RELATED BY SIMILARITY: PORISTIC TRIANGLES AND 3-PERIODICS IN THE ELLIPTIC BILLIARD

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ABSTRACT. Discovered by William Chapple in 1746, the Poristic family is a set of variable-perimeter triangles with common Incircle and Circumcircle. By definition, the family has constant Inradius-to-Circumradius ratio. Interestingly, this invariance also holds for the family of 3-periodics in the Elliptic Billiard, though here Inradius and Circumradius are variable and perimeters are constant. Indeed, we show one family is mapped onto the other via a varying similarity transform. This implies that any scale-free quantities and invariants observed in one family must hold on the other.

Keywords poristic, triangles, invariant, invariance, inconics, circumconics, elliptic, billiard.

MSC 51M04 and 37D50 and 51N20 and 51N35 and 68T20

1. Introduction

The Poristic family was discovered by William Chapple in 1746 and was later studied by Euler and Poncelet [1, 3, 8]. It is a 1d of set of variable-perimeter triangles (blue) with fixed Incircle and Circumcircle. Figure 1. By definition its Inradius-to-Circumradius \( r/R \) ratio is constant. Interestingly, the same invariance holds for the family of constant-perimeter 3-periodics in the Elliptic Billiard [5, 13].

Our main contribution is to show that one family is mapped onto the other via a (varying) similarity transform whose parameters we derive explicitly. Therefore, all scale-free (e.g., area and length ratios) quantities and invariants verified for one family must hold for the other. To show this we study the dimensions (semi-axes and focal lengths) of various circum- and inconics derived from the family remain constant.

Summary of the Paper: we start with preliminaries in Section 2 and then present the following results:

- Theorem 1: The Excentral Caustic is the MacBeath Circumconic to the Excentrals.
- Theorem 2: The moving Inconic to the Excentral centered on its Circumcenter has invariant axes. A proof appears in Appendix C.
- Theorem 3: The Incenter-centered Circumconic has invariant axes identical to the former. A proof appears in Appendix D.
- Theorem 4: Poristic and Elliptic Billiard Triangle families are related by a varying similarity transform.

All figures reference illustrative videos in the format [10, PL/#nn], where “nn” stands for the position within a playlist. For convenience, all videos
Figure 1. Poristic Triangle family (blue): fixed Incircle (green) and Circumcircle (purple). **Left:** With \( d = r, \frac{r}{R} = \sqrt{2} - 1 \), all Poristic triangles are acute (dashed green, \( X_3 \) is interior) except for the one shown (blue) which is a right-triangle. If \( d < r \) (not shown), \( X_3 \) will lie in the Incircle (green) and the whole family is acute. **Right:** If \( d > r \), \( X_3 \) can be both interior or exterior to the triangle, and the family will contain both acute (dashed green) and obtuse (dashed blue) triangles. **Video:** [0x0]10, PL#01.

mentioned are compiled on Table 4 in Section 5. Table 6 in Appendix A lists all symbols used below.

1.1. **Related Work.** Weaver [14] proved the Antiorthic Axis\(^1\) of this family is stationary. In Appendix B we revisit and add to some of Weaver’s original work [14].

Odehnal showed the locus of the Excenters is a circle centered on \( X_{40} \) and of radius \( 2R \) [8]. He also lists several Triangle centers which are stationary. Both circular and elliptic loci are described for Triangle Centers and vertices of derived Triangles. For example, the locus of the Mittenpunkt \( X_9 \) is a circle, of known radius and center [8, page 17]; the locus of the vertices of the Tangential Triangle is an ellipse, etc.

We previously studied and Circum- and Inconics associated with the family of 3-periodics in the Elliptic Billiard [11, 12], identifying certain semi-axes and focal length ratios to be invariant.

2. **Preliminaries**

Given a generic triangle, let \( d = |X_1X_3| \). Let the origin be placed at \( X_3 \), with \( x \) running along \( X_1X_3 \) and \( y \) along \((X_3 - X_1)^t\). Note: in all of our figures, for compactness, \( x \) is shown vertical. First proven by Chapple [1] though known as a theorem by Euler is the relation:

\[
(1) \quad d = \sqrt{R(R - 2r)}
\]

Let \( \rho \) denote the invariant ratio \( r/R \). For \( d \) to be real in (1), \( R/r \geq 2 \), i.e.:

\[
\rho = \frac{r}{R} \in (0, 1/2]
\]

\(^1\)The line passing through the intersections of reference and Excentral sidelines [15].
Figure 2. The centers of the Incircle (solid green) and Circumcircle (purple) are $X_1$ and $X_3$, respectively. A Poristic triangle (solid blue) and its excentral (solid green) with $r/R \approx 0.3266$. The same are shown (dashed) at a distinct configuration. Odehnal observed the locus of the Excenters is a circle (orange) centered on $X_{40}$ with radius $2R$. Also shown (dotted orange) is the Caustic to the Excentrals, which is the MacBeath Inconic with centers and foci at $X_i, i=5,4,3$ of the Excentrals and $X_j, j=3,1,40$ of the Poristics. Notice $X_{40}$ is the reflection of $X_1$ about $X_3$. Video: [10, PL#02].

Proposition 1. The Poristic family will contain obtuse triangles if $d > r$.

This stems from the fact that when $d < r$, $X_3$ is always within the Poristic triangles, Figure 1.

3. Conic Invariants

Some of the results in this section were obtained with the aid of a Computer Algebra System (CAS).

3.1. Excentral Caustic. Let $I_5'$ be the MacBeath Inconic [15] to the Excentral Triangles, with center and foci on the latter’s $X_5', X_4', X_3'$, i.e., $X_3, X_1, X_{40}$, of the Poristic family, Figure 2.

Let $\mu_5'$ and $\nu_5'$ be the major and minor semiaxes of $I_5'$.

Theorem 1. $\mu_5' = R$ and $\nu_5' = \sqrt{R^2 - d^2}$ are invariant and $I_5'$ is stationary, i.e., it is the Caustic to the family of Excentral Triangles.

Proof. The sides of the excentral triangle $\ell_i', i = 1, 2, 3$ are defined in (3). Observe the translation $(d, 0)$ in the parametrization of the vertices $P_i(t), i = 2X_1'$ refers to Triangle Centers of the Excentral Triangle.
1, 2, 3 given by equation (2) and so $X_3 = (d, 0)$. It is straightforward to verify these are tangent to the ellipse:

$$\frac{(x - d)^2}{R^2} + \frac{y^2}{R^2 - d^2} = 1$$

with center $X_3 = (d, 0)$ and foci $X_{40} = (0, 0)$ and $X_1 = (2d, 0)$.

Applying (1) to $\mu'_5/\nu'_5 = R/\sqrt{R^2 - d^2}$ obtain:

Corollary 1. The aspect ratio of $I'_5$ is given by:

$$\frac{\mu'_5}{\nu'_5} = \frac{1}{\sqrt{2\rho}}$$

Let $C'$ be the circle centered on $X_3$ and of radius $2R$. Let $I_5$ be the ellipse centered on $X_5$ and with foci on $X_4$ and $X_4$.

Corollary 2. The conic pair $(C', I_5)$ is associated with a $N = 3$ Poncelet family with stationary $X_5$.

This stems from the fact that this pair is the Excentral and Caustic to the Poristic family (taken as reference triangles).

3.2. Excentral $X_3$-Centered Inconic. Let $I'_3$ be the Inconic to the Excentral Triangles centered on their stationary $X_3$ ($X_{40}$ of the Poristic family). Let $\mu'_3$ and $\nu'_3$ be the major and minor semiaxes of $I'_3$.

Theorem 2. $\mu'_3 = R + d$ and $\nu'_3 = R - d$ are invariant over the Poristic family, i.e., $I'_3$ rigidly rotates about $X_{40}$.

Proof. See Appendix C.

As before, applying (1) to $\mu'_3/\nu'_3 = (R + d)/(R - d)$ yields:

Corollary 3. The aspect ratio of $I'_3$ is invariant and given by:

$$\frac{\mu'_3}{\nu'_3} = 1 + \frac{\sqrt{1 - 2\rho}}{\rho} - 1$$

Proposition 2. The non-concentric conic pair $(C', I'_3)$ is associated with a $N = 3$ Poncelet family with stationary $X_3$.

This fact was made originally in [11]:

Remark 1. $I'_3$ contains $X_{100}$.

3.3. The $X_1$-Centered Circumconic. Let $E_1$ be the Circumconic the Poristic triangles centered on $X_1$.

Let $\eta_1$ and $\zeta_1$ be the major and minor semiaxes of $E_1$.

Theorem 3. $\eta_1 = R + d$ and $\zeta_1 = R - d$ are invariant over the Poristic family, i.e., $E_1$ rigidly rotates about $X_1$.

Proof. See Appendix D

Corollary 4. The aspect ratio of $E_1$ is invariant and identical to the aspect ratio of $I'_3$. 

Figure 3. Inconic Invariants: two configurations shown of the Poristic Triangle family (blue). The Incircle (green) and Circumcircle (purple) are fixed, and \( r/R = 0.3627 \). The Excentral Caustic \( I'_5 \) (dashed green) is the (stationary) MacBeath Inconic with center and foci at \( X_i = 5, 4, 3 \) of the Excentral, i.e., \( X_j, j = 3, 1, 40 \) of the Poristic (blue) triangles. The ratio of its semi-axes \( \mu'_5/\mu_5 = 1/\sqrt{2}p \). Also shown is \( I'_3 \), the Inconic to the Excentrals centered on its \( X_3 \), i.e., \( X_{40} \) of the Poristic family (one of the foci of \( I'_5 \)). Over the family, its semiaxes are invariant at \( R+d \) and \( R-d \), i.e., this is a rigidly-rotating ellipse about \( X_{40} \). Also shown is \( E_1 \) (green ellipse), the \( X_1 \)-centered circumconic, an 90°-rotated copy of \( I'_3 \). Video: [10, PL#03]

Proposition 3. \( E_1 \) contains \( X_{100} \).

Proof. \( E_1 \) is the set of trilinear triples \( p : q : r \) such that:

\[
E_1 : (s_2 + s_3 - s_1)/p + (s_1 + s_3 - s_2)/q + (s_1 + s_2 - s_3)/r = 0.
\]

In trilinear coordinates \( X_{100} = [1/(s_2 - s_3) : 1/(s_3 - s_1) : 1/(s_1 - s_2)] \) and so \( E_1(X_{100}) = 0 \). \( \square \)

Two configurations for \( I'_5, E_1, I'_3 \) are shown in Figure 3.

3.4. \( X_{10} \)-circumconic. Let \( \eta_{10}, \zeta_{10} \) be the major, minor semi-axes of \( E_{10} \), the \( X_{10} \)-centered Circumconic. The locus of \( X_{10} \) over the Poristic family is a circle centered on \( X_{1385} \) with radius \( R/4 - r/2 \) [8, page 56]. Let \( \eta'_5 \) and \( \zeta'_5 \) be the major, minor semi-axes of \( E'_5 \), the Circumconic to the Excentrals centered on their \( X_5 \) (i.e., \( X_3 \) of the Poristics). Referring to Figure 4:

Proposition 4. \( \eta_{10}/\zeta_{10} \) is invariant and equal to \( \eta'_5/\zeta'_5 \). These are given by:

\[
\frac{\eta'_5}{\zeta'_5} = \frac{\eta_{10}}{\zeta_{10}} = \sqrt{\frac{R+d}{R-d}}.
\]

Proof. We used a similar approach: generate candidate ratio at isosceles configuration and verify with CAS the ratio is independent of \( t \). \( \square \)
4. CONNECTION WITH ELLIPTIC BILLIARDS

The Circumbilliard to a generic triangle is the Circumconic which renders the triangle a 3-periodic orbit, i.e., it will be centered on $X_9$. Consider the Circumbilliard of a Poristic triangle, Figure 5 and let its semiaxes be denoted by $a_9, b_9$.

**Proposition 5.** The perimeter $L(t)$ of a Poristic triangle is given by:

$$L(t) = \frac{(3R^2 - 4dR \cos t + d^2) \sqrt{3R^2 + 2dR \cos t - d^2}}{R\sqrt{R^2 - 2dR \cos t + d^2}}$$

**Proof.** Follows directly computing $L(t) = |P_1 - P_2| + |P_2 - P_3| + |P_3 - P_1|$ using equation (2) and the relation $r = (R^2 - d^2)/2R$. The long expressions involving square roots were manipulated using a CAS. □

**Theorem 4.** The 3-periodic family is the image of the Poristic family under a one-dimensional family of similarity transformations (rigid rotation, translation, and uniform dilation).

**Proof.** Let $\Delta(t) = \{P_1(t), P_2(t), P_3(t)\}$ be a Poristic triangle given by (2) translated by $(-d,0)$ and consider the circumellipse $E_9(t)$ centered on $X_9(t) = (x_9(t), y_9(t))$ with $a_9(t)$ and $b_9(t)$ the major, minor semiaxes.

Odehnal showed that the locus of the Mittenpunkt $X_9$ is a circle whose radius is $Rd^2R/(9R^2 - d^2)$ and center is $X_1 + (X_1 - X_3)(2R - r)/(4R + r) = \ldots$
Figure 5. Two Poristic triangles (blue and dashed blue) are shown. Also shown are their Circumbilliards (black and dotted black), centered on $X_9(t)$. The locus of $X_9$ is a circle (red) [8]. It turns out the locus of the CB foci $F$ (cyan) is also a circle centered at $C$ and of radius $r_9$ (see Proposition 7). $F'$ denote the foci of the CB of the second (dashed blue) Poristic triangle. Video: [10, PL#05].

Figure 6. Two configurations (left and right) of the Poristic family (blue) for $R = 1$, $r = 0.36266$. The Incircle and Circumcircle appear green and purple. The Excentral Triangle (green) is shown inscribed in the circular locus (orange) of its vertices [8]. Also shown is $I'_3$ (red, inconic to the Excentrals centered on its Circumcenter) and $E_9$, the Circumbilliard to the Poristic triangles (black). Over the family, (i) $E_9$, $I'_3$, $E_1$ (latter not shown) have invariant aspect ratios, with the latter two identical; (ii) their axes remain parallel; (iii) all meet the Circumcircle at $X_{100}$. Video: [10, PL#06,07].
Poristic family and given by:

\[ \theta \]

The aspect ratio of the Circumbilliard is invariant over the Poristic family and given by:

\[ \text{Proposition 6.} \quad \text{Corollary 5.} \]

\[ \text{Obtain the equation of the Circumellipse } \theta \]

\[ \text{it follows that the angle of rotation } \Delta(t) \text{ fits } \]

\[ \text{parametrized by:} \]

\[ X_9(t) = \left[ \frac{4(4d \cos^2 t (R \cos t - d) - r(3d \cos t + R) - r^2)}{(4R + r)(d \cos t - R + r)}, \frac{4Rd^2 \sin t (R^2 - (2R \cos t - d)^2)}{(R^2 + d^2 - 2R \cos t)(9R^2 - d^2)} \right] \]

Let \( \theta(t) \) be the angle between \( a_9(t) \) and the line \( X_1X_3 \), Figure 6. Using the vertices \( P_1(t), P_2(t), P_3(t) \), translated by \( (-d, 0) \) and the center \( X_9(t) \) we can obtain the equation of the Circumellipse \( E_9(t) \). Developing the calculations it follows that the angle of rotation \( \theta(t) \) is given by:

\[ \tan \theta(t) = \frac{1 - \cos t(R + d - 2R \cos t)}{(2R \cos t + R - d) \sin t} \]

Consider the following transformation:

\[ x = L(t)(\cos \theta(t)u + \sin \theta(t)v + x_9(t)) \]
\[ y = L(t)(-\sin \theta(t)u + \cos \theta(t)v + y_9(t)). \]

By construction, the family of Poristic triangles \( \Delta(t) \) is the image of the 3-periodic family of the elliptic billiard defined by:

\[ E(u, v) = \frac{u^2}{a_9} + \frac{v^2}{b_9} - 1 = 0 \]
\[ a_9 = L(t) \frac{R \sqrt{R^2 - 2Rd - d^2}}{\sqrt{R^2}} = L(t) \frac{2 \sqrt{\sqrt{\rho + 1 + \sqrt{1 - \rho^2}}}}{2 \rho + 8}, \]
\[ b_9 = L(t) \frac{R \sqrt{R - d}}{\sqrt{4R + d(3R - d)}} = L(t) \frac{2 \sqrt{\sqrt{\rho + 1 - \sqrt{1 - \rho^2}}}}{2 \rho + 8}, \]
\[ c_9 = \sqrt{a_9^2 - b_9^2} = L(t) \frac{2R \sqrt{dR}}{9R^2 - d^2}. \]

Therefore, the similarity transform is given by \( \theta(t), X_9(t), L(t). \)

\[ \text{Corollary 5.} \quad \text{The ratios } a_9(t)/L(t), b_9(t)/L(t), \text{ and } c_9(t)/L(t) \text{ are invariant over the Poristic family.} \]

\[ \text{Proposition 6.} \quad \text{The aspect ratio of the Circumbilliard is invariant over the Poristic family and given by:} \]
**Figure 8.** The Circumconic to the Excentral $E'_6$ (olive green), centered on its $X_6$ is concentric and axis-parallel to the CB (black). Both conserve their aspect ratio. The locus of the foci of the former is not an ellipse, whereas that of the CB is. **Video:** [10, PL#08].

$$\frac{a(t)}{b(t)} = \sqrt{\frac{(R + d)(3R - d)}{(R - d)(3R + d)}} = \sqrt{\frac{\rho^2 + 2(\rho + 1)\sqrt{1 - 2\rho} + 2}{\rho(\rho - 4)}}$$

**Proof.** The following expression for $r/R$ was derived for the 3-periodic family of an $a, b$ Elliptic Billiard [5, Equation 7]:

$$\rho = \frac{r}{R} = \frac{2(\delta - b^2)(a^2 - \delta)}{c^4}$$

where $\delta = \sqrt{a^4 - a^2b^2 + b^4}$, and $c^2 = a^2 - b^2$. Solving the above for $a/b$ yields the result.

Figure 7 illustrates the variable perimeter and invariant aspect ratio for the CB of the Poristic family for various values of $r/R$.

**Corollary 6.** The axes of the $I'_3$ are parallel to Circumbilliard’s.

This stems from the fact that $E'_3$ for 3-periodics has parallel axes to the CB [11] and the fact that it will be preserved under the similarity transform.

**Corollary 7.** The axes of the $E_{10}$ and $E'_6$ are parallel to Circumbilliard’s axes.

This stems from the fact that the axes of $E'_6$ are parallel to those of $E_{10}$, and that the latter has parallel axes to the Circumbilliard [12], see Figure 8.

**Corollary 8.** The aspect ratio for $I'_5$ and $I'_3$ is the invariant and the same for both Poristic and Billiard families.
Let $F$ be the Feuerbach Hyperbola and $J_{exc}$ be the Excentral Jerabek Hyperbola, Figure 9. Let their focal lengths be $\gamma$ and $\gamma'$. 

**Corollary 9.** The focal length ratio $\gamma'/\gamma = \sqrt{2}/\rho$ is invariant and the same for both Poristic and Billiard families.

Again, this ratio is invariant for 3-periodics [12] and must be also invariant for the Poristic family.

With the aid of CAS, the following can be shown:

**Proposition 7.** Over the Poristic family, the foci of the Circumbilliard describe a circle with center $[(R - d)d/(3R + d), 0]$ and radius $r_9$ given by:

$$r_9 = \frac{4d(R - d)\sqrt{dR}}{(3R - d)\sqrt{(3R - d)(R + d)}}$$

Let $\eta'_6, \zeta'_6$ be the major, minor of the $E'_6$, the Circumconic to the Excentrals centered on their $X_6$ ($X_9$ of the Poristics. Referring to Figure 8:

**Proposition 8.** The $E'_6$ is concentric an has parallel axes to the Circumbilliard. Furthermore, its aspect ratio is given by:

$$\frac{\eta'_6}{\zeta'_6} = \frac{b_9^2 + \delta}{a_9^2 + \delta} = \frac{(R + d)(3R + d)}{(3R - d)(R - d)}$$

$$\delta = \sqrt{a_9^2 - a_9^2b_9^2 + b_9^2}.$$
Figure 10. **Left:** Poristic triangle (blue), stationary Incircle (green) and Circumcircle (purple). Varying Poristic CB (black), whose aspect ratio is constant. Stationary Excentral MacBeath Inconic and Caustic $I'_5$ (red), circular Excentral locus (orange), and Excentral (MacBeath) Circumconic $E'_6$ (olive green), all with invariant aspect ratios. **Right:** same objects observed on a stationary Elliptic Billiard system: Incircle and Circumcircle are varying (though $r/R$ is invariant). $I'_5$ is moving though its aspect ratio is invariant and equal to its counterpart in the Poristic system. Conversely, $E'_6$ is now stationary and is the locus of the Excenters [4]. Notice the Excentral Circumcircle (orange) is movable. **Video:** [10, PL#09]

This stems from the fact that $E'_6$ for 3-periodics is the locus of the Excenters, shown to be an ellipse with said aspect ratio [4].

5. Conclusion

Table 1 summarizes properties and invariants for the various circum- and inconics mentioned above. A comparison between basic parameters in the Poristic family and 3-Periodics in the Elliptic Billiard appear on Table 2. Finally, shape invariances of conics in either family are compared on Table 3 and illustrated in Figure 10.

Videos mentioned above have been placed on a playlist [10]. Table 4 contains quick-reference links to all videos mentioned, with column “PL/#” providing video number within the playlist.
Table 1. Table of conics, all with mutually parallel axes (except for $I_9'$). Columns “poristic” and “EB” define whether for that family the aspect ratio is invariant. $E_k$ (resp. $I_k'$) stands for the Circumellipse (resp. Inellipse) centered on $X_k$. $E'_k, I'_k$ refer to Excentral conics.

| conic | poristic | EB | $X_{100}$ | ctr | note |
|-------|----------|----|-----------|-----|------|
| $E_1$ | axes     | ratio | y         | $X_1$ | center on $F_{med}$ |
| $E_9$ | ratio    | axes | y         | $X_9$ | (Circum-) EB, center on $F_{med}$ |
| $E_{10}$ | ratio | ratio | y         | $X_{10}$ | center on $F_{med}$ |
| $I_9$ | ratio    | axes | –         | $X_9$ | Mandart Inellipse, EB Caustic |
| $E_1'$ | axes     | ratio | y         | $X_{40}$ | Excentral Circumcircle |
| $E_3'$ | ratio    | ratio | –         | $X_3$ | same ratio as $E_{10}$ |
| $E_6'$ | ratio    | axes | –         | $X_9$ | MacBeath Circumconic |
| $I_3'$ | axes     | ratio | y         | $X_{40}$ | 90°-rotated copy of $E_1$ |
| $I_9'$ | axes     | ratio | –         | $X_3$ | McBeath Inconic, Excentral Caustic |

Table 2. Column “poristic” (resp. EB) indicates if the named quantity is invariant in the given family. Only $r/R$ and $(R + d)/(R - d) = f(r/R)$ are invariant on both.

| qty | poristic | EB |
|-----|----------|----|
| $d$ | y        | –  |
| $r$ | y        | –  |
| $R$ | y        | –  |
| $r/R$ | y   | y  |
| $R\pm d$ | y | –  |
| $R-d$ | y       | y  |
| $L$ | –        | y  |
| $J$ | –        | y  |

Table 3. Various position and axes for conics in each Poristic and 3-periodic (EB) families. A ‘y’ in the “poristic” or “EB” columns indicates shape invariance.

| object   | ctr | semi-axes | poristic | EB | note                  |
|----------|-----|-----------|----------|----|----------------------|
| Incircle | $X_1$ | $r$       | y        | –  |                      |
| Circumcircle | $X_3$ | $R$       | y        | –  |                      |
| $I_3'$ | $X_9$ | $(b_9^2 + \delta)/a_9, (a_9^2 + \delta)/b_9$ | – | y | poristic exc. caustic |
| $E_6'$ | $X_{40}$ | $2R$       | y        | –  |                      |
| Exc. Circumcircle | $X_9$ | $a_9, b_9$ | – | y |                      |
| Elliptic Billiard | $X_9$ | $a_9, b_9$ | – | y |                      |

Table 4. Videos mentioned in the paper. Column “PL#” indicates the entry within the playlist [10].

| PL# | Title                                                | Section |
|-----|------------------------------------------------------|---------|
| 01  | Poristic family, circular locus of excenters, and Antiorthic axis | 1, App. B |
| 02  | Poristic Circumbilliard (CB) has invariant aspect ratio | 3       |
| 03  | $E_1$ and $I_3'$ have constant, parallel, and identical semi-axes | 3       |
| 04  | $E_{10}$ and $E_6'$ have axes parallel to the Poristic CB as well as invariant, identical aspect ratio | 3       |
| 05  | Loci of center and foci of Poristic CB are circles | 4       |
| 06  | $I_3'$ has constant semi-axes, parallel to those of the Poristic CB | 4       |
| 07  | $I_3'$ and $I_9'$ of 3-Periodics in the EB have invariant aspect ratio. | 4       |
| 08  | $E_6'$ has invariant aspect ratio and its axes coincide with those of the Poristic CB | 4       |
| 09  | Side-by-side Poristic and Elliptic Billiard (EB) | 4       |
| 10,11 | $F$ and $J_{exc}$ Circumhyperbolas have invariant focal length ratio over 3-periodic family | 4       |
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Appendix A. Table of Symbols

Tables 5 and 6 lists most Triangle Centers and symbols mentioned in the paper.
| Symbol | Meaning                                                                 | Note |
|--------|-------------------------------------------------------------------------|------|
| $P_i, s_i$ | Vertices and sidelengths of Poristic triangles |       |
| $P'_i$ | Vertices of the Excentral triangle |       |
| $X_i, X'_i$ | Kimberling Center $i$ of Poristic, Excentral |       |
| $a_c, b_c$ | Semi-axes of confocal Caustic |       |
| $r, R, \rho$ | Inradius, Circumradius, $r/R$ | $\rho$ is invariant |
| $a_0, b_0$ | Semi-axes of Poristic CB | $\sqrt{a_0^2 - a_0^2 b_0^2 + b_0^4}$ |
| $d$ | Distance $|X_1 X_3|$ | $\sqrt{R(R - 2r)}$ |
| $\delta$ | Constant associated w/ the CB |       |
| $E_i$ | Circumellipse centered on $X_i$ | Axes parallel to $E_9$ if $X_i$ on $F_{med}$ |
| $E'_i$ | Excentral Circumellipse centered on $X'_i$ |       |
| $\eta_i, \zeta_i$ | Major and minor semiaxis of $E_i$ | Invariant ratio for $i = 1, 3, 9, 10$ |
| $\eta'_i, \zeta'_i$ | Major and minor semiaxis of $E'_i$ | Invariant ratio for $i = 3, 5, 6$ |
| $I_i$ | Inellipse on $X_i$ | $I_3$: MacBeath $I_5$: Mandart $I_9$ |
| $I'_i$ | Excentral Inellipse centered on $X'_i$ | $I'_3$: MacBeath $I'_5$ |
| $\mu_i, \nu_i$ | Major and minor semiaxis of $I_i$ | Invariant ratio for $i = 3, 5$ |
| $\mu'_i, \nu'_i$ | Major and minor semiaxis of $I'_i$ |       |
| $F_{exc}$ | $F$ of Excentral Triangle | Center $X_{3659}$ [7] |
| $J_{exc}$ | $J$ of Excentral Triangle | Center $X_{180}$, Perspector $X_{649}$ |
| $F_{med}$ | $F$ of Medial | Center $X_{3035}$ [7] |
| $\lambda', \lambda$ | Focal lengths of $J_{exc}, F$ | Invariant ratio |

Table 6. Symbols used in paper

| Center | Meaning | Note |
|--------|---------|------|
| $X_1$ | Incenter | Trilinear Pole of $L_1$, focus of $P'_5$ |
| $X_3$ | Circumcenter | Focus of $I_5$ |
| $X_4$ | Orthocenter | Focus of $I_5$ |
| $X_5$ | Center of the 9-Point Circle | Center of $I_5$ |
| $X_6$ | Symmedian Point |       |
| $X_9$ | Mittenpunkt | Center of (Circum)billiard |
| $X_{10}$ | Spieker Point | Incenter of Medial |
| $X_{40}$ | Bevan Point | Focus of $I'_5$ |
| $X_{100}$ | Anticomplement of $X_{11}$ | Lies on $E_i$, $i = 1, 3, 9, 10$ and $P'_5$ |
| $X_{650}$ | Cross-difference of $X_1, X_3$ | Generates $X_1 X_3$ |
| $X_{651}$ | Isogonal Conj. of $X_{650}$ | Trilinear Pole of $L_{650} = X_1 X_3$ |
| $X_{1155}$ | SchrÃ¶der Point | Intersection of $X_1 X_3$ with Antiorthic Axis |
| $L_1$ | Antiorthic Axis | Line $X_{44}X_{515}$ [6] |
| $L_{650}$ | OI Axis | Line $X_1 X_3$ |

Table 5. Kimberling Centers and Central Lines mentioned in paper

APPENDIX B. WEAVER INVARIANTS

B.1. Antiorthic Axis. The Antiorthic axis $L_1$ is stationary, and $X_{1155}$ is stationary intersection of $L_1$ with $L_{650} = X_1 X_3$, Figure 11. The antiorthic axis is given by:

$$x = \frac{3R^2 - d^2}{2d}$$
Figure 11. A result by Weaver [14] is that over the Poristic family, the Antiorthic Axis $L_1$ is invariant. Odehnal observed $X_{1155}$ was one of the many stationary Triangle centers along $L_{663} = X_1X_3$. This point happens to lie at the latter’s intersection with $L_1$. Video: [10, PL#01].

B.2. A possible correction to Weaver’s 2nd order invariant. Assume the origin is on $X_3$. In [14, Theorem III] it is proposed that a circle $C_w$ centered on $[-R,0]$ and of radius $\sqrt{Rd(R + d)(R + d + r)/d}$ has the same power with respect to the Antiorthic axis $L_1$ as the Incircle. We have found this not to be the case. Let $I_1 : (x - d)^2 + y^2 = r^2$ denote the Incircle. Referring to Figure 12(left), let $C'_w$ be circle centered on $[-R,0]$ and of radius:

$$r'_w = \left(\frac{d + R}{2R}\right) \sqrt{\frac{(3R - d)(4R^2 - Rd - d^2)}{d}}$$

Proposition 9. $C'_w$ and $I_1$ have the same power with respect to $L_1$.

Proof. Translate the vertices of the Poristic family in (2) by $(-d,0)$. It is straightforward to show that the Antiorthic axis $L_1$ is given by $x = (3R^2 - d^2)/(2d)$. The power $P_w$ of $P_0 = [(3R^2 - d^2)/(2d),0]$ with respect to the circle $C'_w : (x + R)^2 + y^2 = r'^2$ is given by:

$$P_w(P_0, C'_w) = |P_0 + [-R,0]|^2 - r'^2 = \frac{(d + R)^2(3R - d)^2}{4d^2} - r'_w^2.$$  

Also,

$$P_w(P_0, I_1) = |P_0 + [d,0]|^2 - r^2 = \frac{(R^2 - d^2)^2(9R^2 - d^2)}{4R^2d^2}.$$  

In Weaver’s paper, the origin is on $X_1$, so the center of $C_w$ is at $[-R - d,0]$. 

\[d=0.7000, R=1, r=0.2550, R/r=3.922, \theta=85.0°\]
Therefore, $L_1$ is the radical axis of the pair of circles $C_w'$ and $I_1$ if, and only if,

$$
\frac{(d + R)^2(3R - d)^2}{4d^2} - r_w'^2 = \frac{(R^2 - d^2)^2(9R^2 - d^2)}{4R^2d^2}.
$$

Solving the equation above leads to the result.

Additionally, we derive a circle whose power with respect to $L_\infty$ is equal to the Circumcircle’s, Figure 12(right). Let $C_w''$ be a circle centered on $[-R, 0]$ and of radius:

$$
r_w'' = \sqrt{\frac{(3R - d)(d + R)R}{d}}.
$$

**Proposition 10.** $C_w''$ has the same same power with respect to $L_1$ as (i) the Circumcircle $C$, and (ii) $C_e$, centered on $X_{40}$ and of radius $2R$ (locus of the Excenters).

**Proof.** Translate the vertices of the Poristic family in (2) by $(-d, 0)$. Also, $L_1$ is the radical axis (see [2, Chapter 2]) of the pair of circles

$$
C_e : (x + d)^2 + y^2 = 4R^2, \ C : x^2 + y^2 = R^2.
$$

In fact, the power $P_w$ of $P_0 = ((3R^2 - d^2)/(2d), 0)$ with respect to the circles $C_e$ and $C$ is given by:

$$
P_w(P_0, C_e) = |P_0 - (-d, 0)|^2 - 4R^2 = \frac{Rr(4R + r)}{R - 2r} = P_w(P_0, C).
$$

Consider the pair of circles

$$
C_w'' : (x + R)^2 + y^2 = r_w''^2, \ C : x^2 + y^2 = R^2
$$

Analogously, $P_w(P_0, C_w') = P_w(P_0, C)$ if, and only if, $r_w'' = (3R - d)(d + R)R/d$. 

□

**Figure 12.** Left: Circle $C_w$ proposed in [14, Theorem III] (red) does not have the same power as the Incircle $I_1$ (green) with respect to $L_1$ (blue): rather, its tangency points $T'_1, T'_2$ from from $X_{1155}$ are collinear with $X_3$. We derived a new equal power circle $C_w'$ (green) of radius $r_w'$ (see text): its tangency points $T''_1, T''_2$ are concyclic with $T_1, T_2$. Note: both $C_w, C_w'$ are centered on $C = [-R, 0]$. Right: the following three circles have the same power with respect to $L_1$: (i) the Circumcircle $C$, (ii) $C_e$, the $X_{40}$-centered circular locus of the Excenters of radius $2R$ (orange), and (iii) $C_w''$, centered on $[-R, 0]$ and of radius $r_w''$ (red), see text. Notice tangency point $T_i, T''_i, i = 1, 2$ are concyclic.
**Appendix C. \(I'_3\) Axis Invariance**

Here we reproduce a proof that the axes of \(I'_3\) are invariant and equal to \(R \pm d\), kindly contributed by Odehnal [9].

Let \(X_{40}\) be the origin and the x-axis run along \(X_3X_1\), Figure 13. Parametrize Poristic triangles \(\Delta(t) = P_1P_2P_3\) by their tangency point on the Incircle [8]:

\[
\begin{align*}
P_1 &= \left[ \cos t (d \cos t + r) - \omega \sin t + d, (d \cos t + r) \sin t + \omega \cos t \right] \\
P_2 &= \left[ \cos t (d \cos t + r) + \omega \sin t + d, (d \cos t + r) \sin t - \omega \cos t \right] \\
P_3 &= \left[ \frac{R(2dR - (R^2 + d^2) \cos t)}{R^2 - 2dR \cos t + d^2} + d, \frac{R(d^2 - R^2) \cos t}{R^2 - 2dR \cos t + d^2} \right] \\
\omega &= \sqrt{R^2 - (d \cos t + r)^2}
\end{align*}
\]

**Lemma 1.** Let \(\ell_i : a_i x + b_i y + c_i = 0 \ (i \in \{1, 2, 3\})\) be three tangents lines of a conic \(E : Ax^2 + 2Bxy + Cy^2 + D = 0\) centered at \((0, 0)\). Then, the coefficients \(A, B, C, D\) are given by:

\[
\begin{align*}
A &= a_2a_3c_1^2\delta_{23} - a_1a_3c_2^2\delta_{13} + a_1a_2c_3^2\delta_{12} \\
B &= \frac{1}{4} \left((a_2b_3 + a_3b_2)c_1^2\delta_{23} - (a_1b_3 + a_3b_1)c_2^2\delta_{13} + (a_1b_2 + a_2b_1)c_3^2\delta_{12}\right) \\
C &= b_2b_3c_1^2\delta_{23} - b_1b_3c_2^2\delta_{13} + b_1b_2c_3^2\delta_{12} \\
D &= \frac{1}{4} \left(\delta_{12}\delta_{13}\delta_{23}\right)^{-1} \left(\delta_{23}c_1 + \delta_{13}c_2 - \delta_{12}c_3\right) \left(\delta_{23}c_1 - \delta_{13}c_2 - \delta_{12}c_3\right) \\
&\quad \left(\delta_{23}c_1 - \delta_{13}c_2 + \delta_{12}c_3\right) \left(\delta_{23}c_1 + \delta_{13}c_2 + \delta_{12}c_3\right)
\end{align*}
\]

Here \(\delta_{ij} = a_i b_j - a_j b_i\).

**Proof.** The condition of tangency of \(\ell_i \ (i = 1, 2, 3)\) with \(E\) is given by the discriminant equation

\[(AC - B^2)c_1^2 + (Ab_1^2 - 2Ba_1b_1 + Ca_1^2)D = 0.\]
Solving the system leads to the result stated. \[\square\]

**Lemma 2.** The three sides of the excentral triangle \(\Delta'(t) = \{P'_1(t), P'_2(t), P'_3(t)\}\) are given by the straight lines

\[
\begin{align*}
\ell'_1(t) : & \quad ((d \sin t - \omega) \sin t - r \cos t)x - ((d \cos t + r) \sin t - \omega \cos t)y + R^2 - d^2 = 0 \\
\ell'_2(t) : & \quad ((d \sin t + \omega) \sin t - r \cos t)x - ((d \cos t - r) \sin t + \omega \cos t)y + R^2 - d^2 = 0 \\
\ell'_3(t) : & \quad (R \cos t - d)x + R \sin t y - 2dR \cos t + R^2 + d^2 = 0.
\end{align*}
\]

**Proof.** Direct calculations of the external bisector lines passing through the vertices \(P_1(t), P_2(t)\) and \(P_3(t)\) given by equation (2). \[\square\]

**Proposition 11.** \(I'_3\) is given implicitly by the equation:

\[
I'_3(x, y, t) = ((R^2 - d^2)^2 - 8dR^2(R \cos t - d) \sin^2 t)x^2 + ((R^2 - d^2)^2 - 4dR \cos t((R \cos t - d)^2 - R^2 \sin^2 t))y^2 + 4dR \sin t(2R \cos t - R - d)(2R \cos t + R - d)xy - (R^2 - d^2)^2(R^2 + d^2 - 2dR \cos t) = 0.
\]

**Proof.** Consider the Poristic defined by the circles \((x - d)^2 + y^2 = R^2\) and \((x - 2d)^2 + y^2 = r^2\). By [8] we know that the inconic \(I'_3(t)\), tangent to the sides of the excentral triangle \(\Delta'(t)\), is centered in \(X_{40} = (0, 0)\). Applying lemmas 1 and 2, and using the Euler relation \(R^2 - d^2 = 2rR\), the equation (4) is obtained. This expression was confirmed using CAS.

Consider a rotation of the coordinates of (4) by angle \(\theta\) defined by:

\[
\tan 2\theta = \frac{\sin t(R^2 - (2R \cos t - d)^2)}{\cos t((2R \cos t - d)^2 - 3R^2) + 2dR}.
\]

This re-expresses (4) in canonical form:

\[
(R^2 + d^2 - 2dR \cos t)((R + d)^2u^2 + (R - d)^2v^2 - (R^2 - d^2)^2) = 0.
\]

Clearly, the semiaxis lengths of (5) are \(R \pm d\) which is the goal of this proof.

**Appendix D.** \(E_1\) Axis Invariance

**Proposition 12.** The semiaxes of \(E_1\) are \(\eta_1 = R + d\) and \(\zeta_1 = R - d\).

**Proof.** Using the parametrization of the triangle \(P_1(t), P_2(t), P_3(t)\) given by equation (2) and that \(E_1\) pass through the vertices \(P_i(t)\) (\(i = 1, 2, 3\)) and centered in \(X_1 = (2d, 0)\) it is obtained:
$E_1(x, y) = ((R^2 - d^2)^2 - 4dR \cos t(R \cos t - d)^2 - R^2 \sin^2 t) x^2$
$+ ((R^2 - d^2)^2 - 8dR^2(R \cos t - d) \sin^2 t) y^2$
$- 4dR \sin t(2R \cos t - R - d)(2R \cos t + R - d) xy$
$+ 4d(4dR \cos t((R \cos t - d)^2 - R^2 \sin^2 t) - (R^2 - d^2)^2)x$
$+ 8Rd^2 \sin t(2R \cos t + R - d)(2R \cos t - R - d)y$
$- 2dR \cos t(16d^2R \cos t(R \cos t - d) - (R^2 - d^2)(R^2 + 7d^2))$
$-(R^2 - 3d^2)(R^2 - d^2) = 0$

Proceeding as in the proof of Theorem 2 it is direct to verify that the canonical form of the above equation is $u^2/(R + d)^2 + v^2/(R - d)^2 = 1$.  

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