Quark masses and mixing in $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3$ models.

A. E. Cárcamo Hernández\textsuperscript{a}\textsuperscript{*}, R. Martínez\textsuperscript{b} and F. Ochoa\textsuperscript{b}\textsuperscript{†}

\textsuperscript{a}Universidad Técnica Federico Santa María

and Centro Científico-Tecnológico de Valparaíso

Casilla 110-V, Valparaíso, Chile,

\textsuperscript{b}Departamento de Física,

Universidad Nacional de Colombia,

Ciudad Universitaria, Bogotá D.C., Colombia.

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By imposing a specific $S_3$ discrete symmetry for fermions and scalars, we extend the group of the model based on the gauge symmetry $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$. We found that after the symmetry breaking of the continuous group, all the down type quarks acquire tree level masses, while in the up sector only the top quark and a new heavy $T$ quark obtain masses at tree level. By imposing hierarchical Yukawa couplings between the ordinary down-type quarks and new heavy down-type quarks, we may explain the relation $m_d << m_s << m_b$. The massless up-type quarks get small mass terms through effective six dimensional operators according to the gauge and discrete symmetries. In particular, the smallness of the $u$ quark can be understood by a small deviation in the universality of the dimensionless couplings associated to the effective operators. We also obtain with only four free parameters, the magnitudes of the CKM matrix elements and the Jarlskog invariant, within experimental error values.

I. INTRODUCTION

The ATLAS and CMS experiments at the CERN Large Hadron Collider (LHC) have found a 126 GeV Higgs boson\cite{1-4}, increasing our knowledge of the Electroweak Symmetry Breaking (EWSB) sector and opening a new era in particle physics. Now the priority of the LHC experiments will be to measure precisely the couplings of the new particle to standard model fermions and gauge bosons and to establish its quantum numbers. It also remains to look for further new states associated with the EWSB mechanism which will allow to discriminate among the different theoretical models addressed to explain EWSB.

Despite its great success, the Standard Model (SM) based on the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry is unlikely to be a truly fundamental theory due to having internal problems and unexplained features\cite{5,12}. Most of them are linked to the mechanism responsible for the stabilization of the weak scale, the origin of fermion masses and mixing and the existence of three generations of fermions. Because of these reasons, many people consider the Standard Model to be an effective framework of a more fundamental theory that one expects should have a dynamical explanation for the fermion masses and mixing. The flavour structure of Yukawa interactions is not restricted by gauge invariance, consequently fermion masses and mixing cannot be entirely determined in a gauge theory. In the SM there are in fact arbitrary parameters that must be measured. The lack of predictivity of the fermion masses and mixing in the SM has motivated many models based on extended symmetries, leading to specific textures for the Yukawa couplings. There are models with Multi-Higgs sectors, Grand Unification, Extradimensions and Superstrings,\cite{9-11}. In particular, discrete flavour symmetries may play an important role in models of fermion mixings and many models based on flavour symmetries have been proposed in order to provide an explanation for the current pattern of fermion mixing, for recent reviews see Refs.\cite{6-8}. The understanding of the discrete flavor symmetries hidden in such textures may be useful in the knowledge of the underlying dynamics responsible for quark mass generation and CP violation. One clear and outstanding feature in the pattern of quark masses is that they increase from one generation to the next spreading over a range of five orders of magnitude, and that the mixings from the first to the second and to the third family are in decreasing order\cite{12}. From the phenomenological point of view, it is possible to describe some features of the mass hierarchy by assuming zero-texture Yukawa matrices\cite{10}. Models with spontaneously broken flavor symmetries may also produce hierarchical mass structures. These horizontal symmetries can be continuous and Abelian, as the original Froggatt-Nielsen model\cite{17}, or non-Abelian as for example $SU(3)$
and $SO(3)$ family models \cite{20}. Models with discrete symmetries may also predict mass hierarchies for leptons \cite{19} and quarks \cite{20}. Models with spontaneously broken flavor symmetries may also produce hierarchical mass structures. Recently, discrete groups such as $A_4$, $S_4$, $S_4$, $A_4$ and $\Delta(27)$ have been considered to explain the observed pattern of fermion masses and mixings \cite{15,22}. These models have as a common issue the breaking of the flavor symmetry so that the observed data can be fitted in a natural way. The breaking of the flavor symmetry takes place when the scalar fields acquire vacuum expectation values. Other models with horizontal symmetries have been proposed in the literature \cite{24}.

On the other hand, the origin of the family structure of the fermions can be addressed in family dependent models where a symmetry distinguishes fermions of different families. Alternatively, an explanation to this issue can also be provided by the models based on the gauge symmetry $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$, also called 3-3-1 models, which introduce a family non-universal $U(1)_X$ symmetry \cite{25,28}. These models have a number of phenomenological advantages. First of all, the three family structure in the fermion sector can be understood in the 3-3-1 models from the cancellation of chiral anomalies \cite{23} and asymptotic freedom in QCD. Secondly, the fact that the third family is treated under a different representation, can explain the large mass difference between the heaviest quark family and the two lighter ones \cite{30}. Finally, these models contain a natural Peccei-Quinn symmetry, necessary to solve the strong-CP problem \cite{31}.

The 3-3-1 models extend the scalar sector of the SM into three $SU(3)_L$ scalar triplets: one heavy triplet field with a Vacuum Expectation Value (VEV) at high energy scale $\langle \chi \rangle = v_{\chi}$, which breaks the symmetry $SU(3)_L \otimes U(1)_X$ into the SM electroweak group $SU(2)_L \otimes U(1)_Y$, and two lighter triplets with VEVs at the electroweak scale $\langle \rho \rangle = v_{\rho}$ and $\langle \eta \rangle = v_{\eta}$, which trigger Electroweak Symmetry Breaking. Besides that, as shown in Ref. \cite{22}, the 3-3-1 model can explain the number of events in the $h \rightarrow \gamma \gamma$ decay, recently observed at the LHC, since the heavy exotic quarks, the charged Higgs and the heavy charged gauge bosons contribute to this process. On the other hand, the 3-3-1 model reproduces a specialized Two Higgs Doublet Model type III (2HDM-III) in the low energy limit, where both triplets $\rho$ and $\eta$ are decomposed into two hypercharge-one $SU(2)_L$ doublets plus charged and neutral singlets. Thus, like the 2HDM-III, the 3-3-1 model can predict huge flavor changing neutral currents (FCNC) and CP-violating effects, which are severely suppressed by experimental data at electroweak scales. In the 2HDM-III, for each quark type, up or down, there are two Yukawa couplings. One of the Yukawa couplings is for generating the quark masses, and the other one produces the flavor changing couplings at tree level. One way to remove both the huge FCNC and CP-violating effects, is by imposing discrete symmetries, obtaining two types of 3-3-1 models (type I and II models), which exhibit the same Yukawa interactions as the 2HDM type I and II at low energy where each fermion is coupled at most to one Higgs doublet. In the 3-3-1 model type I, one Higgs triplet (for example, $\rho$) provide masses to the phenomenological up- and down-type quarks, simultaneously. In the type II, one Higgs triplet ($\eta$) gives masses to the up-type quarks and the other triplet ($\rho$) to the down-type quarks. In the type III, both Higgs triplets ($\eta$) and ($\rho$) gives masses to the up-type and down type quarks. Recently, authors in Refs. \cite{33} and \cite{34} discuss the mass structures in the framework of the I-type and II-type 331 model, respectively. In this paper we obtain different structures for the $S_3$ type III-like model. We choose the $S_3$ discrete symmetry since it is the smallest non-Abelian discrete symmetry group that contains a two-dimensional irreducible representation which can connect two maximally mixed generations. Thereby we group the scalar fields into doublet and non trivial $S_3$ singlet representations. Regarding the quark sector, we group left handed quarks as well as the right handed bottom quark into $S_3$ singlet representations while the remaining right handed quarks are assigned to $S_3$ doublet representations. We found that the down type quarks acquire tree level masses while in the up sector only the heavy $T$ quark and the top quark acquire tree level masses. The remaining charm and up quarks acquire masses by six dimensional operators. The breaking of universality in the dimensionless couplings of the six dimensional operators is crucial to generate the up quark mass. Our model describes a realistic pattern of SM quark masses and mixings. Our model is more minimal than the model described in Ref.\cite{22} since the latter has a larger scalar sector and introduces a new charge $U(1)_L$ responsible for lepton number and lepton parity. The lepton parity symmetry introduced in Ref.\cite{22} suppresses the mixing between ordinary quarks and exotic quarks. The work described in Ref.\cite{22} is more focused in the lepton sector, while in the quark sector, the obtained quark mass matrices are diagonal and the quark mixing matrix is trivial.

This paper is organized as follows. In Section 2 we briefly describe some theoretical aspects of the 3-3-1 model and its particle content, as well as the particle assignments under $S_3$ doublets and $S_3$ singlets, in particular in the fermionic and scalar sector in order to obtain the mass spectrum. Section 3 is devoted to discuss the resulting tree level quark mass textures. In Section 4 we employ six dimensional Yukawa operators for the up-type quark sector in order to generate the up and charm quark masses. In Section 5, we present our results in terms of quark mixings, which is followed by a numerical analysis. Finally in Section 6, we state our conclusions.
II. THE FERMION AND SCALAR SECTOR.

We consider the 3-3-1 model where the electric charge is defined by:

\[ Q = T_3 - \frac{1}{\sqrt{3}} T_8 + X, \]

with \( T_3 = \frac{1}{2} \text{Diag}(1, -1, 0) \) and \( T_8 = (\frac{1}{2\sqrt{3}}) \text{Diag}(1, 1, -2). \) In order to avoid chiral anomalies, the model introduces in the fermionic sector the following \((SU(3)_C, SU(3)_L, U(1)_X)\) left- and right-handed representations:

\[
Q^1_L = \begin{pmatrix} U^1 \cr D^1 \end{pmatrix} : (3, 3, 1/3), \quad \begin{pmatrix} U^1_R : (3^*, 1, 2/3) \\
D^1_R : (3^*, 1, -1/3) \\
T_R : (3^*, 1, 2/3) \end{pmatrix}
\]

\[
Q^{2,3}_L = \begin{pmatrix} D^{2,3}_L \\
U^{2,3}_L \end{pmatrix} : (3, 3^*, 0), \quad \begin{pmatrix} D^{2,3}_R : (3^*, -1/3) \\
U^{2,3}_R : (3^*, 1, 2/3) \\
J^{2,3}_R : (3^*, 1, -1/3) \end{pmatrix}
\]

\[
L^{1,2,3}_L = \begin{pmatrix} \rho^{1,2,3}_L \\
\eta^{1,2,3}_L \end{pmatrix} : (1, 3, -1/3), \quad \begin{pmatrix} \rho^{1,2,3}_R : (1, 1, -1) \\
N^{1,2,3}_R : (1, 1, 0) \end{pmatrix}
\]

where \( U^i_L \) and \( D^i_L \) for \( i = 1, 2, 3 \) are three up- and down-type quark components in the flavor basis, while \( \nu^i_L \) and \( e^i_L \) are the neutral and charged lepton families. The right-handed sector transforms as singlets under \( SU(3)_L \) with \( U(1)_X \) quantum numbers equal to the electric charges. In addition, we see that the model introduces heavy fermions with the following properties: a single flavor quark \( T^1 \) with electric charge 2/3, two flavor quarks \( J^{2,3} \) with charge \(-1/3\), three neutral Majorana leptons \( \nu^{1,2,3}_L \) and three right-handed Majorana leptons \( N^{1,2,3}_R \) (recently, a discussion about neutrino masses via double and inverse see-saw mechanism was perform in ref. [21]). On the other hand, the scalar sector introduces one triplet field with VEV \( \langle \chi \rangle_0 = v_\chi \), which provides the masses to the new heavy fermions, and two triplets with VEVs \( \langle \rho \rangle_0 = v_\rho \) and \( \langle \eta \rangle_0 = v_\eta \), which give masses to the SM-fermions at the electroweak scale. The \([SU(3)_L, U(1)_X]\) group structure of the scalar fields are:

\[
\chi = \begin{pmatrix} \chi_1 \\
\chi_2 \\
\xi_\chi \end{pmatrix} : (3, -1/3), \quad \rho = \begin{pmatrix} \rho_1^+ \\
\rho_2^+ \\
\xi_\rho \end{pmatrix} : (3, 2/3), \quad \eta = \begin{pmatrix} \eta_1 \\
\eta_2 \\
\eta_3 \end{pmatrix} : (3, -1/3).
\]

The EWSB follows the scheme \( SU(3)_L \otimes U(1)_X \xrightarrow{\langle \chi \rangle} SU(2)_L \otimes U(1)_Y \xrightarrow{\langle \eta \rangle, \langle \rho \rangle} U(1)_Q \), where the vacuum expectation values satisfy \( v_\chi \gg v_\rho, v_\eta \).

Furthermore, we impose the \( S_3 \) flavor symmetry for fermions and scalars so that the full symmetry of our model is extended to be \( SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3 \). The inclusion of the discrete \( S_3 \) symmetry will allow to reduce the number of parameters in the Yukawa and scalar sector of the \( SU(3)_C \otimes SU(3)_L \otimes U(1)_X \) model making it more predictive. We choose the \( S_3 \) discrete symmetry since it is the smallest non-Abelian discrete symmetry group that contains a two-dimensional irreducible representation which connect two maximally mixed generations. Besides facilitating maximal mixing through its doublet representation, the \( S_3 \) discrete group provide two inequivalent singlet representations which play a crucial role in reproducing fermion masses and mixing [21]. The scalar fields are grouped into doublet and singlet representations of \( S_3 \) as follows:

\[ \Phi = (\eta, \chi) \sim 2, \quad \rho \sim 1'. \]

Notice that the scalar triplet \( \rho \) cannot be assigned to a trivial \( S_3 \) singlet \( 1 \) since in that assignment, the \( S_3 \) symmetry will forbid a trilinear term in the scalar potential crucial to keep the non Standard Model like Higgs heavy. Furthermore, the absence of the trilinear term will imply a breaking of the \( S_3 \) symmetry in order to have the \( \langle \Phi \rangle \) vacuum consistent with the minimization conditions of the scalar potential.
Regarding the quark sector, it is assumed that all quarks are assigned to $S_3$ doublets excepting the $Q_L^1$, $Q_L^2$, $Q_L^3$ and $D_R^1$ fields, which are assigned to $S_3$ singlets:

$$Q_R^\mu = (U_R^\mu, U_R^\mu)^{\dagger} \sim 2, \quad Q_R^{(D)} = (D_R^2, D_R^3)^{\dagger} \sim 2, \quad Q_R^{(J)} = (J_R^2, J_R^3)^{\dagger} \sim 2. \quad (4)$$

$$Q_R^{(T)} = (U_R^T, T_R^T)^{\dagger} \sim 2, \quad Q_L^1 \sim 1, \quad Q_L^2 \sim 1', \quad Q_L^3 \sim 1', \quad D_R^1 \sim 1'. \quad (5)$$

With the above spectrum, we obtain the following $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3$ renormalizable Yukawa part of the model Lagrangian for the quark sector:

$$-\mathcal{L}_Y = h_{\Phi}^U \overline{Q}_L^U (\Phi Q_R^U)^{\dagger} + h_{\Phi}^D \overline{Q}_L^D (\Phi Q_R^D)^{\dagger} + h_{\Phi}^J \overline{Q}_L^J (\Phi^* Q_R^J)^{\dagger} + H.c. \quad (6)$$

where $n = 2, 3$ is the index that label the second and third quark triplet shown in Eq. (11), and $h_{\Phi i j} (f = U, D, T, J)$ are the $i, j$ components of non-diagonal matrices in the flavor space associated with each scalar triplet $\phi : \eta, \rho, \chi$.

The interactions among the scalar fields are contained in the following most general potential that we can construct with three scalar triplets:

$$V_H = \mu_1^2 (\rho^3 \rho) + \mu_2^2 (\Phi_1 \Phi_1) + \lambda_1 (\rho^3 \rho) + \lambda_2 (\Phi_1^\dagger \Phi_1) + \lambda_3 (\Phi_1^\dagger \Phi_1) + \lambda_4 (\Phi_2^\dagger \Phi_2) + \lambda_5 (\rho^3 \rho) + \lambda_6 (\rho^3 \rho) + \epsilon_{ijk} \Phi_i \Phi_j \Phi_k + h.c. \quad (7)$$

where $\Phi_i = (\eta_i, \chi_i)$ is a $S_3$ doublet with $i = 1, 2, 3$.

The $S_3$ group has three irreducible representations: $1, 1'$ and $2$. The multiplication rules of the $S_3$ group for the case of real representations are given by $[3]$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_2 = (x_1 y_1 + x_2 y_2)_1 + (x_1 y_2 - x_2 y_1)_1 + \begin{pmatrix} x_1 y_2 + x_2 y_1 \\ x_1 y_1 - x_2 y_2 \end{pmatrix}_2, \quad (8)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 \otimes (y)_1^1 = \begin{pmatrix} -x_2 y_1 \\ x_1 y_1 \end{pmatrix}_2, \quad (x)_1^1 \otimes (y)_1^1 = (x y)_1^1. \quad (9)$$

Besides that, the scalar potential can be written in terms of the three scalar triplets as follows:

$$V_H = \mu_1^2 (\rho^3 \rho) + \mu_2^2 (\eta^3 \eta + \chi^3 \chi) + \lambda_1 (\rho^3 \rho) + \lambda_2 (\eta^3 \eta) + \lambda_3 (\chi^3 \chi) + \lambda_4 (\eta^3 \eta) + \lambda_5 (\rho^3 \rho) + \lambda_6 (\rho^3 \rho) + \epsilon_{ijk} \eta_i \eta_j \eta_k + h.c. \quad (10)$$

As shown in detail in the Appendix, the $\langle \Phi \rangle$ vacuum is consistent with the minimization conditions of the scalar potential when the following relation is fulfilled:

$$\lambda_2 = -\lambda_4 \frac{f v_\eta}{\sqrt{2 v_\eta v_\chi}}. \quad (11)$$

After the symmetry breaking, as shown in detail in the Appendix, it is found that the mass eigenstates are related to the weak states in the scalar sector by:

$$\begin{pmatrix} H_1^0 \\ h_1^0 \end{pmatrix} \approx \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \xi_\rho \\ \xi_\eta \end{pmatrix}, \quad \begin{pmatrix} H_2^0 \\ G_2^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \eta_3 \\ \chi_1 \end{pmatrix}. \quad (12)$$
\[
\begin{align*}
H_1^\pm & \approx \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \rho_1^\pm \\ \eta_2 \end{pmatrix}, \\
H_2^\pm & \approx \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} \rho_3^\pm \\ \lambda_2 \end{pmatrix}, \\
\begin{pmatrix} A_1^0 \\ G_1^0 \end{pmatrix} & = \begin{pmatrix} \cos \beta & \sin \beta \\ \sin \beta & -\cos \beta \end{pmatrix} \begin{pmatrix} \zeta_\rho \\ \zeta_\eta \end{pmatrix}, \quad H_3^0 \approx \xi_\chi, \quad G_3^0 = -\zeta_\chi,
\end{align*}
\]

where the mixing angles are given by:
\[
\tan \alpha = \frac{v_\rho}{v_\eta}, \quad \tan \gamma = \frac{v_\rho}{v_\chi},
\]

while the masses of the physical scalar fields are:
\[
m_{h^0}^2 = \frac{\lambda_1}{2} v_\rho^2 + \frac{v_\eta^2 + v_\rho^2}{2 \sqrt{2} v_\eta v_\rho} f v_\chi - \frac{1}{2} \sqrt{\left(\frac{v_\rho^2 + v_\eta^2}{\sqrt{2} v_\eta v_\rho} f v_\chi \right)^2 + 2 f v_\chi} \left[ \lambda_1 \frac{v_\rho}{v_\eta} \left( v_\eta^2 - v_\rho^2 \right) - 2 \lambda_5 v_\eta v_\rho \right] + \lambda_1^2 v_\rho^4 + \lambda_2^2 v_\rho^2 v_\eta^2
\]
\[
\approx \frac{\lambda_1}{2} v_\rho^2 + \frac{1}{2} \left(2 \lambda_5 - \lambda_1 \left(1 - \frac{v_\eta^2}{v_\rho^2}\right)\right) \frac{v_\eta^2 v_\rho}{v_\rho^2 + v_\eta^2},
\]
\[
m_{\mu^0}^2 = \frac{\lambda_1}{2} v_\rho^2 + \frac{v_\eta^2 + v_\rho^2}{2 \sqrt{2} v_\eta v_\rho} f v_\chi + \frac{1}{2} \sqrt{\left(\frac{v_\rho^2 + v_\eta^2}{\sqrt{2} v_\eta v_\rho} f v_\chi \right)^2 + 2 f v_\chi} \left[ \lambda_1 \frac{v_\rho}{v_\eta} \left( v_\eta^2 - v_\rho^2 \right) - 2 \lambda_5 v_\eta v_\rho \right] + \lambda_1^2 v_\rho^4 + \lambda_2^2 v_\rho^2 v_\eta^2
\]
\[
\approx \frac{v_\rho^2 + v_\eta^2}{\sqrt{2} v_\eta v_\rho} f v_\chi + \frac{\lambda_1}{2} v_\rho^2 - \frac{1}{2} \left(2 \lambda_5 - \lambda_1 \left(1 - \frac{v_\eta^2}{v_\rho^2}\right)\right) \frac{v_\eta^2 v_\rho}{v_\rho^2 + v_\eta^2},
\]
\[
m_{h^0}^2 = \frac{1}{\sqrt{2}} \left( \frac{v_\chi + v_\eta}{v_\chi - v_\eta} f v_\rho \right) f v_\chi, \quad m_{h^0}^2 \approx \frac{f v_\rho v_\eta}{\sqrt{2} v_\chi},
\]
\[
m^2_{\lambda_1} = \frac{f}{\sqrt{2}} \left( \frac{v_\chi v_\eta + v_\rho v_\chi}{v_\eta} + \frac{v_\rho v_\eta}{v_\chi} \right),
\]
\[
m^2_{\lambda_2} = m^2_{G_1^0} = m^2_{G_2^0} = m^2_{G_3^0} = m^2_{G_4^0} = 0,
\]
\[
m^2_{H_1^\pm} = \frac{\lambda_6}{2} \left( v_\eta^2 + v_\rho^2 \right) + \sqrt{2} \left( \frac{v_\rho}{v_\eta} + \frac{v_\eta}{v_\rho} \right) f v_\chi, \quad m^2_{H_2^\pm} = \frac{\lambda_6}{2} \left( v_\chi^2 + v_\rho^2 \right) + \sqrt{2} \left( \frac{v_\rho}{v_\chi} + \frac{v_\chi}{v_\rho} \right) f v_\eta,
\]
where we have taken into account that \( f \approx v_\chi \gg v_\eta, v_\rho \). Notice that after the spontaneous breaking of the gauge symmetry \( SU(3)_C \otimes U(1)_X \) and rotations into mass eigenstates, the model contains 4 massive charged Higgs (\( H_1^\pm, H_2^\pm \)), one CP-odd Higgs (\( A^0 \)) and 4 neutral CP-even Higgs (\( h^0, H_1^0, H_2^0, H_3^0 \)) bosons. Here we identify the scalar \( h^0 \) with the SM-like 126 GeV Higgs boson observed at the LHC. We recall that the neutral Goldstone bosons \( G_1^0, G_2^0 \) and \( G_3^0 \) correspond to the \( Z \), \( K^0 \) and \( Z' \) gauge bosons, respectively. Furthermore, the charged Goldstone bosons \( G_1^\pm \) and \( G_2^\pm \) correspond to the \( W^\pm \) and \( K^\pm \) gauge bosons, respectively. [25][28].

Using the multiplication rules of the \( S_3 \) group, it follows that the Yukawa part of the model Lagrangian for the quark sector takes the following form:
\[
- \mathcal{L}_Y = h_{\Phi}^{(U)} \overline{Q}_{L\nu} u_{R}^U + h_{\Phi}^{(T)} \overline{Q}_{L\chi} y_{R}^U + h_{\Phi}^{(D)} \overline{Q}_{L\eta} \eta_{R}^D + h_{\Phi}^{(T)} \overline{Q}_{L\chi} T_{R}^U + h_{\Phi}^{(D)} \overline{Q}_{L\eta} \eta_{R}^D + \left[ h_{\eta}^{(U)} \overline{Q}_{L\nu} \eta_{R}^U + h_{\eta}^{(D)} \overline{Q}_{L\chi} \eta_{R}^D + h_{\eta}^{(T)} \overline{Q}_{L\eta} \eta_{R}^T + h_{\eta}^{(D)} \overline{Q}_{L\eta} \eta_{R}^D \right] + H.c.
\]
The Yukawa part of the model Lagrangian for the quark sector can be rewritten as:

$$- \mathcal{L}_Y = - \mathcal{L}_Y^{(\text{mass})} - \mathcal{L}_{\phi QQ}$$

where $- \mathcal{L}_Y^{(\text{mass})}$ correspond to the quark mass terms, while $- \mathcal{L}_{\phi QQ}$ includes the interactions of the quarks with the neutral and charged Higgses and Goldstone bosons. These terms are given by:

$$- \mathcal{L}_Y^{(\text{mass})} = h_{\phi}^{(U)} \frac{\nu_n}{\sqrt{2}} U_L^T U_R^2 + h_{\phi}^{(U)} \frac{\nu_n}{\sqrt{2}} T_L^T U_R^3 + h_{\phi}^{(T)} \frac{\nu_n}{\sqrt{2}} T_L^T U_R^1 + h_{\phi}^{(U)} \frac{\nu_n}{\sqrt{2}} T_L^T T_R^1 + h_{\phi}^{(D)} \frac{\nu_n}{\sqrt{2}} T_L^D_1 D_R^1 + \sum_{n=2}^3 \left[ h_{\phi}^{(D)} \frac{\nu_n}{\sqrt{2}} D_L^1 D_R^3 - h_{\phi}^{(D)} \frac{\nu_n}{\sqrt{2}} T_L^D_1 D_R + h_{\phi}^{(D)} \frac{\nu_n}{\sqrt{2}} D_L^1 J_R^3 - h_{\phi}^{(D)} \frac{\nu_n}{\sqrt{2}} J_L^D_1 J_R^3 \right] + \text{H.c.}$$

$$- \mathcal{L}_{\phi QQ} = \frac{h_{\phi}^{(U)}}{\sqrt{2}} U_L^T [\cos \alpha h^0 - \sin \alpha H_1^0 + i \left( \cos \beta A_1^0 + \sin \beta G_1^0 \right)] U_R^2 + h_{\phi}^{(U)} D_L^1 \left( \sin \beta H_1^- - \cos \beta G_1^- \right) U_R^2$$

$$+ \frac{h_{\phi}^{(T)}}{\sqrt{2}} U_L^T [\cos \alpha h^0 - \sin \alpha H_1^0 + i \left( \cos \beta A_1^0 + \sin \beta G_1^0 \right)] U_R^1 + h_{\phi}^{(T)} D_L^1 \left( \sin \beta H_1^- - \cos \beta G_1^- \right) U_R^1$$

$$+ h_{\phi}^{(T)} T_L^H H_2^0 U_R^1 + \frac{h_{\phi}^{(U)}}{\sqrt{2}} T_L^H H_2^0 U_R^2 + \sum_{n=2}^3 \left( h_{\phi}^{(D)} T_L^H H_2^0 D_R^3 + h_{\phi}^{(D)} T_L^H H_2^0 J_R^3 \right)$$

$$- h_{\phi}^{(U)} U_L^T G_2^0 U_R^3 + h_{\phi}^{(U)} D_L^1 \left( \sin \gamma H_2^- - \cos \gamma G_2^- \right) U_R^3 + \frac{h_{\phi}^{(U)}}{\sqrt{2}} T_L^H (H_3^0 - iG_3^0) U_R^3$$

$$- h_{\phi}^{(T)} U_L^T G_2^0 U_R^2 + h_{\phi}^{(T)} T_L^H \left( \sin \gamma H_2^- - \cos \gamma G_2^- \right) T_R^1 + \frac{h_{\phi}^{(T)}}{\sqrt{2}} T_L^H (H_3^0 - iG_3^0) T_R^1$$

$$+ h_{\phi}^{(D)} T_L^H \left( \cos \beta H_1^+ + \sin \beta G_1^+ \right) D_R^1 + \frac{h_{\phi}^{(D)}}{\sqrt{2}} T_L^H \left( \cos \alpha H + \sin \alpha h + iG_1^0 \right) T_R^1 + h_{\phi}^{(D)} T_L^H \left( \cos \gamma H_2^+ + \sin \gamma G_2^+ \right) D_R^1$$

$$+ \sum_{n=2}^3 \left( \frac{h_{\phi}^{(D)}}{\sqrt{2}} T_L^H \left( \cos \alpha h^0 - \sin \alpha H_1^0 + i \left( \cos \beta A_1^0 + \sin \beta G_1^0 \right) \right) D_R^3 + h_{\phi}^{(D)} U_L^T \left( \sin \beta H_1^+ - \cos \beta G_1^+ \right) D_R^3 \right)$$

$$+ \sum_{n=2}^3 \left( \frac{h_{\phi}^{(D)}}{\sqrt{2}} T_L^H \left( \cos \alpha h^0 - \sin \alpha H_1^0 + i \left( \cos \beta A_1^0 + \sin \beta G_1^0 \right) \right) J_R^1 + h_{\phi}^{(D)} T_L^H \left( \sin \beta H_1^+ - \cos \beta G_1^+ \right) J_R^1 \right)$$

$$- \sum_{n=2}^3 \left[ h_{\phi}^{(D)} T_L^H G_2^0 D_R^3 + h_{\phi}^{(D)} U_L^T \left( \sin \gamma H_2^- - \cos \gamma G_2^- \right) D_R^2 + \frac{h_{\phi}^{(D)}}{\sqrt{2}} T_L^H (H_3^0 - iG_3^0) D_R^2 \right]$$

$$- \sum_{n=2}^3 \left[ h_{\phi}^{(D)} T_L^H G_2^0 J_R^3 + h_{\phi}^{(D)} U_L^T \left( \sin \gamma H_2^- - \cos \gamma G_2^- \right) J_R^2 + \frac{h_{\phi}^{(D)}}{\sqrt{2}} T_L^H (H_3^0 - iG_3^0) J_R^2 \right] + \text{H.c.}$$

### III. Tree Level Quark Mass Textures.

From Eq. 24 it follows that the mass matrices for the up and down type quarks are given by:

$$M_U = \begin{pmatrix} h_{\phi}^{(T)} \frac{\nu}{\sqrt{2}} & h_{\phi}^{(U)} \frac{\nu}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & h_{\phi}^{(U)} \frac{\nu}{\sqrt{2}} & h_{\phi}^{(T)} \frac{\nu}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} S_U & 0_{2\times2} \\
0_{2\times2} & Y_U \end{pmatrix}.$$
In sake of simplicity we assume that the Yukawa couplings $h_{uy}^{(T)}$ and $h_{uy}^{(U)}$ are real. Then, the matrix $M_U$ satisfies the following relation:

$$M_U M_U^T = \begin{pmatrix} (a^2 + b^2) v_\eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (a^2 + b^2) v_\chi^2 \end{pmatrix},$$

where:

$$a = \frac{h_{uy}^{(U)}}{\sqrt{2}}, \quad b = \frac{h_{uy}^{(T)}}{\sqrt{2}}.$$  

Now we identify the top and bottom quarks as $U^1$ and $D^1$, respectively, while the remaining SM quarks are identified as $u = U^2$, $c = U^3$, $d = D^2$, $s = D^3$. Then, it follows that the masses of the up-type quarks are given by:

$$m_u = m_c = 0, \quad m_t = \sqrt{a^2 + b^2} v_\eta, \quad m_T = \sqrt{a^2 + b^2} v_\chi = \frac{v_\chi}{v_\eta} m_t.$$  

Therefore, the up and charm quarks are massless at tree level, while the top quark and exotic $T$ quark get their masses from the $\eta$ and $\chi$ triplets, respectively. Notice that the masslessness of the up and charm quarks at tree level, is a consequence of the absence of mixings among them and between them and the heavy $t$, $T$ quarks.

Regarding the down type quark sector, we have that the matrix $\tilde{M}_D$ can be rewritten as:

$$\tilde{M}_D = \begin{pmatrix} 0 & c v_\eta & 0 & x v_\eta \\ 0 & d v_\eta & 0 & y v_\eta \\ -c v_\chi & 0 & -x v_\chi & 0 \\ -d v_\chi & 0 & -y v_\chi & 0 \end{pmatrix},$$

where:

$$c = \frac{h_{uy}^{(D)}}{\sqrt{2}}, \quad d = \frac{h_{uy}^{(D)}}{\sqrt{2}}, \quad x = \frac{h_{uy}^{(J)}}{\sqrt{2}}, \quad y = \frac{h_{uy}^{(J)}}{\sqrt{2}}.$$  

In the down type quark sector we assume that the Yukawa coupling $h_{uy}^{(D)}$ is real while the Yukawa couplings $h_{uy}^{(D)}$ and $h_{uy}^{(J)}$, $(m = 1, 2)$ are complex. Then, the mass matrix $M_D$ for down-type quarks satisfies the following relation:
\[ M_D M_D^\dagger = \begin{pmatrix} \left( h^{(D)}_p \right)^2 \frac{v^2}{2} & 0 & 0 & 0 \\ 0 & \left( |c| + |x|^2 \right) v^2_\eta & (cd^* + xy^*) v^2_\eta & 0 \\ 0 & (c^d + x^*y) v^2_\eta & \left( |d| + |y|^2 \right) v^2_\eta & 0 \\ 0 & 0 & 0 & \left( |c| + |x|^2 \right) v^2_\chi & (cd^* + xy^*) v^2_\chi \\ 0 & 0 & 0 & (c^d + x^*y) v^2_\chi & \left( |d| + |y|^2 \right) v^2_\chi \end{pmatrix} = \begin{pmatrix} S_D & Y_D \end{pmatrix}, \quad (35) \]

where:

\[ S_D = \begin{pmatrix} h^{(D)}_p \frac{v^2}{2} & 0 & 0 \\ 0 & \frac{pe^2_\eta}{wv^2_\eta e^{-i\omega}} & \frac{wv^2_\eta e^{i\omega}}{rv^2_\eta} \\ 0 & \frac{wv^2_\eta e^{-i\omega}}{rv^2_\eta} & \frac{h^{(D)}_p \frac{v^2}{2}} \end{pmatrix} \quad Y_D = \begin{pmatrix} \frac{pe^2_\chi}{wv^2_\chi e^{i\omega}} & \frac{wv^2_\chi e^{-i\omega}}{rv^2_\chi} \end{pmatrix}. \quad (36) \]

The matrix \( S_D \) is written in the basis \((D^1, D^2, D^3) = (h, d, s)\). Furthermore, the matrix \( S_D \) can be written in the basis \(d, s, b\) as follows:

\[ Z_D = \begin{pmatrix} r^2 v^2_\eta & \frac{wv^2_\eta e^{-i\omega}}{rv^2_\eta} & 0 \\ \frac{wv^2_\eta e^{i\omega}}{rv^2_\eta} & \frac{pe^2_\eta}{wv^2_\eta e^{-i\omega}} & 0 \\ 0 & 0 & h^{(D)}_p \frac{v^2}{2} \end{pmatrix}. \quad (37) \]

The matrices \( Z_D \) and \( Y_D \) can be diagonalized by the rotation matrices \( R_D \) and \( P_D \) according to:

\[ R_D^\dagger Z_D R_D = \text{diag} \left( m^2_d, m^2_s, m^2_b \right), \quad P_D^\dagger Y_D P_D = \text{diag} \left( m^2_{j3}, m^2_{j2} \right) \quad (38) \]

where:

\[ R_D = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\omega} & 0 \\ -\sin \theta e^{i\omega} & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_D = \begin{pmatrix} \cos \theta & -\sin \theta e^{i\omega} \\ \sin \theta e^{-i\omega} & \cos \theta \end{pmatrix}. \quad (39) \]

\[ \tan 2\theta = \frac{2w}{p - r}. \quad (40) \]

Furthermore, the masses of the down-type quarks are given by:

\[ m_D = \sqrt{\frac{p + r - \sqrt{(p - r)^2 + 4w^2}}{2} v_\eta}, \quad m_s = \sqrt{\frac{p + r + \sqrt{(p - r)^2 + 4w^2}}{2} v_\eta}, \quad m_b = h^{(D)}_p \frac{v^2}{\sqrt{2}}. \quad (41) \]

\[ m_{j2} = \sqrt{\frac{p + r - \sqrt{(p - r)^2 + 4w^2}}{2} v_\chi} = \frac{v_\chi}{v_\eta} m_d, \quad m_{j3} = \sqrt{\frac{p + r + \sqrt{(p - r)^2 + 4w^2}}{2} v_\chi} = \frac{v_\chi}{v_\eta} m_s. \quad (42) \]
Therefore, the hierarchy $m_d << m_s << m_b$ can be explained by the inequality $\left| h_{m\Phi}^{(D)} \right| << \left| h_{m\Phi}^{(J)} \right| << \frac{\sqrt{\epsilon_\eta}}{\sqrt{\epsilon_\eta}} \left| h_{\rho\Phi}^{(D)} \right|$, with $m = 1, 2$. Therefore, considering $v_\rho \sim v_\eta$, we obtain that the triplet $\rho$ couples more strongly to the bottom quark than the triplets $\eta$ and $\chi$ to the exotic down type quarks. This assumption implies that the hierarchy $m_b << m_t$ can be explained by one of the inequalities, i.e., $\left| h_{\rho\Phi}^{(D)} \right| << \left| h_{\rho\Phi}^{(T)} \right|$ or $\left| h_{\rho\Phi}^{(D)} \right| << \left| h_{\rho\Phi}^{(U)} \right|$. Furthermore, the bottom quark gets its mass from the triplet $\rho$, while the exotic down type quarks $J^2$ and $J^3$, get their masses from the triplet $\chi$. Unlike the up type quark sector, the light down type quarks $d$ and $s$ get tree level masses from the triplet $\eta$. Besides that, we get a strong hierarchy between the masses of the exotic quarks $J^2$, $J^3$ and $T$, i.e, $m_T >> m_J^3 >> m_J^2$.

IV. ON THE EFFECTS OF LOOP CORRECTIONS AND NON RENORMALIZABLE OPERATORS ON THE UP- TYPE QUARK MASS TEXTURE.

As it can be seen from Eqs. (23) and (30), the vanishing entries of the up type quark mass matrix do not receive one loop level corrections. In the up type quark sector, only some of the non vanishing entries of the corresponding mass matrix receive radiative corrections at one loop level due to virtual neutral scalars and heavy exotic $T$ quark. Notice that charged scalars do not contribute radiatively to the entries of the up type quark mass matrix since $\mathcal{J}_L$ and $\mathcal{J}_U$ do not have interactions neither with the charged Higgs nor with the up type quarks as seen from Eq. (23). Furthermore, $\mathcal{U}_L$ and $\mathcal{U}_U$ do not mix with $T_R$, avoiding that the vanishing entries of the tree level up type quark mass matrix receive loop corrections. Therefore, in our $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3$ model, radiative corrections do not generate masses for the up and charm quarks. Because of this reason, we consider the following non renormalizable $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3$ Yukawa part of the model Lagrangian:

$$
\mathcal{L}_{\text{nonren}} = \frac{1}{\Lambda^2} \sum_{n=2}^3 \left[ \varepsilon_{1n} \mathcal{U}_L^* \left( Q^{(U)}_R \right) \left( \Phi^1 \Phi \right)_2 \right] + \frac{1}{\Lambda^2} \left[ \varepsilon_{2n} \mathcal{U}_L^* \left( Q^{(T)}_R \right) \left( \Phi^1 \Phi \right)_2 \right] + \varepsilon_{3n} \mathcal{U}_L^* \left( Q^{(U)}_R \right) \left( \Phi^1 \Phi \right)_2.
$$

Using the multiplication rules of the $S_3$ group, it follows that the aforementioned non renormalizable Yukawa part of the model Lagrangian can be rewritten as follows:

$$
\mathcal{L}_{\text{nonren}} = \frac{1}{\Lambda^2} \sum_{n=2}^3 \left[ \varepsilon_{1n} \left( \mathcal{U}_L^* U_R \right)^3 \left( \eta^1 \chi + \chi^1 \eta \right) + \left( \mathcal{U}_L^* U_R \right)^3 \left( \eta^1 \eta - \chi^1 \chi \right) \right] + \varepsilon_{2n} \left( \mathcal{U}_L^* U_R \right)^3 \left( \eta^1 \chi + \chi^1 \eta \right) + \left( \mathcal{U}_L^* U_R \right)^3 \left( \eta^1 \eta - \chi^1 \chi \right) + \varepsilon_{3n} \left( \mathcal{U}_L^* U_R \right)^3 \left( \eta^1 \chi + \chi^1 \eta \right) + \left( \mathcal{U}_L^* U_R \right)^3 \left( \eta^1 \eta - \chi^1 \chi \right).
$$

After the scalars triplets get vacuum expectation values, the previous non renormalizable interaction yields the following contributions to the entries of the up type quark mass matrix:

$$
\mathcal{L}_{\text{nonren}} \supset \frac{v_\rho v_\chi v_\eta}{\sqrt{2} \Lambda^2} \sum_{n=2}^3 \left( \varepsilon_{1n} \mathcal{U}_L^* U_R^2 + \varepsilon_{2n} \mathcal{U}_L^* U_R^1 \right) + \varepsilon_{3n} \frac{v_\chi}{\sqrt{2} \Lambda^2} \left( v_\chi \mathcal{U}_L^* U_R^3 - \epsilon v_\eta \mathcal{U}_L^* U_R^2 \right) + \varepsilon_{4n} \frac{v_\chi}{\sqrt{2} \Lambda^2} \left( v_\chi \mathcal{U}_L^* U_R^1 - \epsilon v_\eta \mathcal{U}_L^* U_R^2 \right).
$$

Therefore, the up type quark mass matrix takes the following form:

$$
M_U = \begin{pmatrix}
\kappa_1 v_\eta & \kappa_2 v_\eta & 0 & 0 \\
\kappa_1 v_\eta & f_1 v_\rho & f_2 v_\rho & 0 \\
f_1 v_\rho & f_2 v_\rho & \kappa_3 v_\chi & 0 \\
0 & f_3 v_\rho & f_4 v_\rho & \kappa_4 v_\chi
\end{pmatrix}.
$$

where:
Then, the mass matrix for the up-type quarks satisfies the following relation:

\[ \kappa_1 = \frac{h_\Phi^{(T)}}{\sqrt{2}} - \frac{\varepsilon_4 v_\chi^2}{2\sqrt{2} \Lambda^2}, \quad \kappa_2 = \frac{h_\Phi^{(U)}}{\sqrt{2}} - \frac{\varepsilon_3 v_\chi^2}{2\sqrt{2} \Lambda^2}. \]  

(47)

\[ \kappa_3 = \frac{h_\Phi^{(U)}}{\sqrt{2}} \left[ 1 + \frac{h_\Phi^{(T)} (2\sqrt{2} f v_\eta + \lambda_3 v_\rho^2)}{16\sqrt{2\pi^2 v_\chi m_T}} D_0 \left( \frac{m_{H_0}}{m_T} \right) \right] + \frac{\varepsilon_3 v_\chi^2}{2\sqrt{2} \Lambda^2}. \]

(48)

\[ \kappa_4 = \frac{h_\Phi^{(T)}}{\sqrt{2}} \left[ 1 + \frac{h_\Phi^{(T)} (2\sqrt{2} f v_\eta + \lambda_3 v_\rho^2)}{16\sqrt{2\pi^2 v_\chi m_T}} D_0 \left( \frac{m_{H_0}}{m_T} \right) \right] + \frac{\varepsilon_4 v_\chi^2}{2\sqrt{2} \Lambda^2}. \]

(49)

\[ f_1 = \frac{\varepsilon_2 v_\chi v_\eta}{\sqrt{2} \Lambda^2}, \quad f_2 = \frac{\varepsilon_12}{\varepsilon_2} f_1, \quad f_3 = \frac{\varepsilon_13}{\varepsilon_2} f_1, \quad f_4 = \frac{\varepsilon_13}{\varepsilon_2} f_1. \]  

(50)

We introduced the function \[ [34]\] :

\[ D_0(x) = \frac{-1 + x^2 - \ln x^2}{(1 - x^2)^2}. \]

(51)

Notice that the dimensionless couplings in Eq. \[ [40]\] satisfy the following hierarchy:

\[ f_1 << \kappa_i, \quad i = 1, 2, 3, 4. \]

(52)

Then, the mass matrix for the up-type quarks satisfies the following relation:

\[
M_{U}M_{U}^{T} = \begin{pmatrix}
\left( \kappa_1^2 + \kappa_2^2 \right) v_\eta^2 & (\kappa_1 f_1 + \kappa_2 f_2) v_\eta v_\rho & (\kappa_1 f_3 + \kappa_2 f_4) v_\eta v_\rho & 0 \\
(\kappa_1 f_1 + \kappa_2 f_2) v_\eta v_\rho & \left(f_1^2 + f_2^2\right) v_\rho^2 & (f_1 f_3 + f_2 f_4) v_\rho^2 & 0 \\
(\kappa_1 f_3 + \kappa_2 f_4) v_\eta v_\rho & (f_1 f_3 + f_2 f_4) v_\rho^2 & \left(f_3^2 + f_4^2\right) v_\rho^2 & 0 \\
0 & 0 & 0 & \left(\kappa_3^2 + \kappa_4^2\right) v_\chi^2
\end{pmatrix}
\]

(53)

From Eqs. \[ [52]\] and \[ [53]\] it follows that the dimensionless couplings in Eq. \[ [53]\] satisfy the following hierarchy:

\[ \omega_j << \zeta_l << \sigma, \tau, \quad l = 1, 2, 3, j = 1, 2, 3. \]

(54)

For the sake of simplicity we assume that the dimensionless couplings \( \varepsilon_{sn} \), with \( s = 1, 2 \) and \( n = 2,3 \) are approximately equal. This assumption implies \( \zeta_l \simeq \zeta, \omega_i \simeq \omega \) with \( l = 1, 2 \) and \( i = 1, 2, 3, 4 \). In order to reduce the parameter space in our model, we choose \( \omega_j = \omega (j = 1, 2, 3), \zeta_1 = (1 + \vartheta) \zeta, \zeta_2 = (1 - \vartheta) \zeta \) with \( \vartheta << 1 \). Notice that the parameter \( \vartheta \) controls the breaking of universality in the dimensionless couplings of the six dimensional operators. Then, after the aforementioned choice is made, the expression given by Eq. \[ [53]\] becomes:

\[
M_{U}M_{U}^{T} = \begin{pmatrix}
\sigma v_\eta^2 & (1 + \vartheta) \zeta v_\eta v_\rho & (1 - \vartheta) \zeta v_\eta v_\rho & 0 \\
(1 + \vartheta) \zeta v_\eta v_\rho & \omega v_\rho^2 & \omega v_\rho^2 & 0 \\
(1 - \vartheta) \zeta v_\eta v_\rho & \omega v_\rho^2 & \omega v_\rho^2 & 0 \\
0 & 0 & 0 & \tau v_\chi^2
\end{pmatrix}
\]

(55)

where:

\[
S_U = \begin{pmatrix}
\sigma v_\eta^2 & (1 + \vartheta) \zeta v_\eta v_\rho & (1 - \vartheta) \zeta v_\eta v_\rho \\
(1 + \vartheta) \zeta v_\eta v_\rho & \omega v_\rho^2 & \omega v_\rho^2 \\
(1 - \vartheta) \zeta v_\eta v_\rho & \omega v_\rho^2 & \omega v_\rho^2
\end{pmatrix}, \quad m_T^2 = \tau v_\chi^2. \]  

(56)
The matrix $S_U$ can be diagonalized by a rotation matrix $X_U$, according to:

$$X^T_U S_U X_U = \text{diag} \left( m^2_{t}, m^2_{c}, -m^2_{u} \right), \quad (57)$$

where:

$$X_U = \begin{pmatrix}
\frac{\cos \phi \cos \psi}{\sqrt{2}} & \frac{-\cos \psi \sin \phi}{\sqrt{2}} & -\sin \psi \\
\frac{\sin \phi - \frac{1}{\sqrt{2}} \cos \phi \sin \psi}{\sqrt{2}} & \frac{\cos \phi + \frac{1}{\sqrt{2}} \sin \phi \sin \psi}{\sqrt{2}} & -\sin \phi \\
\frac{\sin \phi + \frac{1}{\sqrt{2}} \cos \phi \sin \psi}{\sqrt{2}} & \frac{\cos \phi - \frac{1}{\sqrt{2}} \sin \phi \sin \psi}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cos \phi
\end{pmatrix}, \quad (58)$$

with

$$\tan 2\phi = \frac{2\sqrt{2} \zeta \eta v_{\rho}}{\sigma + 2\theta^2 \zeta v_{\rho}^2 v_{\eta}^2 - 2\omega v_{\rho}^2}, \quad \tan 2\phi = \frac{-\theta \frac{2\sqrt{2} \zeta \eta}{\sigma v_{\eta}}}. \quad (59)$$

And the masses of the SM up type quarks are given by:

$$m^2_{t} = \frac{1}{2} \left\{ \left[ \sigma + 2\theta^2 \zeta v_{\rho}^2 \right] v_{\eta}^2 + 2\omega v_{\rho}^2 \right\} + \frac{1}{2} \sqrt{\left\{ \left[ \sigma + 2\theta^2 \zeta v_{\rho}^2 \right] v_{\eta}^2 - 2\omega v_{\rho}^2 \right\}^2 + 8\zeta^2 v_{\rho}^2 v_{\eta}^2}, \quad (60)$$

$$m^2_{c} = \frac{1}{2} \left\{ \left[ \sigma + 2\theta^2 \zeta v_{\rho}^2 \right] v_{\eta}^2 + 2\omega v_{\rho}^2 \right\} - \frac{1}{2} \sqrt{\left\{ \left[ \sigma + 2\theta^2 \zeta v_{\rho}^2 \right] v_{\eta}^2 - 2\omega v_{\rho}^2 \right\}^2 + 8\zeta^2 v_{\rho}^2 v_{\eta}^2}, \quad (61)$$

$$m^2_{u} = 2\theta^2 \zeta v_{\rho}^2 \zeta v_{\eta}^2. \quad (62)$$

Notice that the six dimensional operators are crucial to give masses the up and charm quarks. However if universality in the dimensionless couplings of the six dimensional operators were exact, the up quark would be massless. Therefore, the smallness of the up quark mass can be explained by the suppression of universality violation in the dimensionless couplings of six dimensional operators.

On the other hand, in the $u, c, t$ basis, the matrix $X_U$ can be written as:

$$R_U = \begin{pmatrix}
\frac{1}{\sqrt{2}} \cos \psi & \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{\sqrt{2}} \sin \phi \sin \psi & \frac{1}{\sqrt{2}} \sin \phi + \frac{1}{\sqrt{2}} \cos \phi \sin \psi \\
\frac{1}{\sqrt{2}} \cos \phi + \frac{1}{\sqrt{2}} \sin \phi \sin \psi & \frac{1}{\sqrt{2}} \sin \phi - \frac{1}{\sqrt{2}} \cos \phi \sin \psi & \frac{1}{\sqrt{2}} \cos \phi \\
-\sin \phi & -\cos \phi \sin \psi & \cos \phi \cos \psi
\end{pmatrix}, \quad (63)$$

V. QUARK MIXINGS.

With the rotation matrices $R_U$ and $R_D$ for the up and down type quarks, respectively, we find the CKM mixing matrix:

$$K = R^T_U R_D = \begin{pmatrix}
\frac{1}{\sqrt{2}} \cos \psi & \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{\sqrt{2}} \sin \phi \sin \psi & -\frac{1}{\sqrt{2}} \cos \phi - \frac{1}{\sqrt{2}} \sin \phi \sin \psi \\
\frac{1}{\sqrt{2}} \cos \phi - \frac{1}{\sqrt{2}} \sin \phi \sin \psi & \frac{1}{\sqrt{2}} \sin \phi + \frac{1}{\sqrt{2}} \cos \phi \sin \psi & \frac{1}{\sqrt{2}} \cos \phi + \frac{1}{\sqrt{2}} \sin \phi \sin \psi \\
\frac{1}{\sqrt{2}} \sin \phi + \frac{1}{\sqrt{2}} \cos \phi \sin \psi & \frac{1}{\sqrt{2}} \cos \phi + \frac{1}{\sqrt{2}} \sin \phi \sin \psi & \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{\sqrt{2}} \sin \phi \sin \psi
\end{pmatrix} \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\omega} & 0 \\
-\sin \theta e^{i\omega} & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (64)$$

where the CKM matrix elements are given by:

$$K_{ud} = \frac{1}{\sqrt{2}} (\cos \theta + e^{i\omega} \sin \theta) \cos \psi, \quad K_{us} = -\frac{1}{\sqrt{2}} \cos \psi (\cos \theta - e^{-i\omega} \sin \theta), \quad K_{ub} = -\sin \psi, \quad (65)$$
\[ K_{cd} = \frac{1}{\sqrt{2}} \cos \theta \left( \cos \phi - \sin \phi \sin \psi \right) - \frac{1}{\sqrt{2}} e^{i\varpi} \sin \theta \left( \cos \phi + \sin \phi \sin \psi \right), \]  

(66)

\[ K_{cs} = \frac{1}{\sqrt{2}} \cos \theta \left( \cos \phi + \sin \phi \sin \psi \right) + \frac{1}{\sqrt{2}} e^{-i\varpi} \sin \theta \left( \cos \phi - \sin \phi \sin \psi \right), \]  

(67)

\[ K_{cb} = - \cos \psi \sin \phi, \]  

(68)

\[ K_{td} = \frac{1}{\sqrt{2}} \cos \theta \left( \sin \phi + \cos \phi \sin \psi \right) - \frac{1}{\sqrt{2}} e^{i\varpi} \sin \theta \left( \sin \phi - \cos \phi \sin \psi \right), \]  

(69)

\[ K_{ts} = \frac{1}{\sqrt{2}} \cos \theta \left( \sin \phi - \cos \phi \sin \psi \right) + \frac{1}{\sqrt{2}} e^{-i\varpi} \sin \theta \left( \sin \phi + \cos \phi \sin \psi \right), \]  

(70)

\[ K_{tb} = \cos \phi \cos \psi, \]  

(71)

while the Jarlskog invariant which measures the amount of CP violation takes the following form:

\[ J = \text{Im} \left[ K_{us} K_{cb} K_{us}^* K_{cb}^* \right] = -\frac{1}{8} \sin 2\varpi \sin 2\psi \sin 2\phi \cos \psi \sin \varpi. \]  

(72)

Varying the parameters \( \varpi, \theta, \psi \) and \( \phi \) we have fitted the magnitudes of the CKM matrix elements and the Jarlskog invariant \( J \) to the experimental values shown in Tables I, II. The experimental values of CKM magnitudes and Jarlskog invariant are taken from [12]. These tables also show the obtained values of CKM magnitudes and Jarlskog invariant.

The values of the parameters \( \varpi, \theta, \psi \) and \( \phi \) that successfully reproduce the CKM magnitudes and the value of the Jarlskog invariant are given by:

\[ \varpi = 154.2^\circ, \quad \theta = -46.7^\circ, \quad \phi = -2.4^\circ, \quad \psi = 0.19^\circ. \]  

(73)

| CKM matrix element | Obtained Value | Experimental Value |
|--------------------|----------------|--------------------|
| \( |V_{ud}| \)        | 0.974          | 0.97427 ± 0.00015  |
| \( |V_{us}| \)        | 0.225          | 0.22534 ± 0.00065  |
| \( |V_{ub}| \)        | 0.00330        | 0.00351 ± 0.00015  |
| \( |V_{cd}| \)        | 0.225          | 0.22520 ± 0.00065  |
| \( |V_{cs}| \)        | 0.973          | 0.97344 ± 0.00016  |
| \( |V_{cb}| \)        | 0.0412         | 0.0412 ± 0.00014   |
| \( |V_{td}| \)        | 0.00940        | 0.00867 ± 0.00029  |
| \( |V_{ts}| \)        | 0.0403         | 0.0404 ± 0.00013   |
| \( |V_{tb}| \)        | 0.999          | 0.999146 ± 0.000021 |

Table I: Obtained and experimental values of the magnitudes of the CKM matrix elements.

The obtained magnitudes of the CKM matrix elements and Jarlskog invariant are in excellent agreement with the experimental data. Notice that we have assumed that the Yukawa couplings for the up type quarks as well as the couplings of higher dimensional operators are real while the Yukawa couplings for down type quarks are complex with the exception of \( h_D^{(D)} \). It is worth mentioning that with only four effective parameters, i.e, the complex phase \( \varpi \) and the mixing angles \( \theta, \phi \) and \( \psi \), we obtain a realistic pattern of SM quark mixings. Furthermore, the complex phase \( \varpi \) in the mass matrix for down type quarks is the only one responsible for CP violation in the quark sector.
Table II: Obtained and experimental values of the Jarlskog invariant.

| Jarlskog invariant | Obtained Value | Experimental Value |
|--------------------|----------------|--------------------|
| J                 | $2.96 \times 10^{-5}$ | $(2.96^{+0.20}_{-0.15}) \times 10^{-5}$ |

VI. CONCLUSIONS

In this work we proposed a model based on the extended group $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3$. By assuming specific particle assignments into $S_3$ doublets and $S_3$ singlets for scalars and quarks, we obtained a predictive model, where quark masses and mixing can successfully be reproduced. Taking into account the multiplication rules of the $S_3$ discrete group, we obtained the complete mass terms of the scalar and quark sector. In particular, the model reproduces the mass structures of the down sector at tree level, where the light d- and s-quarks acquire masses through one scalar triplet $(\eta)$ and the b-quark obtains mass from the other triplet $(\rho)$, while the two heavy quarks $J^1$ and $J^2$ get large masses from the third scalar triplet $(\chi)$. Furthermore, by assuming that the light Standard Model down-type quarks possess smaller Yukawa couplings than the couplings of the heavy quarks, we may explain the hierarchy $m_d << m_s$. The hierarchy $m_b >> m_s$ can be explained by assuming that the Yukawa coupling of the bottom quark is much larger than the Yukawa couplings of the exotic down type quarks, considering the vacuum expectation values of the triplets $(\rho)$ and $(\eta)$ of the same order. Regarding the up sector, the model predicts at tree level two massless quarks (the u- and c-quarks), one massive quark at the electroweak scale (the t-quark), and a very heavy T-quark. Due to the symmetries of the model, the massless quarks do not get masses from radiative corrections. However, these quarks get masses from non renormalizable operators. In particular, if universality of the non renormalizable couplings is assumed, the u-quark remains massless. Only if the above universality is violated, the lightest quark gets mass. Thus, the smallness of the u-quark can be understood as a consequence of a small deviation in the universality of non renormalizable effective couplings. We have reproduced with only three mixing angles and one complex phase, the magnitudes of the CKM matrix elements and the Jarlskog invariant, which turn out to be in excellent agreement with the experimental data. The complex phase responsible for CP violation in the quark sector has been assumed to come from the down type quark mass matrix.

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Appendix A: Scalar Potential and mass spectrum for the neutral and charged scalar fields.

The interactions among the scalar fields are contained in the following most general potential invariant under $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes S_3$ that we can construct with three scalar triplets:

$$V_H = \mu_\rho^2 (\rho^\dagger \rho) + \mu_\eta^2 \left[ (\eta^\dagger \eta) + (\chi^\dagger \chi) \right] + \lambda_1 (\rho^\dagger \rho)^2 + (\lambda_2 + \lambda_4) \left[ (\chi^\dagger \chi)^2 + (\eta^\dagger \eta)^2 \right] + \lambda_5 \left[ (\rho^\dagger \rho) (\chi^\dagger \chi) + (\rho^\dagger \rho) (\eta^\dagger \eta) \right] + 2 (\lambda_2 - \lambda_4) \left( \chi^\dagger \chi (\eta^\dagger \eta) + 2 (\lambda_4 - \lambda_3) \right) + \lambda_6 \left[ (\chi^\dagger \rho) (\rho^\dagger \chi) + (\rho^\dagger \rho) (\rho^\dagger \eta) \right] + (\lambda_3 + \lambda_4) \left[ (\chi^\dagger \eta)^2 + (\eta^\dagger \chi)^2 \right] + 2 f (\varepsilon^{ijk} \eta_i \chi_j \rho_k + h.c). \quad (A1)$$

From the previous expressions and from the scalar potential minimization conditions, the following relations are obtained:

$$\frac{\partial \langle V_H \rangle}{\partial \psi_\chi} = v_\chi \mu_\rho^2 + (\lambda_2 + \lambda_4) v_\chi^2 + \frac{\lambda_5}{2} v_\chi v_\rho^2 + (\lambda_2 - \lambda_4) v_\chi v_\eta^2 - \sqrt{2} f v_\rho v_\eta = 0, \quad (A2)$$

$$\frac{\partial \langle V_H \rangle}{\partial \psi_\eta} = v_\eta \mu_\rho^2 + (\lambda_2 + \lambda_4) v_\eta^2 + \frac{\lambda_5}{2} v_\eta v_\rho^2 + (\lambda_2 - \lambda_4) v_\eta v_\chi^2 - \sqrt{2} f v_\chi v_\rho = 0, \quad (A3)$$
\[ \frac{\partial (V_H)}{\partial v_\rho} = v_\rho \mu_\rho^2 + \lambda_1 v_\rho^3 + \frac{\lambda_5}{2} v_\rho (v_\chi^2 + v_\eta^2) - \sqrt{2} f v_\chi v_\eta = 0. \] (A4)

From which it follows that:

\[- \mu_\phi^2 = (\lambda_2 + \lambda_4) v_\chi^2 + \frac{\lambda_5}{2} v_\rho^2 + (\lambda_2 - \lambda_4) v_\eta^2 - \frac{\sqrt{2} f v_\chi v_\eta}{v_\rho}, \] (A5)

\[- \mu_\rho^2 = \lambda_3 v_\rho^2 + \frac{\lambda_5}{2} (v_\chi^2 + v_\eta^2) - \frac{\sqrt{2} f v_\eta v_\chi}{v_\rho}. \] (A6)

Then, the following relation is obtained:

\[ \lambda_2 = - \lambda_4 = \frac{f v_\rho}{\sqrt{2} v_\eta v_\chi}. \] (A7)

Furthermore, from the scalar potential it follows that the quartic interactions relevant for the computation of radiative corrections to the up- and down-type quark mass matrices are:

\[ V_H \supset \frac{1}{4} \left[ (\lambda_2 + \lambda_4) \xi_\chi^2 + 2 (\lambda_2 - \lambda_4) \xi_\eta^2 + \lambda_5 \xi_\rho^2 + 4 (\lambda_2 + \lambda_4) \left( (\eta_0^0)^2 + (\chi_0^0)^2 \right) \xi_\chi^2 + \frac{1}{2} \left[ 2 (\lambda_2 - \lambda_4) \eta_2^+ \eta_2^- + \lambda_5 \rho_1^+ \rho_1^- + (\lambda_5 + \lambda_6) \rho_3^+ \rho_3^- + 2 (\lambda_2 + \lambda_4) \chi_2^+ \chi_2^- \right] \xi_\eta^2 + \frac{1}{2} \lambda_6 (\rho_1^+ \chi_2^- + \rho_3^- \chi_2^+) \xi_\eta \xi_\chi + 4 \lambda_4 \xi_0^0 \xi_0^0 \xi_\eta \xi_\chi, \right] \] (A8)

where \( \xi_\rho = v_\rho + \xi_\rho, \xi_\eta = v_\eta + \xi_\eta \) and \( \xi_\chi = v_\chi + \xi_\chi \).

By expanding the scalar potential up to quadratic terms of the neutral scalar fields and using the relations given by Eqs. (A5)-(A6), we obtain the following mass terms for the neutral scalar fields:

\[ - \mathcal{L}_{\text{mass}}^{\text{neutral}} = \left( \lambda_1 v_\rho^2 + \frac{f v_\rho v_\eta}{\sqrt{2} v_\eta} \right) \xi_\rho^2 + \left( \lambda_2 + \lambda_4 \right) v_\eta^2 + \frac{f v_\rho v_\chi}{\sqrt{2} v_\eta} \xi_\eta^2 + \left( \lambda_2 + \lambda_4 \right) v_\chi^2 + \frac{f v_\rho v_\eta}{\sqrt{2} v_\chi} \xi_\rho^2 + \frac{f v_\rho v_\chi}{\sqrt{2} v_\eta} \xi_\eta^2 + \frac{f v_\rho v_\chi}{\sqrt{2} v_\eta} \xi_\chi^2 + 2 (\lambda_2 - \lambda_4) v_\chi v_\eta \xi_\chi \xi_\eta + \lambda_5 v_\chi v_\eta \xi_\rho \xi_\chi + \lambda_5 v_\eta v_\chi \xi_\rho \xi_\eta + 2 \lambda_6 (\rho_1^+ \chi_2^- + \rho_3^- \chi_2^+) \xi_\rho \xi_\eta \xi_\chi + 4 \lambda_4 \xi_0^0 \xi_0^0 \xi_\eta \xi_\chi, \right] \] (A9)

Therefore, the squared mass matrix for the neutral scalar fields in the basis \( \xi_\rho, \xi_\eta, \xi_\chi, \xi_0^0, \chi_0^0, \xi_\rho, \xi_\eta, \xi_\chi \) is given by:

\[ M^2_N = \begin{pmatrix} M^2_{N_a} & 0_{3 \times 2} & 0_{3 \times 3} \\ 0_{2 \times 3} & M^2_{N_b} & 0_{2 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 2} & M^2_{N_c} \end{pmatrix}, \] (A10)
where the squared mass matrices $M_{Na}^2$, $M_{Nb}^2$ and $M_{Nc}^2$ are given by:

$$M_{Na}^2 = \left( \begin{array}{ccc}
\frac{\lambda_3 v^2 + \frac{f_{\nu\nu}}{\sqrt{2} v}}{2 v} & \frac{f_{\nu\nu}}{\sqrt{2} v} v_{\rho} & \frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} \\
\frac{\lambda_2}{2} v_{\rho} & \left( \lambda_2 + \lambda_4 \right) v_{\eta}^2 + \frac{f_{\nu\nu}}{\sqrt{2} v} & \left( \lambda_2 - \lambda_4 \right) v_{\chi} v_{\eta} - \frac{1}{\sqrt{2}} f_{\nu} \\
\frac{\lambda_2}{2} v_{\chi} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} & \left( \lambda_2 - \lambda_4 \right) v_{\chi} v_{\eta} - \frac{1}{\sqrt{2}} f_{\nu} & \left( \lambda_2 + \lambda_4 \right) v_{\chi} + \frac{f_{\nu\nu}}{\sqrt{2} v}
\end{array} \right),$$

(A11)

$$M_{Nb}^2 = \left( \begin{array}{ccc}
2 \lambda_4 v_{\eta}^2 + \frac{f_{\nu\nu}}{\sqrt{2} v} & 2 \lambda_4 v_{\eta} v_{\chi} + \sqrt{2} f_{\nu} \\
\sqrt{2} v_{\eta} & 2 \lambda_4 v_{\eta} v_{\chi} + \sqrt{2} f_{\nu} & 2 \lambda_4 v_{\eta}^2 + \frac{f_{\nu\nu}}{\sqrt{2} v}
\end{array} \right),$$

(A12)

$$M_{Nc}^2 = \left( \begin{array}{ccc}
\frac{f_{\nu\nu}}{\sqrt{2} v} v_{\rho} & \frac{1}{\sqrt{2}} f_{\nu} v_{\eta} & \frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} \\
\frac{1}{\sqrt{2}} f_{\nu} v_{\eta} & \frac{1}{\sqrt{2}} f_{\nu} v_{\rho} & \frac{1}{\sqrt{2}} f_{\nu} v_{\chi}
\end{array} \right) = \frac{1}{\sqrt{2}} f \left( \begin{array}{ccc}
\frac{v_{\rho}}{v} & v_{\chi} & v_{\eta} \\
v_{\eta} & v_{\chi} & v_{\rho} \\
v_{\rho} & v_{\eta} & v_{\chi}
\end{array} \right),$$

(A13)

Here we assume that the mass parameter of the cubic term in the scalar potential satisfies $f \simeq v_{\chi}$. Since $v_{\chi} \gg v_{\eta}, v_{\rho}$, the squared mass matrix $M_{Na}^2$ can be block-diagonalized through the rotation matrix $W$, according to:

$$W^T M_{Na}^2 W \simeq \begin{pmatrix}
M_{Na}^2 & 0_{2 \times 1} \\
0_{1 \times 2} & m_{H_3}^2
\end{pmatrix},$$

(A14)

with

$$W = \begin{pmatrix}
1_{2 \times 2} & B \\
-B^T & 1
\end{pmatrix}, \quad m_{H_3}^2 \simeq \frac{f_{\nu\nu}}{\sqrt{2} v_{\chi}},$$

(A15)

$$\tilde{M}_{Na}^2 = \begin{pmatrix}
\lambda_3 v_{\rho}^2 + \frac{f_{\nu\nu}}{\sqrt{2} v} & \frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} \\
\frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} & \frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu}
\end{pmatrix}.$$  

(A16)

where the relation $\lambda_2 = -\lambda_4$ given by Eq. (A7) has been used.

From the condition of the vanishing of the off-diagonal submatrices in the previous expression, we obtain at leading order in $B$ the following relations:

$$a B + b - m_{H_3}^2 B = 0, \quad B^T a + b^T - m_{H_3}^2 B^T = 0,$$

(A17)

where $a$ and $b$ are given by:

$$a = \begin{pmatrix}
\lambda_3 v_{\rho}^2 + \frac{f_{\nu\nu}}{\sqrt{2} v} & \frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} \\
\frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} & \frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu}
\end{pmatrix}.$$  

(A18)

$$b = \begin{pmatrix}
\frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu} \\
\frac{\lambda_2}{2} v_{\rho} - \frac{1}{\sqrt{2}} f_{\nu}
\end{pmatrix}. $$

(A19)

By using the method of recursive expansion taking into account the hierarchy $m_{H_3}^2 \ll b_{n1} \ll a_{nm}$, $(m, n = 1, 2)$ we find that the submatrix $B$ is approximatively given by:

$$B_{n1} \simeq -\frac{b_{n1}}{a_{nm}} \sim -\frac{b_{n1}}{v_{\chi}},$$

(A20)
The matrix $\tilde{M}^2_{Na}$ is diagonalized by a rotation matrix is diagonalized by a rotation matrix:

$$R = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix},$$

(A21)

which satisfies:

$$R^T \tilde{M}^2_{Na} R = \begin{pmatrix} m_{H_1^0}^2 & 0 \\ 0 & m_{H_1^0}^2 \end{pmatrix},$$

(A22)

with

$$\tan 2\alpha = \frac{\lambda_5 v_\eta v_\rho - \sqrt{2} f \chi}{\sqrt{2} f \chi \left( \frac{v_\eta}{v_\rho} - \frac{v_\rho}{v_\eta} \right) - \lambda_1 v_\rho^2},$$

$$\tan \alpha \simeq \frac{v_\rho}{v_\eta}.$$  

(A23)

$$m_{H_1^0}^2 = \frac{\lambda_1}{2} v_\rho^2 + \frac{v_\rho^2 + v_\eta^2}{2 \sqrt{2} v_\eta v_\rho} f \chi - \frac{1}{2} \left[ \lambda_1 v_\rho^2 + \frac{1}{\sqrt{2}} f \chi \left( \frac{v_\eta}{v_\rho} - \frac{v_\rho}{v_\eta} \right) \right]^2 + \left( \lambda_5 v_\eta v_\rho - \sqrt{2} f \chi \right)^2$$

$$\simeq \frac{\lambda_1}{2} v_\rho^2 + \frac{1}{2} \left[ 2 \lambda_5 - \lambda_1 \left( 1 - \frac{v_\rho^2}{v_\eta^2} \right) \right] \frac{v_\rho^2 v_\eta^2}{v_\rho^2 + v_\eta^2},$$

(A24)

$$m_{H_1^0}^2 = \frac{\lambda_1}{2} v_\rho^2 + \frac{v_\rho^2 + v_\eta^2}{2 \sqrt{2} v_\eta v_\rho} f \chi + \frac{1}{2} \left[ \lambda_1 v_\rho^2 + \frac{1}{\sqrt{2}} f \chi \left( \frac{v_\eta}{v_\rho} - \frac{v_\rho}{v_\eta} \right) \right]^2 + \left( \lambda_5 v_\eta v_\rho - \sqrt{2} f \chi \right)^2$$

$$\simeq \frac{v_\rho^2 + v_\eta^2}{\sqrt{2} v_\eta v_\rho} f \chi + \frac{\lambda_1}{2} v_\rho^2 + \frac{\lambda_1}{2} \left( 1 - \frac{v_\rho^2}{v_\eta^2} \right) \frac{v_\rho^2 v_\eta^2}{v_\rho^2 + v_\eta^2},$$

(A25)

where we have used the Eq. (A7) and the fact that $f \simeq v_\chi \gg v_\eta, v_\rho$.

Therefore, the physical neutral scalar mass eigenstates contained in the squared mass matrix $M^2_{Na}$ are given by:

$$\begin{pmatrix} H_1^0 \\ h_1^0 \end{pmatrix} \simeq \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \xi_\rho \\ \xi_\eta \end{pmatrix},$$

$$H_3^0 \simeq \xi_\chi.$$  

(A26)

The matrix $\tilde{M}^2_{Nb}$ is diagonalized by a rotation matrix is diagonalized by a rotation matrix:

$$S = \begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix},$$

(A27)

which satisfies:

$$S^T \tilde{M}^2_{Nb} S = \begin{pmatrix} -m_{H_2^0}^2 & 0 \\ 0 & m_{H_2^0}^2 \end{pmatrix},$$

(A28)

where
\[ \tan 2\varphi = \frac{2 (2\lambda_4 v_\eta v_\chi + \sqrt{2} f v_\rho)}{2\lambda_4 (v_\eta^2 - v_\chi^2) + \frac{1}{\sqrt{2}} f v_\rho \left( \frac{v_\eta}{v_\chi} - \frac{v_\chi}{v_\eta} \right)} \approx 0, \quad (A29) \]

\[ m_{G_2}^2 = 0, \quad -m_{H_2}^2 = 2\lambda_4 (v_\eta^2 + v_\chi^2) + \frac{1}{\sqrt{2}} f v_\rho \left( \frac{v_\eta}{v_\chi} + \frac{v_\chi}{v_\eta} \right) f v_\rho = -\frac{1}{\sqrt{2}} \left( \frac{v_\eta}{v_\chi} + \frac{v_\chi}{v_\eta} \right) f v_\rho, \quad (A30) \]

where we have used Eq. (A7) and the fact that \( f \approx v_\chi \gg v_\eta, v_\rho \), so that the physical neutral scalar mass eigenstates contained in the matrix \( \tilde{M}_{N_6}^2 \) are given by:

\[ \begin{pmatrix} H_0^0 \\ G_2^0 \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix} \begin{pmatrix} \eta_3^0 \\ \chi_1 \end{pmatrix}, \quad (A31) \]

Now we apply a transformation to the matrix \( M_{N_6}^2 \) as follows:

\[ \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ \sin \beta & -\cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{v_\chi + \sqrt{2} f v_\rho}{v_\eta} \\ \frac{v_\eta}{v_\chi} & v_\eta & \frac{v_\chi + \sqrt{2} f v_\rho}{v_\eta} \\ \frac{v_\eta}{v_\chi} & v_\eta & \frac{v_\chi + \sqrt{2} f v_\rho}{v_\eta} \end{pmatrix} \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ \sin \beta & -\cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{v_\chi + \sqrt{2} f v_\rho}{v_\eta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A32) \]

where

\[ \tan 2\beta = \frac{2 v_\chi v_\rho}{v_\eta^2 - v_\chi^2}, \quad \tan \beta = \frac{v_\rho}{v_\eta}. \quad (A33) \]

Besides that, the following relation is fulfilled:

\[ \begin{pmatrix} \cos \varpi & 0 & \sin \varpi \\ 0 & 1 & 0 \\ \sin \varpi & 0 & -\cos \varpi \end{pmatrix} \begin{pmatrix} \frac{v_\chi (v_\eta^2 + v_\rho^2)}{v_\eta v_\rho} & 0 & \sqrt{v_\rho^2 + v_\eta^2} \\ 0 & 0 & 0 \\ \sqrt{v_\rho^2 + v_\eta^2} & 0 & \frac{v_\chi (v_\eta^2 + v_\rho^2)}{v_\eta v_\rho} \end{pmatrix} \begin{pmatrix} \cos \varpi & 0 & \sin \varpi \\ 0 & 1 & 0 \\ \sin \varpi & 0 & -\cos \varpi \end{pmatrix} = \begin{pmatrix} \frac{v_\chi v_\eta + v_\rho v_\varpi}{v_\eta v_\rho} + \frac{v_\rho v_\eta + v_\varpi v_\chi}{v_\eta v_\chi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A34) \]

where

\[ \tan 2\varpi = \frac{2 \sqrt{v_\rho^2 + v_\eta^2}}{\frac{v_\chi (v_\eta^2 + v_\rho^2)}{v_\eta v_\rho} - \frac{v_\rho v_\eta + v_\varpi v_\chi}{v_\eta v_\chi}}. \quad (A35) \]

Therefore, the matrix \( M_{N_6}^2 \) is diagonalized by a rotation matrix \( P \), according to:

\[ P^T M_{N_6}^2 P = \begin{pmatrix} \frac{v_\chi v_\eta + v_\rho v_\varpi}{v_\eta v_\rho} + \frac{v_\rho v_\eta + v_\varpi v_\chi}{v_\eta v_\chi} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} \cos \beta & \sin \beta & \cos \beta \sin \varpi \\ \cos \varpi & -\cos \varpi & \sin \varpi \sin \varpi \\ 0 & -\cos \varpi & \sin \varpi \sin \varpi \end{pmatrix}, \quad (A36) \]

so that the physical neutral scalar mass eigenstates contained in the matrix \( M_{N_6}^2 \) are given by:

\[ \begin{pmatrix} A_0^0 \\ G_1^0 \\ G_3^0 \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta & 0 \\ \sin \beta & -\cos \beta & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \zeta_\rho \\ \zeta_\eta \\ \zeta_\chi \end{pmatrix}, \quad (A37) \]
where we have taken into account that $\kappa \approx 0$ since $v_\chi \gg v_\eta, v_\rho$.

The masses of these physical neutral scalar mass eigenstates contained in the matrix $M_{Nc}^2$ are:

$$m_{C_0^1}^2 = m_{C_0^2}^2 = 0, \quad m_{A_0}^2 = \frac{f}{\sqrt{2}} \left( \frac{v_\chi v_\eta}{v_\eta} + \frac{v_\rho v_\chi}{v_\rho} \right).$$  \hspace{1cm} (A38)

Now, concerning the charged scalar sector, we expand the scalar potential up to quadratic terms of the charged fields and use the relations given by Eqs. (A5)-(A6), obtaining the following mass terms for the charged scalars:

$$-\mathcal{L}_{mass} = \frac{\lambda_6}{2} \left[ v_\eta^2 \rho_1^+ \rho_1^- + v_\chi^2 \rho_2^+ \rho_3^- + v_\rho^2 \eta_2^+ \eta_2^- + v_\rho^2 \chi_2^+ \chi_2^- \right] + \frac{\sqrt{2} f v_\chi v_\rho}{v_\rho} \left( \rho_1^+ \rho_1^- + \rho_3^+ \rho_3^- \right)$$

$$+ \frac{\sqrt{2} f v_\rho v_\chi}{v_\rho} \eta_2^+ \eta_2^- + \frac{\sqrt{2} f v_\rho v_\eta}{v_\rho} \chi_2^+ \chi_2^- + \frac{\lambda_6}{2} \left[ v_\chi v_\rho \left( \rho_1^+ \chi_2^- + \rho_3^- \chi_2^+ \right) + v_\eta v_\rho \left( \rho_1^+ \eta_2^- + \rho_3^- \eta_2^+ \right) \right]$$

$$+ \sqrt{2} f v_\eta \left( \rho_1^+ \chi_2^- + \rho_3^- \chi_2^+ \right) + \sqrt{2} f v_\chi \left( \rho_1^+ \eta_2^- + \rho_3^- \eta_2^+ \right).$$  \hspace{1cm} (A39)

Therefore, the squared mass matrix for the charged scalar fields in the basis $\eta_2^+, \rho_1^+, \chi_2^+, \rho_3^+$ is given by:

$$M_C^2 = \begin{pmatrix}
\frac{\lambda_6}{2} v_\rho^2 + \frac{\sqrt{2} f v_\rho v_\chi}{v_\rho} & \frac{\lambda_6}{2} v_\eta v_\rho + \frac{\sqrt{2} f v_\rho}{v_\rho} & 0 & 0 \\
\frac{\lambda_6}{2} v_\eta v_\rho + \frac{\sqrt{2} f v_\chi}{v_\rho} & \frac{\lambda_6}{2} v_\eta^2 + \frac{\sqrt{2} f v_\eta v_\rho}{v_\rho} & 0 & 0 \\
0 & 0 & \frac{\lambda_6}{2} v_\eta^2 + \frac{\sqrt{2} f v_\eta v_\rho}{v_\rho} & \frac{\lambda_6}{2} v_\chi v_\rho + \frac{\sqrt{2} f v_\eta}{v_\rho} \\
0 & 0 & \frac{\lambda_6}{2} v_\chi v_\rho + \frac{\sqrt{2} f v_\eta}{v_\rho} & \frac{\lambda_6}{2} v_\chi^2 + \frac{\sqrt{2} f v_\chi v_\rho}{v_\rho}
\end{pmatrix}.$$  \hspace{1cm} (A40)

Hence, the physical charged scalar eigenstates are given by:

$$\begin{pmatrix}
H_1^+ \\
G_1^+
\end{pmatrix} = \begin{pmatrix}
\cos \nu & \sin \nu \\
\sin \nu & -\cos \nu
\end{pmatrix} \begin{pmatrix}
\rho_1^+ \\
\eta_2^+
\end{pmatrix}, \quad \begin{pmatrix}
H_2^+ \\
G_2^+
\end{pmatrix} = \begin{pmatrix}
\cos \gamma & \sin \gamma \\
\sin \gamma & -\cos \gamma
\end{pmatrix} \begin{pmatrix}
\rho_3^+ \\
\chi_2^+
\end{pmatrix},$$  \hspace{1cm} (A41)

where:

$$\tan 2\nu = \frac{\lambda_6 v_\eta v_\rho + 2 \sqrt{2} f v_\chi}{\frac{\lambda_6}{2} \left( v_\eta^2 - v_\rho^2 \right) + \sqrt{2} \left( \frac{v_\eta}{v_\rho} - \frac{v_\rho}{v_\eta} \right) f v_\chi} \approx \frac{2 v_\eta v_\rho}{v_\eta^2 - v_\rho^2} = \tan 2\beta,$$  \hspace{1cm} (A42)

$$\tan 2\gamma = \frac{\lambda_6 v_\chi v_\rho + 2 \sqrt{2} f v_\eta}{\frac{\lambda_6}{2} \left( v_\chi^2 - v_\rho^2 \right) + \sqrt{2} f v_\eta \left( \frac{v_\chi}{v_\rho} - \frac{v_\rho}{v_\chi} \right)} \approx \frac{2 v_\chi v_\rho}{v_\chi^2 - v_\rho^2},$$  \hspace{1cm} (A43)

and the masses of the physical charged scalars are:

$$m_{G_1^+}^2 = m_{G_2^+}^2 = 0, \quad m_{H_1^+}^2 = \frac{\lambda_6}{2} \left( v_\eta^2 + v_\rho^2 \right) + \sqrt{2} \left( \frac{v_\rho}{v_\eta} + \frac{v_\eta}{v_\rho} \right) f v_\chi, \quad m_{H_2^+}^2 = \frac{\lambda_6}{2} \left( v_\chi^2 + v_\rho^2 \right) + \sqrt{2} \left( \frac{v_\rho}{v_\chi} + \frac{v_\chi}{v_\rho} \right) f v_\eta.$$  \hspace{1cm} (A44)
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