Manifest calculation and the finiteness of the superstring Feynman diagrams

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Abstract

The multi-loop amplitudes for the closed, oriented superstring are represented by finite dimensional integrals of explicit functions calculated through the super-Schottky group parameters and interaction vertex coordinates on the supermanifold. The integration region is proposed to be consistent with the group of the local symmetries of the amplitude and with the unitarity equations. It is shown that, besides the $SL(2)$ group, super-Schottky group and modular one, the total group of the local symmetries includes an isomorphism between sets of the forming group transformations, the period matrix to be the same. The singular integration configurations are studied. The calculation of the integrals over the above configurations is developed preserving all the local symmetries of the amplitude, the amplitudes being free from divergences. The nullification of the 0-, 1-, 2- and 3-point amplitudes of massless states is verified. Vanishing the amplitudes for a longitudinal gauge boson is argued.

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1 Introduction

Superstrings are currently considered for the unified interaction particle theory. Nevertheless, despite great efforts during years, the multi-loop superstring interaction amplitudes were not calculated explicitly so as to be suitable for applications.

It has been attempted to construct the integration measures (partition functions) from modular forms on the Riemann surface that requires a complicated module description. Further still, the world-sheet supersymmetry is lost as far as in this case the amplitudes depend on the choice of a basis of the gravitino zero modes. The 0-, 1-, 2- and 3-point massless state amplitudes of higher genera are not seen to be nullified contrary to the requirement following from the space-time supersymmetry. In the supercovariant scheme the calculation is complicated because of Grassmann moduli, especially, in the Ramond case. As the result, for an extended period only the Neveu-Schwarz spin structures where known explicitly.

All the contributions to the amplitude, the Ramond sector included, have been explicitly calculated in. The superstring amplitudes have been given by integrals of a sum over super-spin structures defined by super-Schottky groups on the complex (1|1) supermanifold. Every term of the sum is the integration measure times the vacuum value of the vertex product calculated through the superfield vacuum correlators. The super-spin structures are superconformal extensions of ordinary spin ones. Being an SL(2) transformation, the super-Schottky group element is, generically, determined by its multiplier and limiting points where each are the Grassmann partner of or, respectively, of . The integrations are performed over the above parameters and over coordinates of the interaction vertices on the complex (1|1) supermanifold, any (3|2) variables being fixed due to the SL(2) symmetry.

The integration region over the Schottky parameters was not, however, fully known. Moreover, the calculation of the integrals remained to be solved because of degenerated configurations where the leading approximation integrand is proportional to either a vacuum function, or to the 1-point one. Unlike the genus-1 case, the higher genus functions under discussion do not vanish locally in the space of super-Schottky group parameters. In this case the amplitude seemingly diverges like the boson string amplitude. In the superstring, due to Grassmann integrations, the result is, however, finite or divergent being dependent on the choice of the integration variables (see a discussion of this point in the beginning of Section 7 of the present paper). In this case one must be guided by the requirement to preserve the local symmetries of the amplitude. In particular, if divergences appear, the SL(2) symmetry is broken due to the cut-off parameter. As was observed, the divergences have an evident tendency for the cancellation, if the calculation preserves the explicit SL(2) symmetry. Nevertheless, in the total cancellation of the divergences was not verified. Moreover, an explicit regularization procedure for the integrals is not necessary as far as the result is expected to be finite.

At present paper the integration region over the Schottky parameters is given to be
restricted by the group of the local symmetries of the amplitude and by the unitarity. The above group of the local symmetries is considered. The singular configurations are studied. The calculation of the integrals over the singular configurations is proposed. In this case we, step-by-step, integrate over variables of every handle in such a way that for every step, the integral is convergent. So the multi-loop superstring amplitudes are obtained to be finite. We argue that the calculation preserves all the local symmetries of the amplitude. The finiteness of the superstring amplitudes has been expected [1, 8, 19] from years, but till now it was clearly seen only for the one-loop case. Unlike one-loop amplitudes, the finiteness of the multi-loop ones is provided not by local sum rules but by the vanishing of certain convergent integrals of a sum over super-spin structures. We verify the required nullification of the vacuum amplitude, vacuum-dilaton transition constant and of the 2- and 3-point amplitudes for the massless boson states. We show the vanishing of the amplitudes for the emission of a longitudinal boson due to the gauge invariance.

The present paper provides rich opportunities for applications of the superstring perturbation series. In particular, the obtained expressions can be used for the calculation of the infinity genus amplitudes and for the summation of the perturbation series under various asymptotics conditions. Being calculated in the infra-red energy limit, they give explicit and rather compact amplitudes in any perturbative order for the 10-dimensional IIA super-gravity corresponding to the superstring considered (to be given in another place). So, among other things, the superstring perturbation series is the wonderfully effective instrument for the explicit calculation in the quantum field theory [20]. In this paper the closed, oriented superstring theory is considered. An extension of the results to the open and/or non-oriented superstring will be given elsewhere. The perturbation series can also be constructed in heterotic and compactified superstring theories. Moreover, the series can be built for the Dirichlet boundary conditions. The perturbation series for relevant compactified (super)string can be used for calculations in four-dimensional quantum field theories [20].

As it is known, the group of the local symmetries of the amplitude includes the $SL(2)$ group, changes of any given interaction vertex coordinate by super-Schottky group transformations and the modular group. We show that, in addition, the total genus-$n$ group $R_n$ contains also a group $\{\tilde{G}\}$ of isomorphic replacements $\Gamma_s \rightarrow G_s \Gamma_s G_s^{-1}$ of the given $\{\Gamma_s\}$ set of forming super-Schottky group transformations by the $\{G_s \Gamma_s G_s^{-1}\}$ set. Here $\{G_s\}$ is a relevant set of the super-Schottky group transformations (not every $\{G_s\}$ set originates the isomorphism, and, therefore, is relevant!) Evidently, the considered isomorphism does not touch the period matrix.

For the Riemann surface theory, there are known [21] the constraints making the period matrix to be inside the fundamental domain of the modular group. In the present paper we extend the constraints [21] to the superstring case where the period matrix depends on the super-spin structure [3, 14]. The period matrix being given through the super-Schottky group parameters, the above constraints bound a region for the Schottky variables. The discussed constraints are, however, invariant under the above $\{\tilde{G}\}$ group transformations
preserving the period matrix. Moreover, we demonstrate module configurations destroying the unitarity equations (see Section 4 of the present paper). In this case one of the limiting points of the forming group transformation \( \Gamma_s \) penetrates deep enough into the Schottky circle for the forming transformation \( \Gamma_r \) with \( r \neq s \) while another point lies outside both Schottky circles assigned to \( \Gamma_r \). The considered configurations need to be excluded from the integration region. Evidently, they can not be reduced by the \( \{ \tilde{G} \} \) transformations to those configurations, which are consistent with the unitarity equations. Hence the fundamental domains for the direct product of \( SL(2) \) group times \( \{ \tilde{G} \} \) do not all are equivalent to each other, but they are divided into the classes of the equivalent domains.

The unitarity equations are already saturated due to the region where all the Schottky circles of the forming group transformations are separated from each other, but it is not a fundamental region for the local symmetry group. We argue that the integration region can be restricted by configurations having no group limiting points inside the common interior of any pair of Schottky circles of the forming group transformations. Really the discussed region can be varied within a certain range, as it is usually for a group fundamental region.

The modular symmetry constraints \([21]\), along with the \( \{ \tilde{G} \} \) symmetry, bounds the absolute value of the multiplier of any Schottky group transformation by a magnitude smaller than unity. In this case, for not very high genera, we show that the Schottky circles of the same forming transformation are separated from each other. For the sake of simplicity we assume the same to be in the general case though now we have not a full proof for this.

When the \( k \) multiplier for the boson loop is nullified, the divergence appear for the single spin structure. The divergence is, however, canceled for the sum over relevant spin structures, just as in the Neveu-Schwarz sector \([1]\). Divergences might be also due to singularities in the \( \{ u_s, v_s \} \) limiting points of the forming group transformations. As has been noted above, generally, the discussed singularities for the genus-\( n > 1 \) amplitudes are not canceled locally even for the total sum over the spin structures. Nevertheless, we give the integration procedure calculating the amplitude to be finite and consistent with its local symmetries.

When all the limiting group points lay within a finite domain, the singularities in \( \{ u_s, v_s \} \) do not appear in the integration region until certain of \( (v_s - u_s) \) differences go to zero. Generally, the integrand is, however, singular when all the limiting points of \( n_1 \leq n \) transformations go to the same point \( z_0 \). For the amplitudes having more than three legs, the basic configurations dangerous for the divergences are the configurations with no more than one interaction vertex coordinate going to \( z_0 \).

When limiting points \( u_1 \) and \( v_1 \) of the sole transformation go to each other, the integrand is singular in \( \tilde{u}_1 = v_1 - u_1 \) (or in \( w_1 = \tilde{v}_1 - v_1 \mu_1 \)) at \( \tilde{v}_1 \to 0 \) (or at \( w_1 \to 0 \)). By a direct calculation using manifest expressions, one can, however, verify that the singularity is canceled. The total cancellation of the singularity is achieved after the summation over the spin structures of the considered handle and, if the interaction vertex coordinate goes to \( z_0 \), once certain integrations to be performed, \( u_1 \) or \( v_1 \) being fixed (see the Section 6). Therefore, the contribution to the amplitude of the \( n_1 = 1 \) configuration is finite provided that the integral over \( \tilde{v}_1 \) (or \( w_1 \)) is calculated after integrations over other variables of the
considered configuration to be taken.

For the \( n_1 = 2 \) we consider the integral over variables of the discussed configuration keeping \( \tilde{v}_2 \) (or \( w_2 \)) to be fixed. So the integral is a function of \( \tilde{v}_2 \) (or \( w_2 \)). The integration over \( \tilde{v}_1 \) (or \( w_1 \)) is taken after the integration over the remaining variables of the first handle to be performed. In this case the integral of the sum over the spin structures is convergent due to the cancellation of the singularity at \( \tilde{v}_1 \to 0 \) or \( w_1 \to 0 \) for the \( n_1 = 1 \) integral with fixed \( \tilde{v}_1 \) of the sum over the spin structure of the handle. Indeed, at \( \tilde{v}_1 = 0 \) (or \( w_1 = 0 \)) the integrand of the \( n_1 = 2 \) integral is a sum of terms bi-linear in the \( n_1 = 1 \) integrals (see the proof in the beginning of Section 7). Moreover, we show (Section 7) that the \( n_1 = 2 \) integral is convergent being taken of the sum over the spin structures either of two handles. In the calculation of the amplitude we have deal only with these convergent integrals and with the integral of the total sum over spin structures of both handles, which is convergent, too. As we show, the last integral being a function of \( \tilde{v}_2 \) (or \( w_2 \)), is finite at \( \tilde{v}_2 = 0 \) (or \( w_2 = 0 \)). Thus the contribution to the amplitude from the discussed configuration is finite, as far as this contribution to the amplitude is just obtained by the additional integration over \( \tilde{v}_2 = 0 \) (or \( w_2 = 0 \)) of the discussed integral with \( \tilde{v}_2 \) (or \( w_2 \)) to be fixed.

The required finiteness of the integral with fixed \( \tilde{v}_2 \to 0 \) (or \( w_2 \to 0 \)) at \( \tilde{v}_2 \to 0 \) (\( w_2 \to 0 \)) is manifested through the change of the integration variables by relevant transformations from \( SL(2) \) and \( \{\tilde{G}\} \) groups, as well as by a super-Schottky group transformation of the interaction vertex coordinate (if it goes to \( z_0 \)). The discussed finiteness of the integral with fixed \( \tilde{v}_2 \to 0 \) (or \( w_2 \to 0 \)) at \( \tilde{v}_2 \to 0 \) (\( w_2 \to 0 \)) is again shown to be due to the cancellation of the singularity for the \( n_1 = 1 \) integrals. Apart from \( SL(2) \) transformations, the remaining transformations depend, however, on the spin structure because of fermion-boson mixing, which is present for non-zero \( \{\mu_s, \nu_s\} \) Grassmann parameters. At the same time, we can not perform the transformations of the integration variables for the integral of the single spin structure since this integral is divergent. Nevertheless, the transformations dependent on the spin structure of the first handle can be made for the convergent integral of the sum over the spin structures of the second handle. And the transformations dependent on the spin structure of the second handle can be made for the convergent integral of the sum over the spin structures of the first one. Being a group product of the above transformations, an arbitrary \( \{\tilde{G}\} \) or super-Schottky group transformation can be performed only for the integral of the total sum over the spin structures of the configuration. Hence the cancellation of the singularity at \( \tilde{v}_2 \to 0 \) (\( w_2 \to 0 \)) is verified only for the integral of the total sum over the spin structures. Correspondingly, the finiteness of the contribution to the amplitude from the discussed \( n_1 = 2 \) configuration is achieved only for the integrals of the total sum over the spin structures of the configuration. In this case one calculates the integral over the variable of the 1-st handle keeping \( \tilde{v}_1 \) (or \( w_1 \)) to be fixed. Then one calculate the integral over \( \tilde{v}_1 \) (or \( w_1 \)). After this one calculates the integral over the variable of the 2-nd handle keeping \( \tilde{v}_2 \) (or \( w_2 \)) to be fixed, and after this the integral over \( \tilde{v}_2 \) (or \( w_2 \)) is calculated. The calculation preserves all the local symmetries of the amplitude including the modular symmetry. Being, generally, dependent on the spin structure [12], modular transformations can be performed
only for the integral of the total sum over the spin structures, like the above discussed \( \{ \tilde{G} \} \) and super-Schottky group transformations.

In the general case \( n_1 > 2 \) one integrates, step-by-step, over the limiting points of every forming transformation \( s \) (except the points fixed due to the \( SL(2) \) symmetry) once the summation over its spin structures to be performed. The integral is first calculated with fixed \( \tilde{v}_s = v_s - u_s \) or, on equal terms, with fixed \( w_s = \tilde{v}_s - \nu_s \mu_s \). For every step, the integral being a function of \( \tilde{v}_s \) (or \( w_s \)), is shown to be non-singular at \( \tilde{v}_s = 0 \) (or \( w_s = 0 \)). Hence it can be further integrated over \( \tilde{v}_s \) (or \( w_s \)), the result being finite. When some \( (u_s, v_s) \) pairs go to the infinity the integral is finite again since the infinite point can be reduced to the finite one by the relevant \( L(2) \) transformation. As far as the calculation preserves all the local symmetries, the amplitude is independent of the fundamental region of the \( R_n \) group, which is employed as the integration region. In particular, the amplitude is independent of those the \((3|2)\) variables, which are fixed.

The 0-, 1-, 2- and 3-point massless state amplitude is given by the integrals with the \((3|2)\) fixed parameters \( (u_1|\mu_1) \) and \( (v_1|\nu_1) \) assigned to any given handle along with one of two local limiting parameters \( (u_2 \) or \( v_2) \) of any one of the remaining handles. We show this integral is convergent, and it is zero at \( u_1 \rightarrow v_1 \). Thus, being invariant under the \( SL(2) \) and \( \{ \tilde{G} \} \) group and under the super-Schottky group (for 1-, 2- and 3-point amplitudes), the integral vanishes identically for any \( v_1 \) and \( u_1 \), as it is required.

There are different super-extensions of ordinary spin structures, but not all they are suitable for the superstring, especially, because the space of half-forms does not necessarily have a basis when there are odd moduli \([22]\). The super-Schottky groups for all superspin structures have been constructed in \([11, 12, 23]\). In the Neveu-Schwarz case they have been given before \([5, 6, 24]\). Due to the fermion-boson agitation, the genus-\( n > 1 \) super-spin structures are different from ordinary spin structures \([15]\) where boson fields are single-valued on Riemann surfaces while fermion fields being twisted about \((A, B)\)-cycles, may only receive the sign. Really the super-Schottky group description of the supermanifold is non-split in the sense of \([25]\). The transition to a split description is singular \([20]\), the superstring being non-invariant under the transition discussed. As the result, the world-sheet supersymmetry is lost in \([2]\) where a split module description was implied. Contrary to \([2]\), our calculation preserves the world-sheet supersymmetry, as well as other local symmetries of the amplitude.

In the calculation we use partial functions and superfield vacuum correlators obtained \([11]\) by equations, which were derived from the requirement that multi-loop superstring amplitudes are independent of a choice of both the \emph{vierbein} and the gravitino field. As it is usually, the superfield vacuum correlators is calculated through the holomorphic Green functions. For the Ramond type handle, the round about the Schottky circle being given by a non-split transformation \([11, 12, 23]\), the holomorphic Green functions in the Ramond sector can not be represented by the Poincaré series \([27]\). Correspondingly, the integration measure is not a product over known expressions \([3, 11]\) in terms of super-Schottky group multipliers. Nevertheless, the discussed genus-\( n \) functions can be given \([11]\) by a series over integrals of relevant genus-1 function products. For the Neveu-Schwarz sector the above series can
be reduced to the Poincaré series. Now we continue the study of the holomorphic Green functions and integration measures. In particular, we present them (Appendices B and C of the paper) in the form, which is more convenient for application than the expressions in \[11\]. We also derive them through the functions of lower genera \(n_i \geq 1\). We employ these expressions for the calculation of the integrals over singular configurations.

The paper is organized as it follows. Sections 2 contains a brief review of the super-Schottky group parameterization \[11, 12\] using in the paper. The expression for the multi-loop superstring amplitude is given. In Sections 3 and 4 the constraints on the integration region are discussed. The integration region over the Schottky parameters is proposed. In Section 5 the integration measures and the superfield vacuum correlators are derived through the lower genus functions. The integration measures and the superfield vacuum correlators are calculated for the degenerated configurations dangerous for divergences. In Section 6 the strategy calculating the integrals is discussed. Vanishing the amplitude of the emission of a longitudinally polarized boson is argued. Configurations dangerous for divergences are collected. The cancellation of divergences for easy configurations is demonstrated. In Section 7 the finiteness of the amplitudes is shown. The vanishing of the 0-, 1-, 2- and 3-point functions is verified. The preservation of the local symmetries of the amplitudes is argued.

2 Expression for the multi-loop superstring amplitude

As it was noted, we employ super-Schottky groups variables. The super-Schottky group determines the super-spin structure on the complex \((1|1)\) supermanifold mapped by the \(t = (z|\vartheta)\) coordinate. The genus-\(n\) super-spin structure presents a superconformal extension of the relevant genus-\(n\) spin one given by the set of transformations \(\Gamma^{(0)}_{a,s}(l_{1s})\) and \(\Gamma^{(0)}_{b,s}(l_{2s})\) (where \(s = 1, \ldots, n\)), which correspond to the round of \(A_s\)-cycle and, respectively, of the \(B_s\)-cycle on the Riemann surface. They depend on the theta function characteristics \(l_{1s}\) and \(l_{2s}\), assigned to the given handle \(s\). A discrimination is made only between those (super-)spin structures, for which field vacuum correlators are distinct. So \(l_{1s}\) and \(l_{2s}\) can be restricted by 0 and 1/2. In this case \(l_{1s} = 0\) is assigned to the Neveu-Schwarz handle while \(l_{1s} = 1/2\) is reserved for the Ramond one. In doing so

\[
\Gamma^{(0)}_{b,s}(l_{2s}) = \left\{ z \to \frac{a_s z + b_s}{c_s z + d_s}, \quad \vartheta \to -\frac{(-1)^{2l_{2s}}}{c_s z + d_s} \vartheta \right\}, \quad \Gamma^{(0)}_{a,s}(l_{1s}) = \left\{ z \to z, \quad \vartheta \to (-1)^{2l_{1s}} \vartheta \right\}
\]

(1)

where \(a_s d_s - b_s c_s = 1\). Furthermore,

\[
a_s = \frac{u_s - k_s v_s}{\sqrt{k_s(u_s - v_s)}}, \quad d_s = \frac{k_s u_s - v_s}{\sqrt{k_s(u_s - v_s)}}, \quad c_s = \frac{1 - k_s}{\sqrt{k_s(u_s - v_s)}}
\]

(2)

where \(k_s\) is a complex multiplier and \(|k_s| \leq 1\). Further, \(u_s\) is the attractive limiting (unmoved) point of \([1]\) while \(v_s\) is the repulsive limiting one. The set of transformations \([1]\) along with
their group products form the Schottky group. The $\Gamma^{(0)}_{b,s}(l_{2s})$ transformation in (3) turns the boundary of the Schottky circle $C_{v_s}$ into the boundary of $C_{u_s}$ where

$$C_{v_s} = \{ z : |c_s z + d_s| = 1 \} \quad \text{and} \quad C_{u_s} = \{ z : |-c_s z + a_s| = 1 \} \quad (3)$$

for $s = 1, \ldots, n$. Using (4), one can see that $v_s$ lies inside $C_{v_s}$ and outside $C_{u_s}$. Correspondingly, $u_s$ is inside $C_{u_s}$ and outside $C_{v_s}$. Since $\Gamma^{(0)}_{a,s}(l_{1s})$ in (4) corresponds to the round of the Schottky circle, in the Ramond case a square root cut appears on $z$-plane between $u_s$ and $v_s$. The fundamental region of a group is that one, which does not contain points congruent under the group, and such that the neighborhood of any point on the boundary contains points congruent to the points of the region [28]. In particular, a fundamental region of the Schottky group on the complex $z$ plane is the exterior of all the Schottky circles associated with the given group. The group invariant integral of the conformal $(1,1)$ tensor being calculated over the fundamental region of the group, does not depend on the choice of the fundamental region above [28].

In the superstring theory the forming transformations (4) are replaced by $SL(2)$ transformations $\Gamma_{a,s}(l_{1s})$ and $\Gamma_{b,s}(l_{2s})$ with [11, 12, 23]

$$\Gamma_{a,s}(l_{1s}) = \tilde{\Gamma}_s^{-1}\Gamma^{(0)}_{a,s}(l_{1s})\tilde{\Gamma}_s, \quad \Gamma_{b,s}(l_{2s}) = \tilde{\Gamma}_s^{-1}\Gamma^{(0)}_{b,s}(l_{2s})\tilde{\Gamma}_s \quad (4)$$

where $\Gamma^{(0)}_{b,s}(l_{2s})$ and $\Gamma^{(0)}_{a,s}(l_{1s})$ are given by (4) while $\tilde{\Gamma}_s$ depends, among other things, on two Grassmann parameters $(\mu_s, \nu_s)$ as it follows

$$\tilde{\Gamma}_s = \left\{ z = z^{(s)} + \vartheta^{(s)} \varepsilon_s(z^{(s)}) , \quad \vartheta = \vartheta^{(s)} \left( 1 + \frac{\varepsilon_s \varepsilon_s'}{2} \right) + \varepsilon_s(z^{(s)}) \right\} , \quad (5)$$

$$\varepsilon_s(z) = \frac{\mu_s(z - v_s) - \nu_s(z - u_s)}{u_s - v_s}, \quad \varepsilon_s' = \partial_z \varepsilon_s(z) \quad (6)$$

Thus $(u_s, \mu_s)$ and $(v_s, \nu_s)$ are limiting points of transformations (4). The set of the transformations (4) for $s = 1, \ldots, n$ together with their group products forms the genus-$n$ super-Schottky group. If $l_{1s} = 1/2$, then both transformations (4) are non-split, as it was already discussed in the Introduction. The $\Gamma_{a,s}(l_{1s})$ transformation relates superconformal $p$-tensor $T_p(t)$ with its value $T_p^{(s)}(t)$ obtained from $T_p(t)$ by $2\pi$-twist about $C_{v_s}$-circle (3). So, $T_p(t)$ is changed under the $\Gamma_{a,s}(l_{1s}) = \{ t \to t^a_s \}$ and $\Gamma_{b,s} = \{ t \to t^b_s \}$ transformations as follows

$$T_p(t^a_s) = T_p^{(s)}(t) Q^p_{\Gamma_{a,s}(l_{1s})}(t), \quad T_p(t^b_s) = T_p(t) Q^p_{\Gamma_{b,s}(l_{2s})}(t). \quad (7)$$

Here $Q_G(t)$ is the factor, which the spinor left derivative $D(t)$ receives under the $SL(2)$ transformation $G(t) = \{ t \to t_G = (z_G(t)|\vartheta_G(t)) \}$:

$$Q_G^{-1}(t) = D(t) \partial_G(t); \quad D(t_G) = Q_G(t) D(t), \quad D(t) = \partial \partial z + \partial \vartheta \quad (8)$$

1The congruent points, or curves, or domains are those related by a group transformation other that the identical transformation [23].
It follows from (8) that for the group product \( G = G_1G_2 \),

\[
Q_{G_1G_2}(t) = Q_{G_1}(G_2(t))Q_{G_2}(t)
\]  

(9)

Furthermore, \( \Gamma_{b,s}(l_2s) \) turns the boundary of the \( \hat{C}_{u_s} \) "circle" to the boundary of \( \hat{C}_{v_s} \) where

\[
\hat{C}_{v_s} = \{ t : |c_s z(s) + d_s|^2 = 1 \} \quad \text{and} \quad \hat{C}_{u_s} = \{ t : |c_s z(s) + a_s|^2 = 1 \}
\]  

(10)

and \( z(s) \) is defined by (5). Moreover, the same is true for the "circles"

\[
\hat{C}'_{v_s} = \{ t : |Q_{\Gamma_{b,s}}(t)|^2 = 1 \} \quad \text{and} \quad \hat{C}'_{u_s} = \{ t : |Q_{\Gamma^{-1}_{b,s}}(t)|^2 = 1 \}
\]  

(11)

where the super-derivative factors (8) correspond to \( \Gamma_{b,s}(l_2s) \) and, respectively, its inverse transformation. "Circles" (10) and (11) differ from (3) only in terms proportional to the Grassmann quantities. Being constructed for every group transformation, both the "circles" can be used to define the boundary of the fundamental region. For applications it is useful solely to keep in mind that the fundamental region can be given by the step function factor through relevant functions \( \ell_G(t) \) and \( \ell_G(G(t)) \) as it follows

\[
B_{L,L'}^{(n)}(t, \bar{t}; \{ q, \bar{q} \}) = \prod_G \theta(|\ell_G(t)|^2 - 1)\theta(1 - |\ell_G(G(t))|^2)
\]  

(12)

where \( \theta(x) \) is step function defined to be \( \theta(x) = 1 \) at \( x > 0 \) and \( \theta(x) = 0 \) at \( x < 0 \). Further, \( L = \{ l_1s, l_2s \} \) is the super-spin structure for the right movers while \( L' \) is the same for the left ones. The product is taken over all group products \( G \) of the \( \Gamma_{b,s}(l_1s) \) transformations except \( G = I \). Evidently, one can exclude from \( \{ G \} \) a transformation inverse to the given one of the set. It is implied that all the group limiting points lay exterior to the region, and the region does not contain points related with each other by the group transformation.

The region (12) is relevant for the integration region for the group invariant integral of the \((1/2, 1/2)\) super-tensor. Generically, the argument of step function (12) depends on Grassmann parameters. The expansion in a series over the above Grassmann ones originates \( \delta \)-functions and their derivatives, which give rise to the boundary terms in the integral. The integral is independent of the boundary sharp that can be directly verified for infinitesimal variations of the boundary. As it is usually \cite{28}, one can replace any part of the fundamental region by a congruent part and still have a fundamental region. The integral discussed is independent of the fundamental region, which it is taken over.

The superstring amplitude (14) is calculated by the integration over a fundamental region of the total group \( \mathcal{R} \) of local symmetries, for details see Sections 3 and 4. In this case the \( \{ N_0 \} \) set of \((3|2)\) variables among the group limiting points and interaction vertex coordinates are fixed due to the \( SL(2) \) symmetry. Simultaneously, the integrand is multiplied by a factor \( H(\{ N_0 \}) \). For the sake of simplicity, we assume to be fixed two any variables \( z_1^{(0)} \) and \( z_2^{(0)} \), along with their Grassmann partners \( \vartheta_1^{(0)} \) and \( \vartheta_2^{(0)} \), and one more variable \( z_3^{(0)} \), as well. In
this case [11]

\[ H(\{N_0\}) = (z_1^{(0)} - z_3^{(0)})(z_2^{(0)} - z_3^{(0)}) \left[ 1 - \frac{\vartheta_1^{(0)} \vartheta_3}{2(z_1^{(0)} - z_3^{(0)})} - \frac{\vartheta_2^{(0)} \vartheta_3}{2(z_2^{(0)} - z_3^{(0)})} \right]. \] (13)

So, generically, (13) depends on the integration variable \( \vartheta_3 \), which is the Grassmann partner of \( z_3^{(0)} \). The \( n \)-loop, \( m \)-point amplitude \( A^{(n)}_m(\{p_j, \zeta^{(j)}\}) \) for the interaction states carrying 10-dimensional momenta \( \{p_j\} \) and the polarization tensors \( \zeta^{(j)} \), is given by\[ A^{(n)}_m(\{p_j\}, \zeta^{(j)}) = g^{2n+m-2} 2^{n+1} \int |H(\{N_0\})|^2 \sum_{L,L'} Z^{(n)}_{L,L'}(\{q, \overline{q}\}) < \prod_{r=1}^{m} V(t_r, \overline{t}_r; p_r; \zeta^{(r)}) > \times \tilde{B}^{(n)}_{L,L'}(\{q, \overline{q}\}) \tilde{B}^{(n)}_{L,L'}(\{q, \overline{q}\}) \prod_{j=1}^{m} B^{(n)}_{L,L'}(t_j, \overline{t}_j; \{q, \overline{q}\})(dq d\overline{q} dt d\overline{t}) \] (14)

where \( \{q\} = \{k_s, u_s, \nu_s, \mu_s, \nu_s\} \) is the set of the super-Schottky group parameters, \( g \) is the coupling constant, \( H(\{N_0\}) \) is defined by (13) and \( L \) (\( L' \)) labels the super-spin structures of right (left) movers. Further, \( Z^{(n)}_{L,L'}(\{q, \overline{q}\}) \) is the partition function, and \( < ... > \) denotes the vacuum expectation of the product of the interaction vertices \( V(t_r, \overline{t}_r; p_r; \zeta^{(r)}) \). The integration is performed over those variables, which do not belong to the \( \{N_0\} \) set. The step function factor \( \tilde{B}^{(n)}_{L,L'}(\{q, \overline{q}\}) \) keeps the period matrix to be interior to the fundamental region of the modular group. As far as the period matrix is given through the super-Schottky group parameters \( \{q\} \), this factor restricts the integration region over \( \{q\} \). The discussed factor depends on the spin structures as far as the period matrix depends on the spin structure by terms proportional to Grassmann parameters. Further, \( \tilde{B}^{(n)}_{L,L'}(\{q, \overline{q}\}) \) more bounds the integration region due to the \( \{G\} \) symmetry and the unitarity as it has been mentioned in the Introduction. Both factors are discussed in Sections 3 and 4. The step multiplier \( B^{(n)}_{L,L'}(t_j; \{q, \overline{q}\}) \) is given by (12) at \( t = t_j \). Generically, \( \ell_G(t_j) \) can depend on \( j \). The discussed multiplier is assigned every \( t_j \) including \( t_j \) of the \( \{N_0\} \) set, as well. Indeed, due to the invariance under the super-Schottky group changes of any one vertex coordinate, \( t_j \) can be fixed inside region (12). Really the discussed step factor must be assigned to fixed \( t_j \) of the \( \{N_0\} \) set for the unitarity equations to be true, see Section 4.

The \( 1/2^m \) factor in (14) is due to the symmetry of the integrand under the interchange between \( (u_s|\mu_s) \) and \( (v_s|\nu_s) \), and \( 1/n! \) is due to the symmetry under the interchange of the handles. Both symmetries are particular cases of the modular symmetry [21] (see the next Section). The above factors are consistent with the unitarity equations (as an example, see Appendix A of the paper). For any boson variable \( x \) we define \( dx d\overline{x} = d(Re x)d(Im x)/(4\pi) \). For any Grassmann variable \( \eta \) we define \( \int d\eta \overline{d\eta} = 1 \). The super-spin structure being odd\[ ^2 \) the integrand has some features due to the spinor zero modes, which present in this case.

\[ ^2 \) Through the paper the overline denotes the complex conjugation.

\[ ^3 \) The (super)-spin structure is even (odd), if \( 4l_1l_2 = 4 \sum_{s=1}^{2} l_{1s}l_{2s} \) is even (odd).
We mainly discuss the even super-spin structures as far as the odd super-spin one can be obtained by a factorization of relevant even super-spin structure [13].

Our classification over the super-spin structures implies that

\[ |\arg k_s| \leq \pi, \quad |\arg (u_s - u_r)| \leq \pi, \quad |\arg (u_s - v_r)| \leq \pi, \quad |\arg (v_s - v_r)| \leq \pi, \quad (s \neq r). \tag{15} \]

To be accurate, (15) are given for the "bodies" of the corresponding quantities, as far as they may have "soul" parts proportional to Grassmann parameters. Using the Green functions [14, 12], one can see that \( \arg k_s \leq \pi \) discriminates between the Green functions for \( (l_{1s} = 0, l_{2s} = 0) \) and for \( (l_{1s} = 0, l_{2s} = 1/2) \). The rest constraints discriminate between \( (l_{1s} = l_{1r} = 1/2, l_{2s} = l_{2r}) \) and \( (l_{1s} = l_{1r} = 1/2, l_{2s} = l_{2r} = 1/2) \). The adding of \( \pm 2\pi \) to any one of the quantities in (15) presents a modular transformation [12], see Section 3 below. We use (15) instead of known constrains [24] for the real part of the period matrix.

We consider the massless boson interaction amplitudes. Thus, for the normalization used, the known expression [29] of the interaction vertex through the string superfields \( X^N(t, t') \) for \( N = 0, \ldots 9 \) is as follows

\[ V(t, \bar{t}; p; \zeta) = 4\zeta_{MN}[D(t)X^M(t, \bar{t})][\overline{D(t)}X^N(t, \bar{t})] \exp[ipRX^R(t, \bar{t})] \tag{16} \]

where \( p = \{p^M\} \) is 10-momentum of the interacting boson while \( \zeta_{MN} \) is its polarization tensor, \( p^M\zeta_{MN} = p^N\zeta_{MN} = 0 \), and \( p^2 = 0 \). The spinor derivative \( D(t) \) is defined in (8). The summation over twice repeated indexes is implied. We use the "mostly plus" metric. The dilaton \( \zeta_{MN} \) tensor is equal to the transverse Kronecker symbol \( \delta_{MN} \). The vacuum expectation in (16) is calculated in term of the genus-\( n \) scalar superfield vacuum correlator given through the holomorphic Green function \( R_{L}^{(n)}(t, t'; \{q\}) \) and super-holomorphic functions \( J_r^{(n)}(t; \{q\}; L) \) having periods (here \( r = 1, \ldots n \)). In this case

\[
\begin{align*}
J_r^{(n)}(t^b_s; \{q\}; L) &= J_r^{(n)}(t; \{q\}; L) + 2\pi i \omega_{st}(\{q\}, L), \\
J_r^{(n)}(t^a_s; \{q\}; L) &= J_r^{(n)}(t; \{q\}; L) + 2\pi i \delta_{rs}.
\end{align*}
\tag{17}
\]

where \( t^a_s \) and \( t^b_s \) are the same as in (7) and \( \omega_{st}(\{q\}, L) \) is the corresponding element of the period matrix. The Green function is changed under the transformations (7) as

\[
\begin{align*}
R_L^{(n)}(t^b_s, t'; \{q\}) &= R_L^{(n)}(t, t'; \{q\}) + J_r^{(n)}(t'; \{q\}; L), \\
R_L^{(n)}(t^a_s, t'; \{q\}) &= R_L^{(n)}(t, t'; \{q\}).
\end{align*}
\tag{18}
\]

In this case the Green function is normalized as it follows

\[ R_L^{(n)}(t, t'; \{q\}) = \ln(z - z' - \psi\vartheta') + \tilde{R}_L^{(n)}(t, t'; \{q\}) \tag{19} \]

where \( \tilde{R}_L^{(n)}(t, t'; \{q\}) \) has no a singularity at \( z = z' \). The scalar superfield vacuum correlator \( \tilde{X}_{L,L'}(t, \bar{t}, t', \bar{t}'; \{q\}) \) is given by\(^4\)

\[ 4\tilde{X}_{L,L'}(t, \bar{t}, t', \bar{t}'; \{q\}) = R_L^{(n)}(t, t'; \{q\}) + R_{L'}^{(n)}(t, t'; \{q\}) + I_{LL'}^{(n)}(t, \bar{t}; t', \bar{t}'; \{q\}), \tag{20} \]

\(^4\)The string tension is taken to be 1/\( \pi \)
\[ I^{(n)}_{L,L'}(\bar{t}, \bar{t}'; \{q, \bar{q}\}) = [J^{(n)}_s(t; \{q\}; L) + J^{(n)}_s(t; \{q\}; L')] \Omega^{(n)}_{L,L'}(q, \bar{q})^{-1} \times [J^{(n)}_r(t'; \{q\}; L) + J^{(n)}_r(t'; \{q\}; L')] \]

where \( \Omega^{(n)}_{L,L'}(q, \bar{q}) \) being calculated in terms of the \( \omega^{(n)}(\{q\}, L) \) period matrix, is
\[
\Omega^{(n)}_{L,L'}(q, \bar{q}) = 2\pi i [\omega^{(n)}(\{q\}, \bar{L'}) - \omega^{(n)}(\{q\}, L)].
\]

As it is usually, the Green function at the same point \( z = z' \) is defined to be the \( \hat{F}^{(n)}_L(t, t'; \{q\}) \) at \( z = z' \). The dilaton emission amplitudes include the vacuum pairing \( \hat{I}^{(n)}_{L,L'}(t, \bar{t}; \{q\}) \) of the superfields in front of the exponential in (10). In line with aforesaid
\[
\hat{I}^{(n)}_{L,L'}(t, \bar{t}; \{q\}) = 2D(t)D(t')I^{(n)}_{L,L'}(t, \bar{t}, t', \bar{t}; \{q\})|_{t=t'}
\]

where the definitions are given in (20) and in (21). The dilaton-vacuum transition constant is determined by the integral of (23) over the supermanifold. Integrating by parts the right side of (23) with the following using of the relations (17), one obtains the discussed constant to be \( n \) times the vacuum amplitude \( \Omega \).

Due to a separation in right and left movers, the integration measure in (14) is represented
\[
Z^{(n)}_{L,L'}(\{q, \bar{q}\}) = (8\pi)^5n [\det \Omega^{(n)}_{L,L'}(\{q, \bar{q}\})]^{-5} Z^{(n)}_{L}(\{q\})Z^{(n)}_{L'}(\{q\})
\]

where \( Z^{(n)}_{L}(\{q\}) \) is a holomorphic function of the \( q \) moduli and the \( \Omega^{(n)}_{L,L'}(\{q, \bar{q}\}) \) matrix is given by (22). The holomorphic partition function in (23) is given by
\[
Z^{(n)}_{L}(\{q\}) = \hat{Z}^{(n)}_{L}(\{q\}) \prod_{s=1}^{n} (u_s - v_s - \mu_s v_s)^{-1}
\]

where \( \hat{Z}^{(n)}_{L}(\{q\}) \) is invariant under the \( SL(2) \) transformations as far as
\[
du_s dv_s d\mu_s dv_s/(u_s - v_s - \mu_s v_s)
\]
is \( SL(2) \) invariant. Explicit \( \hat{Z}^{(n)}_{L}(\{q\}) \) and other functions of interest are given in Section 5.

In two following Sections we consider the integration region.

### 3 Modular symmetry constraints

The modular transformation of the supermanifold, generically, presents a globally defined, holomorphic superconformal change \( t \rightarrow \hat{t} \) of the coordinate along with holomorphic changes \( q \rightarrow \hat{q} \) of the super-Schottky group parameters and by a change \( L \rightarrow \hat{L} \) of the super-spin structure. Like the modular transformation of the Riemann surface [21], it determines the
going to a new basis of non-contractable cycles. So the period matrix \( \omega(\{q\}, L) \) is changed, as it is usually, by

\[
\omega(\{q\}, L) = [A \omega(\{\hat{q}\}, \hat{L}) + B][C \omega(\{\hat{q}\}, \hat{L}) + D]^{-1}
\]

(27)

where integer \( A, B, C \) and \( D \) matrices obey [21] the relations

\[
C^T A = A^T C, \quad D^T B = B^T D, \quad D^T A - B^T C = I.
\]

(28)

The right-top "T" symbol labels the transposing. In this case \( t(\hat{t}; \{\hat{q}\}; \hat{L}) \) and \( q(\{\hat{q}\}; \hat{L}) \) both depend on the superspin structure by terms proportional to Grassmann parameters [12].

Then the period matrix also depends on the superspin structure by terms proportional to Grassmann parameters [12].

Indeed, from (18), under the above replacement, \( \delta_{sp} \) is the Kronecker symbol. The summation in (28) is performed over all those the transformations \( g_\Gamma \) of the Schottky group, whose leftmosts are not group powers of \( g_s \), or the rightmosts are not group powers of \( g_s \). Besides, \( g_\Gamma \neq I \), if \( s = p \). So the addition of \( \pm 2\pi \) to \( \arg k_s \) adds \( \pm 1 \) to \( \omega^{(0)}(\{q\}) \). For \( r \neq s \), due to the term with \( \Gamma = I \) in (29), the addition of \( \pm 2\pi \) to the argument of the difference between limiting points in (15) adds \( \pm 1 \) to \( \omega^{(0)}_r(\{q\}) \). Hence the discussed changes of arguments [13] just correspond to transformations of the period matrix by (28) with \( C = 0 \) and \( A = D = I \). Evidently, it is true for non-zero Grassmann parameters too, and we do not enlarge on this matter. We note only that for non-zero Grassmann parameters, the addition of \( \pm 2\pi \) to the argument of the difference between limiting points in (14) includes all the distinct spin structures without a double counting, but constraints (15) are much more convenient for applications than constraints \( |Re \omega^{(0)}_r(\{q\})| \leq 1/2 \).

Further constraints appear due to transformations (29) with

\[
B = C = 0, \quad A = F, \quad D^{-1} = F^T, \quad \det F = \pm 1,
\]

(30)

\( F \) being an integer matrix. For the diagonal \( F \) matrix, \( F_{ss} = -1 \) corresponds to the replacement of the corresponding group transformation by its inverse that interchanges between \( (u_s|\mu_s) \) and \( (v_s|\nu_s) \). Indeed, from (18), under the above replacement, \( J^{(0)}_s(t; \{q\}; L) \) receives the sign. Hence from (17), the period matrix elements \( \omega_{rs}(\{q\}, L) \) with \( r \neq s \) also receive the sign that just corresponds to modular transformation discussed (for zero Grassmann parameters this follows directly from (29)). The \( F \) matrix having a sole non-zero non-diagonal matrix element \( F_{s_1s_2} = F_{s_1s_2} = 1 \) and \( F_{s_1s_1} = F_{s_2s_2} = 0 \), corresponds to the interchange \( s_1 \leftrightarrow s_2 \) between the handles. Thus the Schottky parameters is bounded by constraints
which discriminate between repulsive and attractive limiting points, as well as between the handles. In their stead we prefer to introduce the $1/(2^n n!)$ factor in \([11]\). Remaining $F$ matrices correspond to transitions to new basic cycles, which are certain sums over the former basic ones. For non-zero Grassmann parameters, unlike the case of zero Grassmann ones, both $t$ and $\{q\}$ are changed. Indeed, in this case $\Gamma_{as}(l_1s = 1/2)$ does not commute with transformations \([11]\) assigned to another handle. Hence the group transformations for rounds about resulted $(A,B)$-cycles being a sum of the former ones, do not commute with each other, if $t$ and $\{q\}$ do not changed. The required changes of $t$ and $\{q\}$ could be calculated by the method developed in \([12]\), but it is not a subject of the present paper.

For zero Grassmann parameters, among period matrices related by \([31]\), one takes \([21]\) the matrix having the smallest imaginary part $y_{jj}(\{q, \bar{q}\})$. In this case

$$[Fy(\{q, \bar{q}\})F^T]_{jj} \geq y_{jj}(\{q, \bar{q}\}).$$

\[\text{(31)}\]

As it is usually, $y_{jj}(\{q, \bar{q}\})$ is non-negative. Starting with $j = 1$, one, step-by-step, constructs a constraints for $j = 2, \ldots, n$. Calculating the integral, one sums with the $1/n!$ factor over permutations of the handles. Further constraints are due to transformations with $C \neq 0$. In this case, among the $\omega^{(0)}(\{q\})$ period matrices related by \([27]\), one takes the matrix, which gives the greatest magnitude of $\det y(\{q, \bar{q}\})$. Due to \([21]\) and \([28]\), the corresponding constraints are given \([21]\) by

$$|\det[C\omega^{(0)}(\{q\}) + D]|^2 \geq 1$$

\[\text{(32)}\]

where $C$ and $D$ are the matrices in \([21]\). Important particular constraints are obtained from \([31]\) when, for given $r$ and $j$, one takes $F_{jr} = F_{rj} = \pm 1$ and either $F_{jj} = 0$, or $F_{rr} = 0$. All other non-diagonal elements of $F$ are assumed to be zeros. Then

$$\min[y_{jj}(\{q, \bar{q}\}), y_{rr}(\{q, \bar{q}\})] + 2y_{rs}(\{q, \bar{q}\}) \geq 0.$$  \[\text{(33)}\]

Also, using special $C$ and $D$ in \([32]\), one derives important constraints for the principal minors $[\det(F\omega^{(0)}(\{q\})F^T + \overline{B})]_{s_1 \ldots s_k}$ of the $\det(F\omega^{(0)}(\{q\})F^T + B)$ as it follows

$$[\det(F\omega^{(0)}(\{q\})F^T + \overline{B})]_{s_1 \ldots s_k} \geq 1$$

\[\text{(34)}\]

where $F$ is an integer matrix, det $F = \pm 1$. There is no summation over the $s_1 \ldots s_k$ indices, and the integer matrix $\overline{B}$ is chosen from the condition that $\Re |F\omega^{(0)}(\{q\})F^T + B| \leq 1/2$.

For non-zero Grassmann parameters, $\omega^{(0)}(\{q\})$ is replaced by $\omega(\{q\}, L)$ while $\omega^{(0)}(\{q\})$ is replaced by $\overline{\omega(\{q\}, L')}$. All the other extensions contain $\omega(\{q\}, L)$ and/or $\omega(\{q\}, L')$. Hence they are not consistent with the modular symmetry. Indeed, \([27]\) relates $\omega(\{q\}, L)$ with $\overline{\omega(\{q_L\}, \overline{L})}$, which is different from $\omega(\{qL\}, L)$ due to terms proportional to the Grassmann parameters \([12]\). So the imaginary part of the period matrix is replaced by

$$y(\{q, \bar{q}\}, L, L') = \frac{1}{2t}[\omega(\{q\}, L) - \overline{\omega(\{q\}, L')}]$$

\[\text{(35)}\]
In this case the corresponding step factor in (14) is given by

$$\tilde{B}^{(n)}_{L,L'}(\{q, \bar{q}\}) = \left[ \prod_{C,D} \theta(\det\{C\omega(\{q\}, L + D)[C(\overline{\omega(\{q\}, L')} + \widetilde{B}) + D]\} - 1 \right]$$

$$\times \left[ \sum_F \prod_{j} \theta([Fy(\{q, \bar{q}\}, L, L')y_{jj} - y_{jj}(\{q, \bar{q}\}, L, L')] \prod_{j=1}^{n} \theta(y_{jj}(\{q, \bar{q}\}, L, L')) \right]$$

(36)

where the sum is performed over permutations of the handles while the products are calculated over all the matrices $C$, $D$ and $F$ in (27) and (30). The step function $\theta(x)$ is the same as in (12).

4 Integration region

The period matrix is preserved under isomorphic changes

$$\Gamma_{a,s}(l_{1s}) \to G_s \Gamma_{a,s}(l_{1s})G_s^{-1}, \quad \Gamma_{b,s}(l_{2s}) \to G_s \Gamma_{b,s}(l_{2s})G_s^{-1}$$

(37)

of the set of the forming group transformations (4) where $G_s$ is a relevant transformation from the super-Schottky group. Generically, $G_s$ depends on $s$. The discussed isomorphism only replaces the limiting points $U_s = (u_s | \mu_s)$ and $V_s = (v_s | \nu_s)$ of the transformation (4) by $G_s U_s$ and by $G_s V_s$. As far as this is globally defined holomorphic transformation, the amplitude integral (14) is invariant under the $\{ G \}$ group of the isomorphic changes discussed. In particular, the holomorphic Green function and the scalar functions are not changed since (17) does not touch (17) and (18). Moreover, from the equations (11) for the partition functions, one can derive that the invariant partition function $\tilde{Z}^{(n)}_L(\{q\})$ in (23) is also unchanged. Only the multiplier behind $\tilde{Z}^{(n)}_L(\{q\})$ receives a factor, which is just canceled by the Jacobian of the transformation, as it also follows from the below constructing of the considered group.

In doing so, for given $s_1$ and $s_2 \neq s_1$, one changes the transformations (4) for $s = s_1$ by (37) with $G_{s_1}$ to be any one from transformations (4) for $s = s_2$. The resulted set $S(s_1 | s_2)$ is evidently isomorphic to the former set $S_0$ of transformations (4) for $s = 1, \ldots, n$. Moreover, $U_{s_1}$ and $V_{s_1}$ are changed by the $SL(2)$ transformation independent of $U_{s_1}$ and $V_{s_1}$. Thus factor (20) is unchanged. In this way one obtains the set of different $S(s_1 | s_2)$ sets. Applying this procedure to every set, one builds further sets. As an example, one obtains the $S(s_1, s_2 | s_2, s_1)$ set where the transformations for $s = s_2$ from the $S(s_1 | s_2)$ set are changed by (37) with $G_{s_2}$ to be the $s = s_1$ transformation from the same $S(s_1 | s_2)$ set. In this case the $s_2$ transformation of the $S(s_1, s_2 | s_2, s_1)$ set depends, among other things, on the starting transformation (4) for the same $s = s_2$. By construction, all the sets are isomorphic to each

\[5\text{Not every $\{ \tilde{G}_s \}$ set is relevant to determine the isomorphism. As an example, the set formed by $G_1 G_2 G_1 G_2^{-1} G_1^{-1}$ and $G_2$ is not isomorphic to the $(G_1, G_2)$ set since $G_1$ can not be represented by a group product constructed by using $G_1 G_2 G_1 G_2^{-1} G_1^{-1}$ and $G_2$.} \]
other, the amplitude being invariant under the transformations discussed. So every set can be used as the set of forming group transformations. In this way one constructs the desired infinite (super-)group \( \{ \tilde{G} \} \) generating forming group transformation sets by action of \( \{ \tilde{G} \} \) on the given set \( \{ \tilde{G} \} \).

Since (12) and (36) are preserved under the \( \tilde{G} \) transformations, a further constraint of the integration region is necessary to exclude domains related by \( \tilde{G} \) each to other. In particular, due to the \( \tilde{G} \) symmetry, either one of two limiting points of every forming transformation (4) can be restricted to be the exterior of Schottky circles assigned to all the other forming transformations. For certain configurations, both limiting point of every given forming transformation appear exterior to the Schottky circles above. These configurations can not be obtained by \( \tilde{G} \) transformations of configurations where this is not so. Moreover, the integral over the last configurations destroys the unitarity equations, as demonstrated for the case when all the Schottky multipliers go to zero.

In this case, from (2), the boundaries of Schottky circles (3) are given by the conditions

\[
|z - v_s|^2 = |k_s||u_s - v_s|^2, \quad |z - u_s|^2 = |k_s||u_s - v_s|^2.
\]  

Period matrix elements (29) are reduced to

\[
2\pi i \omega_{jr}^{(0)} \to \ln \frac{(u_j - u_r)(v_j - v_r)}{(u_j - v_r)(v_j - u_r)}, \quad \omega_{jj} \to \ln k_j
\]  

Moreover, from (33), one derives that

\[
2 \left| \ln \frac{(u_j - u_r)(v_j - v_r)}{(u_j - v_r)(v_j - u_r)} \right| \leq \min \left( -\ln |k_j|, -\ln |k_r| \right)
\]  

Below we assume \( j = 1 \) and \( r = 2 \). Constraint (10) allows, for instance, the discussed configuration

\[
|u_1 - u_2| \sim |k_2|^{1-\delta}, \quad |v_1 - u_2| \sim |k_2|^{1/2-\delta}, \quad |k_2| > |k_1|, |v_1 - v_2| \sim |u_1 - v_2| \sim 1
\]  

where \( k_2 \to 0 \) and \( \delta << 1 \). In this case \( v_2 \) is exterior to \( C_{u_1} \) and to \( C_{v_1} \) while \( u_2 \) lies inside \( C_{u_1} \). Other constraints (30) also do not prevent the above configuration. In parallel with (11), the boundary of (10) is reached for only two limited points, say \( u_1 \) and \( u_2 \), to go to each other. In this case \( |u_1 - u_2| \sim \sqrt{|k_2|} \), all the other distances being of order of the unity. Both regions originate discontinuities. To demonstrate this, we discuss the discontinuities in the energy invariant \( s = -(p_1 + p_2)^2 = -(p_1 + p_2)^2 \) for the two-loop scattering amplitude. We ignore specifics due to the Grassmann moduli that is not of very importance for the matter discussed. We fix the complex coordinates \( z_1, z_2 \) and \( z_3 \) of the vertices to be exterior to the Schottky circles. The integration is performed over the Schottky parameters and over \( z_4 = z \), certain group limiting points being assumed to go to \( z_3 \). In addition, \( z \) is assumed nearby one of Schottky circles. If the circle is not nearby \( z_3 \), then \( z \) is changed by a relevant
Schottky group transformation moving \( z \) to the interior of a circle to be nearby \( z_3 \). First, we consider the configuration

\[
|u_1 - z_3| \sim |u_2 - z_3| \sim \min(|k_1|^{1/2-\delta}, |k_2|^{1/2-\delta}), \quad |v_1 - z_3| \sim |v_2 - z_3| \sim |v_1 - v_2| \sim 1 \quad (42)
\]

where \( \delta \) being positive, goes to zero. In this case the Schottky circles have no a common interior. We define variables \( y_1 \sim y \sim 1 \) to be given by \( (u_1 - z_3) = y_1(u_2 - u_1) \) and \( (z-u_1) = y(u_2-u_1) \). The integration over the ”small” variables being taken, the discontinuity is represented by an integral over four complex \( y_1, y, (v_1 - z_3) \) and \( (v_2 - z_3) \). The integral just corresponds to the unitarity equation. Indeed, this unitarity equation is by-linear in 2 \( \to \) 3 tree amplitudes, every amplitude being the integral over two complex variables (an example is given in Appendix A). More discontinuities would appear, if \( z_3 \) is allowed to penetrate into the Schottky circles. For instance, due to configuration where \( z_3 \) being interior to \( C_{v_1} \), \( g_1(z_3) \), is, simultaneously, to be exterior to all the Schottky circles. And \( C_{u_1} \) along with \( C_{u_2} \) both go to \( g_1(z_3) \). In this case the unitarity equations would be broken. Hence constraints \((\mathbb{I}^2)\) for the fixed vertex coordinates hold the unitarity equations.

For the configuration \((\mathbb{I})\), there are only \( (u_1 - z_3) \) and \( (z - z_3) \) to be of the same order magnitudes. So only two variables of order the unity can be constructed. They are \( (v_1 - z_3) \) and \( (\tilde{z} - z_3) \) where \( z - z_3 = (\tilde{z} - z_3)(u_1 - z_3) \). Once the integration over the small variables being performed, the discontinuity is given by the integral over \( v_1 \sim 1 \) and \( \tilde{z} \sim 1 \), which is not a product of two \( 2 \to 3 \) tree amplitudes. Moreover (see Appendix A), for the tachyon-tachyon scattering amplitude from the boson string theory the configuration \((\mathbb{I})\) originates the false threshold at \( s = 6m_{th}^2 \) where \( m_{th}^2 = -8 \) is the square of the tachyon mass. A like discontinuity appears, if \( u_2 \) and \( v_2 \) in \((\mathbb{I})\) both are the \( \{\tilde{G}\} \) image of the limiting points of a forming transformation (instead of to be the limiting points of the forming one). Along with \((\mathbb{I})\), the discussed configuration must be removed from the integration region.

By above, the unitarity equations are saturated due to the region when the Schottky circles of the forming transformations do not overlap each other. But this region being no a fundamental region for the symmetry group, can not be the total integration region. The relevant region for the integration one seems the \( \mathcal{G}_n \) region where all the limiting group points of \( \{\tilde{G}\} \) lay inside the Schottky circles of the forming transformation \((\mathbb{I})\) and outside the common interior of any pair of the circles above. Indeed, we demonstrate that for the integrals of the group covariant expression over the boundary of the region, the boundary can be replaced by pieces, which are congruent to each other under the \( \{\tilde{G}\} \) group. Hence the group invariant integral over the region discussed is the same under those infinitesimal variations of the circles, which are related by the corresponding group transformation \((\mathbb{I})\). In addition, the interior of the region does not contain the points related by the \( \{\tilde{G}\} \) transformations. So the region possesses properties of the fundamental region desired. We demonstrate a range, which the boundary of the region can be varied within.

For this purpose we consider the boundary ”1” with \( u_2 = u_2^{(1)} \) laying on \( C_{u_1} \), and the boundary ”2” including the \( u_2 = u_2^{(2)} \) point on \( C_{v_1} \). The \( u_2^{(2)} \) point is obtained by the \( \Gamma_{b,1}(l_{21}) \) transformation \((\mathbb{I})\) of \( u_2^{(1)} \). On the boundary ”1” the remaining variables cover a domain \( \mathcal{C}[u] \)
and, respectively, over $C[v]$. We add $C[u]$ by the $C^{(1)}[v]$ domain, which is the image of $C[v]$ under that $\hat{G} = G_2$ transformation. The transformation only changes both limiting points $U_2 = (u_2|\mu_2)$ and $V_2 = (v_2|\nu_2)$ by $G_{h_1}(l_2)$. Respectively, we add $C[v]$ by the $C^{(-1)}[u]$ domain to be the image of $C[u]$ under the inverse transformation $G^{-1}$. Due to the $\hat{G}$ symmetry, the integral over $C[v] + C[u]$ differs only by the factor $1/2$ from the integral over $C'[v] + C'[u]$ where $C'[v] = C[v] + C^{(-1)}[u]$ and $C'[u] = C[u] + C^{(1)}[v]$. So we can replace $C[u]$ by $C'[u]$ and $C[v]$ by $C'[v]$ with dividing the integral by $2$. By construction, the boundaries $C'[u]$ and $C'[v]$ are just transformed to each other under the above transformation $G_2$, as it required for the fundamental region boundaries. The kindred consideration can be performed for those boundaries, which include the group limiting points obtained by a $\hat{G}$ change of the forming set. So the boundary of the region can be represented by pieces, which are congruent to each other under the \{\hat{G}\} group. Thus the boundary integral is nullified, the integral over $G_n$ the region discussed is the same under those infinitesimal variations of the circles, which are related by group transformations.

To clarify a range, which the boundary of the region can be varied within, we consider the genus-2 configuration (12) with $|k_2| < |k_1|$. Then the $C[u]$ region restricts $v_2$ to lay between $C_{u_1}$ and some closed curve $\ell_u$ rounding $C_{u_1}$. The above curve is determined by (10). Respectively, $C[v]$ restricts $v_2$ to lay between $C_{v_1}$ and a closed curve $\ell_v$ around $C_{u_1}$, the curve being determined by (10). When $u_2^{(1)}$ penetrates into $C_{u_1}$, the $u_2^{(2)}$ point goes away from $C_{v_1}$. The penetration of $u_2^{(1)}$ into $C_{u_1}$ is allowed until $u_2^{(2)}$ meets $\ell_v$. At this moment, an integration region over $v_2$ can not be deformed continuously that gives a natural restriction for the penetration $u_2^{(1)}$ into $C_{u_1}$. One can see that configuration (11) is not reached. The considered example holds that the $G_n$ region is relevant for the integration one.

The limiting points of any super-Schottky group transformation $G$ are obtained by the action $G^n$ at $n \to \pm \infty$ on an arbitrary point including the limiting points of the transformations (11) to be among them. So, in the $G_n$ region all the group limiting points lay outside the overlapping of Schottky circles assigned to the forming transformations. Using (10), one shows that, if the leftmost of the transformation is a positive (negative) power of $G_{b,s}(l_{2s})$, the attractive limiting point lies inside the $\hat{C}_{u_s}$ circle\(^6\) (the $\hat{C}_{v_s}$ one). The attractive (repulsive) limiting point of $G_{b,s}(l_{2s})$ lies exterior to the $\hat{C}_{u_s}$ circle\(^7\) of any group transformation $G$ having the leftmost no to be a positive (negative) power of $G_{b,s}(l_{2s})$. So, we take $\tilde{B}^{(n)}(\{q, \overline{q}\})$ in (13) as

$$\tilde{B}^{(n)}(\{q, \overline{q}\}) = \prod_{s=1}^{n} \left( \prod_{G \in \{G_s\}} \theta(|\ell_s(U_G)|^2 - 1)\theta(1 - |\ell_s(U_G)|^2) \right)$$

---

\(^6\)As an example, $u_{g_1g_2}$ being the attractive point of the $g_1g_2$ transformation, $|Q_{g_1^{-1}(u_{g_1g_2})}| = |Q_{g_1^{-1}(g_1g_2(u_{g_1g_2}))}| = |Q_{g_1^{-1}(g_2u_{g_1g_2})}| \leq 1$ due to the constraints above, $Q_G(z) = c_Gz + d_G$. Thus $u_{g_1g_2}$ lies inside $C_{v_1}$.

\(^7\)As an example, $|Qg_1g_2(u_{g_2})| = |Q_{g_1}(g_2(u_{g_2}))|Q_{g_2}(u_{g_2})| \geq 1$. 
where $\tilde{\ell}_s(t)$ is circle (10) or (11), or $\tilde{\ell}_s(t)$ is obtained by a continuous deformation of the circle within the region allowed by (36). In this case $\Gamma_s[U_G]$ denotes the $\Gamma_{b,s}(l_{2s})$ transformation (4) of $U_G$. Furthermore, $\{G_s\}$ is formed by those group products of $\Gamma_{b,s}(l_{2r})$, whose leftmost is not a power of $\Gamma_{b,s}(l_{2s})$. Leftmost of any transformation from the $\{G_s'\}$ set is a positive power of $\Gamma_{b,s}(l_{2s})$. Every $\{G_s''\}$ transformation has leftmost to be a negative power of $\Gamma_{b,s}(l_{2s})$. The sets include the given transformation along with its inverse one. Hence in (43) only the attractive limiting points $U_G$ present. They are defined by the condition that $G^n[t] \to U_G$ at $n \to \infty$. In place of (43), one can use any region obtained through a replacement of a part of (44) by the part congruent to it under the $\{G\}$ group transformation.

In the considered region the radius of the Schottky circle of any group transformation $G$ is finite, if its limiting points lay at a finite distance from limiting points of the forming group transformation. Then (34) forbids the multiplier $|k_G| \to 1$. Indeed, choosing relevant $F$ in (34), one obtains that $\omega_{GG}^{(0)}(\{q\}) \geq 1$ where $\omega_{GG}^{(0)}$ is given by (29) for $j = r = G$. Furthermore, $(u_G - v_G) \to 0$ at $k_G \to 1$, as far as the radius of the circles assigned to $G$ is finite. Thus the sum in (29) for the matrix element discussed is nullified forcing $k_G$ to be small. The limiting points going to infinity can be moved to the same finite point by a relevant $L(2)$ transformation that does not change the sum (29). So the sum vanishes, the multiplier being small again. For any integer $\tilde{n}$, the multiplier $k_G \to \exp[2\pi/\tilde{n}]$ is too excluded since the multiplier of $G^{\tilde{n}}$ goes to the unity that is forbidden by above. As far as $\tilde{n}$ can be arbitrary large, $|k_G| \to 1$ is excluded at all. Along with the $k \to 0$, the $k \to 1$ region contributes to the unitarity equations. So it is naturally that this region is a copy of the region where $k \to 0$.

When all the multipliers are small, one see from (29) that

$$2\pi i \omega_{jj}^{(0)}(\{q\}) \approx \ln k_j + 2 \sum_{s \neq j} \frac{k_s(u_s - v_s)^2(u_j - v_j)^2}{(u_j - u_s)(v_j - v_s)(u_j - v_j)(v_j - v_s)} + \ldots$$

(44)

Only the leading term being considered, constraints (34) require $|k_j| \leq 1/230$. In this case Schottky circles (3) of the same transformation to be separated from each other. The leading correction is most when the boundary of (38) is achieved. For $n$ limiting points to be closed to each other, the correction is roughly $\sim n|k|d_{uv}/\hat{d} \sim n\sqrt{|k|}$. Here $\hat{d}$ is a characteristic distance between closed limiting points of distinct forming transformations while $d_{uv}$ is a characteristic size for the $|u_s - v_s|$ distances. For the genera being not much high, the correction is small. In this case the Schottky circles $C_{u_s}$ and $C_{v_s}$ have no the common interior. When the genus is increased, $\hat{d}$ grows that might reduce the correction. In the general case we have not the estimation for the correction term, but it seems natural to expect that constraints (36) and (43) always forbid for $C_{u_s}$ and $C_{v_s}$ to be overlapped.

Constraints (36) and (43) along with (22) fully determine the integration region in (44). One can also use any region obtained by a replacement of an arbitrary part of the given
region by a part congruent under the group \( \mathcal{R} \) of the local symmetries of the amplitude. Subtle details due to divergences in the particular spin structure are discussed in Section 7. The \( \{N_0\} \) set being changed, the integral is reduced to the initial form due to due to the \( SL(2) \) symmetry, the symmetry under the \( \{\tilde{G}\} \) group and the symmetry under the super-Schottky group transformations of any given interaction vertex coordinate. So the amplitude is independent of the choice of the \( \{N_0\} \) set. The local symmetries are employed in the following calculation of the integrals over the degenerated configurations, but details of the integration region will not be important for this purpose.

5 Green functions and the integration measures

In \([11, 12]\) the Green function \( R_L^{(n)}(t, t'; \{q\}) \), the period matrix and the scalar functions have been obtained in terms of genus-1 functions. Now we represent them through the genus-\( n_i \) functions where \( \sum_i n_i = n \) and \( n_i \geq 1 \). In this case \( n \) handles are divided into groups of \( n_i \) ones, every group being given by the set \( \{q\}_i \) of the super-Schottky group parameters and by its super-spin structure \( L_i \) assumed to be even. So \( \{q\} = \{\{q\}_i\} \) and \( L = \{L_i\} \). Using the obtained formulas, we derive convenient expressions of the above quantities for degenerated configurations mentioned in the Introduction. In the following Sections these expressions will be applied to the calculation of integrals over the degenerated configurations of interest.

Along with \( R_L^{(n)}(t, t'; \{q\}) \) of Section 2, we consider \( K_L^{(n)}(t, t'; \{q\}) \) defined to be

\[
K_L^{(n)}(t, t'; \{q\}) = D(t') R_L^{(n)}(t, t'; \{q\})
\]

(45)

where the spinor derivative is defined by \([8]\). Furthermore, we build (see also \([11]\)) a matrix operator \( \tilde{K} = \{\tilde{K}_{rs}\} \) where \( \tilde{K}_{sr} \) is an integral operator vanishing at \( s = r \). For \( s \neq r \), the kernel of \( \tilde{K}_{sr} \) is \( \tilde{K}_{L_s}^{(n_s)}(t, t'; \{q\}_s) dt' \). Here \( \tilde{K}_{L_s}^{(n_s)}(t, t'; \{q\}_s) \) is related by \((13)\) with \( \tilde{R}_{L_s}^{(n_s)}(t, t'; \{q\}_s) \), which is the non-singular part \((19)\) of the Green function. So

\[
K_{L_s}^{(n_s)}(t, t'; \{q\}_s) = \frac{\vartheta - \vartheta'}{z - z'} + \tilde{K}_{L_s}^{(n_s)}(t, t'; \{q\}_s) .
\]

(46)

Like \([11]\), we define kernels together with the differential \( dt' = dz'd\vartheta'/'/2\pi i \). The discussed operator performs the integration with \( \tilde{K}_{L_s}^{(n_s)}(t, t'; \{q\}_s) \) over \( t' \) along C\(_r\)-contour, which surrounds the limiting points associated with the considered group \( r \) of the handles and the cuts between limiting points for the Ramond handles. The desired relation for the Green function is

\[
R_L^{(n)}(t, t'; \{q\}) = \ln(z - z' - \vartheta \vartheta') + \sum_s \tilde{R}_{L_s}^{(n_s)}(t, t'; \{q\}_s)
+ \sum_{r,s} \int_{C_s} [(1 - \tilde{K}^{-1}) \tilde{K}]_{rs}(t, t_1) dt_1 \tilde{R}_{L_r}^{(n_r)}(t_1, t'; \{q\}_r)
\]

(47)
where \([(1 - \hat{K})^{-1} \hat{K}]_{rs}(t, t_1)dt_1\) is the kernel of the operator
\[(1 - \hat{K})^{-1} \hat{K} = \hat{K} + \hat{K}^2 + \ldots\] (48)

In the Neveu-Schwarz sector where Green function (13) has the poles solely [5, 6, 10], the
integrals in (17) are calculated without difficulties. In this case, using for Green functions
(45), one reproduces the series for \(R^{(n)}_L(t, t'; \{q\})\). In fact, to prove
(17), one needs only check that (17) satisfies to (18). In particular, to verify (18) under the
transformations assigned to the \(r\)-th group of the handles, we represent (17) as

\[
R^{(n)}_L(t, t'; \{q\}) = R^{(nr)}_L(t, t'; \{q\}_r) + \sum_{s \neq r} \int_{C_s} K^{(nr)}_L(t, t_1; \{q\}_r) \tilde{R}^{(ns)}_L(t_1, t; \{q\}_s)dt_1
+ \sum_{p \neq r} \sum_s \int_{C_p} K^{(nr)}_L(t, t_1; \{q\}_r)dt_1 \int_{C_s} [(1 - \hat{K})^{-1} \hat{K}]_{ps}(t_1, t_2)dt_2 \tilde{R}^{(ns)}_L(t, t'; \{q\}_s)
\]

where \(R^{(nr)}_L(t, t'; \{q\}_r)\) and \(K^{(nr)}_L(t, t_1; \{q\}_r)\) are total Green functions including the singular
term in (13) and (46). Indeed, once, using (13), one calculates the contribution to (19) of
the pole term in (46), eq. (17) appears. Relations (18) for the transformations of the \(r\)-th
sector discussed are evidently satisfied for (19), the scalar function \(J^{(n)}_{jr}(t; \{q\}; L)\)
associated with the \(j_r\) handle of the \(r\)-th supermanifold being

\[
J^{(n)}_{jr}(t; \{q\}; L) = J^{(nr)}_{jr}(t; \{q\}_r; L_r) + \sum_{s \neq r} \int_{C_s} D(t_1)J^{(nr)}_{jr}(t_1; \{q\}_r; L_r)dt_1 \tilde{R}^{(ns)}_L(t_1, t; \{q\}_s)
+ \sum_{p \neq r} \sum_s \int_{C_p} D(t_1)J^{(nr)}_{jr}(t_1; \{q\}_r; L_r)dt_1 \int_{C_s} [(1 - \hat{K})^{-1} \hat{K}]_{ps}(t_1, t_2)dt_2 \tilde{R}^{(ns)}_L(t, t'; \{q\}_s).
\]

Hence (19) and (17) both are the correct expressions for \(R^{(n)}_L(t, t'; \{q\})\). The period
matrix is calculated from (50). For this purpose one considers the difference \(J^{(n)}_{jr}(t; \{q\}; L) -
J^{(n)}_{jr}(t_0; \{q\}; L)\) where t₀ is a fixed parameter. The above difference is given by

\[
J^{(n)}_{jr}(t; \{q\}; L) - J^{(n)}_{jr}(t_0; \{q\}; L) = J^{(nr)}_{jr}(t; t_0 \{q\}_r; L_r) + \int_{C'_s} D(t_1)J^{(nr)}_{jr}(t_1; \{q\}_r; L_r)dt_1
\times \tilde{R}^{(ns)}_L(t_1, t; t_0; \{q\}_s)
+ \sum_{p \neq r} \int_{C_p} D(t_1)J^{(nr)}_{jr}(t_1; \{q\}_r; L_r)dt_1 \int_{C_s} [(1 - \hat{K})^{-1} \hat{K}]_{ps}(t_1, t_2)dt_2 \tilde{R}^{(ns)}_L(t, t_2; t_0; \{q\}_s)
\]

where both \(z\) and \(z_0\) lay inside the \(C'_s\) contour while

\[
R^{(ns)}_L(t_1, t; t_0; \{q\}_s) = R^{(ns)}_L(t_1, t; \{q\}_s) - R^{(ns)}_L(t_1, t_0; \{q\}_s),
J^{(nr)}_{jr}(t; t_0 \{q\}_r; L_r) = J^{(nr)}_{jr}(t; \{q\}_r; L_r) - J^{(nr)}_{jr}(t_0; \{q\}_r; L_r)
\]

(52)
where \( R_{L_s}^{(n)}(t_1, t; \{q\}_s) \) is the total Green function \((19)\) including the singular term. To prove \((21)\), one, using \((18)\), calculates the contribution from the \( \ln[(z_2 - z - \vartheta_2 \vartheta)/(z_2 - z_0 - \vartheta_2 \vartheta_0)] \) term due to the singularity of the Green function. The corresponding integral is transformed to the one along the cut between \( z_2 = z - \vartheta_2 \vartheta \) and \( z_2 = z_0 - \vartheta_2 \vartheta_0 \). Then it is found to be

\[
\int D(t_2)f(t_2)[\theta(z_2 - z - \vartheta_2 \vartheta) - (z_2 - z_0 - \vartheta_2 \vartheta_0)]dz_2d\vartheta_2 = f(t) - f(t_0)
\]

(53)

where \( f(t) \) denotes either the Green function, or \( J_{j_r}^{(nr)}(t_1; \{q\}_r; L_r) \). As the results, one obtains \((20)\). From \((21)\), the \( \omega_{j_r,j_s}^{(n)}(\{q\}; L) \) element of the period matrix is found to be

\[
2\pi i \omega_{j_r,j_s}^{(n)}(\{q\}; L) = \delta_{j_r,j_s} \ln k_{j_r} + (1 - \delta_{j_r,j_s}) \int_{C_s} D(t)J_{j_r}^{(nr)}(t; \{q\}_r; L_r)dt J_{j_s}^{(ns)}(t; \{q\}_s; L_s) + \sum_p \int_{C_p} D(t)J_{j_r}^{(nr)}(t; \{q\}_r; L_r)dt \int_{C_s} [(1 - \hat{K})^{-1} \hat{K}]_{pr}(t, t')dt' J_{j_s}^{(ns)}(t'; \{q\}_s; L_s).
\]

(54)

For the odd super-spin structure, due to the spinor zero mode, there is no the Green function satisfying \((18)\) and, at the same time, having no non-physical poles. In this case further terms need to be added in \((17)\) providing true properties of \( R_{L_s}^{(n)}(t, t'; \{q\}) \). In particular, these terms appear in the genus-\( n \) Green function given in terms of the genus-1 functions when \( L \) includes genus-1 odd spin structures, see Appendix B of the present paper.

The integration measure is given by \((24)\) and \((25)\) with \( \tilde{Z}_{L_s}^{(n)}(\{q\}) \) to be \((11)\)

\[
\tilde{Z}_{L_s}^{(n)}(\{q\}) = \tilde{Z}^{(n)}(\{q\}, L) \prod_{s=1}^{n} \frac{Z^{(1)}(k_s; l_{1s}, l_{2s})}{k_s^{(3-2l_{1s})/2}}
\]

(55)

where the \( (l_{1s}, l_{2s}) \) theta characteristics are either 0, or 1/2, while \((11)\)

\[
Z^{(1)}(k; l_1, l_2) = (-1)^{2l_1+2l_2-1}16^{2l_1/2} \prod_{p=1}^{\infty} \frac{[1 + (-1)^{2l_2} k^p k^{(2l_1)/2}]^8}{[1 - k^p]^8}.
\]

(56)

The expression of \( \tilde{Z}_{L_s}^{(n)}(\{q\}) \) through the genus-\( n_i \) functions is derived using Appendix C along with the Green functions given in the Appendix B. When all the \( L_i \) super-spin structures are even, the desired expression is given by (see Appendix C for more details)

\[
\ln \tilde{Z}_{L_s}^{(n)}(\{q\}) = \sum_i \ln \tilde{Z}_{L_i}^{(n_i)}(\{q\}_i) - 5 trace \ln(I - \hat{K}) + trace \ln(I - \hat{G})
\]

(57)

where \( \hat{K} \) is the same as in \((17)\). The \( \hat{G} \) operator is constructed similar to \( \hat{K} \), the non-singular part \( \hat{G}_{L_s}^{(n_s)}(t, t'; \{q\}_s) \) of the \( G_{L_s}^{(n)}(t, t'; \{q\}) \) ghost superfield Green function \((11)\) being employed instead of \( \hat{K}_{L_s}^{(n_s)}(t, t'; \{q\}_s) \). In this case

\[
G_{L_s}^{(n)}(t, t'; \{q\}) = \frac{\vartheta - \vartheta'}{z - z'} + \tilde{G}_{L_s}^{(n_s)}(t, t'; \{q\}_s)
\]

(58)
where the last term on the right side has no singularity at \( z = z' \). The Green function is a superconformal 3/2-tensor under transformations (60) of \( t' \), but it is not a superconformal (-1) tensor under the transformations of \( t \). Indeed, in the last case the Green function receives additional terms being the sum of a polynomials in \( t \) multiplied by a relevant superconformal 3/2-tensor in \( t' \) (see eq.(63) in [11] and Appendix C of the present paper).

The obtained expressions can be applied to the degenerated configurations where all the limiting points of \( n_1 \) forming group transformations go to the same point. As the basic case, we consider \( n_1 < n \) forming group ones, the limiting points going to \( z_0 \). We say that they form the degenerated \( n_1 \) configuration, its super-spin structure being \( L_1 \). The remaining \( n_2 = n - n_1 \) handles form the \( n_2 \) configuration, its super-spin structure being \( L_2 \). We assume no more than one interaction vertex to be near \( z_0 \). As was noted in the Introduction, the considered configuration is the main one, which might originate divergences in the group limiting points of the amplitude integral (64).

By Section 4, the Schottky multipliers are not closely to the unity in their absolute values. Moreover, the \( \tilde{G} \) symmetry bounds \( z_0 \) to lay exterior to Schottky circles assigned to the \( n_2 \) configuration. When \( z_0 \) lies at a finite distances from the group limiting points of the \( n_2 \) configuration, we imply that \( \rho_1 \ll \rho \). In this case \( \rho_1 \) is the maximal size of the \( n_1 \) configuration while \( \rho \) is the minimal distance between \( z_0 \) and any point of essence assigned to the \( n_2 \) one. So \( \rho \leq \rho_2 \) where \( \rho_2 \) is the maximal size of the \( n_2 \) configuration. If \( z_0 \to \infty \), then \( \rho_1 \leq \rho \) and \( \rho \gg \rho_2 \). This case is, however, reduced to the finite \( z_0 \) case by a relevant \( L(2) \) transformation. Hence, for brevity, we discuss finite \( z_0 \). The super-spin structures \( L_1 \) and \( L_2 \) are taken to be even. We obtain for this configuration the leading approximated integration measure, vacuum correlator, scalar functions and period matrix, as well as the leading corrections for the above quantities. To clarify the method, we present a more detailed calculation of the leading corrections for the holomorphic partition function (67). In this case the sum on the right side of (67) gives the factorized partition function while two rest terms are corrections. In particular, the correction due to the second term is given by

\[
J_{\text{cor}}^n = -\text{trace} \ln(I - \tilde{K}) = \int_{C_1} \left( \int_{C_{II}} \tilde{K}_{L_1}^{(n_1)}(t_1, t_2) dt_2 \tilde{K}_{L_2}^{(n_2)}(t_2, t_1) \right) dt_1 + \ldots \tag{59}
\]

where \( dt = dz d\vartheta/2\pi \). In this case \( \tilde{K}_{L_1}^{(n_1)}(t, t') \) and \( \tilde{K}_{L_2}^{(n_2)}(t, t') \) are defined by (45) on the genus-\( n_1 \) supermanifold and, respectively, on the genus-\( n_2 \) one. The \( C_I \) contour bounds the domain occupied by the degenerated handles while the \( C_{II} \) contour bounds the domain of the rest handles. To calculate the first term on the right side of (67), the \( C_I \) contour is gone on a distance \( \sim \rho \) from the degenerated handles. Thus \( \tilde{K}_{L_1}^{(n_1)}(t, t') \) can be approximated by its asymptotics, \( z \) and \( z' \) both being far from \( z_0 \). Due to (45), the above asymptotics is related with the asymptotics of \( \tilde{R}_{L}^{(n_1)}(t, t') \) in (44), which are given by

\[
\tilde{R}_L^{(n)}(t, t'; \{q\}) \approx \alpha_L^{(n)}(\{q\})(z' - z_0) - \frac{b_L^{(n)}(\{q\})}{(z - z_0)(z' - z_0)} \left[ \frac{1}{z - z_0} - \frac{1}{z' - z_0} \right]. \tag{60}
\]
We assign a smallness $\sim \sqrt{\rho_1}$ to each of the integrated Grassmann parameters of the degenerated $n_1$ configuration. Along with the estimation $\sim 1/\sqrt{\rho_1}$ for its differential, it correctly determines the magnitude of the integral over the variables discussed. Hence
\begin{equation}
\hat{a}_L^{(n)}(\{q\}) \sim \rho_1^2, \quad \hat{b}_L^{(n)}(\{q\}) \sim \rho_1^2, \quad \hat{\alpha}_L^{(n)}(\{q\}) \sim \rho_1 \sqrt{\rho_1} \quad (61)
\end{equation}

The asymptotics of $K_{L_1}^{(n_1)}(t, t')$ is given by (63) and (60) at $n = n_1$ and $L = L_1$. The integrals over $z_1$ and $z_2$ are calculated in the Cauchy theorem upon going the contours to surround the poles on the right side of (60). Thus
\begin{equation}
J_{\text{cor}}^n \approx \left[ \hat{a}_L^{(n_1)}(\{q\}_1) \partial_z \partial_{z'} - 2\hat{b}_L^{(n_1)}(\{q\}_1) \partial_z \partial_{\vartheta} \partial_{\vartheta'} - 2\hat{\alpha}_L^{(n_1)}(\{q\}_1) \partial_z \partial_{\vartheta} \right] \hat{R}^{(n_2)}_{L_2}(t, t'; \{q\})_2 \quad (62)
\end{equation}

where $z = z' = z_0$ and $\vartheta = \vartheta' = 0$. The indices “1” and “2” are assigned to the corresponding configuration. Due to (64), the terms in (62) are $\sim \rho_1^2 \rho_2^2 / \rho^4$ and $\sim \rho_1 \sqrt{\rho_1} \rho_2 \sqrt{\rho_2} / \rho^3$. The last term in (64) is calculated in the same fashion using Appendix C. Being no more than $\sim \rho_1^3 \sqrt{\rho_1} \rho_2^3 \sqrt{\rho_2} / \rho^7$, it can be neglected. Thus
\begin{equation}
Z_{L_2}^{(n)}(\{q\}) \approx Z_{L_1}^{(n_1)}(\{q\}_1) Z_{L_2}^{(n_2)}(\{q\}_2)[1 + 5 J_{\text{cor}}^n] \quad (63)
\end{equation}

where $J_{\text{cor}}^n$ is given by (62). Other corrections are calculated in the kindred manner. In particular, from (54), the corrections for the $J_{L_2}(\{q\}; L)$ period matrix elements are also proportional to the coefficients in (60). Here $j_2$ labels the cycles assigned to the $n_2$ configuration. Really we shall see in Section 6 that corrections proportional to the coefficients in (60) are too small to originate divergences. More large corrections are determined by the asymptotics at $z \to \infty$ of the scalar function
\begin{equation}
J_r^{(n)}(t; \{q\}; L) \approx \frac{\hat{S}_r^{(n)}(\{q\}; L) + \hat{\Sigma}_r^{(n)}(\{q\}; L) \vartheta}{z - z_0} \equiv \frac{\hat{J}_r^{(n)}(\{q\}; L; \vartheta)}{z - z_0}. \quad (64)
\end{equation}

In this case $\hat{S}_r^{(n)}(\{q\}_1; L_1) \sim \rho_1$ and $\hat{\Sigma}_r^{(n)}(\{q\}_1; L_1) \sim \sqrt{\rho_1}$. Below $j_1$ is reserved for the cycles assigned to the degenerated $n_1$ configurations. Corrections for $\omega_j^{(n_1)}(\{q\}; L)$ are quadratic in the coefficients of (64) since the sole $\sim \rho_1$ term $\sim \hat{S}_j^{(n_1)}(\{q\}_1; L_1) \hat{\Sigma}_j^{(n_1)}(\{q\}_1; L_1)$ vanishes. Indeed, the above term is multiplied by $\partial_{\vartheta} \partial_{\vartheta'} \hat{R}^{(n_2)}_{L_2}(t, t'; \{q\})_2$ at $z = z'$, which is nullified due to the base symmetry of the Green function. So
\begin{equation}
\omega_j^{(n_1)}(\{q\}; L) \approx \omega_{j_1}^{(n_1)}(\{q_1\}_1; L_1), \quad \omega_j^{(n_1)}(\{q\}; L) \approx \omega_{j_2}^{(n_2)}(\{q_2\}_2; L_2),
\end{equation}

\begin{equation}
2\pi i \omega_j^{(n_1)}(\{q\}; L) \approx - \int \hat{j}_j^{(n_1)}(\{q\}_1; L_1; \vartheta_1) d\vartheta_1 D(t_1) \hat{R}_{j_2}^{(n_2)}(t_1; \{q\}_2; L_2) \quad (65)
\end{equation}

where the integral is calculated at $z_1 = z_0$, for further definitions see (64). The Green functions and the scalar functions for $|z - z_0| >> \rho_1$ and $|z' - z_0| >> \rho_1$ are found to be
\begin{equation}
J_{j_2}^{(n_1)}(t; \{q\}; L) \approx J_{j_2}^{(n_2)}(t; \{q\}_2; L_2), \quad R_{L_1}^{(n_1)}(t, t'; \{q\}) \approx R_{L_2}^{(n_2)}(t, t'; \{q\}_2)_2,
\end{equation}

\begin{equation}
J_{j_1}^{(n_1)}(t; \{q\}; L) \approx - \int \hat{j}_{j_1}^{(n_1)}(\{q\}_1; L_1; \vartheta_1) d\vartheta_1 D(t_1) R_{L_2}^{(n_2)}(t_1; t; \{q\}_2)_2 \quad (66)
\end{equation}
where $z_1 = z_0$. The indices \"1\" and \"2\" mark to the corresponding configurations. When $|z - z_0| \sim \rho_1$ and $|z' - z_0| >> \rho_1$ the correction for the Green function is calculated through the asymptotics of $\tilde{R}_L^{(n)}(t, t'; \{ q \})$ at $z' \to \infty$. The above asymptotics is given by

$$\tilde{R}_L^{(n)}(t, t'; \{ q \}) \approx \frac{\tilde{S}^{(n)}(t; \{ q \}; L) + \tilde{\Sigma}^{(n)}(t; \{ q \}, L)\vartheta}{(z' - z_0)} \equiv \frac{\tilde{E}_L^{(n)}(t; \{ q \}; \vartheta')}{(z' - z_0)}. \quad (67)$$

In more details, the coefficients are represented as

$$\tilde{S}^{(n)}(t; \{ q \}, L) = \vartheta\tilde{\Sigma}_1^{(n)}(z; \{ q \}; L) + \tilde{S}_L^{(n)}(t; \{ q \}, L),$$
$$\tilde{\Sigma}^{(n)}(t; \{ q \}, L) = \vartheta\tilde{\Sigma}_2^{(n)}(z; \{ q \}; L) + \tilde{\Sigma}_2^{(n)}(z; \{ q \}; L); \quad \text{(68)}$$

where the \"tilde\" terms determine the asymptotics of the \"tilde\" Green function in (69).

By dimensional reasons, $\tilde{\Sigma}_2^{(n)}(z; \{ q \}; L_1) \sim 1$, $\tilde{\Sigma}_1^{(n)}(z; \{ q \}; L_1) \sim \Sigma_2^{(n)}(z; \{ q \}; L_1) \sim \sqrt{\rho_1}$ and $\tilde{S}_L^{(n)}(t; \{ q \}; L_1) \sim \rho_1$. Further, at $z \to \infty$ the functions discussed are related with the coefficients in (67) by $(z - z_0)\tilde{S}_L^{(n)}(t; \{ q \}, L) \rightarrow \tilde{S}_L^{(n)}(\{ q \})$, $(z - z_0)^2 \tilde{S}_2^{(n)}(t; \{ q \}, L) \rightarrow \tilde{b}_L^{(n)}(\{ q \})$, $(z - z_0)\tilde{\Sigma}_2^{(n)}(t; \{ q \}, L) \rightarrow \tilde{\alpha}_L^{(n)}(\{ q \})$ and $\tilde{\Sigma}_2^{(n)}(t; \{ q \}, L) \rightarrow \tilde{\Sigma}_L^{(n)}(t; \{ q \}, L)$. In the case discussed,

$$R_L^{(n)}(t, t'; \{ q \}) = R_L^{(n)}(t_0, t'; \{ q \} t_2) - \int E_L^{(n)}(t; \{ q \} t_1; \tilde{\vartheta}) d\tilde{\vartheta} \tilde{D}(\tilde{t}) \tilde{R}_L^{(n)}(\tilde{t}, t'; \{ q \} t_2) + \ldots \quad (69)$$

where $\tilde{z} = z_0$, $t_0 = (z_0)|0\rangle$ and, in addition,

$$E_L^{(n)}(t; \{ q \} t_1; \tilde{\vartheta}) = S_L^{(n)}(t; \{ q \} t_1; L_1) + \Sigma_L^{(n)}(t; \{ q \} t_1; L_1)\vartheta,$$  
$$S_L^{(n)}(t; \{ q \} t_1; L_1) = -(z - z_0) + \tilde{S}_L^{(n)}(t; \{ q \} t_1; L_1),$$  
$$\Sigma_L^{(n)}(t; \{ q \} t_1; L_1) = \vartheta + \tilde{\Sigma}_L^{(n)}(t; \{ q \} t_1; L_1) \quad \text{(70)}$$

with definitions given in (67) and in (68). The sum $[\ln(z' - z_0) + E_L^{(n)}(t; \{ q \} t_1; \tilde{\vartheta})]$ gives the asymptotics at $\tilde{z} \to \infty$ of the Green function $R_L^{(n)}(t, t; \{ q \})$ related with $R_L^{(n)}(t, t; \{ q \})$ by (19). The scalar functions at $|z - z_0| \sim \rho_1$ are given by

$$J_1^{(n)}(t; \{ q \}; L) = J_1^{(n)}(t; \{ q \}; L_1) - \int J_1^{(n)}(\{ q \} t_1; L_1; \vartheta) d\vartheta \tilde{D}(\tilde{t}) \tilde{R}_L^{(n)}(\tilde{t}, t; \{ q \} t_2) + \ldots,$$  
$$J_2^{(n)}(t; \{ q \}; L) = J_2^{(n)}(t_0; \{ q \} t_2; L_2) - \int E_L^{(n)}(t; \{ q \} t_1; \tilde{\vartheta}) d\tilde{\vartheta} \tilde{D}(\tilde{t}) J_2^{(n)}(\tilde{t}; \{ q \} t_2; L_2) + \ldots \quad (71)$$

with the definitions given in (67). The correction $\sim \tilde{\Sigma}_L^{(n)}(\{ q \} t_1; L_1)\tilde{\Sigma}_L^{(n)}(\{ q \} t_1; L_1) \sim \rho_1$ in the first line is absent because, being proportional to $\vartheta\vartheta \tilde{R}_L^{(n)}(t, t'; \{ q \} t_2)$ at $z = z'$, it is nullified. The vacuum correlator (21) is given through $X_L^{(n)}(t, \tilde{t}; \vartheta_1; \{ q, \bar{q} \})$ defined to be

$$X_L^{(n)}(t, \tilde{t}; \vartheta_1; \{ q, \bar{q} \}) = E_L^{(n)}(t; \{ q \}; \vartheta_1) + J_1^{(n)}(t; \{ q \}; L_1) + \Xi_L^{(n)}(t, \tilde{t}; \vartheta_1; \{ q, \bar{q} \}) \quad (72)$$

$$\times \Omega_L^{(n)}(\{ q, \bar{q} \}) \equiv X_L^{(n)}(t, \tilde{t}; \vartheta_1; \{ q, \bar{q} \}) = \vartheta + \tilde{\Xi}_L^{(n)}(t, \tilde{t}; \vartheta_1; \{ q, \bar{q} \}). \quad (72)$$
In this case the “tilted” quantities are calculated through the “tilted” ones in (70). From (20), (69) and (71) the correlator is given by

\[ \hat{X}_{L,L'}(t, t'; \{q\}) \approx \hat{X}_{L_2,L_2'}(t_0, t'; \{q\}) - \int [\chi^{(q_1)}_{L_1,L_1'}(t, t'; \vartheta_1; \{q, \bar{q}\})] d\vartheta_1 \times D(t_1) \hat{X}_{L_2,L_2'}(t_1, t'; \{q\}) + \ldots \] (73)

where \( t_0 = (z_0|0) \), \( t_1 = (z_0|\vartheta_1) \) and the ”dots” encode the terms at \( t = t_0 \) due to the leading corrections for the \((j_2,j_2')\) matrix elements of (22) and for the scalar functions carrying the \( j_1 \) index. The above terms are easy found using (22), (65) and (71). From (62) and (65), the integration measure is mainly factorized. For \( n_1 = 1 \), the genus-1 factor

\[ Z^{(1)}_{L_1,L_1'}(\{q, \bar{q}\}) = Z_{tore}(k, \bar{k}; L_1, L_1')|u - v - \mu|^{-2} \] (74)
differs from the integration measure \( Z_{tore}(k, \bar{k}; L_1, L_1') \) on the torus \([1]\) by the \(|u - v - \mu|^{-2}\) multiplier, which is due to the integration over Killing genus-1 modes. Furthermore, only even \( L_1 \) super-spin structures might originate the divergences. Otherwise the integrand is not singular, see Appendix C. Moreover, the contribution to \((14)\) of odd super-spin structures is obtained by the factorization of the relevant even super-spin ones, see the end of Appendix C. So, it is sufficient to check the cancellation of divergences for the even super-spin structures.

In doing so we use eqs. (62) and (63) for the holomorphic partition function, expression (65) for the period matrix, eqs. (66) and (71) for the scalar functions, eq. (69) for the Green function and eq. (73) for the vacuum correlator.

6 Integrals of the superstring theory

The amplitude \((14)\) includes Grassmann integrations along with the ordinary ones. In this case the result is finite or divergent depending on the used integration variables, as this is seen for an easy integral

\[ I_{(ex)} = \int \frac{dxdyda\beta d\alpha d\bar{\beta}}{|z - \alpha\beta|^p} \theta(1 - |z|^2) \] (75)

where \( z = x + iy \) while \( \alpha \) and \( \beta \) are complex Grassmann variables. The complex parameter \( p \) characterizes the strength of the singularity. Integrals of this kind really appear in \((14)\).

In particular, from (23), the integration measure contains singularity \((73)\) with \( p = 2 \). The kindred expression is originated by the singularity \((19)\) at \( z = z' \) of the Green function. In this case \( p = s_{jl}/4 + 2 \) where \( s_{jl} = -(p_j + p_l)^2 \) is the square center mass energy in the given reaction channel and the add 2 is due to the vacuum contractions of the fields in front of the
exponential in (16). For the sake of simplicity, we bound the integration region in (75) by
\[ |z|^2 \leq 1. \]
Once the Grassmann integrations being performed, one obtains the integral
\[ I_{(ex)} = p^2 \int \frac{dxdy}{4|z|^{p+2}} \theta(1 - |z|^2), \]
which is divergent at \( z = 0 \), if \( \text{Re} p > 0 \). On the other side, in (75) one can turn to the
variable \( \tilde{z} = z - \alpha \beta \). Then the Grassmann variables will present only in the step function
\[ \theta(|\tilde{z} + \alpha \beta|^2). \]
After the integration over the Grassmann variables, the integral is turned to
the boundary integral at \( |z|^2 = 1 \). Thus for any \( p \) the result of the integration is finite being
\[ I_{(ex)} = -\int d\tilde{z} d\bar{\alpha} d\bar{\beta} \left[ \delta(|\tilde{z}|^2 - 1) + |\tilde{z}|^2 \frac{d\delta(|\tilde{z}|^2 - 1)}{d|\tilde{z}|^2} \right] = -\frac{\pi p}{2}. \]
So (75) depends on the integration variables, at least for \( \text{Re} p > 0 \). For \( \text{Re} p < 0 \) when
(76) is convergent, (76) and (77) both give the same result. One could, however, change
the integration variable \( z \) by
\[ z + \sum_{i=1}^{N} \delta_i \delta_i^{(1)} \]
where \( \delta_i \) and \( \delta_i^{(1)} \) are arbitrary Grassmann numbers. When \( p \) is not an negative even number, the resulted integrand has the singularity
\[ \sim |z|^{-(p+2+2N)}. \]
So, for \( 2N > -p \), the integral is divergent. For a negative even \( p \) the integral can be reduced to the singular integral by a change \( z = \tilde{z} + \alpha \beta \) with the following \( |\tilde{z}| = |\tilde{z}|^{p_1} \)
and \( \arg \tilde{z} = \arg \hat{z} \) where \( p_1 > 0 \) is no integer or half-integer, the strength of the resulted singularity being \( (p_1 p - 1) \). This integral can be transformed to the divergent one. The result is, however, the same, if the integration variable change remains the integral to be convergent.

Calculating the amplitude, we are guided by preserving its local symmetries. Divergences break the conformal symmetry due to a cutoff parameter, the amplitude depends on \( \{N_0\} \) set (13) that falls the theory. In the proposed calculation all the divergences are cancelled and the local symmetries are preserved.

Due to the singularity (14) at \( z = z' \) of the Green function, the integrand in (14) is singular when \( m_1 > 1 \) interaction vertices go the same point on the complex \( z \) plane. When the vertices are accompanied by degenerated Schottky circles, singularities present also for the integration measure. If \( 1 < m_1 < (m - 1) \) for the \( m > 3 \) point amplitude, then the strength of the singularity depends on the energy 10-invariant of the given reaction channel. The integral is calculated [1] for those energies below the reaction threshold, where it is convergent. The result is analytically continued to energies above the threshold. In this case the amplitude receives singularities required by the unitarity equations (as an example, see Appendix A). Due to the energy-momentum conservation, there is no a domain for the energy 10-invariants where the integrals over all the singular regions are convergent simultaneously. Hence the amplitude is obtained by the summing of the pieces obtained by the analytical continuation from the distinct regions of the 10-invariants. For instance, the calculation of the scattering amplitude includes an analytical continuation from the relevant energy region of the integrals over nodal regions. Each of the integrals gives rise to the cut in \( s, t \) and
configuration are the integration variables while, for the non-singular until no one of the differences \( \tilde{v} \) difference above differences as the integration variables. On equal terms one can also take the super-partner of \( z \) convention for the \( \{ \) point \( z \) from \( n \) where \( n \) \mid \) small corresponding interaction vertex \( j \) to be the vertex, we say this is the vacuum configuration. If the \( z \) follows from (56)). From (22) and (54), the non-holomorphic fact or in (24) is at \( z \) mainly consider finite symmetries of the amplitude. A). The analytical continuation procedure discussed is evidently consistent with the local to each other (see the discussion of the unitarity equations in Section 4 and in Appendix A). The analytical continuation procedure discussed is evidently consistent with the local

Moreover, the amplitudes for the emission of a longitudinal gauge boson are nullified as it is required. Indeed, from (17), in this case the integrated function is the super-derivative in \( t_j \) of a local function of the \( t_j \) coordinate assigned to the longitudinal boson discussed. When \( z_j \) and certain other vertex coordinates both go to the same point, the integral is calculated for those energy invariants, which it is convergent for. Hence the integral is reduced to the integral over boundary of a singular region discussed. The boundary integral is analytically continued to considered energies. Having no singularity discussed, the boundary integral is independent from which energy region it was be continued. Hence the boundary integrals being collected together, are canceled.

By aforesaid, the divergences of the \( m \)-point amplitude may appear only when a number \( m_1 \) of the vertices at the same point is \( m_1 = 0 \), \( m_1 = 1 \) and \( m_1 \geq (m-1) \) case is, however, out of the integration region, if \( \{N_0\} \) set in (14) is formed by (3|2) ones from the vertex coordinates. So first we consider configurations of degenerated Schottky circles, no more than one vertex being nearby. From (24), (25) and (55), the integration measure is singular in Schottky multipliers \( k_s \) and in group limiting points, as well.

Due to (30) and (13), the singularity in \( k_s \) appears only at \( k_s \to 0 \). In this case, from (55) and (30), the holomorphic integration measure is \( \sim k_s^{-(3-2l_1s)}/2 \). Half-integer powers of \( k_s \) at \( l_1s = 0 \) are cancelled after the summation over \( 2l_2s = 0 \) and \( 2l_2s = 1 \) since the sum is unchanged when \( \sqrt{k_s} \to -\sqrt{k_s} \), see Section 3 (for the vacuum amplitude this directly follows from (50)). From (22) and (24), the non-holomorphic factor in (24) is \( \sim 1/(\ln |k_s|)^5 \) at \( k_s \to 0 \). So the integrand (14) at \( k_s \to 0 \) is \( \sim 1/[|k_s|^2(\ln |k_s|)^5] \), the integral (14) over small \( |k_s| \) being finite as it has been observed in [5] for the Neveu-Schwarz sector.

Singularities in the group limiting points are due to the configurations discussed in Section 5 where \( n_1 \) degenerated handles (carrying the even super-spin structure \( L_1 \)) go to the same point \( z_0 \). As far as \( z_0 \to \infty \) is reduced to a finite \( z_0 \) by a relevant \( L(2) \) transformation, we mainly consider finite \( z_0 \). By aforesaid, we assume no more than one vertex nearby \( z_0 \). When no to be the vertex, we say this is the vacuum configuration. If the \( z_j \) coordinate of the corresponding interaction vertex \( j \) goes to \( z_0 \), we say it is the \( j \)-th configuration. With our convention for the \( \{N_0\} \) set (see Section 2) all the Grassmann module parameters of the \( n_1 \) configuration are the integration variables while, for the \( j \)-th configuration, the \( \vartheta_j \) Grassmann partner of \( z_j \) may be fixed. So we discuss the integral over the group limiting points of the \( n_1 \) configuration with given \( z_0 \) and \( \vartheta_j \) (for the \( j \)-th one). We bound from top the size of the \( n_1 \) configuration by the \( \Lambda << \rho \) cut-off where (see Section 5) \( \rho \) is the characteristic distance from \( z_0 \) to points assigned to the \( n_2 \) configuration. Due to (13), and (12), the integrand is non-singular until no one of the differences \( \tilde{v}_s = (v_s - u_s) \) is equal to zero. Hence we take the above differences as the integration variables. On equal terms one can also take the super-difference \( w_s = \tilde{v}_s - \nu_s \mu_s \) instead of \( \tilde{v}_s \). To be detailed, the remaining variables are chosen

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to be $u_s$, $\mu_s$ and $\nu_s$. When $\{\tilde{v}_s\}$ (or $\{w_s\}$) are fixed, the integrals over the above variables are non-singular. By dimensional reasons, the result of the integration may, however, be singular at $\tilde{v}_s \to 0$ (or at $w_s \to 0$) that originates the divergences for the integral over $\{\tilde{v}_s\}$ (or over $\{w_s\}$). All $\tilde{v}_s$ (or $w_s$) being of the same order $\sim \rho_1 \to 0$, the singularity is due to the integration over the region in a size $\sim \rho_1$. The easiest way to estimate the integrals is, as has been noted already, to assign the $\sim \sqrt{\rho_1}$ smallness to each one of the integrated Grassmann variable of the degenerated genus-$n_1$ supermanifold. Simultaneously, the differential of the variable is $\sim 1/\sqrt{\rho_1}$. So leading terms in (28) and (29) might originate the divergence $\sim (\rho/\rho_1)^2$ where $\rho >> \rho_1$ characterizes distances from $z_0$ to points associated with the $n_2$ non-degenerated configuration. The corrections terms might originate the divergences $\sim (\rho/\rho_1)$ and $\sim \ln(\rho/\rho_1)$. We propose, however, a calculation, which avoids the divergences. Before we need to discuss in more details the singular configurations of interest, which are the vacuum configuration and the $j$-th one.

For the vacuum configuration the integrand (14) can be represented as

$$\mathcal{F}^{n_1,n_2}_0(\{q, \bar{q}\}; \{p_r, \zeta^{(r)}\}) = \sum_{P, P'} O^{(n_1)}_0 (P, \overline{P}; \{q, \bar{q}\}_1) Y^{(n_2)}_{P, P'}$$

(78)

where $O^{(n_1)}(P, \overline{P}; \{q\}_1)$ is calculated for the degenerated configuration while $Y^{(n_2)}_{P, P'}$ depends only on $z_0$, parameters of the $n_2$ configuration and on characteristics of the interaction states. As above, $\{q\}_1$ is the set of the module variables assigned to the degenerated configuration. The sum over corrections $(P)$ for holomorphic functions and over corrections $(P')$ for the anti-holomorphic ones, includes $P = 1$ (and $P' = 1$) corresponding to the leading term. Eq. (78) follows directly from the calculation of the holomorphic functions in the previous Section. Among other things, the sum in (78) includes terms due to the corrections for the boundary (33). For the vacuum configuration (78) the leading term $O^{(n_1)}_0 (1, \bar{1}; \{q, \bar{q}\}_1)$ is

$$O^{(n_1)}_0 (1, \bar{1}; \{q, \bar{q}\}_1) = \sum_{L_1, L'_1} \tilde{Z}^{(n_1)}_{L_1, L'_1}(\{q, \bar{q}\}_1)$$

(79)

where zero point function $\tilde{Z}^{(n)}_{L_1, L'}(\{q, \bar{q}\})$ including step functions (36) and (13) is given by

$$\tilde{Z}^{(n)}_{L_1, L'}(\{q, \bar{q}\}) = \frac{g^{2n}}{2^{n} n!} Z^{(n)}_{L_1, L'}(\{q, \bar{q}\}) \hat{B}^{(n)}_{L_1, L'}(\{q, \bar{q}\}) \overline{B}^{(n)}_{L_1, L'}(\{q, \bar{q}\})$$

(80)

integration measure $Z^{(n)}_{L_1, L'}(\{q, \bar{q}\})$ being given by (24) and (23). For the $j$-th configuration containing the dilaton emission vertex, the leading term $O^{(n_1)}(DJ, \overline{DJ}; 1, \bar{1}; \{q, \bar{q}\}_1)$ is due to pairing (23) of fields (10) in front of the exponential calculated for the $n_1$ configuration. In this case

$$O^{(n_1)}_j (DJ, \overline{DJ}; 1, \bar{1}; \{q, \bar{q}\}_1) = \sum_{L_1, L'_1} \tilde{Z}^{(n_1)}_{L_1, L'_1}(\{q, \bar{q}\}_1) \tilde{B}^{(n_1)}_{L_1, L'_1}(t_j, \bar{t}_j; \{q, \bar{q}\}_1) \overline{B}^{(n_1)}_{L_1, L'_1}(t_j, \bar{t}_j; \{q, \bar{q}\}_1)$$

(81)
where $t_j = (z_j | \vartheta_j)$ is the vertex coordinate going to $z_0$, for other definitions see also (23) and (20). We define the integrand together with the step factor (12). The integrand of (14) for the $j$-th configuration (including the corrections) is given by

$$\mathcal{F}_{(j)}^{\alpha_1,\alpha_2}(t_j, \tilde{t}_j; \{q, \tilde{q}\}; \{p, \zeta^{(r)}\}) = \sum_{P, P'} O_j^{(\alpha_1)}(D, \overline{D}; P, P'; \{q, \tilde{q}\}) \tilde{Y}_{P, P'}(j)$$

$$+ \sum_{P, P'} \tilde{O}_j^{(\alpha_1)}(P, P'; \{q, \tilde{q}\}) \tilde{Y}_{P, P'}(j)$$

where $t_0 = (z_0 | 0)$. As above, $P$ lists holomorphic corrections, $P'$ lists anti-holomorphic ones, and $\tilde{Y}_{P, P'}(j)$ depends only on the parameters of the $n_2$ configuration, the interaction particle characteristics and on $z_0$. Every term for the first sum on the right side is proportional to $\tilde{I}_{L_1, L_1'}^{(\alpha_1)}(t_j, \tilde{t}_j; \{q, \tilde{q}\})$. So the first sum presents only when the configuration includes the dilaton emission vertex. The remaining terms are included in the second sum. In the second sum the term $P = 1$ (and $P' = 1$) is absent. When no one of $\{q\}_1$ belongs to the $\{N_0\}$ set, one can take $z_0 = z_j$. Otherwise we identify $z_0$ with a group limiting point. In more details, the $(P, P')$ term in (78) is represented as

$$O_0^{(\alpha_1)}(P, P'; \{q, \tilde{q}\}) = \sum_{L_1, L_1'} P(\{q\}_1; L_1) \tilde{A}_{P, P'}(\{q, \tilde{q}\}_1; L_1, L_1') P^*(\{q\}_1; L_1')$$

where $\tilde{A}_{P, P'}(\{q, \tilde{q}\}_1; L_1, L_1')$ offers the $SL(2)$ and $\{\tilde{G}\}$ symmetries while $P(\{q\}_1; L_1)$ is the function of $\{q\}_1$ in (60) or in (54), or products constructed using the functions above. As it has been noted, $P = 1$ or $P' = 1$ is assigned to the leading term for the corresponding movers. In a like fashion, the term for the first sum on the right side of (82) is given by

$$O_j^{(\alpha_1)}(D, \overline{D}; P, P'; \{q, \tilde{q}\}_1) = \sum_{L_1, L_1'} \sum_{r, s} B_{L_1, L_1'}^{(\alpha_1)}(t_j, \tilde{t}_j; \{q, \tilde{q}\}_1) D(t_j) J^{(\alpha_1)}(t_j; \{q\}_1; L_1)$$

$$\times \tilde{A}_{P, P'}^{(r, s)}(\{q, \tilde{q}\}_1; L_1, L_1') P(t_j; \{q\}_1; L_1) D(t_j) J^{(\alpha_1)}(t_j; \{q\}_1; L_1') P^*(t_j; \{q\}_1; L_1')$$

where $\tilde{A}_{P, P'}^{(r, s)}(\{q, \tilde{q}\}_1; L_1, L_1')$ has the $SL(2)$ and $\{\tilde{G}\}$ symmetries. For the second sum one obtains

$$\tilde{O}_j^{(\alpha_1)}(P, P'; \{q, \tilde{q}\}) = \sum_{L_1, L_1'} P(t_j; \{q\}_1; L_1) \tilde{A}_{P, P'}^{(j)}(\{q, \tilde{q}\}_1; L_1, L_1')$$

$$\times \tilde{A}_{P, P'}^{(j)}(\{q, \tilde{q}\}_1; L_1, L_1') B_{L_1, L_1'}^{(\alpha_1)}(t_j, \tilde{t}_j; \{q, \tilde{q}\}_1)$$

where $\tilde{A}_{P, P'}^{(j)}(\{q, \tilde{q}\}_1; L_1, L_1')$ possesses the $SL(2)$ and $\{\tilde{G}\}$ symmetries. The $P(\{q\}_1; L_1)$ function in (84) and in (85) is the one of $P(\{q\}_1; L_1)$ in (83), or one of functions (70), or it is a certain product of the functions above. For $P$ and $P'$ in (78) and (82) we use the same symbol as for the corresponding function on the right side of (83), (84) or of (83). As an example,
O^{(n_{1})}_{0}(\hat{S}_{r}, P^{t}; \{q, \bar{q}\}_{1}) is given by (83) for P(\{q\}_{1}; L_{1}) = \hat{S}_{r}^{(n_{1})}(\{q\}_{1}; L_{1}), which is defined in (84). The term with P = \Xi in (82) is given by (84) for P(t_{j}; \{q\}_{1}; L_{1}) = \tilde{\Sigma}^{(n_{1})}_{\ell_{1}, L_{1}'}(t, \bar{t}; \{q, \bar{q}\}_{1}) where the right side is defined in (72). The term with P = DX is given by (85) for P(t_{j}; \{q\}_{1}; L_{1}) = D(t_{j})X^{(n_{1})}_{L_{1}, L_{1}'}(t, \bar{t}; \{q, \bar{q}\}_{1}) where X is defined in (72). And so on. For the following, below we collect terms due to holomorphic and anti-holomorphic corrections each are no less than \sim \rho_{1} in respect to the leading term. Since P' repeats P, only P are listed.

The sum in (78) and the first sum in (82) include P = 1, P = \hat{S}_{r}, P = \hat{\Sigma}_{r}, and P = \Sigma_{r}\Sigma_{s} where \Sigma_{r} and S_{r} are defined by (64). These terms are due to corrections for the period matrix (63) and corrections for the correlator (24) when z and z' both are not nearby z_{0}. In (82) the discussed terms are due to, in addition, by the "dots" terms in (73). Evidently, O^{(n_{1})}_{0}(\hat{\Sigma}_{r}, P^{t}; \{q, \bar{q}\}_{1}) in (78) being odd function of \{\mu, \nu\}, is nullified after the integration over the Grassmann variables. The first sum in (82) contains also terms with P = X and P = \Xi due to corrections in (73). Also, it includes the by-linear term with P = \hat{\Sigma}_{r}\Xi. Generally, in (82) we do not assume the integration over \vartheta_{j}, but for the estimation, it is convenient to assign the \sim 1/\sqrt{\rho_{1}} smallness also to \vartheta_{j}. Finally a proportional to \vartheta_{j} term must be multiplied by \sim 1/\sqrt{\rho_{1}}. From Section 5, the correction for the pairing (24) at t = t_{j} is \sim D(t_{j})X^{(n_{1})}_{L_{1}, L_{1}'}(t, \bar{t}; \vartheta_{1}; \{q, \bar{q}\}_{1}) times D(t_{j})X^{(n_{1})}_{L_{1}, L_{1}'}(t, \bar{t}; \vartheta_{2}; \{q, \bar{q}\}_{1}), see eq. (73). Hence P for the second sum in (82) runs P = DX, P = D\Xi, P = \hat{\Sigma}_{r}D\Xi and P = \Xi D\Xi. In this case DX and D\Xi each denote the super-derivative in respect to t_{j} of the corresponding function. All the rest holomorphic (anti-holomorphic) corrections are less than \sim \rho_{1}.

For n_{1} = 1 the singularity of (78) at \tilde{v}_{1} = v_{1} - u_{1} = 0 is canceled locally due to the summation over the spin structures. Indeed, the terms with P = 1 vanish since, from (74), each a term is proportional to the torus partition function, the sum over the torus partition functions being nullified [1]. Terms with P = \hat{S}_{1} and P = \hat{\Sigma}_{1} disappear for the same reason since the genus-1 scalar function (B.4) is independent of the spin structure. As far as the singularity is canceled separately for the right movers and for the left ones, spin dependent corrections proportional to the coefficients in (61) are not singular. For the integrals over (\mu_{1}, \nu_{1}) with w_{1} = v_{1} - u_{1} - \nu_{1}\mu_{1} to be given, the leading singularity is canceled already for every spin structure since the leading terms do not depend on (\mu_{1}, \nu_{1}), but the cancellation of the non-leading singularity occurs only for the sum over the spin structures.

For the j-th configuration one can define boundary (12) of the fundamental region on the complex z_{j} plane using the "circles" (11). In this case \ell_{s}(t) is the same for all spin structures. Then the spin structure independent terms are nullified locally due to the summation over the spin structures. If one uses "circles" (11) dependent on the spin structure, the spin structure independent terms are nullified after the integration over module variables (but \tilde{v}_{1}). Corrections due to the coefficients in (61) are too small to originate singular terms. Further spin structure dependent terms are due to those corrections for the vacuum correlator (73), which include the \tilde{E}_{L_{1}}^{(1)}(t; \{q\}_{1}; \vartheta') function (67), see eq. (72). For n_{1} = 1 the above function is given by (B.17) of Appendix B. From (B.17), the spin structure dependent part of is \sim (\vartheta - \varepsilon(z)). At the same time, from (B.4), it follows that (\vartheta - \varepsilon(z))D(t)J(t) = 0. So the
first sum on the right side of (82) is non-singular. From (B.17), the singularity of the second sum disappears for every spin structure after the integration over the Grassmann variables (with an exception of any one from them) and after the following integration over either \( u_1 \), or over \( z_j \). The integral over \( u_1 \) can be replaced by the integral over \((z_j - u_1)\) since the integrand depends on \( u_1 \) solely through \((u_1 - z_j)\).

Instead of \( \tilde{v}_1 \), one can use the \( w_1 = (v_1 - u_1 - \nu_1 \mu_1) \) variable. To obtain (B.17) in the \((u, w, z)\) variables, the partial derivative \( (\partial_w)_{(u, z)} \) when \((u, z)\) are fixed, is calculated through \((\partial_z)_{(u, w)}\) with \((u, w)\) are fixed. For this purpose one can use the invariance of (B.17) under the special \( SL(2) \) transformations

\[
    z = \tilde{z} + \tilde{\vartheta} \vartheta_0, \quad \vartheta = \tilde{\vartheta} - \vartheta_0
\]

(86)

where \( \vartheta_0 \) is a parameter common for all the variables. The above invariance of (B.17) follows from the invariance under (80) of Green function (B.1). In this case functions (68), which determine the corrections, are found to be

\[
    \tilde{\Sigma}^{(1)}(t; \{q\}_1; L_1) = (\partial - \varepsilon(z))\tilde{W}_1 - w^{-1}\left[\tilde{W}_b(\mu - \nu) - \mu \varrho \partial_z[(z - u)\tilde{W}_1 + \tilde{W}_b]\right],
\]

\[
    \tilde{\Sigma}^{(1)}(t; \{q\}_1; L_1) = \tilde{S}^{(1)}_{\text{inv}}(t; \{q\}_1; L_1) - \tilde{\Sigma}^{(1)}(t; L_1) \tilde{\vartheta},
\]

\[
    \tilde{S}^{(1)}_{\text{inv}}(t; L_1) = \tilde{W}_b - w^{-1}\left[(z - u)\tilde{W}_1 - \tilde{W}_2\tilde{W}_b(\mu - \nu) - \varepsilon(z)\partial_z\tilde{W}_1\right](\vartheta - \mu)
\]

(87)

with \( \{q\}_1 = (u, w, k) \), \( \tilde{W}_1 \equiv W_1(z, u, w; L_1) \), \( \tilde{W}_2 \equiv W_2(z, u, w; L_1) \) and \( \tilde{W}_b \equiv W_b(z, u, w) \). So \( \tilde{W}_b \) does not depend on \( L_1 \). Both \( \tilde{\Sigma}^{(1)}(t; \{q\}_1; L_1) \) and \( \tilde{S}^{(1)}_{\text{inv}}(t; \{q\}_1; L_1) \) are invariant under (86). For the second sum in (82) the singularity at \( w_1 = 0 \) disappears for every spin structure after the integration over \( u_1, \mu_1 \) and \( \nu_1 \) (if we fix \( z_j \) and \( \vartheta_j \)). Really (87) is required only to verify the vanishing of the term proportional to \( D(t_j)\tilde{\Sigma}^{(1)}(t; \{q\}_1; L_1) \). The rest terms have fermi statistics and, in addition, they are invariant under (80) along with step factor (12). After the integration over \( \mu_1 \) and \( \nu_1 \) the integrand appears to be the derivative in \( u_1 \) of the local function (see Appendix D). Thus the singularity disappears after the integration over \( u_1 \). (Generically, the cancellation of the non-leading singularity in \( w \) forces the cancelation of the singularity in \( \tilde{v} \) and vice versa, see the next Section). The leading singularity of the first sum in (82) disappears for the same reasons while the non-leading singularities disappear only after the summation over the spin structures. One can define boundary (12) of the fundamental region using either the ”circles” (14), or (11). The cancellation of the singular terms occurs for both cases.

In addition, the singularity of (82) for \( n_1 = 1 \) disappears after the integration over \( t_j \), the summation over the spin structures being performed. To see this one transforms the integration variables by (3) reducing \( \mu_1 \) and \( \nu_1 \) to zeros. The above change of the variables is correct since the integral is non-singular. The integral vanishes locally in the super-Schottky group parameters due to the known nullification of the 1- and 2-point genus-1 function (1).
7 Finiteness of the multi-loop superstring amplitudes

To clarify the calculation of the integrals for \( n_1 > 1 \), first we consider the \( n_1 = 2 \) configuration, the group limiting points being \( U_s = (u_s|\mu_s) \) and \( V_s = (v_s|\nu_s) \). Here \( s = 1 \) and \( s = 2 \) mark the first handle and the 2-nd one. In the vacuum configuration \( (78) \) we take \( z_0 = u_2 \), while in \( (82) \) we take \( z_0 = z_j \) assuming \( t_j \) to be fixed. As before, the size of the configuration is restricted from top by a cut-off \( \Lambda \ll \rho \) where \( \rho \) is the characteristic distance from \( z_0 \) to points associated with the \( n_2 \) configuration. As has been discussed in the Introduction, we consider the integrals of functions \( (83), (84) \) or \( (85) \), every function being the sum over the spin structures of the configuration discussed. The integrals are taken over the group limiting points of the configuration keeping \( z_0 \) and \( \tilde{v}_2 \) or \( w_2 \) to be fixed. Calculating the integral over the variables of the 1-st handle, we first integrate over \( u_1, \mu_1 \) and \( \nu_1 \). Then the singularity at \( \tilde{v}_1 \to 0 \) disappears due to vanishing the integrals of the \( n_1 = 1 \) functions. Indeed, at \( \tilde{v}_1 \to 0 \), the integrand is given by \( (78) \) or \( (82) \) with \( n_1 = n_2 = 1 \) where \( Y_{P_P}^{(n_2=1)} \) is equal to the corresponding expression among \( (83), (84) \) and \( (85) \) calculated for the handle "2". Therefore, the considered \( n_1 = 2 \) integrals are convergent. And they are functions of \( \tilde{v}_2 \) or \( w_2 \). As it was announced in the Introduction, it will be shown that the integrals have no the singularity at \( \tilde{v}_2 = 0 \) or \( w_2 = 0 \). Thus the contribution to the amplitude from the considered configuration is finite since it is just given by the additional integration over \( \tilde{v}_2 = 0 \) (or \( w_2 = 0 \)) of the integrals considered. Also we show the vanishing of the 0,-1-, 2- and 3-point genus-2 amplitudes.

For the \( j \)-th configuration we perform \( u_2 \)-boost of the integration variables. Then the integral over \( u_2 \) with fixed \( t_j \) is turned to the integral over \( z_j \) with fixed \( u_2 = 0 \) and fixed \( \vartheta_j \), as well. So we shall consider the integral with fixed \( u_2 = 0 \) and \( \vartheta_j \), the integration over \( z_j \) being performed. Often, as it explained below, we shall transform these integrals into the integrals with \( \vartheta_j \) to be the integration variable.

By the reasons of the previous paragraph, the integral with fixed \( \tilde{v}_2 \) (or \( w_2 \)) of the sum over the spin structures of the first handle is convergent even without the summation over the spin structures of the second handle. Moreover, due to the nullification of the integrals of the \( Y_{P_P}^{(n_2=1)} \) functions associated with the 2-nd handle, the integral of the sum over the spin structures of the second handle is convergent for the particular spin structure of the first handle. Hence the integral of either of the partial spin structure sums discussed is convergent, as it has been announced in the Introduction. By the above reasons, the integral with fixed both \( U_2 = (u_2|\mu_2) \) and \( V_2 = (v_2|\nu_2) \) of the sum over the spin structures of the 1-st handle is convergent, as well. If, additionally, the integration over \( t_j = (z_j|\vartheta_j) \) is performed, the integral with fixed \( U_2 \) and \( V_2 \) of the sum over the spin structures of the 2-nd one is convergent, too. Indeed, in this case the divergence at \( \tilde{v}_1 = 0 \) (or \( w_1 = 0 \)) is canceled due to the nullification of the 1- and 2-point genus-1 function (see the end of Section 6). The integrals of the discussed partial spin structure sums both to be convergent, the integral of the total spin structure sum admits changes of the integration variables by the spin structure dependent transformations from the group of its local symmetries can be performed (see a
discussion of this point in the Introduction).

First we consider the integrals of terms in (83) and (84) with \( P = 1 \). For convenience, these terms in (82) can be additionally integrated over \( \vartheta_j \). Indeed, solely the \( \sim \vartheta_j \) piece of the integrand contributes to the integral while the remaining part disappears due to the integration over the Grassmann module variables. By aforesaid, the integral of the sum over the spin structures of either of two handles is convergent. Furthermore, the \( P = 1 \) terms in both (83) and (82) are invariant under transformations (86) of the holomorphic variables (the anti-holomorphic ones may be unchanged). Thus for \( w_2 = v_2 - u_2 - \nu_2 \mu_2 \) to be fixed and once the integration over the holomorphic Grassmann variables to be performed, the integrand is transformed to a sum of derivatives in respect to the boson integration variables, see (D.2) in the Appendix D. Hence the corresponding integral is nullified apart from the non-singular at \( w_2 \to 0 \) terms originated by the \( \Lambda \) cut-off. One could also change the integration variables but \( \mu_2 \) by transformation (86) with \( \vartheta_0 = \mu_2 \) (it remains the same both \( w_2 \) and \( u_2 \)). Then \( \mu_2 \) is removed from the integrand (with an exception of the \( \Lambda \) cut-off boundary). In this case the singularity at \( w_2 = 0 \) vanishes after the integration over \( \mu_2 \). Both calculations give the same result since the integral is not singular. At the same time, the singular integral of the single spin structure of every handle is divergent or finite depending on the variables used. In particular, the last integral appears to be the finite once the above change the integration variables to be performed. For the integral of the sum over the spin structures of any handle the discussed ambiguity is absent.

When \( \tilde{v}_2 \) is fixed instead of \( w_2 \) the integral of function (82) with \( P = 1 \) considered is, for convenience, again additionally integrated over \( \vartheta_j \). To see for no to be the singularity at \( \tilde{v}_2 = 0 \), we represent the discussed integral by the integral over \( u_1, \mu_2 \) and \( \nu_2 \) of the integral \( \mathcal{A}_{1,pr}^{(2)}(u_1, u_2, \tilde{v}_2, \mu_2, \nu_2) \) calculated with fixed \( u_1, \mu_2 \) and \( \nu_2 \). By reasons given in the third paragraph of this Section, this integral is convergent for the sum over the spin structures of either of two handles. The discussed singularity at \( \tilde{v}_2 = 0 \) might appear solely due to the region where \( u_1, u_2 \) and \( \nu_2 \) both are closely to each other since the \( n_1 = 1 \) integral appearing when \( u_2 \to \nu_2 \) and \( u_1 \neq \nu_2 \), has not the singularity discussed. Hence we assume both \( u_1 \) and \( \nu_2 \) to be fixed closely to \( u_2 \). In this case we remove the \( \Lambda \) cut-off because, owing to (13), the integration \( \nu_1 \) variable is bounded inside a small region nearby \( u_2 \). Then, due to the symmetry of the integrand under \( SL(2), \{ \tilde{G} \}_2 \) and super-Schottky group transformations, \( \mathcal{A}_{1,pr}^{(2)}(u_1, u_2, \tilde{v}_2, \mu_2, \nu_2) \) is related with its magnitude \( \mathcal{A}_{1,pr}^{(2)}(u_1, u_2, \tilde{v}_2, 0, 0) \) at \( \mu_2 = \nu_2 = 0 \) by (the proof is given in the next paragraph)

\[
\mathcal{A}_{1,pr}^{(2)}(u_1, u_2, \tilde{v}_2, \mu_2, \nu_2) = \left( 1 - \mu_2 \nu_2 / \tilde{v}_2 \right) \mathcal{A}_{1,pr}^{(2)}(u_1, u_2, \tilde{v}_2, 0, 0). \tag{88}
\]

For the same integral \( \tilde{\mathcal{A}}_{1,pr}^{(2)}(u_1, u_2, w_2, \mu_2, \nu_2) \) to be considered as function of \( w_2 \) (instead of \( \tilde{v}_2 \)) one obtains that

\[
\tilde{\mathcal{A}}_{1,pr}^{(2)}(u_1, u_2, w_2, \mu_2, \nu_2) = \left( 1 - \mu_2 \nu_2 / w_2 - \mu_2 \nu_2 \partial_{w_2} \right) \mathcal{A}_{1,pr}^{(2)}(u_1, u_2, w_2, 0, 0), \tag{89}
\]

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Eq. (39) is obtained by substituting \( \tilde{v}_2 = w_2 - \mu_2 \nu_2 \) into (38). Integrating (39) over \( u_1, \mu_2 \) and \( v_2 \), one sees that the absence of the singularity at \( w_2 = 0 \) for the integral on the left side (proving in the previous paragraph) forces the vanishing at \( w_2 = 0 \) of the integral of \( \mathcal{A}^{(2)}_{1; \nu}; (u_1, u_2, w_2, 0, 0) \) over \( u_1 \). Besides, from (38) and (39), the singularity at \( \tilde{v}_2 = 0 \) disappears for the integral of \( \mathcal{A}^{(2)}_{1; \nu}; (u_1, u_2, \tilde{v}_2, \mu_2, \nu_2) \) over \( u_1, \mu_2 \) and \( \nu_2 \), as it is required. Relation (38) will be proved for the integral of the total spin structure sum. Hence the discussed cancellation of the singularity in \( \tilde{v}_2 \) is shown only for the integrals of the sum over all the spin structures of the configuration considered.

To prove (38), we represent the integrand \( \mathcal{O}(1, P'; \{q, \bar{q}\}_1) \) of the discussed integral as \( [\mathcal{O}(1, P'; \{q, \bar{q}\}_1)H(U_2, V_2, U_1)] \) times \( H^{-1}(U_2, V_2, U_1) \) where \( H(U_2, V_2, U_1) \) is the factor (13) for \( t_1^0 = U_2, t_2^0 = V_2 \) and \( t_3^0 = U_1 \). As before, \( U_2 = (u_2|\mu_2), V_2 = (\nu_2|\nu_2) \) and \( U_1 = (u_1|\mu_1) \). Then we change the holomorphic integration variables by the \( SL(2) \) transformation (E.2) (see Appendix E), which reduces \( \mu_2 \) and \( \nu_2 \) to zeros, but it does not change both \( u_1, u_2 \) and \( v_2 \). Since the expression inside the square brackets is \( SL(2) \) co-variant, the transformation remains it the same, but for \( \mu_2 = \nu_2 = 0 \). Moreover, we obtain that \( H^{-1}(U_2, V_2, U_1) \) in front of the square brackets receives only the \([1 + \mu_2 \nu_2/(u_2 - v_2)]\) multiplier. The \( \sim \mu_1 \) terms in \( H^{-1}(U_2, V_2, U_1) \) do not contribute to the integral since \( [H(U_2, V_2, U_1)\mathcal{O}_0^{(2)}(1; P'; \{q, \bar{q}\})] \) at \( \mu_2 = \nu_2 = 0 \) is even function of \( (\mu_1, \nu_1) \) (after the integration over \( \theta_j \) the \( j \)-th configuration).

The integration region boundary (12) and (13) is non-invariant under the transformation that originates additional boundary integrals. They, however, cancel each other since, on equal terms, the integration region can be bounded by (12) and (13) taken as for the former variables, so for the resulted ones. Indeed, each of the integration regions is the fundamental region of the local symmetry group of the integral. Hence the integral is the same in both cases. As the result, (38) appears. Really the boundary integrals are canceled to be reduced to each other by the change of the integration variables by means spin dependent transformations of the \( \{\hat{G}\}_2 \) group and of the super-Schottky group (for the \( j \)-configuration). As it has been discussed, these transformations can be sure performed only for the integrals of the whole spin structure sum. Hence relation (38) is established only for the integral of the total sum over the spin structures of the discussed configuration.

The \( \mathcal{A}^{(2)}_{1; 1; 1}(u_1, u_2, \tilde{v}_2, 0, 0) \) integral assigned to function (38) (for \( P = P' = 1 \) times \( [(u_1-u_2)(u_1-v_2)]^2 \), is just the genus-2 vacuum amplitude, while the corresponding integral of (34) is the 1-point, genus-2 one. Indeed, every discussed amplitude is given by an integral (14) over \( \mu_1, \nu_1 \) and \( \nu_1 \) and their complex conjugated with \( u_1, \nu_2 \), \( v_2, \mu_2 \) and \( \nu_2 \) to be fixed. The integral does not depend on the fixed variables due to \( SL(2), \hat{G} \) and super-Schottky group symmetries. Taking \( \mu_2 = \nu_2 = 0 \), one just obtains \( \mathcal{A}^{(2)}_{1; 1; 1}(u_1, u_2, \tilde{v}_2, 0, 0) \) required. Calculating its limit at \( v_2 \to u_2 \) under the integral sign, one finds that \( \mathcal{A}^{(2)}_{1; 1; 1}(u_1, u_2, \tilde{v}_2 = 0, 0, 0) \) is nullified owing to the above discussed properties of the \( n_1 = 1 \) functions. Indeed, at \( v_2 \to u_2 \) it is expressed through the \( n_1 = 1 \) integrals (see the first paragraph of this Section). The limit under the integral sign is correct since (see Section 4) at \( v_2 \to u_2 \) both \( u_2 \) and \( v_2 \) lay exterior to the Schottky circles assigned to the handle ”1”. Thus the integral vanishes identically since
it is independent of \( \tilde{v}_2 \). So the 0- and 1-point amplitudes both are nullified. As it has been discussed above, the independence of the integral on the fixed variables implies a possibility to perform spin dependent transformations \( \{G\}_2 \) and, for the 1-point amplitude, the super-Schottky group transformations of the interaction vertex coordinate. These transformations can be surely performed only for the integrals of the whole spin structure sum. Hence the nullification is shown only for the integral of the whole sum over the spin structures. To directly verify the vanishing of the discussed amplitudes \( v_2 = u_2 \) for arbitrary \( (\mu_2, \nu_2) \), it requires the consideration of corrections \( \sim |\tilde{v}_2|^2 \) for each one of movers, which we did not performed.

The integrals of terms with \( P = 1 \) being non-singular at \( w_2 = 0 \) (or \( \tilde{v}_2 = 0 \)), corrections might be singular when both \( P \neq 1 \) and \( P' \neq 1 \). In this case only corrections listed in the previous Sections need to be examined. For \( \tilde{v}_2 \) to be fixed, the integral differs from the corresponding integral with fixed \( w_2 \) by the additional term \( \mu_2 v_2 \partial_{\hat{w}_2} \mathcal{O}_0^{(2)}(P, P'; \{q, \bar{q}\}) \). By the dimensional reasons, the singular part of the integral of \( \mathcal{O}_0^{(2)}(P, P'; \{q, \bar{q}\}) \) at \( \mu_2 = v_2 = 0 \) (when \( P \neq 1 \) and \( P' \neq 1 \)) is \((\pi_2 - \nu_2)^{-1}\) times a non-singular factor. This singularity is week to give the divergence in the amplitude. So it is sufficient to verify the absence of the singularity for only one of \( w_2 \) and \( \tilde{v}_2 \) to be fixed.

The terms of (82) with \( P = \hat{\Sigma}_r \hat{\Sigma}_{2r}, P = \hat{\Sigma}_r \hat{\Xi}_r, P = \hat{\Sigma} \hat{\Xi} \) and \( P = \hat{\Xi} \hat{\Xi} \) (for definitions, see the text just below eq.(83)) can be additionally integrated over \( \vartheta_j \) since, like the terms with \( P = 1 \), the remaining part disappears due to the integration over the Grassmann module variables. The integrals are invariant under (86). Indeed, due to the invariance under (86) of \( \hat{F}_L^{(n)}(t, t'; \{q\}) \), both \( \hat{\Xi}_{L_1, L'_1}^{(n_1)}(t_j, \bar{t}_j; \{q, \bar{q}\})_1 \) and \( D(t_j)\Xi_{L_1, L'_1}^{(n_1)}(t_j, \bar{t}_j; \{q, \bar{q}\})_1 \) (see (72)) are invariant under (86) that, in turn, provides the discussed invariance of the considered terms. Thus the singularity at \( w_2 = 0 \) is nullified for the integral of the sum over the spin structures of any one of the two handles, just as for the \( P = 1 \) terms. So, by the previous paragraph, the singularity at \( \tilde{v}_2 \) disappears, too. Only the integrals (see the previous Section) linear in functions (64) and (29) need to be more examined.

In doing so, for convenience we again transform the integrals of functions (82) with fixed \( \vartheta_j \) to integrals where \( \vartheta_j \) is the integration variable. For \( P = X \), \( P = DX \) and \( P = \hat{S}_r \) we can additionally integrate over \( \vartheta_j \) as far as the \( \sim \mu_1 \nu_1 \mu_2 \nu_2 \) term of the integrand includes \( \vartheta_j \), as well. With \( \vartheta_j \) no to be fixed, the terms with \( P = \hat{\Sigma}_r, P = \hat{\Xi} \) and \( P = \Xi \) disappear after the integration over Grassmann variables as far as there is no a piece proportional to all the Grassmann variables including \( \vartheta_j \). When \( \vartheta_j \) is fixed (this is just assumed), we reduce \( \mu_2 \) to zero by transformation (86) with \( \vartheta_0 = \mu_2 \). In this case we omit the change of the \( \Lambda \) cut-off boundary since it does not originate the singularity at \( w_2 = 0 \). When \( \tilde{v}_2 \) is fixed instead of \( w_2 \), additional terms appear due to this transformation, but they can be omitted. Indeed, by aforesaid, they are non-singular in \( \tilde{v}_2 \). Hence the integrals of terms with \( P = \hat{\Sigma}_r, P = \hat{\Xi} \) or \( P = \Xi \) are turned into the integrals over \( \vartheta_j \) with \( \mu_2 = 0 \) to be fixed. The integral of the \( \vartheta_j \) term in \( \Xi \) (defined by (72)) is nullified being proportional to a vanishing integral of a term with \( P = 1 \).
For definiteness, we discuss the integrals with \( \tilde{v}_2 \) to be fixed. As above, \( u_2 \) is fixed, too (see the second paragraph of this Section). Really in this case the third fixed point being \( \infty \), presents. Indeed, the integrand contains \( P \) to be the asymptotics of the corresponding function. We shall prove that every considered integral \( \mathcal{A}_{P,P'}(u_2, \tilde{v}_2, \infty) \) is related to an integral \( \hat{\mathcal{A}}(u_2, \tilde{v}_2, u_1(0), 0, 0; \hat{P}, \hat{P}') \) with the fixed \( u_1 = u_1(0) \), \( u_2 \), \( \tilde{v}_2 \) and \( \mu_2 = \nu_2 = 0 \) as it follows

\[
\mathcal{A}_{P,P'}(u_2, \tilde{v}_2, \infty) = \frac{1}{|\tilde{v}_2|^2} \hat{\mathcal{A}}(u_2, \tilde{v}_2, u_1(0), 0, 0; \hat{P}, \hat{P}')(u_1(0) - u_2)(u_1(0) - \nu_2)|^2 + \ldots \quad (90)
\]

where the "dots" denote terms, which are non-singular in \( \tilde{v}_2 \) at \( \tilde{v}_2 = 0 \). The integrand for \( \hat{\mathcal{A}}(u_2, \tilde{v}_2, u_1(0), 0, 0; \hat{P}, \hat{P}') \) contains \( D(t'_0)P(t'_0) \) to be the spinor derivative \( (\tilde{s}) \) of the \( P(t'_0) \) function whose asymptotics at \( z'_0 \to \infty \) is proportional to the \( P \) function in \( (83) \), \( (84) \) or \( (85) \). In the above integral the integration over \( t'_0 \) is implied. It will be shown that

\[
\hat{\mathcal{A}}(u_2, \tilde{v}_2, u_1(0), 0, 0; \hat{P}, \hat{P}') \]

is zero at \( \tilde{v}_2 = 0 \) due to vanishing of the \( n_1 = 1 \) integrals discussed in Section 6. Hence the singularity at \( \tilde{v}_2 \) disappears for the considered integral \( \mathcal{A}_{P,P'}(u_2, \tilde{v}_2, \infty) \). Thus (see the Introduction and the beginning of this Section) the contribution to the amplitude from the considered configuration is finite.

Eq. (94) is derived with using a change of the integration variables by the relevant \( SL(2) \) transformation. In doing so the integration region (12) and (13) is changed. To reduce it to the former region, the spin structure dependent transformations of the \( \{G\}_2 \) group and of the super-Schottky group transformations of \( t_j \) (for the \( j \)-th configuration) are necessary. As it has been discussed above, these spin structure dependent transformations can be surely performed only for the integrals of the total sum over the spin structures. Hence eq. (24) is established only for the integrals of the total sum over the spin structures of the \( n_1 = 2 \) configuration discussed.

To derive eq. (90) we consider the integral \( \mathcal{A}(u_2, \tilde{v}_2; \hat{P}(t'_0), \hat{P}'(t'_0)) \), which is obtained by the replacement \( P \to D(t'_0)P(t'_0) \) and \( P' \to D(t'_0)P'(t'_0) \) in \( \mathcal{A}_{P,P'}(u_2, \tilde{v}_2, \infty) \). Here \( D(t'_0)P(t'_0) \) is the spinor derivative of a relevant function \( P(t'_0) \) proportional to \( P \) at \( z'_0 \to \infty \). In particular, the integrals of terms with \( P = \hat{S}_r \) and \( P = \hat{S}_r \) are calculated from the integral of term with \( P(t'_0) = D(t'_0)J_r(t'_0) \) obtained by the replacing of \( P \) in \( (83) \) and \( (84) \) by \( D(t'_0)J_r(t'_0) \{q\}; L_1 \) and \( D(t'_0)J_r(t'_0) \{q\}; L_1 \) by \( D(t'_0)\hat{\mathcal{X}}^{(2)}_{L_1,L'_1}(t_j, t'_j; t'_0, \tilde{t}_0; \{q, \tilde{q}\}_1) \) where the vacuum correlator \( \hat{\mathcal{X}}^{(2)}_{L_1,L'_1}(t_j, t'_j; t'_0, \tilde{t}_0; \{q, \tilde{q}\}_1) \) is given by (24). Correspondingly, the integrals of terms with \( P = DX \) and \( P = D\Xi \) are calculated from the integral containing \( P(t'_0) = DD(t'_0)\mathcal{X}(t'_0) \). In doing so the integrals of terms with \( P = \hat{S}_r, P = X \) and \( P = DX \) are calculated as the \( z'_0 \to \infty \) limit of the corresponding integral multiplied by \( (z'_0 - u_2)(z'_0 - \nu_2) \), the integration over \( \nu'_0 \) being performed. And the integral of terms with \( P = \hat{S}_r, P = \hat{\Xi} \) or \( P = D\Xi \) is the \( z'_0 \to \infty \) limit of the corresponding integral with \( \mu_2 = \nu'_0 = 0 \) multiplied by \( (z'_0 - \nu_2) \). We want to relate the above integrals to the integrals with \( u_1 \) to be fixed instead of \( z'_0 \). AS the first step, we calculate the considered integrals in terms of the integrals with fixed
μ₂ = ν₂ = 0 and with the same u₂, v₁ and zᵣ (in the last integrals the integration over \( v₀' \) is performed). Then we use the relevant \( L(2) \) transformation \( g(z) \) to fix \( u₁ \) instead of \( z₀' \).

To do the first step above, we represent \( A(u₂, \vec{v}_2; \vec{P}(t₀'), \vec{P}'(t₀')) \) by the integral over either \((μ₂, ν₂)\), or \( ν₂ \) (with \( μ₂ = 0 \)) of the \( A(u₂, \vec{v}_2, μ₂, ν₂; \vec{P}(t₀'), \vec{P}'(t₀')) \) integral with \((μ₂, ν₂)\) to be fixed. By the third paragraph of this Section, all the integrals of the sum over the spin structures either of the handles are convergent. Once we calculate the \((μ₂, ν₂)\) dependence of the integrals, the desired \( A(u₂, \vec{v}_2; \vec{P}(t₀'), \vec{P}'(t₀')) \) integrals appear to be given though the \( A(u₂, \vec{v}_2, μ₂, ν₂; \vec{P}(t₀'), \vec{P}'(t₀')) \) integral with \( μ₂ = ν₂ = 0 \), as it is required. To calculate the \((μ₂, ν₂)\) dependence of the \( A(u₂, \vec{v}_2, μ₂, ν₂; \vec{P}(t₀'), \vec{P}'(t₀')) \) integral, we change the integration variables by transformation (E.2) (see Appendix E), which does not change \( u₂, v₁ \) and \( z₀' \).

In doing so we proceed like the deriving of (89).

The considered transformation (E.2) to be performed for the term including \( P(t₀') = D(t₀') \partial^{(2)} \bar{X}_{L₁, L₁′}^{(2)}(t_j, \bar{t}_j; t₀', \bar{t}_₀; \{q, \bar{q}\}) \), the integral receives the add since the vacuum correlator (20) is, generally, not invariant under \( SL(2) \) transformations. Really, the above correlator receives two additional terms, every term being dependent on only one of the points. For arbitrary Grassmann parameters these terms are rather tremendous. To avoid the direct calculation of the additional terms, we replaces the integral by the integral of the difference

\[
\Delta_{L₁, L₁′}(t_j, \bar{t}_j; t₀', \bar{t}_₀; \{q, \bar{q}\}) = D(t₀') \partial^{(2)} \bar{X}_{L₁, L₁′}^{(2)}(t_j, \bar{t}_j; t₀'; \{q, \bar{q}\}) - D(t₀') \partial^{(2)} \bar{X}_{L₁, L₁′}^{(2)}(t_j, \bar{t}_j; t₀'; \{q, \bar{q}\})_{t_j = t₀', \bar{t}_j = \bar{t}_0}
\]

(91)

where the correlator at the same point is defined as usually, see Section 2. Indeed, being at \( z \to \infty \) smaller than \( \rho, \) the last term of the difference does not originate the singularity at \( \vec{v}_2 = 0 \) in the integral. The add to (21) under the \( SL(2) \) change is due to only the singular term due to is absent for the correlator at the same points. So the add to (21) is \( D(t₀') \ln \bar{Q}(t₀') \) where \( \bar{Q}(t₀') \) is given by (8) for the transformation (E.2) considered. Being independent of the module variables, this addition term originates the \( \bar{P} = 1 \) integral, which vanishes. Finally, the correlator at the same point \( t₀' \) can be omitted since it is not contribute to the singular term. Hence all the specifics due to the non-invariance of the vacuum correlator can be neglected.

As before, the \( \Lambda \) cut-off boundary does not contributes to the singularity due to the vanishing of the integrals of the \( n₁ = 1 \) functions. Thus the singular part of the integral associated with \( P = \bar{S}_r, \bar{P} = \bar{X} \) or \( P = \bar{D} \bar{X} \) is, like the \( P = 1 \) case, related by (89) with the corresponding integral at \( μ₂ = ν₂ = 0 \). To calculate \( A(u₂, \vec{v}_2; \vec{P}(t₀'), \vec{P}'(t₀')) \) desired, one integrates the above relation over \((μ₂, ν₂)\). Thus \( A(u₂, \vec{v}_2; \vec{P}(t₀'), \vec{P}'(t₀')) \) appears to be \( A(u₂, \vec{v}_2, 0, 0; \vec{P}(t₀'), \vec{P}'(t₀')) \) times \( 1/|v₂|^2 \). For the integrals with \( \vec{v}_0' = μ₂ = 0 \) we reduce \( ν₂ \) to zero and \( \vec{v}_0' \) to \( \vec{v}' \). In this case the integration over \( ν₂ \) is replaced by the integration over \( \vec{v}' \) and, simultaneously, the integral receives factor \((z₀' - u₂)/(u₂ - v₂)\). In any case the singular at \( v₂ = 0 \) part of the desired \( A_{P, P'}(u₂, \vec{v}_2, \infty) \) integral in (89) is the limit at \( z₀' \to \infty \) of the integral over \( \vec{v}' \) of \( A(u₂, \vec{v}_2, 0, 0; \vec{P}(t₀'), \vec{P}'(t₀')) \) times \(|(z₀' - u₂)(z₀' - v₂)|^2/|v₂|^2\).
The \((z'_0 - u_2)(z'_0 - v_2)\) factor is just the factor \([13]\) for fixed \(z'_0\), \(u_2\), \(v_2\) and \(\mu_2 = \nu_2 = 0\). Furthermore, we can remove the cut-off and, simultaneously, to restrict the integration region by the \(B^{(2)}_0(t'_0, \bar{t}'_0; \{q, \bar{q}\}_1)\) step factor \([12]\). Indeed, like the \(P = 1\) case, the integral may be non-vanishing at \(\bar{v}_2 = 0\) solely due to the region where \(u_1\), \(v_1\) and \(v_2\) both go to \(u_2\). The resulted integral with the above constraint of the integration region is convergent. Truly, when either \(\bar{v}_1 \to 0\), or \(u_1\) and \(v_1\) both go to the infinity, the singular part of the integrand disappears since it is proportional to the vanishing integral of \(n_1 = 1\) function assigned to the handle "\(2\)". When \(z_j \to z'_0\), the integral is again convergent due to the vanishing of the \(P = 1\) integrals. We fix \(u_1 = u^{(0)}_1\) (nearby \(u_2\)) by a relevant \(L(2)\) transformation \(g(z)\) of the integration variables. Then \(z'_0\) turns into \(z'\), and the integration over \(u_1\) is replaced by the integration over \(z'\). Once the \(L(2)\) transformation being performed, the \((z'_0 - u_2)(z'_0 - v_2)\) factor in the integral is replaced by \((u^{(0)}_1 - u_2)(u^{(0)}_1 - v_2)\), and eq.(14) appears.

Except of the integral of the term with either \(P(t'_0)\), or \(P(t'_0)\) to be \(D(t'_0)\mathcal{X}(t'_0)\), the integrand for the right side integral in \([14]\) is obtained by the \(P \to P(t'_0)\) replacement in the integrand for the left side one. Taking the \(\bar{v}_2 \to 0\) limit, one can see that the right side integral vanishes at \(\bar{v}_2 = 0\) due to the nullification of the integrals of the \(n_1 = 1\) function associated with \((u_2, v_2)\) limiting points. The integral is invariant under \(SL(2)\) and \(\bar{G}\) transformations as well the super-Schottky transformations of the interaction vertex coordinate. Hence it vanishes identically in \(\bar{v}_2\).

The integral of the term with \(P(t'_0) = D(t'_0)\mathcal{X}(t'_0)\) is non-invariant under the considered transformation \(g(z) = (az + b)/(cz + d)\). Truly, it receives the additional term

\[-D(t')\bar{X}^{(2)}_{L, 1, i'}(t(\infty), \bar{t}(\infty); t', \bar{t}'; \{q, \bar{q}\}_1),\]

which appears due to the corresponding addition to the vacuum correlator, see Appendix E. In this case \(t' = (z'|\bar{q}'t)\) and \(t(\infty) = (z(\infty)|\bar{q}(\infty))\) with \(z(\infty) = -d/c\) and \(\bar{q}(\infty) = 0\). The discussed \(g(z)\) transformation has \(u_2\) and \(v_2\) to be the limiting points, and the multiplier \(k\) to be \(k = (z'_0 - u_2)(z' - v_2)/(z'_0 - v_2)(z' - u_2)\). So \(z(\infty) = z' + (z' - u_2)(z' - v_2)/(z'_0 - z')\). The integration region is changed under the \(g(z)\) transformation. To calculate the right side integral in \([14]\) at \(\bar{v}_2 = 0\) by taking the \(\bar{v}_2 \to 0\) limit under the integral sign, the integration region needs to be is reduced to the \([12]\) and \([43]\). In doing so \(t(\infty)\) is changed. Nevertheless, since the addition term discussed is invariant under the super-Schottky transformations of \(t(\infty)\), one can replace it by its super-Schottky group image laying interior to region \([12]\). In this case the desired \(\bar{v}_2 \to 0\) limit under the integral sign can be performed. Moreover, the discussed additional term is not singular at \(\bar{v}_2 \to 0\). Thus the integral is nullified at \(\bar{v}_2 \to 0\) due the nullification of the \(n_1 = 1\) function, just as the integrals discussed above. Since considered integral is not \(SL(2)\)-invariant, it does not vanish identically, unlike the integrals considered before. In any case, from \([14]\) the desired integrals of \([78]\) and of \([82]\) have no the singularity at \(\bar{v}_2 \to 0\). So the contribution to the amplitude of the discussed \(n_1 = 2\) configuration is finite.

The nullification of the 2- and 3-point amplitudes is verified as for the 0- and 1-point ones discussed above. Every amplitude is given by the corresponding integral \([14]\) of the 2- or 3-point function. The integral is taken over \(\mu_1, \nu_1\) and \(v_1\) and their complex conjugated, both
\( \mu_e, \nu_2, u_1, u_2 \) and \( v_2 \) being fixed. The integral of the sum over the spin structures of either of two handles is convergent for the given spin structure of the remaining handle. In this case the integral of the spin structure sum of the 2-nd handle is convergent due properties 1-, 2- and 3- point genus-1 functions. The 3-point genus-1 function are examined just as the \( n_1 = 1 \) functions in the previous Section. Since the desired amplitude are independent of the fixed parameters (the total sum over the spin structures is implied), they can be calculated at \( \mu_2 = \nu_2 = 0 \). It allows to avoid a large number of the corrections \( \sim |\tilde{v}_2|^2 \) in each one of movers. The amplitude is nullified at \( v_2 \to u_2 \) due to vanishing the integrals of \( n_1 = 1 \) functions associated with the handle ”2”. Being independent of the fixed variables, the considered amplitude is zero for any \( v_2 \).

For general \( n_1 \), the cancellation of the divergences can be derived by the mathematical induction. Assuming the \( n'_1 \) integrals to be convergent for all \( n'_1 < n_1 \), one verifies the cancellation of the singularity at \( \tilde{v}_{n_1} = 0 \) or \( w_{n_1} = 0 \) for the integrals over variables of the remaining \( (n_1 - 1) \) handles. In doing so the consideration like given for the \( n_1 = 2 \) case, is performed. In particular, it is verified that the integrals of the spin structures of the \( (n_1 - 1) \) handles are convergent for every spin structure of the \( n_1 \)-th handle. And the integrals of the sum over the spin structures of the \( n_1 \)-th handle are convergent for every spin structure of the remaining \( (n_1 - 1) \) handles. As for \( n_1 = 2 \), the integrals with \( P(t'_0) \) instead of \( P \) are considered. Like the \( n_1 = 2 \) case, the total cancellation of the singularity is verified for the whole sum over the spin structures. Step-by-step, the nullification of the 0-, 1, 2- and 3-point amplitudes is verified.

Due to \( L(2) \) symmetry, the cancellation of the divergences for the finite \( z_0 \) forces the same for \( z_0 \to \infty \). This case could also be considered like the finite \( z_0 \) case. When the \{\( \tilde{N}_0 \)\} set in (14) for the \( m > 3 \) point amplitude is formed by the limiting group points, further singular configurations appear to be with either \( (m - 1) \), or \( m \) vertices go to the same point. In this case the leading approximated integrand is proportional either to the 1- function or, respectively, to the 0-point one. By aforesaid, the above 0- and1-point integrals are nullified along with the integrals due to the leading corrections. Hence the considered configurations originate no divergences in the amplitude.

So, the divergences do not appear when one integrate, step-by-step, the sum over the spin structures of the given handle. over its limiting points. For the considered handle, the integration over the difference \( \tilde{v} \) or over the super-difference \( w \) between the limiting points is performed in the last turn. In this case the \( SL(2) \) symmetry is preserved along with the \{\( \tilde{G} \)\} symmetry and with the symmetry under super-Schottky group changes of every particular interaction vertex coordinate. These spin structure dependent transformations can be performed due to the convergence of the integrals over \( n_1 > 1 \) configurations of the above discussed partial sums over the spin structures.

The total group of the local symmetries of the amplitude contains, in addition, the modular group. Naively, the modular symmetry is provided due in this case to terms of the sum over the spin structures in amplitude (14) are correctly transformed into each other (12).

So the invariance under the \( \sqrt{k_s} \to -\sqrt{k_s} \) change (\( k_s \) is the super-Schottky group multiplier)
is evident since the integration variables are not touched. For non-zero Grassmann moduli, all the other modular transformations (including the addition $\pm 2\pi$ to each of the remaining arguments in (15)) are, however, accompanied by the spin structure dependent change of the integration variables. The integral of the single spin structure over the $n_1$ configurations being divergent, both a possibility to perform the modular transformations and the modular invariance need an argumentation. Like $\{G\}$ and super-Schottky group transformations, the modular transformations can be performed due to the convergence of the integrals of the above discussed partial sums over the spin structures. And the modular symmetry is preserved. The argumentation is outlined below.

The transition functions for modular transformations depend on the super-spin structure by terms proportional to Grassmann super-Schottky group parameters. The discussed terms are calculated from the set of integral equations, the kernels being given through ghost Green functions for zero module parameters. The integration is performed along contours where every contour rounds the Schottky circles of the given handle together with the cut between the circles. For the $n_1$ configuration the leading approximated transition functions coincide with the transition functions of the relevant modular transformation of the genus-$n_1$. And the remaining $n_2 = n - n_1$ configuration is changed by the relevant transition functions of the modular genus-$n_2$ transformation. The corrections for the transition functions are due to corrections for the ghost Green functions, which are calculated either using the expressions, or using representation of the Green functions in terms of the genus-1 ones at zero Grassmann module parameters, see Appendix C of the present paper. In the last case the method of Section 4 can be employed. For the $n_2$ configuration, the spin structure dependent corrections are no larger than $\sim \rho_1^3$ for the Grassmann transition function and no larger than $\sim \rho_1^3$ for the boson transition one, $\rho_1 \to 0$ (for definitions, see the previous Section). So, they are negligible.

For the $n_1 = 1$ case ($\tilde{v} \to 0$) the leading approximated transition functions are independent of the spin structure. The spin structure dependent corrections for the Grassmann transition function are not larger than $\sim |\tilde{v}|^3$, and they are not more than $\sim |\tilde{v}|^4$ for the boson transition one. Before the modular transformation to be performed, we can previously cut with below the integration region $\tilde{v} \to 0$ by some cut-off $\tilde{\rho}$ and then to perform the required change of the variables. At $\tilde{\rho} \to 0$ the additional terms due to spin structure dependent corrections are nullified since the integral of these terms is found to be convergent. So the resulted integral coincides with the former one. For $n_1 = 2$ we, as before, consider the integral of the given spin structure of the first handle over its variables, $U_2$ and $V_2$ being fixed. We transform the variables of the 1-st handle for the sum over the spin structures of the second one. Then we transform the variables of the 2-nd handle for the integral of the sum over the spin structures of the first one. Hence for the integral of the total sum over the spin structures of the configuration the modular transformation can be performed, the modular symmetry being preserved. For $n_1 > 2$ the mathematical induction is employed. So all the local symmetries of the amplitude are preserved.
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A Integration region and the unitarity

As an example, we consider the genus-2 forward scattering amplitude. The vacuum expectation of the vertex product is proportional to exp $S$ where, being calculated for zero Grassmann parameters, $S$ is given through the field vacuum correlators (24) of the boson string [31]. The Schottky multipliers to go to zeros, the leading approximated holomorphic Green function $R(z, z')$ is $\ln(z - z')$ while the scalar function $J^{(2)}_p(z; \{q\})$ is $\ln[(z - u_r)/(z - v_r)]$. The period matrix is given by (29). Then for the tachyon-tachyon forward scattering amplitude in the boson string theory

$$S = -\frac{s}{4} \left[ \ln \left( \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z_3)(z_2 - z_3)} \right) - \ln \left( \frac{(z_1 - u_j)(z_3 - v_j)}{(z_1 - v_j)(z_3 - u_j)} \right) \right]$$

$$-2 \ln \left( \frac{(z_1 - u_j)(z - u_j)(z_2 - v_j)(z_3 - v_j)}{(z_1 - v_j)(z - v_j)(z_2 - u_j)(z_3 - u_j)} \right) \frac{\hat{\omega}_{j1}}{\det \hat{\omega}} \ln \left( \frac{(z_1 - u_j)(z - u_j)(z_2 - v_j)(z_3 - v_j)}{(z_1 - v_j)(z - v_j)(z_2 - u_j)(z_3 - u_j)} \right)$$

$$-4 \ln \left( (z_1 - z_2)(z_3 - z) \right)$$

where $s = -(p_1 + p_2)^2 = -(p_3 + p_4)^2$ and $\hat{\omega}_{j1}$ is given by

$$\hat{\omega}_{11} = \ln |k_2|, \quad \hat{\omega}_{22} = \ln |k_1|, \quad \hat{\omega}_{12} = \hat{\omega}_{21} = -\ln \left( \frac{(u_1 - u_2)(v_1 - v_2)}{(u_1 - v_2)(v_1 - u_2)} \right). \quad \text{(A.2)}$$

For the massless boson scattering amplitude only the proportional to $s$ term presents on the right side of (A.1). First we discuss the case where both $v_1$ and $v_2$ does not go to $z_3$. The configurations $|k_1| \geq |k_2|$ and $|k_1| \leq |k_2|$ give the same contribution to the amplitude. So we consider $|k_1| \geq |k_2|$. We define new variables by

$$\ln |k_1| = x, \quad \ln |k_2| = x\alpha, \quad |u_2 - u_1| = |k_1|^\beta |v_1 - z_3|,$$

$$u_1 - z_3 = y_1(u_2 - u_1), \quad z - u_1 = y(u_2 - u_1) \quad \text{(A.3)}$$

where $\alpha > 1$. From (40), it follows that $0 \leq \beta \leq 1/2$. Generically, $y \sim y_1 \sim 1$. In the boson string theory [31] at $k_1 \to 0$ the holomorphic partition function is $\sim |k_1 k_2|^{-2}$ while in the superstring theory (see Section 5 of the present paper) it is $\sim |k_1 k_2|^{-2}$. Terms of the expansion of the integrand (14) over the powers of the small variables corresponds to different thresholds while the expansion over powers of $1/x$ corresponds to the expansion
over powers of the center mass space momentum of the intermediate state. Near the given
threshold the leading approximated amplitude \( A^{(2)}_i(s) \) discussed is found to be

\[
A^{(2)}_i(s) = \frac{(4\pi)^D}{8} \int_1^\infty d\alpha \int_0^{1/2} d\beta \frac{\hat{A}_1(\alpha, \beta) \hat{A}_2(\alpha, \beta)}{[\alpha - \beta^2]^{D/2}} \int_{-\infty}^{-\kappa} e^{\kappa \tilde{S}} dx
\]  

(A.4)

where the variables are defined by (A.3). In this case \( D = 10 \) for the superstring and \( D = 26 \)
for the boson string. The cutoff \( \kappa \gg 1 \) bounds the region of small \( |k_1| \). Furthermore,
\( \hat{A}_1(\alpha, \beta) \) is an integral over real and imaginary parts of both \( y \) and \( y_1 \) while \( \hat{A}_2(\alpha, \beta) \) is an
integral over real and imagery parts of \( v_1 \) and of \( v_2 \). By using (A.1) one find that

\[
\tilde{S} = \left( \beta - \frac{\alpha \beta^2 + \beta^2 - 2\beta^3}{\alpha - \beta^2} \right) \left[ -\frac{s}{4} + \tilde{p}(\alpha, \beta) \left( \beta - \frac{\alpha \beta^2 + \beta^2 - 2\beta^3}{\alpha - \beta^2} \right)^{-1} \right]
\]  

(A.5)

where \( \tilde{p}(\alpha, \beta) \) is a linear function of its arguments with the coefficients depending on
the threshold discussed. In the superstring theory all the coefficients are non-negative. At \( \tilde{S} \geq 0 \)
the integral (A.4) is divergent. Thus the cut begin with that \( s \), which is the minimal value of \( s \) where \( \tilde{S} \) is
nullified in the integration region. To calculate the discontinuity, we go to \( \tilde{x} = -x \tilde{S} \) integrated from \( \kappa \tilde{S} \) till \( \infty \). When \( \tilde{S} \) rounds the \( \tilde{S} = 0 \) point, the initial point \( \kappa \tilde{S} \) of
the integration contour gets about the pole at \( \tilde{x} = 0 \) to be either above the \( \tilde{x} = 0 \) point, or
below it, depending on the sign of the \( Im \tilde{S} \). Thus the discontinuity \( [A^{(2)}_i(s)]_{\text{disc}} \) of \( \hat{A}^{(2)}_i(s) \)
is given by the integral over \( \tilde{x} \) along the closed contour surrounding the \( \tilde{x} = 0 \) point to be

\[
[A^{(2)}_i(s)]_{\text{disc}} = \left( \frac{4\pi)^D}{8} \int_1^\infty d\alpha \int_0^{1/2} d\beta \frac{\hat{A}_1(\alpha, \beta) \hat{A}_2(\alpha, \beta)}{[\alpha - \beta^2]^{D/2}} \theta(-\tilde{S}) \frac{2\pi \tilde{S}^{(D-3)}}{(D-3)!} \right)
\]  

(A.6)

Every threshold \( s = s_i \) determine the minimum of the second term in the square brackets
on the right side of (A.7) at corresponding \( \alpha = \text{alpha}_i \) and \( \beta = \beta_i \), which, among other, can
be on the boundary of the region. At \( (s - s_i) \rightarrow 0 \) only small \( (\alpha - \alpha_i) \) and \( \beta - \beta_i \) contribute to (A.6). Being smooth functions of \( \alpha \) and \( \beta \), in the leading approximation \( \hat{A}_1(\alpha, \beta) \) and
\( \hat{A}_2(\alpha, \beta) \) both are replaced by \( \hat{A}_1(\alpha_i, \beta_i) \) and \( \hat{A}_2(\alpha_i, \beta_i) \). It is naturally to expect that every
one from \( \hat{A}_1(\alpha_i, \beta_i) \) and \( \hat{A}_2(\alpha_i, \beta_i) \) present the threshold values of the corresponding \( 2 \rightarrow 3 \)
amplitude as this is for three tachyon cut of the tachyon-tachyon amplitude in the boson
string theory. In this case \( \tilde{p}(\alpha, \beta) = m_{th}^2(1 + \alpha - \beta)/4 \) where \( m_{th}^2 = -8 \) is the square of
the tachyon mass. As far as \( m_{th}^2 < 0 \), the calculation includes rather subtle matters, which
are not discussed here. Instead we calculate the discontinuity for a \( m_{th}^2 > 0 \) continuing the
obtained result to \( m_{th}^2 = -8 \). There is only one threshold to be at \( s = 9m_{th}^2 \). In this case
\( \beta \approx 1/2 \) and \( \alpha \approx 1 \). So (A.3) is approximated by

\[
\tilde{S} \approx \frac{1}{3} \left( \frac{s - 9m_{th}^2}{6} + 8m_{th}^2((\alpha - 1)^2 - 2(\alpha - 1)((1/2 - \beta) + 2(1/2 - \beta^2)) \right)
\]  

(A.7)
Other factors in (A.6) are taken at $\alpha = 1$ and $\beta = 1/2$. One can check that $\hat{A}_1(1, 1/2) = \hat{A}_2(1, 1/2) = A_s^{(0)}$ is really the $2 \rightarrow 3$ tree tachyon interaction amplitude at $s = 9m_{th}^2$. In this case $\hat{A}_1(1, 1/2)$ given by the integral where three vertex coordinates are fixed to be 0, 1 and $\infty$. In $\hat{A}_2(1, 1/2)$ the fixed coordinates are $z_1$, $z_2$ and $z_3$. The integral is none other than $[A_s^{(0)}]^2$ times the phase volume.

For the configuration $v_2 \rightarrow z_3$, $u_1 \rightarrow z_3$ and $u_2 \rightarrow z_3$ we discuss, as an example, the case $k_1 \geq k_2$. We define new integration variables as it follows

$$\log |k_1| = x, \quad \log |k_2| = x\alpha, \quad |u_2 - u_1| = |k_1|^{|\beta + \delta|} |v_1 - z_3|, \quad |v_2 - u_1| = |k_1|^\delta |v_1 - z_3|,$$

$$|u_1 - z_3| = |k_1|^\gamma |v_1 - z_3|, \quad z - u_1 = |k_1|^\eta (v_1 - z_3)$$

(A.8)

where $1 \leq \alpha$ and $0 \leq \delta \leq \eta$. It follows from (40) that $0 \leq \beta \leq 1/2$. In addition, $0 \leq \eta \leq 1/2$, as far as $z_3$ lies out of the $C_{u_1}$ circle. Then, by using eq. (A.1), for the tachyon-tachyon forward scattering amplitude the expression of $\tilde{S}$ in (A.6) is found to be

$$\tilde{S} = \left( \eta - \frac{\alpha \eta^2 - (\eta - \delta)^2 - 2\beta \eta(\eta - \delta)}{\alpha - \beta^2} \right) \left[ -\frac{s}{4} + \frac{m_{th}^2}{4}(1 + \alpha - \beta) \right] \times \left( \frac{\eta - \alpha \eta^2 - (\eta - \delta)^2 - 2\beta \eta(\eta - \delta)}{\alpha - \beta^2} \right)^{-1}$$

(A.9)

In this case the false threshold appears at $s = 6m_{th}^2$, which corresponds to the minimum of the last term on the right side of (A.3) to be at $\alpha = 1$, $\beta = \eta = 1/2$ and $\delta = \eta(1 - \beta) = 1/4$. The discussed configuration is removed from the integral as it is proposed in Section 4.

### B Explicit Scalar superfield Green functions

Explicitly $R_L^{(n)}(t, t'; \{q\})$ can be given through the genus-1 Green functions $R_{i_s}^{(1)}(t, t'; s)$ calculated for the Schottky parameters $\hat{m}_s = (k_s, u_s, v_s)$ along with the Grassmann ones $\mu_s$ and $\nu_s$, the spin structure being $l_s = (l_{1s}, l_{2s})$. Due to (41), the above genus-1 function is written through $z_s$ and $\vartheta_s$ in (44) using the boson Green function $R_b^{(1)}(z, z' ; \hat{m}_s)$ and the fermion Green one $R_f^{(1)}(z, z' ; \hat{m}_s, l_{1s}, l_{2s})$ as it follows (44)

$$R_{i_s}^{(1)}(t, t'; s) = R_b^{(1)}(z_s, z'_s ; \hat{m}_s) - \vartheta_s \vartheta'_s R_f^{(1)}(z_s, z'_s ; \hat{m}_s, l_{1s}, l_{2s})$$

$$+ \varepsilon'_s \vartheta'_s \Upsilon_s(\infty, z'_s) + \varepsilon_s \vartheta_s \Upsilon_s(z_s, \infty),$$

$$\Upsilon_s(z, z') = (z - z') R_f^{(1)}(z, z' ; \hat{m}_s, l_{1s}, l_{2s})$$

(B.1)

The proportional to $\Upsilon_s$ terms are added to provide decreasing $K_{i_s}^{(1)}(t, t'; s)$ at $z \rightarrow \infty$ or $z' \rightarrow \infty$. The above $K_{i_s}^{(1)}(t, t'; s)$ is related with $R_{i_s}^{(1)}(t, t'; s)$ by (45). The Poincaré series for boson and fermion Green functions are given in the end of this Appendix. For the odd spin structure $l_m = (1/2, 1/2)$ we use (41) the Green function with the property that

$$R_{i_m}^{(1)}(t'_m, t'; m) = R_{i_m}^{(1)}(t, t'; m) + J_m^{(1)}(t') - \varphi_m(t) \varphi_m(t')$$

(B.2)
where \( J_{m}^{(1)}(t) \) the genus-1 scalar function and \( \varphi_{m}(t) \) is the spinor zero mode given by

\[
\varphi_{m}(t) = \frac{\vartheta_{m}(u_{m} - v_{m})^{1/2}}{[(z_{m} - u_{m})(z_{m} - v_{m})]^{1/2}} + \varepsilon'_{m}(u_{m} - v_{m})^{1/2}
\]  
(B.3)

Here \((z_{m}|\theta_{m})\) variables are determined by (5). The last term in (B.3) provides vanishing the spinor zero mode at \( z \to \infty \). The genus-1 scalar function \( J_{s}^{(1)}(t) \) is

\[
J_{s}^{(1)}(t) = \ln \frac{z_{s} - u_{s}}{z_{s} - v_{s}}
\]  
(B.4)

where \( z_{s} \) is given by (3).

For the genus-\( n \) superspin structure without the odd genus-1 spin structures, the desired Green function is directly given by (19) \( \tilde{R}_{L}^{(n)}(t, t'; \{q\}_{r}) \) are replaced by \( \tilde{R}_{L}^{(1)}(t, t'; s) \). The kernel of the operator \( \hat{K} = \hat{K}_{sr} \) for \( s \neq r \) is \( \tilde{R}_{L}^{(1)}(t, t'; s)dt' \), which is the non-singular part of \( K_{L}^{(1)}(t, t'; s) \) related with \( \tilde{R}_{L}^{(1)}(t, t'; s) \) by (19). The non-singular part of the Green function is defined by (19). The integration over \( t' \) is performed along \( C_{r} \)-contour surrounding the limiting points \( u_{r} \) and \( v_{r} \) and, for the Ramond handle, the cut between them.

Below we denote \( \tilde{R}_{L}^{(n)}(t, t'; \{q\}) \) the Green function given it terms of the \( \tilde{R}_{L}^{(1)}(t, t'; s) \) genus-1 functions by (19). If odd spin structure handles present, the change of \( \tilde{R}_{L}^{(1)}(t, t'; s) \) under \( t \to t^{(b)} \) is different from (18) due to the last term in (B.2). We show that in this case the Green function satisfying (18), is given by

\[
\tilde{R}_{L}^{(n)}(t, t'; \{q\}) = \tilde{R}_{L}^{(n)}(t, t'; \{q\}) - \frac{1}{2} \sum_{m, m'} \Phi_{m}(t; L; \{q\}) \hat{V}_{mm'}^{-1} \Phi_{m'}(t'; L; \{q\})
\]  
(B.5)

where the last term appears, if genus-1 odd spin structures present. The \( \hat{V}_{mm'} \) matrix elements are defined only for those \((m, m')\) that label the odd genus-1 spin structures. Both \( \hat{V}_{mm'} \) and \( \Phi_{m}(t; L; \{q\}) \) are calculated in terms of the genus-1 zero spinor modes \( \varphi_{m}(t) \) defined by (B.3). In so doing

\[
\Phi_{m}(t; L; \{q\}) = \varphi_{m}(t) + \sum_{p} \int_{C_{m}} [(1 - \hat{K})^{-1} \hat{K}]_{pm}(t, t')dt' \varphi_{m}(t')
\]  
(B.6)

where \( \delta_{mm'} \) is the Kronecker symbol while the \( \hat{V}_{mm'} \) matrix is found to be\(^8\)

\[
\hat{V}_{mm'} = -\frac{1}{2} \sum_{p \neq m} \int_{C_{p}} D(t) \varphi_{m}(t)dt \int_{C_{m'}} [(1 - \hat{K})^{-1} \hat{K}]_{pm'}(t, t')dt' \varphi_{m'}(t')
\]

\[
\quad -\frac{1}{2} (1 - \delta_{mm'}) \int_{C_{m'}} D(t) \varphi_{m}(t)dt \varphi_{m}(t),
\]  
(B.7)

\(^8\)This matrix is slightly different from the corresponding matrix in (1). Eq. (B.3) can be also obtained from eq.(59) of (11) using the second of eqs.(52) in (11).
To check that (B.5) satisfies eqs.(18), the first term on the right side of (B.3) is presented by (49), and \( \Phi_m(t; L; \{q\}) \) in (B.4) is presented in the like way as

\[
\Phi_m(t; L; \{q\}) = \sum_{p \neq r} \int_{C_p} K^{(1)}_{lr}(t, t_1; r) dt_1 \int_{C_m} [(1 - \hat{K})^{-1} \hat{K}]_{pm}(t_1, t_2) dt_2 \varphi_m(t_2)
\]

\[+ \delta_{mr} \varphi_m(t) + (1 - \delta_{mr}) \int_{C_m} K^{(1)}_{lr}(t, t_1; r) dt_1 \varphi_m(t_1) \quad (B.8)\]

Indeed, calculating the contribution to (B.8) of the pole term in \( K^{(1)}_{lr}(t, t_1; r) \), one obtains (B.9) using (18). One can see from (B.8) the relations (18) to be true, the scalar function \( J^{(n)}_r(t; \{q\}; L) \) in (18) being

\[
J^{(n)}_r(t; \{q\}; L) = \tilde{J}^{(n)}_r(t; \{q\}; L) - \frac{1}{2} \sum_{m, m'} \Phi^{(n)}_{m'}(L; \{q\}) \hat{V}^{-1}_{mm'} \Phi_{m'}(t; L; \{q\}) \quad (B.9)
\]

where \( \tilde{J}^{(n)}_r(t; \{q\}; L) \) is given by (54) through the genus-1 functions, and

\[
\Phi^{(n)}_{m'}(L; \{q\}) = \Phi_{m'}(t^n; L; \{q\}) - \Phi_{m'}(t; L; \{q\}) = (1 - \delta_{mr}) \int_{C_m} D(t_1) J^{(1)}_r(t_1) dt_1 \varphi_m(t_1)
\]

\[+ \sum_{p \neq r} \int_{C_p} D(t_1) J^{(1)}_r(t_1) dt_1 \int_{C_m} [(1 - \hat{K})^{-1} \hat{K}]_{pm}(t_1, t_2) dt_2 \varphi_m(t_2) \quad (B.10)\]

The genus-1 function \( J^{(1)}_r(t) \) is given by (B.4). To calculate the period matrix, one chooses a fixed parameter \( t_0 \), as it is discussed in Section 3. Then one can check that

\[
J^{(n)}_r(t; \{q\}; L) - J^{(n)}_r(t_0; \{q\}; L) = [\tilde{J}^{(n)}_r(t; \{q\}; L) - \tilde{J}^{(n)}_r(t_0; \{q\}; L)]
\]

\[-\frac{1}{2} \sum_{m, m'} \Phi^{(n)}_{m'}(L; \{q\}) \hat{V}^{-1}_{mm'} \{ \Phi_{m'}(t; L; \{q\}) - \Phi_{m'}(t_0; L; \{q\}) \} \quad (B.11)\]

where the term in square brackets is calculated by (51) for \( j_r = r \) through the genus-1 functions. To prove (B.11), one calculates the contribution to (B.11) of the singular term in the Green function by the method given in Section 3. In addition, eq. (B.4) for \( \Phi_{m'}(t; L; \{q\}) \) is used. From (B.11), the period matrix elements are found to be

\[
2\pi i \omega^{(n)}_{rs}(\{q\}; L) = 2\pi i \tilde{\omega}^{(n)}_{rs}(\{q\}; L) - \frac{1}{2} \sum_{m, m'} \Phi^{(n)}_{m'}(L; \{q\}) \hat{V}^{-1}_{mm'} \Phi^{(n)}_{m'}(L; \{q\})
\]

\[+ (1 - \delta_{rs}) \int_{C_s} D(t_1) J^{(1)}_r(t) dt_1 J^{(1)}_s(t) . \quad (B.12)\]

where \( \tilde{\omega}^{(n)}_{rs}(\{q\}; L) \) is presented by (54) at \( j_r = r \) and \( j_s = s \) in terms of the genus-1 functions. The boson Green function \( R^{(1)}_b(z, z'; \hat{m}) \) in (B.1) is given by (the symbol ”s” is omitted)

\[
\partial_{z'} R^{(1)}_b(z, z'; \hat{m}) = -\sum_n \frac{1}{(z - g_n(z'))(c_n z' + d_n)^2} \quad (B.13)
\]
where the sum is performed over the group products of the Schottky transformation \( g(z) \). In so doing \( g_0(z) = z \), and that negative values of \( n \) are associated with the inverse transformations. The fermion Green functions are given by

\[
R_f^{(1)}(z, z'; \hat{m}; l_1, l_2) = \sum_{n=0}^{\infty} (-1)^{(2l_2+1)n} \frac{1}{1 - k^n \frac{(z-v)(z'-u)}{(z-u)(z'-v)}} - \frac{1}{1 - k^n \frac{(z-u)(z'-v)}{(z-v)(z'-u)}} \left( u - v \right)
\]

where the sum is performed over the group products of the Schottky transformation \( g(z) \). At \( z' \to \infty \)

\[
R_f^{(1)}(z, z'; \hat{m}; l_1, l_2) \to \frac{1}{z - z'} + \frac{\hat{W}_1(z; \hat{m}; l_1, l_2)}{z' - u} + \frac{\hat{W}_2(z; \hat{m}; l_1, l_2)}{(z' - u)^2}.
\]

with corresponding \( \hat{W}_1(z; \hat{m}; l_1, l_2) \) and \( \hat{W}_2(z; \hat{m}; l_1, l_2) \). At \( z \to \infty \)

\[
\hat{W}_2(z; \hat{m}; l_1, l_2) \to -(z - u)\hat{W}_1(z; \hat{m}; l_1, l_2), \quad \hat{W}_1(z; \hat{m}; l_1, l_2) \to \frac{\hat{a}(k;l_1,l_2)(u - v)^2}{(z' - u)^2}
\]

where \( \hat{a}(k;l_1,l_2) \) depends on the multiplier and on the spin structure. Up to the unessential constant term, the non-singular part \( (B.13) \) of the genus-1 Green function \( (B.1) \) is given at \( z' \to \infty \) by (see eq.\((5)\) for definitions)

\[
\tilde{R}_f^{(1)}(t, t'; s)(z' - u) = (\vartheta - \varepsilon(z))(1 + \varepsilon'(z'))\left[ \hat{W}_1(z; \hat{m}; l_1, l_2)(\vartheta' - \mu) + \hat{W}_2(z; \hat{m}; l_1, l_2)\vartheta' \right]
\]

\[
+ [\hat{W}_b(z; \hat{m}) - \vartheta \varepsilon(z)\partial_{\vartheta} \hat{W}_b(z; \hat{m})](1 - z'\vartheta')
\]

where \( \hat{W}_b(z; \hat{m}) \) determines the asymptotics of the boson Green function \( (B.13) \).

**C** Integration measures

In this Appendix we reduce to the convenient for application form eq.(127) from \([1]\) expressing \( \hat{Z}^{(n)}(\{q\}, L) \) through genus-1 functions. First, we transform the scalar superfield contribution on the right side of eq.(112) of \([1]\) as it follows (in notations of \([1]\))

\[
trace \ln(I - \hat{K}^{(1)} + \hat{\varphi} \hat{f}) = trace \ln(I - \hat{K}^{(1)}) - trace \ln[I + \hat{f}_1(I - \hat{K}^{(1)})^{-1} \varphi]
\]

\[
= trace \ln(I - \hat{K}^{(1)}) - \ln \det \hat{V}.
\]

We have used that for any operators \( A_1, A_2 \) and for \( A \) with \( \det A \neq 0 \),

\[
trace \ln[A + A_1 A_2] = trace \ln A + trace \ln[1 + A^{-1} A_1 A_2]
\]

\[
= trace \ln A + (-1)^F trace \ln[1 + A_2 A^{-1} A_1]
\]

(C.2)
where $P = 1$ when both $A_1$ and $A_2$ are the Fermi operators, otherwise $P = 0$. In addition, eq.(50) and eq.(52) of [14] are used. The $K^{(1)}$ operator in (C.11) is the same as Appendix B. In addition the ghost contribution we express now in terms of function $\hat{G}^{(1)}$ given below instead of $G^{(1)}(z, z')$ in eq.(127) from [11]. The desired factor in (55) is given by

$$
\ln \tilde{Z}^{(n)}(\{q\}, L) = -5 \text{trace} \ln(I - \hat{K}) + 5 \ln \det \hat{V} + \text{trace} \ln(I - \hat{G}) - \ln \det \hat{U}
$$

(C.3)

where the $\hat{K}$ operator is the same as in (17) and the $\hat{V}$ matrix is defined by (B.7). The matrix operator $\hat{G}$ is defined in terms of a genus-1 ghost correlator $G^{(1)}_m(t, t'; s)$ defined below in the same manner as $\hat{K}$ is given in terms of $K^{(1)}_m(t, t'; s)$. So $\hat{G} = \{G_{sr}\}$ where $G_{sr}$ is an integral operator vanishing at $s = r$. For $s \neq r$, the kernel of $G_{sr}$ is $\hat{G}^{(1)}_m(t, t'; s)dt'$ defined by (58) for the genus-1 case. Like $\hat{V}$, the elements $\hat{U}_{mn}$ of $\hat{U}$ are defined only for $(m, m')$ assigned to the odd genus-1 spin structures. They are given it terms of 3/2 zero modes $\chi^{(1)}_m(t)$ and in terms of -1/2 genus-1 zero modes $\phi^{(1)}_m(t)$ as it follows

$$
\hat{U}_{m', m} = -\frac{1}{2} \sum_{p \neq m} \int_{C_p} \chi^{(1)}_m(t)dt \int_{C_{m'}} [G_{pm'}(t, t')dt'\phi^{(1)}_{m'}(t')]
$$

$$
-\frac{1}{2} (1 - \delta_{m'm}) \int_{C_{m'}} \chi^{(1)}_m(t)dt \phi^{(1)}_{m'}(t).
$$

(C.4)

The above genus-1 zero modes are given by

$$
\chi^{(1)}_m(t) = -\frac{(u_m - v_m)^2}{[(z_m - u_m)(z_m - v_m)Q^2_m(t)]^{3/2}}, \quad \phi^{(1)}_m(t) = \frac{\vartheta_m Q^2_m(t)\sqrt{(z_m - u_m)(z_m - v_m)}}{(u_m - v_m)}
$$

(C.5)

where $Q_m$ is defined in (8) and the $(z_m, \vartheta_m)$ variables are defined by (3) at $s = m$. The genus-1 function $G^{(1)}_m(t, t'; s)$ is given through the boson Green function $G^{(1)}_b(z, z'; \hat{m}_s)$ and the fermion Green one $G^{(1)}_f(z, z'; \hat{m}_s; l_1, l_2)$ as [11]

$$
G^{(1)}_b(z, z'; \hat{m}_s) = Q^2_s(t)\left[G^{(1)}_{b}(z_s, z_s'; \hat{m}_s)\theta'_s + \theta G^{(1)}_f(z_s, z_s'; \hat{m}_s; l_1, l_2)
-\varepsilon'_s \Upsilon^{(gh)}(\infty, z'_s)Q^{-3}_s(t')\right], \quad \Upsilon^{(gh)}(z, z') = (z - z')G^{(1)}_f(z, z'; \hat{m}_s; l_1, l_2)
$$

(C.6)

where $z_s$ and $\vartheta_s$ are defined by (5) while $\hat{m}_s = (k_s, u_s, v_s)$. The proportional to $\Upsilon^{(gh)}_s$ terms provide decreasing $G^{(1)}_b(z, z'; s)$ at $z \to \infty$ or at $z' \to \infty$. The boson part of the ghost Green function in (C.6) is (see eq.(68) in [11])

$$
G^{(1)}_b(z, z'; \hat{m}) = -\sum_n \frac{1}{(z - g_n(z'))(c_n z' + d_n)^4}
$$

(C.7)

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where the summation is performed over the group products of $g(z)$. For even spin structures
the fermion part in (C.9) expressed in terms of (B.14) as

\[
G_f^{(1)}(z, z'; \hat{m}; l_1, l_2) = \frac{(z-u)(z-v)}{(z'-u)(z'-v)} R_f^{(1)}(z, z'; \hat{m}; l_1, l_2)
- \frac{(z-v)\Sigma_1(z'; \hat{m}; 0, l_2) + \Sigma_2(z'; \hat{m}; 0, l_2)}{(z'-u)(z'-v)} \tag{C.8}
\]

where the last term is calculated in terms of (B.15). In this case

\[
\Sigma_1(z; \hat{m}; l_1, l_2) = 1 + \hat{\Sigma}_1(z; \hat{m}; l_1, l_2), \quad \Sigma_2(z; \hat{m}; l_1, l_2) = z - u + \hat{\Sigma}_2(z; \hat{m}; l_1, l_2). \tag{C.9}
\]

One can check that the function (C.8) goes to zero at $z \to \infty$. Furthermore, at $z' \to \infty$

\[
G_f^{(1)}(z, z'; \hat{m}; l_1, l_2) \to \frac{1}{z - z'} + \frac{\Sigma_{gh}(z; \hat{m}; l_1, l_2)}{(z' - u)^3} \tag{C.10}
\]

where the numerator in the last term is a function of $z$. If both $z \to \infty$ and $z' \to \infty$, then

\[
G_f^{(1)}(z, z'; \hat{m}; l_1, l_2) \to \frac{1}{z - z'} - \frac{a_{gh}(k; l_1, l_2)(u - v)^4}{(z - u)(z' - u)^3} \left[ 3 \frac{z'}{z' - u} - \frac{1}{z - u} \right] \tag{C.11}
\]

where $a_{gh}(k; l_1, l_2)$ depends on the multiplier and on the spin structure. Moreover,

\[
(cz + d)G_f^{(1)}(g(z), z'; \hat{m}; l_1, l_2) - G_f^{(1)}(z, z'; \hat{m}; l_1, l_2) + \frac{1 - \sqrt{k}}{\sqrt{k}} p_\mu(z) \chi_\mu(z'; \hat{m}; l_1, l_2)
- (1 - \sqrt{k}) p_\nu(z'; \hat{m}; l_1, l_2) \tag{C.12}
\]

where $p_\mu(z)$ and $p_\nu(z)$ are given by

\[
p_\mu(z) = \frac{2(z - v)}{u - v}, \quad p_\nu(z) = -\frac{2(z - u)}{u - v} \tag{C.13}
\]

while the depending on $z'$ functions are defined to be

\[
\chi_\mu(z; \hat{m}; l_1, l_2)) = -\frac{(u - v)\Sigma_1(z; \hat{m}; l_1, l_2) + \Sigma_2(z; \hat{m}; l_1, l_2)}{2(z - u)(z - v)},

\chi_\nu(z; \hat{m}; l_1, l_2)) = -\frac{\Sigma_2(z; \hat{m}; l_1, l_2)}{2(z - u)(z - v)}. \tag{C.14}
\]

Eq.(C.12) is non other than eq.(63) of [1] in the genus-1 case. For the odd spin structure, due to the $(-1/2)$ mode, there is no the Green function obeying (C.12). In this case we define the ghost Green function by

\[
G_f^{(1)}(z, z'; \hat{m}; 1/2, 1/2) = G_{(\sigma=1)}^{(1)}(z, z') - 2 \left( \sqrt{\frac{z - u}{z - v}} - 1 \right) \chi_\nu(z'; \hat{m}; 1/2, 1/2),

\chi_\nu(z; \hat{m}; 1/2, 1/2) = -\frac{1}{2\sqrt{(z - u)(z - v)}} \sum_n \frac{1}{(c_n z' + d_n)^2} \tag{C.15}
\]

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where \(G_{(\sigma=1)}^{(1)}(z, z')\) is defined by eq.(69) in [11]. At \(z' \to \infty\)
\[
G_{(\sigma=1)}^{(1)}(z, z'; \hat{m}; 1/2, 1/2) \to \frac{1}{z - z'} + \frac{\Sigma_{gh}(z; \hat{m}; 1/2, 1/2)}{(z' - u)^2}.
\] (C.16)

Moreover, at \(z \to \infty\)
\[
\Sigma_{gh}(z; \hat{m}; 1/2, 1/2) \to \frac{(u - v)^2}{8(z - u)}
\] (C.17)

Under the Schottky transformation \(g(z)\) the above function (C.15) is changed as
\[
(cz + d)G_{(f)}^{(1)}(g(z), z'; \hat{m}; 1/2, 1/2) - G_{(f)}^{(1)}(z, z', \hat{m}; 1/2, 1/2) =
\[
\left[\frac{1 - \sqrt{k}}{\sqrt{k}} p_\mu(z) - (1 - \sqrt{k})p_\nu(z)\right] \chi_\nu(z'; \hat{m}; 1/2, 1/2)
\]
\[
- \frac{\sqrt{(z - u)(z - v)(u - v)}}{[(z' - v)(z' - u)]^{3/2}}. 
\] (C.18)

Eq.(C.18) follows from (C.15) along with eq.(82) in [11]. We show that in this case the left side of (C.18) is obtained in a form of series, which are given in terms of (C.18) as it follows
\[
- \frac{(u - v)^2}{[(z - v)(z - u)]^{3/2}} = c \sqrt{\frac{z - v}{z - u}} \sum_{n = -\infty}^{\infty} \frac{1}{k^{(n+1)/2}Q_n(z)Q_{n+1}(z)} + \frac{1}{k^{n/2}Q_n^2(z)Q_{n+1}(z)},
\]
\[
\chi_\nu(z'; \hat{m}; 1/2, 1/2) = \frac{c}{2} \sqrt{\frac{z - v}{z - u}} \sum_{n = -\infty}^{\infty} \frac{1}{k^{(n-1)/2}Q_n(z)Q_{n+1}(z)} + \frac{1}{k^{n/2}Q_n^2(z)Q_{n+1}(z)}. 
\] (C.19)

Here \(Q_n(z) = c_n z + d_n\) for the group product \(g^n\). Indeed, the right side in the first line of (C.13) is \(\sim 1/z^{3/2}\) at \(z \to \infty\), and so it is proportional to \(3/2\)-zero mode on the left side. To verify the coefficient, the both parts are multiplied by \(\sqrt{(z - u)(z - v)}\). Then the equation is integrated along the corresponding Schottky circle. In the second line, the leading at \(z \to \infty\) term on the left side is equal to corresponding term on the right side. So the right side of the discussed relation may differ from its left side only by the term proportional to \(\chi_\mu(z; \hat{m}; 1/2, 1/2)\). To verify the equation, it is again multiplied by \(\sqrt{(z - u)(z - v)}\) and then it is integrated along the Schottky circle. For even spin structures the ghost function (C.18) is related to \(G_{(\sigma=1)}^{(1)}(z, z')\) in [11] by
\[
G_{(\sigma=1)}^{(1)}(z, z') = G_{(f)}^{(1)}(z, z'; \hat{m}; l_1, l_2) - [p_\mu(z)\chi_\mu(z'; \hat{m}; l_1, l_2) + p_\nu(z)\chi_\nu(z'; \hat{m}; l_1, l_2)]
\]
\[
+ \frac{1}{2} \frac{\sqrt{z - u}}{z - v} [p_\mu(z)(3\chi_\mu(z'; \hat{m}; l_1, l_2) - \chi_\mu(z'; \hat{m}; l_1, l_2))
\]
\[
+ p_\nu(z)(\chi_\mu(z'; \hat{m}; l_1, l_2) + \chi_\nu(z'; l_1, l_2))] 
\] (C.20)

with \(p_\mu(z)\) and \(p_\nu(z)\) being defined by (C.13).
To derive the ghost terms in (C.3) one represents \( \hat{S}_{\sigma}^{(1)} \) in eq.(127) of [11] through \( \hat{G}^{(1)} \). For this purpose one uses eq.(81) of [11] along with eqs. (C.13) and (C.20) of the present paper. The result expression being arranged by (C.22), the desired terms in (C.3) are obtained.

To verify eq.(57), we represent \( G_L^{(n)}(t, t'; \{ q \}) \) in (B.3) as it follows

\[
G_L^{(n)}(t, t'; \{ q \}) = \sum_{s=1}^{n} \tilde{G}_t^{(1)}(t, t'; s) + \sum_{r,s} \int_{C_s} [(1 - \tilde{G})^{-1}]_{rs}(t, t_1)dt_1 \tilde{G}_t^{(1)}(t_1, t'; s)
\]

\[
+ \frac{\vartheta - \vartheta'}{z - z'} - \sum_{m, m'} \phi_m^{(n)}(t; \{ q \}) \hat{U}_{mm'}(t'; \{ q \})
\]

(C.21)

where the sum in the last term is performed over genus-1 odd spin handles. The \( \hat{U}_{mm'} \) matrix elements are defined by (C.4) and the functions in the last term on the right side are calculated in terms of the genus-1 zero modes (C.3) by

\[
\chi_m^{(n)}(t; \{ q \}) = \chi_m^{(1)}(t) - \sum_{p,p'} \int_{C_p} \chi_m^{(1)}(t')dt' \int_{C_{p'}} [(1 - \tilde{G})^{-1}]_{pp'}(t', t_1)dt_1 \tilde{G}(t_1, t),
\]

\[
\phi_m^{(n)}(t; \{ q \}) = \tilde{\phi}_m^{(1)}(t) + \sum_p \int_{C_m} [(1 - \tilde{G})^{-1}]_{pm}(t, t')dt' \phi_m^{(1)}(t')
\]

(C.22)

where \( \tilde{\phi}_m^{(1)}(t) \) is as follows

\[
\tilde{\phi}_m^{(1)}(t) = \phi_m^{(1)}(t) - \frac{1}{2} \tilde{\vartheta}_m Q_m^2(t) [2z_m - u - v] - \frac{\varepsilon'}{8}.
\]

(C.23)

Due to (3) and (C.3), \( \tilde{\phi}_m^{(1)}(t) \) vanishes at \( z \to \infty \). To prove (C.21) one uses the same trick as in (B.3). Then one can verify that \( \phi_m^{(n)}(t; \{ q \}) \) is the superconformal 3/2 tensor under the transformations of the super-Schottky group, and that (C.21) obeys the conditions (63) of the paper [11]. Thus (C.21) is the correct expression for \( G_L^{(n)}(t, t'; \{ q \}) \). In (C.4) and (C.22) the \( \phi_m^{(1)}(t) \) zero mode can be replaced by \( \tilde{\phi}_m^{(1)}(t) \). Indeed the difference of the above quantities has no singularities inside the \( C_r \) contour, the integral of it along \( C_r \) being equal to zero. So one substitutes in eq.(57) the Green functions given by (B.3) and by (C.21) through the genus-1 functions. Further one uses (C.2) considering the \( (m, m') \) sum in (B.3) and in (C.21) as the separable operator \( A_1A_2 \). In so doing (C.3) appears. So (57) is proved.

Applying (C.2) to the non-holomorphic term in (24), one can obtain that

\[
5 \text{trace } \ln \hat{V} = -5 \ln \det \Omega_{\{ q, \bar{q} \}}^{(n)} + 5 \text{trace } \ln \hat{V}
\]

(C.24)

where \( V \) is the same as in (C.3) and the summation over \( (r, s) \) is implied. The \( \tilde{\Omega}_{\{ q, \bar{q} \}}^{(n)} \) matrix is calculated for the period one \( \tilde{\omega}_{rs}^{(n)}(\{ q \}; L) \) in (B.12). The \( \hat{V}_{mm'} \) of \( \hat{V} \) is defined by

\[
\hat{V}_{mm'} = \hat{V}_{mm'} + \Phi_m^{(r)}(L; \{ q \})[\tilde{\Omega}_{\{ q, \bar{q} \}}^{(n)}]^{-1}_{rs} \Phi_m^{(s)}(L; \{ q \})
\]

(C.25)
where $\Phi^{(r)}_m(L; \{ q \})$ is defined by (3.10). From (B.7) one can see that $\hat{V}^T = -\hat{V}$ and therefore, $\hat{V}^T = -\hat{V}$. So $\hat{V}_{mm} = \hat{V}_{mm} = 0$. Then for the degenerated configuration of Section 5 one obtains using (B.7), (3.10), that $\det \hat{V} \sim \rho_1/\rho$ when both $L_1$ and $L_2$ are odd super-spin structures. From (C.4) $\det \hat{U}$ in (C.3) is nullified when $1 < n_1 < n - 1$, all the rest factors being finite. In this case the integration measure is $\sim (\rho_1/\rho)^5$. When both $L_1$ and $L_2$ are odd and either $n_1 = 1$, or $n_2 = n - n_1 = 1$, then $\det \hat{U}$ is nullified due to a presence of the ghost (-1/2) zero genus-1 mode. Then $\det \hat{U} \sim (\rho_1/\rho)^3$ when $n_1 = 1$, and $\det \hat{U} \sim (\rho_1/\rho)^2$ for $n_2 = 1$ and $n > 2$. In this case the integration measure is $\sim (\rho_1/\rho)^2$ and, respectively, $\sim (\rho_1/\rho)^3$. Hence the integration measure is nullified when $L_1$ and $L_2$ are odd. Due to the second term on the right side of (B.3), the Green function is $\sim 1/\sqrt{\rho}$ when one of its argument lies near $z_0$ (see Section 5) another argument being at a finite distance from $z_0$. So the integrand decreases also for the configuration where one from the vertex coordinates goes to $z_0$. So $L_1$ being odd, the degenerated configurations discussed are not able to originate divergences, as it has already been noted in the end of Section 5. In the $1 < n_1 < n - 1$ case the $\sim (\rho_1/\rho)^5$ smallness in the integration measure is compensated when a number of the vertices is $\geq 10$ and 5 vertices lay near $z_0$. This case is relevant for obtaining the contribution to (14) of odd super-spin structures [13].

**D Property of the function invariant under the super-boosts**

We consider the function $\psi\left(\{(x_r | \xi_r)\}, \{(w_s | \ell_s)\}\right)$ invariant under change (86) of its $p$ arguments $(x_r | \xi_r)$ where $r = 1, \ldots, p$. Here $\ell_s = \nu_s - \mu_s$ while $w_s = v_s - u_s - \nu_s \mu_s$. Due to the invariance under the boosts, it depends on the differences $\{(x_r - x_s)\}$. We presents the above function as it follows

$$
\psi\left(\{(x_r | \xi_r)\}, \{(w_s | \ell_s)\}\right) = \xi_i \left( \prod_{j=2}^{p} (\xi_j - \xi_1) \right) \psi_0\left(\{x_r\}, \{(w_s | \ell_s)\}\right) + \sum_{i=2}^{p} \left( \prod_{j=2, j \neq i}^{p} (\xi_j - \xi_1) \right) \psi_i\left(\{\tilde{x}_r\}, \{(w_s | \ell_s)\}\right) + \ldots \quad (D.1)
$$

where $\tilde{x}_r = x_r - x_1$ while the dots encode lower powers of $\{\xi_r\}$. Really $\{(x_r | \xi_r)\}$ are identified with $(u_r | \mu_r)$ and with the vertex coordinates of interest. Applying (86) to (D.1), one obtains the desired relation

$$
\psi_0\left(\{x_r\}, \{(w_s | \ell_s)\}\right) = \sum_{j=2}^{p} (-1)^j \partial_{\tilde{x}_j} \psi_j\left(\{\tilde{x}_r\}, \{(w_s | \ell_s)\}\right). \quad (D.2)
$$
E SL(2) transformations

The general super-conformal transformation is given by

\[ z = f(\hat{z}) + f'(\hat{z})\hat{\theta}\xi(\hat{z}), \quad \hat{\theta} = \sqrt{f'(\hat{z})}\left[(1 + \frac{1}{2}\xi^2)\hat{\theta} + \xi(\hat{z})\right] \tag{E.1} \]

where \( f'(z) = \partial_z f(z) \) while \( f(z) \) is the transition function and \( \xi(z) \) is the Grassmann partner.

For the transformation, which preserving \( z_1, z_2 \) and \( z_3 \), reduces \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) to zeros,

\[ f(\hat{z}) = \hat{z} - \frac{(\hat{z} - z_1)(\hat{z} - z_2)}{(z_3 - z_1)(z_3 - z_2)}\hat{\theta}_3\xi_0(z_3), \quad \xi(\hat{z}) = \frac{\hat{\theta}_1(\hat{z} - z_2)}{(z_1 - z_2)\sqrt{f'(z_1)}} - \frac{\hat{\theta}_2(\hat{z} - z_1)}{(z_1 - z_2)\sqrt{f'(z_2)}} \tag{E.2} \]

where \( \xi_0(z) = \left[\hat{\theta}_1(z - z_2) - \hat{\theta}_2(z - z_1)\right]/(z_1 - z_2) \). Evidently, \( f'(z_1)f'(z_2) = 1 \). If \( \hat{\theta}_1 = \hat{\theta}_3 = 0 \), then \( f(\hat{z}) = \hat{z} \) and \( \xi(\hat{z}) = -\hat{\theta}_2(\hat{z} - z_1)/(z_1 - z_2) \).

Under the \( L(2) \) transformation given by (E.1) with \( f(\hat{z}) = g(\hat{z}) \) and \( \xi(\hat{z}) \equiv 0 \), the spinor derivative of the vacuum correlator (20) is changed as

\[ D(\hat{t}')\hat{X}_{L,L'}(t, \overline{t}; t', \overline{t}'; \{q\}) = [g'(\hat{z})]^{-1/2}D(\hat{t}')\left[\hat{X}_{L,L'}(\hat{t}, \overline{t}; t, \overline{t}; \{q\}) - \hat{X}_{L,L'}(\hat{t}(\infty), \overline{t}(\infty); t, \overline{t}; \{q\})\right] \tag{E.3} \]

where \( \hat{t}(\infty) = (\hat{z} = g^{(-1)}(\infty)|\hat{\theta} = 0) \) while \( g^{(-1)} \) is the transformation inverse to \( g \). The addition is determined by the requirement that the right side of (E.3) vanishes at \( z \to \infty \), just as its left part.

References

[1] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, vols.I and II (Cambridge University Press, England, 1987).
[2] E. Verlinde and H. Verlinde, Phys. Lett. B192 (1987) 95.
[3] J. Atick and A. Sen, Nucl.Phys. B. 296 (1988) 157; J. Atick, J. Rabin and A. Sen, Nucl. Phys. B 299 (1988) 279.
[4] G. Moore and A. Morozov, Nucl. Phys. B 306 (1988) 387;
[5] P. Di Vecchia, K. Hornfeck, M. Frau, A. Ledra and S. Sciuto, Phys. Lett. B211 (1988) 301.
[6] J.L. Petersen, J.R. Sidenius and A.K. Tollstén, Phys Lett. B 213 (1988) 30; Nucl. Phys. B 317 (1989) 109.
[7] B.E.W. Nilsson, A.K. Tollstén and A. Wätterstam, Phys Lett. B 222 (1989) 399.

[8] S. Mandelstam, Phys. Lett. B 277 (1992) 82.

[9] E. Martinec, Phys. Lett. B171 (1986) 189.

[10] G.S. Danilov, Sov. J. Nucl. Phys. 52 (1990) 727 [Jadernaja Fizika 52 (1990) 1143]; Phys. Lett. B 257 (1991) 285.

[11] G.S. Danilov, Phys. Rev. D51 (1995) 4359 [Erratum-ibid. D52 (1995) 6201].

[12] G.S. Danilov, Nucl. Phys. B463 (1996) 443.

[13] G.S. Danilov, Phys. Atom. Nucl. 59 (1996) 1774 [Yadernaya Fizika 59 (1996) 1837].

[14] M.A. Baranov and A.S. Schwarz, Pis’ma ZhETF 42 (1985) 340 [JETP Lett. 49 (1986) 419]; D. Friedan, Proc. Santa Barbara Workshop on Unified String theories, eds. D. Gross and M. Green (World Scientific, Singapore, 1986).

[15] N. Seiberg and E. Witten, Nucl. Phys. B276 (1986) 272.

[16] D.V. Volkov, A.A. Zheltukhin and A.I. Pashnev, Sov. J. Nucl. Phys. 27 (1978) 131 [Jadernaya Fizika 27 (1978) 243].

[17] A.A. Belavin and V.G. Knizhnik, Phys. Lett. B 168 (1986) 201; ZhETF 91 (1986) 364.

[18] G.S. Danilov, Surveys in High Energy Physics 14 (1999) 205 (Talk given on the XXX PNPI Winter School, 1998).

[19] N. Berkovits, Nucl. Phys. B408 (1993) 43.

[20] V.S. Kaplunovsky, Nucl. Phys. B307 (1988) 145; Z. Bern and D.A. Kosower, Phys. Rev. D38 (1988) 1888; M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Phys. Lett. B400 (1997) 52. Z. Bern, L. Dixon and D.A. Kosower, Phys. Rev. Lett 70 (1993) 2677; P. Di Vecchia, A. Lerda, L. Magnea and R. Margotta, Phys Lett. B351 (1995) 445; P. Di Vecchia, A. Lerda, L. Magnea R. Margotta and R. Russo, Nucl. Phys. B469 (1996) 235.

[21] C.L. Siegal, Topics in Complex Function Theory, Vol. 3 (New York, Willey, 1973).

[22] L. Hodkin, J. Physics and Gravity, 6 (1989) 333.

[23] G.S. Danilov, JETP Lett. 58 (1993) 796 [Pis’ma JhETF 58 (1993) 790].

[24] E. Martinec, Nucl. Phys. B 281 (1986) 157.
[25] L. Crane and J.M. Rabin, Commun. Math. Phys. 113 (1988) 601; J.D. Cohn, Nucl. Phys. B306 (1988) 239.

[26] G.S. Danilov, Phys. Atom. Nucl. 60 (1997) 1358 [Yadernaya Fizika 60 (1997) 1495].

[27] V. Alessandrini, Nuovo Cim. 2A (1971) 321;

[28] L. Ford, Automorphic Functions (New York, Chelsea 1951).

[29] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.

[30] P. Di Vecchia, M. Frau, A. Ledra and S. Sciuto, Phys. Lett. B 199 (1987) 49.

[31] G.S. Danilov, Sov. J. Nucl. Phys. 49 (1989) 1106 [Jadernaja Fizika 49 (1989) 1787].