Secure Computation on Additive Shares

Zhihua Xia, Member, IEEE, Qi Gu, Wenhao Zhou, Lizhi Xiong, Jian Weng, Member, IEEE

Abstract—The rapid development of cloud computing has probably benefited each of us. However, the privacy risks brought by untrustworthy cloud servers arise the attention of more and more people and legislatures. In the last two decades, plenty of works seek the way of outsourcing various specific tasks while ensuring the security of private data. Although the addition and multiplication are enough for implementing any functions, the direct utilization of existing schemes like homomorphic encryption will lead to significant efficiency and accuracy loss, which is not suitable for outsourcing computation tasks. The tasks to be outsourced are endless, however, the involved calculations are similar. In this paper, inspired by additive secret sharing and multiplicative secret sharing technologies, we construct a series of novel protocols which support the common secure calculations on numbers (e.g., basic elementary functions) or matrices (e.g., solve eigenvectors) in arbitrary $n$ number of servers ($n \geq 2$), and the $n$-party protocols ensure the security of the original data even if $n-1$ servers collude. All protocols we designed only need constant interaction rounds, and we demonstrate them under universally composable security. We believe that these protocols can provide a new basic tool for actual outsourced tasks.

Index Terms—secure computation protocols, additive secret sharing, multiplicative secret sharing, share-transform-reveal

1 INTRODUCTION

With the rapid development of the Internet, we are all in the information explosion era. Ordinary hardware equipment has been incapable of meeting our routine needs in time. Cloud computing provides a perfect solution to this dilemma. They offer powerful cloud servers, store the data, and complete computing resource-intensive tasks for us. At the same time, the mature network makes the real-time requirements can be guaranteed. It seems quite nice, however, the conveniences all come at the cost of our personal data being exposed.

In recent years, a large number of privacy problems caused by cloud servers [1] arouse widespread concern. In this case, some government agencies have set up relevant laws and regulations. A well-known example is General Data Privacy Regulation [2] implemented in the European Union, which clearly defines that the utilization of personal data needs the consent of the data subject. The convenience and privacy are both public demands, therefore, how to complete the daily needs without leaking personal privacy data becomes one of the most urgent problems nowadays.

There are mainly three kind of schemes focusing on coping with secure computation outsourcing problem: differential privacy [3], Homomorphic Encryption (HE) [4], and Secure Multiparty Computation (SMC) [5]. Schemes in differential privacy try to ensure security by adding reasonable noise to the individual data but retain the valid statistical information. However, the significant loss on the accuracy or security [3] and task-related perturbation methods limit their application scenarios.

HE [4] is one of the most typical cryptography tools which support direct computation on the ciphertext. Generally speaking, HE schemes based on one assumption of computational intractability [6], which means the encryption and computation process in their schemes always involve operations on large prime numbers, and the raw data can only appear as integers or bits. Therefore, schemes in this category always cause unacceptable efficiency and a certain degree of accuracy loss. Besides, such schemes are difficult to directly support secure calculations more than linear computation and multiplication.

The above two kinds of schemes can be supported by a single cloud server which is necessary during the early stage of cloud computing. However, after decades of development, more and more cloud computing vendors are willing to provide computation services. The service jointly supported by multiple servers becomes a new possibility and attracts many researchers in recent years. The utilization of multiple servers provides a new stage for the traditional SMC schemes [7], and the avoidance of HE makes recent secure outsourced tasks [8], which utilize SMC technologies [5], gain thousands of time efficiency advantage. Attracted by the huge improvement, we try to expand this kind of scheme in this work.

Additive secret sharing (ASS) which naturally supports linear operations on ciphertext (i.e., share) is one typical technology in SMC. As linear operations are usually the most common operations in real-world tasks, ASS has received great attention in schemes that consider outsourcing secure computing tasks. However, most of the existing schemes [9] still limit in classical conclusions gotten in SMC. For example, utilizing Beaver triples [5] to execute multiplication, and bit-decomposition [10] is used for comparison. In this case, they are fairly passive during facing with more general operations widely existed in practical tasks (e.g., non-linear functions).

Inspired by Multiplicative Secret Sharing (MSS), the recent work [11] combines both ASS and MSS. By switching between two kinds of sharing at the proper time, they
implemented plenty of non-linear protocols based on two non-collusion cloud servers and the introduction of MSS greatly optimizes other existing protocols (e.g., comparison). However, limited by the protocols they designed, there still exists much room for improvement.

The greatest problem that exists in [11] is the assumption of two non-collusion servers. It will lead to the following deficiency naturally: the owner of data may not believe that the two servers will not collude. For instance, the development of federal learning [12] has led to the following demand: plenty of different commercial organizations have some valuable data, they hope to collaboratively train a better model (e.g., neural network) without exposing their data. In this case, it is hard to find two servers that can be trusted by all the participants. More likely, all data providers will insist on participating in the secure calculation as they do not lack communication or computing resources.

The above demand urges us to consider a more general scene: How to outsource computation losslessly to arbitrary $n$ cloud servers ($n \geq 2$), and ensure the security of original data even $n-1$ cloud servers collude. We try to construct efficient secure computation protocols for common functions on numbers and matrices under the above security limitation. In summary, we mainly make the following contributions:

1) Inspired by [11], follow the idea of share-transform-reveal, we construct two novel resharing protocols that support transforming share, which is stored in $n$ servers, in ASS format to that in MSS, and vice versa. Then follow a series of identities, we construct the secure $n$-party computation protocols on all basic fundamental functions and remain the input and output in ASS format. We also design the $n$-party comparison and division protocols. As a theoretical supplement, we also give the corresponding protocols, which keep the input and output in MSS format, in the appendix.

2) We further optimize and expand partial protocols (e.g., comparison) at the cost of extremely lightweight offline work undertaken by the data owner. All $n$-party protocols designed only need constant (i.e., irrelevant to $n$) rounds of interaction.

3) We further construct some common $n$-party computation protocols, such as solving eigenvectors and eigenvalues, on matrices, and remain the input and output in ASS format. The interaction rounds of protocols on matrices are independent of their dimensions.

4) We prove the security of our protocols in the Universally Composability (UC) security model [13], which shows that our protocols can against collusion between $n-1$ servers. We also analyze the communication complexity of all protocols proposed.

The rest of our paper is organized as follows. In section 2, we briefly introduce some typical technologies in SMC. The system and threat model will be given in section 3. Section 4 considers the resharing protocols which can transform share in ASS and MSS format. Then section 5 constructs all the basic elementary functions protocols in our system model. More optimization, expansion and protocols on matrices are considered in section 6. The security and efficiency analysis of all designed protocols are shown in section 7 and section 8. Finally, we make the conclusion in section 9. Some protocols designed for completeness are placed in the Appendix.

2 RELATED WORK

In this section, we first briefly introduce the SMC, then focus on secret sharing technology that is strongly related to this work.

2.1 Secure Multiparty computation

Typical SMC schemes enable a group of participants collaboratively compute a certain function $F(x_1, x_2, \cdots, x_n)$ without revealing their private input $x_i$, here $n$ is the number of participants. SMC is firstly introduced by Yao [14] to cope with the typical Two Millionaires Problem which is essentially a secure comparison computation. After decades of development, there are mainly two kinds of schemes still attracts plenty of researchers: Garbled Circuits (GC) [15] and Secret Sharing [11].

GC [16] introduced by Yao proposed a general scheme for 2-party secure computation. Consider the OR gate on one bit, which means there are four cases in the truth table, and Alice and Bob hope to secretly compute it with their private input. There are mainly three steps: Alice generates the garbled circuit, the bits in all four potential case will be encrypted with symmetric encryption and permutation; Bob evaluate the garbled circuit gotten from Alice, and Bob can secretly get the encrypted version of his input from Alice with the help of oblivious transfer [17]; Bob shares the evaluation results. When the circuit becomes more complex, the values computed from the intermediate circuit will be protected. Due to any discrete function can be composited by basic gate circuits [18], which means GC can be widely used in 2-party secure computation.

Although GC only needs constant rounds interaction, however, there are two reasons that GC is not suitable for outsourced computation in cloud computing. On the one hand, the input and circuit will become much more complex when facing a real outsourced task, which means the communication size will grow significantly [9]. On the other hand, for a GC scheme, each party owns their private data and will get a knowledge (i.e., $F$) after the evaluation. However, in the scene of outsourced computation, data owners do not want the servers to have private data or to get the corresponding knowledge.

The second reason actually reflects the difference between the typical SMC scene and the outsourced computation scene. In this case, the secret sharing technology which is more universal attracts more researchers in recent years.

2.2 Secret sharing

Secret sharing is first introduced by Shamir in [19], which is earlier than SMC [14]. Briefly speaking, secret sharing considers the way splitting secret into different shares to meet the specific access structure [20]. Besides, how to execute computation on secrets [21] by calculating on the shares is also an important issue. Due to the characteristics
of this technology, it is suitable for SMC naturally. For instance, \[17\] utilized it and expand 2-party GC to \(n\)-party.

In \[19\], Shamir constructed a \((k,n)\)-threshold sharing scheme based on polynomial interpolation. Here \((k,n)\)-threshold means that the original secret can be shared into \(n\) different part, and any \(k\) shares can reconstruct the original secret, however, \(k-1\) shares cannot get any information of secret. It is quite useful as both security and robustness are gotten, which can prevent the situation that partial participants suffer from accidents. Please note that the situation in outsourced computation tasks is quite different. The robustness of the cloud server needs no worry, however, the security of secret is quite unbelievable. In this case, we actually need the \((n,n)\)-threshold secret sharing scheme. Interestingly, the ASS which may be the simplest sharing method meets our demands.

### 2.2.1 Additive secret sharing

In ASS, the secret, e.g., \(x\), will be splitted into \(n\) different shares \([x_i]\), \(i \in [1,n]\), here \(\sum_{i=1}^{n} [x_i] = x\). From theorem 1, the additive secret sharing is a \((n,n)\)-threshold as the lack of any share will make the information independent from secret.

**Theorem 1.** The element \(x + r\) is uniformly distributed and independent from \(x\) for any element \(x \in \mathbb{R}\) if the element \(r \in \mathbb{R}\) is also uniformly distributed and independent from \(x\).

**Proof.** If \(r \in \mathbb{R}\) is uniformly distributed and independent from \(x\), then so is \(x + r\), since \(f_r(x) = r + x\) is a bijective mapping for \(\mathbb{R}\).

It should be noted that ASS naturally supports linear operation: \(\sum_{i=1}^{n} (x_i \pm y_i) = x \pm y\). It implies each party can execute \(x_i \pm y_i\) locally, and the result is also in the additive. Similarly, each party can execute constant multiplication without interaction. In this paper, for simplicity and generality, we focus on the secret in the infinite field \(\mathbb{R}\). Although the actual computer is unable to cope with an infinite field, let us put this problem on hold until subsection 8.3.

ASS does not have multiplication homomorphism, which means the interaction is inevitable when executing secure multiplication. Beaver \[5\] creatively proposed a scheme which called Beaver triples, which introduce an offline phase, to compute secure multiplication losslessly in one interaction. We will introduce this fundamental protocol in section 4.

Since the addition and multiplication can fit any other basic computations, therefore, to our knowledge, few later papers pay their attention to construct other losslessly basic numerical calculation protocols (e.g., exponentiation). To comparison protocol, as the Most Significant Bit (MSB) implies the sign of number, plenty of schemes follow \[22\] try to secretly compute the sign of share based on secure multiplication protocol \[3\] or more complex random numbers \[23\] generated during offline phase. However, these schemes either lead to \(O(l)\) rounds of interaction, here \(l\) is the bit-length of secret, or cause too much work during offline phase.

The above deficiencies is essentially due to lack of multiplication homomorphism. In this case, recent work \[11\] introduces MSS which naturally owns multiplication homomorphism and provides a new direction to cope with the above problems.

### 2.2.2 Multiplicative secret sharing

MSS can be seen as a symmetrical technology from the perspective of ASS. In detail, for a secret \(x\), it will be splitted into \(n\) shares \((x_i)\), \(i \in [1,n]\), here \(\prod_{i=1}^{n} x_i\). However, MSS does not strictly satisfy the \((n,n)\)-threshold property.

**Theorem 2.** The nonzero element \(xr\) is uniformly distributed and independent from \(x\) for any element \(x \in \mathbb{R}\) if the element \(r \in \mathbb{R}\) is also uniformly distributed and independent from \(x\).

**Proof.** If \(r \in \mathbb{R}\setminus\{0\}\) is uniformly distributed and independent from \(x\), then so is \(xr\), since \(f_r(x) = r \times x\) is a bijective mapping for \(\mathbb{R}\setminus\{0\}\).

From theorem 2, it is easy to infer that, unfortunately, MSS is a \((n,n)\)-threshold only in \(\mathbb{R}\setminus\{0\}\). Consider the secret equal to zero, then at least one party will have the zero share, and the secret will be exposed to the party. Although it is an inherent deficiency, however, we will show that, basically, it will not influence the following protocols construction.

As described above, MSS has an important property that \(\prod_{i=1}^{n} x_i = xy\), which means each party can execute multiplication without interaction. Although it is difficult to find plenty of practical applications by multiplication homomorphism alone, the combination of ASS and MSS will let share has the property we want at the proper time.

### 3 Problem formulation

In this paper, not limited to any specific outsourced task, we focus on the common basic calculations on numbers or matrices which are gotten from data owner. In detail, the system model and security model are shown as follow.

#### 3.1 System model

As a general model, the proposed system model involves three kinds of entities, i.e., data owner, cloud servers and data user. The system model is shown as Fig. 1.

**Data owner** owns the original data, in this paper, we consider two commonest data types: number and matrix in \(\mathbb{R}\). At the same time, we assume the data owner is trusty to himself \[24\], and he will undertake the tasks of generating random numbers or matrices during offline phase (i.e., before secure computation).

**Cloud servers** undertake the computation tasks gotten from the data owner. In this paper, we consider \(n\) different servers \(S_i\), here \(i \in [1,n]\), \(n\) is a number not less than 2. We
do not limit the actual number of servers as it is usually determined by the specific task. Similarly, to avoid limitations, we focus on the common computations on numbers and matrices, rather than any specific task. By keeping the input and output in the same format (e.g., ASS), plenty of tasks can be executed losslessly by combining our protocols simply.

**Data user** receives the share of results from each $S_i$ and recover the real results from the shared version. In many cases, the data user is also the data owner.

### 3.2 Security model

In this paper, the honest-but-curious (also called passive) cloud servers are considered. It means each server will execute the protocol as the setting but may attempt to analyze the information from data (i.e., share). As described above, the risk that at most $n - 1$ servers collude is considered, please note that the secret without encryption will be inevitably exposed when all the computational parties are involved in the conspiracy, therefore, we actually consider the worst collusion situation which maybe satisfies almost all the real tasks.

### 4 Resharing Protocols

#### 4.1 Secure multiplication

Before considering how to transform share between ASS and MSS, the secure multiplication on shares in ASS should be considered. Actually, Beaver [5] has creatively given the crucial solution in the 1990s. The kernel idea is introducing an offline phase (i.e., before executing secure multiplication), the triple $\{a, b, c | c = ab\}$ should be pre-generated. Then follow the identity

\[(x - a)(y - b) = xy - a(y - b) - b(x - a) - ab, \tag{1}\]

the information of $x$ and $y$ will be covered (e.g., $x - a$) by the triple, however, the share of $xy$ will still be able to recover. In detail, the process of secure multiplication in our system model based on Beaver triples is shown as algorithm [1].

As described above, we assume the data owner will not betray himself, therefore the owner of data can be seen as trusty party $T$, and undertakes the task of generating random numbers during offline phase. In SecMul, each server has one additive share of $x$ and $y$, then server will use $a_i$ and $b_i$ to cover its share, and expose the result $e$ and $f$. Then follow the formula [1] each server can use $[a_i]$, $[b_i]$, and $[c_i]$ to reconstruct additive share of $xy$ from $ef$.

Although the computation of $ef$ can be executed by any server or all servers (e.g., each server computes $\frac{1}{ef}$), for simplicity and efficiency, we always assume $S_1$ to undertake these tasks in this paper. For consistency, all the security proof of protocols will be shown in section [2].

#### 4.2 Multiplicative resharing

Inspired by Beaver triples, to our knowledge, previous work [11] first considered the way converting multiplicative share to additive share in $(2, 2)$-threshold. Here is their idea: following formula [11] consider two servers which have $x$ and $y$, which means $S_1$ actually has $x_1$ and $x_2$, and $S_2$ owns $y_1$ and $y_2$, which is different from the situation in SecMul. In this case, [11] gives a novel allocation scheme of Beaver triples: $(a, [c_1])$ and $(b, [c_2])$ will be sent to corresponding server. It is valid in $(2, 2)$-threshold, however, this numerical trick no longer exists in the $(n, n)$-threshold.

Besides the $(n, n)$-threshold property of MSS, theorem 2 also provides another useful information: if a nonzero secret is multiplied with a random nonzero number, it is impossible to infer the secret from the result. In this case, we can manipulate additive and multiplicative format with the help of ASS and MSS share of a nonzero random number which is generated by $T$ during offline phase. In detail, the secure multiplicative resharing protocol is shown in algorithm [2].

In SecMulRes, each server which has the multiplicative share firstly secretly transforms $x_i$ to $\tilde{x}_i$, then reveal $\tilde{x}_i$, and the servers can finally get additive share of $x$ with the help of $[c_i]$. The key process can be called as STR (Share-Transform-Reveal): on the one hand, transforming the original private data to an irreversible number; on the other hand, the desired results can be secretly gotten based on known information.

#### 4.3 Additive resharing

Previous work [11] constructs their additive resharing protocol based on the inverse process of multiplicative resharing, which is also obviously invalid in the arbitrary $(n, n)$-threshold situation. Besides, their scheme will lead to potential information leakage risk. Symmetric to SecMulRes,
we design a novel additive resharing protocol shown in algorithm 3.

In SecAddRes, based on SecMul, servers firstly transform the \([x]_i\) to \([xc]_i\), then reveal \(xc\), and each server finally gets \(\langle x \rangle_i\) with the help of \(\langle c \rangle_i\). Both SecMulRes and SecAddRes are \((n, n)\)-threshold scheme in \(\mathbb{R} \setminus \{0\}\), however, when \(secret\) is equal to zero, it is inevitably as at least one multiplicative share \(\alpha\) will equal to zero, which means the corresponding server (e.g., \(S_i\)) knows the \(secret\). We will show that it will not cause actual damage to further protocols.

It should be noted that multiplicative share naturally contains the sign of \(secret\), in this case, the secure comparison protocol can be constructed shown in algorithm 4.

As shown in SecCmp, the sign of \(secret\) can be gotten from signs of multiplicative \(shares\). Especially, if one share is zero, then the corresponding numbers are equal. Please note that the servers will get the comparison result, rather than one share, as the comparison result is always serving as control information. In this case, although benefiting from SecAddRes, SecCmp will not expose more information besides the size relationship under any inputs. We will consider the better secure comparison protocol in section 5 now let us turn our attention back to the basic elementary functions.

5 Protocols on Basic Elementary Functions

Interestingly, there are many additive resharing and multiplicative resharing naturally contained in the basic elementary functions, which makes the losslessly secure computation possible. In this section, the \(n\)-party secure computation protocols, whose input and output are both in ASS format, on all basic elementary functions are considered. With the help of protocols on four fundamental operations, all elementary functions can be executed losslessly in theory.

5.1 Exponentiation and Logarithm

Secure exponentiation protocol tries to compute \([a^x]_i\), where \(a\) is a public base number. Follow the identity
\[
a = \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} a x_i, \tag{2}
\]
it is easy to note that the additive share of input will directly become multiplicative share of the output after each party executes exponentiation computation locally. The corresponding protocol is shown as algorithm 5.

Generally speaking, the public base number is assumed in the range \((0, 1) \cup (1, +\infty)\) in plaintext domain. In this case, \(a^x\) is always over zero, which means the SecMulRes will not face zero input. Therefore, SecExp can compute exponentiation losslessly in the \((n, n)\)-threshold.

The situation in logarithm computation, \([\log_a x]_i\), is quite symmetric. Follow the identity
\[
\log_a \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} \log_a [x]_i, \tag{3}
\]
the corresponding protocol is shown as algorithm 6.

In SecLog, to meet formula 3 the ASS of input will be convert to MSS, then each server computes logarithm computation locally, the result is just the additive share of output. Since the \(x\) is assumed in \((0, +\infty)\), the above protocol is losslessly in \((n, n)\)-threshold.

5.2 Power

Secure power protocol tries to compute \([x^a]_i\), where \(a\) is a public number in \(\mathbb{R}\). The following identity
\[
(\prod_{i=1}^{n} u_i)^a = \prod_{i=1}^{n} u_i^a, \tag{4}
\]
should be noticed. However, when the \(a\) is a real number (i.e., non-integer), the computation is a multivalued function \[27\]. In this case, the complex number will be involved.

For continuity, the discussion of this case will be shown in the appendix. Here we only consider that \(a\) is integer. The corresponding protocol is shown in algorithm \[7\]

Generally speaking, \(0^0\) is an undefined value, we believe it should be dependent on the specific task (e.g., 1 or raise wrong), and therefore we skip this situation. The key problem is that when secret is zero, one server (e.g., \(S_i\)) will inevitably know that the real secret.

However, we believe the extreme situation is low-destructive based on two reasons: on the one hand, the power on zero is meaningless, it generally does not contain extra information. For instance, the 0 in neural network is basically gotten from ReLU function, in this case, the exposure of 0 is inevitable and known by all servers naturally.

On the other hand, generally speaking, the multivariate functions (e.g., SecMul) with known input (e.g., 0) will not leak other share of inputs. We will demonstrate it in section \[7\] and it means the exposure of some rare secrets will not infect others. Whatever, we believe that this situation is rare and easy to handle in specific tasks.

As described above, the secure power protocol is dependent on the property of MSS, it further implies that the multiplication of multiple powers can also be computed based on above protocol. Especially, there are two important cases should arise our attention:

(i) **Secure multiple multiplication protocol SecMul**. The protocol tries to compute \(
\prod_{j=1}^{m} x^j
\), \(j \in \{1, m\}\), where \(m \geq 2\). It is easy to notice that when \(m = 2\), the SecMul is enough. However, when \(m\) over 2, it will be expensive to directly expand SecMul (shown in subsection 6.2). The corresponding protocol is shown as algorithm \[8\].

As described above, if one secret is zero, the other secrets will not be exposed.

(ii) **Secure division protocol SecDiv**. This protocol, shown in algorithm \[9\] tries to compute \([xy^{-1}]\), from \([x]_i\) and \([y]_i\), similar, the computation result will expose when \(x\) equal to zero. We will show how to optimize it in subsection 6.1.

### 5.3 Trigonometric Functions

Trigonometric functions involve \(\sin\), \(\cos\), \(\tan\), \(\cot\), \(\csc\) and \(\sec\). Since the later four functions can be gotten by combining the former two and SecDiv, and calculation of \(\cos\) is similar to \(\sin\), only the secure computation process on \(\sin\) is considered.

#### Algorithm 7 Secure power protocol SecPow

**Input:** \(S_i\) has \([x]_i\) and public number \(a\). (\(a\) is integer)

**Output:** \(S_i\) gets \([x^a]_i\).

**Offline Phase** :
1. \(T\) generates enough random numbers the sub-protocols use and sends to \(S_i\).

**Online Phase** :
2. \(S_i\) collaboratively compute \([x]_i = \text{SecAddRes}([x]_i]\).
3. \(S_i\) computes \([x^a]_i = ([x]_i)^a\).
4. \(S_i\) collaboratively compute \([x^a]_i = \text{SecMulRes}([x^a]_i)\).

#### Algorithm 8 Secure multiple multiplication protocol SecMul

**Input:** \(S_i\) has \([x]_i\), \(j \in \{1, m\}\).

**Output:** \(S_i\) gets \([\prod_{j=1}^{m} x^j]_i\).

**Offline Phase** :
1. \(T\) generates enough random numbers the sub-protocols use and sends to \(S_i\).

**Online Phase** :
2. \(S_i\) collaboratively compute \([x^j]_i = \text{SecAddRes}([x]_i]\) for all \(j \in \{1, m\}\).
3. \(S_i\) computes \([\prod_{j=1}^{m} x^j]_i = \prod_{j=1}^{m} [x^j]_i\).
4. \(S_i\) collaboratively compute \([\prod_{j=1}^{m} x^j]_i = \text{SecMulRes}([\prod_{j=1}^{m} x^j]_i)\).

#### Algorithm 9 Secure division protocol SecDiv

**Input:** \(S_i\) has \([x]_i\), and \([y]_i\). (\(y \neq 0\)).

**Output:** \(S_i\) gets \([\frac{x}{y}]_i\).

**Offline Phase** :
1. \(T\) generates enough random numbers the sub-protocol uses and sends to \(S_i\).

**Online Phase** :
2. \(S_i\) collaboratively compute \([x]_i = \text{SecAddRes}([x]_i]\), \([y]_i = \text{SecAddRes}([y]_i]\).
3. \(S_i\) computes \([\frac{x}{y}]_i = (x)_i \times (y)_i^{-1}\).
4. \(S_i\) collaboratively compute \([\frac{x}{y}]_i = \text{SecMulRes}([\frac{x}{y}]_i)\).

For simplicity, we here assume that \(n\) is odd. In this case, there exists the identity

\[
sin(\sum_{i=1}^{n} \theta_i) = \sum_{k \geq 1} (-1)^{\frac{k-1}{2}}\sum_{|A|=k} \prod_{i \in A} \sin \theta_i \prod_{i \notin A} \cos \theta_i. \tag{5}
\]

It could be noted that the additive share of input can be transformed into a series sum of multiplicative shares of output. In this case, by algorithm \[10\] we can execute the secure sine losslessly.

As shown in SecSin, since each multiplication in formula \[15\] is combined by the \(\sin \theta_i\) or \(\cos \theta_i\), here \(i\) takes all the elements in \([1, n]\) once and only once. In this case, each term is actually a part of secret store in \(n\) servers under MSS format. After transforming them to additive shares, the \([\sin \theta_i]\) has been gotten. Please note that each term is a part of true secret, which means even some parts (all parts are impossible from the perspective of \(n-1\) servers due to the property of formula \[15\]) are zero, the secret is still secure. In this case, we assert that SecSin can be computed losslessly in \((n, n)\)-threshold model.

### 5.4 Inverse Trigonometric Functions

The inverse trigonometric functions involves \(\arcsin\), \(\arccos\), \(\arctan\), \(\arccot\), \(\arccsc\) and \(\arcsec\), as each one could compute from \(\arctan\) under the transformation of input, for instance, \(\arcsin(x) = \arctan(\frac{x}{\sqrt{1-x^2}})\). Therefore, we here focus on the secure computation protocol on \(\arctan\):

\(\arctan(x)_i \leftarrow [x]_i\).

To our knowledge, there exists no identity could directly transform input in ASS format to the output in ASS or MSS format. In this case, here construct the output that...
addition and multiplication homomorphism. However, for elementary functions, as computing them will involve both high communication consumption which is shown in table [11], we still suggest using the Taylor series in the scene which does not care about accuracy. 

1, as previous works [11], we still suggest using the Taylor protocol, the above process can be designed as shown in detail.

| Algorithm 10 Secure sin protocol SecSin |
|----------------------------------------|
| **Input:** $S_i$ has $[\theta]_i$.     |
| **Output:** $S_i$ gets $[\sin \theta]_i$. |
| **Offline Phase:** |
| 1: $\mathcal{T}$ generates enough random numbers the sub-protocol uses and sends to $S_i$. |
| **Online Phase:** |
| 2: $S_i$ computes $\prod_{i \in A} \sin [\theta]_i, \prod_{i \in A} \cos [\theta]_i$ for all the potential set $A$ in formula [5], the results compose $\{f^j_i\}_i, j \in [1, 2^n]$. |
| 3: $S_i$ collaboratively compute $[f^j_i]_i = \text{SecMulRes}(\{f^j_i\})$ for the whole $\{f^j_i\}$. |
| 4: $S_i$ computes $[\sin \theta]_i = \sum_{j=1}^{2^n} [f^j_i]$. |

could meet our demand based on existing protocols in ASS. Follow the identity

$$\arctan(u) + \arctan\left(\frac{u}{1+ux}\right) = \arctan(x), \quad (6)$$

where $u \times \frac{u}{1+ux}$ should less than 1. The $u \times \frac{u}{1+ux}$ will not be equal to 1 for the $u$ and $x$ in $\mathbb{R}$. If more than 1, then $\pm \pi$ will be added based on the sign of $u$ (or $\frac{u}{1+ux}$).

First, consider the $\arctan$ in (2, 2)-threshold model: two server owns $[x]_1$ and $[x]_2$. Since we wish the server can own $u$ and $\frac{u}{1+ux}$ so that the output can be locally computed and stored in ASS format. Therefore, the problem here is the way of transforming the additive share of $x$ to $u$ and $\frac{u}{1+ux}$ without leakage of $x$.

To achieve the above goal, the following process can be executed: $S_i$ generates a random number $u$; $S_i$ collaboratively compute the additive share of $x$ to $u$ and $1+ux$; then the division will be secretly computed; finally, $S_i$ sends $[\frac{u}{1+ux}]_1$ to $S_2$, and the transformation has completed. Due to each server still know nothing on $x$, the above process is a (2, 2)-threshold scheme, in this case, we should expand the process to the $(n, n)$-threshold.

It is difficult to directly expand the formula [5], therefore, we seek an alternative method: transform the computation in $(n, n)$-threshold to $(n-1, n-1)$-threshold process. In detail, $n$ servers can be seen as two parts, through above process, $n-1$ servers get one share of $\arctan(x)$, and the other server get another share. If above process leak no information, then we could further decrease $n-1$ to $n-2$, and until the (2, 2)-threshold. Based on the existing protocols, the above process can be designed as shown in algorithm [11] and [12].

Please note that algorithm [11] can be executed in parallel as the inputs are pre-generated in SecArctan, the interaction rounds are same to that in (2, 2)-threshold. Due to the high communication consumption which is shown in table [11] as previous works [11], we still suggest using the Taylor series in the scene which does not care about accuracy.

6 Optimization and expansion on Protocols

In the section [4] follow the idea of STR, we construct two resharing protocol. The resharing between ASS and MSS is quite useful when face non-linear functions (e.g., basic elementary functions), as computing them will involve both addition and multiplication homomorphism. However, for some multivariate functions (e.g., comparison), although it is feasible to compute them as the above section shows, it is unnecessary and these protocols will lead to the potential risk when secret equals zero. In this section, we first optimize comparison and division protocol, then show how to expand typical SecMul, and finally consider the way constructing some efficient basic computation protocols on matrices.

6.1 An optimization on comparison and division

As described in section [4], a nonzero secret multiple by a random nonzero number, the result is still random. In this case, follow the idea of STR, we further use random input to ensure the security of secret, however, ensure the result still has (or can secretly transform back to) the property we want at the same time. To distinguish the protocols designed to follow this idea, we add the suffix "-STR" in the corresponding protocols.

To comparison computation, we need to get the sign of difference of two inputs. It should be noted that the multiplication by a positive number will not change the sign of the original number. Therefore, it is feasible to let $T$ provide a random positive number which directly attends

| Algorithm 11 One iteration of Secure arctan protocol |
|-----------------------------------------------|
| **Input:** $S_i$ has $[x]_i$, $S_j$ has $[u]_j, i \in [1, n], j \in [2, n]$ |
| **Output:** $S_i$ gets $\arctan(\frac{x-u}{1+ux})$, $S_j$ gets the $[u]_j$. |
| **Offline Phase:** |
| 1: $\mathcal{T}$ generates enough random numbers the sub-protocols use and sends to $S_i$. |
| **Online Phase:** |
| 2: $S_i$ computes $[x - u]_i = [x]_i - [u]_i$, here $[u]_i = 0$. |
| 3: $S_i$ collaboratively compute $[xu]_i = \text{SecMul}(\{x\}_i, \{u\}_i)$. |
| 4: $S_i$ collaboratively compute $[\frac{x-u}{1+ux}]_i = \text{SecDiv}(\{x - u\}_i, [1 + xu]_i)$. |
| 5: $S_i$ sends $[\frac{x-u}{1+ux}]_i$ to $S_j$. |
| 6: $S_j$ collaboratively compute $\text{SecCmp}(\text{SecMul}(\{\frac{x-u}{1+ux}\}_i, [u]_j)), [1]_j$, here $[\frac{x-u}{1+ux}]_j = 0$. |
| 7: If $u \times \frac{u}{1+ux}$ more than 1, then $S_i$ computes $\arctan(\frac{x-u}{1+ux}) = \arctan(\frac{x-u}{1+ux}) + \text{sgn}(\frac{x-u}{1+ux})\pi$. |

| Algorithm 12 Secure arctan protocol SecArctan |
|-----------------------------------------------|
| **Input:** $S_i$ has $[x]_i$. |
| **Output:** $S_i$ gets $[\arctan(x)]_i$. |
| **Offline Phase:** |
| 1: $\mathcal{T}$ generates enough random numbers the sub-protocols use and sends to $S_i$. |
| **Online Phase:** |
| 2: $S_i$ sets $[u^1]_i = [x]_i$. |
| 3: for $l = 2 : n$ do |
| 4: $S_j$ generates random numbers $[u^l]_j, j \in [l, n]$. |
| 5: end for |
| 6: for $l = 1 : n - 1$ do |
| 7: $S_j$ collaboratively computes algorithm [11] with input $[u^l]_j$ and $[u^{l+1}]_j, j \in [l, n]$. |
| 8: $S_l$ gets the corresponding $[\arctan(x)]_l$. |
| 9: end for |
Further, the trigonometric functions are all exposed more information besides the size relationship of two schemes in this subsection. We will use multiplication on three numbers to discuss the multiplication on multiple numbers in only one interaction? Is it possible to execute secure multiplication in $n$ rounds of interaction? Let us see formula 1 from the following perspective: with the help of $(x-a)$ and $(y-b)$, the computation of $xy$ can be replaced by $ab$, in other word, the difficulty of multiplication on secrets is undertaken by $T$ during offline phase. Consider the situation that multiplication on three secrets, the $T$ needs to generate enough random numbers which could replace $xyz$. The detailed process is shown in algorithm [15]

As algorithm [15] shows, with the help of $acy$, $bxy$, and $abc$, the $xyz$ can be extracted from $efg$. Since $ab$, $ac$, and $be$ can be generated by $T$, which means line number 5 and 7 can be executed in parallel, therefore, the protocol only needs one interaction. The kernel process of the algorithm is excluding the term, like $axy$, by $e$, $f$, and $g$, in that the computation of $axy$ is still a multiplication on three secrets. Please note that the exclusion method is always existing, which means multiplication on $n$ numbers can be executed in one interaction with the help of multiplication on $n - 1$ numbers.

However, as the communication size will increase significantly (i.e., $O(n!)$), we only suggest using above method on three secrets; the SecMul is suitable for facing four secrets; SecMultMul is still suggested during meeting more secrets.

6.2.2 Dot production

For simplicity, we only consider the $d \times d$ squared matrices in this paper. Please note that the common operations, which are valid in plaintext domain, on non-square matrices can be designed similarly.

The addition and subtraction of matrix is just the composition of the number, the matrix multiplication (i.e., dot production) can also be composed by multiplication on numbers, which means it can be designed shown in algorithm [16]

Due to $(X_1 + X_2) \cdot Z = X \cdot Z$, the dot production has similar property of constant multiplication. In this case, inspired by Beaver triples, previous work [25] constructed SecMatMul from the perspective of matrices, here we further extend it to $(n, n)$-threshold as shown in algorithm [17]

It is easy to notice that SecMatMul2 is similar to SecMul, however, the dot production replaces multiplication. Due to plenty of optimization on matrices, when facing matrices in high dimension, there has some potential efficiency advantage [26]. However, in SecMatMul2, the
Algorithm 16 Secure matrix multiplication protocol SecMatMul1

Input: $S_i$ has $[X^{d \times d}], [Y^{d \times d}]$.
Output: $S_i$ gets $[Z^{d \times d}], Z = X \cdot Y$.

**Offline Phase**:
1. $T$ generates enough random numbers the sub-protocol uses and sends to $S_i$.

**Online Phase**:
2. $S_i$ computes $[xykl] = \text{SecMul}([x_{jk}], [y_{kl}])$ for all $j, k, l \in [1, d]$ in parallel, where $x$ and $y$ is the corresponding value in $X$ and $Y$.
3. $S_i$ computes $[z_{kl}]$ by computing $\sum_{k=1}^{d}[xy_{kl}]$, for all $j, l \in [1, d]$, where $z$ is the corresponding value in $Z$.

Algorithm 17 Secure matrix multiplication protocol SecMatMul2

Input: $S_i$ has $[X], [Y]$.
Output: $S_i$ gets $[X \cdot Y]$.

**Offline Phase**:
1. $T$ generates enough random vectors to compose $A, B$ and computes $C = AB$.
2. $T$ randomly splits $(A, B, C)$ into two additive share $([A], [B], [C])$ and sends the share to corresponding server $S_i$.

**Online Phase**:
3. $S_i$ computes $[E] = [X] - [A]$ and $[F] = [Y] - [B]$.
4. $S_i$ collaboratively recover $E$ and $F$.
5. $S_i$ computes $[X \cdot Y] = E \cdot [B] + [A] \cdot F - [C]$, $S_i$ further computes $[X \cdot Y]_i = [X \cdot Y]_i + E \cdot F$.

6.3 Secure computation on matrix

Similar to SecMatMul1, generally speaking, the operations on the matrix can be composed by the calculations on numbers. However, the direct simulation always causes $O(d)$ rounds of interaction as most operations can not be executed in parallel. To cope with the problems on matrices, we should consider the operation from the view of matrices like SecMatMul2. In this paper, we focus on the following two basic demands on matrices: Compute matrix inversion, Compute eigenvalue and eigenvector. Similar to the operations on numbers, we also consider the dot production on multiple matrices.

6.3.1 Matrix inversion

Following the idea of STR, inspired by [28], to avoid the inversion computation on shares, we first transform the original secret to an irreversible matrix, and compute the inversion of the new matrix, and finally transform it back to the true result. The detailed process is shown in algorithm 18.

As shown in SecMatInv, each server randomly generates an additive share of $Z$ for transforming the secret $X$. Due to $(Z \cdot X)^{-1} \cdot \sum_{i=1}^{m} Z_i = X^{-1}$, the inverse computation of $X$ can actually substitute by inversion of $Z \cdot X$, two times of SecMatMul and the random matrix $Z$ ensure the security and efficiency. Please note that it is almost impossible that $Z$ is not full rank (i.e., the probability is zero if the numbers are in $\mathbb{R}$), and it can be detected and remedied by computing the rank of $Z \cdot X$. Following SecMatInv, the secure dot production on multiple matrices is shown in algorithm 19.

In SecMultMatMul, a series of matrices $\{Z^j\}$ will be generated to cover secrets, here $j$ is in the range $[0, m]$, which is one more than the number of matrices to be processed. To avoid the difficulty of multiple dot production on shares, the inversion of $\{Z\}$ is computed. As shown in line number 6, the dot production will remove the $\{Z^j\}$, where $j$ is in the range $[1, m-1]$. In this case, the real shares of secret can be transformed back with the help of $Z^0$ and $(Z^m)^{-1}$.

6.3.2 Solve eigenvalue and eigenvector

The solutions of eigenvalue or eigenvector in plaintext domain (e.g., QR decomposition) always need to iterate the corresponding matrix multiple times, which means that the solution, from the perspective of numbers, in the encrypted domain may cause $O(d)$ interaction rounds and unnecessary precision loss. In this case, follow the idea of STR, to avoid the difficulty of solving eigenvalues and eigenvectors on shares, the following lemmas should be noticed:
Algorithm 20 Secure eigenvalue and eigenvectors solution

\textbf{Input:} $S_i$ has $[[X^{d \times d}]]$.
\textbf{Output:} $S_i$ gets one share of eigenvalues $\{[\lambda_j]\}$, one share of eigenvectors $[[V^{d \times d}]]_i$, here $j \in [1, d]$.

\textbf{Offline Phase}:
1. $T$ generates enough random numbers and vectors the sub-protocols use and sends to $S_i$.

\textbf{Online Phase}:
2. $S_i$ generates a random square matrix $[[P^{d \times d}]]$, and a random number $[[t]]$.
3. $S_i$ collaboratively compute $[[P^{-1}]] = \text{SecMatInv}([[[P]]])$.
4. $S_i$ collaboratively compute $[[t \cdot \lambda]] = \text{SecDiv}([[[t]], [[\lambda]]])$.
5. $S_i$ collaboratively compute $[[Y]] = \text{SecMul}([[\lambda]], \text{SecMatMul}([[X]]_i, [[P]]))$.
6. $S_i$ sends $[[Y]]$ to $S_i$, then $S_i$ computes the eigenvalues $\{t \cdot \lambda\}$ and eigenvectors of $Y$, and sends back to all $S_i$.
7. $S_i$ computes $[[\lambda]] = [[t \cdot \lambda]]$ for all $j \in [1, d]$.
8. $S_i$ computes $[[V]] = [[P]] \cdot [Y]$.

\textbf{Lemma 1.} If $A$ and $B$ are similar matrices (i.e., $A \sim B$), and $B = P^{-1}AP$, then $A$ and $B$ have the same eigenvalues; If there is an eigenvector $\vec{x}$ under eigenvalue $\lambda$ of matrix $A$, then $P^{-1}\vec{x}$ is an eigenvector of $B$ under eigenvalue $\lambda$.

\textbf{Lemma 2.} If $\lambda$ is an eigenvalue of matrix $A$, then $k \cdot \lambda$ is an eigenvalue of matrix $k \times A$; If there is an eigenvector $\vec{x}$ under eigenvalue $\lambda$ of matrix $A$, then the eigenvector $k \cdot \vec{x}$ is also under eigenvalue $k \cdot \lambda$ of matrix $k \times A$.

The detailed process of solving eigenvalues and eigenvectors is shown in algorithm 20. A random matrix $P$ and a random number $t$ will be generated to cover the secret $X$. The matrix $Y$ with eigenvalues in a constant scaling and similar eigenvectors will be computed and recovered in line number 3 to 6. With the help of $t$ and $Z$, the share of eigenvalues and eigenvectors of $X$ can be secretly transformed back.

7 Security analysis

We prove the security of our protocols under the framework of universal composability framework [13]. The execution of our schemes mainly involves the interaction between servers, and the interaction process is defined as the real experiment. Here, for simplicity, we directly consider the extreme situation, which means $n - 1$ parties attend the collusion. To prove that a protocol is secure, it suffices to show that the view of corrupted party (i.e., arbitrarily $n - 1$ servers) is simulatable. We assume that $S_n$ does not attend the collusion for simplicity.

In the ideal experiment, the simulator $S$ is defined as the one that can simulate the view of the corrupted party with the help of functionality $F$. In this paper, we define $F$ as follows. In the honest-but-curious model, $F$ owns the true input information of each protocol, and generates the random numbers or matrices locally, then completes the calculation as subprotocols. The corresponding view of $S$ will be filled by the calculation results. In the following, we will prove that the view is indistinguishable to that in the real world, and the view will not expose the input information. To simplify the proofs, the following Lemma will be used.

\textbf{Lemma 3.} [29] A protocol is perfectly simulatable if all its sub-protocols are perfectly simulatable.

\textbf{Theorem 3.} The protocol $\text{SecMul}$ is secure in the honest-but-curious model.

\textbf{Proof.} The view of adversary during executing $\text{SecMul}$ is $\{[x]_i\}, \{[y]_i\}, \{[a]_i\}, \{[b]_i\}, \{[c]_i\}, \{[f]_i\}$, where $i \in [1, n], j \in [1, n - 1]$. As described in theorem 1, the recover of $x$ needs $[[x]]_n$. Here $([a]_i), ([b]_i)$ are uniformly random which is simulatable. Due to lack of $[a]_n$, the $[c]_n$ is also totally random which is simulatable. Therefore, the above information can not infer any information of $x$. The situation of $y$ is similar, and the uniformly random view also leads to uniformly random output which is independent from the input. Therefore, both view and output are simulatable by the simulator $S$ and the views of $S$ and $A$ will be computationally indistinguishable. It is easy to note that any $n - 1$ conspiracies will lead to the same situation. □

The $\text{SecMatMul}$ is secure based on Lemma 3; algorithm 15 and $\text{SecMatMul2}$ can be proven in a similar way. Therefore, we omit their proofs for simplicity.

As both ASS and MSS are $(n, n)$-threshold in $\mathbb{R}\not\{0\}$, which means the view computed by $n - 1$ conspiracies locally is obviously random and simulatable, in this case, in the following proof, we focus on the view brought by the communication with the non-conspiracy party (i.e., $S_n$).

\textbf{Theorem 4.} The protocol $\text{SecMulRes}$ with nonzero input is secure in the honest-but-curious model.

\textbf{Proof.} The view brought by communication is $\text{view} = \langle (\alpha)_n \rangle$. Follow theorem 2, due to lack of $[c]_n$, it is impossible to infer $\langle [x]_n \rangle$ from $\langle (\alpha)_n \rangle$. In this case, the view is simulatable and computationally indistinguishable. □

Please note that when secret equal to zero, then the adversary can infer it from $\langle x \rangle_1$.

\textbf{Theorem 5.} The protocol $\text{SecAddRes}$ with nonzero input is secure in the honest-but-curious model.

\textbf{Proof.} Besides those brought by $\text{SecMul}$, the view brought by communication is $\text{view} = \langle xc \rangle$. Follow theorem 2, due to lack of $c$, it is impossible to infer $x$ from $xc$. Therefore, the view is simulatable and computationally indistinguishable. □

Please note that when secret equal to zero, then the adversary can infer it from $\langle x \rangle_1$.

\textbf{Theorem 6.} The protocols $\text{SecExp}, \text{SecLog}$ are secure in the honest-but-curious model.

\textbf{Proof.} The above protocols are all only involves $\text{SecMulRes}, \text{SecAddRes}$, and the input of $\text{SecMulRes}$ is in $\mathbb{R}\not\{0\}$. Therefore, these protocols are secure based on Lemma 3. □

\textbf{Theorem 7.} The protocols $\text{SecPow}, \text{SecDiv}, \text{SecMultiMul}$ are secure in the honest-but-curious model.

\textbf{Proof.} $\text{SecPow}$ only involves $\text{SecMulRes}$ and $\text{SecAddRes}$, and the nonzero input will not lead to zero input of $\text{SecMulRes}$. Therefore, $\text{SecPow}$ is secure based on Lemma 3. □

Please note that when secret equal to zero, then the adversary can infer it from $\langle x \rangle_1$. 

Theorem 8. The protocols SecCmp-STR, SecDiv-STR are secure in the honest-but-curious model.

Proof. To SecCmp-STR, the view brought by the communication is \( \text{view} = \{(x_a)\} \), as \( t \) is a random non-zero number, in this case, follow theorem 2, besides the size relationship, any information of \( x, y \) and \( x - y \) will be secure. To SecDiv-STR, the corresponding view is \( \{(ty)\} \), as both \( t \) and \( y \) is nonzero, any information of \( x \) and \( y \) will not be exposed.

Theorem 9. The protocol SecArctan is secure in the honest-but-curious model.

Proof. As the algorithm 12 is the repeat of algorithm 11, if \( S_z \) does not attend conspiracy, then the security can be directly proven based on Lemma 3. Consider only \( S_z \) is not the conspirator, in this case, the corresponding view brought by communication is \( \text{view} = \{(\frac{x_u}{y_{tu}})\} \). Due to the lack of \( x_n \) and \( u_n \), the information of \( \frac{x_u}{y_{tu}} \) can not infer two unknown numbers. In this case, the simulator \( S \) for the adversary can simulate this random information.

Theorem 10. The protocols SecMatInv-STR, SecMultMatMul are secure in the honest-but-curious model.

Proof. Besides those brought by SecMatMul, the view brought by communication is \( \text{view} = \{(XZ)\} \). Due to lack of information of \( Z \), any full-rank matrix in the corresponding size is potential \( X \), which means \( X \) is uniformly random and simulatable. Similar to SecMultMatMul, any corresponding size full-rank matrix is the potential \( T \).

Theorem 11. The protocols for solving eigenvectors and eigenvalues are secure in the honest-but-curious model.

Proof. Besides the view brought from SecMatInv, SecDiv, SecMul, the view brought by communication is \( \text{view} = \{(Y)\} \). Due to lack of \( P \) and \( t \), any full-rank matrix in the corresponding size will be potential \( X \), which means it is random and simulatable. However, the ratio of eigenvalues and a similar matrix of eigenvectors will be exposed. Please note that, generally speaking, this fuzzy information will not affect specific tasks.

Now we prove that for secure pluralistic functions, generally speaking, the known (e.g., exposed) input will not lead to exposure of unknown inputs.

Theorem 12. The SecMul will not expose unknown secret when the other is known.

Proof. Considering the exposed \( y \), as the view related to \( x \) (i.e., \( e \)) is covered by \( a \), which is irrelevant to \( y \), therefore, the \( x \) will not be influenced by the exposed \( x \).

The proofs for SecMultMatMul, algorithm 15 SecMatMul, SecMultMatMul are similar and we omit them for simplicity.

Theorem 13. The SecDiv and SecDiv-STR will not expose unknown secret when the other is known.

Proof. To SecDiv, as all views related to \( x \) or \( y \) are brought by SecAddRes and SecMulRes independently, the unknown secret is secure. To SecDiv-STR, consider the exposed \( y \), due to the \( ty \) is recovered, the \( t \) is exposed to each server. However, follow theorem 12, the information of \( x \) is still secure; consider the exposed \( x \), follow theorem 12, the conspirators can not infer \( t \), in this case, the information of \( y \) is still secure.

Theorem 14. The SecCmp and SecCmp-STR will not expose unknown secret if it is not equal to the known input.

Table 1: Communication complexity of protocols

| Protocol          | Rounds | Comms(bits) |
|-------------------|--------|-------------|
| SecMul            | 1      | 2n(n-1)t   |
| SecMulRes         | 1      | n(n-1)t    |
| SecAddRes         | 2      | 2(n+1)(n-1)t|
| SecCmp            | 3      | (2n+1)(n-1)t+n(n-1) |
| SecExp            | 1      | n(n-1)t    |
| SecLog            | 1      | (2n+1)(n-1)t|
| SecPow            | 1      | (3n+1)(n-1)t|
| SecMultiMul       | 3      | mn+2n+1(n-1)t|
| SecDiv            | 3      | 4n(n-1)t   |
| SecSin,cos        | 1      | n(n-1)2n-1t|
| tan,cot           | 3      | n(n-1)(2n+5)t|
| csc,sec           | 3      | n(n-1)2n-1t+5n(n-1)t|
| arctan            | 6      | 7n3+n2x2-12nx+12t |
| arcsin,arccos     | 12     | 7n3+2n/2-2n/2-1t|
| arcsec            | 10     | 7n3+12n/2-2n/2-1t|
| arccsc            | 12     | 7n3+17n/2-2n/2-1t|
| arccot            | 8      | 7n3+2n/2-4n/2-1t|
| SecCmp-STR        | 2      | 3n(n-1)t   |
| SecDiv-STR        | 2      | 5n(n-1)t   |
| SecMatMul         | 1      | 2n(n-1)2t |
| SecMatInv         | 2      | 3n(n-1)2t |
| SecMultiMatMul    | 6      | (m+9n)(n-1)2t|
| SecEigenvec       | 7      | (11n+2)(n-1)2t|
| SecEigenval       | 7      | (11n2-10n-1)2t+3n+1d|

Proof. To comparison operation, as discussed above, only the size relationship between two inputs will be exposed to each server. In this case, if they are equal, the exposure of unknown secret is inevitable. However, please note that, on the one hand, generally speaking, the server can know zero secret, which is rare; on the other hand, the comparison calculation in specific tasks is generally used to make a choice, which means the equality situations are few and meaningless.

As proven in Theorem 12 to 14, the exposure of zero secret brought by MSS will not expose other secrets except the extreme situation during comparison.

8 Efficiency analysis

8.1 Communication complexity of our protocols

We summary the rounds and communication size of the above protocols as shown in Table 1, here \( n \) means the number of numbers or matrices, \( d \) represents one dimension of the squared matrix. The protocols listed are all lossless, means the interaction rounds are all irrelevant to the bit-length of share or the dimension of the matrix. Here the communication size is the sum of all servers.

The complexity of SecMultiMul is related to \( m \) as all the number participated need to be transformed into MSS format. The trigonometric functions, such as SecSin, need to execute SecMulRes on each sub-item, therefore, the complexity of communication size is related to \( 2n \). To SecArctan, since our scheme needs to execute algorithm \( \Pi \) in \(-1\) times in parallel, the complexity of communication size reach \( O(n^3) \). Please the SecCmp-STR, SecDiv-STR and algorithm 15 are both utilized in optimizing algorithm 11. The other inverse trigonometric functions, since appropriate initial conversions are needed, the communication size is slightly increasing. The high communication consumption implies that the Taylor series is feasible for the
TABLE 2: Comparison of communication complexities.

| Scheme | SecCmp | SecDiv |
|--------|--------|--------|
| Rounds | Comm(bits) | Rounds | Comm(bits) |
| [10]   | 6lm    | 30lm + 32l | -   | -       |
| [9]    | 15     | 279l + 5 | -   | -       |
| [11]   | l + 3  | 10l − 2 | -   | -       |
| Ours   | 3      | 2l + 2 | 3   | 6l      |

Scenes which do not make excessive demands on accuracy. The complexity of other protocols is easy to infer from the description of algorithms.

8.2 Comparison on protocols

As described above, to our knowledge, few works in SMC pay their attention to computing nonlinear computation losslessly. In this case, we here only give the comparison on SecCmp and SecDiv as shown in table 2. In table 2, $n$ is set as 2 to get the same security level with the other papers. It is easy to note that our scheme has better interaction rounds compared to past works.

8.3 Remark

In the above, we analyze the protocols on $\mathbb{R}$, however, real computers can only process the data in a finite field, like $\mathbb{Z}_2^l$, where $l$ is the bit-length (e.g., 64). It gives two strong limitations: the computer can only cope with floating-point numbers with finite precision; the computer can only cope with the number in certain range size.

To the first limitation, please note that most of the outsourced tasks have strong robustness, for instance, for a classification task, the error caused by computer precision is always too low (e.g., $10^{-8}$) to affect the label. Besides, the drawback can be remedied to some extent by scaling original data up to several times. Previous work [26] also proved that the error caused by ASS is too weak to affect the retrieval accuracy, even after the process of inference of convolutional neural network and principal component analysis.

To the second limitation, please note that, generally speaking, a modern machine can support the floating number in the range $[2^{-64}, 2^{64}]$. The range is enough to contain the secret and cover its range. To avoid share over the range, the servers can execute the following process: when an additive share is over the range $2^{64}$, then the server subtracts $2^{64}$, and the other servers collaboratively add the fixed value. In this case, the servers can always seek the reasonable share as long as the secret is in the reasonable range. It is easy to note that the frequency of this problem depends on the range of random numbers, which is easy to control.

Besides, the finite field $\mathbb{F}$ also makes the above protocols which need servers generate the random numbers have $1/|\mathbb{F}|$ possibility invalid. However, it is still negligible and detectable. Whatever, for the high performance machine, the problem drawn by precision and range is too weak. And the other functional encryption methods, e.g., HE, will be more affected due to their theoretical limitations. Therefore, we assert our scheme is feasible and efficient in the real world.

9 Conclusion and future work

In this paper, we focus on the outsourced computation problem, with the help of secret sharing technology, we construct a series of fundamental computation protocols on numbers or matrices. We do not limit the number (e.g., $n$) of participating servers, and ensure the security of original data even $n − 1$ servers collude. We believe that the protocols provide a potential new tool for plenty of secure outsourced tasks in cloud computing. Due to the universality of secret sharing technology, the traditional SMC tasks will also benefit from these schemes to some extent.

We believe there are still existing many meaningful and challenging problems related to this work as follows:

1) How to let servers undertake the tasks in offline phase in security?
2) Are there more efficient protocols? How to design more secure and extensive protocols on matrices? How to cope with the security problems in the malicious model?
3) Is there any secret sharing technology that can fix the drawback of MSS which brought by zero secret? How to combine it with ASS?
4) How to decrease $n$ to 1?

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References

[1] C. Bo, G. Shen, J. Liu, X.-Y. Li, Y. Zhang, and F. Zhao, “Privacy tag: Privacy concern expressed and respected,” in Proceedings of the 12th ACM conference on embedded network sensor systems, 2014, pp. 163–176.
[2] P. Veigt and A. Von dem Bussche, “The eu general data protection regulation (gdpr),” A Practical Guide, 1st Ed., Cham: Springer International Publishing, 2017.
[3] C. Wei, S. Ji, C. Liu, W. Chen, and T. Wang, “Asgdp: Collecting and generating decentralized attributed graphs with local differential privacy,” IEEE Transactions on Information Forensics and Security, vol. 15, pp. 3239–3254, 2020.
[4] C. Gentry, “Fully homomorphic encryption using ideal lattices,” in Proceedings of the forty-first annual ACM symposium on Theory of computing, 2009, pp. 169–178.
[5] D. Beaver, "Efficient multiparty protocols using circuit randomization," in Annual International Cryptology Conference. Springer, 1991, pp. 420–432.
[6] A. Acar, H. Aksu, A. S. Uluagac, and M. Conti, “A survey on homomorphic encryption schemes: Theory and implementation,” ACM Computing Surveys (CSUR), vol. 51, no. 4, pp. 1–35, 2018.
[7] O. Goldreich, S. Micali, and A. Wigderson, “How to play any mental game,” in Proceedings of the 12th ACM conference on computational complexity, 1991, pp. 26–33.
[8] K. Huang, X. Liu, S. Fu, D. Guo, and M. Xu, “A lightweight privacy-preserving cnn feature extraction framework for mobile sensing,” IEEE Transactions on Dependable and Secure Computing, 2019.
[9] L. Liu, J. Su, X. Liu, R. Chen, K. Huang, R. H. Deng, and X. Wang, “Toward highly secure yet efficient kNN classification scheme on outsourced cloud data,” IEEE Internet of Things Journal, vol. 6, no. 6, pp. 9841–9852, 2019.

[10] T. Nishide and K. Ohta, “Multiparty computation for interval, equality, and comparison without bit-decomposition protocol,” in International Workshop on Public Key Cryptography. Springer, 2007, pp. 334–360.

[11] L. Xiong, W. Zhou, Z. Xia, Q. Gu, and J. Weng, “Efficient privacy-preserving computation based on additive secret sharing,” arXiv preprint arXiv:2009.05356, 2020.

[12] P. Kairouz, H. B. McMahan, B. Avent, A. Bellet, M. Bennis, A. N. Bhagoji, K. Bonawitz, Z. Charles, G. Cormode, R. Cummings et al., “Advances and open problems in federated learning,” arXiv preprint arXiv:1912.04977, 2019.

[13] R. Canetti, “Universally composable security: A new paradigm for cryptographic protocols,” in Proceedings 42nd IEEE Symposium on Foundations of Computer Science. IEEE, 2001, pp. 136–145.

[14] A. C. Yao, “Protocols for secure computations,” in 23rd annual symposium on foundations of computer science (sfcs 1982). IEEE, 1982, pp. 160–164.

[15] Y. Wu, X. Wang, W. Susilo, G. Yang, Z. L. Jiang, Q. Chen, and P. Xu, “Efficient server-aided secure two-party computation in heterogeneous mobile cloud computing,” IEEE Transactions on Dependable and Secure Computing, 2020.

[16] A. C.-C. Yao, “How to generate and exchange secrets,” in 27th Annual Symposium on Foundations of Computer Science (sfcs 1986). IEEE, 1986, pp. 162–167.

[17] G. Asharov, Y. Lindell, T. Schneider, and M. Zohner, “More efficient oblivious transfer and extensions for faster secure computation,” in Proceedings of the 2013 ACM SIGSAC conference on Computer & communications security, 2013, pp. 535–548.

[18] D. Evans, V. Kolesnikov, and M. Rosulek, “A pragmatic introduction to secure multi-party computation,” Foundations and Trends® in Privacy and Security, vol. 2, no. 2-3, 2017.

[19] A. Shamir, “How to share a secret,” Communications of the ACM, vol. 22, no. 11, pp. 612–613, 1979.

[20] A. Beimel, “Secret-sharing schemes: a survey,” in International Conference on Coding and Cryptology. Springer, 2011, pp. 11–46.

[21] M. Ben-Or, S. Goldwasser, and A. Wigderson, “Completeness theorems for non-cryptographic fault-tolerant distributed computation,” in Providing Sound Foundations for Cryptography: On the Work of Shafi Goldwasser and Silvio Micali, 2019, pp. 351–371.

[22] I. Damgård, M. Fitzi, E. Kiltz, J. B. Nielsen, and T. Toft, “Unconditionally secure constant-rounds multi-party computation for equality, comparison, bits and exponentiation,” in Theory of Cryptography Conference. Springer, 2006, pp. 285–304.

[23] H. Morita, N. Attrapadung, T. Teruya, S. Ohata, K. Nuida, and G. Hanaoka, “Constant-round client-aided two-server secure comparison protocol and its applications,” IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, vol. 103, no. 1, pp. 21–32, 2020.

[24] P. Mohassel and Y. Zhang, “Secureml: A system for scalable privacy-preserving machine learning,” in 2017 IEEE Conference on Security and Privacy (SP). IEEE, 2017, pp. 19–39.

[25] S. Wagh, D. Gupta, and N. Chandran, “Securem: Efficient and private neural network training,” IACR Cryptol. ePrint Arch., vol. 2018, p. 442, 2018.

[26] Z. Xia, Q. Gu, L. Xiong, W. Zhou, and J. Weng, “Privacy-preserving image retrieval based on additive secret sharing,” arXiv preprint arXiv:2009.06683, 2020.

[27] “https://en.wikipedia.org/wiki/multivalued_function.”

[28] J. Bar-Ilan and D. Beaver, “Non-cryptographic fault-tolerant computing in constant number of rounds of interaction,” in Proceedings of the eighth annual ACM Symposium on Principles of distributed computing, 1989, pp. 201–209.

[29] D. Bogdanov, S. Laur, and J. Willemsen, “Sharemind: A framework for fast privacy-preserving computations,” in European Symposium on Research in Computer Security. Springer, 2008, pp. 192–206.

[30] D. Bogdanov, M. Niitsoo, T. Toft, and J. Willemsen, “High-performance secure multi-party computation for data mining applications,” International Journal of Information Security, vol. 11, no. 6, pp. 403–418, 2012.
**APPENDIX**

9.1 Power with real exponent

When the exponent $a$ is in the $\mathbb{Q}$, the power function will become a multivalued function. However, in the real task, we believe only the principal value is the desired result. Therefore, when the base number is positive, we here solve the corresponding principal value. However, when the base number is negative, the complex number will be inevitably involved. Whatever, it is a problem that should be defined in the specific task, in this paper, with the help of the identity

$$u^a = (-1)^n|u|^a,$$

where $u$ is negative, the detailed choice of value can be shifted to the choice of $(-1)^a$ and the solution can be defined in the specific task (e.g., define $(-1)^\frac{1}{2} = -1$). The detailed protocol is shown in algorithm [21].

Algorithm 21 Secure Power with Rational Exponent protocol SecPRE

| Input: | $S_i$ has $[x]_i$ and a public rational exponent $a$. ($x \neq 0$) |
| Output: | $S_i$ gets $\llbracket x^a \rrbracket$ |

**Online Phase:**
1: $T$ generates enough random numbers the sub-protocols use and sends to $S_i$.
2: $S_i$ collaboratively compute $\text{SecCmp}([x]_i, 0)$ to judge the sign of $x$.
3: $S_i$ collaboratively compute $\langle x \rangle_i = \text{SecAddRes}(\langle x \rangle_i)$.
4: $S_i$ computes $\llbracket x^a \rrbracket_i = \llbracket [x]^a \rrbracket$. |
5: $S_i$ collaboratively compute $\llbracket [x]^a \rrbracket = \text{SecMulRes}(\llbracket [x]^a \rrbracket)$.
6: $S_i$ compute $\llbracket x^a \rrbracket_i = (-1)^n \times \llbracket [x]^a \rrbracket_i$ if $x$ is negative.

The communication rounds is 5, the communication size is $(6n + 1)(n - 1)l$. A common case in this category is computing the positive square root, in this case, the base number is positive naturally, and the communication rounds and size will decrease to 3 and $(3n + 1)(n - 1)l$.

9.2 Basic computations built on multiplicative share

The input and output of the following protocols are all in MSS format. Due to the inherent defects of MSS, these protocols are $(n, n)$-threshold in $\mathbb{R}\setminus\{0\}$

9.2.1 Secure Addition and subtraction

Algorithm 22 Secure addition/subtraction protocol SecAdd/Sub

| Input: | $S_i$ has $\langle x \rangle_i$, $\langle y \rangle_i$. |
| Output: | $S_i$ gets $\langle x \pm y \rangle_i$. |

**Offline Phase:**
1: $T$ generates enough random numbers the sub-protocols use and sends to $S_i$.

**Online Phase:**
2: $S_i$ collaboratively compute $\llbracket x \rrbracket_i = \text{SecMulRes}(\langle x \rangle_i)$, $\llbracket y \rrbracket_i = \text{SecMulRes}(\langle y \rangle_i)$.
3: $S_i$ collaboratively compute $\langle x \pm y \rangle_i = \text{SecAddRes}(\llbracket x \rrbracket_i + \llbracket y \rrbracket_i)$.

During executing addition/subtraction, the servers first transform the multiplicative share to additive share and transform back after the addition.

9.2.2 Secure Comparison

Although the sign of multiplicative share directly reflects the sign of $\text{secret}$, however, the subtraction before comparison will make extra consumption which is more than that in additive share.

**Algorithm 23** Secure comparison protocol SecCmp-Mul

| Input: | $S_i$ has $\langle x \rangle_i$, $\langle y \rangle_i$. |
| Output: | $S_i$ gets size relationship between $x$ and $y$. |

**Offline Phase:**
1: $T$ generates enough random numbers the sub-protocols use and sends to $S_i$.

**Online Phase:**
2: $S_i$ collaboratively compute $\langle x - y \rangle_i = \text{SecSub}([x]_i, [y]_i)$.
3: $S_i$ sends the sign of $\langle x - y \rangle_i$ to all other $S_j$, where $j \neq i$.

The $\text{sgn}(x - y) = \prod_{j=1}^n \text{sgn}((x - y)_j)$

9.2.3 Secure exponentiation and logarithm

The situation in the exponentiation and logarithm operations are symmetric to that in additive share. For example, in SecExp-Mul, the servers need firstly convert the input to additive share, and the multiplicative share of output is directly gotten after executing exponentiation operation locally.

**Algorithm 24** Secure exponentiation protocol SecExp-Mul

| Input: | $S_i$ has $\langle x \rangle_i$, a base number $a$. |
| Output: | $S_i$ gets $\langle a^x \rangle_i$. |

**Offline Phase:**
1: $T$ enough random numbers the sub-protocols use and sends to $S_i$.

**Online Phase:**
2: $S_i$ collaboratively compute $\llbracket x \rrbracket_i = \text{SecMulRes}(\langle x \rangle_i)$.
3: $S_i$ computes $\langle a^x \rangle_i = \llbracket a^{[x]} \rrbracket_i$.

**Algorithm 25** Secure logarithm protocol SecLog-Mul

| Input: | $S_i$ has $\langle x \rangle_i$, a base number $a$. |
| Output: | $S_i$ gets $\langle \log_a x \rangle_i$. |

**Offline Phase:**
1: $T$ enough random numbers the sub-protocols use and sends to $S_i$.

**Online Phase:**
2: $S_i$ computes $\llbracket \log_a x \rrbracket_i = \log_a \langle x \rangle_i$.
3: $S_i$ collaboratively compute $\langle \log_a x \rangle_i = \text{SecAddRes}(\llbracket \log_a x \rrbracket_i)$.

9.2.4 Secure Trigonometric functions

It seems no useful identity which could directly support the multiplicative share of the input of trigonometric. In this case, we believe that the most reasonable way is still transforming the input to additive share, and get the result
TABLE 3: Communication complexity of protocols in MSS

| Protocol          | Rounds | Comm(bits)       |
|-------------------|--------|------------------|
| SecAdd/SecSub     | 3      | $(4n + 1)(n - 1)$l |
| SecAdd/SecSub     | 3      | $(4n + 1)(n - 1)$l + $n(n - 1)$l |
| SecExp-Mul        | 1      | $n(n - 1)$l      |
| SecLog-Mul        | 1      | $(2n + 1)(n - 1)$l |
| SecPow-Mul        | 0      | 0                |
| SecSin-Mul,cos    | 4      | $n(n - 1)2^{n-1-l} + (3n + 1)(n - 1)$l |
| tan,cot           | 4      | $n(n - 1)2^{n-1-l} + (6n + 2)(n - 1)$l |
| sec,sec           | 4      | $n(n - 1)2^{n-1-l} + (3n + 1)(n - 1)$l |
| SecArctan-Mul     | 9      | $(7n^3 + \frac{7}{2}n^2 - \frac{19}{4}n - 1)$l |
| arccos,arcsin     | 18     | $(7n^3 + \frac{7}{2}n^2 - \frac{19}{4}n - 3)$l |
| arccsc,arcsec     | 16     | $(7n^3 + \frac{7}{2}n^2 - \frac{19}{4}n - 3)$l |
| arccot            | 11     | $(7n^3 + \frac{7}{2}n^2 - \frac{19}{4}n - 1)$l |

with the help of SecAddRes. The total process is shown in algorithm 26. The cosine process will be similar, due to the support of multiplication, and other trigonometric functions can be directly computed based on the results of sine and cosine.

Algorithm 26 Secure sine/arctan protocol

Input: $S_i$ has $\langle \theta \rangle_i$.
Output: $S_i$ gets $\langle \sin/\arctan(\theta) \rangle_i$.

**Offline Phase**:
1. $T$ enough random numbers the sub-protocol uses and sends to $S_i$.

**Online Phase**:
2. $S_i$ collaboratively compute $\langle \theta \rangle_i = \text{SecMulRes}(\theta)_i$.
3. $S_i$ collaboratively compute $\langle \sin/\arctan(\theta) \rangle_i = \text{SecSin/SecArctan}(\langle \theta \rangle_i)$.
4. $S_i$ collaboratively compute $\langle \sin/\arctan(\theta) \rangle_i = \text{SecAddRes}(\langle \sin/\arctan(\theta) \rangle_i)$

The overview of communication complexity in multiplicative share is shown in Table 3. It is easy to note that multiplicative share only gets the advantages in the operations related to the power function. The exponentiation and logarithm are the same consumption as ASS. In this case, the tasks which mainly involve power function (e.g., multiplication or division) will be better when built on MSS. However, we believe it is rare and ASS is one of the most appropriate storage formats for the shares.