DIFFERENTIAL FORMS ON SINGULAR VARIETIES
AND CYCLIC HOMOLOGY.

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by J. P. Brasselet and A. Legrand

Abstract. A classical result of A. Connes asserts that the Frechet algebra of smooth functions on a smooth compact manifold $X$ provides, by a purely algebraic procedure, the de Rham cohomology of $X$. Namely the procedure uses Hochschild and cyclic homology of this algebra.

In the situation of a Thom-Mather stratified variety, we construct a Frechet algebra of functions on the regular part and a module of poles along the singular part. We associate to these objects a complex of differential forms and an Hochschild complex, on the regular part, both with poles along the singular part. The de Rham cohomology of the first complex and the cyclic homology of the second one are related to the intersection homology of the variety, the corresponding perversity is determined by the orders of poles.

The detailed proofs of the results will appear in a forthcoming publication. The first author thanks the organizers of the Terry Wall’s 60th birthday meeting in Liverpool for the invitation to give a lecture during the conference.

1. Introduction.

The aim of this paper is to extend to singular varieties the de Rham and Connes theorems, using the intersection homology, due to Goresky and MacPherson which is more adapted to singularities, instead of the classical homology. Let $X$ be a $C^\infty$ compact manifold, $\Omega^*(X)$ the de Rham complex and $H^*_{{\text{dR}}}(X)$ its cohomology. A. Connes shows that the classical differential de Rham construction

$$X \implies \Omega^*(X) \implies H^*_{{\text{dR}}}(X)$$

can be provided by a purely algebraic process

$$C^\infty(X) \implies HH^*(C^\infty(X)) \implies PHC^*(C^\infty(X))$$

where $HH^*(C^\infty(X))$ is the Hochschild homology of the Frechet algebra of $C^\infty$-functions on $X$ and $PHC^*(C^\infty(X))$ is the periodic cyclic homology [Co].
Now let $X$ be a Thom-Mather stratified variety, with singular part $\Sigma \subset X$, so the regular stratum $X - \Sigma$ is an open manifold. We want to construct an algebra $IC^\infty(X)$ of differentiable functions on $X - \Sigma$ such that

$$C^\infty_c(X - \Sigma) \subset IC^\infty(X) \subset C^\infty(X - \Sigma)$$

and which contains enough geometric informations on the compactification of $X - \Sigma$ by the singular variety $\Sigma$ to fullfill the Connes program in this singular framework. Evidently the two extremal algebras do not fit in, we need to add asymptotic control to keep informations “near $\Sigma$”. We shall proceed of the following way: For each stratum $S_i$ in $\Sigma$, there is a tubular neighborhood $T_i$ of $S_i$ and a distance function $r_i$ to $S_i$, defined in $T_i$. Firstly we define the (Frechet) algebra $A$ of differentiable functions in the variables $r_i$, which are indefinitly logarithmically controlled near $\Sigma$. This will be our asymptotic control “reference algebra”. To be controlled a space should be an $A$-space, so $A$ will be also our “basic ring” for the homological constructions. Then we define an $A$-algebra of controlled differentiable functions on $X - \Sigma$ and an $A$-differentiable module of poles along $\Sigma$. These objects allow us to construct two complexes said $\bar{\beta}$-controlled.

The first one is a complex of differential forms with coefficients in the module of poles and whose cohomology is the Goersky-MacPherson intersection cohomology (Theorem 1). The perversity is related to the orders of poles. The second one is a mixed complex copied from the Connes’s algebraic procedure. Its Hochschild homology is identified with the first complex and its periodic cyclic homology is the intersection cohomology of $X$. This last result is explicit in the case of the cone (Theorem 3).

Let us mention the basic difficulties appearing in Hochschild and cyclic homology in the singular situation, using the intersection homology and a “control near $\Sigma$”. At the Hochschild homology level:

1. Whatever the way to introduce a control (by use of poles or $L^p$-forms for instance), it is not compatible with the product.

At the cyclic homology level:

2. It is clear that any type of control does not agree with the de Rham differential (this is the case for the intersection complex or the $L^p$-forms complex).

3. We need to use an Hochschild complex with coefficients but the cyclic structure does not exist in this case.

To solve these problems we use (cf. §4):
(1) a specific $A$-equivariant Hochschild theory with coefficients in poles associated to a modified differentiable Frechet structure,

(2) a suitable unitarization (theorem 2),

(3) the more general setting given by mixed complexes.

Although we often speak of $A$-objects, our equivariant construction does not agree with equivariant theories developed in [Bry], [BG]. We do not know if they can be adapted to this singular situation and roughly speaking we substitute associated equivariant conditions by adapted semi-norms. We will precise this point in the paragraph 3 (see the “the cone situation”).

2. Definition of intersection homology.

We will consider a singular variety $X$ endowed with a Thom-Mather $C^\infty$ stratification i.e. a filtration of $X$ by closed subsets

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

(*)

where $\Sigma = X_{n-1}$ is the singular part. Each stratum $S_i = X_i - X_{i-1}$ is either an emptyset or an $i$-dimensional $C^\infty$-manifold and there are
- an open neighborhood $T_i$ of $S_i$ in $X$,
- a continuous retraction $\pi_i$ of $T_i$ on $S_i$,
- a continuous function $\rho_i : T_i \to [0, 1]$,

such that $S_i = \{ x \in T_i | \rho_i(x) = 0 \}$ and the $(T_i, \pi_i, \rho_i)$ satisfy the axioms of Mather [Ma].

These data imply the following local triviality condition :

$$\forall x \in S_{n-j}, \ \exists U_x \subset X \text{ and an homeomorphism } \psi_x : U_x \to B^{n-j} \times cL_x$$

where $B^{n-j}$ is the standard open $(n-j)$-dimensional ball and $cL_x$ is the cone over the link $L_x$. The link is assumed to be stratified and independant of the point $x \in S_{n-j}$ and $\psi_x$ preserves the stratifications of $U_x$ (induced by the one of $X$) and the one of the product $B^{n-j} \times cL_x$. The parameter of the cone corresponds to the Mather distance function $\rho_{n-j}$. For a complete definition see for instance [GM2].

Now let us recall the definition of intersection homology due to M. Goresky and R. MacPherson [GM1]. Given a stratified singular variety, the idea of intersection homology is to consider chains and cycles whose intersections with the strata are “not too big”.
The allowed chains and cycles meet the strata with a controlled and fixed defect of transversality. This defect is an integer function, called a \textit{perversity}, increasing with the codimension $j$ of the strata, and denoted $\bar{p} = (p_0, p_1, p_2, \ldots, p_j, \ldots, p_n)$. It satisfies:

$$p_0 = p_1 = p_2 = 0 \quad \text{and} \quad p_j \leq p_{j+1} \leq p_j + 1 \quad \text{for } j \geq 2$$

Let $C_i(X)$ be any “classical” chain complex on $X$ with integer coefficients, we can define the complex:

$$IC_{\bar{p}}^i(X) = \{ \xi \in C_i(X) : \dim(\{\xi| \cap X_{n-j}) \leq i - j + p_j \quad \text{and} \quad \dim(\{\partial \xi| \cap X_{n-j}) \leq i - 1 - j + p_j \} \},$$

the \textit{intersection homology groups} $IH_{\bar{p}}^i(X)$ are homology groups of this complex.

For the zero perversity (all $p_j$ are zero), allowed chains and cycles are transverse to all strata. The \textit{total} perversity $\bar{t}$ is the one such that, for all $j \geq 2, t_j = j - 2$.

We shall be mainly interested by the axiomatic definition of intersection homology. Namely, if a complex of sheaves on $X$ satisfies the so-called \textit{perverse sheaves axioms} [GM2], then the hypercohomology of $(X$ with value in) this perverse sheaf is the intersection homology of $X$. The main axioms of perverse sheaves are issued from the following local computation property (cf [GM2]).

Let $cL$ be the open cone over an $(n-1)$-dimensional manifold $L$, then the perversity depends only on $p_n$ and we have:

$$IH_{\bar{p}}^i(c(L)) \cong \begin{cases} 
H_i(L) & i < n - p_n - 1 \\
0 & i \geq n - p_n - 1 
\end{cases}$$

The intersection homology is the good theory for extending many of classical results from manifolds to singular varieties.

The most important is Poincaré duality (which motives the theory). The intersection of cycles is well defined in intersection homology. More precisely, if $\bar{p}$ and $\bar{q}$ are complementary perversities (this means $\bar{p} + \bar{q} = \bar{t}$), there is a non degenerated bilinear map:

$$IH_{\bar{p}}^i(X; \mathbb{Q}) \times IH_{\bar{q}}^n-i(X; \mathbb{Q}) \to IH_{\bar{t}}^0(X; \mathbb{Q})$$

corresponding to the intersection of cycles, followed by the evaluation map $\varepsilon$. 
If \( \Sigma \subset X_{n-2} \), the Poincaré homomorphism, cap-product by the fundamental class \([X]\), admits the following factorisation, for every perversity \( \bar{p} \):

\[
\begin{array}{ccc}
H^{n-i}(X) & \xrightarrow{\cap [X]} & H_i(X) \\
\downarrow \alpha & & \downarrow \omega \\
IH^\bar{p}_i(X) & \rightarrow & IH^\bar{p}_i(X)
\end{array}
\]

Intersection homology theory is the good context to extend to singular varieties results such that Morse theory [GM3], Lefschetz hyperplane theorem [GM3], hard Lefschetz theorem [BBD], Hodge decomposition [Sa] and de Rham theorem (see the following paragraph).

3. de Rham theorem for stratified varieties and polar forms.

The constructions we will use are taken from those of Cheeger [Ch] and Cheeger-Goresky-MacPherson [CGM], who proved in particular situations the “standard” result concerning \( L^2 \)-cohomology of differential forms, i.e. the isomorphism:

\[
H^{\ast}_2(X - \Sigma) \cong \text{Hom}(IH^\bar{p}_* (X, \mathbb{R}); \mathbb{R}) .
\]

Many authors proved de Rham theorems for \( L^2 \)-forms, or \( L^p \)-forms, in different situations but always in the framework of intersection homology (see [Bra] for a partial survey).

The constructions that we give are also related to the theory of shadow forms [BGM] which is another way to extend the de Rham theorem. In a polyedron \( (K) \) in the euclidean space \( \mathbb{R}^n \), with a given barycentric subdivision \( (K') \), we associate, to each simplex \( \sigma \) in \( (K') \), a differential form \( \omega(\sigma) \) in the interior of simplices of maximal dimension, in a very explicit way. The shadow forms have poles over faces of \( (K) \) : If the defect of transversality of \( \sigma \) with a face \( F \) of a \( (K) \)-simplex is \( q \), then the maximum order of poles of \( \omega(\sigma) \) on this face is \( q \). It can be proved the inclusion quasi isomorphism:

\[
\{ \text{Shadow forms} \} \subset \{ L^p \text{- forms} \}
\]
The cone situation.

We shall begin by defining an intersection complex of differential forms on the cone $cL = [0, 1[\times L / \{0\} \times L$ with smooth $(n - 1)$-dimensional basis $L$ and vertex $\{s\}$. This complex will depend on two positive numbers the “pinching number” $\alpha$ and the “control number” $\beta$ of which we give now interpretations.

Recall that we want to characterize the cone by the behavior of differentiable functions on its open regular stratum $]0, 1[\times L$. The metric $dr^2 + r^{2\alpha}g_L$, where $g_L$ is a metric on the link $L$, separates cylinder ($\alpha = 0$) and cones ($\alpha > 0$), [Ch]. But a metric is applied only to $k$-forms, $k > 0$. To define a suitable action at the functions level it seems natural to use, “near the vertex” $\{s\}$, the “germ” action of the multiplicative group $\mathbb{R}^*_+$ on $cL$ given by $\rho(r, x) = (\rho r, x)$ where $\rho \in \mathbb{R}^*_+$ and $(r, x) \in cL$. Although this is not an isometry in the cone case (i.e. $\alpha > 0$), each $g \in C^\infty(L)$ determines a 1-form $\omega = r^\alpha dg$ whose norm is equivariant : $\|\rho^*\omega\| = \|\omega\|$ and the functions $r^\alpha g$ are equivariant in $C^\infty([0, 1[\times L)$ relatively to the modified germ action

$$(\rho f)(r, x) = \rho^\alpha f(\rho r, x).$$

This action (and the associated equivariant relation) plays the role of a metric at the functions level. But it does not respect the algebra operation ($\rho^\alpha$ acts as a derivation, cf. (1) of introduction). To solve this problem we remark that the $r$-derivatives of the functions $r^\alpha g$ verify the equivariant condition :

$$\text{for any } n \in \mathbb{N}, \ |r^{-\alpha + n}(r \partial_r)^n(r^\alpha g)| \text{ is independent of } r.$$ 

So we modify the $C^\infty$-topology of $C^\infty([0, 1[\times L) \cong C^\infty([0, 1[) \hat{\otimes} C^\infty(L)$ using the following semi-norms on $C^\infty([0, 1[)$

$$\sup_{r \in [0, 1[} |r^{-\alpha + n}(r \partial_r)^n(\ -\ )|$$

This motives the introduction of the “reference algebra” that is the Frechet algebra of the (near $\{s\}$) bounded smooth functions relatively to the multiplicative group $\mathbb{R}^*_+$ :

$$A = A(r) = \{ a \in C^\infty([0, 1[) : \forall k \sup_{r \in [0, 1[} |(r \partial_r)^k a(r)| < +\infty \}.$$ 

Then we substitute equivariant functions by controlled functions, which are bounded relatively to these semi-norms, i.e. the elements of

$$r^\alpha A \hat{\otimes} C^\infty(L)$$
where $\widehat{\otimes}$ is the completed projective tensor product. The lack of algebra structure coming from the equivariant framework disappears when considered in the Frechet algebra framework.

The number $\beta$ controls the order of poles of the forms near the vertex $\{s\}$. The differential $A$-module of poles is:

$$M^*_\beta = r^{-\beta}A \oplus r^{-\beta}A\frac{dr}{r}$$

Remark that for $f_i \in C^\infty(L)$, the equivariant $k$-form $\omega_k = r^{-\beta}(r^\alpha f_0)(r^\alpha df_1) \wedge \cdots \wedge (r^\alpha df_k)$ has a pole of order $\beta - (k + 1)\alpha$ in $\{s\}$.

Now we shall construct the intersection complex. The $A$-module of $\beta$-controlled differential forms on the cone $cL$ is defined by:

$$B^k_\beta(cL) = r^{(k+1)\alpha} [M^*_\beta \widehat{\otimes} \Omega^*(L)]^k$$

Let $\{U_i\}_{i \in I}$ be a locally finite atlas of $L$, we denote by $x = \{x_1, \ldots, x_{n-1}\}$ a system of local coordinates in $U_i$. For $\gamma \in \mathbb{R}$, it can be shown that the $A$-module $r^\gamma A\widehat{\otimes}C^\infty(L)$ of $\gamma$-controlled functions on $cL$, denoted by $C^\infty_\gamma(cL)$, verifies:

$$C^\infty_\gamma(cL) = \{ f \in C^\infty([0,1] \times L) : \forall U_i, \forall s, \forall \lambda = (\lambda_1, \ldots, \lambda_{n-1}),$$

$$\sup_{(r,x) \in [0,1] \times U_i} r^{\gamma + |\lambda|} \left| \frac{\partial^{s+|\lambda|} f(r,x)}{\partial r^s \partial^{\lambda} x} \right| < \infty \}$$

where $s \in \mathbb{N}$, $\lambda_i \in \mathbb{N}$, $\partial^{\lambda} x = (\partial x_1)^{\lambda_1} \cdots (\partial x_{n-1})^{\lambda_{n-1}}$ and $|\lambda| = \lambda_1 + \cdots + \lambda_{n-1}$.

Then $B^k_\beta(cL)$ is the module of differential forms $\omega \in \Omega^k([0,1] \times L)$ whose restriction, in each $U_i$, is a sum of elements of the type $adx_k + b\frac{dr}{r} \wedge dx_{k-1}$ with $a, b \in C^\infty_{(k+1)\alpha - \beta}(cL)$.

It is not a complex (cf. (2) in introduction and also §2) so we define the intersection complex by:

$$IB^k_\beta(cL) = \{ \omega \in B^k_\beta(cL) : d\omega \in B^{k+1}_\beta(cL) \}$$

With minor changes of control parameters, the following result is similar to [BL], Théorème 2.4.

**Theorem 0.** Let $cL$ be a cone over an $(n-1)$-dimensional smooth manifold $L$, and $\bar{p}$ any perversity such that $p_n = n - 2 - \lfloor \beta/\alpha \rfloor$. Suppose that $\beta/\alpha$ is not an integer, then

$$H^k(IB^*_\beta(cL)) \cong \text{Hom}(IH^{\bar{p}}_k(cL, \mathbb{R}); \mathbb{R}) = \begin{cases} H^k_{dR}(L) & \text{if } k \leq \lfloor \beta/\alpha \rfloor - 1 \\ 0 & \text{if } k > \lfloor \beta/\alpha \rfloor - 1 \end{cases}$$
The idea of the proof is the following (cf [Ch]) : firstly by Poincaré Lemma, pointed in the vertex \( \{s\} \), all closed forms on \( cL - \{s\} \) are cohomologous to the extension (constant in \( r \)) to \( cL - \{s\} \) of a closed form on \( L \). Now :
- on one hand, such an extension is controlled only if \( k \leq \lceil \beta/\alpha \rceil - 1 \),
- on the other hand, for high degrees the order of poles of controlled forms decreases with the degree of the form. Namely, for \( k > \lceil \beta/\alpha \rceil - 1 \), the form converges to 0 as \( r \) goes to 0.

**Atlases of iterated cones and \( \bar{\beta} \)-controlled forms.**

In order to generalize the result to stratified varieties, we will use atlases whose charts are iterated cones.

Let \( x \) be a point in a stratum \( S_{n-j_1} \) and \( \psi_x : U_x \xrightarrow{\cong} B^{n-j_1} \times cL_x \) a distinguished open neighborhood as previously described. The link \( L_x \) is a singular variety and is covered by distinguished open sets of the same type. By iteration, we obtain a chart which defines an iterated cone :

\[
W_j = B^{n-j_1} \times c (B^{j_1-1-j_2} \times c (B^{j_2-1-j_3} \times \cdots \times c (B^{j_{\ell-1}}) \cdots))
\]

where \( j = \{n+1 = j_0 > j_1 > j_2 > \cdots > j_{\ell} > j_{\ell+1} = 0\} \) denotes a decreasing sequence of integers and \( B^{j_{i-1}-j_{i+1}} \) is an open ball in \( \mathbb{R}^{j_{i-1}-j_{i+1}} \) (possibly a point).

Via the homeomorphism \( \psi_x \), this chart corresponds to the following chain of elements of the filtration of \( X \) :

\[
\emptyset = X_{n-j_0} \subset X_{n-j_1} \subset X_{n-j_2} \subset \cdots \subset X_{n-j_{\ell}} \subset X = X_n = X_{n-j_{\ell+1}}.
\]

We will denote in the same way the iterated cone and its image in \( X \).

We obtain, by this way, a covering of \( U_x \) by charts which are iterated cones and corresponding to different sequences \( \bar{j} \).

Let us describe the coordinates in \( W_j \). For \( t = 0, \ldots, \ell \), we denote by \( u^t_{a_t} \) the coordinates in \( B^{j_{t}-1-j_{t+1}} \) (\( 1 \leq a_t \leq j_t - 1 - j_{t+1} \)) and \( r_j \), the coordinate of the generatrix of the \( t \)-th cone. So, we have \( \ell \) coordinates of the type \( r_j \) and \( n - \ell \) coordinates of the type \( u^t_{a_t} \). We set :

\[
\bar{r} = (r_{j_1}, \ldots, r_{j_{\ell}}) \quad \bar{u} = (\bar{u}_{j_0}, \ldots, \bar{u}_{j_{\ell}}), \text{ with } \bar{u}_{j_t} = (u^t_{1}, \ldots, u^t_{j_t-1-j_{t+1}})
\]
Given such an atlas on a Thom-Mather stratified space $X$, we can define a metric, called $\bar{\alpha}$-metric on $X - \Sigma$, and more precisely on iterated cones, in the following way: consider a sequence of real numbers $\bar{\alpha} = (\alpha_0, \cdots, \alpha_n)$ associated to the filtration $(\ast)$, each $\alpha_j$ corresponding to the stratum $S_{n-j}$. On each open ball $B^k$, with coordinates $u_1, \ldots, u_k$, the metric is the euclidean one: $du_1^2 + \cdots + du_k^2$. On the regular part of each product $B^j \times c(L)$, the metric is the product metric $(du_1^j)^2 + \cdots + (du_{j-1}^j)^2 + (dr_{j+1})^2 + (r_{j+1})^{2\alpha_j+1}g_L$ where $r_{j+1}$ is the coordinate of the generatrix of the cone and $g_L$ is the metric on the regular part of $L$, defined inductively.

Now we can define controlled functions on iterated cones. Let $\bar{\gamma} = (\gamma_1, \cdots, \gamma_n)$ an $n$-uple of real numbers, we define

$$C^{\infty}_{\bar{\gamma}}(W_j) = (\otimes_t r_j^{\gamma_t}\mathbb{A}(r_j)) \otimes (\otimes_t \infty^\infty(B^j))$$

In an equivalent way, this is the set of functions $f \in C^{\infty}(W_j \cap (X - \Sigma))$ such that for all tuples of positive integers $(\bar{s}, \bar{\lambda}) = (s_1, \ldots, s_\ell, \lambda_0, \ldots, \lambda_\ell)$, with $\lambda_\ell = (\lambda_1, \ldots, \lambda_{j-1-j_{t+1}})$,

$$\sup_{(\bar{r}, \bar{u})}(\bar{r})^{\bar{s}+\bar{\lambda}} \left| \frac{\partial^j(\bar{s}, \bar{\lambda}) f(\bar{r}, \bar{u})}{(\partial \bar{r})^{\bar{s}}(\partial \bar{u})^{\bar{\lambda}}} \right| < +\infty$$

where $(\bar{r})^{\bar{s}} = (r_{j_1})^{\gamma_{j_1}} \cdots (r_{j_\ell})^{\gamma_{j_\ell}}$ and $(\partial \bar{r})^{\bar{s}} = \partial (r_{j_1})^{s_1} \cdots \partial (r_{j_\ell})^{s_\ell}$, and in the same way $(\partial \bar{u})^{\bar{\lambda}} = (\partial \bar{u}_{j_0})^{\lambda_0} \cdots (\partial \bar{u}_{j_\ell})^{\lambda_\ell}$ where $(\partial \bar{u}_{j_0}) = (\partial u_0^1) \lambda_0^1 \cdots (\partial u_{j_1}^{j_1-1-j_{t+1}})$. 

Let $U$ be an open set in $X$, we say that a function $f \in C^{\infty}(U - \Sigma)$ is $\bar{\gamma}$-controlled on $U$ and we denote $f \in C^{\infty}_{\bar{\gamma}}(U)$, if for all $W_j$, we have $f \in C^{\infty}_{\bar{\gamma}}(W_j \cap U)$. The correspondence $U \mapsto C^{\infty}_{\bar{\gamma}}(U)$ defines a presheaf, which is not a sheaf. The associated sheaf, independant of the atlas and denoted by $C^{\infty}_{\bar{\gamma}}$ is defined by

$$C^{\infty}_{\bar{\gamma}}(U) = \{ f \in C^{\infty}(U - \Sigma) : \forall x \in U - \Sigma, \exists V_x \subset U - \Sigma, f \in C^{\infty}_{\bar{\gamma}}(V_x) \}.$$ 

Remark that for each open set $U$ such that $\{s\} \in U$, then $C^{\infty}_{\bar{\gamma}}(U) = \Gamma(U, C^{\infty}_{\bar{\gamma}})$.

A differential form defined in $W_j \cap U - \Sigma$ can be written as a sum of terms of the form:

$$a(\bar{r}, \bar{u})(d\bar{r})^{\bar{\mu}} \wedge (d\bar{u})^{\bar{k}}$$

where $(d\bar{r})^{\bar{\mu}} = (dr_{j_1})^{\mu_1} \wedge \cdots \wedge (dr_{j_\ell})^{\mu_\ell}$, $(\mu_i = 0, 1)$, $(d\bar{u})^{\bar{k}} = (d\bar{u}_{j_0})^{k_0} \wedge \cdots \wedge (d\bar{u}_{j_\ell})^{k_\ell}$ and $(d\bar{u}_{j_\ell})^{k_\ell} = (du_1^1)^{k_1} \wedge \cdots \wedge (du_{j_1-1-j_{t+1}}^{j_1-1-j_{t+1}})^{k_{j_1-1-j_{t+1}}}$. Here, $k_\ell$ is the number of coordinates appearing in $d\bar{u}_{j_\ell}$, i.e. $k_\ell = |\bar{k}_\ell| = k_1^\ell + \cdots + k_{j_1-1-j_{t+1}}^\ell$. 

9
Let $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ be a fixed sequence of strictly positive real numbers.

Let $\omega$ be a differential form defined on $U - \Sigma$, we say that $\omega$ is $\bar{\beta}$-controlled on $U$, relatively to the $\bar{\alpha}$-metric, if for all $W_j$ all the coefficients $a(\bar{r}, \bar{u})$ belong to $C_\infty^\infty(W_j \cap U)$ where, for all $1 \leq t \leq \ell$, $\gamma_{jt} = \beta_{jt} + \mu_t + (k_t + 1)\alpha_{jt}$. In this expression, $\alpha_{jt}$ is the pinching of the corresponding cone, $\beta_{jt}$ determines the order of pole, and $\mu_t$ and $k_t$ are determined by $\omega$.

Let us denote by $IB^k_{\bar{\beta}}(U)$ the space of $k$-differential forms $\omega$ such that $\omega$ and $d\omega$ are $\bar{\beta}$-controlled on $U$. We define the sheaf complex of $\bar{\beta}$-controlled differential forms $IB^*_\bar{\beta}$ as the sheaf associated to the presheaf $U \mapsto IB^k_{\bar{\beta}}(U)$ in the same way that we defined $C_\infty^\infty$.

The proof of the following theorem is similar to [BL], Théorème 3.5, up to minor changes of control parameters.

**Theorem 1.** Let $X$ be a Thom-Mather stratified space, endowed with a covering by iterated cones and an $\bar{\alpha}$-metric. Suppose $\bar{\beta}$ given, satisfying the following perversity condition :

$$
\left[ \frac{\beta_j}{\alpha_j} \right] \leq \left[ \frac{\beta_{j+1}}{\alpha_{j+1}} \right] \leq \left[ \frac{\beta_j}{\alpha_j} \right] + 1 \quad \forall j,
$$

with $\beta_j/\alpha_j$ non integer. Then, there is an isomorphism

$$
H^k(IB^*_\bar{\beta}(X)) \cong \text{Hom}(IH^\bar{p}_k(X, \mathbb{R}); \mathbb{R})
$$

with $\bar{p}_j = j - 2 - \left[ \frac{\beta_j}{\alpha_j} \right]$

Remark : The following inclusions are quasi-isomorphisms (the first one being defined only for polyedra) :

$$
\{ \text{Shadow forms} \} \subset \{ \bar{\beta} \text{-controlled forms} \} \subset \{ \mathcal{L}^p \text{- forms} \}
$$

4. **Hochschild and cyclic homology of controlled functions.**

For the rest of the paper and for simplicity, we turn back to the case of a cone over a smooth manifold. Firstly we show that controlled functions generate the intersection complex (Theorem 2), then we give relation with Hochschild and cyclic homology (Theorem 3).
How to generate controlled forms using controlled functions.

We remark that there are two methods to obtain a complex, starting with \( B^*_\beta \) (cf. (2), introduction) : in the first one, we consider the intersection complex \( IB^*_\beta(cL) \) previously defined, in the second one we stabilize by the de Rham operator, i.e. we add coboundaries. It is not difficult to show directly that the two complexes are quasi-isomorphic. The second one has the important property to be generated by controlled functions.

Firstly, dealing with a complex (namely \( \omega \) and \( d\omega \) are \( \beta \)-controlled) is translated in cyclic homology theory by the use of unitarized algebras. We define the \( A \)-unitarization of the \( A \)-algebra \( r^\alpha A \hat{\otimes} C^\infty(L) \) as the algebraic sum in \( C^\infty([0,1[\times L) \)

\[
IC^\infty_\alpha(cL) = r^\alpha A \hat{\otimes} C^\infty(L) + A
\]

The Frechet \( A \)-algebra structure is provided when we identify \( IC^\infty_\alpha(cL) \) with the quotient algebra \( (r^\alpha A \hat{\otimes} C^\infty(L) \oplus A) / I \) where \( I = \{ f + a : \forall g \in r^\alpha A \hat{\otimes} C^\infty(L), (f + a)g = 0 \} \) is a closed ideal.

Every differential form \( \omega \in \Omega^*([0,1[\times L) \) can be written in an unique way \( \omega = \eta + \frac{dr}{r} \wedge \varphi \) where \( \varphi = i \frac{dr}{r} \omega \) and \( \eta \in \Omega^*_p \), the space of differential forms relatively to the projection \( P : [0,1[\times L \rightarrow [0,1[ \).

Let us denote by \( \Omega^*(IC^\infty_\alpha(cL)) \) the subcomplex of \( \Omega^*([0,1[\times L) \) generated by the functions which belong to \( IC^\infty_\alpha(cL) \). With the previous notation, \( \Omega^*_p(IC^\infty_\alpha(cL)) \) is the complex generated by the element \( \eta \) in \( \Omega^*(IC^\infty_\alpha(cL)) \) and with differential \( d_L \) induced by the de Rham differential \( d = d_L + dr \). If the cone is looked as a family of spaces, \( L_r = L \) for \( r \in [0,1[ \) and \( L_0 = \{ s \} \), (this point of view appears implicitly in the \( A \)-module structure), then \( \Omega^*_p(IC^\infty_\alpha(cL)) \) is the complex of sections of the family of de Rham complexes on \([0,1[\) associated to the \( A \)-algebra \( IC^\infty_\alpha(cL) \).

Using the previous definition of the \( A \)-module of poles \( M^*_\beta \) we define the complex

\[
O_\beta^*(cL) = M^*_\beta \hat{\otimes} A \Omega^*_p(IC^\infty_\alpha(cL))
\]

with differential

\[
\begin{align*}
& r^{-\beta} \Omega^k_p(IC^\infty_\alpha(cL)) \oplus r^{-\beta} \frac{dr}{r} \wedge \Omega^{k-1}_p(IC^\infty_\alpha(cL)) \\
& \downarrow d_L \quad \downarrow d_r \quad \downarrow d_L \\
& r^{-\beta} \Omega^{k+1}_p(IC^\infty_\alpha(cL)) \oplus r^{-\beta} \frac{dr}{r} \wedge \Omega_p \Omega^k(IC^\infty_\alpha(cL))
\end{align*}
\]
We observe that the problem (2) of introduction is solved and that the unitarization with $R$ would not be sufficient to obtain a complex.

**Theorem 2.** There is an isomorphism of complexes:

$$I \Omega^*_\beta(cL) \cong \mathcal{B}^*_\beta + dB^*_\beta$$

In the assumptions of the Theorem 0, we have

$$H^k(I \Omega^*_\beta(cL)) = \text{Hom}(IH^0_k(cL, R); R).$$

**Hochschild and periodic cyclic homology of controlled functions.**

Now, let us give the relation with Hochschild and cyclic homology. Firstly we recall some basic and general properties, the references are [Co] and [Lo]. Let $\Lambda$ be a field and $\mathcal{A}$ be an algebra with unit, so $\Lambda \subset \mathcal{A}$. The Hochschild complex $(C^*_\mathcal{A}, b)$ is defined by $C_k(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^\otimes k$ and the Hochschild boundary is

$$b(a_0 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k + (-1)^k a_k a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

Its homology, called Hochschild homology, is denoted by $HH_*(\mathcal{A})$. The reduced Hochschild complex $(C^*_{\mathcal{A}}^{\text{red}}(A), b)$ is the quotient of the Hochschild complex by the subcomplex generated by the elements $a_0 \otimes \cdots \otimes a_k$ where $a_i \in \Lambda$ for some $i > 0$. The reduced Hochschild complex is quasi-isomorphic to the Hochschild complex.

The Hochschild complex is a cyclic module, i.e. it admits a cyclic action

$$\tau(a_0 \otimes \cdots \otimes a_k) = (-1)^k a_k \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

We have $\tau^{k+1} = id$, so $\tau$ defines an action of $\mathbb{Z}/(k + 1)\mathbb{Z}$.

The Connes cyclic homology is defined in the following way: Consider the situation where $\Lambda$ is a field of characteristic 0. The cyclic homology of $\mathcal{A}$, denoted by $HC_*(\mathcal{A})$, is the homology of the complex $(C_*(\mathcal{A})/(1 - \tau), b)$ where $b$ is induced by the Hochschild boundary. The relation between Hochschild and Connes homology is given by the Connes exact sequence

$$\cdots \to HH_k(\mathcal{A}) \xrightarrow{I} HC_k(\mathcal{A}) \xrightarrow{S} HC_{k-2}(\mathcal{A}) \xrightarrow{B} HH_{k+1}(\mathcal{A}) \to \cdots$$

Using the so called periodicity operator $S$ we define the periodic cyclic homology

$$\text{PHC}_* = \lim_k \left[ HC_k(\mathcal{A}) \xrightarrow{S} HC_{k-2}(\mathcal{A}) \right].$$
The importance of the above definitions appears with the following result. Let \( X \) be a compact \( C^\infty \) manifold, \( \mathcal{A} = C^\infty(X) \) the Frechet algebra of differentiable functions on \( X \) and \( \Omega^\star(X) \) the associated de Rham algebra. Replace everywhere \( \otimes \) by the projective tensor product \( \hat{\otimes} \). Then the application \( \pi : C_k(C^\infty(X)) \to \Omega^k(X) \) defined by \( \pi(f_0 \otimes \cdots \otimes f_k) = f_0 df_1 \wedge \cdots \wedge df_k \) induces the following isomorphisms [Co]:

\[
HH_\star(C^\infty(X)) \cong \Omega^\star(X) ; \quad PHC_{odd}^\star(C^\infty(X)) \cong \bigoplus_{even} H^\star_{dR}(X)
\]

If \( \Lambda \) is a ring, we need a more general setting, the notion of mixed complex that we briefly describe, as it appears in the singular framework.

A mixed complex, \([Ka]\), \((M_\star, b, B)\) is a graded module with two differentials, \( b \) of degree \( -1 \) and \( B \) of degree \( +1 \) such that \( bB + Bb = 0 \). It defines a bicomplex \( M_\star[u] \) with differentials \( b(mu^k) = (bm)u^k \), \( B(mu^k) = (Bm)u^{k-1} \) where degree\((u) = 2 \). We define the Hochschild homology of \((M_\star, b, B)\) as \( H_\star(M_\star, b) \) and the cyclic homology as \( H_\star(M_\star[u], b+B) \). There is again a Connes exact sequence and we can define the periodic cyclic homology.

When \( \Lambda \) is a field of characteristic 0, the relation between the two previous definitions is the following. Replacing the quotient \( C_k(A)/(1 - \tau) \) by a \( \mathbb{Z}/(k + 1)\mathbb{Z} \)-free resolution of \( C_k(A) \), we obtain a bicomplex which is quasi-isomorphic to the bicomplex associated to a mixed complex. The mixed complex structure of \( C_\star(A) \) is given by the Hochschild boundary \( b \) and by the operator \( B \) defined by

\[
B(a_0 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^{kj} 1 \otimes a_j \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{j-1} - (-1)^{k(j-1)} a_{j-1} \otimes 1 \otimes a_j \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{j-2} .
\]

Then the two definitions of cyclic homology agree.

In the following, the cone must be seen as a family of spaces \( L_r = L \) for \( r > 0 \) and \( L_0 = \{s\} \) and we shall use a slight generalization of the previous cyclic construction.

Consider the \( A \)-Hochschild complex

\[
C_k^A(IC^\infty_\alpha(cL)) = IC^\infty_\alpha(cL) \hat{\otimes}_A \cdots \hat{\otimes}_A IC^\infty_\alpha(cL)
\]

((\( k + 1 \)-terms) and denote by \( b_A \) its differential. Define the Hochschild-intersection complex by

\[
IC_k^B(cL) = M_\beta^* \hat{\otimes}_A C_k^{A}_{k-\star}(IC^\infty_\alpha(cL))
\]
The elements of degree $k$ are sum of terms

$$r^{-\beta} f_0 \otimes f_1 \otimes \cdots \otimes f_k + r^{-\beta} \frac{dr}{r} g_0 \otimes g_1 \otimes \cdots \otimes g_{k-1}$$

such that $f_i, g_j \in IC_{\alpha}^\infty(cL)$. The total differential is given by

$$r^{-\beta} C_k^A(IC_{\alpha}^\infty(cL)) \oplus r^{-\beta} A \frac{dr}{r} \otimes_A C_{k-1}^A(IC_{\alpha}^\infty(cL))$$

$$b_{A}^{k+1} \uparrow \downarrow B_{A}^{k} \quad \triangledown d_r \quad b_{A}^{k} \uparrow \downarrow B_{A}^{k-1}$$

$$r^{-\beta} C_{k+1}^A(IC_{\alpha}^\infty(cL)) \oplus r^{-\beta} A \frac{dr}{r} \otimes_A C_k^A(IC_{\alpha}^\infty(cL))$$

where the operator $B_A$ is associated to the cyclic operation (as in the classical case, [Lo])

$$\tau(r^{-\beta} g_0 \otimes \cdots \otimes g_k) = (-1)^k r^{-\beta} g_k \otimes g_0 \otimes \cdots \otimes g_{k-1}$$

(we leave the factor $r^{-\beta}$ in the first term cf. (3), introduction) and $d_r$ corresponds to the r-derivation,

$$d_r(r^{-\beta} f_0 \otimes f_1 \otimes \cdots \otimes f_k) =$$

$$(-1)^k r^{-\beta} \frac{dr}{r} \wedge \left[ \beta f_0 \otimes \cdots \otimes f_k + \sum_{i=0}^{k} f_0 \otimes \cdots \otimes \partial_r f_i \otimes \cdots \otimes f_k \right]$$

So $b_{A}^{*} \oplus b_{A}^{*-1}$ has degree $-1$ and $B_{A}^{*} \oplus B_{A}^{*-1} + d_r$ has degree 1.

**Lemma.** The triple

$$(IC_{\ast}^\beta(cL), b = b_{A}^{*} \oplus b_{A}^{*-1}, B = B_{A}^{*} \oplus B_{A}^{*-1} + d_r)$$

is a mixed complex (i. e. $b^2 = B^2 = bB + Bb = 0$).

**Theorem 3.** i) The Hochschild homology of $IC_{\ast}^\beta(cL)$ (with differential $1 \otimes b_{A}^{*}$) is :

$$HH_k(IC_{\ast}^\beta(cL)) \cong I\Omega_{\beta}^k(cL)$$

ii) the periodic cyclic homology is :

$$PHC_k(IC_{\ast}^\beta(cL)) \cong IH_{\bar{q}}^k(cL) \oplus IH_{\bar{q}}^{k-2}(cL) \oplus \cdots$$

where the perversity $\bar{q}$ satisfies $q_n = \left[ \frac{\beta}{\alpha} \right] - 1$. 

14
Sketch of the proof: i) The terms in $r^{-\beta}$ do not modify the demonstration (they stay as common factor), so we omit them in the proof of part (i). Consider the reduced $A$-Hochschild complex

\[ C_k^{\text{red}} = IC^\infty_\alpha(cL) \widehat{\otimes}_A C^\infty_\alpha(cL) \widehat{\otimes}_A \cdots \widehat{\otimes}_A C^\infty_\alpha(cL) \]

\[ \simeq r^{ka} IC^\infty_\alpha(c(L)) \widehat{\otimes} C^\infty(L^{x+k}) \subset C^\infty(c(L^{x+k+1}) - \{s\}) . \]

Using the lemma below, it suffices to prove that $H_k(C_*^{\text{red}}) \cong \Omega^k_P(\Omega^\infty_\alpha(cL))$. For every open $U \subset [0,1[\), we can define the Frechet module of controlled functions on $U \times L^{x(k+1)}$ in the same way, as $C_k^{\text{red}}$. So we have a presheaf $U \mapsto C_k^{\text{red}}(U)$ which define a fine sheaf $C_k^{\text{red}}$ using the same localization condition as $C^\infty_\gamma$. Its space of sections is the reduced $A$-Hochschild complex $\mathcal{C}^{\text{red}} = C_k^{\text{red}}([0,1])$. We can also associate to $\Omega^k_P(\Omega^\infty_\alpha(cL))$ a fine sheaf which is denoted by $\mathcal{I}\Omega^k$. We define a sheaf morphism $\pi : C_k^{\text{red}} \to \mathcal{I}\Omega^k$ as above: for each $U$ set $\pi(f_0 \otimes f_1 \otimes \cdots \otimes f_k) = f_0 df_1 \wedge \cdots \wedge df_k$ where $f_i$ is controlled on $U$. For each $r \in ]0,1[\), the fiber complex $(C_*^{\text{red}})_r$ is isomorphic to the standard Hochschild complex of $C_*(C^\infty(L))$ and the fiber complex $(\mathcal{I}\Omega^*)_r$ is isomorphic to $\Omega^*(L)$. These isomorphisms are induced by the following one in degree 0:

\[ IC^\infty_\alpha(cL)_r \longrightarrow C^\infty(L) \]

\[ r^\alpha a \otimes f + c \mapsto r^\alpha af + c \]

By Connes’s theorem [Co], for $r > 0$, the Hochschild homology of the fiber complex of $C_*^{\text{red}}$ is isomorphic to $\Omega^*(L)$ so the morphism $\pi$ induces a fiber isomorphism between the homology sheaf $\mathcal{H}_k(C_*^{\text{red}})$ and the sheaf $\mathcal{I}\Omega^*$. This can be extended for $r = 0$ and this implies that ([Go], 4.5):

\[ H_k(\Gamma(cL, C_*^{\text{red}})) \cong \Gamma(cL, \mathcal{H}_k(C_*^{\text{red}})) . \]

ii) By the isomorphism $H_k(C_*^{\text{red}}) \cong \Omega^k_P(\Omega^\infty_\alpha(cL))$, the differential $B$ gives the de Rham differential. Using theorems 1 and 2, the proof of ii) is then similar to the non singular case.

Similar sheaf arguments give the proof of the following lemma (used in the previous demonstration):

Lemma. The $A$-Hochschild complex $C_*^A(\Omega^\infty_\alpha(cL))$ and the reduced $A$-Hochschild complex $C_*^{\text{red}}$ are quasi-isomorphic.
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Jean-Paul Brasselet
IML - CNRS
Luminy Case 930
F-13288 Marseille Cedex 9
e-mail : jpb@iml.univ-mrs.fr

André Legrand
Laboratoire Emile Picard
Université Paul Sabatier
118 Route de Narbonne
F-31062 Toulouse Cedex
e-mail : legrand@picard.ups-tlse.fr