Limitations of scaling and universality in stock market data

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We present evidence, that if a large enough set of high resolution stock market data is analyzed, certain analogies with physics – such as scaling and universality – fail to capture the full complexity of such data. Despite earlier expectations, the mean value per trade, the mean number of trades per minute and the mean trading activity do not show scaling with company capitalization, there is only a non-trivial monotonous dependence. The strength of correlations present in the time series of traded value is found to be non-universal: The Hurst exponent increases logarithmically with capitalization. A similar trend is displayed by intertrade time intervals. This is a clear indication that stylized facts need not be fully universal, but can instead have a well-defined dependence on company size.

In the last decade, an increasing number of physicists is becoming devoted to the study of economic and financial phenomena [1, 2, 3]. One of the reasons for this tendency is that societies or stock markets can be seen as strongly interacting systems. Since the early 70’s, physics has developed a wide range of concepts and models to efficiently treat such topics, these include (fractal and multifractal) scaling, frustrated disordered systems, and far from equilibrium phenomena. To understand how similarly complex patterns arise from human activity, albeit from equilibrium phenomena. To understand how similarly complex patterns arise from human activity, albeit truly challenging, seems a natural continuation of such efforts.

While a remarkable success has been achieved [4, 5, 6], studies in econophysics are often rooted in possible analogies, even though there are important differences between physical and financial systems. Despite the obvious similarities to interacting systems here we would like to emphasize the discrepancy in the levels of description. For example, in the case of a physical system undergoing a second order phase transition, it is natural to assume scaling on profound theoretical grounds and the (experimental or theoretical) determination of, e.g., the critical exponents is a fully justified undertaking. There is no similar theoretical basis for the financial market whatsoever, therefore in this case the assumption of power laws should be considered only as one possible way of fitting fat tailed distributions [7, 8]. Also, the reference to universality should not be plausible as the robustness of qualitative features – like the fat tail of the distributions – is a much weaker property. While we fully acknowledge the process of understanding based on analogies as an important method of scientific progress, we emphasize that special care has to be taken in cases where the theoretical support is sparse.

The aim of this paper is to summarize some recent advances that help to understand these fundamental differences. We present evidence, that the size of companies strongly affects the characteristics of trading activity of their stocks, in a way which is incompatible with the popular assumption of universality in trading dynamics. Instead, certain stylized facts have a well-defined dependence on company capitalization. Therefore, e.g., averaging distributions over companies with very different capitalization is questionable.

The paper is organized as follows. Section II introduces the notations and data that were used. Section III shows that various measures of trading activity depend on capitalization in a non-trivial way. In Sec. IV we analyze the correlations present in traded value time series, and find that the Hurst exponent increases with the mean traded value per minute logarithmically. Section V deals with a similar size-dependence of correlations present in the time intervals between trades. Finally, Section VI concludes.

I. NOTATIONS AND DATA

For time windows of size \( \Delta t \), let us write the total traded value (activity, flow) of the \( i \)-th stock at time \( t \) as

\[
    f_i^{\Delta t}(t) = \sum_{n, t_i(n) \in [t, t+\Delta t]} V_i(n),
\]

where \( t_i(n) \) is the time of the \( n \)-th transaction of the \( i \)-th stock. This corresponds to the coarse-graining of the individual events, or the so-called tick-by-tick data. \( V_i(n) \) is the value traded in transaction \( n \), and it can be calculated as the product of the price \( p \) and the traded volume of stocks \( V \).

\[
    V_i(n) = p_i(n) V_i(n).
\]

Price does not change very much from trade to trade, so the dominant factor in the fluctuations and the statistical properties of \( f \) is given by the variation of the number of stocks exchanged in the transactions, \( V \). Price serves as a conversion factor to a common unit (US dollars), and it makes the comparison of stocks possible, while also automatically corrects the data for stock splits. The statistical properties (normalized distribution, correlations, etc.) are otherwise practically indistinguishable between traded volume and traded value.

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We used empirical data from the TAQ database \[9\] which records all transactions of the New York Stock Exchange and NASDAQ for the years 1993–2003.

Finally, note that throughout the paper we use 10-base logarithms.

II. CAPITALIZATION AFFECTS BASIC MEASURES OF TRADING ACTIVITY

Most previous studies are restricted to an analysis of the stocks of large companies. These are traded frequently, and so price and returns are well defined even on the time scale of a few seconds. Nevertheless, other quantities regarding the activity of trading, such as traded value and volume or the number of trades can be defined, even for those stocks which exist a minimal exchanged value that is still profitable to use due to transaction costs, \(\beta\) cannot decrease indefinitely. On the other hand, once a stock is exchanged more often (the change happens at about \(\langle N \rangle = 0.05\) trades/min), it is no more traded in this minimal profitable unit. With more intensive trading, trades "stick together", liquidity allows the exchange of larger packages. This increase is clear, but not very large, up to one order of magnitude. Although increasing package sizes reduce transaction costs, price impact \[12\] increases, and profits will decrease again. The balance between these two effects can determine package sizes and may play a role in the formation of \[\text{4}\].

III. NON-UNIVERSAL CORRELATIONS OF TRaded VALUE

Scaling methods \[16\] have long been used to characterize stock market time series, including prices and trading volumes \[1\]. In particular, the Hurst exponent \(H(i)\) is often calculated. For the traded value time series \(f^\Delta(t)\) of stock \(i\), it can be defined as

\[
\sigma^2_t(\Delta t) = \left( \langle f^\Delta(t) - \langle f^\Delta(t) \rangle \rangle \right)^2 \propto \Delta t^{2H(i)},
\]

where \(\langle \cdot \rangle\) denotes time averaging with respect to \(t\). The signal is said to be correlated (persistent) when \(H > 0.5\), uncorrelated when \(H = 0.5\), and anticorrelated (anti-persistent) for \(H < 0.5\). It is not a trivial fact, but several recent papers \[16\] point out that the variance on the left hand side exists for any stock's traded value and any time scale \(\Delta t\). Therefore, we carried out measurements of \(H\) on all 2647 stocks that were continuously traded on NYSE in the period 2000–2002. We investigated separately the 4039 stocks that were traded at NASDAQ for the same period.

We find, that stock market activity has a much richer behavior, than simply all stocks having Hurst exponents statistically distributed around an average value, as assumed in Ref. \[21\]. Instead, there is a crossover \[15\] between two types of behavior around the time scale of a few hours to 1 trading day. An essentially uncorrelated regime was found when \(\Delta t < 20\) min for NYSE and \(\Delta t < 2\) min for NASDAQ, while the time series of larger companies become strongly correlated when \(\Delta t > 300\) min for NYSE and \(\Delta t > 60\) min for NASDAQ. As a reference, we also calculated the Hurst exponents \(H_{\text{shuff}}(i)\) of the shuffled time series. The results are plotted in Fig. 2.

One can see, that for shorter time windows, correlations are absent in both markets, \(H(i) \approx 0.51 - 0.53\). For windows longer than a trading day, however, while small \(\langle f \rangle\) stocks again display only very weak correlations, larger ones show up to \(H \approx 0.9\). Furthermore, there is a distinguishable logarithmic trend in the data:

\[
H(i) = H^* + \gamma \log \langle f_i \rangle,
\]

with \(\gamma(\Delta t > 300\text{min}) = 0.06 \pm 0.01\) for NYSE and \(\gamma(\Delta t > 60\text{min}) = 0.05 \pm 0.01\) for NASDAQ. This result can be predicted by a general framework based on a new type of scaling law \[11\]. Shorter time scales correspond to the special case \(\gamma = 0\), there is no systematic trend in \(H\). After shuffling the time series, as expected,
they become uncorrelated and show $H_{\text{stref}}(i) \approx 0.5$ at all time scales and without significant dependence on $\langle f_i \rangle$.

It is to be emphasized, that the crossover is not simply between uncorrelated and correlated regimes. It is instead between homogeneous (all stocks show $H(i) \approx H_1$, $\gamma = 0$) and inhomogeneous ($\gamma > 0$) behavior. One finds $H_1 \approx 0.5$, but very small $\langle f \rangle$ stocks do not depart much from this value even for large time windows. This is a clear relation to company size, as $\langle f \rangle$ is a monotonously growing function of company capitalization (see Sec. IV and Ref. 19).

Dependence of the effect on $\langle f \rangle$ is in fact a dependence on company size. This is a direct evidence of non-universality. The trading mechanism that governs the marketplace depends strongly on the stock that is traded. In a physical sense, there are no universality classes comprising a given group of stocks and characterized by a set of stylized facts, such as Hurst exponents. Instead, there is a continuous spectrum of company sizes and the stylized facts may depend continuously on company size/capitalization.

Systematic dependence of the exponent of the power spectrum of the number of trades on capitalization was previously reported in Ref. 20, based on the study of 88 stocks. That quantity is closely related to the Hurst exponent of the respective time series (see Ref. 22). Direct analysis finds a strong, monotonous increase of the Hurst exponent of $N$ with growing $\langle N \rangle$, but no such clear logarithmic trend as Eq. 4.

IV. NON-UNIVERSAL CORRELATIONS OF INTERTRADE TIMES

To strengthen the arguments of Sec. III we carried out a similar analysis of the intertrade interval series $T_i(n = 1 \ldots N_i - 1)$, defined as the time spacings between the $n$'th and $n+1$'th trade. $N_i$ is the total number of trades for stock $i$ during the period under study.

Previously, Ref. 22 used 30 stocks from the TAQ database for the period 1993–1996 and proposed that $H_T$ has the universal value $0.94 \pm 0.05$.  

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Figure 1: (a)-(c) Capitalization dependence of certain measures of trading activity in the year 2000. The functions are monotonously increasing and can be piecewise approximated by power laws as indicated. All three tendencies break down for large capitalizations. (b) Mean number of trades per minute $\langle N \rangle$. The slope on the left is from a fit to $C < 4.5 \cdot 10^9$ USD, while the one on the right is for $C > 4.5 \cdot 10^9$ USD. (c) Mean trading activity (exchanged value per minute) $\langle f \rangle$ in USD. The plots include 3347 stocks that were continuously available at NYSE during 2000. (d) Plot of mean value per trade $\langle V \rangle$ versus mean number of trades per minute $\langle N \rangle$ for the year 2000 of NYSE. For smaller stocks there is no clear tendency. For the top $\sim 1600$ companies ($\langle N \rangle > 0.05$ trades/min), however, there is scaling with an exponent $\beta = 0.57 \pm 0.08$. 

...and Ref. [19]).

...components. Instead, there is a continuous spectrum of company sizes and the stylized facts may depend continuously on company size/capitalization.

The Hurst exponents for the variance of traded value Hurst exponents can be written, analogously to Eq. (4), as
\[
\sigma^2_i(N) = \left( \frac{\sum_{n=1}^{N} T_i(n) - \left( \sum_{n=1}^{N} T_i(n) \right)}{N} \right)^2 \propto N^{2H_T(i)},
\]
where the series is not defined in time, but instead on a tick-by-tick basis, indexed by the number of transactions.

The data show a crossover, similar to that for the traded value $f$, from a lower to a higher value of $H_T(i)$ when the window size is approximately the daily mean number of trades (for an example, see the inset of Fig. 3). For the restricted set studied in Ref. [22], the value $H_T \approx 0.94 \pm 0.05$ was suggested for window sizes above the crossover.

Similarly to the case of traded value Hurst exponents analyzed in Section III, the inclusion of more stocks [22] reveals the underlying systematic non-universality. Again, less frequently traded stocks appear to have weaker autocorrelations as $H_T$ decreases monotonously with growing $(T)$. One can fit an approximate logarithmic law [30, 31] to characterize the trend:
\[
H_T = H_T^* + \gamma_T \log \langle T \rangle,
\]
where $\gamma_T = -0.10 \pm 0.02$ for the period 1994–1995 (see Fig. 3) and $\gamma_T = -0.08 \pm 0.02$ for the year 2000 [27].

In their recent preprint, Yuen and Ivanov [23] independently show a tendency similar to Eq. (7) for intertrade times of NYSE and NASDAQ in a different set of stocks.

V. CONCLUSIONS

In this paper we have summarized a few recent advances in understanding the role of company size in trading dynamics. We revisited a number of previous studies of stock market data and found that the extension of the range of capitalization of the studied firms reveals a new aspect of stylized facts: The characteristics of trading display a fundamental dependence on capitalization.

We have shown that trading activity $\langle f \rangle$, the number of trades per minute $\langle N \rangle$ and the mean size of transactions $\langle V \rangle$ display non-trivial, monotonous dependence on company capitalization, which cannot be described by a simple power law. On the other hand, for moderate to large companies, a power law gives an acceptable fit for the dependence of the mean transaction size on the trading frequency.

The Hurst exponents for the variance of traded value/intertrade times can be defined and they depend logarithmically on the mean trading activity $\langle f \rangle$/mean intertrade time $\langle T \rangle$.

We analyzed the same database, but included a large number of stocks with very different capitalizations. First it has to be noted that the mean intertrade interval has decreased drastically over the years. In this sense the stock market cannot be considered stationary for periods much longer than one year. We analyzed the two year period 1994–1995 (part of that used in Ref. [22]) and separately the single year 2000. We used all stocks in the TAQ database with $(T) < 10^5$ sec, a total of 3924 and 4044 stocks, respectively.
These findings imply that special care must be taken when the concepts of scaling and universality are applied to financial processes. For the modeling of stock market processes, one should always consider that many characteristic quantities depend strongly on the capitalization. The introduction of such models seems a real challenge at present.

VI. ACKNOWLEDGEMENT

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[28] Note that many minor stocks do not represent actual companies, only different class stocks of a larger firm.

[29] For a reliable calculation of Hurst exponents, we had to discard those stocks that had less than $⟨N⟩ < 10^{-3}$ trades/min for 1994–1995 and $⟨N⟩ < 2 \cdot 10^{-3}$ trades/min for 2000. This filtering leaves 3519 and 3775 stocks, respectively.

[30] As intertrade intervals are closely related to the number of trades per minute $N(t)$, it is not surprising to find the similar tendency for that quantity [26].

[31] Note that for window sizes smaller than the daily mean number of trades, intertrade times are only weakly correlated and the Hurst exponent is nearly independent of $⟨T⟩$. This is analogous to what was seen for traded value records in Sec. III.