Character expansion for HOMFLY polynomials.

III. All 3-Strand braids in the first symmetric representation

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We continue the program of systematic study of extended HOMFLY polynomials, suggested in [1, 2]. Extended polynomials depend on infinitely many time-variables, are close relatives of integrable \( \tau \)-functions, and depend on the choice of the braid representation of the knot. They possess natural character decompositions, with coefficients which can be defined by exhaustively general formula for any particular number \( m \) of strands in the braid and any particular representation \( R \) of the Lie algebra \( GL(\infty) \). Being restricted to "the topological locus" in the space of time-variables, the extended HOMFLY polynomials reproduce the ordinary knot invariants. We derive such a general formula, for \( m = 3 \), when the braid is parameterized by a sequence of integers \((a_1, b_1, a_2, b_2, \ldots)\), and for the first non-fundamental representation \( R = [2] \). Instead of calculating the mixing matrices directly, as suggested in [2], we deduce them from comparison with the known answers for torus and composite knots. A simple reflection symmetry converts the answer for the symmetric representation [2] into that for the antisymmetric one [1, 1]. The result applies, in particular, to the figure eight knot 4_1, and was further extended to superpolynomials in arbitrary symmetric and antisymmetric representations in [3].

1 Introduction

In [1, 2] we started a program to promote the HOMFLY polynomials [4] to character expansions, representing them as linear combinations of the Schur functions \( S_Q(p_k) \) (i.e. the characters of linear groups) [5]. Such an expansion depends on the choice of a braid realization of the knot, thus, its coefficients by themselves are not knot invariants, instead they are pure group theory quantities and possess many nice properties. For an \( m \)-strand braid \( B \) the HOMFLY polynomial in representation \( R \) is expanded as

\[
H_R^B = \text{Tr} R^{\otimes m} \left( (q^p)^{\otimes m} B \right) = \sum_{Q^m[R]} h_{RQ}^B S_Q^*(A)
\]

where

\[
S_Q^*(A) = \text{Tr}_{Q \in R^{\otimes m}} (q^p)^{\otimes m} = S_Q(p_k^*), \quad p_k^* = \frac{\lfloor kN \rfloor}{\lfloor k \rfloor} = \frac{A^k - A^{-k}}{q^k - q^{-k}}
\]

are quantum dimensions of representations \( Q \) of \( SU(N) \), where \( A = q^N \) and \( [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \). The coefficients \( h_{RQ}^B \) do not depend on \( A \), i.e. on \( N \), thus, they can be evaluated from analysis of an arbitrary group \( SU_q(N) \).

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[1] Our calculus is based on the approach by [2], though that of [1, 2, 3] is, by essence, also very close. The first part of this formula is related to Chern-Simons theory in the temporal gauge [10]. The new element is a special emphasis on the character expansion, which allows one to extend the knot polynomials to arbitrary time variables [1] and provides very simple general formulas for entire classes of knots. However, following this line, we omit the additional factor \( (A^\gamma q^\gamma)w^B \) in front on the trace in [1], where \( w^B \) is the writhe number of the braid, while \( \alpha \) and \( \gamma \) depend on normalization of the \( R \)-matrix. Throughout this paper we use a special normalization of \( R \)-matrices, though in our normalization \( \alpha \) and \( \gamma \) are actually non-vanishing: for most purposes (in the standard framing) \( \alpha = -|R| \) and \( \gamma = -4\sqrt{q} \). We discuss this issue in a separate subsection [3, 11] and actually restore the factors in the formulas of the Appendix. To simplify the notations we do not put star on \( H(A) \) in this paper, as we do in [1, 2, 3], because the extended polynomials are almost not mentioned here. They can be, however, obtained from the formulas of the Appendix simply by removing the stars from \( \langle S_Q \rangle \), thus, promoting them from the quantum dimensions to the Schur functions.
Instead they can be represented as traces in auxiliary spaces of intertwining operators $\mathcal{M}_{R=Q}$, whose dimension is the number $\dim \mathcal{M}_{R=Q} = N_{R=Q}$ of times the irreducible representation $Q$ appears in the $m$-th tensor power of representation $R$,

$$R^{\otimes m} = \sum_{Q^{m} \oplus R} \mathcal{M}_{R=Q} \otimes Q$$

(3)

The trace is taken of a product of the diagonal quantum $R$-matrices $\hat{R}$ acting in $\mathcal{M}_{R=Q}$, and the "mixing matrices" intertwining $R$-matrices that act on different pairs of adjacent strands in the braid. These mixing matrices, in their turn, can be represented as products of universal constituents, associated with a switch between two "adjacent" trees describing various decompositions (3).

In [2] we exhaustively described such representations for the coefficients $h^R_{RQ}$ for arbitrary $m = 2, 3, 4$-strand braids and for the simplest fundamental representation $R = |1]$:

$$m = 2, \quad B = R^a : \quad H^{(a)}_{[1]} = q^a S^*_2(A) + \left(-\frac{1}{q}\right)^a S^*_1(A) = q^a S^*_2(A) + \left(q \rightarrow -\frac{1}{q}\right)$$

(4)

$$m = 3, \quad B = (R \otimes I)^{a_1} (I \otimes R)^{b_1} (R \otimes I)^{a_2} (I \otimes R)^{b_2} \ldots$$

$$H^{(a_1, b_1| a_2, b_2| \ldots})_{[1]} = q^{\sum (a_i + b_i)} S^*_3(A) + \left(-\frac{1}{q}\right)^{\sum (a_i + b_i)} S^*_1(A) + \left(\text{Tr}_{2 \times 2} \widehat{R}_{21}^{a_1} U_2 \widehat{R}_{21}^{b_1} U_2 \widehat{R}_{22}^{a_2} U_2 \widehat{R}_{22}^{b_2} U_2 \widehat{R}_{23}^{a_3} U_2 \widehat{R}_{23}^{b_3} U_2 \ldots \right) S^*_2(A) + \left(q \rightarrow -\frac{1}{q}\right)$$

(5)

$$m = 4, \quad B = (R \otimes I)^{a_1} (I \otimes R \otimes I)^{b_1} (R \otimes I \otimes I)^{a_2} (I \otimes R \otimes I)^{b_2} (R \otimes I \otimes I)^{a_3} (I \otimes R \otimes I)^{b_3} \ldots$$

$$H^{(a_1, b_1, c_1| a_2, b_2, c_2| \ldots)}_{[1]} = q^{\sum (a_i + b_i + c_i)} S^*_4(A) + \left(-\frac{1}{q}\right)^{\sum (a_i + b_i + c_i)} S^*_1(A) + \left(\text{Tr}_{2 \times 2} \widehat{R}_{21}^{a_1} U_2 \widehat{R}_{21}^{b_1} U_2 \widehat{R}_{22}^{a_2} U_2 \widehat{R}_{22}^{b_2} U_2 \widehat{R}_{23}^{a_3} U_2 \widehat{R}_{23}^{b_3} U_2 \ldots \right) S^*_3(A) + \left(q \rightarrow -\frac{1}{q}\right)$$

(6)

where

$$\widehat{R}_{2} = \begin{pmatrix} q & 1 \\ -1/q & -1/q \end{pmatrix} \quad \widehat{R}_{3} = \begin{pmatrix} q & 1 \\ -1/q & -1/q \end{pmatrix}$$

$$U_2 = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} \quad U_3 = \begin{pmatrix} 1 & 1 \\ c_2 & s_2 \end{pmatrix} \quad V_3 = \begin{pmatrix} c_3 & s_3 \\ -s_3 & c_3 \end{pmatrix}$$

(7)

Subscripts refer to the size of the matrices, the entries of rotation matrices $U$ and $V$ are given by

$$e_k = \frac{1}{|k|}, \quad s_k = \sqrt{1 - e_k^2} = \sqrt{\frac{|k - 1| |k + 1|}{|k|}}$$

(8)

These formulas provide a very transparent and convenient representation for infinitely many HOMFLY polynomials and seem to be very useful for any theoretical analysis of their general properties, from integrability to linear Virasoro like relations (including $A$-polynomials and spectral curves [11, 12, 13, 14], AMM/EO topological recursion [15 16] etc). Therefore, further insights are important about the structure of these formulas and their generalizations (in [2] the $m = 5$ case was also investigated, and the general formula for the coefficients $h_{[1]}^{[m-1,1]}$ was suggested for all $m$).

The limitations in [2] are pure technical: to make the paper readable and the main ideas understandable, we considered only the implications of the theory of $SU_q(2)$ quantum group: this allowed us to calculate only
contributions of the Young diagrams $Q$ with one and two columns or rows. Including the diagrams with $l$ columns or rows, which arise when the number of strands in the braid is $m \geq 5$, requires the similar use of the $SU_q(l)$ quantum group theory, which is tedious but straightforward, and will be considered in further publications. (We emphasize once again that $l$ has nothing to do with $N$ in $A = q^N$, the relevant $l$ is related to the number $m$ of strands, and for small $m \leq 4$ the smallest $l = 2$ is sufficient to describe all the HOMFLY polynomials with $R = [1]$.)

Another restriction in [2] was to $R = [1]$. It is also partly related to restriction to $SU_q(2)$, but not only. It is the purpose of the present paper to make a first step in the direction towards the colored HOMFLY polynomials with $|R| > 1$, that is, to the symmetric representation $R = [2]$. Instead of performing this calculation directly, a la [2], we use a shortcut: we determine the five parameters (angles) of the three orthogonal matrices $\hat{U}_{[51]}$, $\hat{U}_{[321]}$ and $\hat{U}_{[21]}$ by comparison with the answers for torus and composite knots and links in eq. (24) below. This can be also compared with the results of the long-lasting impressive work by the Indian group [7] using the direct evaluation of the Racah coefficients.

Our goal is to find the necessary ingredients for formulas like (11)-(13), which provide an exhaustive description of all braids with a given number of strands and in a given representation. The final task would be to find general formulas that depend explicitly on all the parameters: the number of strands $m$, the set $(a_1, \ldots, a_m)$, specifying the $m$-strand braid (i.e. on the element of the braid group), and on the Young diagram $R$ labeling the representation. The formula is going to be for a coefficient in front of a particular character, like the Schur function $S_Q(p)$ (or, alternatively, Hall-Littlewood [17] or some other element of an appropriate expansion basis). Now we perform a step of this program: find such parametric formulas for given $m = 3$ and $R = [2]$. For $R = [1]$ and $m = 3, 4$ (and partly $m = 5$) they were already derived in [2].

The formulas of this paper for the mixing matrices, which we obtain here indirectly, can be obtained directly using the representation theory, like it was done in [2]. We consider the mixing matrices for a more generic case in [18].

## 2 The case of 2 strands, $m = 2$

To determine emerging $Q$ in this case, one suffices to expand the product of two symmetric representations:

$$[2] \times [2] = [4] + [31] + [22], \quad [11] \times [11] = [22] + [211] + [1111]$$

(9)

This decomposition can be easily obtained from the decomposition of the characters. Indeed, given $S_{[2]} = \frac{1}{4}p_2 + \frac{1}{3}p_3^2$, $S_{[11]} = -\frac{1}{4}p_2 + \frac{1}{3}p_3^2$ and

$$S_{[4]} = \frac{1}{4}p_2 + \frac{1}{3}p_3p_1 + \frac{1}{8}p_2^2 + \frac{1}{4}p_2p_1^2 + \frac{1}{24}p_1^4,$$

$$S_{[31]} = -\frac{1}{4}p_2 - \frac{1}{8}p_2^2 + \frac{1}{4}p_2p_1^2 + \frac{1}{8}p_1^4,$$

$$S_{[22]} = -\frac{1}{3}p_3p_1 + \frac{1}{4}p_2^2 + \frac{1}{12}p_1^4,$$

$$S_{[211]} = \frac{1}{4}p_2 - \frac{1}{8}p_2^2 - \frac{1}{4}p_2p_1^2 + \frac{1}{8}p_1^4,$$

$$S_{[1111]} = -\frac{1}{4}p_2 + \frac{1}{3}p_3p_1 + \frac{1}{8}p_2^2 - \frac{1}{4}p_2p_1^2 + \frac{1}{24}p_1^4,$$

it is easy to check that

$$S_{[2]}^2 = S_{[4]} + S_{[31]} + S_{[22]}, \quad S_{[11]}^2 = S_{[22]} + S_{[211]} + S_{[1111]}$$

(11)

In particular, for the ordinary dimensions $d_Q = S_Q(p_k = N)$ of representations $Q$ of the $SU(N)$ algebra it reads as

$$\left(\frac{N(N+1)}{2}\right)^2 = \frac{N(N+1)(N+2)(N+3)}{24} + \frac{(N-1)N(N+1)(N+2)}{8} + \frac{(N^2-1)N^2}{12}$$

e.t. and very similarly for the quantum dimensions $D_Q = S_Q(p_k = [N]_q)$:

$$\left(\frac{[N][N+1]}{[2]}\right)^2 = \frac{[N][N+1][N+2][N+3]}{[2][3][4]} + \frac{[N-1][N][N+1][N+2]}{[2][4]} + \frac{[N-1][N]^2[N+1]}{[2][3]}$$

$$\left(\frac{[N-1][N]}{[2]}\right)^2 = \frac{[N-1][N]^2[N+1]}{[2]^2[3]} + \frac{[N-2][N-1][N][N+1]}{[2][4]} + \frac{[N-3][N-2][N-1][N]}{[2][3][4]}$$

(12)
Accordingly, the signs $\mp$ where $H$ respectively. It should not be mixed with the "physical" initial conditions for the imposed at $0 < n < m$.

For links one has instead of \(^{13}\):

$$
\begin{align*}
(\hat{A}d_{1}S_{[1]}\{p\})^{2} &= p_{1}^{2} = S_{[2]} + S_{[11]}, \\
(\hat{A}d_{1}S_{[2]}\{p\})^{2} &= \frac{(p_{2} + p_{1})^{2}}{4} = S_{[4]} + S_{[31]} + S_{[22]}, \\
(\hat{A}d_{1}S_{[11]}\{p\})^{2} &= \frac{(-p_{2} + p_{1})^{2}}{4} = S_{[1111]} + S_{[211]} + S_{[222]}
\end{align*}
$$

Accordingly, the signs $\mp$ at the r.h.s. of \(^{13}\) are minuses and pluses for knots and links ($n$ odd or even) respectively.

In particular, for the unknot with $(m, n) = (2, 1)$

$$
\begin{align*}
H_{[1]}^{2,1} &= qS_{[2]}^{*} - q^{-1}S_{[11]}^{*} = qS_{[1]}^{*} \\
H_{[2]}^{2,1} &= q^{6}S_{[4]}^{*} - q^{-2}S_{[31]}^{*} + S_{[22]}^{*} = A^{2}q^{4}S_{[2]}^{*} \\
H_{[11]}^{2,1} &= q^{-6}S_{[1111]}^{*} - q^{-2}S_{[211]}^{*} + S_{[22]}^{*} = A^{2}q^{4}S_{[11]}^{*}
\end{align*}
$$

for the Hopf link with $(m, n) = (2, 2)$

$$
\begin{align*}
H_{[1]}^{2,2} &= q^{2}S_{[2]}^{*} + q^{-2}S_{[11]}^{*} = ((q^{2} - 1 + q^{-2})A - A^{-1}) S_{[1]}^{*} \\
H_{[2]}^{2,2} &= q^{12}S_{[4]}^{*} + q^{4}S_{[31]}^{*} + S_{[22]}^{*} = ((q^{2} - q^{10} - q^{-8} + 2q^{6} - q^{2} + 1)A^{2} - q^{8} + q^{4} - q^{2} - 1 + q^{2}A^{-2}) qS_{[2]}^{*} \\
H_{[11]}^{2,2} &= q^{-12}S_{[1111]}^{*} + q^{-4}S_{[211]}^{*} + S_{[22]}^{*} = \\
&= ((1 - q^{-2} + 2q^{-6} - q^{-8} - q^{-10} + q^{-12})A^{2} - 1 - q^{-2} + q^{-4} - q^{-8} + q^{-2}A^{-2}) \frac{S_{[1]}^{*}}{q(q)q^{2}}
\end{align*}
$$

\(^{2}\)Similarly to \(^{12}\), we use here $\varkappa_{Q} = -\nu_{Q} + \nu_{Q'}$ with the opposite sign as compared with \(^{20}\).
and for the trefoil with \((m, n) = (2, 3)\)

\[
H_{[1]}^{2,3} = q^3 S_{[2]} - q^{-3} S_{[1]} = ((q^2 + q^{-2})A - A^{-1}) S_{[1]}
\]

\[
H_{[2]}^{2,3} = q^{18} S_{[4]} - q^6 S_{[1]} + S_{[22]} = ((q^{12} + q^6 + q^4 + 1)A^2 - q^8 - q^6 - q^2 - 1 + q^2 A^{-2}) q^4 S_{[2]}
\]

\[
H_{[11]}^{2,3} = q^{-18} S_{[111]} - q^{-6} S_{[211]} + S_{[22]} = ((q^{-12} + q^{-6} + q^{-4} + 1)A^2 - q^{-8} - q^{-6} - q^{-2} - 1 + q^{-2} A^{-2}) \frac{S_{[1]}^4}{q^4}
\]

(19)

3 3 strands, \(m = 3\)

3.1 Structure of the answer

Now

\[
[2] \times [2] \times [2] = ([4] + [31] + [22]) \times [2] =
\]

\[
= ([6] + [51] + [42]) + ([51] + [42] + [411] + [33] + [321]) + ([42] + [321] + [222])
\]

(20)

For example, for the dimensions of \(SU(2)\) representations one has \(3^3 = 27 = (7 + 5 + 3) + (5 + 3 + 0 + 1 + 0) + (3 + 0 + 0)\). Again, this decomposition is obtained as the decomposition of the characters:

\[
S_{[6]} = \frac{1}{6} p_6 + \frac{1}{5} p_5 p_1 + \frac{1}{8} p_4 p_2 + \frac{1}{8} p_4 p_1^2 + \frac{1}{18} p_3^2 + \frac{1}{6} p_3 p_2 p_1 + \frac{1}{18} p_3 p_1^3 + \frac{1}{48} p_2^3 + \frac{1}{16} p_2^2 p_1 + \frac{1}{48} p_2 p_1^2 + \frac{1}{72} p_1^4,
\]

\[
S_{[51]} = -\frac{1}{6} p_6 - \frac{1}{8} p_4 p_2 + \frac{1}{8} p_4 p_1^2 - \frac{1}{18} p_3^2 + \frac{1}{9} p_3 p_1^3 - \frac{1}{48} p_2^3 + \frac{1}{16} p_2^2 p_1 + \frac{1}{16} p_2 p_1^2 + \frac{1}{144} p_1^4,
\]

\[
S_{[42]} = -\frac{1}{5} p_5 p_1 + \frac{1}{8} p_4 p_2 - \frac{1}{8} p_4 p_1^2 + \frac{1}{16} p_3^2 + \frac{1}{16} p_2 p_1^2 + \frac{1}{16} p_2^2 p_1 + \frac{1}{80} p_1^4,
\]

\[
S_{[411]} = \frac{1}{6} p_6 + \frac{1}{18} p_3^2 + \frac{1}{16} p_3 p_2 p_1 + \frac{1}{18} p_3 p_1^3 - \frac{1}{24} p_2^2 - \frac{1}{8} p_2^2 p_1^2 + \frac{1}{24} p_2 p_1^4 + \frac{1}{72} p_1^6,
\]

\[
S_{[33]} = -\frac{1}{8} p_4 p_2 - \frac{1}{8} p_4 p_1^2 + \frac{1}{9} p_3^2 + \frac{1}{6} p_3 p_2 p_1 - \frac{1}{18} p_3 p_1^3 - \frac{1}{16} p_2^2 + \frac{1}{16} p_2^2 p_1 + \frac{1}{48} p_2 p_1^4 + \frac{1}{144} p_1^6,
\]

\[
S_{[321]} = \frac{1}{5} p_5 p_1 - \frac{1}{9} p_3^2 - \frac{1}{9} p_3 p_1^3 + \frac{1}{45} p_1^4,
\]

\[
\ldots
\]

\[
S_{[222]} = -\frac{1}{8} p_4 p_2 + \frac{1}{8} p_4 p_1^2 + \frac{1}{9} p_3^2 - \frac{1}{6} p_3 p_2 p_1 - \frac{1}{18} p_3 p_1^3 + \frac{1}{16} p_2^2 + \frac{1}{16} p_2^2 p_1 - \frac{1}{48} p_2 p_1^4 + \frac{1}{144} p_1^6,
\]

\[
\ldots
\]

(21)

Thus, one needs the 2 \times 2 mixing matrices for representations [51] and [321] and the 3 \times 3 mixing matrix for representation [42].

The answer for the HOMFLY polynomial in the fundamental representation for the generic 3-strand knot \((a_1, b_1|a_2, b_2|\ldots)\) has the following form:

\[
H_{[1]}^{a_1|b_1|a_2|b_2|\ldots} = q^{a_1+b_1+a_2+b_2+\ldots} S_{[6]} + \left(-\frac{1}{q}\right)^{a_1+b_1+a_2+b_2+\ldots} S_{[111]} +
\]

\[
+ \text{tr}_{2 \times 2} \left\{ \mathcal{R}^{a_1}_{[21]} \hat{U}_{[21]} \mathcal{R}^b_{[21]} \hat{U}^\dagger_{[21]} \mathcal{R}^{a_2}_{[21]} \hat{U}_{[21]} \mathcal{R}^b_{[21]} \hat{U}^\dagger_{[21]} \ldots \right\} S_{[21]}
\]

(22)

with

\[
\mathcal{R}_{[21]} = \begin{pmatrix} q^{\sigma[2]} & 0 \\ 0 & -q^{\sigma[11]} \end{pmatrix}, \quad \hat{U}_{[21]} = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}
\]

(23)
Likewise, in the symmetric representation, it is going to be
\[
H_{[2]}^{3,n} = \sum_{Q'\in6} q^{n(3)} C_{[2]}^Q S_Q
\]  
where the coefficients are defined from the Adams rule
\[
\widetilde{A}_{3\times2}[2] = \frac{p_6 + p_3^2}{2} = \sum_{Q'\in6} C_{[2]}^Q S_Q = S_6 - S_{[5,1]} + 0 \cdot S_{[42]} + S_{[411]} + S_{[33]} - S_{[321]} + S_{[222]},
\]
\[
\left(\widetilde{A}_{1\times1}[2]\right)^3 = \frac{(p_2 + p_2^2)^3}{8} = \sum_{Q'\in6} C_{[2]}^Q S_Q = S_6 + 2S_{[5,1]} + 3 \cdot S_{[42]} + S_{[411]} + S_{[33]} + 2S_{[321]} + S_{[222]}
\]  
for knots and links, i.e. for \(n = 1, 2 \) (mod 3) and \(n = 0 \) (mod 3) respectively.

Thus for the knots, \(n = 1, 2 \) (mod 3)
\[
H_{[2]}^{3,n} = q^{10n} S_6 - q^{6n} S_{[5,1]} + 0 \cdot q^{10n/3} S_{[42]} + q^{2n} S_{[411]} + q^{2n} S_{[33]} - S_{[321]} + q^{-2n} S_{[222]} = q^{-2n} \left(q^{12n} S_6 - q^{8n} S_{[5,1]} + 0 \cdot q^{16n/3} S_{[42]} + q^{4n} S_{[411]} + q^{4n} S_{[33]} - q^{2n} S_{[321]} + S_{[222]}\right)
\]  
Note that the only would be contribution with non-integer value of \( \frac{1}{2} S_{[2]} \) (underlined) does not contribute in the case of torus knots: the Adams coefficient \( C_{[2]}^{[42]} = 0 \).

Looking at the coefficients in front of the fully known "singlet" terms \( S_6, S_{[411]}, S_{[33]}, S_{[222]} \), which do not involve yet unknown mixing matrices, we see that eq. (29) differs from the correct expression by a factor of
\[
H_{[2]}^{3,n} = q^{-2n} H_{[2]}^{3,n}
\]
For the generic single-line (symmetric) representations \([p]\) and arbitrary number \(m\) of strands one gets, comparing the coefficients in front of \(S_{[pn]}\):

\[
H_{[p]}^{m,n} = q^{2n} x_{[p]} - n(m-1) x_{[2p]}, \quad H_{[p]}^{m,n} = q^{-n(p-2)p(m-1)} H_{[p]}^{m,n}
\]

so that there is no discrepancy for either the first fundamental representation \(p = 1\) or for the case of \(m = 2\) strands, when all the knots are torus.

For the links, \(n \equiv 0 \pmod{3}\)

\[
H_{[2]}^{3,n} = q^{10n} S_{[6]} + 2q^{8n} S_{[5,1]} + 3 \cdot q^{16n/3} S_{[42]} + q^{2n} S_{[411]} + q^{2n} S_{[33]} + 2S_{[321]} + q^{-2n} S_{[222]} =
\]

\[
= q^{-2n}\left(q^{12n} S_{[6]} + 2q^{8n} S_{[5,1]} + 3 \cdot q^{16n/3} S_{[42]} + q^{4n} S_{[411]} + q^{4n} S_{[33]} + 2q^{2n} S_{[321]} + S_{[222]} \right)
\]

This time the underlined terms are non-vanishing, but since for the links \(n \equiv 3\), the power is integer in this case.

Note that the coefficients are the same for knots and links in front of the terms \(S_{[6]}, S_{[411]}, S_{[33]}\) and \(S_{[222]}\), in full accordance with (24), because for the torus knots and links \(a_i + b_j\) is either 2 or 0, i.e., always even, so that the corresponding signs \(\epsilon_\pm\) can not affect the answers in the torus case (however, they affect the answers for the composite knots, see \([3,6]\) below).

These formulas generalize those for the fundamental representation:

\[
\begin{align*}
H_{[1]}^{3,n} &= q^{2n} S_{[3]} - S_{[21]} + q^{-2n} S_{[111]}, \quad n = 1, 2 \pmod{3} \\
H_{[1]}^{3,n} &= q^{2n} S_{[3]} + 2S_{[21]} + q^{-2n} S_{[111]}, \quad n = 0 \pmod{3}
\end{align*}
\]

considered in \([2]\).

### 3.3 \(2 \times 2\) matrices \(\hat{U}_{[51]}\) and \(\hat{U}_{[321]}\) from the torus knots

When mixing matrix is of the size \(2 \times 2\), it can be parameterized by a single parameter \(s\), sine of the mixing angle, cosine \(c\) being related through \(c^2 + s^2 = 1\). Then we have for an elementary building block

\[
\hat{R}^a \hat{U}^b \hat{R}^c \hat{U}^d = \begin{pmatrix} 0 & c & s \\ -s & c & 0 \end{pmatrix} \begin{pmatrix} q^\chi & 0 \\ 0 & \tilde{q}^\chi \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} q^\chi c^2 + \epsilon q^\chi c^2 + e c^2 & 0 \\ 0 & \tilde{q}^\chi c^2 + e c^2 \end{pmatrix}
\]

In the case of torus knots and links \(a = b = 1\) and this reduces to

\[
\hat{R} \hat{U} \hat{R} \hat{U} = \begin{pmatrix} q^{2\chi} c^2 + (\epsilon \tilde{c}) q^{\chi + \tilde{\chi}} s^2 & -q^{2\chi} + (\epsilon \tilde{c}) q^{\chi + \tilde{\chi}} s^2 \\ q^{2\chi} c^2 + (\epsilon \tilde{c}) q^{\chi + \tilde{\chi}} s^2 & q^{2\chi} c^2 + (\epsilon \tilde{c}) q^{\chi + \tilde{\chi}} s^2 \end{pmatrix}
\]

and

\[
\text{Tr}_{2 \times 2} \hat{R} \hat{U} \hat{R} \hat{U} = (q^{2\chi} + q^{2\tilde{\chi}}) c^2 + 2(\epsilon \tilde{c}) q^{\chi + \tilde{\chi}} s^2 = \left(q^{2\chi} + q^{2\tilde{\chi}} - q^{\chi - \tilde{\chi}} \right) s^2
\]

Now it remains to substitute the relevant values of \(\chi, \tilde{\chi}, \epsilon\) and \(\tilde{\epsilon}\), and compare this trace with the relevant coefficient of the HOMFLY polynomial for the torus knot \(T[3,1]\) (it is essentially the unknot but realized by a non-simplest braid; since we do not need to restrict ourselves to the topological locus here, this expression is not the same as \(S_{[p]}\)). After that one can calculate the traces of powers of this matrix and check that with the same value of \(s\) they reproduce the values of the coefficient for all other torus knots and links \(T[3,n]\) with different \(n\). This is, in fact, not a problem, because one should just check that with the right value of \(s\) the matrix \(\hat{R} \hat{U} \hat{R} \hat{U}\) has appropriate eigenvalues, proportional to the roots of unity. Finally, the same value of \(s\) determines the coefficient for all other 3-strand braids \((a_1, b_1, a_2, b_2, \ldots)\).

**The case of \(R = [1]\) and the term \(S_{[21]}\).** We start with this already known case, \([2]\) for illustrative purposes. One has to substitute \(\chi = \tilde{\chi} = 1, e = 1, \tilde{\epsilon} = -1\) and compare \([3,6]\) with the value of the coefficient in front of \(S_{21}\) in \([23]\), with \(n = 1\), which is \(-1\). This gives:

\[
q^2 + q^{-2} - (q + q^{-1})^2 s^2 = -1 \implies s = \sqrt{q^2 + 1 - q^{-2}} = \sqrt{3} \quad s_2, \quad c = \frac{1}{q + q^{-1}} = \frac{1}{2} c_2
\]
This reproduces the answer \( \{31\} \) for \( U_2 \) from \( \{2\} \).

It is easy to check that, with this values of \( s \) and \( c \),

\[
\det_{2 \times 2} \left( \hat{R} \hat{U} \hat{R}^\dagger - \lambda I \right) = \frac{\lambda^3 - 1}{\lambda - 1} = \lambda^2 + \lambda + 1
\]

(37)
i.e. the two eigenvalues of \( \hat{R} \hat{U} \hat{R}^\dagger \) are \( e^{\pm \frac{2\pi i}{3}} \), so that

\[
\text{Tr}_{2 \times 2} \left( \hat{R} \hat{U} \hat{R}^\dagger \right)^n = \begin{cases} -1 & \text{for } n = 1, 2 \pmod{3} \\ +2 & \text{for } n = 0 \pmod{3} \end{cases}
\]

(38)
in full agreement with \( \{32\} \).

**The case of** \( R = \{2\} \) **and the term** \( S_{\{51\}} \). One has to substitute \( \kappa = \kappa_{\{4\}} = 6, \tilde{\kappa}_{\{31\}} = 2, \epsilon = 1, \tilde{\epsilon} = -1 \) and compare \( \{35\} \) with the value of the coefficient in front of \( S_{\{51\}} \) in \( \{28\} \) with \( n = 1 \), which is \( -q^8 \). This gives:

\[
q^{12} + q^4 - (q^6 + q^2)^2 s^2 = -q^8 \implies s = \sqrt{\frac{q^{2} - q^{-4}}{q^2} + \frac{q^{-2}}{q^2}} = \sqrt[3]{3.q^2}, \quad c = \frac{1}{q^2 + q^{-2}} = \frac{1}{[2]q^2}
\]

(39)
where \( [x]_q^2 \equiv \frac{q^{2x} - q^{-2x}}{q^2 - q^{-2}} \).

Again, it is a simple exercise to check that with these values of \( s \) and \( c \)

\[
\det_{2 \times 2} \left( \hat{R} \hat{U} \hat{R}^\dagger - \lambda I \right) = \lambda^2 + q^8 \lambda + q^{16}
\]

(40)
i.e. the two eigenvalues of \( \hat{R} \hat{U} \hat{R}^\dagger \) are \( q^8 e^{\pm \frac{2\pi i}{3}} \) and

\[
\text{Tr}_{2 \times 2} \left( \hat{R} \hat{U} \hat{R}^\dagger \right)^n = \begin{cases} -q^8 & \text{for } n = 1, 2 \pmod{3} \\ +2q^8 & \text{for } n = 0 \pmod{3} \end{cases}
\]

(41)
in full agreement with \( \{28\} \) and \( \{31\} \).

**The case of** \( R = \{2\} \) **and the term** \( S_{\{321\}} \). One has to substitute \( \kappa = \kappa_{\{31\}} = 2, \tilde{\kappa}_{\{22\}} = 0, \epsilon = -1, \tilde{\epsilon} = 1 \) and compare \( \{35\} \) with the value of the coefficient in front of \( S_{\{321\}} \) in \( \{28\} \) with \( n = 1 \), which is \( -q^2 \). This gives:

\[
q^4 + 1 - (q^2 + 1)^2 s^2 = -q^2 \implies s = \sqrt{\frac{q^2 + 1 + q^{-2}}{q^2}} = \sqrt[3]{3}, \quad c = \frac{1}{q + q^{-1}} = \frac{1}{[2]}
\]

(42)
With these values of \( s \) and \( c \)

\[
\det_{2 \times 2} \left( \hat{R} \hat{U} \hat{R}^\dagger - \lambda I \right) = \lambda^2 + q^2 \lambda + q^4
\]

(43)
i.e. the two eigenvalues of \( \hat{R} \hat{U} \hat{R}^\dagger \) are \( q^2 e^{\pm \frac{2\pi i}{3}} \) and

\[
\text{Tr}_{2 \times 2} \left( \hat{R} \hat{U} \hat{R}^\dagger \right)^n = \begin{cases} -q^2 & \text{for } n = 1, 2 \pmod{3} \\ +2q^2 & \text{for } n = 0 \pmod{3} \end{cases}
\]

(44)
again in excellent agreement with \( \{28\} \) and \( \{31\} \).

**3.4 Constraining the** \( 3 \times 3 \) **matrix** \( \hat{U}_{\{42\}} \) **from the torus knots**

When orthogonal mixing matrix is of the size \( 3 \times 3 \), it can be parameterized by three independent Euler angles, namely by their sines and cosines:

\[
\hat{U} = \begin{pmatrix} c_1 & 0 & s_1 \\ 0 & 1 & 0 \\ -s_1 & 0 & c_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix}
\]

(45)
One now needs to perform the same trick: to compare the traces of powers of $\hat{R}U\hat{R}U^\dagger$, where $\hat{R}$ is given in (25) with the known coefficients in front of $S_{[42]}$ in (28). This comparison tells that

\[
\text{tr}_{3\times 3}(\hat{R}U\hat{R}U^\dagger)^n = \begin{cases} 
0 & \text{for } n = 1, 2 \text{ (mod 3)} \\
3q^{16n/3} & \text{for } n = 0 \text{ (mod 3)}
\end{cases}
\] (46)

for diagonal

\[
\hat{R} = \begin{pmatrix}
q^6 & 0 & 0 \\
0 & -q^2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (47)

The choice of the Euler decomposition in (45) is obviously adjusted to this form of the matrix $\hat{R}$. At $q = 1$ a solution is obvious:

\[
\hat{R} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} \implies \hat{U} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c & s \\
0 & -s & c
\end{pmatrix}, \quad \hat{R}\hat{U}\hat{R}U^\dagger = \begin{pmatrix}
1 & 0 & 0 \\
0 & c^2 - s^2 & -2cs \\
0 & 2cs & c^2 - s^2
\end{pmatrix}
\] (48)

i.e. one gets the rotation matrix with the doubled angle $\phi$, $s = \sin \phi$. Then (49) means that $6\phi = 2\pi k$ with any integer $k$, i.e. $\phi = \frac{\pi k}{3}$, and $c = \pm \frac{1}{2}$, $s = \pm \frac{\sqrt{3}}{2}$. Of course, at $q = 1$ there is a huge degeneracy: any rotation involving only the first and the third lines leaves $\hat{R}(q = 1)$ intact, and one can take many other $\hat{U}$, obtained by such a rotation, for example, $\hat{U} = \begin{pmatrix}
c & s & 0 \\
-\frac{c}{2} & \frac{s}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}$ with the same $c$ and $s$. For $q = 1$ only one of the three Euler angles in $\hat{U}$ is fixed by conditions (46).

At $q \neq 1$ conditions (46) imply that the three eigenvalues of $\hat{R}\hat{U}\hat{R}U^\dagger$ are three cubic roots of unity times $q^{16/3}$, i.e. that

\[
\det_{3\times 3}(\hat{R}\hat{U}\hat{R}U^\dagger - \lambda I) = q^{16} - \lambda^3
\] (49)

Clearly, these are only two conditions, so that only two of the three Euler angles will be fixed by (46). One extra condition, not just coming from the 3-strand torus knot and link polynomials, will be needed to fix $\hat{U}_{[42]}$ unambiguously.

We impose this condition by making an "educated guess" that $c_3 = c_1$ and $s_3 = -s_1$. Then

\[
c_1 = c_3 = \frac{1 + q^4}{\sqrt{(2q^4 - q^2 + 2)(1 + q^2 + q^4)}}, \quad c_2 = \frac{-q^4 - q^2 + 1}{1 + q^4} = -\frac{1 + q^6}{(1 + q^2)(1 + q^4)}
\] (50)

(the sign in $c_2$ is essential).

One can use an alternative parametrization instead of (46):

\[
\hat{U} = \begin{pmatrix}
c'_1 & 0 & s'_1 \\
0 & 1 & 0 \\
-s'_1 & 0 & c'_1
\end{pmatrix} \begin{pmatrix}
c'_2 & s'_2 & 0 \\
-s'_2 & c'_2 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
c'_3 & 0 & s'_3 \\
0 & 1 & 0 \\
-s'_3 & 0 & c'_3
\end{pmatrix}
\] (51)

In this case the Euler angles are given by

\[
c'_1 = c'_3 = \sqrt{\frac{1 - q^{10}}{(2q^4 - q^2 + 2)(1 - q^6)}} = -s_1 = s_3, \quad c'_2 = \frac{-q^4 - q^2 + 1}{1 + q^4} = \frac{1 + q^6}{(1 + q^2)(1 + q^4)} = c_2
\] (52)

i.e. are "dual" to those for (45). In both cases one obtains the same matrix $\hat{U}$, see eq. (54) below.

It remains an open question, whether a nicer decomposition exists for this rather sophisticated mixing matrix.
3.5 The final answer

Substituting into (2.1) the values of the mixing angles, found in the previous subsections one finally obtains for arbitrary 3-strand braid:

$$H_{[2]}^{a_1, b_1; a_2, b_2} = q^{6(a_1+b_1+a_2+b_2+\ldots)} S_{[6]} + (-q^2)^{a_1+b_1+a_2+b_2+\ldots} (S_{[411]} + S_{[33]} + S_{[222]} +$$

$$+ \text{tr}_{2 \times 2} \left\{ \begin{pmatrix} q^6 & 0 \\ 0 & -q^2 \end{pmatrix}^{a_1} \begin{pmatrix} -\frac{1}{|2|q^2} & \sqrt{|3|q^2} \\ -\sqrt{|3|q^2} & -\frac{1}{|2|q^2} \end{pmatrix} \begin{pmatrix} q^6 & 0 \\ 0 & -q^2 \end{pmatrix} \begin{pmatrix} -\frac{1}{|2|q^2} & -\sqrt{|3|q^2} \\ \sqrt{|3|q^2} & -\frac{1}{|2|q^2} \end{pmatrix} \ldots \right\} S_{[51]} +$$

$$+ \text{tr}_{2 \times 2} \left\{ \begin{pmatrix} -q^2 & 0 \\ 0 & 1 \end{pmatrix}^{a_1} \begin{pmatrix} -\frac{1}{|2|q} & \sqrt{|3|q} \\ \sqrt{|3|q} & -\frac{1}{|2|q} \end{pmatrix} \begin{pmatrix} -q^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{|2|q} & \sqrt{|3|q} \\ \sqrt{|3|q} & -\frac{1}{|2|q} \end{pmatrix} \ldots \right\} S_{[321]} +$$

$$+ \text{tr}_{3 \times 3} \left\{ \begin{pmatrix} q^6 \\ -q^2 \\ 1 \end{pmatrix}^{a_1} U_{[42]} \begin{pmatrix} q^6 \\ -q^2 \\ 1 \end{pmatrix} U_{[42]}^T \ldots \right\} S_{[42]}$$

(53)

The matrix $U_{[42]}$ is equal to:

$$\begin{pmatrix}
q^4 & \frac{q^4}{(q^4+1)(q^4+q^2+1)} & \frac{q^4}{q^4+q^2+1} \\
\frac{q^4}{q^4+1} & \frac{-q^4+q^2+1}{q^4+1} & \frac{q}{q^4+q^2+1} \\
\frac{-q^4+q^2+1}{q^4+1} & \frac{q}{q^4+q^2+1} & \frac{q^2}{q^4+q^2+1}
\end{pmatrix} =$$

$$\begin{pmatrix}
\frac{[2]}{|3|} & \frac{[2]}{|4|} & \frac{[2]}{|3|} \\
\frac{[2]}{|4|} & \frac{[6]}{|3|} & \frac{[1]}{|3|} \\
\frac{-\sqrt{5}}{|3|} & \frac{-1}{\sqrt{|3|}} & \frac{1}{|3|}
\end{pmatrix}$$

(54)

This is the same matrix as the matrix of the Racah coefficients, (A.20) in [22].

3.6 Composite knots and links: a check

In this section we perform further checks, making use of topological equivalence between different braids, i.e.

homotopic equivalence of the corresponding knots and links. Accordingly, in this section we can consider only

the ordinary HOMFLY polynomials $H$ reduced to the topological locus

$$p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}}$$

(55)

The case of $b_1 = a_2 = b_2 = \ldots = 0$: In this simplest example, with only one non-vanishing parameter $a_1$, the 3-strand knot/link splits into untied a 2-strand knot/link and the unknot. Accordingly,

$$H_R^{3,(a,0,0,\ldots)} = H_R^{2,a} \cdot H_R^0$$

(56)
At the same time, in this case the HOMFLY polynomials are also drastically simplified: all mixing matrices drop away from the formula and it reduces to just

\[ H_{[2]}^{3,0,0,0,...} = q^{6a}S_{[6]} + (-q^2)^a\left(S_{[411]} + S_{[33]}\right) + S_{[222]} + \]
\[ + \left(q^{6a} + (-q^2)^a\right)S_{[51]} + \left((-q^2)^a + 1\right)S_{[321]} + \left(q^{6a} + (-q^2)^a + 1\right)S_{[42]} \]

(57)

Note that, in variance with expressions for the 3-strand torus knots and links, this formula is sensitive to the sign of the $R$-matrix eigenvalue $-q^2$. It remains to reduce (57) to the topological locus (55), where the Schur functions turn into the quantum dimensions, and check that this coincides with the r.h.s. of (56) with $R = [2]$, where the unknot polynomial is just $H^0_R = S^*_R$ and $H^2,a$ is given by the second line of (13). Of course, such a relation can not be lifted to the entire $p$-space: (57) does not coincide with $H^{2,a}_{[2]}S_{[2]}$ beyond the topological locus (55): one suffices to note that the former depends on $p_0$, while the latter one does not.

3.7 Results

3.7.1 The figure eight knot $4_1$

This knot can be realized with the braid

\[ 4_1 : \quad (a_1, b_1 | a_2, b_2) = (1, -1 | 1, -1), \] (58)

similar to a possible 3-strand realization of the trefoil, which is a torus knot $T[2,3] = T[3,2]$

\[ 3_1 : \quad (1, 1 | 1, 1) \] (59)

In the fundamental representation one had

\[ H^4_{[1]} = S^*_3 + \left(q^2 - 2q^2 + 1 - 2q^{-2} + q^{-4}\right)S^*_{21} + S^*_{111} = \left(A^2 - (q^2 - 1 + q^{-2}) + A^{-2}\right)S^*_{[1]} \] (60)

while

\[ H^3_{[1]} = q^4S^*_3 - S^*_{21} + q^{-4}S^*_{111} = \left((q^2 + q^{-2})A^2 - 1\right)S^*_{[1]} \] (61)

The second expression is highly asymmetric, while the formula for $4_1$ is very symmetric even when expressed in terms of the $A$ variable: this is a specifics of $4_1$.

In the symmetric representation $R = [2]$ the answer is

\[ H^{31}_{[2]} = \left(q^4A^4 - (1 + q^2)(1 - q^2 + q^6)q^{-2}A^2 + (q^6 - q^4 + 3 - q^{-4} + q^{-6})-\right.\]
\[ \left.-(1 + q^{-2})(1 - q^{-2} + q^{-6})q^2A^{-2} + q^{-4}A^{-4}\right)S^*_{[2]} \] (62)

This can be compared with the asymmetric formula for the trefoil $(1, 1, 1, 1)$

\[ H^{31}_{[2]} = q^8 \left(A^4(1 + q^4 + q^6 + q^{12}) - A^2(1 + q^2)(1 + q^6) + q^2\right)S^*_{[2]} \] (63)

Expression (62) certainly coincide with results presented in existing literature, see e.g. [8]. Moreover, it turns out that in the case of $4_1$ one can get the result for any symmetric $[p]$ and antisymmetric $[1p]$ representation [4].

The HOMFLY polynomials in the symmetric representation for other 3-strand knots with no more than 8 crossings are collected in the Appendix.

3.8 Antisymmetric representation

In order to construct the HOMFLY polynomials in the antisymmetric representation $[1, 1]$, one could repeat the standard machinery of the mixing matrices etc we described above. However, the result can be obtained much simpler using a symmetry of the HOMFLY polynomials.

Indeed, the character expansions of the HOMFLY polynomials possess a $Z_2$-symmetry

\[ A, q, S^*_R \leftrightarrow A, -\frac{1}{q}, S^*_R \] (64)

where $R'$ is a transposition of the Young diagram $R$. This symmetry can be easily understood, since $S^*_R(p_k) = S^*_R((-)^{k-1}p_k)$ and $\kappa_R = -\kappa_{R'}$. At the same time, all the $(SU_q(N))$ group representation quantities (in particular, the mixing matrices) are also possess this antipodal symmetry. Hence, one can calculate the HOMFLY polynomials in the antisymmetric representation just making a substitution $q \rightarrow -1/q$ in the HOMFLY polynomials for the symmetric representation obtained in the previous sections.
3.9 Ooguri-Vafa conjecture

In the paper [23], the authors conjectured a connection of the Chern-Simons theory with topological string on the resolution of the conifold. In fact, they proposed that the generating function $Z$ of average of the Wilson loop in different representations is associated with the topological string partition function $Z_{str}$. In accordance with the Ooguri-Vafa result [24] $Z$ is given by the sum

$$Z(q, A, K) = \sum_R \chi_R(p) H^R_N(q, A)$$

where the sum runs over all the irreducible representation of $SU(N)$ ($A = q^N$). Now the topological nature of this object implies that the "connected" correlators $f_R(q, A)$ defined by the expansion

$$\log Z = \sum_{n=0,R} \frac{1}{n} f_R(q^n, A^n) \chi_R(p^{(n)})$$

where the set of variables $p^{(n)}_k = p_{nk}$, has the generic structure

$$f_R(q, A) = \sum_{n,k} N_{R,n,k} \frac{A^n q^k}{q - q^{-1}}$$

$N_{R,n,k}$ are integer and the parity of $n$ in the sum coincides with the parity of $|R|$ while the parity of $k$ is inverse. These numbers are related to the Gopakumar-Vafa integers $n_{\Delta, n, k}$ [24] by the relation

$$n_{\Delta, n, k} = \sum_R \Phi_R(\Delta) N_{R,n,k}$$

where $\Phi_R(\Delta)$ is the character of the symmetric group $S_{\Delta}$. The integers $N_{R,n,k}$ are more refined, since their integrality implies that $n_{\Delta, n, k}$ are integer but not vice versa. In fact, one can consider even more refined integers [25]

$$f_R(q, A) = \sum_{n,k,R_1,R_2} C_{RR_1R_2} \Sigma_{R_1}(q) N_{R_2,n,k} A^n \left( q^{-1} - q \right)^{2k-1}$$

where

$$C_{RR_1R_2} = \sum_{\Delta} \frac{\Phi_R(\Delta) \Phi_{R_1}(\Delta) \Phi_{R_2}(\Delta)}{z_{\Delta}}$$

the Clebsh-Gordon coefficients of the symmetric group, $z_{\Delta}$ is the standard symmetric factor of the Young diagram [5] and $\Sigma_R(q)$ is a monomial non-zero only for the corner Young diagrams $R = [l - d, 1^d]$ and is equal to

$$\Sigma_R(q) = (-1)^d q^{2d-l+1}$$

First few terms for $f_R$ and $N_{R,n,k}$ are

$$f_{[1]}(q, A) = H_{[1]}(q, A)$$

$$f_{[2]}(q, A) = H_{[2]}(q, A) - \frac{1}{2} \left( H_{[1]}(q, A)^2 + H_{[2]}(q^2, A^2) \right)$$

$$f_{[1,1]}(q, A) = H_{[1,1]}(q, A) - \frac{1}{2} \left( H_{[1]}(q, A)^2 - H_{[2]}(q^2, A^2) \right)$$

... and

$$f_{[1]}(q, A) = \sum_{n,k} N_{[1],n,k} \left( q^{-1} - q \right)^{2k-1} A^n$$

$$f_{[2]}(q, A) = \sum_{n,k} \left( q^{-1} N_{[2],n,k} - q N_{[1,1],n,k} \right) \left( q^{-1} - q \right)^{2k-1} A^n$$

$$f_{[1,1]}(q, A) = \sum_{n,k} \left( - q N_{[2],n,k} + q^{-1} N_{[1,1],n,k} \right) \left( q^{-1} - q \right)^{2k-1} A^n$$

...  

---

3 This sum can be obtained as the Chern-Simons average of the Ooguri-Vafa operator $\exp \sum_n \frac{1}{n!} \mathrm{Tr} \left( f_K A dx \right)^n \mathrm{Tr} V^n$, where $A$ is the gauge field, $p_k = \mathrm{Tr} V^k$ are external sources and the traces are taken over the fundamental representation.
We calculate both the Ooguri-Vafa polynomials $f_2(q, A)$ and the numbers $N_{[2], n, k}$ for all 3-strand knots with no more than 8 crossings in the Appendix. The integrality of these numbers and their using in product formulas is discussed in \[26\].

### 3.10 “Special” polynomials

The "special" polynomials are defined \[20\] as the limit of ratio of the HOMFLY polynomials and the quantum dimensions as $q \to 1$:

$$S_R^K(A) = \lim_{q \to 1} \frac{H_R^K(q, A)}{S_R^*(q, A)}$$  \hspace{1cm} (73)

Note that the limit is taken with fixed $A$, and both the HOMFLY polynomial $H_R^K$ and the quantum dimension $S_R^*$ are singular behaving as $(q - q^{-1})^{-|R|}$. Here $|R|$ is the number of boxes in the Young diagram $R$. Note that in this limit

$$\lim_{q \to 1} S_R(A) = d_R S_{[1]}(A)^{|R|}$$ \hspace{1cm} (74)

where

$$d_R = S_R\{p\}_{|p_k = \delta_{k,1}|} = \prod_{(i,j) \in R} \frac{1}{h_{i,j}}$$ \hspace{1cm} (75)

and $h_{i,j}$ is the "hook" length.

The conjectured property of the "special" polynomials reads as \[20\] \[27\]

$$S_R^K(A) = \left( S_{[1]}^K(A) \right)^{|R|}$$ \hspace{1cm} (76)

and is presumably valid for arbitrary $K$ and $R$.

For example,

$$S_{[2]}^3(A) = (2A^2 - 1)^2,$$

$$S_{[1]}^3(A) = 2A^2 - 1,$$

$$S_{[2]}^4(A) = \left( A^2 - 1 + A^{-2} \right)^2,$$

$$S_{[1]}^4(A) = A^2 - 1 + A^{-2}$$ \hspace{1cm} (76)

etc.

This conjecture is an amusing "dual" of a somewhat similar conjecture

$$A_R^K(q) = A_{[1]}^K(q^{|R|})$$ \hspace{1cm} (79)

for the Alexander polynomial

$$A_R^K(q) = \lim_{A \to 1} \frac{H_R^K(q, A)}{S_R^*}$$ \hspace{1cm} (80)

We check these two conjectures for the concrete knots in the Appendix.

Similarly, one can consider the "special" limit of $q \to 1$ for other polynomials, e.g. for the Ooguri-Vafa polynomials $f_R(q, A)$. The special Ooguri-Vafa polynomials $f_R(A) \equiv \lim_{q \to 1} \frac{f_R(q, A)}{S_{[n]}^*(q)}$, however, depend on the representation much less trivially than the "special" and Alexander polynomials (see the Appendix for examples). Note that $f_{[2]}(A) = -f_{[1]}(A)$.

### 3.11 Framing factor

In this text we assume that the $R$-matrix is normalized so that in the channel $Q \in R \otimes R$ its eigenvalue is equal to $\pm q^{a_Q}$ and is independent of $R$. This simplifies our formulas, and this is important for their extension beyond the topological locus \[2\]. However, instead this breaks some properties, important for the knot theory, including topological (ambient isotopy) invariance. Still, this difference is very easy to take into account by adding an overall factor, which is simple, but depends on representation and even on the rank of the group $SU(N)$. This factor is also important in the definition of the Ooguri-Vafa polynomials \[4\] and is ambiguously determined due to

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\[4\] For the fully symmetric knots like the figure eight $4_1$ with the vanishing writhe number the factor is unity and the Ooguri-Vafa polynomials can be easily extended beyond the topological locus \[3\]. For generic knots this extension needs a separate discussion.
the freedom in choosing the framing \[28\]. We choose the standard, or canonical framing. Then, the \(R\)-matrix, which is adequate for knot theory calculations is actually normalized differently:

\[
R_{R \otimes R}^{norm} = A^{-|R|} q^{-4\pi i} R_{R \otimes R}
\]  

(81)

This means that all our answers for the HOMFLY polynomials should be multiplied by the additional factor

\[
H_R^B \rightarrow H_R^K = \left( A^{|R|} q^{4\pi i} \right) ^{-w_R} H_R^B \left( p_k^4 \right)
\]  

(82)

where \(w_R\) is the algebraic number of intersections in the braid \(B\) called the writhe number. We illustrate the significance of this factor by three examples. First of them concerns the topological invariance, the second one for the HOMFLY polynomial, which is adequate for knot theory calculations is actually normalized differently:

The standard way to obtain the colored HOMFLY polynomials is to extract them from those in the fundamental representation, but for different knots and links. Namely, if one needs \(H_R^K\), one considers instead \(H_R^{[\xi_1]}\), where \(\xi_1\) is the cabling of the knot \(K\), obtained by substituting the knot with a set of \(|R|\) parallel ones (a "cable")

\[\text{Example 1:}\]

If the torus knot \(3_1\) is represented by the 2-strand braid \([2,3] = (1)^3 = (1,1,1)\), then one gets for the HOMFLY polynomial

\[
\frac{q^3 \cdot S_{[2]} - q^{-3} \cdot S_{[11]}}{S_{[1]}} = \frac{q^3 (A q - A^{-1} q^{-1}) - q^{-3} (A q^{-1} - A^{-1} q)}{q^2 - q^{-2}} = A(q^2 + q^{-2}) - A^{-1}
\]  

(83)

If the same knot is represented by the 3-strand braid \([3,2] = (1,1)^2 = (1,1|1,1)\), one gets instead

\[
\frac{q^4 \cdot S_{[3]} - \text{tr}_{2 \times 2} \left( \hat{R} \hat{U} \hat{U} \right)^2 \cdot S_{[21]} + q^{-4} \cdot S_{[111]}}{S_{[1]}} = A^2(q^2 + q^{-2}) - 1
\]  

(84)

Clearly, these two expressions do not coincide and differ by a factor of \(A\), which is exactly taken into account by the correction factor \([52]\), because \(w^{[3,2]} = 4\), while \(w^{[2,3]} = 3\), and in this example \(\xi_{[1]} = 0\). For \(p = 2\) the two HOMFLY polynomials, calculated in this paper, differ by a factor of \(A^2 q^4\), which is again nicely eliminated by \([52]\), because \(\xi_{[2]} = 1\). Note that in the Appendix we choose the opposite orientation for the trefoil: \((-1, -1, -1)\) in order to better match formulas from the standard knot tables.

\[\text{Example 2:}\]

In fact, the Alexander polynomials made from our extended HOMFLY polynomials,

\[
\mathcal{A}^{[\xi]}_{[p]}(q) = \lim_{A \rightarrow 1} q^{p(2p-1)n} S_{[2p]} - q^{-p(2p-3)n} S_{[2p-1,1]} S_{[p]}
\]  

(85)

(all other Young diagrams from the decomposition of \([p] \otimes [p]\) do not contribute at \(A = 1\), satisfy

\[
\mathcal{A}^{[\xi]}_{[p]}(q) = q^{2(p-1)n} \mathcal{A}^{[\xi]}_{[1]}(q^2) = q^{4 \xi_{[p]} |w^{[2,n]}|} \mathcal{A}^{[\xi]}_{[1]}(q^2)
\]  

(86)

rather than \([74]\). Unwanted factors in this relation are eliminated after the factor \([52]\) is taken into account.

\[\text{Example 3:}\]

As we know from \([30]\), the torus polynomial character expansion of \([19]\) based on use of the Adams operation, and also suitable for continuation from the topological locus \([2]\) to the entire space of time-variables, differs from ours by a factor of \(q^{-n\pi(m-2)p(p-1)}\). The normalized HOMFLY obtained from ours by the rule \([52]\) should, therefore, differ from that one by a factor of \(q^{-2n(m-1)p(p-1)} \cdot q^{n(m-2)p(p-1)} = q^{-np(p-1)}\) (since \(w^{[m,n]} = (m-1)n\) and \(\xi_{[p]} = \frac{p(p-1)}{2}\)):

\[
H_{[p]}^{m,n} = A^{-(m-1)n} q^{-mp(p-1)} H_{[p]}^{m,n} = A^{-(m-1)n} q^{-2mn \xi_{[p]}} H_{[p]}^{m,n}
\]  

(87)

This is exactly the factor used in \([19]\) for arbitrary representation \(R\). It can deserve noting that \(mn\) is not the writhe number of the braid associated with the torus knot, and coefficient 2 is different from 4 in \([52]\).

3.12 Cabling

The standard way to obtain the colored HOMFLY polynomials is to extract them from those in the fundamental representation, but for different knots and links. Namely, if one needs \(H_R^K\), one considers instead \(H_{[1]}^{K[R]}\), where \(K[R]\) is the cabling of the knot \(K\), obtained by substituting the knot with a set of \(|R|\) parallel ones (a "cable").
i.e. actually a knot $\mathcal{K}$ is substituted by an $|R|$-component link. However, to extract information about an arbitrary $R$ of a given size $|R|$ one should also allow additional intertwinings of the wires inside each cable, which decreases the number of components in the link, so that $\mathcal{K}^{[R]}$ is actually a linear combination of several links, made in this way from the $|R|$-cabled $\mathcal{K}$.

If the knot $\mathcal{K}$ is represented by an $m$-strand braid, the cabling involves $m|R|$ strands. Since general formulas are known [2] for arbitrary $r$-strand knots in the fundamental representations, one can actually demonstrate how the cabling procedure works for arbitrary 2-strand knots in symmetric and antisymmetric representations. For the 3-strand knots in these representations or for 2-strand knots in representations [3], [21], [111] one needs the knowledge of the 6-strand knots in the fundamental representation, which is still not available in full generality. Thus, in the rest of this section we rederive $\mathcal{H}^{[2,n]}_{[2]}$ and $\mathcal{H}^{[2,n]}_{[1]}$ from $\mathcal{H}^{[2,n]}_{[1]}$.

Cabling the unknot

Our first example is actually the 1-strand knot: the unknot. The 2-cabling of a 1-strand braid implies that it is substituted with a 2-strand one: the two unlinked unknots and the answer is

$$H^{[2,0]}_{[1]} \{p_k\} = S^2_{[1]} \{p_k\} = S_{[2]} \{p_k\} + S_{[11]} \{p_k\} = H^{[1,0]}_{[2]} \{p_k\} + H^{[1,0]}_{[11]} \{p_k\}$$

(88)

a linear combination of unknot polynomials in two representations of the size $|R| = 2$. Similarly, the untwisted $p$-cabling gives a linear combination

$$H^{[p,0]}_{[1]} \{p_k\} = S^p_{[1]} \{p_k\} = \sum_{R: |R|=p} \mathcal{H}^{[1,0]}_R \{p_k\}$$

(89)

To extract the individual HOMFLY polynomials for two representations [2] and [11] one needs to consider not only the two non-intersecting strands, but also to allow one intertwining. One would naturally assume that one extra intersection just provides $S_{[2]} - S_{[11]}$, but this is the case only for $q = 1$. If one associates an extra $R$-matrix with this additional intersection, one gets $q$-dependent factors:

$$H^{[2,1]}_{[1]} \{p_k\} \overset{13}{=} qS_{[2]} \{p_k\} - q^{-1} S_{[11]} \{p_k\} = qH^{[1,0]}_{[2]} \{p_k\} - q^{-1} H^{[1,0]}_{[11]} \{p_k\}$$

(90)

and finally the cabling of the unknot implies

$$H^{[1,0]}_{[2]} \{p_k\} = \frac{1}{1 + q^2} H^{[2,0]}_{[1]} \{p_k\} + \frac{q}{1 + q^2} H^{[2,1]}_{[1]} \{p_k\},$$

$$H^{[1,0]}_{[11]} \{p_k\} = \frac{q^2}{1 + q^2} H^{[2,0]}_{[1]} \{p_k\} - \frac{q}{1 + q^2} H^{[2,1]}_{[1]} \{p_k\}$$

(91)

If one restricts the answer to the topological locus and restore the factors, see s.3.11 to make contact with the standard calculations, one would write the same relations as follows:

$$H^{[1,0]}_{[2]}(A|q) = \frac{1}{1 + q^2} H^{[2,0]}_{[1]}(A|q) + \frac{qA}{1 + q^2} \left( A^{-1} H^{[2,1]}_{[1]}(A|q) \right),$$

$$H^{[1,0]}_{[11]}(A|q) = \frac{q^2}{1 + q^2} H^{[2,0]}_{[1]}(A|q) - \frac{qA}{1 + q^2} \left( A^{-1} H^{[2,1]}_{[1]}(A|q) \right)$$

(92)

Formulas (91) and (92) actually define the projectors, specifying the linear combinations of cabled knots with additional twistings [19], which select particular representations [2] and [11]. Since they are actually independent of the knot, the same projectors are used for the same purpose below, when we switch to a little more interesting examples of 2-cabling the 2-strand knots.

2-cabling the 2-strand knots

A new thing as compared to the previous subsection is that one has intersections in original 2-strand braid (there were none in the 1-strand case). Each $\mathcal{R}$-matrix at the 2-strand crossing is substituted by four $\mathcal{R}$-matrices, after lifting to 4 strands:

$$\mathcal{R} \rightarrow \left( \mathcal{R} \otimes I \otimes I \right) \left( \mathcal{R} \otimes \mathcal{R} \right) \left( I \otimes I \otimes \mathcal{R} \right)$$

(93)
so that the 2-strand braid \([2, n]\) is lifted to a 4-strand braid of the type \((0, 1, 1|1, 1, 0)^n\). Moreover, to separate representations \([2]\) and \([11]\) one also needs to allow one twisting between the first two and the last two braids, i.e. to consider the four slightly different links/knots

\[
(0, 1, 1|1, 1, 0)^n, \quad \text{(0, 1, 1|1, 1, 0)}^{(0, 0, 0)}, \quad \text{(0, 1, 1|1, 1, 0)}^{(0, 0, 1)}, \quad \text{(0, 1, 1|1, 1, 0)}^{(0, 1, 1)} \quad \text{(94)}
\]

Making use of projectors \([31]\) and \([32]\), one gets, in somewhat compressed notation:

\[
q^{-2n}H_{[2]}^{[2,n]}\{p_k\} = \frac{1}{(1 + q^2)^2} \left( H_{[1]}^{[4,(000),n]}\{p_k\} + qH_{[1]}^{[4,(001),n]}\{p_k\} + q^2H_{[1]}^{[4,(100),n]}\{p_k\} + q^2H_{[1]}^{[4,(101),n]}\{p_k\} \right);
\]

\[
q^{2n}H_{[1]}^{[2]}\{p_k\} = \frac{1}{(1 + q^2)^2} \left( q^4H_{[1]}^{[4,(000),n]}\{p_k\} - q^3H_{[1]}^{[4,(001),n]}\{p_k\} - q^3H_{[1]}^{[4,(100),n]}\{p_k\} - q^2H_{[1]}^{[4,(101),n]}\{p_k\} \right)
\]

According to \([2]\), substituting the peculiar braid \((a_1b_1c_1|a_2b_2c_2|a_3b_3c_3|\ldots) = \underbrace{(011|110|\ldots|011|110)}_{n \text{ times}}\) into the general formula \([2] \text{ eq.}(65)\) for the 4-strand extended HOMFLY polynomials gives

\[
H_{[1]}^{[4,(000),n]} = H_{[1]}^{[4,(011)]^{(000)}} = q^{4n}S_4 + \text{tr}_{3 \times 3} \left( \hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]} \right)^{n} \cdot S_{[31]} + \left( q \leftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left( \hat{R}_{[22]}\hat{U}_{[22]}\hat{V}_{[22]}\hat{R}_{[22]}\hat{U}_{[22]} \right)^{n} \cdot S_{[22]}
\]

and for the other three twisted cablings:

\[
H_{[1]}^{[4,(100),n]} = q^{4n+1}S_4 + \text{tr}_{3 \times 3} \left( \hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]} \right)^{n} \cdot S_{[31]} + \left( q \leftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left( \hat{R}_{[22]}\hat{U}_{[22]}\hat{V}_{[22]}\hat{R}_{[22]}\hat{U}_{[22]} \right)^{n} \cdot S_{[22]};
\]

\[
H_{[1]}^{[4,(001),n]} = q^{4n+1}S_4 + \text{tr}_{3 \times 3} \left( \hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]} \right)^{n} \cdot S_{[31]} + \left( q \leftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left( \hat{R}_{[22]}\hat{U}_{[22]}\hat{V}_{[22]}\hat{R}_{[22]}\hat{U}_{[22]} \right)^{n} \cdot S_{[22]};
\]

\[
H_{[1]}^{[4,(101),n]} = q^{4n+2}S_4 + \text{tr}_{3 \times 3} \left( \hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]}\hat{R}_{[31]}\hat{V}_{[31]}\hat{U}_{[31]} \right)^{n} \cdot S_{[31]} + \left( q \leftrightarrow -\frac{1}{q} \right) + \text{tr}_{2 \times 2} \left( \hat{R}_{[22]}\hat{U}_{[22]}\hat{V}_{[22]}\hat{R}_{[22]}\hat{U}_{[22]} \right)^{n} \cdot S_{[22]}
\]

Now let us look at the coefficient in front of \(S_4\). The two linear combinations, corresponding to \((95)\) for this coefficient give just

\[
q^{4n} + 2q \cdot q^{4n+1} + q^2 \cdot q^{4n+2} = q^{4n},
\]

\[
q^4 \cdot q^{4n} - 2q^3 \cdot q^{4n+1} + q^2 \cdot q^{4n+2} = 0
\]

Similarly, for two linear combinations in front of \(S_{[22]}\) one has

\[
\frac{(q^{2n} + q^{-2n}) + 2q \cdot (-q^{2n-1} + q^{-2n-1}) + q^2 \cdot (q^{2n-2} + q^{-2n-2})} {(1 + q^2)^2} = q^{-2n},
\]

\[
\frac{q^4 \cdot (q^{2n} + q^{-2n}) - 2q^3 \cdot (-q^{2n-1} + q^{-2n-1}) + q^2 \cdot (q^{2n-2} + q^{-2n-2})} {(1 + q^2)^2} = q^{2n}
\]

and for those in front of \(S_{[31]}\) the intermediate expressions are different for knots and links: for \(n\) odd

\[
\frac{-1 + 2q \cdot (-q) + q^2 \cdot (-q^2)} {(1 + q^2)^2} = -1
\]

\[
\frac{q^4 \cdot (-1) - 2q^3 \cdot (-q) + q^2 \cdot (-q^2)} {(1 + q^2)^2} = 0
\]

while for \(n\) even

\[
\frac{(2q^{2n} + 1) + 2q \cdot (q^{2n+1} - q^{2n-1} + q) + q^2 \cdot (q^2 - 2q^{2n})} {(1 + q^2)^2} = 1
\]

\[
\frac{q^4 \cdot (2q^{2n} + 1) - 2q^3 \cdot (q^{2n+1} - q^{2n-1} + q) + q^2 \cdot (q^2 - 2q^{2n})} {(1 + q^2)^2} = 0
\]
Thus, one finally obtains

\begin{align}
q^{-2n} H_{[2]}^{2,n} &= q^{4n} S_{[4]} \mp S_{[31]} + q^{-2n} S_{[22]} \\
q^{2n} H_{[11]}^{2,n} &= q^{-4n} S_{[111]} \mp S_{[211]} + q^{2n} S_{[22]}
\end{align}

(102)

which coincides with (13).

4 Summary

In this paper we continued our program of constructing simple matrix expressions for the colored HOMFLY polynomials of arbitrary knots/links started in [1,2]. In practice, we always deal with braid representations of knots. Here we considered the symmetric and antisymmetric representations [2] and [1,1] for 3-strand braids. One can construct the result inductively, using the representation group theory, however, in this paper we used an indirect way of using the known answers for the torus knot/link polynomials in order to restore all the necessary ingredients (in particular, the mixing matrices) for the generic answer. We return to using the group theory approach elsewhere [18].

Using the formulas, obtained in this paper (we listed various knot polynomials for the knots that can be described by 3-strand braids with no more than 8 crossings in the Appendix) we tested various conjectures, from the Ooguri-Vafa conjecture [23] and its generalization [25] to the conjecture of the representation dependence of the “special” polynomials. The HOMFLY polynomials calculated in the paper were partly obtained earlier in a series of papers by the Indian group [8,7] within a different though close approach. In these cases our results confirm these earlier calculations.

The results presented here are substantially extended in [18] to include higher symmetric representations but this requires a deeper insight into the structure of the mixing matrices and, hence, is beyond the scope of the present paper.

Note added

After this paper was published there appeared a paper [30] with calculations of the HOMFLY polynomials in the first symmetric representation and of the corresponding Ooguri-Vafa polynomials for various knots and links. Their results for the 3-strand knots coincide with formulas of this paper for the only exception of the HOMFLY polynomial for knot 7_5 where we made a misprint (the Ooguri-Vafa polynomial was written in our paper correctly). We are grateful to the authors of [30] for the correction.

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Appendix. Tables of polynomials

In this Appendix we list the HOMFLY polynomials and related quantities for all 3-strand knots in the fundamental, symmetric [2] and antisymmetric [1, 1] representations. Namely, for each knot, besides HOMFLY, we write down expressions for the Jones \((A = q^2)\), Alexander \((A = 1)\), ”special” \((q \rightarrow 1)\) polynomials, the Ooguri-Vafa polynomials \(f_R(q,A)\) and their ”special” limit \(q \rightarrow 1\), and for the numbers \(N_{R,n,k}\) in (68). Note that all the expressions are listed with the factors (82) taken into account. We also use the notation \(\{x\} \equiv x - x^{-1}\).

**Knot 3\(_1\)**

\((-1, -1| -1, -1)\)

We use here the \((-1, -1| -1, -1)\) representation of the trefoil (in contrast with \((1, 1|1, 1)\) used throughout the main body of the paper) in order to match the standard knot tables.

**HOMFLY polynomials**

\[
\frac{H_{[1]}^{[1]} + \ast S_{[1]}^{[1]}}{\ast S_{[1]}^{[1]}} = A^4 (q^2 + q^{-2}) A^{-2} - 1) = A^4 \left( \ast S_{[0]} q^{-4} - \ast S_{[2]} + \ast S_{[1,1,1]} q^4 \right) \tag{103}
\]

\[
\frac{H_{[1]}^{[1]} + \ast S_{[1]}^{[1]}}{\ast S_{[1]}^{[1,1]}} = A^8 q^{-16} \left( (q^{20} + q^{14} + q^{12} + q^{8}) A^{-4} + (-q^{16} - q^{14} - q^{8}) A^{-2} + q^{10} \right) = \tag{104}

= A^8 q^{-16} \left( \ast S_{[3,3]} - \ast S_{[3,2,1]} q^4 + \ast S_{[3,1,1,1]} q^8 + \ast S_{[2,2,2]} q^8 - \ast S_{[2,1,1,1,1]} q^{16} + \ast S_{[1,1,1,1,1]} q^{24} \right)

\frac{H_{[2]}^{[2]} + \ast S_{[2]}^{[2]}}{\ast S_{[2]}^{[2]}} = A^8 q^{16} \left( (q^{-8} + q^{-12} + q^{-14} + q^{-20}) A^{-4} + (-q^{-8} - q^{-10} - q^{-14} - q^{-16}) A^{-2} + q^{10} \right) = \tag{105}

= A^8 q^{16} \left( \ast S_{[6]} q^{-24} - \ast S_{[5,1]} q^{-16} + \ast S_{[4,1,1]} q^{-8} + \ast S_{[3,3]} q^{-8} - \ast S_{[3,2,1]} q^{-4} + \ast S_{[2,2,2]} \right)

**Alexander polynomials**

\[
\mathfrak{A}_{[1]} = q^2 - 1 + q^{-2} \tag{106}
\]

\[
\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]}(q^2) = q^4 - 1 + q^{-4} \tag{107}
\]

**Jones polynomials**

\[
J_{[1]} = -q^8 + q^6 + q^2 \tag{108}
\]

\[
J_{[1,1]} = 1 \tag{109}
\]

\[
J_{[2]} = q^{22} - q^{20} - q^{18} + q^{16} - q^{14} + q^{10} + q^4 \tag{110}
\]

**Special polynomials**

\[
f_{[1]} = 2A^2 - A^4 \tag{111}
\]

\[
f_{[1,1]} = f_{[1]} = (f_{[1]})^2 = 2A^4 - 4A^6 + A^8 \tag{112}
\]

**Ooguri-Vafa polynomials**

\[
f_{[1,1]} = -A^6(A)^2(A/q^2) \tag{113}
\]

\[
f_{[2]} = q^2 A^6(A)^2(A/q^2) \tag{114}
\]

**Special Ooguri-Vafa polynomials**

\[
f_{[2]} = -f_{[1,1]} = 2A^6(A - A^{-1})^3 \tag{115}
\]
Numbers $N_{R,n,k}$

| $N_{[1]}$ | $k/n$ | 1 | 3 | 5 |
|-----------|-------|---|---|---|
| 0         | 2     | 4 | -16 | 24 | -16 | 4 |
| 1         | 4     | -20 | 32 | -20 | 4 |
| 2         | 1     | -8 | 14 | -8 | 1 |
| 3         | 0     | -1 | 2  | -1 | 0 |

| $N_{[1,1]}$ | $k/n$ | 2 | 4 | 6 | 8 | 10 |
|-------------|-------|---|---|---|---|---|
| 0           | 2     | -8 | 12 | -8 | 2 |
| 1           | 1     | -6 | 10 | -6 | 1 |
| 2           | 0     | -1 | 2  | -1 | 0 |

**Knot 41**

$(1, -1|1, -1)$

**HOMFLY polynomials**

\[
\frac{H_{[1]}}{S_{[1]}} = A^{-2} + (-q^2 + 1 - q^{-2})A^{-2} + (q^4 - q^2 + 3 - q^{-4} + q^{-6})A^2 + q^{-4}A^4 = \frac{1}{S_{[1]}}(S_{[3,3]} + S_{[3,2,1]}(q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4}) + S_{[1,1,1,1,1]} + S_{[2,2,2]} + S_{[2,2,1,1]}(q^{12} - 2q^{10} - q^8 + 4q^6 - 3q^4 - 2q^2 + 6 - 2q^{-2} - 3q^{-4} + 4q^{-6} - q^{-8} - 2q^{-10} + q^{-12}) + S_{[2,1,1,1,1,1]}(q^8 - 2q^4 + 1 - 2q^{-4} + q^{-8}) + S_{[1,1,1,1,1,1,1]})
\]

\[
\frac{H_{[2]}}{S_{[2]}} = q^{-4}A^{-4} + (-q^2 + q^{-2} - q^{-4} - q^{-6})A^{-2} + (q^6 - q^4 + 3 - q^{-4} + q^{-6}) + (-q^6 - q^4 + q^{-2} - q^{-2})A^2 + q^4A^4 = \frac{1}{S_{[2]}}(S_{[4,4]} + S_{[5,3]}(q^8 - 2q^4 + 1 - 2q^{-4} + q^{-8}) + S_{[4,1,1,1]} + S_{[3,2,1]}(q^{12} - 2q^{10} - q^8 + 4q^6 - 3q^4 - 2q^2 + 6 - 2q^{-2} - 3q^{-4} + 4q^{-6} - q^{-8} - 2q^{-10} + q^{-12}) + S_{[4,2,1,1,1]} + S_{[3,2,2]}(q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4}) + S_{[2,2,2,2]})
\]

**Alexander polynomials**

\[
\mathfrak{A}_{[1]} = -q^2 + 3 - q^{-2}
\]
\[
\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[1]}(q^2) = -q^4 + 3 - q^{-4}
\]

**Jones polynomials**

\[
J_{[1]} = q^4 - q^2 + 1 - q^{-2} + q^{-4}
\]
\[
J_{[1,1]} = 1
\]
\[
J_{[2]} = q^{12} - q^{10} - q^8 + 2q^6 - q^4 - q^2 + 3 - q^{-2} - q^{-4} + 2q^{-6} - q^{-8} - q^{-10} + q^{-12}
\]

**Special polynomials**

\[
\delta_{[1]} = A^{-2} - 1 + A^2
\]
\[
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^2 = A^{-4} - 2A^{-2} + 3 - 2A^2 + A^4
\]

**Ooguri-Vafa polynomials**

\[
f_{[1,1]} = \frac{(1)\{ A \} \{ Aq^{-2} \} \{ A^2 q^{-2} \}}{q}
\]
\[
f_{[2]} = \frac{(1)\{ A \} \{ Aq^{-2} \} \{ A^2 q^{-2} \}}{q}
\]
Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1,1]} = (A^2 - A^{-2})(A - A^{-1})^3 \]  

(128)

Numbers \( N_{R,n,k} \)

| \( k \mod n \) | -3 | -1 | 1 | 3 |
|----------------|----|----|----|----|
| \( N_{[1]} \)  | 0  | -1 | 2  | -2 | 1  |
| \( N_{[1,1]} \) | 1  | 0  | 1  | -1 | 0  |

Knot 52

(1, -1|1, 3)

The choice of representation for this knot is different in [29] and in [9]. We here follow the convention of [9].

HOMFLY polynomials

\[ \frac{H_{[1]}}{S_{[1]}} = A^{-4} \left( (q^2 - 1 + q^{-2})A^{-2} + (q^2 - 1 + q^{-2}) - A^2 \right) = \]

\[ = \frac{A^{-4}}{S_{[1]}} \left( *S_{[3]} q^4 + *S_{[2,1]} (-q^4 + q^2 - 1 + q^{-2} - q^{-4}) + *S_{[1,1,1]} q^{-4} \right) \]  

(129)

\[ \frac{H_{[1,1]}}{S_{[1,1]}} = q^{16} A^{-8} \left( (q^{20} - q^{18} - q^{16} + 2q^{14} - q^{10} + q^8)A^{-4} + (q^{20} + q^{18} - 2q^{16} + 3q^{12} - q^{10} - q^8 + q^6)A^{-2} + \right. \]

\[ + (-q^{14} + 2q^{10} - q^8 - q^6 + q^4) + (-q^2 + q^2 - q^6 - q^4)A^2 + q^6 A^4 \) = \]

\[ = \frac{q^{16} A^{-8}}{S_{[1,1]}} \left( *S_{[3,3]} q^4 + *S_{[3,2,1]} (-1 + q^{-2} - q^4 - q^2 - q^6 - q^8) + *S_{[3,1,1,1]} q^{-8} + *S_{[2,2,2]} q^{-8} + \right. \]

\[ + *S_{[2,2,1,1]} (q^{2} - 1 - q^{-2} + 2q^4 - 2q^8 + q^{10} + q^{-12} - 2q^{14} + q^{16} - q^{20} + q^{-22}) + \]

\[ + *S_{[2,1,1,1,1]} (-q^8 + q^{12} - q^{16} + q^{20} - q^{-24}) + *S_{[1,1,1,1,1]} q^{-24} \right. \]

(130)

\[ \frac{H_{[2]}}{S_{[2]}} = q^{-16} A^{-8} \left( (q^8 - q^{10} + 2q^{14} - q^{16} - q^{18} + q^{20})A^{-4} + \right. \]

\[ + (q^4 - q^8 - q^{-10} + 3q^{-12} - 2q^{-16} + q^{-18} + q^{-20})A^{-2} + \]

\[ + (q^4 - q^8 - q^{-10} + 2q^{-12} + q^{-14} + (-q^4 - q^6 + q^8 - q^{-12})A^2 + q^6 A^4 \) = \]

\[ = \frac{q^{-16} A^{-8}}{S_{[2]}} \left( *S_{[6]} q^2 + *S_{[5,1]} (-q^{24} + q^{20} - q^{16} + q^{12} - q^8) + \right. \]

\[ + *S_{[4,2]} (q^{22} - q^{20} + q^{16} - 2q^{14} + q^{12} + q^{10} - 2q^8 + 2q^4 - q^2 - 1 + q^2) + \]

\[ + *S_{[4,1,1]} q^8 + *S_{[3,3]} q^6 + *S_{[3,2,1]} (-q^8 + q^6 - q^4 + q^2 - 1) + *S_{[2,2,2]} \right) \]
Alexander polynomials

\[ A_{[1]} = 2q^2 - 3 + 2q^{-2} \]  
\[ A_{[1,1]} = A_{[2]} = A_{[1]}(q^2) = 2q^4 - 3 + 2q^{-4} \]  

(132)

Jones polynomials

\[ J_{[1]} = q^{-2} - q^{-4} + 2q^{-6} - q^{-8} + q^{-10} - q^{-12} \]  
\[ J_{[1,1]} = 1 \]  
\[ J_{[2]} = q^{-4} - q^{-6} + 3q^{-10} - 2q^{-12} - q^{-14} + 4q^{-16} - 3q^{-18} - q^{-20} + 3q^{-22} - 2q^{-24} - q^{-26} + 2q^{-28} - q^{-30} - q^{-32} + q^{-34} \]  

(133)

(134)

(135)

(136)

Special polynomials

\[ H_{[1]} = -A^{-6} + A^{-4} + A^{-2} \]  
\[ H_{[1,1]} = H_{[2]} = (H_{[1]})^2 = A^{-12} - 2A^{-10} - A^{-8} + 2A^{-6} + A^{-4} \]  

(137)

(138)

Ooguri-Vafa polynomials

\[ f_{[1,1]} = \frac{\{A\}^2\{A/q\}\{Aq\}\{q^2 - 1 + q^{-2}\}(q^9 + q^7 + q^5)A^{-10} + (q^7 + q^5)A^{-8} + q^3A^{-6}}{\{q\}} \]  
\[ f_{[2]} = \frac{\{A\}^2\{A/q\}\{Aq\}\{q^2 - 1 + q^{-2}\}(-q^{-1} - q^{-5} - q^{-9})A^{-10} + (-q^{-1} - q^{-7})A^{-8} + (-q^{-1} + q^3 - q^{-5})A^{-6}}{\{q\}} \]  

(139)

(140)

Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1,1]} = -\frac{(A^4 + 2A^2 + 3)(A - A^{-1})^3}{A^{10}} \]  

(141)

Numbers \( N_{R,n,k} \)

\[
\begin{array}{c|cccc}
k/n & -1 & -3 & -5 & -7 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
k/n & -10 & -8 & -6 & -4 & -2 \\
\hline
0 & 6 & 25 & -10 & 0 & -1 \\
1 & 11 & 60 & -25 & 0 & -1 \\
2 & 6 & 50 & -22 & 0 & 0 \\
3 & 1 & 17 & -8 & 0 & 0 \\
4 & 0 & 2 & -1 & 0 & 0 \\
\end{array}
\]

Knot 62

\((1, -1|1, -3)\)
HOMFLY polynomials

\[ \frac{H_{[1]}}{S_{[1]}} = A^2 ((q^2 + q^{-2})A^{-2} + (-q^4 + q^2 - 2 + q^{-2} - q^{-4}) + (q^2 - 1 + q^{-2})A^2) = \]

\[ = \frac{A^2}{S_{[1]}} \left( * S_{[3]}q^2 + * S_{[2,1]}(q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6}) + * S_{[1,1,1]}q^2 \right) \]

\[ \frac{H_{[1,1]}}{S_{[1,1]}} = A^4 q^{-8} \left( (q^{16} + q^{10} + q^{8} + q^4)A^{-4} + (-q^{18} - q^{16} + q^{14} - q^{12} - 3q^{10} - 2q^4 - q^{-2})A^{-2} + \right. \]

\[ + (q^{18} - q^{16} + 4q^{12} - 3q^8 + 4q^6 - 2q^4 - 2q^{-2} + 2 + q^{-2} - q^{-4} + q^{-6}) + \]

\[ + (-q^{14} + 2q^{10} - 2q^8 - 3q^6 + 3q^4 - 3 + q^2 - q^{-2} - q^{-6})A^2 + (q^8 - q^6 - q^4 + 2q^2 - q^{-2} - q^{-4})A^4) = \]

\[ = \frac{A^4 q^{-8}}{S_{[1,1]}} \left( * S_{[3,3]} + * S_{[3,2,1]}(q^8 - 2q^6 + 2q^4 - 3q^2 - 2 - 2q^{-2} + q^{-4}) + * S_{[3,1,1,1,1]}q^4 + * S_{[2,2,2,2]}q^4 + \right. \]

\[ + * S_{[2,2,2,1]}(q^{24} - 2q^{12} - 2q^8 - 3q^6 + 3q^4 - 2 + 2q^{-2} + q^{-4}) + * S_{[1,1,1,1,1,1]}q^{12} \right) \]

\[ \frac{H_{[2]}}{S_{[2]}} = A^4 q^8 \left( (q^{-4} + q^{-8} + q^{10} + q^{16})A^{-4} + (-q^2 - 2q^{-4} - 3q^{10} - q^{-12} + q^{14} - q^{-16} - q^{-18})A^{-2} + \right. \]

\[ + (q^6 - q^4 + q^2 - 2 - 2q^{-2} - 2q^{-4} + 4q^6 - 4q^8 + 4q^{-12} - q^{-16} + q^{-18}) + \]

\[ + (-q^6 + q^2 - 3 + 3q^4 - 3q^6 - 2q^8 + 2q^{10} - q^{-14})A^2 + (q^4 - q^2 + 2q^{-2} - q^{-4} - q^{-6} + q^{-8})A^4) = \]

\[ = \frac{A^4 q^8}{S_{[2]}} \left( * S_{[5]}q^{-12} + * S_{[5,1]}(q^4 - 2 + 2q^2 - 3q^6 + 2q^{-12} - 2q^{-16} + 2q^{-20}) + \right. \]

\[ + * S_{[4,2]}(q^{12} - 2q^{10} + 3q^6 - 5q^4 + 2q^2 - 3 - 3q^4 - 3q^6 + 2q^8 + 2q^{-14} - 3q^{-16} + 5q^{-18} - 2q^{-20} - 2q^{-22} + q^{-24}) + \]

\[ + * S_{[4,1,1]}q^{-4} + * S_{[3,3]}q^{-4} + * S_{[3,2,1]}(q^4 - 2q^2 - 2 - 3q^2 - 2q^{-2} - 2q^{-4} - q^{-6} + q^{-8}) + * S_{[2,2,2,1]} \right) \]

Alexander polynomials

\[ \mathcal{A}_{[1]} = -q^4 + 3q^2 - 3 + 3q^2 - q^{-4} \]

\[ \mathcal{A}_{[1,1]} = \mathcal{A}_{[2]} - \mathcal{A}_{[1]}(q^2) = -q^8 + 3q^4 - 3 + 3q^4 - q^{-8} \]

Jones polynomials

\[ J_{[1]} = q^{10} - 2q^8 + 2q^6 - 2q^4 + 2q^2 - 1 + q^{-2} \]

\[ J_{[1,1]} = 1 \]

\[ J_{[2]} = q^{28} - 2q^{26} + 4q^{22} - 5q^{20} + 6q^{18} - 6q^{16} + 6q^{14} - 6q^{10} - 3q^8 - 3q^6 - 3q^4 - 1 + 3q^{-2} - q^{-4} - q^{-6} + q^{-8} \]

Special polynomials

\[ S_{[1]} = 2 - 2A^2 + 4 \]

\[ S_{[1,1]} = S_{[2]} = (S_{[1,1]})^2 = 4 - 8A^2 + 8A^4 - 4A^6 + A^8 \]

Ooguri-Vafa polynomials

\[ f_{[1,1]} = \left\{ A \right\}^2 \langle A/q \rangle \langle A \rangle (q^2 - 1 + q^{-2})((q^7 + q^5)A^2 + (q^5 + q^{-3})A^4 + (-q^{-1} - q^{-7})A^6) \]

\[ f_{[2]} = \left\{ A \right\}^2 \langle A/q \rangle \langle A \rangle (q^2 - 1 + q^{-2})((-q^{-1} - q^{-3})A^2 + (-q^{-7} - q^{-1})A^4 + (q^{11} + q^5)A^6) \]

Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1,1]} = 2A^2(A^4 - A^2 - 1)(A - A^{-1})^3 \]
### Numbers $N_{R,n,k}$

| $k \backslash n$ | -1 | 1 | 3 | 5 |
|-----------------|----|---|---|---|
| 0               | -2 | 4 | -3 | 1 |
| 1               | -1 | 4 | -4 | 1 |
| 2               | 0  | 1 | -1 | 0 |

**$N_{[1]}$**

| $k \backslash n$ | -2 | 0 | 2 | 4 | 6 | 8 | 10 |
|-----------------|----|---|---|---|---|---|----|
| 0               | 3  | -14 | 33 | -52 | 53 | -30 | 7 |
| 1               | 4  | -25 | 84 | -178 | 212 | -125 | 28 |
| 2               | 1  | -13 | 82 | -246 | 335 | -201 | 42 |
| 3               | 0  | -2  | 40 | -175 | 267 | -159 | 29 |
| 4               | 0  | 0   | 10 | -67  | 113 | -65  | 9 |
| 5               | 0  | 0   | 1  | -13 | 24  | -13 | 1 |
| 6               | 0  | 0   | 0  | -1  | 2   | -1  | 0 |

**$N_{[1,1]}$**

| $k \backslash n$ | -2 | 0 | 2 | 4 | 6 | 8 | 10 |
|-----------------|----|---|---|---|---|---|----|
| 0               | 0  | 5 | -20 | 35 | -40 | 35 | -20 | 5 |
| 1               | 10 | -45 | 85 | -110 | 110 | -65 | 15 |
| 2               | 6  | -34 | 72 | -113 | 132 | -79 | 16 |
| 3               | 1  | -10 | 25 | -54  | 75  | -44 | 7 |
| 4               | 0  | -1  | 3  | -12 | 20  | -11 | 1 |
| 5               | 0  | 0   | 0  | -1  | 2   | -1  | 0 |

**Knot 63**

$(2, -1 | 1, -2)$

**HOMFLY polynomials**

\[
\frac{H_{[1]}}{S_{[3]}} = (-q^2 + 1 - q^{-2})A^{-2} + (q^4 - q^2 + 3 - q^{-2} + q^{-4}) + (-q^2 + 1 - q^{-2})A^2 = \\
= \frac{1}{S_{[3]}} \left( ^*S_{[3]} + ^*S_{[2,1]}(-q^6 + 2q^4 + 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + ^*S_{[1,1,1]} \right)
\] (155)
\[
\frac{H_{[1,1]}}{S_{[1,1]}} = (q^{10} - q^8 - q^6 + 2q^4 - 1 + q^{-2})A^{-4} + (-q^{12} + 2q^8 - 3q^6 - 3q^4 + 4q^2 - 1 - 4q^{-2} + q^{-4} - q^{-8})A^{-2} + \\
\quad + (q^{12} - q^{10} + q^8 + 4q^6 - 3q^4 - q^2 + 9 - q^{-2} - 3q^{-4} + 4q^{-6} + q^{-8} - q^{-10} + q^{-12}) + \\
\quad + (-q^8 + q^4 - 4q^2 - 1 + 4q^{-2} - 3q^{-4} - 3q^{-6} + 2q^{-8} - q^{-12})A^2 + (q^2 - 1 + 2q^{-4} - q^{-6} - q^{-8} + q^{-10})A^4 = \\
= \frac{1}{S_{[1,1]}} \left( h_{[3,3]} + h_{[3,2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + h_{[3,1,1,1]} + h_{[2,2,2]} + \right. \\
\quad + h_{[2,2,1,1]}(q^{18} - 2q^{16} + 5q^{12} - 6q^{10} - 2q^8 + 12q^6 - 9q^4 - 7q^2 + \\
\quad + 16 - 7q^{-2} - 9q^{-4} + 12q^{-6} - 2q^{-8} - 6q^{-10} + 5q^{-12} - 2q^{-16} + q^{-18}) + \\
\quad \left. + h_{[2,1,1,1,1]}(-q^{12} + 2q^8 - 3q^4 + 3 - 3q^{-4} + 2q^{-8} - q^{-12}) + h_{[1,1,1,1,1,1]} \right) \\
\frac{H_{[2]}}{S_{[2]}} = q^2 - 1 + 2q^{-4} - q^{-6} - q^{-8} + q^{-10})A^{-4} + (-q^8 + q^4 - 4q^2 - 1 + 4q^{-2} - 3q^{-4} - 3q^{-6} + 2q^{-8} - q^{-12})A^{-2} + \\
\quad + (q^{12} - q^{10} + q^8 + 4q^6 - 3q^4 - q^2 + 9 - q^{-2} - 3q^{-4} + 4q^{-6} + q^{-8} - q^{-10} + q^{-12}) + \\
\quad + (-q^{12} + 2q^8 - 3q^6 - 3q^4 + 4q^2 - 1 - 4q^{-2} - 3q^{-4} - 3q^{-6} + 2q^{-8} - q^{-12})A^2 + (q^{10} - q^8 - q^6 + 2q^4 - 1 + q^{-2})A^4 = \\
= \frac{1}{S_{[2]}} \left( h_{[0]} + h_{[5,1]}(-q^{12} + 2q^8 - 3q^4 + 3 - 3q^{-4} + 2q^{-8} - q^{-12}) + \\
\quad + h_{[4,2]}(q^{18} - 2q^{16} + 5q^{12} - 6q^{10} - 2q^8 + 12q^6 - 9q^4 - 7q^2 + \\
\quad + 16 - 7q^{-2} - 9q^{-4} + 12q^{-6} - 2q^{-8} - 6q^{-10} + 5q^{-12} - 2q^{-16} + q^{-18}) + \\
\quad + h_{[4,1,1]} + h_{[3,3]} + \\
\quad + h_{[3,2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + h_{[2,2,2]} \right)
\]

**Alexander polynomials**

\[
\mathcal{A}_{[1]} = q^4 - 3q^2 + 5 - 3q^{-2} + q^{-4} \\
\mathcal{A}_{[1,1]} = \mathcal{A}_{[2]} = \mathcal{A}_{[1]}(q^2) = q^8 - 3q^4 + 5 - 3q^{-4} + q^{-8}
\]

**Jones polynomials**

\[
J_{[1]} = -q^6 + 2q^4 - 2q^2 + 3 - 2q^{-2} + 2q^{-4} - q^{-6} \\
J_{[1,1]} = 1 \\
J_{[2]} = q^{18} - 2q^{16} - q^{14} + 5q^{12} - 4q^{10} - 3q^8 + 9q^6 + 5q^4 - 5q^2 + \\
+ 11 - 5q^{-2} - 5q^{-4} + 9q^{-6} - 3q^{-8} - 4q^{-10} + 5q^{-12} - q^{-14} - 2q^{-16} + q^{-18}
\]

**Special polynomials**

\[
\delta_{[1]} = -A^{-2} + 3 - A^2 \\
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^2 = A^{-4} - 6A^{-2} + 11 - 6A^2 + A^4
\]

**Ooguri-Vafa polynomials**

\[
f_{[1,1]} = \frac{(A^2(A/q)(Aq)^2(q^2 - 1 + q^{-2}) (q^2 + q^{-2} - q^{-4})A^{-2} + q^3 + 2q^1 - 2q^{-1} - q^{-3}) + (q^1 - q^{-3} + q^{-7})A^2}{(q)} \\
f_{[2]} = \frac{(A^2(A/q)(Aq)^2(q^2 - 1 + q^{-2}) (q^2 + q^{-2} - q^{-4})A^{-2} + q^3 + 2q^1 - 2q^{-1} - q^{-3}) + (q^2 - q^3 + q^{-1})A^2}{(q)} \\
\]

**Special Ooguri-Vafa polynomials**

\[
f_{[2]} = -f_{[1,1]} = (A^2 - A^{-2})(A - A^{-1})^3
\]
Numbers $N_{R,n,k}$

| $k \backslash n$ | -3 | -1 | 1 | 3 |
|------------------|----|----|---|---|
| 0                | 1  | -4 | 4 | -1|
| 1                | 1  | -4 | 4 | -1|
| 2                | 0  | -1 | 1 | 0 |

| $k \backslash n$ | -6 | -4 | -2 | 0 | 2 | 4 | 6 |
|------------------|----|----|----|---|---|---|---|
| 0                | 1  | -9 | 28 | -42| 33 | -13| 2 |
| 1                | 5  | -27| 72 | -113|104 | -52|11 |
| 2                | 5  | -26| 65 | -114|127 | -72|15 |
| 3                | 1  | -9 | 24 | -54 | 74 | -43 |7 |
| 4                | 0  | -1 | 3  | -12 | 20 | -11 |1 |
| 5                | 0  | 0  | 0  | -1  |2  | -1 | 0 |

$N_{[1]}$:

$N_{[2]}$:

| $k \backslash n$ | -6 | -4 | -2 | 0 | 2 | 4 | 6 |
|------------------|----|----|----|---|---|---|---|
| 0                | 2  | -13| 33 | -42| 28 | -9 | 1 |
| 1                | 11 | -52|104 | -113|72  | -27| 5 |
| 2                | 15 | -72|127 | -114|65  | -26| 5 |
| 3                | 7  | -43| 74 | -54 |24  | -9 | 1 |
| 4                | 4  | -11|20  | -12 | 3  | -1 |0  |
| 5                | 5  | -1 | 2  | -1  | 0  | 0  | 0 |

Knot $7_3$

$(1, -1|1, 5)$

**HOMFLY polynomials**

$$\frac{H_{[1]}}{S_{[1]}} = A^{-6} \left((-q^2-q^{-2}) A^{-2} + (q^4-q^2+2-q^{-2}+q^{-4}) + (q^4-q^2+1-q^{-2}+q^{-4}) A^2 \right) =$$

$$= A^{-6} \left( S_{[3]} S_{[1]} q^6 + S_{[2]} (q^4+q^2-2q^2+q^{-2}+q^{-4}) + S_{[1,1,1]} q^6 \right)$$

$$H_{[1,1]} = A^{-12} q^{24} \left(q^{-6}+q^{-10}+q^{-12}+q^{-14}\right) A^{-4} +$$

$$+(-q^4-q^{-6}+2q^{-10}-2q^{-12}+q^{-14}+q^{-16}+q^{-18}-q^{-20}+q^{-22}+q^{-24}) A^{-2} +$$

$$+(q^4-q^{-6}+2q^{-10}+2q^{-12}+q^{-16}+3q^{-18}+4q^{-20}+3q^{-22}+q^{-26}) +$$

$$+(q^{-6}-q^{-8}+q^{-10}+q^{-12}+q^{-14}+q^{-16}+q^{-18}+q^{-20}+q^{-22}+q^{-24}+q^{-26}) A^2 +$$

$$+(q^8-q^{-10}+q^{-12}+q^{-14}+q^{-16}+q^{-18}+q^{-20}+q^{-22}+q^{-24}+q^{-26}+q^{-28}+q^{-30}+q^{-32}) A^4 \right) =$$

$$= A^{-12} q^{24} \left( S_{[3]} + S_{[3,2,1]} (q^2-2q^{-2}+3q^{-6}-2q^{-8}+q^{-10}+q^{-12}) + S_{[3,1,1,1]} q^{-12} +$$

$$+ S_{[2,2,2]} q^{-12} +$$

$$+ S_{[2,2,1]} (q^2-1+2q^{-4}-3q^{-6}-q^{-8}+6q^{-10}-5q^{-12}-4q^{-14}) +$$

$$+10q^{-16}+4q^{-18}+5q^{-20}+6q^{-22}+q^{-24}+3q^{-26}+2q^{-28}+q^{-30}+q^{-32}+q^{-34}+$$

$$+S_{[2,1,1,1,1]} (-q^{-12}+q^{-16}+2q^{-20}+3q^{-24}+2q^{-28}+q^{-32}+q^{-36}) +$$

$$+ S_{[1,1,1,1,1]} q^{-36} \right)$$
\[
\frac{H[2]}{S[2]} = A^{-12} q^{-24} \left( (q^{18} + q^{12} + q^{10} + q^6)A^{-4} + 
+ (q^{32} + q^{30} - 2q^{28} + 4q^{24} - 3q^{22} - 3q^{20} + 3q^{18} + q^{16} - 2q^{14} + 2q^{12} - q^8 + q^6)A^2 + 
+ (q^{32} - q^{30} - 2q^{28} + 4q^{26} - 2q^{22} + 2q^{20} + q^{18} - q^{16} + q^{14} - q^{10} + q^8)A^4 \right)
\]

(169)

\[
\text{Alexander polynomials}
\]

\[
\mathfrak{A}[1] = 2q^4 - 3q^3 + 3 - 3q^{-2} + 2q^{-4}
\]

(170)

\[
\mathfrak{A}[1,1] = \mathfrak{A}[2] = \mathfrak{A}[1](q^2) = 2q^3 - 3q^2 + 3 - 3q^{-4} + 2q^{-8}
\]

(171)

\[
\text{Jones polynomials}
\]

\[
J[1] = q^{-4} - q^{-6} + 2q^{-8} - 2q^{-10} + 3q^{-12} - 2q^{-14} + q^{-16} - q^{-18}
\]

(172)

\[
J[1,1] = 1
\]

(173)

\[
J[2] = q^{-8} - q^{-10} + 3q^{-14} - 2q^{-16} + 5q^{-20} - 2q^{-22} - 4q^{-24} + 7q^{-26} - 2q^{-28} - 6q^{-30} + 
+ 8q^{-32} - 2q^{-34} - 5q^{-36} + 5q^{-38} - q^{-40} - 3q^{-42} + 2q^{-44} - q^{-48} + q^{-50}
\]

(174)

\[
\text{Special polynomials}
\]

\[
\mathfrak{B}[1] = -2A^{-8} + 2A^{-6} + A^{-4}
\]

(175)

\[
\mathfrak{B}[1,1] = \mathfrak{B}[2] = (\mathfrak{B}[1])^2 = 4A^{-16} - 8A^{-14} + 4A^{-10} + A^{-8}
\]

(176)

\[
\text{Ooguri-Vafa polynomials}
\]

\[
f_{[1,1]} = \frac{(A)^2(A/q)}{q} \left( (q^{17} + 2q^{13} + q^{11} + 2q^9 + q^7 + 3q^5 + q^3 + 2q^1 + q^{-3})A^{-14} + 
+ (q^{15} + q^{11} + q^9 + q^7 + q^5 + q^3 + q + q^{-1})A^{-12} + (q^{13} - q^{11} + q^9 + 2q^1 - 2q^{-1} + q^{-3})A^{-10} \right)
\]

(177)

\[
f_{[2]} = \frac{(A)^2(A/q)}{q} \left( (-q^3 - 2q^{-1} - q^{-3} - 3q^{-5} - q^{-7} - q^{-9} - q^{-11} - 2q^{-13} - q^{-17})A^{-14} + 
+ (-q^3 + q^1 - q^{-1} - q^{-3} - q^{-5} - q^{-9} - q^{-11} - q^{-15})A^{-12} + (-q^3 + 2q^1 - q^{-1} - q^{-9} + q^{-11} - q^{-13})A^{-10} \right)
\]

(178)

\[
\text{Special Ooguri-Vafa polynomials}
\]

\[
f_{[2]} = -f_{[1,1]} = \frac{-2(A^4 + 3A^2 + 7)(A - A^{-1})^3}{A^{14}}
\]

(179)
**Numbers** $N_{R,n,k}$

| $k \backslash n$ | -9 | -7 | -5 | -3 |
|-----------------|----|----|----|----|
| 0               | 2  | -4 | 1  | 1  |
| 1               | 1  | -4 | 0  | 3  |
| 2               | 0  | -1 | 0  | 1  |

| $N_{[1]}$ | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------|----|----|----|----|----|----|
| 0         | 41 | -146 | 179 | -76 | -1 | -2 |
| 1         | 200 | -755 | 944 | -390 | -8 | -15 |
| 2         | 398 | -1639 | 2131 | -890 | -6 | -35 |
| 3         | 412 | -1917 | 2628 | -1123 | -1 | -28 |
| 4         | 241 | -1320 | 1926 | -847 | 0 | -9 |
| 5         | 80  | -549 | 859 | -390 | 0 | -1 |
| 6         | 14  | -135 | 228 | -107 | 0 | 0 |
| 7         | 1   | -18  | 33  | -16  | 0 | 0 |
| 8         | 0   | -1   | 2   | -1   | 0 | 0 |

| $N_{[2]}$ | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------|----|----|----|----|----|----|
| 0         | 55  | -196 | 241 | -104 | 1 | -4 |
| 1         | 330 | -1231 | 1531 | -634 | -10 | -27 |
| 2         | 821 | -3284 | 4213 | -1751 | -15 | -75 |
| 3         | 1085 | -4787 | 6418 | -2716 | -7 | -85 |
| 4         | 837 | -4202 | 5940 | -2575 | -1 | -45 |
| 5         | 389 | -2314 | 3472 | -1547 | 0 | -11 |
| 6         | 107 | -803 | 1286 | -590 | 0 | -1 |
| 7         | 16  | -170 | 292 | -138 | 0 | 0 |
| 8         | 1   | -20  | 37  | -18  | 0 | 0 |
| 9         | 0   | -1   | 2   | -1   | 0 | 0 |

**Knot 75**

$(-2, 1| -1, -4)$

**HOMFLY polynomials**

$$
\frac{H_{[1]}}{S_{[1]}} = \frac{A^6}{S_{[1]}} \left( (q^4 - q^2 + 2 - q^{-2} + q^{-4}) A^{-2} + (q^4 - 2q^2 + 2 - 2q^{-2} + q^{-4}) + (-q^2 + 1 - q^{-2}) A^2 \right) =
$$

$$
= \frac{A^6}{S_{[1]}} \left( *S_{[3]}q^{-6} + *S_{[2,1]}(-q^6 + 2q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - q^{-6}) + *S_{[1,1]}q^6 \right)
$$

(180)
\[
H_{[1]} / S_{[1,1]} = A_{12} q^{24} \left( (q^{32} - q^{30} - q^{28} + 3q^{26} - 3q^{22} + 3q^{20} + q^{18} - 2q^{16} + 2q^{14} + q^{12} - q^{10} + q^{8}) A^{-4} + \\
+ (q^{32} + 3q^{30} - 3q^{28} + 6q^{26} - 3q^{22} - 5q^{20} + 6q^{18} - 5q^{14} + 3q^{12} - 2q^{8} + q^{6}) A^{-2} + \\
+ (-3q^{26} + 5q^{22} - 5q^{20} - 5q^{18} + 7q^{16} - q^{14} - 5q^{12} + 4q^{10} - 2q^{8} + q^{4}) + \\
+ (-q^{24} + q^{22} + 3q^{20} - 3q^{18} - 2q^{16} + 5q^{14} - q^{12} - 3q^{10} + 2q^{8} - q^{4}) A^{2} + \\
+ (q^{18} - q^{16} + 2q^{14} - 2q^{12} + q^{10} + q^{8}) A^{4} \right) = \\
= A_{12} q^{24} \left( -S_{[3,3]} + -S_{[3,2,1]}(-q^{12} + 2q^{10} - 3q^{8} + 3q^{6} - 3q^{4} + 2q^{2} - 1) + * S_{[3,1,1,1]} q^{12} + * S_{[2,2,2]} q^{12} + \\
+ S_{[2,2,1]}(q^{14} - 2q^{12} + 5q^{10} - 6q^{8} - 2q^{6} + 12q^{4} - 9q^{2} - 7q^{18} + 16q^{16} - 7q^{14} + 9q^{12} + 12q^{10} - 2q^{8} - 6q^{6} + 5q^{4} - 2 - q^{2}) + \\
+ S_{[3,2,1,1]}(-q^{36} + 2q^{32} - 3q^{28} + 3q^{24} - 3q^{20} + 2q^{16} - q^{12}) + * S_{[1,1,1,1,1]} q^{36} \right)
\]

\[
H_{[2]} / S_{[2]} = A_{12} q^{24} \left( (q^{6} - 2q^{8} + 3q^{10} - 5q^{12} - 5q^{14} + 6q^{16} - 5q^{20} - 3q^{22} + 6q^{24} - 3q^{28} + q^{30} + q^{32}) A^{-4} + \\
+ (q^{6} + 4q^{8} - 9q^{10} - 5q^{12} - 9q^{14} - 7q^{16} - 5q^{18} - 5q^{20} + 5q^{22} - 3q^{26}) + \\
+ (-q^{4} + 2q^{6} - 3q^{8} - q^{10} - 2q^{12} + q^{14} + 2q^{16} - 3q^{18} + 3q^{20} + q^{22} + q^{24}) A^{2} + \\
+ (q^{4} - q^{6} + 2q^{12} - q^{14} - q^{16} + q^{18}) A^{4} \right) = \\
= A_{12} q^{24} \left( -S_{[3,3]} q^{36} + -S_{[3,1,1]}(-q^{12} + 2q^{10} - 3q^{8} + 3q^{6} - 3q^{4} + 2q^{2} - 1) + * S_{[3,1,1]} q^{36} + \\
+ S_{[4,2]}(q^{14} - 2q^{12} + 5q^{10} - 6q^{8} - 2q^{6} + 12q^{4} - 9q^{2} - 7q^{18} + 16q^{16} - 7q^{14} + 9q^{12} + 12q^{10} - 2q^{8} - 6q^{6} + 5q^{4} - 2 - q^{2}) + \\
+ S_{[3,2,1]}(-q^{36} + 2q^{32} - 3q^{28} + 3q^{24} - 3q^{20} + 2q^{16} - q^{12}) + * S_{[1,1,1,1,1]} q^{36} \right)
\]

Alexander polynomials

\[
A_{[1]} = 2q^{4} - 4q^{2} + 5 - 4q^{-4} + 2q^{-6}
\]

\[
A_{[1,1]} = A_{[2]} = A_{[1]}(q^{2}) = 2q^{8} - 4q^{4} + 5 - 4q^{-4} + 2q^{-8}
\]

Jones polynomials

\[
J_{[1]} = -q^{18} + 2q^{16} - 3q^{14} + 3q^{12} - 3q^{10} + 3q^{8} - q^{6} + q^{4}
\]

\[
J_{[1,1]} = 1
\]

\[
J_{[2]} = q^{50} - 2q^{48} + 5q^{44} - 6q^{42} - 2q^{40} + 11q^{38} - 9q^{36} - 4q^{34} + 14q^{32} - \\
-10q^{30} - 5q^{28} + 13q^{26} - 7q^{24} - 5q^{22} + 9q^{20} + 3q^{18} - 3q^{16} + 4q^{14} + q^{12} - q^{10} + q^{8}
\]

Special polynomials

\[
\delta_{[1]} = 2A^{4} - A^{6}
\]

\[
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^{2} = 4A^{8} - 4A^{12} + A^{16}
\]

Ooguri-Vafa polynomials

\[
f_{[1,1]} = \frac{A^{2}}{q} \left( (q^{2} - 1 + q^{2}) \left( (-2q^{3} + q^{1} - q^{-1} - 2q^{-3} - q^{-5} - q^{-9}) A^{10} + \\
+ (-2q^{3} - q^{-3} - q^{-5} - q^{-11}) A^{12} + (-q^{3} - q^{-5} - q^{-7} - q^{-13}) A^{14} \right) \right)
\]

\[
f_{[2]} = \frac{A^{2} q^{2}}{q} \left( (q^{2} - 1 + q^{2}) \left( (q^{13} + q^{9} + 2q^{7} + q^{5} - q^{3} + 2q^{1}) A^{10} + \\
+ (q^{15} + 2q^{9} + q^{7} + 2q^{1}) A^{12} + (q^{17} + q^{11} + q^{9} + q^{3}) A^{14} \right) \right)
\]

Special Ooguri-Vafa polynomials

\[
f_{[2]} = -f_{[1,1]} = 2A^{10}(2A^{4} + 3A^{2} + 3)(A - A^{-1})^{3}
\]
Numbers $N_{R,n,k}$

\[
\begin{array}{c|cccc}
 k \backslash n & 3 & 5 & 7 & 9 \\
 \hline
0 & -2 & 2 & 1 & -1 \\
1 & -3 & 1 & 3 & -1 \\
2 & -1 & 0 & 1 & 0 \\
\end{array}
\]

$N_{[1]}$:

\[
\begin{array}{cccccccc}
 k \backslash n & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
 \hline
0 & 17 & -50 & 47 & -28 & 47 & -50 & 17 \\
1 & 71 & -203 & 181 & -218 & 481 & -443 & 131 \\
2 & 118 & -324 & 269 & -736 & 1764 & -1484 & 393 \\
3 & 101 & -261 & 197 & -1383 & 3332 & -2604 & 618 \\
4 & 47 & -112 & 75 & -1563 & 3674 & -2681 & 560 \\
5 & 11 & -24 & 14 & -560 & 1295 & -912 & 176 \\
6 & 1 & -2 & 1 & -299 & 665 & -433 & 67 \\
7 & 0 & 0 & 0 & -92 & 197 & -118 & 13 \\
8 & 0 & 0 & 0 & -15 & 31 & -17 & 1 \\
9 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
\end{array}
\]

$N_{[1,1]}$:

\[
\begin{array}{cccccccc}
 k \backslash n & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
 \hline
0 & 11 & -32 & 31 & -24 & 41 & -40 & 13 \\
1 & 37 & -103 & 91 & -144 & 325 & -289 & 83 \\
2 & 48 & -127 & 101 & -400 & 968 & -793 & 203 \\
3 & 30 & -74 & 52 & -619 & 1482 & -1125 & 254 \\
4 & 9 & -20 & 12 & -560 & 1295 & -912 & 176 \\
5 & 1 & -2 & 1 & -299 & 665 & -433 & 67 \\
6 & 0 & 0 & 0 & -92 & 197 & -118 & 13 \\
7 & 0 & 0 & 0 & -15 & 31 & -17 & 1 \\
8 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
\end{array}
\]

Knot 82

(1, −1|1, −5)

HOMFLY polynomials

\[
\frac{H_{[1]}}{S_{[1]}} = A^4 \left( (q^4 + 1 + q^{-4}) A^{-2} + (-q^6 + q^4 - 2q^2 + 1 - 2q^{-2} + 3q^{-4} - q^{-6}) + (q^4 - q^2 + 1 - q^{-2} + q^{-4}) A^2 \right) =
\]

\[
= A^4 \left( *S_{[3]}q^{-4} + *S_{[2,1]}q^8 - 2q^6 + 3q^4 - 3q^2 + 3 - 3q^{-2} + 2q^{-4} - 2q^{-6} + q^{-8} \right) + *S_{[1,1,1]}q^4
\]

(193)
\[
H_{[1,1]}^{[1,1]} = A^8 q^{-16} \left( (q^{26} + q^{22} + q^{20} + q^{16} + q^{14} + q^{12} + q^{10} + q^8 + q^4) A^{-4} + \right.
\]
\[
+ (-q^{30} - q^{28} - q^{26} - q^{24} - 3q^{22} - 3q^{16} - 2q^{14} - q^{12} - 2q^{10} - q^8 - 2q^6 - q^2 - q^0) A^{-2} + \right.
\]
\[
+ (q^{30} - q^{28} + 4q^{24} - 3q^{20} + 4q^{18} + 3q^{16} - 2q^{14} + q^{12} + 3q^{10} + 2q^6 + q^4 - q^2 - 2 + q^{-2} - q^{-4} + q^{-6}) + \right.
\]
\[
+ (-q^{26} + 2q^{22} - 2q^{20} - 3q^{18} + 3q^{16} + q^{14} - 4q^{12} + 2q^8 - 2q^6 - 2 + q^{-2} - q^{-6}) A^2 + \right.
\]
\[
+ (q^{20} - q^{18} - q^{16} + 2q^{14} - 2q^{10} + q^8 + q^6 - q^2 + q^{-2} + q^{-4}) A^4 \right) =
\]
\[
A^8 q^{-16} \left( *S_{[3,3]} + *S_{[3,2,1]}(q^{12} - 2q^{10} + 2q^8 - 3q^6 + 3q^4 - 3q^2 + 2 - 2q^{-2} + q^{-4}) + *S_{[3,1,1,1]}q^8 + *S_{[2,2,1]}q^8 + \right.
\]
\[
+ *S_{[2,2,1,1]}(q^{30} - 2q^{24} - q^{22} + 2q^{20} - 3q^{18} + 3q^{16} + q^{14} - 4q^{12} + 2q^8 - 2q^6 - 2 + q^{-2} - q^{-6}) A^2 + \right.
\]
\[
+ (q^{20} - q^{18} - q^{16} + 2q^{14} - 2q^{10} + q^8 + q^6 - q^2 + q^{-2} + q^{-4}) A^4 \right)
\]
\[
= A^8 q^{-16} \left( *S_{[6]}q^{-24} + *S_{[4]}(1 - 2q^{-2} + 3q^{-1}) + *S_{[2]}(q^{12} - 2q^{10} + 3q^8 - 3q^6 - 2q^4 - 2q^2 - 2 + q^{-2} + q^{-4}) + 3q^{-6} A^{-10} + \right.
\]
\[
+ 2q^{-12} - 7q^{-14} + 4q^{-16} + 5q^{-18} - 9q^{-20} + 8q^{-22} - 5q^{-24} - 3q^{-26} - 3q^{-28} + 5q^{-30} - q^{-32} + 2q^{-34} + q^{-36} \right) + \right.
\]
\[
+ *S_{[4,1,1]}q^{-8} + *S_{[3,3]}q^{-8} + *S_{[3,2,1]}(q^{-4} - 2q^{-2} + 3q^{-4} - 3q^{-6} + 2q^{-8} - 2q^{-10} + q^{-12}) + *S_{[2,2,2]} \right)
\]

Alexander polynomials
\[
\Xi_{[1]} = -q^6 + 3q^4 - 3q^2 + 3 - 3q^{-2} + 3q^{-4} - q^{-6}
\]
\[
\Xi_{[1,1]} = \Xi_{[2]} = \Xi_{[3]}(q^2) = -q^{12} + 3q^8 - 3q^6 + 3 - 3q^{-4} + 3q^{-8} - q^{-12}
\]

Jones polynomials
\[
J_{[1]} = q^{16} - 2q^{14} + 2q^{12} - 3q^{10} + 3q^8 - 2q^6 + 2q^4 - q^2 + 1
\]
\[
J_{[1,1]} = 1
\]
\[
J_{[2]} = q^{44} - 2q^{42} + 3q^{38} - 4q^{36} + 2q^{34} + 3q^{32} - 6q^{30} + 3q^{28} + 4q^{26} - 7q^{24} + 2q^{22} + \right.
\]
\[
+ 5q^{20} - 7q^{18} + 5q^{16} - 5q^{14} - 5q^{12} + 5q^8 - 3q^6 - q^4 + 3q^2 - 1 - q^{-2} + q^{-4}
\]

Special polynomials
\[
\delta_{[1]} = 3A^2 - 3A^4 + A^6
\]
\[
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1,1]})^2 = 9A^4 - 18A^2 + 15 - 6A^{10} + A^{12}
\]

Ooguri-Vafa polynomials
\[
f_{[1,1]} = \frac{A^2 q A q^2}{A q} \left( q + A^{-1} - q^{-3} - q^{-5} \right) A^6 + \right.
\]
\[
+ (q^9 + q^7 + q^5 + q^3 + q^1) A^8 + (q^{15} - q^{13} - q^{11} - 2q^9 - q^7) A^6 + (q^{19} + q^9 + q^7 - q^5 + q^1) A^{10}
\]
\[
f_{[2]} = \frac{A^2 q A q}{A q} \left( q^2 - 1 - q^{-5} \right) A^6 + (q^9 + q^7 + q^5 - q^3 - q^{-1} - q^{-3} - q^{-5} - q^{-15}) A^{10}
\]

Special Ooguri-Vafa polynomials
\[
f_{[2]} = -f_{[1,1]} = A^6(3A^4 - 6A^2 + 1)(A - A^{-1})^3
\]
Numbers $N_{R,n,k}$

| $k \backslash n$ | 1 | 3 | 5 | 7 |
|---------------|---|---|---|---|
| 0             | -3| 6 | -4| 1 |
| 1             | -4| 11| -10| 3 |
| 2             | -1| 6 | -6| 1 |
| 3             | 0 | 1 | -1| 0 |

$N_{[1]}$:

| $k \backslash n$ | 1 | 3 | 5 | 7 |
|---------------|---|---|---|---|
| 0             | 16| -81| 179| -226| 174| -77| 15 |
| 1             | 45| -320| 964| -1585| 1482| -735| 149 |
| 2             | 43| -526| 2311| -4880| 5319| -2870| 603 |
| 3             | 16| -473| 3217| -8572| 10567| -6017| 1262 |
| 4             | 2 | -254| 2838| -9430| 12860| -7524| 1508 |
| 5             | 0 | -81| 1629| -6760| 10036| -5903| 1079 |
| 6             | 0 | -14| 604| -3197| 5107| -2969| 469 |
| 7             | 0 | -1| 139| -987| 1682| -954| 121 |
| 8             | 0 | 0| 18| -191| 345| -189| 17 |
| 9             | 0 | 0| 1| -21| 40| -21| 1 |
| 10            | 0 | 0| 0| -1| 2| -1| 0 |

$N_{[1,1]}$:

| $k \backslash n$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---------------|---|---|---|---|----|----|----|
| 0             | 15 | -71| 146| -174| 131| -59| 12 |
| 1             | 50 | -281| 706| -1049| 955| -482| 101 |
| 2             | 63 | -434| 1414| -2695| 2911| -1601| 342 |
| 3             | 37 | -332| 1547| -3875| 4835| -2795| 583 |
| 4             | 10 | -132| 1025| -3432| 4817| -2838| 550 |
| 5             | 1 | -26| 427| -1938| 2992| -1754| 298 |
| 6             | 0 | -2| 110| -697| 1164| -667| 92 |
| 7             | 0 | 0| 16| -154| 275| -152| 15 |
| 8             | 0 | 0| 1| -19| 36| -19| 1 |
| 9             | 0 | 0| 0| -1| 2| -1| 0 |

Knot $8_5$

$(-1,3| -1,3)$
HOMFLY polynomials

\[
\frac{H_{[1]}}{S_{[1]}} = A^{-4} \left( (q^4 - q^2 - 2 - q^{-2} + q^{-4}) A^{-2} + (-q^6 + q^4 - 3q^2 + 1 - 3q^{-2} + q^{-4} - q^{-6}) + (q^4 + 2 + q^{-4}) A^2 \right) = \\
= A^{-4} \left( *S_{[3,3]} q^4 + *S_{[2,2]} (q^6 - 2q^2 + 3q^4 - 4q^2 + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[1,1,1,1]} q^{-4} \right)
\]

\[
\frac{H_{[1,1]}^{(1,1)}}{S_{[1,1]}} = A^{-8} q^{16} \left( (q^4 - q^2 + 1 + q^{-2} - 2q^4 + 2q^{-6} + 2q^{-8} - 4q^{-10} + 3q^{-12} + 4q^{-14} - 4q^{-16} + q^{-18} + q^{-20}) A^{-2} + \\
+(-q^6 - 3 + q^{-2} - q^{-4} - 5q^{-6} + 2q^{-8} - q^{-10} - 8q^{-12} - 3q^{-16} - 5q^{-18} + 3q^{-20} + 2q^{-22} - q^{-26}) A^{-2} + \\
+q^6 - 4q^2 + 2 - q^{-2} + 4q^{-4} + 4q^{-6} + 8q^{-10} + 5q^{-12} + \\
-7q^{-16} + 8q^{-18} - 3q^{-20} + q^{-22} + 5q^{-24} - q^{-28} + q^{-30} \right) + \\
+(-q^6 - 3 - 4q^2 - 3q^{-4} - 6q^{-6} - 6q^{-8} - 6q^{-10} - 3q^{-12} - 4q^{-14} + \\
+2q^{-16} - 2q^{-18} - 5q^{-20} - 2q^{-24} - q^{-28} - q^{-30}) A^2 + \\
+(q^4 + 2q^{-8} + q^{-10} + 3q^{-12} + 3q^{-14} + 3q^{-16} + 3q^{-18} + 2q^{-20} + 2q^{-22} + q^{-28}) A^4 \right) = \\
= A^{-8} q^{16} \left( *S_{[3,3]} + *S_{[3,2,1]} (q^4 - 2q^2 + 3 - 4q^2 + 3q^{-4} - 4q^{-6} + 3q^{-8} - 2q^{-10} + q^{-12}) + *S_{[1,1,1,1,1]} q^{-8} + \\
+ *S_{[2,2,2]} q^{-8} + *S_{[2,2,1,1]} (q^4 - 2q^2 + 2q^2 + 2q^{-6} + 4q^{-8} + 8q^{-12} - 8q^{-14} + 2q^{-20} + 10q^{-24} - 8q^{-28} + 2q^{-30} - q^{-32} - 2q^{-34} + 2q^{-36}) + \\
+3q^{-12} - 12q^{-14} + 8q^{-16} + 8q^{-18} - 14q^{-20} + 2q^{-22} + 10q^{-24} - 8q^{-26} + 6q^{-30} - 7q^{-32} - 2q^{-34} - 2q^{-36} + \\
+ *S_{[1,1,1,1,1,1]} (1 - 2q^{-4} + 3q^{-8} - 4q^{-12} + 3q^{-16} - 4q^{-20} + 3q^{-24} - 2q^{-28} + q^{-32}) + *S_{[1,1,1,1,1,1,1]} q^{-24} \right)
\]

\[
\frac{H_{[2]}^{[2]}}{S_{[2]}} = A^{-8} q^{16} \left( (q^{20} - q^{18} - q^{16} + 3q^{14} + 12q^{12} - 3q^{10} + 2q^{8} + 3q^{6} - 2q^{4} - q^{2} + 1 - q^{-2} - q^{-4}) A^{-4} + \\
+(-q^{26} + 2q^{22} - 3q^{20} - 3q^{18} + 3q^{16} - 8q^{12} - 4q^{10} + 2q^{8} - 5q^{6} - 5q^{4} + q^{2} - 3 - q^{-6}) A^{-2} + \\
+(q^{30} - q^{28} + 5q^{24} + q^{22} - 3q^{20} + 8q^{18} + 7q^{16} - q^{14} + 5q^{12} + 8q^{10} + 4q^{8} + 4q^{6} - q^{4} + q^{2} + 2q^{-2} - q^{-4} - q^{-6}) + \\
+(q^{30} - 5q^{28} + q^{26} - 2q^{24} - 2q^{22} - q^{20} - 2q^{18} - 7q^{16} - 4q^{14} - 3q^{12} - 6q^{10} - 3q^{8} - 3q^{6} - 3q^{4} - q^{-2} - q^{-4}) A^2 + \\
+(q^{26} + 4q^{22} + 2q^{20} + 3q^{16} + 2q^{14} + 12q^{12} + 2q^{10} + 2q^{8} + q^{4}) A^4 \right) = \\
= A^{-8} q^{16} \left( *S_{[3,3]} q^{24} + *S_{[2,2,1]} (q^{32} - 2q^{28} + 3q^{24} - 4q^{20} + 3q^{16} - 4q^{12} + 3q^{8} - 2q^{4} + 1) + \\
+ *S_{[4,2]} (q^{36} - 2q^{34} - q^{32} + 6q^{30} - 3q^{28} - 8q^{26} + 10q^{24} + 2q^{22} - 14q^{20} + 8q^{18} + 8q^{16} + 12q^{14} + \\
+ 3q^{12} + 8q^{10} - 9q^{8} + 2q^{6} + 6q^{4} - 8q^{2} + 3 + 4q^{-2} - 6q^{-4} + 2q^{-6} + q^{-8} - 2q^{-10} + q^{-12}) + \\
+ *S_{[1,1,1,1]} q^{8} + *S_{[3,3]} q^{8} + *S_{[2,2,1]} (q^{12} - 2q^{10} + 3q^{8} - 4q^{6} + 3q^{4} - 4q^{2} + 3 - 2q^{-2} + q^{-4}) + *S_{[2,2,2]} \right)
\]

Alexander polynomials

\[
\mathfrak{A}_{[1]} = -q^6 + 3q^4 - 4q^2 - 5 - 4q^{-2} + 3q^{-4} - q^{-6}
\]

\[
\mathfrak{A}_{[1,1]} = \mathfrak{A}_{[2]} = \mathfrak{A}_{[1]} (q^2) = -q^{12} + 3q^{10} - 4q^8 + 5 - 4q^4 + 3q^8 - q^{-12}
\]

Jones polynomials

\[
J_{[1]} = 1 - q^{-2} + 3q^{-4} - 3q^{-6} + 3q^{-8} - 4q^{-10} + 3q^{-12} - 2q^{-14} + q^{-16}
\]

\[
J_{[1,1]} = 1
\]

\[
J_{[2]} = q^4 - q^2 - 1 + 4q^{-2} - q^{-4} - 5q^{-6} + 7q^{-8} - 9q^{-12} + 8q^{-14} + 3q^{-16} - 12q^{-18} + 7q^{-20} + 6q^{-22} - 12q^{-24} + \\
+ 5q^{-26} + 7q^{-28} - 10q^{-30} + 3q^{-32} + 5q^{-34} - 6q^{-36} + 2q^{-38} + q^{-40} - 2q^{-42} + q^{-44}
\]

Special polynomials

\[
\delta_{[1]} = 2A^{-6} - 5A^{-4} + 4A^{-2}
\]

\[
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^2 = 4A^{-12} - 20A^{-10} + 41A^{-8} - 40A^{-6} + 16A^{-4}
\]
Ooguri-Vafa polynomials

\[
f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{aq\}}{\{q\}} \left( q^{19} - q^{17} + 2q^{15} + 2q^{13} + q^7 + 3q^3 - 2q^{-1} + 2q^{-3} \right) A^{-10} + (q^{-15} - q^{-13} + q^{-5} - 2q^{-1} - 4q^{-3} - 2q^{-5} + q^{-7} - q^{-9}) A^{-8} + (q^9 + 2q^7 + q^3 - 2q^{-5} - q^{-7}) A^{-6} \right) 
\]

\[
f_{[2]} = \frac{\{A\}^2 \{A/q\} \{aq\}}{\{q\}} \left( -2q^3 + 2q^1 - q^{-3} - q^{-7} - 2q^{-9} - 2q^{-15} + q^{-17} - q^{-19} \right) A^{-10} + (q^9 - q^7 + 2q^5 - 2q^{-1} + 5q^{-3} + 2q^{-7} + q^{-9} + 2q^{-11} + q^{-15}) A^{-8} + (q^{11} + 2q^5 - q^{-1} - 2q^{-5} - q^{-9}) A^{-6} \right) 
\]

Special Ooguri-Vafa polynomials

\[
f_{[2]} = -f_{[1,1]} = -\frac{(A^4 - 16A^2 + 9)(A - A^{-1})^3}{A^{10}} 
\]

Numbers \(N_{R,n,k}\)

\[
\begin{array}{|c|c|c|c|c|}
\hline
k/n & -7 & -5 & -3 & -1 \\
\hline
0 & -2 & 7 & -9 & 4 \\
1 & -3 & 11 & -12 & 4 \\
2 & -1 & 6 & -6 & 1 \\
3 & 0 & 1 & -1 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
k/n & -14 & -12 & -10 & -8 & -6 & -4 & -2 \\
\hline
0 & 25 & -123 & 262 & -318 & 237 & -103 & 20 \\
1 & 170 & -802 & 1536 & -1573 & 951 & -337 & 55 \\
2 & 475 & -2229 & 3987 & -3529 & 1702 & -470 & 64 \\
3 & 704 & -3421 & 5890 & -4601 & 1733 & -342 & 37 \\
4 & 605 & -3180 & 5402 & -3797 & 1093 & -133 & 10 \\
5 & 310 & -1857 & 3175 & -2043 & 440 & -26 & 1 \\
6 & 93 & -683 & 1194 & -713 & 111 & -2 & 0 \\
7 & 15 & -153 & 277 & -155 & 16 & 0 & 0 \\
8 & 1 & -19 & 36 & -19 & 1 & 0 & 0 \\
9 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
\hline
\end{array}
\]

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\((-2, 1) \mid -1, 4\)

HOMFLY polynomials

\[
\frac{H_{[1]}}{S_{[1]}} = A^{-2} \left( (-q^4 + q^2 - 2 + q^{-2} - q^{-4}) A^{-2} + 
+ (q^{-6} - q^4 + 3q^2 - 2 + 3q^{-2} - q^{-4} + q^{-6}) + (-q^4 + q^{-2} - 1 + q^{-2} - q^{-4}) A^2 \right) =
\]
(219)

\[
\frac{H_{[1]}}{S_{[1]}} = A^{-4} q^8 \left( (q^{10} - q^{-8} + 2q^4 - 2q^{-2} + 4q^{-4} - 2q^{-6} - 6q^{-8} - 9q^{-12} + q^{-14}) A^{-4} + 
+ (q^{-12} + q^8 - 3q^6 + 2q^{-2} - 7 + 5q^{-4} - 8q^{-6} - 4q^{-8} + 
+ 5q^{-10} - 2q^{-12} - 4q^{-14} + q^{-16} - q^{-20}) A^{-2} + 
+ (q^{-12} + q^8 + 3q^6 - 2q^{-2} + 5 - 5q^{-2} + 5q^{-4} + 8q^{-6} - 
- 7q^{-8} + 10q^{-12} - 2q^{-14} - 3q^{-16} + 4q^{-18} + q^{-20} - q^{-22} + q^{-24}) + 
+ (-q^{-8} + q^4 - 3q^6 - q^{-2} - 4q^{-4} + 2q^{-6} - 6q^{-10} + 3q^{-12} - 3q^{-16} - 6q^{-20} + q^{-24}) A^2 + 
+ (q^{-2} - 1 + q^{-4} - q^{-6} + q^{-10} - 2q^{-12} + 2q^{-16} - q^{-18} - q^{-20} + q^{-22}) A^2 \right) =
\]
(220)

\[
\frac{H_{[2]}}{S_{[2]}} = A^{-4} q^8 \left( (q^{14} - q^{-12} + 3q^6 - q^{-2} - 4q^{-4} + 2q^{-6} - 2q^{-2} + 4q^{-6} - q^{-8} + q^{-10}) A^{-4} + 
+ (-q^{20} + q^{-16} - 4q^{-14} - 2q^{-12} + 5q^{-10} - 4q^{-6} - 8q^{-6} + 5q^{-4} - 7 + 2q^{-2} - 3q^{-6} + q^{-8} - q^{-12}) A^{-2} + 
+ (q^{24} - q^{-22} + q^{-20} + 4q^{-18} - 3q^{-16} - 2q^{-14} + 10q^{-12} - 7q^{-10} + 8q^{-6} + 5q^{-4} - 5q^{-2} + 5 + 2q^{-2} - 2q^{-4} + 3q^{-6} + q^{-8} - q^{-10} + q^{-12}) + 
+ (-q^{-24} + 4q^{-20} - 3q^{-18} + 5q^{-16} - 6q^{-14} + 2q^{-12} + 4q^{-10} + 5q^{-8} + 3q^{-4} - q^{-6} + q^{-8}) A^2 + 
+ (q^{-22} - q^{-20} - q^{-18} + 2q^{-16} - 2q^{-14} + q^{-12} + q^{-10} + q^{-8} - q^{-6} + q^{-4} - 1 + q^{-2}) A^2 \right) =
\]
(221)

Alexander polynomials

\[
\mathcal{A}_{[1]} = q^6 - 3q^4 + 5q^2 - 5 + 5q^{-2} - 3q^{-4} + q^{-6}
\]
(222)

\[
\mathcal{A}_{[1,1]} = \mathcal{A}_{[2]} = \mathcal{A}_{[1]}(q^2) = q^{12} - 3q^8 + 5q^4 - 5 + 5q^{-4} - 3q^{-8} + q^{-12}
\]
(223)

Jones polynomials

\[
J_{[1]} = -q^4 + 2q^2 - 2 + 4q^{-2} - 4q^{-4} + 4q^{-6} - 3q^{-8} + 2q^{-10} - q^{-12}
\]
(224)

\[
J_{[1,1]} = 1
\]
(225)

\[
J_{[2]} = q^{14} - 2q^{12} - q^{10} + 5q^8 - 4q^6 - 4q^4 + 10q^2 - 3 + 9q^{-2} + 14q^{-4} - 2q^{-6} - 4q^{-8} + 
+ 16q^{-10} - 16q^{-12} + 14q^{-14} + 9q^{-16} - 12q^{-20} + 9q^{-22} + q^{-24} - 6q^{-26} + 4q^{-28} - 2q^{-32} + q^{-34}
\]
(226)
Special polynomials

\[ \delta_{[1]} = -2A^{-4} + 4A^{-2} - 1 \]  
\[ \delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^2 = 4A^{-8} - 16A^{-6} + 20A^{-4} - 8A^{-2} + 1 \]  

Ooguri-Vafa polynomials

\[ f_{[1,1]} = \{A\}^2 \frac{A}{q} \{Aq\} \left( q^2 - 1 + q^{-2} \right) \left( q^{17} + q^{11} + 3q^5 + q^{-3} \right) A^{-6} + \]  
\[ +( -q^{13} - q^{11} - 2q^7 + 5q - 2q^{-1} + 2q^{-3} + 4q^{-5} \right) A^{-4} + (q - q^{-1} + q^{-5} - q^{-9}) A^{-2} \]  
\[ f_{[2]} = \frac{\{A\}^2 \{A\} / q \{Aq\} \left( q^2 - 1 + q^{-2} \right) \left( -q^7 - q^5 - q^{-7} - q^{-13} \right) A^{-6} + \]  
\[ +( -q^9 - 2q^7 + 2q^5 - q^{-1} + 2q^{-3} + 4q^{-7} + 4q^{-9} ) A^{-4} + (q^{13} - q^9 + q^5 - q^3) A^{-2} \]  

Special Ooguri-Vafa polynomials

\[ \tilde{f}_{[2]} = -\tilde{f}_{[1,1]} = - \frac{2(A^3 - 3)(A - A^{-1})^3}{A^6} \]  

Numbers \( N_{R,n,k} \)

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
k \setminus n & -2 & 0 & 2 & 4 & 6 & 8 & 10 \\
\hline
0 & 39 & -286 & 785 & -1028 & 665 & -194 & 19 \\
1 & 275 & -2553 & 8365 & -12718 & 9477 & -3227 & 381 \\
2 & 853 & -10436 & 41492 & -73336 & 62457 & -24578 & 3548 \\
3 & 1497 & -25234 & 123381 & -252783 & 242378 & -107162 & 17923 \\
4 & 1617 & -39707 & 242296 & -573983 & 610752 & -294896 & 53921 \\
5 & 1103 & -42639 & 331047 & -906126 & 1057291 & -544430 & 103754 \\
6 & 471 & -31998 & 324752 & -1028553 & 1303219 & -701974 & 134083 \\
7 & 121 & -16898 & 232948 & -857227 & 1169635 & -648615 & 120036 \\
8 & 17 & -6230 & 123125 & -530576 & 773886 & -435958 & 75736 \\
9 & 1 & -1564 & 47840 & -244654 & 378959 & -214440 & 33858 \\
10 & -254 & 13479 & -83606 & 136710 & -76977 & 10648 \\
11 & 2675 & -20851 & 35800 & -19901 & 2301 \\
12 & 354 & -3683 & 6608 & -3603 & 325 \\
13 & 28 & -436 & 814 & -433 & 27 \\
14 & 1 & -31 & 60 & -31 & 1 \\
15 & -1 & 2 & -1 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
k \setminus n & -1 & 1 & 3 \\
\hline
0 & 2 & -6 & 5 \\
1 & 3 & -11 & 11 \\
2 & 1 & -6 & 6 \\
3 & 0 & -1 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
k \setminus n & -1 & 1 & 3 \\
\hline
0 & 2 & -6 & 5 \\
1 & 3 & -11 & 11 \\
2 & 1 & -6 & 6 \\
3 & 0 & -1 & 1 \\
\hline
\end{array}
\]
\[ K \backslash n = \begin{array}{cccccccc}
-2 & 0 & 2 & 4 & 6 & 8 & 10 \\
0 & 29 & -216 & 605 & -808 & 531 & -156 & 15 \\
1 & 173 & -1685 & 5733 & -8988 & 6875 & -2397 & 289 \\
2 & 451 & -6003 & 25170 & -46202 & 40551 & -16425 & 2458 \\
3 & 654 & -12559 & 65901 & -141221 & 139859 & -63619 & 289 \\
4 & 570 & -12559 & 65901 & -141221 & 139859 & -63619 & 289 \\
5 & 300 & -15385 & 134426 & -391395 & 473113 & -248899 & 47840 \\
6 & 92 & -9578 & 113396 & -386023 & 507192 & -277756 & 52677 \\
7 & 15 & -4078 & 68991 & -276387 & 391286 & -219388 & 39561 \\
8 & 1 & -1162 & 30345 & -144784 & 219153 & -124072 & 26519 \\
9 & 0 & -211 & 9549 & -55385 & 89010 & -50298 & 7335 \\
10 & 0 & -22 & 2993 & -15273 & 25898 & -14468 & 1772 \\
11 & 0 & -1 & 303 & -2952 & 5252 & -2878 & 276 \\
12 & 0 & 0 & 26 & -379 & 704 & -376 & 25 \\
13 & 0 & 0 & 1 & -29 & 56 & -29 & 1 \\
14 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
\end{array} \]

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\((3, -1|1, -3)\)

**HOMFLY polynomials**

\[
\frac{H_{[1]}}{*S_{[1]}} = (q^4 - q^2 + 2 - q^{-2} + q^{-4})A^{-2} + (-q^6 + q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4} - q^{-6}) + (q^4 - q^2 + 2 - q^{-2} + q^{-4})A^2 = \\
= \frac{1}{*S_{[1]}} \left( *S_{[3]} + *S_{[2,1]}(q^8 - 2q^6 + 3q^4 - 5q^2 + 5 - 5q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + *S_{[1,1,1]} \right) \quad (232)
\]
\[
\frac{H_{[1,1]}}{S_{[1,1]}} = (q^{16} - q^{14} + 3q^{10} - q^8 - 2q^6 + 4q^4 - 2 + 2q^{-2} - q^{-6} + q^{-8})A^{-4} + \\
+(-q^{18} + q^{14} - 4q^{12} - q^{10} + 5q^8 - 6q^6 - 6q^4 + 8q^2 - 2 - 7q^{-2} + 4q^{-4} - 3q^{-8} + 3q^{-10} - q^{-14})A^{-2} + \\
+(q^{18} - q^{16} + q^{14} + 3q^{12} - 3q^{10} + q^8 + 9q^6 - 7q^4 - 3q^2 + 15- \\
-3q^{-2} - 7q^{-4} + 9q^{-6} + q^{-8} - 3q^{-10} + 3q^{-12} + q^{-14} - q^{-16} + q^{-18}) + \\
+(-q^{14} + q^{10} - 3q^8 + 4q^4 - 7q^2 - 2 + 8q^{-2} - 6q^{-4} - 6q^{-6} + 5q^{-8} + q^{-10} - 4q^{-12} + q^{-14} - q^{-18})A^2 + \\
+(q^8 - 6 + 2q^2 - 2 + 4q^{-4} - 2q^{-6} - q^{-8} + 3q^{-10} - q^{-14} + q^{-16})A^4 = \\
(233)
\]

\[
\frac{H_{[2]}}{S_{[2]}} = (q^8 - 6 + 2q^2 - 2 + 4q^{-4} - 2q^{-6} - q^{-8} + 3q^{-10} - q^{-14} + q^{-16})A^4 + \\
+(-q^{14} + q^{10} - 3q^8 + 4q^4 - 7q^2 - 2 + 8q^{-2} - 6q^{-4} - 6q^{-6} + 5q^{-8} - q^{-10} - 4q^{-12} + q^{-14} - q^{-18})A^2 + \\
+(q^{18} - q^{16} + q^{14} + 3q^{12} - 3q^{10} + q^8 + 9q^6 - 7q^4 - 3q^2 + 15- \\
-3q^{-2} - 7q^{-4} + 9q^{-6} + q^{-8} - 3q^{-10} + 3q^{-12} + q^{-14} - q^{-16} + q^{-18}) + \\
+(-q^{18} + 4q^{14} - 4q^{12} - q^{10} + 5q^8 - 6q^6 - 4q^4 + 8q^{-2} - 2 - 7q^{-2} + 4q^{-4} - 3q^{-8} + q^{-10} - q^{-14})A^2 + \\
+(q^{14} - q^{12} + q^8 - 2q^6 + 4q^4 - 2 + 2q^{-2} - q^{-4} + q^{-8})A^4 = \\
(234)
\]

\[
\frac{H_{[3]}}{S_{[3]}} = (q^{16} - 2q^{12} + 3q^8 - 5q^4 + 5 - 5q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8}) + \\
+ S_{[3,2,1]}(q^{24} - 2q^{22} + 4q^{18} - 6q^{16} + 11q^{12} - 13q^{10} - 4q^8 + 24q^6 - 18q^4 - 13q^2+ \\
+(-32 - 13q^{-2} - 18q^{-4} + 24q^{-6} - 4q^{-8} - 13q^{-10} + 11q^{-12} - 6q^{-16} + 4q^{-18} - 2q^{-22} + q^{-24})+ \\
+ S_{[2,1,1,1,1]}(q^{16} - 2q^{12} + 3q^8 - 5q^4 + 5 - 5q^{-4} + 3q^{-8} - 2q^{-12} + q^{-14} - q^{-16}) + \\
+ S_{[1,1,1,1,1,1])}
\]

Alexander polynomials

\[
A_{[1]} = -q^6 + 3q^4 - 5q^2 + 7 - 5q^{-2} + 3q^{-4} - 4q^{-6} (235)
\]

\[
A_{[1,1]} = A_{[2]} = A_{[1]}(q^2) = -q^{12} + 3q^8 - 5q^4 + 7 + 5q^{-4} + 3q^{-8} - q^{-12} (236)
\]

Jones polynomials

\[
J_{[1]} = q^8 - 2q^6 + 3q^4 - 4q^2 + 5 - 4q^{-2} + 3q^{-4} - 2q^{-6} + q^{-8} (237)
\]

\[
J_{[2]} = q^{24} - 2q^{22} + 5q^{18} - 6q^{16} - 2q^{14} + 12q^{12} - 10q^{10} - 7q^8 + 20q^6 - 12q^4 - 11q^2+ \\
+25 - 11q^{-2} - 12q^{-4} + 20q^{-6} - 7q^{-8} - 10q^{-10} + 12q^{-12} - 2q^{-14} - 6q^{-16} + 5q^{-18} - 2q^{-22} + q^{-24} (238)
\]

Special polynomials

\[
\delta_{[1]} = 2A^{-2} - 3 + 2A^2 (240)
\]

\[
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^2 = 4A^{-4} - 12A^{-2} + 17 - 12A^2 + 4A^4 (241)
\]

Ooguri-Vafa polynomials

\[
f_{[1,1]} = \left\{ \frac{A^2}{A/q} \right\}^2 \left( (q^{15} + 2q^9 + q^7 + q^3 + 2q^1 - q^{-1})A^{-2} + \\
+(-q^{11} + q^7 + q^5 - 3q^3 + 3q^1 - q^{-1} - q^{-3} - q^{-5} + q^{-7} + (q^5 - 2q^3 - q^1 - q^{-3} - 2q^{-5} - q^{-7}))A^2 + \\
+(q^{15} + 2q^9 + q^7 + q^3 + q^1 - q^{-1} - q^{-3} - q^{-5} + q^{-7} + (q^{15} + 2q^9 + q^7 + q^3 + 2q^1 - q^{-1}))A^2 \right) (242)
\]

\[
f_{[2]} = \left\{ \frac{A^2}{A/q} \right\}^2 \left( (q^5 - 2q^3 - q^1 - q^{-3} - 2q^{-5} - q^{-7})A^{-2} + \\
+(-q^{11} + q^7 + q^5 - 3q^3 + 3q^1 - q^{-1} - q^{-3} - q^{-5} + q^{-7} + (q^{15} + 2q^9 + q^7 + q^3 + 2q^1 - q^{-1}))A^2 \right) (243)
\]

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Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1.1]} = 6(A^2 - A^{-2})(A - A^{-1})^3 \]  

(244)

Numbers \( N_{R,n,k} \)

**\( N_{[1]} \):**

| \( k \backslash n \) | -3 | -1 | 1 | 3 |
|-----------------|----|----|---|---|
| 0               | -2 | 5  | -5| 2 |
| 1               | -3 | 11 | -11| 3 |
| 2               | -1 | 6  | -6| 1 |
| 3               | 0  | 1  | -1| 0 |

**\( N_{[1,1]} \):**

| \( k \backslash n \) | -6 | -4 | -2 | 0  | 2  | 4  | 6  |
|-----------------|----|----|----|----|----|----|----|
| 0               | 14 | -60| 120| -160| 150| -84| 20 |
| 1               | 59 | -278| 598| -846| 815| -448| 100|
| 2               | 103| -521| 1183| -1803| 1845| -1026| 219|
| 3               | 94 | -514| 1221| -2052| 2288| -1298| 261|
| 4               | 46 | -288| 716| -1377| 1705| -979| 177|
| 5               | 11 | -91 | 239| -561| 781| -446| 67 |
| 6               | 1  | -15 | 42 | -136| 214| -119| 13 |
| 7               | 0  | -1  | 3 | -18 | 32 | -17 | 1  |
| 8               | 0  | 0   | 0 | -1  | 2  | -1  | 0  |

**\( N_{[2]} \):**

| \( k \backslash n \) | -6 | -4 | -2 | 0  | 2  | 4  | 6  |
|-----------------|----|----|----|----|----|----|----|
| 0               | 20 | -84| 150| -160| 120| -60 | 14 |
| 1               | 100| -448| 815| -846| 598| -278| 59 |
| 2               | 219| -1026| 1845| -1803| 1183| -521| 103|
| 3               | 261| -1298| 2288| -2052| 1221| -514| 94 |
| 4               | 177| -979| 1705| -1377| 716| -288| 46 |
| 5               | 67 | -446| 781| -561| 239| -91 | 11 |
| 6               | 13 | -119| 214| -136| 42 | -15 | 1  |
| 7               | 1  | -17 | 32| -18 | 3 | -1  | 0  |
| 8               | 0  | -1  | 2 | -1  | 0 | 0   | 0  |

**Knot 8\_{10}**

\((-2, 2| -1, 3)\)
\[ \frac{H_{[1]}}{\cdot S_{[1]}} = A^{-2} \left( (-q^4 + q^2 - 3 + q^{-2} - q^{-4})A^{-2} + \right. \]
\[ + (q^6 - q^4 + 4q^2 - 2 + 4q^{-2} - q^{-4} + q^{-6}) + \left( -q^4 + q^2 - 2 + q^{-2} - q^{-4} \right) A^2 \right) = \]
\[ = A^{-2} \left( \cdot S_{[3]} q^2 + \cdot S_{[2,1]} (-q^6 + 2q^4 - 4q^2 - 5 + 5q^{-2} - 4q^{-4} + 2q^{-6} - q^{-8}) + \cdot S_{[1,1,1]} q^{-2} \right) \]
\[ \frac{H_{[1]}}{\cdot S_{[1]}^2} = A^{-4} q^8 \left( (-q^{10} - q^6 + q^6 + 3q^2 - 3q^2 + 1 + 7q^{-2} - 3q^{-4} - q^{-6} + 4q^{-8} - q^{-12} + q^{-14}) A^{-4} + \right. \]
\[ + (-q^{12} - 5q^6 + q^2 - 13 - 2q^{-2} + 6q^{-4} - 13q^{-6} - 7q^{-8} + 6q^{-10} - 3q^{-12} - 5q^{-14} + q^{-16} - q^{-20} A^{-2} + \right. \]
\[ + (q^{12} - q^6 + 4q^2 - 2q^4 + 6q^2 + 10 - 6q^{-2} + 10q^{-4} + 15q^{-6} - \right. \]
\[ - 8q^{-8} + q^{-10} + 15q^{-12} - 2q^{-14} - 3q^{-16} + 5q^{-18} + q^{-20} - q^{-22} + q^{-24}) + \right. \]
\[ + (-q^8 - 5q^2 - 9q^{-4} + q^{-6} + 10q^{-10} - 2q^{-12} + 6q^{-14} - 4q^{-16} - 4q^{-18} + 2q^{-20} - q^{-24}) A^2 + \right. \]
\[ + (q^2 - 1 + q^{-2} + 2q^{-4} - 2q^{-6} + q^{-8} + 3q^{-10} - 3q^{-12} + 3q^{-16} - q^{-18} - q^{-20} + q^{-22}) A^4 \right) = \]
\[ = A^{-4} q^8 \left( \cdot S_{[3,3]} + \cdot S_{[3,2,1]} (-q^6 + 2q^4 - 4q^2 + 5 - 5q^{-2} + 5q^{-4} - 4q^{-6} + 2q^{-8} - q^{-10}) + \cdot S_{[3,1,1,1]} q^{-4} + \right. \]
\[ + \cdot S_{[2,2,2]} q^{-4} + \cdot S_{[2,2,1,1]} (-q^6 + q^4 + q^2 - 2q^{10} + 3q^8 + 10q^6 - 18q^2 + 4q^2 - 17 - 28q^{-2} + 8q^{-4} + \right. \]
\[ + 24q^{-6} - 29q^{-8} - q^{-10} + 27q^{-12} - 18q^{-14} - 9q^{-16} + 18q^{-18} - 5q^{-20} - 7q^{-22} + 6q^{-24} - 2q^{-28} + q^{-30} + \right. \]
\[ + \cdot S_{[1,1,1,1,1]} q^{-8} + \cdot S_{[1,1,1,1,1]} q^{-12} \right) \right) \]
\[ \frac{H_{[2]}}{\cdot S_{[2]}} = A^{-4} q^{-8} \left( (-q^{14} - q^{12} + 4q^8 - q^6 - q^4 + 1 - 3q^{-2} + 3q^{-4} + q^{-6} - q^{-8} + q^{-10}) A^{-4} + \right. \]
\[ + (-q^{20} + q^{16} - 5q^{12} + 6q^{10} - 7q^8 - 13q^6 + 6q^4 - 2q^2 - 13 + q^{-2} - 5q^{-6} - q^{-12}) A^{-2} + \right. \]
\[ + (q^4 - 2q^2 + 2q^0 + 5q^{18} - 3q^{16} - 2q^{14} + 15q^{12} - q^{10} - 8q^8 + \right. \]
\[ + 15q^6 + 10q^4 - 6q^2 + 10 + 6q^{-2} - 2q^{-4} + 4q^{-6} + 2q^{-8} - q^{-10} + q^{-12}) + \right. \]
\[ + (-q^2 + 2q^0 - 4q^8 - 4q^8 + 4q^2 - 12 - 10q^{-10} + 3q^6 + q^6 - 9q^4 - 5q^{-2} - q^{-8}) A^2 + \right. \]
\[ + (q^4 - 2q^2 - 2q^0 - 3q^6 - 3q^{12} + 3q^{10} + q^8 - 2q^6 + 2q^4 + q^2 - 1 + q^{-2}) A^4 \right) = \]
\[ = A^{-4} q^{-8} \left( \cdot S_{[6]} q^{12} + \cdot S_{[5,1]} (-q^{24} + 2q^{20} - 4q^{16} + 5q^{12} - 5q^8 + 5q^4 - 4 + 2q^{-4} - q^{-8}) + \right. \]
\[ + \cdot S_{[4,5]} q^{-28} - 2q^{-28} + 6q^{24} - 7q^{22} - 5q^{20} + 18q^{18} - 9q^{16} - 18q^{14} + 27q^{12} - q^{10} - 29q^8 + \right. \]
\[ + 24q^6 + 8q^4 - 28q^2 + 17 + 7q^{-2} - 18q^{-4} + 10q^{-6} + 3q^{-8} - 8q^{-10} + 4q^{-12} + q^{-14} - 2q^{-16} + q^{-18} + \right. \]
\[ + \cdot S_{[1,1,1,1]} q^4 + \cdot S_{[3,3,3]} q^4 + \cdot S_{[3,2,1]} (-q^0 + 2q^2 - 4q^2 + 5q^4 + 5q^2 + 5 - 4q^2 + 2q^{-4} - q^{-6}) + \cdot S_{[2,2,2]} \right) \]

**Alexander polynomials**
\[ \mathcal{A}_{[1]} = q^6 - 3q^4 + 6q^2 - 7 + 6q^{-2} - 3q^{-4} + q^{-6} \]
\[ \mathcal{A}_{[1,1]} = \mathcal{A}_{[2]} = \mathcal{A}_{[1]} (q^2) = q^{12} - 3q^8 + 6q^4 - 7 + 6q^{-4} - 3q^{-8} + q^{-12} \]

**Jones polynomials**
\[ J_{[1]} = -q^4 + 2q^2 - 3 + 5q^{-2} - 4q^{-4} + 5q^{-6} - 4q^{-8} + 2q^{-10} - q^{-12} \]
\[ J_{[1,1]} = 1 \]
\[ J_{[2]} = q^{14} - 2q^{12} - q^{10} + 6q^8 - 5q^6 - 6q^4 + 14q^2 - 5 - 14q^{-2} - 21q^{-4} - 4q^{-6} - 21q^{-8} - 21q^{-10} + \]
\[ + 2q^{-12} - 24q^{-14} + 20q^{-16} + 3q^{-18} - 19q^{-20} + 12q^{-22} + 3q^{-24} - 9q^{-26} + 4q^{-28} + q^{-30} - 2q^{-32} + q^{-34} \]

**Special polynomials**
\[ \mathcal{S}_{[1]} = -3A^{-4} + 6A^{-2} - 2 \]
\[ \mathcal{S}_{[1,1]} = \mathcal{S}_{[2]} = (\mathcal{S}_{[1]})^2 = 9A^{-8} - 36A^{-6} + 48A^{-4} - 24A^{-2} + 4 \]
Ooguri-Vafa polynomials

\[ f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left( (q^{17} - q^{15} + 2q^{13} + 2q^{11} - q^9 + 3q^7 + 6q^5 - 3q^3 + 6q^1 + 2q^{-3} - q^{-5} + q^{-7})A^{-6} + 
\right. \\
\left. +(-q^{13} - q^9 - 5q^7 + 3q^5 - 5q^3 - 4q^{-1} + 5q^{-3} - 4q^{-5} + q^{-7} + q^{-9})A^{-4} + 
\right. \\
\left. + (2q^1 - 4q^{-1} + 3q^{-3} - 2q^{-7} + q^{-11} - q^{-13})A^{-2} \right) \\
\]

\[ f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left( (-q^7 + 2q^3 - 6q^{-1} + 3q^{-3} - 6q^{-5} - 3q^{-7} + q^{-9} - 2q^{-11} - 2q^{-13} + q^{-15} - q^{-17})A^{-6} + 
\right. \\
\left. +(-q^9 - q^7 + 4q^5 - 5q^3 + 4q^1 + 5q^{-3} - 3q^{-5} + 5q^{-7} + q^{-9} + q^{-13})A^{-4} + (q^{13} - q^{11} + 2q^7 - 3q^3 + 4q^1 - 2q^{-1})A^{-2} \right) \\
\] 

Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1,1]} = \frac{6(A^4 + 10A^2 - 17)(A - A^{-1})^3}{A^6} \]

Numbers \( N_{R,n,k} \)

\[
\begin{array}{c|cccc|cccc}
\hline
k \backslash n & -10 & -8 & -6 & -4 & -2 & 0 & 2 \\
\hline
0 & 32  & -155 & 307 & -318 & 182 & -55 & 7  \\
1 & 138 & -665 & 1281 & -1283 & 731 & -240 & 38 \\
2 & 272 & -1316 & 2431 & -2294 & 1281 & -456 & 82 \\
3 & 296 & -1502 & 2680 & -2338 & 1250 & -472 & 86 \\
4 & 187 & -1055 & 1849 & -1468 & 719 & -277 & 45 \\
5 & 68 & -460 & 808 & -576 & 239 & -90 & 11 \\
6 & 13 & -120 & 216 & -137 & 42 & -15 & 1 \\
7 & 1 & -17 & 32 & -18 & 3 & -1 & 0 \\
8 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
\hline
\end{array}
\]
\[
N_{[2]}:
\begin{array}{cccccccc}
K \backslash n & -10 & -8 & -6 & -4 & -2 & 0 & 2 \\
\hline
0 & 49 & -233 & 448 & -442 & 233 & -61 & 6 \\
1 & 246 & -1162 & 2150 & -1979 & 949 & -227 & 23 \\
2 & 576 & -2731 & 4838 & -4065 & 1697 & -352 & 37 \\
3 & 769 & -3782 & 6511 & -4967 & 1728 & -283 & 28 \\
4 & 624 & -3322 & 5649 & -3933 & 1092 & -120 & 9 \\
5 & 312 & -1884 & 3225 & -2069 & 440 & -25 & 1 \\
6 & 93 & -685 & 1198 & -715 & 111 & -2 & 0 \\
7 & 15 & -153 & 277 & -155 & 16 & 0 & 0 \\
8 & 1 & -19 & 36 & -19 & 1 & 0 & 0 \\
9 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
\end{array}
\]

\textbf{Knot 8}_{16}

(1, -1|1, -2|1, -2)

\textbf{HOMFLY polynomials}

\[
\frac{H_{[1]}}{\ast S_{[1]}} = A^2 \left( (-q^4 + 2q^2 - 2 + 2q^{-2} - q^{-4})A^{-2} + (q^6 - 2q^4 + 4q^{-2} - 4 + 4q^{-4} - 2q^{-6} + q^{-8}) \right)
\]

\[
= A^2 \left( \ast S_{[3]} q^{-2} + \ast S_{[2,1]} (-q^8 + 3q^6 - 5q^4 + 6q^2 - 7 + 6q^{-2} - 5q^{-4} + 3q^{-6} - q^{-8}) + \ast S_{[1,1,1]} q^2 \right)
\]

\[
\frac{H_{[1]}}{\ast S_{[1]}} = A^4 q^{-8} \left( (q^{22} - 2q^{20} - 2q^{18} + 5q^{16} - q^{14} - 6q^{12} + 5q^{10} + 2q^8 - 5q^6 + 3q^4 + q^2 - 2 + q^{-2})A^{-4} + \right)
\]

\[
+ (q^{24} + q^{22} + 4q^{20} - 5q^{18} - 4q^{16} + 13q^{14} + q^{12} - 14q^{10} + 
+ 10q^8 + 6q^6 - 12q^4 + 4q^2 + 4 - 6q^{-2} + q^{-4} + q^{-6} + q^{-8})A^{-2} + 
+ (q^{24} - 2q^{22} + 6q^{18} - 8q^{16} - 7q^{14} + 19q^{12} - 4q^{10} - 21q^8 + 
+ 18q^6 + 8q^4 - 18q^2 + 9 + 6q^{-2} - 8q^{-4} + 4q^{-6} + 2q^{-8} - 2q^{-10} + q^{-12}) + 
+ (-q^{20} + q^{18} + 3q^{16} - 6q^{14} - 2q^{12} + 14q^{10} - 6q^8 - 16q^6 + 
+ 16q^4 + 4q^2 - 16 + 6q^{-2} + 4q^{-4} - 6q^{-6} + q^{-8} + q^{-10} - q^{-12})A^2 + 
+ (q^{14} - 2q^{12} - q^{10} + 6q^8 - 3q^6 - 7q^4 + 9q^2 + 1 - 7q^{-2} + 4q^{-4} + q^{-6} - 2q^{-8} + q^{-10})A^4 \right)
\]

\[
= A^4 q^{-8} \left( \ast S_{[3,3]} + \ast S_{[3,2,1]} (-q^{10} + 3q^8 - 5q^6 + 6q^4 - 7q^2 + 6 - 5q^{-2} + 3q^{-4} - q^{-6}) + \ast S_{[3,1,1,1]} q^4 + \ast S_{[2,2,2]} q^4 + 
+ \ast S_{[2,2,1,1]} q^{30} - 3q^{28} + 10q^{24} - 11q^{22} - 9q^{20} + 20q^{18} - 13q^{16} + 30q^{14} + 43q^{12} - q^{10} - 47q^8 + 
+ 40q^6 + 12q^4 - 46q^2 + 28 + 13q^{-2} - 31q^{-4} + 16q^{-6} + 7q^{-8} - 14q^{-10} + 6q^{-12} + 2q^{-14} - 3q^{-16} + q^{-18}) + 
+ \ast S_{[2,1,1,1,1]} (-q^{24} + 3q^{20} - 5q^{16} + 6q^{12} - 7q^8 + 6q^4 - 5 + 3q^{-4} - q^{-8}) + \ast S_{[1,1,1,1,1,1]} q^{12} \right)
\]
\[
\frac{H_{[2]}}{S_{[2]}} = A^4q^8\left((q^2 - 2 + q^{-2} + 3q^{-4} - 5q^{-6} + 2q^{-8} + 5q^{-10} - 6q^{-12} - q^{-14} + 5q^{-16} - 2q^{-18} - 2q^{-20} + q^{-22})A^{-4} +
\right.
\]
\[
+(-q^8 + q^6 - 6q^2 + 4 + 4q^{-2} - 12q^{-4} + 6q^{-6} + 10q^{-8} - 14q^{-10} -
\right)
\]
\[
-q^{-12} - 13q^{-14} - 4q^{-16} - 5q^{-18} + 4q^{-20} + q^{-22} - q^{-24})A^{-2} +
\right)
\]
\[
+(q^{12} - 2q^{10} + 2q^8 + 4q^6 - 8q^4 + 6q^2 + 9 - 18q^{-2} + 8q^{-4} + 18q^{-6} -
\right)
\]
\[
-21q^{-8} - 4q^{-10} + 19q^{-12} - 7q^{-14} - 8q^{-16} + 6q^{-18} - 2q^{-22} + q^{-24} +
\right)
\]
\[
+(-q^{-12} + q^{10} - 6q^6 + 4q^4 + 6q^2 - 16 + 4q^{-2} + 16q^{-4} - 16q^{-6} -
\right)
\]
\[
-6q^{-8} + 14q^{-10} - 2q^{-12} - 6q^{-14} + 3q^{-16} + q^{-18} - q^{-20})A^2 +
\right)
\]
\[
+(q^{10} - 2q^8 + q^6 + 4q^4 - 7q^2 + 1 + 9q^{-2} - 7q^{-4} - 3q^{-6} + 6q^{-8} - q^{-10} - 2q^{-12} + q^{-14})A^4\right)
\]
\[
= \frac{A^4q^8}{*S_{[2]}} \left( \ast S_{[6]}q^{-12} + \ast S_{[5,1]}(-q^8 + 3q^4 - 5 + 6q^{-4} - 7q^{-8} + 6q^{-12} - 5q^{-16} + 3q^{-20} - q^{-24}) +
\right)
\]
\[
+ \ast S_{[4,2]}(q^{18} - 3q^{16} + 2q^{14} + 6q^{12} - 14q^{10} + 7q^8 + 16q^6 - 31q^4 + 13q^2 + 28 - 46q^{-2} + 12q^{-4} +
\right)
\]
\[
+ 40q^{-6} - 47q^{-8} - q^{-10} + 43q^{-12} - 30q^{-14} - 13q^{-16} + 29q^{-18} - 9q^{-20} - 11q^{-22} + 10q^{-24} - 3q^{-28} + q^{-30}) +
\right)
\]
\[
+ \ast S_{[4,1,1]}q^{-4} + \ast S_{[3,3]}q^{-4} + \ast S_{[3,2,1]}(-q^6 + 3q^4 - 5q^2 + 6 - 7q^{-2} + 6q^{-4} - 5q^{-6} + 3q^{-8} - q^{-10} + \ast S_{[2,2,2]})\right)
\]
\]
### Numbers $N_{R,n,k}$

| $k \backslash n$ | $-1$ | $1$ | $3$ | $5$ |
|-----------------|------|------|------|------|
| $N_{[1]}$       |      |      |      |      |
| $0$             | 0    | -2   | 3    | -1   |
| $1$             | 2    | -7   | 7    | -2   |
| $2$             | 1    | -5   | 5    | -1   |
| $3$             | 0    | -1   | 1    | 0    |

| $k \backslash n$ | -2  | 0   | 2   | 4   | 6   | 8   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|
| $N_{[1]}$       |     |     |     |     |     |     |     |
| $0$             | -3  | -1  | 44  | -106| 109 | -53 | 10  |
| $1$             | -4  | -43 | 305 | -685| 721 | -368| 74  |
| $2$             | 11  | -145| 779 | -1824| 2075| -1135| 239 |
| $3$             | 19  | -178| 1012| -2710| 3408| -1958| 407 |
| $4$             | 8   | -96 | 757 | -2510| 3493| -2052| 400 |

| $k \backslash n$ | -2  | 0   | 2   | 4   | 6   | 8   | 10  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|
| $N_{[2]}$       |     |     |     |     |     |     |     |
| $0$             | -6  | 15  | 7   | -58 | 72  | -37 | 7   |
| $1$             | -14 | 20  | 126 | -395| 447 | -229| 45  |
| $2$             | 10  | -87 | 441 | -1013| 1124| -598| 123 |
| $3$             | 43  | -219| 659 | -1337| 1530| -848| 172 |
| $4$             | 34  | -189| 499 | -1013| 1247| -710| 132 |
| $5$             | 10  | -75 | 198 | -456 | 624 | -357| 56  |
| $6$             | 1   | -14 | 39  | -120| 186 | -104| 12  |
| $7$             | 0   | -1  | 3   | -17 | 30  | -16 | 1   |
| $8$             | 0   | 0   | 0   | -1  | 2   | -1  | 0   |

**Knot $8_{17}$**

$(2, -1|1, -1|1, -2)$
HOMFLY polynomials

\[
\frac{H_{[1]}}{S_{[1]}} = (q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4})A^{-2} + \\
+(-q^6 + 2q^4 - 4q^2 + 5 - 4q^{-2} + 2q^{-4} - q^{-6}) + (q^4 - 2q^2 + 3 - 2q^{-2} + q^{-4})A^2 = 
\]

\[
\frac{H_{[1,1]}}{S_{[1,1]}} = (q^{16} - 2q^{14} + 6q^{10} - 5q^8 - 5q^6 + 10q^4 - 2q^2 - 6 + 5q^{-2} - 2q^{-6} + q^{-8})A^{-4} + \\
+(-q^{18} + 2q^{16} - 7q^{12} + q^{10} + 13q^8 - 13q^6 - 10q^4 + 22q^2 - \\
-4 - 15q^{-2} + 11q^{-4} + 2q^{-6} - 6q^{-8} + 2q^{-10} + q^{-12} - q^{-14})A^{-2} + \\
+(q^{18} - 2q^{16} + q^{14} + 5q^{12} - 9q^{10} + q^8 + 18q^6 - 19q^4 - 10q^2 + 31 - \\
-10q^{-2} - 19q^{-4} + 18q^{-6} + q^{-8} - 9q^{-10} + 5q^{-12} + q^{-14} - 2q^{-16} + q^{-18}+) + \\
+(-q^{14} + q^{12} + 2q^{10} - 6q^8 + 2q^6 + 11q^4 - 15q^2 - 4 + 22q^{-2} - \\
-10q^{-4} - 13q^{-6} + 13q^{-8} + q^{-10} - 7q^{-12} + 2q^{-14} + q^{-16} - q^{-18})A^{-2} + \\
+(q^{18} - 2q^{16} + q^{14} + 5q^{12} - 9q^{10} + q^8 + 18q^6 - 19q^4 - 10q^2 + 31 - \\
-10q^{-2} - 19q^{-4} + 18q^{-6} + q^{-8} - 9q^{-10} + 5q^{-12} + q^{-14} - 2q^{-16} + q^{-18}+) + \\
+(-q^{18} + q^{16} + 2q^{14} - 7q^{12} + q^{10} + 13q^8 - 13q^6 - 10q^4 - 2q^2 - 6 + 5q^{-2} - 2q^{-6} + q^{-8})A^4 = 
\]

\[
\frac{H_{[2]}}{S_{[2]}} = (q^8 - 2q^6 + 5q^4 - 6q^2 + 2q^0 + 11q^4 - 15q^2 - 4 + 22q^{-2} - \\
-10q^{-4} - 13q^{-6} + 13q^{-8} + q^{-10} - 7q^{-12} + 2q^{-14} + q^{-16} - q^{-18})A^{-2} + \\
+(q^{18} - 2q^{16} + q^{14} + 5q^{12} - 9q^{10} + q^8 + 18q^6 - 19q^4 - 10q^2 + 31 - \\
-10q^{-2} - 19q^{-4} + 18q^{-6} + q^{-8} - 9q^{-10} + 5q^{-12} + q^{-14} - 2q^{-16} + q^{-18}+) + \\
+(-q^{18} + q^{16} + 2q^{14} - 7q^{12} + q^{10} + 13q^8 - 13q^6 - 10q^4 - 2q^2 - 6 + 5q^{-2} - 2q^{-6} + q^{-8})A^4 = 
\]

Alexander polynomials

\[
\mathcal{A}_{[1]} = -q^6 + 4q^4 - 8q^2 + 11 - 8q^{-2} + 4q^{-4} - q^{-6} 
\]

\[
\mathcal{A}_{[1,1]} = \mathcal{A}_{[2]} = \mathcal{A}_{[1]}(q^2) = -q^{12} + 4q^8 - 8q^4 + 11 - 8q^{-4} + 4q^{-8} - q^{-12} 
\]

Jones polynomials

\[
J_{[1]} = q^8 - 3q^6 + 5q^4 - 6q^2 + 7 - 6q^{-2} + 5q^{-4} - 3q^{-6} + q^{-8} 
\]

\[
J_{[1,1]} = 1 
\]

\[
J_{[2]} = q^{24} - 3q^{22} + q^{20} + 9q^{18} - 14q^{16} - 3q^{14} + 28q^{12} - 25q^{10} - 14q^8 + 47q^6 - 29q^4 - 25q^2 + \\
+ 55 - 25q^{-2} - 29q^{-4} + 47q^{-6} - 14q^{-8} - 25q^{-10} + 28q^{-12} - 3q^{-14} - 14q^{-16} + 9q^{-18} + q^{-20} - 3q^{-22} + q^{-24} 
\]
Special polynomials

\[ H_{[1]} = A - 2 - 1 + A^2 \]  
\[ H_{[1,1]} = H_{[2]} = (H_{[1]})^2 = A - 4 - 2A^{-2} + 3 - 2A^2 + A^4 \]

Ooguri-Vafa polynomials

\[ f_{[1,1]} = \frac{\{A\} \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left( (q^{15} - q^{13} - q^{11} + 4q^9 - 3q^5 + 4q^4 - 2q^{-1})A^{-2} + (-q^{11} + q^9 + 2q^7 - 8q^3 + 8q^1 - 2q^{-3} - q^{-5} + q^{-7}) + (2q^5 - 4q^3 + 3q^{-1} - 4q^{-5} + q^{-7} + q^{-9} - q^{-11})A^2 \right) \]

\[ f_{[2]} = \frac{\{A\} \{A/q\} \{Aq\} (q^2 - 1 + q^{-2})}{\{q\}} \left( (2q^5 - 4q^3 + 3q^{-1} - 4q^{-5} + q^{-7} + q^{-9} - q^{-11})A^{-2} + (-q^{11} + q^9 + 2q^7 - 8q^3 + 8q^1 - 2q^{-3} - q^{-5} + q^{-7}) + (q^{15} - q^{13} - q^{11} + 4q^9 - 3q^5 + 4q^4 - 2q^{-1})A^2 \right) \]

Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1,1]} = 2(A^2 - A^{-2})(A - A^{-1})^3 \]

Numbers \( N_{R,n,k} \)

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( k \setminus n \) & -3 & -1 & 1 & 3 \\
\hline
0 & -1 & 2 & -2 & 1 \\
1 & -2 & 7 & -7 & 2 \\
2 & -1 & 5 & -5 & 1 \\
3 & 0 & 1 & -1 & 0 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( k \setminus n \) & -6 & -4 & -2 & 0 & 2 & 4 & 6 \\
\hline
0 & 6 & -24 & 44 & -56 & 54 & -32 & 8 \\
1 & 29 & -128 & 256 & -350 & 345 & -198 & 46 \\
2 & 56 & -277 & 607 & -899 & 916 & -514 & 111 \\
3 & 60 & -322 & 753 & -1223 & 1328 & -749 & 153 \\
4 & 36 & -213 & 522 & -959 & 1148 & -658 & 124 \\
5 & 10 & -77 & 200 & -444 & 601 & -345 & 55 \\
6 & 1 & -14 & 39 & -119 & 184 & -103 & 12 \\
7 & 0 & -1 & 3 & -17 & 30 & -16 & 1 \\
8 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( k \setminus n \) & -6 & -4 & -2 & 0 & 2 & 4 & 6 \\
\hline
0 & 8 & -32 & 54 & -56 & 44 & -24 & 6 \\
1 & 46 & -198 & 345 & -350 & 256 & -128 & 29 \\
2 & 111 & -514 & 916 & -899 & 607 & -277 & 56 \\
3 & 153 & -749 & 1328 & -1223 & 753 & -322 & 60 \\
4 & 124 & -658 & 1148 & -959 & 522 & -213 & 36 \\
5 & 55 & -345 & 601 & -444 & 200 & -77 & 10 \\
6 & 12 & -103 & 184 & -119 & 39 & -14 & 1 \\
7 & 1 & -16 & 30 & -17 & 3 & -1 & 0 \\
8 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
\hline
\end{tabular}

**Knot 8\_18**

\((1, -1|1, -1|1, -1|1, -1)\)
HOMFLY polynomials

$$\frac{H_{[1]}}{S_{[1]}} = (q^4 - 3q^2 + 3 - 3q^{-2} + q^{-4})A^{-2} + \ldots$$

(285)

$$= \frac{1}{S_{[1]}} \left( ^{*}S_{[3]} + \ ^{*}S_{[2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + \ ^{*}S_{[1,1,1]} \right)$$

$$\frac{H_{[1]}}{S_{[1,1,1]}} = (q^{16} - 3q^{14} + 9q^{10} - 8q^8 - 7q^6 + 15q^4 - 3q^2 - 9 + 8q^{-2} - 3q^{-6} + q^{-8})A^{-4} + \ldots$$

(286)

$$= \frac{1}{S_{[1,1,1]}} \left( ^{*}S_{[3,3]} + \ ^{*}S_{[3,2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + \ ^{*}S_{[1,1,1,1]} + \ ^{*}S_{[2,2,2]} + \ ^{*}S_{[2,1,2]}(q^{24} - 4q^{22} + 2q^{20} + 12q^{18} - 21q^{16} + 40q^{12} - 44q^{10} - 15q^{8} + 76q^{6} - 54q^{4} - 40q^{2} + 94 - 40q^{-2} - 54q^{-4} + 76q^{-6} - 15q^{-8} - 44q^{-10} + 21q^{-16} - 12q^{-18} + 2q^{-20} - 4q^{-22} + q^{-24}) + \ ^{*}S_{[1,2,1,1]}(q^{16} - 4q^{14} + 6q^{8} - 8q^{4} + 9 - 8q^{-4} + 6q^{-8} - 4q^{-12} + q^{-16}) + \ ^{*}S_{[1,1,1,1,1,1]} \right)$$

$$H_{[2]} = (q^8 - 3q^6 + 8q^4 - 9 - 3q^{-2} + 15q^{-4} - 7q^{-6} - 8q^{-8} + 9q^{-10} - 3q^{-14} - q^{-16})A^{-4} + \ldots$$

(287)

$$= \frac{1}{S_{[2]}} \left( ^{*}S_{[6]} + \ ^{*}S_{[5,1]}(q^{16} - 4q^{12} + 6q^{8} - 8q^{4} + 9 - 8q^{-4} + 6q^{-8} - 4q^{-12} + q^{-16}) + \ ^{*}S_{[1,2]}(q^{24} - 4q^{22} + 2q^{20} + 12q^{18} - 21q^{16} + 40q^{12} - 44q^{10} - 15q^{8} + 76q^{6} - 54q^{4} - 40q^{2} + 94 - 40q^{-2} - 54q^{-4} + 76q^{-6} - 15q^{-8} - 44q^{-10} + 21q^{-16} - 12q^{-18} + 2q^{-20} - 4q^{-22} + q^{-24}) + \ ^{*}S_{[1,1,1]} + \ ^{*}S_{[3,3]} + \ ^{*}S_{[3,2,1]}(q^8 - 4q^6 + 6q^4 - 8q^2 + 9 - 8q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}) + \ ^{*}S_{[2,2,2]} \right)$$

Alexander polynomials

$$A_{[1]} = -q^6 + 5q^4 - 10q^2 + 13 - 10q^{-2} + 5q^{-4} - q^{-6}$$

(288)

$$A_{[1,1]} = A_{[2]} = A_{[1]}(q^2) = -q^{12} + 5q^8 - 10q^4 + 13 - 10q^{-4} + 5q^{-8} - q^{-12}$$

Jones polynomials

$$J_{[1]} = q^8 - 4q^6 + 6q^4 - 7q^2 + 9 - 7q^{-2} + 6q^{-4} - 4q^{-6} + q^{-8}$$

(289)

$$J_{[1,1]} = 1$$

(290)

$$J_{[2]} = q^{24} - 4q^{22} + 2q^{20} + 13q^{18} - 21q^{16} - 4q^{14} + 41q^{12} - 38q^{10} - 20q^8 + 69q^6 - 43q^4 - 36q^2 + +81 - 36q^{-2} - 43q^{-4} + 69q^{-6} - 20q^{-8} - 38q^{-10} + 41q^{-12} - 4q^{-14} - 21q^{-16} + 13q^{-18} - 2q^{-20} - 4q^{-22} + q^{-24}$$

(291)
Special polynomials

\[ \delta_{[1]} = -A^{-2} + 3 - A^2 \]  
\[ \delta_{[1, 1]} = \delta_{[2]} = (\delta_{[1]})^2 = A^{-4} - 6A^{-2} + 11 - 6A^2 + A^4 \]  

Ooguri-Vafa polynomials

\[ f_{[1, 1]} = \frac{1}{q} \left( \frac{A^2}{\{A\}} \{Aq\} \{q\} \left( q^{15} - 3q^{13} + q^{11} + 6q^9 - 8q^7 + 2q^5 + 4q^3 + 2q^1 - 9q^{-1} + 9q^{-3} - 3q^{-5} \right) + \)  
\[ + (-3q^{11} + 3q^9 - 2q^5 - 9q^3 + 23q^1 - 23q^{-1} + 9q^{-3} + 2q^{-5} - 3q^{-9} + q^{-11}) + \)  
\[ + (3q^5 - 9q^3 + 9q^1 - 2q^{-1} - 4q^{-3} - 2q^{-5} + 8q^{-7} - 6q^{-9} - q^{-11} + 3q^{-13} - q^{-15})A^2 \)  

\[ f_{[2]} = \frac{1}{q} \left( \frac{A^2}{\{A\}} \{Aq\} \{q\} \left( (3q^5 - 9q^3 + q^1 - 2q^{-1} - 4q^{-3} - 2q^{-5} + 8q^{-7} - 6q^{-9} - q^{-11} + 3q^{-13} - q^{-15})A^{-2} + \)  
\[ + (-q^{11} + 3q^9 - 2q^5 - 9q^3 + 23q^1 - 23q^{-1} - 9q^{-3} + 2q^{-5} - 3q^{-9} + q^{-11}) + \)  
\[ + (q^{15} - 3q^{13} + q^{11} + 6q^9 - 8q^7 + 2q^5 + 4q^3 + 2q^1 - 9q^{-1} + 9q^{-3} - 3q^{-5})A^2 \)  

Special Ooguri-Vafa polynomials

\[ f_{[2]} = -f_{[1, 1]} = 2(A^2 - A^{-2})(A - A^{-1})^3 \]  

Numbers \( N_{R,n,k} \)

| k\n | -3 | -1 | 1 | 3 |
|---|---|---|---|---|
| 0 | 1 | -4 | 4 | -1 |
| 1 | -1 | 2 | -2 | 1 |
| 2 | -1 | 4 | -4 | 1 |
| 3 | 0 | 1 | -1 | 0 |

| k\n | -6 | -4 | -2 | 0 | 2 | 4 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 2 | -6 | 8 | -12 | 18 | -14 | 4 |
| 1 | 4 | -2 | -32 | 54 | -18 | -12 | 6 |
| 2 | 10 | -42 | 54 | -46 | 58 | -44 | 10 |
| 3 | 26 | -131 | 288 | -406 | 388 | -211 | 46 |
| 4 | 26 | -138 | 328 | -542 | 593 | -338 | 71 |
| 5 | 9 | -63 | 161 | -327 | 421 | -244 | 43 |
| 6 | 1 | -13 | 36 | -102 | 154 | -87 | 11 |
| 7 | 0 | -1 | 3 | -16 | 28 | -15 | 1 |
| 8 | 0 | 0 | 0 | -1 | 2 | -1 | 0 |

| k\n | -6 | -4 | -2 | 0 | 2 | 4 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 4 | -14 | 18 | -12 | 8 | -6 | 2 |
| 1 | 6 | -12 | -18 | 54 | -32 | -2 | 4 |
| 2 | 10 | -44 | 58 | -46 | 54 | -42 | 10 |
| 3 | 46 | -211 | 388 | -406 | 288 | -131 | 26 |
| 4 | 71 | -338 | 593 | -542 | 328 | -138 | 26 |
| 5 | 43 | -244 | 421 | -327 | 161 | -63 | 9 |
| 6 | 11 | -87 | 154 | -102 | 36 | -13 | 1 |
| 7 | 1 | -15 | 28 | -16 | 3 | -1 | 0 |
| 8 | 0 | -1 | 2 | -1 | 0 | 0 | 0 |

49
Knot $8_{19}$

$(1,3|1,3) = (1,1|1,1,1,1,1,1)$

**HOMFLY polynomials**

$$\frac{H_{[3]}}{S_{[1]}} = A^{-8} \left( A^{-2} + (-q^4 - q^2 - 1 - q^{-2} - q^{-4}) + (q^6 + q^2 + 1 + q^{-2} + q^{-6}) A^2 \right) =$$

$$= \frac{A^{-8}}{S_{[1]}} \left( *S_{[3]} q^8 + *S_{[2,1]} + *S_{[1,1,1]} A^{-8} \right)$$

(297)

$$\frac{H_{[1,1]}}{S_{[1,1]}} = A^{-16} q^{32} \left( q^{-16} A^{-4} + (-q^{12} - 2q^{-14} - q^{-16} - q^{-18} - 2q^{-20} - q^{-22} - q^{-24} - q^{-26}) A^{-2} + \right.$$

$$+ (2q^{-10} + 2q^{-12} + 3q^{-14} + 3q^{-16} + 5q^{-18} + 4q^{-20} + 4q^{-22} + 3q^{-24} + 3q^{-26} + 2q^{-28} + 2q^{-30} + q^{-32} + q^{-34}) +$$

$$\left. + (-q^{-8} - 2q^{-10} - 2q^{-12} - 3q^{-14} - 5q^{-16} - 5q^{-18} - 4q^{-20} - 5q^{-22} - 5q^{-24} - \right.$$

$$- 4q^{-26} - 4q^{-28} - 3q^{-30} - 2q^{-32} - 2q^{-34} - q^{-36} - q^{-38} - q^{-40}) A^2 +$$

$$+(q^{-8} + q^{-12} + 2q^{-14} + 2q^{-16} - q^{-18} + 3q^{-20} + 2q^{-22} + 2q^{-24} + 2q^{-26} +$$

$$+ 2q^{-28} + q^{-30} + 2q^{-32} + q^{-34} + q^{-36} + q^{-38} + q^{-44}) A^4 \right) =$$

$$= \frac{A^{-16} q^{32}}{S_{[1,1]}} \left( *S_{[3,1]} q^{8} + *S_{[3,2,1]} q^{16} + *S_{[2,1,1,1]} q^{16} - *S_{[2,1,1,1,1]} q^{32} + *S_{[1,1,1,1,1,1]} q^{48} \right)$$

(298)

$$\frac{H_{[2]}}{S_{[2]}} = A^{-16} q^{32} \left( q^{16} A^{-4} + (-q^{26} - q^{24} - 2q^{20} - q^{18} - q^{16} - 2q^{14} - q^{12}) A^{-2} + \right.$$

$$+ (q^{34} + q^{32} + 2q^{30} + 2q^{28} + 2q^{26} + 2q^{24} + 4q^{22} + 4q^{20} + 5q^{18} + 3q^{16} + 3q^{14} + 2q^{12} + 2q^{10}) +$$

$$\left. + (-q^{40} - q^{38} - q^{36} - q^{34} - 2q^{32} - q^{30} - 3q^{28} - 4q^{26} - 5q^{24} - 5q^{22} - 4q^{20} - 5q^{18} - 5q^{16} - 3q^{14} - 2q^{12} - 2q^{10} - q^{8}) A^2 + \right.$$

$$\left. + (q^{44} + q^{36} + q^{34} + 2q^{32} + 2q^{30} + 2q^{28} + 2q^{26} + 2q^{24} + 2q^{22} + 2q^{20} + q^{18} + 2q^{16} + 2q^{14} + 2q^{12} + 2q^{10} A^4 \right) =$$

$$= \frac{A^{-16} q^{32}}{S_{[2]}} \left( *S_{[3]} q^{48} - *S_{[3,1]} q^{22} + *S_{[3,2,1]} q^{16} + *S_{[3,1,1,1]} q^{16} - *S_{[2,1,1,1,1]} q^{32} + *S_{[1,1,1,1,1,1]} q^{48} \right)$$

(299)

**Alexander polynomials**

$$A_{[1]} = q^6 - q^4 + 1 - q^{-4} + q^{-6}$$

(300)

$$A_{[1,1]} = A_{[2]} = A_{[3]}(q^2) = q^{12} - q^8 + 1 - q^{-8} + q^{-12}$$

(301)

**Jones polynomials**

$$J_{[1]} = q^{-6} + q^{-10} - q^{-16}$$

$$J_{[1,1]} = 1$$

(302)

$$J_{[2]} = q^{-12} + q^{-18} + q^{-24} - q^{-26} - q^{-32} - q^{-38} + q^{-40} - q^{-44} + q^{-46}$$

(303)

(304)

**Special polynomials**

$$\delta_{[1]} = A^{-10} - 5A^{-8} + 5A^{-6}$$

(305)

$$\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]}^2) = A^{-20} - 10A^{-18} + 35A^{-16} - 50A^{-14} + 25A^{-12}$$

(306)

**Ooguri-Vafa polynomials**

$$f_{[1,1]} = \left\{ A^2 \{ A/q \} \{ A/q \} \{ q^2 + q^{-2} \} \left( (q^{13} + q^5) A^{-18} + (-q^{17} - q^{15} - q^{11} - 3q^9 - 2q^7 - q^5 - 2q^3 - 2q^1 - q^{-1}) A^{-16} + \right.$$

$$\left. + (q^{19} + q^{17} + 2q^{15} + q^9 + 3q^7 + 3q^5 + q^3 + q^1 + 2q^{-1} + q^{-3} + q^{-5}) A^{-14} \right) \right\}$$

$$\frac{f_{[2]}}{q} = \left\{ A^2 \{ A/q \} \{ A/q \} \{ q^2 + q^{-2} \} \left( (-q^{-5} - q^{-13}) A^{-18} + \right.$$

$$\left. + (q + 2q^{-1} + 2q^{-3} + q^{-5} + 2q^{-7} + 3q^{-9} + q^{-11} + q^{-15} + q^{-17}) A^{-16} + \right.$$\n
$$\left. + (-q^5 - q^3 - q^1 - q^{-3} - q^{-5} - 3q^{-7} - q^{-9} - 3q^{-11} - 2q^{-13} - q^{-15} - q^{-19}) A^{-14} \right) \right\}$$

(307)

(308)
Special Ooguri-Vafa polynomials

\[ f[2] = -f[1,1] = -\frac{4(11A^4 - 7A^2 + 1)(A - A^{-1})^3}{A^{18}} \quad (309) \]

Numbers \( N_{R,n,k} \)

| \( k \backslash n \) | -22 | -20 | -18 | -16 | -14 | -12 | -10 |
|---------------------|-----|-----|-----|-----|-----|-----|-----|
| 0                   | 16  | -146| 540 | -1020| 1040 | -546| 116 |
| 1                   | 80  | -775| 3060| -6090| 6410 | -3395| 710 |
| 2                   | 148 | -1709| 7503| -15954| 17363| -9219| 1868 |
| 3                   | 128 | -2001| 10254| -23804| 26999| -14365| 2789 |
| 4                   | 56  | -1365| 8580| -22281| 26597| -14182| 2595 |
| 5                   | 12  | -560| 4570| -13657| 17333| -9247| 1549 |
| 6                   | 1   | -136| 1556| -5557| 7575| -4029| 590 |
| 7                   | 0   | -18 | 328| -1485| 2196| -1159| 138 |
| 8                   | 0   | -1 | 39| -250| 405| -211| 18 |
| 9                   | 0   | 0 | 2| -24| 43| -22| 1 |
| 10                  | 0   | 0 | 0| -1| 2| -1| 0 |

\( N_{[1]} : \)

| \( k \backslash n \) | -11 | -9 | -7 | -5 |
|---------------------|-----|-----|-----|-----|
| 0                   | -1  | 6  | -10| 5   |
| 1                   | 0   | 5  | -15| 10  |
| 2                   | 0   | 1  | -7 | 6   |
| 3                   | 0   | 0  | -1 | 1   |

\( N_{[1,1]} : \)
\[\begin{array}{cccccccc}
N_{(2)} : \\
K \setminus n &= -22 & -20 & -18 & -16 & -14 & -12 & -10 \\
0 &= 20 & -190 & 720 & -1380 & 1420 & -750 & 160 \\
1 &= 130 & -1245 & 4890 & -9710 & 10220 & -5425 & 1140 \\
2 &= 314 & -3403 & 14459 & -30206 & 32619 & -17325 & 3542 \\
3 &= 367 & -4996 & 24019 & -53906 & 60220 & -32020 & 6316 \\
4 &= 230 & -4367 & 24779 & -61035 & 71182 & -37920 & 7131 \\
5 &= 79 & -2380 & 16653 & -46053 & 56575 & -30177 & 5303 \\
6 &= 14 & -816 & 7422 & -23680 & 30910 & -16472 & 2622 \\
7 &= 1 & -171 & 2177 & -8320 & 11638 & -6175 & 850 \\
8 &= 0 & -20 & 404 & -3963 & 32967 & -1561 & 173 \\
9 &= 0 & 1 & 43 & -297 & 489 & -254 & 20 \\
10 &= 0 & 0 & 2 & -26 & 47 & -24 & 1 \\
11 &= 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
\end{array}\]

**Knot 8\(_{20}\)**

\((-1, -3] \to 1, 3)\)

**HOMFLY polynomials**

\[
\frac{H_{[1]}}{S_{[1]}} = A^2 \left( -q^2 + 1 - q^{-2} \right) A^{-2} + (q^4 + 2 + q^{-4}) + (-q^2 - q^{-2}) A^2 \\
= \frac{A^2}{S_{[1]}} \left( * S_{[3]} q^{-2} + * S_{[2,1]} (-q^6 + q^4 - q^2 + 1 - q^{-2} + q^{-4} - q^{-6}) + * S_{[1,1,1]} q^2 \right) \\
\]

\[
\frac{H_{[1]}}{S_{[1]}} = A^4 q^{-8} \left( -q^{12} + q^{10} + q^8 - q^6 + q^2 \right) A^{-4} + \\
+(-q^{12} - q^{10} + q^8 - q^6 - 3q^4 - q^2 - q^{-2} - q^{-4}) A^{-2} + (q^{16} + q^{12} + q^{10} + 2q^8 + 3q^6 + 3q^4 + q^2 + 3 + 3q^{-2} + q^{-4} + q^{-8}) + \\
+(-q^{12} - q^{10} - q^8 - 2q^6 - 3q^4 - 2q^2 - 3 - 2q^{-2} - q^{-6} - q^{-8}) A^2 + (q^6 + q^2 + 1 + q^{-6}) A^4 = \\
= \frac{A^4 q^{-8}}{S_{[1,1]}} \left( * S_{[3,3]} + * S_{[3,2,1]} (-q^8 + q^6 - q^4 + q^2 - 1 + q^{-2} - q^{-4}) + * S_{[1,1,1,1]} q^4 + * S_{[2,2,2]} q^4 + \\
+ S_{[2,2,1,1]} (q^{22} - q^{20} + q^{16} - q^{14} - q^2 + 1 + q^{-2} - 2q^{-4} + 2q^{-8} - q^{-10} - q^{-12} + q^{-14}) + \\
+ S_{[1,1,1,1,1]} (-q^{20} + q^{16} - q^{12} + q^8 - q^4 + 1 - q^{-4}) + * S_{[1,1,1,1,1,1]} q^{12} \right) 
\]

(310)
\[
\frac{H_{[2]}}{S_{[2]}} = A^4 q^8 \left( q^{-2} - q^{-6} + q^{-8} + q^{-10} - q^{-12} \right) A^{-4} + (-q^4 - q^{-2} - 3q^{-4} - q^{-6} + q^{-8} - q^{-10} - q^{-12}) A^{-2} + \\
\quad + (q^8 + q^4 + 3q^2 + 3 + q^{-2} + 3q^{-4} + 3q^{-6} + 2q^{-8} + q^{-10} + q^{-12} + q^{-16}) + \\
\quad + (-q^8 - q^6 - 2q^2 - 3 - 2q^{-2} - 2q^{-4} - 2q^{-6} - q^{-8} - q^{-10} - q^{-12}) A^2 + (q^8 + 1 + q^{-2} + q^{-6}) A^4 \right) = \\
= A^4 q^8 \left( *S_{[0]} q^{-12} + *S_{[7,1]} (-q^4 + 1 - q^{-4} + q^{-8} - q^{-12} + q^{-16} - q^{-20}) + \\
\quad + *S_{[1,2]} (q^{14} - q^{-12} - q^{-10} + 2q^8 - 2q^4 + q^2 + 1 - q^{-2} - q^{-14} + q^{-16} - q^{-20} + q^{-22}) + *S_{[4,1,1]} q^{-4} + *S_{[3,3]} q^{-4} + \\
\quad + *S_{[3,2,1]} (-q^4 + q^2 - 1 + q^{-2} - q^{-4} + q^{-6} - q^{-8}) + *S_{[2,2,2]} \right) \\
\]

**Alexander polynomials**

\[
\mathcal{A}_{[1]} = q^8 - 2q^4 + 3 - 2q^{-4} + q^{-8} \\
\mathcal{A}_{[1,1]} = \mathcal{A}_{[2]} = \mathcal{A}_{[1]}(q^2) = q^8 - 2q^4 + 3 - 2q^{-4} + q^{-8}
\]

**Jones polynomials**

\[
J_{[1]} = -q^{10} + q^8 - q^6 + 2q^4 - q^2 + 2 - q^{-2} \\
J_{[1,1]} = 1 \\
J_{[2]} = q^{30} - q^{28} - q^{26} + 2q^{24} - q^{22} - 2q^{20} + 2q^{18} - 2q^{14} + 2q^{12} + q^{10} - 2q^8 + q^6 + 2q^3 - 2q^2 + 1 + q^{-2} - q^{-4}
\]

**Special polynomials**

\[
\delta_{[1]} = -1 + 4A^2 - 2A^4 \\
\delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^2 = 1 - 8A^2 + 20A^4 - 16A^6 + 4A^8
\]

**Ooguri-Vafa polynomials**

\[
f_{[1,1]} = \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left( (-q^3 + q^1 - q^{-1}) A^2 + \\
\quad + (q^1 + q^{-1} + q^{-3} + q^{-5} + q^{-7} + q^{-9}) A^4 + (-q^3 - 2q^{-1} - q^{-3} - 2q^{-5} - q^{-7} - q^{-9} - q^{-13}) A^6 \right) \\
\]

\[
f_{[2]} = \frac{\{A\}^2 \{A/q\} \{Aq\}}{\{q\}} \left( (q^1 - q^{-1} + q^{-3}) A^2 + \\
\quad + (-q^9 - q^7 - q^5 - q^3 - q^{-1} - q^{-3}) A^4 + (q^{13} + q^9 + q^7 + 2q^5 + q^3 + 2q^1 + q^{-3}) A^6 \right)
\]

**Special Ooguri-Vafa polynomials**

\[
[f_{[2]}] = -f_{[1,1]} = A^4 (3A^2 - A^{-2})^2 (A - A^{-1})^3
\]

**Numbers** \( N_{R,n,k} \)

| \( k \backslash n \) | -1 | 1 | 3 | 5 |
|------------------------|----|----|----|----|
| 0                      | 1  | -5 | 6  | -2 |
| 1                      | 1  | -5 | 5  | -1 |
| 2                      | 0  | -1 | 1  | 0  |
**Knot 8_{21}**

\((-2, 2) - 1, -3\)

**HOMFLY polynomials**

\[
\frac{H_{[1], [2]}^{N_{[1,1]}}}{S_{[1], [1]}} = A^4 \left((2q^2 - 1 + 2q^{-2})A^{-2} + (-q^4 + q^2 - 3 + q^{-2} - q^{-4}) + (q^2 - 1 + q^{-2})A^2\right) =
\]

\[
= \frac{A^4}{S_{[1], [1]}} \left(\ast S_{[3], [1]} q^{-4} \ast S_{[2], [1]} (q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6}) + \ast S_{[1,1], [1]} q^4\right)
\]

\[
= A^8 q^{-16} \left((q^{22} + 3q^{20} - 2q^{18} + q^{14} - 3q^{10} + 3q^8)A^{-4} +
\right.
\]

\[
+(-q^{24} - 2q^{22} + q^{20} - q^{18} - 2q^{16} - 3q^{14} + 4q^{12} - 6q^{10} - 4q^8 + q^6 - q^4 - q^{-2})A^{-2} +
\]

\[
+(q^{22} + 3q^{18} + 3q^{16} - 3q^{14} + 3q^{12} + 8q^{10} - 3q^8 - q^6 + 4q^4 - 1 + q^{-2}) +
\]

\[
+(-q^{20} - 4q^{18} + 3q^8 - 3q^6 - 2q^4 + 2q^2 - q^{-2})A^2 + (q^{12} - q^{10} + 2q^6 - q^4 - q^2 + 1)A^4 = \]

\[
H_{[1], [2]}^{N_{[2]}} = A^8 q^{16} \left((3q^8 - q^{10} + 5q^{14} - 2q^{18} + 3q^{20} + q^{22})A^{-4} +
\right.
\]

\[
+(-q^2 - q^{-4} + 2q^6 - 4q^8 - 6q^{10} + 4q^{12} - 2q^{14} - 8q^{16} + q^{20} - 2q^{22} - q^{24})A^{-2} +
\]

\[
+(q^4 - 1 + 4q^{-4} - 3q^8 - 8q^{10} + 3q^{12} - 3q^{14} + 3q^{16} + 3q^{18} - q^{22}) +
\]

\[
+(-q^2 + 2q^4 - q^{-6} - 3q^8 - 8q^{10} + 3q^{12} - 3q^{14} + 3q^{16} + 3q^{18} - q^{22}) +
\]

\[
+ (1 - q^2 - q^{-4} + 2q^6 - q^{10} + q^{12})A^4 = \]

\[
\]
\[
\begin{align*}
= A^8q^{16} \left( S_6[q^{-24}] + S_{[1,1]}(q^{-4} - 2q^{-8} + 2q^{-12} - 3q^{-16} + 2q^{-20} - 2q^{-24} + q^{-28}) + \\
+ S_{[4,2]}(q^8 - 2q^4 - q^2 + 5q^2 - 3 - 5q^{-2} + 8q^{-4} - 9q^{-8} + 7q^{-10} + \\
3q^{-12} - 8q^{-14} + 5q^{-16} + 2q^{-18} - 5q^{-20} - 3q^{-22} - 2q^{-26} + q^{-28}) + \\
+ S_{[4,1,1]}q^{-8} + S_{[3,3]}q^{-8} + S_{[3,2,1]}(q^2 - 2 + 2q^{-2} - 3q^{-4} + 2q^{-6} - 2q^{-8} + q^{-10}) + S_{[2,2,2]})
\end{align*}
\]

Alexander polynomials
\[
\begin{align*}
\mathcal{A}_{[1]} &= -q^4 + 4q^2 - 5 + 4q^{-2} - q^{-4} \\
\mathcal{A}_{[1,1]} &= \mathcal{A}_{[2]} = \mathcal{A}_{[1]}(q^2) = -q^8 + 4q^4 - 5 + 4q^{-4} - q^{-8}
\end{align*}
\]

Jones polynomials
\[
\begin{align*}
J_{[1]} &= q^{14} - 2q^{12} + 2q^{10} - 3q^8 + 3q^6 - 2q^4 + 2q^2 \\
J_{[1,1]} &= 1 \\
J_{[2]} &= q^{40} - 2q^{38} - q^{36} + 5q^{34} - 3q^{32} - 4q^{30} + 8q^{28} - 2q^{26} - 8q^{24} + \\
&+ 10q^{22} - q^{20} - 10q^{18} + 10q^{16} - 8q^{12} + 6q^{10} + 9q^8 - 4q^6 + 2q^4 + q^2
\end{align*}
\]

Special polynomials
\[
\begin{align*}
\delta_{[1]} &= 3A^2 - 3A^4 + A^6 \\
\delta_{[1,1]} &= \delta_{[2]} = (\delta_{[1]})^2 = 9A^4 - 18A^6 + 15A^8 - 6A^{10} + A^{12}
\end{align*}
\]

Ooguri-Vafa polynomials
\[
\begin{align*}
f_{[1]} &= \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left( (q^5 - 2q^{-1} + 2q^{-3} - 3q^{-5})A^6 + \\
&+ (q^5 + 2q^{-1} - 2q^{-3} + q^{-5} - q^{-7} + q^{-9} + q^{-11})A^8 + (-q^{-1} - q^{-9} + q^{-13} - q^{-15})A^{10} \right)
\end{align*}
\]

\[
\begin{align*}
f_{[2]} &= \frac{\{A\}^2\{A/q\}\{Aq\}}{\{q\}} \left( (3q^5 - 2q^3 + 2q^1 - q^{-5})A^6 + \\
&+ (-q^{11} - q^9 + q^7 - q^5 - 3q^3 + 2q^1 - 2q^{-1} - q^{-5})A^8 + (q^{15} - q^{13} + q^9 + q^7)A^{10} \right)
\end{align*}
\]

Special Ooguri-Vafa polynomials
\[
\begin{align*}
f_{[2]} = -f_{[1,1]} &= 2A^8(A^2 - 3 + A^{-2})(A - A^{-1})^3
\end{align*}
\]

Numbers \(N_{R,n,k}\)

| \(k\) \(|\) \(n\) | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| 0 | -3 | 6 | -4 | 1 |
| 1 | -2 | 5 | -4 | 1 |
| 2 | 0 | 1 | -1 | 0 |

\[\text{55} \]
Knot $10_{139}$

$(2,3|1,4)$

HOMFLY polynomials

$$
\frac{H_{[1]}^{10}}{S_{[1]}^{10}} = A^{-10} \left( (q^2 - 1 + q^{-2}) A^{-2} + (-q^6 - q^{-2} - 2 - q^{-4} - q^{-6}) + (q^8 + q^{-4} + q^2 - q^{-2} + q^{-4} + q^{-8}) A^2 \right) =
$$

$$
= A^{-10} \left( \ast S_{[3]}^{10} q^{10} + \ast S_{[2,1]}^{10} (-q^4 + q^2 - 1 + q^{-2} - q^{-4}) + \ast S_{[1,1,1]}^{10} q^{-10} \right)
$$

$$
\frac{H_{[1]}^{10}}{S_{[1]}^{10}} = A^{-20} q^{40} \left( (q^{-16} - q^{-18} + 2q^{-22} - q^{-24} - q^{-26} + q^{-28}) A^{-2} +

+(-q^{-12} - q^{-14} - q^{-18} - 3q^{-20} - q^{-22} + q^{-24} - 2q^{-26} - 2q^{-28} - q^{-30} - q^{-32}) A^{-2} +

+(2q^{-10} + q^{-12} + 2q^{-14} + 3q^{-16} + 4q^{-18} + 4q^{-20} + 4q^{-22} + 3q^{-24} + 6q^{-26} + 3q^{-28} + q^{-30} + 3q^{-32} + 4q^{-34} + 2q^{-36} + q^{-38} + q^{-40} + 2q^{-42} + q^{-44} + q^{-46})+

+(q^{-8} - 2q^{-10} - q^{-12} - 3q^{-14} - 5q^{-16} - 3q^{-18} - 5q^{-20} - 6q^{-22} - 5q^{-24} - 6q^{-26} - 4q^{-28} - 4q^{-30} - 6q^{-32} -

- 4q^{-34} - q^{-38} - 3q^{-40} - 2q^{-42} - q^{-44} - 2q^{-46} - q^{-48} - q^{-50} - q^{-52}) A^2 +

+(q^{-8} + q^{-12} + 2q^{-14} + q^{-16} + q^{-18} + 4q^{-20} + q^{-22} + 3q^{-24} + 3q^{-26} + q^{-28} + 3q^{-30} +

+3q^{-32} + 2q^{-34} + 3q^{-34} + q^{-40} + 2q^{-44} + q^{-46} + 2q^{-48} + q^{-50} + q^{-56}) A^4 \right)
$$

$$
= A^{-20} q^{40} \left( \ast S_{[3,1]}^{10} + \ast S_{[2,2,1]}^{10} (-q^4 - q^6 - q^8 - q^{-10} + q^{-12} - q^{-14}) + \ast S_{[3,1,1,1]}^{10} q^{-20} + \ast S_{[2,2,2]}^{10} q^{-20} +

+ \ast S_{[2,2,1,1]}^{10} (q^{-14} - q^{-16} - q^{-18} + 2q^{-20} - 2q^{-22} + q^{-26} + q^{-28} - 2q^{-30} + q^{-32} - q^{-36} + q^{-38}) +

+ \ast S_{[2,1,1,1,1]}^{10} (-q^{-32} + q^{-36} - q^{-40} + q^{-44} - q^{-48}) + \ast S_{[1,1,1,1,1,1]}^{10} q^{-60} \right)
$$
\[
\frac{H[3]}{S[2]} = A^{-20} q^{-40} \left( (q^{28} - q^{26} - q^{4} + 2q^{22} - q^{18} + q^{16}) A^{-4} + \\
+(-q^{38} - q^{36} - 2q^{28} - 2q^{26} + q^{24} - q^{22} - 3q^{20} - q^{18} - q^{14} - q^{12}) A^{-2} + \\
+(q^{46} + q^{44} + 2q^{42} + q^{40} + q^{38} + 2q^{36} + 4q^{34} + 3q^{32} + q^{30} + 3 \\
+q^{28} + 6q^{26} + 3q^{24} + 4q^{22} + 4q^{20} + 4q^{18} + 3q^{16} + 2q^{14} + q^{12} + 2q^{10}) + \\
+(-q^{52} - q^{50} - q^{48} - 2q^{46} - q^{44} - q^{42} - 5q^{40} - 3q^{38} - q^{36} - 4q^{34} - 6q^{32} - 4q^{30} - \\
L - 4q^{28} - 6q^{26} - 5q^{24} - 6q^{22} - 5q^{20} - 3q^{18} - 5q^{16} - 3q^{14} - q^{12} - 2q^{10} - q^{8}) A^{2} + \\
+((56 + q^{50} + q^{48} + q^{46} + 2q^{44} + q^{40} + 3q^{38} + 2q^{36} + 3q^{34} + 3q^{32} + 3q^{30} + \\
+q^{28} + 3q^{26} + 3q^{24} + q^{22} + 4q^{20} + q^{18} + 3q^{16} + 2q^{14} + q^{12} + q^{10}) A^{4} ) = \\
= A^{-20} q^{-40} \left( \* S[6][q] + \* S[5,1][(-q^{48} + q^{44} - q^{40} + q^{36} - q^{32}) + \\
+ \* S[4,2][q^{38} - q^{36} + q^{32} - 2q^{30} + q^{28} + q^{26} - 2q^{24} + 2q^{20} - q^{18} - q^{16} + q^{14}) + \\
+ \* S[4,1,1][q^{20}] + \* S[3,3][q^{20}] + \* S[3,2,1][(-q^{14} + 12q^{10} - q^{10} + q^{8} - q^{6}) + \* S[2,2,2] \right)
\]

**Alexander polynomials**

\[ A_{[1]} = q^{8} - q^{6} + 2q^{2} - 3 + 2q^{-2} - q^{-6} + q^{-8} \]  

\[ A_{[1,1]} = A_{[2]} = A_{[1]}(q^{2}) = q^{16} - q^{12} + 2q^{4} - 3 + 2q^{-4} - q^{-12} + q^{-16} \]

**Jones polynomials**

\[ J_{[1]} = q^{-8} + q^{-12} - q^{-16} + q^{-18} - q^{-20} + q^{-22} - q^{-24} \]

\[ J_{[1,1]} = 1 \]

**Special polynomials**

\[ \delta_{[1]} = A^{-12} - 6A^{-10} + 6A^{-8} \]

\[ \delta_{[1,1]} = \delta_{[2]} = (\delta_{[1]})^{2} = A^{-24} - 12A^{-22} + 48A^{-20} - 72A^{-18} + 36A^{-16} \]

**Ooguri-Vafa polynomials**

\[ f_{[1,1]} = \frac{\{A^{2}/A\}}{\{q\}} \left( (q^{23} - q^{21} + q^{19} + q^{17} + q^{11} + q^{7} + q^{4}) A^{-22} + \\
+(-q^{27} - q^{23} - 2q^{21} - 2q^{19} - 2q^{17} - 3q^{15} - 4q^{13} - 4q^{11} - \\
-4q^{9} - 3q^{7} - 3q^{5} - 3q^{3} - 2q^{1} - 2q^{-1} - 2q^{-3} - q^{-5} - q^{-7}) A^{-20} + \\
+(q^{29} + 2q^{27} + 2q^{25} + 2q^{23} + 3q^{21} + 3q^{19} + 7q^{17} + 4q^{15} + 9q^{13} + 6q^{11} + 8q^{9} + \\
+8q^{7} + 8q^{5} + 5q^{3} + 7q^{1} + 4q^{-1} + 4q^{-3} + 2q^{-5} + 2q^{-7} + q^{-9} + q^{-11} A^{-18} ) \right) \]

\[ f_{[2]} = \frac{\{A^{2}/A\}}{\{q\}} \left( (-q^{-1} - q^{-7} - q^{-11} - q^{-17} - q^{-19} + q^{-21} - q^{-23}) A^{-22} + \\
+(q^{9} + q^{7} + 2q^{5} + 2q^{3} + 2q^{1} + 4q^{-3} + 4q^{-5} + 3q^{-7} + 4q^{-9} + 4q^{-11} + \\
+4q^{-13} + 3q^{-15} + 2q^{-17} + 2q^{-19} + 2q^{-21} + q^{-23} + q^{-27}) A^{-20} + \\
+(-q^{-11} - 9q^{-9} - 2q^{-7} - 2q^{-5} - 4q^{-3} - 4q^{-1} - 7q^{-1} - 5q^{-3} - 8q^{-5} - 8q^{-7} - 8q^{-9} - 6q^{-11} - \\
-9q^{-13} - 4q^{-15} - 7q^{-17} - 3q^{-19} - 3q^{-21} - 2q^{-23} - 2q^{-25} - q^{-29} A^{-18} ) \right) \]

**Special Ooguri-Vafa polynomials**

\[ f_{[1]} = -f_{[1,1]} = -\frac{(159A^{8} - 360A^{6} + 272A^{4} - 76A^{2} + 7)(A - A^{-1})^{3}}{A^{26}} \]
Numbers $N_{R,n,k}$

\[ k \backslash n = \begin{array}{cccccc}
-26 & -24 & -22 & -20 & -18 & -16 & -14 \\
0 & 26 & -271 & 1149 & -2406 & 2644 & -1467 & 325 \\
1 & 283 & -2908 & 12519 & -26725 & 29712 & -16483 & 3602 \\
2 & 1231 & -13215 & 59305 & -130641 & 147643 & -81894 & 17571 \\
3 & 2846 & -33702 & 161724 & -372044 & 429324 & -237978 & 49830 \\
4 & 3939 & -54000 & 283511 & -689018 & 815314 & -451252 & 91506 \\
5 & 3445 & -57565 & 338189 & -878533 & 1070393 & -590788 & 114859 \\
6 & 1939 & -42074 & 283375 & -797042 & 1004073 & -551784 & 101513 \\
7 & 697 & -21353 & 169548 & -524186 & 685766 & -374583 & 64111 \\
8 & 154 & -7507 & 72733 & -251892 & 343856 & -186357 & 29013 \\
9 & 19 & -1792 & 22182 & -88275 & 126395 & -67847 & 9318 \\
10 & 1 & -277 & 4692 & -22269 & 33635 & -17852 & 2070 \\
11 & 0 & -25 & 654 & -3931 & 6302 & -3302 & 302 \\
12 & 0 & -1 & 54 & -460 & 788 & -407 & 26 \\
13 & 0 & 0 & 2 & -32 & 59 & -30 & 1 \\
14 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
\end{array} \]

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N_{[1]}:
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\]

\[
N_{[1,1]}:
\]
| $k \setminus n$ | -26 | -24 | -22 | -20 | -18 | -16 | -14 |
|----------------|-----|-----|-----|-----|-----|-----|-----|
| 0              | 31  | -333| 1434| -3026| 3339| -1857| 412 |
| 1              | 396 | -4107| 17757| -37965| 42249| -23472| 5142 |
| 2              | 2031| -21561| 95976| -210260| 237023| -131551| 28342 |
| 3              | 5548| -63635| 299280| -680083| 780367| -432718| 91241 |
| 4              | 9162| -118624| 602538| -1436213| 1684844| -933053| 191346 |
| 5              | 9738| -148552| 831465| -2101076| 2529808| -1397916| 276533 |
| 6              | 6853| -129374| 814438| -2205943| 2736313| -1506797| 284510 |
| 7              | 3212| -79780| 577794| -1698237| 2178428| -1193693| 212276 |
| 8              | 988 | -35018| 299618| -969806| 1291661| -703214| 115771 |
| 9              | 191 | -10859| 113471| -412141| 572417| -309146| 46067 |
| 10             | 21 | -2323| 31010| -129656| 188672| -100927| 13203 |
| 11             | 1  | -326| 5950| -29730| 45559| -24104| 2650 |
| 12             | 0  | -27| 760| -4821| 7823| -4088| 353 |
| 13             | 0  | -1| 58| -523| 904| -466| 28 |
| 14             | 0  | 0| 2| -34| 63| -32| 1 |
| 15             | 0  | 0| 0| -1| 2| -1| 0 |