RELATIVE K-POLYSTABILITY OF PROJECTIVE BUNDLES OVER A CURVE

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Abstract. Let $\mathbb{P}(E)$ be the projectivization of a holomorphic vector bundle $E$ over a compact complex curve $C$. We characterize the existence of an extremal Kähler metric on $\mathbb{P}(E)$ in terms of relative K-polystability and the fact that $E$ decomposes as a direct sum of stable bundles.

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1. Introduction

Let $M = \mathbb{P}(E)$ be the complex manifold underlying the total space of the projectivization of a holomorphic vector bundle $E \to C$ over a compact complex curve $C$. In this paper, we are interested to understand when $\mathbb{P}(E)$ admits an extremal Kähler metric in the sense of Calabi [8], and if such a special metric does exist, which Kähler classes of $\mathbb{P}(E)$ admit extremal Kähler metrics. A Kähler class $\Omega$ endowed with a Kähler metric is refereed to as an extremal class.

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By the openness of the extremal Kähler classes on $M$ (see [22, 16]) and using the fact that $H^{2,0}(M, \mathbb{C}) = 0$, without loss of generality we can restrict our study to rational classes. Furthermore, as extremal Kähler classes are invariant under a positive rescaling, we can even only consider integral classes, i.e. $\Omega = 2\pi c_1(L)$ for a positive line bundle $L$ on $M$. Such bundles on $M = \mathbb{P}(E) \to C$ are of the form $L = L_{q,p} = O(q)\mathbb{P}(E) \otimes O(p)C$, $q, p \in \mathbb{Z}$, $q > 0$, where $O(1)C$ denotes (the pull-back to $M$) of any holomorphic line bundle over $C$ of degree 1. We note that $L = L_{q,p}$ becomes positive for $p/q \gg 0$ and thus defines a polarization on $M$.

Before discussing the case of extremal metrics, let us recall some results about the particular case of constant scalar curvature Kähler metrics (CSC Kähler for short). In the case of $M = \mathbb{P}(E) \to C$, the existence of CSC Kähler metrics and its link to K-polystability in the sense of [13, 41] are completely settled thanks to the works [3, 32].

**Theorem 1.** [3, 32] Let $E$ a holomorphic vector bundle and let $M = \mathbb{P}(E) \to C$ be its projectivisation. The following three conditions are equivalent:

(i) $M$ admits a CSC Kähler metric in any class $2\pi c_1(L)$;
(ii) $M$ is K-polystable for any polarization $\mathcal{L}$;
(iii) $E$ is polystable, i.e. decomposes as the sum of stable bundles of same slopes;

**Remark 1.1.** The notion of stability for bundles refers here to the classical notion of Mumford–Takemoto stability. The equivalence (iii) $\iff$ (i) is established in [3, Theorem 1] when the base $C$ has genus $g \geq 2$, and in [32, Theorem 5.13] when $g \geq 1$ (by using the result of [33]). This equivalence also holds true for $g = 0$ as a consequence of the Lichnerowicz–Matsushima theorem, by noting that in this case $E$ splits as a direct sum of line bundles over $C = \mathbb{C}P^1$. The equivalence (ii) $\iff$ (iii) follows by [32, Theorem. 5.13], by noting again that the case $g = 0$ can be treated apart by observing that the usual Futaki invariant associated to $2\pi c_1(L)$ vanishes if and only if $E$ is the sum of line bundles over $\mathbb{C}P^1$ of the same degree.

In [3], it was introduced the following conjecture in view of the classification of projective bundles over a curve, admitting extremal metrics.

**Conjecture 1.** [3] Let $E$ a holomorphic vector bundle and let $M = \mathbb{P}(E) \to C$ be its projectivisation. The following three conditions are equivalent:

(a) $M$ admits an extremal Kähler metric;
(b) $M$ is relative K-polystable for a certain polarization $\mathcal{L}$;
(c) $E$ decomposes as a direct sum of stable bundles.

We refer to [36] for the notion of relative K-polystability. Our main result is the following theorem.

**Theorem 2.** Conjecture 1 is true.

We do now some comments. Firstly, Conjecture 1 is almost optimal in view of the existence problem of extremal Kähler metric. Actually in the light of Theorem 1, it is natural to ask if Conjecture 1 could be completed by

(d) $M$ admits an extremal Kähler metric in any Kähler class;
(e) $M$ is relative K-polystable polarization for any polarization $\mathcal{L}$;
But this does not hold. In general, with $E$ direct sum of stable bundles, $M$ may also admit polarizations which are not relatively K-polystable, nor extremal, see e.g. [3, Proposition 5] or [2, Theorem 6]. To strengthen Conjecture 1, it would be natural to ask that conditions (a) and (b) occur precisely for the same classes. This is precisely the Yau–Tian–Donaldson conjecture extended to the setting of extremal Kähler metrics by Székelyhidi [36].

**Conjecture 2** (Relative Yau-Tian-Donaldson conjecture). For any polarization $\mathcal{L}$ on $M = \mathbb{P}(E) \to C$, the following two conditions are equivalent

(a’) $2\pi c_1(\mathcal{L})$ is extremal;

(b’) $(M, \mathcal{L})$ is relatively K-polystable.

By virtue of [35] (see also [10, Theorem 1.2]), we know that (a’) implies (b’) on any polarized variety. In direction of this conjecture, Theorem 2 combined with some previous results allows us to establish the following

**Corollary 1.** Let $E \to C$ a holomorphic vector bundle over a complex curve $C$ and write $E = U_1 \oplus \cdots \oplus U_s$ as a direct sum of indecomposable sub-bundles. The relative Yau–Tian–Donaldson conjecture is true for a polarization $L_{q,p}$ on $\mathbb{P}(E)$ in the following cases:

1. $s = 1$ or $s = 2$;
2. $s \geq 3$ and $E$ is polystable;
3. $s \geq 3$ and one of the $U_i$ is unstable;
4. $s \geq 3$ and $p/q$ is large enough.

**Remark 1.2.** An example from [2] strongly suggests that for reaching the remaining cases ($s \geq 3$, $U_i$ stables of different slopes and $p/q$ not large), one would need to enforce the notion of relative K-polystability of $(M, \mathcal{L})$. This would require to consider test-configurations with “irrational” line bundles (i.e. formal tensor powers of line bundles with real coefficients). There are two current approaches to this. The first is the notion of Kähler relative K-stability, which originates in [32] and was recently developed in [14, 12, 10]. The other one is the notion of uniform relative K-stability, as introduced in [39, 6, 11].

Eventually, one would expect that some of the results discussed above can be extended to projective holomorphic vector bundles $M = \mathbb{P}(E) \to B$ over a base $(B, L_B)$ which itself is a polarized variety admitting an extremal Kähler metric in $2\pi c_1(L_B)$. This is evidenced in the works [20, 7, 25, 21].

We sum up now the general structure of the paper. In Section 2 we present the required material about (relative) Donaldson–Futaki invariant. In Section 3, we construct a test-configuration and compute by two different ways the associated relative Donaldson–Futaki invariant (Sections 3.3, 3.4 and 3.5). Our first approach is based on differential-geometric ingredients from [3] and has the advantage to apply to any Kähler class (rational or not), thus evidencing the Kähler feature of the K-stability in line with the recent work [10]. The second approach is algebro-geometric, following the original arguments in [32], and has the merit to cover the case when the genus of $C$ equals 1. The proof of our main result is then given at the end of Section 3.8. The proof of Corollary 1 is obtained in Section 3.9. The appendix (Section 4) contains certain technical results.

2. Preliminaries

2.1. The Donaldson–Futaki invariant. Suppose $(M, \mathcal{L})$ is a polarized variety endowed with a $C^\infty$ action $\rho$ with a lift to $\mathcal{L}$. Let $A_\rho^L$ be the infinitesimal generator of the induced
linear action on the vector space $H_k := H^0(M, \mathcal{L}^k)$ of holomorphic sections of $\mathcal{L}^k$, and denote by

$$d_k := \dim H_k, \quad w_k(\rho) := \text{Tr} A_k^\rho.$$  

It turns out (by using Riemann–Roch) that for $k \gg 0$, $d_k$ and $w_k$ are polynomials

$$d_k = a_0 k^n + b_0 k^{n-1} + O(k^{n-2}), \quad w_k(\rho) = a(\rho) k^{n+1} + b(\rho) k^n + O(k^{n-1}).$$  

We then define

**Definition 2.1.** The algebraic Donaldson–Futaki invariant of $(M, \mathcal{L}, \rho)$ is

$$\bar{\mathfrak{g}}^{\text{alg}}(\rho) := \frac{a_0 b(\rho) - a(\rho) b_0}{a_0}.$$  

We shall use this definition in the case when $(M, \mathcal{L})$ is a smooth polarized variety. We notice that there are different sign choices in the literature for the infinitesimal generator of the induced linear action on $H_k$, thus introducing a sign difference in the definition of the (algebraic) Donaldson–Futaki invariant, see e.g. [38, p. 141]. We shall use in this paper the following convention, which agrees with [19] and, up to a positive constant, with [38, (7.14)].

**Definition 2.2.** Let $(M, \mathcal{L})$ be a smooth polarized variety endowed with a $\mathbb{C}^\times$ action $\rho$ with a lift to $\mathcal{L}$, denoted by $\hat{\rho}$. We let $e^{\sqrt{-1}t}; t \in \mathbb{R}$ be the circle subgroup of $\mathbb{C}^\times$. Then, the infinitesimal generator $A^\rho$ for the action of $\hat{\rho}$ on the space of smooth sections $\Gamma(\mathcal{L})$ is defined to be

$$(A^\rho(s))(x) := \sqrt{-1} \frac{d}{dt}_{|t=0} \left( \hat{\rho}(e^{\sqrt{-1}t})(s(\rho(e^{-\sqrt{-1}t}}(x))) \right), \quad s \in \Gamma(\mathcal{L}), x \in M.$$  

It is shown in [13] (see also [19, 38]) that the above definition agrees, up to a factor of $\frac{1}{4(2\pi)^n}$, with the differential geometric definition of the Futaki invariant [17], i.e.

$$4(2\pi)^n \bar{\mathfrak{g}}^{\text{alg}}(\rho) = \bar{\mathfrak{g}}_{\text{Fut}}(K_\rho) = \int_M \text{Scal}_g f_\rho v_g,$$

where $n = \dim \mathbb{C} M$, $K_\rho$ is the (real) holomorphic vector field on $M$ induced by the action of $S^1$ via $\rho$, $\Omega = \mathfrak{g}c_1(\mathcal{L})$ is the Kähler class determined by $\mathcal{L}$, $g$ is any $S^1$-invariant Kähler metric in $\Omega$, $\text{Scal}_g$ is its scalar curvature, and $f_\rho$ denotes the Killing potential of mean value zero for $K_\rho$ with respect to $g$.

**2.2. Test configurations and K-polystability.** Recall the following definitions from [40] and [13].

**Definition 2.3.** Let $(M, \mathcal{L})$ be a normal polarized variety. A test configuration for $(M, \mathcal{L})$ is a normal variety $\mathcal{M}$ endowed with a line bundle $\mathcal{P}$ together with

(i) a $\mathbb{C}^\times$ action $\rho$ on $\mathcal{M}$ with a lift to $\mathcal{P}$;

(ii) a $\mathbb{C}^\times$ equivariant map $\pi_\mathbb{C} : \mathcal{M} \to \mathbb{C}$ where $\mathbb{C}^\times$ acts on a standard way on $\mathbb{C}$, such that $\pi_\mathbb{C} : \mathcal{M} \to \mathbb{C}$ is a flat family with $\mathcal{P}$ being relatively ample and, for any $t \neq 0$, the fibre $(\mathcal{M}_t, \mathcal{P}_{\mathcal{M}_t})$ of $\pi_\mathbb{C}$ is isomorphic to $(M, \mathcal{L})$ for some fixed $r \in \mathbb{N}$. The number $r$ is called exponent of the test configuration.

A test configuration is said to be a product configuration if $\mathcal{M} = M \times \mathbb{C}$ and $\rho$ is given by a $\mathbb{C}^\times$ action on $M$ (and scalar multiplication on $\mathbb{C}$).

Notice that for any test configuration $(\mathcal{M}, \mathcal{P}, \rho)$ for $(M, \mathcal{L})$, $\rho$ induces a $\mathbb{C}^\times$ action on the central fibre $(\mathcal{M}_0, \mathcal{L}_0)$ (which we still denote by $\rho$). With our convention in Definition 2.2, we then have
Definition 2.4. [13] The Donaldson–Futaki invariant of the test configuration \((M, \mathcal{L})\) is the Donaldson–Futaki invariant \(\mathfrak{F}^{alg}(\rho)\) of its central fibre \((M_0, \mathcal{L}_0)\). The variety \((M, \mathcal{L})\) is said to be K-polystable (resp. K-stable) if the Donaldson–Futaki invariant of any normal test configuration for \((M, \mathcal{L})\) is non-negative, and equal to zero if and only if the test configuration is a product configuration (resp a trivial test configuration).

This implies in particular that the Donaldson–Futaki invariant of any \(\mathbb{C}^\times\) action on \((M, \mathcal{L})\) must be zero, so the notion is adapted to the study of cscK (in particular Kähler–Einstein) metrics.

2.3. Relative K-polystability. In order to account for the obstructions related to the extremal Kähler metrics, G. Székelyhidi has introduced relative version of the above notions as follows.

Suppose \((M, \mathcal{L})\) is a polarized variety endowed with two commuting \(\mathbb{C}^\times\) actions \(\rho_1\) and \(\rho_2\). We first define an inner product \(\langle \rho_1, \rho_2 \rangle\) for such actions. For that, we take lifts of \(\rho_1\) and \(\rho_2\) to \(\mathcal{L}\) and consider the infinitesimal generators \(A_k^{\rho_1}\) and \(A_k^{\rho_2}\) of the actions on \(H_k\). Then for \(k\) sufficiently large,

\[
\text{Tr}(A_k^{\rho_1} A_k^{\rho_2}) = a(\rho_1, \rho_2)k^{n+2} + O(k^{n+1})
\]

is a polynomial of degree at most \(n + 2\) and we let

Definition 2.5. The inner product of two commuting \(\mathbb{C}^\times\) actions \(\rho_1\) and \(\rho_2\) on \((M, \mathcal{L})\) is defined by

\[
\langle \rho_1, \rho_2 \rangle := a(\rho_1, \rho_2) - \frac{a(\rho_1)a(\rho_2)}{a_0}.
\]

Notice that \(\langle \rho_1, \rho_2 \rangle\) is the leading coefficient of the expansion in \(k\) of \(\text{Tr}(A_k^{\rho_1} A_k^{\rho_2})\) of the traceless parts of \(A_k^{\rho_i}\), so it is independent of the choice of liftings. It is shown in [36] that when \(M\) is smooth, the above definition agrees up to a factor of \(1/(2\pi)^n\), with the Futaki–Mabuchi bilinear form on Killing potentials, i.e. if for any Kähler metric \(g\) in \(\Omega = 2\pi c_1(\mathcal{L})\) which is invariant under the \(S^1\) actions corresponding to \(\rho_1\) and \(\rho_2\) we denote by \(f_{\rho_1}\) and \(f_{\rho_2}\) the Killing potentials of zero mean with respect to \(g\), corresponding the induced Killing vector fields, then

\[
\langle \rho_1, \rho_2 \rangle = \frac{1}{(2\pi)^n} \int_M f_{\rho_1} f_{\rho_2} v_g.
\]

We shall next fix a maximal torus \(\mathbb{T}^\ell\) in the automorphism group \(\text{Aut}(M, \mathcal{L})\) and denote by \(\rho_1, \ldots, \rho_\ell\) the \(\mathbb{C}^\times\) corresponding to the \(S^1\) generators of \(\mathbb{T}^\ell\).

Definition 2.6. The extremal \(\mathbb{C}^\times\) action \(\rho_{\text{ex}}\) of \((M, \mathcal{L}, \mathbb{T}^\ell)\) is a \(\mathbb{C}^\times\) subgroup of the complexification \(\mathbb{T}^\ell\) defined by the system of \(\ell\) linear conditions

\[
\langle \rho_{\text{ex}}, \rho_i \rangle = \mathfrak{F}^{alg}(\rho_i).
\]

Definition 2.7. Let \(\rho_0\) be a distinguished \(\mathbb{C}^\times\) action on the polarized manifold \((M, \mathcal{L})\). The \(\rho_0\)-relative Donaldson–Futaki invariant of \((M, \mathcal{L}, \rho_0)\) is defined for any \(\mathbb{C}^\times\) action \(\rho\) commuting with \(\rho_0\) by

\[
\mathfrak{F}^{alg}_{\rho_0}(\rho) := \mathfrak{F}^{alg}(\rho) - \frac{\langle \rho_0, \rho \rangle}{\langle \rho_0, \rho_0 \rangle} \mathfrak{F}^{alg}(\rho_0).
\]

We now apply the above to a test configuration.
Definition 2.8. [36, 34] A test configuration \((\mathcal{M}, \mathcal{P}, \rho)\) for \((M, \mathcal{L})\) is compatible with a fixed maximal torus \(T^d \subset \text{Aut}(M, \mathcal{L})\) if there is a \(T^d\) action on \((\mathcal{M}, \mathcal{P})\), commuting with \(\rho\) and preserving \(\pi_c : \mathcal{M} \to \mathbb{C}\), which induces the trivial action on \(\mathbb{C}\), and restricted to \((M_t, \mathcal{P}|_{M_t})\) for \(t \neq 0\) coincides with the original \(T^d\) action under the isomorphism with \((M, L)\) via \(\rho\).

In this case, we have an induced action of \(T^d\) on the central fibre \(M_0\), and we denote by \(\rho^M_{\text{ex}}\) the \(\mathbb{C}^x\) action on \(M_0\) corresponding to the extremal \(\mathbb{C}^x\) action \(\rho_{\text{ex}}\) on \((M, L, T^d)\).

Now, the relative Donaldson–Futaki invariant of \((M, L, T^d)\) is defined to be the \(\rho^M_{\text{ex}}\)-relative Donaldson–Futaki invariant of \((M_0, L_0)\) of the induced \(\mathbb{C}^x\) action \(\rho\).

A polarized variety manifold \((M, \mathcal{L})\) is relatively K-polystable (resp. K-stable) with respect to a maximal torus \(T^d \subset \text{Aut}(M, \mathcal{L})\) if the relative Donaldson–Futaki invariant of any normal test configuration \((\mathcal{M}, \mathcal{P}, \rho)\) for \((M, \mathcal{L})\) compatible with \(T^d\) is non-negative, and equal to zero if and only if \((M, \mathcal{P})\) is a product configuration (resp. a trivial configuration).

Remark 2.1. More recently, following the works of Wang [42] and Odaka [30, 29], a topological interpretation of the Donaldson–Futaki invariant was given in terms of an integration over the total space of a given test configuration. Among other applications, this point of view led to the definition of the stronger notion of Kähler (relative) K-polystability in [14, 12, 10], where one also takes in consideration the sign of the (relative) Donaldson–Futaki over “irrational” polarizations \(\mathcal{L}\) of \(M\). We shall not use this point of view explicitly in this paper. However, the Reader could notice that our differential-geometric approach to the computation of the relative Donaldson–Futaki invariant is well-adapted to deal with the Kähler relative K-polystability in the sense of [10].

3. Proof of Theorem 2 and Corollary 1

One direction of Theorem 2, namely \((c) \implies (a)\), follows from the facts that an extremal Kähler metric exists in any polarizations \(\mathcal{L} = \mathcal{L}_{q,p}\) with \(p/q \gg 0\) (see [3, Theorem 3] or [7]). Moreover, \((a) \implies (b)\) is a consequence of the general result of [35], i.e the existence of an extremal Kähler metric in \(2\pi c_1(\mathcal{L})\) implies that \((M, \mathcal{L})\) is relative K-polystable. We shall thus focus on establishing \((b) \implies (c)\). As any vector bundle \(E\) over \(\mathbb{CP}^1\) decomposes as the direct sum of line bundles (which are automatically stable), we shall also assume from now on that the base \(C\) has genus \(g \geq 1\).

Assumption 1. \(C\) is a compact complex curve of genus \(g \geq 1\).

As the cohomology \(H^2(M, \mathbb{R})\) of \(M = \mathbb{P}(E)\) is 2-dimensional, up to rescaling, the Kähler cone of \(M\) is 1-dimensional. Similarly, it is well-known that any holomorphic line bundle \(\mathcal{L}\) on \(M\) can be written as

\[
\mathcal{L} = \mathcal{L}_{q,p} := \mathcal{O}(q)_{\mathbb{P}(E)} \otimes \mathcal{O}(p)_C, \quad p, q \in \mathbb{Z},
\]

where, as usual, \(\mathcal{O}(1)_{\mathbb{P}(E)}\) denotes the anti-tautological line bundle of \(E\) (defined on \(\mathbb{P}(E)\)) and \(\mathcal{O}(1)_C\) stands for (the pull back to \(M\) of) any degree 1 holomorphic line bundle over \(C\), see for instance [26, Section 3] for details. If \(\Omega = 2\pi c_1(\mathcal{L}_{q,p})\) is a Kähler class, evaluation over the fibre of \(\mathbb{P}(E)\) shows that \(q > 0\), thus any polarization on \(M = \mathbb{P}(E)\) can be written as \(\mathcal{L} = \mathcal{L}_{q,p}\) with \(q > 0\) (notice that \(\mathcal{L}_{q,p}\) becomes positive when \(p/q \gg 0\)). Clearly, both properties of existence of extremal Kähler metric and relative K-polystability of the polarization \(\mathcal{L}\) on \(M\) are invariant under taking tensor powers \(\mathcal{L}^{\otimes k} = \mathcal{L}^k\). As it will turn out in our specific situation, the same phenomena happens under changing the polarization
\( \mathcal{O}(1)_C \) of the base curve \( C \). It will be useful to normalize the choice of such polarizations, by introducing the following

**Notation 1.** We let \( \mathcal{L}_m := \mathcal{O}(1)_{\mathbb{P}(E)} \otimes \mathcal{O}(m)_C, m \in \mathbb{Q} \) denote the class of holomorphic line bundles \( \mathcal{L}_{q,p} \) over \( M \) such that \( q > 0 \) and \( p/q = m \).

In all of the arguments below involving \( \mathcal{L}_m \) one can take some (and hence any) line bundle \( \mathcal{L}_{q,p} \) as above.

We denote by \( \text{Aut}^{red}(M) \) the reduced automorphism group of \( M = \mathbb{P}(E) \to C \) (see e.g. [19]) whose Lie algebra \( \mathfrak{h}^{red}(M) \) consists of all holomorphic vector fields with zero on \( M \), and let \( \text{Aut}^{red}_C(M) \) be the subgroup of \( \text{Aut}^{red}(M) \) of elements which preserve \( C \) (i.e. act on each fibre), with Lie algebra \( \mathfrak{h}^{red}_C(M) \). As \( M \) is a locally trivial holomorphic \( \mathbb{C}P^{n-1} \)-fibration over \( C \), we have an exact sequence of Lie algebras

\[
0 \to \mathfrak{h}^{red}_C(M) \to \mathfrak{h}^{red}(M) \to \mathfrak{h}^{red}(C) \to 0,
\]

where \( \mathfrak{h}^{red}(C) \) its the Lie algebra of holomorphic vector fields with zeroes on \( C \). Under the assumption \( g(C) \geq 1 \), we have \( \mathfrak{h}^{red}(C) = 0 \), so that \( \mathfrak{h}^{red}(M) = \mathfrak{h}^{red}_C(M) \). We let \( (\ell - 1) \) with \( \ell \geq 1 \) denote the rank of \( \text{Aut}^{red}(M) \) (which is also the rank of \( \text{Aut}^{red}_C(M) \) by the preceding). Thus, \( \ell \) equals the number of summands in the decomposition

\[
E = \bigoplus_{k=0}^{\ell-1} U_k.
\]

of \( E \) as direct sum of indecomposable holomorphic sub-bundles \( U_k \). We want to show that, in general, each \( U_k \) is stable when \( M \) is relative K-polystable with respect to the polarization \( \mathcal{L}_m \) of \( M \). Without loss, we deal with \( U_0 \) and assume \( \text{rk}(U_0) > 1 \).

### 3.1. Constructing a test configuration.

This construction follows [32, Remark 5.14] and [31, Section 3]. For each strict sub-bundle \( L \subset U_0 \subset E \), we consider the exact sequences of holomorphic vector bundles

\[
0 \to L \to U_0 \to F_0 \to 0,
\]

\[
0 \to L \to E \to F \to 0,
\]

where \( F_0 = U_0/L \) and \( F = E/L \cong F_0 \oplus \left( \bigoplus_{k=1}^{\ell-1} U_k \right) \). Thus, \( E \) is given by an element \( e \in \text{ext}^1(L,F) \), coming from an element (still denoted by \( e \)) of \( \text{ext}^1(L,F_0) \); as \( U_0 \) is indecomposable, \( e \neq 0 \), and one can consider the smooth family \( \mathcal{M} := (M_t,t), t \in \mathbb{C} \), where \( M_t := \mathbb{P}(E_t) \) and \( E_t \) is the extension of \( (L,F) \) corresponding to \( te \in \text{ext}^1(L,F_0) = H^1(M,L \otimes F_0^*) \) for \( t \in \mathbb{C} \). As explained in [31, Section 3.1], \( \mathcal{M} = \mathbb{P}(E) \to C \times \mathbb{C} \) is itself a complex ruled manifold, where \( E \) is a holomorphic vector bundle whose restriction to \( C \times \{t\} \) is \( E_t \). We denote by \( \pi_C : \mathcal{M} \to \mathbb{C} \) the natural holomorphic projection on the \( \mathbb{C} \)-factor. As \( E_t \cong E \) for \( t \neq 0 \), we have that \( \pi^{-1}_C(t) = M_t \cong M \), whereas

\[
\pi_C^{-1}(0) = M_0 = \mathbb{P} \left( \bigoplus_{k=0}^{\ell} V_k \right) \to C,
\]

where we have set \( V_0 := U_0/L, V_1 := L, V_k := U_{k-1}, k = 2, \ldots, \ell \). As shown in [31, Lemma 3.1.1], there is a natural \( \mathbb{C}^\times \) action \( \rho_L \) on \( \mathcal{M} \), making \( \pi_C \) equivariant with respect to the standard action on \( \mathbb{C}^\times \), and which induces a \( \mathbb{C}^\times \) action (still denoted by \( \rho_L \)) on the
central fibre $M_0$, given by the fibre-wise multiplication with $\lambda \in \mathbb{C}^\times$ on the factor $V_1 = L$ in the decomposition

$$V := \bigoplus_{k=0}^\ell V_k.$$ 

Given a polarization $\mathcal{L}_m = \mathcal{O}(1)_{\mathbb{P}(E)} \otimes \mathcal{O}(m)_C$ on $M$ (we can work with any line bundle representing $\mathcal{L}_m$, see Notation 1), consider on $\mathcal{M}$ the (class of rational) holomorphic line bundles $\mathcal{P}_m := \mathcal{O}(1)_{\mathbb{P}(E)} \otimes \mathcal{O}(r)_C$ which restricts to $\mathcal{L}_m^r = \mathcal{O}(1)_{\mathbb{P}(E_1)} \otimes \mathcal{O}(m)_C$ on each fibre $M_t = \pi_C^{-1}(t)$. The $\mathbb{C}^\times$ action of $\mathcal{L}_m$ induces an action of $\text{Aut}_0$ on the (class of rational) holomorphic line bundles $\mathcal{P}_m$ which restricts to $\mathcal{L}_m^r$. Furthermore, the holomorphic line bundle $\mathcal{L}_m^r$ induced on the central fibre $M_0$ must be at least semi-ample. As the condition for $\mathcal{L}_m$ to be ample on $\mathcal{M}$ is relatively open with respect to $m \in \mathbb{Q}$, it follows that $\mathcal{L}_m^r$ must be ample too. We thus conclude that $(\mathcal{M}, \rho_L, \mathcal{P}_m)$ defines a test-configuration for $(M, \mathcal{L}_m^r)$ the flatness of the morphism $\pi_C : (\mathcal{M}, \mathcal{P}_m) \to \mathbb{C}$ is a direct consequence of the surjectivity of $\pi_C$ and the fact that the central fibre is smooth). We finally notice that the rank of the reduced automorphism group of the central fibre $M_0$ is at least $\ell$, whereas the rank of the same group on $M_t$ is $(\ell - 1)$ for $t \neq 0$, showing that the test configuration $\mathcal{M}$ is normal and not a product configuration [24, 34]. We thus have established the following

**Lemma 3.1.** Given a completely decomposable vector bundle $E = \bigoplus_{k=0}^{\ell-1} U_k \to C$, a polarization $\mathcal{L}_m = \mathcal{O}(1)_{\mathbb{P}(E)} \otimes \mathcal{O}(m)_C$ on $M = \mathbb{P}(E)$ and a sub-bundle $L \subset U_0$, the data

$$(\pi_C : \mathcal{M} \to \mathbb{C}, \rho_L, \mathcal{P}_m)$$

define a normal test configuration for $(M = \mathbb{P}(E), \mathcal{L}_m)$ which is not a product configuration and with central fibre $(M_0 = \mathbb{P}(V), \mathcal{L}_m^r)$, where

$$V = \bigoplus_{k=0}^\ell V_k, \ V_0 := U_0/L, \ V_1 := L, \ V_k := U_{k-1}, \ k = 2, \ldots, \ell.$$

The induced $\mathbb{C}^\times$ action $\rho_L$ on $M_0$ is given by

$$\rho_L(\lambda) \cdot (x, [e_0, e_1, \ldots, e_\ell]) = (x, [e_0, \lambda e_1, \ldots, e_\ell]),$$

where $x$ is a point on $C$ and $(v_0, v_1, \ldots, v_\ell)$ is a vector in the fibre $V_x$ of $V$.

### 3.2. Relative Donaldson–Futaki invariant.

The central fibre $M_0$ is a smooth complex variety, endowed with a holomorphic action of the torus $T^\ell$, coming from the diagonal action of $T^{\ell+1}$ on $V = \bigoplus_{k=0}^\ell V_k$. We choose any Kähler metric $g$ on $M_0$ in the Kähler class $\Omega = 2\pi c_1(L_m^0)$, which is invariant under the action of $T^\ell$. The action of the subtorus $T^{\ell-1} \subset T^\ell$ by diagonal multiplications on the factors $V_2, \ldots, V_\ell$ extends to each fibre $M_t, t \neq 0$, and on $\mathcal{M}$. As $T^{\ell-1}$ is a maximal torus in the connected component of identity of $\text{Aut}_0^*(M_t)$ for any $t \neq 0$, it follows that the extremal vector field $K_{\text{ex}}$ of $(M_t, \Omega)$ belongs to $\text{Lie}(T^{\ell-1}) \subset \text{Lie}(T^\ell)$ and is independent of $t$ (as $M_t \cong M'_t$ via $\rho_L$ and the action of $T^{\ell-1}$ on $\mathcal{M}$ commutes with $\rho_L$). We shall denote this vector field by $K^M_{\text{ex}}$ and let $f^M_{\text{ex}}$ be the Killing potential of $K^M_{\text{ex}}$ of zero mean value with respect to $g$. As the central fibre $M_0$ is a smooth variety, the relative Donaldson–Futaki invariant is computed up to a positive
normalization factor by the differential-geometric quantity (see [36] or Section 2)

\[ \mathfrak{F}_{\rho_L}(\rho_L) = \mathfrak{F}(K_L) - \int_{M_0} \frac{f_L f^M_{\text{ex}}}{(f^M_{\text{ex}})^2} v_g \mathfrak{F}(K^M_{\text{ex}}) \]

(4)

\[ = \int_{M_0} \text{Scal}_g f_L v_g - \frac{\int_{M_0} (f_L f^M_{\text{ex}})}{\int_{M_0} (f^M_{\text{ex}})^2} v_g \int_{M_0} \text{Scal}_g f^M_{\text{ex}} v_g, \]

where \( K_L \) denotes the generator for the induced \( S^1 \) action by \( \rho_L \) (again a subgroup of \( T^\ell \)), \( f_L \) is its Killing potential of zero mean value with respect to \( g \). Of course, the r.h.s. of (4) is independent of the choice of the \( T^\ell \)-invariant Kähler metric \( g \) in \( \Omega \).

As explained in the proof of Lemma 3 in [3], one can extend the \( T^\ell \) invariant Kähler metric \((g, \omega)\) on \( M_0 = (M, J_0)\) to a smooth family of \( T^{\ell-1} \) invariant Kähler metrics \((g_t, \omega_t)\) on \( M_t = (M, J_t)\) (at least for \(|t| < \varepsilon\)) and then use the equivariant Moser lemma in order to find a \( T^{\ell-1} \) equivariant family of diffeomorphisms \( \Phi_t \) on \( M \), which send the complex structure \( J_t \) of \( M \cong M_t \) to a complex structure \( \tilde{J}_t \) on \( M \), compatible with the initial symplectic form \( \omega \). As \( \Phi_t \) commutes with the action of \( T^{\ell-1} \) and \( K^M_{\text{ex}} \in \text{Lie}(T^{\ell-1}) \), \( \Phi_t \) preserves \( K^M_{\text{ex}} \). In this symplectic setting, it is shown in [3, Lemma 2] (see also [23]) that \( f^M_{\text{ex}} \) can be obtained as the \( L^2 \)-projection of the scalar curvature of any \( T^{\ell-1} \) invariant Kähler metric compatible with \( \omega \) to the finite dimensional space of normalized hamiltonians for the \( T^{\ell-1} \) action on \((M, \omega)\). In particular, with respect to the initial metric \( g \), we have that \( f^M_{\text{ex}} \) coincides with the \( L^2 \)-projection \( \text{Scal}^{g_{\text{ex}}} \) of \( \text{Scal}_g \) to the space of normalized hamiltonians of \( T^{\ell-1} \subset T^\ell \). In particular, we have

\[ \int_{M_0} \text{Scal}_g f^M_{\text{ex}} v_g = \int_{M_0} (f^M_{\text{ex}})^2 v_g, \]

so that (4) becomes

(5)

\[ \mathfrak{F}_{\rho_L}(\rho_L) = \int_{M_0} \text{Scal}_g f_L v_g - \int_{M_0} (f_L \text{Scal}^{g_{\text{ex}}} v_g. \]

From this point of view, (5) can be entirely computed from the symplectic structure \( \omega \) on \( M_0 \), endowed with the hamiltonian action of \( T^\ell \). We thus have

**Lemma 3.2.** The r.h.s of (5) does not depend on the choice of an \( \omega \) compatible, \( T^\ell \) invariant Kähler metric \((g, J_g)\) on \((M, \omega)\) nor on the choice of a \( T^\ell \) invariant Kähler metric \( \tilde{g} \) on \((M, J_g)\) within the Kähler class \( \Omega = [\omega] \).

### 3.3. Generalized Calabi Ansatz.

When \( C \) is of genus \( g \geq 2 \), by using Lemma 3.2 and the Narasimhan–Ramanan approximation theorem [27], we can compute (5) with respect to an \( \omega \) compatible, \( T^\ell \) invariant complex structure \( \tilde{J} \) on \( M_0 \), corresponding to taking stable holomorphic structures on each \( V_k \), see [3, Lemma 2]. Furthermore, in this case, we can use the generalized Calabi Ansatz of [3] in order to choose a particularly simple metric \( g_c \) in the class \( \Omega = [\omega] \) on \((M, \tilde{J})\), which will make the computation of (5) explicit.

To simplify the notation, we shall assume throughout this section that \( M_0 = (M, J_0) \) is a ruled complex manifold

\[ M_0 = \mathbb{P} \left( \bigoplus_{k=0}^\ell V_k \right) \to C, \]

over a compact complex curve of genus \( g \geq 2 \), and \( V_k \) are stable vector bundle over \( C \). This is a special case of the semi-simple rigid toric fibre-bundles considered in [3], see Sect. 2.2 loc cit.
We introduce a family of Kähler metrics \((g_c, \omega_c)\) on \(M_0\), parametrized by a real constant \(c\), as follows: As each \(V_i\) is a stable and therefore projectively-flat bundle over \(C\), it admits a projectively-flat hermitian metric \(h_i\) whose Chern curvature is \(\mu(V_i)\text{Id} \otimes \omega_C\), where the topological constant
\[
\mu(V_i) := \frac{\deg(V_i)}{\text{rk}(V_i)} = \int_C c_1(V_i)/\text{rk}(V_i)
\]
is the slope of \(V_k\), and \(\omega_C\) is the Kähler form of the metric \(g_C\) on \(C\) of constant scalar curvature \(2(1 - g)\). We denote by \(z_i\) one-half of the square norm function defined by \(h_i\) on \(V_i\). Thus, \(z_i\) is the fibre-wise momentum map for the standard \(U(1)\) action on \(V_i\) by scalar multiplication, with respect to the imaginary part of the hermitian product defined by \(h_i\).
We consider the fibre-wise Kähler quotient at moment value \(z_0 + \cdots + z_\ell = 1\) of
\[
V = \bigoplus_{k=0}^\ell V_k
\]
with respect to the hermitian product \(h = h_0 \oplus \cdots \oplus h_\ell\) and the diagonal \(U(1)\) action on \(V\): this gives the Fubini–Study metric \(g_{FS, V}^{P(V)}\) of scalar curvature \(2n(n - 1)\) on each fibre of \(M_0 = P(V) \to C\). We use the Chern connection of \((V, h)\) (which induces a horizontal distribution on \(TM_0\)) in order to complete trivially \((g_{FS, V}^{P(V)}, \omega_{FS, V}^{P(V)})\) in the horizontal direction, and thus define a Kähler metric on \(M_0\) as follows:
\[
\begin{align*}
g_c & = \left(c - \sum_{k=0}^\ell \mu(V_k)L_k(x)\right)g_C + g_{FS, V}^{P(V)}, \\
\omega_c & = \left(c - \sum_{k=0}^\ell \mu(V_k)L_k(x)\right)\omega_C + \omega_{FS, V}^{P(V)},
\end{align*}
\]
(6)
where:
- the function \(L_j(x)\) is the restrictions of \(z_j\) on the level set \(z_0 + \cdots + z_\ell = 1\) and then quotient to \(M\); letting \(x_i := z_i = L_i(x)\) for \(i = 1, \ldots, \ell\), we then have \(L_0(x) = 1 - \sum_{j=1}^\ell x_j\). Thus, \((x_1, \ldots, x_\ell)\) is the induced (fibre-wise) moment map for the \(T^\ell\) action on \((P(V), \omega_c)\), taking values in the standard simplex \(\Delta \subset \mathbb{R}^\ell\).
- \(c\) is a real constant satisfying
\[
(7) \quad c > \max\{\mu(V_i), i = 0, \ldots, \ell\},
\]
or, equivalently, \((c - \sum_{k=0}^\ell \mu(V_k)L_k(x)) > 0\) on \(M_0\).
- \((g_c, \omega_c)\) is the pull back of the Kähler structure on \(C\) to \(M_0\).

It is not immediately clear from the above description that \(\omega_c\) is a closed form, but for various computational purposes it will be more convenient to describe \((g_c, \omega_c)\) in terms of its pull-back to the the blow-up \(\hat{M}_0\) of \(M_0\) along the sub-manifolds \(S_i = P(\hat{V}_i) \subset P(V)\), which is isomorphic to the total space of the \(\mathbb{CP}^\ell\) fibre-bundle
\[
\hat{M}_0 = \mathbb{P}\left(\mathcal{O}(-1)_{P(\hat{V}_0)} \oplus \cdots \oplus \mathcal{O}(-1)_{P(\hat{V}_\ell)}\right) \to \hat{S},
\]
where
\[
\hat{S} = \mathbb{P}(\hat{V}_0) \times_C \cdots \times_C \mathbb{P}(\hat{V}_\ell) \to C.
\]
We can summarize the setting in the following commutative diagram

$$\begin{align*}
M_0 = \mathbb{P}\left(\mathcal{O}(-1)_{\mathbb{P}(V_0)} \oplus \cdots \oplus \mathcal{O}(-1)_{\mathbb{P}(V_\ell)}\right) & \quad \xrightarrow{\cdot} \quad \hat{S} = \mathbb{P}(V_0) \times_C \cdots \times_C \mathbb{P}(V_\ell) \\
M_0 = \mathbb{P}\left(V_0 \oplus \cdots \oplus V_\ell\right) & \quad \xrightarrow{\cdot} \quad C
\end{align*}$$

Notice that $\hat{S}$ admits a family of (locally symmetric) CSC Kähler metrics of the form

$$g_{a,b}^\hat{S} = \sum_{k=0}^\ell a_k g_{V_i}^\text{FS} + b g_C,$$

where $a = (a_0, \ldots, a_\ell)$ is an $(\ell + 1)$-tuple of positive real numbers, $b > 0$ and $g_{V_i}^\text{FS}$ denotes the Fubini–Study metric of scalar curvature $2\text{rk}(V_i)(\text{rk}(V_i) - 1)$ defined on the fibres of $\mathbb{P}(V_k)$ by using the hermitian product $h_k$, and on $\hat{S}$ by using the projectively flat structure of $(V_k, h_k)$.

We denote by $\hat{\theta}_i$ the (real-valued) connection 1-form on the unitary bundle $P_i \subset \mathcal{O}(-1)_{\mathbb{P}(V_i)}$ with respect to $h_i$, induced via the Chern connection of $(\mathcal{O}(-1)_{\mathbb{P}(V_i)}, h_i)$. Using that the curvature of $(V_i, h_i)$ is $\mu(V_i)\text{Id} \otimes \omega_C$, $\hat{\theta}_i$ satisfies

$$\hat{\theta}_i(K_i) = 1, \quad d\hat{\theta}_i = \omega_{V_i}^\text{FS} - \mu(V_i)\omega_C,$$

where $K_i$ stands for the generator of the standard $S^1$ action on $\mathcal{O}(-1)_{\mathbb{P}(V_i)}$, and $\omega_{V_i}^\text{FS}$ and $\omega_C$ are the $(1, 1)$-forms associated to the tensors $g_{V_i}^\text{FS}$ and $g_C$ on $\hat{S}$, introduced above.

Using arguments identical to [2, Lemma 1] (see also [1, Theorem 2] and [3, Section 2.3]), one can see that the pull-back of $(g_c, \omega_c)$ to $M_0 \to \hat{S}$ is given by

$$\begin{align*}
g_c &= \sum_{k=0}^\ell L_k(x) g_{V_k}^\text{FS} + \left(c - \sum_{k=0}^\ell \mu(V_k)L_k(x)\right) g_C \\
\omega_c &= \sum_{k=0}^\ell L_k(x) \omega_{V_k}^\text{FS} + \left(c - \sum_{k=0}^\ell \mu(V_k)L_k(x)\right) \omega_C
\end{align*}$$

where:

- $L_j(x) = x_j$, $j = 1, \ldots, \ell$; $L_0(x) = 1 - \sum_{j=1}^\ell x_j$ and $x = (x_1, \ldots, x_\ell)$ belongs to the standard simplex $\Delta = \{x : L_k(x) \geq 0, k = 0, \ldots, \ell\} \subset \mathbb{R}^\ell$;
- $\theta_j = \theta_j - \theta_0$, $j = 1, \ldots, \ell$ are the components of a connection 1-form defined on $\mathbb{R}^\ell$ over $\Delta$ such that

$$d\theta_j = \omega_{V_j}^\text{FS} - \omega_{V_0}^\text{FS} + (\mu(V_0) - \mu(V_j)) \omega_C, \quad j = 1, \ldots, \ell;
$$

- $u = \frac{1}{2} \sum_{k=0}^\ell L_k(x) \log L_k(x)$ is the Guillemin potential for the Fubini–Study metric on the $\mathbb{CP}^\ell$-fibre of $M_0 = \mathbb{P}\left(\bigoplus_{k=0}^\ell \mathcal{O}(-1)_{\mathbb{P}(V_k)}\right) \to \hat{S}$.

The metric (9) is a special case of the generalized Calabi construction developed in [1, 3]. For the purpose of computing of Donaldson–Futaki invariant, we shall work with the form (9) of the metric, and this can be merely taken to be its definition: even though (9)-(10) define a degenerate Kähler metric on $M_0$, it is shown in [1, Prop. 2 and Theorem 2] that it is the pull-back of a smooth Kähler metric on $M_0 = \mathbb{P}(V) \to C$, provided that condition (7) is satisfied. The corresponding Kähler class $\Omega_c = [\omega_c]$ on $M$ is called admissible. The definition (6) yields that $(g_c, \omega_c)$ restricts to each fibre of $\mathbb{P}(V)$ to a Fubini–Study metric.
of scalar curvature $2n(n - 1)$, thus showing that $\Omega_c = 2\pi (c_1(\mathcal{O}(1)_{\mathbb{P}(V)}) + mc_1(\mathcal{O}(1)_C))$ for a certain real number $m$. This can also be deduced directly from (9), for instance by integrating $\omega_c^{n-1}$ over a fibre of the fibration (over $C$) $M_0 \to \hat{S} \to C$ (and using Lemma 4.1 below). We claim that $m = c$. To show this we use [32, Lemma 5.16] to compute (denoting $r_V = \text{rk}(V) = n$)

$$\frac{1}{(2\pi)^n} \int_{M_0} \Omega_{c, V}^{r_V} = \frac{1}{r_V!} (c_1(\mathcal{O}(1)_{\mathbb{P}(V)}) + mc_1(\mathcal{O}(1)_C))^n = \frac{1}{r_V!} (r_V m - d_V),$$

on the one hand, and Proposition 3.1 below to get

$$\frac{1}{(2\pi)^n} \int_{M_0} \Omega_{c, V}^{r_V} = \frac{\alpha_0}{\pi_R} = \frac{1}{r_V!} (r_V c - d_V)$$

on the other hand.

Conversely, as $H^2(M_0, \mathbb{R}) \cong \mathbb{R}^2$, any Kähler class $\Omega$ on $M_0$ can be rescaled by a positive real number so as it becomes of the form $[\omega_c]$ for some real number $c$ (possibly not satisfying (7)). However, integrating suitable powers of $\omega_c$ over the sub-manifolds $S_i = \mathbb{P}(V_i) \subset M_0$ ($S_i$ is the pre-image of a vertex of $\Delta$) yields the inequality (7). This shows that any Kähler class on $M_0$ is admissible up to a scale. We have thus established

**Lemma 3.3.** Let $M_0 = \mathbb{P}(V) \to C$ with $V = \bigoplus_{i=0}^\ell V_i$, where $V_i$ are stable vector bundle over a curve $C$. Then, (9)-(10)-(7) introduces Kähler metrics $(g_c, \omega_c)$ on $M_0$, which exhaust the Kähler cone of $M_0$ up to positive scales. The constant $c$ corresponding to a polarization $\mathcal{L}_m^n = \mathcal{O}(1)_{\mathbb{P}(V)} \otimes \mathcal{O}(m)_C$ on $M_0$ is $c = m$.

### 3.4. Computing the relative Donaldson–Futaki invariant via $g_c$.

We shall start this section by fixing some notation.

**Notation 2.** We denote for all $i = 0, \ldots, \ell$

$$r_i = \text{rk}(V_i), \quad d_i = \text{deg}(V_i), \quad \mu_i = \mu(V_i),$$

and

$$r_V = r_0 + \ldots + r_\ell, \quad d_V = \text{deg}(V) = \sum_{i=0}^\ell d_i, \quad \pi_R = (r_0 - 1)! \ldots (r_\ell - 1)!.$$

The volume form $v_{g_c} = \omega_c^n / n!$ (with $n = r_V$ being the complex dimension of $M$) of the metric (9) is given by

$$v_0 = \frac{p_c(x)}{\pi_R} \left[ \omega_C \wedge \left( \bigwedge_{i=0}^\ell (\omega_{V_i}^{\text{FS}})^{(r_i - 1)} \right) \wedge \left( d\mu \wedge \theta_1 \wedge \cdots \wedge \theta_\ell \right) \right],$$

where $d\mu$ is the standard Lebesgue measure on $\mathbb{R}^\ell$ and we have set

$$p_c(x) = \left( c - \sum_{k=0}^\ell \mu_k L_k(x) \right) \prod_{k=0}^\ell (L_k(x))^{(r_k - 1)},$$

The scalar curvature $\text{Scal}_{g_c}$ of the metric (9) is computed in [3] to be

$$\text{Scal}_{g_c} = \sum_{k=0}^\ell \frac{2r_k (r_k - 1)}{L_k(x)} + \frac{4(1 - g)}{c - \sum_{k=0}^\ell \mu_k L_k(x)}$$

$$- \frac{1}{p_c(x)} \sum_{p, q=1}^\ell \frac{\partial^2}{\partial x_p \partial x_q} \left( p_c(z) w^{pq}(x) \right),$$

where $w^{pq}(x)$ are the components of the Kähler form of the metric (9).
where \((u^q(x))\) denotes \((\text{Hess}(u))^{-1}\). We then compute (by using integration by parts, compare with [3, Section 2.5]):

\[
\begin{align*}
\alpha_0 := & \frac{2\pi}{(2\pi)^n} \int_{M_0} v_{g_\varepsilon} = \int_\Delta p_c(x) d\mu, \\
\alpha_r := & \frac{2\pi}{(2\pi)^n} \int_{M_0} x_r v_{g_\varepsilon} = \int_\Delta x_r p_c(x) d\mu, \\
\alpha_{rs} := & \frac{2\pi}{(2\pi)^n} \int_{M_0} x_r x_s v_{g_\varepsilon} = \int_\Delta x_r x_s p_c(x) d\mu, \\
\beta_0 := & \frac{2\pi}{(2\pi)^n} \int_{M_0} \text{Scal}_{g_\varepsilon} v_{g_\varepsilon} \\
& = \int_\Delta \left( \frac{2(1-g)}{c - \sum_{k=0}^{\mu(V_k) L_k(x)} + \sum_{k=0}^{\ell} \frac{\text{rk}(V_k)(\text{rk}(V_k) - 1)}{L_k(x)}} \right) p_c(x) d\mu + \int_{\partial \Delta} p_c(x) d\sigma, \\
\beta_r := & \frac{2\pi}{(2\pi)^n} \int_{M_0} \text{Scal}_{g_\varepsilon} x_r v_{g_\varepsilon} \\
& = \int_\Delta \left( \frac{2(1-g)}{c - \sum_{k=0}^{\mu(V_k) L_k(x)} + \sum_{k=0}^{\ell} \frac{\text{rk}(V_k)(\text{rk}(V_k) - 1)}{L_k(x)}} x_r p_c(x) d\mu + \int_{\partial \Delta} x_r p_c(x) d\sigma,
\end{align*}
\]

(13)

where \(d\sigma\) is the induced measure on the facets of \(\Delta\) by \(u_j \wedge d\sigma_i = -d\mu\) for each facet \(F_j\) with \(u_j = dL_j\) being the inward normal of \(F_j\).

We obtain that the normalized hamiltonians and \(\text{Scal}_{g_\varepsilon}^{-1} = f_{g_\varepsilon}^M\) are given respectively by

\[
f_k = x_k - \frac{\alpha_k}{\alpha_0} f_{g_\varepsilon}^M = a_0 + \sum_{j=2}^\ell a_j x_j
\]

with

\[
\begin{align*}
\sum_{j=2}^\ell a_j \alpha_k & = 2 \beta_k, \quad k = 2, \ldots, \ell, \\
\sum_{j=2}^\ell a_j \alpha_j & = 2 \beta_0.
\end{align*}
\]

(14)

This allows us to obtain from (5)

\[
\frac{\alpha_0 \pi^R}{(2\pi)^n} \tilde{\mathfrak{F}}_{\rho_{\mathfrak{a}}}(\rho_L) = 2 (\alpha_0 \beta_1 - \alpha_1 \beta_0) - \sum_{j=2}^\ell a_j \alpha_0 \alpha_j - \frac{\sum_{j=2}^\ell a_j \alpha_j}{\alpha_0}.
\]

(15)

where we introduced the matrix of size \(\ell - 1\),

\[
A_{ij} = \alpha_{ij} - \frac{\alpha_i \alpha_j}{\alpha_0}, \quad 2 \leq i, j \leq \ell.
\]

We thus have, setting \(\tilde{\mathfrak{F}}_{\rho_{\mathfrak{a}}}(\rho_L) = \frac{1}{2} \frac{\alpha_0 \pi^R}{(2\pi)^n} \tilde{\mathfrak{F}}_{\rho_{\mathfrak{a}}}(\rho_L)\),

**Lemma 3.4.** Let \(M = \mathbb{P}(E) \to C\) with \(E = \bigoplus_{k=0}^{g-1} U_k\) be a projectivisation of vector bundle over a curve \(C\) of genus \(g \geq 2\) and \(L \subset U_0\) a strict sub-bundle of one of the indecomposable components of \(E\). The relative Donaldson–Futaki invariant \(\tilde{\mathfrak{F}}_{\rho_{\mathfrak{a}}}(\rho_L)\) of the induced \(\mathbb{C}^\times\) action \(\rho_L\) on the central fibre \(M_0 = P(V)\) with respect to the test configuration of Lemma 3.1 and a polarization \(\mathcal{L}_m\) is positive multiple of

\[
\tilde{\mathfrak{F}}_{\rho_{\mathfrak{a}}}(\rho_L) := (\alpha_0 \beta_1 - \alpha_1 \beta_0) - \sum_{j,r=2}^\ell (A^{-1})_{rj} \left( \alpha_0 \beta_r - \alpha_r \beta_0 \right) \left( \alpha_{ij} - \frac{\alpha_i \alpha_j}{\alpha_0} \right),
\]

(16)

where \(\alpha_i, \alpha_{ij}\) and \(\beta_i\) are the integrals defined by (13) with \(c = m\), and \(A\) is the matrix (15).
In the remainder of this section, we collect the main technical ingredients allowing to evaluate the sign of the r.h.s. of (16)

**Notation 3.** We denote

- $\kappa = \#\{j, r_j = 1\}$ the number of integers $0 \leq j \leq \ell$ such that $r_j = 1$,
- $\kappa_k = \#\{j \neq k, r_j = 1\}$ the number of integers $0 \leq j \neq k \leq \ell$ such that $r_j = 1$,
- $\kappa_{k_1,k_2} = \#\{j \neq k_1, j \neq k_2, r_j = 1\}$ the number of integers $0 \leq j \neq (k_1,k_2) \leq \ell$ such that $r_j = 1$.

**Proposition 3.1.** With the notations above, $j \neq k$, $0 < j, k < \ell$, we have

$$
\begin{align*}
\alpha_0 &= \frac{\pi R}{r_V!} (cr_V - d_V), \\
\alpha_j &= \frac{\pi R}{(r_V + 1)!} r_j (c(r_V + 1) - d_V - \mu_j), \\
\alpha_{jk} &= \frac{\pi R}{(r_V + 2)!} r_j r_k (c(r_V + 2) - d_V - \mu_j - \mu_k), \\
\alpha_{jj} &= \frac{\pi R}{(r_V + 2)!} r_j (r_j + 1) (c(r_V + 2) - d_V - 2\mu_j), \\
\beta_0 &= \frac{\pi R}{(r_V - 1)!} ((r_V - 1)r_V c + 2(1 - g) - (r_V - 1)d_V), \\
\beta_j &= \frac{\pi R r_j}{r_V!} (r_V (r_V - 1)c + 2(1 - g) - d_V (r_V - 2) - r_V \mu_j).
\end{align*}
$$

**Proof.** This is a direct corollary of Lemmas 4.1 and 4.2 that can be found in the Appendix (Section 4). Actually, we have

$$
\alpha_0 = \frac{\pi R}{(r_V - 1)!} \left( c - \frac{\mu_0 r_0 + \ldots + \mu_\ell r_\ell}{r_V} \right).
$$

For $j > 0$, we get

$$
\alpha_j = \frac{\pi R}{(r_V - 1)! r_V} \frac{r_j}{r_V} \left( c - \frac{\mu_0 r_0 + \ldots + \mu_j (r_j + 1) + \ldots + \mu_\ell r_\ell}{r_V + 1} \right).
$$

If $j \neq k$, then

$$
\alpha_{jk} = \frac{\pi R}{(r_V - 1)! (r_V + 1)!} \frac{r_j r_k}{r_V} \left( c - \frac{\mu_0 r_0 + \ldots + \mu_j (r_j + 1) + \ldots + \mu_k (r_k + 1) + \ldots + \mu_\ell r_\ell}{r_V + 2} \right).
$$

If $j = k$,

$$
\alpha_{jj} = \frac{\pi R}{(r_V - 1)! (r_V + 1)!} \frac{r_j (r_j + 1)}{r_V} \left( c - \frac{\mu_0 r_0 + \ldots + \mu_j (r_j + 2) + \ldots + \mu_\ell r_\ell}{r_V + 2} \right).
$$

Moreover,

$$
\begin{align*}
\beta_0 &= \frac{\pi R}{(r_V - 1)!} \left( 2(1 - g) + (r_V - 1) \sum_{k=0, k \neq 1}^\ell r_k \left( c - \frac{\mu_0 r_0 + \ldots + \mu_k (r_k - 1) + \ldots + \mu_\ell r_\ell}{r_V - 1} \right) \right) \\
&\quad \quad + \frac{\pi R}{(r_V - 1)!} \left( (r_V - 1) \kappa c - \sum_{k=0}^\ell d_k \kappa_k \right) \\
&= \frac{\pi R}{(r_V - 1)!} \left( 2(1 - g) + (r_V - 1) \sum_{k=0}^\ell r_k \left( c - \frac{\mu_0 r_0 + \ldots + \mu_k (r_k - 1) + \ldots + \mu_\ell r_\ell}{r_V - 1} \right) \right)
\end{align*}
$$
We have

\[
\frac{\pi_R}{(r_V + 1)!} \left( (r_V - 1)c_k - \sum_{k=0, r_k=1}^\ell (d_0 + \ldots + d_{k-1} + d_{k+1} + \ldots + d_\ell) \right) 
+ \frac{\pi_R}{(r_V + 1)!} \left( (r_V - 1)c_k - \sum_{k=0}^\ell d_k\kappa_k \right) 
= \frac{\pi_R}{(r_V + 1)!} \left( 2(1 - g) + (r_V - 1)r_vc - r_Vd_V + \sum_{k=0}^\ell r_k\mu_k \right).
\]

Similarly, for \( j > 0 \),

\[
\beta_j = \frac{\pi_R r_j}{r_V!} \left( 2(1 - g) + r_V \sum_{k \neq j} r_k c - \sum_{k \neq j} r_k (d_V - \mu_k + \mu_j) \right) + \frac{\pi_R}{r_V!} (c r_V - d_V) r_j (r_j - 1) 
- \frac{\pi_R r_j}{r_V!} \left( r_V\kappa_k c - \sum_{k=0, k \neq j}^\ell (d_0 + \ldots + d_{k-1} + d_{k+1} + \ldots + d_\ell) \right) 
+ \frac{\pi_R}{r_V!} \left( r_j r_V\kappa_k c - r_j \sum_{k=0}^\ell d_k\kappa_{k,j} - \kappa_j d_j \right) 
= \frac{\pi_R r_j}{r_V!} (2(1 - g) + r_V (r_V - r_j) c - (d_V + \mu_j)(r_V - r_j) + d_V - d_j + (r_j - 1)(c r_V - d_V)).
\]

We need to compute the term \( \alpha_0\alpha_{jk} - \alpha_j\alpha_k \) explicitly in order to get \( \mathfrak{Z}_{p_{\text{can}}} (\rho_L) \). By direct computation from the previous proposition, we obtain

**Lemma 3.5.** Define

\[
\gamma_{jk} = \frac{\pi_R^2 r_j r_k}{(r_V + 1)!^2 (r_V + 2)} \left[- (r_V + 1)(r_V + 2)c^2 + 2(\mu_k + \mu_j + d_V)(1 + r_V)c 
- (\mu_k + \mu_j + d_V)d_V - (r_V + 2)\mu_j\mu_k, \right)
\]

\[
\gamma_j' = \frac{\pi_R^2}{r_V!(r_V + 2)!} r_j [r_V(r_V + 2)c^2 - 2(d_V(r_V + 1) + r_V\mu_j)c + d_V(2\mu_j + d_V)].
\]

Then for \( j \neq k \),

\[
\alpha_0\alpha_{jk} - \alpha_j\alpha_k = \gamma_{jk}, \quad \alpha_0\alpha_{jj} - \alpha_j\alpha_j = \gamma_{jj} + \gamma_j'.
\]

In a similar way, we obtain the following lemma.

**Lemma 3.6.** We have

\[
\beta_k \alpha_0 - \beta_0 \alpha_k = \frac{\pi_R^2}{r_V!(r_V + 1)!} (2r_V (r_k d_V - r_V d_k)c + 2(g - 1 - d_V)(r_k d_V - r_V d_k)),
\]

and in particular

\[
\sum_{k=2}^\ell \beta_k \alpha_0 - \beta_0 \alpha_k = 2c \frac{\pi_R^2}{(r_V - 1)! (r_V + 1)!} (r_V(d_0 + d_1) - d_V(r_0 + r_1)) 
+ 2 \frac{\pi_R^2}{r_V!(r_V + 1)!} (g - 1 - d_V)(r_V(d_0 + d_1) - d_V(r_0 + r_1)).
\]
3.5. Algebraic computation of the relative Donaldson–Futaki invariant. We consider in this section \( M = \mathbb{P}(\bigoplus_{i=0}^{\ell} V_i) \to C \) with no assumptions for \( V_i \) or \( C \), and take a polarization \( \mathcal{L} = \mathcal{L}_{q,p} = \mathcal{O}(q)\mathcal{E}^* \otimes \mathcal{O}(p)_{\mathcal{C}} \). Up to scale, these will only depend on the ratio \( p/q \), so write \( \mathcal{L} = \mathcal{L}_m = \mathcal{O}(1)\mathcal{O}(V) \otimes \mathcal{O}(m)_{\mathcal{C}} \) with \( m \in \mathbb{Q} \). Using that the volume of the sub-variety \( \mathbb{P}(V_i) \subset \mathbb{P}(V) \) with respect to \( \mathcal{L} \) must be positive for \( \mathcal{L} \) to define a polarization, one gets (see e.g. [15, Prop. 1]) that \( m > \mu(V_i) \) for all \( i = 0, \ldots, \ell \), compare with (7).

We denote by \( \rho_i, i = 1, \ldots, r \) the \( \mathbb{C}^\times \) action on \( M \) given by multiplication on \( V_i \) (and acting trivially on the other summands of \( V = \bigoplus_i V_i \)). We want to compute algebraically the relative Donaldson–Futaki invariants of \( \rho_i \) on \( (M, \mathcal{L}) \). This computation for the classical Donaldson–Futaki invariant is a standard procedure and can be done in different ways, see [32, Section 5.4] and [9, 18, 21, 37].

We first compute \( d_k = \chi(\mathbb{P}(V), \mathcal{L}^k) \) for \( k \gg 0 \) using Proposition 4.1. Recall that \( n = \text{dim}(\mathbb{P}(V)) = r_V \) is the dimension of the ruled manifold. With our notations, we have the formula \( \pi_*\mathcal{L}^k = S^k V^* \otimes \mathcal{O}(mk)_{\mathcal{C}} \). In the computations below, we will also use the fact that \( \int_C c_1(C) = 2(1 - g) \) and \( \text{deg}_C \mathcal{O}(1)_{\mathcal{C}} = 1 \). Then, we have

\[
d_k = \int_C ch(S^k V^*) ch(\mathcal{O}(mk)) Todd(C),
\]

\[
= \left( n - 1 + k \right) \left( k(m - \mu(V)) + 1 - g \right),
\]

\[
= \tilde{\alpha}_0 k^n + \tilde{\beta}_0 k^{n-1} + O(k^{n-2}),
\]

with

\[
\tilde{\alpha}_0 = \frac{1}{(n-1)!} (m - \mu(V)) = \frac{1}{r_V!} (r_V m - d_V) = \frac{1}{\pi_R} \alpha_0(m),
\]

\[
\tilde{\beta}_0 = \frac{1}{(n-1)!} \left( 1 - g \right) + \frac{n(n-1)}{2} (m - \mu(V))
\]

\[
= \frac{1}{2(r_V - 1)!} \left( (r_V - 1)(r_V m - d_V) + 2(1 - g) \right) = \frac{1}{2\pi_R} \beta_0(m).
\]

where \( \alpha_i(m), \beta_i(m) \) are given by Proposition 3.1 with \( c = m \).

For a \( \mathbb{C}^\times \) action \( \rho \) on \( (M, \mathcal{L}) \), there is an associated weight \( w_k(\rho) \) given by the trace \( \text{tr}(A_k) \) of the infinitesimal generator \( A_k \) on \( H^0(M, \mathcal{L}^k) \), see Definition 2.2. For \( k \gg 0 \), we let

\[
w_k(\rho_i) = k^{n+1} \tilde{\alpha}_i + k^n \tilde{\beta}_i + O(k^{n-1}).
\]

In order to compute \( w_k(\rho_i) \), we apply the \( S^1 \)-equivariant Riemann–Roch theorem with the Cartan model of equivariant cohomology in order to compute the equivariant characteristic quantities.

For so, let \( T^{r+1} \) denotes the natural \((r+1)\)-dimensional torus action by scalar multiplication on each factor \( V_i \) and \( \rho \) be the \( \mathbb{C}^\times \) action associated to an \( S^1 \) subgroup of \( T^{r+1} \),

\[
\rho(t) \cdot (v_0, \ldots, v_r) = (t^{\lambda_0} v_0, \ldots, t^{\lambda_r} v_r)
\]

for some integer coefficients \( \lambda_i \). Let us fix a \( T^{r+1} \)-invariant hermitian metric \( h = h_0 \oplus \cdots \oplus h_r \) on \( V = \bigoplus_{i=0}^{r} V_i \) (here \( h_i \) is a fixed hermitian metric on \( V_i \)) with \( \rho \) equivariant curvature \( \nabla^V - \Lambda \), where \( F_V = F_0 \oplus \cdots \oplus F_r \) is the usual (non-equivariant) curvature of \( V \) (with \( F_i \) being the curvature of \( (V_i, h_i) \)), and \( \Lambda = \Lambda_0 \) is the endomorphism of \( E \) given by

\[
\Lambda_0 = \Lambda = \lambda_0 Id_0 \oplus \cdots \oplus \lambda_r Id_r.
\]
Notice that the singularity in front of $\Lambda$ of the equivariant curvature corresponds to our convention in Definition 2.2 for the infinitesimal generator of the action $\rho$ on $V$. Thus, $-\frac{1}{2\pi}F_{V} + \Lambda$ is a $\rho$-equivariant curvature for the dual action on $V^*$ (still denoted by $\rho$). Using the identification $\pi_*\mathcal{L}^k = S^kV^* \otimes \mathcal{O}(mk)_C$, we apply the $S^1$-equivariant Riemann–Roch theorem (see [4] and [5]) in order to compute $w_k(\rho)$, as is done in [13]. Since we are dealing with $S^1$-invariance over a base of dimension 1, it is only necessary to compute the $(2, 2)$ part of the $\rho$-equivariant cohomology class

\begin{equation}
ch^\rho(S^kV^*)ch^\rho(\mathcal{O}_C(mk))\text{Todd}^\rho(C)
\end{equation}

in order to get the weight $w_k(\rho)$ by integration. We can apply Proposition 4.1 together with the fact that

$$\text{Todd}^\rho(C) = 1 + \frac{1}{2}c_1(C) = [1 + (1 - g)\frac{\omega_C}{2\pi}],$$

where $\omega_C$ is any representative of $c_1(\mathcal{O}(1)_C)$, in order to expand (17). Then, using equivariant Chern–Weil theory, we can replace before integration the quantities $c^\rho_1(V^*)$, $c^\rho_2(V^*)$, $c^\rho_3(\mathcal{O}(1)_C)$, using the following formulas:

$$c^\rho_1(V^*) = [-\frac{1}{2\pi}\text{tr}(F_V) + \text{tr}(\Lambda)],$$
$$c^\rho_2(V^*) = [-\frac{1}{2\pi}\text{tr}(F_V\Lambda) + \frac{1}{2}\text{tr}(\Lambda^2)],$$
$$c^\rho_3(\mathcal{O}(1)_C) = \frac{1}{2\pi}\omega_C.$$

We obtain, keeping only the terms that can be integrated along $C$,

$$w_k(\rho) = -\frac{1}{2\pi}\int_C \left[ \binom{n+k}{k-1}\text{tr}(F_V\Lambda) + \binom{n-1+k}{k-2}\text{tr}(\Lambda)\text{tr}(F_V) \right]$$
$$- \binom{n-1+k}{k-1}\text{tr}(\Lambda)(1 - g)\omega_C - \binom{n+1+k}{k-1}km\text{tr}(\Lambda)\omega_C \right]$$
$$= -\binom{n+k}{k-1}\left( \sum_{i=0}^r \lambda_i d_i \right) - \binom{n-1+k}{k-2}\left( \sum_{i=0}^r \lambda_i r_i \right) d_V$$
$$+ \binom{n+1+k}{k-1}\left( \sum_{i=0}^r \lambda_i r_i \right) ((1 - g) + km),$$

where $r_j = \text{rk}(V_j)$; $d_j = \text{deg}_{C}(V_j)$. Letting $\rho = \rho_i$ (i.e. $\lambda_j = \delta_{ij}$), we thus get $w_k(\rho_i) = \tilde{\alpha}_i k^{n+1} + \tilde{\beta}_i k^n + O(k^{n-1})$ with

$$\tilde{\alpha}_i = \frac{1}{(n+1)!}(-d_i - r_i d_V + (n+1)m r_i),$$
$$\tilde{\beta}_i = -\frac{1}{(n+1)!}\left( d_i\frac{n(n+1)}{2} + r_i d_V \right) + \frac{1}{n!}\left( r_i(1 - g) + m\frac{n(n-1)}{2} \right),$$

and

$$\alpha_i = \frac{1}{\pi R} \alpha_i(m),$$
$$\beta_i = \frac{1}{2\pi R} \beta_i(m),$$

for $R \equiv R_{V, t}$. And the answer is

$$\alpha_i = \frac{1}{\pi R} \alpha_i(m),$$
$$\beta_i = \frac{1}{2\pi R} \beta_i(m).$$
with $\alpha_j(m), \beta_j(m)$ given by Proposition 3.1 with $c = m$.

Similarly, letting $w_k(\rho_i, \rho_j)$ denote the trace $\text{tr}(A_{k,i}A_{k,j})$ where $A_{k,i}$ is the infinitesimal generator of the actions of $\rho_i$ on $H^0(M, \mathcal{L}^k) \cong H^0(C, S^kV^* \otimes \mathcal{O}(mk_c))$, there is an expansion

$$w_k(\rho_i, \rho_j) = k^{n+2} \alpha_{ij} + O(k^{n+1})$$

which we are going to detail below. We do a similar computation as before but apply the Hirzebruch–Riemann–Roch $S^1 \times S^1$-equivariant Theorem to take into account the two actions $\rho, \rho'$ corresponding to the generators of the $S^1 \times S^1$ action. This time, working on $C \times S^1 \times S^1$, we need to compute the $(3, 3)$ part of

$$\chi^\rho \chi^\rho' \chi^\rho' \chi^\rho' \text{(O}(mk_c))T \text{odd}^\rho \rho'(C)$$

and integrate. This involves to compute the terms $T_1, T_2$ where

$$T_1 = \int_{C \times S^1 \times S^1} \frac{\chi^\rho}{\chi^\rho'} (S^kV^*) \chi^\rho' (\mathcal{O}(mk_c))$$

Since the base manifold is a curve, we use now that

$$\chi^\rho (V^*) = - \frac{1}{2\pi} [\text{tr}(\Lambda_\rho \Lambda_\rho' F_V)],$$

$$\chi^\rho' (V^*) = - \frac{1}{2\pi} [\text{tr}(F_V \Lambda_\rho') \text{tr}(\Lambda_\rho') + \text{tr}(F_V \Lambda_\rho') \text{tr}(\Lambda_\rho) + \text{tr}(\Lambda_\rho \Lambda_\rho') \text{tr}(F_V)],$$

$$\chi^\rho_1 (V^*)^3 = - \frac{1}{2\pi} [6 \text{tr}(\Lambda_\rho') \text{tr}(\Lambda_\rho') \text{tr}(F_V)].$$

We get,

$$T_1 = \int_C \frac{\chi^\rho}{\chi^\rho'} (\mathcal{O}(mk_c)) \left( - \frac{n-1+k}{k-3} \right) \text{tr}(\Lambda_\rho) \text{tr}(\Lambda_\rho') \text{tr}(F_V)$$

$$- \left( \text{tr}(F_V \Lambda_\rho) \text{tr}(\Lambda_\rho') + \text{tr}(F_V \Lambda_\rho') \text{tr}(\Lambda_\rho) + \text{tr}(\Lambda_\rho \Lambda_\rho') \text{tr}(F_V) \right) \left( \frac{n+k}{k-2} \right)$$

$$- \left( \frac{n+1+k}{k-1} + \frac{n+k}{k-2} \right) \text{tr}(\Lambda_\rho \Lambda_\rho' F_V),$$

$$T_2 = \int_C \left( \frac{n+k}{k-1} \right) \left( \frac{\chi^\rho_1}{\chi^\rho'}(mk_\omega + \frac{1}{2} c_1(C)) \right)$$

$$+ \left( \frac{n-1+k}{k-2} \right) \text{tr}(\Lambda_\rho) \text{tr}(\Lambda_\rho') \left( mk_\omega + \frac{1}{2} c_1(C) \right).$$

Thus, the leading term of $\text{tr}(A_kB_k)$ (which equals the leading terms of $T_1 + T_2$) is

$$\text{tr}(A_kB_k) = \frac{k^{n+2}}{2\pi(n+2)!} \int_C t_1 + O(k^{n+1}),$$

$$t_1 = - \text{tr}(\Lambda_\rho) \text{tr}(\Lambda_\rho') \text{tr}(F_E) - \text{tr}(F_E \Lambda_\rho) \text{tr}(\Lambda_\rho') - \text{tr}(F_E \Lambda_\rho') \text{tr}(\Lambda_\rho)$$

$$- \text{tr}(\Lambda_\rho \Lambda_\rho') \text{tr}(F_E) - 2 \text{tr}(\Lambda_\rho \Lambda_\rho' F_E)$$

$$+ m(n+2)(\text{tr}(\Lambda_\rho \Lambda_\rho') + \text{tr}(\Lambda_\rho) \text{tr}(\Lambda_\rho')).$$
Letting $\rho = \rho_i$ and $\rho' = \rho_j$ as for the computation of $w_k(\rho_i)$, we obtain

$$\frac{1}{2\pi} \int_{C} t_1 = - \left( \sum_s \lambda_s r_s \right) (\sum_t \lambda'_s r_t) - \left( \sum_s \lambda_s d_s \right) (\sum_t \lambda'_t r_t) - \left( \sum_s \lambda'_s d_s \right) (\sum_t \lambda_t r_t)$$

$$- \left( \sum_s \lambda'_s r_s \right) d_V - 2 \left( \sum_s \lambda_s r_s \right) d_V$$

$$+ m(n + 2) \left( \sum_s \lambda_s r_s \right) + \left( \sum_s \lambda_s r_s \right) \omega_C.$$  

With $\lambda_s = \delta_{si}$ and $\lambda_t = \delta_{sj}$ we deduce from above,

$$\frac{1}{2\pi} \int_{C} t_1 = - r_i r_j d_V - d_i r_j + m(n + 2) r_i r_j$$

if $i \neq j$,

$$\frac{1}{2\pi} \int_{C} t_1 = - r_j^2 d_V - 2 d_j r_j - r_j d_V - 2 d_j m(n + 2)(d_j + r_j^2)$$

if $i = j$.

Consequently, if $i \neq j$,

$$\tilde{\alpha}_{ij} = \frac{1}{2\pi(n + 2)!} \int_{C} t_1 = \frac{1}{(n + 2)!} r_i r_j [m(n + 2) - d_V - \mu_i - \mu_j],$$

$$= \frac{1}{\pi R} \alpha_{ij}(m);$$

$$\tilde{\alpha}_{jj} = \frac{1}{2\pi(n + 2)!} \int_{C} t_1 = \frac{1}{(n + 2)!} r_j (r_j + 1) [-d_V - 2 \mu_j + m(n + 2)],$$

$$= \frac{1}{\pi R} \alpha_{jj}(m),$$

where, again, $\alpha_{ij}(m)$ are given by Proposition 3.1 with $c = m$.

Recall from the general theory (see Section 2) that the algebraic Donaldson–Futaki invariant $\mathfrak{F}(\rho_i)$ of $\rho_i$ on $(M, L_m)$ is given (up to a normalizing positive factor) by

$$\mathfrak{F}^{alg}(\rho_i) = \beta_1 - \tilde{\alpha}_{ij} \tilde{\beta}_0 / \tilde{\alpha}_0.$$  

If we assume now that $V_i$ are indecomposable and $C$ has genus $g \geq 1$, so as the fibre-wise $\mathbb{T}^\ell$ action generated by $\rho_i$, $i = 1, \ldots, \ell$ corresponds to a maximal torus in $\text{Aut}^{\text{red}}(M)$, the extremal $\mathbb{C}^\times$ action $\rho_{\text{ex}}$ of $(M, L_m)$ is generated by

$$K_{\text{ex}} := \sum_{i = 1}^{\ell} \tilde{\alpha}_i K_i,$$

where $K_i$ is a generator of $\rho_i$ and the rational numbers $\tilde{\alpha}_i$ are given by

$$\langle \rho_i, \rho_{\text{ex}} \rangle = \sum_{k = 1}^{\ell} \tilde{\alpha}_k \left( \tilde{\alpha}_{ik} - \tilde{\alpha}_i \tilde{\alpha}_j / \tilde{\alpha}_0 \right) = \mathfrak{F}^{alg}(\rho_i) = \beta_1 - \tilde{\alpha}_i \tilde{\beta}_0 / \tilde{\alpha}_0,$$

see Section 2.3.

We now consider $M = \mathbb{P}(\bigoplus_{i=0}^{\ell-1} U_i) \to C$ with $U_i$ indecomposable and $C$ of genus $g \geq 1$, endowed with a polarization $L_m = \mathcal{O}(1)_{\mathbb{P}(E)} \otimes \mathcal{O}(m)_C$ and the test configuration $\mathcal{M}$ with central fibre $M_0 = \mathbb{P}(\bigoplus_{i=0}^{\ell} V_i)$ given by Lemma 3.1. We use the above computations in order to express the relative Donaldson–Futaki invariant $\mathfrak{F}^{alg}_{\text{rel}}(\rho_L)$ on the central fibre $M_0$. For consistency, we let $V_{i+1} := U_i, i = 1, \ldots, (\ell - 1)$ and write the generic fibre as
\( M = \mathbb{P}(U_0 \oplus \bigoplus_{i=1}^{\ell} V_i) \to C \). We are also denoting by \( \rho_i, i = 2, \ldots, \ell \) the \( \mathbb{C}^\times \) actions on \( M \) by multiplication on \( V_i \) and let \( K_i, i = 2, \ldots, \ell \) be the corresponding generating vector fields. By the discussion above, \( \rho_{\mathbb{C}^\times}^M \) is the \( \mathbb{C}^\times \) action generated by the vector field \( K_{\mathbb{C}^\times}^M = \sum_{i=2}^{\ell} a_i K_i \) with
\[
\sum_{k=2}^{\ell} \tilde{a}_k \left( \tilde{\alpha}_{ik} - \frac{\tilde{\alpha}_i \tilde{\alpha}_k}{\tilde{\alpha}_0} \right) = \tilde{s}_{\text{alg}}(\rho_i) = \tilde{\beta}_i \frac{\tilde{\alpha}_i \tilde{\beta}_0}{\tilde{\alpha}_0}.
\]
This implies in particular that \( \tilde{a}_k = a_k/4 \) where \( a_k \) satisfies (14).

Now, the \( \mathbb{C}^\times \) actions \( \rho_i \) extend to the central fibre \( M_0 = \mathbb{P}(\bigoplus_{i=0}^{\ell} V_i) \), and the algebraic relative Donaldson–Futaki invariant on the central fibre \( M_0 \) is (see Definition 2.8)
\[
\tilde{s}_{\text{alg}}(\rho_L) = \tilde{s}_{\text{alg}}(\rho_1) = \tilde{s}_{\text{alg}}(\rho_1) - (\rho_1, \rho_{\mathbb{C}^\times}^M), \\
= \tilde{s}_{\text{alg}}(\rho_1) - \sum_{j=2}^{\ell} a_j \left( \tilde{\alpha}_{j1} - \frac{\tilde{\alpha}_j \tilde{\alpha}_1}{\tilde{\alpha}_0} \right), \\
= \frac{1}{2\pi R} \left( \beta_1 - \frac{\alpha_1 \beta_0}{\alpha_0} \right) - \frac{1}{4\pi R} \sum_{j=2}^{\ell} a_j \left( \alpha_{j1} - \frac{\alpha_1 \alpha_j}{\alpha_0} \right), \\
= \frac{1}{4(2\pi)^n} \tilde{s}_{\text{alg}}(\rho_L).
\]

We thus obtain that Lemma 3.4 is true for \( g \geq 1 \) too.

**Proposition 3.2.** Let \( M = \mathbb{P}(E) \to C \) with \( E = \bigoplus_{k=0}^{\ell-1} U_k \) be a projectivisation of vector bundle over a curve \( C \) of genus \( g \geq 1 \) and \( L \subset U_0 \) a sub bundle of one of the indecomposable components of \( E \). The relative Donaldson–Futaki invariant \( \tilde{s}_{\text{alg}}(\rho_L) \) of the induced \( \mathbb{C}^\times \) action \( \rho_L \) on the central fibre \( M_0 = \mathbb{P}(V) \) with respect to the test configuration of Lemma 3.1 and a polarization \( \mathcal{L}_m \) is positive multiple of (16) where \( \alpha_1, \alpha_{ij} \) and \( \beta_i \) are given by Proposition 3.1 with \( c = m \), and \( A \) is the matrix (15).

### 3.6. The case of an indecomposable bundle

This is the case \( \ell = 1 \) in the setting of the previous sections, i.e. \( M = \mathbb{P}(E) \) with \( E \) an indecomposable vector bundle over \( C \), \( L \subset E \) is a sub-bundle, and the central fibre of the test-configuration given by Lemma 3.1 is \( M_0 = \mathbb{P}(F \oplus L) \) with \( F = E/L \). In this case, the reduced automorphisms group \( \text{Aut}_\text{red}(M) \) then has rank 0 and, therefore, the relative Donaldson–Futaki invariant reduces to the usual Futaki invariant \( \tilde{s}(\rho_K) \) on the central fibre \( M_0 \). This is computed (algebraically) in [32, Theorem 5.13] (see also Theorem 1 in the introduction) and it is shown that it is given by a positive multiple of \( (\mu(E) - \mu(L)) \). For the sake of completeness, and to make a better contact between [32] and the setting of this paper, we compute below (16).

As the Donaldson–Futaki invariant (5) reduces to the usual Futaki invariant (i.e. \( f_{\mathbb{C}^\times} = 0 \) in this case), (16) becomes
\[
\tilde{s}_{\text{alg}}(\rho_L) = (\alpha_0 \beta_1 - \alpha_1 \beta_0),
\]
so that, by Lemma 3.6, we obtain
\[
\tilde{s}_{\text{alg}}(\rho_L) = \frac{\pi^2 R}{r_{V_2}(r_V + 1)!} (2r_V(r_L d_V - r_V d_L)c + (g - 1 - d_V)(r_L d_V - r_V d_L)) \\
= \frac{2\pi^2 r_{V_2} r_L}{r_{V_2}(r_V + 1)!} (\mu(E) - \mu(L)) (r_V c - d_V + (g - 1)),
\]
where, we recall, $\pi_R = (r_L - 1)!(r_V - r_L - 1)!$, $c$ is the constant determined by the polarization $L_m$ on $M$ with $m = c$, and the expression $(r_Vc - d_V + (g - 1))$ is manifestly positive by (7).

**Remark 3.1.** In the case $\ell = 1$, the Donaldson–Futaki invariant on $M_0$ is computed (by essentially the same construction) in [2, Prop. 6]): up to a positive constant it is given by the expression $-(c(x)/x)$, where $c(x)$ is related to the strictly negative function $c(s, x)$ appearing on page 575 of [2] by $c(x) = c\left(\frac{2(1-g)}{(\rho(L)-\mu(L))}, x\right)$, and $x = (\rho(F)-\mu(L))$. A straightforward computation shows that the expressions agree (up to multiplication of a positive constant).

**3.7. The case $\ell = 2$.** We now consider the case when $E = U_0 \oplus U_1$ is the direct sum of two indecomposable bundles $U_0, U_1$. This is also equivalent to the assumption that the rank of the reduced group of automorphisms $\text{Aut}^\text{red}(M)$ equals 1.

In this case, the matrix $A$ induced by (15) is a scalar, $A = \left(\frac{\alpha_0}{\alpha_{22} - \alpha_2^2}\right)$ and $\alpha_0\alpha_{22} - \alpha_2^2 > 0$ by Cauchy–Schwarz (this is also a positive multiple of the $L^2$ square norm of the function $f_2$, see Sect. 3.3). Consequently, we can restrict our attention on

$$(\alpha_0\alpha_{22} - \alpha_2^2)\hat{\mathcal{F}}_{\rho_x}(\rho_L) = (\alpha_0\alpha_{22} - \alpha_2^2)(\alpha_0\beta_1 - \alpha_1\beta_0) - (\alpha_0\alpha_{12} - \alpha_1\alpha_2)(\beta_2\alpha_0 - \alpha_2\beta_0).$$

**Proposition 3.3.** For any admissible Kähler class, the relative Donaldson–Futaki invariant $\hat{\mathcal{F}}_{\rho_x}(\rho_L)$ has the sign of $\mu_0 - \mu_1$.

**Proof.** This is a direct computation of the quantities $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \alpha_{12}, \alpha_{22}$ using Lemma 4.1. It is obtained that

$$2\hat{\mathcal{F}}_{\rho_x}(\rho_L) = \Gamma_0(\mu_0 - \mu_1),$$

where

$$\Delta_c = r_Vc - d_V,$$

$$\Gamma_0 = \Delta_c(\Delta_c + g - 1)(\Delta_c + 2(c - \mu_2))\Gamma_1,$$

$$\Gamma_1 = \frac{2\pi^2r_0r_1r_2(r_2 - 1)!(r_1 - 1)!(r_0 - 1)!^2}{(r_V + 2)!}(r_V + 1)!(r_V + 1)!^2 > 0.$$

Note that with (7), one has $\Delta_c > 0$. This finishes the proof. Of course the full expression of $\hat{\mathcal{F}}_{\rho_x}$ can be provided but it is particularly lengthy even in this case.

**Remark 3.2.** Let us mention at that stage that the classical Donaldson–Futaki invariant $\hat{\mathcal{F}}(\rho_L)$ is positive proportional to $\alpha_0\beta_1 - \alpha_1\beta_0$ which is

$$\alpha_0\beta_1 - \alpha_1\beta_0 = \Gamma'_0(\Delta_c + g - 1)(\mu_2 - \mu_1).$$

with $\Gamma'_0 = \frac{2\pi^2r_0r_2(r_2 - 1)!(r_1 - 1)!(r_0 - 1)!^2}{(r_V + 1)!r_V!^2}$. This points out that the computation of the classical Donaldson–Futaki invariant does not bring any information on the stability of $U_0$.

**3.8. The general case and the proof of Theorem 2.** With the notation of Section 3.2, we aim to compute the sign of following quantity

$$2\hat{\mathcal{F}}_{\rho_x}(\rho_L) = 2(\alpha_0\beta_1 - \alpha_1\beta_0) - \sum_{j=2}^{\ell} a_j(\alpha_0\alpha_{j1} - \alpha_1\alpha_j).$$

Both the differential geometric and algebraic approaches lead to the same difficulty of controlling the terms $a_j$. In order to do so, we are going to expand the unknowns $a_j$,
solutions of (14), in Taylor series with respect to the variable $c$ (recall that $c = m$). Our method consists in evaluating the quantities

$$\Sigma_1 = \sum_{j=2}^{\ell} a_j r_j, \quad \Sigma_2 = \sum_{j=2}^{\ell} a_j d_j.$$ 

We write the Taylor expansions

$$\Sigma_1 = \sum_{i=1}^{\infty} u_i c^{-i},$$

$$\Sigma_2 = \sum_{i=1}^{\infty} v_i c^{-i}.$$ 

From the expression of the matrix $A$ in (15) and the asymptotics above, it is clear that $a_j = \sum_{r=2}^{\ell} (A^{-1})_{rj} \left( \beta_r - \frac{\partial}{\partial c} \beta_0 \right)$ is at most $O(1/c)$. Consequently, $\Sigma_1$ and $\Sigma_2$ are at most $O(1/c)$ for $c \to +\infty$. In order to ease the computations, we will assume $d_V = 0$.

This is not a restrictive assumption. Actually, we can tensorize $V$ with the rational line bundle $O(-\mu(V))_C$ and notice that this does not change the underlying variety $M$. It only introduces a translation with $-\mu(V)$ of the parameter $c = m$ of the polarizations on $M$.

**Notation 4.** We denote $\mu_{01} = d_0 + d_1 r_0 + r_1$.

**Lemma 3.7.** With notations as before, the terms $(u_i), (v_i)$ satisfy the system (S)

$$\begin{cases}
    u_1 = 4r_V^2 \mu_{01}, \\
    (r_V + 2)u_2 - 2\mu_{01} u_1 - 2v_1 = 4(r_V + 2)(g - 1) r_V \mu_{01}, \\
    (r_V + 2)u_{i+2} - 2\mu_{01} u_{i+1} - 2v_{i+1} + \frac{(r_V + 2)}{(r_V + 1)} \mu_{01} v_i = 0
\end{cases}$$

In particular, this provides the first term $u_1$ of the expansion of $\Sigma_1$.

**Proof.** Actually, the system (14) implies that for $k \neq 0, k \neq 1$,

$$\sum_{j=2}^{\ell} a_j (\alpha_{jk} \alpha_0 - \alpha_j \alpha_k) = 2(\beta_k \alpha_0 - \beta_0 \alpha_k).$$

We expand each equation using the expressions of $(\alpha_{jk} \alpha_0 - \alpha_j \alpha_k)$. Then we sum the equations from $k = 2$ to $\ell$. This way, we obtain

$$[(r_V + 2)c^2 - 2\mu_{01} c]\Sigma_1$$

$$+ [-2c + \frac{(r_V + 2)}{(r_V + 1)} \mu_{01}] \Sigma_2 = 2\frac{r_V!(r_V + 1)!(r_V + 2)}{\pi^2 r_0 r_1} \sum_{k=2}^{\ell} (\beta_k \alpha_0 - \beta_0 \alpha_k).$$

We use Lemma 3.6. Eventually, we obtain the system by using the Taylor expansions of $\Sigma_1, \Sigma_2$. \qed
We explain briefly how the last result allows us to compute the expansions of \(a_j\). From (21), at a fixed \(k\), we have at first order in \(c\),

\[
\gamma'_ka_k = 2(\beta_k \alpha_0 - \beta_0 \alpha_k) - \sum_{j=2}^{\ell} a_j \gamma_{jk}
\]

\[
= -4 \pi^2_R (r_V - 1)! (r_V + 1)! r_V d_j c + \frac{\pi^2_R}{(r_V + 1)!^2 (r_V + 2)}(r_k (r_V + 2) (r_V + 1) c^2 u_1}{c} + O(1),
\]

where we have used the fact that \(\sum_{j=2}^{\ell} a_j \gamma_{jk}\) can be written in terms of \(\Sigma_1\) and \(\Sigma_2\). This gives from Lemma 3.5,

\[
a_k = 4 r_V (\mu_0 - \mu_k) \frac{1}{c} + O(1/c^2).
\]

From this expression, one can derive the first term of \(\Sigma_2\), by summing, as

\[
(23) \quad v_1 = -4 r_V \left( \frac{(d_0 + d_1)^2}{r_0 + r_1} + \sum_{j=2}^{\ell} \frac{\ell d_j^2}{r_j} \right).
\]

Back to (S), we can deduce from \((u_1, v_1)\) the value of \(u_2\) and apply the same trick recursively to deduce all the values of \((u_i), (v_i)\). This way we get

\[
(24) \quad u_2 = -\frac{8 r_V}{r_V + 2} \left( \frac{(d_0 + d_1)^2}{r_0 + r_1} + \sum_{j=2}^{\ell} \frac{\ell d_j^2}{r_j} \right) + 4 r_V \mu_01 \left( g - 1 + \frac{2 r_V}{r_V + 2} \mu_{01} \right)
\]

We are now ready to prove the main technical result of this section.

**Proposition 3.4.** We have the following asymptotic expansion of the normalized relative Donaldson–Futaki invariant,

\[
\hat{\delta}_{rel} (\rho_L) = (\mu_0 - \mu_1) (Fut_1 c + Fut_2 + \frac{Fut_3}{c} + \frac{Fut_4}{c^2} + ...)
\]

where \(Fut_1 > 0\) and the \(Fut_i\) are explicit for \(i = 1, 2\).

**Proof.** The computation of \(\hat{\delta}_{rel} (\rho_L)\) at first order in the \(c\) variable depends only on the asymptotic expansion of \(\Sigma_1\) at first order. Using the expressions of \(a_0 \beta_1 - \alpha_1 \beta_0\) and \(\sum_{j=2}^{\ell} a_j \gamma_{j1}\), we obtain from Lemma 3.5, Lemma 3.6 and Lemma 3.7 that provides the value of \(u_1\), that

\[
\hat{\delta}_{rel} (\rho_L) = (\alpha_0 \beta_1 - \alpha_1 \beta_0) - \frac{1}{2} \sum_{j=2}^{\ell} a_j (\alpha_j \alpha_0 - \alpha_1 \alpha_j),
\]

\[
= (\alpha_0 \beta_1 - \alpha_1 \beta_0) - \frac{1}{2} \sum_{j=2}^{\ell} a_j \gamma_{j1},
\]

\[
= -2 c \frac{\pi^2_R}{r_V! (r_V + 1)!} r_V d_j + 2 c \frac{\pi^2_R}{(r_V + 1)! r_V} r_V \mu_01 + O(1),
\]

\[
= 2 c \frac{\pi^2_R}{(r_V - 1)! (r_V + 1)! (r_0 + r_1)} (\mu_0 - \mu_1) + O(1).
\]
Thus $Fut_1 = \frac{2r_0r_1\pi_R^2}{(r_V - 1)!(r_V + 1)!(r_0 + r_1)} > 0$. Using (24) and (23), we obtain by brute force

$\mathcal{F}_{\rho_{\alpha}}(\rho_L) = (\mu_0 - \mu_1)Fut_1c + (\mu_0 - \mu_1)Fut_2 + O(1/c)$

with explicitly

$$Fut_2 = \frac{2r_0r_1\pi_R^2}{(r_V - 1)!(r_V + 2)!(r_0 + r_1)}[(r_V + 2)(g - 1) + 2r_0\mu_0].$$

Eventually, we justify that all the other terms of the expansion of $\alpha_0\mathcal{F}_{\rho_{\alpha}}$ are multiples of $(\mu_0 - \mu_1)$. In order to do so, we notice from Lemma 3.6 that these terms are given by the expansion of $-\sum_{i=2}^t a_j\gamma_1 = \sum_{i \geq 1} \frac{\mathcal{F}_{\mu_0}}{c^i}$. This is given, up to a multiplicative factor $\frac{\pi_R^2(r_V+1)!}{(r_V+1)!^2(r_V + 2)!}$, by

$$r_1(r_V + 2)\sum_{i \geq 1} \frac{u_i + 2}{c^i} - 2d_1 \sum_{i \geq 1} \frac{u_i + 1}{c^i} - 2r_1 \sum_{i \geq 1} \frac{v_i + 1}{c^i} + d_1 \frac{(r_V + 2)}{(r_V + 1)} \sum_{i \geq 1} \frac{v_i}{c^i}.$$

Hence, we are doomed to check that

$$[r_1(r_V + 2)(r_V + 1)u_{i+2} - 2(r_V + 1)d_1u_{i+1} - 2(r_V + 1)r_1v_{i+1} + d_1(r_V + 2)v_i],$$

are multiples of $(\mu_0 - \mu_1)$ for $i \geq 1$. We use the last relationship given by (5) in Lemma 3.7. This provides by replacing $u_{i+2}$ that

$$\mathcal{F}_{\mu_0} = \frac{\pi_R^2}{(r_V + 1)!^2(r_V + 2)!}(2(r_V + 1)u_{i+1} - v_i(r_V + 2))(r_1d_0 - d_1r_0).$$

and the conclusion holds as expected with

$$Fut_{i+2} = \frac{\pi_R^2r_1r_0}{2(r_V + 1)!^2(r_V + 2)(r_0 + r_1)}(2(r_V + 1)u_{i+1} - v_i(r_V + 2)),$$

for $i \geq 1$. \qed

We refine Proposition 3.4 and show that the following holds.

**Proposition 3.5.** For $c > \max(\mu(V_i))$.

$$(Fut_1c + Fut_2 + \frac{Fut_3}{c} + \frac{Fut_4}{c^2} + ... > 0.$$ In particular if the relative Donaldson–Futaki $\mathcal{F}_{\rho_{\alpha}}(\rho_L)$ is positive then $\mu_0 - \mu_1 > 0$.

**Proof.** First we remark that from the proof of Proposition 3.4,

$$\sum_{i \geq 1} \frac{Fut_{i+2}}{c^i} = \frac{\pi_R^2r_1r_0}{2(r_V + 1)!^2(r_V + 2)(r_1 + r_0)}[2(r_V + 1)c \sum_{i \geq 1} \frac{u_i + 1}{c^i} - (r_V + 2) \sum_{i \geq 1} \frac{v_i}{c^i}],$$

$$= \frac{\pi_R^2r_1r_0}{2(r_V + 1)!^2(r_V + 2)(r_1 + r_0)}[2(r_V + 1)c\Sigma_1 - (r_V + 2)\Sigma_2]$$

$$- u_1r_1r_0/(r_V + 1)!(r_V + 2)(r_1 + r_0).$$

So we deduce using the expressions of $Fut_1, Fut_2$ computed in the previous proposition

$$\sum_{i \geq -1} \frac{Fut_{i+2}}{c^i} = \frac{r_1r_0\pi_R^2}{2(r_1 + r_0)(r_V + 1)!^2(r_V + 2)}$$

$$\times [4r_V(r_V + 1)(r_V + 2)(r_Vc + g - 1) + 2(r_V + 1)c\Sigma_1 - (r_V + 2)\Sigma_2].$$
By the assumption $d_V = 0$, we have $c > \frac{dv}{r_V} = 0$, so that in order to prove the proposition, we only need to show the positivity of

$$[4r_V(r_V + 1)(r_V + 2)(r_vc + g - 1) + 2(r_V + 1)c\Sigma_1 - (r_V + 2)\Sigma_2].$$

We have a linear relationship between $\Sigma_1$ and $\Sigma_2$ thanks to (22). We seek for a second linear relationship. Firstly, from (21) and Lemmas 3.6 and 3.5, we get using $d_V = 0$,

$$\gamma'_k a_k = \frac{\pi^2_R}{(r_V + 1)!^2(r_V + 2)} \times \left[-4(r_V + 1)(r_V + 2)r_Vdk(r_vc + g - 1) - (r_k(r_V + 1)(r_V + 2)c^2 + 2dk(1 + r_V)c)\Sigma_1 - (2r_k(1 + r_V)c - (r_V + 2)dk)\Sigma_2\right].$$

On another side, from Lemma 3.5,

$$\gamma'_k = \frac{\pi^2_R}{(r_V + 1)!^2(r_V + 2)} (r_V + 1)r_k[r_V(r_V + 2)c^2 - 2r_V\mu_kc].$$

From these two previous equations, we obtain

$$a_k = \frac{-4(r_V + 2)dk(r_vc + g - 1)}{c[\Sigma_2 + 2]} + \frac{\Sigma_1}{r_V - \frac{(2r_k(1 + r_V)c - (r_V + 2)dk)\Sigma_2}{(r_V + 1)r_vc[r_k(r_V + 2)c - 2dk]}.$$}

We multiply this expression by $d_k$ and then sum. This provides

$$\Sigma_2 = -\sum_{k=2}^\ell \frac{4(r_V + 2)d^2_k(r_vc + g - 1)}{c[\Sigma_2 + 2]} + \frac{(d_V - d_0 - d_1)}{r_V} \Sigma_1 - \left(\sum_{k=2}^\ell \frac{d_k(2r_k(1 + r_V)c - (r_V + 2)dk)}{(r_V + 1)r_vc[r_k(r_V + 2)c - 2dk]}\right)\Sigma_2,$$

i.e a second linear relationship between the unknowns $(\Sigma_1, \Sigma_2)$. Hence, using (22), we can identify $\Sigma_2$ as

$$\Sigma_2 = -\sum_{k=2}^\ell \frac{4(r_V + 2)d^2_k(r_vc + g - 1)}{c[\Sigma_2 + 2]} + \frac{4(r_V + 2)(d_0 + d_1)\mu_0 + cr_V + g - 1}{(\mu_0 + 2)c - 2\mu_0} \Sigma_2,$$

where $\Delta_{\Sigma_2}$ is given by

$$\Delta_{\Sigma_2} = \sum_{k=2}^\ell \frac{d_k(2r_k(1 + r_V)c - (r_V + 2)dk)}{(r_V + 1)r_vc[r_k(r_V + 2)c - 2dk]} + \frac{r_V(r_V + 2)(r_V + 1)c^2 - 2(r_V + 1)(r_V - r_0 - r_1)\mu_0 c - (r_V + 2)(d_1 + d_0)\mu_0}{r_V(r_V + 1)(r_V + 2)c - 2\mu_0}. $$

As we said previously, we are looking for the sign of

$$[r_V(r_V + 1)(r_V + 2)(r_vc + c(g - 1)) + 2(r_V + 1)c\Sigma_1 - (r_V + 2)\Sigma_2]$$

$$= 4r_V(r_V + 1)(r_V + 2)^2 \frac{c(r_V + g - 1)}{(r_V + 2)c - 2\mu_0}$$

$$+ (-\Sigma_2) \left(\frac{c^2}{(r_V + 2)c - 2\mu_0}\right).$$

We will show this is positive. Actually, the first term is positive because $c > 0$ and $(r_0 + r_1)c > (d_1 + d_0)$. As $g \geq 1$, and $r_kc > d_k$, the only difficulty is to show that $\Delta_{\Sigma_2}$
is positive, which implies easily $-\Sigma_2 > 0$. The proof of the proposition is complete with Lemmas 3.8 and 3.9 which exhaust all possible cases.

**Lemma 3.8.** Assume as above that $d_V = 0$ and also that $(d_1 + d_0) \geq 0$. Then $\Delta \Sigma_2 > 0$.

*Proof.* We write

$$\Delta \Sigma_2 = \sum_{k=2}^{\ell} \frac{(2r_k(1 + r_V)c - (r_V + 2)d_k)}{(r_V + 1)r_Vc} B_{\Delta \Sigma_2},$$

where one has denoted

$$B_{\Delta \Sigma_2} = \frac{d_k}{r_k(r_V + 2)c - 2d_k}$$

(28) $+ \frac{r_V(r_V + 2)(r_V + 1)c^2 - 2(r_V + 1)(r_V - r_1)\mu_0c - (r_V + 2)\mu_0(d_1 + d_0)}{2(r_V - r_1 - r_0)(r_V + 1)c + (r_V + 2)(d_1 + d_0)[(r_V + 2)c - 2\mu_0].}$

The factor term $(2r_k(1 + r_V)c - (r_V + 2)d_k)$ is positive. Next, we are interested in the term $B_{\Delta \Sigma_2}$. Its denominator is positive as $d_1 + d_0 \geq 0$ and $r_kc > d_k$. Its numerator, after gathering the 2 terms, is given (up to a factor $(r_V + 2)c$) by

$$2(r_V + 1)(r_Vr_k(c - \mu_0) + (r_kc - d_k)(r_0 + r_1))c + (c - \mu_0)(d_1 + d_0)(r_V + 2)r_k$$

$$+ (r_V + 1)r_k[r_V^2 - 2(r_0 + r_1)c^2 + (d_1 + d_0)[r_Vr_kc + (r_V + 2)d_k].$$

The first line is obviously positive. We only need to check that the last line is non negative. To do so, we write

$$(r_V + 1)r_k[r_V^2 - 2(r_0 + r_1)c^2 + (d_1 + d_0)[r_Vr_kc + (r_V + 2)d_k]$$

$$> (r_V + 1)r_k[r_V^2 - 2(r_0 + r_1)c\mu_0c + (d_1 + d_0)[r_Vr_kc + (r_V + 2)d_k],$$

$$= (r_V + 2)(d_1 + d_0) \left( [(r_V + 1)(r_V \frac{r_V}{r_1 + r_0} - 2) + r_V]\frac{r_kc}{r_V + 2} \right).$$

Now as $d_V = 0$,

$$d_k \geq -d^+_V, \text{ with } d^+_V = \sum_{j=0,d_j \geq 0}^{\ell} d_j \geq 0.$$

From the assumption on $c$, $c > \frac{d^+_V}{\sum_{j=0,d_j \geq 0}^{\ell} r_j} \geq \frac{d^+_V}{r_V - 1}$ as there will be at least one subbundle $V_i \subset V$ of negative degree (otherwise $d_0 = d_1 = \cdots = d_k = d^+_V = 0$, the last line vanishes and we are done). We get since $r_k \geq 1$,

$$\left( [(r_V + 1)(r_V \frac{r_V}{r_1 + r_0} - 2) + r_V]\frac{r_kc}{r_V + 2} + d_k \right)$$

$$\geq [(r_V + 1)(r_V \frac{r_V}{r_V - 1} - 2) + r_V]\frac{d^+_V}{r_V - 1}(r_V + 2) - d^+_V,$$

$$\geq \frac{2r_V}{(r_V + 2)(r_V - 1)^2} d^+_V \geq 0.$$

□

**Lemma 3.9.** Assume as above that $d_V = 0$ and also that $(d_1 + d_0) \leq 0$. Then $\Delta \Sigma_2 > 0.$
Proof. This is similar to the previous lemma. First, as $d_1 + d_0 \leq 0$, $c > \frac{d_1^2}{r_v - r_1 - r_0}$ so, 
\begin{align*}
2(r_v - r_1 - r_0)(r_v + 1)c + (r_v + 2)(d_1 + d_0) &> 2(r_v + 1)d_v^d - (r_v + 2)d_v^1 > 0.
\end{align*}
We consider the numerator of the $B_{\Delta_{X_2}}$ term given by (28). Then (29) can be rewritten
\begin{align*}
& r_k(r_v + 1)[2(r_v - r_0 - r_1)]c^2 + 2c(r_v + 1)(r_1 + r_0)(r_k c - d_k) \nonumber \\
& - 2\mu_01(r_v + 1)r_k(r_v - (r_0 + r_1)c - r_k(d_0 + d_1)(r_v + 2)\mu_01 \nonumber \\
& + (r_v + 2)(d_1 + d_0)d_k + r_k(r_v + 1)r_k^2c^2.
\end{align*}
The first terms is positive. The 2nd term is $-\mu_0 r_k[2(r_v + 1)(r_v - r_0 - r_1)c + (d_1 + d_0)(r_v + 2)]$, which is positive from (30). The 3rd term is positive if $d_k \leq 0$. Let us assume $d_k > 0$. Then, using the properties of $c$,
\begin{align*}
(r_v + 2)(d_1 + d_0)d_k + (r_v + 1)r_v(r_v c)(r_k c) &> -(r_v + 2)d_k d_v^d + r_v(r_v + 1)d_k d_v^1, \\
& \geq 0.
\end{align*}
This concludes the proof.

We obtain now the proof of Theorem 2.

Proof of Theorem 2. As we explained at beginning of Section 3, we only need to show that the relative K-polystability of a Kähler class (corresponding to some value of the constant $c$) implies that each indecomposable factor $U_i$ of $E$ is a stable bundle. The test configuration $M$ we defined is normal and not a product test configuration, so we get from Proposition 3.5
\[ \mu_0 - \mu_1 > 0. \]
This means that $V_1 = L$ does not destabilize $U_0$, i.e. $U_0$ is stable. The same reasoning by permuting of the $U_k$ concludes the proof.

3.9. Proof of Corollary 1.

Proof. As we have already mentioned in the introduction, one direction of the relative Yau–Tian–Donaldson Conjecture (see Conjecture 2), namely that the existence of an extremal Kähler metric in $2\pi c_1(L)$ implies the relative K-polystability of $(M, L)$ is established (for any polarized variety) in [35]. We shall thus discuss bellow the other direction of the conjecture in each of the cases (1)–(4) listed in the Corollary 1.

(1) Suppose $s = \ell = 1$. Then the automorphisms group of $(M = \mathbb{P}(E), L)$ has rank 0 (see the beginning of Section 3) unless $E$ has rank 1 and $M = \mathbb{P}^1$. Thus, the relative Donaldson–Futaki invariant of $(M, L)$ coincides with the usual Donaldson–Futaki invariant and it follows from [32, Theorem 5.13] (see also Section 3.6) that $E$ must be stable. By the Narasimhan–Seshadri Theorem [28], $M = \mathbb{P}(E)$ admits a CSC Kähler metric in each Kähler class, in particular in $c_1(L)$.

Suppose $s = \ell = 2$, i.e. $E = U_1 \oplus U_2$. If $(M = \mathbb{P}(E), L)$ is relative K-polystable then, by Theorem 2, $U_1$ and $U_2$ must be stable. In this case, [2, Theorem 1] shows that the existence of an extremal metric in $\Omega = 2\pi c_1(L)$ is equivalent to the positivity of the extremal polynomial $F_{\Omega}(z)$ (see [2, Definition 1]) over the interval $(-1, 1)$. Furthermore, by [2, Theorem 2], the latter is satisfied provided that the Kähler relative K-polystability of $(M, L)$ holds (see [10] for a precise definition) whereas the relative K-polystability of $(M, L)$ insures only that $F_{\Omega}(z) > 0$ on $(-1, 1) \cap \mathbb{Q}$. In the case when the base of $\mathbb{P}(E)$ is a curve,
the explicit form of the extremal polynomial $F_\Omega(z)$ is given at the beginning of Section 3.2 of [2]: it follows that for the rational class $\Omega = 2\pi c_1(\mathcal{L})$ (i.e. an admissible Kähler class corresponding to a rational parameter $x$) the constant $c$ is also rational and one obtains that $F_\Omega(z) > 0$ on $(-1, 1)$ if and only if $F_\Omega(z) > 0$ on $(-1, 1) \cap \mathbb{Q}$. Consequently, one can improve slightly [2, Theorem 2] if the base is a complex curve: the relative K-polystability of $(M, \mathcal{L})$ implies the positivity of the extremal polynomial $P_\Omega(z)$ and thus the existence of an extremal metric in $\Omega = 2\pi c_1(\mathcal{L})$.

(2) By the Narasimhan–Seshadri Theorem [28], $M = \mathbb{P}(E)$ admits a CSC Kähler metric in each Kähler class, in particular in $c_1(\mathcal{L})$.

(3) By Theorem 2, in this case $(M = \mathbb{P}(E), \mathcal{L})$ cannot be relative K-polystable.

(4) If $(M = \mathbb{P}(E), \mathcal{L} = \mathcal{L}_{q,p})$ is relative K-polystable, by Theorem 2, $E = \bigoplus_{s=1}^{k} U_s$ with $U_i$ stable. In this case [3, Theorem 3] or the main result of [7] implies that there exists $c_0 > 0$ such that any Kähler class $2\pi c_1(\mathcal{L}_{q,p})$ with $p/q > c_0$ is extremal (and hence also relatively $K$-polystable). □

4. Appendix

4.1. Integration over the simplex.

**Lemma 4.1.** Let us fix the integers $m_k \geq 0$. We have

$$\int_\Delta \prod_{k=0}^\ell (L_k(x))^{m_k} \, dv = \frac{m_0! \ldots m_\ell!}{(m_0 + \ldots + m_\ell + \ell)!}.$$  

**Proof.** This is elementary but we include a proof as we could not find a reference in the literature. It is not difficult to see that

$$\frac{1}{T^{m+n+1}} \int_0^T y^m (T - y)^n \, dy = \int_0^1 x^m (1 - x)^n \, dx = B(m + 1, n + 1),$$

where $B(., .)$ is the standard Beta function (also called the Euler integral of the first kind). Now, by integrating out one variable at each step, we obtain

$$\int_\Delta \prod_{k=0}^\ell (L_k(x))^{m_k} \, dv$$

$$= \int_0^1 \int_0^{1-x_1} \ldots \int_0^{1-x_1-\ldots-x_{\ell-1}} x_1^{m_1} \ldots x_\ell^{m_\ell} (1 - x_1 - \ldots - x_\ell)^{m_\ell} \, dx_1 \ldots dx_\ell, $$

$$= \int_0^1 x_1^{m_1} \int_0^{1-x_1} \ldots \int_0^{1-x_1-\ldots-x_{\ell-1}} x_\ell^{m_\ell} (1 - x_1 - \ldots - x_\ell)^{m_\ell} \, dx_1 \ldots dx_\ell,$$

$$= \int_0^1 x_1^{m_1} \int_0^{1-x_1} x_2^{m_2} \int_0^{1-x_1-\ldots-x_2} x_\ell^{m_\ell} (1 - x_1 - \ldots - x_\ell)^{m_\ell} \, dx_1 \ldots dx_\ell,$$

$$= \int_0^1 x_1^{m_1} \int_0^{1-x_1} x_2^{m_2} \int_0^{1-x_1-\ldots-x_2} \cdots \int_0^{1-x_1-\ldots-x_{\ell-2}} x_{\ell-2}^{m_{\ell-2}} (1 - x_1 - \ldots - x_{\ell-2})^{m_{\ell-2}} \, dx_1 \ldots dx_{\ell-2}.$$  

and so on, till we get

$$\int_\Delta \prod_{k=0}^\ell (L_k(x))^{m_k} \, dv$$

$$= B(m_0 + 1, m_\ell + 1) B(m_0 + m_\ell + 2, m_{\ell-1} + 1) \ldots$$

$$\times B(m_2 + 1, m_0 + m_\ell + \ldots + m_3 + \ell - 1) B(m_1 + 1, m_0 + m_\ell + \ldots + m_2 + \ell),$$
\[ \frac{m_0!...m_\ell!}{(m_0 + ... + m_\ell + \ell)!} \]

where for the last step we have used the classical relationship between the Beta and the Gamma function. \[ \square \]

We need the following lemma to treat the case of ranks equal to 1.

**Lemma 4.2.** The following relations hold:

\[
\int_{\partial \Delta} p_c(x) d\sigma = \frac{\pi R}{(r_V - 1)!} \left( (r_V - 1) \kappa c - \sum_{k=0}^\ell d_k \kappa_k \right),
\]

\[
\int_{\partial \Delta} x_j p_c(x) d\sigma = \frac{\pi R}{r_V!} \left( r_j r_V \kappa_j c - r_j \sum_{k=0}^\ell d_k \kappa_{k,j} - \kappa_j d_j \right).
\]

**Proof.** We start by computing \( \int_{\partial \Delta} \prod_{k=0}^\ell L_k^{r_k-1} d\sigma \). Denote \( \Delta'_j \) the standard simplex obtained by freezing the coordinate \( L_j(x) = 1 \). Then, applying Lemma (4.1),

\[
\int_{\partial \Delta} \prod_{k=0}^\ell L_k^{r_k-1} d\sigma = \sum_{j=0}^\ell \int_{\Delta'_j} \prod_{k=0,k\neq j}^\ell L_k^{r_k-1} dv,
\]

\[
= \sum_{j=0}^\ell \prod_{k=0,k\neq j}^\ell (r_k - 1)!,
\]

\[
= \frac{\pi R}{(\sum_{k=0}^\ell r_k - 2)!},
\]

\[
= \frac{\pi R}{(\sum_{k=0}^\ell r_k - 2)!} \kappa.
\]

Now, we obtain

\[
\int_{\partial \Delta} p_c(x) d\sigma = c \frac{\pi R}{(r_V - 2)!} \kappa - \sum_{k=0}^\ell \mu_k \frac{\pi R r_k}{(\sum_{k=0}^\ell r_k - 1)!} \kappa_k,
\]

\[
= c \frac{\pi R}{(r_V - 2)!} \kappa - \frac{\pi R}{(r_V - 1)!} \sum_{k=0}^\ell d_k \kappa_k,
\]

which leads to the first result. Now,

\[
\int_{\partial \Delta} x_j p_c(x) d\sigma = c \frac{\pi R}{(r_V - 1)!} r_j \kappa_j - \sum_{k=0}^\ell \mu_k r_k r_j \kappa_{k,j} \frac{\pi R}{r_V!} - \mu_j \frac{\pi R r_j (r_j + 1)}{r_V!} \kappa_j,
\]

and this gives the second result as \( \kappa_{j,j} = \kappa_j \). \[ \square \]

4.2. **Chern characters of symmetric tensor powers of vector bundles.** In this section we gather some technical formulas.
Proposition 4.1. Let $E$ a smooth vector bundle (or locally free sheaf) of rank $r_V$ over a smooth manifold. For $k \geq 1$, we denote the $S^kE$ the symmetric tensor power of order $k$ of $E$. Then,

$$rk(S^kE) = \binom{r_V - 1 + k}{k},$$

$$c_1(S^kE) = \binom{r_V - 1 + k}{k - 1} c_1(E),$$

$$c_2(S^kE) = \binom{r_V + k}{k - 1} c_2(E) + \frac{1}{2} \binom{r_V - 1 + k}{k - 2} c_1(E)^2,$$

$$c_3(S^kE) = \frac{1}{6} \left( \binom{r_V - 1 + k}{k - 3} c_1(E)^3 + \binom{r_V + k}{k - 2} c_1(E) c_2(E) \right) + \left( \binom{r_V + 1 + k}{k - 1} + \binom{r_V + k}{k - 2} \right) c_3(E).$$

Proof. This is done using splitting principle and symmetries. It can be checked easily that the formulas are correct for $E$ direct sum of 2 line bundles. \qed

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