GEOMETRIC HODGE STRUCTURES WITH PRESCRIBED
HODGE NUMBERS.

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Let us call a pure, and necessarily effective polarizable, Hodge structure geometric if it is contained in the cohomology of a smooth complex projective variety. A natural question is which polarizable Hodge structures are geometric? We discuss a potential answer for two dimensional Hodge structures in the last section, following ideas of [FM, Se]. But in general, it seems far from clear how to characterize them.

The main goal of this note is to show that there are no numerical restrictions on geometric Hodge structures. That is for any set of Hodge numbers (subject to the obvious constraints imposed by Hodge symmetry and effectivity), there exists a geometric Hodge structure with precisely these Hodge numbers. Our example is contained in the cohomology of a power $E^N$ of a CM elliptic curve. Since the Hodge conjecture is known for such varieties, we get slightly stronger statement, that our example is the Hodge realization of Grothendieck motive as we will explain.

We should mention that Schreieder [Scr] has solved the related, but more difficult, problem of finding a smooth projective variety with a prescribed set of Hodge numbers in a given degree $k$ (under suitable hypotheses when $k$ is even). This does imply the main result here for odd $k$. Nevertheless, the construction is different and somewhat more involved. So we hope our contribution has independent interest.

We thank Greg Pearlstein for bringing Schreieder’s work to our attention.

1. Main theorem

We will work exclusively with rational Hodge structures below. It is worth observing that since the category of polarizable Hodge structures is semisimple $\mathbf{M}$, we have:

**Proposition 1.1.** A geometric Hodge structure of weight $k$ is a direct summand of $H^k(X, \mathbb{Q})$ for some smooth complex projective variety $X$.

**Theorem 1.1.** Given a set of nonnegative integers $g^{k,0}, g^{k-1,1}, \ldots, g^{0,k}$, satisfying $k \geq 0$ and $g^{p,q} = g^{q,p}$, there exists a geometric rational Hodge structure $H$ with $\dim H^{p,q} = g^{p,q}$.

**Lemma 1.1.** The Tate structure $\mathbb{Q}(-n)$ is geometric for $n \geq 0$. Let $V_1$ and $V_2$ be geometric Hodge structures of weight $n_1$ and $n_2$ respectively, then $V_1 \otimes V_2$ is geometric. If $n_1 = n_2$, then $V_1 \oplus V_2$ is also geometric.

**Proof.** We have $\mathbb{Q}(-n) = H^{2n}(\mathbb{P}^n, \mathbb{Q})$. By assumption, there exists smooth projective varieties $X_i$ such that $V_i$ is a summand of $H^{n_i}(X_i)$. Then $V_1 \otimes V_2$ and $V_1 \oplus V_2$ are summands of $H^k(X_1 \times X_2)$ where $k = n_1n_2$ and $k = n_1 = n_2$ respectively. □
Lemma 1.2. For any $n > 0$, there exists a geometric Hodge structure $G$ with \( \dim G^{n,0} = \dim \Omega^{0,n} = 1 \) and the remaining Hodge numbers equal to zero.

Proof. Although it is not difficult to prove by hand, it is a bit cleaner if we make use of basic facts about Mumford-Tate groups, cf [L, appendix B] or [M]. In order to spell things out as explicitly as possible, we work with the specific elliptic curve $E = \mathbb{C} / \mathbb{Z} i \oplus \mathbb{Z}$. This has complex multiplication by $K = \mathbb{Q} (i)$. Let $V = H^{1} (E, \mathbb{Q})$. We choose a basis $v_1, v_2$ for $V$ dual to the basis $i, 1$ of the lattice $\mathbb{Z} i \oplus \mathbb{Z}$. Then $dz = iv_1 + v_2$ and $d\bar{z} = -iv_1 + v_2$ determines the Hodge decomposition on $V \otimes \mathbb{C}$. We define a homomorphism $h$ of the unit circle into $GL (V \otimes \mathbb{R})$ by rotations

$$h(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Then $h(\theta)dz = e^{i\theta} dz$, and $h(\theta)d\bar{z} = e^{-i\theta} d\bar{z}$. Let $SMT (V)$ denote the Weil restriction

$$Res_{K/\mathbb{Q}} G_m = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid (a, b) \in \mathbb{Q}^2, (a, b) \neq (0, 0) \right\}$$

This is clearly the smallest $\mathbb{Q}$-algebraic group whose real points contain $im h$. In other words, $SMT (V)$ is the special Mumford-Tate group or Hodge group of $V$. The Mumford-Tate group $MT (V)$ is defined similarly, as the smallest $\mathbb{Q}$-algebraic group whose real points contain the image of Deligne’s torus $S(\mathbb{R}) = \mathbb{C}^*$. It works out to the product of $SMT (V)$ and the group of nonzero scalar matrices. The significance of this group comes from the Tannakian interpretation: $MT (V)$ is the group whose category of representations is equivalent to the tensor category generated by $V$. Since $MT (V)$ was described as a matrix group, it comes with an obvious representation $\rho : MT (V) \to GL (V)$. For an integer $n$, let $V_n$ be the representation of this group given by composing $\rho$ with the $n$th power homomorphism.

$MT (V) \to MT (V)$. Now let us suppose that $n > 0$. Then $V_n$ is irreducible, and therefore simple as a Hodge structure. The elements $dz, d\bar{z}$ still give a basis of $V_n \otimes \mathbb{C}$, but now the circle acts by $dz \mapsto e^{in\theta} dz$ and $d\bar{z} \mapsto e^{-in\theta} d\bar{z}$. Thus these vectors span $V_n^{0,0}$ and $V_n^{0,n}$. We have an embedding $V_n \subseteq S^n V \subset V^{\otimes n}$ given by identifying it with the span of $v_1^n$ and $v_2^n$. Therefore the previous lemma implies that $V_n$ is geometric. Thus $G = V_n$ is the desired Hodge structure. \(\square\)

Lemma 1.3. Given integers $p > q \geq 0$, there exists a geometric Hodge structure $H (p, q)$ with $\dim H (p, q)^{p,q} = \dim H (p, q)^{q,p} = 1$ and the remaining Hodge numbers equal to zero.

Proof. Let $G$ be the Hodge structure constructed in the previous lemma with $n = p - q$. Then $H (p, q) = G \otimes \mathbb{Q} (-q)$ will satisfy the above conditions. \(\square\)

Remark 1.2. We can replace $E$ by any CM elliptic curve $E'$, and the same construction works. We denote the corresponding Hodge structure by $H_{E'} (p, q)$. However, the method will fail for non CM curves, because the Mumford-Tate group will no longer be abelian.

Proof of theorem. We get the desired Hodge structure by taking sums of the Hodge structures constructed in lemma [L, appendix B] and the appropriate number of Tate structures when $k$ is even. More explicitly

$$H = \bigoplus_{p>q} H (p, q)^{p,q} \oplus \mathbb{Q} (-k/2)^{g^{k/2},k/2} \quad \text{if } k \text{ even}$$
The proof actually shows that $H \subset H^k(E^N, \mathbb{Q})$ for some $N$. We can use this fact to get the stronger conclusion stated in the introduction; it says roughly that the inclusion is also defined algebro-geometrically. In order to make a precise statement, we recall that an effective Grothendieck motive consists of a smooth projective variety $X$ together with an algebraic cycle $p \in H^*(X \times X, \mathbb{Q})$ such that $p \circ p = p$ [K]. The composition $\circ$ is the usual one for correspondences [F] chap 16. It follows that $p$ is an idempotent of the ring of Hodge endomorphisms $\prod_i \text{End}_{HS}(H^i(X))$. Thus we get a mixed Hodge structure $p(H^k(E^N))$ which is by definition the Hodge realization of the motive $(X, p)$. We quickly run up against fundamental difficulties. For instance, it is unknown, whether $H^k(X)$ is the realization of a motive for arbitrary $X$ and $k$. For this, we would need to know that the Künneth components of the diagonal $\Delta \subset X \times X$ are algebraic, and this is one Grothendieck’s standard conjectures [G]. Fortunately, it is not an issue in our example, because the Hodge conjecture holds for $E^2N$ [L, appendix B, §3] and this implies Grothendieck’s conjecture. Thus $H^k(E^N)$ is the realization of a motive. Since $H$ is a summand of $H_k(E^N)$, it is given by the image of an idempotent in $\text{End}_{HS}(H^k(E^N))$. By the Hodge conjecture, this is algebraic. Thus to summarize:

**Proposition 1.2.** The geometric Hodge structure $H$, given in the theorem, is the Hodge realization of a Grothendieck motive.

2. **Two dimensional geometric Hodge structures**

The main trick in the proof of the theorem was to construct a good supply of two dimensional building blocks $H_E(p, q)$. These examples have CM, and so are rather special. Somewhat surprisingly, it turns out that any simple two dimensional polarizable Hodge structure of even weight has CM. (We learned this fact from Totaro’s paper [T], but it seems to have been previously known to Beauville.) It is not difficult to conclude, as a consequence, that such Hodge structures are of the form $H_E(p, q)$, for some CM elliptic curve $E$. For odd weight this is not true, because we can associate $H^1(E)$ and its Tate twists to any elliptic curve. So we may ask, where do typical examples come from? To exclude the cases already discussed, let us assume the weight is odd and the level is at least two. We recall that the level of $H$ is the largest value of $|p - q|$ such that $H^{pq} \neq 0$. Based on conjectures of Serre [Sa, §4.8] and Fontaine-Mazur [FM] introduction coming from the arithmetic side of the fence, we guess that all two dimensional geometric Hodge structures of odd weight and level at least two arise from modular forms. This statement needs to be unpacked a bit before it really makes sense. But before getting to this, we should point out that the original conjectures were intended for varieties defined over $\overline{\mathbb{Q}}$, yet we have made no such assumption about the underlying variety in our definition of geometric Hodge structure. Our only justification in making this leap is that Griffiths transversality forces the points that correspond to geometric Hodge structures in the relevant period domain to be very rigid (cf. [T]). One might hope this actually forces the corresponding varieties to be arithmetic.

Let us explain how to get a Hodge structure from a modular form. For simplicity, we work with the principal congruence group $\Gamma(n) = \ker[SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/n)]$ with $n \geq 3$. Recall that a weight $k$ cusp form $f \in S(k(\Gamma(n)))$ is given by a holomorphic
function on the upper half plane $\mathbb{H}$ satisfying
\[ f \left( \frac{az+b}{cz+d} \right) = (cz+d)^k f(z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(n) \]
and such that the Fourier expansion $f = \sum a_j q^j$, $q = e^{2\pi i z/n}$, has only positive terms [DS]. The moduli space $Y(n) = \mathbb{H}/\Gamma(n)$ of elliptic curves with a full level $n$ structure is a fine moduli space, so it comes with a universal family $\pi_n : \mathcal{E}(n) \to Y(n)$. This can be completed to a minimal elliptic surface $\overline{\mathcal{E}}(n) \to X(n)$ over the nonsingular compactification $j : Y(n) \hookrightarrow X(n)$. By a theorem of Zucker [Z], the intersection cohomology of the symmetric power
\[ H = IH^1(X(n), S^{k-1} R^1 \pi_{ns}^* \mathbb{Q}) = H^1(X(n), j_* S^{k-1} R^1 \pi_{ns}^* \mathbb{Q}) \]
carries a pure Hodge structure of weight $k$. This Hodge structure turns out to be isomorphic to one constructed by Shimura [Z] §12. It has only $(k,0)$ and $(0,k)$ parts, and the $(k,0)$ part is isomorphic to the space $S_{k+1}(\Gamma(n))$ of cusp forms of weight $k+1$. We note that $H$ is geometric, since it can be shown to lie the cohomology of a desingularization of the $(k-1)$-fold fibre product $\mathcal{E}(n) \times X(n) \times \ldots \mathcal{E}(n)$. The quickest way to see this is by applying the decomposition theorem for Hodge modules [Sa], although we won’t actually need this. The space $X(n)$ has a large collection of commuting self-correspondences called Hecke operators [DS]. These operators act on $H$ [Sc], and they can be used to break it up into pieces. Suppose that $f = \sum a_j q^j \in S_{k+1}(\Gamma(n))$ is a suitably normalized nonzero simultaneous eigenvector for the Hecke operators, and further assume that the coefficients $a_j$ are all rational. Then $f$ and $\bar{f}$ span a two dimensional Hodge structure $H(f) \subset H$. This is geometric since it comes from a motive constructed by Scholl [Sc].

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