Exact solutions for a cosmology within the Israel-Stewart theory

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Abstract: In this paper we find a novel exact analytic solution for a cosmology filled with a dissipative fluid, in the framework of the causal Israel-Stewart theory. We assume a bulk viscous coefficient with the dependence ξ = ξ0ρ1/2, where ρ is the energy density of the fluid. We also consider a relaxation time τ of the form ξ = c_b^2 = η(2 - γ), where c_b^2 is the speed of bulk viscous perturbations, and as well the barotropic EoS p = (γ - 1)ρ. We study the non-trivial particular limit when γ = 1, in order to describe the late time evolution of a universe with a dissipative dark matter. In both cases, we found implicit analytical solutions for the scale factor and for the entropy evolution that lead to an accelerated expansion at late times and, in addition, the consequences that arise from the positiveness of the entropy production along the time evolution. In general, the accelerated expansion at late times is only possible when ε ≥ 1/18, which implies a very large non-adiabatic contribution to the speed of sound. We discuss the physical implications on the structure formation arising from our solution.

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I. INTRODUCTION

It is well accepted that nowadays the cosmological data consistently indicates that the expansion of the universe began to accelerate [1] around z = 0.64 [2]. Thus, every model used to describe the cosmic background evolution must display this transition in its dynamics. Of course, ΛCDM presents this transition as well and it can be understood as the transition between the dark matter (DM) dominant era and the era dominated by the dark energy (DE). Nevertheless, despite the fact that the ΛCDM model has been very successful in explaining the cosmological data, it presents the following weak points from the theoretical point of view: i) Why the observed value of Λ is 120 orders of magnitude smaller than the physically measured value?, This is known as the cosmological constant problem [3], which can be represented mainly by the two following open questions: a) Why does the observed vacuum energy has such an unnaturally small but non vanishing value?, and b) Why do we observe vacuum density to be so close to matter density, even though their ratio can vary up to 120 orders of magnitude during the cosmic evolution? (the coincidence problem) [3]. This model presents also serious specific observational challenges and tensions (for example, see [2] for a brief review).

As an alternative to ΛCDM, the DM unified models do not invoke a cosmological constant. In the framework of general relativity, non perfect fluids drive accelerated expansion due to the negativeness of the viscous pressure, which appears from the presence of bulk viscosity. Therefore, a cold DM viscous component is a kind of unified DM model that could, in principle, explain the above mentioned transition without the inclusion of a DE component. It is worthy mentioning that measurements of the Hubble constant show tension with the values obtained from large scale structure (LSS) and Planck CMB data, which can be alleviated when viscosity is included in the DM component [8]. The new era of gravitational waves detectors has also opened the possibility to detect dissipative effects in DM and DE through the dispersion and dissipation experimented by these waves propagating in a non perfect fluid. Some constraints on those effects were found in [2].

At background level, where a homogeneous and isotropic space describes the universe as a whole, only bulk viscosity is present in the cosmic fluid and the dissipative pressure must be described by some relativistic thermodynamical approach for non perfect fluids. This implies a crucial point in a fully consistent physical description of the expansion of the universe using dissipative processes to generate the transition. Meanwhile, in the ΛCDM model the acceleration is due to a cosmological constant and the entropy remains constant, in the case of non perfect fluids it is necessary to find a solution that not only consistently describes the kinematics of the universe, but also that satisfies the thermodynamical requirements, such as the positiveness of entropy generation. In the case of a description of viscous fluids, the Eckart’s theory [10] has been widely investigated due to its simplicity and became the starting point to shed some light in the behavior of the dissipative effects in the late time cosmology [11] or in inflationary scenarios [12]. Nevertheless, it is a well known result that the Eckart’s theory has non causal behavior, presenting the problem of superluminal propagation velocities and some instabilities. So, from the point of view of a consistent description
of the relativistic thermodynamics of non perfect fluids, it is necessary to include a causal description such as the one given by the Israel-Stewart (IS) theory.

Our aim in this paper is twofold, first of all we present a novel and exact cosmological solution for a universe filled with only one dissipative fluid, that displays a transition between deceleration and acceleration expansions at background level. Secondly, we study the consistence of our solution with the 2nd law of thermodynamics.

We shall assume a barotropic EoS for the fluid that filled the universe, with the expression

\[ p = (\gamma - 1) \rho, \]

where \( p \) is the barotropic pressure, \( \rho \) is the energy density. Since our aim is to describe the evolution of the universe with dissipative normal matter, we shall consider that the EoS parameter lies in the range \( 1 \leq \gamma < 2 \). Furthermore, we will use the following Ansatz for the bulk viscosity coefficient, \( \xi(\rho) \),

\[ \xi = \xi_0 \rho^s, \]

which has been widely considered as a suitable function between the bulk viscosity and the energy density of the main fluid. \( \xi_0 \) is a positive constant because of the second law of thermodynamics. Taking into account the above assumptions, the IS theory leads to a nonlinear ordinary differential equation that has been solved and investigated in many previous works for some particular parameter values. Using, for example, the factorization method, some new exact parametric solutions for different values of the viscous parameter \( s \) were found in [18]. A particular solution for stiff matter and \( s = 1/4 \) was found in [19]. Other exact solutions found in [20] well describe determined periods of inflationary and non inflationary evolutions of the universe. Inflationary solutions and their stability properties were studied in [21].

One important assumption in the thermodynamical approaches of relativistic viscous fluids is the near equilibrium condition, i.e., that the viscous pressure must be lower than the equilibrium pressure of the fluid. In the case of the solutions that present acceleration from the beginning, like the bulk viscous inflation case, or at some stage, like those that could represent the late transition between decelerated and accelerated expansions, the above condition is not fulfilled, therefore the application of these theories is not strictly justified. A non linear extension of IS theory to take into account deviations from the near equilibrium condition was formulated in [22]. This non linear extension was investigated in the context of inflation [23] and also in late time phantom behavior [24].

Our novel solution generalizes the exact solution found in [25], for the particular values \( s = 1/2, \gamma = 1 \), where the expression \( \tau = \xi/\rho \) for the relaxation time was used. In this article, the solution displays a decelerated phase an exponential expansion for late times, corresponding to a de Sitter phase. Moreover, our solution was obtained using the following expression for the relaxation time \( \tau \) [16], derived from the study of the causality and stability of the IS theory in [26],

\[ \frac{\xi}{(\rho + p)^\tau} = c_b^2, \]

where \( c_b \) is the speed of bulk viscous perturbations (non-adiabatic contribution to the speed of sound in a dissipative fluid without heat flux or shear viscosity). Since the dissipative speed of sound \( V \), is given by \( V^2 = c_s^2 + 2c_b^2 \), where \( c_s^2 = (\partial p / \partial \rho)_s \) is the adiabatic contribution, then for a barotropic fluid \( c_s^2 = \gamma - 1 \) and thus \( c_b^2 = \epsilon (2 - \gamma) \) with \( 0 < \epsilon \leq 1 \), known as the causality condition.

In a previous work, which includes Eq. (3) for the relaxation time and a pressureless main fluid, the IS equation was solved using an Ansatz for the viscous pressure [27]. Their conclusions indicates that the full causal theory seem to be disfavored. Nevertheless, in the truncated version of the theory, similar results to those of the \( \Lambda \)CDM model were found for a bulk viscous speed in the interval \( 10^{-11} \ll c_b^2 \lesssim 10^{-8} \). This last constraint on \( c_b^2 \), even though is obtained with a suitable Ansatz and is not an exact solution of the theory, teaches us that the non-adiabatic contribution to the speed of sound must be very small to be consistent with the cosmological data.

In what follows we will discuss our novel solution aiming to obtain a fully physically consistent behavior within the allowed regions of the parameters. Our goal is to find a solution in the framework of unified DM models that can describe consistently well the late transition between decelerated and accelerated expansion, and, in addition, presents a behavior consistent with the second law of thermodynamics, in the context of a linear IS theory. We assume in the case of accelerated expansion that the linear version of the causal theory is valid, which occurs within a range of the parameters involved in our solution.

This paper is organized as follow: In section II we describe briefly the causal Israel-Stewart theory and we find the general differential equation to be solved. In section III we study the possibility to obtain De Sitter like solutions. In section IV we present a new analytic solution for the Israel-Stewart theory and we study its physical properties, computing the corresponding scale factor and the constrains over the free parameters which allowed to have an accelerated expansion at late times and a positive entropy production due to the dissipative effects. In section V we study a particular analytic solution with \( \gamma = 1 \), in order to study the case of a universe filled with dissipative pressureless DM, computing also its scale factor and its entropy production. In section VI we study a special case that emerges from the particular solution. Finally, in section VII, we present some conclusions.

II. ISRAEL-STEWART FORMALISM

In what follows we assume that the universe contains a DM component which experiments dissipative processes during the cosmic evolution. We assume a barotropic
EoS, \( p = (\gamma - 1) \rho \), where \( p \) is the barotropic pressure, \( \rho \) the energy density and \( 1 \leq \gamma < 2 \). For a flat FLRW universe without cosmological constant, the constraint equation can be written, using natural units defined by \( 8\pi G = c = 1 \), as

\[
3H^2 = \rho, \tag{4}
\]

and the Einstein pressure equation is given by

\[
2\dot{H} + 3H^2 = -p - \Pi. \tag{5}
\]

In the IS framework, the transport equation for the viscous pressure \( \Pi \) reads \[14\]

\[
\tau \dot{\Pi} + \Pi = -3\xi H - \frac{1}{2} \frac{\dot{\tau}}{\tau} \left( 3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} \right), \tag{6}
\]

where “dot” accounts for the derivative with respect to the cosmic time, \( \tau \) is the relaxation time, \( \xi(\rho) \) is the bulk viscosity coefficient, for which we assume a dependence with the energy density of DM. \( H \) is the Hubble parameter and \( T \) is the barotropic temperature, which takes the form \( T = T_0 \rho^{(\gamma - 1)/\gamma} \) (Gibbs integrability condition when \( p = (\gamma - 1) \rho \)) with \( T_0 \) being a positive parameter. The DM EoS, \( \xi(\rho) \) and the relaxation time are related by Eq.\[13\].

It is very interesting and always desirable to obtain analytical solutions to cosmological models, as they don’t suffer from the numerical instabilities of numerical solutions nor hide a different underlying behaviour of the dynamical system, implicitly ruled out by the numerical algorithm used. For this aim we have chosen the particular case \( s = 1/2 \) and will show a novel exact solution, discussing its physical properties, in section IV. Thus, from Eq.\[3\] the relaxation time results to be

\[
\tau = \frac{\xi_0}{c\gamma (2 - \gamma)} \rho^{s-1}. \tag{7}
\]

In order to obtain a differential equation in terms of the Hubble parameter, we evaluate the ratios \( \dot{\tau}/\tau, \xi/\xi \) and \( \dot{T}/T \), which appear in Eq.\[10\]. Using Eq.\[4\], we get the following expressions

\[
\frac{\dot{\tau}}{\tau} = 2 (s - 1) \frac{\dot{H}}{H}, \tag{8}
\]

\[
\frac{\xi}{\xi} = 2s \frac{\dot{H}}{H}, \tag{9}
\]

and

\[
\frac{\dot{T}}{T} = 2 \left( \frac{\gamma - 1}{\gamma} \right) \frac{\dot{H}}{H}. \tag{10}
\]

From Eqs.\[11\] and \[12\], we obtain the following expression for the viscous pressure

\[
\Pi = - \left( 2\dot{H} + 3\gamma H^2 \right), \tag{11}
\]

whose time derivative is,

\[
\dot{\Pi} = - \left( 2\ddot{H} + 6\gamma H \dot{H} \right). \tag{12}
\]

Finally, inserting Eqs.\[7\] and \[12\] into Eq.\[10\], we obtain the nonlinear second order differential equation for \( H \), that represents the general differential equation to be solved in this model, which governs the time evolution of the Hubble parameter

\[
\ddot{H} + 3H \dot{H} + (3)^{1-s} \xi_0^{-1} c\gamma (2 - \gamma) H^{2-2s} \dot{H} = \frac{(2\gamma - 1)}{\gamma} H^{-1} \dot{H}^{2} + \frac{9}{4} \gamma [1 - 2\epsilon (2 - \gamma)] H^{3}. \tag{13}
\]

In the special case where \( s = 1/2 \), Eq.\[13\] has a phantom solution of the form \( H (t) = A (t_s - t)^{-1} \), with \( A > 0, \epsilon = 1 \) and the restriction \( 0 < \gamma < 3/2 \). This solution was discussed in \[28\]. Also the solution \( H (t) = A (t - t_s)^{-1} \) can represent accelerated universes if \( A > 1 \), Milne universes if \( A = 1 \) and decelerated universes if \( A < 1 \), all with an initial singularity at \( t = t_s \). \[29\]. It is worthy mentioning that only the decelerated solution satisfies a positive entropy production, therefore there is no transition from a decelerated phase to an accelerated one, as it occurs in the standard model. As we shall see below, the dynamical behavior of an exact solution of a model described by the IS thermodynamic formalism does not necessarily implies that its thermodynamical properties behave physically consistent.

### III. DE SITTER TYPE LIKE SOLUTION

Before to integrate Eq.\[13\] for the special value \( s = 1/2 \) and in order to shed some light in its properties, we will study the possibility of a de Sitter type like solution, which is the asymptotic behavior of the \( \Lambda CDM \) model. We note that \( H = \text{const} \neq 0 \) is a solution of Eq.\[13\]

\[
H = \left\{ \frac{3\xi_0}{2} \left[ \frac{2\epsilon (2 - \gamma) - 1}{c\gamma (2 - \gamma)} \right] \right\} \text{const}. \tag{14}
\]
It is easy to see that there is no de Sitter solution when \( s = 1/2 \) as the exponent flows up. On the other hand, if we require a positive Hubble parameter that represents an expanding universe (or avoids a complex Hubble parameter) we need to impose that the term within parenthesis be positive. Because \( \epsilon = 0 \) and \( \gamma = 2 \) indeterminate the Hubble parameter, we have to restrict the parameters to the regions \( 0 < \epsilon \leq 1 \) and \( 1 \leq \gamma < 2 \). Furthermore as \( \xi_0 > 0 \), an expanding universe requires
\[
\frac{1}{2} \leq \frac{1}{2(2 - \gamma)} < \epsilon \leq 1 \text{ with } 1 \leq \gamma < \frac{3}{2}.
\]

The solution of Eq. (14) was previously found in [28], but the particular value \( \epsilon = 1 \) was used, so the lower bound for \( \epsilon \) displayed in (13) was missing.

IV. A NOVEL NEW ANALYTICAL SOLUTION FOR ARBITRARY \( \gamma \)

A new analytical solution can be found for the Eq. (13) if we consider the particular value \( s = 1/2 \). In fact in this case Eq. (13) goes into
\[
\ddot{H} + d_1 H \dot{H} + d_2 H^3 - d_3 \frac{\dot{H}^2}{H} = 0,
\]
where for simplicity we have defined the constants
\[
d_1 \equiv 3 \left[ 1 + \frac{\epsilon \gamma (2 - \gamma)}{\sqrt{3} \xi_0} \right],
\]
\[
d_2 \equiv \frac{9}{4 \gamma} \left\{ \left[ 1 - 2 \epsilon (2 - \gamma) \right] + \frac{2 \epsilon \gamma (2 - \gamma)}{\sqrt{3} \xi_0} \right\},
\]
\[
d_3 \equiv \frac{2 \gamma - 1}{\gamma}.
\]

In the Eq. (16) we change the variable from the cosmic time \( t \) to \( x = \ln (a) \), and the differential equation takes the form
\[
\frac{d^2 H}{dx^2} + \frac{dH}{dx} + d_2 H + \frac{1 - d_3}{H} \left( \frac{dH}{dx} \right)^2 = 0,
\]
which is a nonlinear second order differential equation. Further using the Ansatz
\[
H(x) = e^{-\frac{d_1 x}{2(2 - \gamma)}} \phi (x),
\]
Eq. (20) goes into the equation
\[
\frac{d^2 \phi}{dx^2} + \left[ d_2 - \frac{d_1^2}{4(2 - \gamma)} \right] \phi + \frac{(1 - d_3)}{\phi} \left( \frac{d\phi}{dx} \right)^2 = 0,
\]
i.e. we have eliminated the linear first derivative term. Now, in order to eliminate the nonlinear term in the above equation, we use a nonlinear second Ansatz
\[
\phi (x) = [\Phi (x)]^{\frac{1}{2 -d_3}},
\]
and Eq. (22) reduces to the following expression
\[
\frac{d^2 \Phi}{dx^2} - \frac{1}{4} \left[ d_1^2 + 4d_2 (d_3 - 2) \right] \Phi = 0,
\]
which is in fact a linear second order differential equation. Thus, the general solution of Eq. (20) can be expressed as
\[
H(x) = e^{-\alpha x} \left[ A \cosh (\beta x) + B \sinh (\beta x) \right]^{\gamma},
\]
where \( A \) and \( B \) are integration constants. For numerical purposes we further define the constants
\[
\alpha = \frac{\sqrt{3} \gamma}{2 \xi_0} \left[ 3 \xi_0 + \epsilon \gamma (2 - \gamma) \right],
\]
\[
\beta = \frac{\sqrt{3}}{2 \xi_0} \left[ 6 \xi_0^2 \epsilon (2 - \gamma) + \epsilon^2 \gamma^2 (2 - \gamma)^2 \right].
\]

For simplicity we can rewrite the solution (25), as a function of the redshift \( z \), in the form
\[
H(z) = C_3 a^{-\alpha} \cosh ^\gamma \left[ \beta (\ln (1 + z) + C_4) \right],
\]
where \( C_3 \) and \( C_4 \) are constants given by
\[
C_3 = \frac{H_0}{\cosh ^\gamma (\beta C_4)} = H_0 \left[ 1 - \frac{(q_0 + 1 - \alpha)^2}{\gamma^2 \beta^2} \right]^{\frac{\gamma}{2}},
\]
\[
C_4 = \frac{1}{\beta} \frac{1}{\cosh ^\gamma (\beta C_4)} \left[ \frac{(q_0 + 1 - \alpha)}{\gamma \beta} \right].
\]

In the above equations \( H_0 \) and \( q_0 \) are the Hubble and the deceleration parameters respectively, at the present time \( t = t_0 \). The parameter \( q \) is defined trough \( q = -1 - H/H^2 \). We have also set the condition \( a_0 = 1 \).

Note from the Eqs. (29) and (30) that for a real Hubble parameter the deceleration parameter \( q_0 \) must fulfill the constraints
\[
(\alpha - \gamma \beta) - 1 < q_0 < (\alpha + \gamma \beta) - 1.
\]
A consequence of the above restriction is that the Hubble parameter given by the Eq. (28) remains positive during the whole cosmic evolution. In Fig. 1 are displayed the allowed regions imposed by the constraint in terms of the parameters \( q_0, \xi_0, \gamma \) and \( \epsilon \).

A. Mathematical properties of the solution

Before studying the behaviour of the Hubble parameter obtained above, it is worthwhile discussing some interesting mathematical properties of the solutions.

Note that the Eq. (20) is scale-invariant, i.e., if we perform the conformal change \( H(x) \rightarrow \sigma H(x) \) for \( \sigma \) constant, then the differential equation remains unchanged. We therefore look for a solution of the form
\[
H(x) = e^{\lambda x},
\]
and
\[
\frac{d^2 \Phi}{dx^2} - \frac{1}{4} \left[ \frac{d_1^2}{4(2 - \gamma)} + 4d_2 (d_3 - 2) \right] \Phi = 0.
\]
which leads to the following condition on the constant \( \lambda \)
\[
\lambda_\pm = \frac{-d_1 \pm \sqrt{d_1^2 + 4d_2(d_3 - 2)}}{2(2-d_3)}.
\] (33)

Because Eq. (20) is a non linear differential equation, then the superposition principle does not hold. Nevertheless, from Eqs. (32) and (33), there are two independent solutions
\[
H_+(x) = e^{\lambda_+ x} \quad \text{and} \quad H_-(x) = e^{\lambda_- x},
\] (34)

but as already mentioned, a linear combination of them does not in general fulfills the differential equation.

![Figure 1](image1.png)

(a) Allowed values of the parameters \( q_0 \) and \( \xi_0 \) for fixed \( \epsilon \) value and \( \gamma = 1.05 \).

(b) Allowed values of the parameters \( q_0 \) and \( \gamma \) for fixed \( \xi_0 \) value and \( \epsilon = 0.1 \).

FIG. 1. Comparative graphics of the allowed values for the free parameters \( \xi_0, \epsilon, q_0 \) and \( \gamma \), compatible with the restriction indicated in Eq. (34).

In order to find a complete solution of the second order differential equation, we need to explore the conditions under which a general linear combination of the solutions \( 34 \) is also a solution. To this aim we consider
\[
H(x) = C_1 H_+(x) + C_2 H_-(x),
\] (35)

and inserting this into Eq. (20) we obtain the following condition on the parameters defined in Eqs. (19) - (21)
\[
(\lambda_+^2 + \lambda_-^2) + d_1 (\lambda_+ + \lambda_-) + 2d_2 + 2(1-d_2)\lambda_+\lambda_- = 0.
\] (36)

This condition does not imply a constraint on the constant \( C_1 \) and \( C_2 \), but leads to a new condition on the free parameters \( \epsilon, \gamma \) and \( \xi_0 \). After some computations, the condition of \( 36 \) can be written as
\[
\frac{(1-d_3)}{2-d_3} \left[ 4d_2 - \frac{d_1^2}{2-d_3} \right] = 0.
\] (37)

From the above equation there are two possibilities. The first on is,
\[
\frac{\epsilon^2 \gamma^2}{6c_0} = -1,
\] (38)

which clearly cannot be fulfilled for real parameters. The second possibility leads to the condition
\[
1 - d_3 = 0,
\] (39)

which implies \( \gamma = 1 \). Thus, the linear combination is a solution of Eq. (20) only when \( \gamma \) has the particular value 1. It is important to mention that, in this particular case, Eq. (20) becomes a second order linear differential equation, whose solutions are given in fact by the linear combination of Eq. (35). This case will be addressed later in section V.

On the other hand, the scale-invariant solution Eq. (32) is structurally stable in the sense we will explain below. We first make a perturbation Ansatz of the solution in Eq. (35) for \( \gamma \) different but close to one
\[
H_\delta(x) = e^{\lambda x} [1 + \delta \omega(x)],
\] (40)

where \( |\delta| \ll 1 \), and from Eq. (20) we obtain the first order perturbative equation correction for the first order \( \omega \)
\[
\frac{d^2 \omega}{dx^2} + \mu \frac{d \omega}{dx} = 0,
\] (41)

where
\[
\mu = \sqrt{d_1^2 + 4d_2(d_3 - 2)}.
\] (42)

The solution of the Eq. (41) is trivial and has the form
\[
\omega(x) = \frac{D}{\mu} e^{-\mu x} + F,
\] (43)

where \( D \) and \( F \) are integration constants. It is worthwhile pointing out that \( \omega \) has the same form as the naked solution \( 32 \). This feature shows that the original solution is structurally stable. In fact, the first correction does not change the behavior of the original solution and therefore the scale-invariant solution \( 32 \) is perturbatively stable in the sense that
\[
\lim_{x \to \infty} \frac{(H_\delta - H_0)(x)}{H_0(x)} = 0.
\] (44)

In other words, nonlinear contributions of Eq. (20) does not change the asymptotic behavior of \( H(x) \) up to first order in \( \delta \).
B. Behavior of the scale factor

In what follows we find an implicit solution for the scale factor \( a(t) \). From the definition \( H = \dot{a}/a \), Eq. (28) leads to the implicit integral

\[
t + C = \frac{1}{C_3} \int \frac{(2a^\beta e^{-\beta C_4})^\gamma da}{a^{1-\alpha} (1 + a^{2\beta} e^{-2\beta C_4})},
\]

where \( C \) is an integration constant. The integral of the above equation can be expressed as an hyper-geometric function \( _2F_1 \). Considering the initial condition \( a(t_0) = 1 \), the scale factor is given by the following implicit expression

\[
a^{\alpha+\gamma \beta} _2F_1 \left[ \gamma, \frac{\alpha + \gamma \beta}{2\beta}, 1 + \frac{\alpha + \gamma \beta}{2\beta}, -a^{2\beta} e^{-2\beta C_4} \right] = 2F_1 \left[ \gamma, \frac{\alpha + \gamma \beta}{2\beta}, 1 + \frac{\alpha + \gamma \beta}{2\beta}, e^{-2\beta C_4} \right] + C_3 (\alpha + \gamma \beta) (t - t_0).
\]

Due to the numerical complexity of the above equation, the behavior of the universe will be studied by considering the dynamic evolution of the Hubble and deceleration parameters. Using the expression for \( H(z) \) given by Eq. (28), \( q(z) \) can be obtained straightforwardly as

\[
q(z) = -1 + \alpha + \gamma \beta \tanh \left[ \beta (\ln (1 + z) + C_4) \right].
\]

We will study the behavior of both \( H \) and \( q \) when \( z \to \infty \), which corresponds to early times, and when \( z \to -1 \), which represents the very far future. From Eqs. (28) and (47) the Hubble and deceleration parameters behave at early times as

\[
H (z \to \infty) \to C_3 \left( \frac{e^{\beta C_4}}{2} \right)^\gamma (1 + z)^{\alpha+\gamma \beta},
\]

\[
q (z \to \infty) \to -1 + (\alpha + \gamma \beta),
\]

while for the very far future they behave as

\[
H (z \to -1) \to C_3 \left( \frac{e^{-\beta C_4}}{2} \right)^\gamma (1 + z)^{\alpha-\gamma \beta},
\]

\[
q (z \to -1) \to -1 + (\alpha - \gamma \beta).
\]

Note that the behaviour of the Eqs. (48) and (49) depends on the exponent \( \alpha + \gamma \beta \) defined by Eqs. (20) and (21), which is always positive. Furthermore, this exponent has the following constraint

\[
\alpha + \gamma \beta \geq 3\gamma \frac{2}{\beta}.
\]

Therefore, the Hubble parameter is positive at early times, and monotonically decreasing with the redshift. This behavior corresponds to a decelerated expansion, as it can be see from Eqs. (19) and (22), which leads to a lower bound for the deceleration parameter \( q \geq 1/2 \).

On the other hand, the behavior of the Eqs. (50) and (51) is driven by the exponent \( (\alpha - \gamma \beta) \), which is positive for

\[
3 [1 - 2\epsilon (2 - \gamma)] \xi_0 + 2\sqrt{3} \epsilon (2 - \gamma) > 0,
\]

where the expressions for \( \alpha \) and \( \beta \) are given by Eqs. (20) and (21) respectively, in terms of the parameters \( \gamma, \epsilon \) and \( \xi_0 \). It follows that for \( \gamma \geq 3/2, \alpha - \gamma \beta \) it is positive, and from Eq. (51) we conclude that the Hubble parameter goes to zero in the infinite cosmological time limit. The same behavior arises when \( 1 \leq \gamma < 3/2 \), with \( 0 < \epsilon \leq \frac{1}{2\gamma - 3} \) and for \( \frac{1}{2\gamma - 3} < \epsilon \leq 1 \), if and only if, \( \xi_0 \) satisfies the additional inequality

\[
\xi_0 < \frac{2\epsilon (2 - \gamma)}{3(2\epsilon (2 - \gamma) - 1)}.
\]

Due to the numerical complexity of the above equation, the behavior of the universe will be studied by considering the dynamic evolution of the Hubble and deceleration parameters. Using the expression for \( H(z) \) given by Eq. (28), \( q(z) \) can be obtained straightforwardly as

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q (z \to \infty) \to -1 + (\alpha + \gamma \beta),
\]

while for the very far future they behave as

\[
H (z \to -1) \to C_3 \left( \frac{e^{-\beta C_4}}{2} \right)^\gamma (1 + z)^{\alpha-\gamma \beta},
\]

\[
q (z \to -1) \to -1 + (\alpha - \gamma \beta).
\]

Note that the behaviour of the Eqs. (48) and (49) depends on the exponent \( \alpha + \gamma \beta \) defined by Eqs. (20) and (21), which is always positive. Furthermore, this exponent has the following constraint

\[
\alpha + \gamma \beta \geq 3\gamma \frac{2}{\beta}.
\]

Therefore, the Hubble parameter is positive at early times, and monotonically decreasing with the redshift. This behavior corresponds to a decelerated expansion, as it can be see from Eqs. (19) and (22), which leads to a lower bound for the deceleration parameter \( q \geq 1/2 \).

On the other hand, the behavior of the Eqs. (50) and (51) is driven by the exponent \( (\alpha - \gamma \beta) \), which is positive for

\[
3 [1 - 2\epsilon (2 - \gamma)] \xi_0 + 2\sqrt{3} \epsilon (2 - \gamma) > 0,
\]

where the expressions for \( \alpha \) and \( \beta \) are given by Eqs. (20) and (21) respectively, in terms of the parameters \( \gamma, \epsilon \) and \( \xi_0 \). It follows that for \( \gamma \geq 3/2, \alpha - \gamma \beta \) it is positive, and from Eq. (51) we conclude that the Hubble parameter goes to zero in the infinite cosmological time limit. The same behavior arises when \( 1 \leq \gamma < 3/2 \), with \( 0 < \epsilon \leq \frac{1}{2\gamma - 3} \) and for \( \frac{1}{2\gamma - 3} < \epsilon \leq 1 \), if and only if, \( \xi_0 \) satisfies the additional inequality

\[
\xi_0 < \frac{2\epsilon (2 - \gamma)}{3(2\epsilon (2 - \gamma) - 1)}.
\]

On the other hand, if \( 1 \leq \gamma < 3/2 \), then \( \alpha - \gamma \beta < 0 \) for \( 0 < \epsilon \leq \frac{1}{2\gamma - 3} \), and if the constraint (54) is not satisfied, the Hubble parameter at late times stop to decrease and start to grow, becoming infinite at \( z = -1 \). Finally, if \( 1 \leq \gamma < 3/2 \), then we have the especial case in what \( \alpha - \gamma \beta = 0 \) when \( 0 < \epsilon \leq \frac{1}{2\gamma - 3} \) and if the inequality in the Eq. (51) becomes an equality, and we will have a constant Hubble parameter at late times.

From Eq. (51) we pointed out that, this last behavior leads to a accelerated expansion, provided the following two conditions are fulfilled

\[
\xi_0 > \epsilon (2 - \gamma) \left( \frac{6\gamma - 4}{\sqrt{3} \gamma} \right) \times
\]

\[
\left[ \frac{3\epsilon^2}{18\epsilon^2 (2 - \gamma) - 9\gamma^2 + 12\gamma - 4} \right],
\]

and

\[
\epsilon > \frac{9\gamma^2 - 12\gamma + 4}{18\gamma^2 (2 - \gamma)}.
\]

If one of the above conditions is not fulfilled, then the behavior of the Hubble parameter leads to a decelerated expansion. The transition between the accelerated expansion to a decelerated one occurs at redshift value

\[
z_{q=0} = -1 + exp \left[ \frac{1}{\beta} \arctanh \left( -\frac{1 - \alpha}{\gamma \beta} \right) - C_4 \right].
\]

It is important to mention that in all cases the constraint of Eq. (51) must be fulfilled. In Fig (2a) we displayed the behavior of the deceleration parameter in terms of the free parameters \( q_0, \xi_0, \gamma \) and \( \epsilon \). Note that only for large values of epsilon and a negative \( q_0 \) it is possible to obtain a transition in the past and for a z value compatible with the observations. In Fig (2b) we have use Eq. (57) to draw the allowed values for \( \xi_0 \) and \( \epsilon \), for fixed gamma, where
we have chosen the evaluated value value from observations for the transition redshift, \( z = 0.64 \), and also the estimated value for \( q_0 \) at the present time: \( q_0 = -0.6 \). It can be also noted that the transition occurs only for large values of \( \epsilon \), or, instead of this, for very large values of \( \xi_0 \).

\[ \frac{dS}{dt} = \frac{3Hn}{nT}. \] (58)

where \( n \) is the number of particles, which has to satisfy the conservation equation

\[ \dot{n} + 3Hn = 0. \] (59)

The solution of the above equation in terms of the scale factor is

\[ n = \frac{n_0}{a^3}. \] (60)

Using Eq. (53), one sees that \( T = T_0 (3H^2)^{(\gamma - 1)/\gamma} \), and from Eqs. (60) and (60) we can rewrite Eq. (58) in the form

\[ \frac{dS}{dt} = \frac{3^{1/\gamma} a^3}{T_0 n_0} H^{2/\gamma} \left(2 \frac{dH}{da} a + 3\gamma H \right). \] (61)

Now using conveniently Eq. (28), the above equation can be finally written as

\[ \frac{dS}{dt} = \frac{3^{1/\gamma} a^3}{T_0 n_0} H^{2+\gamma)/\gamma} \times \]

\[ [-2\beta \tanh [\beta (\ln (1 + z) + C_4)] - 2\alpha + 2\gamma]. \] (62)

Because of the second law of thermodynamics, the entropy production must be a non-negative function of the time. This requirement constraints the parameters of the r.h.s. of the above equation, which leads to the condition

\[ -2\gamma \beta \tanh [\beta (\ln (1 + z) + C_4)] - 2\alpha + 3\gamma \geq 0. \] (63)

Similarly to what was already done for the Hubble parameter, we will analyze the above condition only for early and very far future times. This is why we will only consider the strict inequality of Eq. (63), and we will study its saturation \( ds/dt = 0 \) only if it is required. It is easy to note that if \( z \to \infty \) then the term inside the brackets tends to the constant expression

\[ 3\gamma - 2(\alpha + \gamma \beta) = \frac{\sqrt{3\gamma}}{\xi_0} \times \]

\[ [c\gamma (2 - \gamma) + \sqrt{6\xi_0^2 (2 - \gamma) + c^2 \gamma^2 (2 - \gamma)^2}] < 0. \] (64)

On the other hand, if \( z \to -1 \), the term within the brackets tends to the constant expression

\[ 3\gamma + 2(\gamma \beta - \alpha) = \frac{\sqrt{3\gamma}}{\xi_0} \times \]

\[ \left[\sqrt{6\xi_0^2 (2 - \gamma) + c^2 \gamma^2 (2 - \gamma)^2} - c\gamma (2 - \gamma)\right] > 0. \] (65)

Thus, Eqs. (64) and (65) show that the entropy production is negative at early times and positive for late times, which leads to the conclusion that this model is not fully consistent with the physical requirement of an entropy monotonically growing in the whole range of the cosmological time. Nevertheless, this solution has been considered from the very beginning with only one matter fluid,
which we expect to successfully describes the transition from decelerated to accelerated expansion, but as it does not include the contribution from radiation, that is necessary to consider in order to describe early times of the universe. From Eq. (63) it follows that the change of sign in the entropy production occurs at a redshift value given by

\[ z_{ds/dt=0} = -1 + \exp \left[ \frac{1}{\beta} \arctanh \left( \frac{2\alpha - 3\gamma}{2\gamma \beta} \right) \right] \cdot C_4. \]  

(66)

Therefore, our solution at late times can successfully describe, for certain particular parameters values, the above mentioned transition and furthermore has a positive entropy production. Of course, at late times the dominant fluid is the pressureless DM, and therefore have to analyze the particular solution with \( \gamma = 1 \), which is addressed in the following section. A numerical calculation of Eq. (66) indicates us that the transition from a negative entropy production to a positive one may occurs at \( z \) in the range \( 1 < z < 5 \) choosing values of epsilon between 0.5 and 0.7, and 0.8 < \( \xi_0 < 2 \). In other words, allowed values of the model’s parameters can describe an scenario where the transition from a decelerated expansion to an accelerated one, occurs while the entropy production remains positive.

V. THE PARTICULAR CASE \( \gamma = 1 \)

As it was observed in the section IV.A., when \( \gamma = 1 \) or, in other words, when a pressureless DM is considered as the main material content of the universe, a particular solution of Eq. (10) is obtained by Eq. (35). Considering Eqs. (62) and (63) with \( \gamma = 1 \), and recalling that \( x = \ln (a) = -\ln (1 + z) \), this solution can be written as

\[ H(z) = H_0 \left[ C_1 (1 + z)^{m_1} + C_2 (1 + z)^{m_2} \right], \]

(67)

where \( H_0 \) is the Hubble parameter at the present time \( t = t_0 \), and

\[ m_1 = \frac{\sqrt{3}}{2\xi_0} \left( \sqrt{3\xi_0 + \epsilon + \sqrt{6\xi_0^2 \epsilon + \epsilon^2}} \right), \]

(68)

\[ m_2 = \frac{\sqrt{7}}{2\xi_0} \left( \sqrt{3\xi_0 + \epsilon - \sqrt{6\xi_0^2 \epsilon + \epsilon^2}} \right), \]

(69)

\[ C_1 = \frac{(q_0 + 1) - m_2}{m_1 - m_2}, \]

(70)

\[ C_2 = \frac{m_1 - (q_0 + 1)}{m_1 - m_2}. \]

(71)

In the above equations \( q_0 \) is the deceleration parameter at the present time \( t = t_0 \), and the conditions \( a_0 = 1 \) and \( C_1 + C_2 = 1 \) have been set. This solution was previously found and discussed in [25], but with a particular relation for the relaxation time of the form \( \xi_0 \beta^{s-1} \) (which correspond to \( \alpha = \xi_0 \) for our), instead of the more general relation as Eq. (7), in which the causality condition \( 0 < \epsilon \leq 1 \) is imposed. From a perturbative point of view, it is necessary to have a knowledge of the speed of sound in the fluid, which has to be very close to zero in order to be compatible with the growth of structures. In this sense, imposing from the beginning \( \epsilon = 1 \) leads to possible solutions of the Israel-Stewart equation that could behave reasonable at the background level, but present drawbacks at perturbative level.

Furthermore, in the solution found in [25], \( \Pi \) was used as a second initial condition, instead of using \( q_0 \). The constants \( m_1 \) and \( m_2 \) are reduced to the corresponding constants in [25] by taking the particular value \( \epsilon = 1 \). In the other hand, the Hubble parameter given by the Eq. (28) reduces to the Hubble parameter given by Eq. (67) when one chooses \( \gamma = 1 \). It is important to mention that in this solution there is no restriction upon \( q_0 \), being the most important feature of the new analytical solution.

A. Dynamics of the universe

In this subsection we are interested in characterizing the expansion of the universe according to the particular solution found in section V for the particular value \( \gamma = 1 \), which corresponds to CDM. From the definition \( H = \dot{a}/a \) and from Eq. (67) it follows that

\[ H_0 t + C = \int \frac{a^{m_1 + m_2 - 1}}{C_1 a^{m_2} + C_2 a^{m_1}} da, \]

(72)

where \( C \) is an integration constant. Similarly as the method used in the section IV-B, the integral of the above equation can be expressed as a hyper-geometric function \( _2F_1 \). For the initial condition \( a(t_0) = 1 \), the scale factor is given by the following implicit formula

\[ a^{m_1} _2F_1 \left[ 1, \frac{m_1}{m_1 - m_2}, 1 + \frac{m_1}{m_1 - m_2}, -a^{m_1 - m_2} \frac{C_2}{C_1} \right] = \]

\[ _2F_1 \left[ 1, \frac{m_1}{m_1 - m_2}, 1 + \frac{m_1}{m_1 - m_2}, \frac{C_2}{C_1} \right] + C_1 m_1 H_0 (t - t_0). \]

(73)

As we did with the general solution, the dynamics of the universe will be studied by considering the Hubble parameter expressed by the Eq. (67). The deceleration parameter can be written as

\[ q(z) = -1 + \frac{m_2 C_1 + m_2 C_2 (1 + z)^{m_2 - m_1}}{C_1 + C_2 (1 + z)^{m_2 - m_1}}. \]

(74)

We will study the behavior of both parameters at early times and at very far future. Considering the Eqs. (68) and (69), it follows that \( m_1 > 0 \) and \( m_1 > m_2 \) hold.
Therefore, at early times the Hubble and deceleration parameters behaves following the simple expressions

\[ H(z \to \infty) \to C_1 H_0 (1 + z)^{m_1}, \quad (75) \]

\[ q(z \to \infty) \to -1 + m_1, \quad (76) \]

while for very far future they behave as

\[ H(z \to -1) \to C_2 H_0 (1 + z)^{m_2}, \quad (77) \]

\[ q(z \to -1) \to -1 + m_2. \quad (78) \]

In the latter case, the Hubble parameter is not necessarily positive during the cosmic evolution. In fact, from Eq. (67) one sees that the Hubble parameter is zero for

\[ (1 + z)^{m_2 - m_1} = -\frac{C_1}{C_2}. \quad (79) \]

Because \(1 + z > 0\), from Eqs. (75) and (77), it follows that the Hubble parameter will always be positive for \(C_1 > 0\) and \(C_2 > 0\), and always negative for \(C_1 < 0\) and \(C_2 < 0\). Positive at early times and negative at late times for \(C_1 > 0\) and \(C_2 < 0\), and negative at early times and positive at late times for \(C_1 < 0\) and \(C_2 > 0\). Note that \(C_1 > 0\) requires the following constraint for the deceleration parameter (see Eq. (60))

\[ q_0 > \frac{1}{2} + \frac{\sqrt{3}}{2 \xi_0} \left( \epsilon - \sqrt{6 \xi_0^2 \epsilon + \epsilon^2} \right), \quad (80) \]

and \(C_2 > 0\) requires the following constraint for the deceleration parameter

\[ q_0 < \frac{1}{2} + \frac{\sqrt{3}}{2 \xi_0} \left( \epsilon + \sqrt{6 \xi_0^2 \epsilon + \epsilon^2} \right), \quad (81) \]

Hence, a positive Hubble parameter requires a deceleration parameter bounded according to (see Fig. 3(a))

\[ \epsilon - \sqrt{6 \xi_0^2 \epsilon + \epsilon^2} < \frac{2 \xi_0}{\sqrt{3}} \left( q_0 - \frac{1}{2} \right) < \epsilon + \sqrt{6 \xi_0^2 \epsilon + \epsilon^2}, \quad (82) \]

while a negative Hubble parameter, in the whole region is clearly not possible. On the other hand, a positive Hubble parameter at early times and negative at late times requires that \(q_0\) fulfills the condition of Eq. (80) and

\[ q_0 > \frac{1}{2} + \frac{\sqrt{3}}{2 \xi_0} \left( \epsilon + \sqrt{6 \xi_0^2 \epsilon + \epsilon^2} \right), \quad (83) \]

whose intersection is showed in Fig. 3(b). Finally, a negative Hubble parameter at early times and positive at late times requires that \(q_0\) fulfilled the condition (81) and

\[ q_0 < \frac{1}{2} + \frac{\sqrt{3}}{2 \xi_0} \left( \epsilon - \sqrt{6 \xi_0^2 \epsilon + \epsilon^2} \right), \quad (84) \]

whose intersection is showed in see Fig. 3(c).

FIG. 3. Comparative graphics of the permitted values of the parameters \(q_0\) and \(\xi_0\) for a fixed \(\epsilon\) that leads an always positive (a), positive at early times and negative at late times (b), and negative at early times and positive at late times (c) Hubble parameter.

The behavior of Eq. (75) depends on the exponent \(m_1\) which we already mentioned that it is positive. Furthermore, this exponent has the following constraint

\[ m_1 \geq \frac{3}{2}. \quad (85) \]

Therefore, the Hubble parameter is positive at early
times and decreases when the scale factor grows up to a positive non-zero value. If the condition \( \xi > 0 \) is fulfilled, then the Hubble parameter decreases up to a negative value. On the other hand, if the condition \( \xi < 0 \) is fulfilled, then the Hubble parameter is negative and increases when the scale factor grows up to a positive non-zero value. This behaviors correspond to a decelerated expansion, as can be see from Eqs. (77) and (85), which lead to a value of the deceleration parameter \( q \geq 1/2 \).

On the other hand, the behavior of Eqs. (77) and (78) depends on the exponent \( m_2 \), which will be positive only if

\[
3 \left(1 - 2\epsilon\right) \xi_0 + 2\sqrt{3}\epsilon > 0. 
\]  
(86)

Therefore, if \( 0 < \epsilon < 1/2 \), then \( m_2 > 0 \) and from Eq. (77) we will have a Hubble parameter that at late times continues decreasing, getting closer to zero and for the positives, if we fulfilled with the conditions (82) or (84). If we fulfilled the condition (83), then the Hubble parameter goes to zero at late times but from negatives values. The same behavior for the Hubble parameter at late time is possible when \( 1/2 < \epsilon \leq 1 \), and if only if, \( \xi_0 \) is given by the constraint

\[
\xi_0 < \frac{2\epsilon}{\sqrt{3(2\epsilon - 1)}}. 
\]  
(87)

If \( 0 < \epsilon < 1/2 \), then \( m_2 < 0 \) when the condition (82) is violated and from Eq. (77) we will have a Hubble parameter that at late times tends to positive infinite value, if fulfills the restrictions (82) or (84), or tends to a negative infinite value if the restriction of Eq. (83) is fulfilled. Finally, if \( 0 < \epsilon < 1/2 \), the special case \( m_2 = 0 \) arises, in the particular case where the inequality (87) becomes and equality and from Eq. (77) a constant Hubble parameter at late times is obtained.

From Eq. (78) it follows this last behavior leads to a stage of accelerated expansion when \( m_2 < 1 \), and this is only possible under the condition

\[
\xi_0 > \frac{2\sqrt{3}\epsilon}{18\epsilon - 1}, 
\]  
(88)

for

\[
1/18 < \epsilon \leq 1. 
\]  
(89)

If one of the above conditions is not fulfilled, then the behavior of the Hubble parameter leads to a stage of decelerated expansion. The transition between the accelerated expansion and the decelerated one occurs at the redshift value

\[
z_{q=0} = -1 + \left[ -\frac{C_1 (1 - m_1)}{C_2 (1 - m_2)} \right]^{1/(m_2 - m_1)}. 
\]  
(90)

In Fig. 4 are displayed the behavior of the deceleration parameter in terms of the free parameters \( q_0 \), \( \xi_0 \) and \( \epsilon \) for the Hubble parameter that is always positive.

---

**FIG. 4.** Plot of the deceleration parameter as a function of the redshift (a) and contour plot of the allowed values for the free parameters \( \xi_0 \) and \( \epsilon \) that leads to a transition between the decelerated expansion to a accelerated one (b).

### B. Thermodynamics properties of the solution

For \( \gamma = 1 \) Eq. (61) takes the form

\[
\frac{dS}{dt} = \frac{3H^2a^3}{T_0n_0} \left( 2\frac{dH}{da} + 3H \right), 
\]  
(91)

and using the Eq. (61), this can be written as

\[
\frac{dS}{dt} = \frac{3H_oH^2(1+z)^{-3}}{T_0n_0} \left[C_1 (1+z)^{m_1} (3-2m_1) + C_2 (1+z)^{m_2} (3-2m_2) \right]. 
\]  
(92)

Due to the second law of Thermodynamics, the above derivative must be a non-negative function of the cosmological time. As the first factor in the above expression is positive, a non-negative entropy production requires

\[
[C_1 (1+z)^{m_1} (3-2m_1) + C_2 (1+z)^{m_2} (3-2m_2) ] > 0. 
\]  
(93)
As we have done for the Hubble parameter, we are going to analyzed the above condition only for early and very far future times. This is why we only considered the strict inequality in the Eq. (81). If \( z \to \infty \) the above term in bracket tends to the expression

\[ C_1 (3 - 2m_1) > 0, \]  

(94)

but from Eqs. (68) and (69) it follows

\[ 3 - 2m_1 = \frac{\sqrt{3}}{2\xi_0} \left( \epsilon + \sqrt{6\xi_0^2 \epsilon + \epsilon^2} \right) < 0, \]  

(95)

\[ 3 - 2m_2 = \frac{\sqrt{3}}{2\xi_0} \left( \sqrt{6\xi_0^2 \epsilon + \epsilon^2} - \epsilon \right) > 0, \]  

(96)

so, the Eq. (95) shows that a positive entropy production at early times requires a negative constant \( C_1 \), which contradicts the content of the Eq. (80), i.e., a positive entropy production at early times necessarily implies a positive Hubble parameter. On the other hand, for \( z \to -1 \) the terms in brackets in Eq. (83) tends to the expression

\[ C_2 (3 - 2m_2) > 0, \]  

(97)

and the Eq. (86) shows that a positive entropy production at early times requires a positive constant \( C_2 \), that is the condition indicated in the Eq. (81), i.e., a positive entropy production at late times necessarily requires a positive Hubble parameter at this times. Thus, a positive entropy production for all the cosmic evolution is only possible for a Hubble parameter that is negative at early times and positive at late times. In the other hand, a positive Hubble parameter for all the cosmic evolution leads to a negative entropy production at early times and positive at late times. From Eq. (92) it can be see that the change of sign in the entropy production occurs at redshift given by

\[ z_{ds/dt=0} = -1 + \left[ \frac{C_1 (3 - 2m_1)}{C_2 (3 - 2m_2)} \right]^{1/(m_2-m_1)}. \]  

(98)

A numerical calculation of Eq. (98) indicates us that the transition from a negative entropy production to a positive one may occurs at \( z \) in the range \( 1 < z < 5 \) choosing values of epsilon between 0.5 and 0.7, and 0.8 < \( \xi_0 < 2 \). The result is similar to the case of \( \gamma \neq 1 \) but in this case, the value of the redshift es lower than the value for the general case, for the same values of \( \epsilon \) and \( \xi_0 \); even so, the intervals are the same. The conclusion is this case is similar to the former case \( \gamma \neq 1 \).

C. Special cases of the particular solution

In the Eq. (77) there are two particular cases: i) \( C_1 = 0 \) and \( C_2 = 1 \) and ii) \( C_1 = 1 \) and \( C_2 = 0 \). These particular cases were not addressed so far because they lead to quite different physical scenarios that we will discuss in this section.

From Eq. (70) we see that in the case i) the deceleration parameter has the particular value

\[ q_0 = m_2 - 1, \]  

(99)

which correspond to Eq. (80) but with

\[ q_0 = \frac{1}{2} + \frac{\sqrt{3}}{2\xi_0} \left( \epsilon - \sqrt{6\xi_0^2 \epsilon + \epsilon^2} \right). \]  

(100)

This leads to a Hubble parameter as a function of the scale factor of the form

\[ H(a) = H_0 a^{-m_2}, \]  

(101)

which is always positive during cosmic evolution. The scale factor can be obtained straightforwardly, and is given by

\[ H_0 t + C = \int a^{m_2-1} da, \]  

(102)

where \( C \) is an integration constant.

Taking \( m_2 = 0 \) in Eq. (101) we obtain a de Sitter type expansion with a constant Hubble parameter. The scale factor, with the initial condition \( a(t_0) = 1 \), is given as a function of time by the expression

\[ a(t) = e^{H_0(t-t_0)}. \]  

(103)

For \( m_2 \neq 0 \), the scale factor as a function of the cosmic time is given by

\[ a(t) = \left[ H_0 (t-t_0) m_2 + 1 \right]^{1/m_2}. \]  

(104)

Inserting this expression into Eq. (101), one obtains the following Hubble parameter

\[ H(t) = \frac{H_0}{H_0 (t-t_0) m_2 + 1}. \]  

(105)

In order to avoid nonphysical scale factors. The solution \( H_0 (t-t_0) m_2 + 1 \) represents an universe with an origin at time \( t = t_0 - 1/(H_0 m_2) \) and with an accelerated expansion for \( 0 < m_2 < 1 \) and a decelerated expansion for \( m_2 > 1 \). The case \( m_2 = 1 \) is clearly an universe with constant rate of expansion during the whole cosmic evolution. Finally, for \( m_2 < 0 \) the Eq. (101) can be rewritten as

\[ a(t) = \frac{1}{\left[ 1 - H_0 (t-t_0) \right]^{1/m_2}}, \]  

(106)

where clearly one needs to impose

\[ t < t_0 + \frac{1}{H_0 |m_2|}. \]  

(107)

In this case the Eq. (106) represent an emergent universe with an accelerated expansion at late times and a Big Rip at the time \( t_{BR} = t_0 + 1/(H_0 |m_2|) \).
Let us see now the behavior of the above solution in terms of their entropy production. In the case i) Eq. (32), for the particular values \( C_1 = 0 \) and \( C_2 = 1 \), gives the entropy production as a function of the scale factor, which yields

\[
\frac{dS}{dt} = \frac{3H_0^3}{T_0 \rho_0} (3 - 2m_2) a^{3(1-m_2)}, \quad (108)
\]

which indicates that the entropy production is always positive since \( 3 - 2m_2 > 0 \) by Eq. (35). Within this range of the parameter \( m_2 \) we have the cosmological scenarios with accelerated expansion \((0 < m_2 < 1)\), with decelerated expansion \( (m_2 > 1) \) and expansion at constant rate \( (m_2 = 1) \). These special cases have the particularity of the absence of transition from a decelerated phase to an accelerated expansion. I this sense they present a well behavior in terms of the thermodynamics but they are unable to model the universe like the \( \Lambda CDM \) model where a transition from the DM dominated era to the DE era naturally appears.

The other case, \( C_2 = 0 \) and hence \( C_1 = 1 \) will not be addressed explicitly as it drives to a cosmic evolution with nonphysical negative entropy production.

VI. CONCLUSIONS

We have discussed a novel solution of the equation for evolution of the Hubble parameter in the framework of the full causal thermodynamics of Israel-Stewart. This solution was obtained considering a bulk viscous coefficient with the dependence \( \xi = \xi_0 \rho^{1/2} \), and the general expression given by Eq. (3) for the relaxation time. We have also showed that the particular solution with \( \gamma = 1 \) reduces Eq. (20) to a second order linear equation, and, as it is well known, the general solution can be written as a linear combination of two independent solutions (see Eq. (35)), for which the restrictions of Eq. (34) do not apply.

In the case of the novel solution with arbitrary \( \gamma \), the entropy production is negative at early times and positive for late times with a positive Hubble parameter. On the other hand, a transition from a decelerated phase to an accelerated one is only possible for large values of \( \epsilon \) (see Eqs. (55) and (56)).

The lesson we have learned here is that exact solutions of unified DM models displaying the transition above mentioned, which is an essential feature supported by the observational data, not necessarily satisfy simultaneously the other physical requirements related to a positive entropy production and a very low non-adiabatic contribution to the speed of sound coming from the dissipation.

In the case of the \( \gamma = 1 \)-solution, one sees that the Hubble parameter can be positive or negative at early times as well as late times, depending on the election of the deceleration parameter \( q_0 \). It is worth mentioning that this accelerated expansion is compatible with a positive Hubble parameter at late times, and therefore one needs a \( q_0 \) value that fulfills the conditions indicated above in Eq. (34), which is possible for some values of the free parameters. On the other hand, if \( 0 < \epsilon \leq 1/8 \), then the accelerated expansion will not be possible, independent of the value of \( \xi_0 \). Nevertheless, for \( 1/18 < \epsilon \leq 1 \) it will possible, if and only if, \( \xi_0 \) satisfies the inequality given by Eq. (38). From the thermodynamical point of view, we have found that the entropy production for this model can be positive or negative depending on the Hubble parameter at early and late times. There are three possible scenarios: 1) The case when the Hubble parameter is negative at late times, which leads to a negative entropy production; 2) The case when the Hubble parameter is always positive, the entropy production is negative at early times and positive at late times. This conclusion remains true for arbitrary \( \gamma \) values; and 3) The case with a negative Hubble parameter at early times and positive at late times, which results in a positive entropy production after the late time transition.

Since our model contains only one cold fluid as the main component of the universe, it should only be considered as an adequate approximation for the late time evolution, where cold DM dominates. In this sense, our model cannot expected to be fairly representative of the early times evolution, where ultrarelativistic matter dominates. From the above mentioned cases, only the second one can be considered as a correct scenario of a cosmic evolution: it has a transition between decelerated and accelerated expansions, and at the same time it has a positive entropy production at late times. Its unphysical negative entropy production at early times should not be then a reason to discard it.

On the other side, an accelerated expansion at late times is obtained if \( \epsilon \) has a large value, close to 1, which implies that the non-adiabatic contribution to the speed of sound from dissipation is very close to the speed of light. In [30] a model with dissipation and a cosmological constant was constrained using observations from supernovae SNe Ia. The results obtained indicates a similar behavior in the sense that if \( \epsilon \) is very large, the contribution from the cosmological constant can be neglected. Furthermore, a large value of \( \epsilon \) is an unphysical feature for a fluid like pressureless DM and it is not compatible with the structure formation. In fact, a very small non-adiabatic contribution to the speed of sound, in the range of \( 10^{-11} \ll \epsilon \ll 10^{-8} \), was found in [27], in order to be consistent with the properties of structure formation.

The special case discussed in Sec. V is the only case where there is a positive entropy production during all the cosmic evolution and at the same time a Hubble parameter that remains positive. Nevertheless, this requires a very strong constraint on the deceleration parameter and therefore this solution would not be compatible with the observational value of \( q_0 \). It is important to mention that the De Sitter expansion at late times is possible only for \( \epsilon > 1/2 \), which is in agreement with the De Sitter solution found in section III.

Another interesting feature obtained is that the exact
solution found for pressureless DM, gives the advantage to allow computing the non-adiabatic contribution to the speed of sound coming from its kinematical properties (possibility of transition from a decelerated expansion to an accelerated one), and its thermodynamical behavior, expressed in terms of the sign of the entropy production. In the $\gamma = 1$-limit, we have obtained a solution that displays the above mentioned transition, and also has a positive entropy production, nevertheless it leads to a large non-adiabatic contribution to the speed of sound, which is not compatible with the structure formation. As an overall conclusion of our study, further refinements to models with dissipative DM are still required to obtain more realistic alternatives to $\Lambda \text{CDM}$.

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