SPECTRAL RADIUS ALGEBRAS OF WCE OPERATORS

Y. ESTAREMI AND M. R. JABBARZADEH

Abstract. In this paper, we investigate the spectral radius algebras related to the weighted conditional expectation operators on the Hilbert spaces $L^2(F)$. We give a large classes of operators on $L^2(F)$ that have the same spectral radius algebra. As a consequence we get that the spectral radius algebras of a weighted conditional expectation operator and its Aluthge transformation are equal. Also, we obtain an ideal of the spectral radius algebra related to the rank one operators on the Hilbert space $H$. Finally we get that the operator $T$ majorizes all closed range elements of the spectral radius algebra of $T$, when $T$ is a weighted conditional expectation operator on $L^2(F)$ or a rank one operator on the arbitrary Hilbert space $H$.

1. Introduction

Let $(X, F, \mu)$ be a complete $\sigma$-finite measure space. All sets and functions statements are to be interpreted as holding up to sets of measure zero. For a $\sigma$-subalgebra $A$ of $F$, the conditional expectation operator associated with $A$ is the mapping $f \to E_A f$, defined for all non-negative $f$ as well as for all $f \in L^2(F) = L^2(X, F, \mu)$, where $E_A f$ is the unique $A$-measurable function satisfying $\int_A (E_A f) d\mu = \int_A f d\mu$, for all $A \in A$. We will often write $E$ for $E^A$. The mapping $E$ is a linear orthogonal projection from $L^2(F)$ onto $L^2(A)$. For more details on the properties of $E$ see [14].

We continue our investigation about the class of bounded linear operators on the $L^p$-spaces having the form $M_w E M_u$, where $E$ is the conditional expectation operator, $M_w$ and $M_u$ are (possibly unbounded) multiplication operators and it is called weighted conditional expectation operator. Our interest in operators of the form $M_w E M_u$ stems from the fact that such forms tend to appear often in the study of those operators related to conditional expectation. Weighted conditional expectation operators appeared in [3], where it is shown that every contractive projection on certain $L^1$-spaces can be decomposed into an operator of the form $M_w E M_u$ and a nilpotent operator. For

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more strong results about weighted conditional expectation operators one can see [3, 9, 11, 13]. In these papers one can see that a large classes of operators are of the form of weighted conditional expectation operators.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null-space and range of an operator $T$, respectively. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be invariant for an operator $T \in \mathcal{B}(\mathcal{H})$ if $TM \subseteq \mathcal{M}$. The collection of all invariant subspaces of $T$ is a lattice and it is denoted by $Lat(T)$. If $\mathcal{M}$ is invariant for all operators commute with $T$, then it is called a hyperinvariant subspace for $T$. The description $Lat(T)$ is an open problem. Some author describes $Lat(T)$ in the special case of $T$. In [12], Lambert and Petrovic introduced a modified version of a class of operator algebras that is called spectral radius algebras. Since a spectral radius algebra related to an operator $T \in \mathcal{B}(\mathcal{H})$ ($\mathcal{B}_T$) contains all operators that commute with $T$ ($\{T\}'$), then the invariant subspaces of $\mathcal{B}_T$ are hyperinvariant subspaces of $T$. In [12], the authors established several sufficient conditions for $\mathcal{B}_T$ to have a nontrivial invariant subspace. When $T$ is compact the results of [12] generalizes the Lomonosov’s theorem. In [2], the authors demonstrated that for a subclasses of normal operators $\mathcal{B}_T$ has a nontrivial invariant subspace. Spectral radius algebras for complex symmetric operators are discussed in [10].

In this paper we investigate the spectral radius algebras related to the weighted conditional expectation operators on the Hilbert spaces $L^2(\mathcal{F})$. We will show that there are lots of operators on $L^2(\mathcal{F})$ such as $T$ with $\mathcal{B}_T \neq \{T\}'$. In addition, we obtain an ideal of the spectral radius algebra related to the rank one operators on the Hilbert space $\mathcal{H}$. Finally we get that the operator $T$ majorizes all closed range elements of the spectral radius algebra of $T$, when $T$ is a weighted conditional expectation operator on $L^2(\mathcal{F})$ or a rank one operator on the arbitrary Hilbert space $\mathcal{H}$.

2. SPECTRAL RADIUS ALGEBRAS

For notation and basic terminology concerning spectral radius algebras, we refer the reader to [3, 12].

Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ and let $r(T)$ be the spectral radius of $T$. For $m \geq 1$ we define

$$R_m(T) = R_m := \left( \sum_{n=0}^{\infty} d_n^m T^n T^* T^n \right)^{\frac{1}{2}},$$

(2.1)
where $d_m = \frac{1}{1/m+r(T)}$. Since $d_m \uparrow 1/r(T)$, the sum in \((2.1)\) is norm convergent and for each $m$, $R_m$ is well defined, positive and invertible. The spectral radius algebra $\mathcal{B}_T$ of $T$ consists of all operators $S \in \mathcal{B}({\mathcal{H}})$ such that
\[
\sup_{m \in \mathbb{N}} \| R_m S R_m^{-1} \| < \infty.
\]
$\mathcal{B}_T$ is an algebra and it contains all operators commute with $T$. Throughout this section we assume that $w, u \in D(E) := \{ f \in L^0(\mathcal{F}) : E(|f|) \in L^0(\mathcal{A}) \}$. Now we recall the definition of weighted conditional expectation operators on $L^2(\mathcal{F})$.

**Definition 2.1.** Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and let $\mathcal{A}$ be a $\sigma$-subalgebra of $\mathcal{F}$ such that $(X, \mathcal{A}, \mu_\mathcal{A})$ is also $\sigma$-finite. Let $E$ be the conditional expectation operator relative to $\mathcal{A}$. If $u, w \in L^0(\mathcal{F})$, the spaces of $\mathcal{F}$-measurable functions on $X$, such that $uf$ is conditionable \[?\] and $wE(uf) \in L^2(\mathcal{F})$ for all $f \in \mathcal{D} \subseteq L^2(\mathcal{F})$, where $\mathcal{D}$ is a linear subspace, then the corresponding weighted conditional expectation (or briefly WCE) operator is the linear transformation $M_w EM_u : \mathcal{D} \to L^2(\mathcal{F})$ defined by $f \to wE(uf)$.

As was proved in \[8\] we have an equivalent condition for boundedness of the weighted conditional expectation operators $M_w EM_u$ on $L^2(\mathcal{F})$ as the next theorem.

**Theorem 2.2.** The operator $T = M_w EM_u : L^2(\mathcal{F}) \to L^2(\mathcal{F})$ is bounded if and only if $(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}}) \in L^\infty(\mathcal{A})$, in this case $\|T\| = \|(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}})\|_\infty$.

Let $T = M_w EM_u$ be a bounded operator on $L^2(\mathcal{F})$. Direct computations shows that for every $n \in \mathbb{N}$ (natural numbers) we have
\[
T^n f = (E(uw))^{n-1}wE(uf);
\]
\[
T^n f = (E(uw))^{n-1}\bar{u}E(\bar{w}f).
\]

Since $R_m = R_m(M_w EM_u)$ is positive and invertible operator, we obtain
\[
R_m = \left( I + M_{E(|w|^2)\sum_{n=1}^{\infty} d_n^2 |E(uw)|^{2(n-1)}} M_w EM_u \right)^{\frac{1}{2}}.
\]
It is easy to see that the following equality holds almost every where on $X$.
\[
\sum_{n=1}^{\infty} d_n^2 |E(uw)|^{2(n-1)} = \frac{d_m^2}{1 - d_m^2 |E(uw)|^2}.
\]
If we set
\[
v_m = \frac{d_m^2 |E(|w|^2)}}{1 - d_m^2 |E(uw)|^2},
\]
then we have
\[
R_m = (I + M_{v_m} EM_u)^{\frac{1}{2}}.
\]
Y. ESTAREMI AND M. R. JABBARZADEH

By an elementary technical method we can compute the inverse of $R_m$ as follow:

$$R_m^{-1} = \left( I + M_{v_mE(|u|^2)^{-1}} EM_u \right)^{\frac{1}{2}}.$$

Here we recall a fundamental lemma in operator theory.

**Lemma 2.3.** Let $T$ be a bounded operator on the Hilbert space $\mathcal{H}$ and $\lambda \geq 0$. Then we have

$$\|\lambda I + T^*T\| = \lambda + \|T^*T\| = \lambda + \|T\|^2.$$

Specially, if $T$ is a positive operator, then $\|\lambda I + T\| = \lambda + \|T\|$. 

**Proof.** It is an easy exercise. \qed

From now on, we assume that $E(|u|^2) \in L^\infty(\mathcal{A})$. Now we characterize the spectral radius algebra corresponding to the WCE operator $M_wEM_u$ in the next theorem.

**Theorem 2.4.** Let $S \in \mathcal{B}(L^2(\mathcal{F}))$. Then $S \in \mathcal{B}_{M_wEM_u}$ if and only if $\mathcal{N}(EM_u)$ is invariant under $S$.

**Proof.** Since $R_m$ and $R_m^{-1}$ are positive operators and $(EM_u)^* = M_aE$, then by Lemma 2.3 and Theorem 2.2 we have

$$\|R_m\|^2 = \|R_m\| = 1 + \|E(|u|^2)v_m\|_{\infty}$$

and

$$\|R_m^{-1}\|^2 = \|R_m^{-1}\| = 1 + \|\frac{E(|u|^2)v_m}{v_mE(|u|^2) - 1}\|_{\infty}.$$ 

If we decompose $L^2(\mathcal{F})$ as a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$, in which

$$\mathcal{H}_2 = \mathcal{N}(EM_u) = \{f \in L^2(\mathcal{F}) : E(uf) = 0\}$$

and

$$\mathcal{H}_1 = \mathcal{H}_2^\perp = \bar{u}L^2(\mathcal{A}),$$

then the corresponding block matrix of $R_m$ is

$$R_m = \begin{pmatrix} M_{\frac{q_m}{2}} & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad R_m^{-1} = \begin{pmatrix} M_{\frac{q_m}{2}} & 0 \\ 0 & I \end{pmatrix},$$

where $q_m = 1 + v_mE(|u|^2)$. Notice that for $m > m'$ we have $q_m \geq q_{m'}$ and $\|q_m\|_{\infty} \to \infty$ as $m \to \infty$. If $S \in \mathcal{B}(L^2(\mathcal{F}))$ say $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, the block matrix with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$, then

$$R_mSR_m^{-1} = \begin{pmatrix} M_{\frac{q_m}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_{\frac{q_m}{2}} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} M_{\frac{q_m}{2}}X & M_{\frac{q_m}{2}}Y \\ ZM_{\frac{q_m}{2}} & WM_{\frac{q_m}{2}} \end{pmatrix}.$$
Since \( \|M_{\{q_m\}}^{\frac{1}{2}}XM_{\{q_m\}}^{\frac{1}{2}}\| \leq \|X\| \), then we get that \( \sup_m \|R_mSR_m^{-1}\| < \infty \) if and only if \( \sup_m \|M_{\{q_m\}}^{\frac{1}{2}}Y\| < \infty \). Direct computations shows that \( \sup_m \|M_{\{q_m\}}^{\frac{1}{2}}Y\| < \infty \) if and only if \( Y = 0 \). This means that \( \mathcal{H}_2 \) is an invariant subspace for \( S \).

Therefore by Theorem 2.3 we get that there are many different operators that have the same spectral radius algebra.

**Corollary 2.5.** Let \( w, w', u \in \mathcal{D}(E) \). If \( M_wEM_u \) and \( M_{w'}EM_u \) are bounded operator on the Hilbert space \( L^2(\mathcal{F}) \), then \( \mathcal{B}_{M_wEM_u} = \mathcal{B}_{M_{w'}EM_u} \).

Also in the next corollary we have a sufficient condition for \( \mathcal{B}_{M_wEM_u} \) to be equal to \( \mathcal{B}(L^2(\mathcal{F})) \).

**Corollary 2.6.** If \( \mathcal{N}(EM_u) = \{0\} \), then \( \mathcal{B}_{M_wEM_u} = \mathcal{B}(L^2(\mathcal{F})) \).

In the next Proposition we find some special elements of \( \mathcal{B}_{M_wEM_u} \).

**Proposition 2.7.** If \( a \in L^0(\mathcal{A}) \) such that \( a \geq 0 \) and \( M_{\bar{a}}EM_u \in \mathcal{B}(L^2(\mathcal{F})) \), then \( M_{a\bar{a}}EM_u \in \mathcal{B}_{M_wEM_u} \).

**Proof.** Since \( R_m = (I + M_{\bar{a}}EM_u)^{\frac{1}{2}} \) and \( v_m = \frac{d^2_mE(|u|^2)}{1 - d^2_mE(|uw|)^2} \) is an \( \mathcal{A} \)-measurable function, it holds that \( R_mM_{a\bar{a}}EM_u = M_{a\bar{a}}EM_uR_m \). Therefore we have \( \|R_mM_{a\bar{a}}EM_uR_m^{-1}\| = \|M_{a\bar{a}}EM_u\| \), and so we get that \( M_{a\bar{a}}EM_u \in \mathcal{B}_{M_wEM_u} \).

Every operator \( T \) on a Hilbert space \( \mathcal{H} \) can be decomposed into \( T = U|T| \) with a partial isometry \( U \), where \( |T| = (T^*T)^{\frac{1}{2}} \). \( U \) is determined uniquely by the kernel condition \( \mathcal{N}(U) = \mathcal{N}(|T|) \). Then this decomposition is called the polar decomposition. The Aluthge transformation \( \tilde{T} \) of the operator \( T \) is defined by \( \tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2} \). Here we recall that the Aluthge transformation of \( T = M_wEM_u \) is

\[
\tilde{T}(f) = \frac{\chi_{z_1E(uw)}}{E(|u|^2)}\bar{u}E(uf), \quad f \in L^2(\mathcal{F}),
\]

in which \( z_1 = z(E(|u|^2)) \) (see [8]). Thus \( \tilde{T} = M_{w'}EM_{u'} \) where \( w' = \frac{E(uw)\bar{a}\chi_{z_1}}{E(|u|^2)} \) and \( u' = u \). We recall that \( r(M_wEM_u) = \|E(uw)\|_\infty \) (see [9]). Direct computations shows that \( E(u'w') = E(uw) \). Hence \( r(T) = r(\tilde{T}) \).

Hence by using Proposition 2.7 we have the next corollary.

**Corollary 2.8.** If \( w \) and \( u \) are positive measurable functions, then \( \tilde{T} \in \mathcal{B}_T \) where \( T = M_wEM_u \).

By the proof of Proposition 2.7 we get that the commutant of \( M_wEM_u \) (in symbol \( \{M_wEM_u\}' \)) is a proper subset of \( \mathcal{B}_{M_wEM_u} \) when \( w, u \) are positive and \( w \neq u \). In the next theorem we get that \( \mathcal{B}_T = \mathcal{B}_{\tilde{T}} \) when \( T = M_wEM_u \) and \( w, u \geq 0 \).

**Corollary 2.9.** If \( T = M_wEM_u \) and \( w, u \geq 0 \), then \( \mathcal{B}_T = \mathcal{B}_{\tilde{T}} \).
Recall that for \( f, g \in L^2(\mathcal{F}) \) we can define a rank one operator \( f \otimes g \) on \( L^2(\mathcal{F}) \) by the action \( (f \otimes g)(h) = \langle h, g \rangle f \) for every \( h \in L^2(\mathcal{F}) \), in which \( \langle , \rangle \) is the inner product of the Hilbert space \( L^2(\mathcal{F}) \). In the next proposition we give some conditions under which a rank one operator \( \sigma \) corresponding to \( L \) on \( \langle \cdot, \cdot \rangle \) is an operator \( \sigma \) for which \( \sigma = \sigma^* \). In Remark 2.11. Let \( T = EMu \in \mathcal{B}(L^2(\mathcal{F})) \), \( u \in L^\infty(\mathcal{A}) \) and let \( \mathcal{A}, \mathcal{B} \) be \( \sigma \)-subalgebras of \( \mathcal{F} \) such that \( \mathcal{A} \subseteq \mathcal{B} \). If \( E = E^A \) and \( S \) is an operator for which \( TS = E^B ST \), then \( S \in \mathcal{B}_T \).

**Corollary 2.12.** If \( T = M_uEM_u \in \mathcal{B}(L^2(\mathcal{F})) \) and \( a \in L^\infty(\mathcal{A}) \), then \( M_a \in \mathcal{B}_T \).

Let \( \mathcal{H} \) be a Hilbert space and \( T \in \mathcal{B}(\mathcal{H}) \). Here we recall the definition of \( Q_T \), that is defined in [12], as follows:

\[
Q_T = \{ S \in \mathcal{B}(\mathcal{H}) : \| R_mSR_m^{-1} \| \to 0 \}.
\]

In the next theorem we illustrate \( Q_T \) when \( T = M_uEM_u \in \mathcal{B}(L^2(\mathcal{F})) \).

**Theorem 2.13.** Let \( T = M_uEM_u \) and \( S \in \mathcal{B}(L^2(\mathcal{F})) \). Then \( S \in Q_T \) if and only if \( \mathcal{N}(EM_u) \) is invariant under \( S \) and \( \mathcal{N}(EM_u) \subseteq \mathcal{N}(S) \).

**Proof.** Let \( S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \), the block matrix with respect to the decomposition \( \mathcal{H}_1 \oplus \mathcal{H}_2 \), in which \( \mathcal{H}_2 = \mathcal{N}(EM_u) \) and \( \mathcal{H}_1 = \mathcal{H}_2^\perp \). So similar to the proof of Theorem 2.4 we have

\[
R_mSR_m^{-1} = \begin{pmatrix} \frac{1}{\alpha_m}XM_{(q_m)^\perp}^T & \frac{1}{\alpha_m}M_{(q_m)^\perp}^TY \\ ZM_{(q_m)^\perp}^- & W \end{pmatrix}.
\]
Hence \( \|W\| \leq \|R_m SR_m^{-1}\| \). Since \( S \in Q_T \), then \( Y = 0 \) and \( \|W\| = 0 \). This means that \( \mathcal{H}_2 \) is invariant under \( S \) and \( PSP = 0 \) in which \( P = P_{\mathcal{H}_2} \). Therefore \( SP = PSP = 0 \), and so \( \mathcal{H}_2 \subseteq \mathcal{N}(S) \). Conversely, if \( \mathcal{N}(EM_u) \) is invariant under \( S \) and \( \mathcal{N}(EM_u) \subseteq \mathcal{N}(S) \), then we get that \( W = Y = 0 \) and \( R_m SR_m^{-1} = \begin{pmatrix} M_{(q_m)\frac{1}{2}}X M_{(q_m)\frac{1}{2}} & 0 \\ Z M_{(q_m)\frac{1}{2}} & 0 \end{pmatrix} \).

Hence
\[
\|R_m SR_m^{-1}\| \leq \|M_{(q_m)\frac{1}{2}}X M_{(q_m)\frac{1}{2}}\| + \|Z M_{(q_m)\frac{1}{2}}\|.
\]

Since \( \|M_{(q_m)\frac{1}{2}}\| = \|(q_m)^{\frac{1}{2}}\|_\infty \to 0 \), then \( \|R_m SR_m^{-1}\| \to 0 \) when \( m \to \infty \). This completes the proof. □

Now by using [12, Theorem 2.6] and some information about WCE operators we have an equivalent condition for the spectral radius algebra of a WCE operator to be equal to \( \mathcal{B}(L^2(\mathcal{F})) \).

**Proposition 2.14.** If \( T = M_w EM_u \), then \( \mathcal{B}_T = \mathcal{B}(L^2(\mathcal{F})) \) if and only if
\[
\sup_m (\|E(|u|^2)v_m\|_\infty + \|\frac{E(|u|^2)v_m}{v_m E(|u|^2) - 1}\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)) < \infty,
\]
where \( v_m = \frac{d^2 E(|u|^2)}{1 - d^2 E(uw)^2} \).

**Proof.** It is a direct consequence of [12, Theorem 2.6] and some information of the proof of Theorem 2.4. □

By using Proposition 2.14 and some results of [2] we have an equivalent condition for the WCE operator \( M_w EM_u \) to be a constant multiple of an isometry.

**Theorem 2.15.** If \( T = M_w EM_u \) is a bounded operator on the Hilbert space \( L^2(\mathcal{F}) \), then \( T \) is a constant multiple of an isometry if and only if
\[
\sup_m (\|E(|u|^2)v_m\|_\infty + \|\frac{E(|u|^2)v_m}{v_m E(|u|^2) - 1}\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)) < \infty,
\]
where \( v_m = \frac{d^2 E(|u|^2)}{1 - d^2 E(uw)^2} \).

**Proof.** It is a direct consequence of [2, Theorem 2.7] and Proposition 2.14. □

Now in the next theorem we obtain some sufficient conditions for \( \mathcal{B}_{M_w EM_u} \) to a nontrivial invariant subspace.

**Theorem 2.16.** If the measure space \( (X, \mathcal{A}, \mu) \) is not a non-atomic measure space and \( E(uw) = 0 \), then \( \mathcal{B}_{M_w EM_u} \) has a nontrivial invariant subspace.
Proof. Since $E(uw) = 0$ then $M_wEM_u$ is quasinilpotent. Also since the $\sigma$-algebra $\mathcal{A}$ has at least one atom, then we have a compact multiplication operator $M_a$ for some $a \in L^\infty(\mathcal{A})$. Hence by Corollary 2.12 we have $M_a \in \mathcal{B}_{M_aEM_a}$. Moreover by using \cite{[12], Lemma 3.1} we get that $M_wEM_u \in Q_{M_wEM_u}$. Therefore by \cite{[12], Theorem 3.4} we get the proof. \hfill $\Box$

Here we give a remark on \cite{[12], Proposition 2.8} as follows:

Remark 2.17. For the unit vectors $u, v, w$ of the Hilbert space $\mathcal{H}$ we have $\mathcal{B}_{u \otimes w} = \mathcal{B}_{v \otimes w}$.

In the next theorem we describe $Q_{u \otimes v}$ for a rank one operator $u \otimes v$ in which $u, v$ are in the Hilbert space $\mathcal{H}$.

Theorem 2.18. Let $\mathcal{H}$ be a Hilbert space and $S \in \mathcal{B}(\mathcal{H})$. If $u, v \in \mathcal{H}$, then $S \in Q_{u \otimes v}$ if and only if $S = (I - P)SP$, where $P = P_{\mathcal{H}_1}$ and $\mathcal{H}_1$ is the one-dimensional space spanned by $v$.

Proof. As was computed in \cite{[12], Proposition 2.8} we have

$$R_m^2 = I + \frac{d_m^2}{1 - d_m^2}v \otimes v,$$

in which $r = r(u \otimes v) = |\langle u, v \rangle|$. Let $\lambda_m = \sqrt{1 + \frac{d_m^2}{1 - d_m^2}}$. If $\mathcal{H}_1$ is the one-dimensional space spanned by $v$ and $\mathcal{H}_2 = \mathcal{H}_1^\perp$. For $S \in \mathcal{B}(\mathcal{H})$, we have the corresponding block matrix of $R_m$, $R_m^{-1}$ and $S$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ as follows:

$$R_m = \begin{pmatrix} M_{\lambda_m} & 0 \\ 0 & I \end{pmatrix}, \quad R_m^{-1} = \begin{pmatrix} M_{\lambda_m}^{-1} & 0 \\ 0 & I \end{pmatrix}$$

and

$$S = \begin{pmatrix} PSP & PS(I - P) \\ (I - P)SP & (I - P)S(I - P) \end{pmatrix}.$$

Therefore, we have

$$R_mSR_m^{-1} = \begin{pmatrix} PSP & M_{\lambda_m}PS(I - P) \\ M_{\lambda_m}(I - P)SP & (I - P)S(I - P) \end{pmatrix}.$$

If $S \in Q_{u \otimes v}$, then $S \in \mathcal{B}_{u \otimes v}$. Hence by \cite{[12], Proposition 2.8} we obtain $PS(I - P) = 0$. Since $S \in Q_{u \otimes v}$, $\|PSP\| \leq \|R_mSR_m^{-1}\|$ and $\|(I - P)S(I - P)\| \leq \|R_mSR_m^{-1}\|$, then $\|PSP\| = 0$ and $\|(I - P)S(I - P)\| = 0$. Hence $PSP = 0$ and $(I - P)S(I - P) = 0$. Thus

$$S = \begin{pmatrix} 0 & 0 \\ (I - P)SP & 0 \end{pmatrix} = (I - P)SP.$$

Conversely, If $S = (I - P)SP$, then

$$\|R_mSR_m^{-1}\| = \left\| \begin{pmatrix} 0 & 0 \\ M_{\lambda_m}(I - P)SP & 0 \end{pmatrix} \right\| = \|M_{\lambda_m}(I - P)SP\|.$$
Since $\|M_{\lambda m}(I - P)SP\| \to 0$, then $\|R_m SR^{-1}_m\| \to 0$ as $m \to \infty$. So $S \in Q_{u \otimes v}$. □

Let $X, Y, Z$ be Banach spaces. Assume that $T \in B(X,Y)$ and $S \in B(X,Z)$. Then $T$ majorizes $S$ if there exists $M > 0$ such that
\[
\|Sx\| \leq M\|Tx\|
\]
for all $x \in X$ (see [1]). Here we recall a result of [1] that gives us an equivalent condition for a closed range operator to majorize another bounded operator.

**Remark 2.19.** [1, Proposition 4] Let $X$ be Banach spaces and $T, S \in B(X)$ with $\mathcal{R}(T)$ closed. Then $T$ majorizes $S$ if and only if $\mathcal{N}(T) \subseteq \mathcal{N}(S)$.

Now we recall an assertion about closed range weighted conditional expectation operators.

**Proposition 2.20.** [7, Theorem 2.1] If $z(E(u)) = z(E(|u|^2))$ and for some $\delta > 0$, $E(u) \geq \delta$ on $z(E(|u|^2))$, then the operator $EM_u$ has closed range on $L^2(\mathcal{F})$.

**Proposition 2.21.** Let $T = M_w EM_u$ and $u \geq 0$. If $S \in Q_T$ and $E(u) \geq \delta$, then $EM_u$ majorizes $S$.

**Proof.** Since $u \geq 0$, then $z(E(u)) = z(E(|u|^2))$. Hence by the Remark 2.19, Theorem 2.13 and Proposition 2.20 we get the proof. □

Finally, since the rank one operator $x \otimes y$ has closed range, the we can obtain the next proposition.

**Proposition 2.22.** Let $x, y \in H$. If $T \in Q_{x \otimes y}$, then $x \otimes y$ majorizes $T$.

**Proof.** If $T \in Q_{x \otimes y}$, then by the proof of Theorem 2.18 we have $\mathcal{H}_2 = \mathcal{N}(x \otimes y)$ and $\mathcal{N}(x \otimes y) \subseteq \mathcal{N}(T)$. Since $x \otimes y$ has closed range, then by the Remark 2.19 we conclude that $x \otimes y$ majorizes $T$. □

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Y. Estaremi and M. R. Jabbarzadeh

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