Vacuum Structure around Identity-Based Solutions

Isao KISHIMOTO\(^1\) and Tomohiko TAKAHASHI\(^2\)

\(^1\)Theoretical Physics Laboratory, RIKEN, Wako 351-0198, Japan
\(^2\)Department of Physics, Nara Women’s University, Nara 630-8506, Japan

(Received April 21, 2009)

We explore the vacuum structure in bosonic open string field theory expanded around an identity-based solution parameterized by \(a \geq -1/2\). Analyzing the expanded theory using level truncation approximation up to level 14, we find that the theory has a stable vacuum solution for \(a > -1/2\). The vacuum energy and gauge invariant overlap numerically approach those of the tachyon vacuum solution with increasing truncation level. We also find that, at \(a = -1/2\), there exists an unstable vacuum solution in the expanded theory and it rapidly becomes the trivial zero configuration just above \(a = -1/2\). The numerical behavior of the two gauge invariants suggests that the unstable solution corresponds to the perturbative open string vacuum. These results reasonably support the expectation that the identity-based solution is a trivial pure gauge configuration for \(a > -1/2\), but it can be regarded as the tachyon vacuum solution at \(a = -1/2\).

Subject Index: 126, 128

§1. Introduction

Bosonic open string field theory\(^1\) (SFT) has classical solutions describing a tachyon vacuum where D-branes with attached open strings completely annihilate. A numerical tachyon vacuum solution was first constructed using level truncation approximation in the Siegel gauge.\(^2\)–\(^5\) Then, an analytic classical solution has been constructed by Schnabl\(^6\) and it has been found that the solution possesses several properties of the tachyon vacuum. The vacuum energy of the nontrivial analytic solution exactly cancels the D-brane tension,\(^6\)–\(^8\) and the cohomology of the kinetic operator around the vacuum is trivial.\(^9\) There are also several attempts to construct tachyon vacuum solutions in superstring field theory as an extension of the bosonic solution.\(^10\)–\(^13\)

In SFT, there is another type of analytic solution that was constructed earlier on the basis of the identity string field instead of wedge states used in Schnabl’s solution.\(^14\)–\(^24\) An identity-based solution discussed in Refs. 14), 25)–27) involves one parameter \(a\) larger than or equal to \(-1/2\). It is expected that the solution for \(a > -1/2\) is a trivial pure gauge solution because of the following facts:

1. The solution can be expressed as a pure gauge form connecting to a trivial configuration.\(^14\)
2. The action obtained by expanding around the solution can be transformed back to the action with the original BRST charge.\(^14\)
3. The new BRST charge gives rise to the cohomology, which has one-to-one correspondence to the cohomology of the original BRST charge.\(^25\)
4. The expanded theory reproduces ordinary open string amplitudes.\(^27\)
These are consistent with the expectation that the solution corresponds to a trivial pure gauge. On the other hand, we find completely different properties in the expanded theory around a solution with $a = -1/2$:

5. The solution can be given as a type of singular gauge transformation of the trivial configuration.\textsuperscript{14}

6. The new BRST charge has vanishing cohomology in the Hilbert space with the ghost number one.\textsuperscript{25}

7. The open string scattering amplitudes vanish and the result is consistent with the absence of open string excitations.\textsuperscript{27}

From these facts, it would be reasonable to expect that the identity-based solution at $a = -1/2$ indeed corresponds to the tachyon vacuum solution. Hence, we expect that the identity-based solution corresponds to a trivial pure gauge form for almost all the parameter region and it can be regarded as the tachyon vacuum solution at $a = -1/2$.

The Schnabl solution can also be parameterized by $\lambda$. The solution corresponds to a trivial pure gauge for $-1 \leq \lambda < 1$. However, at $\lambda = 1$, it changes markedly to the tachyon vacuum solution. Thus, the parameter $a$ in the identity-based solution has a property similar to $\lambda$ in the Schnabl solution. The similar dependence of these parameters suggests that the identity-based solution may be gauge-equivalent to the Schnabl solution.

Unfortunately, we have not yet known how to calculate the vacuum energy of the identity-based solution. To calculate the vacuum energy, it is necessary to apply a kind of regularization because a string field consists of an infinite number of component fields. Indeed, the level can be regarded as a regularization parameter for the numerical solution in the Siegel gauge. Moreover, an analytic expression of the Schnabl solution includes a parameter that regularizes the infinite sum of wedge-like states. Hence, the difficulty of calculating the vacuum energy seems to arise from the lack of such a regularization method for the identity-based solution.

However, we can provide indirect evidence that supports the possibility of calculating the vacuum energy. The vacuum structure in the theory expanded around the identity-based solution has been analyzed using level truncation approximation and then we have found the following results:

8. A numerical analysis shows that the nonperturbative vacuum found for $a > -1/2$ disappears as $a$ approaches $-1/2$.\textsuperscript{26}

9. The energy of the nonperturbative vacuum for $a > -1/2$ becomes closer to the value appropriate to cancel the D-brane tension as the truncation level increases.\textsuperscript{26}

These imply that the theory around the identity-based solution for $a > -1/2$ has the tachyon vacuum, but the theory at $a = -1/2$ is stable. From consistency with the theory before expanding a string field, it follows that the vacuum energy of the identity-based solution itself is zero for $a > -1/2$ and it is equal to the tachyon vacuum energy at $a = -1/2$.

The purpose of this paper is to perform additional numerical analysis of vacuum structure in the theory expanded around the identity-based solution and to provide further evidence for the expectation that the identity-based solution can be regarded...
Vacuum Structure around Identity-Based Solutions

The results of 8 and 9 are very encouraging but they were based on a slightly lower level analysis. First, we will raise the level from (6,18) to (14,42) searching the tachyon vacuum in the expanded theory. As a result, we will numerically confirm these results with higher precision.

As the parameter $a$ approaches $-1/2$, the tachyon vacuum solution existing in the expanded theory for $a > -1/2$ annihilates into a trivial zero configuration. From this fact, it is naturally expected that the identity-based solution at $a = -1/2$ can be regarded as the tachyon vacuum solution. If so, it is reasonable to expect that the expanded theory at $a = -1/2$ has an unstable vacuum corresponding to the perturbative open string vacuum in the original theory. The unstable vacuum should emerge as the parameter $a$ approaches $-1/2$. In this paper, we will find this emergence of the unstable vacuum using level truncation approximation.

The paper is organized as follows. In §2, after briefly reviewing the identity-based solution, we will provide results for the annihilation of the stable vacuum in the expanded theory. For each nontrivial solution, we will calculate two gauge invariants: vacuum energy and gauge invariant overlap. For $a > -1/2$, the former invariant should cancel the brane tension and the latter should be nonzero, and both should become trivially zero at $a = -1/2$. We will see that this tendency becomes obvious as the truncation level is increased. In §3, we will consider the emergence of an unstable vacuum. Indeed, using the level truncation up to (16,48), we will find that the unstable vacuum solution does exist in the expanded theory at $a = -1/2$. Then, the vacuum energy for the solution approaches the expected value with increasing level. Moreover, we will show that the gauge invariant overlap is nearly the expected value. In §4, we will give the summary and discussion.

§2. Annihilation of tachyon vacuum

The identity-based solution can be expressed as

$$\Psi_0 = Q_L(e^h - 1)I - C_L((\partial h)^2 e^h)I,$$  

where $I$ is the identity string field associated with the star product, and the half string operators $Q_L$ and $C_L$ are defined using the BRST current $J_B(z)$ and ghost field $c(z)$ as follows,

$$Q_L(f) = \int_{C_{\text{left}}} \frac{dz}{2\pi i} f(z)J_B(z), \quad C_L(g) = \int_{C_{\text{left}}} \frac{dz}{2\pi i} g(z)c(z).$$

Here, $C_{\text{left}}$ denotes the contour along a right semi-unit circle from $-i$ to $i$, which conventionally corresponds to the left half of strings. The function $h(z)$ is defined on the whole unit circle. For the function $h(z)$ satisfying $h(-1/z) = h(z)$ and $h(\pm i) = 0$, the equations of motion, $Q_B\Psi_0 + \Psi_0 * \Psi_0 = 0$, hold for the identity-based solution (2.1).

Expanding the string field $\Psi$ around the solution $\Psi_0$ as $\Psi = \Psi_0 + \Phi$ and subtracting the vacuum energy at $\Psi_0$ from the original action, we can find the action for the
fluctuation string field $\Phi$:

$$S[\Phi] = -\frac{1}{g^2} \int \left( \frac{1}{2} \Phi \ast Q' \Phi + \frac{1}{3} \Phi \ast \Phi \ast \Phi \right), \quad (2.3)$$

where the kinetic operator is given by

$$Q' = Q(e^h) - C((\partial h)^2 e^{h}). \quad (2.4)$$

The operators, $Q(f)$ and $C(g)$, are defined by

$$Q(f) = \oint \frac{dz}{2\pi i} f(z) J_B(z), \quad C(g) = \oint \frac{dz}{2\pi i} g(z) c(z), \quad (2.5)$$

where the integration contour is a unit circle.

Hereafter, we consider the expanded theory around the solution derived from the function

$$h(z) = \log \left( 1 + \frac{a}{2} \left( z + \frac{1}{z} \right)^2 \right) \quad (2.6)$$

$$= -\log(1 - Z(a))^2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} Z(a)^n (z^{2n} + z^{-2n}), \quad (2.7)$$

where $Z(a) = (1 + a - \sqrt{1 + 2a})/a$. For the kinetic operator to be well-defined, the parameter $a$ is larger than or equal to $-1/2$. For this function, the kinetic operator can be expanded as

$$Q' = (1 + a)Q_B + \frac{a}{2} (Q_2 + Q_{-2}) + 4a Z(a)c_0 - 2a Z(a)^2 (c_2 + c_{-2})$$

$$-2a(1 - Z(a)^2) \sum_{n=2}^{\infty} (-1)^n Z(a)^{n-1}(c_{2n} + c_{-2n}), \quad (2.8)$$

where we expand the BRST current and ghost field as $J_B(z) = \sum_n Q_n z^{-n-1}$ and $c(z) = \sum_n c_n z^{-n+1}$, respectively.

The cohomology of this new BRST operator has been investigated in Refs. 14) and 25). For $a > -1/2$, the new BRST operator can be transformed to the original BRST charge by a similarity transformation, and then the cohomology has one-to-one correspondence to the original cohomology. For $a = -1/2$, the new BRST operator has vanishing cohomology in the Hilbert space with ghost number 1. From these facts, it turns out that $a$ has properties similar to the parameter $\lambda$ of the Schnabl solution.*

We consider the tachyon vacuum in the expanded theory around the identity-based solution. The expanded theory $S[\Phi]$ (2.3) has a gauge symmetry under

$$\delta \Phi = Q' \Lambda + \Phi \ast \Lambda - \Lambda \ast \Phi. \quad (2.10)$$

*) The Schnabl solution with a parameter $\lambda$ can be written as

$$\psi = \frac{\lambda \partial_r}{\lambda e^{\psi} - 1} \psi_r |_{r=0}. \quad (2.9)$$
Vacuum Structure around Identity-Based Solutions

Fig. 1. Vacuum structure expected for the theory expanded around the identity-based solution.

(1) For $a > -1/2$, the theory should have a nontrivial vacuum solution, the vacuum energy of which cancels the D-brane tension. (2) At $a = -1/2$, the trivial configuration $\Phi = 0$ should be stable. An unstable solution is expected to exist and its vacuum energy should be equal to the D-brane tension.

To find classical solutions in the theory, we impose the Siegel gauge condition on the fluctuation string field; $b_0 \Phi = 0$. Under the Siegel gauge condition, the potential can be expressed as

$$f_a(\Phi) = 2\pi^2 \left( \frac{1}{2} \langle \Phi, c_0 L(a) \Phi \rangle + \frac{1}{3} \langle \Phi, \Phi \ast \Phi \rangle \right), \quad (2.11)$$

where it is normalized as $-1$ for the tachyon vacuum solution at $a = 0$. Here, the operator $L(a)$ is given by

$$L(a) = (1 + a) L_0 + \frac{a}{2}(L_2 + L_{-2}) + a(q_2 - q_{-2}) + 4(1 + a - \sqrt{1 + 2a}), \quad (2.12)$$

where $L_n$ is a total Virasoro generator and $q_n$ is a mode of the ghost number current.\(^\text{*)}\)

If the identity-based solution corresponds to a trivial pure gauge for $a > -1/2$ and then to the tachyon vacuum for $a = -1/2$, the potential (2.11) should be illustrated as in Fig. 1. For $a > -1/2$, the tachyon vacuum configuration $\Phi_1$ should minimize the potential, since the expanded theory for $\Phi$ is still the theory on the perturbative open string vacuum. However, at $a = -1/2$, the trivial configuration $\Phi = 0$ should be stable since the expanded theory is expected to be already on the tachyon vacuum. In addition, the theory at $a = -1/2$ should have an unstable solution corresponding to the perturbative open string vacuum of the unexpanded original theory. We will discuss the unstable solution $\Phi_2$ in \S 3.

Let us suppose that we obtain the tachyon vacuum solution $\Phi_1$ in the expanded theory and we evaluate its vacuum energy. Then, the vacuum energy $f_a(\Phi_1)$ is given as a function of the parameter $a$. If the potential has such a structure as previously expected, the vacuum energy is $-1$ for $a > -1/2$ but the solution $\Phi_1$ becomes trivial and therefore its vacuum energy annihilates at $a = -1/2$ (see Fig. 2).

\(^{\text{*)}}\) The expression is derived from $L(a) = \{Q', b_0\}$. It can be rewritten only using ghost twisted Virasoro operators as in Ref. 27.)
Now, let us consider the stable solution $\Phi_1$ by level truncation calculation to confirm the above conjecture. Since the level truncation is a good approximation, the vacuum energy for the truncated solution is considered to approach the step function in Fig. 2 as the truncation level is increased. We apply an iterative approximation algorithm as used in Refs. 5) and 28) to find the stable solution. Namely, with an appropriate initial configuration $\Phi(0)$, we solve

$$c_0 b_0 (c_0 L(a) \Phi^{(n+1)} + \Phi^{(n)} \ast \Phi^{(n+1)} + \Phi^{(n+1)} \ast \Phi^{(n)} - \Phi^{(n)} \ast \Phi^{(n)}) = 0, \quad (2.13)$$

$n = 0, 1, 2, \ldots$) in the Siegel gauge. If the iteration converges to a configuration $\Phi(\infty)$, it satisfies

$$c_0 b_0 (Q' \Phi(\infty) + \Phi(\infty) \ast \Phi(\infty)) = 0, \quad (2.14)$$

which is a projected part of the equation of motion for $S[\Phi]$ (2.3). First, we construct a solution for $a = 0$, namely, in the case of $Q' = Q_B$, with the initial configuration:

$$\Phi(0) = \frac{64}{81\sqrt{3}} c_1 |0\rangle, \quad (2.15)$$

which is the level zero solution. In fact, the iteration using (2.13) converges to the stable tachyon vacuum, which we denote as $\Phi_1|_{a=0}$. Next, for $a = \epsilon$ ($0 < |\epsilon| \ll 1$), we use the solution $\Phi_1|_{a=0}$ as the initial configuration for the iteration (2.13) and obtain a solution $\Phi_1|_{a=\epsilon}$. Then, for $a = 2\epsilon$, we use the solution $\Phi_1|_{a=\epsilon}$ as the initial configuration and obtain a solution $\Phi_1|_{a=2\epsilon}$. In this way, we can uniquely construct a solution $\Phi_1|_{a=(n+1)\epsilon}$ from $\Phi_1|_{a=n\epsilon}$ ($n = 0, 1, 2, \ldots$). In the case $\epsilon < 0$, we find that $\Phi_1|_{a=n_0\epsilon}$ reaches the trivial configuration, namely, $\Phi_1|_{a=n_0\epsilon} = 0$, for some $n_0$ such as $n_0\epsilon > -1/2$. Therefore, for $a \leq n_0\epsilon$, the nontrivial stable solution $\Phi_1$ vanishes.

Actually, for each $a$, we continue the iterative procedure until the relative error reaches $10^{-8}$ and we also check the BRST invariance$^{29)}$ of the resulting configuration, $b_0 c_0 (Q' \Phi(\infty) + \Phi(\infty) \ast \Phi(\infty)) = 0$, as in Ref. 28).$^*$ For each calculation, we use the

$^*$ Here, the BRST invariance that we checked is an invariance in the expanded theory. Namely, the BRST transformation is given by $\delta_B \Phi = Q' \Phi + \Phi \ast \Phi$. 
Vacuum Structure around Identity-Based Solutions

Fig. 3. Vacuum energy of the numerical stable solutions in the expanded theory around the identity-based solution. As the truncation level is increased, the resulting plots approach the step function expected as in Fig. 2.

$(L, 3L)$ truncation. Namely, we truncate the string field to level $L \equiv L_0 + 1$ and interaction terms up to total level $3L$.

First, we show plots of the vacuum energy for the resulting solution in Fig. 3. We can find that, for various $a$, the resulting plots approach the tachyon vacuum energy $-1$ as the truncation level is increased. Particularly around $a = 0$, the plots so rapidly approach $-1$ that they are indistinguishable from each other for higher levels than $(8, 24)$. Then, for decreasing $a$ to $-1/2$, the vacuum energy increases rapidly to zero from $-1$. As a whole, the plots become closer to the step function as depicted in Fig. 2 as the truncation level increases. In Fig. 4, we display an enlarged view of the vacuum energy around $a = -0.5$. We find that at the point at which the vacuum energy is zero, the numerical solution itself becomes trivially zero. Namely, in the level truncated theory, the nontrivial stable solution gradually annihilates as the parameter $a$ approaches a critical value. Furthermore, this annihilation point becomes closer to $a = -0.5$ as the level becomes larger. All these results support our expectation for the vacuum structure associated with the stable solution $\Phi_1$.

Next, we consider the gauge invariant overlap for the numerical stable solution. The gauge invariant overlap in the original theory is given as

$$O(\Psi) = N \langle I | V(\pi/2) | \Psi \rangle,$$

where $I$ denotes the identity string field, $V(\pi/2)$ is an onshell closed string vertex operator inserted at a string midpoint, and $N$ is a normalization constant. (Hereafter, we take the same normalization as that in Ref. 28.) In the theory with the ordinary BRST charge, the gauge invariant overlap is to be zero for a trivial pure
I. Kishimoto and T. Takahashi

Fig. 4. Enlarged view of the vacuum energy for $\Phi_1$ around $a = -0.5$.

gauge configuration, but it becomes a nonzero value for the tachyon vacuum solution. These facts have been confirmed by both analytical and numerical calculations.\textsuperscript{32,33)}

The gauge invariant overlap $O(\Psi)$ (2.16) is also left invariant under the gauge transformation (2.10) in the expanded theory.\textsuperscript{*}) Since the level truncation is applicable to the evaluation of the invariant overlap in the original theory,\textsuperscript{32)} it is naturally expected that we can also calculate the overlap for the numerical stable solution in the expanded theory. Contrasting to the vacuum energy, the invariant overlap does not include explicitly the parameter $a$. However, the overlap is given as the function of $a$ since the stable solution itself depends on the parameter. If the expanded theory has the vacuum structure expected above, the overlap should be nonzero for almost all $a$, but it is to be zero at $a = -1/2$. This behavior is considered to be approximately realized for the numerical stable solution.

The plots of the overlap are displayed in Fig. 5 and the enlarged view around $a = -1/2$ is shown in Fig. 6. We find that the resulting plots approach the expected step function as the truncation level is increased. In the enlarged graph, we can find the point at which the solution becomes trivially zero more clearly than in the vacuum energy case. This critical point approaches $a = -1/2$ as the level is increased.

Hence, all the numerical results for the annihilation of the stable solution confirm that the expanded theory has the vacuum structure as depicted in Fig. 2. In this section, we checked the existence of the stable solution for $a > -1/2$ and it annihilates at $a = -1/2$.

\textsuperscript{*}) Note that $\langle I | V(\pi/2)(c_n + (-1)^n c_{-n}) = 0$ and $\langle I | V(\pi/2)(Q_n + (-1)^n Q_{-n}) = 0$ hold.
§3. Emergence of unstable vacuum

In this section, we consider an unstable solution in the theory expanded around the identity-based solution. For $a > -1/2$, the expanded theory is unstable at $\Phi = 0$. However, the expanded theory is expected to be already on the stable
I. Kishimoto and T. Takahashi

Fig. 7. Vacuum energy expected for the unstable solution $\Phi_2$.

vacuum for $a = -1/2$; namely, $\Phi = 0$ is a stable vacuum at $a = -1/2$. If that is the case, the expanded theory for $a = -1/2$ should have a nontrivial unstable solution corresponding to the perturbative vacuum in the original theory.

Since the unstable solution $\Phi_2$ should correspond to the perturbative open string vacuum, the vacuum energy of the unstable solution should be equal to the D-brane tension (not the minus D-brane tension). Therefore, the vacuum energy of $\Phi_2$ is expected to behave as depicted in Fig. 7. $f_a(\Phi_2)$ is trivially zero for $a > -1/2$, but it should be increased to +1 owing to the emergence of the unstable solution at $a = -1/2$. Similarly, we expect that the gauge invariant overlap for $\Phi_2$ should be -1 at $a = -1/2$ and trivially zero for $a > -1/2$.

Now, let us find the unstable solution $\Phi_2$ using level truncation calculation. We also apply the iterative approximation algorithm as used to find the stable solution in §2. At $a = -1/2$, as the initial configuration $\Phi_2(0)$, we take

$$\Phi^{(0)} = -\frac{32}{9\sqrt{3}} c_1 |0\rangle,$$

which is the nontrivial solution in the level $(0, 0)$ truncation. By the iteration (2.13), we obtain a nontrivial solution $\Phi_2|_{a=-1/2}$. Furthermore, for $a = -1/2 + \epsilon$ ($0 < \epsilon \ll 1$), with this solution $\Phi_2|_{a=-1/2}$ as the initial configuration, we can construct a solution $\Phi_2|_{a=-1/2+\epsilon}$ using the iteration (2.13). In the same manner, for $a = -1/2 + (m+1)\epsilon$, we can uniquely construct a solution $\Phi_2|_{a=-1/2+(m+1)\epsilon}$ with the initial configuration $\Phi_2|_{a=-1/2+m\epsilon}$ ($m = 0, 1, 2, \ldots$). We find that $\Phi_2|_{a}$ reaches the trivial configuration for some $m_0$: $\Phi_2|_{a=-1/2+m_0\epsilon} = 0$. Therefore, for $a \geq -1/2 + m_0\epsilon$, the nontrivial unstable solution $\Phi_2$ vanishes.

As in §2, we continue the iterative procedure until the relative error reaches $10^{-8}$ and examine whether the final configuration is truly physical in terms of its BRST invariance. We find that the solutions $\Phi_2$ constructed as above are consistent with the equation of motion $Q'\Phi + \Phi \ast \Phi = 0$ with increasing level.

The vacuum energy and gauge invariant overlap for $\Phi_2$ at $a = -1/2$ are shown in Table. I.\textsuperscript{4} We find that the vacuum energy approaches the expected value of +1

\textsuperscript{4} The vacuum energy of the unstable solution at $a = -1/2$ was firstly written in Ref. 34) up
Table I. Vacuum energy and gauge invariant overlap for the unstable solution $\Phi_2$ at $a = -1/2$.

| level  | vacuum energy | gauge inv. overlap |
|--------|---------------|-------------------|
| (0,0)  | 2.3105796     | -1.0748441        |
| (2,6)  | 2.5641847     | -1.0156983        |
| (4,12) | 1.6550774     | -0.9539832        |
| (6,18) | 1.6727496     | -0.9207572        |
| (8,24) | 1.4193393     | -0.9377548        |
| (10,30)| 1.4168893     | -0.9110994        |
| (12,36)| 1.3035715     | -0.9237917        |
| (14,42)| 1.2986472     | -0.9056729        |
| (16,48)| 1.2357748     | -0.9229035        |

as the truncation level is increased. At level (16,48), the vacuum energy is about 24% over, although it is about 260% at level (2,6). Moreover, the gauge invariant overlap also takes around the expected value of $-1$. These results suggest that the level truncation approximation is also applicable to the analysis of the unstable solution. Although it is not so efficient compared with that of the stable solution, it is reasonably confirmed that the unstable solution does exist as expected in the expanded theory at $a = -1/2$.

Note that the vacuum energy decreases to +1 with the period of level 4. This is a contrasting fact to the monotonically decreasing tachyon vacuum energy in the original theory. Probably, this behavior is considered to be related to the fact that the kinetic operator (2.12) mixes states with level 2 difference.

In Ref. 5), the vacuum energy up to level (18,54) was extrapolated to a higher level on the basis of the study of an effective potential in the level truncated theory. In fact, in the original theory ($a = 0$), if we fit the stable vacuum energy $f_{a=0}(\Phi_1)$ by a function of the level $L$ such as

$$F_N(L) = \sum_{n=0}^{N} \frac{a_n}{(L+1)^n} \quad (3.2)$$

with the data for the level $(L, 3L)$ ($L = 0, 2, 4, 6, 8, 10, 12, 14, 16; N = 9$), we find that the ‘straightforward’ extrapolation gives the vacuum energy at $L = \infty$ as $\hat{E}_{\infty}^{(16)} = F_9(\infty) = -1.00003$. Similarly, at $a = -1/2$, let us consider a fit of the unstable vacuum energy $f_{a=-1/2}(\Phi_2)$ using $F_N(L)$ (1) with the data for the level $(L, 3L)$ ($L = 0, 4, 8, 12, 16; N = 5$) and (2) with the data for the level $(L, 3L)$ ($L = 2, 6, 10, 14; N = 4$). Then, the straightforward extrapolation gives the vacuum energy at $L = \infty$ as (1) $F_5(\infty) = 0.98107$ and (2) $F_4(\infty) = 0.98146$, respectively. Both of them are 98% of the expected value, which seems to be an encouraging result although this extrapolation may lack a rigorous justification.

We should comment on the increase in the gauge invariant overlap beyond the expected value of $-1$. Because we do not have a reasonable fitting method for the gauge invariant overlap, we have to examine higher level behavior to clarify whether the gauge invariant overlap is away from the expected value.

to level 6. It was also discussed in the context of vacuum string field theory also up to level 6. 35)}
Finally, let us consider the unstable solution $\Phi_2$ for various $a (> -1/2)$. The resulting vacuum energy of the unstable solution is depicted in Fig. 8. The vacuum energy is around the expected value for $a = -1/2$, but it decreases rapidly to zero for increasing $a$.\footnote{Note that in Fig. 8 the parameter $a$ ranges from $-0.5$ to $-0.48$ and this range is much narrower than that of Fig. 3.} This zero vacuum energy is due to the fact that the solution

---

Fig. 8. Vacuum energy of the unstable solution for various $a$.

---

Fig. 9. Gauge invariant overlap of the unstable solution for various $a$. 

---
itself becomes a trivial configuration ($\Phi_2 = 0$) for the parameter $a$ over a critical value. For example, at level 14, the solution converges trivially into zero at about $a = -0.494$. This critical value approaches $a = -1/2$ as the truncation level is increased. We can also find that the critical value is nearly identical to that of the stable solution, at which the stable solution becomes trivial for decreasing $a$ as in the previous section. Thus, we find that the vacuum energy of the unstable solution approaches the step function as expected in Fig. 7 for increasing truncation levels. For the gauge invariant overlap of the unstable solution, the resulting plots are displayed in Fig. 9. Similarly to the vacuum energy, the overlap becomes increasingly closer to the expected behavior for higher truncation level.

Although the above results are encouraging for our conjecture, we eventually need more higher level investigation to provide more reliable and quantitative results.

§4. Summary and discussion

We have found that the stable and unstable solutions in the Siegel gauge, $\Phi_1$ and $\Phi_2$, numerically exist in the theory expanded around the identity-based solution. For these solutions, we have evaluated the vacuum energy and the gauge invariant overlap in terms of level truncation approximation up to level (14, 42). As the truncation level is increased, the plots of the two gauge invariants for the parameter $a$ become remarkably closer to the behavior expected from the vacuum structure in Fig. 1. These results strongly support our expectation that the identity-based solution corresponds to a trivial pure gauge configuration for $a > -1/2$, but it can be regarded as the tachyon vacuum solution for $a = -1/2$.

In this paper, we have considered the identity-based solution constructed only by the specific function (2-6). However, it is possible to construct many identity-based solutions for various functions. Concerning the stable solution, it was found that the vacuum energy for various functions behaves similarly to that for the function in this paper. To clarify the nature of the identity-based solution, we should study unstable solutions further for the various functions. Moreover, it would be better to study the extrapolation of our analysis to higher levels. In any case, it is necessary to compute several quantities by higher level truncation.

Our results suggest that it may be possible to evaluate directly the vacuum energy and the gauge invariant overlap of the identity-based solution. While the results are encouraging, the direct calculation remains one of the most difficult issues in string field theory. This difficulty seems to come from the lack of a reasonable regularization scheme for the identity-based solution. However, the success of the numerical analysis is a characteristic feature of the identity-based solution. In addition to the Schnabl-type solutions, the identity-based solution seems to provide complementary approaches to a deeper understanding of the string field theory.

In particular, for the identity-based solution, there is an interesting possibility of understanding closed strings on the tachyon vacuum. The expanded theory

---

*1 We find that the critical values for $\Phi_1$ and $\Phi_2$ exactly coincide with the level 0 approximation.
around the identity-based solution provides a worldsheet picture of perturbative amplitudes.\(^{38},39,41,42\) For \(a > -1/2\), the worldsheet has boundaries and this is consistent with the expectation that the solution corresponds to a trivial pure gauge configuration. However, the boundary existing for \(a > -1/2\) shrinks into a point as \(a\) approaches \(-1/2\), and the worldsheet is given as a closed surface without boundaries. This fact also confirms that the identity-based solution at \(a = -1/2\) can be regarded as the tachyon vacuum where there are no open strings. Consequently, we expect that the worldsheet picture in the expanded theory clarifies the existence of closed strings on the tachyon vacuum and, moreover, it may provide a quantitative computational method of closed string amplitudes via open string fields.

**Acknowledgements**

The work of I. K. was supported in part by a Special Postdoctoral Researchers Program at RIKEN and a Grant-in-Aid for Young Scientists (#19740155) from MEXT of Japan. The work of T. T. was supported in part by a Grant-in-Aid for Young Scientists (#18740152) from MEXT of Japan. The level truncation calculations based on Mathematica were carried out partly on the computer sushiki at Yukawa Institute for Theoretical Physics in Kyoto University.

**References**

1) E. Witten, Nucl. Phys. B 268 (1986), 253.
2) V. A. Kostelecky and S. Samuel, Nucl. Phys. B 336 (1990), 263.
3) A. Sen and B. Zwiebach, J. High Energy Phys. 03 (2000), 002; hep-th/9912249.
4) N. Moeller and W. Taylor, Nucl. Phys. B 583 (2000), 105; hep-th/0002237.
5) D. Gaiotto and L. Rastelli, J. High Energy Phys. 08 (2003), 048; hep-th/0211012.
6) M. Schnabl, Adv. Theor. Math. Phys. 10 (2006), 433; hep-th/0511286.
7) Y. Okawa, J. High Energy Phys. 04 (2006), 055; hep-th/0603159.
8) E. Fuchs and M. Kroyter, J. High Energy Phys. 05 (2006), 006; hep-th/0603195.
9) I. Ellwood and M. Schnabl, J. High Energy Phys. 02 (2007), 096; hep-th/0606142.
10) T. Erler, J. High Energy Phys. 01 (2008), 013; arXiv:0707.4591.
11) I. Y. Aref’eva, R. V. Gorbachev and P. B. Medvedev, arXiv:0804.2017.
12) E. Fuchs and M. Kroyter, J. High Energy Phys. 10 (2008), 054; arXiv:0805.4386.
13) I. Y. Aref’eva, R. V. Gorbachev, D. A. Grigoryev, P. N. Kromov, M. V. Maltsev and P. B. Medvedev, arXiv:0901.4533.
14) T. Takahashi and S. Tanimoto, J. High Energy Phys. 03 (2002), 033; hep-th/0202133.
15) T. Takahashi and S. Tanimoto, Prog. Theor. Phys. 106 (2001), 863; hep-th/0107046.
16) I. Kishimoto and T. Takahashi, J. High Energy Phys. 11 (2005), 051; hep-th/0506240.
17) I. Kishimoto and T. Takahashi, J. High Energy Phys 01 (2006), 013; hep-th/0510224.
18) J. Khuson, J. High Energy Phys. 04 (2002), 043; hep-th/0202045.
19) J. Khuson, hep-th/0205294.
20) J. Khuson, hep-th/0208028.
21) J. Khuson, Int. J. Mod. Phys. A 19 (2004), 4695; hep-th/0209255.
22) J. Khuson, J. High Energy Phys. 12 (2003), 050; hep-th/0303199.
23) O. Lechtenfeld, A. D. Popov and S. Uhlmann, Nucl. Phys. B 637 (2002), 119; hep-th/0204155.
24) M. Sakaguchi, hep-th/0112135.
25) I. Kishimoto and T. Takahashi, Prog. Theor. Phys. 108 (2002), 591; hep-th/0205275.
26) T. Takahashi, Nucl. Phys. B 670 (2003), 161; hep-th/0302182.
27) T. Takahashi and S. Zeze, Prog. Theor. Phys. 110 (2003), 159; hep-th/0304261.
28) I. Kishimoto and T. Takahashi, Prog. Theor. Phys. 121 (2009), 695; arXiv:0902.0445.
29) H. Hata and S. Shinohara, J. High Energy Phys. 09 (2000), 035; hep-th/0009105.
30) B. Zwiebach, Mod. Phys. Lett. A 7 (1992), 1079; hep-th/9202015.
31) A. Hashimoto and N. Itzhaki, J. High Energy Phys. 01 (2002), 028; hep-th/0111092.
32) T. Kawano, I. Kishimoto and T. Takahashi, Nucl. Phys. B 803 (2008), 135; arXiv:0804.1541.
33) I. Ellwood, J. High Energy Phys. 08 (2008), 063; arXiv:0804.1131.
34) S. Zeze, “Exact Solutions and Tachyon Condensation in String Field Theory”, PhD thesis, Osaka City Univ. (2003).
35) N. Drukker and Y. Okawa, J. High Energy Phys. 06 (2005), 032; hep-th/0503068.
36) W. Taylor, J. High Energy Phys. 03 (2003), 029; hep-th/0208149.
37) I. Kishimoto and T. Takahashi, in preparation.
38) S. Zeze, Prog. Theor. Phys. 112 (2004), 863; hep-th/0405097.
39) Y. Igarashi, K. Itoh, F. Katsumata, T. Takahashi and S. Zeze, Prog. Theor. Phys. 114 (2005), 695; hep-th/0502042.
40) Y. Igarashi, K. Itoh, F. Katsumata, T. Takahashi and S. Zeze, Prog. Theor. Phys. 114 (2006), 1269; hep-th/0506083.
41) N. Drukker, Phys. Rev. D 67 (2003), 126004; hep-th/0207266.
42) N. Drukker, J. High Energy Phys. 08 (2003), 017; hep-th/0301079.