Hyperbolic Fourier coefficients of Poincaré series

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In memory of Marvin Knopp

Abstract

Poincaré in 1911 and Petersson in 1932 gave the now classical expression for the parabolic Fourier coefficients of holomorphic Poincaré series in terms of Bessel functions and Kloosterman sums. Later, in 1941, Petersson introduced hyperbolic and elliptic Fourier expansions of modular forms and the associated hyperbolic and elliptic Poincaré series. In this paper we express the hyperbolic Fourier coefficients of Poincaré series, of both parabolic and hyperbolic type, in terms of hypergeometric series and Good’s generalized Kloosterman sums. In an explicit example for the modular group, we see that the hyperbolic Kloosterman sum corresponds to a sum over lattice points on a hyperbola contained in an ellipse. This allows for numerical computation of the hyperbolic Fourier coefficients.

1 Introduction

The group $\text{SL}_2(\mathbb{R})$ acts by linear fractional transformations on $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ with $\mathbb{H}$ denoting the upper half plane. Let $\Gamma \subset \text{SL}_2(\mathbb{R})$ be a Fuchsian group of the first kind, i.e. a discrete subgroup of $\text{SL}_2(\mathbb{R})$ so that $\Gamma \backslash \mathbb{H}$ has finite hyperbolic volume. Write $Z := \{\pm I\} \cap \Gamma$ for $I$ the identity matrix. Elements in $\Gamma \backslash \Gamma$ may be classified as parabolic, elliptic or hyperbolic according to their types of fixed points. A function $f$ on $\mathbb{H}$ transforms with weight $k$ with respect to $\Gamma$ if $(f|_k \gamma)(z) = f(z)$ for all $\gamma \in \Gamma$, where $(f|_k \gamma)(z)$ indicates $j(\gamma, z) - k f(\gamma z)$ for $j(\gamma, z) := cz + d$ when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Unless stated otherwise, we assume throughout that $k$ is even and at least 4.

The usual way to describe such an $f$ is in terms of its Fourier expansion. For example, the modular discriminant function is of weight 12 for $\Gamma = \text{SL}_2(\mathbb{Z})$, see Section 7.1, and its expansion begins

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots \quad (q = e^{2\pi iz}).$$

To describe generalizations of this Fourier expansion, we first review some basic notation and results for modular forms as described in [Shi71], [Ran77] and [Iwa97], for example.

The series (1.1) is the Fourier expansion corresponding to the cusp (parabolic fixed point) at $\infty$. In general, for a cusp $\alpha$ for $\Gamma$, let $\Gamma_\alpha$ be the subgroup fixing $\alpha$. Then $\Gamma_\alpha$ is isomorphic to $\mathbb{Z}$, where the bar means the image under the map $\text{SL}_2(\mathbb{R}) \to \text{SL}_2(\mathbb{R})/\pm I$. This isomorphism can be seen explicitly as there exists a scaling matrix $\sigma_\alpha \in \text{SL}_2(\mathbb{R})$ such that $\sigma_\alpha \infty = \alpha$ and

$$\sigma_\alpha^{-1} \Gamma_\alpha \sigma_\alpha = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\}.$$

The matrix $\sigma_\alpha$ is unique up to multiplication on the right by $\pm \left( \begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix} \right)$ for any $t \in \mathbb{R}$.

**Definition 1.** Let $f$ be holomorphic on $\mathbb{H}$ and of weight $k$ with respect to $\Gamma$. Its Fourier expansion at $\alpha$ is

$$(f|_k \sigma_\alpha)(z) = \sum_{m \in \mathbb{Z}} c_\alpha(m; f)e^{2\pi imz}. \quad (1.2)$$

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Definition 2. Let $S_k(\Gamma)$ be the set of holomorphic functions on $\mathbb{H}$, of weight $k$ with respect to $\Gamma$, such that $y^{k/2}f(x+iy)$ is bounded for all $x+iy \in \mathbb{H}$.

If $\Gamma$ has cusps then $S_k(\Gamma)$ consists of cusp forms $f$ whose coefficients $c_a(m; f)$ are zero at every cusp $a$ when $m \leq 0$, see for example [Iwa97 Sect. 5.1]. Relaxing this condition to allow $c_a(0; f)$ to be non-zero gives the set $M_k(\Gamma)$ of modular forms, and allowing a finite number of $c_a(-m; f)$ to be non-zero for $-m < 0$ gives the set $M^0_k(\Gamma)$ of weakly holomorphic forms.

If $\Gamma$ has no cusps then $\Gamma \backslash \mathbb{H}$ is compact and $S_k(\Gamma)$ is the set of all holomorphic functions on $\mathbb{H}$ with weight $k$, since the condition that $y^{k/2}f(x+iy)$ is bounded is automatically satisfied. Whether $\Gamma$ has cusps or not, $S_k(\Gamma)$ is a finite dimensional vector space over $\mathbb{C}$, equipped with the Petersson inner product given by $(f,g) := \int_{\Gamma \backslash \mathbb{H}} y^k f(z) g(z) \, d\mu_z$ where $d\mu_z := y^{-2} \, dx \, dy$.

Another result of Petersson [Pet41] is that alongside the parabolic expansions (1.2) there are also elliptic Fourier expansions associated to each point in $\mathbb{H}$ and hyperbolic Fourier expansions associated to each pair of hyperbolic fixed points in $\mathbb{R} \cup \{\infty\}$. For example, the elliptic expansion of $\Delta$ at $i \in \mathbb{H}$ is given in [OR13] as

$$\Delta_{12}\sigma_1(z) = -64\Delta(i) \left( 1 - 12\frac{(r_iz)^2}{2!} + 216\frac{(r_iz)^4}{4!} + 10368\frac{(r_iz)^6}{6!} + \ldots \right)$$

where $r_i = -\Gamma(1/4)^4/(8\sqrt{3}\pi^2)$ and $\sigma_1 := \frac{1+\sqrt{3}}{2} \left( \frac{i}{i-1} \right)$.

In this paper we develop the theory of hyperbolic expansions of modular forms, with the aim of expressing the hyperbolic coefficients as explicitly as possible. For example, we show that the expansion of $\Delta$ at the hyperbolic pair $\eta = (\sqrt{2}, -\sqrt{2})$ is given numerically by

$$\frac{\Delta_{12}\sigma_2(\eta)}{1721.23z^{-6}} \approx \cdots -3.47 \times 10^{-7} q^{-4} + 1.20 \times 10^{-7} q^{-3} + 0.00176 q^{-2} - 0.0937 q^{-1} + 1 + 25.31 q^1 + 128.12 q^2 - 3.73 q^3 - 1849.07 q^4 + \cdots \quad (q = z^{2\pi i/\ell_\eta})$$

for the scaling matrix $\sigma_2$ given in (5.1) and $\ell_\eta = 2\log(3+2\sqrt{2})$. (We divided by 1721.23 to make the zeroth coefficient $\approx 1$ and the other coefficients more visible.)

Some examples of hyperbolic expansions have already appeared in the literature. Siegel in [Sie65 Chap. II, Sect. 3] worked out the hyperbolic expansions of parabolic non-holomorphic Eisenstein series in terms of Hecke grossencharakter $L$-functions. In [VPI0 Prop. 4.2.2], von Pippich computed the hyperbolic Fourier coefficients of non-holomorphic Eisenstein series of elliptic type. Legendre functions (examples of $2F_1$ hypergeometric functions) appear in these coefficients. Good, in the book [Goo83], found the hyperbolic expansions of certain non-holomorphic Poincaré series. We will use much of the theory he developed, and expand some of his results that appear there in condensed form. Hiramatsu in [Hir70] worked in the holomorphic setting. He gave the hyperbolic expansion of an $f \in S_k(\Gamma(p, q))$ derived from a Hilbert modular form associated to a real quadratic field. The group $\Gamma(p, q)$ is coming from a quaternion algebra and has no cusps. In [Hir72] he also found basic bounds on the size of hyperbolic coefficients for elements of $S_k(\Gamma)$, as we see in Section 7.2.

1.1 Hyperbolic definitions

For most of the definitions and results in this subsection, see [Kat92], [Pet41], [Hir70] and [IO09]. Let $\eta = (\eta_1, \eta_2)$ be an ordered hyperbolic fixed pair for $\Gamma$, i.e. $\eta_1, \eta_2$ are distinct elements of $\mathbb{R} \cup \{\infty\}$ so that there exists a hyperbolic $\gamma \in \Gamma$ with $\gamma \eta_1 = \eta_1$ and $\gamma \eta_2 = \eta_2$. Let $\Gamma_\eta$ be the subgroup of all such $\gamma$ fixing $\eta_1$ and $\eta_2$. There exists a scaling matrix $\sigma_\eta \in \text{SL}_2(\mathbb{R})$ such that $\sigma_\eta 0 = \eta_1$, $\sigma_\eta \infty = \eta_2$ and $\sigma_\eta$ is unique up to multiplication on the right by $\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$ for any $t \in \mathbb{R} \neq 0$. That $\Gamma_\eta$ is isomorphic to $\mathbb{Z}$ may be seen with

$$\sigma_\eta^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1/t \end{pmatrix} \sigma_\eta = \left\{ \begin{pmatrix} e^{m\ell_\eta/2} & 0 \\ 0 & e^{-m\ell_\eta/2} \end{pmatrix} \right\} \quad m \in \mathbb{Z} \right\}.$$

The number $\ell_\eta$ is the hyperbolic length of the geodesic from $z$ to $\gamma_\eta z$ for any $z \in \mathbb{H}$ where $\gamma_\eta$ is a generator of $\Gamma_\eta$. We also set

$$\varepsilon_\eta := e^{\ell_\eta/2} > 1.$$
If $f$ has weight $k$ then $e^{k\ell\eta w/2} (f|_{k\sigma_\eta}) \left( e^{\ell\eta w} \right)$ has period 1 in $w$ and a Fourier expansion. Rewrite this expansion with $z = e^{\ell\eta w}$ to get the following (here and throughout $z^s$ for $z, s \in \mathbb{C}$ means $e^{s \log z}$ with the principal branch of $\log$).

**Definition 3.** Let $f$ be holomorphic on $\mathbb{H}$ and of weight $k$. Its hyperbolic Fourier expansion at $\eta$ is

\[ (f|_{k\sigma_\eta})(z) = \sum_{m\in \mathbb{Z}} c_\eta(m; f) z^{-k/2+2\pi im/\ell_\eta}, \]  

valid for all $z \in \mathbb{H}$. The coefficients $c_\eta(m; f)$ depend on $\sigma_\eta$ in a simple way:

\[ \sigma_\eta \rightarrow \sigma_\eta \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \quad \implies \quad c_\eta(m; f) \rightarrow c_\eta(m; f) \cdot (t^2)^{2\pi im/\ell_\eta}. \]  

Also note that the expansions at $\eta$ and $\gamma \eta$ for $\gamma \in \Gamma$ might differ by this type of $(t^2)^{2\pi im/\ell_\eta}$ factor unless $\sigma_{\gamma \eta}$ is chosen as $\gamma \sigma_\eta$. For example, with $-\infty < \eta_1 < \eta_2 < \infty$, a simple choice for the scaling matrix is

\[ \tilde{\sigma}_\eta := \frac{1}{\sqrt{\eta_2 - \eta_1}} \begin{pmatrix} \eta_2 & \eta_1 \\ 1 & 1 \end{pmatrix}. \]  

With

\[ c_\eta(m; f) = \int_{w_0}^{w_0+1} e^{k\ell\eta w/2} (f|_{k\sigma_\eta}) \left( e^{\ell\eta w} \right) \cdot e^{-2\pi im w} dw \]

we may recover the hyperbolic coefficients for any $w_0$ satisfying $0 < \text{Im}(w_0) < \pi/\ell_\eta$. Writing this as

\[ c_\eta(m; f) = \int_0^1 (f|_{k\sigma_\eta}) \left( e^{\ell\eta(w_0+t)} \right) \cdot e^{(w_0+t)(k\ell\eta/2-2\pi im)} dt \]

and using the change of variables $r_0 e^{i\theta_0} = e^{\ell\eta w_0}$, $r = r_0 e^{i\theta_0 t}$ then gives (with $\varepsilon_\eta^2 = e^{\ell\eta}$)

\[ c_\eta(m; f) = \frac{e^{i\theta_0(k/2-2\pi im/\ell_\eta)}}{\ell_\eta} \int_{r_0}^{e^{2\pi i r_0}} (f|_{k\sigma_\eta}) \left( e^{i\theta_0} \right) \cdot r^{k/2-2\pi im/\ell_\eta} dr \frac{dr}{r} \]  

valid for arbitrary $r_0 > 0$ and $0 < \theta_0 < \pi$.

**Definition 4.** The (weight $k$) hyperbolic Poincaré series $P_{\eta,m}$ is defined for $m \in \mathbb{Z}$ as

\[ P_{\eta,m}(z) := \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} z^{-k/2+2\pi im/\ell_\eta} |k\sigma_\eta^{-1}\gamma| = \sum_{\gamma \in \Gamma_\eta \backslash \Gamma} \frac{(\sigma_\eta^{-1}\gamma z)^{-k/2+2\pi im/\ell_\eta}}{j(\sigma_\eta^{-1}\gamma, z)^k}. \]  

The convergence is absolute for $k > 2$ and uniform for $z$ in compact sets in $\mathbb{H}$. We have $P_{\eta,m} \in S_k(\Gamma)$ for $m \in \mathbb{Z}$. For $f \in S_k(\Gamma)$ and $m \in \mathbb{Z}$ the Petersson inner product of $f$ with $P_{\eta,m}$ yields

\[ \langle f, P_{\eta,m} \rangle = c_\eta(m; f) \left[ \frac{\pi \Gamma(k-1) \ell_\eta e^{-2\pi m/\ell_\eta}}{2^{k-2} |\Gamma(k/2 + 2\pi im/\ell_\eta)|^2} \right]. \]  

It follows from (1.11) that, for fixed $\eta$ and $m \in \mathbb{Z}$, the series $P_{\eta,m}$ span the space $S_k(\Gamma)$.

These hyperbolic Poincaré series, at least in the case $m = 0$, have appeared for example in the works of Kohnen and Zagier [KZ84] and Katok [Kat83], obtaining hyperbolic rational structures on $S_k(\Gamma)$. See the related discussion in [IO09, Sect. 3]. In [BKK] they discover an interesting generalization of $P_{\eta,0}$ to a locally harmonic hyperbolic Poincaré series of negative weight.

One advantage of the expansion (1.6) and the series (1.10) is that they are always available since $\Gamma$ always has hyperbolic elements and hyperbolic fixed points. If $\Gamma$ has no cusps then there are no expansions of the form (1.2). The more familiar parabolic Poincaré series, defined next, also requires a cusp for its construction.
Definition 5. For \( m \in \mathbb{Z} \), the Poincaré series \( P_{a,m} \) associated to the cusp \( a \) is defined as

\[
P_{a,m}(z) := \sum_{\gamma \in \Gamma \backslash \Gamma} e^{2\pi i mz} |k\sigma_a^{-1}\gamma| = \sum_{\gamma \in \Gamma \backslash \Gamma} \frac{e^{2\pi im(\sigma_a^{-1}\gamma z)}}{|j(\sigma_a^{-1}\gamma, z)|^k}.
\] (1.12)

This series converges absolutely for \( k > 2 \) with the convergence uniform for \( z \) in compact sets in \( \mathbb{H} \). We have \( P_{a,m} \in S_k(\Gamma) \) for \( m \geq 1 \), \( P_{a,0} \in M_k(\Gamma) \) and \( P_{a,m} \in M_k^!(\Gamma) \) if \( m \leq -1 \). For \( f \in S_k(\Gamma) \) and \( m \in \mathbb{Z}_{\geq 1} \)

\[
\langle f, P_{a,m} \rangle = c_a(m; f) \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}
\] (1.13)

and the series \( P_{a,m} \) for fixed \( a \) and \( m \in \mathbb{Z}_{\geq 1} \) span \( S_k(\Gamma) \).

1.2 Main results

In this paper we calculate the parabolic and hyperbolic Fourier expansions of the parabolic and hyperbolic Poincaré series. The parabolic Fourier expansion of \( P_{a,m} \) for \( m \in \mathbb{Z} \) was first found by Poincaré himself in \([\text{Po}11]\) for \( \text{SL}_2(\mathbb{Z}) \), see the discussions in \([\text{K}07, \text{Kow}10]\). This was generalized by Petersson in \([\text{Pet}30, \text{Pet}32]\) to general groups. The coefficients are expressed as series involving Kloosterman sums, denoted \( S_{ab}(m,n;C) \), multiplied by Bessel functions. To establish the first instance of the pattern we will see in the other cases, we rewrite the coefficients in terms of the \( \varphi(1) \) hypergeometric function. Doing this has the added bonus of making the statement very concise, independent of the signs of \( m \) and \( n \). Recall that the general hypergeometric function is given by

\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!},
\] (1.14)

where \((a)_n := a(a+1) \cdots (a+n-1)\) and \(b_1 \not\in \mathbb{Z}_{\leq 0}\). The series \((1.14)\) is absolutely convergent for all \( x \in \mathbb{C} \) if \( p \leq q \), and absolutely convergent for all \( |x| < 1 \) if \( p = q + 1 \). See \([\text{AAR}99, \text{Chap. 2}]\).

Theorem 1.1 (Poincaré, Petersson). For \( m, n \in \mathbb{Z} \), the \( n \)th coefficient in the parabolic Fourier expansion at \( b \) of the parabolic Poincaré series \( P_{a,m} \) is given by

\[
c_b(n; P_{a,m}) = \begin{cases} \frac{(2\pi i)^k n^{k-1}}{\Gamma(k)} \sum_{C \in \mathcal{C}_{ab}} pFq(\frac{a}{c}, \ldots, \frac{b}{c}; \frac{k}{C^2}) \frac{S_{ab}(m,n;C)}{C^k} & \text{if } n \geq 1 \\ + 1 & \text{if } m = n \text{ and } a \equiv b \mod \Gamma \end{cases}
\] (1.15)

where we understand 0 when a condition is not met. Here, if \( a \) and \( b \) are \( \Gamma \)-equivalent we choose \( \sigma_b = \gamma \sigma_a \) for some \( \gamma \in \Gamma \) with \( b = \gamma a \).

See Section 3 for all the details. Petersson worked more generally with real weight \( k \) and an associated multiplier system.

To describe the parabolic Fourier series \( P_{a,m} \) we need the following notation. Put \( C_{\eta a} := \{ac \mid (a,b,c,d) \in \sigma_a^{-1} \Gamma \sigma_a \} \). We will see later that \( 0 \not\in C_{\eta a} \). For \( C \in C_{\eta a} \) and \( e(z) := e^{2\pi iz} \) define

\[
S_{\eta a}(m,n;C) := \sum_{\gamma \in \Gamma \backslash \Gamma/\Gamma_a} \cdot e\left(\frac{m}{l}(\log\left|\frac{a}{c}\right| + n\left(\frac{b}{2a} + \frac{d}{2c}\right))\right).
\] (1.16)

This generalized Kloosterman sum was first identified and studied by Good in \([\text{Goo}83]\). Renormalizing \((1.16)\) by multiplying it by \( \exp(\pi^2 m(\text{sgn}(C) - 1)/l(\eta - \pi in/C)) \) gives the variant

\[
S_{\eta a}^*(m,n;C) := \sum_{\gamma \in \Gamma \backslash \Gamma/\Gamma_a} \cdot e\left(\frac{m}{l}(\log\left|\frac{a}{c}\right| + n\frac{b}{a})\right)
\] (1.17)

where the logarithm takes its principal value. The next theorem is proved in Section 4.
Theorem 1.2. For \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}_{>1} \), the \( n \)th coefficient in the parabolic Fourier expansion at \( \alpha \) of the hyperbolic Poincaré series \( P_{\eta,m} \) has the formula

\[
c_{\alpha}(n; P_{\eta,m}) = \frac{(2\pi i)^k n^{k-1}}{\Gamma(k)} \sum_{C \in C_{\eta_a}} 1F1 \left( \frac{k}{2} + \frac{2\pi in}{\ell_\eta}; \frac{2\pi in}{C} \right) S^*_{\eta_a}(m, n; C). \tag{1.18}
\]

In the case that \( \Gamma = \text{SL}_2(\mathbb{Z}) \), \( a = \infty \) and \( \eta = (-\sqrt{D}, \sqrt{D}) \) for \( D \) a positive integer that is not a perfect square, we can give a very explicit expression for \( S_{\eta_a}(m, n; C) \). First, choose \( \sigma_\eta = \hat{\sigma}_C \) and \( \sigma_\infty = I \) so that \( C_{\eta,\infty} \subset \mathbb{Z}/(2\sqrt{D}) \). Let \( (a_0, c_0) = (a, c) \) be the minimal positive integer solution to Pell’s equation

\[
a^2 - Dc^2 = 1. \tag{1.19}
\]

Such a solution always exists and may be found from the continued fraction expansion of \( \sqrt{D} \). Set \( \varepsilon_D := a_0 + \sqrt{D}c_0 \), \( \ell_\eta := 2\log \varepsilon_D \) and write

\[
\frac{a_0 + 1}{c_0} = \frac{u_+}{v_+}, \quad \frac{a_0 - 1}{c_0} = \frac{u_-}{v_-}
\]

in lowest terms. Also set \( D_+ := u_+^2 - Dv_+^2 \), \( D_- := u_-^2 - Dv_-^2 \) and we will see later that \( D_+ > 0 \) and \( D_- < 0 \). Define

\[
\psi_D(m, n; N) := \begin{cases} (-1)^{m+c_0+n} & \text{if } N = D_+ \text{ or } D_- \\ 0 & \text{otherwise} \end{cases}
\]

and put

\[
R_D^*(N) := \left\{ (e, g) \in \mathbb{Z}^2 \mid \gcd(e, g) = 1, e^2 - Dg^2 = N, e^2 + Dg^2 \leq a_0|N| \right\}. \tag{1.20}
\]

See Figure 1 for an example of \( R_D^*(N) \).

Theorem 1.3. Let \( \infty \) be the cusp and \( \eta = (-\sqrt{D}, \sqrt{D}) \) a hyperbolic fixed pair for \( \text{SL}_2(\mathbb{Z}) \) with scaling matrices \( I \) and \( \hat{\sigma}_\eta \) respectively. Then for all \( m, n \in \mathbb{Z} \)

\[
S_{\eta,\infty}(m, n; \frac{-N}{2\sqrt{D}}) = -\psi_D(m, n; N) + \frac{1}{2} \sum_{(e, g) \in R_D^*(N)} \text{e} \left( \frac{m}{\ell_\eta} \log \left| \frac{e + g\sqrt{D}}{e - g\sqrt{D}} \right| - \frac{\text{neg}^{-1}}{N} \right) \tag{1.21}
\]

where \( g^{-1} \) denotes the inverse of \( g \mod N \). If \( g = 0 \) then \( N = 1 \) and we may set \( g^{-1} = 0 \).

In Theorems 6.4 and 6.5 of Section 6 we also give the hyperbolic expansion of \( P_{a,m} \), which is similar to Theorem 1.2. Section 7 explores our formulas numerically with the result (1.4) calculated there. Finally, in Section 8 the hyperbolic expansion of \( P_{\eta,m} \) is obtained. For this, first put

\[
C_{\eta,\eta} := \left\{ ad \mid (a b \ c d) \in \sigma_\eta^{-1} \Gamma \sigma_\eta', \ abcd \neq 0 \right\}. \tag{1.22}
\]
When $C \in C_{\eta\eta'}$ and $\alpha = \pm 1$ define

$$S^{\ast}_{\eta\eta'}(m, n; C, \alpha) := \sum_{\gamma \in \Gamma_C \setminus \Gamma / \Gamma'_{\eta\eta'}} \left( \frac{a}{c} \right) \log \left( \frac{a}{c} \right) + \frac{n}{\ell_{\eta'}} \log \left( -\frac{c}{d} \right). \quad (1.23)$$

For $\eta = (\eta_1, \eta_2)$, let $\eta^*$ be the reversed pair $(\eta_2, \eta_1)$. It is easy to see that if $\sigma_\eta$ is a scaling matrix for $\eta$, then $\sigma_\eta S$ is a possible scaling matrix for $\eta^*$ where $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Also we recall the beta function $B(u, v) := \Gamma(u) \Gamma(v)/\Gamma(u + v)$.

**Theorem 1.4.** For any $m, n \in \mathbb{Z}$, the $n$th coefficient in the hyperbolic expansion at $\eta'$ of the hyperbolic Poincaré series $P_{\eta, m}$ is given by

$$c_{\eta'}(n; P_{\eta, m}) = \frac{e^{2\pi in/\ell_{\eta'}}}{\ell_{\eta'}} B \left( \frac{k}{2} - \frac{2\pi in}{\ell_{\eta'}}, \frac{k}{2} + \frac{2\pi in}{\ell_{\eta'}} \right) \left( \sum_{1} + \sum_{2} + \sum_{3} \right) \quad (1.24)$$

where

$$\sum_{1} = \sum_{C \in C_{\eta\eta'}, \epsilon(0, 1)} 2F_{1} \left( \frac{k}{2} - \frac{2\pi im}{\ell_{\eta'}}, \frac{k}{2} + \frac{2\pi in}{\ell_{\eta'}} ; 1/\Gamma \right) S^{\ast}_{\eta\eta'}(m, n; C, 1) + S^{\ast}_{\eta\eta'}(m, n; C, -1) / C^{k/2},$$

$$\sum_{2} = \sum_{C \in C_{\eta\eta'} \cap (0, 1)} 2F_{1} \left( \frac{k}{2} - \frac{2\pi im}{\ell_{\eta'}}, \frac{k}{2} + \frac{2\pi in}{\ell_{\eta'}} ; 1/\Gamma \right) S^{\ast}_{\eta\eta'}(m, n; C, 1) / C^{k/2},$$

$$\sum_{3} = \sum_{C \in C_{\eta\eta'} \cap (0, 1)} \left( \frac{C}{C - 1} \right)^{2\pi in/\ell_{\eta'}} 2F_{1} \left( \frac{k}{2} + \frac{2\pi im}{\ell_{\eta'}}, \frac{k}{2} + \frac{2\pi in}{\ell_{\eta'}} ; k/\Gamma - 1 \right) S^{\ast}_{\eta\eta'}(m, n; C, -1) / (C - 1)^{k/2}.$$

The sums $\sum_{2}$ and $\sum_{3}$ are finite. The real numbers $a$ and $b$ in (1.25), (1.26) depend on the choice of scaling matrices $\sigma_\eta$ and $\sigma_{\eta'}$.

**Remark 1.5.** With specific choices of $\sigma_\eta$ and $\sigma_{\eta'}$ we can make $a$ and $b$ in (1.25), (1.26) explicit. For example, suppose $\eta$ and $\eta^*$ are not $\Gamma$-equivalent. If $\eta' = \rho \eta$ for some $\rho \in \Gamma'$ put $\sigma_{\eta'} = \rho \sigma_\eta$ and if $\eta' = \rho \eta^*$ for some $\rho \in \Gamma$ put $\sigma_{\eta'} = \rho \sigma_{\eta^*}$. Then (1.25), (1.26) become

$$+ \left\{ \begin{array}{l} 1 \text{ if } \eta' \equiv \eta \mod \Gamma \text{ and } n = m \\
\left( -1 \right)^{k/2} e^{2\pi in/\ell_{\eta'}} \text{ if } \eta' \equiv \eta^* \mod \Gamma \text{ and } n = -m \end{array} \right.$$
2 Good's generalized Kloosterman sums

The Kloosterman sums that arise in all the cases we need are covered by Good's theory as described in [Goo83]. Following his notation, let $\xi$ and $\chi$ each denote either a cusp such as $a$ or a hyperbolic fixed pair $\eta$. If the object we are defining is independent of the particular cusp or hyperbolic fixed pair we sometimes write $\text{par}$ or $\text{hyp}$, respectively, instead. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \SL_2(\mathbb{R})$ define the functions $\xi \Lambda_{\chi}(M)$ as follows:

$$
\begin{align*}
\text{par} \Lambda_{\text{par}}(M) &:= \frac{a}{c} \\
\text{hyp} \Lambda_{\text{par}}(M) &:= \log \left| \frac{a}{c} \right| \\
\text{par} \Lambda_{\text{hyp}}(M) &:= \frac{a}{2c} + \frac{b}{2d} \\
\text{hyp} \Lambda_{\text{hyp}}(M) &:= \frac{1}{2} \log \left| \frac{ab}{cd} \right|.
\end{align*}
$$

Good parameterized his sums with $\xi \nu_{\chi}(M)$, $\xi \delta_{\chi}(M)$ and $\xi \delta'_{\chi}(M)$, defined as

$$
\begin{align*}
\text{par} \nu_{\text{par}}(M) &:= |c| \\
\text{hyp} \nu_{\text{par}}(M) &:= |2ac|^{1/2} \\
\text{par} \delta_{\text{par}}(M) &:= 0 \\
\text{hyp} \delta_{\text{par}}(M) &:= \frac{1 - \text{sgn}(ac)}{2} \\
\text{par} \delta'_{\text{par}}(M) &:= 0 \\
\text{hyp} \delta'_{\text{par}}(M) &:= \frac{1 + \text{sgn}(cd)}{2} \\
\text{par} \nu_{\text{hyp}}(M) &:= |2cd|^{1/2} \\
\text{hyp} \nu_{\text{hyp}}(M) &:= |ad|^{1/2} + |bc|^{1/2} \\
\text{par} \delta_{\text{hyp}}(M) &:= 0 \\
\text{hyp} \delta_{\text{hyp}}(M) &:= \frac{1 - \text{sgn}(ac)}{2} \\
\text{par} \delta'_{\text{hyp}}(M) &:= 0 \\
\text{hyp} \delta'_{\text{hyp}}(M) &:= \frac{1 + \text{sgn}(cd)}{2}.
\end{align*}
$$

The functions $\xi \Lambda_{\chi}(M)$ and $\xi \nu_{\chi}(M)$ are derived from the geometry of the fixed points of $\SL_2(\mathbb{R})$ in $\mathbb{H}$ and double coset decompositions of $\SL_2(\mathbb{R})$, see [Goo83 Sect. 3]. The Iwasawa and Bruhat decompositions are generalized in Lemma 1 of [Goo83]. The four cases of this Lemma we need are given explicitly in our Lemmas 3.3, 4.5, 6.3 and 8.9.

Let $\ell_{\gamma}$ be as in (1.5) and put $\ell_{a} := 1$ for any cusp $a$. For $\delta, \delta' \in \{0, 1\}$ define the generalized Kloosterman sum, [Goo83 Eq. (5.10)], as

$$
\delta S^\delta_{\chi}(m, n; \nu) := \sum_{\gamma \in \Gamma \xi \backslash \Gamma / \Gamma_{\chi}} e\left(\frac{m}{\ell_{\xi}} \xi \Lambda_{\chi}(M) - \frac{n}{\ell_{\xi}} \chi \Lambda_{\chi}(M^{-1})\right) \quad (2.1)
$$

where the sum is restricted to $M$ such that

$$
\xi \nu_{\chi}(M) = \nu, \quad \xi \delta_{\chi}(M) = \delta, \quad \xi \delta'_{\chi}(M) = \delta'.
$$

The usual Kloosterman sum corresponds to the parabolic/parabolic combination $0 \sigma^0_{a,b}(m, n; \nu)$ in (2.1), see Sections 3.3 and 7.1. We use the three other families of sums with parabolic and hyperbolic combinations in our Fourier expansions in Sections 4, 6 and 8. Including the elliptic case gives five further combinations which Good also fit into the formalism (2.1).

In [Goo83] these generalized Kloosterman sums are required for the Fourier expansions of the non-holomorphic Poincaré series

$$
P_{\xi}(z, s, m) := \sum_{\gamma \in \Gamma \xi \backslash \Gamma} V_{\xi}(\sigma_{\xi}^{-1} \gamma z, s, m/\ell_{\xi})
$$

for $z \in \mathbb{H}$ and $\text{Re}(s) > 0$ where

$$
\begin{align*}
V_{\text{par}}(z, s, \lambda) &:= \frac{1}{i} \int_{-\infty}^{\infty} e(-\lambda \rho) \left(\frac{y}{(\rho + z)(\rho + \bar{z})}\right)^{1-s} d\rho, \\
V_{\text{hyp}}(z, s, \lambda) &:= \frac{1}{i} \int_{-\infty}^{\infty} e(-\lambda \rho) \left(\frac{2ye^\rho}{(ze^\rho - 1)(\bar{z}e^\rho - 1)}\right)^{1-s} d\rho \quad (\text{Re}(z) > 0).
\end{align*}
$$
These series are constructed to be eigenfunctions of the Laplacian:

\[ \Delta P_{\xi}(z, s, m) = -s(1 - s)P_{\xi}(z, s, m) \quad \text{for} \quad \Delta := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad z = x + iy. \]

See [Goo83, Sect. 7] for the details.

## 3 Parabolic Poincaré series and their parabolic Fourier expansions

Let \( \alpha \) and \( \beta \) be two cusps for \( \Gamma \) and let \( m \) and \( n \) be any two integers. In this section we give a detailed review of the computation of the coefficients \( c_b(n; P_{\alpha,m}) \) in the parabolic Fourier expansion of \( P_{\alpha,m} \) at \( \beta \):

\[ (P_{\alpha,m}|\kappa\sigma_\beta)(z) = \sum_{n \in \mathbb{Z}} c_b(n; P_{\alpha,m})e^{2\pi inz}. \]

See for example [Iwa97, Chap. 2, 3] and [Ran77, Chap. 5] for similar treatments. Sections 4, 6 and 8 will extend these calculations to the cases when \( \alpha \) or \( \beta \) equals \( \eta \) or \( \eta' \). We also remark that in [dAP07] the Fourier expansion is computed for a very general kind of parabolic Poincaré series with complex ‘weight’ and separate multiplier system.

### 3.1 An integral for the parabolic/parabolic case

For \( m, n \in \mathbb{Z} \) and \( r \in \mathbb{R}_{\neq 0} \) define

\[ I_{\text{par par}}(m, n; r) := \int_{-\infty+iy}^{\infty+iy} e^{-\frac{m}{r^2u} - nu} u^{-k} \, du \quad (y > 0, \, k > 1). \]  

(3.1)

This is the integral we will need shortly, see (3.24) in the proof of Theorem 1.1, and we study it here first.

**Proposition 3.1.** The integral (3.1) is absolutely convergent. For an implied constant depending only on \( k > 1 \),

\[ I_{\text{par par}}(m, n; r) = 0 \quad (n \leq 0), \]  

(3.2)

\[ I_{\text{par par}}(m, n; r) \ll n^{(k-1)/2} \exp \left( 2\pi n^{1/2} \left( 1 + \frac{|m| - m}{r^2} \right) \right) \quad (n > 0), \]  

(3.3)

\[ I_{\text{par par}}(m, n; r) \ll n^{k-1} \quad (m, n > 0). \]  

(3.4)

**Proof.** Bounding the absolute value of the integrand in (3.1) when \( u = x + iy \) shows

\[ |I_{\text{par par}}(m, n; r)| \leq \exp \left( 2\pi n y + \frac{\pi (|m| - m)}{r^2 y} \right) \int_{-\infty}^{\infty} \frac{dx}{(x^2 + y^2)^{k/2}}. \]  

(3.5)

Clearly, the right side of (3.5) converges for \( k > 1 \). Since the integrand is holomorphic, (3.1) is independent of \( y > 0 \). Letting \( y \to \infty \) in (3.5) yields (3.2). A special case of [GR07, 3.251.11] implies

\[ \int_{-\infty}^{\infty} \frac{dx}{(x^2 + y^2)^s} = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} y^{1-2s} \quad (\text{Re}(s) > 1/2). \]  

(3.6)

Using (3.6) in (3.5) with \( y = 1/\sqrt{m} \) and \( y = 1/n \) proves (3.3) and (3.4) respectively.

Next we evaluate \( I_{\text{par par}}(m, n; r) \) in terms of the confluent hypergeometric function \( 0F_1(; b; z) \). Recall that for each \( b \not\in \mathbb{Z}_{\leq 0} \) it is a holomorphic function of \( z \in \mathbb{C} \).

**Proposition 3.2.** For all \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}_{\geq 1} \)

\[ I_{\text{par par}}(m, n; r) = \frac{(2\pi i)^k n^{k-1}}{\Gamma(k)} 0F_1 \left( \begin{array}{c} k \; 4\pi^2 mn \hfill \frac{r^2}{\pi} \end{array} \right). \]  

(3.7)
Proof. The formula (3.7) follows directly by a change of variables from

\[ 0 F_1 \left( ; b; z \right) = \frac{e^{\pi i b/2} \Gamma \left( b \right)}{2\pi b} \int_{-\infty+it}^{\infty+it} e^{-u + \frac{z}{4\pi^2 u}} u^{-b} du \quad (t > 0, \, \text{Re}(b) > 1). \quad (3.8) \]

We can establish (3.8) by linking it to the integral representation of the \( J \)-Bessel function in \([GR07, 8.412.2]\). Provided that \( \text{Re}(b) > 1 \), we may deform the contour of integration in \([GR07, 8.412.2]\) to a vertical line with positive real part. Multiplying the variable by \( i \) then produces

\[ J_{b-1}(2z) = \frac{z^{b-1}}{2\pi} e^{\pi i b/2} \int_{-\infty+it}^{\infty+it} e^{-u - \frac{z^2}{4\pi^2 u}} u^{-b} du \quad (y > 0). \quad (3.9) \]

See also \([Ran77, \text{p. 156}]\). The \( J \)-Bessel function may be expressed in terms of hypergeometric functions:

\[ J_{b-1}(2z) = \frac{1}{\Gamma(b)} z^{b-1} 0 F_1 \left( ; b; -z^2 \right) \quad (3.10) \]

\[ = \frac{1}{\Gamma(b)} z^{b-1} e^{-2iz} 1 F_1 \left( b - \frac{1}{2}, 2b - 1; 4iz \right) \quad (3.11) \]

as in \([AAR99, \text{p. 200}]\). Formulas (3.9) and (3.10) together prove (3.8).

\[ \square \]

### 3.2 Double cosets in the parabolic/parabolic case

Let \( L \) be a complete set of inequivalent representatives for \( \Gamma_a \backslash \Gamma / \Gamma_b \). Partition \( L \) into two sets:

\[ \Gamma(a, b)_0 := \{ \delta \in L \mid \delta b = a \}, \quad \Gamma(a, b) := \{ \delta \in L \mid \delta b \neq a \}. \]

It is easy to see that \( \Gamma(a, b)_0 \) has at most one element.

**Proposition 3.3.** With the above notation, a complete set of inequivalent representatives for \( \Gamma_a \backslash \Gamma \) is given by

\[ \Gamma(a, b)_0 \cup \{ \delta \tau \mid \delta \in \Gamma(a, b), \tau \in \Gamma_b / Z \}. \quad (3.12) \]

**Proof.** The set \( L' := \{ \delta \tau \mid \delta \in L, \tau \in \Gamma_b / Z \} \) clearly gives a complete set of representatives for \( \Gamma_a \backslash \Gamma_b \), but some of its elements may be equivalent modulo \( \Gamma_a \). Suppose

\[ \Gamma_a \delta \tau = \Gamma_a \delta' \tau' \quad \text{for} \quad \delta, \delta' \in L \quad \text{and} \quad \tau, \tau' \in \Gamma_b / Z. \quad (3.13) \]

We must have \( \delta' = \delta \) because \( L \) is defined as a set of inequivalent representatives. Hence there is a \( \gamma \in \Gamma_a \) so that \( \gamma \delta \tau = \delta \tau' \). It follows that \( \gamma \) fixes \( a \) and \( \delta b \) which can only happen if \( \gamma = \pm 1 \) or if \( \delta b = a \).

If \( \gamma = \pm 1 \) then \( \tau = \tau' \). If \( \delta b = a \) then \( \Gamma_b = \delta^{-1} \Gamma_a \delta \) and any \( \tau \in \Gamma_b \) may be written as \( \delta^{-1} \gamma \delta \) for \( \gamma \in \Gamma_a \). Therefore, for all \( \tau \in \Gamma_b \), \( \Gamma_a \delta \tau = \Gamma_a \delta \delta^{-1} \gamma \delta \delta \Gamma_a \delta \). We have shown that (3.11) implies \( \delta' = \delta \), and then \( \tau = \tau' \) or \( \delta b = a \) and \( \Gamma_a \delta \tau = \Gamma_a \delta \tau' = \Gamma_a \delta \). Hence, with (3.12), we have removed all of the equivalent elements from the set \( L' \) we started with. \( \square \)

We may also characterize the sets \( \Gamma(a, b)_0 \) and \( \Gamma(a, b) \) with

\[ \Gamma(a, b)_0 = \left\{ \delta \in L \mid \sigma_a^{-1} \delta \sigma_b = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c = 0 \right\}, \]

\[ \Gamma(a, b) = \left\{ \delta \in L \mid \sigma_a^{-1} \delta \sigma_b = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \neq 0 \right\} \]

since

\[ c = 0 \iff \sigma_a^{-1} \delta \sigma_b = \infty \iff \delta \sigma_b = \sigma_a \infty \iff \delta b = a. \]

To describe an example of \( \Gamma(a, b) \) more explicitly, we first recall the Bruhat decomposition in the form given by [Goo83, Lemma 1].
Lemma 3.4. For $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R})$ with $c \neq 0$,
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \text{sgn}(c) \left( \begin{array}{cc} 1 & a/c \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} \nu & 0 \\ 0 & 1/\nu \end{array} \right) \left( \begin{array}{cc} 1 & d/c \\ 0 & 1 \end{array} \right)
\]
for $\nu = \text{par}_\nu(\text{par}(M)) = |c|$.

We see that multiplying (3.14) on the left by $\left( \begin{array}{cc} 1 & f \\ 0 & 1 \end{array} \right)$ changes $a/c$ to $\ell + a/c$ and leaves $c$ and $d$ fixed. Similarly, multiplying on the right by $\left( \begin{array}{cc} 1 & f \\ 0 & 1 \end{array} \right)$ changes $d/c$ to $\ell + d/c$ and leaves $a$ and $c$ fixed. Define
\[
R_{ab} := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \sigma_a^{-1}\Gamma \sigma_b \mid c \neq 0, \ 0 \leq a/c < 1, \ 0 \leq d/c < 1 \right\}.
\]

Lemma 3.5. We may take $\sigma_a^{-1}\Gamma(a, b)\sigma_b = R_{ab}/Z$.

Proof. Let $B = \left\{ \left( \begin{array}{cc} 1 & f \\ 0 & 1 \end{array} \right) \mid \ell \in \mathbb{Z} \right\}$ and suppose that $-I \not\in \Gamma$. Then
\[
\sigma_a^{-1}(\Gamma_a \setminus \Gamma/\Gamma_b)\sigma_b = \sigma_a^{-1}\Gamma_a\sigma_a \setminus \sigma_a^{-1}\Gamma_b/\sigma_b = B \setminus \sigma_a^{-1}\Gamma_b/B.
\]
It follows that $R_{ab}$ gives a complete set of representatives for $\sigma_a^{-1}\Gamma(a, b)\sigma_b$. Suppose that two elements $(\begin{array}{cc} a & b \\ c & d \end{array})$, $(\begin{array}{cc} a' & b' \\ c' & d' \end{array})$ of $R_{ab}$ are equivalent, i.e. $(\begin{array}{cc} 1 & f \\ 0 & 1 \end{array})(\begin{array}{cc} a & b \\ c & d \end{array})(\begin{array}{cc} 1 & f' \\ 0 & 1 \end{array}) = (\begin{array}{cc} a' & b' \\ c' & d' \end{array})$. Then $c = c'$ and also $a = a'$, $d = d'$. This proves the lemma when $-I \not\in \Gamma$. If $-I \in \Gamma$ then $\sigma_a^{-1}\Gamma_a\sigma_a = \sigma_b^{-1}\Gamma_b\sigma_b = -B \cup B$. Hence $(\begin{array}{cc} a & b \\ c & d \end{array})$ and $(\begin{array}{cc} -a & -b \\ c & d \end{array})$ are now equivalent in $R_{ab}$. □

We also note that if $(\begin{array}{cc} a & b \\ c & d \end{array}) \in R_{ab}$ then $b$ and $d$ are uniquely determined. To see this, suppose $\gamma = (\begin{array}{cc} a & b \\ c & d \end{array})$ and $\gamma' = (\begin{array}{cc} a' & b' \\ c' & d' \end{array})$ are in $R_{ab}$. Then $(\begin{array}{cc} a & b \\ c & d \end{array})^{-1}(\begin{array}{cc} a' & b' \\ c' & d' \end{array}) = (\begin{array}{cc} c & b \\ a & d \end{array}) \in \sigma_b^{-1}\Gamma\sigma_b$ and we must have $\gamma = \gamma'$. Similarly, if $(\begin{array}{cc} c & b \\ a & d \end{array}) \in R_{ab}$ then $a$ and $b$ are uniquely determined.

3.3 Kloosterman sums

Put
\[
C_{ab} := \left\{ |c| : (\begin{array}{cc} a & b \\ c & d \end{array}) \in \sigma_a^{-1}\Gamma \sigma_b, \ c \neq 0 \right\}.
\]

We use $|c|$ instead of $c$ here because it is convenient that $(\begin{array}{cc} a & b \\ c & d \end{array})$ and $(\begin{array}{cc} -a & -b \\ c & d \end{array})$ if it is in $\sigma_a^{-1}\Gamma \sigma_b$ have the same representative. (We could also have used $c^2$, making the parameter a product of two matrix elements as we do in Sections 4 [5] and 8 but this goes against the conventional notation.) For $C \in C_{ab}$ the Kloosterman sum
\[
S_{ab}(m, n; C) := \sum_{\gamma \in \Gamma_a \setminus \Gamma/\Gamma_b} e(ma/n + nd/c)
\]
is well defined. Since $c \neq 0$ we could equivalently have summed over $\gamma \in \Gamma(a, b)$. In Good’s notation [2.1], we have $S_{ab}(m, n; C) = 0_{a, b}^{0, 0}(m, n; C)$. See also [Iwa97] Eq. (3.13), for example. Note that the sum $S_{ab}(m, n; C)$ depends on the choice of scaling matrices $\sigma_a$ and $\sigma_b$ in a simple way; we assume the choice is fixed for each cusp. Replacing $\gamma$ by $\gamma^{-1}$ in (3.15) shows
\[
S_{ab}(m, n; C) = S_{ba}(-n, -m; C) = S_{ba}(n, m; C).
\]

Now let $N_{ab}(C) := S_{ab}(0, 0; C)$ be the number of terms in the sum (3.15). Then $N_{ab}(C)$ is always finite and in fact, by [Iwa97] Prop. 2.8,
\[
\sum_{C \in C_{ab}, C \leq X} C^{-1}N_{ab}(C) \ll X.
\]

From (3.16) we deduce the bounds
\[
N_{ab}(C) \ll C^2, \quad S_{ab}(m, n; C) \ll C^2, \quad \# \{ C \in C_{ab} : C \leq X \} \ll X^2
\]
with implied constants depending only on $\Gamma$, $\sigma$ and $\sigma_b$. 

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3.4 The parabolic expansion of $P_{a,m}$

Proof of Theorem 1.1  With $z = x + iy \in \mathbb{H}$, use Proposition 3.3 to write

$$(P_{a,m}|_{\Gamma(b)})(z) = \sum_{\gamma \in \Gamma \setminus \Gamma} e^{2 \pi i m \gamma \Gamma(b)} = \sum_{\gamma \in \Gamma \setminus \Gamma} e^{2 \pi i m \gamma \Gamma(b)} = \sum_{\delta \in \Gamma(a,b)_0} e^{2 \pi i m \gamma \Gamma(b)} + \sum_{\gamma \in \Gamma(a,b)} e^{2 \pi i m \gamma \Gamma(b)}. \tag{3.20}$$

The first sum in (3.20) is just $e(mz)$ if $\Gamma(a,b)_0$ is non-empty, which happens exactly when $a$ and $b$ are $\Gamma$-equivalent. Write the second sum as

$$\sum_{C \in \mathcal{C}_{ab}} \sum_{\gamma \in \Gamma(a,b)} e^{2 \pi i m \gamma \Gamma(b)} = \sum_{\gamma \in \Gamma(a,b)} e^{2 \pi i m \gamma \Gamma(b)} \tag{3.21}$$

and the inner sum is

$$\sum_{n \in \mathbb{Z}} e^{2 \pi i m \gamma \Gamma(b)} \phi(z + n) \phi(z + n). \tag{3.22}$$

Since $\sigma_a^{-1} \gamma \sigma_b = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{R})$ with $c \neq 0$ we have

$$\sigma_a^{-1} \gamma \sigma_b z = \frac{a}{c} - \frac{1}{c( tz + d)} \tag{3.23}$$

and so (3.22) equals

$$\sum_{n \in \mathbb{Z}} f(n) \quad \text{for} \quad f(t) := e^{2 \pi i m \gamma \Gamma(b)} \frac{(z - \frac{1}{c})}{\phi(z + n)} \tag{3.24}$$

Poisson summation gives

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} f(n) \quad \text{for} \quad f(n) := \int_{-\infty}^{\infty} f(t) e(-nt) \, dt. \tag{3.25}$$

To check this is valid, we may use the convenient form of [Rad73, Thm. A, p. 71], which requires $f$ to be twice continuously differentiable on $\mathbb{R}$ and that $\int_{-\infty}^{\infty} f(t) \, dt$ and $\int_{-\infty}^{\infty} |f''(t)| \, dt$ exist. It is straightforward to check that for $k > 1$ our $f$ in (3.24) meets these conditions. Hence (3.25) holds.

Substituting $u = z + t + d/c$ and recalling (3.1) shows that (3.21) is now

$$\sum_{C \in \mathcal{C}_{ab}} \sum_{\gamma \in \Gamma(a,b)} e^{2 \pi i m \gamma \Gamma(b)} \phi(z + n) \phi(z + n) \tag{3.26}$$

Taking absolute values and using (3.2), (3.3), we find that (3.26) is majorized by

$$\sum_{C \in \mathcal{C}_{ab}} \sum_{\gamma \in \Gamma(a,b)} \sum_{n=1}^{\infty} e^{-2 \pi n \gamma \Gamma(b)} \frac{n^{(k-1)/2}}{C^k} \exp \left( 2 \pi n \frac{C}{C^2} \left( 1 + \frac{|m| - m}{C^2} \right) \right) \ll \sum_{C \in \mathcal{C}_{ab}} N_{ab}(C'). \tag{3.27}$$

With (3.16) and summation by parts, this last is convergent for $k > 2$. Changing the order of summation in (3.26) is now justified and formula (3.7) completes the proof.
4 Hyperbolic Poincaré series and their parabolic Fourier expansions

Let \( \alpha \) be a cusp and \( \eta \) a hyperbolic fixed point pair for \( \Gamma \). In this section we compute coefficients in the parabolic Fourier expansion of \( P_{\eta,m} \) at \( \alpha \):

\[
(P_{\eta,m}|_{\kappa}\sigma_\alpha)(z) = \sum_{n=1}^{\infty} c_n(n; P_{\eta,m})e^{2\pi inz}.
\]

4.1 The hyperbolic/parabolic integral

For \( m, n \in \mathbb{Z} \) and \( r \in \mathbb{R}_{\neq 0} \), the integral we will need is

\[
I_{\eta par}(m, n; r) := \int_{-\infty+iy}^{\infty+iy} \frac{(\text{sgn}(r)\frac{u-r}{u+r})^{2\pi im/\ell_\eta} e^{-2\pi inu}}{(u-r)^{k/2}(u+r)^{k/2}} du \quad (y > 0, \ k > 1). \tag{4.1}
\]

**Proposition 4.1.** The integral (4.1) is absolutely convergent. We have \( I_{\eta par}(m, n; r) = 0 \) for \( n \leq 0 \) and

\[
I_{\eta par}(m, n; r) \ll n^{k-1} \exp \left( \frac{\pi^2(|m| - m)/\ell_\eta}{2\pi ny} \right) \quad (n > 0)
\]

for an implied constant depending only on \( k > 2 \).

**Proof.** Notice that \( w := \text{sgn}(r)\frac{u-r}{u+r} \in \mathbb{H} \) and for \( y = \text{Im}(u) \) we have the bound

\[
\left| w^{2\pi im/\ell_\eta}e^{-2\pi inu} \right| \leq \exp \left( \frac{\pi^2(|m| - m)/\ell_\eta + 2\pi ny}{2\pi ny} \right).
\]

It follows that (4.1) is absolutely convergent for \( k > 1 \). If we assume \( k > 2 \), write \( u - r = x + iy \), note that \(|u + r|^{-k/2} \leq y^{-k/2}\) and recall (3.6), then

\[
|I_{\eta par}(m, n; r)| \leq \exp \left( \frac{\pi^2(|m| - m)/\ell_\eta + 2\pi ny}{2\pi ny} \right) y^{-k/2} \int_{\infty}^{-\infty} \frac{dx}{(x^2 + y^2)^{k/4}} \leq \exp \left( \frac{\pi^2(|m| - m)/\ell_\eta + 2\pi ny}{2\pi ny} \right) y^{1-k}. \tag{4.3}
\]

Since the integrand in (4.1) is holomorphic, \( I_{\eta par}(m, n; r) \) is independent of \( y > 0 \). Letting \( y \to \infty \) in (4.3) we see that \( I_{\eta par}(m, n; r) = 0 \) for \( n \leq 0 \). For \( n > 0 \) let \( y = 1/n \). \( \Box \)

Bounds for \( I_{\eta par}(m, n; r) \) when \( k \in (1, 2] \) are of course possible. The advantage of (4.2) for \( k > 2 \) is that it does not depend on \( r \), see (4.2).

We assume from here that \( k \geq 4 \) is even. If \( m = 0 \) we can evaluate \( I_{\eta par}(0, n; r) \) for \( n > 0 \) by moving the line of integration down past the poles of order \( k/2 \) at \( u = \pm r \) and letting \( y \to -\infty \). Evaluating the residues we find for \( n > 0 \),

\[
I_{\eta par}(0, n; r) = 2\pi i(-1)^{k/2} \sum_{j=0}^{k/2-1} \frac{(2\pi in)^j}{j!} \left( \frac{k-2-j}{k/2-1} \right) \left( e^{-2\pi inr} \frac{e^{2\pi inr}}{(2r)^{k/2-1-j}} + e^{2\pi inr} \frac{e^{2\pi inr}}{(-2r)^{k/2-1-j}} \right). \tag{4.4}
\]

More generally, we may express \( I_{\eta par}(m, n; r) \) in terms of the confluent hypergeometric function \( {}_1F_1 \).

**Proposition 4.2.** For \( m \in \mathbb{Z} \) and \( n \in \mathbb{Z}_{\geq 1} \), \( I_{\eta par}(m, n; r) \) is real-valued and equals

\[
\frac{(2\pi i)^k nk^{-1}}{\Gamma(k)} \exp \left( \pi^2 \frac{m}{\ell_\eta}(\text{sgn}(r) - 1) - 2\pi inr \right) {}_1F_1 \left( \frac{k}{2} + \frac{2\pi im}{\ell_\eta}; k; 4\pi inr \right). \tag{4.5}
\]
Proof. From [GR07, 3.84.8], we will use the formula

\[ \int_{-\infty}^{\infty} (\beta + ix)^{-\mu} (\gamma + ix)^{-\nu} e^{-ipx} \, dx = \frac{2\pi e^{ip(-p)\mu+\nu}}{\Gamma(\mu+\nu)} \mathbf{1}_{F_1}(\mu; \mu+\nu; (\beta - \gamma)p) \]  

(4.6)

where \( \text{Re}(\beta), \text{Re}(\gamma) > 0, \text{Re}(\mu+\nu) > 1 \) and \( p < 0 \). Rewrite (4.1) by letting \( u = x + iy \), multiplying through by \( i \) and replacing \( x \) by \( -x \) to get

\[ \Im \left( e^{ip(x+iy)} \right)^\gamma = e^{\pi is(1-\sgn(r))/2} (\gamma + ix)^{s} (\beta + ix)^{-s} \quad (x \in \mathbb{R}, s \in \mathbb{C}). \]

To see that \( \mathbf{I}_{\eta \text{par}}(m, n; r) \in \mathbb{R} \), use Kummer’s transformation

\[ 1F_1(a; c; z) = e^{-z} \mathbf{1}_{F_1}(c - a; c; -z), \]

(4.8)

to show that the last part of the right side of (4.5),

\[ e^{-2\pi i nr} 1F_1(k/2 + 2\pi im/\ell_\eta; k; 4\pi i nr), \]

(4.9)
equals its conjugate. (We note that (4.9) takes exactly the form of a Coulomb wave function, used to describe charged particles with a spherically symmetric potential as in [AAR99, p. 199].)

Using (4.11) in (4.5) when \( m = 0 \) shows another version of (4.4):

\[ \mathbf{I}_{\eta \text{par}}(0, n; r) = (2\pi i)^k \frac{\Gamma((k+1)/2)}{\Gamma(k)} \left( \frac{n}{\pi r} \right)^{(k-1)/2} J_{(k-1)/2}(2\pi nr) \quad (n > 0). \]

4.2 Double cosets in the hyperbolic/parabolic case

Lemma 4.3. If \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_\eta^{-1} \Gamma \sigma_a \) then \( ac \neq 0 \).

Proof. Let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma_\eta^{-1} \gamma \sigma_a \). Since \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \begin{pmatrix} a \\ c \end{pmatrix} \) we have

\[ ac = 0 \iff \sigma_\eta^{-1} \gamma \sigma_a \infty = 0 \text{ or } \infty \]

\[ \iff \gamma \sigma_a \infty = \sigma_\eta 0 \text{ or } \sigma_\eta \infty \]

\[ \iff \gamma a = \eta_1 \text{ or } \eta_2. \]

But the cusp \( \gamma a \) cannot be a hyperbolic fixed point, implying \( ac \neq 0 \). \( \square \)

Since \( \gamma a \) cannot be a hyperbolic fixed point, the analog of \( \Gamma(a, b)_0 \) in Proposition 3.3 is empty here. Let \( \Gamma(\eta, a) \) be a complete set of inequivalent representatives for \( \Gamma_\eta \backslash \Gamma / \Gamma_a \).

Proposition 4.4. With the above notation, a complete set of inequivalent representatives for \( \Gamma_\eta \backslash \Gamma \) is given by

\[ \{ \delta \tau \mid \delta \in \Gamma(\eta, a), \tau \in \Gamma_a / Z \}. \]

(4.10)

Proof. The set (4.10) gives a complete set of representatives for \( \Gamma_\eta \backslash \Gamma \). To see that the representatives are also inequivalent modulo \( \Gamma_\eta \), suppose

\[ \Gamma_\eta \delta \tau = \Gamma_\eta \delta' \tau' \quad \text{for} \quad \delta, \delta' \in \Gamma(\eta, a) \text{ and } \tau, \tau' \in \Gamma_a / Z. \]

(4.11)

We must have \( \delta' = \delta \) because \( \Gamma(\eta, a) \) is defined as a set of inequivalent representatives. Hence there is a \( \gamma \in \Gamma_\eta \) so that \( \gamma \delta \tau = \delta \tau' \). It follows that \( \gamma \) fixes \( \eta \) and \( \delta a \). Therefore \( \gamma = \pm I \) and \( \tau = \tau' \). \( \square \)
Lemma 4.5. For \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \) with \( ac \neq 0 \) we have
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\text{sgn}(a)}{\sqrt{2}} \begin{pmatrix} |\frac{a}{c}|^{1/2} & 0 \\ 0 & |\frac{d}{e}|^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & -\text{sgn}(ac) \\ \text{sgn}(ac) & 1 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & 1/\nu \end{pmatrix} \begin{pmatrix} 1 & b \frac{1}{2a} + d \frac{1}{2c} \\ 0 & 1 \end{pmatrix}.
\]
for \( \nu = \text{hyp} \mu_{\text{par}}(M) = |2ac|^{1/2} \).

For a convenient choice of \( \Gamma(\eta, a) \), our representatives for \( \Gamma_{\eta} \backslash \Gamma / \Gamma_a \), we therefore define
\[
R_{\eta a} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_{\eta}^{-1}\Gamma_a \mid \frac{1}{\varepsilon_{\eta}} \leq |\frac{a}{c}| < \varepsilon_{\eta}, 0 \leq b \frac{1}{2a} + d \frac{1}{2c} < 1 \right\}.
\]

Lemma 4.6. We may take \( \sigma_{\eta}^{-1}\Gamma(\eta, a)\sigma_a = R_{\eta a}/\mathbb{Z} \).

Proof. Let \( A = \{ (\varepsilon_0^m, -c m) \mid m \in \mathbb{Z} \} \) for \( \varepsilon = \varepsilon_{\eta} \) and let \( B = \{ \left( \begin{smallmatrix} 1 \\ 0 \\ell \end{smallmatrix} \right) \mid \ell \in \mathbb{Z} \} \) as before. Suppose that
\[
-I \notin \Gamma. \text{ Then }
\sigma_{\eta}^{-1}(\Gamma_{\eta} \backslash \Gamma / \Gamma_a)\sigma_a = \sigma_{\eta}^{-1}\Gamma_{\eta} \sigma_a \backslash \sigma_{\eta}^{-1}\Gamma_a \sigma_a = A \backslash \sigma_{\eta}^{-1}\Gamma_a / B.
\]
Start with any \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_{\eta}^{-1}\Gamma_a \). If we multiply on the left by \( \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix} \) we obtain \( \begin{pmatrix} a \varepsilon & b \varepsilon \\ c \varepsilon & d \varepsilon \end{pmatrix} \) so that \( |a/c| \)
becomes \( \varepsilon^2|a/c| \) and \( b/2a + d/2c \) is unaffected. Multiplying \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) on the right by \( \begin{pmatrix} 1 \\ 0 \\ell \end{pmatrix} \) produces \( \begin{pmatrix} a & b + c \ell \\ c & d + a \ell \end{pmatrix} \).
Then \( b/2a + d/2c \) becomes \( b/2a + d/2c + 1 \) and \( a/c \) remains the same. It follows that every element of
\( \sigma_{\eta}^{-1}\Gamma(\eta, a)\sigma_a \) has a representative in \( R_{\eta a} \) and, as in the proof of Lemma 3.5, the representative is unique.

If \( -I \in \Gamma \) then \( -I \in \Gamma_{\eta} \Gamma_a \) so that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \) are now equivalent in \( R_{\eta a} \).

The reasoning after Lemma 3.5 also shows that if \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{\eta a} \) then \( b \) and \( d \) are uniquely determined.

4.3 The hyperbolic/parabolic Kloosterman sum

Recall the definition of \( S_{\eta a}(m, n; C) \) in (1.16). It is related to Good’s sum (2.1) by
\[
S_{\eta a}(m, n; C) = \delta_{\eta a} G(m, n; 2C)^{1/2} \quad \text{for} \quad \delta = \frac{1 - \text{sgn}(C)}{2}.
\]

Good showed in [Goo83] Lemma 6 that these are finite sums. In this subsection we reprove this and find the analog of the bound (3.13). First set
\[
\mathcal{N}_{\eta a}(C) := S_{\eta a}(0, 0; C)
\]
\[
= \# \left\{ \gamma \in \Gamma_{\eta} \backslash \Gamma / \Gamma_a \mid \sigma_{\eta}^{-1}\gamma\sigma_a = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } ac = C \right\}.
\]

Then \( \mathcal{N}_{\eta a}(C) \) is well defined and independent of the scaling matrices \( \sigma_{\eta} \) and \( \sigma_a \). The next proposition is based on [Iwa97] Prop. 2.8. It requires the existence of \( M_{aa} > 0 \) such that \( |c| \geq M_{aa} \) for all \( c \in C_{aa} \). For this see [Shi71] Lemma 1.25 or [Iwa97] p. 38.

Proposition 4.7. With the above notation
\[
\sum_{C \in C_{\eta a, |C| \leq X}} \mathcal{N}_{\eta a}(C) \ll X.
\]

Proof. We may write \( \mathcal{N}_{\eta a}(C) \) more explicitly as \# \( \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{\eta a} / \mathbb{Z} \mid ac = C \} \). Also let
\[
R(X) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{\eta a} \mid |ac| \leq X \right\} \subset \sigma_{\eta}^{-1}\Gamma\sigma_a.
\]
Suppose \( \gamma = (a/b \ c/d) \) and \( \delta = (a'/b' \ c'/d') \) are in \( R(X) \). Then \( \gamma^{-1}\delta = (a''/b'' \ d'/c') \in \sigma_a^{-1}\Gamma\sigma_a \) for
\[
|c''| = \left| cc' \left( \frac{a}{c} - \frac{a'}{c'} \right) \right|.
\]

If \( c'' = 0 \) then \( \gamma^{-1}\delta \in \sigma_a^{-1}\Gamma\sigma_a \) and so \( \gamma = \delta \). Otherwise we have \( |c''| \geq M_{aa} > 0 \). Hence
\[
\left( \frac{a}{c} - \frac{a'}{c'} \right) \geq \frac{M_{aa}}{|cc'|}. \tag{4.13}
\]

For any \( (a/b \ c/d) \in R(X) \) we have
\[
\frac{1}{\varepsilon_\eta} \leq \left| \frac{a}{c} \right| < \varepsilon_\eta, \quad |ac| \leq X \implies \frac{1}{|c|} \geq \frac{1}{\varepsilon_\eta^{1/2}X^{1/2}}.
\]

Therefore (4.13) implies
\[
\left( \frac{a}{c} - \frac{a'}{c'} \right) \geq \frac{M_{aa}}{\varepsilon_\eta X}. \tag{4.14}
\]

Since each element of \( R(X) \) corresponds to a distinct \( a/c \in [-\varepsilon_\eta, \varepsilon_\eta] \) with the distance between any two bounded from below by (4.14), the set \( R(X) \) is finite and we may list the fractions as \( a_1/c_1 < a_2/c_2 < \cdots < a_n/c_n \). Then
\[
\sum_{j=1}^{n-1} \left( \frac{a_j+1}{c_j+1} - \frac{a_j}{c_j} \right) = \sum_{j=1}^{n-1} \left( \frac{a_j+1}{c_j+1} - \frac{a_j}{c_j} \right) \leq 2\varepsilon_\eta
\]
and combining this with (4.14) shows
\[
\sum_{C \in C_{\eta \alpha}} \sum_{|C| \leq X} \left[ \frac{M_{aa}}{\varepsilon_\eta X} \right] \leq 2\varepsilon_\eta. \tag*{\Box}
\]

As a result of Proposition 4.7, for implied constants depending only on \( \Gamma, \eta \) and \( a \),
\[
N_{\eta \alpha}(C) \ll C, \quad S_{\eta \alpha}(m, n; C) \ll C, \quad \#\{C \in C_{\eta \alpha} : |C| \leq X\} \ll X. \tag{4.15}
\]

4.4 The parabolic expansion of \( P_{\eta, m} \)

**Theorem 4.8.** For \( m, n \in \mathbb{Z} \), the \( n \)th parabolic Fourier coefficient at the cusp \( \alpha \) of the hyperbolic Poincaré series \( P_{\eta, m} \) is given by
\[
c_{\alpha}(n; P_{\eta, m}) = \sum_{C \in C_{\eta \alpha}} C^{-k/2} I_{\eta \alpha}(m, n; C) \cdot S_{\eta \alpha}(m, n; C). \tag{4.16}
\]

**Proof.** With Proposition 4.4 and \( z = x + iy \in \mathbb{H} \), write the absolutely convergent
\[
(P_{\eta, m}|_{k \sigma_\alpha})(z) = \sum_{\gamma \in \Gamma_\eta \setminus \Gamma} \left( \frac{\sigma_\eta^{-1}\gamma\sigma_\alpha z}{j(\sigma_\eta^{-1}\gamma\sigma_\alpha, z)^k} \right).
\]
as
\[
\sum_{C \in C_{\eta \alpha}} \sum_{\gamma \in \Gamma(\eta, \alpha)} \sum_{n \in \mathbb{Z}} \left( \frac{\sigma_\eta^{-1}\gamma\sigma_\alpha(z + n)}{j(\sigma_\eta^{-1}\gamma\sigma_\alpha, z + n)^k} \right) \tag{4.17}
\]
If we let \( \sigma_\eta^{-1}\gamma\sigma_\alpha = (a/b \ c/d) \in SL_2(\mathbb{R}) \) with \( ac \neq 0 \), then the inner sum is \( \sum_{n \in \mathbb{Z}} f_\gamma(n) \) for
\[
f_\gamma(t) = f(t) := \left( \frac{a(z + t) + b}{c(z + t) + d} \right)^{-k/2 + 2\nu \pm \ell_n} \frac{1}{(c(z + t) + d)^k} = \left( \frac{a}{c} - \frac{1}{c(z + t) + d} \right)^{-k/2 + 2\nu \pm \ell_n} \frac{1}{c^k(z + t + d/c)^k}.
\]

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As in the proof of Theorem 1.1 we may apply Poisson summation if \( \int_{-\infty}^{\infty} f(t) \, dt \) and \( \int_{-\infty}^{\infty} |f''(t)| \, dt \) exist. The first integral exists for \( k > 1 \) by Proposition 4.1 with \( n = 0 \). It follows that the second exists too since differentiating logarithmically shows
\[
f''(t) = f(t) \left( \frac{(s+k)(s+k+1)c^2}{(c(z+t)+d)^2} - \frac{2s(s+k)ac}{(a(z+t)+b)(c(z+t)+d)} + \frac{s(s-1)a^2}{(a(z+t)+b)^2} \right)
\]
where \( s = -k/2 + 2\pi im/\ell \). Therefore
\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(t) e(-nt) \, dt.
\]
(4.18)

For the most symmetric result we substitute
\[
u = z + t + \frac{b}{2a} + \frac{d}{2c} = z + t + \frac{d}{c} - \frac{1}{2ac}
\]
and find that the integral in (4.18) equals
\[
e^{\left(n \left(z + \frac{b}{2a} + \frac{d}{2c}\right)\right) \int_{-\infty+iy}^{\infty+iy} \frac{1}{e^{(u+1/(2ac))k} - e^{(u+1/(2ac))k}} - 2\pi im/\ell \eta} e^{-2\pi i nu} \, du
\]
\[
= e^{\left(n \left(z + \frac{b}{2a} + \frac{d}{2c}\right)\right) \int_{-\infty+iy}^{\infty+iy} \frac{1}{(ac)^{k/2} \eta} (u - 1/(2ac))^k (u + 1/(2ac))^{k/2} e^{-2\pi i nu} \, du
\]
\[
= e^{2\pi i nz} \frac{1}{(ac)^{k/2}} e^{\left(m \log |a| |c| + n \left( \frac{b}{2a} + \frac{d}{2c} \right) \right) I_{\eta} \left(m, n; \frac{1}{2ac}\right)} \right).
\]
(4.19)

Therefore (4.17) is now
\[
\sum_{C \in C_{\eta, a}} \sum_{\gamma \in \Gamma(\eta, a)} \sum_{n \in \mathbb{Z}} e^{2\pi i nz} \frac{1}{C^{k/2}} e^{\left(m \log |a| |c| + n \left( \frac{b}{2a} + \frac{d}{2c} \right) \right) I_{\eta} \left(m, n; \frac{1}{2ac}\right)}.
\]
(4.20)

By Proposition 4.1 (4.20) is majorized by
\[
\sum_{C \in C_{\eta, a}} \sum_{\gamma \in \Gamma(\eta, a)} \sum_{n=1}^{\infty} e^{-2\pi n y} \frac{n^{k-1}}{|C|^{k/2}} \exp \left( \pi^2 (|m| - m)/\ell \right) \ll \sum_{C \in C_{\eta, a}} \frac{N_{\eta, a}(C)}{|C|^{k/2}}.
\]
(4.21)

Using Proposition 4.7 and summation by parts shows the last series is convergent for \( k > 2 \). Changing the order of summation in (4.20) is therefore valid, and moving the sum over \( n \) to the outside completes the proof.

Theorem 1.2 follows from Theorem 4.8 and Proposition 4.2

5 An example in \( \Gamma = SL_2(\mathbb{Z}) \)

5.1

Set \( \Gamma = SL_2(\mathbb{Z}) \). We consider the results of the last section in the special case where
\[
a = \infty \quad \text{and} \quad \eta = (\eta_1, \eta_2) = (-\sqrt{D}, \sqrt{D})
\]
for \( D \) a positive integer that is not a perfect square. If \( \gamma = (a \ b \ c \ d) \in SL_2(\mathbb{R}) \) fixes \( \pm \sqrt{D} \) then \( cz^2 + (d-a)z - b \) has \( z = \pm \sqrt{D} \) as its zeros. Therefore \( d-a = 0 \) and \( b/c = D \) so that \( \gamma = (c \ Dc) \). If \( \gamma \in \Gamma \) then \( (a, c) \) is an
integer solution of Pell’s equation \((1.19)\). Let \((a_0, c_0)\) be the positive integer solution of \((1.19)\) minimizing \(a > 1\). Set \(\varepsilon_D := a_0 + \sqrt{D}c_0\) and we see that \(\varepsilon_D > 1\). Choose \(\sigma_\infty = I\) and \(\sigma_\eta = \hat{\sigma}_\eta\) as in \((1.8)\). Then

\[
\sigma_\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} D^{1/4} & -D^{1/4} \\ D^{-1/4} & D^{-1/4} \end{pmatrix}, \quad \sigma_\infty^{-1} \begin{pmatrix} a & Dc \\ c & a \end{pmatrix} \sigma_\eta = \begin{pmatrix} a + \sqrt{D}c & 0 \\ 0 & a - \sqrt{D}c \end{pmatrix} \tag{5.1}
\]

so that

\[
\sigma_\eta^{-1} \Gamma_\sigma \sigma_\eta = \left\{ \begin{pmatrix} \varepsilon_D & 0 \\ 0 & 1/\varepsilon_D \end{pmatrix}, -I \right\} = \left\{ \pm \begin{pmatrix} \varepsilon_D^n & 0 \\ 0 & \varepsilon_D^{-n} \end{pmatrix} \right\} n \in \mathbb{Z}. \tag{5.2}
\]

(The picture for general hyperbolic points of \(SL_2(\mathbb{Z})\) is not much different from the above. See, for example [KZ84, Sect. 3.1].)

For \(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \Gamma, \) write

\[
\sigma_\eta^{-1} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \sigma_\infty = \frac{1}{\sqrt{2}D^{1/4}} \begin{pmatrix} e + g\sqrt{D} & f + h\sqrt{D} \\ -e + g\sqrt{D} & -f + h\sqrt{D} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then \(ac = (e^2 - Dg^2)/(-2\sqrt{D})\) and \(C_\eta \in \mathbb{Z}/(2\sqrt{D})\). Also

\[
\frac{b}{2a} + \frac{d}{2c} = \frac{f + h\sqrt{D}}{2(e + g\sqrt{D})} - \frac{f - h\sqrt{D}}{2(e - g\sqrt{D})} = \frac{ef - ghD}{e^2 - g^2D}.
\]

Set \(R_D := \sigma_\eta R_\eta \sigma_\eta^{-1}\), so that

\[
R_D = \left\{ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \frac{1}{\varepsilon_D} \leq \frac{e + g\sqrt{D}}{e - g\sqrt{D}} < \varepsilon_D, \ 0 \leq \frac{ef - ghD}{e^2 - g^2D} < 1 \right\} \tag{5.3}
\]

and let \(R_D(N)\) be the elements of \(R_D\) with \(e^2 - Dg^2 = N\). Combining Proposition 4.2, Lemma 4.6, and Theorem 4.8 with the work above shows the following.

**Proposition 5.1.** For \(m \in \mathbb{Z}\) and \(n \in \mathbb{Z}_{\geq 1}\)

\[
c_\infty(n; P_\eta, m) = \frac{(2\pi i)^k}{\Gamma(k)} \sum_{N \in \mathbb{Z}_{\neq 0}} \left(-\frac{2\sqrt{D}}{N}\right)^{k/2} \exp \left(-\frac{\pi^2 m}{\ell_\eta} (\text{sgn}(N) + 1) \right) + \frac{2\pi i m \sqrt{D}}{N} \chi_\eta \left(m; -\frac{N}{2\sqrt{D}}\right)
\]

for

\[
S_\infty(m, n; -N/2\sqrt{D}) = \frac{1}{2} \sum_{\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in R_D(N)} e \left(\frac{m}{\ell_\eta} \log \left|\frac{e + g\sqrt{D}}{e - g\sqrt{D}}\right| + \frac{n(ef - ghD)}{N}\right). \tag{5.4}
\]

**5.2 A more explicit form for \(S_\infty(m, n; -N/(2\sqrt{D}))\)**

Recall the statement of Theorem 1.2 and the definitions preceding it. To prove this result, we begin by examining the sets \(R_D\) and \(R_D(N)\) in more detail.

Given \(e, g \in \mathbb{Z}\) with \((e, g) = 1\), how many ways, if any, can we complete the matrix \((\frac{e}{g}, \ast)\) to an element of \(R_D\)? If \(g = 0\) then it can be quickly seen that the only way to complete \((\frac{e}{g}, \ast)\) is to \(\pm I \in R_D(1)\). For \(g \neq 0\) write \(\overline{\sigma}\) for the inverse of \(e \mod |g|\), chosen with \(0 \leq \overline{\sigma} < |g|\) say. We find the solution

\[
(e, f_0, g, h_0) = \left(e, \frac{e\overline{\sigma} - 1}{g}, g, \overline{\sigma}\right) \quad \text{to} \quad eh - fg = 1 \tag{5.5}
\]
and any other solution must be of the form \((e, f_0 + te, g, h_0 + tg)\) for \(t \in \mathbb{Z}\). With these solutions, the second condition in \(R_D\), \([5.3]\), becomes

\[
0 \leq \frac{ef_0 - gh_0D}{e^2 - g^2D} + t < 1.
\]

It follows from the above arguments that there is at most one way to complete \((\frac{e}{g}, *)\) to an element of \(R_D\). The next result gives the summands in \([5.4]\) in terms of just \(e\) and \(g\).

**Lemma 5.2.** For \(g \neq 0\)

\[
e^{\left(n(ef - ghD)\right)N} = e^{-\left(n\text{neg}^{-1}\right)N}
\]

and for \(g = 0\) the left side of \([5.6]\) is 1.

**Proof.** When \(g = 0\) then we must have \(e = \pm 1\) and \(N = 1\) so that \(ef - ghD \equiv 0 \mod N\). When \(g \neq 0\), we have from \([5.5]\) that

\[
ef - ghD \equiv ef_0 - gh_0D \equiv e\frac{e\epsilon - 1}{g} - g\epsilon D \equiv \frac{N\epsilon - e}{g} \mod N.
\]

Note that \((g, N) = 1\) since \((e, g) = 1\) and \(g\) has an inverse \(\mod N\). Writing \([5.7]\) as \(x \mod N\) then implies \(-e \equiv gx \mod N\) and the lemma follows.

We now examine the first condition in \([5.3]\).

**Lemma 5.3.** For \(\epsilon, e, g \in \mathbb{R}\) with \(e^2 - Dg^2 = N \neq 0\) and \(\epsilon > 1\) we have

\[
\frac{1}{\epsilon} \leq \left|\frac{e + g\sqrt{D}}{e - g\sqrt{D}}\right| \leq \epsilon \iff e^2 + Dg^2 \leq \left(\epsilon + \frac{1}{\epsilon}\right)\frac{|N|}{2}.
\]

**Proof.** The right side of \([5.8]\) is equivalent to

\[
(e + g\sqrt{D})^2 + (e - g\sqrt{D})^2 \leq \left(\epsilon + \frac{1}{\epsilon}\right)|N|
\]

\[
\iff \frac{(e + g\sqrt{D})^2}{(e - g\sqrt{D})^2} + 1 \leq \left(\epsilon + \frac{1}{\epsilon}\right)\frac{|N|}{(e - g\sqrt{D})^2}
\]

\[
\iff \left|\frac{e + g\sqrt{D}}{e - g\sqrt{D}}\right| + 1 \leq \left(\epsilon + \frac{1}{\epsilon}\right)\left|\frac{e + g\sqrt{D}}{e - g\sqrt{D}}\right|
\]

which is equivalent to the left side of \([5.8]\).

Since \(\epsilon_D = a_0 + c_0\sqrt{D}\), we may write \(\epsilon_D + 1/\epsilon_D = 2a_0\) in \([5.8]\). Recall \(R_D^*(N)\) defined in \([1.20]\). We see from \([5.3]\) and Lemma \([5.3]\) that \(R_D(N)\) corresponds exactly to all pairs \((e, g) \in R_D^*(N)\) such that

\[
\left|\frac{e + g\sqrt{D}}{e - g\sqrt{D}}\right| \neq \epsilon_D.
\]

**Lemma 5.4.** We have equality in \([5.9]\) if and only if \((e, g) = \pm(u_+, v_+)\) or \(\pm(u_-, v_-)\).

**Proof.** There can be equality in \([5.9]\) only for two possible values of \(N\), as we see next. We have

\[
\frac{e + g\sqrt{D}}{e - g\sqrt{D}} = \pm\epsilon_D = \pm(a_0 + c_0\sqrt{D}) \iff e = \frac{a_0 \pm 1}{c_0} g.
\]

If \(\gcd(e, g) = 1\) then

\[
e = \frac{a_0 + 1}{c_0} g \iff e = \frac{u_+}{v_+} g \iff (e, g) = \pm(u_+, v_+),
\]

\[
e = \frac{a_0 - 1}{c_0} g \iff e = \frac{u_-}{v_-} g \iff (e, g) = \pm(u_-, v_-).
\]

We also note that \(D_+ > 0\) and \(D_- < 0\) since \(D_+ = 2a_0v_+^2/c_0^2, D_- = -2a_0v_-^2/c_0^2\).
The points ±(u⁺, v⁺) lie on both the ellipse \( e^2 + Dg^2 = a_0|N| \) and the hyperbola \( e^2 - Dg^2 = N \) for \( N = D⁺. \) Similarly for ±(u⁻, v⁻) when \( N = D⁻. \) See Figure 1.

**Lemma 5.5.** If \((e, g, N) = (±u⁺, ±v⁺, D⁺) \) or \((±u⁻, ±v⁻, D⁻) \) then

\[
e \left( \frac{m}{\ell_\eta} \log \left| \frac{e + g \sqrt{D}}{e - g \sqrt{D}} \right| - \frac{\text{neg}^{-1}}{N} \right) = (-1)^{m+c_0-n}. \tag{5.11}
\]

**Proof.** Lemma 5.4 implies

\[
e \left( \frac{m}{\ell_\eta} \log \left| \frac{e + g \sqrt{D}}{e - g \sqrt{D}} \right| \right) = e \left( \frac{m}{\ell_\eta} \log e \right) = e \left( \frac{m}{2} \right) = (-1)^m.
\]

Writing \( a_0 + 1 = \lambda u⁺ \) and \( c_0 = \lambda v⁺ \) for \( \lambda \in \mathbb{Z}_{≥1} \) we find

\[
a_0^2 - Dc_0^2 = 1 \implies (\lambda u⁺ - 1)^2 - D\lambda^2(v⁺)^2 = 1 \implies \lambda D⁺ = 2u⁺. \tag{5.12}
\]

Consider \( u⁺(v⁺)^{-1} \mod D⁺. \) If \( c_0 \) is even then \( a_0 \) must be odd and so \( \lambda \) is even. Hence (5.12) implies \( D⁺ \) divides \( u⁺ \) and therefore \( u⁺(v⁺)^{-1} \equiv 0 \mod D⁺ \) and

\[
e \left( - \frac{\text{neg}^{-1}}{N} \right) = e \left( - \frac{nu⁺(v⁺)^{-1}}{D⁺} \right) = 1 = (-1)^{c_0-n}.
\]

On the other hand, if \( c_0 \) is odd then \( \lambda \) is odd and \( D⁺ \) is even. Hence (5.12) implies \( u⁺ \equiv D⁺/2 \mod D⁺. \)

In this case we must also have \( v⁺ \) and \( (v⁺)^{-1} \) odd so that \( u⁺(v⁺)^{-1} \equiv D⁺/2 \mod D⁺ \) and

\[
e \left( - \frac{\text{neg}^{-1}}{N} \right) = e \left( - \frac{nu⁺(v⁺)^{-1}}{D⁺} \right) = (-1)^n = (-1)^{c_0-n}.
\]

This completes the proof for \((e, g, N) = (u⁺, v⁺, D⁺), (u⁻, v⁻, D⁻). \) The proof for \((e, g, N) = (±u⁻, ±v⁻, D⁻) \) is similar. \( \square \)

**Proof of Theorem 4.3** We see now that the sum for \( S_{\eta, \infty} \) in (5.4) may be replaced by the sum over \((e, g) \in R⁺(N) \) in (1.21) except that the extra summands with \((e, g) = ±(u⁺, v⁺), ±(u⁻, v⁻) \) must be removed from \( R⁺(N). \) This is accomplished by the term \( -\psi_D(m, n; N) \) in (1.21). The factor 1/2 in both sums comes from the equivalence of \((e, f) \) and \((-f, -g). \) \( \square \)

### 6 Parabolic Poincaré series and their hyperbolic Fourier expansions

The results in the section are similar to those in Section 4 switching \( \eta \) with \( a. \) Note the relation

\[
c_a(m; P_{\eta,n}) = c_a(n; P_{\eta,m}) \left[ \frac{(2\pi)^k \ell_\eta m^{k-1} e^{-2\pi^2 n/\ell_\eta}}{\Gamma(k/2 + 2\pi in/\ell_\eta)^2} \right] \quad (m \in \mathbb{Z}_{≥0}, n \in \mathbb{Z}) \tag{6.1}
\]

coming from (1.13) and (1.11) applied to \((P_{\eta,m}, P_{\eta,n}) = (P_{\eta,n}, P_{\eta,m}). \) However, (6.1) is not quite symmetrical. For \( m \leq 0 \) we have that \( c_a(m; P_{\eta,n}) = 0 \) since \( P_{\eta,n} \in S_k(\Gamma), \) but we don’t expect \( c_a(n; P_{\eta,m}) \) to be zero since \( P_{\eta,m} \in M_k^1(\Gamma). \)
6.1 The parabolic/hyperbolic integral

For \( m, n \in \mathbb{Z} \) and \( r \in \mathbb{R} \neq 0 \) define

\[
I_{\text{par}}(m, n; r) := \int_{-\infty + iy}^{\infty + iy} e \left( m \left( \frac{\text{sgn}(r) e^n - 1}{2r(\text{sgn}(r) e^n + 1)} \right) \right) e^{u(k/2 - 2\pi n/\ell_n)} \frac{du}{\ell_n} \quad (0 < y < \pi, \ k > 0). \tag{6.2}
\]

This is the integral that appears in the proof of Theorem 6.4 and we develop its properties here first.

**Proposition 6.1.** The integral (6.2) is absolutely convergent and we have the estimates

\[
I_{\text{par}}(m, n; r) \ll \exp\left(\frac{\pi e(|m| - m)/|r|}{\ell_n}\right) \quad (n = 0), \tag{6.3}
\]

\[
I_{\text{par}}(m, n; r) \ll n^{k/2} \exp\left(\frac{\pi^2 n^{1/2} \left( \frac{1}{\ell_n} + \frac{|m| - m}{|r|} \right)}{r} \right) \quad (n > 0), \tag{6.4}
\]

\[
I_{\text{par}}(m, n; r) \ll |n|^{k/2} e^{2\pi^2 n/\ell_n} \exp\left(\frac{\pi^2 n^{1/2} \left( \frac{1}{\ell_n} + \frac{|m| - m}{|r|} \right)}{r} \right) \quad (n < 0), \tag{6.5}
\]

for implied constants depending only on \( k > 0 \).

**Proof.** Let \( u = x + iy \) to get

\[
\left| e \left( m \left( \frac{\text{sgn}(r) e^n - 1}{2r(\text{sgn}(r) e^n + 1)} \right) \right) \right| = \exp \left( \frac{2\pi m}{r |\text{sgn}(r) e^n + 1|^2} \text{Im} \left( \text{sgn}(r) e^{-iy} + 1 \right) \right) = \exp \left( \frac{-2\pi m \sin(y)}{|r|} \frac{e^x}{|e^x + \text{sgn}(r) e^{-iy}|^2} \right). \tag{6.6}
\]

Hence

\[
|I_{\text{par}}(m, n; r)| \leq \frac{e^{2\pi ny/\ell_n}}{\ell_n} \int_{-\infty}^{\infty} \exp \left( \frac{-2\pi m \sin(y)}{|r|} \frac{e^x}{|e^x + \text{sgn}(r) e^{-iy}|^2} \right) \left( \frac{e^x}{|e^x + \text{sgn}(r) e^{-iy}|^2} \right) k/2 \ dx.
\]

Also

\[
\frac{e^x}{|e^x + \text{sgn}(r) e^{-iy}|^2} \leq \begin{cases} e^{1-|x|} & \text{when } |x| \geq 1 \\ e^{-2(y)} & \text{when } |x| \leq 1. \end{cases} \tag{6.7}
\]

It follows that (6.6) is bounded by

\[
\exp \left( \frac{\pi(|m| - m) \sin(y)}{|r|} e^{1-|x|} \right) \quad \text{when } |x| \geq 1, \quad \exp \left( \frac{\pi(|m| - m) \sin(y)}{|r|} e^{2(y)} \right) \quad \text{when } |x| \leq 1.
\]

Altogether, for an implied constant depending only on \( k > 0 \),

\[
I_{\text{par}}(m, n; r) \ll \frac{e^{2\pi ny/\ell_n}}{\ell_n} \left( \exp \left( \frac{\pi(|m| - m) \sin(y)}{|r|} \right) + \exp \left( \frac{\pi(|m| - m)}{|r| \sin(y)} e - \frac{1}{\sin^4(y)} \right) \right) \tag{6.8}
\]

proving convergence. We have that \( (6.2) \) is independent of \( y \) by Cauchy’s theorem. Finally, letting \( y = \pi n^{-1/2}/2, y = \pi/2 \) and \( y = \pi(1 - |n|^{-1/2})/2 \) in (6.8) for \( n > 0, n = 0 \) and \( n < 0 \), respectively, and using

\[
2y/\pi \leq \sin(y), \sin(\pi - y) \leq y \quad \text{for } 0 \leq y \leq \pi/2, \tag{6.9}
\]

completes the proof.

Assume from here that \( k \) is even and at least 4.
Proposition 6.2. For \( m, n \in \mathbb{Z} \) and \( r \in \mathbb{R}_{\neq 0} \), \( I_{par\eta}(m, n; r) \) is real-valued with

\[
I_{par\eta}(m, n; r) = \frac{1}{\ell_{\eta}} \exp \left( \frac{\pi i}{2} \left( \frac{k}{2} - \frac{2\pi im}{\ell_{\eta}} \right) \frac{\pi i m}{r} \right) (1 - \text{sgn}(r)) \frac{\pi i m}{r} \times B \left( \frac{k}{2} + \frac{2\pi im}{\ell_{\eta}}, \frac{k}{2} - \frac{2\pi im}{\ell_{\eta}} \right) \Gamma_1 \left( \frac{k}{2} - \frac{2\pi im}{\ell_{\eta}}; k; \frac{2\pi im}{r} \right). \quad (6.10)
\]

Proof. Let \( u = t + iy \) for \( y = \pi/2 \) and then write \( v = e^t \) so that (6.2) becomes

\[
\exp \left( \frac{\pi i}{2} \left( \frac{k}{2} - \frac{2\pi im}{\ell_{\eta}} \right) \frac{\pi i m}{r} \right) \int_0^\infty \exp \left( -\frac{2\pi im}{r} x \right) x^{k-2} \left( \frac{1 - x}{\text{sgn}(r)i x} \right)^{k/2-2\pi im/\ell_{\eta}-1} dx. \quad (6.11)
\]

Substitute \( x = 1/(\text{sgn}(r)i v + 1) \) and the integral in (6.11) is now

\[
\frac{1}{\text{sgn}(r)i \ell_{\eta}} \int_0^1 \exp \left( -\frac{2\pi im}{r} x \right) x^{k-2} \left( \frac{1 - x}{\text{sgn}(r)i x} \right)^{k/2-2\pi im/\ell_{\eta}-1} dx
\]

where the path of integration is a small curve from 0 to 1. Except for the endpoints, we have \(-\pi/2 < \text{sgn}(r) \arg(x) < 0 \) and \( 0 < \text{sgn}(r) \arg(1 - x) < \pi/2 \). Hence

\[
\left( \frac{1 - x}{\text{sgn}(r)i x} \right)^w = (1 - x)^w (\text{sgn}(r)i)^{-w} x^{-w} \quad (w \in \mathbb{C}).
\]

Finally, move the contour of integration to the interval \([0, 1] \subset \mathbb{R} \) and use

\[
\int_0^1 x^{\mu-1} (1 - x)^{\nu-1} e^{\beta x} dx = B(\mu, \nu) \Gamma_1(\mu; \mu + \nu; \beta)
\]

when \( \text{Re}(\mu), \text{Re}(\nu) > 0 \) from [GR07, 3.383.1], along with an application of Kummer’s transformation (4.8), to show (6.10). It now follows from (6.10), as in Proposition 4.2, that \( I_{par\eta}(m, n; r) \in \mathbb{R} \). \( \square \)

6.2 Double cosets and Kloosterman sums in the parabolic/hyperbolic case

All of the results in Sections 4.2 and 4.3 translate directly to the parabolic/hyperbolic case here by means of the map \( \sigma_{-\gamma}^{-1} \Gamma \sigma_{-\nu} \rightarrow \sigma_{-\gamma}^{-1} \Gamma \sigma_{-\eta} \) given by \( \gamma \mapsto \gamma^{-1} \). We summarize the main things we need:

(i) If \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \sigma_a^{-1} \Gamma \sigma_{a} \) then \( cd \neq 0 \).

(ii) Let \( \Gamma(\alpha, \eta) \) be a complete set of inequivalent representatives for \( \Gamma \alpha \backslash \Gamma / \Gamma_{\eta} \). Then

\[
\left\{ \delta \tau \mid \delta \in \Gamma(\alpha, \eta), \tau \in \Gamma_{\eta}/\mathbb{Z} \right\}
\]

is a complete set of inequivalent representatives for \( \Gamma \alpha \backslash \Gamma \).

(iii) In this case [Goo83, Lemma 1] says:

Lemma 6.3. For \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R}) \) with \( cd \neq 0 \) we have

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \frac{\text{sgn}(d)}{\sqrt{2}} \left( \begin{array}{cc} 1 & a/2c + b/2d \\ 0 & 1/\nu \end{array} \right) \left( \begin{array}{cc} 1/\nu & 0 \\ 0 & \text{sgn}(cd) \end{array} \right) \left( \begin{array}{cc} \left| \frac{d}{c} \right|^{-1/2} & 0 \\ 0 & \left| \frac{d}{c} \right|^{1/2} \end{array} \right)
\]

for \( \nu = \text{par\nu}_{\text{hyp}}(M) = |2cd|^{1/2} \).

(iv) Define

\[
R_{\eta\eta} := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \sigma_a^{-1} \Gamma \sigma_{\eta} \mid 0 \leq a/2c + b/2d < 1, \frac{1}{\varepsilon_{\eta}} \leq \left| \frac{d}{c} \right| < \varepsilon_{\eta} \right\}
\]

and we may take \( \sigma_a^{-1} \Gamma(a, \eta) \sigma_{\eta} = R_{\eta\eta}/\mathbb{Z} \).
(v) Put \( C_{\alpha} = \{ cd \mid (a/b, c/d) \in \sigma_a^{-1} \Gamma \sigma_\eta \} \). We have \( C_{\alpha} = -C_{\eta a} \).

(vi) For \( C \in C_{\alpha} \), define
\[
S_{\alpha}(m, n; C) := \sum_{\gamma \in \Gamma_a \backslash \Gamma_\eta} e\left( m \left( \frac{a}{2c} + \frac{b}{2d} \right) + \frac{n}{\ell_\eta} \log |c/d| \right), \tag{6.14}
\]
It is related to Good's sum by
\[
S_{\alpha}(m, n; C) = \frac{\chi}{a} S_{\eta}(m, n; |2C|^{1/2}) \quad \text{for} \quad \delta' = \frac{1 + \text{sgn}(C)}{2}.
\]

Also
\[
S_{\alpha}(m, n; C) = S_{\eta a}(-n, -m; -C) = S_{\eta a}(m, n; -C), \tag{6.15}
\]
so the formula in Theorem 1.3 for \( a = \infty \) and \( \eta = (\sqrt{D}, \sqrt{D}) \) also evaluates \( S_{\alpha}(m, n; C) \).

(vii) With
\[
N_{\alpha}(C) := S_{\alpha}(0, 0; C) = \# \left\{ \gamma \in \Gamma_a \backslash \Gamma_\eta \mid \sigma_a^{-1} \gamma \sigma_\eta = \left( \begin{array}{cc} a/b & c/d \\ c/d & b/a \end{array} \right) \right\} \] with \( cd = C \)

it is clear that \( N_{\alpha}(C) \leq N_{\eta a}(-C) \). Therefore
\[
\sum_{C \in C_{\alpha} \setminus |C| \leq X} N_{\alpha}(C) \ll X, \tag{6.16}
\]
\[
N_{\alpha}(C) \ll C, \quad S_{\alpha}(m, n; C) \ll C, \quad \# \{ C \in C_{\alpha} \mid |C| \leq X \} \ll X. \tag{6.17}
\]

### 6.3 The hyperbolic expansion of \( P_{a,m} \)

In the next result we prove a formula for \( c_{\eta}(n; P_{a,m}) \) using the same approach as in Theorems 1.1 and 4.8. An alternative derivation may be given using (1.9) as the starting point.

**Theorem 6.4.** For all \( m, n \in \mathbb{Z} \), the \( n \)th hyperbolic Fourier coefficient at \( \eta \) of the parabolic Poincaré series \( P_{a,m} \) is given by
\[
c_{\eta}(n; P_{a,m}) = \sum_{C \in C_{\alpha}} |C|^{-k/2} I_{\alpha}(m, n; C) S_{\alpha}(m, n; C). \tag{6.18}
\]

**Proof.** With (6.12) and \( z \in \mathbb{H} \), write the absolutely convergent
\[
(P_{a,m}|_{k \sigma_\eta})(z) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} e\left( m \left( \frac{\sigma(a^{-1} \gamma \sigma_\eta z)}{\sigma(a^{-1} \gamma \sigma_\eta, z)^k} \right) \right) \tag{6.19}
\]
as
\[
\sum_{C \in C_{\alpha}} \sum_{\gamma \in \Gamma(a, \eta)} \sum_{n \in \mathbb{Z}} \frac{e\left( m \left( \sigma(a^{-1} \gamma \sigma_\eta (e^{n\ell_\eta + A}) \right) \right)}{j\left( \sigma(a^{-1} \gamma \sigma_\eta, e^{n\ell_\eta + A})^k e^{-n\ell_\eta k/2} \right)} \tag{6.20}
\]
where we let \( z = e^A \) for \( 0 < \text{Im} A < \pi \). With \( \sigma_a^{-1} \gamma \sigma_\eta = \left( \begin{array}{cc} a/b & c/d \\ c/d & b/a \end{array} \right) \in \operatorname{SL}_2(\mathbb{R}) \) for \( cd \neq 0 \), the inner sum is \( \sum_{n \in \mathbb{Z}} f_{\gamma}(n) \) for
\[
f_{\gamma}(t) = f(t) := \exp \left( 2\pi i m \frac{ae^t + A + b}{ce^t + A + d} + \ell_\eta k/2 \right) \frac{1}{(ce^t + A + d)^k}.
\]

As in the proof of Theorem 6.1 we may apply Poisson summation if \( \int_{-\infty}^{\infty} f(t) \, dt \) and \( \int_{-\infty}^{\infty} |f''(t)| \, dt \) exist. The first integral exists for \( k > 0 \) by similar arguments to Proposition 6.1 with \( n = 0 \). It follows that the second also exists since, with \( g(t) := ce^{\ell\eta} + d \),
\[
\frac{f''(t)}{t^2} = f(t) \left[ \frac{2\pi i me^{\ell\eta} + A}{g(t)^2} \left( \frac{2\pi i me^{\ell\eta} + A}{g(t)^2} + \frac{2d(k+1)}{g(t)} - k - 1 \right) + \frac{dk(k+1)}{g(t)} \left( \frac{d}{g(t)} + 1 \right) + \frac{k^2}{4} \right].
\]

With Poisson summation, as in (4.18), the inner sum in (6.20) now equals
\[
\sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \mathbf{e} \left( m \left( \frac{a}{2c} + \frac{b}{2d} \right) - n \frac{\ell\eta}{c} \right) e^{\ell\eta (k/2 - 2\pi i n/\ell)} \, dt.
\]

Let \( u = \ell\eta t + A + \log |z| \) in (6.21) and use \( \frac{a}{c} = \frac{a_1}{c_1} + \frac{b_1}{c_1} + \frac{1}{2cd} \) to get
\[
z^{-k/2+2\pi i n/\ell} \left| \frac{d}{c} \right|^{k/2} e \left( m \left( \frac{a}{2c} + \frac{b}{2d} \right) - n \frac{\ell\eta}{c} \log \left| \frac{d}{c} \right| \right) \times \int_{-\infty}^{\infty} e^{u(k/2 - 2\pi i n/\ell)} \, du.
\]

Altogether (6.20) equals
\[
\sum_{C \in C_{a\eta}} \sum_{\gamma \in \Gamma(a, \eta)} \sum_{n \in \mathbb{Z}} z^{-k/2+2\pi i n/\ell} e \left( m \left( \frac{a}{2c} + \frac{b}{2d} \right) - n \frac{\ell\eta}{c} \log \left| \frac{d}{c} \right| \right) \frac{I_{a\eta}(m, n; C)}{|C|^{k/2}}.
\]

For \( z = e^A \) with \( 0 < \Im A < \pi \) as before, we have \( z^{-k/2+2\pi i n/\ell} \ll e^{-2\pi n \Im A/\ell} \). With Proposition 6.1 we then have
\[
z^{-k/2+2\pi i n/\ell} I_{a\eta}(m, n; C) \ll e^{-|n|}
\]
for \( \varepsilon > 0 \), depending on \( z \). Therefore (6.23) is majorized by \( \sum_{C \in C_{a\eta}} |C|^{-k/2} N_{a\eta}(C) \) and thus convergent for \( k > 2 \) by (6.16). This proves that changing the order of summation in (6.23) is valid, and moving the sum over \( n \) to the outside completes the proof.

Set
\[
S_{a\eta}^*(m, n; C) := \exp \left( -\pi^2 n (\text{sgn}(C) + 1)/\ell\eta - \pi i m/C \right) S_{a\eta}(m, n; C)
\]
\[
= \sum_{\gamma \in \Gamma(a) \backslash \Gamma \eta} \mathbf{e} \left( m \frac{b}{d} + n \frac{\ell\eta}{c} \log \left( \frac{d}{c} \right) \right)
\]
and note the relation
\[
S_{a\eta}^*(m, n; C) = S_{a\eta}^*(m, m; -C).
\]

Rewriting \( I_{a\eta}(m, n; C) S_{a\eta}(m, n; C) \) with (6.10) and (6.24) gives a more explicit version of Theorem 6.4.

**Theorem 6.5.** For all \( m, n \in \mathbb{Z} \)
\[
c_{a\eta}(n; P_{a, m}) = e^{2\pi^2 n/\ell\eta} B \left( \frac{k}{2} + \frac{2\pi i n}{\ell\eta} \frac{k}{2} - \frac{2\pi i n}{\ell\eta} \right)
\]
\[
\times \sum_{C \in C_{a\eta}} \binom{1}{F_1} \left( \frac{k}{2} - \frac{2\pi i n}{\ell\eta}; k; \frac{2\pi i m}{C} \right) \frac{S_{a\eta}^*(m, n; C)}{|C|^{k/2}}.
\]

The identity (6.11) may be verified by comparing (6.26) above with (1.18) and using the symmetry (6.25).
7 Computations

In this section we restrict our attention to $\Gamma = \text{SL}_2(\mathbb{Z})$, its cusp at $\infty$ with scaling matrix $\sigma_\infty = I$ and its hyperbolic pairs $\eta = (-\sqrt{D}, \sqrt{D})$ with scaling matrix $\delta_\eta$ given by (1.8).

7.1 Parabolic coefficients

We have $C_{\infty \infty} = \mathbb{Z}_{\geq 1}$ in the notation of Section 3.3. With $c \in C_{\infty \infty}$, using for example Lemma 3.5 and the sentences following it, we obtain the classical Kloosterman sum

$$S_{\infty \infty}(m, n; c) = \sum_{0 \leq d < c, \ (c,d)=1, \ ad \equiv 1 \mod c} \ e\left(\frac{ma + nd}{c}\right).$$

It is necessarily real-valued since each term with $a, d$ gets added to (or equals) its conjugate with $c - a, c - d$. For all $m \in \mathbb{Z}$, Theorem 1.1 then gives

$$c_{\infty}(n; P_{\infty, m}) = \delta_{mn} + \left\{\frac{(2\pi)^k}{\Gamma(k)} \sum_{c=1}^{\infty} 0F1\left(\frac{1}{C}; \frac{4\pi^2 mn}{C^2}\right) S_{\infty \infty}(m, n; C) \right\}_{c} \text{ if } n \geq 1. \quad (7.1)$$

This is usually stated using $J$-Bessel functions (recall (3.10)) when $m \geq 1$ and $I$-Bessel functions when $m \leq -1$. See for example [Ran77, Thm. 5.3.2].

With (7.1), we can investigate the Poincaré series $P_{\infty, m}$ numerically. The coefficient $c_{\infty}(n; P_{\infty, m})$ evaluates to zero when $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{4, 6, 8, 10, 14\}$ since $\dim S_k(\Gamma) = 0$ in these cases. The space $S_{12}(\Gamma)$ is one-dimensional, containing $\Delta(z) := q\prod_{n=1}^{\infty}(1 - q^n)^{24} = \sum_{m=1}^{\infty} \tau(m) q^m$. It follows that, when $k = 12$, each $P_{\infty, m}$ for $m \in \mathbb{Z}_{\geq 1}$ must equal $\lambda_m \Delta$ for some $\lambda_m \in \mathbb{R}$. Since $\lambda_m = c_{\infty}(1; P_{\infty, m})$, we find for example:

$$\lambda_1 \approx 2.840287, \quad \lambda_2 \approx -0.0332846, \quad \lambda_3 \approx 0.004040443, \quad \lambda_4 \approx -0.0009968. \quad (7.2)$$

This is consistent with (1.13), which implies $\lambda_m = \Gamma(11) \tau(m)/(\Delta, \Delta, (4\pi m)^{11})$ for $m \in \mathbb{Z}_{\geq 1}$.

If $m = 0$ then $P_{\infty, 0}$ is the Eisenstein series $E_k(z) := 1 - 2k/B_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ where $B_k$ is the $k$th Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} q^{k-1}$.

For $m \in \mathbb{Z}_{\leq -1}$, each $P_{\infty, m}$ is related to the $j$-function, defined as

$$j(z) := E_4^3(z)/\Delta(z) = q^{-1} + 744 + 196884q + \cdots \in M_0^!(\Gamma).$$

For example, in weight $k = 12$, Rankin [Ran96 (4.4)] constructed $(j(z) + 264) E_6^2(z) = q^{-1} - 598428q + \cdots$ with integer coefficients. Then we must have

$$P_{\infty, -1}(z) = (j(z) + 264) E_6^2(z) + \lambda_{-1} \Delta(z) \quad (7.3)$$

for some $\lambda_{-1} \in \mathbb{R}$. Computing, we find $\lambda_{-1} \approx 600270.8947$, agreeing with [Ran96 (4.12)]. Following Rankin’s method we similarly have

$$P_{\infty, -2}(z) = (j(z)^2 - 480j(z) + 205128) E_6^2(z) + \lambda_{-2} \Delta(z) \quad \text{for } \lambda_{-2} \approx 321214058.075.$$

It would be interesting to identify the $\lambda_m$s resulting from continuing this sequence. For more on parabolic Poincaré series and weakly holomorphic forms, see for example [DJ08], [Rho12] and their contained references.

With Theorems 1.2 and 1.3 we may calculate the parabolic Fourier coefficients at $\infty$ of the series $P_{\eta, m}$ with $\eta = (-\sqrt{D}, \sqrt{D})$. In weight $k = 12$ each is again a constant times $\Delta$. This constant, $c_{\infty}(1; P_{\eta, m})$, is given numerically for $-2 \leq m \leq 2$ and $D = 2, 3, 5$ in Table I. To see why all the entries in the table are real it is simplest to use (4.16) in Theorem 4.8. We know that the factor $I_{\eta \infty}(m, n; C)$ there is real by Proposition 4.2. The factor $S_{\eta \infty}(m, n; C)$ is also real since in the formula (1.21) each term with $e, g$ is added to (or equals) its conjugate with $e, -g$. 

---

24
\[m\] | \[D = 2\] | \[D = 3\] | \[D = 5\]  \\
--- | --- | --- | ---  \\
2 | 23.43 | 7.93 | −130.37  \\
1 | 252.41 | 114.79 | −311.81  \\
0 | 1529.46 | −1665.07 | 1857.25  \\
−1 | −6819.34 | 78417.86 | 9515.95  \\
−2 | 1709726.97 | −1243941.21 | −121422.56

Table 1: Computations of \(c_\infty(1; P_{\eta,m})\) for \(\eta = (−\sqrt{D}, \sqrt{D})\) with \(k = 12\).

7.2 Hyperbolic coefficients

With Theorems 6.4 or 6.5 we may numerically compute the hyperbolic expansion coefficients at \(\eta = (−\sqrt{D}, \sqrt{D})\) of \(P_{\infty,m}\) for \(m \in \mathbb{Z}\). As above, the Kloosterman sums are computed with Theorem 1.3 now combined with the symmetries (6.15) or (6.25), and the coefficients are necessarily real.

The first column of Table 2 shows part of the hyperbolic expansion of the weight \(k = 12\) series \(P_{\infty,1}\) at \(\eta = (−\sqrt{2}, \sqrt{2})\). As we saw earlier, \(P_{\infty,1} = \lambda_1\Delta\) for \(\lambda_1\) given in (7.2). Hence, on renormalizing, we obtain the expansion (1.4). As in the parabolic and elliptic cases, we suspect that these coefficients should have some arithmetic significance, but this remains to be determined.

A noticeable feature of these hyperbolic coefficients, first shown by Hiramatsu in [Hir72, Thm. 1], is that they have exponential decay as \(n \to -\infty\). A slightly more precise version of his result, appearing in [IMO], is that for all \(f \in S_k(\Gamma)\) we have

\[c_{\eta}(m; f) \ll |m|^{k/2} \times \begin{cases} 1 & \text{if } m > 0, \\ e^{-2\pi|m|/\ell_\eta} & \text{if } m < 0. \end{cases} \quad (7.4)\]

This is the analog of the usual Hecke bound for parabolic Fourier coefficients. The second and third columns in Table 2 give the larger hyperbolic coefficients of \(E_{12} \in M_{12}(\Gamma)\) and \(P_{\infty,-1} \in M^!_{12}(\Gamma)\) as seen in (7.3).

8 Hyperbolic Poincaré series and their hyperbolic Fourier expansions

Returning to a general \(\Gamma\), let \(\eta\) and \(\eta'\) be two pairs of hyperbolic fixed points: \(\eta = (\eta_1, \eta_2)\) and \(\eta' = (\eta'_1, \eta'_2)\). We describe the hyperbolic Fourier expansion of \(P_{\eta,m}\) at \(\eta'\),

\[ (P_{\eta,m}|_{k\sigma_{\eta'}})(z) = \sum_{n \in \mathbb{Z}} c_{\eta'}(n; P_{\eta,m}) z^{-k/2 + 2\pi i n/\ell_{\eta'}}, \]

in this section. Here, the group \(\Gamma\) may or may not have parabolic elements.
8.1 The hyperbolic/hyperbolic integral

The integral we will need shortly in (8.42) is the following one. Let \( r \in \mathbb{R} \neq 0,1 \) and \( \alpha, \beta = \pm 1 \) satisfy \( \alpha \beta = \text{sgn}(r) \). Put

\[
I_{\eta \eta'}(m, n; r, \alpha, \beta) := \int_{-\infty+iy}^{\infty+iy} \left( \frac{r-1}{r-1} \right)^{1/2} \frac{\alpha e^{\alpha} + \alpha \text{sgn}(r-1) e^{-\alpha}}{e^{\alpha} + \beta e^{-\alpha}} \frac{2\pi i m/\ell}{\ell_{\eta'} e^{2\pi i m/\ell}} \frac{e^{u(k/2-2\pi in/\ell')}}{\ell_{\eta'}} \, du \tag{8.1}
\]

for \( 0 < y < \pi \) and \( k > 0 \). We next establish good bounds for \( I_{\eta \eta'}(m, n; r, \alpha, \beta) \) with respect to \( n \). These bounds will be required at the end of the proof of Theorem 8.15.

**Proposition 8.1.** The integral (8.1) is absolutely convergent. We have

\[
I_{\eta \eta'}(m, n; r, \alpha, \beta) \ll \exp \left( \frac{\pi^2 (|m| - m)}{\ell_{\eta}} \right) / \ell_{\eta'} \tag{8.2}
\]

\[
I_{\eta \eta'}(m, n; r, \alpha, \beta) \ll n^{k/2} \exp \left( \frac{\pi^2 (|m| - m)}{\ell_{\eta}} + \frac{\pi^2 |n|^{1/2}}{\ell_{\eta'}} \right) / \ell_{\eta'} \tag{8.3}
\]

\[
I_{\eta \eta'}(m, n; r, \alpha, \beta) \ll |u|^{k/2} e^{2\pi^2 n/\ell_{\eta'}} \exp \left( \frac{\pi^2 (|m| - m)}{\ell_{\eta}} + \frac{\pi^2 |n|^{1/2}}{\ell_{\eta'}} \right) / \ell_{\eta'} \tag{8.4}
\]

for implied constants depending only on \( k > 0 \).

**Proof.** Note that \( u := \frac{\alpha e^{\alpha} + \alpha \text{sgn}(r-1) e^{-\alpha}}{e^{\alpha} + \beta e^{-\alpha}} \in \mathbb{H} \) so that \( 0 < \arg u < \pi \). Hence, with \( u = x + iy \),

\[
|u|^{2\pi i m/\ell_{\eta}} e^{-2\pi i m/\ell_{\eta'}} \leq e^{\pi^2 (|m| - m)/\ell_{\eta}} \cdot e^{2\pi ny/\ell_{\eta'}}.
\]

The remaining part of the integrand in (8.1) is bounded by

\[
e^{xk/2}
\]

\[
\left| \frac{r-1}{r-1} e^{x} + \alpha \text{sgn}(r-1) e^{-y} \right|^{k/2} \left| \frac{r-1}{r-1} e^{x} + \beta e^{-y} \right|^{k/2}.
\]

Let \( t = x + \frac{1}{2} \log |r-1| \) and \( u = x - \frac{1}{2} \log |r-1| \). Using (6.7),

\[
e^{x} \left| \frac{r-1}{r-1} e^{x} + \alpha \text{sgn}(r-1) e^{-y} \right|^{2} \leq \frac{r}{r-1} \left| e^{1+|t|} \right|^{1/2} \times \left| e^{1-|u|} \right|^{1/2} \text{ if } |t| \geq 1,
\]

\[
e^{x} \left| \frac{r-1}{r-1} e^{x} + \beta e^{-y} \right|^{2} \leq \frac{r}{r-1} \left| e^{1+|t|} \right|^{1/2} \times \left| e^{1-|u|} \right|^{1/2} \text{ if } |t| \leq 1,
\]

Therefore, (8.5) is bounded by

\[
(e^{2-|t|-|u|})^{k/4} \leq e^{(2-2|t|)k/4} \quad \text{if } |t| \geq 1 \text{ and } |u| \geq 1,
\]

\[
(e^{2 \sin^{-4}(y)})^{k/4} = e^{k/2 \sin^{-k}(y)} \quad \text{if } |t| \leq 1 \text{ or } |u| \leq 1.
\]

Altogether, for an implied constant depending only on \( k > 0 \),

\[
I_{\eta \eta'}(m, n; r, \alpha, \beta) \ll e^{\pi^2 (|m| - m)/\ell_{\eta}} \cdot e^{2\pi ny/\ell_{\eta'}} \left( 1 + \sin^{-k}(y) \right).
\]

Therefore (8.1) is absolutely convergent. Since the integrand is holomorphic, it is independent of \( y \) with \( 0 < y < \pi \). Recalling (6.5) and choosing \( y = \pi n^{1/2}/2, y = \pi/2 \) and \( y = \pi (1 - |n|^{-1/2})/2 \) for \( n > 0 \), \( n = 0 \) and \( n < 0 \), respectively, finishes the proof. \( \Box \)
Proposition 8.2. If $r \not\in (0, 1)$ or if $r \in (0, 1)$ and $\alpha = 1$ then

$$I_{\eta'}(m, n; r, \alpha, \beta) = \frac{\text{sgn}(r)^{k/2}}{\ell_{\eta'}} \left| \frac{r - 1}{r} \right|^{k/4} e^{2\pi i n/\ell_{\eta'}} e \left( \frac{m}{2\ell_{\eta'}} \left[ \log \left| \frac{r - 1}{r} \right| + \pi i (1 - \alpha) \right] + \frac{n}{2\ell_{\eta'}} \left[ \log \left| \frac{r - 1}{r} \right| + \pi i (1 + \beta) \right] \right) \times B \left( \frac{k}{2} - \frac{2 \pi i n}{\ell_{\eta'}}, \frac{k}{2} + \frac{2 \pi i n}{\ell_{\eta'}} \right) \Gamma \left( \frac{k}{2} - \frac{2 \pi i n}{\ell_{\eta'}}, \frac{k}{2} + \frac{2 \pi i n}{\ell_{\eta'}} \right) \frac{1}{1 - r} \right).$$

(8.6)

Also, if $r \not\in (0, 1)$ or if $r \in (0, 1)$ and $\alpha = -1$

$$I_{\eta'}(m, n; r, \alpha, \beta) = \frac{\text{sgn}(r - 1)^{k/2}}{\ell_{\eta'}} \left| \frac{r - 1}{r} \right|^{k/4} e^{2\pi i n/\ell_{\eta'}} e \left( \frac{m}{2\ell_{\eta'}} \left[ \log \left| \frac{r - 1}{r} \right| + \pi i (1 - \alpha) \right] + \frac{n}{2\ell_{\eta'}} \left[ \log \left| \frac{r - 1}{r} \right| + \pi i (1 + \beta) \right] \right) \times B \left( \frac{k}{2} - \frac{2 \pi i n}{\ell_{\eta'}}, \frac{k}{2} + \frac{2 \pi i n}{\ell_{\eta'}} \right) \Gamma \left( \frac{k}{2} - \frac{2 \pi i n}{\ell_{\eta'}}, \frac{k}{2} + \frac{2 \pi i n}{\ell_{\eta'}} \right) \frac{1}{1 - r} \right).$$

(8.7)

Proof. Writing $u = t + i \pi/2$ and then $x = e^t$ in (8.1) gives

$$I_{\eta'}(m, n; r, \alpha, \beta) = \left| \frac{r - 1}{r} \right|^{\pi i n/\ell_{\eta'}} \int_0^{-\infty} \left( \frac{x - \text{sgn}(r-1) |x-\beta|^{1/2} i}{x} \frac{1}{x-\text{sgn}(r-1) |x-\beta|^{1/2} i} \right)^{2\pi i n/\ell_{\eta'}} \frac{x^{k-2\pi i n/\ell_{\eta'}-1}}{x^{k+2\pi i n/\ell_{\eta'}}} \frac{dx}{\ell_{\eta'}}. \right)$$

(8.8)

With $\alpha = \pm 1$ and $x > 0$ as above, we have

$$\left( \frac{x + i u}{x + i v} \right)^w = e^{\pi i w (1-\alpha)/2} (x + i u)^w (x + i v)^{-w} \quad (u, v \in \mathbb{R}, \ w \in \mathbb{C}) \right)$$

(8.9)

if $\alpha \frac{x + i u}{x + i v} \in \mathbb{H}$ since $|\arg(x + i u)|, |\arg(x + i v)| < \pi/2$. We may apply the identity (8.9) to (8.8) since the quotient to be exponentiated is in $\mathbb{H}$, as can be verified by a direct check or by noting that it originates as $\sigma_{\eta}^{-1} \gamma_{\eta'} z$ in the proof of Theorem 8.15.

Therefore

$$I_{\eta, \eta'}(m, n; r, \alpha, \beta) = \frac{(-i a) k/2}{\ell_{\eta'}} e^{\pi i n/\ell_{\eta'}} e \left( \frac{m}{2\ell_{\eta'}} \left[ \log \left| \frac{r - 1}{r} \right| + \pi i (1 - \alpha) \right] \right) \times \int_0^{-\infty} (x + a)^{-k/2+2\pi i n/\ell_{\eta'}} (x + b)^{-k/2-2\pi i n/\ell_{\eta'}} x^{k-2\pi i n/\ell_{\eta'}-1} dx \right)$$

(8.10)

for $a = -\alpha \text{sgn}(r-1) |x-\beta|^{1/2} i$ and $b = -\beta |x-\beta|^{1/2} i$. The evaluation of the integral in (8.10) has some subtleties so we give it in the following lemma. Recall that $2 F_1(a; b; c; 1 - z)$ is a multi-valued function of $z$ in general, and by convention we take the principal branch with $-\pi < \arg z \leq \pi$.

Lemma 8.3. Suppose $a, b \in \mathbb{C} \neq 0$ with $|\arg(a)|, |\arg(b)| < \pi$. For any $\mu, \rho, \nu \in \mathbb{C}$ with $0 < \Re(\nu) < \Re(\mu + \rho)$ we have

$$\int_0^{-\infty} (x + a)^{-\mu} (x + b)^{-\rho} x^{\nu-1} dx = B(\nu, \mu + \rho - \nu) \times \begin{cases} a^{\mu - (\mu + \rho)} 2 F_1(\rho, \mu + \rho - \nu; \mu + \rho; 1 - b/a) & \text{if } -\pi < \arg b - \arg a \leq \pi \\ b^{\mu - (\mu + \rho)} 2 F_1(\mu, \mu + \nu - \rho; \mu + \nu; 1 - a/b) & \text{if } -\pi < \arg a - \arg a < \pi \end{cases}$$

(8.11)

using the principal value of the hypergeometric function in (8.11).
Proof. From [GR07, 3.197.1] we have
\[
\int_0^\infty (x+a)^{-\mu} (x+b)^{-\nu} x^{\nu-1} dx = a^{-\mu} b^{-\nu} \frac{\Gamma(\mu, \mu + \rho - \nu) \Gamma(\nu, \mu + \rho; 1 - b/a)}{\Gamma(\mu + \rho; 1 - b/a)}.
\] (8.12)

If \(\arg b - \arg a \in (-\pi, \pi]\) then the right side of (8.12) requires the principal branch of \(\Gamma_1\). For \(\arg b - \arg a \not\in (-\pi, \pi]\) we require values of \(\Gamma_1\) on the branch reached by crossing the branch-cut from above or below. Applying the Pfaff transformation [AAR99, Thm. 2.2.5] to \(\Gamma_1\) converts (8.12) into
\[
a^{\nu - (\mu + \rho)} B(\mu, \mu + \rho - \nu) \Gamma_1(\mu, \mu + \rho - \nu; \mu + \rho; 1 - b/a) \quad (8.13)
\]
or
\[
b^{\nu - (\mu + \rho)} B(\mu, \mu + \rho - \nu) \Gamma_1(\mu, \mu + \rho - \nu; \mu + \rho; 1 - a/b) \quad (8.14)
\]
by switching \(a\) and \(b\). Clearly we remain in the principal branch of \(\Gamma_1\) in (8.13) for \(-\pi < \arg b - \arg a \leq \pi\) and the principal branch of \(\Gamma_1\) in (8.14) for the overlapping range \(-\pi < \arg a - \arg b \leq \pi\). This proves the lemma.

In our case, with \(a\) and \(b\) given after (8.10), we have \(\arg a, \arg b = \pm \pi/2\). Therefore, \(-\pi < \arg b - \arg a \leq \pi\) unless \(\beta = 1\) and \(\alpha \mathrm{sgn}(r - 1) = -1\) which is equivalent to
\[
\alpha = \beta = -\mathrm{sgn}(r - 1) = 1,
\] (8.15)
since it is not possible to have \(-\alpha = \beta = \mathrm{sgn}(r - 1) = 1\). Note that (8.15) implies \(r\) is in the interval \((0, 1)\). Hence we have \(-\pi < \arg b - \arg a \leq \pi\) if \(r \not\in (0, 1)\) or if \(r \in (0, 1)\) and \(\alpha = -1\). In this case we may evaluate the integral in (8.10) using the top option in (8.11), with for example
\[
a^{\nu - (\mu + \rho)} = (-i\alpha)^{-1/2} \mathrm{sgn}(r - 1)^{1/2} \frac{r}{r - 1}^{(1/2)} \left( \frac{n}{2\ell} \left[ \log \left| \frac{r}{r - 1} \right| + \pi \mathrm{sgn}(r - 1) \right] \right).
\]
The result is (8.7). Similarly, if \(r \not\in (0, 1)\) or if \(r \in (0, 1)\) and \(\alpha = 1\), we may evaluate the integral in (8.10) using the bottom option in (8.11) and the result is (8.6).

8.2 Double cosets in the hyperbolic/hyperbolic case

We need some preliminary material to understand the double cosets appearing in the Kloosterman sum \(S_{\eta\eta^*}\). Let \(L\) be a complete set of inequivalent representatives for \(\Gamma_\eta \backslash \Gamma / \Gamma_{\eta^*}\). Partition \(L\) into two subsets:
\[
\Gamma(\eta, \eta') := \left\{ \delta \in L \mid \delta \eta' = \eta \text{ or } \eta' \right\}, \quad \Gamma(\eta, \eta') := \left\{ \delta \in L \mid \delta \eta' \neq \eta \text{ or } \eta' \right\}.
\]

Lemma 8.4. There exist \(a, b \in \mathbb{R}_{\neq 0}\) such that \(\Gamma(\eta, \eta')_0\) is a subset of
\[
\left\{ \sigma_\eta \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \sigma_{\eta^*}^{-1}, \sigma_\eta \left( \begin{array}{cc} 0 \quad b \\ -a & 0 \end{array} \right) \sigma_{\eta^*}^{-1} \right\}.
\] (8.16)

Then \(\Gamma(\eta, \eta')_0\) contains the first element of (8.16) if \(\eta' \equiv \eta \mod \Gamma\) and the second if \(\eta' \equiv \eta^* \mod \Gamma\). The numbers \(a\) and \(b\) depend on the choice of the scaling matrices \(\sigma_\eta\) and \(\sigma_{\eta^*}\).

Proof. If \(\Gamma(\eta, \eta')_0\) contains \(\delta\) and \(\tau\) such that \(\delta \eta' = \eta\) and \(\tau \eta' = \eta\) then \(\tau^{-1} \delta \in \Gamma_{\eta^*}\) and hence \(\tau = \delta\). Similarly, if \(\gamma, \tau \in \Gamma(\eta, \eta')_0\) with \(\gamma \eta' = \eta^*\) and \(\tau \eta' = \eta^*\) then we must have \(\tau = \gamma\) also. Therefore \(\Gamma(\eta, \eta')_0\) contains at most one element \(\delta\) satisfying \(\delta \eta' = \eta\) and at most one \(\gamma\) satisfying \(\gamma \eta' = \eta^*\). If \(\Gamma(\eta, \eta')_0\) contains such a \(\delta\) and such a \(\gamma\) then they must be distinct since \(\eta \neq \eta^*\). The computations \(\sigma_\eta^{-1} \delta \sigma_{\eta^*} = 0\) and \(\sigma_\eta^{-1} \delta \sigma_{\eta^*} = \infty\) show \(\delta\) takes the form of the first element of (8.16) and similarly for \(\gamma\) taking the form of the second element.

It may be shown that
\[
\ell_\eta = \ell_{\eta^*} \quad \text{if} \quad \eta' \equiv \eta \mod \Gamma \quad \text{or} \quad \eta' \equiv \eta^* \mod \Gamma.
\] (8.17)

Then we see that the effect of a different choice of \(L\) on the \(a\) and \(b\) in Lemma [8.4] is multiplication by a factor of the form \(\pm e^{m\ell_\eta/2}\) for \(m \in \mathbb{Z}\). In other words, for fixed scaling matrices \(\sigma_\eta\) and \(\sigma_{\eta^*}\), the sets \(\log(a^2) + \ell_\eta \mathbb{Z}\) and \(\log(b^2) + \ell_\eta \mathbb{Z}\) are independent of \(L\).
Proposition 8.5. With the above notation,
\[
\Gamma(\eta, \eta')_0 \cup \left\{ \gamma \tau \mid \gamma \in \Gamma(\eta, \eta'), \tau \in \Gamma_{\eta'}/Z \right\}
\] (8.18)
is a complete set of inequivalent representatives for $\Gamma_{\eta}\backslash \Gamma$.

Proof. The set $L' := \{ \delta \tau \mid \delta \in L, \tau \in \Gamma_{\eta'}/Z \}$ clearly gives a complete set of representatives for $\Gamma_{\eta}\backslash \Gamma$. To see which of its elements are equivalent modulo $\Gamma_\eta$, suppose

\[
\Gamma_\eta \delta \tau = \Gamma_\eta \delta' \tau' \quad \text{for} \quad \delta, \delta' \in L \quad \text{and} \quad \tau, \tau' \in \Gamma_{\eta'}/Z.
\] (8.19)

Arguing as in Proposition 3.3, we must have $\eta$ fixes $\eta'$. This can happen if $\gamma = \pm I$, in which case $\tau = \tau'$. The other possibility is that $\delta \eta' = \eta$ or $\eta^*$. In these cases $\Gamma_\eta \delta \tau = \Gamma_\eta \delta' \tau' = \Gamma_\eta \delta$. Hence, with (8.18), we have removed all of the equivalent elements from the set $L'$ we started with. $\blacksquare$

To give another characterization of the sets $\Gamma(\eta, \eta')_0$ and $\Gamma(\eta, \eta')$, we first prove the following two results, contained in [Goo83, Lemma 6 (iii)].

Lemma 8.6. Suppose $\gamma$ and $\delta$ are hyperbolic elements of $\Gamma$ and that $\gamma$ fixes $\eta_1$, $\eta_2$ while $\delta$ fixes $\eta_3$, $\eta_2$. Then $\eta_1 = \eta_3$.

Proof. Suppose $\eta_2 \neq \eta_3$ and let $\eta = (\eta_1, \eta_2)$. We have $\sigma_\eta^{-1} \gamma \sigma_\eta = \left( \begin{array}{cc} u & 0 \\ 0 & 1/u \end{array} \right)$ and $\sigma_\eta^{-1} \delta \sigma_\eta = \left( \begin{array}{cc} v & w \\ 0 & 1/v \end{array} \right)$ for some $u, v, w$ in $\mathbb{R}_{\neq 0}$. Then $\sigma_\eta^{-1} \gamma^k \gamma \sigma_\eta^{-k} \sigma_\eta = \left( \begin{array}{cc} v & u \cdot w \\ 0 & 1/v \end{array} \right)$ for $k \in \mathbb{Z}$. Applying these elements to $i \in \mathbb{H}$ gives $v^2 i + u \cdot w \cdot i^k$, with infinitely many points contained in a compact neighborhood of $v^2 i$. But this is impossible since $\sigma_\eta^{-1} \Gamma \sigma_\eta$ is a discrete group. Hence we must have $\eta_2 = \eta_3$. $\blacksquare$

Lemma 8.7. For $\delta \in \Gamma$, write $\sigma_\eta^{-1} \delta \sigma_{\eta'} = (a \ b \ c \ d)$. Then

\[
\begin{align*}
\text{b = 0 or } c = 0 & \iff b = c = 0 \iff \delta \eta' = \eta & \quad (8.20) \\
\text{a = 0 or } d = 0 & \iff a = d = 0 \iff \delta \eta' = \eta^* & \quad (8.21)
\end{align*}
\]

Proof. Write $\eta = (\eta_1, \eta_2)$, $\eta' = (\eta'_1, \eta'_2)$ and let $\gamma \in \Gamma_\eta$ and $\gamma' \in \Gamma_{\eta'}$. Suppose that $b = 0$. This implies

\[
\sigma_\eta^{-1} \delta \sigma_{\eta'} = 0 \implies \delta \eta' = \eta_1 \implies (\delta \gamma \delta^{-1}) \eta_1 = \eta_1.
\]

Since $\delta \gamma' \delta^{-1}$ and $\gamma$ both fix $\eta_1$, they must both fix $\eta_2$ by Lemma 8.6. It follows that $\delta \eta' = \eta$ and $c = 0$. Similarly, starting with $c = 0$ instead of $b = 0$ we also find that $\delta \eta' = \eta$ and $b = 0$.

Conversely, if $\delta \eta' = \eta$ then $\sigma_{\eta'}$ must be of the form $\delta^{-1} \sigma_\eta \left( \begin{array}{cc} t & 0 \\ 0 & 1/t \end{array} \right)$ for some $t \in \mathbb{R}_{\neq 0}$. Hence $\sigma_\eta^{-1} \delta \sigma_{\eta'}$ has $b = c = 0$. This finishes the proof of (8.20).

If $a = 0$ or $d = 0$ or $\delta \eta' = \eta^*$ we may choose $\sigma_{(\eta')^*} = \sigma_{\eta'} S$. Applying (8.20) to

\[
\sigma_\eta^{-1} \delta \sigma_{(\eta')^*} = \sigma_\eta^{-1} \delta \sigma_{\eta'} S = (a' \ b' \ c' \ d')
\]

implies

\[
\text{b' = 0 or c' = 0} \iff b' = c' = 0 \iff \delta (\eta')^* = \eta
\]

which is equivalent to (8.21). $\blacksquare$

The next corollary follows directly.

Corollary 8.8. We have

\[
\Gamma(\eta, \eta'_0) = \left\{ \delta \in L \mid \sigma_\eta^{-1} \delta \sigma_{\eta'} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \text{ with } abcd = 0 \right\},
\]

\[
\Gamma(\eta, \eta') = \left\{ \delta \in L \mid \sigma_\eta^{-1} \delta \sigma_{\eta'} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \text{ with } abcd \neq 0 \right\}.
\]
Good’s decomposition of $\text{SL}_2(\mathbb{R})$ in this hyperbolic/hyperbolic case, see [Goo83 Lemma 1] and [Goo85 Lemma 1], says the following.

**Lemma 8.9.** Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$.

(i) When $|ad| + |bc| \neq 1$ we have

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\text{sgn}(a)}{2} \begin{pmatrix} |ab|/c & 0 \\ |cd|/c & 0 \end{pmatrix} \begin{pmatrix} 1 & -\text{sgn}(ac) \\ \text{sgn}(ac) & 1 \end{pmatrix} \times \begin{pmatrix} \nu & 0 \\ 0 & 1/\nu \end{pmatrix} \begin{pmatrix} 1 & \text{sgn}(cd) \\ -\text{sgn}(cd) & 1 \end{pmatrix} \begin{pmatrix} |bd|/c & 0 \\ 0 & |bd|/c \end{pmatrix} \tag{8.22}
$$

for $\nu = \text{hyp}_\text{hyp}(M) = |ad|^{1/2} + |bc|^{1/2}$.

(ii) When $|ad| + |bc| = 1$ and $abcd \neq 0$ we have

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\text{sgn}(c) \begin{pmatrix} |ab|/c & 0 \\ |cd|/c & 0 \end{pmatrix} \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix} \begin{pmatrix} |bd|/c & 0 \\ 0 & |bd|/c \end{pmatrix} \tag{8.23}
$$

for $\theta = \theta(M) = 2 \cos^{-1}(-\text{sgn}(ac)|ad|^{1/2})$ and $0 < \theta < 2\pi$.

**Proof.** Let $r = ad$ so that $bc = r - 1$ and $|ad| + |bc| \neq 1$ is equivalent to $r \notin [0, 1]$. The identity $\text{(8.22)}$ in (i) follows from a direct calculation, reducing to $|r - 1| + |bc| = |r|$ or $|r - 1| + |bc(r - 1)| = |r|$. Part (ii) corresponds to $r \in (0, 1)$ and may be easily verified also.

Based on the above decomposition we define

$$R_{\eta \eta'} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_\eta^{-1} \Gamma \sigma_{\eta'} \bigg| abcd \neq 0, \quad \frac{1}{\epsilon_\eta} \leq \frac{|ab|}{\epsilon', |bd|} \leq \frac{1}{\epsilon_\eta'}, \quad \frac{1}{\epsilon_\eta} \leq \frac{|ab|}{|ac|} < \frac{1}{\epsilon_\eta'} \right\} \tag{8.24}
$$

and a similar proof to Lemma 4.6 shows the next result.

**Lemma 8.10.** We may take $\sigma_\eta^{-1} \Gamma(\eta, \eta') \sigma_{\eta'} = R_{\eta \eta'}/\mathbb{Z}$.

### 8.3 The hyperbolic/hyperbolic Kloosterman sum

Recall from (1.22) that $C_{\eta \eta'} = \{ \text{ad} \big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_\eta^{-1} \Gamma \sigma_{\eta'}, \quad abcd \neq 0 \}$. For $C \in C_{\eta \eta'}$ and $\alpha, \beta = \pm 1$ define

$$S_{\eta \eta'}(m, n; C, \alpha, \beta) := \sum_{\gamma \in \Gamma \setminus \Gamma_{\eta \eta'}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \sigma_\eta^{-1} \gamma \sigma_{\eta'}} e\left( \frac{m}{2\ell_\eta} \log \frac{|ab|}{|cd|} + \frac{n}{2\ell_{\eta'}} \log \frac{|ac|}{|bd|} \right) \tag{8.25}
$$

Then $S_{\eta \eta'}(m, n; C, \alpha, \beta)$ is related to Good’s generalized Kloosterman sum (2.1) by

$$S_{\eta \eta'}(m, n; C, \alpha, \beta) = \delta S_{\eta \eta'}^{\delta}(m, n; |C|^{1/2} + |C - 1|^{1/2}) \quad \text{for} \quad \delta = \frac{1 - \alpha}{2}, \quad \delta' = \frac{1 + \beta}{2} \tag{8.26}
$$

when $C$ is not in the interval $(0, 1)$. For $C \in (0, 1)$, Good made the right side of (8.26) zero and treated this case separately with another sum: $S_{\eta \eta'}^{(\delta)}(m, n; \theta)$ for $\theta$ as in (8.23). See [Goo83 (5.11)].

To show that (8.25) is a finite sum, and to bound it, we start with the following analog of [Shi71] Lemma 1.24.

**Lemma 8.11.** Given $M > 0$, there are only finitely many double cosets $\Gamma \gamma \Gamma$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma_\eta^{-1} \gamma \sigma_{\eta'}$ has $|abcd| \leq M$. Note that $|abcd|$ is independent of the double coset representative and also the choice of scaling matrices.
Proof. There are at most two double cosets with $abcd = 0$ by Lemma [8.4] and Corollary [8.5]. Assume now that $abcd \neq 0$. Since $\left\{ \left( \frac{e^m}{e^{-m}} \right) \mid m \in \mathbb{Z} \right\} \subseteq \sigma^{-1}_\eta \Gamma \eta \sigma_\eta$ and $\left\{ \left( \frac{e^n}{e^{-n}} \right) \mid n \in \mathbb{Z} \right\} \subseteq \sigma^{-1}_{\eta'} \Gamma \eta' \sigma_{\eta'}$ we may choose representatives $\delta$ for $\Gamma \eta \gamma \Gamma \eta'$ satisfying

$$\sigma^{-1}_\eta \delta \sigma_{\eta'} = \left( \begin{array}{ccc} \frac{e^m}{e^{-m}} & 0 & 0 \\ 0 & \frac{e^m}{e^{-m}} & 0 \\ \frac{e^n}{e^{-n}} & 0 & \frac{e^n}{e^{-n}} \end{array} \right)$$

so that

$$\sigma^{-1}_\eta \delta \sigma_{\eta'} = \varepsilon_{\eta'}^{2m} a \varepsilon_{\eta'}^{2n} i + b \varepsilon_{\eta'}^{2m} c \varepsilon_{\eta'}^{2n} i + d$$

We will show that distinct double cosets satisfying the statement of the lemma give distinct elements in the discrete group $\sigma^{-1}_\eta \Gamma \eta$ mapping $z_0 := \sigma^{-1}_\eta \sigma_{\eta'} i \in \mathbb{H}$ into a compact set $K \subset \mathbb{H}$ of the form

$$K = \left\{ re^{i\theta} \mid 1 \leq r \leq \varepsilon_{\eta'}^2, \theta_1 \leq \theta \leq \theta_2 \right\}$$

with $\theta_1, \theta_2$ depending only on $\eta'$ and $M$. This forces the number of double cosets to be finite.

Choose $n \in \mathbb{Z}$ so that $\lambda := \varepsilon_{\eta'}^{2n}$ satisfies

$$\frac{bd}{ac}^{1/2} \leq \lambda < \frac{bd}{ac}^{1/2} \varepsilon_{\eta'}^2. \quad \text{(8.27)}$$

We have

$$\arg \left( \frac{a\lambda i + b}{c\lambda i + d} \right) = \arg (ac\lambda + bd/\lambda + i)$$

and, using (8.27),

$$|ac\lambda + bd/\lambda| \leq M^{1/2} (1 + \varepsilon_{\eta'}^2).$$

Hence $\arg(\sigma^{-1}_\eta \delta \sigma_{\eta} z_0)$ is bounded between constants $\theta_1, \theta_2$ that depend only on $\eta'$ and $M$. Choose $m \in \mathbb{Z}$ so that $1 \leq |\sigma^{-1}_\eta \delta \sigma_{\eta} z_0| < \varepsilon_{\eta}$ and $\sigma^{-1}_\eta \delta \sigma_{\eta} z_0$ is contained in the compact set $K$ as required.

Corollary 8.12. Given two hyperbolic fixed point pairs $\eta$ and $\eta'$ for $\Gamma$, there exists $M_{\eta\eta'} > 0$ with the following properties. For all $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \sigma^{-1}_\eta \Gamma \eta'$ we have

(i) $|bc| \geq M_{\eta\eta'}$, if $bc \neq 0$,

(ii) $|ad| \geq M_{\eta\eta'}$, if $ad \neq 0$,

(iii) $|abcd| \geq M^2_{\eta\eta'}$, if $abcd \neq 0$.

Proof. Consider a double coset $\Gamma \eta \gamma \Gamma \eta'$ with $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = \sigma^{-1}_\eta \gamma \sigma_{\eta'}$. If $\delta \in \Gamma \eta \gamma \Gamma \eta'$ has $(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}) = \sigma^{-1}_\eta \delta \sigma_{\eta'}$ then $b'c' = bc$. So distinct values of $bc$ correspond to different double cosets. Take any $N > 0$ and we want to examine the possible values for $|bc| \in [0, N]$ where $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \sigma^{-1}_\eta \Gamma \eta'$. Clearly $|abcd| = |bc(bc + 1)| \leq N(N + 1)$. It follows from Lemma [8.11] that there are only finitely many values for $|bc| \in [0, N]$. Hence the nonzero ones are bounded from below, proving part (i). The proof of (ii) is similar and we may take $M_{\eta\eta'}$ as the smaller of the two lower bounds. Part (iii) is a consequence of (i) and (ii).

We next set

$$\mathcal{N}_{\eta\eta'}(C) := \# \left\{ \gamma \in \Gamma \eta \Gamma \eta' \mid \sigma^{-1}_\eta \gamma \sigma_{\eta'} = \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \text{ with } ad = C \right\}.$$ 

Then $\mathcal{N}_{\eta\eta'}(C)$ is well defined and independent of the scaling matrices $\sigma_\eta$ and $\sigma_{\eta'}$. It bounds the number of terms in the sum (8.25), though at the outset it may be infinite.
Proposition 8.13. With the above notation

$$\sum_{C \in C_{\eta \eta'}} N_{\eta \eta'}(C) \ll X^{3/2}.$$  

Proof. We may write $N_{\eta \eta'}(C)$ more explicitly as $\# \left\{ \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \in R_{\eta \eta'} / Z \mid ad = C \right\}$. Also let

$$R(X) := \left\{ \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \in R_{\eta \eta'} \mid |ad| \leq X \right\} \subset \sigma_{\eta}^{-1} \Gamma_{\sigma_{\eta'}}.$$  

Suppose $\gamma = \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right)$ and $\delta = \left( \begin{array}{c} a' \\ b' \\ c' \\ d' \end{array} \right)$ are in $R(X)$. Then $\gamma \delta^{-1} = \left( \begin{array}{c} a'' \\ b'' \\ c'' \\ d'' \end{array} \right) \in \sigma_{\eta}^{-1} \Gamma_{\sigma_{\eta}}$ for

$$|b'' c''| = \left| dd'' a a' \left( \frac{c}{d} - \frac{c'}{d'} \right) \left( \frac{b}{a} - \frac{b'}{a'} \right) \right|.$$  

If $b'' c'' = 0$ then $\gamma \delta^{-1} \in \sigma_{\eta}^{-1} \Gamma_{\sigma_{\eta}}$ by Lemma 8.7 and so we must have $\gamma = \delta$. Otherwise, it follows that $|b'' c''| \geq M_{\eta \eta'} > 0$ by Corollary 8.12. Hence

$$\left| \frac{c}{d} - \frac{c'}{d'} \right| \left| \frac{b}{a} - \frac{b'}{a'} \right| \geq M_{\eta \eta'}/X^2. \quad (8.28)$$  

We next determine how large $|\frac{c}{d}|$ and $|\frac{b}{a}|$ can be for $\left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \in R(X)$. Combining the inequalities in the definition (8.24) implies

$$\frac{1}{\varepsilon_{\eta \eta'}} \leq \left| \frac{b}{c} \right| \leq \varepsilon_{\eta \eta'}, \quad \frac{1}{\varepsilon_{\eta \eta'}} \leq \left| \frac{a}{d} \right| \leq \varepsilon_{\eta \eta'}. \quad (8.29)$$  

We also know that

$$M_{\eta \eta'} \leq |bc| \leq X + 1, \quad M_{\eta \eta'} \leq |ad| \leq X. \quad (8.30)$$  

Together (8.29) and (8.30) prove

$$\left| \frac{c}{d} \right| , \left| \frac{b}{a} \right| \leq \frac{\varepsilon_{\eta \eta'}(X + 1)^{1/2}}{M_{\eta \eta'}^{1/2}}. \quad (8.31)$$  

Use (8.31) in (8.28) to bound $\left| \frac{b}{a} - \frac{b'}{a'} \right|$ and show

$$\left| \frac{c}{d} - \frac{c'}{d'} \right| \geq \frac{M_{\eta \eta'}/X^2 \cdot M_{\eta \eta'}^{1/2}}{2 \varepsilon_{\eta \eta'}(X + 1)^{1/2}}. \quad (8.32)$$  

for any two distinct $\left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right)$ and $\left( \begin{array}{c} a' \\ b' \\ c' \\ d' \end{array} \right)$ in $R(X)$. Since we have seen in (8.31) that $c/d$ is restricted to a finite interval, it follows from (8.32) that $R(X)$ has a finite number of elements, say $n$. List the corresponding fractions as $c_1/d_1 < c_2/d_2 < \cdots < c_n/d_n$. Then

$$\sum_{j=1}^{n-1} \left| \frac{c_{j+1}}{d_{j+1}} - \frac{c_j}{d_j} \right| = \sum_{j=1}^{n-1} \left( \frac{c_{j+1}}{d_{j+1}} - \frac{c_j}{d_j} \right) \leq 2 \varepsilon_{\eta \eta'}(X + 1)^{1/2}/M_{\eta \eta'}^{1/2} \quad (8.33)$$  

using (8.31). With (8.33) and the inequality of the arithmetic and geometric means, we have

$$\prod_{j=1}^{n-1} \frac{c_{j+1}}{d_{j+1}} - \frac{c_j}{d_j} \ll \left( \frac{(X + 1)^{1/2}}{n - 1} \right)^{n-1} \quad (8.34)$$  

and the same bound holds for $\prod_{j=1}^{n-1} \left| \frac{b_{j+1}}{a_{j+1}} - \frac{b_j}{a_j} \right|$ by a similar argument. Combining these bounds with (8.28) shows

$$\left( \frac{1}{X^2} \right)^{n-1} \ll \left( \frac{X + 1}{(n - 1)^2} \right)^{n-1}$$  

and therefore $n \ll X^{3/2}$, as desired. \(\Box\)

Corollary 8.14. For implied constants depending only on $\Gamma$, $\eta$ and $\eta'$

$$N_{\eta \eta'}(C) \ll C^{3/2}, \quad S_{\eta \eta'}(m, n; C) \ll C^{3/2}, \quad \# \{ C \in C_{\eta \eta'} : |C| \leq X \} \ll X^{3/2}.$$
8.4 The hyperbolic expansion of $P_{\eta,m}$

**Theorem 8.15.** Recall the numbers $a$ and $b$ from Lemma 8.4. For $m, n \in \mathbb{Z}$, the $n$th hyperbolic Fourier coefficient at $\eta'$ of the hyperbolic Poincaré series $P_{\eta,m}$ is given by

$$c_{\eta'}(n; P_{\eta,m}) = \sum_{C \in C_{\eta', \alpha, \beta}^+} I_{\eta'}(m, n; C, \alpha, \beta) \frac{S_{\eta'}(m, n; C, \alpha, \beta)}{|C(C - 1)|^{k/4}}$$

$$+ \left\{ \begin{array}{ll}
(a^2)^{2\pi im/\ell_{\eta'}} & \text{if } \eta' \equiv \eta \mod \Gamma \text{ and } n = m,
\end{array} \right. \quad (8.35)$$

$$+ \left\{ \begin{array}{ll}
(-1)^{k/2}e^{2\pi in/\ell_{\eta'}}(b^2)^{-2\pi im/\ell_{\eta'}} & \text{if } \eta' \equiv \eta^* \mod \Gamma \text{ and } n = -m.
\end{array} \right. \quad (8.36)$$

**Proof.** We have

$$(P_{\eta,m}|_{k\sigma_{\eta'}})(z) = \sum_{\gamma \in \Gamma(n, \eta')} (\sigma_{\eta}^{-1}\gamma\sigma_{\eta'}z)^{-k/2+2\pi im/\ell_{\eta}}$$

$$\cdot j(\sigma_{\eta}^{-1}\gamma\sigma_{\eta'}, z)^k$$

which is absolutely convergent for $z$ in $\mathbb{H}$ and $k > 2$. We use the set of representatives for $\Gamma(n, \eta') \setminus \Gamma$ given by Proposition 8.5. The elements of $\Gamma(n, \eta')_0$, as described in Lemma 8.4, easily yield the contributions (8.35) and (8.36) – using (8.17) and for (8.36) that $(-1/z)^s = e^{s\pi i}z^{-s}$ for all $z \in \mathbb{H}$ and $s \in \mathbb{C}$.

Write the remaining terms in (8.37) as

$$\sum_{\gamma \in \Gamma(n, \eta')} \sum_{\tau \in \Gamma(n, \eta')/Z} \sum_{\gamma \in \Gamma(n, \eta')} (\sigma_{\eta}^{-1}\gamma\tau\sigma_{\eta'}z)^{-k/2+2\pi im/\ell_{\eta}}$$

$$\cdot j(\sigma_{\eta}^{-1}\gamma\tau\sigma_{\eta'}, z)^k$$

$$= \sum_{\gamma \in \Gamma(n, \eta')} \sum_{\tau \in \Gamma(n, \eta')/Z} \sum_{\gamma \in \Gamma(n, \eta')} \sum_{\tau \in \Gamma(n, \eta')/Z} \frac{(\sigma_{\eta}^{-1}\gamma\tau\sigma_{\eta'}z)^{-k/2+2\pi im/\ell_{\eta}}}{j(\sigma_{\eta}^{-1}\gamma\tau\sigma_{\eta'}, z)^k}$$

The inner series is

$$\sum_{n \in \mathbb{Z}} \sum_{\sigma_{\eta}^{-1}\gamma\sigma_{\eta'}(e^{n\ell_{\eta'}}z)}^{-k/2+2\pi im/\ell_{\eta}} \cdot j(\sigma_{\eta}^{-1}\gamma\sigma_{\eta'}, e^{n\ell_{\eta'}}z)^k e^{-n\ell_{\eta'}k/2}$$

$$= \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{(\sigma_{\eta}^{-1}\gamma\sigma_{\eta'}(e^{\ell_{\eta'}t+A}))^{-k/2+2\pi im/\ell_{\eta}}}{j(\sigma_{\eta}^{-1}\gamma\sigma_{\eta'}, e^{\ell_{\eta'}t+A})^k} e^{\ell_{\eta'}tk/2-2\pi int} dt$$

where $z = e^A$ for $0 < \text{Im}A < \pi$ and we used Poisson summation which may be justified as in Theorem 6.4. Here $\sigma_{\eta}^{-1}\gamma\sigma_{\eta'} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ with $abcd \neq 0$ and the integral in (8.39) equals

$$\int_{-\infty}^{\infty} \frac{(ae^{\ell_{\eta'}t+A}+b)^{2\pi im/\ell_{\eta}}}{(ce^{\ell_{\eta'}t+A}+d)^{2\pi im/\ell_{\eta}}} e^{\ell_{\eta'}tk/2-2\pi int} dt.$$ \hspace{1cm} (8.40)

Substitute $u = \ell_{\eta'}t + A + \frac{1}{2} \log \left| \frac{ad}{bc} \right|$ and (8.40) equals

$$z^{-k/2+2\pi im/\ell_{\eta'}} \left| \frac{ac}{bd} \right|^{-k/4} e^{\left( \frac{m}{2\ell_{\eta'}} \log \left| \frac{ab}{cd} \right| + \frac{n}{2\ell_{\eta'}} \log \left| \frac{ac}{bd} \right| \right)}$$

$$\times \int_{-\infty+\text{Im}A}^{\infty+\text{Im}A} \left( \frac{cd}{ab} \right)^{1/2} e^{u} \left( \frac{ac}{bd} \right)^{1/2} e^{u} \left( \frac{a}{b} \right)^{1/2} e^{u+b} \frac{2\pi im/\ell_{\eta}}{e^{u(k-2\pi im/\ell_{\eta'})}} \frac{du}{\ell_{\eta'}}.$$ \hspace{1cm} (8.41)
The integrand is holomorphic for $0 < \text{Im} u < \pi$ and therefore independent of $\text{Im} A$ provided $0 < \text{Im} A < \pi$. The equalities
\[
a \frac{|bd|}{|ac|}^{1/2} e^u + b = \left| \frac{b}{e} \right|^{1/2} |ad|^{1/2} \left( \text{sgn}(a) e^u + \text{sgn}(b) \left| \frac{bc}{ad} \right|^{1/2} \right),
\]
\[
c \frac{|bd|}{|ac|}^{1/2} e^u + d = \left| \frac{d}{a} \right|^{1/2} |bc|^{1/2} \left( \text{sgn}(c) e^u + \text{sgn}(d) \left| \frac{ad}{bc} \right|^{1/2} \right)
\]
show the integral in (8.41) is
\[
\left| \frac{ac}{bd} \right|^{k/4} \frac{1}{|abcd|^{k/4}} \int_{-\infty+iy}^{\infty+iy} \frac{1}{2\pi i} e^{u(k/2 - 2\pi in/\ell_{\eta'})} \frac{du}{\ell_{\eta'}}
\]
for any $y$ with $0 < y < \pi$. Finally, multiplying through by $\text{sgn}(c)$, (8.41) is now
\[
z^{-k/2 + 2\pi in/\ell_{\eta'}} e \left( \frac{m}{2\ell_{\eta}} \log \left| \frac{ab}{cd} \right| + \frac{n}{2\ell_{\eta'}} \log \left| \frac{ac}{bd} \right| \right) \frac{I_{\eta'}(m, n; C, \text{sgn}(ac), \text{sgn}(cd))}{|C(C-1)|^{k/4}}.
\]

Hence (8.38) is
\[
\sum_{C \in C_{\eta'}} \sum_{\alpha, \beta = \pm 1} \sum_{\gamma \in \Gamma(\eta', \eta)} \sum_{\alpha \beta \gamma = \eta} \sum_{n \in \mathbb{Z}} z^{-k/2 + 2\pi in/\ell_{\eta'}}
\times e \left( \frac{m}{2\ell_{\eta}} \log \left| \frac{ab}{cd} \right| + \frac{n}{2\ell_{\eta'}} \log \left| \frac{ac}{bd} \right| \right) \frac{I_{\eta'}(m, n; C, \alpha, \beta)}{|C(C-1)|^{k/4}}.
\]

With Proposition 8.1, we have

\[
z^{-k/2 + 2\pi in/\ell_{\eta'}} I_{\eta'}(m, n; C, \alpha, \beta) \ll e^{-\varepsilon |n|}
\]

for $\varepsilon > 0$ depending on $z$. Therefore (8.44) is majorized by a constant time $\sum_{C \in C_{\eta'}} |C|^{-k/2} N_{\eta'}(C)$ and thus convergent for $k > 3$ by Proposition 8.13. This proves that changing the order of summation in (8.44) is valid. Rearranging completes the proof.

**Proof of Theorem 1.4.**

Set
\[
S_{\eta'}^*(m, n; C, \alpha) := S_{\eta'}(m, n; C, \alpha, \beta)
\times e \left( \frac{m}{2\ell_{\eta}} \left[ \log \left| \frac{C}{C-1} \right| + \pi i (1 - \alpha) \right] + \frac{n}{2\ell_{\eta'}} \left[ \log \left| \frac{C-1}{C} \right| + \pi i (1 + \beta) \right] \right)
\]
for $\beta = \alpha \text{sgn}(C)$ and this agrees with our earlier definition (1.23). Combining Theorem 8.15 with Proposition 8.2 and (8.45) gives Theorem 1.4.

By choosing the scaling matrices $\sigma_{\eta}$ and $\sigma_{\eta'}$ we can make $a$ and $b$ in the statements of Theorems 1.4 and 8.15 explicit as follows.

**Proposition 8.16.** (i) Suppose $\eta$ and $\eta^*$ are not $\Gamma$-equivalent. If $\eta' = \rho \eta$ for some $\rho \in \Gamma$ put $\sigma_{\eta'} := \rho \sigma_{\eta}$ and if $\eta' = \rho \eta^*$ for some $\rho \in \Gamma$ put $\sigma_{\eta'} := \rho \sigma_{\eta} S$. In this case (8.35), (8.36) become

\[
+ \begin{cases} 1 & \text{if } \eta' \equiv \eta \mod \Gamma \text{ and } n = m, \\ (-1)^{k/2} e^{2\pi n/\ell_{\eta'}} & \text{if } \eta' \equiv \eta^* \mod \Gamma \text{ and } n = -m. \end{cases}
\]
(ii) Suppose $\eta$ and $\eta^*$ are $\Gamma$-equivalent. If $\tau\eta^* = \eta$ for $\tau \in \Gamma$. It follows that $\sigma_{\eta}^{-1} \tau \sigma_{\eta} = \left(\begin{array}{c} \frac{e}{f} \\ -\frac{b}{a} \end{array}\right)$ for some $t \in \mathbb{R}_{\geq 0}$. If $\eta' \equiv \eta \mod \Gamma$ with $\eta' = \rho \eta$, choose $\sigma_{\eta'} = \rho \sigma_{\eta}$. Then (8.35), (8.36) become

$$
\begin{align*}
&+ \begin{cases} 1 \text{ if } \eta' \equiv \eta \mod \Gamma \text{ and } n = m, \\
+ \left(-\frac{b}{a}\right) \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix} \end{cases}
\end{align*}
$$

(8.48)

and (8.36) follows from (8.35). To show (8.49), we note that $\gamma \rho \tau \eta^* = \eta^*$ implying $\gamma \rho \tau \in \Gamma$. Hence

$$
\begin{align*}
&= \sigma_{\eta}^{-1} \gamma \rho \tau \sigma_{\eta} \\
&= \sigma_{\eta}^{-1} \gamma \rho \tau \sigma_{\eta} \in \sigma_{\eta}^{-1} \Gamma \eta \sigma_{\eta}
\end{align*}
$$

(8.49)

so that $b^2 = t^2 e^{r \ell_n}$ for some $r \in \mathbb{Z}$. Then (8.49) follows from (8.36). The proof of part (i) is similar.

With (1.11), the identity $(P_{\eta'};n,P_{\eta,m}) = (P_{\eta,m},P_{\eta';n})$ implies

$$
c_{\eta'}(n,P_{\eta,m}) \frac{\ell_{\eta'} e^{-2\pi n / \ell_{\eta'}}}{|\Gamma(\frac{k}{2} + \frac{2\pi m}{\ell_{\eta'}})|^2} = c_{\eta}(m,P_{\eta';n}) \frac{\ell_{\eta} e^{-2\pi m / \ell_{\eta}}}{|\Gamma(\frac{k}{2} + \frac{2\pi m}{\ell_{\eta}})|^2}. 
$$

(8.50)

To check that our formulas satisfy this symmetry, first note that $\eta_{\eta'} = \eta_{\eta'}$ and

$$
S_{\eta_{\eta'}}(m,n,C,\alpha,\beta) = S_{\eta_{\eta}}(n,m,C,-\beta,-\alpha).
$$

It follows that (8.50) is a consequence of Theorem 8.15 if we can show that

$$
I_{\eta_{\eta'}}(m,n,C,\alpha,\beta) \frac{\ell_{\eta'} e^{-2\pi m / \ell_{\eta'}}}{|\Gamma(\frac{k}{2} + \frac{2\pi m}{\ell_{\eta'}})|^2} = I_{\eta_{\eta}}(m,n,C,\beta,-\alpha) \frac{\ell_{\eta} e^{-2\pi m / \ell_{\eta}}}{|\Gamma(\frac{k}{2} + \frac{2\pi m}{\ell_{\eta}})|^2}. 
$$

(8.51)

When $I_{\eta_{\eta'}}(m,n,C,\alpha,\beta)$ is given by (8.6) then (8.51) is straightforward to verify. When $I_{\eta_{\eta'}}(m,n,C,\alpha,\beta)$ is given by (8.7), the final step of the verification of (8.51) requires Euler’s transformation formula, [AAR99 (2.2.7)]:

$$
\begin{align*}
2F_1(a,b;c;1-z) &= z^{c-a-b}2F_1(c-a,c-b;c;1-z) \\
&\quad (-\pi < \arg z \leq \pi).
\end{align*}
$$

8.5 Examples

As in Sections 8 and 7, we take the example $\Gamma = SL_2(\mathbb{Z})$ with $\eta = \eta' = (-\sqrt{D}, \sqrt{D})$ and $\sigma_{\eta} = \sigma_{\eta'}$ given by $\delta_{\eta}$. For $\left(\begin{array}{c} e \\ g \\ h \end{array}\right) \in \Gamma$, write

$$
\begin{align*}
\sigma_{\eta}^{-1} \left(\begin{array}{c} e \\ f \\ g \\ h \end{array}\right) \sigma_{\eta} = \frac{1}{2} \left(\begin{array}{cc} e + g\sqrt{D} + f/\sqrt{D} + h & e - g\sqrt{D} + f/\sqrt{D} + h \\
-e + g\sqrt{D} - f/\sqrt{D} + h & e - g\sqrt{D} - f/\sqrt{D} + h \end{array}\right) = \left(\begin{array}{cc} a & b \\
-c & d \end{array}\right).
\end{align*}
$$

Then

$$
ad = \frac{1}{2} + \frac{1}{4} \left( e^2 - Dg^2 - \frac{f^2 - Dh^2}{D} \right). 
$$

(8.52)
Recall the determination of $\varepsilon_D$ and $\sigma_\eta^{-1} \Gamma_\eta \sigma_\eta$ in (5.2). Set $H_D := \sigma_\eta R_{\eta\eta} \sigma_\eta^{-1}$ (for $R_{\eta\eta}$ defined in (8.24)) to get explicitly

$$H_D = \left\{ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \left| \frac{1}{\varepsilon_D} \left( \begin{matrix} (e + g\sqrt{D})^2 - (f + h\sqrt{D})^2/D \\ (e - g\sqrt{D})^2 - (f - h\sqrt{D})^2/D \end{matrix} \right) < \varepsilon_D \right\} $$

and let $H_D(C)$ be the elements of $H_D$ with $ad$, given by (8.52), equalling $C$.

**Lemma 8.17.** If $\left( \begin{array}{c} e \\ f \\ g \\ h \end{array} \right) \in H_D(C)$ for $C \neq 0, 1$ then

$$e^2 + Dg^2 + \frac{f^2 + Dh^2}{D} \leq (\varepsilon_D^2 + \varepsilon_D^{-2}) (|C| + |C - 1|).$$

**Proof.** As in (8.29), the inequalities in (8.24) imply $\varepsilon_D^{-2} \leq |a/d|, |b/c| \leq \varepsilon_D^2$. Arguing as in Lemma 5.3,

$$\varepsilon_D^{-2} \leq |a/d| \leq \varepsilon_D^2 \iff |a/d|^2 + 1 \leq (\varepsilon_D^2 + \varepsilon_D^{-2}) |a/d| \iff a^2 + d^2 \leq (\varepsilon_D^2 + \varepsilon_D^{-2}) |C|.$$ 

Similarly for $|b/c|$, implying $a^2 + b^2 + c^2 + d^2 \leq (\varepsilon_D^2 + \varepsilon_D^{-2}) (|C| + |C - 1|)$ which is equivalent to (8.54).

So we may calculate the sums in (1.24) as sums over $\left( \begin{array}{c} e \\ f \\ g \\ h \end{array} \right) \in H_D(C)$, restricting our attention to entries satisfying (8.54). For example, the hyperbolic coefficients at $\eta = (-\sqrt{2}, \sqrt{2})$ of $P_{\eta,0}$ with weight $k = 12$ are computed in Table 3 using Theorem 1.4 and summing over all $C$ with $|C - 1/2| \leq 20$. Since $P_{\eta,0} \approx 1529.46\Delta \approx 1529.46 P_{\infty,1}/2.840287$, (using Table 1 and $\lambda_1$ from (1.2)), we may verify that the coefficients in Table 3 and the first column of Tables 2 agree. It would be interesting to see if the sum $S_{\eta\eta'}(m, n; C, \alpha)$ has a simple explicit expression similar to that of $S_{\eta\eta'}(m, n; C)$ in Theorem 1.3.

We finally note that Theorem 1.4 may be used it to detect when the negative Pell equation (1.27) has integer solutions. To explain this, let $\eta = \eta' = (-\sqrt{D}, \sqrt{D})$ for $\Gamma = \text{SL}_2(\mathbb{Z})$, as before, and define $\Phi(D)$ as the right side of (1.24) (without (1.25), (1.26)) for $m = n = 0, k = 10$: 

$$\Phi(D) := \frac{1}{1260 \log \varepsilon_D} \left( \sum_1 + \sum_2 + \sum_3 \right).$$

The fundamental solution $(a_0, c_0)$ to the Pell equation (1.19) is built into $\Phi(D)$ through $\varepsilon_D := a_0 + \sqrt{D} c_0$.

**Proposition 8.18.** The function $\Phi(D)$ takes only the values $0$ and $-1$. The negative Pell equation (1.27) has integer solutions if and only if $\Phi(D) = 0$.

**Proof.** Note that $S_{10}(\Gamma) = \{0\}$ and that the hyperbolic pairs $(-\sqrt{D}, \sqrt{D})$ and $(\sqrt{D}, -\sqrt{D})$ are equivalent in $\text{SL}_2(\mathbb{Z})$ exactly when (1.27) has integer solutions. With Proposition 8.16 Theorem 1.4 yields

$$0 = \Phi(D) + \begin{cases} 1 & \text{if } \eta \not\equiv \eta^* \mod \Gamma, \\
1 + (-1)^5 & \text{if } \eta \equiv \eta^* \mod \Gamma. \end{cases} \quad \square$$
Examples of $\Phi(D)$ for some small values of $D$ are shown in Table 4. They were found by computing $\sum_1$, $\sum_2$ and $\sum_3$ in (1.24) for all $C$ with $|C - 1/2| \leq 2$, using the techniques from earlier in this section. If (1.27) has a solution then there is a fundamental one, $(x_0, y_0)$, and all other solutions $(x_n, y_n)$ are given by $x_n + \sqrt{D} y_n = (x_0 + \sqrt{D} y_0)^{n+1}$ for $n + 1$ odd. See for example [MS12] and its contained references. When $(x_0, y_0)$ exists it is given by
\[
x_0 = \frac{1}{2} \left( \varepsilon_{D}^{1/2} - \varepsilon_{D}^{-1/2} \right), \quad y_0 = \frac{1}{2\sqrt{D}} \left( \varepsilon_{D}^{1/2} + \varepsilon_{D}^{-1/2} \right).
\]

| $D$  | 2   | 3   | 5   | 7   | 11  | 13               |
|------|-----|-----|-----|-----|-----|------------------|
| $\varepsilon_D$ | $3 + 2\sqrt{2}$ | $2 + \sqrt{3}$ | $9 + 4\sqrt{5}$ | $8 + 3\sqrt{7}$ | $10 + 3\sqrt{11}$ | $649 + 180\sqrt{13}$ |
| $\Phi(D)$      | 0.0  | -0.99998 | 0.0  | -1.00005 | -0.99997 | 0.0              |
| $(x_0, y_0)$   | $(1, 1)$ | $(2, 1)$ |      |      |      | $(18, 5)$        |

Table 4: Solutions of the negative Pell equation

Table 4 at least, serves as a check of Theorem 1.4. It is well known that (1.27) has solutions if and only if the continued fraction expansion of $\sqrt{D}$ has an odd period. Recently, a very simple criterion was given in [MS12]: for $D \equiv 1, 2 \pmod{4}$, equation (1.27) has solutions if and only if $a_0 \equiv -1 \pmod{2D}$.

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