Testing the standard model with $D_{(s)} \rightarrow K_1(\rightarrow K\pi\pi)\gamma$ decays

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The photon polarization in $D_{(s)} \rightarrow K_1(\rightarrow K\pi\pi)\gamma$ decays can be extracted from an up-down asymmetry in the $K\pi\pi$ system, along the lines of the method known to $B \rightarrow K_1(\rightarrow K\pi\pi)\gamma$ decays. Charm physics is advantageous as partner decays exist: $D^+ \rightarrow K_1^+(\rightarrow K\pi\pi)\gamma$, which is standard model-like, and $D_s \rightarrow K_1^+(\rightarrow K\pi\pi)\gamma$, which is sensitive to physics beyond the standard model in $|\Delta c| = |\Delta u| = 1$ transitions. The standard model predicts their photon polarizations to be equal up to U-spin breaking corrections, while new physics in the dipole operators can split them apart at order one level. We estimate the proportionality factor in the asymmetry multiplying the polarization parameter from axial vectors $K_1(1270)$ and $K_1(1400)$ to be sizable, up to the few $\mathcal{O}(10\%)$ range. The actual value of the hadronic factor matters for the experimental sensitivity, but is not needed as an input to perform the null test.

I. INTRODUCTION

Charm decay amplitudes are notoriously challenging due to an often overwhelming resonance contribution in addition to poor convergence of the heavy quark expansion. Yet, rare charm decays are of particular importance as they are sensitive to flavor and CP violation in the up-sector, complementary to $K$- and $B$-physics. While the number of radiative and semileptonic $|\Delta c| = |\Delta u| = 1$ modes within reach of the flavor facilities BaBar, Belle, LHCb, BESIII, and Belle II is plenty, it needs dedicated efforts to get sufficient control over hadronic uncertainties to be able to test the standard model (SM). A useful strategy known as well to the presently much more advanced $B$-physics program is to custom-built observables "null tests", exploiting approximate symmetries of the SM, such as lepton universality, CP in $b \rightarrow s$ and $c \rightarrow u$ transitions, or $SU(3)_F$. This allows to bypass a precise, first-principle computation of hadronic matrix elements which presently may not exist.

In this work we provide a detailed study of the up-down asymmetry $A_{UD}$ in the angular
distributions of $D^+ \to K_1^+(\to K\pi\pi)\gamma$ and $D_s \to K_1^+(\to K\pi\pi)\gamma$ decays, as a means to test the SM. Originally proposed for $B$-decays [1] [2], the method is advantageous in charm as one does not have to rely on prior knowledge of the $K\pi\pi$ spectrum and theory predictions of the photon polarization. Instead, one can use the fact that the spectrum is universal and the photon polarizations of $D^+$ and $D_s$ decays in the SM are identical in the U-spin limit [3].

Both $D_{(s)} \to K_1^+\gamma$ decays are color-allowed, and are induced by $W$-exchange "weak annihilation" (WA), which is doubly Cabibbo-suppressed and singly Cabibbo-suppressed in $D^+$ and $D_s$ decays, respectively. Thus, the ratio of their branching fractions $B(D^+ \to K_1^+\gamma)/B(D_s \to K_1^+\gamma) \approx |V_{cd}/V_{cs}|^2(\tau_{D^+}/\tau_{D_s})$ is about 0.1, taking into account the different CKM elements $V_{ij}$ and life times $\tau_{D_{(s)}}$ [4]. While the $D^+$ decay is SM-like, the $D_s$ decay is a flavor changing neutral current (FCNC) process and is sensitive to physics beyond the SM (BSM) in photonic dipole operators, which can alter the polarization. The photon dipole contributions in the SM are negligible due to the Glashow-Iliopoulos-Maiani (GIM) mechanism. The photon polarization in the SM in $c \to u\gamma$ is predominantly left-handed, however, in the $D$-meson decays sizable hadronic corrections are expected [5] [6] [7]. In the proposal discussed in this work the polarization is extracted from the SM-like decay $D^+ \to K_1^+\gamma$. We test the SM by comparison to the photon polarization in $D_s \to K_1^+\gamma$ decays. Methods to look for new physics (NP) with the photon polarization in $c \to u\gamma$ transitions have been studied recently in [3] [8].

The plan of the paper is as follows: General features of the decays $D^+ \to K_1^+\gamma$ and $D_s \to K_1^+\gamma$ are discussed in Sec. II, including angular distributions for an axial-vector $K_1^+$ decaying to $K\pi\pi$. Predictions in the framework of QCD factorization [9] [10] are given, which we use to estimate the NP reach. In Sec. III we analyze $K_1^+ \to K^+\pi^+\pi^-$ and $K_1^+ \to K^0\pi^+\pi^0$ decay chains. Phenomenological profiles of the up-down asymmetry are worked out in Sec. IV. In Sec. V we conclude. Auxiliary information is given in three appendices.

II. THE DECAYS $D^+ \to K_1^+\gamma$ AND $D_s \to K_1^+\gamma$

In Sec. II.A we give the $D_{(s)} \to K_1(\to K\pi\pi)\gamma$ angular distribution that allows to probe the photon polarizations and perform the null test. In Sec. II.B we discuss dominant SM amplitudes and estimate the $D_{(s)} \to K_1(1270)\gamma$ and $D_{(s)} \to K_1(1400)\gamma$ branching ratios. The BSM reach is investigated in Sec. II.C.
A. $D_{(s)} \rightarrow K_{1}(\rightarrow K\pi\pi)\gamma$ angular distribution

The $D_{(s)} \rightarrow K_{1}\gamma$ decay rate, where $K_{1}$ is an axial-vector meson, can be written as \cite{11}

$$\Gamma^{D_{(s)}} = \frac{\alpha_{e}G_{F}^{2}m_{D_{(s)}}^{3}}{32\pi^{4}} \left(1 - \frac{m_{K_{1}}^{2}}{m_{D_{(s)}}^{2}}\right)^{3} \left(|A_{L}^{D_{(s)}}|^{2} + |A_{R}^{D_{(s)}}|^{2}\right),$$

(1)

where $L, R$ refers to the left-handed, right-handed polarization state, respectively, of the photon. Here, $G_{F}$ denotes Fermi’s constant and $\alpha_{e}$ is the fine structure constant. $A_{L,R}^{D_{(s)}}$ denote the $D_{(s)} \rightarrow K_{1}\gamma$ decay amplitudes.

The polarization parameter $\lambda_{\gamma}^{D_{(s)}}$ is defined as

$$\lambda_{\gamma}^{D_{(s)}} = \frac{1 - r_{D_{(s)}}^{2}}{1 + r_{D_{(s)}}^{2}}, \quad r_{D_{(s)}} = \left|\frac{A_{R}^{D_{(s)}}}{A_{L}^{D_{(s)}}}\right|,$$

(2)

and can be extracted from the angular distribution in $D_{(s)} \rightarrow K_{1}(\rightarrow K\pi\pi)\gamma$ decays

$$\frac{d^{4}\Gamma^{D_{(s)}}}{dsd\sin\theta d\cos\theta} \propto \left\{ |\mathcal{J}|^{2}(1 + \cos^{2}\theta) + \lambda_{\gamma}^{D_{(s)}}2\Im\left[\bar{n} \cdot (\vec{J} \times \vec{J}^{\dagger})\right] \cos\theta \right\} \text{PS}^{D_{(s)}},$$

(3)

with the phase space factor

$$\text{PS}^{D_{(s)}} = \frac{1 - s/m_{D_{(s)}}^{2}}{256(2\pi)^{5}m_{D_{(s)}} s}.$$  

(4)

Here, $s$ denotes the $K\pi\pi$ invariant mass squared, needed for finite width effects, $\theta$ is the angle between the normal $\bar{n} = (\bar{p}_{1} \times \bar{p}_{2})/|\bar{p}_{1} \times \bar{p}_{2}|$ and the direction opposite to the photon momentum in the rest frame of the $K_{1}$, and $s_{ij} = (p_{i} + p_{j})^{2}$ with four-momenta $p_{i}$ of the final pseudo-scalars with assignments specified in \cite{18}. Note, $p_{3}$ refers to the $K$’s momentum. Furthermore, $\mathcal{J}$ is a helicity amplitude defined by the decay amplitude $A(K_{1} \rightarrow K\pi\pi) \propto \varepsilon^{\mu}\mathcal{J}_{\mu}$ with a polarization vector $\varepsilon$ of the $K_{1}$, see Sec. \[III\] for details. $\vec{J}$ are the spacial components of the four vector $\mathcal{J}$. $\mathcal{J}$ is a feature of the resonance decay and as such it is universal for $D^{+}$ and $D_{s}$ decays.

From (3) one can define an integrated up-down asymmetry which is proportional to the polarization parameter,

$$\mathcal{A}_{UD}^{D_{(s)}} = \left( \int_{0}^{1} \frac{d^{2}\Gamma}{dsd\cos\theta} d\cos\theta - \int_{-1}^{0} \frac{d^{2}\Gamma}{dsd\cos\theta} d\cos\theta \right)/ \left( \int_{-1}^{1} \frac{d^{2}\Gamma}{dsd\cos\theta} d\cos\theta \right)$$

$$= \frac{3}{4} \left\langle \Im\left[\bar{n} \cdot (\vec{J} \times \vec{J}^{\dagger})\right] |\kappa| \right\rangle \lambda_{\gamma}^{D_{(s)}},$$

(5)

where $\kappa = \text{sgn}[s_{13} - s_{23}]$ for $K_{1}^{+} \rightarrow K^{0}\pi^{+}\pi^{0}$ and $\kappa = 1$ for $K_{1}^{+} \rightarrow K^{+}\pi^{+}\pi^{-}$ . The $\langle \ldots \rangle$-brackets denote integration over $s_{13}$ and $s_{23}$. The reason for introducing $\kappa$ is explained in Sec [III] The
up-down asymmetry is maximal for maximally polarized photons, purely left-handed, \( \lambda_{\gamma}^{D(s)} = -1 \), or purely right-handed ones, \( \lambda_{\gamma}^{D(s)} = +1 \).

It is clear from Eqs. (3) and (5) that the sensitivity to the photon polarization parameter \( \lambda_{\gamma}^{D(s)} \) depends on \( \Im[\mathbf{n} \cdot (\mathbf{J} \times \mathbf{J}^\ast)] \). If this factor is zero, or too small, we have no access to \( \lambda_{\gamma}^{D(s)} \). As the \( \mathbf{J} \)-amplitudes are the same for \( D^+ \) and \( D_s \), the factor drops out from the ratio

\[
\frac{A_{UD}^{(D^+)}}{A_{UD}^{(D_s)}} = \frac{\lambda_{\gamma}^{D^+}}{\lambda_{\gamma}^{D_s}} = \frac{1 - r_{D^+}^2}{1 + r_{D^+}^2} \frac{1 + r_{D_s}^2}{1 - r_{D_s}^2}.
\]

In the SM, this ratio equals one in the U-spin limit. Corrections are discussed in Sec. III B.

In general, there is more than one \( K_1 \) resonance contributing to \( K \pi \pi \), such as \( K_1(1270) \) and \( K_1(1400) \). Note, the phase space suppression for the \( K_1(1400) \)-family and higher with respect to the \( K_1(1270) \) is stronger in charm than in \( B \)-decays. Therefore, a single- or double- resonance ansatz with the \( K_1(1270) \) or \( K_1(1400) \) is in better shape than in the corresponding \( B \to K_1(\to K \pi \pi)\gamma \) decays. In the presence of more than one overlapping \( K_1 \) resonance, beyond the zero-width approximation, the relation between the polarization and the up-down asymmetry gets more complicated than (5). The reason is that, ultimately, \( r_{D(s)} \) and the polarization are different for \( K_1(1270) \) and \( K_1(1400) \), that is, they vary with \( s \), an effect that can be controlled by cuts. The general formula can be seen in Appendix C. What stays intact, however, is the SM prediction, \( \left(A_{UD}^{(D^+)}/A_{UD}^{(D_s)}\right)_{\text{SM}} = 1 \) up to U-spin breaking.

### B. SM

Rare \( c \to u \gamma \) processes can be described by the effective Hamiltonian \ref{eq:2} ,

\[
\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} \left[ \sum_{q=d,s} V_{cq}^* V_{us} \sum_{i=1}^6 C_i O_i^q + \sum_{i=3}^8 C_i O_i + \sum_{i=7}^{10} (C_i O_i + C_i^\prime O_i') \right],
\]

where the operators relevant to this work are defined as follows

\[
O_1^{q=d,s} = (\bar{\pi}_L \gamma_\mu T^a q_L)(\bar{\eta}_L \gamma^\mu T^a c_L), \quad O_2^{q=d,s} = (\bar{\pi}_L \gamma_\mu q_L)(\bar{\eta}_L \gamma^\mu c_L),
\]

\[
O_7 = \frac{e}{16\pi^2} m_c \bar{u}_L \sigma^{\mu\nu} c_R F_{\mu\nu}, \quad O_7' = \frac{e}{16\pi^2} m_c \bar{u}_R \sigma^{\mu\nu} c_L F_{\mu\nu},
\]

with chiral left (right) projectors \( L(R) \), the field strength tensor of the photon, \( F_{\mu\nu} \), and the generators of \( SU(3)_c \), \( T^a \), \( a = 1, 2, 3 \). Contributions to \( D(\to K_1\gamma) \) decays are illustrated in Fig. \ref{fig:1}.

In the SM both four quark operators \( O_{1,2} \) are induced at tree level, and acquire order one coefficients at the charm quark mass \( m_c \). On the other hand, the SM contributions to the dipole operators \( O_7^{q} \) are strongly GIM-suppressed, \( C_7^{\text{eff}} \in [-1.51 - 5.51i, -0.88 - 3.25i] \times 10^{-3} \) at two loop.
Figure 1: Weak annihilation (left) and photon dipole (right) contributions to $D_{(s)} \to K_1 \gamma$ decays. In the weak annihilation diagram the crosses indicate where the photon can be attached.

level $[11]$, and $C_7' \sim m_u/m_c \simeq 0$. The $D^+ \to K_1^+ \gamma$ and $D_s \to K_1^+ \gamma$ decays are therefore expected to be dominated by the four quark operators.

We employ QCD factorization methods $[10]$ to estimate the branching ratios and the BSM sensitivity. The leading SM contribution is shown in the diagram to the left in Fig. 1 with the radiation of the photon from the light quark of the $D_{(s)}$ meson. The other three WA diagrams are suppressed by $\Lambda_{QCD}/m_c$ and are neglected. The corresponding WA amplitudes for $D \to V \gamma$ have been computed in Ref. $[11]$. We obtain

$$A_{\ell,SM}^{D} = -\frac{2\pi^2 Q_d f_{D} f_{K_1} m_{K_1}}{m_D \lambda_D} V_{cd}^* V_{us} C_2 \frac{m_D^2}{m_D^2 - m_{K_1}^2},$$

$$A_{\ell,SM}^{D_s} = -\frac{2\pi^2 Q_d f_{D_s} f_{K_1} m_{K_1}}{m_{D_s} \lambda_{D_s}} V_{cs}^* V_{us} C_2 \frac{m_{D_s}^2}{m_{D_s}^2 - m_{K_1}^2},$$

where $Q_d = -1/3$. We also kept explicitly, i.e., did not expand in $1/m_D$, the factors that correct for the kinematic factors in $\Gamma^{D_{(s)}}$, see $[1]$, corresponding to the matrix elements of dipole operators. Using the range $C_2 \in [1.06, 1.14]$ $[11]$ we find

$$\mathcal{B}(D^+ \to K_1^+ (1270) \gamma) = [(1.3 \pm 0.3), (1.5 \pm 0.4)] \times 10^{-5} \left( \frac{0.1 \text{ GeV}}{\lambda_D} \right)^2,$$

$$\mathcal{B}(D^+ \to K_1^+ (1400) \gamma) = [(1.4 \pm 0.6), (1.6 \pm 0.7)] \times 10^{-5} \left( \frac{0.1 \text{ GeV}}{\lambda_D} \right)^2,$$

$$\mathcal{B}(D_s \to K_1^+ (1270) \gamma) = [(1.9 \pm 0.4), (2.2 \pm 0.5)] \times 10^{-4} \left( \frac{0.1 \text{ GeV}}{\lambda_{D_s}} \right)^2,$$

$$\mathcal{B}(D_s \to K_1^+ (1400) \gamma) = [(2.0 \pm 0.9), (2.4 \pm 1.0)] \times 10^{-4} \left( \frac{0.1 \text{ GeV}}{\lambda_{D_s}} \right)^2,$$

where the first (second) value corresponds to the lower (upper) end of the range for the Wilson coefficient $C_2$. In each case, parametric uncertainties from the $K_1$ decay constants $[A4]$, $D_{(s)}$ decay

$^1$ There is a minus sign for axial vectors relative to vector mesons from the definition of the decay constant.
constants from lattice-QCD \( f_D = (212.15 \pm 1.45) \) MeV and \( f_{D_s} = (248.83 \pm 1.27) \) MeV \cite{13}, masses, life times \cite{14} and CKM elements \cite{14} are taken into account and added in quadrature. The parameter \( \lambda_{D(s)} \sim \Lambda_{QCD} \) is poorly known, and constitutes a major uncertainty to the SM predictions \cite{10}. Data on \( D \rightarrow V\gamma \) branching ratios suggest a rather low value for \( \lambda_D \) \cite{11}. We use 0.1 GeV as benchmark value for both \( D \) and \( D_s \) mesons.

Despite its V-A structure in the SM contributions to right-handed photons are expected, which we denote by \( A_{D(s)}^\text{RSM} \). One possible mechanism responsible for \( \lambda_D \neq -1 \) is a quark loop with an \( \mathcal{O}_{1,2} \) insertion and the photon and a soft gluon attached \cite{15}, at least perturbatively also subject to GIM-suppression \cite{11}. Here we do not need to attempt an estimate of such effects as we take the SM fraction of right- to left-handed photons from a measurement of \( A_{UD}^+ \) in \( D^+ \rightarrow K^+_1\gamma \) decays, which has no FCNC-contribution. (We neglect BSM effects in four quark operators.)

\( U \)-spin breaking between \( D \) and \( D_s \) meson decays can split the photon polarizations in the SM. While obvious sources such as phase space and CKM factors can be taken into account in a straight-forward manner, there are further effects induced by hadronic physics. Examples for parametric input are the decay constants, and \( \lambda_{D(s)} \), as in \cite{9}. The former has known \( U \)-spin splitting of \( \sim 0.15 \) \cite{13}, and for the latter, as not much is known, we assume that the spectator quark flavor does not matter beyond that. A measurement of \( D_s \rightarrow \rho^+\gamma \), which is a Cabibbo and color-allowed SM-like mode with branching ratios of order \( 10^{-3} \) \cite{11} can put this to a test.

Nominal \( U \)-spin breaking in charm is \( \mathcal{O}(0.2-0.3) \), e.g. \cite{10}, however, the situation for the photon polarization is favorable, as only the residual breaking on the ratio of left-handed to right-handed amplitude is relevant for the null test. In the BSM study we work with \( U \)-spin breaking between \( r_{D^+} \) and \( r_{D_s} \) within \( \pm 20\% \).

\[ C_7, C'_7 \lesssim 0.5, \]  
(11)

obtained from \( D \rightarrow \rho^0\gamma \) decays \cite{11,19}, and consistent with limits from \( D \rightarrow \pi^+\mu\mu \) decays \cite{12}.

The corresponding NP contributions to \( D_s \rightarrow K^+_1\gamma \) decays are given as

\[ A_{LNP}^{D_s} = m_c C_7 T_{K_1}, \quad A_{RNP}^{D_s} = m_c C'_7 T_{K_1}, \]
(12)

where \( T_{K_1} = T_{1,0}^{D_s \rightarrow K_1} \) is the form factor for the \( D_s \rightarrow K_1 \) transition, defined in Appendix A.
Figure 2: BSM reach of $\lambda^D_\gamma$ for given $\lambda^D_\gamma$ for NP in $C'_7$ (with $C_7 = 0$, green curves) and NP in $C_7$ (with $C'_7 = 0$ red curves), within (11) for the $K_1(1270)$, central values of input, $f_{K_1} = 170$ MeV, $T^{K_1} = 0.8$ and for $\lambda_{D(\gamma)} = 0.1$ GeV. The black dashed line denotes the SM in the flavor limit, the gray shaded area illustrates $\pm 20\%$ U-spin breaking between $r_{D^+}$ and $r_{D^s}$.

From radiative $B$-decay data [20]

$$B(B \to K^{0\gamma}(892)\gamma) = (41.7 \pm 1.2) \times 10^{-6},$$

(13)

$$B(B^+ \to K_1^+(1270)\gamma) = (43.8^{+7.1}_{-6.3}) \times 10^{-6},$$

(14)

$$B(B^+ \to K_1^+(1400)\gamma) = (9.7^{+5.4}_{-3.8}) \times 10^{-6},$$

(15)

one infers that $T_{1}^{B \to K_1(1400)} / T_{1}^{B \to K_1(1270)} \simeq 0.5$ and $T_{1}^{B \to K^0(1270)} / T_{1}^{B \to K^* (892)} \simeq 1.1$. Using $T_{1}^{D_s \to K^*(892)} \simeq 0.7$ from a compilation in [11] points to $T^{K_1(1270)} \simeq 0.8$ and $T^{K_1(1400)} \simeq 0.4$. We use $T^{K_1(1270)} = 0.8$ and $m_c = 1.27$ GeV to estimate the BSM reach.

The SM plus NP decay amplitudes read

$$A_{L/R}^{D^+} = A_{L/R}^{D^+}_{SM}, \quad A_{L/R}^{D_s} = A_{L/R}^{D_s}_{SM} + A_{L/R}^{D_s}_{NP},$$

(16)

and

$$r_{D^+} = \left| \frac{A_{L/R}^{D^+}}{A_{L/R}^{D^+}_{SM}} \right|, \quad r_{D_s} = \left| \frac{m_c T^{K_1 C_7^T \text{eff}} + A_{L/R}^{D_s}_{SM}}{m_c T^{K_1 C_7^T \text{eff}} + A_{L/R}^{D_s}_{SM}} \right|.$$  

(17)

In Fig. 2 we illustrate BSM effects that show up in $\lambda^D_\gamma$ being different from $\lambda^D_\gamma$ for NP in $C'_7$ with $C_7 = 0$ (green curves) and in $C_7$ with $C'_7 = 0$ (red curves), within the constraints in (11) for
the $K_1(1270)$, central values of input, and for $\lambda_D(s) = 0.1$ GeV. We learn that NP in the left- or right-handed dipole operator can significantly change the polarization in $D^+$ decays from the one in $D_s$ decays. Larger values of $\lambda_D(s)$ and $T_{K_1}$, and smaller values of $f_{K_1}$ enhance the BSM effects.

III. THE $K_1 \to K\pi\pi$ DECAYS

Here we provide input for the $K_1 \to K\pi\pi$ helicity amplitude $J$, which drives the sensitivity to the photon polarization in the up-down asymmetry \cite{3}. After giving a general Lorentz-decomposition we resort to a phenomenological model for the form factors, which allows us to estimate $J$ and sensitivities. This section is based on corresponding studies in $B$ decays \cite{2,21}. While being relevant for the sensitivity, we recall that knowledge of $J$ in charm is not needed as a theory input to perform the SM null test.

We consider two $K_1$ states, $K_1(1270)$ and $K_1(1400)$, with spin parity $J^P = 1^+$. For the charged resonance $K_1^+$ two types of charge combinations exist for the final state, $K_1^+ \to K^0\pi^+\pi^0$ (channel I) and $K_1^+ \to K^+\pi^+\pi^−$ (channel II),

\begin{align*}
I : \ K_1^+(1270/1400) & \to \pi^0(p_1)\pi^+(p_2)K^0(p_3), \\
II : \ K_1^+(1270/1400) & \to \pi^−(p_1)\pi^+(p_2)K^+(p_3),
\end{align*}

both of which we consider in the following.

The $K_1 \to K\pi\pi$ decay amplitude can be written in terms of the helicity amplitude $J$ as

$$M(K_{1L,R} \to K\pi\pi)^{I,II} = \varepsilon_{L,R}^{\mu}J_{\mu}^{I,II},$$

with the $K_1$ polarization vector $\varepsilon_{L,R}^{\mu} = (0, \pm 1, -i, 0)/\sqrt{2}$. For a $1^+$ state $J_{\mu}^{I,II}$ can be parameterized by two functions, $C_{1,2}$, as

$$J_{\mu}^{I,II} = [C_{1,II}^{I}(s, s_{13}, s_{23})p_{1\mu} - C_{2,II}^{I}(s, s_{13}, s_{23})p_{2\mu}]BW_{K_1}(s).$$

From here on assumptions are needed to make progress on the numerical predictions of the phenomenological profiles. First, the $C_{1,2}$-functions are modelled by the quasi-two-body decays $K_1 \to K\rho(\to \pi\pi)$ and $K_1 \to K^*(\to K\pi)\pi$. Taking into account the isospin factors for each charge
mode, \( K_1^+ \to K^0 \pi^+ \pi^0 \) and \( K_1^+ \to K^+ \pi^+ \pi^- \), \( C_{1,2}^{III} \) can be rewritten in the following form \[21\]

\[
C_1^I = \frac{\sqrt{2}}{3} (a_{13}^{K^*} - b_{13}^{K^*}) + \frac{\sqrt{2}}{3} b_{13}^{K^*} + \frac{1}{\sqrt{3}} a_{12}^\rho, \quad C_2^I = \frac{\sqrt{2}}{3} b_{13}^{K^*} - \frac{\sqrt{2}}{3} a_{23}^{K^*} - \frac{1}{\sqrt{3}} b_{12}^\rho, \\
C_1^{II} = -\frac{2}{3} (a_{13}^{K^*} - b_{13}^{K^*}) - \frac{1}{\sqrt{6}} a_{12}^\rho, \quad C_2^{II} = -\frac{2}{3} b_{13}^{K^*} + \frac{1}{\sqrt{6}} b_{12}^\rho, 
\]

(21)

where, using factorization,

\[
a_{ij}^V = g_{VP_{P_j}} BW_V(s_{ij}) [f^V + h^V \sqrt{s}(E_i - E_j) - \Delta_{ij}], \\
b_{ij}^V = g_{VP_{P_j}} BW_V(s_{ij}) [-f^V + h^V \sqrt{s}(E_i - E_j) - \Delta_{ij}],
\]

(22)

with \( \Delta_{ij} = \frac{(m_2^2 - m_1^2)}{m_\rho^2} [f^V + h^V \sqrt{s}(E_i + E_j)], E_i = \frac{(s - s_{23} + m_1^2)}{2\sqrt{s}} \) and the Breit-Wigner shapes \( BW_V(s_{ij}) = (s_{ij} - m_\rho^2 + im_\rho \Gamma \rho)^{-1} \). The definitions of the form factors of the \( K_1 \rightarrow V \rho \) (\( V = K^*, \rho \) and \( P = \pi, K \)) decay, \( f^V, h^V \), and decay constants of the \( V \rightarrow P_P \rho \) decay, \( g_{VP_{P_j}} \), are given in Appendix \[B\]. The form factors are obtained in the Quark-Pair-Creation Model (QPCM) \[25\].

In the presence of two \( K_1 \) states, \( K_1(1270) \) and \( K_1(1400) \), this framework can be extended by adding the contributions weighted by the line-shapes

\[
\mathcal{J}_{III}^I = \sum_{K_{res}=K_1(1270,1400)} \xi_{K_{res}} \left[ C_{1K_{res}}^{III} (s, s_{13}, s_{23}) p_{1\mu} - C_{2K_{res}}^{III} (s, s_{13}, s_{23}) p_{2\mu} \right] BW_{K_{res}}(s),
\]

(23)

and the parameter \( \xi_{K_{res}} \), which allows to switch the states on and off individually. Importantly, in a generic situation with all \( K_1 \)-resonances contributing \( \xi_{K_{res}} \) takes into account the differences in their production in the weak decay. Such effects are induced by the \( K_1 \)-dependence of hadronic matrix elements, such as \( f_{K_1} m_{K_1} \) in \[6\], or \( T_{K_1} \) in \[12\]. For \( f_{K_1(1400)} m_{K_1(1400)} / (f_{K_1(1270)} m_{K_1(1270)}) \sim 1.1 \) and \( T_{K_1(1400)} / T_{K_1(1270)} \sim 0.5 \) this effect is rather mild. The ansatz \[23\], which is an approximation of the general formula \( (3) \), allows to compute \( A_{UD}/\lambda_\chi \) as in \[3\] in Sec. \[IV\] independent of the weak decays. Eq. \[23\] becomes exact, i.e., coincides with \( (3) \) for universal \( \xi_{K_{res}} \).

Due to isospin \( I m[\vec{n} \cdot (\vec{J} \times \vec{J}^*)] \) in the \( K_1^+ \rightarrow K^0 \pi^+ \pi^0 \) channel is antisymmetric in the \( (s_{13}, s_{23}) \)-Dalitz plane. This can be seen explicitly by interchanging \( s_{13} \leftrightarrow s_{23} \) in Eq. \[21\], which implies \( C_1 \leftrightarrow C_2 \) and therefore \( I m[\vec{n} \cdot (\vec{J} \times \vec{J}^*)] \propto I m[C_1 C_2^*] \) changes sign when crossing the \( s_{13} = s_{23} \) line, see the plot to the right in Fig. \[5\]. Therefore, in order to have a non-zero up-down asymmetry after \( s_{13}, s_{23} \)-integration, one has to define the asymmetry with \( \langle \text{sgn}(s_{13} - s_{23}) I m[\vec{n} \cdot (\vec{J} \times \vec{J}^*)] \rangle \) in Eq. \[5\]. In the \( K_1^+ \rightarrow K^+ \pi^- \pi^+ \) channel and with only one \( K_1 \), the border, at which \( A_{UD} \) changes sign, is a straight line in the \( (s_{13}, s_{23}) \)-plane, see the plot to the left in Fig. \[3\] which is described by \( I m[BW_{K^*}(s_{13}) BW_{\rho}^*(s_{12})] = 0 \). The location of this line in the Dalitz plane depends on \( s \) via \( s = s_{12} + s_{23} + s_{13} + 2m_\pi^2 + m_K^2 \).
Figure 3: Dalitz contour plots of $\Im \{ \vec{n} \cdot (\vec{J} \times \vec{J}^*) \}$ for $K^+\pi^+\pi^-$ (plot to the left) and $K^0\pi^+\pi^0$ (plot to the right) at $m_{K\pi\pi}^2 = m_{K1(1270)}^2$. Red (blue) areas correspond to positive (negative) values of $\Im \{ \vec{n} \cdot (\vec{J} \times \vec{J}^*) \}$. Grey bands represent the $K^*(\rho)$ resonance $[(m_{K^*(\rho)} - \Gamma_{K^*(\rho)})^2, (m_{K^*(\rho)} + \Gamma_{K^*(\rho)})^2]$ intervals.

IV. UP-DOWN ASYMMETRY PROFILES

In the following we work out estimates for the up-down asymmetry in units of the photon polarization parameter $A_{UD}/\lambda_\gamma$, as in (5). The crucial ingredient for probing the photon polarization is the hadronic factor $\Im \{ \vec{n} \cdot (\vec{J} \times \vec{J}^*) \}$. Using (23), and for two interfering resonances $a, b$, e.g., $a = K_1(1270)$ and $b = K_1(1400)$, dropping channel $I, II$ superscripts and kinematic variables to ease notation, it reads

$$\Im \{ \vec{n} \cdot (\vec{J} \times \vec{J}^*) \} = -2\Im \left[ \xi_a^2 C_{1a} C_{2a}^* |BW_a|^2 + \xi_b^2 C_{1b} C_{2b}^* |BW_b|^2 \right] + \xi_a \xi_b (C_{1a} C_{2b}^* - C_{1b} C_{2a}^*) |BW_a| |BW_b^*| \left| \vec{p}_1 \times \vec{p}_2 \right|,$$

which shows the necessity of having relative strong phases for a non-zero up-down asymmetry. Such phases can come from the interference between $K^*\pi$ and $K\rho$ channels inside of $C_{1,2}$, as well as from the interference between the $K_1$ resonances. Due to the larger number of interfering amplitudes (18), we quite generally expect larger phases in the $K_1^+ \rightarrow K^0\pi^+\pi^0$ channel. While the $K_1(1270)$ decays both to $K\rho$ and $K^*\pi$, the $K_1(1400)$ decays predominantly to $K^*\pi$. We therefore expect the pure $K_1(1400)$ contribution to $A_{UD}/\lambda_\gamma$ in the $K^+\pi^-\pi^-$ channel to be very small.

In Fig. 4 we show the $m_{K\pi\pi}$ dependence of $|\vec{J}|^2$ (plots to the left) and $A_{UD}/\lambda_\gamma$ (plots to the right). The different colors refer to different ratios of the $K_1(1270)$ and $K_1(1400)$ contributions.
Specifically, black, red, green and magenta lines correspond to $\xi_{K_1(1400)} = 0, +0.5, +1$ and $-1$, respectively, for fixed $\xi_{K_1(1270)} = 1$. The blue curve refers to only the $K_1(1400)$ being present, with
\( \xi_{K_1(1270)} = 0 \). Upper (lower) plots are for channel II (channel I).

The measured invariant mass \( m_{K\pi\pi} \) spectrum in \( B^+ \rightarrow K^+\pi^+\pi^-\gamma \) decays \cite{22,24} exhibits the dominant \( K_1(1270) \)-peak along with a \( K_1(1400) \)-shoulder, plus higher resonances. For our model, these measurements suggest a value of \( \xi_{K_1(1400)}/\xi_{K_1(1270)} \) around +1, see Fig. 4, consistent with expectations based on small \( K_1 \)-dependence, see Sec. III. We also note that resonances higher than the \( K_1(1270) \) and the \( K_1(1400) \) contribute, such as the \( K_2^*(1430)(2^+) \) and the \( K^*(1410)(1^-) \), which are not taken into account in our analysis. Our predictions therefore oversimplify the situation for \( m_{K\pi\pi} \gtrsim 1400 \text{ MeV} \).

Since the up-down asymmetry is sensitive to complex phases in the \( K_1 \) decay amplitudes, we test several possible sources apart from the ones coming from the Breit-Wigner functions of the \( K_1, K^* \) and the \( \rho \). As expected, it turns out that such phases have only a negligible effect on the \( |\vec{J}|^2 \) distributions, and we do not show corresponding plots. The Belle collaboration in the analysis of \( B^+ \rightarrow J/\psi K^+\pi^+\pi^- \) and \( B^+ \rightarrow \psi' K^+\pi^+\pi^- \) decays signals a non-zero phase,

\[
\delta_\rho = \arg \left[ \frac{\mathcal{M}(K_1(1270) \rightarrow (K\rho)_S) \times \mathcal{M}(\rho \rightarrow \pi\pi)}{\mathcal{M}(K_1(1270) \rightarrow (K^*\pi)_S) \times \mathcal{M}(K^* \rightarrow K\pi)} \right],
\]  

(25)

as \( \delta_\rho = -(43.8 \pm 4.0 \pm 7.3)^\circ \) \cite{22}. A similar value was found in the reanalysis of the ACCMOR data \cite{26} by the Babar collaboration, as \( \delta_\rho = (-31 \pm 1)^\circ \) \cite{27}. Therefore, we add an additional phase \( \delta_\rho = -40^\circ \) to the \( K\rho \) S-wave \footnote{Due to the smallness of the \( K\rho \) D-wave amplitude we neglect its contribution in our study.} amplitude and consider it as theoretical uncertainty. The effect of this additional phase in \( \mathcal{A}_{UD} \) (dashed curves) in comparison with the QPCM predictions (solid curves) is presented in Fig. 5. We also investigate the impact of the additional phase \( \delta_D = \arg[\mathcal{M}(K_1(1270) \rightarrow (K^*\pi)_D)/\mathcal{M}(K_1(1270) \rightarrow (K^*\pi)_S)] = 90^\circ \). The result can be seen in

\[\text{Figure 6: The same as Fig. 5 for } \delta_\rho = 0 \text{ and with dotted lines representing the “off-set” phase } \delta_D = \arg[\mathcal{M}(K_1(1270) \rightarrow (K^*\pi)_D)/\mathcal{M}(K_1(1270) \rightarrow (K^*\pi)_S)] = 90^\circ.\]
Fig. 6. Note that $\delta_\rho$ and $\delta_D$ vanish in the QPCM and are therefore termed “off-set” phases.

We learn from Figs. 4 – 6 that $A_{UD}/\lambda_\gamma$ profiles with $\xi_{K_1(1400)} = 0.5, 1$ (red, green curves, respectively) can be of the order $\sim 0.05 - 0.1$ (channel II) and $\sim 0.2 - 0.3$ (channel I), which are, as expected, larger for $K^0\pi^+\pi^0$ than for $K^+\pi^+\pi^-$ final states. Adding phenomenological strong phases such as $\delta_\rho$ and $\delta_D$ has a significant effect for channel II. As zero-crossings can occur it may be disadvantageous to not use $m_{K\pi\pi}$ bins, in particular, for channel II. The position of the zeros, however, cannot be firmly predicted, although the one at $m_{K^+\pi^+\pi^-} \approx 1$ GeV, whose origin is discussed at the end of Sec. III, is quite stable, as well as the one at $m_{K^+\pi^+\pi^-} \approx 1.3$ GeV. The latter stems from $K_1(1270)$ and $K_1(1400)$ interference.

Strong phases and, related to this, $K_1$-mixing, constitute the main sources of uncertainty. Figs. 4 – 6 are obtained for fixed mixing angle $\theta_{K_1} = 59^\circ$, see Appendix B. Varying $\theta_{K_1}$ within its 1 $\sigma$ range, $\pm 10^\circ$, determined within QPCM, as well as $\delta_D \in [0, 2\pi]$ for $\delta_\rho = 0, -40^\circ$, we find for the $m_{K\pi\pi}$-integrated up-down asymmetry assuming $K_1(1270)$ dominance the ranges $[-30, +2]$ % (channel I) and $[+2, +13]$ % (channel II). Recall that the latter exhibits cancellations so that locally the asymmetry can be larger. Our results are compatible with the findings $[-10, -7]$ % (channel I) and $[-13, +24]$ % (channel II) of Ref. [28], which are based on $K_1(1270)$ dominance. Note that Ref. [28] uses $\kappa = \text{sgn}(s_{13} - s_{23})$ for both channels. Our prediction for channel II in this convention reads $[-18, +8]$ %.

We stress that the estimates are subject to sizable uncertainties and serve as a zeroth order study to explore the BSM potential in $D_s \to K_1 \gamma$ decays. $K\pi\pi$ profiles from the $B$-sector can be linked to charm physics, and vice versa.

V. CONCLUSIONS

New physics may be linked to flavor, and $K, D,$ and $B$ systems together are required to decipher its family structure. Irrespective of this global picture, SM tests in semileptonic and radiative $c \to u$ transitions are interesting per se, and quite unexplored territory today: present bounds on short-distance couplings are about two orders of magnitude away from the SM [11, 12].

We study a null test of the SM in radiative rare charm decays based on the comparison of the up-down asymmetry in $D^+ \to K_1^+ (\to K\pi\pi)\gamma$, which is SM-like, to the one in $D_s \to K_1^+ (\to K\pi\pi)\gamma$, which is an FCNC. The up-down asymmetry depends on the photon polarization, subject to BSM effects in the $|\Delta c| = |\Delta u| = 1$ transition.

We find that, model-independently, NP in photonic dipole operators can alter the polarization of
$D_s \rightarrow K_1^+ (\rightarrow K\pi\pi)\gamma$ from the SM value at order one level, see Fig. 2. We estimate the proportionality factor between the integrated up-down asymmetry \cite{5} and the polarization parameter to be up to $\mathcal{O}(5-10)\%$, and 40% in extreme cases, for $K_1^+ \rightarrow K^+\pi^+\pi^-$ and $\mathcal{O}(20-30)\%$ for $K_1^+ \rightarrow K^0\pi^+\pi^0$, respectively, see Figs. 4–6. As in previous studies carried out for $B \rightarrow K_1^+ (\rightarrow K\pi\pi)\gamma$ decays there are sizable uncertainties associated with these estimates. Unlike in $B$-physics, these do not affect the SM null test. With branching ratios \cite{10} of $\mathcal{B}(D^+ \rightarrow K_1^+\gamma)$ of $\mathcal{O}(10^{-5})$ and $\mathcal{B}(D_s \rightarrow K_1^+\gamma)$ of $\mathcal{O}(10^{-4})$ analyses of up-down asymmetries in charm constitute an interesting NP search for current and future flavor facilities.

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Appendix A: Matrix elements

The matrix element of the electromagnetic dipole operator can be parametrized as

$$\langle K_1(\epsilon,k)|\bar{u}\sigma_{\mu\nu}(1 \pm \gamma_5)q^{\nu}c|D_s(p)\rangle = T_2^{K_1}(q^2) \left[ \varepsilon^\mu(p^2 - m_D^2) - (\varepsilon^\mu p + k)^\mu \right]$$

$$+ T_3^{K_1}(q^2)(\varepsilon^\mu p) \left[ q^\mu - \frac{q^2}{m_D^2 - m_{K_1}^2}(p + k)^\mu \right]$$

$$\pm 2T_1^{K_1}(q^2)i\varepsilon_{\mu\rho\sigma\tau}e^{\rho\sigma}p^\tau k^\mu,$$

with $T_1^{K_1}(0) = T_2^{K_1}(0)$.

The $K_1$ and $D_{(s)}$ decay constants are defined as

$$\langle K_1(\epsilon,k)|\bar{u}\gamma_\mu\gamma_5s|0\rangle = f_{K_1}m_{K_1}\varepsilon^\mu,$$

$$\langle 0|\bar{d}(s)\gamma_\mu\gamma_5c|D_{(s)}(p)\rangle = if_{D_{(s)}}p_\mu.$$

We employ the following values for the $K_1$ decay constants

$$f_{K_1(1270)} = (170 \pm 20)\text{ MeV},$$

$$f_{K_1(1400)} = (175 \pm 37)\text{ MeV}.$$

Here, $f_{K_1(1270)}$ is extracted from $\mathcal{B}(\tau^- \rightarrow K_1(1270)^-\nu_\tau)^{\text{exp}} = (4.7 \pm 1.1) \times 10^{-3}$ \cite{4}, as

$$\mathcal{B}(\tau \rightarrow K_1\nu_\tau) = \tau_{\tau} \frac{G_F^2}{16\pi} |V_{us}|^2 f_{K_1}^2 m_\tau^3 \left( 1 + \frac{2m_{K_1}^2}{m_\tau^2} \right) \left( 1 - \frac{m_{K_1}^2}{m_\tau^2} \right)^2.$$
The value of $f_{K_1(1270)}$ from a light cone sum rule calculation \cite{29} is consistent with the data-based value (A4) assuming the SM. The value of $f_{K_1(1400)}$ is taken from Ref. \cite{29}; we added statistical and systematic uncertainties in quadrature and symmetrized the uncertainties. $B(\tau^- \to K_1(1400)\nu_\tau)^{\text{exp}} = (1.7 \pm 2.6) \times 10^{-3}$ \cite{4} has too large uncertainty to allow for an extraction of $f_{K_1(1400)}$, however, yields a 90 % CL upper limit as $|f_{K_1(1400)}| < 235$ MeV, consistent with (A4).

**Appendix B: $K_1 \to VP$ form factors**

The hadronic form factors, $f_V$ and $h_V$, defined as

$$
\mathcal{M}(K_1 \to VP) = \varepsilon_{K_1}^\mu (f^V g_{\mu\nu} + h^V p_{\nu} p_{K_1}) \varepsilon_{V}^\nu
$$

are related to the partial $S,D$ wave amplitudes,

$$
f^V = -A_S^V - \frac{1}{\sqrt{2}} A_D^V,
$$
$$
h^V = \frac{E_V}{\sqrt{s^V P^V}} \left[ \left( 1 - \frac{s^V}{E_V} \right) A_S^V + \left( 1 + 2 \frac{s^V}{E_V} \right) \frac{1}{\sqrt{2}} A_D^V \right].
$$

These partial wave amplitudes are computed in the framework of the $^3P_0$ QPCM \cite{25}. The details of the computation and expressions for $A^K_{S,D} / \rho$ can be found in Ref. \cite{21}. Due to $SU(3)$ breaking, the $K_1(1270)$ and $K_1(1400)$ mesons are an admixture of the spin singlet and triplet $P$-wave states $K_{1B}(1^1P_1)$ and $K_{1A}(1^3P_1)$, respectively,

$$
|K_1(1270)\rangle = |K_{1A}\rangle \sin \theta_{K_1} + |K_{1B}\rangle \cos \theta_{K_1},
$$
$$
|K_1(1400)\rangle = |K_{1A}\rangle \cos \theta_{K_1} - |K_{1B}\rangle \sin \theta_{K_1},
$$

with mixing angle $\theta_{K_1} = (59 \pm 10)^\circ$ \cite{21}, which has been obtained from $K_1 \to VP$ decay data.

**Appendix C: General formula for the up-down asymmetry**

The reduced amplitude of $D_{(s)} \to K_{\text{res}}\gamma \to K\pi\pi\gamma$ decays can be written as the product of the weak decay amplitude $\mathcal{M}_{L/R}^{D_{(s)},K_{\text{res}}}$ and strong decay amplitude $J^K_{\mu,K_{\text{res}}}$ as

$$
G^{D_{(s)},K_{\text{res}}}_{\mu,L/R} = \sum_{K_{\text{res}}} \mathcal{M}_{L/R}^{D_{(s)},K_{\text{res}}} J^K_{\mu,K_{\text{res}}}.
$$

Multiplying $G^{D_{(s)},K_{\text{res}}}_{\mu,L/R}$ by the photon polarization vector and integrating over azimuthal angles, we obtain the general formula for modulus squared of the matrix element

$$
|\mathcal{M}^{D_{(s)}}|^2 \propto \left( |G^{D_{(s)}}_L|^2 + |G^{D_{(s)}}_R|^2 \right) \left( 1 + \cos^2 \theta \right) - 2 \text{Im} \left[ n \cdot \left( G^{D_{(s)}}_L \times G^{D_{(s)}}_L^* - G^{D_{(s)}}_R \times G^{D_{(s)}}_R^* \right) \right] \cos \theta.
$$
This expression holds even beyond (C1), such as for non-resonant contributions, as long as the $K\pi\pi$ system is in the same spin, parity state as $K_{res}$, $1^+$. The up-down asymmetry then reads

$$A_{UD}^{D(s)} = \frac{3}{4} \langle \text{Im} \left[ \vec{n} \cdot \left( \vec{g}_L^{D(s)} \times \vec{g}_L^{D(s)*} - \vec{g}_R^{D(s)} \times \vec{g}_R^{D(s)*} \right) \right] \rangle \langle |\vec{g}_L^{D(s)}|^2 + |\vec{g}_R^{D(s)}|^2 \rangle.$$

(C3)

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