Research Article

New Fixed Point Results in $\mathcal{F}$-Quasi-Metric Spaces and an Application

Ehsan Lotfali Ghasab,¹ Hamid Majani,¹ Erdal Karapinar,²,³ and Ghasem Soleimani Rad⁴

¹Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran
²Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
³Department of Mathematics, Çankaya University, 06790, Etimumet, Ankara, Turkey
⁴Young Researchers and Elite club, West Tehran Branch, Islamic Azad University, Tehran, Iran

Correspondence should be addressed to Hamid Majani; majani.hamid@gmail.com and Erdal Karapinar; erdalkarapinar@yahoo.com

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In the last century, nonlinear functional analysis has experienced many advances. One of these improvements is the introduction of various spaces and is the proof of fixed point results in these spaces along with their applications in engineering science. One of these spaces is function weighted metric space introduced by Jleli and Samet [1]. This is a generalization of metric spaces.

**Definition 1** [1, 2]. A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is named a nondecreasing function if $f(s_1) \leq f(s_2)$ for every $s_1, s_2 \in (0, +\infty)$. Also, $f$ is said to be logarithmic-like when every positive sequence $\{t_n\}$ satisfies $\lim_{n \to \infty} t_n = 0$ iff $\lim_{n \to \infty} f(t_n) = -\infty$.

In the sequel, we apply $\mathcal{F}$ for the set of all nondecreasing functions that are logarithmic-like.

In 2019, some of researchers such as Alqahtani et al. [2], Aydi et al. [3], and Bera et al. [4] discussed on the structure of this space and on the fixed points of mappings satisfying in various contractive conditions.

**Definition 2** [1]. Consider a mapping $\delta : X \times X \rightarrow [0, +\infty)$, a constant $B \in [0, +\infty)$ and a $f \in \mathcal{F}$ so that

$\langle \eta_1 \rangle \delta(x_1, x_2) = 0 \iff x_1 = x_2$ for all $x_1, x_2 \in X$;

$\langle \eta_1 \rangle \delta(x_1, x_2) = \delta(x_1, x_2)$ for all $x_1, x_2 \in X$ and $\langle \eta_2 \rangle \delta(x_1, x_2) > 0$ implies that $f(\delta(x_1, x_2)) \leq f(\sum_{i=1}^{N-1} \delta(v_i, v_{i+1})) + B$ for every $N \in \mathbb{N}$ with $N \geq 2$, for every $x_1, x_2 \in X$, and for all $(v_i)_{i=1}^N \subseteq X$ with $(v_1, v_N) = (x_1, x_2)$.

Then, the function $\delta$ is named as a function weighted metric or a $\mathcal{F}$-metric on $X$, and the pair $(X, \delta)$ is called a $\mathcal{F}$-metric space.

**Definition 3** [5]. Consider a mapping $\delta_q : X \times X \rightarrow [0, +\infty)$ satisfies the properties $\langle \eta_1 \rangle$ and $\langle \eta_3 \rangle$ from the definition of a $\mathcal{F}$-metric. Then, $\delta_q$ is named a $\mathcal{F}$-quasi-metric on $X$ and $(X, \delta_q)$ is called a $\mathcal{F}$-quasi-metric space.

In [5], Karapinar et al. showed that $\delta_q$ generally induces other $\mathcal{F}$-quasi-metrics such as $\delta_q^{-1}, \delta_q^*: X \times X \rightarrow [0, +\infty)$ defined by $\delta_q^{-1}(s, t) = \delta_q(t, s)$ and $\delta_q^*(s, t) = \max \{\delta_q(s, t), \delta_q^{-1}(s, t)\}$. Regarding the discussion above, we conclude that any quasi-metric is a $\mathcal{F}$-quasi-metric by choosing $f(t) = \ln t$ for the axiom $\langle \eta_1 \rangle$ with $B = 0$. 

1. Introduction and Preliminaries

In the last century, nonlinear functional analysis has experienced many advances. One of these improvements is the introduction of various spaces and is the proof of fixed point results in these spaces along with their applications in engineering science. One of these spaces is function weighted metric space introduced by Jleli and Samet [1]. This is a generalization of metric spaces.
Definition 4 [5]. Let \((X, \delta_q)\) be a \(\mathcal{F}\)-quasi-metric space. For \(x \in X\), the right (left) centered ball at \(x\) and of radius \(\epsilon > 0\) is the set \(B_r(s, e) = \{ t \in X : \delta_q(s, t) < \epsilon \} \) \((B_l(s, e) = \{ t \in X : \delta_q(t, s) < \epsilon \})\).

Definition 5 [5]. Consider a \(\mathcal{F}\)-quasi-metric space \((X, \delta_q)\) with a sequence \(\{x_n\}\) therein. Then, \(\{x_n\}\) is said to be a right-convergent sequence (left-convergent sequence) to \(x \in X\) if \(\lim_{n \to \infty} \delta_q(x, x_n) = 0 \) \((\lim_{n \to \infty} \delta_q(x_n, x) = 0)\). Further, \(\{x_n\}\) is said to be a biconvergent sequence (in summary, convergent sequence) when it is both right-convergent and left-convergent.

Proposition 6 [5]. Consider a \(\mathcal{F}\)-quasi-metric space \((X, \delta_q)\) with a sequence \(\{x_n\}\) therein. Also, let \(\lim_{n \to \infty} \delta_q(x, x_n) = \lim_{n \to \infty} \delta_q(x_n, t) = 0\) for every \(s, t \in X\). Then \(s = t\).

Definition 7 [5]. Consider a \(\mathcal{F}\)-quasi-metric space \((X, \delta_q)\) with a sequence \(\{x_n\}\) therein. Then, \(\{x_n\}\) is a right-Cauchy sequence (left-Cauchy sequence) if \(\lim_{n,m \to \infty} \delta_q(x_m, x_n) = 0 \) \((\lim_{n,m \to \infty} \delta_q(x_n, x_m) = 0)\). With this interpretation, \(\{x_n\}\) is bi-Cauchy sequence (in summary, Cauchy sequence) if it is both left-Cauchy sequence and right-Cauchy sequence.

Now, a \(\mathcal{F}\)-quasi-metric space \((X, \delta_q)\) is named right-complete (left-complete) if every right-Cauchy sequence (left-Cauchy sequence) in \(X\) is a right-convergent sequence (left-convergent sequence) in \(X\). Further, \((X, \delta_q)\) is bicomplete (in short, complete) if it is both left-complete and right-complete.

Example 8. Define \(\delta_q : \mathbb{N} \times \mathbb{N} \rightarrow [0, +\infty)\) by

\[
\delta_q(s, t) = \begin{cases} 
0 & \text{when } s = t \\
\epsilon^s + |s - t| & \text{otherwise,}
\end{cases}
\]

for every \(s, t \in \mathbb{N}\). Evidently, \((\mathbb{N}, \delta_q)\) is a bicomplete \(\mathcal{F}\)-quasi-metric space.

On the other hand, Bhaskar and Lakshmikantham [6] defined the notion of coupled fixed point and presented several coupled fixed point propositions for a mixed monotone mapping in partially ordered metric spaces. Also, they studied the existence and uniqueness of a solution to a periodic boundary value problem. For more details on coupled, tripled, and \(n\)-tupled fixed point assertions, one can see [7] and references therein.

Definition 9 [8, 9]. Let \(F : X \times X \rightarrow X\) and \(g : X \rightarrow X\) be two optional mappings. An element \((u, v) \in X \times X\) is said to be a coupled coincidence point of \(F\) and \(g\) if \(F(u, v) = gu\) and \(F(v, u) = gv\). Further, an element \((u, v) \in X \times X\) is named a common fixed point of \(F\) and \(g\) if \(F(u, v) = gu = u\) and \(F(v, u) = gv = v\).

Note that if \(g\) is the identity mapping, then \((x, y)\) is called a coupled fixed point of \(F\) [6].

Definition 10 [9]. Let \(F : X \times X \rightarrow X\) and \(g : X \rightarrow X\) be two optional mappings. Then \(F\) and \(g\) is said to be commutative if \(F(gu, gv) = g(F(u, v))\) for every \(u, v \in X\).

In this paper, we introduce several common fixed point and common coupled fixed point theorems in such spaces and prove them. In Section 2, we prove a common fixed point theorem and a common coupled fixed point result in this space. In Section 3, we obtain a coincidence point result for single-valued and multivalued mappings regarding a Hausdorff \(\delta\)-distance. Ultimately, as an application of these results, the existence of solution of the Volterra-type integral equation is investigated in Section 4.

2. \(\mathcal{F}\)-Quasi-Metric Space and Fixed Point Theory

Theorem 11. Let \((X, \delta_q)\) be a bicomplete \(\mathcal{F}\)-quasi-metric space. Also, let \(g, T : X \rightarrow X\) be two arbitrary mappings so that \(T\) and \(g\) are commutative, \(T(X) \subseteq g(X), g(X)\) is closed, and

\[
\delta_q(Tx, Ty) \leq k\delta_q(gx, gy),
\]

for every \(x, y \in X\), where \(k \in (0, 1)\). Then \(T\) and \(g\) contain a unique common fixed point in \(X\).

Proof. Due to \(T(X) \subseteq g(X)\), we select a point \(x_1 \in X\) such that \(Tx_1 = gx_1\) for a given \(x_1 \in X\). By continuing this process, we can construct a sequence \(y_n\) in \(X\) by \(y_n = Tx_n = gx_{n+1}\) for \(n = 0, 1, \cdots\). First, note that \(T\) and \(g\) possess a unique coincidence point. On the contrary, assume that \(u_1, v_1 \in X\) are two different coincidence points of \(T\) and \(g\). Then, \(\delta_q(u_2, v_2) > 0\) with \(gu_1 = Tu_1 = u_2\) and \(gv_1 = Tv_1 = v_2\). Now, by (2), we get

\[
\delta_q(u_2, v_2) = \delta_q(Tu_1, Tv_1) \leq k\delta_q(gu_1, gv_1) = k\delta_q(u_2, v_2),
\]

which is a contradiction.

Assume \((f, B) \in \mathcal{F} \times [0, +\infty)\) so that \((\eta_\varepsilon)\) is complied. For an arbitrary \(\varepsilon > 0\) and because of \((\eta_\varepsilon)\), there exists \(\gamma > 0\) such that

\[
0 < t < \varepsilon \Rightarrow f(t) < f(\varepsilon) - B.
\]

Now, let \(\{y_n\}\) be a sequence in \(X\). Without loss of totality, suppose that \(\delta_q(Tx_n, Tx_{n+1}) > 0\). Otherwise, \(x_1\) is a coincidence point of \(T\) and \(g\). Now, using (2), we obtain

\[
\delta_q(Tx_n, Tx_{n+1}) \leq k\delta_q(gx_n, gx_{n+1}) = k\delta_q(Tx_{n-1}, Tx_n) \leq k^2\delta_q(gx_{n-1}, gx_n),
\]

for every \(n \in \mathbb{N}\).
which implies by induction that
\[ \delta_q(Tx_n, Tx_{n+1}) \leq k^n \delta_q(Tx_0, Tx_1) \] (6)
for every \( n \) in \( \mathbb{N} \). Hence, for every \( m \) and \( n \) in \( \mathbb{N} \) so that \( m > n \), we get
\[ \sum_{i=n}^{m-1} \delta_q(Tx_i, Tx_{i+1}) \leq \frac{k^n}{1-k} \delta_q(Tx_0, Tx_1). \] (7)
Since
\[ \lim_{n \to \infty} \frac{k^n}{1-k} \delta_q(Tx_0, Tx_1) = 0, \] (8)
there is some \( N \in \mathbb{N} \) so that \( 0 < (k^n/1-k) \delta_q(Tx_0, Tx_1) < \gamma \) for every \( n \geq N \). Hence, from (4) and (\( \eta_i \)), we observe that
\[ f \left( \sum_{i=n}^{m-1} \delta_q(Tx_i, Tx_{i+1}) \right) \leq f \left( \frac{k^n}{1-k} \delta_q(Tx_0, Tx_1) \right) < f(\epsilon) - B, \] (9)
for \( m > n \geq N \). Employing (\( \eta_3 \)) together with (9), we obtain
\[ \delta_q(Tx_n, Tx_m) > 0 \implies f(\delta_q(Tx_n, Tx_m)) \leq f \left( \sum_{i=n}^{m-1} \delta_q(Tx_i, Tx_{i+1}) \right) + B < f(\epsilon). \] (10)

It follows that \( \delta_q(Tx_n, Tx_m) < \epsilon \). Therefore, \( \{y_n\} = \{Tx_n\} \) is right-Cauchy. Similarly, by changing the order of the pairs \((x_{i_1}, x_i)\) in the above process, we conclude that \( \{y_n\} \) is also a left-Cauchy sequence. Hence, it is a Cauchy sequence. Now, since \((X, \delta_q)\) is a bicomplete space, there exists \( z \in X \) such that \( \{y_n\} \) is convergent to \( z \). Since \( \{Tx_n\} = \{g\alpha_{n+1}\} \subset g(X) \) and \( g(X) \) is closed, we have \( \lim_{n \to \infty} \delta\left(g\alpha_{n}, zg\right) = 0 \). As a next step, we show that \( z \) is a coincidence point of \( T \) and \( g \). On the contrary, consider \( \delta_q(Tz, gz) > 0 \). Then we have
\[ f(\delta_q(Tz, gz)) \leq f(\delta_q(Tz, Tx_n) + \delta_q(Tx_n, gz)) + B \leq f(k\delta_q(gz, gx_n) + \delta_q(gx_{n+1}, gz)) + B. \] (11)

As \( n \to \infty \) in the inequality above, we obtain
\[ \lim_{n \to \infty} f(k\delta_q(gz, gx_n) + \delta_q(gx_{n+1}, gz)) + B = -\infty, \] (12)
which is a contradiction. Hence, \( \delta_q(Tz, gz) = 0 \); that is, \( z \) is a unique coincidence point of \( T \) and \( g \). Therefore, \( g \) and \( T \) contain a unique point of coincidence \( w = gz = Tw \). By commutativity of the mapping \( T \) and \( g \), we have \( gw = g(gz) = gT(z) = Tg(z) = Tw \). Hence, \( gw \) is another point of coincidence of \( g \) and \( T \). Now, by the uniqueness of the point of coincidence of \( g \) and \( T \), we have \( w = gw = Tw \); that is, \( g \) and \( T \) contain a unique common fixed point. This completes the proof.

In the sequel, denote for simplicity \( X \times \cdots \times X \) by \( X^n \), where \( X \) is a nonempty set and \( n \in \mathbb{N} \).

**Lemma 12.** Consider a \( \mathcal{F} \)-quasi-metric space \((X, \delta_q)\). Then, the following assertions hold:

1. \((X^n, \Delta_q)\) is a \( \mathcal{F} \)-quasi-metric space with
\[ \Delta_q((u_1, \cdots, u_n), (v_1, \cdots, v_n)) = \max \{\delta_q(u_1, v_1), \delta_q(u_2, v_2), \cdots, \delta_q(u_n, v_n)\}. \] (13)

2. The mapping \( f : X^n \to X \) and \( g : X \to X \) contain an \( n \)-tuple common fixed point iff the mapping \( F : X^n \to X^n \) and \( G : X^n \to X^n \) defined by
\[ F(u_1, u_2, \cdots, u_n) = (f(u_1, u_2, \cdots, u_n), f(u_2, \cdots, u_n, u_1), \cdots, f(u_n, u_1, \cdots, u_{n-1})) \]
\[ G(u_1, u_2, \cdots, u_n) = (gu_1, gu_2, \cdots, gu_n) \] (14)
possess a common fixed point in \( X^n \).

3. \((X, \delta_q)\) is bicomplete iff \((X^n, \Delta_q)\) is bicomplete.

**Proof.** Clearly, \( \Delta_q \) satisfies in (\( \eta_3 \)). We show that \( \Delta_q \) satisfies in (\( \eta_3 \)). For every \((x_{i_1}) \subset X \) for \( 1 \leq i \leq N \) and \( 1 \leq j \leq n \), consider \((x_{i_1}, x_{i_{N+1}}) = (u_1, v_1)\). Suppose that
\[ \delta_q(u_j, v_j) = \max \{\delta_q(u_1, v_1), \delta_q(u_2, v_2), \cdots, \delta_q(u_n, v_n)\}. \] (15)

Then, we have
\[ f_j(\delta_q(u_j, v_j)) \leq f_j \left( \sum_{i=1}^{N-1} \delta_q(x_{i_1}, x_{i_{N+1}}) \right) + B_j, \] (16)
where \( f_j \in \mathcal{F} \) and \( B_j \in [0, +\infty) \). Therefore, we obtain
\[ f_j(\Delta_q((u_1, u_2, \cdots, u_n), (v_1, v_2, \cdots, v_n))) = f_j(\Delta_q((u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n))) \]
\[ = f_j(\delta_q(u_1, v_1)) \leq f_j \left( \sum_{i=1}^{N-1} \delta_q(x_{i_1}, x_{i_{N+1}}) + B_j \right) \]
\[ \leq f_j \left( \sum_{i=1}^{N-1} \Delta_q((x_{i_1}, x_{i+1}), (x_{i_{N+1}}, x_{i_{N+1}})) \right) + B_j. \] (17)

The proofs of (2) and (3) are straightforward and left to the reader.

Remember that Lemma 12 is a two-way relationship. Consequently, we can establish \( n \)-tuple fixed point propositions.
from fixed point assertions and conversely. Now, set \( n = 2 \) in Lemma 12. Then, we have the following theorem.

**Theorem 13.** Let \((X, \delta_q)\) be a bicomplete \(\mathcal{F}\)-quasi-metric space. Also, let \(g : X \to X\) and \(T : X^2 \to X\) be two arbitrary mappings so that \(T\) and \(g\) are commutative, \(T(X^2) \subset g(X)\), \(g(X)\) is closed, and

\[
\delta_q(T(x, y), (x^*, y^*)) \leq \frac{k}{2}(\delta_q(gx, gx^*) + \delta_q(gy, gy^*)),
\]

for all \((x, y)\) and \((x^*, y^*)\) in \(X^2\), where \(k \in (0, 1)\). Then, \(T\) and \(g\) contain a unique common coupled fixed point in \(X^2\).

**Proof.** Let us define \(\Delta_q : X^2 \times X^2 \to [0, \infty)\) by \(\Delta_q((u_1, v_1), (u_2, v_2)) = \max\{\delta_q(u_1, v_1), \delta_q(u_2, v_2)\}\) for all \(u_1, u_2, v_1, v_2 \in X\). Further, consider \(F : X^2 \to X^2\) by \(F(x, y) = (T(x, y), T(y, x))\) and \(G : X^2 \to X^2\) by \(G(x, y) = (gx, gy)\) for all \(x, y \in X\). Using Lemma 12, \((X^2, \Delta_q)\) is a bicomplete \(\mathcal{F}\)-quasi-metric space. Also, \((x, y) \in X^2\) is a common coupled fixed point of \(T\) and \(g\) if and only if it is a common fixed point of \(F \) and \(G\). On the other hand, from (18), we have either

\[
\Delta_q(F(x, y), (x^*, y^*))
\]

\[
= \Delta_q((T(x, y), T(y, x)), (T(x^*, y^*), T(y^*, x^*)))
\]

\[
= \max\{\delta_q(T(x, y), T(x^*, y^*)), \delta_q(T(y, x), T(y^*, x^*))\}
\]

\[
= \delta_q(T(x, y), T(x^*, y^*)) \leq \frac{k}{2}(\delta_q(gx, gx^*) + \delta_q(gy, gy^*))
\]

\[
\leq k \max\{\delta_q(gx, gx^*), \delta_q(gy, gy^*)\}
\]

\[
= k\Delta_q(g(x, y), (x^*, y^*)),
\]

or

\[
\Delta_q(F(x, y), (x^*, y^*))
\]

\[
= \Delta_q((T(x, y), T(y, x)), (T(x^*, y^*), T(y^*, x^*)))
\]

\[
= \max\{\delta_q(T(x, y), T(x^*, y^*)), \delta_q(T(y, x), T(y^*, x^*))\}
\]

\[
= \delta_q(T(y, x), T(y^*, x^*)) \leq \frac{k}{2}(\delta_q(gy, gy^*) + \delta_q(gx, gx^*))
\]

\[
\leq k \max\{\delta_q(gy, gy^*), \delta_q(gx, gx^*)\}
\]

\[
= k\Delta_q(g(y, x), (y^*, x^*)).
\]

Now, by Theorem 11, \(F\) and \(G\) have a common fixed point and by Lemma 12, \(T\) and \(g\) have a common coupled fixed point.

**Example 14.** Let \(X = [0, 1]\). Define \(\delta_q : X \times X \to [0, \infty)\) by

\[
\delta_q(s, t) = \begin{cases} 
0 & \text{when } s = t \\
|s| + |s - t| & \text{otherwise},
\end{cases}
\]

for every \(x, y \in X\). Evidently, \(\delta_q\) is a bicomplete \(\mathcal{F}\)-quasi-metric with \(f(t) = \ln t\) and \(B = 0\). Consider \(T : X^2 \to X\) by \(T(x, y) = (x/2) + (y/2)\) and define \(g : X \to X\) by \(g(x) = 2x\). Clearly \(T\) and \(g\) are commutative. Also, we have

\[
\delta_q(T(x, y), T(x^*, y^*))
\]

\[
= \left(\frac{x}{2} + \frac{y}{2} - \frac{x^* + y^*}{2}\right) + \frac{|x + y|}{2}
\]

\[
= \frac{1}{4}(2|2x - 2x^* + (2y - 2y^*)| + |2x + 2y|)
\]

\[
\leq \frac{1}{4}(2|2x - 2x^*| + |2y - 2y^*| + |2x| + |2y|)
\]

\[
= \frac{1}{4}\delta_q(gx, gx^*) + \frac{1}{4}\delta_q(gy, gy^*).
\]

Therefore, by letting \(k = 1/2\), all of the hypotheses of Theorem 13 hold. Thus, \(T\) and \(g\) possess a common coupled fixed point in \(X^2\).

### 3. Fixed Point Theorem and Hausdorff \(\delta_q\)-Distance

Let us start with the following definition:

Consider a \(\mathcal{F}\)-quasi-metric space \((X, \delta_q)\), and denote the family of all nonempty bounded closed subsets of \(X\) by \(CB(X)\). Then, \(H(\cdot, \cdot)\) is said to be a Hausdorff \(\delta_q\)-distance on \(CB(X)\), if

\[
H_{\delta_q}(A, B) = \max\left\{ \sup_{x \in A} \delta_q(x, B), \sup_{x \in B} \delta_q(A, x) \right\},
\]

where \(\delta_q(x, B) = \inf \{\delta_q(x, y) : y \in B\}\).

**Definition 15** [10]. Let \(X\) be a nonempty set, \(g : X \to X\) be a single-valued mapping, and \(T : X \to CB(X)\) be a multivalued mapping. Also, let \(w = gx \in T_x\) for some \(x \in X\). Then \(w\) is said to be a point of coincidence of \(g\) and \(T\), and \(x\) is said to be a coincidence point of \(g\) and \(T\).

**Theorem 16.** Let \((X, \delta_q)\) be a bicomplete \(\mathcal{F}\)-quasi-metric space. Also, let \(g : X \to X\) be a single-valued mapping and \(T : X \to CB(X)\) be a multivalued mapping so that \(T(X) \subset g(X)\), \(g(X)\) is closed, and \(g\) is continuous. Assume that there exists \(k \in (0, 1)\) such that

\[
H_{\delta_q}(Tx, Ty) \leq k\delta_q(gx, gy),
\]

for all \(x, y \in X\). Then \(T\) and \(g\) have a coincidence point in \(X\).

**Proof.** Due to \(T(X) \subset g(X)\), we select a point \(x_1 \in X\) such that \(gx_1 \in T_{x_1}\) for a given \(x_0 \in X\). By continuing this procedure, we can construct a sequence \(x_n\) in \(X\) such that \(gx_{n+1} \in Tx_n\) for \(n = 0, 1, \ldots\). Suppose that \((f, B) \in \mathcal{F} \times [0, +\infty)\) so that \((\eta_2)\) holds. For an arbitrary \(\varepsilon > 0\) and due to \((\eta_3)\), there exists \(y > 0\) such that
Consider the sequence \( \{g x_n\} \subseteq X \). Now, without loss of generality, suppose that \( H_{\delta_c}(T x_0, T x_1) > 0 \). Otherwise, \( x_1 \) is a coincidence point of \( T \) and \( g \). Now, from (24), we have

\[
\delta_q(g x_{n+1}, g x_{n+2}) \leq H_{\delta_c}(T x_n, T x_{n+1}) \leq k \delta_q(g x_n, g x_{n+1}) \\
\leq k \delta_q(g x_n, g x_n). 
\]

(26)

Hence, we have \( \delta_q(g x_n, g x_{n+1}) \leq k^n \delta_q(g x_0, g x_1) \) for all \( n \in N \). Now, let \( m \) and \( n \) be two natural numbers with \( m > n \). Then, we have

\[
\sum_{i=n}^{m-1} \delta_q(g x_i, g x_{i+1}) \leq \frac{k^n}{1-k} \delta_q(g x_0, g x_1). 
\]

(27)

On the other hand, since \( \lim_{n \to \infty} (k^n/1-k) \delta_q(g x_0, g x_1) = 0 \), there exists \( N \in N \) so that

\[
0 < \frac{k^n}{1-k} \delta_q(g x_0, g x_1) < \gamma, 
\]

(28)

for \( n \geq N \). Hence, by (25) and (\( \eta_1 \)), we have

\[
f \left( \sum_{i=n}^{m-1} \delta_q(g x_i, g x_{i+1}) \right) \leq f \left( \frac{k^n}{1-k} \delta_q(g x_0, g x_1) \right) < f(\varepsilon) - B, 
\]

(29)

for all \( m > n \geq N \). Employing (\( \eta_1 \)) together with (29), we obtain

\[
\delta_q(g x_n, g x_m) > 0 \implies f(\delta_q(g x_n, g x_m)) \leq f \left( \sum_{i=n}^{m-1} \delta_q(g x_i, g x_{i+1}) \right) + B < f(\varepsilon).
\]

(30)

Now, by (\( \eta_1 \)), we have \( \delta_q(g x_n, g x_m) < \varepsilon \). This proves that \( \{g x_n\} \) is right-Cauchy. Similarly, by changing the order of the pairs \( (x_{i+1}, x_i) \) in the above process, we conclude that \( \{g x_n\} \) is also a left-Cauchy sequence. Therefore, it is a Cauchy sequence. Note that \( (X, \delta_q) \) is bicomplete and \( g(X) \) is closed. Thus, there exists \( x \in X \) such that \( \lim_{n \to \infty} g x_n = g x \).

Now, we shall show that \( g x \in T x \). For this purpose, using (24), we have

\[
\delta_q(g x_{n+1}, T x) \leq H_{\delta_c}(T x_n, T x) \leq k \delta_q(g x_n, g x). 
\]

(31)

Thus,

\[
\lim_{n \to \infty} \delta_q(g x_{n+1}, T x) = \delta_q(g x, T x) = 0. 
\]

(32)

Hence, \( g x \in T x \). Consequently, \( T \) and \( g \) possess a point of coincidence.

\textbf{Example 17.} Consider \( X = [0, 1] \). Define \( T : X \to CB(X) \) by \( T x = [0, 1/16 x] \) and \( g : X \to X \) by \( g x = x/2 \). Also, define \( \delta_q : X \times X \to [0, \infty) \) by

\[
\delta_q(s, t) = \begin{cases} 0 & \text{if } s = t \\ |s| + |s-t| & \text{otherwise.} \end{cases}
\]

(33)

Clearly, \( \delta_q \) is a bicomplete \( \mathcal{F} \)-quasi-metric with \( f(t) = \ln t \) and \( B = 0 \). Also, evidently, \( T(X) \subseteq g(X) \) and \( g(X) \) is closed. First, let \( x = y = 0 \). Then

\[
H_{\delta_q}(T x, T y) = 0 \leq k \delta_q(g x, g y). 
\]

(34)

Therefore, we may consider \( x \) and \( y \) are not zero. Without loss of totality, suppose that \( x \leq y \). Then

\[
H_{\delta_q}(T x, T y) = H_{\delta_q} \left( \left[ 0, \frac{1}{16} x \right], \left[ 0, \frac{1}{16} y \right] \right) \\
= \max \sup_{0 \leq a \leq 1/16y} \delta_q \left( a, \left[ 0, \frac{1}{16} y \right] \right) \\
\cdot \sup_{0 \leq b \leq 1/16x} \delta_q \left( \left[ 0, \frac{1}{16} x \right], b \right). 
\]

(35)

Due to \( x \leq y \), we have \( [0, 1/16x] \subseteq [0, 1/16y] \). For every \( a \in [0, 1/16x] \), we have \( \delta_q(a, [0, 1/16y]) = 0 \). Also, for every \( b \in [0, 1/16y] \), we obtain

\[
\delta_q \left( \left[ 0, \frac{1}{16} x \right], b \right) = \begin{cases} 0 & b \leq \frac{x}{16} \\
\left| \left| \frac{x}{8} - 2b \right| \right| & b \geq \frac{x}{16}. \end{cases}
\]

(36)

This yields that

\[
\sup_{0 \leq b \leq 1/16y} \delta_q \left( \left[ 0, \frac{1}{16} x \right], b \right) = \left| \frac{x}{8} - \frac{y}{8} \right|. 
\]

(37)

We deduce that

\[
H_{\delta_q}(T x, T y) = \left| \frac{x}{8} - \frac{y}{8} \right| \leq \frac{1}{4} \left( \left| x \right| + \frac{\left| x - y \right|}{2} \right) = \frac{1}{4} \delta_q(g x, g y). 
\]

(38)

Obviously, all other hypotheses of Theorem 16 hold. Hence, \( g \) and \( T \) possess a coincidence point in \( X \).

\textbf{4. An Application}

As an application of our results, we consider the following Volterra-integral equation:

\[
x(t) = \int_0^t K(t, s, x(s)) ds + v(t), 
\]

(39)

where \( t \in I = [0, 1] \), \( K \in C(I \times I \times R, R) \), and \( v \in C(I, R) \).
Consider a Banach space of all real continuous functions defined on \( I \subseteq (C(I, \mathbb{R})) \) with norm \( \|x\|_\infty = \max \{|x(t)|\} \) for every \( x \in C(I, \mathbb{R}) \). Also, let \( C(I \times I \times C(I, \mathbb{R}), \mathbb{R}) \) be the space of all continuous functions defined on \( I \times I \times C(I, \mathbb{R}) \). On the other hand, the Banach space \( C(I, \mathbb{R}) \) can be dedicated with the Bielecki norm \( \|x\|_B = \sup_{t \in I} \{|x(t)|e^{t\tau}\} \) for all \( x \in C(I, \mathbb{R}) \) and \( \tau > 0 \), and the derived metric \( \delta_B(x, y) = \|x - y\|_B \) for all \( x, y \in C(I, \mathbb{R}) \). Define \( \delta_q : X \times X \to [0, \infty) \) by

\[
\delta_q(x, y) = \sup_{t \in I} \left\{ |x(t) - y(t)| e^{-\|x\|_B t}\right\}.
\]

Also, define \( T : C(I, \mathbb{R}) \to C(I, \mathbb{R}) \) by

\[
Tx(t) = \int_0^t K(t, s, x(s))ds + v(t), \quad v \in C(I, \mathbb{R}).
\]

**Theorem 18.** Consider a bicomplete \( \mathcal{F} \)-quasi-metric space \( (C(I, \mathbb{R}), \delta_B) \) with \( f(t) = \ln t \). Also, let \( T \) be an operator from \( C(I, \mathbb{R}) \) into \( C(I, \mathbb{R}) \) with \( Tx(t) = \int_0^t K(t, s, x(s)) ds + v(t) \), and let \( g_x = f(x) \). Assume that \( K \in C(I \times I \times \mathbb{R}) \) is an operator such that

(i) \( K \) is continuous;

(ii) \( \int_0^t K(t, s, \cdot) \) for all \( t, s \in I \) is increasing; and

(iii) for every \( x \) and \( y \) in \( C(I, \mathbb{R}) \), and \( t \) and \( s \) in \( I \), we have

\[
|K(t, s, x(s)) - K(t, s, y(s))| \leq e^{-\|x\|_B} |x(s) - y(s)|.
\]

Then, the integral equation (39) possesses an answer in \( C(I, \mathbb{R}) \).

**Proof.** By definition of \( T \), we have

\[
\delta_q(Tx, Ty) = \sup_{t \in I} \left\{ \int_0^t K(t, s, x(s))ds - \int_0^t K(t, s, y(s))ds \right\} e^{-\|x\|_B t}
\]

\[
\leq \sup_{t \in I} \left\{ \int_0^t |K(t, s, x(s)) - K(t, s, y(s))| ds \right\} e^{-\|x\|_B t}
\]

\[
\leq \sup_{t \in I} \left\{ \int_0^t e^{-\|x\|_B} |x(s) - y(s)| ds \right\} e^{-\|x\|_B t}
\]

\[
\leq (\|x - y\|_B) \sup_{t \in I} \left\{ \int_0^t e^{-\|x\|_B t} ds \right\} e^{-\|x\|_B t} \delta_q(x, y).
\]

Now, we consider that the function \( f(t) = \ln t \) for every \( t \in I \), \( B = 0 \), and \( k = e^{-\|x\|_B t} \). Therefore, all assertions of Theorem 11 hold. As a result, Theorem 11 confirms the existence of fixed point of \( T \) so that this fixed point is the answer of the integral equation.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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