Abstract. The Ehrhart polynomial of a lattice polytope $P$ encodes information about the number of integer lattice points in positive integral dilates of $P$. The $h^*$-polynomial of $P$ is the numerator polynomial of the generating function of its Ehrhart polynomial. A zonotope is any projection of a higher dimensional cube. We give a combinatorial description of the $h^*$-polynomial of a lattice zonotope in terms of refined descent statistics of permutations and prove that the $h^*$-polynomial of every lattice zonotope has only real roots and therefore unimodal coefficients. Furthermore, we present a closed formula for the $h^*$-polynomial of a zonotope in matroidal terms which is analogous to a result by Stanley (1991) on the Ehrhart polynomial. Our results hold not only for $h^*$-polynomials but carry over to general combinatorial positive valuations. Moreover, we give a complete description of the convex hull of all $h^*$-polynomials of zonotopes in a given dimension: it is a simplicial cone spanned by refined Eulerian polynomials.

1. Introduction

The Ehrhart function $\text{ehr}_P(n)$ of a polytope $P$ records the number of integer lattice points in the $n$-th positive integer dilate of the polytope. If $P$ is a lattice polytope (i.e., the vertices of $P$ have all integer coordinates), Ehrhart [6] showed that this function is in fact a polynomial—the Ehrhart polynomial of the polytope.

A fundamental class of polytopes are zonotopes, which make an appearance in various areas of mathematics. Besides geometry and combinatorics, they play, for example, a role in approximation theory, optimization, and crystallography. Given a set of vectors $V = \{v_1, \ldots, v_n\} \in \mathbb{R}^d$, $Z = \left\{ \sum_{i=1}^{n} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \right\}$, defines the zonotope generated by $V$, and, up to translation, every zonotope is generated that way. Stanley [21] showed that the Ehrhart polynomial of a lattice zonotope (i.e., when $v_1, \ldots, v_n \in \mathbb{Z}^d$) is given by the following beautiful combinatorial formula.

**Theorem 1.1 [21].** Let $Z$ be a lattice zonotope generated by a set of vectors $V \subseteq \mathbb{Z}^d$. Then

$$\text{ehr}_Z(n) = \sum_I g(I) n^{|I|}$$

where $I$ ranges over all linearly independent subsets of $V$, and $g(I)$ denotes the greatest common divisor of all maximal minors of a matrix with column vectors $I$.

A central problem, which is wide open already in dimension 3, is to characterize Ehrhart polynomials. An important tool here is the $h^*$-polynomial of a lattice polytope, which encodes its Ehrhart polynomial in a certain binomial basis (we give the details in Section 2.2 below).
fundamental result of Stanley [24] says that the coefficients of the $h^*$-polynomial are always nonnegative integers. This set the stage for intensive studies on the inequality relations among the coefficients of $h^*$-polynomials, which remains an active area of research.

There is an entire hierarchy of conjectures concerning unimodal $h^*$-polynomials. (A polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ is \textbf{unimodal} if $h_0 \leq h_1 \leq \cdots \leq h_k \geq \cdots \geq h_d$ for some $k \in \{0, 1, \ldots, d\}$.) A well-known conjecture due to Stanley [23] was originally formulated in the language of commutative algebra and implies that the $h^*$-polynomial of any integrally closed lattice polytope has unimodal coefficients. (A lattice polytope $P \in \mathbb{R}^d$ is \textbf{integrally closed} if for all integers $n \geq 1$ and every $p \in nP \cap \mathbb{Z}^d$ there are $p_1, \ldots, p_n \in P \cap \mathbb{Z}^d$ such that $p = p_1 + \cdots + p_n$.) As a first non-trivial instance of Stanley’s conjecture, Schepers and Van Langenhoven [19] proved that the coefficients of the $h^*$-polynomial for a lattice parallelepiped are unimodal. We follow their route and investigate, more generally, $h^*$-polynomials of lattice zonotopes. We give a combinatorial interpretation by showing that the $h^*$-polynomial of any zonotope is a weighted sum of certain polynomials $A_1(d+1, t), \ldots, A_{d+1}(d+1, t)$ originally introduced by Brenti and Welker [4]. These polynomials record the distribution of refined descent statistics on permutations and play a central role for computing $h$-polynomials of barycentric subdivisions; see Section 3.1 for a detailed definition. We give a geometric interpretation for $A_{j+1}(d+1, t)$ as the $h^*$-polynomial of a half-open unit cube with $j$ facets removed (Theorem 4.2 below). We consider, more generally, half-open parallelepipeds and, using a result of Savage and Visontai [18], we prove that the $h^*$-polynomial of every half-open parallelepiped is real-rooted and thus unimodal (Corollary 4.7). Moreover, we show that the peak of unimodality of the $h^*$-vector is in the middle.

Using half-open decompositions, we further show that our results extend to zonotopes.

Our results hold not only for counting lattice points in polytopes, but for $h^*$-polynomials $h^\varphi(P)(t)$ with respect to arbitrary combinatorial positive valuations initiated and studied by Jochemko and Sanyal in [9]; we carefully introduce the relevant terminology in Section 2.2. Our first main theorem is the following.

\textbf{Theorem 1.2.} Let $\varphi$ be a combinatorially positive valuation and let $Z$ be an $r$-dimensional lattice zonotope. Then the $h^*$-polynomial $h^\varphi(Z)(t) = h_0 + h_1 t + \cdots + h_r t^r$ has only real roots. Moreover, $h_0 \leq \cdots \leq h_{\frac{r}{2}} \geq \cdots \geq h_r$ if $r$ is even and $h_0 \leq \cdots \leq h_{\frac{r-1}{2}}$ and $h_{\frac{r+1}{2}} \geq \cdots \geq h_r$ if $r$ is odd.

Our second main result gives a simple description of the convex hull of all $h^*$-polynomials of $d$-dimensional lattice zonotopes.

\textbf{Theorem 1.3.} Let $d \geq 1$. The convex hull of the $h^*$-polynomials of all $d$-dimensional lattice zonotopes (viewed as points in $\mathbb{R}^{d+1}$) and the convex hull of the $h^*$-polynomials of all $d$-dimensional lattice parallelepipeds are both equal to the $d$-dimensional simplicial cone $A_1(d+1, t) + \mathbb{R}_{\geq 0} A_2(d+1, t) + \cdots + \mathbb{R}_{\geq 0} A_{d+1}(d+1, t)$.

Our third line of research concerns type-$B$ zonotopes and coloop-free zonotopes, introduced in Section 5, which we believe are both interesting in their own right. Schepers and Van Langenhoven [19] conjectured that every integrally closed lattice polytope with an interior lattice point has an alternatingly increasing $h^*$-polynomial, a property stronger than unimodality. We give further evidence for their conjecture by proving it for type-$B$ and coloop-free zonotopes (Corollary 5.4 and Theorem 5.11). We relate the $h^*$-polynomial of type-$B$ zonotopes to type-$B$ Eulerian polynomials via discrete geometry. Again our results hold in the more general context.
of translation-invariant valuations. We introduce refined type-$B$ Eulerian polynomials, and by expressing them in terms of $A_1(d + 1, t), \ldots, A_{d+1}(d + 1, t)$ we prove that these refined Eulerian polynomials are real-rooted (Theorem 3.9), generalizing a result of Brenti [3].

Our final main result is a closed formula for the $h^*$-polynomial of a lattice zonotope in the spirit of Theorem 1.1.

**Theorem 1.4.** Let $Z$ be a $d$-dimensional lattice zonotope generated by a set of vectors $V \subset \mathbb{Z}^d$, and let $\varphi$ be a translation-invariant valuation. Then

$$h^\varphi(Z)(t) = \sum_{I \in \mathcal{I}} b_\varphi(I) \sum_{B \subseteq \mathcal{B}} A_{|I \cup IP(B)|+1}(d + 1, t).$$

Here, $\mathcal{I}$ and $\mathcal{B}$ denote the set of independent subsets of $V$ and the bases formed by elements in $V$, respectively. The internally passive elements of a basis $B$ (see Section 5.2 for a definition) are denoted $IP(B)$ and $b_\varphi(I)$ is the value of $\varphi$ on the relative interior of the parallelepiped generated by the vectors in $I$.

2. **Preliminaries**

2.1. **Polynomials.** A polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ of degree $d$ is called **unimodal** if its coefficient vector $h = (h_0, \ldots, h_d)$ is unimodal, that is, if

$$h_0 \leq h_1 \leq \cdots \leq h_k \geq \cdots \geq h_d$$

for some $k \in \{0, 1, \ldots, d\}$. If $h_k$ is a largest coefficient, then we say that $h(t)$ and $h$ have a **peak** at $k$. The polynomial $h(t)$ is called **alternatingly increasing** if its coefficient vector $h$ is such that

$$h_0 \leq h_d \leq h_{d-1} \leq \cdots \leq h_{\lfloor \frac{d+1}{2} \rfloor}.$$

In particular, if $(h_0, h_1, \ldots, h_d)$ is alternatingly increasing, then $(h_0, h_1, \ldots, h_d)$ is unimodal with peak at $\lfloor \frac{d+1}{2} \rfloor$. We call a polynomial $h(t)$ **palindromic** with center of symmetry at $\frac{d}{2}$ if $t^d h\left(\frac{1}{t}\right) = h(t)$. If it is in addition unimodal, then the coefficients closest to the center of symmetry are maximal, i.e., $h(t)$ has a peak at $\frac{d}{2}$ if $d$ is even, and at $\lfloor \frac{d}{2} \rfloor$ and $\lceil \frac{d}{2} \rceil + 1$ if $d$ is odd.

Every polynomial $h(t)$ of degree $d$ can be uniquely decomposed into a sum $h(t) = a(t) + t b(t)$, where $a(t)$ and $b(t)$ are palindromic with $t^d a\left(\frac{1}{t}\right) = a(t)$ and $t^{d-1} b\left(\frac{1}{t}\right) = b(t)$ [25].

**Lemma 2.1.** Let $h(t) = a(t) + t b(t)$ be a polynomial of degree $d$, where $a(t)$ and $b(t)$ are palindromic with center of symmetry $\frac{d}{2}$ and $\frac{d-1}{2}$, respectively. Then $h(t)$ is alternatingly increasing if and only if $a(t)$ and $b(t)$ are unimodal.

**Proof.** Let $h(t) = \sum_{i=0}^{d} h_i t^i$. Since $a(t)$ and $b(t)$ are palindromic, it is easy to check that $h_i \leq h_{d-i}$ for all $i$ if and only if $b(t)$ is unimodal, and $h_{d-i} \leq h_{i+1}$ for all $i$ if and only if $a(t)$ is unimodal.

2.2. **Translation-invariant valuations.** A **lattice polytope** is a polytope with vertices in the integer lattice. The family of all lattice polytopes in $\mathbb{R}^d$ will be denoted by $\mathcal{P}(\mathbb{Z}^d)$. A **valuation** on lattice polytopes is a map $\varphi$ from $\mathcal{P}(\mathbb{Z}^d)$ into some Abelian group $G$ such that $\varphi(\emptyset) = 0$ and

$$\varphi(P \cup Q) = \varphi(P) + \varphi(Q) - \varphi(P \cap Q)$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\mathbb{Z}^d)$. In [15] McMullen showed that every valuation satisfies the **inclusion-exclusion property.** Namely, for lattice polytopes $P_1, \ldots, P_m \in \mathcal{P}(\mathbb{Z}^d)$ such that
\( P_1 \cup \cdots \cup P_n \in \mathcal{P}(\mathbb{Z}^d) \) and \( \bigcap_{i \in I} P_i \in \mathcal{P}(\mathbb{Z}^d) \) for all \( I \subseteq \{1, 2, \ldots, m\} \)

\[
\varphi(P_1 \cup \cdots \cup P_m) = \sum_{\varnothing \neq I} (-1)^{|I|-1} \varphi\left( \bigcap_{i \in I} P_i \right).
\]

This allows for a definition of \( \varphi \) on the relative interior \( \text{relint} P \) of a polytope as

\[
\varphi(\text{relint} P) = \sum_{F} (-1)^{\dim P - \dim F} \varphi(F),
\]

where the sum is taken over all faces of \( P \). We call \( \varphi \) translation-invariant or a \( \mathbb{Z}^d \)-valuation if \( \varphi(P+x) = \varphi(P) \) for all \( x \in \mathbb{Z}^d \) and all \( P \in \mathcal{P}(\mathbb{Z}^d) \). Fundamental examples besides the volume are the Euler characteristic, the discrete volume \( \varepsilon(P) := |P \cap \mathbb{Z}^d| \) and the solid-angle sum (see, e.g., [1]). McMullen [14] proved that for integers \( n \geq 0 \) the value \( \varphi(nP) \) of the \( n \)-th dilate of an \( r \)-dimensional lattice polytope \( P \) is given by a polynomial \( \text{ehr}_P(n) \) of degree at most \( r \) in \( n \). For the discrete volume this was proved by Ehrhart [6]; when \( \varphi = \varepsilon \), we suppress the superscript and call \( \text{ehr}_P(n) \) the Ehrhart polynomial of \( P \). Equivalently, there are \( h_0^P(P), \ldots, h_r^P(P) \in G \) such that

\[
\text{ehr}_P^\varphi(n) = h_0^\varphi(P)\binom{n+r}{r} + h_1^\varphi(P)\binom{n+r-1}{r} + \cdots + h_r^\varphi(P)\binom{n}{r}
\]

for all \( n \geq 0 \). In terms of generating series, this is equivalent to

\[
\text{Ehr}^\varphi(P, t) := \sum_{n \geq 0} \text{ehr}_P^\varphi(n) t^n = \frac{h_0^\varphi(P) + \cdots + h_r^\varphi(P) t^r}{(1-t)^{r+1}}.
\]

In the special case \( \varphi = \varepsilon \), we call \( \text{Ehr}(P, t) \) the Ehrhart series of \( P \). The numerator polynomial \( h^\varphi(P)(t) \) is called the \( \varphi \)-polynomial of \( P \) with respect to \( \varphi \) and the vector \( h^\varphi(P) := (h_0^\varphi(P), \ldots, h_r^\varphi(P)) \) is the \( \varphi \)-vector of \( P \) with respect to \( \varphi \). When \( \varphi = \varepsilon \), we call \( h^\varepsilon(P) \) simply the \( \varepsilon \)-vector; alternative names in this case include \( \delta \)-vector and Ehrhart \( h \)-vector.

For the discrete volume Stanley [21] showed that the entries of \( h^\varphi(P) \) are nonnegative for all lattice polytopes \( P \). For the solid-angle sum this was shown by Beck, Robins and Sam [1]. In [9] Jochemko and Sanyal studied the class of all \( \mathbb{Z}^d \)-valuations into some partially ordered Abelian group such that \( h^\varphi(P)(t) \) has nonnegative entries for every lattice polytope \( P \). They called these valuations combinatorially positive and obtained the following simple characterization.

**Theorem 2.2** [9]. Let \( \varphi \) be a \( \mathbb{Z}^d \)-valuation. Then \( \varphi \) is combinatorially positive if and only if \( \varphi(\text{relint} \Delta) \geq 0 \) for all simplices \( \Delta \in \mathcal{P}(\mathbb{Z}^d) \).

Note that this implies that \( \varphi(\text{relint} P) \geq 0 \) for all \( P \in \mathcal{P}(\mathbb{Z}^d) \) if \( \varphi \) is combinatorially positive.

### 2.3. Half-open polytopes.

To every polytope \( P \in \mathcal{P}(\mathbb{Z}^d) \) and every generic \( q \) in the affine hull \( \text{aff}(P) \) of \( P \) we can associate a half-open polytope \( \mathbb{H}_q P \), defined as the set of points \( p \in P \) such that \( [q, p] \cap P \neq \emptyset \). Thinking of \( q \) as a light source, \( \mathbb{H}_q P \) is the set of all points in \( P \) that are not visible from \( q \). Note that \( \mathbb{H}_q P \) is closed if and only if \( q \in P \) and in this case \( \mathbb{H}_q P = P \). If \( F_1, \ldots, F_m \) are the facets of \( P \), let \( I_q(P) \subseteq \{1, 2, \ldots, m\} \) be the set of facets visible from \( q \). Then

\[
\mathbb{H}_q P = P \setminus \bigcup_{i \in I_q(P)} F_i.
\]

Accordingly, for a valuation \( \varphi \) we define

\[
\varphi(\mathbb{H}_q P) := \varphi(P) - \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|-1} \varphi\left( \bigcap_{i \in I} F_i \right).
\]
In particular, we can consider \( \text{ehr}_q^\oplus P(n) \) and the \( h^* \)-polynomial of \( \mathbb{H}_q P \). For example, it is easy to see that if \( Q \) is a half-open unimodular simplex\(^1\) of dimension \( d \) with \( k \) missing (visible) facets, then \( \text{ehr}_Q(n) = \binom{n+d-k}{d} \), or equivalently, \( \text{Ehr}(Q, t) = \frac{t^k}{(1-t)^{d+1}} \).

**Lemma 2.3** [10, Theorem 3]. Let \( P \) be a polytope, \( P = P_1 \cup \cdots \cup P_k \) a dissection and \( q \in \text{aff}(P) \) generic. Then

\[
\mathbb{H}_q P = \mathbb{H}_q P_1 \cup \cdots \cup \mathbb{H}_q P_k
\]

is a disjoint union of half-open polytopes.

**Corollary 2.4** [9, Corollary 3.2]. Let \( P = P_1 \cup \cdots \cup P_k \) be a dissection with \( P_1, \ldots, P_k \in \mathcal{P}(\mathbb{Z}^d) \). If \( \varphi \) is a valuation, then for a generic \( q \in \text{relint}(P) \)

\[
\varphi(P) = \varphi(\mathbb{H}_q P_1) + \cdots + \varphi(\mathbb{H}_q P_k).
\]

### 3. Descent Statistics

#### 3.1. Type-A. Let \( S_d \) denote the set of all permutations on \([d] = \{1, \ldots, d\}\). For every permutation word \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_d \in S_d \) the descent set is defined by

\[
\text{Des}(\sigma) := \{ i \in [d-1] : \sigma_i > \sigma_{i+1} \}.
\]

The number of descents of \( \sigma \) is denoted by \( \text{des}(\sigma) := |\text{Des}(\sigma)| \). The (type-A) Eulerian number \( a(d, k) \) counts the number of permutations in \( S_d \) with \( k \) descents:

\[
a(d, k) := |\{ \sigma \in S_d : \text{des}(\sigma) = k \}|.
\]

We consider a refinement of the descent statistic: the \((A, j)\)-Eulerian number

\[
a_j(d, k) := |\{ \sigma \in S_d : \sigma_d = d+1-j \text{ and } \text{des}(\sigma) = k \}|
\]

giving the number of permutations \( \sigma \in S_d \) with last letter \( d+1-j \) and exactly \( k \) descents. The corresponding \((A, j)\)-Eulerian polynomial is

\[
A_j(d, t) := \sum_{k=0}^{d-1} a_j(d, k) t^k
\]

Note that by definition \( A_j(d, k) = 0 \) for \( k < 0 \) and \( k > d-1 \). As far as we know, the \((A, j)\)-Eulerian polynomials were first considered by Brenti and Welker [4], though the \((A, j)\)-Eulerian numbers and generalizations of them had been considered earlier (see, e.g., [5, 22]).

#### 3.2. Type-B. A signed permutation on \([d]\) is a pair \((\sigma, \epsilon)\) with \( \sigma \in S_d \) and \( \epsilon \in \{\pm 1\}^d \). To each letter \( \sigma_i \) in the permutation word \( \sigma \) we assign the sign \( \epsilon_i \), the \( i^{\text{th}} \) entry of \( \epsilon \). For a given \( d \), the set of signed permutations is denoted by \( B_d \) and has \( 2^d d! \) elements. We will use one-line notation to denote signed permutation words with the convention that letters associated with a negative sign will be followed by an accent mark. So for \( d = 5 \), \( \sigma = 42135 \) and \( \epsilon = (-1, -1, 1, -1, 1) \) we write \((\sigma, \epsilon) = 4'2'13'5'\).

Set \( \sigma_0 := 0 \) and \( \epsilon_0 := 1 \) for all \((\sigma, \epsilon) \in B_d \) and all \( d \geq 1 \). Then \( i \in [d-1] \cup \{0\} \) is a descent of \((\sigma, \epsilon) \in B_d \) if \( \epsilon_i \sigma_i > \epsilon_{i+1} \sigma_{i+1} \). E.g., 0 and 3 are the descents of 4'2'13'5'. We define the descent set and the descent number of \((\sigma, \epsilon) \in B_d \), respectively, as

\[
\text{Des}(\sigma, \epsilon) := \{ i \in [d-1] \cup \{0\} : \epsilon_i \sigma_i > \epsilon_{i+1} \sigma_{i+1} \} \quad \text{and} \quad \text{des}(\sigma, \epsilon) := |\text{Des}(\sigma, \epsilon)|.
\]

\(^1\) A simplex is a \( d \)-polytope with (the minimal number of) \( d + 1 \) vertices; it is unimodular if these vertices have integer coordinates and the simplex has (minimal) volume \( \frac{1}{d!} \).
Observe that the descent statistic on permutations in $S_d$ agrees with the descent statistic on signed permutations $B_d$ when we fix the sign vector $\epsilon = 1 := (1, 1, \ldots, 1)$. However, since 0 is a possible descent of a signed permutation, $0 \leq \text{des}(\sigma, \epsilon) \leq d$ for all $(\sigma, \epsilon) \in B_d$, in contrast to $0 \leq \text{des}(\sigma, 1) \leq d - 1$ for all $\sigma \in S_d$.

The number of signed permutations on $[d]$ with exactly $k$ descents is the type-B Eulerian number. We write

$$b(d, k) := \left| \{ (\sigma, \epsilon) \in B_d : \text{des}(\sigma, \epsilon) = k \} \right| .$$

The type-B Eulerian polynomial is

$$B(d, t) := \sum_{k=0}^{d} b(d, k) t^k .$$

We also introduce and study $(B, l)$-Eulerian numbers, a refinement of the type-B Eulerian numbers defined by

$$b_l(d, k) := \left| \{ (\sigma, \epsilon) \in B_d : \epsilon_d \sigma_d = d + 1 - l \text{ and } \text{des}(\sigma, \epsilon) = k \} \right| ,$$

where $1 \leq l \leq d$, and define the $(B, l)$-Eulerian polynomial

$$B_l(d, t) := \sum_{k=0}^{d} b_l(d, k) t^k .$$

As far as we know, these have not been studied before.

### 3.3. Unimodality and real-rootedness

A fundamental result of Savage and Visontai [18] implies that the $(A, j)$-Eulerian polynomials have only real roots and are therefore unimodal. In fact, they proved the following stronger result.

**Theorem 3.1** [18]. Let $c_1, \ldots, c_d \geq 0$ be real numbers. Then the polynomial

$$c_1A_1(d, t) + c_2A_2(d, t) + \cdots + c_dA_d(d, t)$$

has only real roots. In particular, its coefficients form a unimodal sequence.

Their inductive proof was based on the following recurrence for $(A, j)$-Eulerian polynomials, which seems to go back to Brenti and Welker [4].

**Lemma 3.2** [4, Lemma 2.5]. For $1 \leq j \leq d + 1$,

$$A_j(d + 1, t) = t \sum_{l=1}^{j-1} A_l(d, t) + \sum_{l=j}^{d} A_l(d, t) .$$

Note that in general, $A_j(d, t)$ is not palindromic. Nevertheless, using the recurrence above together with the following lemma one can determine the exact position of their peaks.

**Lemma 3.3** [4, Lemma 2.5]. For all $d \geq 1$ and $1 \leq j \leq d$,

$$A_j(d, t) = t^{d-1}A_{d+1-j}(d, \frac{1}{t}) .$$

The following theorem is a slight strengthening of [11, Corollary 4.4] by Kubitzke and Nevo. While they used quite heavy algebraic machinery, we give an elementary combinatorial proof.
Theorem 3.4. For all $1 \leq j \leq d$, the coefficients of $A_j(d,t)$ are unimodal. More specifically, if $d$ is even,
\begin{align*}
a_j(d,0) &\leq \cdots \leq a_j(d,\frac{d}{2}-1) \geq \cdots \geq a_j(d,d-1) &\text{if } 1 \leq j \leq \frac{d}{2}, \\
a_j(d,0) &\leq \cdots \leq a_j(d,\frac{d}{2}) \geq \cdots \geq a_j(d,d-1) &\text{if } \frac{d}{2} < j \leq d,
\end{align*}
and if $d \geq 3$ is odd,
\begin{align*}
a_1(d,0) &\leq \cdots \leq a_1(d,\lfloor \frac{d}{2} \rfloor - 1) = a_1(d,\lfloor \frac{d}{2} \rfloor) \geq \cdots \geq a_1(d,d-1) \\
a_d(d,0) &\leq \cdots \leq a_d(d,\lfloor \frac{d}{2} \rfloor) = a_d(d,\lfloor \frac{d}{2} \rfloor + 1) \geq \cdots \geq a_d(d,d-1), \\
a_j(d,0) &\leq \cdots \leq a_j(d,\lfloor \frac{d}{2} \rfloor) \geq \cdots \geq a_j(d,d-1) &\text{if } 2 \leq j \leq d-1.
\end{align*}

Proof. We argue by induction on $d$. When $d = 1$, the claim is trivially true since $A_1(1,t) = 1$. The case $d = 2$ is easily checked.

Let $d + 1$ be even. We distinguish two cases:

CASE: $1 \leq j \leq \frac{d+1}{2}$. Then
\begin{equation*}
A_j(d+1,t) = t \sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d+1-j} A_l(d,t) + \sum_{l=d+2-j}^{d} A_l(d,t)
\end{equation*}
by Lemma 3.2. The first and the third summand added give, by Lemma 3.3, a palindromic polynomial with center of symmetry at $\frac{d}{2}$, which, by induction, has unimodal coefficients with peaks at $\lfloor \frac{d}{2} \rfloor$ and $\lfloor \frac{d}{2} \rfloor + 1$. The second summand has, by induction, unimodal coefficients with peak at $\lfloor \frac{d}{2} \rfloor = \frac{d+1}{2} - 1$.

CASE: $\frac{d+1}{2} < j \leq d + 1$. Then
\begin{equation*}
A_j(d+1,t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + t \sum_{l=d+2-j}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t).
\end{equation*}

The first and the third summand added give a palindromic polynomial with center of symmetry at $\frac{d}{2}$, which has unimodal coefficients with peaks at $\lfloor \frac{d}{2} \rfloor$ and $\lfloor \frac{d}{2} \rfloor + 1$. The coefficients of the second summand form a unimodal sequence with peak at $\lfloor \frac{d}{2} \rfloor + 1 = \frac{d+1}{2}$.

If $d + 1 \geq 3$ is odd, we distinguish again two cases.

CASE: $1 \leq j \leq \frac{d+1}{2}$. By Lemma 3.2,
\begin{equation*}
A_j(d+1,t) = t \sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d+1-j} A_l(d,t) + \sum_{l=d+2-j}^{d} A_l(d,t).
\end{equation*}

The second summand is, by induction and Lemma 3.3, a palindromic polynomial with unimodal coefficients and peaks at $\frac{d}{2} - 1$ and $\frac{d}{2}$. The coefficients of the first and third summand are unimodal with peak at $\frac{d}{2} = \lfloor \frac{d+1}{2} \rfloor$.

CASE: $\frac{d+1}{2} < j \leq d + 1$. Then
\begin{equation*}
A_j(d+1,t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + t \sum_{l=d+2-j}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t).
\end{equation*}

As in the previous case, the coefficients of the summand in the middle are unimodal and palindromic, this time with peaks at $\frac{d}{2}$ and $\frac{d}{2} + 1$. The coefficients of the first and third summand form again a unimodal sequence with peak at $\frac{d}{2} = \lfloor \frac{d+1}{2} \rfloor$. \qed
From the proof of [19, Proposition 2.17], it can moreover be seen that the coefficients of these polynomials are alternatingly increasing for sufficiently large $j$. We formally record this result and give a short proof.

**Lemma 3.5.** For all $d \geq 0$ and $\frac{d+1}{2} < j \leq d+1$, the coefficients of $A_j(d+1, t)$ are alternatingly increasing.

**Proof.** If $\frac{d+1}{2} < j \leq d+1$ then, by Lemma 3.2, we have $A_j(d+1, t) = b(t) + tc(t)$ with

$$b(t) = t \sum_{l=1}^{d+1-j} A_l(d, t) + \sum_{l=j}^{d} A_l(d, t)$$

and

$$c(t) = \sum_{l=d+2-j}^{j-1} A_l(d, t).$$

By Lemmas 3.3 and 3.4 we know $b(t)$ and $c(t)$ are unimodal and $t^d b(\frac{1}{t}) = b(t)$ and $t^{d-1}c(\frac{1}{t}) = c(t)$. Therefore the claim follows with Lemma 2.1. □

**Remark 3.6.** Ehrenborg, Readdy, and Steingrímsson showed in [5] that the $(A, j)$-Eulerian numbers have a geometric meaning as mixed volumes of certain hypersimplices. It would be interesting to see whether this yields a geometric proof of Theorem 3.4 by using, e.g., the Alexandrov–Fenchel inequalities.

Brenti [3] proved that the type-$B$ Eulerian polynomials have only real roots.

**Theorem 3.7** [3, Corollary 3.7]. The type-$B$ Eulerian polynomial $B(d, t)$ has only real roots. In particular, the coefficients of $B(d, t)$ form a unimodal sequence.

In Section 5.1 we prove the following explicit expression of $(B, l)$-Eulerian polynomials in terms of $(A, l)$-Eulerian polynomials.

**Proposition 3.8.**

$$B_{l+1}(d+1, t) = 2^l \sum_{j=0}^{d-l} \begin{pmatrix} d-l \\ j \end{pmatrix} A_{j+l+1}(d+1, t).$$

We obtain the following generalization of Theorem 3.7 as a corollary, which to our knowledge is new.

**Theorem 3.9.** The polynomial $B_l(d, t)$ has only real roots and is alternatingly increasing for all $0 \leq l \leq d$.

**Remark 3.10.** Theorem 3.9 can be generalized further. Indeed, we could have defined $(B, l)$-Eulerian numbers also for integers $l \in [2d+1] \setminus \{d+1\}$. A bijection of Pensyl and Savage [17, Theorem 3] between signed permutations and $s$-lecture hall partitions for $s = (2, 4, \ldots, 2d)$ together with the results from [18] yield that all $(B, l)$-Eulerian polynomials for $|l| \leq d$ are real-rooted.

**Lemma 3.11.** For all $d \geq 1$ and all $1 \leq i, j \leq d$ with $i + j \geq d + 2$,

$$A_i(d+1, t) + A_j(d+1, t)$$

is alternatingly increasing.
Proof. If both $i$ and $j$ are greater than $\frac{d+1}{2}$ then the claim follows from Lemma 3.5. So we may suppose that $i \leq \frac{d+1}{2}$; then we must have $j = d + 2 - i + k > \frac{d+1}{2}$ for some $0 \leq k \leq i - 1$ by assumption. Using the recursions in the proof of Theorem 3.4 we obtain

$$A_i(d + 1, t) + A_j(d + 1, t) = t \sum_{l=i-1}^{i-1} A_i(d, t) + \sum_{l=i}^{d+i-1} A_l(d, t) + \sum_{l=d+i+1}^{d} A_l(d, t) + t \sum_{l=1}^{i-k-1} A_l(d, t).$$

As in the proof of Theorem 3.4, the first, third, fourth, and last summand add up to a palindromic polynomial with center of symmetry at $\frac{d}{2}$. By Lemma 3.3 this is also true for the sum of the second and the sixth sum. The remaining sums yield a polynomial that is palindromic with center of symmetry at $\frac{d+1}{2}$ times a factor $t$. Therefore the claim follows from Lemma 2.1. \hfill \Box

Proof of Theorem 3.9. Real-rootedness follows directly from Proposition 3.8 and Theorem 3.1. Furthermore, $A_{j+i+1}(d + 1, t) + A_{d-j+1}(d + 1, t)$ is alternatingly increasing for all $0 \leq j \leq d - l$ by Lemma 3.11, and so is $B_{i+1}(d + 1, t)$, since it is a positive linear combination of polynomials of that form. \hfill \Box

4. GEOMETRY

4.1. Half-open unit cubes. For $j \in \{0, \ldots, d\}$ we define the half-open unit cube

$$C_j^d = [0, 1]^d \setminus \{x \in \mathbb{R}^d : x_d = x_{d-1} = \cdots = x_{d+1-j} = 1\}.$$ 

The subscript $j$ indicates the number of facets removed from $C^d := [0, 1]^d$. The $j$-descent set $\text{Des}_j(\sigma) \subseteq \{1, \ldots, d\}$ of a permutation $\sigma \in S_d$ is

$$\text{Des}_j(\sigma) := \begin{cases} \text{Des}(\sigma) \cup \{d\} & \text{if } d + 1 - j \leq \sigma_d \leq d, \\ \text{Des}(\sigma) & \text{otherwise}, \end{cases}$$

and the $j$-descent number $\text{des}_j(\sigma) := |\text{Des}_j(\sigma)|$ counts the $j$-descents of $\sigma$. We can describe the $(A, j)$-Eulerian numbers in terms of $j$-descents, as the following lemma shows.

Lemma 4.1. For all $0 \leq j \leq d$ and $0 \leq k \leq d$,

$$|\{\sigma \in S_d : \text{des}_j(\sigma) = k\}| = a_{j+1}(d + 1, k).$$

Proof. For every $\sigma \in S_d$ define $\sigma' \in S_{d+1}$ by

$$\sigma'_i := \begin{cases} \sigma_i & \text{if } \sigma_i < d + 1 - j, \\ \sigma_i + 1 & \text{if } \sigma_i \geq d + 1 - j, \\ \sigma_{d+1} = d + 1 - j & \text{otherwise}. \end{cases}$$

It is straightforward to check that the map $\sigma \mapsto \sigma'$ bijectively maps $S_d$ to the set of permutations on $[d + 1]$ ending with $d + 1 - j$, and thus $\text{des}_j(\sigma) = \text{des}(\sigma')$. \hfill \Box

The discrete volume of half-open unit cubes plays a distinguished role when determining the $h^*$-vector of parallelepipeds with respect to arbitrary $\mathbb{Z}^d$-valuations; this is based on the following result.
Figure 1. Decomposition of $C^2_1$ into half-open unimodular simplices.

**Theorem 4.2.** Let $0 \leq j \leq d$. Then

$$\text{Ehr} \left( C^d_j, t \right) = \frac{A_{j+1}(d+1,t)}{(1-t)^{d+1}}.$$  

In particular, the numerator polynomial is real-rooted and its coefficients form a unimodal sequence.

**Proof.** We will decompose $C^d_j$ into half-open unimodular simplices induced by the arrangement of hyperplanes given by inequalities of the form $x_i = x_j$, $i \neq j$; see also [2, Chapter 6]. For $0 \leq j \leq d$ and $\sigma \in S_d$, define the half-open unimodular simplex

$$\Delta^d_{\sigma,j} := \begin{cases} x \in C^d_j : x_{\sigma_1} \leq x_{\sigma_2} \leq \cdots \leq x_{\sigma_d} \\
\text{with } x_{\sigma_i} < x_{\sigma_{i+1}} \text{ when } i \in \text{Des}(\sigma) \end{cases}$$

$$= \begin{cases} x \in \mathbb{R}^d : 0 \leq x_{\sigma_1} \leq x_{\sigma_2} \leq \cdots \leq x_{\sigma_d} \leq 1 \\
\text{with } x_{\sigma_i} < x_{\sigma_{i+1}} \text{ when } i \in \text{Des}_j(\sigma) \\
\text{and } x_{\sigma_d} < 1 \text{ when } d \in \text{Des}_j(\sigma) \end{cases}.$$  

The closure of $\Delta^d_{\sigma,j}$ is a unimodular simplex for all $j$ and $\sigma$. Each strict inequality corresponds bijectively to a missing facet of the simplex. Therefore, the half-open unimodular simplex $\Delta^d_{\sigma,j}$ has exactly $\text{des}_j(\sigma)$ missing facets. Furthermore,

$$C^d_j = \bigcup_{\sigma \in S_d} \Delta^d_{\sigma,j}$$

is a disjoint union. Therefore,

$$\text{Ehr} \left( C^d_j, t \right) = \sum_{\sigma \in S_d} \text{Ehr} \left( \Delta^d_{\sigma,j}, t \right) = \sum_{\sigma \in S_d} t^{\text{des}_j(\sigma)} \frac{A_{j+1}(d+1,t)}{(1-t)^{d+1}} = \frac{A_{j+1}(d+1,t)}{(1-t)^{d+1}},$$

where the last equality holds by Lemma 4.1.  

The $h^*$-vector of parallelepipeds with respect to arbitrary $\mathbb{Z}^d$-valuations will be treated in the following paragraph.

### 4.2. Half-open parallelepipeds

In the following let $\varphi$ be a $\mathbb{Z}^d$-valuation, and let $v_1, \ldots, v_r \in \mathbb{Z}^d$ be fixed linearly independent vectors. For each $I \subseteq [r]$ we define the (closed) parallelepiped

$$\hat{\diamond}(I) := \left\{ \sum_{i \in I} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \text{ for all } i \in I \right\}$$
and the relatively open parallelepiped
\[ \square(I) := \left\{ \sum_{i \in I} \lambda_i v_i : 0 < \lambda_i < 1 \text{ for all } i \in I \right\}. \]

We set \( b_\varphi(I) := \varphi(\square(I)) \) and observe that if \( \varphi \) is combinatorially positive then, from Theorem 2.2, we obtain \( b_\varphi(I) \geq 0 \) for all \( I \subseteq [r] \).

Further, for each \( I \subseteq [r] \) we define the half-open parallelepipeds
\[ \Phi(I) := \left\{ \sum_{i=1}^{r} \lambda_i v_i : 0 < \lambda_i \leq 1 \text{ for all } i \in I, \ 0 \leq \lambda_i \leq 1 \text{ for all } i \not\in I \right\} \]
and
\[ \Pi(I) := \left\{ \sum_{i \in I} \lambda_i v_i : 0 < \lambda_i \leq 1 \text{ for all } i \in I \right\}. \]

Note that \( \Diamond([r]) = \Diamond(\emptyset); \) we also set \( \Diamond(\emptyset) = \Pi(\emptyset) = \{0\}. \) The following lemma of Schepers and van Langenhoven [19] was originally stated only for discrete volumes. However, their proof works as well for arbitrary \( \mathbb{Z}^d \)-valuations.

**Lemma 4.3** [19, Lemma 2.1]. Let \( \varphi \) be a \( \mathbb{Z}^d \)-valuation and let \( I \subseteq [r] \). Then
\[ \varphi(n \Phi(I)) = \sum_{I \subseteq J} n^{|J|} \varphi(\Pi(J)). \]

**Proof.** To keep this paper self contained we give a proof here (slightly modified from that of [19]).

As \( v_1, \ldots, v_r \) are linearly independent, for every \( x \in \Diamond([r]) \) there are unique \( \lambda_1, \ldots, \lambda_r \in [0, 1] \) such that
\[ x = \sum_{i=1}^{r} \lambda_i v_i. \]

Let \( J_x := \{ i \in [r] : \lambda_i > 0 \} \). Then \( x \in \Pi(J) \) if and only if \( J_x = J \). We observe that \( x \in \Phi(I) \) if and only if \( I \subseteq J_x \), and therefore we can partition
\[ \Phi(I) = \bigsqcup_{I \subseteq J} \Pi(J). \]

Further, for all \( J \subseteq [r] \) and all \( n \geq 1 \) we can tile \( n \Pi(J) \) with \( n^{|J|} \) translates of \( \Pi(J) \). Thus by the translation-invariance of \( \varphi \),
\[ \varphi(n \Phi(I)) = \sum_{I \subseteq J} \varphi(n \Pi(J)) = \sum_{I \subseteq J} n^{|J|} \varphi(\Pi(J)). \]

Applying Lemma 4.3 to the linearly independent standard basis vectors \( e_1, \ldots, e_d \) and the discrete volume, we obtain the following corollary:

**Corollary 4.4.** Let \( 0 \leq j \leq d \). Then the Ehrhart polynomial of the half-open unit cube \( C_d^j \) equals
\[ \text{ehr}_{C_d^j}(n) = \sum_{|j| \subseteq J} n^{|J|} = t^j(1 + t)^{d-j}, \]
where we set \([0] := \emptyset\).

Recall our notation \( b_\varphi(I) = \varphi(\square(I)) \).
Lemma 4.5. Let \( \varphi \) be a \( \mathbb{Z}^d \)-valuation. Then for all \( I \subseteq [r] \),
\[
\varphi(\Pi(I)) = \sum_{J \subseteq I} b_\varphi(J).
\]

Proof. For \( x \in \Pi(I) \) there are unique \( \lambda_i \in (0, 1] \) such that
\[
x = \sum_{i \in I} \lambda_i v_i.
\]
Let \( J_x := \{ i \in I : \lambda_i = 1 \} \subseteq I \). For all \( J \subseteq I \) we have \( J = J_x \) if and only if \( x \in \square(I \setminus J) + \sum_{i \in J} v_i \). Therefore
\[
\Pi(I) = \biguplus_{J \subseteq I} \left( \square(I \setminus J) + \sum_{i \in J} v_i \right),
\]
and the result follows by the translation-invariance of \( \varphi \). \( \square \)

The following theorem generalizes [19, Proposition 2.2].

Theorem 4.6. Let \( \varphi \) be a \( \mathbb{Z}^d \)-valuation. Then for all \( I \subseteq [r] \),
\[
\varphi(\Phi(I), t) = \sum_{K \subseteq [r]} b_\varphi(K) A_{|I \cup K|+1}(r+1,t) \frac{(1-t)^{r+1}}{(1-t)^{r+1}}.
\]

Proof. We follow the line of argumentation in [19, Proposition 2.2]. By Lemmas 4.3 and 4.5,
\[
\varphi(\Phi(I), t) = \sum_{n=0}^{\infty} t^n \sum_{J \geq I} n^{|J|} \varphi(\Pi(J))
\]
\[
= \sum_{n=0}^{\infty} t^n \sum_{J \geq I} n^{|J|} \sum_{K \subseteq J} b_\varphi(K)
\]
\[
= \sum_{K \subseteq [r]} b_\varphi(K) \sum_{n=0}^{\infty} t^n \sum_{J \geq I \cup K} n^{|J|}.
\]

By Corollary 4.4,
\[
\sum_{J \geq I \cup K} n^{|J|} = \text{ehr}_{C_{[I \cup K]}}(n).
\]
The claim now follows from Corollary 4.4. \( \square \)

As a corollary we obtain unimodality of the \( h^* \)-vectors of half-open parallelepipeds in the case that \( \varphi \) is combinatorially positive.

Corollary 4.7. Let \( \varphi \) be a combinatorially positive \( \mathbb{Z}^d \)-valuation and let \( I \subseteq [r] \). Then \( h^*(\Phi(I))(t) \) has only real roots. Moreover, if \( h^*(\Phi(I)) = (h_0, \ldots, h_r, 0, \ldots, 0) \) is the \( h^* \)-vector of \( \Phi(I) \), then
\[
h_0 \leq \cdots \leq h_{\frac{r}{2}} \geq \cdots \geq h_r \quad \text{if } r \text{ is even}
\]
and
\[
h_0 \leq \cdots \leq h_{\frac{r-1}{2}} \quad \text{and} \quad h_{\frac{r+1}{2}} \geq \cdots \geq h_r \quad \text{if } r \text{ is odd}.
\]
Proof. By Theorem 4.6,
\[ h^\varphi(\Phi(I))(t) = \sum_{K \subseteq [r]} b_\varphi(K) A_{|I\cup K|+1}(r + 1, t). \]
As \( \varphi \) is combinatorially positive, \( b_\varphi(K) \geq 0 \) for all \( K \subseteq [r] \) by Theorem 2.2. By Theorem 3.1 \( h^\varphi(\Phi(I))(t) \) is real-rooted. Moreover, by Theorem 3.4, the coefficients of \( A_{|I\cup K|+1}(r + 1, t) \) form a unimodal sequence with peak at \( \lceil \frac{r+1}{2} \rceil = \frac{r}{2} \) if \( r \) is even, and peak at \( \frac{r-1}{2} \) or \( \frac{r+1}{2} \) if \( r \) is odd, and so does any nonnegative linear combination. □

4.3. Zonotopes. The following well-known result is due to Shephard [20].

Theorem 4.8 [20, Theorem 54]. Every (lattice) zonotope has a subdivision into (lattice) parallelepipeds.

Proposition 4.9. Let \( Z \) be an \( r \)-dimensional zonotope. Then \( Z \) can be partitioned into \( r \)-dimensional half-open parallelepipeds in the sense of Section 4.2.

Proof. Let \( Z = P_1 \cup \cdots \cup P_k \) be a dissection of \( Z \) into parallelepipeds, which exists by Theorem 4.8, and let \( q \in Z \) be a generic point. Then, by Lemma 2.3, \( Z = H_q P_1 \cup \cdots \cup H_q P_k \) is a partition into half-open polytopes. Moreover, every \( H_q P_i \) equals, up to translation, some \( \Phi(I) \): Let \( F_1, \ldots, F_m \) be the facets of \( P_k \) visible from \( q \). Since at most one of two parallel facets can be visible, either \( H_q P_i = P_i \) if \( q \in P_i \), or \( \bigcap_{i=1}^m F_i \neq \emptyset \). In particular, \( 0 \leq m \leq r \). Let \( w \in \bigcap_{i=1}^m F_i \) be a vertex of \( P_i \). As \( P_i \) is a simple polytope, the vertex figure at \( w \) is a simplex, i.e., every facet containing \( w \) is uniquely determined by the neighbor vertex of \( w \) it does not contain. For \( 1 \leq i \leq m \), let \( w_i \) be the neighbor of \( w \) that is not contained in \( F_i \) and let \( w_{m+1}, \ldots, w_r \) be the other neighbors of \( w \). Let \( v_i := w_i - w \) for all \( 1 \leq i \leq r \). Then with \( I = \{1, \ldots, m\} \) we obtain
\[ H_q P_i = P_i \setminus \bigcup_{i=1}^m F_i = \Phi(I) + w. \]
As a consequence we deduce one of our main theorems.

Proof of Theorem 1.2. By Proposition 4.9 there is a partition \( Z = H_q P_1 \cup \cdots \cup H_q P_k \) into half-open zonotopes, where each \( H_q P_i \) equals, up to translation, some \( \Phi(I) \). By Corollary 4.7 and Theorem 3.1, \( h^\varphi(Z)(t) = \sum_i h^\varphi(H_q P_i)(t) \) is real-rooted. Moreover, every \( H_q P_i \) has a unimodal \( h^\varphi \)-vector with peak at \( \frac{r}{2} \) if \( r \) is even and peak at \( \frac{r-1}{2} \) or \( \frac{r+1}{2} \) if \( r \) is odd. The same is true for \( h^\varphi(Z) \) since it is a nonnegative linear combination of \( h^\varphi(H_q P_1), \ldots, h^\varphi(H_q P_k) \). □

Figure 2. A zonotope and one of its half-open decomposition into parallelepipeds.
4.4. **Ehrhart $h^*$-vectors of zonotopes.** Theorem 4.6 allows us to explicitly describe the convex hull of all Ehrhart $h^*$-vectors of zonotopes. Let
\[
Z_d = \text{conv}\{h^\varepsilon(P) : P \text{ $d$-dimensional lattice zonotope}\}
\]
and
\[
P_d = \text{conv}\{h^\varepsilon(P) : P \text{ $d$-dimensional lattice parallelepiped}\}.
\]
For all $1 \leq j \leq d$ set $A_j(d) = (A_j(d,0), A_j(d,1), \ldots, A_j(d,d))$. Theorem 1.3 says that
\[
Z_d = P_d = A_1(d+1) + \mathbb{R}_{\geq 0} A_2(d+1) + \cdots + \mathbb{R}_{\geq 0} A_{d+1}(d+1),
\]
and that this cone is simplicial.

**Proof of Theorem 1.3.** From the proof of Theorem 1.2 we see that the $h^*$-polynomial of every zonotope is a nonnegative linear combination of $(A, j)$-Eulerian polynomials. Moreover, since in every half-open decomposition used in the proof, there is at most one closed parallelepiped, we see from Theorem 4.6 that the multiplicity of $A_1(d+1)$ equals $b(\emptyset) = 1$. Therefore, $Z_d$ and $P_d$ are contained in $A_1(d+1) + \mathbb{R}_{\geq 0} A_2(d+1) + \cdots + \mathbb{R}_{\geq 0} A_{d+1}(d+1)$.

For the reverse inclusions, since $P_d \subseteq Z_d$, it suffices to prove that for every integer $m$ and every $2 \leq k \leq d+1$ there is a parallelepiped $P_{k,m}$ with $h^\varepsilon(P_{k,m}) = A_1(d+1) + m A_k(d+1)$. Consider the parallelepiped
\[
P_{k,m} = \left\{ \sum_{i=1}^{d} \lambda_i v_i : 0 \leq \lambda_i \leq 1 \right\}
\]
with $v_k = e_1 + \cdots + e_{k-1} + (m+1)e_k$ and $v_i = e_i$ for all other $i$. From the proof of [24, Theorem 2.2] it follows that $|\Pi(I) \cap \mathbb{Z}^d|$ equals the absolute value of the maximal minor of the matrix with columns $\{v_i\}_{i \in I}$. It is therefore not hard to calculate that $|\Pi(I) \cap \mathbb{Z}^d| = m + 1$ if and only if $[k] \subseteq I$, and $|\Pi(I) \cap \mathbb{Z}^d| = 1$ otherwise. Thus, by Lemma 4.5 and Möbius inversion,
\[
b_{c}(I) = \begin{cases} 
    m & \text{if } I = [k], \\
    1 & \text{if } I = \emptyset, \\
    0 & \text{otherwise.}
\end{cases}
\]

Theorem 4.6 now gives $P_{k,m} = A_1(d+1) + m A_k(d+1)$.

Moreover, the vectors $A_1(d+1), \ldots, A_{d+1}(d+1)$ are linearly independent: by Corollary 4.4, $\text{ehr}_{C_j}(n) = t^j(1 + t)^{d-j}$, and the polynomials $\{t^j(1 + t)^{d-j}\}_{j=0,\ldots,d}$ form a basis of the space of polynomials of degree at most $d$. Considering instead the $h^*$-polynomial defines a basis transformation, and $h^\varepsilon(C_j(t)) = A_{j+1}(d+1, t)$ by Theorem 4.2, so the polynomials $\{A_{j+1}(d+1, t)\}_{j \in [d]}$ also define a basis.

Theorem 1.3 naturally gives rise to the following open problem.

**Problem 1.** *Characterize the sets of all $h^*$-vectors of $d$-dimensional parallelepipeds/zonotopes.*

We suspect that this problem is quite nontrivial. From the proof of Theorem 1.2 we see, that every such $h^*$-vector is contained in $A_1(d+1) + \mathbb{Z}_{\geq 0} A_2(d+1) + \cdots + \mathbb{Z}_{\geq 0} A_{d+1}(d+1)$. However, it is easy to check that already for $d = 2$,
\[
A_1(d+1) + \mathbb{Z}_{\geq 0} A_2(d+1) + \cdots + \mathbb{Z}_{\geq 0} A_{d+1}(d+1) \nsubseteq C^d \cap \mathbb{Z}^d,
\]
and so $C^d \cap \mathbb{Z}^d$ cannot be the right answer. But $A_1(d+1) + \mathbb{Z}_{\geq 0} A_2(d+1) + \cdots + \mathbb{Z}_{\geq 0} A_{d+1}(d+1)$ does not characterize all $h^*$-vectors of $d$-zonotopes either, since with Lemma 3.3 it is not hard to see that this affine semigroup contains infinitely many symmetric vectors, i.e., vectors of the form $(a_0, a_1, \ldots, a_d)$ with $a_i = a_{d-i}$ for all $0 \leq i \leq d$. By a theorem of Hibi [7], the
corresponding polytopes are reflexive, however, there are only finitely many reflexive polytopes in each dimension, by a result of Lagarias and Ziegler [12]. Although a complete solution of Problem 1 might be out of reach, it would be very interesting to see whether the set of \( h^* \)-vectors exhibits interesting combinatorial or algebraic structures.

5. Alternatingly increasing \( h^* \)-vectors

In [19] Schepers and Van Langenhoven conjectured that the Ehrhart \( h^* \)-vector of every integrally closed polytope that has an interior lattice point is alternatingly increasing.

**Conjecture 1** [19]. Let \( P \) be an \( r \)-dimensional integrally closed polytope with a lattice point in its relative interior. Then

\[
0 \leq h^*_0(P) \leq h^*_1(P) \leq h^*_r(P) \leq \cdots \leq h^*_{\lfloor \frac{r+1}{2} \rfloor}(P).
\]

Schepers and Van Langenhoven proved their conjecture for lattice parallelepipeds with an interior lattice point. Towards that conjecture, we prove that the \( h^* \)-vector with respect to any combinatorially positive valuation of any lattice centrally symmetric and coloop-free zonotope is alternatingly increasing.

5.1. Type-B zonotopes. A type-B zonotope (or lattice centrally symmetric zonotope) in \( \mathbb{R}^d \) is a lattice zonotope that arises as a projection of \([-1,1]^m\) for some \( m \geq d \). Equivalently, a zonotope \( Z \) is lattice centrally symmetric if there are vectors \( v_1, \ldots, v_m \in \mathbb{Z}^d \) such that

\[
Z = \left\{ \sum_{i=1}^m \lambda_i v_i : -1 \leq \lambda_i \leq 1 \right\}
\]

up to translation by an element in \( \mathbb{Z}^d \). A central role in determining the \( h^* \)-polynomial of lattice centrally symmetric zonotopes is played by the type-B Eulerian polynomials. To that end we define the half-open \pm 1-cube \([-1,1]^l_1\), where \( d \geq 1 \) and \( 0 \leq l \leq d \), by

\[
[-1,1]^l_1 := [-1,1]^d \setminus \left\{ x_d = 1, x_{d-1} = 1, \ldots, x_{d+1-l} = 1 \right\},
\]

and the \( l \)-descent set, \( \text{Des}_l(\pi, \epsilon) \subseteq \{0,1,\ldots,d\} \), of signed permutation \( (\pi, \epsilon) \in B_d \) by

\[
\text{Des}_l(\pi, \epsilon) := \begin{cases} 
\text{Des}(\pi, \epsilon) \cup \{d\} & \text{if } d + 1 - l \leq \epsilon_d \pi_d \leq d, \\
\text{Des}(\pi, \epsilon) & \text{otherwise}.
\end{cases}
\]

The cardinality of this set is the (natural) \( l \)-descent number of \( (\pi, \epsilon) \), denoted by

\[
\text{des}_l(\pi, \epsilon) := |\text{Des}_l(\pi, \epsilon)|.
\]

**Theorem 5.1.**

\[
\text{Ehr} \left( [-1,1]^d_l, t \right) = B_{l+1}(d+1,t) \frac{(t^d+1)}{(1-t)^{d+1}}.
\]

To prove the theorem we will use the following lemma which parallels Lemma 4.1.

**Lemma 5.2.** Let \( d \geq 1 \) and \( 0 \leq l \leq d \). Then

\[
b_{l+1}(d+1,k) = \left| \left\{ (\pi, \epsilon) \in B_d : \text{des}_l(\pi, \epsilon) = k \right\} \right|.
\]

**Proof of Theorem 5.1.** We use a half-open decomposition of \([-1,1]^d\) induced by the hyperplanes given by \( x_i = \pm x_j \) and \( x_i = 0 \), for \( 0 \leq i < j \leq d \). As before, we obtain a decomposition into half-open unimodular simplices...
The number of missing facets of $\Delta_{(\pi, \epsilon)}^{d,l}$ is therefore $\text{des}_i(\pi, \epsilon)$. Observe that

$$[-1, 1]^d = \bigcup_{(\pi, \epsilon) \in B_d} \Delta_{(\pi, \epsilon)}^{d,l}$$

as a disjoint union. Therefore,

$$\text{Ehr} \left([-1, 1]^d, t\right) = \sum_{\sigma \in S_d} \text{Ehr} \left(\Delta_{(\pi, \epsilon)}^{d,l}, t\right) = \sum_{\sigma \in S_d} t^{\text{des}_i(\pi, \epsilon)} = \frac{B_{t+1}(d + 1, t)}{(1 - t)^{d+1}},$$

where the last equality holds by Lemma 5.2.

Proof of Proposition 3.8. By Theorem 5.1, the Ehrhart $h^*$-polynomial of $[-1, 1]^d$ equals $B_{t+1}(d + 1, t)$. Theorem 4.6 allows us to give an expression in terms of $(A, j)$-Eulerian polynomials: observe that $[-1, 1]^d$ is lattice isomorphic to $\Phi([l])$ for $v_1 = 2e_1, v_2 = 2e_2, \ldots, v_d = 2e_d$. Since $|\square(K) \cap \mathbb{Z}^d| = 1$ for all $K \subseteq [d]$ and by simple counting,

$$B_{t+1}(d + 1, t) = \sum_{K \subseteq [d] \cap \mathbb{Z}^d} B_{|\square(K) \cup \mathbb{Z}^d|+1}(d + 1, t) = 2^d \sum_{j=0}^{d-1} \binom{d - l}{j} A_{j+t+1}(d + 1, t).$$

Up to lattice translation, every half-open type-$B$ parallelepiped equals $2 \Phi(I)$ for some $v_1, \ldots, v_d \in \mathbb{Z}^d$.

Theorem 5.3. Let $\varphi$ be a $\mathbb{Z}^d$-valuation. Then

$$\text{Ehr}^\varphi \left(2 \Phi(I), t\right) = \frac{\sum_{K \subseteq [d]} b_{\varphi}(K) B_{|I \cup K|+1}(d + 1, t)}{(1 - t)^{d+1}}.$$ 

Proof. Since $[-1, 1]^d$ is lattice isomorphic to $2C_d$, by Corollary 4.4

$$\text{ehr}_{[-1, 1]^d}(n) = \sum_{|I| \leq J \subseteq [d]} (2n)^{|J|}.$$ 

By Lemmas 4.3 and 4.5,

$$\text{Ehr}^\varphi \left(2 \Phi(I), t\right) = \sum_{n=0}^{\infty} t^n \sum_{J \supseteq I} (2n)^{|J|} \varphi(\Pi(J))$$

$$= \sum_{n=0}^{\infty} t^n \sum_{J \supseteq I} (2n)^{|J|} \sum_{K \subseteq J} b_{\varphi}(K)$$

$$= \sum_{K \subseteq [r]} b_{\varphi}(K) \sum_{n=0}^{\infty} t^n \sum_{J \supseteq I \cup K} (2n)^{|J|},$$

and so the claim follows from Theorem 5.1. □
Corollary 5.4. The $h^*$-polynomial with respect to any combinatorially positive valuation of a half-open type-$B$ parallelepiped has alternatingly increasing coefficients.

Proof. Since $\varphi$ is combinatorially positive, by Theorems 2.2 and 5.3, the $h^*$-polynomial is a non-negative linear combination of $(B,l)$-Eulerian polynomials. These polynomials are alternatingly increasing (Theorem 3.9), and so is every nonnegative linear combination. \hfill $\Box$

Corollary 5.5. The $h^*$-polynomial of a type-$B$ zonotope has alternatingly increasing coefficients.

Proof. From Theorem 4.8 and Corollary 4.9 we see that every type-$B$ zonotope can be partitioned into half-open type-$B$ parallelepipeds, which are up to translation all of the form $2\Phi(I)$ for some spanning vectors. By Corollary 5.5 every such half-open type-$B$ parallelepiped has an alternatingly increasing $h^*$-polynomial, and so does their sum which equals the $h^*$-polynomial of the zonotope. \hfill $\Box$

5.2. Matroidal aspects of $h^*$-polynomials. The terminology used in this section originates from matroid theory; see, e.g., [16]. Let $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{Z}^d$ be a set of vectors, and $v_1 < \cdots < v_n$ be a fixed order on the elements of $V$. Let $\mathcal{I}$ denote the set of independent subsets of $V$ and $\mathcal{B}$ be the collection of maximally independent subsets (i.e., bases). Without loss of generality, in the sequel we assume that $V$ spans $\mathbb{R}^d$. For simplicity, we identify $v_i$ with $i \in [n]$ in the sequel. The order on $[n]$ induces an order on $\mathcal{B}$, namely $B_1 < B_2$ whenever $B_1$ is lexicographically smaller than $B_2$. For every independent set $I \in \mathcal{I}$, we define $|I| := \min_{B \in \mathcal{B}} \{|I| \subseteq B\}$, i.e., $|I|$ is the smallest basis that contains $I$. An element $i$ in a basis $B$ is called internally passive if there is an element $j < i$ with $j \notin B$, such that $\{j\} \cup B \setminus \{i\}$ is a basis. In other words, $i$ can be exchanged with a smaller element $j$. We denote the set of internally passive elements of $B$ by $\text{IP}(B)$. Note that, in particular, $\text{IP}(B) \subseteq B$.

Lemma 5.6. Let $I \in \mathcal{I}$ and $B \in \mathcal{B}$. If $|I| \neq B$ then there is an $i \in \text{IP}(B)$ such that $I \subseteq B \setminus \{i\}$.

Proof. Without loss of generality we may assume that $|I| \cap B = I$. Since $|I| \neq B$, there is $j \in |I| \setminus I$ that is smaller than all elements in $B \setminus I$. Since $j \cup I \in \mathcal{I}$, by Steinitz’s Exchange Lemma, there exists $i \in B \setminus I$ such that $j \cup B \setminus \{i\}$ is a basis, in particular, $i \in \text{IP}(B)$ and moreover $I \subseteq B \setminus \{i\}$. \hfill $\Box$

Lemma 5.7. Let $I \in \mathcal{I}$ and $B \in \mathcal{B}$ such that $I \subseteq B$. Then

$$|I| = B \iff \text{IP}(B) \subseteq I.$$ 

Proof. For the forward direction, suppose that there is an $i \in \text{IP}(B)$ that is not contained in $I$. Since $i \in \text{IP}(B)$ we obtain $[B \setminus \{i\}] \neq B$. Because $I \subseteq B \setminus \{i\}$ it also follows that $|I| \neq B$, which is a contradiction.

For the backward direction, it suffices to prove $|\text{IP}(B)| = B$. Suppose $|\text{IP}(B)| \neq B$, then by Lemma 5.6 there exists an $i \in \text{IP}(B)$ with $\text{IP}(B) \subseteq B \setminus \{i\}$, a contradiction. \hfill $\Box$

In the sequel we freely make use of the notation from Section 4.2. The following lemma by Stanley [24] (with a proof by Ziegler) was used to prove Theorem 1.1.

Lemma 5.8 [24, Lemma 2.1]. Let $Z$ be the zonotope generated by $V$. Then

$$Z = \bigcup_{I \in \mathcal{I}} \Pi(I)$$

up to translation of each half-open parallelepiped on the right-hand side.
With Lemma 5.8, [24, Theorem 2.2] extends immediately to calculating $\text{ehr}^\varphi_Z(n)$ for arbitrary $\mathbb{Z}^d$-valuations. We formally record this with the following proposition.

**Proposition 5.9.** Let $Z$ be the zonotope generated by $V$, and let $\varphi$ be a $\mathbb{Z}^d$-valuation. Then $$\text{ehr}^\varphi_Z(n) = \sum_{I \in \mathcal{I}} \varphi(\Pi(I))n^{|I|}.$$ 

We are now able to prove Theorem 1.4.

**Proof of Theorem 1.4.** By Lemmas 4.3 and 5.7, up to translation of the half-open parallelepipeds, $$Z = \biguplus_{B \in \mathcal{B}} \biguplus_{I : |I| = B} \Pi(I) = \biguplus_{B \in \mathcal{B}} \Phi(\text{IP}(B)).$$ The claim now follows from Theorem 4.6. \hfill \Box

If the vectors in $V$ are the columns of a totally unimodular matrix, and $\varphi = \varepsilon$, the result simplifies.

**Corollary 5.10.** Let $Z$ be the zonotope generated by the columns of a totally unimodular matrix. Then $$h^{\varepsilon}(Z)(t) = \sum_{B \in \mathcal{B}} A_{|\text{IP}(B)|+1}(d+1,t).$$

A set of vectors $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ is called **coloop free** if there is no $v_i$ that is contained in every maximally independent subset of $V$. Equivalently, $V$ is coloop free if every linearly independent subset of $V$ that is not a basis is contained in at least two different bases formed by elements in $V$.

**Theorem 5.11.** Let $V \subset \mathbb{Z}^d$ be coloop free, let $Z$ be the zonotope generated by $V$ and let $\varphi$ be a combinatorially positive valuation. Then $h^\varphi(Z)(t)$ is alternatingly increasing.

**Proof.** For all $B \in \mathcal{B}$ let $\text{IP}(B)$ denote the internally passive elements of $B$ with respect to the reverse ordering $v_n < \ldots < v_1$. By Theorem 4.6, $$h^\varphi(Z)(t) = \frac{1}{2} \sum_{B \in \mathcal{B}} \sum_{I \subseteq B} b_\varphi(I) \left( A_{|I \cup \text{IP}(B)|+1}(d+1,t) + A_{|I \cup \text{IP}(B)|+1}(d+1,t) \right).$$

Since $V$ is coloop free, for every $B \in \mathcal{B}$ and every $i \in B$ there is $j \neq i$ such that $j \cup B \setminus i$ is a basis. Since either $j < i$ or $j > i$ we obtain $i \in \text{IP}(B) \cup \overline{\text{IP}(B)}$. Therefore, $|\text{IP}(B)| + |\overline{\text{IP}(B)}| \geq |B| = d$ for every $B \in \mathcal{B}$, and hence $$|I \cup \text{IP}(B)| + 1 + |I \cup \overline{\text{IP}(B)}| + 1 \geq d + 2$$ for all $I \subseteq B$. Since $\varphi$ is combinatorially positive, $b_\varphi(I) \geq 0$ for all $I \subseteq B$ and therefore the claim follows from Lemma 3.11, since every nonnegative linear combination of alternatingly increasing sequences is itself alternatingly increasing. \hfill \Box

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