Group theory analysis of electrons and phonons in N-layer graphene systems

L. M. Malard, D. L. Mafra, M. H. D. Guimarães, M. S. C. Mazzoni and A. Jorio
Departamento de Física, Universidade Federal de Minas Gerais, 30123-970, Belo Horizonte, Brazil
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In this work we study the symmetry properties of electrons and phonons in graphene systems as function of the number of layers. We derive the selection rules for the electron-radiation and for the electron-phonon interactions at all points in the Brillouin zone. By considering these selection rules, we address the double resonance Raman scattering process. The monolayer and bilayer graphene in the presence of an applied electric field are also discussed.

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I. INTRODUCTION

The current interest on graphene and its multilayered materials has been stimulated by various experimental and theoretical works addressing the physics of Dirac fermions and the potential for device applications ([1, 2] and references therein). Group theory is a powerful theoretical tool to determine eigenvectors, the number and the degeneracies of eigenvalues and to obtain and understand the selection rules governing, for example, electron-radiation and electron-phonon interactions. Although the symmetry aspects of mono-layer graphene and graphite have been largely discussed in the literature [3], the recent findings generate interest in a group theory analysis depending on the number of graphene layers.

This work presents a group theory analysis for electrons and phonons in mono-, bi- and tri-layer graphene, extending for N layers depending if N is even or odd. The selection rules for electron-radiation interaction within the dipole approximation and for electron scattering by phonons are derived. With these selection rules, we discuss the double-resonance Raman (DRR) scattering process, which has been widely used to characterize the number of layers [4, 5, 6] and to probe their electronic and vibrational properties [7, 8, 9]. Finally, we also discuss the differences when mono- and bi-layer graphene are exposed to external electrical fields, giving insight on the gap opening in the biased bilayer graphene [10, 11, 12, 13] and different selection rules for the electron-phonon scattering (EPS) process.

Section II gives the symmetry properties for monolayer and N layer graphene depending if N is even or odd. The monolayer and bilayer graphene are also considered in the presence of an electric field perpendicular to the graphene plane. The notation adopted is related to the space group symmetry, and conversion to the point group notation can be found in the appendix. Section III presents the selection rules for the electron-radiation interaction. Section IV shows the Γ point Raman and infrared active modes, and we extend the electron-phonon selection rules to points in the interior of the Brillouin zone in section V. Considering both sections II and IV we address the DRR process for mono-, bi- and tri-layer graphene in section VI. The main findings are summarized in section VII.

II. SYMMETRY PROPERTIES

A. Group of wavevector

Figure I(a,d) shows the hexagonal real space for the monolayer graphene with two inequivalent atoms in the unit cell. The origin is set at the highest symmetry point, i.e. at the center of a hexagon. The reciprocal space is shown in Fig. I(g) highlighting the high symmetry points Γ, K, K', M and lines T, T', Σ. Any other generic point outside the high symmetry lines and points is named here u. The monolayer graphene on an isotropic medium has the space group P6/mmm (D_{3h}^1) in the Hermann-Mauguin notation. At the Γ point, the group of wavevector (GWV) is isomorphic to the point group D_{6h} (the Schoenflies character tables for the point groups can be found in Ref. [3]).

The real space for bilayer and trilayer graphene with AB Bernal stacking are show in Figs. I(b,e) and (c,f), respectively. The symmetries for N-layer graphene, with N even or odd (from now on, N ≠ 1), are the same of bilayer and trilayer graphene, respectively. The main symmetry operation distinguishing the point groups between even and odd layers are the horizontal mirror plane, which is absent for N even, and the inversion, which is absent for N odd. The point groups isomorphic to the GWV for mono-, N-layer graphene (N even and odd), and for N infinite (graphite) are listed in table II, for all points and lines in the first Brillouin Zone (BZ). The GWV for N-layers graphene are subgroups of the GWV for single layer graphene. The direct product between the group from N even and N odd gives the graphene GWV, i.e.

\[ \{G_{even}|0\} \otimes \{G_{odd}|0\} = \{G_{monolayer}|0\}. \]

On graphite, the wavevector point groups are isomorphic to the wavevector point groups of monolayer graphite, but differ fundamentally for some classes where a translation of \( c/2 \) is present, graphite belonging to the P6_3/mmc (D_{6h}^4) non-symorphic space group.
Two bands are degenerated at the K point (see Table III and Fig. 2(b)) and the other two have a gap of $2\gamma_1$, where $\gamma_1$ is the Slonczewski-Weiss-McClure parameter [18, 19] that have a experimental value of $\sim 0.3 - 0.4$eV [7, 10, 20].

Trilayer graphene in the ABA Bernal stacking (see Fig. 1(c,f)), belongs to the $D_{3h}$ point group and Fig. 2(c) shows its electronic dispersion. The K point of trilayer is isomorphic to $C_{3h}$. In Tables III and IV $K_1^+$ and $K_1'^+$ are the two one-dimensional representations of the $K_1^+$ representation, where * means the complex conjugate. The same happens for the $K_2^-$ representation. The electron representations will be given by $\Gamma_{K} = K_1^+ + 2K_1^- + K_2'^+ + K_2'^- + K_2^*$ for the K point and $\Gamma_{K'} = K_1'^+ + 2K_1'^- + K_2^+ + K_2'^- + K_2^*$ for K' point. Although time reversal symmetry can imply degeneracy between complex conjugate representations in cyclic groups, in graphene the complex conjugation also takes K into K' point and, consequently, there are no degenerated bands at the K (K') point, in agreement with tight-binding calculations when including the $\gamma_2$ and $\gamma_3$ next-nearest-layer coupling parameters [21, 22]. This energy gap is also obtained from ab initio calculations (see the inset of Fig. 2(c) and Ref. [23]).

C. Gated mono- and bi-layer graphene

If the monolayer graphene is in the presence of a perpendicular electric field (gated graphene), the Fermi level changes. The presence of charge inhomogeneity caused by substrate and/or absorbed water can generate the same effect as verified in transport [24, 25] and Raman measurements [26], where the Dirac point is shifted from
the neutrality point. In this case, the $\pi$ electrons lose the horizontal mirror plane and the inversion symmetry, and the system is isomorphic to the point group $C_{6v}$. The irreducible representations for the $\Gamma_{\pi}$ for the gated graphene can be found in Table IV. There is no gap opening at $K$ point for a perfect perpendicular electric field effect.

The biased bilayer graphene have attracted a lot of attention recently because it is the only material known to have a tunable energy gap $[10,11,12,13]$, promising for applications on devices and lasers with tunable energy. The mechanism behind this feature is based on applying an electric field perpendicular to the graphene layers, so that the two layers will be under an inequivalent potential. Then it is possible to open a gap at the $K$ point, breaking the double degenerated $K_{\text{D}}$ irreducible representation into two one-dimensional irreducible representations. Since the biased bilayer graphene brakes the inversion center symmetry, the group of the wavevector at $\Gamma$ for perfect perpendicular electric field is isomorphic to $C_{3v}$. Table IV shows that the biased bilayer contains the two one-dimensional representations $K_2$ and $K_3$ at the $K$ point, then a gap opening is expected on the basis of symmetry arguments of inequivalent layers.

The representations for the $\Gamma_{\text{lat.sub.}}$ of the gated monolayer (or biased bilayer) are the same as the monolayer (or bilayer) in an isotropic medium, given in table III. The electric field does not affect the symmetries of the phonons.

### III. SELECTION RULES FOR ELECTRON-RADIATION INTERACTION

The symmetry properties described in the previous section will now be applied to physical processes. In this section we discuss the selection rules for electron-radiation interaction in the dipole approximation, with emphasis on the high symmetry lines $T$ and $T'$ in the electronic dispersion, where interesting phenomena occur.

In the dipole approximation, the absorption of light in a material is related to the wave functions of the electron states in the valence ($\psi^v(k)$) and conduction ($\psi^c(k)$) bands and the polarization of the incoming light ($\mathbf{P}$) by

$$W(k) \propto |\mathbf{P} \cdot \langle \psi^c(k)|\mathbf{v}\psi^v(k)\rangle|^2.$$  

Knowing the symmetry of the initial an final states, and the representation that generates the basis function of the light polarization vector ($x$, $y$, or $z$), group theory can be used to compute wether $W(k)$ is null or not. The results are summarized in Table IV considering graphene layers laying in the $(x,y)$ plane and light propagating along $z$. In the case of graphene, the light absorption up to 3 eV occurs only at $T$, $T'$ and $u$ points.

It is important to highlight some results given in Table IV. In the case of monolayer graphene on an isotropic medium, numerical calculations show an anisotropy in the optical absorption $[28,29,30]$. This anisotropy has indeed a symmetry basis, as clearly seen when analyzing the selection rules at the $T$ line. Absorption by visible light has to couple $T_2$ and $T_3 \pi$ electron symmetries (see Fig. 2(a)). For the $T$ line direction along $\hat{y}$, the only

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**TABLE II:** The $\Gamma_{\text{lat.sub.}}$ wavevector point-group representations for mono- and N-layer graphene at all points in the BZ.

| Layer   | N even | N odd |
|---------|--------|-------|
| $\Gamma$ | $\Gamma + \Gamma_1$ | $(N-1)\Gamma_1^+ + (N+1)\Gamma_1^-$ |
| $K$     | $K^+ + K_-^+ + K_3^-$ | $N(K_1 + K_2 + 2K_3)$ |
| $M$     | $M^+ + M^0 + M^- + M_1^+ + M_1^- + M_2^+ + M_2^- + 2M_3^+$ | $2N M_1 + (N-1)M_2 + (N+1)M_3 + 2NM_4$ |
| $U$     | $2\Sigma_2 + 2\Sigma_3 + 2\Sigma_4$ | $2N \Sigma_4 + (N-1)\Sigma_2 + (N+1)\Sigma_3 + 2N\Sigma_4$ |

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**TABLE III:** The $\Gamma_{\pi}$ wavevector point-group representations for mono- and N-layer graphene at all points in the BZ.

| Layer   | N even | N odd |
|---------|--------|-------|
| $\Gamma$ | $\Gamma + \Gamma_1$ | $(N-1)\Gamma_1^+ + (N+1)\Gamma_1^-$ |
| $K(K')$ | $K^0$ | $(N\Gamma_1^+ + \Gamma_1^-)K_1^+ + (N\Gamma_1^- + \Gamma_1^+)K_1^- + g(N)K_2^+ + g(N-2)K_2^- + g(N)K_3^+ + g(N+2)K_3^-)$ |
| $M$     | $M^+ + M_1^+$ | $N(M_1^+ + M_1^-)$ |
| $T(T')$ | $T_2 + T_4$ | $(N-1)\Sigma_4 + (N+1)\Sigma_4$ |
| $\Sigma$ | $2\Sigma_4$ | $2N \Sigma_4$ |

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**TABLE IV:** The group of wavevector and its $\Gamma_{\pi}$ representations for gated monolayer and biased bilayer graphene.

| Layer   | Gated monolayer | Biased Bilayer |
|---------|-----------------|----------------|
| $\Gamma$ | $\Gamma_{\pi}$ | $\Gamma_{\pi}$ |
| $K$     | $C_{3v}$ | $C_{3v}$ |
| $M$     | $M_1 + M_4$ | $C_{1v}$ |
| $T$     | $C_{1v}$ | $C_{1v}$ |
| $u$     | $C_1$ | $C_1$ |

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*Where $g(N) = \sum_{m=0}^{\infty} \Theta(N - 4m - 2)$, where $\Theta(x)$ is equal to 0 if $x < 0$ and equal 1 otherwise.*
Local Density Approximation (LDA) with the Ceperley-Alder parametrization for the exchange-correlation functional, implemented in the SIESTA program [16, 17]. We used a basis set composed of pseudo atomic orbitals of finite range and the

\[
W(k) = \sum_{x,y} \langle \phi_x | \hat{\mathbf{T}} \cdot \hat{\mathbf{T}} \cdot \hat{\mathbf{T}} \cdot \hat{\mathbf{T}} | \phi_y \rangle \delta(k - k')
\]

for (a) monolayer, (b) bilayer and (c) trilayer along the KΓMK directions. The calculation was done via pseudopotential DFT [15] as allowed absorption is for light polarized along the \(\hat{x}\) direction. For incident light polarization along the \(\hat{y}\) direction, no absorption will occur along KΓ direction, giving rise to the optical absorption anisotropy on graphene [28, 29, 30]. Outside the high symmetry T line there is a non-zero probability of absorption and the anisotropy is obtained by defining orthogonal basis, as shown in Ref. [28].

When the monolayer graphene is on top of a substrate, with the influence of the environment changing the Fermi level, there will be no change in the selection rules for electron-radiation interaction. Along the T line, the \(\pi\) electrons are described by \(T_1\) and \(T_2\) representations,

\[
\begin{align*}
\text{monolayer} & : x \in T_2^\pm, y \in T_2^\pm; T_2 \otimes T_2 \otimes T_2 \otimes T_2 \text{ non null} \\
& : T_2 \otimes T_2 \otimes T_2 \otimes T_2 \text{ null} \\
& : x, y \in u^s; u \otimes u \otimes u \otimes u \text{ non null} \\
\text{gated monolayer} & : x \in T_2^\pm, y \in T_1; T_2 \otimes T_2 \otimes T_2 \otimes T_2 \text{ null} \\
& : T_2 \otimes T_2 \otimes T_2 \otimes T_2 \text{ null} \\
& : x, y \in u; u \otimes u \otimes u \otimes u \text{ non null} \\
\text{bilayer (N-even)} & : x \in T_2^\pm, y \in T_1; T_2 \otimes T_2 \otimes T_2 \otimes T_2 \text{ null} \\
& : T_2 \otimes T_2 \otimes T_2 \otimes T_2 \text{ null} \\
& : x, y \in u; u \otimes u \otimes u \otimes u \text{ non null} \\
\text{biased bilayer} & : x, y \in T; T \otimes T \otimes T \otimes T \text{ non null} \\
& : u \otimes u \otimes u \otimes u \text{ non null} \\
\text{trilayer (N-odd)} & : x, y \in T^\pm; T^\pm \otimes T^\pm \otimes T^\pm \text{ non null} \\
& : T^\pm \otimes T^\pm \otimes T^\pm \text{ null} \\
& : u^s \otimes u^s \otimes u^s \otimes u^s \text{ non null} \\
& : u^s \otimes u^s \otimes u^s \otimes u^s \text{ null} \\
\end{align*}
\]

where \(T_2\) and \(T_1\) contain \(x\) and \(y\) basis functions, respectively. Again there will be no absorption for \(y\) polarization.

The bilayer graphene is composed by four electronic bands at the T line, belonging to two \(T_1\) and two \(T_2\) irreducible representations. The four possible transitions are illustrated in Fig. 3 (a,b). In this case both \(x\) and \(y\) polarized light can be absorbed. For the biased bilayer graphene, all electronic representations are the same, and it contains both \(x\) and \(y\) basis functions for light polarizations. Thus, all the four transitions are allowed connecting all the four bands by the same light polarization, different from the unbiased bilayer case where the light polarization selects the pair of bands that can be connected.

The trilayer graphene will have more possibilities for light induced transitions, since there are more possib-
IV. SELECTION RULES FOR THE FIRST-ORDER RAMAN SCATTERING AND INFRARED ABSORPTION PROCESSES

The first-order Raman scattering process is limited to phonons at the center of BZ (Γ point) due to momentum conservation requirement (phonon wavevector q = 0). In monolayer graphene the first-order Raman spectra is composed by the G band vibrational mode, which is doubly degenerated at the Γ point with Γ^+ symmetry. The Raman active modes depending on N (N>1) (without acoustic modes) are:

\[ \Gamma_{\text{Raman}} = N(\Gamma_3^+ + \Gamma_1^+), \text{ for } N \text{ even} \]
\[ \Gamma_{\text{Raman}} = N\Gamma_3^+ + (N-1)(\Gamma_3^- + \Gamma_1^+), \text{ for } N \text{ odd} \]

For even number of layers the G band belongs to the Γ^+_3. There is a low frequency Γ^+_3 mode with frequency depending on the number of layers (35-53 cm\(^{-1}\))\(^{31}\). Two new Raman active modes near ~80 cm\(^{-1}\) and ~900 cm\(^{-1}\) appear with Γ^+_3 irreducible representations \(^{31,32}\). For odd number of layers the G band is assigned as a combination of Γ^+_3 and Γ^-_3 representations, also the smaller wavenumber component is active in Raman by a Γ^+_1 representation.

For monolayer graphene there is an infrared (IR) active mode belonging to the Γ^-_3 representation, giving rise to an absorption near ~ 870 cm\(^{-1}\). The IR active modes for N>1 are:

\[ \Gamma_{\text{IR}} = (N-1)\Gamma^-_3 + (N-1)(\Gamma^-_3 + \Gamma^-_1), \text{ for } N \text{ even} \]
\[ \Gamma_{\text{IR}} = N(\Gamma^+_3 + \Gamma^-_2), \text{ for } N \text{ odd} \]

For even number of layers the active modes belong to the Γ^-_2 and Γ^-_3 representations, the latter one referring to the ~ 1590 cm\(^{-1}\) frequency vibration \(^{31,32}\). The infrared active modes for odd layer number belong to Γ^-_2 and Γ^-_3 which are also Raman active.

V. ELECTRON SCATTERING BY q ≠ 0 PHONONS

The electron-phonon scattering (EPS) is calculated from the initial and final electron wave functions coupled by the phonon eigenvector \(^{33,34}\) using the phonon-induced deformation potential. Therefore, the selection rules of the EPS processes are obtained by the direct product of the symmetries of the initial and final electronic states and the symmetry of the phonon involved in the process. The allowed electron-phonon scattering processes for monolayer, gated monolayer, bilayer, biased bilayer an trilayer graphene along the KF and KM directions (T and T' lines respectively) and at a generic u point are summarized in Table VI.

| BZ point | phonon | allowed scattering |
|----------|--------|--------------------|
| monolayer | T(T') | T_1 → T_2, T_4 → T_4 |
| u        | u^+   | u → u |
| gated     | T(T') | T_1 → T_1, T_2 → T_2 |
| u        | u     | u → u |
| bilayer   | T(T') | T_1 → T_1, T_2 → T_2 |
| u        | u     | u → u |
| biased    | T(T') | T^- → T^- |
| u        | u^+   | u^+ → u^+, u^- → u^- |
| trilayer  | T(T') | T^- → T^- |
| u        | u^-   | u^- → u^- |

VI. DOUBLE RESONANCE RAMAN SCATTERING PROCESS

One example of explicit use of the electron-radiation and EPS selection rules is the double resonance Raman scattering process \(^{35,36}\), in which an electron in the conduction band is scattered by a phonon with wavevector outside the Γ point in an intervalley (connecting electronic states near the K and K' points) or in an intravalley (connecting electronic state near the same K or K' point) process. The G' Raman band (~ 2700 cm\(^{-1}\)) comes from an intervalley process in which the electron is scattered by an in-plane transversal optic (iTO) phonon. We will discuss in details the G' scattering for mono- and multi-layer graphene.

For the monolayer graphene, the possible scattering is illustrate in Fig. 4 \(^{37}\). The iTO phonon at the KM (T') direction presents a T_1 symmetry \(^{37}\), which can only connect two electrons with the same symmetry. Many other similar scattering events are allowed by symmetry, involving electron in the K' (T) direction or at any general u point inside the circle defined by the T_3 photon energy. However, the matrix element has a strong angular dependence and the scattering is dominated by the T electrons, as discussed in Ref. \(^{9}\). Therefore, the G' Raman band has only one peak, with full width at half maximum (FWHM) of ~ 24 cm\(^{-1}\) (see Fig. 4(a)). For the graphene on top of a substrate, the same selection rules apply, and the expected number of G' peaks is the same as for the isolated monolayer graphene on an
FIG. 4: The most efficient DRR process in graphene showing the absorption of light with polarization symmetry $T_3$ followed by electron scattering by a phonon with $T_1$ symmetry.

FIG. 5: The four allowed DRR processes in bilayer graphene, taking into account the optical anisotropy.

isotropic medium.

For the bilayer graphene, the number of allowed DRR processes predicted by group theory will be larger, since both electronic and phonon branches are doubled. Along the T line, there is polarization dependence for the absorption linking different electronic bands, as discussed in section III. For wavevectors in the range of visible light energy, the electron dispersion are almost linear, then optical anisotropy can be applied here as for monolayer graphene [28]. Now, for computing the number of resonant conditions involved in the DRR process, we are left with only two excited electronic bands with symmetries $T_1$ and $T_2$, which corresponds to Fig. 3 (a). The iTO phonons for bilayer graphene have $T_1$ and $T_2$ symmetries. For the electron scattering by a $T_1$ phonon, the allowed process are between K and K’ electronic bands with same symmetry ($T_1 \rightarrow T_1$ or $T_2 \rightarrow T_2$). The same happens with the electron scattering by a $T_2$ phonon, but it connects conduction bands of different symmetries, i.e. $T_1 \rightarrow T_2$. This gives rise to four possible DRR processes, as shown in Fig. 5 (a). The Raman spectra can then be used to differentiate mono- and bi-layer graphene (see Fig. 6 (a)-(c)).

In the case of biased bilayer graphene, there are no selection rules involving different photon polarizations.

FIG. 6: The measured Raman spectra of the $G'$ band of a (a) mono- (b) bi- and (c) trilayer graphene. The samples were made by exfoliating graphite on top of a 100 nm silicon oxide substrate using 2.41 eV laser energy. The $G'$ band for mono- bi- and tri-layer graphene were fitted with 1, 4 and 6 Lorentzians, respectively, with a FWHM of 24 cm$^{-1}$.

The biased bilayer can have photon absorption linking all valence and conduction bands. This leads to eight possible transitions considering EPS selection rules for $T_1$ and $T_2$ phonons.

For the trilayer graphene, the DRR process will have again more contributions because each phonon and electron band will be split in three levels. Along the T line, there are five possibilities linking the electronic bands between the K and K’ points with a $T^+$ phonon, and four possibilities for the $T^-$ phonon. The total number of DRR process predicted by group theory will be fifteen. However, the FWHM is large when compared to the energy splitting between the $G'$ Raman peaks, and when one makes measurements of the $G'$ Raman band, these fifteen peaks cannot be distinguished, as illustrated in Fig. 6. Similar problem should happen for $N \geq 4$.

VII. SUMMARY

In this work we analyzed the symmetry aspects related to electrons and phonons at each point in the BZ of graphene, depending on the number of layers. The symmetry aspects can be generalized to any value of N, differing for N even or odd. For monolayer and bilayer we consider both an isotropic and an anisotropic medium. We derived the selection rules for electron-radiation and electron-phonon interactions. Some specific findings can be remarked:

- For the monolayer graphene, the predicted optical anisotropy [28] comes out directly from group theory analysis. The electron-phonon scattering process is allowed by symmetry at any generic point ($u$) in the Dirac cone, and the observation of a single Lorentzian in the $G'$ Raman band comes from a strong anisotropy in the electron-phonon matrix element [4].
• The gated graphene have lower symmetry, but the optical anisotropy is still present, and for the DRR process, the symmetry considerations are the same as graphene on an isotropic medium.
• In the case of bilayer graphene, the optical anisotropy is also present and there are four dominant processes in the DRR. This number increases to eight on biased bilayer.
• In trilayer graphene, the number of possible DRR processes is fifteen. However, the 15 processes are not distinguishable and the G' Raman band can be nicely fit with 6 Lorentzians. Similar situation is expected for larger number of layers.

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APPENDIX: NOTATION CONVERSION FROM SPACE GROUP TO POINT GROUP IRREDUCIBLE REPRESENTATIONS

In this work we derived the $\Gamma_\pi$ and $\Gamma_{lat, vib}$ for all points in the first BZ of multilayer graphene maintaining the notation of space group (SG) for the irreducible representations. The conversion to point group (PG) representation is obtained considering that (a) superscript sign $^+$ or $^-$ applies if the character of the horizontal mirror ($\sigma_h$) or inversion ($i$) is positive or negative, respectively; (b) the subscript number is given following the order of the point group irreducible representations: (c) two representations can only have the same number if they have superscript with positive or negative signs. As an example we give in Table VII the $\Gamma$ point space group notation conversion to the $D_{3h}$ (N-odd) and $D_{3d}$ (N-even) point groups and for the $K$ point space group to the $C_{3h}$ (N-odd) and $D_3$ (N-even) point groups.

| $\Gamma$ point | $K$ point |
|---------------|-----------|
| $D_{3h}$ | $D_{3d}$ | $C_{3h}$ | $D_3$ |
| SG | PG | SG | PG | SG | PG | SG | PG |
| $\Gamma_\pi'$ | $A'$ | $\Gamma_{18}'$ | $K_1'$ | $A'$ | $K_1$ | $A_1$ |
| $\Gamma_\pi''$ | $A''$ | $\Gamma_{18}''$ | $K_2''$ | $A''$ | $K_2$ | $A_2$ |
| $\Gamma_\pi''$ | $E''$ | $K_3$ | $E''$ |
| $\Gamma_\pi'''$ | $E''$ | $K_3$ | $E''$ |

TABLE VII: Example of irreducible representation notation conversion from the $\Gamma$ point space group to $D_{3h}$ and $D_{3d}$ point groups, and from the $K$ point space group to $C_{3h}$ and $D_3$ point groups.

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