The trace on the K-theory of group $C^*$-algebras

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Abstract

The canonical trace on the reduced $C^*$-algebra of a discrete group gives rise to a homomorphism from the K-theory of this $C^*$-algebra to the real numbers. This paper studies the range of this homomorphism. For torsion free groups, the Baum-Connes conjecture together with Atiyah’s $L^2$-index theorem implies that the range consists of the integers.

We give a direct and elementary proof that if $G$ acts on a tree and admits a homomorphism $\alpha$ to another group $H$ whose restriction $\alpha|_{G_v}$ to every stabilizer group of a vertex is injective, then

$$\text{tr}_G(K(C^*_r G)) \subset \text{tr}_H(K(C^*_r H)).$$

This follows from a general relative Fredholm module technique.

Examples are in particular HNN-extensions of $H$ where the stable letter acts by conjugation with an element of $H$, or amalgamated free products $G = H*_{U} H$ of two copies of the same groups along a subgroup $U$.

MSC: 19K (primary); 19K14, 19K35, 19K56 (secondary)

1 Introduction

Let $G$ be a discrete group. All discrete groups considered in this paper are assumed to be countable. The trace $\text{tr}_G: CG \to \mathbb{C}: \sum_{g \in G} \lambda_g g \mapsto \lambda_1$ (where 1 is the neutral element of $G$) extends to a trace on the reduced $C^*$-algebra of $G$ and therefore gives rise to a homomorphism

$$\text{tr}_G: K_0(C^*_r G) \to \mathbb{R}.$$
If $G$ is torsion free, we have the commutative diagram

$$
\begin{array}{ccc}
K_0(BG) & \xrightarrow{A} & K_0(C^*_rG) \\
\downarrow \text{ind}_G & & \downarrow \text{tr}_G \\
\mathbb{Z} & \longrightarrow & \mathbb{R},
\end{array}
$$

where $A$ is the Baum-Connes assembly map. The Baum-Connes conjecture says that $A$ is an isomorphism. We denote by $\text{ind}_G$ Atiyah’s $L^2$-index, which coincides with the ordinary index \[\square\] and therefore takes values in the integers. Surjectivity of $A$ of course implies that $\text{tr}_G$ is also integer valued. We will denote this consequence of the Baum-Connes conjecture as the trace conjecture. It implies by a standard argument that there are no nontrivial projections in $C^*_rG$.

The trace conjecture was verified directly for free groups using a special Fredholm module, which can be assigned to free groups, cf. e.g. \[\square\]. Based on the ideas of this proof we get the following result.

1.1. **Theorem.** Let $H, G$ be discrete countable groups and assume

$$\text{tr}_H(K(C^*_rH)) \subset A \subset \mathbb{R}.$$ 

Let $\Omega$ and $\Delta$ be sets with commuting $G$-action from the left and $H$-action from the right such that $\Omega$ and $\Delta$ are free $H$-sets. Let $\Omega = \Omega' \cup X$ and assume $\Delta$ and $\Omega'$ are free $G$-sets and $X$ consists of $1 \leq r < \infty$ $H$-orbits. Assume there is a bijective right $H$-map $\phi: \Delta \to \Omega$. Suppose that for every $g \in G$ the set

$$R_g := \{ x \in \Delta \mid \phi(gx) \neq g\phi(x) \}$$

is contained in the union of finitely many $H$-orbits of $\Delta$.

Then

$$\text{tr}_G(K(C^*_rG)) \subset \frac{1}{r} A.$$ 

It remains now of course to give interesting examples of Theorem 1.1.

For this, we use results of Dicks-Schick \[\square\] and the following definition:

1.2. **Definition.** Let $G$ be a group which acts on a tree. We say $G$ is **subduced** by a group $H$ if there is a homomorphism $\alpha: G \to H$ whose restriction $\alpha|_{G_v}$ to the stabilizer group $G_v := \{ g \in G \mid gv = v \}$ is injective for each vertex $v$ of the tree.

Remember that one can translate between groups acting on trees and fundamental groups of graphs in such a way that conjugacy classes of vertex stabilizers correspond to vertex groups.
1.3. Example. (1) If $G = H *_U H$, then $G$ is the fundamental group of a graph consisting of two vertices joined by an edge. Hence there is an action of $G$ on a tree such that the stabilizer group of every vertex is conjugate to one of the two copies of $H$. Using the obvious projection $G \to H$ which is the identity on both factors we see that $G$ is subdued by $H$.

(2) Assume $U < H$ and $g \in H$. Let $\phi: U \to U^g$ be given by conjugation with $g$. Let $G$ be the HNN-extension of $H$ along $\phi$. Then $G$ is the fundamental group of a graph of groups with one vertex (the vertex group being $H$) and therefore acts on a tree such that each vertex stabilizer is conjugate to $H$. We define a homomorphism $\alpha: G \to H$ such that the restriction of $\alpha$ to $H$ is the identity and maps the stable letter $t$ to $g$. This implies that $G$ is subdued by $H$.

1.4. Theorem. Assume $G$ is the fundamental group of a graph of groups which is subdued by another group $H$. Then

$$\text{tr}_G (K_0(C^*_r[G])) \subset \text{tr}_H (K_0(C^*_r[H])).$$

In particular, this applies to the situations of Example 1.3.

If $\text{tr}_H (K_0(C^*_r[H])) \subset \mathbb{Z}$ then this implies the trace conjecture for $G$. Baum and Connes conjecture [2, p.32] that for groups with torsion the range should be contained in $\mathbb{Q}$. Our results support also this assertion (Baum and Connes make in fact a more precise and stronger conjecture, which however was disproved by Roy [17]).

1.5. Remark. In some cases, it is possible to derive the conclusions of Theorem 1.4 from elaborate K-theory calculations. One can use the exact sequence for the fundamental group of a graph of groups [16, Theorem 18]. In some cases elementary properties of the trace then imply Theorem 1.4.

One might hope to give a general treatment of the range of the traces in these cases as is done for certain HNN-extensions in [15] and [8]. However, even in the case of HNN-extensions, in general those results are difficult to interpret and it is not clear that [15] or [8] implies Example 1.3 (1).

Moreover, observe that Pimsner uses deep KK-theoretic methods to derive the exact sequence for a graph of groups. In contrast, our derivation is elementary.

1.6. Remark. To apply Theorem 1.4, essentially we have to know the trace conjecture for $H$. The obvious sufficient condition is that $H$ fulfills the Baum-Connes conjecture, but it is also enough that $H$ is a subgroup of such a group.

If $H$ satisfies the Baum-Connes conjecture with coefficients, then the same is true for each group $G(v)$ (since they are subgroups of $H$), hence by
\[ G \text{ fulfills the Baum-Connes conjecture, and the statement of Theorem 1.4 follows also immediately from this fact.} \]

However, these arguments do not apply to the Baum-Connes conjecture without coefficients. Lafforgue proves that cocompact discrete subgroups, e.g. of \( SL_3(\mathbb{R}) \) or \( Sp(n,1) \) satisfy the Baum-Connes conjecture without coefficients \[ \text{[11]} \]. However, it is unknown whether the Baum-Connes conjecture with coefficients is true for these groups. Therefore, the consequences of Theorem 1.4 are not included in the knowledge about the Baum-Connes conjecture for a non-cocompact subgroup \( H \) of \( SL_3(\mathbb{R}) \) or \( Sp(n,1) \) contained in a cocompact torsion-free subgroup.

In particular, for such an \( H \) every \( H \ast U \) fulfills the trace conjecture, but it is not clear whether it fulfills the Baum-Connes conjecture.

The method described in \[ \text{[7]} \] was used by Linnell \[ \text{[12, 13]} \] to prove the Atiyah conjecture about the integrality of \( L^2 \)-Betti numbers for free groups, and starting with this for a lot of other groups. We investigate the Atiyah conjecture and obtain generalizations of Linnell’s results in \[ \text{[18, 5]} \].

\section{The trace conjecture for the K-theory of group \( C^* \)-algebras}

In this section we prove Theorem 1.1 and Theorem 1.4.

We first show how Theorem 1.4 follows from Theorem 1.1. The method for this is developed by Dicks and Schick in \[ \text{[5]} \]. For the convenience of the reader we repeat the easy proof of the special case we are concerned with here.

\textit{Proof of Theorem 1.4.} Assume \( G \) acts on the tree \( T \) with set of vertices \( V \) and set of edges \( E \). We choose an arbitrary \( v_0 \in V \).

Let \{\ast\} be a trivial \( G \)-set. Let \( \tilde{\phi} : V \to E \cup \{\ast\} \) denote the map which assigns to each \( v \in V \) the last edge in the \( T \)-geodesic from \( v_0 \) to \( v \), where this is taken to be \( \ast \) if \( v = v_0 \). By Julg-Valette \[ \text{[9]} \], \( \tilde{\phi} \) is bijective, and for all \( v \in V, g \in G \), we have \( \tilde{\phi}(gv) = g\tilde{\phi}(v) \) if and only if \( v \) is not in the \( T \)-geodesic from \( v_0 \) to \( g^{-1}v_0 \).

Define now \( \Delta := V \times H, \Omega' := E \times H \) and \( \Omega := (E \amalg \{\ast\}) \times H \). We define the \( G \) and the \( H \)-action on \( \Delta \) and \( \Omega' \) be setting

\[ g(x, u)h := (gx, \alpha(g)uh) \quad \forall g \in G, \ x \in T, u, h \in H. \]

Since the restriction of \( \alpha \) to each stabilizer group is injective, this is a free \( G \)- and of course also a free \( H \)-action, and they commute. Extend the action to \( \Omega \) by \( g(\ast, u)h = (\ast, uh) \) for \( g \in G \) and \( u, h \in H \).

We define \( \phi : \Delta = V \times H \to \Omega = (E \amalg \{\ast\}) \times H \) by

\[ \phi(v, u) = (\tilde{\phi}(v), u). \]
By linearity we get
\[ G \]
Since the actions of \( g \) each \( t \)
\[ H \]
union of the finitely many \( H \)-orbits of \( \Delta \) determined by the \( T \)-geodesic from \( v_0 \) to \( g^{-1}v_0 \). Theorem 1.4 now follows from Theorem 1.1.

\[ \Box \]

We are now going to prove Theorem 1.4. We use the language of Hilbert \( A \)-modules (for a \( C^* \)-algebra \( A \)), compare e.g. [3, Section 13].

2.1. Definition. Let \( T \) be a free set of generators of the free \( H \)-set \( \Delta \). Then we can form the Hilbert \( C^*_r H \)-module \( E := l^2(T) \otimes_{\mathbb{C}} C^*_r H \). Of course, \( E \) is nothing but the Hilbert sum of copies of \( C^*_r H \) indexed by \( T \) (in the sense of Hilbert \( C^*_r H \)-modules). Moreover, \( E \otimes_{C^*_r H} l^2 H \cong l^2(\Delta) \), and \( E \) is in a natural way a subset of \( l^2(\Delta) \) because \( C^*_r H \subset l^2 H \). The module \( E \) as a subset of \( l^2(\Delta) \) does not depend on the choice of the choice of \( T \).

Let \( B(E) \) be the set of bounded adjointable Hilbert \( C^*_r H \)-module homomorphisms. The map
\[ B(E) \to B(l^2(\Delta)): A \mapsto A_{\Delta} := A \otimes 1 \]
is an injective algebra homomorphism by [3, p. 111], therefore an isometric injection of \( C^* \)-algebras [10, 4.1.9]. Observe that the image of \( B(E) \) commutes with the right action of \( H \) on \( l^2(\Delta) \) and therefore is contained in the corresponding von Neumann algebra with its canonical trace. We will prove in Lemma 2.2 that \( CG \subset B(E) \). It follows that the closure of \( CG \) in \( B(E) \) and \( B(l^2(\Delta)) \) coincides. Since \( \Delta \) is a free \( G \)-set, this closure is isomorphic to \( C^*_r G \).

2.2. Lemma. We have a natural inclusion \( CG \subset B(E) \) which extends the action of \( G \) on \( \mathbb{C} \Delta \subset E \), in particular it is compatible with the injection \( CG \subset B(l^2 \Delta) \).

Proof. First observe that \( E \) is a closure of the algebraic tensor product of \( l^2(T) \) and \( C^*_r H \). Moreover, \( \mathbb{C}[T] \) is dense in \( l^2(T) \) and \( \mathbb{C} H \) is dense in \( C^*_r H \). Therefore \( \mathbb{C} \Delta = \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C} H \) is a dense subset of \( E \).

Now fix \( g \in G \). Since \( \Delta = \bigcup_{t \in T} tH \), where the union is disjoint, for each \( t \in T \) we get unique elements \( t_{g,t} \in T \), \( h_{g,t} \in H \) such that \( gt = t_{g,t}h_{g,t} \). Since the actions of \( G \) and \( H \) commute, \( gt = gt'h \) implies \( t = t' \). The map \( \alpha_g: T \to T; t \mapsto t_{g,t} \) therefore is a bijection. Pick
\[ x = \sum_{t \in T} tv_t, \quad x' = \sum_{t \in T} tv'_t \in \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C} H, \quad \text{with} \quad v_t, v'_t \in \mathbb{C} H. \]
By linearity we get
\[ gx = \sum_{t \in T} t_{g,t}h_{g,t}v_t \quad \text{and} \quad gx' = \sum_{t \in T} t_{g,t}h_{g,t}v'_t. \]
As is the convention in the theory of Hilbert $A$-modules, all of our inner products are linear in the second variable. Taking now the $C^*_r H$-valued inner product we get (adjoint and products of elements in $CH \subset C^*_r H$ are taken in the sense of the $C^*$-algebra)

$$\langle x, x' \rangle_{C^*_r H} = \sum_{t \in T} v_t^* v_t' = \sum_{t \in T} (h_{g,t} v_t)^* (h_{g,t} v_t')$$

$$= \sum_{t \in T} (h_{g, a^{-1}_g(t) v_{a^{-1}_g(t)}^*})^* (h_{g, a^{-1}_g(t)} v_t') = \langle gx, gx' \rangle_{C^*_r H}$$

(here we used that $H$ acts unitarily, i.e. $h_{g,t}^* = h_{g,t}^{-1}$). Hence $G$ acts $C^*_r H$-isometrically and this action extends to $E$. By linearity we get an $*$-algebra homomorphism $\mathbb{C}G \to \mathcal{B}(E) \to \mathcal{B}(l^2 \Delta)$. The composition is injective, therefore the same is true for the first map.

The above reasoning implies in the same way:

2.3. Lemma. For $n \in \mathbb{N}$ we have canonical injections of $*$-algebras

$$M_n(\mathbb{C}G) \subset M_n(C^*_r G) \subset \mathcal{B}(E^n) \subset \mathcal{B}(l^2 \Delta^n)^H,$$

where $\mathcal{B}(l^2 \Delta^n)^H$ denotes the operators which commute with the right action of $H$.

We have to compute the $G$-trace of operators in $M_n(C^*_r G)$, and we want to express this in terms of the $H$-trace of a suitable other operator. For this end, we repeat the following definition:

2.4. Definition. An operator $A \in \mathcal{B}(E)$ is of $H$-trace class if its image $A_{l^2 \Delta} \in \mathcal{B}(l^2 (\Delta))^H$ is of $H$-trace class in the sense of the von Neumann algebra (compare e.g. [19, 2.1]), i.e. if (with $|A| = \sqrt{AA^*}$)

$$\sum_{t \in T} \langle t, |A_{l^2 \Delta}| t \rangle_{l^2 (\Delta)} < \infty.$$

Then also $\sum_{t \in T} \langle t, A_{l^2 \Delta} t \rangle_{l^2 (\Delta)}$ converges and we set

$$\text{tr}_H(A) := \sum_{t \in T} \langle t, A_{l^2 \Delta} t \rangle_{l^2 (\Delta)}.$$

For $A = (A_{ij}) \in \mathcal{B}(E^n) = M_n(\mathcal{B}(E))$ we set

$$\text{tr}_H(A) = \sum_{i=1}^n \text{tr}_H(A_{ii}),$$

if $|A_{l^2 \Delta}|$ of $H$-trace class (with the obvious definition for this).
2.5. Lemma. Let $A, B, C \in B(E^n)$ and $A$ be of $H$-trace class. The trace class operators form an ideal inside $B(E^n)$ and we have $\text{tr}_H(AB) = \text{tr}_H(BA)$. Moreover

$$|\text{tr}_H(A + B)| \leq \text{tr}_H(|A|) + \text{tr}_H(|B|)$$

$$\text{tr}_H(|CAB|) \leq \|C\| \cdot \|B\| \cdot \text{tr}_H(|A|).$$

Proof. These are standard properties of the (von Neumann) trace, compare [19, 2.3], [6, Théorème 8 and Corollaire 2 on p. 106].

Set $S := \phi(T)$. This is an $H$-basis for the free $H$-set $\Omega$. Similarly to $E$ we can build $F := l^2(S) \otimes _CC_r^*H \subset l^2(\Omega)$. If $S' := S \cap \Omega'$ and $S'' := S \cap X$ (i.e. $S'$ is an $H$-basis for $\Omega'$ and $S''$ is an $H$-basis for $X$) then with $F' := l^2(S') \otimes _CC_r^*H$ and $F'' := l^2(S'') \otimes _CC_r^*H$ we get a direct sum decomposition of Hilbert $C_r^*H$-modules

$$F = F' \oplus F''.$$

As in the case of $E$ we get an canonical inclusion

$$C_rG \subset C_r^*G \subset B(F') \subset B(F) \subset B(l^2\Omega)$$

(we extend the action of $C_r^*G$ to all of $F$ by setting it zero on $F''$). This composition is a non-unital $*$-algebra homomorphism.

Corresponding statements hold for matrices.

Denote the image of $A \in M_n(C_r^*G)$ in $B(E^n)$ with $A_{\Delta}$ and in $B(F^n)$ with $A_{\Omega}$. We therefore have $A_{\Omega} = A_{\Omega'} \oplus 0$ with the obvious notation.

The bijection $\phi: \Delta \rightarrow \Omega$ induces a unitary map of Hilbert spaces $\phi: l^2(\Delta)^n \rightarrow l^2(\Omega)^n$. Since $\phi: \Delta \rightarrow \Omega$ is $H$-equivariant, the same is true for the unitary map. Moreover, we get a Hilbert $C_r^*H$-module unitary map $\phi: E^n \rightarrow F^n$.

One key observation is now (this is an extension of the corresponding observation in the classical proof for the free group):

2.6. Lemma. Suppose $A \in M_n(C_r^*G) \subset B(E^n)$ is such that the Hilbert $C_r^*H$-module morphism $A_{\Delta} - \phi^*A_{\Omega}\phi: E^n \rightarrow E^n$ is of $H$-trace class. Then

$$\text{tr}_G(A) = \frac{1}{r} \text{tr}_H(A_{\Delta} - \phi^*A_{\Omega}\phi).$$

Proof. Observe that $\phi$ is diagonal and traces are the sum over the diagonal entries. Therefore we may assume that $n = 1$. Since $\Delta$ is a free $G$-module, for every $x \in \Delta$ (which we identify with the element of $l^2(\Delta)$ which is 1 at $x$ and zero everywhere else) we have

$$\langle x, A_{\Delta}x \rangle_{l^2(\Delta)} = \text{tr}_G(A)$$
(simply identify $Gx$ with $G$ and the left and right hand side become identical). Similarly, since $A_{\Omega} = A_{\Omega'} \oplus 0$ on $l^2(\Omega') \oplus l^2(X)\oplus l^2(X)\oplus l^2(X)$

$$\langle \phi x, A_{\Omega} \phi x \rangle_{l^2(\Omega)} = \begin{cases} \text{tr}_G(A) & \text{if } \phi(x) \in \Omega' \\
0 & \text{if } \phi(x) \in X. \end{cases}$$

Moreover

$$\text{tr}_H(A_{\Delta} - \phi^* A_{\Delta} \phi) = \sum_{t \in T} \langle t, A_{\Delta} t \rangle |t^2(\Delta) - \langle t, \phi^* A_{\Delta} \phi(t) \rangle |t^2(\Delta)$$

$$= \sum_{\phi(t) \in X \cap S^0} \text{tr}_G(A) = r \text{tr}_G(A)$$

since $|X \cap S^0|$ is the number of $H$-orbits in $X$, i.e. $r$. All other summands cancel each other out. \qed

Because $K(C^*_r G)$ is generated by projections $P \in M_n(C^*_r G) \subset M_n(NG)$ and the trace we have to compute is exactly $\text{tr}_G(P)$, we are tempted to apply Lemma 2.6 to such a $P$. A problem is that it is hard to check whether the trace class condition is fulfilled in general.

To circumvent these difficulties recall the following fact (compare e.g. [4, III.3, Proposition 3]):

2.7. Proposition. Let $B$ be a $C^*$-algebra and $U \subset B$ a dense $*$-subalgebra that is closed under holomorphic functional calculus. Then the inclusion induces an isomorphism

$$K(U) \cong K(B).$$

In particular, if $B = C^*_r G$ and $U$ is closed under holomorphic functional calculus and contains $CG$, then the ranges of the canonical trace applied to $K(C^*_r G)$ and $K(U)$ coincide.

As algebra $U$ we will use the closure under holomorphic functional calculus of $CG \subset B(E)$. This of course fulfills the conditions of Proposition 2.7. It remains to check:

2.8. Lemma. Let $x \in M_n(U) \subset M_n(C^*_r G) \subset B(E^n)$, where $U$ is the closure under holomorphic functional calculus of $CG$ in $C^*_r G$. Then $x - \phi^* x \phi: E \to E$ is of $H$-trace class and $C^*_r H$-compact.

Proof. Start with $g \in G \subset CG$. Since $R_g$ as defined in Theorem 1.1 is contained in finitely many $H$-orbits, $gt = \phi^* g \phi t$ for all but a finitely many $t \in T$. In particular $g - \phi^* g \phi$ is zero outside the $C^*_r H$-submodule of $E$ spanned by this finite number of elements of $T$, and is nonzero only on the complement, which is isomorphic to $(C^*_r H)^N$ for some $N \in \mathbb{N}$. Since $id: C^*_r H^N \to C^*_r H^N$ is of finite rank in the sense of Hilbert $C^*_r H$-module
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morphisms, the same is true for \( g - \phi^* g \phi \). Finite rank operators form a subspace, therefore the same is true if we replace \( g \) by \( v \in CG \subset B(E) \). Passage to finite matrices does preserve the finite rank property. Finite rank implies \( H \)-trace class and \( C^*_f H \)-compactness. In particular, all operators in \( M_n(CG) \) give rise to \( C^*_f H \)-compact operators, which also are of \( H \)-trace class. The map \( x \mapsto x - \phi^* x \phi \) is norm continuous and the compact operators form a closed ideal, therefore \( x - \phi^* x \phi \) is compact even for arbitrary \( x \in M_n(C^*_f G) \).

Assume \( A \in M_n(C^*_f H) \) and \( 0 \neq \xi \notin \text{Spec}(A) \). Since the homomorphism \( A \mapsto A_\Delta \) is unital, we get

\[
((\xi - A)^{-1})_\Delta = (\xi - A_\Delta)^{-1}.
\]

Similarly \( ((\xi - A)^{-1})_\Omega' = (\xi - A_{\Omega'})^{-1} \). Consequently

\[
((\xi - A)^{-1})_\Omega = ((\xi - A)^{-1})_\Omega' \oplus 0 = (\xi - A_\Omega)^{-1} - \xi^{-1} P
\]

where \( P : F \to F \) is the projection onto \( F'' \). Note that \( (\xi - A_\Omega)^{-1} \) acts by multiplication with \( \xi^{-1} \) on \( F'' \). Here we need the assumption \( \xi \neq 0 \). Since \( T'' \) is finite, \( P \) is of finite rank as Hilbert \( C^*_f H \)-module morphism.

Suppose now \( f \) is a function that is holomorphic in a neighborhood of \( \text{Spec}(A) \). Let \( \Gamma \) be a loop around \( \text{Spec}(A) \) and choose \( \Gamma \) so that it does not meet \( 0 \in C \). Then

\[
f(A)_\Delta = \left( \int_\Gamma f(\xi)(\xi - A)^{-1} d\xi \right)_\Delta = \int_\Gamma f(\xi)(\xi - A_\Delta)^{-1} d\xi
\]

\[
f(A)_\Omega = \int_\Gamma f(\xi) ((\xi - A_\Omega)^{-1} - \xi^{-1} P) \, d\xi.
\]

For \( u, v \in B(E) \) and \( \xi \notin \text{Spec}(u) \cup \text{Spec}(v) \) we have

\[
(\xi - u)^{-1} - (\xi - v)^{-1} = (\xi - u)^{-1}(\xi - v - (\xi - u))(\xi - v)^{-1}
\]

\[
= (\xi - u)^{-1}(u - v)(\xi - v)^{-1}.
\]

Therefore

\[
f(A)_\Delta - \phi^* f(A)_\Omega \phi = \phi^* P \phi \int_\Gamma f(\xi) \xi^{-1} d\xi
\]

\[
+ \int_\Gamma \underbrace{f(\xi)(\xi - A_\Delta)^{-1}}_{=: f_\Delta(\xi)} \underbrace{\left( \Omega A - \phi^* A_\Omega \phi \right)}_{=: A_0} \underbrace{\left( \Omega - \phi^* A_\Omega \phi \right)^{-1}}_{=: f_\Omega(\xi)} \, d\xi.
\]

As a consequence of Lemma 2.3 we have

\[
\text{tr}_H \left| \int_\Gamma f_\Delta(\xi) A_0 f_\Omega(\xi) \, d\xi \right| \leq \int_\Gamma \text{tr}_H |f_\Delta(\xi) A_0 f_\Omega(\xi)| \, d\xi
\]

\[
\leq \int \|f_\Delta(\xi)\| \cdot \|f_\Omega(\xi)\| \cdot \text{tr}_H(\|A_0\|).
\]
Since the operator valued functions $f_{\Delta}$ and $f_{\Omega}$ are norm-continuous, $f(A)_{\Delta} - \phi^*f(A)_{\Omega}\phi$ is of $H$-trace class if $A_0$ is of $H$-trace class, in particular if $A \in M^\infty(CG)$. This concludes the proof. \hfill \Box

We determine now the range of the trace on the dense and holomorphically closed subalgebra $U$ of $C^*_r G$. Since the trace class condition is fulfilled, it only remains to calculate $\text{tr}_H(P_{\Delta} - \phi^*P_{\Omega}\phi)$ for a projection over $U$, and Theorem 1.1 follows from Lemma 2.6.

2.9. Lemma. Let $E$ be the Hilbert $C^*_r H$-module introduced in Definition 2.4 and $P, Q \in B(E)$ be projections such that $P - Q$ is of $H$-trace class and compact in the sense of Hilbert $C^*_r H$-module morphisms. Then

\[ w := \left( (PE \oplus QE), 1, \begin{pmatrix} 0 & FQ \\ QP & 0 \end{pmatrix} \right) \]

is a Kasparov triple (in the sense of [3, 17.1.1]) representing an element in $KK(\mathbb{C}, C^*_r H) \cong K_0(C^*_r H)$ and

\[ \text{tr}_H(P - Q) = \text{ind}_H(w) = \text{tr}_H([w]), \]

where $\text{tr}_H(P - Q)$ is to be understood in the sense of Definition 2.4, whereas $\text{tr}_H([w])$ is the canonical trace defined on $K_0(C^*_r H)$.

Proof. Using Lemma 2.5 and the fact that $P^2 = P, Q^2 = Q, (1 - P)P = 0$, and $\text{tr}_H(XY) = \text{tr}_H(YX)$, we conclude

\[ \text{tr}_H(P - Q) = \text{tr}_H(P^2(P - Q)) + \text{tr}_H((1 - P)(P - Q^2)) \]
\[ = \text{tr}_H(P(P - Q)P) - \text{tr}_H(Q(1 - P)Q) \]
\[ = \text{tr}_H(P - PQP) - \text{tr}_H(Q - QPQ). \]

Let $\alpha_P : PE \to PE$ be the orthogonal projection with image $PE \cap \ker(QP)$, and $\alpha_Q : QE \to QE$ the orthogonal projection with image $QE \cap \ker(PQ)$. Observe that $\alpha_P = (P - PQP)\alpha_P$. Therefore $\alpha_P$ is of $H$-trace class, since the same is true for $P - PQP$. In the same way we see that $\alpha_Q$ is of $H$-trace class. Set

\[ T_0 := \text{id}_{PE} - PQP - \alpha_P : PE \to PE \]
\[ T_1 := \text{id}_{QE} - QPQ - \alpha_Q : QE \to QE. \]

Then

\[ \text{tr}_H(P - Q) = \text{tr}_H(P - PQP) - \text{tr}_H(Q - QPQ) \]
\[ = \text{tr}_H(T_0) - \text{tr}_H(T_1) + \text{tr}_H(\alpha_P) - \text{tr}_H(\alpha_Q). \] (2.10)

Now $QP : PE \to QE$ is a bounded operator with adjoint $PQ : QE \to PE$, and

\[ \text{tr}_H(\alpha_P) = \dim_H(\ker(QP : PE \to QE)) \]
\[ \text{tr}_H(\alpha_Q) = \dim_H(\ker(PQ : QE \to PE)) \]
\[ = \dim_H(\text{coker}(QP : PE \to QE)). \] (2.11)
For a complemented submodule $X$, one defines $\dim_H(X) := \text{tr}_H(\text{pr}_X)$, where $\text{pr}_X$ is the orthogonal projection onto $X$. Since $QP\alpha P = 0$ and $PQ\alpha Q = 0$, and the latter implies $0 = (PQ\alpha Q)^* = \alpha QQP$, we have

$$QPT_0 = QP(id_{PE} - PQP - \alpha P) = \text{id}_{QE} QP - PQP^2 P - \alpha QQP = T_1 QP.$$ 

Moreover, $\ker(QP: PE \to QE) \subset \ker(T_0)$, since $QP(Px) = 0$ implies $\alpha P(Px) = Px$, and in the same way we conclude $\ker(PQ) = \ker((PQ)^*) \subset \ker(T_1) = \ker(T_1^*)$. It follows that $QP$ “conjugates” $T_0$ and $T_1$, and by [19, Proposition 2.6] (which goes back to a corresponding result in [1, p. 67]) that

$$\text{tr}_H(T_0) = \text{tr}_H(T_1).$$

Using Equation (2.10) and Equation (2.11) we arrive at

$$\text{tr}_H(P - Q) = \text{ind}_H(QP: PE \to QE)$$

with the obvious definition of $\text{ind}_H$. This is exactly the $H$-index in the graded sense of the operator $F := \begin{pmatrix} 0 & PQ \\ PQ & 0 \end{pmatrix}: PE \oplus QE \to PE \oplus QE$ (where $PE$ is the positive and $QE$ the negative part of the graded Hilbert $C^*_r H$-module $PE \oplus QE$).

The only thing it remains to check is whether $w$ fulfills all the axioms of Kasparov triples. Since the action of $\mathbb{C}$ is unital and the operator is self adjoint, this amounts to check that $1 - F^* F$ and $1 - FF^*$ are compact in the sense of Hilbert $C^*_r H$-module morphisms. Now $F^* F = F^2 = FF^* = \begin{pmatrix} PQP & 0 \\ 0 & \alpha QQP \end{pmatrix}$. Since $P - Q$ is compact, the same is true for $P(P - Q)P = P - PQP: E \to E$. Then also the composition with the inclusion of $PE$ into $E$ and the projection $P: E \to PE$ is compact. This operator coincides with $1 - PQP: PE \to PE$. Similarly $1 - QPQ: QE \to QE$ is compact. This concludes the proof.

To finish the proof of Theorem 1.1 observe that by Proposition 2.7 it suffices to compute $\text{tr}_G(P)$ if $P \in M_n(U)$ is a projection, where $U$ is the holomorphic closure of $\mathbb{C}G \subset C^*_r G$. Since $A \to A_\Delta$ and $A \to A_\Omega$ are $*$-algebra homomorphisms, $P_\Delta$ and $P_\Omega$ are projections. Now Lemma 2.3 implies that we can apply Lemma 2.9 to $P_\Delta - \phi^* P_\Omega \phi$. By assumption $\text{tr}_H(K_0(C^*_r H)) \subset A$, therefore $\text{tr}_H(P_\Delta - \phi^* P_\Omega \phi) \in A$. By Lemma 2.4 then $\text{tr}_G(P) \in \frac{1}{2} A$, and this concludes the proof of Theorem 1.1.

2.12. Remark. We use the language of Hilbert modules and Kasparov triples only for convenience. Observe that we don’t use much more than the definition: the single theorem we use is that our Kasparov triples indeed give rise to K-theory elements, and this is not very deep. By [2, 17.5.5] $KK(\mathbb{C}, C^*_r G)$
and $K_0(C^*_rG)$ are isomorphic, but to construct the map much less is needed.
(Essentially we only have to perturb $QP$ such that kernel and cokernel are
finitely generated projective modules over $C^*_rG \otimes \mathbb{K}$.)

3 Final remarks

We hope that Proposition 1.1 can be applied to more situations than the
one described in Theorem 1.4. However, in [18] the situation where $H$
is trivial (and consequently $X$ is finite) is classified. It turns out that in this
setting the assumptions of Theorem 1.1 can be fulfilled exactly if $G$ is a
finite extension of a free group. But then one has a transfer homomorphism
for the $K$-theory of the reduced $C^*$-algebras relating the trace for $G$ to
the trace of the free subgroup of finite index. One easily computes the range
of the trace using this (and the known trace conjecture for the free group).
The range is $\frac{d}{2} \mathbb{Z}$ where $d$ is the smallest index of a free subgroup. Therefore
it is not necessary to give details of the approach using Theorem 1.1 which
gives the same result.

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