EXTENDED DECAY PROPERTIES FOR GENERALIZED BBM EQUATIONS

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Abstract. In this note we show that all small solutions of the BBM equation must decay to zero as $t \to +\infty$ in large portions of the physical space, extending previous known results, and only assuming data in the energy space. Our results also include decay on the left portion of the physical line, unlike the standard KdV dynamics.

1. Introduction and Main Results

1.1. Setting of the problem. In this note we shall consider nonlinear scattering and decay properties for the one-dimensional generalized Benjamin, Bona and Mahony (gBBM) equation [4] (or regularized long wave equation) in the energy space:

$$(1 - \partial_x^2)u_t + (u + u^p)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad p = 2, 3, 4, \ldots$$

Here $u = u(t, x)$ is a real-valued scalar function. The original BBM equation, which is the case $p = 2$ above, was originally derived by Benjamin, Bona and Mahony [4] and Peregrine [32] as a model for the uni-directional propagation of long-crested, surface water waves. It also arises mathematically as a regularized version of the KdV equation, obtained by performing the standard "Boussinesq trick". This leads to simpler well-posedness and better dynamical properties compared with the original KdV equation. Moreover, BBM is not integrable, unlike KdV [10, 26].

It is well-known (see [11]) that (1.1) for $p = 2$ is globally well-posed in $H^s$, $s \geq 0$, and weakly ill-posed for $s < 0$. As for the remaining cases $p = 3, 4, \ldots$, gBBM is globally well-posed in $H^1$ [4], thanks to the preservation of the mass and energy

$$(1.2) \quad M[u](t) := \frac{1}{2} \int (u^2 + u_x^2) (t, x) dx,$$

$$(1.3) \quad E[u](t) := \int \left( \frac{1}{2} u^2 + \frac{u^{p+1}}{p+1} \right) (t, x) dx.$$

Since now, we will identify $H^1$ as the standard energy space for (1.1).

1.2. Main result. In this note we consider the problem of decay for small solutions to gBBM (1.1). Let $b > 0$ and $a > \frac{1}{s}$ be any positive numbers, and $I(t)$ be given by

$$(1.4) \quad I(t) := (-\infty, -at) \cup ((1 + b)t, \infty), \quad t > 0.$$

Theorem 1.1. Let $u_0 \in H^1$ be such that, for some $\varepsilon = \varepsilon(b) > 0$ small, one has

$$(1.5) \quad \|u_0\|_{H^1} < \varepsilon.$$

Let $u \in C(\mathbb{R}, H^1)$ be the corresponding global (small) solution of (1.1) with initial data $u(t = 0) = u_0$. Then, for $I(t)$ as in (1.4), there is strong decay to zero:

$$(1.6) \quad \lim_{t \to \infty} \|u(t)\|_{H^1(I(t))} = 0.$$

Additionally, one has the mild rate of decay

$$(1.7) \quad \int_2^\infty \int e^{-c_0 |x+\sigma t|} (u^2 + u_x^2) (t, x) dxd\sigma \lesssim c_0 \varepsilon^2,$$

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where $\sigma$ is fixed and such that $\sigma > \frac{1}{8}$ or $\sigma = -(1 + b)$.

**Remark 1.1.** The case of decay inside the interval $((1 + b)t, +\infty)$ is probably well-known in the literature, coming from arguments similar to those exposed by El-Dika and Martel in [15]. However, decay for the left portion $(-\infty, -\frac{1}{8} t)$ seems completely new as far as we understand, and it is in strong contrast with the similar decay problem for the KdV equation on the left, which has not been rigorously proved yet.

**Remark 1.2.** Note that our results also consider the cases $p = 2$ and $p = 3$, which are difficult to attain using standard scattering techniques because of very weak linear decay estimates, and the presence of long range nonlinearities. Recall that the standard linear decay estimates for BMM are $O(t^{-1/3})$ [1].

**Remark 1.3.** Theorem 1.1 is in concordance with the existence of solitary waves for (1.1) [15]. Indeed, for any $c > 1$,

$$u(t, x) := (c - 1)^{1/(p - 1)} Q \left( \sqrt{\frac{c - 1}{c}} (x - ct) \right), \quad Q(s) := \left( \frac{p + 1}{2 \cosh^2(\frac{1}{2c} s)} \right)^{1/(p - 1)},$$

is a solitary wave solution of (1.1), moving to the right with speed $c > 1$. Small solitary waves in the energy space have $c \sim 1$ ($p < 5$), which explains the emergence of the coefficient $b$ in (1.4). Also, (1.1) has solitary waves with negative speed: for $c > 0$ and $p$ even,

$$u(t, x) := -(c + 1)^{1/(p - 1)} Q \left( \sqrt{\frac{c + 1}{c}} (x + ct) \right),$$

is solitary wave for (1.1), but it is never small in the energy space. The stability problem for these solitary waves is well-known: it was studied in [6, 34, 33, 9]. Indeed, solitary waves are stable for $p = 2, 3, 4, 5$, and stable/unstable for $p > 5$, depending on the speed $c$. See also [26] for the study of the collision problem for $p = 2$.

**Remark 1.4.** The extension of this result to the case of perturbations of solitary waves is an interesting open problem, which will be treated elsewhere.

### 1.3 About the literature.

Albert [1] showed scattering in the $L^\infty$ norm for solutions of (1.1) provided $p > 4$, with resulting global decay $O(t^{-1/3})$. Here the power 4 is important to close the nonlinear estimates, based in weighted Sobolev and Lebesgue spaces. Biler et. al [5] showed decay in several space dimensions, using similar techniques. Hayashi and Naumkin [17] considered BBM with a diffusion term, proving asymptotics for small solutions. Our result improves [1, 17] in the sense that it also considers the cases $p = 2$ and 3, which are not part of the standard scattering theory, and it does not requires a damping term to be valid.

Concerning asymptotic regimes around solitary waves, the fundamental work of Miller and Weinstein [28] showed asymptotic stability of the BBM solitary wave in exponentially weighted Sobolev spaces. El-Dika [13, 14] proved asymptotic stability properties of the BBM solitary wave in the energy space. El-Dika and Martel [15] showed stability an asymptotic stability fo the sum of $N$ solitary waves. See also Mizumachi [31] for similar results. All these results are proved on the right of the main part of the solution itself, and no information is given on the remaining left part. Theorem 1.1 is new in the sense that it also gives information on the left portion of the space.

### 1.4 About the proof.

In order to prove Theorem 1.1, we follow the ideas of the proof described in [22], where decay for an $abcd$-Boussinesq system [12, 7, 8] was considered. The main tool in [22] was the construction of a suitable virial functional for which the dynamics is converging to zero when integrated in time. In this paper, this construction is somehow simpler but still interesting enough, because it allows to consider two different regions of the physical space, on the left (dispersive) and on the right (soliton region), unlike KdV for which virial estimates only reach the soliton region [23, 24, 25]. The virial that we use here is also partly inspired in the ones introduced in [18, 19, 20], and previously in [23, 27]. See also [2, 29, 16] for similar results.

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2. Proof of Theorem 1.1

Let $L > 0$ be large, and $\varphi = \varphi(x)$ be a smooth, bounded weight function, to be chosen later. For each $t, \sigma \in \mathbb{R}$, we consider the following functionals (see [15] for similar choices):

$$(2.1) \quad I(t) := \frac{1}{2} \int \varphi \left( \frac{x + \sigma t}{L} \right) \left( u^2 + u_x^2 \right)(t, x) dx,$$

and

$$(2.2) \quad J(t) := \int \varphi \left( \frac{x + \sigma t}{L} \right) \left( \frac{1}{2} u^2 + \frac{u^{p+1}}{p+1} \right)(t, x) dx.$$

Clearly each functional above is well-defined for $H^1$ functions. Using (1.1) and integration by parts, we have the following standard result (see also [15, 22] for similar computations).

**Lemma 2.1.** For any $t \in \mathbb{R}$, we have

$$(2.3) \quad \frac{d}{dt} I(t) = \frac{\sigma}{2L} \int \varphi' u_x^2 + \frac{1}{2L} (\sigma - 1) \int \varphi' u^2 + \frac{1}{L} \int \varphi' u(1 - \partial_x^2)^{-1} u$$

and if $v := (1 - \partial_x^2)^{-1} (u + w)$,

$$(2.4) \quad \frac{d}{dt} J(t) = \frac{\sigma}{2L} \int \varphi' \left( u^2 + \frac{2}{p+1} u^{p+1} \right) + \frac{1}{2L} \int \varphi' (v^2 - v_x^2).$$

**Proof.** Proof of (2.3). We compute:

$$\frac{d}{dt} I(t) = \frac{\sigma}{2L} \int \varphi' (u^2 + u_x^2) + \int \varphi (w_{tt} + w_{tx}).$$

Replacing (1.1), and integrating by parts, we get

$$\frac{d}{dt} I(t) = \frac{\sigma}{2L} \int \varphi' (u^2 + u_x^2) - \int \varphi u (u + w)_x + \frac{1}{L} \int (\varphi' u)_{xx} (1 - \partial_x^2)^{-1} (u + w)$$

$$= I_1 + I_2 + I_3.$$

$I_1$ is already done. On the other hand,

$$I_2 = - \int \varphi (w_{ux} + pw_{ux}) = \frac{1}{L} \int \varphi' \left( \frac{1}{2} u^2 + \frac{p}{p+1} u^{p+1} \right).$$

Finally,

$$I_3 = - \int \varphi (u + w)_x = \frac{1}{L} \int \varphi' (u^2 + u^{p+1})$$

and

$$= - \frac{1}{L} \int \varphi' (u^2 + u^{p+1}) + \frac{1}{L} \int \varphi' (1 - \partial_x^2)^{-1} (u + w).$$

We conclude that

$$\frac{d}{dt} I(t) = \frac{\sigma}{2L} \int \varphi' u_x^2 + \frac{1}{2L} (\sigma - 1) \int \varphi' u^2 + \frac{1}{L} \int \varphi' u(1 - \partial_x^2)^{-1} u$$

and

$$- \frac{1}{L(p+1)} \int \varphi' u^{p+1} + \frac{1}{L} \int \varphi' (1 - \partial_x^2)^{-1} (u + w).$$

This last equality proves (2.3).

**Proof of (2.4).** We compute:

$$\frac{d}{dt} J(t) = \frac{\sigma}{2L} \int \varphi' \left( u^2 + \frac{2}{p+1} u^{p+1} \right) + \int \varphi (u + w)u_t$$

and

$$= \frac{\sigma}{2L} \int \varphi' \left( u^2 + \frac{2}{p+1} u^{p+1} \right) - \int \varphi (u + w) \partial_x (1 - \partial_x^2)^{-1} (u + w).$$

Recall that $v = (1 - \partial_x^2)^{-1} (u + w)$. Then

$$- \int \varphi (u + w) \partial_x (1 - \partial_x^2)^{-1} (u + w) = - \int \varphi (1 - \partial_x^2) w v_x = \frac{1}{2L} \int \varphi' (v^2 - v_x^2).$$
Replacing these values in (2.8), we easily have
\[ H(t) := H_\alpha(t) := I(t) + \alpha J(t). \]
From Lemma 2.1, we get (recall that \( v = (1 - \partial_x^2)^{-1}(u + u^p) \))
\[ \frac{d}{dt} H(t) = \frac{\sigma}{2L} \int \varphi' u_x^2 + \frac{1}{2L} (\sigma(1 + \alpha) - 1) \int \varphi' u^2 + \frac{1}{2L} \int \varphi' u(1 - \partial_x^2)^{-1} u \\
+ \frac{\alpha}{2L} \int \varphi'(v^2 - v_x^2) - \frac{1}{(\alpha \sigma - 1)} \int \varphi' u^{p+1} + \frac{1}{L} \int \varphi' u(1 - \partial_x^2)^{-1} (u^p). \]
Let also, for \( u \in H^1 \),
\[ f := (1 - \partial_x^2)^{-1} u \in H^3. \]
We have
\[ \int \varphi' u_x^2 = \int \varphi' (f_x^2 + 2f_x^2) - \frac{1}{L^2} \int \varphi'' f^2, \]
\[ \int \varphi' u_x^2 = \int \varphi' (f_x^2 + 2f_x^2) - \frac{1}{L^2} \int \varphi'' f_x^2, \]
and
\[ \int \varphi' u(1 - \partial_x^2)^{-1} u = \int \varphi' (f_x^2 + f_x^2) - \frac{1}{2L^2} \int \varphi'' f^2. \]
Additionally, we easily have
\[ v^2 = f^2 + 2f(1 - \partial_x^2)^{-1}(u^p) + ((1 - \partial_x^2)^{-1} u^p)^2, \]
and similarly,
\[ v_x = f_x^2 + 2f_x(1 - \partial_x^2)^{-1}(u^p)_x + (\partial_x(1 - \partial_x^2)^{-1} u^p)^2. \]
Replacing these values in (2.6) and rearranging terms, we get
\[ \frac{d}{dt} H(t) = Q(t) + S(t) + N(t), \]
where
\[ Q(t) := \frac{\sigma}{2L} (1 + \sigma)(1 + \alpha) \int \varphi' f^2 + \frac{1}{2L} (\sigma(3 + 2\alpha) - \alpha) \int \varphi' f_x^2 \\
+ \frac{1}{2L} ((3 + \alpha)\sigma - 1) \int \varphi' f_x^2 + \frac{\sigma}{2L} \int \varphi' f_x^2 \]
\[ S(t) := -\frac{1}{2L^3} (\sigma(1 + \alpha) - 1) \int \varphi'' f^2 - \frac{\sigma}{2L^3} \int \varphi'' f_x^2 - \frac{1}{2L^3} \int \varphi'' f^2, \]
and
\[ N(t) := \frac{\alpha}{2L} \int \varphi' (2f(1 - \partial_x^2)^{-1}(u^p) + ((1 - \partial_x^2)^{-1} u^p)^2) \\
- \frac{\alpha}{2L} \int \varphi' (2f_x(1 - \partial_x^2)^{-1}(u^p)_x + (\partial_x(1 - \partial_x^2)^{-1} u^p)^2) \\
- \frac{1}{(\alpha \sigma - 1)} \int \varphi' u^{p+1} + \frac{1}{L} \int \varphi' u(1 - \partial_x^2)^{-1} (u^p). \]
Now we consider two different cases.

Case \( x > 0 \). This is the simpler case. We choose \( \varphi := \tanh \), \( \alpha = 0 \), and \( \sigma = -(1 + b) < 0 \), for \( b \) any fixed positive number. Note that \( \varphi' = \sech^2 > 0 \). Then
\[ Q(t) = -\frac{1}{2L} b \int \varphi' f^2 - \frac{3}{2L} (1 + b) \int \varphi' f_x^2 - \frac{1}{2L} (4 + 3b) \int \varphi' f_x^2 - \frac{1}{2L} (1 + b) \int \varphi' f_x^2. \]
Now we recall the following result.
Lemma 2.2 (Equivalence of local $H^1$ norms, [22]). Let $f$ be as in (2.7). Let $\phi$ be a smooth, bounded positive weight satisfying $|\phi''| \leq \lambda \phi$ for some small but fixed $0 < \lambda \ll 1$. Then, for any $a_1, a_2, a_3, a_4 > 0$, there exist $c_1, C_1 > 0$, depending on $a_j$ and $\lambda > 0$, such that

\begin{equation}
(2.14) \quad c_1 \int \phi(u^2 + u_x^2) \leq \int \phi(a_1 f^2 + a_2 f_x^2 + a_3 f_{xx}^2 + a_4 f_{xxx}^2) \leq C_1 \int \phi(u^2 + u_x^2).
\end{equation}

Thanks to this lemma, we get for this case

\begin{equation}
(2.15) \quad Q(t) \lesssim_{b, L} - \int \varphi'(f^2 + f_x^2 + f_{xx}^2 + f_{xxx}^2) \sim - \int \varphi'(u_x^2 + u^2).
\end{equation}

Case $x < 0$. Here we need different estimates. In (2.10), we will impose

$$\sigma = \frac{1}{8} (1 + \tilde{\sigma}), \quad \tilde{\sigma} > 0, \quad \text{and} \quad \alpha = 1.$$ 

We choose now $\varphi := - \tanh$. Note that $\varphi' = - \sech^2 < 0$. Then we have

$$-16LQ(t) = 2(9 + \tilde{\sigma}) \int |\varphi'| f^2 + (-3 + 5\tilde{\sigma}) \int |\varphi'| f_x^2 + 4(-1 + \tilde{\sigma}) \int |\varphi'| f_{xx}^2 + (1 + \tilde{\sigma}) \int |\varphi'| f_{xxx}^2.$$ 

Define $g := |\varphi'|^{1/2} f = \sech(x + \alpha t) f$. Then we have the following easy identities

$$g_x = \sech\left( \frac{x + \alpha t}{L} \right) f_x + \frac{1}{L} \sech'(\frac{x + \alpha t}{L}) f,$$

$$g_{xx} = \sech\left( \frac{x + \alpha t}{L} \right) f_{xx} + \frac{2}{L} \sech'(\frac{x + \alpha t}{L}) f_x + \frac{1}{L^2} \sech''\left( \frac{x + \alpha t}{L} \right) f,$$

and

$$g_{xxx} = \sech\left( \frac{x + \alpha t}{L} \right) f_{xxx} + \frac{3}{L} \sech'(\frac{x + \alpha t}{L}) f_{xx} + \frac{3}{L^2} \sech''\left( \frac{x + \alpha t}{L} \right) f_x + \frac{1}{L^3} \sech'''\left( \frac{x + \alpha t}{L} \right) f.$$ 

Therefore,

$$g_x = -\frac{1}{L^2} g - \frac{2}{L} \tanh\left( \frac{x + \alpha t}{L} \right) g_x + \sech\left( \frac{x + \alpha t}{L} \right) f_x$$

and

$$g_{xx} = -\frac{1}{L^2} \tanh\left( \frac{x + \alpha t}{L} \right) g - \frac{3}{L^2} g_{xx} - \frac{3}{L} \tanh\left( \frac{x + \alpha t}{L} \right) g_{xx} + \sech\left( \frac{x + \alpha t}{L} \right) f_{xxx}.$$ 

Consequently, for $L$ large enough,

$$-16LQ(t) = 2(9 + \tilde{\sigma}) \int g^2 + (-3 + 5\tilde{\sigma}) \int g_x^2 + 4(-1 + \tilde{\sigma}) \int g_{xx}^2 + O\left( \frac{1}{L} \int (g^2 + g_x^2 + g_{xx}^2) \right).$$ 

Now we have for $g \in H^3$,

$$\int (g_{xxx} - \sqrt{2} g_{xx} + 3 g_x - 3\sqrt{2}g)^2 \geq 0.$$ 

Expanding terms and integrating by parts,

$$\int g_{xxx}^2 - 4 \int g_{xx}^2 - 3 \int g_x^2 + 18 \int g^2 \geq 0.$$ 

We conclude that for $\tilde{\sigma} > 0$ fixed and $L$ large enough,

$$-16LQ(t) \geq \tilde{\sigma} \int g^2 + 4\tilde{\sigma} \int g_x^2 + 3\tilde{\sigma} \int g_{xx}^2 + \frac{1}{2} \tilde{\sigma} \int g_{xxx}^2.$$
Coming back to the variable $f$, we obtain for $L$ even larger if necessary,

$$-16LQ(t) \geq \frac{1}{2} \tilde{\sigma} \int |\varphi'| f^2 + 3 \tilde{\sigma} \int |\varphi'| f_x^2 + 2 \tilde{\sigma} \int |\varphi'| f_{xx}^2 + \frac{1}{4} \tilde{\sigma} \int |\varphi'| f_{xxx}^2,$$

Then we have

$$Q(t) \lesssim_{\sigma,L} - \int |\varphi'| (f^2 + f_x^2 + f_{xx}^2 + f_{xxx}^2) \sim - \int |\varphi'| (u_x^2 + u^2).$$

From (2.15) and (2.16) we conclude that

$$Q(t) \lesssim - \int |\varphi'| (u_x^2 + u^2),$$

provided $\sigma = -(1 + b)$, $b > 0$, or $\sigma > \frac{1}{2}$. The terms in (2.11) can be absorbed by this last term using $L > 0$ large and the fact that $|\varphi''| \lesssim |\varphi'|$. Finally, (2.12) can be absorbed by (2.17) using (1.5) (provided $\varepsilon$ is small enough compared with $b$), just as in [14, 22]. See Appendix A for more details. We get

$$\frac{d}{dt} \mathcal{H}(t) \lesssim - \int |\varphi'| (u_x^2 + u^2).$$

Therefore, we conclude that

$$\int_2^\infty \int \operatorname{sech}^{2} \left(\frac{x + \sigma t}{L}\right) (u^2 + u_x^2) (t, x) \, dx \, dt \lesssim_L \varepsilon^2.$$

This proves (1.7). As an immediate consequence, there exists an increasing sequence of time $t_n \to \infty$ as $n \to \infty$ such that

$$\int_2^\infty \int \operatorname{sech}^{2} \left(\frac{x + \sigma t_n}{L}\right) (u^2 + u_x^2) (t, x) \, dx \, dt \to 0 \text{ as } n \to \infty.$$

### 2.1. End of proof of the Theorem 1.1.

Consider $I(t)$ in (2.1). Choose now $\varphi := \frac{1}{2} (1 + \tanh)$ (for the right side) and $\varphi := \frac{1}{2} (1 - \tanh)$ (for the left hand side) in (2.1). The conclusion (1.6) follows directly from the ideas in [25]. Indeed, for the right side (i.e. $(1 + b)t, \infty$, $b > 0$ fixed), we choose $\tilde{b} = \frac{1}{2}$ and fix $t_0 > 2$. For $2 < t \leq t_0$ and large $L \gg 1$ (to make all estimates above hold), we consider the functional $I_{t_0}(t)$ by

$$I_{t_0}(t) := \frac{1}{2} \int \varphi \left(\frac{x + \sigma t_0 - \tilde{\sigma}(t_0 - t)}{L}\right) (u^2 + u_x^2) (t, x) \, dx,$$

where $\sigma = -(1 + b)$ and $\tilde{\sigma} = -(1 + \tilde{b})$. From Lemma 2.1, (2.13) with $\tilde{b} > 0$ and the smallness condition (1.5), we have

$$\frac{d}{dt} I_{t_0}(t) \lesssim_{\tilde{b},L} - \int \operatorname{sech}^{2} \left(\frac{x + \sigma t_0 - \tilde{\sigma}(t_0 - t)}{L}\right) (u^2 + u_x^2) \leq 0,$$

which shows that the new functional $I_{t_0}(t)$ is decreasing on $[2, t_0]$. On the other hand, since $\lim_{x \to -\infty} \varphi(x) = 0$, we have

$$\limsup_{t \to \infty} \int \varphi \left(\frac{x - \beta t - \gamma}{L}\right) (u^2 + u_x^2) (\delta, x) \, dx = 0,$$

for any fixed $\beta, \gamma, \delta > 0$. Together with all above, for any $2 < t_0$, we have

$$0 \leq \int \varphi \left(\frac{x - (1 + b)t_0}{L}\right) (u^2 + u_x^2) (t_0, x) \, dx \leq \int \varphi \left(\frac{x - (b - \tilde{b})t_0 - 2(1 + \tilde{b})}{L}\right) (u^2 + u_x^2) (2, x) \, dx,$$

which implies

$$\limsup_{t \to \infty} \int \varphi \left(\frac{x - (1 + b)t}{L}\right) (u^2 + u_x^2) (t, x) \, dx = 0.$$
APPENDIX A. ABOUT THE PROOF OF (2.18)

In this section we estimate the nonlinear term
\[
\mathcal{N}(t) = \frac{\alpha}{2} \int \varphi' \left( 2f(1-\partial_x^2)^{-1}(u^p) + ((1-\partial_x^2)^{-1}u^p)^2 \right) - \frac{\alpha}{2} \int \varphi' \left( 2f_x(1-\partial_x^2)^{-1}(u^p)_x + (\partial_x(1-\partial_x^2)^{-1}u^p)^2 \right) - \frac{1}{p+1}(\alpha \sigma - 1) \int \varphi'u^{p+1} + \int \varphi'u(1-\partial_x^2)^{-1}(u^p).
\]

Clearly,
\[
\left| \frac{1}{p+1}(\alpha \sigma - 1) \int \varphi'u^{p+1} \right| \lesssim \varepsilon^{p-1} \int |\varphi'|u^2,
\]
which is enough. Now, recall the following results.

Lemma A.1 ([14]). The operator \((1-\partial_x^2)^{-1}\) satisfies the following comparison principle: for any \(u, v \in H^1\),
\[
(A.1) \quad v \leq w \implies (1-\partial_x^2)^{-1}v \leq (1-\partial_x^2)^{-1}w.
\]

Also,

Lemma A.2 ([14, 22]). Suppose that \(\phi = \phi(x)\) is such that
\[
(A.2) \quad (1-\partial_x^2)^{-1}\phi(x) \lesssim \phi(x), \quad x \in \mathbb{R},
\]
for \(\phi(x) > 0\) satisfying \(|\phi^{(n)}(x)| \lesssim \phi(x), n \geq 0\). Then, for \(v, w \in H^1\), we have
\[
(A.3) \quad \int \phi^{(n)}v(1-\partial_x^2)^{-1}(u^p) \lesssim \|v\|_{H^1} \|w\|_{H^{p-2}} \int \phi w^2
\]
and
\[
(A.4) \quad \int \phi v_x(1-\partial_x^2)^{-1}(u^p)_x \lesssim \|v\|_{H^1} \|w\|_{H^{p-2}} \int \phi (w^2 + u^2_2).
\]

Using (A.3) with \(n = 0\),
\[
\left| \int \varphi'u(1-\partial_x^2)^{-1}(u^p) \right| \lesssim \varepsilon \int |\varphi'|\|u^p\| \lesssim \varepsilon^{p-1} \int |\varphi'|u^2.
\]

Using (A.3) with \(n = 0\) and (A.4), we also have from \(\|f\|_{L^\infty}, \|f_x\|_{L^\infty} \lesssim \|u\|_{H^1}\) that
\[
\left| \int \varphi'f(1-\partial_x^2)^{-1}(u^p) \right| \lesssim \varepsilon \int |\varphi'|\|u^p\| \lesssim \varepsilon^{p-1} \int |\varphi'|u^2
\]
and
\[
\left| \int \varphi'f_x(1-\partial_x^2)^{-1}(u^p)_x \right| \lesssim \varepsilon \int |\varphi'|((u^p)_x) \lesssim \varepsilon^{p-1} \int |\varphi'|(u^2 + u^2_2).
\]

For the rest terms, using (A.3) with \(n = 0\) and (A.4),
\[
\left| \int |\varphi'|((1-\partial_x^2)^{-1}(u^p)^2 \right| \lesssim \left| (1-\partial_x^2)^{-1}(u^p)\right|_{H^1} \varepsilon^{p-2} \int |\varphi'|u^2
\]
and
\[
\left| \int |\varphi'|((1-\partial_x^2)^{-1}\partial_x(u^p))^2 \right| \lesssim \left| (1-\partial_x^2)^{-1}(u^p)\right|_{H^1} \varepsilon^{p-2} \int |\varphi'|(u^2 + u^2_2).
\]

Finally, \(\|(1-\partial_x^2)^{-1}(u^p)\|_{H^1} \lesssim \|u^p\|_{H^{-1}} \lesssim \varepsilon^p\). Gathering these estimates, we get for some \(\delta\) small enough,
\[
|\mathcal{N}(t)| \lesssim \delta \int |\varphi'|(u^2 + u^2_2).
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