ISOMETRIES BETWEEN LEAF SPACES

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Abstract. In this paper we prove that an isometry between orbit spaces of two proper isometric actions is smooth if it preserves the codimension of the orbits or if the orbit spaces have no boundary. In other words, we generalize Myers-Steenrod’s theorem for orbit spaces. These results are proved in the more general context of singular Riemannian foliations.

1. Introduction

In this section, we recall some definitions and state our main results in Theorem 1.2 and Theorem 1.6.

Definition 1.1 (SRF). A partition $F$ of a complete Riemannian manifold $M$ by connected immersed submanifolds (the leaves) is called a singular Riemannian foliation (SRF for short) if it satisfies condition (1) and (2):

1. $F$ is a singular foliation, i.e., for each leaf $L$ and each $v \in TL$ with footpoint $p$, there is a smooth vector field $X$ with $X(p) = v$ that is tangent at each point to the corresponding leaf.

2. $F$ is Riemannian, i.e., every geodesic perpendicular to one leaf is perpendicular to every leaf it meets.

A leaf $L$ of $F$ (and each point in $L$) is called regular if the dimension of $L$ is maximal, otherwise $L$ is called singular. In addition a regular leaf is called principal if it has trivial holonomy. A typical example of a singular Riemannian foliation is the partition of a Riemannian manifold into the orbits of an isometric action. In this case the principal leaves coincide with the principal orbits.

Let us consider a SRF $(M, F)$ with closed leaves. The quotient $M/F$ is equipped with the natural quotient metric and a natural quotient $C^k$ structure. The $C^k$ structure on $M/F$ is given by the sheaf of $C^k$ basic functions on $M$. One says that a map $\varphi : M_1/F_1 \to M_2/F_2$ between two leaf spaces of SRF’s is of class $C^k$ if the pull-back by $\varphi$ sends smooth functions on $M_2/F_2$ to functions on $M_1/F_1$ of class $C^k$. In the case of group action and with $\varphi$ smooth, this definition coincides with the definition of Schwarz [7].

As pointed out in [1], it is natural to ask if an isometry between two orbit spaces, or more general an isometry between the leaf spaces of SRF’s, is smooth. In other words, if a version of Myers-Steenrod theorem is still valid for quotient spaces.

In this paper we prove the above conjecture in two special cases.

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**Theorem 1.2.** Let $M_1$ and $M_2$ be complete Riemannian manifolds and $(M_1, F_1)$ and $(M_2, F_2)$ SRF’s with closed leaves. Assume that there exists an isometry $\varphi : M_1/F_1 \to M_2/F_2$ that preserves the codimension of the leaves. Then $\varphi$ is a smooth map.

**Example 1.3.** Let us illustrate the above result with an example constructed by suspending two homomorphisms. In this example $M_1/F_1$ and $M_2/F_2$ are isometric but $(M_1, F_1)$ and $(M_2, F_2)$ are not foliated diffeomorphic to each other. Consider two manifolds $Q_1$, $Q_2$ with the same fundamental group but not homeomorphic to each other. Let $V$ be Riemannian manifolds with a SRF $\tilde{F}_0$, e.g., orbits of an isometric action. Consider a homomorphism $\rho : \pi_1(Q_i, q_0) \to K \subset \text{Iso}(V)$. Here $K$ must act on $V$ preserving the foliation $\tilde{F}_0$, i.e., it sends leaves to leaves. For example, $V$ can be a product $V_1 \times V_2$, each leaf $L_{(x,y)}$ is contained in $V_1 \times \{ y \}$ and $K \subset \text{Iso}(V_2)$. Let $\tilde{P}_i : \tilde{Q}_i \to Q_i$ be the projection of the universal cover of $Q_i$. Then there is an action of $\pi_1(Q_i, q_0)$ on $\tilde{M}_i = \tilde{Q}_i \times V$ given by

$$(\hat{q}, v) \cdot [\alpha] = (\hat{q} \cdot [\alpha], \rho(\alpha^{-1}) \cdot v),$$

where $\hat{q} \cdot [\alpha]$ denotes the deck transformation associated to $[\alpha]$ applied to a point $\hat{q} \in \tilde{Q}$. We denote by $\tilde{M}_i$ the set of orbits of this action, which is clearly a manifold, and let $\Pi_i : \tilde{M}_i \to M_i$ be the canonical projection. Define a map $\mathcal{P}_i$ by

$$\mathcal{P}_i : M_i \longrightarrow Q_i,$$

$$\Pi_i(\hat{q}, v) \longmapsto \tilde{P}_i(\hat{q}).$$

It follows that $M_i$ is the total space of a fiber bundle, and $\mathcal{P}_i$ is its projection over the base space $Q_i$. In addition, the fiber is $V$ and the structural group is given by the image of $\rho$. Finally, set $F_i = \Pi_i(\tilde{Q}_i \times \tilde{F}_0)$.

**Example 1.4.** Another typical example of isometry between leaf spaces that preserves the codimension of the leaves is the isometry induced by the singular holonomy, or more generally by the flow of a continuous transversal Killing vector field that is tangent to the closure of the leaves of a SRF.

Consider for example a SRF with locally closed leaves, i.e., for each (singular) point $q$ there exists a neighborhood $U$ such that the intersection $F_U := F \cap U$ is a closed SRF. It can happen that two different leaves of $F \cap U$ are contained in the same leaf of $F$. It is possible to prove that the closure of the leaves is a partition of $M$ into submanifolds that is Riemannian; see Molino [4]. It is also possible to prove that there exists a continuous transversal Killing vector field $X$ tangent to the closure of the leaves. Let $\varphi : (-\epsilon, \epsilon) \times U/F_U \to U/F_U$ the projection of the flow of $X$.

The above theorem implies that $\varphi \epsilon$ is smooth.

**Remark 1.5.** The above theorem implies that if $M_i/F_i$ are isometric orbifolds then they are diffeomorphic in the sense of Schwarz and hence in the classical sense, see, e.g., Strub [8] and Swartz [9, Lemma 1].

A small modification of the proof of Theorem 1.2 also allow us to prove the smoothness of isometries between orbit spaces without boundary; for definitions see Section 3.

**Theorem 1.6.** Let $\varphi : M_1/F_1 \to M_2/F_2$ be an isometry between the leaf spaces of two SRF’s $M_i/F_i$, and suppose that $M_1/F_1$ has no boundary. Then $\varphi$ is smooth.
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2. Proof of Theorem 1.2

In order to avoid cumbersome notations, we will denote each basic function \( f : M \to \mathbb{R} \) and the induced function on \( M/\mathcal{F} \) by the same letter \( f \).

Before we go through the details of the proof, let us briefly sketch the main ideas. First we will note that the main problem can be reduced to a problem in Euclidean space. In Proposition 2.4 we will prove that an isometry between quotient spaces preserves the projection of these vector fields. Finally, in Proposition 2.5 we will prove that an isometry between quotient spaces preserves the mean curvature vector field of the principal leaves is basic, and an isometry between quotient spaces preserves the projection of these vector fields. Finally, in Proposition 2.6 we will prove that an isometry between quotient spaces preserves the basic mean curvature vector field must be smooth. In order to prove this result, we need to show that each isometry between quotient spaces is at least of class \( C^1 \); see Proposition 2.3.

Let us first observe that the main problem can be reduced to a problem in Euclidean space, following standard arguments from the theory of SRF’s; see [6, 2, 5].

Recall that the restriction of \( \mathcal{F} \) to a slice \( S_p \) is diffeomorphic to a SRF \( \mathcal{F}_p \) on the normal space \( \nu_p L_p = T_p S_p \) with respect to the flat metric \( g_p \). This foliation \( \mathcal{F}_p \) is called infinitesimal foliation at \( p \).

Let \( p_1 \in M_1 \) and \( p_2 \in M_2 \) so that \( \varphi(\pi_1(p_1)) = \pi_2(p_2) \), where \( \pi_i : M_i \to M_i/\mathcal{F}_i \) is the quotient map. Recall that \( \pi_i(S_{p_i}) = (\nu_{p_i} L_{p_i}/\mathcal{F}_{p_i})/K_i \) where \( \mathcal{F}_{p_i} \) is the infinitesimal foliation at \( p_i \) and \( K_i \) is a finite subgroup of isometries of \( (\nu_{p_i} L_{p_i}, g_{p_i}) \) that preserves the foliation \( \mathcal{F}_{p_i} \); the existence of \( K_i \) can be proved using the linearization of vector fields or as in Section 2.2 of [5]. Also recall that the flat metrics \( g_{p_i} \) are the limit of metrics \( g_{\lambda} = \lambda^* \frac{1}{2} h_\lambda^* \exp_{p_i}^* g_i \), where \( h_\lambda \) denotes the homothetic transformation with respect to zero. Therefore the isometry \( \varphi \) induces an isometry \( \varphi : (\nu_{p_1} L_{p_1}/\mathcal{F}_{p_1})/K_1 \to (\nu_{p_2} L_{p_2}/\mathcal{F}_{p_2})/K_2 \).

The above considerations lead us to deal with actions of non-connected subgroups of the isometry group of Euclidean spaces or more general with SRF’s with non-connected leaves on Euclidean spaces, concept that we now review.

Let \( \tilde{\mathcal{F}} = \{ \tilde{L} \} \) be a SRF on a Riemannian manifold \( M \). Assume that there exists a discrete group \( K \) of the isometry group of \( M \) that preserves the foliation \( \tilde{\mathcal{F}} \). With \( (M, \tilde{\mathcal{F}}, K) \) we can construct a new partition \( \mathcal{F} = \{ L \} \) of \( M \) into non-connected submanifolds (the non-connected leaves) as follows. Two leaves \( \tilde{L}_p, \tilde{L}_q \in \tilde{\mathcal{F}} \) are contained in the same non-connected leaf \( L \in \mathcal{F} \) if there exists an isometry \( k \in K \) with \( k \cdot \tilde{L}_p = \tilde{L}_q \).

A typical example of a SRF with non-connected leaves on a Euclidean space is the action of the isotropy group \( G_p \) on the tangent space \( T_p S_p \) of a slice \( S_p \). In this case \( K = G_p/(G_p)_0 \) where \( (G_p)_0 \) denotes the connected component of the identity of \( G_p \) and the non-connected leaves are the orbits of \( G_p \).

A leaf \( L \) of \( \mathcal{F} \) is called principal leaf if it satisfies the following conditions:

(1) Each leaf \( \tilde{L} \) of \( \tilde{\mathcal{F}} \) contained in \( L \) is a principal leaf of \( \tilde{\mathcal{F}} \).
If there exists an isometry $k \in K$ which fixes a leaf $\hat{L}$ that is contained in $L$ then $k$ acts as the identity on the normal space of $\hat{L}$.

From now on, $(M, \mathcal{F})$ will denote a SRF with possible non-connected leaves, and we will simply call it again a SRF.

We continue the proof of Theorem 1.2 stressing a special property of SRF’s on Euclidean space.

**Proposition 2.1.** Let $(\mathbb{R}^n, \mathcal{F}_1)$, $(\mathbb{R}^{n+k}, \mathcal{F}_2)$ be two (possibly non-connected) SRF’s with closed leaves. If there exists an isometry $\varphi : \mathbb{R}^n/\mathcal{F}_1 \to \mathbb{R}^{n+k}/\mathcal{F}_2$ that preserves the codimension of the leaves, then the mean curvature vector fields of the corresponding principal leaves are basic and $\varphi$ preserves the projections of those vector fields.

**Proof.** The proposition was proved in Gromoll and Walschap [4, Theorem 4.1.1] in the case of regular Riemannian foliations. In what follows we will explain how that proof can be adapted in the case of SRF.

Take $p \in \mathbb{R}^n$, $\hat{p} \in \mathbb{R}^{n+k}$ principal points such that $\varphi(\pi_1(p)) = \pi_2(\hat{p})$. Take $x \in H_p$, $\hat{x} \in H_{\hat{p}}$, such that $\varphi_* \pi_2(x) = \pi_2, \hat{x}$. Finally, define $\gamma(t) = p + tx$, and $\hat{\gamma}(t) = \hat{p} + t\hat{x}$. It is easy to see that $\varphi(\pi_1(\gamma)) = \pi_2(\hat{\gamma})$.

We need to show that $\text{tr}(S_x) = \text{tr}(\hat{S}_x)$. We will actually show something stronger, namely that every nonzero eigenvalue of $S_x$ is an eigenvalue of $\hat{S}_x$ of the same multiplicity.

Let us denote $E_\lambda$ (respectively $\hat{E}_\lambda$) the eigenspace of $S_x$ (respectively $\hat{S}_x$) related to an eigenvalue $\lambda$ (respectively $\hat{\lambda}$). For each fixed eigenvalue $\lambda$ of $S_x$ we want to prove that $\hat{\lambda}$ is also an eigenvalue of $\hat{S}_x$ and that that $\dim(E_\lambda) = \dim(\hat{E}_\lambda)$. Along $\gamma$ and $\hat{\gamma}$, define two spaces of holonomy Jacobi fields, i.e., vertical $L_\mu$-Jacobi fields and $L_{\hat{\mu}}$-Jacobi fields, (see [4, Definition 1.4.3])

\[
K_\lambda = \left\{ J \mid J \text{ is holonomy Jacobi field along } \gamma \text{ and } J(1/\lambda) = 0 \right\},
\]

\[
\hat{K}_\hat{\lambda} = \left\{ \hat{J} \mid \hat{J} \text{ is holonomy Jacobi field along } \hat{\gamma} \text{ and } \hat{J}(1/\hat{\lambda}) = 0 \right\}.
\]

Notice that

\[
\dim K_\lambda = \dim L_{\gamma(0)} - \dim L_{\gamma(1/\lambda)}, \quad \dim(\hat{K}_\hat{\lambda}) = \dim L_{\hat{\gamma}(0)} - \dim L_{\hat{\gamma}(1/\hat{\lambda})},
\]

and $\dim(K_\lambda) = \dim(\hat{K}_\hat{\lambda})$. Since $p$, $\hat{p}$ are regular, $\dim K_\lambda = \dim K_\lambda(0)$ and $\dim \hat{K}_\hat{\lambda} = \dim \hat{K}_\hat{\lambda}(0)$; see [2]. Also note that $K_\lambda(0) \subseteq E_\lambda$ and $\hat{K}_\hat{\lambda}(0) \subseteq \hat{E}_\lambda$. Therefore it’s enough to show that

\[
\dim E_\lambda/K_\lambda(0) = \dim \hat{E}_\lambda/\hat{K}_\hat{\lambda}(0).
\]

Given $[v] \in E_\lambda/K_\lambda(0)$, pick a representative $v$ in $E_\lambda$. Take $J_v$ the projectable Jacobi field along $\gamma(t)$ such that $J_v(0) = v$, $J'_v(0) = -S_x v = -\lambda v$. Since we are in $\mathbb{R}^n$, this Jacobi field is

\[
J_v(t) = J_v(0) + tJ'_v(0) = (1 - \lambda t)v.
\]

We can now look at the projected vector field $\pi_1 J_v$ along $\pi_1(\gamma)$, and notice that $\lim_{t \to 1/\lambda} \|\pi_1 J_v(t)\| = 0$. Notice, moreover, that $\pi_1 J_v$ doesn’t depend on the choice of representative $v$ we started with. Applying $\varphi$, we obtain a Jacobi field $\overline{J}_v$ along $\varphi(\pi_1(\gamma)) = \pi_2(\hat{\gamma})$ such that $\lim_{t \to 1/\lambda} \|\overline{J}_v(t)\| = 0$.

In what follows we will prove that there exists a projectable Jacobi field $\hat{J}_v$ along $\hat{\gamma}$ that projects to $\overline{J}_v$, such that $\hat{J}_v(1/\hat{\lambda}) = 0$. 


Let $P$ be a projectable Jacobi field that projects to $\mathcal{J}_\gamma$. We claim that $P$ is tangent to the leaf $L_{\bar{\gamma}(1/\lambda)}$. This is clear if $\bar{\gamma}(1/\lambda)$ is a regular point. So assume that this is a singular point. Let $P^v$ be the component of $P^\gamma$ that is tangent to the slice at $\bar{\gamma}(1/\lambda)$; here $P^v$ denotes the vertical component of $P$. The claim is equivalent to saying that $P^v(1/\lambda) = 0$. This, on the other hand, follows from equation [4, page 32, Eq.(1.6.1)] below (2.1) \[(P^\gamma)'v = -S_{\bar{\gamma}}P^v - A_{\bar{\gamma}}P^h,\] and from the fact that principal curvatures of spheres centered at $\bar{\gamma}(1/\lambda)$ increase when the radii decrease. Here $A_{\bar{\gamma}}X = \frac{1}{2}[X,Y]^v$ (where $X$ and $Y$ are horizontal vector fields), and we have used the fact that the vertical and horizontal distributions, as well as $A_{\bar{\gamma}}$, are well defined along $\bar{\gamma}$ even at singular points; see Wilking [10] or [4, Section 1.7].

Finally, since $P$ is tangent to the leaf $L_{\bar{\gamma}(1/\lambda)}$, there exists a holonomy Jacobi field $Y$ so that $Y(1/\lambda) = -P(1/\lambda)$; see [2]. Set $\bar{J}_v = P + Y$. By construction, the projectable Jacobi field $\bar{J}_v$ along $\bar{\gamma}$ projects to $\mathcal{J}_v$, and $\bar{J}_v(1/\lambda) = 0$.

Such a $\bar{J}_v$ is defined, up to a Jacobi field in $\mathcal{K}_\lambda$. In particular, $\bar{J}_v(0)$ is an eigenvector of $S_{\bar{\gamma}}$ with eigenvalue $\lambda$, and it is well defined up to an element in $\mathcal{K}_\lambda(0)$. Therefore, the map $\psi : E/\mathcal{K}(0) \longrightarrow \bar{E}/\mathcal{K}(0)$

\[
[v] \longmapsto [\bar{J}_v(0)]
\]
is well defined. The linearity of the map $\psi$ can be proved using the fact that the space of the projected Jacobi fields $\pi_2_*J_v$ is a vector space. The map $\psi$ has an inverse obtained by inverting the roles of $p$ and $\bar{p}$. Thus the two spaces have the same dimension, which is what we wanted to prove.

\[\square\]

Remark 2.2. The above proposition implies that, given a SRF $\mathcal{F}$ on $\mathbb{R}^n$, then each principal leaf $L$ of $\mathcal{F}$ is a generalized isoparametric submanifold, i.e., the principal curvatures along a basic vector field of $L$ are constant.

Due to the above discussion, the proof of the theorem will be concluded, if we can show that an isometry that preserves the (basic) mean curvature vector fields is smooth. In order to do this, we show:

**Proposition 2.3.** Let $M_1$ and $M_2$ be complete Riemannian manifolds and $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be SRF’s with closed leaves. Then an isometry $\varphi : M_1/\mathcal{F}_1 \rightarrow M_2/\mathcal{F}_2$ is of class $C^1$.

**Proof.** First we assume that there exists only two strata. As pointed out in the beginning of the proof of the theorem, it suffices to assume that $M_1 = \mathbb{R}^n = M_2$ and that the singular stratum $Y$ is a stratum of leaves that are points. Let $X$ denote the orthogonal complement of $Y$. Note that $\varphi|_Y : Y \rightarrow Y$ is smooth and $\varphi$ sends geodesics orthogonal to the minimal stratum to geodesics orthogonal to the minimal stratum.

Let $f$ be a smooth basic function on $M_2$. We want to prove that $\varphi^*f$ is a basic function on $M_1$ of class $C^1$.

Since $f$ is smooth, it follows from Taylor’s formula that

\[f(x, y) = f(0, y) + R(x, y),\]
consider a sequence \( p \) for each sequence \( v \) strata we can also assume that the sequence is contained in a stratum \( \Sigma \).

\[ \lim_{t \to 0} (x(t), y(t)) = (x, y) \]

Equations (2.4) and (2.5) imply (2.3) and this concludes the case where there exist only two strata. Now the general case will follow by induction.

Indeed assume that the proposition is true when we have \( i_0 - 1 \) strata. As before we can assume that the minimal stratum \( Y \) is a stratum of leaves that are points.

As discussed above it suffices to prove that

\[ \lim_{n \to \infty} v_n \cdot R(y_n) = 0, \]

for each sequence \( q_n \to (0, y) \) and \( \|v_n\| \leq 1 \).

Equations (2.4) and (2.5) imply (2.3) and this concludes the case where there exist only two strata. Now the general case will follow by induction.

The next proposition concludes the proof of Theorem 1.2.
Proposition 2.4. Let $M_1$ and $M_2$ be complete Riemannian manifolds and $(M_i, \mathcal{F}_i)$ SRF’s with closed leaves such that the mean curvature vector fields $H_i$ of the corresponding principal leaves are basic. Assume that there exists an isometry $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ that preserves the mean curvature fields restricted to the principal stratum. Then $\varphi$ is a smooth map.

Proof. Recall that $H_i$ also denotes the projection onto $M_i^0/\mathcal{F}_i$, where $M_i^0$ denotes the set of principal points. We want to prove that $\varphi^* f$ is a smooth basic function of $(M_1, \mathcal{F}_1)$.

Let $f$ be a smooth basic function of $(M_2, \mathcal{F}_2)$. We want to prove that $\varphi^* f$ is a smooth basic function of $(M_1, \mathcal{F}_1)$.

We start by recalling that (see e.g., [3] page 53).

$$\Delta_{M_i^0} f = \Delta_{M_i^0} f - g_i(\nabla f, H_i).$$

Set $u := \Delta f$. Equation (2.8) implies that $u$ is a smooth basic function of $(M_2, \mathcal{F}_2)$. Our hypothesis assure us that $\varphi : M_1^0/\mathcal{F}_1 \to M_2^0/\mathcal{F}_2$ is an isometry and that $d\varphi H_1(\cdot) = H_2 \varphi(\cdot)$. By straightforward calculations, we infer from those hypothesis and Equation (2.8) that

$$\Delta \varphi^* f|_{M_i^0} = \varphi^* u|_{M_i^0}.$$

Now we fix a singular point $p \in M_1$ and consider a small neighborhood $U$ of $p$ diffeomorphic to a ball in Euclidean space.

Lemma 2.5.

$$\int_U \varphi^* f \, \Delta h = \int_U \varphi^* u \, h$$

for each smooth function $h$ with compact support on $U$.

Proof. As we will see below, the proof will follow from equation (2.11) and from the fact that $\varphi^* f$ and $\varphi^* u$ are functions of class $C^1$ (recall Proposition 2.3).

Let $W$ be a neighborhood of the stratum $\Sigma_1 \cup \ldots \cup \Sigma_n$ contained in $U$. Assume that $\partial W$ is a smooth hypersurface. Then Green’s second identity assure us that

$$\int_{U-W} (\varphi^* f \, h) - (\varphi^* f) \, \Delta h = \int_{\partial W} h \, g_1(\nabla \varphi^* f, \eta) - \varphi^* f \, g_1(\nabla h, \eta),$$

where $\eta$ is the normal vector field of $\partial W$.

From equation (2.11), we infer

$$\int_{U-W} (\varphi^* u \, h) - (\varphi^* u) \, \Delta h = \int_{\partial W} h \, g_1(\nabla \varphi^* f, \eta) - \varphi^* f \, g_1(\nabla h, \eta).$$

Fixed an arbitrary positive $\epsilon$, it is possible to choose a small neighborhood $W$ so that

$$\left| \int_U (\varphi^* u) \, h - \int_{U-W} (\varphi^* u) \, h \right| < \frac{\epsilon}{3},$$

$$\left| \int_U (\varphi^* f) \, \Delta h - \int_{U-W} (\varphi^* f) \, \Delta h \right| < \frac{\epsilon}{3},$$

One can also choose $W$ such that

$$\left| \int_{\partial W} h \, g_1(\nabla \varphi^* f, \eta) - \varphi^* f \, g_1(\nabla h, \eta) \right| < \frac{\epsilon}{3}.$$
In fact, in order to get equation (2.13) consider the union of strata \( \Sigma_1 \cup \ldots \cup \Sigma_{n-1} \) that have codimension greater than 1. Then we can find a neighborhood \( \tilde{W} \) of \( \Sigma_1 \cup \ldots \cup \Sigma_{n-1} \) so that

\[
\int_{\partial \tilde{W}} \frac{1}{c} < \frac{c}{6c},
\]

where \( c = \sup_{\Sigma_1} |h g_1(\nabla \varphi^* f, \eta) - \varphi^* f g_1(\nabla h, \eta)| \). If there is no stratum \( \Sigma_n \) of codimension 1, then (2.14) implies (2.13). Therefore, assume that there exists a stratum \( \Sigma_n \) of codimension 1. From [6] it follows that this stratum has only regular leaves with nontrivial holonomy. Set \( K_r := \partial \text{Tub}_r(\Sigma_n - (\Sigma_n \cap \tilde{W})) \). Note that \( \rho : K_r \to \Sigma_n - (\Sigma_n \cap \tilde{W}) \) is a double covering and the points contained in the preimage of each point of \( \Sigma_n - (\Sigma_n \cap \tilde{W}) \) have normal vectors pointing in opposite directions. Therefore

\[
\lim_{r \to 0} \int_{K_r} h g_1(\nabla \varphi^* f, \eta) - \varphi^* f g_1(\nabla h, \eta) = 0.
\]

Due to equations (2.14) and (2.15) one can choose a neighborhood \( W \) of \( \Sigma_1 \cup \ldots \cup \Sigma_n \) such that

- \( \partial W \) contains \( \partial \tilde{W} \) apart from a region of \( \partial \tilde{W} \) with small volume,
- \( \partial W \) contains \( K_r \) apart from a region of \( K_r \) with small volume,
- \( \partial W \) fulfills equation (2.13) and \( W \) fulfill equations (2.11) and (2.12).

Finally equations (2.10), (2.11), (2.12) and (2.13) imply

\[
|\int_U \varphi^* f \cdot \Delta h - \int_U \varphi^* u h| < \epsilon.
\]

This equation and the arbitrary choice of \( \epsilon \) conclude the proof of the lemma.

Proposition 2.4 follows from the fact that \( \varphi^* u \) is a function of class \( C^1 \) (recall Proposition 2.3), from Lemma 2.5 and from regularity theory of solutions of linear elliptic equations (see e.g., the proof of Theorem 3, section 6.3.1 of Evans[3]).

Proof of Theorem 1.6

Before starting the proof of Theorem 1.6 recall that any closed SRF \((M^n, \mathcal{F})\) has a canonical projection \( \pi : M \to M/\mathcal{F} \), and that the image of a stratum \( \Sigma \) is an orbifold of dimension \( \dim \pi(\Sigma) = \dim \Sigma - \dim \mathcal{F}|_\Sigma \). If \( M_{\text{reg}} \) denotes the regular stratum, the quotient \textit{codimension} of \( \Sigma \) is

\[
\text{qcodim}(\Sigma) = \dim \pi(M_{\text{reg}}) - \dim \pi(\Sigma) = \dim M - \dim \mathcal{F} - \dim \Sigma + \dim \mathcal{F}|_\Sigma.
\]

To say that \( M/\mathcal{F} \) has no boundary is equivalent to requiring that \( \text{qcodim}(\Sigma) > 1 \) for every singular stratum.

Lemma 3.1. Suppose \( \Sigma \) is a stratum with \( \text{qcodim}(\Sigma) > 1 \). Then for almost every regular point \( p \) and almost every unit-length horizontal vector \( x \in \nu_p L \), the geodesic with initial conditions \((p, x)\) never meets \( \Sigma \).
Proof. This result is proved by Lytchak and Thorbergsson in [1, Lemma 4.1]. For the sake of completeness we review this proof.

Consider the unit horizontal bundle $\nu^1F|_{M_{\text{reg}}} \to M_{\text{reg}}$ given by $(\nu^1F|_{M_{\text{reg}}})_p = \{ x \in \nu_pL_p, \| x \| = 1 \}$. In $\nu^1F|_{M_{\text{reg}}}$ consider the subset $\Omega_\Sigma$ of horizontal vectors $(p, x)$ such that the geodesic with initial conditions $(p, x)$ never meets $\Sigma$. We want to prove that $\Omega_\Sigma$ is full measure in $\nu^1F|_{M_{\text{reg}}}$. Consider the vector bundle $E \to \Sigma$ given by $E_p = \nu_qL_p$. We have rank $E = \dim M - \dim F|_{\Sigma}$, and therefore $\dim E = \dim \Sigma + \dim M - \dim F|_{\Sigma}$. There is an map $G : U \subset E \to \nu^1F|_{M_{\text{reg}}}$ that takes $(p, x)$ with $x \in \nu_pL_p$ to $\gamma'(|x|)$, where $\gamma$ is the unit speed geodesic with initial conditions $(p, x)$. Here $U$ is an open dense set of $E$. The map $G$ is smooth, and by construction the image of $G$ is exactly $\Omega_\Sigma$.

The dimension of $\nu^1F|_{M_{\text{reg}}}$ is $2\dim M - \dim F - 1$, and therefore

$$\dim \nu^1F|_{M_{\text{reg}}} - \dim E = \dim M - \dim F - \dim \Sigma + \dim F|_{\Sigma} - 1 = q\text{codim}(\Sigma) - 1 > 0.$$  

The last inequality is strict, and therefore $\dim E < \dim \nu^1F|_{M_{\text{reg}}}$. By Sard’s Theorem, $\Omega_\Sigma = \text{Im}(G)^c$ is full measure in $\nu^1F|_{M_{\text{reg}}}$, that is what we wanted to prove.

We can now prove Theorem 1.6

Consider the map $\varphi : M_1/F_1 \to M_2/F_2$, where $M_1/F_1$ has no boundary. By localizing around a point, we can assume we are in the Euclidean case $\varphi : \mathbb{R}^n_1/F_1 \to \mathbb{R}^n_2/F_2$, where $(\mathbb{R}^n_1, F_1)$ are possibly non-connected SRF’s. Because of Proposition 2.3 and Theorem 2.4 it is enough to show that $\varphi : \mathbb{R}^n_1/F_1 \to \mathbb{R}^n_2/F_2$ preserves the mean curvature vector field.

For $i = 1, 2$, let $\Omega_i \subseteq \nu^1F|_{\mathbb{R}^n_i}$ be the set of unit vectors $(p, x) \in T\mathbb{R}^n_i$ whose corresponding geodesics never meet any singular point. This set can be written as

$$\Omega_i = \cap_{\Sigma^{ij}_i} \Omega_{\Sigma^{ij}_i}$$

where the intersection runs over (the finitely many) singular strata $\Sigma^{ij}_i$ of $F_i$, and $\Omega_{\Sigma^{ij}_i}$ is the set of unit vectors $(p, x) \in \nu^1F|_{\mathbb{R}^n_i}$, whose corresponding geodesics never meet $\Sigma^{ij}_i$. Since there is no boundary, every stratum has $q\text{codim}(\Sigma^{ij}_i) > 0$ and by Lemma 3.1 the corresponding $\Omega_{\Sigma^{ij}_i}$ is open and dense. It follows that $\Omega_i$ is open and dense as well.

Take $(p, x) \in \Omega_1$, and consider $(\tilde{p}, \tilde{x}) \in \nu^1F|_{\mathbb{R}^n_2}$ such that $\varphi(\pi_1(p)) = \pi_2(\tilde{p})$, and $\varphi_{\ast}(\pi_1,x) = \pi_2, \tilde{x}$. As in Proposition 2.4 let $\gamma(t) = p + tx$, and $\tilde{\gamma}(t) = \tilde{p} + t\tilde{x}$. Since $(p, x) \in \Omega_1$, the geodesic $\pi_1(\gamma(t))$ stays in the regular part of $\mathbb{R}^n_1/F_1$, and since $\pi_2(\tilde{\gamma}) = \varphi(\pi_1(\gamma))$, the same holds for $\tilde{\gamma}$. In particular, $(\tilde{p}, \tilde{x}) \in \Omega_2$.

As in Gromoll and Walschap [1, Theorem 4.1.1], one can prove that if $(p, x) \in \Omega_1$ then every nonzero eigenvalue of $S_x$ is also an eigenvalue of $S_{\tilde{x}}$ with the same multiplicity. Of course this implies that $\text{tr}(S_x) = \text{tr}(S_{\tilde{x}})$ and since this holds for a dense set of vectors, $\varphi$ preserves the mean curvature vector field, and the theorem is proved.

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