SUBSPACES OF MONOTONICALLY NORMAL COMPACTA

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Abstract. We analyze the structure of topological spaces having monotonically normal compactifications. By [7] and [2], it is immediate that under PFA, the class $\mathcal{M}$ of uncountable subspaces of monotonically normal compacta has a 3-element basis. We show that the same holds under $\text{MA}_{\omega_1}$. Variants of the “canonical” basis are exhibited, further elucidating the structure of $\mathcal{M}$. Our main tools are Nikiel’s conjecture [16] and a combinatorial lemma we establish for continuous images of compact ordered spaces. Not surprisingly, the double arrow space figures centrally in this setting.

Introduction

In [19], Todorcevic set out an outstanding program to inspect basis problems in combinatorial set theory. The blueprint can be described as follows: given a class $\mathcal{S}$ of structures, identify a collection of critical members of $\mathcal{S}$ which is in some sense complete (in other words, a basis). If, for instance, $\mathcal{S}$ is a class of topological spaces, and a basis being complete is interpreted in terms of embedding, then a basis for $\mathcal{S}$ is a subclass $\mathcal{S}_0$ such that each element of $\mathcal{S}$ contains a (homeomorphic) copy of an element of $\mathcal{S}_0$.

Let $\mathcal{S}$ be the class of uncountable first countable regular spaces. In the context of his study of perfectly normal compacta, Gruenhage [8] first formulated what could thus be called a basis conjecture for $\mathcal{S}$. Aptly named, the “3-element basis conjecture” for uncountable first-countable regular spaces states that it is consistent that each such space contains a set of reals of cardinality $\omega_1$ with either the metric, the Sorgenfrey, or the discrete topology, thus postulating a 3-element basis for $\mathcal{S}$. If true, this conjecture would afford a deep understanding of the structure of regular spaces, and even partial instances of it would have far-reaching consequences and settle long-open problems in set-theoretic topology.

Conjecture (19). Assume PFA. The class of uncountable first countable regular spaces has a three element basis consisting of $D(\omega_1)$, $B$ and $B \times \{0\}$, where $D(\omega_1)$ is the discrete space on $\omega_1$, $B$ is some uncountable subset of the unit interval, and $B \times \{0\}$ is considered as a subspace of the double arrow space.

A revision of the basis conjecture, background information, related open problems, and trends of research can be found in the excellent survey article [9]. A natural approach is, rather than work in the class of regular spaces, to try to prove weakenings of the basis conjecture for certain classes of spaces where conditions stronger than regularity are imposed. Thus, Gruenhage confirmed the basis conjecture for the class of cometrizable spaces (under PFA [7], Todorcevic later under OCA [18]). Todorcevic gave a ZFC positive answer in the class of subspaces of...
Rosenthal compacta [20]. Restricted to the class of subspaces of perfectly normal compacta, Gruenhage observed that the basis conjecture is equivalent, under PFA, to the fundamental conjecture about this class, due to Fremlin: every perfectly normal compactum admits an at most 2-to-1 continuous map onto a compact metric space (which in some sense says that in ZFC the only perfectly normal non-metrizable compacta are “relatives” of the double arrow space).

Apart from the paradigmatic [19], the reader is encouraged to check [18] and notice how several of our results are essentially restatements for monotonically normal compacta of results in [18] about Rosenthal compacta, and notice as well how other results contrast with their analogues in [18]. To emphasize, we excerpt some of the structure results we obtain.

**Theorem.** Suppose that $X$ is a monotonically normal compactum which does not contain an uncountable discrete subspace. The following are true:

1. Assume $\text{MA}_{\aleph_1}$. Then $X$ is separable, and $X$ admits a continuous and at most 2-to-1 map onto a compact metric space which is the union of a locally connected metric continuum and countably many isolated points. In particular, $X$ admits a continuous and at most 3-to-1 map $f$ onto a locally connected metric continuum $M$ satisfying $|\{m \in M : |f^{-1}(m)| = 3\}| \leq \aleph_0$.
2. $X$ contains an uncountable subspace of real type or $X$ contains a copy of the double arrow space.
3. If $X$ is not metrizable, then $X$ contains a subspace homeomorphic to $I_S$, for some uncountable $S \subseteq (0, 1)$ (check preliminaries). If $X$ has weight $\aleph_1$, then it contains a subspace homeomorphic to $I_{\aleph_1}$.

Hence the basis conjecture is true under $\text{MA}_{\aleph_1}$ for the class of subspaces of monotonically normal compacta, or what is the same, the class of spaces $\mathcal{M}$, where $X \in \mathcal{M}$ if and only if $X$ has a monotonically normal compactification. We note that a proof of this fact under PFA is immediate by [7] and [2]. The additional results provide sharpenings of this conclusion, further illuminating the structure of monotonically normal compacta. Our main tool is Nikiel’s conjecture, proved by M.E. Rudin.

**Theorem 0.1** (Nikiel’s conjecture [14], Rudin [16]). *The class of monotonically normal compacta coincides with the class of Hausdorff spaces which are continuous images of compact linearly ordered spaces.*

1. **Preliminaries**

All spaces are assumed uncountable. A *compactum* is a compact Hausdorff space. A *continuum* is a connected compactum. A *Peano continuum* is a locally connected metric continuum. An *arc* is a space homeomorphic to unit interval $[0, 1]$. Recall that a Hausdorff space is a continuous image of an arc if and only if it is a Peano continuum (the classical Hahn-Mazurkiewicz theorem), and that a Peano continuum is arcwise connected (see e.g. [3]).

A *linearly ordered space*, in short, an *ordered space*, is a linearly ordered set equipped with the open interval topology. A separable ordered continuum is homeomorphic to $[0, 1]$. The double arrow space, also called the split interval, is the space $[0, 1] \times \{0, 1\}$, lexicographically ordered and equipped with the interval topology. For $S \subseteq (0, 1)$, the space obtained by taking the lexicographic order topology on
[0,1] \times \{0\} \cup S \times \{1\}
will be denoted by \(I_S\). Notice that the double arrow space is
just \(I_{(0,1)}\).

Let \((L, \leq)\) be an ordered space. For \(x \in L\), let
\(L(x) = \{a \in L : a \leq x\}\) and
\(M(x) = \{a \in L : x \leq a\}\). A subset \(C \subseteq L\) is said to be convex if whenever
\(a, b, c \in L\) are such that \(a, b \in C\) and \(a < c < b\), then \(c \in C\). An order component
of a subset \(A\) of \(L\) is a subset \(C \subseteq L\) which is maximal with respect to the
properties “\(C \subseteq A\), \(C\) is convex”. A mapping \(f : L \to X\) is said to be order-light if
for each \(x \in X\), every order component of \(f^{-1}(x)\) is degenerate. A jump in \(L\) is an
element \(\{r, s\} \in [L]^2\) such that there is no point of \(L\) which lies between \(r\) and \(s\).

A space is submetrizable if it has a weaker metric topology. A space is cotmetrizable
if it has a weaker metric topology such that each point has a neighborhood
base consisting of sets closed in the metric topology. A space is orderable if it is
homeomorphic to a linearly ordered space. A space is suborderable if it can be
embedded in a linearly ordered space.

A map \(f : X \to Y\) is said to be (at most) \(\kappa\)-to-1 at \(x\) if \(|f^{-1}(f(x))| = \kappa\) \((\leq \kappa)\).
\(f\) is said to be (at most) \(\kappa\)-to-1 if for each \(x \in X\), \(f\) is (at most) \(\kappa\)-to-1 at \(x\).

We will utilize Nikiel’s conjecture without reference.

2. Results

Recall the following theorem of Eisworth [2], which, he informs in the intro-
duction, grew out of his attempt at understanding M.E. Rudin’s proof of Nikiel’s
conjecture: each separable monotonically normal compactum admits an at most
2-to-1 continuous map onto a compact metric space. We will prove a more general
version of this theorem.

A Hausdorff space which is a continuous image of a compact ordered space is
the continuous image of a compact ordered space under a map which is irreducible
and order-light. This is a lemma which appears, with somewhat long proofs, in [13]
and [21]. We extend this lemma by adding a condition crucial for our later results.

**Lemma 2.1.** Let \(f : K \to X\) be a continuous map of an ordered compactum \(K\) onto
a Hausdorff space \(X\). Then there are an ordered compactum \(K’\) and a continuous
map \(f’ : K’ \to X\) such that \(f’\) is surjective, irreducible, order-light, and \(f’\) restricted
to the union of all jumps of \(K’\) is 1-to-1.

**Proof.** That \(f’\) may be taken to be irreducible follows by an easy application of
Zorn’s lemma (we refer the reader to [21]). We derive all the remaining conditions
in one go, utilizing Wallman’s representation theorem (see e.g. [17]). Let \(B\) be
a lattice-base for the closed subsets of \(X\) of minimal size (that is, the weight of
\(X\)), and identify it with its copy \(\{f^1[B] : BB\} \in 2^K\). Denote by \(H\) the set of
all sublattices \(L\) of \(2^K\) containing \(B\), and generated by a subbasis of the form
\(\{L(x)\}_{x \in w(L)} \cup \{M(x)\}_{x \in w(L)}\), where \(w(L)\) denotes the Wallman space of \(L\) (this
condition just specifies that \(w(L)\) is a linearly ordered compactum; there should
be no trouble seeing that this is possible, for, we may even take the sublattices to
be elementary substructures of \(2^K\) with the requirement that their Wallman spaces
are ordered spaces). Let \(J\) be a non-empty chain in \(H\). It is not hard to see that
\(J = \bigcap J \in H\). One way of seeing this is to look at the Wallman spaces. Let
\(\sup(J)\) be the direct system of the Wallman spaces of elements of \(J\), and take the
set-theoretic direct limit \(Z\) of \(w(J)\) with the final topology. It is readily verified
that \(Z\) is an ordered compactum, and that it is the Wallman space of \(J\). This shows
that $J \in \mathcal{H}$. Hence, by Zorn’s lemma there exists a minimal element $H$ of $\mathcal{H}$. Put $K' = w(H)$.

That $K'$ is a compact ordered space follows immediately from construction. Since $B \subset H$, $K'$ admits an continuous map $f'$ onto $X$. Suppose there are two points $a, b \in K'$ such that $f'(a) = f'(b)$ such that there is no point $z$ between $a$ and $b$ satisfying $f(z) = f(a)$. Let $A$ be the quotient of $K'$ obtaining by collapsing the interval between $a$ and $b$ to one point. The $A$ is a linearly ordered compactum which admits a continuous map onto $X$, and $2^A$ is a proper subset of $2^{K'}$, a contradiction. Hence $f'$ is order-light. Now suppose that there are two point $a, b \in K'$ such that each of $a$ and $b$ belongs to a jump, and $f'(a) = f'(b)$. It doesn’t take a lot of effort to observe that we can take a reordering of $K'$ and a quotient of that reordering by identifying $a$ and $b$ in that reordering so as to obtain a compact ordered space $A$ which is a quotient of $K'$, and which again satisfies that $2^A$ is a proper subset of $2^{K'}$. This finishes the proof.\[\Box\]

Lemma 2.1 enables us to obtain a short proof of a generalization of Eisworth’s result [2].

**Theorem 2.1.** Every dense-in-itself separable monotonically normal compactum admits a continuous and at most 2-to-1 map onto a locally connected metric continuum.

**Proof.** Let $X$ be a dense-in-itself separable monotonically normal compactum. By Nikiel’s conjecture, $X$ is an image of a compact ordered space $K$ under a continuous map $f$. Let $D$ be a countable dense subset of $X$, and let $D'$ a countable subset of $K$ which is mapped onto $D$ by $f$. Then $\overline{D'}$ is a separable compact ordered space, and the restriction of $f$ to $\overline{D'}$ maps continuously onto $X$. Hence, we may assume without loss of generality that $K$ is separable. Assume also that $f$ and $K$ satisfy the conditions of lemma 2.1. Identify now each two points of $K$ that are the endpoints of a jump in $K$. The resulting space $K'$ is connected and so is a separable ordered continuum; that exactly means that $K'$ is homeomorphic to $[0,1]$. Identify then each two points of $X$ that are respective images of two points of $K$ which form a jump in $K$. The quotient map is obviously at most 2-to-1, and it is easy to see that the quotient space $X'$ is an image of $[0,1]$. Hence, by the classical Hahn-Mazurkiewicz theorem, $X'$ is a locally connected metric continuum.\[\Box\]

Since a separable space can have only countably many isolated points, we have the following.

**Corollary 2.2.** Every separable monotonically normal compactum admits a continuous and at most 2-to-1 map onto a locally connected metric space which is the union of a locally connected metric continuum and countably many isolated points.

One additional reading of this result is: each separable monotonically normal compactum admits an at most 3-to-1 continuous map $f$ onto a locally connected metric continuum $M$ satisfying $|\{m \in M : |f^{-1}(m)| = 3\}| \leq \aleph_0$.

Now we turn our attention to studying the structure of monotonically normal compacta.

**Theorem 2.3.** Assume $\text{MA}_{\aleph_1}$. The three element basis conjecture holds for the class of subspaces of monotonically normal compacta.

**Remark 2.4.** Before we begin the proof, we note the following:
(1) Theorem 2.3 is not subsumed by known results. The double arrow space does not have a weaker metric topology; hence it is not even submetrizable. On the other hand, there is a continuous image of the double arrow space (hence a compact monotonically normal space) which is not Rosenthal compact [5].

(2) MA$_{\aleph_1}$ is necessary. A left separated subspace of a Souslin line is a counterexample; MA$_{\aleph_1}$ negates the existence of Souslin lines.

(3) For the same reason as (2), the 3-element basis conjecture is equivalent, under MA$_{\aleph_1}$, to the following conjecture: each uncountable first countable regular space contains an uncountable suborderable subspace [8].

Proof of theorem 2.3. First, we reduce the problem to one about separable monotonically normal compacta. Let $X$ be a monotonically normal compactum. Then $X$ contains a dense orderable subspace $D$ [22]. If $D$ is uncountable, then by part (3) of the above remark, MA$_{\aleph_1}$ implies that the basis conjecture holds for $X$. If $D$ is countable, $X$ is separable.

The proof will now follow by an application of the above theorem. Since each uncountable compact metric space contains a homeomorphic copy of the Cantor set, it follows that a separable monotonically normal compactum must contain either an uncountable subspace of the real line or an uncountable subspace of the Sorgenfrey line. For, let $f$ be a continuous and at most 2-to-1 map of $X$ onto a compact metric space, and let $g$ be a continuous map of a quotient $H$ of the double arrow space onto $X$ satisfying the conclusion of lemma 2.1 (we are able to obtain this since each separable continuous image of a compact ordered space is an image of the double arrow space). If $f$ is 1-to-1 on an uncountable subset of $X$, then it is easy to see that $X$ contains an uncountable subspace of the unit interval. If $X$ contains no uncountable metrizable subspace, then we can extract an uncountable subspace of the Sorgenfrey line by inspecting the map $g$. It is not hard to calculate this on fingers, given the properties of the maps $f$ and $g$, specifically utilizing that $g$ is 1-to-1 on the union of all jumps in $H$. Hence, we have proved that the three element basis conjecture holds for monotonically normal compacta. Now, since each subspace of a monotonically normal compactum is a space with a monotonically normal compactification (i.e., it is an uncountable dense subspace of a monotonically normal compactum), it is easy to see that the conclusion actually holds for subspaces of monotonically normal compacta. This completes the proof.

Next, we state some consequences which provide us with alternative 3-element bases for our class $M$ of uncountable subspaces of monotonically normal compacta.

**Theorem 2.5.** Assume MA$_{\aleph_1}$. Every monotonically normal compactum $X$ satisfies one of the following alternatives:

(i) $X$ contains an uncountable discrete subspace.

(ii) $X$ is separable.

**Proof.** If $X$ is not ccc, then it contains an uncountable discrete subspace. If $X$ is ccc, then $X$ is hereditary Lindelöf [15], and hence perfectly normal. MA$_{\aleph_1}$ then implies that $X$ is separable [11].

**Theorem 2.6.** Every separable monotonically normal compactum $X$ satisfies one of the following alternatives:
(i) \(X\) contains an uncountable subspace of real type.
(ii) \(X\) contains a copy of the double arrow space.

Proof. Suppose \(X\) does not contain an uncountable subspace of real type. Applying theorem 2.1, we obtain a subset \(A\) of \(X\) that admits an at-most 2-to-1 continuous map \(f\) onto the Cantor set. \(f\) can not be 1-to-1 on an uncountable subset of \(A\), and \(A\) contains at most countably many isolated points. Let \(A'\) be the set obtained from \(A\) by deleting the set of all the points of \(A\) which are either isolated, or on which \(f\) is 1-to-1. Clearly, \(f\) restricted to \(A'\) is exactly 2-to-1 and \(f(A')\) contains a Cantor set. By deleting countably many points of the preimage of that Cantor set, we can obtain a copy of the double arrow space. \(\square\)

Theorem 2.7. Every separable monotonically normal compactum \(X\) satisfies one of the following alternatives:

(i) \(X\) is a metric compactum.
(iii) \(X\) contains a subspace of the form \(I_S\), for some uncountable \(S \subseteq (0,1)\).

Proof. Same ideas apply as in the last proof. The only difference is that here, \(f\) can not be 1-to-1 on a co-countable subset of \(A\) (i.e. the set of points on which \(f\) is not 1-to-1 is uncountable). \(\square\)

Recall that a set \(B \subseteq \mathbb{R}\) is said to be \(\kappa\)-dense if every interval of \(\mathbb{R}\) meets \(B\) in \(\kappa\) many points.

Theorem 2.8. Every separable monotonically normal compactum \(X\) of weight \(\aleph_1\) contains a subspace of the form \(I_S\), where \(S\) is \(\aleph_1\)-dense in \((0,1)\).

Proof. Again, what we need to notice is that we may take a subset of \(A\) which is \(\aleph_1\)-dense in \(A\), and on which \(f\) is exactly 2-to-1. \(\square\)

PFA implies that every pair of \(\aleph_1\)-dense subsets of \(\mathbb{R}\) are order-isomorphic \([1]\). Hence, all spaces of the form \(I_S\), where \(S\) is an \(\aleph_1\)-dense subset of \((0,1)\), are homeomorphic. In other words, there is a unique homogeneous separable linearly ordered compactum of weight \(\aleph_1\). We denote it by \(I_{\aleph_1}\).

Corollary 2.9. Assume PFA. Every separable monotonically normal compactum \(X\) of weight \(\aleph_1\) contains a subspace homeomorphic to \(I_{\aleph_1}\).

Note that the Cantor set is just the space \(I_{\aleph_0}\), where we blow up a countable dense subset of \((0,1)\).

Theorem 2.10. Every separable monotonically normal compactum contains a subspace of the form \(I_S\), \(S \subseteq (0,1)\).

An arc is said to be \(C^1\) if it has a continuously differentiable arc length parametrization. In alternative (iii) of corollary 2.6, \(I_S\) was discovered by choosing an uncountable subset of \(R \subseteq M\), from alternative (iii) of theorem 2.4, lying on an arc \(I\) in \(M\). A pleasant observation is that, under PFA, the arc \(I \subseteq M\) may be chosen to be a \(C^1\) arc (follows from \([10]\)).

Theorem 2.11. Assume PFA. Let \(X\) be a monotonically normal compactum that contains no uncountable discrete subspace and has weight \(\aleph_1\). Then \(X\) is the union of a metric space and countably many copies of \(I_{\aleph_1}\).
3. A Question

In our work, we inherently used the proximity between monotonically normal compacta and linearly ordered compacta. It is natural to ask whether it is consistent that there is a 3-element basis for the class of uncountable monotonically normal spaces. This makes sense, from the point of view of the conjecture itself, because monotone normality is a severe restriction in terms of separation axioms when compared with regularity. Testing the conjecture on this class will hopefully give some insight on possible approaches to the conjecture in its full generality.

**Question.** Is it consistent (with MA$_{\aleph_1}$) that there is an uncountable ccc monotonically normal space that does not have a monotonically normal compactification, and that does not contain a set of reals of cardinality $\aleph_1$ with either the metric or the Sorgenfrey topology?

A positive answer to this question implies that for monotonically normal spaces, we can’t do much without (some sort of) compactness. A negative answer implies of course that the basis conjecture is true for uncountable monotonically normal spaces.

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