MATHEMATICAL ANALYSIS OF BUMP TO BUCKET PROBLEM

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Abstract. In this article, several systems of equations which model surface water waves generated by a sudden bottom deformation (bump) are studied. Because the effect of such deformation are often approximated by assuming the initial water surface has a deformation (bucket), this procedure is investigated and we prove rigorously that by using the correct bucket, the solutions of the regularized bump problems converge to the solution of the bucket problem.

1. Introduction. The aim of this investigation is to study several well known equations when it is used to describe surface water waves generated by an instantaneous moving bottom. Because the equation is posed on a domain involving a free surface $\eta(x, y, t)$ on top, which is an unknown, and a singular moving bottom $h(x, y, t) = \chi(t)B(x, y)$ (Bump), where $\chi(t)$ is the heaviside function, it is rather challenging to investigate both theoretically and numerically. When the movement of the bottom is instantaneous, it is conceivable that the effect is similar if not the same to the effect of starting the wave with an initial condition (Bucket). Therefore, many numerical simulations are carried out by using initial wave profile to approximate the effect of bottom deformation. We will analyze the “Bump” problem and compare the solutions of “Bump” problem with a related “Bucket” problem (B2B). Our investigation here is restricted to relatively long and small amplitude waves in one space dimension, so the result here will shed some light on the complicated problem related to the full equation. For the small amplitude and long waves,
the free surface and moving domain problem where the fluid occupies the domain \((-\infty, \infty) \times (-1 + h(x,t)), \eta(x,t))\) can be transformed into the Boussinesq systems

\[
\begin{align*}
\eta_t + u_x + ((\eta + h)u)_x + au_{xxx} - bh_{xxx} &= -h_t + \mu h_{xxt}, \\
u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} &= -(P_0)_x + (1 - \theta)h_{xxt},
\end{align*}
\]

(see [4] and the references in there). In the above formula, the length scale is taken to be \(h_0\) (the average depth of still water) and the time scale is taken to be \(h_0/c_0\) with \(c_0 = \sqrt{gh_0}\), where \(g\) is the gravitational acceleration constant. There is a genuine ambiguity on how the vertical coordinate \(z\) should be defined, so we are trying to be precise. The vertical origin is taken at the still water level, red where \(\eta(x,t)\) is the scaled displacement of water surface from its equilibrium position \(z = 0\) and \(u(x,t)\) is the scaled velocity at \(z = -(1 + h(x,t)) + \theta(\eta(x,t) + 1 + h(x,t))\). The constants \(a, b, c, d\) and \(\mu\) are given by:

\[
\begin{align*}
a &= \left(\frac{\theta^2}{2} - \frac{1}{6}\right) \lambda, \quad b = \left(\frac{\theta^2}{2} - \frac{1}{6}\right) (1 - \lambda), \quad c = \frac{1 - \theta^2}{2} \nu, \\
d &= \frac{1 - \theta^2}{2} (1 - \nu), \quad \mu = \frac{1}{2} \left(1 - \lambda\right) \left(\frac{\theta^2}{3} - \frac{1}{3}\right) - 2\theta + 1.
\end{align*}
\]

The parameters \(\theta\) is in \([0, 1]\), \(\lambda\) and \(\nu\) are free and can be chosen to fit certain requirements. For example, the classical Boussinesq system, the Bona-Smith system, the Kaup and the BBM-BBM system are systems in (1.1) with \((\theta^2, \lambda, \nu)\) equals to \((\frac{1}{3}, \mathbb{R}, 0), ((\frac{2}{3} - \nu)/(2 - \nu), 0, \nu < 0), (1, 1, \mathbb{R}), (\frac{2}{3}, 0, 0)\) respectively (cf. [2, 3]).

When there is a sudden (instantaneous) change on the bottom, such as in the case of an earthquake which may generate tsunami, \(h(x,t) = \chi(t)B(x)\) in (1.1) and the problem is denoted as “Bump” problem. It can be viewed as a problem with a singular additive forcing and a multiplicative forcing. The related “Bucket” problem is that \(h(x,t)\) in (1.1) is taken to be zero in the equations in (1.1) and the function \(B(x)\) which is related to the sudden movement of the bottom appears in initial data. In a case which is in between, namely \(B(x)\) appears in the initial data and in the equation, the problem is denoted as “quasi-Bucket”. Study a bump problem is to investigate a system with singular forcing and study the “Bump to Bucket” (B2B) problem resembles an inverse problem, namely by giving the solution of the bump problem and we are looking for the right bucket (initial data) such that the initial value problem has that as its solution.

The bucket problem is a standard initial value problem and there has been extensive studies on it (see [2, 3] and reference therein). In comparison, there are much less results on the Bump problems and most of the numerical and theoretical investigations are placed on the linear models. The “Bump” models (e.g. [4, 11, 12, 13, 14, 15, 17, 20, 22, 23, 24, 25]) proposed and applied in the generation of tsunamis during the past decades have limited rigorous theoretical analysis. To our knowledge, there are not much investigation on the theoretical comparison between the “Bump” and the “Bucket” problems on the nonlinear dispersive models.

In a series of works (e.g. [5, 6, 7, 8, 9, 10, 16, 17, 19]) Dutykh, Dias and others studied the “Bump to Bucket” problem. They refered the “B2B” problem as the “moving bottom solution” (Bump) and the “passive generation solution” (Bucket). Their numerical estimates on the linear Euler system show [8] that the wave amplitude of the “moving bottom solution” is underestimated when it is compared to the “passive generation solution” and the difference is typically of the order of 20% . In
addition, they have also discussed the B2B problem by comparing them on various aspects, such as velocity and energy, confirming that there are differences between the two approaches. Nersisyan, Dutykh, and Zuazua [17] studied a “Bump” model based on a nonlinear BBM-type equation with moving disturbance on the bottom. Iguchi [12] studied the generation of tsunami (moving bottom) and intended to improve the pure “Bucket” model. He derived a related shallow-water system that reads as

\[
\begin{align*}
\eta_t + \nabla \cdot \left( (h + \eta - b_1)u \right) &= 0, \\
u_t + (u \cdot \nabla)u + g\nabla \eta &= 0,
\end{align*}
\tag{1.2}
\]

where \( h \) is the depth of water, \( \eta \) denotes the free surface, \( u \) denotes the velocity, \( b = b(x, t) \) denotes the bottom during the deformation, \( b_0 = b_0(x) \) denotes the one before the deformation and \( b_1 = b_1(x) \) denotes the one afterward. We may refer the model of this kind (i.e., \(1.2\)) as a “quasi-Bucket” model since the bottom deformation, \( b_1 \), are involved both in the system and in the initial conditions. Although the “quasi-Bucket” model \(1.2\) still initiated with a “Bucket” at the surface (non-zero initial conditions), Iguchi [12] had mathematically justified that the model is indeed a proper approximation of the “Bump” problem with shallow water assumptions and other necessary assumptions. Later, Iguchi [11] derived another shallow wave system containing the moving bottom (a pure “Bump” problem),

\[
\begin{align*}
\eta_t + \nabla \cdot \left( (1 + \eta - b)u \right) &= b_t, \\
u_t + (u \cdot \nabla)u + g\nabla \eta &= 0.
\end{align*}
\tag{1.3}
\]

Tulin et al in [25] also derived a Boussinesq-type system in \(xz\)-plane to study solitary waves generating by moving disturbances at the seabed,

\[
\begin{align*}
\eta_t + ((h + \eta)u)_x &= -h_t, \\
u_t + uu_x + g\eta_x &= \frac{1}{2}h(h_t + (hu)_x)_{xt} - \frac{1}{6}h^2u_{xxx},
\end{align*}
\tag{1.4}
\]

where the undisturbed free surface is located at \( z = 0, -\infty < x < \infty \), for all \( t \geq 0 \). Here, the bottom is at \( z = -(h(x, t) = -h_0 + h_0b(x, t) \) and it indicates a change of \( h_0b(x, t) \) acting over the bottom floor at depth \( h_0 \) when undisturbed (\(h_0\) being constants). Equation \(1.4\) was adapted to derive a KdV-type equation (uni-directional wave model), as well as a Boussinesq-type equation by Lee [15]. In the small amplitude and long wave region and with the same non-dimensionalization, and drop the higher order terms, system \(1.4\) becomes

\[
\begin{align*}
\eta_t + [(1 - b + \eta)u]_x &= b_t, \\
u_t + uu_x + \eta_x - \frac{1}{3}u_{xxx} &= -\frac{1}{2}b_{xxt}.
\end{align*}
\tag{1.5}
\]

This is not a system in \(1.1\) and variable \( u \) in \(1.5\) is the averaged velocity across the vertical line, but it is formally equivalent to \(1.1\) (see [4] for detail). In this paper, we mainly follow from \(1.1\) but use \( h(x, t) = -b(x, t) \) as in \(1.5\), so the bottom is at \(-1 + b(x)\) instead of at \(-1 + b(x)\). This seems to be a more popular choice. In reality, the instantaneous lift off is not really “instantaneous”, so a particular choice of “Bump” problems (regularized bumps) that is given by the approximation of an instantaneous “lift-up” of the bottom is discussed.
This article is organized as follows: section 2 is devoted to the leading order of the B2B problem, the linear wave system with bumps, and the comparison between the Bump solutions and the bucket solutions. In section 3, we address the B2B for a BBM type nonlinear dispersive equation. In section 4, we address the comparison on a nonlinear bi-directional model.

2. **Bump to Bucket of a linear system.** To provide a guidance for the investigation of the B2B problem, we start with the leading order approximation of (1.1), the linear wave equation with bumps. The order of each term can be made clear with the scales as in [4]. We now study this linear “Bump” problem and the related standard linear “Bucket” problem.

By assuming an instantaneous “lift-up” at the bottom, \( b(x,t) = \chi(t)B(x) \), the “Bump” problem reads

\[
\begin{align*}
\eta_t + u_x &= b_t(x,t) = \delta_0(t)B(x), \\
u_t + \eta_x &= 0, \\
\eta(x,0) &= 0, \quad u(x,0) = 0,
\end{align*}
\]

(2.1)

where \( B(x) \) is the elevation of the bottom, and \( \delta_0 \) is the Dirac mass. Since the system (2.1) is a singular problem containing \( \delta_0 \), we investigate a sequence of smooth approximate systems (regularized bump) that reads

\[
\begin{align*}
\eta^\varepsilon_t + u^\varepsilon_x &= b^\varepsilon_t(x,t) := \varphi^\varepsilon(t)B(x), \\
u^\varepsilon_t + \eta^\varepsilon_x &= 0, \\
\eta^\varepsilon(x,0) &= 0, \quad u^\varepsilon(x,0) = 0,
\end{align*}
\]

(2.2)

where \( \varphi^\varepsilon(t)B(x) \) is a smooth approximation of the Dirac mass, \( \delta_0(t)B(x) \) in (2.1). Throughout this article, we set \( \varphi^\varepsilon(t) \) to be a suitable approximation of the Heaviside step function,

\[
\chi(t) = \begin{cases} 
1, & t > 0, \\
0, & t \leq 0.
\end{cases}
\]

More precisely, we take \( \varphi^\varepsilon(t) = \varphi(\frac{t}{\varepsilon}) \) where \( \varphi \) is a smooth increasing function with

\[
\varphi(t) = \begin{cases} 
1, & t \geq 2, \\
0, & t \leq 1.
\end{cases}
\]

(2.3)

**Theorem 2.1.** For \( B(x) \in L^2(\mathbb{R}) \), let \((\eta^\varepsilon, u^\varepsilon)\) be the solution of the regularized “Bump” problem (2.2) where \( \varphi^\varepsilon(t) \to \chi(t) \) as \( \varepsilon \to 0 \), and let \((\eta^0, u^0)\) be the solution of the “Bucket” problem,

\[
\begin{align*}
\eta^0_t + u^0_x &= 0, \\
u^0_t + \eta^0_x &= 0, \\
\eta^0(x,0) &= B(x), \quad u^0(x,0) = 0.
\end{align*}
\]

(2.4)

Then,

\((\eta^\varepsilon, u^\varepsilon) \to (\eta^0, u^0)\)

in \( L^q(0,T; L^2(\mathbb{R})^2) \) for any \( q < +\infty \) and \( T > 0 \) as \( \varepsilon \to 0 \).

**Remark 2.2.** A boundary layer occurs at \( t = 0 \) and we cannot expect the convergence in \( L^\infty(0,T; L^2(\mathbb{R})^2) \).
Proof. System (2.2) is equivalent to the system of equations, with $\pm = +$ or $-$,

\[
(\eta^\varepsilon + u^\varepsilon)_t + (\eta^\varepsilon + u^\varepsilon)_x = \varphi'_\varepsilon(t)B(x).
\]

(2.5)

Applying the Fourier transform in $x$ on this system yields

\[
(\hat{\eta}^\varepsilon + \hat{u}^\varepsilon)_t + i\xi(\hat{\eta}^\varepsilon + \hat{u}^\varepsilon) = \varphi'_\varepsilon(t)\hat{B}(\xi).
\]

(2.6)

Solving this system of ODE (with null initial data) leads to

\[
(\hat{\eta}^\varepsilon + \hat{u}^\varepsilon)(t, \xi) = \left(\int_0^t e^{\pm i(s-t)\xi}\varphi'_\varepsilon(s)ds\right)\hat{B}(\xi).
\]

(2.7)

Similarly, solving (2.4) yields,

\[
\hat{\eta}^0 + \hat{u}^0 = e^{\mp i\xi}\hat{B}(\xi).
\]

(2.8)

By observing $\varphi'_\varepsilon \to \delta_0$, it is then expected that as $\varepsilon \to 0$,

\[
(\eta^\varepsilon, u^\varepsilon) \to (\eta^0, u^0)
\]

in certain spaces. This can be shown rigorously by setting

\[
\rho(t, \xi) = \left((\hat{\eta}^\varepsilon - \hat{\eta}^0) \pm (\hat{u}^\varepsilon - \hat{u}^0)\right)(t, \xi).
\]

(2.9)

We then infer from (2.7)-(2.8) that,

\[
|\rho(t, \xi)| \leq \left|\int_0^t e^{\pm i\xi}\varphi'_\varepsilon(s)ds - 1\right|\hat{B}(\xi).
\]

(2.10)

This leads to the first estimate

\[
\|\rho\|_{L^\infty(L^2(\mathbb{R}))} \leq 2\|\hat{B}\|_{L^2(\mathbb{R})}.
\]

(2.11)

Besides, (2.10) yields that

\[
|\rho(t, \xi)| \leq \left(\left|\int_0^t (e^{\pm i\xi} - 1)\varphi'_\varepsilon(s)ds\right| + |\varphi'_\varepsilon(t) - 1|\right)\|\hat{B}\|_{L^2(\mathbb{R})},
\]

(2.12)

then, using that $|e^{\pm i\xi} - 1| \leq |s\xi|$, we can infer that

\[
\|\rho(t)\|_{L^2(\mathbb{R})} \leq \left|\int_0^t s\varphi'_\varepsilon(s)ds\right|\|\hat{B}\|_{L^2(\mathbb{R})} + |\varphi'_\varepsilon(t) - 1|\|\hat{B}\|_{L^2(\mathbb{R})}
\]

\[
\leq 2\varepsilon\|\xi\|_{L^\infty(\mathbb{R})} + |\varphi'_\varepsilon(t) - 1|\|\hat{B}\|_{L^2(\mathbb{R})}.
\]

(2.13)

It implies that for any $q < +\infty$,

\[
\|\rho\|_{L^q(0,T;L^q(\mathbb{R}))} \leq c(\varepsilon T^\frac{1}{q}\|B\|_{H^1(\mathbb{R})} + \|B\|_{L^2(\mathbb{R})}\|\varphi'_\varepsilon - 1\|_{L^q(0,T)}),
\]

(2.14)

since $\|\varphi'_\varepsilon - 1\|_{L^q(0,T)}$ converges towards $0$ when $\varepsilon$ converges to $0$. Thus, we have the result when $B \in H^1(\mathbb{R})$. It is worth to note that the result is false if $q = +\infty$ since $\|\varphi'_\varepsilon - 1\|_{L^\infty(0,T)} = 1$.

Now we want to relax the assumption $B \in H^1(\mathbb{R})$ into $B \in L^2(\mathbb{R})$. To achieve that, we use a density argument by introducing $B_\alpha \in H^1(\mathbb{R})$ that is close to $B \in L^2(\mathbb{R})$ such that $\|B_\alpha - B\|_{L^2(\mathbb{R})}$ converges 0 as $\alpha \to 0$. Using (2.11) for $\rho - \rho_\alpha$ and $B - B_\alpha$ leads to

\[
\int_0^T \|\rho(t)\|^2_{L^2(\mathbb{R})}dt \leq 4T\|\hat{B} - \hat{B}_\alpha\|^2_{L^2(\mathbb{R})} + \int_0^T \|\rho_\alpha(t)\|^2_{L^2(\mathbb{R})}dt,
\]

(2.15)
where $\rho_\alpha$ is the error corresponding to the solution where $B$ is replaced by $B_\alpha$. Using (2.13) for $B_\alpha$ (and letting $\varepsilon$ goes to 0), $\alpha$ being fixed, we deduce from (2.15) that

$$\limsup_{\varepsilon \to 0} \|\rho\|_{L^2(0,T;L^2(\mathbb{R}))} \leq 4T\|\hat{B}_\alpha - \hat{B}\|_{L^2(\mathbb{R})}.$$  \hfill (2.16)

Therefore, $\alpha \to 0$ concludes the proof. \hfill \Box

3. **Bump to Bucket of a BBM type single equation.** We now move to a BBM type equation with an instantaneous lift-up on the bottom (Bump), which reads

$$\eta_t - \eta_{ext} + \eta_x - \kappa(b\eta)_x + \eta\eta_x = b_t, \quad \kappa = 0, 1$$ \hfill (3.1)

where $b = \chi(t)B(x)$ and $b_t = \delta_0(t)B(x)$.

**Remark 3.1.** When $\kappa = 0$, the “Bump” problem is compared with a pure “Bucket” problem. When $\kappa = 1$, the “Bump” problem is compared with a “quasi-Bucket”, since both the equation and the initial condition involves the bottom deformation.

Following the idea from the linear problem in Section 2, we will compare the regularized (Bump) problem

$$\eta_t^\varepsilon - \eta_{ext}^\varepsilon + \eta_x^\varepsilon - \kappa(b^\varepsilon \eta^\varepsilon)_x + \eta^\varepsilon\eta_x^\varepsilon = b_t^\varepsilon, \quad \kappa = 0, 1$$ \hfill (3.2)

where $b^\varepsilon(x,t) = \varphi_\varepsilon(t)B(x)$ with a related nonlinear “Bucket” problem.

**Theorem 3.2.** For any $T > 0$ and $B \in L^2(\mathbb{R})$, the regularized “Bump” problem

$$\begin{cases}
\eta_t^\varepsilon - \eta_{ext}^\varepsilon + \eta_x^\varepsilon - \kappa(b^\varepsilon \eta^\varepsilon)_x + \eta^\varepsilon\eta_x^\varepsilon = b_t^\varepsilon, & \kappa = 0, 1, \\
\eta^\varepsilon(x,0) = 0,
\end{cases} \hfill (3.3)$$

and the “Bucket” problem

$$\begin{cases}
\eta_t^0 - \eta_{ext}^0 + \eta_x^0 - \kappa(B\eta^0)_x + \eta^0\eta_x^0 = 0, & \kappa = 0, 1, \\
\eta^0(x,0) = D(x),
\end{cases} \hfill (3.4)$$

where $D(x)$ is the solution in $H^2(\mathbb{R})$ of

$$D(x) - \Delta D(x) = B(x)$$

that vanishes at $\pm \infty$, admit unique solutions $\eta^\varepsilon$ and $\eta^0$ in $C([0,T];H^1(\mathbb{R}))$ with

$$\|\eta^\varepsilon(\cdot,t)\|_{H^1(\mathbb{R})} \leq e^{\frac{1}{2}T\|B\|_{L^2(\mathbb{R})}} \|D\|_{H^1(\mathbb{R})},$$ \hfill (3.5)

$$\|\eta^0(\cdot,t)\|_{H^1(\mathbb{R})} \leq e^{\frac{1}{2}T\|B\|_{L^2(\mathbb{R})}} \|D\|_{H^1(\mathbb{R})}. \hfill (3.6)$$

Moreover, as $\varepsilon \to 0$, $\eta^\varepsilon$ converges to $\eta^0$ in $L^q([0,T];H^1(\mathbb{R}))$ with $1 \leq q < \infty$. More precisely, there exists a constant $C(D,T)$ (that could increase as an exponential function of both $\|D\|_{H^1(\mathbb{R})}$ and $T$) such that for any $q \geq 1$

$$\|\eta^\varepsilon - \eta^0\|_{L^q([0,T];H^1(\mathbb{R}))} \leq C(D,T)e^{\frac{1}{2}} \varepsilon. \hfill (3.7)$$

**Remark 3.3.** A boundary layer is likely to occur at $t = 0$ and we cannot expect the convergence in $L^\infty$. 

Proof. We assume $\kappa = 1$ in the following arguments. The proof for $\kappa = 0$ case is similar to $\kappa = 1$ case and therefore is omitted.

First step: we prove the uniqueness of solution $\eta^\varepsilon$ of (3.3). For a given $\varepsilon > 0$, we consider $w = \eta^\varepsilon - v^\varepsilon$, the difference between two solutions of (3.3). Then

$$w_t - w_{xx} + w_x - (b^\varepsilon w)_x + \left(\frac{\eta^\varepsilon + v^\varepsilon}{2}w\right)_x = 0. \tag{3.8}$$

Multiply both side by $w$ and integrate with respect to $x$ yields

$$\frac{1}{2} \frac{d}{dt} \|w\|_{H^1(\mathbb{R})}^2 - \frac{1}{2} \int_{\mathbb{R}} b^\varepsilon_x w^2 dx + \int_{\mathbb{R}} \frac{\eta^\varepsilon_x + v^\varepsilon_x}{4} w^2 dx = 0. \tag{3.9}$$

Recall that $b^\varepsilon_x = \varphi^\varepsilon(t) B'(x)$ is bounded in $L^\infty(0, T; H^{-1}(\mathbb{R}))$, then

$$\int_{\mathbb{R}} b^\varepsilon_x w^2 dx \leq C \|b^\varepsilon_x\|_{H^{-1}(\mathbb{R})} \|w\|_{H^1(\mathbb{R})}^2 \leq C \|b^\varepsilon_x\|_{H^{-1}(\mathbb{R})} \|w\|_{H^1(\mathbb{R})}^2, \tag{3.10}$$

and

$$\int_{\mathbb{R}} \frac{\eta^\varepsilon_x + v^\varepsilon_x}{4} w^2 dx \leq C (\|\eta^\varepsilon\|_{L^2(\mathbb{R})} + \|v^\varepsilon\|_{L^2(\mathbb{R})}) \|w\|_{H^1(\mathbb{R})}^2 \leq C (\|\eta^\varepsilon\|_{L^2(\mathbb{R})} + \|v^\varepsilon\|_{L^2(\mathbb{R})}) \|w\|_{H^1(\mathbb{R})}^2,$$

since $H^1(\mathbb{R})$ is an algebra. We have that,

$$\frac{1}{2} \frac{d}{dt} \|w\|_{H^1(\mathbb{R})}^2 \leq K \|w\|_{H^1(\mathbb{R})}. \tag{3.11}$$

Applying the Gronwall Lemma, it follows

$$\|w\|_{H^1(\mathbb{R})}^2 \leq e^{2Kt} \|w(x, 0)\|_{H^1(\mathbb{R})}^2 = 0. \tag{3.12}$$

Hence, we have the uniqueness of the solution.

Second step: We construct a truncated equation of (3.3) and then prove the existence of this solution. Introduce the usual classical bounded skew-symmetric operator $A = \partial_x (I - \Delta)^{-1}$. Also, setting $D(x)$ to be the solution to $D - D'' = B$, and denoting $\eta = \eta^\varepsilon$ for simplicity, then the equation in (3.3) reads

$$\eta_t + A \left(\eta - b^\varepsilon \eta + \frac{\eta^2}{2}\right) = \varphi^\varepsilon(t) D(x). \tag{3.13}$$

Integrating in time between 0 and $t < T$, we are led to the integral equation

$$\eta(t) = -\int_0^t A \left(\eta - b^\varepsilon \eta + \frac{\eta^2}{2}\right) ds + \varphi^\varepsilon(t) D(x). \tag{3.14}$$

Here and in the sequel we have assumed w.l.o.g. that $\varphi^\varepsilon(t) = 0$ for non positive $t$.

We now introduce a truncated equation. For any given large $R > 0$, we replace the nonlinearity $F(\eta) = \frac{\eta^2}{2}$ by a globally Lipschitz map $F_R(\eta) = \frac{\eta}{2} \min(R, |\eta|)$. Then $F_R$ is Lipschitz with constant $L_R = \frac{1}{2} R$ and since $||\eta||_{L^\infty(\mathbb{R})} \leq ||\eta||_{H^1(\mathbb{R})}$ then for $||\eta||_{H^1(\mathbb{R})} \leq R$ we have $F_R(\eta) = F(\eta)$.

We then set

$$\eta(t) = T(\eta) := -\int_0^t A(\eta - b^\varepsilon \eta + F_R(\eta(s))) ds + \varphi^\varepsilon(t) D(x), \tag{3.15}$$

and solve (3.15) by a fixed point argument. Let us observe that

$$\|A(b^\varepsilon \eta)||_{H^1(\mathbb{R})} \leq \|b^\varepsilon \eta||_{L^2(\mathbb{R})} \leq \|B||_{L^2(\mathbb{R})} ||\eta||_{H^1(\mathbb{R})}. $$
Recalling that $A$ is a bounded operator in any $H^s$ space with norm $\frac{1}{2}$, we have that
\[
\|T(\eta) - T(v)\|_{H^1(\mathbb{R})} \leq \left( 1 + \|B\|_{L^2(\mathbb{R})} + \frac{L_R}{2} \right) \int_0^t \|\eta - v\|_{H^1(\mathbb{R})} ds,
\] (3.16)
where $L_R$ is the Lipschitz constant of $F_R$. Set $\tilde{L}_R = (1 + \|B\|_{L^2(\mathbb{R})} + \frac{L_R}{2})$. Consider now two sequences $\eta^{k+1} = T(\eta^k)$ and $v^{k+1} = T(v^k)$. Set $\gamma_k(t) = \sup_{s \in [0, t]} \|\eta^k - v^k\|_{H^1(\mathbb{R})}$. We prove recursively that
\[
\gamma_k(t) \leq \left( \frac{\tilde{L}_R t}{k!} \right)^k \gamma_0(T)
\]
using
\[
\gamma_{k+1}(t) \leq \tilde{L}_R \int_0^t \gamma_k(s) ds.
\]
Hence for $k$ large enough depending on $T$ the map $T^k$ is a contraction and we have a unique $\eta_R$ in $C([0, T]; H^1(\mathbb{R}))$ that solves (3.15).

Third step: we bound above the $H^1$ norm of $\eta_R$ and consider the convergence of $\eta_R$ as $R \to +\infty$. We have that $\eta_R$ is the solution to
\[
(\eta_R)_t - (\eta_R)_{xxx} + (\eta_R - b^r \eta_R + F_R(\eta_R))_x = \varphi'_e(t)B(x).
\] (3.17)

Multiplying this equation by $\eta_R$ leads to
\[
\frac{d}{dt} \|\eta_R\|_{H^1(\mathbb{R})}^2 = \int \varphi_e B' \eta_R^2 dx + 2 \varphi'_e(t) \int B \eta_R dx.
\] (3.18)

Using $D - D'' = B$ and $\varphi'_e(t)$ is nonnegative, we have
\[
\frac{d}{dt} \|\eta_R\|_{H^1(\mathbb{R})}^2 \leq \|B\|_{L^2(\mathbb{R})} \|\eta_R\|_{H^1(\mathbb{R})}^2 + 2 \varphi'_e(t) \|D\|_{H^1(\mathbb{R})} \|\eta_R\|_{H^1(\mathbb{R})}.
\] (3.19)

Canceling a $2\|\eta_R\|_{H^1(\mathbb{R})}$ on both sides yields
\[
\frac{d}{dt} \left( e^{-\frac{2}{3} \|B\|_{L^2(\mathbb{R})} \|\eta_R\|_{H^1(\mathbb{R})}} \right) \leq \varphi'_e(t) \|D\|_{H^1(\mathbb{R})}.
\] (3.20)

Integrating (3.20) in time leads to (3.5) for $\eta_R$.

We now prove the convergence of $\eta_R$. Let us set
\[
R_D = e^{\frac{2}{3} \|B\|_{L^2(\mathbb{R})} \|D\|_{H^1(\mathbb{R})}},
\] (3.21)

As soon as $R \geq R_D$, $F_R(\eta_R) = F_{R_D}(\eta_R) = F(\eta_R)$ and by uniqueness $\eta_R = \eta_{R_D} = \eta^r$.

Fourth step: we prove the existence of solution $\eta^0$ of (3.4) and its estimate (3.6). According to (3.4), one has the integrated equation
\[
\eta^0 = - \int_0^t A \left( \eta^0 - B\eta^0 + \frac{1}{2} (\eta^0)^2 \right) ds + D(x).
\] (3.22)

In addition, multiplying $\eta^0$ to both sides of (3.4) leads to
\[
\frac{d}{dt} \|\eta^0\|_{H^1(\mathbb{R})}^2 = \int B'(\eta^0)^2 dx \leq \|\eta^0\|_{H^1(\mathbb{R})}^2 \|B\|_{L^2(\mathbb{R})}.
\] (3.23)

Therefore, the proof will be similar to the ones in the previous steps.

Fifth step: We prove the convergence of the sequence $\eta^r$. Let us denote $L_D = \frac{1}{2} R_D$ the Lipschitz constant of $F_{R_D}$ that depends only on $D$, $T$ and $B$. According to (3.3) and (3.4), we have
\[
\eta^0(t) - \eta^r(t)
\]
\[- \int_0^t A \left( \left( 1 - b^\varepsilon + \frac{\eta^0 + \eta^\varepsilon}{2} \right) \left( \eta^0 - \eta^\varepsilon - \eta^0 (b^\varepsilon - B) \right) \right) ds + (1 - \varphi_\varepsilon(t)) D(x). \]  (3.24)

Using (3.5) and (3.6), there exists a constant \( M \) that depends on \( B, T, L_D \) such that
\[
\| \eta^0(t) - \eta^\varepsilon(t) \|_{H^1(\mathbb{R})} \\
\leq M \left( \int_0^t \| \eta^0(s) - \eta^\varepsilon(s) \|_{H^1(\mathbb{R})} ds + \int_0^t 1 - \varphi_\varepsilon(s) ds + (1 - \varphi_\varepsilon(t)) \right). \]  (3.25)

Setting \( \ell(t) = \int_0^t \| \eta^0(s) - \eta^\varepsilon(s) \|_{H^1(\mathbb{R})} ds \), we then have
\[
\dot{\ell} \leq M \ell + M (1 - \varphi_\varepsilon(t)) + M \int_0^t 1 - \varphi_\varepsilon(s) ds. \]  (3.26)

Setting \( \theta(t) = \int_0^t 1 - \varphi_\varepsilon(s) ds \), we infer from the previous estimate that
\[
\frac{d}{dt} (e^{-Mt} \ell) \leq Me^{-M(t-\varepsilon)} \frac{d}{dt} (e^\theta(t)) \leq M \frac{d}{dt} (e^\theta(t)). \]  (3.27)

This leads to
\[
\ell(t) \leq Me^{Mt+\varepsilon} \int_0^t 1 - \varphi_\varepsilon(s) ds. \]  (3.28)

Since \( \varphi_\varepsilon(s) = 1 \) for \( s \geq 2\varepsilon \) then for any \( t \geq 0 \)
\[
\theta(t) \leq \int_0^{2\varepsilon} 1 - \varphi_\varepsilon(s) ds \leq 2\varepsilon.
\]

Therefore,
\[
\ell(T) \leq 2e^{MT+T} M\varepsilon. \]  (3.29)

This provides the result in (3.7) for \( q = 1 \). Gathering this estimate with (3.26) we have, for \( t \leq T \)
\[
\| \eta^0(s) - \eta^\varepsilon(s) \|_{H^1(\mathbb{R})} = \dot{\ell}(t) \leq M (\ell(t) + \theta(t) + \dot{\theta}(t)) \leq 2M\varepsilon (1 + e^{MT+T} M) + M.
\]

This provides the result of (3.7) for \( q = \infty \) and, for \( 1 < q < \infty \). The estimate (3.7) for other spaces can be obtained by interpolation between \( L^1 \) and \( L^\infty \).

**Remark 3.4.** From (3.14), one has
\[
\lim_{t \to 0^+} \lim_{\varepsilon \to 0^+} \eta^\varepsilon(x, t) = D(x).
\]

Therefore, in order to have \( \eta^\varepsilon \) converges towards to \( \eta^0 \) in the desired space, the initial condition of the “Bucket” problem is desired to have
\[
\eta^0(x, 0) = D(x).
\]

**Remark 3.5.** We cannot prove that the \( L^\infty(0, T; H^1) \) norm of \( \eta^\varepsilon - \eta^0 \) converges to 0 since we have a singular limit problem where
\[
\lim_{\varepsilon \to 0^+} \lim_{t \to 0^+} \eta^\varepsilon(t) \neq \lim_{t \to 0^+} \lim_{\varepsilon \to 0^+} \eta^\varepsilon(t).
\]
4. Bump to Bucket of a bidirectional Boussinesq system. We now investigate the Bump to Bucket problem of a nonlinear Boussinesq system (1.1), written as,

\[
\begin{align*}
\eta_t + u_x + (\eta + h)u_x + au_{xxx} - bh_{xxx} &= -h_t + \mu h_{xxx}, \\
u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxx} &= -(P_0)_x + (1 - \theta)h_{xxx},
\end{align*}
\]

in the particular case where \( P_0 = 0 \) and the parameters are fixed as follows: \( \lambda = 0 \) and \( \theta = 1 \), that is,

\[
\begin{align*}
\eta_t + u_x + (\eta + h)u_x - \frac{1}{3}\eta_{xxx} &= -h_t - \frac{1}{6}h_{xxx}, \\
u_t + \eta_x + uu_x &= 0.
\end{align*}
\]  

(4.1)

When \( h = 0 \), namely when there is no bump, the system is locally nonlinearly wellposed, see equation (3.20) and Theorem 3.11 in [3]. Here, choosing \( \theta = 1 \) implies that there is no \( h_{xxx} \) in the resulting equation (4.1). This avoid the drawbacks to compute derivatives of \( \delta_i = 0 \). The case \( \theta \neq 1 \) will be studied in a future work with other approximations of \( \delta_i = 0 \).

Adjusting constants to be one and introducing the new variable \( v = \eta + h = \eta - b \) as in Section 2 and 3, the system becomes

\[
\begin{align*}
v_t^\varepsilon + \left((1 + v^\varepsilon)u^\varepsilon\right)_x - v^\varepsilon_{xxx} &= b^\varepsilon_{xxx}, \\
u^\varepsilon_t + v^\varepsilon_x + u^\varepsilon u^\varepsilon_x &= -b^\varepsilon_x, \\
u^\varepsilon(x,0) &= 0, \quad v^\varepsilon(x,0) = 0.
\end{align*}
\]  

(4.2)

The quasi-Bucket problem related to (4.2) reads

\[
\begin{align*}
v^0_t + \left((1 + v^0)u^0\right)_x - v^0_{xxx} &= 0, \\
u^0_t + v^0_x + u^0 u^0_x &= -B', \\
u^0(x,0) &= 0, \quad v^0(x,0) = (1 - \Delta)^{-1}B''(x).
\end{align*}
\]  

(4.3)

Theorem 4.1. Assume \( B \) belongs to \( H^3(\mathbb{R}) \). There exists a \( T(B) \) that depends only on \( B \) such that problems (4.2) and (4.3) have respectively a unique solution \( (u,v) \) in \( L^\infty(0,T(B));H^2(\mathbb{R}) \times H^3(\mathbb{R}) \). Moreover there exists a constant \( C \) that depends on \( B \) such that

\[
\|u^0 - u^\varepsilon\|_{L^1(0,T(B);L^2(\mathbb{R}))} + \|v^0 - v^\varepsilon\|_{L^1(0,T(B);H^1(\mathbb{R}))} \leq C\varepsilon.
\]

Remark 4.2. Once again, we cannot expect an estimate in \( L^\infty(0,T) \) due to a boundary layer at \( t = 0 \).

Proof. We first address the initial value problem for (4.2). Let us begin with the uniqueness result, denote \( (u_1,v_1) \) and \( (u_2,v_2) \) be two solution of (4.2), that satisfy

\[
\begin{align*}
v_t + u_x + (uv + vu)_x - v_{xxx} &= 0, \\
u_t + v_x + (vu)_x &= 0,
\end{align*}
\]  

(4.4)

where \( v, u \) are the difference of two different solutions, and where

\[
\bar{u} = \frac{1}{2}(u_1 + u_2) \quad \text{and} \quad \bar{v} = \frac{1}{2}(v_1 + v_2).
\]

Multiplying the first equation by \( v \), the second by \( u \) and integrating with respect to \( x \), these lead to

\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2(\mathbb{R})}^2 + \|v\|_{H^1(\mathbb{R})}^2 \right) = - \int_\mathbb{R} \left( \left(\frac{u}{2} + \frac{v}{2}\right)v_x - \frac{u}{2} u^2 \right) dx.
\]  

(4.5)
Then there exists a constant $M$ that depends on the bound of the solution in $L^\infty(0,T(B);H^1(\mathbb{R}))$ such that
\[
\frac{d}{dt} \left( \|u\|_{L^2(\mathbb{R})}^2 + \|v\|_{H^1(\mathbb{R})}^2 \right) \leq M \left( \|u\|_{L^2(\mathbb{R})}^2 + \|v\|_{H^1(\mathbb{R})}^2 \right).
\] (4.6)

Integrating in time gives the uniqueness result. The proof is similar for (4.3) and then omitted.

For the existence result, we proceed as in [3] even the system (4.2) is not covered by the case at Section 3.3. in [3]. To begin with, for $\gamma > 0$, we introduce a regularized version of (4.2) that reads,
\[
\begin{aligned}
\frac{v_t^{\varepsilon,\gamma}}{\varepsilon} + ((1 + v^{\varepsilon,\gamma}) u^{\varepsilon,\gamma})_x - v^{\varepsilon,\gamma}_{xxt} &= b^{\varepsilon}_{xxt}, \\
\frac{u_t^{\varepsilon,\gamma}}{\varepsilon} + v^{\varepsilon,\gamma}_x u^{\varepsilon,\gamma}_x - \gamma u^{\varepsilon,\gamma}_{xxt} &= -b^{\varepsilon}_x,
\end{aligned}
\] (4.7)

with initial conditions $v^{\varepsilon,\gamma}(x,0) = v^{\varepsilon,\gamma}(x,0) = 0$. Uniqueness and existence results for (4.7) are standard and then omitted (see Theorem 2.1 of [3]). We now seek for a priori estimate, and consider smooth solutions to (4.7).

**Claim 4.3.** There exist $T(B), C(B)$ depending on $B$, and that are independent of $\gamma$ and $\varepsilon$, such that for any $t$ in $[0,T(B)]$,
\[
\|u^{\varepsilon,\gamma}\|_{H^2(\mathbb{R})} + \|v^{\varepsilon,\gamma}\|_{H^3(\mathbb{R})} \leq C(B).
\] (4.8)

**Proof of the claim.** Multiplying the first equation in (4.7) by $v^{\varepsilon,\gamma}$ and the second one by $u^{\varepsilon,\gamma}$, integrating and summing the resulting equations lead to
\[
\frac{1}{2} \frac{d}{dt} \left[ \|u^{\varepsilon,\gamma}\|_{L^2(\mathbb{R})}^2 + \|v^{\varepsilon,\gamma}\|_{L^2(\mathbb{R})}^2 + \|u^{\varepsilon,\gamma}\|_{H^1(\mathbb{R})}^2 \right]
\leq \int_{\mathbb{R}} u^{\varepsilon,\gamma} v^{\varepsilon,\gamma} v^{\varepsilon,\gamma}_x dx + \left( \|u^{\varepsilon,\gamma}\|_{L^2(\mathbb{R})} + \|v^{\varepsilon,\gamma}\|_{L^2(\mathbb{R})} \right) \|B'\|_{L^2(\mathbb{R})}.
\] (4.9)

Introducing $E = \left[ \|u^{\varepsilon,\gamma}\|_{L^2(\mathbb{R})}^2 + \|v^{\varepsilon,\gamma}\|_{H^1(\mathbb{R})}^2 \right]^{\frac{1}{2}}$ and using
\[
\left| \int_{\mathbb{R}} u^{\varepsilon,\gamma} v^{\varepsilon,\gamma} v^{\varepsilon,\gamma}_x dx \right| \leq \|u^{\varepsilon,\gamma}\|_{L^\infty(\mathbb{R})} \|u^{\varepsilon,\gamma}\|_{L^2(\mathbb{R})} \|v^{\varepsilon,\gamma}_x\|_{L^2(\mathbb{R})} \leq E^3,
\]
we infer from (4.9) that
\[
\frac{d}{dt} E \leq E^2 + \left( \frac{1}{\varepsilon} \varphi' \left( \frac{t}{\varepsilon} \right) + 1 \right) \|B'\|_{L^2(\mathbb{R})}.
\] (4.10)

Let us recall that $E(0) = 0$. Introduce the stopping time
\[
T_0 = \inf\{t > 0; E(t) > 2\|B'\|_{L^2(\mathbb{R})}\},
\]
we then integrate (4.10) between 0 and $T_0$, assuming that $T_0 < +\infty$,
\[
2\|B'\|_{L^2(\mathbb{R})} = E(T_0) \leq \|B'\|_{L^2(\mathbb{R})} + T_0 \left( \|B'\|_{L^2(\mathbb{R})} + \|B'\|_{H^1(\mathbb{R})} \right).
\] (4.11)
Therefore $T_0 \geq T(B) = \frac{1}{4\|B'\|_{L^2(\mathbb{R})}} > 0$.

Multiplying the first equation in (4.7) by $v^{\varepsilon,\gamma}_{xx}$ and the second one by $u^{\varepsilon,\gamma}_{xx}$, integrating and summing the resulting equations leads to
\[
\frac{1}{2} \frac{d}{dt} \left[ \|u^{\varepsilon,\gamma}_{xx}\|_{L^2(\mathbb{R})}^2 + \|v^{\varepsilon,\gamma}_{xx}\|_{L^2(\mathbb{R})}^2 + \|v^{\varepsilon,\gamma}_{x}\|_{H^1(\mathbb{R})}^2 \right]
\]
where $D$ in a bounded subset of $L^2$ we then have, integrating by parts twice

$V(t) + \int_0^t (I-\Delta)^{-1} \partial_x \left( U + \frac{u^\varepsilon + u^0}{2} V + \frac{v^\varepsilon + v^0}{2} U \right) ds = - \left( 1 - \varphi \left( \frac{t}{\varepsilon} \right) \right) D'$, \hspace{1cm} (4.18)

where $D$ is defined in (3.13). This yields

\[ \begin{align*}
- \int_R (u^\varepsilon v^\varepsilon) x v^\varepsilon dx &= \int_R u^\varepsilon v^\varepsilon u_x^\varepsilon dx \\
+ \left( \| u_x^\varepsilon \|_{L^2(R)} + \frac{1}{\varepsilon} \varphi \left( \frac{t}{\varepsilon} \right) \right) \| v^\varepsilon \|_{L^2(R)} \| B^3 \|_{L^2(R)}. \tag{4.12}
\end{align*} \]

Introducing

\[ E_1 = \left[ E^2 + \| u_x^\varepsilon \|_{L^2(R)}^2 + \| v^\varepsilon \|_{L^2(R)}^2 \right] \frac{1}{2}, \]

we then have, integrating by parts twice

\[ \left| \int_R u^\varepsilon v^\varepsilon u_x^\varepsilon dx \right| = \frac{5}{2} \int_R (u_x^\varepsilon)^2 u_x^\varepsilon dx \leq \frac{5}{2} E_1^3. \tag{4.13} \]

Analogously we may prove, integrating by parts once

\[ \int_R u^\varepsilon v^\varepsilon u_x^\varepsilon dx \leq E_1^3. \tag{4.14} \]

From the a priori estimates, we can prove that the sequence $(u^\varepsilon, v^\varepsilon)$ remains in a bounded subset of $L^\infty(0, T(B); H^2(R) \times H^3(R))$ and the sequence $(u^\varepsilon_t, v^\varepsilon_t)$ remains in a bounded sequence of $L^2(0, T(B); L^2(R) \times H^1(R))$. Then the sequence $(u^\varepsilon, v^\varepsilon)$ remains in a compact subset of $L^2(0, T(B); H^1_{loc}(R) \times H^2_{loc}(R))$. Let $\gamma$ converges to 0, we have a solution to (4.2). The proof of the existence result for (4.3) is similar and then omitted.

We then handle the comparison between the solutions of (4.2) - (4.3). Let us denote $V = v^\varepsilon - v^0$ and $U = u^\varepsilon - u^0$. Considering the difference of the second equations in both (4.2) and (4.3), we have

\[ U_t + V_x + \frac{(u^\varepsilon + u^0) U}{2} = \left( 1 - \varphi \left( \frac{t}{\varepsilon} \right) \right) B'. \tag{4.15} \]

Multiplying this equation by $U$ leads to, after mere computations

\[ \frac{d}{dt} \| U \|_{L^2(R)} \leq \left( 1 - \varphi \left( \frac{t}{\varepsilon} \right) \right) \| B' \|_{L^2(R)} + M \| U \|_{L^2(R)} + \| V_x \|_{L^2(R)}, \tag{4.16} \]

where $M$ depends on the bound for $u^\varepsilon, u^0$ in $L^\infty(0, T(B), H^2(R))$. Integrating in time and applying Gronwall Lemma yield, for some constant $C$ that depends on $B$ and $T(B)$,

\[ \| U(t) \|_{L^2(R)} \leq C \left( \varepsilon + \int_0^t \| V_x(s) \|_{L^2(R)} ds \right). \tag{4.17} \]

On the other hand, integrating the difference of the first equations in both (4.2) and (4.3) in time, we have

\[ \begin{align*}
\frac{d}{dt} \| U \|_{L^2(R)} &\leq \left( 1 - \varphi \left( \frac{t}{\varepsilon} \right) \right) \| B' \|_{L^2(R)} \| U \|_{L^2(R)} + \| V_x \|_{L^2(R)}, \tag{4.16} \\
&\leq \left( 1 - \varphi \left( \frac{t}{\varepsilon} \right) \right) \| B' \|_{L^2(R)} \| U \|_{L^2(R)} + \| V_x \|_{L^2(R)}, \tag{4.16}
\end{align*} \]

where $D$ is defined in (3.13). This yields
\[ \|V(t)\|_{H^1(\mathbb{R})} \leq M \int_0^t \left( \|U(s)\|_{L^2(\mathbb{R})} + \|V(s)\|_{L^2(\mathbb{R})} \right) ds + \left(1 - \frac{t}{\epsilon} \right) \|B\|_{H^1(\mathbb{R})}. \]  

(4.19)

Combining (4.17) and (4.19), then proceeding as in Section 3 we obtain

\[ \|U\|_{L^1(0,T;L^2(\mathbb{R}))} + \|V\|_{L^1(0,T;H^1(\mathbb{R}))} \leq C(T) \epsilon, \]  

(4.20)

that completes the proof of the Theorem.

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