A DYNAMICAL BOREL-CANTELLI LEMMA
VIA IMPROVEMENTS TO DIRICHLET’S THEOREM

DMITRY KLEINBOCK AND SHUCHENG YU

ABSTRACT. Let $X \cong \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z})$ be the space of unimodular lattices in $\mathbb{R}^2$, and for any $r \geq 0$ denote by $K_r \subset X$ the set of lattices such that all its nonzero vectors have supremum norm at least $e^{-r}$. These are compact nested subsets of $X$, with $K_0 = \bigcap_r K_r$ being the union of two closed horocycles. We use an explicit second moment formula for the Siegel transform of the indicator functions of squares in $\mathbb{R}^2$ centered at the origin to derive an asymptotic formula for the volume of sets $K_r$ as $r \to 0$. Combined with a zero-one law for the set of the $\psi$-Dirichlet numbers established by Kleinbock and Wadleigh [KW18], this gives a new dynamical Borel-Cantelli lemma for the geodesic flow on $X$ with respect to the family of shrinking targets $\{K_r\}$.

1. Introduction

Let $(X, \mu)$ be a probability space, and let $\{a_s\}_{s \in \mathbb{R}}$ be a one-parameter measure-preserving flow on $X$. Given a family of measurable subsets $\{B_s\}_{s > 0}$ of $X$ with $\mu(B_s) \to 0$ as $s \to \infty$ (called shrinking targets), the shrinking targets problem asks for a dichotomy on whether generic orbits of $\{a_s\}_{s > 0}$ would hit the shrinking targets indefinitely. That is, we are looking for a zero-one law for the measure of the limsup set
\[ B_\infty := \lim_{s \to \infty} a_{-s}B_s = \{x \in X \mid a_s x \in B_s \text{ for an unbounded set of } s > 0 \}. \]

For any $n \in \mathbb{N}$ let
\[ \tilde{B}_n := \bigcup_{0 \leq s < 1} a_{-s}B_{n+s} \]
be the thickening of the shrinking targets $\{B_s\}_{n \leq s < n+1}$ along the flow $\{a_{-s}\}_{0 \leq s < 1}$. Note that $a_n x \in \tilde{B}_n$ if and only if there exists some $s \in [n, n+1)$ such that $a_s x \in B_s$. We thus have
\[ B_\infty = \lim_{n \to \infty} a_{-n}\tilde{B}_n = \{x \in X \mid a_n x \in \tilde{B}_n \text{ infinitely often} \}, \]
and the classical Borel-Cantelli lemma implies that
\[ \sum_{n} \mu(\tilde{B}_n) < \infty \quad \implies \quad \mu(B_\infty) = 0. \]

On the other hand, following the terminology of [CK01] we say the family of shrinking targets $\{B_s\}_{s > 0}$ is Borel-Cantelli (BC) for the flow $\{a_s\}_{s > 0}$ if $\mu(B_\infty) = 1$. Thus a necessary condition for $\{B_s\}_{s > 0}$ to be BC for $\{a_s\}_{s > 0}$ is that the sequence of its thickenings has divergent sum of

Date: January 16, 2020.

D.K. has been supported by NSF grants DMS-1600814 and DMS-1900560. S.Y. acknowledges that this project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No. 754475).
measures, and we say \( \{B_s\}_{s>0} \) satisfies a dynamical Borel-Cantelli lemma for \( \{a_s\}_{s>0} \) if this is also a sufficient condition.

The shrinking targets problem for continuous time flow in the context of homogeneous spaces was first studied by Sullivan in [Sul82], where he established a logarithm law for the fastest rate of geodesic cusp excursions in finite-volume hyperbolic manifolds. Later using the exponential mixing rate and a smooth approximation argument, the first-named author and Margulis [KM99] proved that the family of cusp neighborhoods \( \{\Phi^{-1}(r(s), \infty)\}_{s>0} \) with divergent sum of measures is BC for any diagonalizable flow on \((G/\Gamma, \mu)\), where \( G \) is a connected semisimple Lie group without compact factors, \( \Gamma < G \) is an irreducible lattice, and \( \mu \) is the probability measure on \( X = G/\Gamma \) coming from a Haar measure on \( G \). Here \( \Phi \) is a distance-like function on \( X \) [KM99, Definition 1.6] and \( r(\cdot) \) is a quasi-increasing function [KM99, Section 2.4]. Later Maucourant [Mau06] obtained a similar dynamical Borel-Cantelli lemma for geodesic flows making excursions into shrinking hyperbolic balls (with a fixed center) on a finite-volume hyperbolic manifold. See [Ath09] for a survey on shrinking targets problems in dynamical systems.

One main reason that such dynamical Borel-Cantelli lemmas have gained much attention is due to their connections to metric number theory which were first explored by Sullivan in [Sul82]. Such connections were made more apparent later in [KM99]. Let \( m, l \) be two positive integers and let \( M_{m,l}(\mathbb{R}) \) be the space of \( m \times l \) real matrices. Given \( \psi : [t_0, \infty) \to (0, \infty) \) a continuous non-increasing function, let us define \( W(\psi) \subset M_{m,l}(\mathbb{R}) \), the set of \( \psi \)-approximable \( m \times l \) real matrices such that \( A \in W(\psi) \) if and only if there are infinitely many \( q \in \mathbb{Z}^l \) satisfying

\[
||A q - p||^m < \psi \left( \|q\|^l \right)
\]

for some \( p \in \mathbb{Z}^m \), where \( ||\cdot|| \) is the supremum norm on respective Euclidean spaces. The classical Khinchin-Groshev theorem gives an exact criterion on when \( W(\psi) \) has full or zero Lebesgue measure.

**Theorem KG (Khinchin-Groshev).** Given a continuous non-increasing \( \psi \), \( W(\psi) \) has full (resp. zero) Lebesgue measure if and only if the series \( \sum_k \psi(k) \) diverges (resp. converges).

See [Sch80] for more details. On the other hand, let \( X = SL_{m+l}(\mathbb{R})/SL_{m+l}(\mathbb{Z}) \) be the space of unimodular lattices in \( \mathbb{R}^{m+l} \) and let \( \Delta : X \to [0, \infty) \) be the function on \( X \)

\[
\Delta(A) := \sup_{v \in A \setminus \{0\}} \log \left( \frac{1}{\|v\|} \right).
\]

Note that \( \Delta(A) \geq 0 \) for any \( A \in X \) due to Minkowski’s Convex Body Theorem, and for all \( r \geq 0 \) the sets

\[
K_r := \Delta^{-1}([0, r])
\]

(of lattices such that all its nonzero vectors have supremum norm at least \( e^{-r} \)) are compact due to Mahler’s Compactness Criterion, see e.g. [Cas97]. Following ideas of Dani [Dan85], it was shown in [KM99] that there exists a unique function \( r = r_\psi : [s_0, \infty) \to \mathbb{R} \) depending on \( \psi \) (this was referred to as the Dani Correspondence) such that \( A \in M_{m,l}(\mathbb{R}) \) is \( \psi \)-approximable if and only if the events \( a_s \Lambda_A \in \Delta^{-1}(r(s), \infty) \) happen for an unbounded set of \( s > s_0 \), where

\[
a_s = \text{diag}(e^{s/m}, \ldots, e^{s/m}, e^{-s/l}, \ldots, e^{-s/l})
\]
with $m$ copies of $e^{s/m}$ and $l$ copies of $e^{-s/l}$, and $\Lambda_A = \left( \begin{array}{cc} I_m & A \\ 0 & I_l \end{array} \right) \mathbb{Z}^{m+l} \in X$. This way the first-named author and Margulis showed Theorem KG to be equivalent to a dynamical Borel-Cantelli lemma for the $a_s$-orbits making excursions into the cusp neighborhoods $\Delta^{-1}(r(s), \infty)_{s > s_0}$, and used this to give an alternative dynamical proof of Theorem KG based on mixing properties of the $a_s$-action on $X$, see [KM99, KM18].

More recently, for a given $\psi$ as above, the first-named author and Wadleigh [KW18] studied the finer problem of improvements to Dirichlet’s Theorem. See [DS70, DS70] for the history of the problem of improving Dirichlet’s Theorem. Following the definition in [KW18] an $m$ by $l$ real matrix $A$ is called $\psi$-Dirichlet if the system of inequalities

$$
\|Aq - p\|_m^m < \psi(t) \quad \text{and} \quad \|q\|_l^l < t
$$

has solutions in $(p, q) \in \mathbb{Z}^m \times (\mathbb{Z}^l \setminus \{0\})$ for all sufficiently large $t$. Following the general scheme developed in [KM99] they gave a dynamical interpretation of $\psi$-Dirichlet matrices. Namely, they showed that $A \in M_{m,l}(\mathbb{R})$ is not $\psi$-Dirichlet if and only if the events

$$
a_s \Lambda_A \in K_r(s)
$$

happen for an unbounded set of $s > s_0$, where $a_s, \Lambda_A$ and $r = r_\psi$ are all as above. Hence in this case the family of shrinking targets is given by $\{K_r(s)\}_{s > s_0}$, and one is naturally interested in whether this family of shrinking targets is BC for the flow $\{a_s\}_{s > 0}$.

However this dynamical interpretation is not helpful when it comes to determining necessary and sufficient conditions on $\psi$ guaranteeing that almost every (almost no) $A$ is $\psi$-Dirichlet. One of the main difficulties is that the shrinking targets $K_r(s)$ are far away from being $\text{SO}_{m+l}(\mathbb{R})$-invariant, and thus when applying the mixing properties of the $a_s$-action it will involve certain Sobolev norms which are hard to control. Still, using a different method based on continued fractions the aforementioned conditions were found in [KW18] for the case $m = l = 1$. Namely, the following was proved:

**Theorem KW (Kleinbock-Wadleigh).** Let $\psi : [t_0, \infty) \to (0, \infty)$ be a continuous, non-increasing function satisfying

(1.6) \hspace{1cm} \text{the function } t \mapsto t\psi(t) \text{ is non-decreasing}

and

(1.7) \hspace{1cm} t\psi(t) < 1 \quad \text{for all } t \geq t_0.

Then if the series

(1.8) \hspace{1cm} \sum_{n} -\frac{(1 - n\psi(n)) \log (1 - n\psi(n))}{n}

diverges (resp. converges), then Lebesgue-a.e. $x \in \mathbb{R}$ is not (resp. is) $\psi$-Dirichlet.

In this paper we use the above theorem to derive a dynamical Borel-Cantelli lemma for the diagonal flow $a_s := \text{diag}(e^s, e^{-s})$ on $X := \text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z})$. Let $\mu$ be the probability Haar measure on $X$, consider the function $\Delta$ on $X$ as in (1.4), and define the sets $K_r$ as in (1.5).

We now state our dynamical Borel-Cantelli lemma.
Theorem 1.1. Let \( r : [s_0, \infty) \to (0, \infty) \) be a continuous and non-increasing function. Let \( B_s = K_{r(s)} \) and let \( B_\infty = \lim_{t \to \infty} a_{-s} B_s \). Then we have
\[
\sum_n r(n) \log \left( \frac{1}{r(n)} \right) < \infty \implies \mu(B_\infty) = 0.
\]
If in addition we assume that the function \( s \mapsto s + r(s) \) is non-decreasing, then we have
\[
\sum_n r(n) \log \left( \frac{1}{r(n)} \right) = \infty \implies \mu(B_\infty) = 1.
\]

Comparing the statement of the above theorem with (1.3), one can guess that it can be approached by studying the thickenings
\[(1.9) \quad \tilde{B}_n = \bigcup_{0 \leq s < 1} a_{-s} B_{n+s} = \bigcup_{0 \leq s < 1} a_{-s} K_{r(n+s)}
\]
as in (1.1). We do it in several steps. In the beginning of §3 we prove an asymptotic measure formula for the sets \( K_r \) where \( r \) is small:

**Theorem 1.2.** For any \( 0 < r < \frac{1}{2} \log 2 \) we have
\[
\mu(K_r) = \frac{4r^2 \log \left( \frac{1}{r} \right)}{\zeta(2)} + O(r^2),
\]
where \( \zeta(2) = \frac{\pi^2}{6} \) is the value of the Riemann zeta function at 2.

Here and hereafter for two positive quantities \( A \) and \( B \), we will use the notation \( A \ll B \) or \( A = O(B) \) to mean that there is a constant \( c > 0 \) such that \( A \leq cB \), and we will use subscripts to indicate the dependence of the constant on parameters. We will write \( A \asymp B \) for \( A \ll B \ll A \).

The next step is to use Theorem 1.2 to estimate the measure of the thickening of \( K_r \) along the flow \( \{a_{-s}\}_{0 \leq s < 1} \) by bounding it from above and below by a finite union of \( a_{-s} \)-translates of \( K_r \). This is also done in §3 and yields the following result:

**Theorem 1.3.** For any \( 0 < r < \log 1.01 \) we have
\[
\mu \left( \bigcup_{0 \leq s < 1} a_{-s} K_r \right) \asymp r \log \left( \frac{1}{r} \right).
\]

The above asymptotic equality shows that the series appearing in Theorem 1.1 converges/diverges iff so does the series \( \sum_n \mu(\tilde{B}_n) \), where \( \tilde{B}_n \) is as in (1.9):

**Corollary 1.4.** Let \( r : [s_0, \infty) \to (0, \infty) \) be a non-increasing function, and let \( \tilde{B}_n \) be as in (1.9). Then we have
\[
\sum_n \mu(\tilde{B}_n) = \infty \quad \text{if and only if} \quad \sum_n r(n) \log \left( \frac{1}{r(n)} \right) = \infty.
\]

Therefore, in view of (1.2) and (1.3), the convergence part of Theorem 1.1 is immediate from the Borel-Cantelli lemma. The divergence part however is trickier. Instead of using a dynamical approach as in [KM99], our proof in §4 is non-dynamical and relies on Theorem KW and the Dani Correspondence.
It remains to comment on our proof of Theorem 1.2. Instead of trying to describe the sets $K_r$ explicitly in terms of coordinates and compute their measures directly, we adapt an indirect approach which relies on an explicit second moment formula of the Siegel transform of certain indicator functions. Recall that if $f$ is a function on $\mathbb{R}^2$, its primitive Siegel transform is the function on $X$ given by

$$\hat{f}(\Lambda) := \sum_{v \in \Lambda_{pr}} f(v),$$

where $\Lambda_{pr}$ is the set of primitive vectors of $\Lambda$. Clearly $\hat{f}(\Lambda) = \#(\Lambda_{pr} \cap S)$ when $f$ is the indicator function of a subset $S$ of $\mathbb{R}^2$.

Let us briefly describe the history of the problem. The Siegel transform was originally defined by Siegel [Sie45] as the sum over all nonzero lattice point for unimodular lattices of any rank. In the same paper Siegel proved a Mean Value Theorem for the Siegel transform, which in the primitive set-up amounts to

$$(1.10) \int_X \hat{f}(\Lambda) \, d\mu(\Lambda) = \frac{1}{\zeta(2)} \int_{\mathbb{R}^2} f(x) \, dx.$$

for any bounded compactly supported $f$ on $\mathbb{R}^2$. Since then there has been much work extending his result to higher moments. For example, in [Rog55] Rogers proved a series of higher moment formulas, which in particular includes a second moment formula for the Siegel transform defined on the space of unimodular lattices of rank greater than 2. However, his result did not give a second moment formula on $X$ as in our setting. For this setting, Schmidt [Sch60] proved an upper bound for the second moment of the primitive Siegel transform of indicator functions on $\mathbb{R}^2$. His bound was later logarithmically improved by Randol [Ran70] for discs centered at the origin and by Athreya and Margulis [AM09] for general indicator functions building on Randol’s bound. Athreya and Konstantoulas [AK16] obtained similar bounds on the space of general symplectic lattices for certain family of indicator functions. Continuing [AK16], Kelmer and the second-named author [KY19] proved a second moment formula on the space of symplectic lattices $Y_n := \text{Sp}(2n, \mathbb{R})/\text{Sp}(2n, \mathbb{Z})$. In particular, when $n = 1$ we have $Y_1 = X$ and their formula also applies to our setting\(^\dagger\). However, for our applications all these formulas are not explicit enough.

We now state an explicit second moment formula which we use to derive Theorem 1.2.

**Theorem 1.5.** For any $r \geq 0$ let $S_r$ be the open square with vertices given by $(\pm e^{-r}, \pm e^{-r})$, and let $f_r$ be the indicator function of $S_r$. Then we have

$$(1.11) \|\hat{f}_r\|_2^2 = \frac{8}{\zeta(2)} \left( e^{-2r} + \int_{D_r} \left( \frac{e^{-r}}{x_1} + \frac{e^{-r}}{x_2} - \frac{1}{x_1 x_2} \right) dx_1 dx_2 \right),$$

where

$$D_r := \{ x = (x_1, x_2) \in S_r \mid x_1 > 0, x_2 > 0, x_1 + x_2 > e^r \},$$

and $\|\cdot\|_2$ stands for the $L^2$-norm with respect to $\mu$.

**Remark 1.6.** When $r \geq \frac{1}{2} \log 2$ the region $D_r$ is empty, and equation (1.11) simply reads as $\|\hat{f}_r\|_2^2 = \frac{8e^{-2r}}{\zeta(2)}$. We note that the latter equality in fact already follows from Siegel’s Mean Value Theorem, since in this case for any unimodular lattice there can only be at most one

\(^\dagger\)See also [Fai19] for moment formulas of the Siegel-Veech transform recently obtained by Fairchild.
pair of primitive lattice points allowed in \( S_r \), which implies that \( \frac{1}{2} \hat{f} \) is an indicator function on \( X \). When \( 0 \leq r < \frac{1}{2} \log 2 \), the region \( D_r \) is not empty, and it is not hard to compute the integral in (1.11) explicitly, see (3.5) below. In particular, plugging \( r = 0 \) into (1.11) we have \( \| \hat{f} \|_2^2 = (\frac{12}{\pi})^2 - 8 \approx 6.59 \).

In §2 we prove a much more general second moment formula, see Theorem 2.1, with an arbitrary bounded measurable subset \( S \) of \( \mathbb{R}^2 \) in place of \( S_r \). Theorem 1.5 is derived from Theorem 2.1 by taking \( S = S_r \).

**Acknowledgements.** The authors would like to thank Anurag Rao, Nick Wadleigh and Cheng Zheng for many helpful conversations. Thanks are also due to the anonymous referee for a quick and careful report.

### 2. The second moment formula

In this section, we prove Theorem 1.5 by establishing the following second moment formula for quite general subsets of \( \mathbb{R}^2 \).

**Theorem 2.1.** Let \( S \) be a measurable bounded subset of \( \mathbb{R}^2 \), and let \( f \) be the indicator function of \( S \). Let \( \tilde{S} = \{ x \in \mathbb{R}^2 \mid -x \in S \} \). Then we have

\[
\| \hat{f} \|_2^2 = \frac{1}{\zeta(2)} \left( \text{area}(S) + \text{area}(S \cap \tilde{S}) + \sum_{n \neq 0} \frac{\varphi(|n|)}{|n|} \int_S |I_n^x| \, dx \right),
\]

where \( \varphi \) is the Euler’s totient function, \( I_n^x \subset \mathbb{R} \) is defined by

\[
I_n^x := \left\{ t \in \mathbb{R} \left| n \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) + t(x_1, x_2) \in S \right. \right\},
\]

and \( |I_n^x| \) is the length of \( I_n^x \) with respect to the Lebesgue measure on \( \mathbb{R} \).

Before giving the proof let us make a few remarks about Theorem 2.1. First we note that for any bounded \( S \) there exists a sufficiently large \( T > 0 \) depending on \( S \) such that for any \( |n| > T \) the set \( I_n^x \) is empty for all \( x \in S \). Thus the series on the right hand side of (2.1) is a finite sum. Next we note that if we further assume \( S \) is symmetric with respect to the origin, then by symmetry we have \( S \cap \tilde{S} = S \) and \( |I_n^x| = |I_{-n}^x| \) for any \( n \neq 0 \). In particular, for such \( S \) we have the following slightly simpler formula

\[
\| \hat{f} \|_2^2 = \frac{2}{\zeta(2)} \left( \text{area}(S) + \sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \int_S |I_n^x| \, dx \right).
\]

Finally we note that for any \( \Lambda \in X \) and \( f \) as in Theorem 2.1 we have

\[
\left( \hat{f}(\Lambda) \right)^2 = \hat{f}(\Lambda) + \hat{\chi_{S \cap \tilde{S}}}(\Lambda) + \sum_{v_1, v_2 \in \Lambda_{pr} \text{ linearly independent}} f(v_1) f(v_2).
\]

Thus Theorem 2.1 together with (1.10) implies that

\[
\int_X \sum_{v_1, v_2 \in \Lambda_{pr} \text{ linearly independent}} f(v_1) f(v_2) \, d\mu(\Lambda) = \frac{1}{\zeta(2)} \sum_{n \neq 0} \frac{\varphi(|n|)}{|n|} \int_S |I_n^x| \, dx.
\]
It is worth pointing out that the above formula can be compared to its higher-dimensional analogue: when $f$ is an indicator function of a bounded measurable subset $S$ of $\mathbb{R}^k$ with $k \geq 3$, $X = \text{SL}_k(\mathbb{R})/\text{SL}_k(\mathbb{Z})$ and $\mu$ is the Haar probability measure on $X$, according to Rogers' second moment formula [Rog55] the left hand side of (2.2) equals $\left(\frac{\text{vol}(S)}{\zeta(k)}\right)^2$. However, as we can see here the $k = 2$ case is much more complicated, with the answer depending on both the shape and the position of $S$.

2.1. Coordinates and measures. We fix coordinates on $G = \text{SL}_2(\mathbb{R})$ via the Iwasawa decomposition $G = KAN$ with

$$K = \{ k_\theta \mid 0 \leq \theta < 2\pi \}, \quad A = \{ a_s \mid s \in \mathbb{R} \} \quad \text{and} \quad N = \{ u_t \mid t \in \mathbb{R} \},$$

where $k_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $a_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}$ and $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Explicitly, under coordinates $g = k_\theta a_s u_t$, $\mu$ is given by

$$d\mu(g) = \frac{1}{\zeta(2)} e^{2s} d\theta ds dt. \tag{2.3}$$

There is a natural identification between the homogeneous space $G/N$ and $\mathbb{R}^2 \setminus \{0\}$ induced by the map $G \to \mathbb{R}^2 \setminus \{0\}$ sending $g = k_\theta a_s u_t \in G$ to,

$$x(g) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^s \cos \theta \\ e^s \sin \theta \end{pmatrix},$$

the left column of $g$. The Lebesgue measure, $dx$, on $\mathbb{R}^2 \setminus \{0\} \cong G/N$ can be expressed via the polar coordinates $(s, \theta)$ as

$$dx(k_\theta a_s) = e^{2s} d\theta ds. \tag{2.5}$$

2.2. The second moment formula. In this subsection we prove Theorem 2.1, and with some more analysis we prove Theorem 1.5. As the first step of our computation we recall the following preliminary identity which relies on a standard unfolding argument. We note that one can find it in [Lan75, Chapter VIII, §1], and we include a short proof here to make the paper self-contained. See also [KY19, Proposition 2.3] for a generalization to the space of symplectic lattices.

**Lemma 2.2.** For any bounded and compactly supported function $f$ on $\mathbb{R}^2$ and for any bounded $F \in L^2(X, \mu)$ we have

$$\langle \hat{f}, F \rangle = \frac{1}{\zeta(2)} \int_{-\infty}^{\infty} \int_0^{2\pi} f(x(k_\theta a_s)) \overline{\mathcal{P}_F(x(k_\theta a_s))} e^{2s} d\theta ds,$$

where $\mathcal{P}_F$ is defined by

$$\mathcal{P}_F(x(k_\theta a_s)) = \int_0^1 F(k_\theta a_s u_t Z^2) dt$$

with $k_\theta, a_s$ and $u_t$ as above, and $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(X, \mu)$.\]
Proof. Let \( \Gamma = \text{SL}_2(\mathbb{Z}) \) and let \( \Gamma_\infty = \Gamma \cap N \). Recall that there is an identification between \( \Gamma/\Gamma_\infty \) and \( Z^2_{pr} \) sending \( \gamma \Gamma_\infty \) to \( \gamma \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \). Using this identification, for any \( \Lambda = gZ^2 \) with \( g \in \text{SL}_2(\mathbb{R}) \) we can write
\[
\hat{f}(\Lambda) = \sum_{v \in \Lambda_{pr}} f(v) = \sum_{w \in Z^2_{pr}} f(gw) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \hat{f}(g\gamma),
\]
(2.6) where \( \hat{f}(g) := f \left( g \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right) \). We note that \( \hat{f} \) is a right \( N \)-invariant function on \( G \). Let \( \mathcal{F}_\Gamma \) be a fundamental domain for \( X = G/\Gamma \), and let \( \mathcal{F}_\infty \) be a fundamental domain for \( G/\Gamma_\infty \). Note that using the Iwasawa decomposition \( G = KAN \) we can choose
\[
\mathcal{F}_\infty = \{k_\theta a_s u_t \mid 0 < \theta < 2\pi, s \in \mathbb{R}, 0 < t < 1\}.
\]
Moreover, fix a set of coset representatives \( \Sigma_\infty \subset \Gamma \) for \( \Gamma/\Gamma_\infty \), and note that \( \bigcup_{\gamma \in \Sigma_\infty} \mathcal{F}_\Gamma \gamma \) is a disjoint union and forms a fundamental domain for \( G/\Gamma_\infty \). Now for any bounded \( F \in L^2(X, \mu) \), using (2.3), (2.6), (2.7) and the facts that \( F \) is right \( \Gamma \)-invariant and \( \hat{f} \) is right \( N \)-invariant, we have
\[
\langle \hat{f}, F \rangle := \int_{\mathcal{F}_\Gamma} \hat{f}(gZ^2) F(gZ^2) d\mu(g) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \int_{\mathcal{F}_\Gamma} \hat{f}(g\gamma) F(gZ^2) d\mu(g)
\]
\[
= \sum_{\gamma \in \Sigma_\infty} \int_{\mathcal{F}_\Gamma \gamma} \hat{f}(g) F(gZ^2) d\mu(g) = \int_{\bigcup_{\gamma \in \Sigma_\infty} \mathcal{F}_\Gamma \gamma} \hat{f}(g) F(gZ^2) d\mu(g)
\]
\[
= \int_{\mathcal{F}_\infty} \hat{f}(g) F(gZ^2) d\mu(g) = \frac{1}{\zeta(2)} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \int_{0}^{1} \hat{f}(k_\theta a_s u_t) F(k_\theta a_s u_t Z^2) e^{2s} dtd\theta ds
\]
\[
= \frac{1}{\zeta(2)} \int_{-\infty}^{\infty} \int_{0}^{2\pi} \hat{f}(\mathbf{x}(k_\theta a_s)) \int_{0}^{1} F(k_\theta a_s u_t Z^2) dt e^{2s} d\theta ds.
\]
Finally, we note that the above equalities can be justified since \( F \) is bounded and the defining series for \( \hat{f} \) is absolutely convergent (see [Vee98, Lemma 16.10]). \( \square \)

With this preliminary identity, we can now give the

Proof of Theorem 2.1. Using the relation (2.5) and Lemma 2.2 we have
\[
\|\hat{f}\|_2^2 = \frac{1}{\zeta(2)} \int_{\mathbb{R}^2} f(\mathbf{x}(k_\theta a_s)) \mathcal{P}_{\hat{f}}(\mathbf{x}(k_\theta a_s)) d\mathbf{x} = \frac{1}{\zeta(2)} \int_{S} \mathcal{P}_{\hat{f}}(\mathbf{x}(k_\theta a_s)) d\mathbf{x},
\]
(2.8) where
\[
\mathcal{P}_{\hat{f}}(\mathbf{x}(k_\theta a_s)) = \int_{0}^{1} \hat{f}(k_\theta a_s u_t Z^2) dt
\]
with \( k_\theta, a_s \) and \( u_t \) as before. First, by the definition of the primitive Siegel transform we have
\[
\hat{f}(k_\theta a_s u_t Z^2) = \# \left\{ (m, n) \in Z^2_{pr} \mid k_\theta a_s u_t \left( \begin{array}{c} m \\ n \end{array} \right) \in S \right\}.
\]
Thus for \( \mathbf{x}(k_\theta a_s) \in S \) and \( 0 \leq t < 1 \) we have
\[
\hat{f}(k_\theta a_s u_t Z^2) = \sum_{(m,n) \in Z^2_{pr}} \chi_{f(\mathbf{x}(k_\theta a_s))}(t),
\]
where
\[ I_{x(k_0a_s)}^{(m,n)} := \left\{ 0 \leq t < 1 \mid k_0a_s u_t \left( \frac{m}{n} \right) \in \mathcal{S} \right\}, \]
implicating that
\[ P_f(x(k_0a_s)) = \sum_{(m,n) \in \mathbb{Z}_n^2} I_{x(k_0a_s)}^{(m,n)} = \left| I_{x(k_0a_s)}^{(1,0)} \right| + \left| I_{x(k_0a_s)}^{(-1,0)} \right| + \sum_{(m,n) \in \mathbb{Z}_n^2, n \neq 0} I_{x(k_0a_s)}^{(m,n)} \cdot \]

Next, by direct computation we have for \( x(k_0a_s) = (x_1, x_2) = (e^s \cos \theta, e^s \sin \theta) \in \mathcal{S}, \)
\[ (2.9) \quad k_0a_s u_t \left( \frac{m}{n} \right) = n \left( -e^{-s} \sin \theta \right) + (m + nt) \left( e^s \cos \theta \right) = n \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) + (m + nt) \left( x_1 \right) \cdot \]

When \( (m,n) = (1,0) \) we have for \( x(k_0a_s) \in \mathcal{S}, \ k_0a_s u_t \left( \frac{1}{0} \right) = \left( x_1 \right) \) is contained in \( \mathcal{S} \) for any \( 0 \leq t < 1. \) Thus \( I_{x(k_0a_s)}^{(1,0)} = [0, 1) \) and \( \left| I_{x(k_0a_s)}^{(1,0)} \right| = 1 \) for any \( x(k_0a_s) \in \mathcal{S}. \) Similarly, when \( (m,n) = (-1,0) \) we have for \( x(k_0a_s) \in \mathcal{S}, \ k_0a_s u_t \left( -1 \right) = \left( \frac{-x_1}{-x_2} \right) \) is contained in \( \mathcal{S} \) if and only if \( x \in \mathcal{S} \cap \tilde{\mathcal{S}} \) with \( \tilde{\mathcal{S}} \) as in the theorem, implying that \( I_{x(k_0a_s)}^{(-1,0)} = [0, 1) \) whenever \( x \in \mathcal{S} \cap \tilde{\mathcal{S}}. \)

When \( n \neq 0 \) by (2.9) we have for any integer \( m \) coprime to \( n \)
\[ \left| I_{x(k_0a_s)}^{(m,n)} \right| = \left\{ 0 \leq t < 1 \mid n \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) + (m + nt)(x_1, x_2) \in \mathcal{S} \right\} \]
\[ = \left\{ \frac{m}{n} \leq t < 1 + \frac{m}{n} \mid n \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) + nt(x_1, x_2) \in \mathcal{S} \right\}. \]

We note that as \( m \) runs through all the integers in each congruence class in \((\mathbb{Z}/|n|\mathbb{Z})^\times, \) the intervals \([\frac{m}{n}, 1 + \frac{m}{n}) \) cover \( \mathbb{R} \) exactly once. Thus for \( n \neq 0 \)
\[ \sum_{m \in \mathbb{Z}, (m,n)=1} \left| I_{x(k_0a_s)}^{(m,n)} \right| = \varphi(|n|) \left\{ t \in \mathbb{R} \mid n \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) + nt(x_1, x_2) \in \mathcal{S} \right\} = \frac{\varphi(|n|)}{|n|} |I^n_x|, \]
where \( \varphi \) is the Euler’s totient function and \( I^n_x \) is as in Theorem 2.1. We thus have for \( x \in \mathcal{S} \)
\[ P_f(x) = 1 + \chi_{\mathcal{S} \cap \tilde{\mathcal{S}}}(x) + \sum_{n \neq 0} \frac{\varphi(|n|)}{|n|} |I^n_x|. \]

We conclude the proof by plugging the above equation into (2.8).

**Proof of Theorem 1.5.** To simplify notation for any \( x \in \mathbb{R}^2, \ t \in \mathbb{R} \) and \( n \geq 1 \) let
\[ v(x,t,n) := n \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right) + t(x_1, x_2). \]
First we note that $\|\mathbf{v}(\mathbf{x}, t, n)\|_2^2 = \frac{n^2}{x_1^2 + x_2^2} + t^2(x_1^2 + x_2^2) \geq \frac{n^2}{x_1^2 + x_2^2}$, where $\|\cdot\|_2$ stands for the standard Euclidean norm on $\mathbb{R}^2$. Thus for $\mathbf{x} \in S_r$ and $n \geq 2$ we have

$$\|\mathbf{v}(\mathbf{x}, t, n)\| \geq \frac{\sqrt{2}}{2} \|\mathbf{v}(\mathbf{x}, t, n)\|_2 \geq \frac{\sqrt{2}}{\|\mathbf{x}\|_2} > e^r \geq e^{-r},$$

implying that $\mathcal{I}_x^n$ is empty for any $\mathbf{x} \in S_r$ and any $n \geq 2$. Here $\|\cdot\|$ stands for the supremum norm on $\mathbb{R}^2$, and for the third inequality we used the fact that $\|\mathbf{x}\|_2 < \sqrt{2}e^{-r}$, which follows from $\mathbf{x}$ being an element of $S_r$. Since $S_r$ is symmetric with respect to the origin, applying (2.1) to $f = f_r$, we get

$$\langle f_r \rangle_2 = \frac{8e^{-2r}}{\zeta(2)} + \frac{2}{\zeta(2)} \int_{S_r} |\mathcal{I}_x^1| \, d\mathbf{x} = \frac{8e^{-2r}}{\zeta(2)} + \frac{8}{\zeta(2)} \int_{S_r^+} |\mathcal{I}_x^1| \, d\mathbf{x},$$

where $S_r^+$ is the intersection of $S_r$ with the first quadrant, and for the second equality we used the fact that $|\mathcal{I}_{x_1, x_2}^1| = |\mathcal{I}_{(x_1, x_2)}^1|$ which follows from the invariance of $S_r$ under reflections around the coordinate axes. We note that for $\mathbf{x} \in S_r^+, (\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}) + t(x_1, x_2) \in S_r$ if and only if

$$-e^{-r} + \frac{x_2}{x_1(x_1^2 + x_2^2)} < t < e^{-r} + \frac{x_2}{x_1(x_1^2 + x_2^2)},$$

and

$$-e^{-r} - \frac{x_1}{x_2(x_1^2 + x_2^2)} < t < e^{-r} - \frac{x_1}{x_2(x_1^2 + x_2^2)}.$$

By direct computation if $r \geq \frac{1}{2} \log 2$ then there is no $t \in \mathbb{R}$ satisfying above inequalities. Thus $\mathcal{I}_x^1$ is empty, and the integral in the right hand side of (2.10) is zero. If $0 \leq r < \frac{1}{2} \log 2$, we define for any $\mathbf{x} \in S_r^+$,

$$L(\mathbf{x}) := \max \left\{ -e^{-r} + \frac{x_2}{x_1(x_1^2 + x_2^2)}, -e^{-r} - \frac{x_1}{x_2(x_1^2 + x_2^2)} \right\}$$

and

$$U(\mathbf{x}) := \min \left\{ e^{-r} + \frac{x_2}{x_1(x_1^2 + x_2^2)}, e^{-r} - \frac{x_1}{x_2(x_1^2 + x_2^2)} \right\}.$$

It is not hard to verify that as long as $0 \leq r < \frac{1}{2} \log 2$, for $\mathbf{x} \in S_r^+$ we have

$$L(\mathbf{x}) = -\frac{e^{-r}}{x_1} + \frac{x_2}{x_1(x_1^2 + x_2^2)} \quad \text{and} \quad U(\mathbf{x}) = \frac{e^{-r}}{x_2} - \frac{x_1}{x_2(x_1^2 + x_2^2)}.$$ 

Thus $\mathcal{I}_x^1$ is nonempty if and only if $L(\mathbf{x}) < U(\mathbf{x})$ and whenever it is nonempty we have $\mathcal{I}_x^1 = \left( -\frac{e^{-r}}{x_1} + \frac{x_2}{x_1(x_1^2 + x_2^2)}, \frac{e^{-r}}{x_2} - \frac{x_1}{x_2(x_1^2 + x_2^2)} \right)$. By direct computation we have $L(\mathbf{x}) < U(\mathbf{x})$ if and only if $\mathbf{x} \in D_r := \{(x_1, x_2) \in S_r^+ \mid x_1 + x_2 > e^r \}$. Hence

$$\langle \hat{f}_r \rangle_2 = \frac{8e^{-2r}}{\zeta(2)} + \frac{8}{\zeta(2)} \int_{D_r} \left( \frac{e^{-r}}{x_2} - \frac{x_1}{x_2(x_1^2 + x_2^2)} \right) - \left( \frac{e^{-r}}{x_1} + \frac{x_2}{x_1(x_1^2 + x_2^2)} \right) \, dx_1 dx_2$$

$$= \frac{8e^{-2r}}{\zeta(2)} + \frac{8}{\zeta(2)} \int_{D_r} \left( \frac{e^{-r}}{x_1} + \frac{e^{-r}}{x_2} - \frac{1}{10x_1 x_2} \right) \, dx_1 dx_2. \quad \square$$
Besides the sets $S_r$, another natural candidate to test formula (2.1) is the family of indicator functions of balls. For any $R > 0$ let $B_R$ be the open ball of radius $R$ centered at the origin, and let $h_R$ be the indicator function of $B_R$. We note that Randol [Ran70] established an asymptotic formula for $\|\hat{h}_R\|_2^2$ for large $R$, and here we prove the following formula for $\|\hat{h}_R\|_2^2$.

**Corollary 2.3.** For any $R > 0$ let $h_R$ be as above. Then we have

$$\|\hat{h}_R\|_2^2 = \frac{12R^2}{\pi} + \frac{48}{\pi} \sum_{n=1}^{\lfloor R^2 \rfloor} \varphi(n) \left( \sqrt{R^4 - n^2} + \arcsin \left( \frac{n}{R^2} \right) - \frac{\pi}{2} \right).$$

**Proof.** Since $B_R$ is symmetric with respect to the origin, we can apply (2.1) to $\|\hat{h}_R\|_2$, and use $\varphi(2) = \frac{x^2}{6}$ to get

$$\|\hat{h}_R\|_2^2 = \frac{12R^2}{\pi} + \frac{12}{\pi^2} \varphi(n) \int_{B_R} \|T^0_{x_1}\|_2 dx,$$

where

$$T^0_{x_1} := \left\{ t \in \mathbb{R} \mid \|n \left( \frac{-x_2}{x_1 + x_2}, \frac{x_1}{x_1 + x_2} \right) + t(x_1, x_2) \|_2 < R \right\}.$$

Using the polar coordinates, for any $(x_2, x_2) = (r \cos \theta, r \sin \theta) \in B_R$ and $n \geq Rr$ we can write

$$\|n \left( \frac{-x_2}{x_1 + x_2}, \frac{x_1}{x_1 + x_2} \right) + t(x_1, x_2) \|_2^2 = \frac{n^2}{r^2} + t^2 r^2 \geq R^2,$$

implying that $T^0_{x_1}$ is empty whenever $n \geq Rr = R\|x\|_2$. In particular, $T^0_{x_1}$ is empty for any $x \in B_R$ if $n \geq R^2$. Similarly for any $1 \leq n \leq \lfloor R^2 \rfloor$, $T^0_{x_1}$ is empty if $\|x\|_2 \leq \frac{n}{R}$, and $T^0_{x_1} = \left( -\sqrt{R^2 + n^2}, \sqrt{R^2 + n^2} \right)$ if $\frac{n}{R} < \|x\|_2 < R$. Hence

$$\|\hat{h}_R\|_2^2 = \frac{12R^2}{\pi} + \frac{12}{\pi^2} \sum_{n=1}^{\lfloor R^2 \rfloor} \varphi(n) \int_0^{2\pi} \int_0^R \frac{2\sqrt{R^2 r^2 - n^2}}{r^2} dr d\theta$$

$$= \frac{12R^2}{\pi} + \frac{48}{\pi} \sum_{n=1}^{\lfloor R^2 \rfloor} \varphi(n) \int_1^{R^2} \sqrt{1 - r^{-2}} dr$$

$$= \frac{12R^2}{\pi} + \frac{48}{\pi} \sum_{n=1}^{\lfloor R^2 \rfloor} \varphi(n) \left( \sqrt{R^4 - n^2} + \arcsin \left( \frac{n}{R^2} \right) - \frac{\pi}{2} \right),$$

where for the second equality we applied a change of variable $\frac{R}{n} r \mapsto r$, and for the last equality we used the fact that $\int \sqrt{1 - r^{-2}} dr = \sqrt{r^2 - 1} + \arcsin \left( \frac{1}{r} \right) + C$ for $r \geq 1$. \[\square\]

3. **Measure estimates of the shrinking targets**

In this section, using the methods developed in the previous section, we prove Theorem 1.2 and then use it to derive Theorem 1.3 and Corollary 1.4.
Proof of Theorem 1.2. For any \( r > 0 \), let \( f_r \) be the indicator function of \( S_r \) as before. For any integer \( k \geq 0 \), let \( B^k_r \subset X \) be the set of unimodular lattices having \( 2k \) nonzero primitive points in \( S_r \). First, we note that \( K_r = B^0_r \) consists of lattices with no nonzero points in \( S_r \). Moreover, for any \( \Lambda \in X \), there are at most two linearly independent primitive points of \( \Lambda \) inside \( S_r \). We thus have for any \( r > 0 \)

\[
\sum_{k=0}^{2} \mu(B^k_r) = 1,
\]

and

\[
\hat{f}_r = 2\chi_{B^1_r} + 4\chi_{B^2_r}.
\]

Thus we can take the first moment and apply (1.10) to get

\[
\mu(B^1_r) + 2\mu(B^2_r) = \frac{1}{2} \int_X \hat{f}_r(\Lambda) d\mu(\Lambda) = \frac{2e^{-2r}}{\zeta(2)}.
\]

Taking the second moment of \( \hat{f}_r \) we get

\[
4\mu(B^1_r) + 16\mu(B^2_r) = \|\hat{f}_r\|^2.
\]

Solving equations (3.1), (3.2) and (3.3) and applying Theorem 1.5 to (3.3), we get

\[
\mu(K_r) = \mu(B^0_r) = 1 - \frac{2e^{-2r}}{\zeta(2)} + \frac{1}{\zeta(2)} \int_{D_r} \left( \frac{e^{-r}}{x_1} + \frac{e^{-r}}{x_2} - \frac{1}{x_1 x_2} \right) dx_1 dx_2.
\]

By direct computation we have for \( 0 < r < \frac{1}{2} \log 2 \)

\[
\int_{D_r} \left( \frac{e^{-r}}{x_1} + \frac{e^{-r}}{x_2} - \frac{1}{x_1 x_2} \right) dx_1 dx_2
\]

\[
= 2(1 - r)(2e^{-2r} - 1 + r) + (2 - 2e^{-2r} - 2r) \log(1 - e^{-2r}) - 2r^2 + \int_{1-e^{-2r}}^{e^{-2r}} \frac{\log t}{1 - t} dt
\]

\[
= 2(1 - r)(2e^{-2r} - 1 + r) + (2 - 2e^{-2r} - 2r) \log(1 - e^{-2r}) - 2r^2 + \text{Li}_2(1 - 2e^{-2r}) - \text{Li}_2(e^{-2r}),
\]

where \( \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \) is the polylogarithm function. Now for the term \( \log(1 - e^{-2r}) \), using the Taylor expansion \( e^{-2r} = 1 - 2r + 2r^2 + O(r^3) \), we get

\[
\log(1 - e^{-2r}) = \log(2r) + \log(1 - r + O(r^2)) = \log(2r) - r + O(r^2).
\]

Using the series representation \( \text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \) we get that \( \text{Li}_2(1 - 2e^{-2r}) = 2r + O(r^2) \). Finally for the term \( \text{Li}_2(e^{-2r}) \) we have the expansion (see [Woo92, Equation (9.7)])

\[
\text{Li}_2(e^{-2r}) = -2r(1 - \log(2r)) + \zeta(2) + O(r^2).
\]

Plugging these into (3.4) and using the expansion \( e^{-2r} = 1 - 2r + 2r^2 + O(r^3) \), we get

\[
\int_{D_r} \left( \frac{e^{-r}}{x_1} + \frac{e^{-r}}{x_2} - \frac{1}{x_1 x_2} \right) dx_1 dx_2 = 2 - \zeta(2) - 4r - 4r^2 \log r + O(r^2),
\]
implying that
\[
\mu(K_r) = 1 - \frac{2e^{-2r}}{\zeta(2)} + \frac{1}{\zeta(2)}(2 - \zeta(2) - 4r - 4r^2 \log r + O(r^2))
\]
\[
= -\frac{4r^2 \log r}{\zeta(2)} + O(r^2)
\]
finishing the proof. \(\square\)

To estimate the measure of the thickening, we will need the following two preliminary lemmas. We note that by Hajos-Minkowski Theorem (see [Cas97, IX.1.3]) we have
\[
K_0 = \Delta^{-1}\{0\} = \bigcup_{x \in [0,1)} \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \mathbb{Z}^2 \bigcup \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) \mathbb{Z}^2.
\]

A simple observation is that any \(\Lambda \in K_0\) contains either the point \((1,0)\) or the point \((0,1)\). Thus intuitively one shall expect that when \(r\) is small, lattices in \(K_r\) contain points close to either \((1, 0)\) or \((0, 1)\). For any \(r > 0\), let \(A_r \subset \mathbb{R}^2\) be the closed rectangle with vertices \((\pm \sqrt{e^{2r} - 1}, e^{-r})\) and \((\pm \sqrt{e^{2r} - 1}, e^{-r})\) and let \(C_r\) be the closed rectangle with vertices \((e^{-r}, \pm \sqrt{e^{2r} - 1})\) and \((e^{-r}, \pm \sqrt{e^{2r} - 1})\), see Figure 1. The following lemma asserts that when \(r\) is small, then any \(\Lambda \in K_r\) contains points either in \(A_r\) or in \(C_r\) (noting that \(A_r\) is a small rectangle containing \((0, 1)\) and \(C_r\) is a small rectangle containing \((1, 0)\)).

**Figure 1.** The square \(S_r\) (red), the rectangles \(A_r\) (green) and \(C_r\) (blue).

**Figure 2.** The square \(S_r\) (red), the rectangles \(U_r\) (green) and \(R_r\) (blue).

**Lemma 3.1.** Let \(A_r\) and \(C_r\) be as above. For any \(0 < r < \log 1.01\) and for any \(\Lambda \in K_r\), we have \(\Lambda_{pr} \cap (A_r \cup C_r) \neq \emptyset\).

**Proof.** Let \(U_r\) be the closed rectangle with vertices \((\pm e^{-r}, e^{-r})\) and \((\pm e^{-r}, e^{-r})\), and let \(R_r\) be the closed rectangle with vertices \((e^{-r}, \pm e^{-r})\) and \((e^{-r}, \pm e^{-r})\), see Figure 2. Let
\[
\tilde{U}_r := \{x \in \mathbb{R}^2 \mid -x \in U_r\}.
\]
Consider the rectangle \(U_r \cup S_r \cup \tilde{U}_r\) and note that it has area 4. For any \(\varepsilon > 0\) let \(U_{r,\varepsilon}\) be the open rectangle with vertices \((\pm e^{-r}, \pm (e^{-r} + \varepsilon))\). Applying the Minkowski’s Convex Body Theorem to \(U_{r,\varepsilon}\) and letting \(\varepsilon\) approach zero, we see that for any \(\Lambda \in X\), \(\Lambda_{pr}\) intersects \(U_r \cup S_r \cup \tilde{U}_r\) nontrivially. Now let \(\Lambda \in K_r\); since \(\Lambda\) has no nonzero point in \(S_r\) and \(\Lambda_{pr}\) is invariant under inversion, we have \(\Lambda_{pr} \cap U_r \neq \emptyset\). Similarly we also have \(\Lambda_{pr} \cap R_r \neq \emptyset\). Moreover, we note that for \(0 < r < \log 1.01\), we have \(\Lambda \cap U_r = \Lambda_{pr} \cap U_r\) and \(\Lambda \cap R_r = \Lambda_{pr} \cap R_r\).
This is because otherwise there would be some nonzero point \( \mathbf{v} \in \Lambda \cap (U_r \cup R_r) \) and some integer \( k \geq 2 \) such that \( \frac{v}{k} \notin \Lambda_{pr} \), but \( \mathbf{v} \in U_r \cup R_r \) and \( k \geq 2 \) imply that \( \frac{v}{k} \notin S_r \), contradicting the assumption that \( \Lambda_{pr} \cap S_r = \emptyset \). Let \( \mathbf{v}_1 = (t_1, 1 + v_1) \) be a point in \( \Lambda_{pr} \cap U_r \) that is closest to the \( y \)-axis and let \( \mathbf{v}_2 = (1 + v_2, t_2) \) be a point in \( \Lambda_{pr} \cap R_r \) that is closest to the \( x \)-axis. We thus have \( |t_i| \leq e^{-r} \) and \( e^{-r} \leq 1 + v_i \leq e^r \) for \( i = 1, 2 \).

Let \( P_{\mathbf{v}_1, \mathbf{v}_2} \) be the parallelogram spanned by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Then we have for \( 0 < r < \log 1.01 \)
\[
|P_{\mathbf{v}_1, \mathbf{v}_2}| = |(1 + v_1)(1 + v_2) - t_1 t_2| = |(1 + v_1)(1 + v_2) - t_1 t_2| \leq e^{2r} + e^{-2r} < 3,
\]
where \( |P_{\mathbf{v}_1, \mathbf{v}_2}| \) denote the area of \( P_{\mathbf{v}_1, \mathbf{v}_2} \), and for the second equality we used that
\[
(1 + v_1)(1 + v_2) \geq e^{-2r} \leq |t_1 t_2|.
\]
Thus \( |P_{\mathbf{v}_1, \mathbf{v}_2}| \) equals 1 or 2. We claim that \( |P_{\mathbf{v}_1, \mathbf{v}_2}| = 1 \). Suppose not, then \( |P_{\mathbf{v}_1, \mathbf{v}_2}| = 2 \) and we have for \( 0 < r < \log 1.01 \)
\[
t_1 t_2 = v_1 + v_2 + v_1 v_2 - 1 \leq 2(e^r - 1) + (e^r - 1)^2 - 1 < 0
\]
and
\[
|t_1 t_2| = 1 - v_1 - v_2 - v_1 v_2 \geq 1 - 2(e^r - 1) - (e^r - 1)^2 = 2 - 2e^r > 0.9.
\]
This implies that \( \min\{|t_1|, |t_2|\} > \frac{0.9}{e^r} > 0.9 \). Since \( t_1 t_2 < 0 \), without loss of generality we may assume that \( t_2 < 0 \). Then we have \( -e^{-r} \leq t_2 < -0.9 \). On one hand, since \( |P_{\mathbf{v}_1, \mathbf{v}_2}| = 2 \) and \( \mathbf{v}_1, \mathbf{v}_2 \in \Lambda_{pr} \), we have
\[
\mathbf{w} := \frac{\mathbf{v}_1 + \mathbf{v}_2}{2} = \left( \frac{t_1 + 1 + v_2}{2}, \frac{t_2 + 1 + v_1}{2} \right) \in \Lambda.
\]
On the other hand, we have \( 0 < \frac{t_1 + 1 + v_2}{2} \leq \frac{e^{-r} + e^r}{2} < e^r \), \( 0 < \frac{t_2 + 1 + v_1}{2} < \frac{1 + v_1}{2} \leq \frac{e^r}{2} < e^{-r} \) and \( \mathbf{w} \notin S_r \) implying that \( \mathbf{w} \in R_r \). Thus \( \mathbf{w} \in \Lambda \cap R_r = \Lambda_{pr} \cap R_r \) is also a primitive vector of \( \Lambda \).
Moreover, since \( -e^{-r} \leq t_2 < -0.9 \), we have
\[
0 < \frac{t_2 + 1 + v_1}{2} < \frac{e^r - 0.9}{2} < \frac{1.01 - 0.9}{2} = 0.055 < |t_2|,
\]
contradicting the assumption that \( \mathbf{v}_2 \) is the closest point in \( \Lambda_{pr} \cap R_r \) to the \( x \)-axis. We thus have proved the claim, and it implies that
\[
|t_1 t_2| = |v_1 + v_2 + v_1 v_2| \leq 2(e^r - 1) + (e^r - 1)^2 = e^{2r} - 1.
\]
Hence we have \( \min\{|t_1|, |t_2|\} \leq \sqrt{|t_1 t_2|} \leq \sqrt{e^{2r} - 1} \) which implies that \( \Lambda_{pr} \cap (A_r \cup C_r) \neq \emptyset \) finishing the proof.

The following lemma states that for \( r > 0 \) small, the orbits \( a_s K_r \) will completely leave the set \( K_r \) very shortly, and will remain separated for quite a long time.

**Lemma 3.2.** For any \( 0 < r < \log 1.01 \) and any \( 6r \leq |s| \leq \log 1.9 \), we have
\[
a_s K_r \cap K_r = \emptyset.
\]

**Proof.** Suppose not, then there exists some \( \Lambda \in a_s K_r \cap K_r \), and by definition the intersection of \( \Lambda_{pr} \) with \( S_r \cup a_s S_r \) is empty. Without loss of generality we may assume that \( s > 0 \). By Lemma 3.1 we have \( \Lambda_{pr} \cap (A_r \cup C_r) \neq \emptyset \) and similarly, \( \Lambda_{pr} \cap (a_s A_r \cup a_s C_r) \neq \emptyset \). We note that \( a_s A_r \) is the rectangle with vertices \((\pm e^s \sqrt{e^{2r} - 1}, e^{r-s})\) and \((\pm e^s \sqrt{e^{2r} - 1}, e^{-r-s})\). Since \( e^{6r} \leq e^s \leq 1.9 \) we have \( a_s A_r \subseteq S_r \), implying that \( \Lambda_{pr} \cap a_s C_r \neq \emptyset \). Similarly, we have \( C_r \subseteq a_s S_r \) and this implies that \( \Lambda_{pr} \cap A_r \neq \emptyset \) (see Figure 4). Let \( \mathbf{v}_1 \in \Lambda_{pr} \cap A_r \)
and \( \mathbf{v}_2 \in \Lambda_{pr} \cap a_s \mathcal{C}_r \), and let \( P_{\mathbf{v}_1, \mathbf{v}_2} \) be the parallelogram spanned by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Then for \( 0 < r < \log 1.01 \) and \( 6r \leq s \leq \log 1.9 \) we have
\[
1 < e^{s - 2r} - (e^{2r} - 1)e^{-s} \leq |P_{\mathbf{v}_1, \mathbf{v}_2}| \leq e^{s + 2r} + (e^{2r} - 1)e^{-s} < 2
\]
contradicting the fact that \( |P_{\mathbf{v}_1, \mathbf{v}_2}| \) is a positive integer.

\[
\square
\]

**Figure 3.** Figure 1 under the flow \( a_s \): The rectangles \( a_s \mathcal{S}_r \) (orange), \( a_s \mathcal{A}_r \) (brown) and \( a_s \mathcal{C}_r \) (purple).

**Figure 4.** Figure 1 and Figure 3 in one picture: The rectangle \( a_s \mathcal{A}_r \) (brown) is contained in \( \mathcal{S}_r \) (red), the rectangle \( \mathcal{C}_r \) (blue) is contained in \( a_s \mathcal{S}_r \) (orange).

We can now give the Proof of Theorem 1.3. We prove the upper and lower bounds separately. For the upper bound, we first note that for any \( \mathbf{v} \in \mathbb{R}^2 \), \( e^{-|s|} \| \mathbf{v} \| \leq \| a_s \mathbf{v} \| \leq e^{|s|} \| \mathbf{v} \| \). Hence for any \( \Lambda \in X \) we have
\[
|\Delta (a_s \Lambda) - \Delta (\Lambda)| \leq |s|.
\]
This implies that for any \( s \in \mathbb{R} \) and any \( r > 0 \)
\[
(3.6) \hspace{1cm} a_s K_r \subset K_{r + |s|}.
\]
Let \( N = \lceil \frac{1}{r} \rceil \). Using (3.6) and the fact that \( \frac{1}{N} \leq r \) we can estimate
\[
\bigcup_{0 \leq s < 1} a_{-s} K_r = \bigcup_{0 \leq i < N} \bigcup_{0 \leq t < \frac{1}{N}} a_{-\frac{i}{N}} a_{-t} K_r \subset \bigcup_{0 \leq i < N} a_{-\frac{i}{N}} K_{2r}.
\]
Hence by Theorem 1.2 and since \( N > \frac{1}{r} \) we have
\[
\mu \left( \bigcup_{0 \leq s < 1} a_{-s} K_r \right) \leq \sum_{i=0}^{N-1} \mu \left( a_{-\frac{i}{N}} K_{2r} \right) \approx r \log \left( \frac{1}{r} \right).
\]
For the lower bound, for \( 0 < r < \log 1.01 \) let \( N = \lfloor \frac{1}{6r} \rfloor \). First we have
\[
\bigcup_{0 \leq i < \lfloor N \log 1.9 \rfloor} a_{-\frac{i}{N}} K_r \subset \bigcup_{0 \leq s < 1} a_{-s} K_r.
\]
Moreover, for each \( 0 \leq i < j < \lfloor N \log 1.9 \rfloor \), \( 6r \leq \frac{1}{N} \leq \frac{j-i}{N} < \log 1.9 \), thus by Lemma 3.2 we have
\[
a_{-\frac{i}{N}} K_r \cap a_{-\frac{j}{N}} K_r = a_{-\frac{j-i}{N}} (a_{-\frac{i}{N}} K_r \cap K_r) = \emptyset.
\]
Thus the union $\bigcup_{0 \leq s < 1} a_{-s} K_r$ is disjoint and, again applying Theorem 1.2 and noting that $N \asymp \frac{1}{r}$ we can estimate
\[ \mu \left( \bigcup_{0 \leq s < 1} a_{-s} K_r \right) \geq \sum_{i=0}^{\left\lfloor N \log 1.9 \right\rfloor - 1} \mu \left( a_{-\frac{i}{N}} K_r \right) \asymp r \log \left( \frac{1}{r} \right), \]
finishing the proof. \hfill \Box

**Proof of Corollary 1.4.** First we note that we can assume that $\lim_{s \to \infty} r(s) = 0$ since otherwise both series would diverge. It follows that there exists $N > 0$ such that for any $n > N$, $0 < r(n) < \log 1.01$. Next, since $r(\cdot)$ is non-increasing, for any $n > N$ we have
\[ \bigcup_{0 \leq s < 1} a_{-s} K_{r(n+1)} \subset \tilde{B}_n \subset \bigcup_{0 \leq s < 1} a_{-s} K_{r(n)}. \]
Moreover, since $n > N$ we have $0 < r(n+1) \leq r(n) < \log 1.01$. Applying Theorem 1.3 to the left and right hand sides of the above inclusion relations we get
\[ r(n+1) \log \left( \frac{1}{r(n+1)} \right) \ll \mu \left( \tilde{B}_n \right) \ll r(n) \log \left( \frac{1}{r(n)} \right) \]
which finishes the proof. \hfill \Box

4. **The dynamical Borel-Cantelli lemma**

In this section we give the proof of Theorem 1.1 based on Theorem KW. Recall that for a given function $\psi : [t_0, \infty) \to (0, \infty)$ with $t_0 \geq 1$ fixed, we say a real number $x \in \mathbb{R}$ is $\psi$-Dirichlet if the system of inequalities
\[ |qx - p| < \psi(t) \quad \text{and} \quad |q| < t \]
has a solution in $(p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ for all sufficiently large $t$. Let us denote by $D(\psi)$ the set of all $\psi$-Dirichlet numbers. Theorem KW gives a zero-one law for the Lebesgue measure of $D(\psi)$ as follows: if $\psi : [t_0, \infty) \to (0, \infty)$ is a continuous, non-increasing function satisfying (1.6) and (1.7), then the series (1.8) diverges (resp. converges) if and only if the Lebesgue measure of $D(\psi)$ (resp. of $D(\psi)^c$) is zero.

For our purpose, we prove the following slightly modified version of Dani Correspondence.

**Lemma 4.1.** Let $\psi : [t_0, \infty) \to (0, \infty)$ be a continuous, non-increasing function satisfying (1.6) and (1.7). Then there exists a unique continuous, non-increasing function
\[ r = r_\psi : [s_0, \infty) \to (0, \infty), \quad \text{where} \quad s_0 = \frac{1}{2} \log t_0 - \frac{1}{2} \log \psi(t_0), \]
such that
\[ \psi(e^{s-r(s)}) = e^{-s-r(s)} \text{ for all } s \geq s_0. \]
Conversely, given a continuous, non-increasing function $r : [s_0, \infty) \to (0, \infty)$ satisfying (4.1), then there exists a unique continuous, non-increasing function $\psi = \psi_r : [t_0, \infty) \to (0, \infty)$ with $t_0 = e^{s_0-r(s_0)}$ satisfying (1.6), (1.7) and (4.2). Furthermore, if we assume
$\lim_{t \to \infty} \psi(t) = 1$ (or equivalently, $\lim_{n \to \infty} r(n) = 0$), then the series in (1.8) diverges if and only if the series

\begin{equation}
\sum_{n} r(n) \log \left( \frac{1}{r(n)} \right)
\end{equation}

diverges.

Proof. The correspondence between $\psi = \psi_r$ and $r = r_\psi$ follows from the exact same construction as in \cite[Lemma 8.3]{KM99}, where $\psi(\cdot)$ and $r(\cdot)$ determine each other with the relations

$$e^{s} \psi(t) = e^{-r(s)} = e^{-st}$$

with $s$ and $t$ satisfying $s = \frac{1}{2} \log t - \frac{1}{2} \log \psi(t)$. The only difference is that here we require the two extra assumptions (1.6) and (1.7) on $\psi$ which are respectively equivalent to the assumptions that $r(\cdot)$ is non-increasing and $r(\cdot)$ is positive. We refer the reader to \cite[Lemma 8.3]{KM99} for more details about this correspondence.

For the furthermore part, first we claim that the series in (1.8) diverges if and only if the integral

\begin{equation}
\int_{t_0}^{\infty} \frac{-(1-t\psi(t)) \log (1-t\psi(t))}{t} dt
\end{equation}

diverges. It suffices to show the function $G(t) := -\log (1-t\psi(t)) (1-t\psi(t))$ is eventually non-increasing in $t$. Note that the function $T \mapsto -T \log T$ is strictly increasing on the interval $(0, e^{-1})$. Since $\lim_{t \to \infty} t\psi(t) = 1$ and $\psi(t) < 1$ for all $t \geq t_0$, there exists some $T_0 > t_0$ such that for all $t > T_0$, $0 < 1 - t\psi(t) < e^{-1}$. Moreover, together with the assumption (1.6) we get that $G(t)$ is non-increasing in $t$ for any $T > T_0$, finishing the proof the claim. Next, since $r(\cdot)$ is positive and non-increasing, we have $0 < r(s) \leq r(s_0)$. Thus there exist constants $0 < c_1 < c_2$ such that for all $s \geq s_0$ and all $t \geq t_0$ with $s = \frac{1}{2} \log t - \frac{1}{2} \log \psi(t)$, we have

$$c_1 r(s) \leq 1 - t\psi(t) = 1 - e^{-2r(s)} \leq c_2 r(s).$$

This also implies that

$$-\log (1-t\psi(t)) = -\log (r(s)) + O_{c_1, c_2}(1) \approx c_1, c_2 - \log (r(s)),$$

where for the second estimate we used that $\lim\limits_{s \to \infty} r(s) = 0$. Moreover, since $r(\cdot)$ is non-increasing and continuous, it is differentiable at Lebesgue almost every $s \in \mathbb{R}$, and we denote by $r'(s)$ for its derivative at $s \in \mathbb{R}$ whenever it exists. Using the relation $t = e^{s-r(s)}$ we get $\frac{dt}{t} = (1 - r'(s)) ds$ for Lebesgue almost every $s \in \mathbb{R}$. We thus have

$$\int_{t_0}^{\infty} \frac{-(1-t\psi(t)) \log (1-t\psi(t))}{t} dt \approx c_1, c_2 \int_{s_0}^{\infty} -r(s) \log (r(s)) (1-r'(s)) ds$$

$$\approx \int_{s_0}^{\infty} -r(s) \log (r(s)) ds,$$

where for the second estimate we used that $1 \leq 1 - r'(s) \leq 2$ for Lebesgue almost every $s \in \mathbb{R}$ which comes from the assumption (4.1) and that $r(\cdot)$ is non-increasing. Finally, we conclude the proof by noting that the integral $\int_{s_0}^{\infty} -r(s) \log (r(s)) ds$ diverges if and only if the series $\sum_{n} -r(n) \log (r(n))$ diverges since $\lim_{n \to \infty} r(s) = 0$ and $r(\cdot)$ is non-increasing which imply that the function $s \mapsto -r(s) \log (r(s))$ is eventually non-increasing in $s$. \hfill $\square$
As mentioned in the introduction, we have the following dynamical interpretation of $\psi$-Dirichlet numbers.

**Lemma 4.2.** ([KW18, Proposition 4.5]) Let $\psi : [t_0, \infty) \to (0, \infty)$ be a continuous and non-increasing function satisfying (1.6) and (1.7). Let $r = r_\psi$ be as in Lemma 4.1. Then $x \in D(\psi)^c$ if and only if

\begin{equation}
(4.5) \quad a_s \Lambda_x \in K_{r(s)} \quad \text{for an unbounded set of } s,
\end{equation}

where $a_s = \text{diag}(e^s, e^{-s})$ and $\Lambda_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 \in X$ are as before.

Combining Theorem KW with Lemmas 4.1 and 4.2, we immediately have the following zero-one law.

**Proposition 4.3.** Let $r : [s_0, \infty) \to (0, \infty)$ be continuous, non-increasing, satisfying (4.1) and such that $\lim_{s \to \infty} r(s) = 0$. Then (4.5) holds for Lebesgue almost every (resp. almost no) $x \in \mathbb{R}$ provided that the series (4.3) diverges (resp. converges).

To connect the above proposition with the corresponding property of almost every $\Lambda \in X$, we need an auxiliary lemma, which borrows some ideas from the work [KR19] of the first-named author with Anurag Rao.

**Lemma 4.4.** Let $r(\cdot)$ be as in Proposition 4.3. For any $c \in \mathbb{R}$ and $\lambda > 0$ let

\[ r_{c,\lambda}(s) := r(s + c) - \lambda e^{-2(s+c)}, \]

and define

\[ D_{c,\lambda} := \{ x \in \mathbb{R} \mid a_s \Lambda_x \in K_{r_{c,\lambda}(s)} \quad \text{for an unbounded set of } s \} \cdot \]

If the series (4.3) diverges, then the set

\[ D := \bigcap_{c \in \mathbb{R}} \bigcap_{\lambda > 0} D_{c,\lambda} \]

has full Lebesgue measure.

**Remark 4.5.** We note that by our assumption $r_{c,\lambda}(\cdot)$ is not necessarily always positive, and the set $K_{r_{c,\lambda}(s)}$ is empty whenever $r_{c,\lambda}(s)$ is negative.

**Proof of Lemma 4.4.** For any function $f : [s_f, \infty) \to (0, \infty)$ with $s_f \geq 1$ we denote

\[ A_{\infty,f} := \{ x \in \mathbb{R} \mid a_s \Lambda_x \in K_f(s) \quad \text{for an unbounded set of } s > s_f \} \]

and $N_f := \sum_{n \geq s_f} f(n) \log\left(\frac{1}{f(n)}\right)$. First we note that the divergence of the series $N_r$ is equivalent to the divergence of the series $N_{r_{c,\lambda}}$ for any $c \in \mathbb{R}$, where $r_c(s) := r(s + c) = r_{c,0}(s)$. Moreover, it is clear that $\frac{1}{2}r_{c,\lambda}(\cdot)$ satisfies the assumptions in Proposition 4.3. Thus, by Proposition 4.3, if the series $N_r$ diverges, then the set $A_{\infty,\frac{1}{2}r_c}$ is of full Lebesgue measure for any $c \in \mathbb{R}$. On the other hand, for any $c \in \mathbb{R}$ and $\lambda > 0$ let $f_{c,\lambda}(s) = \lambda e^{-2(s+c)}$. It is easy to check that $f_{c,\lambda} \mid_{[s_{c,\lambda}, \infty)}$ satisfies the assumptions in Proposition 4.3 with $s_{c,\lambda} := \frac{1}{2} r_{c,\lambda}$ and $\frac{1}{2}r_{c,\lambda}(\cdot)$.
max\{\log(2\lambda) - c, 1\}, and the series \(N_{f_c,\lambda}\) converges for any \(c \in \mathbb{R}\) and \(\lambda > 0\). Thus by Proposition 4.3 the set \(A_{\infty, f_c, \lambda}\) is of zero Lebesgue measure for any \(c \in \mathbb{R}\) and \(\lambda > 0\). Define

\[
\overline{A} := \bigcap_{c \in \mathbb{R}} A_{\infty, \frac{1}{2}r_c} \quad \text{and} \quad \overline{A} := \bigcup_{c \in \mathbb{R}, \lambda > 0} A_{\infty, f_c, \lambda}.
\]

We note that since \(r(\cdot)\) is non-increasing, for any \(c_1 < c_2\) we have \(\frac{1}{2}r_{c_1} \geq \frac{1}{2}r_{c_2}\) implying that \(A_{\infty, \frac{1}{2}r_{c_2}} \subset A_{\infty, \frac{1}{2}r_{c_1}}\). Hence the family of sets \(\{A_{\infty, \frac{1}{2}r_c}\}_{c \in \mathbb{R}}\) is nested and \(\overline{A} = \lim_{c \to \infty} A_{\infty, \frac{1}{2}r_c}\) is of full Lebesgue measure. Similarly, the family of sets \(\{A_{\infty, f_c, \lambda}\}_{c \in \mathbb{R}, \lambda > 0}\) is also nested and the set \(\overline{A} = \lim_{\lambda \to \infty} \lim_{c \to -\infty} A_{\infty, f_c, \lambda}\) is of zero Lebesgue measure. Thus the set \(\overline{A} \setminus \overline{A}\) is of full Lebesgue measure and it suffices to show that \(\overline{A} \setminus \overline{A} \subset D\). That is, for any \(x \in \overline{A} \setminus \overline{A}\) we want to show that for any \(c \in \mathbb{R}\) and any \(\lambda > 0\) the events \(a_s \Lambda x \in K_{r_c,\lambda}(s)\) happen for an unbounded set of \(s\). First we note that \(x \in \overline{A}\) means that for any \(c \in \mathbb{R}\) there exists an unbounded subset \(S_c \subset \mathbb{R}\) such that \(a_s \Lambda x \in K_{\frac{1}{2}r_c}(s)\) for any \(s \in S_c\). Secondly, we note that \(x \notin \overline{A}\) means that for any \(c \in \mathbb{R}\) and \(\lambda > 0\) there exists some constant \(T_{c,\lambda} > 0\) such that for any \(s \geq T_{c,\lambda}\) we have \(a_s \Lambda x \in \Delta^{-1}(f_{c,\lambda}(s), \infty)\). In particular, for any \(s \in S_c \cap (T_{c,\lambda}, \infty)\) we have

\[
f_{c,\lambda}(s) < \Delta(a_s \Lambda x) \leq \frac{1}{2}r_c(s).
\]

This implies that \(0 < \Delta(a_s \Lambda x) \leq \frac{1}{2}r_c(s) < \frac{1}{2}r_c(s) + \frac{1}{2}r_c(s) - f_{c,\lambda}(s) = r_{c,\lambda}(s)\) for any \(s \in S_c \cap (T_{c,\lambda}, \infty)\). Finally, we finish the proof by noting that since \(S_c\) is unbounded, the set \(S_c \cap (T_{c,\lambda}, \infty)\) is also unbounded. \(\square\)

We can now give the

**Proof of Theorem 1.1.** The convergent case follows directly from Corollary 1.4 and the classical Borel-Cantelli lemma, and we thus only need to prove the divergent case. Let \(r : [s_0, \infty) \to (0, \infty)\) be continuous, non-increasing, satisfying (4.3) and such that the series (4.3) diverges; we want to show that \(\mu(B_\infty) = 1\). First we note that we can assume that \(\lim_{s \to \infty} r(s) = 0\), since otherwise the result would follow from the ergodicity of the flow \(\{a_s\}_{s > 0}\) on \(X\). Let \(D := \bigcap_{c \in \mathbb{R}} \bigcap_{\lambda > 0} D_{c,\lambda}\) be as in Lemma 4.4 and define \(B \subset X\) such that

\[
B = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \Lambda x \in X \right| b \in \mathbb{R}, \ a > 0, \ x \in D \right\}.
\]

We note that by Lemma 4.4 the set \(D\) has full Lebesgue measure. Thus the set \(B \subset X\) is also of full measure (with respect to \(\mu\)) and it suffices to show that \(B \subset B_\infty\). First, by direct computation for \(\Lambda = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}\) we have

\[
a_s \Lambda = \begin{pmatrix} e^{-2s} a^{-1} b & 0 \\ 1 & 1 \end{pmatrix} a_s \log a \Lambda x.
\]

Next, for any \(y \in \mathbb{R}\) let \(u_y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\). Note that for any \(v \in \mathbb{R}^2, \|u_y v\| \leq (|y| + 1)\|v\|\). This implies that for any \(\Lambda \in X\)

\[
|\Delta(u_y^- \Lambda) - \Delta(\Lambda)| \leq \log(1 + |y|).
\]
Using the above inequality, the relation (4.6) and the inequality \( \log(1+x) < 2x \) for all \( x > 0 \), we get

\[
|\Delta(a_s \Lambda) - \Delta(a_{s+\log a \Lambda x})| \leq 2a^{-1}|b|e^{-2x}.
\]

Since \( x \in D \) we have for any \( c \in \mathbb{R} \) and any \( \lambda > 0 \), \( a_s \Lambda x \in K_{r_{c,\lambda}}(s) \) for an unbounded set of \( s \). In particular, taking \( c = -\log a \), \( \lambda = 2a^{-1}|b| \) we get

\[
0 \leq \Delta(a_s \Lambda) \leq \Delta(a_{s-c} \Lambda x) + \lambda e^{-2s} \leq r_{c,\lambda}(s-c) + \lambda e^{-2s} = r(s)
\]

for an unbounded set of \( s \), finishing the proof. \( \square \)

References

[AK16] J. S Athreya and I. Konstantoulas. Discrepancy of general symplectic lattices. \textit{arXiv preprint arXiv:1611.07146}, 2016. 5

[AM09] J. S. Athreya and G. A. Margulis. Logarithm laws for unipotent flows. I. \textit{J. Mod. Dyn.}, 3(3):359–378, 2009. 5

[Ath09] J. S. Athreya. Logarithm laws and shrinking target properties. \textit{Proc. Indian Acad. Sci. Math. Sci.}, 119(4):541–557, 2009. 2

[Cas97] J. W. S. Cassels. \textit{An introduction to the geometry of numbers}. Classics in Mathematics. Springer-Verlag, Berlin, 1997. Corrected reprint of the 1971 edition. 2, 13

[CK01] N. Chernov and D. Y. Kleinbock. Dynamical Borel-Cantelli lemmas for Gibbs measures. \textit{Israel J. Math.}, 122:1–27, 2001. 1

[Dan85] S. G. Dani. Divergent trajectories of flows on homogeneous spaces and Diophantine approximation. \textit{J. Reine Angew. Math.}, 359:55–89, 1985. 2

[DS70] H. Davenport and W. M. Schmidt. Dirichlet’s theorem on diophantine approximation. In \textit{Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)}, pages 113–132. Academic Press, London, 1970. 3

[Fai19] S. K Fairchild. A higher moment formula for the Siegel–Veech transform over quotients by Hecke triangle groups. \textit{arXiv preprint arXiv:1901.10115}, 2019. 5

[KM99] D. Y. Kleinbock and G. A. Margulis. Logarithm laws for flows on homogeneous spaces. \textit{Invent. Math.}, 138(3):451–494, 1999. 2, 3, 4, 17

[KM18] D. Y. Kleinbock and G. A. Margulis. Erratum to: Logarithm laws for flows on homogeneous spaces. \textit{Invent. Math.}, 211(2):855–862, 2018. 3

[KR19] D. Y. Kleinbock and A. Rao. A zero-one law for uniform Diophantine approximation in Euclidean norm. \textit{arXiv preprint arXiv:1910.00126}, 2019. 18

[KW18] D. Y. Kleinbock and N. Wadleigh. A zero-one law for improvements to Dirichlet’s Theorem. \textit{Proc. Amer. Math. Soc.}, 146(5):1833–1844, 2018. 1, 3, 18

[KY19] D. Kelmer and S. Yu. The second moment of the Siegel transform in the space of symplectic lattices. \textit{Int. Math. Res. Not. IMRN}, 02 2019. rnz027. 5, 7

[Lan75] S. Lang. \textit{SL_2(R)}, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975. 7

[Maucourant] F. Maucourant. Dynamical Borel-Cantelli lemma for hyperbolic spaces. \textit{Israel J. Math.}, 152:143–155, 2006. 2

[Ran70] B. Randol. A group-theoretic lattice-point problem. In \textit{Problems in analysis (papers dedicated to Salomon Bochner, 1969)}, pages 291–295. Princeton Univ. Press, Princeton, N.J., 1970. 5, 11

[Rog55] C. A. Rogers. Mean values over the space of lattices. \textit{Acta Math.}, 94:249–287, 1955. 5, 7

[Sch60] W. M. Schmidt. A metrical theorem in geometry of numbers. \textit{Trans. Amer. Math. Soc.}, 95:516–529, 1960. 5

[Sch80] W. M. Schmidt. \textit{Diophantine approximation}, volume 785 of \textit{Lecture Notes in Mathematics}. Springer, Berlin, 1980. 2

[Sie45] C. L. Siegel. A mean value theorem in geometry of numbers. \textit{Ann. of Math. (2)}, 46:340–347, 1945. 5
[Sul82] D. Sullivan. Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics. *Acta Math.*, 149(3-4):215–237, 1982. 2

[Vee98] W. A. Veech. Siegel measures. *Ann. of Math. (2)*, 148(3):895–944, 1998. 8

[Woo92] D. Wood. The computation of polylogarithms. Technical Report 15-92*, University of Kent, Computing Laboratory, Canterbury, UK, June 1992. 12

Brandeis University, Waltham MA, USA, 02454-9110
*E-mail address*: kleinboc@brandeis.edu

Department of Mathematics, Technion, Haifa, Israel
*E-mail address*: yushucheng@campus.technion.ac.il

21