Stability problem in dynamo
M. Reshetnyak

Institute of the Physics of the Earth, B.Gruzinskaya 10,
Moscow, Russia
email: m.reshetnyak@gmail.com

Abstract
It is shown, that the saturated $\alpha$-effect taken from the non-
linear dynamo equations for the thin disk can still produce
exponentially growing magnetic field in the case, when
this field does not feed back on the $\alpha$. For negative dy-
amo number (stationary regime) stability is defined by
the structure of the spectra of the linear problem for the
positive dynamo numbers. Stability condition for the os-
cillatory solution (positive dynamo number) is also ob-
tained and related to the phase shift of the original mag-
etic field, which produced saturated $\alpha$ and magnetic field
in the kinematic regime. Results can be used for expla-
nation of the similar effect observed in the shell models
simulations as well in the 3D dynamo models in the plane
layer and sphere.

1 Introduction
It is believed, that variety of the magnetic fields observed
in astrophysics and technics can be explained in terms of
the dynamo theory, e.g. (Hollerbach & Rüdiger, 2004).
The main idea is that kinetic energy of the conductive mo-
tions is transformed into the energy of the magnetic field.
Magnetic field generation is the threshold phenomenon:
the magnetic Reynolds number $R_m$ reaches its
critical value $R_m^{cr}$. After that magnetic field grows ex-
ponentially up to the moment, when it already can feed back
on the flow. This influence does not come to the simple
suppression of the motions and reducing of $R_m$, rather to
the change of the spectra of the fields closely connected to
constraints caused by conservation of the magnetic energy
and helicity (Brandenburg & Subramanian, 2005). The other
important point is effects of the phase shift and co-
herence of the physical fields before and after onset of
quenching discussed in (Tilgner & Brandenburg, 2008).

As a result, even after quenching the saturated velocity
field is still large enough, so that $R_m \gg R_m^{cr}$. Moreover,
velocity field taken from the nonlinear problem (when
the exponential growth of the magnetic field stopped)
can still generate exponentially growing magnetic field
providing that feed back of the magnetic field on the flow is
omitted (kinematic dynamo regime) (Cattaneo & Tobias,
2009, Tilgner, 2008, Tilgner & Brandenburg, 2008,
Schrinner, Schmidt, Cameron, 2009). In other words, the
problem of stability of the full dynamo equations includ-
ing induction equation, the Navier-Stokes equation with
the Lorentz force differs from the stability problem of the
single induction equation with the given saturated veloc-
ity field taken from the full dynamo solution: stability of
the first problem does not provide stability of the second
one.

Here we consider effect of such kind of stability on an
example of the model of galactic dynamo in the thin disk,
as well as some applications to the dynamo in the sphere.

2 Dynamo in the thin disk
One of the simplest galactic dynamo models
is a one-dimensional model in the thin disk
(Ruzmaikin, Shukurov, Sokoloff, 1988):
\[
\frac{\partial A}{\partial t} = \alpha B + A''
\]
\[
\frac{\partial B}{\partial t} = -D A' + B''
\]
(1)
where $A$ and $B$ are azimuthal components of the vector
potential and magnetic field, $\alpha(z)$ is a kinetic helicity, $D$
is a dynamo number, which is a product of the amplitudes
of the $\alpha$- and $\omega$-effects and primes denote derivatives with
respect to a cylindrical polar coordinate $z$. Equation (1)
is solved in the interval $-1 \leq z \leq 1$ with the boundary
conditions $B = 0$ and $A' = 0$ at $z = \pm1$. We look for a
solution of the form
\[
\begin{pmatrix} A \\ B \end{pmatrix} = e^{i\gamma}(\mathcal{A}(z), \mathcal{B}(z)).
\]
(2)
Substituting (2) in (1) yields the following eigenvalue
problem:
\[
\gamma \mathcal{A} = \alpha \mathcal{B} + \mathcal{A}''
\]
\[
\gamma \mathcal{B} = -D \mathcal{A}' + \mathcal{B}''
\]
(3)
where the constant \( \gamma \) is the growth rate. So as \( \alpha(z) = -\alpha(z) \) is odd function of \( z \), the generation equations have an important property: system (3) is invariant under transformation \( z \to -z \) when  

\[
\mathcal{A}(-z) = \mathcal{A}(z), \quad \mathcal{B}(-z) = -\mathcal{B}(z)
\]

or

\[
\mathcal{A}(-z) = -\mathcal{A}(z), \quad \mathcal{B}(-z) = \mathcal{B}(z).
\]  

Therefore, all solutions may be divided into two groups: odd on \( \mathcal{B}(z) \), dipole (D), and even, quadrupole on \( \mathcal{B}(z) \). Then we can replace \( -1 \leq z \leq 1 \) with the interval \( 0 \leq z \leq 1 \) and the following boundary conditions at \( z = 0 \): \( \mathcal{A}' = 0, \mathcal{B} = 0 \) (D) and \( \mathcal{A} = 0, \mathcal{B}' = 0 \) (Q). Usually, \( \alpha = \alpha_0 \) with \( \alpha_0(z) = \sin(\pi z) \) is used, see also  

Soward, 1978

for \( \alpha_0(z) = z \) dependence, more appropriate for analytical applications.

System (3) has growing solution, \( \Re \gamma > 0 \), when \( |D| > |Df^1| \). For \( D < 0 \) the first exciting mode is quadrupole with \( Df^1 \approx -8 \) and \( \Im \gamma = 0 \): solution is non-oscillatory. For \( D > 0 \) the leading mode is oscillatory dipole, \( \Im \gamma \neq 0 \) with higher threshold of generation: \( Df^2 \approx 200 \). Putting nonlinearity of the form

\[
\alpha(z) = \frac{\alpha_0(z)}{1 + E_m} \quad \text{for} \quad |B| \gg 1
\]  

in (1), where \( E_m = (B^2 + A^2)/2 \) is a magnetic energy, gives stationary solutions for \( \mathcal{Q} \)-kind of symmetry and quasi-stationary solutions for \( \mathcal{D} \), see about various forms of nonlinearities in  

Beck, Brandenburg, Moss, Shukurov, Sokoloff, 1996.

The property of the nonlinear solution is mostly defined by the form of the first eigenfunction.

Now, in the spirit of  

Cattaneo & Tobias, 2009

Tilgner & Brandenburg, 2008

we add to (1) equations for the new magnetic field \((\hat{A}, \hat{B})\) with the same \( \alpha \) (5), which depends on \((A, B)\) and does not depend on \((\hat{A}, \hat{B})\):

\[
\frac{\partial \hat{A}}{\partial t} = \alpha \hat{B} + A''
\]

\[
\frac{\partial \hat{B}}{\partial t} = -DA' + B''
\]

and

\[
\frac{\partial A}{\partial t} = \alpha B + A''
\]

\[
\frac{\partial B}{\partial t} = -DA' + B''
\]

\[
\frac{\partial \hat{A}}{\partial t} = \alpha \hat{B} + \hat{A}''
\]

\[
\frac{\partial \hat{B}}{\partial t} = -D\hat{A}' + \hat{B}'' .
\]

Numerical simulations demonstrate that for the negative \( \mathcal{D} \) the both \((A, B)\) and \((\hat{A}, \hat{B})\) are steady, however the final magnitudes of \((A, B)\) depend on the initial conditions for \((\hat{A}, \hat{B})\), see Fig. 1. The procedure was the following: equations (1, 5) for \((A, B)\) were integrated up to the moment \( t = t_0 \), then the full system (6) was simulated with the initial conditions for \((A, \hat{B})\) in the form: \((\hat{A}, \hat{B})_{t=t_0} = (A, B)_{t=t_0}(1 + C\varepsilon)\), where \( \varepsilon \in [-0.5, 0.5] \) is a random variable and \( C \) is a constant. The both vectors \((A, B)\) and \((\hat{A}, \hat{B})\) are stable in time, however the final magnitude of \( \hat{E}_m \) for \( C \neq 0 \) slightly depends on \( C \). Presence of alignment of the fields \((A, B)\) and \((\hat{A}, \hat{B})\) follows from linearity and homogeneity of equations for \((\hat{A}, \hat{B})\), where \( \alpha(z, E_m) \) is given. Latter we consider stability of \((\hat{A}, \hat{B})\) in more details.

For \( \mathcal{D} > 0 \) situation is different, resembling that one of instability described in  

Cattaneo & Tobias, 2009

Tilgner, 2008

Tilgner & Brandenburg, 2008

Schrinner, Schmidt, Cameron, 2009

for more sophisticated models: field \((A, B)\) oscillates and starts to grow exponentially, see Fig. 2. Note, that no regime in oscillations for \((\hat{A}, \hat{B})\) is observed. The other specific feature is delay of \((\hat{A}, \hat{B})\) relative to \((A, B)\): \( \theta \approx -\frac{\pi}{3} \).

If \( E_m \) in (5) is averaged over the space, so that \( \alpha \) is steady, then instability dissapears. The question arises: does instability depends on stationarity, either it depends on something else?

It is known, that for \( \mathcal{D} < 0 \) stability of the system  

(3, 5), which has stationary solution, is tightly bound to behaviour of the linear solution of (3) for \( \mathcal{D} > 0 \)  

Reshetnyak, Sokoloff, Shukurov, 1992.

Note, that for the complex form of (3) it is equivalent to the solution of the conjugate problem.
Let $\tilde{(A, B)} = (\mathcal{A} + a, B + b)$, where $(\mathcal{A}, B)$ is a solution of the nonlinear problem and $(a, b)$ is a perturbation with the same boundary conditions as for $(\mathcal{A}, B)$. Putting $\tilde{(A, B)}$ in (3) with $\alpha = a_0 + \frac{\partial a}{\partial B} b$ yields equations for $(a, b)$

$$
\begin{align*}
\gamma a &= a^\varepsilon b + a'' \\
\gamma b &= -\mathcal{D} a' + b'' ,
\end{align*}
$$

(7)

where $a^\varepsilon = a + \frac{\partial a}{\partial B} B$ for $a = \frac{a_0}{1 + B^2}$ is

$$
a^\varepsilon = \frac{1 - B^2}{(1 + B^2)^2} a_0 \sim \frac{a_0}{B^2} \text{ for } |B| \gg 1. \quad \text{(8)}
$$

Behaviour of $\alpha \omega$-dynamo (3) is defined by the sign of $\mathcal{D} a_0$, and its change in the perturbed equations (7) is important. In other words, instead of nonlinear equations (3) we come to the linear problem (3) with given $a = a(\zeta, E_m)$ and effective dynamo number $\mathcal{D}' = -\frac{\mathcal{D} a_0}{B^2}$. Then stability of fields $\tilde{(A, B)}$ for the negative $\mathcal{D}$ can be explained as follows. For negative $\mathcal{D}$ solution $(\tilde{A}, \tilde{B})$ is finite and stable, because the threshold of generation $\mathcal{D}'_e$ for (3) is much larger than $\mathcal{D}'$, $\mathcal{D}'_e \gg \mathcal{D}'$. Field $(\tilde{A}, \tilde{B})$ is defined up to an arbitrary factor, what corresponds to alignment of the vectors $(A, B)$ and $(\tilde{A}, \tilde{B})$. Note, that $\mathcal{D}_e' \ll \mathcal{D}'$ does not guarantee, that $(\tilde{A}, \tilde{B})$ will grow exponentially due to nonlinearity (5).

It is worthy of note that nonlinear solution of (15) and (15) demonstrates similar stationary behaviour even for $\mathcal{D} \sim -10^3$ in spite of the fact, that $\mathcal{D}_e'$ for the quadrupole oscillatory mode for positive $\mathcal{D}$ is $\sim 200$. The reason is that dynamo system tends to the state of the strong magnetic filed with $B \sim \mathcal{D}_{1/2}$, so that $\alpha \sim \frac{1}{B^2}$, leaving $\mathcal{D}'$ at the level of the first mode’s threshold of generation.

For positive $\mathcal{D}$ $(A, B)$, and therefore $\alpha(B)$, oscillate and one needs additional information on correlation of the waves. Here, instead of (5) we get $\alpha^\varepsilon \sim -\frac{a_0 B}{|B|^3}$. If phase shift between $B$ and $\tilde{B}$ is negligible, then $\alpha$-effect is saturated and time evolution of $(A, B)$ and $(\tilde{A}, \tilde{B})$ is similar.

2It is usually supposed that in $\alpha \omega$-dynamo models $B \gg A'$.
between other prediction of the linear analysis is the phase shift
If 
original field \((A, B)\) changes production of \(\hat{A}^2 + \hat{B}^2\) near the threshold of generation \(D^2\). We start from the linear analysis of the system in the form:

\[
i \omega \hat{A} = \alpha \hat{B} - k^2 \hat{A}
\]

\[
i \omega \hat{B} = -iD^2 k \hat{A} - k^2 \hat{B}.
\]

(9)

From condition of solvability for (9): \((k^2 + i \omega)^2 = -iD^2 k \omega_0\) with \(\alpha = \omega_0\) follows that \(\omega^2 = k^4 = 1\). The other prediction of the linear analysis is the phase shift \(\phi\) between \(\hat{A}\) and \(\hat{B}: \phi = \pm \frac{\pi}{2}\), what is twice smaller than for the nonlinear regime \(\tilde{A}\) and \(\tilde{B}: \phi = \pm \pi\), so that for the nonlinear regime the maximal \(\hat{A}\) is when \(\hat{B}\) is zero and quenching is absent.

Then putting in (5) \(B = b \sin(x - t), \hat{A} = \sin(x - t + \varphi + \theta), \hat{B} = \sin(x - t + \theta)\), and \(\alpha = \frac{1}{1 + B^2}\) we get how generation depends on \(\theta\). Equation for \(\hat{B}\) does not include original field \((A, B)\), so we consider only production of \(\tilde{A}^2\). Then \(\delta \hat{A}(\varphi, \theta) = \alpha_0 \frac{2 \pi \hat{B} A}{1 + B^2} dt\). If \(|\Pi| \gg 1\), where

\[
\Pi = \frac{\delta \hat{A}(\varphi, \theta)}{\delta \hat{A}(\varphi, 0)}
\]

then \((\hat{A}, \hat{B})\) is unstable.

The exact equation for \(\delta \hat{A}\) is:

\[
\delta \hat{A}(\varphi, \theta) = h_1 + h_2 \tan(\varphi),
\]

\[
h_1 = \frac{\cos(\theta)(4 - 2\sin^2(\theta) - 2 \sin(2\theta) - 4)}{2 \sin(2\theta)(2\sin^2(\theta) - 1)},
\]

\[
h_2 = \frac{\sin(2\theta)(3\sin^2(\theta) - 4)}{2 \sin(2\theta)(2\sin^2(\theta) - 1)}.
\]

If \(\theta = 0\) then \(h_2 = 0\) and \(\hat{A}(\varphi, 0) = h_1 = 2\sin(\theta) - 4\). Then, for \(\theta = \frac{\pi}{2}\)

\[
h_1 = \frac{2\sin(\pi/2) - 10}{4}, h_2 = \frac{-3\sin(\pi/2)}{4}(2\sin^2(\theta) - 1), \Pi\) at \(\varphi = \pm \pi/2\) is singular and instability appears.

Summarizing results for the steady and oscillatory dynamos we have the following predictions for stability of field \((\hat{A}, \hat{B})\). For \(D < 0\) \((A, B)\) is steady and \((\hat{A}, \hat{B})\) is unstable when \(|D| \gg 1|\hat{B}|^2|\).

When \((A,B)\) oscillates then \((\hat{A}, \hat{B})\) continues to oscillate with \((A, B)\) increasing the phase shift between \((\hat{A}, \hat{B})\) and \((A, B)\). Then instability caused by the parameter resonance may arise.

3 Conclusions
Here we argue, that stability of the kinematic \(\alpha \omega\)-dynamo problem with the \(\alpha\)-effect taken from the the weakly-nonlinear regime near the threshold of generation can be predicted from the knowledge on the threshold of generation of the linear problem with the opposite sign of the dynamo number. It appears, that in spite of the fact, that the magnetic field already saturated \(\alpha\), it still can generate magnetic field if spectra of linear problem are similar for dynamo number \(D\) with the opposite sign. So, as \(D\) depends on the product of the \(\alpha\) and \(\omega\) effects similar analysis can be performed with the \(\omega\)-quenching, usually used in geodynamo models, see e.g. (Soward, 1978), as well as with the feed back of the magnetic field on diffusion. It is likely, that for the more complex systems, velocity field, taken from the saturated regime, with many exited modes will always generate magnetic field if the Lorentz force would be omitted.

So as nonlinearity (5) has quite a general form, we consider applications of these results to some other dynamos models.

Linear analysis of the axi-symmetrical \(\alpha \omega\)-equations gives the following, see (Moffatt, 1978) and references therein: for positive \(D\) (which is believed to be in the Earth) in presence of the meridional velocity \(U_p\) the first exciting mode is dipole with \(\mathcal{J}_\gamma = 0\). Reducing of \(U_p\) firstly leads to oscillatory dipole solution (regime of the frequent reversals (Braginskii, 1964)). The further reduce of \(U_p\) gives the quadrupole oscillatory regime with larger value of \(D\). For negative \(D\) and \(U_p \neq 0\) the first mode is quadrupole with \(\mathcal{J}_\gamma = 0\). \(U_p \rightarrow 0\) gives non-oscillatory dipole mode with decreased \(D\), see for more details (Meunier, Proctor, Sokoloff, Soward, Tobias, 1997). In contrast to the dynamo in the disk the thresholds of generation for positive and negative \(D\) in the sphere are of the same order and situation with stability of the field \(B\)
is uncertain, and can depend on the particular form of the \( \alpha \)- and \( \omega \)-effects. Anyway, stability of \( \hat{B} \) for the steady regime is more expected.

In accordance with (Cattaneo & Tobias, 2009) shell models of turbulence demonstrate exponential growth of the magnetic field. This case, as well as 3D simulations of the turbulence in the box, which have the same instabilities, correspond to the oscillatory regimes and using our predictions should be unstable.

In the case of the 3D dynamo in the sphere simulations demonstrate different behaviour of \( \hat{B} \) (Tilgner, 2008; Schrinner, Schmidt, Cameron, 2009). For small Rayleigh numbers, when the preferred solutions is dipole (in oscillations) and close to the single mode structure (Case 1 in Christensen, Aubert, Cardin, Dormy, Gibbons et al., 2001)), \( \hat{B} \) is finite. Increase of the Rossby number (Schrinner, Schmidt, Cameron, 2009) leads to the turbulent state and \( \hat{B} \) becomes unstable.

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