Asymptotic Chern–Simons formulation of spacelike stretched AdS gravity

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Abstract
We show that the asymptotic structure of topologically massive gravity in the spacelike stretched AdS sector can be faithfully represented by an $SL(2, \mathbb{R}) \times U(1)$ Chern–Simons gauge theory, by adopting a natural correspondence between their fields and coupling constants.

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1. Introduction

In three-dimensional (3D) Einstein’s gravity with a cosmological constant ($\text{GR}_{\Lambda}$), the AdS asymptotic structure is described by two independent Virasoro algebras with classical central charges [1], for a review of the subject see [2]. Of particular importance for our understanding of the corresponding gravitational dynamics, at both classical and quantum level, is the fact that $\text{GR}_{\Lambda}$ can be represented as an ordinary gauge theory—the Chern–Simons (CS) theory based on the internal $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ gauge group [3, 4].

Adding the gravitational CS term to $\text{GR}_{\Lambda}$ substantially changes its dynamical structure: while $\text{GR}_{\Lambda}$ is a topological theory with no dynamical degrees of freedom, the new theory, known as topologically massive gravity with a cosmological constant ($\text{TMG}_{\Lambda}$), is a truly dynamical theory with one degree of freedom, the massive graviton [5]. In $\text{GR}_{\Lambda}$, the AdS sector is defined around the maximally symmetric vacuum AdS$_3$, and it contains the BTZ black hole with interesting thermodynamic properties [6, 7]. Since the same AdS$_3$ is automatically a solution of $\text{TMG}_{\Lambda}$, one can define the AdS sector also in $\text{TMG}_{\Lambda}$. However, for generic values of the coupling constants, the physical interpretation of this sector suffers from serious difficulties: for the usual sign of the gravitational coupling constant $G > 0$, massive excitations around AdS$_3$ carry negative energies [5], while for $G < 0$, the black hole mass becomes negative [8, 9].

In an interesting attempt to find a resolution of this inconsistency, Li et al [9] studied the chiral version of $\text{TMG}_{\Lambda}$, defined by a specific relation between the graviton mass and the cosmological constant. However, we shall focus our attention to another promising
idea: Anninos et al [10] suggested that choosing a new vacuum, the spacelike stretched AdS3, could lead to a stable ground state of TMG\(_{\Lambda}\). Geometrically, choosing the spacelike stretched AdS3 as the ground state corresponds to a deformation of the AdS3 isometry group \(SL(2, R) \times SL(2, R)\) to its four-parameter subgroup \(SL(2, R) \times U(1)\) [11, 12].

The constrained Hamiltonian analysis of the full TMG\(_{\Lambda}\) was carried out recently in [13], see also [14], leading to a clear and precise picture of its gauge and dynamical features. An important step toward a proper understanding of the spacelike stretched AdS asymptotic structure was achieved by constructing a set of suitable asymptotic conditions [15, 16]. The resulting asymptotic symmetry was shown to be centrally extended semidirect sum of a Virasoro and a \(u(1)\) Kac–Moody algebra, \(Virasoro \oplus u(1)_{KM}\). This whole sector of TMG\(_{\Lambda}\) will be called shortly the spacelike stretched AdS gravity. Motivated by the experience stemming from GR\(_{\Lambda}\), the goal of the present paper is to improve our understanding of the spacelike stretched AdS gravity by showing that its asymptotic structure can be faithfully represented by an \(SL(2, R) \times U(1)\) Chern–Simons gauge theory.

Let us mention here a highly interesting hypothesis formulated by Anninos et al [10], according to which the spacelike stretched AdS boundary dynamics is characterized by a two-dimensional conformal symmetry with central charges. The proof of this hypothesis, presented recently in [16], is based on the asymptotic Virasoro \(\oplus u(1)_{KM}\) canonical algebra and the (algebraic) Sugawara construction, see also [17]. We expect that the asymptotic CS representation of the spacelike stretched AdS gravity will be a useful tool in clarifying the boundary conformal structure lying behind the Sugawara construction.

The paper is organized as follows. In sections 2 and 3, we use the canonical approach to study the asymptotic structure of the \(SL(2, R) \times U(1)\) CS gauge theory, a natural counterpart of the spacelike stretched AdS gravity. The result of this analysis is the Virasoro \(\oplus u(1)_{KM}\) Poisson bracket algebra of the canonical generators. Then, in section 4, we give a brief overview of the basic asymptotic features of the spacelike stretched AdS gravity, including its asymptotic canonical algebra, Virasoro \(\oplus u(1)_{KM}\). In section 5 we introduce specific asymptotic conditions, find a new form of the canonical surface terms and count and identify boundary degrees of freedom of the spacelike stretched AdS gravity. In section 6, we compare the resulting gravitational asymptotic structure with the one found in the CS theory and find, by a natural identification of the corresponding coupling constants and dynamical fields, that they are identical. Section 7 is devoted to concluding remarks, while appendices contain some technical details.

Our conventions are the same as in [16]: the Latin indices refer to both the basis of \(sl(2, R)\) and the local Lorentz frame. The Greek indices refer to the coordinate frame; the middle alphabet letters \((i, j, k, \ldots; \mu, \nu, \lambda, \ldots)\) run over 0,1,2, and the first letters of the Greek alphabet \((\alpha, \beta, \gamma, \ldots)\) run over 1,2; the metric components in the local Lorentz frame are \(\eta_{ij} = (+, -, -)\). A totally antisymmetric tensor \(\epsilon^{ijk}\) and the related tensor density \(\epsilon^{\mu\nu\rho}\) are both normalized as \(\epsilon^{012} = 1\).

2. \(SL(2, R) \times U(1)\) Chern–Simons gauge theory

Motivated by the asymptotic structure of TMG\(_{\Lambda}\) in the spacelike stretched AdS sector [16], we discuss here the corresponding aspects of the \(SL(2, R) \times U(1)\) CS gauge theory.

2.1. The action and boundary conditions

Consider the CS gauge theory defined by the action

\[
I_{CS} = -\kappa \int_M \left( A' \cdot dA' + \frac{1}{3} \epsilon_{ijk} A' A^i A^j \right) + \bar{\kappa} \int_M \bar{A} \cdot d\bar{A}.
\]
Here, $M$ is a spacetime manifold with topology $R \times \Sigma$, where $R$ is interpreted as time and $\Sigma$ is a spatial manifold whose boundary is topologically a circle (which may be located at infinity). $A^\mu = A^\mu_\nu \, dx^\nu$ and $\bar{A} = \bar{A}_\mu \, dx^\mu$ (1-forms) are the $SL(2, R)$ and $U(1)$ gauge potentials, respectively, and $\epsilon_{ijk}$ are the structure constants of $sl(2, R)$ (appendix A). The action is invariant under the infinitesimal gauge transformations

$$\delta_0 A^i = \nabla u^i := du^i + \epsilon^{ijk} A^j u^k, \quad \delta_0 \bar{A} = d\bar{u},$$

where $u^i$ and $\bar{u}$ are the gauge parameters.

We assume the existence of the Schwarzschild-like coordinates $x^\mu = (t, \rho, \phi)$ on $M$, such that the boundary $\partial \Sigma$ is described by the standard angular coordinate $\phi$. Having in mind the fact that the asymptotic parameters in the spacelike stretched AdS sector of TMG are time independent, we choose the CS boundary conditions as

$$A^0 = 0, \quad \bar{A} = \bar{a}_0 \quad \text{at} \quad \partial \Sigma,$$  \label{2.2}

since they imply $\partial_\mu u^\mu = 0, \partial_0 \bar{u} = 0$ at the boundary. Although the choice (2.2) leads to a nontrivial boundary term $\delta B$ in the variation of the action, this can be corrected by introducing the improved action $\tilde{I}$:

$$\tilde{I} := I - B,$$  \label{2.3}

Consequently, the Lie algebra-valued gauge potentials are locally trivial:

$$A^\mu = G^{-1} \partial^\mu G,$$\quad $\bar{A}^\mu = \bar{G}^{-1} \partial^\mu \bar{G},$$

where $G$ and $\bar{G}$ are the elements of $SL(2, R)$ and $U(1)$, respectively.

2.2. The canonical structure

Now, we analyze the symmetry structure of our CS theory by using the canonical formalism.

**Gauge generator.** Introducing the canonical momenta $(\pi^i_\mu, \bar{\pi}^\mu)$ corresponding to the Lagrangian variables $(A^\mu, \bar{A}^\mu)$, one obtains the primary constraints

$$\pi^0_i \approx 0, \quad \phi^\alpha_i := \pi_i^\alpha + \kappa \epsilon_{\alpha \beta} A^\beta_i \approx 0,$$

$$\bar{\pi}^0 \approx 0, \quad \bar{\phi}^\alpha := \bar{\pi}^\alpha - \bar{\kappa} \epsilon_{\alpha \beta} \bar{A}_\beta \approx 0,$$

where $\alpha, \beta = 1, 2$. The secondary constraints have the form

$$\mathcal{H}_i := \kappa \epsilon_{\alpha \beta} F^\alpha_{i \beta} - \nabla_\alpha \phi_i^\alpha \approx 0,$$

$$\mathcal{\bar{H}} := -\bar{\kappa} \epsilon_{\alpha \beta} \bar{F}^\alpha_{\alpha \beta} - \partial_\alpha \bar{\phi}^\alpha \approx 0,$$

and the total Hamiltonian (up to an irrelevant divergence) is given by

$$\mathcal{H}_T = A^0_\mu A^\mu_i \mathcal{H}_i + \bar{A}^0_\mu \bar{\mathcal{H}} + w^i_0 \pi^0_i + \bar{w} \bar{\phi}^0,$$

where $w^i$ and $\bar{w}$ are the arbitrary multipliers.

The constraints $(\pi^0_i, \mathcal{H}_i, \bar{\phi}^0, \bar{\mathcal{H}})$ are first class while $(\phi^\alpha_i, \bar{\phi}^\alpha)$ are second class. Using Castelmanni’s procedure [18], one finds the form of the canonical gauge generator:

$$G = \int d^2x \left[ (\nabla_0 u^i) \pi^0_i + u^i \mathcal{H}_i \right] + \int d^2x [ (\partial_0 \bar{u}) \bar{\pi}^0 + \bar{w} \bar{\mathcal{H}}].$$  \label{2.4}
Fixing the gauge. We have found two sets of the first class constraints, \((\pi^{0}_{i}, \bar{\pi}^{0})\) and \((\mathcal{H}_{i}, \bar{\mathcal{H}}_{i})\). The first set of the corresponding gauge conditions is chosen so as to extend the boundary conditions (2.2) to the whole spacetime:

\[
A'_{0} = 0, \quad \bar{A}_0 = \bar{a}_0. \tag{2.5a}
\]

Using the notation \(A_{\mu} = A'_{\mu}T_{\mu}\), where \(T_{\mu}\) is a basis of the \(sl(2, \mathbb{R})\) Lie algebra (appendix A), the second set of gauge conditions is defined by restricting \(A_{1}\) and \(\bar{A}_{1}\) to be the functions of the radial coordinate only:

\[
A_{1} \approx b^{-1}(\rho)\partial_{\rho}b(\rho), \quad \bar{A}_{1} \approx \bar{b}^{-1}(\rho)\partial_{\rho}\bar{b}(\rho),
\]

where \(b\) and \(\bar{b}\) are in \(SL(2, \mathbb{R})\) and \(U(1)\), respectively. By a suitable choice of the radial coordinate, we can write

\[
A_{1} = a_{1}, \quad \bar{A}_{1} = \bar{a}_{1}, \tag{2.5b}
\]

where \(a_{1} = a'_{1}T_{1}\) and \(\bar{a}_{1}\) are the constant elements of the corresponding Lie algebras. The gauge conditions (2.5b) are conserved in time.

The constraints \(F_{12} \approx 0, \bar{F}_{12} \approx 0\) imply

\[
A_{2} \approx b^{-1}\hat{A}_{2}(t, \varphi)b, \quad \bar{A}_{2} \approx \hat{A}_{2}(t, \varphi),
\]

where the field equations \(F_{02} = 0, \bar{F}_{02} = 0\) lead to

\[
\hat{A}_{2} = \hat{A}_{2}(\varphi), \quad \bar{A}_{2} = \bar{A}_{2}(\varphi).
\]

The residual gauge symmetry has the form

\[
\delta \hat{A}_{2} = \partial_{2}\hat{u}', \quad \delta \bar{A}_{2} = \partial_{2}\bar{u}, \tag{2.6}
\]

where \(u = : b^{-1}\hat{u}(\varphi)b\) and \(\bar{u} = \bar{u}(\varphi)\).

The improved generator. After adopting the gauge conditions (2.5), the effective gauge generator can be written as

\[
G = \int d^{2}x u^{i}H_{i} + \int d^{2}x \bar{u}\bar{H}.\]

This expression is not differentiable. Indeed, when the gauge parameters are independent of field derivatives, the variation of \(G\) contains certain boundary contributions:

\[
\delta G = -\delta \Gamma_{L}[u] - \delta \Gamma_{R}[\bar{u}] + R,
\]

\[
\delta \Gamma_{L}[u] = \oint d_{f_{\alpha}}u^{i}(2\kappa \varepsilon^{0a\beta} \delta A_{i\beta} + \delta \phi_{i}u^{a}), \tag{2.7a}
\]

\[
\delta \Gamma_{R}[\bar{u}] = \oint d_{f_{\alpha}}\bar{u}(2\kappa \varepsilon^{0a\beta} \delta \bar{A}_{\beta} + \delta \bar{\phi}^{a}).
\]

Here, \(R\) are the regular terms, which correspond to well-defined functional derivatives, \(\delta \Gamma_{L}, \delta \Gamma_{R}\) are the boundary integrals obtained with the help of the Stokes theorem and \(d_{f_{\alpha}} = \varepsilon_{0a\beta}dx^{\beta}\) represents the line element on the spatial boundary. If we can integrate \(\delta \Gamma_{L}[u]\) and \(\delta \Gamma_{R}[\bar{u}]\) to find \(\Gamma_{L}[u]\) and \(\Gamma_{R}[\bar{u}]\), the improved gauge generator takes the form

\[
\tilde{G}[u, \bar{u}] = G[u, \bar{u}] + \Gamma_{L}[u] + \Gamma_{R}[\bar{u}]. \tag{2.7b}
\]

3. Asymptotic symmetries

The form of the improved canonical generator \(\tilde{G}\), which gives a complete description of the boundary symmetry, depends on the boundary conditions imposed on \(u'\) and \(\bar{u}\).
3.1. Kac–Moody extension of \( \text{sl}(2, \mathbb{R}) \oplus \text{u}(1) \)

The simplest boundary conditions on the gauge parameters that allow us to find an explicit form of the surface terms are defined as follows:

- \( u \) and \( \bar{u} \) are independent of the fields (at the boundary).

On the subspace defined by \( \phi^i, \bar{\phi}^a \), \( \phi^i \approx 0 \), relations (2.7a) imply

\[
\Gamma_L[\tau] \approx -2\kappa \oint d^\beta \tau^i A_{i\beta}, \\
\Gamma_R[\bar{\tau}] \approx 2\bar{\kappa} \oint d^\beta \bar{\tau} \bar{A}_\beta.
\]

(3.1)

To find the PB algebra of the improved generator (2.7b), we use the relation

\[
[\tilde{G}[\tau, \bar{\tau}], \tilde{G}[\lambda, \bar{\lambda}]] \approx \delta_{\lambda}/\Gamma_1L[\tau] + \delta_{\bar{\lambda}}/\Gamma_1R[\bar{\tau}],
\]

where

\[
\sigma^i = \varepsilon^{ijk} \lambda^j \tau^k
\]

and obtain

\[
[\tilde{G}[\tau, \bar{\tau}], \tilde{G}[\lambda, \bar{\lambda}]] = \tilde{G}[\sigma, 0] - 2\kappa \oint d^\beta \tau^i \partial_\beta \lambda^i + 2\bar{\kappa} \oint d^\beta \bar{\tau} \partial_\beta \bar{\lambda}.
\]

(3.2a)

After introducing the Fourier modes

\[
J_{im} := \tilde{G}[\hat{\tau}^i = e^{-im\phi}, \hat{\bar{\tau}} = 0] \approx 2\kappa (\hat{A}_2)_m,
\]

\[
K_m := \tilde{G}[\hat{\bar{\tau}} = 0, \hat{\tau} = e^{-im\phi}] \approx 2\bar{\kappa}(\hat{\bar{A}}_2)_m,
\]

the PB algebra (3.2a) takes the form of a Kac–Moody extension of the \( \text{sl}(2, \mathbb{R}) \oplus \text{u}(1) \) Lie algebra:

\[
i \{J^i_m, J^j_n\} = i\varepsilon^{ijkl} J^k_{m+n} + 4\pi \kappa m\eta_j \delta_{m,-n},
\]

\[
i \{K_m, K_n\} = -4\pi \bar{\kappa} m \delta_{m,-n}.
\]

(3.2b)

3.2. Semidirect sum of Virasoro and \( \text{u}(1)_{\text{KM}} \)

We now wish to examine another set of boundary conditions on \( u' \) and \( \bar{u} \):

- \( u' = -\theta^i - \xi^\rho A^\rho_{i'} \) and \( \bar{u} = -\xi^\rho \bar{A}^\rho \) (at the boundary).

These conditions are analogous to those used in the AdS sector of Einstein’s 3D gravity, but not identical [4]; the presence of the additional \( \theta^i \) term in \( u' \) will become clear soon. We begin the analysis by discussing the symmetry structure of the \( \text{SL}(2, \mathbb{R}) \) sector.

\textit{SL}(2, \mathbb{R}) \textit{sector}. Imposing the adopted gauge conditions (2.5), the form of \( \delta \Gamma_L[u] \) in (2.7a) leads to

\[
\Gamma_L = 2\kappa \int_0^{2\pi} d\phi \left[(\bar{\theta} + \xi^1 A_1)A_2 + \frac{1}{2}\bar{\varepsilon}^2(\bar{A}_2)^2\right],
\]

where \( \theta = b^{-1}\bar{\theta}b \).

To proceed, we impose two additional requirements:

\[
\hat{A}_1 = 0, \quad \hat{A}_2 = -2C.
\]

(3.4a)
They are of the same form as in the AdS sector of 3D gravity [19], but now, $C$ is an arbitrary constant. Using the residual gauge symmetry (2.6), the invariance of these requirements implies

$$\begin{align*}
\bar{\theta}^-=\xi^1a_i^-=0, \\
\bar{\theta}^+=\xi^1a_i^+=-\partial_2\xi^2, \\
C(\bar{\theta}^++\xi^1a_i^+)=\partial_2^2\xi^2.
\end{align*}$$

The last equation shows why the additional $\theta^i$ term is needed: with the usual GR$_A$ choice $\theta^i=0$ and for the standard ‘gravitational’ value $a_i^+=0$ [4], we would have a too strong restriction $\partial^2_2\xi^2=0$. Note that this is true even in GR$_A$. As a consequence of (3.4b), the integrand in (3.3) is linearized:

$$\Gamma_L[\xi]=-\int d\varphi \, \xi^2 M_L, \quad M_L:=2\kappa C \hat{A}^2. \quad \text{(3.5a)}$$

The canonical algebra can now be derived using the transformation rule

$$\delta_\eta M_L=-2(\partial_2\eta^2)M_L-\eta^2\partial_2M_L-2\kappa \partial_2^2\eta^2. \quad \text{(3.5b)}$$

Indeed, this rule implies

$$\delta_\eta \Gamma_L[\xi]=\Gamma_L[\sigma]+2\kappa \int d\varphi \, \xi^2 \partial_2^3\eta^2,$$

with $\sigma^2=\eta^2\partial_2^2-x^2\partial_2\eta^2$, and consequently,

$$\{\tilde{G}_L[\xi],\tilde{G}_L[\eta]\}=[\tilde{G}_L[\xi],\tilde{G}_L[\eta]]+2\kappa \int d\varphi \, \xi^2 \partial_2^3\eta^2. \quad \text{(3.6a)}$$

Expressed in terms of the Fourier modes

$$L'_m=\tilde{G}[\xi^2=e^{-im\varphi}]\approx-(M_L)_m,$$

the canonical algebra takes the form of a Virasoro algebra with classical central charge:

$$i\{L'_m,L'_n\}=(m-n)\kappa L'_{m+n}+4\pi \kappa m^3\delta_{m,-n} \quad \text{(3.6b)}$$

The complete theory. Going now to the $U(1)$ sector with $\hat{a}=-\xi^0\hat{A}\rho$ and imposing the additional restriction

$$\bar{a}_1=0,$$

we obtain

$$\begin{align*}
\Gamma_R=-2\kappa \int_0^{2\pi} d\varphi \left[\xi^0\partial_0\bar{A}_2+\frac{1}{2}\xi^2(\bar{A}_2)^2\right], \\
\delta_\eta \bar{A}_2=-\partial_2(\eta^2\bar{A}_2)-\bar{a}_0\partial_2\eta^2. \quad \text{(3.7)}
\end{align*}$$

Combining (3.7) with (3.5), we find that the complete surface term has the form

$$\begin{align*}
\Gamma[\xi]:&=\Gamma_L[\xi]+\Gamma_R[\xi]=-\int d\varphi \, \xi^0\mathcal{E}-\int d\varphi \, \xi^2 \mathcal{M}, \\
\mathcal{E}&=2\kappa \partial_0\bar{A}_2, \quad \mathcal{M}=M_L+\kappa (\bar{A}_2)^2. \quad \text{(3.8a)}
\end{align*}$$

As before, the transformation rules

$$\begin{align*}
\delta_\eta \mathcal{M}&=-2(\partial_2\eta^2)\mathcal{M}-\eta^2\partial_2\mathcal{M}-2\kappa \partial_2^2\eta^2-(\partial_2\eta^0)\mathcal{E}, \\
\delta_\eta \mathcal{E}&=-(\partial_2\eta^2)\mathcal{E}-\eta^2\partial_2\mathcal{E}-2\kappa \partial_2^2\partial_2\eta^0. \quad \text{(3.8b)}
\end{align*}$$
define the form of the complete PB algebra:

\[ \{ \tilde{G}[\xi], \tilde{G}[\eta] \} = \tilde{G}[\sigma] + 2 \kappa \int d\varphi \, \xi^2 \partial_2^2 \eta^2 + 2 \tilde{\kappa} \tilde{a}_0^2 \int d\varphi \, \xi^0 \partial_2^0 \eta^0, \quad (3.9) \]

where \( \sigma^\alpha = \eta^2 \partial_2^2 \xi^\alpha - \xi^2 \partial_2^2 \eta^\alpha, \alpha = 0, 2 \). Expressed in terms of the Fourier modes:

\[ K_n = \tilde{G}[\xi^0 = e^{-in\varphi}, \xi^2 = 0] = -E_n, \]
\[ L_m = \tilde{G}[\xi^0 = 0, \xi^2 = e^{-im\varphi}] = -M_m, \]

the above PB algebra takes the form of the semidirect sum Virasoro \( \oplus \) \( \text{sd}(1) \) KM:

\[ i\{L_m, L_n\} = (m - n)L_{m+n} + 4\pi \kappa m^3 \delta_{m,-n}, \]
\[ i\{L_m, K_n\} = -nK_{m+n}, \]
\[ i\{K_m, K_m\} = -4\pi \tilde{\kappa} \tilde{a}_0^2 m \delta_{m,-n}. \]

The gauge conditions (2.5a) and (2.5b) in conjunction with the additional requirements (3.4a) imply that the original set of 9 + 3 gauge potentials \( A^\mu_1 \) and \( \tilde{A}_1 \) is reduced to just two independent boundary degrees of freedom, \( \hat{A}_2(\varphi) \) and \( \tilde{A}_2(\varphi) \). These are the only modes that appear in the CS surface term (3.8a).

The basic content of our analysis is encoded in the form of the surface term (3.8a) and the PB algebra of the asymptotic generators (3.10). These results will be compared to those found in the asymptotic region of the spacelike stretched AdS gravity.

4. Spacelike stretched AdS gravity

We now turn our attention to TMG, defined by the Lagrangian

\[ L_{\text{TMG}} = 2abiR_i - \frac{\Lambda}{3} \epsilon_{ijk}b^i b^j b^k + a\mu^{-1}L_{\text{CS}}(\omega) + \lambda^i T_i, \quad (4.1) \]

where the notation is the same as in [16]: \( \omega^i \) is the Lorentz connection and \( b^i \) is the orthonormal coframe, \( R^i \) and \( T^i \) are their associated field strengths, the curvature and torsion, \( L_{\text{CS}}(\omega) = \omega^i d\omega_i + \frac{1}{6} \epsilon_{ijk} \omega^i \omega^j \omega^k \) is the CS Lagrangian for the connection, \( \lambda^i \) is the Lagrange multiplier that ensures the vanishing of torsion and \( a = 1/16\pi G \). We assume that \( G \) is positive, while \( \mu \) remains arbitrary. By construction, TMG is invariant under the local Poincaré transformations.

The variation of the action with respect to \( b^i, \omega^i \) and \( \lambda^i \) yields the gravitational field equations:

\[ 2aR_i - \Lambda\epsilon_{ijk}b^j b^k + \nabla \lambda^i = 0, \quad (4.2a) \]
\[ 2aT_i + 2a\mu^{-1}R_i + \epsilon_{imn}\lambda^m b^n = 0, \quad (4.2b) \]
\[ T_i = 0, \quad (4.2c) \]

where \( \nabla \lambda^i = d\lambda^i + \epsilon_{ijk} \omega^j \lambda^k \) is the covariant derivative of \( \lambda_i \).

In order to prepare a comparison between our CS theory and TMG, we now give a brief account of the asymptotic structure of TMG and derive a new form of the surface terms.

4.1. Spacelike stretched AdS asymptotics

The spacelike stretched black hole is a solution of TMG (appendix B), which can be constructed as a discrete quotient space of the spacelike stretched AdS vacuum. This black hole generates an interesting set of asymptotic states, the structure of which is defined, in analogy with the AdS case, by the following requirements:
(a) asymptotic configurations should include spacelike stretched black hole geometries;
(b) they should be invariant under the action of $SL(2, R) \times U(1)$, the isometry group of the
spacelike stretched AdS$_3$;
(c) asymptotic symmetries should have well-defined canonical generators.

In [16], this general approach was used to derive asymptotic properties of the fields and
gauge parameters. Based on the requirements (a) and (b), the gravitational fields $b^i_\mu$, $\omega^i_\mu$ and
$\lambda^i_\mu$ are found to have the following asymptotic form:

\[ b^i_\mu = \bar{b}^i_\mu + B^i_\mu, \quad B^i_\mu := \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_0 \end{pmatrix}, \tag{4.3a} \]

\[ \omega^i_\mu = \bar{\omega}^i_\mu + \Omega^i_\mu, \quad \Omega^i_\mu := \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_0 \end{pmatrix}, \tag{4.3b} \]

\[ \lambda^i_\mu = \bar{\lambda}^i_\mu + \Lambda^i_\mu, \quad \Lambda^i_\mu := \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_0 \end{pmatrix}, \tag{4.3c} \]

where $\mathcal{O}_n$ is a term that tends to zero as $r^{-n}$ or faster. Expansion (4.3) is an asymptotic
expansion around the spacelike stretched black hole vacuum $(\bar{b}^i_\mu, \bar{\omega}^i_\mu, \bar{\lambda}^i_\mu)$, displayed in
appendix B.

The subset of Poincaré gauge transformations that leave the asymptotic configurations
(4.3) invariant defines the asymptotic symmetry. The invariance of (4.3) restricts the gauge
parameters $\xi^\mu$ (translations) and $\theta^i$ (Lorentz rotations) to have the following form:

\[ \xi^0 = \ell T(\varphi) - \frac{4\ell^2 v}{(v^2 + 3)^2} \frac{1}{r^2} \partial_2^2 S + \mathcal{O}_2, \quad \xi^1 = -r \partial_2 S(\varphi) + \mathcal{O}_0(\varphi), \quad \xi^2 = -r \partial_2 S(\varphi) + \mathcal{O}_0(\varphi), \tag{4.4a} \]

\[ \theta^0 = -\frac{2\ell}{\sqrt{3(v^2 + 3)(v^2 - 1)}} \partial_2 T(\varphi) + \mathcal{O}_2, \quad \theta^1 = -\frac{2\ell^2}{(v^2 + 3)^2} \frac{1}{r} \partial_2 T(\varphi) + \mathcal{O}_2, \quad \theta^2 = -\frac{4\ell v}{(v^2 + 3)^2} \frac{1}{r} \partial_2^2 S(\varphi) + \mathcal{O}_2. \tag{4.4b} \]

The leading-order terms in (4.4), which are determined by just two functions, $T(\varphi)$ and
$S(\varphi)$, define the $(T, S)$ transformations; their time independence closely corresponds to the
CS boundary conditions (2.2). The sub-leading terms, those that remain after imposing
$T = S = 0$, define the residual (or pure) gauge transformations. The asymptotic symmetry is
defined by the $(T, S)$ pair, ignoring all the residual gauge parameters.

In expressions (4.3) and (4.4), some typos appearing in [16] are corrected.
4.2. Canonical PB algebra

Using the adopted asymptotic conditions, the improved canonical generator is given as [16]

\[ \tilde{G} = G + \Gamma, \]
\[ \Gamma := -\int_0^{2\pi} d\phi (\ell T E^1 + S M^1), \]

where

\[ E^1 = b_0' \left[ \frac{4a}{3} \omega_2 + \lambda_{i2} - \frac{a}{3\ell v} (2\nu^2 + 3) b_{i2} \right], \]
\[ M^1 = b_2'(2a\omega_2 + \lambda_{i2}) + \frac{a\ell}{3v} \omega_2 \omega_2. \] 

The improved generator \( \tilde{G} \) is differentiable and has a finite value. For \( T = 1 \) and \( S = 1 \), the surface term \( \Gamma \) defines the conserved canonical charges, energy and angular momentum (for a background independent approach to the conserved quantities, see [20]).

The corresponding PB algebra is determined by the transformation laws

\[ \delta_0 E^1 = -S \partial_2 E^1 - (\partial_2 S) E^1 - \frac{2a(\nu^2 + 3)}{3v} \partial_2 T, \]
\[ \delta_0 M^1 = -2(\partial_2 S) M^1 - S \partial_2 M^1 - (\ell \partial_2 T) E^1 - \frac{2a(\nu^2 + 3)}{3v} \partial_2 S. \]

Expressed in terms of the Fourier modes,

\[ K_n := \tilde{G}(T = e^{-i\nu}, S = 0) \approx -\ell E_n^1, \]
\[ L_n := \tilde{G}(T = 0, S = e^{-i\nu}) \approx -M_n^1, \]

the asymptotic canonical algebra of the spacelike stretched AdS gravity reads

\[ i[L_m, L_n] = (m - n) L_{m+n} + \frac{c_V}{12} m^3 \delta_{m, -n}, \]
\[ i[L_m, K_n] = -n K_{m+n}, \]
\[ i[K_m, K_n] = -c_K m \delta_{m, -n}, \]

where

\[ c_V = \frac{(\nu^2 + 3)\ell}{Gv(\nu^2 + 3)}, \]
\[ c_K = \frac{(\nu^2 + 3)\ell}{Gv}. \]

One should note that this algebra is of the same form as the corresponding CS algebra: Virasoro \( \oplus_{sd} u(1)_{KM}. \)

5. Boundary structure of the spacelike stretched AdS gravity

In this section, we introduce a set of specific asymptotic conditions for the spacelike stretched AdS gravity, corresponding to the gauge conditions introduced in the CS theory; then, we derive a new form of the surface terms and discuss the boundary degrees of freedom.

5.1. Specific asymptotic conditions

In section 4.1, our intention was to construct the most general set of asymptotic conditions based on the requirements (a) and (b). Here, we introduce a set of the specific (refined) asymptotic conditions, compatible with the general asymptotic structure.
We begin by noting that neither the black hole solution nor the leading-order asymptotic parameters depend on time. These properties can be naturally extended by introducing the following refined asymptotic conditions:

\[(b^i_{\mu}, \omega^i_{\mu}, \lambda^i_{\mu}) \text{ and } (\xi^i, \theta^i)\]

are time independent.\(^{(5.1)}\)

In particular, \((5.1)\) implies that pure gauge parameters are time independent.

Next, motivated again by the properties of the black hole solution (to leading order, it is represented by the black hole vacuum \((B.2)\)), we adopt the following conditions:

\[\nu \ell b^0 + \omega^0 = 0, \frac{a}{\mu \ell^2} (4\nu^2 - 3) b^0 - \lambda^0 = 0.\]

\(^{(5.2)}\)

One can verify that these conditions do not lead to any restriction on the asymptotic parameters. They can be considered as the gauge conditions that are compatible with the spacelike stretched AdS asymptotics, while canonically, conditions \((5.2)\) are associated with the first class constraints \((\pi_i, \Pi_i)\) in TMG\(^{13}\). Note the analogy between \((5.1), (5.2)\) and the CS boundary conditions \((2.2)\) or gauge conditions \((2.5a)\).

In the standard AdS gravity, the BTZ black hole satisfies the condition \(b^i/\ell + \omega^i = 0\), which is similar to the first condition in \((5.2)\). The difference stems from different asymptotic conditions in the standard and spacelike stretched AdS gravity: using the CS variables, these conditions can be expressed as \(A_+ = 0\) and \(A_0 = 0\), respectively.

\[\text{5.2. A new form of the surface terms}\]

The surface terms \(\mathcal{E}^1\) and \(\mathcal{M}^1\) can be written in a form which closely resembles the corresponding CS expressions \((3.8a)\). Indeed, conditions \((5.1)\) and the equations of motion \((C.1a), (C.2a)\) and \((C.3a)\) imply

\[\mathcal{E}^1 = -\frac{a[3(\nu^2 - 1)]^{3/2}}{3\nu \ell} \hat{B}^2_0,\]

\[\mathcal{M}^1 = \mathcal{M}^+ + \frac{12\pi \ell^2}{cK} (\mathcal{E}^1)^2,\]

\(^{(5.3a)}\)

where

\[\mathcal{M}^+ := -a\sqrt{3(\nu^2 - 1)} \left(\frac{\hat{B}^2_2}{\nu \ell} + \frac{2\sqrt{\nu^2 + 3} \hat{B}^2_2 + \frac{4}{3} \hat{\Omega}^2_2}{3 \nu} + \frac{1}{a} \hat{A}^2_2 + \frac{2\sqrt{\nu^2 + 3} \hat{\Omega}^2_0}{3 \nu} \right).\]

\(^{(5.3b)}\)

Here, the first-/second-order sub-leading terms in the asymptotic expansion of the fields are denoted by single/double hats. For instance, the relation \(B^2_0 = \mathcal{O}_1\) in \((4.3)\) is written as

\[B^2_0 = \frac{\hat{B}^2_0}{r} + \frac{\hat{B}^2_0}{r^2} + \cdots,\]

and similarly for the other field components.

\[\text{5.3. Boundary degrees of freedom}\]

We are now going to prove the following statement:

- in the spacelike stretched AdS sector of TMG\(_{\Lambda}\), there are two independent boundary degrees of freedom.
Let us first note that the only sub-leading field modes that contribute to the values of the conserved charges are those of the order $O_0$ and $O_1$. The connection $\omega^i$, and the multiplier field $\lambda^i$, can be expressed in terms of the triad by using the asymptotic expansion of the equations of motion. The only second-order triad modes that appear in these asymptotic relations are $\hat{B}^0_0$, $\hat{B}^2_0$, $\hat{B}^2_2$ and $\hat{B}^4_1$ (see appendix C). Thus, the number of boundary modes is defined by the nine first-order modes $\hat{B}^i_{\mu}$ plus the additional four second-order modes.

However, not all of these modes are independent: there are four constraints (C.1), and seven modes can be fixed by fixing the residual gauge symmetry, defined by seven residual gauge parameters in (D.1). Thus, the number of independent boundary degrees of freedom is $13 - 4 - 7 = 2$. They can be identified with $E^1$ and $\mathcal{M}^+$, the surface terms of the canonical generator, which are invariant under the residual gauge transformations, see (D.3).

The boundary degrees of freedom should not be confused with the propagating degrees of freedom. Thus, for instance, Einstein’s 3D gravity is a topological theory without propagating degrees of freedom, but its AdS sector possesses two boundary degrees of freedom.

6. Asymptotic relation between CS theory and TMG$_\Lambda$

We are now ready to establish a remarkable asymptotic relation between the $SL(2, R) \times U(1)$ CS gauge theory and TMG$_\Lambda$:

- asymptotic structures of the spacelike stretched AdS gravity and the $SL(2, R) \times U(1)$ CS gauge theory can be identified by adopting a natural asymptotic correspondence between their field variables and coupling constants.

The result holds when the gauge conditions (2.5b) have the form $a_1 = T_1, \bar{a}_1 = 0$, and for a specific value of the constant $C$ in (3.4a).

To prove the statement, we compare the asymptotic canonical algebras (3.10) and (4.8) and the corresponding surface terms (3.8) and (4.5) of the two theories, and find that these structures coincide if we adopt the following asymptotic correspondence:

$$4\pi \kappa \bar{a}_0^2 \sim \frac{c_k}{12}, \quad 4\pi \kappa \sim \frac{c_v}{12}.$$  \hspace{1cm} (6.1)

$$\mathcal{E} \sim \mathcal{E}^1, \quad \mathcal{M} \sim \mathcal{M}^1.$$  \hspace{1cm} (6.1)

Taking into account (5.3), the correspondence between the surface terms reads

$$2\kappa \bar{a}_0 \bar{A}_2 \sim \mathcal{E}^1, \quad 2\kappa C \bar{A}^+_2 \sim \mathcal{M}^+,$$  \hspace{1cm} (6.2)

or equivalently, when expressed in terms of the boundary modes:

$$\bar{\bar{A}}_2 \sim -\frac{a^3(3v^2 - 1)^{3/2}}{6\kappa \bar{a}_0 v \ell} \bar{B}^2_0,$$

$$\bar{A}^+_2 \sim -\frac{a\sqrt{3(v^2 - 1)}}{4\kappa C} \left( \frac{\hat{B}^2_2}{v \ell} + \frac{2\sqrt{v^2 + 3}}{3\ell} \hat{B}^0_2 + \frac{4\hat{\Omega}^2_2}{3} + \frac{2\sqrt{v^2 + 3}}{3\ell} \hat{\Omega}^2_0 + \frac{2\sqrt{v^2 + 3}}{3\ell} \hat{\Omega}^2_0 + \frac{4\hat{\Lambda}^2_2}{a}\right).$$

It is interesting to note that, under the adopted gauge and boundary conditions, this correspondence can also be rewritten in a covariant form (appendix E):

$$\bar{A}^i_{\mu} \sim \alpha^i_{\mu} + \frac{3v}{2(2v + \sqrt{v^2 + 3})} \left( \frac{3 + 2v \sqrt{v^2 + 3}}{3\ell v} b^i_{\mu} + \frac{1}{a} \lambda^i_{\mu} \right),$$  \hspace{1cm} (6.3a)

$$\bar{A}_{\mu} \sim \frac{v}{2\kappa \bar{a}_0} b^0_{\mu} \left( \frac{4a}{3} \omega_{\mu} + \lambda_{\mu} \right) - \frac{a}{3\ell} \frac{2v^2 + 3}{v} b_{\mu}. \hspace{1cm} (6.3b)$$
The transformation laws of $A^i_{\mu}$ and $\bar{A}^i_{\mu}$, induced by (6.3), take the expected form:

$$
\delta_0 A^i_{\mu} \sim - \nabla_{\mu} \theta^i - (\partial_{\mu} \xi^\rho) A^i_{\rho} - \xi^\rho \partial_{\rho} A^i_{\mu}, \quad (6.4a)
$$

$$
\delta_0 \bar{A}^i_{\mu} \sim - (\partial_{\mu} \xi^\rho) \bar{A}^i_{\rho} - \xi^\rho \partial_{\rho} \bar{A}^i_{\mu}. \quad (6.4b)
$$

To illustrate practical aspects of the established correspondence, we note that in thermodynamic applications, one needs an action which is both finite and differentiable [21]. These properties are ensured by the following procedure: first, if the value $I_{bh}$ of the action $I$ at the black hole configuration is divergent, we apply a suitable regularization to define $I_r$, a finite piece of $I$, and second, we construct the improved action $\tilde{I} = I_r + B$, where $B$ is a surface term that ensures the differentiability of $\tilde{I}$ under the adopted boundary conditions.

Let us now apply this procedure to the CS action (2.1). We begin by noting that in the spacelike stretched AdS sector, the asymptotic relations (6.3) imply the gauge conditions (2.5) and the relations $\partial_0 A_2 = \partial_0 \bar{A}_2 = 0$. Then, one finds that $(I_{CS})_{bh} \approx (\kappa/3) \int \varepsilon_{ijk} A^i A^j A^k \approx 0$, so that there is no need for any regularization: $(I_{CS})_r = I_{CS}$. After that, the improved action $\tilde{I}_{CS}$ is found to be of the form

$$
\tilde{I}_{CS} = I_{CS} + B_{CS}, \quad B_{CS} := -\tilde{k} a_0 \int_{\partial M} dt \ d\phi \ \bar{A}_2.
$$

In the Euclidean spacetime with the periodic time coordinate, $B_{CS}$ is finite.

When the same procedure is applied to TMG, using (4.1) and the spacelike stretched boundary conditions, we find

$$
(I_{TMG})_r = I_{TMG} + \int dt \ d\phi \ a(y^2 + 3) 2\ell^{-1} r_\infty,
$$

$$
\tilde{I}_{TMG} = (I_{TMG})_r + B_{TMG}, \quad B_{TMG} = -\frac{1}{2} \int_{\partial M} dt \ d\phi \ \varepsilon^1,
$$

where $r_\infty$ is the value of $r$ at the boundary. Then, in view of the asymptotic correspondence (6.2), the CS boundary term is seen to coincide with its gravitational counterpart:

$$
B_{CS} = B_{TMG}. \quad (6.5)
$$

and the on-shell values of the improved actions $\tilde{I}_{CS}$ and $\tilde{I}_{TMG}$ are identical. This result gives a deeper insight into the correspondence of the two theories, extending it from an asymptotic relation between fields and coupling constants, to the level of equality of the boundary terms needed to improve the regularized actions. Equality (6.5) might lead to a simplified approach to the gravitational entropy, which is, on the other hand, closely related to the question of warped AdS/CFT correspondence in TMG [10, 16, 17].

7. Concluding remarks

In this paper, we compared the asymptotic structures of the spacelike stretched AdS gravity and the $SL(2, R) \times U(1)$ CS gauge theory.

We studied the asymptotic properties of the $SL(2, R) \times U(1)$ CS gauge theory in the canonical formalism. By imposing a suitable set of the gauge and boundary conditions, we calculated two conserved charges and found that the boundary symmetry is described by the Virasoro $\mathfrak{so}_d U(1)_{KM}$ PB algebra with central charges.

This result shows a remarkable resemblance with the properties of the spacelike stretched AdS gravity. Indeed, by comparing the boundary canonical algebras and the surface terms of the improved generators in the two theories, one finds that their asymptotic structures
can be identified by adopting a natural mapping between the respective coupling constants and field variables. Thus, in spite of the fact that TMG, is not a topological theory, the asymptotic structure of its spacelike stretched AdS sector can be faithfully represented by the $SL(2, R) \times U(1)$ CS gauge theory. Note that this result holds only asymptotically, not in the bulk. It represents a natural extension of the known asymptotic correspondence between the AdS sector of TMG and another topological gauge theory—the Mielke–Baekler 3D gravity [13], or, equivalently, the $SL(2, R) \times SL(2, R)$ CS gauge theory [22].

As indicated by equality of the boundary terms needed to improve the regularized CS and TMG Euclidean actions, the asymptotic CS representation of TMG might be a useful tool in clarifying the status of the hypothesis conjectured by Anninos et al [10].

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Appendix A. The $sl(2, R)$ Lie algebra: conventions

For the basis of the fundamental matrix representation of the $sl(2, R)$ Lie algebra (real, traceless, $2 \times 2$ matrices), we choose

$T_0 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

In this basis, the components of the Cartan metric are \( \eta_{ij} = -2T_iT_j = (+, -, -) \), and the form of the Lie algebra is \([T_i, T_j] = \varepsilon_{ijk}T_k\), with \(\varepsilon_{012} = +1\). In these conventions, the gauge potential can be represented as

\[ A = A^i T_i = \frac{1}{2} \begin{pmatrix} A^1 & A^+ \\ -A^- & -A^1 \end{pmatrix}, \]

where \(A^\pm = A^0 \pm A^2\) are the light-cone components of \(A^i\).

Appendix B. The spacelike stretched black hole

The spacelike stretched black hole is a solution of TMG, which represents a discrete quotient of the spacelike stretched AdS3 vacuum. Using the notation \(\Lambda = -a/\ell^2\), \(v = \mu \ell/3\), the metric of the black hole in Schwarzschild-like coordinates is given by

\[ ds^2 = N^2 dr^2 - B^2 dr^2 - K^2 (d\varphi + N_\varphi dr)^2, \]

where

\[ N^2 = \frac{(v^2 + 3)(r - r_s)(r - r_-)}{4K^2}, \quad B^2 = \frac{4N^2K^2}{\ell^2}, \quad K^2 = \frac{r}{4} \left[ 3(v^2 - 1)r + (v^2 + 3)(r_+ + r_-) - 4v\sqrt{r_+r_-}(v^2 + 3) \right], \]

\[ N_\varphi = \frac{2vr - \sqrt{r_+r_-}(v^2 + 3)}{2K^2}. \]

The metric is defined for \(v^2 > 1\).
As shown in [16], we can use (B.1) to calculate the simple diagonal form of the triad field \( b^i \); then, the connection \( \omega^i \) is determined by the condition of vanishing torsion, and finally, the solution for the multiplier \( \lambda^i \) is found from (4.2b). The triple \( (b^i, \omega^i, \lambda^i) \) defined in this way represents the spacelike stretched black hole in the first-order formalism. The corresponding black hole vacuum \( (\bar{b}^i, \bar{\omega}^i, \bar{\lambda}^i) \) is defined by the conditions \( r_+ = r_- = 0 \):

\[
\bar{b}^\mu = \begin{pmatrix}
\frac{\nu^2 + 3}{3(\nu^2 - 1)} & 0 & 0 \\
0 & \frac{\ell \nu}{\sqrt{\nu^2 + 3}} & 0 \\
\frac{2\nu}{\sqrt{3(\nu^2 - 1)}} & 0 & \frac{\sqrt{3(\nu^2 - 1)}}{2}
\end{pmatrix}, \tag{B.2a}
\]

\[
\bar{\omega}^i = \begin{pmatrix}
-\frac{\nu}{\ell} \frac{\nu^2 + 3}{3(\nu^2 - 1)} & 0 & -\sqrt{3(\nu^2 + 3)(\nu^2 - 1)} \frac{r}{2\ell} \\
0 & \frac{\nu}{\sqrt{\nu^2 + 3}} & 0 \\
-\frac{2\nu^2}{\ell \sqrt{3(\nu^2 - 1)}} & 0 & -\nu \sqrt{3(\nu^2 - 1)} \frac{r}{2\ell}
\end{pmatrix}, \tag{B.2b}
\]

\[
\bar{\lambda}^i = \frac{2a}{\mu} \begin{pmatrix}
\frac{4\nu^2 - 3}{2\ell^2} \frac{\nu^2 + 3}{3(\nu^2 - 1)} & 0 & \frac{\nu}{\ell^2} \sqrt{3(\nu^2 + 3)(\nu^2 - 1)} r \\
0 & 3 - 2\nu^2 \frac{1}{2\ell \sqrt{\nu^2 + 3}} r & 0 \\
\frac{(4\nu^2 - 3)\nu}{\ell^2 \sqrt{3(\nu^2 - 1)}} & 0 & 3(2\nu^2 + 1) \frac{1}{4\ell^2} \sqrt{3(\nu^2 - 1)} r
\end{pmatrix}. \tag{B.2c}
\]

Appendix C. Asymptotic expansion of the equations of motion

Let us now explore the equations of motion in the asymptotic region.

We start with equation (4.2c) which, in conjunction with the specific asymptotic conditions (5.1) and (5.2), leads to four constraints on the triad modes. Two of them involve the first-order modes,

\[
\dot{B}_0^0 = -\frac{2\nu}{\sqrt{\nu^2 + 3}} \dot{B}_0^0 = 0, \tag{C.1a}
\]

\[
\dot{B}_1^1 + 2\nu \frac{\ell}{\sqrt{\nu^2 + 3}} \dot{B}_0^0 + \frac{2\ell}{\sqrt{3(\nu^2 - 1)(\nu^2 + 3)}} \dot{B}_2^2 = 0,
\]

and the remaining two contain second-order modes:

\[
\dot{B}_0^0 = -\frac{2\nu}{\sqrt{\nu^2 + 3}} \dot{B}_0^0 + \frac{1}{2} \left[ \frac{3(\nu^2 - 1)}{\nu^2 + 3} \right]^{3/2} \left( \dot{B}_0^0 \right)^2 = 0,
\]

\[
\dot{B}_1^1 + \frac{\ell}{\sqrt{\nu^2 + 3}} \dot{B}_0^0 + \frac{2\ell}{\sqrt{3(\nu^2 - 1)(\nu^2 + 3)}} \dot{B}_2^2 = -\frac{4\nu}{\sqrt{(\nu^2 + 3)^3}} \dot{B}_0^0 \dot{B}_2^2 = 0. \tag{C.1b}
\]
The remaining equations are algebraic relations containing the connection modes. For the
first-order modes, one obtains

\[
\begin{align*}
\dot{\Omega}_2 &= \frac{3(v^2 - 1)}{2\ell} \hat{B}_0^0 + \frac{\nu v^2 + 3}{\ell} \hat{B}_2^0 = 0, \\
\dot{\Omega}_2 &= \frac{3(v^2 - 1)}{2\ell} \hat{B}_0^2 + \frac{\nu}{\ell} \hat{B}_2^2 = 0, \\
\dot{\Omega}_1 &= \frac{\nu}{\ell} \hat{B}_1^1 - \sqrt{\frac{3(v^2 - 1)}{v^2 + 3}} \hat{B}_2^0 = 0, \\
2\hat{B}_1^0 + \frac{2\nu}{\sqrt{3(v^2 - 1)}} \left( \frac{\nu}{\ell} \hat{B}_0^0 - \hat{\Omega}_1^0 \right) - \sqrt{\frac{v^2 + 3}{3(v^2 - 1)}} \left( \frac{\nu}{\ell} \hat{B}_2^0 - \hat{\Omega}_2^1 \right) &= 0, \\
2\hat{B}_1^2 - \frac{\nu}{\sqrt{v^2 + 3}} \left( \frac{\nu}{\ell} \hat{B}_2^0 + \hat{\Omega}_0^2 \right) - \frac{\nu}{\ell} \hat{B}_0^2 &= 0, \\
\end{align*}
\]  

\text{(C.2a)}

and the equations containing the second-order modes are

\[
\begin{align*}
2\hat{B}_0^0 + \frac{2\nu}{\sqrt{3(v^2 - 1)}} \left( \frac{\nu}{\ell} \hat{B}_1^1 - \hat{\Omega}_1^1 \right) + \hat{B}_2^0 \left( \frac{\nu}{\ell} \hat{B}_1^1 - \hat{\Omega}_1^1 \right) &= 0, \\
\sqrt{\frac{3(v^2 + 3)(v^2 - 1)}{2\ell}} \hat{B}_1^1 + \hat{B}_2^2 - \frac{\ell}{\sqrt{v^2 + 3}} \left( \frac{\nu}{\ell} \hat{B}_0^0 + \hat{\Omega}_0^2 \right) - \hat{B}_1^0 \hat{\Omega}_2^2 &= 0, \\
- \frac{3(v^2 - 1)}{2} \left( \frac{\nu}{\ell} \hat{B}_1^1 - \hat{\Omega}_1^1 \right) + \frac{\ell}{\sqrt{v^2 + 3}} \left( \frac{\nu}{\ell} \hat{B}_2^0 + \hat{\Omega}_0^2 \right) &= 0, \\
\end{align*}
\]

\text{(C.2b)}

Similarly, starting from equation (4.2b), we find the following independent algebraic
relations for the first-order multiplier modes:

\[
\begin{align*}
\dot{A}_2^0 &= -2a \left[ \frac{3(v^2 - 1)(5v^2 + 3)}{4v} \hat{B}_0^0 + 2\nu (v^2 + 3) \hat{B}_2^0 \right] = 0, \\
\dot{A}_2^2 &= -2a \left[ \frac{3(v^2 - 1)\sqrt{v^2 + 3} \hat{B}_0^0 + 3(2v^2 + 1) \hat{B}_2^0}{2} \right] = 0, \\
\dot{A}_1^2 &= \frac{a}{3v} (2v^2 - 3) \hat{B}_1^2 = 0, \\
-\frac{2(v^2 - 1)}{\ell} \hat{B}_1^0 + \frac{(2v^2 - 3)}{3v} \hat{B}_1^2 + \frac{1}{a} \hat{A}_2^1 = 0, \\
\frac{2v^2 - 3}{3v} \left( \sqrt{v^2 + 3} \hat{B}_2^1 - 2v \hat{B}_0^0 \right) + \frac{\nu v^2 + 3}{a} \hat{A}_1^2 - \frac{2\nu}{a} \hat{A}_1^0 = 0, \\
-\frac{3(v^2 - 1)}{6} \left[ \frac{3(2v^2 + 1)}{2v} \hat{B}_0^0 - \hat{\Omega}_1^0 - \frac{\nu v^2 + 3}{\nu} \left( \frac{2\nu}{\ell} \hat{B}_1^1 - \hat{\Omega}_2^1 \right) \right] \\
+ \frac{\ell}{3v} (\hat{\Omega}_1^1 + \hat{\Omega}_2^1) = -\frac{3(v^2 - 1)}{4a} \hat{A}_1^0 = 0, \\
\end{align*}
\]

\text{(C.3a)}
and similarly for the second-order modes:

\[
\hat{A}_1 + \frac{a(2v^2 - 3)}{3v \ell} \hat{B}_1 = 0,
\]

\[
\frac{(2v^2 + 1)\sqrt{3(v^2 - 1)}}{4v \ell} \hat{B}^1_1 + \frac{(3 - 2v^2)}{6v\sqrt{v^2 + 3}} \hat{B}^2_2 = \frac{\sqrt{3(v^2 - 1)}}{6} \Omega_1^a
\]

\[
- \frac{\ell}{3v} \hat{\Omega}_2^0 + \frac{\ell}{3\sqrt{v^2 + 3}} \hat{\Omega}_2^2 + \frac{\sqrt{3(v^2 - 1)}}{4a} \hat{A}_1 + \frac{\ell}{2a\sqrt{v^2 + 3}} \hat{A}_2^2
\]

\[
+ \frac{\ell}{3v} \hat{\Omega}_1^1 \hat{\Omega}_2^2 + \frac{1}{2a} \hat{B}^1_2 \hat{A}_1^1 + \frac{1}{2a} \hat{B}_1^1 \hat{A}_2^2 = 0,
\]

\[
- \frac{2\sqrt{3(v^2 + 3)(v^2 - 1)}}{3\ell} \hat{B}^1 \hat{B}^2 - \frac{\ell}{3v} \hat{\Omega}_2^2 - \frac{\ell}{a\sqrt{v^2 + 3}} \hat{A}_2^2 + \frac{2v^2 - 3}{3\sqrt{v^2 + 3}} \hat{B}^0_2
\]

\[
- \frac{\sqrt{3(v^2 - 1)}}{6} \left( \hat{\Omega}_1^a \hat{\Omega}_2^0 - \frac{\sqrt{v^2 + 3}}{v} \hat{\Omega}_2^1 \right) - \frac{1}{a} \hat{B}^1_1 \hat{A}^2_2 = 0. \tag{C.3b}
\]

The remaining equations (4.2a) do not lead to any new relations. Thus, we see that \(\hat{\Omega}^a_\mu, \hat{\Omega}^b_\mu\) and \(\hat{A}^a_\mu, \hat{A}^b_\mu\) can be expressed in terms of \(\hat{B}^a_\mu, \hat{B}^b_\mu\).

### Appendix D. Residual gauge transformations

In this appendix, we calculate the action of the residual gauge transformations on the triad modes. These transformations are defined by (4.4) with \(T = S = 0\) and are denoted by \(\hat{\delta}_0\).

For the first-order triad modes we have

\[
\hat{\delta}_0 \hat{B}^0_0 = \hat{\delta}_0 \hat{B}^2_0 = 0,
\]

\[
\hat{\delta}_0 \hat{B}^1_0 = - \frac{\ell}{\sqrt{v^2 + 3}} \hat{\theta}^2 + 2 \sqrt{\frac{v^2 + 3}{3(v^2 - 1)}} \hat{\xi}^0,
\]

\[
\hat{\delta}_0 \hat{B}^2_0 = \frac{\sqrt{3(v^2 - 1)}}{2} \hat{\theta}^1,
\]

\[
\hat{\delta}_0 \hat{B}^1_0 = \frac{2v}{\sqrt{3(v^2 - 1)}} \hat{\theta}^0 - \sqrt{\frac{v^2 + 3}{3(v^2 - 1)}} \hat{\theta}^2,
\]

\[
\hat{\delta}_0 \hat{B}^1_2 = \frac{\ell}{\sqrt{v^2 + 3}} \hat{\xi}^1,
\]

\[
\hat{\delta}_0 \hat{B}^2_1 = \frac{\sqrt{3(v^2 - 1)}}{2} \hat{\theta}^0 - \frac{\ell}{\sqrt{v^2 + 3}} \hat{\theta}^2 + \frac{4v}{\sqrt{3(v^2 - 1)}} \hat{\xi}^0 + \frac{3}{2} \sqrt{3(v^2 - 1)} \hat{\xi}^2,
\]

\[
\hat{\delta}_0 \hat{B}^2_2 = - \frac{\sqrt{3(v^2 - 1)}}{2} \hat{\xi}^1.
\]

Hence, there is only one component \(\hat{B}^2_0\) (or equivalently \(\hat{B}^0_0\), because of (C.1a)) which remains invariant under the residual gauge transformations.
For the second-order triad modes (with non-zero vacuum values), the transformation laws read
\[ \hat{\delta}_0 \hat{\theta}_0^0 = \frac{2\nu}{\sqrt{3(\nu^2 - 1)}} \hat{\theta}_1 + \xi^1 \hat{B}_0^0, \]
\[ \hat{\delta}_0 \hat{\theta}_2^0 = \frac{\sqrt{\nu^2 + 3}}{3(\nu^2 - 1)} \hat{\theta}_1 + \xi^1 \hat{B}_0^2, \]
\[ \hat{\delta}_0 \hat{\theta}_0^1 = \frac{2\ell}{\sqrt{\nu^2 + 3}} \xi^1 + 2\xi^1 \hat{B}_0^1, \]
\[ \hat{\delta}_0 \hat{\theta}_2^2 = -\frac{\sqrt{3(\nu^2 - 1)}}{2} \xi^1. \]

In a similar way, we can find the residual gauge transformations for the connection and the multiplier modes. The modes that appear in the expressions for the asymptotic charges (5.3) transform in the following manner:
\[ \hat{\delta}_0 \hat{\Lambda}_2^2 = -\frac{\sqrt{3(\nu^2 - 1)(\nu^2 + 3)}}{2\ell} \hat{\Lambda}_2^2, \]
\[ \hat{\delta}_0 \hat{\Lambda}_0^2 = -\frac{\nu\sqrt{3(\nu^2 - 1)}}{2\ell} \hat{\Lambda}_0^2 + \frac{\sqrt{3(\nu^2 + 3)(\nu^2 - 1)}}{2\ell} \xi^1, \]
\[ \hat{\delta}_0 \hat{\Lambda}_2^2 = \frac{2\nu\sqrt{3(\nu^2 - 1)(\nu^2 + 3)}}{3\ell} \hat{\Lambda}_0^2 - \frac{a(2\nu^2 + 1)\sqrt{3(\nu^2 - 1)}}{2\ell} \xi^1. \]

Using these results, one can verify that the asymptotic charges are invariant under the residual gauge transformations:
\[ \hat{\delta}_0 \hat{E}_1^1 = 0, \quad \hat{\delta}_0 \hat{M}_1^1 = 0. \]

Indeed, the invariance of \( \hat{E}_1^1 \) follows from \( \hat{\delta}_0 \hat{\theta}_0^2 = 0 \), while the transformation laws for \( \hat{\theta}_0^0 \) and \( \hat{\theta}_0^2 \) in (D.1), together with the relations (D.2), imply \( \hat{\delta}_0 \hat{M}_2^2 = 0 \); hence \( \hat{\delta}_0 \hat{M}_1^1 = 0 \).

Appendix E. Derivation of the asymptotic relations (6.3)

In this appendix, we derive the asymptotic formulas (6.3), relating the field variables of the CS theory to those of the spacelike stretched AdS gravity, in the asymptotic region.

\( SL(2, R) \) sector

Let us first consider the \( SL(2, R) \) sector of the theory. Radial coordinates in the CS theory and TMG, \( \rho \) and \( r \) respectively, are not identical. They are connected by \( \ell e^\rho / \ell \sim r \). Hence, for \( a_1 = \ell_1 \), we have
\[ b = e^{\rho_1 / \ell_1} = \begin{pmatrix} \sqrt{\ell} / \ell & 0 \\ 0 & \sqrt{\ell} / \ell \end{pmatrix}. \]

Using the gravitational radial coordinate \( r \) also in the CS theory, we find
\[ A^r_i = \frac{\delta_i^r}{r}, \quad \hat{A}^r_+ = \frac{r}{\ell} A^r_+, \quad \hat{A}^r_- = \frac{\ell}{r} A^r_-, \]
and consequently
\[ A^r_+ \sim -\frac{a\ell\sqrt{3(\nu^2 - 1)}}{4\kappa C} \left( \frac{b_0^2}{\nu \ell} + \frac{2\sqrt{\nu^2 + 3}}{3\ell} \hat{b}_0^2 + \frac{4}{3} \omega_2^2 + \frac{2\sqrt{\nu^2 + 3}}{3\nu} \omega_0^2 + \frac{\lambda^2}{a} \right). \]
Next, we note that \((C_{1b}), (C_{2b})\) and \((C_{3b})\) imply
\[
A_{2}^+ \sim -\frac{a\ell\sqrt{3(v^2 - 1)}}{4\kappa} C \left[ \left( \frac{1}{v\ell} + \frac{2\sqrt{v^2 + 3}}{3\ell} \right) b^+_2 + \left( \frac{4}{3} + \frac{2\sqrt{v^2 + 3}}{3v} \right) \omega^+_2 + \frac{\lambda^+_2}{a} \right].
\]
This result motivates us to assume the following general correspondence:
\[
A_{\mu} \sim -\frac{a\ell\sqrt{3(v^2 - 1)}}{4\kappa} C \left[ \left( \frac{1}{v\ell} + \frac{2\sqrt{v^2 + 3}}{3\ell} \right) b_{\mu 2} + \left( \frac{4}{3} + \frac{2\sqrt{v^2 + 3}}{3v} \right) \omega_{\mu 2} + \frac{\lambda_{\mu 2}}{a} \right],
\]
where \(4\kappa = cV/12\pi\). To prove this assumption, we examine its validity for all values of the indices, using the adopted gauge and asymptotic conditions.

For \(\mu = 0\), \(A_{0}^+\) vanishes as a consequence of (2.5a), while (5.2) implies that the rhs of (E.2) also vanishes.

For \(\mu = 2\) and \(i = 1\), the first additional requirement in (3.4a) yields \(A_{12}^+ = \hat{A}_{2}^1 = 0\), while (4.3) implies that the rhs of (E.2) \(\sim O_2\). For \(\mu = 2\) and \(i = -\), we use (4.3) and the second requirement in (3.4a), which imply that (E.2) is satisfied for
\[
C = \mp \frac{(v^2 + 3)\sqrt{3(v^2 - 1)}}{2(2\nu + \sqrt{v^2 + 3})}.
\]
Choosing the negative value of \(C\), (E.2) takes the form (6.3a).

Finally, for \(\mu = 1\), conditions (4.3) imply that (6.3a) is identically satisfied.

The transformation law (6.4a) of \(A'_{\mu}\), induced by relation (6.3a), would not have been correct if we had chosen the plus sign for \(C\).

\(U(1)\) sector

By using (4.6a), the asymptotic expression for \(\hat{A}_{2}\) can be written as
\[
\hat{A}_{2} \sim \frac{\ell b_{0}^j}{2k\hat{a}_{0}} \left( \frac{4a}{3} \omega_{2}^j + \frac{a}{3} \frac{2v^2 + 3}{v} b_{2}^j \right).
\]
This relation can be consistently generalized to (6.3b). Indeed, for \(\mu = 1\), the rhs of (6.3b) \(\sim O_2\) as a consequence of (4.3), in agreement with the gauge condition \(\hat{A}_1 = 0\). Similarly, using the refined asymptotic conditions (5.2), we find the expected result:
\[
\hat{A}_0 \sim -\frac{a\ell(v^2 + 3)}{3v\kappa\hat{a}_0} g_{00} = \bar{a}_0.
\]

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18
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