Geodesics and geodesic deviation in a two-dimensional black hole

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Abstract

We introduce an exactly solvable example of timelike geodesic motion and geodesic deviation in the background geometry of a well-known two-dimensional black hole spacetime. The effective potential for geodesic motion turns out to be either a harmonic oscillator or an inverted harmonic oscillator or a linear function of the spatial variable, corresponding to the three different domains of a constant of the motion. The geodesic deviation equation also is exactly solvable. The corresponding deviation vector is obtained and the nature of the deviation is briefly discussed by highlighting a specific case.
I. INTRODUCTION

Learning to solve the geodesic and geodesic deviation equations is an integral part of a first course on general relativity. However, most texts on general relativity do not contain a sufficient number of solvable examples that illustrate the behavior of geodesics in a nontrivial geometry. This lack is largely due to the fact that the geodesic equations as well as the geodesic deviation equations are not easy to solve exactly in most of the standard geometries of interest. A traditional example is the case of the line element on the surface of a two-dimensional sphere for which the geodesic and the geodesic deviation equations can be solved analytically. Some other exactly solvable examples in two dimensions include Rindler spacetime, the line element on a cylinder and the hyperbolic plane.

In this article, we will discuss a two-dimensional example (with a diagonal line element of Lorentzian signature $(-+)\), i.e. of the form $ds^2 = g_{00}dt^2 + g_{11}dx^2$, with $g_{00}$ everywhere negative and $g_{11}$, positive), for which, like the sphere, there are exact solutions for geodesics as well as geodesic deviation. The line element for this example arises in the context of the stringy black holes. In fact, this is the geometry of the first stringy black hole discovered in the early nineties by Mandal et al. We will not go into the details of how this line element is obtained. Interested readers may read the original references to enlighten themselves on this aspect. For our purposes here, the line element represents a geometry that is given to us, and we see that there are exact solutions for geodesics and geodesic deviation for test particles moving in this geometry.

II. THE LINE ELEMENT

We begin by writing down the line element of interest:

$$ds^2 = -(1 - \frac{M}{r})dt^2 + \frac{kdr^2}{4r^2(1 - \frac{M}{r})},$$  \hspace{1cm} (1)

where $M$ is equivalent to $GM/c^2$ with the choice of units $G = 1$ and $c = 1$. Similarly $k$ is required to have dimensions of length squared in order to render the metric coefficient $g_{rr}$ dimensionless.

The domain for the radial coordinate is $r > M$ and $t$ is allowed to vary from $-\infty < t < \infty$. The physical meaning of the constants $M$ and $k$ can be explained in the context of two-dimensional string theory. For completeness (and also with the risk of introducing some
unexplained jargon), we mention that $M$ is related to mass and $k$ to the central charge parameter.

Several purely geometrical facts about this line element are worth noting. At $r = M$, $g_{tt} \to 0$ and $g_{rr} \to \infty$. If $r < M$, the signs of $g_{tt}$ and $g_{rr}$ are interchanged. Readers familiar with some general relativity (say, the Schwarzschild solution) would call $r = M$ the horizon and thereby, characterize the line element as a black hole provided there exists a singularity (which is the case here, as we shall see later) inside the horizon (that is, in the region $r < M$). As the title of this article suggests, the geometry represents a two-dimensional black hole. As is common knowledge, a black hole is a region from where nothing can escape. If an object falls into a black hole (that is, the object crosses the horizon), the object ends up getting stretched and torn apart by the gravitational forces within. This distortion is quantified through the deviation of geodesics. The ultimate fate is collapse into a singularity where the matter density is infinitely large, thereby being a region of extreme curvature. We shall concern ourselves with the $r > M$ region.

The pathology in the line element (that is, $g_{rr} \to \infty$) at $r = M$ is entirely a coordinate artifact (a poor choice of coordinates). In addition, it is worth noting, that unlike the case for similar but four-dimensional line elements where the horizon is a sphere or some other two-dimensional surface, here it is a point and falling into a black hole is manifest in crossing this point and entering the region of no return. The spacetime here is undoubtedly very similar to the two-dimensional $r$–$t$ section of the static, spherically symmetric four-dimensional black holes (like the Schwarzschild metric). Many of the features of black holes can be understood via this example. In fact an extensive research literature exists on two-dimensional black holes where interesting phenomena such as Hawking radiation and information loss have been discussed (see Ref. 2 and references therein and Ref. 3).

The line element in Eq. (1) also can be written in the following form:\footnote{3}

\begin{equation}
\begin{aligned}
    ds^2 &= -\tanh^2\left(\frac{Mx}{2\sqrt{k}}\right)dt^2 + dx^2, \\
    &= -\tanh^2\left(\frac{Mx}{2\sqrt{k}}\right)dt^2 + dx^2,
\end{aligned}
\end{equation}

where we have used the coordinate transformation $x = \frac{\sqrt{k}}{M} \cosh^{-1}(\frac{2r}{M} - 1) > 0$. Notice that the singular nature of the metric function $g_{11}$ at $r = M$ is no longer there in this form of the line element. A third form of the line element is the Kruskal extension or the maximal analytical extension done exactly in the same way as for the Schwarzschild metric. (For details on how to construct Kruskal coordinates see any standard text on general relativity.)
If we introduce the coordinates $u$ and $v$ defined by:

\begin{align*}
    u &= t - \frac{\sqrt{k}}{2} \ln |r - M| \quad \text{(3a)} \\
    v &= t + \frac{\sqrt{k}}{2} \ln |r - M|, \quad \text{(3b)}
\end{align*}

and $U = -\sqrt{k} \exp(-u/\sqrt{k})$, $V = \sqrt{k} \exp(v/\sqrt{k})$, we can rewrite the line element as:

\[ ds^2 = \frac{-dUdV}{M \sqrt{k} - UV} \quad \text{(4)} \]

In Eq. (3) the coordinates $u$ and $v$ run from $-\infty$ to $\infty$ and $-\infty < U < 0$, $0 < V < \infty$. It is possible to further rewrite the line element using coordinates $2T = V + U$, $2X = V - U$. The Kruskal form of the line element is an ‘extension’ over the whole domain of $r$ and $t$. The geometry of this Kruskal extension has features similar to that of the Schwarzschild.

### III. TIMELIKE GEODESICS

Our main goal is to find timelike geodesics (that is, trajectories of test particles of nonzero rest mass) in the line element given in Eq. (1). The equations and constraint for timelike geodesics are given as:

\[ \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0 \quad \text{(5a)} \\
    g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1, \quad \text{(5b)} \]

where $x^\mu$ are the coordinates (here $r$ and $t$), and $\mu$ takes on values 0 and 1 for $t$ and $r$ respectively. Equation (5b) is the timelike constraint on the trajectories.

The differentiation (indicated by dots) in Eq. (5) is with respect to the affine parameter (proper time) $\lambda$. The Christofel symbol $\Gamma^\mu_{\nu\lambda}$ is given as:

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} [g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}], \quad \text{(6)} \]

where a comma denotes ordinary differentiation with respect to coordinate $x^\mu$. The nonzero components of $\Gamma^\mu_{\nu\lambda}$ for the line element in Eq. (1) are

\begin{align*}
    \Gamma^0_{10} &= \Gamma^0_{01} = \frac{M}{2r(r - M)} \quad \text{(7a)} \\
    \Gamma^1_{00} &= \frac{2M(r - M)}{kr} \quad \text{(7b)} \\
    \Gamma^1_{11} &= -\frac{(2r - M)}{2r(r - M)} \quad \text{(7c)}
\end{align*}
The reader might ask why aren’t we interested in null geodesics (that is, trajectories of test particles having zero rest mass, such as light rays)? In two dimensions null geodesics are the same as those for the flat Minkowski line element. Note that any two-dimensional line element (with Euclidean or Lorentzian signature) can be written as $ds^2 = \Omega^2(x, t)[-dt^2 + dx^2]$, where $\Omega^2(x, t)$ is a nonzero, positive definite function of $x$ and $t$ and is called the conformal factor. Null geodesics require that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, which implies that in two dimensions, null geodesics for Minkowski spacetime are the same as those for any other nontrivially curved spacetime.

The timelike constraint in Eq. (5) for the line element of interest here (Eq. (1)), is given as:

$$-(1 - \frac{M}{r})\dot{t}^2 + \frac{k}{4r^2(1 - \frac{M}{r})}\dot{r}^2 = -1.$$  \hspace{1cm} (8)

The geodesic equations (one each for $r$ and $t$) are

$$\dot{t} = \frac{E}{2(1 - \frac{M}{r})}$$  \hspace{1cm} (9)

$$\ddot{r} - \frac{1}{2}[\frac{2r - M}{r^2 - Mr}]\dot{r}^2 + \frac{E^2Mr}{2k(r - M)} = 0,$$  \hspace{1cm} (10)

where in Eq. (9) we have substituted for $\dot{t}$ from Eq. (9). $E$ in the above is a constant of the motion and arises due to the absence of the coordinate $t$ in the metric coefficients. If we substitute $\dot{t}$ from Eq. (9) in the timelike constraint in Eq. (8), we obtain

$$\dot{r}^2 = \frac{1}{k}[((E^2 - 4)r^2 + 4Mr] = -V(r) + E_0,$$  \hspace{1cm} (11)

where $E_0$ is a constant and $V(r)$ is like an effective potential given by:

$$V(r) = E_0 - \frac{1}{k}[((E^2 - 4)r^2 + 4Mr].$$  \hspace{1cm} (12)

Notice that the effective potential is like that of the harmonic oscillator for $E^2 < 4$ and like that of an inverted harmonic oscillator for $E^2 > 4$. For $E^2 = 4$ it is linear with a negative slope. We can make a coordinate change (translation) in order to see the harmonic oscillator/inverted harmonic oscillator forms explicitly. The potentials for the three cases are shown in Fig. 1.

If we substitute $\dot{r}$ and its derivative with respect to $\lambda$ from the timelike constraint (11) in Eq. (10) for $r$, the equation is satisfied identically. Therefore, it is enough to just obtain
$r(\lambda)$ by integrating the timelike constraint for the three cases ($E^2 < 4$, $E^2 = 4$, $E^2 > 4$) and obtain $r(\lambda)$ and subsequently $t(\lambda)$. These results are given below.

**Case 1: $E^2 < 4$.**

$$r(\lambda) = \frac{B}{2A}(1 + \cos a\lambda) \quad (13a)$$

$$t(\lambda) = \frac{E}{2} \left[ \lambda + \frac{1}{qa} \ln \frac{q + \tan c\lambda}{q - \tan c\lambda} \right], \quad (13b)$$

where $q^2 = (1 + f)/(1 - f)$, $A = 4 - E^2$, $a = \sqrt{A/k}$, $c = a/2$, $f = 1 - A/2$, and $B = 4M$.

**Case 2: $E^2 = 4$.**

$$r(\lambda) = \frac{M}{k} \lambda^2 \quad (14a)$$

$$t(\lambda) = \lambda + \sqrt{\frac{k}{4}} \ln \left( \frac{\lambda - \sqrt{k}}{\lambda + \sqrt{k}} \right). \quad (14b)$$

**Case 3: $E^2 > 4$.**

$$r(\lambda) = \frac{B}{2A}(-1 + \cosh a\lambda) \quad (15a)$$

$$t(\lambda) = \frac{E}{2} \left[ \lambda + \frac{q}{a} \ln \frac{\tanh c\lambda - q}{\tanh c\lambda + q} \right], \quad (15b)$$

where $q^2 = (f - 1)/(f + 1)$, $A = E^2 - 4$, $a = \sqrt{A/k}$, $c = a/2$, $f = 1 + A/2$, and $B = 4M$.

In each case, the functional form of $r(t)$ can be obtained by utilizing the parametric forms $r(\lambda), t(\lambda)$. Figures 2–4 illustrate the above function. The origin of the coordinates in each of these figures is chosen to be $r = 1$ and $t = 0$. In particular, Fig. 4 shows the actual radial trajectory $r(t)$ as a function of time. It is not possible to write $r$ as an explicit function of $t$ because of the nature of the functional dependence of $t$ on $\lambda$. However, we may write $t(r)$ by inverting $r(\lambda)$ to $\lambda(r)$ and then using this function in $t(\lambda(r))$. Notice that for $E^2 < 4$, the trajectory hovers around the horizon (extending only up to $r = 2M = 2$), reaching the value $r = M = 1$ only asymptotically. However, for $E^2 \geq 4$ there is no such restriction and the trajectory can extend from $M$ to $\infty$.

It should be mentioned that the “discontinuities” in some of the plots arise because of the choice of the parameter ($M = 1$ implies that $r > 1$) and, consequently, the allowed domain of the independent variable. They are not real discontinuities of the functions, as is apparent from the solutions.
IV. GEODESIC DEVIATION

We now consider the geodesic deviation equation in order to analyze the behavior of the spreading of the geodesic curves obtained in Sec. III. Before we discuss the geodesic deviation in the black hole line element, we briefly recall some of the basic notions of geodesic deviation. In a curved spacetime geodesics are not the straight lines of our Euclidean intuition. For instance, the geodesics on the surface of a sphere are the great circles. To understand how geodesics are spread out in a spacetime of nontrivial curvature, we consider a bundle of geodesics around a specific geodesic (called the central geodesic). At every point on this central geodesic we erect little normals. This normal is the connecting vector, so-called because as we move along the connecting vector, we reach neighboring geodesics. The reader might ask why normal deformations only? The answer is that tangential deformations are taken into account through the reparametrisation of the affine parameter \( \lambda \), or, in other words, they do not deform the trajectory to a neighboring one.

It is obvious that the nature of the connecting vector (or deviation vector) in a nontrivial spacetime depends crucially on the properties of the spacetime, or more importantly, as it turns out, on its curvature properties. For instance, on the surface of a sphere, if we look at two neighboring great circle arcs connecting the north and south poles, the distance between them increases, reaches a certain value at the equator, and then decreases until it reaches a zero value at the other pole. The deviation vector essentially measures this effect, which is dictated by the properties of the Riemann tensor \( R^\mu_{\nu\lambda\rho} \) given by:

\[
R^\mu_{\nu\lambda\rho} = \Gamma^\mu_{\nu\rho,\lambda} - \Gamma^\mu_{\nu\lambda,\rho} + \Gamma^\mu_{\sigma\lambda} \Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho} \Gamma^\sigma_{\nu\lambda}.
\]  

(16)

The Ricci tensor and the Ricci scalar are obtained from the above Riemann tensor by appropriate contractions with the metric tensor (Ricci tensor, \( R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \) and Ricci scalar, \( R = g^{\mu\nu} R_{\mu\nu} \)).

We know from Einstein’s general relativity that geometry is equivalent to the presence of a gravitational field. Thus the deviation of trajectories from each other is a measure of the relative gravitational force between the objects moving on these separate, neighboring, trajectories. We now state the geodesic deviation equation and solve it for the geodesics in the black hole line element, Eq. (1), discussed here.
Given a geodesic curve, we identify the tangent and normal to it as the vectors $e^\mu$ and $n^\mu$ respectively. These satisfy the orthonormality conditions:

\begin{align}
  g_{\mu\nu}e^\mu e^\nu &= -1 \\
  g_{\mu\nu}n^\mu n^\nu &= 1 \\
  g_{\mu\nu}e^\mu n^\nu &= 0.
\end{align} (17a) (17b) (17c)

We make the assumptions:

\begin{align}
  e^\mu &\equiv (\dot{t}, \dot{r}) \\
  n^\mu &\equiv (f(r)\dot{r}, g(r)\dot{t}).
\end{align} (18a) (18b)

Note that other choices can be made and thus the above definitions in Eq. (18) are by no means unique. If we substitute the assumptions in Eq. (18) into the orthonormality conditions Eq. (17), we obtain the following relations for $f(r)$ and $g(r)$:

\begin{align}
  f(r) &= \sqrt{g_{11} - g_{00}} \\
  g(r) &= \sqrt{-\frac{g_{00}}{g_{11}}}. (19a) (19b)
\end{align}

Therefore the normal vector $n^\mu$ takes the form:

\begin{align}
  n^0 &= \frac{\sqrt{k}}{2r(1 - \frac{M}{r})}\dot{r} \\
  n^1 &= \frac{2r(1 - \frac{M}{r})}{\sqrt{k}}\dot{t}. (20a) (20b)
\end{align}

The equation for the normal deformations $\eta$ (where $\eta^\mu = \eta n^\mu$, $\eta^\mu$ being the deviation vector) is given by:

\[ \frac{d^2\eta}{d\lambda^2} + R_{\mu\nu\rho\sigma} e^\mu n^\nu e^\rho n^\sigma \eta = 0. \] (21)

The nonzero components of Riemann tensor in the coordinate frame are given by:

\begin{align}
  R_{0101} &= R_{1010} = -\frac{M}{2r^3} \\
  R_{0110} &= R_{1001} = \frac{M}{2r^3}. (22a) (22b)
\end{align}

There is actually only one independent component of the Riemann tensor in two dimensions (recall that in $n$ dimensions, the number of independent components is $n^2(n^2 - 1)/12$).
The nonzero Ricci tensor components in the coordinate frame \( R_{\mu \nu} = R^\alpha_{\mu \alpha \nu} \) and the Ricci scalar \( R = g^{\mu \nu} R_{\mu \nu} \) are

\[
R_{00} = -\frac{2M(r - M)}{kr^2} \tag{23a}
\]

\[
R_{11} = \frac{M}{2r^2(r - M)} \tag{23b}
\]

\[
R = \frac{4M}{kr} \tag{23c}
\]

The Riemann tensor components and Ricci scalar diverge as \( r \to 0 \). The geometry becomes singular as \( r \to 0 \). However, \( r \to M \) does not seem to be a preferred point (in comparison to \( r = 0 \) say), a fact that re-emphasizes the point that the pathology at \( r = M \) in the line element in Eq. (1) is largely a coordinate artifact. Of course, the special nature of \( r = M \) is that it is a horizon with the metric signature changing sign as we move from \( r > M \) to \( r < M \).

If we substitute the Riemann tensor components for the two-dimensional black hole line element and use the normal and tangent mentioned in Eq. (18), we find that the deviation equation reduces to the following equations for the three cases discussed earlier.

**Case 1:** \( E^2 < 4 \).

\[
\frac{d^2 \eta}{d \lambda^2} - 2 \sec^2 \bar{\lambda} \eta = 0, \tag{24}
\]

where \( \bar{\lambda} = \frac{1}{2} \sqrt{\frac{4 - E^2}{k}} \lambda \). The two linearly independent solutions to Eq. (24) are

\[
\eta(\bar{\lambda}) = C_1 \tan \bar{\lambda} + C_2(\bar{\lambda} \tan \bar{\lambda} + 1). \tag{25}
\]

**Case 2:** \( E^2 = 4 \).

\[
\frac{d^2 \eta}{d \lambda^2} - \frac{2}{\lambda^2} \eta = 0. \tag{26}
\]

The solutions are

\[
\eta(\lambda) = C_1 \lambda^2 + C_2 \frac{1}{\lambda}. \tag{27}
\]

**Case 3:** \( E^2 > 4 \).

\[
\frac{d^2 \eta}{d \lambda^2} - 2\cosech^2 \bar{\lambda} \eta = 0, \tag{28}
\]

where \( \bar{\lambda} = \frac{1}{2} \sqrt{\frac{E^2-4}{k}} \lambda \). In this case, the solutions are

\[
\eta(\bar{\lambda}) = C_1 \coth \bar{\lambda} + C_2(\bar{\lambda} \coth \bar{\lambda} - 1). \tag{29}
\]
In Eqs. (25), (27), and (29), $C_1$ and $C_2$ are separate arbitrary constants in each case. These constants should be determined by the initial conditions (that is, the value of the initial separation between a pair of geodesics at a specific value of the affine parameter and the value of the rate of change of the initial separation at the same value of $\lambda$, similar to the conditions for obtaining particular solutions of second-order, linear, ordinary differential equations).

Now that we have obtained $\eta$ for each case, it is easy to write down the deviation vectors, which are given by $\eta^\mu = \eta^\mu$ (where $\mu = 0, 1$). Let us now discuss a representative case, say $E^2 = 4$. The range of $\lambda$ (for $r > M = 1$) is between $-\infty < \lambda < -1$ and $1 < \lambda < \infty$. If we consider a pair of neighboring geodesics at say $\lambda = 1.5$ and assume $\dot{\eta}$ is positive, then we are forced to choose the solution for $\eta$ to be proportional to $\lambda^2$. The evolution of $\eta$ then suggests that as we move further and further to larger positive $\lambda$, the geodesics spread out (diverge). On the other hand, if we assume $\dot{\eta}$ to be negative, then we must choose the $\frac{1}{\lambda}$ solution, which says that for larger $\lambda$ (which corresponds to larger $r$ as well), the geodesics converge, at least locally. However, as we go to larger $\lambda$, $\dot{\eta}$ also becomes smaller and approaches zero as $\lambda \to \infty$ ($r \to \infty$). Therefore, ultimately the geodesics become parallel to each other.

It is also useful to note that $d^2\eta/d\lambda^2$, the measure of the gravitational force, is like a relative force which acts to change the separation between a pair of neighboring geodesics. We can easily substitute the values of $\lambda$ (or $\bar{\lambda}$) from the expressions of $r(\lambda)$ in the geodesic deviation equations (24), (26), and (28) to analyze the nature of the force. As an example, let us consider the case $E^2 = 4$. For $\eta = \lambda^2$, the relative force is constant throughout and is given by:

$$\frac{d^2\eta}{d\lambda^2} = 2,$$  

(30)

whereas for $\eta = 1/\lambda$, we have

$$\frac{d^2\eta}{d\lambda^2} = 2 \left(\frac{M}{kr}\right)^{3/2}.$$  

(31)

Equation (31) tells us that the force vanishes as $r$ goes to infinity (recall that the geometry is flat in this limit and the geodesics will become parallel to each other), whereas for $r \to M$, it has the value $2/k^{3/2}$. A similar analysis can be carried out for the cases with $E^2 < 4$ and $E^2 > 4$.

A point worth mentioning here (without proof) is that in the above geometry, the geodesics do not seem to converge to a point within a finite value of $\lambda$ in any of the cases.
because of the fact that the Ricci scalar is non-negative (it is zero only when \( r \to \infty \)). In two dimensions, timelike geodesics do not converge to a point (focus) within a finite value of the affine parameter unless \( R \leq 0.6 \).

V. CONCLUSIONS

In summary, the exact solutions of the geodesic deviation equation can provide us with a better understanding of the nature of the separation between geodesics. Readers can further investigate the deviation vectors in the cases not discussed above in order to improve their understanding of geodesic deviation. Trajectories in two dimensions are of the form \( r(t) \). The nontrivial curvature of the spacetime is responsible for their difference from the usual ‘straight lines’ in Minkowski spacetime—the solutions to the geodesic equation obtained in the above line element provide examples. If we take pairs of trajectories it is obvious that they may converge towards or diverge away from each other—a measure of this effect is geodesic deviation. We have illustrated this effect in our example above.

Our aim has been to provide an example where the geodesic and geodesic deviation equations can be exactly solved. It has been shown that for the line element of a two-dimensional black hole, exact, reasonably simple solutions do exist. We hope that the relevance of the line element, as well as the simplicity of the solutions will attract students and teachers to use this example when teaching a first course in general relativity. We also mention that, as far as we know, the exact solutions to the geodesic and the geodesic deviation equations do not exist in the literature and might be of some interest to researchers as well. There are innumerable solutions representing black holes and cosmologies in two-dimensional gravity, most of which have appeared largely in the last decade and a half. We believe that a carefully chosen section of these line elements can be used as worthwhile teaching tools in a first course on general relativity.
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1 A few of the standard texts on general relativity include R. D'Inverno, *Introducing Einstein's Relativity* (Oxford University Press, 1995); B. Schutz, *A First Course in General Relativity* (Cambridge University Press, 1990); W. Rindler, *Relativity* (Oxford University Press, 2001); S. Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, 1971); C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1972); L. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Pergamom Press, Oxford, 1962); R. M. Wald, *General Relativity* (University of Chicago Press, 1985).

2 For an extensive review of two-dimensional gravity, see D. Gromiller, W. Kummer and D. Vassilevich, “Dilaton gravity in two dimensions,” bhep-th/0204253 (to appear in Physics Reports) and references therein; A lucid introduction to string theory is available at <www.theory.caltech.edu/people/patricia (home page of Patricia Schwarz)>; More extensive texts on string theory are M. S. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, 1987), Vols. I and II; J. Polchinski, *String Theory* (Cambridge University Press, 1998), Vols. I and II.

3 J. Harvey and A. Strominger, “Quantum aspects of black holes in string theory and quantum gravity,” in *String Theory and Quantum Gravity '92*, edited by J. Harvey et al. (World Scientific, 1993).

4 The original papers where this line element was first proposed are G. Mandal, A. M. Sengupta, and S. R. Wadia, “Classical solutions of two dimensional string theory,” Mod. Phys. Letts. A 6, 1685–1692 (1991); E. Witten, “On string theory and black holes,” Phys. Rev. D 44, 314–324 (1991).

5 A good introduction to geodesics and geodesic deviation in particular is available in the text by
The statement made here is based on advanced topics such as the Raychaudhuri equation and geodesic focusing theorems (a good, though advanced discussion on these topics is available in the text by Wald referred in Ref. 1). In a nutshell, convergence of timelike geodesics (that is, focusing) within a finite value of the affine parameter occurs only when the timelike convergence condition $R_{\mu\nu} \xi^\mu \xi^\nu \geq 0$ is obeyed. In two dimensions $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$ always and $g_{\mu\nu} \xi^\mu \xi^\nu = -1$ (timelike condition). Therefore convergence/focusing occurs within a finite $\lambda$ only if $R \leq 0$. For some details on this (particularly in two dimensions and in the context of stringy black holes), see S. Kar, “Stringy black holes and energy conditions,” Phys. Rev. D 55, 4872–4879 (1997).
Figure Captions

**Fig. 1:** Effective potential ($y$-axis) as a function of $r$ ($x$-axis) for three cases $E^2 = 1$ (Fig. 1a), 4 (Fig. 1b), 7 (Fig. 1c) respectively; $E_0 = 10$, $k = 1$, $M = 1$ for all three cases.

**Fig. 2:** $r(\lambda)$ ($y$-axis) versus $\lambda$ ($x$-axis) as in Eqs. 13a (Fig. 2a), 14a (Fig. 2b) and 15a (Fig. 2c) with $A = 3$, $k = 1$, and $M = 1$.

**Fig. 3:** $t(\lambda)$ ($y$-axis) versus $\lambda$ ($x$-axis) as in Eqs. 13b (Fig. 3a), 14b (Fig. 3b) and 15b (Fig. 3c) with $A = 3$, $k = 1$, and $M = 1$.

**Fig. 4:** $r(t)$ ($y$-axis) versus $t$ ($x$-axis) from Eqs. 13a and 13b (Fig. 4a), 14a and 14b (Fig. 4b), 15a and 15b (Fig. 4c) with $A = 3$, $k = 1$, and $M = 1$. 

Fig. 1: Effective potential ($y$-axis) as a function of $r$ ($x$-axis) for three cases $E^2 = 1$ (Fig. 1a), 4 (Fig. 1b), 7 (Fig. 1c) respectively; $E_0 = 10$, $k = 1$, $M = 1$ for all three cases.
Fig. 2a: $E^2 < 4$

Fig. 2b: $E^2 = 4$

Fig. 2c: $E^2 > 4$

**Fig. 2:** $r(\lambda)$ (y-axis) versus $\lambda$ (x-axis) as in Eqs. 13a (Fig. 2a), 14a (Fig. 2b) and 15a (Fig. 2c) with $A = 3$, $k = 1$, and $M = 1$. 
Fig. 3a: $E^2 < 4$

Fig. 3b: $E^2 = 4$

Fig. 3c: $E^2 > 4$

Fig.3: $t(\lambda)$ (y-axis) versus $\lambda$ (x-axis) as in Eqs. 13b (Fig. 3a), 14b (Fig. 3b) and 15b (Fig. 3c) with $A = 3$, $k = 1$, and $M = 1$. 
Fig. 4a: $E^2 < 4$

Fig. 4b: $E^2 = 4$

Fig. 4c: $E^2 > 4$

Fig. 4: $r(t)$ (y-axis) versus $t$ (x-axis) from Eqs. 13a and 13b (Fig. 4a), 14a and 14b (Fig. 4b), 15a and 15b (Fig. 4c) with $A = 3$, $k = 1$, and $M = 1$ in all three cases.