Some Studies on Multidimensional Fourier Theory for Hilbert Transform, Analytic Signal and AM–FM Representation

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Abstract
In this paper, we propose the notion of Fourier frequency vector (FFV) which is inherently associated with the multidimensional (MD) Fourier representation (FR) of a signal. The proposed FFV provides physical meaning to the so-called negative frequencies in the MD-FR that in turn yields MD spatial and MD space-time series analysis. The one-dimensional Hilbert transform (1D-HT) and associated 1D analytic signal (1D-AS) of an 1D signal are well established; however, their true generalization to an MD signal, which possess all the properties of 1D case, are not available in the literature. To achieve this, we observe that in MD-FR the complex exponential representation of a sinusoidal function always yields two frequencies, namely negative frequency corresponding to positive frequency and vice versa. Thus, using the MD-FR, we propose MD-HT and associated MD analytic signal (AS) as a true generalization of the 1D-HT and 1D-AS, respectively, and obtain an explicit expression for the analytic image computation by 2D discrete Fourier transform (2D-DFT). We also extend the Fourier decomposition method for 2D signals that decomposes an image into a set of amplitude-modulated and frequency-modulated (AM–FM) image components. We finally propose a single-orthant Fourier transform (FT) of real MD signals which computes FT in the first orthant, and values in rest of the orthants are obtained by simple conjugation defined in this study.

Keywords Fourier representation (FR) and Fourier frequency vector (FFV) · Hilbert transform (HT) and analytic signal (AS) · Single-orthant Fourier transform (SOFT) · Fourier decomposition method (FDM) · Linearly independent non-orthogonal yet energy-preserving (LINOEP) vectors
1 Introduction

The Fourier representation (FR) of a signal is the most important mathematical tool for modeling and analysis of engineering systems and physical phenomena. The FRs for the four distinct classes of signals are: (1) Fourier series (FS) for periodic continuous-time signals, (2) FT for aperiodic continuous-time signals, (3) discrete-time FT (DTFT) for aperiodic discrete-time signals, and (4) DTFS for periodic discrete-time signals. The one-dimensional (1D) Hilbert transform (1D-HT) and 1D analytic signal (1D-AS) [8] have been used extensively in signal processing and information theory since their introduction. Practically, the 1D-AS of a periodic signal is obtained by suppressing negative frequencies in the complex FS representation. Let \( g(x) \) be a periodic function of time (and/or space), i.e., \( g(x) = g(x + T_1), \forall x \in \mathbb{R} \), then the FS of \( g(x) \) is defined as

\[
g(x) = \sum_{k=0}^{\infty} [a_k \cos(k\omega_1 x) + b_k \sin(k\omega_1 x)],
\]

where \( \omega_1 = \frac{2\pi}{T_1} \), \( a_0 = \frac{1}{T_1} \int_{\frac{T_1}{2}}^{\frac{T_1}{2}} g(x) \, dx \), \( b_0 = 0 \), \( a_k = \frac{2}{T_1} \int_{\frac{T_1}{2}}^{\frac{T_1}{2}} g(x) \cos(k\omega_1 x) \, dx \) and \( b_k = \frac{2}{T_1} \int_{\frac{T_1}{2}}^{\frac{T_1}{2}} g(x) \sin(k\omega_1 x) \, dx \) for \( k \geq 1 \). The Fourier amplitude spectrum for a real-valued function is given by \( |F_0| = |a_0| \), \( |F_k| = \sqrt{a_k^2 + b_k^2} \) and phase spectrum by \( \angle F_k = \phi_k = \tan^{-1}(-b_k/a_k) \), which implies that \( F_k \) can be written as

\[
F_0 = a_0, \quad F_k = a_k - jb_k = \frac{2}{T_1} \int_{\frac{T_1}{2}}^{\frac{T_1}{2}} g(x)e^{-j\omega_1 x} \, dx.
\]

Thus, \( F_k = |F_k|e^{j\phi_k} \Leftrightarrow a_k = |F_k|\cos(\phi_k), b_k = |F_k|\sin(\phi_k) \) and (1) can be written as \( g(x) = \sum_{k=0}^{\infty} |F_k|\cos(k\omega_1 x - \phi_k) \).

In the Euler’s formula \( e^{j\phi} = \cos(\phi) + j\sin(\phi) \), the Hilbert observed and proved that (with \( \phi = \omega t \)) the function \( \sin(\phi) \) is the HT of function \( \cos(\phi) \). This yields the \( \pm T_2 \) radian phase-shift operator, which is a basic property of the HT. Using the Euler’s formula, \( \cos(\phi) = \frac{1}{2}[e^{j\phi} + e^{-j\phi}] \) and \( \sin(\phi) = \frac{1}{2j}[e^{j\phi} - e^{-j\phi}] \), (1) can be written as

\[
g(x) = \sum_{k=0}^{\infty} [c_k \exp(jk\omega_1 x) + c^*_k \exp(-jk\omega_1 x)] = \sum_{k=-\infty}^{\infty} [c_k \exp(jk\omega_1 x)],
\]

where \( c_0 = a_0 \) and \( c_k = \frac{(a_k - jb_k)}{2} = \frac{1}{T_1} \int_{\frac{T_1}{2}}^{\frac{T_1}{2}} g(x) \exp(-jk\omega_1 x) \, dx \) for \( k \neq 0 \). From (3), it is clear that the extra term, \( \exp(-jk\omega_1 x) \), which here corresponds to negative frequency, is introduced only due to complex exponential representation (CER) of sinusoids in (1); otherwise, this extra and redundant term is not required.

We consider the HT, for the sinusoidal functions, as the \( \frac{T}{2} \) radian phase delay operator, i.e.,

\[\Phi\]
\[ H\{\cos(\phi)\} = \cos\left(\phi - \frac{\pi}{2}\right) = \sin(\phi), \]
\[ H\{\sin(\phi)\} = \sin\left(\phi - \frac{\pi}{2}\right) = -\cos(\phi), \]

where \( \phi = k\omega_1 x \) for 1D, \( \phi = k\omega_1 x + l\omega_2 y \) for 2D and \( \phi = k_1\omega_1 x + \ldots + k_M\omega_M x_M \) for multidimensional (MD) function, the HT of a function \( g(x) \) is denoted as \( H\{g(x)\} = \hat{g}(x) \). The HT of constant \( (a_0) \) is zero, and it can be easily verified by (4) as \( H\{a_0\cos(0)\} = a_0\sin(0) = 0 \) and \( H\{a_0\sin(\pi/2)\} = -a_0\cos(\pi/2) = 0 \). The HT of complex sinusoidal (complex exponential) signal is obtained by the multiplication of \(-j\) with signal itself, e.g., \( H\{\exp(j\omega t)\} = \exp(j[\omega t - \pi/2]) = -j\exp(j\omega t) \). By using (4), we evaluate the HT of (1) and obtain

\[ \hat{g}(x) = \sum_{k=0}^{\infty} [a_k \sin(k\omega_1 x) - b_k \cos(k\omega_1 x)]. \]  

Using (1) and (5), or from (3), the AS is defined as [31]

\[ z(x) = 2\sum_{k=0}^{\infty} [c_k \exp(jk\omega_1 x)] = g(x) + j\hat{g}(x) = r(x)e^{j\phi(x)}, \]

where \( r(x) = [g^2(x) + \hat{g}^2(x)]^{1/2} \) is the magnitude envelope of a signal \( g(x) \), \( \phi(x) = \tan^{-1}\left[\frac{\hat{g}(x)}{g(x)}\right] \), real part of AS is the original signal, imaginary part \( \hat{g}(x) \) of AS is the HT of original signal \( g(x) \), real and imaginary parts of AS are orthogonal, i.e., \( \int_{-T_1/2}^{T_1/2} g(x)\hat{g}(x) \, dx = 0 \).

This 1D-AS (6) satisfies the following properties or conditions: (C1) the extra and redundant negative frequencies, introduced due to CER of sinusoids, are suppressed, i.e., \( k \) takes only positive values in (6), (C2) real part of AS is original signal and imaginary part \( \hat{g}(x) \) of AS is the HT of original signal \( g(x) \), real and imaginary parts of AS are orthogonal, and (C4) the magnitude envelope of a real-valued signal is obtained as the magnitude of its associated AS, which is the instantaneous amplitude of the AS.

**Observation** We observe that for real-valued periodic function \( g(x) \)

\[ \sum_{k=-\infty}^{\infty} [a_k \sin(k\omega_1 x) - b_k \cos(k\omega_1 x)] = 0, \]

and this can be easily proved from (3) by equating imaginary part to zero. Notice the difference between (5) which is the HT of a signal under consideration and obtained by summing over the range of values of \( k \) from zero to infinite, and (7) which is essentially zero when summation over \( k \) is from minus infinity to infinity.

The frequency, in Hertz (Hz), is defined as the number of events or cycles per second. The fundamental period, \( T_1 \), is the duration of one cycle and is the reciprocal of the fundamental frequency \( f_1 \), i.e., \( f_1 = \frac{1}{T_1} \). Thus, by definition, the frequency is a positive physical quantity. In real world, the negative frequency does not exist and the meaning of negative frequencies is only mathematical, and not physical.
The extension of the HT and AS to the two-dimensional (2D) case and their applications to image processing has been limited, due to the non-uniqueness of the MD-HT and MD-AS. This fact has led to a variety of definitions with different approaches, which tried to satisfy the 1D conditions in the 2D case, such as the conventional 2D-HT in the spatial domain [17] where the negative frequencies are not suppressed, directional HT and quaternionic 2D-AS [3,5,19], single-orthant 2D-HT [12] where real part of AS is not original signal, monogenic signal that is based on the Riesz transform instead of the HT [6], generalized radial HT [27], 2D-HT and its corresponding AS based on a combination of 1D-HTs [21]. These approaches have some useful applications; however, the 2D-AS obtained with them does not satisfy the all properties of 1D-AS. There are many interesting applications of the 2D-HT and the corresponding AS, such as edge detection [27], corner detection [17], phase congruency calculations [20], AM–FM image models [15], nonlinear spatiotemporal processing [7], biomedical image computing using AM–FM methods and models, image retrieval in ophthalmology [1], analysis [18,38], tracking [24,28,29], image interpolation [2], image retrieval in digital libraries [13], classification of fingerprints [25], repairing of damaged image textures [14], image segmentation [26], reconstruction [22,37] and medical imaging [23] including others.

The fundamental property of the AS, especially from the viewpoint of image processing and recognition, is the split of identity [6]. The AS in polar representation yields two local features, the instantaneous amplitude and instantaneous phase. These local features fulfill the property of invariance and equivariance [9], i.e., the local phase depends only on the local structure, and the local amplitude depends only on the local energy (square of amplitude). If these local features are a complete description of a signal, they are said to perform a split of identity [4]. The split of identity is valid only for band-limited signals with local zero mean property [6]. Thus, the split of identity is valid for all the sinusoidal functions and, hence, valid for the Fourier representation of a function. Therefore, the Fourier theory yields the AS representation of a signal that relies on an orthogonal decomposition of the structural information (local phase), and the energetic information (local amplitude).

In this study, for an MD signal analysis, we present the MD trigonometric Fourier series, as well as propose generalized MD-AS and MD-HT using the Fourier theory which satisfy the following properties: (P1) in the Fourier spectrum of the MD-AS, extra and redundant positive, negative or both frequencies introduced due to the CER of multidimensional sinusoids, are suppressed; (P2) in the MD-AS, real part is the original signal and imaginary part is the HT of real part; (P3) the real and imaginary parts of MD-AS are orthogonal, and (P4) the magnitude envelope of an MD real-valued signal is obtained as the magnitude of its associated MD-AS, which is the instantaneous amplitude of the MD-AS. These properties (P1 to P4) of the MD-AS are true and desired extension of the 1D-AS properties (C1 to C4). An explicit expression for the analytic image computation by 2D-DFT is presented in Appendix 1.

In the literature, there are various methods and applications [10,11,16,30,32–35] of 1D nonlinear and non-stationary time series representation. Recently, based on the Fourier theory, the Fourier decomposition method (FDM) for nonlinear and non-stationary time series analysis is proposed in [31,36]. We present an extension of the
FDM for image signal (2D data) and refer to it as 2D-FDM that yields multicomponent AM–FM image model.

In the MD signals, there are $2^M$ orthants and the traditional MD-FT computes values in $2^M/2$ orthants and values in other $2^M/2$ orthants can be obtained by simple conjugation. In this work, we propose MD single-orthant FT (SOFT) that computes values in the first orthant only and values in all other $(2^M-1)$ orthants can be obtained by simple conjugation defined in this study.

This paper is organized as follows: the multidimensional Fourier series, HT and AS are discussed in Sect. 2. The 2D-DTFT and associated AS are discussed in Sect. 3. The 2D-FDM for AM–FM image model is discussed in Sect. 4. Section 5 introduces the single-orthant MD Fourier transform. Numerical results and discussions are presented in Sect. 6. Section 7 presents conclusions.

2 The Multidimensional Fourier Series, Hilbert Transform and Analytic Signal

In this section, we discuss the 2D Fourier Series (2D-FS) and MD-FS and introduce the concept of the Fourier frequency vector (FFV) and obtain associated AS. Let $g(x, y)$ be a periodic (with period $T_1, T_2$) function of 2D space (or 1D space-time), i.e., $g(x + T_1, y) = g(x, y + T_2) = g(x, y)$, $\forall x, y \in \mathbb{R}$, then the 2D Fourier series (2D-FS) of $g(x, y)$ can be defined as

$$g(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left[ a_{k,l} \cos(k\omega_1 x + l\omega_2 y) + b_{k,l} \sin(k\omega_1 x + l\omega_2 y) \right]$$

$$+ \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} \left[ a_{k,l} \cos(k\omega_1 x + l\omega_2 y) + b_{k,l} \sin(k\omega_1 x + l\omega_2 y) \right], \quad (8)$$

where $\omega_1 = \frac{2\pi}{T_1} = 2\pi u$, $\omega_2 = \frac{2\pi}{T_2} = 2\pi v$. Since $l$ takes both negative and positive values, therefore individual spatial frequencies $l\omega_2$ can be positive as well as negative, but over all resultant spatial frequency (rad/m) is always positive, which is given by $\omega = \sqrt{(k\omega_1)^2 + (l\omega_2)^2}$ and negative sign only helps in determining the direction, $\theta = \tan^{-1}(k\omega_1/l\omega_2)$, of wave. Hence, with respect to Fourier theory, we refer to them $(k\omega_1, l\omega_2)$ as FFV that has a magnitude as well as direction and can be written as $\omega = [k\omega_1 \ l\omega_2]^T$. (In physics, it is similar to a ‘wave vector’ in multidimensional systems.) The coefficients $a_{0,0}$, $a_{k,l}$ and $b_{k,l}$ can be obtained by

$$a_{0,0} = \frac{1}{T_1T_2} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \int_{-\frac{T_2}{2}}^{\frac{T_2}{2}} g(x, y) \, dx \, dy,$$

$$a_{k,l} = \frac{2}{T_1T_2} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \int_{-\frac{T_2}{2}}^{\frac{T_2}{2}} g(x, y) \cos(k\omega_1 x + l\omega_2 y) \, dx \, dy,$$

$$b_{k,l} = \frac{2}{T_1T_2} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \int_{-\frac{T_2}{2}}^{\frac{T_2}{2}} g(x, y) \sin(k\omega_1 x + l\omega_2 y) \, dx \, dy. \quad (9)$$
The Fourier amplitude spectrum for a real-valued function is given by 

\[ |F_{0,0}| = |a_0,0|, |F_{k,l}| = \sqrt{a_{k,l}^2 + b_{k,l}^2} \]

and phase spectrum by \( \angle F_{k,l} = \phi_{k,l} = \tan^{-1}(-b_{k,l}/a_{k,l}) \), which implies that \( F_{k,l} \) can be written as

\[
F_{k,l} = \frac{2}{T_1 T_2} \int_{-T_2/2}^{T_2/2} \int_{-T_1/2}^{T_1/2} g(x, y) e^{-j(k\omega_1 x + l\omega_2 y)} \, dx \, dy = a_{k,l} - jb_{k,l}, \quad F_{0,0} = a_{0,0}.
\]

We can also consider (8) with (a) limits of double sum from \( k = -\infty \) to \( \infty, l = 0 \) to \( \infty \) and results would be same as there is no change in (9), (b) limits of double sum from \( k = -\infty \) to \( \infty, l = -\infty \) to \( \infty \) and values of \( a_{k,l} \) and \( b_{k,l} \) in (9) would get divided by two.

Discussion

There is a need for defining FFV in multidimensional Fourier representations. In order to demonstrate this need, let \( g(x, y) = \cos(3\omega_1 x - 4\omega_2 y) \) be a periodic signal (standing wave in 2D space, like image). From (9), we find only two solutions, \( a_{k,l} = 1 \) if \( (k = 3, l = -4) \) or \( (k = -3, l = 4) \); \( b_{k,l} = 0 \) for all \( k, l \). Interestingly, both solutions are same as \( \cos(3\omega_1 x - 4\omega_2 y) = \cos(-3\omega_1 x + 4\omega_2 y) \). In situation like this, we cannot avoid negative values of \( l \) (or \( k \)), because there are no solutions with only positive values (i.e., nonnegative integers \( N_0 = \{0, 1, 2, \ldots\} \)) of \( k \) and \( l \), which represent Fourier frequencies. This kind of situation does not arise in 1D Fourier representation as \( k \) takes values only zero onward in (1). Thus, for the multidimensional Fourier representation, we define the concept of FFV and in this case FFV is \( \omega = [3\omega_1 - 4\omega_2]^T \), which has a magnitude and direction, like any other vector. The magnitude of FFV is frequency \( \omega = |\omega| = \sqrt{(3\omega_1)^2 + (4\omega_2)^2} \), which is always positive by definition itself. The CER of this function, \( \cos(3\omega_1 x - 4\omega_2 y) = e^{j(3\omega_1 x - 4\omega_2 y)} + e^{-j(3\omega_1 x - 4\omega_2 y)} \), introduces the extra frequencies by the second term \( e^{-j(3\omega_1 x - 4\omega_2 y)} \).

Observation

We can also use (8) for space-time \((x, t)\) series analysis, e.g., 1D wave equation, \( g(x, t) = \cos(k\omega_1 t - l\omega_2 x) \), where \( k\omega_1 = \frac{2\pi k}{T_1} = 2\pi kf_1 = 2\pi f \), wave vector (or FFV) \( \omega = l\omega_2 \), wave number (or spatial frequency) \( |\omega| = \frac{2\pi|l|}{\lambda_2} = \frac{2\pi f}{\lambda} \) and phase velocity \( v_p = \frac{k\omega_1}{|\omega|} = f\lambda \).

By the Euler’s formula, we know that \( \cos(\phi) = \frac{1}{2}[e^{j\phi} + e^{-j\phi}] \) and \( \sin(\phi) = \frac{1}{2j}[e^{j\phi} - e^{-j\phi}] \); using these values, we can write (8) as

\[
g(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left[ \frac{(a_{k,l} - jb_{k,l})}{2} \exp[j(k\omega_1 x + l\omega_2 y)] + \frac{(a_{k,l} + jb_{k,l})}{2} \exp[-j(k\omega_1 x + l\omega_2 y)] \right]
\]

\[
+ \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} \left[ \frac{(a_{k,l} - jb_{k,l})}{2} \exp[j(k\omega_1 x + l\omega_2 y)] + \frac{(a_{k,l} + jb_{k,l})}{2} \exp[-j(k\omega_1 x + l\omega_2 y)] \right].
\]
This equation can be written as

\[
g(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left[ c_{k,l} \exp[j(k \omega_1 x + l \omega_2 y)] + c_{k,l}^* \exp[-j(k \omega_1 x + l \omega_2 y)] \right]
\]

\[
+ \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} \left[ c_{k,l} \exp[j(k \omega_1 x + l \omega_2 y)] + c_{k,l}^* \exp[-j(k \omega_1 x + l \omega_2 y)] \right]
\]

\[
\Leftrightarrow g(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{k,l} \exp[j(k \omega_1 x + l \omega_2 y)], \quad (12)
\]

where \( c_{0,0} = a_{0,0} \) and \( c_{k,l} = \frac{(a_{k,j} - j b_{k,j})}{2} \) for \( k, l = -\infty, \ldots, 0, 1, \ldots, \infty \). Hence, from (9) we can write

\[
c_{k,l} = \frac{1}{T_1 T_2} \int_{-T_2}^{T_2} \int_{-T_1}^{T_1} g(x, y) \exp(-j[k \omega_1 x + l \omega_2 y]) \, dx \, dy. \quad (13)
\]

From (12), we observe that the extra term, \( \exp[-j(k \omega_1 x + l \omega_2 y)] \), which may correspond to positive, negative, or both frequencies, is introduced only due to CER of sinusoids in (8); otherwise, this extra and redundant term is not required.

By using (4), we evaluate the MD-HT of (8) and obtain

\[
\hat{g}(x, y) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left[ a_{k,l} \sin(k \omega_1 x + l \omega_2 y) - b_{k,l} \cos(k \omega_1 x + l \omega_2 y) \right]
\]

\[
+ \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} \left[ a_{k,l} \sin(k \omega_1 x + l \omega_2 y) - b_{k,l} \cos(k \omega_1 x + l \omega_2 y) \right]. \quad (14)
\]

Using (8) and (14), or from (11), we define multidimensional AS (MD-AS)

\[
z(x, y) = 2 \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} c_{k,l} \exp[j(k \omega_1 x + l \omega_2 y)]
\]

\[
+ 2 \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} c_{k,l} \exp[j(k \omega_1 x + l \omega_2 y)],
\]

\[
= g(x, y) + j \hat{g}(x, y) = r(x, y)e^{j \phi(x, y)}, \quad (15)
\]

where \( r(x, y) = \left[ g^2(x, y) + \hat{g}^2(x, y) \right]^{1/2} \) is a magnitude envelope of a signal \( g(x, y) \), \( \phi(x, y) = \tan^{-1}\left( \frac{\hat{g}(x, y)}{g(x, y)} \right) \), real part of MD-AS is original signal, imaginary part is the MD-HT of original signal, and MD-AS has only originally present frequencies. Clearly, the CER of sinusoidal function always yields two frequencies, negative frequency corresponding to positive frequency and vice versa, in the Fourier spectrum. Hence, MD-AS suppresses the extra and redundant positive, negative, or both fre-
quencies, introduced due to CER of multidimensional Fourier spectrum and satisfies all the MD-AS properties P1 to P4.

**Observation** We observe that for real-valued periodic function \( g(x, y) \)

\[
\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left[ a_{k,l} \sin(k\omega_1 x + l\omega_2 y) - b_{k,l} \cos(k\omega_1 x + l\omega_2 y) \right] = 0, \tag{16}
\]

and this can be easily proved from (12) by equating imaginary part to zero.

The above discussion is extended for MD-FS and presented in Appendix 2. It is very convenient to take the first frequency always positive (may be related to 1D time or space) and rest of the frequencies (related to MD space) may be positive or negative or both, which is also clear from (1), (8), (44) and (47), and this convention has been used throughout in this paper.

From the above discussions (1D equations (1) and (6); 2D equations (8) and (15)), we propose the following result:

**Proposition 1** Let \( g(x) \) be a real-valued signal and \( z(x) = g(x) + j\hat{g}(x) \) is the AS representation of \( g(x) \). Then real-valued Fourier spectrum (RVFS) of \( g(x) \) and complex-valued Fourier spectrum (CVFS) of \( z(x) \) are same, i.e., \( G_r(\omega) = Z(\omega) \), where \( G_r(\omega) \) is RVFS of \( g(x) \), \( Z(\omega) \) is CVFS of \( z(x) \), \( x = [x_1 \ldots x_M]^T \) and \( \omega = [\omega_1 \ldots \omega_M]^T \).

For a 2D periodic signal, the RVFR is given by (8) and (9), the CVFR is given by (12) and (13).

**Example 1** Let \( g(x, y) = \cos(\omega_1 x + \omega_2 y) \) be a 2D periodic signal. We can write \( \cos(\omega_1 x + \omega_2 y) = \frac{1}{2} [e^{j(\omega_1 x+\omega_2 y)} + e^{-j(\omega_1 x+\omega_2 y)}] \), \( e^{j(\omega_1 x+\omega_2 y)} \), imaginary part is phase delayed (by \( \pi/2 \) radian) version of the real part, and \( e^{-j(\omega_1 x+\omega_2 y)} \), imaginary part is phase advanced (by \( \pi/2 \) radian) version of the real part. All cases, for example, are shown in Fig. 1 for the first quadrant: real part \( \rightarrow \) phase delay \( (\omega_1 x + \omega_2 y - \pi/2) \) \( \rightarrow \) imaginary part; the third quadrant: real part \( \rightarrow \) phase delay \( (-\omega_1 x - \omega_2 y - \pi/2) \) \( \rightarrow \) imaginary part; the fourth quadrant: real part \( \rightarrow \) phase delay \( (-\omega_1 x + \omega_2 y - \pi/2) \) or phase advance \( (\omega_1 x - \omega_2 y + \pi/2) \) \( \rightarrow \) imaginary part; and the second quadrant: real part \( \rightarrow \) phase delay \( (-\omega_1 x + \omega_2 y - \pi/2) \) or phase advance \( (\omega_1 x - \omega_2 y + \pi/2) \) \( \rightarrow \) imaginary part.

### 3 The 2D-DTFT and Associated Analytic Signal

Let \( g[m, n] \) be a non-periodic and real-valued function of time/space, then the 2D discrete-time FT (2D-DTFT) of \( g[m, n] \) is defined as

\[
G(\omega_1, \omega_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g[m, n] \exp(-j[\omega_1 m + \omega_2 n]) = G_r(\omega_1, \omega_2) + jG_i(\omega_1, \omega_2), \tag{17}
\]

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Fig. 1 (Section 2) Relationship between real and imaginary parts in four quadrants (I, II, III and IV) for the Fourier expansion Example 1: (a) I quadrant, \( \sin(\omega_1 x + \omega_2 y) = \cos(\omega_1 x + \omega_2 y - \pi/2) \), (b) II quadrant, \(-\sin(\omega_1 x - \omega_2 y) = \cos(\omega_1 x - \omega_2 y + \pi/2)\), (c) III quadrant, \(-\sin(\omega_1 x + \omega_2 y) = \cos(\omega_1 x + \omega_2 y + \pi/2)\), and (d) IV quadrant, \(\sin(\omega_1 x - \omega_2 y) = \cos(\omega_1 x - \omega_2 y - \pi/2)\). Clearly, in the I and IV (II and III) quadrants, the imaginary part is \(\pi/2\) phase delayed (advanced) version of the real part of the complex exponential.

where \(G_r(\omega_1, \omega_2)\) and \(G_i(\omega_1, \omega_2)\) are real and imaginary parts of the 2D-DTFT, respectively. The 2D inverse DTFT (2D-IDTFT) is defined as

\[
g[m, n] = \frac{1}{2\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2. \tag{18}
\]

It is easy to show that, from (17), \(G(-\omega_1, -\omega_2) = G^*(\omega_1, \omega_2), G(-\omega_1, \omega_2) = G^*(\omega_1, -\omega_2)\). We rewrite (18) as

\[
g[m, n] = \frac{1}{2\pi} \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} \int_{0}^{\pi} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2 \\
+ \int_{0}^{\pi} \int_{0}^{\pi} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2 \\
+ \int_{0}^{\pi} \int_{-\pi}^{0} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2 \\
+ \int_{-\pi}^{0} \int_{-\pi}^{0} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2 \right]. \tag{19}
\]

In this equation, the first term (denoted as \(z_1[m, n]\) and \(0 \leq \omega_1 \leq \pi, 0 \leq \omega_2 \leq \pi\)) is complex conjugate of the third term \((z_3^*[m, n]\) and \(-\pi \leq \omega_1 \leq 0, -\pi \leq \omega_2 \leq 0\) and the second term (denoted as \(z_2[m, n]\) and \(-\pi \leq \omega_2 \leq 0, 0 \leq \omega_1 \leq \pi\)) is complex conjugate of the fourth term \((z_4^*[m, n]\) and \(0 \leq \omega_2 \leq \pi, -\pi \leq \omega_1 \leq 0\)). As \(g[m, n]\) is real-valued function, we can write

\[
g[m, n] = Re\{z_{14}[m, n]\} = Re\{z_{23}[m, n]\} = Re\{z_{12}[m, n]\} = Re\{z_{34}[m, n]\}, \tag{20}
\]
where \( Re[.] \) denotes real part of AS

\[
\begin{align*}
    z_{14}[m, n] &= 2(z_1[m, n] + z_2[m, n]) = g[m, n] + j\hat{g}_{14}[m, n], \\
    z_{23}[m, n] &= 2(z_1^*[m, n] + z_2^*[m, n]) = g[m, n] + j\hat{g}_{23}[m, n], \\
    z_{12}[m, n] &= 2(z_1[m, n] + z_2^*[m, n]) = g[m, n] + j\hat{g}_{12}[m, n], \\
    z_{34}[m, n] &= 2(z_1^*[m, n] + z_2[m, n]) = g[m, n] + j\hat{g}_{34}[m, n],
\end{align*}
\]  

(21)

where \( \hat{g}_{14}[m, n] = -\hat{g}_{23}[m, n] \), \( \hat{g}_{12}[m, n] = -\hat{g}_{34}[m, n] \) and subscripts denote the quadrants of the Fourier domain considered in AS representation. If we write \( z_1[m, n] = x_1[m, n] + j\hat{g}_1[m, n] \) and \( z_2[m, n] = x_2[m, n] + j\hat{g}_2[m, n] \), then \( z_{12}[m, n] = (x_1[m, n] + x_2[m, n]) + j(\hat{g}_1[m, n] + \hat{g}_2[m, n]) \) and \( z_{14}[m, n] = (x_1[m, n] + x_2[m, n]) + j(\hat{g}_1[m, n] - \hat{g}_2[m, n]) \), where \( g[m, n] = (x_1[m, n] + x_2[m, n]) \), \( \hat{g}_{12}[m, n] = (\hat{g}_1[m, n] + \hat{g}_2[m, n]) \) and \( \hat{g}_{14}[m, n] = (\hat{g}_1[m, n] - \hat{g}_2[m, n]) \).

The DTFT of these AS, with their frequency supports, can be written as

\[
\begin{align*}
    Z_{14}(\omega_1, \omega_2), \quad &\omega_1 \in [0, \pi], \quad \omega_2 \in [-\pi, \pi], \\
    Z_{23}(\omega_1, \omega_2), \quad &\omega_1 \in [-\pi, 0], \quad \omega_2 \in [-\pi, \pi], \\
    Z_{12}(\omega_1, \omega_2), \quad &\omega_1 \in [-\pi, \pi], \quad \omega_2 \in [0, \pi], \\
    Z_{34}(\omega_1, \omega_2), \quad &\omega_1 \in [-\pi, \pi], \quad \omega_2 \in [-\pi, 0].
\end{align*}
\]

(22)

Notice that with this definition of AS, the extra negative frequencies \( \omega_1 \) and \( \omega_2 \) are suppressed in AS \( z_{14}[m, n] \) and \( z_{12}[m, n] \), respectively, and this preserves the desired property for the AS. The AS \( z_{23}[m, n] \) and \( z_{34}[m, n] \) are the complex conjugate of \( z_{14}[m, n] \) and \( z_{12}[m, n] \), respectively. From the analytic image point of view, all four imaginary parts of AS in (21) yield different images. To obtain the original signal, we need to know the values of 2D-DTFT in first and second or third and fourth, first and fourth or second and third quadrants, which is also clear from the above discussions with Eqs. (21) and (22). It is to be noted that the only AS \( z_{14}[m, n] \) coincides with the proposed AS in (15) and hence its imaginary part, \( Im[z_{14}[m, n]] \), coincides with the proposed HT in (14).

**Example 2** Time/space frequency analysis of 2D unit sample sequence. The unit sample sequence defined as \( \delta[m-m_0, n-n_0] = 1 \) at \( m = m_0, n = n_0 \) and zero otherwise. We obtain the analytic representation of \( g[m, n] = \delta[m-m_0, n-n_0] \iff G(\omega_1, \omega_2) = \exp(-j[\omega_1 m_0 + \omega_2 n_0]) \) as \( z_{14}[m, n] = 2z_1[m, n] + 2z_2[m, n] = \left[ \frac{\sin(\pi(m-m_0))}{\pi(m-m_0)} \right] \left[ \frac{\sin(\pi(n-n_0))}{\pi(n-n_0)} \right] + j\left[ \frac{(1-\cos(\pi(m-m_0)))\sin(\pi(n-n_0))}{\pi(m-m_0)\pi(n-n_0)} \right] \), as given in (20), its real part is original signal, i.e., \( g[m, n] = \left[ \frac{\sin(\pi(m-m_0))}{\pi(m-m_0)} \right] \left[ \frac{\sin(\pi(n-n_0))}{\pi(n-n_0)} \right] = \delta[m-m_0, n-n_0] \).

We obtain the phase of \( z_{14} \) as \( \phi[m, n] = \frac{\pi}{2}(m-m_0) \) and hence \( \omega_1[m, n] = \frac{\pi}{2} \), which corresponds to half of the Nyquist frequency, and other frequency is \( \omega_2[m, n] = 0 \). Similarly, we obtain AS \( z_{12}[m, n] = 2z_1[m, n] + 2z_2^*[m, n] = \left[ \frac{\sin(\pi(m-m_0))}{\pi(m-m_0)} \right] \left[ \frac{\sin(\pi(n-n_0))}{\pi(n-n_0)} \right] + j\left[ \frac{\sin(\pi(m-m_0))(1-\cos(\pi(n-n_0)))}{\pi(m-m_0)\pi(n-n_0)} \right] \), as given in (20), its real part is original signal, i.e., \( g[m, n] = \left[ \frac{\sin(\pi(m-m_0))}{\pi(m-m_0)} \right] \left[ \frac{\sin(\pi(n-n_0))}{\pi(n-n_0)} \right] = \delta[m-m_0, n-n_0] \).

We obtain the phase of \( z_{12} \) as \( \phi[m, n] = \frac{\pi}{2}(n-n_0) \) and hence \( \omega_2[m, n] = \frac{\pi}{2} \), which corresponds to half of the Nyquist frequency, and other frequency is \( \omega_1[m, n] = 0 \).
If $z[m, n]$ is defined as four times of the first quadrant, where $0 \leq \omega_1 \leq \pi$, $0 \leq \omega_2 \leq \pi$, then we can observe that

$$g[m, n] \neq Re[z[m, n]],$$

(23)

where $Re[.]$ denotes real part of $AS$

$$z[m, n] = \frac{1}{\pi} \int_{0}^{\pi} \int_{0}^{\pi} G(\omega_1, \omega_2) \exp(j\omega_1 m + j\omega_2 n) d\omega_1 d\omega_2.$$

(24)

Notice that with this definition of $AS$, the negative frequencies are suppressed but real part of $AS$ is not original signal and this does not provide a phase delay of $-\pi/2$ radian.

4 The 2D Fourier Decomposition Method and AM–FM Image Model

We propose to use the following 2D Fourier decomposition method (2D-FDM) to obtain multicomponent AM–FM image model defined as

$$g(x, y) = \sum_{i=1}^{M} g_i(x, y) + w(x, y),$$

(25)

where $w(x, y)$ is a noise representing any residue (constant or trend) components, and the $g_i(x, y) = a_i(x, y) \cos(\phi_i(x, y))$ are $M$ monocomponent (zero mean narrow band) non-stationary image signals that represent the 2D Fourier intrinsic band functions (2D-FIBFs) for an image signal. The analytic representation of (25), by excluding $w(x, y)$, can be written as

$$z(x, y) = \sum_{i=1}^{M} \left[ g_i(x, y) + j\hat{g}_i(x, y) \right] = \sum_{i=1}^{M} a_i(x, y) e^{j\phi_i(x, y)}.$$

(26)

We define 2D monocomponent signals that follow either or both of the following conditions

$$\omega_{1i}(x, y) = \frac{\partial \phi_{i}(x, y)}{\partial x} \geq 0, \quad \forall x, y,$$

(27a)

$$\omega_{2i}(x, y) = \frac{\partial \phi_{i}(x, y)}{\partial y} \geq 0, \quad \forall x, y.$$

(27b)

**Example 3** For example, we consider two 2D monocomponent signals (a) $g(x, y) = \cos(3\omega_1 x - 4\omega_2 y)$ which follows (27a) or (27b), and (b) $g(x, y) = \cos(3\omega_1 x + 4\omega_2 y)$ which follows both (27a) and (27b).

Using the analytic function $Z_{14}[m, n]$ from (21), we obtain discrete version of (26) as
\[
\frac{1}{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2 \\
= \sum_{i=1}^{M} a_i[m, n] \exp(j\phi_i[m, n]), \tag{28}
\]

where (with \(\omega_{10} = 0, \omega_{1M} = \pi\))

\[
a_i[m, n] \exp(j\phi_i[m, n]) = \frac{1}{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{\omega_{1(i-1)}}^{\omega_{1i}} G(\omega_1, \omega_2) \exp(j[\omega_1 m + \omega_2 n]) \, d\omega_1 \, d\omega_2, \tag{29}
\]

for \(i = 1, \ldots, M\). To obtain minimum number of analytic Fourier intrinsic band functions (AFIBFs) in low- to high-frequency scan (LTH-FS), for each \(i\), start with \(\omega_1(i-1)\), increase and select the maximum value of \(\omega_{1i}\) such that \(\omega_1(i-1) \leq \omega_{1i} \leq \pi\) and phase \(\phi_i[m, n]\) is a monotonically increasing function with respect to (w.r.t.) \(\omega_{1i}[m, n]\), i.e.,

\[
a_i[m, n] \geq 0, \quad \omega_{1i}[m, n] = (\phi_i[m + 1, n] - \phi_i[m, n]) \geq 0, \tag{30}
\]

for all \(m, n\). This decomposition is steered in the \(x\) direction as many frequency components are responding to \(x\) direction (as it can be seen in Fig. 9). Similarly, in high- to low-frequency scan (HTL-FS), the lower and upper limits of integration in (29) will change to \(\omega_{1i}\) to \(\omega_1(i-1)\), respectively, with \(\omega_{10} = \pi, \omega_{1M} = 0\), and we can obtain minimum number of AFIBFs by selecting the minimum value of \(\omega_{1i}\) such that \(0 \leq \omega_{1i} \leq \omega_1(i-1)\) and (30) is satisfied. As we have represented \(Z_{14}[m, n]\) by Eqs. (28), (29), (30) and considered the decomposition w.r.t. only \(\omega_1\), the same way, we can represent \(Z_{12}[m, n]\) and consider decomposition w.r.t. only \(\omega_2\).

We can obtain decomposition w.r.t. \(\omega_1\) and \(\omega_2\) simultaneously by

\[
a_i[m, n] \exp(j\phi_i[m, n]) = \text{IDTFT}\{G(\omega_1, \omega_2)H_i(\omega_1, \omega_2)\}, \tag{31}
\]

where \(H_i(\omega_1, \omega_2) \in \mathbb{R}\) is the \(i\)th zero-phase low-pass filter (ZPLPF) or zero-phase high-pass filter (ZPHPF) in the frequency domain which can be a Butterworth, Gaussian, rectangular or ideal filter. This decomposition is not steered to any direction (as it can be seen in Fig. 10). The multivariate FDM (MFDM) for 1D signals, based on zero-phase filtering (ZPF), is proposed in [31]. Here, we propose a 2D-FDM Algorithm 2 to decompose an image into a set of linearly independent, non-orthogonal yet energy-preserving (LINOEP) [33,36] or orthogonal AM–FM components such that they follow the model defined in (25). The desired cutoff frequencies \(f_{ci}\) are used in ZPLPF or ZPHPF for the signal decomposition.

### 5 Single-Orthant Fourier Transform

In this section, we consider multidimensional (2D and its extension to general MD) DTFT of real-valued multidimensional signals. This approach evaluates DTFT of a
real-valued signal in only the first orthant, and values in rest of the orthant are obtained by simple conjugation defined (for 2D and MD cases) as follows: Let \( g[m, n] \) be a non-periodic and real-valued function of time/space, then we define the 2D single-orthant discrete-time FT (2D-SODTFT) of \( g[m, n] \) as

\[
X(\omega_1, \omega_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g[m, n] \exp(-i\omega_1 m - j\omega_2 n) \tag{32}
\]

with

\[
i^2 = -1, \quad j^2 = -1 \quad \text{and} \quad ij = ji, \tag{33}
\]

where \( i \) and \( j \) are purely imaginary numbers and represent phase corresponding to frequencies \( \omega_1 \) and \( \omega_2 \), respectively.

Since there are two frequencies \( \omega_1 \) and \( \omega_2 \) and integration of frequency is phase, it is logical and natural to define two phases corresponding to two frequencies. Here, we define two conjugation, bar (\( \overline{\cdot} \)) and star (\( * \)) corresponding to \( i \) and \( j \), respectively. Hence, we write (32) as

\[
X(\omega_1, \omega_2) = X_r(\omega_1, \omega_2) + ij X_{ij}(\omega_1, \omega_2) + iX_i(\omega_1, \omega_2) + jX_j(\omega_1, \omega_2) \tag{34}
\]

where the first term of the right side of this equation is real part, the second term is product of two imaginary parts corresponding to \( i \) and \( j \), third and fourth terms are imaginary parts. For real-valued \( g[m, n] \), it is easy to show, from (32), that (a) \( \overline{X}(\omega_1, \omega_2) = X(-\omega_1, \omega_2) \), (b) \( \overline{X}^*(\omega_1, \omega_2) = X(-\omega_1, -\omega_2) \), (c) \( X^*(\omega_1, \omega_2) = X(\omega_1, -\omega_2) \). This clearly indicates that, if we know the value of \( X(\omega_1, \omega_2) \) in the first quadrant \((0 \leq \omega_1 \leq \pi \text{ and } 0 \leq \omega_2 \leq \pi)\) only, then we can obtain the value of \( X(\omega_1, \omega_2) \) in all the quadrants.

It is easy to observe that the traditional 2D-DTFT (17) and proposed 2D-SODTFT (32) are same when \( i = j \). Using (17), (33) and (34), we obtain the following relations (one-to-one and onto mapping) for each of the four quadrants \( G_1(\omega_1, \omega_2), G_2(\omega_1, \omega_2), G_3(\omega_1, \omega_2) \) and \( G_4(\omega_1, \omega_2) \) of traditional 2D-DTFT in terms of the proposed 2D-SODTFT as

\[
G_1(\omega_1, \omega_2) = [X_r(\omega_1, \omega_2) - X_{ij}(\omega_1, \omega_2)] + j[X_i(\omega_1, \omega_2) + X_j(\omega_1, \omega_2)],
\]

\[
0 \leq \omega_1 \leq \pi, \quad 0 \leq \omega_2 \leq \pi,
\]

\[
G_2(\omega_1, \omega_2) = [X_r(\omega_1, \omega_2) + X_{ij}(\omega_1, \omega_2)] + j[-X_i(\omega_1, \omega_2) + X_j(\omega_1, \omega_2)],
\]

\[
-\pi \leq \omega_1 \leq 0, \quad 0 \leq \omega_2 \leq \pi,
\]

\[
G_3(\omega_1, \omega_2) = [X_r(\omega_1, \omega_2) - X_{ij}(\omega_1, \omega_2)] - j[X_i(\omega_1, \omega_2) + X_j(\omega_1, \omega_2)],
\]

\[
-\pi \leq \omega_1 \leq 0, \quad -\pi \leq \omega_2 \leq 0,
\]

\[
G_4(\omega_1, \omega_2) = [X_r(\omega_1, \omega_2) + X_{ij}(\omega_1, \omega_2)] - j[-X_i(\omega_1, \omega_2) + X_j(\omega_1, \omega_2)],
\]

\[
0 \leq \omega_1 \leq \pi, \quad -\pi \leq \omega_2 \leq 0,
\]

(35)

where the third quadrant is complex conjugate of the first one, and the fourth quadrant is complex conjugate of the second one.
Now, it is straightforward to generalize the single-orthant 2D-DTFT case and obtain the single-orthant MD-DTFT. Let \( g[n_1, \ldots, n_M] \) be a non-periodic and real-valued function, then we define the MD discrete-time FT (MD-DTFT) of \( g[n_1, \ldots, n_M] \) as

\[
X(\omega_1, \ldots, \omega_M) = \sum_{n_1=-\infty}^{\infty} \ldots \sum_{n_M=-\infty}^{\infty} g[n_1, \ldots, n_M] \exp(-j\omega_1 n_1-\ldots-jM\omega_M n_M)
\]  

such that

\[
j_l^2 = -1, \quad l = 1, \ldots, M,
\]

where \( j_1 \) to \( j_M \) are purely imaginary numbers and represent phase corresponding to frequencies \( \omega_1 \) to \( \omega_M \), respectively. Since, there are \( M \) frequencies \( \omega_1 \) to \( \omega_M \) and integration of frequency is phase, hence it is logical and natural to define \( M \) phases corresponding to \( M \) frequencies. Here, we define \( M \) conjugations for each \( j_l \) to \( j_M \), respectively.

For MD signals, there would be \( 2^M \) orthants and the traditional MD-FT requires computation of values in \( 2^M/2 \) orthants as values in other \( 2^M/2 \) orthants can be obtained by simple conjugation. In the proposed MD single-orthant FT (SOFT), we are required to compute values in the first orthant only and values in all other orthants can be obtained by simple conjugation as defined above.

It is to be noted that all the above discussions have been done with MD-FS, MD-DTFT and associated MD-AS. It is very easy to extend this discussion for all other variants of FRs and associated HTs.

6 Numerical Results and Discussions

To demonstrate efficacy of the proposed HT and associated AS, which are defined through the FR in this paper and satisfy all the properties of 1D-AS, we performed the following simulations and compare the results with some HTs available in the literature. We also present the decomposition of an image into AM–FM components using the 2D-FDM and finally show the computation of SOFT with a simple example.

6.1 Amplitude Demodulation Using the Hilbert Transform and Fourier Representation

Let \( g_m(x, y) = A_m \cos(3\omega_1 x - 4\omega_2 y) \) be a modulating signal and \( g_c(x, y) = A_c \cos(10\omega_1 x + 8\omega_2 y) \) be a carrier signal, Fig. 2 (middle), with \( \omega_1 = \frac{2\pi}{T_1} \), \( \omega_2 = \frac{2\pi}{T_2} \), \( T_1 = 3 \), \( T_2 = 4 \), \( A_c = 1 \), \( A_m = 1 \) and spatial sampling frequency \( \omega_s = 2\pi f_s = 2\pi \times 30 \) (rad/m). The AM-modulated signal can be written as \( g_{AM}(x, y) = [A_m + g_m(x, y)]g_c(x, y) \), shown in Fig. 2 (bottom). The envelope of this modulated signal is given by \( g_{env}(x, y) = [A_m + g_m(x, y)] \), which is shown in Fig. 2 (top). Figure 3 shows the perfectly recovered AM signal \( g_{AM}(x, y) \) (top), the HT of AM signal \( g_H(x, y) \) (middle) and the envelope signal \( \sqrt{g_{AM}^2(x, y) + g_H^2(x, y)} \) (bottom). This example clearly demonstrates that the proposed HT based on the Fourier
representation can be used to recover envelope and hence modulating signal from the AM signal.

### 6.2 Examples of 2D Analytic Signal Representations

First, we consider the two-dimensional harmonic signal \( g(x_1, x_2) = \cos(\omega_1 x_1) \cos(\omega_2 x_2) \). Using (8), the Fourier series expansion of this signal is given by
\[
g(x_1, x_2) = \frac{1}{2} \cos(\omega_1 x_1 + \omega_2 x_2) + \frac{1}{2} \cos(\omega_1 x_1 - \omega_2 x_2),
\]
which implies that \( g(x_1, x_2) \) signal is sum of two sinusoidal signals of phases \( (\omega_1 x_1 + \omega_2 x_2) \) and \( (\omega_1 x_1 - \omega_2 x_2) \), respectively. Using (4) or (14), and (15), we obtain 2D-AS \( z(x_1, x_2) = \frac{1}{2} e^{j(\omega_1 x_1 + \omega_2 x_2)} + \frac{1}{2} e^{j(\omega_1 x_1 - \omega_2 x_2)}. \)
This 2D-AS $z(x_1, x_2)$ has amplitude dependent on $x_1, x_2$ and a nonlinear phase. This shows that the superposition of constant amplitude and linear phase signals result in a signal with variable amplitude and a nonlinear phase, whereas the 2D-AS with single-quadrant spectra \cite{12} is given by $\psi(x_1, x_2) = e^{j(\omega_1 x_1 + \omega_2 x_2)}$ that shows $g(x_1, x_2)$ has a constant amplitude and a linear phase in both coordinates.

Some other examples are presented in Tables 1 and 2 for the comparative study of the proposed HTs $\hat{g}(x_1, x_2)$, which are based on the Fourier theory and phase delay, with the four HTs \cite{12}, namely the partial HT (PHT) in $x_1$ direction $\hat{g}_{x_1}(x_1, x_2)$ with 2D-AS $z(x_1, x_2) = g(x_1, x_2)$, partial HT (PHT) in $x_2$ direction $\hat{g}_{x_2}(x_1, x_2)$ with 2D-AS $z(x_1, x_2) = g(x_1, x_2)$, total HT (THT) $\hat{g}_T(x_1, x_2)$ with 2D-AS $z(x_1, x_2) = g(x_1, x_2)$, and single-orthant HT (SOHT) $\hat{g}_{SO}(x_1, x_2)$ with 2D-AS $z(x_1, x_2) = g(x_1, x_2) - \hat{g}_T(x_1, x_2) + j[\hat{g}_{x_1}(x_1, x_2) + \hat{g}_{x_2}(x_1, x_2)]$. From these examples, it is clear that the only proposed HT with 2D-AS, presented in Table 2, is providing correct amplitude and phase in all the cases.

Figure 4 shows a signal $g(x, y)$ and its proposed HT $\hat{g}(x, y)$, partial HT in $x$ direction $\hat{g}_x(x, y)$ and partial HT in $y$ direction $\hat{g}_y(x, y)$. This example clearly demonstrates that the proposed HT is not oriented to any direction, unlike PHTs which are oriented to respective directions.

From Tables 1 and 2, we observe that the proposed HT $\hat{g}(x, y)$ is close to PHT in $x$ direction $\hat{g}_x(x, y)$. To observe the difference between these two, we rewrite (8) as

$$
g(x, y) = a_{0,0} + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ a_{k,l} \cos(k\omega_1 x + l\omega_2 y) + b_{k,l} \sin(k\omega_1 x + l\omega_2 y) \right] + \sum_{k=1}^{\infty} \left[ a_{k,0} \cos(k\omega_1 x) + b_{k,0} \sin(k\omega_1 x) \right] + \sum_{l=1}^{\infty} \left[ a_{0,l} \cos(l\omega_2 y) + b_{0,l} \sin(l\omega_2 y) \right] + \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} \left[ a_{k,l} \cos(k\omega_1 x + l\omega_2 y) + b_{k,l} \sin(k\omega_1 x + l\omega_2 y) \right], \quad (38)$$

and obtain a partial HT in $x$ direction as

$$
\hat{g}_x(x, y) = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left[ a_{k,l} \sin(k\omega_1 x + l\omega_2 y) - b_{k,l} \cos(k\omega_1 x + l\omega_2 y) \right] + \sum_{k=1}^{\infty} \left[ a_{k,0} \sin(k\omega_1 x) - b_{k,0} \cos(k\omega_1 x) \right] + \sum_{l=-\infty}^{-1} \sum_{k=1}^{\infty} \left[ a_{k,l} \sin(k\omega_1 x + l\omega_2 y) - b_{k,l} \cos(k\omega_1 x + l\omega_2 y) \right]. \quad (39)
$$

Equations (14) and (39) provide clear differences between the proposed HT and PHT in $x$ direction.
Table 1 (Section 6.2) The partial HT (HT) in $x_1$ direction $\hat{g}_{x_1}(x_1, x_2)$, PHT in $x_2$ direction $\hat{g}_{x_2}(x_1, x_2)$, total HT (THT) $\hat{g}_T(x_1, x_2)$ and single-orthant HT (SOHT) $\hat{g}_{SO}(x_1, x_2)$

| S.N. | Function $g(x_1, x_2)$ | $\hat{g}_{x_1}(x_1, x_2)$ | $\hat{g}_{x_2}(x_1, x_2)$ | $\hat{g}_T(x_1, x_2)$ | $\hat{g}_{SO}(x_1, x_2)$ |
|------|------------------------|-----------------|-----------------|-----------------|-----------------|
| 1    | Constant ($a_0$)       | 0               | 0               | 0               | 0               |
| 2    | $\sin(\omega_2 x_2)\sin(\omega_1 x_1)$ | $-\sin(\omega_2 x_2)\cos(\omega_1 x_1)$ | $-\cos(\omega_2 x_2)\sin(\omega_1 x_1)$ | $\cos(\omega_2 x_2)\cos(\omega_1 x_1)$ | $-\sin(\omega_1 x_1 + \omega_2 x_2)$ |
| 3    | $\cos(\omega_2 x_2)\sin(\omega_1 x_1)$ | $-\cos(\omega_2 x_2)\cos(\omega_1 x_1)$ | $\sin(\omega_2 x_2)\sin(\omega_1 x_1)$ | $-\sin(\omega_2 x_2)\cos(\omega_1 x_1)$ | $-\cos(\omega_1 x_1 + \omega_2 x_2)$ |
| 4    | $\sin(\omega_2 x_2)\cos(\omega_1 x_1)$ | $\sin(\omega_2 x_2)\sin(\omega_1 x_1)$ | $-\cos(\omega_2 x_2)\cos(\omega_1 x_1)$ | $-\cos(\omega_2 x_2)\sin(\omega_1 x_1)$ | $-\cos(\omega_1 x_1 + \omega_2 x_2)$ |
| 5    | $\cos(\omega_1 x_1 + \omega_2 x_2)$ | $\sin(\omega_1 x_1 + \omega_2 x_2)$ | $\sin(\omega_1 x_1 + \omega_2 x_2)$ | $-\cos(\omega_1 x_1 + \omega_2 x_2)$ | $2\sin(\omega_1 x_1 + \omega_2 x_2)$ |
| 6    | $\cos(\omega_1 x_1 - \omega_2 x_2)$ | $\sin(\omega_1 x_1 - \omega_2 x_2)$ | $-\sin(\omega_1 x_1 - \omega_2 x_2)$ | $\cos(\omega_1 x_1 - \omega_2 x_2)$ | $0$ |
| 7    | $\sin(\omega_1 x_1 + \omega_2 x_2)$ | $-\cos(\omega_1 x_1 + \omega_2 x_2)$ | $-\cos(\omega_1 x_1 + \omega_2 x_2)$ | $-\sin(\omega_1 x_1 + \omega_2 x_2)$ | $-2\cos(\omega_1 x_1 + \omega_2 x_2)$ |
| 8    | $\sin(\omega_1 x_1 - \omega_2 x_2)$ | $-\cos(\omega_1 x_1 - \omega_2 x_2)$ | $\cos(\omega_1 x_1 - \omega_2 x_2)$ | $\sin(\omega_1 x_1 - \omega_2 x_2)$ | $0$ |
| 9    | $\cos(\omega_1 x_1)$ | $\sin(\omega_1 x_1)$ | $0$ | $0$ | $\sin(\omega_1 x_1)$ |
| 10   | $\cos(\omega_2 x_2)$ | $0$ | $\sin(\omega_2 x_2)$ | $0$ | $\sin(\omega_2 x_2)$ |
| 11   | $\sin(\omega_1 x_1)$ | $-\cos(\omega_1 x_1)$ | $0$ | $0$ | $-\cos(\omega_1 x_1)$ |
| 12   | $\sin(\omega_2 x_2)$ | $0$ | $-\cos(\omega_2 x_2)$ | $0$ | $-\cos(\omega_2 x_2)$ |
Table 2 (Section 6.2) The proposed Fourier theory and phase delay-based 2D-HT \( \hat{g}(x_1, x_2) \) and 2D-AS \( z(x_1, x_2) \)

| S.N. | Function \( g(x_1, x_2) \) | Proposed \( \hat{g}(x_1, x_2) \) | Proposed 2D-AS \( z(x_1, x_2) \) |
|------|-------------------|-------------------|-------------------|
| 1    | Constant \( (a_0) \) | 0                 | \( a_0 \)          |
| 2    | \( \sin(\omega_1 x_2) \sin(\omega_1 x_1) \) | \( -\sin(\omega_2 x_2) \cos(\omega_1 x_1) \) | \( \frac{1}{2}[e^{j(\omega_1 x_1 - \omega_2 x_2)} - e^{j(\omega_1 x_1 + \omega_2 x_2)}] \) |
| 3    | \( \cos(\omega_1 x_2) \sin(\omega_1 x_1) \) | \( \cos(\omega_2 x_2) \cos(\omega_1 x_1) \) | \( \frac{1}{2}[e^{j(\omega_1 x_1 - \omega_2 x_2 - \frac{\pi}{2})} + e^{j(\omega_1 x_1 + \omega_2 x_2 - \frac{\pi}{2})}] \) |
| 4    | \( \sin(\omega_1 x_2) \cos(\omega_1 x_1) \) | \( \sin(\omega_2 x_2) \sin(\omega_1 x_1) \) | \( \frac{1}{2}[e^{j(\omega_1 x_1 - \omega_2 x_2 + \frac{\pi}{2})} + e^{j(\omega_1 x_1 + \omega_2 x_2 - \frac{\pi}{2})}] \) |
| 5    | \( \cos(\omega_1 x_1 + \omega_2 x_2) \) | \( \sin(\omega_1 x_1 + \omega_2 x_2) \) | \( e^{j(\omega_1 x_1 + \omega_2 x_2)} \) |
| 6    | \( \cos(\omega_1 x_1 - \omega_2 x_2) \) | \( -\cos(\omega_1 x_1 - \omega_2 x_2) \) | \( e^{j(\omega_1 x_1 - \omega_2 x_2)} \) |
| 7    | \( \sin(\omega_1 x_1 + \omega_2 x_2) \) | \( -\cos(\omega_1 x_1 + \omega_2 x_2) \) | \( e^{j(\omega_1 x_1 + \omega_2 x_2 - \frac{\pi}{2})} \) |
| 8    | \( \cos(\omega_1 x_1 - \omega_2 x_2) \) | \( \sin(\omega_1 x_1) \) | \( e^{j(\omega_1 x_1)} \) |
| 9    | \( \cos(\omega_2 x_2) \) | \( \sin(\omega_2 x_2) \) | \( e^{j(\omega_2 x_2)} \) |
| 10   | \( \sin(\omega_1 x_1) \) | \( -\cos(\omega_1 x_1) \) | \( e^{j(\omega_1 x_1 - \frac{\pi}{2})} \) |
| 11   | \( \sin(\omega_2 x_2) \) | \( -\cos(\omega_2 x_2) \) | \( e^{j(\omega_2 x_2 - \frac{\pi}{2})} \) |

Fig. 4 (Section 6.2) Original signal \( g(x, y) \) and its proposed HT \( \hat{g}(x, y) \) which is not oriented to any direction; partial HT in \( x \) direction \( \hat{g}_x(x, y) \) and partial HT in \( y \) direction \( \hat{g}_y(x, y) \) which are oriented to respective directions

6.3 Edge Detection

The HT can be seen as an edge detector. Figures 5 and 6 show examples of gray level checkerboard image and natural Lena image, respectively. We have calculated
the directional HTs (DHTs), corresponding to first and fourth quadrants (DHT-1-4), second and third quadrants (DHT-2-3), first and second quadrants (DHT-1-2), third and fourth quadrants (DHT-3-4) using FFT-based algorithm. It appears that these HTs act as edge detection steered in the $x$ and $y$ directions. If we compare the two Hilbert components, we can see different kind of edges responding to the $x$ and $y$ directions. To observe the differences between the directional HTs and proposed HTs, we evaluated the proposed HTs, AS, analytic phase and gradient of same gray level checkerboard image and natural Lena image in Figs. 7 and 8, respectively. In these figures, clear visible differences between proposed HTs and DHTs are captured in the analytic phase and gradient of DHTs (e.g., Fig. 5 [(h) to (o)]) and proposed HT (e.g., Fig. 7 [(e) to (h)]).

### 6.4 Decomposition of an Image Into AM–FM Components Using the 2D FDM

The Lena image and its decomposition into a set of orthogonal 2D FIBFs are shown in Fig. 9 by considering first and fourth quadrants, i.e., decomposition of image w.r.t. $\omega_1$ by (29). It appears that this decomposition is steered in the $x$ direction as many frequency components are responding to the $x$ direction. The FIBF-1 captures the
Fig. 6 (Section 6.3) Directional HT (DHT), a Lena image $g[m, n]$, b DHT $\hat{g}_{14}[m, n]$, c DHT $\hat{g}_{23}[m, n]$, d DHT $\hat{g}_{12}[m, n]$, e DHT $\hat{g}_{34}[m, n]$, f analytic amplitude $|z_{14}[m, n]|$, g analytic amplitude $|z_{12}[m, n]|$, h analytic phase $\angle z_{14}[m, n]$ (AP-1-4), i analytic phase $\angle z_{12}[m, n]$ (AP-1-2), j gradient $\phi_m[m, n]$ of AP-1-4, k gradient $\phi_n[m, n]$ of AP-1-4, l $\sqrt{\phi_m^2[m, n] + \phi_n^2[m, n]}$ of quadrant 1-4, m gradient $\phi_m[m, n]$ of AP-1-2, n gradient $\phi_n[m, n]$ of AP-1-2, o $\sqrt{\phi_m^2[m, n] + \phi_n^2[m, n]}$ of quadrant 1-2

Fig. 7 (Section 6.3) Proposed HT, a Checkerboard image $g[m, n]$, b HT $\hat{g}_{14}[m, n]$, c HT $\hat{g}_{23}[m, n]$, d analytic amplitude $|z_{14}[m, n]|$, e analytic phase $\angle z_{14}[m, n]$ (AP-1-4), f gradient $\phi_m[m, n]$ of AP-1-4, g gradient $\phi_n[m, n]$ of AP-1-4, h $\sqrt{\phi_m^2[m, n] + \phi_n^2[m, n]}$ of quadrant 1-4.
Fig. 8 (Section 6.3) Proposed HT, a Lena image $g[m, n]$, b HT $\hat{g}_{14}[m, n]$, c HT $\hat{g}_{23}[m, n]$, d analytic amplitude $|z_{14}[m, n]|$, e analytic phase $\angle z_{14}[m, n]$ (AP-1-4), f gradient $\phi_m[m, n]$ of AP-1-4, g gradient $\phi_n[m, n]$ of AP-1-4, h $\sqrt{\phi_m^2[m, n] + \phi_n^2[m, n]}$ of quadrant 1-4

Fig. 9 (Section 6.4) Lena image and its decomposition, w.r.t. $\omega_1$ using (29), into a set of orthogonal 2D FIBF-1 to FIBF-7 in order of increasing frequency components by 2D-FDM such that Lena image is sum of all FIBFs. The FIBF-1 and FIBF-7 represent the lowest and highest frequency components of Lena image, respectively. The Lena image is the exact sum of all the FIBFs (FIBF-1 to FIBF-7).

Figure 10 shows decomposition of Lena image using Fourier-based zero-phase filtering (31) and Algorithm 2, into a set of orthogonal 2D FIBF-1 to FIBF-7 in order of lowest components, and FIBF-7 captures the highest frequency component of image.
of increasing frequency components by 2D-FDM. The FIBF-1 and FIBF-7 represent the lowest and highest frequency components in the AM–FM decomposition of image, respectively. In this case as well, the Lena image is the exact sum of all the FIBFs. Thus, it is clear that this decomposition, as shown in Fig. 10, is not steered to any direction.

6.5 Computation of Single-Orthant Fourier Transform (SOFT)

In order to show how single-orthant Fourier transform works, we consider an example signal \( g[m, n] \) which is shown in Table 3a. The DFT of this signal is shown in Table 3b. The proposed single-orthant DFT, \( \mathcal{X}[k, l] \), is shown in Table 3c where first-orthant DFT (SODFT) elements are shown in bold numbers and values in rest of the orthant are obtained by simple conjugations. The elements of DFT Table 3b can be obtained by the elements of SODFT Table 3c from the relation \( G[k, l] = (\mathcal{X}_r[k, l] + \mathcal{X}_{ij}[k, l]) + j(\mathcal{X}_i[k, l] + \mathcal{X}_j[k, l]). \)

7 Conclusion

Salient contributions of this study can be summarized as follows: (1) the notion of Fourier frequency vector (FFV) inherently associated with multidimensional Fourier representation (MD-FR) is introduced. It is observed that the proposed FFV provides physical meaning to the so-called negative frequencies in MD-FR. This in turn provides a tool for multidimensional spatial and space-time series analysis. (2) Using the MD-FR, MD Hilbert transform (MD-HT) and associated MD analytic signal (MD-AS) are
In this study, we have considered limited number of examples, applications and compared the results with directional as well as total HTs, our future direction of the research would be to perform comparative study with quaternion-based HT and other existing methods using numerical analysis, and explore new applications of the proposed methodologies. Moreover, the proposed SOFT is valid only for a real-valued signal, so we would like to develop and extend it for MD complex-valued signals and investigate possible applications in image processing.

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**Author contributions** P. Singh conceived and designed the study, carried out the simulation work, participated in data analysis, and drafted the manuscript; S. D. Joshi discussed and checked the mathematical analyses, coordinated the study and helped in drafting the manuscript. All authors commented and gave final approval for publication.

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**Data Availability** The FDM MATLAB code is publicly available for download at https://www.researchgate.net/publication/319877224_MATLABCodeOfFDMforImageDecomposition, and A MATLAB code for the proposed analytic image and 2D Hilbert transform computation by 2D-DFT is included in Appendix of the paper itself.
Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Permission to Carry out Fieldwork No permissions were required prior to conducting this research.

1 An Analytic Image Computation Using the 2D-DFT

In this appendix, we derive analytic image representation by 2D-DFT as follows. Let \( g[m, n] \) be a real-valued function, then the 2D-DFT of \( g[m, n] \) is defined as

\[
G[k, l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g[m, n]e^{-j\left(\frac{km}{M} + \frac{ln}{N}\right)} \tag{40}
\]

and 2D-IDFT is defined as

\[
g[m, n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} G[k, l]e^{j\left(\frac{km}{M} + \frac{ln}{N}\right)} \tag{41}
\]

In (40), for odd numbers \((M = N)\), there is only one real term \(G[0, 0]\) and \((MN - 1)/2\) terms are complex conjugate of the rest \((MN - 1)/2\) terms; and for even numbers \((M = N)\), there are only four real terms \(G[0, 0], G[0, N/2], G[M/2, 0], G[M/2, N/2]\) and \((MN - 4)/2\) terms are complex conjugate of the rest \((MN - 4)/2\) terms. From the above discussions and using the conjugate symmetry of 2D-DFT, we obtain 2D-AS for odd numbers \((M = N)\) as

\[
z_{14}[m, n] = 2\sum_{k=0}^{(M-1)/2} \sum_{l=0}^{(N-1)/2} G[k, l]e^{j\left(\frac{km}{M} + \frac{ln}{N}\right)} + 2\sum_{k=(M+1)/2}^{(M-1)} \sum_{l=1}^{(N-1)/2} G[k, l]e^{j\left(\frac{km}{M} + \frac{ln}{N}\right)} - G(0, 0) \tag{42}
\]

and for even numbers \((M = N)\) as

\[
z_{14}[m, n] = 2\sum_{k=0}^{M/2} \sum_{l=0}^{N/2} G[k, l]e^{j\left(\frac{km}{M} + \frac{ln}{N}\right)} + 2\sum_{k=M/2+1}^{M-1} \sum_{l=1}^{N/2-1} G[k, l]e^{j\left(\frac{km}{M} + \frac{ln}{N}\right)} - G[0, 0] - G[0, N/2] - G[M/2, 0] - G[M/2, N/2] \tag{43}
\]

where real part of AS is original signal (i.e., \(g[m, n] = Re\{z_{14}[m, n]\}\)) and imaginary part of AS is the HT of original signal (i.e., \(\hat{g}[m, n] = Im\{z_{14}[m, n]\}\)). This 2D-AS has been obtained by considering the first and fourth quadrants of 2D DFT. These 2D-AS (analytic image) computation and 2D-HT can be easily implemented with 2D-FFT algorithms, e.g., A MATLAB implementation is presented in Algorithm 1.
Algorithm 1: (Appendix 1) A MATLAB code for the proposed analytic image and 2D Hilbert transform computation by 2D-FFT.

```matlab
% Case 1: when M and N are even numbers;
X = fft2(x); [M, N] = size(x) % compute 2D-FFT and size;
% create mask;
Xm = zeros(M, N);
Xm(1:(M/2)+1, 1:(N/2)+1) = 2;
Xm((M/2)+2:M, 2:(N/2)) = 2; % take 1st & 4th quadrant;
Xm(1, 1) = 1; Xm(1, N/2+1) = 1; Xm(M/2+1, 1) = 1; Xm(M/2+1, N/2+1) = 1;
% mask 2D-FFT and compute inverse to obtain AS;
tmp = Xm.*X; z14 = ifft2(tmp);

% Case 2: when M and N are odd numbers;
X = fft2(x); [M, N] = size(x) % compute 2D-FFT and size;
% create mask;
Xm = zeros(M, N); Xm(1:(M+1)/2, 1:(N+1)/2) = 2; Xm(1, 1) = 1;
Xm((M+3)/2:M, 2:(N+1)/2) = 2; % take 1st & 4th quadrant;
Xm(1, 1) = 1; Xm(1, N/2+1) = 1; Xm(M/2+1, 1) = 1; Xm(M/2+1, N/2+1) = 1;
% mask 2D-FFT and compute inverse to obtain AS;
tmp = Xm.*X; z14 = ifft2(tmp);
```

Algorithm 2: (Section 4) A 2D-FDM algorithm to decompose an image \( g \) into a set of LINOEP vectors \( g_i \) such that

\[
g(x_1, \ldots, x_M) = \sum_{k_1=-\infty}^{\infty} \ldots \sum_{k_M=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left[ a_{k_1, \ldots, k_M} \cos(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \\
+ b_{k_1, \ldots, k_M} \sin(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \right] \\
+ \sum_{k_1=-\infty}^{\infty} \ldots \sum_{k_3=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_1=1}^{\infty} \left[ a_{k_1, \ldots, k_M} \cos(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \\
+ b_{k_1, \ldots, k_M} \sin(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \right],
\]

\( (44) \)

2 MD-FS, MD-HT and MD-AS Representation

The 2D-FS discussion presented in Sect. 2 can be easily extended for MD-FS as follows

\[
g(x_1, \ldots, x_M) = \sum_{k_M=-\infty}^{\infty} \ldots \sum_{k_1=1}^{\infty} \sum_{k_1=1}^{\infty} \left[ a_{k_1, \ldots, k_M} \cos(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \\
+ b_{k_1, \ldots, k_M} \sin(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \right] \\
+ \sum_{k_1=-\infty}^{\infty} \ldots \sum_{k_3=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_1=1}^{\infty} \left[ a_{k_1, \ldots, k_M} \cos(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \\
+ b_{k_1, \ldots, k_M} \sin(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M) \right],
\]

\( (44) \)
\[ g(x_1, \ldots, x_M) = \sum_{k_M=-\infty}^{\infty} \ldots \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} c_{k_1,\ldots,k_M} \exp\{j(k_1 \omega_1 x_1 + \ldots + k_M \omega_M x_M)\}. \]

where \( c_{k_1,\ldots,k_M} = (a_{k_1,\ldots,k_M} - jb_{k_1,\ldots,k_M})/2 \). We can easily obtain MD-HT by replacing \( \cos \) with \( \sin \) and \( \sin \) with \( -\cos \) in (44) and obtain AS such that real part of AS is original signal, \( \text{AS has positive resultant frequency } \omega = |\omega| = (k_1 \omega_1)^2 + \ldots + (k_M \omega_M)^2 \) and its FFV can be written as \( \omega = [k_1 \omega_1 \ldots k_M \omega_M]^T \). The prowess and efficacy of the Fourier theory can be realized from the fact that the HT and analytic representation of a signal are, inherently, present in the Fourier representation.

**Observation** We can also use (44) for \((M-1)D\) space-time series \((x_1, \ldots, x_{(M-1)}, t)\) analysis, e.g., 2D wave equation, \( g(x, y, t) = \cos(k_1 \omega_1 t - k_2 \omega_2 x - k_3 \omega_3 y) \), where \( k_1 \omega_1 = \frac{2\pi k_1}{T_1} = 2\pi k_1 f_1 = 2\pi f, \) wave vector (or FFV) \( \omega = [k_2 \omega_2 k_3 \omega_3]^T, \) wave number (or spatial frequency) \( |\omega| = \sqrt{(k_2 \omega_2)^2 + (k_3 \omega_3)^2} = \frac{2\pi}{\lambda} \) and phase velocity \( v_p = c_{k_1 \omega_1} = f \lambda. \)

For non-periodic MD signal, \( g(x_1, \ldots, x_M) \), the MD-FT and MD inverse FT (MD-IFT) are defined as

\[
C[f_1, \ldots, f_M] = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(x_1, \ldots, x_M) \exp\{-j(\omega_1 x_1 + \ldots + \omega_M x_M)\} \, dx_1 \ldots dx_M,
\]

\[
g(x_1, \ldots, x_M) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} C[f_1, \ldots, f_M] \exp\{j(\omega_1 x_1 + \ldots + \omega_M x_M)\} \, df_1 \ldots df_M,
\]

respectively. We define MD-AS, for real-valued signal \( g(x_1, \ldots, x_M) \), as

\[
z(x_1, \ldots, x_M) = 2 \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \int_{0}^{\infty} C[f_1, \ldots, f_M] \exp\{j(\omega_1 x_1 + \ldots + \omega_M x_M)\} \, df_1 \ldots df_M.
\]

where its real part is original signal and imaginary part is HT of original signal.

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