be the chosen \( \theta \)-structure on \( D \) and let \( \ell_{\mathbb{R} \times D} \) be the \( \theta \)-structure on \( \mathbb{R} \times D \) induced by \( \ell_D \).

**Definition 4.2.3.** We define a sequence of subcategories of \( \text{Cob}_\theta^\ell \) as follows:

(i) The topological subcategory \( \text{Cob}_\theta^{\ell,D} \subseteq \text{Cob}_\theta^\ell \) has as its objects those \((M,\ell,L)\) with \((M,\ell) \in \text{Ob} \text{Cob}_\theta^D\). Similarly, it has as its morphisms those \((t,W,\ell)\) that give a morphism in \( \text{Cob}_\theta^D\).

(ii) Let \( l \in \mathbb{Z}_{\geq -1} \). The topological subcategory \( \text{Cob}_\theta^{\ell,l} \subseteq \text{Cob}_\theta^{\ell,D} \) is the full-subcategory on those objects \((M,\ell,L)\) such that \( M \) is \( l \)-connected, or in other words \((M,\ell) \in \text{Ob} \text{Cob}_\theta^l\).

In other words, the categories \( \text{Cob}_\theta^{\ell,D} \) and \( \text{Cob}_\theta^{\ell,l} \) are the evident pull-backs.

**Observation 4.2.4.** The forgetful functor \( \text{Cob}_\theta^{\ell,n} \to \text{Cob}_\theta^n \) is clearly an isomorphism, and thus \( \text{Cob}_\theta^{\ell,n} \) can be considered a subcategory of \( \text{Cob}_\theta \).

**4.3. The technical theorems.** We now state our results about the category \( B\text{Cob}_\theta^\ell \) and granting them for the moment deduce the results in the introduction from them. Their proofs occupy the remaining sections, roughly in order. Let again \( \theta : B \to BO(2n+1)/n \) be a map with \( n \geq 4, n \neq 7 \).

The first result occupies Section 5. To state it recall that Nguyen [Ng 17] constructed a \( \Gamma \)-space structure on \( B\text{Cob}_\theta \) underlain by disjoint union for which the equivalence to \( \Omega^{\infty-1}MT\theta \) becomes one of infinite loopspaces.

**Theorem 4.3.1.** The operation of disjoint union gives \( B\text{Cob}_\theta^\ell \), the structure of a special \( \Gamma \)-space, such that the forgetful functor

\[
\text{Cob}_\theta^\ell \to \text{Cob}_\theta
\]

induces a map of \( \Gamma \)-spaces. In particular, \( B\text{Cob}_\theta^{\ell,0} \) carries the structure of an infinite loopspace.

**Remark 4.3.2.** One may wonder whether \( B\text{Cob}_\theta^\ell \) itself is an infinite loopspace, the only question being whether the \( \Gamma \)-space structure from the above theorem makes it grouplike. This is indeed the case, but a proof is most readily given by showing \( \text{Cob}_\theta^\ell \) to be equivalent to a cobordism category with no connectivity assumption on the morphisms (following the procedure in [GaRW 14]), where it is then immediate that the components form a group. Since we will not make use of this more general assertion we have omitted it.
The next result is proven using the same ideas as [GaRW 14, Corollary 2.17], but in a different set-up. We indicate the necessary changes in Section 6.4.

**Theorem 4.3.3.** The inclusion $B\text{Cob}_{\theta}^{L,D} \hookrightarrow B\text{Cob}_{\theta}^{L}$ is a weak homotopy equivalence.

The next theorem is proven in Section 8. It is the first result of the paper whose proof requires a substantial amount of technical work. The constructions that go into it, however, closely resemble those from [GaRW 14].

**Theorem 4.3.4.** Let $l \leq n-1$ and assume that $B$ is $l$-connected and $\pi_{l+1}(B)$ is finitely generated. Then the inclusion $B\text{Cob}_{\theta}^{L,l} \hookrightarrow B\text{Cob}_{\theta}^{L,l-1}$ is a weak homotopy equivalence.

By combining the theorems stated above, we obtain the weak homotopy equivalence

$$B\text{Cob}_{\theta}^{L,n-1} \simeq B\text{Cob}_{\theta}^{L}$$

in analogy with (3.3.2) whenever $B$ is $(n-1)$-connected and $\pi_{n}(B)$ finitely generated. Finally we have:

**Theorem 4.3.5.** Suppose that $\theta$ is weakly once-stable and that $B$ is $n$-connected. Then the inclusion $B\text{Cob}_{\theta}^{L,n} \hookrightarrow B\text{Cob}_{\theta}^{L,n-1}$ is a weak homotopy equivalence.

We again emphasize that this theorem is in stark contrast to the situation for cobordism categories without Lagrangians. The proof will occupy Sections 9 and 10. Via the isomorphism $\text{Cob}_{\theta}^{L,n} \cong \text{Cob}_{\theta}^{n}$ the above theorems imply that there is a weak homotopy equivalence $B\text{Cob}_{\theta}^{n} \simeq B\text{Cob}_{\theta}^{L}$ whenever $\theta$ satisfies the conditions of Theorem 4.3.5.

4.4. **Deduction of the main results.** Supposing these four theorems, we now proceed to prove all results stated in the introduction. Let $n \geq 4$ be an integer except 7 and specialise to $\theta^n$, the $n$-connected cover $BO(2n+1)\langle n \rangle \to BO(2n+1)$. Theorem A, follows immediately by combining the weak homotopy equivalence $B\text{Cob}_{\theta}^{n} \simeq B\text{Cob}_{\theta}^{L}$ with the weak homotopy equivalences

$$BM_{2n+1} \simeq BM_{\theta^n} \simeq B\text{Cob}_{\theta}^{n,0},$$

proven in Section 3.2. Corollary B then follows by combining this with Theorem 4.3.1. Corollary C follows by combining the weak homotopy equivalence $BM_{2n+1} \simeq B\text{Cob}_{\theta}^{n,0}$, with the homology equivalence $B_{\infty} \to \Omega_0 BM_{2n+1}$ established in Section 3.2. To obtain Corollary D we do the following. Consider the commutative diagram

$$\begin{array}{ccc}
\coprod_{W} \text{BDiff}(W,D^{2n+1}) & \longrightarrow & \Omega_0 B\text{Cob}_{\theta^n} \\
\downarrow & & \downarrow \\
B_{\infty} \simeq H_{*} & \longrightarrow & \Omega_0 B\text{Cob}_{\theta}^{L} \\
\end{array}$$

and notice that the downwards left arrow (which is given by the inclusions of the terms of the colimit sequence into the colimit) is surjective on homology and thus injective in cohomology (with rational...
coefficients). The kernel of the left hand vertical map in cohomology is therefore the same as that of the top horizontal arrow. Composing with the weak homotopy equivalence \( \Omega B \text{Cob}_\theta \to \Omega^\infty M \theta \) yields the claim.

5. The infinite loop space structure

In this section we equip \( B \text{Cob}_\theta^L \) with the structure of a a \( \Gamma \)-space, proving Theorem 4.3.1. In order to do this we first need to introduce a more convenient model for the nerve of \( \text{Cob}_\theta^L \).

5.1. A model for the nerve. Let \( \theta: B \to BO(2n + 1)(n) \) be a tangential structure, with \( n \geq 4 \), \( n \neq 7 \).

Definition 5.1.1. For each \( p \in \mathbb{Z}_{\geq 0} \), \( C_p \) and \( \hat{C}_p \), respectively, are defined to be the spaces of tuples \((a, \varepsilon, W, \ell)\) where:

- \( a \in \mathbb{R}^{p+1} \), \( \varepsilon \in (0, \infty)^{p+1} \) are \((p+1)\)-tuples with the property that \( a_i + \varepsilon_i < a_{i+1} + \varepsilon_{i+1} \), and \( a_i \leq a_{i+1} \) for all \( i = 0, \ldots, p-1 \), respectively;
- \((W, \ell)\) is an element of \( \psi_\theta(\infty, 1) \) with the property that \((W, \ell)\left|_{(a_i - \varepsilon_i, a_{i+1} + \varepsilon_{i+1})}\right.\) is cylindrical for all \( i = 0, \ldots, p \), i.e. equal to a cylinder on \( W_{[a_i]} \).

Both \( C_p \) and \( \hat{C}_p \) are topologized as subspaces of \( \mathbb{R}^{p+1} \times \mathbb{R}^{p+1} \times \psi_\theta(\infty, 1) \) and the assignments \([p] \mapsto C_p \) and \([p] \mapsto \hat{C}_p \)
yield semi-simplicial spaces with \( i \)-th face map given by deleting \( a_i \) and \( \varepsilon_i \). \( \hat{C}_\bullet \) is in fact a simplicial space with \( i \)-th degeneracy map doubling \( a_i \) and \( \varepsilon_i \).

\( C_\bullet \) appears in the proof of \([\text{GaRW 14}, \text{Proposition 2.14}]\) as the nerve of a certain poset \( D^\perp_\theta \) (a slightly different poset appears in the proof of \([\text{GaRW 10}, \text{Theorem 3.9}]\) under the same name; we, however, want to reserve the letter \( D \) for yet another object that incorporates Lagrangians and will be made use of in the same way \( D \) is used in the later parts of \([\text{GaRW 14}]\); see for example Section 8).

For each \( p \)-simplex \((a, \varepsilon, W, \ell) \in C_p \), the tuple \((a, W, \ell)\) determines a unique element in \( N_p \text{Cob}_\theta \) by remembering only the various \( W_{[a_i, a_{i+1}]} \) (appropriately translated and with infinite collars attached) with their induced \( \theta \)-structures. This correspondence defines a semi-simplicial map \( C_\bullet \to N_\bullet \text{Cob}_\theta \) which is easily shown to be a level-wise weak homotopy equivalence (see \([\text{GaRW 10}, \text{Theorem 3.9}]\) for details).

The reason for including the version with degeneracies is as follows: We shall momentarily construct a simplicial \( \Gamma \)-space using (Lagrangian enhancements of) the spaces \( \hat{C}_p \) that is levelwise special. The speciality is then automatically inherited by its realization and will give the main result of the section; the same is not true when working with \( C_p \) and its associated semi-simplicial \( \Gamma \)-space.

We thank Johannes Ebert for pointing out this oversight in a previous version. We shall furthermore use the spaces \( \hat{C}_\bullet \) to correct a small mistake in the proofs of \([\text{GaRW 14}, \text{Theorems 4.5 & 5.14}]\) in turn, see \([\text{5.2.5}]\). To compare the two versions we have:
Lemma 5.1.2. The simplicial space $\hat{\mathcal{C}}_\bullet$ is good, i.e. its degeneracies are closed cofibrations, and the inclusion $\mathcal{C}_\bullet \hookrightarrow \hat{\mathcal{C}}_\bullet$ is a levelwise equivalence and thus in particular realizes to a weak equivalence.

By the first part it does not matter whether we use the thin or the thick realisation for the target, as they are equivalent by [Se 74, Proposition A.1, (iv)].

Proof. To see that $s_i$ is a closed cofibration it is enough (e.g. by combining [tDKaPu 70, Satz 3.13 and Satz 3.26]) to show that (i) $s_i(\hat{\mathcal{C}}_p) \subset \hat{\mathcal{C}}_{p+1}$ is given as the vanishing set of some real-valued function and (ii) admits a halo $U$ that contracts onto it (recall that any neighbourhood given by a strict inequality for some positive function to the real numbers is a halo [tDKaPu 70, Definition 3.1]).

For property (i) consider $(a,\varepsilon,W,\ell) \mapsto |\varepsilon_i - \varepsilon_{i+1}| + a_{i+1} - a_i$. Property (ii) is witnessed by the neighborhood

$$U = \left\{(a,\varepsilon,W,\ell) \in \hat{\mathcal{C}}_{p+1} \mid \frac{a_{i+1} - a_i}{\min\{\varepsilon_i, \varepsilon_{i+1}\}} < 1\right\},$$

with the deformation retraction moving $a_{i+1}$ towards $a_i$ (using the fact that $W$ is cylindrical between $a_i$ and $a_{i+1}$ for elements in $U$) and shrinking $\varepsilon_i$ and $\varepsilon_{i+1}$ as required.

The second part is trivial. □

Definition 5.1.3. For each $p \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}^L_p$ and $\hat{\mathcal{C}}^L_p$, respectively, are the spaces of tuples $(a,\varepsilon,W,\ell,L)$ subject to the following conditions:

(a) The tuple $(a,\varepsilon,W,\ell)$ is an element of $\mathcal{C}_p$ or $\hat{\mathcal{C}}_p$, respectively. Furthermore, for each $i = 0,\ldots,p$, the pair $(W|_{a_i,a_{i+1}}, W|_{a_{i+1}})$ is $(n-1)$-connected,

(b) $L = (L_0,\ldots,L_p)$ is a $(p+1)$-tuple with $L_i \leq H_n(W|_{a_i})$ a Lagrangian subspace for each $i = 0,\ldots,p$.

(c) For each $i = 0,\ldots,p$, we require $\iota^\text{in}_i(L_i) = \iota^\text{out}_{i+1}(L_{i+1})$ where

$$\iota^\text{in}_i : H_n(W|_{a_i}) \longrightarrow H_n(W|_{a_i,a_{i+1}}) \quad \text{and} \quad \iota^\text{out}_{i+1} : H_n(W|_{a_{i+1}}) \longrightarrow H_n(W|_{a_i,a_{i+1}})$$

are the maps induced by inclusion.

Each space $\mathcal{C}^L_p$ is topologized as a subspace of the product $\mathcal{C}_p \times (\text{Ob} \mathcal{C}^L)_{p+1}$. The assignment $[p] \mapsto \mathcal{C}^L_p$ defines a semi-simplicial space with face maps defined similarly to the previous definition and similarly for the simplicial version. As in the case with $\mathcal{C}_\bullet$, there is a semi-simplicial map $\mathcal{C}^L_\bullet \rightarrow N_\bullet \mathcal{C}^L_\theta$ and applying the same arguments as before we obtain weak equivalences

$$\hat{\mathcal{C}}^L_\bullet \leftrightarrow |\mathcal{C}^L_\bullet| \longrightarrow B\mathcal{C}^L_\theta.$$

Corresponding to the sequence of subcategories from Definition 4.2.3 we have a sequence of sub-semi-simplicial spaces

$$\mathcal{C}^L_{\bullet,1} \subset \cdots \subset \mathcal{C}^L_{\bullet,-1} \subset \mathcal{C}^L_{\bullet,D} \subset \mathcal{C}^L_{\bullet}.$$
defined analogously to the subcategories of Definition 4.2.3. Applying the construction from [GaRW 10, Theorem 3.9] again, we also obtain weak homotopy equivalences,
\[ \big| \mathcal{C}_\bullet^\mathcal{L} \big| \simeq \text{BCob}_0^\mathcal{L}, \quad \big| \mathcal{C}_\bullet^\mathcal{L,D} \big| \simeq \text{BCob}_0^{\mathcal{L,D}}, \quad \big| \mathcal{C}_\bullet^\mathcal{L,I} \big| \simeq \text{BCob}_0^{\mathcal{L,I}}. \]

The analogous result using the full simplicial spaces also holds, but we shall have no need for it.

5.2. The \(\Gamma\)-space. We now proceed to give \( \hat{\mathcal{C}}_\bullet^\mathcal{L} \) the structure of a \(\Gamma\)-space. We follow [Ng 17] and make every \( \hat{\mathcal{C}}_p^\mathcal{L} \) into the underlying space of a \(\Gamma\)-space \( A_p \), which assemble into a simplicial \(\Gamma\)-space \( A_\bullet \). By 5.1.4 the realization (in the simplicial direction) is then a \(\Gamma\)-space underlain by \( \text{BCob}_0^\mathcal{L} \).

Construction 5.2.1. Let \( S \) be a finite set. We define \( A_p(S) \) to be the subset of \( (\hat{\mathcal{C}}_p^\mathcal{L})^S \) consisting of those tuples
\[ (a, \varepsilon, W, \ell, L) \]
that satisfy for all \( s, s' \in S \):
\[
\begin{align*}
(a) & \quad a_s = a_{s'} \text{ and } \varepsilon_s = \varepsilon_{s'} \\
(b) & \quad \text{the submanifolds, } W_s \text{ and } W_{s'} \subset \mathbb{R} \times (-1,1)^\infty, \text{ are disjoint.}
\end{align*}
\]

For the rest of this section, we will suppress \( \varepsilon \) and \( \ell \) from notation as they will play no role. Given a morphism \( \phi : S \to T \) (in the category \( \Gamma \)), the map
\[ A_p(\phi) : A_p(T) \to A_p(S) \]
is defined by sending a tuple \( (a, W, L) \in A_p(T) \) to the element in \( A_p(S) \) whose entry in the \( s \)-th spot is:
\[
\left( a, \bigcup_{t \in \phi(s)} W_t, \sum_{t \in \phi(s)} L_t \right).
\]

It is readily checked that this really makes \( A_p \) into a \(\Gamma\)-space and indeed \( A_\bullet \) into a simplicial \(\Gamma\)-space.

Note that \( A_\bullet(1) = \hat{\mathcal{C}}_\bullet^\mathcal{L} \) and therefore
\[ |A_\bullet|(1) = |\hat{\mathcal{C}}_\bullet^\mathcal{L}| \simeq \text{BCob}_0^\mathcal{L}. \]

Lemma 5.2.3. The \(\Gamma\)-space \( A_p \) is special for every \( p \).

Proof. We need to show that the Segal maps
\[ p_k : A_p(k) \to A_p(1)^k = (\hat{\mathcal{C}}_p^\mathcal{L})^k \]
are weak homotopy equivalences. Note that \( p_k \) is just the inclusion of disjoint \( k \)-tuples of manifolds (with equal \( a \)'s and \( \varepsilon \)'s) into the space of all \( k \)-tuples. But clearly an arbitrary \( k \)-tuple can be made disjoint by shrinking the \( i \)-th manifold for example into \( \mathbb{R} \times \left( \frac{i}{k}, \frac{i+1}{k} \right) \times (-1,1)^{\infty-1} \) and so the fact that the space of embeddings of a manifold into \( \mathbb{R}^\infty \) is contractible gives the claim, compare e.g. [Ng 17, Proposition 5.10]. □
Proposition 5.2.5. The geometric realization $B\text{Cob}_\theta^c \simeq |\hat{C}_L^c|$ has the structure of a special $\Gamma$-space.

Proof. All that remains to check is that

$$p_k : |A_\bullet|(k) \rightarrow |A_\bullet|(1)^k$$

is a weak equivalence. But the same argument as in [5.1.2] shows that the simplicial space $A_\bullet(k)$ is good for every $k$, whence the claim follows from [5.2.3] since levelwise weak equivalences between good simplicial spaces are preserved by realization.

Alternatively, we could have used the thick realisation of the simplicial space $A_\bullet(S)$ and applied [EbRW 17, Theorem 7.2] bypassing the goodness of $\hat{C}_L^c$.

Corollary 5.2.6. The space $\Omega B\text{Cob}_\theta^c$ admits an infinite loop space structure.

Proof. Note only that the identity component of a special $\Gamma$-space is automatically very special and thus an infinite loop space by [Se 74, Proposition 1.4].

It is evident straight from the definitions that the $\Gamma$-space structure just constructed is compatible with that of [Ng 17].

6. Alternate Models for the Nerve

The category defined in Section 4.2 is difficult to analyze directly, as is $C_L^c$. In order to prove Theorems 4.3.3 - 4.3.5 we will need to work with a more flexible substitute $X_L^c$ for the semi-simplicial nerve. In fact we will have three models $C_L^c$, $D_L^c$ and $X_L^c$ by the end of this section; we named these by analogy with the objects from [GaRW 14]. The middle model is forced on us by the definition of $X_\bullet$ from [GaRW 14, Definition 2.8]: Its constituent manifolds no longer have preferred slices in which to place Lagrangians. The semi-simplicial space $D_L^c$ therefore reformulates the Lagrangians as objects on the entirety of a long manifold rather than its slices. The relation between $D_L^c$ and $X_L^c$ will then be the same as that between $D_\bullet$ and $X_\bullet$ in [GaRW 14].

6.1. Models with spread-out Lagrangians. Recall the semi-simplicial space $C_\bullet$ from Definition 5.1.1 and $C_L^c$ from Definition 5.1.3.

Definition 6.1.1. For $p \in \mathbb{Z}_{\geq 0}$, $D_L^c$ is defined to be the space of tuples $(a, \varepsilon, (W, \ell), V)$ subject to the following conditions:

(i) The tuple $(a, \varepsilon, (W, \ell))$ is an element of $C_p$ with the property that the pair $(W|_{(a_i, a_{i+1})}, W|_{a_{i+1}})$ is $(n-1)$-connected for all $i = 0, \ldots, p-1$.

(ii) $V = (V_0, \ldots, V_p)$ is a $p$-tuple of subspaces

$$V_i \leq H_{n+1}^U(W|_{(a_0-\varepsilon_0, a_p+\varepsilon_p)})$$

for $i = 0, \ldots, p$,

subject to the following conditions:
To topologize $D_p^C$ we use the following construction. Choose once and for all a family of increasing diffeomorphisms

$$\psi = \psi(a_0, \varepsilon_0, a_p, \varepsilon_p) : (0, 1) \to (a_0 - \varepsilon_0, a_p + \varepsilon_p),$$

varying smoothly in the data $(a_0, \varepsilon_0, a_p, \varepsilon_p)$. We then let

$$\bar{\psi} = \bar{\psi}(a_0, \varepsilon_0, a_p, \varepsilon_p) : (0, 1) \times \mathbb{R}^\infty \to (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty$$

be the smooth family of diffeomorphisms given by the product, $\psi \times \text{Id}_{\mathbb{R}^\infty}$. For each $i$, we define a map $\pi_i : D_p^C \to \Psi^\Delta_i((0, 1) \times \mathbb{R}^\infty)$ by

$$(a, \varepsilon, W, \ell, V) \mapsto \left( \bar{\psi}^{-1}(W|_{(a_0, \varepsilon_0, a_p, \varepsilon_p)}), \ell|_{(a_0, \varepsilon_0, a_p, \varepsilon_p)} \circ D\bar{\psi}, \bar{\psi}^{-1}(V_i) \right).$$

Using these maps we obtain an embedding $D_p^C \hookrightarrow C_p \times \Psi^\Delta_i((0, 1) \times \mathbb{R}^\infty)^{p+1}$ defined by the formula

$$(a, \varepsilon, W, \ell, V) \mapsto ((a, \varepsilon, W, \ell), \pi_0(a, \varepsilon, W, \ell, V), \ldots, \pi_p(a, \varepsilon, W, \ell, V)).$$

By this embedding we topologize $D_p^C$ as a subspace of $C_p \times \Psi^\Delta_i((0, 1) \times \mathbb{R}^\infty)^{p+1}$. For $0 < i < p$, the face maps $d_i : D_p^C \to D_{p-1}^C$ are defined by

$$d_i(a, \varepsilon, W, \ell, V) = (a(i), \varepsilon(i), W, \ell, V(i))$$

where $a(i)$, $\varepsilon(i)$, and $V(i)$ are the $(p - 1)$-tuples obtained by removing the $i$-th entry from the $p$-tuples $a$, $\varepsilon$, and $V$ respectively. For the face maps $d_0, d_p : D_p^C \to D_{p-1}^C$, a small change is needed in the definition. The map $d_0$ is defined by

$$d_0(a, \varepsilon, W, \ell, V) = (a(0), \varepsilon(0), W, \ell, V(0)|_{(a_1 - \varepsilon_1, a_p + \varepsilon_p)}).$$

and $d_p$ is defined by

$$d_p(a, \varepsilon, W, \ell, V) = (a(p), \varepsilon(p), W, \ell, V(p)|_{(a_0 - \varepsilon_0, a_{p-1} + \varepsilon_{p-1})}).$$

These face maps are continuous as a consequence of Proposition 2.3.3 and the assignment $[p] \mapsto D_p^C$ makes $D_p^C$ into a semi-simplicial space.

Since any two families of diffeomorphisms as in (6.1.2) are isotopic the topology on $D_p^C$ is independent of the choice of $\psi$. The reason for having distinct subspaces $V_i$ (instead of a single one that restricts to a Lagrangian for every slice) is technical in nature: We do not know how to show that $|C_i^C| \simeq |D_i^C|$ for that variant of the definition.

We filter $D_p^C$ by a sequence of sub-semi-simplicial spaces

$$D_p^{C_n} \subset \cdots \subset D_p^{C_1} \subset D_p^{C_0} \subset D_p^C,$$

defined analogously to (6.1.3).

In order to compare $D_p^C$ to $C_p^C$, we need the following proposition.

(a) For each $i$, the restriction $V_i|_{a_i} \leq H_n(W|_{a_i})$ is a Lagrangian subspace (recall from Section 2.2 that $V_i|_{a_i}$ is the image of $V_i$ under the map $j_i : H_{n+1}^F(W) \to H_n(W|_{a_i})$).

(b) Let $i \neq j$. Then the subspace $V_i|_{a_j} \subseteq H_n(W|_{a_j})$ is contained in the subspace $V_j|_{a_j}$.
Proposition 6.1.4. Let \( p \in \mathbb{Z}_{\geq 0} \). For any \((a, \varepsilon, (W, \ell), V) \in \mathbf{D}_p^C\) the associated tuple

\[
(a, \varepsilon, (W, \ell), V_0|_{a_0}, \ldots, V_p|_{a_p})
\]

is an element of \( \mathbf{C}_p^C \). Thus the correspondence

\[
(a, \varepsilon, (W, \ell), V) \mapsto (a, \varepsilon, (W, \ell), V_0|_{a_0}, \ldots, V_p|_{a_p})
\]

yields a well defined semi-simplicial map \( \mathcal{F}: \mathbf{D}_p^C \to \mathbf{C}_p^C \).

Proof. Let \((a, \varepsilon, (W, \ell), V) \in \mathbf{D}_p^C\) with \( V = (V_0, \ldots, V_p) \). We need to show that for all \( 0 \leq i < p \)

\[
\iota^{\text{in}}(V_i|_{a_i}) = \iota^{\text{out}}(V_{i+1}|_{a_{i+1}}),
\]

where

\[
\iota^{\text{in}}: H_n(W|_{a_i}) \to H_n(W|_{a_i, a_{i+1}})
\]

and

\[
\iota^{\text{out}}: H_n(W|_{a_{i+1}}) \to H_n(W|_{a_i, a_{i+1}})
\]

are the maps induced by inclusion. Let \( x \in V_i|_{a_i} \) and choose \( v \in V_i \) such that \( v|_{a_i} = x \). By Definition 6.1.1 (condition (ii), part (b)), we have \( v|_{a_{i+1}} \in V_{i+1}|_{a_{i+1}} \). Let

\[
\overline{v} \in H_{n+1}(W|_{a_i, a_{i+1}}, W|_{a_i} \sqcup W|_{a_{i+1}})
\]

denote the image of \( v \) under

\[
H_{n+1}^f(W) \xrightarrow{-|a_i, a_{i+1}|} H_{n+1}(W|_{a_i, a_{i+1}}, W|_{a_i} \sqcup W|_{a_{i+1}})
\]

One readily checks from the definition that the diagram

\[
\begin{array}{ccc}
H_{n+1}(W|_{a_i, a_{i+1}}, W|_{a_i} \sqcup W|_{a_{i+1}}) & \xrightarrow{\partial} & H_n(W|_{a_i} \sqcup W|_{a_{i+1}}) \\
\downarrow \iota^{\text{in}} & & \downarrow \iota^{\text{out}} \\
H_{n+1}(W) & \xrightarrow{-|a_i, a_{i+1}|} & H_n(W|_{a_i, a_{i+1}})
\end{array}
\]

commutes up to the sign \((-1)^{\nu+1}\) for \( \nu = 0, 1 \). It follows that

\[
\partial(\overline{v}) = v|_{a_{i+1}} - v|_{a_i}.
\]

By exactness of

\[
H_{n+1}(W|_{a_i, a_{i+1}}, W|_{a_i} \sqcup W|_{a_{i+1}}) \xrightarrow{\partial} H_n(W|_{a_i} \sqcup W|_{a_{i+1}}) \xrightarrow{\iota^{\text{in}} \cdot \iota^{\text{out}}} H_n(W|_{a_i, a_{i+1}}),
\]

we then find

\[
\iota^{\text{in}}(x) = \iota^{\text{in}}(v|_{a_i}) = \iota^{\text{out}}(v|_{a_{i+1}}),
\]

so \( \iota^{\text{in}}(x) \in \iota^{\text{out}}(V_{i+1}|_{a_{i+1}}) \). Thus

\[
\iota^{\text{in}}(V_i|_{a_i}) \leq \iota^{\text{out}}(V_{i+1}|_{a_{i+1}}),
\]

and exchanging indices shows that \( \iota^{\text{in}}(V_i|_{a_i}) \geq \iota^{\text{out}}(V_{i+1}|_{a_{i+1}}) \). \( \square \)

The following theorem is the main result of this section, its proof occupies the entire next section.
Theorem 6.1.7. The semi-simplicial map \( F: D^\ell_* \rightarrow C^\ell_* \) induces the weak homotopy equivalences
\[
|D^\ell_*| \simeq |C^\ell_*|, \quad |D^\ell_{*,D}| \simeq |C^\ell_{*,D}|, \quad |D^\ell_{*,1}| \simeq |C^\ell_{*,1}|
\]
for all \( l \in \mathbb{Z}_{\geq 0} \).

6.2. Proof of Theorem 6.1.7. We will only explicitly prove the weak homotopy equivalence \(|D^\ell_*| \simeq |C^\ell_*|\). The other weak homotopy equivalences asserted in the theorem are established by repeating the exact same argument, which is largely formal. The proof makes use of the simplicial technique introduced in [GaRW 14, Section 6.2] and largely follows [GaRW 14, Section 6.3 & 6.4]. The first step is to define an augmented bi-semi-simplicial space \( \mathcal{C}^\ell_* \rightarrow \mathcal{C}^\ell_{*,-1} \), with \( \mathcal{C}^\ell_{*,-1} = \mathcal{C}^\ell_* \).

Definition 6.2.1. Let \( p \in \mathbb{Z}_{\geq 0} \) and let \( x = (a, \varepsilon, (W, \ell), L) \in \mathcal{C}^\ell_p \). For each \( q \in \mathbb{Z}_{\geq -1} \), we define \( \mathcal{Z}_q(x) \) to be the set of tuples \( (V^0, \ldots, V^q) \) subject to the following conditions:

(i) Each \((a, \varepsilon, (W, \ell), V^j)\) is an element of \( D^\ell_p \). In other words for each \( j \), \( V^j = (V^j_0, \ldots, V^j_p) \) is a \((p+1)\)-tuple of subspaces of \( H_{n+1}^W(a_{0-\varepsilon_0, a_0+\varepsilon_0}) \), subject to the conditions from Definition 6.1.1.

(ii) The equality
\[
V^j_i|_{a_i} = L_i
\]
holds for all \( j = 0, \ldots, q \) and \( i = 0, \ldots, p \). In other words, \( F(a, \varepsilon, W, \ell, V^j) = (a, \varepsilon, W, \ell, L) \) for all \( j = 0, \ldots, q \), where recall that \( F \) is the map from [6.1.3].

For \( p, q \in \mathbb{Z}_{\geq -1} \), the space \( \mathcal{C}^\ell_{p,q} \) is defined by
\[
\mathcal{C}^\ell_{p,q} = \{(x, y) \mid x \in \mathcal{C}^\ell_p \text{ and } y \in \mathcal{Z}_q(x)\}
\]
and topologized as a subspace of \( (D^\ell_p)^q+1 \). The assignment \([p, q] \mapsto \mathcal{C}^\ell_{p,q} \) defines a bi-semi-simplicial space \( \mathcal{C}^\ell_* \). The forgetful maps
\[
\mathcal{C}^\ell_{p,q} \rightarrow \mathcal{C}^\ell_p, \quad (x, y) \mapsto x,
\]
define an augmented bi-semi-simplicial space \( \mathcal{C}^\ell_* \rightarrow \mathcal{C}^\ell_{*,-1} = \mathcal{C}^\ell_* \).

Observation 6.2.2. By condition (i) in the above definition, it follows that \( \mathcal{Z}_q(x) \simeq [\mathcal{Z}_0(x)]^{q+1} \) for all \( x \). It follows that the semi-simplicial set given by the correspondence \([q] \mapsto \mathcal{Z}_q(x)\) is contractible whenever it is non-empty.

Notice that the semi-simplicial space \( \mathcal{C}^\ell_{*,0} \) is nothing but \( D^\ell_* \). Under this identification the forgetful map \( \mathcal{C}^\ell_{*,0} \rightarrow \mathcal{C}^\ell_* \) coincides with the map \( F: D^\ell_* \rightarrow C^\ell_* \). Inclusion of zero-simplices yields an embedding \(|D^\ell_*| = |C^\ell_{*,0}| \hookrightarrow |C^\ell_*|\). To prove Theorem 6.1.7 it will suffice to prove that the maps
\[
|C^\ell_{*,0}| \hookrightarrow |C^\ell_*| \rightarrow |C^\ell_*|
\]
are both weak homotopy equivalences, where the first map is given by inclusion of zero-simplices and the second is induced by the augmentation. We break this up into two steps, Lemma 6.2.3 and Lemma 6.2.5.
Lemma 6.2.3. The map $|C^L_{p,*}| \to |C^L_p|$ induced by the augmentation is a weak homotopy equivalence.

Proof. We will apply [GaRW 14, Theorem 6.2] for each $p \in \mathbb{Z}_{\geq 0}$ to show that the induced maps $|C^L_{p,*}| \to |C^L_p|$ are weak homotopy equivalences for each $p \in \mathbb{Z}_{\geq 0}$. Geometrically realizing the first coordinate will then imply the lemma. We thus need to verify conditions (i), (ii), and (iii) from [GaRW 14, Theorem 6.2] (it is clear that $C^L_{p,*} \to C^L_{p}$ is an augmented topological flag complex as in [GaRW 14, Definition 6.1]).

Condition (i) is proven similarly to [GaRW 14, Proposition 6.10] and so we omit the proof. Condition (iii) is trivial (see Observation 6.2.2). We proceed to verify condition (ii).

Let $x = (a, \varepsilon, (W, \ell), L) \in C^L_p$ be with $L = (L_0, \ldots, L_p)$. We will need to show that $Z_0(x)$ is non-empty. By the definition of $C^L_p$, we have

$$ (6.2.4) \quad \ell^{\text{in}}(L_0) = \ell^{\text{out}}(L_1) $$

as subspaces of $H_n(W|_{a_0,a_1})$, where $\ell^{\text{in}}$ and $\ell^{\text{out}}$ are the maps induced by the inclusions

$$ W|_{a_0} \hookrightarrow W|_{a_0,a_1} \hookleftarrow W|_{a_1}. $$

Let $x_1, \ldots, x_k \in L_0$ be a set of generators. By (6.2.4), for each $i = 1, \ldots, k$, we may choose $y_i \in L_1$ such that

$$ \ell^{\text{in}}(x_i) = \ell^{\text{out}}(y_i). $$

By exactness of

$$ H_{n+1}(W|_{a_0,a_1}), W|_{a_0} \cup W|_{a_1} \xrightarrow{\partial} H_n(W|_{a_0} \cup W|_{a_1}) \xrightarrow{\ell^{\text{in}} + \ell^{\text{out}}} H_n(W|_{a_0,a_1}), $$

it follows that for each $i = 1, \ldots, k$, there exists a class $w_i \in H_{n+1}(W|_{a_0,a_1}, W|_{a_0} \cup W|_{a_1})$ such that

$$ \partial w_i = x_i - y_i. $$

Since $y_i \in L_1$ for all $i$, we can similarly find classes $v_i \in H_{n+1}(W|_{a_1,a_2}, W|_{a_1} \cup W|_{a_2})$, with $\partial_v(v_i) = y_i$ and $\partial_{w_i}(v_i) = -z_i$ for some classes $z_i \in H_n(W|_{a_2})$ with $\ell^{\text{in}}(y_i) = \ell^{\text{out}}(z_i)$. Now consider the element $j_0(w_i) + j_1(v_i) \in H_{n+1}(W|_{a_0,a_2}, W|_{a_0} \cup W|_{a_1} \cup W|_{a_2})$ where

$$ j_\nu : H_{n+1}((W|_{a,\nu+1}), W|_{a_0} \cup W|_{a_1} \cup W|_{a_2}) \to H_{n+1}(W|_{a_0,a_2}, W|_{a_0} \cup W|_{a_1} \cup W|_{a_2}) $$

is the obvious inclusion for $\nu = 0, 1$. In the long exact sequence

$$ \cdots \to H_{n+1}(W|_{a_0,a_2}, W|_{a_0} \cup W|_{a_2}) $$$$ H_{n+1}(W|_{a_0,a_2}, W|_{a_0} \cup W|_{a_1} \cup W|_{a_2}) $$

$$ \partial \to H_n(W|_{a_0} \cup W|_{a_1} \cup W|_{a_2}, W|_{a_0} \cup W|_{a_2}) \to \cdots $$
of the triple \((W|_{a_0} \sqcup W|_{a_2}, W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2}, W|_{[a_0,a_2]})\) it is clearly mapped to zero:

\[
\partial (j_0(w_i) + j_1(v_i)) = \partial \text{in}(w_i) + \partial \text{out}(w_i) + \partial \text{in}(v_i) + \partial \text{out}(v_i) = c(x_i - y_i + y_i - z_i) = 0
\]

where \(c : H_n(W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2}) \to H_n(W|_{a_0} \sqcup W|_{a_1} \sqcup W|_{a_2})\) given by considering \(p,q\) each \((6.2.6)\)

comes from \(H_n(W|_{[a_0,a_2]}, W|_{a_0} \sqcup W|_{a_2})\), as \(c(x_i - z_i)\) comes from \(H_n(W|_{a_0} \sqcup W|_{a_2})\). We can therefore pick preimages \(u_i \in H_{n+1}(W|_{[a_0,a_2]}, W|_{a_0} \sqcup W|_{a_2})\) of \(j_0(w_i) + j_1(v_i)\). These satisfy \(\partial \text{out} u_i = z_i\) and so we can repeat the process until we have constructed a subspace

\[
V^0 \subseteq H_{n+1}(W|_{[a_0,a_p]}, W|_{a_0} \sqcup W|_{a_p}) \cong H_{n+1}^R(W|_{(a_0 - \varepsilon, a_p + \varepsilon)})
\]

By construction and signed commutativity of the diagram

\[
\begin{tikzcd}
H_{n+1}(W|_{[a_0,a_i]}, W|_{a_0} \sqcup W|_{a_i}) \arrow{r}{\partial} \arrow{d}{\cong} & H_n(W|_{a_0} \sqcup W|_{a_i}) \arrow{d}{pr_{a_\nu}} \\
H_{n+1}(W|_{[a_0,a_i]}, W|_{a_0} \sqcup W|_{a_i}) & H_n(W|_{a_\nu})
\end{tikzcd}
\]

for \(\nu = 0, i\) we find \(V^0|_{a_0} = L_0\) and \(V^0|_{a_i} \leq L_i\) for all \(i \neq 0\).

The other required subspaces \(V^i\) are constructed in an entirely analogous fashion. \(\square\)

**Lemma 6.2.5.** The map \(|D^L_*| = |C^L_{*,0}| \leftrightarrow |C^L_{*,*}|\) induced by inclusion of zero-simplices is a weak homotopy equivalence.

*Proof.* The proof of this lemma follows the argument from [GaRW 14, Page 327]. We spell it out as there is a small oversight in the final argument in [GaRW 14], which we correct.

To begin, we resurrect the simplicial space \(\tilde{C}^L_*\) from [5.1.1]; recall that this meant that the inequalities \(a_i + \varepsilon_i < a_{i+1} - \varepsilon_{i+1}\) are replaced by \(a_i \leq a_{i+1}\). There is an evident augmented bi-semi-simplicial space \(\tilde{C}^L_* \to \tilde{C}^L_{*,-1}\) defined similarly with \(\tilde{C}^L_{*,-1} = \tilde{C}^L_*\). By 5.1.2 it suffices to show that \(|\tilde{C}^L_{*,0}| \leftrightarrow |\tilde{C}^L_{*,*}|\) is a weak homotopy equivalence.

We will define a retraction \(r : |\tilde{C}^L_*| \to |\tilde{C}^L_{*,0}|\) which is a weak homotopy equivalence. For each \(p,q \in \mathbb{Z}_{\geq 0}\) there is a map

\[
h_{p,q} : \tilde{C}_{p,q} \to \tilde{C}_{(p+1)(q+1) - 1,0}
\]

given by considering \(p + 1\) regular values, each equipped with \((q + 1)\) collections of subspaces of \(H^R_{n+1}(W)\), as \((p + 1)(q + 1)\) not-necessarily distinct regular values each equipped with a single collection of subspaces of \(H^R_{n+1}(W)\). For example, in the case that \(p = 1\) and \(q = 2\), the map \(\tilde{C}_{1,2} \to \tilde{C}_{5,0}\) is given by sending

\[
((a_0,a_1), (W, \ell), (L_0, L_1), (V^0_0, V^0_1), (V^1_0, V^1_1), (V^2_0, V^2_1))
\]
to the element
\[(a_0, a_0, a_1, a_1), (W, \ell), (L_0, L_0, L_1, L_1), (V^0_0, V^0_1, V^2_1, V^1_0, V^1_1, V^2_1)\],

where we have dropped the data \(\varepsilon = (\varepsilon_0, \varepsilon_1)\) from the notation to save space. Being able to do this is the very reason for having distinct subspaces \(V_i\) for every slice \(W|_{a_i}\). There is also a map
\[(6.2.7) \quad \rho_{p,q} : \Delta^p \times \Delta^q \to \Delta^{(p+1)(q+1)-1} \subset \mathbb{R}^{(p+1)(q+1)}\]

with \((i + (q + 1)j)\)th coordinate given by \((t, s) \mapsto t_j s_i\). Taking the product of these maps yields
\[(6.2.8) \quad r_{p,q} : \hat{C}_{p,q} \times \Delta^p \times \Delta^q \to \hat{C}_{(p+1)(q+1)-1,0} \times \Delta^{(p+1)(q+1)-1}\]

which glue together to give a map \(r : \hat{C}_{\bullet, \bullet} \to \hat{C}_{\bullet, 0}\). It is clear that this map is a retraction and thus the induced map on homotopy groups is surjective. Consider the augmentation map \(|\varepsilon| : \hat{C}_{\bullet, \bullet} \to \hat{C}_{\bullet}| \) in the second bi-semi-simplicial coordinate. By Lemma 6.2.3, this map is a weak homotopy equivalence. The fact that \(r\) induces an injection on homotopy groups will follow once we prove that \(|\varepsilon| : \hat{C}_{\bullet, \bullet} \to \hat{C}_{\bullet}|\) induces the same map on homotopy groups as
\[(6.2.9) \quad \hat{C}_{\bullet, \bullet} \xrightarrow{r} \hat{C}_{\bullet, 0} \xrightarrow{|\varepsilon|} \hat{C}_{\bullet}|\].

In [GaRW 14] it is claimed that the (analogous) two maps are in fact equal, but this is not true: Chasing an element through the composition shows only that it is a degeneracy of its image under \(|\varepsilon|\), as its entries have been repeated as indicated in the example above. It can, however, be checked by hand that the diagram
\[(6.2.10) \quad \hat{C}_{\bullet, \bullet} \xrightarrow{r} \hat{C}_{\bullet, 0} \xrightarrow{|\varepsilon|} \hat{C}_{\bullet} \xrightarrow{p} \hat{C}_{\bullet}^{\text{th}}\]

is commutative, where \(|-|^{\text{th}}\) denotes the thin realization collapsing degenerate simplices. From 5.1.2 we find that \(p\) induces an isomorphism in homotopy groups which completes the argument. \(\square\)

Again the goodness of \(\hat{C}_{\bullet}\) can be avoided by instead noting that the two maps \(|\varepsilon|\) and \(|\varepsilon| \circ r\) are homotopic through a straight line homotopy (through the degenerate simplices by whose appearance they differ). With the two lemmas above it follows that the maps \(|C_{\bullet,0}^\ell| \hookrightarrow |C_{\bullet, \bullet}^\ell| \to |C_{\bullet}^\ell|\) are weak homotopy equivalences. Identifying \(|D_{\bullet}^\ell| = |C_{\bullet,0}^\ell|\), establishes the weak homotopy equivalence \(|D_{\bullet}^\ell| \simeq |C_{\bullet}^\ell|\) and completes the proof of Theorem 6.1.7.

6.3. Flexible models. The following definition builds on [GaRW 14, Definition 2.8]. We do, however, need a slightly more general version than provided by loc.cit. for the proof of 6.3.3 at the end of this section.
Definition 6.3.1. Define $X^E_p$ to be the semi-simplicial space with $p$-simplices consisting of certain tuples $(a, \varepsilon, (W, \ell), (V_0, \ldots, V_p))$ with $a \in \mathbb{R}^{p+1}$, $\varepsilon \in \mathbb{R}_{\geq 0}^{p+1}$, and

$$(W, \ell, V_i) \in \Psi^p_\Delta \left( (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty \right) \quad \text{for all } i = 0, \ldots, p,$$

subject to the following conditions:

(i) $W$ is contained in $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times (-1, 1)^\infty$;
(ii) $a_{i-1} + \varepsilon_{i-1} < a_i - \varepsilon_i$ for $i = 1, \ldots, p$;
(iii) For any two regular values $b < c \in \bigcup_{i=0}^p (a_0 - \varepsilon_0, a_p + \varepsilon_p)$ of the height function $W \rightarrow \mathbb{R}$, the pair $(W|_{[b,c]}, W|_c)$ is $(n-1)$-connected.
(iv) Let $i = 0, \ldots, p$. If $c \in (a_i - \varepsilon_i, a_i + \varepsilon_i)$ is a regular value for the height function $W \rightarrow \mathbb{R}$, then $V_i|c \leq H_n(W|_c)$ is a Lagrangian subspace and $V_j|c \leq V_i|c$ for all other $j = 0, \ldots, p$.

The space $X^E_p$ is topologized in the same way as the space $D^E_p$ and possesses evident face maps $d_i : X^E_p \rightarrow X^E_{p-1}$ for $0 \leq i \leq p$, by forgetting the $i$-th piece of data (and appropriately restricting $V$ and $W$ for $i = 0, p$). Proposition 2.3.3 implies that these maps are continuous and we obtain a semi-simplicial space $X^E_\bullet$. Notice that the principal difference between $X^E_\bullet$ and $D^E_\bullet$ is that for any $(a, \varepsilon, (W, \ell), V) \in X^E_\bullet$ the manifold $W$ is not required to be cylindrical over the intervals $(a_i - \varepsilon_i, a_i + \varepsilon_i)$. Furthermore, these intervals need not even be comprised entirely of regular values for the height function. The formula

$$(6.3.2) \quad (a, \varepsilon, W, \ell, V) \mapsto (a, \varepsilon, W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, \ell|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}, V)$$

induces a semi-simplicial map $D^E_\bullet \rightarrow X^E_\bullet$ which is continuous by Proposition 2.3.3.

Definition 6.3.3. We define a sequence of sub-semi-simplicial spaces of $X^E_\bullet$ (compare (6.1.3)):

(a) $X^{E,D}_\bullet \subset X^E_\bullet$ has as its $p$-simplices those $(a, \varepsilon, (W, \ell), V)$ such that $W$ contains

$$(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times D$$

and such that the restriction of $\ell$ to $(a_0 - \varepsilon_0, a_p + \varepsilon_p) \times D$ agrees with the structure $\ell|_{\mathbb{R} \times D}$ used in the definition of $\text{Cob}^D_\bullet$.

(b) Let $l \in \mathbb{Z}_{\geq -1}$. $X^{E,l}_\bullet \subset X^{E,D}_\bullet$ has as its $p$-simplices those $(a, \varepsilon, (W, \ell), V)$ with the property that for any regular value $c \in \bigcup_{i=0}^p (a_i - \varepsilon_i, a_i + \varepsilon_i)$ of the height function, the manifold $W|c$ is $l$-connected.

Proposition 6.3.4. The map from (6.3.2) induces a weak homotopy equivalence, $|D^E_\bullet| \simeq |X^E_\bullet|$. Similarly, it induces weak equivalences $|D^{E,D}_\bullet| \simeq |X^{E,D}_\bullet|$ and $|D^{E,l}_\bullet| \simeq |X^{E,l}_\bullet|$.

Proof sketch. Let $D^{E,D}_{\bullet,R}$ be the semi-simplicial space with $p$-simplices given by tuples $(a, \varepsilon, (W, \ell), V)$ as in the definition of $D^E_\bullet$, but instead of requiring $W$ to be cylindrical over the intervals $(a_i - \varepsilon_i, a_i + \varepsilon_i)$, we only require $W$ to be regular over the intervals $(a_i - \varepsilon_i, a_i + \varepsilon_i)$. By this we mean that we require each interval $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ to consist entirely of regular values for the height function on
W. There is an embedding $D^L_{\bullet} \hookrightarrow D^L_{\bullet,\alpha}$ and the map from the statement of the proposition factors as the composite,

$$|D^L_{\bullet}| \xrightarrow{(1)} |D^L_{\bullet,\alpha}| \xrightarrow{(2)} |X^L_{\bullet}|.$$

The proposition follows from the fact that both maps in this composition are weak homotopy equivalences. The proof that map (1) is a weak homotopy equivalence proceeds exactly as [GaRW 10, Theorem 3.9], while the proof that map (2) is a weak homotopy equivalence goes through in the same way as [GaRW 14, Proposition 2.20]. □

6.4. Proof of Theorem 4.3.3. Finally, we give the proof of Theorem 4.3.3 which asserts the weak homotopy equivalence $B\text{Cob}^L_{\theta,D} \simeq B\text{Cob}^L_{\theta}$.

Proof of 4.3.3. By the results above it will suffice to prove that the inclusion of semi-simplicial spaces $X^L_{\bullet,D} \hookrightarrow X^L_{\bullet}$ induces a weak homotopy equivalence on geometric realization. This is proven in essentially the same way as [GaRW 14, Proposition 2.16]. In loc.cit. the proof is carried out at the level of spaces of manifolds by construction of a homotopy inverse $r : \psi(\infty, 1) \to \psi_{\theta,D}(\infty, 1)$ to the inclusion. The map $r$ squeezes a given manifold $W \subseteq \mathbb{R} \times (-1, 1)^\infty$ into $\mathbb{R} \times (1/2, 1) \times (-1, 1)^\infty$ and then adds the subset $\mathbb{R} \times S \subset \mathbb{R} \times (-1/2, 1/2) \times (-1, 1)^\infty$, where $S$ is a sphere with $S \cap (-1/2, 0] \times (-1, 1)^\infty = D$. Clearly this procedure provides maps

$$r_p : X^L_{\bullet} \to X^L_{\bullet,D}$$

as well (the Lagrangian being transported in the evident manner) and the homotopies described by Galatius and Randal-Williams show $r_p$ a levelwise homotopy inverse to the inclusion. □

The same argument would not work at the level of $C$- or $D$-spaces as the homotopies move critical points through the sets $(a_i - \epsilon_i, a_i + \epsilon_i)$ and we do not have analogues of the $\psi$-spaces.

7. SURGERY ON MANIFOLDS EQUIPPED WITH A LAGRANGIAN SUBSPACE

In this section we develop some technical results about surgery on manifolds that will be needed in Sections 8 and 9. Without loss of continuity, the reader can skip this section and come back to it when its results are used.

7.1. Transport of Lagrangians. For what follows, let $M$ be a closed, $2n$-dimensional, oriented manifold and let

$$(7.1.1) \quad L \leq H_n(M)$$

be a Lagrangian subspace with respect to the intersection form $(H_n(M), \lambda, \mu)$. Let

$$(7.1.2) \quad \phi : S^k \times D^{2n-k} \to M$$
be an embedding. Let $\widetilde{M}$ denote the manifold obtained by performing surgery on $M$ along the embedding $\phi$ and $M'$ the complement $M \setminus \phi(S^k \times \text{Int}(D^{2n-k}))$. Let finally

$$H_n(M) \xrightarrow{\alpha} H_n(M') \xrightarrow{\beta} H_n(\widetilde{M})$$

denote the maps induced by inclusion. Consider the subspaces,

(7.1.3) \[ L' := \alpha^{-1}(L) \leq H_n(M') \quad \text{and} \quad \tilde{L} := \beta(\alpha^{-1}(L)) \leq H_n(\widetilde{M}). \]

We will use this same notation throughout the rest of the section, which is devoted to proving the following:

**Theorem 7.1.4.** Let $\phi : S^k \times D^{2n-k} \to M$ be an (orientation preserving) embedding as in (7.1.2) and suppose that $k < n$. Then the subspace $\tilde{L} \leq H_n(\widetilde{M})$ is a Lagrangian subspace with respect to $(H_n(\widetilde{M}), \lambda, \mu)$.

The reader willing to believe this result may well skip the rest of this section. The verification is a lengthy but routine homology calculation. We begin by proving the special case of this theorem when $k < n - 1$. This proof of this special case is significantly easier than the $k = n - 1$ case.

**Proof of Theorem 7.1.4 for $k < n - 1$.** By excision we have isomorphisms

(7.1.5) \[ H_*(M, M') \cong H_*(S^k \times D^{2n-k}, S^k \times S^{2n-k-1}), \]

\[ H_*(\widetilde{M}, M') \cong H_*(D^{k+1} \times S^{2n-k-1}, S^k \times S^{2n-k-1}), \]

and since $k \leq n - 2$ it follows that

$$0 = H_n(M, M') = H_{n+1}(M, M') = H_n(\widetilde{M}, M') = H_{n+1}(\widetilde{M}, M').$$

From the long exact sequence associated to the pairs $(M, M')$ and $(\widetilde{M}, M')$ it follows that the maps

$$H_n(M) \xrightarrow{\alpha} H_n(M') \xrightarrow{\beta} H_n(\widetilde{M})$$

denote both isomorphisms. The maps $\alpha$ and $\beta$ are codimension-0 embeddings and thus they preserve both the intersection pairing $\lambda$ and its refinement $\mu$. It follows that $\tilde{L} = \beta(\alpha^{-1}(L)) \leq H_n(\widetilde{M})$ is a Lagrangian subspace. \qed

### 7.2. **Proof of 7.1.4**

We now focus on proving Theorem 7.1.4 in the harder case that $k = n - 1$. Let $\phi : S^k \times D^{2n-k} \to M$ be as in (7.1.2) and let $x \in H_k(M)$ be the class determined by $\phi|_{S^k \times \{0\}} : S^k \to M$. The following is [KeMi 63, Lemma 5.6].

**Proposition 7.2.1.** Let $j : H_{2n-k}(M) \to H_{2n-k}(M, M')$ be the map induced by inclusion and let $\alpha_{2n-k} \in H_{2n-k}(M, M') \cong \mathbb{Z}$ be the generator induced by the orientation on $(D^{2n-k}, S^{2n-k-1})$. The map $j$ is given by the formula

$$j(y) = \lambda(x, y) \cdot \alpha_{2n-k}.$$
for all $y \in H_{2n-k}(M)$, where $\lambda : H_k(M) \otimes H_{2n-k}(M) \to \mathbb{Z}$ is the intersection pairing.

**Corollary 7.2.2.** Let $y \in H_{2n-k}(M)$ be a class such that $\lambda(x, y) = 0$. Then the class $y$ is in the image of the map $H_{2n-k}(M') \to H_{2n-k}(M)$ induced by inclusion.

**Proof.** Immediate from the long exact sequence associated to the pair $(M, M')$. \qed

Now consider the situation of Corollary 7.2.3 again.

**Lemma 7.2.3.** The subspace $L' = \alpha^{-1}(L) \leq H_n(M')$ is a Lagrangian.

**Proof.** Let us start with the inclusion $(L')^\perp \leq L'$. Let $v \in (L')^\perp$ and let $w \in L$. By surjectivity of $\alpha : H_n(M') \to H_n(M)$, we choose $w' \in L' = \alpha^{-1}(L)$ such that $\alpha(w') = w$. Since $v \in (L')^\perp$ we have

$$0 = \lambda(v, w') = \lambda(\alpha(v), w).$$

Since $w$ was arbitrary $\alpha(v) \in L^\perp$ and since $L$ is a Lagrangian it follows that $\alpha(v) \in L$ and so $v \in L' = \alpha^{-1}(L)$. This proves $(L')^\perp \leq L'$.

For the other inclusion suppose that $v, w \in L'$. Since $\alpha$ preserves the intersection pairing we have $\lambda(v, w) = \lambda(\alpha(v), \alpha(w))$. Since $\alpha(v), \alpha(w) \in L$ and $L$ is Lagrangian it follows that $\lambda(v, w) = \lambda(\alpha(v), \alpha(w)) = 0$. This proves that $L'$ is isotropic. The same argument shows that $\mu$ vanishes on $L'$. \qed

**Proof of Theorem 7.1.4** for $k = n - 1$. Let $x \in H_{n-1}(M)$ denote the class represented by the embedding $\phi|_{S^{n-1} \times \{0\}} : S^{n-1} \to M$. Let $x' \in H_{n-1}(M')$ be the unique class that maps to $x$ under the map $H_{n-1}(M') \to H_{n-1}(M)$ induced by inclusion, which is an isomorphism by the long exact sequence associated to the pair $(M, M')$. The proof breaks down into two cases: the case where $x$ is of infinite order and the case where $x$ is of finite order.

**Case 1:** Suppose that the class $x \in H_{n-1}(M)$ has infinite order. By 7.2.3 it will suffice to prove that the map $\beta : H_n(M') \to H_n(\#)$ is an isomorphism. Since $x$ has infinite order it follows that $x' \in H_{n-1}(M')$ has infinite order as well. Since the boundary map $H_n(\# , M') \to H_{n-1}(M')$ of the long exact sequence for the pair $(\#, M')$ sends a generator to $x'$ it is injective. It follows that $\beta : H_n(M') \to H_n(\#)$ is surjective. Since $H_{n+1}(\#, M') = 0$, it follows $\beta$ is injective as well and thus an isomorphism.

**Case 2:** Suppose that $x$ is of order $m < \infty$. It follows that the class $x' \in H_{n-1}(M')$ (that maps to $x$) has order $m < \infty$ as well. As before $x'$ generates the image of $H_n(\#, M') \to H_{n-1}(M')$ and using the same exact sequence as before we obtain

$$0 \longrightarrow H_n(M') \overset{\beta}{\longrightarrow} H_n(\#) \longrightarrow \text{Ker}(\partial) \cong m \cdot \mathbb{Z} \longrightarrow 0.$$ 

Let now $\alpha_{n+1} \in H_{n+1}(M, M') \cong \mathbb{Z}$ denote the generator from Lemma 7.2.1 and let $y' \in H_n(M')$ denote the class $\partial(\alpha_{n+1})$, where $\partial : H_{n+1}(M, M') \to H_n(M')$ is the boundary map (it is represented by $\phi : \{0\} \times S^{n+1} \to M'$). We will need to use the following basic property about $y'$, whose proof we postpone until after the proof of the current proposition.
Claim. The class $y'$ has infinite order. Furthermore, $y' \in L'$ and $\lambda(y', v) = 0$ for all $v \in H_n(M')$.

Let $\tilde{y} = \beta(y')$ for $\beta : H_n(M') \to H_n(\widetilde{M})$. Since $\beta$ is injective it follows that $\tilde{y}$ has infinite order. Moreover, it follows that $\tilde{y} \in \widetilde{L} = \beta(L')$ by the claim. We make one more observation about the class $\tilde{y}$: $\langle \tilde{y} \rangle^\perp = \text{im}(\beta)$.

Indeed, the map $\lambda(\tilde{y}, \cdot) : H_n(\widetilde{M}) \to \mathbb{Z}$ annihilates the image of $\beta$ by the claim above (giving one inclusion) and therefore factors over $\ker(\partial) \cong m \cdot \mathbb{Z}$ by exactness of (7.2.4). Also, it cannot be the null map, as the intersection pairing on $H_n(\widetilde{M})$ is non-degenerate. As a non-zero homomorphism from one infinite cyclic group to another it is injective. These two facts imply that $\lambda(\tilde{y}, v) = 0$ if and only if the image of $v$ under $H_n(\widetilde{M}) \to \ker(\partial) \subset H_n(\widetilde{M}, M')$ is equal to zero, which gives the other inclusion.

We are finally in a position to show that $\widetilde{L}$ is a Lagrangian subspace. Let $w \in \widetilde{L}^\perp$. Since $\tilde{y} \in \widetilde{L}$, we have $\lambda(\tilde{y}, w) = 0$ and thus $w = \beta(w')$ for some $w' \in H_n(M')$. Since $\beta$ preserves the intersection pairing it follows that $w' \in (L')^\perp$. By Lemma 7.2.3 $L'$ is a Lagrangian subspace, so $w' \in L'$ which yields $w \in \widetilde{L}$.

This proves that $\widetilde{L}^\perp \leq \widetilde{L}$. Since $\widetilde{L}$ is by definition equal to $\beta(L')$, $\beta$ preserves the intersection pairing, and $L'$ is an isotropic subspace (i.e. $L' \leq (L')^\perp$), it follows that $\widetilde{L}$ is an isotropic subspace as well, so indeed $\widetilde{L}^\perp = \widetilde{L}$. The fact that $\mu$ vanishes on $\widetilde{L}$ also follows by the selfintersection form being preserved by $\beta$. 

It remains to verify the claim.

**Proof of the Claim.** We begin by showing that the class $y' = \partial(\alpha_{n+1}) \in H_n(M')$ has infinite order. By assumption, the class $x \in H_{n-1}(M)$ has finite order. It follows that $\lambda(x, v) = 0$ for all $v \in H_{n+1}(M)$. It then follows from Lemma 7.2.1 that the map $H_{n+1}(M) \to H_{n+1}(M, M')$ is the zero map. By exactness the boundary map

$$\partial : H_{n+1}(M, M') \to H_n(M')$$

is then injective. Since $y' = \partial(\alpha_{n+1})$ (where $\alpha_{n+1} \in H_{n+1}(M, M') \cong \mathbb{Z}$ is the generator) it follows that $y'$ has infinite order.

Since $y'$ is in the image of the boundary map $\partial$, it follows by exactness that $y'$ is in the kernel of $\alpha : H_n(M') \to H_n(M)$. It follows from this that $y' \in \alpha^{-1}(L) = L'$, since $\alpha^{-1}(L)$ contains the kernel of $\alpha$. This establishes the third assertion of Claim 7.2. Let $v \in H_n(M')$. We have

$$\lambda(v, y') = \lambda(\alpha(v), \alpha(y')) = \lambda(\alpha(v'), 0) = 0.$$

This proves that $\lambda(v, y') = 0$ for all $v \in H_n(M')$. 

8. Surgery on Objects Below the Middle Dimension

Let $l \in \mathbb{Z}_{\geq -1}$. We proceed to prove Theorem 4.3.4 which asserts that there is a weak homotopy equivalence $B\text{Cob}_\theta^{L,l-1} \simeq B\text{Cob}_\theta^{L,l}$ whenever $l \leq n - 1$ and the tangential structure $\theta : B \to$
$BO(2n+1)$ is such that $B$ is $l$-connected and of type $F_{l+1}$. By Theorem 6.1.7 it will suffice to prove the weak homotopy equivalence $|D_{*,0}^{C,l-1}| \simeq |D_{*,l}^{C,l-1}|$. The proof will closely follow [GaRW 14 Section 4], so closely in fact, that we shall forego spelling out the construction of the surgery moves and instead ask the reader to have his copy of [GaRW 14] at the ready. In particular, we will reuse the constructions and notation of [GaRW 14] and only indicate the differences and extra steps that have to be taken.

To give an outline, one considers a bi-semi-simplicial resolution $|D_{*,l}^{C,l}| \to |D_{*,l-1}^{C,l-1}|$, in which a $(p,q)$-simplex consists of an element of $D_{p}^{C,l-1}$ together with $q + 1$ disjoint pieces of surgery data, together with a ‘perform surgery map’ $|D_{*,l}^{C,l}| \to |D_{*,l-1}^{C,l-1}|$ and shows that both of these are weak equivalences. In truth, just as in [GaRW 14], the second map is, however, only defined with source and target replaced by weakly equivalent spaces and the proofs for the maps being weak equivalences are intertwined with these auxiliary spaces as well.

### 8.1. A semi-simplicial resolution.
To conform with the notation of [GaRW 14 Section 4], set $d = 2n+1$, $\kappa = n-1$, $N = \infty$, $L = D$ and fix a positive integer $l < n$. Then the semi-simplicial space $D_{*,l}^{C,l}$ of 6.1.1 agrees with the nerve of the topological poset $D_{\kappa,l}^{\theta,L}$ of [GaRW 14 Section 2.6], except for the appearance of Lagrangians (of course) and our requirement that cobordisms be cylindrical in the $\epsilon$-neighbourhood of the boundary whereas Galatius and Randal-Williams only require the projection onto the first coordinate to not have critical points in the same $\epsilon$-neighbourhood. The second difference will play no role throughout the rest of this paragraph as it is preserved by all constructions to come.

For $x = (a, \varepsilon, (W, \ell_W), V) \in D_{p}^{C,l-1}$, put $x_u = (a, \varepsilon, (W, \ell_W))$ and consider the semi-simplicial space $Y_q^l(x_u)$ of surgery data from [GaRW 14 Definition 4.3]. In correspondence with [GaRW 14 Definition 4.4], set

$$D_{p,q}^{C,l} = \{x \in D_{p}^{C,l-1}, y \in Y_q^l(x_u)\}.$$  

Forgetting the surgery data produces an augmentation $D_{p,q}^{C,l} \to D_{p}^{C,l-1}$. Let us emphasize that the subspaces $(V_0, \ldots, V_p)$ associated to an element $x \in D_{p}^{C,l-1}$ play no role in the resolution. Therefore, the verification of the following result corresponds to that of its counterpart [GaRW 14 Theorem 4.5] (given in [GaRW 14 Section 6]) verbatim.

**Theorem 8.1.1.** Let $l \leq n-1$ and suppose that $\theta : B \to BO(2n+1)$ is such that $B$ is $l$-connected and of type $F_{l+1}$. Then there are weak homotopy equivalences

$$|D_{*,0}^{C,l}| \xrightarrow{\simeq} |D_{*,l}^{C,l}| \xrightarrow{\simeq} |D_{*,l-1}^{C,l-1}|,$$

where the first map is induced by inclusion of zero-simplices and the second is induced by the augmentation.

### 8.2. A surgery move.
To implement the surgery we resurrect the homotopy

$$S : [0,1] \times |D_{\theta,L}^{r,l}(\mathbb{R}^N)_{*,0}| \to |X_{*,l-1}^{l-1}|$$
from [GaRW 14, Lemma 4.7], starting at the forgetful map (followed by the inclusion $D_{\theta,L}^{n,l-1} \subseteq X^{n,l-1}$) and ending at a map that factors through the inclusion $X_{\bullet}^{n,l} \subseteq X_{\bullet}^{n,l-1}$. We would like to extend it to a commutative diagram

$$
\begin{array}{ccc}
[0,1] \times |D_{\bullet,0}^{\mathbb{C},l}| & \xrightarrow{\mathcal{F}} & |X_{\bullet}^{\mathbb{C},l-1}| \\
\downarrow & & \downarrow \\
[0,1] \times |D_{\theta,L}^{n,l}(|\mathbb{R}^{N}|)_{\bullet,0}| & \xrightarrow{\mathcal{S}} & |X_{\bullet}^{\mathbb{C},l-1}| 
\end{array}
$$

with the corresponding properties (and forgetful vertical maps).

Comparing the definitions of $X_{\bullet}^{\mathbb{C},l-1}$ and $X_{\bullet}^{n,l-1}$ all that remains is to produce the homological data on the underlying manifolds given by $S$. To do so recall that $S$ is glued from maps

$$
S_p : [0,1]^{p+1} \times D_{\theta,L}^{n,l}(|\mathbb{R}^{N}|)_{p,0} \rightarrow X_{p}^{n,l-1}
$$
given by

$$(t, (a, \epsilon, (W, l_W)), (e, \ell)) \mapsto (a, \epsilon/2, \mathcal{K}_{e,\ell}^{t}(W, l_W))$$

(after suppressing tangential structures) for $(e, \ell) \in Y_{0}^{t}(a, \epsilon, W)$ and the family

$$
\mathcal{K}_{e}^{t}(W) \in \Psi_{\theta}((a_0 - \epsilon_0, a_p, \epsilon_p) \times \mathbb{R}^{N})
$$

from [GaRW 14, Lemma 4.6]. It therefore suffices to lift these maps $S_p$ to maps

$$
\mathcal{F}_{p} : [0,1]^{p+1} \times D_{p,0}^{\mathbb{C},l} \rightarrow X_{p}^{\mathbb{C},l-1}.
$$

To do so we need to describe how to transport the subspaces $V_0, \ldots, V_p \leq H^{\mathbb{C},l}_{n+1}(W|_{a_0 - \epsilon_0, a_p + \epsilon_p})$ over to the homology group $H^{\mathbb{C},l}_{n+1}(\mathcal{K}_{e_i,\ell_i}(W, l_W))$ for every constituent $(e_i, \ell_i)$ of $(e, \ell)$. Let $W'$ denote the complement $W|_{a_0 - \epsilon_0, a_p + \epsilon_p} \setminus \text{Int(Im}(e_i))$. For each $t \in [0,1]$, let

$$
H^{\mathbb{C},l}_{n+1}(W|_{a_0 - \epsilon_0, a_p + \epsilon_p}) \xrightarrow{\alpha} H^{\mathbb{C},l}_{n+1}(W') \xrightarrow{\beta} H^{\mathbb{C},l}_{n+1}(\mathcal{K}_{e_i,\ell_i}(W, l_W))
$$

denote the maps induced by inclusion. The inclusions of $W'$ are both proper maps and so the homomorphisms $\alpha$ and $\beta_t$ are indeed well-defined.

For $t \in [0,1]$ and $j = \{0, \ldots, p\}$ let

$$
(8.2.1) \quad V^{t}_{j} \leq H^{\mathbb{C},l}_{n+1}(\mathcal{K}_{e_i,\ell_i}(W, l_W))
$$

be the subspace given by

$$
\beta^{t}(\alpha^{-1}(V^{t}_{j}|_{a_0 - \epsilon_0, a_p + \epsilon_p})) \leq H^{\mathbb{C},l}_{n+1}(\mathcal{K}_{e_i,\ell_i}(W, l_W)).
$$

**Proposition 8.2.2.** The above construction defines a (continuous) map

$$
[0,1] \times D_{p,0}^{\mathbb{C},l} \rightarrow \Psi_{\theta}^{\mathbb{C}}((a_0 - \epsilon_0, a_p + \epsilon_p) \times \mathbb{R}^{\infty})
$$

$$(t, (W, l_W), (a, \ell)) \mapsto (\mathcal{K}_{e_i,\ell_i}(W, l_W), V^{t}_{j}),$$
with initial value given by

\[ (K^0_{\epsilon_i, \ell_i}(W, \ell_W), V^0_j) = (W|_{(a_0-\epsilon_0, a_0+\epsilon_0)}, \ell_W|_{(a_0-\epsilon_0, a_0+\epsilon_0)}, V_j|_{(a_0-\epsilon_0, a_0+\epsilon_0)}). \]

For the verification we need:

**Lemma 8.2.3.** The map \( \alpha : H^\text{lf}_{n+1}(W') \rightarrow H^\text{lf}_{n+1}(W|_{(a_0-\epsilon_0, a_0+\epsilon_0)}) \) is an isomorphism in the case that \( l < n-1 \), and is surjective in the case that \( l = n-1 \).

**Proof.** Consider the exact sequence on \( H^\text{lf}_k \) associated to the pair \((W,W')\):

\[ \cdots \rightarrow H^\text{lf}_{n+2}(W,W') \xrightarrow{\partial} H^\text{lf}_{n+1}(W') \xrightarrow{\alpha} H^\text{lf}_{n+1}(W) \rightarrow H^\text{lf}_{n+1}(W,W') \rightarrow \cdots \]

Let \( P_i \) and \( P_i^\theta \) denote the manifolds

\[ P_i = \Lambda_i \times (a_i - \epsilon_i, a_i + \epsilon_i) \times S^l \times \mathbb{R}^{2n-l}, \]

\[ P_i^\theta = \Lambda_i \times (a_i - \epsilon_i, a_i + \epsilon_i) \times S^l \times (\mathbb{R}^{2n-l} \setminus \text{Int}(D^{2n-l})). \]

A simple calculation gives

\[ H^\text{lf}_k(P_i, P_i^\theta) = 0 \quad \text{for all } k < 2n - l + 1. \]

and via the embedding \( e_i \) we have

\[ H^\text{lf}_k(W, W') \cong H^\text{lf}_k(P_i, P_i^\theta) \quad \text{for all } k. \]

by excision. Using this equation in the exact sequence gives the claim. \( \square \)

**Proof of 8.2.2.** The claim of continuity follows immediately from the family \( \mathcal{K}^l \) being locally generated by vector fields (see 2.3.5), which is readily checked from the construction (and is in fact needed to check that the underlying family of manifolds is continuous in [GaRW 14]) and 2.3.8. The manifold part of the initial condition is immediate from the construction of \( \mathcal{K}^l \) (it follows from [GaRW 14 Proposition 4.2 (i)]) and by definition of \( V_j^0 \) we have

\[ V_j^0 = \beta^0(\alpha^{-1}(V_j|_{(a_0-\epsilon_0, a_0+\epsilon_0)})). \]

The maps \( \beta^0 \) and \( \alpha \) agree and so \( V_j^0 = \alpha(\alpha^{-1}(V_j|_{(a_0-\epsilon_0, a_0+\epsilon_0)})) \). To prove that

\[ V_j^0 = V_j|_{(a_0-\epsilon_0, a_0+\epsilon_0)}, \]

it will therefore suffice to show that \( \alpha : H^\text{lf}_{n+1}(W') \rightarrow H^\text{lf}_{n+1}(W|_{(a_0-\epsilon_0, a_0+\epsilon_0)}) \) maps surjectively onto the subspace \( V_j|_{(a_0-\epsilon_0, a_0+\epsilon_0)} \), which follows from the lemma above. \( \square \)

We shall now verify that these subspaces \( V_j^t \) indeed let us define the map \( F_p \)

\[ ((a, \ell, W, \ell_W, V), e, \ell) \in D_p^{\ell, l-1}. \]

Fix \( i \in \{0, \ldots, p\} \) and let \((W_i^t, \ell_i^t)\) denote the family of \( \theta \)-manifolds \( \mathcal{K}^l_{\epsilon_i, \ell_i}(W, \ell_W) \). For each \( j = 0, \ldots, p \) thus

\[ V_j^t \leq H^\text{lf}_{n+1}(W_i^t). \]
Finally, let us denote by \( h : W^i_l \rightarrow \mathbb{R} \) the height function on \( W^i_l \) given by projecting \( W^i_l \subset (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^{\infty-1} \) onto the first coordinate of the ambient space and abuse notation by setting

\[ W := W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)} \]

Lemma 8.2.8. Let \( c \in \bigcup_{k=0}^p (a_k - \varepsilon_k, a_k + \varepsilon_k) \) be a regular value for the height function \( h : W_l^i \rightarrow \mathbb{R} \).

Let

\[
\begin{array}{ccc}
H_n(W|_c) & \xrightarrow{\alpha_c} & H_n(W'|_c) \\
& \beta_c \downarrow & \beta'_c \downarrow \\
& H_n(W^i_l|_c) & H_n(W^i_l|_c)
\end{array}
\]

denote the maps induced by inclusion. Then for any \( j = 0, \ldots, p \), the two subspaces

\[ V^j_l|_c \leq H_n(W^i_l|_c) \quad \text{and} \quad \beta'_c(\alpha_c^{-1}(V^j_l|_c)) \leq H_n(W^i_l|_c) \]

are equal.

Proof. Let

\[
\pi_c : H_{n+1}^l(W) \rightarrow H_n(W|_c) \quad \text{and} \quad \pi'_c : H_{n+1}^l(W') \rightarrow H_n(W'|_c)
\]

denote the restriction maps. To prove the lemma it will suffice to show that

\[ \alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)) \]

The result is then immediate from the commutativity of the diagram

\[
\begin{array}{ccc}
H_{n+1}^l(W') & \xrightarrow{\beta'_c} & H_{n+1}^l(W^i_l) \\
\pi'_c \downarrow & & \beta'_c \downarrow \\
H_n(W'|_c) & \xrightarrow{\beta'_c} & H_n(W^i_l|_c).
\end{array}
\]

To show the equality \( \alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)) \) we need to make some calculations. Recall from Lemma 8.2.3 that \( \alpha : H_{n+1}^l(W') \rightarrow H_{n+1}^l(W) \) is an isomorphism \( l < n - 1 \) and surjective when \( l = n - 1 \). The verification of the equality \( \alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)) \) breaks down into two cases:

Case 1: Suppose \( l < n - 1 \). We desire to show that \( \alpha_c : H_n(W'|_c) \rightarrow H_n(W|_c) \) is an isomorphism. With this, the equality \( \alpha^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)) \) will follow from the commutative diagram

\[
\begin{array}{ccc}
H_{n+1}^l(W) & \xrightarrow{\alpha_c^{-1}} & H_{n+1}^l(W') \\
\pi_c \downarrow & & \pi'_c \downarrow \\
H_n(W|_c) & \xrightarrow{\alpha_c^{-1}} & H_n(W'|_c).
\end{array}
\]

To prove that \( \alpha_c \) is an isomorphism, we need to analyze the pair \( (W|_c, W'|_c) \). This pair takes on two forms depending on whether or not \( c \) is contained in the interval \( (a_i - \varepsilon_i, a_p + \varepsilon_p) \).
Let us first suppose that \( c \in (a_i - \varepsilon_i, a_p + \varepsilon_p) \). In this case we have
\[
P_i|_c = \Lambda_i \times \{c\} \times S^l \times \mathbb{R}^{2n-l},
\]
\[
P_{i|_c} = \Lambda_i \times \{c\} \times S^l \times (\mathbb{R}^{2n-l} \setminus \text{Int}(D^{2n-l})),
\]
where \( P \) and \( P_{\partial} \) are from (8.2.3). Since \( l < n-1 \), it follows that \( H_k(P_i|_c, P_{i|_c}^\partial) = 0 \) for all \( k \leq n+1 \).

Excision for the pair \((W|_c, W'|_c)\) yields
\[
H_k(W|_c, W'|_c) \cong H_k(P|_c, P^\partial|_c) \quad \text{for all } k,
\]
and thus we obtain
\[
H_{n+1}(W|_c, W'|_c) \quad \text{for all } k \leq n + 1.
\]

From the exact sequence associated to \((W|_c, W'|_c)\) it follows that
\[
\alpha_c : H_n(W'|_c) \xrightarrow{\cong} H_n(W|_c)
\]
is an isomorphism whenever \( c \in (a_i - \varepsilon_i, a_p + \varepsilon_p) \) (assuming \( l < n - 1 \)).

For \( c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p) \), we have \( W|_c = W'|_c \) and \( \alpha_c \) is the identity so there is nothing to show.

**Case 2:** Suppose that \( l = n - 1 \). In this case the maps \( \alpha \) and \( \alpha_c \) are not necessarily isomorphisms and so we cannot employ the same argument used above. Consider the commutative diagram
\[
\begin{array}{ccc}
0 & \to & H_{n+1}^H(W) \to H_{n+1}^H(W') \to H_{n+2}^H(W,W') \\
\pi_c & & \alpha_c \downarrow \quad \partial_c \downarrow \\
0 & \to & H_n(W|_c) \to H_n(W'|_c) \to H_{n+1}(W|_c, W'|_c)
\end{array}
\]
which has exact rows. To establish \( \alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)) \), it will suffice to prove that the right-vertical map \( \pi_c \) is surjective: Indeed, the equality \( \alpha_c^{-1}(\pi_c(V_j)) = \pi'_c(\alpha^{-1}(V_j)) \) can then be verified through a simple diagram chase. The map \( \pi_c \) takes on two forms depending on whether or not \( c \) is contained in the interval \((a_i - \varepsilon_i, a_p + \varepsilon_p)\).

So assume that \( c \in (a_i - \varepsilon_i, a_p + \varepsilon_p) \). By (8.2.4) and (8.2.9) it follows that the restriction map
\[
H_{n+2}^H(P_i, P_{i|_c}^\partial) \to H_{n+1}^H(P_i|_c, P_{i|_c}^\partial)
\]
is an isomorphism. By the commutativity of the diagram
\[
\begin{array}{ccc}
H_{n+2}^H(W, W') & \xrightarrow{\cong} & H_{n+2}(P_i, P_{i|_c}^\partial) \\
\pi_c & & \cong \\
H_{n+1}(W|_c, W'|_c) & \xrightarrow{\cong} & H_{n+1}(P_i|_c, P_{i|_c}^\partial)
\end{array}
\]
it follows that \( \pi_c \) is an isomorphism, and hence surjective. This establishes the first case.
For \( c \notin (a_i - \varepsilon_i, a_p + \varepsilon_p) \) we again have \( W'_i|_c = W|_c \) making \( \pi_c \) is surjective since its target vanishes.

The next proposition requires the use of Lemma S.2.8 and the results of Section 7.

**Proposition 8.2.11.** Fix \( j \in \{0, \ldots, p\} \). If \( c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j) \) is a regular value for the height function \( h \), then the submodule \( V_j|_c \leq H_n(W_t|_c) \) is a Lagrangian subspace. Furthermore, for all \( k = 0, \ldots, p \), we have \( V_k|_c \leq V_j|_c \).

**Proof.** Let \( c \in (a_j - \frac{1}{2}\varepsilon_j, a_j + \frac{1}{2}\varepsilon_j) \) is a regular value for the height function \( h \). Proving that \( V_j|_c \) is Lagrangian breaks down into two cases depending on the form the level set \( W_t|_c \) takes: \( c \) is automatically a regular value for \( h : W \to \mathbb{R} \) and by design (compare [GaRW 14, Proposition 4.2 (iv)]) either

(a) there is a diffeomorphism \( W_t|_c \cong W|_c \), rel \( W'|_c \), or
(b) \( W_t|_c \) is obtained from \( W|_c \) by a collection of \( \theta \)-surgeries of degree \( l \).

**Case (a):** Since the diffeomorphism is relative to \( W' \) we obtain a commutative diagram

\[
\begin{array}{c}
H_n(W|_c) \\
\downarrow \varphi \\
H_n(W_t|_c), \\
\end{array}
\begin{array}{c}
\cong \\
\alpha_c \\
\beta_c, \\
\end{array}
\begin{array}{c}
H_n(W'|_c) \\
\downarrow \\
\end{array}\]

where the diagonal map \( \varphi \) is the isomorphism induced by the diffeomorphism

\( W_t|_c \cong W|_c \), rel \( W'|_c \).

Since \( \varphi \) is an isomorphism that preserves the intersection form on \( H_n(W_t|_c) \), it follows that

\( \varphi(V_j|_c) \leq H_n(W_t|_c) \)

is a Lagrangian subspace. Now, \( \alpha_c \) maps surjectively onto the subspace \( V_j|_c \). This fact together with commutativity of the above diagram implies that \( \beta_c(\alpha_c^{-1}(V_j|_c)) = \varphi(V_j|_c) \). By Lemma S.2.8 we have \( \beta_c(\alpha_c^{-1}(V_j|_c)) = V_j|_c \), and thus \( V_j|_c \) is a Lagrangian as well.

**Case (b):** In this case, Theorem 7.1.4 implies that the subspace \( \beta_c(\alpha_c^{-1}(V_j|_c)) \leq H_n(W_t|_c) \) is a Lagrangian. Again, by Lemma S.2.8 we have

\( \beta_c(\alpha_c^{-1}(V_j|_c)) = V_j|_c \),

and thus \( V_j|_c \) is Lagrangian.

To obtain the addendum note that by definition of \( X^L_{c,l} \), we have \( V_k|_c \leq V_j|_c \). Thus, for all \( t \) we have

\( \beta_t(\alpha_c^{-1}(V_k|_c)) \leq \beta_t(\alpha_c^{-1}(V_j|_c)) \).

Another application of Lemma S.2.8 finishes the proof.
With these properties established we can define
\[ F_p : [0, 1]^{p+1} \times D_{p,0}^{L,l} \to X_p^{L,l-1} \]
by
\[(t, (a, \epsilon, (W, \ell_W), V), (e, \ell)) \mapsto (a, \epsilon/2, K_{t,e}(W, \ell_W), V^t),\]
where \( V^t \) is obtained by iterating the above construction just as \( K_{t,e} \) is iteratively built from the various \( K_{t,e_i} \)’s.

The proof of Theorem 4.3.4, which is the goal of this entire section, may now be concluded just as that of [GaRW 14, Theorem 4.1] is in [GaRW 14, Sections 3.3 & 4.4] upon replacing \( S_p \) by \( F_p \). First, the 'perform surgery'-map \( F(1, -) : |D_{*,0}^{L,l}| \to |X_{*,l-1}^{L,l}| \) factors through the inclusion \( i \) of \( |X_{*,l-1}^{L,l}| \) into the target. Secondly, the composition of \( F(1, -) : |D_{*,0}^{L,l}| \to |X_{*,l}^{L,l}| \) with the inclusion \( |D_{*,l}^{L,l}| \to |D_{*,0}^{L,l}| \) given by the empty set of surgery data is a weak equivalence by 6.3.4 and therefore \( F(1, -) \) is surjective on homotopy groups. Thirdly, regarded as a map \( |D_{*,0}^{L,l}| \to |X_{*,l-1}^{L,l}| \), \( F(1, -) \) is homotopic to \( F(0, -) \), which is just forgetful map \( |D_{*,0}^{L,l}| \to |X_{*,l-1}^{L,l}| \), and thus a weak equivalence by 8.1.1. This means that \( F(1, -) : |D_{*,0}^{L,l}| \to |X_{*,l}^{L,l}| \) is also injective on homotopy groups and so finally \( i \) has to be weak equivalence as well.

9. Surgery on Objects in the Middle Dimension

In this section and the next we prove Theorem 4.3.5 which asserts that there are weak homotopy equivalences \( BCob_{g}^{L,n} \to BCob_{g}^{L,n-1} \) whenever \( n \geq 4 \) and \( n \neq 7 \), the tangential structure \( \theta : B \to BO(2n+1)\langle n \rangle \) is weakly once-stable and \( B \) \( n \)-connected. The section is structure similarly to 8, but the Lagrangians really come into play now so many of the geometric arguments are necessarily different from those of [GaRW 14]. The present section essentially contains the formal outline and those statements which do immediately follow from [GaRW 14, Section 5], which we again mimic closely, and some necessary homological arguments, while the new geometric arguments are relegated to the next section.

9.1. A semi-simplicial resolution. We want to consider a semi-simplicial space \( Y_n(x) \) of middle dimensional surgery data on some element \( x \in D_{*,n-1}^{L,n-1} \) à la [GaRW 14, Definition 5.13]. There are two main differences to be taken into account: The minor one is that the entirety of [GaRW 14, Section 5] is written for the case of a \( 2n \)-cobordism and \( n-1 \)-surgeries. Adopting the construction for \( d = 2n+1 \) and \( n \)-surgeries, however, requires nothing more than careful remixing of the numbers \( n-1, n \) and \( n+1 \) that appear as sub- and superscripts. After these adaptions the major point is that we have to add a condition to the definition of \( Y_n(x) \) taking the Lagrangian on \( x \) into account. This condition is in fact the entire raison d’être for carrying the Langrangians through the surgery process.

To keep the section somewhat readable we decided against including an exhaustive list of the numerical changes and only indicate the most pertinent ones. In that spirit we alter the definition of
\[ Y^n_q(x) \] from [GaRW 14, Definition 5.13] for \( x = (a, \varepsilon, (W, \ell_W), V) \in D^{\mathcal{L}, n-1}_p \) as follows: As we want to perform \( n \)-surgeries, the embedding \( e \) is to be of the form

\[
\Lambda \times \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1} \rightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{\infty-1}.
\]

Conditions i) to iv) require no further change. Condition v) should be altered to condition v') below, but the main distinction with the even-dimensional case is the inclusion of condition vi'), the meaning of which is explained in the next lemma:

\[ v') \text{ the manifold arising from } M_i = W|_{a_i} \text{ by surgery along } \partial e_i \text{ is } n \text{-connected.} \]

\[ vi') \text{ The subspace } \sum_{i=0}^p V_i \leq H_n(W'_{(a_0-\varepsilon_0, a_p+\varepsilon_p)}) \text{ is contained in the image of} \alpha : H_n(W') \rightarrow H_n(W'_{(a_0-\varepsilon_0, a_p+\varepsilon_p)}), \]

where \( \alpha \) is induced by the inclusion \( W' \supset W'_{(a_0-\varepsilon_0, a_p+\varepsilon_p)} \) for \( W' = W'_{(a_0-\varepsilon_0, a_p+\varepsilon_p)} \setminus \text{Int(Im(e))} \).

Condition vi') will be crucial in proving that these complexes of surgery data have contractible realizations; this is false for the complexes of surgery data without Lagrangians. vi') is most easily thought of via its following consequence:

**Lemma 9.1.1.** We have \( V_i|M_i \leq \text{im}(H_n(M'_i) \rightarrow H_n(M_i)) \). In particular, for each \( \lambda \in \Lambda_{i,j} \), the homology class represented by the embedding

\[ \partial e_{i,j}^\lambda : \{\lambda\} \times \{0\} \times \{a_i\} \times \{0\} \times \partial D^{n+1} \rightarrow M_i \]

(obtained by restricting \( \partial e_i \)) is contained in \( V_i|M_i \).

**Proof.** The first claim is trivial. By Lemma 7.2.1 we have an exact sequence

\[
H_n(M'_i) \longrightarrow H_n(M_i) \overset{\lambda(\cdot, [\partial e_i^\lambda])}{\longrightarrow} \mathbb{Z}.
\]

Therefore \( [\partial e_i^\lambda] \) pairs trivially with the entire image of \( H_n(M'_i) \), in particular with \( V_i|M_i \), by condition vi'). Since \( V_i|M_i \) is a Langrangian we find \( [\partial e_i^\lambda] \in V_i|M_i \). \( \square \)

Now, just as in [GaRW 14, Definition 5.13] put

\[
D^{\mathcal{L}, n}_{p,q} = \{ x \in D^{\mathcal{L}, n-1}_p, y \in Y^n_q(x) \}
\]

augmented over \( D^{\mathcal{L}, n-1}_p \) by forgetting \( y \).

**Theorem 9.1.2.** Let \( n \geq 4, n \neq 7 \) Then the maps

\[
|D^{\mathcal{L}, n}_{\bullet, 0}| \rightarrow |D^{\mathcal{L}, n}_{\bullet, \bullet}| \rightarrow |D^{\mathcal{L}, n-1}_{\bullet, \bullet}|,
\]

induced by inclusion of zero-simplices and the augmentation, respectively, are weak equivalences.
The claim for the second map will be taken up in the next section. Granting this for now, it follows that the former map is an equivalence from the argument provided at the beginning of [GaRW 14, Section 6.1], once the correction we spelled out in 6.2.5 is taken into account.

9.2. A surgery move. To conclude the proof of 4.3.5 assuming 9.1.2 we proceed just as in the previous section by resurrecting the surgery move of [GaRW 14, Section 5.2]. This requires another bout of index adjustments to obtain the family that gives the surgery move and then some homological calculations to verify that it interacts correctly with the homological data.

To start the former we follow [GaRW 14, Section 5.2] and consider the submanifold

\[ K = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |y|^2 = \rho(|x|^2 - 1)\}. \]

Using this a starting point the construction of [GaRW 14, Sections 4.4 & 5.2] we obtain for each \((a, \varepsilon, (W, \ell_W), V, (e, \ell)) \in D_{p,0}^{n-1}\) a continuous family of manifolds

\[ K_{(e, \ell)}^t(W, \ell_W) \in \Psi_{\theta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty) \]

and from it we want to define maps

\[ F_p: [0, 1]^{p+1} \times D_{p,0}^{n-1} \to X_p^{\mathcal{L}, n-1} \]

just as in [GaRW 14, Sections 4.4 & 5.4].

To this end we need to check that \(K_{(e, \ell)}^t(W, \ell_W)\) preserves the connectivity assumption on \(W\), i.e. condition iii) of 6.3 on the one hand and produce new homological data on \(K_{(e, \ell)}^t(W, \ell_W)\) from \(V\) on the other. The former is handled by [GaRW 14, Proposition 5.12 (iii)] whose claim is not affected by the change in \(K\): The critical points of the Morse function arising in the proof are now of index \(n\) and \(n+1\), and this increase cancels against the dimension increase from \(2n\) to \(2n+1\).

To construct the homological data for \(K_{(e, \ell)}^t(W, \ell_W)\) we use the same formula as in (8.2.1): Given subspaces \(V_0, \ldots, V_p \leq H^\text{hf}_{n+1}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})\) we again set

\[ V_j^t \leq H^\text{hf}_{n+1}(K_{e_i, \ell_i}^t(W, \ell_W)) \]

equal to

\[ \beta^t(\alpha^{-1}(V_j|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)})) \leq H^\text{hf}_{n+1}(K_{e_i, \ell_i}^t(W, \ell_W)), \]

where \(W'\) again denotes the complement \(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)} \setminus \text{Int}(\text{Im}(e_i))\) and

\[ H^\text{hf}_{n+1}(W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}) \xrightarrow{\alpha} H^\text{hf}_{n+1}(W') \xrightarrow{\beta^t} H^\text{hf}_{n+1}(K_{e_i, \ell_i}^t(W, \ell_W)) \]

denote the maps induced by inclusion. The following proposition is the analogue of Proposition 8.2.2.

Proposition 9.2.1. The above construction defines a (continuous) map

\[ [0, 1] \times D_{p,0}^{n} \to \Psi_{\theta}^t((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^\infty) \]

\[ (t, (W, \ell_W), V, (e, \ell)) \mapsto (K_{e_i, \ell_i}^t(W, \ell_W), V_j^t), \]
with initial value given by
\[(K_{ε_1,ε_2}^0(W, tW), V_j^0) = (W|_{(a_0-ε_0, a_p+ε_p)}, t|_{(a_0-ε_0, a_p+ε_p)}, V_j|_{(a_0-ε_0, a_p+ε_p)}) .\]

**Proof.** This is entirely analogous to 8.2.2 except that the surjectivity of α is directly implied by condition vi') from the definition of \( Y_n \).

To verify that the \( V_j^i \) are indeed eligible subspace we again need:

**Lemma 9.2.2.** Let \( c \in \bigcup_{k=0}^{p}(a_k-ε_k, a_k+ε_k) \) be a regular value for the height function \( h : W_t^i \to \mathbb{R} \).

Let
\[
\begin{array}{c}
H_n(W|_c) \xrightarrow{α_c} H_n(W'|_c) \xrightarrow{β_c^i} H_n(W''|_c)
\end{array}
\]
denote the maps induced by inclusion. Then for any \( j = 0, \ldots, p \), the two subspaces
\[ V_j^i|_c \leq H_n(W_t^i|_c) \quad \text{and} \quad β_c^i(α_c^{-1}(V_j|_c)) \leq H_n(W_t^i|_c) \]
are equal.

**Proof.** Consider the commutative diagram
\[(9.2.3) \quad \begin{array}{ccc}
H_n(W'|_c) & \xrightarrow{α_c} & H_n(W''|_c)
\end{array}
\]
\[ \begin{array}{c}
H_n(W|_c) \xrightarrow{π_c} \end{array} \]

As in the proof of Lemma 8.2.2, it will suffice to prove the equality \( α_c^{-1}(π_c(V_j)) = π_c^i(α_1^{-1}(V_j)) \). As in the proof of Lemma 8.2.2, let \( P_i \) and \( P_i^0 \) be the manifolds
\[(9.2.4) \quad P_i = \Lambda_i \times (a_i-ε_i, a_p+ε_p) \times S^n \times \mathbb{R}^n,
\]
\[ P_i^0 = \Lambda_i \times (a_i-ε_i, a_p+ε_p) \times S^n \times (\mathbb{R}^n \setminus \text{Int}(D^n)).\]

As before, we have excision isomorphisms
\[ H_k(W, W') \cong H_k(P_i, P_i^0) \]
\[ H_k(W|_c, W'|_c) \cong \begin{cases}
H_k(P_i|_c, P_i^0|_c) & \text{if } c \in (a_i-ε_i, a_p+ε_p),
0 & \text{if } c \notin (a_i-ε_i, a_p+ε_p).
\end{cases} \]

It follows that \( H_k^i(W, W') \) is trivial in all degrees other than \( (n+1) \) and that \( H_k(W|_c, W'|_c) \) is trivial in all degrees other than \( n \) (if \( c \notin (a_i-ε_i, a_p+ε_p) \) then \( H_k(W|_c, W'|_c) \) is trivial in all degrees). Using this together with the long exact sequences on \( H_k^i \) and \( H_* \) associated to the pairs \( (W, W') \) and \( (W|_c, W'|_c) \), it follows that both maps \( α \) and \( α_c \) are injective. By condition vi'), every element of \( V_j \) lies in the image of \( α \). Using these facts, a simple diagram chase in (9.2.3) proves that \( α_c^{-1}(π_c(V_j)) = π_c^i(α_1^{-1}(V_j)) \).

The next proposition is the analogue of Proposition 8.2.11.
Proposition 9.2.5. Let $j = 0, \ldots, p$. If $c \in (a_j - \frac{1}{2} \varepsilon_j, a_j + \frac{1}{2} \varepsilon_j)$ is a regular value for the height function $h$, then the submodule $V^t_j|c \leq H_n(W^t_i|c)$ is a Lagrangian subspace. Furthermore, $V^t_k|c \leq V^t_j|c$ for $k = 0, \ldots, p$.

Proof. By design of the surgery move there are two cases depending on the form that the level set $W^t_i|c$ takes. To see that $V^t_j|c$ is indeed a Lagrangian we separate the cases again:

Case 1: There is a diffeomorphism $W^t_i|c \cong W|c$, rel $W'\underset{c}{\times}$. This is entirely the same as Case (a) in [8.2.11] except that $V_j|c$ lies in the image of $\alpha_c$ directly by condition vi'). The property that $V^t_k|c \leq V^t_j|c$ for $k = 0, \ldots, p$ also follows just as in [8.2.11] using [9.2.2] instead of [8.2.8].

Case 2: The manifold $W^t_i|c$ is obtained from $W|c$ by a collection of surgeries in degree $n$ and $W^t_i|c$ is $n$-connected. It follows that $H_n(W^t_i|c) = 0$ and thus $V^t_j|c$ is automatically Lagrangian and the claimed containment is a trivial equality. □

With these proposition established the proof is again concluded just as in [GaRW 14, Sections 4.4 & 5.4] (outlined at the end of [8.2]).

10. Contractibility of the Space of Surgery Data

We finally prove Theorem [0.1.2] which asserts that there are weak homotopy equivalences,

$$|D^{\mathcal{L},n}| \rightarrow |D^{\mathcal{L},n-1}|,$$

where the first map is induced by the inclusion of zero-simplices and the second is induced by the augmentation. As mentioned before, the first homotopy equivalence $|D^{\mathcal{L},n}| \rightarrow |D^{\mathcal{L},n-1}|$ is deduced from the second just as in Lemma [6.2.5] (see also [GaRW 14, Page 327]) and so we omit the proof of this and focus on establishing the second weak homotopy equivalence, $|D^{\mathcal{L},n}| \rightarrow |D^{\mathcal{L},n-1}|$. We would like to apply Theorem [GaRW 14, Theorem 6.2] again but it turns out that a slightly stronger version is required, even in [GaRW 14] as pointed out in the erratum [GaRW 14e].

10.1. A stronger simplicial technique. It was observed by the second author that property iii) of [GaRW 14, Theorem 6.2] does not hold in the case of middle dimensional surgery, already in the even dimensional case. This oversight was corrected in the erratum [GaRW 14e] and we will need to use the following strengthening from [BoPe 15, Theorem 6.4], which was abstracted from [GaRW 14e].

Consider a symmetric relation $\mathcal{R}$ that is open and dense as a subset of the fibred product $X_0 \times_{X_{-1}} X_0$.

Theorem 10.1.1. Let $X_\bullet \rightarrow X_{-1}$ be an augmented topological flag complex that satisfies conditions (i) and (ii) of [GaRW 14, Theorem 6.2]. Let $\mathcal{R} \subset X_0 \times_{X_{-1}} X_0$ be an open and dense symmetric relation with the property that $X_1 \subset \mathcal{R}$. Suppose that $X_\bullet \rightarrow X_{-1}$ satisfies the following further condition:

(iii)* Let $x \in X_{-1}$. Given:
• a non-empty subset, \( \{v_1, \ldots, v_m\} \subset \varepsilon^{-1}(x) \), whose elements are pairwise related by \( R \), and

• an arbitrary subset, \( \{w_1, \ldots, w_k\} \subset \varepsilon^{-1}(x) \), such that \((v_i, w_j) \in X_1\) for all \( i, j \), there exists \( v \in \varepsilon^{-1}(x) \) such that \((v, v_i) \in X_1\) and \((v, w_j) \in X_1\) for all \( i, j \).

If condition (iii)* is satisfied for all \( x \in X_{-1} \) then the induced map \( |X_\bullet| \to X_{-1} \) is a weak homotopy equivalence.

10.2. **Proof of Theorem 9.1.2.** To apply \[10.1.1\] we will yet another modification of \( D*_{\bullet, n} \). Define the core of the surgery tube to be

\[
C = \{0\} \times (-6, -2) \times \{0\} \times D^{n+1} \subset \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1}.
\]

**Definition 10.2.2.** For \( x = (a, \varepsilon, (W, \ell_W), V) \in D*_{\bullet, n-1} \), let \( \bar{Y}_\bullet(x) \) be the semi-simplicial space from Section 9.1 except now we only ask the map \( e \) to be a smooth embedding on a neighborhood of the subset

\[
\Lambda \times C \subset \Lambda \times (\mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1}).
\]

Furthermore, define a bi-semi-simplicial space by

\[
\hat{D}_{p,q}^{L,n} = \{(x, y) \mid x \in D_{p}^{L,n-1}, y \in \bar{Y}_q(x)\}.
\]

Using the projection we obtain an augmented bi-semi-simplicial space \( \hat{D}_{\bullet, n}^{L,n} \to \hat{D}_{\bullet, n}^{L,n} \) with \( \hat{D}_{\bullet, -1}^{L,n} = \hat{D}_{\bullet, n}^{L,n} \). Let \( \mathcal{T} \subset \hat{D}_{p,0}^{L,n} \times \hat{D}_{p,-1}^{L,n} \) be the subset consisting of those

\[
((a, \varepsilon, (W, \ell_W), V), (\Lambda_1, \delta_1, e_1, \ell_1), (\Lambda_2, \delta_2, e_2, \ell_2))
\]

such that the embeddings \( e_1|_{\Lambda_1 \times C} \) and \( e_2|_{\Lambda_2 \times C} \) are transverse. This subset \( \mathcal{T} \) is clearly a symmetric and open relation. By the Thom transversality theorem applied to each of the fibres over \( \hat{D}_{p,-1}^{L,n} \) we see that it is a dense subset of the fibred product \( \mathcal{T} \subset \hat{D}_{p,0}^{L,n} \times \hat{D}_{p,-1}^{L,n} \).

Just as in \[GalRW 14\] Proposition 6.15 it follows that the inclusion \( \hat{D}_{\bullet, n}^{L,n} \to D_{\bullet, n}^{L,n} \) is a level-wise weak homotopy equivalence and thus it induces a weak homotopy equivalence \( \hat{D}_{\bullet, n}^{L,n} \simeq |D_{\bullet, n}^{L,n}| \). To prove Theorem 9.1.2 we will as before show that for each \( p \in \mathbb{Z}_{\geq 0} \), the augmented topological flag complex

\[
(10.2.3) \quad \hat{D}_{p,\bullet}^{L,n} \to \hat{D}_{p}^{L,n-1}
\]

induces a weak homotopy equivalence \( |\hat{D}_{p,\bullet}^{L,n}| \simeq \hat{D}_{p}^{L,n-1} \). We shall do so by applying \[10.1.1\] for the relation \( \mathcal{T} \).

10.3. **Verification of Condition (iii)*.** The main technical tool that we use to establish condition (iii) is the proposition stated below. For what follows let \( n \geq 4 \), \( M_0 \) and \( M_1 \) two \((n-1)\)-connected, 2n-dimensional, closed manifolds, and \( W \) a cobordism between \( M_0 \) and \( M_1 \) that is \((n-1)\)-connected as well.
Proposition 10.3.1. Let
\[ f, g_1, \ldots, g_k : (S^n \times [0,1], S^n \times \{0,1\}) \to (W, M_0 \sqcup M_1) \]
be a collection of embeddings and let \( x, y_1, \ldots, y_k \in H_n(M_0) \) denote the classes represented by
\[
\left. f \right|_{S^n \times \{0\}}, \left. g_1 \right|_{S^n \times \{0\}}, \ldots, \left. g_k \right|_{S^n \times \{0\}}
\]
respectively. Let \( K_1, \ldots, K_l \subset W \) be a collection of pairwise transverse submanifolds of codimension \( \geq 3 \), and let \( K \) denote the union \( \bigcup_{i=1}^l K_i \). Suppose that the following conditions are met:

(a) \( \lambda(x, y_i) = 0 \) for all \( i = 1, \ldots, k \);
(b) the embeddings \( g_1, \ldots, g_k \) are pairwise transverse.
(c) the images of \( f \) and \( g_1, \ldots, g_k \) are contained in the complement, \( W \setminus K \).

Then there exists an isotopy \( f_t : (S^n \times [0,1], S^n \times \{0,1\}) \to (W, M_0 \sqcup M_1) \) with \( t \in [0,1] \), that satisfies:

- \( f_0 = f \),
- \( f_t(S^n \times [0,1]) \subset W \setminus K \) for all \( t \in [0,1] \),
- \( f_1(S^n \times [0,1]) \cap g_i(S^n \times [0,1]) = \emptyset \) for all \( i = 1, \ldots, k \).

Suppose further that \( f \) is such that \( f(S^n \times \{0\}) \cap g_i(S^n \times \{0\}) = \emptyset \) for all \( i = 1, \ldots, k \). Then the isotopy \( f_t \) can be chosen so that \( f_t|_{S^n \times \{0\}} = f|_{S^n \times \{0\}} \) for all \( t \in [0,1] \).

Proof. By condition (a) we may apply the Whitney trick [Mi 65, Theorem 6.6] inductively to obtain an isotopy of \( f|_{S^n \times \{0\}} \), that pushes \( f(S^n \times \{0\}) \) off of the submanifolds \( g_1(S^n \times \{0\}), \ldots, g_k(S^n \times \{0\}) \subset M_0 \), while staying in the complement, \( M_0 \setminus (M_0 \cap K) \). Thus we reduce to the case where
\[
 f(S^n \times \{0\}) \cap g_i(S^n \times \{0\}) = \emptyset \quad \text{for all} \quad i = 1, \ldots, k.
\]
We remark that in order to inductively apply the Whitney trick as we did above, it is necessary that the submanifolds \( g_1(S^n \times \{0\}), \ldots, g_k(S^n \times \{0\}) \subset M_0 \) be pairwise transverse, see [BoPe 15, Proposition 6.9]. We also remark that in order to apply Whitney trick in this situation it is necessary that the submanifolds \( K_1, \ldots, K_l \subset W \) be pairwise transverse and have codimension \( \geq 3 \). Indeed, the lower bound on the codimension, together with the pairwise transversality, ensures that the complement \( W \setminus K \) will be simply connected. Simple-connectivity of ambient space is required to apply the Whitney trick.

Let \( W', M_0' \), and \( M_1' \) denote the complements \( W \setminus K, M_0 \setminus (M_0 \cap K) \), and \( M_1 \setminus (M_1 \cap K) \). Since the co-dimension of \( K \) is greater than or equal to 3, it follows that the pair \((W', M_0')\) is 2-connected. To prove the corollary, it will suffice to construct an isotopy
\[
 f_t : (S^n \times [0,1], S^n \times \{0,1\}) \to (W', M_0' \sqcup M_1'), \quad t \in [0,1],
\]
with \( f_0 = f \) and \( f_t|_{S^n \times \{0\}} = f|_{S^n \times \{0\}} \) for all \( t \in [0,1] \), such that
\[
 f(S^n \times [0,1]) \cap g_i(S^n \times [0,1]) = \emptyset
\]
for all $i = 1, \ldots, k$. Such an isotopy exists by inductive application of higher dimensional half-Whitney trick from [BoPe 15] Theorem C.3 using the same inductive argument employed in [BoPe 15] Corollary 5.10.1. We note that here as in [BoPe 15] Corollary 5.10.1, it is essential that the embeddings $g_1, \ldots, g_n$ are pairwise transverse.

The next proposition establishes condition (iii)*. The proof is very similar to [BoPe 15] Lemma 5.11] and [GaRW 14] Proposition 6.19] except in our situation we must take the intersection form and the Lagrangian subspaces into account.

**Proposition 10.3.2.** For $p \in \mathbb{Z}_{\geq 0}$, let $x = (a, \varepsilon, (W, \ell, W), V) \in \mathbb{D}^{C,n-1}_p$.

- Let $\{v_1, \ldots, v_k\} \in \tilde{Y}_0(x)$ be a non-empty collection of elements that are pairwise transverse.
- Let $\{w_1, \ldots, w_s\} \in \tilde{Y}_0(x)$ be a collection of elements with $(v_i, w_j) \in \tilde{Y}_1(x)$ for all $i, j$.

Then there exists $u \in \tilde{Y}_0(x)$ such that $(u, v_j) \in \tilde{Y}_1(x)$ and $(u, w_j) \in \tilde{Y}_1(x)$ for all $i, j$.

**Proof.** For each $j = 1, \ldots, k$, let $(\Lambda_j^e, \delta_j^e, e_j^e, \ell_j^e)$ denote the element $v_j$, and for $r = 1, \ldots, s$ let $(\Lambda_r^w, \delta_r^w, e_r^w, \ell_r^w)$ denote the element $w_r$. We temporarily set $u = v_1$ and write $u = (\Lambda, \delta, e, \ell)$. Since $(w_r, v_j) \in \tilde{Y}_1(x)$ for all $r, j$, it follows that

$$e_j^e(\Lambda_j^e \times C) \cap e_r^w(\Lambda_r^w \times C) = \emptyset \quad \text{for all } j, r,$$

where $C$ is the core from Definition 10.2.2. Let $K \subset W$ denote the union $\bigcup_{r=1}^s e_r^w(C) \subset W$ and let $W'$ denote $W \setminus K$. We have $e(\Lambda \times C) \subset W'$ and $e_j^e(\Lambda_j^e \times C) \subset W'$ for all $j = 1, \ldots, k$.

By Remark 9.1.11 the homology classes in $H_n(W|_{a_0})$ determined by the submanifolds

$$e_j^e(\Lambda_j^e \times C) \cap W|_{a_0} \subset W|_{a_0} \quad j = 1, \ldots, k,$$

are all contained in the subspace $V_0|_{a_0} \subset H_n(W|_{a_0})$, which is Lagrangian by definition. Similarly, for each $\lambda \in \delta^{-1}(i)$, the homology class determined by the submanifold

$$e(\Lambda \times C) \cap W|_{a_0} \subset W|_{a_0},$$

is contained in $V_i|_{a_0} \subset H_n(W|_{a_0})$ as well. Since the intersection form vanishes on $V_0$, and the set $\{v_1, \ldots, v_k\}$ is in general position, it follows from Proposition 10.3.1 that for each $\lambda \in \Lambda$, there exists an isotopy of $e(\Lambda \times C)$ that pushes $e(\Lambda \times C) \cap W|_{[a_0,a_1]}$ off of $e_j^e(\Lambda_j^e \times C) \cap W|_{[a_0,a_1]}$ for all $j = 1, \ldots, k$, and keeps $e(\Lambda \times C)$ inside of $W' = W \setminus K \subset W$. The rest of the proof of [GaRW 14] Proposition 6.19] (or [BoPe 15] Lemma 5.11]) now applies.

10.4. **Verification of Condition (ii).** To prove Condition (ii), it will suffice to show that $\tilde{Y}_0(x)$ is non-empty for any $p$-simplex $x \in \mathbb{D}^{C,n-1}_p$. This will require us to develop some preliminary results. The technical tools involved include the embedding theorems of Haefliger and Hudson from [Ha 61] and [Hu 69]. Both of these embedding theorems require the manifolds involved to be above a certain dimension, and this requirement is the source of our condition on the integer $n$.

Let $n \geq 4$ and $M_0, M_1$, and $W$ be as in the last section.
Proposition 10.4.1. Every relative homology class

\[ x \in H_{n+1}(W, M_0 \sqcup M_1) \]

is represented by an embedding \((S^n \times [0,1], S^n \times \{0,1\}) \to (W, M_0 \sqcup M_1)\).

Proof. Let \(x \in H_{n+1}(W, M_0 \sqcup M_1)\) be as in the statement of the proposition. Consider the boundary map

\[ \partial : H_{n+1}(W, M_0 \sqcup M_1) \to H_n(M_0 \sqcup M_1), \]

and let \(y\) denote the class \(\partial(x)\). By the Hurewicz theorem (applied to \(\pi_n(M_0)\) and \(\pi_n(M_1)\)), the class \(y\) is represented by a map, \(\phi : S^n \times \{0,1\} \to M_0 \sqcup M_1\) sending \(S^n \times \{i\}\) into \(M_i\) for \(i = 0,1\). Let \(\iota_i : M_i \hookrightarrow W\) denote the inclusion. By exactness, the class \(y\) maps to zero under \(H_n(M_0 \sqcup M_1) \to H_n(W)\). It follows that the maps,

\[ \iota_0 \circ \phi|_{S^n \times \{0\}}, \quad -\iota_1 \circ \phi|_{S^n \times \{1\}} : S^n \to W, \]

are homotopic, where \(-\iota_1 \circ \phi|_{S^n \times \{1\}}\) denotes the pre-composition of \(\iota_1 \circ \phi|_{S^n \times \{1\}}\) with some reflection (which reverses orientation). It then follows that there exists a map

\[ \Phi : (S^n \times [0,1], S^n \times \{0,1\}) \to (W, M_0 \sqcup M_1) \]

such that

\[ \Phi|_{S^n \times \{0\}} = \iota_0 \circ \phi|_{S^n \times \{0\}} \quad \text{and} \quad \Phi|_{S^n \times \{1\}} = -\iota_1 \circ \phi|_{S^n \times \{1\}}. \]

Using Haefliger’s and Hudson’s embedding results from [Ha 61] and [Hu 69], we may deform \(\Phi\), to a new map \(\Phi'\) such that \(\Phi'\) is an embedding. We note that the use of these embedding theorems is precisely where the assumption \(n \geq 4\) comes into play.

Now, let \(w \in H_{n+1}(W, M_0 \sqcup M_1)\) denote the class represented by this embedding \(\Phi'\). It follows that, \(\partial w = y = \partial x\). Let \(v\) denote the difference \(w - x \in H_{n+1}(W, M_0 \sqcup M_1)\). The class \(v\) is in the kernel of \(\partial\) and thus is in the image of \(H_{n+1}(W) \to H_{n+1}(W, M_0 \sqcup M_1)\). Since \(W\) is \((n-1)\)-connected, by Haefliger’s embedding theorem [Ha 61] we may assume that the map \(h : S^{n+1} \to W\) represents the class \(w - v = w - (w - x) = x\). The map \(\Phi'\) may not be embedding because the image of \(\Phi'\) may have non-empty intersection with the image of \(h\). However, with the map \(\Psi\) constructed, we may apply Hudson’s theorem again [Hu 69 Theorem 1] to find a homotopy of \(\Psi\) to a new map \(\hat{\Psi}\) which is an embedding. By Hudson’s theorem, this homotopy may not be fixed on the boundary of \(S^n \times [0,1]\), none-the-less the resulting map \(\hat{\Psi}\) still represents the class \(x \in H_{n+1}(W, M_0 \sqcup M_1)\). This concludes the proof of the proposition. □

Theorem 10.4.3. For \(p \in \mathbb{Z}_{\geq 0}\), let \(x = (a, \varepsilon, (W, \ell_W), V) \in D_{p,-1}^{\mathbb{C}, n}\). The set \(\hat{Y}_0(x)\) is non-empty.
By carrying out the exact same construction for all \( i \in \{0, \ldots, p\} \), we obtain embeddings \( \hat{\varphi}_i \) as in (10.4.5). Applying Proposition 10.3.1, we may arrange for

\[
\varphi_i : \Lambda_i \times [a_i, a_p] \times S^n \longrightarrow W_{[a_i, a_p]}
\]

with the following properties:

(a) The homology classes represented by the embeddings

\[
\varphi_i|_{\lambda \times [a_i, a_p] \times S^n} : \lambda \times [a_i, a_p] \times S^n \longrightarrow W_{[a_i, a_p]}, \quad \lambda \in \Lambda_i,
\]

are all contained in the subspace \( V_i[a_i, a_p] \leq H_{n+1}(W_{[a_i, a_p]}, M_i \cup M_p) \).

(b) The collection of embeddings

\[
\varphi_i|_{\lambda \times \{a_i\} \times S^n} : \Lambda_i \times \{a_i\} \times S^n \longrightarrow M_i, \quad \lambda \in \Lambda_i,
\]

yields a basis for the subspace \( V_i \leq H_n(M_i) \).

(c) The restriction \( \varphi_i|_{\Lambda_i \times \{a_i\} \times S^n} : \Lambda_i \times \{a_i\} \times S^n \longrightarrow M_i \) extends to an embedding

\[
\varphi' : \Lambda_i \times \{a_i\} \times \mathbb{R}^n \times S^n \longrightarrow M_i,
\]

with the property that the induced bundle map

\[
T(\Lambda_i \times \{a_i\} \times \mathbb{R}^n \times S^n) \oplus \epsilon^1 \longrightarrow TM_i \times \epsilon^1 \longrightarrow T_{\varphi_i|_{\Lambda_i \times \{a_i\} \times S^n}} \longrightarrow \theta^* \gamma^{2n+1},
\]

admits an extension to a \( \theta \)-structure on \( \Lambda_i \times \{a_i\} \times \mathbb{R}^n \times D^{n+1} \).

The map \( (\Lambda_i \times \{a_i\} \times \mathbb{R}^n \times S^n) \cup (\Lambda_i \times [a_i, a_p] \times S^n) \rightarrow W_{[a_i, a_p]} \), obtained by combining \( \varphi_i \) and \( \varphi'_i \), extends to an embedding \( \hat{\varphi}_i : \Lambda_i \times [a_i, a_p] \times \mathbb{R}^n \times S^n \longrightarrow W_{[a_i, a_p]} \), which in turn extends to an embedding

\[
(10.4.5) \quad \hat{\varphi}_i : \Lambda_i \times [a_i - \varepsilon_i, a_p + \varepsilon_p] \times \mathbb{R}^n \times D^{n+1} \longrightarrow \mathbb{R} \times (-1, 1)^{\infty-1}.
\]

By carrying out the exact same construction for all \( i = 0, \ldots, p \), we obtain embeddings \( \hat{\varphi}_0, \ldots, \hat{\varphi}_p \) as in (10.4.5). Applying Proposition 10.3.1, we may arrange for

\[
\hat{\varphi}_i (\Lambda_i \times \mathbb{R} \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \{0\} \times S^n) \cap \hat{\varphi}_j (\Lambda_j \times \mathbb{R} \times (a_j - \varepsilon_j, a_p + \varepsilon_p) \times \{0\} \times S^n) = \emptyset
\]

for all \( i, j = 0, \ldots, p \). Let \( \Lambda = \bigsqcup_{i=0}^p \Lambda_i \). By forming the disjoint union of the embeddings \( \hat{\varphi}_0, \ldots, \hat{\varphi}_p \) and applying a reparametrization \( (a_i - \varepsilon_i, a_p + \varepsilon_p) \cong (-6, -2) \), we obtain an embedding

\[
e : \Lambda \times \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^{n+1} \longrightarrow \mathbb{R} \times (-1, 1)^{\infty-1}
\]
This embedding $e$ determines part of the data of an element of $\tilde{Y}_0(x)$. The other part of the necessary data is a $\theta$-structure $\ell$ on $\Lambda \times K|_{(-6, 2)}$ that restricts to the $\theta$-structure on $\Lambda \times K|_{(-6, -2)}$ given by the composition

$$T(\Lambda \times K|_{(-6, -2)}) \xrightarrow{De} W \xrightarrow{W} \theta^* \gamma^{2n+1}.$$  

This extension can be obtained by same argument employed in the proof of [GaRW 14, Proposition 6.22, page 350] and then by construction $(e, \ell) \in \tilde{Y}_0(x)$.

Its hypotheses established, we can now apply 10.1.1 to obtain Theorem 9.1.2 as described at the beginning of this section.

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