Intrinsic Geometry and Analysis of Diffusion Processes and $L^\infty$-Variational Problems

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Abstract  The aim of this paper is two-fold:

First, we obtain a better understanding of the intrinsic distance of diffusion processes. Precisely, (i) for all $n \geq 1$, the diffusion matrix $A$ is weak upper semicontinuous on $\Omega$ if and only if the intrinsic differential and the local intrinsic distance structures coincide; (ii) if $n = 1$, or if $n \geq 2$ and $A$ is weak upper semicontinuous on $\Omega$, the intrinsic distance and differential structures always coincide; (iii) if $n \geq 2$ and $A$ fails to be weak upper semicontinuous on $\Omega$, the (non-) coincidence of the intrinsic distance and differential structures depend on the geometry of the non-weak-upper-semicontinuity set of $A$.

Second, for an arbitrary diffusion matrix $A$, we show that the intrinsic distance completely determines the absolute minimizer of the corresponding $L^\infty$-variational problem, and then obtain the existence and uniqueness for given boundary data. We also give an example of a diffusion matrix $A$ for which there is an absolute minimizer that is not of class $C^1$. When $A$ is continuous, we also obtain the linear approximation property of the absolute minimizer.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain (connected open subset). Denote by $\mathcal{A}(\Omega)$ the collection of all matrix-valued measurable maps $A = (a_{ij})_{1 \leq i, j \leq n} : \Omega \to \mathbb{R}^{n \times n}$, which are elliptic, that is,
for each $A \in \mathcal{A}(\Omega)$, there exists a continuous function $\lambda : \Omega \to [1, \infty)$ such that

$$\frac{1}{\lambda(x)}|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda(x)|\xi|^2$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, where

$$\langle A(x)\xi, \xi \rangle = \sum_{i,j=1}^{n} \xi_i a_{ij} \xi_j.$$ 

An Hamiltonian associated to $A$ is given by $H(x, \xi) = \langle A(x)\xi, \xi \rangle$. 

Associated to each diffusion matrix $A \in \mathcal{A}(\Omega)$, there is a “Riemannian metric” (differential structure) on $\Omega$: for all $x \in \Omega$ and for each vector $\xi \in T_x \Omega$, the length of $\xi$ is given by $\sqrt{H(x, \xi)}$. The corresponding differential operator $L_A u = \text{div}(A\nabla u)$ generates a regular, strongly local bilinear Dirichlet energy form $\mathcal{E}_A$ with domain $\mathcal{D}(\mathcal{E}_A)$ in $L^2(\Omega)$. Notice that $C^\infty_c(\Omega)$ is a core of $\mathcal{E}_A$, $\mathcal{D}_{\text{loc}}(\mathcal{E}_A) = W^{1,2}_{\text{loc}}(\Omega)$, and for all $f, h \in \mathcal{D}(\mathcal{E}_A)$,

$$\mathcal{E}_A(f, h) = \int_{\Omega} \langle A(x)\nabla f(x), \nabla h(x) \rangle \, dx.$$ 

For the details see for example [14]. Moreover, the intrinsic distance $d_A$ associated to $A$ is defined by

$$d_A(x, y) = \sup\{u(x) - u(y)\}$$

for all $x, y \in \Omega$, where the supremum is taken over all $u \in C(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)$ such that $H(x, \nabla u(x)) \leq 1$ almost everywhere. The ellipticity implies that $d_A$ is locally comparable to the Euclidean distance. We define the pointwise Lipschitz constant by setting

$$\text{Lip}_{d_A} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_A(x, y)},$$

and its local variant

$$|D u|_{d_A}(x) = \lim_{r \to 0} \text{Lip}_{d_A}(u, B(x, r)),$$

where and in what follows

$$\text{Lip}_{d_A}(u, K) = \sup_{x, y \in K, y \neq x} \frac{|u(y) - u(x)|}{d_A(x, y)}.$$ 

Then $\text{Lip}_{d_A}(K)$ denotes the collection of all $u$ with $\text{Lip}_{d_A}(u, K) < \infty$. When $A = I_n$, $d_A$ is the Euclidean distance, and it is always omitted in the above notation.

Motivated by the work of Norris [22], who showed that the intrinsic distance determines the small time asymptotics of heat kernel, Sturm [27] asked the following question: Is a diffusion process determined by the intrinsic distance? In other words, do the intrinsic differential and distance structures coincide in the sense that for $u \in \text{Lip}_{d_A}(\Omega)$,

$$H(x, \nabla u(x)) = (\text{Lip}_{d_A} u(x))^2$$

almost everywhere?
The answer to this question is not always in the positive as shown by Sturm’s construction [27, Theorem 2]: for each $A \in \mathcal{A}(\Omega)$, there exists a $\tilde{A} \in \mathcal{A}(\Omega)$ with $d_{\tilde{A}} = d_A$ but
\[ \langle \tilde{A}(x)\xi, \xi \rangle < \langle A(x)\xi, \xi \rangle \]
for all $\xi \in \mathbb{R}^n \setminus \{0\}$; see also [20] for a different example. On the other hand, with the additional assumption that $A$ is continuous, Sturm [27, Proposition 4] proved that the intrinsic differential and distance structures coincide.

The first aim of this paper is to obtain a better understanding on the properties of $A$ that determine the (non-)coincidence of intrinsic differential and distance structures. It turns out that weak upper semicontinuity plays a critical role. A function $u$ is said to be \textit{weak upper semicontinuous} at $x \in \Omega$ if there exists a set $E$ with $|E| = 0$ such that
\[ u(x) \geq \limsup_{(\Omega \setminus E) \ni y \to x} u(y), \]
and is said to be weak upper semicontinuous on $\Omega$ if it is weak upper semicontinuous at almost all $x \in \Omega$. A diffusion matrix $A$ is said to be \textit{weak upper semicontinuous} at $x \in \Omega$ (resp. on $\Omega$) if for every $\xi \in S^{n-1}$, $\langle A(\cdot)\xi, \xi \rangle$ is weak upper semicontinuous at $x$ (resp. at almost all $x \in \Omega$). Denote by $\mathcal{A}_{\text{wusc}}(\Omega)$ the collection of all $A \in \mathcal{A}(\Omega)$ that are weak upper semicontinuous on $\Omega$. We prove the following results.

(i) For all $n \geq 1$, the diffusion matrix $A$ belongs to $\mathcal{A}_{\text{wusc}}(\Omega)$ if and only if the intrinsic differential and the local intrinsic distance structures coincide in the sense that for all $u \in C^1_{\text{loc}}(\Omega)$, $|Du|_{d_A}^2(x) = H(x, \nabla u(x))$ almost everywhere; see Theorem 2.6.

(ii) If $n = 1$, or if $n \geq 2$ and $A \in \mathcal{A}_{\text{wusc}}(\Omega)$, then the intrinsic distance and differential structures always coincide, that is, for all $u \in \text{Lip}(\Omega)$, $(\text{Lip}_{d_A} u(x))^2 = H(x, \nabla u(x))$ almost everywhere; see Theorems 2.1 and 2.2.

(iii) If $n \geq 2$ and $A \notin \mathcal{A}_{\text{wusc}}(\Omega)$, the (non-) coincidence of the intrinsic distance and differential structures depend on the geometry of the non-weak-upper-semicontinuity set of $A$. Indeed, we construct two examples via a large Cantor set and a large Sierpinski carpet to show that both coincidence and noncoincidence may happen; see, respectively, Theorem 3.4 and Proposition 3.1.

The proofs of Theorems 2.2 and 2.6 rely on the (key) Lemmas 2.3 and 2.4. The proof of Theorem 3.4 is more intricate; we use an approximation of the distance by Norris [22] to derive some careful estimates on a good set of the distance function based on geometric properties of our Sierpinski carpet. Proposition 3.1 uses the geometry of the complement of our large Cantor set.

We also consider the $L^\infty$-variational problem associated with an arbitrary matrix-valued map $A \in \mathcal{A}(\Omega)$: the goal is to study the local minimizers of the functional
\[ F(u; U) = \text{esssup}_{x \in U} H(x, \nabla u(x)) \]
over the class of Lipschitz functions on $U \Subset \Omega$ with a given boundary data. This study was initiated by Aronsson [2, 3, 4, 5] in the case $H(x, \xi) = |\xi|^2$, that is, $A = I_n$. He introduced the idea of absolute minimizer, that is, minimize $F$ on all open subset of $U$. To
be precise, let $U$ be an open subset such that $\overline{U} \subset \Omega$. A function $u \in \text{Lip}(U)$ is said to be an absolute minimizer for $H$ on $U$ if for every open subset $V \subset U$ and $v \in \text{Lip}(V) \cap C(\overline{V})$ with $u|_{\partial V} = v|_{\partial V}$, we have

$$\text{esssup}_{x \in V} H(x, \nabla u(x)) \leq \text{esssup}_{x \in V} H(x, \nabla v(x)).$$

Moreover, given a function $f \in \text{Lip}(\partial U)$, $u \in \text{Lip}(U)$ is said to be an absolutely minimizing Lipschitz extension of $f$ if $u$ is an absolute minimizer for $H$ and $u|_{\partial U} = f$. In recent years, the study of the $L^\infty$-variational problem, even for more general Hamiltonians but with some smoothness, has advanced significantly; see [6] for a survey and [10, 17] for some seminal works. The $L^\infty$-variational problem is still interesting even if the Hamiltonian is not smooth or even continuous. See for example [6, 15, 7, 8] and the reference therein.

In this case, one cannot always derive an Aronsson equation from the $L^\infty$-variational problem.

Our results concerning absolute minimizer are as follows:

(iv) For arbitrary $A \in \mathcal{A}(\Omega)$ and the Hamiltonian $H(x, \xi) = \langle A(x)\xi, \xi \rangle$, we show that the absolute minimizer is completely determined by the intrinsic distance, and then obtain the existence and uniqueness of the absolute minimizer given a boundary data; see Theorem 4.1. Consequently, if $A, \tilde{A} \in \mathcal{A}(\Omega)$ and $d_A = d_{\tilde{A}}$, then given the boundary data, the absolute minimizers associated to $A$ and $\tilde{A}$ coincide.

(v) Associated to the diffusion matrix $A / \notin \mathcal{A}_{\text{wusc}}(\mathbb{R}^n)$ given in Subsection 3.2, we show in Proposition 4.2 that there is an absolute minimizer $u$ on $(0, 1)^n$ which fails to be $C^1$. This example indicates that perhaps weak upper semicontinuity of $A$ is needed in order for the corresponding absolute minimizers to be of class $C^1$.

(vi) We obtain in Theorem 5.1 the linear approximation property of the absolute minimizer at all points of continuity of $A$, and hence at all points when $A$ is continuous on $\Omega$.

The proof of Theorem 4.1 relies on the (crucial) Lemma 2.3 and Lemma 4.4, which allows us to describe the absolute minimizer via the pointwise Lipschitz constant. Then the existence of the absolute minimizer follows from [19], while the uniqueness will be proved following the idea of [1] (see [23] for an earlier proof via the tug of war). Proposition 4.2 follows from Theorem 3.4 and properties of absolute minimizers. The proof of Theorem 5.1 borrows the blow-up ideas of [10], but due to the change of distance in the blow-up process, a detailed study is necessary.

The $C^1$-regularity of the absolute minimizer is still open except for the case $n = 2$ and $A = I_n$. Precisely, if $A = I_n$, Savin [24] obtained the $C^1$-regularity of the absolute minimizer when $n = 2$ (see also [28] for a homogeneous norm and [12]) while Evans-Smart [13] obtained the everywhere differentiability when $n \geq 3$. All the proofs in [24, 12, 13, 28] rely on the linear approximation property; indeed, controlling the convergence of different sequences appearing in the linear approximation. For an arbitrary continuous or even $C^1$-continuous $A$, we do not know if it is possible to obtain the everywhere differentiability by controlling the linear approximation process provided in Theorem 5.1 as done in [24, 12, 13, 28].
Finally, we state some conventions. Throughout the paper, we denote by $C$ a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as $C_0$, do not change in different occurrences. The notation $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. Denote by $\mathbb{N}$ the set of positive integers. If $V$ is a bounded open set with $\nabla \subset U$, we simply write $V \Subset U$. We use $C(\Omega)$ to denote the continuous function on $\Omega$ while $C^1(\Omega)$ the function with continuous gradient on $\Omega$. For any locally integrable function $f$, we denote by $\int_E f \, d\mu$ the average of $f$ on $E$, namely, $\int_E f \, d\mu \equiv \frac{1}{\mu(E)} \int_E f \, d\mu$.

2 Case $n = 1$ or $A \in \mathcal{A}_{\text{wusc}}(\Omega)$: $H(\cdot, \nabla u) = (\text{Lip}_{d_A} u)^2$

We first show that if $n = 1$, or if $n \geq 2$ and $A \in \mathcal{A}_{\text{wusc}}(\Omega)$, then the intrinsic distance and differential structures always coincide in the sense that for all $u \in \text{Lip}(\Omega)$, $(\text{Lip}_{d_A} u(x))^2 = H(x, \nabla u(x))$ almost everywhere; see Theorems 2.1 and 2.2. Then, for all $n \geq 1$, we prove that $A \in \mathcal{A}_{\text{wusc}}(\Omega)$ if and only if the intrinsic differential and the local intrinsic distance structures coincide in the sense that for all $u \in C^1_{\text{loc}}(\Omega)$, $|Du|_{d_A}^2 (x) = H(x, \nabla u(x))$ almost everywhere; see Theorem 2.6.

**Theorem 2.1.** If $n = 1$, then for all $u \in \text{Lip}_{d_A}(\Omega)$, $\text{Lip}_{d_A} u = \sqrt{A} |u'|$ almost everywhere.

**Proof.** By the continuity of $\lambda$ associated with the ellipticity condition of $A$, we have $\text{Lip}_{d_A}(U) = \text{Lip}(U)$ for $U \Subset \Omega$. To prove that $\text{Lip}_{d_A} u(x) \leq \sqrt{A(x)} |u'(x)|$ for almost all $x \in \Omega$, notice that

\[
\text{Lip}_{d_A} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_A(x, y)} \\
\leq \limsup_{y \to x} \frac{|u(y) - u(x)|}{|x - y|} \limsup_{y \to x} \frac{|x - y|}{d_A(x, y)} \\
= |u'(x)| \limsup_{y \to x} \frac{|x - y|}{d_A(x, y)}.
\]

Here we used the Rademacher theorem, according to which locally Lipschitz continuous functions on $\mathbb{R}$ are differentiable almost everywhere. Thus it suffices to check that for almost all $x \in \Omega$,

\[
(2.1) \quad \limsup_{y \to x} \frac{|x - y|}{d_A(x, y)} \leq \sqrt{A(x)}.
\]

This is reduced to showing that, for any $\epsilon > 0$, there exists a constant $\delta > 0$ and a Lipschitz continuous function $w$ such that $A(x) |w'(x)|^2 \leq 1$ and for all $y \in (x - \delta, x + \delta)$,

\[
(2.2) \quad |y - x| \leq (1 + \epsilon) \sqrt{A(x)} |w(y) - w(x)|.
\]

Indeed, from this and the definition of $d_A$, we know that

\[
\sqrt{A(x)} d_A(x, y) \geq \sqrt{A(x)} |w(y) - w(x)| \geq \frac{1}{1 + \epsilon} |y - x|,
\]
which implies (2.1) by the arbitrariness of \( \epsilon \). Towards (2.2), take

\[
    w(z) = \int_x^z \frac{1}{\sqrt{A(s)}} \, ds
\]

for \( z \in \Omega \). Notice that the lower bound of \( A \) guarantees that \( \frac{1}{\sqrt{A}} \in L^1_{\text{loc}}(\Omega) \), and so \( w'(z) = A(z)^{-1/2} \) for almost all \( z \in \mathbb{R} \). Let \( I_{x,y} = [x, y] \) if \( x < y \) ([\( y, x \]) if \( x > y \)). By Lebesgue’s differentiation theorem, for almost all \( x \in \Omega \), we can find \( \delta > 0 \) such that whenever \( y \in (x - \delta, x + \delta) \),

\[
    \frac{|w(y) - w(x)|}{|x - y|} = \int_{I_{x,y}} \frac{1}{\sqrt{A(s)}} \, ds \geq \frac{1}{(1 + \epsilon)\sqrt{A(x)}},
\]

which implies (2.2).

On the other hand, to prove \( \sqrt{A(x)}|u'(x)| \leq \text{Lip}_{d_A} u(x) \) for almost all \( x \in \Omega \), observe that at points \( x \) of differentiability of \( u \) (by the classical Rademacher’s theorem, almost every \( x \) is such a point),

\[
    |u'(x)| = \lim_{y \to x} \frac{|u(y) - u(x)|}{|x - y|} \\
    \leq \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_A(x, y)} \limsup_{y \to x} \frac{d_A(x, y)}{|x - y|} \\
    = \text{Lip}_{d_A} u(x) \limsup_{y \to x} \frac{d_A(x, y)}{|x - y|}.
\]

By the definition of \( d_A \), for any \( \epsilon > 0 \) and any fixed \( y \), there exists a function \( v \) such that \( A(z)|v'(z)|^2 \leq 1 \) for almost all \( z \in \Omega \) and \( d_A(x, y) \leq (1 + \epsilon)|v(x) - v(y)| \), which implies that

\[
    \frac{d_A(x, y)}{|x - y|} \leq (1 + \epsilon) \frac{|v(x) - v(y)|}{|x - y|} \leq (1 + \epsilon) \int_{I_{x,y}} |v'(s)| \, ds \leq (1 + \epsilon) \int_{I_{x,y}} \frac{1}{\sqrt{A(s)}} \, ds.
\]

If \( x \) is a Lebesgue point of \( \frac{1}{\sqrt{A}} \), there exists a \( \delta > 0 \) such that whenever \( y \in (x - \delta, x + \delta) \),

\[
    \frac{d_A(x, y)}{|x - y|} \leq (1 + \epsilon)^2 \frac{1}{\sqrt{A(x)}},
\]

which is as desired.

\[\square\]

**Theorem 2.2.** If \( n \geq 2 \) and \( A \in \mathcal{A}_{\text{wusc}}(\Omega) \), then the intrinsic distance and differential structures coincide. That is, given Lipschitz function \( u \) on \( \Omega \) (with respect to the Euclidean metric), for almost every \( x \in \Omega \) we have

\[
    (\text{Lip}_{d_A} u(x))^2 = \langle A(x) \nabla u(x), \nabla u(x) \rangle.
\]
To prove Theorem 2.2, we first notice that the distance $d_A$ is locally comparable to the Euclidean distance, and hence $(\Omega, d_A, dx)$ satisfies the local doubling property in the sense that if $U$ is open and $U \subseteq \Omega$, there exists a constant depending on $U$ and $A$ such that for each $x \in U$ and $0 < r < \min\{\text{diam}(U), d_A(x, \Omega^{\delta})\}/4$,

$$\mu(B_{d_A}(x, 2r)) \leq C(A, U)\mu(B_{d_A}(x, r)).$$

Here and in what follows, $d_A(x, K) = \inf_{z \in K} d_A(x, z)$ and if $d_A$ is the Euclidean distance, we use the notation $d_{\mathbb{R}^n}(x, K)$.

Therefore, applying [20, Theorem 2.1] and its remark, we conclude with the following.

**Lemma 2.3.** For each $u \in \text{Lip}_d(\Omega)$, $H(x, \nabla u(x)) \leq (\text{Lip}_d u(x))^2$ for almost all $x \in \Omega$.

To obtain the reverse relation, we need the following result.

**Lemma 2.4.** Assume that $n \geq 2$. Let $x_0 \in \Omega$ and $0 < r < d_{\mathbb{R}^n}(x_0, \Omega^{\delta})$. If the diffusion matrix $\tilde{A}$ is a constant positive definite symmetric matrix $A$ on the Euclidean ball $B(x_0, r)$, then, for the function $u(y) = |A^{1/2}\xi|^{-1}\langle \xi, y \rangle$ with $\xi \in S^{n-1}$, we have $\text{Lip}_{d_A} u \leq 1$ on $B(x_0, r)$.

**Proof.** It suffices to show that, for every $x \in B(x_0, r)$, there exists $\delta \in (0, r - |x - x_0|)$ such that for each fixed $y \in B(x, \delta)$, we can find a function $\tilde{u}_{x,y}$ on $\Omega$ satisfying

(i) $\tilde{u}_{x,y}(z) = u(z)$ for all $z$ in the line segment joining $x$ and $y$,

(ii) $|\tilde{A}(z)\nabla \tilde{u}_{x,y}(z), \nabla \tilde{u}_{x,y}(z)| \leq 1$ for almost all $z \in \Omega$.

Indeed, from the definition of the intrinsic distance, the existence of such a function will lead to $|\tilde{u}_{x,y}(z) - \tilde{u}_{x,y}(w)| \leq d_{\tilde{A}}(z, w)$ for all $z, w \in \Omega$, which gives the desired inequality $|u(x) - u(y)| \leq d_{\tilde{A}}(x, y)$ when choosing $z, w$ as $x, y$. Hence we have $\text{Lip}_{d_A} u \leq 1$ for all $x \in B(x_0, r)$.

To this end, we first consider the case $A = \lambda I_n$. Set

$$\tilde{u}_{x,y}(z) = \begin{cases} 
[u(y) - u(x)]\|z - x\|^{-1} + u(x) & \text{if } |z - x| \leq |y - x| \\
u(y) & \text{if } |z - x| \geq |y - x|.
\end{cases}$$

Obviously, $\tilde{u}_{x,y}$ satisfies (i). To see (ii), for $x \neq z \in B(x, |x - y|) \subset B(x, \delta) \subset B(x_0, r)$, we have

$$|A^{1/2}\nabla \tilde{u}_{x,y}(z)| = \frac{|u(y) - u(x)|}{|y - x|} \frac{|A^{1/2}(z - x)|}{|z - x|} = \frac{|\langle \xi, y - x \rangle|}{|A^{1/2}\xi|} \frac{|A^{1/2}(z - x)|}{|y - x||z - x|} \leq 1;$$

while when $z \in \Omega \setminus \overline{B}(x, |x - y|)$ we have $|A^{1/2}\nabla \tilde{u}_{x,y}(z)| = 0 \leq 1$.

For a more general constant positive definite symmetric matrix $A$, we modify the above construction as follows, following the idea given above. Notice that there exist $\delta_1, \delta \in (0, r)$ such that $A^{1/2}B(0, \delta_1) \subset B(0, r)$ and $A^{-1/2}B(0, \delta) \subset B(0, \delta_1)$. Thus, for every $y \in B(x, \delta)$, we have $|A^{-1/2}(y - x)| < \delta_1$ and hence

$$\{z \in \mathbb{R}^n : |A^{-1/2}(z - x)| \leq |A^{-1/2}(y - x)|\} \subset A^{1/2}B(0, \delta_1) + \{x\} \subset B(x_0, r).$$
For a given pair $x, y$, set
\begin{equation}
\tilde{u}_{x,y}(z) = \begin{cases} 
[u(y) - u(x)]\frac{|A^{-1/2}(z-x)|}{|A^{-1/2}(y-x)|} + u(x) & \text{if } |A^{-1/2}(z-x)| \leq |A^{-1/2}(y-x)|, \\
[u(y)] & \text{if } |A^{-1/2}(z-x)| \geq |A^{-1/2}(y-x)|.
\end{cases}
\end{equation}

We still need to check that if $|A^{-1/2}(z-x)| \leq |A^{-1/2}(y-x)|$, then $|A^{1/2}\nabla \tilde{u}_{x,y}(z)| \leq 1$. Indeed, notice that for $y \in \mathbb{R}^n$,
\[
\langle A^{1/2}\xi, A^{-1/2}y \rangle = \langle A^{1/2}\xi, A^{1/2}A^{-1}y \rangle = \langle \xi, A^{-1}y \rangle = \langle \xi, y \rangle,
\]
and so $\langle \xi, y \rangle = \langle A^{1/2}\xi, A^{-1/2}y \rangle$. Furthermore, for $z \in S^{n-1}$,
\[

abla |A^{-1/2}z| = \frac{A^{-1}z}{|A^{-1/2}z|},
\]
and so by the Cauchy-Schwarz inequality, we have
\[
|A^{1/2}\nabla \tilde{u}_{x,y}(z)| = \frac{|\langle \xi, y-x \rangle|}{|A^{1/2}\xi|} \frac{|A^{1/2}A^{-1}(z-x)|}{|A^{-1/2}(y-x)||A^{-1/2}(z-x)|} \leq 1,
\]
as desired. \qed

**Lemma 2.5.** Given $A \in \mathcal{M}(\Omega)$ and $x \in \Omega$, we have
\[
\lim_{x \neq y \to x} \inf \frac{d_A(y,x)}{|y-x|} \geq \frac{1}{\sqrt{\lambda(x)}}.
\]
Furthermore,
\[
\lim_{x \neq y \to x} \sup \frac{d_A(y,x)}{|y-x|} \leq \sqrt{\lambda(x)}.
\]

**Proof.** We fix $x \in \Omega$ and $r > 0$ such that $\overline{B}(x,r) \subset \Omega$. Set
\[
\lambda_x(r) = \sup_{z \in B(x,r)} \lambda(z).
\]
Notice that, by the continuity of $\lambda$, we have $\lim_{r \to 0} \lambda_x(r) = \lambda(x)$. For the function $u_x$ given by
\[
u_x(z) = \frac{1}{\sqrt{\lambda_x(r)}}(r - |z - x|)_+, \]
we have that $\nabla u_x = 0$ on $\Omega \setminus \overline{B}(x,r)$ and so $H(z, \nabla u_x(z)) = 0 \leq 1$ when $z \in \Omega \setminus \overline{B}(x,r)$. When $x \neq z \in B(x,r)$, we have $\nabla u_x(z) = \lambda_x(r)^{-1/2}|z - x|^{-1}(z - x)$. Therefore, by the ellipticity condition of $A$, $H(z, \nabla u_x(z)) = \langle A(z)\nabla u_x(z), \nabla u_x(z) \rangle \leq 1$. It follows that, by the definition of $d_A$, when $x \neq y \in B(x,r)$,
\[
d_A(x, y) \geq |u_x(y) - u_x(x)| = \frac{1}{\sqrt{\lambda_x(r)}}|y - x|,
\]
from which the first part of the claim follows.

For the second part, notice that if \( x, y \in \Omega \), then there is a function \( w \) on \( \Omega \) with \( d_A(x, y) \leq |w(y) - w(x)| + \epsilon \) and \( H(z, \nabla w(z)) \leq 1 \) for almost every \( z \in \Omega \). Let \( E \) be the set of points at which this inequality fails. Then the Lebesgue measure of \( E \) is zero. By the ellipticity property of \( A \), it follows that for almost every \( z \in \Omega \setminus E \), \( |\nabla w(z)|^2 \leq \lambda(z) \). By an argument using Fubini’s theorem, for each \( \eta > 0 \) there is a point \( y_\eta \in B(y, \eta) \) and a point \( x_\eta \in B(x, \eta) \) such that the intersection of the Euclidean line segment \([x_\eta, y_\eta]\) connecting \( x_\eta \) to \( y_\eta \) with \( E \) has 1-dimensional Hausdorff measure zero. Thus

\[
|w(y_\eta) - w(x_\eta)| \leq \int_{[x_\eta, y_\eta]} |\nabla w| \, ds \leq \left[ \sup_{p \in B(x, 2d_A(x, y))} \lambda(p) \right] |x_\eta - y_\eta|.
\]

Letting \( \eta \to 0 \) and using the fact that \( w \) is continuous, we obtain

\[
d_A(x, y) \leq \left[ \sup_{p \in B(x, 2d_A(x, y))} \lambda(p) \right] |x - y| + \epsilon.
\]

Letting \( \epsilon \to 0 \) we obtain

\[
d_A(x, y) \leq \left[ \sup_{p \in B(x, 2d_A(x, y))} \lambda(p) \right] |x - y|,
\]

from which the second part of the claim follows.

**Proof of Theorem 2.2.** Let \( u \in \text{Lip}_{d_A}(\Omega) \). Then, \( u \) is also locally Lipschitz continuous with respect to the Euclidean metric on \( \Omega \). Notice that by Lemma 2.3, we always have \( H(x, \nabla u(x)) \leq (\text{Lip}_{d_A}(u))^2 \) for almost all \( x \in \Omega \). Now we will show that \((\text{Lip}_{d_A}(u))^2 \leq H(x, \nabla u(x)) \) for every \( x \in \Omega \) at which \( A \) is weak upper semicontinuous and \( u \) is differentiable. Fix such an \( x \in \Omega \). If \( \nabla u(x) = 0 \), then \( |u(x) - u(y)| = o(|x - y|) \), which implies by Lemma 2.5 that

\[
\text{Lip}_{d_A} u(x) \leq \text{Lip} u(x) \sqrt{\lambda(x)} = 0 = H(x, \nabla u(x)).
\]

If \( \nabla u(x) \neq 0 \), take \( \xi = \frac{\sum u(x)}{|A^{1/2}(x)\nabla u(x)|} \). Then, by Lemma 2.5 again, together with the fact that \( u \) is differentiable at \( x \),

\[
(2.5) \text{Lip}_{d_A} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_A(x, y)} \\
\leq \limsup_{y \to x} \frac{|u(y) - u(x) - \langle \nabla u(x), y - x \rangle|}{d_A(x, y)} + \limsup_{y \to x} \frac{|\langle \nabla u(x), y - x \rangle|}{d_A(x, y)} \\
\leq \sqrt{\lambda(x)} \lim_{y \to x} \frac{|u(y) - u(x) - \langle \nabla u(x), y - x \rangle|}{|x - y|} + \limsup_{y \to x} \frac{|\langle \nabla u(x), y - x \rangle|}{d_A(x, y)} \\
\leq 0 + |A^{1/2}(x)\nabla u(x)| \limsup_{y \to x} \frac{|\xi, y - x \rangle}{d_A(x, y)}.
\]
Observe that

\[ |A^{1/2}(x)\nabla u(x)|^2 = \langle A^{1/2}(x)\nabla u(x), A^{1/2}(x)\nabla u(x) \rangle = H(x, \nabla u(x)). \]

Let \( w(y) = \langle \xi, y \rangle \). It suffices to prove that

\[
\limsup_{y \to x} \frac{|w(y) - w(x)|}{d_A(x, y)} \leq 1. \tag{2.6}
\]

To this end, notice that \( \nabla w(y) = \xi \), and hence \( H(y, \nabla w(y)) = H(y, \xi) \) for all \( y \in \Omega \).

By the weak upper semicontinuity of \( A \) at \( x \), there exists \( \delta \in (0, r) \) such that for all \( y \in B(x, \delta) \),

\[ H(y, \xi) \leq (1 + \epsilon)H(x, \xi) = (1 + \epsilon). \]

Set \( \tilde{A}(z) = (1+\epsilon)A(x) \) for \( z \in B(x, \delta) \) and \( \tilde{A}(z) = A(z) \) for \( z \notin B(x, \delta) \). It can be directly seen that \( d_A \leq d_{\tilde{A}} \). We consider the function \( v(y) = \frac{1}{\sqrt{1+\epsilon}} w(y) \) and \( \eta = |\xi|^{-1} \xi \in S^{n-1} \); to this choice we apply Lemma 2.4 to obtain

\[ \limsup_{y \to z} \frac{|w(y) - w(z)|}{d_A(z, y)} \leq \limsup_{y \to z} \frac{|w(y) - w(z)|}{d_{\tilde{A}}(z, y)} \leq \sqrt{1+\epsilon}. \]

The arbitrariness of \( \epsilon > 0 \) leads to (2.6). \( \square \)

**Theorem 2.6.** A diffusion matrix \( A \) belongs to \( \mathcal{A}_{wusc}(\Omega) \) if and only if \( |Du|_{d_A}^2 = H(\cdot, \nabla u) \) almost everywhere for all \( u \in C^1(\Omega) \).

**Proof.** Suppose that whenever \( u \in C^1(\Omega) \) we have \( |Du|_{d_A}^2(x) = H(x, \nabla u(x)) \) almost everywhere. For each \( \xi \in S^{n-1} \), taking \( u(x) = \langle \xi, x \rangle \), we know that \( H(x, \xi) = |Du|_{d_A}^2(x) \) almost everywhere, and hence it is weak upper semicontinuous on \( \Omega \) because \( x \mapsto |Du|_{d_A}^2(x) \) is upper semicontinuous.

Suppose now that \( A \in \mathcal{A}_{wusc}(\Omega) \). Then by Theorem 2.1 and Theorem 2.2, for all \( u \in \operatorname{Lip}_{d_A}(\Omega) \) and almost all \( x \in \Omega \) we have

\[ H(x, \nabla u(x)) \leq (\operatorname{Lip}_{d_A} u(x))^2 \leq |Du|_{d_A}^2(x). \]

To see that \( |Du|_{d_A}^2(x) \leq H(x, \nabla u(x)) \) almost everywhere for \( u \in C^1(\Omega) \), we give a 3-step argument.

**Step 1.** Let \( \xi \in S^{n-1} \) and consider the function \( u(y) = \langle \xi, y \rangle \). Then \( \nabla u(x) = \xi \). To prove that \( |Du|_{d_A}^2(x) \leq H(x, \xi) \), it suffices to check that for almost every \( x \in \Omega \),

\[
|Du|_{d_A}^2(x) \leq (1 + \epsilon)H(x, \xi) \tag{2.7}
\]

for any \( \epsilon > 0 \). The following argument is similar to that of Theorem 2.2. Let \( x \) be a point of weak upper semicontinuity of \( A \). For each fixed \( \epsilon > 0 \), we know that there exists \( r > 0 \) such that for almost all \( y \in B(x, r) \), \( H(y, \xi) \leq (1 + \epsilon)H(x, \xi) \). Let

\[ \tilde{A}(y) = (1 + \epsilon)A(x)1_{B(x, r)}(y)I_n + A(y)1_{\Omega \setminus B(x, r)}. \]
The corresponding intrinsic distance $d_A^-$ is no more than $d_A$, and hence $|Du|_{d_A} \leq |Du|_{d_A^-}$ everywhere. By Lemma 2.4, for any $y, z \in B(x, r/4)$, we have

$$|u(y) - u(z)| \leq d_A^-(z, y)$$

and hence, $|Du|^2_{d_A^-}(x) \leq (A(x)\xi, \xi)$, which implies (2.7) as desired.

**Step 2.** If $u \in C^1(\Omega)$ and $\nabla u(x) = 0$, then for any $\epsilon > 0$, by the continuity of $\nabla u$, there exists a ball $B(x, r)$ such that $\|\nabla u\|_{L^\infty(B(x, r))} \leq \epsilon$. With the aid of Lemma 2.5, we obtain

$$|u(y) - u(z)| \leq \epsilon|y - z| \lesssim \epsilon d_A(y, z)$$

which implies that $|Du|_{d_A^-}(x) \lesssim \epsilon$, and hence $|Du|_{d_A^-}(x) = 0$ due to the arbitrariness of $\epsilon > 0$.

**Step 3.** If $u \in C^1_{loc}(\Omega)$ with $\nabla u(x) \neq 0$, then let

$$\tilde{u}(y) = u(y) - \langle \nabla u(x), y \rangle$$

for $y \in \Omega$. Then $\tilde{u}$ is of class $C^1(\Omega)$ with $\nabla \tilde{u}(x) = 0$, which implies that $|D\tilde{u}|_{d_A^-}(x) = 0$ by Step 2. Moreover, since $\langle A(y)\nabla u(x), \nabla u(x) \rangle$ is weak upper semicontinuous at $y = x$, by Step 1 we have

$$|Du|_{d_A^-}(x) \leq |D\tilde{u}|_{d_A^-}(x) + |D(\langle \nabla u(x), \cdot \rangle)|_{d_A^-}(x) \leq \sqrt{H(x, \nabla u(x))}$$

as desired. \(\square\)

**Remark 2.7.** (i) We cannot replace the function class $C^1(\Omega)$ by $\text{Lip}_{loc}(\Omega)$ in the above Theorem 2.6. Indeed, there exists a function $u \in \text{Lip}(\Omega)$ such that $|\nabla u|^2$ is not weak upper semicontinuous on $\Omega$, hence the above theorem fails for $A = I_n$. For example

$$u(x_1, x_2) = \int_0^{x_1} 1_{C_a}(z_1) \, dz_1,$$

where $C_a \subset \mathbb{R}$ is a Cantor set of positive 1-dimensional Lebesgue measure, and $1_{C_a} : \mathbb{R} \rightarrow \mathbb{R}$ is the characteristic function of $C_a$. For a construction of such a Cantor set $C_a$ see Section 3.

(ii) Generally, for every open set $U$ with $\overline{U} \subset \Omega$ and $u \in \text{Lip}(\Omega)$, we have $|Du|_{d_A} \leq C_U \text{Lip}_{d_A}u$ almost everywhere on $U$, where $C_U \geq 1$ is a constant. The proof of this is not trivial.

Finally, we point out a relation between weak upper semicontinuity and the Eikonal equation. The Eikonal equation is necessary to obtain the coincidence of the intrinsic differential and distance structures. The Eikonal equation states that

$$\langle A(\cdot)\nabla d_A; x_0(\cdot) \rangle = \nabla d_A; x_0(\cdot) \rangle = 1$$

almost everywhere for each $x_0 \in \Omega$, where $d_A; x_0(x) = d_A(x, x_0)$. When $A = I_n$ the statement of the Eikonal equation is that whenever $x_0 \in \mathbb{R}^n$, the function $d_A(\cdot, x_0)$ satisfies $|\nabla d_A(\cdot, x_0)| = 1$ almost everywhere in $\mathbb{R}^n$ (indeed, everywhere in $\mathbb{R}^n \setminus \{x_0\}$).
Proposition 2.8. For each $x_0 \in \Omega$, $H(x, \nabla d_{A;x_0}(x)) = 1$ almost everywhere if and only if $H(\cdot, \nabla d_{A;x_0})$ is weak upper semicontinuous on $\Omega$.

Proof. If $H(x, \nabla d_{A;x_0}(x)) = 1$ almost everywhere, then obviously it is weak upper semicontinuous. Conversely, assume that $H(\cdot, \nabla d_{A;x_0})$ is weak upper semicontinuous on $\Omega$. It suffices to show that for each point $x \in \Omega$ and all sufficient small $r > 0$,

$$\|H(\cdot, \nabla d_{A;x_0})\|_{L^\infty(B_{d_A}(x, r))} = 1. \tag{2.8}$$

Indeed, if this is true, then for almost all $x$, the weak upper semicontinuity leads to

$$1 \geq H(x, \nabla d_{A;x_0}(x)) \geq \limsup_{r \to 0} \|H(\cdot, \nabla d_{A;x_0})\|_{L^\infty(B_{d_A}(x, r))} = 1.$$

We prove (2.8) by contradiction. Assume that (2.8) fails for some $x_0 \in \Omega$ and some decreasing sequence $\{r_k\}$ which converges to 0 as $k \to \infty$. By Lemma 2.5 and its proof, $d_A$ is comparable to the Euclidean distance. Hence for sufficiently large $k$ we have $B_{d_A}(x_0, r_k) \subset \Omega$. Moreover, since we already know from Lemma 2.3 applied to the function $d_A(\cdot, x_0)$ that $\|H(\cdot, \nabla d_{A;x_0})\|_{L^\infty(B_{d_A}(x, r_k))} \leq 1$, by our assumption there must be a positive number $\epsilon_k < 1$ such that

$$\|H(\cdot, \nabla d_{A;x_0})\|_{L^\infty(B_{d_A}(x, r_k))} \leq 1 - \epsilon_k.$$

Taking

$$u_k(x) = \frac{1}{\sqrt{1 - \epsilon_k}} \min\{d_A(x_0, x), r_k\},$$

we have $u_k \in \text{Lip}(\Omega)$ with $\|H(\cdot, \nabla u_k)\|_{L^\infty(B_{d_A}(x_0, r_k))} \leq 1$ and $H(z, \nabla u_k(z)) = 0$ for $z \in \Omega \setminus B_{d_A}(x_0, r_k)$. Hence $u_k$ satisfies the conditions in the definition of $d_A(x, x_0)$, and so

$$d_A(x_0, x) \geq u_k(x) - u_k(x_0) = \frac{1}{\sqrt{1 - \epsilon_k}}d_A(x_0, x),$$

which is a contradiction. \hfill \Box

Remark 2.9. As shown in [20, Section 7], the Eikonal equation determines the asymptotic behavior of the gradient of heat kernel for a regular, strongly local Dirichlet form on a compact underlying space. We do not know if it is possible to deduce the coincidence of the intrinsic differential and distance structures from the Eikonal equation.

3 Case $n \geq 2$ and $A \notin \mathcal{A}_{wusc}(\Omega)$: $H(\cdot, \nabla u) = (\neq)(\text{Lip}_{d_A} u)^2$

In this section, we always assume that $n \geq 2$ and $\Omega = \mathbb{R}^n$. From Theorem 2.2 we know that when $A \in \mathcal{A}_{wusc}(\Omega)$, for locally Lipschitz functions $u$ on $\Omega$ we have $(\text{Lip}_{d_A} u)^2 = H(x, \nabla u)$. In this section we show that when $A \notin \mathcal{A}_{wusc}(\Omega)$, the (non-)coincidence of the above intrinsic distance and differential structures depends on the geometry of the set where $A$ fails to be weak-upper semicontinuous. Indeed, we construct two examples based on
a Cantor set and a Sierpinski carpet to show that both coincidence and noncoincidence may happen; see, respectively, Theorem 3.4 and Proposition 3.1. We consider the simple $A \notin \mathcal{A}_{\text{wusc}}(\Omega)$ defined by

$$A_{E, \delta}(x) = (1 - \delta 1_E)I_n,$$

where $\delta \in (0, 1)$ and $E$ is a closed subset of $\mathbb{R}^n$ with positive measure and empty interior. Obviously, $A_{E, \delta}$ fails to be weak upper semicontinuous at each $x \in E$, and hence $A \notin \mathcal{A}_{\text{wusc}}(\mathbb{R}^n)$. If $E$ is a suitable large Cantor set, then the intrinsic distance and differential structures never coincide. If $E$ is a suitable large Sierpinski carpet, then the intrinsic distance and differential structures do coincide. Recall that for $\delta \in (-\infty, 0]$, $A_{E, \delta} = (1 - \delta 1_E)I_n \in \mathcal{A}_{\text{wusc}}(\mathbb{R}^n)$, and hence the associated intrinsic distance and differential structures always coincide by Theorem 2.2.

### 3.1 The large Cantor set $C_a$

Let $a = \{a_j\}$ with $0 < a_j < 1$. Then the associated Cantor set $C_a$ is constructed as follows: $I_i$, $i = 1, 2$, are the two closed intervals obtained by removing the middle open interval with length $a_1$ from $I = [0, 1]$ and are ordered from left to right; when $m \geq 2$, the subintervals $I_{i_1 \cdots i_m}$, $i_m = 1, 2$, are the two closed intervals obtained by removing the middle open interval with length $a_m |I_{i_1 \cdots i_{m-1}}|$ from $I_{i_1 \cdots i_{m-1}}$, and are ordered from left to right; finally set

$$C_a \equiv \cap_{m \in \mathbb{N}} \bigcup_{i_1 \cdots i_m \in \{1, 2\}} I_{i_1 \cdots i_m},$$

and $C_a^{(n)} \equiv C_a \times \cdots \times C_a$.

Notice that $C_a^{(n)}$ is closed and has empty interior, and that $C_a^{(n)}$ has positive $n$-dimensional Lebesgue measure if and only if $C_a$ has positive 1-dimensional Lebesgue measure. Moreover,

$$|C_a| = \lim_{m \to \infty} (1 - a_1) \cdots (1 - a_m),$$

and by taking logarithms, $|C_a| > 0$ if and only if $a \in \ell^1$. Thus, the $n$-dimensional Lebesgue measure of $C_a^{(n)}$ is positive if and only if $a \in \ell^1$.

**Proposition 3.1.** Assume that $a \in \ell^1$ with $0 < a_j < 1$ for all $j \in \mathbb{N}$. If $\delta \in (-\infty, 0]$, then the associated intrinsic length and differential structure of $A_{C_a^{(n)}, \delta}$ do coincide; while if $\delta \in (0, 1)$, then the associated intrinsic distance and differential structures never coincide.

To prove Proposition 3.1, we need the following (geometric) property of $C_a^{(n)}$ that holds even if $a \notin \ell^1$. To simplify our notation, we set $d_{C_a^{(n)}, \delta}(x, y) := d_{A_{C_a^{(n)}, \delta}}(x, y)$.

**Lemma 3.2.** Let $\delta \in (0, 1)$. Then for any $x \in C_a^{(n)}$ and any $y \in x + \mathbb{R}e_1$, we have $d_{C_a^{(n)}, \delta}(x, y) = |x - y|$.

**Proof.** Notice that, from the definition, we have $d_{C_a^{(n)}, \delta}(x, y) \geq |x - y|$ for each pair $x, y \in \mathbb{R}^n$. Let $x \in C_a^{(n)}$ and $y = x + re_1$ for some $r > 0$. It suffices to show that $d_{C_a^{(n)}, \delta}(x, y) \leq |x - y|$.
Recall that if $n = 2$, it is already proved in [20, Proposition 3.1] that for every pair $x, y \in \mathbb{R}^n$, there exists a curve $\gamma$ such that except for its endpoints, $\gamma \subset \mathbb{R}^n \setminus C^{(2)}_a$, and with Euclidean length $\ell_{\mathbb{R}^2}(\gamma) \leq C_{C^{(2)}_a} |x - y|$. Hence $d_{C^{(2)}_a, \delta}(x, y) \leq C_{C^{(2)}_a} |x - y|$, where $C_{C^{(2)}_a}$ is a constant determined by $C^{(2)}_a$ and independent of $\delta$. If $n \geq 2$, similar arguments still apply and hence for every pair $x, y \in \mathbb{R}^n$, $d_{C^{(n)}_a, \delta}(x, y) \leq C_{C^{(n)}_a} |x - y|$. By the construction of $C_a$, for each $k \in \mathbb{N}$ we can find $x_k \in \mathbb{R}^n \setminus C^{(n)}_a$ such that $\langle x - x_k, e_1 \rangle = 0$ and

$$d_{C^{(n)}_a, \delta}(x, x_k) \leq C_{C^{(n)}_a} |x - x_k| \leq \frac{1}{2k} |x - y|.$$ 

Take $y_k = y - (x - x_k) = x_k + re_1$. Then $y_k \in \mathbb{R}^n \setminus C^{(n)}_a$. Moreover, the line segment $x_k + [0, r]e_1$ is contained in $\mathbb{R}^n \setminus C^{(n)}_a$, which implies that

$$d_{C^{(n)}_a, \delta}(x_k, y_k) = |x_k - y_k| = |x - y|.$$ 

Therefore,

$$d_{C^{(n)}_a, \delta}(x, y) \leq d_{C^{(n)}_a, \delta}(x, y_k) + d_{C^{(n)}_a, \delta}(x, x_k) + d_{C^{(n)}_a, \delta}(y, y_k) \leq (1 + \frac{1}{k}) |x - y|,$$

which implies that $d_{C^{(n)}_a, \delta}(x, y) \leq |x - y|$ by letting $k \to \infty$. \hfill $\Box$

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** Take $u(z) = \langle e_1, z \rangle$ for $z \in \mathbb{R}^n$. For each $\delta \in (0, 1)$ and each $x \in C^{(n)}_a$, by Lemma 3.2, we have that

$$\text{Lip}_{d_{C^{(n)}_a, \delta}} u(x) \geq \limsup_{y \in x + \mathbb{R}e_1} \frac{|u(x) - u(y)|}{d_{C^{(n)}_a, \delta}(x, y)} = \limsup_{y \in x + \mathbb{R}e_1} \frac{|u(x) - u(y)|}{|x - y|} = 1 > \sqrt{1 - \delta} |\nabla u(x)|,$$

which is as desired because the Cantor set $C^{(n)}_a$ has positive measure. \hfill $\Box$

From the above proof, we also conclude the following corollary.

**Corollary 3.3.** Let $E$ be a closed subset of $\mathbb{R}^n$. Assume that $E$ has positive measure and empty interior.

(i) If there is a constant $C_E$ such that for each pair $x, y \in \mathbb{R}^n$ we can find a curve $\gamma$ such that all of the curve except for perhaps countably many points lies in $\mathbb{R}^n \setminus E$ and $gz$ satisfies $\ell_{\mathbb{R}^n}(\gamma) \leq C_E |x - y|$, then the intrinsic distance and differential structures associated to $A_{E, \delta} = (1 - \delta 1_E)I_n$ with $\delta \in (\frac{1}{C_E}, 1)$, do not coincide.

(ii) If there exists $\xi \in S^{n-1}$ such that for all $x \in E$, we can find a sequence of $y \in x + \mathbb{R}\xi$ satisfying $d_{E, \delta}(x, y) = |x - y|$, then the intrinsic distance and differential structures associated to $A_{E, \delta} = (1 - \delta 1_E)I_n$ with $\delta \in (0, 1)$, do not coincide.
3.2 The large Sierpinski carpet $S_a$

Let $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}}$ with $a_j \in \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots \}$, that is, $a_j$ is the reciprocal of an odd integer strictly greater than one. A modified Sierpinski carpet $S_a$ is constructed as follows. First, divide the unit cube $T = [0,1]^n$ into $a_1^{-n}$ essentially disjoint closed congruent subcubes, and remove the interior of the central one; denote the central one by $T_{a_1^{-n}}$ and the others by $T_{k_1}$ with $1 \leq k_1 \leq a_1^{-n} - 1$. When $m \geq 2$, divide each $T_{k_1,\ldots,k_{m-1}}$ into $a_m^{-n}$ essentially disjoint closed congruent subcubes and remove the interior of the central one; denote the central one by $T_{k_1,\ldots,k_{m-1},a_m^{-n}}$ and the others by $T_{k_1,\ldots,k_{m}}$ with $1 \leq k_m \leq a_m^{-n} - 1$. For each $m \in \mathbb{N}$, define the level $m$ precarpet by

$$S_{a,m} = \bigcup_{k_1} \cdots \bigcup_{k_m} T_{k_1,\ldots,k_m}.$$

The modified Sierpinski carpet $S_a$ is defined as the limit of precarpets $S_{a,m}$, that is, $S_a = \cap_{m \in \mathbb{N}} S_{a,m}$.

Obviously, $S_a$ is closed, has empty interior, and $S_a$ has positive $n$-dimensional Lebesgue measure if and only if $\mathbf{a} \in \ell^n$. Indeed, the $n$-dimensional Lebesgue measure of the precarpet $S_{a,m}$ is

$$|S_{a,m}| = (1 - a_1^n) \cdots (1 - a_m^n).$$

Thus, by taking logarithms, $|S_a| = \lim_{m \to \infty} |S_{a,m}| > 0$ if and only if $\mathbf{a} \in \ell^n$.

**Theorem 3.4.** Let $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}} \in \ell^n$ with $a_j \in \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots \}$. Then for all $\delta \in (-\infty, 1)$, the associated intrinsic distance and differential structures of $A_{S_a,\delta}$ do coincide.

We employ the following geometric property of $S_a$ to prove Theorem 3.4. We only need to consider the case $0 < \delta < 1$.

**Lemma 3.5.** Let $\delta \in (0, 1)$ and $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}} \in \ell^n$ with $a_j \in \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots \}$. Then there exists a subset $E \subset S_a$ with measure zero such that for any $\epsilon > 0$ and each $x \in S_a \setminus E$, we can find $r = r(x, \delta, \epsilon) > 0$ which satisfies: for all $y \in B(x, r)$,

$$d_{S_a,\delta}(x, y) \geq (1 - \epsilon) \frac{1}{\sqrt{1 - \delta}} |x - y|.$$

(3.1)

With Lemma 3.5, we can prove Theorem 3.4 easily.

**Proof of Theorem 3.4.** Obviously, Lemma 2.3 yields $H(x, \nabla u(x)) \leq (\text{Lip}_{d_{S_{a,\delta}}} u(x))^2$ for almost all $x \in \Omega$. We now need to show that $(\text{Lip}_{d_{S_{a,\delta}}} u(x))^2 \leq H(x, \nabla u(x))$ almost everywhere. To this end, it suffices consider the cases $x \in \mathbb{R}^n \setminus S_a$ and $x \in E \subset S_a$, where $E$ is as in Lemma 3.5.

**Case 1:** $x \in \mathbb{R}^n \setminus S_a$. It suffices to show that if $r < d_{\mathbb{R}^n}(x, S_a)/2$ and $y \in B(x, r)$ we have

$$d_{S_a,\delta}(x, y) = |x - y|.$$  

(3.2)

Indeed, (3.2) will give

$$H(x, \nabla u(x)) = |\nabla u(x)|^2 = (\text{Lip} u(x))^2 = (\text{Lip}_{d_{S_{a,\delta}}} u(x))^2$$
by the definition of the pointwise Lipschitz constant. The verification of (3.2) is done as in the proof of Lemma 2.5, with \( \lambda : \mathbb{R}^n \to [1, \infty) \) given as a continuous function that satisfies \( \lambda(y) = 1 \) when \( y \in B(x, 3^r/2) \) and \( \lambda(y) = 1/\sqrt{1 - \delta} \) when \( y \in S_n \).

Case 2: \( x \in S_n \setminus E \). In this case, (3.1) implies that

\[
\limsup_{y \to x} \frac{|y - x|}{d_{S_n}(x, y)} \leq \sqrt{1 - \delta}.
\]

With this, if \( u \) is differentiable at \( x \), we have

\[
\text{Lip}_{d_{S_n}} u(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d_{S_n}(x, y)} \\
\leq \limsup_{y \to x} \frac{|u(x) - u(y) - \langle \nabla u(x), y - x \rangle|}{d_{S_n}(x, y)} + \limsup_{y \to x} \frac{|\nabla u(x), y - x|}{d_{S_n}(x, y)} \\
\leq 0 \cdot \limsup_{y \to x} \frac{|y - x|}{d_{S_n}(x, y)} + |\nabla u(x)| \limsup_{y \to x} \frac{|y - x|}{d_{S_n}(x, y)} \\
\leq \sqrt{1 - \delta} |\nabla u(x)| = \sqrt{H(x, \nabla u(x))}.
\]

This proves Theorem 3.4. \( \square \)

Finally, we prove Lemma 3.5. Notice that Lemma 3.5 is much stronger than Lemma 2.5; see Remark 3.7 below. The proof of Lemma 3.5 relies on the following approximation of distance established by Norris [22]. Let \( \Phi \in C_c^\infty(\mathbb{R}^n) \) be such that \( \int_{\mathbb{R}^n} \Phi(x) \, dx = 1 \), \( \text{supp} \Phi \subset B(0, 2) \) and \( 0 \leq \Phi(x) \leq 1 \) for all \( x \in \mathbb{R}^n \). For \( t > 0 \), let \( \Phi_t(x) = t^{-n} \Phi(t^{-1} x) \). Standard analysis arguments show that \( \Phi_t * f \to f \) almost everywhere when \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). The following lemma is due to Norris [22].

**Lemma 3.6.** Let \( A_E, \delta, t = [\Phi_t * \left( \frac{1}{1 - \delta 1_E} \right)]^{-1} I_n \) and denote by \( d_{E, \delta, t} \) the associated intrinsic distance. Then \( d_{E, \delta, t}(x, y) \to d_{E, \delta}(x, y) \) as \( t \to 0 \) for all \( x, y \in \mathbb{R}^n \). Moreover,

\[
(d_{E, \delta, t}(x, y))^2 = \inf_{\gamma} \int_0^1 \Phi_t * \left( \frac{1}{1 - \delta 1_E} \right)(\gamma(s)) \left| \frac{d}{ds} \gamma(s) \right|^2 ds.
\]

**Proof of Lemma 3.5.** We divide the proof into 6 steps.

**Step 1.** Recall that for each \( m \), \( T_{k_1, \ldots, k_{m-1}, a_m} \) is the central cube whose interior has been removed from the cube \( T_{k_1, \ldots, k_{m-1}} \) at step \( m \) when constructing the Sierpinski Carpet \( S_n \). For each fixed \( N \in \mathbb{N} \cup \{0\} \), denote by \( J_{N,m} \) the collection of all \( (k_1, \ldots, k_{m-1}, j) \) such that \( T_{k_1, \ldots, k_{m-1}, j} \) is \( N \)-close to the central cube \( T_{k_1, \ldots, k_{m-1}, a_m} \) in the sense that there exists a sequence \( i_0, i_1, \ldots, i_M \) with \( 1 \leq M \leq N \) such that \( i_0 = j \), \( i_M = a_m \), \( i_l \neq i_s \) if \( i \neq s \), and for \( 0 \leq s < M - 1 \),

\[
T_{k_1, \ldots, k_{m-1}, i_s} \cap T_{k_1, \ldots, k_{m-1}, i_{s+1}} \neq \emptyset.
\]

Let

\[
E_{N,m} = \bigcup_{(k_1, \ldots, k_{m-1}, j) \in J_{N,m}} T_{k_1, \ldots, k_{m-1}, j}.
\]
If $N \geq a_m^{-1}$, then $E_N,m = T_{k_1,\ldots,k_{m-1}}$. Recall that we assume $a \in \ell^n$. It follows that for sufficiently large $m$ we have $N < a_m^{-1}$. In this case, we see that $E_N,m$ is a cube centered at the center of $T_{k_1,\ldots,k_{m-1},a_m^{-1}}$ of edge length the $(2N+1)a_m$-fraction of the edge length of the cube $T_{k_1,\ldots,k_{m-1}}$. Observe that this fraction tends to zero as $m \to \infty$. We set

$$E_N = \bigcap_{m \in \mathbb{N}} \bigcup_{\ell \geq m} E_{N,\ell} = \bigcap_{N \geq m > N} \bigcup_{N \geq \ell \geq m} E_{N,\ell},$$

Let $F$ be the union of all the $(n-1)$-dimensional faces of all the cubes that were removed in the construction of $S_a$, and

$$E = F \cup \left( \bigcup_{N \in \mathbb{N}} E_N \right).$$

We claim that $|E| = 0$. It is easy to see that $|F| = 0$. From the above discussion,

$$|E_{N,m}| \leq (2N+1)^n(1 - a_1^n)\cdots(1 - a_{m-1}^n)a_m^n.$$ 

From this, it follows that

$$\left| \bigcup_{\ell \geq m} E_{N,\ell} \right| \leq (2N+1)^n \sum_{\ell \geq m} (1 - a_1^n)\cdots(1 - a_{\ell-1}^n)a_{\ell}^n \leq (2N+1)^n \sum_{\ell \geq m} a_{\ell}^n,$$

which converges to zero as $m \to \infty$ because $a \in \ell^n$. This implies that $E_N$ with $N \in \mathbb{N}$, and hence $E$, has measure zero.

**Step 2.** For any $\epsilon > 0$, we choose $\bar{N}_e, N_e \in \mathbb{N}$ such that

$$\bar{N}_e \geq \frac{100^a n^{2m}}{(1 - \delta)\epsilon}$$

and $N_e \geq (\bar{N}_e)^{n+1}$. For each fixed $x \in S_a \setminus E$, recall that $x \in S_a \setminus E_N$ for all $N \in \mathbb{N}$. Since

$$S_a \setminus E_{N_e} = \left( S_a \setminus \left( \bigcap_{m \in \mathbb{N}} \bigcup_{\ell \geq m} E_{N,\ell} \right) \right) = \bigcup_{m \in \mathbb{N}} \left( S_a \setminus \left( \bigcup_{\ell \geq m} E_{N,\ell} \right) \right)$$

there exists an $m_x \in \mathbb{N}$ such that $x \in S_a \setminus \left( \bigcup_{\ell \geq m} E_{N,\ell} \right)$ for all $m \geq m_x$. We also let $r_x > 0$ be the Euclidean distance from $x$ to the union of all removed $T_{k_1,\ldots,k_{m-1},a_m^{-1}}$ with $m \leq m_x - 1$. Since $x \notin F$, we see that $r_x > 0$. Because $a \in \ell^n$, we can further find $\bar{m}_x \geq m_x$ such that for all $m \geq \bar{m}_x$,

$$(3.3) \quad a_m \leq (1 - \delta) r_x / 2N_{\ell}^2.$$

**Step 3.** For each $m > \bar{m}_x$ and

$$y \in B(x, \bar{N}_e a_1 \cdots a_m) \setminus B(x, \bar{N}_e a_1 \cdots a_{m+1}),$$
we are going to estimate $d_{S_n, \delta}(x, y)$ from below. By Lemma 3.6, we know that $d_{S_n, \delta, t}(x, y)$ converges to $d_{S_n, \delta}(x, y)$ as $t \to 0$. So it suffices to estimate $d_{S_n, \delta, t}(x, y)$ (for simplicity, we denote this quantity $d_t(x, y)$) from below for all sufficiently small $t$. By Lemma 3.6 again, we have

$$
(d_t(x, y))^2 = \inf_{\gamma} \int_0^1 \Phi_t \ast \left( \frac{1}{1 - \delta 1_{S_n}} \right) (\gamma(s)) \left| \frac{d}{ds} \gamma(s) \right|^2 ds
$$

and for each $t \in (0, 1)$, we can find a rectifiable curve $\gamma_{x,y,t}$ joining $x$ and $y$ such that the above infimum is reached, that is

$$
(d_t(x, y))^2 = \int_0^1 \Phi_t \ast \left( \frac{1}{1 - \delta 1_{S_n}} \right) (\gamma_{x,y,t}(s)) \left| \frac{d}{ds} \gamma_{x,y,t}(s) \right|^2 ds,
$$

and hence, by Hölder’s inequality,

$$
d_t(x, y) \geq \int_0^1 \Phi_t \ast \left( \frac{1}{1 - \delta 1_{S_n}} \right) (\gamma_{x,y,t}(s)) \left| \frac{d}{ds} \gamma_{x,y,t}(s) \right| ds.
$$

Observe that for each $t \in (0, 1)$ and every pair $z, w \in \mathbb{R}^n$,

$$
|z - w| \leq d_t(z, w) \leq \frac{1}{1 - \delta} |z - w|,
$$

which follows from

$$
1 \leq \Phi_t \ast \left( \frac{1}{1 - \delta 1_{S_n}} \right) \leq \frac{1}{1 - \delta}.
$$

Hence the curves $\gamma_{x,y,t}$ are Lipschitz with respect to the Euclidean metric under a suitable parametrization, and moreover, with a normalization, we can assume that for the Euclidean derivative, $|\frac{d}{ds} \gamma_{x,y,t}(s)| = 1$ for almost all $s \in [0, \ell_{\mathbb{R}^n}(\gamma_{x,y,t})]$. Hence

$$
d_t(x, y) \geq \int_0^{\ell_{\mathbb{R}^n}(\gamma_{x,y,t})} \Phi_t \ast \left( \frac{1}{1 - \delta 1_{S_n}} \right) (\gamma_{x,y,t}(s)) ds.
$$

**Step 4.** To estimate $d_t(x, y)$, we only need to know the length of the set

$$
L_t = \left\{ s \in [0, \ell_{\mathbb{R}^n}(\gamma_{x,y,t})] : \Phi_t \ast \left( \frac{1}{1 - \delta 1_{S_n}} \right) (\gamma_{x,y,t}(s)) \geq (1 - \epsilon) \frac{1}{1 - \delta} \right\}.
$$

To this end, observe that if $t = a_1 \cdots a_{\ell}$ for any large $\ell > m$ and $z \in B(x, \tilde{N}_0^{-1} a_1 \cdots a_m)$ but does not belong to the double enlargement of the (removed) cube $T_{k_1, \ldots, k_{i-1}, a_i^{-1}}$ with $m \leq i \leq \ell$, by $N_{\epsilon} \geq 4\tilde{N}_0^{-1} \frac{1}{(1 - \delta)}$, we have

$$
B(z, 2t) \subset B \left( x, \frac{4\tilde{N}_0^{-1}}{1 - \delta} a_1 \cdots a_m \right) \subset B(x, N_{\epsilon} a_1 \cdots a_m).
$$
Hence
\[ \Phi_t \ast \left( \frac{1}{1 - \delta} S_n \right) (z) \geq \frac{1}{1 - \delta} \left[ 1 - \frac{c_n}{|B(z, t)|} \right] B(z, 2t) \cap \left( \bigcup_{j \geq \ell} T_{k_1, \ldots, k_{j-1}, a_j^{-n}} \right)]. \]

Note that \( a \in \ell^n \) implies \( |\bigcup_{j \geq \ell} T_{k_1, \ldots, k_{j-1}, a_j^{-n}}| \to 0 \) as \( \ell \to \infty \). For every \( \epsilon > 0 \) there exists \( \ell_0 \in \mathbb{N} \) which depends only on \( \epsilon \) such that for all \( \ell \geq \ell_0 \),
\[ \left| \bigcup_{j \geq \ell} T_{k_1, \ldots, k_{j-1}, a_j^{-n}} \right| \leq \epsilon |B(z, t)|. \]
Therefore, if \( \ell > \max\{\ell_0, m\} \), for the above \( z \), we have
\[ \Phi_t \ast \left( \frac{1}{1 - \delta} S_n \right) (z) \geq \frac{1 - \epsilon}{1 - \delta}. \]

On the other hand, by the choice of \( r_x \) and \( N_{\epsilon} \) at Step 2, when \( t = a_1 \cdots a_\ell \leq a_\ell \leq r_x/10 \) we have \( \ell_{\mathbb{R}^n}(\gamma_{\{x, y, t\}}) \leq r_x/10 \). So for \( i \leq \tilde{m}_x - 1 \),
\[ \text{dist}_{\mathbb{R}^n}(\gamma_{\{x, y, t\}}, T_{k_1, \ldots, k_{i-1}, a_i^{-n}}) \geq \text{dist}_{\mathbb{R}^n}(x, T_{k_1, \ldots, k_{i-1}, a_i^{-n}}) - \ell_{\mathbb{R}^n}(\gamma_{\{x, y, t\}}) \geq 2t. \]
Therefore \( T_{k_1, \ldots, k_{i-1}, a_i^{-n}} \) makes no contribution when we estimate \( \Phi_t \ast \left( \frac{1}{1 - \delta} S_n \right) (z) \) from below for \( z \in \gamma_{\{x, y, t\}} \). This also holds for \( \tilde{m}_x \leq i \leq m - 1 \) by a similar argument. Indeed, for \( \tilde{m}_x \leq i \leq m - 1 \), we also have \( t = a_1 \cdots a_\ell \leq a_1 \cdots a_{i-1}/10 \) and \( \ell_{\mathbb{R}^n}(\gamma_{\{x, y, t\}}) \leq a_1 \cdots a_{i-1}/10 \), and hence, in this case the Euclidean distance from \( \gamma_{\{x, y, t\}} \) to each \( T_{k_1, \ldots, k_{i-1}, a_i^{-n}} \) is at least \( 2t \).

Based on the above argument, the lower bound estimate of the length of \( L_t \) is transferred to the upper bound estimate of the length of
\[ \tilde{L}_t = \left\{ s \in [0, \ell_{\mathbb{R}^n}(\gamma_{\{x, y, t\}})]: \gamma_{\{x, y, t\}}(s) \in \bigcup_{m \leq i \leq \ell - 1} \bigcup_{k_1, \ldots, k_i} 2T_{k_1, \ldots, k_{i-1}, a_i^{-n}} \right\}. \]
Here, \( \ell \) is the positive integer such that \( t = a_1 \cdots a_\ell \); keep in mind that \( \ell > m \).

Step 5. To estimate \( \tilde{L}_t \), we need the following key observations.
(i) Since \( |x - y| \leq d_{t}(x, y) \leq \frac{1}{1 - \delta} |x - y| \), we have
\[ (3.5) \quad \gamma_{\{x, y, t\}} \subset B \left( x, \frac{\tilde{N}_t}{1 - \delta} a_1 \cdots a_m \right). \]
Recall that \( x \) is not in any \( N_{\epsilon} \)-close cube of \( T_{k_1, \ldots, k_{m-1}, a_m^{-n}} \) whenever \( m \geq m_x \). Since \( N_{\epsilon} \geq 2n\tilde{N}_t \frac{1}{(1 - \delta)^x} \), we have that
\[ 2T_{k_1, \ldots, k_{m-1}, a_m^{-n}} \cap \gamma_{\{x, y, t\}} = \emptyset. \]
(ii) If $|x-y| \leq N_\epsilon a_1 \cdots a_{m+1}$, then by (3.3), $|x-y| \leq a_1 \cdots a_m$, and hence there are at most $2^n$ many cubes $T_{k_1, \ldots, k_m}$ with $k_m < a_{m+1}^{-n}$ that overlaps with $\gamma_{\{x,y,t\}}$, and hence, there are at most $2^n$ many $T_{k_1, \ldots, k_m, a_{m+1}^{-n}}$ such that its twice-enlargement overlapped with $\gamma_{\{x,y,t\}}$. Moreover, up to a modification of the curve $\gamma_{\{x,y,t\}}$ without increasing the $d_t$-length of $\gamma_{\{x,y,t\}}$, we may assume that the Euclidean length of $\gamma_{\{x,y,t\}} \cap 2T_{k_1, \ldots, k_m, a_{m+1}^{-n}}$ is less than $4\sqrt{n}a_1 \cdots a_{m+1}$. Thus by $|x-y| \geq N_\epsilon a_1 \cdots a_{m+1}$,

$$
\hat{L}_{t,m} = \left\{ s \in [0, \ell_{\mathbb{R}^n}(\gamma_{\{x,y,t\}})], \gamma_{\{x,y,t\}}(s) \in \bigcup_{k_1, \ldots, k_m} 2T_{k_1, \ldots, k_m, a_{m+1}^{-n}} \right\}
$$

$$
\leq 4\sqrt{n}2^n a_1 \cdots a_{m+1} \leq 4\sqrt{n}2^n \frac{1}{N_\epsilon} |x-y| \leq \epsilon |x-y|
$$

If $N_\epsilon a_1 \cdots a_{m+1} \leq |x-y| \leq N_\epsilon a_1 \cdots a_m$, since $\ell_{\mathbb{R}^n}(\gamma_{\{x,y,t\}}) \leq \frac{N_\epsilon}{1-\delta} a_1 \cdots a_m$, there are at most $(2\frac{N_\epsilon}{1-\delta})^n$ many $T_{k_1, \ldots, k_m}$ with $k_m < a_{m+1}^{-n}$ such that

$$
T_{k_1, \ldots, k_m} \cap \gamma_{\{x,y,t\}} \neq \emptyset,
$$

and hence, at most $(2\frac{N_\epsilon}{1-\delta})^n$ many $T_{k_1, \ldots, k_m, a_{m+1}^{-n}}$ such that their twice-enlargement overlap with $\gamma_{\{x,y,t\}}$. Notice that $4\sqrt{n}(2\frac{N_\epsilon}{1-\delta})^n \leq \epsilon N_\epsilon$ and $a_{m+1} \leq (1-\delta)r_x/2N_\epsilon^2 \leq 1/2N_\epsilon^2$. With a similar argument on $\gamma_{\{x,y,t\}}$ as above, we have

$$
\tilde{L}_{t,m} = \left\{ s \in [0, \ell_{\mathbb{R}^n}(\gamma_{\{x,y,t\}})], \gamma_{\{x,y,t\}}(s) \in \bigcup_{k_1, \ldots, k_m} 2T_{k_1, \ldots, k_m, a_{m+1}^{-n}} \right\}
$$

$$
\leq 2\sqrt{n}(2\frac{N_\epsilon}{1-\delta})^n a_1 \cdots a_{m+1} \leq 2\sqrt{n}(2\frac{N_\epsilon}{1-\delta})^n \frac{1}{N_\epsilon} |x-y| \leq \epsilon |x-y|
$$

(iii) For each $i \geq m+1$, the numbers of $T_{k_1, \ldots, k_{i-1}, a_{i+1}^{-n}}$, whose twice-enlargement overlapped with $\gamma$, is no more that the numbers of $T_{k_1, \ldots, k_{i}}$ with $k_{i} < a_{i}^{-n}$ which overlaps with $\gamma$. By induction and similar argument as above,

$$
\tilde{L}_{t,i} = \left\{ s \in [0, \ell_{\mathbb{R}^n}(\gamma_{\{x,y,t\}})], \gamma_{\{x,y,t\}}(s) \in \bigcup_{k_1, \ldots, k_{i}} 2T_{k_1, \ldots, k_{i}, a_{i+1}^{-n}} \right\}
$$

$$
\leq 8\sqrt{n}a_{i+1} \left\{ s \in [0, \ell_{\mathbb{R}^n}(\gamma_{\{x,y,t\}})], \gamma_{\{x,y,t\}}(s) \in \bigcup_{k_1, \ldots, k_{i}} T_{k_1, \ldots, k_{i}} \right\}
$$

$$
\leq (8\sqrt{n})^{i-m} a_{i+1} \cdots a_{m+1} \epsilon |x-y|.
$$

**Step 6.** The three observations above yield that

$$
|\tilde{L}_t| \leq \sum_{i=m}^{t} \tilde{L}_{t,i} \leq \epsilon |x-y| + \sum_{i \geq m+1} (8\sqrt{n})^{i-m} a_{i+1} \cdots a_{m+1} \epsilon |x-y| \leq 3\epsilon |x-y|,
$$
and hence,

\[ |L_t| \geq \ell_{\mathbb{R}^n}(\gamma_{\{x, y, t\}}) - |\tilde{L}_t| \geq (1 - 3\epsilon)|x - y|. \]

Noticing that \( \ell_{\mathbb{R}^n}(\gamma_{\{x, y, t\}}) \geq |x - y| \), we have

\[ d_t(x, y) \geq \frac{\sqrt{1 - \epsilon}}{\sqrt{1 - \delta}}(1 - 3\epsilon)|x - y| \geq \frac{(1 - 4\epsilon)}{\sqrt{1 - \delta}}|x - y|. \]

By the arbitrariness of \( \epsilon \), we conclude (3.1).

\[ \square \]

**Remark 3.7.** (i) Notice that Lemma 3.5 is much stronger than Lemma 2.5. To see this, let \( \lambda \) be a positive continuous on \( \mathbb{R}^n \) such that (1.1) holds when \( A = (1 - \delta)S_{\alpha, \delta}I_n \), that is,

\[ \frac{1}{\lambda(x)}|\xi|^2 \leq \langle A(x)\xi, \xi \rangle = (1 - \delta)S_{\alpha, \delta})|\xi|^2 \leq \lambda(x)|\xi|^2 \]

for all \( x \in \mathbb{R}^n \) and \( \xi \in \mathbb{R}^n \setminus \{0\} \). From this, when \( x \in S_{\alpha} \), it follows that \( \frac{1}{\lambda(x)} \leq 1 - \delta \leq \lambda(x) \), which yields \( \lambda(x) \geq \frac{1}{1 - \delta} \geq 1 \). Without loss of generality, we may let that \( \lambda(x) = \frac{1}{1 - \delta} \) for \( x \in S_{\alpha} \). Then by Lemma 2.5, we always have that

\[ \liminf_{x \neq y \to x} \frac{d_{S_{\alpha}, \delta}(y, x)}{|y - x|} \geq \frac{1}{\sqrt[4]{\lambda(x)}} = \sqrt{1 - \delta}, \]

which is equivalent to that for any \( \epsilon > 0 \), we can find \( r > 0 \) such that for all \( y \in B(x, r) \),

\[ d_{S_{\alpha}, \delta}(x, y) \geq (1 - \epsilon)\sqrt{1 - \delta}|x - y|; \]

and that

\[ \limsup_{x \neq y \to x} \frac{d_{S_{\alpha}, \delta}(y, x)}{|y - x|} \leq \sqrt[4]{\lambda(x)} = \frac{1}{\sqrt{1 - \delta}}, \]

which is equivalent to that for any \( \epsilon > 0 \), we can find \( r > 0 \) such that for all \( y \in B(x, r) \),

\[ d_{S_{\alpha}, \delta}(x, y) \leq (1 + \epsilon)\frac{1}{\sqrt{1 - \delta}}|x - y|. \]

Obviously, we cannot obtain (3.1) from (3.6) and (3.7), and hence cannot obtain (3.1) from Lemma 2.5. Indeed, (3.1) is much stronger than (3.6).

(ii) The reason why our Cantor set and Sierpinski carpet give entirely different outcomes is the different behavior of \( d_{S_{\alpha}, \delta} \) and \( d_{C_{\alpha}, \delta} \) when \( \delta \in (0, 1) \). Indeed, as the proof of Lemma 3.5 shows, given almost every point \( x \in S_{\alpha} \) and every close-by point \( y \), any curve that connects \( x \) to \( y \) with length comparable to \( |x - y| \) lives in \( S_{\alpha} \) for a significant fraction of the time, and sees \( x \) as a linear density point of \( S_{\alpha} \). In comparison, almost every point \( x \) in the Cantor set \( C_{\alpha} \) can be connected to some near-by point by a curve of length comparable to the Euclidean distance between the two points while avoiding \( C_{\alpha} \) for almost all of the time.
Remark 3.8. Given an $A \in \mathcal{A}(\Omega)$ with the intrinsic distance $d_A$, by Lemma 2.3, for all $u \in \text{Lip}(\Omega)$, we always have

$$\langle A \nabla u, \nabla u \rangle \leq (\text{Lip} d_A u)^2 \leq |Du|_{d_A}^2$$

almost everywhere. If $A \in \mathcal{A}_{wusc}(\Omega)$, then the first “≤” is actually “=”, if $A \in \mathcal{A}_{wusc}(\Omega)$ and $u \in C^1_{\text{loc}}(\Omega)$, the second “≤” is actually “=”. However if $A \notin \mathcal{A}_{wusc}(\Omega)$, then the first “≤” may be “<” on some set with positive measure as shown by Proposition 3.1; the second “≤” may be “<” on some set with positive measure as shown by Theorem 3.4 even for $u \in C^1_{\text{loc}}(\Omega)$.

4 $L^\infty$-Variational problem for arbitrary $A \in \mathcal{A}(\Omega)$

In this section, we assume that $n \geq 2$. Let $A \in \mathcal{A}(\Omega)$ and $U \Subset \Omega$ be a bounded open subset. We obtain the following existence and uniqueness of the absolute minimizer given a boundary data (see Section 1 for the definition of absolute minimizers).

Theorem 4.1. (i) For every $f \in \text{Lip}(\partial U)$, there exists a unique absolutely minimizing Lipschitz extension.

(ii) The absolute minimizer is completely determined by the intrinsic distance in the following sense: Let $A, \tilde{A} \in \mathcal{A}(\Omega)$ and denote by $d_A, d_{\tilde{A}}$ (resp. $H, \tilde{H}$) the corresponding intrinsic distance (resp. Hamiltonian). If

$$\lim_{x \neq y \to x} \frac{d_A(x, y)}{d_{\tilde{A}}(x, y)} = 1$$

for almost all $x \in U$, then $u$ is an absolute minimizer on $U$ for the Hamiltonian $H$ if and only if $u$ is an absolute minimizer on $U$ for the Hamiltonian $\tilde{H}$.

A special case of (4.1) is that for almost every $x \in U$, there exists $r_x > 0$ such that $d_A(x, y) = d_{\tilde{A}}(x, y)$ for all $y \in B_d(x, r_x)$.

We do not know whether if weak upper semicontinuity of $A$ could guarantee $C^1$-regularity for the associated minimizers. However, we have the following negative result for the $C^1$-regularity of absolute minimizers.

Proposition 4.2. Let $A = 1 - \delta 1_{S_a}$ be as in Subsection 3.2, where $a \in \ell^n$ with $a_j \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ and $\delta \in (0, 1)$. Then there is an absolute minimizer on $U = (0, 1)^n$ associated to a related $L^\infty$-variational problem that is not $C^1$-regular on $U$.

Now we prove Theorem 4.1 and Proposition 4.2. Observe that the relative compactness of $U$ implies that the function $\lambda$ appearing (1.1) is bounded from above on $U$. Without loss of generality, we may assume that $\Omega = \mathbb{R}^n$ and that the diffusion matrix $A$ satisfies

$$\frac{1}{\lambda} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda |\xi|^2$$

for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where $\lambda \geq 1$ is a fixed constant. Observe that $(\mathbb{R}^n, d_A)$ is a length space (see for example [25]), and hence a geodesic space due to its local compactness.
Since we have no regularity of continuity assumption on the diffusion matrix, the approach of using the Aronsson equations is not applicable. Instead of this we characterize absolute minimizers via intrinsic distance; see for example [15, 16, 7, 11]. The following Lemma 4.3 connects the absolute minimizer with a description via pointwise Lipschitz constants; its proof relies on (the key) Lemma 4.4.

**Lemma 4.3.** Let $u \in \text{Lip}(U)$. Then $u$ is an absolute minimizer on $U$ if and only if for each bounded open subset $V \subseteq U$ and all $v \in \text{Lip}(V) \cap C(\overline{V})$ with $u|_{\partial V} = v|_{\partial V}$, one (both) of the following holds:

(i) $\text{esssup}_{x \in V} \text{Lip}_{d_A} u(x) \leq \text{esssup}_{x \in V} \text{Lip}_{d_A} v(x)$;

(ii) $\sup_{x \in V} \text{Lip}_{d_A} u(x) \leq \sup_{x \in V} \text{Lip}_{d_A} v(x)$.

**Lemma 4.4.** For every bounded open set $V \subseteq \mathbb{R}^n$ and every $u \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$, we have

$$\text{esssup}_{x \in V} \sqrt{H(x, \nabla u(x))} = \text{esssup}_{x \in V} \text{Lip}_{d_A} u(x) = \sup_{x \in V} \text{Lip}_{d_A} u(x).$$

**Proof.** By Lemma 2.3, we always have

$$\sqrt{F(u, V)} := \text{esssup}_{x \in V} \sqrt{H(x, \nabla u(x))} \leq \text{esssup}_{x \in V} \text{Lip}_{d_A} u(x) \leq \sup_{x \in V} \text{Lip}_{d_A} u(x).$$

It then suffices to show that $\sup_{x \in V} \text{Lip}_{d_A} u(x) \leq \sqrt{F(u, V)}$, which is further reduced to showing that for every $x \in V$, there exists $r_x < d_A(x, \partial V)$ such that for all $y \in B_{d_A}(x, r_x)$,

$$|u(x) - u(y)| \leq \sqrt{F(u, V)}d_A(x, y).$$

We divide the proof of (4.4) into 4 steps.

**Step 1.** Fix $x \in V$ and $0 < r < \frac{1}{4}d_A(x, \partial V)$. Extend $u$ from $B_{d_A}(x, r)$ to $\mathbb{R}^n$ via a McShane extension as follows:

$$u_{x,r}(y) = \inf_{z \in B_{d_A}(x, r)} \{u(z) + \text{Lip}_{d_A}(u, B_{d_A}(x, r))d_A(z, y)\}.$$

Then $u_{x,r} = u$ on $B_{d_A}(x, r)$, and

$$H(z, \nabla u_{x,r}(z)) = H(z, \nabla u(z)) \leq F(u, B_{d_A}(x, r)) \leq F(u, V)$$

for almost all $z \in B_{d_A}(x, r)$, and by Lemma 2.3,

$$\sqrt{H(z, \nabla u_{x,r}(z))} \leq \text{Lip}_{d_A} u_{x,r}(z) \leq \text{Lip}_{d_A}(u, B_{d_A}(x, r))$$

for almost all $z \notin B_{d_A}(x, r)$.

**Step 2.** By the proof of Lemma 2.5 and the ellipticity condition (4.2), we have $d_A(x, y) \geq |x - y|/\sqrt{\lambda}$. Hence it follows from the last part of Step 1 above that for almost all $z \in \mathbb{R}^n \setminus B_{d_A}(x, r)$,

$$\sqrt{H(z, \nabla u_{x,r}(z))} \leq \text{Lip}_{d_A}(u, B_{d_A}(x, r)) \leq \sqrt{\lambda} \text{Lip}(u, B_{d_A}(x, r))$$
and denote by \( \tilde{z} \) each \( z, y \)

\[
\tilde{z} = \sqrt{A} \sup_{B_{d_A}(x, r)} |\nabla u| \leq \lambda \sqrt{F(u, V)}.
\]

**Step 3.** Now we set

\[
\tilde{A} = 1_{B_{d_A}(x, r)} A + \frac{1}{2\lambda} 1_{\mathbb{R}^n \setminus B_{d_A}(x, r)} A
\]

and denote by \( \tilde{H} \) and \( d_{\tilde{A}} \) the corresponding Hamiltonian and intrinsic distance. Then for each \( z \in B_{d_A}(x, r) \), there exists \( 0 < r_z < r - d_A(z, x) \) such that whenever \( d_A(z, y) < r_z \), we have

\[
d_{\tilde{A}}(z, y) = d_A(z, y).
\]

This is seen by modifying the proof of Lemma 2.5 by replacing the Euclidean metric with the metric \( d_A \). Indeed, notice that there exists a constant \( \mathcal{C} \geq 1 \) such that for all \( z, y \in \mathbb{R}^n \),

\[
\frac{1}{\mathcal{C}} d_A(z, y) \leq d_{\tilde{A}}(z, y) \leq \mathcal{C} d_A(z, y).
\]

For \( r_z \leq \frac{1}{\mathcal{C}} (r - d_A(z, x)) \), we have

\[
B_{d_A}(z, r_z) \subset B_{d_{\tilde{A}}}(z, \mathcal{C} r_z) \subset B_{d_{\tilde{A}}}(z, \mathcal{C}^2 r_z) \subset B_{d_A}(x, r).
\]

Set

\[
v_{z, r_z}(y) = \min\{d_{\tilde{A}}(z, y), r_z\}.
\]

We see that

\[
H(y, \nabla v_{z, r_z}(y)) = \tilde{H}(y, \nabla v_{z, r_z}(y)) \leq (\text{Lip}_{d_{\tilde{A}}} v_{z, r_z}(y))^2 \leq 1
\]

for almost all \( y \in B_{d_{\tilde{A}}}(z, \mathcal{C} r_z) \) and

\[
H(y, \nabla v_{z, r_z}(y)) = \tilde{H}(y, \nabla v_{z, r_z}(y)) = 0
\]

for almost all \( y \notin B_{d_{\tilde{A}}}(z, \mathcal{C} r_z) \). Hence using \( v_{z, r_z} \) in the definition of \( d_A \), we have \( d_A(z, y) \geq d_{\tilde{A}}(z, y) \) for all \( y \in B_{d_{\tilde{A}}}(z, \mathcal{C} r_z) \) and hence \( y \in B_{d_{\tilde{A}}}(z, r_z) \). Similar argument show to that \( d_A(z, y) \geq d_{\tilde{A}}(z, y) \) for all \( y \in B_{d_A}(z, r_z) \). We conclude (4.5) from these inequalities.

**Step 4.** From the discussion in Steps 1 and 2,

\[
\tilde{H}(z, \nabla u_{x, r}(z)) \leq F(u, V)
\]

for almost every \( z \). Hence, using \( \frac{1}{\sqrt{F(u, V)}} u_{x, r} \) in the definition of \( d_{\tilde{A}} \), we see that for all \( z, y \in \mathbb{R}^n \),

\[
\frac{1}{\sqrt{F(u, V)}} |u_{x, r}(z) - u_{x, r}(y)| \leq d_{\tilde{A}}(z, y).
\]

In particular, for all \( z, y \in B_{d_A}(x, r) \), since \( u(z) = u_{x, r}(z) \) and \( u(y) = u_{x, r}(y) \), we have

\[
|u(z) - u(y)| \leq \sqrt{F(u, V)} d_{\tilde{A}}(z, y) = \sqrt{F(u, V)} d_A(z, y).
\]

Applying this with \( z = x \) and \( y \in B_{d_A}(x, r) \), we obtain (4.4). \( \square \)
Remark 4.5. By Lemma 2.3 above, we always have $H(x, \nabla u(x)) \leq (\text{Lip}_{d_A} u(x))^2$ almost everywhere. But Proposition 3.1 shows that it may happen that $H(x, \nabla u(x)) < (\text{Lip}_{d_A} u(x))^2$ on a set with positive measure, and hence we cannot expect $H(x, \nabla u(x)) = (\text{Lip}_{d_A} u(x))^2$ almost everywhere. But as a compensation, Lemma 4.4 provides a weak variant for this:

$$\text{esssup}_{z \in B(x, r)} H(z, \nabla u(z)) = \text{esssup}_{x \in B(x, r)} (\text{Lip}_{d_A} u(x))^2 = \sup_{x \in B(x, r)} (\text{Lip}_{d_A} u(x))^2$$

for all $x$ and small $r$. This phenomenon persists in the setting of general regular, strongly local Dirichlet forms; see the companion paper [21] of this paper.

Notice that our concept of absolutely minimizing Lipschitz extension defined in Section 1 corresponds to the strongly absolutely minimizing Lipschitz extension in [19]. Recall that (4.2) implies that $(\mathbb{R}^n, d_A, dx)$ is a doubling metric measure space supporting a $(1,1)$-Poincaré inequality. By Lemma 4.3 and [19, Theorem 3.1], we have the following existence result.

Lemma 4.6. For every $f \in \text{Lip}(\partial U)$, there exists an absolutely minimizing Lipschitz extension.

The uniqueness of an absolute minimizing Lipschitz extension will follow from the comparison formula.

Lemma 4.7. Let $u, v \in \text{Lip}(U) \cap C(\overline{U})$ be absolute minimizers on $U$. Then

$$\max_{x \in U} [u(x) - v(x)] = \max_{x \in \partial U} [u(x) - v(x)].$$

To prove Lemma 4.7, we need the following lemmas. First, as a consequence of Lemma 4.3, we have the following result.

Lemma 4.8. If $u$ is an absolute minimizer on $U$, then for all open subsets $V \Subset U$, $\text{Lip}_{d_A}(u, V) = \text{Lip}_{d_A}(u, \partial V)$.

Proof. Notice that for every pair $x, y \in \partial V$ with $x \neq y$, by the continuity of $d_A$ we can find $x_n, y_n \in V$ such that $x_n \to x$ and $y_n \to y$. By the continuity of $u$, $\frac{|u(x_n) - u(y_n)|}{d_A(x_n, y_n)} \to \frac{|u(x) - u(y)|}{d_A(x, y)}$ and hence $\text{Lip}_{d_A}(u, V) \geq \text{Lip}_{d_A}(u, \partial V)$. Thus it suffices to prove the converse.

For $x \in \mathbb{R}^n$, set

$$w(x) = \sup_{z \in \partial V} [u(z) + \text{Lip}_{d_A}(u, \partial V)d_A(x, z)].$$

Then $\text{Lip}_{d_A}(w, \mathbb{R}^n) = \text{Lip}_{d_A}(u, \partial V)$ and $w = u$ on $\partial V$. Applying Lemma 4.3, we have

$$\sup_{x \in V} \text{Lip}_{d_A} u(x) \leq \sup_{x \in V} \text{Lip}_{d_A} w(x) \leq \text{Lip}_{d_A}(u, \partial V).$$

Now, given a pair of points $x, y \in U$, let $\gamma$ be a $d_A$-geodesic curve joining $x$ and $y$. The existence of $\gamma$ is guaranteed by the fact that $(\mathbb{R}^n, d_A)$ is a geodesic space [25]. If $\gamma \subset V$, then

$$|u(x) - u(y)| \leq \int_{\gamma} \text{Lip} u(z) d_A z \leq d_A(x, y) \sup_{x \in V} \text{Lip}_{d_A} u(x) \leq d_A(x, y) \text{Lip}_{d_A}(u, \partial V).$$
Here $d_A z$ denotes arc-length integral on $\gamma$ with respect to the metric $d_A$. If $\gamma \not\subset V$, denote by $\hat{x}$ and $\hat{y} \in \gamma \cap \partial V$ points that have shortest distance to $x$ and $y$, respectively. Then

$$|u(x) - u(y)| \leq |u(x) - u(\hat{x})| + |u(\hat{x}) - u(\hat{y})| + |u(\hat{y}) - u(y)|$$

$$\leq [d_A(x, \hat{x}) + d_A(\hat{y}, y)] \sup_{x \in V} \text{Lip}_{d_A} u(x) + d_A(\hat{x}, \hat{y}) \text{Lip}_{d_A} u, \partial V$$

$$\leq d_A(x, y) \text{Lip}_{d_A} u, \partial V.$$ 

In either case, we have the inequality

$$\frac{|u(x) - u(y)|}{d_A(x, y)} \leq \text{Lip}_{d_A} u, \partial V.$$ 

This means that $\text{Lip}_{d_A}(u, V) \leq \text{Lip}_{d_A}(u, \partial V)$.

A function $u \in C(U)$ is said to satisfy the property of comparison with cones if for all each subset $V \Subset U$, and for all $a \geq 0$, $b \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n \setminus V$, we have

(i) $\max_{x \in \partial V} |u(x) - C_{b,a,x_0}(x)| \leq 0$ implies $\max_{x \in V} |u(x) - C_{b,a,x_0}(x)| \leq 0$;

(ii) $\max_{x \in \partial V} |u(x) - C_{b,a,x_0}(x)| \geq 0$ implies $\max_{x \in V} |u(x) - C_{b,a,x_0}(x)| \geq 0,$

where the cone function is defined by $C_{b,a,x_0}(x) = b + a d_A(x, x_0)$. It is known that an absolute minimizer satisfies the comparison property with cones; see [10] for Euclidean case and [6, 19, 16, 7, 11] for the setting of metric spaces that are length spaces. For the sake of completeness, we sketch a proof below.

**Lemma 4.9.** An absolute minimizer satisfies the property of comparison with cones.

**Proof.** We prove Condition (i), with the proof of Condition (ii) similar (and left to the interested reader). Let $u$ be an absolute minimizer and assume that $\max_{x \in \partial V} |u(x) - C_{b,a,x_0}(x)| \leq 0$. Suppose that the condition $\max_{x \in V} |u(x) - C_{b,a,x_0}(x)| \leq 0$ is not true. Denote by $W$ the open set of all $x \in V$ such that $u(x) > C_{b,a,x_0}(x)$. By the above supposition, $W$ is not empty. We can see that $u = C_{b,a,x_0}$ on $\partial W$. Since $W \subset V \Subset U$, by Corollary 4.8 we have $\text{Lip}_{d_A}(u, W) = \text{Lip}_{d_A}(u, V) = \text{Lip}_{d_A}(u, \partial W) = a$. For $x \in W$, let $\gamma$ be the $d_A$-geodesic curve joining $x$ and $x_0$, and take $z \in \partial V \cap \gamma$ be a closest point to $x$. Then

$$u(x) - u(z) > C_{b,a,x_0}(x) - C_{b,a,x_0}(z) = ad_A(x_0, x) - ad_A(x_0, z) = ad_A(x, z),$$

which implies that $\text{Lip}_{d_A}(u, W) > a$. This is a contradiction. So $W$ must be empty. \qed

With the aid of Lemma 4.3 and Lemma 4.9, Lemma 4.7 will be proved by following the procedure from [1]. Since the proof in [1] is for the case $A = I^n$, we write down the details below for the reader’s convenience. For $x \in U_r = \{z \in U : B_{\text{d}_{d_A}(z, r)} \subset U\}$ with $r > 0$, we set $u^r(x) = \sup_{d_A(z, x) \leq r} u(z)$ and $u_r(x) = \inf_{d_A(z, x) \leq r} u(z)$ and also

$$S_r^+ u(x) = \frac{u^r(x) - u(x)}{r}, \quad S_r^- u(x) = \frac{u(x) - u_r(x)}{r}.$$
Proof of Lemma 4.7. First we claim that for $x \in U_{2r}$,\textflush
\begin{equation}
(4.6) \quad S^-_r u^r(x) - S^+_r u^r(x) \leq 0 \leq S^-_r v_r(x) - S^+_r v_r(x).
\end{equation}
Indeed, let $y \in \overline{B_dA(x, r)}$ and $z \in \overline{B_dA(x, 2r)}$ such that $u^r(x) = u(y)$ and $(u^r)'(x) = u^2r(x) = u(z)$. Observe that $(u^r)_r(x) \geq u(x)$. We then have\textflush
$$S^-_r u^r(x) - S^+_r u^r(x) = \frac{1}{r} [2u^r(x) - (u^r)'(x) - (u^r)_r(x)] \leq \frac{1}{r} [2u(y) - u(z) - u(x)].$$
For $w \in \Omega$ such that $d_A(x, w) = 2r$, we have\textflush
$$u(w) \leq u(z) = u(x) + [u(z) - u(x)] = u(x) + \frac{[u(z) - u(x)]}{2r} d_A(w, x).$$
Thus the comparison with cones property of $u$ implies that the inequality\textflush
$$u(w) \leq u(x) + \frac{[u(z) - u(x)]}{2r} d_A(w, x)$$
holds for all $w \in \Omega$ with $d_A(x, w) \leq 2r$. In particular, taking $w = y$ and by $d_A(y, x) \leq r$, we have\textflush
$$u(y) \leq u(x) + \frac{[u(z) - u(x)]}{2r} d_A(y, x) \leq u(x) + \frac{1}{2} [u(z) - u(x)] = \frac{1}{2} [u(z) + u(x)],$$
which implies the first inequality of (4.6). The second inequality of (4.6) follows similarly. Notice that (4.6) further implies\textflush
\begin{equation}
(4.7) \quad \sup_{x \in U_r} [u^r(x) - v_r(x)] = \sup_{x \in U_r \setminus U_{2r}} [u^r(x) - v_r(x)].
\end{equation}
Given the above, letting $r \to 0$ in (4.7), we obtain Lemma 4.7. Thus it remains only to prove (4.7). Assume that (4.7) is not true. Then there is some $r > 0$ for which\textflush\textflush
$$\sup_{x \in U_r \setminus U_{2r}} [u^r(x) - v_r(x)] > \sup_{x \in U_r \setminus U_{2r}} [u^r(x) - v_r(x)].$$
By the continuity of $u^r - v_r$, there must exist some $y \in \overline{U_r}$ such that $[u^r(y) - v_r(y)] = \sup_{x \in U_r} [u^r(x) - v_r(x)]$. Because (4.7) fails, we know that $y \in U_{2r}$. Denote by $E$ all such $y$ and set $F = \{ x \in E : u^r(x) = \max_{z \in E} u^r(z) \}$. Then $F$ is a closed subset of $U_{2r}$ by the continuity of $u^r$ again. Choose $x_0 \in \partial F$. Then because $x_0 \in E$, for every $x \in U$ we have\textflush
$$u^r(x_0) - v_r(x_0) \geq u^r(x) - v_r(x),$$
and it follows from the fact that $x_0 \in U_{2r}$ and for every $x \in B_dA(x_0, r)$,\textflush
$$u^r(x_0) - v_r(x_0) \geq \inf_{z \in B_dA(x_0, r)} u^r(z) - v_r(x) = (u^r)_r(x_0) - v_r(x).$$
Taking the infimum over $x \in B_dA(x_0, r)$, we obtain $S^-_r v_r(x_0) \leq S^-_r u^r(x_0)$. 


Case 1: \( S^+_r u'(x_0) = 0 \). Then by (4.6), we have \( S^-_r u'(x_0) \leq 0 \) and hence \( S^-_r u'(x_0) = 0 \), which together with \( S^+_r v_r(x_0) \leq S^-_r u'(x_0) \) implies that \( S^+_r v_r(x_0) = 0 \). By (4.6) again, we have \( S^+_r v_r(x_0) \leq 0 \) and hence \( S^+_r v_r(x_0) = 0 \). So \( u^r \) and \( v_r \) are constant on \( B_{d_A}(x_0, r) \). This contradicts \( x_0 \in \partial F \).

Case 2: \( S^+_r u'(x_0) > 0 \). Choose \( z \in \overline{B}_{d_A}(x_0, r) \) such that \( r S^+_r u'(x_0) = u^r(z) - u^r(x_0) \). Since \( u^r(z) > u^r(x_0) \) and \( x_0 \in F \), we know that \( z \notin E \). Therefore \( u^r(x_0) - v_r(x_0) > u^r(z) - v_r(z) \), and hence

\[
r S^+_r v_r(x_0) \geq v_r(z) - v_r(x_0) > u^r(z) - u^r(x_0) = r S^+_r u(x_0),
\]

which together with \( S^+_r v_r(x_0) \leq S^-_r u'(x_0) \) again yields that

\[
S^+_r v_r(x_0) - S^-_r v_r(x_0) \geq S^+_r u^r(x_0) - S^-_r u^r(x_0).
\]

This is a contradiction of (4.6).

So our assumption is not correct and hence (4.7) is true.

Now we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Theorem 4.1 (i) follows from Lemmas 4.6 and 4.7. Theorem 4.1 (ii) follows from Lemma 4.3 with the observation that under the assumption (4.1), \( \text{Lip}_{d_A} u = \text{Lip}_{d_A} u \) almost everywhere for every \( u \in \text{Lip}_{\text{loc}}(\mathbb{R}^n) \).

Proposition 4.2 will follow from Theorem 3.4 and the following result; see [19, Lemma 5.6] for the proof.

**Lemma 4.10.** Let \( u \in \text{Lip}(U) \) be an absolute minimizer. Then for all \( x \in U \), we have

\[
\text{Lip}_{d_A} u(x) = \lim_{r \to 0} S^+_r u(x) = - \lim_{r \to 0} S^-_r u(x).
\]

Moreover, \( S^+_r u(x) \) and \( S^-_r u(x) \) are increasing function with respect to \( r \in (0, d_A(x, \partial U)) \).

**Proof of Proposition 4.2.** We prove Proposition 4.2 on \( U = [0, 1]^n \supset S_A \). The proof is in two steps.

**Step 1:** In this step we show that each absolute minimizer on \( U \) that is of class \( C^1 \) must satisfy \( \nabla u = 0 \) on \( S_A \). Suppose that \( u \) is an absolute minimizer on \( U \) that is of class \( C^1 \). Observe that \( \text{Lip}_{d_A} u \) is upper semicontinuous on \( U \), that is, for any \( x \in U \), we have \( \text{Lip}_{d_A} u(x) \geq \limsup_{z \to x} \text{Lip}_{d_A} u(z) \). To see this, recall that the upper semicontinuity is equivalent to the property that for all \( L > 0 \), \( E_L = \{x \in U : \text{Lip}_{d_A} u(x) < L\} \) is open. Without loss of generality, we may assume that \( E_L \) is not empty. Let \( x \in E_L \). By Lemma 4.10, we know that for all \( z \in U \),

\[
(4.8) \quad \text{Lip}_{d_A} u(z) = \inf_{0 < r < d_A(z, \partial U)} S^+_r u(z) = \inf_{0 < r < d_A(z, \partial U)} \frac{u^r(z) - u(z)}{r}.
\]

It is easy to see that for each \( r > 0 \), \( u^r \) and hence \( S^+_r u \) is continuous on \( U_r = \{y \in U : d_A(y, \partial U) > r\} \). Since \( x \in E_L \), applying (4.8), we have \( S^+_r u(x) < L \) for some
0 < r < d_A(x, \partial U)/2. By the continuity of \( S_+^+ u \) at \( x \in U \), we know that there exists \( r > \delta > 0 \) such that \( B_{d_A}(x, \delta) \subset U \) and \( S_+^+ u(z) < L \) for all \( z \in B_{d_A}(x, \delta) \). Since \( d_A(z, \partial U) > r \), by (4.8) again, we have \( \text{Lip}_{d_A} u(z) \leq S_+^+ u(z) < L \). This means \( E_L \) is open and hence gives the upper semicontinuity of \( \text{Lip}_{d_A} u \) as desired.

Applying Theorem 3.4, we further obtain that \( \text{Lip}_{d_A} u(x) = \sqrt{H(x, \nabla u(x))} \) for almost all \( x \in U \), which yields that \( x \mapsto H(x, \nabla u(x)) \) is weak upper semicontinuous on \( U \). In particular, for almost all \( x \in K \), there exists a set \( E \) (which may depend on \( x \)) with measure zero such that

\[
H(x, \nabla u(x)) \geq \limsup_{y \in U \setminus E, \ y \to x} H(y, \nabla u(y)) \geq \limsup_{y \in U \setminus (S_a \cup E), \ y \to x} H(y, \nabla u(y)).
\]

Note by the construction of \( S_a \) that whenever \( x \in S_a \) and \( r > 0 \), the Lebesgue measure of \( B(x, r) \setminus S_a \) is positive. Therefore \( x \) is a cluster point of \( \mathbb{R}^n \setminus (S_a \cup E) \), and hence the above limit supremum makes sense. This, together with the continuity of \( \nabla u \), implies that

\[
(1 - \delta) |\nabla u(x)|^2 \geq \limsup_{y \in U \setminus (S_a \cup E), \ y \to x} |\nabla u(y)|^2 = |\nabla u(x)|^2.
\]

This is a contradiction if \( \nabla u(x) \neq 0 \) and \( \delta > 0 \). Hence it follows that each absolute minimizer \( u \) on \( U \) must satisfy \( \nabla u = 0 \) on \( S_a \) if \( \nabla u \) is continuous on \( U \).

**Step 2:** Now we show the existence of an absolute minimizer \( u \) on \( U \) which is either not of class \( C^1 \), or else satisfies \( \nabla u \neq 0 \) on some set \( K \) of \( S_a \) with positive measure. Consider the absolute minimizer \( u \) on \( U \) with boundary data \( f(x) = x_1 \). Assume that \( u \in C^1(U) \). Due to the continuity of \( \nabla u \), it suffices to show that \( \nabla u(x) \neq 0 \) for some \( x \in S_a \). We prove this by contradiction. Assume that \( \nabla u(x) = 0 \) for all \( x \in S_a \). Then for any \( \epsilon > 0 \), there exists \( 0 < \epsilon' < \epsilon \) such that for all \( x \in U_\epsilon = \{ y \in [\epsilon, 1 - \epsilon] \times [0, 1]^{n-1} : \text{dist}_{\mathbb{R}^n}(y, S_a) < \epsilon' \} \) we have \( |\nabla u(x)| \leq \epsilon \). Fix \( x' \in \mathbb{R} \) such that \( |x'| < \epsilon' \), and choose \( x, y \in \partial U \) such that \( x = (0, x'), y = (1, x') \) and let \( \gamma \) be the line segment joining \( x, y \). From the construction of \( S_a \), we see that \( \gamma \cap [\epsilon, 1 - \epsilon] \times [0, 1]^{n-1} \subset U_\epsilon \) when \( \epsilon' \) is small enough and hence,

\[
\int_{\epsilon}^{1-\epsilon} |\nabla u((\gamma(t)))| \, dt \leq \epsilon(1 - 2\epsilon).
\]

Moreover, since

\[
|\nabla u(z)| = \text{Lip} u(z) \leq \frac{1}{\sqrt{1 - \delta}} \text{Lip}_{d_{S_a}} u, U
\]

\[
\leq \frac{1}{\sqrt{1 - \delta}} \text{Lip}_{d_{S_a}} (f, \partial U) \leq \frac{1}{1 - \delta} \text{Lip}(f, \partial U) = \frac{1}{1 - \delta}
\]

for all \( z \in U \), we have

\[
\left( \int_{0}^{\epsilon} + \int_{1-\epsilon}^{1} \right) |\nabla u((\gamma(t)))| \, dt \leq 2\epsilon \frac{1}{1 - \delta}.
\]

Thus

\[
1 = |u(x) - u(y)| = \left| \int_{\gamma} (u \circ \gamma)'(t) \, dt \right| \leq \int_{0}^{1} |\nabla u((\gamma(t)))| \, dt \leq 2\epsilon \frac{1}{1 - \delta} + \epsilon(1 - 2\epsilon).
\]
Taking $\epsilon$ small enough, the term $2r \frac{1}{1-3} + \epsilon(1 - 2\epsilon) < 1$, which is a contradiction. So the assumption is not true and $\nabla u(x) \neq 0$ for some $x \in S_n$, which contradicts Step 1 above. Therefore $u$ is not of class $C^1$ on $U$. This proves Proposition 4.2.

Finally, for later use, we list some more characterizations of absolute minimizers.

**Lemma 4.11.** The following conditions on $u$ are equivalent:

(i) $u$ is an absolute minimizer on $U$.

(ii) $u$ satisfies the property of comparison with cones.

(iii) for all open sets $V \subseteq U$, $\text{Lip}_{d_A}(u, V) = \text{Lip}_{d_A}(u, \partial V)$.

**Proof.** From Lemmas 4.8 and 4.9, it follows that (i)$\Rightarrow$(ii), (iii). To obtain (ii)$\Rightarrow$(i), we only need to notice that, with the help of Lemma 4.4, the argument provided by the proof of [19, Proposition 5.8] still works here, without the additional weak Fubini property required in [19]; see also [6]. The proof of (iii)$\Rightarrow$(ii) follows directly from the proof of Lemma 4.9. 

5 Linear approximation when $A$ is continuous on $\Omega$

In this section we only consider $n \geq 2$.

**Theorem 5.1.** Let $A \in \mathcal{A}(\Omega)$, $U \subseteq \Omega$ and $u$ be an absolute minimizer on $U$. If $A$ is continuous at $x \in U$, then for every sequence $\{r_j\}_{j \in \mathbb{N}}$ that converges to 0, there exists a subsequence $r = \{r_{j_k}\}_{k \in \mathbb{N}}$ and a vector $e_{x, r}$ such that

\[
(5.1) \quad \lim_{k \to \infty} \left| \frac{u(x + r_{j_k}y) - u(x)}{r_{j_k}} - \langle e_{x, r}, y \rangle \right| = 0
\]

and $H(x, e_{x, r}) = \text{Lip}_{d_A} u(x)$. Consequently, if $A$ is continuous on $U$, then (5.1) holds for all $x \in U$.

To prove Theorem 5.1, we need the following auxiliary lemmas. We first look at the case $x = 0 \in U$, $u(0) = 0$ and $\text{Lip}_{d_A} u(0) \neq 0$. For any $r_0 \in (0, d_A(0, \partial U))$, we know that $u$ is an absolute minimizer on $B(0, r_0) \subseteq U$. Moreover, $\nabla u \in L^\infty(B(0, r_0))$ and the ellipticity function $\lambda$ of (1.1) is bounded on $B(0, r_0)$. In what follows, we fix such a radius $r_0$, and without loss of generality, we write $U = B(0, r_0)$ and assume that $r_{j+1} < r_j < r_0$ for all $j$.

For each $j \in \mathbb{N}$ we scale the absolute minimizer $u$ by setting

\[
u_j(y) = \frac{u(r_jy)}{r_j}\]

for all $y \in \frac{1}{r_j}U = \{\frac{1}{r_j}x, x \in U\}$. For each $j \in \mathbb{N}$, points $x \in \frac{1}{r_j}U$, and $\xi \in \mathbb{R}^n$, set $A_j(x) = A(r_jx)$, and also set $A_\infty(x) = A(0)$. Furthermore, for vectors $\xi \in S^{n-1}$, set $H_\infty(\xi) = \langle A(0)\xi, \xi \rangle$. Denote by $d_j$ the intrinsic distance of $A_j$ for $j \in \mathbb{N} \cup \{\infty\}$.

**Lemma 5.2.** There exists $u_\infty \in W^{1, \infty}(\mathbb{R}^n)$ and a subsequence $\{r_{j_k}\}_{k \in \mathbb{N}}$ of $\{r_j\}_{j \in \mathbb{N}}$ such that $u_{j_k}$ converges to $u_\infty$ locally uniformly and in weak $W^{1, \infty}(\mathbb{R}^n)$.
Proof. Since $\nabla u \in L^\infty(U)$ and $U = B(0, r_0)$ is convex, we know that Lip($u, U$) $\leq$ $\|\nabla u\|_{L^\infty(U)}$. Extend $u$ to $\mathbb{R}^n$ by the McShane extension, that is, set

$$\tilde{u}(x) = \sup_{z \in U} \{u(z) + \text{Lip}(u, U)|x - z|\}$$

for all $x \in \mathbb{R}^n$. Moreover, for each $j \in \mathbb{N}$ and all $x \in \mathbb{R}^n$, let $\tilde{u}_j(x) = \frac{\tilde{u}(r_j x)}{r_j}$ is such a McShane extension from $B(0, r_0/r_j)$ to $\mathbb{R}^n$.

On $\mathbb{R}^n$ we have $\nabla \tilde{u}_j(y) = (\nabla \tilde{u})(r_j y)$ and $\nabla \tilde{u} \in L^\infty(\mathbb{R}^n)$, so it follows that $\tilde{u}_j \in W^{1, \infty}(\mathbb{R}^n)$ with $\|\nabla \tilde{u}_j\|_{L^\infty(\mathbb{R}^n)} = \|\nabla \tilde{u}\|_{L^\infty(\mathbb{R}^n)} = \|\nabla u\|_{L^\infty(B(0, r_0))} < \infty$. Therefore, by the Arzela-Ascoli theorem, there exists a subsequence $\{j_k\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ and $u_\infty \in W^{1, \infty}(\mathbb{R}^n)$ such that $\tilde{u}_{j_k}$ converges to $u_\infty$ locally uniformly and in weak $W^{1, \infty}(\mathbb{R}^n)$. This means that for each $(n + 1)$-tuple of compactly supported continuous functions $(\phi_0, \phi_1, \ldots, \phi_n)$, we have

$$\lim_k \int_{\mathbb{R}^n} \left[ \tilde{u}_{j_k}(x)\phi_0(x) + \sum_{i=1}^n \phi_i(x) \partial_i \tilde{u}_{j_k}(x) \right] dx = \int_{\mathbb{R}^n} \left[ u_\infty(x)\phi_0(x) + \sum_{i=1}^n \phi_i(x) \partial_i u_\infty(x) \right] dx.$$

This weak convergence (strictly, to be called weak-* convergence), follows from the Banach-Alaoglu theorem upon noting that $W^{1, \infty}(\mathbb{R}^n)$ is a subset of the dual of the Banach space $(L^1(\mathbb{R}^n))^n$, together with Mazur’s lemma.

Notice that $u_j(x) = \tilde{u}_j(x)$ whenever $x \in \frac{1}{r_j}U = B(0, r_0/r_j)$ for all $j \in \mathbb{N}$. Given a compact set $K$, there exists a constant $j_K$ such that for all $j \geq j_K$, $K \subset \frac{1}{r_j}U$. Therefore $\tilde{u}_j$ converges to $u_\infty$ on $K$ uniformly implies that $u_j$ converges to $u_\infty$ uniformly as $j_K \leq j \to \infty$. \hfill $\square$

In what follows, for simplicity, we always write the subsequence $\{j_k\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ obtained in above Lemma 5.2 as $\mathbb{N}$ by abuse of notation.

**Lemma 5.3.** (i) For all $j \in \mathbb{N}$, $u_j$ is an absolute minimizer on $\frac{1}{r_j}U$ associated to the Hamiltonian $H_j$ which corresponds to $A_j$.

(ii) If $A$ is continuous at $0$, then $u_\infty$ is an absolute minimizer on $\mathbb{R}^n$ associated to the Hamiltonian $H_\infty$.

To prove this, we need two facts given in the following; the second one relies on the continuity of $A$ at $0$. We postpone the proof of Lemma 5.4 until after the proof of Lemma 5.3.

**Lemma 5.4.** (i) For $j \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$, $r_j d_j(x, y) = d_A(r_j x, r_j y)$.

(ii) Assume that $A$ is continuous at $0$. Given a compact set $K$ and $x \in \mathbb{R}^n$, for every $\epsilon > 0$, there exists $j(x, \epsilon, K) \in \mathbb{N}$ such that for all $j \geq j(x, \epsilon, K)$ and all $y \in K$,

$$(1 - \epsilon)d_\infty(x, y) \leq d_j(x, y) \leq (1 + \epsilon)d_\infty(x, y).$$

**Proof of Lemma 5.3.** Proof of (i): Let $j \in \mathbb{N}$. It suffices to show that for all open subsets $V \subset \frac{1}{r_j}U$, Lip$_{d_j}(u_j, V) = \text{Lip}_{d_j}(u_j, \partial V)$. By Lemma 5.4 (i) and observing that $x, y \in V$ implies $r_j x, r_j y \in r_j V \subset U$, we have

$$\frac{u_j(x) - u_j(y)}{d_j(x, y)} = \frac{u(r_j x) - u(r_j y)}{d_A(r_j x, r_j y)},$$

for all $x, y \in V$.\hfill $\square$
which yields $\text{Lip}_d(u_j, V) = \text{Lip}_d(u, r_j V)$. Similarly, $\text{Lip}_d(u_j, \partial V) = \text{Lip}_d(u, \partial(r_j V))$ with the help of $\partial r_j V = r_j \partial V$. Thus by $\text{Lip}_d(u, r_j V) = \text{Lip}_d(u, \partial(r_j V))$, we obtain $\text{Lip}_d(u_j, V) = \text{Lip}_d(u_j, \partial V)$. Thus the claim follows from Lemma 4.11.

Proof of (ii): It suffices to show that $u_\infty$ satisfies the comparison property with cones. Let $V \subset \mathbb{R}^n$ and assume that for each $z \in \partial V$,

$$u_\infty(z) \leq b + \text{ad}_\infty(z_0, z)$$

for some $z_0 \notin V$, $b \in \mathbb{R}$ and $a > 0$ which are independent of $z$. By Lemma 5.4 (ii), for every $\epsilon > 0$, there exists $\epsilon$ such that whenever $j \geq \epsilon$ and $z \in \overline{V}$, we have $V \subset \frac{1}{r_j} U$,

$$(1 - \epsilon)d(0, z) \leq d_j(z_0, z) \leq (1 + \epsilon)d(0, z)$$

and because $u_j \to u_\infty$ uniformly on the compact set $\overline{V}$, we also have

$$u_\infty(z) - \epsilon \leq u_j(z) \leq u_\infty(z) + \epsilon.$$

Thus by (5.2),

$$u_j(z) \leq (b + \epsilon) + \frac{a}{(1 - \epsilon)}d_j(z_0, z)$$

for all $z \in \partial V$. Since $u_j$ is an absolute minimizer on $\frac{1}{r_j} U$ associated to $H_j$ and $V \subset \frac{1}{r_j} U$, we have

$$u_j(z) \leq (b + \epsilon) + \frac{a}{(1 - \epsilon)}d_j(z_0, z)$$

for all $z \in V$, which further implies that

$$u_\infty(z) \leq (b + 2\epsilon) + \frac{a(1 + \epsilon)}{(1 - \epsilon)}d_\infty(z_0, z)$$

for all $z \in V$. Due to the arbitrariness of $\epsilon$, we finally have $u_\infty(z) \leq b + \text{ad}_\infty(z_0, z)$ for all $z \in V$.

Similar argument also holds for $-u_\infty$. We omit the details. So, by Lemma 4.11, $u_\infty$ is an absolute minimizer on $\frac{1}{r_j} U$ associated to the Hamiltonian $H_\infty$ for each $r_j$, and hence on $\mathbb{R}^n$.

Proof of Lemma 5.4. (i) Let $v$ be a locally Lipschitz function on $U$ such that $H(x, \nabla v(x)) \leq 1$ for almost every $x \in U$, and let $v_j(z) = \frac{1}{r_j} v(r_j z)$. Since $H(z, \nabla v(z)) \leq 1$ for almost all $z \in \mathbb{R}^n$, we have

$$H_j(z, \nabla v_j(z)) = H(r_j z, (\nabla v)(r_j z)) \leq 1$$

and hence

$$d_j(x, y) \geq v_j(y) - v_j(x)$$

Taking the supremum over all such $v$, we see that $r_j d_j(x, y) \geq d_A(r_j x, r_j y)$. The inequality $r_j d_j(x, y) \leq d_A(r_j x, r_j y)$ can be obtained similarly.

(ii) Given a compact set $K$ and $x \in \mathbb{R}^n$, let $R > 0$ and $j_x \in \mathbb{N}$ be such that for all $j \geq j_x$, we have $K \cup \{x\} \subset B_{d_j}(0, R) \subset B_{d_j}(0, 2R) \subset B(0, CR) \subset \frac{1}{r_j} U$, where $C > 1$ is a constant
depending on the lower and upper bounds of $\lambda$ on $U$. We set $v_j(z) = \min\{d_j(x, z), R\}$ for all $z \in \mathbb{R}^n$. Note that by Lemma 2.3, $\langle A_j(z)\nabla v_j(z), \nabla v_j(z) \rangle \leq 1$ for all $z \in \mathbb{R}^n$, and that $\nabla v_j(z) = 0$ for $z \notin B(0, CR)$. Moreover, since $A$ is continuous at 0, for sufficiently large $j_\varepsilon > j_\omega$ we have that for all $z \in B(0, CR)$,

$$|A_j(z) - A(0)| = |A(r_j z) - A(0)| < \varepsilon,$$

where we consider the operator norm on $A_j(z) - A(0)$. So for almost every $z \in B(0, CR)$,

$$\langle A(0)\nabla v_j(z), \nabla v_j(z) \rangle = \langle [A(0) - A_j(z)]\nabla v_j(z), \nabla v_j(z) \rangle + \langle \nabla A_j(z)\nabla v_j(z), \nabla v_j(z) \rangle \leq L\varepsilon + 1,$$

where $L > 0$ is a constant related to the bound of the ellipticity function $\lambda$ on $B(0, CR)$ such that $|\nabla v| \leq L$ on $B(0, CR)$. It follows that

$$w_j = \frac{1}{\sqrt{L\varepsilon + 1}} v_j$$

can be used to compute $d_\infty$ on $B(0, CR)$. Thus for $y \in K \subset B(0, CR)$,

$$d_\infty(x, y) \geq w_j(x) - w_j(y) = \frac{v_j(x) - v_j(y)}{\sqrt{L\varepsilon + 1}},$$

that is

$$d_\infty(x, y) \geq \frac{d_j(x, y)}{\sqrt{L\varepsilon + 1}}.$$

Now let $w(z) = \min\{d_\infty(x, z), R\}$ for $z \in \mathbb{R}^n$. An argument similar to above yields that for $j \geq j_\varepsilon$,

$$\langle A_j(z)\nabla w(z), \nabla w(z) \rangle \leq L\varepsilon + 1,$$

and so we obtain the reverse inequality

$$d_j(x, y) \geq \frac{d_\infty(x, y)}{\sqrt{L\varepsilon + 1}}.$$

The conclusion of (ii) of the lemma follows. \hfill \Box

In what follows, $S^+_\varepsilon u(x)$ is as in Section 4 and by Lemma 4.10, when $u$ is an absolute minimizer associated to the Hamiltonian $H$ that corresponds to $A$, we know that $\text{Lip}_{d_\varepsilon} u(x) = \lim_{r\to 0} S^+_\varepsilon u(x)$.

**Lemma 5.5.** Assume that $A$ is continuous at 0. Then

(i) For all $r > 0$, $S^+_\varepsilon u_\infty(0) = \text{Lip}_{d_\varepsilon} u_\infty(0) = \text{Lip}_{d_\varepsilon} u(0)$ and $\sup_{x \in \mathbb{R}^n} S^+_\varepsilon u_\infty(x) \leq \text{Lip}_{d_\varepsilon} u_\infty(0)$.

(ii) $\sup_{x \in \mathbb{R}^n} \text{Lip}_{d_\varepsilon} u_\infty(x) = \text{Lip}_{d_\varepsilon}(u_\infty, \mathbb{R}^n) = \text{Lip}_{d_\varepsilon} u_\infty(0)$.

**Proof.** By Lemma 5.4, we know that $u_\infty$ is an absolute minimizer associated with $H_\infty$. Hence by Lemma 4.10 and the claim (i) of this lemma, the claim (ii) will follow. Hence it suffices to prove the claim (i). We first observe that

$$\text{Lip}_{d_\varepsilon}(u_\infty, \mathbb{R}^n) \leq \text{Lip}_{d_\varepsilon} u(0).$$

(5.3)
Indeed, for all \( x, y \in \mathbb{R}^n \) with \( x \neq y \), by \( r_jd_j(x, y) = d_A(r_jx, r_jy) \), we have

\[
(5.4) \quad \frac{|u_\infty(x) - u_\infty(y)|}{d_\infty(x, y)} = \lim_{j \to \infty} \frac{|u_j(x) - u_j(y)|}{d_j(x, y)} = \lim_{j \to \infty} \frac{|u(r_jx) - u(r_jy)|}{d_A(r_jx, r_jy)}.
\]

letting \( \gamma \) be the geodesic curve in the metric \( d_j \) (and hence in the metric \( d_A \)) joining \( r_jx, r_jy \), (such \( \gamma \) exists when \( j \) large enough because then \( r_jx, r_jy \in B(0, R) \)), we obtain

\[
|u(r_jx) - u(r_jy)| \leq \int_\gamma \text{Lip}_d u(z)dz \leq \sup_{z \in \gamma} \text{Lip}_d u(z)d_A(r_jx, r_jy).
\]

Thus

\[
\frac{|u_\infty(x) - u_\infty(y)|}{d_\infty(x, y)} \leq \lim_{j \to \infty} \sup_{z \in B_A(0, d_A(r_jx, r_jy))} \text{Lip}_d u(z).
\]

Observing that \( \text{Lip}_d u \) is upper semicontinuous (for details see the proof of Proposition 4.2, in particular, (4.8)), and by \( d_A(r_jx, r_jy) \to 0 \) as \( j \to \infty \), we arrive at

\[
\frac{|u_\infty(x) - u_\infty(y)|}{d_\infty(x, y)} \leq \text{Lip}_d u(0).
\]

This proves (5.3).

From (5.4) and Lemma 4.10, it also follows that \( \text{Lip}_d u_\infty(x) \leq S^+_r u_\infty(x) \leq \text{Lip}_d u(0) \) for all \( x \in \mathbb{R}^n \) and \( r > 0 \). Moreover, we will show below that

\[
(5.5) \quad S^+_r u_\infty(0) \geq \text{Lip}_d u(0).
\]

From this, and applying the above discussion to \( x = 0 \), by Lemma 4.10 we have

\[
S^+_r u_\infty(0) = \text{Lip}_d u_\infty(0) = \text{Lip}_d u(0)
\]

for all \( r > 0 \). Since we already have \( \text{Lip}_d u_\infty(x) \leq \text{Lip}_d (u_\infty, \mathbb{R}^n) \) for all \( x \in \mathbb{R}^n \), we obtain (i). This proves Lemma 5.5. Hence we end the proof of Lemma 5.5 by establishing (5.5).

By the continuity of \( u_\infty \), for every \( \epsilon > 0 \) there exists \( 0 < \delta_0 < 1/4 \) such that whenever \( \delta \in (0, \delta_0) \),

\[
(5.6) \quad \frac{\sup_{d_\infty(0, y) \leq r} u_\infty(y)}{r} \geq \frac{\sup_{d_\infty(0, y) \leq (1 + \delta)r} u_\infty(y)}{(1 + \delta)r} - \epsilon = S^+_{(1 + \delta)r} u_\infty(0) - \epsilon;
\]

and since \( u_j \to u_\infty \) locally uniformly as \( j \to \infty \), there exists \( j_\epsilon \in \mathbb{N} \) such that for all \( j \geq j_\epsilon \) and \( y \in B_{d_\infty}(0, 2r) \setminus B_{d_\infty}(0, r/2) \),

\[
(5.7) \quad \frac{u_\infty(y) - u_\infty(0)}{d_\infty(0, y)} \geq \frac{u_j(y) - u_j(0)}{d_j(0, y)} - \epsilon = \frac{u(r_jy) - u(0)}{d_A(0, r_jy)} - \epsilon.
\]

Moreover, by Lemma 5.4 (ii), for any \( \delta \in (0, \delta_0) \), there exists \( j_\delta \) such that such that for all \( j \geq j_\delta \) and \( y \in B_{d_\infty}(0, 2r) \),

\[
(5.8) \quad (1 - \delta)d_j(0, y) \leq d_\infty(0, y) \leq (1 + \delta)d_j(0, y).
\]
Let $z_j \in \mathbb{R}^n$ such that $d_A(0, z_j) \leq r_j r$ and
\begin{equation}
(5.9) \quad \frac{u(z_j) - u(0)}{r_j r} = \max_{d_A(z, 0) \leq r_j r} \frac{u(z) - u(0)}{r_j r} = S_{r_j r}^+ u(0) \geq \operatorname{Lip}_{d_A} u(0).
\end{equation}
By the comparison Lemma 4.7, we know that $d_A(0, z_j) = r_j r$. Let $y_j = z_j/r_j$. Observe that whenever $j \geq j_\delta$, (5.8) implies that
\[ d_\infty(0, y_j) \leq (1 + \delta) d_j(0, y_j) \leq (1 + \delta) \frac{1}{r_j} d_A(0, z_j) = (1 + \delta) r \]
and similarly, $(1 - \delta) r \leq d_\infty(0, y_j)$. By this, the increasing property of $S_{1 (1 + \delta) r}^+ u_\infty(0)$ with respect to $r$ given by Lemma 4.10, (5.7), Lemma 5.4 (i) and (5.9), we have whenever $j \geq \max\{j_\delta, j_\epsilon\}$,
\[ S_{1 (1 + \delta) r}^+ u_\infty(0) \geq S_{d_\infty(0, y_j)}^+ u_\infty(0) \geq \frac{u_\infty(y_j) - u_\infty(0)}{d_\infty(0, y_j)} \geq \frac{u_j(y_j) - u_j(0)}{d_j(0, y_j)} - \epsilon = \frac{u(z_j) - u(0)}{d_A(0, z_j)} - \epsilon \geq \operatorname{Lip}_{d_A} u(0) - \epsilon, \]
which together with (5.6) implies $S_{r}^+ u_\infty(0) \geq \operatorname{Lip}_{d_A} u(0) - 2\epsilon$. From this, we conclude that $S_{r}^+ u_\infty(0) \geq S^+ u(0)$, and hence (5.5).

**Lemma 5.6.** Assume that $A$ is continuous at 0. There exists $e \in \mathbb{R}^n$ such that $u_\infty(x) = \langle e, x \rangle$ for all $x \in \mathbb{R}^n$ and $H_\infty(x, e) = \operatorname{Lip}_{d_\infty} u_\infty(0)$.

**Proof.** Notice that by Lemma 5.3, $u_\infty$ is an absolute minimizer on $\mathbb{R}^n$ associated to the Hamiltonian $H_\infty$. Moreover $u_\infty$ satisfies Lemma 5.5 (i) and (ii). If $A(0) = I_n$, then $H_\infty(\xi) = \langle \xi, \xi \rangle$ and hence Lemma 5.6 follows from [10]. Generally, Lemma 5.6 follows from Lemma 3.4 of [28], where $H_\infty$ satisfies the conditions required there.

**Proof of Theorem 5.1.** Without loss of generality, we may assume that $x = 0$, $u(x) = 0$ and $\operatorname{Lip}_{d_A} u(0) > 0$. Indeed, set $\tilde{u}(z) = u(x + z) - u(x)$ for $z \in \mathbb{R}^n$, and $\tilde{H}(z, \xi) = \langle A(x + z)\xi, \xi \rangle$. Then $\tilde{u}$ is an absolute minimizer of $\tilde{H}$ if and only if $u$ is an absolute minimizer of $H$; Theorem 5.1 holds for $\tilde{u}$ at 0 if and only if Theorem 5.1 holds for $u$ at $x$. But $\tilde{u}(0) = 0$. We also notice that $\operatorname{Lip}_{d_A} u(0) = 0$ together with the equivalence of $d$ and the Euclidean distance yields that $\operatorname{Lip} u(0) = 0$, and hence, $u$ is differentiable at 0 with $\nabla u(0) = 0$. This means that (5.1) holds with $e = 0$ and $\operatorname{Lip}_{d_A} u(0) = H(0, e) = 0$.

Now we consider the scaling of the absolute minimizer $u$ by $u_j(y) = \frac{u(r_j y)}{r_j}$ as above. $u_\infty$ is the limit of some subsequence of $u_j$, which is still denoted by $u_j$ for simple. Then (5.1) is reduced to showing $u_\infty(z) = \langle e, z \rangle$ for some vector $e \in \mathbb{R}^n$ and $H_\infty(e) = \operatorname{Lip}_{d_A} u(0)$. But this follows from Lemma 5.6 and Lemma 5.5.

**Remark 5.7.** (i) In the above proof, we do need the continuity of $A$ at $x$ to conclude that $d_\infty$ is the intrinsic distance associated to $H_\infty$. We do not know what happens if $A$ is only assumed to be weak upper semicontinuous at $x$. 
(ii) We expect that the above linear approximation property provided by Theorem 5.1 may help to understand the $C^1$-regularity or the differentiability everywhere of the absolute minimizer associated to a continuous diffusion matrix $A$, see [24, 12, 13, 28] in the case $A = I_n$.

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