Perturbative and non-perturbative aspects of the two-dimensional string/Yang-Mills correspondence

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Abstract: It is known that YM$_2$ with gauge group $SU(N)$ is equivalent to a string theory with coupling $g_s = 1/N$, order by order in the $1/N$ expansion. We show how this results can be obtained from the bosonization of the fermionic formulation of YM$_2$, improving on results in the literature, and we examine a number of non-perturbative aspects of this string/YM correspondence. We find contributions to the YM$_2$ partition function of order $\exp\{-kA/(\pi\alpha'g_s)\}$ with $k$ an integer and $A$ the area of the target space, which would correspond, in the string interpretation, to D1-branes. Effects which could be interpreted as D0-branes are instead strictly absent, suggesting a non-perturbative structure typical of type 0B string theories. We discuss effects from the YM side that are interpreted in terms of the stringy exclusion principle of Maldacena and Strominger. We also find numerically an interesting phase structure, with a region where YM$_2$ is described by a perturbative string theory separated from a region where it is described by a topological string theory.

Keywords: String-YM correspondence, two-dimensional Yang-Mills
1. Introduction

In the early nineties Gross [1] and Gross and Taylor [2] showed that two-dimensional pure YM theory with gauge group $SU(N)$ on a euclidean manifold of arbitrary topology is equivalent, order by order in the large $N$ expansion, to a string theory with coupling $g_s = 1/N$ (see e.g. refs. [3]–[11] for further developements).

In the light of the recent advances on string/YM correspondence it is interesting to go back to this result, for a number of reasons. First, in this two-dimensional setting the correspondence can be proven mathematically, at least at the level of perturbation theory. This comes from the remarkable fact that the partition function of YM$_2$ on an arbitrary euclidean manifold, with gauge group $U(N)$ or $SU(N)$ and $N$ generic, can be computed exactly. Second, this theory has no space-time supersymmetry, suggesting that supersymmetry is not a necessary ingredient for a string/YM correspondence to hold– a fact of obvious importance if one hopes to apply the correspondence to QCD. And finally, in the years after refs. [1, 2] came out, D-branes have been introduced and the understanding of non-perturbative string theory has developed greatly, so it becomes possible to ask whether this correspondence holds even beyond perturbation theory.

In this paper we consider some aspects of the relation between YM$_2$ and string theory. In sect. 2 we briefly recall the main results of refs. [1, 2, 3, 4], where it is shown that the $1/N$ expansion of YM$_2$ can be interpreted geometrically in terms of a theory of maps
from a two-dimensional world-sheet to a two-dimensional target space. We also recall the result of Minahan and Polychronakos [5], who showed that this expansion can be elegantly summarized in terms of a “string field theory” Hamiltonian, i.e. a Hamiltonian containing operators that create and destroy strings with a given winding over the cycles of the target manifold. This Hamiltonian, for $U(N)$, consists of a term $O(1)$ plus a term $O(1/N)$ (for $SU(N)$, there is also a term $O(1/N^2)$) and all other perturbative corrections to it in powers of $1/N$ are exactly zero; the full and complicated $1/N$ expansion of the $YM_2$ partition function is completely reproduced by the expansion of the exponential of this Hamiltonian, traced over a multistring Fock space. Thus this Hamiltonian summarizes very compactly all the perturbative expansion, and is useful to clarify the physical meaning of this two-dimensional string-$YM$ correspondence.

In sect. 3 we show how this Hamiltonian can be rigorously derived from a bosonization of the fermionic formulation of $YM_2$. The idea behind the computation has been described by Douglas [6, 7]. However, strictly speaking the derivation of refs. [6, 7] only shows that the Hamiltonian of Minahan and Polychronakos is obtained as the leading term in the large $N$ limit, while we will see explicitly that it is exact, i.e. all its further perturbative corrections in $1/N$ vanish. This completes a simple and rather elegant proof of the perturbative correspondence.

In sect. 4 we examine some non-perturbative aspects of the correspondence. The expansion of the $YM_2$ partition function at large $N$ has in fact also terms $e^{-O(N)}$, already noted by Gross [1], which should match with contribution $e^{-O(1/g_s)}$ of the corresponding string theory, if the correspondence holds even beyond the perturbative level. Indeed we will find that, from the YM side, there is a set of contributions proportional to $e^{-kA/(\pi\alpha'g_s)}$, with $k$ an integer, $A$ the target space area, $\alpha'$ the string tension of the string theory (fixed by the ’t Hooft coupling of the YM theory, see below) and $g_s = 1/N$. The factor $1/g_s$ at the exponent is suggestive of $D$-branes. More precisely, the proportionality to the area of the target space is just what one would expect from $D1$-branes in this string theory. In fact, the strings corresponding to $YM_2$ have the peculiar properties of having no foldings [1], i.e. their world-sheet area is an integer times the target space area. It is then natural to expect the same for the $D1$-branes, and indeed the factor $kA$ in the exponent can be interpreted as the world-sheet area of a $D1$-brane wrapping $k$ times over the target space without foldings, and $\tau_1 = 1/(\pi\alpha'g_s)$ can be interpreted as the $D1$-brane tension. We will see that instead there is no effect that has an interpretation in terms of $D0$-branes. We therefore find a non-perturbative structure typical of type B string theories: $p$-branes with $p$ even are absent and with $p$ odd are present.

We will also find that a non-perturbative string effect as the stringy exclusion principle of Maldacena and Strominger [12] appears from the $YM_2$ side, as a very simple consequence of the fermionic formulation of $YM_2$. We will then discuss our attempts to evaluate numerically the non-perturbative effects in $YM_2$, and we will find an interesting structure in the plane $(g_s, a)$, where $a = \lambda A/2$, $\lambda = e^2N$ is the ’t Hooft coupling of the YM theory and $A$ is the area of the target space.
2. The large-$N$ expansion of YM$_2$

We consider pure Yang-Mills theory on a two dimensional euclidean manifold $\mathcal{M}$ of arbitrary topology, with gauge group $U(N)$ or $SU(N)$ and charge $e$. The partition function can be written as a sum over all representations $R$ of the gauge group [13, 14]

$$Z_{\text{YM}} \equiv \int [DA^\mu] \exp\left\{ -\frac{1}{4e^2} \int_{\mathcal{M}} d^2x \sqrt{g} \text{Tr}F^{\mu\nu}F_{\mu\nu} \right\} = \sum_R (\dim R)^{2-2G} e^{-2G} \frac{A}{N} C_2(R), \quad (2.1)$$

where $G$ is the genus of $\mathcal{M}$, $A$ its area, $\lambda = e^2 N$ is the 't Hooft coupling, to be held fixed in the large $N$ expansion, and $C_2(R)$ is the quadratic Casimir in the representation $R$.

The representations $R$ of $U(N)$ or $SU(N)$ are given by the Young diagrams with $m$ rows, with $m \leq N$ for $U(N)$ and $m < N$ for $SU(N)$. Denoting by $h_i$, $i = 1, \ldots, m$, the number of boxes in the $i$-th row (with $h_N = 0$ for $SU(N)$) and by $c_j$ the number of boxes in the $j$-th column, the quadratic Casimir can be written as [1]

$$C_2^{U(N)}(R) = Nn + \tilde{C}(R), \quad (2.2)$$
$$C_2^{SU(N)}(R) = Nn + \tilde{C}(R) - \frac{n^2}{N}, \quad (2.3)$$

where $n = \sum_{i=1}^{N} h_i = \sum_{j=1}^{\infty} c_j$ is the total number of boxes in the Young diagram, and

$$\tilde{C}(R) = \sum_{i=1}^{N} h_i^2 - \sum_{j=1}^{\infty} c_j^2. \quad (2.4)$$

Observe that each of the $h_i$ takes values in the range $0 \leq h_i < \infty$ and its index $i$ takes the values $i = 1, \ldots, N$, i.e. the number of rows is limited by $N$ (with $h_N = 0$ for $SU(N)$) but the rows can be arbitrarily long. Instead $0 \leq c_j \leq N$, with $j = 1, \ldots, \infty$, corresponding to the fact that the length of the columns is limited by $N$ (by $N - 1$ for $SU(N)$) but the number of columns is arbitrary. This asymmetry between the $h_i$ and the $c_j$ is important when one considers non-perturbative effects, as we shall see.

The dimension of the representation, $\dim R$, has also a closed form in terms of the $h_i$ and therefore one has a very explicit expression for the partition function, which can be expanded in powers of $1/N$.

The beautiful result of Gross [4] is that, order by order in $1/N$, all terms in the expansion of the logarithm of $Z_{\text{YM}}$ can be interpreted geometrically as a sum of contributions due to maps from a two dimensional world-sheets to the target space $\mathcal{M}$, or, more precisely, as a sum over all possible branched coverings of $\mathcal{M}$, so that one can identify $\log Z_{\text{YM}}$ with the partition function of a string theory with coupling $g_s = 1/N$ and string tension $\alpha' \sim 1/\lambda$ (recall that in two dimensions the electric charge $e$ has dimensions of mass, so $\lambda = e^2 N$ is a mass squared):

$$\log Z_{\text{YM}}[G, A, \lambda, N] = Z_{\text{string}} \left[ g_s = \frac{1}{N}, \alpha' = \frac{1}{\pi \lambda} \right]. \quad (2.5)$$

The details of this identification, fully worked out in refs. [6, 13, 3], are quite intricated, but basically one finds that the terms in the expansion of the left-hand side are weighted
by a factor $\exp(-n\lambda A/2)$, with $n$ a summation index which is interpreted as the number of sheets of the covering, so that the factor $nA$ has the geometric interpretation of the area of the world-sheet of a string which has no foldings, and $\lambda/2$ is then identified with the string tension $1/(2\pi\alpha')$; the identification of $g_s$ with $1/N$ comes from the presence of factors $N^\chi$, with $\chi$ equal to the Euler characteristic of the branched covers (which includes the contribution of the singularities of the branched cover); furthermore, the overall coefficient associated to each contribution of the sum (i.e. to each branched cover) turns out to be related to the number of topologically inequivalent maps from the given branched cover to the target space. Therefore $\log Z_{YM}$ has a full geometric interpretation, and has the structure of the partition function of a theory of maps.

The relation $Z_{YM} = \exp(Z_{\text{string}})$ is of course the same relation that one has between the partition function of a first quantized particle, $Z_{S^1} = \int Dx^\mu e^{-S}$, computed integrating over all trajectories $x^\mu(\tau)$ with the topology of the circle, and the partition function of the corresponding field theory, $Z_{\text{vac}} = \exp(Z_{S^1})$. So eq. (2.5) means that $YM_2$ is rather a string field theory.

This point becomes evident when one realizes that the whole complicated $1/N$ expansion can be summarized very concisely in terms of a Hamiltonian acting on a Fock space generated by operators that create and destroy strings [5]. To understand this, one observes first of all that the $YM_2$ partition function on a surface of arbitrary genus can be obtained from the partition function on the cylinder by using the gluing property [16], so we can limit ourselves to the partition function on a cylinder of circumference $L$ and length $T$. To quantize $YM_2$ on a cylinder one chooses the gauge $A_0 = 0$ and is therefore left with wave-functionals $\Psi[A_1(x)]$. The constraint obtained varying with respect to $A_0$ imposes that $\Psi[A_1(x)]$ actually depends only on the holonomy $U = P \exp[i \int_0^L dA_1]$. The Hilbert space of states can therefore be labelled by the holonomies, $|U\rangle$ [3].

We then introduce the Fock space generated by the operators $\alpha_n$, with $[\alpha_n, \alpha_m] = n\delta_{n+m}$. Physically $\alpha_n$ with $n > 0$ destroys a string winding $n$ times in the clockwise direction around the cylinder and $\alpha_{-n}$ creates it. We also introduce a second set $\tilde{\alpha}_n$ creating and destroying strings winding in the counterclockwise direction. A generic multistring state is therefore of the form [3]

$$|\vec{k}, \vec{l}\rangle = \prod_{i>0} (\alpha_{-i})^{k_i} \prod_{j>0} (\tilde{\alpha}_{-j})^{l_j} |0\rangle.$$ (2.6)

Now we consider the $YM_2$ partition function on a cylinder, with holonomies $U_1, U_2$ at the boundaries,

$$Z_{\text{cyl}} = \langle U_1 | e^{-HT} | U_2 \rangle,$$ (2.7)

where $H$ is the $YM_2$ Hamiltonian. On the one hand, this can be computed exactly in closed form, similarly to (2.1). On the other hand, we can rewrite it as

$$Z_{\text{cyl}} = \sum_{s,s'} \langle U_1 | s \rangle \langle s | e^{-HT} | s' \rangle \langle s' | U_2 \rangle,$$ (2.8)

where $|s\rangle, |s'\rangle$ are a complete set of multistring states of the type (2.6). The matrix elements $\langle U | s \rangle$ are fixed requiring that eq. (2.8) reproduces the dependence of $Z_{\text{cyl}}$ on the holonomies.
When the state $|s\rangle$ is constructed only from operators $\alpha_{-n}$ (or only from $\tilde{\alpha}_{-n}$) the result is especially simple\[^1\],

$$
\langle U|\tilde{k}\rangle = \prod_{j=1}^{\infty} (\text{Tr} U^j)^{k_j}.
$$

(2.9)

Then the Hamiltonian $H$ in the string basis is fixed requiring that, when inserted into eq. (2.8), it reproduces the full $1/N$ expansion of the cylinder amplitude, and for $SU(N)$ it turns out to be \[^5\]

$$
H = \frac{\lambda L}{2} \left\{ \left( N + \bar{N} \right) - \frac{1}{N^2} (N - \bar{N})^2 + \frac{1}{N} \sum_{n,l>0} \left( \alpha_{-n-l}\alpha_{n}\alpha_{l} + \alpha_{-n}\alpha_{-l}\alpha_{n+l} \right) + \frac{1}{N} \sum_{n,l>0} \left( \tilde{\alpha}_{-n-l}\tilde{\alpha}_{n}\tilde{\alpha}_{l} + \tilde{\alpha}_{-n}\tilde{\alpha}_{-l}\tilde{\alpha}_{n+l} \right) \right\},
$$

(2.10)

where $N = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_{n}$.

Eq. (2.10) shows in the clearest way that YM$_2$ is equivalent to a string field theory, since all matrix elements can be computed in terms of a Hamiltonian and a Fock space constructed using operators that create and destroy strings with a given winding number around the cylinder.

Considerable effort has gone into trying to reproduce $Z_{\text{string}}$ in eq. (2.5) from the path integral over a suitable string action \[^3, 4, 8\], in order to make contact with the standard first-quantized formalism of string theory. It appears, however, that if such a formulation exists at all, it is very complicated, except in the limit of vanishing target-space area, $A \to 0$, where one finds a topological string theory. On the other hand, at least at the perturbative level, a first quantized formulation is not really necessary, since in this case we are in the rather unique situation of having already at our disposal a second quantized string theory, defined by the Hamiltonian (2.10), which furthermore has an extremely simple form, with just a free piece plus cubic and quartic interaction terms, and, at least at the level of perturbation theory, contains all the informations that we need on the stringy description of YM$_2$.

The Hamiltonian (2.10) was first found \[^5\] as a sort of bookkeeping device that summarizes the whole $1/N$ expansion of $Z_{YM}$. One can ask whether it can be derived directly from the YM$_2$ action, shortcutting the highly elaborated procedure of the $1/N$ expansion. In fact this is possible, if one starts from the fermionic formulation of YM$_2$ and then bosonizes it, as was understood by Douglas \[^6, 7\]. Actually, while we can see, following refs. \[^3, 4\], that the Hamiltonian $H$ emerges from this bosonization procedure in the large $N$ limit, a little more care is needed to make sure that $H$ in eq. (2.10) is reproduced exactly, with no further subleading term in $1/N$. Since the great power of the Hamiltonian (2.10) is just that it is exact at all orders in $1/N$, we find useful in the next section to perform the calculation carefully, verifying explicitly the cancellation of the subleading terms. We will also find the expression for the $U(N)$ Hamiltonian, which is not correctly given in the literature.

\[^1\]For the most general case, see ref. \[^3\], sect. 4.7.1.
3. The string Hamiltonian

3.1 Fermionic representation of YM\(_2\)

The starting point is the description of YM\(_2\) in terms of free non-relativistic fermions \(\text{[17, 6]}\) (see also ref. \([3]\) for review). We have seen that in the functional Schroedinger equation the wave-functional \(\Psi\) depends only on the holonomies \(U\); by gauge invariance, it must indeed be a class function, i.e. \(\Psi[U] = \Psi[gu^{-1}]\) with \(g \in U(N)\) or \(SU(N)\). Class functions depends only on their value on the maximal torus, whose elements can be parametrized as diag\((e^{i\theta_1}, \ldots, e^{i\theta_N})\) (with the further constraint \(\sum_i \theta_i = 0\) for \(SU(N)\)). Then \(\Psi = \Psi[\vec{\theta}]\) and, by Weyl symmetry, is symmetric under exchange of any two \(\theta_i\). The inner product on class function is fixed by the invariant measure over the group and is

\[
(\Psi, \Psi) = \int \prod d\vec{\theta} \tilde{\Delta}(\vec{\theta})^2 |\Psi(\vec{\theta})|^2 ,
\]

with \(\tilde{\Delta} = \prod_{i<j} \sin[(\theta_i - \theta_j)/2]\). The YM\(_2\) Hamiltonian acting on \(\Psi[\vec{\theta}]\) is, for \(U(N)\),

\[
H_{U(N)} = \frac{e^2 L}{2} \frac{1}{\Delta(\vec{\theta})} \left[ \sum_i -\frac{d^2}{d\theta_i^2} - \frac{N}{12}(N^2 - 1) \right] \tilde{\Delta}(\vec{\theta}) ,
\]

while \(H_{SU(N)} = H_{U(N)} - (e^2 L/2)Q^2/N\), with \(Q\) the \(U(1)\) generator, see below. We can therefore work with a new wave-functional \(\psi[\vec{\theta}] = \tilde{\Delta}(\vec{\theta})\Psi[\vec{\theta}]\), in terms of which both the inner product and the functional Schroedinger equation are those of a free theory. However, since \(\Psi\) is symmetric and \(\tilde{\Delta}\) antisymmetric, \(\psi\) is antisymmetric, and the YM\(_2\) theory with gauge group \(U(N)\) or \(SU(N)\) is therefore reduced to the quantum mechanics of \(N\) free non-relativistic fermions, with each fermion described by a coordinate \(\theta_i, i = 1, \ldots, N\) and therefore living on the circle, and with the further constraint \(\sum_i \theta_i = 0\) for \(SU(N)\).

The generic state of this fermionic system is labelled as

\[
|n_1, \ldots, n_N\rangle
\]

with \(n_i \in \mathbb{Z}\) and \(n_1 > n_2 > \ldots n_N\), by the exclusion principle. The energy of such a state is read from eq. (3.2) and is

\[
E_{U(N)} = \frac{e^2 L}{2} \left[ \sum_{i=1}^N n_i^2 - \frac{N}{12}(N^2 - 1) \right]
\]

while the \(U(1)\) charge is easily seen to be \(Q = \sum_{i=1}^N n_i\). The ground state, restricting for simplicity to \(N\) odd,\(^2\) is obtained filling all levels from \(-n_F\) to \(n_F\), see fig. \([4]\), with the Fermi surface at

\[
n_F = \frac{N - 1}{2} .
\]

For this state

\[
\sum_{i=1}^N n_i^2 = 2 \sum_{i=1}^{n_F} i^2 = \frac{N}{12}(N^2 - 1)
\]

\(^2\)The analysis that we will discuss can be repeated with very minor modifications for \(N\) even. In order not to burden all arguments, repeating them for \(N\) even and \(N\) odd, we will just restrict to \(N\) odd. No interesting new feature appears for \(N\) even.
and therefore the energy (3.4) is zero. Each fermionic configuration \( \{n_i\} \) corresponds to a Young diagram with rows of length \( \beta \)

\[
h_i = n_i + i - 1 - n_F,
\]

and therefore the partition function (2.1) is immediately rewritten as a sum over all fermionic configurations. For \( U(N) \), the representation is labelled also by the \( U(1) \) charge. For \( SU(N) \), two fermionic configurations correspond to the same Young diagrams if they are related by a global shift of the \( n_i \), \( n_i \to n_i + b, b \in \mathbb{Z} \). We can use this freedom to set \( n_N = -n_F \).

Even if the total number of fermions, \( N \), is fixed for a given \( U(N) \) or \( SU(N) \) YM theory, it turns out to be convenient to introduce a second quantization formalism, defining \( B_n \) (with \( n \in \mathbb{Z} \)) as the operator that destroys a fermion in the state \( |n\rangle \) and \( B_n^\dagger \) as the creation operator, with \( \{B_n, B_m^\dagger\} = \delta_{n,m} \). The number operator is therefore \( \hat{N} = \sum_{n=-\infty}^{\infty} B_n^\dagger B_n \). The vacuum \( |0\rangle \) is defined by \( B_n|0\rangle = 0 \) for all \( n \). However, it is not a state of the \( U(N) \) or \( SU(N) \) theory, since it does not have \( N \) occupied levels. We instead define the Fermi vacuum \( |0_F\rangle \) from

\[
\begin{align*}
B_n|0_F\rangle & = 0 \text{ if } |n| > n_F \quad (3.8) \\
B_n^\dagger|0_F\rangle & = 0 \text{ if } |n| \leq n_F \quad (3.9)
\end{align*}
\]

**Figure 1:** The filled fermionic levels in the ground state of \( SU(N) \) YM\(_2\) (when \( N \) is odd).

We use \( B_n, B_n^\dagger \) to define operators in which the mode number is measured with reference to the two Fermi surfaces at \( n = \pm n_F \):

\[
\begin{align*}
c_n & = B_{n_F+1-n}^\dagger \\
b_n & = B_{n_F+1+n}
\end{align*}
\]

\[
\begin{align*}
c_n & = \delta_{n,m} \\
b_n & = B_{n_F+1+n}^\dagger
\end{align*}
\]

(3.11)

and

\[
\begin{align*}
c_n & = B_{-(n_F+1)-n}^\dagger \\
b_n & = B_{-(n_F+1)-n}
\end{align*}
\]

\[
\begin{align*}
c_n & = \delta_{n,m} \\
b_n & = B_{n_F+1+n}^\dagger
\end{align*}
\]

If we would extend the definitions of \( b_n, c_n \) and \( \tilde{b}_n, \tilde{c}_n \) at \( |n| > n_F \) then they would not be independent, since e.g. the same operator \( B_m \) would be assigned both to one of the \( b_n \) and to one of the \( \tilde{b}_n \); we find simpler to put a cutoff on the mode number \( n \) and work with independent quantities.\(^3\) In terms of the \( B_n, B_n^\dagger \) the cutoff is such that are included all operators \( B_n, B_n^\dagger \) with \( -N \leq n \leq N \).

With our definition, the operators \( B_0, B_0^\dagger \) are not assigned neither to the \( bc \) sector nor to the \( \tilde{b}\tilde{c} \) sector. On a generic state, the operator \( B_0^\dagger B_0 \) takes the values \( \beta = 0, 1 \) depending on whether the level \( n = 0 \) is empty or filled. While it takes no effort to keep \( \beta \) generic.

\(^3\)In principle, one might decide to use an asymmetric cutoff; for instance, in \( b_n, \tilde{b}_n \) all we really need is a lower bound on \( n \) for both \( b_n \) and \( \tilde{b}_n \), so that the \( b_n \) and the \( \tilde{b}_n \) do not ‘collide’ with each other; an upper bound like \( n \leq n_F \) in \( b_n, \tilde{b}_n \) or a lower bound for \( n \) in \( c_n, \tilde{c}_n \) are not necessary. It is however slightly simpler to put the cutoff symmetrically, which means that we forbid very high excitations like \( B_n^\dagger|0\rangle \) with \( n > N \). As we will see, at all orders in perturbation theory in \( 1/N \), these definitions are equivalent.
in the calculations, this is not really necessary, since the configurations in which the level \( n = 0 \) is empty have a Casimir \( O(N^2) \) and therefore do not contribute in perturbation theory, as we see from eq. (2.24). In this section we limit ourselves to the perturbative equivalence, and we can therefore restrict to the case \( B_0^1 B_0^0 = 1 \).

With these definitions,
\[
\{ b_n, c_m \} = \delta_{n+m} \tag{3.13}
\]
and
\[
c_n |0\>_F = 0 \quad n > 0, \tag{3.14}
\]
\[
b_n |0\>_F = 0 \quad n \geq 0. \tag{3.15}
\]

We now introduce an auxiliary complex variable \( z \) and we arrange \( b_n, c_n \) into the modes of two holomorphic fields \( b(z), c(z) \):
\[
b(z) \equiv \sum_{n=-n_F}^{n_F} \frac{b_n}{z^{n+1}}, \tag{3.16}
\]
\[
c(z) \equiv \sum_{n=-n_F}^{n_F} \frac{c_n}{z^n}. \tag{3.17}
\]

Eqs. (3.13) to (3.17) defines a \( bc \) theory with \( \lambda = 1 \) (see e.g. ref. [18], sect. 2.7), with a cutoff at \( \|n\| = n_F \). The Fermi vacuum \( |0\>_F \) corresponds, in the notation of ref. [18], to the vacuum state \( |↓\>_F \) of the \( bc \) theory. Similarly, for the modes \( \tilde{b}_n, \tilde{c}_n \) it follows from the definition that
\[
\tilde{c}_n |0\>_F = 0 \quad n > 0, \tag{3.18}
\]
\[
\tilde{b}_n |0\>_F = 0 \quad n \geq 0, \tag{3.19}
\]
and \( \{ \tilde{b}_n, \tilde{c}_m \} = \delta_{n+m} \). It is convenient to arrange them into two antiholomorphic fields,
\[
\tilde{b}(\bar{z}) \equiv \sum_{n=-n_F}^{n_F} \frac{\tilde{b}_n}{\bar{z}^{n+1}}, \tag{3.20}
\]
\[
\tilde{c}(\bar{z}) \equiv \sum_{n=-n_F}^{n_F} \frac{\tilde{c}_n}{\bar{z}^n}. \tag{3.21}
\]

The fields \( b(z), c(z) \) (and similarly for \( \tilde{b}, \tilde{c} \)) are just useful bookkeeping devices for assembling together the modes \( b_n, c_n \), and there is nothing special in the choice \( \lambda = 1 \). We could as well assemble them into a \( bc \) theory with \( \lambda \) generic,
\[
b(z) \equiv \sum_{n=-n_F}^{n_F} \frac{b_n}{z^{n+\lambda}}, \quad c(z) \equiv \sum_{n=-n_F}^{n_F} \frac{c_n}{z^{n+1-\lambda}} \tag{3.22}
\]
(and similarly for the \( \tilde{b}\tilde{c} \) theory). However, the calculation of the bosonized form of the YM hamiltonian that we will perform below turns out to be slightly simpler when \( \lambda = 1 \),
so we will restrict to this choice. In appendix A we will check that the same final result for the string hamiltonian is obtained for $\lambda$ arbitrary.

A point to be kept in mind is that our $bc$ and $\tilde{b}\tilde{c}$ theories depend on $N$ through the cutoff, $|n| \leq n_F$. Furthermore, the two theories are coupled by the constraint $\sum_n B_n^\dagger B_n = N$, which can be rewritten as

$$N = B_0^\dagger B_0 + \sum_{n=1}^N B_n^\dagger B_n + \sum_{n=-N}^{-1} B_n^\dagger B_n = 1 + \sum_{n=-n_F}^{n_F} (c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n),$$

(3.23)

where we used the fact that $B_0^\dagger B_0 = 1$ on perturbative states. We now define $\langle \ldots \rangle :$ as the normal ordering with respect to $|0\rangle_F$, i.e. we anticommutate the operators $b_n, c_m$ and $\tilde{b}_n, \tilde{c}_m$ until all destructors with respect to $|0\rangle_F$ (i.e. $c_n, \tilde{c}_n$ with $n > 0$ and $b_n, \tilde{b}_n$ with $n \geq 0$) are to the right. Of course, this is different from the normal ordering with respect to $|0\rangle$. Then in eq. (3.23) the normal ordering exchanges all terms with $n = -n_F, \ldots, -1$ both in $c_{-n} b_n$ and in $\tilde{c}_{-n} \tilde{b}_n$, and therefore eq. (3.23) can be written as

$$N = 1 + 2n_F + \sum_n : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n :,$$

(3.24)

and, since $n_F = (N - 1)/2$, we get

$$\sum_n : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n : = 0.$$  

(3.25)

### 3.2 Bosonization

The bosonization of the $bc$ theory with $\lambda = 1$ is known to be given by the linear dilaton theory \[18\]. However we have seen that, at finite $N$, YM$_2$ is not exactly given by the product of a $bc$ theory and a $\tilde{b}\tilde{c}$ theory, but there is also an $N$-dependence which enters through the cutoff on the mode number; furthermore the $bc$ and $\tilde{b}\tilde{c}$ theories are coupled through the constraint (3.25).

As far as the cutoff is concerned, however, we can see that if in eqs. (3.16) and (3.17) we send the cutoff to infinity, writing

$$b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+1}}, \quad c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^n},$$

(3.26)

the error that we are doing is exponentially small in $N$, and therefore is irrelevant in the $1/N$ expansion. In fact, the fermionic configurations in which some of the states with $|n| > n_F$ (where $n$ is the index of $b_n, c_n$, i.e. it measures the excitation above the Fermi surface) are occupied correspond, through eq. (3.7), to Young diagrams with lines longer than $n_F$. From eqs. (2.2) to (2.4) we see that the quadratic Casimir of these diagrams are $O(N^2)$ and therefore, from eq. (2.1), the contribution of these fermionic configurations to the partition function is $O(\exp\{-cN^2\})$, with $c$ some positive constant. These “long” Young diagram give therefore contributions that are non-perturbative in the $1/N$ expansion. These will be the subject of sect. 4. In this section we limit ourselves to perturbation theory. This
means that, in bosonizing the \(bc\) theory, we can use the results valid in the infinity cutoff limit, and set to zero all modes \(b_n, c_n\) with \(|n| > n_F\).

The \(bc\) theory can then be bosonized using the standard formulas, in terms of a holomorphic field \(X_L(z)\) (see e.g. \[18\], sect. 10.3),

\[
b = : e^{iX_L} :, \quad c = : e^{-iX_L} :, \quad : bc := i\partial X_L. \tag{3.27}
\]

The normal ordering in this standard formula is just the normal ordering with respect to \(|0\rangle_F\) that we have used above (for a \(bc\) theory with \(\lambda \neq 1\) this is actually not true, as discussed in appendix A, and one must be more careful). Defining the modes \(\alpha_m\) of \(X_L\) from

\[
\partial X_L = i \sum_m \alpha_m, \tag{3.28}
\]

eq (3.27) gives

\[
\alpha_m = \sum_{n=-\infty}^{\infty} : c_{m-n} b_n : = \sum_{n=-n_F}^{n_F} : c_{m-n} b_n : . \tag{3.29}
\]

We have used the fact that perturbatively we can set \(b_n = 0\) for \(|n| > n_F\), to restrict the sum over \(-n_F \leq n \leq n_F\). Furthermore we can also restrict \(|m-n| \leq n_F\), that implies \(-(N-1) \leq m \leq (N-1)\).

The energy-momentum tensor of the \(bc\) theory with \(\lambda = 1\) can be written in terms of \(X_L\) as \[18\]

\[
: (\partial b) c : - \partial : bc := -\frac{1}{2} : \partial X_L \partial X_L : - \frac{i}{2} \partial^2 X_L \tag{3.30}
\]

The right-hand side is the energy-momentum tensor of a linear dilaton CFT. In terms of the Virasoro operators, we have \(L^{(bc)}_m = L^{(X)}_m\), with

\[
L^{(bc)}_m = \sum_{n=-\infty}^{\infty} (m-n) : b_n c_{m-n} :, \tag{3.31}
\]

\[
L^{(X)}_m = \frac{1}{2} \left( \sum_{n=-\infty}^{\infty} : \alpha_{m-n} \alpha_n : \right) - \frac{1}{2} (m+1) \alpha_m. \tag{3.32}
\]

In particular, for \(m = 0\) we have

\[
L_0 = \sum_{n=-\infty}^{\infty} n : c_{-n} b_n : = \frac{1}{2} \alpha_0^2 - \frac{1}{2} \alpha_0 + \sum_{n=1}^{\infty} : \alpha_{-n} \alpha_n : . \tag{3.33}
\]

The \(\tilde{b}\tilde{c}\) theory is bosonized similarly, in terms of an antiholomorphic field \(X_R(\bar{z})\), whose modes we denote by \(\tilde{\alpha}_m\), with

\[
\tilde{\alpha}_m = \sum_{n=-n_F}^{n_F} : \tilde{c}_{m-n} \tilde{b}_n : . \tag{3.34}
\]

The constraint (3.27) that relates the \(bc\) and \(\tilde{b}\tilde{c}\) theories now becomes simply

\[
\alpha_0 + \tilde{\alpha}_0 = 0, \tag{3.35}
\]
so it is a constraint between the holomorphic and antiholomorphic parts of $X = X_L + X_R$.

The winding number $w$ of $X$ is defined as usual, $w = \alpha_0 - \tilde{\alpha}_0$. Eq. (3.35) then means that

$$\alpha_0 = \frac{w}{2}, \quad \tilde{\alpha}_0 = -\frac{w}{2}.$$  

Writing

$$\alpha_0 = \sum_{n=-n_F}^{n_F} c_{-n} b_n := \left( \sum_{n=-n_F}^{n_F} c_{-n} b_n \right) - n_F = \left( \sum_{n=1}^{N} B_n^\dagger B_n \right) - n_F, \quad (3.37)$$  
\[ \tilde{\alpha}_0 = \sum_{n=-n_F}^{n_F} \tilde{c}_{-n} \tilde{b}_n := \left( \sum_{n=-n_F}^{n_F} \tilde{c}_{-n} \tilde{b}_n \right) - n_F = \left( \sum_{n=-N}^{-1} B_n^\dagger B_n \right) - n_F, \quad (3.38)\]

we see that

$$w = (\text{filled levels with } n > 0) - (\text{filled levels with } n < 0). \quad (3.39)$$

For $SU(N)$, we have seen that representations that differ by an overall shift of the $n_i$, $n_i \to n_i + b$, with $b$ integer, are equivalent. We can use this freedom to set $w = 0$ and therefore $\alpha_0 = \tilde{\alpha}_0 = 0$. For $U(N)$ instead this is not so, because the $U(1)$ generator $Q = \sum_i n_i$ is not invariant under the shift, and therefore we must keep $w$ generic. We can further notice that, since the number of fermions is fixed to be $N$, there can be at most $N$ filled fermionic modes with $n > 0$, in which case there are none with $n < 0$ and $w$ reaches its maximum values $w = N$, while in the opposite case all fermions have $n < 0$ and $w$ reaches its minimum value, $w = -N$. Therefore for $U(N)$

$$-N \leq w \leq N. \quad (3.40)$$

We now want to write the hamiltonian for the $U(N)$ YM theory, as well as the $U(1)$ charge, in terms of $\alpha_n$, $\tilde{\alpha}_n$. The $U(1)$ charge $Q = \sum_i n_i$ in the second quantization formalism is

$$Q = \sum_{n=-N}^{N} n B_n^\dagger B_n. \quad (3.41)$$

We rewrite it as

$$Q = \sum_{n=1}^{N} n B_n^\dagger B_n + \sum_{n=-N}^{-1} n B_n^\dagger B_n = \sum_{n=-n_F}^{n_F} (n_F + 1 + n)c_{-n}b_n - \sum_{n=-n_F}^{n_F} (n_F + 1 + n)\tilde{c}_{-n}\tilde{b}_n =$$

$$= (n_F + 1) \sum_{n=-n_F}^{n_F} (c_{-n}b_n - \tilde{c}_{-n}\tilde{b}_n) + \sum_{n=-n_F}^{n_F} n(c_{-n}b_n - \tilde{c}_{-n}\tilde{b}_n) =$$

$$= (n_F + 1)(\alpha_0 - \tilde{\alpha}_0) + (L_0 - \tilde{L}_0) = \left( n_F + \frac{1}{2} \right) w + \sum_{n=1}^{N-1} (\alpha_n \alpha_n - \tilde{\alpha}_n \tilde{\alpha}_n), \quad (3.42)$$
where in the last line we have used eqs. (3.33) and (3.36). So we find

\[ Q = \frac{N}{2} w + \sum_{n=1}^{N-1} (\alpha_{-n} \alpha_n - \tilde{\alpha}_{-n} \tilde{\alpha}_n). \tag{3.43} \]

The dependence on \( w \) can be easily understood noting that, under the constant shift \( n_i \to n_i + b \), with \( b \) an integer, \( Q = \sum_i n_i \to Q + Nb \). This is correctly reproduced by eq. (3.43), since under \( n_i \to n_i + b \) we have \( w \to w + 2b \), as we see from eq. (3.39).

We now perform the bosonization of the \( U(N) \) hamiltonian. The \( U(N) \) hamiltonian in second quantization reads

\[ H_{U(N)} = \frac{e^2 L}{2} \left[ \sum_{n=-N}^{N} n^2 B_n^\dagger B_n - \frac{N}{12} (N^2 - 1) \right] = \frac{e^2 L}{2} \sum_{n=-N}^{N} n^2 : B_n^\dagger B_n : \tag{3.44} \]

since \( N(N^2 - 1)/12 \) is just the normal ordering constant, see eq. (3.4). We write

\[ \sum_{n=-N}^{N} n^2 : B_n^\dagger B_n : = \sum_{n=-n_F}^{n_F} (n_F + 1 + n)^2 : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n : = \]

\[ = (n_F + 1)^2 \sum_{n=-n_F}^{n_F} : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n : + 2(n_F + 1) \sum_{n=-n_F}^{n_F} n : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n : + \]

\[ + \sum_{n=-n_F}^{n_F} n^2 : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n : . \tag{3.45} \]

The first sum vanishes because of eq. (3.23). The second sum is just \( L_0 + \tilde{L}_0 \), and it is immediately written in terms of \( \alpha_n, \tilde{\alpha}_n \) using eq. (3.33). Therefore

\[ H_{U(N)} = \frac{e^2 L}{2} \left[ (N + 1)(L_0 + \tilde{L}_0) + \sum_{n=-n_F}^{n_F} n^2 : c_{-n} b_n + \tilde{c}_{-n} \tilde{b}_n : \right] . \tag{3.46} \]

The last term in eq. (3.46) can be bosonized using the identity

\[ \oint \frac{dz}{2\pi i} z^2 : \partial c \partial b : = - \sum_n n^2 : c_{-n} b_n : - \sum_n n : c_{-n} b_n : , \tag{3.47} \]

which is easily checked substituting the mode expansion of the \( b, c \) fields into the left-hand side. Similarly

\[ \oint \frac{d\tilde{z}}{2\pi i} \tilde{z}^2 : \partial \tilde{c} \partial \tilde{b} : = + \sum_m m^2 : \tilde{c}_{-m} \tilde{b}_m : + \sum_m m : \tilde{c}_{-m} \tilde{b}_m : . \tag{3.48} \]
Eq. (3.46) then becomes
\[ H_{U(N)} = \frac{e^2 L}{2} \left[ N (L_0 + \tilde{L}_0) - \oint \frac{dz}{2\pi i} z^2 : \partial \bar{c} \partial b : + \oint \frac{d\bar{z}}{2\pi i} \bar{z}^2 : \partial \bar{c} \partial b : \right]. \] (3.49)

Computing the OPE \( \partial \bar{b}(z) \partial c(0) \) we can derive the relation
\[ : \partial \bar{b} \partial c : = \frac{i}{3} : (\partial X_L)^3 : + \frac{i}{6} : \partial^3 X_L : . \] (3.50)

Then
\[ - \oint \frac{dz}{2\pi i} z^2 : \partial c \partial \bar{b} : + \oint \frac{d\bar{z}}{2\pi i} \bar{z}^2 : \partial \bar{c} \partial b : = \frac{i}{3} \int \frac{dz}{2\pi i} z^2 : (\partial X_L)^3 : - \oint \frac{d\bar{z}}{2\pi i} \bar{z}^2 : (\partial X_R)^3 : + \]
\[ + \frac{i}{6} \left[ \int \frac{dz}{2\pi i} z^2 : \partial^3 X_L : - \oint \frac{d\bar{z}}{2\pi i} \bar{z}^2 : \partial^3 X_R : \right]. \] (3.51)

Substituting the mode expansion for \( X \) into the above expression, the second bracket gives \((-1/3)(\alpha_0 + \bar{\alpha}_0)\), which vanishes because of the constraint (3.33). The first bracket gives instead a term cubic in \( \alpha, \bar{\alpha} \), and we get
\[ H_{U(N)} = \frac{e^2 L}{2} \left[ N (L_0 + \tilde{L}_0) + \frac{1}{3} \sum_{mnp} \delta_{m+n+p} : \alpha_m \alpha_n \alpha_p + \bar{\alpha}_m \bar{\alpha}_n \bar{\alpha}_p : \right]. \] (3.52)

In \( \sum_{mnp} \) we separate from the rest terms where \( \alpha_0, \bar{\alpha}_0 \) appear,
\[
\frac{1}{3} \sum_{mnp} \delta_{m+n+p} : \alpha_m \alpha_n \alpha_p + \bar{\alpha}_m \bar{\alpha}_n \bar{\alpha}_p : =
\sum_{m,n>0} : \alpha_m \alpha_n \bar{\alpha}_{-m-n} : + \sum_{m,n<0} : \alpha_m \alpha_n \bar{\alpha}_{-m-n} : +
+ \alpha_0 \sum_{m \neq 0} : \alpha_m \bar{\alpha}_m : + \bar{\alpha}_0 \sum_{m \neq 0} : \bar{\alpha}_m \bar{\alpha}_m : + \frac{1}{3} (\alpha_0^3 + \bar{\alpha}_0^3). \] (3.53)

Using eqs. (3.33) and (3.36), and introducing \( \lambda = e^2 N \), which is the coupling to be held fixed in the \( 1/N \) expansion, our final result for the \( U(N) \) hamiltonian reads\(^5\)
\[ H_{U(N)} = \frac{\lambda L}{2} \left[ \frac{w^2}{4} + \sum_{n=1}^{N-1} (\alpha_n \alpha_n + \bar{\alpha}_n \bar{\alpha}_n) + \frac{1}{N} w \sum_{n=1}^{N-1} (\alpha_{-n} \alpha_{n} - \bar{\alpha}_{-n} \bar{\alpha}_n) + \right. \]
\[ + \left. \frac{1}{N} \left( \sum_{m,n>0} + \sum_{m,n<0} \right) : \alpha_m \alpha_n \bar{\alpha}_{-m-n} : + \bar{\alpha}_m \bar{\alpha}_n \bar{\alpha}_{-m-n} : \right]. \] (3.54)

The Hamiltonian for \( SU(N) \) is obtained subtracting \((e^2 L/2)Q^2/N\) from eq. (3.54). Using eq. (3.43) we find
\[ H_{SU(N)} = \frac{\lambda L}{2} \left[ \sum_{n=1}^{N-1} (\alpha_n \alpha_n + \bar{\alpha}_n \bar{\alpha}_n) - \frac{1}{N^2} \left( \sum_{n=1}^{N-1} (\alpha_{-n} \alpha_{n} - \bar{\alpha}_{-n} \bar{\alpha}_n) \right)^2 \right. \]
\[ + \left. \frac{1}{N} \left( \sum_{m,n>0} + \sum_{m,n<0} \right) : \alpha_m \alpha_n \bar{\alpha}_{-m-n} : + \bar{\alpha}_m \bar{\alpha}_n \bar{\alpha}_{-m-n} : \right]. \] (3.55)

\(^5\)Our result disagrees with eq. (4.48) of ref. [9], where the dependence on \( w \) has been lost.
Observe that for $SU(N)$ the dependence on the winding number $w$ cancels, as it should, since we have seen that for $SU(N)$ we could have set $w = 0$ from the beginning. The cancellation is however a check of the correctness of eqs. (3.54) and (3.43).

Eqs. (3.54) and (3.55) prove that, at least in perturbation theory in $1/N$, YM$_2$ with gauge group $U(N)$ or $SU(N)$ is equivalent to a string field theory, described by a string field $X(z, \bar{z})$, and governed by an Hamiltonian consisting of terms $O(1)$ and $O(1/N)$ (and, for $SU(N)$, a quartic term $O(1/N^2)$), describing the creation and annihilation of strings. We have seen explicitly that, at least perturbatively, eqs. (3.54) and (3.55) are exact, i.e. there are no further terms suppressed by powers of $1/N$. Eq. (3.55) coincides with eq. (2.10) and correctly reproduces the $1/N$ expansion of $SU(N)$ YM$_2$.

4. The non-perturbative correspondence

4.1 D-branes from YM$_2$

From eq. (2.1) we see that Young diagrams with a quadratic Casimir $C_2 = O(N^2)$ give contributions to $Z_{YM}$ proportional to $\exp\{-O(N)\}$; limiting ourselves for simplicity to a torus target space (so that $(\dim R)^2 - 2G = 1$ in eq. (2.1)), the structure of $Z_{YM}$ is

$$Z_{YM} = [O(1) + O(1/N^2) + \ldots] + O(e^{-O(N)})$$

where the bracket represents the perturbative expansion discussed above. If the string-YM correspondence holds even beyond perturbation theory in $1/N$, the terms $e^{-O(N)}$ should correspond to terms $e^{-O(1/g_s)}$ on the string theory side. In the following, for definiteness, we will consider the case of $SU(N)$.

An exact evaluation of the contributions $e^{-O(N)}$ to $Z_{YM}$ seems to be a quite formidable task. However, there is a large class of diagrams that we are able to evaluate, and which will turn out to give a rather interesting result. These are the Young diagrams in which one or more lines have more than $N$ boxes and the remaining part of the diagram has a number of boxes $O(1)$, see figs. 2 and 3. Thus, we find useful to introduce a distinction

**Figure 2:** Diagram with the first line longer than $N$.

**Figure 3:** Generic diagram with $k$ lines with more than $N$ boxes.

between “bounded” diagrams, defined as those diagrams in which all lines have less than $N$ boxes, and “long” diagrams, i.e. those in which at least the first line, and possibly more lines, are longer or equal to $N$. In particular one can consider long diagrams with $k$ long
lines, and long diagrams with \( N - k \) long lines: the contribution to the partition function of the two groups of diagrams is the same, since each diagram of the second group has the same Casimir of a complementary diagram of the first group, where the correspondence is the one shown in fig. 4.

\[ \begin{array}{c}
\text{Figure 4: Diagram with } O(N) \text{ long lines and its complementary, both with the same Casimir value.}
\end{array} \]

From the expression of the Casimir, eq. (2.3), one sees immediately that long diagrams are interesting candidates for non-perturbative contributions, since they have \( C_2 = O(N^2) \) and therefore their contributions to \( \exp\{-\lambda A/2N\}C_2 \) is \( \exp\{-O(N)\} \). However, they certainly do not exhaust the class of all Young diagrams with \( C_2 = O(N^2) \), since in general diagrams with \( O(N) \) boxes in the bounded part \( R' \) can have \( C_2 = O(N^2) \), independently of whether they have long lines or not. While the contribution of diagrams with \( O(N) \) boxes in the bounded part \( R' \) is difficult to evaluate, the contribution of long diagrams with \( O(1) \) boxes in \( R' \) can be evaluated as follows.

Consider first a long diagram as the one shown in fig. 3, with the first line of length \( N + m \), with \( m = 0, 1, \ldots \infty \) generic, and \( O(1) \) boxes in the remaining part. Since we are considering \( SU(N) \), there are at most \( N - 1 \) lines in total. Eliminating the first line, we are left with a Young diagram corresponding to a generic representation \( R' \) of a chiral sector (in the sense of 2) of \( SU(N - 1) \). Let again \( h_i \) be the number of boxes in the \( i \)-th line, \( n = \sum_{i=1}^{N-1} h_i \) the total number of boxes in the diagram \( R \), and let \( n' = \sum_{i=2}^{N} h_i \) be the total number of boxes in \( R' \). Simple algebra shows that

\[
C_2(R) = C_2(R') + m \left( 3N - 3 - \frac{2n'}{N} \right) + m^2 \left( 1 - \frac{1}{N} \right) + 2N^2 - 2N - 4n'.
\]

(4.2)

Since \( n' = O(1) \), the leading terms in \( C_2(R) \) are:

\[
C_2(R) \simeq C_2(R') + 2N^2 + N(3m - 2) + m^2.
\]

(4.3)

The great simplification in eq. (4.3) is that \( n' \) does not appear explicitly and all dependence on \( R' \) is through \( C_2(R') \). This allows to factorize the contributions of the subdiagram \( R' \). In fact, summing over all \( m = 0, \ldots, \infty \) and over all representations \( R' \) with \( n' = O(1) \), and defining

\[
a \equiv \frac{\lambda A}{2},
\]

we find that the contribution to \( Z_{YM} \) of this class of diagrams is

\[
\sum_{R'} \sum_{m=0}^{\infty} e^{-\frac{\lambda}{N} \left[ C_2(R') + 2N^2 + N(3m-2) + m^2 \right]} \left( 1 + O \left( \frac{1}{N} \right) \right) =
\]

\[
e^{2a} e^{-2aN} \left[ \sum_{R'} e^{-\frac{\lambda}{N} C_2(R')} \sum_{m=0}^{\infty} e^{-\frac{\lambda}{N} (3mN + m^2)} \left( 1 + O \left( \frac{1}{N} \right) \right) \right] =
\]

\[
e^{-2aN} \left[ \left( Z_{chir}^{SU(N-1)} \right) \frac{e^{2a}}{1 - e^{-3a}} + O \left( \frac{1}{N} \right) \right].
\]

(4.5)
We have denoted by $Z_{\text{chir}}^{\text{SU}(N-1)}$ the chiral partition function of $SU(N-1)$ (see ref. [2]).

It is not difficult to extend this result to diagrams with $k > 1$ long lines, with $k \ll N$ (see fig. 3). Let the length of the long lines be $h_i = N + m_i$, with $m_1 \geq m_2 \geq \ldots \geq m_k$. As discussed above, an identical contribution comes from diagrams with $N - k$ long lines; then, with the same approximations used in eq. (4.3), we find

$$C_2(R) \simeq C_2(R') + 2kN^2 + (3 \sum_{i=1}^{k} m_i - 2k^2)N + \sum_{i=1}^{k} m_i^2. \quad (4.6)$$

The resummation of all contributions with $k$ long lines, with $k \ll N$, gives therefore

$$\sum_{R'} \sum_{m_1, \ldots, m_k} e^{-\frac{\pi^2}{4 \alpha'} \left[ C_2(R') \right]} e^{2kN^2 + (3 \sum_{i=1}^{k} m_i - 2k^2)N + \sum_{i=1}^{k} m_i^2} \left( 1 + O \left( \frac{1}{N} \right) \right) = e^{-2\pi \alpha'} \prod_{m=1}^{k} \frac{1}{1 - e^{-3\alpha m}} + O \left( \frac{1}{N} \right). \quad (4.7)$$

(where in the first line $\sum_{m_1, \ldots, m_k}$ runs over all $m_i = 0, \ldots \infty$ with the condition $m_1 \geq m_2 \geq \ldots \geq m_k$). Thus, we have been able to resum a very large class of diagrams, and the result is quite interesting: the resummation of all diagrams with $k$ “long” lines, i.e. with $k$ lines longer than $N$, and with all possible chiral subdiagrams $R'$ with $O(1)$ boxes gives a contribution proportional to

$$e^{-2\pi \alpha' \alpha N}, \quad k = 1, 2, \ldots \quad (4.8)$$

An equal contribution comes from the resummation of the diagrams with $N - k$ long lines.

Recalling that $N = 1/\alpha_s$, $a = \lambda A/2$ and that $\lambda$ is related to the string tension of the string theory by $\lambda = 1/(\pi \alpha')$ (see the discussion below eq. (2.5)), we see that these contributions are just of the form $e^{-S_{D1}}$ with

$$S_{D1} = \tau_1 kA, \quad (4.9)$$

and

$$\tau_1 = \frac{1}{\pi \alpha' \alpha_s}. \quad (4.10)$$

Now, $\tau_1$ has exactly the form expected for the tension of a D1-brane, modulo a numerical factor which depends on the specific theory (for instance in type IIB in 10 dimensions, $\tau_1 = 1/(2\pi \alpha' g_s)$ [18]). The dependence on the target space area is also what we would expect from D1-branes. Indeed, recall that the string theory equivalent to YM$_2$ is quite peculiar because it describe a string with no foldings [1], i.e. a string whose world-sheet area is an integer times the target space area. The integer then counts the number of times that the string world-sheet covers the target space. If this theory has D1-branes, it is therefore natural to expect that they, too, have no foldings, and indeed the factor $kA$

---

[6]We should remark that the factorization of the contribution of the representations $R'$ takes place only at leading order. The $O(1/N)$ corrections in eq. (4.3) are not simply proportional to $Z_{\text{chir}}^{\text{SU}(N-1)}$. 
in eq. (4.9) can be interpreted as the world-sheet area of a D1-brane wrapping \( k \) times, without foldings, over the target space.

Instead, it is clear that (independently of any approximation) there is no contribution that could be interpreted as \( e^{-S_{D0}} \), with \( S_{D0} \) the action of a D0-brane. In fact, \( S_{D0} \) would rather be proportional to the length of the world-line of the D0-brane. However, \( Z_{YM} \) is a function only of the area of the target space, which has no relation to the world-line length and therefore such terms are absent.

Thus, we have a non-perturbative structure in which terms that allow an interpretation as \( Dp \)-brane with \( p \) odd (i.e. \( p = 1 \) because we are in two dimensions) are present, while with \( p \) even (i.e. \( p = 0 \)) they are absent. This is the typical structure of a type B string theory. Since we have no spacetime supersymmetry, it is quite natural to identify the theory with a sort of type 0B string theory.

Finally, it is interesting to recall that perturbation theory is really an expansion in \( 1/N^2 \), i.e. in \( g_s^2 \), rather than in \( 1/N \). It was in fact observed by Gross \[1\] that the terms with odd powers in \( 1/N \) are zero because of a cancellation between a Young diagram \( R \) and its conjugate \( \bar{R} \) which has its rows and columns interchanged. However, when we consider “long” diagrams, i.e. diagrams with lines longer than \( N \), the conjugate diagram does not exist, because we cannot have columns with more than \( N \) boxes. So the cancellation does not take place, and in the non-perturbative sectors the corrections have the form

\[
e^{-\frac{k^2 a_s}{gs}} \left( 1 + O(g_s) \right),
\]

rather than \( e^{-\frac{k^2 a_s}{gs}} \left( 1 + O(g_s^2) \right) \).

### 4.2 The stringy exclusion principle

In the previous section we have understood the effect of “long” Young diagrams: we have seen that the Young diagrams with \( k \) (or \( N - k \)) lines longer than \( N \) give the contribution that, in string theory, would be expected from a D1-brane wrapping \( k \) times over the target space. We now turn our attention to the non-perturbative effects in \( Z_{SU(N)}^{\text{bounded}} \), again limiting ourselves to the torus. Following ref. \[1\], we consider the contribution of the (bounded) diagrams in which the total number of boxes \( n \) is \( O(1) \), rather than \( O(N) \).\(^7\) Then in the Casimir \( \tilde{C} \) the term \( \tilde{C} \) is \( O(1) \) while \( n^2/N = O(1/N) \), so they can both be neglected compared to \( nN \). Therefore in this approximation \[1\]

\[
Z_{YM}^{G=1} \simeq \sum_{\{h_i\}} e^{-a \sum_{i=1}^{N-1} h_i},
\]

where as usual \( h_i \) denote the length of the \( i \)-th row and therefore \( \sum_{\{h_i\}} \) runs over the domain \( h_1 \geq h_2 \geq \ldots \geq h_{N-1} \geq 0 \). The sum is performed \[1\] introducing \( k_1 = h_1 - h_2, k_2 = \ldots \)

\(^7\)In the language of sect. \[2,3\] this means that we are restricting to excitations around the Fermi surface at \( +n_F \). A similar contribution comes from the excitations around \( -n_F \). For the torus, at leading order, this just results in an overall factor of 2, which is not important for our purposes. In the notation of \[2,3\], we are restricting to one chiral sector.
\[ h_2 - h_3, \ldots, k_{N-2} = h_{N-2} - h_{N-1}, k_{N-1} = h_{N-1}. \] Then \( \sum_i h_i = \sum_j jk_j \) and

\[ Z_{YM}^{G=1} \simeq \sum_{k_1=0}^{\infty} \ldots \sum_{k_{N-1}=0}^{\infty} e^{-a \sum_{j=1}^{N-1} jk_j} = \prod_{m_1=1}^{N-1} \left( \sum_{k=0}^{\infty} e^{-am_1 k} \right). \] (4.13)

Then

\[ Z_{YM}^{G=1} \simeq \prod_{m_1=1}^{N-1} \frac{1}{1 - e^{-am_1}}. \] (4.14)

From the non-perturbative point of view, the interesting aspect of this result is that the product over \( m \) runs only from \( m = 1 \) to \( m = N - 1 \), rather than up to \( m = \infty \). The reason, of course, is that there are only \( N - 1 \) variables \( k_j \) because the Young diagrams of \( SU(N) \) have at most \( N - 1 \) lines. Taking the logarithm and expanding it,

\[ \ln Z_{YM}^{G=1} \simeq - \sum_{m_1=1}^{N-1} \ln(1 - e^{-am_1}) = \sum_{m_1=1}^{N-1} \sum_{m_2=1}^{\infty} \frac{1}{m_2} e^{-am_1 m_2}. \] (4.15)

Inserting \( 1 = \sum_{n=1}^{\infty} \delta_{m_1 m_2 n} \),

\[ \ln Z_{YM}^{G=1} \simeq \sum_{n=1}^{\infty} c(n) e^{-an}, \] (4.16)

with

\[ c(n) = \sum_{m_1=1}^{N-1} \sum_{m_2=1}^{\infty} \frac{1}{m_2} \delta_{m_1 m_2 n}. \] (4.17)

Eqs. (4.16) and (4.17) show clearly the geometric interpretation in terms of a theory of maps. In fact, \( e^{-an} = \exp\{-\lambda/2 An\} \) is just the factor expected from a string without foldings that wraps \( n \) times around the target space, with string tension \( 1/(2\pi\alpha') = \lambda/2 \), while it is possible to show that \( c(n) \) is just the number of coverings of the torus by a torus with \( n \) sheets [1, 15]. Thus we have a mapping from a world-sheet to a target space, i.e. a string, and in this interpretation \( m_1, m_2 \) are the number of times that the string world-sheet winds around the two cycles of the torus.

The surprise, in eq. (4.17), is that the winding over one of the cycles, \( m_1 \), is limited by \( N - 1 \) for \( SU(N) \) (or by \( N \) if we repeat the calculation for \( U(N) \)). So, first of all, there is an asymmetry between \( m_1 \) and \( m_2 \), which instead ranges from 1 to \( \infty \). Technically this came out because \( m_1 \) and \( m_2 \) have a very different origin: \( m_1 \) labels the variables \( k_j \) and therefore the lines in a Young diagram, and then it cannot exceed \( N - 1 \). Instead \( m_2 \) appeared from the Taylor expansion of the logarithm in eq. (4.15). However, it is clear that this asymmetry must be an artefact of our approximations, i.e. of restricting to the class of Young diagrams such that \( \tilde{C} \) can be neglected, and if one would be able to compute exactly the non-perturbative contributions the symmetry should be restored. Our expectation is that both cycles will then be limited by \( N - 1 \). Therefore, the number of times that the string winds on the target space torus is limited by a value \( N - 1 \) for \( SU(N) \) or \( N = 1/g_s \) for \( U(N) \). This is clearly a non-perturbative limitation, and it is very similar to the stringy exclusion principle found by Maldacena and Strominger [12] in the context of AdS₃.
4.3 Numerical investigation of the non-perturbative phase structure

Given the difficulty of an exact analytical investigation of the non-perturbative contributions, one might consider a numerical study. Actually, the “long” diagrams discussed in sect. 4.1 would be difficult to study numerically, because even for fixed $N$ there is an infinite number of them; however, we have shown that these diagrams can be resummed and can be well understood analytically. A complete analytic understanding is instead more difficult for the “bounded” diagrams but since, at fixed $N$, there is a finite number of them, one could try to compute their effect numerically. In particular, one might try a strategy borrowed from lattice gauge theory simulations: evaluate the partition function \((2.1)\) numerically, restricting the sum to the bounded diagrams; subtract the perturbative contribution, evaluated to a sufficiently large order, chosen such that, numerically, the exponential terms can be extracted by a fit against $N$. Furthermore, the perturbative contribution to the torus partition function have already been computed explicitly to very large order in ref. \[19\].

This strategy however meets an instructive problem. Fig. 5 shows the “bounded” partition function of the torus, evaluated numerically for different values of $N$ and $a$, and compares it with the perturbative expansion of ref. \[13\], pushed up to 6th order. As we expect, for sufficiently large $N$ the two coincide, and there is a critical value $N_c$ below which they start to diverge; actually, at $N < N_c$ even the qualitative behaviour of $Z_{\text{bounded}}$ has nothing to do with its perturbative expansion.

Numerically, we have found that the critical value $N_c$ is a decreasing function of $a$, roughly given by $aN_c(a) \sim \gamma$, with $\gamma$ a numerical constant. This means that the perturbative expansion is a good approximation only for $aN \gg \gamma$; Taking as a typical reference value the non-perturbative contributions $\sim e^{-2aN}$ found from long diagrams, we see that, when the perturbative expansion starts to be in rough agreement with the exact result, a term of this type would be already suppressed at least by a factor $e^{-2aN} \sim e^{-2\gamma} = O(10^{-18})$ compared to the perturbative term which is $O(1)$, and it is therefore numerically invisible.

However, the fact that $aN_c(a) \sim \gamma$ is of some interest in itself. It means that in the plane $(g_s, a)$ there is a non-trivial phase structure. When $g_s \ll a/\gamma$, perturbative string theory is a good approximation to the full theory (at least if at the same time \(g_s \leq 1/2\), because $g_s = 1/N$ and $N \geq 2$). Instead, when $g_s \sim a/\gamma$ we enter into a qualitatively different regime, as we see from fig. 4, where the perturbative expansion is of no use and strong coupling effects are dominant.

If we take the limit $a \rightarrow 0$ at fixed $g_s$, we always end up in this strong coupling domain, for all non-zero values of $g_s$. The limit $a \rightarrow 0$ has been studied in ref. \[3\], where it is found that YM$_2$ becomes a topological string theory. This therefore clarifies the nature of the theory in the strong coupling phase $g_s \sim a/\gamma$. On the other hand, this also means that from

---

8Of course $Z_{\text{bounded}}$ and the full partition function $Z_{YM}$ differ only by the contribution of “long” diagrams, so they have the same perturbative expansion.

9The numerical value of $\gamma$ depends of course on the precise definition of $N_c$. For instance, if $N_c$ is defined as the point where the 6th order perturbative series and the numerical result differ by 5%, then $\gamma = O(20)$. Also, the precise functional form of $N_c(a)$ is not exactly $\sim 1/a$. However, the only important point for us is simply that there are two qualitatively different regions separated by a curve $N_c(a)$.
Figure 5: Numerical evaluation of $Z_N$ for $a = 2.0, 1.8, 1.6, 1.4, 1.2$. Dashed lines are the plots of the perturbative series.

In the limit $a \to 0$ we cannot learn anything about the perturbative string theory, since the two regimes are qualitatively different.

5. Conclusions

We have examined various aspects of the string/YM correspondence in two dimensions. At the perturbative level we have shown how, from the bosonization of the fermionic formulation of YM$_2$, one can derive rigorously the string field theory Hamiltonian which reproduces the full $1/N$ expansion of the theory. At the non-perturbative level, we have found that the YM$_2$ partition function reproduces a number of non-perturbative effects which should be expected in the corresponding string theory. In particular, we have identified representations of $SU(N)$ that would correspond to $D1$-branes in the string formulation, while terms that could be identified with $D0$-branes are absent; this suggests that the correspondence holds even non-perturbatively, and that the non-perturbative structure is typical of a type 0B string theory.

We conclude with some conjectural remarks. If the interpretation in terms of some form of type 0B theory on the cylinder is correct, it is natural to ask what happens if we perform a T-duality transformation along the compact spatial direction of the cylinder, and it is natural to expect to get a type 0A string theory; the $D1$-branes would then become $D0$-branes. Such a string theory would not have a direct relation with a two-dimensional YM, since we have seen that in YM$_2$ the partition function depends only on the area of the target space, and cannot account for the effect of $D0$-branes.

However, a type A theory, and $D0$-branes, could be the signal of the non-perturbative opening up of a third dimension, with size $R_3 \sim g_s \alpha'^{1/2}$. Of course, since we have no space-time supersymmetry, the possibility of the opening of a third dimension, and correspondingly the existence of a three-dimensional M-theory, should be taken with the same caveats that hold for the bosonic string in 26 dimensions. Even in that case, however, there are arguments suggesting the existence of a 27-dimensional M-theory [20].
If these conjectures are correct, there should be a 3-dimensional M-theory which reduces to a two-dimensional string theory at weak coupling, when the third dimension becomes unaccessible. It is quite tempting to conjecture that such an M-theory could be a Chern-Simons (CS) theory on a suitable manifold with a boundary. This is suggested by the well known fact that a CS theory on a three-dimensional manifold with a boundary induces a current algebra on the boundary [21], and indeed CS theory can be used to produce in this way all rational CFT [22]. Furthermore, it is possible to construct string theories, which have the peculiarity that the matter and ghost sectors do not decouple, which have the target space interpretation of a CS theory [23].

A. Bosonization and string hamiltonian for \( \lambda \) generic

In this appendix we repeat the calculations that led to the string hamiltonian starting from eq. (3.22) with \( \lambda \) generic. This is an useful check of the correctness of the result, and will reveal some small subtlety in the computation, especially concerning the relevant definition of normal ordering.

The formulas for the bosonization of the \( bc \) (\( \tilde{b}\tilde{c} \)) theory are the standard ones used in sect. [3]:

\[
b = : e^{iX_L} c, \quad c = : e^{-iX_L} c, \quad bc = i\partial X_L.
\]  

(A.1)

(and similar ones for the \( \tilde{b}\tilde{c} \) fields, with an antiholomorphic bosonic field \( X_R(z) \)). However, it is important to observe that the normal ordering in this relations is the conformal one, that in the \( bc \) theory is related to the annihilation-creation one by (see e.g. [18], chapt. 2):

\[
: b(z)c(z') : c = : b(z)c(z') : + (z/z')^{1-\lambda - 1} \frac{1}{z-z'}
\]  

(A.2)

from which one can derive:

\[
: b(z)c(z) : c = : b(z)c(z) : + \frac{1-\lambda}{z}
\]  

(A.3)

We see that for \( \lambda = 1 \) they are equal; therefore in this special case we could neglect the distinction between the two.

For the bosonic theory instead the normal ordering of annihilation-creation (with respect to the standard vacuum) is identical to the conformal one.

Developing the \( X \) field in modes as in the \( \lambda = 1 \) case (eq. (3.28)), using eq. (A.1) and the analogous ones for the \( \tilde{b}\tilde{c} \) fields) and the relation (A.3) between the normal orderings, we obtain:

\[
\alpha_m = \sum_{n=-n_F}^{n_F} : c_{m-n} b_n : + (\lambda - 1) \delta_{m,0}
\]  

(A.4)

\[
\tilde{\alpha}_m = \sum_{n=-n_F}^{n_F} : \tilde{c}_{m-n} \tilde{b}_n : + (\lambda - 1) \delta_{m,0}
\]
In particular the constraint (3.25) becomes:
\[ \alpha_0 + \bar{\alpha}_0 = 2(\lambda - 1) \]  
(A.5)

In the general case the Virasoro generators of the \( bc \) theory are:
\[ L_m^{(bc)} = \sum_n (m\lambda - n) : b_n c_{m-n} : + \frac{\lambda (1 - \lambda)}{2} \delta_{m,0} \]  
(A.6)

In particular:
\[ L_0^{(bc)} = \sum_n n : c_{-n} b_n : + \frac{\lambda (1 - \lambda)}{2} \]  
(A.7)

The Virasoro generators of the bosonic theory are:
\[ L_m^{(X)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \alpha_n : \left( \alpha - \frac{1}{2} \alpha_m \right) (m+1) \alpha_m . \]  
(A.8)

and in particular:
\[ L_0^{(X)} = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n : - \left( \alpha - \frac{1}{2} \right) \alpha_0 \]  
(A.9)

The antiholomorphic field obeys similar formulas. To obtain the hamiltonian (for \( U(N) \) and \( SU(N) \)) we can easily generalize the calculations of sect. 3.

For what concern the \( U(1) \) charge, following the same steps of eq. (3.42), we obtain:
\[ Q = (n_F + 1)(\alpha_0 - \bar{\alpha}_0) + (L_0 - \bar{L}_0) = (n_F + 1)(\alpha_0 - \bar{\alpha}_0) + (\alpha_0 - \bar{\alpha}_0)(\lambda - 1) + \]  
\[ - (\lambda - \frac{1}{2})(\alpha_0 - \bar{\alpha}_0) + \sum_{n=1}^{N-1} (\alpha_{-n} \alpha_n - \bar{\alpha}_{-n} \bar{\alpha}_n) \]  
(A.10)

where we have used eq. (A.9) and eq. (A.5). So we finally obtain the same result of the \( \lambda = 1 \) case, as we expected:
\[ Q = \frac{N}{2} w + \sum_{n=1}^{N-1} (\alpha_{-n} \alpha_n - \bar{\alpha}_{-n} \bar{\alpha}_n) . \]  
(A.11)

The \( U(N) \) hamiltonian, given by eq. (3.45), can be written as:
\[ H_{U(N)} = |e^2| \sum_{n=-n_F}^{n_F} n^2 : c_{-n} b_n : + : \bar{c}_{-n} \bar{b}_n : + (N+1)(L_0 + \bar{L}_0) - (N+1)\lambda(1-\lambda) \]  
(A.12)

For \( \lambda \) generic we must use the relation:
\[ \oint \frac{dz}{2\pi i} z^2 : \partial c \partial b : = - \sum_n n^2 : c_{-n} b_n : + (1 - 2\lambda) \sum_n n : c_{-n} b_n : + \]  
\[ + \lambda (1 - \lambda) \sum_n : c_{-n} b_n : \]  
(A.13)
and the similar one for tilded fields:

$$\oint \frac{dz}{2\pi i} z^2 : \partial \tilde{c} \partial b : = + \sum_n n^2 : \tilde{c}_{-n} \tilde{b}_n : - (1 - 2\lambda) \sum_n n : \tilde{c}_{-n} \tilde{b}_n :$$

$$- \lambda (1 - \lambda) \sum_n \tilde{c}_{-n} \tilde{b}_n :$$

(A.14)

Thus the Hamiltonian becomes:

$$H_{U(N)} = \frac{e^2}{2} L \left[ - \oint \frac{dz}{2\pi i} z^2 : \partial c \partial b : + \oint \frac{dz}{2\pi i} z^2 : \partial \tilde{c} \partial \tilde{b} : + \right.$$

$$\left. (N + 2 - 2\lambda)(L_0 + \tilde{L}_0) - (N + 2 - 2\lambda)\lambda (1 - \lambda) \right]$$

(A.15)

(\text{where we have used eq. (3.25) and eq. (A.7)). Using eq. (A.2) we can derive the relation:

$$: \partial c(z) \partial b(z) : = : \partial c(z) \partial b(z) : + \frac{\lambda^3 - 3\lambda^2 + 2\lambda}{3z^3}$$

(A.16)

(\text{and analogously for the} \tilde{b} \tilde{c} \text{fields}).

Using eq. (A.16), then eq. (3.50) and finally the mode expansion of the bosonic field, we obtain:

$$H_{U(N)} = \frac{e^2}{2} L \left[ \sum_{m,n>0} + \sum_{m,n<0} \alpha_m \alpha_n \alpha_{-m-n} + \tilde{\alpha}_m \tilde{\alpha}_n \tilde{\alpha}_{-m-n} : + 2\alpha_0 \sum_{n=1}^{N-1} \alpha_{-n} \alpha_n + \right.$$

$$+ 2\tilde{\alpha}_0 \sum_{n=1}^{N-1} \tilde{\alpha}_{-n} \tilde{\alpha}_n + \frac{1}{3} \alpha_0^3 + \frac{1}{3} \tilde{\alpha}_0^3 - \frac{1}{3} (\alpha_0 + \tilde{\alpha}_0) + (N + 2 - 2\lambda)(L_0 + \tilde{L}_0) +$$

$$- (N + 2 - 2\lambda)\lambda (1 - \lambda) - \frac{2}{3} (\lambda^3 - 3\lambda^2 + 2\lambda) \right]$$

(A.17)

which using eq. (A.3) and eq. (A.3) becomes:

$$H_{U(N)} = \frac{(e^2 N)L}{2} \left[ \sum_{m,n>0} + \sum_{m,n<0} (\alpha_n \alpha_{-n} + \tilde{\alpha}_n \tilde{\alpha}_{-n}) + \frac{1}{N} w \sum_{n=1}^{N-1} (\alpha_{-n} \alpha_n - \tilde{\alpha}_{-n} \tilde{\alpha}_n) + 

+ \frac{1}{N} \left( \sum_{m,n>0} + \sum_{m,n<0} \alpha_m \alpha_n \alpha_{-m-n} + \tilde{\alpha}_m \tilde{\alpha}_n \tilde{\alpha}_{-m-n} : \right. \right.$$

(A.18)

which is exactly eq. (3.54) as we expected. Finally it is obvious that, being \textit{H}_{U(N)} and \textit{Q} the same ones of the \lambda = 1 case, \textit{H}_{SU(N)} is given by eq. (3.55).

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