1. Introduction

The development of a good moduli theory consists of four basic steps.

1. Identify a class of objects whose moduli theory is nice.

In some cases the answer is obvious, for instance, we should study the moduli theory of smooth projective curves. In other cases, it took some time to understand what the correct objects should be: Abelian varieties should be replaced by polarized Abelian varieties, K3 surfaces by marked K3 surfaces and vector bundles by stable vector bundles. We see later that smooth, projective varieties of general type should be replaced by their canonical models. We discuss this in Section 2.

In all these cases, we get non-proper moduli spaces. This is inconvenient for many reasons, for instance it is hard to count objects with various properties. To remedy this, one should look for a compactification whose points have clear geometric meaning.

2. Choose a larger class of objects to form a proper moduli space.

The choice of these objects is usually neither obvious nor unique. It was not until the 1960’s that the importance of this step was understood and stable curves and semi-stable sheaves were identified and studied in detail. For surfaces of general type the right class was described in [KSB88] and for polarized Abelian varieties in [Ale02]. The solution for varieties of general type is treated in Section 3.

3. Establish the correct moduli functor.

Once a class of objects $V$ is established, the corresponding moduli functor is usually declared to consist of all flat families whose fibers are in $V$. However, for varieties of general type, allowing all flat families gives the wrong moduli functor. The problem and the solution are analyzed in Section 4.
Study the resulting moduli spaces and their applications.

The moduli of curves and the moduli of semi-stable vector bundles on curves appear in many contexts and by now established themselves as one of the richest applications of algebraic geometry. We are only at the beginning of the development of the moduli of higher dimensional varieties; the basic results are outlined in Section 5. I hope to see many more applications in the future.

These notes are intended to give a survey of the subject, stressing key examples and results. The forthcoming book [Kol10b] aims to give a complete treatment.

Definition 2 (Moduli functors). Let \( V \) be a “reasonable” class of projective varieties (or schemes, or sheaves, or ...). For the next definition we only need to assume that if \( K \supset k \) is a field extension then a \( k \)-variety \( X_k \) is in \( V \) iff \( X_K := X_k \times_{\text{Spec} k} \text{Spec} K \) is in \( V \). Define the corresponding moduli functor as

\[
\text{Varieties}_V(T) := \begin{cases} 
\text{Flat families } X \to T \text{ such that } \\
\text{every fiber is in } V, \\
\text{modulo isomorphisms over } T. 
\end{cases}
\]

(As noted in (1.3), we will need to impose additional restrictions eventually, but for now let us ignore these.)

3 (Moduli spaces). We say that a scheme \( \text{Moduli}_V \), or, more precisely, a flat morphism

\[
u : \text{Univ}_V \to \text{Moduli}_V
\]

is a fine moduli space for the functor \( \text{Varieties}_V \) if, for every scheme \( T \), pulling back gives an equality

\[
\text{Varieties}_V(T) = \text{Mor}(T, \text{Moduli}_V).
\]

Our aim is to understand all families whose fibers are in \( V \) and a fine moduli space presents the answer in the most succinct way.

Applying the definition to \( T = \text{Spec} K \), where \( K \) is a field, we see that every fiber of \( u : \text{Univ}_V \to \text{Moduli}_V \) is in \( V \) and the \( K \)-points of \( \text{Moduli}_V \) are in one-to-one correspondence with the \( K \)-isomorphism classes of objects in \( V \).

We consider the existence of a fine moduli space as the ideal possibility. Unfortunately, it is rarely achieved. When there is no fine moduli space, we still can ask for a scheme that best approximates its properties. Therefore, we look for schemes \( M \) for which there is a natural transformation of functors

\[
T_M : \text{Varieties}_V(*) \longrightarrow \text{Mor}(*, M).
\]

Such schemes certainly exist, for instance, if we work over a field \( k \) then \( M = \text{Spec} k \). All schemes \( M \) for which \( T_M \) exists form an inverse system which is closed under fiber products. Thus, as long as we are not unlucky, there is a universal (or largest) scheme with this property. Though it is not usually done, it should be called the categorical moduli space.

This object can be rather useless in general. For instance, fix \( n, d \) and let \( H_{n,d} \) be the class of all hypersurfaces of degree \( d \) in \( \mathbb{P}^{n+1}_k \) up to isomorphisms. One can see that a categorical moduli space exists and it is \( \text{Spec} k \).

In order to get something reasonable, we impose extra conditions. A scheme \( \text{Moduli}_V \) is a coarse moduli space for \( V \) if the following hold.
(1) There is a natural transformation of functors

\[ \text{ModMap} : \text{Varieties}_V(*) \to \text{Mor}(\ast, \text{Moduli}_V), \]

(2) \text{Moduli}_V is universal satisfying (1), and

(3) for any algebraically closed field \( K \supset k \),

\[ \text{ModMap} : \text{Varieties}_V(\text{Spec } K) \to \text{Mor}(\text{Spec } K, \text{Moduli}_V) = \text{Moduli}_V(K) \]

is an isomorphism (of sets).

In many cases, the naturally occurring moduli spaces have a further very useful property.

(4) There is a \( V_U \to U \) in \( V \) such that the corresponding moduli map \( U \to \text{Moduli}_V \) is open and quasi finite.

Following woodworking terminology, I propose to call a moduli space satisfying conditions (1–4) a \textit{bastard} moduli space.

4 (Problems with the moduli of smooth varieties).

In contrast with curves, the moduli theory for higher dimensional smooth varieties can be very badly behaved, as shown by the following examples.

(4.1) (Ruled surfaces) Let \( C \) be a smooth curve and \( L \) a line bundle on \( C \) that is generated by 2 sections \( f, g \). On \( S := C \times \mathbb{A}^1_t \), with first projection \( \pi_1 \), consider the exact sequence

\[ 0 \to \pi_1^*L^{-1} \to \pi_1^*L^{-1} + \mathcal{O}_S + \mathcal{O}_S \to Q \to 0. \]

\( Q \) is a rank 2 vector bundle on \( C \times \mathbb{A}^1 \). We can view \( \mathbb{P}_{C \times \mathbb{A}^1}Q \) is a \( \mathbb{P}^1 \)-bundle over \( S \), or as a flat family of ruled surfaces over \( \mathbb{A}^1 \).

If \( t \neq 0 \) then \( t : \pi_1^*L^{-1} \to \pi_1^*L^{-1} \) is an isomorphism, thus \( Q_t \cong \mathcal{O}_C + \mathcal{O}_C \). If \( t = 0 \) then we get

\[ Q_0 \cong L^{-1} + \text{coker}[L^{-1} : \mathcal{O}_C + \mathcal{O}_C] \cong L^{-1} + L. \]

Thus we get a flat family of smooth ruled surfaces whose general member is \( \mathbb{P}^1 \times C \) and whose special member is \( \mathbb{P}_C(L^{-1} + L) \). In a coarse moduli space over \( C \) both of these should correspond to \( C \)-points, but the above family shows that the moduli point \( [\mathbb{P}_C(L^{-1} + L)] \) is in the closure of the moduli point \( [\mathbb{P}^1 \times C] \). This is impossible (at least for schemes but not for stacks).

One can be even more specific for \( C = \mathbb{P}^1 \). Minimal ruled surfaces over \( \mathbb{P}^1 \) are \( F_m := \mathbb{P}^1 \left( \mathcal{O}_y + \mathcal{O}_m(m) \right) \) for \( m \geq 0 \). The “moduli space” has 2 connected components, corresponding to even and odd values of \( m \).

There are no closed points in this “moduli space”; the closure of \( [F_m] \) consists of all the points

\[ \{ [F_m], [F_{m+2}], [F_{m+4}], \ldots \}. \]

(4.2) (Abelian, elliptic and K3 surfaces)

A general problem in all these cases is that a typical deformation of such an algebraic surface over \( C \) is a non-algebraic complex analytic surface. Thus any algebraic theory captures only a small part of the full moduli theory.

(4.3) For Abelian varieties and for K3 surfaces, the moduli spaces look very strange topologically. For instance, the 3-dimensional space of Kummer surfaces is dense in the 20-dimensional space of all K3 surfaces \([\text{PS}71]\).
This can be corrected by fixing a basis in $H^2(\ast, \mathbb{Z})$, but it is not clear how similar tricks work in general. Also, as it happens already for stable curves, we would like to consider families where not all fibers are homeomorphic to each other. Then it is no clear what one means by “fixing a basis in $H^2(\ast, \mathbb{Z})”.

(4.4) (Repeated blow-ups lead to non-separatedness)

Let $f : X \to B$ be a smooth family of projective surfaces over a smooth (affine) pointed curve $b \in B$. Let $C_1, C_2, C_3 \subset X$ be three sections of $f$, all passing through a point $x_b \in X_b$ that intersect pairwise transversally at $x_b$ and are disjoint elsewhere.

Set $X^1 := B_{C_1}B_{C_2}B_{C_3}X$, where we first blow-up $C_3 \subset X$, then the birational transform of $C_2$ in $B_{C_1}X$ and finally the birational transform of $C_1$ in $B_{C_2}B_{C_3}X$. Similarly, set $X^2 := B_{C_1}B_{C_2}B_{C_3}X$. Since the $C_i$ are sections, all these blow-ups are smooth families of projective surfaces over $B$. It is easy to check that

(1) all the fibers are smooth, projective surfaces of general type,
(2) $X^1 \to B$ and $X^2 \to B$ are isomorphic over $B \setminus \{b\}$,
(3) the fibers $X^1_b$ and $X^2_b$ are not isomorphic.

This type of behavior happens every time we look at deformations of a surface with at least 3 points blown-up.

(4.5) (Non-separatedness for minimal resolutions.)

Let $X_0 := \{(f(x_1, \ldots, x_4) = 0) \subset \mathbb{P}^4$ be a surface of degree $n$ that has an ordinary double point at $p = (0:0:0:1)$ as its sole singularity and contains the pair of lines $(x_1x_2 = x_3 = 0)$. Let $g$ be homogeneous of degree $n - 1$ such that $x_n^{n-1}$ appears in $X$ with nonzero coefficient. Consider the family of surfaces

$$X := (f(x_1, \ldots, x_4) + tx_3g(x_1, \ldots, x_4) = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1.$$  

Note that $X_t$ is smooth for general $t \neq 0$ and $X$ contains the pair of smooth surfaces $(x_1x_2 = x_3 = 0)$.

For $i = 1, 2$, let $X^i := B_{(x_i,x_3)}X$ denote the blow-up of $(x_i = x_3 = 0)$ with induced morphisms $\pi_i : X^i \to X$ and $f_i : X^i \to \mathbb{A}^1$. There is a natural birational map $\phi := \pi_2^{-1} \circ \pi_1 : X^1 \dasharrow X^2$. Let $B_pX$ denote the blow-up of $p = ((0:0:0:1),0)$ with exceptional divisor $E \subset B_pX$. One checks that

(1) all the fibers are smooth, projective minimal models,
(2) $X^1 \to B$ and $X^2 \to B$ are isomorphic over $B \setminus \{b\}$,
(3) the fibers $X^1_b$ and $X^2_b$ are isomorphic, but
(4) $X^1 \to B$ and $X^2 \to B$ are not isomorphic.

While it is not clear from our construction, similar problems happen for any smooth family of surfaces where the general fiber has ample canonical class and a special fiber has nef (but not ample) canonical class, see [Art74, Bri68, Rei80].

5 (Answers to these problems). The problems (4.1–3) come from the global geometry of the varieties that we work with.

The current assumption is that the moduli problem of uniruled varieties is usually pathological. Furthermore, any general attempt to create a good moduli functor ends up with a theory that is not compatible with birational equivalence. (Although there are examples, like the moduli of smooth hypersurfaces of degree $n$ in $\mathbb{P}^n$ for $n \geq 4$, where the biregular moduli theory ends up being birationally invariant. However, even in these cases, it is not clear that a sensible compactification exists.)

For varieties with trivial canonical class one should get a nice moduli theory only after some suitable “rigidification.” This can consist of choosing a basis in
of K3 surfaces is yet to be found. For instance, a geometrically meaningful compactification of the moduli of K3 surfaces is yet to be found.

The problems (4.4–5) are more local. The aim of these notes is to explain how to deal with them for varieties of general type. The solution is to work with proper families of dimension $n$.

Smooth, proper families $X \to S$ modulo birational equivalence over a base scheme.

Proposition 6. Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve $B$ such that the canonical classes $K_{X^i}$ are $f_i$-ample. Then every isomorphism between the generic fibers $\phi : X^1_{k(B)} \cong X^2_{k(B)}$ extends to an isomorphism $\Phi : X^1 \cong X^2$.

Proof. Let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of $\phi$. Let $Y \to \Gamma$ be the normalization, with projections $p_i : Y \to X^i$ and $f : Y \to B$. We use the canonical class to compare the $X^i$. Since the $X^i$ are smooth,

$$K_Y \sim p^*_i K_{X^i} + E_i$$ where $E_i$ is effective and $p_i$-exceptional. (5.1)

Since $(p_i)_* \mathcal{O}_Y(mE_i) = \mathcal{O}_{X^i}$ for every $m \geq 0$, we get that

$$(f_i)_*(p_i)_* \mathcal{O}_{X^i}(mK_{X^i}) = (f_i)_*(mp_i^* K_{X^i}) = (f_i)_*(p_i)_* \mathcal{O}_{X^i}(mp_i^* K_{X^i} + mE_i) = (f_i)_*(p_i)_* \mathcal{O}_Y(mK_Y) = f_* \mathcal{O}_Y(mK_Y).$$

Since the $K_{X^i}$ are $f_i$-ample, $X^i = \text{Proj}_B \sum_{m \geq 0} (f_i)_* \mathcal{O}_{X^i}(mK_{X^i})$. Putting these together, we get the isomorphism

$$\Phi : X^1 \cong \text{Proj}_B \sum_{m \geq 0} (f_i)_* \mathcal{O}_{X^i}(mK_{X^i}) \cong \text{Proj}_B \sum_{m \geq 0} f_* \mathcal{O}_Y(mK_Y) \cong \text{Proj}_B \sum_{m \geq 0} (f_2)_* \mathcal{O}_{X^2}(mK_{X^2}) \cong X^2. \quad \square$$
Note that the smoothness of the $X^i$ is used only through the pull-back formula (6.1). This leads to the first major definition:

**Preliminary Definition 7.** Let $B$ be a smooth curve, $X$ a normal variety and $f : X \to B$ a non-constant projective morphism. Assume that

1. $m_0K_X$ is Cartier for some $m_0 > 0$. (This is needed to make sense of (2) and also to define the pull-back in (3).)
2. $m_0K_X$ is $f$-ample.
3. If $p : Y \to X$ is a resolution of singularities then
   
   $$m_0K_Y \sim p^*(m_0K_X) + E(m_0)$$
   
   where $E(m_0)$ is effective and $p$-exceptional.

We do not want to keep carrying the $m_0$ along, thus we switch to $\mathbb{Q}$-divisors and $\mathbb{Q}$-linear equivalence and write (3) as

$$m_0K_Y \sim_{\mathbb{Q}} p^*K_X + E$$

where $E$ is an effective, $p$-exceptional $\mathbb{Q}$-divisor.

With these assumptions we can make the following informal definitions:

4. A “general” fiber of $f$ is a **canonical model**.
5. A “special” fiber of $f$ is a **semi log canonical model** if (1–3’) continue to hold for every base change $X' := X \times_B B' \to B'$, for every smooth curve $B'$.

In this area, “semi” refers to allowing non-normal schemes and “log” refers to allowing exceptional divisors with coefficients $\geq -1$, see [1].

(Note that (1–2) are inherited by every base change, thus the only question is (3’). Already for curves, this condition is necessary. Indeed, let $(f(x,y,z) = 0)$ be any plane curve with isolated singularities and $(g(x,y,z) = 0)$ a general smooth plane curve. Then

$$\left(f(x,y,z) + tg(x,y,z) = 0\right) \subset \mathbb{P}^2 \times \mathbb{A}_t^1$$

is smooth near the $t = 0$ fiber. After the base change $t = s^m$ we get

$$\left(f(x,y,z) + s^m g(x,y,z) = 0\right) \subset \mathbb{P}^2 \times \mathbb{A}_s^1$$

and now (3’) is satisfied for $m \geq 3$ iff $(f(x,y,z) = 0)$ has only ordinary nodes.)

**Basic principles.**

The moduli theory of higher dimensional varieties of general type is governed by the following four basic principles.

**Principle 8.** **Canonical models are the correct higher dimensional analogs of smooth projective curves of genus $\geq 2$.**

**Principle 9.** **Semi log canonical models are the correct higher dimensional analogs of stable projective curves of genus $\geq 2$.**

**Principle 10.** **Flat families of canonical models form the correct higher dimensional open moduli problem.**

**Principle 11.** **Flat families of semi log canonical models do not form the correct higher dimensional compactified moduli problem.**
2. Canonical models

As noted in [8], canonical models are the basic objects of our moduli theory.

**Definition 12.** A normal projective variety $X$ is a *canonical model* iff the following hold.

1. $m_0K_X$ is Cartier for some $m_0 > 0$.
2. for every (equivalently, for one) resolution of singularities $f : X' \to X$ there is an effective, $f$-exceptional $\mathbb{Q}$-divisor $E$ such that
   
   $K_{X'} \sim_\mathbb{Q} f^*K_X + E$.
3. $K_X$ is ample.

Singularities satisfying (1–2) are called *canonical*.

This assumption implies that the *canonical ring* of $X$

\[ R(X, K_X) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)) \]

is isomorphic to the canonical ring of $X'$

\[ R(X', K_{X'}) := \sum_{m \geq 0} H^0(X', \mathcal{O}_{X'}(mK_{X'})) \].

In particular, $X = \text{Proj}_k R(X, K_X) = \text{Proj}_k R(X', K_{X'})$ is the unique canonical model in the birational equivalence class of $X$.

Now we know [BCHM06, Siu08] that the canonical ring $R(X, K_X)$ of a smooth projective variety of general type is always finitely generated, thus $X_{\text{can}}$ is a projective variety. It is not obvious, but true, that $X_{\text{can}}$ is a canonical model [Rei80].

**Definition 13** (Moduli functor of canonical models). The moduli functor of canonical models is

\[
\text{CanMod}(S) := \left\{ \begin{array}{l}
\text{Flat, proper families } X \to S, \\
\text{every fiber is a canonical model,} \\
\text{modulo isomorphisms over } S. 
\end{array} \right. \] (13.1)

This is an improved version of the birational moduli functor $\text{GenType}_{\text{bir}}(*)$ (5.1).

(Traditionally this was considered to be the obviously correct definition, but, in view of Principle 11, it needs an explanation. For details, see (30).)

By a theorem of [Siu98], in a smooth, proper family of varieties of general type the canonical rings form a flat family and so do the canonical models. Thus there is a natural transformation

\[ T_{\text{CanMod}} : \text{GenType}_{\text{bir}}(*) \to \text{CanMod}(*). \]

By definition and by (13), if $X_i \to S$ are two smooth, proper families of varieties of general type then

\[ T_{\text{CanMod}}(X_1/S) = T_{\text{CanMod}}(X_2/S) \iff X_1 \text{ and } X_2 \text{ are birational,} \]

thus $T_{\text{CanMod}}$ is injective. It is, however, not surjective, but we have the following partial surjectivity statement.
Let $Y \to S$ be a flat family of canonical models. Then there is a dense open subset $S^0 \subset S$ and a smooth, proper family of varieties of general type $Y^0 \to S^0$ such that

$$T_{\text{CanMod}}(Y^0/S^0) = [X^0/S^0].$$

Some of the obstruction to surjectivity are obvious but some, as in (4.3), are quite subtle.

Canonical curves are exactly the smooth, projective curves of genus $\geq 2$. Canonical surfaces have at most Du Val singularities (also called rational double points). Starting with dimension 3, we get more complicated singularities. For instance, $(x_0^{a_0} + \cdots + x_n^{a_n} = 0) \subset \mathbb{A}^{n+1}$ is canonical iff $\sum a_i > 1$ and a quotient of $\mathbb{A}^n$ be a finite subgroup $G$ without quasi-reflections is canonical iff for every $g \in G$ ($g \neq 1$) with eigenvalues $e^{2\pi ic_j}$ (where $0 \leq c_j < 1$) we have $\sum_j c_j \geq 1$.

The most important general property of canonical singularities is the following. For short proofs see [KM98, Sec.5.1] or [Kol10a].

**Theorem 14. [Elk81]** Let $X$ be a canonical model over a field of characteristic 0. Then $X$ has rational singularities. That is, $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$ for every resolution of singularities $f : Y \to X$.

Using the covering trick of [Rei80] this implies that the reflexive hulls $\omega^m_X$ are CM (=Cohen-Macaulay) for every $m$.

### 3. Semi log canonical models

First we translate (7.5) into a proper definition.

Let $B$ be a smooth curve, $X$ a normal variety and $f : X \to B$ a non-constant projective morphism. When is the fiber $X_b$ a semi log canonical model?

By [KKMSD73] there is a smooth pointed curve $b' \in B'$ and a finite morphism $(b' \in B') \to (b \in B)$ such that the base change $f' : X' := X \times_B B' \to B'$ has a resolution $\pi : Y' \to X'$ such that $(f' \circ \pi)^{-1}(b') \subset Y'$ is a reduced simple normal crossing divisor. We can also assume that the birational transform $Y'_{b'} := \pi^{-1}_* X'_{b'}$ is smooth. Thus $(f' \circ \pi)^{-1}(b') = Y'_{b'} + F$ where $F$ is a reduced simple normal crossing divisor. By (7.3),

$$K_{Y'} \sim_Q \pi^* K_{X'} + E'$$

where $E'$ is effective. Since $Y'_b + F' = \pi^*(X'_{b'})$, we can rewrite the above as

$$K_{Y'} + Y'_b \sim \pi^*(K_{X'} + X'_{b'}) + E' - F.$$

Restricting to $Y'_b$ we get

$$K_{Y'_b} \sim \pi^* K_{X'_{b'}} + (E' - F)|_{Y'_b}.$$  

Since $E'$ is effective, its restriction is again effective, but the restriction of $-F$ brings in negative coefficients. However, none of these is smaller than $-1$ since $F$ and $Y'_b$ intersect transversally.

**Definition 15.** A reduced, projective variety $X$ is a semi log canonical model or slc model iff the following hold.

1. $m_0 K_X$ is Cartier for some $m_0 > 0$.
2. $X$ has only ordinary nodes in codimension 1.
(3) For every resolution of singularities $f : X' \to X$ there is an $f$-exceptional $\mathbb{Q}$-divisor $E = \sum a_i E_i$ such that
\[ K_{X'} \sim f^* K_X + \sum a_i E_i \] and $a_i \geq -1$ for every $i$.

(4) $K_X$ is ample.

One should think of this as combining a global condition (4) with purely local conditions (1–3). Singularities satisfying (1–3) are called semi log canonical or slc.

For slc models it is usually better to use semi resolutions, that is, a proper birational morphism $g : X^s \to X$ such that $X^s$ has only double normal crossing points ($xy = 0 \subset \mathbb{C}^{n+1}$ and pinch points ($x^2 = y^2 z \subset \mathbb{C}^{n+1}$) and $g$ maps the double locus of $X^s$ birationally on the double locus of $X$. Let $E$ denote the (reduced) exceptional divisor of $g$. Then the canonical ring of $X$
\[ R(X, K_X) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X)) \]
is isomorphic to the semi log canonical ring of $X^s$
\[ R(X^s, K_{X^s} + E) := \sum_{m \geq 0} H^0(X^s, \mathcal{O}_{X^s}(mK_{X^s} + mE)). \]

This actually creates a lot of problems since semi log canonical rings are not always finitely generated [Kol07].

It is a quite subtle theorem that semi log canonical models actually satisfy the preliminary definition (7.5). This is proved in [K+92, 17.4] and [Kaw07].

To get a feeling for semi log canonical, let us review the classification of slc surface singularities.

**Singularities of semi log canonical surfaces.**

It is convenient to describe the singularities of log canonical surfaces by the dual graph of their minimal resolution. That is, given a singularity ($s \in S$) with minimal resolution $g : X \to S$ we draw a graph $\Gamma$ whose vertices are the $g$-exceptional curves and two vertices are connected by an edge iff the corresponding curves intersect. We use the number $-(E_i \cdot E_j)$ to represent a vertex. In our examples, save in (16.4.a), all the exceptional curves are isomorphic to $\mathbb{P}^1$.

Let $\det(\Gamma)$ denote the determinant of the negative of the intersection matrix of the dual graph. This matrix is positive definite for exceptional curves. For instance, if $\Gamma = \{2 - 2 - 2\}$ then
\[
\det(\Gamma) = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4.
\]

**16 (List of log canonical surface singularities).**

Each case includes all previous ones.

- **16.1** Terminal = smooth.
- **16.2** Canonical = Du Val (= rational double point).
- **16.3** Log terminal = quotient of $\mathbb{C}^2$ by a finite subgroup of $GL(2, \mathbb{C})$ that acts freely outside the origin. The order of the group is $\det(\Gamma)$. A more detailed list is the following:
(a) (Cyclic quotient) \[ c_1 - \cdots - c_n \]

(b) (Dihedral quotient) Here \( n \geq 2 \) with dual graph

\[ \begin{array}{c}
\Gamma_1 - c_0 - \Gamma_2 \\
\Gamma_3
\end{array} \]

(c) (Other quotients) The dual graph has 1 fork

\[ \Gamma_1 - c_0 - \Gamma_2 \]

\[ \Gamma_3 \]

with 3 cases for \((\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3)):\)

(Tetrahedral) (2,3,3)
(Octahedral) (2,3,4)
(Icosahedral) (2,3,5).

16.4 Log canonical
(a) (Simple elliptic) \( \Gamma = \{ E \} \) has a single vertex which is a smooth elliptic curve with self intersection \( \leq -1 \).

(b) (Cusp) \( \Gamma \) is a circle of smooth rational curves, at least one of them with with \( c_i \geq 3 \). (The cases \( n = 1, 2 \) are somewhat special.)

\[ \begin{array}{c}
c_n - \cdots - c_{r+1} \\
c_1 \\
c_2 - \cdots - c_{r-1}
\end{array} \]

(c) (\( \mathbb{Z}/2 \)-quotient of a cusp or simple elliptic) \( \Gamma \) has 2 forks.

\[ \begin{array}{c}
c_1 - \cdots - c_n \\
2 \\
\end{array} \]

(d) (Other quotients of a simple elliptic) The dual graph is as in 16.3.c with 3 possibilities for \((\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3)):\)

(\( \mathbb{Z}/3 \)-quotient) (3,3,3)
(\( \mathbb{Z}/4 \)-quotient) (2,4,4)
(\( \mathbb{Z}/6 \)-quotient) (2,3,6).

If \( X \) is a non-normal semi log canonical surface singularity, then we describe its normalization \( \bar{X} \) together with the preimage of the double curve \( \bar{B} \subset \bar{X} \).

The extended dual graph \((\Gamma, \bar{B})\) has an additional vertex (repesented by \( \bullet \)) for each local branch of \( \bar{B} \) connected to \( C_i \) if \((\bar{B} \cdot C_i) \neq 0 \).

17 (List of semi log canonical surface singularities). There are 3 irreducible cases. (The number on some edges is the different, which we do not define here [K+92 Sec.16]. Their role is explained in 17.4).
(17.1) (Cyclic quotient, one branch of $\bar{B}$)
$$\begin{align*}
&\bullet \quad 1 - \frac{1}{c_1} - \cdots - c_n \\
&\end{align*}$$

(17.2) (Cyclic quotient, two branches of $\bar{B}$)
$$\begin{align*}
&\bullet \quad 1 - \frac{1}{c_1} - \cdots - c_n \cdot \frac{1}{\bullet} \\
&\end{align*}$$

(17.3) (Dihedral quotient) Here $n \geq 2$ with dual graph
$$\begin{align*}
&\bullet \quad 1 - c_1 - \cdots - c_n \quad 2 \\
&\quad / \\
&\quad \bullet \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 2 \\
&\end{align*}$$

(17.4) (Reducible cases) We can take several components as above and glue them together along two local branches of $\bar{B}$. The gluing is allowed only if we see the same numbers on the edges.

Thus we can glue 2 copies as in (17.1) as long as both have the same $\det(\Gamma)$.

Or we can take any number of those in (17.2), make a chain out of them and then either turn the chain into a circle or end it with a copy of (17.3).

We are also allowed to glue a local branch of $\bar{B}$ to itself by an involution. For instance, $\bullet - 1$ glued to itself gives the pinch point $(x^2 = y^2z) \subset A^3$.

Note: The above dual graphs are correct in any characteristic, the descriptions as quotients are correct as long as the the characteristic does not divide the order of the group mentioned.

**Du Bois singularities.**

Semi log canonical singularities need not be rational, not even CM (=Cohen-Macaulay) and their most important property is that they are Du Bois. After some examples, we discuss Du Bois singularities and their useful properties.

**Example 18.** It is easy to see that a cone over a smooth variety $X \subset \mathbb{P}^N$ is log canonical iff $K_X \sim_\mathbb{Q} r \cdot H$ for some $r \leq 0$ where $H$ is the hyperplane class. For us the interesting case is when $K_X \sim_\mathbb{Q} 0$ (hence $r = 0$). For these, the cone is CM (resp. rational) iff $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$ (resp. for $0 < i \leq \dim X$). Thus we see the following.

1. A cone over an Abelian variety $A$ is CM iff $\dim A = 1$.
2. A cone over a K3 surface is CM but not rational.
3. A cone over an Enriques surface is CM and rational.
4. A cone over a smooth Calabi-Yau complete intersection is CM but not rational.

The concept of Du Bois singularities was introduced by Steenbrink in [Ste83] as a weakening of rationality. The precise definition is rather involved, but our main applications rely only on the following consequence.

**Theorem 19.** [KK09, Kol10b Chap3] Let $X$ be a proper slc scheme over $\mathbb{C}$. Then the natural map
$$H^i(X^{an}, \mathbb{C}) \to H^i(X^{an}, \mathcal{O}_{X^{an}}) \cong H^i(X, \mathcal{O}_X)$$
is surjective for all $i$. (In fact, with a functorial splitting.)

In studying moduli questions, it is very useful to know that certain numerical invariants are locally constant. All of these follow from the Du Bois property, via the following base-change theorem [DJ74] [DBS1].

**Proposition 20.** Let $f : X \to S$ be a flat, proper morphism. Assume that the fiber $X_s$ is Du Bois for some $s \in S$. Then there is an open neighborhood $s \in S^0 \subset S$ such that, for all $i$,

1. $R^i f_*\mathcal{O}_X$ is locally free and compatible with base change over $S^0$ and
2. $s \mapsto H^i(X_s, \mathcal{O}_{X_s})$ is a locally constant function on $S^0$.

Proof. By Cohomology and Base Change [Har77, III.12.11], the theorem is equivalent to proving that the restriction maps

$$\phi^i_s : R^i f_* \mathcal{O}_X \to H^i(X_s, \mathcal{O}_{X_s}) \quad (20.3)$$

are surjective for every $i$. By the Theorem on Formal Functions [Har77, III.11.1], it is enough to prove this when $S$ is replaced by any 0-dimensional scheme $S_n$ whose closed point is $s$.

Thus assume from now on that we have a flat, proper morphism $f_n : X_n \to S_n$, $s \in S_n$ is the only closed point and $X_s$ is Du Bois. Then $H^0(S_n, R^i f_* \mathcal{O}_X) = H^i(X_n, \mathcal{O}_{X_n})$, hence we can identify the $\phi^i_s$ with the maps

$$\psi^i : H^i(X_n, \mathcal{O}_{X_n}) \to H^i(X_s, \mathcal{O}_{X_s}) \quad (20.4)$$

By the Lefschetz principle we may assume that everything is defined over $\mathbb{C}$. By GAGA (cf. [Har77, App.B]), both sides of (20.4) are unchanged if we replace $X_n$ by the corresponding analytic space $X^a_n$. Let $\mathbb{C}_{X_n}$ (resp. $\mathcal{C}_{X_n}$) denote the sheaf of locally constant functions on $X_n$ (resp. $X_s$) and $j_n : \mathbb{C}_{X_n} \to \mathcal{O}_{X_n}$ (resp. $j_s : \mathbb{C}_{X_s} \to \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$
\begin{array}{ccc}
H^i(X_n, \mathbb{C}_{X_n}) & \xrightarrow{\alpha^i} & H^i(X_s, \mathbb{C}_{X_s}) \\
\downarrow j^n_i & & \downarrow j^s_i \\
H^i(X_n, \mathcal{O}_{X_n}) & \xrightarrow{\psi^i} & H^i(X_s, \mathcal{O}_{X_s})
\end{array}
$$

Note that $\alpha^i$ is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism and $j^s_i$ is surjective since $X_s$ is Du Bois. Thus $\psi^i$ is also surjective. \hfill \Box

A line bundle $L$ on $X$ is called $f$-semi ample if there is an $m > 0$ such that $L^m$ is $f$-generated by global sections. Using cyclic coverings [KM98, Sec.2.4], there is a finite morphism $\pi : Y \to X$ such that $\pi_* \mathcal{O}_Y = \sum_{r=0}^{m-1} L^{-r}$ and $f \circ \pi : Y \to S$ also has slc fiber over $s$. Thus (20) implies the following.

**Corollary 21.** Let $f : X \to S$ be a proper and flat morphism with slc fibers over closed points; $S$ connected. Let $L$ be an $f$-semi ample line bundle on $X$. Then, for all $i$,

1. $R^i f_*(L^{-1})$ is locally free and compatible with base change and
2. $H^i(X_s, L^{-1})$ is independent of $s \in S$. \hfill \Box

Choose $L$ to be $f$-ample above. By [KM98 5.72], $X_s$ is CM iff $H^i(X_s, L^{-m}) = 0$ for all $m \gg 1$ and $i < \dim X$. The latter properties are deformation invariant for slc fibers by (21). Thus we conclude:
Corollary 22. Let \( f : X \to S \) be a projective and flat morphism with slc fibers over closed points; \( S \) connected. Then, if one fiber of \( f \) is CM then all fibers of \( f \) are CM. \( \square \)

(Note that for arbitrary flat, projective morphisms \( f : X \to S \), the set of points \( s \in S \) such that the fiber \( X_s \) is CM is open, but usually not closed.)

The next example shows that non-CM varieties occur among the irreducible components of smoothable, CM and slc varieties.

Example 23. Here is an example of a stable family of projective varieties \( \{ Y_t : t \in T \} \) such that

1. \( Y_t \) is smooth, projective for \( t \neq 0 \),
2. \( K_{Y_t} \) is ample and Cartier for every \( t \),
3. \( Y_0 \) is slc and CM,
4. the irreducible components of \( Y_0 \) are normal, but
5. one of the irreducible components of \( Y_0 \) is not CM.

Let \( Z \) be a smooth Fano variety of dimension \( n \geq 2 \) such that \( -K_Z \) is very ample, for instance \( Z = \mathbb{P}^2 \). Set \( X := \mathbb{P}^1 \times Z \) and view it as embedded by \( |-K_X| \) into \( \mathbb{P}^N \) for suitable \( N \). Let \( C(X) \subset \mathbb{P}^{N+1} \) be the cone over \( X \).

Let \( M \in |-K_Z| \) be a smooth member and consider the following divisors in \( X \):

\[ D_0 := \{(0 : 1) \} \times Z, \quad D_1 := \{(1 : 0) \} \times Z \quad \text{and} \quad D_2 := \mathbb{P}^1 \times M. \]

Note that \( D_0 + D_1 + D_2 \sim -K_X \). Let \( E_1 \subset C(X) \) denote the cone over \( D_i \). Then \( E_0 + E_1 + E_2 \) is a hyperplane section of \( C(X) \) and \( (C(X), E_0 + E_1 + E_2) \) is lc.

For some \( m > 0 \), let \( H_m \subset C(X) \) be a general intersection with a degree \( m \) hypersurface. Then \( (C(X), E_0 + E_1 + E_2 + H_m) \) is snc outside the vertex and is lc at the vertex. Set \( Y_0 := E_0 + E_1 + E_2 + H_m \) since \( Y_0 \sim O_{C(X)}(m+1) \), we can view it as a slc limit of a family of smooth hypersurface sections \( Y_t \subset C(X) \).

The cone over \( X \) is CM, hence its hyperplane section \( E_0 + E_1 + E_2 + H_m \) is also CM. However, \( E_2 \) is not CM. To see this, note that \( E_2 \) is the cone over \( \mathbb{P}^1 \times M \) and, by the Künneth formula,

\[
H^i(\mathbb{P}^1 \times M, O_{\mathbb{P}^1 \times M}) = H^i(M, O_M) = \begin{cases} 
  k & \text{if } i = 0, n-1, \\
  0 & \text{otherwise}.
\end{cases}
\]

Thus \( E_2 \) is not CM.

As in the proof of \cite[III.9.9]{Har77}, we get from (21) the following.

Proposition 24. Let \( f : X \to S \) be a projective, flat morphism with slc fibers over closed points. Then \( \omega_{X/S} \) exists and is compatible with base change. That is, for any \( g : T \to S \) the natural map

\[ g^* \omega_{X/S} \to \omega_{X_T/T} \]

is an isomorphism

where \( g_X : X_T := X \times_S T \to X \) is the first projection. \( \square \)

(This seems like a very complicated way to prove that \( \omega_{X/S} \) behaves as expected, but, as far as I can tell, this was not known before. A proof for non-projective algebraic maps is given in \cite{Kol10a}. I do not know how to prove (24) for analytic morphisms \( f : X \to S \).)
If the fibers $X_s$ are CM, then $H^i(X_s, \omega_{X_s} \otimes L_s)$ is dual to $H^{n-i}(X_s, L_s^{-1})$, and the following is clear. In general, a more detailed inductive argument is needed [Kol10b, Chap.4].

**Corollary 25.** Let $f : X \to S$ be a projective, flat morphism with slc fibers over closed points; $S$ connected. Then, for all $i$,

1. $R^if_*\omega_{X/S}$ is locally free and compatible with base change and
2. $H^i(X_s, \omega_{X_s})$ is independent of $s \in S$. □

4. Modular of semi log canonical models

Let us illustrate (11) with an example of a flat, projective family of surfaces with log canonical singularities over the pair of lines $(xy = 0) \subset \mathbb{C}^2$ such that over one line we have smooth surfaces with ample canonical class and over the other line we have smooth elliptic surfaces.

**Example 26 (Jump of Kodaira dimension).** There are 2 families of nondegenerate degree 4 smooth surfaces in $\mathbb{P}^5$.

One family consists of Veronese surfaces $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2)$. The general member of the other family is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2, 1)$, special members are embeddings of the ruled surface $\mathbb{F}_2$. The two families are distinct since $K^2_{\mathbb{P}^2} = 9$ and $K^2_{\mathbb{P}^1 \times \mathbb{P}^1} = 8$.

For both of these surface, a smooth hyperplane section gives a degree 4 rational normal curve in $\mathbb{P}^4$.

Let $T_0 \subset \mathbb{P}^5$ be the cone over the degree 4 rational normal curve in $\mathbb{P}^4$. $T_0$ has a log canonical (even log terminal) singularity and $K^2_{T_0} = 9$.

For us the interesting feature is that one can write $T_0$ as a limit of smooth surfaces in two distinct ways, corresponding to the two possibilities of writing the degree 4 rational normal curve in $\mathbb{P}^4$ as a hyperplane section of a surface.

From the first family, we get $T_0$ as the special fiber of a flat family whose general fiber is $\mathbb{P}^2$. This family is denoted by $\{T_t : t \in \mathbb{C}\}$. From the second family, we get $T_0$ as the special fiber of a flat family whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. This family is denoted by $\{T'_t : t \in \mathbb{C}\}$. (In general, one needs to worry about the possibility of getting embedded points at the vertex, but in both cases the special fiber is indeed $T_0$.)

Note that $K^2$ is constant in the family $\{T_t : t \in \mathbb{C}\}$ but jumps at $t = 0$ in the family $\{T'_t : t \in \mathbb{C}\}$.

Next we take a suitable cyclic cover of the two families to get similar examples with ample canonical class.

Let $\pi_0 : S_0 \to T_0$ be a double cover, ramified along a smooth quartic hypersurface section. Note that $K_{T_0} \sim_{\mathbb{Q}} -\frac{1}{2}H$ where $H$ is the hyperplane class. Thus, by the Hurwitz formula,

$$K_{S_0} \sim_{\mathbb{Q}} \pi_0^*(K_{T_0} + 2H) \sim_{\mathbb{Q}} \frac{1}{2}\pi_0^*H.$$  

So $S_0$ has ample canonical class and $K^2_{S_0} = 2$. Since $\pi_0$ is étale over the vertex of $T_0$, $S_0$ has 2 singular points, locally (in the analytic or étale topology) isomorphic to the singularity on $T_0$. Thus $S_0$ is a log canonical surface.

Both of the smoothings lift to smoothings of $S_0$. 

From the family \( \{ T_t : t \in \mathbb{C} \} \) we get a smoothing \( \{ S_t : t \in \mathbb{C} \} \) where \( \pi_t : S_t \to \mathbb{P}^2 \) is a double cover, ramified along a smooth octic. Thus \( S_t \) is smooth, \( K_{S_t} \sim_Q \pi_t^* \mathcal{O}_{\mathbb{P}^2}(1) \) is ample and \( K_{S_t}^2 = 2 \).

From the family \( \{ T'_t : t \in \mathbb{C} \} \) we get a smoothing \( \{ S'_t : t \in \mathbb{C} \} \) where \( \pi'_t : S'_t \to \mathbb{P}^1 \times \mathbb{P}^1 \) is a double cover, ramified along a smooth curve of bidegree \((8, 4)\). One of the families of lines on \( \mathbb{P}^1 \times \mathbb{P}^1 \) pulls back to an elliptic pencil on \( S'_t \) and \( K_{S'_t}^2 = 0 \).

In order to exclude such examples, we concentrate on the Hilbert function of a slc model.

**Definition 27** (Hilbert function of slc models). Let \( X \) be an slc model. Note that \( \omega_X \) is locally free outside a subscheme \( Z \subset X \) such that \( Z \) has codimension \( \geq 2 \). Hence the reflexive hull \( \omega_X^{[m]} := (\omega_X^{\otimes m})^{**} \) is isomorphic to \( \omega_X^{\otimes m} \) over \( X \setminus Z \). The Hilbert function of \( X \) is
\[
H_X(m) := \chi(X, \omega_X^{[m]}).
\]
If \( \omega_X^{[N]} \) is locally free, then
\[
\omega_X^{[m_0 + mN]} \cong \omega_X^{[m_0]} \otimes \omega_X^{[mN]},
\]
thus \( H_X(m_0 + mN) \) is a polynomial in \( m \). Thus we can view \( H_X(m) \) as a polynomial in \( m \) whose coefficients are periodic functions (with period \( N \)).

We view \( \chi(X, \omega_X^{[m]} \) as the basic numerical invariants of \( X \). It is then natural to insist that they stay constant in “good” families of slc models. Over a reduced base, this is enough to get the correct definition.

**Definition 28** (Moduli of slc models over reduced bases). Let \( H(m) \) be an integer valued function. On reduced schemes, the moduli functor of semi log canonical models with Hilbert function \( H \) is
\[
\text{SlcMod}_H(S) := \left\{ \begin{array}{l}
\text{Flat, proper families } X \to S, \text{ fibers are slc models with} \\
\text{ample canonical class and Hilbert function } H(m),
\end{array} \right. \mod \text{modulo isomorphisms over } S.
\]

Over an arbitrary base, let \( f : X \to S \) be a flat, proper family of slc models. Note that \( \omega_{X/S} \) is locally free outside a subscheme \( Z \subset X \) such that \( Z \cap X_s \) has codimension \( \geq 2 \) in each fiber. Each \( \omega_{X/S}^{\otimes m} \) is also locally free on \( X \setminus Z \), hence it has a reflexive hull \( \omega_{X/S}^{[m]} \).

If \( s \in S \) is a general point, then \( \omega_{X/S}^{[m]}|_{X_s} \cong \omega_{X_s}^{[m]} \) but for an arbitrary \( s \in S \) we only have a restriction map
\[
r_s : \omega_{X/S}^{[m]}|_{X_s} \to \omega_{X_s}^{[m]}
\]
which is, in general, neither injective nor surjective. The best way to ensure that every fiber of \( X_s \) has the same Hilbert function is to require these restriction maps to be isomorphisms for every \( s \in S \). (It turns out that this is the only way, that is, the kernel and the cokernel of \( r_s \) can nor cancel each other for every \( s \), unless they are both zero.) This leads to our final definition.
Definition 29 (Moduli of slc models). Let $H(m)$ be an integer valued function. The moduli functor of semi log canonical models with Hilbert function $H$ is

$$SlcMod_H(S) := \begin{cases} \text{Flat, proper families } X \to S, \text{ fibers are slc models with} \\
\omega_X^{[m]} \text{ is flat over } S \text{ and commutes with base change,} \\
\text{modulo isomorphisms over } S. \end{cases}$$

Aside 30. We can now explain Principle [11]. The reason is that for flat families of canonical models, $\omega_X^{[m]}$ is automatically flat over $S$ and commutes with base change. This follows from two special properties of canonical singularities. For simplicity, consider a flat family $X \to \text{Spec } k[\epsilon]$ whose special fiber $X_0$ is affine.

First we use that, as a result of the classification of canonical surface singularities [10(2)], there is an open subset $j : U \to X$ whose complement $Z$ has codimension $\geq 3$ such that $\omega_{U_0}$ is locally free. Thus we have an exact sequence

$$0 \to \epsilon \cdot \omega_{U_0}^m \to \omega_U^m \to \omega_{U_0}^m \to 0.$$  

By pushing it forward, we get

$$0 \to \epsilon \cdot j_* \omega_U^m \to j_* \omega_U^m \to j_* \omega_{U_0}^m \to \epsilon \cdot R^1 j_* \omega_{U_0}^m$$

As noted after [14], $\omega_X^{[m]}$ is a CM sheaf, hence has depth $\geq 3$ at every point of $Z$. Therefore,

$$R^1 j_* \omega_{U_0}^m = H^1(U_0, \omega_{U_0}^m) = H^2_{Z_0}(X_0, \omega_X^{[m]}) = 0.$$  

This implies that $\omega_X^{[m]}$ equals $j_* \omega_U^m$ and it is flat over $k[\epsilon]$.

Now that we have the correct definition, we need to prove that the corresponding deformation theory is reasonable. The key result is the following.

Theorem 31. Let $f : X \to S$ be flat, projective morphism whose fibers are slc models. Let $H$ be an integer valued function.

Then there is a locally closed embedding $S_H \hookrightarrow S$ such that a morphism $g : T \to S$ factors through $S_H$ iff $X \times_S T \to T$ is in $SlcMod_H(T)$.

For surfaces, a proof of this is outlined in [Hac03], a general solution is in [AH09]. The following general theory of hulls [Kol08] applies in many similar contexts as well.

Definition 32. Let $X$ be a scheme over a field $k$ and $F$ a coherent sheaf on $X$. Set $n := \dim \text{Supp } F$. The hull of $F$ is the unique $q : F \to F^{[*]}$ such that

1. $q$ is an isomorphism at all generic points of $\text{Supp } F$,
2. $q$ is surjective at all codimension 1 points of $\text{Supp } F$,
3. $F^{[*]}$ is $S_2$.

If $X$ itself is normal, $F$ is coherent and $\text{Supp } F = X$, then $F^{[*]}$ is the usual reflexive hull $F^{**}$ of $F$. The hull of a nonzero sheaf is also nonzero, in contrast with the reflexive hull which kills all torsion sheaves.

One can construct $F^{[*]}$ as follows. First replace $F$ by $F/\text{tors}_{n-1}(F)$ where $\text{tors}_{n-1}(F)$ is the largest subsheaf whose support has dimension $\leq n - 1$. Then there is a closed subscheme $Z \subset \text{Supp } F$ of codimension $\geq 2$ such that $F/\text{tors}(F)$ is $S_2$ on $X \setminus Z$. Let $j : X \setminus Z \to X$ be the open embedding and take

$$F^{[*]} = j_* \left( (F/\text{tors}(F))|_{X \setminus Z} \right).$$
Definition 33. Let $f : X \to S$ be a morphism and $F$ a coherent sheaf. A hull of $F$ is a coherent sheaf $G$ together with a map $q : F \to G$ such that,

1. $G$ is flat over $S$ and
2. for every $s \in S$, the induced map $q_s : F_s \to G_s$ is a hull.

It is easy to see that a hull is unique if it exists.

It is clear from the definition that hulls are preserved by base change. That is, if $g : T \to S$ is a morphism, $X_T := X \times_S T$ and $g_X : X_T \to X$ the first projection then $g^*_X q : g^*_X F \to g^*_X G$ is also a hull.

Definition 34. Let $f : X \to S$ be a projective morphism and $F$ a coherent sheaf on $X$. For a scheme $g : T \to S$ set $\text{Hull}(F)(T) = 1$ if $g^*_X F$ has a hull and $\text{Hull}(F)(T) = \emptyset$ if $g^*_X F$ does not have a hull, where $g_X : T \times_S X \to X$ is the projection.

The main existence theorem is the following.

Theorem 35 (Flattening decomposition for hulls). Let $f : X \to S$ be a projective morphism and $F$ a coherent sheaf on $X$. Then

1. $\text{Hull}(F)$ has a fine moduli space $\text{Hull}(F)$.
2. The structure map $\eta : \text{Hull}(F) \to S$ is a locally closed decomposition, that is, $\eta$ is one-to-one and onto on geometric points and a locally closed embedding on every connected component.

Applying (35) to the relative dualizing sheaf gives the following result.

Corollary 36. Let $f : X \to S$ be projective and equidimensional. Assume that there is a closed subscheme $Z \subset X$ such that $\text{codim}(X_s, Z \cap X_s) \geq 2$ for every $s \in S$, $(X \setminus Z) \to S$ is flat and $\omega_{X/S}$ is locally free on $X \setminus Z$. Then, for any $m$ there is a locally closed decomposition $S_m \to S$ such that for any $g : T \to S$ the following are equivalent

1. $\omega_{X \times_S T/T}^m$ is flat over $T$ and commutes with base change.
2. $g$ factors through $S_m \to S$.

Proof. We claim that $S_m = \text{Hull}(\omega_{X/S}^m)$. Given $g : T \to S$, let $j_T : X \times_S T \setminus Z \times_S T \to X \times_S T$ be the inclusion. Then

$$\omega_{X \times_S T/T}^m = (j_T)_* g_X^* \omega_{X \setminus Z/S}^m.$$ 

If $T \mapsto \omega_{X \times_S T/T}^m$ commutes with restrictions to the fibers of $X \times_S T \to T$, then $\omega_{X \times_S T/T}^m$ has $S_2$ fibers, hence $\omega_{X \times_S T/T}^m$ is the hull of $\omega_{X \times_S T/T}^m$.

Conversely, if $\omega_{X \times_S T/T}^m$ has a hull then it is $\omega_{X \times_S T/T}^m$ and it commutes with further base changes by (33). \hfill \square

In order to prove (31), choose $N$ such that $\omega_{X_s}^{[N]}$ is locally free for every $s \in S$. For $1 \leq i \leq N$, let $S_i \to S$ be as in (36). Take $T$ to be the fiber product $S_1 \times_S \cdots \times_S S_N \to S$. Then $\omega_{X \times_S T/T}^m$ is flat over $T$ and commutes with base change for every $m$. Thus $S_H$ is the disjoint union of those connected components of $T$ where the Hilbert function is $H$. \hfill \square
5. Coarse moduli spaces

Having defined the correct moduli functor for slc models, we can now get down
to studying its properties and the corresponding moduli spaces.

37 (Valuative criterion of separatedness). The functor $\text{SlcMod}$ satisfies the valuative criterion of separatedness, which is essentially (36). This was built into our construction.

38 (Valuative criterion of properness). The short answer is that $\text{SlcMod}$ satisfies the valuative criterion of properness, but some warnings are in order.

We proceed very much as for curves. We start with a family of canonical models over an open curve $X^0 \to B^0$. By the semi-stable reduction theorem of [KKMSD73], after a base change $C^0 \to B^0$ and extending the family over a proper curve $C \supset C^0$, there is a resolution $g: Y \to C$ all of whose fibers are reduced simple normal crossing divisors. Finally we replace $Y$ by its relative canonical model

$$Y^c := \text{Proj}_C \sum_{m \geq 0} g_* (\omega^m_{Y/C}).$$

It is not hard to see that $Y^c \to C$ is in $\text{SlcMod}(C)$ extending $X \times_{B^0} C^0 \to C^0$.

This establishes the valuative criterion of properness if canonical models are dense in the moduli of slc models. We probably mostly care about the irreducible components where canonical models are dense, so we could take this as the final answer.

However, not all irreducible components are such, and it would be better to understand all of them.

So let us start with a family of slc models $X^0 \to B^0$. We can proceed as above, but instead of the relative canonical model of $Y$ we need to take the relative semi log canonical model. As we noted, semi log canonical rings are not always finitely generated [Kol07].

Here the solution is to normalize the family, construct the models of the normalization over $C$ and then try to reconstruct the desired extension of the original family. This is actually quite subtle, see [Kol10b, Chap.3].

39 (Existence of coarse moduli spaces). Fix a function $H$ and an integer $m$. Let $\text{SlcMod}_{H,m}$ be the moduli functor of slc models with Hilbert function $H$ for which $\omega^{[m]}$ is locally free, very ample and has no higher cohomologies. All of these thus embed into $\mathbb{P}^N$ for $N = H(m) - 1$. We use a variant of (31) to show that there is a locally closed subscheme $S_{H,m}$ of the Hilbert scheme $\text{Hilb}(\mathbb{P}^N)$ that parametrizes families of $m$-canonically embedded slc models with Hilbert function $H$.

The general quotient theorems of [Kol97], [KM97] apply and we obtain the coarse moduli space $\text{SlcMod}_{H,m}$ of $\text{SlcMod}_{H,m}$ as the geometric quotient $S_{H,m}/\text{Aut}(\mathbb{P}^N)$.

Finally we let $m$ run through the sequence $2!, 3!, 4!, \ldots$ to get an increasing sequence of coarse moduli spaces whose union is the coarse moduli space $\text{SlcMod}_H$. For now we know only that it is a separated algebraic space which is locally of finite type.

40 (Properness). We saw that $\text{SlcMod}_H$ satisfies the valuative criterion of properness, hence it is proper if it is of finite type.

The components where the canonical models are dense were studied by [Kar00]. He proves that one can control the procedure outlined in (38) uniformly. Thus every such component is of finite type, hence proper.
With some modifications, this implies that every irreducible component of \(\text{SlcMod}_H\) is proper.

Thus the only remaining question is: can there by infinitely many irreducible components?

To illustrate some of the difficulties, let us consider a much simpler question: can we bound the number of irreducible components of a slc surface \(S\) with Hilbert function \(H\)?

For curves the answer is easy. If \(C = \cup_i C_i\) then
\[
2g(C) - 2 = \deg \omega_C = \sum_i \deg(\omega_{C_i}).
\]
Each \(C_i\) on the right hand side contributes at least 1, hence there are at most \(2g - 2\) irreducible components. In the surface case, we have something very similar. If \(S = \cup_i S_i\) then we can compute the self intersection of the canonical class as
\[
(K_S \cdot K_S) = \sum_i (K_{S_i} \cdot K_{S_i}).
\]
The unexpected problem is that \(K_S\) is only a \(\mathbb{Q}\)-Cartier divisor, hence each summand on the right hand side is a positive rational number, not an integer.

We are, however, saved if the contributions on the right are bounded away from 0. This, and much more that is needed for boundedness was proved in [Ale94] and improved in [AM04]. The lower bound \(\frac{1}{164}\) was established in [Kol94]. (I do not know the optimal bound, but \((\mathbb{P}(3, 4, 5), (x^3y+y^2z+z^2x=0))\) has \((K_S + D)^2 = \frac{1}{164}\).

In higher dimensions, recent work of Hacon-McKernan-Xu establishes a lower bound for \((K_X|_{X_i})^n\); the methods are likely to give boundedness as well.

41 (Projectivity). The method of [Kol90] shows that every proper subscheme of \(\text{SlcMod}\) is projective. For \(m\) sufficiently divisible, the 1-dimensional vector spaces \(\det H^0(X, \omega_X^{[m]})\) naturally glue together to an ample line bundle.

The proof uses the Nakai-Moishezon ampleness criterion, thus it works only for proper schemes.

It seems very hard to give quasi-projectivity criteria. For instance, [Kol07] gives an example of a normal crossing surface \(S\) with a line bundle \(L\) and normalization \(\pi : \bar{S} \to S\) such that \(\pi^*L\) is ample yet \(L\) is not ample, in fact no power of \(L\) is generated by global sections.

6. MODULI OF SLC PAIRS

In dimension 1, it is useful to consider not just the moduli of curves but also the moduli of pointed curves. Similarly, in higher dimensions, one should consider the moduli of pairs \((X, \Delta)\) where \(\Delta = \sum a_i D_i\) is a linear combination of divisors with coefficients \(0 \leq a_i \leq 1\). These were first considered in [Ale96].

The first task is to define slc singularities of pairs. This is actually quite natural, see [K+92].

By contrast, finding the correct analog of [29] turns out to be a quite thorny problem. Instead of going into details, let me just present a key example, due to Hassett, which shows that in general we can not view a deformation of a pair as first a deformation of \(X\) and then a deformation of the \(D_i\). One must view \((X, \Delta)\) as an inseparable unit.

Example 42. Let \(S \subset \mathbb{P}^5\) be the cone over the degree 4 rational normal curve. Fix \(r \geq 1\) and let \(D_S\) be the sum of \(2r\) lines. Then \((S, \frac{1}{r}D_S)\) is lc and \((K_S + \frac{1}{r}D_S)^2 = 4\).

There are two different deformations of the pair \((S, D_S)\).
First, set $P := \mathbb{P}^2$ and let $D_P$ be the sum of $r$ general lines. Then $(P, \frac{1}{r} D_P)$ is lc (even canonical if $r \geq 2$) and $(K_P + \frac{1}{r} D_P)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^2$ to the Veronese surface gives a family $f : (X, D_X) \rightarrow \mathbb{P}^1$ with general fiber $(P, D_P)$ and special fiber $(S, D_S)$. We can concretely realize this as deforming $(P, D_P) \subset \mathbb{P}^3$ to the cone over a general hyperplane section. Note that for any general $D_S$ there is a choice of lines $D_P$ such that the above limit is exactly $D_S$.

The total space $(X, D_X)$ is the cone over $(P, D_P)$ (blown up along curve) and $X$ is $\mathbb{Q}$-factorial. The structure sheaf of an effective divisor on $X$ is CM.

In particular, $D_S$ is a flat limit of $D_P$. Since the $D_P$ is a plane curve of degree $r$, we conclude that

$$\chi(\mathcal{O}_{D_S}) = \chi(\mathcal{O}_{D_P}) = -\frac{r(r-3)}{2}.$$ 

Second, set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and let $A, B$ denote the classes of the 2 rulings. Let $D_Q$ be the sum of $r$ lines from the $A$-family. Then $(Q, \frac{1}{r} D_Q)$ is canonical and $(K_Q + \frac{1}{r} D_Q)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $H := A + 2B$ gives a family $g : (Y, D_Y) \rightarrow \mathbb{P}^1$ with general fiber $(Q, D_Q)$ and special fiber $(S, D_S)$. We can concretely realize this as deforming $(Q, D_Q) \subset \mathbb{P}^3$ to the cone over a general hyperplane section. 

The total space $(Y, D_Y)$ is the cone over $(Q, D_Q)$ (blown up along curve) and $Y$ is not $\mathbb{Q}$-factorial. However, $K_Q + \frac{1}{r} D_Q \sim_{\mathbb{Q}} -H$, thus $K_Y + \frac{1}{r} D_Y$ is $\mathbb{Q}$-Cartier and $(Y, \frac{1}{r} D_Y)$ is lc.

In this case, however, $D_Q$ is not a flat limit of $D_P$ for $r > 1$. Thus follows, for instance, from comparing their Euler characteristic:

$$\chi(\mathcal{O}_{D_S}) = -\frac{r(r-3)}{2} \quad \text{and} \quad \chi(\mathcal{O}_{D_Q}) = r.$$ 

Because of their role in the canonical algebra, we are also interested in the sheaves $\mathcal{O}(mK + [\frac{m}{r} D])$.

Let $H_P$ be the hyperplane class of $P \subset \mathbb{P}^5$ (that is, 2 times a line $L \subset P$) and write $m = br + a$ where $0 \leq a < r$. One computes that

$$\chi(P, \mathcal{O}_P(mK_P + [\frac{m}{r} D_P] + nH_P)) = \binom{2n-2m+2}{2} - a(2n - 2m + 1) + \binom{a}{2},$$

$$\chi(S, \mathcal{O}_S(mK_S + [\frac{m}{r} D_S] + nH_S)) = \binom{2n-2m+2}{2} - a(2n - 2m + 1) + \binom{a}{2},$$

$$\chi(Q, \mathcal{O}_Q(mK_Q + [\frac{m}{r} D_Q] + nH_Q)) = \binom{2n-2m+2}{2} - a(2n - 2m + 1).$$

From this we conclude that the restriction of $\mathcal{O}_Y(mK_Y + [mD_Y])$ to the central fiber $S$ agrees with $\mathcal{O}_S(mK_S + [mD_S])$ only if $a \in \{0, 1\}$, that is when $m \equiv 0, 1 \mod r$. The if part was clear from the beginning. Indeed, if $a = 0$ then $\mathcal{O}_Y(mK_Y + [mD_Y])$ is locally free and if $a = 1$ then $\mathcal{O}_Y(mK_Y + [mD_Y])$ is $\mathcal{O}_Y(K_Y)$ tensored with a locally free sheaf. Both of these commute with restrictions.

In the other cases we only get an injection

$$\mathcal{O}_Y(mK_Y + [mD_Y])|_S \hookrightarrow \mathcal{O}_S(mK_S + [mD_S])$$

whose quotient is a torsion sheaf of length $\binom{a}{2}$ supported at the vertex.

There are several ways to overcome these problems; all them will be discussed in [Kol10b].

1. Embedded points do not appear if all the coefficients $a_i$ are $> \frac{1}{2}$.
2. By wiggling the $a_i$ suitably, one again avoids embedded points.
(3) Fix $m$ such that $\mathcal{O}_X(mK_X + m\Delta)$ is locally free. One can identify a pair $(X, \Delta)$ with the corresponding map $\omega^m_X \to \mathcal{O}_X(mK_X + m\Delta)$. It turns out to be easier to deal with the moduli of triples $(X, \omega^m_X \to L)$ for some line bundle $L$.

(4) The branch varieties of [AK06] give another approach.

Acknowledgments. Parts of this paper were written in connection with a lecture series on moduli at IHP, Paris. I thank my audience and especially C. Voisin for the invitation, useful comments and corrections. Partial financial support was provided by the NSF under grant number DMS-0758275.

References

[AH09] Dan Abramovich and Brendan Hassett, Stable varieties with a twist, http://www.citebase.org/abstract?id=oai:arXiv.org:0904.2797, 2009.

[AK06] Valery Alexeev and Allen Knutson, Complete moduli spaces of branchvarieties, 2006.

[Ale94] Valery Alexeev, Boundedness and $K^2$ for log surfaces, Internat. J. Math. 5 (1994), no. 6, 779–810. MR MR1298994 (95k:14048)

[Ale96] Valery Alexeev, Moduli spaces $M_{g,n}(W)$ for surfaces, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22. MR MR1463171 (99b:14010)

[Ale02] Valery Alexeev, Complete moduli in the presence of semiabelian group action, Ann. of Math. (2) 155 (2002), no. 3, 611–708. MR MR1923963 (2003g:14059)

[AM04] Valery Alexeev and Shigefumi Mori, Existence of minimal models for varieties of log general type, http://www.citebase.org/abstract?id=oai:arXiv.org:math/0610203, 2006.

[Bir68] Egbert Brieskorn, Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, Math. Ann. 178 (1968), 255–270. MR MR0233819 (38 #2140)

[DB81] Philippe Du Bois, Complexe de de Rham filtré d’une variété singulière, Bull. Soc. Math. France 109 (1981), no. 1, 41–81. MR MR613848 (82j:14006)

[DJ74] Philippe Dubois and Pierre Jarraud, Une propriété de commutation au changement de base des images directes supérieures du faisceau structural, C. R. Acad. Sci. Paris Sér. A 279 (1974), 745–747. MR MR0376678 (82j:14063)

[Elk81] R. Elkik, Rationalité des singularités canoniques, Inv. Math. 64 (1981), 1–6.

[Hac04] Paul Hacking, Compact moduli of plane curves, Duke Math. J. 124 (2004), no. 2, 213–257. MR MR2078368 (2005f:14056)

[Kaw07] Massayuki Kawakita, Inversion of adjunction on log canonicity, Invent. Math. 167 (2007), no. 1, 129–183. MR MR2264806 (2008a:14025)

[KK09] János Kollár and Sándor J. Kovács, Log canonical singularities are Du Bois, http://www.citebase.org/abstract?id=oai:arXiv.org:0902.0648, 2009.

[KKMSD73] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal embeddings. I, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973. MR MR0355518 (49 #299)

[KM97] Seán Keel and Shigefumi Mori, Quotients by groupoids, Ann. of Math. (2) 145 (1997), no. 1, 193–213. MR MR1432041 (97m:14014)
János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR MR1658959 (2000b:14018)

János Kollár, *Projectivity of complete moduli*, J. Differential Geom. 32 (1990), no. 1, 235–268. MR MR1064874 (92e:14008)

János Kollár, *Log surfaces of general type; some conjectures*, Classification of algebraic varieties (L'Aquila, 1992), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 261–275. MR MR1272703 (95c:14042)

János Kollár, *Quotient spaces modulo algebraic groups*, Ann. of Math. (2) 145 (1997), no. 1, 33–79. MR MR1432036 (97m:14013)

János Kollár, *Two examples of surfaces with normal crossing singularities*, http://www.citebase.org/abstract?id=oai:arXiv.org:0705.0926, 2007.

János Kollár, *Hulls and husks*, http://www.citebase.org/abstract?id=oai:arXiv.org:0805.0576, 2008.

János Kollár, *A local version of the Kawamata-Viehweg vanishing theorem*, http://www.citebase.org/abstract?id=oai:arXiv.org:1005.4843, 2010.

János Kollár, *Moduli of varieties of general type*, (book in preparation), 2010.

J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. 91 (1988), no. 2, 299–338. MR MR922803 (88m:14022)

Robert Lazarsfeld, *Positivity in algebraic geometry. I-II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48–49, Springer-Verlag, Berlin, 2004. MR MR2095471 (2005k:14001a)

I. I. Pjateckiǐ-Šapiro and I. R. Šafareviĉ, *Torelli's theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572. MR MR0284440 (44 #1666)

Miles Reid, *Canonical 3-folds*, Journées de Géometrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 273–310. MR MR605348 (82i:14025)

Yum-Tong Siu, *Invariance of plurigenera*, Invent. Math. 134 (1998), no. 3, 661–673. MR MR1660941 (99i:32035)

Yum-Tong Siu, *Finite generation of canonical ring by analytic method*, Sci. China Ser. A 51 (2008), no. 4, 481–502. MR MR2395400

J. H. M. Steenbrink, *Mixed Hodge structures associated with isolated singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 513–536. MR MR713277 (85d:32044)

Princeton University, Princeton NJ 08544-1000
kollar@math.princeton.edu