High-Energy Physics with Gravitational-Wave Experiments

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We discuss the possible relevance of gravitational-wave (GW) experiments for physics at very high energy. We examine whether, from the experience gained with the computations of various specific relic GW backgrounds, we can extract statements and order of magnitude estimates that are as much as possible model-independent, and we try to distinguish between general conclusions and results related to specific cosmological mechanisms. We examine the statement that the Virgo/LIGO experiments probe the Universe at temperatures $T \sim 10^5 - 10^{10}$ GeV (or timescales $t \sim 10^{-20} - 10^{-26}$ sec) and we consider the possibility that they could actually probe the Universe at much higher energy scales, including the typical scales of grand unification, string theory and quantum gravity. We consider possible scenarios, depending on how the inflationary paradigm is implemented. We discuss the prospects for detection with present and planned experiments. In particular, a second Virgo interferometer correlated with the planned one, and located within a few tens of kilometers from the first, could reach an interesting sensitivity for stochastic GWs of cosmological origin.

I. INTRODUCTION

The energy range between the grand unification scale $M_{\text{GUT}} \sim 10^{16}$ GeV and the Planck scale $M_{\text{Pl}} \simeq 1.22 \times 10^{19}$ GeV is crucial for fundamental physical questions and for testing current ideas about grand unification, quantum gravity, string theory. Experimental results in this energy range are of course very difficult to obtain. From the particle physics point of view, there are basically two important experimental results that can be translated into statements about this energy range (see e.g. for recent reviews): (i) the accurate measurement of gauge coupling constants at LEP that, combined with their running with energy, shows the unification of the couplings at the scale $M_{\text{GUT}}$, provided that the running is computed including the supersymmetric particles in the low energy spectrum. And (ii) the negative results on proton decay; the lower limit on the inverse of the partial decay width for the processes $p \rightarrow e^+\pi^0$ and $p \rightarrow K^+\bar{\nu}$ are $5.5 \times 10^{32}$ yr and $1.0 \times 10^{32}$ yr, respectively, and imply a lower bound $M_{\text{GUT}} \gtrsim 10^{15}$ GeV, which excludes non-supersymmetric SU(5) unification. Further improvement is expected from the SuperKamiokande experiment, which should probe lifetimes $\sim 10^{35}$ yr.

From the cosmological point of view, informations on this energy range can only come from particles which decoupled from the primordial plasma at very early time. Particles which stay in thermal equilibrium down to a decoupling temperature $T_{\text{dec}}$ can only carry informations on the state of the Universe at $E \sim T_{\text{dec}}$. All informations on physics at higher energies has in fact been obliterated by the successive interactions.

The condition for thermal equilibrium is that the rate $\Gamma$ of the processes that maintain equilibrium be larger than the rate of expansion of the Universe, as measured by the Hubble parameter $H$. The rate is given by $\Gamma = n \sigma |v|$ where $n$ is the number density of the particle in question, and for massless or light particles in equilibrium at a temperature $T$, $n \sim T^3$; $|v| \sim 1$ is the typical velocity and $\sigma$ is the cross-section of the process. Consider for instance the weakly interacting neutrinos. In this case the equilibrium is maintained, e.g., by electron-neutrino scattering, and at energies below the $W$ mass $\sigma \sim G_F^2 \langle E^2 \rangle \sim G_F^2 T^3$ where $G_F$ is the Fermi constant and $\langle E^2 \rangle$ is the average energy squared. The Hubble parameter during the radiation dominated era is related to the temperature by $H \sim T^2/M_{\text{Pl}}$. Therefore

$$\left( \frac{\Gamma}{H} \right)_{\text{neutrino}} \sim \frac{G_F^2 T^5}{T^2/M_{\text{Pl}}} \simeq \left( \frac{T}{1 \text{MeV}} \right)^3.$$

Even the weakly interacting neutrinos, therefore, cannot carry informations on the state of the Universe at temperatures larger than approximately 1 MeV. If we repeat the above computation for gravitons, the Fermi constant $G_F$
is replaced by Newton constant $G = 1/M_{Pl}^2$ (we always use units $\hbar = c = k_B = 1$) and at energies below the Planck mass

$$\left( \frac{\Gamma}{\Pi} \right)_{\text{graviton}} \sim \left( \frac{T}{M_{Pl}} \right)^3.$$  

The gravitons are therefore decoupled below the Planck scale. (At the Planck scale the above estimate of the cross section is not valid and nothing can be said without a quantum theory of gravity). It follows that relic gravitational waves are a potential source of informations on very high-energy physics. Gravitational waves produced in the very early Universe have not lost memory of the conditions in which they have been produced, as it happened to all other particles, but still retain in their spectrum, typical frequency and intensity, important informations on the state of the very early Universe, and therefore on physics at correspondingly high energies, which cannot be accessed experimentally in any other way. It is also clear that the property of gravitational waves that makes them so interesting, i.e. their extremely small cross section, is also responsible for the difficulties of the experimental detection.

With the very limited experimental informations that we have on the very high energy region, $M_{GUT} \ll E \ll M_{Pl}$, it is unlikely that theorists will be able to foresee all the interesting sources of relic stochastic background, let alone to compute their spectra. This is particularly clear in the Planckian or string theory domain where, even if we succeed in predicting some interesting physical effects, in general we cannot compute them reliably. So, despite the large efforts that have been devoted to understanding possible sources, it is still quite possible that, if a relic background of gravitational waves will be detected, its physical origin will be a surprise. In this case a model-independent analysis of what we can expect might be useful.

In this paper we discuss whether, from the experience gained with various specific computations of relic backgrounds, it is possible to extract statements or order of magnitude estimates which are as much as possible model-independent. These estimates would constitute a sort of minimal set of naive expectations, that could give some orientation, independently of the uncertainties and intricacies of the specific cosmological models. We discuss typical values of the frequencies involved and of the expected intensity of the background gravitational radiation, and we try to distinguish between statements that are relatively model-independent and results specific to given models.

The paper is written having in mind a reader interested in gravitational-wave detection but not necessarily competent in early Universe cosmology nor in physics at the string or Planck scale, and a number of more technical remarks are relegated in footnotes and in an appendix. We have also tried to be self-contained and we have attempted to summarize and occasionally clean up many formulas and numerical estimates appearing in the literature. The organization of the paper is as follows. In sect. 2 we introduce the variables most commonly used to describe a stochastic background of gravitational waves. We give a detailed derivation of the relation between exact formulas for the signal-to-noise ratio, and approximate but simpler characterizations of the characteristic amplitude and of the noise. The former variables are convenient in theoretical computations while the latter are commonly used by experimentalists, so it is worthwhile to understand in some details their relations. In sect. 3 we apply these formulas to compute the sensitivity to a stochastic background that could be obtained with a second Virgo interferometer correlated with the first, and we compare with various others detectors. We find that in the Virgo-Virgo case the noise which would give the dominant limitation to the measurement of a stochastic background is the mirror thermal noise, and we give the sensitivity for different forms of the relic GW spectrum. In sect. 4 and 5 we discuss estimates of the typical frequency scales. We examine the statements leading to the conclusion that Virgo/LIGO will explore the Universe at temperatures $T \sim 10^7$ GeV (sect. 4), and in the appendix we discuss some qualifications to this statement. In sect. 5 we discuss the possibility to reach much higher energy scales, including the typical scales of grand unification and quantum gravity. In sect. 6 we discuss different scenarios, depending on how the inflationary paradigm is implemented. Characteristic values of the intensity of the spectrum and existing limits and predictions are discussed in sect. 7. Sect. 8 contains the conclusions.

*Thinking in terms of cross-sections, one is lead to ask how comes that gravitons could be detectable altogether, since the graviton-matter cross section is smaller than the neutrino-matter cross section, at energies below the $W$-mass, by a factor $G^2/G_p^2 \sim 10^{-67}$ and neutrinos are already so difficult to detect. The answer is that gravitons are bosons, and therefore their occupation number per cell of phase space can be $n_k \gg 1$; we will see below that in interesting cases, in the relic stochastic background we can have $n_k \sim 10^{69}$ or larger, and the squared amplitude for exciting a given mode of the detector grows as $n_k^2$. So, we will never really detect gravitons, but rather classical gravitational waves. Neutrinos, in contrast, are fermions and for them $n_k \leq 1$. 

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II. DEFINITIONS

A. $\Omega_{gw}(f)$ and the optimal SNR

The intensity of a stochastic background of gravitational waves (GWs) can be characterized by the dimensionless quantity

$$\Omega_{gw}(f) = \frac{1}{\rho_c} \frac{d\rho_{gw}}{d\log f},$$

where $\rho_{gw}$ is the energy density of the stochastic background of gravitational waves, $f$ is the frequency ($\omega = 2\pi f$) and $\rho_c$ is the present value of the critical energy density for closing the Universe. In terms of the present value of the Hubble constant $H_0$, the critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G}.$$  \hspace{1cm} (4)

The value of $H_0$ is usually written as $H_0 = h_0 \times 100$ km/(sec–Mpc), where $h_0$ parametrizes the existing experimental uncertainty. Ref. [3] gives a value $0.5 < h_0 < 0.85$. In the last few years there has been a constant trend toward lower values of $h_0$ and typical estimates are now in the range $0.55 < h_0 < 0.60$ or, more conservatively, $0.50 < h_0 < 0.65$. For instance ref. [4], using the method of type IA supernovae, gives two slightly different estimates $h_0 = 0.56 \pm 0.04$ and $h_0 = 0.58 \pm 0.04$. Ref. [5], with the same method, finds $h_0 = 0.60 \pm 0.05$ and ref. [6], using a gravitational lens, finds $h_0 = 0.51 \pm 0.14$. The spread of values obtained gives an idea of the systematic errors involved.

It is not very convenient to normalize $\rho_{gw}$ to a quantity, $\rho_c$, which is uncertain: this uncertainty would appear in all the subsequent formulas, although it has nothing to do with the uncertainties on the GW background. Therefore, we rather characterize the stochastic GW background with the quantity $h_0^2 \Omega_{gw}(f)$, which is independent of $h_0$. All theoretical computations of a relic GW spectrum are actually computations of $d\rho_{gw}/d\log f$ and are independent of the uncertainty on $H_0$. Therefore the result of these computations is expressed in terms of $h_0^2 \Omega_{gw}$, rather than of $\Omega_{gw}$.

To detect a stochastic GW background the optimal strategy consists in performing a correlation between two (or more) detectors, since, as we will discuss below, the signal will be far too low to exceed the noise level in any existing or planned single detector (with the exception of the space interferometer LISA, see below). The strategy has been discussed in refs. [7–10], and a clear review is ref. [11]. Let us recall the main points of the analysis. The output of any single detector is of the form $s_i(t) = h_i(t) + n_i(t)$, where $i = 1, 2$ labels the detector, and the output $s_i(t)$ is made up of a noise $n_i(t)$ and possibly a signal $h_i(t)$. In the typical situation, $h_i \ll n_i$. We can correlate the two outputs defining

$$S = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' s_1(t)s_2(t')Q(t-t'),$$

where $T$ is the total integration time (e.g. one year) and $Q$ a filter function. If the noises in the two detectors are uncorrelated, the ensemble average of the Fourier components of the noise satisfies

$$\langle \tilde{n}_i(f)\tilde{n}_j(f')\rangle = \delta(f-f')\delta_{ij} \frac{1}{2} S_n^{(i)}(|f|).$$

The above equation defines the functions $S_n^{(i)}(|f|)$, with dimensions Hz$^{-1}$. The factor $1/2$ is conventionally inserted in the definition so that the total noise power is obtained integrating $S_n(f)$ over the physical range $0 \leq f < \infty$, rather than from $-\infty$ to $\infty$. The noise level of the detector labelled by $i$ is therefore measured by $\tilde{h}_j^{(i)} \equiv \sqrt{S_n^{(i)}}$, with dimensions Hz$^{-1/2}$. The function $S_n$ is known as the square spectral noise density.$^\dagger$

\footnote{This simple point has occasionally been missed in the literature, where one can find the statement that, for small values of $H_0$, $\Omega_{gw}$ is larger and therefore easier to detect. Of course, it is larger only because it has been normalized using a smaller quantity.}

\footnote{Unfortunately there is not much agreement about notations in the literature. The spectral noise density, that we denote by $S_n(f)$ following e.g. ref. [8], is called $P(f)$ in ref. [11]. Other authors use the notation $S_h(f)$, which we instead reserve for the spectral density of the signal. To make things worse, $S_n$ is sometime defined with or without the factor $1/2$ in eq. (6).}
As discussed in refs. [11], in the limit $h_i \ll n_i$, for any given form of the signal, i.e. for any given functional form of $h_0^2 \Omega_{gw}(f)$, it is possible to find explicitly the filter function $Q(t)$ which maximizes the signal-to-noise ratio (SNR). In the case of L-shaped interferometers the corresponding value of the optimal SNR turns out to be (see e.g. ref. [11], eq.(43))

$$\text{SNR} = \left( \frac{9H_0^4}{50\pi^4} \right) T \int_0^{\infty} df \frac{\gamma^2(f) \Omega_{gw}^2(f)}{f^6 S_n(1)(f) S_n(2)(f)} \right)^{1/4}.$$  \hspace{1cm} (7)

We have taken into account the fact that what has been called $S$ in eq. (3) is quadratic in the signals and, with usual definitions, it contributes to the SNR squared. This differs from the convention used in ref. [11]. The function $\gamma(f)$ is called the overlap function. It takes into account the difference in location and orientation of the two detectors. It has been computed for the various pairs of LIGO1, LIGO2, Virgo and GEO detectors [9]. For detectors very close and parallel, $\gamma(f) = 1$. Basically, $\gamma(f)$ cuts off the integrand in eq. (7) at a frequency $2\pi f$ of the order of the inverse separation between the two detectors. For the two LIGO detectors, this cutoff is around 60 Hz. We will discuss $\gamma(f)$ in sect. 2C, where we will also comment on the modifications needed for different geometries.

In principle the expression for the SNR, eq. (7), is all that we need in order to discuss the possibility of detection of a given GW background. However it is useful, for order of magnitude estimates and for intuitive understanding, to express the SNR in terms of a characteristic amplitude of the stochastic GW background and of a characteristic noise level, although, as we will see, the latter is a quantity that describes the noise only approximately, in contrast to eq. (8) which is exact. We will introduce these quantities in the next two subsections.

B. The characteristic amplitude

A stochastic GW at a given point $\vec{x} = 0$ can be expanded, in the transverse traceless gauge, as (we follow the notations of ref. [11], app.A)

$$h_{ab}(t) = \sum_{A=+,-} \int_{-\infty}^{\infty} df \int d\Omega \tilde{h}_A(f,\Omega) \exp(2\pi ift) c_{ab}^A(\Omega),$$ \hspace{1cm} (8)

where $\tilde{h}_A(-f,\Omega) = \tilde{h}_A^*(f,\Omega)$. $\Omega$ is a unit vector representing the direction of propagation of the wave and $d\Omega = d\cos \theta d\phi$. The polarization tensors can be written as $c_{ab}^+(\Omega) = ma_m b_n - m_a n_b$ and $c_{ab}^x(\Omega) = m_a n_b + n_a m_b$, with $\hat{m}, \hat{n}$ unit vectors orthogonal to $\Omega$ and to each other. With these definitions, $c_{ab}^A(\Omega)c_{ab}^{A'}(\Omega) = 2\delta^{AA'}$. For a stochastic background, assumed to be isotropic, unpolarized and stationary (see [11] for a discussion of these assumptions) the ensemble average of the Fourier amplitudes can be written as

$$\langle \tilde{h}_A^*(f,\Omega) \tilde{h}_A'(f',\Omega') \rangle = \delta(f - f') \frac{1}{4\pi} \delta^2(\Omega,\Omega') \delta_{AA'} \frac{1}{2} S_h(f),$$ \hspace{1cm} (9)

where $\delta(\Omega,\Omega') = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta')$. The function $S_h(f)$ defined by the above equation has dimensions $Hz^{-1}$ and satisfies $S_h(f) = S_h(-f)$. The factor $1/2$ is conventionally inserted in the definition of $S_h$ in order to compensate for the fact that the integration variable $f$ in eq. [11] ranges between $-\infty$ and $+\infty$ rather than over the physical domain $0 \leq f < \infty$. The factor $1/(4\pi)$ is inserted so that, integrating the left-hand side over $d\Omega$ and over $d\Omega'$, we get $\delta(f - f')\delta_{AA'}(1/2)S_h(f)$. With this normalization, $S_h(f)$ is therefore the quantity to be compared with the noise level $S_n(f)$ defined in eq. [11]. Using eqs. (8,9) we get

$$\langle h_{ab}(t) h^{ab}(t) \rangle = 2 \int_{-\infty}^{\infty} df \ S_h(f) = 4 \int_{f=0}^{f=\infty} d(log f) \ f S_h(f).$$ \hspace{1cm} (10)

We now define the characteristic amplitude $h_c(f)$ from

$$\langle h_{ab}(t) h^{ab}(t) \rangle = 2 \int_{f=0}^{f=\infty} d(log f) \ h^2_c(f).$$ \hspace{1cm} (11)

Note that $h_c(f)$ is dimensionless, and represents a characteristic value of the amplitude, per unit logarithmic interval of frequency. The factor of two on the right-hand side of eq. (11) is inserted for the following reason. The response of the
detector to a single wave with amplitudes $h_+, h_\times$ is of the form (Ref. [13], eqs. (26) and (103)) $h(t) = F_+ h_+ + F_\times h_\times$. For an interferometer, $h(t) = \Delta l(t)/L$ where $l$ is the difference in arm lengths. The function $h(t)$ is also called the gravitational wave strain acting on the detector. The functions $F_+, F_\times$ are known as detector pattern functions, and $0 \leq |F_+, F_\times| \leq 1$. They depend on three angles, that determine the direction of arrival of the GW and its polarization. For any quadrupole-beam-pattern GW detector we have (Ref. [13], pg. 369) $\langle F_+^2 \rangle_\Omega = \langle F_\times^2 \rangle_\Omega$ and $\langle F_+ F_\times \rangle_\Omega = 0$, where

$$\langle \ldots \rangle_\Omega = \int \frac{d\Omega}{4\pi} \frac{d\psi}{2\pi} \langle \ldots \rangle$$

denotes an average over the direction of propagation of the wave, $d\Omega = d\cos \theta d\phi$, and over the angle $\psi$ that gives the preferred frame $(x', y')$ where the wave in the transverse traceless gauge takes the simple form $h_{x'x'} = -h_{y'y'}$, $h_{x'y'} = h_{y'x'}$ (see [13], pg. 367). This average should not to be confused with $\langle \ldots \rangle$ which is the time average, i.e., in Fourier space, the ensemble average of eq. (4). Therefore, $\langle h^2(t) \rangle_\Omega = \langle F_+^2 \rangle_\Omega h_+^2 + \langle F_\times^2 \rangle_\Omega h_\times^2$. In a unpolarized stochastic background $\langle h_+^2 \rangle = \langle h_\times^2 \rangle$ (where $\langle \ldots \rangle$ is the ensemble average of eq. (4)) and therefore, since $\langle F_+^2 \rangle_\Omega = \langle F_\times^2 \rangle_\Omega$, the two terms in $h^2(t)$, after averaging over the angles and over time, give the same contribution. This motivates the factor of two in the definition of $h_+^2(f)$, eq. (4). For the same reason, we could insert a factor $\langle F_+^2 \rangle_\Omega$ in the definition of $h_\times^2(f)$. However, what is really meaningful is not a characterization of the signal $h_+(f)$ nor of the noise level $h_\times(f)$ separately, but rather the signal-to-noise ratio discussed in the previous subsection. We are trying to express the SNR in terms of a ratio of two quantities, $h_L(f)/h_n(f)$. So, we are free to arbitrarily move factors from $h_+$ to $h_\times$ as long as $h_+ / h_\times$ is unchanged. However, it is convenient to collect in $h_\times$ all factors related to the source, and in $h_\times$ all factors related to the detectors. We do not insert a factor $\langle F_+^2 \rangle_\Omega^{1/2}$ in the definition of $h_\times$, and therefore, automatically, we will obtain a factor $\langle F_+^2 \rangle_\Omega^{-1/2}$ in $h_n$, see eqs. (23,24) below. The same is true for numerical factors like the factor of two just discussed. We could as well have neglected it in $h_\times^2$ and we would find an additional factor $1/\sqrt{2}$ in $h_n$. We will discuss $h_n$ in the next subsection.

Comparing eqs. (9) and (13), we get

$$h_+^2(f) = 2f S_h(f).$$

We now wish to relate $h_+(f)$ and $h_\times^2 \Omega_{gw}(f)$. The starting point is the expression for the energy density of gravitational waves, given by the 00-component of the energy-momentum tensor. The energy-momentum tensor of a GW cannot be localized inside a single wavelength (see e.g. ref. [13], sects. 20.4 and 35.7 for a careful discussion) but it can be defined with a spatial averaging over several wavelengths:

$$\rho_{gw} = \frac{1}{32\pi G} \langle h_{ab} h^{ab} \rangle.$$  

For a stochastic background, the spatial average over a few wavelengths is the same as a time average at a given point, which, in Fourier space, is the ensemble average performed using eq. (4). We therefore insert eq. (9) into eq. (4) and use eq. (4). The result is

$$\rho_{gw} = \frac{4}{32\pi G} \int_{f=0}^{f=\infty} d(\log f) \ f(2\pi f)^2 S_h(f),$$

so that

$$\frac{d\rho_{gw}}{d\log f} = \frac{\pi}{2G} f^3 S_h(f).$$

Comparing eqs. (10) and (13) we get the important relation

$$\frac{d\rho_{gw}}{d\log f} = \frac{\pi}{4G} f^2 h_+^2(f),$$

or, dividing by the critical density $\rho_c$,

$$\Omega_{gw}(f) = \frac{2\pi^2}{3H_0^2} f^2 h_+^2(f).$$

Inserting the numerical value of $H_0$, we find (Ref. [13], eq. (65))

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\[ h_c(f) \approx 1.263 \times 10^{-18} \left( \frac{1\text{Hz}}{f} \right) \sqrt{h_0^2 \Omega_{\text{gw}}(f)}. \]  

Using eqs. (13,18) we can also write \( \Omega_{\text{gw}}(f) = (4\pi^2/3H_0^2)f^3S_h(f) \). Using this relation, and defining \( S_n(f) = (S_n^0(f)/S_n^0(f))^{1/2} \), eq. (13) can be written in a more transparent form,

\[
\text{SNR} = \left[ 2T \int_0^\infty df \left( \frac{2\gamma(f)}{5} \right)^2 \frac{S^2_{\Omega}(f)}{S^2_n(f)} \right]^{1/4}.
\]

The physical reason for the appearance of the factor 2/5 in the above formula will be clear from eq. (24) below. (The number 2/5 is specific to L-shaped interferometers, see below). The factor of 2 in front of the integral can instead be understood from \( \int_{-\infty}^{\infty} df = 2 \int_0^\infty df \).

Finally, we mention another useful formula which expresses \( h_0^2 \Omega_{\text{gw}}(f) \) in terms of the number of gravitons per cell of the phase space, \( n(\vec{x},\vec{k}) \). For an isotropic stochastic background \( n(\vec{x},\vec{k}) = n_f \) depends only on the frequency \( f = |\vec{k}|/(2\pi) \), and \( \rho_{\text{gw}} = 2 \int n_f 2\pi f d^3k/(2\pi)^3 = 16\pi^2 \int_0^\infty d(\log f)n_f f^4 \). Therefore \( d\rho_{\text{gw}}/d\log f = 16\pi^2 n_f f^4 \), and

\[
h_0^2 \Omega_{\text{gw}}(f) \approx 3.6 \left( \frac{n_f}{10^{37}} \right) \left( \frac{f}{1\text{kHz}} \right)^4.
\]

As we will discuss below, to be observable at the LIGO/Virgo interferometers, we should have at least \( h_0^2 \Omega_{\text{gw}} \sim 10^{-6} \) between 1 Hz and 1 kHz, corresponding to \( n_f \) of order 10^{31} at 1 kHz and \( n_f \sim 10^{43} \) at 1 Hz. A detectable stochastic GW background is therefore exceedingly classical, \( n_k \gg 1 \).

### C. The characteristic noise level

We have seen in the previous section that there is a very natural definition of the characteristic amplitude of the signal, given by \( h_c(f) \), which contains all the informations on the physical effects, and is independent of the apparatus.

We can therefore associate to \( h_c(f) \) a corresponding noise amplitude \( h_n(f) \), that embodies all the informations on the apparatus, defining \( h_c(f)/h_n(f) \) in terms of the optimal SNR.

If, in the integral giving the optimal SNR, eq. (7) or eq. (20), we consider only a range of frequencies \( \Delta f \) such that the integrand is approximately constant, we can write

\[
\text{SNR} \sim \left[ \frac{8}{25} T \Delta f \frac{\gamma^2(f)S^2_n(f)}{S^2_{\Omega}(f)} \right]^{1/4} = \left[ \frac{2T \Delta f \gamma^2(f)h^4_c(f)}{25f^3S^2_n(f)} \right]^{1/4}.
\]

The right-hand side of eq. (22) is proportional to \( h_c(f) \), and we can therefore define \( h_n(f) \) equating the right-hand side of eq. (22) to \( h_c(f)/h_n(f) \), so that

\[
h_n(f) = \left( \frac{1}{2T \Delta f} \right)^{1/4} \left[ \frac{1}{2} f S_n(f) \right]^{1/2} \left( \frac{5}{\gamma(f)} \right)^{1/2}.
\]

For \( L \)-shaped interferometers, the overlap function \( \gamma(f) \) is defined in terms of the detector pattern functions \( F^{(i)}_{\pm,\times} \), where \( i = 1, 2 \) labels the detector, as \( 13 \)

\[
\gamma(f) = \frac{5}{2} \int \frac{d\Omega}{4\pi} e^{-2\pi i f \Omega \cdot \Delta x} \left( F^{(1)}_+ F^{(2)}_+ + F^{(1)}_x F^{(2)}_x \right).
\]

If the separation \( \Delta x \) between the two detectors is very small, \( 2\pi f \Delta x \ll 1 \) and if the detectors have the same orientation, so that \( F^{(1)}_{\pm,\times} = F^{(2)}_{\pm,\times} \), then (averaging also over the angle \( \psi \) discussed in the previous subsection)

\[
\gamma(f) \approx \frac{5}{2} \left( \langle F^2_{\pm} \rangle_{\Omega} + \langle F^2_{\times} \rangle_{\Omega} \right) = 5\langle F^2_{\pm} \rangle_{\Omega}.
\]

For \( L \)-shaped interferometers \( \langle F^2_{\times} \rangle_{\Omega} = 1/5 \) (see ref. [13], eq. (110), or ref. [14]). We see that the factor 5/2 in the definition of \( \gamma(f) \), eq. (24), has been inserted so that for parallel detectors at the same site \( \gamma(f) = 1 \). Therefore, the
factor $5^{1/2}$ in eq. (23) measures the increase in the noise level due to the fact that stochastic GWs hit the detectors from all directions, rather than just from the direction where the sensitivity is optimal, while $1/\gamma^{1/2}$ measures the decrease in sensitivity due to the detectors separation.

For a generic geometry, the factors 5 in the above formulas are replaced by $1/(F^2_{\perp})_{\Omega}$. In the special case $\gamma(f) = 1$ (detectors at the same site) the factor $(5/\gamma)^{1/2}$ in eq. (23) becomes therefore $1/(F^2_{\perp})_{\Omega}^{1/2}$, and we recover eq. (66) of ref. 13 (for comparison, note that the quantity denoted $S_h(f)$ in 13 is called here $S_n(f)/2$).

From the derivation of eq. (23) we can understand the limitations implicit in the use of $S_n(f)$. It gives a measure of the noise level only under the approximation that leads from eq. (24), which is exact (in the limit $n_i \ll n$, to eq. (22). This means that $\Delta f$ must be small enough compared to the scale on which the integrand in eq. (21) changes, so that $\gamma(f)_{S_h(f)}/S_n(f)$ is approximately constant. In a large bandwidth this is non trivial, and of course depends also on the form of the signal; for instance, if $h_0^{3/2}\Omega_{gw}$ is flat, then $S_h(f) \sim 1/f^3$. For accurate estimates of the SNR at a wideband detector there is no substitute for a numerical integration of eq. (7) or eq. (23). However, for order of magnitude estimates, eq. (19) for $h_c(f)$ and eq. (23) for $h_n(f)$ are simpler to use, and they have the advantage of clearly separating the physical effect, which is described by $h_c(f)$, from the properties of the detectors, that enter only in $h_n(f)$.

Eq. (23) also shows very clearly the advantage of correlating two detectors compared with the use of a single detector. With a single detector, the minimum observable signal, at SNR=1, is given by the condition $S_h(f) \geq S_n(f)$. This means, from eq. (3), a minimum detectable value for $h_c(f)$ given by $h_{\text{min}} = (2FS_n(f))^{1/2}$. The superscript 1d reminds that this quantity refers to a single detector. From eq. (23) we find for the minimum detectable value with two interferometers in coincidence, $h_{\text{min}}^{2d}$,

$$h_{\text{min}}^{2d}(f) = \left(\frac{1}{(\hat{T}\Delta f)^{1/4}}\right)^{1/2}h_{\text{min}}^{1d}(f) \left(\frac{5}{\gamma(f)}\right)^{1/2} \simeq 1.77 \times 10^{-2} h_{\text{min}}^{1d}(f) \left(\frac{1 \text{ Hz}}{\Delta f}\right)^{1/4} \left(\frac{1 \text{ yr}}{T}\right)^{1/4} \frac{1}{\gamma^{1/2}(f)}.$$ (26)

Of course, the reduction factor in the noise level is larger if the integration time is larger, and if we increase the bandwidth $\Delta f$ over which a useful coincidence (i.e. $\gamma(f) \sim 1$) is possible. Note that $h_0^{3/2}\Omega_{gw}$ is quadratic in $h_c(f)$, so that an improvement in sensitivity by two orders of magnitude in $h_c$ means four orders of magnitude in $h_0^{3/2}\Omega_{gw}$.

### III. APPLICATION TO VARIOUS DETECTORS

**Single detectors.** To better appreciate the importance of correlating two detectors, it is instructive to consider first the sensitivity that can be obtained using only one detector. In this case a hypothetical signal would manifest itself as an excess noise, and should therefore satisfy $S_h(f) \geq S_n(f)$. Using eqs. (13,18) we write $S_h$ in terms of $h_0^{3/2}\Omega_{gw}$, and we write $S_n(f) = \tilde{h}_f^2$. For the minimum detectable value of $h_0^{3/2}\Omega_{gw}$ we get

$$h_{\text{min}}^{2\Omega_{gw}}(f) \simeq 10^{-2} \left(\frac{f}{100 \text{ Hz}}\right)^3 \left(\frac{\tilde{h}_f}{10^{-22} \text{Hz}^{-1/2}}\right)^2.$$ (27)
accounted for. This might not be a great problem if the SNR is very large, but certainly with a single detector we cannot make a reliable detection at SNR of order one, so that the above estimates (which have been obtained setting SNR=1) are really overestimates.

**Virgo-Virgo.** We now consider the sensitivity that could be obtained at Virgo if the planned interferometer were correlated with a second identical interferometer located at a few tens of kilometers from the first, and with the same orientation.

This distance would be optimal from the point of view of the stochastic background, since it should be sufficient to decorrelate local noises like, e.g., the seismic noise and local electromagnetic disturbances, but still the two interferometers would be close enough so that the overlap function does not cut off the high frequency range, as it happens, instead, correlating the two LIGOs.

Let us first give a rough estimate of the sensitivity using \( h_c(f), h_n(f) \). From fig. 1 we see that we can take, for our estimate, \( h_f \sim 10^{-22}\text{Hz}^{-1/2} \) over a bandwidth \( \Delta f \sim 1 \text{kHz} \). Using \( T = 1 \text{yr} \), eq. (23) gives

\[
h_n(f) \sim 4.5 \times 10^{-24} \left( \frac{f}{100 \text{Hz}} \right)^{1/2} \left( \frac{\tilde{h}_f}{10^{-22} \text{Hz}^{-1/2}} \right).
\]

Requiring for instance SNR=1.65 (this corresponds to 90% confidence level; a more precise discussion of the statistical significance, including the effect of the false alarm rate can be found in ref. [12]) gives an estimate for the minimum detectable value of \( h_0^2\Omega_{gw}(f) \),

\[
h_0^2\Omega_{gw}^{\text{min}}(f) \sim 3 \times 10^{-7} \left( \frac{f}{100 \text{Hz}} \right)^3 \left( \frac{\tilde{h}_f}{10^{-22} \text{Hz}^{-1/2}} \right)^2.
\]

This suggests that correlating two Virgo interferometers we can detect a relic spectrum with \( h_0^2\Omega_{gw}(100\text{Hz}) \sim 3 \times 10^{-7} \) at SNR=1.65, or \( 1 \times 10^{-7} \) at SNR=1. Compared to the case of a single interferometer with SNR=1, eq. (27), we

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\[^{8}\text{Correlations between two interferometers have already been carried out using prototypes operated by the groups in Glasgow and at the Max Planck Institute for Quantum Optics, with an effective coincident observing period of 62 hours [13]. Although the sensitivity of course is not yet significant, they demonstrate the possibility of making long-term coincident observations with interferometers.}\]
gain five orders of magnitude. As already discussed, to obtain a precise numerical value one must however resort to eq. (9). This involves an integral over all frequencies, (that replaces the somewhat arbitrary choice of $\Delta f$ made above) and depends on the functional form of $h_0^2 \Omega_{gw}(f)$. If for instance $h_0^2 \Omega_{gw}(f)$ is independent of the frequency, using the numerical values of $\hat{h}_f$ plotted in fig. 1 (see ref. [14]) and performing the numerical integral we get for the minimum detectable $h_0^2 \Omega_{gw}$ (we give the result for a generic value of the SNR and of the integration time)

$$h_0^2 \Omega_{gw}^{\text{min}} \simeq 2 \times 10^{-7} \left( \frac{\text{SNR}}{1.65} \right)^2 \left( \frac{1 \text{yr}}{T} \right)^{1/2} \left( h_0^2 \Omega_{gw}(f) = \text{const.} \right). \quad (30)$$

We see that this number is quite consistent with the approximate estimate (29), and with the value $2 \times 10^{-7}$ reported in ref. [21]. Stretching the parameters to SNR=1 (68% c.l.) and $T = 4$ years, the value goes down at $(3 - 4) \times 10^{-8}$. It is interesting to note that the main contribution to the integral comes from the region $f < 100$ Hz. In fact, neglecting the contribution to the integral of the region $f > 100$ Hz, the result for $h_0^2 \Omega_{gw}^{\text{min}}$ changes only by approximately 2%. Also, the lower part of the accessible frequency range is not crucial. Restricting for instance to the region $20$ Hz $\leq f \leq 200$ Hz, the sensitivity on $h_0^2 \Omega_{gw}$ degrades by less than 1%, while restricting to the region $30$ Hz $\leq f \leq 100$ Hz, the sensitivity on $h_0^2 \Omega_{gw}$ degrades by approximately 10%. Then, from fig. 1 we conclude that by far the most important source of noise for the measurement of a flat stochastic background is the thermal noise. In particular, the sensitivity to a stochastic background is limited basically by the mirror thermal noise, which dominates in the region $40$ Hz $\leq f \leq 200$ Hz, while the pendulum thermal noise dominates below approximately $40$ Hz.

The sensitivity depends however on the functional form of $\Omega_{gw}(f)$. Suppose for instance that in the Virgo frequency band we can approximate the signal as

$$\Omega_{gw}(f) = \Omega_{gw}(1\text{kHz}) \left( \frac{f}{1\text{kHz}} \right)^{\alpha}. \quad (31)$$

For $\alpha = 1$ we find that the spectrum is detectable at SNR=1.65 if $h_0^2 \Omega_{gw}(1\text{kHz}) \simeq 3.6 \times 10^{-6}$. For $\alpha = -1$ we find (taking $f = 5$ Hz as lower limit in the integration) $h_0^2 \Omega_{gw}(1\text{kHz}) \simeq 6 \times 10^{-9}$. Note however that in this case, since $\alpha < 0$, the spectrum is peaked at low frequencies, and $h_0^2 \Omega_{gw}(5\text{Hz}) \simeq 1 \times 10^{-6}$. So, both for increasing or decreasing spectra, to be detectable $h_0^2 \Omega_{gw}$ must have a peak value, within the Virgo band, of order a few $\times 10^{-6}$ in the case $\alpha = \pm 1$, while a constant spectrum can be detected at the level $2 \times 10^{-7}$. Clearly, for detecting increasing (decreasing) spectra, the upper (lower) part of the frequency band becomes more important, and this is the reason why the sensitivity degrades compared to flat spectra, since for increasing or decreasing spectra the maximum of the signal is at the edges of the accessible frequency band, where the interferometer sensitivity is worse.

**LIGO-LIGO.** The two LIGO detectors are under construction at a large distance from each other, $\sim 3000$ km. This choice optimizes the possibility of detecting the direction of arrival of GWs from astrophysical sources, but it is not optimal from the point of view of the stochastic background, since the overlap functions cuts off the integrand in eq. (7) around 60 Hz. The sensitivity to a stochastic background of the LIGO-LIGO detectors has been computed in refs. [21,22,23]. The result, for $\Omega_{gw}(f)$ independent of $f$, is

$$h_0^2 \Omega_{gw} \simeq (5 - 6) \times 10^{-6} \quad (32)$$

for the initial LIGO. The advanced LIGO aims at $5 \times 10^{-11}$. These numbers are given at 90% c.l. in [11], and a detailed analysis of the statistical significance is given in [12].

**Resonant masses and Resonant mass-Interferometer.** Resonant mass detectors includes bars like NAUTILUS, EXPLORER and AURIGA (see e.g. refs. [7,22] for reviews). Spherical [23,24] and truncated icosahedron (TIGA) [25] resonant masses are also being developed or studied. The correlation between two resonant bars and

**Note also that it is not very meaningful to give more decimal figures in the minimum detectable value of $h_0^2 \Omega_{gw}^{\text{min}}$. Apart from the various uncertainties which enter the computation of the sensitivity curve shown in fig. 1, a trivial source of uncertainty is the fact that the computation of the thermal noises are performed using a temperature of 300K. A 5% variation, corresponding to an equally plausible value of the temperature, gives a 5% difference in $h_0^2 \Omega_{gw}^{\text{min}}$. Quoting more figures is especially meaningless when the minimum detectable $h_0^2 \Omega_{gw}$ is estimated using the approximate quantities $h_c(f), h_n(f)$, i.e. approximating the integrand of eq. (6) with a constant over a bandwidth $\Delta f$. For a broadband detector these estimates typically give results which agree with the exact numerical integration of eq. (6) at best within a factor of two.**
between a bar and an interferometer has been considered in refs. [10,21,24,28]. The values quoted in ref. [10] are as follows (using 1 year of integration time and SNR=1; the minimum detectable value of $h_0^2\Omega_{gw}$ grows as (SNR)$^2$, so the minimum detectable value at SNR=1.65 is about a factor of 3 larger). Correlating the AURIGA and NAUTILUS detectors, with the present orientation, we can detect $h_0^2\Omega_{gw} \approx 2 \times 10^{-4}$. Reorienting the detectors for optimal correlation, which is technically feasible, we can reach $h_0^2\Omega_{gw} \approx 8 \times 10^{-5}$. For AURIGA-VIRGO, with present orientation, $h_0^2\Omega_{gw} \approx 2 \times 10^{-4}$, and for NAUTILUS-VIRGO, $h_0^2\Omega_{gw} \approx 3.5 \times 10^{-4}$. A three detectors correlation AURIGA-NAUTILUS-VIRGO, with present orientation, would reach $h_0^2\Omega_{gw} \approx 1 \times 10^{-4}$, and with optimal orientation $h_0^2\Omega_{gw} \approx 6 \times 10^{-5}$. Although the improvement in sensitivity in a bar-bar-interferometer correlation is not large compared to a bar-bar or bar-interferometer correlation, a three detectors correlation would be important in ruling out spurious effects [10]. Preliminary results on a NAUTILUS-EXPLORER correlation, using 12 hours of data, have been reported in [24], and give a bound $h_0^2\Omega_{gw} \sim 120$.

Using resonant optical techniques, it is possible to improve the sensitivity of interferometers at special values of the frequency, at the expense of their broad-band sensitivity. Since bars have a narrow band anyway, narrow-banding the interferometer improves the sensitivity of a bar-interferometer correlation by about one order of magnitude [21].

While resonant bars have been taking data for years, spherical detectors are at the moment still at the stage of theoretical studies (although prototypes might be built in the near future), but could reach extremely interesting sensitivities. In particular, two spheres with a diameter of 3 meters, made of Al5056, and located at the same site, could reach a sensitivity $h_0^2\Omega_{gw} \sim 4 \times 10^{-7}$ [10]. This figure improves using a more dense material or increasing the sphere diameter, but it might be difficult to build a heavier sphere. Another very promising possibility is given by hollow spheres [24]. The theoretical studies of ref. [24] suggest a minimum detectable value $h_0^2\Omega_{gw} \sim 10^{-9}$ at $f = 200$ Hz.

IV. THE ENERGY SCALES PROBED BY RELIC GRAVITATIONAL WAVES

Let us consider the standard Friedmann-Robertson-Walker (FRW) cosmological model, consisting of a radiation-dominated (RD) phase followed by the present matter-dominated (MD) phase, and let us call $a(t)$ the FRW scale factor. The RD phase goes backward in time until some new regime sets in. This could be an inflationary epoch, e.g. at the grand unification scale, or the RD phase could go back in time until Planckian energies are reached and quantum gravity sets in, i.e., until $t \sim t_{Pl} \simeq 5 \times 10^{-44}$ s. If the correct theory of quantum gravity is provided by string theory, the characteristic mass scale is the string mass which is somewhat smaller than the Planck mass and is presumably in the $10^{17} - 10^{18}$ GeV region, and the corresponding characteristic time is therefore one or two orders of magnitude larger than $t_{Pl}$. The transition between the RD and MD phases takes place at $t = t_{eq}$, when the temperature of the Universe is of the order of only a few eV, so we are interested in graviton production which takes place well within the RD phase, or possibly at Planckian energies.

A graviton produced with a frequency $f_*$ at a time $t = t_*$, within the RD phase has today ($t = t_0$) a red-shifted frequency $f_0$ given by $f_0 = f_*a(t_0)/a(t_*)$. To compute the ratio $a(t_0)/a(t_*)$ one uses the fact that during the standard RD and MD phases the Universe expands adiabatically. The entropy per unit comoving volume is $S = \text{const.}g_S(T)a^3(t)T^3$, where $g_S(T)$ counts the effective number of species [2]. In the standard model, at $T \gtrsim 300$ GeV, $g_S(T)$ becomes constant and has the value $g_S(T_eq) = 106.75$, while today $g_S(T_0) = 3.91$ and $T_0 = 2.728\pm0.002$K [30]. Using $g_S(T_*)a^3(t_*)T_0^3 = g_S(T_0)a^3(t_0)T_0^3$ one finds

$$f_0 = f_*\frac{a(t_*)}{a(t_0)} \simeq 8.0 \times 10^{-14} f_* \left( \frac{100}{g_S(T_*)} \right)^{1/3} \left( \frac{1\text{GeV}}{T_*} \right).$$

The first point to be addressed is what is the characteristic value of the frequency $f_*$ produced at time $t_*$, when the temperature was $T_*$. One of the relevant parameters in this estimate is certainly the Hubble parameter at time of production, $H(t_*) \equiv H_*$. This comes from the fact that $H_*^{-1}$ is the size of the horizon at time $t_*$. The horizon size, physically, is the length scale beyond which causal microphysics cannot operate (see e.g. [2], ch. 8.4), and therefore, for causality reasons, we expect that the characteristic wavelength of gravitons or any other particles produced at time $t_*$ will be of order $H_*^{-1}$ or smaller[11].

[11] On a more technical side, the deeper reason has really to do with the invariance of general relativity under coordinate transformations, combined with the expansion over a fixed, non-uniform, background. Consider for instance a scalar field $\phi(x)$
Therefore, we write \( \lambda_\epsilon = \epsilon H_*^{-1} \). The above argument suggests \( \epsilon \leq 1 \). During RD, \( H_*^2 = (8\pi/3)G\rho_{\text{rad}} \). The contribution to \( \rho \) from a single species of relativistic particle with \( g_i \) internal states (helicity, color, etc.) is \( \rho_{\text{rad}} = g_i(\pi^2/30)T_i^4 \) for a boson an \( \rho = (7/8)g_i(\pi^2/30)T_i^4 \) for a fermion. Taking into account that the \( i \)-th species has in general a temperature \( T_i \neq T \) if it already dropped out of equilibrium, we can define a function \( g(T) \) from \( \rho_{\text{rad}} = (\pi^2/30)g(T)T^4 \). Then \[ g(T) = \sum_{i=\text{bosons}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left( \frac{T_i}{T} \right)^4 \] (34)
The sum runs over relativistic species. This holds if a species is in thermal equilibrium at the temperature \( T_i \). If instead it does not have a thermal spectrum (which in general is the case for gravitons) we can still use the above equation, where for this species \( T_i \) does not represent a temperature but is defined (for bosons) from \( \rho_i = g_i(\pi^2/30)T_i^4 \), where \( \rho_i \) is the energy density of this species. The quantity \( g_S(T) \) used before for the entropy is given by the same expression as \( g(T) \), with \( (T/T_i)^4 \) replaced by \( (T_i/T)^3 \). We see that both \( g(T) \) and \( g_S(T) \) give a measure of the effective number of species. For most of the early history of the Universe, \( g(T) = g_S(T) \), and in the standard model at \( T \gtrsim 300 \) GeV they have the common value \( g_* = 106.75 \), while today \( g(T_0) = 3.36 \). Therefore

\[ H_*^2 = \frac{8\pi^3 g_*T_*^4}{90M_P^2}, \] (35)

and, using \( f_* = H_*/\epsilon \), eq. (33) can be written as \( 31 \)

\[ f_0 \approx 1.65 \times 10^{-12} \frac{1}{\epsilon} \left( \frac{T_*}{\text{MeV}} \right) \left( \frac{g_*}{100} \right)^{1/6} \text{Hz}. \] (36)

This simple equation allows to understand a number of important points concerning the energy scales that can be probed in GW experiments. The simplest estimate of \( f_* \) corresponds to taking \( \epsilon = 1 \) in eq. (33). In this case, we would find that a graviton observed today at the frequency \( f_0 = 1 \) Hz was produced when the Universe had a temperature \( T_* \sim 6 \times 10^8 \) GeV. Using the relation between time and temperature in the RD phase,

\[ t \approx \frac{2.42}{g_*^{1/2}} \left( \frac{\text{MeV}}{T} \right)^2 \text{sec}, \] (37)

we find that this corresponds to a production time \( t_* \sim 7 \times 10^{-21} \) sec, and at time of production this graviton had an energy \( E_* \sim 0.3 \) MeV. At \( f_0 = 100 \) Hz we get \( t_* \sim 7 \times 10^{-25} \) sec, \( T_* \sim 6 \times 10^8 \) GeV and \( E_* \sim 3 \) GeV. These would be, therefore, the scales relevant to Virgo and LIGO. For a frequency \( f_0 = 10^{-4} \) Hz, relevant to LISA, we would get instead \( t_* \approx 7 \times 10^{-13} \) sec, \( T_* \sim 600 \) GeV.

However, the estimate \( \lambda_* \sim H^{-1} \), or \( \epsilon \sim 1 \), can sometimes be incorrect even as an order of magnitude estimate. In the Appendix we illustrate this point with two specific examples, one in which the assumption \( \lambda_* \sim H^{-1} \) turns out to be basically correct, and one in which it can miss by several orders of magnitudes. Both examples will in general illustrate the fact that the argument does not involve only kinematics, but also the dynamics of the production mechanism.

From eq. (33) we see that the temperatures of the early Universe explored detecting today relic GWs at a frequency \( f_0 \) are smaller by a factor approximately equal (for constant \( g_* \)) to \( \epsilon \), compared to the estimate with \( \epsilon = 1 \). Equivalently,
a signal produced at a given temperature $T_\ast$ could in principle show up today in the Virgo/LIGO frequency band when a naive estimate with $\epsilon = 1$ suggests that it falls at lower frequencies.

There is however another effect, which instead gives hopes of exploring the Universe at much higher temperatures than naively expected, using GW experiments. In fact, the characteristic frequency that we have discussed is the value of the cutoff frequency in the graviton spectrum. Above this frequency, the spectrum decreases exponentially [33], and no signal can be detected. Below this frequency, however, the form of the spectrum is not fixed by general arguments. Thermal spectra have a low frequency behaviour $\Omega_{\text{gw}}(f) \sim f^3$, as we read from eq. (36) inserting a Bose-Einstein distribution for $n_f$, so that, at low $f$, $n_f \sim 1/f$. However, below the Planck scale gravitons interact too weakly to thermalize, and there is no a priori reason for a $\sim f^3$ dependence. The gravitons will retain the form of the spectrum that they had at time of production, and this is a very model dependent feature. However, from a number of explicit examples and general arguments that we will discuss below, we learn that spectra flat or almost flat over a large range of frequencies seem to be not at all unusual.

This fact has potentially important consequences. It means that, even if a spectrum of gravitons produced during the Planck era has a cutoff at frequencies much larger than the Virgo/LIGO frequency band, still we can hope to observe in the 10Hz–1kHz region the low-frequency part of these spectra. In the next subsection we will therefore discuss what signals can be expected from the Universe at extremely high (Planckian, string or GUT) temperatures.

V. TOWARD THE PLANCK ERA?

The scale of quantum gravity is given by the Planck mass, related to Newton constant by $G = 1/M_{\text{Pl}}^2$. More precisely, since in the gravitational action enters the combination $8\pi G$, we expect that the relevant scale is the reduced Planck mass $M = M_{\text{Pl}}/(8\pi)^{1/2} \approx 2.44 \times 10^{18}$ GeV. Using eq. (38) with $T_\ast = M$ and $\epsilon = 1$ gives

$$f_0 \sim 400 \left( \frac{g_*}{100} \right)^{1/6} \text{ GHz}, \quad (T_\ast = M_{\text{Pl}}/\sqrt{8\pi}).$$

The dependence on $g_\ast$ is rather weak because of the power 1/6 in eq. (38). For $g_\ast = 1000$, $f_0$ increases by a factor $\sim 1.5$ relative to $g_\ast = 100$. For $T_\ast = M_{\text{GUT}} \sim 10^{16}$ GeV, and $g_\ast \simeq 220$ (likely values for a supersymmetric unification),

$$f_0 \sim 2 \text{ GHz}, \quad (T_\ast = M_{\text{GUT}}).$$

Using instead the typical scale of string theory, $M_S$, the characteristic frequency is between these two values, since $M_S$ is expected to be approximately in the range $10^{-2} \lesssim M_S/M_{\text{Pl}} \lesssim 10^{-1}$ [34]. Actually, in the estimate of $f_0$, precise numbers depend on details of the criteria used. For instance, in the context of string cosmology [35,36] it might be more meaningful to perform the estimate setting $H_\ast \sim M_S$ rather than $T_\ast \sim M_S$, since the Hubble constant during the large curvature regime is fixed by stringy corrections, whose scale is given by $M_S$ [17]. In this case, from eq. (35), the estimate for $T_\ast$ is $T_\ast \sim (90/8\pi^3 g_\ast)^{1/4} (M_{\text{Pl}}/M_S)^{1/2}$ instead of $T_\ast \sim M_S$. Thus, the dependence of $f_0$ on $g_\ast$ is different ($\sim g_\ast^{-1/12}$ instead of the factor $g_\ast^{1/6}$ that appears in eq. (38)), and there appears also a dependence on $M_S/M_{\text{Pl}}$ [36]. However, numerically the difference is not very significant, since if say $g_\ast = 100$, then $(90/8\pi^3 g_\ast)^{1/4} \simeq 0.25$ while for $g_\ast = 1000$, $(90/8\pi^3 g_\ast)^{1/4} \simeq 0.14$, and this number is partially compensated by $(M_{\text{Pl}}/M_S)^{1/2}$, that ranges between approximately 10 and 3; thus, for $f_0$ the overall number with the estimate $T_\ast \sim M_S$ or with $H_\ast \sim M_S$ can differ, say, by a factor of two, which is anyway beyond the accuracy of these order of magnitude estimates.

The typical frequency region for GUT/string/Planck physics is therefore between the GHz and a few hundreds GHz. This is very far from the region accessible to interferometers, which arrive at most to 10 kHz, see fig. 1.

One might consider the possibility of building a detector in the GHz region to search for a signal from the Planck era. GW detectors based on microwave superconducting cavities that could operate at these frequencies have indeed been studied in the literature [38,39] in the years 1978-1980. However, from eq. (14) we see that a level $h_0^2 \Omega_{\text{gw}} \sim 10^{-6}$ corresponds, at $f = 1$ GHz, to $h_c \sim 10^{-20}$, which is many orders of magnitude below the sensitivities discussed in [38]. Still, given the great interest that a detector in this frequency range could have, it could be worthwhile to investigate them further.\footnote{These cavities could also be constructed so that they operate in the MHz region [40], where the requirements on $h_c$ are less stringent, and the wavelength is long enough to perform a correlation between two detectors. A relic background detected at these frequencies would be extremely interesting since it would be unambiguously of cosmological origin, see sect. 7B.}
It is clear from the above estimates of the characteristic frequency, that a necessary condition for observing at Virgo or LIGO a signal from the Planckian era is that we have a relic gravitational spectrum which is not peaked around the cutoff frequency, but rather is almost flat from the GHz region, where it has the cutoff, down to at least the kHz region, where interferometers can operate.

A thermal spectrum behaves as $h^2\Omega_{\text{gw}} \sim f^3$ and the value in the kHz region would be a factor $\sim 10^{-18}$ smaller than a peak value at the GHz, and therefore would be obviously unobservable. However, as already remarked, gravitons do not thermalize and there is no a priori reason for a $\sim f^3$ spectrum. On the contrary, there are good general reasons and various explicit examples favoring almost flat spectra, at least over some range of frequencies.

A simple and instructive example has been given by Krauss [1]. Consider a global phase transition in the early Universe, associated with some scalar field $\phi$ that, below a critical temperature, gets a vacuum expectation value $\langle \phi \rangle$. For causality reasons, in an expanding Universe the scalar field cannot have a correlation length larger than the horizon size. So, even if the configuration which minimizes the energy is a constant field, the field will be constant only over a horizon distance. During the RD phase the horizon expands, and the field will relax to a spatially uniform configuration within the new horizon distance. This relaxation process will in general produce gravitational waves, since there is no reason to expect $\langle T_{\mu\nu} \rangle$ for the classical field just entering the horizon to be spherically symmetric. A simple estimate of the spectrum produced makes use of the fact that the characteristic scale of spatial or temporal variation of the field is given by the inverse of the Hubble constant, $H^{-1}$, at the value of time under consideration. Then, the energy density of the field is $\rho \sim (\partial\phi)^2 \sim \langle \phi \rangle^2 H^2$. The total energy in a Hubble volume is $\rho H^{-3} \sim \langle \phi \rangle^2 H^{-1}$ and the corresponding quadrupole moment is $Q \sim (\rho H^{-3}) H^{-2} \sim \langle \phi \rangle^2 H^{-3}$, times a constant smaller than one, which measures the non-sphericity of the configuration, and which is not very relevant for order of magnitude estimates of the frequency dependence. The energy liberated in GWs in a horizon time is then given by [1]

$$\Delta E \sim H^{-1} \times \text{Luminosity} \sim H^{-1} G \left( \frac{d^2 Q}{dt^2} \right)^2 \sim G \langle \phi \rangle^4 H^{-1},$$

and the energy density of GW produced is

$$\frac{d\rho_{\text{gw}}}{d\log f} \sim \frac{\Delta E}{H^{-3}} \sim G \langle \phi \rangle^4 H^2.$$

This is the energy density at a frequency $f \sim H$, at a value of time $t = t(f)$ which is the time when the mode with frequency $f$ enters the horizon. So, $d\rho/d\log f$ in the above equation is evaluated at different values of time for different frequencies. However, in the RD phase, the energy density scales like $\rho \sim 1/a^4(t) \sim t^{-2}$ and $H \sim t^{-1}$ so that $\rho$ scales as $H^2$. Therefore, eq. (11) means that, if we evaluate $d\rho/d\log f$ at the same value of time, for all frequencies which at this value of time are inside the horizon, the spectrum is flat, $h^2\Omega_{\text{gw}}(f) \sim \text{const.}$ This is a rather nice example of how a flat spectrum can follow simply from the requirement of causality and simple dimensional estimates.

The best known example of an (almost) flat spectrum is the amplification of quantum fluctuations in the inflation-RD phase transition [12, 14]. In this case, during the inflationary phase the horizon size $H^{-1}$ is approximately constant and perturbations exit from the horizon, since the physical wavelength scales as $\sim a(t)$. The spectrum of quantum fluctuations for gravitational waves during inflation is (see e.g. [3], pg.289) $\Delta h_{+x}(f) \sim H/M_{\text{Pl}}$, where $H$ is the Hubble constant at time of crossing. Different wavelengths cross at different time, but since $H$ does not vary during inflation, this is really independent of $f$. When the mode is outside the horizon, its amplitude stays constant, again for causality reasons. During the RD phase the horizon size grows faster than $a(t)$ and so finally the perturbation reenters the horizon, and its energy density appears in the form of gravitons. Its energy density as it crosses back is $d\rho/d\log f \sim h^2 \sim f^2 (\Delta h)^2$. As in the previous example, this is the value of $d\rho/d\log f$ at frequency $f$, at the value of time when this wavelength crosses back inside the horizon, so it is evaluated at different times for different frequencies, and a dependence $\sim f^2$ means that the spectrum, when $d\rho/d\log f$ is evaluated at the same time for all frequencies, is flat. So, again in this example a basic ingredient for a flat spectrum was causality, together with the fact that $H$ is constant during inflation.

The same mechanism of amplification of vacuum fluctuations produces an interesting stochastic background in string cosmology. In recent years Gasperini and Veneziano have proposed an inflationary model based on string theory [35]. In this model the Universe starts at low curvatures; at this stage it is described by the lowest order effective action of string theory. Then it evolves toward a regime of large curvature, where higher order corrections to the string effective action stabilize the growth of the curvature [37, 41], which is now of order one in string units, and the evolution should then be matched to the standard RD phase [48]. The big-bang, i.e. a state of the Universe with Planckian energies and curvature, rather than being a postulated initial condition, is the end result of an evolution (the ‘pre-big-bang’ phase) which started from a cold Universe at low curvature.
The form of the spectrum of relic GWs has been discussed in various recent papers [43, 54, 58] and depends on the rate of growth of the curvature and of the dilaton field in the various phases of the model. Very low frequencies are sensitive to the behavior in the low curvature pre-big-bang regime. In this regime both the curvature and the dilaton are growing and the spectrum turns out to grow as $h_0^2 \Omega_{gw} \sim f^3 \log^2 f$. In the large curvature regime, according to a possible scenario [17], the Hubble parameter is constant, while the dilaton can grow, with a derivative parametrized by an unknown constant. In the limit in which the derivative of the dilaton in this phase goes to zero, the spectrum in the region $f_s < f < f_{\text{peak}}$ is approximately flat. The value of $f_s$ is determined by the duration of this intermediate large curvature phase ($f_{\text{peak}}/f_s$ is in fact the total red-shift during this phase), and it is not calculable without a detailed knowledge of string phenomena. With a sufficiently long string phase, it is possible to have $f_s$ smaller than, say, 1kHz. The cutoff value $f_{\text{peak}}$ is instead in the region of a few hundred GHz, as already discussed. Therefore, if in the large curvature regime we have a long inflationary phase with an almost constant dilaton, we have again a spectrum which is practically flat over a very large range of frequencies.

Another example of relic GW spectrum which is almost flat over a huge range of frequencies is provided by GWs produced by the decay of cosmic strings [52]. In this case the spectrum has a peak around $f \sim 10^{-12}$ Hz, where $h_0^2 \Omega_{gw}$ can reach a few times $10^{-6}$, and then it is almost flat over a huge range of frequencies, from $f \sim 10^{-8}$ Hz to the GHz region, where it has the cutoff (fixed by the arguments previously discussed, considering that the relevant scale for cosmic string is $M_{\text{GUT}}$). The basic reason for such a behavior is the scaling property of the string network, which says that a single lengthscale, the Hubble length, characterizes all properties of the string network [53, 54]. The network of strings evolves toward a self-similar configuration, with small loops being chopped off very long strings, and the typical radiation emitted by a single loop has a wavelength related to the length of the loop.

In conclusion, from these examples we learn that GW spectra almost flat over a large range of frequencies are not at all unusual in cosmology, and often the approximate flatness of the spectrum is a consequence of rather general principles, like causality, or the existence of a single lengthscale in the problem. Therefore, there is some hope that we need not go in the GHz region to explore Planck scale physics.

VI. THE EFFECTS OF INFLATION

In spite of the fact that the standard cosmological model based on a RD and a MD phase is not contradicted by any experimental data, inflation (i.e., a stage of exponential expansion, or more generally a period characterized by the condition $\dot{a}(t) > 0$ on the FRW scale factor $a(t)$) has many compelling features, see e.g. refs. [44, 56, 57] for reviews. At present, there is no ‘standard model’ of inflation. The mechanism is very general, and can be implemented in different models. In the models which have been ‘traditionally’ studied, inflation is driven by the potential energy of a scalar field.

Independently of the details, these models predict an almost flat gravitational wave spectrum extending from some high-frequency cutoff down to extremely small frequencies, $f \sim 10^{-16}$ Hz, and rising as $1/f^2$ as we go at even lower frequencies, such as $f \sim 10^{-18}$ Hz, that corresponds to wavelengths of the order of the present Hubble distance $H^{-1}(t_0)$. The value of $h_0^2 \Omega_{gw}(f)$ for this spectrum depends on the Hubble constant during inflation, $H_{DS}$, which in turn is related to the energy scale $M_{\text{inf}}$ at which inflation takes place, $H_{DS} \sim M_{\text{inf}}^2/M_{Pl}$. A strong signal in GWs at wavelengths $\lambda \sim H^{-1}(t_0)$ would produce a stochastic red-shift on the frequencies of the photons of the 2.7K radiation, and a corresponding fluctuation $\delta T/T$ in their temperature. The measurements of $\delta T/T$ from COBE therefore put an upper bound on the intensity of the relic GWs produced at these long wavelengths, that translates into an upper bound on the Hubble constant during inflation and on the scale $M_{\text{inf}}$ where inflation occurs [43, 58]. Using COBE four-year data (and taking properly into account the relation between the amplitude at low frequency and the tilt of the spectrum, which is not exactly flat) gives an upper bound [58, 57]

$$M_{\text{inf}} < 3.4 \times 10^{16}\text{GeV}. \quad (42)$$

A crucial consequence of this bound would be that we cannot see today a cosmological signal from the Planck scale. Every particle production which took place at Planckian times has been exponentially diluted by the subsequent

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§§Note that, while in standard inflationary models a long inflationary phase at large curvature would provide too much GWs at the frequencies relevant for the COBE bound (see below), this is not the case for string cosmology because of the behaviour $\sim f^3 \log^2 f$ that sets in at low $f$. 

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inflationary stage. In this case the maximum energy that can be explored by cosmological observations is the reheating temperature after inflation, which is equal to $M_{\text{inf}}$ in the case of a perfectly efficient reheating, and is in general lower.

However, the above conclusion is not unescapable. We have already considered above the inflationary model based on string theory proposed by Gasperini and Veneziano [35]. In this model we have automatically an inflationary evolution during the pre-big-bang phase, triggered by the kinetic energy of the dilaton field. The bound (12) in this case does not apply because the spectrum of relic gravitational waves is not flat over the whole frequency range, from COBE frequencies $f \sim 10^{-16}$ up to the cutoff frequency, as in standard inflation, but rather goes like $\sim f^3 \log^2 f$ at sufficiently small $f$, thereby automatically avoiding the COBE bound by many orders of magnitude [49].

Refs. [35,59] discuss whether this inflationary phase can solve the kinematical problems of standard cosmology without fine-tuned initial conditions. In this paper we do not want to enter a discussion on the positive and negative aspects of any specific model. We rather take the model of refs. [35] as an indication that the bound (12) is not necessarily unescapable, and we discuss in subsections A and B two different scenarios which depend on whether this bound is obeyed or not. The amplification of vacuum fluctuations is common to both scenarios and is discussed in subsection C, where we also consider the possibility of combining the pre-big-bang scenario with potential-driven inflation.

### A. Cosmological signals from the reheating era

If we do have an inflationary phase at a scale $M_{\text{inf}}$ that satisfies the bound (12), there is no gravitational analog of the 2.7K radiation. At the Planck scale photons and gravitons can be produced simply by thermal collisions, with probably similar production rates. However, the particles produced in this way at the Planck (or string) scale are exponentially diluted by the subsequent inflationary phase, and the photons that we see today in the cosmic microwave background radiation (CMBR) have been produced during the reheating era which terminated the inflationary phase. The highest energy scale that can be probed by cosmological observations in this case is given by the reheating temperature (the amplification of vacuum fluctuations can provide an exception to this statement, see subsect. C).

Since the reheating temperature $T_{\text{rh}} < M_{\text{inf}} \ll M_{\text{Pl}}$, thermal collisions are by now unable to produce a substantial amount of gravitons. However, in this case relic GWs can be produced through some non-equilibrium phenomenon connected with the reheating process. For a process that takes place at the reheating temperature $T_{\text{rh}}$ the estimate of the characteristic frequency is given by eq. (29) with $T_* = T_{\text{rh}}$. In principle the reheating temperature can be between a minimum value at the TeV scale (since this is the last chance for baryogenesis, via the anomaly in the electroweak theory) and a maximum value $T_{\text{rh}} \sim M_{\text{inf}} \leq 3 \times 10^{16}$ GeV. In supersymmetric theories, the gravitino problem give a further constraint $T_{\text{rh}} \lesssim 10^9$ GeV [50]. More precisely, ref. [50] gives a bound on $T_{\text{rh}} \sim 10^6 - 10^9$ GeV for a gravitino mass 100 GeV–1 TeV, and a bound $10^{11} - 10^{12}$ GeV independent of the gravitino mass.

So, phenomena occurring at reheating will manifest themselves with cutoff frequencies between $10^{-4}$ and $10^9$ Hz, if we set $\epsilon = 1$ in eq. (39). In typical non-supersymmetric models the reheating temperature is large, say $10^{14}$ GeV, corresponding to $f_0 \sim (1/\epsilon)10^7$ Hz, and in this case interferometers or resonant masses could only look for their low-frequency tails. If instead the reheating temperature is $T_{\text{rh}} \sim 10^9$ GeV, non-equilibrium phenomena taking place during reheating would give a signal just in the LIGO/Virgo frequency band.

A promising mechanism for GW generation during reheating is bubbles collision, in the case when inflation terminates with a first order phase transition [51]. In this case the relevant estimate is $f_0 \sim (1/\epsilon)10^7$ Hz. One should however keep in mind the possibility that bubble nucleation occurs before the end of inflation, and then the cutoff frequency would be red-shifted by the subsequent inflationary evolution toward lower values, possibly within the Virgo/LIGO frequency range (12).

The other possibility is that reheating occurs through the decay of the inflaton field. In this case, in a very general class of models, there is first an explosive stage called preheating, where the inflaton field decays through a non-perturbative process known as parametric resonance [13], and at this stage there are mechanisms that can produce GWs, see [14,53]. We believe that in these cases a reliable estimate of the characteristic frequency is really very difficult to obtain, since the relevant parameter, which is the value of the Hubble constant at time of production, depends on the complicated dynamics of preheating and on the specific inflationary model considered. Ref. [54] examines models where the characteristic frequency turns out to be in the region between a few tens and a few hundreds kHz.
B. Cosmological signals from the Planck era

The second possibility is that there is no inflationary phase after the Planck era. This could happen if there is no inflationary phase at all (although in this way we would lose the advantages of inflation, so that this possibility does not seem theoretically very appealing), or if inflation is realized in the ‘pre-big-bang’ phase of string cosmology, as suggested in [13]. Independently of other considerations, the pre-big-bang model is interesting in this context because it is an explicit example of the fact that physics at the Planck or string scale can invalidate the considerations leading to the bound (42), without necessarily giving up the idea of inflation.

If there is no inflationary phase after the Planck era, the photons of the CMBR have been produced at the Planck (or string) scale, when gravitons are also produced with probably similar rates, and therefore we expect a GW analog of the 2.7 K radiation. At Planckian times both gravitons and photon will have approximately the same characteristic energy. Then, gravitons decouple and their characteristic energy scales like $1/a(t)$, while photons are in thermal equilibrium and their temperature evolves so that the entropy $\sim g_* S(T) T^3 a^3(t)$ is constant. Therefore the characteristic energy of the gravitons today, $T_G$, is related to the present temperature of the photon, $T_0 = 2.728 \pm 0.002$, by [2]

$$T_G = \left( \frac{3.91}{g_*} \right)^{1/3} T_0 \approx 0.93 \left( \frac{100}{g_*} \right)^{1/3} \text{K},$$

where we used $g_*(T_0) \approx 3.91$ [2] and $g_*$ is the effective number of degrees of freedom at time of production. As we already discussed, in the case of the gravitons $T_G$ should not be considered a temperature, since there is no a priori reason why gravitons should have a thermal spectrum, but rather is a characteristic energy, close to the high-energy cutoff in the spectrum. The corresponding frequency is $f_0 \approx 120 (100/g_*)^{1/3}$ GHz; this is the cutoff of a spectrum which extends toward lower frequencies. If its behaviour is, say, $h_0^3 \Omega_{gw} \sim f^3$ there is no hope of observing it in the kHz region, while if it is flatter it can be more interesting for present experiments.

Actually, since in this case we are looking for a signal coming from the quantum gravity regime, the situation can be richer. For instance, the correct theory of quantum gravity is provided by string theory, it may happen that massive string modes are highly excited in the large curvature regime, as indeed happens in the scenario discussed in ref. [15]. In this case the decay of excited string modes can be an interesting mechanism for production of GWs. In fact, for large level number $N$ the decay width of the string is dominated by transitions of the form $N\rightarrow N-1$ with emission of a graviton or another massless particle [63]. The mass levels of the string depend on $N$ as $M_N^2 \sim N$ so that, in string units, $\Delta M_N \sim 1/\sqrt{N}$, and therefore these decays would produce a set of peaks at today frequencies $f_N = f_0/\sqrt{N}$. For large $N$ these frequencies are close to each other and it is possible that the width will be large, so that we will not have lines but rather a continuous spectrum, but in any case this mechanism shifts some of the energy from the GHz region toward a more accessible frequency range.

Note also that the value $f_0 \approx 120(100/g_*)^{1/3}$ GHz is really only an upper bound, since the production through decay of excited string levels would take place before the onset of the RD phase, and a further red-shift should be taken into account. This cannot be computed in the absence of a detailed knowledge of the cosmological evolution in the large curvature regime, but in general it might be possible that the effect of the redshift and/or the factor $1/\sqrt{N}$ shift this radiation in an interesting frequency band [47].

C. Amplification of vacuum fluctuations

A very general and well studied mechanism for the production of relic GWs is the amplification of vacuum fluctuations due to a non-adiabatic transition between different regimes for the evolution of the scale factor. The mechanism is common to the scenarios discussed in the two previous subsections.

A well known example is the transition between a (potential-driven) inflationary phase and the standard RD phase [12, 46]. At first sight, it is not promising for detection, since once the bound (42) is obeyed, at LIGO/Virgo frequencies one gets $h_0^3 \Omega_{gw} \sim 10^{-15}$ [3], very far from the sensitivity of first and even of the planned second generation experiments. This is due to the fact that in this model $h_0^3 \Omega_{gw}$ is almost flat at all frequencies, from COBE frequencies up to the cutoff of the spectrum, and therefore the bound imposed by COBE at very low frequencies gives a bound at all higher frequencies. One should however keep in mind that these computations are performed assuming that before the transition the Universe is in the DeSitter invariant vacuum. If instead before the transition we have $n_k^0$ gravitons per cell of the phase space, the total number of gravitons after the transition is
where $\beta_k$ is the relevant Bogoliubov coefficient (we are considering the case $\beta_k k^4 = \beta_k \delta_{kk'}$, as is usually the case in cosmological backgrounds). If $n_k^0 = 0$ we have amplification of the ‘half quantum’ of vacuum fluctuations, with an amplification factor $2|\beta_k|^2$. But also any preexisting non-vanishing $n_k^0$ is amplified. One can expect that before the transition $n_k^0 \ll 1$ because a phase space density of order one would correspond to a large energy density, since at early time these modes have very high frequency [44]. However, the argument need not be true if there has been some mechanism which pumped the energy into these modes, as may happen for instance with a previous amplification of vacuum fluctuations at the string scale, as in [19]. For a transition between a DeSitter phase ($H=\text{const.}$) and RD, the Bogoliubov coefficients depend on $k$ as $\beta_k \sim 1/k^2$ [12–14]; if before the transition $n_k^0 = 0$, then after the transition $n_k = |\beta_k|^2 \sim 1/k^4 \sim 1/f^4$ and eq. (21) shows that the spectrum for $h_0^2 \Omega_{\text{gw}}(f)$ is flat (actually, during slow-roll inflation, the Hubble parameter is not exactly constant, and this results in a small tilt in the spectrum, see e.g. [28]). If instead for some $k$ we have $n_k^0 \neq 0$ and $|\beta_k| > 1$, the dependence of $h_0^2 \Omega_{\text{gw}}$ on $f$ is of the form $h_0^2 \Omega_{\text{gw}}(f) \sim n_k^0 + (1/2)$, since the factor $f^4$ in eq. (2) is cancelled by $|\beta_k|^2 \sim 1/f^4$. We see that the spectrum of vacuum fluctuations after an inflationary phase is determined by the occupation numbers before the onset of inflation.

This is an example (actually, the only example of which we are aware) of the fact that an inflationary phase does not erase every information about the previous stage of the Universe. A measurement of a relic GW spectrum, and hence of $n_k$, if the Universe underwent a stage of inflation, would allow us to reconstruct the state of the Universe (i.e. $n_k^0$) before inflation. The technical reason behind this is that the quantity which is amplified is a number of particle per unit cell of the phase space, and the volume of a cell of the phase space, $d^3 x d^3 k/(2\pi)^3$, is unaffected by the expansion of the Universe, contrarily to a physical spatial volume $d^3 x \sim a^3(t)$.

These considerations can be applied to the case in which the pre-big-bang model of Gasperini and Veneziano is followed, rather than by the standard RD cosmology, by a phase of inflation, driven by the potential energy of some inflaton field. Indeed, it was stressed already in the first papers on pre-big-bang cosmology [22] that the pre-big-bang scenario is by no means to be regarded as an alternative to standard (even inflationary) cosmology, but rather it is a completion of the standard picture beyond the Planck era. While most efforts to date have been concentrated in matching the pre-big-bang phase to a RD phase, one can certainly consider the matching to, say, a stage of chaotic inflation. This could have some attractive features, since the inflationary stage that takes place during the prebig-bang phase, independently of whether it is long enough to solve the horizon/flatness problems [41], can anyway prepare sufficiently homogeneous patches that are need to start chaotic inflation. The spectrum of relic GWs in a model with a pre-big-bang phase and a successive stage of potential-driven inflation would still retain at low $f$ the form $h_0^2 \Omega_{\text{gw}} \sim f^3 \log^2 f$ obtained in string cosmology, and therefore the comparison of the relic GW spectrum with COBE anisotropies would give no bound on the scale $M_{\text{inf}}$ of the potential-driven inflation.

Another interesting example, considered in [59], is the case in which, before the onset of an inflationary phase, the Universe was already in a state of thermal equilibrium, so that $n_k^0$ is given by a Bose-Einstein distribution. This is a natural initial condition when the inflationary phase is triggered by a first order phase transition, since this requires a homogeneous thermal state as initial condition. This example works in the opposite direction compared to the case of string cosmology followed by potential inflation. In fact, now at small $k$ we have $n_k = (\exp(E_k/T) - 1)^{-1} \sim 1/k$, and the spectrum is enhanced at low frequency, and totally negligible at Virgo/LIGO frequencies. This also results in a stronger bound on the inflation scale compared to eq. (12). [19]

A different possibility is that the equation of state in the early Universe is modified, and this can result in growing spectra so that again we can obey the COBE bound at $f \sim 10^{-16}$ Hz, but still we could have a large signal at higher frequencies. Ref. [71] considers the case of a phase of the early Universe with scale factor expanding as $a(\eta) \sim |\eta|^{1+\beta}$, where $\eta$ is conformal time. This corresponds to an equation of state $p = \gamma \rho$ with $\gamma = (1-\beta)/(3+3\beta)$. Inflation corresponds to $\gamma = -1$ and then to $\beta = -2$. At frequencies $f > 10^{-18}$ Hz the amplification of vacuum fluctuations at the transition between this phase and the standard RD phase gives a spectrum with $h_0(f) \sim f^{\beta+1}$ and therefore, from eq. (3), $h_0^2 \Omega_{\text{gw}} \sim f^{2\beta+4}$, which of course reproduces a flat spectrum in the inflationary case, $\beta = -2$. The parameter $\beta$ is related to the spectral index of density perturbations $n$ by $n = 2\beta + 5$, so that $\beta = -2$ corresponds to $n = 1$, the Harrison-Zeldovich spectrum. The value of $n$ can in principle be derived from COBE observations. Two different groups, analyzing the same data set in different ways, have produced the results $n = 1.2 \pm 0.3$ [71] (therefore consistent with a Harrison-Zeldovich spectrum) and $n = 1.84 \pm 0.29$ [22]. So, systematic errors presumably dominate the analysis, but there is the possibility that $n > 1$ and therefore $\beta > -2$, which would give a growing GW spectrum $h_0^2 \Omega_{\text{gw}}(f) \sim f^{n-1}$. However, if $n$ is too large (like the value $n = 1.84$) one should also find a way to cutoff the spectrum before it grows too large and exceeds existing bounds like the nucleosynthesis bound, see sect. 7.

If instead there is no phase of potential-driven inflation, while there is an inflationary pre-big-bang phase as suggested...
in [43], than one finds a relic GW spectrum that grows like $f^3 \log^2 f$ at small $f$, thereby avoiding the COBE bound, and which has the cutoff in the GHz region, as we already discussed. Whether the spectrum is approximately flat from the GHz down to the kHz region relevant for Virgo/LIGO depends on some unknown parameters of the large curvature phase, in particular the rate of growth of the dilaton and the duration of this phase. For some values of these parameters (almost constant dilaton and long large curvature phase) the spectrum is almost flat in a large range of frequencies and it might be relevant at the frequencies of present experiments [44, 54, 38].

VII. CHARACTERISTIC INTENSITIES

A. The nucleosynthesis bound

Nucleosynthesis successfully predicts the primordial abundances of deuterium, $^3$He, $^4$He and $^7$Li in terms of one cosmological parameter $\eta$, the baryon to photon ratio. In the prediction enter also parameters of the underlying particle theory, which are therefore constrained in order not to spoil the agreement. In particular, the prediction is sensitive to the effective number of species at time of nucleosynthesis, $g_\ast = g(T \simeq \text{MeV})$. With some simplifications, the dependence on $g_\ast$ can be understood as follows. A crucial parameter in the computations of nucleosynthesis is the ratio of the number density of neutrons, $n_n$, to the number density of protons, $n_p$. As long as thermal equilibrium is maintained we have (for non-relativistic nucleons, as appropriate at $T \sim \text{MeV}$, when nucleosynthesis takes place) $n_n/n_p = \exp(-Q/T)$ where $Q = m_n - m_p \simeq 1.3 \text{ MeV}$. Equilibrium is maintained by the process $pe \leftrightarrow n\nu$, with width $\Gamma_{pe\rightarrow n\nu}$, as long as $\Gamma_{pe\rightarrow n\nu} > H$. When the rate drops below the Hubble constant $H$, the process cannot compete anymore with the expansion of the Universe and, apart from occasional weak processes, dominated by the decay of free neutrons, the ratio $n_n/n_p$ remains frozen at the value $\exp(-Q/T_i)$, where $T_i$ is the value of the temperature at time of freeze-out. This number therefore determines the density of neutrons available for nucleosynthesis, and since practically all neutrons available will eventually form $^4$He, the final primordial abundance of $^4$He is very sensitive to the freeze-out temperature $T_i$. Let us take for simplicity $\Gamma_{pe\rightarrow n\nu} \simeq G_F^2 T_i^5$ (which is really appropriate only in the limit $T \gg Q$). The Hubble constant is given by $H^2 = (8\pi/3)G\rho$, where $\rho$ include all form of energy density at time of nucleosynthesis, and therefore also the contribution of primordial GWs. As usual, it is convenient to write the total energy density $\rho$ in terms of $g_\ast$, see eqs. (43), as $\rho = (\pi^2/30)g_\ast T_i^4$. We recall that, for gravitons, the quantity $T_i$ entering eq. (44) is defined by $\rho_{gw} = 2(\pi^2/30)T_i^4$, and this does not imply a thermal spectrum. Then $T_i$ is determined by the condition

$$G_F^2 T_i^5 \simeq \left(\frac{8\pi^3 g_\ast}{90}\right)^{1/2} \frac{T_i^2}{M_{Pl}}.$$  

(45)

This shows that $T_i \sim g_\ast^{1/6}$, at least with the approximation that we used for $\Gamma_{pe\rightarrow n\nu}$. A large energy density in relic gravitons gives a large contribution to the total density $\rho$ and therefore to $g_\ast$. This results in a larger freeze-out temperature, more available neutrons and then in overproduction of $^4$He. This is the idea behind the nucleosynthesis bound [43]. More precisely, since the density of $^4$He increases also with the baryon to photon ratio $\eta$, we could compensate an increase in $g_\ast$ with a decrease in $\eta$, and therefore we also need a lower limit on $\eta$, which is provided by the comparison with the abundance of deuterium and $^3$He.

Rather than $g_\ast$, it is often used an ‘effective number of neutrino species’ $N_\nu$ defined as follows. In the standard model, at $T \sim \text{a few MeV}$, the active degrees of freedom are the photon, $e^\pm$, neutrinos and antineutrinos, and they have the same temperature, $T_i = T$. Then, for $N_\nu$ families of light neutrinos, $g_\ast(N_\nu) = 2 + (7/8)(4 + 2N_\nu)$, where the factor of 2 comes from the two elicity states of the photon, 4 from $e^\pm$ in the two elicity states, and 2$N_\nu$ counts the $N_\nu$ neutrinos and the $N_\nu$ antineutrinos, each with their single elicity state. For the Standard Model, $N_\nu = 3$ and therefore $g_\ast = 43/4$. So we can define an ‘effective number of neutrino species’ $N_\nu$ from

$$g_\ast(N_\nu) = \frac{43}{4} + \sum_{i=\text{extra bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i=\text{extra fermions}} g_i \left(\frac{T_i}{T}\right)^4.$$  

(46)

One extra species of light neutrino, at the same temperature as the photons, would contribute one unit to $N_\nu$, but all species, weighted with their energy density, contribute to $N_\nu$, which of course in general is not an integer. For $i = \text{gravitons}$, we have $g_i = 2$ and $(T_i/T)^4 = \rho_{gw}/\rho_\gamma$, where $\rho_\gamma = 2(\pi^2/30)T^4$ is the photon energy density. If gravitational waves give the only extra contribution to $N_\nu$, compared to the standard model with $N_\nu = 3$, using $g_\ast(N_\nu) = 2 + (7/8)(4 + 2N_\nu)$, the above equation gives immediately
\[
\left(\frac{\rho_{\text{gw}}}{\rho_{\gamma}}\right)_{\text{NS}} = \frac{7}{8}(N_{\nu} - 3),
\]

where the subscript NS reminds that this equality holds at time of nucleosynthesis. If more extra species, not included in the standard model, contribute to \(g_s(N_{\nu})\), then the equal sign in the above equation is replaced by lower or equal. The same happens if there is a contribution from any other form of energy present at the time of nucleosynthesis and not included in the energy density of radiation, like, e.g., primordial black holes.

To obtain a bound on the energy density at the present time, we note that from the time of nucleosynthesis to the present time \(\rho_{\text{gw}}\) scaled as \(1/a^4\), while the photon temperature evolved following \(g_s(T)T^4a^3 = \text{const.}\) (i.e. constant entropy) and \(\rho_{\gamma} \sim T^4 \sim 1/(a^4g_s^{4/3})\). Therefore

\[
\left(\frac{\rho_{\text{gw}}}{\rho_{\gamma}}\right)_{0} = \left(\frac{\rho_{\text{gw}}}{\rho_{\gamma}}\right)_{\text{NS}} \frac{g_s(T_0)}{g_s(1\text{ MeV})} \frac{4/3}{4/3} = \left(\frac{\rho_{\text{gw}}}{\rho_{\gamma}}\right)_{\text{NS}} \frac{3.913}{10.75},
\]

where the subscript zero denotes present time. Therefore we get the nucleosynthesis bound at the present time,

\[
\left(\frac{\rho_{\text{gw}}}{\rho_{\gamma}}\right)_{0} \leq 0.227(N_{\nu} - 3).
\]

Of course this bound holds only for GWs that were already produced at time of nucleosynthesis \((T \sim \text{MeV}, t \sim \text{sec})\). It does not apply to any stochastic background produced later, like backgrounds of astrophysical origin. Note that this is a bound on the total energy density in gravitational waves, integrated over all frequencies. Writing \(\rho_{\text{gw}} = f d\log f d\rho_{\text{gw}}/d\log f\), multiplying both \(\rho_{\text{gw}}\) and \(\rho_{\gamma}\) in eq. (49) by \(h_0^2/\rho_c\) and inserting the numerical value \(h_0^2/\rho_c \approx 2.481 \times 10^{-5}\) \(\frac{\text{yr}}{\text{sec}}\), we get

\[
\int_{f=0}^{f=\infty} d\log f \ h_0^2 \Omega_{\text{gw}}(f) \leq 5.6 \times 10^{-6}(N_{\nu} - 3).
\]

The bound on \(N_{\nu}\) from nucleosynthesis is subject to various systematic errors in the analysis, which have to do mainly with the issues of how much of the observed \(^4\text{He}\) abundance is of primordial origin, and of the nuclear processing of \(^3\text{He}\) in stars, and as a consequence over the last five years have been quoted limits on \(N_{\nu}\) ranging from 3.04 to around 5. The situation has been recently reviewed in ref. [74]. The conclusions of ref. [74] is that, until current astrophysical uncertainties are clarified, \(N_{\nu} < 4\) is a conservative limit. Using extreme assumptions, a meaningful limit \(N_{\nu} < 5\) still exists, showing the robustness of the argument. Correspondingly, the right-hand side of eq. (50) is, conservatively, of order \(5 \times 10^{-6}\) and anyway cannot exceed \(10^{-5}\).

If the integral cannot exceed these values, also its positive definite integrand \(h_0^2 \Omega_{\text{gw}}(f)\) cannot exceed it over an appreciable interval of frequencies, \(\Delta\log f \sim 1\). One might still have, in principle, a very narrow peak in \(h_0^2 \Omega_{\text{gw}}(f)\) at some frequency \(f\), with a peak value larger than say \(10^{-5}\), while still its contribution to the integral could be small enough. But, apart from the fact that such a behaviour seems rather implausible, or at least is not suggested by any cosmological mechanism, it would also be probably of little help in the detection at broadband detectors like Virgo, because even if we gain in the height of the signal we lose because of the reduction of the useful frequency band \(\Delta f\), see eq. (5).

These numbers therefore give a first idea of what can be considered an interesting detection level for \(h_0^2 \Omega_{\text{gw}}(f)\), which should be at least a few times \(10^{-6}\), especially considering that the bound \(\frac{16}{27}\) refers not only to gravitational waves, but to all possible sources of energy which have not been included, like particles beyond the standard model, primordial black holes, etc.

**B. The astrophysical background**

GWs of astrophysical origin are not subject to the nucleosynthesis bound, and therefore our first concern is whether a stochastic background due to a large number of unresolved astrophysical sources can give a contribution to \(h_0^2 \Omega_{\text{gw}}(f)\)

***To compare with eq. (56) of ref. [11], note that while we use \(\rho_{\gamma}\), in ref. [1] the bound is written in terms of the total energy density in radiation at time of nucleosynthesis, which includes also the contribution of \(e^\pm\), neutrinos and antineutrinos, \((\rho_{\text{rad}})_{\text{NS}} = [1 + (7/8)(2 + 3)](\rho_{\gamma})_{\text{NS}}\).***
larger than the bound (50), or anyway larger than the expected relic signal, thereby masking the background of cosmological origin.

A first observation is that there is a maximum frequency at which astrophysical sources can radiate. This comes from the fact that a source of mass $M$, even if very compact, will be at least as large as its gravitational radius $2GM$, the bound being saturated by black holes. Even if its surface were rotating at the speed of light, its rotation period would be at least $4\pi GM$, and the source cannot emit waves with a period much shorter than that. Therefore we have a maximum frequency \( f \leq \frac{1}{4\pi GM} \sim 10^4 \frac{M_{\odot}}{M} \text{ Hz}. \) (51)

To emit near this maximum frequency an object must presumably have a mass of the order of the Chandrasekhar limit \( \sim 1.2M_{\odot} \), which gives a maximum frequency of order 10 kHz \( [72] \), and this limit can be saturated only by very compact objects (see ref. [70] for a recent review of GWs emitted in the gravitational collapse to black holes, with typical frequencies \( f \lesssim 5 \text{ kHz} \)). The same numbers, apart from factors of order one, can be obtained using the fact that for a self-gravitating Newtonian system with density $\rho$, radius $R$ and mass $M = \rho(4/3)\pi R^3$, there is a natural dynamical frequency \( \frac{\pi}{2} \frac{GM}{R^3} \).

\[ f_{\text{dyn}} = \frac{1}{2\pi} (\pi G \rho)^{1/2} = \left( \frac{3GM}{16\pi^2 R^3} \right)^{1/2}. \] (52)

With $R \geq 2GM$ we recover the same order of magnitude estimate apart from a factor $(3/8)^{1/2} \approx 0.6$. This is already an encouraging result, because it shows that the natural frequency domains of cosmological and astrophysical sources can be very different. We have seen that the natural frequency scale for Planckian physics is the GHz, while no astrophysical objects can emit above, say, 6-10 kHz. A stochastic background detected above these frequencies would be unambiguously of cosmological origin.

However, ground based interferometers have their maximum sensitivity around 100 Hz, where astrophysical sources hopefully produce interesting radiation (since these sources were the original motivation for the construction of interferometers). The radiation from a single source is not a problem, since it is easily distinguished from a stochastic background. The problem arises if there are many unresolved sources. (Of course this is a problem from the point of view of the cosmological background, but the observation of the astrophysical background would be very interesting in itself; techniques for the detection of this background with a single interferometer using the fact that it is not isotropic and exploiting the sidereal modulation of the signal have been discussed in ref. [74]).

The stochastic background from rotating neutron stars has been discussed in [73] and references therein. The main uncertainty comes from the estimate of the typical ellipticity $\epsilon$ of the neutron star, which measures its deviation from sphericity. An upper bound on $\epsilon$ can be obtained assuming that the observed slowing down of the period of known pulsars is interely due to the emission of gravitational radiation. This is almost certainly a gross overestimate, since most of the spin down is probably due to electromagnetic losses, at least for Crab-like pulsars. With realistic estimates for $\epsilon$, ref. [73] gives, at $f = 100$ Hz, a value of $h_\epsilon(f) \sim 5 \times 10^{-28}$, that, using eq. (19), corresponds to $h_0^2 \Omega_{\text{gw}}(100\text{Hz}) \sim 10^{-15}$. This is very far from the sensitivity of even the advanced experiments. An absolute upper bound can be obtained assuming that the spin down is due only to gravitational losses, and this gives $h_0^2 \Omega_{\text{gw}}(100\text{Hz}) \sim 10^{-7}$, but again this value is probably a gross overestimate.

The stochastic background from supernovae is studied in ref. [80]. The expected frequencies in this case are of the order of the kHz or lower (down to 500-600 Hz, depending on the redshift when these objects are produced). Using the observational data on the star formation rate, it turns out that the duty cycle, i.e. the ratio between the duration of a typical burst and the typical time interval between successive bursts is low, of order 0.01. Therefore this background is not stochastic, but rather like a ‘pop noise’, and can be distinguished from a really stochastic background. The value of $h_0^2 \Omega_{\text{gw}}$ for the background from supernovae have been computed in [80] assuming axially symmetric collapse, and assuming that all sources have the same value of $a = J/(GM^2)$, where $J$ is the angular momentum. The results depend on the value of $a$, and on $h_0$. For typical choices, one gets values of order $h_0^2 \Omega_{\text{gw}} \sim 10^{-14} - 10^{-11}$ at say $f = 65$ Hz, rising up to $h_0^2 \Omega_{\text{gw}} \sim 10^{-9} - 10^{-8}$ around 3kHz, where the spectrum has the maximum [80].

These results suggest that astrophysical backgrounds might not be a problem for the detection of a relic background at LIGO/Virgo frequencies. The situation is different in the LISA frequency band [73,75,77,10,19]. LISA can reach a sensitivity of order $h_0^2 \Omega_{\text{gw}} \sim$ a few $10^{-13}$ at $f \sim 10^{-3}$ Hz (see fig.1.3 of ref. [19]). However, for frequencies below a few mHz, one expects a stochastic background due to a large number of galactic white dwarf binaries. The estimate of this background depends on the rate of white dwarf mergers, which is uncertain. With rates of order $4 \times 10^{-3}$ per year (which should be a secure upper limit [72]), the background can be as large as $h_0^2 \Omega_{\text{gw}} \sim 10^{-8}$ at $f = 10^{-3}$ Hz.
This number is actually quite uncertain, and in ref. [19], fig. 1.3, it is used another plausible rate, which gives for instance \( h_0^2 \Omega_{\text{gw}} \sim 10^{-11} - 10^{-10} \) at \( f = 10^{-3} \) Hz. Above a frequency of order a few times \( 10^{-2} \) Hz, most signals from galactic binaries can be resolved individually and no continuous background of galactic origin is presently known at the level of sensitivity of LISA.

It should be observed that, even if an astrophysical background is present, and masks a relic background, not all hopes are lost. If we understand well enough the astrophysical background, we can subtract it, and the relic background would still be observable if it is much larger than the uncertainty that we have on the astrophysical background. In fact, LISA should be able to subtract the background due to white dwarf binaries, since there is a large number of binaries close enough to be individually resolvable [19]. This should allow to predict with some accuracy the space density of white dwarf binaries in other parts of the Galaxy, and therefore to compute the stochastic background that they produce. Furthermore, any background of galactic origin is likely to be concentrated near the galactic plane, and this is another handle for its identification and subtraction. The situation is more uncertain for the contribution of extragalactic binaries, which again can be relevant at LISA frequencies. The uncertainty in the merging rate is such that it cannot be predicted reliably, but it is believed to be lower than the galactic background [79]. In this case the only handle for the subtraction would be the form of the spectrum. In fact, even if the strength is quite uncertain, the form of the spectrum may be known quite well [19].

C. Cosmological predictions for the intensity

The next question is whether it is reasonable to expect that some cosmological production mechanism saturates the nucleosynthesis bound. This of course depends on the production mechanism, but some general considerations are possible. First of all, it is clear that, if GWs are produced at the Planck scale by collisions and decays, toghether with the photons that we observe today in the CMBR, and there is not an inflationary phase at later time (the scenario of sect. 6B), we expect roughly \( \rho_{\text{gw}} \sim \rho_{\gamma} \), so that the bound \ref{eq:inequality} is approximately saturated.

If the mechanism that produces GWs is different from the mechanism that produces the photons in the CMBR, often it is still possible to relate the respective energy densities, simply because the scales of the two processes are related. For instance, ref. [66] discusses the spectrum of relic GWs produced in string cosmology, using only general arguments. The peak of the spectrum, \( f_1 \), is fixed with the criterium \( H_* \sim M_S \), which as discussed in sect. 5, corresponds to an effective ‘temperature’ at time of production \( T_* \sim (M_S M_{Pl})^{1/2} = (M_S/M_{Pl})^{1/2} M_{Pl} \). Redshifting \( T_* \) we get a characteristic frequency today \( f_1 \sim (M_S/M_{Pl})^{1/2} T_0 \). Fixing the peak energy from the ‘one graviton level’ \ref{eq:one_graviton}, \( n_f \sim 1 \), eq. \ref{eq:energy_density_relation} gives a peak energy density \( d\rho_d/d\log f \sim f_1^2 \), so that at the peak frequency we have

\[
\frac{d\rho_{\text{gw}}}{d\log f} \sim \left( \frac{M_S}{M_{Pl}} \right)^2 \frac{d\rho_{\gamma}}{d\log f},
\]

so that \( \rho_{\text{gw}} \) is related to \( \rho_{\gamma} \), although with a suppression factor \((M_S/M_{Pl})^2\) which is expected to range between \( 10^{-4} \) and \( 10^{-2} \), and numerical factors.

Similarly, in the production of GWs through bubble collisions when inflation terminates with a first order phase transition, there is only one energy scale, the vacuum energy density \( M \) during inflation (so that \( M^4 \) is the false-vacuum energy density). The value of \( M \) fixes the reheating temperature, and therefore \( \rho_{\gamma} \), from \((\pi^2 g_* / 30)T_{rh}^4 \sim M^4 \), and fixes also the energy liberated in gravitational waves. The latter can be estimated as follows \ref{eq:energy_density_relation}. The typical wavelength \( \lambda_* \) produced in the collision of two bubbles will be of the order of the radius of the bubbles when they collide, which in turn is a fraction of the horizon scale \( H_{H}^{-1} \). As in sect. 4, we write \( \lambda_* = c H^{-1} \), and \( H_* \sim M^2/M_{Pl} \). The energy liberated in GWs in the collision of two bubbles is of order \( E_{GW} \sim GM_{B}^2 / \lambda_* \) where \( M_B \sim M^4 \lambda_*^4 \) is the energy of a typical bubble. The fraction of the false vacuum energy that goes into GWs instead of going into

\[
\rho_{\text{gw}} \sim \epsilon^2 \rho_{\gamma}.
\]

Again \( \rho_{\text{gw}} \) is related to \( \rho_{\gamma} \), and the basic reason is that there is essentially only one dimensionful parameter that enters in the estimate.

From these explicit examples, we see that independently of the production mechanism, eq. \ref{eq:energy_density_relation} is quite general. The scale for the intensity of a relic GW background is indeed fixed in many cases by \( \rho_{\gamma} \), which therefore gives a first order of magnitude estimate of the effect. However, there are also suppression factors, like the factor \( \epsilon^2 \) in eq. \ref{eq:energy_density_relation} or \((M_S/M_{Pl})^2\) in eq. \ref{eq:energy_density_relation}, that, toghether with the exact numerical coefficients, are crucial for the detection at
present experiments. A value $\epsilon \sim 0.1$ would allow detection at the level $h_0^2\Omega_{gw}\sim 10^{-7}$ while $\epsilon \sim 10^{-2}$ would require $h_0^2\Omega_{gw}\sim 10^{-9}$, which is beyond the possibilities of first generation experiments.

For GWs produced by collisions at the Planck era we can even expect $\epsilon \sim 1$. However, another type of suppression is present if we have a spectrum of relic GWs with a total energy density $\rho_{gw}\sim \rho_\gamma$ with a cutoff in the GHz, and we want to observe it in the kHz region. In this case, as we discussed, we must hope that the spectrum is practically flat between the kHz and the GHz. While we have seen in sect. 5 that there are good reasons for considering such a behaviour, the fact that the energy density is spread over a wide interval of frequencies diminishes of course the value of $h_0^2\Omega_{gw}(f)$ at a given frequency. The nucleosynthesis bound then gives a maximum value at 1kHz

$$h_0^2\Omega_{gw}(1kHz) \leq \frac{5.6 \times 10^{-6}(N_\nu-3)}{\log \left(\frac{1GHz}{1kHz}\right)} \simeq 4 \times 10^{-7}(N_\nu-3).$$

If the spectrum extends further toward lower frequencies, the maximum value of $h_0^2\Omega_{gw}$ decreases accordingly, with a factor $\log(1GHz/f_{min})$ instead of $\log(1GHz/1kHz)$.

In production mechanisms where there is no obvious relation between $\rho_{gw}$ and $\rho_\gamma$, there is no general reason suggesting a value of $\rho_{gw}$ close to the nucleosynthesis bound, but still we can have rather large relic backgrounds. An important example is provided by cosmic strings (see \[55\] and references therein). Cosmic strings might form during phase transitions, but whether they actually form or not depends on the precise nature of the transition. Therefore, as remarked in ref. [11], the computation of the GW spectra from cosmic strings must be considered as illustrative rather than realistic (as on the other hand happens in most of the examples that we have discussed). Anyway, at LIGO/Virgo frequencies, the typical estimate of the intensity is of order $h_0^2\Omega_{gw} \sim 10^{-8} - 10^{-7}$. (Similar results are obtained from hybrid topological defects [22].) The scale for these numbers is given first of all by the combination $(G\mu)^2$, where $\mu$ is the mass per unit length of the string. It comes from the fact that a loop radiates with a power $P \sim \gamma G\mu^2$, ($\gamma$ is a dimensionless constant) while, evaluating $h_0^2\Omega_{gw}$, another factor of $G$ comes from $1/\rho_c$. For strings created at the GUT scale $G\mu \sim 10^{-6}$. Then, there are also large numbers related to the number of strings per horizon volume, which is thought to be of order 50, to the fact that also $\gamma \sim 50$, and to the size of the loop at formation time [11], that finally give a value $h_0^2\Omega_{gw} \sim 10^{-8} - 10^{-7}$. It is clear that in this case the right order of magnitude can only be obtained with a detailed understanding of the dynamics of the cosmic string network.

**VIII. CONCLUSIONS**

Present GW experiments have not been designed especially for the detection of GW backgrounds of cosmological origin. Nevertheless, there are chances that in their frequency window there might be a cosmological signal. The most naive estimate of the frequency range for signals from the very early Universe singles out the GHz region, very far from the region accessible to ground based interferometers, $f < \text{a few kHz}$, or to resonant masses. To have a signal in the accessible region, one of these two conditions should be met: either we find a spectrum with a long low-frequency tail, that extends from the kHz down to the kHz region, or we have some explosive production mechanism much below the Planck scale. As we have discussed, both situations seem to be not at all unusual, at least in the examples that have been worked out to date. The crucial point is the value of the intensity of the background.

With a very optimistic attitude, one could hope for a signal, present just in the 10Hz-1 kHz band, with the maximum intensity compatible with the nucleosynthesis bound, $h_0^2\Omega_{gw} \sim \text{a few} \times 10^{-6}$ (or even $10^{-5}$, stretching all parameters to the maximum limit). Such an option is not excluded, and the fact that such a background is not predicted by the mechanisms that have been investigated to date is probably not a very strong objection, given our theoretical ignorance of physics at the Planck scale and the rate at which new production mechanisms have been proposed in recent years, see the reference list. However, with more realistic estimates, on general grounds it appears difficult to predict a background that in the kHz region exceeds the level $h_0^2\Omega_{gw} \sim \text{a few} \times 10^{-7}$, independently of the production mechanism. This should be considered the minimum detection level at which a significant search can start. Such a level is beyond the sensitivity of first generation experiments, unless a ground based interferometer is correlated with a second interferometer, located at a distance small enough so that a significant correlation is possible, and large enough to decorrelate local noises. A few tens of kilometers would probably be the right order of magnitude. In this case we could detect a signal at the level $2 \times 10^{-7}$ in one year of integration time, with SNR=1.65, and the level of a few $\times 10^{-8}$ could be reached with longer integration time (and possibly allowing for a slightly worse confidence level; this could make some sense in a stochastic search because the SNR increases with time, and the hint of a signal at low SNR would provide a strong motivation for pursuing the search). The difficulties of such a detection are clear,
but it should be stressed that the payoff of a positive result would be enormous, opening up a window in the Universe and in fundamental high-energy physics that will never be reached with particle physics experiments.

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APPENDIX A:

In this Appendix we discuss the estimate of the quantity $\epsilon$, that enters eq. (39), in two specific examples. The discussion is meant to illustrate the uncertainty inherent in the determination of $\epsilon$ and the range of values that it can take.

i) Inflation-RD transition. Quantum particle creation due to vacuum fluctuations is associated with a sudden change of the scale factor, e.g. from a DeSitter inflationary phase to the RD phase (23,36). Let us consider a typical inflationary scenario (2): at some time $t_i$ the potential energy $V(\phi)$ of some scalar field $\phi$ (the ‘inflaton’ field) starts to trigger inflation, and the Universe expands exponentially. This inflationary stage ends at time $t_f$, when the so-called slow-roll conditions on $V(\phi)$ are not anymore satisfied. To solve the problems of standard cosmology, $t_f - t_i$ should be at least of order $60H_{DS}^{-1}$, where $H_{DS}$ is the Hubble constant during the DeSitter inflationary phase. At $t = t_f$ the temperature of the Universe is exponentially small, and a reheating period must take place. In the simplest scenarios, reheating lasts for a time interval $\Delta t \sim \Gamma_{\phi}^{-1}$, where $\Gamma_{\phi}$ is the decay width of the inflaton, and the decay of the inflaton reheats the Universe, which then enters the standard RD phase.

The cutoff of the spectrum at large frequency is determined by how fast the transition between the two phases takes place (12,16). Therefore the maximum value of the frequency produced is $f_* \sim 1/\Delta t \sim \Gamma_{\phi}$. The gravitons have been produced at a time $t_\ast$, with $t_f \leq t_\ast \leq t_f + \Delta t$. If the transition proceeds slowly, $\Delta t \gg t_f$, i.e. if $\Gamma_{\phi}$ is sufficiently small compared to $\sim H_{DS}$, then $t_\ast + \Delta t \sim \Delta t$ and we can take $t_\ast \sim \Delta t \sim \Gamma_{\phi}^{-1}$. Therefore in this case the statement $f_* \sim H(t_\ast) = 1/(2t_\ast)$ is indeed correct. However, if $\Gamma_{\phi}$ is sufficiently large, so that the transition takes place on an interval $\Delta t \ll t_f$, the value of time explored detecting these gravitons is $t_\ast \simeq t_f$, while the characteristic frequency produced is still $f_* \sim \Gamma_{\phi}$. In this case $f_* \gg 1/t_\ast$, or $\lambda \ll H_{DS}^{-1}$.

However, in most models $\Gamma_{\phi} \ll H_{DS}$. For instance, in models derived from supergravity, a large $\Gamma_{\phi}$ would produce a large reheating temperature $T_h$, and a value $T_h > 10^9$ GeV is forbidden because of the gravitino problem (see e.g. ref. [2], pg. 298). And indeed small values of $\Gamma_{\phi}$ come out quite naturally in supersymmetric models. For instance, in the model of ref. [34], $\Gamma_{\phi} \sim \Delta^5/\Lambda^2$ where $\Lambda = M_{P}(\sqrt{8\pi}$ and $\Delta \simeq 3 \times 10^{-5}M$ is a mass scale presumably related to supersymmetry breaking. Therefore in this case $\Gamma_{\phi} \sim 10^{-27}M$ is extremely small.

Not all models of inflation satisfy $\Gamma_{\phi} \ll H_{DS}$ automatically. For instance, in the old Coleman-Weinberg model $\Gamma_{\phi} > H_{DS}$, but the model is not viable. So, in the case of quantum production of particles after an inflationary phase, the estimate $f_* \sim 1/t_\ast$, and therefore $\epsilon \sim 1$, is correct, at least in typical viable models.

ii) Phase transitions. Another important example is given by phase transitions in the early Universe, as the QCD or the electroweak phase transition. These are believed to proceed through bubbles nucleation of the low-temperature phase, when the Universe cools below the critical temperature for the phase transition, $T_c$. If the transition occurs explosively, the collisions of different bubbles produce gravitational waves (34,35). The characteristic wavelength of these GWs is of the order of the typical radius $R$ of the bubbles when they collide. We write again $\lambda_{peak}(t_\ast) \sim R = \epsilon H_{DS}^{-1}$. Plausible values of $\epsilon$ have been discussed in detail by Hogan (34) and by Witten (ref. [35], app. B). In the case of the QCD transition $\epsilon \sim 1$ is excluded because otherwise in the bubble collisions there would be overproduction of primordial black holes. Production of primordial black holes is severely constrained because the energy density of primordial black holes scales like $1/\alpha^3$ in the RD phase, and therefore they would come to dominate the energy density of the Universe. In particular, the Universe would not be radiation dominated during nucleosynthesis. Actually, primordial black holes can evaporate through emission of Hawking radiation. However, in order to evaporate before nucleosynthesis, the scale at which the phase transition takes place must be higher than $10^{11}$ GeV (61). Therefore $\epsilon$ could be of order one in phase transitions which take place at these temperatures, but not in the QCD or electroweak phase transition.

To estimate $\epsilon$ it is necessary to make assumptions about how the phase transition is nucleated. If it is nucleated by thermal fluctuations, ref. [34] suggests an upper bound $\epsilon \lesssim 10^{-2}$. For nucleation of bubbles via quantum tunneling
A detailed analysis has been done in ref. [86]. The relevant parameter is the nucleation rate, which again is a model-dependent quantity, but for a wide class of models one typically finds \( \epsilon \sim 10^{-3} \cdots 10^{-2} \) [86,87,31]. If the transition is nucleated by impurities (which is most often the case, except in very pure and homogeneous samples) the issue is much more complicated. For instance, the impurities could be given by turbulent motion of the cosmic fluid generated prior to the QCD epoch; they would depend on the detailed spectrum of the fluid motion, or in general on the characteristic distance between impurities, and in this case \( \epsilon \) is basically impossible to estimate, even as an order of magnitude.

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