Stochastic field theory for a Dirac particle propagating in gauge field disorder

T. Guhr, T. Wilke, and H.A. Weidenmüller
Max–Planck–Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany
(March 28, 2022)

Recent theoretical and numerical developments show analogies between quantum chromodynamics (QCD) and disordered systems in condensed matter physics. We study the spectral fluctuations of a Dirac particle propagating in a finite four dimensional box in the presence of gauge fields. We construct a model which combines Efetov's approach to disordered systems with the principles of chiral symmetry and QCD. To this end, the gauge fields are replaced with a stochastic white noise potential, the gauge field disorder. Effective supersymmetric non-linear $\sigma$-models are obtained. Spontaneous breaking of supersymmetry is found. We rigorously derive the equivalent of the Thouless energy in QCD. Connections to other low–energy effective theories, in particular the Nambu–Jona-Lasinio model and chiral perturbation theory, are found.

PACS numbers: 12.40.Ee, 11.30.Rd, 12.38.Lg, 05.45.+b

The propagation of an electron in a finite sample, such as a piece of wire, becomes diffusive due to multiple scatterings at the impurities. The ensemble of the impurities is referred to as disorder. The diffusion constant of this process determines, together with the size of the sample, a universal energy scale, the Thouless energy. This, in turn, measured in units of the single particle mean level spacing yields the dimensionless conductance of the wire, see the review in Ref. 1. Recently, it has been argued that the Gell-Mann–Oakes–Renner (GOR) relation has for QCD an analogous meaning 2–4, implying the existence of a Thouless energy in QCD. Since the Thouless energy sets the scale within which the fluctuation properties are fully of Wigner–Dyson type, it can be found by analyzing the spectral statistics. Indeed, this scale was identified in data of lattice gauge calculations 5,6.

Our goal is the description of the diffusion process for QCD by a stochastic field theory. To this end, we merge Efetov’s supersymmetric approach to disordered systems 2,3 with the principles of QCD and chiral symmetry. This extends and complements the semi–classical reasoning of Refs. 2–4. We will not use formal analogies to the GOR relation as in Refs. 2–4, rather we will derive the Thouless energy as a natural result from our stochastic model. Anticipating our analytical findings, we draw an intuitive picture in Fig. 1. Due to spontaneous breaking of chiral symmetry, the gluonic gauge fields in the Yang–Mills action generate closed gluon and fermion loops in the QCD vacuum. We may view these vacuum fluctuations as “gauge field disorder” for the propagation of a constituent quark or, more generally, of a Dirac particle in a four dimensional box of finite volume $V_4$. The multiple scattering at the vacuum fluctuations renders the motion of the quark diffusive. In this way, the pion decay constant and the constituent quark mass are generated.

For our stochastic model, we introduce the Euclidean Dirac operator in four dimensions

$$iD[u] = \begin{pmatrix} 0 & \sigma_\mu \partial_\mu - ig\sigma_\mu u_\mu(x) \\ \sigma_\mu \partial_\mu - ig\sigma_\mu u_\mu(x) & 0 \end{pmatrix},$$

where $\sigma_i$, $i = 1, 2, 3$ are the Pauli matrices, $\sigma_4 = 1_2$, and $g$ is the gauge coupling. The complex fields $u_\mu(x)$ describe the gauge field disorder. We choose them as stochastic white noise potentials with moments $\langle u_\mu(x) \rangle_u = 0$ and $\langle u_\mu(x) u_\nu(y) \rangle_u = \gamma \delta_{\mu\nu} \delta^{(4)}(x-y)$. The strength constant $\gamma$ will be determined later. Thus, the average of a quantity $\mathcal{R}[u]$ is given by

$$\langle \mathcal{R}[u] \rangle_u = \int d|u|\mathcal{R}[u] \exp \left( -\frac{1}{\gamma} \int d^4x u_\mu(x) u_\mu^*(x) \right).$$

This replaces in our model the average over the non–Abelian gauge fields in the Yang–Mills action.
Thus, we abandon gauge invariance, as can be seen from Eq. (2). We are free to do so because we are exclusively interested in stochastic features of the spectrum. The structure of the original gauge group enters in the model only through the assumption that the Dirac operator is proportional to the unit matrix \( \mathbb{1}_{N_c} \) in color space. This is fully consistent with standard models of QCD.

The eigenvalue equation \( iD[u]\psi_i = \lambda_i[u]\psi_i \) defines for each realization of the disorder potential \( u(x) \) a spectrum \( \{\lambda_i[u]\} \). We wish to discuss the spectral correlation functions of \( k \) eigenvalues \( \lambda_p, p = 1, \ldots, k \)

\[
\tilde{R}_k(\lambda_1, \ldots, \lambda_k) = \frac{1}{(2\pi)^k} \prod_{p=1}^{k} \frac{\partial}{\partial \lambda_p} \left. Z_k(\lambda + J) \right|_{\lambda = 0}
\]

(4)

where we have introduced \( \lambda = \text{diag}(\lambda_1, \ldots, \lambda_k) \otimes \mathbb{1}_2 \) and \( J = \text{diag}(J_1, \ldots, J_k) \otimes \hat{k} \) with \( \hat{k} = \text{diag}(-1, +1) \). In the supersymmetry method, the generating or partition function is expressed as a functional integral over the 4 \( \cdot 2k \) component and space time dependent superfields \( \chi(x) \) through

\[
Z_k(\lambda + J) = \langle \det g D[u; \lambda + J] \rangle_u
\]

\[
= \left\langle \int d[\chi] \exp \left( i \int d^4x \chi^\dagger(x) D[u; \lambda + J][\chi(x)] \right) \right\rangle_u
\]

(5)

with \( \det g \) denoting the superdeterminant. Here, we have defined

\[
D[u; \lambda + J] = (\lambda + J) \otimes \mathbb{1}_4 - \mathbb{1}_{2k} \otimes iD[u].
\]

(6)

We suppress a Kronecker product with \( \mathbb{1}_{N_c} \). The entries of \( \lambda + J \) can be viewed as virtual valence quark masses. Due to the chiral structure of the Dirac operator the superfields \( \chi(x) \) can be decomposed in left and right handed parts \( \chi_{L,R}(x) = P_{L,R}\chi(x) \), where \( P_{L,R} \) are the left and right handed projectors acting on the Dirac spinors. The disorder average according to Eq. (3) can now be performed and yields a field theory quartic in \( \chi(x) \). Similar to the steps taken in Ref. [10], it can be decoupled via a Hubbard–Stratonovich transformation by introducing \( 2k \times 2k \) complex supermatrix fields \( \sigma(x) \) such that the fields \( \chi(x) \) can be integrated out, namely

\[
Z_k(\lambda + J) = 2^{2k^2} \int d[\sigma]d[\sigma^\dagger] \exp \left( -F[\sigma, \sigma^\dagger; \lambda + J] \right)
\]

(7)

Both, \( \sigma(x) \) and its complex conjugate \( \sigma^\dagger(x) \) are independent variables. This reflects that \( \chi_L(x) \) and \( \chi_R(x) \) are independent degrees of freedom in the chiral symmetric phase. The free energy reads

\[
F[\sigma, \sigma^\dagger; \lambda + J] = \text{tr} \int d^4x \left( \frac{1}{4g^2} \sigma(x)\sigma^\dagger(x) \right)
\]

\[
+ \text{tr} \log \det D[\sigma, \sigma^\dagger; \lambda + J].
\]

(8)

The trace \( \text{tr} = \text{tr}_{\text{Dirac}} \text{tr}_{\text{color}} \) in front of the logarithm is taken over both Dirac and color indices. The supertrace \( \text{tr} \) is defined in the space of supermatrices. The supermatrix analogue of the Dirac operator is given by

\[
D[\sigma, \sigma^\dagger; \lambda + J] = \sigma(x) \otimes P_R + \sigma^\dagger(x) \otimes P_L
\]

\[
+ (\lambda + J) \otimes \mathbb{1}_4 - \mathbb{1}_{2k} \otimes (i\emptyset),
\]

(9)

and describes a Dirac particle coupled to the supermatrix fields \( \sigma(x) \). An important observation from Eq. (9) is that \( \sigma(x) \) is coupled only to the right handed modes of the Dirac spinors and \( \sigma^\dagger(x) \) only to the left handed modes. This implies chiral symmetry, since \( \sigma(x) \) and \( \sigma^\dagger(x) \) are independent variables.

In contrast to supersymmetry in standard high energy physics, neither \( \chi(x) \) nor \( \sigma(x) \) directly represent physical particles, such as quarks or mesons. Rather, these fields are the degrees of freedom in the stochastic model. Nevertheless, the functionals defined in Eqs. (8) and (9) possess certain symmetries which can be interpreted as the supersymmetric version of particle symmetries known from QCD. The Lagrangian in Eq. (3) is invariant under the chiral transformations

\[
\chi_L(x) \rightarrow u_L \chi_L(x), \quad u_L u_L^\dagger = u_L^\dagger u_L = \mathbb{1}_{2k},
\]

\[
\chi_R(x) \rightarrow u_R \chi_R(x), \quad u_R u_R^\dagger = u_R^\dagger u_R = \mathbb{1}_{2k},
\]

(10)

with \( u_{L,R} \in \text{SU}(k/k) \). We notice that \( \text{U}(k/k) \) is likely to lead to axial anomalies which we do not discuss here. The corresponding supersymmetric version of particle symmetries is \( \text{SU}_L(k/k) \otimes \text{SU}_R(k/k) \). The generating functional (10) exhibits again chiral invariance

\[
\sigma(x) \rightarrow u_L \sigma(x) u_R \quad \text{and} \quad \sigma^\dagger(x) \rightarrow u_R^\dagger \sigma^\dagger(x) u_L^\dagger,
\]

(11)

if the explicit symmetry breaking is turned off, i.e. \( \lambda + J = 0 \). This is not true for non-zero values of the virtual quark masses, \( \lambda + J \neq 0 \) which explicitly breaks chiral symmetry. However, there is a remnant symmetry. Consider the case of \( k \) degenerate imaginary valence quark masses, \( \lambda_1 = \ldots = \lambda_k = \lambda \), i.e. \( \lambda + J = \lambda \mathbb{1}_{2k} \), where the sources are set to zero. Then the functional is invariant under the transformation

\[
\sigma(x) \rightarrow u \sigma(x) u^\dagger \quad \text{and} \quad \sigma^\dagger(x) \rightarrow u^\dagger \sigma^\dagger(x) u^\dagger,
\]

(12)

with \( u \in \text{SU}(k/k) \).
We observe a close relation between the one–point function in the microscopic regime (near zero virtuality) and the two–point function in the bulk (far away from zero virtuality). In both cases, the symmetry breaking is controlled by a single parameter, although the symmetry groups are different. In the bulk, the symmetry is broken by non–degenerate eigenvalues $\lambda_1$ and $\lambda_2$, and the symmetry breaking parameter is $\omega = \lambda_2 - \lambda_1$. In the microscopic region, the symmetry is already broken by a non–zero eigenvalue $\lambda = \lambda_1$.

In disordered systems, to make analytical progress, one considers the field theory via a saddle point approximation in the limit of weak disorder, i.e. for a relatively low density of impurities. This corresponds to a Born approximation of the self–energy where diagrams with crossed interaction lines are neglected [7]. Here, we proceed similarly by a saddle point approximation of the partition function in the limit of weak gauge field disorder. This coincides with a $1/N_c$ expansion, which is a standard asymptotic limit in many QCD models. These two limits coincide because for $N_c \to \infty$ and $N_c g^2 = \text{const}$, the relevant diagrams contributing in leading order to the self–energy within our model are analogous to the ones in disordered systems. This shows the equivalence of the $1/N_c$ expansion and the weak disorder limit. From the saddle point condition $\delta F[\sigma, \sigma^\dagger; \lambda + J] = 0$, we find the self–consistent matrix equation in momentum representation

$$\Sigma_0^\dagger = -8\gamma N_c g^2 (\Sigma_0 + \Sigma_0^\dagger + \Xi)$$

$$\times \int \frac{d^4 p}{(2\pi)^4} \left[ (\Sigma_0 + \Xi)(\Sigma_0^\dagger + \Xi) - \Pi_{2k} \otimes \Pi_{4p} \right]^{-1}$$

(13)

with $\Xi = (\lambda + J) \otimes \mathbb{1}_4$ and $\Sigma_0 = \sigma_0 \otimes P_R + \sigma_0^\dagger \otimes P_L$, where $\sigma_0$ and $\sigma_0^\dagger$ are the fields at the saddle point. A cutoff $\Lambda_{\text{cut}}$ has to be introduced to regularize the divergent momentum integration. For scalar fields, the saddle point equation (13) coincides with the gap equation of the Nambu–Jona-Lasinio (NJL) model [13]. The NJL model is a theory for chiral fermions coupled via a four–point interaction. The gap equation describes the self–energy of a quark calculated in the self–consistent Hartree approximation obtained from a $1/N_c$ expansion [8]. For vanishing virtual quark masses, $\lambda + J = 0$, there exists only a non–trivial solution, indicating chiral symmetry breaking, if the coupling exceeds a critical value $\gamma g^2 > 2\pi^2/N_c A^2_{\text{cut}}$. For non–vanishing masses, however, one complex solution always exists, which we denote by $M_\lambda$. Since the spectral density per four–volume is given by $\nu(\lambda) = \text{Im} R(\lambda)$, see Eq. (3), the value of the constant $\gamma$ in Eq. (3) is now fixed. From Eq. (13) we find

$$\gamma = \text{Im} M_\lambda / (\pi^2 \nu(\lambda)).$$

In contrast to the situation in disordered systems, we have to distinguish the microscopic and the bulk region. In the latter, the symmetries of the saddle point solution are still the same as given in Eq. (12). A solution of the saddle point equation is

$$\sigma_0 = \sigma_0^\dagger = \text{Im} M_\lambda^* L,$$

(14)

where $L = \mathbb{1}_4 \otimes \hat{k}$. On the other hand, in the microscopic region the chiral symmetry defined in Eqs. (11) to (11) is not preserved at the saddle point. The solutions read

$$\sigma_0(x) = i M^* \tilde{U}_0(x) \quad \text{and} \quad \sigma_0^\dagger(x) = i M^* \tilde{U}_0^\dagger(x)$$

(15)

where $\tilde{U}_0(x)$ is $2 \times 2$ unitary supermatrix field and $M^*$ is the non–trivial scalar solution of the gap equation (13) for $\lambda + J = 0$. We conclude that chiral symmetry is spontaneously broken such that

$$\text{SU}_R(k/k) \otimes \text{SU}_L(k/k) \longrightarrow \text{SU}(k/k).$$

(16)

This is the supersymmetric analogue of spontaneous breaking of chiral symmetry in our stochastic model.

By integrating out the quadratic fluctuations around the saddle points, we obtain, first, a theory for the two–point correlations in the bulk and, second, a theory for the one–point function in the microscopic region. In both cases, the zeroth order term is expanded up to linear order in the symmetry breaking parameter, i.e. $\omega$ in the bulk and $\lambda$ in the microscopic region, respectively. Moreover, we restrict ourselves to fields varying slowly in space–time, allowing for a gradient expansion of the quadratic fluctuations around the saddle points. In the bulk we obtain for the generating functional of the two–point function

$$Z_2(\lambda, \omega, J) = \int d\mu(Q) \exp (-F[Q; \lambda, \omega, J])$$

(17)

with

$$F[Q; \lambda, \omega, J] = \frac{\pi}{2} \nu(\lambda) \text{tr} \int d^4 x \left(D(\lambda) \partial_\mu Q(x) \partial_\mu Q(x) - 2i \left(\omega + i\epsilon \right) \partial_\mu J(x) \partial_\mu Q(x) \right),$$

(18)

where $\lambda = (\lambda_1 + \lambda_2)/2$. The fields $Q(x)$ are the Goldstone modes of the saddle point manifold. They are in the coset manifold $U(1,1/2)/(U(1) \otimes U(1,1))$. Here, $L$ is the symmetry breaking matrix and $i\epsilon$ a proper imaginary increment to ensure convergence. This formally extends Efetov’s result [7] to four space time dimensions. In particular, the coset spaces coincide. Chiral symmetry is not present. This is in agreement with its explicit breaking by the virtual current mass $iL$. The diffusion coefficient $D(\lambda)$, however, a main object of our interest, must be different from disordered systems because we start out from the Dirac equation. It is given as the second moment

$$\text{tr} \int d^4 y (x - y)^2 S(x, y; \lambda + i\text{Im} M_\lambda^* ) S(y, x; \lambda - i\text{Im} M_\lambda^* )$$

$$= \frac{2\pi \nu(\lambda) D(\lambda)}{(\text{Im} M_\lambda^*)^2}$$

(19)
where $S(x, y; \lambda - i\text{Im}M^2_X)$ is the Green function to the mean field Dirac operator $\lambda + i\text{Im}M^2_X - i\hat{\theta}$.

From the competition between the kinetic and the symmetry breaking term one can infer the Thouless energy in the bulk. In a finite volume $V_4$, the longest wavelength which fits inside the Euclidean box is given by the inverse of the linear extension of the box, i.e. $1/(V_4)^{1/4}$. Thus, the kinetic energy cannot be smaller than $\mathcal{D}(\lambda)/\sqrt{V_4}$. For smaller energies, the wavelength exceeds the extension of the box and does not resolve details inside it. In that case, the fields become constant and the functional is dominated by the constant modes. Hence, only the symmetry breaking term gives a contribution. Then the level statistics is of Wigner–Dyson type. If the energy exceeds this scale, the kinetic term becomes more and more important, leading to deviations from the Wigner–Dyson statistics. Thus, the equivalent of the Thouless energy in the bulk is given by

$$\lambda^\text{bulk}_c \sim \mathcal{D}(\lambda)/\sqrt{V_4}.$$  \hspace{1cm} (20)

In the chiral limit, $\lambda \to 0$, the diffusion coefficient (19) in momentum representation reduces to

$$\mathcal{D}(0) = \frac{8N_c(M^*)^2}{|\langle \bar{\psi}\psi \rangle|} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + (M^*)^2)^2}.$$  \hspace{1cm} (21)

where $\langle \bar{\psi}\psi \rangle$ is the chiral condensate. Obtained from the Banks–Casher relation, $\langle \bar{\psi}\psi \rangle = -\pi \lim_{V_4 \to \infty} \nu(0)$. For this derivation, we used only the basic assumption (11) and chiral symmetry. Low energy theorems of QCD imply that the integral in Eq. (21) turns out to be proportional to the square of the pion decay constant $f_\pi^2$,

$$f_\pi^2 = 4N_c(M^*)^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + (M^*)^2)^2},$$  \hspace{1cm} (22)

for a detailed discussion see Ref. [8]. With this additional information, we arrive at

$$\mathcal{D}(0) = 2f_\pi^2/|\langle \bar{\psi}\psi \rangle|.$$  \hspace{1cm} (23)

In Refs. [2, 4], this formula was obtained from the GOR relation by using formal analogies and semi–classical considerations. Here, we gave a rigorous derivation of the diffusion constant (21) from our stochastic model. Finally, we obtain for the generating functional of the one–point function in the microscopic regime

$$Z_1(\lambda, J) = \int d\mu(U) \exp\left(-F[U; \lambda + J]\right),$$  \hspace{1cm} (24)

with

$$F[U; \lambda + J] = |\langle \bar{\psi}\psi \rangle| \text{tr} \int dx \left(\frac{f_\pi^2}{|\langle \bar{\psi}\psi \rangle|} \partial_\mu U(x) \partial_\mu U^\dagger(x) - \frac{i}{2}(\lambda + J\tilde{K})(U(x) + U^\dagger(x))\right).$$  \hspace{1cm} (25)

The supermatrices $U(x)$ are $2 \times 2$ unitary. Again, the functional is dominated by the constant modes for energies smaller than $f_\pi^2/|\langle \bar{\psi}\psi \rangle|\sqrt{V_4})$. We conclude that the equivalent of a Thouless energy in the microscopic region is given by

$$\lambda^\text{micro}_c \sim \mathcal{D}(0)/\sqrt{V_4} = 2f_\pi^2/|\langle \bar{\psi}\psi \rangle|\sqrt{V_4}.$$  \hspace{1cm} (26)

We note that the microscopic functional reduces, up to irrelevant numerical factors, to the partially quenched chiral perturbation theory (pqCPT). This is consistent with the considerations of Refs. [4, 5], were pqCPT was used to identify the equivalent of the Thouless energy in the Dirac spectrum.

In summary, we presented a stochastic field theory by merging Efetov’s approach to disordered systems with the standard principles of QCD and chiral symmetry. Due to the latter, we obtain different theories in the microscopic and the bulk region which connect to the NJL model and pqCPT. The Thouless energy is a natural consequence of our model. In conclusion, we wish to make a proposal: since we now have a detailed and, importantly, quantitative description of the spectral fluctuations in QCD, it would be worthwhile to use this insight as an input for lattice QCD, either by constructing effective models or by modifying the algorithms accordingly. It would be highly desirable if this made lattice QCD more efficient.

We thank R. Baltin, S.P. Klevansky, E. Lutz, D. Toublan, and J.J.M. Verbaarschot for fruitful discussions. TG acknowledges financial support from the Heisenberg foundation.

[1] T. Guhr, A. Müller–Groeling and H.A. Weidenmüller, Phys. Rep. 299, 189 (1998).
[2] R.A. Janik, M.A. Nowak, G. Papp, and I. Zahed, Phys. Rev. Lett. 81, 264 (1998).
[3] J.C. Osborn and J.J.M. Verbaarschot, Phys. Rev. Lett. 81, 268 (1998).
[4] J.C. Osborn and J.J.M. Verbaarschot, Nucl. Phys. B525, 738 (1998).
[5] M.E. Berbenni-Bitsch et al., Phys. Lett. B438, 14 (1998).
[6] T. Guhr, J.-Z. Ma, S. Meyer, and T. Wilke, Phys. Rev. D59, 054501 (1999).
[7] K.B. Efetov, Adv. Phys. 32, 53 (1983).
[8] S.P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).
[9] T. Schäfer and E. Shuryak, Rev. Mod. Phys. 70, 323 (1998).
[10] T. Guhr and T. Wettig, Nucl. Phys. B506, 589 (1997).
[11] G. ’t Hooft, Nucl. Phys. B75, 461 (1974).
[12] E. Witten, Nucl. Phys. B160, 57 (1979).
[13] V. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); 124, 246 (1961).
[14] J.C. Osborn, D. Toublan, and J.J.M. Verbaarschot, Nucl. Phys. B540, 317 (1999).
[15] P.H. Damgaard, J.C. Osborn, D. Toublan, J.J.M. Verbaarschot, Nucl. Phys. B547, 305 (1999).