FINITE RANK TOEPLITZ OPERATORS IN BERGMAN SPACES

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ABSTRACT. We discuss recent developments in the problem of description of finite rank Toeplitz operators in different Bergman spaces and give some applications

1. Introduction

Toeplitz operators arise in many fields of Analysis and have been an object of active study for many years. Quite a lot of questions can be asked about these operators, and these questions depend on the field where Toeplitz operators are applied.

The classical Toeplitz operator $T_f$ in the Hardy space $H^2(S^1)$ is defined as

$$T_fu = Pfu,$$  

(1.1)

for $u \in H^2(S^1)$, where $f$ is a bounded function on $S^1$ (the weight function) and $P$ is the Riesz projection, the orthogonal projection $P : L_2(S^1) \to H^2(S^1)$. Such operators are often called Riesz-Toeplitz or Hardy-Toeplitz operators (see, [15], for more detail). More generally, for a Hilbert space $\mathcal{H}$ of functions and a closed subspace $\mathcal{L} \subset \mathcal{H}$, the Toeplitz operator $T_f$ in $\mathcal{L}$ acts as in (1.1), where $P$ is the projection $P : \mathcal{H} \to \mathcal{L}$. In particular, in the case when $\mathcal{H}$ is the space $L_2(\Omega, \rho)$ for some domain $\Omega \subset \mathbb{C}^d$ and some measure $\rho$ and $\mathcal{L}$ is the Bergman space $\mathcal{B}^2 = \mathcal{B}^2(\Omega, \rho)$ of analytical functions in $\mathcal{H}$, such operator is called Bergman-Toeplitz; we will denote it by $T_f$.

Among many interesting properties of Riesz-Toeplitz operators, we mention the following cut-off one. If $f$ is a bounded function and the operator $T_f$ is compact then $f$ should be zero. For many other classes of operators a similar cut-off on some level is also observed. The natural question arises, whether there is a kind of cut-off property for Bergman-Toeplitz operators. Quite long ago it became a common knowledge that at least direct analogy does not take place. In the paper [13], the conditions were found on the function $f$ in the unit disk $\Omega = D$ guaranteeing that the operator $T_f$ in $\mathcal{B}^2(D, \lambda)$ with Lebesgue measure $\lambda$ belongs to the Schatten class $S_p$. So, the natural question came up: probably, it is on the finite rank level that the cut-off takes
place. In other words, if a Bergman-Toeplitz operator has finite rank it should be zero.

It was known long ago that the Schatten class behavior of $T_f$ is determined by the rate of convergence to zero at the boundary of the function $f$. Therefore the finite rank (FR) hypothesis deals with functions $f$ with compact support not touching the boundary of $\Omega$. In this setting the FR hypothesis is equivalent to the one for Toeplitz operators on the Bargmann (Fock, Segal) space consisting of analytical functions in $\mathbb{C}$, square summable with a Gaussian weight. A proof of the FR hypothesis appeared in the same paper [13], about twenty lines long. Unfortunately, there was an unrepairable fault in the proof, so the FR remained unsettled.

It was only in 2007 that the proof of the FR hypothesis was finally found, even in a more general form. The Bergman projection $P : L_2 \rightarrow B$ can be extended to an operator from the space of distributions $D'(\Omega)$ to $B^2(\Omega, \lambda)$. Let $\mu$ be a regular complex Borel measure with compact support in $\Omega$. With $\mu$ we associate the Toeplitz operator $T_\mu : u \mapsto Pu \mu$ in $B^2(\Omega, \lambda)$.

In the paper [14] the following result was established.

**Theorem 1.1.** Suppose that the Toeplitz operator $T_\mu$ in $B^2(\Omega, \lambda)$, $\Omega \subset \mathbb{C}$ has finite rank $r$. Then the measure $\mu$ is the sum of $r$ point masses,

$$\mu = \sum_{k=1}^{r} C_k \delta_{z_j}, \; z_j \in \Omega. \quad (1.2)$$

The publication of the proof of Theorem 1.1 induced an activity around it. In two years to follow several papers appeared, where the FR theorem was generalized in different directions, and interesting applications were found in Analysis and Mathematical Physics.

In this paper we aim for collecting and systematizing the existing results on the finite rank problem and their applications. We also present several new theorems generalizing and extending these results.

## 2. Problem setting

Let $\Omega$ be a domain in $\mathbb{R}^d$ or $\mathbb{C}^d$. We suppose that a measure $\rho$ is defined on $\Omega$, jointly absolutely continuous with Lebesgue measure. Suppose that $\mathcal{L}$ is a closed subspace in $\mathcal{H} = L_2(\Omega, \rho)$, consisting of smooth functions, $\mathcal{L} \subset C^\infty(\Omega)$. In this case the orthogonal projection $P : \mathcal{L} \rightarrow \mathcal{H}$ is an integral operator with smooth kernel,

$$Pu(x) = \int P(x, y)u(y)d\rho(x). \quad (2.1)$$

We will call $P$ the Bergman projection and $P(x, y)$ the Bergman kernel (corresponding to the subspace $\mathcal{L}$).
Let $F$ be a distribution, compactly supported in $\Omega$, $F \in \mathcal{E}'(\Omega)$. We will denote by $\langle F, \phi \rangle$ the action of the distribution $F$ on the function $\phi \in \mathcal{E}$. Then one can define the Toeplitz operator in $L$ with weight $F$:

$$(T_Fu)(x) = \langle F, P(x, \cdot)u(\cdot) \rangle.$$  \hspace{1cm} (2.2)

The formula (2.2) can be also understood in the following way. The operator $P$ considered as an operator $P : \mathcal{H} \to \mathcal{L}$ has an adjoint, $P' : \mathcal{L}' \to \mathcal{H}$, so $PP'$ is the extension of $P$ to the operator $\mathcal{L}' \to \mathcal{L}$, in particular, $P$ extends as an operator from $\mathcal{E}'(\Omega)$ to $\mathcal{L}$. In this setting, $Fu \in \mathcal{E}'(\Omega)$ for $u \in \mathcal{E}(\Omega)$ and the Toeplitz operator has the form

$$T_Fu = PFu,$$  \hspace{1cm} (2.3)

consistently with the traditional definition of Toeplitz operators.

It is more convenient to use the description of the Toeplitz operator by means of the sesquilinear form. For $u, v \in \mathcal{L}$, we have

$$(T_Fu, v) = \langle PFu, v \rangle = \langle \sigma Fu, \overline{Pv} \rangle = \langle \sigma F, uv \rangle,$$  \hspace{1cm} (2.4)

where $\sigma$ is the Radon-Nicodim derivative of $\rho$ with respect to the Lebesgue measure. In particular, if $F$ is a regular Borel complex measure $F = \mu$, the corresponding Toeplitz operator acts as

$$T_\mu u(x) = \int_\Omega P(x,y)u(x)d\mu(x),$$  \hspace{1cm} (2.5)

and the quadratic form is

$$(T_Fu, v) = \int_\Omega uv\sigma d\mu(x).$$

Finally, when $F$ is a bounded function, the formula (2.4) takes the form

$$(T_Fu, v) = \int_\Omega uvF(x)d\rho(x).$$  \hspace{1cm} (2.6)

Classical examples of Bergman spaces and corresponding Toeplitz operators are produced by solutions of elliptic equations and systems.

**Example 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}$, $\rho = \lambda$ be the Lebesgue measure, $\mathcal{L} = \mathcal{B}^2(\Omega)$ be the space of $L^2$ functions analytical in $\Omega$. This is the classical Bergman space.

**Example 2.2.** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^d$, $d > 1$, with Lebesgue measure $\rho$ and let the space $\mathcal{L}$ consist of $L^2$ functions analytical in $\Omega$. This is also a classical Bergman space. Here, and in Example 2.1, measures different from the Lebesgue one are also considered, especially when $\Omega$ is a ball or a (poly)disk.

**Example 2.3.** For a bounded domain $\Omega \subset \mathbb{R}^d$, we set $\mathcal{L}$ to be the space of $L^2$ solutions of the equation $Lu = 0$, where $L$ is an elliptic differential operator with constant coefficients. In particular, if $L$ is the Laplacian, the space $\mathcal{L}$ is called the harmonic Bergman space.
Example 2.4. If $\Omega$ is a bounded domain in $\mathbb{R}^d$ with even $d = 2m$, and $\mathbb{R}^d$ is identified with $\mathbb{C}^m$ with variables $z_j = (x_j, y_j), j = 1, \ldots, m$, the Bergman space of functions which are harmonic with respect to each pair $(x_j, y_j)$ is called $m$-harmonic Bergman space; if on the other hand, the space of functions $u(z)$ such that $u_\zeta(\xi_1, \xi_2) = u(\zeta(\xi_1 + i\xi_2))$ is harmonic as a function of variables $\xi_1, \xi_2$ for any $\zeta \in \mathbb{C}^m \setminus \{0\}$, is called pluriharmonic Bergman space.

Example 2.5. Let $\Omega$ be the whole of $\mathbb{C}^m = \mathbb{R}^d$, with the Gaussian measure $d\rho = \exp(-|z|^2/2)d\lambda$. The subspace $L \subset L^2(\mathbb{C}^m, \rho)$ of entire analytical functions in $\mathbb{C}^m$ is called Fock or Segal-Bargmann space.

The study of Toeplitz operators in many cases is based upon the consideration of associated infinite matrices.

Let $\mathcal{F}_1 = \{f_j(x), x \in \Omega\}, \mathcal{F}_2 = \{g_j(x), x \in \Omega\}$ be two infinite systems of functions in $\mathcal{L}$. With these systems and a distribution $F \in \mathcal{E}'(\Omega)$ we associate the matrix

$$A = A(F) = A(F, \mathcal{F}_1, \mathcal{F}_2, \Omega, \rho) = (T_F f_j, g_k)_{j,k=1,...} = (\langle \sigma F, f_j \bar{g}_k \rangle).$$

(2.7)

So, the matrix $A$ is the matrix of the sesquilinear form of the operator $T_F$ on the systems $\mathcal{F}_1, \mathcal{F}_2$. We formulate the obvious but important statement.

**Proposition 2.6.** Suppose that the Toeplitz operator $T_F$ has finite rank $r$. Then the matrix $A$ also has finite rank, moreover $\text{rank}(A) \leq r$.

The use of matrices of the form (2.7) enables one to perform important reductions. In particular, since the domain $\Omega$ does not enter explicitly into the matrix, the rank of this matrix does not depend on the domain $\Omega$, as long as one can choose the systems $\mathcal{F}_1, \mathcal{F}_2$ dense simultaneously in the Bergman spaces in different domains. Thus, in particular, the FR problems for the analytical Bergman spaces in bounded domains and for the Fock space are equivalent (see the discussion in [19].)

3. **Theorem of D. Luecking. Extensions in dimension 1**

In this section we present the original proof given by D. Luecking in [14], and give extensions in several directions.

**Theorem 3.1.** Let $\Omega \subset \mathbb{C}$ be a bounded domain, with Lebesgue measure. Suppose that for some regular complex Borel measure $\mu$, absolutely continuous with respect to the Lebesgue measure, with compact support in $\Omega$, the Toeplitz operator $T_\mu$ in the Bergman space of analytical functions has finite rank $r$. Then $\mu = 0$.

We formulate and prove here Luecking’s theorem only in in the case of an absolutely continuous measure; the case of more singular measures will be taken care of later, as a part of the general distributional setting.
In the proof, which follows [14], we separate a lemma that will be used further on.

**Lemma 3.2.** Let \( \phi \) be a linear functional on polynomials in \( z, \bar{z} \). Denote by \( A(\phi) \) the matrix with elements \( \phi(z^j\bar{z}^k) \). Then the following are equivalent:

1. the matrix \( A(\phi) \) has finite rank not greater than \( r \);
2. for any collections of nonnegative integers \( J = \{j_0, \ldots, j_r\} \), \( K = \{k_0, \ldots, k_r\} \),
   \[ \phi^\otimes N (\prod_{i \in (0, r)} z_i^{j_i} \det \bar{z}_i^{k_i}) = 0, \]
   where \( N = r + 1 \).

**Proof.** Since passing to linear combinations of rows and columns does not increase the rank of the matrix, it follows that for any polynomials \( f_j(z), g_k(z) \), with \( j, k = 0, \ldots, r \), the determinant \( \text{Det} (\phi(f_j\bar{g}_k)) \) vanishes.

The determinant is linear in each column and \( \phi \) is a linear functional, so we can write

\[
\phi \left( f_0(z) \times \begin{bmatrix} g_0(z) & \mu(f_1\bar{g}_0) & \cdots & \phi(f_r\bar{g}_0) \\ g_1(z) & \phi(f_1\bar{g}_1) & \cdots & \phi(f_r\bar{g}_1) \\ \vdots & \vdots & \ddots & \vdots \\ g_r(z) & \phi(f_1\bar{g}_r) & \cdots & \phi(f_r\bar{g}_r) \end{bmatrix} \right) = 0
\]

We introduce the variable \( z_0 \) in place of \( z \) above and use \( \phi_0 \) for \( \phi \) acting in the variable \( z_0 \). We repeat this process in each column (using the variable \( z_j \) in column \( j \)) to obtain

\[
\phi_0 \left( \phi_1 \left( \cdots \phi_r \left( \prod_{k=0}^{r} f_k(z_k) \det (g_j(z_k)) \right) \right) \right) = 0 \quad (3.2)
\]

We now specialize to the case where each \( f_i = z^{j_i}, g_i = z^{k_i} \) and arrive at (3.1), thus proving the implication \( 1 \Rightarrow 2 \). The converse implication follows by going along the above reasoning in the opposite direction.

**Proof of Theorem 3.1.** We identify \( \mathbb{C} \) and \( \mathbb{R}^2 \) with co-ordinates \( z = x + iy \). Consider the functional \( \phi(f) = \phi_\mu(f) = \int f(z)d\mu(z) \). Write \( Z \) for the \( N \)-tuple \( (z_0, z_1, \ldots, z_r) \) and \( V_J(Z) \) for the determinant \( \det \left( z_i^{k_j} \right) \).

By Lemma 3.2,

\[
\phi^\otimes N \left( Z^K V_J(Z) \right) = 0. \quad (3.3)
\]

Taking finite sums of equations (3.3), we get for any polynomial \( P(Z) \) in \( N \) variables:

\[
\phi^\otimes N \left( P(Z) V_J(Z) \right) = 0. \quad (3.4)
\]
By taking linear combinations of antisymmetric polynomials $V_j(Z)$ one can obtain any antisymmetric polynomial $Q(Z)$ (see [14]) for details). Thus

$$\phi^{\otimes N} \left( P(Z)\overline{Q(Z)} \right) = 0$$

for any polynomial $P(Z)$ and any antisymmetric polynomial $Q(Z)$. In its turn, the polynomial $Q(Z)$ is divisible by the lowest degree antisymmetric polynomial, the Vandermonde polynomial $V(Z) = \prod_{0 \leq j \leq r}(z_j - z_k)$, $Q(Z) = Q_1(Z)V(Z)$ with a symmetric polynomial $Q_1(Z)$. We write (3.5) for $Q$ of this form and $P$ having the form $P(Z) = P_1(Z)V(Z)$. So we arrive at

$$\phi^{\otimes N} \left( P_1(Z)\overline{Q_1(Z)}|V(Z)|^2 \right) = 0 \quad \text{for all symmetric } P_1 \text{ and } Q_1. \quad (3.6)$$

It is clear that finite sums of products of the form $P_1(Z)\overline{Q_1(Z)}$ (with $P_1$ and $Q_1$ symmetric) form an algebra $\mathcal{A}$ of functions on $\mathbb{C}$ which contains the constants and is closed under conjugation. It doesn’t separate points because each element is constant on sets of points that are permutations of one another. Therefore we define an equivalence relation $\sim$ on $\mathbb{C}^N: Z_1 \sim Z_2$ if and only if $Z_2 = \pi(Z_1)$ for some permutation $\pi$. Let $Z = (z_0, \ldots, z_r)$ and $W = (w_0, \ldots, w_r)$. If $Z \not\sim W$ then the polynomials $p(t) = \prod(t - z_j)$ and $q(t) = \prod(t - w_j)$ have different zeros (or the same zeros with different orders). This implies that the coefficient of some power of $t$ in $p(t)$ differs from the corresponding coefficient in $q(t)$. Thus there is an elementary symmetric function that differs at $Z$ and $W$. Consequently, $\mathcal{A}$ separates equivalence classes.

We give the quotient space $\mathbb{C}^N/\sim$ the standard quotient space topology. If $K$ is any compact set in $\mathbb{C}^N$ that is invariant with respect to $\sim$, then $K/\sim$ is compact and Hausdorff. Also, any symmetric continuous function on $\mathbb{C}^N$ induces a continuous function on $\mathbb{C}^N/\sim$ (and conversely). Thus we can apply the Stone-Weierstrass theorem (on $K/\sim$) to conclude that $\mathcal{A}$ is dense in the space of continuous symmetric functions, in the topology of uniform convergence on any compact set. Therefore, for any continuous symmetric function $f(Z)$

$$\int_{\mathbb{C}^N} f(Z)|V(Z)|^2 \, d\mu^{\otimes N}(Z) = 0 \quad (3.7)$$

If $f$ is an arbitrary continuous function, the above integral will be the same as the corresponding integral with the symmetrization of $f$ replacing $f$. This is because the function $|V(Z)|^2$ and the product measure $\mu^{\otimes N}$ are both invariant under permutations of the coordinates. We conclude that this integral vanishes for any continuous $f$ and so the measure $|V(Z)|^2 \, d\mu^{\otimes N}(Z)$ must be zero. Thus, $\mu^{\otimes N}$ is supported on the set where $V$ vanishes, i.e. on the set of Lebesgue measure zero. Since $\mu^{\otimes N}$ is absolutely continuous, it must be zero. \qed
The initial setting of Theorem 3.1 dealt with arbitrary measures, as it is explained in the Introduction. A more advanced result was obtained in [2], where Luecking’s theorem was carried over to distributions.

**Theorem 3.3.** Suppose that \( F \in \mathcal{E}'(\Omega) \) is a distribution with compact support in \( \Omega \subset \mathbb{C} \) and the Toeplitz operator \( T_F \) has finite rank \( r \). Then the distribution \( F \) is a finite combination of \( \delta \)-distributions at some points in \( \Omega \) and their derivatives,

\[
F = \sum_{j \leq r} L_j \delta(z - z_j),
\]

(3.8)

\( L_j \) being differential operators.

We start with some observations about distributions in \( \mathcal{E}'(\mathbb{C}) \). For such distribution we denote by \( \text{psupp} F \) the complement of the unbounded component of the complement of \( \text{supp} F \).

**Lemma 3.4.** Let \( F \in \mathcal{E}'(\mathbb{C}) \). Then the following two statements are equivalent:

a) there exists a distribution \( G \in \mathcal{E}'(\mathbb{C}) \) such that \( \frac{\partial G}{\partial \overline{z}} = F \), moreover \( \text{supp} G \subset \text{psupp} F \);

b) \( F \) is orthogonal to all polynomials of \( z \) variable, i.e. \( \langle F, z^k \rangle = 0 \) for all \( k \in \mathbb{Z}_+ \).

**Proof.** The implication \( a) \implies b) \) follows from the relation

\[
\langle F, z^k \rangle = \langle \frac{\partial G}{\partial \overline{z}}, z^k \rangle = \langle G, \frac{\partial z^k}{\partial \overline{z}} \rangle = 0.
\]

(3.9)

We prove that \( b) \implies a) \). Put \( G := F * \frac{1}{\pi z} \in \mathcal{S}'(\mathbb{C}) \), the convolution being well-defined because \( F \) has compact support. Since \( \frac{1}{\pi z} \) is the fundamental solution of the Cauchy-Riemann operator \( \frac{\partial}{\partial \overline{z}} \), we have \( \frac{\partial G}{\partial \overline{z}} = F \) (cf., for example, [10], Theorem 1.2.2). By the ellipticity of the Cauchy-Riemann operator, \( \text{singsupp} G \subset \text{singsupp} F \subset \text{supp} F \), in particular, this means that \( G \) is a smooth function outside \( \text{psupp} F \), moreover, \( G \) is analytic outside \( \text{psupp} F \) (by \( \text{singsupp} F \) we denote the singular support of the distribution \( F \), see, e.g., [10], the largest open set where the distribution coincides with a smooth function). Additionally, \( G(z) = \langle F, \frac{1}{\pi(z-w)} \rangle = \pi^{-1} \sum_{k=0}^{\infty} z^{-k-1} \langle F, w^k \rangle = 0 \) if \( |z| > R \) and \( R \) is sufficiently large. By analyticity this implies \( G(z) = 0 \) for all \( z \) outside \( \text{psupp} F \).

**Proof of Theorem 3.3.** The distribution \( F \), as any distribution with compact support, is of finite order, therefore it belongs to some Sobolev space, \( F \in H^s \) for certain \( s \in \mathbb{R}^1 \). If \( s \geq 0 \), \( F \) is a function and must be zero by Luecking’s theorem. So, suppose that \( s < 0 \).

Consider the first \( r + 1 \) columns in the matrix \( \mathbf{A}(F) \), i.e.

\[
a_{kl} = (T_F z^k, z^l) = \langle \sigma F, z^k z^l \rangle, l = 0, \ldots, r; \quad k = 0, \ldots, (3.10)
\]
Since the rank of the matrix $A(F)$ is not greater than $r$, the columns are linearly dependent, in other words, there exist coefficients $c_0, \ldots, c_r$ such that $\sum_{l=0}^r a_{kl} c_l = 0$ for any $k \geq 0$. This relation can be written as

$$\langle F, z^k h_1(\bar{z}) \rangle = \langle h_1(\bar{z}) F, z^k \rangle = 0, \quad h_1(\bar{z}) = \sum_{k=0}^r c_k \bar{z}^k. \quad (3.11)$$

Therefore, the distribution $h_1(\bar{z}) F \in H^s$ satisfies the conditions of Lemma 3.4 and hence there exists a compactly supported distribution $F^{(1)}$ such that $\frac{\partial F^{(1)}}{\partial z} = h_1 F$. By the ellipticity of the Cauchy-Riemann operator, the distribution $F^{(1)}$ is less singular than $F$, $F^{(1)} \in H^{s+1}$. At the same time,

$$\langle F^{(1)}, z^k \bar{z}^l \rangle = (l+1)^{-1} \langle F^{(1)}, \frac{\partial z^k \bar{z}^{l+1}}{\partial \bar{z}} \rangle$$

$$= (l+1)^{-1} \langle h_1(\bar{z}) F, z^k \bar{z}^l \rangle = (l+1)^{-1} \langle F, z^k \bar{z}^l h_1(\bar{z}) \rangle, \quad (3.12)$$

and therefore the rank of the matrix $A(F^{(1)})$ does not exceed the rank of the matrix $A(F)$.

We repeat this procedure sufficiently many (say, $n = [-s] + 1$) times and arrive at the distribution $F^{(n)}$ which is, in fact, a function in $L_2$, for which the corresponding matrix $A(F^{(n)})$ has finite rank. By Luecking’s theorem, this may happen only if $F^{(n)} = 0$.

Now we go back to the initial distribution $F$. Since, by our construction, $\frac{\partial F^{(n)}}{\partial \bar{z}} = h_n(\bar{z}) F^{(n-1)}$, we have that $h_n(\bar{z}) F^{(n-1)} = 0$ and therefore $\text{supp } F^{(n-1)}$ is a subset of the set of zeroes of the polynomial $h_n(\bar{z})$. On the next step, since $\frac{\partial F^{(n-1)}}{\partial \bar{z}} = h_{n-1}(\bar{z}) F^{(n-2)}$, we obtain that $\text{supp } F^{(n-2)}$ lies in the union of sets of zeroes of polynomials $h_{n-1}(\bar{z})$ and $h_n(\bar{z})$. After having gone all the way back to $F$, we obtain that its support is a finite set of points lying in the union of zero sets of polynomials $h_j$. A distribution with such support must be a linear combination of $\delta$-distributions in these points and their derivatives, $F = \sum L_q \delta(z - q)$, where $L_q = L_q(D)$ are some differential operators. Finally, to show that the number of points $z_q$ does not exceed $r$, we construct for each of them the interpolating polynomial $f_q(z)$ such that $L_q(-D)|f_q|^2 \neq 0$ at the point $z_q$ while at the points $z_{q'}$, $q' \neq q$, the polynomial $f_q$ has zero of sufficiently high order, higher than the order of $L_{q'}$, so that $L_{q'}(f_q g)(z_{q'}) = 0$ for any smooth function $g$. With such choice of polynomials, the matrix with entries $\langle F, f_q f_{q'} \rangle$ is the diagonal matrix with nonzero entries on the diagonal, and therefore its size (that equals the number of the points $z_q$) cannot be greater than the rank of the whole matrix $A(F)$, i.e., cannot be greater than $r$. □

Remark 3.5. The attempt to extend directly the original proof of Theorem 3.1 to the distributional case would probably meet certain complications. The following property is crucial in this proof: the algebra generated by polynomials of the form $P_1(Z)Q_1(Z)$ with symmetric
$P_1, Q_1$ is dense (in the sense of the uniform convergence on compacts) in the space of symmetric continuous functions. This latter property is proved above by a reduction to the Stone-Weierstrass theorem.

Now, if $F$ is a distribution that is not a measure, the analogy of reasoning in the proof would require a similar density property, however not in the sense of the uniform convergence on compacts, but in a stronger sense, the uniform convergence together with derivatives up to some fixed order (depending on the order of the distribution $F$.) The Stone-Weierstrass theorem seems not to help here since it deals with the uniform convergence only. Moreover, the required more general density statement itself is \textit{wrong}, which follows from the construction below (see [2]).

\textbf{Example 3.6.} The algebra generated by the functions having the form $P_1(Z)Q_1(Z)$, where $P_1, Q_1$ are symmetric polynomials of the variables $Z = (z_0, \ldots, z_N)$ is not dense in the sense of the uniform $C^l$-convergence on compact sets in the space of $C^l$-differentiable symmetric functions, as long as $l \geq N(N-1)$. To show this, consider the differential operator $V(D) = \prod_{j<k} (D_j - D_k)$, $D_j = \frac{\partial}{\partial z_j}$. It is easy to check that $V(D)H$ is symmetric for any antisymmetric function $H(Z)$ and $V(D)H$ is antisymmetric for any symmetric function $H(Z)$. Further on, consider any function $H(Z)$ of the form $H(Z) = P_1(Z)Q_1(Z)$ where $P_1(Z), Q_1(Z)$ are analytic polynomials. If at least one of them is symmetric, we have

$$(V(D)V(\bar{D}))H(0) = 0.$$  \hfill (3.13)

In fact, $V(D)V(\bar{D})P_1(Z)Q_1(Z) = \{|V(D)|P_1(Z)||V(D)|Q_1(Z)|].$ In the last expression, for the symmetric polynomial $P_1$, the corresponding polynomial $V(D)P_1(Z)$ is antisymmetric, and therefore equals zero for $Z = 0$. Now consider the symmetric function $|V(Z)|^2 = V(Z)\overline{V(Z)}$. We have

$$V(D)V(\bar{D})V(Z)\overline{V(Z)} = [V(D)V(Z)][V(\bar{D})\overline{V(Z)}].$$

Now note that $V(Z) = \sum_\kappa C_\kappa \prod z_j^{\kappa_j} \prod$ where the summing goes over multi-indices $\kappa = (\kappa_1, \ldots, \kappa_N)$, $|\kappa| = N$ and not all of real coefficients $C_\kappa$ are zeros. Simultaneously, $V(D) = \sum_\kappa C_\kappa \prod D_j^{\kappa_j}$ with the same coefficients. We recall now that $\prod D_j^{\kappa_j} \prod z_j^{\kappa_j} = 0$ if $|\kappa| = |\kappa'|$, $\kappa \neq \kappa'$ and it equals $\kappa!$ if $\kappa = \kappa'$. Therefore, $V(D)V(Z) = \sum_\kappa C_\kappa^2 \kappa!$ is a positive constant. In this way we have constructed the differential operator $V(D)V(\bar{D})$ of order $N(N-1)$, satisfying (3.13) for any function of the form $H(Z) = P_1(Z)Q_1(Z)$ with symmetric $P_1, Q_1$, and not vanishing on some symmetric differentiable function $|V(Z)|^2$. Therefore the function $|V(Z)|^2$ cannot be approximated by linear combinations of the functions $H(Z) = P_1(Z)Q_1(Z)$ in the sense of the uniform $C^{N(N-1)}$ convergence on compacts.
D. Luecking’s theorem was extended in a different direction by T. Le in [11]. For the particular system of functions \( f_k = z^k \) used above for the construction of the matrix \( \mathcal{A} \), it turns out that its rank may even be infinite, but the assertion of the theorem still holds, as long as the range of the operator avoids sufficiently many analytical functions.

We will say that the set of indices \( J = \{ n_j \} \subset \mathbb{Z}_+ = \{ n \in \mathbb{Z}, n \geq 0 \} \) is sparse if the series \( \sum_{n \in J} (n+1)^{-1} \) converges.

**Theorem 3.7.** ([11]) Suppose that \( \mu \) is a regular complex Borel measure and that \( J = \{ n_j \} \subset \mathbb{Z}_+ \) is sparse, \( J' = \mathbb{Z}_+ \setminus J \). Consider the reduced matrix \( \mathcal{A}^J \) consisting of \( a_{jk} : j, k \in J' \). Suppose that the rank \( r \) of \( \mathcal{A}^J \) is finite. Then the support of \( \mu \) consists of no more than \( r + 1 \) points.

The original formulation of this theorem in [11] is given in the terms of the Toeplitz operator itself. Denote by \( \mathcal{M}, \mathcal{N} \) the space of polynomials spanned by monomials \( z^j \) with \( j \), respectively, in \( J, J' \). In Theorem 3.7 it is supposed that the operator \( T_\mu \), being restricted to \( \mathcal{M} \), has range in the linear span of \( \overline{\mathcal{M}} \) and some finite-dimensional subspace in \( \mathcal{N} \).

In the next section we will establish a result that generalizes Theorem 3.7 in three directions: the multidimensional case will be considered, any distribution with compact support will replace the measure \( \mu \), and the condition of sparseness will be considerably relaxed.

**4. The multidimensional case**

In this Section we extend our main Theorem 3.3 to the case of Toeplitz operators in Bergman spaces of analytical functions of several variables. For the case of a measure acting as weight, there exist two ways of proving this result, in [5] and [19], [2]. The first approach generalizes the one used in [14] in proving Theorem 3.1, the other one uses the induction on dimension. As it follows from Remark 3.5, for the case of distribution the approach of [5] is likely to meet some complications. We present here the proof given in [2], with some modifications.

**Theorem 4.1.** Let \( F \) be a distribution in \( \mathcal{E}'(\mathbb{C}^d) \). Consider the matrix

\[
\mathcal{A}(F) = (a_{\alpha \beta})_{\alpha, \beta \in \mathbb{Z}_d^d}; \quad a_{\alpha \beta} = \langle F, z^\alpha \overline{z}^\beta \rangle, \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d.
\]

Suppose that the matrix \( \mathcal{A}(F) \) has finite rank \( r \). Then \( \text{card supp } F \leq r \) and \( F = \sum L_q \delta(z - z_q) \), where \( L_q \) are differential operators and \( z_q, 1 \leq q \leq r \), are some points in \( \mathbb{C}^d \).

We notice first, following [19], that if the function \( g \) is analytical and bounded in some polydisk neighborhood of \( \text{supp } F \) and \( F_g \) is the distribution \( |g|^2 F \) then \( \text{rank } \mathcal{A}(F_g) \leq \text{rank } \mathcal{A}(F) \). To show this, we denote by \( M_g \) the bounded operator acting in \( \mathcal{B}^2 \) by multiplication by \( g \). The adjoint operator \( M_g^* \) is, of course, bounded as well. Consider the quadratic form of the operator \( T_{F_g} \): for \( u \in \mathcal{B}^2 \),

\[
(T_{F_g} u, u) = \langle F, g u \overline{u} \rangle = (T_F (gu), gu) = (T_F M_g u, M_g u) = (M_g^* T_F M_g u, u).
\]
So we see that the operator $u \mapsto T_{F_g}u$ coincides with $M_g^* T_{F} M_g$. The multiplication by bounded operators does not increase the rank of an operator, and the property follows.

Thus, it is sufficient to prove the statement that is, actually, only formally weaker than Theorem 4.1.

**Theorem 4.2.** Suppose that for any function $g(z)$, analytic and bounded in a polydisk neighborhood of the support of the distribution $F$, the conditions of Theorem 4.1 are fulfilled with the distribution $F$ replaced by $|g(z)|^2 F \equiv F_g$. Then $\text{card supp } F \leq r$ and $F = \sum_{1 \leq q \leq r} L_q \delta(z - z_q)$, where $L_q$ are differential operators.

**Proof.** We use the induction on $d$. For $d = 1$ the statement of Theorem 4.2 coincides with the one of Theorem 3.3 that was proved in Sect. 3. We suppose that we have established our statement in the complex dimension $d - 1$ and consider the $d$-dimensional case. We denote the variables as $z = (z_1, z')$, $z' \in \mathbb{C}^{d-1}$.

For a fixed function $g(z)$ we denote by $G(g) = \pi_* F_g$ the distribution in $\mathcal{E}'(\mathbb{C}^{d-1})$ induced from $F_g$ by the projection $\pi : z \mapsto z'$: for $u \in C^\infty(\mathbb{C}^{d-1})$,

$$\langle G(g), u \rangle = \langle F_g, 1_{\mathbb{C}^1} \otimes u \rangle. \quad (4.2)$$

Although the function $g$ is defined only in a polydisk, the distribution (4.2) is well defined since this polydisk contains supp $F$.

Consider the submatrix $\mathcal{A}'(F_g)$ in the matrix $\mathcal{A}(F_g)$ consisting only of those $a_{\alpha \beta} = \langle |g|^2 F, z^\alpha \overline{z}^\beta \rangle$ for which $\alpha_1 = \beta_1 = 0$. It follows from (4.2), that the matrix $\mathcal{A}'(F_g)$ coincides with the matrix $\mathcal{A}(G(g))$ constructed for the distribution $G(g)$ in dimension $d - 1$. Thus, the matrix $\mathcal{A}(G(g))$, being a submatrix of a finite rank matrix, has a finite rank itself, moreover, $\text{rank } \mathcal{A}(G(g)) \leq r$. By the inductive assumption, this implies that the distribution $G(g)$ has finite support consisting of $r(g) \leq r$ points $\zeta_1(g), \ldots, \zeta_r(g)$; $\zeta_q(g) \in \mathbb{C}^{d-1}$ (the notation reflects the fact that both the points and their quantity may depend on the function $g$). Among all functions $g$, we can find the one, $g = g_0$, for which $r(g)$ attains its maximal value $r_0 \leq r$. Without losing in generality, we can assume that $g_0 \equiv 1$.

Fix an $\epsilon > 0$, sufficiently small, so that $2\epsilon$-neighborhoods of $\zeta_q(1)$ are disjoint, and consider the functions $\varphi_q(z') \in C^\infty(\mathbb{C}^{d-1})$, $q = 1, \ldots, r_0$ such that $\text{supp } \varphi_q$ lies in the $\epsilon$-neighborhood of the point $\zeta_q(1)$ and $\varphi_q(z') = 1$ in the $\frac{\epsilon}{2}$-neighborhood of $\zeta_q(1)$. We fix an analytic function $g(z)$ and consider for any $q$ the distribution $\Phi_q(t, g) \in \mathcal{E}'(\mathbb{C}^d)$, $\Phi_q(t, g) = |1 + tg(z)|^2 \varphi_q(z') F = \varphi_q(z') F_{1+tg}$. For $t = 0$, $\Phi_q(t, g) = \varphi_q(z') F$, the point $\zeta_q(1)$ belongs to the support of $\pi_* \Phi_q(0, g)$, and therefore for some function $u \in C^\infty(\mathbb{C}^{d-1})$, $\langle \pi_* \Phi_q(0, g), u \rangle \neq 0$. By continuity, for $|t|$ small enough, we still have $\langle \pi_* \Phi_q(t, g), u \rangle \neq 0$, which means that the $\epsilon$-neighborhood of the point $\zeta_q(1)$ contains at least one point in the support of the distribution $G(1+tg)$. Altogether, we have not less than
points of the support of $G(1 + tg)$ in the union of $\epsilon$-neighborhoods of the points $\zeta_j(1)$. However, recall, the support of $G(1 + tg)$ can never contain more than $r_0$ points, so we deduce that for $t$ small enough, there are no points of the support of $G(1 + tg)$ outside the $\epsilon$-neighborhoods of the points $\zeta_q(1)$, for $|t|$ small enough (depending on $g$.) Thus the support of the distribution $G(1 + tg)$ is contained in the $\epsilon$-neighborhood of the set of points $\zeta_j(1)$ for any $g$.

Now we introduce a function $\psi \in C^\infty(C^{d-1})$ that equals 1 outside $2\epsilon$-neighborhoods of the points $\zeta_q(1)$ and vanishes in $\epsilon$-neighborhoods of these points. By the above reasoning, the distribution $\psi G(1 + tg)$ equals zero for any $g$, for $t$ small enough. In particular, applying this distribution to the function $u = 1$, we obtain

$$\langle \psi G(1 + tg), 1 \rangle = \langle \psi F, |1 + tg|^2 \rangle = \langle \psi F, 1 + 2t \Re g + t^2 |g|^2 \rangle = 0.$$  \hspace{1cm} (4.3)

By the arbitrariness of $t$ in a small interval, (4.3) implies that $\langle \psi F, |g|^2 \rangle = 0$ for any $g$. By standard polarization, this implies that for any functions $g_1, g_2$ analytical in a polydisk neighborhood of $\text{supp } F$.

$$\langle \psi F, g_1 \bar{g}_2 \rangle = 0.$$ \hspace{1cm} (4.4)

Any polynomial $p(z, \bar{z})$ can be represented as a linear combination of functions of the form $g_1 \bar{g}_2$, so, (4.4) gives

$$\langle \psi F, p(z, \bar{z}) \rangle = 0.$$ \hspace{1cm} (4.5)

Now we take any function $f \in C^\infty(C^d)$ supported in the neighborhood $U$ of $\text{supp } F$ such that $f = 0$ on the support of $\psi$. We can approximate $f$ by polynomials of the form $p(z, \bar{z})$ uniformly on $U$ in the sense of $C^l$, where $l$ is the order of the distribution $F$. Passing to the limit in (4.5), we obtain $\langle \psi F, f \rangle = \langle F, f \rangle = 0$.

The latter relation shows that $\text{supp } F \subset \bigcup_q \{z : |z' - \zeta_q(1)| < 2\epsilon\}$. Since $\epsilon > 0$ is arbitrary, this implies that $\text{supp } F$ lies in the union of affine subspaces $z' = \zeta_j$, $j = 1, \ldots, r_0$ of complex dimension 1.

Now we repeat the same reasoning having chosen instead of $z = (z_1, z')$ another decomposition of the complex variable $z$: $z = (z'', z_d)$. We obtain that for some points $\xi_k \in C^{d-1}$, no more than $r$ of them, the support of $F$ lies in the union of subspaces $z'' = \xi_k$. Taken together, this means that, actually, $\text{supp } F$ lies in the intersection of these two systems of subspaces, which consists of no more than $r^2$ points $z_s$. The number of points is finally reduced to $r_0 \leq r$ in the same way as in Theorem 3.3, by choosing a special system of interpolation functions.

The theorem just proved can be extended to the case of a sparse range, following the pattern of Theorem 3.7.

**Definition 4.3.** Let $J \subset \mathbb{Z}_+^d$ be a set of multiindices and $\gamma \in \mathbb{Z}_+^d$ be a fixed multiindex. We say that the set $J$ is $N$- sparse in the direction
γ if for any $\alpha \in \mathbb{Z}_+^d$,
\[
\limsup_{n \to \infty} n^{-1} \# \{ (\alpha + ([0, n] \gamma)) \cap J \} < N^{-1}.
\] (4.6)

In other words, the fact that the set $J$ is sparse in direction $\gamma$ means that along any half-line starting at some point of $\mathbb{Z}_+^d$ and going in the direction $\gamma$, the density of the points of $J$ on this line is less than $N^{-1}$.

For a multiindex $\gamma = (k_1, \ldots, k_d)$ we denote by $n(\gamma)$ the set of indices $j$ such that $k_j = 0$. We introduce the set $J' = \mathbb{Z}_+^d \setminus J$ and the reduced matrix $A^J(F)$ consisting of $a_{\alpha,\beta} : \alpha, \beta \in J'$.

**Theorem 4.4.** Suppose that for a distribution $F \in \mathcal{E}'(\mathbb{C}^d)$ the reduced matrix $A^J(F)$ has finite rank $r$, and for some $\epsilon > 0$, the set $J$ is $2r + 2 + \epsilon$-sparse in some directions $\gamma_l$, $l = 1, \ldots, l$ such that
\[
\bigcup n(\gamma_l) = \{1, \ldots, d\}.
\] (4.7)

Then the distribution $F$ has finite support consisting of no more than $r + 1$ points. In the particular case when $F$ is a function, the condition (4.7) can be dropped and if $J$ is $2r + 2 + \epsilon$-sparse just in one, arbitrary, direction then $F = 0$.

**Remark 4.5.** For the case of measures, a version of Theorem 4.4 with a more restrictive notion of sparseness, has been proved in [12].

**Remark 4.6.** The condition (4.7) is sharp in the sense that if it is violated, the support of $F$ can be infinite. Consider the set $J$ consisting of multiindices having zero in the first position. This set $J$ is $N$-sparse for any $N$ in any direction $\gamma_l$, with a non-zero in the first position. For such directions $\gamma_l$, we have $1 \notin \bigcup n(\gamma_l)$. Let the distribution $F \in \mathcal{E}'(\mathbb{C}^d)$ be a measure supported in the subspace $\{z_1 = 0\}$. Then for any $\alpha, \beta \notin J$, the functions $z^\alpha \bar{z}^\beta$ vanish on $\{z_1 = 0\}$ and thus $\langle F, z^\alpha \bar{z}^\beta \rangle = 0$, so the statement on the finiteness of support becomes wrong. Of course, due to the second part of Theorem 4.4, such examples are impossible in the case when the distribution is, in fact, a function.

**Proof.** The proof follows the structure of the proof of Theorem 3.7 in [11], with two main ingredients replaced by their multi-dimensional analogies and with the extension to distributions and a more general notion of sparse sets.

First, similar to Lemma 3.2, the following two properties are equivalent:

1. the matrix $A(F)$ has finite rank not greater than $r$;
2. for any collections of $2N = 2r + 2$ multiindices $\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_r$
\[
\phi^{\otimes N}(\prod_{i \in (0,r)} z_i^{\alpha_i} \det z_i^{\beta_i}) = 0.
\] (4.8)

The proof of this fact is quite similar to the proof of Lemma 3.2; one can also see details in [5], P. 215.
Now we fix the multiindices $\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_r$ and, supposing that the set $J$ is $2r + 2 + \epsilon$-sparse in the direction $\gamma$, consider the set of multiindices 

$$\mathcal{Z} = \mathbb{Z}^d_+ \setminus \left( (\cup_{j=0}^r (J - \alpha_j)) \cup (\cup_{j=0}^r (J - \beta_j)) \right).$$

(4.9)

The set $\mathbb{Z}^d_+ \setminus \mathcal{Z}$ thus consists of $2^N$ shifts of the set $J$, therefore 

$$\limsup_{n \to \infty} \left\{ n^{-1} \# \{ \mathcal{Z} \cap \gamma[0, n] \} \right\} > \epsilon > 0.$$ 

In other words, the set of integers $n$ such that $\alpha_j + n\gamma, \beta_j + n\gamma \notin J$ for all $j$ has positive density in $\mathbb{Z}_+$. In particular, this means that 

$$\sum_{n: \gamma n \in \mathcal{Z}} (n + 1)^{-1} = \infty.$$ 

(4.10)

Now we consider the function of the complex variable $w$: 

$$\Phi(w) = \langle F^\otimes N, \prod z_j^{\alpha_j + \gamma w} \det(z_j^{\beta_j + \gamma w}) \rangle$$ 

$$= \langle F^\otimes N, \prod z_j^{\alpha_j} \det(z_j^{\beta_j}) \prod z_j^{2Nw} \rangle.$$ 

The function onto which the distribution $F^\otimes N$ acts, is not smooth for non-integer $w$, but we will take care of this in the following way. The distribution $F$, having compact support, must have finite order, $\kappa \geq 0$. Thus $F$ can be extended as a functional on $\kappa$ times differentiable functions. The function $|\prod z_j|^{2Nw}$ belongs to $C^\kappa$ for $2N \Re w \geq \kappa$, so, in the half-plane $\mathbb{K} = \{ \Re w > \frac{\kappa}{2N} \}$ the function $\Phi(w)$ is well defined, analytical, and it is continuous in $\mathbb{K}$. Therefore, if the support of $F$ lies in the ball $|z| < R$, the function $\Psi(w) = R^{-N} \Phi(w)$ is a bounded analytical function in $\mathbb{K}$. Since, by our construction, all $\alpha_j + \gamma n, \beta_n + \gamma n$ belong to $\mathbb{Z}_+ \setminus J$ for all $n \in \mathcal{Z}$, we have 

$$\Phi(n) = \Psi(n) = 0, \; n \in \mathcal{Z}. \quad (4.11)$$

Now let $H(\zeta) = \Psi\left(\frac{1+(\zeta+\kappa)}{1-(\zeta+\kappa)}\right)$. Then $H$ is a bounded analytical function on the unit disk. For any $n \in \mathcal{Z}$, the equation (4.11) implies that 

$$H\left(\frac{n-1-\kappa}{n+1+\kappa}\right) = 0.$$ 

Now 

$$\sum_{n \in \mathcal{Z}} \left(1 - \frac{n - 1 - \kappa}{n + 1 + \kappa}\right) = \sum_{n \in \mathcal{Z}} \frac{2\kappa + 2}{n + 1 + \kappa} = \infty$$ 

by (4.10). The Corollary to Theorem 15.23 in [20] shows that in this case $H$ should be identically zero on the unit disk, and therefore $\Phi(w) = 0, \Re w > k$. By continuity, $\Phi(\kappa) = 0$, and this means, in particular that 

$$|z|^2 \langle F^\otimes N, \prod z_j^{\alpha_j} \det(z_j^{\beta_j}) \rangle = 0.$$
In the reasoning above, the multiindices $\alpha_j, \beta_j$ are arbitrary, this means that (4.8) holds for the distribution $|z^\gamma|^{2k}F$, and therefore, by Theorem 4.1, this distribution must have a support consisting of a finite number of points. Therefore the support of $F$ itself is contained in the union of the above points and the subset where $z^\gamma = 0$. The latter subset is the union of the subspaces $z_m = 0$ for those $m$ that do not belong to $\mathbf{n}(\gamma)$. In particular, if $F$ is a function, this implies that $F = 0$. In the general case, we repeat the reasoning in the proof for any direction $\gamma_l$. Since, by the conditions of the theorem, $\bigcup \mathbf{n}(\gamma_l) = \{1, \ldots, d\}$, the intersection of zero sets of $z^\gamma_l$ consists only of the point 0, and this proves our statement. □

5. Applications

In this section we give some applications of the results on the finite rank Toeplitz operators in analytical Bergman spaces.

5.1. Approximation. For a subset $Q \subset C(\Omega)$, we denote by $\mathcal{Z}(Q)$ the set of common zeros of functions in $Q$. Conversely, for a subset $E$ of $\Omega$, we denote by $J(E)$ the ideal in $C(\Omega)$ consisting of all functions vanishing on $E$. Given a subspace $W$ in the Bergman space $\mathcal{B}(\Omega)$ of analytical functions, we denote by $\hat{W}$ the closure in $C(\Omega)$ of the span of functions of the form $h(z) = \overline{f(g)}$, $f \in \mathcal{B}(\Omega), g \in W$ in the topology of uniform convergence on compacts in $\Omega$.

**Theorem 5.1.** ([5]). Let $W$ be a subspace in $\mathcal{B}(\Omega)$ with finite codimension. Then $\mathcal{Z}(W)$ is a finite set and $\hat{W} = J(\mathcal{Z}(W))$. In particular, if $\mathcal{Z}(W) = \emptyset$ then $\hat{W} = C(\Omega)$.

**Proof.** Endowed with the topology of uniform convergence on compact sets, the space $C(\Omega)$ is locally convex and its continuous linear functionals are identified with complex Borel measures supported on compact sets in $\Omega$. Let $Y$ be the space of measures orthogonal to $\hat{W}$. If $Y \neq \emptyset$ there should exist a complex Borel measure $\mu \neq 0$ supported on a compact set in $\Omega$ such that

$$0 = \int_{\Omega} f \bar{g} d\mu = \int_{\Omega} (T_\mu f) \bar{g} d\lambda \quad (5.1)$$

for all $f \in \mathcal{B}, g \in W$. This shows that $T_\mu \mathcal{B}$ is contained in $W^\perp$, which is finite dimensional. By Theorem 3.3, $\mu$ must be supported on some finite set in $\Omega$, say $E_\mu$. It follows that $T_\mu \mathcal{B}$ is spanned by finitely many kernel functions $P(\cdot, w), w \in E_\mu$. By (5.1), we have that these functions $P(\cdot, w)$ lie in $W^\perp$. Since these functions are also linearly independent, the union $E$ of the sets $E_\mu$, $\mu \in Y$, must be finite. By the reproducing property, $E \subset \mathcal{Z}(W)$. Moreover, we have $E = \mathcal{Z}(W)$, because point masses at the points of $\mathcal{Z}(W)$ belong to $Y$. Now, since each $\mu \in Y$ is supported in $E$, the ideal $J(E) = J(\mathcal{Z}(W))$ is annihilated by all
\( \mu \in Y. \) Thus \( J(\mathcal{Z}(W)) \subset \hat{W} \), and the converse inclusion is obvious. The case \( Y = \emptyset \) is easily treated by the fact that \( Y = \emptyset \) if and only if \( \mathcal{Z}(W) = \emptyset \), which one may see from the proof above. \( \square \)

Theorem 5.1 can be understood as saying that the linear combinations of the functions of the form \( f\bar{g}, f \in W, g \in \mathcal{B}, \) can approximate any continuous function uniformly on any compact not containing points from some finite set. By means of the more general Theorem 4.4, we can extend this approximation result in several directions.

**Theorem 5.2.** Let \( \Omega \subset \mathbb{C}^d \) let \( J \subset \mathbb{Z}^d_+ \) be some set of multi-indices, \( J' = \mathbb{Z}^d_+ \setminus J \), satisfying the conditions of Theorem 4.4 with \( r < 2N + 1 \), for some \( N \). Denote by \( \mathcal{P} = \mathcal{P}(J) \) the space of polynomials of the form \( p(z) = \sum c_\alpha z^\alpha, \alpha \in J' \), Let \( U, V \) be linear subspaces in \( \mathcal{P}(J) \) with codimension not greater than \( N \). Then there are no more than \( 2N + 1 \) points \( w_k \in \Omega \) such that for any \( n \), the space \( \mathcal{R} \) of linear combinations of functions of the form \( p(z)q(z), p(z) \in U, q(z) \in V, \) is dense in \( C^N(\Omega) \) in the sense of uniform convergence of all derivatives of order not higher than \( n \), on any compact \( K \subset \Omega \) not containing the points \( w_k \).

Compared with Theorem 5.1, this theorem takes care of a more strong type of convergence, while the approximating set is considerably smaller.

**Proof.** Suppose that on some compact \( K \) the functions \( p(z)q(z) \) are not dense in the sense of uniform convergence on \( K \) with derivatives of order up to \( n \). This means that there exists a distribution \( F \) with support in \( K \) such that \( \langle F, p(z)q(z) \rangle = 0 \) for all \( p(z) \in U, q(z) \in V \).

Since \( V \) has finite codimension in \( \mathcal{P}(J) \), there exist no more than \( N \) polynomials \( \phi_0, \ldots, \phi_N, N_0 < N, \) so that \( \mathcal{P}(J) = V + \text{Span}(\phi_0, \ldots, \phi_N) \). Similarly, there exist no more than \( N \) polynomials \( \psi_0, \ldots, \psi_{N_1}, N_1 < N, \) so that \( \Psi(J) = U + \text{Span}(\psi_0, \ldots, \psi_{N_1}) \).

We choose some basis \( p_i(z) \) in \( U \) and some basis \( q_j(z) \) in \( V \). Consider the infinite matrix \( \mathcal{C}_0 \) consisting of elements \( b_{ij} = \langle F, p_i(z)q_j(z) \rangle \), which are, of course, all zeros. Now, we append the matrix \( \mathcal{C}_0 \) by \( N_0 \) columns \( b_{i,-s} = \langle F, p_i(z)\bar{\phi}_s(z) \rangle, s = 0, \ldots, N_0 \) and then by \( N_1 \) horizontal rows \( b_{-t,j} = \langle F, \psi_t(z)q_j(z) \rangle, b_{-t,-s} = \langle F, \psi_t(z)\bar{\phi}_s(z) \rangle, t = 0, \ldots, N_1, \) thus obtaining the matrix \( \mathcal{C} \). Each of these two operations increases the rank of the matrix no more than by \( N \), so \( \text{rank}(\mathcal{C}) \leq 2N \). Now, since any monomial \( z^k, k \in J' \) is a finite linear combination of polynomials \( p_i, \psi_t \) and any monomial \( z^l, l \in J' \) is a finite linear combination of polynomials \( q_s, \phi_s \), the matrix \( \mathcal{A}^F(F) \) consisting of \( a_{k,l} = \langle F, z^kz^l \rangle \), \( k, l \in J' \), has rank not greater than \( \text{rank}(\mathcal{C}) \), i.e., \( \text{rank}(\mathcal{A}^F(F)) \leq 2N \).

Now Theorem 4.2 implies that the distribution \( F \) has support consisting of no more than \( 2N + 1 \) points; we denote this set \( \mathcal{Z}(F) \). For some other distribution \( G \), also vanishing on all functions of the form
\( p(z)q(z) \), \( p(z) \in U, q(z) \in V \), the support \( \mathcal{Z}(G) \), by the same reasoning also consists of no more than \( 2N + 1 \) points. By considering a linear combination of \( F \) and \( G \), we see that still \( \#(\mathcal{Z}(F) \cup \mathcal{Z}(G)) \leq 2N + 1 \).

So, the support of any distributions vanishing on \( p(z)q(z) \), \( p(z) \in U, q(z) \in V \), is a part of some set \( E \subset K' \subset K \), that has no more than \( 2N + 1 \) points. Therefore, any function in \( C^n \) can be approximated by the functions of the form \( p(z)q(z) \), \( p(z) \in U, q(z) \in V \), in \( C^n \) uniformly on any compact \( K' \subset K \), not containing these points. Finally, the compact \( K \) in our construction can be chosen arbitrarily large, while the set \( E \) can never have more than \( 2N + 1 \) points and thus can be taken independently of \( K \).

\[ \square \]

We give an example of the application of Theorem 5.2. For the sake of simplicity, we take \( J = \emptyset \). For multiindices \( \alpha, k \in \mathbb{Z}^d_+ \), we write \( \alpha \prec k \) if each component of \( \alpha \) is not greater than the corresponding component of \( k \), while at least one component is strictly less.

**Example 5.3.** Suppose that for any multi-index \( k \in \mathbb{Z}^d_+, |k| > m \), two polynomials \( p_k(z) \) and \( q_k(z), z \in \mathbb{C}^d \) are given, of the form \( p_k(z) = z^k + \sum_{\alpha < k} c_{\alpha,k}z^\alpha \), resp., \( q_k(z) = z^k + \sum_{\beta < k} p_{\beta,k}z^\beta \). These sets of polynomials have codimension not greater than \( N = \binom{d+m}{d} \) in the space of all polynomials. Thus, Theorem 5.2 guarantees that there are no more than \( 2N + 2 \) points \( w_\kappa \) such that any \( C^n \) function can be approximated by linear combinations of \( p_k(z)q_l(z) \) on any compact not containing these points.

### 5.2. Products of Toeplitz operators

It has been known since long ago that the product of two Toeplitz operators in the Hardy space on a circle can be zero only in the case one of them is zero, see [3]. This result has been gradually extended to an arbitrary finite product of operators on a circle (see [1]) and to the multi-dimensional case, i.e. operators in the Hardy space on the torus, where the product of up to six Toeplitz operators is taken care of, see [8].

Much less understandable is the situation with Toeplitz operators in the Bergman space; even for the case of a disk it is still not known if it is true that for \( f, g \in L_\infty \), the relation \( T_fT_g = 0 \), or, more generally \( \text{rank}(T_fT_g) < \infty \), implies vanishing of \( g \) or \( f \). Affirmative answers to this problem, as well as to its multidimensional versions, have been obtained only in rather special cases, say, under the assumption that the functions \( f, g \) are harmonic or \( m \)-harmonic (see [9], [6] where extensive references can also be found.)

We present here some very recent results on the finite rank product problem, essentially obtained by T. Le [11].

**Theorem 5.4.** Let \( D \) be the unit disk in \( \mathbb{C}^1 \). Suppose that the function \( f(z) \in L_2(D), |z| \leq 1 \) has in the expansion in polar coordinates the
form
\[ f(re^{i\theta}) = \sum_{n=-\infty}^{M} f_n(r)e^{in\theta}, \quad (5.2) \]
and \( \hat{f}_M(l) = \int_{0}^{1} f_M(r)r^l dr \neq 0 \) for all \( l \) large enough, \( l > l_0 \). If for some distribution \( G \in \mathcal{E}'(D) \), the product \( T_f T_f \) has finite rank, the distribution \( G \) must have finite support. In particular, if \( G \) is a function then \( G = 0 \).

**Proof.** The proof follows mostly the one in [11], with modifications allowed by more advanced finite rank theorems. First, recall that the Bergman space \( \mathcal{B}^2 \) on the disk has a natural orthonormal basis
\[ e_s(z) = \sqrt{s + 1}z^s, \quad s = 0, 1, \ldots \quad (5.3) \]
The matrix representation of the operator \( T_f \) in this basis has the form
\[ (T_f e_k, e_l) = C_{k,l} \int_{0}^{1} f_{l-k}(r)r^{k+l+1}, \quad k, l \geq 0, \]
where
\[ C_{k,l} = 2\sqrt{(k + 1)(l + 1)}. \]
By our assumption about \( f \), we have \( (T_f e_k, e_l) = 0 \) whenever \( l - k \leq M \). Thus for \( k \in \mathbb{Z}_+ \), we can write
\[ T_f e_k = \sum_{l=0}^{k+M} (T_f e_k, e_l)e_l = C_{k,k+M} \hat{f}_M(2k + M + 1)e_{k+M} + \sum_{l=0}^{k+M+1} C_{k,l} \hat{f}_{l-k}(k + l + 1)e_l. \]
This shows that when \( k + M \geq 1 \) and \( 2k + M + 1 > l_0 \), the function \( e_{k+M} \) can be expressed as a linear combination of \( T_f e_k \) and \( e_l, l < M + k \).

Now suppose that \( T_f T_f \) has finite rank \( r \) and let \( \phi_1, \ldots, \phi_r \) be some basis in the range of \( T_f T_f \). Then for any nonnegative integer \( k \) such that \( k + M \geq 1 \) and \( 2k + M + 1 > l_0 \), the function \( T_f e_{k+M} \) is a linear combination of \( \phi_1, \ldots, \phi_r \) and \( T_f e_l, l \leq k + M \). We substitute consecutively this expression for \( T_f e_{k+M} \) into the similar expression for \( T_f e_{k'+M} \), for \( k' > k \). Thus all functions \( T_f e_{k'+M}, k' + M > 1, 2k' + M + 1 > l_0 \), will be expressed as linear combinations of functions \( T_f e_{k+M} \) with \( 2k + M + 1 \leq l_0 \) and the finite set of functions \( \phi_1, \ldots, \phi_r \). This means that the matrix with entries \( (T_f e_{k+M}, e_l) \) has finite rank.

Now we can apply Theorem 4.4 that grants the required properties for \( G \).

Since for any monomial \( z^k \overline{z}^l, r^{k+l}(m) \) is never zero, the conditions of Theorem 5.4 are fulfilled for any \( f \) having the form \( f(z) = p(z, \overline{z}) + h(z) \) where \( p \) is a nonzero polynomial of \( z, \overline{z} \) and \( h \) is a bounded analytical function.
Another type of results on finite rank products of Toeplitz operators in the analytical Bergman space in the unit disk or polydisk, established in [11], [12], covers the case when all Toeplitz weights, except one, are functions of a special form. We present here the formulation of the general theorem proved in [12], generalized to cover the case of distributional weights.

**Theorem 5.5.** Let $f_1, \ldots, f_{m_1+m_2}$ be bounded functions in the polydisk $D^d \subset \mathbb{C}^d$ such that each of them is radial, $f_j(z_1, \ldots, z_d) = f_j(|z_1|, \ldots, |z_d|)$ and none is identically zero. For a collection of multiindices $\alpha_j, \beta_j \in \mathbb{Z}_d^+,$ $j = 1, \ldots, m_1 + m_2$ we set $g_j(z) = f_j(z)|z|^\alpha_j \bar{z}^\beta_j$. Suppose that $F$ is a distribution with compact support in $D^d$ and the operator

$$A = T_{g_1} \cdots T_{g_{m_1}} T_F T_{g_{m_1+1}} \cdots T_{g_{m_1+m_2}} \quad (5.4)$$

has finite rank. Then $F$ has finite support. In particular, if $F$ is a function, $F$ is zero.

The proof is based upon the consideration of the kernel of the product of the operators $S_1 = T_{g_1} \cdots T_{g_{m_1}}$ and the range of $S_2 = T_{g_{m_1+1}} \cdots T_{g_{m_1+m_2}}$. The action of these operators is explicitly described in the natural basis in the Bergman space (and it is here the geometry of the polydisk is crucial.) The set of multiindices numbering the basis functions in the kernel of $S_1$ and in the cokernel of $S_2$ turns out to be sparse. Therefore, the finiteness of the rank of $S_1 T_F S_2$ leads to the finiteness of the rank of the properly restricted operator $T_F$. The reasoning concludes by the application of Theorem 4.4.

To demonstrate the idea, not going into complicated details, we present the proof, borrowed from [11], [12], for the most simple case, when $d = 1$, $m_1 = 1$, $m_2 = 0$, so the operator $A$ in (5.4) has the form $A = T_g T_F$.

**Proof.** As in the proof of Theorem 5.4, we consider the standard orthogonal basis $e_s$ in the Bergman space, given by (5.3). In this base, the action of the operator $T_g$ for $g(z) = f(|z|)z^\alpha \bar{z}^\beta$ is easily calculated,

$$T_g e_s = \begin{cases} 0, & s < \alpha - \beta \\ C \hat{f}(2s + 2\alpha + 1)e_{s+\alpha-\beta}, & s \geq \alpha - \beta \end{cases}, \quad (5.5)$$

with some positive constants $C$, depending on all indices and exponents in (5.5).

Denote $\mathcal{J} = \{s : s < \alpha + \beta\} \cup \{s : \hat{f}(2s + 2\alpha + 1) = 0\}$. Since the function $f$ is nonzero, the set $\mathcal{J}$ is sparse by Müntz-Szasz theorem. For $s \not\in \mathcal{J}$, we see from (5.5) that $T_g e_s \neq 0$ and $e_{s+\alpha-\beta}$ is a multiple of $T_g e_s$. Suppose that $\varphi \in \mathcal{B}$ is a function such that $T_g \varphi = 0$. Then

$$0 = T_g \varphi = T_g \left( \sum_s (\varphi, e_s) e_s \right) = \sum_s (\varphi, e_s) T_g e_s.$$
By (5.5), this implies that \((\varphi, e_s) = 0\) for all \(s \notin J\). Therefore, \(\text{Ker} \mathcal{T}_g\) is contained in the closed span of \(\{e_s, s \in J\}\). It follows that the rank of the matrix \((\mathcal{T}_F e_s, e_t), s, t \notin J\) is not greater than the rank of \(\mathcal{T}_g \mathcal{T}_F\), thus it is finite. Finally, Theorem 4.4 applies.

5.3. Sums of products of Toeplitz operators. Another interesting problem in the theory of Bergman spaces consists in determining the condition for some algebraic expression involving Toeplitz operators to be a Toeplitz operator again. The results existing by now concern only Toeplitz operators with weights of some special form.

In [7] this problem has been considered in the following setting. Suppose that \(u_j, v_j, j = 1, \ldots, n\), and \(w\) are \(m\)-harmonic functions in the polydisk \(D_m \subset \mathbb{C}^m\).

**Theorem 5.6.** ([7]) The necessary and sufficient condition for the operator \(S = \mathcal{T}_w + \sum \mathcal{T}_{u_j} \mathcal{T}_{v_j}\) to have finite rank, \(S = \sum_{l=1}^r (\cdot, g_l) f_l\) with some analytical functions \(f_l, g_l\) is

\[
\sum_{j=1}^m u_j v_j + w = \prod_{k=1}^m (1 - |z_k|^2)^2 \sum_{l=1}^r f_l g_l
\]

and

\[
w + \sum \overline{P u_j} P v_j i \text{mharmonic.}
\]

The proof of this result, as well as other ones in [7], is based upon the finite rank theorems.

5.4. Landau Hamiltonian and Landau-Toeplitz operators. This topic was the source of the initial interest of the author in Bergman-Toeplitz operators. About the Landau Hamiltonian one can find a detailed information in [16], [17], [4], and references therein. It is a second order differential operator \(H\) in \(L_2(\mathbb{C}^1) = L_2(\mathbb{R}^2)\) that describes the dynamics of a quantum particle confined to a plane, under the action of the uniform magnetic field \(B\) acting orthogonal to the plane. The operator has spectrum consisting of eigenvalues \(\Lambda_q = (2q+1)B, q = 0, 1, \ldots\), called Landau levels, with corresponding spectral subspaces \(X_q\) having infinite dimension. The subspace \(X_0\) is closely related with the Fock space: \(X_0\) consists of the functions \(u(z) \in L_2, z \in \mathbb{C}^1\), having the form \(u(z) = \exp(-\frac{B|x|^2}{4}) f(z)\), where \(f(z)\) is an entire analytical function.

The other spectral subspaces \(X_q\) are obtained from \(X_0\) by the action of the so called creation operator \(\overline{Q} = (2i)^{-1}(\partial + \frac{B}{2}(x_2 - ix_1))\):

\[
X_q = \overline{Q}^q X_0.
\]

Under the perturbation by the operator of multiplication by a real valued function \(V(x)\), tending to zero at infinity, the spectrum, generally, splits into clusters around Landau levels, and a lot of interesting results were obtained, describing in detail the properties of these clusters. In particular, for the case of the perturbation \(V\) having compact
support, the infiniteness of the clusters has been proved only under the condition that \( V \) has constant sign (or its minor generalizations). It remained unclear whether it is possible that some perturbation would not split some (or all) clusters.

The methods of the papers cited above relate this question with the following one: is it possible that for some function \( V \) with compact support, the Landau-Toeplitz operator \( T_q(V) \) in \( X_q \) has finite rank. Here \( T_q(V) \) is the operator \( T_q(V)u = P_qVu \) in \( X_q \), where \( P_q \) is the orthogonal projection onto \( X_q \). More exactly, the eigenvalues of \( T_q(V) \) give the main contribution to the spectrum of \( H + V \) near \( \Lambda_q \). If \( T_q(V) \) has finite rank, this does not immediately mean that the Landau level \( \Lambda_q \) does not split, but only that it may split by the interaction with \( L_{q'} \), \( q' \neq q \). In such case we will say that there is no principal splitting.

The case \( q = 0 \) can be treated directly by means of Luecking’s theorem. The subspace \( X_0 \) consists of analytical function multiplied by a Gaussian weight. Therefore, the infinite matrix \( A(V) \) constructed by means of the functions \( f_j = \exp(-\frac{B|z|^2}{4})z^j \) is the same as the matrix \( A(F) \) for the function \( F = \exp(-\frac{B|z|^2}{2})V \), constructed by means of monomials \( g_j = z^j \); these two matrices have a finite rank simultaneously. Thus, by Theorem 3.1, the function \( F \), and consequently the function \( V \), should be zero. In the initial terms, this means that the lowest Landau level \( \Lambda_0 \) necessarily principally splits into an infinite cluster, as soon as the perturbation \( V \) is nonzero.

A more advanced technic is needed for higher Landau levels. The subspaces \( X_q \) in which the Toeplitz operators \( T_q(V) \) act, do not fit into the framework of Luecking’s theorem. We can, however, use the relation (5.6) between different Landau subspaces.

The following fact has been established in [4], Corollary 9.3.

**Proposition 5.7.** Let \( V \) be a bounded function with compact support. Then for any \( q \) the Toeplitz operator \( T_q(V) \) is unitary equivalent to the operator \( T_0(W) \), where \( W = D_q(\Delta)V \), \( D_q \) being a polynomial of degree \( q \) with positive coefficients.

To be exact, the statement was proved in [4] for smooth functions \( V \), but the proof extends automatically to the case of bounded functions \( V \), and, of course, the expression \( D_q(\Delta)V \) should be understood in the sense of distribution.

Now, suppose that for some \( q \) the operator \( T_q(V) \) has finite rank. By Proposition 5.7, the Toeplitz operator \( T_0(W) \) has finite rank as well, and we can apply Theorem 3.3. So, the distribution \( W = D_q(\Delta)V \) must be a combination of a finite number of the \( \delta \)-distributions and their derivatives. Therefore, the Fourier transform of \( W \) is a polynomial, the Fourier transform of \( V \) must be a rational function, and such \( V \) cannot be a function with compact support, by the analytic hypoellipticity.

We arrived at the following result.
Theorem 5.8. Suppose that under the perturbation of the Landau Hamiltonian by a bounded function $V$ with compact support there is no principal splitting for one of Landau levels. Then the perturbation is zero, and therefore there is no principal splitting on other Landau levels either.

6. Other Bergman Spaces

The analytical Bergman spaces, considered above, have a vast advantage, the multiplicative structure that is used all the time. For other types of Bergman spaces, without the multiplicative structure, the results are therefore less extensive. An exception is constituted by the harmonic Bergman spaces in an even-dimensional space, due to their close relation to analytical functions.

6.1. Harmonic Bergman spaces. The aim of this section is to establish finite rank results for Toeplitz operators in Bergman spaces of harmonic functions. The results presented here generalize the ones in [2].

We start with the even-dimensional case, $d = 2m$. Here the problem with harmonic spaces reduces easily to the analytical Bergman spaces.

For a distribution $F \in \mathcal{E}'(\mathbb{R}^d)$ we consider a matrix $H(F)$ consisting of elements $\langle F, f_j \overline{f_k} \rangle$, where $f_j$ is some complete system of homogeneous harmonic polynomials in $\mathbb{R}^d = \mathbb{C}^m$. It is convenient (but not obligatory) to suppose that real and imaginary parts of analytic monomials, $\text{Re}(z^\alpha), \text{Im}(z^\alpha), \alpha \in \mathbb{Z}_+^d$, are among the polynomials $f_j$. For some subset $J \subset \mathbb{Z}_+^d$, we denote by $H^J(F)$ the matrix with entries $\langle F, f_j \overline{f_k} \rangle$, with $\text{Re}(z^\alpha), \text{Im}(z^\alpha), \alpha \in J$ removed.

Theorem 6.1. Let $d = 2m$ be an even integer. Suppose that for some $N$, the set $J$ satisfies the conditions of Theorem 4.4, and for a distribution $F \in \mathcal{E}'(\mathbb{R}^n)$ the matrix $H^J(F)$ has rank $r \leq N$. Then the distribution $F$ is a sum of $m \leq r + 1$ terms, each supported at one point: $F = \sum L_q \delta(x - x_q), x_q \in \mathbb{R}^d, L_q$ are differential operators in $\mathbb{R}^d$.

Proof. We identify the space $\mathbb{R}^d$ with the complex space $\mathbb{C}^m$. Since the functions $z^\alpha, \overline{z}^\beta$ are harmonic, the matrix $\mathcal{A}^J(F)$ (defined in Section 4) can be considered as a submatrix of $H^J(F)$, and therefore it has rank not greater than $r$. It remains to apply Theorem 4.4 to establish that the distribution $F$ has the required form, with no more than $r + 1$ points $x_q$. \qed

The same reasoning establishes similar properties for the Bergman spaces of pluriharmonic and $m$-harmonic functions.

The odd-dimensional case requires considerably more work, and the results are less complete. We will use again a kind of dimension reduction, as in Theorem 3.3, however, unlike the analytic case, we will...
need projections of the distribution to one-dimensional subspaces. We have to restrict our considerations to distributions being regular complex Borel measures (so we will use the notation \( \mu \) instead of \( F \)) and from now on we will not consider the generalizations related with the removal of sparse subsets \( J \).

Let \( S \) denote the unit sphere in \( \mathbb{R}^d \), \( S = \{ \zeta \in \mathbb{R}^d : |\zeta| = 1 \} \) and let \( \sigma \) be the Lebesgue measure on \( S \). For \( \zeta \in S \), we denote by \( \mathcal{L}_\zeta \) the one-dimensional subspace in \( \mathbb{R}^d \) passing through \( \zeta \), \( \mathcal{L}_\zeta = \zeta \mathbb{R}^1 \). For a measure with compact support \( \mu \) on \( \mathbb{R}^d \) we define the measure \( \mu_\zeta \) on \( \mathbb{R}^1 \) by setting \( \langle \mu_\zeta, \phi \rangle = \langle \mu, \phi \rangle \), where \( \phi \in \mathcal{C}(\mathbb{R}^d) \) is \( \phi(x) = \phi(x \cdot \zeta) \). The measure \( \mu_\zeta \) can be understood as result of projecting of \( \mu \) to \( \mathcal{L}_\zeta \) with further transplantation of the projection, \( \pi^\mathcal{L}_\zeta \mu \), from the line \( \mathcal{L}_\zeta \) to the standard line \( \mathbb{R}^1 \). The Fourier transform \( \mathcal{F}_\mu_\zeta \) of \( \mu_\zeta \) is closely related with \( \mathcal{F}_\mu \):

\[
\mathcal{F}(\mu_\zeta)(t) = (\mathcal{F}_\mu)(t \zeta).
\]  

The following fact in the harmonic analysis of measures was established in [2].

**Proposition 6.2.** For a finite complex Borel measure \( \mu \) with compact support in \( \mathbb{R}^d \) the following three statements are equivalent:

a) \( \mu \) is discrete;

b) \( \mu_\zeta \) is discrete for all \( \zeta \in S \);

c) \( \mu_\zeta \) is discrete for \( \sigma \)-almost all \( \zeta \in S \).

The proof of Proposition 6.2 can be found in [2], see Corollary 5.3 there.

Now we return to our finite rank problem.

**Theorem 6.3.** Let \( d \geq 3 \) be an odd integer, \( d = 2m + 1 \). Let \( \mu \) be a finite complex Borel measure in \( \mathbb{R}^d \) with compact support. Suppose that the matrix \( H(\mu) \) has finite rank \( r \). Then \( \text{supp} \mu \) consists of no more than \( r \) points.

**Proof.** Fix some \( \zeta \in S \) and choose some \( d - 1 = 2m \)-dimensional linear subspace \( \mathcal{L} \subset \mathbb{R}^d \) containing \( \mathcal{L}_\zeta \). We choose the co-ordinate system \( x = (x_1, \ldots, x_d) \) in \( \mathbb{R}^d \) so that the subspace \( \mathcal{L} \) coincides with \( \{ x : x_d = 0 \} \). The even-dimensional real space \( \mathcal{L} \) can be considered as the \( m \)-dimensional complex space \( \mathbb{C}^m \) with co-ordinates \( z = (z_1, \ldots, z_m) \), \( z_j = x_{2j-1} + i x_{2j}, j = 1, \ldots, m \). The functions \( (z, x_d) \mapsto z^\alpha, (z, x_d) \mapsto z^\beta, \alpha, \beta \in (\mathbb{Z}^+)^d \), are harmonic polynomials in \( \mathbb{C}^d \times \mathbb{R}^1 \). Moreover, by definition, \( \langle \mu, z^\alpha z^\beta \rangle = \langle \pi_\zeta^\mathcal{L} \mu, z^\alpha z^\beta \rangle \). Hence, the matrix \( \mathcal{A}(\pi_\zeta^\mathcal{L} \mu) \) is a submatrix of the matrix \( H(\mu) \), and the former has not greater rank than the latter, \( \text{rank}(\mathcal{A}(\pi_\zeta^\mathcal{L} \mu)) \leq r \). So we can apply Theorem 6.1 and obtain that the measure \( \pi_\zeta^\mathcal{L} \mu \) is discrete and its support contains not more than \( r \) points. Now we project the measure \( \pi_\zeta^\mathcal{L} \mu \) to the real one-dimensional linear subspace \( \mathcal{L}_\zeta \) in \( \mathcal{L} \). We obtain the same measure as if we had projected \( \mu \) to \( \mathcal{L}_\zeta \) from the very beginning, and not in two
steps i.e., \( \pi_{\zeta}^* \mu \). As a projection of a discrete measure, \( \pi_{\zeta}^* \mu \) is discrete and has no more than \( r \) points in the support. By our definition of the measure \( \mu_{\zeta} \) as \( \pi_{\zeta}^* \mu \) transplanted to \( \mathbb{R}^1 \), this means that \( \mu_{\zeta} \) is discrete.

Due to the arbitrariness of the choice of \( \zeta \in S \), we obtain that all measures \( \mu_{\zeta} \) are discrete. Now we can apply Proposition 6.2 which implies that the measure \( \mu \) is discrete itself. Finally, in order to show that the number of points in \( \text{supp} \mu \) does not exceed \( r \), we chose \( \zeta \in S \) such that no two points in \( \text{supp} \mu \) project to the same point in \( L_{\zeta} \). Then the point masses of \( \mu \) cannot cancel each other under the projection, and thus card \( \text{supp} \mu = \text{card} \, \text{supp} \mu_{\zeta} \leq r \).

The number of points in the support of \( \mu \) is estimated in the same way as in Theorem 3.3. \( \square \)

The analysis of the reasoning in the proof shows that the only essential obstacle for extending Theorem 6.3 to the case of distributions is the limitation set by Proposition 6.2. If we were able to prove this proposition for distributions, all other steps in the proof of Theorem 6.3 would go through without essential changes. However, it turns out that not only the proof of Proposition 6.2 cannot be carried over to the distributional case, but, moreover, the Corollary itself becomes wrong. The example, that can be found in [2], does not disprove Theorem 6.3 for distributions, however it indicates that the proof, if exists, should involve some other ideas.

6.2. Helmholtz Bergman spaces. We consider now the Helmholtz equation

\[
\Delta u + k^2 u = 0, \tag{6.2}
\]

in \( \Omega \subset \mathbb{R}^d \), where \( k > 0 \) (we set \( k^2 = 1 \), without losing in generality). Let \( \Omega \) be a bounded domain and \( F \) be a distribution with compact support in \( \Omega \). We denote by \( \mathcal{H} = \mathcal{H}(\Omega) \) the space of solutions of (6.2) in \( \Omega \) belonging to \( L_2(\Omega) \) (we consider the Lebesgue measure here). We will call such solutions Helmholtz functions. For a distribution \( F \in \mathcal{E}'(\Omega) \) we, as usual, define the Toeplitz operator \( \mathcal{T}_F : \mathcal{H} \rightarrow \mathcal{H} \), by means of the quadratic form \( (\mathcal{T}_F u, v) = \langle F, uv \rangle \), \( u, v \in \mathcal{H} \). For any two systems of linearly independent functions \( \Sigma_1 = \{f_j\}, \Sigma_2 = \{g_k\} \subset \mathcal{H} \), we consider the matrix \( \mathcal{A} = \mathcal{A}(F; \Sigma_1, \Sigma_2) : \)

\[
\mathcal{A}(F; \Sigma_1, \Sigma_2) = (\langle F, f_j(x)g_k(x) \rangle), f_j \in \Sigma_1, g_k \in \Sigma_2. \tag{6.3}
\]

If the Toeplitz operator \( \mathcal{T}_F \) has finite rank \( r \), the matrix (6.3) has rank not greater than \( r \) for any \( \Sigma_1, \Sigma_2 \). Moreover,

\[
\text{rank} \mathcal{T}_F = \max_{\Sigma_1, \Sigma_2} \text{rank}(\mathcal{A}(F; \Sigma_1, \Sigma_2)).
\]

**Theorem 6.4.** Let \( d \geq 3 \) and let \( F \) be a function with compact support. Suppose that the Toeplitz operator \( \mathcal{T}_F \) in the space of Helmholtz functions has finite rank. Then \( F = 0 \).
Proof. Consider the systems $\Sigma_1, \Sigma_2$ consisting of functions having the form $f_j(x) = e^{-ix_1}h_j(x')$, $g_k(x) = e^{ix_1}h_k(x')$, where $h_j(x')$ is an arbitrary system of harmonic functions of the variable $x'$ in the subspace $L \subset \mathbb{R}^d: x_1 = 0$. Then the expression in (6.3) takes the form

$$\langle F, f_j(x)\overline{g_k(x)} \rangle = \int \int F(x_1, x') e^{-2ix_1}dx_1 h_j(x') \overline{h_k(x')} dx'.$$  

(6.4)

This matrix has finite rank, not greater than $r$. Now we are in the conditions of Theorem 6.3 or Theorem 6.1, depending on whether $d$ is even or odd, in dimension $d - 1 \geq 2$, applied to the function $\tilde{F}(x') = \int F(x_1, x') e^{-2ix_1}dx_1$, i.e., the partial Fourier transform of $F$ in $x_1$ variable, calculated in the point $\xi_1 = 2$. Since the matrix $A(F; \Sigma_1, \Sigma_2)$ has finite rank for arbitrary system of harmonic functions $h_j(x')$, by the above finite rank theorems about harmonic Bergman spaces, the function $\tilde{F}(x')$ must be zero. We make the Fourier transform of $\tilde{F}$ in the remaining variables and obtain that the Fourier transform $\hat{F}(\xi)$ of $F(x)$ equals zero for all $\xi$ having the first component equal to 2.

Next we fix some $\omega \in \mathbb{R}^d$, $|\omega| = 1$ and consider the system $\Sigma_1 = \Sigma_2$ consisting of the functions having the form $f_j(x) = e^{-i\omega x}h_j(x')$, $g_k(x) = e^{i\omega x}h_k(x')$ where $x'$ is the variable in the subspace $L(\omega) \subset \mathbb{R}^d$, orthogonal to $\omega$, and $h_j$ are arbitrary harmonic functions. We repeat the reasoning above to obtain that $\hat{F}(\xi) = 0$ for all $\xi$ having the component in the direction of $\omega$ equal to 2. Now note that for any $\xi \in \mathbb{R}^d$, $|\xi| \geq 2$, it is possible to find such $\omega$, $|\omega| = 1$ that $\xi$ has $\omega$-component equal to 2. Therefore we obtain that $\hat{F}(\xi) = 0$ for all $|\xi| \geq 2$. So, we obtained that $\hat{F}$ has compact support. But, recall, $F$ also has compact support. Therefore $F$ must be zero. \hfill \Box

6.3. An application: the Born approximation. In the quantum scattering theory one of the main objects to consider is the scattering matrix; details can be found in many books on the scattering theory, e.g., [21], [18]. We consider the Born approximation, which (up to a constant factor) is the integral operator $K$ with kernel

$$K(\omega, \varsigma) = \int_{\mathbb{R}^d} F(x)e^{i\varsigma(x - \varsigma)} dx,$$  

(6.5)

where $|\omega|^2 = |\varsigma|^2 = E > 0$, $F(x)$ is the potential (decaying at infinity sufficiently fast) and the operator acts on the sphere $S^{d-1}: |\omega|^2 = E$. Further on, we suppose that $E = 1$.

The expression (6.5) coincides with the quadratic form of the Toeplitz operator $T_F$ in the space of solutions of the Helmholtz equation, considered on the systems of functions $e^{i\omega x}, e^{i\varsigma x}$. We consider the case when the operator $K$ has finite rank. This implies that the matrix (6.3) has finite rank. So, applying Theorem 6.4, we obtain the following result.
Theorem 6.5. Let \( d \geq 3 \) and let \( F \) be a function with compact support. Suppose that the Born approximation operator \( K \) has finite rank. Then \( F = 0 \).

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