Vanishing theorems for locally conformal hyperkähler manifolds

Misha Verbitsky,\(^1\)

verbit@maths.gla.ac.uk, verbit@mccme.ru

Abstract

Let \(M\) be a compact locally conformal hyperkähler manifold. We prove a version of Kodaira-Nakano vanishing theorem for \(M\). This is used to show that \(M\) admits no holomorphic differential forms, and the cohomology of the structure sheaf \(H^i(\mathcal{O}_M)\) vanishes for \(i > 1\). We also prove that the first Betti number of \(M\) is 1. This leads to a structure theorem for locally conformally hyperkähler manifolds, describing them in terms of 3-Sasakian geometry. Similar results are proven for compact Einstein-Weyl locally conformal Kähler manifolds.

Contents

1 Introduction 2
1.1 Locally conformal hyperkähler manifolds, definition and examples . . 3
1.2 Vanishing theorems for LCHK manifolds . . . . . . . . . . . . . . . . 4
1.3 Geometry of LCHK manifolds . . . . . . . . . . . . . . . . . . . . . . 4
1.4 3-Sasakian geometry and structure theorem for LCHK manifolds . . 6
1.5 Subvarieties of Vaisman manifolds . . . . . . . . . . . . . . . . . . . 6

2 Locally conformal hyperkähler manifolds 7
2.1 Weyl structures . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
2.2 Locally conformal hyperkähler (LCHK) manifolds: the definition . . 9

3 Vaisman manifolds 10

4 Kähler potential on Vaisman manifolds 12

5 Einstein-Weyl LCK manifolds 13

6 The form \(\omega_0\) on Vaisman manifolds 15
6.1 The form \(\omega_0\): definition and eigenvalues . . . . . . . . . . . . 16
6.2 The form \(\omega_0\) and the canonical foliation . . . . . . . . . . . . . 17
6.3 Curvature of a weight bundle . . . . . . . . . . . . . . . . . . . . . . 19

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1 Introduction

The locally conformal Kähler manifolds and locally conformally hyperkähler manifolds were intensively studied throughout the last 30 years; see [DO] and [OR] for a survey of known results and further reference. The key notion in this study is the notion of a Vaisman manifold, also known as generalized Hopf manifold (Definition 3.7). These manifolds were discovered by I. Vaisman and studied in a big series of papers in early 1980-es (see [V1], [V2], and the bibliography in [DO]).

We prove vanishing theorems for locally conformally hyperkähler manifolds and locally conformal Kähler Einstein-Weyl manifolds and use these results to obtain structure theorems. Similar vanishing theorems were obtained in [AI] using estimates on Ricci curvature.
In physics, the vanishing theorems for locally conformal Kähler manifolds can be interpreted as conditions for existence of certain string compactifications ([Str], [IP1], [IP2]).

1.1 Locally conformal hyperkähler manifolds, definition and examples

Let $M$ be a smooth manifold equipped with operators $I, J, K \in \text{End}(TM)$ satisfying the quaternion relations

$$I^2 = J^2 = K^2 = -1, I \circ J = -J \circ I = K.$$ 

Assume that the operators $I, J, K$ induce integrable complex structures on $M$. Then $M$ is called hypercomplex. By a theorem of Obata ([Ob]), a hypercomplex manifold admits a unique torsion-free connection preserving $I, J, K$. If the Obata connection preserves the metric, $(M, g)$ is called hyperkähler. Hyperkähler manifolds were introduced by E. Calabi ([Ca]), and hypercomplex manifolds, much later, by C.P.Boyer ([Bo]).

A hypercomplex manifold $M$ is called locally conformal hyperkähler (LCHK) if the covering of $M$ is hyperkähler, and the monodromy transform preserves the conformal class of a hyperkähler metric. For a differently worded definition, see Subsection 2.2.

The most elementary example of an LCHK manifold is the Hopf manifold, defined as follows. Fix a quaternion number $q \in \mathbb{H}$, $|q| > 1$. Consider the manifold $\tilde{M} = \mathbb{H}^n \setminus 0$, and let $M := \tilde{M}/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\tilde{M}$ by right quaternionic dilatations as

$$(i, z) \mapsto z \cdot q^i, \quad z \in \mathbb{H}^n, i \in \mathbb{Z}.$$ 

This map is clearly compatible with the hypercomplex structure and the conformal structure on $\mathbb{H}^n$. Therefore, the manifold $M$ is locally conformal hyperkähler.

More examples of LCHK manifolds are provided by the quaternionic Kähler geometry (see e.g. [BCS]). We shall not use these examples; the reader not versed in quaternionic Kähler geometry may skip the next paragraph.

Given a quaternionic Kähler manifold $Q$ with positive scalar curvature (e.g. the quaternionic projective space $\mathbb{H}P^n$) one considers its Swann bundle $U(Q)$, which fibers on $Q$ with a fiber $\mathbb{C}^2 \setminus 0$ if $Q$ is Spin, or $\mathbb{C}^2 \setminus 0/\{\pm 1\}$ if $Q$ is not Spin ([Sw]). The total space of $U(Q)$ is hyperkähler, and the natural dilatation map $p_t$ acts on $U(Q)$ preserving the hypercomplex structure, and multiplies the metric by a number. Take a quotient $M := U(Q)/p_{q^i}$, $i \in \mathbb{Z}$,
where \( q > 1 \) is a fixed real number. If \( Q \) is compact, then \( M \) is also compact. By construction, \( M \) is locally conformal hyperkähler. When \( Q = \mathbb{HP}^n \), \( \mathcal{U}(Q) = \mathbb{C}^{2n}\setminus 0 \), and \( M = \mathbb{C}^{2n}\setminus 0/\{q^n\} \) is the Hopf manifold.

### 1.2 Vanishing theorems for LCHK manifolds

In this paper we obtain several vanishing results for LCHK (locally conformal hyperkähler) manifolds based on the same analytic arguments as the Kodaira-Nakano vanishing theorem. In particular, we obtain

**Theorem 1.1:** Let \( M \) be a compact LCHK manifold which is not hyperkähler. Consider \( M \) as a complex manifold, with the complex structure \( I \) induced by the hypercomplex structure. Then

(i) \( M \) does not admit non-trivial global holomorphic forms

(ii) The cohomology of the structure sheaf of \( M \) satisfy \( H^i(\mathcal{O}_M) = 0 \) for \( i > 1 \), \( \dim H^1(\mathcal{O}_M) = 1 \).

(iii) The first Betti number of \( M \) is 1.

**Proof:** This is Theorem 9.7, Theorem 8.4, Theorem 9.8.

Using the Dolbeault spectral sequence, it is easy to deduce Theorem 1.1 (iii) from (i) and (ii). Theorem 1.1 (iii) can be proven directly by a simple geometric argument (see Section 12).

The same results are true for compact Einstein-Weyl locally conformally Kähler manifolds.

### 1.3 Geometry of LCHK manifolds

Consider a compact LCHK manifold \( M \). Then \( M \) admits a special metric, discovered by P. Gauduchon (Definition 3.4). The Gauduchon metric is defined as follows.

Let \( \tilde{M} \) be the hyperkähler covering of \( M \). Since the deck transform acts on \( \tilde{M} \) preserving the conformal class of the metric, the manifold \( M \) is equipped with the canonical conformal structure \([g]\). The Obata connection \( \nabla \) on a hyperkähler manifold coincides with the Levi-Civita connection. Therefore, \( \nabla \) preserves the conformal class \([g]\). We obtain that a parallel transport along \( \nabla \) multiplies a metric \( g \in [g] \) by a number. Therefore, \( \nabla(g) = g \otimes \theta \), where \( \theta \) is a 1-form, called the Lee form of \((M, g)\).
A metric $g \in [g]$ is called a Gauduchon metric if $\theta$ is co-closed. Gauduchon proved that this metric exists and is unique up to a constant multiplier if we fix the connection and the conformal class (Lemma 3.3).

If $M$ is an LCHK manifold equipped with a Gauduchon metric, then the Lee form $\theta$ is parallel with respect to the Levi-Civita connection (Theorem 3.6). The corresponding vector field $\theta^\sharp$ is obviously Killing. Moreover, the flow, associated with $\theta^\sharp$, is compatible with the hypercomplex structure on $M$ (Proposition 4.1). If we lift $\theta^\sharp$ to a hyperkähler covering $\tilde{M}$, the corresponding flow multiplies the hyperkähler metric by constant. This allows one to construct the Kähler potential on $\tilde{M}$ explicitly in terms of the Lee form (Proposition 4.4).

Applying Proposition 4.4 to different induced complex structures, we find that $M$ is equipped with a hyperkähler potential, that is, a function which serves as a Kähler potential for all induced complex structures.

Hyperkähler manifolds admitting hyperkähler potential were studied by A. Swann [Sw]. These manifolds are deeply related to quaternionic Kähler geometry. Take a 4-dimensional foliation $\Phi$ generated by the gradient $\theta^\sharp$ of the hyperkähler potential and $I(\theta^\sharp), J(\theta^\sharp), K(\theta^\sharp)$. This foliation is integrable, flat and completely geodesic. The leaf space of $\Phi$ is quaternionic Kähler. This leads to nice structure theorems for hyperkähler manifolds admitting hyperkähler potential (see [Sw]).

The LCHK geometry is much more delicate, due to possible global irregularities of the foliation $\Phi$.

Let $M$ be an LCHK manifold equipped with a Gauduchon metric, $\theta$ is Lee form, and $\Phi$ the 4-dimensional foliation generated by $\theta^\sharp, I(\theta^\sharp), J(\theta^\sharp), K(\theta^\sharp)$. One can speak of the leaf space of $\Phi$ if every point $x \in M$ has a neighbourhood $U \subset M$ such that every leaf of $\Phi$ meets $U$ in finitely many connected components. In this case the foliation $\Phi$ and the manifold $M$ is called quasiregular. The leaf space $Y$ of a quasiregular foliation is an orbifold. Theorem 1.1 (iii) (the equality $h^1(M) = 1$) is proven for quasiregular LCHK manifolds (OP).

Given a compact quasiregular LCHK manifold, the leaf space $Q$ of $\Phi$ is a quaternionic Kähler orbifold, and $M$ is fibered over $Q$ with fibers which are isomorphic to Hopf surfaces (OP).

In this paper we give a similar structure theorem for LCHK manifolds with no quasiregularity assumption.
1.4 3-Sasakian geometry and structure theorem for LCHK manifolds

It is more convenient to speak of 3-Sasakian manifolds than of quaternionic Kähler orbifolds. A 3-Sasakian manifold is locally a bundle over a quaternionic Kähler orbifold, with a fiber isomorphic to $SU(2)/\Gamma$, where $\Gamma \subset SU(2)$ is a finite group.

One defines 3-Sasakian manifolds as follows.

Given a Riemannian manifold $(X, g)$, the cone $C(X)$ of $X$ is defined as a Riemannian manifold $X \times \mathbb{R}^>0$ with the metric $t^2g + dt^2$, where $t$ is the parameter in $\mathbb{R}^>0$.

A 3-Sasakian structure on $X$ is a hyperkähler structure on $C(X)$, defined in such a way the map $(x, t) \rightarrow x, qt$ is holomorphic, for all $q \in \mathbb{R}^>0$.

3-Sasakian manifolds were discovered in late 1960-ies (see [U] and [K]), and studied extensively in mid-1990-ies by Boyer, Galicki and Mann ([BGM]); see the excellent survey [BG1].

The 3-Sasakian (and, more generally, Einstein-Sasakian) manifolds can also be obtained as circle bundles over Einstein Fano orbifolds. This gives a way to construct extensive lists of examples of Einstein-Sasakian manifolds (see e.g. [BG2] [BGN2]).

Given a compact LCHK manifold $M$, its universal covering $\tilde{M}$ is hyperkähler. Using the explicit description of the hyperkähler metric in terms of the Gauduchon metric [Proposition 4.4], we find that $\tilde{M}$ is a cone manifold: $\tilde{M} = C(X)$, where $X$ is 3-Sasakian [Proposition 11.1].

Fix $q \in \mathbb{R}^>1$. Consider the equivalence relation on $C(X)$ generated by $(x, t) \sim (x, qt)$. The quotient $C(X)/\sim_q$ is clearly an LCHK manifold.

The structure theorem [Theorem 11.6] describes any compact LCHK manifold in terms of 3-Sasakian manifolds, as follows. We show that $M \cong C(X)/\sim_{\varphi,q}$, where $q \in \mathbb{R}^>1$, $\varphi : X \rightarrow X$ is a 3-Sasakian isometry, and $\sim_{\varphi,q}$ an equivalence relation on $C(X)$ generated by $(x, t) \sim (\varphi(x), qt)$.

1.5 Subvarieties of Vaisman manifolds

We also obtain the following application.

**Proposition 1.2:** Let $M$ be a compact Vaisman manifold, and $X \subset M$ a closed complex subvariety. Then $X$ is tangent to the canonical foliation $\Xi$ in all its smooth points. In particular, if $X$ is smooth, then $X$ is also a Vaisman manifold.
Proof: See Proposition 6.5.

For smooth manifolds, Proposition 1.2 is proven by K. Tsukada, see [Ts2].

2 Locally conformal hyperkähler manifolds

In this Section we give the definitions and site some results related to Weyl geometry and locally conformal hyperkähler manifolds. We follow [Or].

2.1 Weyl structures

For an introduction to Weyl geometry and further reference on the subject see e.g. [CP].

Definition 2.1: Let $(M,g)$ be a Riemannian manifold, and $\nabla$ a torsion-free connection on $M$. Assume that $\nabla$ preserves the conformal class of $g$, that is,

$$\nabla g = g \otimes \theta \in S^2T^*M \otimes T^*M,$$

(2.1)

where $\theta$ is a 1-form. Then $(M,g,\nabla,\theta)$ is called a Weyl manifold. If $d\theta = 0$, $M$ is called a closed Weyl manifold. The form $\theta$ is called the Lee form, or the Higgs field of $M$. A torsion-free connection which satisfies (2.1) is called a Weyl connection.

Remark 2.2: Let $M$ be a Riemannian manifold equipped with a Levi-Civita connection. Then $M$ is a Weyl manifold, with trivial $\theta$.

Given a Weyl manifold $(M,g,\nabla,\theta)$ and a function $\zeta : M \rightarrow \mathbb{R}$, we observe that

$$(M,e^\zeta g,\nabla,\theta + d\zeta)$$

is also a Weyl manifold. Indeed,

$$\nabla(e^\zeta g) = e^\zeta \nabla(g) + e^\zeta g \otimes d\zeta = e^\zeta g \otimes (\theta + d\zeta).$$

Definition 2.3: In the above assumptions, the Weyl manifolds

$$(M,g,\nabla,\theta) \quad \text{and} \quad (M,e^\zeta g,\nabla,\theta + d\zeta)$$

are called globally conformal equivalent.
Let \((M, g, \nabla, \theta)\) be a Weyl manifold, and \(L \subset \mathbb{R}\) a trivial 1-dimensional real bundle on \(M\). Denote the trivial connection on \(L\) by \(\nabla_{\text{tr}}\). Consider a connection \(L \nabla := \nabla_{\text{tr}} - \frac{\theta^2}{2}\) on \(L\). If \((M, g, \nabla, \theta)\) is closed, the bundle \((L, L \nabla)\) is flat, because \((L \nabla)^2 = d\theta = 0\). Denote by \(L\) the complexification of \(L \subset \mathbb{R}\), with the induced connection.

**Definition 2.4:** The bundle \((L, L \nabla)\) is called the **weight bundle** of a Weyl manifold \(M\). The weight bundle is equipped with a trivialization \(\lambda\).

**Claim 2.5:** Let \((M, g, \nabla, \theta)\) and \((M', g', \nabla, \theta')\) be conformal equivalent closed Weyl manifolds, and \(L, L'\) the corresponding weight bundles. Then \(L, L'\) are isomorphic as flat bundles.

**Proof:** Write \(g' = e^\zeta g\), \(\theta' = \theta + d\zeta\). Let \(\lambda, \lambda'\) be the sections of \(L, L'\) inducing the standard trivialization. Then \(e^\zeta \lambda\) satisfies

\[
L \nabla (e^\zeta \lambda) = \theta + d\zeta (e^\zeta \lambda) = \theta' + L \nabla (e^\zeta \lambda).
\]

\[\blacksquare\]

**Remark 2.6:** Using the trivialization of \(L\), we may consider \(g\) as a 2-form on \(M\) with values in \(L \otimes 2\). Then \(g\) is a parallel section of \(\text{S}^2 T^*M \otimes L \otimes 2\).

**Remark 2.7:** Let \(n = \dim \mathbb{R} M\). Denote by \(P\) the principal \(GL(n)\)-bundle \(P\) associated with \(TM\). Clearly, \(L\) is a line bundle associated with the representation \((\det T^*M)^{-\frac{1}{n}}\) of \(P\).

Let \((M, g, \nabla, \theta)\) be a closed Weyl manifold, and \(L\) its weight bundle. The natural connection \(L \nabla\) in \(L\) is flat because \(M\) is closed. Consider a covering \((\tilde{M}, \tilde{g}, \tilde{\nabla}, \tilde{\theta})\) of \(M\), such that the lift \(\tilde{L}\) of \(L\) to \(\tilde{M}\) has trivial monodromy. Let \(\zeta_0\) be a \(L \nabla\)-parallel section of \(\tilde{L}\), \(\zeta_0 \neq 0\), and \(\lambda\) the trivialization defined on \(L\) as on a weight bundle. The quotient \(\zeta := \frac{\zeta_0}{\lambda}\) satisfies

\[
d\zeta = \frac{\nabla_{\text{tr}}(\zeta_0)}{\lambda} = \frac{\theta}{2}
\]

because \(\nabla_{\text{tr}} - L \nabla = \frac{\theta}{2}\). Therefore, the manifolds \((M, g, \nabla, \theta)\) and \((M, e^{2\zeta} g, \nabla, 0)\) are conformally equivalent. We obtain the following claim.
Claim 2.8: Let \((M, g, \nabla, \theta)\) be a closed Weyl manifold, and \((L, L\nabla)\) its weight bundle. Assume that the monodromy of \(L\) is trivial. Then \((M, g, \nabla, \theta)\) is conformal equivalent to a Riemannian manifold equipped with a Levi-Civita connection. Conversely, if \((M, g, \nabla, \theta)\) is conformal equivalent to a Riemannian manifold, then its weight bundle \(L\) has trivial monodromy.

2.2 Locally conformal hyperkähler (LCHK) manifolds: the definition

Definition 2.9: Let \(M\) be a hypercomplex manifold, and \(\nabla\) the Obata connection on \(M\) (see Subsection 1.1). Assume that \(M\) admits a quaternionic Hermitian metric \(g\) and a closed form \(\theta\), such that \((M, g, \nabla, \theta)\) is a closed Weyl manifold. Then \(M\) is called locally conformal hyperkähler manifold, or LCHK manifold.

Let \(M\) be an LCHK manifold, \(L\) the corresponding weight bundle, and \((\tilde{M}, \tilde{g}, \tilde{\nabla})\) the covering associated with the monodromy representation of \(L\). In Claim 2.8 we constructed a Riemannian metric \(g' = e^{2\zeta} \tilde{g}\) which is preserved by \(\tilde{\nabla}\). Then \(\tilde{\nabla}\) is the Levi-Civita connection for \((\tilde{M}, g')\). On the other hand, \(\tilde{\nabla}\) preserves the quaternion action. As we have mentioned in the Introduction (Subsection 1.1), this implies that, the manifold \((\tilde{M}, g')\) is hyperkähler. We obtain

Claim 2.10: Let \(M\) be an LCHK (locally conformal hyperkähler) manifold, \(L\) its weight bundle, and \(\tilde{M}\) the covering of \(M\) associated with its monodromy. Then \(\tilde{M}\) is equipped with a hyperkähler metrics, which is determined uniquely up to a constant multiplier.

The converse statement is also true.

Proposition 2.11: Let \(M\) be a hypercomplex manifold, and \(\tilde{M}\) its universal covering. Assume that \(\tilde{M}\) is equipped with a hyperkähler metric \(g'\) in such a way that for any \(\gamma \in \pi_1(M)\) the corresponding deck transform \(k_\gamma : \tilde{M} \to \tilde{M}\) multiplies \(g'\) by a scalar \(c(\gamma) \in \mathbb{R}^{>0}\). Then \(M\) admits a LCHK metric, which is determined uniquely up to conformal equivalence.
Proof: The map $\gamma \rightarrow c(\gamma)$ defines a 1-dimensional representation $c : \pi_1(M) \rightarrow \mathbb{R}^{>0}$. Let $L$ be the corresponding flat bundle on $M$. Since $\mathbb{R}^{>0}$ is contractible, $L$ is topologically trivial. Pick a nowhere degenerate section $\lambda$ of $L$. The hyperkähler metrics $\tilde{g}$ on $\tilde{M}$ can be considered as a map $S^2T^*M \rightarrow L^{\otimes 2}$. Using $\lambda^2$ as a trivialization of $L^{\otimes 2}$, we obtain a hypercomplex Hermitian metrics $g_\lambda$ on $M$. Clearly,

$$\nabla_{g_\lambda} = \nabla (\lambda^2 \tilde{g}) = \lambda^2 \tilde{g} \otimes \nabla (\lambda^2) = g_\lambda \otimes (-2\theta_L),$$

where $\theta_L$ is the connection form of $L$ associated with the trivialization $\lambda$. Therefore, $(M, g_\lambda, \nabla, -2\theta_L)$ is an LCHK manifold. ■

Remark 2.12: Proposition 2.11 gives a nice interpretation of LCHK geometry, which is much more clear than the usual approach. We shall sometimes implicitly use Proposition 2.11 instead of the definition.

3 Vaisman manifolds

In this Section we present some introductory material in locally conformal Kähler geometry. For more details and a bibliography the reader is referred to [DO].

Definition 3.1: Let $M$ be a complex manifold equipped with a Hermitian metric $\omega$ and a closed Weyl connection $\nabla$,

$$\nabla(\omega) = \omega \otimes \theta, \quad \theta \in \Lambda^1(M).$$

Assume that $\nabla$ preserves the complex structure operator: $\nabla(I) = 0$. Then $M$ is called locally conformal Kähler (LCK) manifold, and $\theta$ is called the Lee form of $M$.

Remark 3.2: Given an LCHK (locally conformal hyperkähler) manifold $M$, we obtain an LCK structure $(M, J)$ for every quaternion $J, J^2 = -1$.

The basic working tool of locally conformal Kähler geometry is the following lemma of P. Gauduchon ([G1]).

Lemma 3.3: [G1] Let $M$ be a compact, oriented conformal manifold, $\dim_{\mathbb{R}} M > 2$. For any Weyl connection preserving the conformal structure, there exists a metric $g_0$ in this conformal class, such that the corresponding
Lee form $\theta$ is co-closed with respect to $g_0$. This metric is unique up to a constant multiplier.

**Definition 3.4:** Let $M$ be a conformal manifold equipped with a Weyl connection preserving the conformal structure $[g]$, and $g_0$ the metric in the conformal class $[g]$, such that the corresponding Lee form $\theta$ is co-closed. Then $g_0$ is called a Gauduchon metric.

**Remark 3.5:** If $(M, g, \nabla, \theta)$ is a closed Weyl manifold, and $g$ is a Gauduchon metric, then $\theta$ is harmonic.

For compact LCHK manifolds, the Gauduchon metric has parallel Lee form, in the following sense.

**Theorem 3.6:** [PPS] Let $M$ be an LCHK manifold equipped with a Gauduchon metric $g$, and $\theta$ its Lee form. Then $\nabla_g(\theta) = 0$, where $\nabla_g$ is the Levi-Civita connection associated with $g$.

**Definition 3.7:** Let $(M, g, \nabla, \theta)$ be an LCK manifold, and $\nabla_g$ its Levi-Civita connection. We say that $M$ is an LCK manifold with parallel Lee form, or Vaisman manifold, if $\nabla_g(\theta) = 0$. If $\theta \neq 0$, then after rescaling, we may always assume that $|\theta| = 1$. Unless otherwise stated, we shall assume implicitly that $|\theta| = 1$ for all Vaisman manifolds we consider.

Vaisman manifolds were introduced by I. Vaisman under the name "generalized Hopf manifolds" in a big series of papers (see e.g. [V1], [V2]) and studied extensively since then.

**Remark 3.8:** If $\nabla_g \theta = 0$, then $\theta$ is harmonic with respect to $g$. Therefore, the metric $g$ is automatically a Gauduchon metric.

**Example 3.9:** Fix a quaternion $q \in \mathbb{H}$, $|q| > 1$. Let $M := \mathbb{H}^n \setminus 0 / \sim_q$, where $\sim_q$ is an equivalence relation generated by $z \sim_q qz$. This manifold is called a Hopf manifold. Since the multiplication by $q$ preserves the flat connection $\nabla_{fl}$, this connection can be obtained as a pullback of a connection $\nabla$ on $M$. Since $\nabla_{fl}$ preserves the conformal class of a flat metric, $(M, \nabla_{fl})$ is a Weyl manifold. On the other hand, $M$ is by construction a hypercomplex manifold, and its covering a hyperkähler one. By Proposition 2.11
this implies that $M$ is an LCHK (locally conformal hyperkähler) manifold. Topologically, we have $M = S^{4n-1} \times S^1$. The Gauduchon metric is the standard metric on $S^{4n-1} \times S^1$, and the form $\theta$ is the coordinate form lifted from $S^1$. Therefore, $\theta$ is parallel.

For other examples of Vaisman manifolds, see e.g. [GO], [Bel], [KO].

4 Kähler potential on Vaisman manifolds

We construct a Kähler potential on a Vaisman manifold with exact Lee form (see [V2]).

**Proposition 4.1:** Let $M$ be an LCK manifold with parallel Lee form $\theta$, and $\theta^\sharp$ the vector field dual to $\theta$. Consider a diffeomorphism flow $\psi_t$ associated with $\theta^\sharp$. Then $\psi_t$ acts on $M$ preserving the LCK structure.

**Proof:** For a more detailed proof see e.g. [DO].

Since $\theta$ is parallel, $\theta^\sharp$ is a parallel vector field. Therefore, $\theta^\sharp$ is Killing, and $\psi_t$ acts on $M$ by isometries.

On the other hand, $\psi_t$ is a geodesic flow along $\theta^\sharp$, therefore its differential $d\psi_t : T_xX \to T_{\psi_t(x)}X$ is equal to the parallel transport along the geodesics associated with $\theta^\sharp$. Since the holonomy of $\nabla_g$ is contained in $U(n) \cdot \mathbb{R} \subset GL(n, \mathbb{C})$, we obtain that $d\psi_t$ is $\mathbb{C}$-linear. Therefore, $\psi_t$ is holomorphic. We find that $\psi_t$ preserves the complex and the Hermitian structure on $M$. The Weyl connection $\nabla$ can be written explicitly in terms of the Levi-Civita connection and the Lee form (see e.g. [Or], Definition 1.1)

\[
\nabla = \nabla_g - \frac{1}{2}(\theta \otimes Id + Id \otimes \theta - g \otimes \theta^\sharp),
\]

where $\nabla_g$ is the Levi-Civita connection on $M$. Since $\psi_t$ preserves $\theta$, $g$ and $\nabla_g$, we obtain that $\psi_t$ preserves $\nabla$. Proposition 4.1 is proven.

The Lee form $\theta$ is by definition closed. Passing to a covering if necessary, we may assume that it is exact: $\theta = dt$. Write $r = e^{-t}$. In the Example 3.9 $r$ is the radius function. In this case, $r$ is obviously a Kähler potential of $M$.

**Definition 4.2:** Let $M$ be an LCK manifold with exact Lee form $\theta = dt$. The function $r := e^{-t}$ is called the potential of $M$. Clearly, $r$ is defined uniquely, up to a positive constant multiplier.
Claim 4.3: Let \((M,g,\nabla)\) be an LCK manifold with exact Lee form \(\theta\), \(r\) its potential and \(\omega \in \Lambda^{1,1}(M)\) the Hermitian form of \((M,g)\). Then \(r\omega\) is a Kähler form.

Proof: Clearly,
\[
d(r\omega) = -\theta \wedge r\omega + r \cdot d\omega = -\theta \wedge r\omega + r\theta \wedge \omega = 0
\]
On the other hand, \(r\omega\) is positive definite, because \(r\) is a positive function. A closed positive definite \((1,1)\)-form is Kähler. \(\square\)

Proposition 4.4: Let \(M\) be an LCK manifold with parallel Lee form \(\theta\). Assume that \(\theta\) is exact, and let \(r\) be the corresponding potential function. Then \(r\) is the Kähler potential for the Kähler form \(r\omega\).

Proof: Let \(\text{Lie}_\theta\) be the operator of Lie derivative along the vector field \(\theta^\sharp\) dual to \(\theta\). Then \(\text{Lie}_\theta \omega = 0\) by Proposition 4.1. Similarly,
\[
\text{Lie}_\theta r = dr \cdot \theta^\sharp = -r\theta \cdot \theta^\sharp = -r.
\]
Therefore, \(\text{Lie}_\theta (r\omega) = -r\omega\). On the other hand, \(r\omega\) is closed: \(d(r\omega) = 0\). We obtain
\[
r\omega = -\text{Lie}_\theta (r\omega) = d(r\omega \cdot \theta^\sharp).
\]
Let \(d^c = I \circ d \circ I\) be the twisted de Rham differential. The function \(r\) is a Kähler potential for the form \(r\omega\) if \(r\omega = dd^c r\). As we have seen above, \(r\omega = d(r\omega \cdot \theta^\sharp)\). Therefore, \(r\omega = dd^c r\) is implied by
\[
r\omega \cdot \theta^\sharp = d^c r. \tag{4.2}
\]
To prove (4.2), notice that \(\omega \cdot \theta^\sharp = I(\theta)\), and \(dr = r\theta\). Therefore,
\[
r\omega \cdot \theta^\sharp = rI(\theta) = I(dr) = d^c r. \tag{4.3}
\]
This proves (4.2). Proposition 4.4 is proven. \(\square\)

5 Einstein-Weyl LCK manifolds

In the Section we relate the definition and basic results on the geometry of Einstein-Weyl LCK manifolds. We follow [Or].
Definition 5.1: Let $M$ be a Weyl manifold, $[g]$ its conformal class, $
abla$ its Weyl connection and $R$ the Ricci curvature of $\nabla$, $R = \text{Tr}_{jl} \Theta^l_{ijk}$, where $\Theta \in \Lambda^2(M) \otimes \Lambda^1(M) \otimes TM$ is the curvature of $\nabla$. Assume that the symmetric part of $R$ is proportional to $[g]$. Then $M$ is called an Einstein-Weyl manifold.

The following important result was proven by P. Gauduchon.

Theorem 5.2: Let $M$ be a compact Einstein-Weyl manifold with closed Lee form, and $g$ the Gauduchon metric on $M$ (Definition 3.4). Denote by $\theta$ the corresponding Lee form. Assume that $\theta$ is not exact. Then

(i) $\theta$ is parallel with respect to the Levi-Civita connection associated with $g$.

(ii) The Ricci curvature of the Weyl connection $\nabla$ vanishes.

Remark 5.3: By Theorem 5.2 (i), any Einstein-Weyl LCK manifold is also a Vaisman manifold.

The following claim is quite obvious from the definitions.

Claim 5.4: Let $M$ be a compact Einstein-Weyl manifold equipped with a Gauduchon metric, and $L$ the corresponding weight bundle, equipped with a canonical flat connection. Assume that $L$ has trivial monodromy, that is, the form $\theta$ is exact: $\theta = d(\zeta)$. Let $e^{-\zeta} \omega$ be the corresponding Kähler form associated with $\zeta$ as in Claim 2.8. Then the following conditions are equivalent

(i) The manifold $M$ is Einstein-Weyl

(ii) The Kähler metric $e^{-\zeta} \omega$ is Calabi-Yau, that is, the corresponding Levi-Civita connection is Ricci-flat.

Proof: Let $\nabla$ be the Weyl connection on $M$. Since $\nabla$ is torsion free and $\nabla$ preserves $e^{-\zeta} \omega$, $\nabla$ is the Levi-Civita connection associated with $e^{-\zeta} \omega$. The metric $e^{-\zeta} \omega$ is Calabi-Yau if and only if $\nabla$ is Ricci-flat.

Corollary 5.5: Let $M$ be a locally conformal hyperkähler (LCHK) manifold. Then $M$ is Einstein-Weyl.
**Proof:** By [Claim 2.10](#), the universal covering $\tilde{M}$ is hyperkähler. Therefore, $\tilde{M}$ is Calabi-Yau. Now [Claim 5.4](#) implies that $M$ is Einstein-Weyl.

Further on, we shall need the following proposition.

**Proposition 5.6:** Let $M$ be an Einstein-Weyl LCK manifold, and $L$ its weight bundle. Then $L^n = K^{-1}$, where $K$ is the canonical bundle of $M$ and $n = \dim\mathbb{C} M$.

**Proof:** Let $\tilde{M}$ be a universal covering of $M$. By [Claim 5.4](#), $\tilde{M}$ is equipped with a Ricci-flat Kähler metric. Therefore, the holonomy $\text{Hol}(\tilde{M})$ is contained in $SU(n)$. Since the monodromy of $M$ preserves the conformal class of the metric, we have $\text{Hol}(M) \subset SU(n) \cdot \mathbb{R}^{>0}$, where $\mathbb{R}^{>0}$ denotes the group of positive real numbers. By definition, the weight bundle corresponds to a representation $\det_{\mathbb{R}}(TM)^{\frac{1}{n}}$, $n = \dim\mathbb{C} M$ (see [Remark 2.7](#)). Therefore, the monodromy group of $L$ coincides with the quotient $\text{Hol}(M)/\text{Hol}(\tilde{M})$ of full holonomy by the local holonomy. Let $\alpha$ be a non-zero element of the monodromy group $G$ of $L$, $G = \text{Hol}(M)/\text{Hol}(\tilde{M}) \subset \mathbb{R}^{>0}$. Then $\alpha$ acts on $L$ as $(\alpha, l) \rightarrow \alpha l$, $\alpha \in \mathbb{R}^{>0}, l \in L$.

The action of $\text{Hol}(M)$ on $K(M)$ factors through

$$G = \text{Hol}(M)/\text{Hol}(\tilde{M}),$$

because $\text{Hol}(\tilde{M}) \subset SU(n)$. The quotient

$$\text{Hol}(M)/\text{Hol}(\tilde{M}) \subset \mathbb{R}^{>0}$$

acts on $K(M) = \det(A^{1,0}(M))$ as

$$(\alpha, \eta) \rightarrow \alpha^{-n} \eta, \quad \alpha \in \mathbb{R}^{>0}, \eta \in \det(A^{1,0}(M)).$$

This relates the monodromy of $K(M)$ and the action of holonomy. The bundles $L^n$, $K^{-1}$ are flat, and their monodromy is equal. Therefore, these bundles are isomorphic.

---

6 **The form $\omega_0$ on Vaisman manifolds**

In this Section we present some basic results and calculations on the geometry of Vaisman manifolds.

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15
6.1 The form $\omega_0$: definition and eigenvalues

Let $M$ be an LCK manifold. Consider the form

$$\omega_0 := d^e \theta$$

on $M$. We have $d^e = \frac{\partial - \overline{\partial}}{\sqrt{-1}}$. Therefore,

$$\omega_0 = \frac{\partial - \overline{\partial}}{\sqrt{-1}} \theta.$$

Write the Hodge decomposition of $\theta$ as $\theta = \theta^{1,0} + \theta^{0,1}$. Since $\theta$ is closed, we have

$$\partial \theta^{1,0} = \overline{\partial} \theta^{0,1} = 0, \quad \partial \theta^{0,1} = -\overline{\partial} \theta^{1,0}.$$ 

This implies

$$\omega_0 = -2\sqrt{-1} \partial \theta^{0,1}.$$

**Proposition 6.1:** Let $M$ be a Vaisman manifold, that is, an LCK manifold with parallel Lee form $\theta$. Consider the form $\omega_0 := d^e \theta$. Chose an orthonormal basis

$$\xi_1, ..., \xi_{n-1}, \sqrt{2} \cdot \theta^{1,0}$$

in $T^{1,0}M$, where $\theta^{1,0}$ is the $(0,1)$-part of $\theta$, and let

$$\omega = \sqrt{-1} (\xi_1 \wedge \overline{\xi}_1 + \xi_2 \wedge \overline{\xi}_2 + ... + \xi_{n-1} \wedge \overline{\xi}_{n-1} + 2 \theta^{1,0} \wedge \theta^{0,1})$$

be the Hermitian form on $M$. Then

$$\omega_0 = \sqrt{-1} (\xi_1 \wedge \overline{\xi}_1 + \xi_2 \wedge \overline{\xi}_2 + ... + \xi_{n-1} \wedge \overline{\xi}_{n-1}).$$

In particular, all eigenvalues of $\omega_0$ are positive except the one corresponding to $\theta$, which is equal zero.

**Proof:** Passing to a covering, we can always assume that $\theta$ is exact. Let $r$ be the potential of $M$. By (4.3), we have

$$d^e (r \theta) = r \omega.$$ 

(6.3)

On the other hand,

$$d^e (r \theta) = rd^e \theta + d^e r \wedge \theta = rd^e \theta + r I(\theta) \wedge \theta.$$ 

(6.4)

$^1$Since $|\theta| = 1$, we have $|\theta^{1,0}| = \frac{1}{\sqrt{2}}$. 

16
Comparing (6.3) and (6.4), we find \( \omega = d\theta + \theta \wedge I(\theta) \), and
\[
\omega_0 = \omega - 2\sqrt{-1} \theta^{1,0} \wedge \theta^{0,1}.
\] (6.5)

This proves Proposition 6.1.

The following claim is obvious.

**Claim 6.2:** The form \( \omega_0 \) is exact.

**Proof:** By definition, we have
\[
\omega_0 = -Id(I(\theta)).
\] (6.6)

On the other hand, \( \omega_0 \) is a (1,1)-form, hence \( I(\omega_0) = \omega_0 \). Comparing this with (6.6), we find
\[
\omega_0 = -d(I(\theta)).
\]

\[
\omega_0 \wedge I(\theta) = 0
\] (6.7)

### 6.2 The form \( \omega_0 \) and the canonical foliation

Let \( M \) be a Vaisman manifold, that is, an LCK manifold with a parallel Lee form \( \theta \). Consider a 2-dimensional real foliation \( \Xi \subset TM \) generated by \( \theta^\sharp, I(\theta^\sharp) \). The vector field \( \theta^\sharp \) is holomorphic (Proposition 4.1). Therefore, \( \Xi \) is integrable and holomorphic.

**Definition 6.3:** The foliation \( \Xi \) is called the **canonical foliation** of a Vaisman manifold \( M \).

The canonical foliation is related to the form \( \omega_0 \) in the following way.

**Proposition 6.4:** Let \( M \) be a Vaisman manifold, \( \Xi \) the foliation defined above and \( \omega_0 = d\theta \) the standard (1,1)-form. Assume that the space of leaves of \( \Xi \) is well defined, and let \( f : M \to Y \) the the corresponding quotient map. Then \( Y \) is equipped with a natural Kähler form \( \omega_Y \), in such a way that \( f^*\omega_Y = \omega_0 \).

**Proof:** To check that \( \omega_0 = f^*\omega_Y \) for some 2-form \( \omega_Y \), we need to show that
\[
\omega_0 \wedge I(\theta^\sharp) = \omega_0 \wedge I(\theta^\sharp) = 0
\] (6.7)
and

\[ \text{Lie}_{\theta^*} \omega_0 = \text{Lie}_{I(\theta^*)} \omega_0 = 0. \]

(6.8)

The equation (6.7) follows immediately from Proposition 6.1. Using the Cartan formula for the Lie derivative and (6.7), we obtain

\[ \text{Lie}_{\theta^*} \omega_0 = (d\omega_0) \cdot \theta^* = 0 \]

and

\[ \text{Lie}_{I(\theta^*)} \omega_0 = (d\omega_0) \cdot I(\theta^*) = 0. \]

This implies that \( \omega_0 = f^* \omega_Y \) for some 2-form \( \omega_Y \) on \( Y \). Since \( \omega_0 \) has \( \dim_{\mathbb{C}} Y = n - 1 \) positive eigenvalues, the form \( \omega_Y \) is positive definite. Therefore, \( \omega_Y \) is a Kähler form on \( Y \).

The form \( \omega_0 \) is exact (Claim 6.2). The following proposition immediately follows from this observation.

**Proposition 6.5:** Let \( M \) be a compact Vaisman manifold, and \( X \subset M \) a closed complex subvariety. Then \( X \) is tangent to the canonical foliation \( \Xi \) in all its smooth points. In particular, if \( X \) is smooth, then \( X \) is also a Vaisman manifold.

**Proof:** The form \( \omega_0 \) is positive in the sense of distribution theory; that is, \( \omega_0 \) is a real \((1,1)\)-form with non-negative eigenvalues. It is well known that

\[ \int_X \omega_0^k \geq 0 \]

(6.9)

for all complex subvarieties \( X \subset M \), \( \dim_{\mathbb{C}} X = k \), and all positive forms \( \omega_0 \). Moreover, the integral (6.9) vanishes only if \( X \) is tangent to the null-space foliation of \( \omega_0 \).

Since \( \omega_0 \) is exact, the integral (6.9) vanishes. Therefore, \( X \) is tangent to the null-space foliation of \( \omega_0 \). As we have seen above, the null-space of \( \omega_0 \) is \( \Xi \). This implies \( X \) is tangent to the canonical foliation.

If \( X \) is smooth, it is clearly an LCK manifold. To prove that \( X \) is a Vaisman submanifold, we use the following theorem of Kamishima and Ornea ([KO]).

**Theorem 6.6:** ([KO] Let \( X \) be a compact LCK manifold admitting a conformal holomorphic flow which is not conformal equivalent to isometry. Then \( X \) is a Vaisman manifold.
Consider the holomorphic flow $\psi_t$ associated with the Lee field on $M$ [Proposition 4.1]. Since $X$ is tangent to $\Xi$, $\psi_t$ preserves $X$. It is trivial to check that $\psi_t$ is not conformal equivalent to isometry. Applying Theorem 6.6 we find that $X$ is Vaisman. Proposition 6.5 is proven. ■

For smooth manifolds, Proposition 6.5 is proven by K. Tsukada, see [Ts2].

6.3 Curvature of a weight bundle

Let $M$ be an LCK manifold and $L$ its weight bundle. By construction, $L$ is a complex vector bundle equipped with a flat connection $L\nabla$ and a nowhere degenerate section $\lambda$, such that

$$L\nabla(\lambda) = \lambda \otimes \left(-\frac{1}{2} \theta\right).$$

A $(0,1)$-part of $L\nabla$ gives a holomorphic structure on $L$. Throughout this Subsection, we shall consider $L$ as a holomorphic bundle. Consider a Hermitian structure $g_L$ on $L$, defined in such a way that $|\lambda|_{g_L} = 1$.

**Theorem 6.7:** Let $M$ be an LCK manifold, and $L$ the corresponding weight bundle equipped with a holomorphic and a Hermitian structure as above. Let $C\nabla$ be the standard Hermitian connection on $L$ (so-called Chern connection), and $C$ its curvature. Then $C = -2\sqrt{-1} \omega_0$, where $\omega_0 = d^c \theta$ is the standard 2-form on $M$.

**Proof:** By definition of the Chern connection, we have

$$C\nabla^{0,1}(\lambda) = L\nabla^{0,1}(\lambda) = \lambda \otimes \left(-\frac{1}{2} \theta^{0,1}\right),$$

and

$$(C\nabla^{1,0}(\lambda), \lambda)_H = -(\lambda, C\nabla^{0,1}(\lambda))_H,$$

where $(\cdot, \cdot)_H$ is the Hermitian form on $L$. From (6.10), we obtain

$$C\nabla = \nabla_{tr} + \frac{\theta^{0,1} - \theta^{1,0}}{2},$$

where $\nabla_{tr}$ is the trivial connection on $L$ fixing $\lambda$. From (6.11), we obtain

$$C = \frac{\partial \theta^{0,1} - \bar{\partial} \theta^{0,1}}{2}.$$
On the other hand, $\theta$ is closed, and therefore $\partial\theta^{0,1} = -\overline{\partial}\theta^{0,1}$. Comparing this with (6.12), we obtain

$$C = \partial\theta^{0,1} = -2\sqrt{-1}\omega_0.$$  

(the last equation holds by (6.1)). We proved Theorem 6.7.

7 Kähler geometry of the form $\omega_0$

The form $\omega_0$ behaves, in many ways, as a surrogate Kähler form on $M$. In this Section we prove the $\omega_0$-version of the Kodaira relations and apply it to obtain the standard identities for Laplace operators with coefficients in a bundle. This leads to Kodaira-Nakano-type vanishing theorems.

7.1 The $SL(2)$-triple $L_0, \Lambda_0, H_0$

In this Subsection, we study the Lefschetz-type $SL(2)$-action associated with $\omega_0$. Let $M$ be a Vaisman manifold and $\omega_0$ the standard 2-form (Section 6). Denote by $L_0$ the operator $\eta \rightarrow \eta \wedge \omega_0$, and by $\Lambda_0$ the Hermitian adjoint operator. Using the coordinate expression of $\omega_0$ given in Proposition 6.1, we find that $L_0, \Lambda_0, H_0 := [L_0, \Lambda_0]$ form an $SL(2)$-triple (exactly the same argument is used in the proof of Lefschetz theorem via the $SL(2)$-action on cohomology, see [GH]).

Locally, the operator $H_0$ can be expressed as follows. Let

$$\xi_1, \xi_2, \ldots, \xi_{n-1}, \sqrt{2}\theta^{1,0}, \overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_{n-1}, \sqrt{2}\theta^{0,1} \in \Lambda^1(M)$$

be an orthonormal frame in the bundle of forms. Consider a monomial

$$\lambda = \xi_{i_1} \wedge \xi_{i_2} \wedge \ldots \wedge \xi_{i_k} \wedge \overline{\xi}_{i_{k+1}} \wedge \overline{\xi}_{i_{k+2}} \wedge \ldots \wedge \xi_{i_p} \wedge R \quad (7.1)$$

where $R$ is a monomial of $\theta^{0,1}, \theta^{1,0}$. Then

$$H_0(\lambda) = (p - n + 1)\lambda. \quad (7.2)$$

where $p$ is the number of $\xi_{i_l}, \overline{\xi}_{i_l}$ in $\lambda$ and $n = \dim_{\mathbb{C}} M$.

The equation (7.2) is proved in the same way as the explicit form of the operator $H$ in Hodge theory ([GH]).
7.2 Kodaira identities for the differential $\partial_0$

Let $M$ be a Vaisman manifold (an LCK manifold with a parallel Lee form). Consider the $\omega_0$-Lefschetz triple $L_0$, $\Lambda_0$, $H_0$ acting on $\Lambda^*(M)$ as above, and let

$$\Lambda^*(M) = \bigoplus_i \Lambda^*_i(M)$$

be the weight decomposition associated with $H_0$, in such a way that the monomial (7.1) has weight $p$. Clearly,

$$\Lambda^p(M) = \Lambda^p_p(M) \oplus \Lambda^p_{p-1}(M) \oplus \Lambda^p_{p-2}(M).$$

(7.3)

Denote by $d_0$ the weight 1 component of the de Rham differential:

$$d_0 : \Lambda^p(M) \longrightarrow \Lambda^{p+1}(M).$$

(7.4)

**Remark 7.1:** By (6.5), we have $d(\omega_0) = \omega_0 \wedge \theta$. Therefore, $d_0(\omega_0) = 0$. Similarly, one can check by elementary computations that $d_0 \theta = 0$ and $d_0(I(\theta)) = 0$.

**Remark 7.2:** The differential $d_0$ satisfies the Leibniz rule.

The (1,0)-part and (0,1)-part of $d_0$ are denoted by $\partial_0$, $\overline{\partial}_0$ as usual.

**Definition 7.3:** The operator $d_0$ is called the $\omega_0$-de Rham differential, the operators $\partial_0$, $\overline{\partial}_0$ the $\omega_0$-Dolbeault operators.

**Theorem 7.4:** (Kodaira identities for $\partial_0$, $\overline{\partial}_0$). Let $M$ be a Vaisman manifold, and $\partial_0$, $\overline{\partial}_0$, $L_0$, $\Lambda_0$ the operators defined above. Then

$$[\Lambda_0, \partial_0] = \sqrt{-1} \overline{\partial}_0, \quad [L_0, \overline{\partial}_0] = -\sqrt{-1} \partial_0^*,$$

$$[\Lambda_0, \overline{\partial}_0^*] = -\sqrt{-1} \overline{\partial}_0, \quad [L_0, \partial_0^*] = \sqrt{-1} \partial_0,$$

(7.5)

where $\partial_0^*$, $\overline{\partial}_0^*$ are the Hermitian adjoint operators to $\partial_0$, $\overline{\partial}_0$.

**Proof:** Further on, we shall prove

$$[L_0, d_0^*] = -I \circ d_0 \circ I.$$

(7.6)

Taking the (0,1) and (1,0)-parts of (7.6), we obtain the bottom line of (7.5). Taking Hermitian adjoint, we obtain the top line of (7.5). Therefore, (7.6) implies the Kodaira relations for $\partial_0$. 

21
We prove (7.6) in an algebraic fashion similar to the proof of Kodaira relations in HKT geometry ([Ve]). Consider $L_0$, $d_0$, etc. as operators on the algebra of differential forms. A. Grothendieck gave a general recursive definition of differential operators on an algebra (see, e.g., [Ve]). Then $L_0$ is a 0-th order algebraic differential operator, and $d_0^*$ a second order algebraic differential operator on $\Lambda^*(M)$ (this is straightforward; see the full argument in [Ve]). Therefore, the commutator $[L_0, d_0^*]$ is a first order algebraic differential operator. Since $-I \circ d_0 \circ I$ satisfies the Leibniz rule, it is also a first order algebraic differential operator. A first order algebraic differential operator $D$ satisfies

$$D(ax) = (-1)^{\deg D} d^{\deg a} D(a) x + D(a) x - D(1) ax.$$  (7.7)

This is clear from the definition; see, again, [Ve]. From (7.7), we obtain that the first order algebraic differential operator is determined by the values taken on any set of generators of the algebra. Since both sides of (7.6) are first order algebraic differential operators, it suffices to check that they are equal on some subspace $V \subset \Lambda^*(M)$ which generates the algebra $\Lambda^*(M)$.

On 0-forms, (7.6) is clear:

$$d_0^*(f \omega_0) = \omega_0 \cdot (d_0 f)^2 = -I(d_0 f)$$

where $v^\sharp$, as usually, denotes the vector field associated with a form $v$.

We are going to construct a subspace $V \subset \Lambda^1(M)$ generating $\Lambda^1(M)$ over $C^\infty(M)$ such that the operators on both sides of equation (7.6) are equal on $V$. As we have mentioned above, this is sufficient for the proof of Theorem 7.4.

The statement of Theorem 7.4 is local. Passing to an open neighbourhood if necessary, we may assume that the space $Y$ of leaves of $\Xi$ is well defined. Let $f : M \to Y$ be the corresponding quotient map. By Proposition 6.4 the manifold $Y$ is equipped with a natural Kähler form $\omega_Y$, in such a way that $f^* \omega_Y = \omega_0$. Let $d_Y$ be the de Rham operator on $Y$, and $d_Y^*$ its Hermitian adjoint. Denote by $L_Y$ the Hodge operator on $Y$, $L_Y(\eta) = \eta \wedge \omega_Y$. Since $Y$ is Kähler, the usual Kodaira identity holds:

$$[L_Y, d_Y^*] = -I \circ d_Y \circ I,$$  (7.8)
Lifting (7.8) to $M$, we obtain that (7.6) holds on all forms $\eta = f^*\eta_Y$ which are obtained as a pullback.

Let us check (7.6) on the 2-dimensional space $\langle \theta, I(\theta) \rangle \subset \Lambda^1(M)$ generated by $\theta$ and $I(\theta)$. We have $-I \circ d_0 \circ I(\theta) = 0$ because $\theta$ is $d_0$-closed (Remark 7.1). The same argument proves $d_0^*(\omega^k \wedge I(\theta)) = *d_0(\omega^{n-k} \wedge \theta) = 0$ because $d_0(\theta) = d_0(\omega) = 0$ (see Remark 7.1). Therefore, the operators $[L_0, d_0^*]$ and $-I \circ d_0 \circ I$ vanish on $I(\theta)$.

Similarly, we have $-I \circ d_0 \circ I(\theta) = \omega_0$, and $[L_0, d_0^*](\theta) = 0$.

We have shown that the operators $[L_0, d_0^*]$ and $-I \circ d_0 \circ I$ are equal on the space $V \subset \Lambda^1(M)$ generated by $\theta, I(\theta)$ and the pullbacks of differential forms from $Y$. Clearly, $C^\infty(M) \cdot V = \Lambda^1(M)$. Therefore, the first order algebraic differential operators $[L_0, d_0^*]$ and $-I \circ d_0 \circ I$ are equal on a set of generators of $\Lambda^*(M)$. This proves (7.6). Theorem 7.4 is proven.

A similar argument proves the following

**Claim 7.5:** Let $M$ be a Vaisman manifold, and $d_0, \partial_0, \overline{\partial}_0$ the differential operators defined above. Then

$$d_0^2 = \partial_0^2 = \overline{\partial}_0^2 = 0.$$

**Proof:** Since $\partial_0^2$ and $\overline{\partial}_0^2$ are $(2,0)$ and $(0,2)$-parts of $d_0^2$, it suffices to show that $d_0^2 = 0$. We use the same algebraic argument as in the proof of Theorem 7.4. The anticommutator $d_0^2 = \{d_0, d_0\}$ is a first order algebraic differential operator, because it is a supercommutator of two first order algebraic differential operators. Therefore, it suffices to prove that $d_0^2 = 0$ on some set of generators of $\Lambda^*(M)$.

All 1-forms have weight $\leq 1$ with respect to the decomposition (7.3). Therefore, the restriction $d^2|_{\Lambda^0(M)}$ has components of weight 0 and 1 only (no weight 2 component). Therefore, the weight 2 component of $d^2|_{\Lambda^0(M)}$ is equal to $d_0^2$. Since $d^2 = 0$, we have

$$d_0^2|_{\Lambda^0(M)} = 0.$$

Passing to a local neighbourhood, we may always assume that the space $Y$ of leaves of $\Xi$ is well defined. Clearly, $d_0 = d$ on all forms lifted from $Y$. Therefore, $d_0^2 = d^2 = 0$ on such forms.
Finally, \( d_0 = 0 \) on the space generated by \( \theta, I(\theta) \). Hence \( d_0^2 = 0 \) on this space. We obtain that \( d_0^2 = 0 \) on a set of generators of the algebra \( \Lambda^*(M) \). This implies \( d_0^2 = 0 \). We proved Claim 7.5.

7.3 Laplace operators with coefficients in a bundle

Let \( M \) be a Vaisman manifold, \( B \) a holomorphic vector bundle equipped with a Hermitian form, and

\[
\nabla : \Lambda^p(M) \otimes B \rightarrow \Lambda^{p+1}(M) \otimes B
\]

the Chern connection on \( B \). Denote by \( d_0 \) the weight 1 component of \( \nabla \), taken with respect to the \( H_0 \)-action (see (7.1)). Let \( \partial_0, \overline{\partial}_0 \) be the \((1,0)\)- and \((0,1)\)-parts of \( d_0 \), and \( \partial^*_0, \overline{\partial}^*_0 \) the Hermitian adjoint operators.

**Proposition 7.6**: (Kodaira identities for \( \partial_0, \overline{\partial}_0 \) with coefficients in a Hermitian bundle). Let \( M \) be a Vaisman manifold, \( B \) a Hermitian holomorphic bundle and

\[
\partial_0, \overline{\partial}_0, L_0, \Lambda_0 : \Lambda^*(M) \otimes B \rightarrow \Lambda^*(M) \otimes B
\]

the operators defined above. Then

\[
\begin{align*}
[\Lambda_0, \partial_0] &= \sqrt{-1} \partial^*_0, \quad [L_0, \overline{\partial}_0] = -\sqrt{-1} \partial^*_0, \\
[\Lambda_0, \overline{\partial}_0] &= -\sqrt{-1} \partial_0, \quad [L_0, \partial^*_0] = \sqrt{-1} \overline{\partial}_0. 
\end{align*}
\] (7.9)

**Proof**: The proof of Proposition 7.6 is essentially the same as the proof of the Kodaira identities with coefficients in a bundle on a Kähler manifold. We deduce Proposition 7.6 from the usual (coefficient-less) Kodaira identities, proven in Theorem 7.4.

Write the Chern connection in \( B \) as

\[
\nabla = \partial^{tr} + \overline{\partial}^{tr} + \eta + \overline{\eta},
\] (7.10)

where \( \partial^{tr} + \overline{\partial}^{tr} \) is the trivial connection fixing the Hermitian structure, and \( \eta \in \Lambda^{1,0}(M) \otimes \text{End} B \) a \((1,0)\)-form. Let \( \eta_0 := \Pi(\eta) \) be the projection of \( \eta \) to \( \Lambda^0_0(M) \otimes \text{End} B \subset \Lambda^{1}(M) \otimes \text{End} B \). We consider \( \eta, \eta_0 \) as operators on differential forms. Denote by \( \eta^*, \eta^*_0 \) the Hermitian adjoint operators. Using the coordinate expression of \( \omega_0 \) given in Proposition 6.1, we find

\[
\begin{align*}
[\Lambda_0, \eta_0] &= \sqrt{-1} \overline{\eta}_0, \quad [L_0, \overline{\eta}_0] = -\sqrt{-1} \eta^*_0, \\
[\Lambda_0, \eta^*_0] &= -\sqrt{-1} \eta_0, \quad [L_0, \eta^*_0] = \sqrt{-1} \overline{\eta}_0. 
\end{align*}
\] (7.11)
Adding (7.11) and (7.5) (which holds for the trivial connection $\nabla^{tr} = \overline{\nabla}^{tr} + \overline{\partial}^{tr}$) termwise, we obtain (7.9). This proves Proposition 7.6.

**Theorem 7.7:** Let $M$ be a Vaisman manifold, $B$ a Hermitian holomorphic bundle and

$$\partial_0, \overline{\partial}_0, L_0, \Lambda_0 : \Lambda^r(M) \otimes B \longrightarrow \Lambda^s(M) \otimes B$$

the operators defined above. Consider the Laplacians $\Delta_{\partial_0} = \partial_0 \partial_0^* + \partial_0^* \partial_0$, $\Delta_{\overline{\partial}_0} = \overline{\partial}_0 \overline{\partial}_0^* + \overline{\partial}_0^* \overline{\partial}_0$. Then

$$\Delta_{\partial_0} - \Delta_{\overline{\partial}_0} = \sqrt{-1} [\Theta_B, \Lambda_0],$$

(7.12)

where $\Theta_B : \Lambda^p(M) \longrightarrow \Lambda^{p+2}(M) \otimes B$ is the curvature operator of $B$.

**Proof:** Theorem 7.7 is a formal consequence of the Kodaira relations (7.9) (see e.g. [GH] for a detailed proof).

### 7.4 Serre’s duality for $\overline{\partial}_0$-cohomology

Further on, we shall use the following version of Serre’s duality. Since the operator $\overline{\partial}_0$ is not elliptic, the $\overline{\partial}_0$-cohomology can be infinite-dimensional. Therefore, it is more convenient to state the Serre’s duality as an isomorphism of vector spaces.

**Theorem 7.8:** (Serre’s duality for $\overline{\partial}_0$-cohomology) Let $M$ be a compact Vaisman manifold, $\dim_{\mathbb{C}} M = n$, $B$ a holomorphic Hermitian bundle, and $K$ the canonical bundle. Then there exists an isomorphism of the spaces of $\overline{\partial}_0$-harmonic forms with coefficients in $B$, $B^* \otimes K$

$$\mathcal{H}_{\overline{\partial}_0}^p(B) \cong \mathcal{H}_{\overline{\partial}_0}^{n-p}(B^* \otimes K),$$

where $\overline{V}$ denotes the complex conjugate vector space to $V$, that is, the same real space with the opposite complex structure.

**Proof:** Consider the Hodge $*$ operator acting on the differential forms with coefficients in a bundle. Given a $\overline{\partial}_0$-harmonic form $\eta \in \Lambda^{0,p}(M, B)$, the form $*\eta$ is a $\partial_0$-harmonic $(n-p, n)$-form with coefficients in $\overline{B}$. This is clearly the same as $\partial_0$-harmonic $(n-p, 0)$-form with coefficients in $\mathbb{K} \otimes \overline{B}$. Taking a complex conjugate, we obtain that $\overline{*\eta}$ can be considered as a $\overline{\partial}_0$-harmonic form with coefficients in $B^* \otimes K$. This proves Theorem 7.8.
8 Vanishing theorems for Vaisman manifolds

8.1 Vanishing theorem for the differential $\partial_0$

As one does in the proof of Kodaira-Nakano-type vanishing theorem, the Kodaira relations (7.12) can be used to obtain various vanishing results for $\partial_0$-cohomology of holomorphic vector bundles. We do not need the whole spectrum of vanishing theorems in this paper; they can be stated and proven in the same way as the usual vanishing theorems. For our present purposes, we need only the following result.

**Theorem 8.1:** Let $M$ be a compact Vaisman manifold, that is, an LCK manifold with parallel Lee form $\theta$, and $V$ a positive tensor power of the weight bundle [Definition 2.4], equipped with a Hermitian structure as in Subsection 6.3. Consider the $\omega_0$-differential $\partial_0 : \Lambda^0,0(M) \otimes V \longrightarrow \Lambda^0,1(M) \otimes V$ associated with the Chern connection in $V$ (see Subsection 7.3), and let $\eta \in \Lambda^{0,p}(M) \otimes V$ be a non-zero $\partial_0$-harmonic form. Then $p \geq \dim \mathbb{C} M - 1$. Moreover, if $p = \dim \mathbb{C} M - 1$, then $\eta \cdot \theta^2 = 0$, where $\theta^2$ is the vector field dual to $\theta$.

**Remark 8.2:** An elementary linear-algebraic argument implies that all $(n-1,0)$-forms satisfying $\eta \cdot \theta^2 = 0$ are proportional to $\mathcal{L} \cdot \theta^2$, where $\mathcal{L}$ is a non-degenerate $(n,0)$-form on $M$.

**Proof of Theorem 8.1** By Theorem 7.7, we have

$$\Delta_{\partial_0} - \Delta_{\partial_0} = \sqrt{-1} [\Theta_V, \Lambda_0],$$

where $\Theta_V$ is the curvature operator of $V$. By Theorem 6.7, $\sqrt{-1} \Theta_V = \omega_0$, $c > 0$. Therefore,

$$\Delta_{\partial_0} = \Delta_{\partial_0} + cH_0,$$

where $H_0 = [L_0, \Lambda_0]$. By (7.2), $H_0$ is positive definite on $(r,0)$-forms for $r < n - 1$. By (8.1), the Laplace operator $\Delta_{\partial_0}$ is a sum of a positive semidefinite operator $\Delta_{\partial_0}$ and positive definite $cH_0$. Therefore, all eigenvalues of $\Delta_{\partial_0}$ are strictly positive, and there are no harmonic $(r,0)$-forms $r < n - 1$.

Similarly, $H_0$ is positive semidefinite on $(n-1,0)$-forms, and its only zero eigenvalue corresponds to the form

$$\nu = \xi_1 \wedge \xi_2 \wedge ... \wedge \xi_{n-1}.$$
Vanishing theorems for LCHK manifolds

M. Verbitsky, 15 February 2003

(we use the notation of Proposition 6.1 here). Therefore, if $\Delta_{\partial^0_0}(\eta) = 0$, then $\Delta_{\partial^0_0}(\eta) = 0$ and $H_0(\eta) = 0$, and $\eta$ is proportional to $\nu$. We proved Theorem 8.1.

8.2 Basic cohomology

To be able to use the vanishing theorem obtained above, we need a way to compare the cohomology of $\partial^0_0$ and the Dolbeault cohomology. This comparison is obtained from the results of K. Tsukada ([Ts1]).

Let $M$ be a manifold equipped with a foliation $\Xi$. A form $\eta$ is called basic if for all vector fields $v \in \Xi$, we have $\eta \lrcorner v = 0$, $\text{Lie}_v \eta = 0$. Locally, such forms are lifted from the space of leaves of $\Xi$, if this space is defined. Clearly, $d\eta$ is basic if $\eta$ is basic. The de Rham cohomology of basic forms is called the basic cohomology of the foliation $\Xi$ ([To], [NT]).

Given a holomorphic foliation on a complex manifold, we can also define the basic Dolbeault cohomology, as the cohomology of the differential $\overline{\partial}$ on the basic forms. Clearly, on basic forms, the differential $\overline{\partial}$ is equal to the differential $\partial^0_0$ defined above.

**Theorem 8.3:** Let $M$ be an $n$-dimensional compact Vaisman manifold, and $\eta$ a $(p,q)$-form, with $p + q \leq n - 1$. Denote by $\theta^{0,1}$ the $(0,1)$-part of the Lee form, and let $\Lambda_0$ be the Hodge operator associated with the form $\omega_0$ (Subsection 7.2). Then the following conditions are equivalent.

(i) The form $\eta$ is $\overline{\partial}$-harmonic: $\overline{\partial}^\dagger \eta + \overline{\partial}^\ast \overline{\partial} \eta = 0$.

(ii) $\eta$ has a decomposition $\eta = \theta^{0,1} \wedge \alpha + \beta$, where $\alpha$, $\beta$ are basic forms of the canonical foliation $\Xi$ which are $\overline{\partial}_0$-harmonic and satisfy $\Lambda_0 \alpha = \Lambda_0 \beta = 0$.

**Proof:** This is [Ts1], Theorem 3.2.

8.3 The cohomology of the structure sheaf

The vanishing result of Subsection 8.1 can be used to prove the following theorem.

**Theorem 8.4:** Let $M$ be a compact Vaisman manifold, such that the
Vanishing theorems for LCHK manifolds

M. Verbitsky, 15 February 2003

The canonical bundle $K(M)$ is a negative power of the weight bundle\(^1\). Then the holomorphic cohomology of the structure sheaf satisfy $H^i(\mathcal{O}_M) = 0$ for $i > 1$, $\dim H^1(\mathcal{O}_M) = 1$.

**Proof:** [Theorem 8.4] is a consequence of [Theorem 8.1] and [Theorem 8.3]. Indeed, [Theorem 8.3] implies that to prove that $H^i(\mathcal{O}_M) = 0$ for $i > 1$, $\dim H^1(\mathcal{O}_M) = 1$, it suffices to show that all basic $\overline{\partial}$-harmonic $(0,p)$-forms vanish, for $p > 0$. A basic form is $\overline{\partial}$-harmonic if and only if it is $\overline{\partial}_0$-harmonic. By Serre’s duality,

$$H^p_{\overline{\partial}_0}(\mathcal{O}_M) \cong H^{n-p}_{\overline{\partial}_0}(K),$$

([Theorem 7.8]). By [Theorem 8.1] $H^{n-p}_{\overline{\partial}_0}(K) = 0$ for $p > 1$, and for $p = 1$ all non-trivial classes $\eta \in H^{n-1}_{\overline{\partial}_0}(K)$ satisfy $\eta \cdot \theta^{0,1} = 0$. Applying the Serre’s duality again, we obtain that any $\overline{\partial}_0$-harmonic $(0,p)$-form $\eta \in H^p_{\overline{\partial}_0}(\mathcal{O}_M)$ vanishes for $p > 1$, and for $p = 1$, $\eta$ is proportional to $\theta^{0,1}$. Therefore, for $p = 1$, $\eta$ cannot be basic, and we obtain that there are no non-trivial basic $\overline{\partial}$-harmonic $(0,p)$-forms. This proves [Theorem 8.3].

A similar theorem was obtained in [AI] using an estimate of Ricci curvature.

9 Holomorphic forms on Einstein-Weyl LCK manifolds

In this Section, we prove that all holomorphic $(p,0)$-forms on a compact Einstein-Weyl LCK manifold vanish, for $p > 0$.

9.1 The $\omega_0$-Yang-Mills bundles

Let $M$ be a Vaisman manifold, that is, an LCK manifold with parallel Lee form $\theta$, and $\omega_0 = d^c(\theta)$ the standard 2-form (Section 6). For our purposes, $\omega_0$ plays the role of a Kähler form on $M$. It is natural to study the Yang-Mills geometry associated with $\omega_0$.

**Definition 9.1:** In the above assumptions, let $B$ be a holomorphic Hermitian bundle equipped with the standard (Chern) connection, $\Theta_B$ its cur-

\(^1\)By Proposition 5.6 this holds for all Einstein-Weyl manifolds. Hence, any LCHK manifold satisfies this assumption.
 curvature, and \( \Lambda_0 : \Lambda^p(M) \to \Lambda^{p-2}(M) \) the Hodge-type operator associated with \( \omega_0 \) (see Subsection 7.2). We say that \( B \) is \( \omega_0 \)-Yang-Mills if

\[
\Lambda_0(\Theta_B) = -c\sqrt{-1} \text{Id}_B,
\]

where \( \text{Id}_B \) is the identity section of \( \text{End}(B) \), and \( c \) a constant. We call \( c \) the Yang-Mills constant of \( B \).

The \( \omega_0 \)-Yang-Mills bundles satisfy the same elementary properties as the usual Yang-Mills bundles. In particular, any tensor power of \( \omega_0 \)-Yang-Mills bundles is again \( \omega_0 \)-Yang-Mills. The following theorem provides us with an example of a \( \omega_0 \)-Yang-Mills bundle.

**Proposition 9.2:** Let \( M \) be an Einstein-Weyl LCK manifold with parallel Lee form.\(^1\) Consider the tangent bundle \( TM \) as a holomorphic Hermitian bundle. Then \( TM \) is \( \omega_0 \)-Yang-Mills, with positive Yang-Mills constant.

**Proof:** Let \( \nabla_W \) be the Weyl connection on \( TM \), and \( \nabla_C \) the Chern connection. By definition, \( \nabla_W^{0,1} = \nabla_C^{0,1} \). Consider the weight bundle \((L, \nabla_L)\) on \( M \). Since \( TM \) is equipped with a metric with values in \( L^{\otimes 2} \), the natural connection \( \nabla_{W,L} \) on \( TM \otimes (L^*)^{\otimes 2} \) induced by \( \nabla_W, \nabla_L \), is Hermitian. Therefore, it is a Chern connection on the holomorphic Hermitian vector bundle \( TM \otimes L^{\otimes 2} \).

Since \( \nabla_L \) is flat, we have \( \nabla_{W,L}^2 = \nabla_W^2 \). The Chern connection on \( TM \otimes (L^*)^{\otimes 2} \) is obtained as a tensor product of the Chern connection on \( TM \) and that on \( L \). We obtain

\[
\nabla_W^2 = \nabla_{W,L}^2 = \nabla_C^2 - 2C,
\]

where \( C = -2\sqrt{-1} \omega_0 \) is the curvature of the Chern connection on \( L \) (see Theorem 6.7). This gives

\[
\nabla_C^2 = 2C + \nabla_W^2.
\]

Since \( \Lambda_0(C) \) is a constant, to prove Proposition 9.2 it remains to show that

\[
\Lambda_0(\Theta_W) = 0,
\]

where \( \Theta_W \in \Lambda^2(M) \otimes \text{End}(TM) \) is the curvature of the Weyl connection.

---

\(^1\)By Theorem 5.2 a compact Einstein-Weyl LCK manifold with the Gauduchon metric has parallel Lee form.
The following lemma is clear.

**Lemma 9.3:** Let $M$ be an Einstein-Weyl LCK manifold equipped with a Gauduchon metric and $\omega$ the Hermitian form on $M$. Denote by $\Lambda_\omega : \Lambda^p(M) \to \Lambda^{p-2}(M)$ the Hermitian adjoint operator to $\eta \to \eta \wedge \omega$. Then $\Lambda_\omega(\Theta_W) = 0$.

**Proof:** Let $\widetilde{M}$ be the universal covering of $M$, and $rg$ the Calabi-Yau metric on $\widetilde{M}$. The form $\Theta_W$ is the curvature of the Levi-Civita connection of the Kähler metric on $\widetilde{M}$. Therefore, $TM$ is Yang-Mills with respect to this metric, and $\Theta_W$ is orthogonal to the Kähler form $r\omega$. This implies that $\Theta_W$ is orthogonal to $\omega$. We obtain $\Lambda_\omega(\Theta_W) = 0$.

Return to the proof of Proposition 9.2. By Lemma 9.3, we have $\Lambda_\omega(\Theta_W) = 0$, and we need to show $\Lambda_0(\Theta_W) = 0$.

By Proposition 6.1, we have

$$\Lambda_\omega \Theta_W - \Lambda_0 \Theta_W = \Theta_W(\theta^2, I(\theta^2)).$$

Therefore, to prove $\Lambda_\omega \Theta_W = \Lambda_0 \Theta_W = 0$, we need to show that the curvature $\Theta_W$ restricted to the canonical foliation $\Xi$ vanishes. The following lemma finishes the proof of Proposition 9.2.

**Lemma 9.4:** Let $M$ be a Vaisman manifold, $\Xi \subset TM$ the canonical foliation, and $\nabla$ the Weyl connection. Then $\nabla$ is flat on the leaves of $\Xi$.

**Proof:** Using (4.1) and $\nabla_\theta \theta^2 = 0$, we obtain

$$\nabla_X \theta^2 = X,$$

for all vector fields $X \in TM$. Therefore,

$$\nabla_X \nabla_Y \theta^2 = \nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

We find that $\Theta_W(X, Y, \theta^2) = 0$ for all $X, Y \in TM$. Using the symmetries of a curvature tensor, we obtain that $\Theta_W(X, \theta^2, Y) = 0$ identically. This proves Lemma 9.4. Proposition 9.2 is proven. ■
9.2 Vanishing theorem for negative bundles

In this Subsection we present another vanishing theorem based on Theorem 7.7. More general results are possible, in line with the standard vanishing theorems in algebraic geometry.

**Theorem 9.5:** Let \( M \) be a compact Vaisman manifold, \( B \) a holomorphic Hermitian bundle and \( \Theta_B \in \Lambda^{1,1}(M) \otimes \text{End}(B) \) its curvature. Consider the \( \omega_0 \)-Hodge operator

\[
\Lambda_0 : \Lambda^{1,1}(M) \otimes \text{End}(B) \rightarrow \text{End}(B)
\]

(Subsection [22]). Assume that the self-adjoint operator \( \sqrt{-1} \Lambda_0(\Theta_B) \) is strictly negative everywhere in \( M \). Then \( B \) has no non-zero holomorphic sections.

**Proof:** Let \( \beta \in B \) be a holomorphic section of \( B \). Then \( \nabla^{0,1}(\beta) = 0 \). From the definition of \( \overline{\partial}_0 \), we obtain that \( \overline{\partial}_0 \beta = 0 \). Clearly, \( \overline{\partial}_0^* \beta = 0 \) as well. Therefore, \( \beta \) is \( \overline{\partial}_0 \)-harmonic. From the equality (7.12), we find immediately that

\[
\left( \sqrt{-1} \Lambda_0 \Theta_B(\beta), \beta \right) > 0.
\]

This is impossible, because the operator \( \sqrt{-1} \Lambda_0(\Theta_B) \) is strictly negative. \( \blacksquare \)

**Corollary 9.6:** Let \( M \) be a compact Vaisman manifold, not Kaehler, and \( B \) a \( \omega_0 \)-Yang-Mills bundle with negative Yang-Mills constant. Then \( B \) has no global holomorphic sections.

**Proof:** Follows immediately from Theorem 9.5. \( \blacksquare \)

9.3 Vanishing for holomorphic forms

The main result of this section is the following theorem.

**Theorem 9.7:** Let \( M \) be a compact Einstein-Weyl LCK manifold. Assume that \( M \) is not Kähler. Then all holomorphic \( p \)-forms on \( M \) vanish, for all \( p > 0 \).

**Proof:** Follows from Proposition 9.2 and Corollary 9.6. \( \blacksquare \)
A similar theorem was obtained in [AI] using an estimate of Ricci curvature.

This result has the following topological implications.

**Theorem 9.8**: Let $M$ be a compact Einstein-Weyl LCK manifold. Assume that $M$ is not Kähler. Then the first Betti number of $M$ is 1:

$$h^1(M) = 1.$$ 

**Proof**: Consider the Dolbeault spectral sequence $E^{p,q}_r$ associated with $M$. Then $E^{p,q}_2 = H^q(\Omega^p(M))$. By Theorem 9.7 we have $H^0(\Omega^1(M)) = 0$, by Theorem 8.4 we have $H^1(\Omega^0(M)) = \mathbb{C}$. Since the Dolbeault spectral sequence converges to de Rham cohomology, we obtain that

$$h^1(M) \leq \dim H^0(\Omega^1(M)) + \dim H^1(\Omega^0(M)) = 1.$$ 

It remains to prove that $h^1(M) \neq 0$. The monodromy of the weight bundle lies in $\mathbb{R}^>0$, hence it is abelian and torsion-free. If $h^1(M) = 0$, the weight bundle has trivial monodromy. By Claim 2.8 an LCK manifold with trivial weight bundle is Kähler. Since $M$ is not Kaehler, we have $h^1(M) > 0$. This proves Theorem 9.8.

A similar theorem was obtained in [AI] using an estimate of Ricci curvature.

10 **Sasakian geometry: an introduction**

Sasakian manifolds were introduced by S. Sasaki ([Sa]). In this Section we reproduce the definition and some basic results on 1-Sasakian and 3-Sasakian manifolds. For a survey and reference of Sasakian geometry, the reader should consult [LGT]

**Definition 10.1**: Let $(X, g)$ be a Riemannian manifold. A cone of $X$ is a Riemannian manifold $C(X) := (X \times \mathbb{R}^>0, dt^2 + t^2 g)$, where $t$ is the parameter on $\mathbb{R}^>0$. For any $\lambda \in \mathbb{R}^>0$, the map

$$\tau_\lambda : C(X) \rightarrow C(X), \quad (x, t) \rightarrow (x, \lambda t)$$ 

multiplies the metric by $\lambda^2$.

**Definition 10.2**: Let $X$ be a Riemannian manifold. A 1-Sasakian structure in $X$ is a complex structure on $C(X)$ satisfying the following
(i) The metric on $C(X)$ is Kähler

(ii) The map $\tau_\lambda : C(X) \to C(X)$ is holomorphic, for all $\lambda \in \mathbb{R}^>0$.

A 3-Sasakian structure on $X$ is a hypercomplex structure on $C(X)$ such that

(i) The metric on $C(X)$ is hyperkähler

(ii) The map $\tau_\lambda : C(X) \to C(X)$ is compatible with the hypercomplex structure, for all $\lambda \in \mathbb{R}^>0$.

**Remark 10.3:** A 3-Sasakian manifold is equipped with a 1-Sasakian structure, for any quaternion $L \in \mathbb{H}$, $L^2 = -1$.

One can define 1-Sasakian and 3-Sasakian structures in terms of vector fields $I(dt^z) \in TX$ and $I(dt^z)$, $J(dt^z)$, $K(dt^z) \in TX$, and the associated contact structures (see [BG1]). Historically, these manifolds were defined in this fashion. The Sasakian geometry relates to contact geometry in exactly the same way as Kähler geometry is related to symplectic geometry.

**Definition 10.4:** Let $X$ be a 1-Sasakian manifold, such that the Riemannian metric on $X$ satisfies Einstein equation. Then $X$ is called **Sasakian-Einstein**.

Sasakian-Einstein manifolds can be characterized in terms of their cones, as follows.

**Proposition 10.5:** Let $X$ be a 1-Sasakian manifold, $\dim_{\mathbb{R}} X = 2n + 1$. Then $X$ is Sasakian-Einstein if and only if its cone is Ricci-flat. In this case the Einstein constant of $X$ is equal to $2n$.

**Proof:** See e.g. [BG1], Proposition 1.1.9.

**Remark 10.6:** A cone of a 3-Sasakian manifold is hyperkähler, hence Ricci-flat. Therefore, a 3-Sasakian manifold is always Einstein.

**Remark 10.7:** By Myers’ theorem, a complete Einstein manifold with positive Einstein constant is compact and has finite fundamental group.
11 Vaisman manifolds with $h^1(M) = 1$ and Sasakian geometry

11.1 Vaisman manifolds with exact Lee form

Vaisman manifolds are intimately related to Sasakian geometry, as the following proposition shows.

**Proposition 11.1:** ([KO], [GOP]) Let $M$ be a Vaisman manifold, that is, an LCK manifold with a parallel Lee form $\theta$. Assume that $\theta \neq 0$ and $\theta$ is exact: $\theta = d\zeta$. Assume, moreover, that the Gauduchon metric on $M$ is complete. Consider the function $\zeta$ as a map $\zeta : M \to \mathbb{R}$. Then $M$ is isometric to a product $X \times \mathbb{R}$, where $X$ a complete 1-Sasakian manifold, with the projection to $M = X \times \mathbb{R} \to \mathbb{R}$ given by $\zeta$. Moreover, if $M$ is LCHK, then $X$ is naturally a 3-Sasakian manifold.

**Proof:** Consider the vector field $\theta^\sharp$ dual to $\theta$. By definition, $\theta^\sharp$ is a parallel vector field on $M$, and $\theta^\sharp$ is equal to the gradient of $\zeta$. Therefore, the gradient flow $\Phi_\lambda$ associated with $\zeta$ is an isometry. The map $\Phi_\lambda$ is well defined because $M$ is complete. Being a gradient flow, $\Phi_\lambda$ commutes with $\zeta$: 
\[ \zeta(\Phi_\lambda(x)) = \lambda + \zeta(x). \] (11.1)
Therefore, $\Phi_\lambda$ induces an isometry on the fibers of $\zeta(x)$. Denote by $X$ the fiber $\zeta^{-1}(0)$. Let $R : X \times \mathbb{R} \to M$ map $(t,x)$ into $\Phi_t(x)$. Clearly, $R$ is an isometry. Denote the Riemannian metric on $M$ by $g$. By [Claim 4.3], the metric $e^{2t}g$ is Kähler. After a reparametrization $t \to e^t$, we obtain that the metric $(e^{2t}g, (de^t)^2)$ is the cone metric on $M = X \times \mathbb{R}^{>0}$. Therefore, $X$ is 1-Sasakian. If $M$ is LCHK, then $C(X)$ is hyperkähler (Claim 2.10), hence $X$ is 3-Sasakian.

**Remark 11.2:** A covering of a compact manifold is complete. Therefore, Proposition 11.1 holds for a covering $\tilde{M}$ of any compact Vaisman manifold, if $\tilde{M}$ has exact Lee form.

11.2 Structure theorem for Vaisman manifolds with the first Betti number 1

It is possible to classify the Vaisman manifolds with the first Betti number 1, in terms of 1-Sasakian geometry, as follows.
Let $X$ be a 1-Sasakian manifold, and $C(X)$ its cone. Given a number $q \in \mathbb{R}$, $q > 1$, consider an equivalence relation $\sim_q$ on $C(X) = X \times \mathbb{R}^\geq 0$ generated by $(x, t) \sim (x, qt)$. Since the map $(x, t) \mapsto (x, qt)$ multiplies the metric by $q^2$, the quotient $M = C(X)/ \sim_q$ is an LCK manifold. Moreover, $M$ is a Vaisman manifold, with the Gauduchon metric provided by an isomorphism $M \cong X \times S^1$.

This construction can be generalized as follows. Let $\varphi : X \rightarrow X$ be an automorphism of a Sasakian structure. The map $(x, t) \mapsto \varphi_q(x, qt)$ is compatible with the complex structure and multiplies the metric by $q^2$. Therefore, the quotient $M_{\varphi, q}$ of $C(X)$ by the corresponding equivalence relation $\sim_{\varphi, q}$ is an LCK manifold. The following theorem is quite elementary.

**Theorem 11.3:** (Structure theorem for Vaisman manifolds with the first Betti number 1). Let $X$ be a compact 1-Sasakian manifold, $\varphi : X \rightarrow X$ a 1-Sasakian automorphism, and $M_{\varphi, q}$ an LCK manifold constructed above. Then $M_{\varphi, q}$ is a Vaisman manifold satisfying the following conditions.

(i) $h^1(M_{\varphi, q}) = 1 \iff h^1(X) = 0$

(ii) The 1-Sasakian manifold $X$, together with the automorphism $\varphi$, is uniquely (up to a scaling) determined by the LCK structure on $M_{\varphi, q}$.

(iii) Any compact Vaisman manifold $M$, $h^1(M) = 1$ can be constructed this way.

**Remark 11.4:** In [GOP], it was shown that $M_{\varphi, q}$ is a Vaisman manifold (Proposition 7.4).

**Proof of Theorem 11.3:** To show that $M_{\varphi, q}$ is a Vaisman manifold, consider the map $\text{Id} \times \log : C(X) \rightarrow X \times \mathbb{R}$, $(x, t) \mapsto (x, \log t)$. Let $\tilde{g}$ be the product metric on $X \times \mathbb{R}$ pulled back to $C(X)$. The map $\varphi_q$ induces an isometry, hence $\tilde{g}$ corresponds to a metric $g$ on $M_{\varphi, q}$. By construction, $g$ belongs to the same conformal class as the LCK structure on $M_{\varphi, q}$. An elementary computation shows that $\nabla(\tilde{g}) = g \otimes dt$, where $t$ is the parameter on $X \times \mathbb{R}$ corresponding to the second component. Therefore, the Lee form of $g$ is parallel.

To find $H^1(M_{\varphi, q})$, consider the natural projection $r : C(X) \rightarrow \mathbb{R}^\geq 0$, and let $\zeta = \log r$. The map $\zeta$ sends the points $(x, t) \sim_{\varphi, q} (x', t')$ to

$$\log t, \log t + k \log q, \quad k \in \mathbb{Z}.$$
Therefore, \( \zeta \) induces a map \( \zeta_q : M_{\varphi,q} \to \mathbb{R}/(\log q)\mathbb{Z} \) from \( M_{\varphi,q} \) to a circle. By construction, \( d\zeta_q = \theta \), where \( \theta \) is the Lee form. Therefore,

\[
\zeta_q : M_{\varphi,q} \to S^1
\]  

(11.2)
is a smooth fibration, with the fiber \( X \).

Consider the Serre’s spectral sequence \( E_r^{p,q} \) for the fibration (11.2). Since \( H^i(S^1) = 0 \) for \( i > 0 \), the spectral sequence \( E_r^{p,q} \) degenerates in \( E_2 \). Therefore,

\[
h^1(M_{\varphi,q}) = h^1(S^1) + h^1(X) = h^1(X) + 1
\]  

(11.3)
This proves \textbf{Theorem 11.3} (i).

To recover \( (X, \varphi) \) from the LCK structure on \( M = M_{\varphi,q} \), notice that the Gauduchon metric on \( M \) is unique, hence the form \( \theta \) is determined uniquely from the LCK geometry. Applying \textbf{Proposition 11.1} to \( \tilde{M} = C(X) \), we reconstruct the 1-Sasakian structure on \( X \), together with an isomorphism \( \tilde{M} \cong C(X) \). The deck transform of \( \tilde{M} \) induces an automorphism \( \varphi \) of \( X \). This allows one to recover \( (X, \varphi) \) from the LCK structure on \( M \).

It remains to prove \textbf{Theorem 11.3} (iii).

Let \( M \) be a compact Vaisman manifold, \( h^1(M) = 1 \), \( L \) its weight bundle and \( \tilde{M} \) the covering associated with the monodromy group \( G \) of \( L \). Since \( h^1(M) = 1 \), \( M \) is non-Kähler, hence \( L \) is non-trivial. The monodromy group \( G \) is naturally a subgroup of \( \mathbb{R}^>0 \). Therefore \( G \) is torsion-free and abelian. Since \( h^1(M) = 1 \), \( G = \mathbb{Z} \).

We find that \( \tilde{M} = \tilde{M}/\mathbb{Z} \). Let \( \zeta : \tilde{M} \to \mathbb{R} \) be the function satisfying \( d\zeta = \theta \). By \textbf{Proposition 11.1} \( \tilde{M} = X \times \mathbb{R} \), where \( X \) is a complete 1-Sasakian manifold, and the projection to the second component given by \( \zeta \). Fix a point \( x_0 \in \tilde{M} \). Given \( y \in \tilde{M} \) and a path \( \gamma \) from \( x_0 \) to \( y \), we have

\[
\zeta(x_0) - \zeta(y) = \int_\gamma \theta
\]

(11.4)
by the Stoke’s formula. Let \( \gamma_0 \) be the generator of the monodromy group of \( L \) and \( w \) the integral \( w := \int_{\gamma_0} \theta \). Denote by \( R : \tilde{M} \to \tilde{M} \) be the monodromy transform of \( \tilde{M} \) associated with \( \gamma_0 \in G \). Using an isomorphism \( \tilde{M} = X \times \mathbb{R} \) and (11.4), we obtain that \( R \) maps \( (x, t) \) to \( (x_1, t + w) \). This gives a map \( \varphi : X \to X \), \( x \to x_1 \). Clearly, \( M = C(X)/\sim_{\varphi,e^w} \). This proves \textbf{Theorem 11.3} (iii).
11.3 Structure theorem for Einstein-Weyl LCK and LCHK manifolds

Comparing Theorem 9.8 and Theorem 11.3 we immediately obtain the following structure theorems.

**Theorem 11.5:** Let $M$ be a compact Einstein-Weyl LCK manifold which is not Kähler. Then there exists an Einstein Sasakian manifold $X$ and a Sasakian automorphism $\varphi : X \to X$ such that $M \cong C(X)/\sim_{\varphi,q}$, for some $q \in \mathbb{R}, q > 1$, where $\sim_{\varphi,q}$ is an equivalence relation generated by $(x,t) \sim_{\varphi,q} (\varphi(x),qt)$. Moreover, the manifold $X$ and an automorphism $\varphi$ are determined uniquely from the LCK structure on $M$, up to a rescaling of a metric on $X$.

**Proof:** By Proposition 11.1 the covering $\tilde{M}$ is isomorphic to $C(X)$, for a 1-Sasakian manifold $X$. By Claim 5.4 $C(X)$ is equipped with a Ricci-flat Kähler metric. By Proposition 10.5 $X$ is Sasakian-Einstein. By Theorem 9.8 $h^1(M) = 1$. By Theorem 11.3 $M \cong C(X)/\sim_{\varphi,q}$ and $X, \varphi$ are determined uniquely.

**Theorem 11.6:** Let $M$ be a compact LCHK manifold which is not hyperkähler. Then there exists a 3-Sasakian manifold $X$ and a 3-Sasakian automorphism $\varphi : X \to X$ such that $M \cong C(X)/\sim_{\varphi,q}$, for some $q \in \mathbb{R}, q > 1$, where $\sim_{\varphi,q}$ is an equivalence relation generated by $(x,t) \sim_{\varphi,q} (\varphi(x),qt)$. Moreover, the manifold $X$ and an automorphism $\varphi$ are determined uniquely from the LCHK structure on $M$, up to a rescaling of a metric on $X$.

**Proof:** Since $M$ is LCHK, it is an Einstein-Weyl manifold. By Theorem 9.8 $h^1(M) = 1$. By Theorem 11.3 $M \cong C(X)/\sim_{\varphi,q}$ and $X, \varphi$ are determined uniquely. The covering $\tilde{M} \cong C(X)$ is by definition hyperkähler, hence $X$ is 3-Sasakian.

11.4 Quasiregular LCHK manifolds

**Definition 11.7:** Let $M$ be a Vaisman manifold, and $\theta^\sharp$ the vector field dual to the Lee form $\theta$. Consider the flow $\Phi_t$ associated with $\theta^\sharp$. The manifold $M$ is called quasiregular if for all compact sets $K \subset M$ and all points $x \in M$, the intersection of the orbit

$$V_x = \{\Phi_t(x), t \in \mathbb{R}\}$$
with $K$ is compact. In other words, $M$ is quasiregular if the set $V_x$ does not have concentration points outside itself, for all $x \in M$.

Given an LCHK manifold $M$, one can construct a number of foliations on $M$, similar to the canonical foliation $\Xi$. The leaf space of these foliations will be an orbifold if $M$ is quasiregular. This way, we can reduce a quasiregular LCHK manifold to

(a) A 3-Sasakian orbifold (by taking a leaf space of the real 1-dimensional foliation generated by $\theta^\sharp$).

(b) A holomorphic contact Kähler-Einstein orbifold (by taking a leaf space of $\Xi$)

(c) A quaternionic Kähler orbifold (by taking a leaf space of the real 4-dimensional foliation generated by $\theta^\sharp$, $I(\theta^\sharp)$, $J(\theta^\sharp)$, $K(\theta^\sharp)$).

For details of these constructions and further results see [Or], [OP].

Using the structure theorem (Theorem 11.6), it is possible to determine the class of quasiregular LCHK manifolds in terms of 3-Sasakian fibrations. The following claim is clear.

Claim 11.8: Let $M$ be a compact LCHK manifold, obtained as in Theorem 11.6 from a 3-Sasakian manifold $X$ and a 3-Sasakian automorphism $\varphi : X \to X$. Then $M$ is quasiregular if and only if $\varphi$ is a finite order automorphism.

12 Appendix A. $h^1(M) = 1$ for Einstein-Weyl LCK manifolds

In this Appendix, we give a direct proof of a version of Theorem 9.8.

Let $M$ be a complete Vaisman manifold, and $\tilde{M}$ the covering associated with the monodromy of the weight bundle. Then $\tilde{M} = C(X)$, for a complete 1-Sasakian manifold $X$ (Proposition 11.1). Whenever $M$ is Einstein-Weyl, the manifold $X$ becomes Sasakian-Einstein (this is implied immediately by Claim 5.4, Proposition 10.5). By Myers’ Theorem (Remark 10.7), then, $X$ is compact, and $\pi_1(X)$ is finite. Therefore, the following theorem implies Theorem 9.8.
**Theorem 12.1:** Let $M$ be a compact Vaisman manifold, and $\tilde{M}$ its covering associated with the monodromy $G$ of the weight bundle. We have $\tilde{M} \cong C(X)$, for some 1-Sasakian manifold $X$ [Proposition 11.1]. Assume that $X$ is compact and $h^1(X) = 0$. Then $h^1(M) = 1$.

**Proof:** By definition, $M = \tilde{M}/G$. Let $\zeta : \tilde{M} \to \mathbb{R}$ be a function such that $d\zeta = \theta$. Denote by $\chi : G \to \mathbb{R}$ is the group homomorphism $\gamma \to \int_\gamma \theta$. It is easy to check that this map is a logarithm of the monodromy map $G \to \mathbb{R}^\times$. By (11.4), the monodromy acts in such a way that for all $\gamma \in G$ we have

$$\zeta(\gamma x) = \chi(\gamma) + \zeta(x).$$

(12.1)

Since $G$ is the monodromy of $L$, the map $\chi$ is a monomorphism. Now, either $G = \mathbb{Z}$, and in this case $M$ is fibered over a circle with fibers $X$, hence $h^1(M) = h^1(X) + 1 = 1$ (see (11.3)); or $\chi(G)$ is dense in $\mathbb{R}$. To prove Theorem 12.1 it remains to show that $\chi(G)$ cannot be dense.

Consider an interval $[0, 1] \subset \mathbb{R}$, and let $\tilde{M}_0 := \zeta^{-1}([0, 1])$ be the corresponding subset in $\tilde{M}$. Clearly $\tilde{M}_0 = X \times [0, 1]$ is compact. Given a point $x \in \tilde{M}$, $\zeta(x) = 0$, let $Gx$ denote its orbit with respect to the monodromy action. By (12.1), $Gx$ meets $\tilde{M}_0$ for all $\gamma \in G$ such that $\chi(\gamma) \in [0, 1]$. If $\chi(G)$ is dense, this set is infinite. We obtain that $Gx \cap \tilde{M}_0$ is infinite. Since this set is compact, $Gx$ has concentration points. This is clearly impossible, because $\tilde{M} \to \tilde{M}/G$ is a covering. Therefore, $\chi(G)$ cannot be dense. We proved Theorem 12.1. 

13 Appendix B. Counterexamples

For a general Vaisman manifold (without the Einstein-Weyl assumption), Theorem 9.7 and Theorem 9.8 are false, as the following example shows.

Let $S$ be an elliptic curve and $B$ a negative holomorphic line bundle. Assume that $B$ is equipped with a Hermitian metric in such a way that the corresponding Chern connection has curvature

$$\Theta_B = c\sqrt{-1}\omega,$$

(13.1)

where $c > 0$ is a positive constant, and $\omega$ the Kähler form of $S$. Consider the function $r : \text{Tot } B \to \mathbb{R}, v \to |v|^2$. Using (13.1), it is easy to check that $r$ is a Kähler potential on $\text{Tot } B$ (see [Bes] (15.19)). Let

$$X := \{v \in \text{Tot } B \mid |v|^2 = 1\}$$

39
be the circle bundle over $S$ corresponding to $B$. Denote by $\text{Tot}_0(B)$ the space of non-zero vectors in $B$. Then, $\text{Tot}_0(B) \cong C(X)$. Since $\text{Tot}_0(B)$ is Kähler, $X$ is 1-Sasakian.

Considering $X$ as a circle bundle over $S$ and using the Serre’s spectral sequence, we find that $h^1(X) = 2$.

Fix $q \in \mathbb{R}^{>1}$. Let $M = \text{Tot}_0(B)/\sim_q$, where $\sim_q$ is the equivalence relation generated by $v \sim qv$. Clearly, $M$ is an LCK manifold. Using the product metric on $M \cong X \times S^1$, we find that $M$ is actually a Vaisman manifold.

By (11.3), $h^1(M) = h^1(X) + 1 = 3$. This gives a counterexample to Theorem 9.8 (without the Einstein-Weyl assumption).

The manifold $M$ is equipped with a natural holomorphic projection $\pi : M \to S$. Clearly, $S$ admits a non-trivial holomorphic 1-form. Lifting this form to $M$, we obtain a non-trivial holomorphic form on a Vaisman manifold. Therefore, for Theorem 9.7 the Einstein-Weyl assumption is also essential.

This example appears (in another language) in [V1], under the name of induced Hopf bundle.

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References

[AI] Alexandrov, B., Ivanov, St., Weyl structures with positive Ricci tensor, math.DG/9902033, version 2 (9 Jan 2003)

[Bel] Belgun, F. A., On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000), no. 1, 1–40.

[Bes] Besse, A., Einstein Manifolds, Springer-Verlag, New York (1987)

[Bo] Boyer, C. P., A note on hyper-Hermitian four-manifolds. Proc. Amer. Math. Soc. 102 (1988), no. 1, 157–164.

[BG1] Boyer, C. P., Galicki, K., 3-Sasakian Manifolds, hep-th/9810250 also published in Surveys Diff.Geom. 7 (1999) 123-184

[BG2] Boyer, C. P., Galicki, K., New Einstein Metrics in Dimension Five, math.DG/0003174 J. Diff. Geom. 57 (2001), 443-463.

[BGM] Boyer, C. P., Galicki, K., Mann, B. M., The geometry and topology of 3-Sasakian manifolds, J. Reine Angew. Math. 455 (1994), 183-220.
Vanishing theorems for LCHK manifolds

M. Verbitsky, 15 February 2003

[BGN1] C. P. Boyer, K. Galicki, and M. Nakamaye, On the Geometry of Sasakian-Einstein 5-Manifolds, math.DG/0012047, to appear in Math. Ann.

[BGN2] Boyer, C. P., Galicki, K., Nakamaye, M., Sasakian Geometry, Homotopy Spheres and Positive Ricci Curvature, 22 pages, math.DG/0201147

[Ca] Calabi, E., Métriques kählériennes et fibrés holomorphes, Ann. Ecol. Norm. Sup. 12 (1979), 269-294.

[CP] Calderbank, D., Pedersen, H., Einstein-Weyl geometry, in "Surveys in differential geometry: Essays on Einstein Manifolds", M. Wang and C. LeBrun eds., International Press 2000, 387-423.

[DO] Dragomir, S., Ornea, L., Locally conformal Kähler geometry, Progress in Mathematics, 155. Birkhäuser, Boston, MA, 1998.

[G1] Gauduchon, P. La 1-forme de torsion d’une variété hermitienne compacte, Math. Ann., 267 (1984), 495-518.

[G2] Gauduchon, P. Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, J. Reine Angew. Math. 469 (1995), 1-50.

[GO] Gauduchon, P., Ornea, L., Locally conformly Kähler metrics on Hopf surfaces, Ann. Inst. Fourier (Grenoble) 48 (1998), no. 4, 1107–1127.

[GOP] Gini, R., Ornea, L., Parton, M., Locally conformal Kaehler reduction, math.DG/0208208, 25 pages.

[GH] Griffiths, Ph., Harris, J., Principles of Algebraic Geometry, Wiley-Interscience, New York, 1978.

[IP1] Ivanov, St., Papadopoulos, G., A no-go theorem for string warped compactifications, hep-th/0008232, published in: Phys. Lett. B497 (2001) 309-316

[IP2] Ivanov, St., Papadopoulos, G., Vanishing Theorems and String Backgrounds, math.DG/0010038, published in: Class. Quant. Grav. 18 (2001) 1089-1110

[KO] Kamishima, Y., Ornea, L., Geometric flow on compact locally conformally Kähler manifolds, math.DG/0105040, 21 pages.

[K] Kuo, Y.-Y., On almost contact 3-structure, Tohoku Math. J. 22 (1970), 325-332.

[NT] Nishikawa, S., Tondeur, Ph., Transversal infinitesimal automorphisms for harmonic Kähler foliations, Tohoku Math. J. (2) 40 (1988), no. 4, 599–611.

[Ob] Obata, M., Affine connections on manifolds with almost complex, quaternionic or Hermitian structure, Jap. J. Math., 26 (1955), 43-79.

[Or] Ornea, L., Weyl structures on quaternionic manifolds. A state of the art, math.DG/0105041, also in: Selected Topics in Geometry and Mathematical Physics, vol. 1, 2002, 43-80, E. Barletta ed., Univ. della Basilicata (Potenza).
Vanishing theorems for LCHK manifolds

M. Verbitsky, 15 February 2003

[OP] Ornea, L., Piccinni, P., Locally conformal Kahler structures in quaternionic geometry, Trans. Am. Math. Soc. 349 (1997), 641-655.

[PPS] Pedersen, H., Poon, Y. S., Swann, A., The Einstein-Weyl equations in complex and quaternionic geometry, Diff. Geom. Appl. 3 (1993), 309-321.

[Sa] Sasaki, S., On differentiable manifolds with certain structures which are closely related to almost contact structure, Tohoku Math. J. 2 (1960), 459-476.

[Str] Strominger, A., Superstrings with torsion, Nucl. Phys. B274 (1986) 253.

[Sw] Swann, A., Hyper-Kahler and quaternionic Kahler geometry, Math. Ann. 289 (1991), no. 3, 421-450.

[To] Tondeur, Ph., Foliations on Riemannian manifolds, Universitext, Springer-Verlag, New York, 1988, 247 pp.

[Ts1] Tsukada, K., Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds, Compositio Math. 93 (1994), no. 1, 1–22.

[Ts2] Tsukada, K., Holomorphic maps of compact generalized Hopf manifolds, Geom. Dedicata 68 (1997), no. 1, 61–71.

[U] Udrişte, C. Structures presque coquaternioniennes, Bull. Math. de la Soc. Sci. Math. de Roumanie 12 (1969), 487-507.

[V1] Vaisman, I. Generalized Hopf manifolds, Geom. Dedicata 13 (1982), no. 3, 231–255.

[V2] Vaisman, I. A survey of generalized Hopf manifolds, Rend. Sem. Mat. Torino, Special issue (1984), 205-221.

[Ve] Verbitsky, M., Hyperkähler manifolds with torsion, supersymmetry and Hodge theory, math.AG/0112215, 47 pages (Asian J. of Math., Vol. 6 (4), December 2002).