Groups of proper homotopy equivalences of graphs and Nielsen Realization

Yael Algom-Kfir and Mladen Bestvina

Abstract. For a locally finite connected graph $X$ we consider the group $\text{Maps}(X)$ of proper homotopy equivalences of $X$. We show that it has a natural Polish group topology, and we propose these groups as an analog of big mapping class groups. We prove the Nielsen Realization theorem: if $H$ is a compact subgroup of $\text{Maps}(X)$ then $X$ is proper homotopy equivalent to a graph $Y$ so that $H$ is realized by simplicial isomorphisms of $Y$.

1. Introduction

The group $\text{Out}(F_n)$ of outer automorphisms of the free group of rank $n$ can be thought of as the group of homotopy equivalences of a finite graph $X$ with $\pi_1(X) \cong F_n$, up to homotopy. In this paper we begin the study of the analogous group associated with a locally finite graph $X$.

Definition 1.1. Let $X$ be a locally finite connected graph. The mapping class group $\text{Maps}(X)$ of $X$ is the group of proper homotopy equivalences of $X$, up to proper homotopy.

Recall that $f : X \to Y$ is a proper homotopy equivalence if it is proper and there is a proper map $g : Y \to X$ such that both $fg$ and $gf$ are properly homotopic to the identity. For an example of a proper map which is homotopy equivalence but not a proper homotopy equivalence see Example 4.1.

We will equip $\text{Maps}(X)$ with a natural topology which will make it a Polish group (recall that this means that the underlying topological space is separable and admits a complete metric). See Section 4. We thus propose $\text{Maps}(X)$ as the “big $\text{Out}(F_n)$” equivalent of mapping class groups of surfaces of infinite type (or “big mapping class groups”), for a survey of the subject see [2]. Comparison with mapping class groups has shown to be very useful in the study of $\text{Out}(F_n)$, and we expect that comparison between $\text{Maps}(X)$ and big mapping class groups will likewise prove fruitful. We remark that the group of all automorphisms $\text{Aut}(F_\infty)$ of the free group of countable rank has a natural structure of a Polish group (e.g. it is a closed subgroup of the Polish group of all permutations of $F_\infty$), and so

2020 Mathematics Subject Classification. 57S99, 20F65.

The second author gratefully acknowledges the support by the National Science Foundation under the grant number DMS-1905720.

©2009 American Mathematical Society
does $Out(F_\infty)$ since the group of inner automorphisms is discrete. However, the groups $Maps(X)$ seem more appealing as they have a more topological flavor and come in great variety since they depend on the graph $X$. Even when $X$ is a tree the group $Maps(X)$ of homeomorphisms of the space of ends $\partial X$ of $X$ (see Corollary 2.3). Note that if $h : X \to Y$ is a proper homotopy equivalence with inverse $h' : Y \to X$, then $f \mapsto hfh'$ induces an isomorphism $Maps(X) \to Maps(Y)$, which will turn out to be an isomorphism of topological groups, see Corollary 4.5.

In this paper we focus on compact subgroups of $Maps(X)$ and prove the following version of the Nielsen Realization theorem.

**Main Theorem.** Let $H$ be a compact subgroup of $Maps(X)$. Then there is a locally finite graph $Y$ proper homotopy equivalent to $X$ so that under the induced isomorphism $Maps(X) \cong Maps(Y)$ the group $H$ is realized as a group of simplicial isomorphisms of $Y$.

Recall that the original Nielsen Realization theorem for finite type surfaces with negative Euler characteristic was proved by Kerckhoff [18], stating that any finite subgroup of the mapping class group of a surface of negative Euler characteristic can be realized by isometries of a complete hyperbolic metric with finite area. The version for $Out(F_n)$, proved in [22, 5, 19, 15], states that a finite subgroup of $Out(F_n)$ can be realized as a group of simplicial isomorphisms of a finite graph with fundamental group $F_n$. For big mapping class groups Nielsen Realization was proved recently by Afton-Calegari-Chen-Lyman [1]. Among the consequences is that compact subgroups of big mapping class groups are finite. This is not the case for $Maps(X)$. For example, let $X$ be the graph obtained from $[0, \infty)$ by attaching two loops at every integer point. The group of symmetries of this graph is the compact group $H^\infty = \prod_{i=1}^\infty H$, where $H$ is the group of symmetries of order 8 of the wedge of two circles. Note that $X$ is proper homotopy equivalent to the graph $Y$ obtained from $[0, \infty)$ by attaching three circles at every integer point, and the group of symmetries of $Y$ is $G^\infty$, where $G$ is the group of order 48 of symmetries of the wedge of three circles. The groups $Maps(X)$ and $Maps(Y)$ are isomorphic as topological groups, but the realization using different graphs displays different compact subgroups.

In addition to big mapping class groups $Mod(\Sigma)$ and groups $Maps(X)$, the groups $Homeo(Z)$ of homeomorphism groups of compact totally disconnected metrizable spaces with compact-open topology have many similar properties and are studied more classically. Excluding the cases when $\Sigma$ and $X$ have finite type and $Z$ is finite, all these groups have underlying space homeomorphic to the irrationals, they all admit clopen subgroups forming a basis of neighborhoods of the identity, and they all satisfy the Nielsen Realization theorem (for $Homeo(Z)$ this is proved quickly in Section 6; it states that for any compact subgroup $H < Homeo(Z)$ there is a metric on $Z$ so that $H$ consists of isometries).

Since this paper was first circulated, Domat, Hoganson and Kwak [6] investigated the coarse geometry of the pure subgroup $PMaps(X)$ of $Maps(X)$. Their work points out the differences between these three classes of topological groups.
Plan of the paper. We start by recalling the Classification theorem for locally finite graphs in Section 2. We also introduce the notation and review the homotopy extension theorem in our setting and some of its consequences. In Section 3, we explore the natural homomorphism $\text{Maps}(X) \to \text{Out}(\pi_1(X))$ and in particular we look for conditions that guarantee that a proper map $X \to X$ is properly homotopic to the identity. In the case when $X$ is a core graph (i.e. it is the union of immersed loops) the criterion is particularly simple: if $f$ induces the identity in $\text{Out}(\pi_1(X))$, it is properly homotopic to the identity. In the other extreme, when $X$ is a tree, $f$ is properly homotopic to the identity whenever it fixes the end space $\partial X$. The general case is more complicated since rays attached to the core graph could wrap around the core, but we show that if $f$ is identity in $\pi_1$, fixes the ends, and preserves proper lines, then $f$ is properly homotopic to the identity.

Section 4 is devoted to defining the topology on $\text{Maps}(X)$ and establishing that it is a Polish group. As long as $X$ is of infinite type, we show that the underlying topological space of $\text{Maps}(X)$ is homeomorphic to the set of irrationals, but of course the group structure will depend on $X$.

The remainder of the paper is devoted to the proof of the Main Theorem. This is also divided into cases, with the two extremes of $X$ being a core graph and being a tree discussed first. When $X$ is a tree, by averaging we find an $H$-invariant metric on the space of ends $\partial X$. From this we construct $H$-invariant finite covers by disjoint clopen sets that refine each other and with mesh going to 0. The mapping telescope of this sequence is the desired tree $Y$.

The heart of the argument is the case when $X$ is a core graph. For concreteness imagine that $X$ is the graph obtained from the ray $[0, \infty)$ by attaching a circle at every integer point. We then cover $[0, \infty)$ by large intervals $J_1, J_2, \ldots$ so that $J_i \cap J_j = \emptyset$ if $|i - j| > 1$ and so that $J_i \cap J_{i+1}$ are large as well, controlling the properness of elements of $H$. Each $J_i$ and $J_i \cap J_{i+1}$ defines a subgraph of $X$ and a free factor of $\pi_1(X)$. By intersecting the $H$-translates of these free factors we obtain $H$-invariant free factors $F_i^*$ and $F^*_{i,i+1}$ respectively. Using Nielsen Realization in finite rank we find finite graphs $\Gamma_{i,i+1}$ where $H$ acts by simplicial isomorphisms realizing $F^*_{i,i+1}$. We then use the Relative Nielsen Realization, due to Hensel-Kielak [14], to construct finite graphs $\Gamma_i$ where $H$ acts by simplicial isomorphisms realizing $F^*_i$, and that contain $\Gamma_{i-1,i}$ and $\Gamma_{i,i+1}$ as disjoint invariant subgraphs. Finally, we glue the $\Gamma_i$’s along these subgraphs to obtain $Y$. In general, when $X$ is a tree with circles attached at vertices, the above outline still works, but instead of free factors we have to consider free factor systems which makes the notation a bit more complicated.

In the general case, we first use the case of core graphs to reduce to the situation where $H$ is already acting on the core by simplicial isomorphisms. The graph $X$ is obtained from the core by attaching trees, and the central part of the proof in this case is to see how to attach new trees in an equivariant fashion. The new trees are going to be mapping telescopes made of partition elements (as in the tree case) subordinate to suitable clopen sets in $\partial X$. To accomplish this we prove a fixed point theorem (see Lemma 7.3) that provides a suitable point in the core where these telescopes are attached.

In order to verify that the action of $H$ on the new graph $Y$ is conjugate to the given action on $X$ we use the machinery developed in Section 3, see Corollary 3.6.
2. The classification of locally finite connected graphs

Let $X$ be a locally finite infinite connected graph. The fundamental group $\pi_1(X)$ is free and we denote its rank by $g(X) \in \{0, 1, 2, \cdots, \infty\}$ and think of it as the “genus” of $X$. Let 

$$\partial X = \lim_{\leftarrow K \subset X} \pi_0(X \setminus K)$$

be the space of ends of $X$ with its usual inverse limit topology, where the limit runs over all compact subsets $K \subset X$. Then $\partial X$ is a totally disconnected compact metrizable space (recall that these are precisely the spaces homeomorphic to a closed subset of the Cantor set). The union $\hat{X} = X \sqcup \partial X$ has a natural topology that makes it compact; it is the Freudenthal (or end) compactification of $X$. The basis of open sets in $\hat{X}$ consists of open sets in $X$, and for every compact $K \subset X$ and every component $U$ of $X \setminus K$ the set $\hat{U}$ which is the union of $U$ and the set of ends that map to $U$. We will sometimes abuse notation and talk about a neighborhood $U$ in $X$ of an end $\beta \in \partial X$; what we mean is the intersection of such a neighborhood $\hat{U}$ in $\hat{X}$ with $X$. The end compactification can also be constructed in the same way for connected, locally finite cell complexes. Every proper map $f : X \to Y$ between such complexes extends continuously to a map $\hat{X} \to \hat{Y}$ between their end compactifications. For simplicity we will usually denote this extension, as well as its restriction $\partial X \to \partial Y$, by $f$ as well. Properly homotopic maps induce the same map between the boundaries.

Denote by $X_\text{g} \subset X$ the core of $X$, i.e. the smallest subgraph that contains all immersed loops. Thus $X_\text{g} = \emptyset$ precisely when $X$ is a tree. Let $\partial X_\text{g} \subset \partial X$ be the space of ends of $X_\text{g}$; it is a closed subspace of $\partial X$ (and consists of ends “accumulated by genus”). Thus $\partial X_\text{g} = \emptyset$ precisely when $g(X) < \infty$. In general, $X$ is either a tree or is obtained from $X_\text{g}$ by attaching trees.

**Definition 2.1 (Characteristic pairs).** If $g(X) < \infty$, its characteristic pair is $(\partial X, g(X))$, otherwise its characteristic pair is $(\partial X, \partial X_\text{g})$.

The following is the analog of Kerékjártó’s classification theorem for surfaces and was proved by Ayala-Domínguez-Marquez-Quintero [3].

**Theorem 2.2.** Let $X, Y$ be locally finite connected graphs. Then a homeomorphism of characteristic pairs extends to a proper homotopy equivalence. If $X$ and $Y$ are trees the extension is unique up to proper homotopy.

In the case when the genus is finite, a homeomorphism of characteristic pairs means a homeomorphism between the spaces of ends together with the information that the genera are equal.

If $f : X \to Y$ is a proper homotopy equivalence, then the extension $\partial X \to \partial X$ is a homeomorphism which preserves $\partial X_\text{g}$. Thus we have a well defined homomorphism

$$\sigma : Maps(X) \to \text{Homeo}(\partial X, \partial X_\text{g})$$

to the group of homeomorphisms of the pair $(\partial X, \partial X_\text{g})$. The following is then an immediate corollary of the classification theorem.

**Corollary 2.3.** The homomorphism $\sigma$ is always surjective. If $X$ is a tree the map $\sigma : Maps(X) \to \text{Homeo}(\partial X)$ is an isomorphism.

In light of this we will usually focus on the kernel of $\sigma$, which is the pure mapping class group of $X$, and we denote it by $PMaps(X)$. 


Figure 1. A graph with a finitely generated fundamental group. In this case \( \partial X_g = \emptyset \) and the space of ends can be any closed subset of the Cantor set.

Figure 2. This is a core graph, \( \partial X = \partial X_g \). By deleting a part of the Cantor tree we get a standard model for \((A, A)\) where \( A \) is any closed subspace of the Cantor set.

Figure 3. In this case both \( \partial X_g \) and its complement \( DX := \partial X \setminus \partial X_g \) are non-empty. By deleting loops/subtrees from the space in Figure 2 we get a model for any pair of closed subsets \( \partial X = A \supset B = \partial X_g \) of the Cantor set.

**Definition 2.4.** The pure group \( PMaps(X) \) is the subgroup of \( Maps(X) \) consisting of \( f \in Maps(X) \) so that \( f : \partial X \to \partial X \) is the identity.

We shall make use of the following concepts and lemmas from [3].

**Definition 2.5 (Standard Models).** The Cantor tree \( T \) is the rooted binary tree embedded in the plane so that its boundary is the standard trinary Cantor set in \([0, 1] \times \{0\}\). For each closed subset \( B \) of the Cantor set, let \( T_B \) be the union of the set of rays in \( T \) initiating at the root and terminating in \( B \). For a characteristic pair \((B, g)\) where \( B \) is a closed subset of the Cantor set and \( g \) is a natural number let \( X_{(B, g)} \) be the tree \( T_B \) with \( g \) loops attached at the root. The characteristic set of \( X_{(B, g)} \) is \((B, g)\). For two closed non-empty subsets \( A \subseteq B \) of the Cantor set, let \( X_{(B, A)} \) be the tree \( T_B \) with a one edge loop attached at each vertex of the subtree \( T_A \). Again, the characteristic set of \( T_{(B, A)} \) is \((B, A)\). These trees are called the Standard Models.

**Corollary 2.6.** Every locally finite connected infinite graph is proper homotopy equivalent to a Standard Model.
In particular, we can assume that $X$ has no valence 1 vertices, and that $X = X_g$ if $\partial X_g = \partial X$.

The following is the standard Homotopy Extension Theorem, see [133]. We will use it often, usually without saying it. Given a subgraph $Y \subset X$, the frontier of $Y$ is the set of vertices in $Y - \text{int}(Y)$ denoted $Fr(Y)$.

**Proposition 2.7.** Let $Y$ be a subgraph of $X$, let $H : Y \times I \to X$ be a proper homotopy from $h$ to $f$. Let $u : X \to X$ be a proper map so that $u|_Y = h$. Then there is a proper homotopy $H' : X \times I \to X$ extending $H$ and $u$, i.e. $H'(x,0) = u(x)$. Moreover, if $K \subset X$ is a subset so that $H(Fr(Y) \times I) \cap K = \emptyset$ and $u(X \setminus \overline{Y}) \cap K = \emptyset$, then $H'((X \times Y) \times I) \cap K = \emptyset$.

**Proof.** First define the extension on the vertices $v$ of $X$ outside of $Y$ by $H'(v,t) = u(v)$. Now let $e$ be an edge with endpoints $a, b$. If both $a, b$ are outside of $Y$ define $H'$ on $e \times I$ to be stationary as well: $H'(x,t) = u(x)$ for $x \in e$. Otherwise let $\alpha : [0,1] \to e$ be a parametrization of $e$. Let $P$ be a retraction from $I \times I$ to $((0) \times I) \cup (I \times (0)) \cup ((1) \times I)$. Then $e \times I$ can be identified with $I \times I$ (with $0 \times I$ and $1 \times I$ identified if $e$ is a loop). The homotopy is already defined on $((0) \times I) \cup (I \times (0)) \cup ((1) \times I)$ and we extend it to $I \times I$ by composing with $P$.

We leave the verification that $H'$ is proper and the last sentence to the reader. □

We will also sometimes have a need to restrict $f \in Maps(X)$ to an “invariant” subgraph $Y \subset X$ e.g. $Y = X_g$. Since $f$ is defined only up to proper homotopy, $Y$ will usually not satisfy $f(Y) \subseteq Y$, but this will be true after a proper homotopy $H$. We will also want to know that the proper homotopy class of the restriction is independent of the choice of $H$. A bad example to keep in mind is the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$. There are proper maps $f : \mathbb{R} \to \mathbb{R}^2$ such that $\pi f$ is not proper (and there are analogous examples with Cayley graphs $\text{Cay}(\mathbb{Z}) \to \text{Cay}(\mathbb{Z}^2)$), and there are pairs of properly homotopic maps $f, g : \mathbb{R} \to \mathbb{R}^2$ such that $\pi f$ and $\pi g$ are proper, but not properly homotopic.

**Lemma 2.8.** Let $Y$ be a subgraph of $X$ such that inclusion $Y \hookrightarrow X$ induces an injection $\partial Y \hookrightarrow \partial X$. Let $\pi : X \to Y$ be a retraction with the following property: for every $\beta \in \partial Y$ and every neighborhood $U$ of $\beta$ in $Y$ there is a neighborhood $V$ of $\beta$ in $X$ so that $\pi(V) \subseteq U$. Let $Z$ be a locally compact metrizable space.

(i) If $f : Z \to X$ is a proper map such that $f(\partial Z) \subseteq \partial Y$ then $\pi f : Z \to Y$ is proper.

(ii) If $f, g : Z \to Y$ are proper maps such that they are properly homotopic within $X$, then they are properly homotopic within $Y$.

**Proof.** To prove (i), note that if $\gamma \in \partial Z$ then $f(\gamma) = \beta \in \partial Y \subseteq \partial X$. If $U$ is a neighborhood of $\beta$ in $Y$, there is a neighborhood $V$ of $\beta$ in $X$ so that $\pi(V) \subseteq U$. Since $f$ is proper, there is a neighborhood $W$ of $\gamma$ in $Z$ so that $f(W) \subseteq V$. It follows that $\pi f(W) \subseteq U$, so $\pi f$ is proper. For (ii), apply (i) to a proper homotopy $H : Z \times I \to X$. The fact that $H(\partial(Z \times I)) \subseteq \partial Y$ follows from the assumption that $\partial Y \subseteq \partial X$. □

**Proposition 2.9.** Suppose $X, Y$ are two locally finite connected graphs, $f : X_g \to Y$ a proper map that induces $\overline{f} : \partial X_g \to \partial Y$ and let $\overline{F} : \partial X \to \partial Y$ be an extension of $\overline{f}$. Then there is a proper map $F : X \to Y$ that extends $f$ and induces $\overline{F}$. 
PROOF. First consider the case when $X$ and $Y$ are trees, and choose base vertices $x_0 \in X$, $y_0 \in Y$. We are given $F : \partial X \to \partial Y$ and we have to construct a proper map $F : X \to Y$. If $F$ is a homeomorphism the existence of $F$ follows from the Classification theorem (and in fact it is unique up to proper homotopy). In general, we can construct $F$ as follows. When $v$ is a vertex of $X$ let the shadow $\text{Sh}_X(v) \subseteq \partial X$ be the set of endpoints of rays that start at $x_0$ and pass through $v$. When $A \subseteq \partial X$ contains at least two points, let $\sup A$ be the vertex $v \in X$ with the largest distance $|v|$ from $x_0$ satisfying $\text{Sh}_X(v) \supseteq A$. If $A = \{\beta\} \subseteq \partial X$ is a single point, define $\sup A$ to be $\beta$ and let $|\beta| = \infty$. Make the similar definition for subsets of $\partial Y$. For a vertex $v \in X$ consider $\sup F(\text{Sh}_X(v))$. If this is a vertex $w$ at distance $|w| \leq |v|$ from $y_0$ then define $F(v) = w$, and otherwise define $F(v)$ as the vertex at distance $|v|$ from $y_0$ along the segment (or ray) $[y_0, w]$. Extend $F$ linearly to the edges of $X$.

Now consider the general case. We may assume that $f$ sends vertices to vertices. First suppose that there is a maximal tree $T \subseteq Y$ such that $\partial T = \partial Y$. Since $X$ is simplicial and locally finite, $X \setminus X_g$ is a countable (or finite) union of trees, and let $T_i$ for $i \in \mathbb{N}$ be the closure of a component of $X \setminus X_g$. We denote by $x_i$ the point of intersection of $T_i$ and $X_g$, so $x_i$ is the root of $T_i$. For each $i \in \mathbb{N}$ let $F_i : T_i \to T \subseteq Y$ be the map constructed in the first paragraph so that $F_i(x_i) = f(x_i)$ and $\partial F_i = F|_{\partial T_i}$. We define $F$ by gluing the maps $F_i$ for all $i \in \mathbb{N}$.

One way to avoid constructing a special maximal tree is as follows. Let $Z$ be a Standard Model proper homotopy equivalent to $Y$. Then the underlying tree in $Z$ has the same ends. So we may apply the above paragraph to the composition $X_g \to Y \to Z$ and get an extension $X \to Z$, which we then compose with the inverse proper homotopy equivalence $Z \to Y$ to get $F : X \to Y$. The map $F$ may not agree with $f$ on $X_g$ but it is properly homotopic to it, so we conclude by applying the Homotopy Extension Theorem (Proposition 2.7).

3. Relationship with $\text{Out}(\pi_1(X))$.

In this section we will investigate the relationship between $\text{Maps}(X)$ and $\text{Out}(\pi_1(X))$. First, there is a natural homomorphism

$$\Psi : \text{Maps}(X) \to \text{Out}(\pi_1(X))$$

that sends $h \in \text{Maps}(X)$ to (the outer automorphism class of) $h_* : \pi_1(X) \to \pi_1(X)$. If the genus of $X$ is infinite, $\Psi$ is not onto since there are automorphisms not realized by proper maps. On the other hand, $\Psi$ is onto when $g(X) < \infty$. We will show first that $\Psi$ is injective if $X$ is a core graph (meaning $X = X_g$), and in general we will describe the kernel of $\Psi$. The next theorem is analogous to the fact that a homeomorphism of a surface with nonabelian fundamental group that induces identity in $\pi_1$ is isotopic to the identity, see [9, 16].

**Theorem 3.1.** Suppose $X$ is a core graph and let $f : X \to X$ be a proper map so that $f_* = \text{id} \in \text{Out}(\pi_1(X))$. Then $f$ is properly homotopic to the identity on $X$. In other words, $\Psi$ is injective.

**Proof.** In the proof we will use the fundamental property of graphs that disjoint nontrivial loops are not homotopic and that nullhomotopic loops can be nullhomotoped within their images. We may assume that $X$ is a Standard Model. Note that $f$ necessarily induces the identity on the space of ends. Indeed, if $f(\beta) \neq \beta$, ...
β for an end β, there will be an immersed loop α in X near β such that f(α) is
disjoint from α, and in particular f does not fix the conjugacy class of α.

Next, we can assume, by applying a proper homotopy (using Proposition 2.7)
that f fixes all vertices and moreover, by homotoping the root v around a loop,
that \( f_* : \pi_1(X, v) \to \pi_1(X, v) \) is the identity.

We will now construct a proper homotopy between the identity and f. If \( w \)
is a vertex, let \( e_1 e_2 \cdots e_k \) be the edge path in the underlying tree \( T \)
from \( v \) to \( w \) and define \( H : \{ w \} \times I \to X \) to be the tightened path \( \bar{e}_k \cdots \bar{e}_1 f(e_1) \cdots f(e_k) \). Also
define \( H \) on \( X \times \{ 0 \} \) to be identity and on \( X \times \{ 1 \} \) to be \( f \) (see Figure 4). We will
argue below that \( H \) defined so far is proper. If \( e \) is an edge in \( T \) then \( H \) is defined
on \( \partial(e \times I) \) and is nullhomotopic on this loop. Thus we may extend \( H \) to all such
2-cells keeping the image contained in the image of \( \partial(e \times I) \). Finally, \( H \)
extends to the cylinders \( x_w \times I \), where \( x_w \) is the loop attached at \( w \), using the fact that
\( f(e_1 \cdots e_k x_w \bar{e}_k \cdots \bar{e}_1) = e_1 \cdots e_k x_w \bar{e}_k \cdots \bar{e}_1 \). We again ensure that the image of the extension is contained in the image of the boundary of the 2-cell, so the extended \( H \) will be proper.

![Figure 4](image_url)

**Figure 4.** The homotopy \( H \). The brackets signify that we take
the immersed path homotopic to the given one rel endpoints.

It remains to show that \( H \) defined on the 1-skeleton is proper. This is where we
will use the assumption that \( X \) is a core graph. Let \( K \subseteq X \) be a finite subgraph.
Since \( f \) is proper, there is a finite subgraph \( L \subseteq X \) such that \( f(X \setminus L) \subseteq X \setminus K \).
Let \( w \) be a vertex outside of \( L \), \( e_1 \cdots e_k \) the edge path from \( v \) to \( w \) in \( T \), and \( x_w \)
the loop attached to \( w \). Thus \( f(x_w) \cap K = \emptyset \). From the fact that
\[
\begin{align*}
f(e_1 \cdots e_k) f(x_w) f(\bar{e}_k \cdots \bar{e}_1) &\simeq e_1 \cdots e_k x_w \bar{e}_k \cdots \bar{e}_1 \\
\end{align*}
\]
we see that after tightening \( f(e_1 \cdots e_k) \) does not cross any loops attached to vertices
in \( K \). For example, the fundamental group can be thought of as the free group on
the attached loops, and if \( y \) is the last loop in \( K \) crossed by \( [f(e_1 \cdots e_k)] \) the word
\[
[f(e_1 \cdots e_k)] \cdot [f(x_w)] \cdot [f(\bar{e}_k \cdots \bar{e}_1)]
\]
could be tightened by tightening the portion between the corresponding \( y \) and \( y \),
and would not yield the trivial word. Therefore \( H(\{ w \} \times I) \) is disjoint from \( K \).

The fundamental group of \( X \) does not “see” the ends of \( X \) not accumulated
by genus. For example, if \( X \) is a tree the mapping class group \( Maps(X) \) is isomorphic
to the homeomorphism group \( Homeo(\partial X) \) and may be quite nontrivial, while
\( \pi_1(X) = 1 \). It is therefore natural, when studying the kernel of \( \Psi \), to restrict to the
pure mapping class group \( PMaps(X) \).
We will now describe the kernel of the restriction
\[ \Psi_P : PMaps(X) \to Out(\pi_1(X)) \]
of \( \Psi \). This is well-known when \( X \) is obtained from a finite graph, say of rank \( r \) so that \( \pi_1(X) = F_r \), by attaching a finite number, say \( n \), of rays. These rays can be thought of equivalently as distinguished points in the finite graph. When \( n = 1 \) we have \( Maps(X) \cong Aut(F_r) \) and when \( n > 1 \) then \( PMaps(X) \cong Aut(F_r) \times F_r^{n-1} \) with the natural diagonal action of \( Aut(F_r) \) on \( F_r^{n-1} \). The \( F_r^{n-1} \) factor can be thought of as measuring the marking of \((n-1)\) distinguished points with respect to the remaining distinguished point, which is considered to be the basepoint. This will be generalized in Corollary \ref{cor:kernel}. When \( r \geq 2 \) the kernel of \( \Psi_P \) is isomorphic to \( F_r^{n-1} \) represented by maps that are identity on the finite graph and send each ray \( R \) to a ray of the form \( R \) for some loop \( w_R \) in the finite graph.

Let \( X \) be a locally finite graph and we assume \( DX := \partial X - \partial X_g \neq \emptyset \) and choose \( \alpha_0 \in DX \). This will be the basepoint “at infinity”. Let \( \pi_1(X, \alpha_0) \) be the set of proper homotopy classes of lines \( \sigma : \mathbb{R} \to X \) so that \( \lim_{t \to -\infty} \sigma(t) = \lim_{t \to \infty} \sigma(t) = \alpha_0 \), with concatenation as the group operation. Notice that concatenation makes sense since \( \alpha_0 \in DX \) and any two rays limiting to \( \alpha_0 \) eventually coincide, up to a proper homotopy. Given \( x_0 \in X \), there is an isomorphism
\[ \pi_1(X, x_0) \to \pi_1(X, \alpha_0) \]
given by \( \gamma \to \bar{\rho}_{x_0} \gamma \rho_{x_0} \) where \( \rho_{x_0} \) is a fixed ray in \( X \) from \( x_0 \) to \( \alpha_0 \). Moreover, if \( f \in Maps(X) \) fixes \( \alpha_0 \) then \( f \) induces a map \( f_0 \in Aut(\pi_1(X, \alpha_0)) \).

We first consider the case of the graph \( X = X_g^* \) obtained from a core graph (or a point) \( X_g \) by attaching a single ray.

**Lemma 3.2.** For \( X = X_g^* \) the kernel of \( \Psi : Maps(X) \to Out(\pi_1(X)) \) (or \( \Psi_P : PMaps(X) \to Out(\pi_1(X)) \)) is isomorphic to \( \pi_1(X) = \pi_1(X_g) \) when this group is nonabelian, and otherwise it is trivial.

When the genus \( n = g(X) \) is finite, the lemma says that the kernel \( Aut(F_n) \to Out(F_n) \) is \( F_n \) for \( n > 1 \) and otherwise it is trivial.

**Proof.** Let \( f \in Maps(X) \) induce identity in \( Out(\pi_1(X)) \). Using Lemma \ref{lem:kernel} applied to the nearest point projection \( \pi : X \to X_g \) we see that after a proper homotopy we may assume that \( f \) preserves the core \( X_g \), and thus by Theorem \ref{thm:kernel} we may assume \( f \) is identity on \( X_g \). Let \( \rho_0 \) denote the geodesic ray in \( X \) that intersects \( X_g \) at one point and such that \( X = X_g \cup \rho_0 \). Let \( c(f) \) be the homotopy class of \( \rho_0 f(\rho_0) \) in \( \pi_1(X, \alpha_0) \). Then \( f_0 \in Aut(\pi_1(X, \alpha_0)) \) is just conjugation by \( c(f) \).

The map \( f \mapsto c(f) \) is a homomorphism \( Ker(\Psi) \to \pi_1(X, \alpha_0) \). When \( \pi_1(X) \) is nonabelian it is an isomorphism. If \( X_g \) is a circle, \( f \) can be homotoped to the identity by a homotopy that rotates the circle to unwind the attached ray, so \( Ker(\Psi) \) is trivial.

We now consider the general case. Let \( X \) be a Standard Model which is not a tree, let \( X_g \) be the core subgraph of \( X \), and we assume \( DX \neq \emptyset \). Fix \( \alpha_0 \in DX \) and let \( \rho_0 \) be the ray in \( X \) intersecting \( X_g \) in a point and limiting to \( \alpha_0 \). Let \( T \subset X \) be the underlying tree, and let \( T_g = X_g \cap T \) be the underlying tree in the core, and likewise let \( T_g^* = T_g \cup \rho_0 \) be the underlying tree in \( X_g^* = X_g \cup \rho_0 \).
Thus \( X_g \subset X_g^* \subset X \) and both inclusions are homotopy equivalences. We note that restriction maps
\[
PMaps(X) \to PMaps(X_g^*) \to PMaps(X_g)
\]
are well-defined by Lemma 2.8, where for the retraction \( \pi \) we take the nearest point projection. In fact, we have a factorization of \( PMaps(X) \to Out(\pi_1(X)) \) as
\[
PMaps(X) \to PMaps(X_g^*) \to PMaps(X_g) \to Out(\pi_1(X))
\]
Our next goal is to describe \( Ker(PMaps(X) \to PMaps(X_g^*)) \). To that end, we introduce the group \( \mathcal{R} \) that measures how the rays towards the ends in \( DX \) wrap around the loops in the core graph \( X_g \).

**Definition 3.3.** The group \( \mathcal{R} \) as a set is the collection of functions \( h : DX \to \pi_1(X_g^*, \alpha_0) \) satisfying

(R0) \( h(\alpha_0) = 1 \).

(R1) \( h \) is locally constant.

(R2) For all \( \beta \in \partial X_g \) and every neighborhood \( U \) of \( \beta \) in \( X \) there exists a ray \( \rho_U \) from a point in \( U \) to \( \alpha_0 \) which is the concatenation of a segment in \( T \) and \( \rho_0 \) so that if \( \{ \beta_i \}_{i=1}^\infty \subset DX \) limits to \( \beta \) then for large enough \( i \),
\[
h(\beta_i) = \rho_U * \gamma_i * \rho_U \quad \text{where} \quad \gamma_i \text{ is a loop contained in } U.
\]

The group operation in \( \mathcal{R} \) is pointwise multiplication in \( \pi_1(X, \alpha_0) \).

**Definition 3.4.** We start by assigning an element \( \Phi_T(f) \in \mathcal{R} \) to certain proper maps \( f : X \to X \). More precisely assume:

(i) \( f : \partial X \to \partial X \) is the identity, and

(ii) either \( DX \) is compact or \( f_* : \pi_1(X, \alpha_0) \to \pi_1(X, \alpha_0) \) is the identity.

Note that we do not assume that \( f \) is a proper homotopy equivalence, cf. Example 4.1.

If \( \beta \in DX \setminus \{ \alpha_0 \} \) let \( \ell_\beta \) be the bi-infinite line in \( T \) connecting \( \alpha_0 \) to \( \beta \). Define the map
\[
\Phi_T(f) = h : DX \to \pi_1(X, \alpha_0)
\]
\[
h(\beta) = \begin{cases} 
\ell_\beta f(\ell_\beta) & \beta \in DX \setminus \{ \alpha_0 \}, \\
1 & \beta = \alpha_0
\end{cases}
\]

We claim that indeed \( h \in \mathcal{R} \). That \( h \) is locally constant follows from the observation that if \( \beta \in DX \) there is a neighborhood \( U \subset DX \) of \( \beta \) so that if \( f \) is a line joining two distinct points in \( U \) then \( f(\ell) \) is properly homotopic to \( \ell \).

Condition (R2) is vacuous if \( DX \) is compact so suppose \( f_* = id \). Thus we may assume that \( f \) is identity on \( X_g^* \). Let \( \ell_\beta \) be the line in \( T \) from \( \alpha_0 \) to \( \beta \), and similarly let \( \ell_{\beta_i} \) be the line in \( T \) from \( \alpha_0 \) to \( \beta_i \). Note that the lines \( \ell_{\beta_i} \) converge \( \ell_\beta \). The map \( f \) fixes \( \ell_\beta \) and thus for large \( i \) takes \( \ell_{\beta_i} \) to a line that agrees with \( \ell_{\beta_i} \) outside a given neighborhood \( U \) of \( \beta \). This proves (R2).

**Theorem 3.5.** Assume \( f : X \to X \) is proper, \( \alpha_0 \in DX \), and \( f_* = id : \pi_1(X, \alpha_0) \to \pi_1(X, \alpha_0) \). If \( f : \partial X \to \partial X \) is identity and \( \Phi_T(f) = 1 \) then \( f \) is properly homotopic to the identity. Moreover,
\[
\Phi_T : Ker(PMaps(X) \to PMaps(X_g^*)) \to \mathcal{R}
\]
is an isomorphism.
PROOF. Let $K = Ker(PMaps(X) \to PMaps(X^*_g))$. We start by arguing that $\Phi_T : K \to \mathcal{R}$ is a homomorphism. Let $f, g \in K$, and we assume that $f, g$ are identity on $X^*_g$. Let $\ell_\beta$ be the line as in the definition of $\Phi_T$. We have

$$\Phi_T(gf)(\beta) = \ell_\beta \cdot g f(\ell_\beta) = \ell_\beta \cdot g(\ell_\beta) \cdot g(\ell_\beta) = \Phi_T(g)(\beta) \cdot g(\Phi_T(f)(\beta))$$

which equals $\Phi_T(g)(\beta) \cdot \Phi_T(f)(\beta)$ since $g$ acts as the identity on $\pi_1(X, o_0)$.

We next show that if $\Phi_T(f) = 1$, then $f \simeq id$. We may assume $f$ is identity on $X^*_g$. Consider the universal cover $\hat{X}$ of $X$ and let $\hat{X}_g$ be the preimage of $X_g$ to $\hat{X}$ (which is connected). Let $\tilde{f}$ be the lift of $f$ that restricts to the identity on $\hat{X}_g$. The assumption that $\Phi_T(f)(\beta) = 1$ for every $\beta \in DX$ amounts to saying that $\tilde{f}$ fixes the ends of $\hat{X}$. The straight line homotopy $\tilde{H}$ from $\tilde{f}$ to $id$ is equivariant with respect to the deck group and descends to the homotopy $H : X \times I \to X$ from $f$ to $id$. It remains to show that $H$ is proper. It is useful to describe $H$ directly. If $x \in X$ consider a ray $\rho_x$ from $x$ to $o_0$ in $X$. The path $H(\{x\} \times I)$ is the tightened path $f(\rho_x)\bar{\rho}_x$.

Let $K$ be a compact set in $X$. As in the proof of Theorem 3.1 it is enough to show that there is a compact set $S$ so that for each vertex $v$ outside $S$, $H(\{v\} \times I) \cap K = \emptyset$.

Assume $x \in X$ is in a small neighborhood of an end $\beta \in \partial X$. If $\beta \in \partial X_g$ then $f(\rho_x)$ and $\rho_x$ agree outside a bit larger neighborhood by the assumption that $f$ fixes $\beta$ and $X^*_g$, so the path $H(\{x\} \times I)$ is also close to $\beta$. If $\beta \in DX$ consider the concatenation $\bar{\rho}_x \gamma$ where $\gamma$ is the ray from $x$ to $\beta$ in $T$. This concatenation is properly homotopic to the line $\ell_\beta$ from the definition of $\Phi_T$, and so by our assumption that $\Phi_T(f) = 1$ we have $f(\bar{\rho}_x \gamma) \simeq \bar{\rho}_x \gamma$. Since $f$ fixes $\beta$, $f(\gamma)$ is in a bit larger neighborhood of $\gamma$, so $f(\rho_x)\bar{\rho}_x \simeq f(\gamma)\bar{\gamma}$ is also close to $\beta$.

Finally, we argue that $\Phi_T : K \to \mathcal{R}$ is onto. Let $h \in \mathcal{R}$. We define a proper homotopy equivalence $f : X \to X$. Let $f|_{X^*_g} = id$. Let $S_w$ be the tree attached to a vertex $w \in X^*_g$, i.e. the closure of a component of $T \setminus T^*_g$. Then $\partial S_w$ is compact and from the fact that $h$ is locally constant we see that there is some distance $C$ so that for every edge $e \subset S_w$ at distance $C$ from $w$, $h$ is constant on the ends of the unbounded component of $S \setminus e$. Define $f|S_w$ to be identity on all edges of $S$ other than those at distance $C$ from $w$. On such an edge $e$ define $f$ to be the immersed path with the same endpoints as $e$ and so that if $\ell_\beta$ is a line that crosses $e$ then $f(\ell_\beta) \ell_\beta \in \pi_1(X, o_0)$ represents $h(\beta)$ (equivalently, $\bar{\rho}_x \ell_\beta \rho_x$ represents $h(\beta)$ for a suitable ray $\rho_x$ in $T$ with $\rho_x$ oriented towards $\beta$). Since $h(\alpha_0) = 1$ the map $f$ will be identity on $X^*_g$, as no edges $e$ where we modify the identity are along the ray to $o_0$. We must show that $f$ is a proper map. Let $\beta \in \partial X$ and assume that the edge $e$ as above is close to $\beta$. This means that $\beta \in \partial X_g$ (ends in $DX$ have a neighborhood not containing any edges $e$ as above). It follows that $f(e)$ is close to $\beta$ by (R2). Thus $f$ is a proper map. Let $g$ be constructed similarly for $h^{-1} \in \mathcal{R}$. Then $gf \in PMaps(X)$ has the property that $\Phi_T(gf) = \Phi_T(g)\Phi_T(f) = h^{-1}h = id$, so $gf \simeq id$ by the injectivity of $\Phi_T$, and similarly $fg \simeq id$.

We now have a useful criterion when a proper map $X \to X$ is properly homotopic to the identity, without assuming it is a proper homotopy equivalence.

**Corollary 3.6.** Let $f : X \to X$ be proper. Then $f$ is properly homotopic to the identity if and only if it preserves the homotopy class of every oriented closed
curve and the proper homotopy class of every oriented proper line in $X$ that in each direction converges to an end in $DX = \partial X \setminus \partial X_g$.

**Proof.** If $X$ is a core graph we are assuming that $f_* \in Out(\pi_1(X))$ preserves all conjugacy classes in $\pi_1(X)$. This implies that $f_* = id$ and the conclusion follows from Theorem 3.1. If $DX \neq \emptyset$ choose some $\alpha_0 \in DX$. We now see that $f_* : \pi_1(X, \alpha_0) \to \pi_1(X, \alpha_0)$ is identity, since $f$ preserves lines that start and end at $\alpha_0$, and similarly $f$ fixes all ends of $X$. Finally, we see that $\Phi_T(f) = 1$ since $f$ preserves all lines joining $\alpha_0$ with any $\beta \in DX$, so the statement follows from Theorem 3.5. 

**Corollary 3.7.** Suppose $f : X \to Y$ is proper, induces a homeomorphism $\partial X \to \partial Y$ and the restriction $X_g \to Y_g$ is a proper homotopy equivalence. Then $f$ is a proper homotopy equivalence.

**Proof.** Using Proposition 2.9 we have a proper map $g : Y \to X$ so that the restriction $Y_g \to X_g$ is the homotopy inverse to $f : X_g \to Y_g$ and so that $fg$ and $gf$ are identity on the boundaries. But then both are proper homotopy equivalences by Theorem 3.5 and thus both $f$ and $g$ are as well. 

Recall that $\mathcal{R} \cong Ker(PMaps(X) \to PMaps(X_g^*))$.

**Corollary 3.8.** If $\alpha_0 \in DX$ and if $\pi_1(X, \alpha_0)$ is nonabelian then $K = Ker(PMaps(X) \to PMaps(X_g))$ fits in an exact sequence

$$1 \to \mathcal{R} \to K \to \pi_1(X, \alpha_0) \to 1$$

If $X_g = S^1$ then $Ker(PMaps(X) \to PMaps(X_g) = \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathcal{R}$.

**Proof.** We focus on the first statement. The horizontal and vertical sequences in the commutative diagram below are exact by Theorem 3.5 and Lemma 3.2. The diagonal sequence is exact by the definition of $K$. The construction of the red arrows and the exactness of the sequence are a diagram chase.

When $DX$ is compact there is a more refined statement. Note that in that case condition (R2) in the definition of $\mathcal{R}$ is vacuous.
Corollary 3.9. If $DX$ is compact and nonempty then

$$PMaps(X) \cong \mathcal{R} \rtimes PMaps(X^*_g)$$

where $PMaps(X^*_g)$ acts on $\mathcal{R}$ by $g \cdot h(\beta) = g_*(h(\beta))$, where $g_* : \pi_1(X^*_g, \alpha_0) \to \pi_1(X^*_g, \alpha_0)$ is the homomorphism induced by $g$.

Proof. By Definition 3.4, we have $\Phi_T : PMaps(X) \to \mathcal{R}$, and let $R : PMaps(X) \to PMaps(X^*_g)$ be the restriction. Thus we have a function

$$(\Phi_T \times R) : PMaps(X) \to \mathcal{R} \rtimes PMaps(X^*_g)$$

That this is a homomorphism follows from the displayed calculation in the proof of Theorem 3.5. That the map is 1-1 and onto follows from Theorem 3.5 plus the observation that $R$ is onto. $\square$

When $DX$ is not compact, $\Phi_T$ may not be a well-defined function to $\mathcal{R}$ since $(R^2)$ may fail.

4. Topology

It takes a bit of care to define the topology on $Maps(X)$. Let $\hat{X}$ denote the Freudenthal compactification $X \cup \partial X$ by the ends of $X$. If a map $f : X \to X$ is a proper homotopy equivalence then it induces an isomorphism of $\pi_1(X)$ and it extends to a continuous map $\hat{X} \to \hat{X}$ that restricts to a homeomorphism of $\partial X$. However, the converse of this statement is false.

Example 4.1. Let $X$ be the ray $[0, \infty)$ with the circle $x_n$ attached at $n$. Then there is a proper map whose action on $\pi_1$ is $x_0 \mapsto x_0$ and $x_n \mapsto x_nx_{n-1}$ for $n > 0$. The inverse sends every $x_n$ to a word that involves $x_0$ so it cannot be realized by a proper map.

To circumvent this pathology, we will consider the space of pairs of maps which are proper homotopy inverses of each other. This way the inverse is “built in”, cf. the proof that inversion is continuous, Proposition 4.4. More precisely, let $(\hat{X} \to \hat{X})$ be the space of all continuous maps $\hat{X} \to \hat{X}$ equipped with compact open topology. If we fix a metric $d$ on $\hat{X}$ then $(\hat{X} \to \hat{X})$ has the associated sup metric which we also denote by $d$. This metric is also complete, and composition is continuous, and $(\hat{X} \to \hat{X})$ is separable (see e.g. [17, Theorem 4.19]).

Next, we look at the space $PH(X) \subset (\hat{X} \to \hat{X})^2$ consisting of pairs $(\hat{f}, \hat{g})$ such that:

$\hat{f}, \hat{g}$ are extensions to $\hat{X}$ of proper homotopy equivalences $f, g : X \to X$ that are each other’s inverses.

In particular, $\hat{f}, \hat{g}$ are homeomorphisms when restricted to $\partial X$ and they are each other’s inverses. We put the product topology on $(\hat{X} \to \hat{X})^2$ and the subspace topology on $PH(X)$.

Now define the function $\pi : PH(X) \to Maps(X)$ by

$$\pi(\hat{f}, \hat{g}) = [f]$$

where $[f]$ is the proper homotopy class of the restriction of $\hat{f}$ to $X$. This function is surjective and we put the quotient topology on $Maps(X)$.

Proposition 4.2. The quotient map $\pi$ is an open map.
Proof. Let \( U \subseteq PH(X) \) be open; we need to show that \( \pi^{-1}(U) \subseteq PH(X) \) is open. Let \( (\hat{f}, \hat{g}) \in \pi^{-1}(U) \). Thus we have proper homotopies \( H, K \) from \( f, g \) to \( h, k \) respectively, and \( (h, k) \in U \). Therefore there is \( \epsilon > 0 \) such that if \( (h', k') \in PH(X) \) and \( d(h, h') < \epsilon \), \( d(k, k') < \epsilon \) then \( (h', k') \in U \). We now claim that there is \( \delta > 0 \) such that if \( (\hat{f}', \hat{g}') \in PH(X) \), \( d(\hat{f}, \hat{f}') < \delta \), \( d(\hat{g}, \hat{g}') < \delta \), then there are homotopies of \( f', g' \) to maps \( h', k' \) as above, and this will show that \( (\hat{f}', \hat{g}') \in \pi^{-1}(U) \), finishing the proof.

We will prove the claim using Proposition 2.7. Note that all proper homotopies between maps on \( X \) extend continuously to \( \partial X \) and are stationary on all points of \( \partial X \). It follows that in the complement of a sufficiently large finite subgraph the tracks (i.e. paths traversed by points) of each such homotopy are as small as we like. In addition, the edges outside a large finite subgraph are as small as we like.

Choose a large finite subgraph \( L \subseteq X \) and choose \( \delta > 0 \) so that if \( d(\hat{f}, \hat{f}') < \delta \) then we have a homotopy between \( f'\{|L\} \) and \( f\{|L\} \) whose tracks have small size, and we also have a homotopy between \( f \) and \( h \) with small tracks for points in \( X \setminus L \). Applying Proposition 2.7 we get a homotopy from \( f' \) to some map \( h' \) that agrees with \( h \) on \( L \) and whose tracks of points outside \( L \) are small. Thus \( h' \) is close to \( f \) outside \( L \), which in turn is close to \( h \) outside \( L \). Thus \( h' \) is close to \( h \) everywhere. In a similar way we find \( \delta_g > 0 \) and a homotopy from \( g' \) to \( k' \). Then we set \( \delta = \min\{\delta_f, \delta_g\} \).

Corollary 4.3. Let PHE(\( X \)) \( \subseteq (\hat{X} \to \hat{X}) \) be the subspace of maps \( \hat{f} \) that are extensions to \( \hat{X} \) of proper homotopy equivalences \( f: X \to X \). Then the map \( q: PHE(X) \to Maps(X) \) that to \( \hat{f} \) assigns \( [f] \) is open. Thus alternatively we could use this map to define the quotient topology on Maps(\( X \)).

Proof. The projection \( (\hat{X} \to \hat{X})^2 \to (\hat{X} \to \hat{X}) \) to the first coordinate restricts to a continuous map \( PH(X) \to PHE(X) \). The composition with \( q: PHE(X) \to Maps(X) \) is open by the proposition, so \( q \) is open.

Proposition 4.4. With this topology Maps(\( X \)) is \( T_1 \) and it is a topological group.

Proof. We first show Maps(\( X \)) is a topological group. That composition is continuous follows from the fact that the map \( PH(X) \times PH(X) \to PH(X) \) defined by \((f, g) \mapsto (f \circ g, g)\) is continuous, being the restriction of the analogous map \((X \to \hat{X})^4 \to (X \to \hat{X})^2 \).

\[
\begin{array}{ccc}
PH(X) \times PH(X) & \longrightarrow & PH(X) \\
\downarrow \pi \times \pi & & \downarrow \pi \\
Maps(X) \times Maps(X) & \longrightarrow & Maps(X)
\end{array}
\]

Product of open maps is open so both vertical arrows are quotient maps. Continuity of the bottom horizontal map now follows.

For the inverse we consider the map \( PH(X) \to PH(X), (\hat{f}, \hat{g}) \mapsto (\hat{g}, \hat{f}) \) and the argument is similar.
To prove that \( \text{Maps}(X) \) is \( T_1 \) it suffices to show that the identity point is closed, since \( \text{Maps}(X) \) is a topological group. This amounts to showing that the space of \( \hat{f} : \hat{X} \to \hat{X} \) so that \( f \) is properly homotopic to the identity is a closed set. This follows from Corollary 3.6 since preserving a loop or a line is a closed condition. \( \square \)

**Corollary 4.5.** Let \( X, Y \) be two connected locally finite graphs that are proper homotopy equivalent. Then \( \text{Maps}(X) \) and \( \text{Maps}(Y) \) are isomorphic as topological groups.

**Proof.** Let \( F : X \to Y \) and \( G : Y \to X \) be proper homotopy equivalences with \( FG \) and \( GF \) properly homotopic to the identity. Then \( F, G \) extend to maps \( \hat{F} : \hat{X} \to \hat{Y} \) and \( \hat{G} : \hat{Y} \to \hat{X} \) and we have the induced maps \( (\hat{X} \to \hat{X}) \to (\hat{Y} \to \hat{Y}) \), \( \hat{f} \mapsto \hat{F} \circ \hat{f} \circ \hat{G} \) and \( (\hat{Y} \to \hat{Y}) \to (\hat{X} \to \hat{X}) \), \( \hat{g} \mapsto \hat{G} \circ \hat{g} \circ \hat{F} \). These maps restrict to \( \text{PHE}(X) \to \text{PHE}(Y) \) and \( \text{PHE}(Y) \to \text{PHE}(X) \), which induce homomorphisms \( \text{Maps}(X) \to \text{Maps}(Y) \) and \( \text{Maps}(Y) \to \text{Maps}(X) \). These are each other’s inverses and they are both continuous by Corollary 4.3. \( \square \)

### 4.1. Clopen subgroups

We next show that \( \text{Maps}(X) \) has a countable basis consisting of clopen sets, which are in fact cosets of subgroups. We will use this to show that \( \text{Maps}(X) \) is homeomorphic to \( \mathbb{Z}^\infty := \prod_{i=1}^{\infty} \mathbb{Z} \), i.e. the irrationals, and is hence a totally disconnected Polish group.

We define clopen subgroups of \( \text{Maps}(X) \), analogs of pointwise stabilizers of compact subsurfaces in big mapping class groups. Let \( K \subset X \) be a finite subgraph and define \( U_K \) to be the set of equivalence classes \([f] \in \text{Maps}(X)\) with a representative such that:

1. \( f = \text{id} \) on \( K \),
2. \( f \) preserves each complementary component of \( K \),
3. there is a representative \( g \) of \([f]^{-1}\) that also satisfies (i) and (ii),
4. there are proper homotopies \( gf \simeq 1 \) and \( fg \simeq 1 \) that are stationary on \( K \) and preserve complementary components of \( K \).

**Lemma 4.6.** \( U_K \) is open.

**Proof.** We must show that \( \pi^{-1}(U_K) \) is open. Let \( \pi(h, k) \in U_K \). Let \( f, g \) be the representatives satisfying (i)-(iv). We then have proper homotopies \( H, H' : X \times I \to X \) with \( H(x, 0) = h(x), H(x, 1) = f(x), H'(x, 0) = k(x), H'(x, 1) = g(x) \). Choose a compact subgraph \( L \subset K \) such that both \( H \) and \( H' \) at all times map \( X \setminus L \) into \( X \setminus K \). Finally, choose \( \epsilon > 0 \) such that if \( d(h', h) < \epsilon \) and \( d(k', k) < \epsilon \) then \( h', k' \) send \( X \setminus L \) to \( X \setminus K \) and \( h, h' \) (\( k, k' \)) restricted to \( L \) are homotopic by a homotopy that doesn’t move points in \( Fr(L) \) into \( K \).

Now use Proposition 2.7 a total of 4 times to prove that \( \pi(h', k') \in U_K \). First, we have a homotopy from \( h' \) to a map \( h'' \) extending \( h|L \), and second, we have a homotopy from \( h'' \) to a map \( f' \) extending \( f|L \). Thus \( f'|K = 1 \) and preparations above show that \( f' \) maps \( X \setminus L \) to \( X \setminus K \). Two more homotopy extensions yield a similar homotopy from \( k' \) to \( g' \). Since \( (f', g') \) satisfies (i)-(iv) we conclude that \( \pi(h', k') \in U_K \). \( \square \)

**Proposition 4.7.** \( U_K \) is a clopen subgroup. For every neighborhood \( U \) of \( 1 \in \text{Maps}(X) \) there is some \( K \) so that \( U_K \subseteq U \).

**Proof.** To prove that \( U_K \) is a subgroup, we must prove that if \([f_1], [f_2] \in U_K\) with \( f_1, f_2 \) preferred representatives, then \([f_1][f_2] \in U_K\). Indeed, \([f_1][f_2] = [f_1 \circ f_2]\)
and \( f_1 \circ f_2 \) is the identity on \( K \) and preserves complementary components. The same is true for its homotopy inverse \( g_0 \circ g_1 \) as well as for both homotopies to \( id_K \).

We proved in Lemma 4.6 that \( U_K \) is open. Thus \( U_K \) is also closed since its complement is a union of cosets, which are also open.

Finally, we must show that for every open \( U \subset \text{Maps}(X) \) containing 1, there is a compact \( K \) such that \( U_K \subset U \). We have \( \pi^{-1}U \) is an open set in \( PH(X) \) containing \((1, 1)\). Let \( \epsilon > 0 \) be such that if \((f, g) \in PH(X) \) and \( d(1, \hat{f}) < \epsilon \), \( d(1, \hat{g}) < \epsilon \) then \((\hat{f}, \hat{g}) \in \pi^{-1}(U) \). Let \( K \subset X \) be compact so that all complementary components of \( K \) have diameter \( < \epsilon \). Then for preferred representatives \((f, g) \) of \([f] \in U_K \) we have \((f, g) \in \pi^{-1}(U) \), so \( U_K \subseteq \pi \pi^{-1}(U) = U \). □

**Corollary 4.8.** Maps\((X)\) has a countable basis of clopen sets, it is separable, metrizable and totally disconnected.

**Proof.** Maps\((X)\) is separable since it is the continuous image of \( PH(X) \) which is separable, being a subspace of a separable metric space. So in particular each open subgroup \( U_K \) has at most countably many cosets. Choose an exhaustion \( K_i \) of \( X \); then \( U_{K_i} \) and their cosets form a countable basis of clopen sets. Next, \( T_1 \) plus a basis of clopen sets implies regular (proof: let \( x \notin A \) with \( A \) closed; then there is a basis element \( V \) with \( x \in V \) and \( V \cap A = \emptyset \) and so \( V, V^c \) is the required separation). Then countable basis plus regular implies normal \([8 \ 1.5.16]\), and finally countable basis plus normal implies metrizable (Urysohn metrization theorem, \([20 \ Theorem 34.1]\)). □

Recall that \( X \) has **finite type** if \( \pi_1(X) \) is finitely generated and \( \partial X \) is finite; otherwise \( X \) has **infinite type**.

**Lemma 4.9.** Suppose that \( X \) has infinite type. Then for every finite subgraph \( K \subset X \) there is a finite subgraph \( L \subset X \) such that \( L \supset K \) and \( U_L < U_K \) has infinite (countable) index.

**Proof.** First recall that an infinite totally disconnected compact metrizable space either has infinitely many isolated points or else it is the disjoint union of a Cantor set and finitely many isolated points. Consider the complementary components of \( K \). If one of them has genus \( N > 1 \) then there is \( L \supset K \) such that \( U_K \) contains a subgroup \( H \) isomorphic to the infinite group \( Aut(F_N) \), while \( H \cap U_L = 1 \). The elements of \( H \) are realized by homotopy equivalences supported on a finite subgraph of genus \( N \). Otherwise, after perhaps enlarging \( K \), we may assume that all complementary components are trees, and in this case \( \partial X \) is infinite. According to the dichotomy above, we can find \( L \supset K \) such that one of the following holds:

- There is a complementary component of \( K \) whose boundary at infinity contains an infinite set of isolated points, and this set can be written nontrivially as \( A \cup B \) where \( A \) and \( B \) are boundary points of two distinct complementary components of \( L \).

- There is a complementary component of \( K \) whose boundary at infinity contains a Cantor set as a clopen subset, and this Cantor set can be written nontrivially as \( A \cup B \) with both \( A, B \) clopen and both boundary points of two distinct complementary components of \( L \).

In the first case, \( U_K \) contains the entire group \( \text{Perm}_0(A \cup B) \) of finitely supported permutations of \( A \cup B \) (cf. the classification theorem), while the intersection
of this group with \( U_L \) contains only permutations that preserve \( A \) and \( B \). Since either \( A \) or \( B \) is infinite, this subgroup has infinite index.

In the second case \( U_K \) contains the entire group \( H = \text{Homeo}(A \cup B) \), while the intersection \( H \cap U_L \) contains only homeomorphisms that preserve \( A \) and \( B \), and this again has infinite index.

**Lemma 4.10.** \( PH(X) \subset (\hat{X} \to \hat{X})^2 \) is a \( G_\delta \)-subset.

**Proof.** A pair \((f, \hat{g}) \in (\hat{X} \to \hat{X})^2 \) is in \( PH(X) \) iff (see Corollary 3.6):

1. \( \hat{f}(\partial X) \subseteq \partial X \), \( \hat{g}(\partial X) \subseteq \partial X \),
2. \( \hat{f}(X) \subseteq X \), \( \hat{g}(X) \subseteq X \),
3. \( \hat{f} \) and \( \hat{g} \) are identity on \( \partial X \),
4. \( \hat{f} \) and \( \hat{g} \) restricted to \( X \) preserve the homotopy classes of oriented loops and proper homotopy classes of oriented proper lines joining points of \( DX \).

Conditions (1), (3) and (4) are closed conditions and (2) is \( G_\delta \): (2) can be written as countably many conditions \( \hat{f}(K_n) \subset X \), \( \hat{g}(K_n) \subset X \) for an exhaustion \( \{K_n\}, n = 1, 2, \cdots \) and these are all open.

When \( X \) has finite type, \( \text{Maps}(X) \) is countable and discrete. Otherwise we have:

**Proposition 4.11.** Suppose \( X \) has infinite type. Then \( \text{Maps}(X) \) is a Polish group with the underlying space homeomorphic to \( \mathbb{Z}^\infty \), i.e. to the set of irrationals.

**Proof.** There is a theorem of Sierpinski that if \( f : X \to Y \) is an open surjective map between separable metric spaces and \( X \) is complete, then \( Y \) is completely metrizable (see [8] Exercise 5.5.8.(d))). Since \( PH(X) \) is a \( G_\delta \) subset of \((\hat{X} \to \hat{X})^2\), it is completely metrizable (see [8] 4.3.23)) and therefore \( \text{Maps}(X) \) is completely metrizable. Then the fact that \( \text{Maps}(X) \) is homeomorphic to the irrationals follows from a theorem of Hausdorff (see e.g. [7]): If \( Z \) is separable, completely metrizable, zero dimensional (i.e. has a basis of clopen sets), and every compact subset has empty interior, then \( Z \) is homeomorphic to the irrationals. To finish the proof, if \( \text{Maps}(X) \) had a compact subset with nonempty interior, then some \( U_K \) would be compact. But this contradicts Lemma 1.9 since \( U_K \) is covered by the pairwise disjoint cosets of \( U_L \) and this cover doesn’t have a finite subcover.

We finish this section by considering continuity properties of homomorphisms studied in Section 3.

Recall the surjective homomorphism \( \sigma : \text{Maps}(X) \to \text{Homeo}(\partial X, \partial X_g) \) from Corollary 2.3. The group \( \text{Homeo}(\partial X, \partial X_g) \) is equipped with the compact-open topology. This means that a basis of neighborhoods of the identity is defined by clopen subgroups \( V_P \) where \( P \) is a finite partition of \( \partial X \) into clopen subsets and \( V_P \) consists of the elements of \( \text{Homeo}(\partial X, \partial X_g) \) that leave the partition elements invariant. Refining the partition yields a smaller clopen subgroup.

**Corollary 4.12.** The homomorphism \( \sigma \) is continuous and open. In particular, when \( X \) is a tree, \( \sigma : \text{Maps}(X) \to \text{Homeo}(\partial X) \) is an isomorphism of topological groups.

**Proof.** We may assume that \( X \) is a Standard Model. We will consider finite subgraphs \( K \subset X \) consisting of a subtree in the underlying tree together with all circles attached to it. The complementary components of \( K \) determine a partition
$\mathcal{P}_K$ of $\partial X$. Since every partition $\mathcal{P}$ is refined by some $\mathcal{P}_K$ and $\sigma(U_K) \subseteq V_{\mathcal{P}_K}$ it follows that $\sigma$ is continuous. To prove that $\sigma$ is open it suffices to argue that $\sigma(U_K) = V_{\mathcal{P}_K}$. Let $W$ be a complementary component of $K$. Thus $\partial W$ is one of the partition elements $A_W$ of $\mathcal{P}_K$ together with one point $v$ corresponding to the vertex of intersection $W \cap K$. Given a homeomorphism $h$ of $(A_W, A_W \cap \partial X_f)$, extend it by $v \mapsto v$ and view it as a homeomorphism of $(\partial W, \partial W_f)$. By the Classification Theorem there is $f_h \in Maps(W)$ that induces $h : \partial W \to \partial W$. Now define $f \in Maps(X)$ as the identity on $K$ and as $f_h$ on $W$, for each complementary component $W$, and observe that $\sigma(f)$ is the given homeomorphism in $V_{\mathcal{P}_K}$.

Next, recall the homomorphism $\Psi : Maps(X) \to Out(\pi_1(X))$ to the Polish group $Out(\pi_1(X))$. It is injective when $X$ is a core graph (Theorem 3.1).

**PROPOSITION 4.13.** The homomorphism $\Psi$ is continuous. If the genus of $X$ is infinite, the image is not a closed subgroup. If in addition $X$ is a core graph then $\Psi$ is injective but it is not a homeomorphism onto its image.

**PROOF.** The topology on $Aut(\pi_1(X))$ is defined as a subgroup of the symmetric group $S_X$ on the countable set $\pi_1(X)$, so an automorphism is close to the identity if it fixes a large finite set. The group $Out(\pi_1(X))$ is equipped with the quotient topology. If $U$ is an open neighborhood of the identity in $Out(\pi_1(X))$, its preimage in $Aut(\pi_1(X))$ will contain all automorphisms that fix a certain finite set $F$. The elements of $F$ are realized inside some compact subgraph $K \subseteq X$ and it follows that $\Psi(U_K) \subseteq U$, so $\Psi$ is continuous.

Consider $f : X \to X$ from Example 4.1. Let $f_n : X \to X$ be defined by $f_n(x_0) = x_0$, $f_n(x_k) = x_k x_{k-1}$ for $k \leq n$ and $f_n(x_k) = x_k$ for $k > n$. Then $\Phi(f_n) \to f_* \in Out(\pi_1(X))$, but $f_*$ is not in the image of $\Phi$.

Similarly, consider $g_n : X \to X$ defined by $g_n(x_k) = x_k$ when $k \leq n$ or $k \geq 2n$, $g_n(x_k) = x_k x_1$ when $n < k < 2n$. Then $\Psi(g_n) \to id$ but the sequence $g_n$ does not converge to $id$ (or anywhere). So $\Psi$ is not a homeomorphism onto its image.

Generalizing these examples to other graphs is left to the reader.

Finally, we have the following statement, whose proof is left to the reader.

**PROPOSITION 4.14.** The restriction epimorphisms $P Maps(X) \rightarrow P Maps(X_f)$ and $P Maps(X) \rightarrow P Maps(X_{g_f})$ are continuous and open.

5. **Proof of Main Theorem for core graphs**

5.1. **Free factor systems.** Let $F$ be a free group, possibly of infinite rank. Recall that a nontrivial subgroup $A < F$ is a free factor of $F$ if there is a subgroup $B < F$ such that $A * B = F$. We will only consider free factors of finite rank, and only conjugacy classes $[A]$ of such free factors. To simplify notation we will usually omit the brackets. Topologically, a (conjugacy class of a) nontrivial subgroup is a free factor if there is a graph $\Gamma$ with $\pi_1(\Gamma) = F$ and with $A$ represented by a subgraph. Similarly, a finite collection $\mathcal{F}$ of (conjugacy classes of) finitely generated free factors is a free factor system if there are representatives $A_1, A_2, \ldots, A_n$ and a subgroup $B < F$ such that $A_1 * A_2 * \cdots * A_n * B = F$. Topologically, there is a graph $\Gamma$ with $\pi_1(\Gamma) = F$ and with the $A_i$s represented by pairwise disjoint subgraphs.

If $\mathcal{F}$ and $\mathcal{F}'$ are two free factor systems, the intersection $\mathcal{F} \cap \mathcal{F}'$ is naturally a free factor system. It consists of conjugacy classes of nontrivial subgroups obtained by intersecting a representative of a conjugacy class in $\mathcal{F}$ with a representative of a
conjugacy class in $\mathcal{F}'$. Topologically, one can represent $\mathcal{F}$ and $\mathcal{F}'$ by immersions of finite graphs $\Gamma_\mathcal{F} \to \Gamma$ and $\Gamma_\mathcal{F}' \to \Gamma$, form the pull-back (see \cite{21}) and discard the contractible components to get an immersion representing the intersection.

**Example 5.1.** Let $\mathcal{F} = \langle a, b, c \rangle$, $A = \langle a, b \rangle$, $B = \langle a, c b a^{-1} \rangle$. Then $A$ and $B$ are free factors of $\mathcal{F}$, while their intersection is the free factor system consisting of two rank 1 free factors $\langle a \rangle$ and $\langle b \rangle$. The intersection of $A$ and $\langle c \rangle$ is the empty free factor system.

To see that the intersection $\mathcal{F} \cap \mathcal{F}'$ is a free factor system, one can arrange that one of them is represented by subgraphs of $\Gamma$ and then the pullback will be represented by subgraphs of the other one. It is also possible to compute finite intersections of free factor systems by a pull-back of several immersions.

Finally, we write $\mathcal{F} < \mathcal{F}'$ if every group (representing a conjugacy class) in $\mathcal{F}$ is contained in a group in $\mathcal{F}'$. For example, $\mathcal{F} \cap \mathcal{F}' < \mathcal{F}$.

**5.2. Tree of groups.** We now assume that $X$ is a core graph and is a Standard Model. Thus $X$ is a tree $T$ with a root vertex $v$ and with a loop attached at every vertex. We assign length 1 to each edge and let $D_0 : T \to [0, \infty)$ be the distance function from $v$. We extend $D_0$ to all of $X$ so that it is constant on each attached loop. Our first task is to control the sizes of maps, measured in $[0, \infty)$, represented by subgraphs of the other one. It is also possible to compute finite intersections of free factor systems by a pull-back of several immersions.

To see that the intersection $\mathcal{F} \cap \mathcal{F}'$ is a free factor system, one can arrange that one of them is represented by subgraphs of $\Gamma$ and then the pullback will be represented by subgraphs of the other one. It is also possible to compute finite intersections of free factor systems by a pull-back of several immersions.

Finally, we write $\mathcal{F} < \mathcal{F}'$ if every group (representing a conjugacy class) in $\mathcal{F}$ is contained in a group in $\mathcal{F}'$. For example, $\mathcal{F} \cap \mathcal{F}' < \mathcal{F}$.

**Proposition 5.2.** Let $H < Maps(X)$ be a compact subgroup. There is a sequence of integers $0 = r_0 < r_1 < r_2 < \cdots$ and for every $n \geq 0$ and every $|h| \in H$ there is a representative $h$ satisfying

(*) $h$ maps every element of the closed cover $C(r_1, r_2, \cdots, r_n)$ of $X$ consisting of the sets

$$D_0^{-1}[r_0, r_1], D_0^{-1}[r_1, r_2], \cdots, D_0^{-1}[r_{n-1}, r_n], D_0^{-1}[r_n, \infty)$$

to the union of the same element with the one or two adjacent elements.

**Proof.** We construct the numbers inductively, starting with $r_1 = 1$. Then (*) is vacuous.

Suppose that $r_n$ has been constructed satisfying (*). Note that by properness for every $|h| \in H$ (and every representative $h$ that exists by induction) there is some $r_{n+1} > r_n$ so that (*) holds for the cover $C(r_1, r_2, \cdots, r_n, r_{n+1})$ and this $h$. Moreover, the same $r_{n+1}$ will also work in a neighborhood of $|h|$ by choosing representatives of the form $hu$ where $|u| \in U_{D_0^{-1}[0, r_{n+1}]}$ i.e. $u$ fixes $D_0^{-1}[0, r_{n+1}]$ and leaves the complementary components invariant. Now by compactness of $H$, there is a finite cover of $H$ by such open sets and the maximal $r_{n+1}$ will then satisfy the requirements.

It will be convenient to introduce the following notation. First, let $\rho : [0, \infty) \to [0, \infty)$ be a homeomorphism such that $\rho(r_n) = n$ for $n = 0, 1, \cdots$ and let $D = \rho D_0 : X \to [0, \infty)$. Thus $D^{-1}([m, n]) = D_0^{-1}([r_m, r_n])$. We think of $D$ as a “control function”. For example, (*) says that for every $n$ every element of $H$ has a representative $h$ that “moves points $< 2^n$” i.e. $|D(x) - D(h(x))| < 2^n$ for every $x \in D^{-1}[0, n]$. The next proposition says that homotopies “move points $< 3^n$".
Proposition 5.3. Let $0 < r_1 < r_2 < \cdots$ be as in Proposition 5.2. Fix $n$ and let $h, h'$ be the representatives of two elements of $H$ that are inverses of each other as in Proposition 5.2. Then there is a proper homotopy between the identity and $h'h$ that moves each element of the cover $\mathcal{C}(r_1, \cdots, r_n)$ to the union of at most 5 elements, namely the 2-neighborhood of the given element.

Proof. First note that there is a canonical proper homotopy between the identity and $h'h$: lift the given homotopy to the universal cover extending the identity map, and then replace it by the straight line homotopy. We now argue that this homotopy moves within 2-neighborhoods. Fix a component $P$ of an element of the cover and let $\tilde{P}$ be the component of the 2-neighborhood that contains it. Since a loop in $P$ cannot be mapped by $h'h$ disjointly (since otherwise $h'h$ would not be homotopic to the identity) we see that $h'h(P) \subseteq \tilde{P}$. By lifting to the covering space of $X$ corresponding to $\pi_1(\tilde{P})$ and then retracting to the core $\tilde{P}$, we see that $h'h|P : P \to \tilde{P}$ is homotopic to inclusion $i : P \hookrightarrow \tilde{P}$ within $\tilde{P}$. Now note that any homotopy from $i$ to $h'h|P$ has tracks that are nullhomotopic loops (they have to represent $\pi_1$-elements that commute with $\pi_1(P)$, but since $\pi_1(P)$ and $\pi_1(\tilde{P})$ are nonabelian free groups this forces these loops to be trivial). It follows that the tracks described by the straight line homotopy are homotopic to paths in $\tilde{P}$, but since they are immersed, they must be contained in $\tilde{P}$. □

If $J \subset [0, \infty)$ is a closed interval with integer endpoints, write $\mathcal{F}(J)$ for the free factor system represented by $D^{-1}(J)$. Thus the number of free factors in $\mathcal{F}(J)$ is equal to the number of components of $D^{-1}(J)$. When $J$ is a degenerate interval (a single integer point) then each factor in $\mathcal{F}(J)$ has rank 1. We denote by $|J|$ the length of the interval.

We also set
$$\mathcal{F}'(J) = \bigcap_{h \in H} h_* (\mathcal{F}(J))$$
where $h_* : \pi_1(X) \to \pi_1(X)$ is the automorphism induced by $h$ (defined up to conjugation). This is really only a finite intersection since when $h$ is close to the identity we will have $h_* (\mathcal{F}(J)) = \mathcal{F}(J)$, so it suffices to intersect over finitely many coset representatives. Thus $\mathcal{F}'(J)$ is an $H$-invariant free factor system.

When $J = [a, b] \subset [0, \infty)$ with integer endpoints and with $b - a \geq 4$ we set $J^+ = [a - 2, b + 2] \cap [0, \infty)$ to be the 2-neighborhood of $J$, and likewise $J^- = [a', b - 2]$ where $a' = 0$ if $a = 0$ and otherwise $a' = a + 2$ (so $J^-$ is obtained from $J$ by subtracting the 2-neighborhood of the complement). Note that by our assumptions on the sequence $r_n$ we have that
$$\mathcal{F}(J^-) < \mathcal{F}'(J) < \mathcal{F}(J^+)$$
We now show that each group in $\mathcal{F}'(J)$ either contains a group in $\mathcal{F}(J^-)$ or it has trivial intersection with all of them.

Lemma 5.4. Let $A$ be a free factor in $\mathcal{F}'(J)$. If $A$ contains a nontrivial element $\alpha$ that also belongs to a free factor $B$ in $\mathcal{F}(J^-)$ then $B < A$ (up to conjugacy).

Proof. Represent different $h_* (\mathcal{F}(J))$, $h \in H$, by immersions of finite (possibly disconnected) graphs into $X$. The non-tree components of the pull-back then represent the free factors in $\mathcal{F}'(J)$. Since free factors (and free factor systems) are malnormal, if an immersion to $X$ lifts to the pull-back, it does so uniquely. Since an immersion representing $B$ lifts, it must lift to the component representing $A$, since this is where $\alpha$ lifts. □
We now set $\mathcal{F}^*(J)$ to be the free factor system consisting of the free factors in $\mathcal{F}'(J)$ that contain a free factor in $\mathcal{F}(J^-)$. Thus we still have

$$\mathcal{F}(J^-) < \mathcal{F}^*(J) < \mathcal{F}(J^+)$$

and also $J \subset J'$ implies $\mathcal{F}^*(J) < \mathcal{F}^*(J')$.

**Lemma 5.5.** If $|J| \geq 8$ then $\mathcal{F}^*(J)$ is $\mathcal{H}$-invariant.

**Proof.** Take a free factor $A$ in $\mathcal{F}^*(J)$. It will contain an element $\alpha$ corresponding to a loop in $D^{-1}(t)$ for any $t \in J$ whose distance to each endpoint is $\geq 4$. For any $h \in \mathcal{H}$ we have $h_*(\alpha)$ is an element in a free factor of $\mathcal{F}([t-2,t+2]) < \mathcal{F}(J^-)$, and the free factor of the latter that contains it is contained in a free factor $B$ of $\mathcal{F}^*(J)$ by Lemma 5.4 and $h_*(A) = B$. □

Now fix a sequence of intervals $J_1, J_2, \cdots$ that cover $[0, \infty)$ and so that $J_n \cap J_m = \emptyset$ when $|n - m| > 1$ and $J_{n,n+1} := J_n \cap J_{n+1}$ is an interval of length $\geq 22$ for $i = 1, 2, \cdots$. Now construct the following tree of groups $T$. The vertices of the tree are the free factors in $\mathcal{F}(J_n)$ (or equivalently the components of $D^{-1}(J_n)$), $n = 1, 2, \cdots$. The group associated to a vertex is the underlying free factor. The edges are the free factors in $\mathcal{F}(J_{n,n+1})$ (components of $D^{-1}(J_{n,n+1})$), again with the associated group the underlying free factor. Incidence relation is inclusion. The underlying graph is a tree, the nerve of the cover of $X$ by the components of $D^{-1}(J_n)$, $n \geq 1$.

**Lemma 5.6.** $\pi_1(T) \cong \pi_1(X)$.

**Proof.** By induction, the subtree of groups corresponding to the first $n$ intervals has the fundamental group of the corresponding subgraph of $X$. □

In a similar way we construct a tree of groups $T^*$, which will be $\mathcal{H}$-invariant. A vertex of height $n$ is a free factor in $\mathcal{F}^*(J_n)$, with this factor as the vertex group. An edge of height $[n, n+1]$ is a free factor in $\mathcal{F}^*(J_{n,n+1})$, with this factor as the edge group. Such a factor is contained in a unique vertex group at height $n$ and a unique vertex group at height $n + 1$ by Lemma 5.4 and this gives incidence and edge-to-vertex inclusions. Thus $T^*$ is a graph of groups and it is $\mathcal{H}$-invariant by construction. Below we will show that $T^*$ is a tree and $\pi_1(T^*) \cong \pi_1(T)$.

**Lemma 5.7.** If $C$ is an edge group in $T^*$ with $A, B$ the incident vertex groups of heights $n, n + 1$ resp., then $C$ is one of the free factors in $A \cap B$.

**Proof.** We have that $C$ is contained in some group in $A \cap B$ by construction. The free factor $A$ is a free factor in the free factor system $\mathcal{F}'(J_n)$ that contains a factor in $\mathcal{F}(J_n^-)$ and similarly for $B$. The intersection $A \cap B$ consists of free factors in $\mathcal{F}'(J_{n,n+1})$ and one of them contains $C$, which is also a free factor in $\mathcal{F}'(J_{n,n+1})$, so equality holds. □

Note here that in principle the intersection of $A$ and $B$ can consist of several free factors, i.e. the vertices might be joined by several edges. We will rule this out in Lemma 5.9.

There is a natural morphism (vertices to vertices and edges to edges) $\pi : T^* \to T$ that sends a factor in $\mathcal{F}^*(J_i)$ to the factor in $\mathcal{F}(J_i)$ that contains it, and similarly for the edges. Note that we have a height function on both trees (sending factors in $T^*(J_i)$, respectively in $\mathcal{F}(J_i)$ to $i$) that commutes with this map.
In the sequel it will be convenient to abuse the terminology and conflate a subcomplex of \( X \) and its fundamental group, and likewise a “component” and a “free factor” in a free factor system.

**Lemma 5.8.** Every vertex of \( \mathcal{T}^* \) at height \( n + 1 > 0 \) is connected by an edge to a vertex at height \( n \). There is a unique vertex of \( \mathcal{T}^* \) at height 0. In particular, \( \mathcal{T}^* \) is connected.

**Proof.** Suppose the vertex is \( A \), so it contains (possibly more than one) free factor \( B \) in \( \mathcal{F}(J_{n+1}) \). Every component of \( D^{-1}(J_{n+1}) \) contains a (unique) component of \( D^{-1}(J_{n,n+1}) \) and this component is contained in a unique free factor of \( \mathcal{F}^*(J_{n,n+1}) \), which represents an edge at height \([n, n + 1]\) attached to \( A \).

Since \( D^{-1}(J_1^-) \) is connected (recall that \( J_1^- \) contains \( \{0\} \)) and every vertex at height 0 must contain a component of it, it follows that there is only one height 0 vertex in \( \mathcal{T}^* \).

Note that a vertex at height \( n \) may not be connected to any vertices at height \( n + 1 \) since a component of \( D^{-1}(J_n^-) \) may not contain any components of \( D^{-1}(J_{n,n+1}^-) \).

**Lemma 5.9.** Let \( e \) be an edge in \( \mathcal{T} \) with height in \([n, n + 1]\) and consider its preimage \( \pi^{-1}(e) \) in \( \mathcal{T}^* \). After removing isolated vertices from \( \pi^{-1}(e) \), it is a tree with one vertex \( w \) at height \( n \) and all other vertices at height \( n + 1 \), and these are all connected to \( w \) by a unique edge. In particular, \( \mathcal{T}^* \) is a tree.

**Proof.** Let \( J = J_{n,n+1} \). The statement that all edges in the preimage of \( e \) have the same vertex at height \( n \) follows from the following fact. If two components of \( D^{-1}(J) \) are contained in the same component of \( D^{-1}(J_n^-) \) then they are contained in the same component of \( D^{-1}(J_{n,n+1}^-) \) (and this is not true if \( J_n^- \) is replaced by \( J_{n+1}^- \) and there may be several vertices at height \( n + 1 \).

We now argue that the height \( n + 1 \) vertices of all these edges in the preimage of \( e \) are distinct. Fix some integer \( k \in J \) at distance \( \geq 9 \) from the endpoints and let \( x, x' \) be two loops in \( D^{-1}(J) \) that map to \( k \). They will lift to unique components of \( \mathcal{F}^*(J) \) and any two components are determined in this way. If they lift to the same component of \( \mathcal{F}^*(J_{n+1}) \) then there is an immersion \( q : \Gamma \to D^{-1}(J_{n+1}) \) of a barbell (two disjoint loops connected by an edge) sending one loop to \( x \) and the other to \( x' \) and so that \( hq \) can be homotoped into \( D^{-1}(J_{n+1}) \) for every \( h \in H \). Thus \( q \) is kind of a “witness” that \( x, x' \) lift to the same component of \( \mathcal{F}^*(J) \). We need a similar witness that they lift to the same component of \( \mathcal{F}^*(J) \). The map \( q \) may not work, since its image may contain points of \( D^{-1}(J_{n+1} \setminus J) \), and we will perform a kind of surgery on \( q \) to get a better map.

Fix \( h \in H \). By perturbing if necessary we may assume that \( hq \) doesn’t collapse any edges and is simplicial with respect to suitable subdivisions. Then the statement that \( hq \) can be homotoped into \( \mathcal{F}(J_{n+1}) \) is equivalent to saying that after folding and replacing \( hq \) by an immersion, the image of the core subgraph is contained in \( D^{-1}(J_{n+1}) \). This same \( q \) may not map into \( D^{-1}(J) \) since it may map around loops in \( D^{-1}(J_{n+1}) \setminus D^{-1}(J) \), so we will modify it to \( q' : \Gamma' \to D^{-1}(J) \). First we analyze \( q \).

Recall that a vanishing path for \( hq \) is an immersion \( \nu : I \to \Gamma \) such that \( hq \nu : I \to X \) is a nullhomotopic closed path. There are only finitely many maximal vanishing paths and the folding process can be thought of as folding maximal vanishing paths one at a time. We now claim that \( hq \nu \) has \( D \)-size \( < 10 \) (i.e.
diam $Im(Dhqv) < 10$) when $h$ is as in Proposition 5.2 (see Figure 4 for an illustration). Indeed, $h'hqv$ is also a closed nullhomotopic path (where $h'$ is as in Proposition 5.3) and there is a homotopy of $h'hqv$ to $qv$ that moves the endpoints by $< 3$ measured by $D$. Thus the immersed path $qv$ gets closed up to a nullhomotopic loop by a path of $D$-size $< 6$, so it must itself have $D$-size $< 6$, and so $hqv$ has $D$-size $< 10$.

**Figure 5.** $H : I \times I \to X$ denotes the homotopy from $qv$ to $h'hqv$. The path denoted by $\beta$ is mapped by $H$ to a path homotopic to $qv$. Since $H$ maps vertical segments $\{t\} \times I$ to paths whose images have $D$-length less than 3, the dotted subpaths are mapped by $H$ to paths with $D$-length smaller than 3. The dashed part of $\beta$ is null homotopic. Since $qv$ is immersed, its $D$-length is $< 6$.

We now observe that after folding $hq$ the tree components of the complement of the core have $D$-size $< 10$. Indeed, choose any point $p \in \Gamma$. First fold all vanishing paths that do not contain $p$. After this, $p$ is still in the core. Finally, fold the remaining vanishing paths – this operation changes only the neighborhood of $p$ of $D$-size $< 10$.

Now consider $(Dq)^{-1}[k+1, \infty) \subset \Gamma$. It is a disjoint union of (possibly degenerate) closed intervals in the interior of the separating arc of $\Gamma$. Form a new graph $\tilde{\Gamma}$ by attaching an edge $E_a$ to $\Gamma$ for every nondegenerate arc $a$ in this disjoint union, with $\partial E_a = \partial a$. Note that $q$ sends both endpoints of $a$ to the same vertex (at $D$-height $k+1$, i.e. distance $r_{k+1}$ from the root vertex). Extend $q$ to an immersion $\tilde{q} : \tilde{\Gamma} \to X$ by sending each $E_a$ to a loop based at this vertex of combinatorial length $\leq 3$ (for example, one can send it either to the attached loop based at that vertex or to the loop of the form $dcd^{-1}$ where $d$ is an edge that increases the distance from the root and $c$ is the loop attached at the terminal vertex of $d$, see Figure 5). Let $\Gamma' \subset \tilde{\Gamma}$ be the barbell obtained by deleting the interiors of the arcs $a$ as above, and let $q' : \Gamma' \to X$ be the restriction of $\tilde{q}$.

We now claim that $h\tilde{q}$ is homotopic into $D^{-1}(J_{n+1})$ for every $h \in H$. We can fold $h\tilde{q}$ by first folding $hq$, which produces a core graph with trees attached, and then adding the edges $E_a$. They could be attached to points in the attached trees, but all such attached trees have $D$-size $< 10$ and map to $(k+1-10, k+1+10) \subset J_{n+1}$ by $D$. So after removing the attached trees to which no $E_a$’s are attached, the image is entirely contained in $D^{-1}(J_{n+1})$, which proves the claim.

In particular, $hq' : \Gamma' \to X$ is homotopic into $D^{-1}(J_{n+1})$, so $q'$ is also a witness to the fact that $x, x'$ lift to the same component of $F^*(J_{n+1})$. By construction, the image of $q'$ does not exceed the $D$-height $k+2$, so $hq'$ does not exceed the $D$-height $k+4$, and we see that $hq'$ is contained (even without homotopies) in $D^{-1}(J_{n})$. It
Figure 6. The barbell graph on the left is the graph $\Gamma$. It is mapped via $hq$ to $X$. The graph $\tilde{\Gamma}$ is $\Gamma$ union the edge $E_a$. The arc labeled $a$ is mapped into $D^{-1}[k+1, \infty)$. Suppose the initial edge of of $a$ is mapped to the loop $b$ and the terminal edge is mapped to $\bar{b}$. Then in order to define $\tilde{q}$ on $\tilde{\Gamma}$ so that it will be an immersion we will let $E_a$ map to $dcd\bar{d}$ where $d$ is the edge to the right of $b$, and $c$ is the one edge loop based at the endpoint of $d$.

follows that $hq'$ is homotopic into $D^{-1}(J_{n,n+1})$, and so the lifts of $x, x'$ in $F^*(J)$ are in the same component.

To see that $T^*$ has no loops, we note that if there were an embedded loop then let $u \in V(T^*)$ be the vertex of the loop of maximal height. Since no two vertices of the same height are connected then the loop gives us two vertices of the same height attached to $u$ which is a contradiction. □

Our next goal is to verify that $T^* \to T$ induces an isomorphism between the fundamental groups of these graphs of groups. Our method is to find a sequence of folds that converts $T^*$ to $T$. We will do this through an intermediate tree of groups $T^* \to T^{**} \to T$. Only $T^*$ will be $H$-invariant.

Recall the following folding moves on simplicial $G$-trees $T$ [4]. If $e_1, e_2$ are two oriented edges with the common initial vertex $v$ such that $e_1 \cup e_2$ embeds in the quotient $T/G$, then we may construct a new $G$-tree $T'$ by identifying $e_1$ and $e_2$ in an equivariant fashion, i.e. we identify $g(e_1)$ and $g(e_2)$ for every $g \in G$. The stabilizer of the new edge $e_1 = e_2$ is the group generated by $\text{Stab}(e_1)$ and $\text{Stab}(e_2)$, and similarly for the terminal vertices of $e_1$ and $e_2$. The effect in the quotient graph is to fold the images of $e_1$ and $e_2$. This is called Move IA in [4].

Similarly, suppose $e_1, e_2$ are two oriented edges with the common initial vertex $v$, each edge embeds in the quotient $T/G$, but they have the same images in $T/G$. This means that $g(e_1) = e_2$ for some $g \in \text{Stab}(v)$, so $\text{Stab}(e_2) = g\text{Stab}(e_1)g^{-1}$. The equivariant folding operation has the effect that the underlying quotient graph is unchanged, but the stabilizer of $e_1 = e_2$ is now the group generated by $\text{Stab}(e_1)$ and $g$, and similarly for the terminal vertex. This is called Move IIA, and we think of it as pulling the element $g \in \text{Stab}(v)$ across the image edge to the terminal vertex and enlarging the stabilizers by this $g$. In a similar way we can pull finitely generated subgroups (or think of it as several Moves IIA performed in sequence).
Let $T^{**}$ be the tree of groups obtained from $T^{*}$ by folding each preimage of an edge to an edge, so that there is a morphism $T^{**} \to T$. This amounts to performing infinitely many Moves IA, but they are all independent and can be performed simultaneously. The resulting morphism $T^{**} \to T$ is an isomorphism of underlying trees.

It will be convenient to denote by $T(e)$ the group associated to an edge $e$ of $T$, and similarly for the vertices, and for the trees $T^{*}$ and $T^{**}$.

**Lemma 5.10.** After independent Moves IIA, the morphism $T^{**} \to T$ becomes an isomorphism of graphs of groups.

**Proof.** The moves consist of pulling across an edge $e$ from an endpoint $w$ the subgroup $T^{**}(w) \cap T(e)$, simultaneously for all $(w, e)$. Since $J_{n,n+1} \setminus J_n \subset J_{n+1}^{-}$ then $F(J_{n,n+1})$ is generated by elements of $F(J_n)$ and $F(J_{n+1})$ which are contained in $F^e(J_n)$ and $F^e(J_{n+1})$ respectively. Therefore the group $T(e)$ is generated by elements in $T^*(w), T^*(v)$ for the endpoints $w, v$ of $e$. Thus by applying IIA moves we can promote $T^{**}(e)$ to $T(e)$. Similarly, $J_n \subset J_n \cup J_{n-1,n} \cup J_{n,n+1}^*$ hence $T(w)$ is generated by elements in $T^{**}(w)$ and $(T^{**}(e) \mid w$ is an endpoint of $e}$). Therefore we can promote $T^{**}(w)$ to $T(w)$ using IIA moves.

When $\mathcal{Y}$ is a locally finite graph of groups with all vertex and edge stabilizers finite rank free groups we define the geometric realization $GR(\mathcal{Y})$. This is the 2-complex constructed by taking a finite graph $\Gamma_w$ for every vertex $w$ so that $\pi_1(\Gamma_w) = \mathcal{Y}(w)$, and similarly taking a finite graph $\Gamma_e$ for every edge $e$ so that $\pi_1(\Gamma_e) = \mathcal{Y}(e)$, and gluing $\Gamma_e \times [0, 1]$ according to inclusion homomorphisms. Up to a proper homotopy equivalence, $GR(\mathcal{Y})$ is independent of the choices. From the lemmas above we see that the fundamental groups of graphs of groups $T, T^{*}, T^{**}$ are all isomorphic to $\pi_1(X)$. We now upgrade this to proper homotopy equivalences of geometric realizations.

**Lemma 5.11.** $X, GR(T), GR(T^*), GR(T^{**})$ are all proper homotopy equivalent.

**Proof.** $GR(T)$ can be built as a subspace of $X \times [0, \infty)$:

$$GR(T) = \bigcup_{n=0}^{\infty} (D^{-1}(J_n) \times \{n\} \cup D^{-1}(J_n \cap J_{n+1}) \times [n, n+1])$$

The map $GR(T) \to X$ is the projection, and $X \to GR(T)$ is the map $x \mapsto (x, \phi(x))$, where $\phi$ equals $n$ on $J_n \setminus (J_{n-1} \cup J_{n+1})$ and is in $[n, n+1]$ on $J_n \cap J_{n+1}$. These are each other’s proper homotopy inverses by homotoping along the second coordinate.

That $GR(T^*) \to GR(T^{**}) \to GR(T)$ are proper homotopy equivalences follows from the fact that Moves IA as well as IIA consisting of pulling finitely generated subgroups are proper homotopy equivalences of geometric realizations.

To finish, we need the relative version of Nielsen Realization for graphs, proved by Hensel-Kielak.

**Theorem 5.12.** Let $H < Out(F_n)$ be a finite subgroup and $F$ an $H$-invariant free factor system. Suppose the action of $H$ on $F$ is realized as a simplicial action of $H$ on a finite graph $\Gamma_0$ whose fundamental group is identified with $F$ (so the components of $\Gamma_0$ correspond to the free factors in $F$). Then there is a finite graph $\Gamma$, a simplicial action of $H$ on $\Gamma$, an $H$-equivariant embedding $\Gamma_0 \hookrightarrow \Gamma$, and an identification $\pi_1(\Gamma) \cong F_n$ so that the induced $H \to Out(F_n)$ is the given embedding $H < Out(F_n)$. 

When $\mathcal{F}$ is empty, we have the (absolute) Nielsen Realization [22, 5, 19, 15].

To apply this, we note:

**Lemma 5.13.** For every vertex $w$ in $\mathcal{T}^*$ the incident edge groups form a free factor system in $\mathcal{T}^*(w)$.

**Proof.** This is true for the tree $\mathcal{T}$ by construction. The statement then follows from the fact that intersections of free factor systems are free factor systems. □

Now we build a graph $Y$. We first construct graphs associated to the edges. Note that all orbits of edges are finite. For an edge $e$ of $\mathcal{T}^*$ choose a graph $\Gamma_e$ with $\pi_1(\Gamma_e) = \mathcal{T}^*(e)$ where $\text{Stab}_H(e)$ acts inducing the given action on $\mathcal{T}^*(e)$. Of course, $\text{Stab}_H(e)$ is a compact group, but the action on $\mathcal{T}^*(e)$ factors through a finite group, so we can apply the Nielsen Realization theorem. We associate the same graph to all edges in the orbit of $e$, with suitable identifications on $\pi_1$, so that $H$ now acts on the disjoint union of these graphs with the given action on $\pi_1$.

Now consider a vertex $w$. We have that $\text{Stab}_H(w)$ acts on $\mathcal{T}^*(w)$ and this action factors through a finite group, which also acts on the free factor system defined by the incident edges. This action is realized by the action of $\text{Stab}_H(w)$ on the disjoint union of the graphs representing the edge spaces, so the Relative Nielsen Realization provides a finite graph $\Gamma_w$ that contains this disjoint union and an extension of this action. Associate such graphs to the vertices equivariantly. The union along the subgraphs associated to the edges is the desired graph $Y$. Thus $H$ acts on $Y$ simplicially. The following lemma finishes the proof of the Main Theorem in the core graph case.

**Lemma 5.14.** There is a proper homotopy equivalence $Y \to X$ that commutes with the action of $H$.

**Proof.** Using the same graphs to represent vertex and edge groups, the geometric realization $\text{GR}(\mathcal{T}^*)$, after collapsing the $I$-factors, becomes $Y$, and this is a proper homotopy equivalence. By composing with proper homotopy equivalences from Lemma [5.11] we have $f : X \to Y$ and $g : Y \to X$, which are each other’s inverses. If $h \in H$ then by construction $h : X \to X$ and $ghf : X \to X$ induce the same element of $\text{Out}(\pi_1(X))$. It then follows from Theorem [3.1] applied to $ghf \cdot h^{-1}$ that they are properly homotopic. □

6. Proof for trees

We next prove Nielsen realization for trees.

**Theorem 6.1.** Suppose the graph $X$ is a tree and let $H < \text{Maps}(X)$ be a compact subgroup. Then there is a tree $Y \simeq X$ where $H$ acts by simplicial isomorphisms.

Note that by Corollary [4.12] $\text{Maps}(X) = \text{Homeo}(\partial X)$. Fix a metric $d$ on $\partial X$.

**Step 1.** We replace $d$ by an $H$-invariant metric $d'$. Let $\nu$ be a Haar measure on $H$ and define

$$d'(p, q) = \int_H d(h(p), h(q)) \, d\nu$$

This is an $H$-invariant metric. We drop the prime and assume $d$ is $H$-invariant.

**Step 2.** We now build equivariant finite partitions of $\partial X$ into clopen sets. Let $\epsilon > 0$. Say $p, q \in \partial X$ are $\epsilon$-path connected if there is a sequence $p = z_0, z_1, \cdots, z_n =$
q so that \(d(z_i, z_{i+1}) < \epsilon\) for all \(i = 0, \cdots, n - 1\). The equivalence classes form the desired partition \(\mathcal{P}_\epsilon\). Note that if \(\epsilon < \epsilon'\) then \(\mathcal{P}_\epsilon\) refines \(\mathcal{P}_{\epsilon'}\) and if \(\mathcal{P}\) is an arbitrary finite partition into clopen sets, there is \(\epsilon > 0\) so that \(\mathcal{P}_\epsilon\) refines \(\mathcal{P}\).

**Step 3.** Finally we build \(Y\) as the **mapping telescope** of a sequence of partitions from Step 2. Fix a decreasing sequence \(\epsilon_n \to 0\) with \(n = 1, 2, \cdots\) and let \(\mathcal{P}_n := \mathcal{P}_{\epsilon_n}\). We also set \(\mathcal{P}_0\) to be the trivial partition \(\{\partial X\}\). Since \(\mathcal{P}_{n+1}\) refines \(\mathcal{P}_n\) we have a natural surjection \(\mathcal{P}_{n+1} \to \mathcal{P}_n\) induced by inclusion of sets. Now let \(Y\) be the mapping telescope of this sequence. More concretely, the set of vertices is the disjoint union \(\sqcup_{n=0}^\infty \mathcal{P}_n \times \{n\}\), and there is an edge from \(P \times \{n+1\}\) to \(Q \times \{n\}\) whenever \(P \subseteq Q\) (here \(P \in \mathcal{P}_{n+1}\) and \(Q \in \mathcal{P}_n\)). Then \(Y\) is a tree and \(\partial Y\) is naturally (and \(H\)-equivariantly) homeomorphic to \(\partial X\) by the homeomorphism that sends a branch \((\mathcal{P}_n)_n\) of \(Y\) to the point \(\cap_n \mathcal{P}_n\) in \(\partial X\). The theorem is now proved since we have natural identifications

\[
Maps(X) = \text{Homeo}(\partial X) = \text{Homeo}(\partial Y) = Maps(Y)
\]

and \(H\) acts simplicially on \(Y\).

7. **Proof in general**

Let \(X\) be a locally finite graph which is not a tree and assume that a compact group \(H\) is acting on \(X\) by proper homotopy equivalences. The action then restricts to the core \(X_g\) (see Lemma 2.8) and by the special case of core graphs there is a core graph \(Y_g\), an action of \(H\) by simplicial isomorphisms on \(Y_g\), and an \(H\)-equivariant proper homotopy equivalence \(f: X_g \to Y_g\).

**Lemma 7.1.** There is a locally finite graph \(Y \supseteq Y_g\) and a proper homotopy equivalence \(X \to Y\) that extends \(f\).

**Proof.** Form the mapping cylinder \(M = X_g \times [0,1] \cup Y_g/\sim f(x)\) of \(f\). Since \(f\) is a proper homotopy equivalence, both 0 and 1-levels of \(M\) (which can be identified with \(X_g\) and \(Y_g\)) are proper strong deformation retracts of \(M\). For \(Y_g\) this can be seen by deforming along the mapping cylinder lines. For \(X_g\), without the word “proper”, this is a theorem of Ralph Fox [11], see also [12], but their proofs work just as well in the proper category. The statement can also be deduced from the Whitehead theorem, see [13], and [10] for the proper version. Now \(X\) is obtained from \(X_g\) by attaching trees \(T_v\) along vertices \(v \in X_g\). Attach products \(T_v \times I\) to \(M\) along the natural copies of \(v \times I\) to obtain a space \(Z\) and note that both \(X\) and the space \(Y\) (obtained from \(Y_g\) by attaching trees \(T_v\) along \(f(v)\)) are proper strong deformation retracts of \(Z\) and this gives the desired proper homotopy equivalence \(X \to Y\).

We will now revert to the original notation and simply assume that \(H\) is acting by simplicial isomorphisms on \(X_g\).

By the **convex hull** of a nonempty subset of a simplicial tree we mean the smallest simplicial subtree that contains the set. The following fixed point fact is well known.

**Lemma 7.2.** Suppose a compact group \(H\) acts continuously on a simplicial tree. Then \(H\) fixes a point in the convex hull of any orbit.

**Proof.** The convex hull is \(H\)-invariant and it is a tree of finite diameter. Iteratively remove all edges that contain a valence 1 vertex until the tree that’s left
lifts are not asymptotic and hence not properly homotopic. Choose one such lift \( \tilde{\beta} \) to point \( \rho \).

We will now use this fact to prove the following fixed point theorem, which is really the heart of the argument in this case.

**Lemma 7.3.** Suppose \( H \) fixes a point \( \beta \) in \( DX = \partial X \setminus \partial X_g \). Then \( H \) fixes a point \( \rho(\beta) \) in \( X_g \) and there is a ray (called the Nielsen ray) \( r \) from \( \rho(\beta) \) to \( \beta \) such that \( h(r) \) and \( r \) are properly homotopic rel \( \rho(\beta) \) for every \( h \in H \).

**Proof.** Let \( \tilde{X} \) be the universal cover of \( X \). Let \( r \) be a ray in \( X \) converging to \( \beta \). The deck group acts simply transitively on the set of lifts of \( r \) and distinct lifts are not asymptotic and hence not properly homotopic. Choose one such lift \( \tilde{r} \). Every \( h \in H \) has a unique lift to \( \tilde{X} \) that fixes the asymptotic class of rays \([\tilde{r}]\) and the set of these lifts defines an action of \( H \) on \( \tilde{X} \) by proper homotopy equivalences. We will prove that the action is continuous in the next paragraph. The lifted group \( H \) preserves the preimage \( \tilde{X}_g \) of \( X_g \), which is a tree, and this defines an action of \( H \) on \( \tilde{X}_g \). By Lemma 7.2 it fixes a point \( z \). The image of \( z \) in \( X_g \) is the desired fixed point and the image of the ray that starts at \( z \) and is asymptotic to \( \tilde{r} \) is the Nielsen ray.

The action is continuous: if \( h \in H \) is close to the identity, we can choose a representative in its proper homotopy class that fixes a large compact set \( K \subset X \) as well as the ray \( r \), and preserves the complementary components of \( K \). We can also arrange that \( K \cup r \) is connected. Then the lift of \( h \) to \( \tilde{X} \) will fix the preimage \( \tilde{K} \) and will preserve its complementary components. Since \( K \) can be chosen so that \( \tilde{K} \) contains any given compact set, the lift of \( h \) will be close to the identity. □

Let \( d \) be an \( H \)-invariant metric on \( \partial X \) (see Step 1 in Section 6) and let \( \mathcal{P}_r \) be the partition of \( \partial X \) as in Step 2 in Section 6. Again fix a decreasing sequence \( \epsilon_n \to 0 \) and set \( \mathcal{P}_n := \mathcal{P}_{r_n} \). Let \( \pi' : X \cup DX \to X_g \) denote the nearest point projection (this is not equivariant).

Fix an \( H \)-equivariant exhaustion \( \emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots \) of \( X_g \) by finite connected subgraphs so that if \( \beta \in X \cup DX \) and \( \pi'(\beta) \notin K_{i+1} \) then \( \pi'(h(\beta)) \notin K_i \) for every \( h \in H \).

Call an element \( P \in \mathcal{P}_n \) good if the following holds:

- \( P \subset DX \),
- \( \pi'(P) \) is a point,
- \( \text{Stab}_H(P) \) fixes a point \( \rho(P) \) in \( X_g \); moreover, if \( \pi'(P) \) is disjoint from \( K_{i+1} \) then \( \rho(P) \) and \( \pi'(P) \) are in the same component of \( X_g \setminus K_i \),
- for every \( x \in P \) there is a ray \( r_x \) from \( \rho(P) \) to \( x \) so that all these rays (for all \( x \in P \)) agree along \( X_g \) and further they are permuted up to proper homotopy by \( \text{Stab}_H(P) \).

So in particular \( r_x \) is a Nielsen ray with respect to \( \text{Stab}_H(x) < \text{Stab}_H(P) \). We will also call the rays \( r_x \) Nielsen rays.

**Lemma 7.4.** For every \( \beta \in DX \) there is \( n_0 \) so that for every \( n \geq n_0 \) the element \( P \in \mathcal{P}_n \) containing \( \beta \) is good.

**Proof.** We first observe that \( \text{Stab}_H(\beta) \) fixes a point in \( X_g \) by applying Lemma 7.3 to the induced action of \( \text{Stab}_H(\beta) \) on the graph \( X^*_g = X_g \cup \rho_\beta \) (see Lemma 2.8). By our assumption on the exhaustion, if \( \pi'(\beta) \) misses \( K_{i+1} \) then the action restricts
to the complementary component of \( K_1 \) that contains \( \pi'(\beta) \), so in this case the fixed point \( \rho(\beta) \) can be found there. Now notice that the stabilizer of a point in \( X_g \) is a clopen subgroup of \( H \), so when \( n \) is large the stabilizer of \( P_n \in \mathcal{P}_n \) that contains \( \beta \) will also fix the same point. (Since \( H \) permutes the partition elements in \( \mathcal{P}_n \), \( \text{Stab}_H(\beta) \) will leave \( P_n \) invariant and we see that \( \text{Stab}_H(\beta) = \cap_n \text{Stab}_H(P_n) \) is the intersection of clopen subgroups. By compactness we have \( \text{Stab}_H(P_n) \subseteq \text{Stab}_H(\rho(\beta)) \) for large \( n \).) We will of course also have \( P \subseteq DX \), \( \pi'(P) \) is a point, and \( h(\ell) \cap X_g = \emptyset \) for every line \( \ell \) joining two points of \( P \) and every \( h \in H \). \( \square \)

Now we construct an \( H \)-equivariant cover \( \mathcal{N} \) by pairwise disjoint good partition elements. Say an \( H \)-orbit in \( \mathcal{P}_n \) (which is finite) is good if every (any) element in it is good. Then let \( \mathcal{N} \) consist of good orbits in \( \mathcal{P}_1 \) as well as those good orbits in \( \mathcal{P}_n \), \( n = 2, 3, \ldots \) whose union is not contained in the union of any good orbit in \( \mathcal{P}_{n-1} \). Define an equivariant map \( \rho : \mathcal{N} \to X_g \) by letting \( \rho \) be as in the definition of a good partition element on a representative of the orbit, and then extend it equivariantly. Thus we still have the Nielsen rays for all elements of \( \mathcal{N} \).

We now construct a graph \( Y \) by attaching trees to \( Y_g = X_g \). For every \( N \in \mathcal{N} \) we build a tree \( T_N \) as in Step 3 of Section 6 for \( \text{Stab}_H(N) \), namely the mapping telescope with base vertex \( N \) and the other vertices all the partition elements contained in \( N \). We identify \( \partial T_N \) with \( N \). We then attach \( T_N \) to \( X_g \) by identifying the base vertex \( N \) with the point \( \rho(N) \in X_g \). Doing this for all \( N \in \mathcal{N} \) produces the desired graph \( Y \). By construction \( H \) acts on \( Y \) by simplicial isomorphisms.

**Lemma 7.5.** There is a proper homotopy equivalence \( F : Y \to X \) such that
(a) \( F \) is identity on \( X_g \) and on \( DX = DY \),
(b) \( F \) sends the rays in \( T_N \) based at \( N \) to the Nielsen rays \( r_x \) from \( \rho(N) \) to \( \partial N \) preserving the endpoints,
(c) \( F \) is \( H \)-equivariant.

**Proof.** The map \( F \) is uniquely defined on each \( T_N \) by (a)-(c). That this map is proper as a map \( Y \to X \) follows from the fact that if \( N_i \in \mathcal{N} \) converge to \( \beta \in \partial X_g \), then \( \rho(N_i) \to \beta \). Thus \( F \) is a proper homotopy equivalence by Corollary 3.7.

Finally we argue \( H \)-equivariance. Denote by \( F' \) the proper homotopy inverse of \( F \) which is identity on \( X_g \). If \( h \in H \) consider \( F'hF' \cdot h^{-1} : Y \to Y \). This is identity on \( X_g \) and on \( \partial X \). By Corollary 3.6 it suffices to argue that this map preserves oriented loops and lines connecting points of \( DX \). For loops this is clear since the map is identity on \( X_g \). It also preserves lines joining points of \( DX \) since such lines can be written as a concatenation \( r^{-1}sr' \) where \( r, r' \) are Nielsen rays and \( s \) is a segment in \( X_g \). Finally, it preserves lines that connect distinct points of some \( N \in \mathcal{N} \). \( \square \)

This finishes the proof of the Main Theorem.

**References**

1. Santana Afton, Danny Calegari, Lvzhou Chen, and Rylee Alanza Lyman. Nielsen realization for infinite-type surfaces. [arXiv:2002.09760](https://arxiv.org/abs/2002.09760).
2. Javier Aramayona and Nicholas G. Vlamis. Big mapping class groups: an overview. https://arxiv.org/abs/2003.07950.
3. R. Ayala, E. Dominguez, A. Marquez, and A. Quintero. Proper homotopy classification of graphs. *Bull. London Math. Soc.*, 22(5):417–421, 1990.
4. Mladen Bestvina and Mark Feighn. Bounding the complexity of simplicial group actions on trees. *Invent. Math.*, 103(3):449–469, 1991.

5. Marc Culler. Finite groups of outer automorphisms of a free group. In *Contributions to group theory*, volume 33 of *Contemp. Math.*, pages 197–207. Amer. Math. Soc., Providence, RI, 1984.

6. George Domat, Hannah Hoganson, and Sanghoon Kwak. Coarse geometry of pure mapping class groups of infinite graphs. *Adv. Math.*, 413:Paper No. 108836, 57, 2023.

7. Carl Eberhart. Some remarks on the irrational and rational numbers. *Amer. Math. Monthly*, 84(1):32–35, 1977.

8. Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

9. D. B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Math.*, 115:83–107, 1966.

10. F. T. Farrell, L. R. Taylor, and J. B. Wagoner. The Whitehead theorem in the proper category. *Compositio Math.*, 27:1–23, 1973.

11. R. H. Fox. On homotopy type and deformation retracts. *Ann. of Math. (2)*, 44:40–50, 1943.

12. Martin Fuchs. A note on mapping cylinders. *Michigan Math. J.*, 18:289–290, 1971.

13. Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

14. Sebastian Hensel and Dawid Kielak. Nielsen realization by gluing: limit groups and free products. *Michigan Math. J.*, 67(1):199–223, 2018.

15. Sebastian Hensel, Damian Osajda, and Piotr Przytycki. Realisation and dismantlability. *Geom. Topol.*, 18(4):2079–2126, 2014.

16. Jesús Hernández Hernández, Israel Morales, and Ferrán Valdez. The Alexander method for infinite-type surfaces. *Michigan Math. J.*, 68(4):743–753, 2019.

17. Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

18. Steven P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

19. D. G. Khramtsov. Finite groups of automorphisms of free groups. *Mat. Zametki*, 38(3):386–392, 476, 1985.

20. James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [MR0464128].

21. John R. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.

22. Bruno Zimmermann. Über Homöomorphismen $n$-dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen. *Comment. Math. Helv.*, 56(3):474–486, 1981.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, 199 ABBA KHOUSHY AVENUE MOUNT CARMEL, HAIFA, 3498838 ISRAEL

Email address: yalgom@univ.haifa.ac.il

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 S 1400 E, RM 233 SALT LAKE CITY, UT 84112, USA

Email address: mladen.bestvina@utah.edu