A pure Skyrme instanton

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Abstract

The nuclear Skyrme model is considered in the extreme limit where the nucleon radius tends to infinity. In this limit only the Skyrme term in the action is significant. The model is then conformally invariant in dimension 4, and supports an instanton solution which can be constructed explicitly. The construction uses the conformal invariance and a certain symmetry reduction to reduce the model to the static $\phi^4$ model in one dimension. The $\phi^4$ kink solution gives the radial profile of the instanton, the kink position zero-mode corresponding to the instanton width.

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The Skyrme model is a low-energy effective theory of nuclear physics possessing a Lie group valued field $U: \mathbb{R}^{3+1} \to SU(2)$. The action is usually taken to be

$$S = \int_{\mathbb{R}^{3+1}} \frac{F_{\pi}^2}{16} \text{tr} (L^\mu L^\dagger_\mu) + \frac{1}{32e^2} \text{tr} ([L_\mu, L_\nu][L^\mu, L^\nu])$$

where $L_\mu = U^{-1} \partial_\mu U$ is the left invariant current, Greek indices run over 0, 1, 2, 3, and we have given spacetime the signature $+ - - -$. This model possesses topological solitons, labelled topologically by their class $B \in \pi_3(SU(2)) \cong \mathbb{Z}$. These are thought to model light atomic nuclei, $B$ being identified with the number of nucleons in the nucleus. This physical interpretation is used to set the values of the coupling constants $F_\pi$ and $e$. $F_\pi$ is the pion decay constant, and the nucleon radius is proportional to $(eF_\pi)^{-1}$. The purpose of this letter is to show that in the extreme limit $F_\pi e \to 0$, where only the second term in $S$ is important, the model possesses instantons, that is finite action solutions of the Euclideanized model on $\mathbb{R}^4$. We call this limit, which corresponds physically to the limit of large nucleon radius, the pure Skyrme model. The instantons can be constructed explicitly by working within a rotationally equivariant ansatz for which the model reduces to the static $\phi^4$ model (in one dimension). The radial profile of the instanton is directly related to the $\phi^4$ kink profile. The instanton is labelled topologically by its class in $\pi_4(SU(2)) \cong \mathbb{Z}_2$. Hence it coincides with the anti-instanton (the concatenation of two instantons is null-homotopic).
The action of interest is \( S = \frac{1}{2\pi} E_4 \) where

\[
E_4 = \frac{1}{16} \int_{\mathbb{R}^4} \text{tr} ([L_\mu, L_\nu][L_\mu, L_\nu]).
\] (2)

The notation \( E_4 \) signifies that we are thinking of the functional as the potential energy of a static field on \( \mathbb{R}^4 \), the subscript 4 denoting that the energy density is quartic in spatial derivatives. For our purposes it is convenient to use Manton’s geometric formulation of the Skyrme term \( \mathbb{M} \). Recall this makes sense for maps \( U : M \rightarrow N \) between any pair of Riemannian manifolds \((M, g), (N, h)\), interpreted as physical space and target space respectively. In our case \( M = \mathbb{R}^4 \) with the Euclidean metric, and \( N = SU(2) \cong S^3 \) with the round metric of unit radius. Associated to any \( U : \mathbb{R}^4 \rightarrow SU(2) \) there is a symmetric \((1,1)\) tensor \( D \) on \( \mathbb{R}^4 \), that is, a self-adjoint linear map \( D_p : T_p \mathbb{R}^4 \rightarrow T_p \mathbb{R}^4 \) for each \( p \in \mathbb{R}^4 \), called the strain tensor, defined by

\[
g_p(X, D_p Y) = h_{U(p)}(dU_p X, dU_p Y)
\] (3)

for all \( X, Y \in T_p \mathbb{R}^4 \). Manton showed that the Skyrme term \( E_4 \) is, in this language,

\[
E_4 = \frac{1}{2} \int_M (\text{tr} D)^2 - \text{tr} (D^2).
\] (4)

Under a conformal change in the metric on \( M \), \( g \mapsto \tilde{g} = e^{2f} g \) where \( f \in C^\infty(M) \), the strain tensor of a given configuration \( U \) transforms as \( D \mapsto \tilde{D} = e^{-2f} D \), and the volume form on \( M \) transforms as \( \text{vol} \mapsto \tilde{\text{vol}} = e^{\text{dim} \mathbb{R}^4 f} \text{vol} \). So the energy functional \( E_4 \) is conformally invariant if (and only if) \( M \) has dimension 4, the case of interest here. In this sense, the pure Skyrme model is similar to Yang-Mills theory.

We will seek critical points of \( E_4 \) within a radially symmetric ansatz, defined as follows. Split \( \mathbb{R}^4 \setminus \{0\} \) into a family of concentric three-spheres labelled by radial coordinate \( r \). Identify each \( S^3 \) with \( SU(2) \) in the usual way (see equation (3), below). Let \( g_{SU(2)} \) be the usual bi-invariant metric on \( SU(2) \) (of unit radius). Then the Euclidean metric on \( \mathbb{R}^4 \setminus \{0\} \) is

\[
g = dr^2 + r^2 g_{SU(2)} = e^{2s}(ds^2 + g_{SU(2)})
\] (5)

where \( s := \log r \). Since \( \mathbb{R}^4 \setminus \{0\} \) is conformal to \( \mathbb{R} \times SU(2) \) and \( E_4 \) is conformally invariant, we can equally well solve the model on \( \mathbb{R} \times SU(2) \), with the product metric (we must check that the decay of \( U \) as \( |s| \rightarrow \infty \) is sufficiently fast for the energy to converge as \( r \rightarrow 0 \) and \( r \rightarrow \infty \)). Now consider fields of the form

\[
U : \mathbb{R} \times SU(2) \rightarrow SU(2), \quad U(s, q) = q\eta(s)q^{-1}
\] (6)

where \( \eta : \mathbb{R} \rightarrow SU(2) \) is any curve in \( SU(2) \) satisfying the boundary conditions \( \eta(-\infty) = -I \), \( \eta(\infty) = I \). As we shall see in the next paragraph, for fixed \( s \), \( U(s, q) \) lies on a two-sphere in \( SU(2) \) for all \( q \in SU(2) \). Fields within this ansatz are precisely those which are invariant under the left action of \( SU(2) \) defined by

\[
U(s, q) \mapsto V U(s, V^{-1}q)V^{-1}.
\] (7)
This action maps $\mathcal{D}(s, q) \mapsto \mathcal{D}(s, V^{-1} q)$ and hence leaves $E_4$ invariant. Hence, by the principle of symmetric criticality, critical points of the restriction of $E_4$ to fields of this form are solutions of the full variational problem for $E_4$ [4], so the ansatz (6) is guaranteed to be consistent with the variational equation for $E_4$.

We claim that, provided the curve $\eta$ avoids the poles $\eta = \pm \mathbb{I}$ and satisfies the boundary conditions $\lim_{s \to \pm \infty} \eta = \pm \mathbb{I}$, the corresponding field $U(s, q) = q(\eta(s))^{-1}$ (or, more precisely, its unique continuous extension to $S^4$) lies in the nontrivial class of $\pi_4(SU(2)) \cong \mathbb{Z}_2$. To see this, consider for each fixed $s \in \mathbb{R}$ the map $U(s, \cdot) : SU(2) \to SU(2)$. We can identify the target $SU(2)$ with the unit three-sphere in $\mathbb{R}^4$ using

$$\begin{pmatrix} a_0, a_1, a_2, a_3 \end{pmatrix} \mapsto a_0 \mathbb{I} + ia_1 \tau_1 + ia_2 \tau_2 + ia_3 \tau_3 = \begin{bmatrix} a_0 + ia_3 & -a_1 + ia_2 & \cr a_1 + ia_2 & a_0 - ia_3 & \cr \end{bmatrix} ,$$

where $\tau_1, \tau_2, \tau_3$ are the Pauli spin matrices. Then $U(s, \cdot)$ is a Hopf map from $S^3$ onto the two-sphere obtained by intersecting $S^3$ with the hyperplane $a_0 = \eta_0(s)$, where

$$\eta(s) = \eta_0(s) \mathbb{I} + i \eta(s) \cdot \tau .$$

Hence the extension of $U(s, q)$ to $S^4$ is topologically a suspension of the Hopf map $S^3 \to S^2$. Now the Hopf map generates $\pi_3(S^2)$, and suspension induces a surjective homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ by the Freudenthal suspension theorem [4], so the extension of $U$ is not null-homotopic. This conclusion agrees with the results of Williams [9] who considered the topology of similar maps $S^4 \to S^3$.

It remains to substitute the ansatz (6) into $E_4$ and vary the curve $\eta$. Let us introduce an orthonormal frame for $M = \mathbb{R} \times SU(2)$ consisting of the vector field $e_0 = \frac{\partial}{\partial s}$ and the left invariant vector fields $e_1, e_2, e_3$ on $SU(2)$ whose values at $\mathbb{I}$ coincide with $i\tau_1, i\tau_2, i\tau_3$. Note that $e_1, e_2, e_3$ are twice the usual left invariant vector fields $\theta_1, \theta_2, \theta_3$ on $SU(2)$, so our orthonormal frame on $M$ is

$$e_0 = \frac{\partial}{\partial s}, \quad e_1 = 2 \theta_1, \quad e_2 = 2 \theta_2, \quad e_3 = 2 \theta_3 .$$

Relative to this basis the strain tensor of a field of form (6) has matrix representative

$$\mathcal{D} = \begin{bmatrix} \dot{\eta}_0^2 + |\eta|^2 & \frac{-2(\eta \times \dot{\eta})^T}{2|\eta|^2} & \cr \frac{-2(\eta \times \dot{\eta})}{2|\eta|^2} & 4(|\eta|^2 \mathbb{I}_3 - \eta \otimes \eta^T) & \cr \end{bmatrix} ,$$

where $\dot{}$ denotes differentiation with respect to $s$ and $\times$ is the $\mathbb{R}^3$ vector product. Hence the energy functional to be minimized is

$$E_4 = 2\pi^2 \int_{-\infty}^{\infty} ds \left\{ (1 + |\eta|^2) \dot{\eta}_0^2 + |\eta|^2 |\dot{\eta}|^2 + 4|\eta|^4 \right\}$$

subject to the constraint $\eta_0^2 + |\eta|^2 = 1$. Clearly this functional is invariant under the discrete symmetry

$$(\eta_0, \eta_1, \eta_2, \eta_3) \mapsto (\eta_0, -\eta_1, -\eta_2, \eta_3) .$$

(13)
Hence, by the principle of symmetric criticality again, we may seek solutions which are fixed by this symmetry, that is, we can restrict attention to curves of the form $(\eta_0(s), 0, 0, \eta_3(s))$ with $\eta_0^2 + \eta_3^2 \equiv 1$. For such curves $E_4$ reduces to

$$E_4 = 8\pi^2 \int_{-\infty}^{\infty} ds \left\{ \frac{1}{2} \dot{\eta}_0^2 + (1 - \eta_0^2)^2 \right\}.$$  

(14)

Note that this is precisely the potential energy of the $\phi^4$ model in one dimension. We seek solutions of this model interpolating between $\eta_0 = -1$ and $\eta_0 = 1$. There is a one-parameter family of such solutions, the kinks, parametrized by position on the line $s_0 \in \mathbb{R}$,

$$\eta_0(s) = \tanh \sqrt{2}(s - s_0).$$  

(15)

Since the $\phi^4$ kink is known to have finite energy, it is immediate that the decay of $U(s, q) = q\eta(s)q^{-1}$ as $|s| \to \infty$ is fast enough to ensure finite total $E_4$. In fact $E_4 = 32\sqrt{2}\pi^2/3$. Transforming back to polar coordinates on $\mathbb{R}^4 \setminus \{0\}$, we find that

$$U(r, q) = q \left\{ \begin{array}{cc} \frac{1}{(r/r_0)^2\sqrt{2} + 1} \left[ \begin{array}{cc} (r/r_0)^2\sqrt{2} + 2i(r/r_0)\sqrt{2} - 1 \\ 0 \end{array} \right] \\ \begin{array}{cc} 0 \\ (r/r_0)^2\sqrt{2} - 2i(r/r_0)\sqrt{2} - 1 \end{array} \right\} \right\} q^{-1},$$  

where $r_0 = e^{s_0}$. Note that the kink position parameter $s_0$ becomes the width of the instanton $r_0$ on $\mathbb{R}^3$, a free parameter of the solution, reflecting the conformal invariance of $E_4$. Note also that the instanton is smooth away from the origin but only $C^1$ at the origin. Alternatively, we can interpret $U(s, q)$ as a globally smooth, spatially homogeneous, instanton on (Euclideanized) spacetime $\mathbb{R} \times S^3$.

By the usual Bogomol'nyi argument applied to the $\phi^4$ model, we know that the instanton (16) minimizes $E_4$ among all equivariant maps in its homotopy class. It is an open question whether the instanton is a true minimum of $E_4$ among all maps in its homotopy class: perhaps there is a globally smooth minimizer outside the equivariant ansatz.

As a check on our construction, we can verify directly that the mapping constructed satisfies the field equation for action $E_4$. This is

$$\partial_\alpha([L_\beta, [L^\beta, L^\alpha]]) = 0$$  

(17)

where $x^\alpha$ are Cartesian coordinates on $\mathbb{R}^4$ and $L_\alpha = U^{-1}\partial_\alpha U$ is (as before) the left-invariant current [6]. While one could compute the left hand side of (17) by brute force using, for example, Maple, it is more satisfactory to complete the calculation by hand, and for this it is useful to re-write equation (17) in coordinate independent language.

Let $\mu$ be the left-invariant Maurer-Cartan form on $SU(2)$, that is, the $\mathfrak{su}(2)$ valued 1-form which assigns to any vector $X \in T_qSU(2)$ the element $X \in \mathfrak{su}(2) = T_qSU(2)$ whose left translate by $q$ coincides with $X$. Then $L_\alpha = (U^*\mu)(\partial_\alpha)$ (so we are interpreting the left-invariant current as a 1-form on $M$, the pullback by the field $U$ of the Maurer-Cartan form). Further, we define a $\mathfrak{su}(2)$ valued 1-form $\nu$ on $M$ by

$$\nu(X) = \sum_i [U^*\mu(e_i), [U^*\mu(e_i), U^*\mu(X)]]$$  

(18)
where \( \{ e_i \} \) is any orthonormal frame on \( M \), and \( X \) is any vector on \( M \). Then (17) may be rewritten

\[
\delta \nu = 0
\]

where \( \delta \) is the coderivative on \( \mathfrak{su}(2) \otimes T^* M \). (More concretely, \( \delta \nu = \sum_{a=1}^{3} (- \ast d \ast \nu_a) i \tau_a \) where \( \nu = \sum_a \nu_a (i \tau_a) \), so \( \nu_a \) are real 1-forms on \( M \).) This gives an alternative geometric formulation of the variational problem: the 1-form \( \nu \), constructed from \( U^* \mu \), must be coclosed.

Now, for fields within the equivariant ansatz (6), we have

\[
F = i \eta 3 \bigg( \sum_{a=1}^{3} \frac{i}{2} \tau_a q [q,q] \bigg) q^\dagger.
\]

It may seem that the limit \( F_\pi e \rightarrow 0 \) is rather artificial. Certainly, if a multiple of the quadratic term

\[
E_2 = \int_{\mathbb{R}^4} \frac{1}{2} \text{tr} (L_\mu L_\mu^\dagger) = \int_{\mathbb{R}^4} \text{tr } \Phi
\]

(26)
is added to $E_4$ then no instanton solution on $\mathbb{R}^4$ is possible, by Derrick’s theorem \cite{Derrick}. The point is that, under a scaling variation $U(x) \mapsto U(\lambda x)$, where $\lambda > 0$ is a positive constant, $E_2 \mapsto \lambda^{-2}E_2$, while $E_4 \mapsto E_4$, so a nonconstant configuration can always lose energy by shrinking. The modern viewpoint of the Skyrme model, however, is to think of it as an expansion in derivatives of a more fundamental action coming from QCD, truncated at the quartic term, so that sextic and higher terms in the action density are neglected \cite{Manton}. The main reason for truncating at the quartic term is that this gives the simplest theory which evades Derrick’s theorem in dimension 3 and hence allows topological solitons. In principle, there is no reason why terms sextic (and higher) in spatial derivatives should not also be included. One possibility which has been examined in detail in $\mathbb{R}^{3+1}$ \cite{Floratos} is the term
\begin{equation}
E_6 = \int_{\mathbb{R}^4} \text{tr} \{ [L_\mu, L_\nu] [L_\nu, L_\lambda] [L_\lambda, L_\mu] \}.
\end{equation}

Generalized Skyrme models of this type evade Derrick’s theorem in dimensions 3 and 4, so they could quite possibly support both Skyrmions and instantons. Note that $E_2$ and $E_6$ are, like $E_4$, invariant under the $SU(2)$ action \cite{Hatcher}, so the ansatz \cite{Krusch} works in this more general setting too. Conformal invariance is lost, however, as is the reduction to $\phi^4$ theory, and the differential equation for the curve $\eta(r)$ does not appear to be analytically tractable. Nevertheless, one would hope that instanton solutions exist within this ansatz, and could be constructed at least numerically.

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