ATTRACTORS FOR WEAKLY DAMPED BEAM EQUATIONS WITH p-LAPLACIAN

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Abstract. This paper is concerned with a class of weakly damped one-dimensional beam equations with lower order perturbation of p-Laplacian type

\[ u_{tt} + u_{xxxx} - (\sigma(u_x))_x + ku_t + f(u) = h \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \]

where \( \sigma(z) = |z|^{p-2}z, \ p \geq 2, \ k > 0 \) and \( f(u) \) and \( h(x) \) are forcing terms. Well-posedness, exponential stability and existence of a finite-dimensional attractor are proved.

1. Introduction. In this paper we consider the long-time behavior of solutions to the equation

\[ u_{tt} + u_{xxxx} - (\sigma(u_x))_x + ku_t + f(u) = h \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \quad (1) \]

with simply supported boundary condition

\[ u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0, \quad t \geq 0, \quad (2) \]

and initial condition

\[ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in (0, L), \quad (3) \]

where \( \sigma(z) = |z|^{p-2}z, \ p \geq 2 \) and \( k > 0 \). It was motivated by a series of papers by Yang et al [12, 13, 14, 15], where problems like

\[ u_{tt} + \Delta^2 u - \text{div}(\nabla u |\nabla u|^{m-2} \nabla u) - \Delta u_t + h_1(u_t) + h_2(u) = h_3(x), \quad (4) \]

were considered. Their results are mainly concerned with global solvability and long-time behavior of solutions. The main physical justifications come from a model of elastoplastic microstructure flows

\[ u_{tt} + u_{xxxx} = a(u_x^2)_x, \quad a < 0, \]

considered by An and Peirce [1], and a class of Kirchhoff-Boussinesq models

\[ u_{tt} + \Delta^2 u + ku_t = \text{div}(\nabla u |\nabla u|^{2} \nabla u) + \Delta(f_1(u)) - f_2(u), \]

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considered by Chueshov and Lasiecka [5, 6]. A related problem involving a memory term was considered by Andrade et al [2]. In the above cited works we note that a strong damping $-\Delta u_t$ is always assumed if space dimension $N \geq 3$ and $p > 2$ (cf. [2, 12, 13, 14, 15]). In dimension $N = 2$, weaker frictional damping $u_t$ was considered with $p = 4$ (cf. [5, 6]).

Our main result establishes the existence of a finite-dimensional global attractor to the system (1)-(3) with a weak damping. For the $N$-dimensional problem a corresponding result with strong damping $-\Delta u_t$ was proved by Yang [14]. We also present a complete proof of uniqueness for weak solutions which seems to be new in the context of $p$-Laplacian wave equations. In addition, with respect to global existence, we notice the Yang [12] studied problem (1)-(3) with weak damping $u_t$ by assuming

$$
\sigma(0) = \sigma'(0) = \sigma''(0) = 0 \quad \text{and} \quad \sigma''' \text{ is locally Lipschitz,}
$$

which implies that $p \geq 5$. In our case, we prove uniqueness and continuous dependence of initial data for $p \geq 3$. However, for the existence of global attractors we assume $p \geq 4$. This restriction can be weakened to $p \geq 2$ if a strong damping $-u_{txx}$ is added in the system.

Our work is organized as follows. In Section 2 we present the assumptions and the main results. In Section 3 we prove the existence of weak and stronger solutions. In Section 4 we prove the existence of a finite dimensional attractor.

2. Assumptions and main results. Let us assume $f \in C^1(\mathbb{R})$ and that there exists a constant $\rho > 0$ such that

$$
-\rho \leq \hat{f}(s) \leq f(s)s, \quad \forall s \in \mathbb{R}, \quad (5)
$$

where $\hat{f}(s) = \int_0^s f(\tau) d\tau$. In this paper we use standard notations for Sobolev spaces as in the book by Lions [10]. In $L^p(0,L)$, we denote the usual norm by $\|u\|_p = \int_0^L |u|^p dx$. Then the energy of the system (1)-(3) is written as

$$
E(t) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|u_{xx}(t)\|_2^2 + \frac{1}{p} \|u_x(t)\|_p^p + \int_0^L \hat{f}(u(t)) dx - \int_0^L hu(t) dx. \quad (6)
$$

For notation convenience, let us define

$$
H_6^0 = \{u \in H^3(0, L) | u(0) = u(L) = u_{xx}(0) = u_{xx}(L) = 0\} \quad \text{and} \quad \|u\|_{H_6^3} = \|u_{xxx}\|_2.
$$

**Theorem 2.1.** Assume that (5) holds, $p \geq 2$ and $h \in L^2(0, L)$.

(i) If the initial data $(u^0, u^1) \in (H^2(0, L) \cap H_0^1(0, L)) \times L^2(0, L)$, then problem (1)-(3) has a weak solution

$$
u \in C(\mathbb{R}^+; H^2(0, L) \cap H_0^1(0, L)) \cap C^1(\mathbb{R}^+; L^2(0, L)). \quad (7)
$$

(ii) If the initial data $(u^0, u^1) \in H_6^0(0, L) \times H_0^1(0, L)$, the above solution satisfies

$$
\nu \in L^\infty_{loc}(\mathbb{R}^+; H_6^0(0, L)), \quad \nu_{tt} \in L^\infty_{loc}(\mathbb{R}^+; H_0^1(0, L)), \quad u_{tt} \in L^\infty_{loc}(\mathbb{R}^+; H^{-1}(0, L)). \quad (8)
$$

(iii) If $p \geq 3$ then problem (1)-(3) is well-posed with respect to weak solutions.

The well-posedness of weak solutions asserted in Theorem 2.1 shows that problem (1)-(3) corresponds to a nonlinear $C_0$-semigroup $S(t)$ on the phase space

$$
\mathcal{H} = (H^2(0, L) \cap H_0^1(0, L)) \times L^2(0, L),
$$
Now, noting that for
\[ u \text{ where } C \text{ where } E \text{ defined by } \]
the corresponding dynamical system
Theorem 2.2. Assume the hypotheses of Theorem 2.1 hold with \( p \geq 4 \). Then the corresponding dynamical system \((\mathcal{H}, S(t))\) has a compact attractor with finite fractal dimension.

3. Well-posedness. In this Section we prove Theorem 2.1. The existence of a global solution is given by using Faedo-Galerkin method. The uniqueness of weak solutions is proved with a regularization argument.

Proof of Theorem 2.1. This will be done through several steps. The proof of existence of weak solutions is presented briefly.

Step 1. Approximate Problem. Let \( \{w_j\}_{j \in \mathbb{N}} \) be a Galerkin basis given by the eigenfunctions of \( u^m = \lambda u, u(0) = u(L) = u''(0) = u''(L) = 0 \), and let
\[
V^m = \text{span}\{w^1, w^2, \ldots, w^m\}.
\]
Given initial data \( u_0 \in H^2(0, L) \cap H^1_0(0, L) \) and \( u_1 \in L^2(0, L) \), we search an approximate solution of the form
\[
u^m(t) = \sum_{j=1}^{m} y^{mj}(t)w^j, \quad t \geq 0,
\]
which satisfies the approximate problem
\[
(u^m(t), w^j) + (u''^m(t), w^j) + (\sigma(u^m(t)), w^j) + (ku^m(t), w^j) + (f(u^m(t)) - h, w^j) = 0, \quad 1 \leq j \leq m,
\]
with initial condition
\[
u^m(0) = u^0, \quad u'^m(0) = u^1,
\]
where \( u^0 \) and \( u^1 \) are chosen such that
\[
u^0 \rightarrow u_0 \text{ in } H^2(0, L) \cap H_0^1(0, L) \quad \text{and} \quad u^1 \rightarrow u_1 \text{ in } L^2(0, L).
\]
By standard ODE theory, problem (10)-(12) has a local solution \( \nu^m(t) \). The estimate below will allow the local solutions be extended to an interval \([0, T]\), for any given \( T > 0 \).

Step 2. A Priori Estimate. Replacing \( w_j \) by \( u'^m(t) \) in (10) we get by integration over \([0, L]\),
\[
\frac{d}{dt}E^m(t) = -k\|u'^m(t)\|_2^2, \quad t \geq 0,
\]
where \( E^m(t) \) is the approximate energy defined from (6) with \( u \) replaced by \( u^m \). Now, noting that for \( u \in H^2(0, L) \cap H^1_0(0, L) \) one has \( \|u\|_2 \leq L^2\pi^{-2}\|u_{xx}\|_2 \), assumption (5) implies
\[
\frac{1}{4}\|u^m_x\|_2^2 + \int_{0}^{L} (f(u^m) - hu^m)dx + C_{h_p} \geq 0,
\]
where \( C_{h_p} = \rho L + L^2\pi^{-2}\|h\|_2^2 \). Then integrating (13) from 0 to \( t \) we conclude that
\[
\|u'^m(t)\|_2^2 + \|u''^m(t)\|_2^2 + \|u''^m(t)\|_p^p + \int_{0}^{t} \|u^m''(s)\|_2^2 ds \leq M_1, \quad t \geq 0,
\]
for some \( M_1 = M_1(\|u^0_{xx}\|_2, \|u^1\|_2) \).
Step 3. Passage to the Limit. From estimate (15), going to a subsequence if necessary, we infer that
\[ u^m \to u \text{ weakly star in } L^\infty(0,T; H^2(0,L) \cap H^1_0(0,L)), \]
\[ u^m_\alpha \to u_\alpha \text{ weakly star in } L^\infty(0,T; L^2(0,L) \cap L^2(0,T; L^2(0,L))). \]
Then we obtain from Aubin-Lions theorem,
\[ u^m \to u \text{ in } L^2(0,T; H^1_0(0,L)). \]
We also obtain
\[ u^m \to u \text{ in } C([0,T]; H^1_0(0,L)), \]
as proved, for instance, in Kim [8], Lemma 1.4. Then we can pass to the limit the approximate problem (10)-(11) in order to get a weak solution of problem (1)-(3). Then it follows that weak solutions of problem (1)-(3) have higher regularity (8). This essentially proves Theorem 2.1 (i).

Step 4. Regularity. Let us now consider initial data \( u^0 \in H^2_0(0,L) \) and \( u^1 \in H^1_0(0,L) \). The choice of the basis \( \{w^j\}_{j \in \mathbb{N}} \) implies that \( u^m_{\alpha x}(t) \in V^m \). Then by replacing \( w^j \) by \( u^m_{\alpha xx} \) in the approximate problem we can show that
\[ \|u^m_{\alpha x}(t)\|_2^2 + \|u^m_{xx}(t)\|_2^2 \leq M_2, \quad 0 \leq t \leq T, \] (16)
where \( M_2 = M_2(\|u^0_{xx}\|_2, \|u_1\|_2, M_1, T) \). Then it follows that weak solutions of problem (1)-(3) have higher regularity (8). This essentially proves Theorem 2.1 (ii).

Step 5. Uniqueness of Weak Solutions. We apply the classical regularization method of Vishik-Ladyzenskaya [10]. Let \( u, v \) be two weak solutions of problem (1)-(3). Then \( w = u - v \) is a weak solution of
\[ w_{tt} + w_{xxxx} = \sigma(u_{xx})_x - \sigma(v_{xx})_x - kw_t - f(u) + f(v), \] (17)
with boundary condition \( w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0 \) and null initial condition. For a fixed \( s \in [0,T] \) we define
\[ \psi(t) = \begin{cases} -\int_t^s w(\xi) d\xi & \text{if } 0 \leq t \leq s \leq T, \\ 0 & \text{if } 0 \leq s \leq t \leq T. \end{cases} \] (18)
Then we see that
\[ \psi \in C^1([0,T]; H^2(0,L) \cap H^1_0(0,L)), \quad \psi'(t) = w(t), \quad \psi(s) = 0. \]
With the above regularity for \( \psi(t) \), we can multiply equation (17) by \( \psi(t) \) and integrate with respect to \( x \) and \( t \). We have
\[ \int_0^s (w_{tt}(t) + w_{xxxx}(t), \psi(t)) dt = \int_0^s (\sigma(u_{xx}(t))_x - \sigma(v_{xx}(t))_x, \psi(t)) dt \\
- \int_0^s (kw_t(t), \psi(t)) dt \\
- \int_0^s (f(u(t)) - f(v(t)), \psi(t)) dt. \] (19)
Denoting \( w_1(t) = \int_0^t w(\xi) d\xi \) we have \( w(t) = w_1(t) - w_1(s) \), \( 0 < t < s \). Then, arguing as is in Lions [10], we obtain the following standard estimates:
\[ \left| \int_0^s (w_{tt}(t) + w_{xxxx}(t), \psi(t)) dt \right| = \frac{1}{2} (\|w(s)\|_2^2 + \|w_{1xx}(s)\|_2^2). \]
\[
\int_0^s \left( f(u(t)) - f(v(t)), \psi(t) \right) dt \leq \frac{1}{8} \|w_{1xx}(s)\|_2^2 + C_0 \int_0^s \left( \|w(t)\|_2^2 + \|w_{1xx}(t)\|_2^2 \right) dt,
\]
\[
\int_0^s k(w_1(t), \psi(t)) dt = k \int_0^s \|w(t)\|_2^2 dt,
\]
where \(C_0 > 0\) denotes a generic constant depending on the initial data. It remains
to estimate the term with \(p\)-Laplacian. First we note that
\[
\sigma(u_x) - \sigma(v_x) = \int_0^1 \frac{d}{d\varepsilon} \sigma(v_x + \varepsilon(u_x - v_x)) d\varepsilon = w_x \int_0^1 \sigma'(v_x + \varepsilon(u_x - v_x)) d\varepsilon.
\]
Defining
\[
z = \int_0^1 \sigma'(v_x + \varepsilon(u_x - v_x)) d\varepsilon, \quad z = z(x, t),
\]
we get from a priori estimate (15) that \(\|z(t)\|_\infty \leq C_0\). In addition, since \(p \geq 3\)
implies that \(w''\) is locally bounded, we also get \(\|z_x(t)\|_\infty \leq C_0\). Therefore
\[
\left| \int_0^L (\sigma(u_x(t))_x - \sigma(v_x(t))_x) \psi(t) \right| dx = \left| \int_0^L (\sigma(u_x(t)) - \sigma(v_x(t))) \psi_x(t) dx \right|
\]
\[
= \left| \int_0^L w_x(t) z(t) \psi_x(t) dx \right|
\]
\[
= \left| \int_0^L w(t)(z_x(t) \psi_x(t) + z(t) \psi_{xx}(t)) dx \right|
\]
\[
\leq C_0 \|w(t)\|_2 \|\psi_{xx}(t)\|_2.
\]
Then we infer that
\[
\left| \int_0^s (\sigma(u_x(t))_x - \sigma(v_x(t))_x, \psi(t)) dt \right| \leq C_0 \int_0^s \|w(t)\|_2 \|w_{1xx}(t) - w_{1xx}(s)\|_2 dt
\]
\[
\leq C_0 \int_0^s \left( \|w(t)\|_2^2 + \|w_{1xx}(t)\|_2^2 \right) dt + \frac{1}{8} \|w_{1xx}(s)\|_2^2.
\]
Therefore combining the above estimates with (19) yields
\[
\|w(s)\|_2^2 + \|w_{1xx}(s)\|_2^2 \leq C_0 \int_0^s \left( \|w(t)\|_2^2 + \|w_{1xx}(t)\|_2^2 \right) dt, \quad 0 \leq t \leq T.
\]
Since \(\|w(0)\|_2 = \|w_{1xx}(0)\|_2 = 0\), Gronwall inequality implies that
\[
w(s) = 0 \quad \text{in} \quad L^2(0, L), \quad \forall s \in (0, T),
\]
and therefore \(u = v\).

**Step 6. Continuous Dependence on Initial Data.** Since we have proved the
uniqueness of weak solutions, we can work on stronger solutions satisfying (8)
and then extend the conclusion to weak solutions by standard density arguments. Given
initial data \((u^0, u^1)\) and \((v^0, v^1)\) in \(H^1_0(0, L) \times H^1_0(0, L)\), we write \(w = u - v\) where
The corresponding regular solutions. Then \( w \) satisfies equation (17) with initial data \( w(0) = u^0 - v^0 \) and \( w_t(0) = u^1 - v^1 \). In addition,

\[
\frac{1}{2} \frac{d}{dt} \left( \| w_t(t) \|_2^2 + \| w_{xx}(t) \|_2^2 \right) = -k \| w_t(t) \|_2^2 - \int_0^L (f(u(t)) - f(v(t))) w_t(t) \, dx \\
+ \int_0^L \left( \sigma(u_x(t))_x - \sigma(v_x(t))_x \right) w_t(t) \, dx.
\]

(20)

Now,

\[
\int_0^L \left( \sigma(u_x(t))_x - \sigma(v_x(t))_x \right) w_t(t) \, dx = \int_0^L \sigma'(u_x) w_{xx} w_t \, dx \\
+ \int_0^L \left( \sigma'(u_x) - \sigma'(v_x) \right) v_{xx} w_t \, dx.
\]

Using again that \( \sigma'' \) is locally bounded \( (p \geq 3) \), we get

\[
\int_0^L \left( \sigma(u_x(t))_x - \sigma(v_x(t))_x \right) w_t(t) \, dx \leq C_0 \left( \| w_t(t) \|_2^2 + \| w_{xx}(t) \|_2^2 \right),
\]

where \( C_0 > 0 \) is a constant depending on the initial data. Then as before see that

\[
\frac{d}{dt} \left( \| w_t(t) \|_2^2 + \| w_{xx}(t) \|_2^2 \right) \leq C_0 \left( \| w_t(t) \|_2^2 + \| w_{xx}(t) \|_2^2 \right), \quad t \geq 0.
\]

Hence, given \( T > 0 \), there exists a constant \( c_0 > 0 \) such that

\[
\| u_{xx}(t) - v_{xx}(t) \|_2^2 + \| u_t(t) - v_t(t) \|_2^2 \leq (\| u_{xx}^0 - v_{xx}^0 \|_2^2 + \| u^1 - v^1 \|_2^2) e^{c_0 T}, \quad (21)
\]

for all \( t \in [0, T] \). By density, inequality (21) holds for weak solutions, which proves the continuous dependence of weak solutions on the initial data. Then the proof of Theorem 2.1 (iii) is complete. \( \square \)

4. Global attractors. The definitions and classical results to global attractors of infinite dimensional dynamical systems can be found, e.g., in the books by Babin & Vishik [3], Hale [7], Ladyzhenskaya [9] and Teman [11]. We follow closely the book by Chueshov & Lasiecka [4], Chapter 7.

Our framework is that of dissipative dynamical systems. One says that \( (\mathcal{H}, S(t)) \) is dissipative if it possesses an absorbing set, that is, a bounded set \( B \subset \mathcal{H} \) such that for any bounded set \( B \subset \mathcal{H} \) there exists \( t_B \geq 0 \) satisfying

\[
S(t)B \subset \mathcal{B}, \quad \forall t \geq t_B.
\]

Let \( \mathcal{H} = X \times Y \) with \( X \) compactly embedded in \( Y \). Suppose that \( (\mathcal{H}, S(t)) \) is a dynamical system given by an evolution operator

\[
S(t)w = (u(t), u_t(t)), \quad t \geq 0, \quad w = (u(0), u_t(0)) \in \mathcal{H},
\]

(22)

where the function \( u \) has regularity

\[
u \in C(\mathbb{R}^+; X) \cap C^1(\mathbb{R}^+; Y).
\]

(23)

We recall that a seminorm \( n_X(x) \) defined on \( X \) is compact if any sequence such that \( x_j \to 0 \) weakly in \( X \) implies that \( n_X(x_j) \to 0 \). Then one says that \( (\mathcal{H}, S(t)) \) is quasi-stable on a set \( B \subset \mathcal{H} \) if there exists a compact seminorm \( n_X(x) \) on \( X \) and nonnegative scalar functions \( a(t) \) and \( c(t) \), locally bounded in \([0, \infty)\), and \( b(t) \in L^1(\mathbb{R}^+) \) with \( \lim_{t \to \infty} b(t) = 0 \), such that

\[
\| S(t)w^1 - S(t)w^2 \|^2_H \leq a(t)\| w^1 - w^2 \|^2_H
\]

(24)
and
\[ \|S(t)w^1 - S(t)w^2\|_{H^4}^2 \leq b(t)\|w^1 - w^2\|_{H^4}^2 + c(t) \sup_{0<\tau<t} |n_X(u^1(s) - u^2(s))|^2, \] (25)
for any \( w^1, w^2 \in B \). The inequality (25) is often called stabilizability inequality. In this context, the following result is proved in Chueshov & Lasiecka [4], Corollary 7.9.5 and Theorem 7.9.6.

**Theorem 4.1.** Assume that \((\mathcal{H}, S(t))\) is a dissipative dynamical system of the form (22) and satisfying (23). Assume in addition that the system is quasi-stable on any bounded positively invariant set. Then \((\mathcal{H}, S(t))\) has compact a global attractor with finite fractal dimension.

In order to apply Theorem 4.1 we prove the following two lemmas.

**Lemma 4.2 (Absorbing Set).** The system \((S(t), \mathcal{H})\) has an absorbing set.

**Proof.** Given \( \varepsilon > 0 \) let
\[ E_{\varepsilon}(t) = E(t) + \varepsilon \Psi(t) \quad \text{where} \quad \Psi(t) = \int_0^L u_t(t) u(t) dx. \]
From (14) we see that
\[ E(t) \geq \frac{1}{4} (\|u_{xx}(t)\|_2^2 + \|u_t(t)\|_2^2) - C_h\rho. \] (26)
Then there exists \( \varepsilon_0 > 0 \) such that
\[ \frac{1}{2} E(t) - \frac{1}{2} C_h\rho \leq E_{\varepsilon}(t) \leq \frac{3}{2} E(t) + \frac{1}{2} C_h\rho, \quad \forall \ t \geq 0, \ \forall \ \varepsilon \in (0, \varepsilon_0). \] (27)
Next we show that there exists \( \varepsilon_1 > 0 \) such that
\[ E_{\varepsilon}(t) \leq -\varepsilon E(t), \quad \forall \ t \geq 0, \ \forall \ \varepsilon \in (0, \varepsilon_1). \] (28)
By density arguments, we can assume that solutions are regular. Then using equation (1) and adding and subtracting \( E(t) \) we get
\[
\Psi'(t) = \quad -E(t) + \frac{3}{2} \|u_t(t)\|_2^2 - \frac{1}{2} \|u_{xx}(t)\|_2^2 - \left(1 - \frac{1}{p}\right) \|u_x(t)\|_p^p
\]
\[ \quad - \int_0^L ku(t)u_t(t) dx + \int_0^L (\hat{f}(u(t)) - f(u(t))u(t)) dx. \]
Since \( (ku(t), u_t) \leq \frac{1}{2} \|u_{xx}(t)\|_2^2 + C_0\|u_t(t)\|_2^2 \), we conclude in view of (5) that
\[ \Psi'(t) \leq -E(t) + \left(\frac{3}{2} + C_0\right) \|u_t(t)\|_2^2. \]
Now choosing \( \varepsilon < \min\{\varepsilon_0, \varepsilon_1\} \) such that \( \varepsilon \left(\frac{3}{2} + C_0\right) \leq k \), the above inequality implies (28). Then combining (28) with (27) yields
\[ E(t) \leq (3E(0) + C_h\rho) e^{-\frac{\varepsilon t}{2}} + 2C_h\rho, \quad t \geq 0. \] (29)
From estimate (26) we see that
\[ \|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 \leq (12E(0) + 4C_h\rho)e^{-\frac{\varepsilon t}{2}} + 12C_h\rho, \quad t \geq 0. \]
Taking \( R > (12C_h\rho)^{1/2} \) the ball \( B(0, R) \subset \mathcal{H} \) is an absorbing set. \( \square \)
Remark 1. From the proof of the Lemma 4.2 we see that if $h = \rho = 0$ then the system’s energy $E(t)$ decays exponentially. More precisely, from (29),

$$E(t) \leq 3E(0)e^{-\frac{\mu}{2}t}, \ t \geq 0.$$ 

Lemma 4.3 (Stabilizability Inequality). Assume $p \geq 4$. Given a bounded invariant set $B \subset H$ and initial data $z^1 = (u^0, u^1)$ and $z^2 = (v^0, v^1)$ in $B$, there exists $\mu > 0$ such that, for all $t \geq 0$,

$$\|S(t)z^1 \! - \! S(t)z^2\|_H^2 \leq C_B e^{-\mu t} \|z^1 \! - \! z^2\|_H^2 + C_B \int_0^t e^{-\mu(t-s)} \|u_x(s) \! - \! v_x(s)\|_2^2 \, ds,$$  

(30)

where $u, v$ are the corresponding weak solutions of (1)-(3) and $C_B > 0$ is a constant depending on $B$ but not on $t$.

Proof. Let us write $w = u - v$ where $u, v$ are the corresponding solutions. Then $w$ satisfies equation (17) with initial data $w(0) = u^0 - v^0$ and $w_t(0) = u^1 - v^1$. By density, we can assume $u, v$ regular. Let us define

$$F(t) = \frac{1}{2} \|w_{xx}(t)\|_2^2 + \frac{1}{2} \|w_t(t)\|_2^2 + \frac{1}{2} \int_0^L \sigma'(u_x(t))w_x^2(t) \, dx,$$

and

$$F_\eta(t) = F(t) + \eta \phi(t) \text{ where } \phi(t) = \int_0^L w(t)w_t(t) \, dt.$$ 

As in the proof of (27), it is easy to see that for $\eta > 0$ small,

$$\frac{1}{2} F(t) \leq F_\eta(t) \leq \frac{3}{2} F(t), \ \forall t \geq 0.$$  

(31)

Let us show that

$$F_\eta'(t) + \frac{2\eta}{3} F_\eta(t) \leq C_0 \|w_x(t)\|_2^2,$$  

(32)

for a constant $C_0 > 0$ dependent of $B$ but not of $t \geq 0$.

In fact, from equation (17) we obtain (20). To estimate the right hand side of (20) we see that

$$\left| \int_0^L (f(u(t)) - f(v(t)))w_t(t) \, dx \right| \leq \frac{k}{4} \|w_t(t)\|_2^2 + C_0 \|w_x(t)\|_2^2.$$ 

Now,

$$\int_0^L (\sigma'(u_x)x - \sigma'(v_x)x)w_t \, dx = \int_0^L [\sigma'(u_x)x - \sigma'(v_x)x]w_t \, dx$$

$$= \int_0^L \sigma'(u_x)xw_xw_t \, dx + \int_0^L \sigma''(\xi)xw_xw_t \, dx,$$

where $\xi = \theta u_x + (1-\theta)v_x$, $0 \leq \theta \leq 1$, is given by the Mean Value Theorem. Since $p \geq 4$, $|\sigma''|$ and $|\sigma'''|$ are locally bounded. Then

$$|B| \leq C_0 \|w_t(t)\|_2 \|w_x(t)\|_2,$$
and

\[ A = \sigma'(u_x(t))w_t(t)w_x(t)\frac{d}{dt} - \int_0^L \sigma'(u_x(t))w_t(t)xw_x(t)\,dx \]

\[ = -\int_0^L \sigma'(u_x(t))w_{xt}(t)w_x(t)\,dx - \int_0^L \sigma''(u_x(t))u_{xx}(t)w_t(t)w_x(t)\,dx \]

\[ \leq -\int_0^L \sigma'(u_x(t))w_{xt}(t)w_x(t)\,dx + C_0\|w_t(t)\|_2\|w_x(t)\|_2. \]

On other hand

\[-\int_0^L \sigma'(u_x(t))w_{xt}(t)w_x(t)\,dx = -\frac{1}{2} \frac{d}{dt} \int_0^L \sigma'(u_x(t))w_x^2(t)\,dx + \frac{1}{2} \int_0^L \sigma''(u_x(t))u_{xx}(t)w_x^2(t)\,dx, \]

and

\[ \frac{1}{2} \int_0^L \sigma''(u_x(t))u_{xx}w_x^2 \,dx = -\frac{1}{2} \int_0^L u_t(t)\sigma'''(u_x(t))u_{xx}(t)w_x^2(t)\,dx \]

\[ -\frac{1}{2} \int_0^L u_t(t)\sigma''(u_x(t))2w_x(t)w_{xx}(t)\,dx \]

\[ \leq \delta\|w_{xx}(t)\|_2^2 + C_0\|w_x(t)\|_2^2, \]

where \( \delta > 0 \) is a small parameter to be fixed later. Combining these estimates we get

\[ \int_0^L (\sigma(u_x(t))_x - \sigma(v_x(t))_x)w_t(t)\,dx \leq -\frac{1}{2} \frac{d}{dt} \int_0^L \sigma'(u_x(t))w_x^2(t)\,dx \]

\[ + \frac{k}{4} \|w_t(t)\|_2^2 + \delta\|w_{xx}(t)\|_2^2 + C_0\|w_x(t)\|_2^2. \]

Therefore from (20)

\[ F'(t) \leq -\frac{k}{4} \|w_t(t)\|_2^2 + \delta\|w_{xx}(t)\|_2^2 + C_0\|w_x(t)\|_2^2. \]

Similar estimates give

\[ \phi'(t) + F(t) = \frac{3}{2} \|w_t(t)\|_2^2 + \frac{1}{2} \|w_{xx}(t)\|_2^2 + \int_0^L w(t)w_{tt}(t)\,dx \]

\[ \leq \|w_t(t)\|_2^2 - \frac{1}{2} \|w_{xx}(t)\|_2^2 + C_0\|w_x(t)\|_2^2. \]

Then choosing \( 0 < \eta < k/4 \) and \( 0 < \delta < \eta/2 \) we see that

\[ F'(t) + \eta F(t) \leq C_0\|w_x(t)\|_2^2. \]

Taking a smaller \( \eta > 0 \) if necessary, inequality (31) also holds, and therefore we get (32). Then we infer that

\[ F(t) \leq 3F(0)e^{-\frac{2\mu}{k}t} + C_0\int_0^t e^{-\frac{2\mu}{k}(t-s)}\|w_x(s)\|_2^2\,ds, \quad t \geq 0. \]

From definition of \( F(t) \) we obtain (30) with \( \mu = 2\eta/3 \).

**Proof of Theorem 2.2.** Our dynamical system is defined by the evolution operator (9) and has regularity (7). Therefore (22) and (23) hold. We also see that \((H, S(t))\)
is quasi-stable in bounded positively invariant sets. Indeed, condition (24) comes from (21) and condition (25) follows promptly from (30) with $n_X(u) = \|u_x\|_2$. Then taking into account that Lemma 4.2 implies that $(\mathcal{H}, S(t))$ is dissipative, we conclude from Theorem 4.1 that our system has a compact global attractor with finite fractal dimension. This ends the proof of Theorem 2.2.

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