On the thermal description of the BTZ black holes

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We investigate the limitations on the thermal description of three dimensional BTZ black holes. We derive on physical grounds three basic mass scales that are relevant to characterize these limitations. The Planck mass in 2+1 dimensions indicate the limits where the black hole can emit Hawking’s radiation. We show that the back reaction is meaningless for spinless BTZ black hole. For stationary BTZ black holes the nearly extreme case is analyzed showing that may occur a break down of its description as a thermal object.

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I. INTRODUCTION

An important point in discussion is if the extreme black holes behave as thermal objects. More than a decade ago, Preskill \textit{et al.} pointed out that the thermal description of a black hole becomes ill-defined as the black hole approaches the extreme limit. In its treatment the use the criterion that the standard semiclassical description, which neglects the back reaction, is not self-consistent if the emission of a typical quantum radiation changes the temperature by an amount comparable to the value of the temperature. For a Schwarzschild black hole, the thermal description breaks down simultaneously as the classical description of spacetime does. At the final stages of the evaporation, the black hole field has a high curvature and the radius of the horizon is of the order of the Planck length, besides, as the temperature rises as the mass diminishes, the problem of the back reaction becomes unavoidable. Thus, the Planck mass is the only relevant scale of mass that is needed to take account in a full theory of quantum gravity. Nevertheless, for a Kerr-Newman black hole the breaks down occurs near the extreme case when the mass is much greater than the Planck mass and the corrections due to quantum gravity are expected to be negligible.

In 2 + 1 dimensional gravity the BTZ black hole solution is a spacetime of constant negative curvature, but it differs from anti-de Sitter (AdS) space in its global properties. The related investigations of the thermodynamics properties of this solution have not included considerations about the nature of a possible break down of its thermal description. A obvious question is if this type of black hole experiment, as its four dimensional counterpart, a break down of its thermal description and what is the relevant mass scale involved. It was found by Reznik that the Planck mass in 2 + 1 dimensions has a physical significance related to the description of the thermodynamics of the Hawking emission process, for static BTZ black holes. This is a key point when we try to shed some light in the behavior of BTZ black holes in relation to limits to its description as a thermal objects. The purpose of this article is to investigate the limits in the thermal description, using the approach describe in\textit{i}, for static and rotating BTZ black holes.

In Section II we expose briefly the principal parameters related with the thermodynamics of the BTZ black hole. In Section III we show the existence of three mass scales: $m_P$, $m_\lambda$, and $m_T$ and discuss its physical meaning. The mass scale $m_T$ has not been previously discussed in the literature; it appears when the approach describe in\textit{i} is applied to the thermodynamics of BTZ black holes. We have restricted our discussion to the case when the semiclassical description of the BTZ spacetime is valid. In Section IV we discuss the particular limitations that experiment the thermodynamical description of BTZ black holes. We show the roll played by the above three masses in these limitations. In particular, we explicit calculate the conditions imposed on the black hole parameters in order to have a well description of the thermodynamics of the Hawking emission process. The problem of the back reaction is

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considered for spinless BTZ black holes. The near extreme black hole is discussed and the breakdown of its thermal description, in the sense discussed in [1], is analyzed. Finally, in Section V some concluding remarks are presented.

II. THE BTZ BLACK HOLE

In this work we adopt units such that \( c = 1 = k_B \), where \( k_B \) is the Boltzmann constant. We keep the \( G \) constant explicitly in the above expressions. \( MG \) is the geometrized mass, which have no dimensions in \( 2+1 \) gravity. In the following section we will discuss the mass scales relevant in the thermodynamics of the BTZ black hole and the role of \( G^{-1} \) as Planck mass. We also keep \( \hbar \) in the following thermodynamics parameters, since exist another scale which contain explicitly this constant.

The action considered in [2] is
\[
I = \frac{1}{2\pi G} \int \sqrt{-g} (R + 2\ell^{-2}) d^2 x dt + B,
\]
where \( B \) is a surface term, and the radius of curvature \( \ell = (-\Lambda)^{-1/2} \) provides the length scale necessary to have a horizon (\( \Lambda \) is the cosmological constant).

The axially symmetric BTZ black hole written in stationary coordinates is

\[
ds^2 = -N^2 (r) dt^2 + f^{-2} (r) dr^2 + r^2 (N^\phi (r) dt + d\phi)^2,
\]
where the lapse \( N (r) \) and the angular shift \( N^\phi (r) \) are given by
\[
N^2 (r) = f^2 (r) = -MG + \frac{r^2}{\ell^2} + \frac{(JG)^2}{4r^2} \quad \text{and} \quad N^\phi (r) = -\frac{JG}{2r^2},
\]
with \( -\infty < t < \infty , 0 < r < \infty , \) and \( 0 \leq \phi < 2\pi \). The constant of integration \( M \) is the conserved charge associated with asymptotic invariance under time displacements and \( J \) (angular momentum) is that associated with rotational invariance. The lapse function vanishes for two values of \( r \) given by
\[
r_{\pm} (x) = \ell (MG)^2 \left[ \frac{1}{2} (1 \pm \sqrt{x}) \right]^{1/2},
\]
where \( x \) is defined by the relation
\[
x \equiv 1 - \frac{J^2}{(\ell M)^2}.
\]
The black hole horizon is given by \( r_+ (x) \), which exist only for \( M > 0 \) and \( |J| \leq \ell M \). We will denote the horizon of a spinless black hole by \( r_+ (1) \equiv r_+ (x = 1) = \ell (MG)^{1/2} \). The entropy of this black hole is given by
\[
S = \left( \frac{1}{k_B} \right) 4\pi r_+(x),
\]
\{From Eq.(6) it is possible to write the entropy, \( S \), as a function of \( M \) and \( J \) and obtain the first law of the thermodynamics \}
\[
dM = TdS + \Omega dJ,
\]
where \( T \) is the Hawking’s temperature and \( \Omega \) is the angular velocity of the black hole horizon. Both parameters can be expressed in terms of the variable \( x \), yielding for the temperature
\[
T (x) = T (1) \sqrt{\frac{2x}{1 + \sqrt{x}}},
\]
where \( T(1) \) is the temperature of an spinless BTZ black hole given by
\[
T (1) = \frac{\hbar G r_+ (1)}{2\pi \ell^2}.
\]
The corresponding expression for $\Omega$ is

$$\Omega = \frac{1}{\ell} \sqrt{1 - x} \quad (10)$$

Important aspects of the thermodynamics systems need to know the heat capacities $C_z$, where $z$ denote the set of parameters held constant. A useful expression, that we will use latter, for the heat capacity at constant angular momentum $J$ is given by

$$C_J = \left( \frac{\partial S}{\partial T} \right)_J = C_0 \frac{1}{2 - \sqrt{x}} \left[ \frac{(1 + \sqrt{x})x}{2} \right]^{1/2}, \quad (11)$$

where $C_0 = \frac{(\hbar G)^{-1}}{4\pi r_+}$ is the heat capacity at $J = 0$. The expression for $C_\Omega$ is given by

$$C_\Omega = T \left( \frac{\partial S}{\partial T} \right)_\Omega = \frac{4\pi \ell}{\hbar G} \left( \frac{MG}{2} \right)^{1/2} \left[ 1 + \sqrt{x} \right]^{1/2}. \quad (12)$$

Is direct to verify that $C_J$ and $C_\Omega$ are always positive, independently of the values of the parameters $M$ and $J$, for the BTZ black holes. These type of black holes never experiment a phase transition. In the case of two independent thermodynamics variables $Z_1, Z_2$, stability requires that

$$\frac{\partial Z_1}{\partial Z_1} S \leq 0, \quad \frac{\partial Z_2}{\partial Z_2} S \leq 0, \quad \left( \frac{\partial Z_1}{\partial Z_2} \right)_S \leq 0 \quad (5).$$

It is straightforward to prove that the above conditions are satisfied taking $Z_1 = M, Z_2 = J$, for static and rotating BTZ black holes, which ensured the thermal stability of this type of black holes. These conditions are always satisfied for any $M$ and $J$.

### III. MASS SCALES

We will show in this section that the thermal description of a spinless BTZ black hole is ill defined when the black hole mass approaches to a new mass scale, not discussed previously in the literature. For this reason we first discuss the relevant mass scales in the theory of black hole in $2+1$ dimensions.

For a Schwarzschild black hole, at the Planck scale, the fluctuations of the geometry become important and this occurs when its radius becomes comparable to the Compton wavelength, i.e., $r_{\text{horizon}} \sim \lambda_{\text{Compton}}$. From this relationship we obtain the expression for the Planck mass, $m_P$. Nevertheless, we can also obtain the Planck mass imposing that $r_{\text{horizon}} \sim \ell_P$, where $\ell_P$ is the Planck length. In addition, we can define $m_P$ as $\hbar/\ell_P$ from a straightforward dimensional analysis. In four dimensions, these criteria leads to a unique mass scale, the Planck mass, given by

$$m_P^{3+1} = \frac{(\hbar G)^{-1}}{2}.$$

The situation is quite different in three dimensional gravity. In general, for any dimension, the Planck mass, $m_P$, is defined from the relation $m_P \ell_P \sim \hbar$, where $\ell_P$ is fundamental length scale which is obtained imposing that the action in $D$ dimensions

$$I \sim \frac{1}{G} \int d^Dx \sqrt{-g}R, \quad (13)$$

be of the order of $\hbar$. For $2+1$ dimensions the fundamental unit of length is given by

$$\ell_P = \hbar G, \quad (14)$$

which leads to the corresponding Planck mass in $2+1$ dimensions

$$m_P = \frac{1}{G}. \quad (15)$$

We restrict our discussion to the case when the semi-classical description of the BTZ spacetime is valid, i.e., when the condition $S_{2+1}/\hbar > 1$ holds or, in other words, when $\ell > \ell_P$.

Since the Planck mass in $2+1$ dimensions is a classical mass unit (contains only the gravitational constant), it is not surprising that appears in classical phenomena associated with $2+1$ dimensional fluids in hydrostatic equilibrium. If the cosmological constant is not included, classical results show that there exist a universal mass, in the sense that all rotationally invariant structures in hydrostatic equilibrium have a mass that is proportional to $m_P$. In this case there are no black hole solution and the possibility of collapse is clearly forbidden. Nevertheless, the study of
the structures, with a mass $M$ and a radius $R$, in hydrostatic equilibrium in AdS gravity leads to an upper bound on the ratio $M/R$ similar to the four dimensional case. This result shows that exist the possibility of collapse for matter distributions that have the ratio $M/R$ over the above upper bound. Black holes with large masses are possible to exist (large with respect to $m_P$), even more, any fluid distribution in hydrostatic equilibrium has necessarily a mass greater than $m_P$. With the assumption that the BTZ black hole is the end of a collapse, the classical mass $m_P$ represents the lowest mass of any 2 + 1 black hole.[5]

For the Planck mass, $m_P$, $r_+ \sim \ell$, i.e., the size of the horizon is comparable with the associated length of the space-time curvature $\ell$. Since $\ell > \ell_P$, the fluctuations of the black hole geometry are not important at this mass scale.

A remarkable result, as we shall see below, is that without the length scale provided by the cosmological constant it is not possible to build a unit mass containing $\bar{\hbar}$. Quantum phenomena are presents in 2 + 1 dimensions with a length scale that can provide horizons and the basic mass units related contain both, the Planck and the cosmological constant.

The fluctuations of the black hole geometry become important when $r_+ \sim \lambda_{\text{Compton}}$, i.e., when the radius of the black hole becomes comparable to the Compton wavelength. This yield the mass scale, $m_\lambda$, given by

$$m_\lambda = \left(\frac{\hbar^2}{\ell^2 G}\right)^{1/3}.$$  

(16)

For this reason Reznik[3] identify the Planck mass in 2 + 1 dimensions with $m_\lambda$.

We can obtain another mass scale when the limitations of the thermal description of a black hole are studied. In the following analysis we will consider a classical background geometry, ignoring the back reaction. As it was pointed out in [1], the semiclassical description of the black hole evaporation is not self consistent if the emission of a typical quantum radiation changes the temperature by an amount comparable to the value of the temperature. If $T$ is the energy of the quantum and if $\Delta T$ is the change of temperature that experiment the black hole after the emission, then from the relation $C_J \Delta T = T$, where $C_J$ is specific heat at $J = \text{const.}$, the condition for the thermal description to be self-consistent is

$$|T\left(\frac{\partial T}{\partial (MG)}\right)_J| << |T|,$$  

(17)

which is equivalent to impose

$$\frac{\partial T}{\partial (MG)} = C^{-1}_J = C^{-1}_0 \left(2 - \sqrt{x}\right) \left[\frac{2}{(1 + \sqrt{x})x}\right]^{1/2} << 1.$$  

(18)

For the particular case of a static BTZ black hole, $x = 1$, the heat capacity given by

$$C_0 = 4\pi \sqrt{\frac{M}{m_T}},$$  

(19)

where mass scale, $m_T$, has the following expression

$$m_T = \frac{\hbar^2 G}{\ell^2}.$$  

(20)

The breakdown occurs when the black hole mass $M$ satisfy $M \sim m_T$. At this scale $r_{\text{horizon}} \sim \ell_P$ and corrections due to quantum gravity are expected to be very important. Note that this mass satisfy (up to a numerical factor), $T(M = m_T) \sim m_T$. For Schwarzschild black holes the corresponding relation is $T(M = m_{3+1}) \sim m_{3+1}$; i.e., black holes with masses of the order of the Planck mass radiates at Planck temperature.

IV. LIMITATIONS ON THE THERMAL DESCRIPTION

A. The static black hole

A crucial point of the characteristic behavior of thermodynamics of the spinless BTZ black hole was pointed out by Reznik[3]. He indicated that the physical significance of the mass unit $m_P$ is that for $M > m_P$ (or $M < m_P$) the wavelength $\lambda$ of the Hawking radiation satisfies $\lambda < r_+$ ($\lambda > r_+$).

Since the process of energy emission can be thermodynamically well described when a typical wavelength of the Hawking radiation satisfy $\lambda \lesssim r_+$, we calculate for rotating BTZ black hole the conditions on the parameters $M$ and
\[ J \text{ in order to satisfy this restriction. We have the following relation for a typical wavelength, } \lambda, \text{ related to Hawking's temperature} \]
\[ T(x)\ell^{-1} \sim \lambda^{-1}, \]
(21)

The requirement for the wavelength is
\[ \lambda \lesssim r_+(x), \]
(22)

\[ \text{From the relations } (21) \text{ and } (22) \text{ we obtain that} \]
\[ \left( \frac{r_+(1)}{\ell} \right)^2 \gtrsim \frac{1}{\sqrt{x}}. \]
(23)

\[ \text{From this inequality we obtain, in terms of the dimensionless parameters } j \equiv JG/\ell \text{ and } m \equiv M/m_P, \text{ that} \]
\[ j \lesssim m \left( 1 - \frac{1}{m^2} \right)^{1/2}. \]
(24)

Notice that for a static BTZ black hole \( x = 1 \) and the inequality \( (23) \) yields \( M \gtrsim m_P, \) since \( r_+(1)/\ell = M/m_P. \) This was the result obtained in [3].

Let us discuss first the case of a spinless BTZ black hole. Since we only consider the regime where \( \ell \gg \ell_P \) the mass scales obtained on physical grounds satisfy the relation \( m_T < m_\lambda < m_P. \) We have argue that for this black hole the Hawking’s radiations cannot be emitted if \( M < m_P. \) Notice that this is an effect due to the finite size of our system, which has no equivalent in the case of a Schwarzschild or Kerr black holes. The other mass scales founded allow to characterize further the limitations in the description of the thermodynamical behavior of the BTZ black hole.

At the scale of \( m_P \) and unlike the Schwarzschild black hole, where the curvature of the spacetime is associated with the size of the black hole horizon, the BTZ black hole is an AdS space of constant curvature, which no changes through all the evaporation process, neglecting the back reaction, with a radius of curvature given by \( \ell. \) On the other hand, \( r_+(1) > \lambda_{\text{Compton}} (m_\lambda < m_P) \) which means that the fluctuations of the black hole geometry never becomes important for the spinless BTZ black hole.

Contrary to the Schwarzschild black hole, at the scale of \( m_P \) the temperature obey the relation \( T(M = m_P) < m_P. \) At this scale the energy emitted by the black hole is no important with respect to the energy of the black hole itself. This is consistent with the fact that the thermal description is well defined only for black hole masses greater than \( m_T (m_T < m_P). \) Or in other words, there is no breakdown of the thermal description, in the sense discussed in [1], for the spinless BTZ black hole.

Notice that the fluctuations, both in temperature and entropy, are important at mass scale \( m_T. \) The equilibrium thermodynamics fluctuations of BTZ black holes in the microcanonical ensemble, canonical ensemble and grand canonical ensemble was studied in [3]. The fluctuations in the temperature (microcanonical ensemble) are given by
\[ \frac{\langle \delta T(1) \delta T(1) \rangle}{T^2(1)} = C_0^{-1} = \frac{1}{4\pi} \sqrt{\frac{m_T}{M}}, \]
(25)

and in the entropy (canonical ensemble) by
\[ \langle \delta S \delta S \rangle = C_0 = 4\pi \sqrt{\frac{M}{m_T}}. \]
(26)

The above equations allow us to say that at the Planck scale, the fluctuations of the temperature are no important and that the system have many states sampled spontaneously.

These results have important consequences for the back reaction problem. Martínez and Zanelli [3] have evaluated the back reaction of a massless conformal scalar field on the geometry of the spinless BTZ black hole. The authors calculate the \( O(hv^{-1}) \) corrections to the metric, which do not change the value of the ADM mass. The corrections to the temperature and entropy are linear in a function \( F(M) \sim e^{-\pi M/m_P}, \) which implies that the back reaction becomes large only for small masses compared to the Planck mass. These results are consistent with the fact that at the mass scales lower than the Planck mass, for example, \( m_T, \) the fluctuations of the temperature are important. Nevertheless, since only has physical sense to consider black holes above the Planck scale calculations relative to the back reaction are meaningless.

For \( \ell = \ell_P, \) there is a unique mass scale, since \( m_P \sim m_\lambda \sim m_T. \) In this case is no longer valid the semiclassical description of the AdS spacetime due to the fluctuations of the geometry becomes important, independent of the black hole radius. It has no physical sense to consider the limitations on the thermal description.
B. The rotating black hole

Is direct to see that the rotating BTZ black hole has a very different behavior with respect to its four dimensional counterparts. Rewritten Eq. (23) in terms of the black hole mass we obtain

\[ M \gtrsim m_P x^{-1/4}. \]  

(27)

For a rotating black hole \( x < 1 \), which implies that the allowed minimum mass is always greater than the Planck mass. On the other hand, if, for example, \( M \approx m_P \), Eq. (24) imposes an upper bound on the angular momentum of the black hole, which in this case indicates that \( j \ll m \). In simple words, for black hole with a mass near to the Planck mass the upper bound on the angular momentum is very well.

As the black hole mass increases the upper bound on the angular momentum allowed also increases. For a massive rotating black hole, i.e., \( m \gg 1 \), or equivalently, \( M \gg m_P \), we obtain that \( j \lesssim m \). This is the only type of rotating BTZ black hole that can approach to the extreme case. To investigate the breakdown of the thermal description, in the sense discussed in \[1\], we approach to the the extreme case, \( |J| \leq \ell M \), taking \( x \approx 0 \) in Eq. (15), which yields

\[ \frac{\partial T}{\partial (MG)} \approx \frac{1}{2\pi} \left( \frac{2m_T}{mx} \right)^{1/2} \lesssim 1. \]  

(28)

Notice that for near extreme BTZ black holes, the thermal description breaks down when \( Mx \sim m_T \), which implies that this breakdown may occur within the regime \( M \gg m_T \). In this case it is straightforward to see that with an adequate ratio \( \ell/\ell_P \) it is possible that a massive rotating black hole experiment a breakdown of its thermal description. The temperature fluctuations (microcanonical ensemble) can be evaluated from the relation

\[ \frac{\langle \delta T(x)\delta T(x) \rangle}{T^2(x)} = C_j^{-1}, \]  

(29)

which according to the rhs. of Eq. (28) implies that as the near extreme case is approached, the temperature fluctuations become larger to the temperature itself. The entropy fluctuations (canonical ensemble) are given by

\[ \langle \delta S\delta S \rangle = C_j \approx 2\pi \left( \frac{Mx}{2m_T} \right)^{1/2}. \]  

(30)

So near the extreme case few states are sampled spontaneously. Notice the above arguments are valid if \( Mx \to 0 \) as the the black hole mass increases, which is correct since Eq. (24) implies that \( x \sim m^{-4} \) for massive black holes.

In this scenario, as we explained above, the constant curvature of the AdS spacetime at the horizon is small in Planck units, and corrections due to quantum gravity are expected to be negligible. For extreme Kerr-Newman black holes it was found in \[1\] that thermal description breaks down when \( M \gg m_P \), so the situation is equivalent in the sense that we are in a scenario in which it is no necessary to include quantum gravity.

V. CONCLUSIONS

We have shown the existence of three mass scales: \( m_P, m_\lambda, \) and \( m_T \), which have a clear physical meaning. We have restricted our discussion to the case when the semiclassical description of the BTZ spacetime is valid, i.e., when the condition \( S_{2+1}/\hbar > 1 \) holds or, in other words, when \( \ell > \ell_P \). In this case the masses satisfy the relation \( m_T < m_\lambda < m_P \).

We have argued that the process of energy emission can be thermodynamically well described when a typical wavelength of the Hawking radiation satisfy \( \lambda \lesssim r_+ \).

We have shown that the radiation process take place only for static black holes with masses above the Planck scale.

At this stage the size of the horizon is comparable with the associated length of the spacetime curvature and the fluctuations of the black hole geometry are not important. This means that in the evaporation of the BTZ black hole the horizon never lost its classical meaning. On the other hand, we have proved that the breakdown of the thermal description, in the sense discussed by Preskill \textit{et al} \[1\], never occurs. This is consistent with the result found in \[8\], in which the back reaction becomes large for small masses compared to the Planck mass. In this sense, the BTZ black hole never experiment a back reaction comparatively important with respect to the background geometry.

We have shown that for rotating BTZ black holes the allowed minimum mass is always greater than the Planck mass. Exist, besides, an upper bound on the angular momentum of the black hole which depends on the black hole mass. Only for massive rotating black holes it is possible \( j \lesssim m \), and the near extreme case can be reached. In this case the description of black holes as thermal objects may experiment a break down. This also occurs for Kerr-Newman black holes in 3 + 1 dimensions \[1\].
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