$L^p$ ($p > 1$) solutions of BSDEs with generators satisfying some non-uniform conditions in $t$ and $\omega$

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Abstract

This paper is devoted to the $L^p$ ($p > 1$) solutions of one-dimensional backward stochastic differential equations (BSDEs for short) with general time intervals and generators satisfying some non-uniform conditions in $t$ and $\omega$. An existence and uniqueness result, a comparison theorem and an existence result for the minimal solutions are respectively obtained, which considerably improve some known works. Some classical techniques used to deal with the existence and uniqueness of $L^p$ ($p > 1$) solutions of BSDEs with Lipschitz or linear-growth generators are also developed in this paper.

Keywords: Backward stochastic differential equation, Existence and uniqueness, Comparison theorem, Minimal solution, Non-uniform condition in $(t, \omega)$

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1. Introduction

Let us fix a extended real number $0 \leq T \leq +\infty$, which can be finite or infinite. Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$ be the natural $\sigma$-algebra generated by $(B_t)_{t \geq 0}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous and complete. In this paper, we are concerned with the following one-dimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dB_s, \quad t \in [0, T],$$

(1.1)

where the extended real number $T$ is called the terminal time, $\xi$ is a one-dimensional $\mathcal{F}_T$-measurable random variable called the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is $(\mathcal{F}_t)$-progressively measurable for each $(y, z)$ called the generator of BSDE (1.1). The solution $(y_t, z_t)_{t \in [0, T]}$ is a pair of $(\mathcal{F}_t)$-progressively measurable processes and the triple $(\xi, T, g)$ is called the parameters of BSDE (1.1). BSDE with the parameters $(\xi, T, g)$ is usually denoted by BSDE $(\xi, T, g)$.

The nonlinear BSDEs were initially introduced by Pardoux and Peng (1990). They proved an existence and uniqueness result for $L^2$ solutions of multidimensional BSDEs. In their work, the assumptions of generator $g$ is Lipschitz continuous with respect to $(y, z)$ uniformly in $(t, \omega)$, and the terminal time $T$...
is finite, the terminal condition $\xi$ and the process $\{g(t, 0, 0)\}_{t \in [0, T]}$ are square integrable. From then on, BSDEs have been extensively studied and many applications have been found in mathematical finance, stochastic control, partial differential equations and so on (see El Karoui, Peng and Quenez (1997) and Morlais (2009) for details). On the other hand, many papers have been devoted to relaxing the uniform Lipschitz condition on the generator $g$, improving the finite terminal time into the infinite case and studying the solutions under non-square integrable parameters.

Many works including Mao (1995), Lepeltier and San Martin (1997), Bahlali (2001), Briand, Delyon, Hu, Pardoux and Peng (2003), Hamadène (2003), Briand, Lepeltier and San Martin (2007), Briand and Confortola (2008), Wang and Huang (2009), Chen (2010), Delbaen, Hu and Bao (2011), Ma, Fan and Song (2013), Hu and Tang (2015) and Fan (2016), see also the references therein, weakened the uniform Lipschitz condition on the generator $g$, and some of them investigated the $L^p$ ($p > 1$) solution of BSDE (1.1). Chen and Wang (2000) first improved the result of Pardoux and Peng (1990) to the infinite time interval case and proved an existence and uniqueness result for the $L^2$ solution of BSDE (1.1) where the generator $g$ is Lipschitz continuous in $(y, z)$ non-uniformly with respect to $t$. Furthermore, Fan, and Jiang (2010) and Fan, Jiang and Tian (2011) relaxed the Lipschitz condition of Chen and Wang (2000) and obtained two existence and uniqueness results for the $L^2$ solution of BSDE (1.1) with finite and infinite time intervals, which also generalizes the results of Mao (1995) and Lepeltier and San Martin (1997) respectively.

We especially mention that El Karoui and Huang (1997) first introduced a stochastic Lipschitz condition of the generator $g$ in $(y, z)$, where the Lipschitz constant depends also on $(t, \omega)$. They investigated a general time interval BSDE driven by a general càdlàg martingale, and some stronger integrability conditions on the generator and terminal condition as well as on the solutions make it possible to replace the uniform Lipschitz condition by a stochastic one. In this spirit, Bender and Kohlmann (2000) and Wang, Ran and Chen (2007) respectively proved an existence and uniqueness result for the $L^2$ solution and $L^p$ ($p > 1$) solution of BSDE (1.1) with a general time horizon. After that, Briand and Confortola (2008) introduced another stochastic Lipschitz condition involving a bounded mean oscillation martingale and investigated the $L^p$ (for some certain $p > 1$) solution of an infinite dimensional BSDE, where some new higher order integrability conditions on the generator and terminal condition (see their assumptions A3 and A4 for details) need to be satisfied.

Motivated by these results, in this paper, we first put forward a new stochastic Lipschitz condition (see (H1) in Section 3) and prove an existence and uniqueness result of the $L^p$ ($p > 1$) solution of BSDE (1.1) with a finite and infinite time interval (see Theorem 3.1). We do not impose any stronger integrability conditions to the parameters $(\xi, g)$ and the solution $(y, z)$ as made in El Karoui and Huang (1997), Bender and Kohlmann (2000) and Wang, Ran and Chen (2007), and the integrability condition (3.1) is the only requirement in (H1). By introducing an example, we also show that our stochastic Lipschitz condition is strictly weaker than the Lipschitz condition non-uniformly in $t$ used in Chen and Wang (2000) (see Example 3.1). And by using stopping times to subdivide the interval $[0, T]$, we successfully overcome a new difficulty arisen naturally in our framework, see the proof of Theorem 3.1. Furthermore, in Section 4, by developing a method employed in Fan, Jiang and Tian (2011) and Ma, Fan and Song (2013) we establish a general comparison theorem for the $L^p$ ($p > 1$) solutions of BSDEs when one of
generators satisfies a monotonicity condition in $y$ and a uniform continuity condition in $z$, which are both non-uniform in $(t, \omega)$ (see Theorem 4.1). Finally, in Section 5, we prove an existence result of the minimal $L^p$ ($p > 1$) solution for BSDE (1.1) when the generator $g$ is continuous and has a linear growth in $(y, z)$ non-uniform in $(t, \omega)$ (see Theorem 5.1), by improving the method used in Izumi (2013) to prove in a direct way that the sequence of solutions of the BSDEs approximated by Lipschitz generators is a Cauchy sequence in $S^p \times M^p$. And, based on Theorem 5.1 together with Theorem 4.1, we will also give a new comparison theorem of the minimal $L^p$ ($p > 1$) solutions of BSDEs (see Theorem 5.2), and a general existence and uniqueness theorem of $L^p$ ($p > 1$) solutions of BSDEs (see Theorem 5.3).

We would like to mention that our results considerably improve some known works including those obtained in Pardoux and Peng (1990), Chen and Wang (2000), Briand, Lepeltier and San Martin (2007), Chen (2010) and Fan, Jiang and Tian (2011) etc. And, some classical techniques used to deal with the existence and uniqueness of $L^p$ ($p > 1$) solutions of BSDEs with Lipschitz or linear-growth generators are also developed in this paper.

2. Notations and lemmas

In this section, we introduce some basic notations and definitions, which will be used in this paper. First, we use $| \cdot |$ to denote the norm of Euclidean space $\mathbb{R}^d$. For each subset $A \subset \Omega \times [0, T]$, let $1_A = 1$ in case of $(\omega, t) \in A$, otherwise, let $1_A = 0$. For each real number $p > 1$, let $L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ be the set of all $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variables $\xi$ such that $\mathbb{E}[|\xi|^{p}] < +\infty$, and $S^p(0, T; \mathbb{R})$ (or $S^p$ simply) denote the set of $\mathbb{R}$-valued, adapted and continuous processes $(Y_t)_{t \in [0, T]}$ such that

$$\|Y\|_{S^p} := \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^p \right] \right)^{\frac{1}{p}} < +\infty.$$ 

In the whole paper, let $M^p(0, T; \mathbb{R}^d)$ (or $M^p$ simply) denote the set of ($\mathcal{F}_t$)-progressively measurable $\mathbb{R}^d$-valued processes $(Z_t)_{t \in [0, T]}$ such that

$$\|Z\|_{M^p} := \left( \mathbb{E} \left[ \left( \int_0^T |Z_t|^2 dt \right)^{\frac{p}{2}} \right] \right)^{\frac{1}{p}} < +\infty.$$ 

Obviously, both $S^p$ and $M^p$ are Banach spaces for each $p > 1$.

Finally, let $S$ be the set of all nondecreasing continuous functions $\phi(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\phi(0) = 0$ and $\phi(x) > 0$ for all $x \in \mathbb{R}^+$, here and hereafter $\mathbb{R}^+ := [0, +\infty)$.

**Definition 2.1.** A pair of processes $(y_t, z_t)_{t \in [0, T]}$ taking values in $\mathbb{R} \times \mathbb{R}^d$ is called a $L^p$ solution of BSDE (1.1) for some $p > 1$, if $(y_t, z_t)_{t \in [0, T]} \in S^p(0, T; \mathbb{R}) \times M^p(0, T; \mathbb{R}^d)$ and $d\mathbb{P}$ - a.s., BSDE (1.1) holds true for each $t \in [0, T]$.

Let us introduce the following Lemma 2.1, which will be used in Section 3 and Section 5.

**Lemma 2.1.** Let $p > 1$, $0 \leq T \leq +\infty$, and $(g_t)_{t \in [0, T]}$ is a ($\mathcal{F}_t$)-progressively measurable process such that $\int_0^T g_t dt < +\infty$, $d\mathbb{P}$ - a.s.. If $(Y_t, Z_t)_{t \in [0, T]}$ is a $L^p$ solution to the following BSDE:

$$Y_t = Y_T + \int_t^T g_s ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T],$$

(2.1)
then there exists a positive constant $C_p$ depending only on $p$ such that for each $t \in [0, T],

\begin{align*}
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] & \leq C_p \mathbb{E} \left[ |Y_T|^p + \int_t^T \left( |Y_s|^{p-1} |g_s| \right) \, ds \right], \tag{2.2}
\end{align*}

\begin{align*}
\mathbb{E} \left[ \left( \int_t^T |Z_s|^2 \, ds \right)^{\frac{p}{2}} \right] & \leq C_p \left\{ \mathbb{E} \left[ |Y_T|^p + \left( \int_t^T \left( |Y_s| |g_s| \right) \, ds \right)^{\frac{p}{2}} \right] + \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] \right\}. \tag{2.3}
\end{align*}

Moreover, there exists a positive constant $\tilde{C}_p$ depending only on $p$ such that for each $t \in [0, T],

\begin{align*}
\mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \mathbb{E} \left[ \left( \int_t^T |Z_s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq \tilde{C}_p \mathbb{E} \left[ |Y_T|^p + \left( \int_t^T |g_s| \, ds \right)^p \right], \tag{2.4}
\end{align*}

Proof. In the same way as Proposition 2.4 in Izumi (2013), we can prove (2.2) and (2.3). It remains to show (2.4). In fact, by basic inequality $2ab \leq a^2 + b^2$ and Young’s inequality we have, for each constant $\tilde{C}_p > 0,

\begin{align*}
\tilde{C}_p \mathbb{E} \left[ \int_t^T \left( |Y_s|^{p-1} |g_s| \right) \, ds \right] & \leq \tilde{C}_p \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^{p-1} \cdot \int_t^T |g_s| \, ds \right]
\end{align*}

\begin{align*}
& \leq \frac{1}{2} \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \frac{1}{p} \left( \frac{2(p-1)}{p} \tilde{C}_p \right)^{p} \mathbb{E} \left[ \left( \int_t^T |g_s| \, ds \right)^p \right] \tag{2.5}
\end{align*}

and

\begin{align*}
\mathbb{E} \left[ \left( \int_t^T \left( |Y_s| |g_s| \right) \, ds \right)^{\frac{p}{2}} \right] & \leq \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \cdot \left( \int_t^T |g_s| \, ds \right)^{\frac{p}{2}} \right]
\end{align*}

\begin{align*}
& \leq \frac{1}{2} \mathbb{E} \left[ \sup_{s \in [t, T]} |Y_s|^p \right] + \frac{1}{2} \mathbb{E} \left[ \left( \int_t^T |g_s| \, ds \right)^p \right]. \tag{2.6}
\end{align*}

Thus, (2.4) follows immediately from (2.2), (2.3), (2.5) and (2.6). \qed

The following technical Lemma 2.2 comes from Lemma 4 in Fan and Jiang (2011), which will be used in Section 4. It gives a sequence of upper bounds for functions of linear growth.

**Lemma 2.2.** Let $\Psi(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a nondecreasing function of linear growth, which means that $\Psi(x) \leq K(x+1)$ ($K > 0$) holds true for all $x \in \mathbb{R}^+$. Then for each $n \geq 1,

$$
\Psi(x) \leq (n+2K)x + \Psi \left( \frac{2K}{n+2K} \right)
$$

holds true for each $x \in \mathbb{R}^+$.

3. An existence and uniqueness result

In this section, we will use a stopping time technique involved in subdividing the time interval $[0, T]$ to prove a general existence and uniqueness result for the $L^p$ ($p > 1$) solution of BSDE (1.1), and introduce an example to show that our stochastic Lipschitz condition is strictly weaker than the Lipschitz condition non-uniformly in $t$ used in Chen and Wang (2000). First, let us introduce the following assumptions with the generator $g$, where $0 \leq T \leq +\infty$ and $p > 1$. 
(H1) $g$ is Lipschitz continuous in $(y, z)$ non-uniformly with respect to both $t$ and $\omega$, i.e., there exist two $(\mathcal{F}_t)$-progressively measurable nonnegative processes $\{u_t(\omega)\}_{t \in [0, T]}$ and $\{v_t(\omega)\}_{t \in [0, T]}$ satisfying
\[
\int_0^T [u_t(\omega) + v_t^2(\omega)] \, dt \leq M, \quad dP - a.s. \tag{3.1}
\]
for some constant $M > 0$ such that $dP \times dt$ a.e., for each $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,
\[
|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq u_t(\omega)|y_1 - y_2| + v_t(\omega)|z_1 - z_2|;
\]
(H2) $E \left[ \left( \int_0^T |g(\omega, t, 0, 0)| \, dt \right)^p \right] < +\infty.$

Remark 3.1. It is worth noting that the above (3.1) is equivalent to $\left\| \int_0^T u_t(\omega) + v_t^2(\omega) \, dt \right\|_\infty \leq M$. For the sake of convenience, the $\omega$ in $u_t(\omega)$ and $v_t(\omega)$ is usually omitted without confusion.

The following Theorem 3.1 shows an existence and uniqueness result for $L^p(p > 1)$ solutions of BSDEs under assumptions (H1) and (H2), which could be seen as a generalization of the results obtained in Pardoux and Peng (1990) and Chen and Wang (2000), where the $u_t$ and $v_t$ in (H1) do not depend on $\omega$.

Theorem 3.1. Assume that $p > 1$, $0 \leq T < +\infty$ and the generator $g$ satisfies assumptions (H1) and (H2). Then for each $\xi \in L^p(\mathbb{Q}, \mathbb{F}_T, P; \mathbb{R})$, BSDE $(\xi, T, g)$ admits a unique $L^p$ solution.

Proof. Assume that $(y_t, z_t)_{t \in [0, T]} \in S^p(0, T; \mathbb{R}) \times M^p(0, T; \mathbb{R}^d)$. It follows from (H1) that $|g(s, y_s, z_s)| \leq |g(s, 0, 0)| + u_s |y_s| + v_s |z_s|$, then from inequality $(a + b + c)p \leq 3p(a^p + b^p + c^p)$, Hölder’s inequality and (H2), we have
\[
E \left[ \left( \int_0^T |g(s, y_s, z_s)| \, ds \right)^p \right] \leq 3^p E \left[ \left( \int_0^T |g(s, 0, 0)| \, ds \right)^p \right] + (3M)^p E \left[ \sup_{s \in [0, T]} |y_s|^p \right] + 3^p M^\frac{p}{2} E \left[ \left( \int_0^T |z_s|^2 \, ds \right)^{\frac{p}{2}} \right] < +\infty.
\]
As a result, the process $\left( E \left[ \xi + \int_0^T g(s, y_s, z_s) \, ds \mid \mathcal{F}_t \right] \right)_{0 \leq t \leq T}$ is a $L^p$ martingale. It then follows from the martingale representation theorem that there exists a unique process $Z_t \in M^p(0, T; \mathbb{R}^d)$ such that
\[
E \left[ \xi + \int_0^T g(s, y_s, z_s) \, ds \mid \mathcal{F}_t \right] = E \left[ \xi + \int_0^T g(s, y_s, z_s) \, ds \right] + \int_0^t Z_s \cdot dB_s, \quad 0 \leq t \leq T.
\]
Let $Y_t := E \left[ \xi + \int_0^T g(s, y_s, z_s) \, ds \mid \mathcal{F}_t \right]$, $0 \leq t \leq T$. Obviously, $Y_t \in S^p(0, T; \mathbb{R})$, and it is not difficult to verify that the $(y_t, z_t)_{t \in [0, T]}$ is just the unique $L^p$ solution to the following equation:
\[
Y_t = \xi + \int_0^T g(s, y_s, z_s) \, ds - \int_0^T Z_s \cdot dB_s, \quad t \in [0, T]. \tag{3.2}
\]
Thus, we have constructed a mapping from $S^p(0, T; \mathbb{R}) \times M^p(0, T; \mathbb{R}^d)$ to itself. Denote this mapping by $I : (y_t, z_t)_{t \in [0, T]} \rightarrow (Y_t, Z_t)_{t \in [0, T]}$.

Now, suppose that $(y_t^i, z_t^i)_{t \in [0, T]} \in S^p(0, T; \mathbb{R}) \times M^p(0, T; \mathbb{R}^d)$, and let $(Y_t^i, Z_t^i)_{t \in [0, T]}$ be the mapping of $(y_t^i, z_t^i)_{t \in [0, T]}$, $i = 1, 2$, that is, $I(y_t^i, z_t^i)_{t \in [0, T]} = (Y_t^i, Z_t^i)_{t \in [0, T]}$, $i = 1, 2$. We denote
\[
\dot{Y}_t := Y_t^1 - Y_t^2, \quad \dot{Z}_t := Z_t^1 - Z_t^2, \quad \dot{y}_t := y_t^1 - y_t^2, \quad \dot{z}_t := z_t^1 - z_t^2,
\]
\[
\dot{g}_t := g(t, y_t^1, z_t^1) - g(t, y_t^2, z_t^2), \quad t \in [0, T].
\]
Then \((\hat{Y}_t, \hat{Z}_t)_{t\in[0,T]}\) is a \(L^p\) solution of the following BSDE:

\[
\hat{Y}_t = \int_t^T \hat{g}_s \, ds - \int_t^T \hat{Z}_s \cdot dB_s, \quad t \in [0,T].
\]

Furthermore, (2.4) of Lemma 2.1 yields that there exists a constant \(c_p > 0\) depending only on \(p\) such that for each \(t \in [0,T]\),

\[
E \left[ \sup_{s \in [t,T]} |\hat{Y}_s|^p + \left( \int_t^T |\hat{Z}_s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq c_p E \left[ \left( \int_t^T |\hat{g}_s|^2 \, ds \right)^{\frac{p}{2}} \right].
\]

Thus, by virtue of (H1) and Hölder’s inequality we can deduce that for each \(t \in [0,T]\),

\[
E \left[ \sup_{s \in [t,T]} |\hat{Y}_s|^p + \left( \int_t^T |\hat{Z}_s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq c_p E \left[ \left( \int_t^T u_s \, ds \right)^{\frac{p}{2}} + \left( \int_t^T v_s^2 \, ds \right)^{\frac{p}{2}} \right] \left( \sup_{s \in [t,T]} |\hat{g}_s|^p + \left( \int_t^T |\hat{z}_s|^2 \, ds \right)^{\frac{p}{2}} \right). \tag{3.3}
\]

In the sequel, we choose a large sufficiently number \(N\) such that

\[
\frac{M}{N} \leq \frac{1}{(4c_p)^{1/p}} \wedge \frac{1}{(4c_p)^{2/2}},
\]

and subdivide the interval \([0, T]\) into some small stochastic intervals like \([T_{i-1}, T_i]\), \(i = 1, \cdots N\), by defining the following \((\mathcal{F}_t)\)-stopping times:

\[
T_0 = 0;
\]

\[
T_1 = \inf \left\{ t \geq 0 : \int_0^t (u_s + v_s^2) \, ds \geq \frac{M}{N} \right\} \wedge T;
\]

\[
T_i = \inf \left\{ t \geq T_{i-1} : \int_0^t (u_s + v_s^2) \, ds \geq \frac{iM}{N} \right\} \wedge T;
\]

\[
T_N = \inf \left\{ t \geq T_{N-1} : \int_0^t (u_s + v_s^2) \, ds \geq \frac{NM}{N} \right\} \wedge T = T.
\]

Thus, for any \([T_{i-1}, T_i] \subset [0, T], i = 1, \cdots N\), it follows that

\[
\left( \int_{T_{i-1}}^{T_i} u_s \, ds \right)^{\frac{p}{2}} + \left( \int_{T_{i-1}}^{T_i} v_s^2 \, ds \right)^{\frac{p}{2}} \leq \frac{1}{2c_p}. \tag{3.4}
\]

Now, with the help of inequality (3.3), we have

\[
E \left[ \sup_{s \in [T_{N-1}, T]} |\hat{Y}_s|^p + \left( \int_{T_{N-1}}^{T} |\hat{Z}_s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq \frac{1}{2} E \left[ \sup_{s \in [T_{N-1}, T]} |\hat{g}_s|^p + \left( \int_{T_{N-1}}^{T} |\hat{z}_s|^2 \, ds \right)^{\frac{p}{2}} \right],
\]

which means that \(I\) is a strict contraction from \(S^p(T_{N-1}, T; \mathbb{R}) \times M^p(T_{N-1}, T; \mathbb{R}^d)\) into itself. Then \(I\) admits a unique fixed point in this space. It follows that there exists a unique \((y_t, z_t)_{t \in [T_{N-1}, T]} \in S^p(T_{N-1}, T; \mathbb{R}) \times M^p(T_{N-1}, T; \mathbb{R}^d)\) satisfying BSDE \((\xi, T, g)\) on \([T_{N-1}, T]\). That is to say, BSDE \((\xi, T, g)\) admits a unique \(L^p\) solution on \([T_{N-1}, T]\).
Finally, note that (3.4) holds true for \( i = N - 1 \). By replacing \( T_{N-1}, T \) and \( \xi \) by \( T_{N-2}, T_{N-1} \) and \( y_{T_{N-1}} \), respectively, in the above proof, we can obtain the existence and uniqueness for the \( L^p \) solution of BSDE \((\xi, T, g)\) on \([T_{N-2}, T_{N-1}]\). Furthermore, repeating the above procedure and making use of (3.4), we deduce the existence and uniqueness for the \( L^p \) solution of BSDE \((\xi, T, g)\) on \([T_{N-3}, T_{N-2}], \ldots, [0, T]\).

The proof of Theorem 3.1 is then completed. \(\square\)

**Remark 3.2.** It is easy to see that Theorem 3.1 holds still true for multidimensional BSDEs.

The following example shows that assumption (H1) is strictly weaker than the corresponding assumption in Chen and Wang (2000). For readers’ convenience, we list the assumption of Chen and Wang (2000) as the following (H1’):

(H1’) \( g \) is Lipschitz continuous in \((y, z)\), non-uniformly in \( t \), i.e., there exist two functions \( \bar{u}(t), \bar{v}(t) : [0, T] \to \mathbb{R}^+ \) satisfying

\[
\int_0^T [\bar{u}(t) + \bar{v}^2(t)] \, dt < +\infty
\]

such that \( dP \times dt \) a.e., for each \( y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d \),

\[
|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq \bar{u}(t)|y_1 - y_2| + \bar{v}(t)|z_1 - z_2|.
\]

**Example 3.1** Let \( 0 \leq T \leq +\infty \), and for each \( t_0 \in (0, T) \), define the following two stopping times:

\[
\tau_1(\omega) = \inf \{ t > t_0 : |B_{t_0}(\omega)|(t - t_0) \geq M/2 \} \land T, \\
\tau_2(\omega) = \inf \{ t > t_0 : |B_{t_0}(\omega)|^2(t - t_0) \geq M/2 \} \land T.
\]

Consider the generator \( \tilde{g}(\omega, t, y, z) := \bar{u}_t(\omega)|y| + \bar{v}_t(\omega)|z| \), where

\[
\tilde{u}_t(\omega) = |B_{t_0}(\omega)|1_{(t_0, \tau_1(\omega))}(\omega, t), \\
\tilde{v}_t(\omega) = |B_{t_0}(\omega)|1_{(t_0, \tau_2(\omega))}(\omega, t), \\
t, \omega \in [0, T] \times \Omega.
\]

It is clear that \( \tilde{g} \) satisfies assumptions (H1) and (H2) with \( u_t = \tilde{u}_t \) and \( v(t) = \tilde{v}_t \). Then, by Theorem 3.1 we know that for each \( p > 1 \) and each \( \xi \in L^p(\Omega, F_T, P; \mathbb{R}) \), BSDE \((\xi, T, \tilde{g})\) admits a unique \( L^p \) solution.

We especially mention that this \( \tilde{g} \) does not satisfy the above assumption (H1’). In fact, if assumption (H1’) holds true for \( \tilde{g} \), then there exist two deterministic functions \( \bar{u}(t), \bar{v}(t) : [0, T] \to \mathbb{R}^+ \) such that

\[
\bar{u}_t(\omega) \leq \bar{u}(t), \\
\bar{v}_t(\omega) \leq \bar{v}(t), \\
dP \times dt \text{ a.e.}
\]

and

\[
\int_0^T [\bar{u}(t) + \bar{v}^2(t)] \, dt < +\infty.
\]

This yields a contradiction which will be shown below. Note first that for each \( t \in (t_0, T) \), we have

\[
\{ \omega : \tilde{u}_t(\omega) > \bar{u}(t) \} = \{ \omega : t \leq \tau_1(\omega) \text{ and } |B_{t_0}(\omega)| > \bar{u}(t) \}
\]

\[
= \left\{ \omega : |B_{t_0}(\omega)| \leq \frac{M}{2(t - t_0)} \text{ and } |B_{t_0}(\omega)| > \bar{u}(t) \right\},
\]

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and note that \( B_{t_0}(\omega) \) is a normal random variable with zero-expected value and \( t_0 \)-variance values. If \( \bar{u}(t) < \frac{M}{2(t-t_0)} \) for some \( t \in (t_0, T) \), then \( P(\{\omega : \bar{u}_t(\omega) > \bar{u}(t)\}) > 0 \). Using this fact and (3.5) we can conclude that

\[
\bar{u}_t \geq \frac{M}{2(t-t_0)} \quad dt \text{ a.e. in } (t_0, T).
\]

Thus,

\[
\int_0^T \bar{u}(t)dt \geq \frac{M}{2} \int_{t_0}^T \frac{1}{t-t_0}dt = +\infty,
\]

which contradicts with (3.6).

Hence, our assumption (H1) is strictly weaker than (H1’) used in Chen and Wang (2000).

4. A general comparison theorem

In this section, by developing a method employed in Fan, Jiang and Tian (2011) and Ma, Fan and Song (2013) we will prove a general comparison theorem for the \( L^p \) (\( p > 1 \)) solution of BSDE (1.1). Let us first introduce the following assumptions, where \( 0 \leq T \leq +\infty \).

(H3) \( g \) is monotonic in \( y \), non-uniformly with respect to both \( t \) and \( \omega \), i.e., there exists a \( (\mathcal{F}_t) \)-progressively measurable nonnegative process \( \{u_t(\omega)\}_{t \in [0,T]} \) satisfying

\[
\int_0^T u_t(\omega)dt \leq M, \quad dP - a.s.
\]

for some constant \( M > 0 \) such that \( dP \times dt \text{ a.e.} \), for each \( y_1, y_2 \in \mathbb{R} \), \( z_1, z_2 \in \mathbb{R}^d \),

\[
\text{sgn}(y_1 - y_2) (g(\omega, t, y_1, z) - g(\omega, t, y_2, z)) \leq u_t(\omega) |y_1 - y_2|;
\]

(H4) \( g \) is uniformly continuous in \( z \), non-uniformly with respect to both \( t \) and \( \omega \), i.e., there exist a linear-growth function \( \phi(\cdot) \in \mathbf{S} \) and a \( (\mathcal{F}_t) \)-progressively measurable nonnegative process \( \{v_t(\omega)\}_{t \in [0,T]} \) satisfying

\[
\int_0^T v_t^2(\omega)dt \leq M, \quad dP - a.s.
\]

such that \( dP \times dt \text{ a.e.}, \) for each \( y_1, y_2 \in \mathbb{R} \), \( z_1, z_2 \in \mathbb{R}^d \),

\[
|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq v_t(\omega) \phi(|z_1 - z_2|).
\]

Here and henceforth, we always assume that \( 0 \leq \phi(x) \leq ax + b \) for all \( x \in \mathbb{R}^d \). Furthermore, when \( b \neq 0 \), we also assume that \( \int_0^T v_t(\omega)dt \leq M, \ dP - a.s. \), where \( M \) is defined in (H3).

The following Theorem 4.1 establishes a general comparison theorem for BSDEs under assumptions (H3) and (H4), which generalizes partly Theorem 2 in Fan, Jiang and Tian (2011), where the \( u_t(\omega) \) and \( v_t(\omega) \) in (H3) and (H4) do not depend on \( \omega \) and \( p = 2 \), and Lemma 1 in Ma, Fan and Song (2013), where the \( u_t(\omega) \) and \( v_t(\omega) \) need to be bounded processes and \( T < +\infty \).
Theorem 4.1. Let $p > 1$, $0 \leq T \leq +\infty$, $\xi, \xi' \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R})$, $g$ and $g'$ be two generators of BSDEs, and let $(y_t, z_t)_{t \in [0,T]}$ and $(y'_t, z'_t)_{t \in [0,T]}$ be, respectively, a $L^p$ solution to BSDE $(\xi, T, g)$ and BSDE $(\xi', T, g')$. If $dP - a.s., \xi \leq \xi'$, $g$ (resp. $g'$) satisfies (H3) and (H4) and $dP \times dt - a.e., g(t, y_t, z_t) \leq g'(t, y'_t, z'_t)$ (resp. $g(t, y_t, z_t) \leq g'(t, y'_t, z'_t)$), then for each $t \in [0, T]$, we have

$$dP - a.s., \ y_t \leq y'_t.$$  

Proof. Assume that $dP - a.s., \xi \leq \xi'$, $g$ satisfies (H3) and (H4) and $dP \times dt - a.e., g(t, y_t, z_t) \leq g'(t, y'_t, z'_t)$. Setting $\hat{y}_t = y_t - y'_t$, $\hat{z}_t = z_t - z'_t$, $\hat{\xi} = \xi - \xi'$, since $g(s, y_s, z_s) - g'(s, y'_s, z'_s)$ is non-positive, we have

$$g(s, y_s, z_s) - g'(s, y'_s, z'_s) = g(s, y_s, z_s) - g(s, y'_s, z'_s) + g(s, y'_s, z'_s) - g'(s, y'_s, z'_s) \leq g(s, y_s, z_s) - g(s, y'_s, z'_s) + g(s, y'_s, z'_s) - g'(s, y'_s, z'_s)$$

and we deduce, using assumptions (H3) and (H4), that

$$\mathbb{I}_{\hat{y}_t > 0}[g(s, y_s, z_s) - g'(s, y'_s, z'_s)] \leq u_s \hat{y}_s^+ + \mathbb{I}_{\hat{y}_t > 0}v_s \phi(|\hat{z}_s|). \quad (4.1)$$

Thus Tanaka’s formula with (4.1) leads to the following inequality, with $A_t := \int_0^t u_s ds$,

$$e^{A_t} \hat{y}_t^+ \leq e^{A_T} \hat{\xi}_+ + \int_0^T e^{A_s} \mathbb{I}_{\hat{y}_s > 0}[g(s, y_s, z_s) - g'(s, y'_s, z'_s)] - u_s \hat{y}_s^+ ds - \int_0^T e^{A_s} \mathbb{I}_{\hat{y}_s > 0}z_s \cdot dB_s, \quad t \in [0, T]. \quad (4.2)$$

Furthermore, note that Lemma 2.2 with $\Psi(\cdot) = \phi(\cdot)$ and $K = c := a + b$ yields that

$$\forall \ n \geq 1, \ x \in \mathbb{R}^+, \ \phi(x) \leq (n + 2c)x + \mathbb{1}_{b \neq 0} \phi \left( \frac{2c}{n + 2c} \right), \quad (4.3)$$

where $\mathbb{1}_{b \neq 0} = 1$ if $b \neq 0$ and $\mathbb{1}_{b \neq 0} = 0$ if $b = 0$. By (4.1)-(4.3), we get that, for each $n \geq 1$ and each $t \in [0, T]$,

$$e^{A_t} \hat{y}_t^+ \leq a_n + \int_0^T e^{A_s} \mathbb{I}_{\hat{y}_s > 0}(n + 2c) v_s |\hat{z}_s| ds - \int_0^T e^{A_s} \mathbb{I}_{\hat{y}_s > 0}z_s \cdot dB_s,$n

$$= a_n - \int_0^T e^{A_s} \mathbb{I}_{\hat{y}_s > 0}z_s : \left[ (n + 2c) v_s \frac{\hat{z}_s}{|\hat{z}_s|} \mathbb{1}_{|\hat{z}_s| \neq 0} ds + dB_s \right], \quad (4.4)$$

where, by (H4),

$$a_n = \mathbb{1}_{b \neq 0} \phi \left( \frac{2c}{n + 2c} \right) \left\| \int_0^T e^{A_s} v_s ds \right\|_\infty \leq \mathbb{1}_{b \neq 0} \phi \left( \frac{2c}{n + 2c} \right) \cdot M \cdot e^M \to 0 \ as \ n \to \infty. \quad (4.5)$$

In the sequel, let $P_n$ be the probability on $(\Omega, \mathcal{F})$ which is equivalent to $P$ and defined by

$$\frac{dP_n}{dP} := \exp \left\{ (n + 2c) \int_0^T v_s \frac{\hat{z}_s}{|\hat{z}_s|} \mathbb{1}_{|\hat{z}_s| \neq 0} \cdot dB_s - \frac{1}{2} (n + 2c)^2 \int_0^T \mathbb{1}_{|\hat{z}_s| \neq 0} v_s^2 ds \right\}.$$  

It is worth noting that $dP_n/dP$ has moments of all orders since $\int_0^T v^2(s) ds \leq M, dP - a.s..$ By Girsanov’s theorem, under $P_n$ the process

$$B_n(t) = B_t - \int_0^t (n + 2c) v_s \frac{\hat{z}_s}{|\hat{z}_s|} \mathbb{1}_{|\hat{z}_s| \neq 0} ds, \quad t \in [0, T]$$
is Brownian motion. Moreover, the process \( \left( \int_0^t e^{A_t} \mathbb{1}_{y_s > 0} \cdot dB_n(s) \right)_{t \in [0,T]} \) is a \( (\mathcal{F}_n, P_n) \)-martingale. Indeed, let \( E_n[X|\mathcal{F}_t] \) represent the conditional expectation of random variable \( X \) with respect to \( \mathcal{F}_t \) under \( P_n \) and let \( E_n[X] = E_n[X|\mathcal{F}_0] \), then, from the Burkholder-Davis-Gundy (BDG) inequality and Hölder's inequality, we have

\[
E_n \left[ \sup_{0 \leq t \leq T} \left| \int_0^t e^{A_t} \mathbb{1}_{y_s > 0} \cdot dB_n(s) \right| \right] \leq 4e^{M}E_n \left[ \int_0^T \sup_{0 \leq s \leq t} |\hat{z}_s|^2 ds \right] \leq 4e^{M}E \left[ \left( \int_0^T \sup_{0 \leq s \leq t} |\hat{z}_s|^2 ds \right)^{\frac{1}{2}} \right] < +\infty.
\]

Thus, by taking the conditional expectation with respect to \( \mathcal{F}_t \) under \( P_n \) in (4.4), we obtain that for each \( n \geq 1 \) and \( t \in [0,T] \),

\[
e^{A_t} \hat{y}_t^+ \leq a_n, \quad dP - a.s. \tag{4.6}
\]

And in view of (4.5), it follows that for each \( t \in [0,T] \), \( dP - a.s. \), \( y_t \leq y_t' \).

Now, let us assume that \( dP - a.s. \), \( \xi \leq \xi' \), \( g' \) satisfies (H3) and (H4) and \( dP \times dt - a.e., g(t,y_t,z_t) \leq g'(t,y_t,z_t) \). Then, since \( g(s,y_s,z_s) - g'(s,y_s,z_s) \) is non-positive, we have

\[
g(s,y_s,z_s) - g'(s,y'_s,z'_s) = g(s,y_s,z_s) - g'(s,y_s,z_s) + g'(s,y_s,z_s) - g'(s,y'_s,z'_s) \\
\leq g'(s,y_s,z_s) - g'(s,y'_s,z'_s) + g'(s,y'_s,z'_s) - g'(s,y'_s,z'_s),
\]

and using (H3) and (H4), we know that inequality (4.1) holds still true. Therefore, the same proof as above yields that for each \( t \in [0,T] \), \( dP - a.s. \), \( y_t \leq y_t' \). Theorem 4.1 is proved.

From Theorem 4.1, the following corollary is immediate.

**Corollary 4.1.** Let \( p > 1 \), \( 0 \leq T \leq +\infty \), \( \xi, \xi' \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R}) \), one of generators \( g \) and \( g' \) satisfy assumptions (H3) and (H4), and \( (y_t,z_t)_{t \in [0,T]} \) and \( (y'_t,z'_t)_{t \in [0,T]} \) be, respectively, a \( L^p \) solution to BSDE \((\xi,T,g)\) and BSDE \((\xi',T,g')\). If \( dP - a.s. \), \( \xi \leq \xi' \), and \( dP \times dt - a.e., g(t,y,z) \leq g'(t,y,z) \) for any \( (y,z) \in \mathbb{R} \times \mathbb{R}^d \), then for each \( t \in [0,T] \), \( dP - a.s. \), \( y_t \leq y_t' \).

5. **An existence result of the minimal solutions**

In this section, we will put forward and prove an existence result of the minimal \( L^p \) \( (p > 1) \) solution for BSDE (1.1)—Theorem 5.1, by improving the method used in Izumi (2013) to prove in a direct way that the sequence of solutions of the BSDEs approximated by the Lipschitz generators is a Cauchy sequence in \( S^p \times M^p \). And, based on Theorem 5.1 together with Theorem 4.1, we will also give a new comparison theorem of the minimal \( L^p \) \( (p > 1) \) solutions of BSDEs (see Theorem 5.2), and a general existence and uniqueness theorem of \( L^p \) \( (p > 1) \) solutions of BSDEs (see Theorem 5.3). First, we introduce the following assumptions with respect to the generator \( g \), where \( 0 \leq T \leq +\infty \).

(H5) \( g \) has a linear growth in \((y,z)\), non-uniformly with respect to both \( t \) and \( \omega \), i.e., there exist three \((\mathcal{F}_t)\)-progressively measurable nonnegative processes \( \{u_t(\omega)\}_{t \in [0,T]} \), \( \{v_t(\omega)\}_{t \in [0,T]} \) and
\{f_t(\omega)\}_{t \in [0, T]} \text{ satisfying} \\
E \left( \left( \int_0^T f_t(\omega) dt \right)^p \right) < +\infty, \\
and \\
\int_0^T \left[ u_t(\omega) + v_t^2(\omega) \right] dt \leq M, \ dP - a.s.,

for some constant $M > 0$ such that $dP \times dt - a.e.$, for each $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, 

$$
|g(\omega, t, y, z)| \leq f_t(\omega) + u_t(\omega)|y| + v_t(\omega)|z|; 
$$

(H6) $dP \times dt - a.e., g(\omega, t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ is a continuous function.

The following Proposition 5.1 will play an important role in the proof of Theorem 5.1. Its proof is analogous to Lemma 1 in Lepeltier and San Martin (1997), so we omit it here.

**Proposition 5.1.** Assume that the generator $g$ satisfies assumptions (H5) and (H6). Let $g_n$ be the function defined as follows:

$$
g_n(\omega, t, y, z) := \inf_{(\bar{y}, \bar{z}) \in \mathbb{R}^d + \mathbb{R}^d} \left\{ g(\omega, t, \bar{y}, \bar{z}) + nu_t(\omega)|y - \bar{y}| + nv_t(\omega)|z - \bar{z}| \right\}.
$$

Then the sequence of function $g_n$ is well defined, for each $n \geq 1$, $g_n(\omega, t, y, z)$ is ($\mathcal{F}_t$)-progressively measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, and it satisfies, $dP \times dt - a.e.,$

(i) **Stochastic linear growth:** \( \forall \, y, z, \, |g_n(\omega, t, y, z)| \leq f_t(\omega) + u_t(\omega)|y| + v_t(\omega)|z|; \)

(ii) **Monotonicity in $n$:** \( \forall \, y, z, \, g_n(\omega, t, y, z) \) increases in $n$;

(iii) **Lipschitz condition:** \( \forall \, y_1, y_2, z_1, z_2, \, \text{we have} \)

$$
|g_n(\omega, t, y_1, z_1) - g_n(\omega, t, y_2, z_2)| \leq nu_t(\omega)|y_1 - y_2| + nv_t(\omega)|z_1 - z_2|; 
$$

(iv) **Convergence:** If $(y_n, z_n) \to (y, z)$, then $g_n(\omega, t, y_n, z_n) \to g(\omega, t, y, z)$, as $n \to \infty$.

Now we state the main result of this section —Theorem 5.1. It improves Theorem 1 in Fan, Jiang and Tian (2011), where the $u_t(\omega)$ and $v_t(\omega)$ in (H5) do not depend on $\omega$, and $p = 2$, and Theorem 3.3 in Izumi (2013), where the $u_t(\omega)$ and $v_t(\omega)$ need to be bounded processes and $T < +\infty$.

**Theorem 5.1.** Assume that $p > 1$, $0 \leq T \leq +\infty$ and that the generator $g$ satisfies (H5) and (H6). Then for each $\xi \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R})$, BSDE $(\xi, T, g)$ admits a minimal $L^p$ solution $(y_t, z_t)_{t \in [0, T]}$, which means that if $(\bar{y}_t, \bar{z}_t)_{t \in [0, T]}$ is any $L^p$ solution to BSDE $(\xi, T, g)$, then for each $t \in [0, T]$, $dP - a.s., y_t \leq \bar{y}_t$.

**Proof.** Let $g_n$ be defined as in Proposition 5.1. In view of (i) of Proposition 5.1, for each $n \geq 1$, we have

$$
E \left( \left( \int_0^T |g_n(s, 0, 0)| ds \right)^p \right) \leq E \left( \left( \int_0^T f_s ds \right)^p \right) < +\infty.
$$

In view of (iii) of Proposition 5.1 and (H5), it follows from Theorem 3.1, that for each $n \geq 1$, BSDE $(\xi, T, g_n)$ and BSDE $(\xi, T, h)$ admit unique $L^p$ solutions $(y^n_t, z^n_t)_{t \in [0, T]}$ and $(Y_t, Z_t)_{t \in [0, T]}$, respectively,
where \( h(\omega, t, y, z) := f_t(\omega) + u_t(\omega)|y| + v_t(\omega)|z| \) for each \((\omega, t, y, z)\). And in view of (ii) of Proposition 5.1, Corollary 4.1 yields that for each \( n \geq 1 \) and \( t \in [0, T] \), \( y_t^n(\omega) \leq y_t^n(\omega) \leq y_t^{n+1}(\omega) \leq Y_t(\omega), dP-a.s. \). Thus, there must exist a \((\mathcal{F}_t)\)-progressively measurable process \((y_t)_{t \in [0, T]}\) satisfying that for each \( t \in [0, T] \),

\[
\lim_{n \to +\infty} y_t^n(\omega) = y_t(\omega), \quad dP - a.s.,
\]

and for each \( n \geq 1 \),

\[
|y_t^n(\omega)| \leq |y_t^1(\omega)| + |Y_t(\omega)|, \quad dP - a.s. \tag{5.1}
\]

Now, let \( G(\omega) = \sup_{t \in [0, T]}(|y_t^1(\omega)| + |Y_t(\omega)|) \), we have

\[
E \left[ \sup_{t \in [0, T]} |y_t|^p \right] \leq E [G^p] < +\infty. \tag{5.2}
\]

Furthermore, it follows form (2.3) of Lemma 2.1 together with (5.1) and (5.2) that there exists a constant \( C_p > 0 \) depending only on \( p \) such that for each \( n \geq 1 \),

\[
E \left[ \left( \int_0^T |z_s^n|^2 ds \right)^{\frac{p}{2}} \right] \leq C_p E \left[ |\xi|^p + \left( \int_0^T (|y_s^n| |g_n(s, y_s^n, z_s^n)|) ds \right)^{\frac{p}{2}} \right] + C_p E [G^p]. \tag{5.3}
\]

On the other hand, in view of (i) of Proposition 5.1 and by inequalities \((a + b + c)p \leq 3p(a^p + b^p + c^p), ab \leq \varepsilon a^2 + b^2/\varepsilon \) and Hölder’s inequality, we can deduce that for each \( n \geq 1 \) and \( \varepsilon > 0 \),

\[
E \left[ \left( \int_0^T (|y_s^n| |g_n(s, y_s^n, z_s^n)|) ds \right)^{\frac{p}{2}} \right] \leq 3^{\frac{p}{2}} E \left[ \left( \int_0^T (|y_s^n| f_s ds \right)^{\frac{p}{2}} + \left( \int_0^T (|y_s^n|^2 u_s ds \right)^{\frac{p}{2}} + \left( \int_0^T (|y_s^n| |z_s^n| v_s ds \right)^{\frac{p}{2}} \right]
\]

\[
\leq 3^{\frac{p}{2}} \left\{ E \left[ \sup_{|t| \leq T} |y_t^n|^p \right] + \frac{1}{2} E \left[ \left( \int_0^T f_s ds \right)^p \right] + \frac{1}{2} M^p + \left( \frac{1}{\varepsilon} \right)^{\frac{p}{2}} M^{\frac{p}{2}} E \left[ \sup_{|t| \leq T} |y_t^n|^p \right] \right\} + (3\varepsilon)^{\frac{p}{2}} E \left[ \left( \int_0^T |z_s^n|^2 ds \right)^{\frac{p}{2}} \right]. \tag{5.4}
\]

Now choosing \( \varepsilon > 0 \) such that \( C_p(3\varepsilon)^{\frac{p}{2}} = \frac{1}{2} \), from (5.3)-(5.4) together with (5.1) and (5.2), we can conclude that

\[
\sup_{n \geq 1} \|z_s^n\|^p_{L^p} = \sup_{n \geq 1} E \left[ \left( \int_0^T |z_s^n|^2 ds \right)^{\frac{p}{2}} \right] < +\infty. \tag{5.5}
\]

In the sequel, we will show the \((y_t^n)_{t \in [0, T]}\) is a Cauchy sequence in space \( S^p(0, T; \mathbb{R}) \). Note that \((y^m - y^n, z^m - z^n)\) satisfies the following equation:

\[
y_t^m - y_t^n = \int_t^T \left[ g_m(s, y_s^m, z_s^m) - g_n(s, y_s^n, z_s^n) \right] ds - \int_t^T (z_s^m - z_s^n) \cdot dB_s, \quad t \in [0, T],
\]

for each \( m, n \geq 1 \). In view of (H5) and (2.2) of Lemma 2.1, we obtain that there exists a constant \( c_p \) such that

\[
\|y^m - y^n\|_{S^p} \leq 2c_p E \left[ \int_0^T [ |y_s^m - y_s^n|^{p-1} f_s ds \right] + c_p E \left[ \int_0^T \left[ |y_s^m - y_s^n|^{p-1} u_s (|y_s^m| + |y_s^n|) \right] ds \right]
\]

\[
+ c_p E \left[ \int_0^T \left[ |y_s^m - y_s^n|^{p-1} v_s (|z_s^m| + |z_s^n|) \right] ds \right]. \tag{5.6}
\]

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We can prove that the three terms of right-hand side of the previous inequality tend to zero as \( m, n \to \infty \) respectively. Indeed, by (H5), Hölder’s inequality and (5.2), note that

\[
E \left[ \int_0^T (G^{p-1} f_s) \, ds \right] = E \left[ G^{p-1} \int_0^T f_s \, ds \right] \leq (E[G^p])^{\frac{p-1}{p}} \left( E \left[ \left( \int_0^T f_s \, ds \right)^p \right] \right)^{\frac{1}{p}} < +\infty,
\]

\[
E \left[ \left( \int_0^T (G^{p-1} u_s) \, ds \right)^{\frac{p}{p-1}} \right] = E \left[ G^p \left( \int_0^T u_s \, ds \right)^{\frac{p}{p-1}} \right] \leq E[G^p] M^{\frac{p}{p-1}} < +\infty,
\]

\[
E \left[ \left( \int_0^T (G^{2p-2} v_s^2) \, ds \right)^{\frac{p}{p-2}} \right] = E \left[ G^p \left( \int_0^T v_s^2 \, ds \right)^{\frac{p}{p-2}} \right] \leq E[G^p] M^{\frac{p}{p-2}} < +\infty.
\]

Since for each \( m, n \geq 1 \) and \( s \in [0, T] \), \( dP - a.s., |y^m_s(\omega) - y^m_s(\omega)|^{p-1} \leq 2^{p-1}G^{p-1}(\omega) \), and \( dP \times dt - a.e., y^n \to y \) as \( n \to +\infty \), by Lebesgue’s dominated convergence theorem we deduce that, as \( m, n \to \infty \),

\[
E \left[ \int_0^T |y^m_s - y^n_s|^{p-1} \, ds \right] \to 0, \quad E \left[ \left( \int_0^T |y^m_s - y^n_s|^{p-1} \, ds \right)^{\frac{p}{p-1}} \right] \to 0,
\]

\[
E \left[ \left( \int_0^T (|y^m_s - y^n_s|^{2p-2} v_s^2) \, ds \right)^{\frac{p}{p-2}} \right] \to 0.
\]

Thus, in view of (5.1), (5.2), (5.5) and (5.7), it follows from Hölder’s inequality that, as \( m, n \to \infty \),

\[
E \left[ \int_0^T |y^m - y^n|^{p-1} u_s(|y^m_s| + |y^n_s|) \, ds \right] \leq 2 \left( E[G^p] \right)^{\frac{p}{p-1}} \left( E \left[ \left( \int_0^T |y^m_s - y^n_s|^{p-1} \, ds \right)^{\frac{p}{p-1}} \right] \right)^{\frac{p}{p-1}} \to 0
\]

and

\[
E \left[ \int_0^T |y^m - y^n|^{p-1} v_s(|z^m_s| + |z^n_s|) \, ds \right] \leq E \left[ \left( \int_0^T |y^m - y^n|^{2p-2} v_s^2 \, ds \right)^{\frac{p}{p-2}} \right] \cdot E \left[ \left( \int_0^T |z^m_s|^2 + |z^n_s|^2 \, ds \right)^{\frac{p}{p-1}} \right] \to 0.
\]

Hence, combining (5.6)-(5.9), we obtain that

\[
\lim_{n \to \infty} \|y^n - y\|_{S^p} = 0.
\]

Furthermore, we prove that \((z^n_t)_{t \in [0, T]}\) is a Cauchy sequence in space \( M^p(0, T; \mathbb{R}^d) \). In fact, by (2.3) of Lemma 2.1, we know the existence of a constant \( C_p \) depending only on \( p \) such that for each \( m, n \geq 1 \),

\[
\|z^m - z^n\|_{M^p} \leq C_p E \left[ \left( \int_0^T \|y^m_s - y^n_s\|g_0(s, y^m_s, z^m_s) - g_0(s, y^n_s, z^n_s) \, ds \right)^{\frac{p}{2}} \right] + C_p \|y^m - y^n\|_{S^p}^p.
\]
On the other hand, by (H5), inequality \((a + b + c)^p \leq 3^p(a^p + b^p + c^p)\) and Hölder’s inequality, we deduce that
\[
E \left[ \left( \int_0^T \left| y_n^m - y_s^n \right| g_m(s, y_s^n, z_s^n) - g_n(s, y_s^n, z_s^n) \right| \, ds \right]^\frac{p}{2} 
\leq E \left[ \left( \int_0^T \left| y_n^m - y_s^n \right| \left( 2f_s + u_s(|y_s^n| + |y_s^n|) + v_s(|z_s^n| + |z_s^n|) \right) \, ds \right]^\frac{p}{2} 
\leq 3 \frac{p}{2} \| y_n^m - y_s^n \|_{L_p} \cdot \left\{ 2^{\frac{p}{2}} E \left[ \left( \int_0^T f_s \, ds \right)^\frac{p}{2} \right] + 2^\frac{p}{2} E[G^p] \frac{p}{2} \cdot M^\frac{p}{2} \right\} 
+ 3 \frac{p}{2} \| y_n^m - y_s^n \|_{L_p} \cdot E \left[ \left( \int_0^T |z_s^n| + |z_s^n| \, ds \right)^\frac{p}{2} \right] \cdot M^\frac{p}{2}. \tag{5.12}
\]
Thus, combining (5.5), (5.10), (5.11) and (5.12), we can conclude that there exists a process \(z \in M^P(0, T; \mathbb{R}^d)\) such that
\[
\lim_{n \to \infty} \| z_n^m - z \|_{M^P} = 0. \tag{5.13}
\]
Now, we can choose a subsequence of \(\{z_n^m\}\), still denote by itself, such that \(\| z_n^m - z \|_{M^P} \leq \frac{1}{n} \) for each \(n \geq 1\). Then
\[
\left\| \sup_n |z_n^m| \right\|_{M^P} \leq \left\| \sup_n \left( |z_n^m - z| + \| z \|_{M^P} \right) \right\|_{M^P} \leq \left\| \sum_{n=1}^\infty |z_n^m - z| \right\|_{M^P} + \| z \|_{M^P} \leq \sum_{n=1}^\infty \| z_n^m - z \|_{M^P} + \| z \|_{M^P} \leq 1 + \| z \|_{M^P} < +\infty. \tag{5.14}
\]
Denote \(H_t(\omega) := f_t(\omega) + u_t(\omega)G(\omega) + v_t(\omega) \sup_{n} |z_t^n(\omega)|\). By (H5), (5.1), (5.2) and (i) of Proposition 5.1, we know that for each \(n \geq 1\), \(dP \times dt - a.e.,\)
\[
|g_n(t, y_t^n, z_t^n) - g(t, y, z)| \leq 2H_t. \tag{5.15}
\]
And by Hölder’s inequality together with (5.2) and (5.14), we have
\[
E \left[ \left( \int_0^T |H_s| \, ds \right)^p \right] \leq 3^p E \left[ \left( \int_0^T f_s \, ds \right)^p \right] + 3^p E[G^p] M^p \left[ \int_0^T \sup_{n \geq 1} |z_s^n|^2 \, ds \right]^\frac{p}{2} \cdot M^\frac{p}{2} < +\infty. \tag{5.16}
\]
On the other hand, in view of (5.10), (5.13) and (iv) of Proposition 5.1, we can assume that, choosing a subsequence if necessary, as \(n \to \infty,\)
\[
g_n(t, y_t^n, z_t^n) \to g(t, y, z), \quad dP \times dt - a.e.. \tag{5.17}
\]
Thus, by (5.15)-(5.17), it follows from Lesbesgue’s dominated convergence theorem that
\[
\lim_{n \to \infty} E \left[ \left( \int_0^T g_n(s, y_s^n, z_s^n) - g(s, y_s, z_s) \, ds \right)^p \right] = 0.
\]
Finally, taking limits in BSDE \((\xi, T, g_n)\) yields that \((y_t, z_t)_{t \in [0,T]}\) is a \(L^p\) solution of BSDE \((\xi, T, g)\).

It remains to prove that \((y_\cdot, z_\cdot)\) is the minimal \(L^p\) solution of BSDE \((\xi, T, g)\), let \((\hat{y}_t, \hat{z}_t)_{t \in [0,T]}\) be any solution of BSDE \((\xi, T, g)\). In view of (ii) and (iii) of Proposition 5.1, by Corollary 4.1, we obtain that 
\[dP - a.s., \quad y^n_t \leq \hat{y}_t \text{ for each } t \in [0,T] \quad \text{and } n \geq 1,\]
from which and by letting \(n \to \infty\) we get that for each 
\[t \in [0,T], \quad dP - a.s., \quad y_t \leq \hat{y}_t.\]
The proof of Theorem 5.1 is then complete.

\[\square\]

**Remark 5.1.** In the same way as in Theorem 5.1, we can prove the existence of the maximal \(L^p\) \((p > 1)\) solution of BSDE \((1.1)\) under assumptions (H5) and (H6).

By Theorem 4.1 and the proof of Theorem 5.1, we can easily get the following comparison theorem on the minimal (resp. maximal) \(L^p\) solutions of BSDEs.

**Theorem 5.2.** Assume that \(p > 1, 0 \leq T \leq +\infty, \xi, \xi' \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R}),\) and both generators \(g\) and \(g'\) satisfy (H5) and (H6). Let \((y_\cdot, z_\cdot)\) and \((y'_\cdot, z'_\cdot)\) be, respectively, the minimal (resp. maximal) \(L^p\) solution of BSDE \((\xi, T, g)\) and BSDE \((\xi', T, g')\) (recall Theorem 5.1 and Remark 5.1). If 
\[dP - a.s., \xi \leq \xi' \quad \text{and} \quad dP \times dt - a.e., g(\omega, t, y, z) \leq g'(\omega, t, y, z) \quad \text{for each } (y, z) \in \mathbb{R} \times \mathbb{R}^d, \quad \text{then for each } t \in [0, T],\]
\[dP - a.s., \quad y_t \leq y'_t.\]

By Theorem 5.1 and Theorem 4.1, the following Theorem 5.3 follows immediately, which generalizes Theorem 3.1 in Section 3.

**Theorem 5.3.** Assume that \(p > 1, 0 \leq T \leq +\infty,\) and the generator \(g\) satisfies assumption (H2) and the following assumption (H7):

\((H7)\) \(g\) is Lipschitz continuous in \(y\) and uniformly continuous in \(z,\) non-uniformly with respect to both \(t\) and \(\omega\), i.e., there exist a linear-growth function \(\phi(\cdot) \in S\) and two \((\mathcal{F}_t)\)-progressively measurable nonnegative processes \(\{u_t(\omega)\}_{t \in [0,T]}\) and \(\{v_t(\omega)\}_{t \in [0,T]}\) satisfying
\[
\int_0^T [u_t(\omega) + v_t^2(\omega)] \ dt \leq M, \quad dP - a.s.
\]
for some constant \(M > 0\) such that \(dP \times dt - a.e.,\) for each \(y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d,\)
\[|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq u_t(\omega)|y_1 - y_2| + v_t(\omega)\phi(|z_1 - z_2|).
\]

Then for each \(\xi \in L^p(\Omega, \mathcal{F}_T, P; \mathbb{R}),\) BSDE \((\xi, T, g)\) admits a unique \(L^p\) solution.

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