Hypergeometric functions as generalized Stieltjes transforms

D. Karp* and E. Prilepkina†

Abstract. In this paper we apply generalized Stieltjes transform representation to study the generalized hypergeometric function. Among the results thus proved are new integral representations, inequalities, properties of the Padé table and the properties of the generalized hypergeometric function as a conformal map.

Keywords: Generalized Stieltjes function, moment problem, generalized hypergeometric function, hypergeometric inequality, Padé approximation

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1. Introduction. Functions representable in one of the forms

\[ f(z) = C_1 + \int_{[0,\infty)} \frac{\mu(du)}{(u+z)^\alpha} = \int_{[0,\infty)} \frac{\rho(dt)}{(1+tz)^\alpha} + C_2 z^\alpha, \]

are known as generalized Stieltjes functions. Here \( \alpha > 0 \), \( \mu \) and \( \rho \) are non-negative measures supported on \([0,\infty)\), \( C_1 \geq 0 \), \( C_2 \geq 0 \) are constants and we always choose the principal branch of the power function. The measures \( \mu \) and \( \rho \) are assumed to produce convergent integrals \((1)\) for each \( z \in \mathbb{C}\setminus(-\infty,0] \) so that the function \( f \) is holomorphic in \( \mathbb{C}\setminus(-\infty,0] \).

Generalized Stieltjes functions have been studied by a number of authors including [19, 20], [24, Section 8], [23, Chapter VIII]. For more detailed overview of the properties of generalized Stieltjes functions and related bibliography see our recent paper [8]. In the same paper we introduced the notion of the exact Stieltjes order as follows. If we define \( S_\alpha \) to be the class of functions representable by \((1)\) then one can show that \( S_\alpha \subset S_\beta \) when \( \alpha < \beta \). We will say that \( f \) is of the exact Stieltjes order \( \alpha^* \) if \( f \in \bigcup_{\alpha>0} S_\alpha \) and

\[ \alpha^* = \inf \{ \alpha : f \in S_\alpha \}. \]  

Using Sokal’s characterization of \( S_\alpha \) found in [19] it is not difficult to see that \( f \in S_{\alpha^*} \). Moreover, in [8] we gave a criterion of exactness leading to some simple sufficient conditions. In particular, we will need the following result contained in [8, Corollary 1].

*Far Eastern Federal University, Vladivostok, Russia, e-mail: dimkrp@gmail.com
†Far Eastern Federal University, Vladivostok, Russia, e-mail: pril-elena@yandex.ru
Theorem 1 Suppose $f \in S_\alpha$ and for sufficiently small $\varepsilon > 0$

$$\lim_{y \to +\infty} \frac{\Phi_\varepsilon(2y)}{\Phi_\varepsilon(y)} < 1,$$

where

$$\Phi_\varepsilon(y) = \int_{(0,y)} \frac{\mu(du)}{(y-u)\varepsilon}.$$  \hspace{1cm} (3)

Then $\alpha$ is the exact Stieltjes order of $f$.

In this paper we aim to apply the results of [8] to study the generalized hypergeometric function defined by the series

$$q+1F_q \left( \begin{array}{c} \sigma, A \cr B \end{array} \bigg| z \right) = q+1F_q (\sigma, A; B; z) := \sum_{n=0}^{\infty} \frac{(\sigma)_n}{(b_1)_n \cdots (b_q)_n n!} \frac{(a_1)_n \cdots (a_q)_n}{(a_1)_n \cdots (a_q)_n} z^n,$$  \hspace{1cm} (4)

where we write $A = (a_1, a_2, \ldots, a_q)$, $B = (b_1, b_2, \ldots, b_q)$ for brevity and $(a)_0 = 1$, $(a)_n = a(a+1) \cdots (a+n-1)$, $n \geq 1$, denotes the rising factorial. The series (4) converges in the unit disk and its sum can be extended analytically to the whole complex plane cut along the ray $[1, \infty)$. See details in [2,12,18].

Euler’s integral representation [2, Theorem 2.2.1]

$$2F_1 (\sigma, a; b; -z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} dt \frac{1}{(1+zt)^\sigma} = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_1^\infty u^{\sigma-b} (u-1)^{b-a-1} du \frac{1}{(u+z)^\sigma}$$

for the Gauss hypergeometric function $2F_1$ shows that it is a generalized Stieltjes function at least when $b > a > 0$ and $\sigma > 0$. In her book [12] Virginia Kiryakova gave the representation

$$q+1F_q \left( \begin{array}{c} \sigma, A \cr B \end{array} \bigg| -z \right) = \int_0^1 \frac{\rho(s) ds}{(1+sz)^\sigma}$$

under the constraints $b_k > a_k > 0$, $k = 1,2,\ldots, q$, and with $\rho$ expressed in terms of Meijer’s G-function (see [10] below). In [9] Karp and Sitnik established the same formula but with $\rho$ expressed by a multidimensional integral which is manifestly positive under the same constraints. In this work we generalize both these results by stating necessary and sufficient conditions for the above representation to hold and sufficient conditions for the weight $\rho$ to be non-negative (the latter conditions are also believed to be necessary but we have no proof of this claim). We find the exact Stieltjes order of $q+1F_q$ and give a number of consequences, including new integral representations, inequalities, properties of the Padé table and properties of $q+1F_q$ as a conformal map.

2. The exact Stieltjes order of $q+1F_q$. We will need a particular case of the Meijer’s G-function defined by (see [12,18])

$$G_{p,q}^{q,0} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| z \right) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(b_1+s) \cdots \Gamma(b_q+s)}{\Gamma(a_1+s) \cdots \Gamma(a_p+s)} z^{-s} ds,$$  \hspace{1cm} (5)
where $c > -\min(\Re b_1, \Re b_2, \ldots, \Re b_q)$. Since the gamma function is real symmetric, $\Gamma(\tau) = \overline{\Gamma(z)}$, the function $G_{p,q}^{a_0}$ is real if all parameters $a_i$, $b_i$ are real. Define

$$\psi := \sum_{k=1}^{q} (b_k - a_k).$$  \hfill (6)

**Lemma 1** Set $A = (a_1, \ldots, a_q)$, $B = (b_1, \ldots, b_q)$. If

$$\Re(\psi) > 0,$$  \hfill (7)

then

$$G_{p,q}^{a_0} \left( \begin{array}{c} x \\ A \end{array} \right) = 0 \text{ for } x > 1.$$ \hfill (8)

**Proof.** From (5) we have

$$G_{p,q}^{a_0} \left( \begin{array}{c} x \\ A \end{array} \right) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{c-iR}^{c+iR} \frac{\Gamma(a_1 + ) \cdots \Gamma(a_q + )}{\Gamma(b_1 + ) \cdots \Gamma(b_q + )} e^{-s\ln x} ds.$$  

Expression under the integral sign has no poles inside the closed contour starting at the point $c - iR$, tracing the semicircle $c + Re^{i\varphi}$, $-\pi/2 \leq \varphi \leq \pi/2$, up to the point $c + iR$ and then back to $c - iR$ along the line segment $c + it$, $-R \leq t \leq R$. Hence, we have by the Cauchy theorem:

$$I(R) := \frac{1}{2\pi} \int_{-R}^{R} \frac{\Gamma(a_1 + c + it) \cdots \Gamma(a_q + c + it)}{\Gamma(b_1 + c + it) \cdots \Gamma(b_q + c + it)} e^{-c\ln x} dt =$$

$$= -\frac{R}{2\pi} e^{-c\ln x} \int_{-\pi/2}^{\pi/2} \frac{\Gamma(a_1 + c + Re^{i\varphi}) \cdots \Gamma(a_q + c + Re^{i\varphi})}{\Gamma(b_1 + c + Re^{i\varphi}) \cdots \Gamma(b_q + c + Re^{i\varphi})} e^{i(R\ln x \sin \varphi + \varphi)} e^{-R\ln x \cos \varphi} d\varphi.$$  

Set $z = Re^{i\varphi}$. Using Stirling’s asymptotic formula (see, for instance, [2, Theorem 1.4.2]) we get the relation

$$\log \left\{ \frac{\Gamma(a_1 + c + z) \cdots \Gamma(a_q + c + z)}{\Gamma(b_1 + c + z) \cdots \Gamma(b_q + c + z)} \right\} = -\psi \log(z) + O(1/z) \text{ as } |z| \to \infty,$$

which holds uniformly in the sector $|\arg z| \leq \pi - \delta$, for each $\delta \in (0, \pi)$. Hence,

$$\left| \frac{\Gamma(a_1 + c + z) \cdots \Gamma(a_q + c + z)}{\Gamma(b_1 + c + z) \cdots \Gamma(b_q + c + z)} \right| = R^{-\Re(\psi)}(1 + O(1/R)), \quad R \to \infty.$$  

Consequently,

$$|I(R)| = O \left( R^{-\Re(\psi)+1} \right) \int_{-\pi/2}^{\pi/2} e^{-R\ln x \cos \varphi} d\varphi \quad \text{as } R \to \infty.$$
Applying the inequality \( \cos \varphi \geq 1 - \frac{2}{\pi} \varphi \), \( 0 \leq \varphi \leq \pi/2 \), we obtain (recall that \( x > 1 \))

\[
\int_{0}^{\pi/2} e^{-R \ln x \cos \varphi} d\varphi \leq \int_{0}^{\pi/2} e^{-R(1 - \frac{2}{\pi} \varphi)} \ln x d\varphi = e^{-R \ln x} \int_{0}^{\pi/2} \frac{2}{\pi} R \varphi \ln x d\varphi = \frac{\pi}{2R \ln x} (1 - e^{-R \ln x}).
\]

Combining this estimate with the previous relation we see that

\[
\lim_{R \to \infty} I(R) = 0 \text{ for each } x > 1. \quad \square
\]

**Remark.** Formula (8) is given in [18, formula (8.2.2.2)] under more restrictive conditions then (7). For this reason we decided to include a direct proof.

**Theorem 2** Suppose \( |\arg(1 + z)| < \pi \) and \( \sigma \) is an arbitrary complex number. Representation

\[
q + 1 F_q \left( \sigma, A \mid -z \right) = \frac{1}{s} \frac{\rho(s) ds}{(1 + sz)^\sigma}
\]

with a summable on \([0,1]\) function \( \rho \) holds true if and only if \( \Re a_i > 0 \) for \( i = 1, \ldots, q \) and \( \Re \psi > 0 \), where \( \psi \) is defined in (6). Under these conditions

\[
\rho(s) = \left( \prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)} \right) \frac{1}{s} G_{q,0}^{q,q} \left( s \Big| B \right). \quad (10)
\]

**Remark.** Representation (9) after change of variable \( t = 1/s \) can also be written as

\[
q + 1 F_q \left( \sigma, A \mid -z \right) = \int_{1}^{\infty} \frac{\mu(t) dt}{(t + z)^\sigma}, \quad (11)
\]

\[
\mu(t) = \left( \prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)} \right) t^{\sigma-1} G_{q,0}^{q,q} \left( 1/t \mid B \right) \quad (12)
\]

- a form which we will also use.

**Proof.** Suppose first that \( \Re a_i > 0 \) for \( i = 1, \ldots, q \) and \( \Re \psi > 0 \). Consider the right-hand side of (9) with \( \rho \) given by (10). Applying the binomial expansion to \((1 + sz)^{-\sigma}\) and integrating term by term we immediately obtain the left-hand side of (9) since

\[
\int_{0}^{1} s^k \rho(s) ds = \int_{0}^{\infty} s^k \rho(s) ds = \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_q)_k}.
\]

The first equality here is due to Lemma 1. The second equality expresses the basic property of the Meijer’s \( G \)-function: its Mellin transform is equal to the ratio of the appropriate gamma functions (see, for instance, [18, formula 2.24.2.1] or [12, formula (A.25), p.319]).

The integral converges uniformly in \( k \) in the neighbourhood of \( s = 0 \) since

\[
G_{q,0}^{q,q} \left( s \mid B \right) = O \left( s^a \ln^{m-1}(1/s) \right), \quad s \to 0, \quad (13)
\]
where \( a = \min(\Re(a_1), \ldots, \Re(a_q)) > 0 \) by assumption and the minimum is taken over those \( a_i \) for which there is no \( b_j = a_i - l \) for some \( l \in \mathbb{N}_0 \). The minimum can be attained for several different numbers \( a_i \) and then \( m \) is the maximal multiplicity among these numbers. This formula follows from [11, Corollary 1.12.1] or [7, formula (11)]. The integral converges uniformly in \( k \) in the neighbourhood of \( s = 1 \) because, the function \( G_{q,q}^{0,0} \) has a singularity of the magnitude \((1 - s)^{\Re(\psi)} - 1\) possibly multiplied by logarithmic terms if \( \Re(\psi) \leq 1 \) and is bounded if \( \Re(\psi) > 1 \) (see [18, 8.2.59]). Hence, condition (7) guarantees uniform integrability of \( \rho \) in the neighbourhood of \( s = 1 \). Uniform integrability justifies the interchange of summation and integration.

To prove necessity suppose that (7) holds with a summable function \( \rho \). Then

\[
\int_0^1 s^k \rho(s) ds = \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_q)_k} \tag{14}
\]

by termwise integration and comparing with (4). We aim to show that \( \Re a_i > 0 \) for \( i = 1, \ldots, q \) and \( \Re \psi > 0 \). Assume first that \( \Re a_i \leq 0 \) for some \( i \) while \( \Re \psi > 0 \). The asymptotic formula (13) combined with Lemma 1 shows that

\[
\left( \prod_{i=1}^q \frac{\Gamma(b_i)}{\Gamma(a_i)} \right) \int_0^1 s^{k-1} G_{q,q}^{0,0}(s) B_A ds = \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_q)_k}
\]

for \( k > -a \), where as before \( a = \min(\Re(a_1), \ldots, \Re(a_q)) \). Hence all moments of the functions \( s^{[-a]+1} \rho \) and \( s^{[-a]} G_{q,q}^{0,0} \) coincide. This implies that \( \rho \) must be given by (10) by the determinacy of the moment problem on a finite interval. But then the integral in (9) must diverge by (13). A contradiction.

If \( \Re \psi < 0 \) the sequence

\[
\frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_q)_k}
\]

is unbounded and cannot serve as a moment sequence of a signed measure on \([0, 1]\), so that (14) is impossible and hence so is (9). Finally, if \( \Re \psi = 0 \) a careful application of Stirling’s formula shows that this sequence tends to a non-zero constant as \( k \to \infty \) (see [11, formula (1.2.5)]) while the left-hand side of (14) must tend to zero for any summable function \( \rho \), so again a contradiction. □

Remark. Formula (9) has been discovered by Kiryakova in [12] by iterative fractional integrations under additional assumption that all parameters are real and \( b_k > a_k > 0 \), \( k = 1, 2, \ldots, q \). The elementary proof included here is not contained in this reference.

In the sequel we will need the notion of majorization [14, Definition A.2, formula (12)]. It is said that \( B = (b_1, \ldots, b_q) \) is weakly supermajorized by \( A = (a_1, \ldots, a_q) \) (symbolized by \( B \prec^W A \)) if

\[
0 < a_1 \leq a_2 \leq \cdots \leq a_q, \quad 0 < b_1 \leq b_2 \leq \cdots \leq b_q,
\]

\[
\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i \quad \text{for} \quad k = 1, 2, \ldots, q.
\tag{15}
\]

If in addition \( \psi(= \sum_{i=1}^q (b_i - a_i)) = 0 \) then \( B \) is said to be majorized by \( A \), \( B \prec A \).
Lemma 2 Suppose that $B \prec^W A$ but not $B \prec A$ (that is $\psi > 0$). Then for all $0 < s < 1$

$$G_{q,q}^{a,0} \left( s \left| \frac{B}{A} \right. \right) \geq 0. \quad (16)$$

Proof. Alzer showed in [1, Theorem 10] that the function

$$x \rightarrow \prod_{i=1}^{q} \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)}$$

is completely monotonic on $(0, \infty)$ if $B \prec^W A$. This implies that the sequence

$$\left\{ \prod_{i=1}^{q} \frac{\Gamma(n + a_i)}{\Gamma(n + b_i)} \right\}, \quad n = 0, 1, 2, \ldots,$$

is a completely monotonic sequence. Hence by the Hausdorff theorem there exists a unique non-negative measure $d\nu$ supported on $[0, 1]$ such that

$$\int_{[0,1]} s^n d\nu(s) = \prod_{i=1}^{q} \frac{\Gamma(n + a_i)}{\Gamma(n + b_i)}.$$

On the other hand if $\psi > 0$

$$\int_{0}^{1} s^{n-1} G_{q,q}^{a,0} \left( s \left| \frac{B}{A} \right. \right) ds = \prod_{i=1}^{q} \frac{\Gamma(n + a_i)}{\Gamma(n + b_i)},$$

so that by determinacy of the Haaroff moment problem

$$d\nu(s) = \frac{1}{s} G_{q,q}^{a,0} \left( s \left| \frac{B}{A} \right. \right) ds.$$

Non-negativity of the measure completes the proof. □

Remark. According to Bernstein’s theorem every completely monotonic function on $(0, \infty)$ is the Laplace transform of a non-negative measure. The proof of Lemma 2 shows that the representing measure in Alzer’s theorem 10 from [1] is given by

$$\prod_{i=1}^{q} \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)} = \int_{0}^{\infty} e^{-tx} G_{q,q}^{a,0} \left( e^{-t} \left| \frac{B}{A} \right. \right) dt.$$

Remark. By taking the Mellin transform on both sides and changing variables one can show that for $x > 0$

$$G_{q,q}^{a,0} \left( x \left| \frac{B}{A} \right. \right) = \prod_{i=1}^{q} \frac{x^{a_i}}{\Gamma(b_i - a_i)} \times \int_{\Lambda_q(x)} \left[ 1 - x/(t_2 \cdots t_q) \right]^{b_1 - a_1 - 1} \prod_{k=2}^{q} t_k^{a_k - a_1 - 1} (1 - t_k)^{b_k - a_k - 1} dt_2 \cdots dt_q, \quad (17)$$
if \( \Re(b_k - a_k) > 0, k = 1, 2, \ldots, q, q \geq 2 \). Here the domain of integration is given by

\[
\Lambda_q(x) = [0, 1]^{q-1} \cap \{ t_2, \ldots, t_q : t_2 \cdots t_q > x \}.
\]  

This formula shows the positivity of \( G_{q,0}^{q,q} \) under the conditions \( b_k > a_k > 0, k = 1, 2, \ldots, q \), which are manifestly more restrictive than \( B \prec W A \) and \( \psi > 0 \). Formula (17) is implicit in [9].

**Theorem 3** Suppose \( 0 < \sigma \leq \min(a_1, \ldots, a_q) \) and \( B \prec W A \). Then \( f := q+1 F_q(\sigma, A; B; -z) \) is a generalized Stieltjes function of the exact order \( \sigma \). In particular, \( f \) is completely monotonic.

**Proof.** Assume first that \( \psi(= \sum_{i=1}^{q} (b_i - a_i)) > 0 \). Then by Theorem 2 \( f \) is represented by (11) with the measure \( \mu \) non-negative by Lemma 2. Hence, \( f \in S_{\sigma} \). To show that \( \sigma \) is exact we will apply Theorem 1. Fixing \( \varepsilon > 0 \) compute

\[
\Phi_\varepsilon(y) := \int_1^y \frac{\mu(u) du}{(y-u)^\varepsilon},
\]

where \( \mu(u) \) is given by (12). Changing variable \( \tau = 1/u \) and manipulating a little we obtain

\[
\Phi_\varepsilon(y) = \frac{\Gamma(1-\varepsilon)}{y^\varepsilon} y^\sigma G_{q+1,q+1}^{q+1,0} \begin{pmatrix} B \cr A \end{pmatrix} d\tau.
\]

According to [18, formula (2.24.3)] combined with (8) we get

\[
\Phi_\varepsilon(y) = \frac{\Gamma(1-\varepsilon)}{y^\varepsilon} y^\sigma G_{q+1,q+1}^{q+1,0} \begin{pmatrix} B \cr A \end{pmatrix}.
\]

Using (13) for the main asymptotic term of \( G_{q+1,q+1}^{q+1,0} \) we immediately arrive at

\[
\lim_{y \to +\infty} \frac{\Phi_\varepsilon(2y)}{\Phi_\varepsilon(y)} = 2^{-\varepsilon} < 1.
\]

Hence, by Theorem 1 the order \( \sigma \) is exact.

Next, suppose that \( B \prec A \), i.e. (15) holds with \( \psi = 0 \). By Alzer’s theorem the sequence on the right of (14) is still a moment sequence of a non-negative measure (see proof of Lemma 2) which shows that \( f \in S_{\sigma} \). We will, however, give another proof of this fact which will extend to a proof of the exactness of \( \sigma \). Consider the sequence

\[
f_m(z) = q+1 F_q(\sigma, A; B', b_q + 1/m; -z), \quad B' = (b_1, \ldots, b_{q-1}).
\]

According to what we have just proved each \( f_m \in S_{\sigma} \) and the order is exact. We aim to apply [8, Theorem 10] to show that \( f \in S_{\sigma} \). To this end we need to demonstrate that \( f_m(x) \to f(x) \) for all \( x > 0 \). If \( |z| < 1 \) then

\[
|f(z) - f_m(z)| \leq \sum_{k=0}^{\infty} \frac{(\sigma)(a_1) \cdots (a_q)|z|^k}{(b_1) \cdots (b_{q-1}) k!} \left[ \frac{1}{(b_q)k} - \frac{1}{(b_q + 1/m)k} \right] \to 0
\]

as \( m \to \infty \).
due to uniform in $m$ convergence of the series. The convergence can be extended to all $z \in \mathbb{C} \setminus (-\infty, -1]$ using Vitali-Porter (or Stieltjes-Vitali) theorem on induced convergence [4, Corollary 7.5]. This theorem requires the set $\{f_m\}$ to be locally uniformly bounded in $\mathbb{C} \setminus (-\infty, -1]$. This boundedness can be seen from the easily verifiable contiguous relation

$$q_{+1}F_q(\sigma, A; B', b_q + 1/m; -z) = q_{+1}F_q(\sigma, A; B', b_q + 1 + 1/m; -z)$$

$$- \frac{z u \prod_{i=1}^{q} a_i}{(b_q + 1/m)(b_q + 1 + 1/m) \prod_{i=1}^{q} b_i} q_{+1}F_q(\sigma, A + 1; B' + 1, b_q + 2 + 1/m; -z), \quad (21)$$

where both functions on the right are bounded uniformly in $m$ due to representation (9). This proves that $f \in S_{\sigma}$. Finally, we need to demonstrate that the order $\sigma$ is exact for $f$. The distribution function of the representing measure of $f_m$ is given by

$$F_m(y) = \prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)} \int_{[1, y]} t^{-\sigma} G_{q, 0}^{q, 0} \left(1/t \left| \begin{array}{c} B', b_q + 1/m \nonumber \end{array} \right| A \right) dt$$

$$= \prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)} y^\sigma G_{q+1, 0}^{q+1, 0} \left(1/y \left| 1 + \sigma, B', b_q + 1/m \nonumber \right| \sigma, A \right),$$

where we again used [18, formula (2.24.3)] combined with [8]. Taking limit as $m \to \infty$ we obtain the distribution function of the measure representing $f$ in the form

$$\prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)} y^\sigma G_{q+1, 0}^{q+1, 0} \left(1/y \left| 1 + \sigma, B \nonumber \right| \sigma, A \right).$$

Comparing this formula with (19) for $\psi = 0$ we see that the distribution function does not change its form whether $\psi > 0$ or $\psi = 0$. This implies that the function $\Phi_\varepsilon(y)$ is again expressed by (19) when $\psi = 0$ (since $\Phi_\varepsilon$ is proportional to the fractional derivative of order $\varepsilon$ of the distribution function). Hence, the limit in (20) is again less than 1 which according to Theorem [10] proves the exactness of the order $\sigma$. □

**Remark.** If $\psi > 0$ then the representing measure in the above theorem is given in (9) or (11). However, if $B \prec A$ (i.e. $\psi = 0$) then Theorem [3] leaves the question of finding the representing measure open. For $q = 1$ the answer is obvious:

$$2F_1(\sigma, a; a; -z) = \frac{1}{(1 + z)^\sigma}$$

by the binomial theorem, so that the representing measure is $\delta_1$ (the Dirac measure concentrated at 1). For $q = 2$ representation (9) reduces to (see [9, Lemma 2])

$$3F_2(\sigma, a_1, a_2; b_1, b_2; -z) = \frac{\Gamma(b_1) \Gamma(b_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(b_1 + b_2 - a_1 - a_2)}$$

$$\times \int_0^1 t^{a_2 - 1}(1 - t)^{b_1 + b_2 - a_1 - a_2 - 1} \left(1 + zt\right)^\sigma 2F_1(b_1 - a_1, b_2 - a_1; b_1 + b_2 - a_1 - a_2; 1 - t) dt \quad (22)$$
valid if \( a_1, a_2 > 0, b_1 + b_2 > a_1 + a_2 \). To compute the limiting measure when \( b_1 + b_2 = a_1 + a_2 \) we put \( \epsilon = b_1 + b_2 - a_1 - a_2, \varphi(t) = (1 + zt)^\sigma \) and let \( \epsilon \to 0 \in 

\[
\frac{1}{\Gamma(\epsilon)} \int_0^1 t^{a_2-1}(1-t)^{\epsilon-1} \, _2F_1(b_1 - a_1, b_2 - a_1; \epsilon; 1-t) \varphi(t)dt = \frac{1}{\Gamma(\epsilon)} \int_0^1 (1-u)^{a_2-1}u^{\epsilon-1} \, _2F_1(b_1 - a_1, b_2 - a_1; \epsilon; u) \psi(u)du,
\]

where \( t = 1-u \) and \( \psi(u) := \varphi(1-u) \). We have

\[
_2F_1(b_2 - a_1, b_1 - a_1; \epsilon; u) = 1 + \frac{\Gamma(\epsilon)}{\Gamma(b_2 - a_1)\Gamma(b_1 - a_1)} \sum_{k=1}^{\infty} \frac{\Gamma(b_2 - a_1 + k)\Gamma(b_1 - a_1 + k)}{\Gamma(\epsilon + k)k!} u^k.
\]

Hence,

\[
\lim_{\epsilon\to 0} \frac{1}{\Gamma(\epsilon)} \int_0^1 (1-u)^{a_2-1}u^{\epsilon-1} \, _2F_1(b_1 - a_1, b_2 - a_1; \epsilon; u) \psi(u)du
\]

\[
= \lim_{\epsilon\to 0} \frac{1}{\Gamma(\epsilon)} \int_0^1 (1-u)^{a_2-1}u^{\epsilon-1} \left[ 1 + \frac{\Gamma(\epsilon)}{\Gamma(b_2 - a_1)\Gamma(b_1 - a_1)} \sum_{k=1}^{\infty} \frac{\Gamma(b_2 - a_1 + k)\Gamma(b_1 - a_1 + k)}{\Gamma(\epsilon + k)k!} u^k \right] \psi(u)du
\]

\[
= \psi(0) \lim_{\epsilon\to 0} \frac{\Gamma(a_2)\Gamma(\epsilon)}{\Gamma(a_2 + \epsilon)\Gamma(\epsilon)} + \psi'(0) \lim_{\epsilon\to 0} \frac{\Gamma(a_2)\Gamma(\epsilon + 1)}{\Gamma(a_2 + 1)\Gamma(\epsilon)} + \cdots
\]

\[
+ \frac{1}{\Gamma(b_2 - a_1)\Gamma(b_1 - a_1)} \int_0^1 (1-u)^{a_2-1} \left[ \sum_{k=1}^{\infty} \frac{\Gamma(b_2 - a_1 + k)\Gamma(b_1 - a_1 + k)}{(k-1)!k!} u^{k-1} \right] \psi(u)du
\]

\[
= \psi(0) + (b_2 - a_2)(b_1 - a_1) \int_0^1 (1-u)^{a_2-1} \left[ \sum_{k=0}^{\infty} \frac{(b_2 - a_1 + 1)(b_1 - a_1 + 1)}{(2)_k k!} u^k \right] \psi(u)du.
\]

Summing the series we get

\[
\psi(0) + (b_2 - a_2)(b_1 - a_1) \int_0^1 (1-u)^{a_2-1} \, _2F_1(b_1 - a_1 + 1, b_2 - a_1 + 1; 2; u) \psi(u)du.
\]
So we have the following result: if \( b_1 + b_2 = a_1 + a_2 \) then the representing measure has an atom at \( t = 1 \) (\( \psi(0) = \varphi(1) \)) and a continuous part given above, so that

\[
3F_2(\sigma, a_1, a_2; b_1, b_2, -z) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)} \left\{ \frac{1}{(1+z)^\sigma} \right. \\
+ \int_0^1 \frac{(b_2-a_1)(b_1-a_1)t^{a_2-1}}{(1+zt)^\sigma} {}_2F_1(b_1-a_1+1, b_2-a_1+1; 1-t)dt \left. \right\}.
\]

This formula can also be proved by comparing power series coefficients on both sides and using the Gauss summation theorem. Finding the representing measure for general \( q \) remains an interesting open problem we plan to deal with in a separate publication.

**Corollary 1** Suppose \( B \prec^W A \) with \( \psi > 0, \sigma \geq 2 \) and \( |\arg(z)| < \pi/\sigma \). Then

\[
q+1F_q(\sigma, A, B; -z) = \int_0^\infty \varphi(y)dy, \quad \text{where}
\]

\[
\varphi(y) = \frac{\sigma y^{\sigma-1}}{\pi} \left( \prod_{i=1}^q \frac{\Gamma(b_i)}{\Gamma(a_i)} \right) \int_0^1 \frac{\sin \left\{ \sigma \arctan \left( \frac{ty\sin(\pi/\sigma)}{1+ty\cos(\pi/\sigma)} \right) \right\}}{t} \frac{y^2}{(1+2ty\cos(\pi/\sigma) + t^2y^2)^{\sigma/2}} G_{q,0}^{q,0} \left( t \mid B \right) \left( A \right) dt.
\]

**Proof.** According to [8, Theorem 13] combined with Theorem 3 above the function \( q+1F_q(\sigma, A; B; -z^{1/\sigma}) \) belongs to \( S_1 \) for \( \sigma > 1 \) under the assumptions of the corollary. According to the Stieltjes inversion formula [23, Chapter VIII, Theorem 7b] the density of the representing measure for \( f \in S_1 \) is found from \((x>0)\):

\[
\frac{1}{2\pi i} \lim_{\varepsilon \to 0} [f(-x-i\varepsilon) - f(-x+i\varepsilon)].
\]

Substituting the first formula (9) for \( f \) and computing the limit we arrive at (23). □

**Remark.** For \( 1 < \sigma < 2 \) a similar formula can be obtained. However, since it’s more cumbersome than (23) we decided to omit it.

**Remark.** Using the identity \( \sin(2\arctan(s)) = 2s/(1 + s^2) \) formula (23) for \( \sigma = 2 \) simplifies to \((|\arg(z)| < \pi/2)

\[
q+1F_q(2, A, B; -z) = \int_0^\infty \varphi(y)dy, \quad \text{where}
\]

\[
\varphi(y) = \frac{4}{\pi} \left( \prod_{i=1}^q \frac{\Gamma(b_i)}{\Gamma(a_i)} \right) \int_0^1 \frac{y^2}{(1+t^2y^2)^{2\sigma/2}} G_{q,0}^{q,0} \left( t \mid B \right) \left( A \right) dt.
\]

**Remark.** For the Gauss hypergeometric function formula (23) reduces to \((|\arg(z)| < \pi/2)

\[
2F_1(a, b; c; -z) = \int_0^\infty \frac{\varphi(y)dy}{y^a + z^a}, \quad c > b > 0, \quad a \geq 2.
\]
$\varphi(y) = \frac{a\Gamma(c)y^{a-1}}{\pi\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}\sin\left\{a \arctan\left(\frac{ty\sin(\pi/a)}{1+ty\cos(\pi/a)}\right)\right\}}{(1+2ty\cos(\pi/a)+t^2y^2)^{a/2}} dt.$

In particular, for $a = 2$ we obtain:

$$2F_1(2, b; c; -z) = \frac{4b}{\pi c} \int_0^\infty \frac{y^2 F_2(2, (b+1)/2, (b+2)/2; (c+1)/2, (c+2)/2; -y^2) dy}{y^2 + z^2},$$

where we have used

$$\int_0^1 \frac{t^b(1-t)^{c-b-1}}{(1+t^2y^2)^2} dt = \frac{\Gamma(b+1)\Gamma(c-b)}{\Gamma(c+1)} F_2(2, (b+1)/2, (b+2)/2; (c+1)/2, (c+2)/2; -y^2).$$

Using some known results and techniques representation (9) together with Lemma 2 and Theorem 3 leads to a number of implications for generalized hypergeometric function which we present in the subsequent sections. All statements presented below are believed to be new.

3. Inequalities for $q+1 F_q$. Many results of [9] are based on representation (9) with non-negative $\rho$. However, the inequality $\rho \geq 0$ has only been proved in this reference for $b_k > a_k > 0$, $k = 1, 2, \ldots, q$. Theorem 3 combined with some results of [8] allow us to extend the results of [9] to all values of $a_k$, $b_k$ satisfying (15). In particular, we get the following statements.

**Theorem 4** Suppose $B \prec W A$ and $\delta > 0$. Then the function

$$x \rightarrow \frac{q+1 F_q(\sigma, A + \delta; B + \delta; -x)}{q+1 F_q(\sigma, A; B; -x)}$$

is monotone decreasing on $(-1, \infty)$ if $\sigma > 0$ and monotone increasing if $\sigma < 0$.

The proof of this result in [9, Theorem 1] is based on representation (9) with non-negative $\rho$ and so it applies to our situation here if $B \prec W A$ and $\psi > 0$. The claim is then extended by continuity to $\psi = 0$.

Next, we obtain a lower bound.

**Theorem 5** Suppose $B \prec W A$ and $\sigma > 0$. Then for all $x > -1$ the inequality

$$\frac{1}{(1 + x \prod_{i=1}^q (a_i/b_i))^{\sigma}} \leq q+1 F_q(\sigma, A; B; -x)$$

holds true with equality only for $x = 0$.

**Proof.** Consider the case $0 < \sigma \leq 1$ first. Then according to Theorem 3 and 8 Theorem 12 the condition $B \prec W A$ implies that the function $[q+1 F_q(\sigma, A; B; -x)]^{1/\sigma}$ belongs to $S_1$. Note that the condition $\sigma \leq \min(a_1, \ldots, a_q)$ from Theorem 3 is not required to make this conclusion. It is immediate to check that

$$\frac{1}{1 + x \prod_{i=1}^q (a_i/b_i)}.$$
is the Padé approximation to \([q+1]F_q(\sigma, (a_q); (b_q); -x)]^{1/\sigma} at \(x = 0\) of order \([0/1]\). This implies \((26)\) for all \(x > -1\) by Stieltjes inequalities \([6, \text{formulas } (3), (4)]\).

Next, suppose that \(\sigma > 1\). Then \((26)\) can be derived from Theorem 4 by repeating the proof of \([9, \text{Theorem } 3]\) word for word. \(\square\)

Inequality \((26)\) was probably first obtained by Luke in \([13]\) for \(x > 0\) and \(b_k \geq a_k > 0\). Theorem 5 extends his result to all \(x > -1\) and parameters satisfying much weaker restrictions \(B \prec_W A\). An extension of \([9, \text{Theorem } 4]\) reads:

Theorem 6 Suppose \(B \prec_W A\) and \(a_1, b_1 > 1\). Then for \(x > 0\) and \(0 < \sigma \leq 1\) the inequality

\[
q+1F_q(\sigma, A; B; -x) < \frac{1}{(1 + x \prod_{i=1}^q [(a_i - 1)/(b_i - 1)])^{\sigma}} \tag{27}
\]

holds.

In \([10]\) Karp and Sitnik gave sufficient conditions for absolute monotonicty of certain product differences of the functions \(q+1F_q\). This type of absolute monotonicty immediately implies log-convexity or log-concavity of \(\sigma \to q+1F_q(\sigma, A; B; x)\) for \(0 < x < 1\). Representation \((9)\) allows for extension of log-convexity to \(x < 0\) under the restriction \(B \prec_W A\).

Theorem 7 Suppose \(B \prec_W A\). Then the function

\[\sigma \to q+1F_q(\sigma, A; B; x) =: f(\sigma)\]

is log-convex on \([0, \infty)\) for each \(x < 1\).

Proof. Take \(\sigma_2 > \sigma_1 \geq 0\) and arbitrary \(\delta > 0\). The inequality

\[f(\sigma_1 + \delta)f(\sigma_2) \leq f(\sigma_1)f(\sigma_2 + \delta)\]

is equivalent to log-convexity for continuous functions (and is stronger in general, see \([15, \text{Chapter I.4}]\)), so it suffices to prove this inequality. Substituting \((9)\) for \(f(\sigma)\) we see that the above inequality is an instance of the Chebyshev inequality \([15, \text{Chapter IX, formula } (1.1)]\) if we choose

\[p(s) = \frac{\rho(s)}{(1 - sx)^{\sigma_1}}, \quad f(s) = \frac{1}{(1 - sx)^{\sigma_2 - \sigma_1}}, \quad g(s) = \frac{1}{(1 - sx)^{\delta}}.\]

Indeed, \(p(s) \geq 0\) and both \(f(s)\) and \(g(s)\) are decreasing on \((0, 1)\) if \(x < 0\) and increasing if \(0 < x < 1\). \(\square\)

Some comments are in order here. Using a completely different approach Karp and Sitnik proved Theorem 7 in \([10]\) for \(0 < x < 1\) under the following conditions on parameters:

\[\frac{e_q(b_1, \ldots, b_q)}{e_q(a_1, \ldots, a_q)} \geq \frac{e_{q-1}(b_1, \ldots, b_q)}{e_{q-1}(a_1, \ldots, a_q)} \geq \cdots \geq \frac{e_1(b_1, \ldots, b_q)}{e_1(a_1, \ldots, a_q)} \geq 1 \tag{28}\]

where

\[e_k(x_1, \ldots, x_q) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq q} x_{j_1}x_{j_2} \cdots x_{j_k}\]

is \(k\)-th elementary symmetric polynomial. It is curious to compare the conditions \([15]\) and \((28)\). The essential part of this comparison was done by Issai Schur in 1923. More precisely, we have
Lemma 3 Suppose \( B \prec W A \). Then (28) holds.

Proof. According to [14, 3.A.8] \( B \prec W A \) implies that \( \phi(A) \leq \phi(B) \) if and only if \( \phi(x) \) is Schur-concave and increasing in each variable. Inequalities (28) can alternatively be written as

\[
\frac{e_k(a_1, \ldots, a_q)}{e_{k-1}(a_1, \ldots, a_q)} \leq \frac{e_k(b_1, \ldots, b_q)}{e_{k-1}(b_1, \ldots, b_q)}, \quad k = 1, 2, \ldots, q.
\]

So we should choose

\[
\phi_k(x_1, \ldots, x_q) = \frac{e_k(x_1, \ldots, x_q)}{e_{k-1}(x_1, \ldots, x_q)}, \quad k = 1, 2, \ldots, q.
\]

Schur-concavity of these functions has been proved by Schur (1923) - see [14, 3.F.3]. It is left to show that \( \phi_k \) is increasing in each variable. Due to symmetry we can take \( x_1 \) to be variable thinking of \( x_2, \ldots, x_q \) as being fixed. Using the definition of elementary symmetric polynomials we see that for \( k \geq 2 \)

\[
\phi_k(x_1, \ldots, x_q) = \frac{x_1 e_{k-1}(x_2, \ldots, x_q) + e_k(x_2, \ldots, x_q)}{x_1 e_{k-2}(x_2, \ldots, x_q) + e_{k-1}(x_2, \ldots, x_q)}.
\]

So taking derivative with respect to \( x_1 \) we obtain \( (e_m = e_m(x_2, \ldots, x_q) \) for brevity):

\[
\frac{\partial \phi_k(x_1, \ldots, x_q)}{\partial x_1} = \frac{e_{k-1}(x_1 e_{k-2} + e_{k-1}) - e_{k-2}(x_1 e_{k-1} + e_k)}{[x_1 e_{k-2} + e_{k-1}]^2} = \frac{e_{k-1}^2 - e_k e_{k-2}}{[x_1 e_{k-2} + e_{k-1}]^2} \geq 0.
\]

Non-negativity holds by Newton’s inequalities. \( \square \)

Remark. Since the reverse implication in Lemma 3 is clearly not true, we see that the log-convexity of \( q+1 F_q(\sigma, A; B; x) \) in \( \sigma \) holds for \( x < 0 \) under the conditions \( B \prec W A \) and for \( 0 \leq x < 1 \) under weaker conditions (28). Numerical experiments show that the log-convexity indeed does not hold for \( x < 0 \) under conditions (28) if we violate \( B \prec W A \).

4. Padé approximation to \( q+1 F_q \). Theorem 3 together with [8, Theorem 3] imply that for \( B \prec W A \) and \( 0 < \sigma \leq 1 \) the function

\[
z \rightarrow q+1 F_q(\sigma, A; B; -z) := q+1 F_q(-z)
\]

belongs to the Stieltjes cone \( S_1 \) with the representing measure \( \rho \) supported on \([0, 1]\). This fact has a number of consequences for the Padé table of \( q+1 F_q \). Before stating them we give an explicit expression for the density which follows directly from Theorem 2.

Theorem 8 Suppose \( \Re \left( \sum_{i=1}^{q} (b_i - a_i) \right) + 1 > \Re(\sigma) \) and \( \Re(a_i) > 0, i = 1, \ldots, q \). Then

\[
q+1 F_q \left( \begin{array}{c} \sigma, A \\ B \end{array} \right) (-z) = \int_0^1 \frac{\rho_1(s) ds}{1 + sz}, \tag{29}
\]

with

\[
\rho_1(s) = \frac{1}{\Gamma(\sigma)} \prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)} \left( \frac{1}{s} \right)^{q+1,0} C_{q+1, q+1}^{1,0} \left( s \left| \begin{array}{c} 1, B \\ \sigma, A \end{array} \right. \right).
\]
Proof. Write
\[ q+1F_q \left( \frac{\sigma, A}{B} \bigg| -z \right) = q+2F_q+1 \left( \frac{1, \sigma, A}{1, B} \bigg| -z \right) \]
and apply Theorem 2. □

Theorem 9 Suppose \( B \prec^W A \) and \( 0 < \sigma \leq 1 \). Then for all integer \( m, n \geq 0 \) the Padé approximant \([m/n]\) to \( q+1F_q(-z) \) at \( z = 0 \) is normal.

Proof. Follows from representation (29) by \[\text{[5, Theorem 4.2.3].} \] □

Remark. Let us remind the reader that a Padé approximant is called normal if it occupies precisely one entry in the Padé table.

Theorem 10 Suppose \( B \prec^W A \) and \( 0 < \sigma \leq 1 \). Then the Padé approximants \([m+j/m]\), \( j \geq -1 \), converge to \( q+1F_q(-z) \) uniformly on every compact subset of \( \mathbb{C} \setminus (-\infty, -1] \) as \( m \to \infty \).

Proof. Follows from representation (29) by \[\text{[3, Theorem 5.4.2].} \] □

Theorem 11 Suppose \( B \prec^W A \), \( \psi > 0 \) and \( 0 < \sigma \leq 1 \). Then the Padé approximants \([m+j/m]\), \( j \geq -1 \), to \( q+1F_q(-z) \) have the form
\[ P_{m+j/m}^{[m+j/m]}(z) = \frac{P_{m+j/m}^{[m+j/m]}(z)}{(-z)^m \pi_m^j(-1/z)}, \]
where \( \pi_m^j(s) \) are polynomials orthogonal with respect to the following inner product:
\[ \int_0^1 \pi_m^j(s) \pi_n^j(s) s^j Q_{q+1,q+1}^{[m+1,0]} \left( s \bigg| \frac{1, B}{\sigma, A} \right) ds = \text{const} \times \delta_{mn}. \]
The numerator polynomials \( P_{m+j/m}^{[m+j/m]}(z) \) are found from
\[ q+1F_q(\sigma, A; B; -z) Q_{m+j/m}^{[m+j/m]}(z) - P_{m+j/m}^{[m+j/m]}(z) = O(z^{2m+j+1}), \quad z \to 0. \]

Proof. Follows from representation \[\text{(29), Lemma 2 and [3, Chapter 5, formula (3.21)]}. \] □

5. Mapping properties of \( q+1F_q \). There is a vast literature dedicated to the mapping properties of the Gauss hypergeometric function \( _2F_1 \). However, the mapping properties of the functions \( q+1F_q(z) \) and \( z_{q+1}F_q(z) \) for \( q \geq 2 \) have been only considered by a few authors \[\text{[16, 17].} \] A combination of \[\text{[8, Theorem 13, Remark 7] with Theorem 3}. \] immediately yields

Theorem 12 Suppose \( B \prec^W A \) and \( \sigma \geq 1 \). Then the function \( q+1F_q(\sigma, A; B; -z) \) maps the sector \( 0 < \arg(z) < \pi/\sigma \) into the lower half-plane \( \Im(z) < 0 \).

Here we only demonstrate the direct consequences of Theorem 3 when it is combined with the results of Thale \[\text{[21]} \] and Wirths \[\text{[22]} \].
Theorem 13 Suppose $B \triangleleft W A$ and $0 < \sigma \leq 1$. Then the functions

$$z \rightarrow q_{+1} F_q(\sigma, A; B; z) \quad \text{and} \quad z \rightarrow z q_{+1} F_q(\sigma, A; B; z)$$

are univalent in the half-plane $\Re(z) < 1$. The second function is also starlike in the disk $|z| < r^*$, where

$$r^* = \sqrt{13\sqrt{13} - 46} \approx 0.934.$$ 

The proof of the first claim follows from representation (9) combined with [21, Theorems 2.1, 2.2] or [22, Satz 2.2]. The second claim follows from [22, Satz 2.4]. □

Remark. The constant $r^*$ above looks different from the (much more cumbersome) constant given in [22] but a simple calculation shows that they are equal.

Theorem 14 Suppose $B \triangleleft W A$ and $0 < \sigma \leq 2$. Then the function

$$z \rightarrow z q_{+1} F_q(\sigma, A; B; z)$$

is univalent in the disk $|z| < r^* := \sqrt{32} - 5 \approx 0.81$.

The claim follows from representation (9) combined with [22, Satz 3.2].

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