Order-Preserving Freiman Isomorphisms

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Abstract

An order-preserving Freiman 2-isomorphism is a map \( \phi : X \to R \) such that \( \phi(a) < \phi(b) \) if and only if \( a < b \) and \( \phi(a) + \phi(b) = \phi(c) + \phi(d) \) if and only if \( a + b = c + d \) for any \( a, b, c, d \in X \).

We show that for any \( A \subseteq \mathbb{Z} \), if \( |A + A| \leq K|A| \), then there exists a subset \( A' \subseteq A \) such that the following holds: \( |A'| \gg_K |A| \) and there exists an order-preserving Freiman 2-isomorphism \( \phi : A' \to [-c|A|, c|A|] \cap \mathbb{Z} \) where \( c \) depends only on \( K \). Several applications are also presented.

1 Introduction

Let \( G \) and \( H \) be additive groups, and let \( A \subseteq G \) and \( B \subseteq H \). A Freiman \( k \)-homomorphism is a map \( \phi : A \to B \) such that

\[
\phi(x_1) + \ldots + \phi(x_k) = \phi(y_1) + \ldots + \phi(y_k)
\]

whenever

\[
x_1 + \ldots + x_k = y_1 + \ldots + y_k.
\]

Such a map \( \phi \) is called a Freiman \( k \)-isomorphism if the converse holds as well. If \( A \) and \( B \) have an ordering, then \( \phi \) is order-preserving when

\[
\phi(a) < \phi(b) \text{ if and only if } a < b.
\]

A Freiman 2-isomorphism will frequently be referred to as just a Freiman isomorphism. Freiman isomorphisms are used to transfer an additive set \( A \) in some arbitrary abelian group \( G \) into a more amenable ambient group or set (such as \( \mathbb{R} \), \( \mathbb{Z}_N \), or \( [1, n] \)) while preserving the additive structure of \( A \). We refer the interested reader to Chapter 5 of [7] for a detailed exposition on the various uses of Freiman isomorphisms.

The main tool we introduce in this paper allows one to find an order-preserving Freiman isomorphism from a set of \( n \) integers to the interval \( [-cn, cn] \cap \mathbb{Z} \) where \( c \) is not too large provided that the original set is additively structured. We call this tool a ‘Condensing Lemma’ since, in a sense, it allows one to view sets with small doubling as dense subsets of an interval.

Theorem 1. [Condensing Lemma] For any \( K > 0 \), there exists a \( c_1, c_2 \) such that if \( A \subseteq \mathbb{Z} \) is such that \( |A + A| \leq K|A| \) then the following holds: there exists \( A' \subseteq A \) with \( |A'| \geq c_1|A| \), and there exists an order-preserving Freiman 2-isomorphism \( \phi : A' \to [-c_2|A'|, c_2|A'|] \cap \mathbb{Z} \).

Since the constants \( c_1 \) and \( c_2 \) depend exponentially on \( K \), we do not bother specifying their exact value. In order to prove the Condensing Lemma, we need a version of the so-called Bogolyubov-Ruzsa lemma which guarantees us a large generalized arithmetic progression \( P \) in \( 2A - 2A \) when \( A \) has a small doubling. For more on this important result, we refer the reader to the recent work by Sanders [5] who gives the best-known bounds for the constants \( c_1, c_2, \) and \( c_3 \) stated below.
Theorem 2 (Bogolyubov-Ruzsa Lemma). Suppose \( A \subseteq \mathbb{Z} \) satisfies \( |A + A| \leq K|A| \). Then, there exists absolute constants \( c_1, c_2, c_3 \) dependent only on \( K \) such that \( 2A - 2A \) contains a proper, symmetric, generalized arithmetic progression \( G \) of dimension at most \( c_1 \) and size at least \( c_2|A| \). Moreover, for each \( x \in G \), there are at least \( c_3|A|^3 \) quadruples \( (a, b, c, d) \in A^4 \) with \( x = a + b - c - d \).

Along with the Plünnecke-Ruzsa estimates, one can also deduce that \( |G| \leq K^4|A| \).

Theorem 3 (Plünnecke-Ruzsa Inequality [7]). If \( |A + A| \leq K|A| \), then for any positive integers \( \ell, m \), we have that \( |\ell A - m A| \leq K^{\ell + m}|A| \).

The proof of the Condensing Lemma consists of first applying Sanders’ theorem so that we may approximate \( A \) by a generalized arithmetic progression \( G = \{ \sum_{i=1}^{k} x_i d_i : |x_i| \leq L_i \} \). Then, after passing to certain subsets, we use some techniques from convex geometry to show that there is a generalized arithmetic progression \( G' = \{ \sum_{i=1}^{k} x_i d'_i : |x_i| \leq L_i/4 \} \) that shares the additive properties of \( G \), but is contained in an interval of length \( O(|G|) \).

After we prove the Condensing Lemma, we provide some applications. Let \( A = \{ a_1 < a_2 < \ldots < a_n \} \) be a finite subset of the integers, and denote the indexed energy of \( A \) as

\[
EI(A) := \{ (i, j, k, l) : a_i + a_j = a_k + a_l \text{ and } i + j = k + l \}.
\]

The reader may be more familiar with the additive energy of a set which can be used to control the size of the sumset:

\[
E(A) = |\{(i, j, k, l) : a_i + a_j = a_k + a_l\}| \geq \frac{|A|^4}{|A + A|}.
\]

We determine the precise relationship between \( E(A) \) and \( EI(A) \). Although the indexed energy of a set has not been directly studied, the additive properties of a set and how they interact with the related indices has appeared in various forms. Solymosi [6] studied the situation when \( a_i + a_j \neq a_k + a_l \) for \( i - j = k - l = c \) for a fixed constant \( c \), and in particular when a set \( A \) has the property that \( a_{i+1} + a_i \neq a_{j+1} + a_j \) for all pairs \( i, j \). Brown et al [3] asked if one finitely colors the integers \( \{1, \ldots, n\} \), must one be forced to find a monochromatic ‘double’ 3-term arithmetic progression \( a_i + a_j = 2a_k \) where \( i + j = 2k \)?

Layout and Notation. In section 2, we state some basic notions from convex geometry, and then we prove the Condensing Lemma. In section 3, we study the indexed energy of a set, providing both an extremal construction of a set with large additive energy and small indexed energy as well as proving a Balog-Szemeredi-Gowers type theorem to find a subset with large indexed energy. Section 4 contains further applications and conjectures related to the Condensing Lemma as well as the indexed energy.

We write \([a, b]\) for \([a, b] \cap \mathbb{Z}\), and similarly for \([a, b), (a, b), \) and \((a, b]\). For two functions \( f, g \), we write \( f \gg g \) if \( f(n) \geq cg(n) \) for some constant \( c \) and \( n \) sufficiently large. We write \( f \gg_K g \) if \( c \) is allowed to depend on \( K \). The doubling constant of a set \( A \) is \( \frac{|A + A|}{|A|} \). A set has small doubling if its doubling constant is \( O(1) \). A generalised arithmetic progression \( G \) is a set \( \{ a + x_1 d_1 + \ldots + x_k d_k : |x_i| \leq L_i \} \); we call \( k \) the dimension of \( G \); \( |G| \) is the volume of \( G \). Moreover, \( G \) is proper if the volume of \( G \) is maximal – \( (2L_i + 1)^k \).

2 Condensing Lemma

The following lemma in conjunction with Theorem 2 will allow us to prove Theorem 1.
Lemma 4. Let $G$ be a proper generalized arithmetic progression of the form

$$G := \{ \sum_{i=1}^{k} a_i d_i : |a_i| \leq L_i \}$$

such that

$$G' := \{ \sum_{i=1}^{k} a_i d_i : |a_i| \leq 4L_i \}$$

is also a proper generalized arithmetic progression. Then, there exists a constant $c = c(k), d'_1, \ldots, d'_k$, and a map $\phi$ with the following properties:

1. $\phi(\sum_{i=1}^{k} a_i d_i) = \sum_{i=1}^{k} a_i d'_i$.
2. $\phi$ is an order-preserving Freiman 2-isomorphism.
3. For any $x \in G$, $|\phi(x)| \leq c|G|$.

In order to prove this lemma we need some definitions and results from convex geometry, from which we refer the reader to [2] as a reference.

2.1 Convex Geometry

A set $K \subset \mathbb{R}^n$ is said to be a convex cone if for all $\alpha, \beta \geq 0$ and $x, y \in K$ we have $\alpha x + \beta y \in K$.

Fact 5. The set of solutions to the system of linear inequalities

$$\sum_{i=1}^{k} a_{i,j} x_i > 0; a_{i,j} \in \mathbb{R} \text{ and } j = 1, \ldots, n$$

is a convex cone.

Proof. Let $x$ and $y$ be solutions to the system of linear inequalities defined above and let $\alpha, \beta \geq 0$. It is trivial to verify that $\alpha x$ and $x + y$ are also solutions to (1). \hfill \Box

For points $x_1, \ldots, x_m \in \mathbb{R}^n$ and non-negative real numbers $\alpha_1, \ldots, \alpha_m$, the point

$$x = \sum_{i=1}^{m} \alpha_i x_i$$

is called a conic combination of the points $x_1, \ldots, x_m$. The set $\text{co}(D)$ is defined as all conic combinations of points in $D \subset \mathbb{R}^n$ and is called the convex hull of the set $D$. For a non-zero $x \in \mathbb{R}^n$ the convex hull of $x$ is called a ray spanned by $x$. A ray $R$ of the cone $K$ is called an extreme ray if whenever $\alpha x + \beta y \in R$ for $\alpha > 0, \beta > 0$ and $x, y \in K$ then $x, y \in R$. An extreme ray is a 1-dimensional face of the cone. A set $B \subset K$ is called a base of $K$ if $0 \notin B$ and for every point $x \in K, x \neq 0$, there is a unique representation $x = \lambda y$ with $y \in B$ and $\lambda > 0$.

Fact 6. Let

$$A := \{ \sum_{i=1}^{k} a_{i,j} x_i > 0 : a_{i,j} \in \mathbb{R} \}$$

be a system of linear inequalities in $\mathbb{R}^k$ with a nonempty set of solution space in the positive quadrant of $\mathbb{R}^k$. Then, the closure of $A$ has a compact base.
Proof. Observe that $A$ is an open set, and since there is at least one solution, it is nonempty. By Fact 5, $A$ is also a convex cone. Let $c(A)$ be the closure of $A$, and since there is at least one solution, it is nonempty. By Fact 5, $A$ is also a convex cone. Let $\text{cl}(A)$ be the closure of $A$, and let $H := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 + \ldots + x_k = 1\}$. We claim that $B := \text{cl}(A) \cap H$ is a compact base of $\text{cl}(A)$. Clearly $B$ is a subset of $\text{cl}(A) - \{0\}$. Let $y \in \text{cl}(A)$ and consider the line $\lambda y$. Since $A$ is a convex cone, this line is contained in $\text{cl}(A)$ for all $\lambda \geq 0$. If this line intersects $B$, then $B$ must be a compact base, but clearly it does at $\lambda = \frac{1}{y_1 + \ldots + y_k}$.

**Theorem 7** (Cor. 8.5 [2]). If $K$ is a convex cone with a compact base. Then every point $x \in K$ can be written as a conic combination

$$x = \sum_{i=1}^{m} \lambda_i x_i, \quad \lambda_i \geq 0, \quad i = 1, \ldots, m,$$

where the $x_i$ each span an extreme ray of $K$.

Lastly, we need the well-known linear algebraic result known as Cramer’s rule.

**Theorem 8** (Cramer’s Rule). Let $A$ be a $k \times k$ matrix over a field $F$ with nonzero determinant. Then, $Ax = b$ has a unique solution given by

$$x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1, \ldots, k$$

where $A_i$ is obtained by replacing the $i$th column in $A$ with $b$.

The broad idea of the proof of Lemma 4 is as follows. We are given a generalized arithmetic progression $G := \left\{ \sum_{i=1}^{k} a_i d_i : -L_i \leq y_i \leq L_i \right\}$. In a sense, this can be identified with the point $(d_1, \ldots, d_k)$. What we would like to find is another generalized arithmetic progression, $H := \left\{ \sum_{i=1}^{k} b_i d'_i : -L'_i \leq b_i \leq L'_i \right\}$ which maintains the same additive structure as $G$, but is much more compact. Viewed another way, we want to find a point $(d'_1, \ldots, d'_k)$ much closer to the origin than $(d_1, \ldots, d_k)$ that also satisfies certain inequalities (these are what maintain the additive structure). Hence, we reduce our problem to finding an integer solution, relatively close to the origin, to a set of linear inequalities.

### 2.2 Proof of the Condensing Lemma

The crux in the proof of the Condensing Lemma is to first prove it for generalized arithmetic progressions; that is, to first prove Lemma 4.

**Proof of Lemma 4.** Given $G$ as in the statement of the Lemma, consider the following set of inequalities:

$$\left\{ \sum_{i=1}^{k} a_i x_i > 0 : a_1 d_1 + \ldots + a_k d_k > 0; -4L_i \leq a_i \leq 4L_i \right\}. \quad (2)$$

We will first prove that if $(d'_1, \ldots, d'_k)$ is an integer solution to the above system of inequalities, then the map $\phi : G \to \mathbb{Z}$ defined by

$$\phi \left( \sum_{i=1}^{k} a_i d_i \right) = \sum_{i=1}^{k} a_i d'_i$$

is an order-preserving Freiman 2-isomorphism.
To see that $\phi$ is order-preserving, if
\[ \sum_{i=1}^{k} a_id_i < \sum_{i=1}^{k} b_id_i \]
for two elements in $G$, then
\[ \sum_{i=1}^{k} (b_i - a_i)x_i > 0 \]
is one of the inequalities in (2) that $(d'_1, \ldots, d'_k)$ must satisfy; so
\[ \phi \left( \sum_{i=1}^{k} a_id_i \right) = \sum_{i=1}^{k} a_id'_i < \sum_{i=1}^{k} b_id'_i = \phi \left( \sum_{i=1}^{k} b_id_i \right). \]
For the converse, if
\[ \sum_{i=1}^{k} a_id'_i < \sum_{i=1}^{k} b_id'_i \] (3)
and
\[ \sum_{i=1}^{k} (b_i - a_i)d_i \leq 0, \]
then we get a contradiction as follows. First, if
\[ \sum_{i=1}^{k} (b_i - a_i)d_i = 0, \]
then $b_i = a_i$ because $G$ is a proper generalized arithmetic progression. Hence, (3) cannot hold in this case. If
\[ \sum_{i=1}^{k} (b_i - a_i)d_i < 0, \text{ then } \sum_{i=1}^{k} (a_i - b_i)d_i > 0 \]
which implies that
\[ \sum_{i=1}^{k} (a_i - b_i)x_i > 0 \]
is an inequality in (2) satisfied by $(d'_1, \ldots, d'_k)$, again contradicting (3).

If we have points in $G$ such that
\[ \sum_{i=1}^{k} a_id_i + \sum_{i=1}^{k} b_id_i = \sum_{i=1}^{k} s_id_i + \sum_{i=1}^{k} t_id_i \]
then
\[ \sum_{i=1}^{k} (a_i + b_i)d_i = \sum_{i=1}^{k} (s_i + t_i)d_i. \] (4)
Moreover, $|a_i + b_i|, |s_i + t_i| \leq 2L_i$. Hence, each side of (4) corresponds to an element in $G'$, and by the fact that $G'$ is proper, we must have that $a_i + b_i = s_i + t_i$ for $i = 1, \ldots, k$. This implies that indeed, $\phi$ is a Freiman 2-homomorphism:
\[ \sum_{i=1}^{k} a_id'_i + \sum_{i=1}^{k} b_id'_i = \sum_{i=1}^{k} s_id'_i + \sum_{i=1}^{k} t_id'_i. \] (5)
For the converse, if (5) holds and (4) does not, then without loss of generality, we may assume

$$\sum_{i=1}^{k} (a_i + b_i - s_i - t_i) d_i > 0.$$ 

However, $a_i + b_i - s_i - t_i \in [-4L_i, 4L_i]$, and so the inequality

$$\sum_{i=1}^{k} (a_i + b_i - s_i - t_i) x_i > 0$$

is satisfied by $(d'_1, \ldots, d'_k)$ which contradicts (5). This proves $\phi$ is a Freiman 2-isomorphism.

Now, we bound the image of $\phi$. Consider the system of inequalities defined in (2); by Fact 5 the solution space forms a convex cone. Moreover, this interior is nonempty since there is a solution – $(d_1, \ldots, d_k)$. Also, $x_i > 0$ is one of our inequalities for all $i = 1, \ldots, k$ so the solution space is in the positive quadrant of $\mathbb{R}^k$. Let $K$ be the closure of the cone defined by the inequalities in (2). By Fact 6, $K$ has a compact base. So, we may apply Theorem 7 to conclude that each $x \in K$ can be represented as conic combinations of the points on its extreme rays. Because all extreme rays have dimension 1, they are each intersections of $k - 1$ linearly independent hyperplanes corresponding to the system ((2)). For each extreme ray, we show how to find an integer point on it; then, taking a conic combination of these integer points will allow us to find an integer point in the interior of the cone.

Let the following hyperplanes define one of our extreme rays:

$$\{a_{i,1} x_1 + \ldots + a_{i,k} x_k = 0 : i = 1, \ldots, k - 1\}. \tag{6}$$

This system of equations will have all the points along our extreme ray as a solution. Hence, we may treat one of the variables $x_i$ as a free variable while the other variables depend on it. Without loss of generality, assume that $x_k$ is the free variable, and let us solve the system for the case when $x_k = 1$. We will use Cramer’s rule. Let

$$\Delta := \begin{vmatrix} a_{1,1} & \cdots & a_{1,k-1} \\ a_{2,1} & \cdots & a_{2,k-1} \\ \vdots \\ a_{k-1,1} & \cdots & a_{k-1,k-1} \end{vmatrix}$$

and let $\Delta_i$ be the determinant of the same matrix with the $i$th row and column replaced by $-a_{j,k}$ for $j = 1, \ldots, k - 1$:

$$\Delta_i := \begin{vmatrix} a_{1,1} & \cdots & a_{1,i-1} & -a_{1,k} & a_{1,i+1} & \cdots & a_{1,k-1} \\ \vdots \\ a_{k-1,1} & \cdots & a_{k-1,i-1} & -a_{k-1,k} & a_{k-1,i+1} & \cdots & a_{k-1,k-1} \end{vmatrix}.$$ 

By Cramer’s rule, the solution to the system is given by $x_i = \frac{\Delta_i}{\Delta}$ for $i = 1, \ldots, k - 1$. By instead choosing $x_k = c$ instead of $x_k = 1$, we see that we can require that any multiple of this is also a solution to (6). Hence, $(|\Delta_1|, \ldots, |\Delta_{k-1}|, |\Delta|)$ is an integer solution to our system that lies along our edge. For convenience, let $\Delta_k := \Delta$.

Now, we may get such an integer solution for each of our extreme rays. Because cone $K$ has interior points, then not all extreme rays belong to the same face, in particular, we may take a set
of $k + 1$ of such rays that do not all lie along the same face and get $k + 1$ integer solutions as we did above. Call these solutions $p_1, \ldots, p_{k+1}$. We can bound the entries of $p_i$ by using a trivial bound on the determinant of our matrices formed above. We have that for $i = 1, \ldots, k$, since each entry $|a_{i,j}| \leq 4L_j$, the determinant is bounded as follows:

$$|\Delta_i| \leq 4^k k! \prod_{j \neq i} L_j$$

Moreover, the sum, $p_1 + \ldots + p_{k+1} =: (d'_1, \ldots, d'_{k+1})$ does not belong to any of the faces of $K$; so, it belongs to the interior of the cone, and hence, satisfies (2). Lastly, this implies that the image of $\phi$ is bounded as follows:

$$|\phi(\sum_{i=1}^{k} y_id_i)| = |\sum_{i=1}^{k} y_id'_i| \leq \sum_{i=1}^{k} L_id'_i \leq \sum_{i=1}^{k} L_i(k+1)(4^k k! \prod_{j \neq i} L_j) \leq (k+1)!4^k \prod_{j=1}^{k} L_j.$$ 

So if $g \in G'$, $\phi(g) \in [-4^k(k+1)!|G|, 4^k(k+1)!|G|]$. □

The proof of Theorem 1 follows easily from applying Theorem 2 to a set with small doubling. We need the following trivial fact.

**Fact 9.** Let $\phi_1$ be an order-preserving Freiman isomorphism, and let $\phi_2(x) = x + a$. Then $\phi_2, \phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ are order-preserving Freiman isomorphisms.

**Proof of Theorem 1.** Let $A \subseteq \mathbb{Z}$ be such that $|A + A| \leq K|A|$. All constants $c_i$ in the following depend only on $K$. We may apply Theorem 2 to $A$ to get a generalized arithmetic progression $G \subseteq 2A - 2A$ with $|G| \geq c_1|A|$, dimension at most $c_2$, and for each $x \in G$, there are at least $c_3|A|^3$ quadruples $(a, b, c, d) \in A^4$ with $x = a + b - (c + d)$. Hence,

$$|\{(a, b, c, d) \in A^4 : a + b - (c + d) \in G\}| \geq c_3|A|^3|G|.$$ 

So, we can find a triple $(b, c, d)$ such that

$$|\{a \in A : a + b - (c + d) \in G\}| \geq c_3|G|.$$ 

Let $A' := \{a \in A : a + b - c - d \in G\}$. Let $G' = G - b + c + d$. So, $A' \subseteq G'$, $|A'| \geq c_3|G'|$, and $G'$ is a proper generalized arithmetic progression of the same size and dimension as $G$.

Denote $G'$ as

$$G' = \{u + \sum_{i=1}^{k} x_id_i : |x_i| \leq L_i\}.$$ 

By Fact 9, we may assume $u = 0$, else simply shift everything in $A'$ and $G'$ by $-u$, and work with those sets instead. Let

$$G'' := \left\{ \sum_{i=1}^{k} x_id_i : |x_i| \leq \lfloor L_i/4 \rfloor \right\}.$$ 

Apply Lemma 4 to $G''$ to get an order-preserving Freiman isomorphism $\phi : G'' \to [-c_4|G''|, c_4|G''|]$. We have that $A' \subseteq G'$, but $A' \cap G''$ may not be large. However, by considering the $4^k$ different translates, $G'' + v$, where $v = j\lfloor L_i/4 \rfloor$ for $j = 0, 1, 2, 3$, $i = 1, \ldots, k$ there exists an integer $v$ such that

$$|A' \cap (G'' + v)| = |(A' - v) \cap G''| \gg_k |A'|.$$ 

Let $A'' := A' \cap (G'' + v)$. So, $\phi$ is an order-preserving Freiman isomorphism from $A'' - v$ to $[-c_4|G''|, c_4|G''|]$, and by Fact 9, $\phi_0(x) := \phi(x) - v$ is an order-preserving Freiman isomorphism from $A''$ to $[-c_4|G''|, c_4|G''|]$. By Theorem 3, since $G \subseteq 2A - 2A$ and $|A + A| \leq K|A|$, we must have $|G| \ll_K |A|$, and so $[-c_5|A''|, c_5|A''|] = [-c_5|A''|, c_5|A''|]$, proving the lemma. □
3 Indexed Energy

One always has the following relationship between the additive energy and indexed energy:

\[ |A|^2 \leq EI(A) \leq E(A) \leq |A|^3. \]

If \( A \) is an arithmetic progression the relationship is strengthened to \( EI(A) = E(A) \). Moreover, for an arithmetic progression \( A \), \( E(A) \) is maximized. Thus, it is natural to wonder if one loosens the restriction to \( E(A) \gg |A|^3 \) then is \( EI(A) \gg |A|^3 \)? We provide a counterexample to show that this is false.

**Theorem 10.** There exists an integer \( N \) such that for every \( n \geq N \), there exists \( A \subseteq [n] \) such that, \( E(A) \geq \frac{1}{18}|A|^3 \) and \( EI(A) \leq 2000|A|^2(\log |A|)^2 \).

Thus, one can indeed have the additive energy \( \Omega(|A|^3) \) while the indexed energy is \( O(|A| \log |A|)^2 \). However, when the additive energy is large, it turns out that one can still pass to a large subset \( A' \subseteq A \), \( |A'| = \Omega(|A|) \), which has indexed energy \( \Omega(|A'|^3) \). We note that when passing to a subset, the subset does not inherit the same indices as the superset, but rather it is reindexed in the natural way. Hence, \( EI(A') \) is not bounded from above or below by \( EI(A) \).

**Theorem 11.** For any \( K > 0 \), there exists \( c_1, c_2 \) dependent only on \( K \) such that if \( A \) is a finite set of integers with \( |A + A| \leq K|A| \) then the following holds. There exists an \( A' \subseteq A \) such that \( EI(A') \geq c_1|A'|^3 \) and \( |A'| \geq c_2|A| \).

We mention in passing that the condition that \( |A + A| \ll K |A| \) may easily be loosened to \( E(A) \gg K |A|^3 \) by applying the following well-known result of Balog-Szemerédi [1] and Gowers [4] to pass to a subset with small doubling.

**Theorem 12** (Balog-Szemerédi[1], Gowers[4]). For any \( K > 0 \), there exists \( c_1, c_2 \) such that if \( A \subseteq \mathbb{Z} \) is such that \( E(A) \geq K|A|^3 \) then there exists \( A' \subseteq A \) with \( |A'| \geq c_1|A| \) and \( |A' + A'| \leq c_2|A'| \).

3.1 Indexed energy in subsets of \([1, n]\)

It turns out that if \( A \) is a dense subset of an interval, then there is a simple algorithm that can find a subset \( A' \subseteq A \) with \( |A'| \gg |A| \) and \( EI(A') \gg |A'|^3 \). Thus, the general case may then be quickly deduced by applying the Condensing Lemma. We first begin with a lemma that states, loosely speaking, that if \( A \) is a dense subset of \([1, n]\), then one can choose a large subset \( A' \subseteq A \) that is equidistributed over the interval.

**Lemma 13.** For every \( \delta > 0 \), there exists \( c_1, c_2, c_3, N \) such that if \( A \subseteq [1, n] \) with \( n > N \) and \( |A| = \delta n \), then the following holds. There exists an \( A' \subseteq A \), \( |A'| \geq c_1|A| \) and for \( c_3|A|^2 \) pairs of integers \( 0 \leq i, j < n/c_2 \), we have that

\[ |A' \cap [ic_2, jc_2]| = j - i. \]  

(7)

It is easy to establish that a set with property (7) has large indexed energy.

**Lemma 14.** For every \( \delta > 0 \), there exists \( c_0, c_1, N \) such that if \( A \subseteq [1, n] \) with \( n > N \) sufficiently large and \( |A| = \delta n \), then \( A \) has a subset \( A' \subseteq A \) with \( |A'| \geq c_1|A| \) and \( EI(A') \geq c_0|A|^3 \).
Proof of Lemma 13. It suffices to prove that there exists an $A' \subseteq A$ and $c_1, c_2, c_3$ dependent on $\delta$ such that the following holds: $|A'| \geq c_1|A|$, for $c_3|A|$ integers $0 \leq i < n/c_2$,

$$|A' \cap [0, ic_2)| = i.$$ 

Once this statement is established, then for any pair of integers $i, j$ satisfying the above, we have $|A' \cap [ic_2, jc_2)| = j - i$. This would prove the statement of the lemma.

Denote $A = \{a_1 < a_2 < \ldots < a_{\delta n}\}$. Let $d = \lceil \frac{n}{\delta} \rceil$. We may assume $d|n$, if not, replace $n$ with $n' \leq 2n$ were $d|n'$. Such an $n'$ exists if $n$ is sufficiently large, and the proof will proceed in the same manner with only a slight modification in our constants $c_1, c_2, c_3$. Let $I_j = ([j - 1)d, jd)$ for all $j = 1, \ldots, \frac{n}{d}$. Let $A_j = A \cap I_j$. We pick our subset $A'$ as follows:

- **Step 1:** If $A_1 \neq \emptyset$ then let $X_1 = \{a_1\}$. Else, $X_1 = \emptyset$.
- **Step k:** If $|A_k \cup X_{k-1}| \leq k$, then $X_k := A_k \cup X_{k-1}$. Else, arbitrarily choose $Y \subseteq A_k$ so that $|Y \cup X_{k-1}| = k$ and then let $X_k := Y \cup X_{k-1}$.

It is clear this algorithm ends after $\frac{n}{d}$ steps. Let $A' = X_{\frac{n}{d}}$.

To prove that $A'$ satisfies the conclusion of the lemma, we analyze the algorithm as follows. First, note that $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_{\frac{n}{d}} = A'$ and $|X_i| \leq i$ for all $i$. Now, the sets $X_i$ for which $|X_i| = i$ we will call good, and the others we will call bad. Note that if $X_i$ is good, then $|A' \cap [0, id)| = i$; hence, showing that lots of $X_i$ are good will prove the lemma. Let $J = \{j_1, j_2, \ldots, j_k\}$ be the set of indices such that $X_{j_i}$ is good. Observe that for indices between $j_i$ and $j_{i+1}$, we must not have enough elements to make any of those corresponding sets good. More precisely,

$$|A_{j_i+k}| \leq k + \sum_{s=1}^{k-1} |A_{j_i+s}|.$$ 

This implies that

$$\left| \bigcup_{k=1}^{j_{i+1} - j_i - 1} A_{j_i+k} \right| \leq j_{i+1} - j_i - 2.$$

So, we must have that

$$\left| \bigcup_{i=1}^{k-1} \bigcup_{s=1}^{j_i+1 - j_i - 1} A_{j_i+s} \right| \leq \sum_{i=1}^{k-1} (j_{i+1} - j_i - 2) = j_k - j_1 - 2(k - 1) \leq j_k \leq \frac{n}{d}.$$ 

Thus, we have that $\delta n - \frac{n}{d} \geq \delta n/2$ elements of $A$ are distributed over good intervals. Since each interval is of length $d$, then we must have that $k$, the number of good intervals, is at least

$$\frac{\delta n}{2d} \geq \frac{n\delta^2}{4}.$$ 

This in turn gives us a lower bound on $|A'| = j_k \geq k \geq \frac{n\delta^2}{4} = \frac{4}{\delta}|A|$.

Proof of Lemma 14. Apply Lemma 13 to $A$ to get $A'$, $c_1, c_2, c_3$ as in the lemma. Let $A' = \{b_1 < b_2 < \ldots < b_{\delta n}\}$. Let $J = \{j : |A' \cap [0, c_2j)| = j\}$. We know that $|J| \geq c_2|A|$. Now, let $A'' = \{b_j : j \in J\}$. Since $EI(A') \geq |\{(i, j, k, l) \in J^4 : b_i + b_j = b_k + b_l \text{ and } i + j = k + l\}|$, we will simply work with these quadruples from $A''$. However, our final set will still be $A'$ since we need to keep the indices of elements the same as they were in $A'$. 

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For all of the following, $b_i$ will be assumed to be from $A''$. Let $t \in \{2, \ldots, 2m\}$. For $t \leq m$, there are $t - 1$ pairs $(i, j) \in [1, m] \times [1, m]$ such that $i + j = t$. For $t > m$, there are $2m - (t - 1)$ pairs $(i, j) \in [1, m] \times [1, m]$ such that $i + j = t$. Let $\alpha_t$ be defined so that for $t \in \{2, \ldots, 2m\}$ there are $\alpha_t(t - 1)$ pairs $(i, j) \in J \times J$ with $i + j = t$ and there are $\alpha_t(2m - (t - 1))$ such pairs for $t \in \{m + 1, \ldots, 2m\}$. Observe that for such pairs $(i, j) \in J \times J$, we have $b_i + b_j \in [(t - 2)d, td]$. Thus, there are only $2d$ values that $b_i + b_j$ can take. For every $i \in [0, 2d - 1]$, let $t_i$ denote the number of pairs $(i, j) \in J \times J$ with $b_i + b_j = (t - 2)d + i$. We can bound the indexed energy of $A'$ as follows:

$$EI(A') \geq \sum_{t} \sum_{i=0}^{2d-1} t_i^2 = \sum_{t=2}^{m} \sum_{i=0}^{t-1} t_i^2 + \sum_{t=m+1}^{2m} \sum_{i} t_i^2$$

Using Cauchy-Schwarz, one has

$$\geq \frac{1}{2d} \left( \sum_{t=2}^{m} (\alpha_t(t - 1))^2 + \sum_{t=m+1}^{2m} (\alpha_t(2m - t + 1))^2 \right)$$

Using Cauchy-Shwarz again,

$$\geq \frac{1}{2d \cdot m} \left( \left( \sum_{t=2}^{m} \alpha_t(t - 1) \right)^2 + \left( \sum_{t=m+1}^{2m} \alpha_t(2m - t + 1) \right)^2 \right)$$

Since

$$\sum_{t=2}^{m} \alpha_t(t - 1) + \sum_{t=m+1}^{2m} \alpha_t(2m - t + 1) = |J|^2$$

one of the sums must be at least $|J|^2/2$. Hence, we have that

$$EI(A') \geq \frac{|J|^4}{2md} = c_0 |A|^3$$

for some constant $c_0$ depending only on $\delta$. \qed

Now, we are ready to prove Theorem 11.

**Proof of Theorem 11.** Let $A$ be a finite subset of integers with $|A + A| \leq c|A|$. All constants $c_i$ in the following depend only on $c$. Apply Theorem 1 to $A$ to get a set $A' \subseteq A$ with $|A'| \geq c_1|A|$ and an order-preserving Freiman $\phi : A' \to [\pm c_2|A'|, c_2|A'|]$. We may assume at least one third of the elements are in $[1, c_2|A'|]$ or simply shift $A'$ by $v = c_2|A'|$. Apply Lemma 14 to $\phi(A')$ to conclude that $EI(\phi(A')) \geq c_3|\phi(A')|^3 = c_3|A'|^3$. It is easy to see that $EI(\phi(A')) = EI(A')$ since $\phi$ is an order-preserving Freiman 2-isomorphism, so the result follows. \qed

### 3.2 An Extremal Construction

The proof of Theorem 10 follows from the following lemma.

**Lemma 15.** Let $n \in \mathbb{N}$, and let $p \in (1, 2)$ and denote $p = 1 + \epsilon$. Let $A = \{ [a^p] : 1 \leq a \leq \lfloor n^{1/p} \rfloor \}$. Then, $EI(A) \leq 16\epsilon^{-1}n^2 \log n$. 

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Proof of Lemma 15. Let \( x, y \in [1, \lfloor n^{1/p} \rfloor] \) with \( x + 1 < y \). The main part of the argument is to establish the following bound:

\[
x^p + y^p - (x + 1)^p - (y - 1)^p > \frac{\epsilon(y - x)}{2y}
\]

(8)

For now, assume (8) holds. If \( x + y = z + w \), then by convexity, \( x^p + y^p \neq z^p + w^p \) unless \( z = x \) and \( y = w \) or vice versa. However, it may happen that \( x + y = z + w \) and \( [x^p] + [y^p] = [z^p] + [w^p] \). Since \( [a^p] = a^p - \lfloor a \rfloor \), where \( [a^p] \) is the noninteger part of \( a^p \), we must have that if \( x + y = z + w \) and

\[
[2^p] + [y^p] = [z^p] + [w^p]
\]

then

\[
|x^p + y^p - z^p - w^p| < 2.
\]

So, fixing an \( x \) and a \( y \), we can bound how many other pairs \( z \) and \( w \) can have \( z + w = x + y \) and \( [z^p] + [w^p] = [x^p] + [y^p] \). More specifically, we find the largest \( t \) such that

\[
x^p + y^p - (x + t)^p - (y - t)^p < 2.
\]

Using (8), the triangle inequality, and letting \( k = y - x \) we get that

\[
x^p + y^p - (x + t)^p - (y - t)^p \geq \frac{\epsilon k}{2y} + \frac{\epsilon(k + 2)}{2(y - 1)} + \ldots + \frac{\epsilon(k + 2(t - 1))}{2(y - (t - 1))}
\]

Each term in the sum is greater than or equal to \( \frac{\epsilon k}{2y} \), so we get a lower bound of \( \frac{4k}{2y} \). So, if \( t \geq \frac{4y}{\epsilon(y - x)} \), then we cannot have

\[
[x^p] + [y^p] = [(x + t)^p] + [(y - t)^p].
\]

This allows us to conclude that any quadruple \((x, y, z, w)\) with \( x + y = z + w \), with \( x < z < w < y \), \( z < x < y < w \), \( w < y < x < z \), or \( y < w < z < x \) we must have that \( |z - x| < \frac{4y}{\epsilon(y - x)} \). Accounting for an extra factor of 2 for when \( x < w < z < y \) and so on, we can bound the indexed energy of \( A \)

\[
EI(A) \leq 2 \sum_{y} \sum_{x < y} \frac{4y}{\epsilon(y - x)}
\]

Estimating this summation by using the harmonic series gets us that

\[
EI(A) \leq \frac{16}{\epsilon} n^2 \log n
\]

concluding the proof assuming that (8) holds.

Now, we work to establish (8). First, since \( f(x) = x^p \) is convex for \( p > 1 \), it is easy to establish the following bound for any \( k > 1 \):

\[
p(x + k)^{p-1} > (x + 1)^p - x^p > px^{p-1}
\]

Assuming \( p = 1 + \epsilon < 2 \), we have that \( x^{p-1} \) is concave. Doing a similar analysis for \( g(x) = x^{p-1} \), we get that

\[
(p - 1)x^{p-2} > (x + k)^{p-1} - x^{p-1} > (p - 1)(x + k)^{p-2},
\]

Let \( k = y - x \), and we have

\[
x^p + y^p - (x + 1)^p - (y - 1)^p =
\]
Since $x = y - k$, we have

$$p((y - 1)^{p-1} - (y - k + 1)^{p-1} > p((k - 2)(p - 1)(y - 1)^{p-2}) > \frac{\varepsilon k}{2y}$$

where we remind the reader $p = 1 + \epsilon, \epsilon \in (0, 1)$.

Theorem 10 follows by letting $\epsilon = \frac{1}{\log n}$.

**Proof of Theorem 10.** Let $A$ be as in the above lemma, let $\epsilon = \frac{1}{\log n}$. Then, for $n$ sufficiently large

$$|A| = \lfloor n^{1+\epsilon} \rfloor = \left\lfloor n^{\frac{1}{1+\epsilon}} \right\rfloor = \left\lfloor \frac{n}{e} \cdot n^{\frac{1}{1+\log n}} \right\rfloor \geq \left\lfloor \frac{n}{e^2} \right\rfloor \geq \frac{n}{9}.$$ 

So, $A \subseteq [1, n]$, $|A| = \frac{n}{9}$, and $A + A \subseteq [1, 2n]$. Thus, $|A + A| \leq 2n \leq 18|A|$. Hence,

$$E(A) \geq \frac{|A|^4}{|A + A|} \geq \frac{|A|^3}{18}.$$ 

By the lemma above, for $A$ sufficiently large,

$$EI(A) \leq 16n^2(\log n)^2 \leq 16 \cdot (9|A|)^2(\log 9|A|)^2 \leq 1296|A|^2(\log 9|A|)^2 \leq 2000|A|^2(\log |A|)^2.$$

\[\square\]

### 4 Further Applications and Conjectures

Since $|(A \times B) + (A \times B)| = |A + A||B + B|$, it is obvious that if $|A + A| \leq K|A|$ and $|B + B| \leq K|A|$, then for any $C \subseteq A \times B$ of size $\delta|A||B|$, one has $|C + C| \ll_K |C|$. However, if $|C| = O(\sqrt{|A||B|})$, one has little control of $|C + C|$. Does there exist a $C \subseteq A \times B$ with $|C| = c\sqrt{|A||B|}$, and $|C + C| \ll_K |C|$? Clearly one could simply take $C = \{(a, b) : a \in A\}$ for a fixed $b \in B$. If we forbid such sets lying on vertical or horizontal lines by additionally requiring that for any distinct $(x, y), (z, w) \in C$ we have $(x-z)(y-w) > 0$, the answer is not as obvious.

For a set $C \subseteq A_1 \times \ldots \times A_k$, call $C$ a **diagonal set** if for any distinct pairs of elements $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in C$, one has $x_i - y_i > 0$ for all $i$ or $x_i - y_i < 0$ for all $i$.

**Theorem 16.** For any $k, K \in \mathbb{N}$, there exists $c_1, c_2$ such that the following holds. Let $A_1, \ldots, A_k \subseteq \mathbb{Z}$ be sufficiently large sets of size $n$ such that $|A_i + A_i| \leq K|A_i|$ for all $i = 1, \ldots, k$. Then, there exists a diagonal set $C \subseteq A_1 \times \ldots \times A_k$ such that $|C + C| \leq c_1|C|$ and $|C| = c_2(|A_1| \ldots |A_k|)^{1/k}$.

**Proof.** We may apply the Condensing Lemma to each $A_i$ individually to find constants $c_{1,i}, c_{2,i}$ depending on $K$ such that there exists a subset $A'_i \subseteq A_i$ that is Freiman isomorphic to a set $B_i \subseteq [0, c_1_kn]$, and $|A'_i| \geq c_2,i n$. Let $c_1$ be the maximum of $\{c_{1,i} : i = 1, \ldots, k\}$ and let $c_2$ be the minimum of $\{c_{2,i} : i = 1, \ldots, k\}$. So, we may view all the $B_i$ as being dense in the interval $[0, c_1 n]$. Next, we claim that there exists $t_1, \ldots, t_k \in \mathbb{Z}$ such that

$$\left|\sum_{i=1}^k (B_i + t_i)\right| \geq \frac{c_2^k}{2^{k-1}} n.$$
We prove this by induction on \(k\). For \(k = 1\), it is trivial. For the induction step, let \(X, Y \subseteq \{1, n\}\) be of size \(\delta_1 n\) and \(\delta_2 n\) respectively. Then,
\[
\sum_{t = -(n-1)}^{n-1} |X + t \cap Y| = |X||Y| = \delta_1\delta_2 n^2.
\]
Hence, there exists a \(t\) such that
\[
|(X + t) \cap Y| \geq \frac{\delta_1\delta_2}{2} n.
\]
Letting \(X := B_k\) and \(Y := \cap_{i=1}^{k-1} B_i + t_i\) finishes the inductive argument. Now, let \(C' = \cap_{i=1}^{k} B_i + t_i\) for such a set of \(t_i, i = 1, \ldots, k\). Denote \(C := \{x_1 < \cdots < x_m\}\). We let \(B\) be the following set:
\[
B := \{(x_i - t_1, x_i - t_2, \ldots, x_i - t_m) : i = 1, \ldots, m\}.
\]
Since \(x_i - t_j \in B_j\), we have that \(C \subseteq B_1 \times \cdots \times B_k\). Since \(x_i - t_j > x_\ell - t_j\) for \(i > \ell\), \(C\) must be diagonal. Also, \(|C| = |C'| \in \left[\frac{k^2}{2} n, n\right]\). Lastly, it is easy to see that
\[
|C + C'| = |C' + C'| \leq 2n = \frac{2^k}{c_2^2} |C'|.
\]

Although the above application is similar in spirit to the indexed energy problem – letting \(A \times B := A \times \{1, |A|\}\) – there are several subtle differences. Mainly, in the indexed energy problem, when we pass to a subset, we are forced to reindex the set in a very specific way. The following conjecture however would be general enough to imply Theorem 11.

**Conjecture 17.** Let \(A, B \subseteq \mathbb{Z}\) be sets of size \(N\) such that \(|A + A|, |B + B| \leq KN\). Then, there exists \(c_1, c_2\) depending only on \(K\) such that the following holds. There exists an \(A' \subseteq A\) with \(|A'| \geq c_1|A|\), and if we denote \(A' := \{a_1' < \cdots < a_k'\}\) and \(B := \{b_1 < \cdots < b_n\}\), then
\[
|\{(a_1', a_2', a_3', \ldots, a_k') : a_i' + a_j' = a_k' + a_\ell' \text{ and } b_i + b_j = b_k + b_\ell\}| \geq c_2|A'|^3.
\]

Conjecture 17 is true in the case where \(B = \{1, N\}\) (or any arithmetic progression of size \(N\)) since this then becomes the indexed energy result. It would be interesting to know whether the conjecture is even true in the case where \(B\) is a generalized arithmetic progression of dimension 2.

Another problem closely related to the indexed energy problem is as follows. Let \(A \subseteq \mathbb{Z}\) and let \(f : A \to \mathbb{Z}\) be such that \(|f(A) + f(A)| \leq c|A|\), and \(|A + A| \leq c|A|\). Let
\[
E_f(A) := \{(a, b, c, d) : a + b = c + d, f(a) + f(b) = f(c) + f(d)\}.
\]
When \(f\) is the indexing function, \(E_f(A)\) becomes \(EI(A)\). What is the relation between \(E_f(A)\) and \(E(A)\)? Does there always exist an \(A' \subseteq A\) with \(|A'| \gg |A|\), and \(E_f(A') \gg |A|^3\)? Here, we point out to the reader a subtle but important difference between this problem and the indexed energy problem: when passing to a subset, there is a natural way to reindex a set which is distinctly different than how a function restricted to a subset behaves. Therefore, \(E_f(A)\) is not a generalization of \(EI(A)\), but instead, it is a different quantity altogether. There is not always an \(A' \subseteq A\) with \(E_f(A') \gg_K |A|^3\) when \(E(A) \geq K|A|^3\). For instance, let \(f\) be the indexing function, let \(A\) be as in Theorem 10, and since sets are not reindexed
\[
E_f(A') \leq EI(A) \ll_K |A|^2 \log |A|.
\]
Moreover, $|(a + a', f(a) + f(a')) : a, a' \in A| \gg |A|^2 / \log |A|$. As an openended question, we ask if there are any reasonable conditions that we can impose on $f$ or $A$ to arrive at a different conclusion?

Lastly, we remark that the content of Lemma 13 is making a statement about equidistribution of a set in an interval. This has been a well-studied topic in discrepancy theory; however, we are not aware of it appearing in this specific, combinatorial form – where one is allowed to pass to a subset of the original set, and one only requires that for lots of interval, the subset is well-distributed. We tepidly conjecture a generalization of Lemma 13 to higher dimensions, but it would also be interesting if a counterexample was found.

**Conjecture 18.** Let $A \subseteq [1, n] \times [1, n]$ be of size $|A| = \delta n^2$. There exists constants $c_1, c_2, c_3$ depending only on $\delta$ such that the following holds. There exists an $A' \subseteq A$ such that $|A'| \geq c_1 |A|$ and for $c_2 n^2$ pairs $0 \leq i, j \leq n/c_3$, $|A' \cap [0, ic_3) \times [0, jc_3)| = ij$.

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