INTERTWINING RELATIONS FOR THE MATRIX CALOGERO-LIKE MODELS: SUPERSYMMETRY AND SHAPE INVARIANCE.

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Abstract. Intertwining relations for \( N \)-particle Calogero-like models with internal degrees of freedom are investigated. Starting from the well known Dunkl-Polychronakos operators, we construct new kind of local (without exchange operation) differential operators. These operators intertwine the matrix Hamiltonians corresponding to irreducible representations of the permutation group \( S_N \). In particular cases, this method allows to construct a new class of exactly solvable Dirac-like equations and a new class of matrix models with shape invariance. The connection with approach of multidimensional supersymmetric quantum mechanics is established.

1. Introduction

The exactly solvable quantum \( N \)-body problems have provided useful tools to investigate both formal algebraic and analytic properties with applications to different branches of Physics. The most intensively studied models are the Calogero model (many-body extension of the one-dimensional singular harmonic oscillator) \[1\], \[2\] and its various generalizations, so called Calogero-like models. The latter ones have either scalar \[3\]-\[5\] or matrix (with internal degrees of freedom) \[6\]-\[9\] nature. The Calogero-like models have been widely developed incorporating many-body forces \[10\], different root systems \[11\],\[12\] and multi-dimensions \[13\]. The supersymmetric extensions of Calogero-like models \[14\]-\[18\] also seem to be very promising.

In the papers \[19\], \[20\], \[6\], \[11\], \[21\], \[22\], \[7\] different types of Dunkl operators for the investigation of Calogero-like models were used\[\textsuperscript{a}\]. These operators intertwine

\[\textsuperscript{a}\]We call the one-particle operators constructed in \[20\],\[6\] the Dunkl-Polychronakos operators (DP) to distinguish them from the genuine Dunkl operators set forth in the papers \[19\], \[21\], \[22\], which are slightly different.
Calogero-like Hamiltonians and therefore allow one to construct the integrals of motion and the eigenfunctions (if they exist) for these models.

The characteristic trait of DP operators and of the corresponding integrals of motion derived from them is that they involve the coordinate exchange operators, and thus are nonlocal\(^b\). These techniques are briefly outlined in Subsection 2.1 of the paper.

On the other hand, the multidimensional supersymmetric quantum mechanics (SUSY QM) \(^{23}\), applied to the Calogero-like models \(^{14}-^{17},^{24}\) provides one with another set of the intertwining relations, where the matrix Calogero-like Hamiltonians of a specific type \(^{18}\), are intertwined by the the supercharge operators. In that approach, both the Hamiltonians and the supercharge operators are local. The intertwining relations are the most important part of the SUSY QM algebra, which is clear from a number of generalizations of the standard SUSY QM, e.g. \(^{25},^{26}\).

In the rest of Section 2 these two approaches will be unified: from the DP operators set forth in \(^{20},^{6}\) we construct new local operators of the first order in derivatives, that play the role of intertwining operators between the matrix Calogero-like Hamiltonians. For the Calogero-Sutherland model \(^3\) this leads to a new class of exactly solvable Dirac-like (matrix and of the first order in derivatives) Hamiltonians. For the Calogero model with oscillator terms the new intertwining relations give a new implementation of the shape invariance condition \(^{27},^{28},^{15},^{26}\).

In Section 3 we consider the particular case of three-particle Calogero-like models, which is the simplest nontrivial realization of the method introduced in Section 2. For the models without oscillator terms (OT) the above mentioned Dirac-like Hamiltonians coincide with the conventional Dirac Hamiltonians for a massless particle in magnetic field. In that case, the \(2 \times 2\) matrix Calogero-like Hamiltonians can be interpreted as the Pauli Hamiltonians for the same system.

In Section 4 the class of the Calogero-like matrix Hamiltonians described in \(^{18}\) is considered. For such models the intertwining relations derived in Section 2 are reduced to the well-known SUSY QM relations \(^{23}\). However, there is a wide class of models for which the intertwining relations introduced in Section 2 are not reduced to any previously known ones. Clearly, the SUSY QM is valid not only for the Calogero-like models, but for many other multidimensional and multiparticle ones \(^{23}\). The question as to how far the generalization of SUSY QM constructed below can be extended\(^c\) to non Calogero-like models, deserves further attention.

A possible way of extension of the formalism presented below is to consider the generalizations of the Calogero-like models incorporating many-body forces \(^{10}\) and

\(^b\)By the words "nonlocal" and "local" we mean "containing exchange operators" and "containing no exchange operators".

\(^c\)One example of such extension (for the three-particle case) will be given in the first footnote on the page 11.
different root systems \cite{12}, for which the Dunkl operators also exist \cite{11,12}, and therefore one can construct from them local intertwining operators analogous to those of the present paper.

2. Intertwining operators of first order in derivatives.

2.1. Dunkl-Polychronakos operators \cite{20,6} for Calogero-like models.

Let us consider a one-dimensional quantum system of \( N \) particles with coordinates \( x_i \). Let \( M_{ij} \) be the operator that exchanges the coordinates of the \( i \)-th and \( j \)-th particle. The DP operators are defined \cite{20,6} as:

\[
\pi_i = -i \partial_i + i \sum_{j \neq i} V_{ij} M_{ij} = \pi_i^\dagger \quad V_{ij} \equiv V(x_i - x_j);
\]

\[
\partial_i \equiv \frac{\partial}{\partial x_i}; \quad V(x) = V(-x) = V^*(x).
\]

The operators \( \pi_i \) are one-particle ones, i.e.

\[
M_{ij} \pi_i = \pi_j M_{ij}; \quad [M_{ij}, \pi_k] = 0, \quad k \neq i, j.
\]

Their commutators can be written as:

\[
[\pi_i, \pi_j] = \sum_{k \neq i,j} V_{ijk} [M_{ijk} - M_{jik}], \quad \text{where}
\]

\[
V_{ijk} \equiv V_{ij} V_{jk} + V_{jk} V_{ki} + V_{ki} V_{ij},
\]

and \( M_{ijk} \) are the operators of cyclic permutations in three indices:

\[
M_{ijk} \equiv M_{ij} M_{jk} = M_{kji} = M_{kij} = M_{jik}^\dagger.
\]

In the cases both of the Calogero-Sutherland (CS) models

\[
V(x) = l \cot x \quad \text{(trigonometric or TCS model)},
\]

\[
V(x) = l \coth \quad \text{(hyperbolic)},
\]

and of the delta-function model

\[
V(x) = l \operatorname{sign} x,
\]

\textsuperscript{d}The small latin indices range from 1 to \( N \) everywhere.
the function $V_{ijk} = l^2$, so that

$$[\pi_i, \pi_j] = l^2 \sum_{k \neq i,j} [M_{ijk} - M_{jik}]. \quad (6)$$

The Hamiltonians for these models are

$$H = -\Delta + \sum_{i \neq j} \left[ V'_{ij} M_{ij} + V_{ij}^2 \right] = \sum_i \pi_i^2 + \frac{l^2}{3} \sum_{i \neq j \neq k \neq i} M_{ijk}, \quad (7)$$

where $\Delta \equiv \sum_i \partial_i \partial_i$ and $V'_{ij} \equiv V'(x_i - x_j)$. It is known [20],[5] that in this case

$$[\pi_i, H] = 0. \quad (8)$$

In the case of the Calogero model,

$$V(x) = l/x; \quad V_{ijk} = 0; \quad [\pi_i, \pi_j] = 0, \quad (9)$$

and the equations (7),(8) remain valid. What is more, the DP operators themselves mutually commute. However, this model doesn't have a discrete spectrum.

The Calogero model is usually considered in a harmonic confining potential (we abbreviate it as CO: Calogero-Oscillator). For this model, the following operators should be introduced [20],[6], (see also [11],[21],[22],[7]):

$$a_i^\pm = \pi_i \pm i \omega x_i; \quad (a_i^+) = a_i^- \quad (10)$$

The Hamiltonian can be written as:

$$H_{CO} = \sum_i a_i^+ a_i^- + \omega \sum_{i \neq j} M_{ij} = -\Delta + \omega^2 \sum_i x_i^2 + \sum_{i \neq j} \frac{l(l - M_{ij})}{(x_i - x_j)^2} + N\omega. \quad (11)$$

It has been proven [6] that the operators $H_{CO}, a_i^\pm$ form the oscillator algebra:

$$[H_{CO}, a_i^\pm] = \pm 2\omega a_j^\pm. \quad (12)$$

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*The following procedure is applicable to arbitrary set of operators $\pi_i$, provided they satisfy (4),(5) and the Hamiltonian is given by the second equality of (7).*
2.2. The local form of the Hamiltonians.

Let us consider an irreducible representation $A$ of the permutation group $S_N$ realized on real vector functions $f_{\alpha}(x_1, ..., x_N); \alpha = 1, ..., \dim A$ by the matrices $T_{ij}^A$.

$$M_{ij}f_\alpha = (T_{ij}^A)_{\beta\alpha}f_\beta,$$

(13)

where $(T_{ij}^A)_{\beta\alpha} = (T_{ij}^A)_{\alpha\beta}$ are the (constant) matrix elements of the permutation operator $M_{ij}$ in the representation $A$. Below we will assume summation over the repeated indices, unless specified otherwise. We will also use the fact $[30]$ that $T_{ij}^A$ are real symmetric orthogonal matrices.

It will be useful to introduce the vector notations:

$$f = e_\alpha f_\alpha,$$

(14)

where the constant vectors $e_\alpha$ ($\alpha = 1, ..., \dim A$) form a basis in the space of the representation $A$. Then it is also helpful to define the operator $T_{ij}^A$ in the vector form:

$$T_{ij}^A e_\alpha = e_\beta (T_{ij}^A)_{\beta\alpha} e_\beta \equiv (T_{ij}^A)_{\alpha\beta} e_\beta.$$ 

(15)

Multiplying (13) onto $e_\alpha$ and using (15), we get:

$$M_{ij} f = T_{ij}^A f,$$

(16)

where $M_{ij}$ act only on the arguments of $f_\alpha$ and $T_{ij}^A$ only on the vectors $e_\alpha$.

All the Hamiltonians $H$ from the previous Subsection can be written as $H = H_{scal} + V'_{ij} M_{ij}$, where $H_{scal}$ are scalar operators containing no exchange operator terms:

$$H_{scal} = -\Delta + \sum_{i \neq j} V_{ij}^2$$

(17)

for the models without OT (4)-(5), and

$$H_{scal} = -\Delta + \omega x_i^2 + \frac{l^2}{(x_i - x_j)^2} + N\omega$$

(18)

for the CO model (11). The Hamiltonians $H$ act on the functions $f$ from the representation $A$ as

$$Hf_\alpha = H_{\beta\alpha} f_\beta; \quad H_{\beta\alpha}^A \equiv H_{scal}\delta_{\alpha\beta} + \sum_{i \neq j} V'_{ij}(T_{ij}^A)_{\beta\alpha} = H_{\alpha\beta}^A,$$

(19)

$f$Note that the functions $f$ are not necessarily eigenfunctions of $H$. We do not discuss in the present paper the symmetry properties of the eigenfunctions of the Calogero-like models.
or, in the vector form,
\[ Hf = Hf. \]

Note that if \( f \) satisfies (10) then \( Hf \) satisfies it too, since Eq. (10) is equivalent to the condition: \( T_{ij}^A M_{ij} f = f \) (no summation over \( i, j \)). The latter condition is satisfied for \( Hf \), because \( [H, T_{ij}^A M_{ij}] = 0 \).

### 2.3. The intertwining operators in the local form.

The matrix Hamiltonians (19) do not contain exchange operators \( M_{ij} \) explicitly. Our aim now is to get rid of the \( M_{ij} \) in the DP operators (1), (10) too, and rewrite the Eqs. (8), (12) in terms of local operators only.

Let us study the action of the DP operators on the symmetric functions satisfying (13), (16). The expression \( \pi_i f_\alpha \) no longer satisfies (13) even when \( f_\alpha \) satisfies it. Instead, \( \pi_i f_\alpha \) transforms under the action of \( M_{ij} \) as an object from the direct product of representations for \( \pi_i \) and \( f_\alpha \). Of course, the DP operators transform under \( S_N \) in accordance with (2). However, the \( \pi_i \) belong to a reducible representation of \( S_N \) because \( \pi_1 + \ldots + \pi_N \) realizes the absolutely symmetric representation. Therefore, it is helpful to go to the well-known Jacobi coordinates [29]:

\[
y_\xi = \frac{1}{\sqrt{\xi(\xi + 1)}} (x_1 + \ldots + x_\xi - \xi x_{\xi+1}); \quad 1 \leq \xi \leq N - 1 \tag{20}
\]

\[
y_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i,
\]

or \( y_k = R_{km} x_m \), where the orthogonal matrix \( R \) is determined by (21). The derivatives are connected by the same matrix: \( \partial/\partial y_k = R_{km} \partial/\partial x_m \), because \( R \) is an orthogonal matrix. Similarly, we can write the DP operators in the Jacobi variables:

\[
\rho_\xi = \frac{1}{\sqrt{\xi(\xi + 1)}} (\pi_1 + \ldots + \pi_\xi - \xi \pi_{\xi+1}); \quad 1 \leq \xi \leq N - 1
\]

\[
\rho_N = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \pi_j = -\frac{i}{\sqrt{N}} (\partial_1 + \ldots + \partial_N),
\]

\[9\]The indices of the Jacobi variables denoted by Greek letters range from 1 to N-1; those denoted by Latin letters range from 1 to N.
or $\rho_k = R_{km}\pi_m$. The operators $\rho_\xi$ now transform under $S_N$ as the irreducible representation $\Gamma$ with the Young tableau $(N-1,1)$. Similarly to (13) this fact can be written as:

$$M_{ij}\rho_\xi = (T^\Gamma_{ij})_{\lambda\xi}\rho_\lambda.$$  

(21)

This is a property of the Jacobi variables (see e.g. [18]).

The object $\rho_\xi f_\alpha$ transforms under the action of $S_N$ as the interior product $\Gamma \times A$ of the representations $A$ and $\Gamma$, or, in more detail, in accordance with the formulae (13),(21), as

$$M_{ij}\rho_\xi f_\alpha = (T^\Gamma_{ij})_{\lambda\xi}(T^A_{ij})_{\beta\alpha}\rho_\lambda f_\beta.$$  

As outlined in the book [30](chapter 7, §13), the interior product $\Gamma \times A$ contains only the irreducible representations of $S_N$, whose Young tableaux differ from the tableau for $A$ by no more than the position of one cell (but not necessarily all of them). For example, for the absolutely symmetric representation of $S_N$ with the Young tableau $(N)$, obviously,

$$\Gamma \times (N) = \Gamma,$$

and the result does not contain $(N)$.

Let $B$ be some irreducible representation that appears in $\Gamma \times A$. Then we can extract its contribution to $\Gamma \times A$ with the help of the Clebsch- Gordan coefficients $(\Gamma_\xi,A_\alpha|B_\sigma) \equiv (\xi\alpha|\sigma)$:

$$g_\sigma = (\xi\alpha|\sigma) \rho_\xi f_\alpha = D_{\sigma\alpha}f_\alpha; \quad D_{\sigma\alpha} \equiv (\xi\alpha|\sigma) \rho_\xi.$$  

(22)

The resulting function $g_\sigma$ satisfies the analog of (13) for the representation $B$:

$$M_{ij}g_\sigma = (T^B_{ij})_{\delta\sigma}g_\delta.$$  

(23)

This can be checked directly by substituting (22) into (23) and making use of the expression:

$$(T^\Gamma_{ij})_{\lambda\xi}(T^A_{ij})_{\beta\alpha}(\xi\alpha|\sigma) = (T^B_{ij})_{\delta\sigma}(\lambda\beta|\delta),$$  

(24)

and the fact that $T_{ij}$ are hermitean matrices.

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$^h$The standard notation [30] for a Young diagram containing $\lambda_i$ cells in the $i$-th line is $(\lambda_1, \ldots, \lambda_n)$; if the diagram contains $m$ identical lines with $\mu$ cells, it is denoted by $(\ldots, \mu^m, \ldots)$.

$^i$The expression (24) (see [31], formula (5.114)) is actually a necessary and sufficient condition for $(C_\xi,A_\alpha|B_\sigma)$ to be a Clebsch-Gordan coefficient for arbitrary representations $A, B$ and $C \in A \times B$. 

On the functions that satisfy (13) the operator $D_{\sigma \alpha}$ acts as:

\[ D_{\sigma \alpha} f_{\alpha} = D_{\sigma \alpha}^A f_{\alpha}; \]

\[ D_{\sigma \alpha}^A = (\xi \beta | \sigma)(\rho_{\xi}^A)_{\beta \alpha} = (\xi \beta | \sigma) R_{\xi k}(\pi_k^A)_{\beta \alpha}; \]  \hspace{1cm} (25)

\[ (\pi_k^A)_{\beta \alpha} = -i \partial_k \delta_{\beta \alpha} + i \sum_{m \neq k} V_{km}(T_{km}^A)_{\beta \alpha}. \]  \hspace{1cm} (26)

For the models without OT $[H, \pi_i] = 0$, so, $[H, D_{\sigma \alpha}] = 0$. Hence, for any $f_{\alpha}$

\[ HD_{\sigma \alpha} f_{\alpha} = D_{\sigma \alpha} H f_{\alpha}. \]  \hspace{1cm} (27)

For all $f_{\alpha}$ which satisfy (13), $H_{\beta \alpha} f_{\beta}$ satisfies (13) (see the end of the previous Subsection), and $D_{\sigma \alpha} f_{\alpha}$ satisfies (23). Using these symmetry properties, we can expand the sides of the equation (27) as

\[ \text{(l.h.s.)} = H_{\delta \sigma}^B D_{\delta \alpha} f_{\alpha} = H_{\delta \sigma}^B D_{\delta \alpha}^A f_{\alpha}; \]

\[ \text{(r.h.s.)} = D_{\sigma \alpha} H_{\beta \alpha}^A f_{\beta} = D_{\sigma \alpha} H_{\beta \alpha}^A f_{\beta}. \]

Taking into account that $H^A, H^B$ are symmetric matrices in the internal indices, we can see that on the functions satisfying (13)

\[ H_{\delta \sigma}^B D_{\delta \beta}^A = D_{\sigma \alpha}^A H_{\alpha \beta}^A. \]  \hspace{1cm} (28)

Similarly to the above, for the CO model from the Eqs. (12) it follows that

$HD_{\sigma \alpha}^\pm = D_{\sigma \alpha}^\pm (H \pm \omega)$. Following the same route that led us from (27) to (28), we can conclude that on the functions satisfying (13),

\[ H_{\delta \sigma}^B D_{\delta \beta}^{A \pm} = D_{\sigma \alpha}^{A \pm} (H_{\alpha \beta}^A \pm 2 \omega \delta_{\alpha \beta}), \]  \hspace{1cm} (29)

where

\[ D_{\sigma \alpha}^{A \pm} = (\xi \beta | \sigma) R_{\xi j}[\pi_j^A]_{\beta \alpha} \pm i \omega x_j \delta_{\alpha \beta}; \]  \hspace{1cm} (30)

\[ (\pi_j^A)_{\beta \alpha} = -i \partial_j \delta_{\beta \alpha} + il \sum_{m \neq j} \frac{(T_{jm}^A)_{\beta \alpha}}{x_j - x_m}. \]

Note that all terms in the Eqs. (28), (29) are local: they contain no exchange operators $M_{ij}$. 
2.4. The operatorial nature of the intertwining relations.

The Eqs. (28), (29) are not yet operatorial intertwining relations such as, for example, the SUSY QM ones (see [27], [31], [23]), because the former are valid only on the functions that satisfy the symmetry condition (13). On the functions outside that class the Eqs. (28), (29), generally speaking, may no longer be satisfied.

However, we can prove, that they are satisfied on all functions and thus are operatorial intertwining relations, by making use of the following

**Theorem 1:** Let $A$ be some representation of $S_N$. Let $L_{\alpha\beta}$ be some linear differential operator of finite order with the coefficients being rational matrix functions of the variables $x_i$, or $\sin x_i, \cos x_i$, or $\sinh x_i, \cosh x_i$ (but not of any two of them simultaneously), singular at

$$U = \{ \{x \exists i, j : i \neq j, \ x_i = x_j\} \} \text{ at most.}$$

The coefficients are matrices of dimension $\dim A \times \dim A$. Then, if

$$L_{\alpha\beta}f_{\beta} = 0$$

for all $f_{\beta}$ satisfying (13), then $L \equiv 0$ as an operator.

The proof of this Theorem can be found in the Appendix 1. Using this Theorem 1 for the difference of the left and right parts of the Eqs. (28), (29) we can conclude that the latter are satisfied operatorially.

In particular, it means that that when the initial representation $A$ coincides with the resulting representation $B$, the operators $D^A$ are integrals of motion for the (trigonometric and hyperbolic) matrix CS models. Therefore, each CS matrix model corresponding to a representation $A$ such that $A \in \Gamma \times A$ has a local integral of motion $D^A$ of the first order in derivatives. An example of model from this class will be given in the Section 3. Note that e.g. the models with $A = (N)$ or $A = (1^N)$ lie outside this class.

With periodic boundary conditions on a unit circle $S$, the Hamiltonians $H^A$ for the TCS system are exactly solvable (see e.g. [8]) and have discrete spectrum and finite dimensional degeneracy of levels. The fact that (28), (29) are satisfied operatorially when $B = A$ allows us to find also the spectrum and the eigenfunctions of the $D^A$, i.e. the normalizable functions $f_\alpha(x), \alpha = 1, ..., \dim A$:

$$D^A_{\alpha\beta}f_\beta = \epsilon f_\alpha.$$
The operators $D^A$ for the TCS system may be considered here as Dirac-like Hamiltonians of first order in derivatives:

$$D^A_{\sigma\alpha} = (\xi^\beta|\sigma)R_{\xi^\beta} \left[ -i\partial_j \delta_{\beta\alpha} + i\sum_{m \neq j} \cot (x_j - x_m) (T^A_{jm})_{\beta\alpha} \right].$$

\[2\]

From the commutation relations (28) it follows that for a given $A$, the operators $H^A$ and $D^A$ can be diagonalized simultaneously. In more detail, if $f^{(n)}_b$ are degenerate eigenstates of $H^A$ with energy $E_n$:

$$H^A f^{(n)}_b = E_n f^{(n)}_b,$$

and $b = 1, ..., J$ ($J$ is the degree of degeneracy), then after acting by $D^A$ on both sides of the equality (31) we see that $D^A f^{(n)}_b$ satisfies (31) too. Hence, there exists a constant $J \times J$ matrix $F^{(n)}$:

$$D^A f^{(n)}_b = F^{(n)}_{bc} f^{(n)}_c.$$

Because $H^A$ is hermitean, we can choose $f^{(n)}_b$ that constitute a basis of periodic vector functions with dim $A$ components on $S^N$. Because $D^A$ is hermitean, one can check that $F^{(n)}$ is also hermitean, and it can be diagonalized by a unitary rotation:

$$U^{(n)}_{ab} F^{(n)}_{bc} U^{(n)\ast}_{dc} = \delta^{(n)}_{ac},$$

and the functions

$$g^{(n)}_a(x) = U^{(n)}_{ab} f^{(n)}_b(x)$$

are eigenfunctions of the operator $D^A$ with energies $\epsilon^{(n)}_a$. Because $U^{(n)}_{ab}$ are unitary matrices, the functions $g^{(n)}_a$ constitute a basis of the periodic vector functions on $S^N$. Hence, they form a full set of eigenfunctions of $D^A$.

For the CO model the case of $B = A$ is interesting too. The Eqs. (29) will then take the form:

$$H^A_{\delta\sigma} D^A_{\delta\beta} = D^A_{\sigma\alpha} (H^A_{\alpha\beta} \pm 2\omega \delta_{\alpha\beta}),$$

where $D^A_{\pm}$ are still defined by (30), and will be satisfied operatorially. They are the relations of the oscillator-like algebra (similar to (12)). In the language of SUSY QM

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$m$See [32]; compare also with the SUSY QM intertwining relations which were used to find a part of spectrum of Hamiltonians in one [33, 34] or two [26] dimensions.

$n$We assume for simplicity that the action of $D^A$ does not destroy the normalizability of $f^{(n)}_b$.

$a$Note that the matrix elements $F^{(n)}_{bc}$ are NOT dim $A \times$ dim $A$ matrices, but just scalar constants. In other words, they do not affect the vector structure of $f^{(n)}_b$. 
it means that each CO matrix model for a representation \( A \in \Gamma \times A \) obeys shape invariance (SI) in \( N - 1 \) dimensions (because the center of mass motion is decoupled). Let us stress that this is the first example of SI in several dimensions realised by local operators of the first order in derivatives (cf. the attempts in [13]). The nonlocal SI of the Calogero model (see the Eq. (12)) was described in [20], [1], [21], [22], [7]; a model with a two-dimensional SI of the second order in derivatives was proposed in [26].

### 3. Examples with \( N = 3 \).

Assume that \( N = 3 \), and both representations \( A \) and \( B \) are equal to \( \Gamma = (2, 1) \). Then the functions that satisfy (16) are:

\[
\mathbf{f} = \begin{pmatrix} f_{211} \\ f_{121} \end{pmatrix},
\]

where the index \( \alpha = (211), (121) \) enumerates the partitions of the Young tableau \( (2, 1) \) (see [30] or [35], the end of chapter 1):

\[
\begin{align*}
P^+ f_{211} &= P^- f_{211} = 0; \\
P^+ f_{121} &= P^- f_{121} = 0; \\
M_{12} f_{121} &= -f_{121}; \\
P^\pm &= 1 + M_{12} M_{13} + M_{13} M_{12} \pm (M_{12} + M_{23} + M_{13}).
\end{align*}
\]

\( P^\pm \) are the projectors onto the symmetric/antisymmetric representations of \( S_3 \). The matrices of the representation \( \Gamma \) for \( N = 3 \) have the form:

\[
T^\Gamma_{12} = \sigma_3; \\
T^\Gamma_{23} = \frac{\sqrt{3}}{2} \sigma_1 - \frac{1}{2} \sigma_3; \\
T^\Gamma_{31} = -\frac{\sqrt{3}}{2} \sigma_1 - \frac{1}{2} \sigma_3,
\]

(33)

where \( \sigma_i \) are the Pauli matrices. The Hamiltonians for the Calogero-like systems without OT for the representation \( \Gamma \) have the form (19):

\[
H^\Gamma = -\Delta + \sum_{i\neq j} \left[ V_{ij}^2 + V'_{ij} T^\Gamma_{ij} \right].
\]

(34)

As to the DP operators in the Jacobi variables, one can check that \( \rho_1 \) has the same symmetry as \( f_{121} \), and \( \rho_2 \) - the same as \( f_{211} \). The Clebsch- Gordan coefficients for \( \Gamma \times \Gamma \rightarrow \Gamma \) are written in the end of chapter 7 of the book [30]. Plugging them and the \( T^\Gamma_{ij} \) from (33) into the definition of \( D^\Gamma \) (expressions (23), (26)), we can conclude after some algebra that for the Calogero-like models without OT

\[
D^\Gamma = \frac{1}{\sqrt{2}} \left[ -i \sigma_3 \frac{\partial}{\partial y_2} + i \sigma_1 \frac{\partial}{\partial y_1} - \sqrt{2} \sigma_2 (V_{12} + V_{23} + V_{31}) \right],
\]

(35)
where \( y \) are the Jacobi variables (20). The Eq. (28) for \( H^\Gamma \) from (34) and \( D^\Gamma \) from (35) is satisfied operatorially, i.e. \( [H^\Gamma, D^\Gamma] = 0 \). In fact, a stronger statement for \( N = 3 \) can be proven by direct calculation:

\[
(D^\Gamma)^2 = \frac{1}{2} [H^\Gamma + \partial^2 / \partial y_3^2] + C,
\]

where \( C \) is a real constant. What is more\footnote{One could even replace the term \( V_{12} + V_{23} + V_{31} \) in (35) by an arbitrary function \( v(y_1, y_2) \). Then the square of \( D^\Gamma \) would remain the sum of the Laplacian and a momentum independent \( 2 \times 2 \) matrix potential. However, we do not know any cases when this sum is an exactly solvable Hamiltonian, except for those given in this text.}, it is true operatorially even for the case \( V(x) = l \text{ sign } x \). In that case we were unable to prove (28) for arbitrary representation \( A \), but for \( N = 3 \) and \( A = \Gamma \) it turns out to be true.

Eq. (36) signifies that the operator \( D^\Gamma \) realizes a sort\footnote{The operator (35) can be viewed as a Dirac operator for a massless fermion in three dimensions \((y_1, y_2, y_3)\) in the magnetic field that does not depend on \( y_3 \) and is orthogonal to the axis \( y_3 \). The component of the fermion’s momentum along the axis \( y_3 \) should be zero. The Hamiltonian (34) is then the Pauli Hamiltonian for the same system \([30],[31]\).} of a ”square root” of \( H^\Gamma \).

This, in particular, means that the spectrum and eigenfunctions of \( D^\Gamma \) itself can be found easier than in the general case described in Section 2 of this paper, if the spectrum and eigenfunctions of \( H^\Gamma \) are known.

For the TCS model the operator \( D^\Gamma \) (35) has the form:

\[
D^\Gamma = \frac{1}{\sqrt{2}} \left[ -i\sigma_3 \frac{\partial}{\partial y_2} + i\sigma_1 \frac{\partial}{\partial y_1} - l\sqrt{2}\sigma_2 \left( \cot(\sqrt{2}y_1) + \cot\left(-\frac{\sqrt{2}}{2}y_1 + \frac{3}{2}y_2\right) + \right. \\
\left. + \cot\left(-\frac{\sqrt{2}}{2}y_1 - \frac{3}{2}y_2\right) \right) \right].
\]

The eigenfunctions of this operator are 2-component column functions \( g(y_1, y_2) \):

\[
D^\Gamma g(y_1, y_2) = \epsilon g(y_1, y_2).
\]

In this case it follows from (36) that \( g \) is also an eigenfunction of \( H^\Gamma \) with energy \( E = 2(\epsilon^2 - C) \).

Let us prove now that all the eigenfunctions of \( D^\Gamma \) can be obtained from the ones of \( H^\Gamma \). Let \( f(x) \) be an eigenfunction of \( H^\Gamma \) with energy \( E: H^\Gamma f = Ef \) and zero total momentum. Then the following alternative should be considered:

a) \( f \) itself is already an eigenfunction of \( D^\Gamma \), i.e. it satisfies (37). Then the corresponding eigenvalue is: \( \epsilon = \pm \sqrt{E/2 + C} \).

b) \( D^\Gamma f \equiv u \) and \( f(x) \) are linearly independent. Then one can check that \( D^\Gamma u = (C + E/2)f \), and \( u(x) \) is also an eigenfunction of \( H^\Gamma \) with energy \( E \). Thus, the
eigenfunctions of $H^\Gamma$, that are not the ones of $D^\Gamma$, form pairs in which $D^\Gamma$ transforms each member into another. If $f$ is normalizable then $u$ is too, because

$$<u|u> = <f|(D^\Gamma)^2|f> = (E/2 + C) <f|f> < +\infty.$$  

\(\text{From each such pair one can construct two eigenfunctions of } D^\Gamma:\)

$$f \pm (C + E/2)^{-1/2} u$$

with energies \(\epsilon = \pm \sqrt{C + E/2}.

Thus, when taken in the above form, the sets of eigenfunctions of $H^\Gamma + \frac{\partial^2}{\partial y^2}$ and $D^\Gamma$ concide.

For the CO model the Hamiltonian (19) for the representation $\Gamma$ will have the form:

$$H^\Gamma = -\Delta + \omega^2 \sum_i x_i^2 + \sum_{i \neq j} \frac{l(l - T^\Gamma_{ij})}{(x_i - x_j)^2} + 3\omega,$$

where $T^\Gamma_{ij}$ are defined in (33). The intertwining operators (30) can be rewritten as:

$$D^\Gamma_{\pm} = 1 \sqrt{2} \left[ -i\sigma_3 \left( \frac{\partial}{\partial y_2} \mp \omega y_2 \right) + i\sigma_1 \left( \frac{\partial}{\partial y_1} \mp \omega y_1 \right) - \sqrt{2}\sigma_2 \left( \frac{1}{y_1} + \frac{1}{-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2} + \frac{1}{-\frac{1}{2}y_1 - \frac{\sqrt{3}}{2}y_2} \right) \right].$$

The operatorial relations (32) of the oscillator-like algebra

$$[H^\Gamma, D^\Gamma_{\pm}] = \pm 2\omega D^\Gamma_{\pm}$$

correspond to the SI of the matrix $2 \times 2$ Hamiltonian $H^\Gamma$ in two dimensions (because the center of mass motion is decoupled).

4. Connection with the ordinary SUSY QM.

In this Section we will restrict ourselves to the class of representations with Young tableaux of the form

$$A = (N - n, 1^n); \quad n = 1, ..., N. \quad \quad \quad (38)$$

\(\text{The situation } C + E/2 < 0 \text{ is impossible because otherwise the Eq. (37) would remain valid, but with imaginary } \epsilon. \ The \ operator } D^\Gamma \text{ is hermitean, so } C + E/2 \geq 0. \)
It was proven in [18] that for this class of representations we can choose a basis $e_\alpha$ (see (14)) with the help of the fermionic creation/annihilation operators $\psi_i, \psi_i^+$; $i = 1...N$:

\[
\{\psi_i, \psi_j\} = 0, \quad \{\psi_i^+, \psi_j^+\} = 0, \quad \{\psi_i, \psi_j^+\} = \delta_{ij}; \quad (39)
\]

$\psi_i|0\rangle = 0; \quad i, j = 1...N; \quad <0|0> = 1.$

It is useful to introduce also the fermionic analogues $\phi_k^+$ of the Jacobi variables (20) (see [18]):

$$
\phi_k^+ = R_{km} \psi_m^+; \quad \phi_k = R_{km} \psi_m,
$$

where $R_{km}$ are defined in (20). The fermionic Jacobi variables obey anticommutation relations similar to (39):

\[
\{\phi_k, \phi_m\} = 0, \quad \{\phi_k^+, \phi_m^+\} = 0, \quad \{\phi_k, \phi_m^+\} = \delta_{km}. \quad (40)
\]

$$
\phi_k|0\rangle = 0; \quad k, m = 1...N.
$$

Now we can define the basis $e_\alpha$:

$$
e_\alpha = \phi_{\alpha_1}^+...\phi_{\alpha_n}^+|0\rangle \equiv |\alpha_1...\alpha_n > \quad \alpha_i = 1, ..., N - 1, \quad (41)
$$

where $\alpha \equiv (\alpha_1...\alpha_n)$ is a multiindex with values in the fermionic number space, and

\[
(e_\alpha)^\dagger e_\alpha = <\alpha_n...\alpha_1|\alpha_1...\alpha_n >= 1;
\]

$$
(e_\alpha)^\dagger = (|\alpha_1...\alpha_n >)^\dagger = <0|\phi_{\alpha_n}...\phi_{\alpha_1} \equiv <\alpha_n...\alpha_1| \quad (42)
$$

(no summation over $\alpha$ is implied).

Because of (40), it is sufficient to include into the basis only the vectors $e_\alpha$ with, say, $\alpha_1 < ... < \alpha_n$, and the summation over $\alpha$ will be done over such vectors only.

It was also proven in [18], that for the basis (11) the operator $T_{ij}^A$ in the vector form (13) can be realized as

$$
(T_{ij}^A)_{\alpha\beta} e_\alpha = T_{ij}^A e_\alpha = T_{ij}^A |\alpha_1...\alpha_n > = K_{ij} |\alpha_1...\alpha_n >, \quad (43)
$$

where

$$
\hat{K}_{ij} \equiv \psi_i^+ \psi_j + \psi_j^+ \psi_i - \psi_i^+ \psi_i - \psi_j^+ \psi_j + 1 = 1 - (\psi_i^+ - \psi_j^+)(\psi_i - \psi_j) = \\
= \hat{K}_{ji} = (\hat{K}_{ij})^\dagger. \quad (44)
$$

\*The fermionic operators $\phi_N, \phi_N^+$ do not enter into $e_\alpha$ because they correspond to the center of mass degree of freedom which is decoupled.
It follows from (43) that all the Hamiltonians $H_A$ (19) with $A$ from the class (38) take the same form in the basis (41):

$$H = H_{\text{scal}} + \sum_{i \neq j} V'_{ij} K_{ij}. \quad (45)$$

Let us consider the intertwining relations (28) for the Calogero-like models without OT for $A$ from the class (38). From (45), (17) it follows that the Hamiltonians $H_A, H_B$ in (28) will have the form:

$$H = -\Delta + \sum_{i \neq j} \left[ V'_{ij} K_{ij} + V^2_{ij} \right]. \quad (46)$$

One may notice that this Hamiltonian is a particular case of the Superhamiltonian given in [18], up to the sign of $V_{ij}$ and an additive scalar constant.

Because the Young tableaux for $B$ and $A$ belong to the class (38) and can differ by no more than the position of one cell (see [30], chapter 7, §13), $B$ can either coincide with $A$ or have the form $(N - n, 1^k)$.

Let us consider the case $B = (N - n - 1, 1^{n+1})$, for which we can realize (cf. [18]) the Clebsch-Gordan coefficients in the operators $D_A$ as:

$$(\xi | \alpha \sigma) = <\alpha_n \ldots \alpha_1 | \sigma_1 \ldots \sigma_{n+1} > = <\sigma_{n+1} \ldots \sigma_1 | \xi \alpha_1 \ldots \alpha_n >. \quad (47)$$

One can check that the Clebsch-Gordan coefficients, defined by (47), satisfy (24), and therefore they correctly connect the representations $A = (N - n, 1^n), \Gamma = (N - 1, 1)$ and $B = (N - n - 1, 1^{n+1})$. These Clebsch-Gordan coefficients may differ from the standard ones (see [30]) by an inessential overall factor.

Now we can express the intertwining operators $D_A$ (25) in terms of the fermionic operators defined above:

$$D_A e_\beta = e_\sigma (e_\sigma)^\dagger D_A e_\beta = e_\sigma A_{\sigma \beta} =$$

$$= |\sigma_1 \ldots \sigma_{n+1} > <\sigma_{n+1} \ldots \sigma_1 | \xi \alpha_1 \ldots \alpha_n > R_{\xi k} \left[ -i \partial_k \delta_{\beta \alpha} + i \sum_{m \neq k} V_{km} (T^A_{km})_{\beta \alpha} \right] =$$

$$= R_{\xi k} \left[ -i \partial_k |\beta_1 \ldots \beta_n > + i \sum_{m \neq k} V_{km} (T^A_{km})_{\beta \alpha} |\alpha_1 \ldots \alpha_n > \right] =$$

$$= R_{\xi k} \phi_\xi^+ \left[ -i \partial_k |\beta_1 \ldots \beta_n > + i \sum_{m \neq k} V_{km} (T^A_{km})_{\beta \alpha} |\alpha_1 \ldots \alpha_n > \right] =$$

$$= R_{\xi k} \phi_\xi^+ \left[ -i \partial_k + i \sum_{m \neq k} V_{km} K_{km} \right] |\beta_1 \ldots \beta_n >.$$
So, we can conclude that
\[ D^A = R_{\xi k} \phi_\xi^+ \left[ -i \partial_k + i \sum_{m \neq k} V_{km} K_{km} \right]. \] (48)

The operator \( D^A \) turns out not to depend on the specific choice of \( A \) from the class \( \{38\} \), i.e. on the fermionic number. It augments the fermionic number by 1; that is related to the fact that \( D^A \) changes the position of one cell in the Young tableau for \( A \) (see \( \{38\} \)).

One can notice that
\[ R_{\xi k} \phi_\xi^+ = \psi_k^+ - \frac{1}{N} \sum_m \psi_m^+, \] (49)
and after some algebra one can deduce from (48):
\[ D^A = i \phi_N^+ \frac{\partial}{\partial y_N} + \psi_k^+ \left[ -i \partial_k + i \sum_{m \neq k} V_{km} K_{km} \right]. \] (50)

To further simplify the form of the intertwining operator \( D^A \), one can use the following

**Theorem 2:** For all \( V_{km} \) with \( V_{km} = -V_{mk} \) the following equation is satisfied:
\[ \sum_{m \neq k} \psi_i^+ V_{km} K_{km} = \sum_{m \neq k} \psi_i^+ V_{km}, \]
where \( K_{km} \) is the fermionic permutation operator \( \{44\} \).

The proof of the Theorem 2 can be found in the Appendix 2.

Now one can rewrite the Eq. (51) as:
\[ D^A = -iq^+; \quad q^+ \equiv -\phi_N^+ \frac{\partial}{\partial y_N} + \psi_k^+ \left[ \partial_k - \sum_{m \neq k} V_{km} \right]. \] (51)

Therefore, the Eq. (28) with the Hamiltonian \( \{46\} \) takes the form:
\[ [H, q^+] = 0, \] (52)
and its hermitean conjugation\(^\dagger\) is:
\[ [H, q^-] = 0; \quad q^- \equiv (q^+)^\dagger. \] (53)

\(^\dagger\)The Eq. (53) could also be obtained in another way: we could consider the Eq. (28) with \( A = (N - n, 1^n) \) but with \( B = (N - n + 1, 1^{n-1}) \). Using formulae similar to (47)-(51) one can check that then \( D^A = iq^- = i(q^+)^\dagger \).
The operators $q^\pm$ (51) coincide with the supercharge operators $u^\pm$ for the Calogero-like models given in [18], except for the sign of $V_{ij}$, which is determined by the sign of the constant $l$ in (1)-(9). Therefore, if we replace $l$ by $-l$ in the SUSY QM relations of [18], they can be rewritten as:

$$\{q^-, q^+\} = H + \frac{\partial^2}{\partial y^2_N} + C; \quad (q^+)^2 = (q^-)^2 = 0. \quad (54)$$

The term $\frac{\partial^2}{\partial y^2_N}$ in (54) is unimportant because it commutes with $q^\pm$ and $H$.

The commutation relations (52),(53) may be considered as the SUSY QM commutation relations, corresponding to the algebra (54) with the supercharges $q^\pm$ and the Superhamiltonian $H + \frac{\partial^2}{\partial y^2_N} + C$.

We can conclude that for the models without OT the intertwining relations (28) for the representations $A,B$ from the class (38) turn into the relations of SUSY QM [23],[18]. For other $A$ and $B$ (28) can be considered as a generalization of the SUSY QM intertwining relations.

Now let us turn to the CO model and the intertwining relations (29) with $A$ from the class (38). From (13),(18) it follows that the Hamiltonians $H^A, H^B$ in (28) are:

$$H = -\Delta + \omega^2 \sum_i x_i^2 + \sum_{i \neq j} \frac{l(l - K_{ij})}{(x_i - x_j)^2} + N\omega. \quad (55)$$

This Hamiltonian, analogously to the previous case (46), has the same form for all representations $A$ from the class (38). However, the Hamiltonian (55) differs slightly from the corresponding Calogero Hamiltonian given in [18], as will be explained below.

The intertwining operators $D^{A\pm}$ can be treated similarly to the case without OT, the only difference being that one should write $\partial_i \mp \omega x_i$ instead of $\partial_i$ everywhere. In particular, for the case with $B = (N - n - 1, 1^{n+1})$ the definition (30) leads to the following analog of the formula (18) for $D^{A\pm}$ (the same for all $A$ from the class (38)):

$$D^{A\pm} = R_{\xi k} \phi^+_{\xi} \left[ -i \partial_k \pm i \omega x_k + i \sum_{m \neq k} (x_k - x_m)^{-1} K_{km} \right] =$$

$$= R_{\xi k} \phi^+_{\xi} \left[ -i \partial_k + i \sum_{m \neq k} \left( \pm \frac{\omega}{N} (x_k - x_m) + (x_k - x_m)^{-1} K_{km} \right) \right]. \quad (56)$$

*a The operator $q^+$ from (51) differs from the standard supercharge operator for the TCS model [17] by the term $-\phi_N \frac{\partial}{\partial y_N}$ that cancels the dependence of the supercharge on the center of mass coordinates $y_N, \phi_N$.

*v The intertwining relations are the most important part of the SUSY QM algebra, which is clear from a number of generalizations of the standard SUSY QM: e.g., [25],[26].
Making use of the Eq. (49) and of the Theorem 2, we can obtain that, similarly to (51):

\[
D^{A+} = -i q^+; \quad q^+ \equiv -\phi_N \frac{\partial}{\partial y_N} + \psi_k \left[ \partial_k - \sum_{m \neq k} W_{km} \right]; \quad (57)
\]

\[
W_{km} = W(x_k - x_m); \quad W(x) \equiv \frac{\omega}{N} x + \frac{l}{x};
\]

\[
D^{A-} = -i \tilde{q}^+; \quad \tilde{q}^+ \equiv -\phi_N \frac{\partial}{\partial y_N} + \psi_k \left[ \partial_k - \sum_{m \neq k} \tilde{W}_{km} \right];
\]

\[
\tilde{W}_{km} = \tilde{W}(x_k - x_m); \quad \tilde{W}(x) \equiv -\frac{\omega}{N} x + \frac{l}{x}.
\]

Taking into account the formulae (57), (55), we can rewrite the Eq. (29) for \(D^{A+}\) and \(A\) from the class (38) as:

\[
[H, q^+] = 2 \omega q^+; \quad [H, \tilde{q}^+] = -2 \omega \tilde{q}^+;
\]

and its hermitean conjugation:

\[
[H, q^-] = -2 \omega q^-; \quad [H, \tilde{q}^-] = 2 \omega \tilde{q}^-;
\]

\[
q^- = (q^+)^\dagger = \phi_N \frac{\partial}{\partial y_N} + \psi_k \left[ -\partial_k - \sum_{m \neq k} W_{km} \right]; \quad (60)
\]

\[
\tilde{q}^- = (q^+)^\dagger = \phi_N \frac{\partial}{\partial y_N} + \psi_k \left[ -\partial_k - \sum_{m \neq k} \tilde{W}_{km} \right].
\]

The operators \(q^\pm\) from (57), (60) are similar to the supercharge operators \(q^\pm\) given in [18] for the Calogero model with OT, up to a redefinition of constants. Therefore, we can construct the following SUSY algebra:

\[
\{q^-, q^+\} = h; \quad (q^+)^2 = (q^-)^2 = 0,
\]

with the Superhamiltonian \(h\):

\[
h = H + 2 \omega \psi_k \psi_k - H_N; \quad H_N = -\frac{\partial^2}{\partial y_N^2} + \omega^2 y_N^2 + 2 \omega \phi_N \psi_N + C,
\]

\[\text{The Eqs. (59) could also be obtained in another way: we could consider the Eq. (29) with } A = (N - n, 1^n) \text{ but with } B = (N - n + 1, 1^{n-1}). \text{ Using formulae similar to (47), (51), (57) one can check that then } D^{A+} = i \tilde{q}^- \text{, where } q^- \text{ is defined in (60), and } D^{A-} = i q^+.
\]

\[\text{The operator } q^+ \text{ from (57) differs from the standard supercharge operator for the TCS model [14] by the term } -\phi_N \frac{\partial}{\partial y_N} \text{ that cancels the dependence of the supercharge on the center of mass coordinates } y_N \text{ and } \phi_N.\]
where $H$ is defined in (55), and $C$ is a scalar constant. The term $H_N$ is unimportant because it commutes with $q^\pm$ and $H$. The operators $\tilde{q}^\pm$ form an algebra similar to (61) but the sign of $\omega$ in the Superhamiltonian (62) should be different (cf. [37]):

$$\{\tilde{q}^-, \tilde{q}^+\} = \tilde{h}; \quad (\tilde{q}^+)^2 = (\tilde{q}^-)^2 = 0;$$

$$\tilde{h} = H - 2\omega \psi_k \psi_k - \tilde{H}_N; \quad \tilde{H}_N = -\frac{\partial^2}{\partial y_N^2} + \omega^2 y_N^2 - 2\omega \phi_N \phi_N + \tilde{C}.$$ 

¿From the SUSY algebrae (61), (63) one can deduce the commutation relations that can be shown to be equivalent to (58), (59):

$$[h, q^\pm] = 0; \quad [\tilde{h}, \tilde{q}^\pm] = 0.$$

The Eqs. (28), (29) in the case of $A, B$ from the class (38) are reduced to the ordinary multidimensional SUSY QM [23] for the Calogero-like models [18], [15]. However, for $A$ or $B$ outside that class the Eqs. (28), (29) describe a generalization of the SUSY QM intertwining relations that has not been known before. Clearly, the SUSY QM is valid not only for the Calogero-like models, but for many others [23]. The question as to how far the generalization of SUSY QM constructed above can be extended to other, non Calogero-like models, deserves further attention.
Appendix 1.

In this Appendix, we prove the following

**Theorem 1:** Let $A$ be some representation of $S_N$. Let $L_{\alpha\beta}$ be some linear differential operator of finite order with the coefficients being rational matrix functions of the variables $x_i$, or $\sin x_i$, $\cos x_i$, or $\sinh x_i$, $\cosh x_i$ (but not of any two of them simultaneously), singular at $U = \{x|\exists i, j : i \neq j, x_i = x_j\}$ at most. The coefficients are matrices of dimension $\dim A \times \dim A$. Then, if

$$L_{\alpha\beta}f_{\beta} = 0$$

for all $f_{\beta}$ satisfying (13), then $L \equiv 0$ as an operator.

**Proof:** Consider the principal Veyl chamber: $\{x : x_1 < ... < x_N\}$. Every function defined on this chamber can be continued onto the rest of $\mathbb{R}^{\dim A}$ by using (13). The result will obviously satisfy (13), so it is annihilated by $L$. Hence, $L$ annihilates all functions on the principal Veyl chamber. The same can be stated about every other Veyl chamber: $\{x : x_{i_1} < ... < x_{i_N}\}$. Hence, $L$ annihilates all functions on $\mathbb{R}^{\dim A} \setminus U$. From the fact that the coefficients of $L$ are rational functions of $x_i$, or $\sin x_i$, $\cos x_i$, or $\sinh x_i$, $\cosh x_i$, it then follows that they are zero identically.

Appendix 2.

In this Appendix, we prove the following

**Theorem 2:** For all $V_{km}$ such that $V_{km} = -V_{mk}$,

$$\sum_{m \neq k} \psi_k^+ V_{km} K_{km} = \sum_{m \neq k} \psi_k^+ V_{km},$$

(63)

where $K_{km}$ is the fermionic permutation operator defined in (44).

**Proof:** taking into account the definition (44), we can check that

$$\psi_k^+ K_{km} = \psi_k^+ + \psi_k^+ \psi_m^+ \psi_m + \psi_m^+ \psi_k^+ \psi_k.$$ (64)

In the Eq. (64) no summation over either index is implied. Substituting (64) into the left side of (63) we see that

$$\sum_{m \neq k} \psi_k^+ V_{km} K_{km} = \sum_{m \neq k} V_{km} [\psi_k^+ + \psi_k^+ \psi_m^+ \psi_m + \psi_m^+ \psi_k^+ \psi_k] =$$

$$= \sum_{m \neq k} [V_{km} \psi_k^+ + V_{km} \psi_k^+ \psi_m^+ \psi_m + V_{mk} \psi_k^+ \psi_m^+ \psi_m] = \sum_{m \neq k} \psi_k^+ V_{km}.$$
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