On quantum mechanics as constrained $N = 2$ 
supersymmetric classical dynamics

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Abstract

The Schrödinger equation is shown to be equivalent to a constrained Liouville equation
under the assumption that phase space is extended to Grassmann algebra valued vari-
ables. For one-dimensional systems, the underlying Hamiltonian dynamics has a $N = 2$
supersymmetry. Potential applications to more realistic theories are briefly discussed.

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1 Introduction

Since its very beginnings, there have been speculations on the possibility of deriving quantum
theory from more fundamental dynamical structures, possibly deterministic ones. Famous is the
discussion by Einstein, Podolsky and Rosen. This lead to the EPR paradox, which in turn was
interpreted by its authors as indicating the need for a more complete fundamental theory [1].
However, just as numerous have been attempts to prove no-go theorems prohibiting exactly
such "fundamentalism", especially in local theories. This culminated in the studies of Bell,
leading to the Bell inequalities [2]. The paradox as well as the inequalities have come under
experimental scrutiny in recent years. Here, and in general, no disagreement with quantum
mechanics has been observed in the laboratory experiments on scales very large compared to
the Planck scale.

However, to this day, the feasible experiments cannot rule out the possibility that quantum
mechanics emerges as an effective theory only on sufficiently large scales and can indeed be
based on more fundamental models.

Motivated by the unreconciled clash between general relativity and quantum theory, 't Hooft
has argued in favour of model building in this context [3] (see also further references therein).

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In various examples, the emergence of the usual Hilbert space structure and unitary evolution in deterministic classical models has been demonstrated in an appropriate large-scale limit. Particular emphasis has been placed on the observation that it is fairly simple to arrive at a Hilbert space formulation of the classical dynamics of systems with Hamiltonians which are linear in the momenta. However, at the same time, it is difficult to assure that the resulting emergent quantum models possess a well-defined groundstate, i.e., that their energy spectra are bounded from below.

A new kind of gauge fixing or constraints implementing “information loss” at a fundamental level have been invoked here. However, a unifying dynamical principle for the necessary truncation of the Hilbert space is still missing. Therefore, these models have to be constrained or discretized case by case [3, 4, 5].

Various further arguments for deterministically induced quantum features have recently been proposed – see, for example, the works collected in Part III of Ref. [6], or Refs. [4, 7, 8, 9], concerning discrete time models, statistical and/or dissipative systems, quantum gravity, and matrix models, among others.

Most of these attempts to base quantum theory on a classical footing can be seen as variants of the earlier stochastic quantization procedures of Nelson [10] and of Parisi and Wu [11], often accompanied by the problematic analytic continuation from imaginary (Euclidean) to real time (“Wick rotation”), in order to describe evolving systems. In distinction, ’t Hooft’s work points towards a truly dynamical understanding of the origin of quantum phenomena.

In this note, my aim is to report on a large class of deterministic classical systems where the quantum mechanical features emerge from constrained classical dynamics. In particular, based on the extension of classical phase space to variables which take their values in a Grassmann algebra, one obtains the Schrödinger equation with standard Hamiltonians. Extension of this work to interacting field theories is possible and will be considered elsewhere. The natural appearance of supersymmetry in this framework certainly deserves further study as well.

Among the conceptual issues touched here, clearly the interpretation of the measurement process, of the “collapse of the wave function” in particular, must figure prominently, together with the quantum indeterminism and the wider philosophical implications of the algorithmic rules comprising quantum theory as a whole [12]. It is left for future studies to find out, how the deterministic framework introduced here allows to see them in a new light.

This letter is organized as follows. The (dis)similarity of the Liouville and the Schrödinger equations is demonstrated in Section 2. In Section 3, the discrepancy between both is overcome by introducing an extended phase space based on Grassmann algebra. This gives the Schrödinger equation a form which is suitable for reconstructing the underlying supersymmetric classical model in Section 4. In the concluding section, I discuss some problems left to be clarified as well as interesting topics for further exploration.

2 (Dis)similarity of Liouville and Schrödinger equations

It will be convenient for the present argument to recall the Hilbert space formulation of classical statistical mechanics developed by Koopman and von Neumann [13].

Beginning with a \((2n)\)-dimensional classical phase space \(\mathcal{M}\), the coordinates are collectively denoted by \(\varphi^a \equiv (q^1, \ldots, q^n; p^1, \ldots, p^n)\), \(a = 1, \ldots, 2n\), where \(q, p\) stand for the usual coordi-
nates and conjugate momenta. Given the time independent Hamiltonian $H(\varphi)$, the equations of motion are:

$$\frac{\partial}{\partial t} \varphi^a = \omega^{ab} \frac{\partial}{\partial \varphi^b} H(\varphi) ,$$

(1)

where $\omega^{ab}$ is the standard symplectic matrix, with a summation over indices appearing twice.

Considering an ensemble of initial conditions, the evolution of the corresponding phase space density $\rho$ of a conservative system, which follows from Eqs. (1), is described by the Liouville equation:

$$0 = i \frac{d}{dt} \rho = i \partial_t \rho - \hat{L} \rho ,$$

(2)

where a convenient overall factor $i$ has been introduced, and the Liouville operator is:

$$- \hat{L} \equiv \partial_p H \cdot i \partial_q - \partial_q H \cdot i \partial_p ,$$

(3)

in terms of partial derivatives with respect to phase space coordinates.

In order to reformulate standard statistical mechanics in Hilbert space, the following two postulates are put forth: I) the density can be factorized as $\rho \equiv \Psi^* \Psi$, II) the complex valued amplitude or “state” function $\Psi \in L^2$ itself obeys the Liouville equation (2).

Furthermore, with the inner product defined by $\langle \Psi | \Phi \rangle \equiv \int d^n q d^n p \; \Psi^* \Phi$, the Liouville operator is hermitean and the overlap $\langle \Psi | \Psi \rangle$ is a conserved quantity. Then, the Liouville equation also applies to $\rho = |\Psi|^2$, due to its linearity, and $\rho$ is consistently interpreted as probability density [13].

These results certainly remind one of the usual quantum mechanical formalism. In order to expose more clearly the similarity as well as a crucial difference, two further transformations of the Liouville equation for the state function $\Psi$ are useful.

First of all, Fourier transformation replaces the momenta $p$ by new coordinates $Q$, $\Psi(q, p; t) = \int d^n Q \; \Psi(q, Q; t) \exp(-iQp)$, which yields:

$$i \partial_t \Psi = \left\{ (-i \partial_Q) \cdot (-i \partial_q) + V'(q) \cdot Q \right\} \Psi ,$$

(4)

where $V'(x) \equiv (d/dx)V(x)$, and a Hamiltonian with quadratic kinetic term has been assumed, in order to be explicit.

Secondly, motivated by the definition of the quantum mechanical Wigner function, the following coordinate transformation is performed:

$$\sigma \equiv (q + Q)/\sqrt{2} , \quad \delta \equiv (q - Q)/\sqrt{2} .$$

(5)

Thus, one obtains:

$$i \partial_t \Psi = \hat{H} \Psi \equiv \left\{ - \frac{1}{2} (\partial_\sigma^2 - \partial_\delta^2) + V'(\frac{\sigma + \delta}{\sqrt{2}}) \cdot \frac{\sigma - \delta}{\sqrt{2}} \right\} \Psi .$$

(6)

This equation seems as close as one can get in a few steps from the classical Liouville equation to the Schrödinger equation.

However, besides the characteristic doubling of the number of degrees of freedom, and their coupling in a particular form, there is a crucial difference between Eq. (6) and the Schrödinger
equation. The spectrum of the effective Hamiltonian $\hat{H}$, generally, will not be bounded from below. This is related to the fact that $\hat{H} \rightarrow -\hat{H}$ under the interchange $\sigma \leftrightarrow \delta$.

In any case, therefore, the above transformed classical theory, which is presented here in an appropriate Hilbert space form, lacks a stable groundstate and, therefore, does not qualify as a classical theory underlying quantum mechanics as an emergent description.

This may suffice as a brief introduction of putting standard classical mechanics into Hilbert space form. Clearly, this is not limited to systems with a finite number of degrees of freedom.

3 Extending phase space over Grassmann variables

It is obvious that the “no-groundstate” problem, which is encountered when trying to bridge the gap between Liouville and Schrödinger equations, requires a deep modification of the former, in order to be overcome.

The following derivation will newly make use of “pseudoclassical mechanics”. This notion has first been introduced in conjunction with the work of Casalbuoni and of Berezin and Marinov, who considered a Grassmann variant of classical mechanics in studying the classical dynamics of spin degrees of freedom as well as its quantized counterpart [14].

Classical mechanics based on Grassmann algebras has more recently found much attention, in order to elucidate the zerodimensional limit of classical and quantized supersymmetric field theories, see Refs. [15, 16] and further references therein.

In all cases, so far, quantization is a second step, following a standard algorithm when applied to a given classical system. In distinction, the present work is concerned with the attempt to show that quantum mechanics emerges more directly from suitable classical structures without need for any of the known quantization procedures.

The considerations here will be based on the Grassmann algebra $\Lambda_2$. It is generated by two real odd (“fermionic”) elements $o_1, o_2$ obeying:

$$o_1^2 = o_2^2 = 0 = \{o_1, o_2\} ,$$  

where the bracket denotes the anticommutator, $\{A, B\} \equiv AB + BA$. In addition, there are two even (“bosonic”) elements $e_1, e_2$ which are given by:

$$e_1 \equiv 1 , \quad e_2 \equiv o_1 o_2 .$$

The even elements commute among themselves and with all other elements of the algebra. Furthermore, the definition of $e_2$ implies the nilpotency also of this element, $e_2^2 = 0$. The algebra $\Lambda_2$ is the simplest one which supports the concept of Fourier transformations with respect to even and odd supernumbers, to be employed in Section 4.1.

The extension of the phase space pertaining to one classical degree of freedom is now introduced by the “$\Lambda_2$-postulate” that

- the variables $\sigma$ and $\delta$ take their values in the Grassmann algebra $\Lambda_2$ and are Grassmann even and odd, respectively:

$$\sigma \equiv \sigma_i e_i , \quad \delta \equiv \delta_i o_i ,$$

where summation over $i = 1, 2$ is implied, and with $\sigma_i, \delta_i \in R$.

1Grassmann algebra and analysis over supernumbers are presented in detail by DeWitt [15].
Furthermore, the classical Liouville equation in the form of Eq. (6) is now replaced by:

\[ i\partial_t \Psi = \left\{ -\frac{1}{2}(\partial_\sigma^2 - \partial_\delta^2) + V'(\sigma + \delta) \cdot (\sigma - \delta) \right\}_{\text{even}} \Psi, \quad (10) \]

where all terms are considered as Grassmann algebra valued; factors \(1/\sqrt{2}\) multiplying \(V'\) and its argument in Eq. (6) have been absorbed conveniently into the definition of the potential. Furthermore, as indicated by the subscript \(\{...\}_{\text{even}}\), only the Grassmann even part of the operator in brackets is taken here.

These modifications of phase space and the evolution equation are related to one-dimensional quantum mechanics, as will be shown next.\(^2\)

In order to explore consequences of Eqs. (7)–(10), it is useful to expand the state function \(\Psi\), incorporating the nilpotency of the odd Grassmann elements: \(\Psi(\sigma, \delta) \equiv \psi(\sigma) + \phi(\sigma)\delta\), where \(\psi, \phi\) are Grassmann even yet possibly complex valued functions.

Then, incorporating right derivatives, as discussed in Refs. [15], it follows that \(\partial_\sigma \Psi = \psi' + \phi' \delta, \partial_\delta \Psi = \phi, \partial_\sigma^2 \Psi = \psi'' + \phi'' \delta, \partial_\delta^2 \Psi = 0,\) and \(\partial_\sigma \partial_\delta \Psi = \phi' = \partial_\delta \partial_\sigma \Psi,\) where the primes denote ordinary derivatives, which are defined by the corresponding Taylor series, or similar, of the functions restricted to real arguments. Henceforth, all derivatives will be right derivatives, unless stated otherwise.

Applying these derivatives and the expansion of the state function in Eq. (10), the resulting equation can be decomposed with the help of the Grassmann algebra. Thus, one obtains two decoupled equations for the “wave function” \(\psi\) and its “shadow” \(\phi\):

\[ i\partial_t \psi(\sigma) = -\frac{1}{2} \psi''(\sigma) + V'(\sigma)\sigma \psi(\sigma) , \quad (11) \]
\[ i\partial_t \phi(\sigma) = -\frac{1}{2} \phi''(\sigma) + V'(\sigma)\sigma \phi(\sigma) , \quad (12) \]

where it has also been assumed that \(V(\sigma)\) is Grassmann even.

Indeed, the Schrödinger equation and a formally identical shadow equation are obtained. They could naturally be combined into two-component form. This result followed by construction from the modification of the classical Liouville equation together with the extension of phase space over Grassmann variables.

In itself, this may not be surprising. However, it will be demonstrated in Section 4 that this theory presents a classical statistical mechanical description of a supersymmetric Hamiltonian system. Thus, quantum mechanics emerges from an underlying deterministic dynamics.

Up to this point, the result is independent of the particular choice of \(\Lambda_2\). Further decomposing both equations, making use of \(\sigma = \sigma_1 e_1 = \sigma_1 + \sigma_2 e_2\), reproduces Eqs. (11), (12) with \(\sigma\) replaced by \(\sigma_1\), its real “body” \(\sigma_1\) with \(\sigma_1\) replaced by \(\sigma_1\), its real “body” \(\sigma_1\), and yields additional higher-order derivative forms thereof, corresponding to applying \(\partial_\sigma_1\) to both equations. The Schrödinger equation restricted to real variables, and correspondingly the usual quantum mechanical observables, are thus contained in the present framework.

Several further remarks are in order here:

\(^2\)The analogous pointwise extension for field theories will be considered elsewhere.
There is no \( \hbar \) in Eqs. (11), (12) or, equivalently, units are such that \( \hbar = 1 \). Thus, if introduced once and for all model potentials \( V \) alike, it would merely act as a conversion factor of units. – It is interesting to compare the present situation to the various points of view concerning the status of fundamental constants expressed in Ref. [17].

The system has a stable groundstate, in particular for all potentials \( V \), such that the onedimensional potential \( xV'(x) \) yields bound states in quantum mechanics.

There is no coupling between wave function and shadow. Such a coupling would be introduced, however, by a Grassmann odd contribution to the operator on the right-hand side of Eq. (10).

Furthermore, a probability amplitude interpretation of the state function, \( \Psi(\sigma, \delta) \equiv \psi(\sigma) + \phi(\sigma)\delta \), related to the wave function \( \psi \) and its shadow \( \phi \), is compatible with Eqs. (11), (12) in the following sense. The normalization of \( \Psi \) is time independent,

\[
N \equiv \int \Psi^* \Psi \, dM \equiv \int \Psi^* \Psi (a + b\delta) \, d\sigma d\delta = Z \int \{ b\psi^* \psi + a(\phi^* \psi + \phi \psi^*) \} \, d\sigma = \text{const} ,
\]

since the wave and shadow function can be expanded in terms of the same set of orthonormal stationary states. A general measure in terms of two constants \( a, b \in \mathbb{C} \) has been assumed and the rules for integration over Grassmann odd variables have been applied: \( \int d\delta = 0, \int \delta d\delta = Z \), with \( Z \in \mathbb{C} \) a conventional factor [15].

The normalization can be chosen real, with \( 0 \leq N \leq 1 \), for \( \psi \) and \( \phi \) properly normalized to unity, by setting \( b = (2Z)^{-1} = 2a \). For \( \phi = \psi \), one has \( N = 1 \), while in all other cases probability appears to be missing. This might have phenomenological implications, similar to the “negative probability” considered by Feynman [18].

The (pseudo-)Liouville Eq. (10) will be the starting point of the reconstruction of the classical mechanics which underlies the Schrödinger and shadow equations, which follows next.

4 Supersymmetric Hamiltonian dynamics beneath Schrödinger and shadow equations

The strategy here is simple. As close as possible, the derivations of Section 2, which led from the classical equations of motion (1) to the Liouville equation (6) in Hilbert space, will be reversed, duly taking the \( \Lambda_2 \)-postulate (9) of Section 3 into account. In this way, a classical dynamics will be found for which the analogous Liouville equation is Eq. (10), i.e., is equivalent to the Schrödinger and shadow equations, Eqs. (11), (12) respectively.

To begin with, the coordinate transformation (5) is undone by introducing:

\[
q \equiv (\sigma + \delta)/\sqrt{2} , \quad Q \equiv (\sigma - \delta)/\sqrt{2} , \quad (14)
\]

where \( \sigma, \delta \) are the Grassmann even and odd variables defined in Eqs. (9). Note that \([q, Q] = 0\). Similarly, the derivatives, \( \partial_q(Q) \equiv (\partial_\sigma + (-)\partial_\delta)/\sqrt{2} \), commute with each other.

\footnote{It will be interesting to explore such a possibility with regard to the (breaking of) the supersymmetry of the underlying classical model (see Section 4).}
Incorporating this transformation, the Liouville Eq. (10) turns into the Grassmann analogue of Eq. (11), which retains its form. However, being defined as sum and difference of the same Grassmann even and odd terms in Eqs. (14), this specific structure of the variables \( q, Q \) has to be enforced by constraints. They can be stated in different ways.

A geometric set of constraints is: \((q - Q)^2 = 0\) ("distance zero"), \( q - Q = q^* - Q^* \) ("reality"), and \( qQ = (q + Q)^2/4 \) ("geometric mean squared = arithmetic mean squared"). Equivalently, one may demand \([q, Q] = 0\) instead of the last requirement.

More simply, however, one may require \( q_e = Q_e \) and \( q_o = -Q_o \), where the subscripts "e, o" refer to even and odd components of the respective variable. Thus, at the end of the present section, the constraints will be implemented by integrating the Liouville equation derived in the following, or, equivalently, the Grassmann analogue of Eq. (4), with Dirac \( \delta \)-function weights:

\[
\int ... \delta(q_e - Q_e)\delta(q_o + Q_o) dQ_e dQ_o ,
\]

where the order of Grassmann odd terms is important.

The \( \delta \)-function (distribution) for Grassmann algebra valued variables forms the basis for the related generalized theory of Fourier transformation [15].

Next, in fact, the Fourier transformation preceding Eq. (4) will be undone. Properly defining the Fourier transformation over Grassmann variables needs some care and has been elaborated by DeWitt [15]. One has to proceed in two steps:

\[
f(q, Q) \equiv f(q, Q_e, Q_o) = \int f(q, Q_e; p_o) \exp(ip_oQ_o) \frac{dp_o}{\sqrt{2\pi}}
\]

\[
= \int \exp(ip_eQ_e + ip_oQ_o) f(q; p_e, p_o) \frac{dp_e}{2\pi} \frac{dp_o}{\sqrt{2\pi}} .
\]

Then, employing the appropriate partial integrations [15] where necessary, one calculates:

\[
Q\Psi(q, Q) \equiv (Q_e + Q_o)\Psi(q, Q_e, Q_o)
\]

\[
= \int \exp(ip_eQ_e + ip_oQ_o)(i\partial_{p_e} + i\partial_{p_o})\Psi(q; p_e, p_o) \frac{dp_e}{2\pi} \frac{dp_o}{\sqrt{2\pi}}
\]

(16)

\[
\partial_Q\Psi(q, Q) \equiv (\partial_{Q_e} + \partial_{Q_o})\Psi(q, Q_e, Q_o)
\]

\[
= \int \exp(ip_eQ_e + ip_oQ_o)i(p_e - p_o)\Psi(q; p_e, p_o) \frac{dp_e}{2\pi} \frac{dp_o}{\sqrt{2\pi}} .
\]

(17)

Generally, the ordering of factors is important, due to Grassmann integration and algebra.

Incorporating the sequence of transformations discussed in this section, so far, the Liouville equation (11) attains the more familiar looking form:

\[
\partial_t \Psi = -\{p_e\partial_{q_e} - p_o\partial_{q_o} - V'(q_e)\partial_{p_e} - V''(q_e)q_o\partial_{p_o}\} \Psi ,
\]

(18)

where \( \Psi \equiv \Psi(q_e, q_o; p_e, p_o; t) \). Note that the operator on the right-hand side comprises only Grassmann even terms, as demanded before.

It will now be shown that Eq. (18) is, in fact, the Liouville equation for the Lagrangian:

\[
L \equiv \dot{q}_e\dot{q}_o - V'(q_e)q_o ,
\]

(19)
with $V'(q_e)$ Grassmann even.\footnote{This Lagrangian apparently has not been studied before, which might be related to the fact that the action obtained by integrating $L$ over real time is Grassmann odd. However, in line with the present attempt to find a classical structure \textit{beneath} quantum mechanics, no (path integral) quantization of the model is intended.}

Introducing the canonical momenta,

$$P_{e,o} \equiv \frac{\partial L}{\partial \dot{q}_{e,o}} = \dot{q}_{o,e},$$

(20)

the Hamiltonian becomes:

$$H \equiv P_e \dot{q}_e + P_o \dot{q}_o - L = P_e P_o + V'(q_e) q_o.$$  

(21)

In turn, this leads to the equations of motion:

$$\dot{q}_e = \frac{\partial H}{\partial P_e} = P_o, \quad \dot{q}_o = \frac{\partial H}{\partial P_o} = P_e,$$

(22)

$$\dot{P}_e = -\frac{\partial H}{\partial q_e} = -V''(q_e) q_o, \quad \dot{P}_o = -\frac{\partial H}{\partial q_o} = -V'(q_e).$$

(23)

or, in second-order form:

$$\ddot{q}_e = -V'(q_e), \quad \ddot{q}_o = -V''(q_e) q_o.$$  

(24)

These equations imply that the Hamiltonian is a constant of motion, as expected.

Furthermore, identifying $p_e \equiv P_o$ and $-p_o \equiv P_e$, consistently with the Grassmann nature of each variable, it is now seen that the Liouville equation \cite{18} is indeed of the usual form:

$$\partial_t \Psi = -\{ \frac{\partial H}{\partial p_e} \dot{q}_e + \frac{\partial H}{\partial p_o} \dot{q}_o - \frac{\partial H}{\partial q_e} \dot{p}_e - \frac{\partial H}{\partial q_o} \dot{p}_o \} \Psi = -\{ H, \Psi \} \text{PB},$$

(25)

cf. Section 2. Here the graded antisymmetric Poisson bracket is introduced \cite{15}. For any pair of dynamical variables $A, B$ it is defined by:

$$\{ A, B \} \text{PB} \equiv A \{ \dot{\tilde{P}}_e \tilde{q}_e + \dot{\tilde{P}}_o \tilde{q}_o - \dot{\tilde{q}}_e \tilde{P}_e - \dot{\tilde{q}}_o \tilde{P}_o \} B,$$

(26)

where left and right derivatives are indicated explicitly.

The action associated with the above Lagrangian and the equations of motion have various interesting symmetry properties.

Using the decomposition $q_e(t) = q_1(t) + q_2(t)e_2$, together with the defining properties \cite{17} and \cite{18} of $A_2$, one finds that $L$ does not contain derivatives of $q_2$. Therefore, it presents a real parameter, which can be eliminated by a translation.\footnote{Replacing the Grassmann algebra $A_2$ by $A_3$, for example, all coordinates would be dynamical. However, elimination of $q_2$ and, thus, of $e_2$ may be welcome, since $e_2$ is imaginary \cite{16}. This renders $L, H$, etc. real.} Thus, the first of Eqs. (24) becomes an ordinary equation of motion, while the second equation describes a parametrically coupled “fermionic” oscillator.

For the harmonic oscillator, with $V'(q_e) = \bar{V} q_e$, and for a constant potential the Eqs. (24) decouple and are supersymmetric under the discrete interchange $q_e \leftrightarrow q_o$. 

4 This Lagrangian apparently has not been studied before, which might be related to the fact that the action obtained by integrating $L$ over real time is Grassmann odd. However, in line with the present attempt to find a classical structure \textit{beneath} quantum mechanics, no (path integral) quantization of the model is intended.
This suggests to look for supersymmetry also in the case of an arbitrary potential. Indeed, the system described by the Lagrangian $L$ of Eq. (19) is invariant under the transformation:

$$q_e \rightarrow q_e + \epsilon q_o ,$$

(27)

where $\epsilon$ is a Grassmann even infinitesimal parameter. To this is associated a conserved even "charge" $C$, which is obtained by the usual Noether method:

$$C = q_o \dot{q}_o = q_o P_e .$$

(28)

Similarly, there is a second transformation which leaves the dynamics invariant:

$$q_o \rightarrow q_o + \epsilon \dot{q}_e ,$$

(29)

with associated Noether charge:

$$H_e = \frac{1}{2} \dot{q}_e^2 + V(q_e) = \frac{1}{2} P_o^2 + V(q_e) ,$$

(30)

i.e., the energy of the even degree of freedom, particularly when $q_e = q_1 \in \mathbb{R}$, as discussed.

Summarizing the symmetry properties, the above constants of motion satisfy the following $N = 2$ supersymmetry algebra generated by the two charges:

$$\{ C, H_e \}_{PB} = H ,$$

(31)

with the Poisson bracket of Eq. (25). Furthermore, for all combinations of $A, B \in \{ C, H_e, H \}$ other than the above, one finds $\{ A, B \}_{PB} = 0$.

At last, the constraints still should be implemented on the Liouville equation (18), or Eq. (25), in more compact form. As mentioned before, this is achieved by integrating this equation with suitable $\delta$-function weights:

$$\int \left( \int \exp(\text{i}p_e q_e + \text{i}p_o q_o) \left[ \partial_t \Psi + \{ H, \Psi \}_{PB} \right] \frac{dp_e}{2\pi} \frac{dp_o}{2\pi} \right) \delta(q_e - Q_e) \delta(q_o + Q_o) dQ_e dQ_o$$

$$= \int \exp(\text{i}p_e q_e + \text{i}p_o q_o) \left[ \partial_t \Psi + \{ H, \Psi \}_{PB} \right] \frac{dp_e}{2\pi} \frac{dp_o}{2\pi} = 0 ,$$

(32)

where $p_e = P_o$ and $-p_o = P_e$, as before. Note that the Fourier integrals were eliminated in the derivation of Eq. (18) by the inverse transformation. Here, they have to be kept, since the dependence on the coordinates $Q_e, Q_o$ enters the integrations effecting the constraints.

The equations (32) (remaining variables $q_e, q_o, t$) present the main result of this section. While the left-hand side shows the implementation of the local constraints on the coordinates $q, Q$, the second expression involves an integro-differential operator, of course, due to Fourier transformation. The resulting equation is automatically solved by all solutions $\Psi$ of the classical Liouville equation. – It is conceivable that further solutions exist corresponding to solutions of the Liouville equation with source terms $s$, such that $\int \exp(\text{i}p_e q_e + \text{i}p_o q_o) s(q_e, q_o, p_e, p_o; t) dp_e dp_o = 0$. Study of their existence, properties, and relevance is left as an important topic for future work.

Summarizing, it has been shown here that the constrained (pseudo-)Liouville equation (32), pertaining to the deterministic supersymmetric Hamiltonian dynamics of the model defined by the Lagrangian (19), follows from the Schrödinger and shadow equations (11) and (12). Inversely, the quantum mechanical equations follow from the classical Liouville equation (18), or Eq. (25), again implementing the constraints at the end.
5 Conclusions

Presently, it has been shown how the Schrödinger and shadow equations emerge from underlying classical dynamics. This may certainly be questioned in many respects. It might violate one or the other of the assumptions of existing no-go theorems relating to hidden variables theories. However, it is unknown whether those assumptions will be relevant for a future fundamental theory of physics at the Planck scale. Therefore, it is a valid option to try and reconstruct quantum theory as an emergent or effective theory for presently accessible scales [3]–[9].

It seems interesting to further explore the demonstrated connection between quantum mechanical and deterministic classical structures which makes no use of any of the known quantization procedures. Here, instead, the quantum mechanical features arise in a constrained phase space description of dynamics which is based on the Grassmann algebra valued variables of an underlying classical supersymmetric model.

The wave function and its shadow appear together in Eqs. (11) and (12), respectively. Does the shadow contribute to observables? What is the interpretation of observables related to the “soul” [15] of the Grassmann algebra valued variables? Detailed solution of the dynamics of quantum mechanics textbook examples should and will be repeated elsewhere, in order to further illucidate the description by the pseudoclassical Liouville equation (18), or Eq. (25), and especially by its constrained version, Eq. (32).

The extension to higher-dimensional classical models or field theories seems straightforward. However, what are the physical implications of supersymmetry (or its breaking) of the underlying classical system, as seen in our formalism? Where do physical fermions come in? Most interestingly, in emergent quantum field theories, is there a relation of their typical divergences to the necessary constraints on the Grassmann structure of the relevant phase space variables, cf. Eqs. (14) and the ensuing remarks? How do these constraints interfere with intrinsic constraints, for example, Gauss’ law in gauge theories or M(atrix) theory [3, 9, 10]? Clearly, there is room for further work. In the long run, this could lead to a reassessment of the fundamental role played by quantum theory in our description of natural phenomena. One may also ponder anew about the conceptual issues, briefly alluded to in the introduction, which surround quantum theory in its present form.

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