BUMPY SPIN-DOWN OF ANOMALOUS X-RAY PULSARS: THE LINK WITH MAGNETARS

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ABSTRACT

The two anomalous X-ray pulsars (AXPs) with well-sampled timing histories, 1E 1048.1−5937 and 1E 2259+586, are known to spin down irregularly, with ‘bumps’ superimposed on an overall linear trend. Here we show that if AXPs are non-accreting magnetars, i.e. isolated neutron stars with surface magnetic fields $B_0 \gtrsim 10^{10} \, \text{T}$, then they spin down electromagnetically in exactly the manner observed, due to an effect called ‘radiative precession’. Internal hydromagnetic stresses deform the star, creating a fractional difference $\epsilon = (I_3 - I_1)/I_1 \sim 10^{-8}$ between the principal moments of inertia $I_1$ and $I_3$; the resulting Eulerian precession couples to an oscillating component of the electromagnetic torque associated with the near-zone radiation fields, and the star executes an anharmonic wobble with period $\tau_{pr} \sim 2\pi/\epsilon \Omega(t) \sim 10 \, \text{yr}$, where $\Omega(t)$ is the rotation frequency as a function of time $t$. We solve Euler’s equations for a biaxial magnet rotating in vacuo; show that the computed $\Omega(t)$ matches the measured timing histories of 1E 1048.1−5937 and 1E 2259+586; predict $\Omega(t)$ for the next 20 years for both objects; predict a statistical relation between $\langle d\Omega/dt \rangle$ and $\tau_{pr}$, to be tested as the population of known AXPs grows; and hypothesize that radiative precession will be observed in future X-ray timing of soft gamma-ray repeaters (SGRs).

Subject headings: pulsars: individual: 1E 1048.1−5937 — pulsars: individual: 1E 2259+586 — stars: neutron — stars: rotation — X-rays: stars

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1. INTRODUCTION

Anomalous X-ray pulsars (AXPs) are a sub-class of X-ray pulsars distinguished by pulse periods lying in a narrow range between 6s and 12s, low X-ray luminosities, soft X-ray spectra, no detected optical counterparts, no detected orbital Doppler shifts of pulse arrival times, and associations with shell-type supernova remnants (Mereghetti & Stella 1995; van Paradijs, Taam & van den Heuvel 1995). The nature of AXPs is uncertain; possibilities include an accreting neutron star in a binary, with a low-mass white dwarf or He-burning star as a companion (Mereghetti & Stella 1995; Baykal & Swank 1996; Mereghetti, Israel & Stella 1998), an isolated neutron star accreting from a residual disk following a phase of common-envelope evolution (van Paradijs et al. 1995; Ghosh, Angelini & White 1997), a non-accreting massive white dwarf (Morini et al. 1988; Paczyński 1990; Usov 1994), and a non-accreting magnetar (Thompson & Duncan 1996; Heyl & Hernquist 1998; Kouveliotou et al. 1998).

The two AXPs with well-sampled timing histories, 1E 1048.1−5937 and 1E 2259+586, are observed to spin down irregularly: the rotation frequency Ω of each object decreases linearly with time $t$ on average, but there are ‘bumps’ superimposed on this average trend during which $d\Omega/dt < 0$ fluctuates by a factor of between two and five every five to ten years (Mereghetti 1995; Baykal & Swank 1996; Baykal et al. 1998; Oosterbroek et al. 1998; and references therein). In accreting-star models of AXPs, the average spin-down trend is attributed to the accretion torque acting on a near-equilibrium rotator, and the bumps are the result of white torque noise (e.g. due to disk inhomogeneities) as in ordinary binary X-ray pulsars (Baykal & Swank 1996). In isolated-star models, the spin-down trend is attributed to magnetic-dipole braking, and the bumps are analogous to glitches observed in rotation-powered pulsars like Vela (Usov 1994; Heyl & Hernquist 1998).

In this Letter, we present unequivocal new evidence that AXPs are non-accreting magnetars. In §2, it is shown that as a magnetar spins down it wobbles anharmonically, with a period of five to ten years, due to an effect called ‘radiative precession’. The spin-down signature of the wobble, calculated theoretically in §3, matches closely the bumpy timing histories of 1E 1048.1−5937 and 1E 2259+586. The theory is used to predict Ω($t$) over the next 20 years for both objects and yields a statistical relation between bump recurrence time and average spin-down rate for the AXP population as a whole. Implications for the internal and magnetospheric structures of magnetars, including AXPs and soft gamma-ray repeaters (SGRs), are explored in §4.

2. RADIATIVE PRECESSION

A magnetar is a triaxial body in general. Hydromagnetic stresses arising from non-radial gradients of the superstrong internal magnetic field, e.g. between the magnetic poles and equator if the field is dipolar, deform the star, inducing matter-density perturbations $\delta\rho \sim B_{\text{in}}^2/\mu_0c^2$ and hence a fractional difference $\epsilon \sim \delta\rho R^5/I_1 \approx 2 \times 10^{-9}(B_{\text{in}}/10^{10} \text{T})^2$ between the principal
moments of inertia (Goldreich 1970; de Campli 1980; Melatos 1998). Here, $B_{\text{in}}$ is the characteristic magnitude of the internal magnetic field, $c_s$ is the isothermal sound speed ($c_s \approx 3^{-1/2}c$), and $R$ is the stellar radius; one has $B_{\text{in}} \approx B_0$ if the internal field is confined to the stellar crust and $B_{\text{in}} \geq B_0$ if it is generated deep inside the star, e.g. in a convective dynamo (Thompson & Duncan 1993). In a rotation-powered pulsar with $B_0 \lesssim 10^9$ T, the hydromagnetic deformation is comparable to the elastic deformation arising from shear stresses in the crystalline stellar crust, and the principal axes of inertia are oriented arbitrarily with respect to the axis $m$ of the external magnetic dipole (Goldreich 1970; de Campli 1980; Melatos 1998). In a magnetar, however, the hydromagnetic deformation is much larger, and $m$ is approximately parallel to one of the principal axes ($e_3$, say); the alignment is not exact due to the complicated structure of the internal field near its generation site, cf. the non-axisymmetric, magnetically modified Taylor columns seen in simulations of the geodynamo (Glatzmaier & Roberts 1996). Provided that the rotation axis $\Omega$ is not parallel to $e_3 \approx m$, as is usually the case for rotation-powered pulsars, the star precesses about $e_3$ with period $\tau_{\text{pr}} \sim 2\pi/\epsilon \Omega \approx 85(B_{\text{in}}/10^{10}$ T)$^{-2}$(\Omega/1 rad s$^{-1}$)$^{-1}$ yr.

The Eulerian precession is not free. It couples to a component of the vacuum magnetic-dipole torque associated with the asymmetric inertia of the near-zone radiation fields outside the magnetar; the electromagnetic energy density (and hence inertia) of the near-zone radiation fields is greater at the magnetic poles than at the equator by an amount $\sim B_0^2/\mu_0$, resulting in an oscillatory, precession-inducing torque (Goldreich 1970; Good & Ng 1985; Melatos 1998). The near-field torque exceeds the familiar braking torque ($\propto \Omega^3$) by a factor $c/\Omega R \gg 1$ and acts on a commensurately shorter time-scale, $\tau_{\text{nf}} \sim \tau_0 \Omega R/c \approx 6(B_0/10^{10}$ T)$^{-2}$(\Omega/1 rad s$^{-1}$)$^{-1}$ yr, where $\tau_0 = \mu_0 c^3 I_1/2\pi B_0^2 R^6 \Omega^2 \approx 2 \times 10^5(B_0/10^{10}$ T)$^{-2}$(\Omega/1 rad s$^{-1}$)$^{-2}$ yr is the characteristic electromagnetic braking time. Since the near-field torque is directed along $\Omega \times m$, it does not change $\Omega$ for a spherical star. Nor does it change $\Omega$ for an aspherical star, provided $\epsilon$ is large enough to give $\tau_{\text{pr}} \ll \tau_{\text{nf}}$, so that the near-field torque averages to zero over one precession period. When the dominant deformation is hydromagnetic, however, one finds $\tau_{\text{pr}}/\tau_{\text{nf}} \approx 14(B_0/B_{\text{in}})^2$, close to unity (i.e. strong coupling) provided $B_{\text{in}}$ is moderately larger than $B_0$ as expected (Thompson & Duncan 1993). Under these circumstances, the star executes an anharmonic wobble, called radiative precession, with period $\tau_{\text{pr}} \approx \tau_{\text{nf}}$, and the near-field torque changes $\Omega$ on the precession time-scale (Melatos 1998).

3. THEORY OF BUMPY SPIN-DOWN

3.1. Solution of Euler’s Equations of Motion: Past and Future $\Omega(t)$

We now show that the timing signature of radiative precession matches the observed bumpy spin-down of AXPs by solving numerically Euler’s equations of motion for a biaxial, dipole magnet rotating in vacuo,

$$\dot{\Omega}_1 = -\epsilon \Omega_2 \Omega_3 + \Omega_0^{-2} \tau_0^{-1} \cos \chi [a \Omega^2 (-\Omega_1 \cos \chi + \Omega_3 \sin \chi) + b \Omega_2 (\Omega_1 \sin \chi + \Omega_3 \cos \chi)],$$

(1)
\[ \dot{\Omega}_2 = \epsilon \Omega_1 \Omega_3 + \Omega_0^2 \tau_0^{-1} \left[ -a \Omega^2 \Omega_2 + b(-\Omega_1 \cos \chi + \Omega_3 \sin \chi)(\Omega_1 \sin \chi + \Omega_3 \cos \chi) \right], \quad (2) \]
\[ \dot{\Omega}_3 = -\Omega_0^2 \tau_0^{-1} \sin \chi \left[ a \Omega^2 (-\Omega_1 \cos \chi + \Omega_3 \sin \chi) + b \Omega_2 (\Omega_1 \sin \chi + \Omega_3 \cos \chi) \right]. \quad (3) \]

In (1)-(3), subscripts denote components along the principal axes of inertia, \( \chi \) is the angle between \( \mathbf{m} \) and \( \mathbf{e}_3 \), and we have \( a = 0.33, \ b = 0.094c/\Omega_0 R \), and \( \Omega_0 = \Omega(t = t_0) \), where \( t_0 \) is an arbitrary origin; for derivations of the equations, see Goldreich (1970) and Melatos (1998). Terms in (1)-(3) proportional to \( \epsilon \) give rise to Eulerian precession, terms proportional to \( b \tau_0^{-1} \) are associated with the near-field torque, and terms proportional to \( a \tau_0^{-1} \) cause secular braking. A biaxial, dipole magnet rotating in vacuo is an idealized model of a hydromagnetically deformed magnetar. In reality, such a body is triaxial (if it is indeed rigid), has high-order and/or off-centered multipoles contributing to the near-zone magnetic field, and is surrounded by a plasma magnetosphere. The values of \( a \) and \( b \) reflect, in a coarse way, the magnetization state of the stellar interior and the distribution of magnetospheric currents (Melatos 1998).

In Figures 1 and 2, we plot the computed \( \Omega(t) \) from (1)-(3) on top of the X-ray timing histories of 1E 1048.1−5937 and 1E 2259+586 respectively, extending the theoretical curves 20 years beyond the present as a testable prediction. The timing history of 1E 2259+586 is sufficiently well-sampled that only one good fit is possible. In the case of 1E 1048.1−5937, the solid curve is the favored fit, with \( \tau_{pr} \approx 18 \) yr, but an alternative fit, with similar average slope and \( \tau_{pr} \approx 6 \) yr, is also acceptable. Multiple good fits are hard to find.

Euler’s equations of motion contain just three unknown parameters: \( \epsilon, \tau_0, \) and \( \chi \). It is significant that the theory agrees with the observations as well as it does despite its idealized nature — particularly as only two of the three parameters are truly free, since one needs strong torque coupling \( (\tau_{pr} \sim \tau_{nf}, \text{ or equivalently } \epsilon \Omega_0 \tau_0 \sim c/\Omega_0 R) \) in order to get any bumps at all. Moreover, the best agreement with observations is achieved for parameter values that are consistent with general physical considerations. If AXPs are hydromagnetically deformed magnetars with \( B_{in} \gtrsim B_0 \gtrsim 3 \times 10^{10} \) T, one expects \( \epsilon \sim 10^{-8}, \Omega_0 \tau_0 \sim 10^{11}, \) and \( \chi \) relatively small (cf. \( \chi \approx 11^\circ \) for the Earth), as discussed above.

The theoretical curves do not pass exactly through every available data point, and more departures are expected in the future, e.g. the slope between the last two data points in Figure 1 is \( \approx 0.6 \) times the solid-curve theoretical slope at that epoch. Indeed, a formal estimate of the chi-square for the solid curve in Figure 1, taking published timing uncertainties at face value, implies that the fit is poor: one finds a chi-square of \( \approx 8 \times 10^{3} \) with 10 degrees of freedom, notably inferior to multiple-glitch models, for example (Heyl & Hernquist 1998). This is because a biaxial, dipole magnet is an over-idealized model of an AXP, as noted above; the chi-square likelihood improves dramatically for a more realistic model with just two extra parameters, e.g. a non-dipolar near-zone magnetic field and a triaxial ellipsoid of inertia. However, our aim in this paper is not to model \( \Omega(t) \) in detail, but to account for key gross features of the data — the average spin-down rate, bump recurrence time, and bump amplitude — with a simple physical theory. In this regard, the agreement with observations is good.
One data point in the timing history of 1E 2259+586, at $t - t_0 = 15.4$ yr, represents a spin-up event lasting at most 0.8 yr at the 1σ level of uncertainty (Baykal & Swank 1996). Spin-up cannot be explained by radiative precession because (i) $\dot{\Omega}$ is always negative, even while $|\dot{\Omega}|$ decreases by up to a factor of five during a bump, and (ii) there is no natural way to explain a 0.8 yr time-scale. If taken at face value, the spin-up event must be a different phenomenon, e.g. a discontinuous change of internal structure analogous to the glitches of rotation-powered pulsars like Vela (Usov 1994; Heyl & Hernquist 1998).

### 3.2. Predicted Population Statistics

Numerical studies show that the bump recurrence time and average spin-down rate satisfy

$$\tau_{pr} \propto B_{in}^{-2} \Omega^{-1} f(\chi)$$

and

$$\langle \dot{\Omega} \rangle \propto B_0^2 \Omega^3 g(\chi),$$

with $1 \leq f(\chi), g(\chi) \leq 10$. In other words, the greater the magnetic field of an AXP, the shorter is its precession period and the greater is its spin-down rate, with

$$\langle \dot{\Omega} \rangle \approx -2 \times 10^{-4} (B_0/B_{in})^2 (\Omega/1 \text{ rad s}^{-1})^2 (\tau_{pr}/1 \text{ yr})^{-1} \text{ rad s}^{-1} \text{ yr}^{-1}. \tag{4}$$

In addition, one finds that $\dot{\Omega}$ increases from $\dot{\Omega} \approx \langle \dot{\Omega} \rangle$ to $\dot{\Omega} \approx 0$ over a time $\approx 0.25 \tau_{pr}$ during the course of a bump, yielding a bump amplitude

$$\Delta \Omega_{pr}/\Omega \approx 5 \times 10^{-5} (B_0/B_{in})^2 (\Omega/1 \text{ rad s}^{-1}); \tag{5}$$

this number is similar for all AXPs. Both the relations (4) and (5) will be subject to observational testing as the population of AXPs with measured timing histories swells over time. Note that they are statistical relations, with scatter expected about an overall trend. The detailed structure of the magnetic field inside an AXP — which does not enter into the idealized theory presented here, except through $\chi$ — is likely to differ from object to object, affecting $\tau_{pr}$, $\langle \dot{\Omega} \rangle$, and $\Delta \Omega_{pr}$ significantly, as the broad ranges of $f(\chi)$ and $g(\chi)$ attest.

### 4. DISCUSSION

Why is radiative precession not observed in rotation-powered pulsars with $B_0 \lesssim 10^9 \text{T}$? These objects are deformed hydromagnetically like magnetars, with an added elastic deformation, and they spin down electromagnetically. Young pulsars like the Crab, with $\tau_0 \approx 10^3 \text{ yr}$ and $\Omega R/c \approx 10^{-2}$, ought to precess with period $\tau_{pr} \approx 10 \text{ yr}$, yet there is no clean evidence of bumpy spin-down in radio timing data, nor of concomitant changes in pulse profile (e.g. relative height and separation of conal components) and polarization characteristics (e.g. position-angle swing). Lyne, Pritchard & Smith (1988) reported a quasi-sinusoidal variation in Crab timing residuals with a period of $\approx 20$ months but judged it likely to be an artifact of an overlooked glitch. (See also Melatos 1998.) The only reliable detection of pulsar precession to date has been the general-relativistic geodetic precession of the binary pulsar PSR B1913+16 (Weisberg, Romani & Taylor 1989; Kramer 1998).
One possible explanation is that radiative precession is damped in pulsars with $B_0 \lesssim 10^9$ T. Frictional dissipation inside the star, due to time-dependent elastic strains in the crust and/or imperfect crust-core coupling, is thought to occur on a time-scale $\lesssim 1$ yr (Link, Epstein & Baym 1993), rapidly aligning $\Omega$ with $e_3$. (In the case of the Earth, dissipation restricts $\Omega$ to fluctuating within just $1''$ of $e_3$ under the action of solar and lunar tides; see Burša & Pěč 1993). In a magnetar, where the magnetic energy exceeds the mechanical energy of rotation, the stiffening effect of the superstrong magnetic field may hinder the elastic strains and/or sheared fluid flows responsible for internal damping. A second possible explanation is that conduction currents in the magnetosphere of a pulsar with $B_0 \lesssim 10^9$ T, where electron-positron pair production is plentiful, modify the electromagnetic torque in such a way as to suppress the precession-inducing near-field component. In a magnetar, where it is thought that pair production is quenched, e.g. by positronium formation (Usov & Melrose 1996; J. Heyl 1999, private communication) or photon splitting (Baring & Harding 1998), the vacuum magnetic-dipole torque, with its unmodified near-field component, is a closer approximation to reality. Both explanations imply that radiative precession will not be suppressed in SGRs, where one has $B_0 \sim 10^{11}$ T (Kouveliotou et al. 1998).

What does radiative precession teach us about strongly magnetized neutron stars themselves? Firstly, the close agreement between theory and observation implies that AXPs are indeed non-accreting magnetars, and that accretion is not needed to explain fluctuations in $\dot{\Omega}$, contrary to claims in the literature (e.g. Mereghetti 1995). Secondly, if SGRs are magnetars too, as indicated by recent X-ray timing data (Kouveliotou et al. 1998), they ought to exhibit bumpy spin-down just like AXPs, perhaps punctuated by brief, glitch-like spin-up events during the gamma-ray bursts themselves. This constitutes a direct test of the magnetar model for SGRs. Thirdly, the fact that bumpy spin-down is seen in AXPs implies that $B_{\text{in}}$ is at most a few times $B_0$ to ensure strong coupling, i.e. $\tau_{\text{pr}} \sim \tau_{\text{nf}}$, as discussed above. This is indirect evidence that the magnetic field of these objects is generated relatively near the stellar surface. Fourthly, radiative precession places an upper limit on the fluid viscosity $\eta$ of a newly born magnetar; if $\eta$ is too high, $\Omega$ aligns with $e_3 \approx m$ before the stellar crust crystallizes, and there is no precession (Melatos 1998). The upper limit obtained in this way — that the viscous damping time is greater than the crust crystallization time of $\sim 1$ yr — validates certain semi-quantitative calculations of $\eta$ for pulsars with $B_0 \lesssim 10^9$ T by Cutler & Lindblom (1987). We remark in closing that the predicted hydromagnetic deformation ($\epsilon \sim 10^{-8}$) constitutes a misaligned mass quadrupole which generates gravitational radiation. However, the radiation is too weak to be detected by planned gravitational-wave interferometers like LIGO and VIRGO, because AXPs are slow rotators and the gravitational-wave amplitude is a strongly rising function of $\Omega$.

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REFERENCES

Baring, M. G., & Harding, A. K. 1998, ApJ, 507, L55
Baykal, A., & Swank, J. 1996, ApJ, 460, 470
Baykal, A., Swank, J. H., Strohmayer, T., & Stark, M. J. 1998, A&A, 336, 173
Burša, M., & Pěč, K. 1993, Gravity Field and Dynamics of the Earth (Berlin: Springer-Verlag)
Cutler, C., & Lindblom, L. 1987, ApJ, 314, 234
de Campli, W. M. 1980, ApJ, 242, 306
Ghosh, P., Angelini, L., & White, N. E. 1997, ApJ, 478, 713
Glatzmaier, G. A., & Roberts, P. H. 1996, Science, 274, 1887
Goldreich, P. 1970, ApJ, 160, L11
Good, M. L., & Ng, K. K. 1985, ApJ, 299, 706
Heyl, J. S., & Hernquist, L. 1998, MNRAS, submitted (preprint: astro-ph/9808125)
Kouveliotou, C., et al. 1998, Nature, 393, 235
Kramer, M. 1998, ApJ, 509, 856
Link, B., Epstein, R. I., & Baym, G. 1993, ApJ, 403, 285
Lyne, A. G., Pritchard, R. S., & Smith, F. G. 1988, MNRAS, 233, 667
Melatos, A. 1998, MNRAS, submitted
Mereghetti, S. 1995, ApJ, 455, 598
Mereghetti, S., Israel, G. L., & Stella, L. 1998, MNRAS, 296, 689
Mereghetti, S., & Stella, L. 1995, ApJ, 442, L17
Morini, M., Robba, N. R., Smith, A., & van der Klis, M. 1988, ApJ, 333, 777
Oosterbroek, T., Parmar, A. N., Mereghetti, S., & Israel, G. L. 1998, A&A, 334, 925
Paczynski, B. 1990, ApJ, 365, L9
Thompson, C., & Duncan, R. C. 1993, ApJ, 408, 194
Thompson, C., & Duncan, R. C. 1996, ApJ, 473, 322
Usov, V. V. 1994, ApJ, 427, 984
Usov, V. V., & Melrose, D. B. 1996, ApJ, 464, 306
van Paradijs, J., Taam, R. E., & van den Heuvel, E. P. J. 1995, A&A, 299, L41
Weisberg, J. M., Romani, R. W., & Taylor, J. H. 1989, ApJ, 347, 1030

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Fig. 1.— Rotation frequency $\Omega$ versus time $t$ for the AXP 1E 1048.1–5937, with $t_0 = JD 2444000$. The squares and accompanying $1\sigma$ error bars are X-ray timing measurements made over the last 20 years by the satellites *Einstein, EXOSAT, Ginga, ROSAT, ASCA, RXTE*, and *BeppoSAX* (Oosterbroek et al. 1998, and references therein). The solid curve is the solution to Euler’s equations of motion (1)–(3) for $\Omega_0 = 0.976 \text{ rad s}^{-1}$, $\Omega_0 \tau_0 = 3.8 \times 10^{10}$, $\epsilon = 6.4 \times 10^{-8}$, and $\chi = 3.5^\circ$, with initial conditions $\Omega_{1,0} = 0.476 \Omega_0$ and $\Omega_{2,0} = -0.568 \Omega_0$. The broken curve is the solution for $\Omega_0 = 0.9762 \text{ rad s}^{-1}$, $\Omega_0 \tau_0 = 3.5 \times 10^{10}$, $\epsilon = 6.3 \times 10^{-8}$, and $\chi = 18^\circ$, with the same initial conditions. The initial conditions determine the initial phase of the oscillation and are otherwise insignificant.
Fig. 2.— Rotation frequency $\Omega$ versus time $t$ for the AXP 1E 2259+586, with $t_0 = JD 2443000$. The squares and 1σ error bars are X-ray timing data from the satellites cited in Figure 1 as well as from HEAO 1 and Tenma (Baykal et al. 1998, and references therein). The solid curve is the solution to Euler’s equations of motion (1)–(3) for $\Omega_0 = 0.900356 \text{ rad s}^{-1}$, $\Omega_0 \tau_0 = 2.35 \times 10^{12}$, $\epsilon = 3.4 \times 10^{-8}$, and $\chi = 13^\circ$, with initial conditions $\Omega_{2,0} = -0.668\Omega_0$ and $\Omega_{3,0} = 0.658\Omega_0$. 

\[ t-t_0 \text{ (yr)} \]

\[ \Omega \text{ (rad s}^{-1}) \]