The killed Brox diffusion

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Abstract
We are proposing to study a diffusion in random environment confined in bounded interval. As for many standard diffusions, we build in natural way a stochastic process with bounded domain, but with addition of considering a random environment. We carry out this construction using the Brox diffusion and applying well-known diffusion theory in a quenched fashion, which is a natural way to deal with random environment. The outcome of this procedure is an object that we may call the killed Brox diffusion. Since the generator of this process is initially an ill-posed expression we develop a Sturm–Liouville theory for one-dimensional second-order differential operators with white-noise coefficients. Our first main result is to give a close form of the Green operator associated to the generator, i.e., the inverse of the generator. We do so by setting the Lagrange identity in this context. Then, we give explicit expressions in quenched form of the probability density function of the process; such an object is given theoretically in terms of the spectral decomposition using the eigenvalues and eigenfunctions of the infinitesimal generator of the diffusion. Moreover, we characterize the eigenvalues and eigenfunctions using some integro-differential equations.

1. Introduction
Probabilistic models with random environment are important models to capture some kind of external randomness that there exists in the medium. From this type of modeling different mathematical tools have been devised to analyze their complexities; probably a good starting point to initiate being acquainted with the random-environment phenomena is the book by B.D. Hughes [19].

On another hand, when having a standard diffusion, such as the Brownian motion, sometimes we want to build a new diffusion confined in a finite domain, then one applies well-known tools to build such a process. In this article, we focus on building such a process starting with a diffusion in a random environment, in our case the Brox diffusion. Let us mention why this kind of processes are important.
A model that has been considered before is a particle moving in a potential over a bounded set specified by the following random operator, see e.g., \cite{12,16},

$$[Lf](x) = \frac{1}{2} \{ f''(x) - V(x)f(x) \}, \ x \in [0, 1],$$

where $V$ represents a white-noise potential. According to Halperin\cite{16} it is called the Schrödinger particle.

A popular model with random medium is what some people call the Sinai’s random walk, also called the Temkin’s model, which run in discrete time and discrete state space see\cite{34}. The continuous time-space idealization of the Temkin’s model is what people call the Brox diffusion, precisely because Brox\cite{6} carefully studied it in 1989, and proved the same asymptotic behavior as Y.G. Sinai did for the Temkin’s model in 1982. In\cite{33}, this model was also studied as a diffusion with random coefficients.

Different studies have been carried out to have more understanding of the Brox diffusion, let us give a brief account. For instance in\cite{22}, it is analyzed the limit behavior of the process as time evolves. Asymptotic behavior regarding the first passage time has been also studied in\cite{23}. In\cite{32}, it is studied some asymptotics of the local time, and in\cite{18} more ideas on sample path asymptotic are carried out. Interesting formulas were discovered in\cite{7,14,15} in connection with functional of the environment. Yet, without conditioning on the environment more asymptotic information about the process was found in\cite{35} when there is a drift in the environment. In\cite{8}, one finds some understanding on the paths behavior. In\cite{21}, limit behavior about occupation time is worked out. A relevant detail analysis on asymptotic dynamics of the local time is done in\cite{11}, and more recently in\cite{17} it is proposed stochastic differential equations (SDEs) driven by the Brox process where the local times play an important tool. We also mention an interesting study in\cite{5} of the Sinai’s walk truncated in a finite interval, where the authors also carry out an analysis of the eigenvalues and the relation with the local minimum in the potential.

There is also plenty of research about diffusions with generalized drift, where it is used theory of distributions to deal with a general function, including the white-noise. Let us mention some examples of this approach. A series papers were published in the 70s by N.I. Portenko to deal with SDEs or partial differential equations (PDEs) where the general drift term appears. The reader might consult\cite{27} which contains much of this research. In\cite{11}, using theory of distributions, the authors carried out an analysis of SDEs with general drift term, in turn, they also deal with the associated generator where the domain of functions satisfies initial conditions; this work includes solutions of certain equations. More recently, by making use of local times, in\cite{3} they are able to push forwards a generalization of the
drift term to give conditions for the solubility of the SDEs. In other recent papers, such as \cite{10,31}, important progress has been made for SDEs and PDEs with general drift, especially the time-dependence case; again, the domain functions are specified with initial conditions. With similar tool as the mentioned articles, one can see \cite{30} where they construct stochastic processes associated to generators with general drift.

In contrast of these mentioned works, a feature of the model we analyze here is that the associated PDE is specified with boundary conditions in an interval. Previous works seem not to cover this case to give the kind of solutions that we provide here, for instance, in \cite{31} they also invert a similar operator but with different domain given with initial conditions.

To summarize our approach, we mention that our starting point is a bonafide stochastic process in a random environment that we built from the Brox diffusion. The corresponding generator itself gives rise to the stochastic equations that we tackle using adapted theory of Sturm–Liouville. This analysis enables to invert the infinitesimal generator with a well-defined integral operator with no necessity to use the theory of distribution. We believe this approach which enables the constructions of further equations that help in particular to deal with the probability density functions.

Strictly speaking, the Brox diffusion becomes a diffusion only after freezing an environment (i.e., conditioning on one realization of the environment), which gives rise to the so-called quenched case, and without conditioning it is called the annealed case. Informally speaking, the generator $L$ acting on $f$ has the form

$$[Lf](x) = \frac{1}{2} \left\{ f''(x) - W'(x)f'(x) \right\},$$

where $W$ represents the Brownian motion. Since in the quenched case one is dealing with a bonafide diffusion, the whole apparatus of diffusions, for instance the one manufactured by K. Itô and H.P. McKean\cite{20}, can be used to settle this model in firm ground. Moreover, one knows that behind the theory of diffusions it is lurking the Sturm–Liouville theory of second order linear operators.

Although the generator of the Brox diffusion does not have differentiable coefficients as the classical ones, one can adapt many of the results in second order operators, which helps to say a lot about the generator, and ultimately on the stochastic process. It turns out that one can deal with the generator using the inner product to define it rigorously, this kind idea have been use before, say in \cite{25}, to define the so-called Dirichlet forms associated to the generator. Moreover, in \cite{13,26}, the authors also use inner products to work with the operator in a weak sense, and they arrive to similar expressions to invert a random operator.
Due to the importance of analyzing models with random medium in a bounded state space, at this stage, we propose in this article to build the killed version of the Brox diffusion, which ends up having a bounded state space. This version of the Brox process allows us to recast results from the theory of Sturm–Liouville and analyze the generator in such a way that it becomes feasible to write down an spectral representation of the probability density function. Such representation, as one might expect, is in terms of the eigenfunctions and eigenvalues of the generator. It should be mention that an important tool in this analysis is the so-called Green operator, which is the inverse of the generator. We are able to find explicitly the Green operator, which become of tremendous help at the time of analyzing the eigenfunctions of the generator; for instance, Corollary 5.1. is such an elegant form to say that the generator has a discrete set of eigenvalues. For the moment, we have concentrated to the quenched case, leaving for further investigation the annealed case. One can notice that the analysis we carry out here is for a model with state space being a compact interval. Nevertheless our approach leaves room to consider other models in an infinite interval, for instance, references such as [36,37] might be useful to extend the work.

Let us explain how this article is organized. In the coming section, we present the original construction of the Brox diffusion, making emphasis on the domain of the generator, which we need to know for our purposes. In Section 3, we build the killed Brox process and give results on its generator, in particular we make use of what people call the Wronskian to analyze solutions of certain related equation. In Section 4, we find the Green operator; additionally we need to establish the validity of the Lagrange identity in this context. Section 5 has the spectral representation of the density function, which is in terms of the eigenvalues and eigenfunctions. Precisely, Section 6 has a dissection of the eigenfunctions and eigenvalues, providing a way to characterize these objects. In doing so, we establish a version of an oscillation theorem suited for our generator. There is one appendix at the end with the proof using the classic Prüfer method to study zeros of eigenfunctions, again, such a proof is tailored to deal with our generator.

### 2. Preliminaries: the Brox diffusion

The Brox diffusion is sometimes described with the following SDE

\[
    dX_t = -\frac{1}{2} W'(X_t)dt + dB_t,
\]

where \( B := \{B_t : t \geq 0\} \) is the standard Brownian motion, and \( W := \{W(x) : x \in \mathbb{R}\} \) is a two-sided Brownian motion, and they both are
independent from each other. Here \( W' \) denotes the derivative of \( W \), usually called the white noise.

The expression (1) needs to have a rigorous meaning. It happens that one can use the associated generator in order to construct properly the process. When looking the Equation (1) one could say that the process \( X := \{X_t : t \geq 0\} \) has associated the infinitesimal operator given by

\[
L_f(x) := \frac{1}{2} e^{W(x)} \frac{d}{dx} \left( e^{-W(x)} \frac{df(x)}{dx} \right).
\]

In the context of diffusions, this corresponds to considering the scale function

\[
s(x) := \int_0^x e^{W(y)} dy,
\]

and the speed measure

\[
m(A) := \int_A 2e^{-W(y)} dy, \text{ for Borel sets } A \subseteq \mathbb{R}.
\]

Then one can rigorously consider the operator \( L \) as \( \frac{d}{dm} \frac{d}{ds} f \), where

\[
\frac{df(x)}{ds} := \lim_{h \to 0} \frac{f(x + h) - f(x)}{s(x + h) - s(x)} \quad \text{and} \quad \frac{df(x)}{dm} := \lim_{h \to 0} \frac{f(x + h) - f(x)}{m(x, x + h)}.
\]

The equivalence of the expressions for \( L \) is shown in the proof of Proposition 3.2., indeed it holds that

\[
\frac{d}{dm} \frac{d}{ds} f(x) = \frac{e^{W(x)}}{2} \left( e^{-W(x)} f'(x) \right)'.
\]

To learn about the derivatives with respect functions or measures we refer the reader to\(^{[20,28]}\).

Thanks to Itô–McKean’s construction of Feller-diffusion processes from a Brownian motion via scale transformation and time change, the Brox process \( X_t \) can be explicitly given by (see\(^{[6]}\))

\[
X_t = s^{-1}(B_{T_t^{-1}}),
\]

where \( s \) is the scale function given in (2) and \( T \) is the time change function defined by

\[
T_t := \int_0^t e^{-2W(s^{-1}(B_u))} du.
\]

Let us now describe the domain of the generator.

**Theorem 2.1.** For any environment \( W \), the domain \( D(L) \) is contained in the space of continuously differentiable functions \( C^1(\mathbb{R}) \). Moreover the
derivative of a function $f$ in $D(L)$ takes the form $f'(x) = e^{W(x)}g(x)$ with $g \in C^1(\mathbb{R})$.

**Proof.** According to Mandl\[24\], p.22, if $h(x) := Lf(x)$ for $f \in D(L)$, then
\[
f(x) = \int_x^y h(z)dm(z)ds(z) + f(a) + [s(x) - s(a)]\frac{df}{ds}(a),
\]
where
\[
\frac{df}{ds}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{s(a+h) - s(a)} = \frac{f'(a)}{e^{W(a)}}.
\]

Thus, using (4) and (5), as well as the speed measure (3) and the scale function (2), we have that
\[
f(x) = 2\int_x^y h(z)e^{-W(z)}e^{W(y)}dzdy + f(a) + \int_x^y e^{W(z)}dz.
\]

Therefore we obtain that $f \in C^1(\mathbb{R})$. Now, if we calculate explicitly the derivative of $f$, we arrive to
\[
f'(x) = e^{W(x)} \left[ 2\int_x^y h(z)e^{-W(z)}dz + \frac{f'(a)}{e^{W(a)}} \right].
\]

Then we have that $f'(x) = e^{W(x)}g(x)$ with $g \in C^1(\mathbb{R})$.

We use the previous construction of the Brox process to construct the Brox process with killing at $a$ and $b$.

**3. Brox process with killing**

Let us now construct the process that we will study along the article. We start by defining first the infinitesimal generator, including the domain. At this point, it is known *a priori* the type of behavior at the boundaries. Nevertheless, after this analytic description, we present the path representation of the process in terms of a Brownian motion with killing.

Consider the infinitesimal operator
\[
\bar{L}f(x) = \frac{d}{dm} \frac{d}{ds}f(x),
\]
where $m$ and $s$ are the speed measure and the scale function of the Brox process, and the domain $D(\bar{L})$ is the set of functions $f$ such that

- $f(a) = 0$,
- $f(b) = 0$,
- $f'(x) = e^{W(x)}g(x)$, $g \in C^1(\mathbb{R})$. 

Then, from general theory of diffusions, see e.g., \cite{4,20}, we know that for each fix environment $W$ there is a bonafide diffusion, call it $\bar{X}$, living inside $[a, b]$ and such that it is killed when hitting $a$ or $b$.

Let us see the representation of such a process.

**Proposition 3.1.** Denote by $\bar{B}$ to the Brownian motion with killing at $s(a)$ and $s(b)$, and consider

$$
\bar{T}_t := \int_0^t e^{-2W(s^{-1}(\bar{B}_u)))} du.
$$

Then one has the representation

$$
\bar{X}_t = s^{-1}(\bar{B}_{\bar{T}_t}),
$$

where $\bar{\gamma}$ is the inverse of $\bar{T}$.

**Proof.** As mentioned before, we already know that $X$ is killed at $a$ and $b$.

On the other hand, it is also known that $Y_t = s(\bar{X}_t)$ is in its natural scale, see for instance Theorem 46.12 in \cite{29}, page 275. Moreover, since $X$ is killed at $a$ and $b$, $Y$ is killed at $s(a)$ and $s(b)$.

Moreover, by Theorem 47.1 in \cite{29}, page 277, we have that $Y_t = \bar{B}_{\bar{\gamma}_t}$, where $\bar{\gamma}_t$ is the inverse of

$$
\bar{T}_t = \int_0^t e^{-2W(s^{-1}(\bar{B}_u)))} du. \tag{8}
$$

Originally, $\bar{T}$ is defined through the local time, but after one can use the identity in (47.27), page 283 in \cite{29} to end up with previous expression. Notice that by construction $\bar{B}$ is killed by $s(a)$ and $s(b)$.

Therefore, we finally end up with $\bar{X}_t = s^{-1}(Y_t) = s^{-1}(\bar{B}_{\bar{\gamma}_t})$. \hfill \Box

We do not go into details but it turns out that the converse of previous theorem is also true. That is to say, on can go from the path representation to the generator and its domain.

The previous way of thinking was briefly mentioned in \cite{6}, page 1216, where he considered the process with reflexive boundaries at $a$ and $b$.

In general, $\bar{L}f$ is well defined whenever $e^{-W(x)f'(x)}$ is still differentiable, we denote this as

$$
D_0 = \left\{ f \text{ differentiable : } e^{-W(x)f'(x)} \text{ is differentiable} \right\}.
$$

Now we present a couple of results regarding the generator and solutions of the so-called eigenfunction equation. These results will be very useful for the rest of the article. In the following result, we regard $\bar{L}$ as an operator acting not just on $D(\bar{L})$ but on any function in $D_0$. Abusing the notation,
sometimes we write $f(x, \lambda)$ in lieu of $f(x)$ to emphasize the dependence with the spectral parameter $\lambda$.

**Proposition 3.2.**

(i) The operator $\tilde{L}$ can be applied as

$$\tilde{L}f(x) = \frac{e^{W(x)}}{2} \left( e^{-W(x)} f'(x) \right)'$$

for any differentiable functions $f \in D_0$.

(ii) For any $\lambda > 0$, the problem

$$\tilde{L}g + \lambda g = 0, \text{ given } g(a, \lambda) = 0 \text{ and } g'(a, \lambda) = 1,$

admits a solution in $D_0$, where $g'$ means the derivative with respect to $x$. Moreover, such solution satisfies the equation

$$g(x, \lambda) = -2\lambda \int_a^x \int_a^y g(z, \lambda) e^{-W(z)} e^{W(y)} dz \, dy + \frac{1}{e^{W(a)}} \int_a^x e^{W(z)} dz. \quad (9)$$

(iii) The reciprocal of ii) is also true. That is, if $g$ satisfies (9), then $g$ solves the problem $\tilde{L}g + \lambda g = 0$, $g(a, \lambda) = 0$ and $g'(a, \lambda) = 1$.

**Proof.**

(i) We simply calculate $\frac{d}{dm} \frac{d}{ds}f(x)$ using the speed measure $m$ and the scale function $s$. Then we have

$$\frac{d}{dm} \frac{d}{ds}f(x) = \frac{d}{dm} \left( \lim_{h \to 0} \frac{f(x + h) - f(x)}{\int_x^{x+h} e^{W(y)} dy} \right)$$

$$= \frac{d}{dm} \left( e^{-W(x)} f'(x) \right)$$

$$= \lim_{h \to 0} \frac{e^{-W(x+h)} f'(x + h) - e^{-W(x)} f'(x)}{\int_x^{x+h} 2e^{-W(y)} dy}$$

$$= \frac{e^{W(x)}}{2} \left( e^{-W(x)} f'(x) \right)'$$

(ii) The existence of a solution comes from general theory of diffusions, see e.g. [20], specifically in Section 4.6, page 128. To prove the second statement, we apply twice a fundamental theorem of calculus adapted to derivatives with respect a measure, see e.g., [28]. Indeed, using the definition of $\tilde{L}f(x) = \frac{d}{dm} \frac{d}{ds}f(x)$ in the equation $\tilde{L}g + \lambda g = 0$, we calculate first the integral with respect to $m$ and later with respect to $s$, from where Equation (9) arises.

(iii) The proof of this fact is simply to calculate on the equation the derivative with respect to $s$ and later with respect a $m$. \qed
In the theory of differential equations, the so-called Wronskian is an important tool to detect whether there is dependence among solutions of a differential equation. As one might expect, it turns out that in this context with a random coefficients the Wronskian can be calculated and it is used as well for the same purpose. It is defined as the following determinant,

\[
wf, g(x) := \begin{vmatrix}
g(x) & f(x) \\
\frac{dg}{dx}(x) & \frac{df}{dx}(x)
\end{vmatrix} = \frac{f'(x)g(x) - f(x)g'(x)}{e^{W(x)}},
\]

where \(f\) and \(g\) are two functions, and \(e^{W(x)}\) is the density of the scale function associated to the Brox process.

**Proposition 3.3.** Let \(f\) and \(g\) be two solutions of \(\bar{L}\psi + \lambda\psi = 0\), with \(x \in [a, b]\), such that \(f(a, \lambda) = 0\) and \(g(a, \lambda) = 0\). Then \(\frac{dw_{f,g}(x)}{dx} = 0\), which implies that there exists a constant \(C\) such that \(f = Cg\).

**Proof.** Using Proposition 3.2., if \(f\) and \(g\) are solutions of \(\bar{L}\psi + \lambda\psi = 0\), then

\[
\frac{e^{W(x)}}{2} \left( e^{-W(x)}\psi'(x) \right)' + \lambda\psi(x) = 0,
\]

which implies

\[
\left( \frac{\psi'(x)}{e^{W(x)}} \right)' = (-2) \frac{\lambda\psi(x)}{e^{W(x)}}.
\]

Hence, we can substitute into the derivative of the Wronskian to see that

\[
\frac{dw_{f,g}(x)}{dx} = \left( \frac{f'(x)g(x)}{e^{W(x)}} - \frac{f(x)g'(x)}{e^{W(x)}} \right)'
\]

\[
= \left( \frac{f'(x)}{e^{W(x)}} \right)' g(x) - \left( \frac{g'(x)}{e^{W(x)}} \right)' f(x)
\]

\[
= 0
\]

This implies that \(w_{f,g}(x) = M\) for some constant \(M\), so

\[
f'(x)g(x) - g'(x)f(x) = Me^{W(x)}.
\]

Using that \(f(a, \lambda) = 0\) and \(g(a, \lambda) = 0\), we have that \(M = 0\), then for all \(x \in [a, b]\) we have that

\[
w_{f,g}(x) = f'(x)g(x) - g'(x)f(x) = 0.
\]

Consider the set \(A := \{x : g(x) \neq 0\}\). For all \(x \in A\) we have

\[
\frac{f'(x)g(x)}{g^2(x)} - \frac{g'(x)f(x)}{g^2(x)} = 0,
\]
this implies
\[
\left( \frac{f(x)}{g(x)} \right)' = 0.
\]

Then we have that there exists a constant \( C \) such that \( f(x) = Cg(x) \) for all \( x \in A \), which by (11) also holds for \( x \in A^c \). To see that, let \( B := \{ x : g'(x) \neq 0 \} \). Then, using the equation (11), we have for all \( x \in B \),
\[
f(x) = \frac{f'(x)}{g'(x)} g(x).
\]

We now show that \( \frac{f'(x)}{g'(x)} \) is a constant. To do that, we calculate the derivative. Using again the fact that \( f \) and \( g \) are solutions of (10) and taking into account (11), we have
\[
\left( \frac{f'(x)}{g'(x)} \right)' = \left( \frac{f'(x)e^{-W(x)}}{g'(x)e^{-W(x)}} \right)' = \frac{g'(x)e^{-W(x)} \cdot (f'(x)e^{-W(x)})' - f'(x)e^{-W(x)} \cdot (g'(x)e^{-W(x)})'}{(g'(x)e^{-W(x)})^2} = \frac{g'(x)e^{-W(x)} \cdot (-2\lambda f(x)e^{-W(x)}) - f'(x)e^{-W(x)} \cdot (-2\lambda g(x)e^{-W(x)})}{(g'(x)e^{-W(x)})^2} = 0.
\]

Then for all \( x \in B \) there exists a constant \( K \) such that \( f(x) = Kg(x) \). We have that \( C = K \). To do that, it is sufficient to show that \( g \) and \( g' \) do not have common zeros. This is true by using the formulae (6) and (7).

Therefore for all \( x \in [a, b] \) we have
\[
f(x) = Cg(x). \quad \text{(12)}
\]

And the proof is done.

To facilitate the notation, from now on we use \( L \) alone instead of \( \bar{L} \), and \( X \) instead of \( \bar{X} \).

**4. Green operator**

From the previous section, we have that the infinitesimal operator associated with the Brox diffusion with killing on \( a \) and \( b \) is given by
\[
Lf(x) = \frac{e^{W(x)}}{2} \left( e^{-W(x)}f'(x) \right)',
\]
whose domain are functions \( f \in D_0 \) such that \( f(a) = f(b) = 0 \), i.e., \( f \in D(L) \).
We want to construct the so-called Green operator, which is actually the inverse operator of $L$. First, consider the following identity known as the Lagrange identity.

**Lemma 4.1.** Let $f, g$ in $D_0$, then $e^{-W(x)}(f'(x)g(x) - f(x)g'(x))$ is indeed differentiable, and in fact it holds

$$2e^{-W(x)}[g(x)Lf(x) - f(x)Lg(x)] = \left[e^{-W(x)}(f'(x)g(x) - f(x)g'(x))\right]'.$$

**Proof.** Let $h_1(x) = e^{-W(x)}f'(x)$ and $h_2(x) = e^{-W(x)}g'(x)$. Then

$$\left[e^{-W(x)}(f'(x)g(x) - f(x)g'(x))\right]' = [h_1(x)g(x) - h_2(x)f(x)]'$$

$$= h'_1(x)g(x) + g'(x)h_1(x) - h'_2(x)f(x) - f'(x)h_2(x)$$

$$= h'_1(x)g(x) - h'_2(x)f(x)$$

$$= 2e^{-W(x)}[g(x)Lf(x) - f(x)Lg(x)].$$

From previous identity, after integrating we also obtain

**Corollary 4.1.** Let $\alpha, \beta$ be such that $a \leq \alpha < \beta \leq b$. And $\lambda_1$ and $\lambda_2$ with $Lf + \lambda_1 f = 0$, $Lg + \lambda_2 g = 0$. Then

$$\left[e^{-W(x)}(f'(x)g(x) - f(x)g'(x))\right]_{\alpha}^{\beta} = 2(\lambda_2 - \lambda_1) \int_{\alpha}^{\beta} e^{-W(x)}f(x)g(x)dx.$$

We now construct the so-called Green operator.

**Definition 4.1.** Let

$$g(x, \xi) := \begin{cases} -Cu(x)v(\xi), & a \leq x \leq \xi; \\ -Cu(\xi)v(x), & \xi \leq x \leq b, \end{cases}$$

where

$$C := \int_{a}^{b} e^{W(z)}dz, \quad u(x) := \frac{1}{C} \int_{a}^{x} e^{W(z)}dz, \quad v(x) := \frac{1}{C} \int_{x}^{b} e^{W(z)}dz.$$

We then define the Green operator as

$$Tf(x) := \int_{a}^{b} 2e^{-W(z)}g(z, x)f(z)dz,$$

for any $f \in D(L)$.

Let us now study the kernel $g$. Fix $\xi$ and note that $g$ satisfies

$$g(a, \xi) = 0, \quad g(b, \xi) = 0.$$
We also see that \( Lg(x, \xi) = 0 \) for \( x \neq \xi \), where \( \xi \) is any fixed value, and with the understanding that \( g'(x, \xi) \) is the derivative of \( g(x, \xi) \) with respect to the first argument \( x \), with \( x \neq \xi \). To do that, we consider the derivatives from right and left of \( \xi \). Then we obtain 
\[
\begin{aligned}
g'(x, \xi) := \begin{cases} 
-e^{W(x)} v(\xi), & a \leq x \leq \xi; \\
e^{W(x)} u(\xi), & \xi \leq x \leq b.
\end{cases}
\end{aligned}
\]

Then, after multiplying by \( e^{-W(x)} \) there is no more dependence on \( x \), therefore the second derivative gives 0. This implies that \( Lg(x, \xi) = 0 \) for \( x \in [a, \xi) \) and \( x \in (\xi, b] \).

Now we use the Lagrange identity with the function \( f \) and \( g \), using that \( Lg(x, \xi) = 0 \). Then for \( x \in [a, \xi) \) and \( x \in (\xi, b] \) we have
\[
2e^{-W(x)} [g(x, \xi) Lf(x)] = \left[ e^{-W(x)} (f'(x)g(x, \xi) - f(x)g'(x, \xi)) \right]'.
\] (14)

On integrating both sides of (14) on the intervals \((a, \xi^-)\) and \((\xi^+, b)\), where \( \xi^- := \xi - \epsilon \) and \( \xi^+ := \xi + \epsilon \). Then, to calculate \( Tlf \), we have the following two equalities
\[
\begin{aligned}
\int_a^{\xi^-} 2e^{-W(x)} g(x, \xi) Lf(x) dx &= \left[ e^{-W(x)} (f'(x)g(x, \xi) - f(x)g'(x, \xi)) \right]_{\xi^-}^a, \quad (15) \\
\int_{\xi^+}^b 2e^{-W(x)} g(x, \xi) Lf(x) dx &= \left[ e^{-W(x)} (f'(x)g(x, \xi) - f(x)g'(x, \xi)) \right]_{\xi^+}^b. \quad (16)
\end{aligned}
\]

By adding (15) and (16), and using that \( f(a) = f(b) = g(a, \xi) = g(b, \xi) = 0 \), we have
\[
\begin{aligned}
\int_a^b 2e^{-W(x)} g(x, \xi) Lf(x) dx &= \int_{-\epsilon}^\epsilon 2e^{-W(x)} g(x, \xi) Lf(x) dx \\
&= e^{-W(\xi^-)} (f'(\xi^-)g(\xi^-, \xi) - f(\xi^-)g'(\xi^-, \xi)) \\
&\quad - e^{-W(\xi^+)} (f'(\xi^+)g(\xi^+, \xi) - f(\xi^+)g'(\xi^+, \xi)).
\end{aligned}
\]

After expanding we end up with four terms. From the continuity of \( W, f' \) and \( g \), the first and third terms cancel each other when \( \epsilon \to 0 \).

Since \( g' \) is not continuous, the second and fourth terms do not vanish. These terms are
\[
-e^{-W(\xi^-)} f(\xi^-)g'(\xi^-, \xi) + e^{-W(\xi^+)} f(\xi^+)g'(\xi^+, \xi).
\]

Taking the discontinuity into account, previous display is
\[
e^{-W(\xi^-)} f(\xi^-)e^{W(\xi^-)} v(\xi) + e^{-W(\xi^+)} f(\xi^+)e^{W(\xi^+)} u(\xi).
\]
Then, when $\epsilon \to 0$, it becomes

$$e^{-W(\xi)}f(\xi)e^{W(\xi)}u(\xi) + e^{-W(\xi)}f(\xi)\nu(\xi).$$

Since $u(\xi) + \nu(\xi) = 1$,

$$TLf(\xi) = \int_{a}^{b} 2e^{-W(x)}g(x, \xi)Lf(x)dx = f(\xi).$$

Therefore, we conclude that $T(Lf)(x) = f(x)$.

Now, using that

$$Th(x) = -2Cv(x)\int_{a}^{x} e^{-W(z)}u(z)h(z)dz - 2Cu(x)\int_{x}^{b} e^{-W(z)}\nu(z)h(z)dz,$$

one can apply $L$ to verify that $LTh = h$.

The conclusion is that $T$ is the inverse operator of $L$, more precisely:

**Theorem 4.1.** Let $T$ given in Definition (7). Then $T$ satisfies $T(Lf)(x) = f(x)$ for all $f \in D(L)$, and $L(Th)(x) = h(x)$ for all $h \in L^{2}([a, b])$.

### 5. Toward the density

In the theory of Markov processes, it is well known that spectral information of the generator helps to study the transition probability functions of the stochastic process. In particular, notice that after fixing an environment $W$, the process we are dealing with is a diffusion, thus we may use theory of Markov processes to analyze further. In turn, one can use the eigenvalues and eigenfunctions to give expressions for the probability density. In fact, we can identify the eigenvalues of the generator of the killed Brox process with the eigenvalues of the Green operator $T$ of Theorem 4.1., precisely because $T$ is the inverse of $L$.

**Corollary 5.1.** The operator $L$ has almost surely a countable set of eigenvalues.

**Proof.** This comes from the fact that for almost every trajectory of $W$, the operator $T$ is a compact operator, thus it has a countable set of eigenvalues. Then, if $(\lambda, f)$ is an eigenpair of $T$, then $Tf + \lambda f = 0$. Thus, $f = LTf = -\lambda Lf$, i.e., $(1/\lambda, f)$ is an eigenpair of $L$. \qed

Now we know that the generator of $X$ has a discrete spectrum given by the eigenvalues $\lambda_{n}$, and each one has associated an eigenfunction $\phi_{n}$. Thus, at a theoretical point of view, it is just a matter to join pieces to have the spectral decomposition of the probability transition function. Indeed, there are results in the literature regarding It\'s diffusions where one can write down a representation for the transition probability functions using spectral
information of the generator. Here we use the ideas in [2] to deal with our diffusion in random environment.

Notice that a priori we do not know if the transition probabilities are absolutely continuous with respect to the Lebesgue measure, however, it is indeed the case.

**Theorem 5.1.** If we leave fixed an environment $W$, then for all $x, y \in (a, b)$ we have

$$p(t, x, y) = 2e^{-W(y)} \sum_{n=1}^{\infty} e^{\lambda_n t} \phi_n(x) \phi_n(y),$$

where $p(t, x, y)$ is the density function of $X_t$ given that $X_0 = x$, and $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$ are the eigenvalues and eigenfunctions of $L$.

**Proof.** Due to the Corollary 5.1. we a priori know that the set of eigenvalues and eigenfunctions are indeed countable. It turns out that the set $\{\phi_n\}_{n=1}^{\infty}$ forms a basis for the space $L^2([a, b], 2e^{-W(x)})$, where the inner product is given by $\langle f, g \rangle := \int_a^b f(x)g(x)2e^{-W(x)}dx$, this fact can be found in [37], point (5) of Theorem 4.6.2.

Let us see now that the operator $L$ is self-adjoint and non-positive on $D(L)$, which are two important ingredients to carry out the spectral representation. Using integration by parts and the fact that $f(a) = f(b) = g(a) = g(b) = 0$, we obtain

$$\langle Lf, g \rangle = \int_a^b (e^{-W(x)}f'(x))'g(x)dx = -\int_a^b e^{-W(x)}f'(x)g'(x)dx$$

$$= \int_a^b (e^{-W(x)}g'(x))'f(x)dx,$$

which is precisely $\langle f, Lg \rangle$, i.e., $L$ is self-adjoint. Now, from the previous display we can also see the non-positivity:

$$\langle Lf, f \rangle = -\int_a^b e^{-W(x)}(f'(x))^2dx \leq 0.$$  

From this point, we can carry on with the proof in [2], pages 408–410, where based on the facts mention above it is possible to establish the spectral representation of the transition probability density. Notice that the non-positivity helps to see that the eigenvalues are negatives, which guarantees the converges of the series.

6. **Spectral analysis of the generator**

In previous section, we have shown, at least at a theoretical level, how one can give a spectral decomposition for the densities of $X$. Let us go further
to try to characterize the components of such representation, that is to say
the eigenvalues and the eigenfunctions.

Apart from using the spectral information for the density, we are inter-
ested in the connection with the environment. For instance, in \cite{5}, they ana-
lyze a discrete model with random environment and notice that the
eigenvalues of the generator are influenced by the minimum values of the
environment. We hope that our analysis can bring some light to this kind
of phenomena for the continuous model we study here.

We deal first with the eigenfunctions and after with the eigenvalues. We
will keep noticing how the Green operator $T$ of Theorem 4.1. will be useful
for our analysis.

6.1. Eigenfunctions

Let $\phi$ be an eigenfunction and $\lambda$ an eigenvalue of the generator $L$,
then it holds

$$L\phi + \lambda \phi = 0 \text{ with } \phi(a) = \phi(b) = 0.$$ 

The Green operator gives the identity $TL\phi = \phi = -\lambda T\phi$, that is

$$\phi(x) = -2\lambda \int_a^b e^{-W(z)} \phi(z) g(z, x) dz.$$ 

From the definition of $g$ previous display becomes

$$\phi(x) = 2C\lambda v(x) \int_a^x u(z) e^{-W(z)} \phi(z) dz + 2C\lambda u(x) \int_x^b v(z) e^{-W(z)} \phi(z) dz.$$ 

After taking the derivative, a cancelation occurs that yields

the following equation that gives characterization in the quenched case.

**Proposition 6.1.1.** For the fixed trajectory $W$ of the environment, suppose
that $\lambda$ is an eigenvalue of the generator. Then, the corresponding eigenfunc-
tion $\phi$ is solution of the equation

$$\phi'(x) = 2\lambda e^{W(x)} \left[ \int_a^b v(z) e^{-W(z)} \phi(z) dz - \int_a^x u(z) e^{-W(z)} \phi(z) dz \right].$$ 

(18)

6.2. Eigenvalues

In this section, we give a method to deal with the eigenvalues. To do that,
with the aid of the Sturm–Liouville theory, we develop few results tailored
to work with our operator $L$. 

Theorem 6.2.1. Consider functions $f \in D_0$. Define the following two operators for $a < x < b$, 
\[
L_1 f(x) := \left( e^{-W(x)} f'(x) \right)' + 2\lambda_1 e^{-W(x)} f(x),
\]
\[
L_2 f(x) := \left( e^{-W(x)} f'(x) \right)' + 2\lambda_2 e^{-W(x)} f(x),
\]
where $\lambda_2 > \lambda_1$. Let $\phi_1$ and $\phi_2$ such that $L_1 \phi_1 = L_2 \phi_2 = 0$. Then, between two zeros of $\phi_1$ there is a zero of $\phi_2$. Moreover, if $\phi_1(a) = \phi_2(a) = 0$, then $\phi_2$ has at least as many zeros as $\phi_1$ on $[a,b]$.

Proof. Suppose that $x_1$ and $x_2$ are two successive zeros of $\phi_1$, and that $\phi_2(x) \neq 0$ for any $x \in (x_1, x_2)$. Without loss of generality we assume that $\phi_1(x) > 0$ and $\phi_2(x) > 0$ for any $x \in (x_1, x_2)$.

The Lagrange’s identity in Corollary 4.1. gives
\[
\left[ e^{-W(x)} (\phi_1'(x)\phi_2(x) - \phi_1(x)\phi_2'(x)) \right]_{x_1}^{x_2} = 2(\lambda_2 - \lambda_1) \int_{x_1}^{x_2} e^{-W(x)} \phi_1(x)\phi_2(x)dx.
\]

Note that the right-hand side is strictly positive. However, the left-hand side reduces to
\[
e^{-W(x_2)} \phi_1'(x_2)\phi_2(x_2) - e^{-W(x_1)} \phi_1'(x_1)\phi_2(x_1).
\]

Using the assumptions of $\phi$ on $x_1$ and $x_2$, we observe that $\phi_2(x_2) \geq 0$, $\phi_1'(x_2) \leq 0$, $\phi_2(x_1) \geq 0$ and $\phi_1'(x_1) \geq 0$, then the above expression is less or equal to 0, giving a contradiction. Therefore $\phi_2$ has a zero between $x_1$ and $x_2$.

In particular, if $\phi_1(a) = \phi_2(a) = 0$ and $\phi_1(x_1) = 0$ with $a < x_1 < b$, then there exists $z$, with $a < z < x_1$ such that $\phi_2(z) = 0$. Thus $\phi_2$ has at least as many zeros as $\phi_1$ on $[a,b]$. \hfill \Box

Corollary 6.2.1. If $\phi_n$ is an eigenfunction of $L$ associated with $\lambda_n$, with $n = 1, 2,...$, then $\phi_n$ has exactly $n - 1$ zeros in the interval $(a, b)$.

We left the proof of this corollary in the Appendix. The classic proof of this result uses the so-called method of Prüfer which is based in a change of coordinates. The original proof for the standard equation is difficult to find in the literature, one can find it thought in $^9$, from where we adapted to our situation.

Using previous two results we can to show the following theorem.

Theorem 6.2.2. Let $\lambda \in \mathbb{R}$ be fixed, and let $\psi(x, \lambda)$ be solution of
\[
L\psi(x, \lambda) + \lambda \psi(x, \lambda) = 0, \quad x \in (a, b),
\]
that satisfies $\psi(a, \lambda) = 0$ and $\psi'(a, \lambda) = 1$. Then the number of zeros of the map $x \mapsto \psi(x, \lambda)$ on $(a, b]$ equals the number of eigenvalues of $L$ less or equal to $\lambda$. 

Proof. First, from Proposition 3.2., we know that such function $\psi$ really exists.

The proof relies on Theorem 6.2.1. and Corollary 6.2.1. In what follows $\phi_n$ is the eigenfunction associated with the $n$-eigenvalue $\lambda_n$, i.e., $L\phi_n + \lambda_n\phi_n = 0$, with $\phi_n(a) = \phi_n(b) = 0$.

Fix $\lambda$. We first suppose that there exist only $n$ eigenvalues less or equal to $\lambda$, i.e., $\lambda_1 < ... < \lambda_n \leq \lambda < \lambda_{n+1}$, and let us prove that the map $x \mapsto \psi(x, \lambda)$ has exactly $n$ zeros in $(a, b]$. By the Corollary 6.2.1., $\phi_n$ has exactly $n-1$ zeros on the open interval $(a, b)$, thus it has $n+1$ zeros in $[a, b]$. Since $\lambda_n \leq \lambda$, by Theorem 6.2.1. we know that between two consecutive zeros of $\phi_n$ there is one zero for $\psi$. Then $\psi$ has at least $n$ zeros on $(a, b)$, i.e., it has at least $n+1$ zeros on $[a, b]$. However, if $\psi$ had $n+1$ zeros on $(a, b)$, again using Theorem 6.2.1. with $\lambda < \lambda_{n+1}$, the $n+2$ zeros of $\psi$ on $[a, b]$ would imply that $\phi_{n+1}$ had $n+1$ zeros on $(a, b)$, which is not the case. We conclude that $\psi$ has exactly $n$ zeros on $(a, b]$.

On the other hand, we now suppose that $\psi$ has exactly $n$ zeros in $(a, b]$. Let us now show that there exist only $n$ eigenvalues less or equal that $\lambda$. Suppose by contradiction that the eigenvalue $n+1$ is less or equal to $\lambda$, i.e., $\lambda_{n+1} \leq \lambda$. If $\lambda = \lambda_{n+1}$ we have that $\psi = \phi_{n+1}$, and by the Corollary 6.2.1. $\psi$ has $n$ zeros in $(a, b)$, and since $\phi_{n+1}(b) = 0$, we obtain that $\psi$ has $n+1$ zeros in $(a, b)$, which is a contradiction. If $\lambda_{n+1} < \lambda$, by the Corollary 6.2.1. we know that $\phi_{n+1}$ has $n+2$ zeros in $[a, b]$. Now, by Theorem 6.2.1. we have that $\psi$ should have at least $n+2$ in $[a, b]$, this implies that $\psi$ has $n+1$ zeros or more in $[a, b]$, which is again a contradiction.

We know now that if $\psi$ has $n$ zeros in $(a, b)$, the $n+1$ eigenvalue satisfies $\lambda < \lambda_{n+1}$. We will now show that $\lambda_n \leq \lambda$, i.e., there exist only $n$ eigenvalues less or equal to $\lambda$.

Suppose that $\lambda < \lambda_n$. Recall that we are supposing that $\psi$ has $n$ zeros in $(a, b)$, since $\psi(a) = 0$, it has $n+1$ zeros in $[a, b]$. Again, appealing to the Theorem 6.2.1., since $\phi_n(a) = \phi_n(b) = 0$, we have that $\phi_n$ has at least $n+2$ zeros in $[a, b]$. We are saying that $\phi_n$ has $n$ zeros or more in $(a, b)$, which contradicts Corollary 6.2.1. And the proof is completed.

Remark 6.2.1. Let us give a characterization of function $\psi$ of previous theorem, i.e., $\psi$ such that

$$L\psi(x, \lambda) + \lambda\psi(x, \lambda) = 0, \quad x \in (a, b)$$

with $\psi(a, \lambda) = 0$ and $\psi'(a, \lambda) = 1$.

From ii) of Proposition 3.2., $\psi$ is solution of the equation

$$\psi(x, \lambda) = -2\lambda \int_a^x \int_a^y \psi(z, \lambda)e^{-W(z)}e^{W(y)}dzdy + \psi(a, \lambda) + \frac{\psi'(a, \lambda)}{e^{W(a)}} \int_a^x e^{W(z)}dz.$$
By differentiating and taking into account the initial conditions, we have the following two equations,

\[
\psi(x, \lambda) = -2\lambda \int_a^x \psi(z, \lambda) e^{-W(z)} e^{W(y)} dz dy + \frac{1}{e^{W(a)}} \int_a^x e^{W(z)} dz, \\
\psi'(x, \lambda) = e^{W(x)} \left[ -2\lambda \int_a^x \psi(z, \lambda) e^{-W(z)} dz + \frac{1}{e^{W(a)}} \right].
\] (19)

We finally arrive to the point where it is possible to identify the eigenvalues of the generator of \(X\).

**Theorem 6.2.3.** Considering the function \(\psi\) in (19), then we have that \(\lambda\) is an eigenvalue of \(L\) if and only if \(\psi(b, \lambda) = 0\).

**Proof.** Since it holds \(L\psi + \lambda \psi = 0\) and \(\psi(a, \lambda) = 0\), if we are told that \(\psi(b, \lambda) = 0\), then \(\psi\) would be an eigenfunction, consequently \(\lambda\) would be an eigenvalue.

Let us now suppose that \(\lambda\) is an eigenvalue of \(L\). If that is the case, then there exists an eigenfunction \(\varphi\), thus it holds that \(L\varphi + \lambda \varphi = 0\), \(\varphi(a, \lambda) = 0\), \(\varphi(b, \lambda) = 0\). From iii) of Proposition 3.2., we know that \(\psi\) also satisfies \(L\psi + \lambda \psi = 0\), \(\psi(a, \lambda) = 0\) and \(\psi'(a, \lambda) = 1\). And by Proposition 3.3., there exists a constant \(C\) such that

\[\varphi(x, \lambda) = C\psi(x, \lambda),\]
which implies that \(\psi(b, \lambda) = 0\). \(\square\)

In the end, in order to recover spectral information of the stochastic process, we can summarize the section with two ideas. Theoretically speaking, one can first use the previous theorem to run function \(\lambda \mapsto \psi(b, \lambda)\) to find the eigenvalues each time when \(\psi(b, \lambda) = 0\). Secondly, once having an eigenvalue \(\lambda\), one can use Proposition 6.1.1. to find the corresponding eigenfunction.

**Appendix**

**Proof of Corollary 6.2.1**

We want to analyze the equation \(Lx + \lambda x = 0\). We consider the equation

\[
2e^{-W(t)} [Lx(t) + \lambda x(t)] = (e^{-W(t)} x'(t))^' + 2\lambda e^{-W(t)} x(t) = 0, \ a < t < b,
\]
with \(x(a) = x(b) = 0\) \(\) (20)

We are going to use the Prüfer method, which we adapt following the arguments in \cite{9}. In this method, one first defines \(y(t) = e^{-W(t)} x'(t)\). Using (20) we have

\[
x'(t) = \frac{y(t)}{e^{-W(t)}}, \quad y'(t) = -2\lambda e^{-W(t)} x(t).
\] (21)
Notice that even though \( y(t) \) is in terms of the BM \( W(t) \), the derivative \( y'(t) \) is well defined.

The trick in this classic method is to propose the following change of coordinates

\[
  x(t) = r(t) \sin(w(t)), \quad y(t) = r(t) \cos(w(t)).
\]

(22)

where \( r(t) \geq 0 \). Differentiating equation (22) with respect to \( t \) we have

\[
  x'(t) = r'(t) \sin(w(t)) + r(t) \cos(w(t)) \cdot w(t),
\]

\[
  y'(t) = r'(t) \cos(w(t)) - r(t) \sin(w(t)) \cdot w(t).
\]

We now use (21), and solving for \( r' \) and \( w' \), we obtain

\[
  r'(t) = \left( \frac{1}{e^{-W(t)}} - 2\lambda e^{-W(t)} \right) r(t) \sin(w(t)) \cos(w(t)),
\]

(23)

and

\[
  w'(t) = \frac{1}{e^{-W(t)}} \cos^2(w(t)) + 2\lambda e^{-W(t)} \sin^2(w(t)).
\]

(24)

These equations with initial conditions have unique solution. If \( \phi \) is a solution of (20) then \( \phi \) has the form \( \phi(r) = r(t) \sin(w(t)) \). Equation (23) is of the form \( r'(t) = h(t)r(t) \), so the solution \( r \) is \( r(t) = r(a)e^\int_0^t h(u)du \).

We now show that \( r(a) \neq 0 \). Suppose that \( r(a) = 0 \) and let \( x_j(t) \) be an eigenfunction associated with \( \lambda \). Note that \( \lambda^2 = x_j^2(a) + y_j^2(a) = x_j^2(a) + e^{-2W(a)}(x'_j(a))^2 \). This implies that \( x_j(a) = x'_j(a) = 0 \).

It is known from the theory of ordinary differential equations that the only solution of \( Lx(t) + \lambda x(t) = 0 \) with \( x(a) = x'(a) = 0 \) is the trivial solution. Therefore \( x_j \equiv 0 \), which is a contradiction. Then we have that \( r(a) \neq 0 \), and since \( r(t) = r(a)e^\int_0^t h(u)du \), this implies that \( r(t) > 0 \).

A consequence of this is that \( \phi \) vanishes only when \( w \) is a multiple of \( \pi \).

Taking into account the conditions

\[
  x(a) = 0, \quad x(b) = 0,
\]

(25)

let \( \phi(t, \lambda) \) be a nontrivial solution of (20) and (25).

To analyze \( \phi \), we now give some properties of \( w \), defined in (24).

First of all, it holds \( w(a, \lambda) = 0 \). This is because from formulas (21) and (22), one has

\[
  w(a, \lambda) = \tan^{-1}\left( \frac{\phi(a, \lambda)}{e^{-W(a)} \phi'(a, \lambda)} \right) = 0.
\]

Second, it turns out that for \( \lambda \) fixed, \( w \) is increasing function of \( t \). To prove this, let us see that, using equation (24), the derivative is positive. This is the case if the eigenvalues are positive, let us check this fact. Let \( (x, \lambda) \) be an eigenpair of the generator, thus

\[
  (e^{-W(t)}x'(t))^t + 2\lambda e^{-W(t)}x(t) = 0.
\]

(26)

Multiplying (26) by \( x(t) \) we obtain

\[
  x(t)(e^{-W(t)}x'(t))^t + 2\lambda e^{-W(t)}x^2(t) = 0.
\]
Integrating and solving for \( \lambda \) we arrive at
\[
\lambda = \frac{-\int_a^b x(t)(e^{-W(y)}x'(y))'dt}{\int_a^b 2e^{-W(y)}x^2(y)dt}.
\]

Integrating by parts and using that \( x(a) = x(b) = 0 \) we have
\[
\lambda = \frac{\int_a^b e^{-W(t)}(x'(t))^2 dt}{\int_a^b 2e^{-W(t)}x^2(t)dt} > 0.
\]

Now, for fixed \( t \), let us see that \( w(t, \lambda) \) is monotone increasing function of \( \lambda \). This is actually a consequence of the following theorem (for a proof see p.210 from \( ^9 \)).

**Theorem 1.** Let \( L_i x := (p_i x')' + g_i x \). And let \( p_i \) and \( g_i \) be continuous functions on \([a, b]\), such that
\[
0 < p_2(t) \leq p_1(t), g_2(t) \geq g_1(t).
\]

Let \( L_1 \phi_1 = 0 \) and \( L_2 \phi_2 = 0 \) and using the Prüfer method take \( w_1 \) and \( w_2 \) as in (24) with \( w_2(a) \geq w_1(a) \). Then
\[
w_2(t) \geq w_1(t), \ a \leq t \leq b.
\]

In our case, \( p_1(t) = p_2(t) = e^{-W(t)} \) and the function \( g(t) = 2\lambda e^{-W(t)} \) is increasing in \( \lambda \).

Finally, we have this property of \( w \):

**Lemma 18.** \( w(b, \lambda) \to \infty \) as \( \lambda \to \infty \).

**Proof.** We consider the Equation (20). Let \( P, G \) be constants such that for all \( t \in [a, b] \)
\[
e^{-W(t)} \leq P, \quad 2\lambda e^{-W(t)} \geq G.
\]

Now we consider the equation
\[
P x''(t) + 2\lambda G x(t) = 0, \quad (27)
\]
and take \( v \) to be the analogous of \( w \) with the condition \( v(a, \lambda) = w(a, \lambda) \). Hence, from
**Theorem 1** we have that
\[
w(t, \lambda) \geq v(t, \lambda).
\]

On the other hand, if \( f \) is a solution of the **Equation (27)**, then we have that for large \( \lambda \), \( f \) is of the form
\[
f(t) = A \cos \left( \sqrt{\frac{2G}{P}} t \right) + B \sin \left( \sqrt{\frac{2G}{P}} t \right),
\]
where \( A \) and \( B \) are constants. This solution implies that the zeros of \( f \) increase in number when \( \lambda \) is large, because the periodicity increases. The only way to have that is because \( v \) hit multiples of \( \pi \) many times. In particular, we have that \( v(b, \lambda) \to \infty \) as \( \lambda \to \infty \). Since \( w(b, \lambda) \geq v(b, \lambda) \), then we obtain the Lemma.

We continue with the proof of the corollary. Let us see that \( w(b, 0) < \pi \). If \( w(b, 0) = \pi \), the associated function \( \phi(t, 0) \) should be an eigenfunction of the eigenvalue \( \lambda = 0 \). However, as we mentioned before the eigenvalues are strictly positive. If \( w(b, 0) > \pi \), then by continuity there exists \( b' < b \) such that \( w(b', 0) = \pi \), then the associated function \( \phi(t, 0) \) should be an eigenfunction of the eigenvalue \( \lambda = 0 \), which is a contradiction. Therefore \( w(b, 0) < \pi \).
Now, since \( w(t,0) \) is a continuous function strictly increasing and \( w(a,0) = 0 \), then we conclude that \( 0 < w(b,0) < \pi \).

On the other hand, since \( w(b, \lambda) \) is increasing in \( \lambda \) and \( w(b, \lambda) \to \infty \) as \( \lambda \to \infty \), therefore there exists a first value \( \lambda_1 > 0 \) such that \( w(b, \lambda_1) = \pi \).

Since \( w \) is increasing in \( t \) we have that

\[
0 = w(a, \lambda_1) < w(t, \lambda_1) < w(b, \lambda_1) = \pi, \quad a < t < b.
\]

Then we have that \( w(t, \lambda_1) \) is not a multiple of \( \pi \), hence the solution \( \phi(t, \lambda_1) \) does not vanish in \( (a, b) \), and this function \( \phi(t, \lambda_1) \) is the eigenfunction associated to the first eigenvalue \( \lambda_1 \).

In the same way, there exists \( \lambda_2 > \lambda_1 \) such that \( w(b, \lambda_2) = 2\pi \). Then the function \( \phi(t, \lambda_2) \) is the eigenfunction associated with the eigenvalue \( \lambda_2 \) and has only one zero in \( (a, b) \), precisely because \( w \) touches only once the value \( \pi \). And the very same reasoning follows to conclude that the \( n \)th eigenfunction has exactly \( n - 1 \) zeros in \( (a, b) \). This concludes the proof of the Corollary.

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