Discrete States in Two-Dimensional Quantum Gravity

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Abstract

We review the recent developments in the two-dimensional (super)gravity coupled to $c(\hat{c}) \leq 1$ (super)conformal matter in the conformal gauge. Starting from a pedagogical account of the conformal anomaly in such a system, we show how the system is transformed into the representation in terms of the (free) Liouville field. Some perturbative justification is given to this procedure. The physical states are then examined both for the bosonic and supersymmetric theories, using the BRST formulation. It is explained how new discrete states arise together with some examples. We also discuss the relation with the results for $c = -2$ “topological” gravity. The vertex operator representations for the discrete states are summarized for $c = 1$ theory and are used to examine the interactions of these states. It is found that the states with nontrivial ghost number have interactions governed by the area-preserving diffeomorphism similar to those with vanishing ghost number. The resulting effective action has a BRST-like symmetry.

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1 Introduction

The last few years have witnessed remarkable progress in the attempts to treat two-dimensional (2D) quantum gravity nonperturbatively. This has been initiated by the discovery of the double scaling limit in the matrix models, which enables us to go beyond genus expansion by means of the differential equations satisfied by the nonperturbative partition function [1-3].

These advances have spurred much progress in the continuum approach using the Liouville theory. As is well known [4-6], 2D gravity coupled to conformal matter in the conformal gauge reduces to the Liouville theory with complicated nonlinear dynamics through conformal anomaly. However, inspired by the recent exact solution of the Liouville system in the light cone gauge [7], a method based on conformal field theory has been well developed and this allows us to treat the Liouville field as almost free field [8-10]. Various quantities such as correlation functions have been successfully computed in this approach [11]. The consistancy of whole this procedure requires that the conformal anomaly for the total system (conformal matter coupled to the Liouville theory) vanish!! So far this approach makes sense only when the matter conformal field theory (CFT) has the central charge $c^M \leq 1$.

In this approach, the system may be regarded effectively as a critical string theory in two dimensions, since the Liouville field provides a “time-like” dimension in addition to the space coordinate representing the conformal matter. It is then expected that there will be no degrees of freedom beyond that corresponding to the center of mass since there are no transverse directions; the center of mass motion of the string gives rise to a scalar particle, which in our case becomes massless but is usually called “tachyon” in analogy to the usual critical string. However, it has been found both in the matrix model [3] and Liouville approaches [12-19] that there exist an infinite number of extra degrees of freedom at discrete values of momenta. It is difficult, if not impossible, to understand the origin and the role of these “extra states” in the matrix models since it is not clear what characterize these states. It is thus important to try to understand these issues in
In this article, we will try to give a detailed and pedagogical account of the Liouville approach to the 2D quantum gravity and, in particular, clarify the origin and interactions of these states.

In sect. 2, we begin by reviewing how conformal anomaly arises in the 2D gravity coupled to conformal matter. This discussion shows that the 2D gravity which appears to have no degrees of freedom actually leaves its trace as the Liouville theory. In sect. 3, we introduce the DDK ansatz [8] to reduce the theory to the “free” Liouville theory coupled to the conformal matter such that the total central charge vanishes, giving a conformally invariant system. In sect. 4, some perturbative justification of this trick is discussed [20]. In sect. 5, we start analyzing the physical states using the BRST formalism [21-26]. After some preparation in sect. 5.1, we examine some simple examples in the bosonic theory and give an idea on the general mechanism of the origin of the extra discrete states in sect. 5.2. We then discuss the general case in sect. 5.3. The supersymmetric case is next briefly summarized in sect. 6. Since the general idea is explained in the bosonic case, we only sketch the main steps. In sect. 7, we discuss the relation of our approach to the $c = -2$ theory as considered “topological gravity” [27-30]. In sect. 8, we give a summary of the vertex operator representations of the extra state for $c(\hat{c}) = 1$ theory and check their BRST invariance. We use these representations in sect. 9 to examine the interactions of these discrete states with and without ghost number and show that their interactions are governed by the symmetry of area-preserving diffeomorphism [15,16,19,31-35]. Finally sect. 10 is devoted to discussions and future prospects.

For other approach to the 2D gravity using the collective coordinates and supersymmetric case, we refer the reader to refs. [36, 37, 38].

## 2 Conformal anomaly and Liouville theory

Let us consider the 2D gravity coupled to a CFT with central charge $c^M$. The partition function is given by

$$Z = \int \frac{DgD_\phi X}{V(Diff)} e^{-S(X,g)-S(g)},$$

(2.1)
where \( g_{ab} \) is the two-dimensional metric and \( X \) represents the matter field. We assume that the action \( S(X,g) \) is invariant under the diffeomorphism as well as the Weyl rescaling of the metric \( g \to e^\sigma g \):

\[
S(X,e^\sigma g) = S(X,g).
\]  

(2.2)

The action for gravity is just given by the cosmological term

\[
S(g) = \mu_0^2 \int d^2\xi \sqrt{g},
\]  

(2.3)

where \( \xi^a \) \((a = 1, 2)\) are the coordinates.

It is generally believed that there is no physical degrees of freedom in the metric in two dimensions.\(^1\) We are now going to see that this is violated by the quantum effects called conformal anomaly and this leaves nontrivial dynamics unless certain conditions are satisfied [4, 5]. Let us first choose the conformal gauge

\[
g_{ab} = e^{\phi_0} \hat{g}_{ab},
\]  

(2.4)

where \( \hat{g}_{ab} \) is a reference metric conformally equivalent to Euclidean metric \( \delta_{ab} \). In this gauge, it is convenient to use a complex coordinate \( z \) in place of two real \( \xi^a : z = \xi^1 + i\xi^2 \). We then have (for \( \hat{g}_{ab} = \delta_{ab} \))

\[
g_{ab}(\xi) d\xi^a d\xi^b = e^{\phi_0} d^2 z,
\]  

(2.5a)

\[
R = e^{-\phi_0}(-4\partial_z \partial_{\bar{z}} \phi_0).
\]  

(2.5b)

Namely, \( g_{ab} \) have components \( g_{zz} = g_{\bar{z}\bar{z}} = \frac{1}{2} e^{\phi_0} \) on this basis.

In order to separate the volume of diffeomorphism \( V(\text{Diff}) \) from the integral over the metric, we consider an infinitesimal transformation \( \delta z = v^z(z, \bar{z}) \). This induces a variation of the metric by

\[
(g_{ab} + \delta g_{ab}) d\xi^a d\xi^b = g_{ab}(\xi + \delta\xi) d(\xi^a + \delta\xi^a) d(\xi^b + \delta\xi^b).
\]  

(2.6)

\(^1\) The degrees of freedom in the metric in \( N \) dimensions are obtained as follows. The second-rank symmetric tensor has \( N(N+1)/2 \) components. The diffeomorphism invariance subtracts \( N \) and the gauge fixing subtracts another \( N \), leaving \( N(N-3)/2 \) degrees of freedom (Recall the similar counting in the gauge theory). This gives \(-1\) for \( N = 2 \), meaning no degrees of freedom in two dimensions. This is reflected in the fact that the usual Einstein action is a total divergence and gives just the Euler number

\[
\frac{1}{16\pi} \int d^2 z \sqrt{g} R = -2(h - 1),
\]

where \( h \) is the number of handles.
We thus get
\[
\delta g_{zz} = \partial_z (g_{zz} v^z) + \partial_{v^z} (g_{zz} v^z),
\]
\[
\delta g_{z\bar{z}} = 2 g_{zz} \partial_z v^z, \quad \delta g_{\bar{z}z} = 2 g_{zz} \partial_{\bar{z}} v^z.
\] (2.7)

Using the fact that \( \frac{\partial}{\partial \phi_0} g_{zz} = \frac{\partial}{\partial v^z} g_{zz} = 0 \) and \( \frac{\partial}{\partial \phi_0} g_{z\bar{z}} \) is essentially identity, we can rewrite the integral over the metric as
\[
\mathcal{D} g = \mathcal{D} g_{zz} \mathcal{D} g_{z\bar{z}} \mathcal{D} g_{\bar{z}z}
\]
\[
= \mathcal{D} g v^z \mathcal{D} g_{zz} \mathcal{D} g_{z\bar{z}} \frac{\partial (g_{zz}, g_{z\bar{z}}, g_{\bar{z}z})}{\partial (v^z, v^\bar{z}, \phi_0)}
\]
\[
\sim \mathcal{D} g v^z \mathcal{D} g_{zz} \mathcal{D} g_{z\bar{z}} \phi_0 \det(\partial^z) \det(\partial_{\bar{z}}).
\] (2.8)
The first two integral over \( v^z \) and \( v^\bar{z} \) gives the volume \( V(Diff) \) and the last two determinants can be written in terms of ghosts \( b_{zz} \) and \( c^z \). In this way we obtain
\[
Z = \int \mathcal{D} g_0 \mathcal{D} g \mathcal{D} b \mathcal{D} c \mathcal{D} g X e^{-S(X,g) - S(g) - S(g,b,c)},
\] (2.9)
where
\[
S(g, b, c) = \int \frac{d^2 \xi}{2\pi} \sqrt{g} (b_{zz} \partial^z c^z + c.c.).
\] (2.10)

Similar to the matter action, this is conformally invariant:
\[
S(e^{\phi_0} \hat{g}, b, c) = S(\hat{g}, b, c).
\] (2.11)

Thus the conformal field \( \phi_0(\xi) \) appears to decouple from the system. However, it does not because the volume element \( \mathcal{D} g X \mathcal{D} g b \mathcal{D} g c \) depends on \( \phi_0 \).

If we define an action \( \tilde{S}(g) \) by
\[
e^{-\tilde{S}(g)} \equiv e^{-S(g)} \int \mathcal{D} g b \mathcal{D} g c \mathcal{D} g X e^{-S(g,b,c) - S(X,g)},
\] (2.12)
the partition function is given by
\[
Z = e^{-\tilde{S}(\hat{g})} \int \mathcal{D} g_0 e^{-S_{eff}(\hat{g}, \phi)} ,
\] (2.13)
where
\[
S_{eff}(\hat{g}, \phi_0) = \tilde{S}(e^{\phi_0} \hat{g}) - \tilde{S}(\hat{g}).
\] (2.14)
Our next task is to find an explicit formula for $S_{\text{eff}}(\hat{g}, \phi_0)$. For this purpose, we make the stress-energy tensor by varying the effective action $\Gamma$ with respect to the metric:

$$
T_{zz} = -\frac{2\pi}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{zz}} = \frac{2\pi}{\sqrt{g}} \frac{\delta W}{\delta g_{zz}},
$$

$$
T_{\bar{z}z} = -\frac{2\pi}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\bar{z}z}} = \frac{2\pi}{\sqrt{g}} \frac{\delta W}{\delta g_{\bar{z}z}},
$$

(2.15)

where $W$ is the usual generating functional for connected diagrams:

$$
e^W = \int \mathcal{D}_g \mathcal{D}_g c \mathcal{D}_g X e^{-S(X,g)-S(g,b,c)-S(g)+(\chi,X)+(\beta,b)+(\gamma,c)}.\tag{2.16}
$$

Here $(\ ,\ )$ means integral over the product. Using (2.12), we have

$$
W = -\tilde{S}(g) + \text{(terms involving sources)}.
$$

(2.17)

Since only the first term depends on $\phi_0$, we have

$$
T_{\bar{z}z} = \frac{2\pi}{\sqrt{g}} \frac{\delta \tilde{S}}{\delta \phi_0} g_{\bar{z}z}.
$$

(2.18)

The traceless part $T_{zz}$, on the other hand, has the part which involves $X$, $b$ and $c$ but does not depend on $\phi_0$, and the rest $T^\phi_{zz}$ which depends on $\phi_0$. The first part obeys the conservation law by itself. We thus get from the conservation law of the stress-energy

$$
\partial_z T_{zz}^\phi + \partial_{\bar{z}} \left( \frac{2\pi}{\sqrt{g}} \frac{\delta \tilde{S}}{\delta \phi_0} \right) = 0.
$$

(2.19)

The only local quantity of rank 1 involving the metric is $\partial_z R$, hence we find

$$
\partial_{\bar{z}} T_{zz} = \partial_{\bar{z}} T_{zz}^\phi = -\frac{\lambda}{24} \partial_z R,
$$

(2.20)

where $\lambda$ is a constant to be determined. Combined with (2.19), eq. (2.20) yields

$$
\frac{\delta \tilde{S}}{\delta \phi_0} = \frac{\lambda}{48\pi \sqrt{g}} (R + \mu^2).
$$

(2.21)

Integrating (2.21) using (2.5), we obtain

$$
S_{\text{eff}}(\hat{g}, \phi_0) = \frac{\lambda}{48\pi} \int d^2\xi \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{ab} \partial_a \phi_0 \partial_b \phi_0 + \hat{R} \phi_0 + \mu^2 e^{\phi_0} - \mu^2 \right),
$$

(2.22)
which is the desired Liouville action.

To determine \( \lambda \), we use the Ward identity involving (2.20). If we vary (2.20) with a variation \( \delta g^{ww} \) and use the variational formula

\[
\delta R = (-2\partial_z \partial_{\bar{z}} - R)\delta \phi_0 + \partial_z \partial_{\bar{z}} \delta g_{zz} - \partial_{\bar{z}} \partial_z \delta g^{zz},
\]

we obtain

\[
\frac{1}{2\pi} \partial^2 T_{zz} T_{ww} = \frac{\lambda}{24} \partial_z^3 \delta(z - w) + \text{(less singular terms)},
\]

leading to

\[
T_{zz} T_{ww} \sim \frac{\lambda}{12} \frac{1}{z - w} + \cdots = -\frac{\lambda}{2} \frac{1}{(z - w)^4} + \cdots.
\]

Thus this \( \lambda \) must be minus of the central charge of the matter and \( b, c \) ghosts:

\[
\lambda = 26 - c^M.
\]

This completes the reduction of the 2D gravity to the Liouville theory coupled to a CFT. If \( c^M = 26 \) as in the critical string, the Liouville field is decoupled from the system and one may discard it. However, in non-critical case one has to incorporate the effects of the Liouville field. This is very difficult for the following reason.

The measure for the path integral over \( \phi_0 \) in (2.13) is defined by the complicated norm

\[
\int d^2 \xi \sqrt{g} (\delta \phi_0)^2 = \int d^2 \xi \sqrt{\hat{g}} e^{\phi_0} (\delta \phi_0)^2,
\]

which depends on \( \phi_0 \) itself nonlinearly. This has prevented us from proper treatment of the quantum theory of the Liouville field and non-critical strings. We now turn to a very interesting approach based on conformal field theory which enables us to transform the theory into a more tractable form.

### 3 Liouville theory as free field

The conclusion obtained in the previous section may be summarized as follow. The partition function in the conformal gauge is originally given by (2.9):

\[
Z = \int D_g \phi_0 D_g b D_g c D_g X e^{-S(X,g) - S(g) - S(g,b,c)}.
\]
This was rewritten as (2.13)

\[ Z = \int \mathcal{D}e^{\phi_0} \mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g X e^{-S(X,g) - S(\hat{g},b,c) - \frac{\lambda}{48\pi} S_L(\hat{g},\phi_0)}, \quad (3.2) \]

which is obtained from (2.12), (2.13) and (2.22). Here we have defined the standard Liouville action \( S_L(\hat{g},\phi_0) \) by

\[ S_L(\hat{g},\phi_0) = \int d^2 \xi \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{ab} \partial_a \phi_0 \partial_b \phi_0 + \hat{R} \phi_0 + \mu^2 e^{\phi_0} \right). \quad (3.3) \]

As it stands, the path integral over \( \phi_0 \) is very complicated and it is very difficult to make sense of this theory.

In order to put this system in a more tractable form, we make the change of variables such that the measure is independent of \( \phi_0 \). This will produce a Jacobian as

\[ \mathcal{D}_g \phi_0 \mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g X = \mathcal{D}_g \phi \mathcal{D}_g b \mathcal{D}_g c \mathcal{D}_g X e^{J(\phi,\hat{g})}, \quad (3.4) \]

where \( \mathcal{D}_g \phi \) is the free field measure defined by the norm

\[ \int d^2 \xi \sqrt{\hat{g}} (\delta \phi)^2. \quad (3.5) \]

This procedure is similar to the rewriting of eq. (3.1) as (3.2), where the “difference” (the Jacobian) between the two is given by the Liouville action (3.3). In analogy to this, we assume here that the Jacobian is given by the exponential of a renormalizable local action similar to the Liouville one (3.3) [8]:

\[ e^{J(\phi,\hat{g})} = e^{-S(\phi,\hat{g})}, \quad (3.6a) \]

\[ S(\phi,\hat{g}) = \frac{1}{8\pi} \int d^2 \xi \sqrt{\hat{g}} (\hat{g}^{ab} \partial_a \phi \partial_b \phi - 2Q \hat{R} \phi + 4\mu^2 e^{\alpha \phi}) 
\]

\[ = \frac{1}{2\pi} \int d^2 z (\partial \phi \bar{\partial} \phi - \frac{1}{2} Q \sqrt{\hat{g}} \hat{R} \phi + \mu^2 \sqrt{\hat{g}} e^{\alpha \phi}), \quad (3.6b) \]

where \( Q, \mu' \) and \( \alpha \) are unknown coefficients due to quantum effects.

These parameters are determined by the consistency of the above ansatz. First, let us choose the bare cosmological constant \( \mu_0 \) so as to cancel \( \mu' \). Next, to determine \( Q \), notice that the original theory depends only on \( g = e^{\alpha \phi} \hat{g} \) and so is invariant under

\[ \hat{g} \rightarrow e^{\sigma} \hat{g}, \quad \phi \rightarrow \phi - \sigma / \alpha. \quad (3.7) \]
This means

\[ D_\phi \phi D_\phi b D_\phi c D_\phi X e^{-S(\phi, \hat{g})} = D_\sigma \sigma (\phi - \sigma/\alpha) D_\sigma b D_\sigma c D_\sigma X e^{-S(\phi - \sigma/\alpha, \sigma \hat{g})}. \]  

(3.8)

Since \((\phi - \sigma/\alpha)\) is an integration variable, we may write this as

\[ D_\sigma \sigma \phi D_\sigma b D_\sigma c D_\sigma X e^{-S(\phi, \sigma \hat{g})}, \]  

(3.9)

where we have used the fact that the measure for \(\phi\) is not changed under the shift of \(\phi\) for (3.5). Eqs. (3.8) and (3.9) imply that the total conformal anomaly vanishes! Notice that if we simply disregard the integration over the Liouville mode \(\phi_0\) in eqs. (3.1) and (3.2), we have the Liouville action as a conformal anomaly. The important point here is that the inclusion of the (new) Liouville field \(\phi\) recovers the invariance.

The stress-energy tensor for the Liouville field is given by

\[ T^\phi = -\frac{1}{2} (\partial \phi)^2 - Q \partial^2 \phi, \]  

(3.10)

which tells us that it has the central charge \(c^L = 1 + 12Q^2\). Hence the vanishing condition of the conformal anomaly or total central charge reads

\[ c_M - 26 + 1 + 12Q^2 = 0. \]  

(3.11)

The other parameter \(\alpha\) is determined by demanding that \(g = e^{\alpha \phi} \hat{g}\) be invariant under conformal transformation, or \(e^{\alpha \phi}\) be a conformal tensor of dimension (1,1). The dimension of this operator is given by \(-\frac{1}{2} \alpha (\alpha + 2Q)\), and we get

\[ \alpha_{\pm} = -Q \pm \sqrt{Q^2 - 2} = \frac{-\sqrt{25 - c^M} \pm \sqrt{1 - c^M}}{2\sqrt{3}} \]  

(3.12)

We are now faced with the question which solution yields a theory equivalent to 2D quantum gravity. The consistency with the semiclassical limit \((c^\mu \to -\infty)\) [39] tells us that we should choose \(\alpha_+\).

The above whole argument can be easily extended to supersymmetric case [9].
4 Perturbative calculation of the Jacobian

In sect. 3, we have shown that once we assume the local form of the Jacobian (3.6), consistency uniquely fixes its precise form. In this section, we will describe a justification of the local form of the Jacobian in the perturbative approach [20].

The transformation from the measure defined by (2.26) to the measure by (3.5) produces the formal Jacobian

\[ e^{\tilde{J}(\tilde{g}, \phi)} = |\text{Det}(e^{\phi(z_1)/2 + \phi(z_2)/2 \delta(z_1 - z_2)})|^{1/2}. \] (4.1)

In order to calculate this Jacobian, we have to regularize it. For this purpose, we consider a family of metrics

\[ g(x) = e^{x\phi} \hat{g}, \quad 0 \leq x \leq 1. \] (4.2)

The infinitesimal contribution to the Jacobian \( \delta J[\hat{g}, \phi, x] \) as the metric charges from \( g(x) \) to \( g(x - \delta x) \) is, up to \( \delta x^2 \),

\[ e^{\delta J[\hat{g}, \phi, x]} = |\text{Det}(e^{\delta x\phi(z_1)/2 + \phi(z_2)/2} \delta_g(x)(z_1 - z_2))|^{1/2}. \] (4.3)

With the help of the coordinate basis with states \( |z, x> \) normalized as

\[ <z_1, x|z_2, x> = \delta_{g(x)}(z_1 - z_2) = \frac{1}{\sqrt{g(x)}} \delta(z_1 - z_2), \] (4.4)

eq (4.3) is cast into

\[ \delta \tilde{J}[\hat{g}, \phi, x] = \frac{1}{2} \int \sqrt{g(x)} \delta x \phi(z) <z, x|z, x>. \] (4.5)

This expression is ill-defined because \( <z, x|z, x> \sim \delta(0) \). Let us then regularize this by using the heat kernel as

\[ \delta \tilde{J}_\epsilon[\hat{g}, \phi, x] = \frac{1}{2} \int \sqrt{g(x)} \delta x \phi(z) <z, x|e^{-\epsilon \Delta_{g(x)}}|z, x>, \] (4.6)

where \( \epsilon \) is the regulator and \( \Delta_{g(x)} \) is the Laplacian for the metric \( g(x) \).

The evolution operator \( G(z, z', \epsilon) = <z, x|e^{-\epsilon \Delta_{g(x)}}|z', x> \) satisfies

\[ \lim_{\epsilon \to 0} G(z, z', \epsilon) = \delta_{g(x)}(z - z'), \] (4.7)

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and is a solution of the differential equation
\[
\left( \frac{\partial}{\partial t} + \Delta g(x) \right) G(z, z', t) = 0.
\]
(4.8)

The solution to this equation is well known and its short time expansion is given as
\[
< z, x | e^{-\epsilon \Delta g(x)} | z, x > = \frac{1}{4\pi \epsilon} + \frac{1}{12\pi} R g(x) + O(\epsilon),
\]
(4.9)
giving
\[
\delta \tilde{J} [\hat{g}, \phi, x] = \frac{1}{24\pi} \int \sqrt{\hat{g}} \phi [R_{\hat{g}} - 2x\Delta_{\hat{g}} \phi] \delta x + \frac{1}{8\pi \epsilon} \int \sqrt{\hat{g}} e^{x\phi} \delta x.
\]
(4.10)
The last area term may be renormalized into the cosmological constant term in perturbation theory. Integrating (4.10) over \(x\) gives the local action assumed in (3.6).

It seems that no ambiguity appears in the above “derivation”. However, the heat kernel regularization used above is not unique; for instance, one may use the heat kernel
\[
e^{-\epsilon(\Delta g(x) + \beta R g(x))},
\]
(4.11)
which is the most general expression that is diffeomorphism invariant. The constant \(\beta\) then introduces the ambiguity in the coefficient of \(R\phi\) term which is left in (3.6). Moreover, we should remember that the the reasoning and techniques used above implicitly assume the validity of the usual perturbation. The Jacobian (4.1) has short distance singularity and so it is to be expected that the result is a local expression. Thus it would be fair to say that we have partial support for the ansatz (3.6) at present.

5 Discrete states in the bosonic Liouville theory

5.1 Preliminaries

After this long preparation, we now come to the analysis of the physical states in the 2D gravity coupled to \(e^M \leq 1\) CFT. We will treat this system as a free Liouville scalar field coupled to CFT in a conformally invariant manner, regarding the Liouville
\[R_{e^\phi \hat{g}} = e^{-x\phi} (R_{\hat{g}} - 2x\Delta_{\hat{g}} \phi).\]
exponential interaction (the cosmological constant term) as a marginal deformation; the effects of the cosmological constant will be incorporated perturbatively.

We use the free field realization for the conformal matter. The stress-energy tensor is given by

\[ T^X = -\frac{1}{2}(\partial X)^2 - i\lambda^X \partial^2 X, \tag{5.1} \]

where scalar field \( X \) has the mode expansion

\[ X(z) = q^X - i(p^X - \lambda^X) \ln z + i\sum_{n\neq0} \frac{\alpha_n}{n} z^{-n}, \tag{5.2} \]

with the commutation relation

\[ [\alpha_n, \alpha_m] = n\delta_{n+m,0}, \quad [q^X, p^X] = i. \tag{5.3} \]

It satisfies the Virasoro algebra with the central charge \( c_M = 1 - 12(\lambda^X)^2 \). This is very similar to the Liouville theory (3.10) with \( \lambda^X \) replaced with \( \lambda^L = -iQ \).

The conformal invariance of the whole system may be succinctly summarized by using the BRST charge

\[ Q_B = \oint dz \frac{dz}{2\pi i} c(z)(T^X(z) + T^\phi(z) + \frac{1}{2} T^{bc}(z)), \tag{5.4} \]

where \( T^{bc}(z) \) is the stress-energy tensor for the ghosts. The condition that the total central charge add up to zero becomes

\[ (\lambda^X)^2 + (\lambda^L)^2 = -2, \tag{5.5} \]

which is equivalent to the nilpotency of the BRST charge.\(^3\) In terms of the mode operators, the BRST charge is given by

\[ Q_B = \sum_n c_{-n} L_n X_\phi - \frac{1}{2} \sum_{n,m} (n-m) : c_{-n} c_{-m} b_{n+m} :. \]

\(^3\) The total Virasoro generator \( L_n = L_n^{X,\phi} + L_n^{b,c} \) is given by the anticommutator \( L_n = \{b_n, Q_B\} \).

Hence one has

\[ [L_n, L_m] = [L_n, \{b_m, Q_B\}] = -\{Q_B, [b_m, L_n]\} + \{b_m, [L_n, Q_B]\} = (n-m)(Q_B, b_{m+n}) + \{b_m, [\{b_n, Q_B\}, Q_B]\} = (n-m)L_{m+n} + \text{(terms involving } Q_B^2\text{)}, \]

where we have used the relation \([b_m, L_n] = (m-n)b_{m+n}\). This means that the nilpotency of \( Q_B \) is equivalent to zero total central charge.
\[ L_n^{X,\phi} = \frac{1}{2} \sum_n : (\alpha_n \alpha_{n-m} + \phi_n \phi_{n-m}) : + (n+1)(\lambda^X \alpha_n + \lambda^L \phi_n), \]  

(5.6)

where \( \phi_n, c_n \), and \( b_n \) are the mode operators for the Liouville and ghost fields, and \( \alpha_0 = p^X - \lambda^X, \phi_0 = p^L - \lambda^L \). As usual, the BRST charge is decomposed with respect to the ghost zero modes:

\[ Q_B = c_0 L_0 - b_0 M + d. \]  

(5.7)

The physical states are defined to be nontrivial ones satisfying

\[ Q_B|_{\text{phys}} > = 0. \]  

(5.8)

Any BRST-exact state \((Q_B \chi)\) is trivial in the sense that it trivially satisfies (5.8) and is excluded. This is what is called the BRST cohomology. Since \( L_0 = \{b_0, Q_B\} \), these physical states satisfy

\[ L_0|_{\text{phys}} > = Q_B b_0|_{\text{phys}} >. \]  

(5.9)

Therefore, any physical states are BRST-exact unless they satisfy the on-shell condition \( L_0 = 0 \). It is convenient to reduce the zero eigenspace of \( L_0 \) by restricting to the states annihilated by \( b_0 \). In this space the physical state condition (5.8) reduces to

\[ L_0|_{\text{phys}} > = b_0|_{\text{phys}} > = d|_{\text{phys}} > = 0. \]  

(5.10)

Note that in this space \( d^2 = 0 \).

We now examine a few examples to reveal the general mechanism of the origin of extra physical states.

### 5.2 Examples and the general mechanism

To see how extra states arises, let us consider all possible states at level 1:

\[ N_{FP} = -1 : b_{-1}|p, \downarrow>, \]
\[ N_{FP} = 0 : b_{-1}|p, \uparrow>, \alpha_{-1}|p, \downarrow>, \phi_{-1}|p, \downarrow>, \]
\[ N_{FP} = 1 : \alpha_{-1}|p, \uparrow>, \phi_{-1}|p, \uparrow>, c_{-1}|p, \downarrow>, \]
\[ N_{FP} = 2 : c_{-1}|p, \uparrow>, \]

(5.11)
where the ground state $|p, \downarrow\rangle$ carries the momenta $p^X, p^L$ and is annihilated by $b_0$. We apply $Q_B$ on these to find

$$Q_B b_{-1} |p, \downarrow\rangle = [(p^X - \lambda^X) \alpha_{-1} + (p^L - \lambda^L) \phi_{-1}] |p, \downarrow\rangle,$$
$$Q_B b_{-1} |p, \uparrow\rangle = [(p^X - \lambda^X) \alpha_{-1} + (p^L - \lambda^L) \phi_{-1}] |p, \uparrow\rangle + 2c_{-1} |p, \downarrow\rangle,$$
$$Q_B \alpha_{-1} |p, \downarrow\rangle = (p^X + \lambda^X) c_{-1} |p, \downarrow\rangle,$$
$$Q_B \phi_{-1} |p, \downarrow\rangle = (p^L + \lambda^L) c_{-1} |p, \downarrow\rangle,$$
$$Q_B b_{-1} |p, \uparrow\rangle = (p^X + \lambda^X) c_{-1} |p, \uparrow\rangle,$$
$$Q_B \phi_{-1} |p, \uparrow\rangle = (p^L + \lambda^L) c_{-1} |p, \uparrow\rangle,$$
$$Q_B \alpha_{-1} |p, \uparrow\rangle = (p^X + \lambda^X) c_{-1} |p, \uparrow\rangle,$$
$$Q_B c_{-1} |p, \downarrow\rangle = 0,$$
$$Q_B c_{-1} |p, \uparrow\rangle = 0.$$  \tag{5.12}

The on-shell condition reads

$$- \frac{1}{2} [(p^X)^2 + (p^L)^2] = 1.$$ \tag{5.13}

For general momenta, eq. (5.12) shows that there are no nontrivial physical states. However, for the special values of momenta $p^X = \lambda^X, p^L = \lambda^L$, which are compatible with (5.13) and (5.5), eq. (5.12) becomes

$$Q_B b_{-1} |p, \downarrow\rangle = 0, \quad Q_B b_{-1} |p, \uparrow\rangle = 2c_{-1} |p, \downarrow\rangle,$$
$$Q_B \alpha_{-1} |p, \downarrow\rangle = 2\lambda^X c_{-1} |p, \downarrow\rangle, \quad Q_B \phi_{-1} |p, \downarrow\rangle = 2\lambda^L c_{-1} |p, \down\rangle,$$
$$Q_B \phi_{-1} |p, \uparrow\rangle = 2\lambda^L c_{-1} |p, \uparrow\rangle, \quad Q_B \alpha_{-1} |p, \uparrow\rangle = 2\lambda^X c_{-1} |p, \uparrow\rangle,$$
$$Q_B c_{-1} |p, \down\rangle = 0, \quad Q_B c_{-1} |p, \up\rangle = 0.$$  \tag{5.14}

From these relations, we find that the following states are nontrivial physical states:

$$b_{-1} |p, \down\rangle; \quad (\lambda^L \alpha_{-1} - \lambda^X \phi_{-1}) |p, \down\rangle;$$
$$(\lambda^X \alpha_{-1} + \lambda^L \phi_{-1}) |p, \down\rangle + 2b_{-1} |p, \up\rangle; \quad (\lambda^L \alpha_{-1} - \lambda^X \phi_{-1}) |p, \up\rangle.$$ \tag{5.15}

Others are either BRST-exact or do not vanish.

There is another set of momenta at which similar miracle happens; $p^X = -\lambda^X, p^L = -\lambda^L$. We find in this case that the nontrivial states are given as

$$\alpha_{-1} |p, \down\rangle; \quad c_{-1} |p, \down\rangle;$$
$$\alpha_{-1} |p, \up\rangle; \quad c_{-1} |p, \up\rangle.$$  \tag{5.16}
We can repeat similar analysis at level 2 and higher. In this way we find there appear several nontrivial states at the discrete values of momenta: these states appear only at the fixed values of momenta and hence are quite different from the usual particles.

The above examples already involve the general mechanism of how these extra states appear. Under the action of $Q_B$, a state $|\alpha, p >$ transforms into another state $|\beta, p >$ with a coefficient which is a function of momenta

$$Q_B|\alpha, p > = f(p^X, p^L)|\beta, p >.$$  \hspace{1cm} (5.17)

For general momenta, $f(p_X, p_L)$ does not vanish and these states form the BRST doublet and are unphysical. If, however, $f(p_X, p_L)$ happens to vanish, $|\alpha, p >$ becomes a nontrivial physical state. Since $Q_B|\beta, p > = 0$ for general momenta, we also have a physical state. Thus the extra states appear by the “decomposition” of the BRST doublets into singlets. This also explains why such states always appear in the adjacent values of the ghost number, as in (5.15) and (5.16), and in the general case we discuss in the next subsection.

For example, take the first state in eq. (5.12). Using

$$p^L = i\sqrt{(p^X)^2 + 2}, \lambda^L = i\sqrt{(\lambda^X)^2 + 2},$$ \hspace{1cm} (5.18)

which are obtained from (5.12) and (5.5), we find that it is rewritten as

$$Q_B^{\phi_{-1}}|p, \downarrow > = (p^X - \lambda^X) \left[ \alpha_{-1} - \frac{p^X + \lambda^X}{p^L + \lambda^L} \phi_{-1} \right] |p, \downarrow >.$$ \hspace{1cm} (5.19)

If we put $p^X = \lambda^X, p^L = \lambda^L$, we find two physical states $b_{-1}|p, \downarrow >$ and $(\lambda^L \alpha_{-1} - \lambda^X \phi_{-1})|p, \downarrow >$ in (5.15) by this mechanism. We can show that all other states also appear in this way. As we will discuss in sect. 10, this is related to the vanishing of the null states and this decomposition occurs in general at the levels where null states in the minimal model exist; in this example, the state on the r.h.s. of eq. (5.19) for general momenta is a null state.
5.3 General case

Having got general idea how these extra states appear, the only remaining task is to enumerate all possible cases when this happens.

For this purpose, it is convenient to rewrite the BRST charge in the “lightcone-like” variables defined as

\[ P^\pm(n) = \frac{1}{\sqrt{2}}[(p^X + n\lambda^X) \pm i(p^L + n\lambda^L)], \]

\[ p^\pm \equiv P^\pm(0) = \frac{1}{\sqrt{2}}(p^X \pm ip^L), \]

\[ q^\pm = \frac{1}{\sqrt{2}}(q^X \pm iq^L), \quad \alpha^\pm_n = \frac{1}{\sqrt{2}}(\alpha_n \pm i\phi_n). \]  

(5.20)

We then assign the degrees to the mode operators as follows:

\[ \text{deg} \left( \alpha^+_n, c_n \right) = +1, \quad \text{deg} \left( \alpha^-_n, b_n \right) = -1 \]  

(5.21)

and 0 to the ground state. All the states then carry definite degrees, and the cohomology operator \( d \) in (5.7) is decomposed into three parts with definite degrees.

\[ d = d_0 + d_1 + d_2. \]  

(5.22)

where \( d_0 \) is given by

\[ d_0 = \sum_{n \neq 0} P^+(n)c_{-n}\alpha^-_n. \]  

(5.23)

In our Fock space with definite degrees, each term with definite degrees in \( d^2 = (d_0 + d_1 + d_2)^2 = 0 \) is separately zero. As a result, we have

\[ d_0^2 = d_1^2 = 0, \quad \{d_0, d_1\} = \{d_1, d_2\} = 0, \quad d_1^2 + \{d_0, d_2\} = 0. \]  

(5.24)

Our strategy for examining the cohomology problem consists of the following two steps.

1. Enumerate all possible nontrivial states satisfying

\[ d_0 |\psi > = 0. \]  

(5.25)
2. Examine if it is possible to extend the above obtained states to satisfy

\[ d|\tilde{\psi} >= 0 \]  

by adding higher degree terms.

The reason why we first search for the nontrivial states (5.25) is that any nontrivial state satisfying (5.26) has \( d_0 \)-nontrivial state as its lowest degree term. Indeed, if we decompose \( |\tilde{\psi} > \) into terms of definite degree, \( |\tilde{\psi} >= |\psi_k > + |\psi_{k+1} > + \cdots \), eq. (5.26) gives \( d_0 |\psi_k >= 0 \) as the degree \( k \) term. Hence we must start from \( d_0 \)-nontrivial states in order to construct \( d \)-nontrivial states.

Now our first problem is quite easy. There are two possible cases to be examined.

Case I. \( P^+(n) \neq 0, P^-(n) \neq 0 \) for all \( n \in \mathbb{Z} \).

In this case, if we define

\[ K \equiv \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha^+_n b_n, \]  

\( \{d_0, K\} \equiv \hat{N} \) becomes the number operator for the oscillators. This means that any state satisfying (5.25) with nonzero \( \hat{N} \) is trivial; the argument is similar to the on-shell condition (5.9). Hence the only nontrivial state is the ground state without mode oscillator:

\[ |p^X, p^L >, \]  

satisfying on-shell condition \( L_0 = p^+ p^- + \hat{N} = 0 \), i.e.

\[ -p^+ p^- = -\frac{1}{2} [(p^X)^2 + (p^L)^2] = 0. \]  

(5.29)

This is what is called the “tachyon”.

Case II. \( P^+(j) = P^-(k) = 0 \) for some integers \( j, k \neq 0 \).

From the linearity of \( P^\pm(n) \) in \( n \), we have

\[ P^+(n) = \frac{1}{\sqrt{2}} (\lambda^X + i \lambda^L)(n - j) = \frac{1}{\sqrt{2}} t_+^X (j - n), \]  

\[ P^-(m) = \frac{1}{\sqrt{2}} (\lambda^X - i \lambda^L)(m - k) = \frac{1}{\sqrt{2}} t_-^X (k - n), \]  

\[ p^+ = \frac{1}{\sqrt{2}} t_+^X j, \quad p^- = \frac{1}{\sqrt{2}} t_-^X k, \]  

(5.30)
where \( t^X_\pm = -\lambda^X \pm \sqrt{(\lambda^X)^2 + 2} \) (we have chosen \( \lambda_L = i\sqrt{(\lambda^X)^2 + 2} \)). The on-sell condition (5.10) then tells us that \( d_0 \)-nontrivial states exist only at the level \( \tilde{N} = -p^+ p^- = -P^+(0) P^-(0) = jk \), and hence \( j, k > 0 \) or \( j, k < 0 \). The question is what are the candidates for such states.

If we define
\[
K_j = \sum_{n \neq 0, j} \frac{1}{P^+(n)} \alpha^+_n b_n,
\]
\( \tilde{N}_{0,j} = \{d_0, K_j\} \) becomes the operator for all the mode operators but \( \alpha^+_j, c_{-j} \) for \( j > 0 \) \( (\alpha^-_j, b_j \) for \( j < 0 \). These are the only operators that can produce nontrivial states. At level \( jk \), the nontrivial states are
\[
(\alpha^-_j)^k |p^X, p^L > , \quad c_{-j} (\alpha^+_j)^{k-1} |p^X, p^L >,
\]
for \( j, k > 0 \) and
\[
(\alpha^-_j)^{-k} |p^X, p^L > , \quad b_j (\alpha^-_j)^{-k-1} |p^X, p^L >,
\]
for \( j, k < 0 \).

This exhausts all possible \( d_0 \)-nontrivial states. For example, if \( P^+(j) = 0 \) but \( P^-(n) \neq 0 \), we have
\[
P^-(n) = \frac{1}{\sqrt{2}} (\lambda^X - i\lambda^L) (n - \alpha),
\]
with \( \alpha \) being not integer. This gives \( p^+ p^- = P^+(0) P^-(0) = -j\alpha \), which implies that the nontrivial state is possible only at \( j\alpha \). However, the only available oscillators have level \( j \), which cannot produce states at this level.

Having completed step 1, we now proceed to step 2. Without detailed proof (for which we refer the reader to refs. [14, 18]), we summarize the main lemma necessary to understand the final results.

**Lemma.** If, for each ghost number \( N_{FP} \), the cohomology of \( d_0 \) is nontrivial for at most one fixed degree independent of \( N_{FP} \), then we can construct unique (up to \( d \)-exact term) \( d \)-nontrivial state satisfying (5.26) from the \( d_0 \)-nontrivial state by adding higher degree terms.
Therefore we have discrete physical states whose lowest degree terms are given in (5.32) and (5.33). It is clear that these states for \( j = k = 1 \) correspond some of the states in (5.15) and (5.16).\(^4\) Note that these discrete states have adjacent values of the ghost number \( [N_{FP} = 0, 1 \text{ for (5.32)} \) and \( N_{FP} = 0, -1 \text{ for (5.33)}] \), as we pointed out in sect. 5.2. These physical states exist at level \( jk \) for any given integers \( j \) and \( k \), and there are an infinite number of them. The levels at which they exist are precisely those where null states in the minimal models exist (see, for instance, refs. \([10, 11]\)). This is no accident. We will discuss why this is so in sect. 10.

6 Discrete states in the super-Liouville theory

In this section, we briefly discuss the supersymmetric extension of the results in the previous section. The extension involves the introduction of the additional fermionic partners \((\psi, \xi)\) and \((\beta, \gamma)\) of \((X, \phi)\) and \((b, c)\). Using the lightcone-like variables similar to (5.20), the BRST charge takes the form

\[
Q_B = c_0 L - b_0 M + d \left( -\frac{1}{2} \gamma_0 F + 2\beta_0 K - \frac{1}{4} b_0 \gamma_0^2 \right) \quad (6.1)
\]

for the NS (R) sector, where

\[
L_0 = p^+ p^- + \hat{N},
\]

\[
d = d_0 + d_1 + d_2,
\]

\[
d_0 = \sum_{n \neq 0} P^+(n)c_n \alpha_n - \frac{1}{2} \sum_r P^+(2r) \gamma_{-r} \psi_r. \quad (6.2)
\]

The explicit forms of other operators are not necessary except that \( d^2 = 0 \) in the Fock space with states of definite degrees. The nilpotency of the BRST charge now gives the condition

\[
(\lambda^X)^2 + (\lambda^L)^2 = -1. \quad (6.3)
\]

\(^4\)Here we only discussed the relative cohomology defined by (5.10), and the states in sect. 5.2 contain absolute cohomology defined without the restriction \( b_0 |\text{phys} > 0 \). Hence there are more states in sect. 5.2.
Our strategy for finding nontrivial physical states is the same as in the bosonic case. The only difference is that we have extra degrees of freedom coming from the supersymmetric partner. These appear in the second term in $d_0 (6.2)$.

Let us first examine the NS sector. Due to the additional fermionic term, there are many possibilities to be examined. Case I is the same as the bosonic Liouville and we have only ground state $|p^X, p^L >$ with $p^+ p^- = 0$. Case II has the following two possibilities:
(i) Even $j$.

In this case, it appears that $\alpha_{j}^{+}$ and $c_{-j}$ ($\alpha_{j}^{-}$ and $b_{j}$) for $j,k > 0$ ($j,k < 0$) can produce nontrivial states, but the on-shell condition tells us that nontrivial states are possible only at the level

$$\hat{N} = -p^{+}p^{-} = -P^{+}(0)P^{-}(0) = \frac{1}{2}jk.$$  \hspace{1cm} (6.4)

and hence $j,k > 0$ or $j,k < 0$. For odd $k$ the above mode operators cannot produce states at this level. For even $k$, the nontrivial states are

$$\left(\alpha_{-j}^{-}\right)^{k/2}|p^{X},p^{L}> \ , \ c_{-j}\left(\alpha_{-j}^{+}\right)^{k/2-1}|p^{X},p^{L}> ,$$  \hspace{1cm} (6.5)

for $j,k > 0$ and

$$\left(\alpha_{j}^{+}\right)^{-k/2}|p^{X},p^{L}> \ , \ b_{j}\left(\alpha_{j}^{-}\right)^{-k/2-1}|p^{X},p^{L}> .$$  \hspace{1cm} (6.6)

for $j,k < 0$.

(ii) Odd $j$.

Similarly the nontrivial states may be created by $\alpha_{-j}^{+}, c_{-j}, \psi_{-j/2}^{+}$ and $\gamma_{-j/2}^{-}, b_{j}, \psi_{j/2}^{-}$ and $\beta_{j/2}$) for $j,k < 0$ ($j,k < 0$) at level $\frac{1}{2}jk$. There are many states starting from

$$\left(\gamma_{-j/2}\right)^{k}|p^{X},p^{L}> , \ \psi_{-j/2}^{+}\left(\gamma_{-j/2}\right)^{k-1}|p^{X},p^{L}> ,$$  \hspace{1cm} (6.7)

the rest being obtained by replacing two $\gamma$’s by either $c_{-j}$ or $\alpha_{+j}^{+}$ and so on for $j,k > 0$ (and similarly for $j,k < 0$). How this table ends depends on whether $k$ is even or odd, but there is no essential difference in these two cases.

Just as in the bosonic case, there is no other case with nontrivial $d_{0}$ cohomology.

Using the lemma in the previous section, we see that the ground state tachyon for Case I and the states in (6.5) and (6.6) for Case II (i) with even $j$ and odd $k$ can be extended to nontrivial elements of $d$ cohomology. However, this lemma does not apply to the states in case (ii) since there are many nontrivial states. We have in this case:

**Lemma’**. If a $d_{0}$-nontrivial state transforms into another $d_{0}$-nontrivial one under the action of $d$, those two states cannot give rise to $d$-nontrivial state, since these form BRST doublets \[15, 19\].
Examining how the states in Case II (ii) transform into one another, we can easily find that only the states
\[ \psi^+_{-j/2}(\alpha^-_{-j})^{(k-1)/2}|p^X, p^L >, \]
\[ [ (\alpha^+_{-j})^{(k-1)/2} \gamma_{-j/2} - j(k-1) c_{-j} \psi^+_{-j/2}(\alpha^+_{-j})^{(k-3)/2}] |p^X, p^L >, \] (6.8)
for odd \( j, k > 0 \) and
\[ \psi^-_{j/2}(\alpha^-_{j})^{-\frac{1}{2}(k+1)}|p^X, p^L >, \]
\[ [ (\alpha^-_{j})^{-\frac{1}{2}(k+1)} \beta_{j/2} - \frac{1}{2} b_j \psi^-_{j/2}(\alpha^-_{j})^{-\frac{1}{2}(k+3)}] |p^X, p^L >, \] (6.9)
for odd \( j, k < 0 \), are singlets and can produce nontrivial \( d \)-cohomology. (If \( k \) is even, there is no singlet and hence no nontrivial state.)

To summarize, we have found that there are discrete states at level \( \frac{1}{2} j k \) generated from (6.5), (6.6), (6.8) and (6.9) for \( j - k = \) even. We note that these are precisely the conditions for the null states to exist [41] and that these states again have adjacent values of the ghost number.

The R sector may be similarly examined [18]. After exactly the same procedure, it is easy to show that there are nontrivial states at level \( \frac{1}{2} j k \) with \( j - k = \) odd for given integers \( j \) and \( k \). The nontrivial states are generated form (6.5) and (6.6) for odd \( j \) and even \( k \), and from (6.8) and (6.9) for even \( j \) and odd \( k \). Again the above conditions are those for the existence of the null states.

7 \( c^M = -2 \) topological gravity

If the conformal matter coupled to the 2D gravity has \( c^M = -2 \), it has been noted that the system is quite similar to the topological gravity [27-30].

In our notation using free scalar fields, the stress-energy tensor for such a system is given by
\[ T = -\frac{1}{2}(\partial X)^2 - \frac{i}{2} \partial^2 X - \frac{1}{2}(\partial \phi)^2 + \frac{3}{2} \partial^2 \phi, \] (7.1)
If we use the standard bosonization for the bosonic ghost \[42\]

\[
\beta = -ie^{-iX+\phi} \partial X, \quad \gamma = e^{iX-\phi},
\]

(7.2)
eq (7.1) is rewritten as

\[
T = -2\beta \partial \gamma - \partial \beta \gamma.
\]

(7.3)

The conformal dimensions of these “ghosts” are \(\text{dim}(\beta, \gamma) = (2, -1)\) and this system has the central charge 26 which cancels against -26 from the reparametrization ghosts \((b, c)\).

In the representation by \(\beta\) and \(\gamma\), the Virasoro generator \(L_0\) is just the number operator for all the nonzero mode operators. According to eq. (5.9), this means that the nonzero mode operators cannot produce nontrivial cohomology; only zero modes can give rise to such states.

The mode expansion of the ghosts are given by

\[
\gamma(z) = \sum_n \gamma_n z^{-n+1}, \quad \beta(z) = \sum_n \beta_n z^{-n-2},
\]

(7.4)

and the conformal vacuum satisfies

\[
\gamma_n|0> = \beta_m|0> = 0 \quad \text{for} \quad n \geq 2, \ m \geq -1.
\]

(7.5)

The usual choice of the ground state for the \(\beta - \gamma\) system is

\[
\gamma_n|\tilde{0}> = \beta_m|\tilde{0}> = 0 \quad \text{for} \quad n \geq 1, \ m \geq 0,
\]

(7.6)

which are related to (7.5) by

\[
|\tilde{0}> = e^{\phi(0)}|0>.
\]

(7.7)

According to refs. \[29, 30\], the nontrivial cohomology is given by the states

\[
\gamma_n e^{\phi(0)}|0> = \int \frac{dz}{2\pi i} z^{n-2} \gamma(z) e^{\phi(0)}|0> = \int \frac{dz}{2\pi i} z^{n-2} e^{iX-\phi}(z) e^{\phi(0)}|0> = \int \frac{dz}{2\pi i} z^{n-1} :e^{iX(z)-\phi(z)+\phi(0)}: |0>
\]

which vanishes for \(n > 0\).
This space has only one-dimensional extension (depends only on single integer), in contrast to our analysis in the previous section.

In order to compare these results with ours, let us examine what momenta \((p^X, p^L)\) these states carry. From the correspondence (7.2), we see that \(\gamma_0\) carries momenta \((p^X - \frac{1}{2}, p^L - \frac{3}{2}i) = (1, i)\), and hence the first state in (7.8) has the momenta \[
\left( p^X - \frac{1}{2}, p^L - \frac{3}{2}i \right) = (l, i(l - 1)).
\]

(7.9)

On the other hand, since \(\lambda^X = \frac{1}{2}\) and \(\lambda^L = \frac{3}{2}i\) \((t^X_+ = 1, t^X_- = -2)\), our analysis in sect. 5 indicates that there exist extra discrete states at \[\text{see eq. (5.30)}\]

\[
p^X = \frac{1}{\sqrt{2}}(p^+ + p^-) = \frac{1}{2}(j - 2k), \quad p^L = \frac{1}{\sqrt{2i}}(p^+ - p^-) = \frac{i}{2}(j + 2k).
\]

(7.10)

Comparing (7.10) with (7.9), this seems to indicate \(j = 0\) and \(2k = -2l - 1\). The latter condition contradicts the fact that \(k\) and \(l\) are integers. The resolution of this inconsistency lies probably in the fact that the vacuum \(|\tilde{0}\rangle\) is not unique; for example, if we take \(|\tilde{0}\rangle = e^{\phi^{(0)} + (\phi^{(0)} - iX^{(0)})/2}|0\rangle\), which still satisfies (7.6), we get \(j = 0\) and \(k = -l\). Thus these states are at level 0 in our notation.

It is clear that our analysis allows for the wider space than that considered in the representation in terms of \(\beta\) and \(\gamma\); ours includes states connected by the exponentials of \(\phi_1\) and \(\phi_2\), which cannot be reached by simply multiplying \(\beta\) or \(\gamma\). This wider space has been known as the “picture changed one” in superstring theory \[12\]. If we have an additional constraint that we should identify the picture changed states as in superstring, then the nontrivial cohomology is exhausted by (7.8). It is not clear to us at present whether we should impose such condition in the \(c^M = -2\) theory.

8 Vertex operator representations for \(c^M = 1\) gravity

The analysis in sects. 5 and 6 shows that there are an infinite number of discrete physical states. However, their concrete representations have not been given. Such representations are, in fact, complicated for \(c^M < 1\) gravity and the complete representations
for all of them have not been known although some of them have been given in terms of Schur polynomials \[14, 17\]. Fortunately it has been pointed out that \( c^M = 1 \) gravity, the most interesting case from the physical point of view, allows for rather simple representations by means of the vertex operators \[14-19\]. In this section, we summarize the representations and show the BRST invariance of the states.

For \( c^M < 1 \) theory, the representations of the physical states are characterized by the screening operators with dimension one \[41, 43\]:

\[
S_\pm(z) = e^{i \mu_\pm X(z)}, \quad t^M_\pm = -\lambda X \pm \sqrt{(\lambda X)^2 + 2}. \tag{8.1}
\]

In the limit \( c^M \to 1 \) (\( \lambda X \to 0 \)), these constitute the \( SU(2) \) current algebra with an additional generator \( J^0 \):

\[
J^\pm(z) = e^{\pm i \sqrt{2} X(z)}, \quad J^0(z) = \frac{1}{\sqrt{2}} i \partial X(z), \tag{8.2}
\]

which satisfy the operator product expansion with the level \( \kappa = 1 \):

\[
J^+(z)J^-(w) \sim \frac{\kappa}{(z-w)^2} + \frac{2}{z-w} J^0(w),
\]

\[
J^0(z)J^\pm(w) \sim \pm \frac{1}{z-w} J^\pm(w),
\]

\[
J^0(z)J^0(w) \sim \frac{\kappa/2}{(z-w)^2}. \tag{8.3}
\]

The key of the simplicity of the construction of the discrete states for \( c^M = 1 \) is that these currents form a closed algebra. Thus all the physical states in the \( c^M = 1 \) matter theory belong to representations of the \( SU(2) \) current algebra, which are well known \[44\].

It is easy to see that \( e^{i \sqrt{2} X(z)} \) transforms as the highest weight of spin \( J \) representation. By repeatedly acting with the lowering operator \( J^- \), we get the spin \( J \) multiplet \( V_{J,m}, (m = J, J-1, \ldots, -J) \). Note that for \( |m| = J \), \( V_{J,m} \) are the standard tachyon operator \( e^{i \sqrt{2} m X(z)} \) at particular momenta. The conformal dimension of \( V_{J,m} \) is \( J^2 \).

These primary fields receive some gravitational dressing. We demand that the dressed fields have conformal dimension \((1,1)\) since then it makes sense to integrate them over the surface. The operator \( e^{\alpha \phi} \) has the conformal dimension \(-\frac{1}{2} \alpha (\alpha - 2 \sqrt{2})\), (note: \( \lambda^L = i \sqrt{2} \)).
We see that there are two possibilities:

\[ V_{J,m}(z)e^{\sqrt{2}(1\pm J)\phi(z)}. \]  

(8.4)

The SU(2) quantum numbers \((J, m)\) are related to the integers \((j, k)\) introduced in sect. 5. To see this, note that the operator carries momenta \((p^X, p^L - i\sqrt{2}) = (\sqrt{2}m, -i\sqrt{2}(1\pm J))\). On the other hand, we have \(t^X_\pm = \pm\sqrt{2}\) and \(t^L_\pm = -i\sqrt{2}\). Combined with (5.30), we get

\[ J \equiv \left| \frac{j + k}{2} \right|, \quad m \equiv \frac{j - k}{2}. \]  

(8.5)

Similarly we have constructed all the states found in the analysis in sect. 5. The result is summarized as follows [18].

(1) For \(j, k \in \mathbb{Z}_+\) and \(N_{FP} = 0\),

\[ \Psi_{J,m}^-(z) = (J^-_0)^{J-m}e^{i\sqrt{2}J\phi^M(z)}e^{\sqrt{2}(1+J)\phi^L(z)}, \]

where

\[ J^-_0 \equiv \oint_{C_z} \frac{d\zeta}{2\pi i} J^-(\zeta). \]  

(8.6)

(2) For \(j, k \in \mathbb{Z}_+\) and \(N_{FP} = 1\),

\[ \tilde{\Psi}_{J-1,m}^-(z) = (J^-_0)^{J-m-1} \oint_{C_z} \frac{d\zeta}{2\pi i} \frac{K(\zeta)}{\zeta - z} e^{i\sqrt{2}J\phi^M(z)}e^{\sqrt{2}(1+J)\phi^L(z)}, \]

where

\[ K(z) \equiv \hat{c}(z)J^-(z). \]  

(8.7)

(3) For \(j, k \in \mathbb{Z}_-\) and \(N_{FP} = 0\),

\[ \Psi_{J,m}^+(z) = (J^-_0)^{J-m}e^{i\sqrt{2}J\phi^M(z)}e^{\sqrt{2}(1-J)\phi^L(z)}. \]  

(8.8)

(4) For \(j, k \in \mathbb{Z}_-\) and \(N_{FP} = -1\),

\[ \tilde{\Psi}_{J-1,m}^+(z) = (J^-_0)^{J-m-1} \oint_{C_z} \frac{d\zeta}{2\pi i} L(\zeta)e^{i\sqrt{2}(J-1/2)\phi^M(z)}e^{\sqrt{2}(3/2-J)\phi^L(z)}, \]

where

\[ \hat{c}(z) \equiv \partial c(z) \Psi_{J,m}^-(z), \]  

(8.9)

(5) The caret on \(c\) in (8.9) means that the zero mode \(c_0\) is removed. The term proportional to \(c_0\) in (8.8) (if we included \(c_0\)) gives actually \(c_0\Psi_{J,m}^-(z)\), which has spin \(J\) and should be subtracted. This can be done alternatively by subtracting \(\partial c(z)\Psi_{J,m}^-(z)\) (namely, without using \(\hat{c}\)).
They satisfy

\[ L(z) \equiv b(z) e^{-i\phi^M(z)/\sqrt{\mathbb{T}}} e^{-\phi^L(z)/\sqrt{\mathbb{T}}}. \] (8.12)

The states representing the nontrivial cohomology classes are obtained by acting the above operators (with \( z = 0 \)) on the physical vacuum \( |\lambda > \equiv |\lambda^X = 0 > \otimes |\lambda^L = i\sqrt{2} > \otimes c_1|0 >_{bc} \). Notice that \( \Psi_{j,m}^{(\pm)} \) have spin \( J \) whereas \( \tilde{\Psi}_{j-1,m}^{(\pm)} \) have spin \( (J - 1) \).

These states may also be written in terms of the Schur polynomials defined by

\[ \sum_{k \geq 0} S_k(x) z^k = \exp \left( \sum_{k \geq 1} x_k z^k \right). \] (8.13)

They satisfy

\[ \frac{\partial}{\partial x_j} S_k(x) = S_{k-j}(x), \] (8.14a)

\[ \sum_{m=1}^{k-j} mx_m S_{k-j-m}(x) = (k-j) S_{k-j}(x), \] (8.14b)

\[ S_k(x + y) = \sum_{j=0}^{k} S_j(x) S_{k-j}(y), \] (8.14c)

which can be proved using (8.13). From (8.11), we have for case (4)

\[ \tilde{\Psi}_{j-1,J-1}(0)|\lambda > = \oint_{0} \frac{d\zeta}{2\pi i} \zeta^{2-2J} \sum_{n \leq 1} b_n \zeta^{n-2} e^{-iX_+(\zeta)/\sqrt{\mathbb{T}}} e^{-\phi_+(\zeta)/\sqrt{\mathbb{T}}} |\lambda >, \] (8.15)

where we have denoted the creation operator terms by \( X_+(\zeta) \) and \( \phi_+(\zeta) \), and \( p^X_{\text{max}} = \sqrt{2}J, p^L = -i\sqrt{2}(1 - J) \). Noting that \( iX_+(\zeta) + \phi_+(\zeta) = \sum_{n>0} \frac{1}{n} \alpha_n \zeta^n \), we get

\[ \tilde{\Psi}_{j-1,J-1}(0)|\lambda > = \oint_{0} \frac{d\zeta}{2\pi i} \sum_{n \leq 1, k \geq 0} b_n \zeta^{n-2} S_k \left( -\frac{\alpha^-_m}{m} \right) \zeta^k \left| p^X_{\text{max}}, p^L > \right. \]

\[ = \sum_{n \geq 1} b_{-n} S_{2J-n-1} \left( -\frac{\alpha^-_m}{m} \right) \left| p^X_{\text{max}}, p^L > \right. . \] (8.16)

In this way, all the states created by (8.6)-(8.12) may also be written as

1. \( \Psi_{j,m}^{(-)}(0)|\lambda > = (J^-_0)^{J-m} |p^X_{\text{max}}, p^L >, \quad (J = (j + k)/\sqrt{2}) \),
2. \( \tilde{\Psi}_{j-1,m}^{(-)}(0)|\lambda > = (J^-_0)^{J-m-1} \sum_{n=1}^{2J-1} c_{-n} S_{2J-1-n} \left( -\sqrt{2}\frac{\alpha^-_m}{m} \right) \left| p^X_{\text{max}} - \sqrt{2}, p^L > \right. ,
3. \( \Psi_{j,m}^{(+)}(0)|\lambda > = (J^-_0)^{J-m} |p^X_{\text{max}}, p^L >, \quad (J = -(j + k)/\sqrt{2}) \),
4. \( \tilde{\Psi}_{j-1,m}^{(+)}(0)|\lambda > = (J^-_0)^{J-m-1} \sum_{n=1}^{2J-1} b_{-n} S_{2J-1-n} \left( -\frac{\alpha^-_m}{m} \right) \left| p^X_{\text{max}} - \sqrt{2}, p^L > \right. , \) (8.17)
where $p_{\text{max}}^X = \sqrt{2} J = |j + k|/\sqrt{2}$ and $p^L = -i^{2j+k}/\sqrt[4]{2}$, in agreement with ref. [14]. [Note the difference in the arguments of Schur polynomials in cases (2) and (4).]

It is instructive to check explicitly that these states are BRST invariant. Take, for example, the state (2). Since $J_0^-$ commutes with the BRST charge, it is enough to show this for $m = J - 1$. Applying the BRST charge on the state, we get

$$Q_B \tilde{\Psi}^{(\cdot)}_{J-1,J-1}(0)|\lambda >$$

$$= \left[ \sum_{n,l>0} c_{-m} c_{-n} (n,m) : c_{-m} c_{-n} : c_{-l} S_{2J-1-l} \right] \left[ p^X, p^L \right]$$

$$= \left[ \sum_{n,l>0} c_{-m} c_{-n} (n,m) : c_{-m} c_{-n} : c_{-l} S_{2J-1-l} \right] \left[ p^X, p^L \right].$$

Since $S_{2J-1-l}$ does not depend on $\phi_n$, we get for $n > 0$

$$L_n X_n \phi_{2J-1-l}(x) \left[ p^X, p^L \right]$$

$$= \left\{ \frac{1}{\sqrt{2}} [P^+(n) + P^-(n)] \alpha_n + \frac{1}{2} \sum_{m=1}^{n-1} \alpha_n \alpha_m + \sum_{m>0} \alpha_m \alpha_{m+n} \right\} S_{2J-1-l}(x) \left[ p^X, p^L \right],$$

(8.19)

where $x_n \equiv -\sqrt{2} \frac{\alpha_n}{\alpha_n}$. Using $J = \frac{i+k}{2}, m = \frac{i-k}{2} = J - 1$, we have $j = 2J - 1$ and $k = 1$, and hence from (5.30)

$$P^+(n) = -(n - 2J + 1), \quad P^-(n) = n - 1.$$ (8.20)

With the help of (8.14), the second and third terms in (8.19) may be transformed as

$$\sum_{m=1}^{n-1} \frac{d}{dx_{m+n}} S_{2J-1-l}(x) = \sum_{m=1}^{n-1} S_{2J-1-l-n}(x) = (n-1) S_{2J-1-l-n}(x),$$

$$\sum_{m>0} m x_m \frac{d}{dx_{m+n}} S_{2J-1-l}(x) = \sum_{m>0} m x_m S_{2J-1-l-m-n}(x)$$

$$= (2J - 1 - l - n) S_{2J-1-l-n}(x).$$ (8.21)

Substituting these into (8.19) yields

$$\sqrt{2}(J-1)(-\sqrt{2}) \frac{d}{dx_n} S_{2J-1-l}(x) + (n-1) S_{2J-1-l-n}(x) + (2J - 1 - l - n) S_{2J-1-l-n}(x)$$

$$= -l S_{2J-1-l-n}(x).$$ (8.22)

Note: $\alpha_m = \frac{d}{dx_m} = -\sqrt{2} \frac{d}{dx_m}$. 

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Putting this into (8.18), we are left with

\[
Q_B \tilde{\Psi}_{J-1,J-1}(0)|\lambda> = \left[ - \sum_{n,l>0} l c_{-n} c_{-l} S_{2J-1-l-n} + \frac{1}{2} \sum_{n,m>0} (m-n)c_{-n} c_{-m} S_{2J-1-m-n} \right] |p^X, p^L>
\]

\[
= \frac{1}{2} \sum_{m,n>0} (m+n)c_{-n} c_{-m} S_{2J-1-m-n} |p^X, p^L> = 0.
\]  

(8.23)

The last equality follows from the symmetry of the sum. It is easy to check the BRST invariance of the other states.

Finally we summarize the representation for \(\hat{c}^M = 1\) super-Liouville theory [18]. These are obtained by noting that the physical states form again representations of the \(SU(2)\) current algebra generated by

\[
J^\pm(z) =: \sqrt{2} \psi(z) e^{\pm iX(z)} :, \quad J^0(z) = i \partial X(z),
\]

which satisfy (8.3) with level \(\kappa = 2\).

For the NS sector, they are generated by the following operators:

(1) For \(j,k \in \mathbb{Z}_+\) and \(N_{FP} = 0\),

\[
\Psi_{J,m}(z) = (J_0^-)^{J-m-1} \int_{C_\delta} \frac{d\zeta}{2\pi i} \frac{K(\zeta)}{\zeta - z} e^{iJX(z)} e^{(1-J)\phi(z)}.
\]  

(8.25)

(2) For \(j,k \in \mathbb{Z}_+\) and \(N_{FP} = 1\),

\[
\bar{\Psi}_{J-1,m}(z) = (J_0^-)^{J-m-1} \int_{C_\delta} \frac{d\zeta}{2\pi i} \frac{K(\zeta)}{\zeta - z} e^{iJX(z)} e^{(1+J)\phi(z)},
\]

where

\[
K(z) \equiv \frac{1}{2} \gamma(z) + c(z) \psi(z) e^{-iX(z)}.
\]  

(8.27)

(3) For \(j,k \in \mathbb{Z}_-\) and \(N_{FP} = 0\),

\[
\Psi_{J,m}(z) = (J_0^-)^{J-m} e^{iJX(z)} e^{(1-J)\phi(z)}.
\]  

(8.28)

Here \(J \equiv \frac{j+k}{2}\) and \(m \equiv \frac{j-k}{2}\) are integers. We do not have explicit representation for case (4) at present. It seems necessary to use complicated picture changing to construct the
states in (4). In terms of the Schur polynomials, these are written as follows:

\begin{align}
(1) \quad (J^0_0)^{J-m}|p^X_{\text{max}}, p^L >, \quad (J = (j + k) / \sqrt{2}), \\
(2) \quad (J^0_0)^{J-m-1} \left[ \frac{1}{2} \sum_{r>0} \gamma_r S_{J-r-r} - \frac{1}{2} \right] (-\frac{\alpha-m}{m}) \\
+ \sum_{n,r>0} c_n \psi_{r} S_{J-n-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \right] |p^X_{\text{max}} - 1, p^L >, \\
(3) \quad (J^0_0)^{J-m}|p^X_{\text{max}}, p^L >, \quad (J = -(j + k) / \sqrt{2}),
\end{align}

where \( p^X_{\text{max}} = J \) and \( p^L = -i(1 + J) \).

The BRST invariance of these states may be similarly checked. For example, if we apply the BRST charge to the state in (2), we get, after some algebra,

\begin{align}
Q_B \bar{\Psi}_{-1,1}(0)|\lambda > \\
= \left[ \frac{1}{2} \sum_{n,r>0} c_n \gamma_r (J - 1) \alpha_n S_{J-r-r} - \frac{1}{2} \right] (-\frac{\alpha-m}{m}) \\
+ \frac{1}{4} \sum_{n,r>0} \gamma_r \psi_{s} \alpha_n S_{J-r} - \frac{1}{2} \right] (-\frac{\alpha-m}{m}) \\
+ \frac{1}{4} \sum_{n,r,s} \gamma_{s} \gamma_{r} \psi_{s} \psi_{-s} : \psi_{r} S_{J-n-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \\
+ \frac{1}{2} \sum_{n,r>0} \psi_{q} \psi_{r} (J - 1) \psi_{s} S_{J-r} - \frac{1}{2} \right] (-\frac{\alpha-m}{m}) \\
+ \frac{1}{2} \sum_{n,r,s} \gamma_{s} \gamma_{r} \psi_{s} \psi_{-s} \psi_{-r} S_{J-n-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \\
- \frac{1}{4} \sum_{n,r,t} \gamma_{s} \gamma_{r} \psi_{-t} S_{J-n-r-t-\frac{1}{2}} (-\frac{\alpha-m}{m}) \right] |p^X, p^L >.
\end{align}

We collect the terms involving \( c_n \gamma_r \) and use eqs. (8.14) to find

\begin{align}
- \frac{1}{2} \sum_{n,r>0} (J - 1) c_n \gamma_r S_{J-n-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \\
- \frac{1}{2} \sum_{n,r,q} c_n \gamma_r \alpha_q S_{J-n-q-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \\
- \frac{1}{2} \sum_{n,r,s} c_n \gamma_r \psi_{s} \psi_{-s} S_{J-n-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \\
+ \frac{1}{2} \sum_{n,r,s} c_n \gamma_r \psi_{s} \psi_{-s} S_{J-n-r-\frac{1}{2}} (-\frac{\alpha-m}{m}) \\
= - \frac{1}{2} \sum_{n,r,s} c_n \gamma_r \psi_{s} \psi_{-s} S_{J-n-r-s-t-\frac{1}{2}} (-\frac{\alpha-m}{m}) = 0,
\end{align}

where the last equality follows from the symmetry. It is easy to show that the remaining terms also vanish. We can similarly show the BRST invariance of other states.
States for the R sector contain the two-dimensional spinors from the fermion zero-modes \( \psi_0^\pm = \frac{1}{\sqrt{2}}(\psi_0 \pm i\xi_0) \equiv \frac{1}{2}(\sigma_1 \pm i\sigma_2) \). From the condition \( F \equiv 2[\beta_0, Q_B] = 0 \), we find the spinor structure of the state \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) for cases (1) and (2) and \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) for case (3) [18]. So we may take the vacuum \( \begin{pmatrix} 0 \\ |\lambda> \end{pmatrix} \), with \( \beta_0|\lambda> = 0 \), and create the representatives of nontrivial cohomology classes by using the same operators as in the NS sector given in (8.25,26) for the cases (1) and (2) (but with half-odd-integers \( J \) and \( m \)). For case (3), we should take the vacuum \( \begin{pmatrix} |\lambda> \\ 0 \end{pmatrix} \). Of course, the mode expansions should be modified accordingly. We can similarly check the BRST invariance of these states.

### 9 Interactions of the discrete states

Using the vertex operator representations given in the previous section, it is easy to examine the three-point interactions of the discrete states for \( c^M = 1 \) theory [15,16,18,19,31-35]. These can be most easily obtained from the operator product expansion

\[
\Psi^{(+)}_{J_1,m_1}(z)\Psi^{(+)}_{J_2,m_2}(0) = \cdots + \frac{1}{z} C_{J_1,m_1,J_2,m_2}^{J_3,m_3} g(J_1, J_2) \Psi^{(+)}_{J_3,m_3}(0) + \cdots, \tag{9.1}
\]

where \( C \) are the Clebsch-Gordan coefficients and \( g(J_1, J_2) \) is an unknown function. From the dependence of the zero modes of \( X \) and \( \phi \), \( J_3 \) and \( m_3 \) are determined to be \( J_3 = J_1 + J_2 - 1, \ m_3 = m_1 + m_2 \). For these values, the coefficients become

\[
C_{J_1,m_1,J_2,m_2}^{J_3,m_3} = \frac{N(J_3,m_3) J_2 m_1 - J_1 m_2}{N(J_1,m_1) N(J_2,m_2) \sqrt{J_3(J_3+1)}}, \tag{9.2a}
\]

\[
N(J,m) = \sqrt{(J-m)!(J+m)!}, \tag{9.2b}
\]

To determine \( g(J_1, J_2) \), we may compute the operator product for \( m_1 = J_1 - 1 \) and \( m_2 = J_2 \). In this way, one finds

\[
\Psi^{(+)}_{J_1,m_1}(z)\Psi^{(+)}_{J_2,m_2}(0) = \cdots \frac{\sqrt{2J_3(2J_3-1)!}}{z \sqrt{J_1J_2(2J_1-1)!(2J_2-1)!}} \frac{N(J_3,m_3)(J_2 m_1 - J_1 m_2)}{N(J_1,m_1) N(J_2,m_2)} \Psi^{(+)}_{J_3,m_3}(0) + \cdots. \tag{9.3}
\]
Hence after appropriate change of the normalization, the algebra is given by

\[ \Psi_{J_1,\, m_1}^{(+)}(z)\Psi_{J_2,\, m_2}^{(+)}(0) = \cdots + \frac{1}{z}(J_2m_1 - J_1m_2)\Psi_{J_3,\, m_3}^{(+)}(0) + \cdots. \tag{9.4} \]

The others may be computed similarly, with the result

\[ \begin{align*}
\Psi_{J_1,\, m_1}^{(-)}(z)\Psi_{J_2,\, m_2}^{(-)}(0) &= 0, \\
\Psi_{J_1,\, m_1}^{(+)}(z)\Psi_{J_2,\, m_2}^{(-)}(0) &= 0, \ (J_1 \geq J_2 + 1), \\
\Psi_{J_1,\, m_1}^{(-)}(z)\Psi_{J_3,\, m_3}^{(-)}(0) &= \frac{1}{z}(J_2m_1 - J_1m_2)\Psi_{J_2,\, -m_2}^{(-)}(0). \tag{9.5} \end{align*} \]

These relations are equivalent to the following three-point function:

\[ <0|\Psi_{J_2,\, m_2}^{(+)}(z_1)c(z_1)\Psi_{J_1,\, m_1}^{(+)}(z_2)c(z_2)\Psi_{J_3,\, -m_3}^{(-)}(0)c(0)|0> = (J_2m_1 - J_1m_2). \tag{9.6} \]

The coefficient appearing in the algebra (9.4) is known to be the structure constant of the area-preserving diffeomorphism [45].

The algebra for the states with nonzero ghost number may be computed similarly [36]. We have

\[ \begin{align*}
\tilde{\Psi}_{J_1-1,\, m_1}^{(+)}(z)\tilde{\Psi}_{J_3-1,\, -m_3}^{(-)}(0) &= \cdots + \frac{1}{z}F_{J_1-1,\, m_1,\, J_3-1,\, -m_3}^{J_2,\, -m_2}\tilde{\Psi}_{J_2,\, -m_2}(0) + \cdots, \\
F_{J_1-1,\, m_1,\, J_3-1,\, -m_3}^{J_2,\, -m_2} &= C_{J_1-1,\, m_1,\, J_3-1,\, -m_3}^{J_2,\, -m_2}g(J_1, \, J_2), \tag{9.7} \end{align*} \]

where

\[ C_{J_1-1,\, m_1,\, J_3-1,\, -m_3}^{J_2,\, -m_2} = \frac{(-1)^{J_1-1-m_1}N(J_3-1, \, m_3)}{N(J_1-1, \, m_1)N(J_2, \, m_2)}[m_1J_2-m_2(J_1-1)]. \tag{9.8} \]

To determine \( g(J_1, \, J_2) \), we consider the special case \( m_1 = -J_1 + 2 \) and \( m_2 = -J_2 \). After some calculation, one finds

\[ F = \frac{(-1)^{J_1-1-m_1}N(J_3-1, \, m_3)}{N(J_1-1, \, m_1)N(J_2, \, m_2)} \frac{(2J_3-1)!}{(2J_1-3)!(2J_2-1)!\sqrt{2J_2(J_1-1)(J_3-1)}}[m_1J_2-m_2(J_1-1)]. \tag{9.9} \]

Instead of continuing this line, we have computed the three-point function

\[ <0|\Psi_{J_2,\, m_2}^{(+)}(z_1)c(z_1)\tilde{\Psi}_{J_1-1,\, m_1}^{(+)}(z_2)c(z_2)\tilde{\Psi}_{J_3-1,\, -m_3}^{(-)}(0)c(0)|0> \tag{9.10} \]
in ref. [35] and found that this is given precisely by the constant $F$ in (9.9). After changing the normalization, one finds
\[
<0|\Psi_{J_{2,m_{2}}}(z_{1})c(z_{1})\bar{\Psi}_{J_{1-1,m_{1}}(z_{2})c(z_{2})\bar{\Psi}_{J_{1,-m_{3}}}(0)c(0)|0> = J_{2}m_{1} - (J_{1} - 1)m_{2}, \tag{9.11}
\]
again the structure constant of the area-preserving diffeomorphism.

The results (9.6) and (9.11) may be summarized by the effective action for the three-point interactions
\[
S_{3} = g_{0} \sum_{J_{1,m_{1}},J_{2,m_{2}},A,B,C} f^{ABC} \left\{ \frac{1}{2} J_{2}m_{1} - J_{1}m_{2} \right\} \bar{g}^{(-)A}_{J_{1}+J_{2}+1,m_{1}+m_{2}} \bar{g}^{(+),B}_{J_{1},m_{1}} \bar{g}^{(+),C}_{J_{2},m_{2}} \right\} \int d\phi, \tag{9.12}
\]
where we have introduced the open string coupling constant $g_{0}$ and the Chan-Paton index $A$ in the adjoint representation of a Lie group, and associated coupling constants $\bar{g}^{(s),A}_{J,m}(\bar{g}^{(s),A}_{J,m})$ with vertex operators $\Psi^{(s),A}_{J,m}(\bar{\Psi}^{(s),A}_{J,m})$ with $s = \pm$.

This can be rewritten in terms of fields defined as
\[
\Phi(\phi, \theta, \varphi) = \sum_{s,A,J,m} T^{A}g^{(s),A}_{J,m}M^{s}(J,m)D_{m,0}^{J}(\varphi, \theta, 0)e^{(sJ-1)\phi},
\]
\[
\bar{\Phi}^{(\pm)}(\phi, \theta, \varphi) = \sum_{A,J,m} T^{A}g^{(\pm)A}_{J,m}M^{\pm}(J-1,m)D_{m,0}^{J-1}(\varphi, \theta, 0)e^{(\pm J-1)\phi}, \tag{9.13}
\]
where $M^{s}$ are normalization constants defined by
\[
M^{+}(J,m) = \frac{(J-1)!}{\sqrt{(2J-1)!}}N(J,m), \quad M^{-}(J,m) = \frac{(-1)^{m}}{4\pi}\frac{(2J+1)!}{(J-1)!N(J,m)}. \tag{9.14}
\]
Note that the fields with $N_{FP} \neq 0$ have opposite statistics to those with $N_{FP} = 0$. If we use Poisson brackets for the rotation matrix,
\[
\{D_{m_{1}0}^{J_{1}}, D_{m_{2}0}^{J_{2}}\} = i \frac{N(J_{3},m_{3})}{N(J_{1},m_{1})N(J_{2},m_{2})} \sqrt{\frac{(2J_{1}-1)! (2J_{2}-1)!}{(2J_{3}-1)! (J_{3}-1)!}} \frac{(J_{3}-1)!}{(J_{1}-1)! (J_{2}-1)!} \frac{(J_{2}m_{1} - J_{1}m_{2})D_{m_{3},0}^{J_{3}}}{(9.15)}
\]
it is easy to show that $S_3$ can be written as

$$S_3 = ig_0 \int d\phi e^{2\phi} \int_{S^2} d^2\kappa e^{ij} \text{Tr} \left( \frac{1}{3} \Phi \frac{\partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^j} - 2\Phi^{(-)} \frac{\partial \Phi^{(+)}(+) \partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^j} \right), \quad (9.16)$$

where $x^i = (\theta, \varphi)$.

This form of the effective action reminds us of the similar structure of the nonabelian gauge theory. Following this analogy, it is natural to look for similar "BRST-like" symmetry in the action. For this purpose, it turns out to be convenient to write the action for the states without ghost number in terms of the fields

$$\Phi^{(\pm)}(\phi, \theta, \varphi) = \sum_{s,A,J,m} T_A g^{(\pm)A}_{J,m} M^s(J,m) D_{J,m}(\varphi, \theta, 0) e^{(\pm)}(\pm J-1)$$  \quad (9.17)

We have

$$S_3 = ig_0 \int d\phi e^{2\phi} \int_{S^2} d^2\kappa e^{ij} \text{Tr} \left( \Phi^{(-)} \frac{\partial \Phi^{(+)}(+) \partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^j} - 2\Phi^{(-)} \frac{\partial \Phi^{(+)}(+) \partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^j} \right), \quad (9.18)$$

which has the symmetry under

$$\delta \Phi^{(+) = \lambda \Phi^{(+)}, \delta \Phi^{(-)} = \lambda \Phi^{(-)},$$

$$\delta \Phi^{(+) = 0}, \delta \Phi^{(-)} = 0. \quad (9.19)$$

Note that is a nilpotent transformation $\delta^2 = 0$, similar to the BRST transformation. This transformation is similar to that generated by the operator $L(z)$ defined in eq.(8.7) except that this changes spins of the states; the generator is given by

$$Q = \oint \frac{dz}{2\pi i} b(z) e^{-iX(z)/\sqrt{z}} - \sqrt{z}, \quad (9.20)$$

which is clearly nilpotent. Indeed, it is easy to see

$$Q\Psi^{(+)_{J,m} = \Psi^{(+)_{J+1,\frac{1}{2},m-\frac{1}{2}}}} \quad Q\Psi^{(-)_{J-1,m} = \Psi^{(-)_{J-\frac{1}{2},m-\frac{1}{2}}}}. \quad (9.21)$$

---

8 The factors in the effective action (9.16) are obtained as follows. Since the $\phi$ integration restricts the superscripts to $(-, +, +)$ and produces factor three, the first term becomes

$$\text{Tr} \left( \Phi^{(-)} \left[ \frac{\partial \Phi^{(+)}}{\partial \theta} \frac{\partial \Phi^{(+)}}{\partial \varphi} - \frac{\partial \Phi^{(+)}}{\partial \varphi} \frac{\partial \Phi^{(+)}}{\partial \theta} \right] \right) = \text{Tr} \left( \Phi^{(-)} A \left[ \frac{\partial \Phi^{(+)}}{\partial \theta} \frac{\partial \Phi^{(+)}}{\partial \varphi} - \frac{\partial \Phi^{(+)}}{\partial \varphi} \frac{\partial \Phi^{(+)}}{\partial \theta} \right] T^A \frac{1}{2} [T^B, T^C] \right),$$

which gives the first term in (9.12). The second term is obtained similarly.
The action can then be written as

\[ S_3 = i g_0 \int d \phi e^{2 \phi} \int d^2 x \varepsilon_{ij} \delta \text{Tr} \left[ \Phi(-) \frac{\partial \Phi(+) \Phi(+) \Phi(-) \Phi(+) \Phi(+) \Phi(-)}{\partial x} \right]. \]  

(9.22)

What does this symmetry imply? We have no definite answer yet. However this strongly suggests that these “ghost degrees of freedom” play the role of the ghosts in the usual string and cancel part of the contribution from the \( N_{FP} = 0 \) states.

There is some support to this conjecture. Bershadsky and Klebanov \cite{11} have recently computed the one-loop partition function in \( c^M = 1 \) gravity. The result is

\[ \frac{Z}{V_L} = \frac{1}{4\pi \sqrt{2}} \int d^2 \tau \frac{|\eta(q)|^2}{\tau_2^{3/2}} Z_M(\tau, \bar{\tau}), \]  

(9.23)

where \( V_L = |\ln \mu|/\sqrt{2} \), \( \tau \) is the moduli parameter integrated over the fundamental domain, \( q = e^{2\pi i \tau} \), and \( Z_M(\tau, \bar{\tau}) \) is the matter partition function given by

\[ Z_M(\tau, \bar{\tau}) = \text{Tr} q^{L_0^{(M)} - c^M/24} \bar{q}^{\bar{L}_0^{(M)} - c^M/24} \]
\[ = \frac{1}{|\eta(q)|^2} \sum_{s,t} q^{(s \sqrt{2}/R + tR/\sqrt{2})^2/4} \bar{q}^{(s \sqrt{2}/R - tR/\sqrt{2})^2/4} \]
\[ = \frac{R}{\sqrt{2} \tau_2 |\eta(q)|^2} \sum_{n,m} \exp \left( -\frac{\pi R^2 |n - m\tau|^2}{2\tau_2} \right), \]  

(9.24)

for a scalar field compactified on a circle of radius \( R \). This partition function contains the contribution from the primary fields

\[ \exp[i k X(z) + i \bar{k} \bar{X}(\bar{z})], \quad (k, \bar{k}) = \left( \frac{s \sqrt{2}}{R} + \frac{tR}{\sqrt{2}}, \frac{s \sqrt{2}}{R} - \frac{tR}{\sqrt{2}} \right). \]  

(9.25)

If \( |k| \) or \( |\bar{k}| \) is \( n/\sqrt{2} \), there are additional special primary fields \((8.4)\) whose existence is connected with the vanishing of the null states (see the example \((5.19)\) and the discussion in sect. 10).

For \( |k| \neq n/\sqrt{2} \), there are no null states and no special primary fields, giving the Virasoro character \( X_k = q^{k^2/2}/\eta(q) \). For \( |k| = n/\sqrt{2} \), the primary field \( \exp(inX/\sqrt{2}) \) has a null descendant (which actually vanishes) of dimension \((n + 2)^2/4\). Thus we must subtract the latter contribution:

\[ X_{n,0} = \frac{q^{(n+2)^2/4} - q^{n^2/4}}{\eta(q)}. \]  

(9.26)
However we have another primary field (of the same dimension) which itself has a vanishing descendant. They give the character

\[
\chi_{n,1} = \frac{q^{(n+2)^2/4} - q^{(n+4)^2/4}}{\eta(q)}.
\]  

(9.27)

We have the similar sequence of the characters. If we add these characters, all the contributions from the extra primary fields expect those of the form (9.25) cancel out, giving

\[
\sum_{l=0}^{\infty} \chi_{n,l}(q) = \frac{q^{n^2/4}}{\eta(q)}.
\]  

(9.28)

and it gives the result (9.24). Namely, the partition function looks as if the only primary fields are (9.25), and their Virasoro modules do not contain vanishing descendant.

This result suggests that the “ghost states” with spin \((J - 1)\) cancel against the contributions of the discrete states with \(N_{FP} = 0\) and spin \(J\) except those of the “boundary states” \(V_{J,\pm J} = \exp(\pm iJX/\sqrt{2})\). To really check this possibility in our approach, we have to compute the one-loop partition function with our effective action.

It is possible to extend the above analysis to the super-Liouville theory [18, 19, 31, 32, 33]. There again we find the interactions of the discrete states (in the NS sector) are governed by the same area-preserving diffeomorphism. We refer the reader to the second reference in [18] for the details of computation.

10 Discussions

We have shown in sect. 5 and 6 that there are extra discrete states at the fixed values of momenta. These momenta precisely correspond to the values at which special states with respect to the Virasoro algebra appear. We now discuss why this is so.

For this purpose, let us use a free field realization of the Virasoro algebra. In this representation, the Virasoro generators are expressed in terms of the oscillators. Hence the states generated by the Virasoro generators are rewritten by the oscillators. At level \(N\), the relation can be written as

\[
L^{-I}(\lambda)|t + \lambda> = \sum_J C_{IJ}(t + \lambda, \lambda)\alpha^{-J}|t + \lambda>
\]  

(10.1)
where \( |t + \lambda > \) is a Fock vacuum with the momentum \( p = t + \lambda \), \( L_{-I}(\lambda) \) and \( \alpha_{-J} \) stand for all the independent combinations of the Virasoro generators, and \( C_{IJ}(t + \lambda, \lambda) \) is a matrix depending on \( t \) and \( \lambda \). Clearly the oscillator Fock space is much larger than the space spanned by the Virasoro generators. This means that \( C_{IJ} \) does not necessarily have inverse. The criterion when this happens is given by the Kac determinant formula:

\[
det[C(t + \lambda, \lambda)] = \text{const.} \times \prod_{j,k > 0, 1 \leq jk \leq N} (t - t_{(-j,-k)})^{P(N-jk)}, \tag{10.2a}
\]

\[
det[C(t + \lambda, -\lambda)] = \text{const.} \times \prod_{j,k > 0, 1 \leq jk \leq N} (t - t_{(j,k)})^{P(N-jk)}. \tag{10.2b}
\]

The first equation means that for the particular values of \( t = t_{(-j,-k)} \equiv \frac{1}{2} t_+ + \frac{1}{2} t_- \) (\( t_\pm \equiv -\lambda \pm \sqrt{\lambda^2 + 2} \)), there are states at the level \( jk \) in the oscillator Fock space which cannot be constructed by the Virasoro generators:

\[
\sum_I L^{-I} |t + \lambda > = 0, \tag{10.3}
\]

which give physical states. At the zeros of (10.2b), there are again physical primary states ("null states" for \( c^M < 1 \)) at the same level \( jk \). This is the origin why such discrete states exist at these levels, where the decomposition of the BRST doublets into singlets takes place owing to (10.3).

We have thus given a summary of the Liouville approach to the 2D gravity. There are still some unsolved problems. For example, we might say that we now understand the origin of the discrete states fairly well. However, the role of these states, in particular, those with ghost number is not understood yet.

We hope that through this analysis, we may get new insight into the 2D quantum gravity.

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References

[1] E. Brézin and V. Kazakov, Phys. Lett. **B236** (1990) 144;
    M. R. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635;
    D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64** (1990) 127.

[2] G. Parisi, Phys. Lett. **B238** (1990) 209;
    D. J. Gross and N. Miljković, Phys. Lett. **B238** (1990) 217;
    E. Brézin, V. Kazakov and Al. B. Zamolodchikov, Nucl. Phys. **B338** (1990) 637;
    P. Ginsparg and J. Zinn-Justin, Phys. Lett. **B240** (1990) 333;
    J. Ambjørn, J. Jurkiewicz and A. Krzywicki, Phys. Lett. **B243** (1990) 209, 213.

[3] D. J. Gross and I. Klebanov, Nucl. Phys. **B344** (1990) 475;
    D. J. Gross, I. Klebanov and M. Newmann, Nucl. Phys. **B350** (1990) 333;
    A. M. Sengupta and S. R. Wadia, Int. Jour. Mod. Phys. **A6** (1991) 1961;
    G. Mandal, A. Sengupta and S. R. Wadia, Mod. Phys. Lett. **A6** (1991) 1465;
    U. H. Danielsson and D. J. Gross, Nucl. Phys. **B366** (1991) 3.

[4] A. M. Polyakov, Phys. Lett. **103B** (1981) 207.

[5] D. Friedan, in *Recent Advances in Field Theories and Statistical Mechanics*, 1982
    Les Houches summer school, ed. J. B. Zuber and R. Stora, North-Holland (1984) p. 839;
    O. Alvarez, Nucl. Phys. **B216** (1983) 125.

[6] T. L. Curtright and C. B. Thorn, Phys. Rev. Lett. **48** (1982) 1309;
    E. Braaten, T. Curtright and C. Thorn, Phys. Lett. **118B** (1982) 115; Ann. Phys. **147** (1983) 365;
    E. Braaten, T. Curtright, G. Ghandour and C. Thorn, Phys. Rev. Lett. **51** (1983) 19; Ann. Phys. **153** (1984) 147.
[7] A. M. Polyakov, Mod. Phys. Lett. A2 (1987) 893;
   V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[8] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509;
   F. David, Mod. Phys. Lett. A3 (1989) 1651.

[9] J. Distler, Z. Hlousek and H. Kawai, Int. Jour. Mod. Phys. A5 (1990) 375.

[10] N. Seiberg, Prog. Theor. Phys. Suppl. 102 (1991) 319;
    J. Polchinski, Nucl. Phys. B346 (1990) 253; in Strings ’90, ed. R. Arnowitt et al.,
    World Scientific (1991) p.62; Nucl. Phys. B357 (1991) 241.

[11] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051;
    A. Gupta, S. Trivedi and M. Wise, Nucl. Phys. B340 (1990) 475;
    P. Di Francesco and D. Kutasov, Phys. Lett. B261 (1991) 385;
    Y. Kitazawa, Phys. Lett. 265B (1991) 262;
    N. Sakai and Y. Tanii, Prog. Theor. Phys. 86 (1991) 547; Int. Jour. Mod. Phys. A6 (1991) 2743;
    Phys. Lett. B276 (1992) 41;
    Vl. S. Dotsenko, Mod. Phys. Lett. A6 (1991) 3601.

[12] B. H. Lian and G. J. Zuckerman, Phys. Lett. B254 (1991) 417; Phys. Lett. B266 (1991) 21.

[13] A. M. Polyakov, Mod. Phys. Lett. A6 (1991) 635.

[14] P. Bouwknegt, J. M. McCarthy and K. Pilch, CERN preprints, CERN-TH.6162/91 (1991); TH.6279/91 (1991).

[15] E. Witten, Nucl. Phys. B373 (1992) 187.

[16] I. Klebanov and A. M. Polyakov, Mod. Phys. Lett. A6 (1991) 3273.
[17] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. B266 (1991) 337;
    C. Imbimbo, S. Mahapatra and S. Mukhi, Nucl. Phys. B375 (1992) 399;
    S. Mukhi, Tata Inst. preprint, TIFR/TH/91-50 (1991).

[18] K. Itoh and N. Ohta, Fermilab preprint, FERMILAB-PUB-91/228-T (1991), to
    appear in Nucl. Phys. B; Osaka preprint, OS-GE 22-91 (1991), to appear in Prog.
    Theor. Phys. Suppl. 115.

[19] P. Bouwknegt, J. M. McCarthy and K. Pilch, CERN preprint, CERN-TH.6346/91
    (1991).

[20] N. Marvomatos and J. Miramontes, Mod. Phys. Lett. A4 (1989) 1849;
    E. D’Hoker and P. S. Kurzepa, Mod. Phys. Lett. A5 (1990) 1411;
    E. D’Hoker, Mod. Phys. Lett. A6 (1991) 745.

[21] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. 66 (1979) 1.

[22] M. Kato and K. Ogawa, Nucl. Phys. B212 (1983) 443.

[23] N. Ohta, Phys. Rev. D33 (1986) 1681; Phys. Lett. B179 (1986) 347.

[24] M. Ito, T. Morozumi, S. Nojiri and S. Uehara, Prog. Theor. Phys. 75 (1986) 934.

[25] K. Itoh, Nucl. Phys. 342 (1990) 449;
    T. Kuramoto, Phys. Lett. B233 (1989) 363.

[26] K. Itoh, Int. Jour. Mod. Phys. A6 (1991) 1233;
    T. Kuramoto, Nucl. Phys. B346 (1990) 527.

[27] E. Witten, Comm. Math. Phys. 118 (1988) 411; Nucl. Phys. B340 (1990) 281;
    J. M. F. Labastida, M. Pernici and E. Witten, Nucl. Phys. B310 (1988) 611;
    D. Montano and J. Sonnenschein, Nucl. Phys. B313 (1989) 258;
    R. C. Myers and V. Periwal, Nucl. Phys. B333 (1990) 536.

[28] J. Distler, Nucl. Phys. B342 (1990) 523.
[29] E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457.

[30] K. Fujikawa and H. Suzuki, Nucl. Phys. B361 (1991) 539.

[31] D. Kutasov, E. Martinec and N. Seiberg, Phys. Lett. B276 (1992) 437.

[32] N. Ohta, H. Suzuki and H. Tanaka, unpublished (1991), as cited in the second ref. in [18].

[33] B. Sazdović, private communication.

[34] Y. Matsumura, N. Sakai and Y. Tanii, Tokyo Inst. of Tech. preprint, TIT/HEP-186 (1992).

[35] N. Ohta and H. Suzuki, Osaka preprint, OS-GE 25-92 (1992).

[36] S. R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639;
K. Demeterfi, A. Jevicki and J. P. Rodrigues, Mod. Phys. Lett. A6 (1990) 3199.

[37] A. Jevicki and J. P. Rodrigues, Phys. Lett. B268 (1991) 317;
E. Marinari and G. Parisi, Phys. Lett. B240 (1990) 375;
S. Bellucci, T. R. Govindrajan, A. Kumar and R. N. Oerter Phys. Lett. B249 (1990) 49;
L. Alvarez-Gaumé and J. L. Manes, Mod. Phys. Lett. A6 (1991) 2039;
A. Dabholkar, Rutgers preprint RU-91-20 (1991).

[38] S. R. Das, A. Dhar, G. Mandal and S. R. Wadia, Mod. Phys. Lett. A7 (1992) 71;
IAS preprints, IASSNS-HEP-91/52, 79 (1991)

[39] A. B. Zamolodchikov, Phys. Lett. B117 (1982) 87.

[40] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (19984) 333.

[41] M. Kato and S. Matsuda, Adv. Studies in Pure Math. 16 (1988) 205.
[42] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.

[43] V. S. Dotsenko and V. A. Fateev, Nucl. Phys. B240 (1984) 312; Nucl. Phys. B251 (1985) 691.

[44] M. Wakimoto, Comm. Math. Phys. 104 (1986) 605;
    A. B. Zamolodchikov, unpublished;
    D. Nemeschansky, Phys. Lett. B224 (1989) 121;
    A. Bilal, Phys. Lett. B226 (1989) 272;
    J. Distler and Z. Qui, Nucl. Phys. B340 (1990);
    M. Bershadsky and H. Ooguri, Comm. Math. Phys. 126 (1989) 49;
    A. Gerasimov, A. Morozov, M. Olshanetskky, A. Marshakov and S. Shatashvili, Int. Jour. Mod. Phys. A5 (1990) 2495;
    N. Ohta and H. Suzuki, Nucl. Phys. B332 (1990) 146;
    M. Kuwahara, N. Ohta and H. Suzuki, Phys. Lett. B235 (1990) 57; Nucl. Phys. B340 (1990) 448.

[45] I. Bakas, Phys. Lett. B228 (1989) 57;
    C. Pope, L. Romans and X. Sen, Nucl. Phys. B339 (1990) 191;
    E. Bergshoeff, M. P. Blencowe and K. S. Stelle, Comm. Math. Phys. 128 (1990) 213.

[46] M. Bershadsky and I. R. Klebanov, Nucl. Phys. B360 (1991) 559.