A local asymptotic expansion for a local solution of the Stokes system

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Abstract

We consider solutions of the Stokes system in a neighborhood of a point in which the velocity $u$ vanishes of order $d$. We prove that there exists a divergence-free polynomial $P$ in $x$ with $t$-dependent coefficients of degree $d$ which approximates the solution $u$ of order $d + \alpha$ for certain $\alpha > 0$. The polynomial $P$ satisfies a Stokes equation with a forcing term which is a sum of two polynomials in $x$ of degrees $d - 1$ and $d$. The results extend to Oseen systems and to the Navier-Stokes equation.

1 Introduction

In this paper, we study local asymptotic development of solutions of local solutions for the Stokes equation in the unit cylinder. Namely, given $f = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))$ we seek a polynomial in $x$ which approximates a solution $u = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ of the system

$$\begin{align*}
    u_t - \Delta u + \nabla p &= f, \\
    \nabla \cdot u &= 0,
\end{align*}$$

around a point where the solution vanishes of order $d$. The solution is not assumed to have a high degree of regularity and thus the Taylor expansion is not available. Replacing the force with a matrix of functions in the divergence form we also obtain development for solutions of the Navier-Stokes equations around a vanishing point as a consequence.

Fabre and Lebeau in [FL1, FL2] showed that the system (1.1)–(1.2) has a unique continuation property, i.e., local solutions of (1.1)–(1.2) can not vanish to infinite order unless they vanish identically. Having a priori estimates on solutions with respect to their vanishing order is considered a crucial step in many applications. For instance, using a priori estimates on asymptotic polynomials Han [H2] improves the classical Schauder estimates in a way that the estimates of solutions and their derivatives at one point depend on the coefficient and the nonhomogeneous terms at that particular point. Also, Hardt and Simon [HS] applied an estimate of Donnelly and Fefferman for the order of vanishing of eigenfunctions to find an asymptotic bound of the $(n-1)$-dimensional measure of $v_j^{-1}(0)$, where $v_j$ is an eigenfunction corresponding to the $j$-th eigenvalue of the Laplacian on a compact Riemannian manifold.
The method we use in proving the main theorem was introduced by Q. Han, who in [H2] found an asymptotic development of a solution of a parabolic equation of an arbitrary degree (cf. also [H1] for the elliptic case). The main idea in [H2] is based on a local expansion of the corresponding fundamental solution of the global linear equation.

There are several key difficulties when trying to extend the results to the Stokes equation (1.1)–(1.2). First, due to presence of the pressure, it is not reasonable to expect that the velocity and the pressure would vanish at the same point (for instance, the unique continuation result of Fabre and Lebeau gives a unique continuation property for \( u \) and not for the pair \((u, p)\)). Thus in our main result we do require \( p \) to vanish. The second difficulty is the lack of smoothing in the time variable in the system, which is a well-known problem for local solutions of the Stokes and Navier-Stokes systems. Indeed, taking the divergence of the evolution equation for the velocity gives

\[
\triangle p = \nabla \cdot f. \tag{1.3}
\]

which does not contain any smoothing in the time variable. The third difficulty is the nonlocal nature of the Stokes kernel, which in particular causes the Stokes kernel to decay polynomially, rather than exponentially as it is the case for the scalar equations.

We note here that there have been many works on unique continuation of elliptic and parabolic equations showing that, under various assumptions on coefficients, no solution can vanish to infinite order (cf. [AE, AMRV, CRV, DF, EFV, EV, GL, JK, KT, SS1, SS2] for instance); for more complete reviews, see [K1, K2, V]. Unique continuation questions for the Stokes and Navier-Stokes systems were addressed in [CK, FL1, FL2, Ku].

The paper is organized as follows. In Section 2 we state the main results, Theorem 2.1 and 2.3, addressing the forces in standard and divergence forms respectively. We also state the two corollaries concerning the Navier-Stokes and Oseen systems. In Section 3 we recall the properties of the Stokes kernel, while the last part contains a construction of a particular solution vanishing of order \( d \) as well as the proof of Theorem 2.1.

## 2 Notation and the main result on the asymptotic expansion

In this paper, we consider a solution \((u, p)\) of the Stokes system (1.1)–(1.2) in an open set containing \((0, 0)\) (which can always be assumed using translation). For any \((x, t) \in \mathbb{R}^n \times \mathbb{R}\) and \(r > 0\) we denote the parabolic cylinder label by \((x, t)\) with radius \(r > 0\) by

\[
Q_r(x, t) = \{ (y, s) \in \mathbb{R}^n \times \mathbb{R} : |y - x| < r, -r^2 < s - t < 0 \}.
\]

The corresponding parabolic norm for \((x, t) \in \mathbb{R}^n \times \mathbb{R}\) is given by

\[
|(x, t)| = (|x|^2 + |t|)^{1/2}.
\]

Denote by \(W^m_0(Q_1)\) the Sobolev space of \(L^q(Q_1)\) functions whose all the \(x\)-derivatives up to \(m\)-th order and \(t\)-derivative of first order belong to \(L^q(Q_1)\).
Theorem 2.1. Let $q > 1 + n/2$. Suppose that $f_j \in L^q(Q_1)$, for $j = 1, 2, \ldots, n$ satisfy
\[ \|f_j\|_{L^q(Q_1)} \leq \gamma r^{d-2+\alpha+n+2/q}, \quad r \leq 1 \]
for some constants $\gamma > 0$ and $\alpha \in (0, 1)$. Then for any solution $u = (u_1, \ldots, u_n) \in W^{2,1}_q(Q_1)$ of (1.1)–(1.2) there exists $P = (P^1_{d,t}, \ldots, P^n_{d,t})$, whose each component $P^i_{d,t}$ is a polynomial in $x$ of degree less than or equal to $d$, such that
\[ |u_j(x,t) - P^i_{d,t}(x,t)| \leq C \left( \gamma + \sum_{k=1}^n \|u_k\|_{W^{2,1}_q(Q_1)} \right) |(x,t)|^{d+\alpha}, \quad (x,t) \in Q_{1/2}, \]
where $C$ is a positive constant depending on $n$, $q$, $d$, and $\alpha$. Moreover, $P$ satisfies the Stokes system
\[ \partial_t P^i_{d,t}(x) - \Delta P^i_{d,t}(x) + \partial_j R(x) = \sum_{i=d-1}^d \sum_{|\alpha|=i} C^{i}_{\alpha,\gamma} x^{\alpha} \alpha! \quad j = 1, \ldots, d \]
\[ \nabla \cdot P = 0, \]
where $R$ is the corresponding pressure.

Remark 2.2. The pressure term is found explicitly in the proof of the main theorem and is given by
\[ R(x,t) = \sum_{i=0}^{d-2} \sum_{|\alpha|=i} D^\alpha_x (p - \Delta^{-1} \nabla \cdot f) \]
for $(x,t) \in Q_1$.

As we are interested in obtaining estimates in $Q_1$, we assume without loss of generality that
\[ f(x,t) = 0, \quad |(x,t)| \geq 1. \]

In the case when the function on the right side of (1.1) is in the divergence form, the Stokes system reads as
\[ \partial_t u_k - \Delta u_k + \partial_k p = \partial_j g_{jk}, \quad k = 1, \ldots, n \]
\[ \nabla \cdot u = 0, \]
for some function $g = (g_{jk})_{j,k=1}^n \in W^{1,0}_q(Q_1)$. Here also we may assume without loss of generality that
\[ g(x,t) = 0, \quad |(x,t)| > 1. \]

Then we have the following variant of Theorem 2.1.

Theorem 2.3. Assume that $q > 1 + n/2$. Let $g = [g_{jk}] \in W^{2,1}_q$ be an $n \times n$ matrix of functions that satisfies
\[ |g_{jk}(x,t)| \leq \gamma |(x,t)|^{d-1+\alpha}, \quad |(x,t)| < 1, \quad j, k = 1, \ldots, n 

for some constants $\gamma > 0$ and $\alpha \in (0, 1)$. Then for any solution $u = (u_1, u_2, \ldots, u_n) \in W^{2,1}_q$ of (2.6) – (2.7) there exists $P = (P^1_{d,t}, \ldots, P^n_{d,t})$ whose each component $P^j_{d,t}$ is a polynomial in $x$ of degree less than or equal to $d$ such that

$$|u_j(x, t) - P^j_{d,t}(x, t)| \leq C \left( \gamma + \sum_{k=1}^n \|u_k\|_{W^{2,1}_q(Q_t)} \right) |(x, t)|^{d+\alpha},$$  

(2.10)

for any $(x, t) \in Q_{1/2}$, where $C$ is a positive constant depending on $n$, $q$, $d$, $\alpha$. Also, $P$ satisfies the Stokes system

$$\partial_t P^j_{d,t} - \Delta P^j_{d,t} + \partial_j R(x) = \sum_{i=d-1}^d \sum_{|\alpha| = i} C_{\alpha,t} \frac{x^\alpha}{\alpha!}, \quad j = 1, \ldots, n$$

(2.11)

$$\nabla \cdot P = 0$$

(2.12)

where $R$ is the corresponding pressure.

Having a force in divergence form on the right side of (2.7) allows us to apply the above results to the solutions of the Navier-Stokes equations.

**Corollary 2.4.** Let $q > 1 + n/2$. Suppose that $u = (u_1, \ldots, u_n) \in W^{2,1}_q(Q_1)$ solves the Navier-Stokes equations

$$\partial_t u - \Delta u + \nabla(u \otimes u) + \nabla p = 0,$$

(2.13)

$$\nabla \cdot u = 0.$$  

(2.14)

Also, assume that $u$ vanishes of the order at least $d \geq 2$. Then there exists $P = (P^1_{d,t}, \ldots, P^n_{d,t})$ whose each component $P^j_{d,t}$ is a polynomial in $x$ of degree less than or equal to $d$ such that

$$|u_j(x, t) - P^j_{d,t}(x)| \leq C |(x, t)|^{d+1}, \quad (x, t) \in Q_{1/2},$$

(2.15)

for $j = 1, \ldots, n$, where $C$ is a positive constant depending on $n$, $d$, $q$, and $u$. Moreover, $P$ satisfies the Stokes system

$$\partial_t P^j_{d} - \Delta P^j_{d} + \partial_j R(x) = \sum_{i=d-1}^d \sum_{|\alpha| = i} C_{\alpha,t} \frac{x^\alpha}{\alpha!}, \quad j = 1, \ldots, n$$

(2.16)

$$\nabla \cdot P = 0,$$

(2.17)

where $R$ a suitable pressure term depending on $u$ and $p$.

We note that $u$ is not assumed to be smooth in the space or time variable. Therefore, the inequality in (2.15) can not be obtained by expanding the solution in the Taylor series.

The result in Theorem 2.1 can also be applied to the Oseen system considered in [FL1].

**Corollary 2.5.** Let $q > 1 + n/2$. Suppose that $u = (u_1, \ldots, u_n) \in W^{2,1}_q(Q_1)$ solves the Oseen system

$$\partial_t u - \Delta u + (a \nabla)u + \nabla p = 0,$$

(2.18)

$$\nabla \cdot u = 0.$$  

(2.19)
where \( a = (a_1, \ldots, a_n) \in L^\infty(Q_1) \). Also, assume that \( u \) vanishes of the order at least \( d \geq 2 \). Then there exists \( P = (P^1_{d,t}, \ldots, P^m_{d,t}) \) whose each component \( P^j_{d,t} \) is a polynomial in \( x \) of degree less than or equal to \( d \) such that

\[
|u_j(x, t) - P^j_{d,t}(x)| \leq C |(x, t)|^{d+\alpha}, \quad (x, t) \in Q_{1/2},
\]

for \( j = 1, \ldots, n \), where \( \alpha \in (0, 1) \) and \( C \) is a positive constant depending on \( n, d, q, \alpha, \) and \( u \). Moreover, \( P \) satisfies the Stokes system

\[
\partial_t P^j_d - \Delta P^j_d + \partial_j R(x) = \sum_{i=d-1}^d \sum_{\beta \mid \beta_1 = i} C^{\beta,t} x^{\beta} \beta! , \quad j = 1, \ldots, n
\]

\[
\nabla \cdot P = 0,
\]

where \( R \) a suitable pressure term depending on \( u \) and \( p \).

### 3 The basic results

We start by recalling pointwise estimates on the derivatives of solutions to the homogeneous heat equation

\[
\partial_t u - \Delta u = 0.
\]

The fundamental solution is given by \( \Gamma(x, t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right) \) for \( t > 0 \) and \( \Gamma(x, t) = 0 \) for \( t \leq 0 \). Recall that the derivatives are bounded as

\[
|\partial_x^l \partial_t^\mu \Gamma(x, t)| \leq \frac{C(\mu, l)}{|x| + \sqrt{t}} |x|^{\mu + 2l} e^{-|x|^2/8t}, \quad l \in \mathbb{N}_0, \quad \mu \in \mathbb{N}_0^n.
\]

First recall that for any solution \( u \) of (3.1) we have

\[
|D^\mu_x D^l_t u(x, t)| \leq \frac{C}{(R - |(x, t)|)^{n+2l}} \sup_{Q_R} |u|, \quad (x, t) \in Q_{R/2}
\]

where \( C \) depends on \( |\mu| + 2l \).

For completeness, we briefly recall the derivation of the fundamental solution to the Stokes system. Let \( u(0, \cdot) = u_0 \) be the initial condition. By uniqueness of solutions, we have

\[
u_k(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)u_0(y) \, dy + \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s)f_k(y, s) \, dy \, ds
\]

\[ - \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s)\partial_k p(y, s) \, dy \, ds,
\]

for \( k = 1, \ldots, n \). Using the Fourier transform of both sides in (1.3), we get

\[
\partial_k p = -R_j R_k f_j, \quad k = 1, \ldots, n,
\]

where

\[
R_j g = \left( \frac{\xi_j}{|\xi|^2} \right) \hat{g}.
\]
denotes the $j$-th Riesz transform, using the Fourier transform $\hat{f}(\xi) = \int f(x)e^{-i\xi \cdot x} \, dx$. Thus (3.4) can be written as

$$u_k(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t) u_0(y) \, dy + \int_0^t \int_{\mathbb{R}^n} K_{jk}(x - y, t - s) f_j(y, s) \, dy \, ds,$$

where

$$K_{jk}(x, t) = \delta_{jk} \Gamma(x, t) + R_j R_k \Gamma(x, t), \quad j, k = 1, \ldots, n.$$  

For each $j, k = 1, \ldots, n$, the function $K_{jk}$ solves the heat equation, i.e.,

$$\partial_t K_{jk}(x, t) - \Delta K_{jk}(x, t) = 0, \quad t > 0.$$  

Also,

$$\partial_j K_{jk}(x, t) = 0, \quad j = 1, \ldots, n.$$  

Furthermore, we have the estimate

$$|D^{\mu} D_t^l K_{jk}(x, t)| \leq \frac{C}{|x, t|^{n+|\mu|+2l}}, \quad l \in \mathbb{N}_0, \quad \mu \in \mathbb{N}_0^n$$

where $C$ depends on $|\mu|$ and $l$.  

4 Proof of the Main Theorem

In the next lemma, we construct a solution of the system (1.1)–(1.2) which vanishes with a certain prescribed degree.

**Lemma 4.1.** Assume that $f = (f_1, \ldots, f_n) \in L^q(Q_1)$, where $q > 1 + n/2$, satisfies (2.1). Then there exists $u = (u_1, \ldots, u_n) \in W^{2,1}_q(Q_1)$ which solves (1.1)–(1.2) and satisfies

$$\|u\|_{L^q(Q_1)}, \|u\|_{W^{2,1}_q(Q_{3/4})} \leq C\gamma.$$  

Furthermore,

$$|u_k(x, t)| \leq C\gamma |(x, t)|^{d+\alpha}, \quad |(x, t)| < 1/2, \quad k = 1, \ldots, n$$

where the constant $C$ depends on $n$, $d$, and $\alpha$.

**Proof of Lemma 4.1.** We start by setting

$$w_k(x, t) = \int K_{jk}(x - y, t - s) f_j(y, s) \, dy \, ds$$

and

$$p = -\Delta^{-1} \nabla \cdot f = (\xi_j \hat{f}_j / |\xi|^2).$$
Then we have
\[ \frac{\partial}{\partial t} w_k - \Delta w_k + \partial_k p = f_k, \quad k = 1, \ldots, n \] (4.5)
and
\[ \|w\|_{W^{2,1}_x(Q_1)} \leq C\|f\|_{L^q(Q_1)} \leq C\gamma. \] (4.6)

Now, we consider the Taylor expansion of \( K_{jk}(x - y, t - s) \) around \((0,0)\). Let \(|(y,s)| < 1\) be such that \(s \neq 0\). Denote by \( K_{jk}^m \) the \( m \)-th order terms, i.e.,

\[ K_{jk}^m(x, y; t, s) = \sum_{|\mu| + 2l = m} D_x^\mu D_t^l K_{jk}(-y, -s) \frac{x^\mu t^l}{\mu!l!}. \] (4.7)

It is easy to check that \( K_{jk}^m \) solves the heat equation for each \( j, k \), i.e.,
\[ \frac{\partial}{\partial t} K_{jk}^m(x, y; t, s) - \Delta_x K_{jk}^m(x, y; t, s) = 0 \]
for any \((y,s)\). Let
\[ v_k(x, t) = \int_{|(y,s)| < 1} d \sum_{m=0}^d K_{jk}^m(x, y; t, s) f_j(y, s) \, dy \, ds, \quad k = 1, \ldots, n. \] (4.8)
Each \( v_k \) is a polynomial of degree less than or equal to \( d \) and it satisfies
\[ \frac{\partial}{\partial t} v_k(x, t) - \Delta v_k(x, t) = 0. \] (4.9)

Moreover, we have
\[ \nabla \cdot v = 0 \] (4.10)
Indeed, we may write
\[ \frac{\partial}{\partial t} v_k = \int_{|(y,s)| < 1} d \sum_{m=1}^d \sum_{|\mu| + 2l = m} \sum_{\mu_k > 0} D_x^\mu D_t^l K_{jk}(-y, -s) \frac{x^\mu t^l}{(\mu - e_k)!(\mu_k - e_k)_l!} f_j(y, s) \, dy \, ds, \]
where \( e_k \) is the standard \( k \)-th unit vector in \( \mathbb{R}^n \). Note that
\[ \sum_{|\mu| + 2l = m} D_x^\mu D_t^l K_{jk}(-y, -s) \frac{x^\mu t^l}{(\mu - e_k)!(\mu_k - e_k)_l!} = \sum_{|\mu| + 2l = m-1} \frac{\partial}{\partial x^\mu} D_t^l K_{jk}(-y, -s) \frac{x^\mu t^l}{\mu!l!}. \]
Using \( \frac{\partial}{\partial x} K_{jk} = 0 \) for \( j = 1, \ldots, n \), we get \( \nabla \cdot v = 0 \). Now, set
\[ u_k(x, t) = w_k(x, t) - v_k(x, t) = \int_{|(y,s)| < 1} \left( K_{jk}(x - y, t - s) - \sum_{m=0}^d K_{jk}^m(x, y; t, s) \right) f_j(y, s) \, dy \, ds, \quad k = 1, \ldots, n \] (4.11)
and note that we have
\[ \frac{\partial}{\partial t} u_k - \Delta u_k + \partial_k p = f_k, \quad k = 1, \ldots, n. \] (4.12)
We now check the condition (1.2). Since
\[ \partial_k w_k = \int_{\{y, s\} < 1} \partial_k K_{jk}(x - y, t - s) f_j(y, s) \, dy \, ds = 0, \]
where we used \( \partial_k K_{jk} = 0 \) for \( j = 1, \ldots, n \), we get \( \nabla \cdot u = 0 \).

Next, we claim that
\[ |u(x, t)| \leq C_\gamma |(x, t)|^{d+\alpha}, \quad |(x, t)| \leq \frac{1}{2}. \] (4.13)

Fixing \( |(x, t)| \leq 1/2 \), we split the integral on the far right side of (4.11) into three parts
\[ I_1 = \int_{|(y, s)| \leq 2(x, t)} K_{jk}(x - y, t - s) f_j(y, s) \, dy \, ds, \]
\[ I_2 = -\int_{|(y, s)| \leq 2(x, t)} \sum_{m=0}^{d} K_{jk}^m(x, y; t, s) f_j(y, s) \, dy \, ds, \]
\[ I_3 = \int_{2(x, t) < |(y, s)| < 1} \left( K_{jk}(x - y, t - s) - \sum_{m=0}^{d} K_{jk}^m(x, y; t, s) \right) f_j(y, s) \, dy \, ds. \]

By a hypothesis, \( q > 1 + n/2 \). Therefore, by Hölder’s inequality and (3.9),
\[ |I_1| \leq C \sum_{j=1}^{n} \left( \int_{|(y, s)| \leq 2(x, t)} \frac{dy \, ds}{|(x - y, t - s)|^{n q'}} \right)^{1/q} \left( \int_{|(y, s)| \leq 2(x, t)} |f_j(y, s)|^q \, dy \, ds \right)^{1/q} \]
\[ \leq C \sum_{j=1}^{n} \left( \int_{|(y, s)| < 3(x, t)} \frac{dy \, ds}{|(y, s)|^{n q'}} \right)^{1/q} \left( \int_{|(y, s)| < 2(x, t)} |f_j(y, s)|^q \, dy \, ds \right)^{1/q} \]
\[ \leq C_\gamma |(x, t)|^{(n+2)/q' - n} |(x, t)|^{d - 2 + \alpha + (n+2)/q} \]
\[ = C_\gamma |(x, t)|^{d + \alpha}, \]
where \( q' = (q - 1)/q \). Similarly, using (5.9) we estimate
\[ |I_2| \leq C \sum_{j=1}^{n} \sum_{k=0}^{d} \int_{|(y, s)| < 2(x, t)} \frac{|f_j(y, s)|}{|(y, s)|^{n+k}} \, dy \, ds \]
\[ \leq C \sum_{j=1}^{n} \sum_{k=0}^{d} |(x, t)|^k \int_{i=0}^{\infty} \int_{|x, t|/2^i < |(y, s)| < |x, t|/2^{i-1}} \frac{|f_j(y, s)|}{|(y, s)|^{n+k}} \, dy \, ds \]
\[ \leq C_\gamma \sum_{k=0}^{d} |(x, t)|^k \int_{i=0}^{\infty} \left( \frac{|x, t|}{2^i} \right)^{d-k+\alpha} \]
\[ \leq C_\gamma |(x, t)|^{d + \alpha}. \] (4.14)

In order to estimate \( I_3 \), we expand \( K_{jk}(x - y, t - s) \) into Taylor series around \( (x, t) = (0, 0) \). For each \( j, k = 1, \ldots, n \), we have
\[ K_{jk}(x - y, t - s) = \sum_{i=0}^{d} \sum_{|\mu|, |l| = i} D^{\mu, l} K_{jk}(y, t - s) \frac{x^\mu y^l}{\mu!} + \sum_{|\mu| + l = d+1} D^{\mu, l} K_{jk}(x - y, s) \frac{x^\mu y^l}{\mu!}, \] (4.15)
where \(0 < \xi = \xi(x, t; y, s), \eta = \eta(x, t; y, s) < 1\). Therefore,

\[
K_{jk}(x - y, t - s) - \sum_{m=0}^{d} K_{jk}^{m}(x, t; s)
\]

\[
= \sum_{i=0}^{d} \sum_{|\mu_i|+i=d+1} D^{\mu_i} K_{jk}(-y, -s)^{\frac{x_{\mu_i} t_{\mu_i}}{\mu_i!!}} + \sum_{|\mu_i|=d+1} D^{\mu_i} K_{jk}(\xi x - y, \eta t - s)^{\frac{x_{\mu_i} t_{\mu_i}}{\mu_i!!}}.
\]

Using the bound on \(\partial_{x_t}^2 K_{jk}(x, t)\), the difference above can be estimated with

\[
C \sum_{i=d+1}^{2(d+1)} \left( \frac{1}{|(y, s)|^{n+1}} + \frac{1}{|(\xi x - y, \eta t - s)|^{n+1}} \right) |(x, t)|^i.
\]

(4.16)

As we assumed \(2|(x, t)| < |(y, s)|\), we get

\[
2|\xi x - y, \eta t - s| > |(y, s)|.
\]

Now we may estimate

\[
|I_3| \leq \int_{2[(x, t)|<(y, s)|<1]} \left| K_{jk}(x - y, t - s) - \sum_{m=0}^{d} K_{jk}^{m}(x, t; s) \right| |f_j(y, s)| \, dy \, ds,
\]

\[
\leq C \sum_{i=d+1}^{2(d+1)} |(x, t)|^i \int_{2[(x, t)|<(y, s)|<1]} \sum_{j=1}^{n} \left| \frac{|f_j(y, s)|}{|(y, s)|^{n+1}} \right| \, dy \, ds,
\]

\[
\leq C \sum_{i=d+1}^{2(d+1)} |(x, t)|^i \sum_{u=1}^{M} \int_{2^{u}|(x, t)|<(y, s)|<2^{u+1}} \sum_{j=1}^{n} \left| \frac{|f_j(y, s)|}{|(y, s)|^{n+1}} \right| \, dy \, ds,
\]

\[
\leq C \sum_{i=d+1}^{2(d+1)} |(x, t)|^i \sum_{u=1}^{M} \left( 2^u |(x, t)|^{-(n+1)} \sum_{j=1}^{n} \|f_j\|_{L^\infty(Q_{2^{u+1}}(x, t))} \right) \int_{2^{u}|(x, t)|<(y, s)|<2^{u+1}} \, dy \, ds
\]

from where

\[
|I_3| \leq \sum_{i=d+1}^{2(d+1)} |(x, t)|^i \sum_{u=1}^{M} \|f_j\|_{L^\infty(Q_{2^{u+1}}(x, t))} |(x, t)|^{d+\alpha - i},
\]

\[
\leq C_\gamma |(x, t)|^{d+\alpha} \sum_{i=d+1}^{2(d+1)} \sum_{u=1}^{M} \frac{1}{(2^{d-\alpha})^u},
\]

\[
\leq C_\gamma |(x, t)|^{d+\alpha},
\]

which proves (4.2). For (4.1) recall that we have the same estimate on \(\|w_k\|_{L^\infty(Q_1)}\) by (4.6). Furthermore, we may show that for \(k = 1, \ldots, n\)

\[
|v(x, t)| \leq C_\gamma, \quad (x, t) \in Q_1,
\]

(4.17)

by following the same approach we took in estimating \(I_2\). Combining (4.17) with (3.3) and (4.9), we get

\[
\|v\|_{W^{2,1}_{\alpha+1}(Q_r)} \leq C(r) \gamma, \quad r < 1,
\]

(4.18)

which completes the proof of (4.1).
Proof of Theorem 2.1. Suppose \((u, p)\) solves \((1.1)–(1.2)\). In Lemma 4.1, we have already constructed a solution \((\tilde{u}, \tilde{p})\) of \((1.1)–(1.2)\) with \(\tilde{u}_k \in W^{2,1}_q(Q_1)\) for each \(k = 1, \ldots, n\), such that
\[
|\tilde{u}_k(x, t)| \leq C\gamma |(x, t)|^{d + \alpha}, \quad (x, t) \in Q_{1/2}
\]
for each \(k = 1, \ldots, n\), where \(C\) is a constant depending on \(n, d, \) and \(\alpha\). Also, we have
\[
\|\tilde{u}_k\|_{L^q(Q_1)} , \|\tilde{u}_k\|_{W^{2,1}_q(Q_{1/2})} \leq C\gamma, \quad k = 1, \ldots, n.
\]
(4.19)
Then we set \(U = u - \tilde{u}\). Note that \(U\) solves the system
\[
U_t - \Delta U + \nabla(p - \tilde{p}) = 0, \quad (4.20)
\]
\[
\nabla \cdot U = 0. \quad (4.21)
\]
Furthermore, we consider the vorticity equation. Let \(W = \nabla \times U\) denote the curl of \(U\), i.e., \(W_{i,j} = \partial_i U_j - \partial_j U_i\). Then \(W = [W_{i,j}]_{n \times n}\) satisfies the heat equation
\[
W_t - \Delta W = 0.
\]
As \(W_{i,j} = \partial_i (u_j - \tilde{u}_j) - \partial_j (u_i - \tilde{u}_i)\), we obtain
\[
\sum_{i,j=1}^n \|W_{i,j}\|_{L^r(Q_r)} \leq \sum_{k=1}^n \|u_k\|_{W^{2,1}_q(Q_r)} + \sum_{k=1}^n \|\tilde{u}_k\|_{W^{2,1}_q(Q_r)} \leq C\gamma + \sum_{k=1}^n \|u_k\|_{W^{2,1}_q(Q_1)}
\]
(4.22)
for any \(r < 1\). By expanding \(U_k\) into Taylor series in \(x\), we obtain
\[
U_k(x, t) = P^k_{d,t}(x) + R^k_{d,t}(x), \quad k = 1, \ldots, n,
\]
(4.23)
where
\[
P^k_{d,t}(x) = \sum_{i=0}^d \sum_{|\alpha|=i} D_x^\alpha U_k(0, t) \frac{x^\alpha}{\alpha!}
\]
(4.24)
and
\[
R^k_{d,t}(x) = \sum_{|\alpha|=d+1} D_x^\alpha U_k(\xi(x, t), t) \frac{x^\alpha}{\alpha!}
\]
(4.25)
where \(0 < \xi(x, t) \leq 1\). Let \((x, t) \in Q_{1/2}\). Note that
\[
|R^k_{d,t}(x)| \leq C \sum_{|\alpha|=d+1} |D_x^\alpha U_k(\xi(x, t), t)||x^\alpha|
\]
\[
\leq C|x|^{d+1} \sum_{|\alpha|=d+1} |D_x^\alpha U_k(\xi(x, t), t)|
\]
\[
\leq C|(x, t)|^{d+1} \sum_{|\alpha|=d+1} |D_x^\alpha U_k(\xi(x, t), t)|.
\]
Also, selecting $1/2 < r_1 < r_2 < 3/4$, we get
\[
|D^\alpha U_k(x, t)| \leq \|D^\alpha U_k(\cdot, t)\|_{L^\infty((0, 0))} + C\|U_k(\cdot, t)\|_{L^2((0, 0))} + C\|\partial_\alpha U_k(\cdot, t)\|_{L^2((0, 0))}
\]
for any multiindex $\beta$. By (1.14) and (1.15) we have the desired bound on $R^{k}_{d,t}$. Going back to (122), we get
\[
|u_k(x, t) - P^{k}_{d,t}(x)| \leq |\bar{u}_k(x, t)| + |R^{k}_{d,t}(x, t)|
\]
\[
\leq C\|(x, t)^{d+\alpha} + C\left(\sum_{k=1}^{n} u_k \|W^{d+\alpha}_{k, t}\|_{L^2(Q_t)}\right)(x, t)^{d+1},
\]
for any $|(x, t)| \leq 1/2$. Furthermore, $P$ satisfies (1.23) and (1.24). Taking the divergence of (1.24) we get
\[
\partial_k P^{k}_{d,t} = \sum_{i=0}^{d-1} \sum_{|\alpha| = i-1} \partial_k D^\alpha U_k(0, t) \frac{x^\alpha}{\alpha!} = 0,
\]
as $U = u - \bar{u}$ is divergence free. Similarly,
\[
\partial_k P^{k}_{d,t}(x) - \triangle P^{k}_{d,t}(x)
\]
\[
= \sum_{i=0}^{d} \sum_{|\alpha| = i} D^\alpha U_k(0, t) \frac{x^\alpha}{\alpha!} - \sum_{i=0}^{d-2} \sum_{|\alpha| = i} \partial_\alpha D^\alpha U_k(0, t) \frac{x^\alpha}{\alpha!}
\]
\[
= \left(\sum_{i=0}^{d-2} \sum_{|\alpha| = i} D^\alpha(\partial_k p - \partial_k \bar{p})\right) + \sum_{i=d-1}^{d} \sum_{|\alpha| = i} D^\alpha U_k(0, t) \frac{x^\alpha}{\alpha!},
\]
which proves (2.3). \qed

*Proof of Theorem 2.1* The proof of this result follows that of Theorem 2.1 and it is thus omitted. \qed

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