Hyperkahler Singularities in Superstrings Compactification

and 2d \( N = 4 \) Conformal Field Theory

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Abstract

We study the singularities of the Higgs branch of supersymmetric \( U(1)^r \) gauge theories with eight supercharges. We derive new solutions for the moduli space of vacua preserving manifestly the eight supercharges by using a geometric realization of the \( SU(2)_R \) symmetry and a separation procedure of the gauge and \( SU(2)_R \) charges, which allow us to put the hypermultiplet vacua in a form depending on a parameter \( \gamma \). For \( \gamma = 1 \), we obtain new models which flow in the infrared to 2d \( N = (4,4) \) conformal models and we show that the classical moduli spaces are given by intersecting cotangent weighted complex projective spaces containing the small instanton singularity, discussed in [17], as a leading special case. We also make comments regarding
the 2d $N = 4$ conformal Liouville description of the Higgs branch throat by following the analysis of [18]. Other features are also discussed.
1 Introduction

Over the few past years, there has been an increasing interest in studying the moduli space of vacua of the Coulomb and Higgs branches of supersymmetric gauge theories with eight supercharges in various dimensions. This interest is mostly due to the fact that extended supersymmetry severely restricts the quantum corrections to the moduli space metric and allows to make many exact computations [1,2,3]. A large class of these gauge theories can be realized by brane configurations in type II strings on Calabi Yau manifolds by using either the Hanany -Witten method [4,5] or the geometric engineering approach introduced and developed by Klemm , Lerche , Mayr, Vafa and Warner [6] and their collaborators; see [7,8,9,10,11]; see also [12,13,14,15,16].

Recently, a special interest has been given to the analysis of the hypermultiplet gauge invariant moduli space near the Higgs branch singularity. This analysis has been shown to be relevant for the study of many aspects in supersymmetric gauge theories with eight supercharges; in particular in the understanding of the asymptotic regions of the infrared IR low energy limits of the $N = (4,4)$ supersymmetric gauge theories in two dimensions especially within the so called throats of the Coulomb and Higgs branches where the theories are typically described by two $2d\ N = 4$ conformal Liouville theories with different central charges. In this context, it was shown in [17]; see also [18,19,20,21,22,23], that the IR limits of the Coulomb and Higgs branches have isomorphic throat regions associated with different small $2d\ N = 4$ subalgebras of the $N = 4$ superconformal symmetry in two dimensions [17,18,19,20] see also [24-28].

Vector and hypermultiplet moduli spaces have been also much studied in strings compactification to four dimensions where the low energy supergravity has scalar fields in both vector multiplets and hypermultiplets. The basic example of such compactification
is given by type IIA string on $R^{1,3}$ times Calabi Yau threefolds which is believed to be dual to heterotic on $R^{1,3} \times K3 \times T^2$, where the type IIA dilaton is in a hypermultiplet and the heterotic one is in a vector multiplet \cite{29,30}. Using local mirror symmetry and toric geometry methods, the absence of the type IIA dilaton in the vector multiplet has been exploited in \cite{8} to derive exact results in the Coulomb branch of type IIA on local Calabi Yau threefolds. In the same spirit, the hypermultiplet moduli space is independent of the heterotic string coupling and hence can be determined exactly from heterotic conformal field theory near the $K3$ ADE singularity. This issue has been analysed recently in \cite{31} and it was suggested that, in absence of small instantons, the hyperKahler moduli space for the heterotic string near the $K3$ ADE singularity is just the moduli space of vacua of a pure supersymmetric gauge theory in three dimensions with eight supercharges and ADE gauge group \cite{32,33}. Matter adjunction has been considered in \cite{34,35}. In the presence of small instantons, the hyperKahler moduli space has singularities which generally are interpreted in terms of singular conformal field theories or non-perturbative massless particles. Note that contrary to the singularity of the Coulomb branch generated by the one loop quantum corrections, the singularity of the Higgs branch of supersymmetric gauge theories with eight supercharges is not generated by quantum mechanics. It has been first suspected when tempting to understand the breakdown of string perturbation theory in type IIA on $A_r$ ALE surface, where the non-perturbative phenomenon cannot be avoided by making the string coupling constant smaller \cite{17,36,37}. In lower dimensions, the Higgs branch singularity was also motivated by using duality between Higgs and Coulomb of $N = (4,4)$ supersymmetric gauge theories in two dimensions \cite{22,23}. More convincing and rigorous arguments for the existence of the Higgs branch throat are obtained from the study of the low energy physics of the D1/D5 system on $X$, which is equivalent to type
IIB String theory on $AdS_3 \times S^3 \times X$ where $X$ is either $T^4$ or $K3$ [17,18,20,38].

In four dimensions, hypermultiplets moduli are moreover involved in the study of stringy instanton moduli space which is given by hyperkahler deformations of the classical instanton moduli space with non zero B field and small instanton singularites eliminated. Stringy instantons with non zero B field, including hyperkahler deformations resolving small instanton singularities, have been suggested in [39] to be equally described as instantons on a non commutative space [40,41]. Furthermore analysis involving hypermultiplets moduli is also encountered in the study of CFT’s obtained from compactifications of superstrings on Calabi Yau fourfolds and too particularly in the compactification of M-Theory on a Calabi Yau four-folds near the so called hyperkahler singularity [42].

In most of all of these studies, one of the basic eqs describing the moduli space of the hypermultiplets vacua reads, in the sigma model approach, as:

$$\sum_{i=1}^{n} q_i^a [\phi_i^\alpha \phi_i^\beta + \phi_i^\beta \phi_i^\alpha] = \xi_i^a \sigma_i^\alpha \quad ; a = 1, \ldots, r.$$  \hspace{1cm} (1)

Other basic eqs describing the throat region of the Higgs branch are given in section 5; see eqs(65,66). In eqs(1), the $\phi_i^\alpha$’s form a set $\{\phi_i^\alpha; 1 \leq i \leq n\}$ of $n$ component fields doublets $\phi_i^\alpha$ belonging to hypermultiplets and transforming in the $(n,2)$ representations of $G \times SU(2)_R$ group, where the group G will be specified later on. The $\xi_i^a$’s are a collection of $r$ FI coupling 3-vectors, each of it transforms as a triplet under the usual $SU(2)_R$ symmetry rotating the eight supercharges. The $q_i^a$ parameters are the charges of $\phi_i^\alpha$’s under the $U(1)^r$ gauge group of the underlying supersymmetric gauge theory. For later use it is interesting to note that eqs(1) have a formal analogy with the following sigma model vaccum eqs of $2d N = 2$ supersymmetric $U(1)^r$ gauge theory involved in the analysis of the Coulomb branch of IIA superstrings on Calabi Yau threefolds with ADE
In eqs (2), the $X_i$’s are complex scalar fields, the $R_a$’s are FI couplings and the $q_a^i$’s are the $U(1)^r$ charges of the $X_i$’s which, for reference, read in the case of $SU(n)$ singularity as:

$$q_a^i = -2\delta_a^i + \delta_a^{i-1} + \delta_a^{i+1},$$

with the remarkable equality

$$\sum_i q_a^i = 0. \quad (4)$$

It is interesting to note here that in the 2d gauge theories with four supercharges, the above constraint eqs (4) is the condition under which the gauge theory flows in the infrared to 2d $N = 2$ superconformal field theory [43,44]. It is also the condition to have Kahler Calabi Yau backgrounds [8,45] involved in superstring compactifications. Concerning eqs(1), we will see in section 4 that, under some assumptions, one can works out two remarkable classes of gauge invariant solutions of eqs (1) preserving the eight supercharges. The first class leads to the obtention of new singularities extending the usual $N = 2$ ADE ones which are recovered as a special solutions by partial breaking of 2d $N = 4$ supersymmetry down to 2d $N = 2$. As preliminary results we find the following singular hypersurfaces:

$$A_{n-1} : \quad U^{n\frac{(n+1)}{2}} - V^{n\frac{(n+1)}{2}} = [Z^{n+(n+1)}]^n$$

$$D_n : \quad (x^{++})^n + x^{++}(y^{(n-1)})^2 + (z^n)^2 = 0$$

$$E_6 : \quad (x^{+6})^2 + (y^{+4})^3 + (z^{+3})^4 = 0$$

$$E_7 : \quad (x^{+9})^2 + (y^{+6})^3 + y^{+6}(z^{+4})^3 = 0$$

$$E_8 : \quad (x^{+15})^2 + (y^{+10})^3 + (z^{+6})^5 = 0.$$
usual ADE complex surfaces of the $N = 2$ backgrounds. For more details, see eqs (35-36).

In the second class of gauge invariant solutions, we will show that eqs (4) are no longer constraint eqs; they are replaced by the following remarkable identity:

$$\sum_i q^i_a + \sum_i (-q^i_a) = 0,$$

which is usually fulfilled whatever the values of the $q^i_a$'s are. In the low energy limit, the above eqs lead then to $2d \ N = 4$ conformal models going beyond the $N = 2$ ADE conformal ones.

The resemblance between eqs (1) and (2) is only formal, but turns out however to be very useful for the analysis of eqs(1) as well as their solving. Eqs(1) and (2) carry different meanings amongst which we quote the four followings:

(a) Eqs (2) describe the Coulomb branch leading to the well known gauge invariant Kahler moduli space while eqs (1) deal with the hypermultiplet branch and give a gauge invariant hyperkahler moduli space.

(b) For each $U(1)$ factor of the $U(1)^r$ gauge group, eqs (1) involve a triplet of FI parameters whereas eq (2) has only one. This feature goes with the previous one as it is related with the number of complex structures of Kahler and hyperkahler manifolds.

(c) Eqs (1) have a manifest $SU(2)_R$ symmetry which is absent in eqs (2). The latters have a $U(1)$ R symmetry. Eqs (1) are more restrictive than eqs(2) since they are the vacuum eqs of $2d \ N = 4$ supersymmetric linear $\sigma$ models. More precisely $N = 4$ models in two dimensions involve three times the number of the D-flatness eqs of the $N = 2$ models.

This feature is easily seen on the space of the FI couplings which, for $N = 4$, belong to $(R^3)^r \approx R^+_r \times (S^2)^r$ while, for $N = 2$, belong to $R^r$. The extra eqs associated with the moduli space $(S^2)^r$ are necessary conditions to have $N = 4$ supersymmetry; without these constraints $N = 4$ supersymmetry is partially broken down to $N = 2$. 

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(d) It is now quite well established that eqs (1) and (2) hide moreover a comparable behaviour between the Coulomb and Higgs branches even near the singularity where eqs (1) and eqs (2) cease to be valid. We have already mentioned the duality between the two branches and their algebraic descriptions in terms of subalgebras of $2d N = 4$ conformal invariance. Later on, we shall give other arguments, geometrical and field theoretical, showing that in absence of FI couplings, $\theta$ terms and RR fields, both Coulomb and Higgs branches are described by singular CFT’s which seems to have something to do with $2d N = 4$ superconformal ADE Toda theories.

The formal similarity between eqs (1) and (2) together with the abovementioned features show that one may obtain new solutions of the gauge invariant moduli space of vacua of eqs (1) by using $SU(2)_R$ harmonic analysis and generalisations of methods of $2d N = 2$ supersymmetric linear sigma models.

The aim of this paper is to study these solutions and give interpretations in terms of blown up of singularities given by intersections of cotangent complex $n$ dimensional weighted projective spaces. Actually this study extends the results obtained for Coulomb branch of supersymetric gauge theories with four supercharges. Concerning the infrared dynamics of two dimensional $N = (4,4)$ gauge theories, we give also comments on the $N = 4$ conformal Liouville description of the region in the vicinity of the singularity of the metric of the $2d N = 4$ Higgs branch generally interpreted as a semi infinite throat where the string coupling constant $g_s = e^\phi$ blows up as the Liouville field $\phi$ goes to infinity [17,18].

Moreover, in an attempt towards an interpretation of the degenerate $A_r$ singularity carried by eqs (1), we give a field theoretical argument in favor to the hypothesis according to which the metric of the moduli space near the Higgs singularity might be described by a $N = 4$ conformal $SU(r + 1)$ Toda theory in two dimensions. Of course this is just an
observation which deserves in its own right a detailed study.

The presentation of this paper is as follows: In section 2, we review briefly the standard way used in handling eqs (1) where only a $A_1$ singularity has been considered, using the standard $2d \, N = 2$ supersymmetric analysis in which half of the eight supersymmetries are manifest. In this way of doing the $SU(2)_R$ symmetry is broken down to $U(1)_R$, a feature which is exploited in [31,42] by making an appropriate choice of the FI 3-vector coupling where only one parameter is non zero. The two others are put to zero. In section 3, we develop a new way of doing by keeping all the three FI parameters non zero and the eight supercharges manifest. In this approach the $SU(2)_R$ symmetry is apparent but explicitly broken by the non zero FI terms. Our method enables us to exhibit manifestly the role of the three Kahler structures of the gauge invariant hyperkahler moduli spaces and permits moreover to go beyond the $A_1$ singularity analysis of [31,42]. Our way in handling eqs(1) involves two steps based on a geometric realisation of the $SU(2)_R$ symmetry and on the separation of the charges of the gauge and R-symmetries. The first step of this programme is described in Sectin 3 while the second step is studied in Section 4. The gauge and $SU(2)_R$ charge separation of the hypermultiplets moduli involves a parameter $\gamma$ taking the values $\gamma = 0$ or $\gamma = 1$ which distinguish two classes of solutions of eqs(1) both preserving the eight supercharges. For $\gamma = 0$, we obtain a generalisation of the ADE complex surfaces reproducing the standard ones by partial breaking of $2d \, N = 4$ supersymmetry down to $2d \, N = 2$. For $\gamma = 1$, we find new models which flow in the infrared to $2d \, N = (4,4)$ scale invaraint models. In section 5, we study the moduli space of vacua of models with $\gamma = 1$ by distinguishing the two cases $\sum_i q^i_a = 0$ and $\sum_i q^i_a \neq 0$. We show by explicit computation that the hyperKahler moduli space, associated with eqs (1), is given by weighted complex projective spaces. Moreover, we study the Liouville
description of the small instanton conformal theory near the singularity by using the field theoretical approach of Aharony and Berkooz [18] and make comments regarding the $A_r$ singularity of eqs (1). In section 6, we give our conclusion.

2 Hyperkahler moduli space

In this section we review briefly the example of the hyperkahler cotangent bundle of complex projective space: $T^*(CP^2)$; considered in the study of M theory on Calabi Yau fourfold after what we give the $2n$ complex dimension hyperkahler space $T^*(CP^n), n \geq 1$, describing the instantons moduli space of one instanton on $R^4$ with gauge group $U(n)$ [39,46]. To begin, note first of all that a Calabi Yau fourfolds can develop singularities of many types; this includes the $C_4/Z_4$ orbifold, the ADE hypersurface singularities considered recently in [42] in the context of derivations of $2d$ CFT’s from type IIA string compactifications on Calabi Yau fourfolds, and the so called hyperkahler singularity we are interested in here. To describe the $T^*(CP^2)$ bundle, we consider $2d$ $N = 4$ supersymmetric $U(1)$ gauge theory with one isotriplet FI coupling $\vec{\xi} = (\xi^1, \xi^2, \xi^3)$ and three hypermultiplets of charges $q^i_a = q^i = 1; i = 1, 2, 3$ and taking $G$ as $SU(3)$. The zero energy states of this gauge model are obtained by solving

$$\sum_{i=1}^{3} [\phi_{i\alpha} \phi_{i+}^{\beta} + \phi_{i\alpha} \phi_{i+}^{\beta}] = \xi^\alpha \sigma^\beta_{\alpha},$$

which by the way is just a special situation of eqs (1) where all gauge charges $q^i_a$ are equal to one. Eqs (6) is a system of three eqs which, up to replacing the Pauli matrices by their expressions and using the $SU(2)_R$ transformations $\phi^\alpha = \varepsilon^{\alpha\beta} \phi^\beta$ with $\varepsilon_{12} = \varepsilon^{21} = 1$ and $\overline{\phi^\alpha} = \phi_{\alpha}$, split as follows:
\[ \sum_{j=1}^{3} (|\varphi_j|^2 - |\varphi_j'|^2) = \xi_3 \quad (a) \]
\[ \sum_{j=1}^{3} \varphi_j \varphi_j' = \xi_1 + i\xi_2 \quad (b) \]
\[ \sum_{j=1}^{3} \varphi_j' \varphi_j = \xi_1 - i\xi_2 \quad (c) \]

The moduli space of zero energy states of the classical gauge theory is the space of the solutions of eqs (7-8) divided by the action of the \( U(1) \) gauge group. The solutions of eqs (7) depend on the values of the FI couplings. For the case where \( \xi_1 = \xi_2 = \xi_3 = 0 \), the moduli space has an \( SU(3) \times SU(2)_R \) symmetry; it is a cone over a seven manifold described by the eqs:

\[ \sum_{i=1}^{3} (\varphi_{\alpha i} \varphi_{\beta i}' - \varphi_{\beta i} \varphi_{\alpha i}') = \delta_{\alpha \beta}. \quad (9) \]

For the case \( \vec{\xi} \neq 0 \), the abovementioned \( SU(3) \times SU(2)_R \) symmetry is explicitly broken down to \( SU(3) \times U(1)_R \). In the remarkable case where \( \xi_1 \neq \xi_2 = 0 \) and \( \xi_3 \) positive definite, it is not difficult to see that eqs (7) describe the cotangent bundle of \( CP^2 \). Indeed making the change

\[ \psi_i = \frac{\varphi_i}{\left[ \sum_{j=1}^{3} |\varphi_j|^2 + \xi_3 \right]^\frac{1}{2}}, \quad (10) \]

and putting back into eq (8.a), one discovers that \( \psi_i \)'s satisfy \( \sum_i |\psi_i|^2 = 1 \). The \( \psi_i \)'s parametrize the \( CP^2 \) space. On the other hand with \( \xi_1 = \xi_2 = 0 \) conditions, eqs (8.b-c) may be interpreted to mean that \( \varphi_{2j} \) lies in the cotangent space to \( CP^2 \) at the point determined by \( \psi_i \). Although we are usually allowed to make the choice \( \xi_1 = \xi_2 = 0 \) by using an appropriate \( SU(2)_R \) transformation, we shall consider in sections 4 and 5, the generic cases where \( \xi_1, \xi_2 \) and \( \xi_3 \) are all of them non zero as they form altogether the three Kahler parameters of hyperkahler manifolds. For the time being let us note that the previous analysis may be extended to the cases of \( 2d \ N = 4 \) supersymmetric \( U(1) \) gauge linear sigma model involving \( n + 1 \) hypermutiplets with charges \( q_i = 1; \ i = 1, ..., n + 1 \).
and transforming in the fundamental representation of $SU(n + 1)$. The vacuum energy

equations of this $U(1)$ gauge model read as :

$$
\sum_{j=1}^{n+1} (|\varphi_j|^2 - |\varphi_{2j}|^2) = \xi^3
$$

$$
\sum_{j=1}^{n+1} (\varphi_j^1 \varphi_{2j}^1) = \xi^1 + i\xi^2.
$$

For $\xi^1 = \xi^2 = 0$ and $\xi^3$ positive definite, the classical moduli space of the classical gauge
theory is given by the cotangent bundle of complex $n$ projective space: $T^* CP^n$. For
$\xi^1 = \xi^2 = \xi^3 = 0$, one has just the conifold singularity of $n$ dimensional complex manifolds.

Note also that near this singularity the low energy limits of this gauge theory is described
by 2d $N=(4,4)$ superconformal field theory of central charge $C = 6(n + 1 - 1) = 6n$. In
section 5, we shall turn to this point and describe the nature of the conformal field theory
one has in the nearby of the Higgs branch singularity.

3 More on the Eqs (1)

Eqs (1) is a system of $3r$ equations or more precisely $r$ isovector eqs, each of which
shares some general features with the usual 2d $N = 2$ supersymmetic D-flatness conditions
eqs (2); but in addition to the gauge charges, it carries a $SU(2)_R$ charge. To solve eqs (1),
we shall use a different method than that used in [31,39,42]. This method reproduces the
solutions of the abovementioned study as particular cases and offers moreover a possiblity
to address the question of multi instantons in $R^4$. Our approach is done in two steps and
is based on the two following: First we combine methods of 2d $N = 2$ supersymmetric
sigma models, as used in describing the Kahler Coulomb branch of type IIA string on
$K3$, and $SU(2)_R$ harmonic analysis allowing us to interpret $SU(2)_R$ representations as
special functions on $S^3_R = SU(2)_R$. The index R carried by $S^3_R$, $S^2_R$ and $U(1)_R$ refers to

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the $SU(2)_R$. This step allows us to put eqs(1) into a manageable form which exhibit many similarities with eqs (2). Second, we introduce a convenient change of variables based on separating the $U(1)_G$ gauge charges and the $U(1)_R$ ones. This change of variables allows us to benefit from the similarities with the 2d $N = 2$ supersymmetric gauge invariant backgrounds in order to study and solve eqs (1). In this section, we describe the first step of this programme; the charge separation of $U(1)_G$ and $U(1)_R$ symmetry factors will be studied in the next section. Our main purpose in what follows is to establish first that, up to $SU(2)_R$ transformations; eqs (1) can be rewritten in the following remarkable form:

$$\sum_j q^+_j \varphi_j^+ + \varphi_j^- = -i\xi^{++}_a; \quad a = 1, \ldots, r; \quad (12)$$

which, abstraction done of the plus indices describing the $U(1)_R$ Cartan charges carried by the $\varphi_j$’s and the FI couplings, is comparable to eqs (2). In eqs (12); the moduli $\varphi_j^+$, $\varphi_j^-$ and $\xi^{++}_a$ are related to $\varphi^+_{\alpha j}$, $\varphi^-_{\alpha j}$ and $\xi^{(\alpha\beta)}_a$; they will be specified later on. To establish eqs (12) let first note that one may use the isomorphisms $SU(2)_R = S^3_R$ and $SU(2)_R \approx SU(1)_R \approx S^2_R$ to describe $SU(2)_R$ representations (both reducible and irreducible) as harmonic functions on the sphere $S^2_R$ with definite $U(1)_R$ charge. This way of doing is well known in the study of $SU(2)_R$ representation theory; the main idea behind this construction may be summarized as follows: First, consider the following $2 \times 2$ matrix

$$U = \begin{pmatrix} u^+_1 & u^+_2 \\ u^-_1 & u^-_2 \end{pmatrix} \quad (13)$$

and solve the isospin $\frac{1}{2}$ $SU(2)_R$ representation constraints namely the unimodularity $detU = 1$ and the unitarity $U^+U = U^+U = I$ conditions. Straightforward algebra leads to:[47,48,49]
\[ u^{\pm \alpha} = \epsilon^{\alpha \beta} u_{\beta}^{\pm}; \quad \pi^{+ \alpha} = u_{\alpha}^{-}; \quad \epsilon_{\alpha \beta} = -\epsilon_{\beta \alpha} \]

\[ u^{+ \alpha} u_{\alpha}^{-} = 1, \quad u^{+ \alpha} u_{\alpha}^{+} = u^{- \alpha} u_{\alpha}^{-} = 0. \]  

Recall in passing that the \( u_{\alpha}^{\pm} \) harmonic variables are bosonic \( SU(2)_{R} \) doublets which parametrize the unit \( S_{3}^{3} \), they may be solved in terms of the standard \( S_{3}^{3} \) variables \( \psi, \theta \) and \( \phi \) as

\[ u_{1}^{+} = \cos \frac{\theta}{2} \exp \frac{i}{2} (\psi + \phi) \]
\[ u_{2}^{+} = \sin \frac{\theta}{2} \exp \frac{i}{2} (\psi - \phi) \]
\[ u_{1}^{-} = \sin \frac{\theta}{2} \exp \frac{-i}{2} (\psi + \phi) \]
\[ u_{2}^{-} = \cos \frac{\theta}{2} \exp \frac{-i}{2} (\psi - \phi). \]  

We shall not use this realization hereafter as we shall take \( u_{\alpha}^{\pm} \) as our basic variables. Moreover, using the \( u_{\alpha}^{\pm} \) variables, the \( SU(2)_{R} \) algebra is realized as differential operators on the space of harmonic functions on \( S_{3}^{3} \):

\[ D^{++} = u^{+ \alpha} \frac{\partial}{\partial u^{- \alpha}}; \quad D^{--} = u^{- \alpha} \frac{\partial}{\partial u^{+ \alpha}} \]
\[ 2D^{++} = [D^{0}, D^{++}]; \quad -2D^{--} = [D^{0}, D^{--}] \]
\[ D^{0} = [D^{++}, D^{--}] = u^{+ \alpha} \frac{\partial}{\partial u^{- \alpha}} - u^{- \alpha} \frac{\partial}{\partial u^{+ \alpha}} \]  

To study the \( SU(2)_{R} \) representations by using the harmonic variables, it is more convenient to consider harmonic functions \( F^{q}(u_{\alpha}^{\pm}) \) with definite \( U(1)_{R} \) charge \( q \); that is functions \( F^{q}(u_{\alpha}^{\pm}) \) satisfying the eigenfunction eq

\[ [D^{0}, F^{q}] = qF^{q}. \]  

These functions \( F^{q} \) have a global harmonic expansion of total charge \( q \) and carry \( SU(2)_{R} \) representations. For example, taking \( q = 2 \) and choosing \( F^{++} \) as:

\[ F^{++}(u_{\alpha}^{\pm}) = u_{(\alpha}^{+} u_{\beta)}^{+} F^{(\alpha \beta)}; \]
one sees that $F^{++}$ is the highest state of the isovector representation of $SU(2)_R$. This is also seen from the following eqs defining the highest states of $SU(2)_R$ of $U(1)_R$ charge equal to $q$

$$[D^0, F^q] = qF^q$$

$$(19)$$

$$[D^{++}, F^q] = 0.$$  

Thus the harmonic functions $F^{++}$ altogether with $F^0$ and $F^{--}$, defined as

$$F^0 = [D^{--}, F^{++}] = u^+_{(\alpha} u^-_{\beta)} F^{(\alpha\beta)}$$

$$F^{--} = [D^{--}, F^0] = u^-_{(\alpha} u^-_{\beta)} F^{(\alpha\beta)},$$

form the three states of the isotriplet representation of the algebra (16). In connection with the isotriplet representation \{ $F^{++}, F^0, F^{--}$ \}, there is an interesting feature that we want to give at this level and which we will use later on when studying the solutions of eqs (12). This feature concerns the fact that one can usually realize $F^q, q = 0, \pm 2$ as bilinears of isospinors $f^+$ and $\overline{f}^+$ as follows

$$F^{++} = if^+ \overline{f}^+ = iu^+_{(\alpha} u^-_{\beta)} f^{(\alpha} \overline{f}^{\beta)}$$

$$F^0 = \frac{i}{2} (f^+ \overline{f}^- + f^- \overline{f}^+) = \frac{i}{2} u^+_{(\alpha} u^-_{\beta)} f^{(\alpha} \overline{f}^{\beta)}$$

$$F^{--} = if^- \overline{f}^- = iu^-_{(\alpha} u^-_{\beta)} f^{(\alpha} \overline{f}^{\beta)}.$$  

(21)

The complex number $i$ in front of the the factor of the right hand side of the above eqs ensures the reality condition of the isotriplet representation. Moreover the realization of $F^q, q = 0, \pm 2$ as bilinears of isospinors reflects too simply the fact that the $SU(2)_R$ isovectors may be built from the symmetric product of the isospin $\frac{1}{2}$ representation and
its conjugate. After this digression on the $SU(2)_R$ harmonic analysis, we turn now to eqs (1) which we write as:

$$\sum_j q^+_j \varphi^+_j \overline{\varphi}^+_j = -i \xi^{++}_a \quad (a)$$
$$\sum_j q^+_j (\varphi^+_j \overline{\varphi}^-_j + \varphi^-_j \overline{\varphi}^+_j) = -2i \xi^0_a \quad (b)$$
$$\sum_j q^+_j \varphi^-_j \overline{\varphi}^-_j = -i \xi^{--}_a \quad (c).$$

These eqs are obtained from eqs (1) by multiplying their both sides by $u^+_{(\alpha} u^+_{\beta)}$, $u^+_{(\alpha} u^-_{\beta)}$ and $u^-_{(\alpha} u^-_{\beta)}$ respectively. Eqs (22) are also the D-flatness eqs one gets if one is using the $2d N = (4, 4)$ harmonic superspace formulation of $2d N = 4$ gauge theories [50,51]. Thus like for eqs (1), eqs (22) form altogether a system of $r$ isovector eqs of the $SU(2)_R$ algebra (16); but with the remarkable difference that now it is enough to focus attention on the highest weight states eqs (22.a). Knowing the solutions $\varphi^+_j$ and $\overline{\varphi}^+_j$ of eqs (20-a), one can also get the solutions of $\varphi^-_j$ and $\overline{\varphi}^-_j$ by acting on $\varphi^+$ and $\overline{\varphi}^+$ by $D^{--}$; namely:

$$\varphi^-_j = [D^{--}, \varphi^+_j]$$
$$\overline{\varphi}^-_j = [D^{--}, \overline{\varphi}^+_j].$$

In the end of this section, we would like to make two comments. The first comment is that one can use the isospinor bilinear realization of isotriplets eqs (21) to represent the Kahler parameters $\xi^{++}_a$ as follows:

$$\xi^{++}_a = i \zeta^+_a \overline{\zeta}^+_a = i u^+_{(\alpha} u^+_{\beta)} \zeta^{(\alpha\beta)}_a \xi^{(\alpha\beta)}_a$$

$$\zeta^+_a = u^+_a \zeta^a_a; \quad \overline{\zeta}^+_a = u^+_a \overline{\zeta}^a_a,$$

where $\zeta^a_a$ and $\overline{\zeta}^a_a$ may, roughly speaking, be viewed as the square roots of the FI couplings $\xi^{(\alpha\beta)}_a$. Similar relations involving $\zeta^+_a$ and $\overline{\zeta}^+_a$ for $\xi^0_a$ and $\xi^{--}_a$ may be also written down.

Putting back these relations in eqs (22.a), one gets

$$\sum_j q^+_j \varphi^+_j \overline{\varphi}^+_j = \zeta^+_a \overline{\zeta}^+_a = u^+_{(\alpha} u^+_{\beta)} \zeta^{(\alpha\beta)}_a \xi^{(\alpha\beta)}_a.$$
The second comment we want to do is that one can simplify further eqs (25) by making an extra change of variables which turns out to be convenient when discussing the moduli space of gauge invariant vacua of eqs (1). This extra change consists of the mapping

\[ R^3 = R^+ \times S^2 \]

to write the FI isovectors \( \xi^{++}_a \) as

\[ \xi^{++}_a = R_a \eta^+_a \bar{\eta}^+_a = r^2_a \eta^+_a \bar{\eta}^+_a, \]  

or equivalently by using the isospinors \( \zeta^+_a \) and \( \bar{\zeta}^+_a \) introduced previously:

\[ 
\begin{align*}
\zeta^+_a &= u^+_a \zeta^+_a; & \bar{\zeta}^+_a &= u^+_a \bar{\zeta}^+_a \quad (a) \\
\zeta^+_a &= r_a \eta^+_a; & \bar{\zeta}^+_a &= r_a \bar{\eta}^+_a \\
\zeta^+_a \bar{\zeta}^+_a &= r^2_a \geq 0 \quad (c).
\end{align*}
\]

Eqs (26) and (27) tell us that the \( R^+_a \)'s \( (R^+_a = r^2_a \geq 0) \) are the radial variables and the \( \eta^+_a \)'s and \( \bar{\eta}^+_a \)'s, which satisfy

\[ \eta^+_a \bar{\eta}^+_a = 1; \quad \eta^+_a \bar{\eta}^+_a = \eta^+_a \bar{\eta}^+_a = 0, \]  

parametrize the two spheres \( S^2_a \). The \( r^2_a \) and \( \eta^+_a \) and \( \bar{\eta}^+_a \) are in one to one correspondence with the \( r \) FI isovectors. In other words, eqs (28) describe a collection of \( r \) unit two spheres which together with the \( r_a \) conical variables of \( (R^3)^r \) give the \( 3r \) parameters of the \( r \) FI isovector couplings.

4 Separation of the charges of the \( U(1)_G \) gauge and the \( U(1)_R \) symmetries

Here we describe the separation of the gauge and \( U(1)_R \) charges of the hypermultiplet scalar moduli \( \varphi^+_j \) and \( \bar{\varphi}^+_j \). As we mentioned earlier, this charge separation is the second
step in our programme of finding the zero energy states of the classical 2d $N = 4$ supersymmetric $U(1)^r$ gauge theory. Recall that the first step described in section 3 consists to interpret $SU(2)_R$ field representations as harmonic functions on $S^2_R$, fact which allowed us to put eqs (1) in a form similar to eqs (2) as shown on eqs (25). However the field variables of eqs (25) still carry both $U(1)_R$ and $U(1)^r$ charges; these make their geometrical interpretations difficult and moreover do not allow to take advantage with their similarities with the $N = 2$ supersymmetric D-flatness eqs (2) in looking for the solutions. Motivated by these two features, we have been lead to look for a way of rewriting eqs (25) so that the known results of 2d $N = 2$ linear sigma models, including the geometric interpretation, can be exploited. We have found that this way may be achieved by introducing a parametrization of the hyperkahler moduli where the gauge charges and the $U(1)_R$ ones are separated but 2d $N = 4$ supersymmetry still preserved. Note in passing that the idea of separating composite quantum numbers of fields is not a new idea; and corresponds just to a special feature of group representations theory. In the physics literature, the separation of the different charges of quantum fields has been used successfully in various occasions in the past, in particular in coset models of 2d conformal field theory and in strongly correlated electrons models of low dimensional systems. One of the well known examples concerns complex spinors which may be separated into a spinon and a holon. For a review see [52]. Thus our major aim in this section is to use this idea and try to solve these eqs by introducing the method of factorization of the two kinds of charges carried by the hyperKahler moduli. There are various ways one may follows in order to perform the separation of gauge and $U(1)_R$ charges carried by $\varphi_j^+$'s and $\bar{\varphi}_j^+$'s. A quite general way which allows to fulfill the four following requirements ((a), (b), (c) and (d)) is given by splitting $\varphi_j^+$'s and $\bar{\varphi}_j^+$'s as shown on eqs (29). These requirements are natural and may
be stated as follows:

(a) The splitting should preserves supersymmetry, that is preserving the eight supercharges of the gauge theory.

(b) It should recover the results of [31,42] summarized in section 2 but also extend the ADE models of 2d \(N = 2\) supersymmetric backgrounds [8], see also [53].

(c) It should give the standard ADE results up on breaking half of the eight supercharges.

(d) It should has a geometrical interpretation.

The factorization we propose is given by:

\[
\varphi_j^+ = X_j \eta_j^+ + \gamma Y_j \eta_j^+ \\
\varphi_j^- = X_j \eta_j^- - \gamma Y_j \eta_j^-,
\]

where \(\eta_j^+\) and \(\eta_j^-\) are as in eqs (27); that is:

\[
\eta_j^+ = u_\alpha^+ \eta_j^\alpha; \quad \eta_j^- = u_\alpha^- \eta_j^\alpha
\]

\[
\eta_j^\alpha \eta_{\alpha j} = 1; \quad \eta_j^0 \eta_{\alpha j} = \eta_j^0 \eta_{\alpha j} = 0
\]  

and where \(X_j\) and \(Y_j, j = 1, \ldots, n\) are complex fields carrying no \(U(1)_R\) charges. The parameter \(\gamma\) takes the values \(\gamma = 0\) or \(\gamma = 1\) and distinguish the two classes of solutions we will give hereafter. Note that similar decompositions to eqs (29) are also valid for \(\varphi_j^-\) and \(\varphi_j^-\) and may be obtained from eqs (29) by acting on them by \(D^-\) as in eqs (23).

We will not use them in this discussion and then one can ignore them for the moment.

Moreover the quantities \(X_j\), \(Y_j\) and \(\eta_j^+\) and \(\eta_j^-\) of the splitting (29) behave under \(U(1)_R\)
gauge and $U(1)_R$ transformations as follows:

$$
U(1)^r : X_j \rightarrow X'_j = \lambda^{q_j^a} X_j \\
Y_j \rightarrow Y'_j = \lambda^{q_j^a} y_j \\
\eta^+_j \rightarrow \eta'^+_j = \eta^+_j \\
\bar{\eta}^+_j \rightarrow \bar{\eta}'^+_j = \bar{\eta}^+_j;
$$

and

$$
U(1)_R : X_j \rightarrow X'_j = X_j \\
Y_j \rightarrow Y'_j = Y_j \\
\eta^+_j \rightarrow \eta'^+_j = e^{i\theta} \eta^+_j \\
\bar{\eta}^+_j \rightarrow \bar{\eta}'^+_j = e^{i\theta} \bar{\eta}^+_j.
$$

Actually eqs (31-32) define the factorization of the gauge charges and $U(1)_R$ ones. In what follows we shall use the splitting (29) to solve eqs (1) which, by help of the analysis of section 3, is also given by

$$
\sum_j q^j_a \varphi^+_j \bar{\varphi}^+_j = r^2_a \eta^+_a \bar{\eta}^+_a.
$$

We shall give two classes of solutions of these eqs; the first class described a generalisation of the usual $N = 2$ ADE ALE surfaces and the second class describes new models which flow in the infrared to 2$d$ $N = (4, 4)$ conformal field theories; the latters satisfy eqs (6).

### 4.1 Generalized ADE hypersurfaces

Here we would like to use the similarity between eqs (1) and eqs (2) in order to look for special solutions of eqs (1). These solutions are expected to describe generalisations of the standard eqs of ADE singularities associated with 2$d$ $N = 4$ linear sigma models. We find indeed that the moduli space of gauge invariant vacua of eqs (33) is appropriatly formulated in terms of harmonic variables of $SU(2)_R$ symmetry and the hypermultiplets vacua; see
for instance eqs (35-36) for the case of a $SU(n)$ singularity. As a check consistency of of our results, we show that under some assumptions to be specified later on, the generalized ADE hypersurfaces we have obtained may be brought to the well known ADE models of $2d\ N=2$ supersymmetric linear models. We show also, by explicit computation, that for the case of a $SU(n)$ singularity, the moduli space of gauge invariant vacua of eqs (33) is given by the usual ALE space with a $SU(n)$ singularity times the 2-sphere to power $2n$; i.e $(S^2)^{2n}$. Under the abovementioned assumption, this reduces to the usual ALE background times a 2-sphere. A similar result is also valid for the other singularities. To do so, let us first note that up on imposing the condition of ADE models

$$\sum_j q^j_a = 0, \quad a = 1, \ldots, n - 1,$$  

(34)

one can imitate the analysis of $2d\ N=2$ linear $\sigma$ models and build the gauge invariant moduli in terms of the $\varphi^+_j$ fields. In the $SU(n)$ case for instance where $q^j_a$ is given by eq(3); there are three gauge invariant moduli; $U^+\frac{n(n+1)}{2}$, $V^+\frac{n(n+1)}{2}$ and $Z^{+(n+1)}$ carrying $\frac{n(n+1)}{2}$, $\frac{n(n+1)}{2}$ and $(n+1)$ $U(1)_R$ Cartan charges respectively. They are given by:

$$U^+\frac{n(n+1)}{2} = \prod_{j=0}^n (\varphi^+_j)^{n-j},$$

$$V^+\frac{n(n+1)}{2} = \prod_{j=0}^n (\varphi^+_j)^j, $$

(35)

$$Z^{+(n+1)} = \prod_{j=0}^n (\varphi^+_j).$$

They satisfy the following remarkable equation

$$U^+\frac{n(n+1)}{2}V^+\frac{n(n+1)}{2} = [Z^{+(n+1)}]^n$$  

(36)

Eq (36) generalizes the usual equation of the ALE surface with $SU(n)$ singularity which, for later use, we recall it herebelow:

$$uv = z^n.$$  

(37)
To better see the structure of eq (36), we use the splitting method of the charges of the \( \varphi^\pm_j \)'s we have described earlier. Taking \( \gamma = 0 \), the general splitting eqs (29) reduces to:

\[
\varphi^+_j = x_j \eta^+_j; \quad \varphi^-_j = \bar{x}_j \bar{\eta}^+_j,
\]

(38)

where \( X_j \) and \( \eta^+_j \) behave under gauge and \( U(1)_R \) transformations as in eqs (31,32). Note by the way that like \( \varphi^\pm_j \), the realization \( X_j \eta^\pm_j \) carries, for each value of \( j \), four real degrees of freedom; two degrees come from \( X_j \) and the two others from the parameters the 2-sphere described by \( \eta^\pm_j \); eqs (30). Under \( 2dN = 4 \) supersymmetric transformations which may be conveniently expressed as \( 4dN = 2 \) supersymmetric transformations of fermionic parameters \( \epsilon^\pm \) and \( \bar{\epsilon}^\pm \), we have:

\[
\delta \varphi^+_j = \epsilon^+ \psi_j + \bar{\epsilon}^+ \bar{\psi}_j;
\]

(39)

where \( \psi_j \) and \( \bar{\psi}_j \) are the Fermi partners of the \( \varphi^\pm_j \) scalars. \( (\varphi^\pm_j, \psi_j, \bar{\psi}_j) \) constitute altogether the \( 4dN = 2 \) free hypermultiplets. Using the splitting principle by factorizing \( \epsilon^+ \) as \( \epsilon \eta^+ \) and \( \bar{\epsilon}^+ = \bar{\epsilon} \eta^+ \), and using eqs (38) and (39); we get

\[
\eta^+_j \delta X_j + X_j \delta \eta^+_j = \eta^+ \epsilon \psi_j + \bar{\eta}^+ \bar{\epsilon} \bar{\psi}_j,
\]

(40)

or equivalently

\[
\eta^0_j \delta X_j + X_j \delta \eta^0_j = \eta^0 \epsilon \psi_j + \bar{\eta}^0 \bar{\epsilon} \bar{\psi}_j.
\]

(41)

Putting eqs (38) back into eqs (33), we get

\[
\sum_j q^+_a |X_j|^2 \eta^+_j \bar{\eta}^+_a = r^+_a \eta^+_a \bar{\eta}^+_a
\]

(42)

and

\[
U^+ \frac{n(n+1)}{2} = uM^+ \frac{n(n+1)}{2}
\]

\[
V^+ \frac{n(n+1)}{2} = vN^+ \frac{n(n+1)}{2}
\]

\[
Z^{+(n+1)} = zS^{+(n+1)};
\]

(43)
where \( u, v, z \) and \( M^{\frac{n(n+1)}{2}}, N^{\frac{n(n+1)}{2}} \) and \( S^{+(n+1)} \) are gauge invariants given by

\[
\begin{align*}
 u &= \prod_{j=0}^{n} X_j^j ; \quad N^{\frac{n(n+1)}{2}} = \prod_{j=0}^{n} (\eta_j^+)^{n-j} \\
 v &= \prod_{j=0}^{n} X_j^{n-j} ; \quad M^{\frac{n(n+1)}{2}} = \prod_{j=0}^{n} (\eta_j^+)^j \\
 z &= \prod_{j=0}^{n} X_j ; \quad S^{+(n+1)} = \prod_{j=0}^{n} \eta_j^+.
\end{align*}
\]

(44)

Note that \( u, v \) and \( z \) verify the relation (37) and \( M^{\frac{n(n+1)}{2}}, N^{\frac{n(n+1)}{2}} \) and \( S^{+(n+1)} \) satisfy eq (36). Eqs (42,43) may be brought to more familiar forms if we require moreover that the three following types of 2-spheres are identified:

(i) The \((n + 1)\) 2-spheres parametrized by the \(\eta_j\)'s.

(ii) The \((n - 1)\) \(\eta_a\) 2-spheres used in the parametrization of the FI couplings eqs (28).

(iii) The \(\eta^+\) 2-sphere involved in the factorization of the supersymmetric parameter \(\epsilon^+\) eq(40).

In other words, we require the following identity:

\[
\eta_j^+ = \eta_a^+ = \eta^+.
\]

(45)

With this identification eqs, (42) reduce to the well known D-flatness conditions of the \(U(1)_R\) gauge theory with four supercharges; namely:

\[
\sum_j q_a^j |X_j|^2 = r_a^2.
\]

(46)

Moreover eq (36) reduce to the usual ALE surface with \(SU(n)\) singularity eq (37) since the \( M^{\frac{n(n+1)}{2}}, N^{\frac{n(n+1)}{2}} \) and \( S^{+(n+1)} \) gauge invariant become trivial as they are given by powers of \(\eta^+\) as shown here below.

\[
\begin{align*}
 M^{\frac{n(n+1)}{2}} &= (\eta^+)^{\frac{n(n+1)}{2}} = N^{\frac{n(n+1)}{2}} \\
 S^{+(n+1)} &= (\eta^+)^{n+1}.
\end{align*}
\]

(47)

In the general case where the gauge charges \(q_a^j\) of the \(X_j\)'s satisfy the constraints (4), eqs (46) is the vacuum energy of 2d \(N = 2\) supersymmetric linear \(sigma\) models. Thus the
classical moduli space $M$ of the gauge invariant vacua of eqs (45,46) is then given by the 2-sphere parametrized by $\eta^+$; eq (45), times the moduli space of the gauge invariant solutions of $2d \, N = 2$ supersymmetric vacuum energy states. In other words:

$$M = \frac{C^{n+1}}{C^{*n-1}} \times S^2.$$  

Note that the identification constraint eq (45) has a nice interpretation; it breaks explicitly half of the eight supersymmetries leaving then four supercharges preserved. These four supercharges are behind the reduction of eqs (42) down to eqs (46) leading to the standard ADE models. This feature is immediately derived by combining eqs (41) and (45) as follows:

$$\eta^\alpha \delta X_j + X_j \delta \eta^\alpha = \eta^\alpha \epsilon \psi_j + \eta^\alpha \overline{\epsilon} \overline{\psi}_j,$$  

Then multiplying both sides of this identity by $\eta^\alpha$; one gets, after using eqs (30):

$$\delta X_j = \epsilon \psi_j,$$  

giving the usual supersymmetric transformations of the complex scalars of the $2d \, N = 2$ chiral multiplets. This completes the check of consistency of the generalised $SU(n)$ hypersurface singularity (36). Before going ahead let us summarize in few words what we have done until now. Starting from eqs (1), we have shown that it is possible to put them into their equivalent form (33). The corresponding moduli space of gauge invariant vacua is given by eq (36) which reduces to the standard ALE space with $A_{n-1}$ singularity up on imposing the factorization eqs (38) and the conditions (45). The latter breaks four supercharges among the original eight ones. The factorization (38) offers in turns a method for a geometric representation of hyperKahler backgrounds with eight supercharges. However we have not succeeded to solve directly eqs (33) nor (42) without breaking the eight supercharges. We will see later that it still possible to work out solutions with eight supercharges.
by using the general splitting (29) instead of the factorization (38) but still imposing eqs (45). Indeed to restore the eight supersymmetries by still using the constraint (45) we should take $\gamma$ non zero; say $\gamma = 1$. Non zero $\gamma$ brings four extra supercharges which add to the old four existing ones carried by eq (38). This is easily seen from the above analysis and the splitting (38) where each part of the two terms of the right hand of eqs (39-41) carries four supersymmetries. We shall return to this feature with more details in the next subsection; for the time being we would like to make two comments regarding eqs (33).

(1) A naive analysis of eqs (33) suggests that the gauge invariant moduli space of vacua $M$ of eq (38) is given, for the generalized $SU(n)$ singularity eq (36), by the usual ALE space with $SU(n)$ singularity times $2n$ two-spheres. In other words:

$$M = \frac{C^{n+1}}{C_{n+1}} \times (S^2)^{2n},$$

where $(n + 1)$ two spheres come from the $\phi_j$’s as shown in eqs (38) and $(n - 1)$ two spheres come from the FI couplings.

(2) As far eq (36) is concerned, one can also write down the generalized ADE models extending the usual $N = 2$ ones. In addition to eq (36) which generalizes eq (37), we have also

$$(x^{++})^n + x^{++}(y^{+(n-1)})^2 + (z^+)^2 = 0,$$

describing the generalized $D_n$ singularity extending the standard ALE one namely:

$$x^n + xy^2 + z^2 = 0.$$
\[(x^{+15})^2 + (y^{+10})^3 + (z^{+6})^5 = 0.\]

These eqs extend respectively the following exceptional singularities

\[E_6 : \quad x^2 + y^3 + z^4 = 0\]

\[E_7 : \quad x^2 + y^3 + yz^3 = 0\]

\[E_8 : \quad x^2 + y^3 + z^5 = 0.\]

More informations about these extensions will be given in a future occasion.

4.2 Solutions with \(\gamma = 1\)

Choosing \(\gamma = 1\) in the eqs (29) and putting back into eqs (33), we get a system of three eqs given by:

\[
\sum_j q^i_a (|X_j|^2 - |Y_j|^2) \eta^+_j \bar{\eta}^+_j = r^2_a \eta^+_a \bar{\eta}^+_a \quad (a)
\]

\[
\sum_j q^i_a (X_j Y_j) \eta^+_j \bar{\eta}^+_j = 0 \quad (b)
\]

\[
\sum_j q^i_a (\bar{X}_j Y_j) \bar{\eta}^+_j \eta^+_j = 0. \quad (c)
\]

At this level no constraint has been imposed yet on the FI couplings contrary to the analysis of ref [31,42] summarized in section 2. If moreover we require that all the two sphere \(\eta^+_j\) and \(\eta^+_a\) are identified as in eq (45); the above system reduces to

\[
\sum_j q^i_a (|X_j|^2 - |Y_j|^2) = r^2_a \quad (a)
\]

\[
\sum_j q^i_a (X_j Y_j) = 0 \quad (b)
\]

\[
\sum_j q^i_a (\bar{X}_j Y_j) = 0. \quad (c)
\]

Eqs (51) have some remarkable features which have nice interpretations. Though the \(q^i_a\) gauge charges of the hypermultiplets are not required to add to zero as in eq (4), eq (51.a) behave exactly as the D-flatness condition of 2d \(N = 2\) supersymmetric \(U(1)^r\) gauge...
theory. The point is that eqs (51.a) involve twice the number of fields of eqs (2), but with opposite charges $q^j_a$. Put differently; eq (51) involve two sets of fields $X_j$ and $Y_j$ of charge $q^j_a$ and $(-q^j_a)$ respectively. The sum of gauge charges of the $X_j$’s and $Y_j$’s add automatically to zero even though eq (4) is not fulfilled. Thus models with $\gamma = 1$ flow in the IR to a 2d $N = (4,4)$ superconformal models extending the usual 2d $N = (2,2)$ ADE ones since the identities

$$\sum_j q^j_a + \sum_j (-q^j_a) = 0 \quad (52)$$

go beyond the constraint eqs (4). Moreover eqs (51) may be fulfilled in different ways; either by taking all charges $q^j_a$ of the $U(1)^r$ gauge theory to be positive; say $q^j_a = 1; a = 1,...,r; j = 1,...,n$, or part of the $q^j_a$’s are positive and the remaining ones are negative. In the case of a $U(1)$ gauge theory with $(n+1)$ hypermultiplets with gauge charges equal to one, eqs (51.a) describe a $CP^n$ manifold whereas eqs (51.b-c) which read as

$$\sum_j X_j Y_j = 0, \quad (53)$$

together with their complex conjugate, show that the $Y_j$’s are in the cotangent space of $CP^n$ at the point $x_j = X_j/\left[\sum_{i} |Y_i|^2 + r^2_a\right]^{-\frac{1}{2}}$. Observe in passing that in case where some of the positive charges $q^j_a$ of $U(1)^r$ gauge theory are not equal to one, the corresponding moduli space is just the cotangent bundle of some weighted complex projective space, $T^*(WP^n)$. Observe moreover that in the infrared limit this gauge theory flows to a 2d $N = (4,4)$ conformal field theory with central charge $C = 6n$. In the next section we shall give some illustrating examples.
5 Moduli space of vacua of models with $\gamma = 1$

In this section we want to study two types of vacua of the D-flatness conditions of 2d $N = 4$ supersymmetric $U(1)^r$ gauge theory depending on the manner we deal with eqs(52). In other words starting from eqs(52), we develop hereafter two different, but equivalent, ways to solve them. These ways are associated with the value of the sum over the $U(1)^r$ charges of the hypermultiplet moduli that is; $\sum q^i_a \neq 0$ or $\sum q^i_a = 0$. To do so, we shall first study the case $\sum q^i_a \neq 0$. We start by describing explicitly two examples after what we give the general sigma model result we have obtained and give also comments regarding the 2d $N = 4$ Liouville description in the vicinity of these singularities. A similar analysis will be made for the other case $\sum q^i_a = 0$.

5.1 $\sum q^i_a \neq 0$

A priori there are many ways to choose the $q^i_a$ charges such that $\sum q^i_a \neq 0$; each of which corresponds to a definite model. A simple and instructive model is to consider a 2d $N = 4$ supersymmetric abelian gauge theory with $(r + 1)$ hypermultiplets whose scalar fields are denoted as $\varphi^+_j$ and $\overline{\varphi}^+_j ; j = 0, 1, ..., r$. Using the splitting method described previously, we write the $\varphi^+_j$’s and $\overline{\varphi}^+_j$’s as

$$\varphi^+_j = X_j \eta^+ + Y_j \overline{\eta}^+,$$

$$\overline{\varphi}^+_j = -\overline{X}_j \eta^+ + \overline{Y}_j \overline{\eta}^+,$$  \hspace{1cm} (54)

where their $U(1)^r$ charges are choosen as

$$q^i_a = \delta^i_{a-1} + \delta^i_a.$$ \hspace{1cm} (55)
Putting eqs (54) back into the D-flatness conditions (51) we get the following system of algebraic eqs

\[ |X_{a-1}|^2 + |X_a|^2 - (|Y_{a-1}|^2 + |Y_a|^2) = R_a \]
\[ \sum_j q_a^j X_j Y_j = X_{a-1} Y_{a-1} - X_a Y_a = 0 \]
\[ \sum_j q_a^j X_j Y_j = X_{a-1} Y_{a-1} + X_a Y_a = 0. \]  

(56)

For later use, let us rewrite the two leading blocks of eqs of the above system, describing respectively models with \( U(1) \) and \( U(1)^2 \) gauge groups associated with the values \( r = 1 \) and \( r = 2 \). For \( r = 1 \), eqs (56) reduce to the three following eqs:

\[ |X_0|^2 + |X_1|^2 - (|Y_0|^2 + |Y_1|^2) = R_1 \]  
\[ X_0 \bar{Y}_0 + X_1 \bar{Y}_1 = 0 \]  
\[ \bar{X}_0 Y_0 + \bar{X}_1 Y_1 = 0. \]  

(57)

Similarly we have, for the \( U(1)^2 \) gauge model, a system of six equations; three of them coincide with those given by eqs (57); the others are as follows:

\[ |X_1|^2 + |X_2|^2 - (|Y_1|^2 + |Y_2|^2) = R_2 \]  
\[ X_1 \bar{Y}_1 + X_2 \bar{Y}_2 = 0 \]  
\[ \bar{X}_1 Y_1 + \bar{X}_2 Y_2 = 0 \]  

(58)

To solve eqs (56), we shall adopt the following strategy. We shall first consider the solving of eqs (57), then we treat both eqs (57) and (58), after what we give the general solutions for eqs (56) and finally make some comments regarding the Liouville description of the singularities of the metric of the Higgs branch. For the 2d \( N = 4 \) supersymmetric model with one \( U(1) \) gauge factor, one should note first of all that the moduli space of gauge invariant vacua is a complex surface which becomes singular when \( R_1 \) vanishes. It is just the cotangent line bundle of the two- sphere \( S^2; T^*(CP^1) \). A naive way to see this feature is to set \( Y_0 = Y_1 = Y \); a choice which reduces eq (57.a) to the following well known eq of
\[ N = 2 \text{ linear sigma models with four supercharges} \]

\[ |X_0|^2 + |X_1|^2 - 2|Y|^2 = R. \]  \hspace{1cm} (59)

This eq describes the blow up of the SU(2) singularity of the ALE complex surface \( \mathbb{C}^2/\mathbb{Z}_2 \).

An other way to deal with this singularity is to make the change of variables preserving the eight supercharges

\[
x_0 = X_0[R_1 + |Y_0|^2 + |Y_1|^2]^{-\frac{1}{2}}
\]

\[
x_1 = X_1[R_1 + |Y_0|^2 + |Y_1|^2]^{-\frac{1}{2}}, \] \hspace{1cm} (60)

leading to

\[
|x_0|^2 + |y_1|^2 = 1 \hspace{1cm} (a)
\]

\[
x_0\overline{y}_0 + x_1\overline{y}_1 = 0 \hspace{1cm} (b)
\]

\[
x_0y_0 + \overline{x}_1\overline{y}_1 = 0 \hspace{1cm} (c). \] \hspace{1cm} (61)

Eqs (57.b-c), which by the way, are exchanged under complex conjugation, have a geometric meaning; they show that at each point \((x_0, x_1)\) of the base manifold \(B_1\) there is an orthogonal fiber \(F_1\) parametrized by \((\overline{Y}_0, \overline{Y}_1)\), defining altogether the cotangent line bundle \(T^*\mathbb{C}P^1\). For non zero values of \(R_1\) where the change (60) is well defined, eq (57.a) is a 2-sphere and then the bundle is smooth. For \(R_1\) equals to zero, the change (60) falls down at the origin \(X_0 = X_1 = Y_0 = Y_1 = 0\) and the bundle becomes singular. Note that according to the ADHM construction, the moduli space of gauge invariant vacua of the \(U(1)\) gauge model with two (or more) hypermultiplets is just the moduli space of small instantons on \(R^4\). For \(R_1\) positive definite, the small instanton singularity is blown up and in the limit \(R_1 = 0\) the singularity is recovered. In two dimensions, it has been shown moreover that in this limit the fields \(X_0, X_1, Y_0\) and \(Y_1\) do not give a good description of the small instanton conformal theory near the singularity. The appropriate variables in this region turns out to be those of a 2d \(N = 4\) conformal Liouville field theory [17,18...
To see this remarkable feature, it is interesting to use the field theoretical approach of Aharony and Berkooz [18] regarding the study of the low energy limits of 2d $N = 4$ gauge theories whose lagrangian $L = L_{\text{gauge}} + L_H$ reads as:

$$L = \frac{1}{4g_{YM}^2} \int d^2x \text{tr}(F_{\mu \nu}^2 + (D_\mu V) + [V, V]^2 + \bar{\psi}_V \gamma^\mu D_\mu \psi_V + \bar{\psi}_V [V, \psi_V] + \bar{D}^2) +$$

$$\int d^2x \sum_{\text{hypermult}} (|D_\mu \phi_H|^2 + |V \phi_H|^2 + \bar{\psi}_H \gamma^\mu D_\mu \psi_H + \bar{\psi}_H V \psi_H + \bar{\psi}_V V \psi_H) \quad (62)$$

In this formal eq $D_\mu = (\partial_\mu + A_\mu)$ is the covariant derivative, $(V_{\dot{A}A}, A_\mu, D^{(\alpha \beta)})$ and $(\psi^A, \bar{\psi}^A)$ are respectively the bosonic and fermionic fields of the vector multiplet carrying amongst others quantum charges of the $SO(4) \times SU(2)_R \approx SU(2)_r \times SU(2)_l \times SU(2)_R$. Note that $SU(2)_r \times SU(2)_l \times SU(2)_R$ is the R-symmetry of the gauge theory with eight supercharge in two dimensions. Note also that the indices $A, \dot{A}$ and $\alpha$ refer to the isospin $\frac{1}{2}$ representation of $SU(2)_l, SU(2)_r$ and $SU(2)_R$ respectively. Note Moreover that the $\phi_H$ scalars and their fermionic partners $(\psi_H^L, \psi_H^R)$ stand for the fields of the hypermutiplets. Following [18], the low energy limit of this gauge theory, which involves taking $g_{YM} \to \infty$, is described by two decoupled 2d $N = (4, 4)$ superconformal field theories; one describing the Higgs branch whose central charge $C_H = 6(n_H - n_V)$ where $n_H$ the number of hypermultiplets and $n_V$ is the number of vector multiplets. The other conformal field theory corresponds to the Coulomb Branch of central charge $C_V = 6$. An argument supporting this particular feature comes from the analysis of the R-symmetries of the $N = (4, 4)$ superconformal algebra which includes left and right moving $su(2)$ Kac Moody subalgebras. The R-symmetry of the Higgs branch is exactly $SU(2)_l \times SU(2)_r \approx SO(4)$ encountered earlier while the R-symmetry of the Coulomb branch is given by a non visible group $SO(4) \approx SU(2)_l \times SU(2)_r$, containing $SU(2)_R$ as a diagonal subgroup. Since the Coulomb and Higgs superconformal theories have different R-symmetries; they cannot be identified. Moreover taking the naive
limit \( g_{YM} \to \infty \) in eq (62), one sees \( L_{\text{gauge}} \) is removed and the lagrangian of the low energy gauge theory is reduced to \( L_H \); the lagrangian of the Higgs branch namely:

\[
L_H = \int d^2x \sum_{\text{hypermult}} [\left| D_\mu \varphi_H \right|^2 + \left| V \varphi_H \right|^2 + \bar{\psi}_H \gamma^\mu D_\mu \psi_H + \bar{\psi}_H V \psi_H + \bar{\psi}_V V \psi_H
+ \bar{\varphi}_H D \varphi_H] \tag{63}
\]

where now the vector multiplet fields are auxiliary fields which may be eliminated through their eqs of motion. However following [18] see also [54], it more useful to regard the vector multiplet fields as the basic objects instead of the matter fields and integrating over the hypermultiplet fields in order to describe the behavior near the singularity in the moduli space. In this lagrangian approach, one obtains an induced effective action of the vector mutiplet fields which describe the region near the singularity of the Higgs branch (\( \varphi \to 0 \) or \( V \to \infty \)). In the case of supersymmetric \( U(1) \) gauge theory with \( N_f \) hypermultiplets and one vector multiplet, supersymmetry and \( SO(4) \) symmetry constraint the metric of the four gauge scalar fields \((V_i) = (V_1, V_2, V_3, V_4)\) in the vector multiplet to be of the form:

\[
ds^2 = N_f \frac{1}{[(V_1)^2 + V_2^2 + V_3^2 + V_4^2]}[(dV_1^2) + (dV_2^2) + (dV_3^2) + (dV_4^2)]. \tag{64}
\]

or equivalently by changing to radial coordinates \( \sum_{m=1}^{4} (dV_m)^2 = dv^2 + v^2 \sum_{i=1}^{3} (d\Omega_i)^2 \), and defining a new variable \( \phi = \sqrt{\frac{N_f}{2}} \log(\frac{v}{M}) \) for some mass scale \( M \):

\[
ds^2 = d\phi^2 + \frac{N_f}{2} \sum_{i=1}^{3} (d\Omega_i)^2, \tag{65}
\]

together with the 3-form torsion \( H \) given by \((-N_f)\) times the volume form of the 3-sphere namely:

\[
H = -N_f d\Omega_1 d\Omega_2 d\Omega_3 = -N_f d\Omega \tag{66}
\]

The effective theory in the region of large \( V \) is described by a Liouville field \( \phi \), its fermionic partner \( \psi_\phi \) and a supersymmetric level \( N_f \) \( SU(2) \) WZW model generated by the usual
currents $J^\pm$ and $J^3$ which may be rewritten as the sum of a bosonic level \((N_f - 2)\) \(SU(2)\) WZW model plus three free fermions, \(\psi^\pm_{SU(2)}\) and \(\psi^3_{SU(2)}\). Altogether these fields give a realization of the \(N = 4\) conformal field theory of the central charge \(C = 6(N_f - 1)\) as shown on the following central charge counting

\[
6(N_f - 1) = 2 + \frac{3(N_f - 2)}{N_f} + (1 + 3Q^2);
\]  

(67)

where \(Q = (N_f - 1)\sqrt{\frac{2}{N_f}}\). In the end of this digression on the physics in the throat of the Higgs branch, note that the Liouville field \(\phi\) is intimately related with the four scalars \(\{V_m\}\) of the vector multiplet and then with the abelian \(U(1)\) gauge factor as shown on the following eq.

\[
\exp\left(\frac{\sqrt{2}}{N_f} \phi\right) \sim \sqrt{V_1^2 + V_2^2 + V_3^2 + V_4^2}.
\]  

(68)

Therefore there is one to one correspondance between the liouville field \(\phi\) the vector multiplet of the 2d \(N = 4\) \(U(1)\) gauge theory In other words the field \(\phi\) is one to one correspondance with the \(U(1)\) factor of the gauge theory. In this regards, one ask the following question. What happens if, instead of 2d \(N = 4\) supersymmetric \(U(1)\) gauge theory, we consider a \(U(1)^r\) gauge theory involving \(r\) abelian \(U(1)\) factors and then \(4r\) scalars \(V_{m,a}\); \(a = 1, ..., r; m = 1, 2, 3, 4\)? Before discussing the answer to this question, let us first consider the linear sigma model solutions for a typical \(U(1)^r\) D-flatness eqs.

This concerns for example of type eqs (57-58) which are associated with a \(U(1) \times U(1)\) supersymmetric linear sigma model. Following the same steps we described above, one can solve these eqs in a similar way as for the \(U(1)\) theory. The result is that eqs (57-58) describe a two dimensional complex surface given by two intersecting smooth \(T^*CP^1\)'s of base

\[
|x_0|^2 + |x_1|^2 = 1
\]

\[
|\mathfrak{t}_1|^2 + |\mathfrak{t}_2|^2 = 1,
\]  

(69)
where $|\mathfrak{T}_1|$ and $|\mathfrak{T}_2|$ are obtained from eqs (57-58) and analogous changes as in eq (60).

Using the results of [17,18] and the discussions made in the end of the previous example, one sees that here also the fields $X_i$ and $Y_i$ could not be the appropriate variables in the vicinity of the singularity. Since this singularity is a degenerate singularity of type $A_2$, we expect to have more than one Liouville mode in this region and then a more general $2dN = 4$ conformal field theory with background charges. A naive way to see this feature is to use the radial coordinates change of the $U(1)$ gauge theory which allowed us to put eq(64) into its equivalent form eqs (65,66). Since in the $U(1) \times U(1)$ gauge theory we are discussing we have two kinds of scalar fields $V_{1,m}$ and $V_{2,m}$ corresponding to each $U(1)$ factor of the $U(1) \times U(1)$ group, one is tempted to extend the above radial charge to lead to a $N = 4$ conformal $su(3)$ Toda theory. Indeed, starting from the radial parametrization

$$\sum_{m=1}^{4} |dV_{\rho,m}|^2 = (dv_\rho)^2 + v_\rho^2 \sum_{i=1}^{3} (d\Omega_{\rho,i})^2; \rho = 1, 2$$

(70)

and introducing two scalar fields $\phi_\rho$:

$$\phi_\rho = a_\rho \log \frac{v_\rho}{M}$$

(71)

where the coefficients $a_\rho$ should be determined by $2d N = 4$ conformal invariance; one can write down an extension of eqs (65,66). Supersymmetry and $SO(4)$ invariance suggest the following extension :

$$ds^2 = \frac{1}{2}K_{\rho\sigma}[d\phi_\rho d\phi_\sigma + a_\rho a_\sigma \sum_{i=1}^{3} d\Omega_{\rho,i} d\Omega_{\sigma,i}];$$

$$H = -2a_\rho d\Omega_\rho,$$

(72)

where $K_{\rho\sigma}$ is $su(3)$ the Cartan matrix. More generally, this analysis may be extended in a natural way to any $2d N = 4 U(1)^r$ gauge theory $r \geq 1$ in presence of $N_{f,r}$ hypermultiplets. To do so one should first note that for a $2dN = 4$ supersymmetric $U(1)^r$ linear $\sigma$ model with $(r + 1)$ hypermultiplets the moduli space is given by the intersection of $r T^*CP^1$’s.
When all the FI coupling variables vanish simultaneously, the physics within the Higgs branch throat is expected to be described by a general 2d $N = 4$ superconformal Toda theory. In this region the metric is expected to have a form like that given by eqs(65,66). Progress in this direction will be reported elsewhere [55].

5.2 $\sum_i q^i_a = 0$

This situation is the relevant one in the analysis of the moduli space of gauge invariant vacua of 2d$N = 2$ supersymmetric linear sigma models. It ensures that in the infrared, the gauge theory flows to a superconformal one and plays a crucial role in the study of superstrings compactifications on local Calabi Yau manifolds with ADE singularities. The $q^i_a$'s satisfying the relation $\sum_i q^i_a = 0$ are also one of the main ingredient in toric geometry especially in the toric construction of Calabi Yau manifold and their mirrors [56].

In the case of 2d $N = 4$ supersymmetric linear sigma models we have been studying, the sum over the $q^i_a$ charges is automatically fulfilled as shown on eq (52) and one might conclude that it is not necessary to distinguish the two scenarios described in paragraphs 5.1 and 5.2. Though this remark is partially true, there are however some remarkable subtleties we will comment in a moment. Moreover, distinguishing the two scenarios is also relevant for studying $N = 4$ supersymmetric backgrounds by mimicking methods of 2d $N = 2$ supersymmetric linear models as we have done in subsection 4.2. In what follows we give two examples illustrating the above remarks. In the first example we consider 2d $N = 4$ $U(1)^2$ linear sigma model with three hypermultiplets of $q^i_a$ charges chosen as:

$$q^i_a = \delta^i_a - 1, \quad \sum_{j=0}^2 q^j_a = 0.$$  \(73\)
The D-flatness conditions, which may be deduced from eqs (56) and (73) read as

\[ (|X_0|^2 - |X_1|^2) - (|Y_0|^2 - |Y_1|^2) = R_1 \tag{74} \]

\[ X_0 \overline{Y}_0 - X_1 \overline{Y}_1 = \overline{X}_0 Y_0 - \overline{X}_1 Y_1 = 0, \]

together with

\[ (|X_1|^2 - |X_2|^2) - (|Y_1|^2 - |Y_2|^2) = R_2 \tag{75} \]

\[ X_1 \overline{Y}_1 - X_2 \overline{Y}_2 = \overline{X}_1 Y_1 - \overline{X}_2 Y_2 = 0. \]

A way to handle these eqs is to note that they are quite similar to eqs (57,58) up to permutating the roles of \(X_1, X_2\) and \(Y_1\) and \(Y_2\) respectively. From this viewpoint eqs (74) may be rewritten as

\[ (|X_0|^2 + |Y_1|^2) - (|Y_0|^2 + |X_1|^2) = R_1 \tag{76} \]

\[ X_0 \overline{Y}_0 + X_1 (-\overline{Y}_2) = \overline{X}_0 Y_0 + \overline{X}_1 (-Y_2) = 0. \]

For later use it is interesting to rename the field variables of eqs (76) as \(X_0 = Z_0, (-Y_1) = Z_1; Y_0 = W_0\) and \(X_1 = W_1\). Putting this change in the above eqs, one sees that the resulting relations are comparable to those given by eqs (58,61). Thus eqs (76) describe just a cotangent bundle of \(CP^1\). The base \(B_1\) and the fiber \(F_1\) are respectively parametrized by the local coordinates \((z_0, z_1)\) and \((w_0, w_1)\) where the \(z_i\)’s and \(w_i\)’s are related to \(Z_i\)’s and \(W_i\)’s by analogous formulas to those given by eqs (60). In the case of the 2d \(N = 4\) supersymmetric \(U(1) \times U(1)\) gauge theory, we have to solve the system of eqs (74) and (75) which we rewrite for convenience as

\[ (|Z_0|^2 + |Z_1|^2) - (|W_0|^2 + |W_1|^2) = R_1 \tag{77} \]

\[ Z_0 W_0 + Z_1 W_1 = \overline{Z}_0 \overline{W}_0 + \overline{Z}_1 \overline{W}_1 = 0, \]

eqs (77) coincide with eqs (76) we have considered above while eqs (75) read now as

\[ (|W_1|^2 + |Z_2|^2) - (|Z_1|^2 + |W_2|^2) = R_2 \tag{78} \]

\[ Z_1 W_1 + Z_2 W_2 = \overline{Z}_1 \overline{W}_1 + \overline{Z}_2 \overline{W}_2 = 0, \]
where we have set $Y_2 = Z_2$ and $X_2 = W_2$. For positive definite values of $R_2$, if we take $Z_1 = W_2 = 0$, one sees that the complex coordinates $(W_1, Z_2)$ parametrize a $CP^1$ complex curve which is isomorphism to a real 2-sphere of radius $\sqrt{R_2}$. In the limit when $R_2$ goes to zero, this two sphere collapse and one ends with a $SU(2)$ singularity. For generic values of $Z_1$ and $W_2$, eqs (78) describe a cotangent bundle: $T^*CP^1$ exactly as for eqs (77), the radius of the base $B_1$ of $T^*CP^1$ is proportional to $\sqrt{R_1}$. Eqs (77) and (78) describe then two intersecting cotangent $CP^1$’s whose bases $B_1$ and $B_2$ as well as fibers $F_1$ and $F_2$ are roughly speaking parametrized by $(Z_0, Z_1), (W_1, Z_2), (W_0, Z_1)$ and $(W_2, Z_1)$ respectively. The second example we want to give deals with the case of a 2d $N = 4$ supersymmetric $U(1)^r$ gauge theory with $r + 2$ hypermultiplets $\varphi^+_j$ of charges $q^j_a$ given by

$$q^j_a = -2\delta^j_a + \delta^{j-1}_a + \delta^{j+1}_a,$$

satisfying the identity

$$\sum_{i=0}^{r+1} q^j_i = 0.$$  \hspace{1cm} (80)

Using the splitting (29) with $\gamma = 1$, one sees that the $X_j$’s and $Y_j$’s transform under the $C^{sr}$ actions in the same manner as the $\varphi^+_j$ namely:

$$X_j \rightarrow \lambda^{q^j} X_j,$$

$$Y_j \rightarrow \lambda^{q^j} Y_j;$$

where $\lambda$ is non zero complex parameter. Putting eqs (79) into the D-flatness eqs (56), one gets the following system of 3$r$ eqs:

$$\left|X_{a-1}\right|^2 + \left|X_{a+1}\right|^2 - 2\left|X_a\right|^2 - \left|Y_{a-1}\right|^2 + \left|Y_{a+1}\right|^2 - 2\left|Y_a\right|^2 = R_a$$ \hspace{1cm} (a)

$$X_{a-1}\overline{Y}_{a-1} + X_{a+1}\overline{Y}_{a+1} - 2X_a\overline{Y}_a = 0$$ \hspace{1cm} (b)

$$\overline{X}_{a-1}Y_{a-1} + \overline{X}_{a+1}Y_{a+1} - 2\overline{X}_aY_a = 0.$$ \hspace{1cm} (c)

37
For the simple example of the $U(1)$ gauge theory, one can check by following the same procedure we described in the previous example that the system of eqs given herebelow describe the cotangent bundle of the complex two dimensions weighted complex projective space $WP^2_{1,2,1}$. For the general $U(1)^r$ gauge theory, if all the $R_a$'s are non zero, eqs (82) describe the intersection of $r WP^2_{1,2,1}$ cotangent bundles.

In the end of this discussion we would like to make a comment regarding the second example. This concerns the link between our present analysis and the usual ADE $N = 2$ supersymmetric models. Starting from the splitting (29) of the hypermultiplet moduli, one may view the $X_j$’s and $Y_j$’s as vacua of the moduli space of two orthogonal copies of $2d N = 2$ supersymmetric $A_r$ models. This feature is easily seen on the $C^{*r}$ action on these fields as shown on eqs (81) and may be rendered more manifest by analysing the D-flatness eqs (82). Ignoring for the moment eqs (82.b-c) and setting $R_a = A_a - B_a$ with $A_a \geq B_a$, one may put eqs (82.a) into the following remarkable form describing two copies of $A_r$ models

$$ (|X_{a-1}|^2 + |X_{a+1}|^2 - 2|X_a|^2) = A_a $$

$$ (|Y_{a-1}|^2 + |Y_{a+1}|^2 - 2|Y_a|^2) = B_a $$

For $A_a = 0$; $B_a$ positive definite or $A_a$ positive definite; $B_a = 0$, one of the $A_r$ models is singular while $A_a = B_a = 0$ both of them are singular. For $A_a$ positive definite; $B_a$ positive definite, eqs( 84) describe the blown up of the two $A_r$ singularities. Note that each of the $A_r$ models has $N = 2$ supersymmetry whereas the original eq (82) from which they come have $N = 4$ supersymmetry. This means that eqs( 82.b-c) are the lacking piece that rotates the two orthogonal $N = 2$ supersymmetries. Eqs (82.b-c) reflect the
necessary conditions to get $N = 4$ supersymmetry in two dimensions starting from two orthogonal blocks of $N = 2$ supersymmetric models. From this naive parametization of the two intersecting $T^*\mathbb{CP}^1$’s, one sees that the base $B_1$ intersects the fiber $F_2$ along the $Z_1$ direction and the fiber $F_1$ intersects the base $B_2$ along the $\omega_1$ direction.

6 Conclusion

In this paper we have studied two main things. First, we have developed the analysis of the resolution of ADE singularities of hyperKahler manifolds involved in strings compactification. This concerns too particularly the moduli spaces of the Higgs branch of supersymmetric $U(1)^r$ gauge theories with eight supercharges. Second, we have initiated the analysis of singular CFT’s with higher order degeneracies by using the field theoretical approach of Aharony and Berkooz. Actually this study may be viewed as an extension of the recent works dealing with the leading $A_1$ singularity.

Concerning the first part, we have studied the solutions of the D-flatness eqs of supersymmetric $U(1)^r$ gauge theories with eight supercharges by using the linear sigma model approach. We have given, amongst others, a geometrical interpretation of the blown up singularities as a collection of intersecting cotangent complex dimensional weighted projective spaces depending on the number of hypermultiplets and gauge supermultiplets. This examination extends the standard linear sigma model analysis performed for the Kahler Coulomb branch of supersymmetric gauge theories with four supercharges. Our way of doing go beyond literature analysis where only half of the eight supersymmetries are manifest. Our method preserves manifesty all the eight supersymmetries and is realised in two steps based on a geometric realization of the $SU(2)_R$ symmetry on one hand and on a
separation of the charges of the gauge and \( R \)-symmetries on the other hand. The factorisation of the gauge and \( SU(2)_R \) charges of the hypermultiplets moduli involves a parameter \( \gamma \) taking the values \( \gamma = 0 \) or \( \gamma = 1 \) which distinguish two classes of solutions of eqs(1) both preserving the eight supercharges. For \( \gamma = 0 \), we have obtained a generalisation of the ADE complex surfaces reproducing the standard ones by partial breaking of \( 2d \, N = 4 \) supersymmetry down to \( 2d \, N = 2 \). For \( \gamma = 1 \), we have found new models which flow in the infrared to \( 2d \, N = (4,4) \) scale invariant models. In this context several examples are given and classified according the manner one solves eqs(6). In the second part of this paper, we have studied the infrared dynamics of two dimensional \( N = (4,4) \) gauge theories using field theoretical methods. We have made comments regarding the \( N = 4 \) conformal Liouville description of the region in the vicinity of the singularity of the metric of the \( 2d \, N = 4 \) Higgs branch. In this region, the string coupling constant \( g_s = e^{\phi} \) blows up as the Liouville field \( \phi \) goes to infinity [17,18]. In an attempt towards an interpretation of the degenerate \( A_r \) singularity carried by eqs (1), we have given field theoretical arguments suggesting that the metric of the moduli space near the Higgs singularity might be described by a \( N = 4 \) conformal \( SU(r+1) \) Toda theory in two dimensions. This observation needs however a detailed study. In this regards a project of checking this observation for the case of the \( sp(2) \) gauge group is understudy [55].

\textbf{Acknowledgements}

One of us(EHS) would like to thank Profs J.Shinar and J.Vary for kind hospitality at IITAP, Iowa State University; where a part of this work is done. AB would like to thank the organizers of the Spring Workshop on Superstrings and related Matters (March 1999), The Abdus Salam International Centre for Theoretical Physics Trieste ,Italy, for hospitality.
This research work has been supported by the program PARS number 372-98 CNR.

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