Modification of the Abel-Plana formula for functions with non-integrable branch-points

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Abstract
The Abel-Plana formula (APF) is a widely used tool for calculations in Casimir-type problems. In this paper, we present a particular explicit modification of the generalized APF for the functions with non-integrable branch-point singularities.

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1. Introduction
Among numerous methods used for calculations in quantum field theory (QFT), an important role is played by those exploiting the analytical properties of functions. One such method is the summation formula of Abel-Plana (APF) [1] which is widely used for calculations in Casimir problems in different configurations [2], and connected issues [3].

The most frequently used form of the APF is the following [4]:

\[
\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x) \, dx + \frac{1}{2} f(0) + i \int_0^{\infty} \frac{f(ix) - f(-ix)}{e^{\pi x} - 1} \, dx.
\] (1)

It is applicable to functions satisfying the following convergence condition

\[
\lim_{y \to \infty} e^{-2\pi y} |f(x + iy)| = 0
\] (2)

uniformly on any finite interval of \(x\). This condition is naturally met within the framework of QFT even for formally divergent series when (any) appropriate regularization scheme is applied.

There is another important condition of validity of (1)—the analyticity of \(f(x)\) in the right half-plane. This condition however cannot be equally naturally satisfied in field theoretical calculations and should be addressed independently in each case. Unfortunately, in the literature (see for instance [4]), it is not always paid enough (if any) attention for verification of this condition. On the other hand, one can easily see that direct application of (1) to such ‘trivial’ functions as

\[
f = \frac{1}{n^2 + a^2}, \quad \frac{1}{n^4 + a^2n^2 + b^2}
\] (3)

leads to incorrect answers.

The most major research ever made on the APF and its generalizations to different classes of functions is presented in [5, 6]. There is not only a treatment of some of the elementary functions present, but Bessel functions are also considered.

However, it is not always easy to apply a cumbersome generalized formula in particular cases, and explicit expressions are needed. In particular, for the functions possessing non-integrable branch-point singularities on the imaginary axe, an explicit form of the APF is missing. Summation of such functions appears in the calculation of the Casimir energy in the electromagnetic case with a semi-transparent cylindrical shell [7].

In this paper, we present a derivation of the explicit summation formula for this case.

2. The APF for non-integrable branch-point singularities
The derivation of a generalized APF is based on the integration of a pair of functions along a contour in a complex plane. The contour goes in part along the imaginary axe (for details see [6]). Consequently, any singularities at \(z = ia\),...
a \in \mathbb{R} need a careful and detailed study. Integrable branch points as well as normal poles (of arbitrary order) do not bring particular problems as they can be expressed in terms of straightforward integrals or residues. However, the merging of two types of singularities must be treated independently. Such a behavior of a function could be represented as:

\[ f(x) = \frac{1}{(x^2 + q^2)^{k+1/2}}, \quad k = 1, 2, \ldots \] (4)

We consider here the simplest case of a ‘naked’ singularity as the summand function, but further generalizations are immediate.

A direct study of the above mentioned contour integral is rather cumbersome, and it comes out easier to start with the following form of the APF:

\[ \sum_{n=1}^{\infty} \frac{1}{(n^2 + q^2)^{k+1/2} - w^2} = -\frac{1}{2w^{2k} - i\pi \text{Res}} \left( \frac{g(z)}{\sqrt{z^2 - w^2}} \right) \]

\[ + 2(-1)^k \int_w^{\infty} h(x) \frac{dx}{(x^2 + q^2)^{k+1/2} - w} \] (5)

which can be derived from a combination of (3.22) and (3.33) [6] and is valid for \( w > q > 0 \). For convenience, we have introduced the following notation:

\[ g(x) = -\frac{2}{(x + iq)^{k+1/2}e^{-2\pi ix}} - 1, \]

\[ h(x) = -\frac{(j)^{k+1/2}}{2} g^{(j)}(ix). \] (6)

Let one consider the limit \( q \to w \). The lhs of (5) is perfectly convergent in this limit, and so must be the rhs according to the analytical continuation principle.

To investigate the rhs behavior in detail, we first construct explicitly the residue at \( z = iq \). Decomposing \( \frac{g(z)}{\sqrt{z^2 - w^2}} \) into a Taylor series at this point and exploiting an obvious connection between derivatives of \( g \) and \( h \) of arbitrary order \( j \):

\[ h^{(j)}(x) = -\frac{(j)^{k+1/2}}{2} g^{(j)}(ix), \]

we can write for the residue

\[ (-1)^{k-1} \sqrt{\pi} \sum_{j=0}^{k-1} h^{(j)}(q) \frac{\Gamma(k - j - 1/2)}{\Gamma(k - j)\Gamma(j + 1)} \frac{1}{(w - q)^{k-j-1}}. \] (7)

On the other hand, for the integral part of rhs in (5), we can construct the following decomposition:

\[ I = \int_w^{\infty} \left( h(x) - [h(x)]_q^{k-1} \right) \frac{dx}{(x^2 + q^2)^{k+1/2} - w} \]

\[ + \int_w^{\infty} [h(x)]_q^{k-1} \frac{dx}{(x^2 + q^2)^{k+1/2} - w}, \] (8)

where we have subtracted the first \( k \) Taylor terms of \( h(x) \) at \( x = q \)

\[ [h(x)]_q^{(k-1)} = h(q) + \ldots + \frac{h^{(k-1)}(q)}{(k-1)!} (x - q)^{k-1}. \] (9)

Then, the first term in (8) is finite in the limit \( q \to w \) and the second one can be integrated explicitly

\[ \int_w^{\infty} \frac{[h(x)]_q^{k-1} dx}{(x^2 + q^2)^{k+1/2} - w} \]

\[ = \sum_{j=0}^{k-1} \frac{h^{(j)}(q)}{j!} \int_0^{\infty} dx \frac{\sqrt{\pi}}{(x^2 + q^2)^{k-j-1}} \frac{1}{\Gamma(k - j)}. \] (10)

One can easily see that the divergent (as \( q \to w \)) part of \( I \) exactly cancels the residue part (7) and the summation formula in the limit of \( q = w \) takes the form

\[ \sum_{n=1}^{\infty} \frac{1}{(n^2 + w^2)^{k+1/2} - w^2} = -\frac{1}{2w^{2k} + 2(-1)^k \int_w^{\infty} \Delta h(x) \frac{dx}{(x^2 + w^2)^{k+1/2}}} \]

\[ = -\frac{1}{2w^{2k+1}} + 2(-1)^k \int_w^{\infty} \Delta h(x) \frac{dx}{(x^2 + w^2)^{k+1/2}}. \] (11)

where

\[ \Delta h(x) = h(x) - [h(x)]_w^{k-1}. \] (12)

A more general case is also valid

\[ \sum_{n=1}^{\infty} \frac{\tilde{f}(n)}{(n^2 + w^2)^{k+1/2}} = -\int_0^{\infty} \frac{\tilde{f}(x) dx}{(x^2 + w^2)^{k+1/2}} \]

\[ = -i\pi \text{Res} \left( \frac{\tilde{f}(z)}{(z^2 + w^2)^{k+1/2}} \frac{1}{1 - e^{-2\pi i z}} \right) \]

\[ + 2(-1)^k \int_w^{\infty} \frac{\Delta \tilde{f}(x) \frac{dx}{(x^2 + w^2)^{k+1/2}}}{\Delta h(x)} \]

where \( \Delta h(x) \) is a polynomial function of appropriate order not to break (2), and

\[ h(x) = \frac{\tilde{f}(x e^{i\pi/2})}{(x + w)^{k+1/2}} \frac{1}{e^{2\pi i x} - 1}. \]

Further generalizations of this formula for functions with two non-integrable singularities and/or their combination with other known forms of the APF is straightforward.

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