CLASICAL SIMPLE LIE 2-ALGEBRAS OF TORAL RANK 3
AND A CONTRAGREDIENT LIE 2-ALGEBRA OF TORAL
RANK 4.

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Abstract. In this paper we show there are no classical type simple Lie 2-algebras with toral rank odd and we also show that the simple contragredient Lie 2-algebra $G(F_{4, a})$ of dimension 34 has toral rank 4, and we give the Cartan decomposition of $G(F_{4, a})$.

Introduction

The classification of the simple Lie algebras over an algebraically closed field of characteristic $p$ with $p \in \{2, 3\}$ is still an open problem. In characteristic 2, S. Skryabin in [5] showed that all simple Lie algebras on an algebraically closed field of characteristic 2 have absolute toral rank greater than or equal to 2 (see also [2]). Later, A. Premet and A. Grishkov classified Lie algebras of absolute toral rank 2. They announced in [1] (work in progress) the following result: All finite dimensional simple Lie algebra over an algebraically closed field of characteristic 2 of absolute toral rank 2 are classical of dimension 3, 8, 14 or 26. In particular, all finite dimensional simple Lie 2-algebra over a field of characteristic 2 of (relative) toral rank 2 is isomorphic to $A_2$, $G_2$ or $D_4$. When the rank absolute is greater than or equal to 3 the problem of classification is still open. The main obstacle in this problem is the lack of examples.

In this paper we calculate the toral rank of the Classical simple Lie 2-algebras of type $X_l \in \{A_l, B_l, C_l, D_l, F_4, E_6, E_7, E_8\}$, i.e., quotients of Chevalley algebras over a field of characteristic 2, module the center. As a consequence, we obtain our first main result:

Theorem 1. There are no classical type simple Lie 2-algebra of odd toral rank. In particular, there are no classical type simple Lie 2-algebra of odd toral rank.

V. Kac in [10] showed that for $p > 3$ every simple finite dimensional contragredient Lie algebra is isomorphic to one of the simple Lie algebras of the classical type. If $p = 2$, this is no longer true and the classification of simple finite dimensional contragredient Lie algebras is still considered an open problem. In the last section we proved that the simple contragredient Lie 2-algebra of dimension 34 constructed by V. G. Kac and V. Veisfeiler in [9] has toral rank 4 and we calculate the dimension of the root spaces of this contragredient Lie algebra. More specifically, we have:

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Theorem 2. The simple contragredient Lie 2-algebra of dimension 34 with Cartan matrix

\[ F_{4,a} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]

which is denoted by \( G(F_{4,a}) \), has toral rank 4. Furthermore, the Cartan descomposition of \( G(F_{4,a}) \) with respect to the 4-dimensional torus \( T(\mathfrak{h}) \) is

\[ G(F_{4,a}) = T(\mathfrak{h}) \oplus \left( \bigoplus_{\xi \in G} \mathfrak{g}_\xi \right) \]

where \( G := \langle \alpha, \beta, \gamma, \lambda \rangle \) is an elementary abelian group of order 16, and \( \dim_K(\mathfrak{g}_\xi) = 2 \), for all \( \xi \in G \).

The only classical type simple Lie 2-algebra of toral rank 4 over an algebraically closed field of characteristic 2 are \( \mathfrak{sl}_5(K) \), \( \mathfrak{psl}_6(K) \), \( \mathfrak{sp}_{10}(K) \), and \( \mathfrak{sp}_{12}(K)^{\mathfrak{z}}(\mathfrak{gl}_{12}(K)) \) (see Corollary 5.6). Theorem (2) gives us an example of a non-classical simple Lie 2-algebra, which should be taken into account in future investigations related to the problem of classifying the simple Lie 2-algebras of toral rank 4.

In section 1 we present some basic definitions and well-known results that will be used throughout the work. In Section 2 and 3 we show that the linear special Lie algebra \( \mathfrak{sl}_{n+1}(K) \) and the symplectic Lie algebra \( \mathfrak{sp}_{2m}(K) \) over an algebraically closed field of characteristic 2 are Lie 2-algebra, and we study the simplicity of these algebra (Theorem 2.2 and 3.4). In section 4 we show that the orthogonal Lie algebra \( \mathfrak{o}_n(K)^{(1)} \) is not a Lie 2-algebra. In section 5 we list all classical type simple Lie 2-algebras, we calculate its toral rank, and we conclude that there are no classical type simple Lie 2-algebras with odd toral rank. Finally, in last section we show that the simple Lie 2-algebra of dimension 34 constructed by V. G. Kac and V. Veselkter in \([9]\) has toral rank 4, and we also give the Cartan decomposition of this algebra.

1. Preliminarías.

Throughout this paper all algebras are defined over a fixed algebraically closed field \( K \) of characteristic 2 containing the prime field \( \mathbb{F}_2 \) and \( \mathfrak{g} \) is any Lie algebra of finite dimension on \( K \). We start with some basic definitions and known facts.

1.1. Simple Lie 2-algebra.

Definition 1.1. A Lie 2-algebra is a pair \((\mathfrak{g}, [2])\) where \( \mathfrak{g} \) is a Lie algebra over \( K \), and \([2] : \mathfrak{g} \to \mathfrak{g}, a \mapsto a^{[2]}\) is a map (called 2-map) such that:

1. \((a + b)^{[2]} = a^{[2]} + b^{[2]} + [a, b], \quad \forall \ a, b \in \mathfrak{g}.
2. \((\lambda a)^{[2]} = \lambda^2 a^{[2]}, \quad \forall \ \lambda \in K, \quad \forall \ a \in \mathfrak{g}.
3. \(\text{ad}(b^{[2]}) = (\text{ad}(b))^2, \quad \forall \ b \in \mathfrak{g}.

If the center, \( Z(\mathfrak{g}) \), of \( \mathfrak{g} \) is zero and a 2-map \([2] : \mathfrak{g} \to \mathfrak{g}\) exists, it is unique. A Lie 2-algebra \((\mathfrak{g}, [2])\) is called a simple Lie 2-algebra, if \( \mathfrak{g} \) is a simple Lie algebra on \( K \).
Example 1.2. Let $A$ be an associative algebra and let $\text{Lie}(A)$ be the Lie algebra with bracket $[x, y] = x \circ y - y \circ x$ for $x, y \in \text{Lie}(A)$ associated with $A$. Then, $\text{Lie}(A)$ is a Lie 2-algebra with $a^{[2]} := a^2$. In particular, $\text{Lie}(\text{End}(V)) := \mathfrak{gl}(V)$ is a Lie 2-algebra, where $\text{End}(V)$ is the associative algebra of $K$-endomorphism on $V$.

Example 1.3. Let $b : V \times V \to K$ be a bilinear form and consider the subset $\mathfrak{g}(V, b)$ of $\mathfrak{gl}(V)$ defined by

$$\mathfrak{g}(V, b) := \{ f \in \mathfrak{gl}(V) : b(f(u), v) + b(u, f(v)) = 0 \ \forall u, v \in V \}.$$ 

Then, $(\mathfrak{g}(V, b), [2])$ is a Lie 2-subalgebra of $(\mathfrak{g}(V), [2])$. Indeed, take $f, g \in \mathfrak{gl}(V, b)$ and $v, w \in V$. Then,

$$b([f, g](v), w) = b((fg)(v), w) - b((gf)(v), w) = -b(g(v), f(w)) - b(f(v), g(w))$$

$$= -b(v, g(f(w))) + b(v, f(g(w))) = b(v, [f, g](w)).$$

This fact shows that $\mathfrak{g}(V, b)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Moreover,

$$b(f^{[2]}(v), w) = b(f(f(v)), w) = b(f(v), f(w)) = b(v, f(f(w))) = b(v, f^{[2]}(w)).$$

Therefore, $f^{[2]} \in \mathfrak{g}(V, b)$, for all $f \in \mathfrak{gl}(V)$.

It will be useful to have the matricial version of $\mathfrak{g}(V, b)$. Given $A \in \mathfrak{gl}_n(K)$, consider

$$\mathfrak{g}(A) = \{ X \in \mathfrak{gl}_n(K) : X^T A = AX \}.$$ 

Let $\Theta = \{ v_1, v_2, \ldots, v_n \}$ be a basis of $V$ and assume that $A \in \mathfrak{gl}_n(K)$ is the Gram matrix of $b$ with respect to the basis $\Theta$, that is,

$$a_{ij} = b(v_i, v_j) \quad 1 \leq i, j \leq n.$$ 

So, $\mathfrak{g}(V, b)$ and $\mathfrak{g}(A)$ are isomorphic as Lie 2-algebra.

Two matrices $A, B \in \mathfrak{gl}_n(K)$, are said to be congruent if there is $S \in \text{GL}_n(K)$ such that

$$S^T AS = B.$$ 

In this case, the map $\mathfrak{g}(A) \to \mathfrak{g}(B)$ given by $X \mapsto S^{-1} XS$ is a Lie 2-isomorphism.

1.2. Maximal Tori and Toral Rank.

Definition 1.4. Let $(\mathfrak{g}, [2])$ be a Lie 2-algebra. An element $t \in \mathfrak{g}$ is said to be a toral element if $t^{[2]} = t$. A subalgebra $t$ of $(\mathfrak{g}, [2])$ is called toral (or a torus of $\mathfrak{g}$) if the 2-mapping is invertible on $t$.

Any toral subalgebra of $\mathfrak{g}$ is abelian and admits a basis consisting of toral elements (see eg. [3]). A torus $t$ of $\mathfrak{g}$ is called maximal if the inclusion $t \subseteq t'$ with $t'$ toral implies $t' = t$.

Let $(\mathfrak{g}, [2])$ be a simple Lie 2-algebra over an algebraically closed field $K$ and let $\mathfrak{h}$ be a Cartan subalgebra. The set of toroidal elements in $\mathfrak{h}$ generates a torus. We denote this torus by the symbol $T(\mathfrak{h})$. The torus $T(\mathfrak{h})$ is maximal in $(\mathfrak{g}, [2])$ (see [8], Lemma 4. ).

Definition 1.5. (See [9]). The toral rank of a Lie 2-algebra $(\mathfrak{g}, [2])$ is

$$MT(\mathfrak{g}) := \max \{ \dim_K(t) \mid t \text{ is a torus in } \mathfrak{g} \}.$$
2. Special Linear Lie 2-algebra \(\mathfrak{sl}_{n+1}(K), [2]\).

In this section we consider the Lie algebra consisting of matrices of trace zero over \(K\), and we study some properties concerning about simplicity of this algebra.

It is a known fact that the commutator of the Lie general algebra \(\mathfrak{gl}_{n+1}(K)\) is a Lie subalgebra of \(\mathfrak{gl}_{n+1}(K)\). This algebra is called the Lie special algebra, and it is denoted by \(\mathfrak{sl}_{n+1}(K)\), That is,
\[
\mathfrak{sl}_{n+1}(K) := [\mathfrak{gl}_{n+1}(K), \mathfrak{gl}_{n+1}(K)] = \{A \in \mathfrak{gl}_{n+1}(K) : \text{tr}(A) = 0\}.
\]

It is easy to prove that
\[
\mathfrak{sl}_{n+1}(K) = \{A \in \mathfrak{gl}_{n+1}(K) : \text{tr}(A) = 0\}.
\]

A basis for \(\mathfrak{sl}_{n+1}(K)\) is the following:
\[
h_k := e_{kk} + e_{k+1,k+1}, \quad k = 1, ..., n; \quad e_{ij}, \quad i \neq j = 1, 2, ..., n + 1.
\]

Let us consider the 2-map \([2]\) : \(\mathfrak{sl}_{n+1}(K) \to \mathfrak{sl}_{n+1}(K)\) given by \(A[2] := A^2\).

**Remark 2.1.** The Lie 2-algebra \(\mathfrak{sl}_2(K)\) is not simple, since
\[
\mathfrak{sl}_2(K)^{(1)} = [\mathfrak{sl}_2(K), \mathfrak{sl}_2(K)] = \text{span}\{h_1\}
\]
then \(\mathfrak{sl}_2(K)^{(1)}\) is a nontrivial ideal of \(\mathfrak{sl}_2(K)\). In next theorem, we consider the case where \(n \geq 2\).

**Theorem 2.2.** The special Lie algebra \(\mathfrak{sl}_{n+1}(K)\) has the following properties:

1. \((\mathfrak{sl}_{n+1}(K), [2])\) is a Lie 2-algebra.
2. If \(n \geq 2\) and \((n + 1) \not\equiv 0(\text{mod} 2)\), then \(\mathfrak{sl}_{n+1}(K)\) is a simple Lie 2-algebra.
3. \(\mathfrak{psl}_{2n}(K) := \mathfrak{sl}_{2n}(K) / \mathfrak{z}(\mathfrak{gl}_{2n}(K)), n \geq 1\) is a simple Lie 2-algebra.

**Proof.** In order to prove (1), it is enough to see that \(\mathfrak{sl}_n(K)\) is closed by the 2-map \([2]\) : \(\mathfrak{sl}_{n+1}(K) \to \mathfrak{sl}_{n+1}(K)\). But, it is an immediate consequence of the fact that \(\text{tr}(A^2) = \text{tr}(A)^2\), for all \(A \in \mathfrak{sl}_{n+1}(K)\).

Let us prove (2). Firstly, we show that if \((n + 1) \not\equiv 0(\text{mod} 2)\), then
\[
\mathfrak{gl}_{n+1}(K) = \mathfrak{sl}_{n+1}(K) \oplus \mathfrak{z}(\mathfrak{gl}_{n+1}(K)).
\]

Indeed, if \(A \in \mathfrak{gl}_{n+1}(K)\) and \(\text{tr}(A) = \lambda\), then
\[
A = (A - \frac{\lambda}{n+1} I_{n+1}) + \frac{\lambda}{n+1} I_{n+1},
\]
where \((A - \frac{\lambda}{n+1} I_{n+1}) \in \mathfrak{sl}_{n+1}(K)\) and \(\frac{\lambda}{n+1} I_{n+1} \in \mathfrak{z}(\mathfrak{gl}_{n+1}(K))\). If \(A \in \mathfrak{sl}_{n+1}(K) \cap \mathfrak{z}(\mathfrak{gl}_{n+1}(K))\), then \(A = \lambda I_{n+1}\) and \(\text{tr}(A) = (n + 1)\lambda = 0\). Since \(n + 1\) is not a multiple of \(2\), we have \(\lambda = 0\). Therefore, \(\mathfrak{sl}_{n+1}(K) \cap \mathfrak{z}(\mathfrak{gl}_{n+1}(K)) = \{0\}\). Now, let \(I\) be an ideal of \(\mathfrak{sl}_{n+1}(K)\). Then
\[
[\mathfrak{gl}_n(K), I] = [\mathfrak{sl}_{n+1}(K) \oplus \mathfrak{z}(\mathfrak{gl}_{n+1}(K)), I] = [\mathfrak{sl}_{n+1}(K), I] + [\mathfrak{z}(\mathfrak{gl}_{n+1}(K)), I] \subset I + 0 = I.
\]

Therefore, \(I\) is also an ideal of \(\mathfrak{gl}_n(K)\). However, the only ideals of \(\mathfrak{gl}_n(K)\) contained in \(\mathfrak{sl}_{n+1}(K)\) are \(\{0\}\) and \(\mathfrak{sl}_{n+1}(K)\) (see [3]). Then, \(I = \{0\}\) and \(I = \mathfrak{sl}_{n+1}(K)\). Hence, \(\mathfrak{sl}_{n+1}(K)\) is a simple Lie 2-algebra.
We now prove (3). If \((n + 1) \equiv 0(\text{mod } 2)\), then \(\mathfrak{z}(\mathfrak{gl}_{n+1}(K)) \subseteq \mathfrak{sl}_{n+1}(K)\) is an ideal of \(\mathfrak{sl}_{n+1}(K)\). Therefore, \(\mathfrak{sl}_{n+1}(K)/\mathfrak{z}(\mathfrak{gl}_{n+1}(K))\) is a Lie 2-algebra with 2-map given by

\[
(x + \mathfrak{z}(\mathfrak{gl}_{n+1}(K)))[2] := x[2] + \mathfrak{z}(\mathfrak{gl}_{n+1}(K)), \quad \text{for all } x \in \mathfrak{sl}_{n+1}(K).
\]

Now, if \(J\) is another ideal of \(\mathfrak{sl}_{n+1}(K)/\mathfrak{z}(\mathfrak{gl}_{n+1}(K))\), then \(J = I/\mathfrak{z}(\mathfrak{gl}_{n+1}(K))\), where \(I\) is an ideal of \(\mathfrak{sl}_{n+1}(K)\) and \(\mathfrak{z}(\mathfrak{gl}_{n+1}(K)) \subseteq I\). Suppose that \(I \neq \mathfrak{z}(\mathfrak{gl}_{n+1}(K))\). Then, by direct computations, we find that \(e_{kl} \in I\), with \(k \neq l\). Using the identities

\[
[e_{kl}, e_{lk}] := e_{kk} + e_{ll}, \quad k \neq l
\]

we obtain that \(h_k := e_{k,k} + e_{k+1,k+1} \in I\) for all \(k = 1, 2, \ldots, n\). Therefore, \(I = \mathfrak{sl}_{n+1}(K)\), and \(\mathfrak{sl}_{n+1}(K)/\mathfrak{z}(\mathfrak{gl}_{n+1}(K))\) is a simple Lie 2-algebra.

Recall some well known facts about quadratic forms over an algebraically closed field of characteristic 2 and its corresponding Lie algebras. Let \(V\) be a \(n\)-dimensional \(K\)-space and \(b : V \times V \to K\) be a non-degenerate symmetric bilinear form. This means that \(b(x, y) = b(y, x)\), for all \(x, y \in V\) and \(b(x, V) = 0\) implies \(x = 0\). A non-degenerate symmetric bilinear form \(b\) is called symplectic if \(b(x, x) = 0\). Otherwise, it is called an orthogonal bilinear form.

3. The Lie 2-algebra \((\mathfrak{g}(V, b), [2])\) with \(b\) symplectic bilinear form

In this section we study the simplicity of Lie algebra which preserves a bilinear symplectic form over \(K\).

Let \(b : V \times V \to K\) be a symplectic bilinear form. From Example 1.2, we have that \(\mathfrak{g}(V, b)\) is a Lie 2-algebra. We denote this algebra by \(\mathfrak{sp}(V, b)\), and is called the symplectic Lie 2-algebra. In [11], it is shown that the dimension of \(V\) is even, that is, \(n = 2m\) and there exists a basis \(\beta\) of \(V\) in which \(b\) has Gram matrix

\[
J_{2m} := \begin{pmatrix}
0 & I_m \\
-I_m & 0
\end{pmatrix}.
\]

The Lie 2-algebra \(\mathfrak{sp}(V, b)\) is isomorphic to the Lie 2-algebra

\[
\mathfrak{sp}_{2m}(K) := \mathfrak{g}(J_{2m}) = \left\{ \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} : a, b, c \in \mathfrak{gl}_m(K), \ b, c \text{ symmetric} \right\},
\]

which has dimension \(2m^2 + m\) and a basis consisting of the following elements:

\[
d_i := e_{ii} + e_{m+i,m+i}, \quad (1 \leq i \leq m).
\]

\[
a_{ij} := e_{ij} + e_{m+j,m+i}, \quad (1 \leq i, j \leq m, i \neq j).
\]

\[
b_{ij} := e_{i,j+m} + e_{j,i+m}, \quad (1 \leq i < j \leq m, i \neq j).
\]

\[
b_i := e_{i,i+m}, \quad (1 \leq i \leq m).
\]

\[
e_{ij} := e_{i+m,j} + e_{j+m,i}, \quad (1 \leq i < j \leq m, i \neq j).
\]

\[
e_i := e_{i+m,i}, \quad (1 \leq i \leq m).
\]
If Lemma 3.2.

where $\text{Alt}$ symmetric matrices, such that:

We now calculate the derived algebras of $\mathfrak{sp}_{2m}(K)$, and then, we show that the second derived algebra is a Lie 2-algebra whenever 2 does not divided $m$ and $m \geq 3$.

Remark 3.1. For $m = 1$, we have $\mathfrak{sp}_2(K) = \text{span}\{d_1, b_{12}, c_{21}\} = \mathfrak{sl}_2(K)$. Then

$$\mathfrak{sp}_2(K)^{(1)} = \text{span}\{h_1\} = \mathfrak{sl}_2(K)$$ and $$\mathfrak{sp}_2(K)^{(2)} = \{0\},$$

and for $m = 2$, we have $\mathfrak{sp}_4(K) = \text{span}\{d_1, d_2, a_{12}, a_{21}, b_{12}, b_1, b_2, c_{12}, c_1, c_2\}$. By direct computations we obtain that

$$\mathfrak{sp}_4(K)^{(1)} = \text{span}\{d_1, d_2, a_{12}, a_{21}, b_{12}, b_1, b_2, c_{12}, c_1, c_2\}.$$ $$\mathfrak{sp}_4(K)^{(2)} = \text{span}\{d_1 + d_2, a_{12}, a_{21}, b_{12}, c_{12}\}.$$ $$\mathfrak{sp}_4(K)^{(3)} = \mathfrak{sl}_4(K)$$ and $$\mathfrak{sp}_4(K)^{(4)} = \{0\}.$$

Therefore, if $m = 1, 2$ then $\mathfrak{sp}_{2m}(K)$ is a solvable Lie 2-algebra.

**Lemma 3.2.** If $m \geq 3$, then:

1. $\mathfrak{sp}_{2m}(K)^{(1)} = \left\{ \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} : b, c \in \text{Alt}_m(K), a \in \mathfrak{gl}_m(K) \right\}.$
2. $\mathfrak{sp}_{2m}(K)^{(2)} = \left\{ \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} : b, c \in \text{Alt}_m(K) \right\}$ and $\text{tr}(a) = 0$.
3. $\mathfrak{sp}_{2m}(K)^{(3)} = \mathfrak{sp}_{2m}(K)^{(2)}$,

where $\text{Alt}_m(K)$ is the set of alternating $m \times m$-matrices with entries in $K$.

**Proof.** To prove (1), set $\mathfrak{g}_1 := \left\{ \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} : b, c \in \text{Alt}_m(K) \right\}$ and and take $\alpha, \beta$ in $\mathfrak{sp}_{2m}(K)$. Then there are $a, \bar{a}, b, \bar{b}, c$ and $\bar{c}$ in $\mathfrak{gl}_m(K)$, where $b, \bar{b} c$ and $\bar{c}$ are symmetric matrices, such that:

$$\alpha = \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{a}^T \end{pmatrix}.$$

Then

$$[\alpha, \beta] = \begin{pmatrix} a\bar{a} + \bar{b}c + \bar{a}a + \bar{c}c & \bar{ab} + \bar{ba}^T + \bar{a}b + \bar{ba}^T \\ c\bar{a} + a^T\bar{c} + \bar{c}a + a^Tc & \bar{cb} + a^T\bar{a}^T + \bar{cb} + a^T\bar{a}^T \end{pmatrix}.$$

Since $b y \bar{b}$ are symmetric matrices, we have

$$(a\bar{a} + \bar{b}c + \bar{a}a + \bar{c}c)^T = \bar{cb} + a^T\bar{a}^T + \bar{cb} + a^T\bar{a}^T$$

and

$$(\bar{ab} + \bar{ba}^T + \bar{ba}^T)^T = \bar{ba} + a^T\bar{a} + \bar{ba} + ba^T$$

|   | $d_{ij}$ | $a_{ij}$ | $b_{ij}$ | $c_{ij}$ | $b_{i}$ | $c_{i}$ |
|---|---|---|---|---|---|---|
| $d_{ij}$ | $d_{ij}$ | $a_{ij}$ | $b_{ij}$ | $c_{ij}$ | 0 | 0 |
| $a_{ij}$ | $a_{ij}$ | 0 | 0 | 0 | 0 | $a_{ij}$ |
| $b_{ij}$ | $b_{ij}$ | 0 | 0 | $d_{i} + d_{j}$ | 0 | $a_{ij}$ |
| $c_{ij}$ | $c_{ij}$ | 0 | $d_{i} + d_{j}$ | 0 | $a_{ij}$ | 0 |
| $b_{i}$ | 0 | 0 | 0 | $a_{ij}$ | 0 | $d_{i}$ |
| $c_{i}$ | 0 | $c_{ij}$ | $a_{ij}$ | 0 | $d_{i}$ | 0 |

**Table 1.** The Lie 2-algebra $\mathfrak{sp}_{2m}(K)$. 

$2$ when $m = 2$.

By direct computations we obtain that

$$\mathfrak{sp}_4(K)^{(1)} = \text{span}\{d_1, d_2, a_{12}, a_{21}, b_{12}, b_1, b_2, c_{12}, c_1, c_2\}.$$ $$\mathfrak{sp}_4(K)^{(2)} = \text{span}\{d_1 + d_2, a_{12}, a_{21}, b_{12}, c_{12}\}.$$ $$\mathfrak{sp}_4(K)^{(3)} = \mathfrak{sl}_4(K)$$ and $$\mathfrak{sp}_4(K)^{(4)} = \{0\}.$$
Similarly, we prove that 
\[(a\bar{b} + b\bar{a}^T + \bar{a}b + \bar{a}b^T)_{ij} = \sum_{j=1}^{m} (a_{ij} b_{ji} + b_{ij} a_{ij} + \bar{a}_{ij} b_{ji} + b_{ij} a_{ij}) = 0.\]

Analogously, the symmetry of \(c\) and \(\bar{c}\) imply
\[
(c\bar{a} + a^{T}\bar{c} + \bar{c}a + a^{T}c)^{T} = c\bar{a} + a^{T}\bar{c} + \bar{c}a + a^{T}c,
\]
\[
(c\bar{b} + a^{T}\bar{a} + \bar{c}b + a^{T}a)^{T} = \bar{c}b + a^{T}\bar{a} + \bar{c}b + a^{T}a,
\]
\[
\left(\bar{c}a + a^{T}\bar{c} + \bar{c}a + a^{T}c\right)_{ij} = 0.
\]
Therefore, \((a\bar{b} + b\bar{a}^T + \bar{a}b + \bar{a}b^T)\) and \((c\bar{a} + a^{T}\bar{c} + \bar{c}a + a^{T}c)\) belong to \(\text{Alt}_m(K)\). So, \(\mathfrak{sp}_{2m}(K)^{(1)} \subseteq \mathfrak{g}_1\).

Now, we show that \(\mathfrak{g}_1 \subseteq \mathfrak{sp}_{2m}(K)^{(1)}\). Given \(a = (x_{ij}) \in \mathfrak{gl}_m(K)\) we have
\[
\begin{pmatrix}
0 & b \\
0 & 0
\end{pmatrix} = \sum_{i=1}^{m} x_{ii} [b_i, c_i] + \sum_{i \neq j} x_{ij} [b_i, c_{ij}] \in \mathfrak{sp}_{2m}(K)^{(1)}.
\]

Let us consider the linear map \(\varphi : \mathfrak{gl}_m(K) \rightarrow \text{Alt}_m(K)\) given by \(a \mapsto a + a^{T}\). Since 
\[
\text{Ker}(\varphi) = \{a \in \mathfrak{gl}_m(K) : a \text{ is symmetric} \}\text{ and } \dim_K(\text{Im}(\varphi)) = m^2 - \frac{m(m+1)}{2} = \frac{m(m-1)}{2} = \dim_K(\text{Alt}_m(K)),
\]
we conclude that \(\text{Im}(\varphi) = \text{Alt}_m(K)\). That is, \(\varphi\) is a surjective map. Then, given \(b \in \text{Alt}_m(K)\) there exists \(a \in \mathfrak{gl}_m(K)\) such that, \(a + a^{T} = b\). Hence,
\[
\begin{pmatrix}
0 & b \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & a + a^{T} \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
a & 0 \\
0 & a^{T}
\end{pmatrix}, & \begin{pmatrix}
0 & I \\
0 & 0
\end{pmatrix}
\end{pmatrix} \in \mathfrak{sp}_{2m}(K)^{(1)}.
\]

Similarly, we prove that \(\begin{pmatrix}
0 & 0 \\
c & 0
\end{pmatrix} \in \mathfrak{sp}_{2m}(K)^{(1)}\). Therefore, \(\mathfrak{g}_1 \subseteq \mathfrak{sp}_{2m}(K)^{(1)}\).

To prove (2), let \(\mathfrak{g}_2 = \left\{ \begin{pmatrix}
a & b \\
c & a^{T}
\end{pmatrix} : b, c \in \text{Alt}_m(K) \text{ and } \text{tr}(a) = 0 \right\}\). We will prove that \(\mathfrak{sp}_{2m}(K)^{(2)} = \mathfrak{g}_2\). From the description of \(\mathfrak{sp}_{2m}(K)^{(1)}\) in (1), we deduce that the Lie algebra \(\mathfrak{sp}_{2m}(K)^{(1)}\) is generated by \(a_{ij}, b_{ij}, c_{ij}\) and \(d_i\) for \(1 \leq i, j \leq m\). Therefore, \(\mathfrak{sp}_{2m}(K)^{(2)} = [\mathfrak{sp}_{2m}(K)^{(1)}, \mathfrak{sp}_{2m}(K)^{(1)}]\) is generated by \(a_{ij}, b_{ij}, c_{ij}\) and \(d_i + d_j\) for \(1 \leq i, j \leq m\). Since all of these elements belong to \(\mathfrak{g}_2\), we conclude that \(\mathfrak{sp}_{2m}(K)^{(2)} \subseteq \mathfrak{g}_2\). The another inclusion is established reasoning in a similar way to the proof of (1).

Finally, we prove (3). In the proof of (2), it is proven that \(\mathfrak{sp}_{2m}(K)^{(2)}\) is generated by \(d_i + d_j\) for \(1 \leq i, j \leq m\). Therefore,
\[
\mathfrak{sp}_{2m}(K)^{(3)} = \text{span}\{[x, y] : x, y \in \{a_{ij}, b_{ij}, c_{ij}, d_i + d_j : 1 \leq i, j \leq m\}\}.
\]
From Table 3.1 we conclude that
\[
\mathfrak{sp}_{2m}(K)^{(3)} = \text{span}\{a_{ij}, b_{ij}, c_{ij}, d_i + d_j : 1 \leq i, j \leq m\} = \mathfrak{sp}_{2m}(K)^{(2)}.
\]

\[\square\]

**Lemma 3.3.** Let \(I\) be a nontrivial ideal of \(\mathfrak{sp}_{2m}(K)^{(2)}\), then \(c_{ij}, b_{ij} \notin I\), for all \(i, j\).
Proof. Let $1 \leq i \neq j \leq n$ fixed. If $c_{ij} \in I$, then for all $k \neq j$, the relations $[c_{ij}, b_{lj}] = d_i + d_j$, $[d_i + d_j, c_{ik}] = a_{ik}$, $[d_i + d_j, c_{jk}] = c_{ik}$, and $[d_i + d_j, b_{ik}] = b_{jk}$ imply $a_{ik}, b_{ik}, d_i + d_k$ belong to $I$ for all $1 \leq k \leq n$. Since $I$ is an ideal of $\mathfrak{sp}_{2m}(K)^{(2)}$, for all $k, l$ with $l \neq k$, we have $[a_{il}, c_{jk}] = c_{ik}, [b_{il}, c_{jk}] = a_{ik}$ and $[d_i + d_j, b_{ik}] = b_{jk}$ belong to $I$. Therefore, $I = \mathfrak{sp}_{2m}(K)^{(2)}$ which is a contradiction. Similarly, if suppose that $b_{ij} \in I$, we arrive to a contradiction. Hence, $c_{ij}, b_{ij} \notin I$ for all $i, j$. \hfill $\square$

**Theorem 3.4.** Let $b : V \times V \to K$ be a symplectic bilinear form and let $\mathfrak{sp}(V, b)$ be the symplectic Lie algebra associated to $b$. Suppose that $\dim_k(V) = n > 4$, then:

1. $\mathfrak{sp}(V, b)^{(2)}$ is a Lie 2-algebra.
2. If $4 \nmid n$, then $\mathfrak{sp}(V, b)^{(2)}/\mathfrak{sl}(V)$ is simple.
3. If $4 \mid n$, then $\mathfrak{sp}(V, b)^{(2)}$ is simple.

**Proof.** In order to prove (1), we need only to prove that $\mathfrak{sp}_{2m}(K)^{(2)}$ is closed under the 2-map. Let $n = 2m$ and $\alpha = \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} \in \mathfrak{sp}_{2m}(K)^{(2)}$. Then $b, c \in \text{Alt}_m(K)$ and $\text{tr}(\alpha) = 0$. Since $b$ and $c$ are symmetric matrices, we obtain that

$$\alpha^2 = \begin{pmatrix} a^2 + bc & ab + ba^T \\ ca + a^T c & cb + (a^T)^2 \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + ba^T \\ ca + a^T c & (bc + a^T)^2 \end{pmatrix},$$

and $(ab + ba^T)^T = ab + ba^T$, and $(ca + a^T c)^T = ca + a^T c$. Moreover, by using the equalities $b_{ii} = 0$, $c_{ii} = 0$ and $\text{tr}(\alpha) = 0$, we have

$$(ca + a^T c)_{ii} = 0, \quad (ab + ba^T)_{ii} = 0, \quad \text{tr}(a^2 + bc) = 0.$$

Therefore, $\alpha^2 \in \mathfrak{sp}(V, b)^{(2)}$ for all $\alpha \in \mathfrak{sp}(V, b)^{(2)}$. Hence $\mathfrak{sp}_{2m}(K)^{(2)}$ is a Lie 2-algebra.

Let us prove (2). If $4 \mid 2m$, then $\mathfrak{sl}(2m_2(K)) \subseteq \mathfrak{sp}_{2m}(K)^{(2)}$ is an ideal of $\mathfrak{sp}_{2m}(K)^{(2)}$. Let $J$ be an ideal of $\mathfrak{sp}_{2m}(K)^{(2)}/\mathfrak{sl}(2m_2(K))$, then $J = I/\mathfrak{sl}(2m_2(K))$, where $I$ is an ideal of $\mathfrak{sp}_{2m}(K)^{(2)}$ and $\mathfrak{sl}(2m_2(K)) \subseteq I$. Suppose that $I \neq \mathfrak{sp}_{2m}(K)^{(2)}$. By Lemma 3.3, we have $c_{ij}, b_{ij} \notin I$, therefore given $\alpha \in I$, there exists $a = (a_{ij}) \in \mathfrak{g}_m(K)$ with $\text{tr}(\alpha) = 0$ such that

$$\alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & a^T \end{pmatrix}.$$  

Now, since $I$ is an ideal of $\mathfrak{sp}_{2m}(K)^{(2)}$, we get that

$$[\alpha, X] \in I, \quad \forall X \in \mathfrak{sp}_{2m}(K)^{(2)}.$$  

In particular, for

$$X = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

with $b := c_{ij} + e_{ji} \in \text{Alt}_m(K)$, we have $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = a_{11}$ for all $i$. Then $\alpha = a_{ii}I_{2m} \in \mathfrak{sl}(2m_2(K))$. Hence, $I = \mathfrak{sl}(2m_2(K))$ and $\mathfrak{sp}_{2m}(K)^{(2)}/\mathfrak{sl}(2m_2(K))$ is simple.

Finally we prove (3). Let $I$ be an ideal of $\mathfrak{sp}_{2m}(K)^{(2)}$, $I \neq \mathfrak{sp}_{2m}(K)^{(2)}$. Reasoning as the proof of item (2), we get that $a = \lambda \mathbb{I}_m$ with $m$ odd. As $\text{tr}(\alpha) = 0$, we have $\lambda = 0$. Then $\alpha = 0$ and $I = \{0\}$. Therefore, $\mathfrak{sp}_{2m}(K)^{(2)}$ is simple. \hfill $\square$
4. Lie 2-algebras \((\mathfrak{g}(V, b), [2])\) with \(b\) orthogonal bilinear form

In this section we show that the Lie algebra which preserve the orthogonal linear form over \(K\) is not a Lie 2-subalgebra of \(\mathfrak{gl}_n(K)\).

Suppose that \(b : V \times V \to K\) is an orthogonal bilinear form, and let \(\mathfrak{o}(V, b)\) be the Lie 2-algebra associated to \(b\). In [11] (Theorem 20), it is shown that there exists a basis of \(V\) in which \(b\) has Gram matrix \(D = \text{diag}(d_1, d_2, \ldots, d_n)\), where \(0 \neq d_i \in K\) for all \(i\), then

\[
\mathfrak{g}(D) := \{ A \in \mathfrak{gl}_n(K) : d_i a_{ij} = d_j a_{ji}, \ 1 \leq i, j \leq n \}. 
\]

Since \(K\) is an algebraically closed field, we have that \(K^2 = K\), this is, every element of \(K\) is a square. Then, we can assume that \(D = I_n\), then

\[
\mathfrak{o}_n(K) := \mathfrak{g}(I_n) = \{ A \in \mathfrak{gl}_n(K) : A \text{ is symmetric} \}
\]

is a Lie 2-algebra with basis \(\{e_{ii} \cup \{e_{ij} := e_{ij} + e_{ji}\}, 1 \leq i < j \leq n\}\) and whose Lie bracket is given by:

\[
\begin{align*}
[e_{ii}, e_{jj}] &= e_{ij}, \ 1 \leq i < j \leq n, \\
[e_{ii}, e_{kk}] &= 0, \ i \neq k, \\
[e_{ij}, e_{kl}] &= \delta_{ik} e_{jl} + \delta_{il} e_{kj} + \delta_{jk} e_{il} + \delta_{jl} e_{ik} \ \forall i < j, k < l.
\end{align*}
\]

Moreover, \(e_{ij}^2 = e_{ii} + e_{jj}\), and \(\dim_K(\mathfrak{o}_n(K)) = \frac{n(n+1)}{2}\).

**Lemma 4.1.** \(\mathfrak{o}_n(K)^{(1)} = \text{Alt}_n(K)\) and \(\dim_K(\mathfrak{o}_n(K)^{(1)}) = \frac{n(n-1)}{2}\).

**Proof.** Let \(a, b \in \mathfrak{o}_n(K)\), then

\[
[a, b]^T = (ab - ba)^T = b^T a^T - a^T b^T = -(ab - ba) = [a, b].
\]

Thus, \([a, b]\) is a symmetric matrix. Moreover, by the symmetry of \(a\) and \(b\), we have \([a, b]_{ii} = \sum_j (a_{ij} b_{ji} - b_{ij} a_{ji}) = 0\). Therefore, \(\mathfrak{o}_n(K)^{(1)} \subseteq \text{Alt}_n(K)\). Reciprocally the matrices \(\tilde{e}_{ij} := e_{ij} + e_{ji}\), where \(1 \leq i < j \leq n\), form a basis of \(\text{Alt}_n(K)\).

Now, since \(e_{ij} = [\tilde{e}_{ik}, e_{kj}] \in \mathfrak{o}_n(K)^{(1)}\) for all \(i, j\), we have \(\mathfrak{o}_n(K)^{(1)} = \text{Alt}_n(K)\) and \(\dim_K(\mathfrak{o}_n(K)^{(1)}) = \dim_K(\text{Alt}_n(K)) = n = \frac{n(n+1)}{2}\).

**Remark 4.2.** From Lemma 4.1 it follows that the following elements

\[
e_{ij} = e_{ij} + e_{ji}
\]

for \(i \neq j\) form a basis of \(\mathfrak{o}_n(K)^{(1)}\). Now, since \(e_{ij}^2 = e_{ii} + e_{jj}\) does not belong to \(\mathfrak{o}_n(K)^{(1)}\), we have \(\mathfrak{o}_n(K)^{(1)}\) is not a Lie 2-algebra with respect to the 2-map \([2] : \mathfrak{o}_n(K) \to \mathfrak{o}_n(K)\) defined by \(a \mapsto a^2\).

5. Classical type simple Lie 2-algebra and their toral rank.

W. Killing and E. Cartan show that all simple Lie algebra over an algebraically closed field of zero characteristic is isomorphic to one of the Classical algebras of Lie \(A_n\) \((n \geq 1)\), \(B_n\) \((n \geq 2)\), \(C_n\) \((n \geq 3)\), \(D_n\) \((n \geq 4)\) or to the Exceptional Lie algebras, \(\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\) (see [3]), but in characteristic 2, it seems that many new phenomena arise, for instance, these are not necessarily simple, or some of them are isomorphic and, and therefore, the classification of simple Lie algebras over the field \(K\) will be different from those of the characteristics 0 and \(p \geq 5\). In this section, we calculate the toral rank of the simple 2-Lie algebra of the classical type and we conclude that
there are no classical type simple Lie 2-algebra of odd toral rank. In particular, there are no classical type simple Lie 2-algebra of toral rank 3.

**Definition 5.1.** Given an irreducible root system of type \(X_l\) and its corresponding Chevalley algebra \(g(X_l, K)\) over the field \(K\), the quotient
\[
\overline{g(X_l, K)} := g(X_l, K)/\mathfrak{z}(g(X_l, K)),
\]
where \(\mathfrak{z}(g(X_l, K))\) is the center of \(g(X_l, K)\), is usually called the *classical Lie algebra of type* \(X_l\).

**Remark 5.2.** This definition is exactly the same as Steinberg’s, but Steinberg excluded some types of characteristic 2 and 3.

The simplicity of the classical type Lie algebras in characteristic 2 have been determined by Hogeweij in [6], as indicated in the following theorem.

**Proposition 5.3.** Suppose that \(X_l\) is a Lie algebra which is not of type \(A_1\), \(B_l\), \(C_l\), or \(f_4\). Then \(\overline{g(X_l, K)}\) is a simple Lie 2-algebra.

So, from Proposition 5.3. Theorem 2.2 and Theorem 3.4 it follows that the classical type simple Lie 2-algebra are:

**Corollary 5.4.** The classic type simple Lie 2-algebra are:

1. Type \(A_n\):
\[
\overline{g(A_n, K)} \cong \mathfrak{sl}_{n+1}(K) \text{ if } 2 \nmid (n+1),
\]
\[
\overline{g(A_n, K)} \cong \mathfrak{psl}_{2n}(K) \text{ if } 2 \mid (n+1).
\]

2. Type \(D_n\):
\[
\overline{g(D_n, K)} \cong \mathfrak{sp}_{2n}(K)^{(2)} \text{ if } n \text{ is odd},
\]
\[
\overline{g(D_n, K)} \cong \mathfrak{sp}_{2n}(K)^{(2)}/\mathfrak{z}(\mathfrak{gl}_n(K)) \text{ if } n \text{ is even}.
\]

3. Type \(g_2\):
\[
\overline{g(g_2, K)} = g(g_2, K).
\]

4. Type \(e_6\):
\[
\overline{g(e_6, K)} = g(e_6, K).
\]

5. Type \(e_7\):
\[
\overline{g(e_7, K)}
\]

6. Type \(e_8\):
\[
\overline{g(e_8, K)} = g(e_8, K).
\]

**Theorem 5.5.** Let \(g\) be a classical type simple Lie 2-algebra and \(h\) be a Cartan subalgebra of \(g\). Then
\[
MT(g) = \dim_K(h).
\]

**Proof.** Let \(g\) be a classical type simple Lie 2-algebra. Then from Corollary 5.4 it follows that \(g = \overline{g(X_l, K)}\) with \(X_l \neq A_1, B_l, C_l, f_4\). Hence, any quotient of the form
\[
\overline{h(X_l, K)} = h(X_l, K)/\mathfrak{z}(g(X_l, K)),
\]
where \( \mathfrak{h}(X_i, K) \) is a Cartan subalgebra of the Chevalley \( K \)-algebra \( \mathfrak{g}(X_i, K) \) is a Cartan subalgebra of \( \mathfrak{g} \). Since \( \mathfrak{h}(X_i, K) = \text{span}\{h_i \otimes 1 : h_i \in \mathfrak{h}_X\} \) and \( \mathfrak{h}_X \) is the subalgebra of diagonal matrices of \( \mathfrak{sl}_{l+1}(K) \), we obtain \( (h_i \otimes 1)^{[2]} = h_i \otimes 1 \), for each \( h_i \in \mathfrak{h}_X \). Thus, the equality \( ([h_i \otimes 1])^{[2]} = [h_i \otimes 1] \) implies that \( \mathfrak{h}(X_i, K) \subseteq T(h(X_i, K)) \), and as \( T(h(X_i, K)) \subseteq \mathfrak{h}(X_i, K) \), we have that \( \mathfrak{h}(X_i, K) = T(h(X_i, K)) \). Since any pair of Cartan Lie subalgebra of a finite-dimensional classical type Lie algebra \( \mathfrak{g} \) over \( K \) are conjugate (see [12]), there exists an automorphism \( \sigma \in \text{Aut}(\mathfrak{g}) \) such that \( \mathfrak{h} = \sigma(h(X_i, K)) \). Then, from Lemma 5 (see [9]), we obtain
\[
T(\mathfrak{h}) = \sigma(T(h(X_i, K))) = \sigma(h(X_i, K)) = \mathfrak{h}.
\]
Then any Cartan subalgebra of a simple Lie 2-algebra \( \mathfrak{g} \) of classical type is a maximal tori in \( \mathfrak{g} \), hence
\[
MT(\mathfrak{g}) = \dim_K(\mathfrak{h}).
\]

A direct consequence of Theorem 5.5 is the following.

**Corollary 5.6.** The toral rank of the classical type simple Lie 2-algebras is:

1. \( MT(A_n) = n \), if \( 2 \nmid (n+1) \), \( l > 1 \).
2. \( MT(A_n) = n-1 \), if \( 2 \mid (n+1) \), \( n > 1 \).
3. \( MT(D_n) = n-1 \), if \( n \) is odd, \( n \geq 3 \).
4. \( MT(D_n) = n-2 \), if \( n \) is even, \( n \geq 3 \).
5. \( MT(\mathfrak{g}_2) = 2 \).
6. \( MT(\mathfrak{e}_6) = 6 \).
7. \( MT(\mathfrak{e}_7) = 6 \).
8. \( MT(\mathfrak{e}_8) = 8 \).

From Corollary 5.6, it follows the following result.

**Theorem 5.7.** There are no classical type simple Lie 2-algebra of odd toral rank.

6. **A (Contragredient) Simple Lie 2-Algebra of Dimension 34 and Toral Rank 4.**

In this section we show that the contragredient Lie 2-algebra \( G(F_{4,1}) \) constructed by V. Kac and V. Veisfeiler (see [9]) has toral rank 4, and we obtain the Cartan decomposition of this algebra.

**Definition 6.1.** Given an \( (n \times n) \)-matrix \( A = (a_{ij}) \) with elements in \( K \), we denote by \( \bar{G}(A) \) the Lie algebra determined by the system of generators \( e_i, f_i, h_i, i = 1, ..., n \), and the system of relations
\[
[e_i, f_j] = \delta_{ij}h_j, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j,
\]
for \( 1 \leq i, j \leq n \). We set \( \deg e_i = 1, \deg f_i = -1 \) and \( \deg h_i = 0, i = 1, 2, ..., n \). Thus, the algebra \( \bar{G}(A) \) becomes into a graded Lie algebra, \( \bar{G}(A) = \bigoplus_{i \in \mathbb{Z}} \bar{G}_i \). Let \( J(A) \) be a maximal homogeneous ideal in \( \bar{G}(A) \) such that \( J(A) \cap (\bar{G}_{-1} \oplus \bar{G}_0 \oplus \bar{G}_1) = 0 \). The Lie algebra \( G(A) := \bar{G}(A)/J(A) \) is called a contragredient Lie algebra and \( A \) is its Cartan matrix.
In [9], V. Kac and V. Veisfeiler considered the algebra
\[ G(F_{4,a}) := \widetilde{G}(F_{4,a})/J(F_{4,a}), \]
where
\[ F_{4,a} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]
a \in K \setminus \{0, 1\} and \( J(F_{4,a}) \) is the only maximal homogeneous ideal in \( \widetilde{G}(F_{4,a}) \) such that
\[
\begin{align*}
J(F_{4,a}) \cap \text{span}\{h_1, h_2, h_3, h_4\} &= 0 \\
J(F_{4,a}) \cap \text{span}\{e_i, f_i\} &= 0.
\end{align*}
\]

They proved that \( G(F_{4,a}) \) is a simple Lie 2-algebra of dimension 34 with Cartan matrix \( F_{4,a} \) with \( a \in K \setminus \{0, 1\} \) (see [9], Proposition 3.6). We now prove that this 2-algebra Lie has toral rank 4 and, furthermore, we give its Cartan decomposition.

From (6.1), we conclude that
\[ h := \text{span}\{\bar{h}_i := h_i + J(F_{4,a}) : i \in I_4\} \]
is a Cartan subalgebra of \( G(F_{a,4}) \). We now explicitly describe the maximal tori \( T(h) \) consisting of toroidal elements in \( h \).

Since \( h_1^{[2]} \in h \), we have \( h_1^{[2]} = 0 \), and \( h_2^{[2]} = h_3^{[2]} = h_4^{[2]} = 0 \) and \( h_1^{[2]} = 0 \). Similarly, we obtain \( \delta_2 = \delta_3 = \delta_4 = 0 \) and \( \delta_1 = 1 \). So, \( h_1^{[2]} = 1 \)

We also find \( h_3^{[2]} = h_3 \), \( h_4^{[2]} = h_4 \) and \( h_2^{[2]} = ah_2 + \bar{a}h_4 \), where \( \bar{a} = a + 1 \)

Let \( t_2 := xh_2 + yh_4 \), with \( x, y \in K \). If the equality \( t_2^{[2]} = t_2 \) holds true, then \( x, y \) satisfy the following system of equations
\[
\begin{align*}
ax^2 + x &= 0 \\
\bar{a}x^2 + y^2 + y &= 0,
\end{align*}
\]
whose solution set is
\[ \{(0, 0), (0, 1), (\frac{1}{a}, \frac{1}{a}), (\frac{1}{a}, \bar{a})\}. \]
First two solutions give \( t_2 = 0 \), and \( t_2 = h_4 \) respectively, and with the last two solutions we obtain \( t_2 = \frac{1}{a}(h_2 + h_4) \) and \( t_2 = \frac{1}{a}(h_2 + \bar{a}h_4) \). Since \( \frac{1}{a}(h_2 + h_4) = \frac{1}{a}(h_2 + \bar{a}h_4) = h_4 \), we have
\[
T(h) := \text{span}\{\bar{h}_1, \bar{h}_3, \bar{h}_4, \frac{1}{a}(h_2 + h_4) + J(F_{4,a})\}
\]
and \( \dim_K(T(h)) = 4 \). This fact shows that \( MT(G(F_{4,a})) = 4 \).

We now find the Cartan decomposition of \( G(F_{4,a}) \) with respect to \( T(h) \). By definition of the ideal \( J(F_{4,a}) \), the elements \( e_i, f_i \) and \( h_i \) for \( 1 \leq i \leq 4 \) does not belong to \( J(F_{4,a}) \). Therefore, the classes \( \overline{e}_i = e_i + J(F_{4,a}) \), \( \overline{f}_i = f_i + J(F_{4,a}) \) and \( \overline{h}_i = h_i + J(F_{4,a}) \) for \( 1 \leq i \leq 4 \), belong to a basis for \( G(F_{4,a}) \). Now, to complete a basis for \( G(F_{4,a}) \), we consider the product \( xy := [x, y] \). The products \( xy \), where \( x \) and \( y \) are generators of \( G(F_{4,a}) \), and some of them does not
belong to \( \{e_i, f_i : 1 \leq i \leq 4\} \) are zero or belong to span\( \{h_i, e_i, f_i : 1 \leq i \leq 4\} \). Thus, the only products of two generators that give us new generators are \( e_i e_j \) and \( f_i f_j \) with \( i < j \). So, the elements \( \overline{e_i} = e_i e_j + J(F_{4,a}) \) and \( \overline{f_i} = f_i f_j + J(F_{4,a}) \) with \( i < j \) are also generators of \( G(F_{4,a}) \), which are linearly independent with \( e_i e_j + J(F_{4,a}) \) and \( f_i f_j + J(F_{4,a}) \). Reasoning in a similar way we obtain that the elements \( (e_1 e_2) e_3, (e_1 e_2) e_4, (e_1 e_3) e_4, (e_1 e_3) e_4, ((e_1 e_2) e_3) e_4 \) modulo \( J(F_{4,a}) \) and \( (f_1 f_2) f_3, (f_1 f_2) f_4, (f_1 f_3) f_4, (f_2 f_3) f_4, ((f_1 f_2) f_3) f_4 \) modulo \( J(F_{4,a}) \) complete a basis for \( G(F_{4,a}) \). We denote this basis by \( \Phi \).

Next, we calculate the weights for each element of the basis \( \Phi \) of \( \widetilde{G}(F_{4,a}) \) with respect to

\[
\mathfrak{t}_1 := \{h_1, \frac{1}{a} (h_2 + h_4), h_3, h_4\}
\]

are:

- \([e_1, h_1] = a_{11} e_1 = 0 e_1,
  \]
  \( [e_1, \frac{1}{a} (h_2 + h_4)] = \frac{1}{a} ([e_1, h_2] + [e_1, h_4]) = \frac{1}{a} (a e_1) = 1 e_1, \]
  \( [e_1, h_3] = a_{31} e_1 = 0 e_1, \]
  \( [e_1, h_4] = a_{41} e_1 = 0 e_1. \]

Then, the weight of \( \mathfrak{t}_1 \) is \( \beta := (0, 1, 0, 0). \)

- \([e_2, h_1] = a_{12} e_2 = 1 e_2,
  \]
  \( [e_2, \frac{1}{a} (h_2 + h_4)] = \frac{1}{a} ([e_2, h_2] + [e_2, h_4]) = 0 e_2, \]
  \( [e_2, h_3] = a_{32} e_2 = 1 e_2, \]
  \( [e_2, h_4] = a_{42} e_1 = 0 e_1. \]

The weight the \( \mathfrak{t}_2 \) is \( \alpha + \gamma := (1, 0, 1, 0). \)

- \([e_3, h_1] = a_{13} e_3 = 0 e_3,
  \]
  \( [e_3, \frac{1}{a} (h_2 + h_4)] = \frac{1}{a} ([e_3, h_2] + [e_3, h_4]) = 0 e_3, \]
  \( [e_3, h_3] = a_{33} e_3 = 0 e_3, \]
  \( [e_3, h_4] = a_{43} e_3 = 1 e_3. \]

The weight the \( \mathfrak{t}_3 \) is \( \lambda := (0, 0, 0, 1). \)

- \([e_4, h_1] = a_{14} e_4 = 0 e_4,
  \]
  \( [e_4, \frac{1}{a} (h_2 + h_4)] = \frac{1}{a} ([e_4, h_2] + [e_4, h_4]) = 0 e_4, \]
  \( [e_4, h_3] = a_{34} e_4 = 1 e_4, \]
  \( [e_4, h_4] = a_{44} e_4 = 0 e_4. \]

Then, the weight of \( \mathfrak{t}_4 \) is \( \gamma := (0, 0, 1, 0). \)

By the similarity in the definition of the bracket \( \{h_i, f_j\} = a_{ij} f_j \) with the bracket \( [h_i, e_j] = a_{ij} e_j \), we deduce that \( \mathfrak{t}_i \) and \( \mathfrak{j}_i \), for \( 1 \leq i \leq 4 \), have the same weight. On the other hand, by using \( [\mathfrak{g}_\xi, \mathfrak{g}_\mu] \leq \mathfrak{g}_{\xi + \mu} \), we obtain that the remaining elements of \( \Phi \) have the weights given in Table 2 where we use the notation \( \overline{e_{ijk}} = (e_i e_j) e_k + J(F_{4,a}) \), \( \overline{f_{ijk}} = (f_i f_j) f_k + J(F_{4,a}) \), \( \overline{e_{1234}} = ((e_1 e_2) e_3) e_4 + J(F_{4,a}) \) and \( \overline{f_{1234}} = ((f_1 f_2) f_3) f_4 + J(F_{4,a}) \).

Therefore, the Cartan decomposition of \( G(F_{4,a}) \) with respect to \( T(\mathfrak{h}) \) is

\[
G(F_{4,a}) = T(\mathfrak{h}) \oplus \left( \oplus_{\xi \in G} \mathfrak{g}_\xi \right),
\]

where \( G := (\alpha, \beta, \gamma, \lambda) \) is an elementary abelian group of order 16, and \( \dim_k(\mathfrak{g}_\xi) = 2 \), for all \( \xi \in G \). Therefore, we have:
Theorem 6.2. The contragredient Lie algebra $G(F_{4,a})$ on $K$ with Cartan matrix

$$F_{4,a} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

have the following properties:

1. $G(F_{4,a})$ is a simple Lie 2-algebra of dimension 34.
2. $MT(G(F_{4,a})) = 4$.
3. the Cartan descomposition of $G(F_{4,a})$ with respect to $T(\mathfrak{h})$ is

$$G(F_{4,a}) = T(\mathfrak{h}) \oplus \left( \oplus_{\xi \in G} \mathfrak{g}_\xi \right),$$

where $G := \langle \alpha, \beta, \gamma, \lambda \rangle$ is an elementary abelian group of order 16, and $\dim_k(\mathfrak{g}_\xi) = 2$, for all $\xi \in G$.

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