Fragmented and Single Condensate Ground States of Spin-1 Bose Gas

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We show that the ground state of a spin-1 Bose gas with an antiferromagnetic interaction is a fragmented condensate in uniform magnetic fields. The number fluctuations in each spin component change rapidly from being enormous (order \(N\)) to exceedingly small (order 1) as the magnetization of the system increases. A fragmented condensate can be turned into single condensate state by magnetic field gradients. The conditions for existence and method of detecting fragmented states are presented.

Bose-Einstein condensation (BEC) was first introduced as a phenomenon in non-interacting Bose systems. The concept of BEC was generalized to interacting Bose systems in 1956 by Penrose and Onsager. A system of \(N\) Boson is considered BE condensed if its single particle density matrix has one and only one macroscopic eigenvalue (i.e. of order \(N\)). The corresponding eigenfunction is identified as the quantum state macroscopically occupied. This characterization is in good agreement with the recent BEC experiments on magnetically trapped alkali atoms, which are effectively scalar Bosons since their spins are frozen. Recently, optical trapping of Bose condensates has become possible. An optical trap confines all spin states, and the nature of the condensate depends on the magnetic interaction. In the case of \(^{23}\text{Na}\) where the interaction is antiferromagnetic, the single condensate interpretation appears to agree with experiments. Since \(^{23}\text{Na}\) is a spin-1 Boson, the macroscopically occupied state is a three component spinor.

However, in a recent paper, Law, Pu, and Bigelow have pointed out that the Hamiltonian in ref. in zero magnetic field has a singlet ground state, with properties drastically different from those of spinor condensates. They, however, did not discuss how their results will reconcile with the MIT experiment. The spin singlet turns out to be a “fragmented” condensate, meaning that its density matrix has more than one macroscopic eigenvalue. The possibility of fragmented condensate was first discussed by Nozieres and Saint James (NS) who concluded that it cannot occur in homogenous optical Bose systems. The NS conclusion was confirmed by performing Stern-Gerlach experiments along different axes. The origin of the fragmented state turns out to spin conservation. As a result, magnetic field gradients which destroy spin conservation can deform fragmented states toward single condensate states. The degree of deformation depends on particle number and the strength of the field gradient. In the following, we first consider homogeneous Bose gas, where “fragmentation” can be discussed most efficiently. Discussions on trapped gases will follow.

I. Homogenous Spin-1 Bose Gas : Consider a spin-1 Bose gas with Hamiltonian \(\hat{H} = \hat{h} + \hat{V}, \hat{h} = \int \psi^\dagger h(x) \psi \mu \psi^\dagger \psi \mu, h(x) = -\frac{k^2}{2} \nabla^2 + U(x)\delta_{\mu \nu} - \gamma \mathbf{B} \cdot \mathbf{S}_\mu \mathbf{S}_\nu, \hat{V} = \frac{1}{2} \int \psi^\dagger a^\dagger \psi \beta \psi^\dagger \psi \beta [c_0 \delta_{\alpha \beta} + c_2 \mathbf{S}_\alpha \mathbf{S}_\beta \cdot \mathbf{S}_\mu \mathbf{S}_\nu], \) where \(c_0, c_2 > 0, M \) and \(\gamma\) are the mass and gyromagnetic ratio of the Bose gas respectively, and \(\mathbf{B}\) is a uniform magnetic field. The field operator \(\psi^\dagger \psi \mu \) \((\mu = \pm 1, 0)\) can be expanded as \(\phi_\mu (x) = \Omega^{-1/2} \sum_{k \neq 0} e^{i k \cdot x} a_\mu (k)\) where \(\Omega\) is the volume of the system. For simplicity, we shall denote \(a_\mu (k = 0)\) simply as \(a_\mu\). Denoting the part of \(\hat{H}\) containing \(a_\mu\) alone as \(\hat{H}_\mu\) and the rest as \(\hat{H}_{ex}\), we have

\[
\hat{H}_\mu = \frac{c_2}{2 \Omega} (S^2 - 2N) - \gamma \mathbf{B} \cdot \mathbf{S} + C
\]

where \(S = \hat{S}_\mu a_\mu, N = \hat{a}_\mu^\dagger a_\mu\), and \(C = \frac{\gamma}{2 gN} [N^2 - N]\).

To find the ground state \(|\psi\rangle\), we first find the ground state of \(\hat{H}_0\) (denoted as \(|F\rangle\)) and then study condensate depletion effects due to \(\hat{H}_{ex}\). From eq.\(\|\), we see that \(|F\rangle = |S^{total} = S\rangle\), where \(S\) is the integer closest to \((\gamma \mathbf{B} \cdot \mathbf{S})/c_2\), which is the minimum of \(\langle \hat{H}_0 \rangle_F = \langle c_2/2 \rangle S(S+1) - \gamma BS - c_2 S^2 + C\). The equilibrium magnetization is \(S_0 = S/N = \gamma B/c_2 + 0(N^{-1})\) and \(N = N/\Omega\). In the following, we shall only consider the case \(\gamma B < c_2 N\), where \(N/S\) ranges from 0 to 1.

In contrast, the optimum single condensate state in a magnetic field is

\[
|SC\rangle = \frac{1}{\sqrt{N!}} \sum_{N_{1,1}} \left( \frac{N_1}{N} a_{1}^\dagger + e^{i \chi} \frac{N_{-1}}{N} a_{-1}^\dagger \right)^N |\text{vac}\rangle,
\]

where \(\chi\) is a relative phase, \(N_{1,1} = N(1 \pm y)/2\), and \(y\) is the magnetization. (We shall from now on absorb \(\chi\) into the operators for simplicity. The importance of \(\chi\) will be discussed at the end.) The equilibrium magnetization \(y_0\) is obtained by minimizing the energy \(\langle \hat{H}_0 \rangle_{SC} = \frac{c_2}{2 \Omega} N(N - 1)y^2 - \gamma BN y + C \equiv G(y)\), and is \(y_0 = \gamma B/[c_2(N - 1)]\). The energy
difference between $|F\rangle$ and $|SC\rangle$ is $\Delta E = \langle H_o \rangle_{SC} - \langle H_o \rangle_F = G(y_o) - G(s_o) + (c_2 N/2(2 - s_o^2))$. Since $s_o < 1$, the last term in $\Delta E$ is positive. Moreover, $y_o - s_o \approx 0(N^{-1})$, we have $G(y_o) - G(s_o) = \frac{N(N-1)}{2} y_o^2 \sim \frac{\sigma}{\sqrt{N}}$, which is smaller than the last term in $\Delta E$ by a factor of $N$ and can therefore be ignored. This shows that for a homogeneous Bose gas, the states $|F\rangle$ and $|SC\rangle$ are degenerate in the thermodynamic limit ($N \to \infty$, $\Omega \to \infty$, $N/\Omega \to$ finite), since their energies are of order $N$ but their difference ($c_2 N$) is of order 1. The relative stability between $|F\rangle$ over $|SC\rangle$ is therefore very delicate. To discuss this stability, it is necessary to understand the structure of $|F\rangle$, which turns out to be very remarkable.

II. Super- and Coherent- Fragmentation : A simple exercise shows that $\Theta^\dagger = -2a_1^\dagger a_0^2 + a_0^2$ creates a singlet pair of spin-1 Bosons. The ground state $|F\rangle = |S; S\rangle$ is therefore given by

$$|S; S\rangle = \frac{1}{\sqrt{f(Q; S)}} a_1^S \Theta^{1Q} |\text{vac}\rangle, \quad Q = (N - S)/2, \quad (3)$$

where $f(Q; S)$ is the normalization constant

$$f(Q; S) = S!Q!2Q^2(2Q + 2S + 1)!!/(2S + 1)!!.$$

Using eq. (4), it is easy to show that the single particle density matrix of $|F\rangle$ is diagonal, $(\rho F)_{\alpha \beta} = \langle a_\beta^\dagger a_\alpha \rangle_{F} = N_\alpha \delta_{\alpha \beta}$, with

$$N_1 = \frac{N(S + 1) + S(S + 2)}{2S + 3}, \quad N_{-1} = \frac{(N - S)(S + 1)}{2S + 3}, \quad N_0 = \frac{N - S}{2S + 3}. \quad (5)$$

Since $\rho F$ has more than one macroscopic eigenvalue, $|F\rangle$ is a fragmented condensate for all $S < N$. Again using eq. (4), the squared number fluctuation $\langle (\Delta \tilde{N}_1)^2 \rangle = \langle (a_1^\dagger a_1 - \langle a_1^\dagger a_1 \rangle)^2 \rangle$ for spin $\mu = 1$ can be shown to be

$$\langle (\Delta \tilde{N}_1)^2 \rangle = \left( \frac{N}{2S + 3} \right)^2 \left( \frac{S + 1}{2S + 5} \right)^2 + \frac{3N}{2S + 3} \left( \frac{S + 1}{2S + 5} \right)^2.$$

Moreover, we have $\langle (\Delta \tilde{N}_j)^2 \rangle = \langle (\Delta \tilde{N}_0)^2 \rangle/4$ from the relations $\tilde{N}_j = N_j - S$, and $\tilde{N}_0 = N + S - 2N_1$.

Eqs. (3) and (4) show that when $S = 0$, the system has $N_1 = N_0 = N_{-1} = N/3$ with enormous fluctuations $\Delta \tilde{N}_0 \sim N$. On the other hand, both $N_0$ and $\Delta \tilde{N}_0$ shrink rapidly as $S$ increases. When $S$ becomes macroscopic, $N_0$ and $\Delta \tilde{N}_0$ become order 1, whereas $N_{-1}$ remains macroscopic, $N_{-1} \rightarrow (N \pm S)/2$. The exceedingly small fluctuations $\Delta \tilde{N}_0$ means that state $|S; S\rangle$ can be well approximated by

$$|S; S\rangle \sim |N_1, N_{-1}\rangle \equiv \frac{a_1^{N_1} a_{-1}^{N_{-1}}}{\sqrt{N_1! N_{-1}!}} |\text{vac}\rangle \quad (7)$$

To distinguish different fragmented states, we call those with $(\Delta \tilde{N}_0) \sim N$ “super”-fragmented states, and those with $(\Delta \tilde{N}_0) \sim 1$ “coherent”-fragmented state. In the thermodynamic limit, super-fragmented is a singularity which occurs only at $S/N = 0$. All other states with $S/N \neq 0$ are coherent-fragmented.

The origin of the small fluctuations in the coherent-fragmented state can be understood as follows. Noting that $S_{\pm} = \sqrt{2}(a_1^\dagger a_0 + a_0^\dagger a_{-1})$, eq.(8) can be written as $H_F = H_A + H_B$.

$$\hat{H}_A = \frac{c_2}{\Omega} \left[ S_{+}^2 + N \left( 1 + N_{-1}\right) \right] - \gamma BS_{\pm}, \quad (8)$$

and $\hat{H}_B = \frac{c_2}{\Omega} (a_0^\dagger a_0 a_1 + h.c.)$. $\hat{H}_B$ contains all terms which are responsible for transformation among spin species. In $H_A$ we have dropped terms depending only on $N$. Clearly, $H_A$ is minimized when $N_0 = 0$. If $|N_1, N_{-1}, N_0\rangle$ denotes the state with $N_0$ Bosons with spin $\mu$, the ground state of $\hat{H}_A$ is $|N_1, N_{-1}, 0\rangle = |N_1, N_{-1}\rangle$, where $N_{+1} = (N + S)/2$, and $S = \gamma B \Omega/2c_2$. The ground state energy is $E_0 = -c^2 S^2/(2S)$. The effect of $\hat{H}_B$ is to mix in $N_0 \neq 0$ states $|q\rangle = |N_1 - q, N_{-1} + q, 2q\rangle$. To leading order in $N$, we have $\hat{H}_0 = E_0 + H$.

$$\hat{H} = \frac{c_2 N}{\Omega} \sum_{q = 0, 1, 2\ldots} \left[ 2q|q\rangle \langle q + 1| \langle q + 1| + h.c. \right]. \quad (9)$$

where $\lambda_q = \sqrt{(N_1 - q)(N_{-1} - q)}/(2S + 1)$. For $1 < q << N_1, N_{-1}$, $\lambda_q = x(q + 1)$, $x = x(1 - S/N^2)$. To estimate the energy of eq.(9), we consider the following variational state $|\Psi\rangle = \sqrt{\alpha} \sum_k (-1)^k e^{-aq/2}$ with energy $E = \sqrt{\alpha}(\frac{1 - xe^{\alpha/2}}{\alpha} \frac{1 + xe^{-2\alpha}}{1 + 1}).$ The minimum condition is $e^{\alpha/2} = x(1 + \frac{2}{\alpha} + \frac{1}{\alpha^2})$. To leading order in $S/N$, we have $\alpha = \sqrt{2/(S/N)}$. The energy correction $\langle H \rangle_\Psi$ is therefore of order $\sqrt{\alpha}$, which is lower than $E_0$ by a factor of $N$. Thus, to the leading order in $N$, $\hat{H}_0$ can be replaced by $\hat{H}_A$ [eq.(8)] with $N_0 = 0$, with ground state eq.(8).

For later discussions, it is useful to compare the coherent-fragmented state eq.(8) with the single condensate state $|SC\rangle \equiv \langle \Omega \rangle$. Writing $|SC\rangle = \sum_{\ell = -N_1}^{N_1} \left( N_{+\ell} N_{-\ell} \right) \left( N_{+\ell} + N_{-\ell} \right) \left( N_{+\ell} - N_{-\ell} \right) \ell \sqrt{\ell} |\ell\rangle$, where $|\ell\rangle \equiv |N_1 + \ell, N_{-1} - \ell\rangle$, and using the Stirling formula, it is straightforward to show that the single condensate is a Gaussian sum of coherent fragmented states

$$|SC\rangle \approx \frac{(\pi \sigma^2)^{1/4}}{\sqrt{2} \pi \gamma} \sum_{\ell = -N_1}^{N_1} e^{-\ell^2/2\sigma^2} e^{\ell \xi} |\ell\rangle \quad (10)$$

where $\sigma^2 = 2N_1 N_{-1}/N, \xi = (N_{-1} - N_1)/4 \sqrt{\frac{N}{N} \frac{N_{-1}}{N_{-1}}}$. Within the space of $\mu = \pm 1$, the density matrix of $|SC\rangle$ is

$$\rho_{\mu \mu}^{SC} \equiv \langle a_\mu^\dagger a_\mu \rangle_{SC} = \left( \frac{N_1}{\sqrt{N_1 N_{-1} N_{-1}}} \right). \quad (11)$$
The off-diagonal elements $\sqrt{N_1 N_{-1}}$ are absent in $\rho^F$.

**III. Persistence of Fragmentation**: So far, we have ignored condensate depletion (i.e. due to $H_{ex}$), and various realistic atomic physics effects, see below. In the following, we shall focus on coherent-fragmented states as they are most common. Following the method of Huang and Yang \[10\] to extract the dominate condensate depletion for both single condensate and fragmented state have identical Bogoliubov form, which is obtained by replacing the operators $a_{+1}^\dagger (k = 0)$ by the c-number $\sqrt{N_{+1}}$. This means that the thermodynamic degeneracy of $|F\rangle$ and $|SC\rangle$ cannot be lifted by condensate depletion, as it gives rise to same energy change in both cases. This result also applies to the trapped Bosons, in which case the states $(k, -k)$ are replaced by opposite angular momentum states.) As for atomic physics effects such as mixing of different hyperfine states and quadratic Zeeman effects, they all respect spin rotational symmetry. The natural candidate is a magnetic field $B$. Since spin conservation protects an eigenstate of $\hat{H}$, and various realistic atomic physics effects, such as mixing of different hyperfine states and quadratic Zeeman effects, they all respect spin rotational symmetry. The natural candidate is a magnetic field $B$. Since spin conservation protects an eigenstate of $\hat{H}$, this is done by performing a unitary transformation $\hat{U} = \prod_{i=1}^N e^{-i\theta(x_i)s_i}$, where $\theta = \hat{z} \times \hat{B} = G' x \hat{y} + 0(G'^2)$. The interaction $V$ is invariant under $\hat{U}$ because it is a spin conserving contact interaction. However, $\hat{U} \hat{p} \hat{U}^{-1} = \hat{p} + \hbar G' \hat{s}_z \hat{x} + 0(G'^2)$. This causes $\hat{H}_o \rightarrow (\hat{U}^\dagger \hat{H}_o \hat{U})_o = \hat{H}_o + \hat{H}_1$, where $\hat{H}_1 = \epsilon \sum_{i=1}^N (S_i^3)^2$ and $\epsilon = k_B T/\hbar$. When operated on the coherent fragmented states eq.\[12\], $\hat{H}_1$ is reduced to $\hat{H}_1 = -\epsilon / 2 (a_1 a_{-1} + a_1^\dagger a_{-1}^\dagger) + \epsilon N / 2$. The effect of $\hat{H}_1$ on the state $|N_1, N_{-1}\rangle$ is to generate the set $\{|\ell\rangle \equiv |N_1 + \ell, N_{-1} - \ell\}$. Within this set, $\hat{H}_o$ has the tight-binding form

$$\hat{H}'_o = \sum_{\ell = 0, \pm 1, \ldots} \left[ \frac{2\epsilon c \ell^2}{\Omega} |\ell\rangle \langle \ell| - \frac{t_\ell}{2} (|\ell + 1\rangle \langle \ell| + \text{h.c.}) \right]$$

(12)

where $t_\ell \equiv \epsilon \sqrt{(N_1 + \ell + 1)(N_{-1} - \ell)}$. The eigenstates of eq.\[12\], $|\Psi\rangle = \sum_{\ell} \Psi_{\ell} |\ell\rangle$, satisfies the Schrödinger equation $E \Psi_{\ell} = (2c^2/\Omega) \ell^2 \Psi_{\ell} - (t_\ell + t_{\ell+1} + t_{\ell-1}) \Psi_{\ell-1}/2$. Although this equation can be solved numerically, it is more illuminating to consider the following analytic approximation, which turns out to be very accurate. As we shall verify later, the number of $\ell$ terms in the ground state $|\Psi\rangle$ is much less than the typical value of $N_1$ and $N_{-1}$, so that we can replace $(t_\ell + t_{\ell+1})/2 \sim \epsilon \sqrt{N_1 N_{-1}}$, and $(t_\ell - t_{\ell-1})/2 \sim -\epsilon \sqrt{N_1 N_{-1}} \xi$, where $\xi = (N_1^{-1} - N_{-1}^{-1})/4$ as defined before. The Schrödinger equation in the continuum limit is then $(\frac{3\Omega}{2c^2} + \eta^4) \Psi_{\ell} = -\frac{\eta^4}{2} \Psi_{\ell}^{'2} + \eta^4 \Omega \xi \Psi_{\ell-1}^{'2} + \frac{\eta^4}{2} \Psi_{\ell}$, where

$$\eta^4 \equiv \epsilon \sqrt{N_1 N_{-1}} \Omega / (4c^2)$$

(13)

The normalized ground state is

$$|\Psi\rangle = \left( \pi \eta^2 \right)^{-1/4} \sum_{\ell} e^{-\ell^2/2\eta^2} e^{i\ell |\ell\rangle}$$

(14)

with a density matrix $\langle \hat{\rho}_\Psi \rangle^{(\alpha, \beta)} = \langle \Psi | \hat{a}_\alpha^\dagger \hat{a}_\beta | \Psi \rangle (\alpha, \beta = \pm 1)$,

$$\langle \hat{\rho}_\Psi \rangle^{(\alpha, \beta)} = \left( \frac{N_1}{\sqrt{N_1 N_{-1}} e^{-1/4} \eta^2} \right)$$

(15)

The eigenvalues of $\langle \hat{\rho}_\Psi \rangle$ are

$$\lambda_\pm = \frac{1}{2} \left[ N \pm \sqrt{N^2 e^{-1/2} + 2^2 (1 - e^{-1/2})} \right]$$

(16)

For zero field gradient, $\eta \rightarrow 0$, $\lambda_+ \rightarrow \frac{1}{2} (N \pm S)$. For large field gradients, $\eta \gg 1$, eq.\[16\] reduces to eq.\[14\], and $\lambda_+ \rightarrow N$, and $\lambda_- \rightarrow 0$. The systems turns into a single condensate. Note that even for $\eta \sim 5$, the system is essentially a single condensate state. Using the expression of $c_2$ in ref.\[3\], $c_2 = 4 \pi \hbar^2 \Delta_{asc} / M$, $\Delta_{asc} = (a_2 - a_0) / 3$, we have $\eta \equiv \eta_0 (1 - (S / N))^{1/3}$, and $\eta_0 = (\epsilon / \Omega N / (4c^2))^{1/4} = (G'^2 \hbar^2 / 32 \pi (\Delta_{asc} N^{7/3}))^{1/4}$. Since $\eta \sim N^{1/3}$, arbitrarily small field gradients will change a fragmented state into a single condensate for homogenous Bose gas in the thermodynamic limit.

However, trapped Bose gases with $N \lesssim 10^6$ are in mesoscopic rather than thermodynamic limit. In this limit, super- and coherent-fragmented states are no longer singularities in the phase space $(S / N, \epsilon)$. Instead, they occupy a finite region in the phase space which crosses over to single condensate states in a continuous manner. This implies the possibility of observing fragmented condensates in trapped gases.

**V. Trapped Spin-1 Bose Gas**: When $B \neq 0$, different spin components have different spatial extents. For cylindrical traps, we write $\psi_\mu(x) = f_\mu(x) (x) + e^{i\mu x} \phi_\mu(x)$, where $f_\mu$'s are normalized and cylindrically symmetric wavefunction of the condensate with spin $\mu$, and $\phi_\mu$ is the non-condensate part of $\psi_\mu$, satisfying $\int f_\mu \phi = 0$. The exact forms of $f_\mu$ are determined by energy minimization. As in the homogeneous case, we write $\hat{H} = \hat{H}_o + \hat{H}_{ex}$ where $\hat{H}_o$ contains only $\alpha, \beta, \hat{H}_o$ can again be written as $\hat{H}_A + \hat{H}_B$, where $\hat{H}_A$ contains only $S_z, N_1 + N_{-1}, N_0$; and $\hat{H}_B$ has the same form as before but with a different coefficient. $\hat{H}_A$ is again minimized at $N_0 = 0$. The effect of $\hat{H}_B$ is
again to give rise to eq. (10) with a weaker hopping term. Our previous results therefore apply, i.e. to leading order in \( N \), \( H_B \) can be ignored and the ground state with macroscopic magnetization is accurately given by

\[
|F\rangle = |N_1, N_{-1} \rangle, \quad N_{\pm 1} = (N \pm S)/2.
\]

The functions \( f_1 \) and \( f_{-1} \) are determined by minimizing \( E = \langle H_a \rangle_F \). In the presence of a field gradient, a repeat of our previous calculation shows that the Hamiltonian \( \hat{H}_a \) is again given by eq. (13), with \( \Omega^{-1} \rightarrow \frac{1}{N} \int \left[ (f_1^2 + f_{-1}^2) + \frac{a_2}{c_2} (f_1^2 - f_{-1}^2) \right] \), and \( \epsilon \rightarrow \epsilon \int f_1 f_{-1} \).

To estimate the field gradients \( G' \) and magnetization \( S' N \) needed for observing fragmented states, we recall from eq. (13) that

\[
\eta^4 = \frac{NQG^2}{2}\pi^2a_c^2 (1 - \left( \frac{N}{\lambda} \right)^2)^{1/2}
\]

where we have used \( c_2 = 4\pi h^2 \Delta a_{ac}/M \). Next, we note that the maximum and minimum value of \( \lambda_+ \) in eq. (13) is \( N \) and \( (N + S)/2 \). Taking the mean of these values \( \lambda^* = (N + (N + S)/2)/2 \) as a specification of the crossover region between single condensate and fragmented state, we find that fragmented states with \( \lambda^* < \lambda^* \) emerges when \( G' < \left( \frac{8\pi a_c}{\sqrt{N(N+1)} |f_1 f_{-1}|} \right)^{1/2} \). For a \( ^{23}\text{Na} \) gas \( (\Delta a_{ac} \sim 10^{-8}\text{cm}) \) with \( N = 10^6 \), \( S/N = 0.2 \), we have \( \lambda^*/N = 0.8 \). Since \( \Omega \) is roughly the volume of the system, if we take for \( \Omega \int f_1 f_{-1} \sim 10^{-13}\text{cm}^4 \), we have \( \lambda^* < \lambda^* \) when \( G' < 0.46\text{cm}^{-1} \) (13).

Finally, we note that coherent fragmented states cross over to superfragmented states as \( S \) decreases. Setting \( \Delta N_a \sim \sqrt{N} \) as the crossover estimate from super- to coherent- fragmentation. Eq. (13) implies that (for \( N, S >> 1 \)) superfragmentation emerges when \( S/N < 1/\sqrt{N} \). For \( N = 1800 \), it means \( S/N < 0.016 \). Thus, superfragmentation can only be achieved when \( S/N \) is very close to zero. On the other hand, coherent fragmented states are more generic and easier to realize.

VI. Distinguishing fragmented states from single condensate states: Because of the difference in density matrices, the states \( |S, S \rangle \) and \( |F| \) have different rotational properties. Under a spin rotation (with Euler angles \( (\alpha, \beta, \gamma) \)), the density matrix \( \rho_{\mu \nu} = \langle \nu |\rho D^\dagger |\mu \rangle \) becomes \( D\rho D^\dagger \), where \( D(\alpha, \beta, \gamma) \) is the rotational matrix. The number of spin-\( \mu \) particles changes from \( N_\mu = \rho_{\mu \mu} \) to \( N'_\mu = (D\rho D^\dagger)_{\mu \mu} \). For single condensate state of the form eq. (4) before rotation, we have

\[
N'_{+1} = c^4 N_{+1} + s^4 N_{+1} + 2c^2 s^2 \sqrt{N_1 N_{-1}} \cos 2\tilde{\chi}
\]

\[
N'_0 = 2s^2 c^2 (N_1 + N_{-1}) - 4c^2 s^2 \sqrt{N_1 N_{-1}} \cos 2\tilde{\chi}
\]

where \( \tilde{\chi} = \gamma - \chi/2, s = \sin \beta, c = \cos \beta \), and \( \beta \) is the angle between the original and the new quantization axis.

For fragmented states with occupations numbers \( N_1, N_0, N_{-1} \) before rotation, we have

\[
N'_{+1} = c^4 N_{+1} + s^4 N_{+1} + 2s^2 c^2 N_0
\]

\[
N'_0 = 2s^2 c^2 (N_1 + N_{-1}) - N_0 (c^2 - s^2)
\]

In the coherent-fragmented regime where \( N_{\pm 1} >> N_0 \), eqs. (17) to (20) show that the difference in \( N'_{+1} \) between single condensate state and fragmented state is the phase coherence term \( \propto \sin^2 \beta \sqrt{N_1 N_{-1}} \cos 2\tilde{\chi} \). This difference can be detected by first preparing a sample with finite magnetization along a specified axis and then performing a sequence of Stern-Gerlach experiments along a different axis, as well as different axes.

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\[7\] To derive eq. (10), we define \( A_x = (\alpha_1 + \alpha_2)/\sqrt{2} \), \( A_y = \alpha_1(\alpha_1 + \alpha_2)/\sqrt{2} \). Then we have \( A^\dagger \rightarrow \Theta \).
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\[9\] The correct normalization factor should include a factor \( e^{i\sigma} \). However, this factor can be ignored as \( \sigma < 1 \).
\[10\] K. Huang and C.N. Yang, Phys. Rev. 105, 767 (1957).
\[11\] For sufficiently large fields, the quadratic Zeeman effect can force the fragmented state into a single condensate state with only \( N_0 \neq 0 \). This way of recovering the single condensate state is rather trivial. We shall only consider the subtle case where the quadratic Zeeman effect is weak.
\[12\] These statements can be seen most clearly in the local density / Thomas-Fermi approximation. The ratio between \( \lambda_0 \) and the diagonal “potential” term is given by, for \( q < < N_{-1}, N_1 \), \( \sqrt{N_{-1}N_1} \int f_{-1} \rho_{-1} \cdot f_1 / \int f_0 [f_{-1} f_{-1}^2 + N_{-1} / f_{-1}^2] < 1 \).
\[13\] The effect of quadratic Zeeman effect on the crossover region will be studied elsewhere.