An affine version of a theorem of Nagata

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Abstract  Let $R$ be an affine $k$-domain over the field $k$. The paper’s main result is that if $R$ admits a nontrivial embedding in a polynomial ring $K[s]$ for some field $K$ containing $k$, then $R$ can be embedded in a polynomial ring $F[t]$ which extends $R$ algebraically. This theorem can be applied to subrings of a ring which admits a nonzero locally nilpotent derivation. In this way, we obtain a concise new proof of the cancellation theorem for rings of transcendence degree one for fields of characteristic 0.

1. Introduction

If $F \subset E$ are fields and $x \in E$, then the subfield of $E$ generated by $F$ and $x$ is denoted by $F(x)$. If $x$ is transcendental over $F$, then $F(x)$ is isomorphic to the field of rational functions in one variable over $F$, and we write $F(x) \cong F(1)$. In his 1967 paper, Nagata [15] proved the following fundamental result for fields.

THEOREM 1.1 (\cite[THEOREM 2 AND 17, THEOREM 5.2]{15})

Let $k,K,L$ be fields such that

(a) $k \subset K$ and $k \subset L \subset K^{(1)}$;

(b) $K$ is finitely generated over $k$;

(c) $L \not\subset K$.

Then there exists a finite algebraic extension of the form $L \subset M^{(1)}$ for some field $M$ containing $k$.

This result extends the famous theorem of Lüroth, which asserts that if $k \subset L \subset k(x)$ are fields with $k \neq L$ and $x$ transcendental over $k$, then there exists $y \in k(x)$ with $L = k(y)$. By combining the theorems of Lüroth and Nagata, we get an even stronger statement for fields of transcendence degree one over $k$ (see the Appendix).

We consider an analogous situation for integral domains. The polynomial ring in one variable $x$ over the field $F$ is denoted by $F[x] = F^{[1]}$. For the integral domain $R$, we seek criteria to determine when $R = F^{[1]}$ or when $R \subset F^{[1]}$ with $F^{[1]}$ algebraic over $R$. Our main result is Theorem 2.1, which may be regarded as an affine version of Nagata’s theorem:
Let $k$ be a field, and let $R$ be an affine $k$-algebra. Suppose that there exists a field $K$ with $R \subset K[1]$ and $R \not\subset K$. Then there exists a field $F$ and an algebraic extension $R \subset F[1]$.

This result is of particular interest in the setting of locally nilpotent derivations, where we assume that the ground field $k$ is of characteristic 0. If an integral $k$-domain $B$ admits a nonzero locally nilpotent derivation $D$, then $B \subset K[1]$, where $K$ is the field of fractions of the kernel of $D$ and $s$ is a local slice. Thus, any affine subalgebra $R \subset B$ not contained in the Makar-Limanov invariant of $B$ is isomorphic to a nontrivial subring of $F[s]$ for some field $F$, where $F[s]$ is algebraic over $R$.

For rings of transcendence degree one over $k$, Theorem 3.1 gives an even stronger conclusion.

Let $k$ be a field, and let $R$ be a $k$-algebra with $\text{tr.deg}_k R = 1$. Suppose that there exists a field $K$ with $R \subset K[1]$ and $R \not\subset K$. Then $R$ is $k$-affine and there exists a field $F$ algebraic over $k$ with $R \subset F[1]$. If $k$ is algebraically closed, then there exists $t \in \text{frac}(R)$ with $R \subset k[t]$.

Abhyankar, Eakin, and Heinzer [1] proved that if $R, S$ are integral domains of transcendence degree one over a field $k$ such that the polynomial rings $R[x_1, \ldots, x_n]$ and $S[y_1, \ldots, y_n]$ are isomorphic $k$-algebras, then $R$ and $S$ are isomorphic. In Section 4, we apply Theorem 3.1, together with the well-known theorems of Seidenberg and Vasconcelos on derivations, to obtain a short proof of this result in the case in which $k$ is of characteristic 0. Makar-Limanov [14] gave a proof of this result for $k = \mathbb{C}$, and we follow his idea to use the Makar-Limanov invariant. Other proofs are given in [5] for perfect fields and in [6] for the case in which $k$ is algebraically closed.

### 1.1. Background

Lüroth’s theorem was proved by Lüroth [13] for the field $k = \mathbb{C}$ in 1876 and for all fields by Steinitz [22] in 1910. One generalization states that if $k \subset L \subset k(x_1, \ldots, x_n)$ and $L$ is of transcendence degree one over $k$, then $L = k(y)$. This was proved by Gordan [10] for $k = \mathbb{C}$ in 1887 and for all fields by Igusa [12] in 1951; other proofs appear in [15] and [20]. In 1894, Castelnuovo [2] showed that if $\mathbb{C} \subset L \subset \mathbb{C}(x_1, \ldots, x_n)$ and $L$ is of transcendence degree two over $\mathbb{C}$, then $L = \mathbb{C}(y_1, y_2)$. Castelnuovo’s result does not extend to nonalgebraically closed ground fields or to fields $L$ of higher transcendence degree. An excellent account of ruled fields and their variants can be found in [17], including the theorem of Nagata [17, Theorem 5.2].

For polynomial rings, Evyatar and Zaks [7] showed that if $R$ is a PID and $k \subset R \subset k[x_1, \ldots, x_n]$, then $R = k[1]$; Zaks [24] generalized this to the case in which $R$ is a Dedekind domain. Abhyankar, Eakin, and Heinzer [1, (2.5)] showed that if $k \subset R \subset k[x_1, \ldots, x_n]$ and $R$ is of transcendence degree one over $k$, then $R$ is isomorphic to a subring of $k[1]$. Theorem 3.1 below generalizes these earlier results. It should be noted that Makar-Limanov [14, Lemma 14] proved a result equivalent to Theorem 3.1 for the field $k = \mathbb{C}$, which is stated in the language of locally nilpotent derivations (see also [5, Lemma 5.3] and [6, Lemma 4.2]).
The case when $R$ is normal is of particular interest: If $k \subset R \subset k[[n]]$ and $R$ is normal of transcendence degree one over $k$, then $R = k[[1]]$. This was shown by Cohn [3, Proposition 2.1] for $n = 1$, and in the stated form by Abhyankar, Eakin, and Heinzer [1, (2.6), (2.7)].

One generalization deals with the case where the field $k$ is replaced by a unique factorization domain (UFD). Abhyankar, Eakin, and Heinzer [1, Theorem 4.1] treat this case: If, for some integer $n \geq 1$, $A \subset D \subset A[[n]]$ are UFDs such that the transcendence degree of $D$ over $A$ is one, then $D = A[[1]]$. See also [19, Corollary 3.4] and [1, Proposition 4.8]. Connell and Zweibel [4, Theorem 4.1] presented what they call “an affine version of Lüroth’s theorem,” namely: If $A$ is a UFD and $A \subset B \subset A[[x]] = A[[1]]$ for a ring $B$, then $\text{frac}(B) \cap A[[x]] = A[[v]]$ for some $v \in A[[x]]$, where $\text{frac}(B)$ denotes the field of fractions of $B$. The authors state that their result “is just an abstraction of what is proved in the proof of Theorem 2” in the paper of Formanek [8].

The Makar-Limanov invariant of a commutative ring (defined below) was introduced by Makar-Limanov in the mid-1990s, and he called it the ring of absolute constants. It is an important invariant in the study of affine rings, affine varieties, and their automorphisms.

1.2. Preliminaries

If $B$ is an integral domain, then $\text{frac}(B)$ is the quotient field of $B$, and $B[[n]]$ is the polynomial ring in $n$ variables over $B$. Given $f \in B$, $B_f$ denotes the localization $B[1/f]$. The set of derivations $D : B \to B$ is $\text{Der}(B)$.

If $A \subset B$ is a subring, then the transcendence degree of $B$ over $A$, denoted $\text{tr.deg}_A B$, will mean the transcendence degree of $\text{frac}(B)$ over $\text{frac}(A)$. The set of elements of $B$ algebraic over $A$ is denoted by $\text{Alg}_A B$.

Let $k$ be a field of characteristic zero, and let $B$ be an integral domain containing $k$. The set of $k$-derivations $D : B \to B$ is denoted by $\text{Der}_k(B)$, and $D$ is said to be locally nilpotent if, for each $b \in B$, there exists $n \in \mathbb{N}$ with $D^n b = 0$. The set of locally nilpotent derivations of $B$ is denoted by $\text{LND}(B)$. If $D \in \text{LND}(B)$ is nonzero and $A$ is the kernel of $D$, then $A$ is algebraically closed in $B$ and $\text{tr.deg}_A(B) = 1$.

The Makar-Limanov invariant of $B$ is the intersection of all kernels of locally nilpotent derivations of $B$, denoted $\text{ML}(B)$. Note that $\text{ML}(B)$ is a $k$-algebra which is algebraically closed in $B$, and note that any automorphism of $B$ maps $\text{ML}(B)$ into itself.

An element $s \in B$ is a local slice of $D$ if $D^2 s = 0$ and $D s \neq 0$. Note that every nonzero element of $\text{LND}(B)$ admits a local slice. If $s \in B$ is a local slice of $D$, then

$$B_{Ds} = A_{Ds}[s] = (A_{Ds})[[1]].$$

This implies the following property: If $Df \in fB$ for some $f \in B$, then $Df = 0$. The reader is referred to [9] for further details regarding locally nilpotent derivations.

We also need the following.
PROPOSITION 1.1 ([16, PROPOSITION 5.1.2])

Let \( k \) be a field, and let \( A \) be a commutative \( k \)-algebra. Then, for any field extension \( L/k \), \( A \) is finitely generated over \( k \) if and only if \( L \otimes_k A \) is finitely generated over \( L \).

2. Main theorem

For a field \( K \), the polynomial ring \( K[s] = K[1] \) is naturally \( \mathbb{Z} \)-graded over \( K \), where \( s \) is homogeneous of degree one. Let \( \deg \) be the associated degree function in \( s \) over \( K \). A subring \( R \subset K[s] \) is homogeneous if the \( \mathbb{Z} \)-grading restricts to \( R \).

LEMMA 2.1

Suppose that \( K \) is a field, and suppose that \( R \subset K[s] = K[1] \) is a homogeneous subring with \( R \not\subset K \). Let \( L = \text{frac}(R) \cap K \), and let \( \hat{L}, \hat{K} \) denote the algebraic closures of \( L \) and \( K \), respectively. Then there exist \( c \in \hat{K} \) and integer \( d \geq 1 \) such that \( R \subset \hat{L}[cs^d] \) and \( \hat{L}[cs^d] \) is algebraic over \( R \).

Proof

Define the integer

\[
   d = \gcd\{\deg r \mid r \in R, r \neq 0\}.
\]

Let homogeneous \( r \in R \) of positive degree be given. Then there exist \( \kappa \in K \) and positive \( e \in \mathbb{Z} \) with \( r = \kappa s^e \). Let \( c \in \hat{K} \) be such that \( c^e = \kappa \). Then \( r = (cs^d)^e \).

If \( \rho \in R \) is any other homogeneous element of positive degree, then \( \rho = (c's^d)^{e'} \) for \( c' \in \hat{K} \) and positive \( e' \in \mathbb{Z} \). We have that

\[
   \frac{\rho^{c'}}{\rho^e} = \left(\frac{(cs^d)^e}{(c's^d)^{e'}}\right)^{c'} = \left(\frac{c}{c'}\right)^{ee'} \in L \quad \Rightarrow \quad \frac{c}{c'} \in \hat{L} \quad \Rightarrow \quad \hat{L}[c's^d] = \hat{L}[cs^d].
\]

It follows that \( R \subset \hat{L}[cs^d] \). \( \square \)

THEOREM 2.1

Let \( k \) be a field, and let \( R \) be an affine \( k \)-algebra. Suppose that there exists a field \( K \) with \( R \subset K[1] \) and \( R \not\subset K \). Then there exist a field \( F \) and an algebraic extension \( R \subset F[1] \).

Proof

Suppose that \( R \subset K[s] = K[1] \). For each \( g \in K[s] \), let \( \bar{g} \) denote the highest-degree homogeneous summand of \( g \) as a polynomial in \( s \). Define the set

\[
   \bar{R} = \{ \bar{r} \mid r \in R, r \neq 0 \}.
\]

Then \( k[\bar{R}] \) is a homogeneous subalgebra of \( K[s] \) not contained in \( K \).

By Lemma 2.1, if \( L = \text{frac}(k[\bar{R}]) \cap K \) and if \( \hat{L}, \hat{K} \) are the algebraic closures of \( L \) and \( K \), respectively, then

\[
   k[\bar{R}] \subset \hat{L}[cs^d] \quad (c \in \hat{K}, d \geq 1).
\]
By hypothesis, there exist \(w_1, \ldots, w_m \in R\) \((m \geq 1)\) such that \(R = k[w_1, \ldots, w_m]\). Given \(i\), assume that \(w_i = \sum_{j=0}^{n_i} c_{ij} s^j\), where \(c_{ij} \in K\). Define \(A \subset \hat{K}\) and \(B \subset \hat{K}[s]\) by
\[
A = \hat{L}[c, c_{ij} \mid 1 \leq i \leq m, 0 \leq j \leq n_i] \quad \text{and} \quad B = A[s] = A^{[1]}.
\]
Then \(R \subset B\), \(A\) is finitely generated over \(\hat{L}\), and the Jacobson radical of \(A\) is trivial. Choose a maximal ideal \(m\) of \(A\) not containing \(e\).

If \(R \cap mB \neq (0)\), then let nonzero \(r \in R \cap mB\) be given. Since \(mB = m[s]\), we have \(r = \sum_{0 \leq i \leq e} a_i s^i\), where \(a_i \in m\) for each \(i\). Note that \(e \geq 1\), since \(\hat{L} \cap m = (0)\). Therefore, by (2), there exist \(\epsilon \geq 1\) and nonzero \(\lambda \in \hat{L}\) such that
\[
\bar{r} = a_\epsilon s^\epsilon = \lambda (cs^\epsilon)^\epsilon.
\]
But then \(e \in m\), a contradiction. Therefore, \(R \cap mB = (0)\).

Let \(\pi : B \to B/mB\) be the canonical surjection of \(\hat{L}\)-algebras, noting that
\[
B/mB = (A/mA)[\pi(s)] = \hat{L}^{[1]}.
\]
Since \(\pi(cs^d) = \pi(c)\pi(s)^d\), where \(\pi(c) \neq 0\), we see that \(\pi|_R\) is a degree-preserving isomorphism. It follows that \(R\) is a subring of \(\hat{L}^{[1]}\) via \(\pi\).

It remains to show that \(R\) and \(\hat{L}^{[1]}\) have the same transcendence degree over \(k\). Since \(R \subset \hat{L}^{[1]}\), it will suffice to show that \(\text{tr.deg}_k \hat{L}^{[1]} \leq \text{tr.deg}_k R\). By Lemma 2.1, we see that \(\text{tr.deg}_k \hat{L}^{[1]} = \text{tr.deg}_k k[\hat{R}]\), so it will suffice to show that \(\text{tr.deg}_k k[\hat{R}] \leq \text{tr.deg}_k R\).

Let \(n = \dim_k R\), and let \(r_1, \ldots, r_{n+1} \in R\) be given. Then there exists a polynomial \(h \in k[x_1, \ldots, x_{n+1}] = k^{[n+1]}\) with \(h(r_1, \ldots, r_{n+1}) = 0\). If \(k[x_1, \ldots, x_{n+1}]\) is \(\mathbb{Z}\)-graded in such a way that each \(x_i\) is homogeneous and the degree of \(x_i\) is \(\text{deg} r_i\), then \(H(r_1, \ldots, r_{n+1}) = 0\), where \(H\) is the highest-degree homogeneous summand of \(h\). We have thus shown that any subset of \(n + 1\) elements in a generating set for \(k[\hat{R}]\) is algebraically dependent over \(k\). Therefore, the transcendence degree of \(k[\hat{R}]\) over \(k\) is at most \(n\). This completes the proof of the theorem. \(\square\)

3. Rings of transcendence degree one

THEOREM 3.1

Let \(k\) be a field, and let \(R\) be a \(k\)-algebra with \(\text{tr.deg}_k R = 1\). Suppose that there exists a field \(K\) with \(R \subset K^{[1]}\) and \(R \not\subset K\). Then \(R\) is \(k\)-affine and there exists a field \(F\) algebraic over \(k\) with \(R \subset F^{[1]}\). If \(k\) is algebraically closed, then there exists \(t \in \text{frac}(R)\) with \(R \subset k[t]\).

Proof

Suppose that \(R \subset K[s] = K^{[1]}\), and let \(\text{deg}\) be the associated degree function in \(s\) over \(K\).

Consider first the case in which \(k\) is algebraically closed. The set
\[
\Sigma := \{\text{deg} w \mid w \in R, w \neq 0\} \subset \mathbb{N}
\]
is a semigroup and is therefore finitely generated as a semigroup. Let \(w_1, \ldots, w_m \in R\) be such that \(\Sigma = \langle \text{deg} w_1, \ldots, \text{deg} w_m \rangle\), and define \(S = k[w_1, \ldots, w_m] \subset R\).
Then, given \( v \in R \), there exists \( u \in S \) such that \( \text{deg} \, u = \text{deg} \, v \). Assume that \( \text{deg} \, v \geq 1 \).

As in the preceding proof, since \( u \) and \( v \) are algebraically dependent over \( k \), \( \bar{u} \) and \( \bar{v} \) are also algebraically dependent over \( k \). Since \( u \) and \( v \) have the same degree, there exists \( P \in k[x, y] = k^{[2]} \) which is homogeneous relative to the standard \( \mathbb{Z} \)-grading of \( k[x, y] \) such that \( P(\bar{u}, \bar{v}) = 0 \). Write \( P(x, y) = \prod_{1 \leq i \leq \ell} (\alpha_i x + \beta_i y) \), where \( \ell \) is a positive integer and \( \alpha_i, \beta_i \in k^* \) (\( 1 \leq i \leq \ell \)). Then \( \alpha_i \bar{u} + \beta_i \bar{v} = 0 \) for some \( i \).

Therefore, \( \text{deg}(\alpha_i u + \beta_i v) < \text{deg} \, v \) for some \( i \). By induction on degrees, we can assume that \( \alpha, u + \beta v \in S \), which implies that \( v \in S \), and \( R = S \). Therefore, \( R \) is finitely generated over \( k \) when \( k \) is algebraically closed.

For general \( k \), let \( \hat{k} \) and \( \hat{K} \) denote the algebraic closures of \( k \) and \( K \), respectively. Set \( \hat{R} = \hat{k} \otimes_k R \). Then \( \text{tr.deg} \, \hat{k} / R = 1 \), \( \hat{R} \subset \hat{K}^{[1]} \), and \( \hat{R} \not\subset \hat{K} \). By what was shown above, we conclude that \( \hat{R} \) is affine over \( \hat{k} \). Therefore, Proposition 1.1 implies that \( R \) is affine over \( k \).

By Theorem 2.1, there exists a field \( F \) algebraic over \( k \) with \( R \subset F^{[1]} \). If \( k \) is algebraically closed, then \( F = k \) and \( k \subset R \subset k[s] \) for some \( s \) transcendental over \( k \). If \( \mathcal{O} \) is the integral closure of \( R \) in \( \text{frac}(R) \), then since \( k[s] \) is integrally closed, we have that \( k \subset R \subset \mathcal{O} \subset k[s] \). In this situation, it is known that \( \mathcal{O} = k[\theta] \) for some \( \theta \in k[s] \) (see [3, Proposition 2.1]).

**COROLLARY 3.1 (SEE [14, LEMMA 14])**

Let \( k \) be an algebraically closed field of characteristic 0, and let \( B \) be a commutative \( k \)-domain. Given \( r \in B \), if \( r \not\in \text{ML}(B) \), then there exists \( t \in \text{frac}(\text{Alg}_{k[r]} B) \) such that \( \text{Alg}_{k[r]} B \subset k[t] \).

**Proof**

By hypothesis, there exists \( D \in \text{LND}(B) \) with \( Dr \neq 0 \). If \( A = \ker D \) and \( K = \text{frac}(A) \), then \( K \otimes_k B = K^{[1]} \) by (1). We therefore have \( \text{Alg}_{k[r]} B \subset K^{[1]} \), and \( r \not\in K \). The result now follows by Theorem 3.1.

Makar-Limanov [14, p. 39] asked whether this result generalizes to rings of transcendence degree two. Let \( k \) be an algebraically closed field of characteristic 0, and let \( B \) be a commutative \( k \)-domain. Given \( x, y \in B \), does the implication

\[
\text{Alg}_{k[x,y]} B \cap \text{ML}(B) = k \implies \text{Alg}_{k[x,y]} B \subset k^{[2]}
\]

hold?

**EXAMPLE 3.1**

Let \( k \) and \( K \) be fields with \( k \subset K \), where \( K = k[\alpha] \) is a simple algebraic extension of \( k \), and \( [K : k] \geq 2 \). Define

\[
R = k[u, v] \subset K[s] = K^{[1]},
\]

where \( u = \alpha s^2 \) and \( v = \alpha s^3 \). Since \( s = v/u \) and \( \alpha = u^3/v^2 \), we see that \( \text{frac}(R) = K(s) \). If \( R \subset k[t] \) for \( t \in \text{frac}(R) \), then \( k(t) = \text{frac}(R) = K(s) \), which is not possible. Therefore, the ring \( R \) cannot be embedded in \( k^{[1]} \). This shows that the
hypothesis that the field $k$ is algebraically closed is necessary in the last statement of Theorem 3.1.

EXAMPLE 3.2
As an illustration of Corollary 3.1, let $k[x, y] = \mathcal{O}_1$, and write $k[x, y] = \bigoplus_{i \geq 0} V_i$, where $V_i$ is the vector space of binary forms of degree $i$ over $k$. Define $D \in \text{LND}(k[x, y])$ by $D = x \frac{\partial}{\partial y}$. Then $D$ is linear, meaning that $D(V_i) \subset V_i$ for each $i$.

Therefore, if $B = k[V_2, V_3]$, then $D$ restricts to $B$. Let $R$ be the algebraic closure of $k[y^2]$ in $B$, noting that $D(y^2) \neq 0$. Then $R = k[y^2, y^3]$ and $\text{frac}(R) = k(y)$.

4. Cancellation theorem for rings of transcendence degree one

4.1. Integral extensions and the conductor ideal

DEFINITION 4.1
Let $A$ and $B$ be integral domains with $A \subset B$. The conductor of $B$ in $A$ is

$$\mathcal{C}_A(B) = \{a \in A \mid aB \subset A\}.$$ 

If $\mathcal{O}$ is the integral closure of $A$ in $\text{frac}(A)$, then the conductor ideal of $A$ is $\mathcal{C}_A(\mathcal{O})$.

Note that $\mathcal{C}_A(B)$ is an ideal of both $A$ and $B$, and is the largest ideal of $B$ contained in $A$. The following two properties of the conductor are easily verified:

- (C.1) $\mathcal{C}_{A[n]}(B[n]) = \mathcal{C}_A(B) \cdot B[n]$ for every $n \geq 0$;
- (C.2) $D\mathcal{C}_A(B) \subset \mathcal{C}_A(B)$ for every $D \in \text{Der}(B)$ restricting to $A$.

LEMMA 4.1
Let $k$ be a field, let $A$ be an integral domain containing $k$, and let $\mathcal{C} \subset A$ be the conductor ideal of $A$. If $A$ is affine over $k$, then $\mathcal{C} \neq (0)$.

Proof
Since $A$ is affine over $k$, its normalization $\mathcal{O}$ is also affine over $k$, and is finitely generated as an $A$-module (see [11, Chapter I, Theorem 3.9A]). Let $\{\omega_1, \ldots, \omega_n\}$ be a generating set for $\mathcal{O}$ as an $A$-module, and let nonzero $a \in A$ be such that $a\omega_1, \ldots, a\omega_n \in A$. Then $a \in \mathcal{C}$. □

THEOREM 4.1 (SEIDENBERG [21])
Let $A$ be a Noetherian integral domain containing $\mathbb{Q}$, and let $\mathcal{O}$ be the integral closure of $A$ in $\text{frac}(A)$. Then every $D \in \text{Der}(A)$ extends to $\mathcal{O}$.

THEOREM 4.2 (VASCONCELOS [23])
Let $A$ and $A'$ be integral domains containing $\mathbb{Q}$ with $A \subset A'$, where $A'$ is an integral extension of $A$. If $D \in \text{LND}(A)$ extends to $D' \in \text{Der}(A')$, then $D' \in \text{LND}(A')$. 

4.2. The theorem of Abhyankar, Eakin, and Heinzer

**THEOREM 4.3** (SEE [1, (3.3)])

Let $k$ be a field, and let $R, S$ be integral $k$-domains of transcendence degree one over $k$. If $R^{[n]} \cong_k S^{[n]}$ for some $n \geq 0$, then $R \cong_k S$.

**Proof**

Characteristic $k = 0$. Since $R$ is algebraically closed in $R^{[n]}$, we have that

$$\text{Alg}_k(R^{[n]}) = \text{Alg}_k(R).$$

Let $\alpha : R^{[n]} \to S^{[n]}$ be an isomorphism of $k$-algebras. If $k' = \text{Alg}_k(R)$, then $\alpha(k') = \text{Alg}_k(S)$, since $S$ is algebraically closed in $S^{[n]}$. Therefore, identifying $k'$ and $\alpha(k')$, we can view $R$ and $S$ as $k'$-algebras, and $\alpha$ as a $k'$-isomorphism. It thus suffices to assume that $k$ is algebraically closed in $R$.

Since $\text{ML}(R^{[n]}) \subset \text{ML}(R)$, we see that $\text{ML}(R^{[n]})$ is an algebraically closed subalgebra of $R$. Therefore, either $\text{ML}(R^{[n]}) = R$ or $\text{ML}(R^{[n]}) = k$.

**Case 1**: $\text{ML}(R^{[n]}) = R$. In this case, we also must have $\text{ML}(S^{[n]}) = S$. Since $\alpha$ maps the Makar-Limanov invariant onto itself, we conclude that $\alpha(R) = S$.

**Case 2**: $\text{ML}(R^{[n]}) = k$. We will show that $R = k^{[1]}$ in this case. It suffices to assume that $k$ is an algebraically closed field: if $\hat{k}$ is the algebraic closure of $k$ and $\hat{R} = \hat{k} \otimes_k R$, then $\text{ML}(\hat{R}^{[n]}) = \hat{k}$. If this implies $\hat{R} = \hat{k}^{[1]}$, then $R = k^{[1]}$. (All forms of the affine line over a perfect field are trivial; see [18].)

So assume that $k$ is algebraically closed. By hypothesis, there exists $D \in \text{LND}(R^{[n]})$ with $DR \neq 0$. If $\mathcal{O}$ is the integral closure of $R$ in $\text{frac}(R)$, then $\mathcal{O}^{[n]}$ is the integral closure of $R^{[n]}$ in $\text{frac}(R^{[n]})$. By property (C.1), if $\mathcal{C}$ is the conductor ideal of $R$, then $\mathcal{C} \cdot \mathcal{O}^{[n]}$ is the conductor ideal of $R^{[n]}$.

Let $s$ be a local slice of $D$, and let $K = \text{frac}(\ker D)$. Then by (1), $R \subset K[s]$ and $R \not\subset K$. By Theorem 3.1, $R$ is $k$-affine, and there exists $t \in \text{frac}(R)$ such that $\mathcal{O} = k[t]$. By the theorems of Seidenberg [21] and Vasconcelos [23], $D$ extends to a locally nilpotent derivation of $\mathcal{O}^{[n]}$; and by property (C.2), $D(\mathcal{C} \cdot \mathcal{O}^{[n]}) \subset \mathcal{C} \cdot \mathcal{O}^{[n]}$.

By Lemma 4.1, $\mathcal{C} \neq 0$. Since $\mathcal{C}$ is an ideal of $k[t]$, there exists a nonzero $h \in R$ with $\mathcal{C} = h \cdot k[t]$. Thus, $\mathcal{C} \cdot \mathcal{O}^{[n]} = h \cdot \mathcal{O}^{[n]}$ and $D(h \cdot \mathcal{O}^{[n]}) \subset h \cdot \mathcal{O}^{[n]}$. Therefore, $Dh = 0$. If $h \notin k$, then $h[h] \subset \ker D$ implies that $R \subset \ker D$, which is not the case. Therefore, $h \in k^*$ and $R = k^{[1]}$. By symmetry, $S = k^{[1]}$.

**REMARK 4.1**

The Makar-Limanov invariant can be defined for $k$-algebras over a field $k$ of any characteristic. This was done in [5], where it is defined to be the intersection of all invariant rings of actions of the additive group of $k$ on the ring. This is equivalent to the definition given above when the characteristic of $k$ is zero. Crachiola and Makar-Limanov [6, Corollary 3.2] use this approach to prove Theorem 4.3 in the case in which $k$ is algebraically closed. However, the theorems of Seidenberg [21] and Vasconcelos [23] are not available in positive characteristic, since they are valid for $\mathbb{Q}$-algebras.
Appendix

Combining the theorems of Lüroth [13] and Nagata [15] gives the following corollary.

**COROLLARY A.1**

Suppose that $k$ and $L$ are fields with $k \subset L$, where $k$ is algebraically closed, $L$ is finitely generated over $k$, and $\text{tr.deg}_k L = 1$. If there exists a field $E$ containing $k$ such that $L \subset E^{(1)}$ and $L \not\subset E$, then $L = k^{(1)}$.

**Proof**

Assume that $L \subset E(s) = E^{(1)}$. Let $\alpha_1, \ldots, \alpha_n \in L$ be such that $L = k(\alpha_1, \ldots, \alpha_n)$. Choose $f_i(s), g_i(s) \in E[s]$ such that $\alpha_i = f_i/g_i$, and let $K$ be the subfield of $E$ generated by the coefficients of $f_i$ and $g_i$, $1 \leq i \leq n$. Then $K$ is finitely generated over $k$, and $L \subset K(s)$. By Nagata’s theorem [15], there exists a finite algebraic extension $L \subset M^{(1)}$ for some field $M$ containing $k$. Since the transcendence degree of $L$ over $k$ is one, we see that $M$ is algebraic over $k$, that is, $M = k$. The corollary now follows by Lüroth’s theorem [13]. □

We conclude by asking if the analogue of Theorem 2.1 holds for Laurent polynomial rings. Let $k$ be a field, and let $R$ be an affine $k$-algebra. Suppose that there exists a field $K$ with $R \subset K^{[\pm 1]}$ and $R \not\subset K$. Does it follow that there exist a field $F$ and an algebraic extension $R \subset F^{[\pm 1]}$?

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