Flow-Cut Gaps for Integer and Fractional Multiflows

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Abstract

Consider a routing problem instance consisting of a demand graph \( H = (V, E(H)) \) and a supply graph \( G = (V, E(G)) \). If the pair obeys the cut condition, then the flow-cut gap for this instance is the minimum value \( C \) such that there exists a feasible multiflow for \( H \) if each edge of \( G \) is given capacity \( C \). It is well-known that the flow-cut gap may be greater than 1 even in the case where \( G \) is the (series-parallel) graph \( K_{2,3} \). In this paper we are primarily interested in the “integer” flow-cut gap. What is the minimum value \( C \) such that there exists a feasible integer valued multiflow for \( H \) if each edge of \( G \) is given capacity \( C \)? We formulate a conjecture that states that the integer flow-cut gap is quantitatively related to the fractional flow-cut gap. In particular this strengthens the well-known conjecture that the flow-cut gap in planar and minor-free graphs is \( O(1) \) [14] to suggest that the integer flow-cut gap is \( O(1) \). We give several technical tools and results on non-trivial special classes of graphs to give evidence for the conjecture and further explore the “primal” method for understanding flow-cut gaps; this is in contrast to and orthogonal to the highly successful metric embeddings approach. Our results include the following:

• Let \( G \) be obtained by series-parallel operations starting from an edge \( st \), and consider orienting all edges in \( G \) in the direction from \( s \) to \( t \). A demand is compliant if its endpoints are joined by a directed path in the resulting oriented graph. We show that if the cut condition holds for a compliant instance and \( G + H \) is Eulerian, then an integral routing of \( H \) exists. This result includes as a special case, routing on a ring but is not a special case of the Okamura-Seymour theorem.

• Using the above result, we show that the integer flow-cut gap in series-parallel graphs is 5. We also give an explicit class of routing instances that shows via elementary calculations that the flow-cut gap in series-parallel graphs is at least \( 2 - o(1) \); this is motivated by and simplifies the proof by Lee and Raghavendra [20].

• The integer flow-cut gap in \( k \)-Outerplanar graphs is \( cO(k) \) for some fixed constant \( c \).

• A simple proof that the flow-cut gap is \( O(\log k^*) \) where \( k^* \) is the size of a node-cover in \( H \); this was previously shown by Günlük via a more intricate proof [13].

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1 Introduction

Given a (undirected) graph $G = (V, E)$ a routing or multiflow consists of an assignment $f: \mathcal{P} \to \mathbb{R}^+$ where $\mathcal{P}$ is the set of simple paths in $G$ and such that for each edge $e$, $\sum_{P \in \mathcal{P}(e)} f_P \leq 1$, where $\mathcal{P}(e)$ denotes the set of paths containing $e$. Given a demand graph $H = (V, E(H))$ such a routing satisfies $H$ if $\sum_{P \in \mathcal{P}(u,v)} f_P = 1$ for each $g = uv \in E(H)$, where $\mathcal{P}(u,v)$ denotes paths with endpoints $u$ and $v$ (one may assume a simple demand graph without loss of generality). If such a flow exists, we call the instance routable, or say $H$ is routable in $G$. Edges of $G$ and $H$ are called supply edges and demand edges respectively. The above notions extend naturally if each supply edge $e$ is equipped with a capacity $u_e$ and each demand edge $g$ is equipped with a demand $d_g$. If $u$ is an integral vector, we denote by $G_u$, the graph obtained by making $u_e$ copies of each edge $e$. $H_d$ is defined similarly. We call the routing $f$ integral (resp. half-integral) if each $f_P$ (resp. $2f_P$) is an integer.

For any set $S \subseteq V$ we denote by $\delta_G(S)$ the set of edges with exactly one end in $S$, and the other in $V - S$. We define $\delta_H(S)$ similarly. (For graph theory notation we primarily follow Bondy and Murty [4].) The supply graph $G$ satisfies the cut condition for the demand graph $H$ if $|\delta_G(S)| \geq |\delta_H(S)|$ for each $S \subset V$. We sometimes say that the pair $G, H$ satisfies the cut-condition. Clearly the cut condition is a necessary condition for the routability of $H$ in $G$. The cut-condition is not sufficient as shown by the well-known example where $G = K_{2,3}$ is a series-parallel graph with a demand graph (in dotted edges) as shown in the figure.

Given a graph $G$ and a real number $\alpha > 0$ we use $\alpha G$ to refer to the graph obtained from $G$ by multiplying the capacity of each edge of $G$ by $\alpha$. Given an instance $G, H$ that satisfies the cut-condition, the flow-cut gap is defined as the smallest $\alpha \geq 1$ such that $H$ is routable in $\alpha G$; we also refer to $\alpha$ as the congestion. We denote this quantity by $\alpha(G, H)$. Traditional combinatorial optimization literature has focused on characterizing conditions under which the cut-condition is sufficient for (fractional, integral or half-integral) routing, in other words the setting in which $\alpha(G, H) = 1$; see [29] for a comprehensive survey of known results. Typically, these characterizations involve both the supply and demand graphs. A prototypical result is the Okamura-Seymour Theorem [24] that states that the cut-condition is sufficient for a half-integral routing if $G$ is a planar graph and all edges of $H$ are between the nodes of a single face of $G$ in some planar embedding. The proof of such result rely on what we will term “primal-methods” in that they try to directly exhibit routings of the demands, rather than appealing to dual solutions.

On the other hand, since the seminal work of Leighton and Rao [19] on flow-cut gaps for uniform and product multiflow instances, there has been an intense focus in the algorithms and theoretical computer science community on understanding flow-cut gap results for classes of graphs. This was originally motivated by the problem of finding (approximate) sparse cuts. A fundamental and important connection was established in [22, 3] between flow-cut gaps and metric-embeddings. More specifically, for a graph $G$, let $\alpha(G)$ be the largest flow-cut gap over all possible capacities on the edges of $G$ and all possible demand graphs $H$. Also let $c_1(G)$ denote the maximum, over all possible edge lengths on $G$, of the minimum distortion required to embed the finite metric on the nodes of $G$ (induced by the edge lengths) into an $l_1$-space. Then the results in [22, 3] showed that $\alpha(G) \leq c_1(G)$ and subsequently [14] showed that $\alpha(G) = c_1(G)$. Using Bourgain’s result that $c_1(G) = O(\log |V|)$ for all $G$, [22, 3] showed that $\alpha(G) = O(\log |V(G)|)$, and further refined it to prove that $\alpha(G, H) = O(\log |E_H|)$. Numerous subsequent results have explored this connection to obtain a variety of flow-cut gap results. The proofs via metric-embeddings are “dual”-methods since they work by embedding the metric induced by the dual of the linear program for the maximum concurrent multicommodity flow. The embedding approach has been successful in obtaining flow-cut gap results (amongst several other algorithmic applications) as well as forging deep connections between various areas of discrete and continuous mathematics. However, this approach does not directly give us integral routings even in situations when they do exist.

In this paper we are interested in the integer flow-cut gap in undirected graphs. Given $G, H$ that satisfy the cut-condition, what is the smallest $\alpha$ such that $H$ can be integrally routed in $\alpha G$? Is there a relationship between
the (fractional) flow-cut gap and the integer flow-cut gap? A result of Nagamochi and Ibaraki relates the two gaps in directed graphs. Let \( G = (V, A) \) and \( H = (V, R) \) be a supply and demand digraph, respectively. We call \((G, H)\) \textit{cut-sufficient} if for each capacity function \( u : A \to \mathbb{Z}^+ \) and demand function \( d : R \to \mathbb{Z}^+ \), \( G_u, H_d \) satisfying the cut-condition implies the existence of a fractional multiflow for \( H_d \) in \( G_u \).

**Theorem 1.1** (\cite{23}) If \((G, H)\) is cut-sufficient, then for any integer capacity vector \( u \) and integer demand vector \( d \) such that \( G_u, H_d \) satisfy the cut condition, there is an integer multiflow for \( H_d \) in \( G_u \).

The above theorem does not extend to the undirected case. Consider taking \( G \) to be a cycle and \( H \) to be a complete graph. Then it is known that \((G, H)\) is cut-sufficient but we are not guaranteed an integral flow for integer valued \( u \) and \( d \); for example when \( G \) is a 4 cycle with unit capacities and \( H \) consists of two crossing edges with unit demands. For integer valued \( u \) and \( d \), however, there is always a half-integral routing of \( H_d \) in \( G_u \) whenever \((G_u, H_d)\) satisfies the cut-condition. We may therefore ask if a weaker form of Theorem 1.1 holds in undirected graphs. Namely, where we only ask for half-integral flow instead of integral flows. One case where one does get such a half-integral routing in undirected graphs is the following. Consider the case when \( G = H \); if the pair \((G, G)\) is cut-sufficient we simply say that \( G \) is cut-sufficient. It turns out that this is precisely the class of \( K_5 \)-minor free graphs (Seymour \cite{30}; cf. Corollary 75.4d \cite{29}). Moreover we have the following.

**Theorem 1.2** (Seymour) If \( G \) is cut-sufficient, then for any nonnegative integer weightings \( u, d \) on \( E(G) \) for which \( G_u, G_d \) satisfies the cut condition, there is a half-integral routing of \( G_d \) in \( G_u \). Moreover, if \( G_u + G_d \) is Eulerian, then there is an integral routing of \( G_d \).

In this paper we ask more broadly, whether the fractional and integral flow-cut gaps are related even in settings where the flow-cut gap is greater than 1. We formulate the conjecture below.

**Conjecture 1.3** (Gap-Conjecture) Does there exist a global constant \( C \) that satisfies the following. Let \( G = (V, E) \) and \( H = (V, R) \) be a supply and demand graph respectively. Suppose that for each capacity function \( u : A \to \mathbb{Z}^+ \) and demand requirement \( d : R \to \mathbb{Z}^+ \), \( G_u, H_d \) satisfy the cut-condition implies that there is a fractional multiflow for \( H_d \) in \( G_d \) with congestion \( \beta \). Then this implies that there is an integer multiflow for \( H_d \) in \( G_d \) with congestion \( C\beta \).

We do not currently know if the statement holds for \( C = 2 \) in instances with \( \beta = 1 \) (thus generalizing Seymour’s theorem mentioned above). There are several natural weakenings of the conjecture that are already unknown. For instance, one may allow \( C \) to depend on a class of instances (such as planar or series parallel supply graphs). More generally, it would be of interest to bound the integer flow cut gap as some \( g(\beta) \), e.g., \( g(\beta) = \mathcal{O}(\text{poly}(\beta)) \).

Previously other conjectures relating fractional and integer multiflows were shown to be false. For instance, Seymour conjectured that if there is a fractional multiflow for \( G, H \), then it implies a half-integer multiflow. These conjectures have been strongly disproved (see \cite{29}). Note that our conjecture differs from the previous ones in that we relate the flow-cut gap values for hereditary classes of instances on \( G, H \).

The Gap-Conjecture has several important implications. First, it would give structural insights into flows and cuts in graphs. Second, it would allow fractional flow-cut gap results obtained via the embedding-based approaches to be translated into integer flow-cut gap results. Finally, it would also shed light on the approximability of the congestion minimization problem in special classes of graphs. In congestion minimization we are given \( G, H \) and are interested in the least \( \alpha \) such that \( \alpha G \) has an integer routing for \( H \). Clearly, the congestion required for a fractional routing is a lower bound on \( \alpha \); moreover this lower bound can be computed in polynomial time via linear programming. Almost all the known approximation guarantees are with respect to this lower bound; even in directed graphs an \( \mathcal{O}(\log n / \log \log n) \) approximation is known via randomized rounding \cite{26}. In general undirected graphs, this problem is hard to approximate to within an \( \Omega(\log \log n) \)-factor \cite{11}. However, for planar graphs and graphs that exclude a fixed minor, it is speculated that the problem may admit
an $O(1)$ approximation. The Gap-Conjecture relates this to the conjecture of Gupta et al. \cite{14} that states that the fractional flow-cut gap is $O(1)$ for all graphs that exclude a fixed minor. Thus the congestion minimization problem has an $O(1)$ approximation in planar graphs if the Gupta et al. conjecture and the Gap-Conjecture are both true. We also note that an $O(1)$ gap between fractional and integer multiflows in planar graphs (or other families of graphs) would shed light on the Gap-Conjecture.

Our current techniques seem inadequate to resolve the Gap-Conjecture. It is therefore natural to prove the Gap-Conjecture in those settings where we do have interesting and non-trivial upper bounds on the (fractional) flow-cut gap. Note that the conjecture follows easily when $G$ and $H$ are unrestricted (complete graphs); in this case the flow-cut gap is $\Omega(\log n)$; one may consider $G$, a bounded degree expander, with $H$, a uniform multiflow \cite{19}. On the other side, randomized rounding shows that the integer flow-cut gap is $O(\log n)$. Now, if $G$ is a complete graph and $H$ is a complete graph on a subset of $k$ nodes of $G$ then the flow-cut gap for such instances is $\Omega(\log k)$; this easily follows from the above mentioned general result. For these instances the fractional flow-cut gap improves to $O(\log k)$, as shown in \cite{22,3}. One can obtain an improvement for the integer flow-cut gap but one cannot employ simple randomized rounding; in \cite{10} it is shown that the integer flow-cut gap for these instances is $O(\text{polylog}(k))$; this relies on the results in \cite{8,16}.

In a sense, the Gap-Conjecture is perhaps more relevant and interesting in those cases where the flow-cut gap is $O(1)$. We focus on series-parallel graphs and $k$-Outerplanar graphs for which we know flow-cut gaps of $2 \cdot 6$ and $c^k$ (for some universal constant $c$) \cite{7} respectively. Proving flow-cut gaps for even these restricted families of graphs has taken substantial technical effort. In this paper we affirm that one can prove similar bounds for these graphs for the integer flow-cut gap. For instance, in series parallel instances, we show that the integer flow-cut gap is at most 5 (and we conjecture it is 2).

**Overview of results and techniques:** In this paper we focus especially on applying primal methods to two classes of graphs for which the flow-cut gap is known to be $O(1)$: series parallel graphs and $k$-Outerplanar graphs.

The first proof that series parallel instances had a constant flow-cut was given in \cite{14}; subsequently a gap of 2 was shown in \cite{6}. This latter upper bound is tight since it is shown in \cite{20} that there are instances where the gap is arbitrarily close to 2. We give a simpler proof of the lower bound in this paper that is based on an explicit (recursive) instance and elementary calculations — our proof is inspired by \cite{20} but avoids their advanced metric-embeddings machinery.

In Section 4.1 we show that for series-parallel graphs the integer flow-cut gap is at most 5. The primal-method has generally been successful in identifying restrictions on demand graphs for which the cut-condition implies routability. We follow that approach and identify several classes of demands in series-parallel graphs for which cut-condition implies routability (see Sections 3.1 - 3.4). The main class exploited to obtain the congestion 5 result, are the so-called compliant demands (Section 3.4). However, the critical base case for compliant demands boils down to determining classes of demands on $K_{2m}$ instances for which the cut condition implies routability. In fact, for $K_{2m}$ instances, we are able to give a complete characterization of demand graphs $H$ for which $(K_{2m}, H)$ is cut sufficient - see Section 3.3. Moreover, we conjecture that this forbidden minor characterization carries over for all series-parallel graphs (it does not completely characterize cut-sufficiency in general however).

One ingredient we use is a general proof technique for “pushing” demands similar to what has been used in previous primal proofs; for instance in the proof of the Okamura-Seymour theorem \cite{24}. We try to replace a demand edge $uv$ by a pair of edges $ux, xv$ to make the instance simpler (we call this *pushing to x*). Failing to push, identifies some tight cuts and sometimes these tight cuts can be used to shrink to obtain an instance for which we know a routing exists. This contradiction means that we could have pushed in the first place.

In \cite{7}, an upper bound of $c^k$ (for some constant $c$) is given for the flow-cut gap in $k$-Outerplanar graphs. In this paper (Section 4.3), we show that the integer flow cut gap in this case is $c^{O(k)}$. In this effort, we explicitly employ a second proof ingredient which is a simple rerouting lemma that was stated and used in \cite{9} (see Section 4.2). Informally speaking the lemma says the following. Suppose $H$ is a demand graph and for simplicity assume it consists of pairs $s_1t_1, \ldots, s_kt_k$. Suppose we are able to route the demand graph $H'$
consisting of the edges \( s_1s'_1, t_1t'_1, \ldots, s_k s'_k, t_k t'_k \) in \( G \) where \( s'_1, t'_1, \ldots, s'_k, t'_k \) are some arbitrary intermediate nodes. Let \( H'' \) be the demand graph consisting of \( s'_1, t'_1, \ldots, s'_k, t'_k \). The lemma states that if \( G, H \) satisfies the cut-condition and the aforementioned routing exists in \( G \) then \( 2G, H'' \) satisfies the cut-condition. Clearly we can compose the routings for \( H' \) and \( H'' \) to route \( H \). The advantage of the lemma is that it allows us to reduce the routing problem on \( H \) to that in \( H'' \) by choosing \( H' \) appropriately. This simple lemma and its variants give a way to prove approximate flow-cut gaps effectively.

The rerouting lemma sometimes leads to very simple and insightful proofs for certain results that may be difficult to prove via other means — see \[29\]. In this paper we give two applications of the lemma. We give (in Section 4.4) a very short and simple proof of a result of Günlük \[13\]; he refined the result of \[22, 4\] and showed that \( \alpha(G, H) = O(\log k^*) \) where \( k^* \) is the node-cover size of \( H \). Clearly \( k^* \leq |E_H| \) and can be much smaller. We also show that the integer flow-cut gap for \( k \)-Outerplanar graphs is \( cO(k) \) for some universal constant \( c \); in fact we show a slightly stronger result (see Section 4.3). Previously it was known that the (fractional) flow-cut gap for \( k \)-Outerplanar graphs is \( c^k \) \[7\].

Our integer flow-cut gap results imply corresponding new approximation algorithms for the congestion minimization problem on the graph classes considered. Apart from this immediate benefit, we feel that it is important to complement the embedding-based approaches to simultaneously develop and understand corresponding tools and techniques from the primal point of view. As an example, Khandekar, Rao and Vazirani \[16\], and subsequently \[25\], gave a primal-proof of the Leighton-Rao result on product multicommodity flows \[19\]. This new proof had applications to fast algorithms for finding sparse cuts \[16, 25\] as well as approximation algorithms for the maximum edge-disjoint path problem \[28\].

2 Basics and Notation

We first discuss some basic and standard reduction operations in primal proofs for flow-cut gaps and also set up the necessary notation for series-parallel graphs.

2.1 Some Basic Operations Preserving the Cut Condition

We present several operations that turn an instance \( G, H \) satisfying the cut condition into smaller instances with the same property. We call an instance \( G, H \) Eulerian if \( G + H \) is Eulerian; we also seek to preserve this property.

For \( S \subseteq V \), the capacity of the cut \( \delta_G(S) \), is just \( |\delta_G(S)| \) (or sum of capacities if edges have capacities). Similarly, the demand of such a cut is \( |\delta_H(S)| \). Hence the surplus is \( \sigma(S) = |\delta_G(S)| - |\delta_H(S)| \). The set \( S \), and cut \( \delta(S) \), is called tight if \( \sigma(S) = 0 \). The cut condition is then satisfied for an instance \( G, H \) if \( \sigma(S) \geq 0 \) for all sets \( S \). One may naturally obtain “smaller” routing instances from \( G, H \) by performing a contraction of a subgraph of \( G \) (not necessarily a connected subgraph) and removing loops from the resulting \( G' \), and in the resulting demand graph \( H' \). It is easily checked that if \( G, H \) has the cut condition, then so does any contracted instance.

We call a subset \( A \subseteq V(G) \) central if both \( G[A] \) and \( G[V - A] \) are connected. The following is well-known cf. \[29\].

**Lemma 2.1** \( G, H \) satisfy the cut condition if and only if the surplus of every central set is nonnegative.

1-cut reduction: This operation takes an instance where \( G \) has a cut node \( v \) and consists of splitting \( G \) into nontrivial pieces determined by the components of \( G - v \). Demand edges \( f \) with endpoints \( x, y \) in distinct components are replaced by two demands \( xv, yv \) and given over to the obvious instance. One easily checks that each resulting instance again satisfies the cut condition. A simple argument also shows that the Eulerian property is maintained in each instance if the original instance was Eulerian.
Parallel reduction: This takes as input an instance with a demand edge \( f \) and supply edge \( e \), with the same endpoints. The reduced instance is obtained by simply removing \( f, e \) from \( H \) and \( G \) respectively. Trivially the new instance satisfies the cut condition and is Eulerian if \( G, H \) was.

Slack reduction: This works on an instance where some edge \( e \) (in \( G \) or \( H \)) does not lie in any tight cut. In this case, if \( e \in G \), we may remove \( e \) from \( G \) and add it to \( H \). If \( e \in H \), we may add two more copies of \( e \) to \( H \). Again, this trivially maintains the cut condition and the Eulerian property.

Push operations: Such an operation is usually applied to a demand edge \( xy \) whose endpoints lie in distinct components of \( G - \{u, v\} \) for some 2-cut \( u, v \). Pushing a demand \( xy \) to \( u \) involves replacing the demand edge \( xy \) by the two new demands \( xu, uy \). Such an operation clearly maintains the Eulerian property but it may not maintain the cut condition. We have

Fact 2.2 Pushing a demand \( xy \) to \( u \) maintains the cut condition in an Eulerian instance if and only if there is no tight cut \( \delta(S) \) that contains \( u \) but none of \( x, y, v \).

We call the preceding four operations basic, and we generally assume throughout that our instance is reduced in that we cannot apply any of these operations. In particular, we may generally assume that \( G \) is 2-node connected.

2.2 Series Parallel Instances

A graph is series-parallel if it can be obtained from a single edge graph \( st \) by repeated application of two operations: series and parallel operations. A parallel operation on an edge \( e \) in graph \( G = (V, E) \) consists of replacing \( e \) by \( k \geq 1 \) new edges with the same endpoints as \( e \). A series operation on an edge consists of replacing \( e \) by a path of length \( k > 1 \) between the same endpoints. Series-parallel graphs can also be characterized as graphs that do not contain \( K_4 \) as minor.

A capacitated graph refers to a graph where each edge also has an associated positive integer capacity. For purposes of routing, any such edge may be viewed a collection of parallel edges. Conversely, we may also choose to identify a collection of parallel edges as a single capacitated edge. In either case, for a pair of nodes \( u, v \), we refer to the capacity between them as the sum of the capacities of edges with \( u, v \) as endpoints. For a pair of nodes \( u, v \) a bridge is either a (possibly capacitated) edge between \( u, v \) or it is a subgraph obtained from a connected component of \( G - \{u, v\} \) by adding back in \( u, v \) with all edges between \( u, v \) and the component. In the latter case, the bridge is nontrivial. A strict cut is a pair of nodes \( u, v \) with at least 2 nontrivial bridges and at least 3 bridges.

Lemma 2.3 If \( G \) is a 2-node-connected series-parallel graph, then either it is a capacitated ring, or it has a strict 2-cut.

Proof: Suppose that \( G \) has no strict 2-cut. Let \( e_1, e_2, \ldots, e_k \) be the result of the first parallel operation on the original edge \( st \). Let \( P_1 = a_1, a_2, \ldots, a_l \) be the path obtained after subdividing \( e_1 \) if there is ever a nontrivial series operation applied to \( e_1 \) (where the \( a_j \)'s are the edges). Since \( G \) is 2-node-connected, \( k \geq 2 \). If \( k = 2 \), then each edge of \( P_1, P_2 \) results in a capacitated edge in \( G \) and hence \( G \) is a capacitated ring. If this were not the case, then some \( a_j \) has a parallel operation followed by a series operation, and hence the ends of \( a_j \) would form a strict cut in \( G \). So suppose that \( k \geq 3 \). Clearly at most one \( e_i \) is subdivided, say \( P_1 \), or else \( s, t \) is a strict cut. Again, either each edge of \( P_1 \) becomes a capacitated edge. Otherwise any series operation to some \( a_p \) results in its endpoints inducing a strict 2-cut.

The following lemma is useful in applying the push operation (cf. Fact 2.2).

Lemma 2.4 Let \( u, v \) be a pair of nodes in a series parallel graph, and let \( l, r \) be a 2-cut separating \( u \) from \( v \). Let \( L \) be a central set containing \( l \), but not \( u, r \) and \( v \); and let \( R \) be a central set containing \( r \), but not \( u, l \) and \( v \). Then \( L \setminus R \) and \( R \setminus L \) are central.
**Proof:** First, we prove that $V \setminus (R \setminus L) = (V \setminus R) \cup L$ is connected. This is the case since $(V \setminus R)$ and $L$ are each connected, and both contain $l$.

Then, we prove that $R \setminus L$ is connected. Let $R_1$ be the connected component of $R \setminus L$ that contains $r$. Assume $R \setminus L$ contains another connected component $R_2$. The 2-cut $l, r$ separates $u$ from $v$, so $V - l, r$ contains at least two connected components $C_l$ and $C_r$, containing $u$ and $v$ respectively. Since $R_2$ contains neither $l$ nor $r$, it is entirely in a connected component $C$ of $V - l, r$. Assume without loss of generality that $C$ is not $C_u$. Since $R$ is connected, there is a path $P$ in $R$ from $R_1$ to $R_2$; choose $R_2$ so that $P$ is minimal. It follows that $P = (r_1, Q, r_2)$ where $r_1 \in R_1$, and $Q$ is a subpath of $R \cap L$. Note that $P \cup R_2$ is disjoint from $C_u$.

Since $R_1$ and $R_2$ are outside of $L$, there is a path $P'$ outside of $L$ connecting them. This path may have the form $a_1, Q_1, I_2, Q_2, \ldots, Q_p, a_2$ where $a_i \in R_i$, each $Q_i$ is a path in $V \setminus (R \cup L)$ and each $I_j$ is a path in some component (other than $R_1, R_2$ in the graph induced by $R \setminus L$). Once again $P'$ is disjoint from $C_u$, since it contains neither $l, r$ and hence is included in $C$. Since the internal nodes of $P'$ lie entirely in $V \setminus (R \cup L)$ and $P'$'s lie in $R \cap L$, the paths are internally node-disjoint paths connecting $R_1, R_2$ in the graph $G - C_u$.

Since $G[L]$ is connected, there is a path $P_1$ joining $l$ to an internal node of $P$ within $L$; since $G[V \setminus R]$ is connected, there is a path $P_2$ in this graph connecting $l$ to some internal node of $P'$. Choose these paths minimal, and let $x$ be the first point of intersection of $P_1$ with $P$, and $y$ be the first intersection of $P_2$ with $P'$. Once again, note that $P_1 \cup P_2 \setminus l$ does not contain $l, r$ and hence is contained in $G - C_u$ since $y, x \in C$. Choose $z \in P_1$ such that the subpath of $P_2$ from $z$ to $y$ does not intersect $P_1$. Clearly $z \in L \setminus R$ by construction. We now have constructed a $K_4 - e$ minor (in fact a homeomorph) on the nodes $z, x, y$ and $r \in R_1$. This graph is also contained within $C$, except for $r$ and possibly $z$ if $z = l$. We may now extend this to a $K_4$ minor using $C_u$ and the subpath of $P_1$ joining $l, z$.

So $R \setminus L$ is connected, so it is central. $L \setminus R$ is central by the same argument.

3 **Instances where the Cut Condition is Sufficient for Routing**

3.1 **Fully Compliant Instances**

Let $G$ be a series-parallel supply graph and $H$ a demand graph defined on the same set of nodes. An edge $e$ of $H$ is fully compliant if $G + e$ is also series-parallel. An instance $G, H$ is fully compliant if $G + e$ is series-parallel for each $e \in E(H)$. We note that $H$ itself may not be series-parallel in fully compliant instances. For instance, we could take $G$ to be a ring and $H$ to be the complete demand graph.

In this section we prove that fully compliant instances $G, H$ are integrally routable if they satisfy the cut condition and $G + H$ is Eulerian. This forms one base case in showing that compliant instances (introduced in Section [3.4]) are routable, which in turn will yield our congestion 5 routing result for general series-parallel instances.

We start with several technical lemmas.

**Lemma 3.1** Let $G$ be 2-node-connected series-parallel graph. A demand edge $uv$ is fully compliant if and only if there is an edge $uv$ in $G$, or $u, v$ is a 2-cut in $G$.

**Proof:** Suppose that there is no edge $uv$ in $G$, and that $u, v$ is not a 2-cut in $G$. Since $G$ is 2-connected, it contains at least 2 node-disjoint paths from $u$ to $v$. Since there is no edge $uv$, each of these paths contain at least one node apart from $u$ and $v$. And since $u, v$ is not a 2-cut, there is a path connecting these two paths in $G - \{u, v\}$. Therefore, $uv$ is not fully compliant, because $G + uv$ contains a $K_4$.
Suppose \( G + uv \) contains a \( K_4 \). Then there is no edge \( uv \) in \( G \), since it is series-parallel. Let \( S_1, S_2, S_3 \) and \( S_4 \) be the sets of nodes in \( G + uv \) that form the \( K_4 \) minor. Without loss of generality \( u \) and \( v \) are in \( S_1 \) and \( S_2 \) respectively; they have to be in different sets for otherwise \( G \) will have a \( K_4 \) minor. Since \( S_3 \) and \( S_4 \) are adjacent, they are contained in the same connected component of \( G - \{u, v\} \), say \( C \). We claim that there cannot be another connected component \( C' \) in \( G - \{u, v\} \). For if it did, then since \( G \) is 2-node-connected, both \( u \) and \( v \) have an edge to \( C' \). This implies that there is a path between \( S_1 \) and \( S_2 \) in \( G \) that avoids \( C \) (and hence \( S_3 \) and \( S_4 \)); then \( S_1, S_2, S_3, S_4 \) would form a \( K_4 \) minor in \( G \) which is impossible since \( G \) is series-parallel. Therefore, there is no other connected component in \( G - \{u, v\} \), and \( u, v \) is not a 2-cut.

\[ \square \]

**Lemma 3.2** Let \( G \) be 2-node-connected series-parallel graph. If an edge \( uv \) is not fully compliant, then there is a 2-cut separating \( u \) from \( v \).

**Proof:** Suppose an edge \( uv \) is not fully compliant, and there is no 2-cut separating \( u \) from \( v \). Then there are three node disjoint paths connecting \( u \) and \( v \). Since \( uv \) is not fully compliant, there is no edge \( uv \) and each of these paths contain at least one node apart from \( u \) and \( v \). By Lemma 3.1, \( u, v \) is not a 2-cut, so there are paths connecting these paths in \( G - \{u, v\} \). This creates a \( K_4 \) minor in \( G \), which is impossible.

\[ \square \]

**A 2-Cut Reduction:** A partition of \( G \) is any pair of graphs \( (G_1, G_2) \) such that: (i) \( V(G_1) \cap V(G_2) = \{u, v\} \), for distinct nodes \( u, v \); (ii) \( E(G) \) is the disjoint union of \( E(G_1), E(G_2) \) and (iii) \( |V(G_i)| \geq 3 \) for each \( i \). Thus any 2-cut admits possibly several partitions, and we refer to any such as a partition for \( \{u, v\} \). We say that a demand graph \( H \) has no demands crossing a partition for \( u, v \), if \( H \) can be written as a disjoint union \( H_1 \cup H_2 \) where for \( i = 1, 2, H_i \) is a subgraph of \( H[V(G_i)] \), the demand graph induced by one side of the partition. Note that even if \( G, H \) satisfy the cut condition, it may not be the case that \( G_i, H_i \) does. It is easily seen however that we may always add some number \( k \) of parallel edges between \( u, v \) in each \( G_i \) so that \( G_i, H_i \) does have the cut condition. For \( i = 1, 2 \) the smallest such number is called the deficit of the reduced instance \( G_i, H_i \). One easily checks that the deficit of at least one of the reduced instances is at most 0 if \( G, H \) satisfies the cut condition.

**Lemma 3.3** Let \( G, H \) satisfy the cut condition and let \( (G_1, G_2) \) be a partition for 2-cut \( \{u, v\} \) such that \( H \) has no demands crossing the partition. Let \( k \) be the deficit of \( G_1, H_1 \); without loss of generality \( k \geq 0 = k_2 \). Let \( H' \) be obtained by adding \( k \) demand edges between \( u, v \) in \( H_2 \). We also let \( H'_1 = H_1 \). Let \( G' \) be obtained by adding \( k \) supply edges between \( u, v \) to \( G_1 \); we also let \( G'_2 = G_2 \). Then \( G'_1, H'_1 \) satisfies the cut condition

\[ \square \]

**Proof:** First, let \( S \subset V(G_1) \) which defines \( G_1 \)'s deficit. That is the number of demand edges in \( \delta_{H_1}(S) \) is \( k_1 \) greater than \( |\delta_{G_1}(S)| \). Without loss of generality, \( u \in S \). Also, we must have \( v \notin S \), for otherwise \( S \cup V(G_2) \) violates the cut condition for \( G, H \). Clearly, \( G'_1, H_1 \) now satisfies the cut condition. Next suppose that \( G_2, H_2 \) does not obey the cut condition. Then there exists some \( S' \) containing \( u \) and not \( v \), such that

\[ |\delta_{H_2}(S')| + k_1 > |\delta_{G_2}(S')| \]

But then \( S \cup S' \) violates the cut condition for \( G, H \).

Let \( G + H \) be Eulerian, and note that all nodes except possibly \( u, v \) have even degree in \( G'_1 + H'_1 \) (and in \( G_1 + H_1 \)). It is thus sufficient to show that \( u, v \) also have even degree. Let \( p \) be the parity of \( u \) in \( G_1 + H_1 \). Since any graph has an even number of odd-degree nodes, \( v \) must also have parity \( p \) in \( G_1 + H_1 \). Let \( s = |\delta_{G_1}(S)|, d = |\delta_{H_1}(S)| \) and so \( k_1 = d - s \). Since \( S \) separates \( u, v \) we have that \( d + s = |\delta_{G_1 + H_1}(S)| \) has parity \( p \) and hence \( k_1 = d - s = d + s - 2s \) does as well. That is, the deficit \( k_1 \) has parity \( p \) and so \( u \) and \( v \) have even degree in \( G'_1 + H'_1 \). This immediately implies the same for \( G'_2 + H'_2 \).

The last part of the lemma is immediate, since each demand from \( H \) is routed in \( G_1 \cup G_2 \) together with the \( k_1 \) new supply edges in \( G_1 \). However, any such new edge was routed in \( G_2 \) together with the demands in \( H_2 \), so we can simulate the edge by routing through \( G_2 \).

\[ \square \]
**Fully Compliant instances are routable:** Let \( G, H \) satisfy the cut condition where \( G \) is series parallel, and each edge in \( H \) is compliant (with \( G \)). We now show that if \( G + H \) is Eulerian, then there is an integral routing of \( H \) in \( G \). The proof is algorithmic and proceeds by repeatedly applying the reduction described above and those from Section 2.1. In particular, at any point if there is a slack, parallel or 1-cut reduction we apply the appropriate operation. Thus we may assume that \( G \) is 2-node connected, and that each demand or supply edge lies in a tight cut, and no demand edge is parallel to a supply edge.

If \( G \) is a capacitated ring, then the result follows from the Okamura-Seymour theorem. Otherwise by Lemma 2.3, there is some strict cut \( u, v \). Thus \( G \) has 3 node-disjoint paths \( P_1, P_2, P_3 \) between \( u, v \) and so there is a partition \( G_1, G_2 \) for \( u, v \) such that either \( G_1 \) or \( G_2 \) has 2 node-disjoint paths between \( u, v \). Without loss of generality, for \( i = 1, 2 \) \( P_i \) is contained in \( G_i \). But then there could not be any demand edge crossing the partition. Since if \( f \in E(H) \) has one end in \( G_1 - \{u, v\} \) and the other in \( G_2 - \{u, v\} \), then we would have a \( K_4 \) minor in \( G + f \). Thus we may decompose using Lemma 3.3 to produce two smaller instances and inductively find routings for them. These two routings yield a routing for \( G, H \) by the last part of Lemma 3.3.

### 3.2 Routable \( K_{2m} \), Instances

A \( K_{2m} \)-instance consists of a supply graph \( G = K_{2m} \) with a 2-cut \( s, t \) and \( m \) nodes \( v_1, \ldots, v_m \) of degree two, each adjacent to \( s, t \). We may possibly also have an edge between \( s, t \). We also have a demand graph \( H = (V = V(G), F) \) on the same node set \( V \), and edge capacities \( u \) on \( G \)'s edges. A path-bipartite instance is one where the demands with both ends in the \( m \) degree 2 nodes form a bipartite graph. One special case is a so-called tri-source instance, where if \( v_i v_j \in F \), then either \( i \) or \( j \) is 1. The figure shows a tri-source and a path-bipartite instance.

**Lemma 3.4** If \( G, H \) is a path-bipartite instance satisfying the cut condition and \( G + H \) is Eulerian, then there is an integral routing of \( H \) in \( G \).

**Proof:** If any demand edge \( f \) is parallel to an edge \( e \in G \), then we may delete \( f \) and set \( u_e = u - 1 \) to obtain a new Eulerian instance that one easily checks must satisfy the cut condition. Note that we may also assume that either there are no \( st \) demands, or no \( st \) supply for otherwise we could reduce the instance. Thus we may assume that every demand edge either joins \( s, t \) or is of the form \( v_i v_j \) for some \( i \neq j \). Suppose first that some node \( v_i \) does not define a tight cut. Consider the new instance obtained by adding a new supply edge between \( s, t \) and remove one unit of capacity from each edge incident to \( v_i \). The only central cuts whose supply is reduced is the cut induced by \( v_i \); as this cut was not tight, it still satisfies the Eulerian and cut conditions. The new instance is also smaller in our measure and so we assume that each \( \delta(v_i) \) is tight.

Let \( X, Y \) be a bipartition of the degree two nodes for the demands amongst them. Let \( r_j = u_{v_j t} \) and \( l_j = u_{av_j} \) for each \( j \). For a subset \( S \) of the degree two nodes, we also let \( r(S) = \sum_{v_i \in S} r_i \) (similarly for \( l(S) \)). Hence actual supply out of \( \delta_G(X) \) is just \( r(X) + l(X) \). We also let \( d(i) \) denote the total demand out of \( v_i \); hence we have that \( d(i) = r_i + l_i \) for each \( i \). In particular, any subset of \( X \) or of \( Y \) is tight. Thus if \( r(Y) < r(X) \), then \( X \cup t \) is a violated cut and so \( r(Y) \geq r(X) \). Similarly, \( r(X) \geq r(Y) \). Thus \( r(X) = r(Y) \) and the same reasoning shows that \( l(X) = l(Y) \). Moreover, the above argument shows that there are no \( st \) demands or else \( X \cup t \) is a violated cut.
Assume that there are two tight cuts $S$ and $T$ separating $s$ from $t$. We group the nodes of degree two and contract them in eight nodes, by inclusion in $X$ or $Y$, $S$ or $V \setminus S$, $T$ or $V \setminus T$. Since nodes in $X$ (or $Y$) are not adjacent to each other, each node is still tight. The result is shown in the adjacent figure.

Some nodes and demand edges may not actually exist.

We denote by $l_x$ (resp. $r_x$) the capacity of the supply edge between a node $x$ and $s$ (resp. $t$). We denote by $d_{xy}$ the demand between nodes $x$ and $y$. We denote by $u$ the capacity of the $st$ supply edge.

The cut induced by $S$ and $T$ are tight which implies:

$$l_a + l_b + l_c + l_d + r_e + r_f + r_g + r_h - d_{af} - d_{ah} - d_{be} - d_{bg} - d_{cf} - d_{ch} - d_{de} - d_{dg} + u = 0,$$

$$l_a + l_b + r_c + r_d + l_e + l_f + r_g + r_h - d_{ad} - d_{ah} - d_{bc} - d_{bg} - d_{cf} - d_{ch} - d_{de} - d_{fg} + u = 0.$$

The surplus of cuts induced by $\{s, d, f, g, h\}$ and $\{s, c, e, g, h\}$ must be positive:

$$l_a + l_b + l_c + r_d + l_e + r_f + r_g + r_h - d_{ad} - d_{af} - d_{ah} - d_{cd} - d_{cf} - d_{ch} - d_{de} - d_{ef} - d_{eh} - d_{bg} + u \geq 0,$$

$$l_a + l_b + r_c + l_d + r_e + l_f + r_g + r_h - d_{ah} - d_{bc} - d_{cd} - d_{cf} - d_{de} - d_{df} - d_{bg} - d_{dg} - d_{fg} + u \geq 0.$$

If we add these two inequalities and subtract them by the two previous equalities, we get:

$$-2d_{cd} - 2d_{cf} \geq 0,$$

and so $d_{cd} + d_{cf} = 0$. Therefore, for any two distinct $st$ cuts, there is never a demand edge which is on the left of one and on the right of the other.

So any demand edge can be pushed to $s$ or $t$. By induction, any path-bipartite instance is routable. ■

### 3.3 $K_{2m}$ instances are cut-sufficient if and only if they have no odd $K_{2p}$ minor

While the preceding result on “bipartite” $K_{2m}$ instances will be sufficient to deduce our congestion 5 result later, we delve a little more deeply into the structure of general $K_{2m}$ instances. This is because they suggest a very appealing conjecture on cut-sufficient series parallel instances (given below) and could very likely form a basis for settling that conjecture.

Recall that a $K_{2m}$-instance is one whose supply graph is of the form $G = K_{2m}$ with a 2-cut $s$, $t$ and $m$ nodes $v_1, \ldots, v_m$ of degree two, each adjacent to $s$, $t$. We say an instance has an odd $K_{2p}$ minor if it is possible to delete or contract supply edges of the instance, and delete demand edges, until we get an instance that is an odd $K_{2p}$, that is, a $K_{2p}$ supply graph, with $p$ odd, such that the two nodes of degree $p$ are connected by a demand edge, and such that the $p$ nodes of degree 2 are connected by an odd cycle of demand edges. One easily sees that an odd $K_{2p}$ instance is not cut-sufficient (refer to the introduction for the definition of cut-sufficiency). Indeed, setting all demands and capacities to 1 defines an instance that satisfies both Eulerian and cut condition, but is not routable. In general, if we only obtain this structure as a minor, we can think of assigning 0 demand (respectively capacity) to the deleted demand (resp. supply) edges, and assign infinite capacity to any contracted supply edge. Hence for any pair $(G, H)$, if it is cut-sufficient, then it cannot contain an odd $K_{2p}$ minor. For general graphs this is not sufficient (cf. examples in [29]). However we conjecture the following.

**Conjecture 3.5** Let $G$ be series parallel. Then a pair $(G, H)$ is cut-sufficient if and only if it has no odd $K_{2p}$ minor.
Together with the results of the previous section, we now prove that the conjecture holds for $K_{2m}$ instances. In other words, if such an instance has no odd $K_{2p}$ minor, then it is routable. We prove a slightly stronger statement. Let $(G, H)$ be a $K_{2m}$ instance that satisfies the cut condition, is Eulerian and has no odd $K_{2p}$ minor, then there is an integral routing for $H$ in $G$.

The graphs $G$ and $H$ are capacitated. The proof is by induction on total demand. In the following, we may assume that no demand edge is parallel to a supply edge. Furthermore, we assume as in the proof of Lemma 3.4, that any cut induced by a middle node $v$ is tight. If the demand graph does not contain an odd cycle, then it is bipartite and routable by Lemma 3.4. Let us therefore assume that the demand graph contains at least one odd cycle. As the graph should not have any odd $K_{2p}$ minor, there is no demand edge from $s$ to $t$. We will subsequently prove that at least one demand edge $f = v_iv_j$ can be pushed to $s$ or $t$, by assuming that none can, and deducing a contradiction. Here, by pushing the edge $f = v_iv_j$ to $s$ we mean that one unit of demand on $f$ can be removed and new demand edges $sv_i$ and $sv_j$ with unit demand are created (pushing to $t$ is similar). Pushing allows us to complete the proof by induction as follows. Once we push a demand edge $f$ to $s$ or $t$, the resulting instance could possibly have an odd $K_{2p}$ minor. However, the demand edges thus created are parallel to some supply edges, and have only one unit of demand. If we reduce at once the resulting new demand edges with the parallel supply edges, we get a new $K_{2m}$ instance with supply and demand edges that all existed in the original one, and so does not have an odd $K_{2p}$ minor either. As the new instance has a smaller total demand, we can apply induction.

Now we prove the existence of a demand edge that can be pushed. We call a demand edge a leaf demand edge if it is the unique demand edge incident to some $v_i$ node. The following lemma is useful in showing that such demand edges can be pushed.

**Lemma 3.6** Let $(G, H)$ be a $K_{2m}$-instance that satisfies the cut condition and such that any cut induced by a $v_i$ is tight. Then any tight cut separating $s$ from $t$ separates every node $v_i$ from at least one of its adjacent nodes in the demand graph.

**Proof:** Suppose there is some tight cut $S$ which separates a degree 2 node $v$ from none of its adjacent nodes in the demand graph. We have $|\delta_G(S)| = |\delta_H(S)|$. Let us consider the cut $S'$ obtained by flipping the side of $v$ in the cut $S$. Then $|\delta_H(S')| = |\delta_H(S)| + |\delta_H(v)|$, but $|\delta_G(S')| < |\delta_G(S)| + |\delta_G(v)|$, because $S$ separates $s$ from $t$. Since by assumption $|\delta_H(v)| = |\delta_G(v)|$, the cut condition is not satisfied on $S'$, and so we have a contradiction.

As a corollary of the previous lemma, any leaf demand edge in a $K_{2m}$ instance can be pushed to $s$ (or to $t$), since no tight cut can separate both its end points from $s$ (or from $t$).

Recall that the demand graph has an odd cycle. We consider the structure of the demand graph. We call a graph an edge-$K_{2m}$-cycle if it can be obtained from a cycle by replacing some edges by $K_{2m}$ subgraphs.

**Lemma 3.7** If a $K_{2m}$ instance does not have an odd $K_{2p}$ minor, and the demand graph contains an odd cycle, but no leaf demand edge, then the demand graph is either an edge-$K_{2m}$-cycle, or $K_4$.

**Proof:** Let $C$ be the shortest odd cycle contained in the demand graph. We claim that $V(C)$, the nodes in $C$, cover all demand edges. For if a demand edge $v_iv_j$ is not covered, we can get an odd $K_{2p}$ minor by contracting supply edges incident to $v_i$ and $v_j$. Further, if $v_iv_j$ is a demand edge with only one of $v_i, v_j$ on $C$, the edge is a leaf demand edge and hence can be pushed. Therefore we can assume any node $v_i$ not in $V(C)$ is adjacent to at least two nodes of $C$ in the demand graph.

First, suppose that $C$ is a triangle. Then any node $v_i \notin V(C)$ adjacent to two nodes of $C$ creates another triangle. It is easy to see that any two nodes $v_i, v_j \notin V(C)$ adjacent to two nodes of $C$ must be adjacent to the same two nodes of $C$, otherwise we can find a triangle and an additional edge not connected to it, which forms an odd $K_{2p}$ minor. Also, if $v_i, v_j \notin V(C)$ are adjacent to the same two nodes of $C$, neither of $v_i, v_j$ is adjacent to the third node of $C$, because this would again form an odd $K_{2p}$ minor. So there are only two
posibilities: Either the demand graph consists of many triangles all sharing the same edge of $C$, which is an edge-$K_{2m}$-cycle, or it is $K_4$.

Now, suppose $C$ is a odd cycle of length at least 5. Suppose a node $v \notin V(C)$ is adjacent to two distinct nodes $v_1, v_2 \in V(C)$ in the demand graph. Then $v_1v_2$ is not an edge of the demand graph for otherwise $v$ along with $v_1, v_2$ would form a triangle in the demand graph contradicting the fact that $C$ is the shortest odd cycle. Suppose the shortest path in $C$ connecting $v_1$ and $v_2$ has length 3 or more. Then the edges $vv_1$ and $vv_2$ create a shortcut for $C$, which creates shorter cycles with each of the two paths in $C$ from $v_1$ to $v_2$, one of them odd, contradicting the choice of $C$. Therefore, the shortest path in $C$ connecting $v_1$ and $v_2$ must have length 2. This also implies that $v$ has degree 2, because three nodes in $C$ cannot all be at distance 2 of each other.

So any node $v \notin V(C)$ forms a bridge of length 2 between two nodes $v_1, v_2$ of $C$, with $v_1$ and $v_2$ being at distance 2 in $C$. Let $v'$ be the middle node on the length 2 path between $v_1$ and $v_2$ in $C$. We claim that $v'$ has degree 2 in the demand graph. To see this, note that the cycle $C'$ obtained by replacing the path $v_1, v', v_2$ in $C$ by the path $v_1, v, v_2$ has the same length as $C$ and $v' \notin V(C')$. By the argument in the previous paragraph, $v'$ has degree 2. It follows that $v_1$ and $v_2$ form a 2-cut for the demand graph, and all but one of the connected components obtained by removing $v_1$ and $v_2$ are singletons. As any node $v \notin C$ is adjacent to exactly two nodes $v_1$ and $v_2$ for which this is true, the demand graph is an edge-$K_{2m}$-cycle.

It is known that all demand graphs on less than five nodes are routable in any graph that satisfies the cut condition for it (see [29]), and hence $K_4$ is routable. So if we assume that a $K_{2m}$ instance has no odd $K_{2p}$ minor, but that no demand edge can be pushed, then the demand graph must be an edge-$K_{2m}$-cycle. We now show a property of the capacity and demand vectors of such instances.

**Lemma 3.8** Let $(G, H)$ be a $K_{2m}$ in which $H$ is an edge-$K_{2m}$-cycle, no demand edge can be pushed, and any cut induced by a $v_i$ is tight. Then for any node $v$ that has degree 2 in both the demand and supply graph, the capacity and demand of the four edges incident to $v$ in both graphs is the same.

**Proof:** Let $v$ be a node of degree 2 in the demand graph, connected to $v_1$ and $v_2$. By assumption, no edge of the demand graph can be pushed. So there are tight cuts $S_1, T_1$ separating $vv_1$ from $s$ and $t$ respectively. Similarly, there are tight cuts $S_2, T_2$ separating $vv_2$ from $s$ and $t$ respectively. By Lemma 3.6 the cuts $S_1$ and $T_1$ must both separate $vv_1$ from $v_2$. Similarly, $S_2$ and $T_2$ must separate $vv_2$ from $v_1$. Let us consider $S_1$ and $T_1$, and flip the side of $v$ in both. The total capacity $|\delta_H(S_1)| + |\delta_H(T_1)|$ of both cuts is the same as before, but the total demand is modified by replacing the demand of $vv_2$ by that of $vv_1$. Since $S_1$ and $T_1$ were already tight, this means the demand of $vv_1$ is no greater than that of $vv_2$. Symmetrically, starting from $S_2$ and $T_2$, we prove that the demand of $vv_2$ is no greater than that of $vv_1$, and so the demand on both edges is the same. By repeating the argument on the pair of cuts $S_1$ and $S_2$, and the pair $T_1$ and $T_2$, we prove that the capacity of the edge $sv$ is the same as that of $vt$. Since the cut induced by $v$ is tight, the capacities and demands are all equal.

In the following, we call a node $v$ *bracketed* by $v_1$ and $v_2$ if it is of degree 2 in the demand graph, and connected to $v_1$ and $v_2$. The previous lemma has the following easy corollary:

**Corollary 3.9** Let $(G, H)$ be a $K_{2m}$ in which $H$ is an edge-$K_{2m}$-cycle, no demand edge can be pushed, and any cut induced by a $v_i$ is tight. Let $v$ be a node bracketed by $v_1$ and $v_2$. Let $S$ be a tight cut separating $s$ from $t$. Then either $S$ separates $v$ from both $v_1$ and $v_2$, or $S$ separates $v_1$ from $v_2$, in which case the cut obtained by flipping the side of $v$ in $S$ is also tight.

**Proof:** If $S$ does not separate $v_1$ from $v_2$, then it must separate them from any node they bracket, because otherwise we get a tighter cut by flipping the bracketed node. If $S$ does separate $v_1$ from $v_2$, then by Lemma 3.8 flipping $v$ does not change the surplus of the cut.

The above corollary implies that for any tight cut separating $s$ from $t$, there is another tight cut such that all nodes bracketed by a pair $v_1$ and $v_2$ are on the same side of the cut. Let us now consider any demand edge. If it cannot be pushed then there are tight cuts $S$ and $T$ separating the end points of the demand edge from $s$ and
This can be extended to a directed edge that is not fully compliant. Then there is by Lemma 3.2 a directed -cut, say \( l, r \), say. However, this would clearly be the case here, and so we have a contradiction. Indeed, \( P \) can be decomposed as follows: \( P' = (P_1, e_1, P_2, e_2, P_3, e_3, P_4) \) with \( P_1 \) starting from \( s \) and \( P_4 \) ending at \( t \). Without loss of generality, we can assume \( P_1 \) and \( P_3 \) do not intersect \( S \). \( P_2 \) is contained in \( S \), and \( P_4 \) is partially in \( S \).

We show these four parts form a \( K_4 \), which is a contradiction. Indeed, \( P_1 \) and \( P_3 \) are connected by a path which does not cross \( \delta(S) \), because \( S \) is central. Similarly, \( P_2 \) and \( P_4 \) are connected by a path contained in \( S \). Since \( S \) is a series-parallel graph created from an \( st \) edge, adding an \( st \) edge should not create a \( K_4 \) minor. However, this would clearly be the case here, and so we have a contradiction.

**Lemma 3.12** Let \( G \) be a 2-connected series parallel graph obtained from an edge \( st \). Let \( uv \) be a compliant edge that is not fully compliant. Then there is a directed \( s-t \) path \( P \) that contains \( u, v \) and in addition, there is a 2-cut \( l, r \) in \( G \) that separates \( u, v \) such that \( l, r \) lie on \( P \). Moreover, if \( P \) traverses \( s, l, u, r, v, t \) in that order, then \( l, r \) can be chosen to separate \( u \) from both \( s \) and \( t \) (and symmetrically, if \( P \) traverses \( s, u, r, v, l, t \) we can separate \( v \) from \( s, t \)).

**Proof:** Since \( uv \) is compliant, there is a directed path which without loss of generality traverses \( u \) and then \( v \). This can be extended to a directed \( s-t \) path \( P \) traversing \( s, u, v, t \) in that order. Since \( uv \) is not fully compliant, there is by Lemma 3.2 a 2-cut \( l, r \) in \( G \) separating \( u \) from \( v \). One node of the 2-cut, say \( r \), has to be on the...
path from \( u \) to \( v \). So \( P \) traverses \( s - u - r - v - t \) in that order. Suppose \( l \) is not on \( P \); then \( l, r \) separates \( s \) from \( t \) (otherwise, there would be a path \( u - s - t - v \) in \( G - \{l, r\} \)). Since \( G \) is 2-node connected, there are two nodes-disjoint paths from \( s \) to \( t \), one containing \( l \) and the other containing \( r \). This implies that \( s, t \) is also a 2-cut separating \( l \) from \( r \). If this were not the case, then \( G - \{s, t\} \) would contain a path joining \( l, r \). But then \( G + st \) would clearly contain a \( K_4 \) minor, a contradiction. We now claim that \( s, r \) is a 2-cut that separates \( u, v \), and hence choosing \( l = s \) gives the desired cut. Suppose not, then there is a \( u-v \) path \( Q \) in \( G - \{s, r\} \); this path \( Q \) necessarily has to contain \( l \) since \( l, r \) is a 2-cut for \( u, v \). Let \( Q_1 \) be the sub-path of \( Q \) from \( u \) to \( l \). If \( Q_1 \) contains \( t \), then we have a \( u - v \) path that avoids \( l, r \), by following \( Q_1 \) and then the portion of \( P \) between \( t \) and \( v \), a contradiction. Otherwise, \( Q_1 \) combined with the portion of \( P \) from \( u \) to \( r \) is an \( l - r \) path that avoids \( s, t \), again a contradiction. Hence we can assume that \( l \) also lies on \( P \) as claimed.

For the second part, assume that \( P \) traverses \( s, l, u, r, v, t \) in order. Clearly, there cannot be a path from \( u \) to \( t \) in \( G' = G - \{l, r\} \) for otherwise it can be combined with \( P \) to find a \( u - v \) path that avoids \( l, r \). Suppose \( u \) is connected to \( s \neq l \) in \( G' \). Since \( G \) is 2-node-connected, there are two oriented nodes-disjoint \( s-t \) paths in \( G \) and one of them avoids \( l, r \); otherwise one contains \( l \) and the other \( r \). Hence these two paths along with the portion of \( P \) from \( l \) to \( r \) yield a \( K_3 \) in \( G + st \). Thus \( s, t \) are connected in \( G' \) and hence if \( u \) can reach \( s \) in \( G' \) it can reach \( t \), and hence \( v \) as well, again contradicting that \( l \) is a 2-cut for \( u, v \).

We are now ready to prove Theorem 3.10

**Proof:** Let \( uv \) be a demand edge which is compliant, but not fully compliant. We show that we can push \( uv \) into a series of fully compliant demand edges, maintaining the hypotheses for the new instance.

By Lemma 3.12, we have, without loss of generality, a directed \( s-t \) path \( P \) that traverses nodes \( s, l, u, r, v, t \) in that order (possibly \( s = l \)) where the component of \( G - l, r \) containing \( u \) does not contain \( v, s \) or \( t \).

![Figure 1: u and v are separated by the 2-cut l, r, and the demand uv cannot be pushed to l or r.](image)

We show that we can push the edge \( uv \) to \( l \) or \( r \). Suppose this is not the case, then since \( l, r \) is a 2-cut, by Fact 2.2, there is a (central) tight cut \( \delta(L) \) separating \( l \) from \( u, r \) and \( v \), and a (central) tight cut \( \delta(R) \) separating \( r \) from \( u, l \) and \( v \). We show that this is not possible, by contracting the graph into a tri-source or a path-bipartite instance which again contains tight cuts corresponding to \( \delta(L) \) and \( \delta(R) \). Since we know that such instances are routable, these tight cuts could not exist.

Recall that \( L \setminus R \) contains \( l \) but not \( r \), and \( R \setminus L \) contains \( r \) but not \( l \). Hence we can contract \( L \setminus R \) and \( R \setminus L \) and label the nodes \( l' \) and \( r' \) respectively. (By Lemma 2.4, this is actually a contraction on connected subgraphs, although it is not critical at this point in the argument that we have a minor as such.) Denote the resulting instance (after removing loops) by \( G^*, H^* \).

Since \( \{l, r\} \) is a 2-cut separating \( u, v \) we have that the graph induced by the nodes \( V \setminus (L \cup R) \) has at least 2 components (one containing \( u \) and one containing \( v \)). Let \( C_i \) denote the components in this graph and let \( X_i = R \setminus L \cap C_i, Y_i = C_i \cap (V \setminus (R \cup L)) \). Consider any component \( K \) of \( G[Y_i] \). Since \( G - L \) is connected, there is a path from \( K \) to \( R \setminus L \) in the graph induced by \( K \cup (R \setminus L) \). Similarly, there is a path from \( K \) to \( L \setminus R \).
in \(G[K \cup (L \setminus R)]\), since \(G - R\) is connected. Analogously, if \(K\) is a component of \(G[X_i]\), then since \(G[R]\) is connected, there must be a path from \(K\) to \(R\setminus L\) in \(G[K \cup (R \setminus L)]\). Similarly there is a path from \(K\) to \(L \setminus R\).

Now if both \(X_i, Y_i\) are nonempty, then we may choose a pair of components \(K, K'\) from \(X_i, Y_i\) respectively, and a path joining them in \(C_i\), so that we can form \(K_4 = e\) in the minor whose nodes are \(l', r', K, K'\). This together with a path through some other \(C_j\) yields a \(K_4\). Hence for each \(i\), at most one of \(X_i\) or \(Y_i\) is nonempty. Moreover, if we shrink each \(C_i\) to a node \(c_i\), then we have edges \(c_il, c_ir\). The shrunken graph is therefore of type \(K_{2m}\), with possibly an edge from \(l'\) to \(r'\). Since each \(c_i\) was either of “type” \(R \cap L\), or type \(V \setminus (R \cup L)\), in the shrunken graph we can identify tight cuts induced by sets \(L', R'\) associated with our original pair \(L, R\).

We next claim that neither \(s\) nor \(t\) is in \(R\) (and hence \(R \setminus L\)); to see this, note that the \(s\)-\(t\) path \(P\) goes successively through \(s, u, r, v, t\) and hence if either \(s\) or \(t\) is in \(R\), then \(P\) would cross \(\delta(R)\) three or more times.

We now examine two cases based on whether \(t\) is in \(L\) or not. In each case we examine the structure of the demand edges in the shrunken graph.

1. \(t\) is not in \(L\). So \(t\) is contained in some \(C_i\), say \(C_1\). Suppose first that \(s\) lies in some \(C_i\) different from \(C_1\).

   By Lemma 3.12, \(l, r\) separates \(u\) from \(s, t\); so \(u \not\in C_i \cup C_1\). But then adding \(st\) (i.e., \(c_ic_1\)) to the shrunken graph (which maintains the series-parallel property), would create a \(K_4\) on the nodes \(l', r', c_i, c_1\), by considering the \(l' - r'\) path through the component \(C_j\) containing \(u\). Hence we may assume that \(s\) is either in \(L\) or in \(C_1\).

   Consider next some compliant demand edge \(u'v'\) in the shrunken instance, and suppose that neither of its endpoints lie in \(C_1\); say the endpoints are in \(C_1, C_j\). Consider the directed \(s\)-\(t\) path \(P'\) associated with this demand where \(P'\) traverses \(s, u', v', t\) in that order. In the shrunken graph it must cross the cuts induced by \(C_1, C_1, C_j\) at least 5 times. Since every edge in any of these cuts lies in either \(\delta(L')\) or \(\delta(R')\), one of these two cuts is crossed three times, a contradiction. Hence, any compliant edge that remains in the shrunken graph, must have one endpoint in \(C_1\). Thus the shrunken graph is a tri-source instance where \(l', r'\) and \(c_1\) are the three sources.

2. \(t\) is in \(L\). We claim that \(s\) is also in \(L\); \(P\) goes successively through \(l, u, t\), and hence if \(s\) is not in \(L\), \(P\) would cross \(\delta(L)\) three times.

   Recall that any directed path can be extended into a path starting at \(s\) and ending at \(t\), and that this path should cross the cut of any central set at most twice by Lemma 3.11. We also use Lemma 2.4 to obtain that both \(R \setminus L, L \setminus R\) are central sets. Thus since \(s\) and \(t\) are both in \(L \setminus R\), there are two types of directed \(s\)-\(t\) paths. The first type does not ever enter \(R \setminus L\). If such a path ever leaves \(L \setminus R\), then it must enter some \(C_i\). It must then leave \(C_i\) to reach \(t\), and hence it enters \(L \setminus R\) again. Thus the path has crossed the cut of \(L \setminus R\) twice, and hence it can not leave it again. The second type of path may traverse \(R \setminus L\). This is the only type that can traverse more than one \(C_i\). We claim that a directed \(s\)-\(t\) path of the second type goes from \(L \setminus R\) to \(R \setminus L\), traversing at most one \(C_i\) on the way, then goes back from \(R \setminus L\) to \(L \setminus R\), traversing at most one \(C_i\) on the way. This follows since any \(s\)-\(t\) directed path cannot cross \(\delta(R \setminus L)\) more than twice, so it can enter at most once, and leave at most once.

   Therefore, a directed \(s\)-\(t\) path of the second type leaves \(L \setminus R\), possibly traverses a \(C_i\), enters \(R \setminus L\), leaves \(R \setminus L\), possibly traverses a \(C_i\), and enters \(L \setminus R\) in that order. It cannot leave \(L \setminus R\) again after, as it has crossed its cut twice. As a consequence, no directed path can traverse more than two \(C_i\)'s.

   We now show that for any \(C_i\), the arcs (oriented edges of \(G\)) connecting \(C_i\) to \(R \setminus L\) are all oriented in the same direction. Assume there is a \(C_i\), say \(C_1\), with an arc \(e_1\) entering and an arc \(e_2\) leaving \(R \setminus L\).

   We can extend \(e_1\) into a directed path \(P_1\) starting with \(s\). Since \(P_1\) ends by entering \(R \setminus L\) with \(e_1\), it did not leave \(R \setminus L\) before, so it entered \(C_1\) from \(L \setminus R\). We can also extend \(e_2\) into a directed path \(P_2\) ending at \(t\). Since \(P_2\) starts by leaving \(R \setminus L\) with \(e_2\), it can not enter \(R \setminus L\) again, so it leaves \(C_1\) for \(L \setminus R\).
We now show $P_1$ and $P_2$ are node-disjoint inside $C_1$. Suppose that it is not the case, and that there is some $x \in C_1$ contained both in $P_1$ and $P_2$. Then we can create a directed path leaving $R \setminus L$ then entering it again, by following $P_2$ from $e_2$ to $x$, then following $P_1$ from $x$ to $e_1$. The directed path is simple, since the graph is acyclic. We can extend it to a directed $s$-$t$ path which then contradicts our previous claims. Hence $P_1$ and $P_2$ are node-disjoint inside $C_1$. But since $C_1$ is connected, these paths are connected by some path. That is, for any $x_1 \in C_1$ on $P_1$ and any $x_2 \in C_1$ on $P_2$, there is a path inside $C_1$ connecting $x_1$ to $x_2$. Also, since $u$ and $v$ are not in the same $C_i$, there is at least one other $C_i$ linking $L \setminus R$ and $R \setminus L$. This now creates a $K_4$ in the graph (with $l'$, $r'$ shrunk) on the nodes $l'$, $r'$, $x_1$, $x_2$, a contradiction.

Thus for any $C_i$, the arcs connecting $C_i$ to $R \setminus L$ are all oriented in the same direction. Therefore the $C_i$’s are partitioned into two types: out components are those with arcs going into $R \setminus L$, and in components with arcs entering from $R \setminus L$. It follows that any directed path traversing two $C_i$’s traverses one of each type. Any compliant edge $u'v'$ that remains in the shrunken graph that is not incident to $l'$ or $r'$, must admit a directed $u'v'$ with its endpoints in distinct components. Hence exactly one of $u'$, $v'$ lies in an in component, and the other lies in an out component. Thus the shrunken graph is a path-bipartite instance.

In either case, the instance $G^*, H^*$ obtained is routable by Lemma 3.4. Therefore, the cuts $\delta(L)$ and $\delta(R)$ could not have both been tight, and so we can push $uv$ to either $l$ or $r$. By induction, we can push any compliant edge into a series of fully compliant edges.

4 Integral Routing with Congestion

4.1 Routing in Congestion 5 in Series Parallel Graphs

Theorem 4.1 Suppose $G, H$ is a series-parallel instance satisfying the cut condition. Then $H$ has an integral routing with edge congestion 5.

Proof: By the result of [20], there is a fractional routing $f$ of $H$ with congestion 2. For any demand edge $xy$, suppose that $s', l'$ induce the highest level (with respect to the decomposition of $G$ starting from $sl$) 2-cut separating $x, y$. Then at least half of any fractional flow for $xy$ has to go either via $s'$ or $l'$. Without loss of generality, assume it is $s'$. We push the $xy$ demand edge to $s'$, i.e., we create demand edges $xs', ys'$ and remove $xy$. We do this simultaneously for all demand edges - we are pushing demands based on the fractional flow $f$. The new instance is compliant. Let us call $H'$ the new demand graph. By construction, there is a feasible fractional flow of $1/2$ of each demand from $H'$ in $2G$. This implies that $4G$ satisfies the cut condition for $H'$. In order to make the graph Eulerian, we can add a $T$-join, where $T$ is the set of odd degree nodes in $4G + H'$. Since we can assume that $G$ is connected by previous reductions, we may choose such a $T$-join as a subset of $E(G)$. It follows that we can create an Eulerian, compliant instance $G', H'$ that satisfies the cut condition, and $G'$ is a subgraph of $5G$. Hence by Theorem 8.10 we may integrally route $H'$, and hence $H$ in the graph $5G$.

We believe that the above result can be strengthened substantially and postulate the following:

Conjecture 4.2 Let $G, H$ satisfy the cut condition where $G$ is series-parallel and $G + H$ is Eulerian. Then there is a congestion 2 integral routing for $H$.

4.2 Rerouting Lemma from [9]

We state the rerouting lemma that we referred to in the introduction. It is useful to refer to the informal version we described earlier. Let $D$ be a demand matrix in a graph $G$ and let $f: V \to V$ be a mapping. We define a demand matrix $D_f$ as follows:
\[ D_f(xy) = \sum_{uv : f(u) = x, f(v) = y} D(uv). \]

In other words the demand \( D(uv) \) for a pair of nodes \( uv \) is transferred in \( D_f \) to the pair \( f(u)f(v) \). Thus the total demand transferred from \( u \) to \( f(u) \) is \( \sum_v D(uv) \). We define another demand matrix \( D'_f \) which essentially asks that each node \( u \) can send this amount of flow to \( f(u) \).

\[ D'_f(uf(u)) = \sum_v D(uv). \]

**Proposition 4.3** If \( D'_f \) is (integrally) routable in \( G \) with congestion \( a \), and \( D_f \) is (integrally) routable in \( G \) with congestion \( b \), then \( D \) is (integrally) routable with congestion \( a + b \) in \( G \).

We need a cut condition given by the simple lemma below. For completeness, we include a proof in the appendix [A.1].

**Lemma 4.4 ([9])** Let \( D \) be a demand matrix on a given graph \( G \) and let \( f : V \to V \) be a mapping. If the cut condition is satisfied for \( D \), and \( D'_f \) is routable in \( \gamma G \), then the cut condition is satisfied for \( D_f \) in \( (\gamma + 1) \cdot G \).

We give a useful corollary of the above lemma.

**Corollary 4.5** Let \( G = (V, E) \) satisfy the cut-condition for \( H = (V, E_H) \) and let \( A \subseteq V \) be a node-cover in \( H \). Then there exists a demand graph \( I = (A, F) \) such that \( 2G \) satisfies the cut-condition for \( I \). Moreover, if \( I \) is (integrally) routable in \( 2G \) with congestion \( \alpha \), then \( H \) is (integrally) routable in \( G \) with congestion \((1 + 2\alpha)\).

**Proof:** Assume for simplicity that \( G \) and \( H \) have unit capacities. Let \( A \subset V \) be a node-cover in \( H \). Shrink \( A \) to a node \( a \) to obtain a new supply graph \( G' \) and a new demand graph \( H' \). Since \( A \) is a node-cover all demand edges in \( H' \) are incident to \( a \), and so \( H' \) is a star, and \( G', H' \) is a single-source instance. For simplicity assume that there are no parallel edges in \( H' \); if node \( u \) has \( d > 1 \) parallel edges to \( a \), then add \( d \) dummy terminals connected to \( u \) with infinite capacity edges in \( G' \) and replace each \((u, a)\) edge by an edge from a dummy terminal to \( a \). Let \( S \subseteq V \setminus A \) be the set of nodes that have a demand edge to \( a \) in \( H' \). Note that \( G' \) satisfies the cut-condition for \( H' \). Therefore by the maxflow-mincut theorem (or Menger’s theorem) \( H' \) is routable in \( G' \) with congestion 1 and by our assumption that the demands are unit valued and capacities are integer valued, the flow corresponds to \(|S|\) paths, one from each node in \( S \) to \( a \). Now unshrink \( a \) to \( A \); thus the flow corresponds to paths from \( S \) to \( A \) in \( G \). Define a mapping \( f : V \to V \) where \( f(u) = u \) if \( u \in V \setminus S \) (we only care about \( u \in A \)), and if \( u \in S \) then \( f(u) = v \) if the path from \( u \) to \( A \) ends in \( v \in A \). Let \( D \) be the demand matrix corresponding to \( H \) and \( D_f \) be demand matrix induced by the mapping \( f \). Let \( I = (A, F) \) be the demand graph induced by \( f \). Note then that \( D'_f \) corresponds to the single-sink flow problem determined by \( H' \). Hence by Lemma 4.4 \( D'_f \), and hence \( I \), satisfies the cut condition in \( 2G \). We then apply Proposition 4.3 to see that if \( I \) is (integrally) routable in \( 2G \) with congestion \( \alpha \) (which is the same as \( I \) being routable in \( G \) with congestion \( 2\alpha \)), then \( H \) is (integrally) routable in \( G \) with congestion \((1 + 2\alpha)\) since \( H' \) is integrally routable in \( G \) with congestion 1.

\[ \square \]

### 4.3 k-Outerplanar and k-shell Instances

Let \( G = (V, E) \) be an embedded planar graph. We define the outer layer or the 1-layer of \( G \) to be the nodes of \( G \) that are on the outer face of \( G \). The \( k \)-th layer of \( G \) is the set of nodes of \( V \) that are on the outer face of \( G \) after the nodes in the first \( k-1 \) layers have been removed. A \( k \)-outerplanar graph is a planar graph that can be embedded with at most \( k \) layers. We let \( V_i \) denote the nodes on the \( i \)-th layer. We are interested in multilows in planar graphs when at least one terminal of each demand pair is lies in one of the outer \( k \) layers; we call such instances \( k \)-shell instances. Let \( H = (V, F) \) be a demand graph.
Theorem 4.6 (Okamura-Seymour [24]) Let $G$ be a planar graph and $H$ be a demand graph with all terminals on a single face. If $H$ satisfies the cut-condition, then there is a half-integral routing of $H$ in $G$. Moreover if $G+H$ is Eulerian, $H$ is integrally routable in $G$.

Theorem 4.7 ([7]) If $G$ is a $k$-Outerplanar graph and $H$ satisfies the cut-condition, then $H$ is fractionally routable in $G$ with congestion $c_k$ for some universal constant $c$.

We can strengthen the above theorem to prove the following on $k$-shell instances.

Theorem 4.8 Let $G$ be a planar graph and let $V' = \bigcup_{i=1}^{k} V_i$ be the set of nodes in the outer $k$ layers of a planar embedding of $G$. Suppose $H = (V, F)$ is a demand graph where for each demand edge at least one of the end points is in $V'$. If $G, H$ satisfy the cut condition, then $H$ can be fractionally routed in $G$ with congestion $c_k$ for some universal constant $c$.

The proof of Theorem 4.7 relies on machinery from metric embeddings. Our proof of Theorem 4.8 also relies on metric embeddings, and in particular uses recent results [21] and [12]. Since these techniques are somewhat orthogonal to the primal methods that we use in this paper, we describe a proof of Theorem 4.8 in a separate manuscript [11]. Below we relate the integer and fractional flow-cut gaps for $k$-shell instances.

Theorem 4.9 Let $G$ be a planar graph and let $V' = \bigcup_{i=1}^{k} V_i$ be the set of nodes in the outer $k$ layers of a planar embedding of $G$. Suppose $H = (V, F)$ is a demand graph where for each demand edge at least one of the end points is in $V'$. If $H$ is fractionally routable in $G$, then it can be integrally routed in $G$ with congestion $6^k$.

Combining the above two theorems, we have:

Corollary 4.10 If $G, H$ is a $k$-shell instance that satisfies the cut-condition, then $H$ is integrally routable in $G$ with congestion $c_k$ for some universal constant $c$.

We need the following claim as a base case to prove Theorem 4.9.

Claim 4.11 Let $G$ be a planar graph and $H = (V, F)$ be a demand graph such that for each demand edge at least one end point is on the outer face $V_1$. If $G, H$ satisfy the cut-condition, then there is an integral routing of $H$ in $G$ with congestion 5.

Proof: We observe that $V_1$ is a node cover in $H = (V, F)$. Therefore, by Corollary 4.5 there is a demand graph $I = (V_1, F')$ such that $2G, I$ satisfy the cut condition. By the Okamura-Seymour theorem (Theorem 4.6), $I$ is integrally routable in $2G$ with congestion 2. Therefore, by Corollary 4.5, $H$ is integrally routable in $G$ with congestion 5.

Proof: [of Theorem 4.9] We prove the theorem by induction on $k$. The base case of $k = 1$ follows from Claim 4.11.

Assuming the hypothesis for $j < k$, we prove it for $j = k$. Let $H = (V, F)$ be a demand graph that is fractionally routable in $G$ and such that each demand edge is incident to a node in the outer $k$ layers. Let $H_k = (V, F_k)$ be the subgraph of $H$ induced by the demand edges $F_k \subseteq F$ that are incident to at least one node in $V_k$ and moreover the other end point is not in $V_1 \cup \ldots \cup V_{k-1}$. We obtain a new supply graph $G'$ by shrinking the nodes in $\bigcup_{i=1}^{k-1} V_i$ to a single node $v$. Note that $H_k$ is fractionally routable in $G'$ as well. Fix some arbitrary routing of $H_k$ in $G'$. Partition $F_k$ into $F_a$ and $F_b$ as follows. $F_a$ is the set of all demands that route at least half their flow through $v$ in $G'$. $F_b = F_k \setminus F_a$. Thus a demand in $F_b$ routes at least half its flow in the graph $G'' = G[V \setminus \bigcup_{i=1}^{k-1} V_i]$. We claim that the demand graph $H_k = (V \setminus \bigcup_{i=1}^{k-1} V_i, F_b)$ is integrally routable in $G''$ with congestion 10. For this we note that $H_b$ is fractionally routable in $2G''$ which implies that $(2G'', H_b)$ satisfies the cut-condition. Moreover $V_k$ is the outer-face of $G''$ and each demand in $F_b$ has at least one end point in $V_k$ and hence we can apply Claim 4.11.
Now consider demands in $F_k^a$ and their flow in $G'$. For simplicity assume that the end points of $F_k^a$ are disjoint and let $T$ be the set of end points. By doubling the fractional flow in $G'$ of each demand $f \in F_k^a$ we get a feasible routing for sending one unit of flow from each $t \in T$ to $v$. Thus in $G'$ there are paths $P_t$, $t \in T$ where $P_t$ is a path from $t$ to $v$ and no edge has more than two paths using it. These paths imply that the terminals in $T$ can be integrally routed to nodes in $\bigcup_{i=1}^{k-1} V_i$ with congestion 2 in $G$ (simply unshrink $v$). For each demand $f = (u, v) \in F_k^a$ let $f' = (u', v')$ be a new demand where $u'$ and $v'$ are the nodes in $\bigcup_{i=1}^{k-1} V_i$ that $u$ and $v$ are routed to. Let these demands be $F_k^c$. We claim that $F_k^c$ is fractionally routable in $G$ with congestion 3 - simply concatenate the routing of $F_k^a$ with the paths that generated $F_k^a$ from $F_k^c$. Now consider the demand graph $H' = (V, (F \setminus F_k) \cup F_k^c)$. Since $H = (V, F)$ is fractionally routable in $G$ and $F_k^c$ is fractionally routable in $3G$, we have that $H'$ is fractionally routable in $4G$. Also, each demand in $H'$ has an end point in $\bigcup_{i=1}^{k-1} V_i$. Therefore, by the induction hypothesis, $H'$ is integrally routable in $G$ with congestion $4 \cdot 6^{k-1}$.

Routing $H$ as above consists of routing $H'$, the routing of $F_k^a$, and the routing of the demands in $F_k^b$ to the outer $k - 1$ layers; adding up the congestion for each of these routings as shown above, we see that $H$ is routable in $G$ with congestion $4 \cdot 6^{k-1} + 10 + 2 \leq 6^k$ for $k \geq 2$. This proves the hypothesis for $k$.

**4.4 Flow-Cut Gap and Node Cover size of Demand Graph**

Linial, London and Rabinovich [22] and Aumann and Rabani [3] showed that if the supply graph $G = (V, E)$ satisfies the cut condition for a demand graph $H = (V, E_H)$, then $H$ is routable in $G$ with congestion $O(\log k)$ where $k = |E_H|$; to obtain this refined result (instead of an $O(\log n)$ bound), [22, 3] rely on Bourgain’s proof of the distortion required to embed a finite metric into $\ell_1$. G"{u}nl"{u}k [13] further refined the bound and showed that the flow-cut gap is $O(\log k^*)$ where $k^*$ is the size of the smallest node cover in $H$; recall a node cover is a subset $S$ of nodes for which every edge of $H$ has at least one endpoint in $S$. For example if $k^* = 1$, then $H$ induces a single-source problem for which the flow-cut gap is 1. G"{u}nl"{u}k’s argument requires a fair amount of technical reworking of Bourgain’s proof. Here we give a simple and insightful proof via Lemma 4.4 in particular Corollary 4.5.

**Theorem 4.12** Let $G = (V, E)$ satisfy the cut-condition for $H = (V, E_H)$ such that $H$ has a node-cover of size $k^*$. Then $H$ is routable in $G$ with congestion $O(\log k^*)$.

**Proof:** Let $A \subset V$ be a node-cover in $H$ such that $|A| = k^*$. We now apply Corollary 4.5 which implies that there is a demand graph $I = (A, F)$ such that $2G$ satisfies the cut-condition for $I$. Moreover if $I$ is routable in $2G$ with congestion $\alpha$ then $H$ is routable in $G$ with congestion $(1 + 2\alpha)$. Note that $I$ is a demand graph with at most $(k^*)^2$ edges, therefore, it is routable in $2G$ with congestion $O(\log k^*)$ [22, 3]. Hence $H$ is routable in $G$ with congestion $O(\log k^*)$.

**4.5 Multiflows with terminals on $k$ faces of a planar graph**

Lee and Sidiropoulos [21] recently gave a powerful methodology via their peeling lemma to reduce the flow-cut gap question for a class of instances to other potentially simpler class of instances. Using this they reduced the flow-cut gap question for minor-free graphs to planar graphs and graphs closed under bounded clique sums. One of the applications of their peeling lemma is the following result. Let $G$ be an embedded planar graph and $H$ be a demand graph such that the endpoints of edges in $H$ lie on at most $k$ faces of $G$. If $G$ satisfies the cut-condition, then $H$ is (fractionally) routable in $G$ with congestion $e^{O(k)}$. Their proof extends to graphs of bounded genus and relies on the non-trivial peeling lemma. Here we give a simple proof with a stronger guarantee for the planar case, again using Lemma 4.4.

**Theorem 4.13** Let $G = (V, E)$ be an embedded planar graph and $H = (V, F)$ be a demand graph such that the endpoints of edges in $H$ lie on at most $k$ faces of $G$. If $G$ satisfies the cut-condition, then $H$ is routable in $G$ with congestion $3k$. Moreover if $G$ and $H$ have integer capacities and demands respectively, then there is an integral flow with congestion $5k$. 

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Proof: Let $V_1, \ldots, V_k$ be the node sets of the $k$ faces on which the demand edges are incident to. Let $F_i \subseteq F$ be the edges in $H$ that have at least one end point incident to a node in $V_i$ and let $H_i = (V_i, F_i)$ be the demand graph induced by $F_i$. Note that $G$ satisfies the cut-condition for $H_i$. Clearly $V_i$ is a node-cover for $H_i = (V, F_i)$ and hence by Corollary 4.5, there is a demand graph $I_i = (V_i', F_i')$ such that $2G$ satisfies the cut-condition for $I_i$. Note, however, that $I_i$ is an Okamura-Seymour instance in that all terminals lie on a single face. Hence $I_i$ is routable in $2G$ with congestion 1 and is integrally routable in $2G$ with congestion 2. Hence, by Corollary 4.5, $H_i$ is routable in $G$ with congestion 3 and integrally routable in $G$ with congestion 5. By considering $H_1, \ldots, H_k$ separately, $H$ is routable in $G$ with congestion $3k$ and integrally with congestion $5k$.

Remark: We observe that a bound of $k$ is easy via the Okamura-Seymour theorem if for each demand edge the two end points are incident to the same face. The rerouting lemma allows us to easily handle the case when the end points may be on different faces; in fact the proof extends to the case when nodes on $k$ faces in $G$ form a node cover for the demand graph $H$. The above proof can be easily extended to graphs embedded on a surface of genus $g$ to show a flow-cut gap of $3\alpha_g k$ where $\alpha_g$ is the gap for instances in which all terminals are on a single face.

5 Lower-bound on flow-cut gap in series-parallel graphs

Lee and Raghavendra have shown in [20] that the flow-cut gap in series-parallel graph can be arbitrarily close to 2; this lower bound matches an upper bound found previously [6]. They introduce a family of supply graphs and prove the required congestion using the theory of metric embeddings. In particular, their lower bound is shown by a construction of a series parallel graph and a lower bound on the distortion required to embed the shortest path metric on the nodes of the graph into $\ell_1$.

In this section we give a different proof of the lower bound. We use the same class of graphs and the recursive construction of Lee and Raghavendra [20]. However, we use a primal and direct approach to proving the lower bound by constructing demand graphs that satisfy the cut-condition but cannot be routed. We lower bound the congestion required for the instances by exhibiting a feasible solution to the dual of the linear program for the maximum concurrent multicommodity flow in a given instance. This is captured by the standard lemma below which follows easily from LP duality.

Lemma 5.1 Let $G = (V, E)$ and $H = (V, F)$ be undirected supply and demand graphs respectively with $c_e$ denoting the capacity of $e \in E$ and $d_f$ denoting the demand of $f \in F$. Then the minimum congestion $r$ required to route $H$ in $G$ is given by

$$\min_{\ell : E \rightarrow \mathbb{R}^+} \frac{\sum_{f \in F} d_f \ell_f}{\sum_{e \in E} c_e \ell_e}$$

where $\ell_f$ is the shortest path distance between the end points of $f$ in $G$ according to the non-negative length function $\ell : E \rightarrow \mathbb{R}^+$.

Corollary 5.2 By setting $\ell_e = 1$ for each $e \in E$, the minimum congestion required to route $H$ in $G$ is at least $D/C$ where $C = \sum_{e \in E} c_e$, and $D = \sum_{f \in F} d_f \ell_f$, where $\ell_f$ is the shortest path distance in $G$ between the end points of $f$ (with unit-lengths on the edges).

We refer to $C$ and $D$ as defined in the corollary above as the total capacity and the total demand length respectively. We refer to the lower bound $D/C$ as the standard lower bound.

Construction of the Lower Bound Instances: The family of graphs presented in [20] is obtained from a single edge by a simple operation, which consists in replacing every edge by a $K_{2,m}$ graph, as shown in Figure 2.

Assuming $m$ is even, consider the following demands and capacities for a $K_{2,m}$ graph: a complete demand graph on the $m$ nodes of degree 2, with demand 1 on each edge, and a capacity $(m-1)/2$ on each supply edge.
Lemma 5.3 The instance $I_c(s, t)$ has the following properties:

1. It has $2m$ supply edges, each of capacity $\frac{c}{m(m-2)/4}$, and it has $\frac{m-1}{2}$ demand edges, each with demand $\frac{c}{m(m-2)/4}$.

2. The standard lower bound for $I_c(s, t)$ is 1 and the instance is routable.

3. The minimum surplus of any cut separating $s$ from $t$ is equal to $c$.

4. Adding an $st$ demand edge of demand $c$ to $I_c(s, t)$ preserves the cut condition but the standard lower bound for this modified instance is $(1 + \frac{m-2}{2(m-1)})$.

Proof: The first three facts follow directly from the description of $I_c(s, t)$. The surplus of any $st$ cut in $I_c(s, t)$ is $c$, and $I_c(s, t)$ satisfies the cut condition. Hence, it follows that adding an $st$ demand edges with demand $c$ preserves the cut condition. We note that for $I_c(s, t)$, the total capacity $C = 2m \cdot \frac{c}{m(m-2)/4} \cdot \frac{m-1}{2} = \frac{4c(m-1)}{(m-2)}$, this is also equal to the total demand length $D$ for $I_c(s, t)$. Adding an $st$ demand edge with demand value $c$ increases the total demand length by $2c$, and therefore the standard lower bound for the modified instance is $\frac{C+2c}{C} = (1 + \frac{m-2}{2(m-1)})$.

As in [20], the goal is to show that the congestion lower bound obtained in the construction above can be amplified by iteration. As a first step, the next lemma shows that replacing an edge of the supply graph by an instance of type $I_c(s, t)$ maintains the cut condition. Given a multiflow instance $G, H$ and a supply edge $uv$ in
Proof: The surplus of any cut separating $u$ from $v$ in $I'$ is at least as big as the surplus of the corresponding cut in $I$, since the surplus of any cut separating $s$ from $t$ in $I_c(s, t)$ is at least $c$. Also, the surplus of any cut in $I'$ that does not separate $u$ from $v$ is equal to the surplus of the corresponding cut in $I$, plus the nonnegative surplus of some cut in $I_c(u, v)$. 

From any instance, we build an instance requiring a larger congestion using the following transformation: We replace each supply edge by an instance of type $I_c(u, v)$. The next theorem gives a lower bound on the required congestion in the transformed instance.

Theorem 5.5 Let $I = G, H$ be an instance verifying the cut condition, with total capacity $C$ and total demand length $D$. Let $I'$ be the instance obtained by replacing each supply edge $uv$ of capacity $c$ by an instance $I_{cuv}(u, v)$. Then the transformed instance $I' = G', H'$ verifies the cut condition, and the standard lower bound for $I'$ is $1 + \frac{D}{C} \cdot \frac{m-2}{2(m-1)}$.

Proof: The fact that $I'$ verifies the cut condition is a direct consequence of **Lemma 5.4**.

Let us first compute the total capacity of $G'$. Since each supply edge of capacity $c$ in $I$ is replaced with $2m$ supply edges of capacity $\frac{c}{m(m-2)/4} \cdot \frac{m-1}{2}$, the total capacity of $G'$ is $C' = C \cdot \frac{2m}{m(m-2)/4} \cdot \frac{m-1}{2} = C \cdot \frac{4(m-1)}{m-2}$.

The demand edges in $H'$ either exist in $H$, or are added by the transformation, i.e. they are internal to some $I_c(u, v)$ instance that replaces a supply edge $uv$. The total demand length in $I'$ can therefore be decomposed into the part corresponding to the demand edges that exist in $I$, and the total demands internal to each $I_c(u, v)$ instance. In each $I_c(u, v)$ instance, the total demand length is equal to the total capacity. So the sum of demand lengths internal to each $I_c(u, v)$ instance is equal to the total capacity, which is the total capacity of $G'$. On the other hand, the demand edges that exist in $I$ have the same demand, but the shortest path of each such edge has exactly doubled in the transformed instance. It follows that $D' = C' + 2D$.

Therefore, the standard lower bound for $I'$ is $\frac{D'}{C'} = \frac{C' + 2D}{C'} = 1 + \frac{2D}{C'} \cdot \frac{m-2}{4(m-1)} = 1 + \frac{D}{C} \cdot \frac{m-2}{2(m-1)}$. 

![Figure 4: The flow-cur gap r grows bigger with each iteration.](image)

Thus, **Theorem 5.5** can be used to amplify the flow-cut gap. In particular, if the standard lower bound yields a gap of $x$ for an instance $I$ then, one obtains an instance with standard lower bound yielding a gap of $f(x) = 1 + x \cdot \frac{m-2}{2(m-1)}$. We can iterate this process $k$ times yielding instances with flow-cut gap $f^{(k)}(x)$ where $f^{(k)}$ is the function $f$ composed $k$ times. We note that

$$f^{(k)}(x) = 1 + \frac{(m-2)}{2(m-1)} + \cdots + \left(\frac{(m-2)}{2(m-1)}\right)^{k-1} + x \left(\frac{(m-2)}{2(m-1)}\right)^k,$$
and that this converges to
\[
\lim_{k \to \infty} f^{(k)}(x) = 2 - \frac{2}{m},
\]
for any \(x\).

**Theorem 5.6** For any \(\varepsilon > 0\), there is a series-parallel graph instance for which the flow-cut gap is \(2 - \varepsilon\).

**Proof:** We apply the iterated construction starting with the instance consisting of a single edge of capacity 1 as the supply graph, and a demand graph consisting of the same edge with demand 1; clearly, the standard lower bound for this instance is 1. See Figure 4. We observe that the construction preserves the property that the supply graphs are series parallel. Thus, if we set \(m = \lceil 4/\varepsilon \rceil\) in \(I(s, t)\), the iterated construction yields a series parallel graph instance with flow-cut gap arbitrarily close to \(2 - \varepsilon/2\), and hence one can choose a sufficiently large \(k\) such that iterating the construction \(k\) times gives an instance with gap at least \(2 - \varepsilon\). \(\blacksquare\)

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A Appendix

A.1 Proof of the Rerouting Lemma [9]

For completeness we give the proof from [9].

Proof: Consider a cut $\delta_G(S)$ in $G$ for some $S \subset V$. Let $D(S)$ denote the total demand across this cut. Since $D$ satisfies the cut condition $|\delta_G(S)| \geq D(S)$ for all $S \subset V$. Also, since $D_f'$ is routable in $\gamma G$, $\gamma|\delta_G(S)| \geq D_f'(S)$ for all $S \subset V$.

To prove the lemma we need to show that $(\gamma + 1)|\delta_G(S)| \geq D_f(S)$. From the above inequalities, it suffices to show that $D_f(S) \leq D(S) + D_f'(S)$. Let $X_S$ denote the set of all unordered pairs of nodes $uv$ such that $u$ and $v$ are separated by $S$, that is $|\{u, v\} \cap S| = 1$. We can write $D_f(S)$ as $\sum_{uv,f(u)f(v) \in X_S} D(uv)$. For each pair $uv$ such that $f(u)f(v) \in X_S$, we charge $D(uv)$ to either $D(S)$ or $D_f'(S)$ such that there is no overcharge. This will complete the argument.

We consider two cases. If $uv \in X_S$ then we charge $D(uv)$ to $D(S)$. Note that $\sum_{uv \in X_S} D(uv) = D(S)$ and hence we do not overcharge $D(S)$. If $uv \notin X_S$, then either $uf(u) \in X_S$ or $vf(v) \in X_S$ but not both. In $uf(u) \in X_S$ we charge $D(uv)$ to $u$, otherwise to $v$. We observe that the total charge to a node $u$ is at most $D_f'(uf(u))$ and it is charged only if $uf(u) \in X_S$. Hence the total charge to $D_f'(S)$ is not exceeded either. □