Compact Objects for Everyone I: White Dwarf Stars

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Abstract

Based upon previous discussions on the structure of compact stars geared towards undergraduate physics students, a real experiment involving two upper-level undergraduate physics students, a beginning physics graduate, and two advanced graduate students was conducted. A recent addition to the physics curriculum at Florida State University, The Physics of Stars, sparked quite a few students’ interests in the subject matter involving stellar structure. This, coupled with Stars and Statistical Physics by Balian and Blaizot [1] and Neutron Stars for Undergraduates by Silbar and Reddy [2], is the cornerstone of this small research group who tackled solving the structure equations for compact objects in the Summer of 2004. Through the use of a simple finite-difference algorithm coupled to Microsoft Excel and Maple, solutions to the equations for stellar structure are presented in the Newtonian regime appropriate to the physics of white dwarf stars.

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I. OVERVIEW OF THE PROJECT

It is the central tenet of the present “experiment” that advances in both algorithms and computer architecture bring the once-challenging problem of the structure of compact objects within the reach of beginning undergraduate students — and even high-school students. After a brief historical review in Sec. II, a synopsis of stellar evolution is presented in Sec. III that culminates with a detailed description of the physics of collapsed stars. Following this background information, the equations of hydrostatic equilibrium for Newtonian stars are derived in Sec. IV. In particular, the need for an equation of state and the enormous advantage of scaling the equations are emphasized. The section concludes with the presentation of mass-vs-radius relationships for white dwarf stars obtained using both Excel and Maple. The special role played by special relativity for the existence of a limiting mass, the Chandrasekhar limit, is strongly emphasized. Finally, a summary and some concluding remarks are offered in Sec. VI.

II. HISTORICAL PERSPECTIVE

The story of collapsed stars starts in earnest with Subrahmanyan Chandrasekhar in the early 1930’s. As a young man of 20, he was embarking from his native India to Cambridge University to start life as a graduate student under Fowler’s supervision. By 1926 Fowler had already explained the structure of white dwarf stars by using the electron degenerate pressure only a few months after the formulation of the Fermi-Dirac statistics. However, Chandrasekhar noticed a critical ingredient missing from Fowler’s analysis: special relativity. Chandrasekhar discovered that as the density of the star increases and the momentum of the electrons becomes comparable with and later exceeds $mc$, where $m$ is the rest mass and $c$ the speed of light, then their ability to support the star against gravity weakens. Chandrasekhar concluded that stars with masses above $M_{Ch} = 1.44M_{\odot}$ ($M_{\odot}$ a solar mass) cannot cool down but will continue to contract and heat up. This limiting mass $M_{Ch}$ is fittingly known as the Chandrasekhar mass. However, it was not all smooth sailing for Chandrasekhar. One pre-eminent figure in the field, Sir Arthur Eddington, opposed Chandrasekhar publicly and privately with great vigor, even claiming — by arguments found generally difficult to follow — that the non-relativistic pressure-density relation should be used at all densities.

Why wouldn’t Chandrasekhar silence his critics by revealing the ultimate fate of a heavy ($M > M_{Ch}$) star? After all, we now know that such a star will collapse into a neutron star, or if too heavy into a black hole. Unfortunately for Chandrasekhar, at the time of his ground-breaking discovery it was impossible for him (or for anyone else for that matter) to have predicted the existence of neutron stars. It was one year later in 1932 that Chadwick proved the existence of the neutron. From that point on things developed very quickly, culminating with the 1933 proposal by Baade and Zwicky that supernovae are created by the collapse of a “normal” star to form a neutron star. Chandrasekhar was thus vindicated and awarded the Nobel Prize in Physics in 1983 for his lifetime contributions to the physical processes of importance to the structure and evolution of the stars.

The process of discovery of these two classes of compact objects (white dwarf and neutron stars) is radically different. In the case of white dwarf stars, observation predated Fowler’s theoretical explanation by more than 10 years. Indeed, in 1915 at the Mount Wilson Observatory in California a group of astronomers headed by Walter Sydney Adams discovered...
that Sirius B — the companion of the brightest star in the night sky — was a white-dwarf star, the first to be discovered. On the other hand, it took more than 30 years since the bold prediction by Baade and Zwicky [6] to discover neutron stars. As is often the case in science, this discovery marks one of the greatest examples of serendipity. Antony Hewish and his team at the Cavendish Laboratory built a radio telescope to study some of the most energetic quasi-stellar objects in the universe (quasars). Among the members of the team was a young research student by the name of Jocelyn Bell [7]. Shortly after the telescope started gathering data in 1967, Bell observed a signal of seemingly unknown origin (a “scruff”). Particularly puzzling was that the signal showed at a remarkably precise pulse rate of 1.33 seconds [8]. So unexpected was the signal, that after a month of futile attempts at understanding it, it was dubbed the “Little Green Men.” However, by the beginning of 1968 Hewish, Bell, and collaborators had found three additional pulsating sources of radio waves, or pulsars [9]. The final explanation of these enigmatic sources was due to Gold shortly after Hewish and Bell published their findings [10, 11]. Gold suggested that the radio signals were due to rapidly rotating neutron stars that rather than emitting pulses of radiation, could emit a steady radio signal that it swept around in circles. When the pulsar “lighthouse” was pointing in the direction of the telescope, the signal will indeed show up as the short “pulses” that Bell had discovered. It is not our intention to present here a comprehensive review of either the history or the fascinating phenomena behind pulsars. For a recent account see Ref. [12].

III. STELLAR EVOLUTION – A SYNOPSIS

Stellar objects are dynamical systems involving a symbiotic relationship between matter and radiation creating enough pressure to oppose gravitational contraction. Thermonuclear fusion, the thermally-induced combining of nuclei as they tunnel through the Coulomb barrier, is initially responsible for supporting stars against gravitational contraction. The ultimate fate of the star depends upon its remaining mass once thermonuclear fusion can no longer provide the pressure required to counteract gravity.

Thermonuclear fusion drives stars through many stages of combustion; the hot center of the star allows hydrogen to fuse into helium. Once the core has burned all available hydrogen, it will contract until another source of support becomes available. As the core contracts and heats, transforming gravitational energy into kinetic (or thermal) energy, the burning of the helium ashes begins. For stars to burn heavier elements, higher temperatures are necessary to overcome the increasing Coulomb repulsion and allow fusion through quantum-mechanical tunneling. Thermonuclear burning continues until the formation of an iron core. Once iron — the most stable of nuclei — is reached, fusion becomes an endothermic process. However, combustion to iron is only possible for the most massive of stars. When thermonuclear fusion can no longer support the star against gravitational collapse, either because they are not massive enough (like our Sun) or because they have developed an iron core, the star dies and a compact object is ultimately formed. The three final possible stages a star can take is a white dwarf, a neutron star, or a black hole. In this our first contribution — one that should be accessible to motivated high-school students — we focus exclusively on the physics of white dwarf stars. The fascinating topic of neutron stars requires the use of general relativity and is therefore reserved for a more advanced forthcoming publication.

Our Sun will die as a white dwarf star once all of the hydrogen and helium in the core has been burned. Towards the final stages of burning, the star will expand and expel most
of the outer matter to create a planetary nebula. At the beginning, the non-degenerate core contracts and heats up through conversion of gravitational energy into thermal kinetic energy. However, at some point the Fermi pressure of the degenerate electrons begins to dominate, the contraction is slowed up, and the core becomes a compact object known as a white dwarf, cooling steadily towards the ultimate cold, dark, static black dwarf state. On the other hand, neutron stars result from one of the most cataclysmic events in the universe, the death of a star with a mass much greater than that of our Sun. Electrons in these stars behave ultra-relativistically, and as pointed out by Chandrasekhar, hydrostatic equilibrium as a cold body becomes impossible to achieve when $M > M_{\text{Ch}}$. However, during the collapse of the core, a supernovae shock develops ejecting most of the mass of the star into the interstellar space and leaving behind an extremely dense core — the neutron star. As the star collapses, it becomes energetically favorable for electrons to be captured by protons, making neutrons and neutrinos. The neutrinos carry away 99% of the gravitational binding energy of the compact object, leaving neutrons behind to support the star against further collapse. The pressure provided by the degenerate neutrons, like degenerate electron pressure for white dwarf stars, has a limit on the mass it can bear. Beyond this limiting mass, no source of pressure exists that can prevent gravitational contraction. If such is the case, then the star will continue to collapse into an object of zero radius: a black hole. There is a large number of excellent textbooks on the birth, life, and death of stars. The following are some references used in this work [13, 14, 15, 16, 17, 18].

### A. The Physics of Collapsed Stars

It is a remarkable fact that quantum mechanics and special relativity — both theories perceived as of the very small — play such a crucial role in the dynamics of stars, the former in preventing low-mass stars from collapsing into black holes, while the latter in driving the collapse of massive stars. In this research experiment we limit ourselves to the study of white dwarf (or Newtonian) stars. Further, we assume that white dwarf stars are “cold”, spherically symmetric, non-rotating objects in hydrostatic equilibrium.

In dealing with white dwarf stars, the system may be approximated as a plasma containing positively charged nuclei and electrons, with the nuclei providing (almost) all the mass and none of the pressure and the electrons providing all the pressure and none of the mass. This state of matter corresponds to a gas that is electrically neutral on a global scale, but locally composed of positively charged ions (nuclei) and the negatively charged electrons. Note that even in a zero-temperature, black dwarf state, the matter is effectively ionized. For a free atom, the Heisenberg Uncertainty Relation ensures that in the state of minimum total energy — kinetic plus electric potential energy — the electrons occupy a finite volume of dimension given by the Fermi-Thomas distance, the analogue for more massive atoms of the Bohr radius for hydrogen. The density of a collapsed star is so high that the mean distance between nuclei is less than the Fermi-Thomas distance: the matter is ‘pressure-ionized’, with the electrons forming an effectively free gas. However, all identical fermions (such as electrons, protons and neutrons) obey Fermi-Dirac statistics in that the occupation of states is governed by the Pauli Exclusion Principle — no two fermions can exist in the same quantum state. For a zero temperature Fermi gas, all available electron states below the Fermi energy are filled, while the rest are empty. The Fermi energy is determined solely by the electron number density and rest mass. For a compact object such as a white dwarf star, the number density is very high and so is the Fermi energy. Typical electron Fermi
energies are of the order of 1 MeV which correspond to a \textit{Fermi temperature} of $T_F \simeq 10^{10}$ K. As the temperature of the system is increased (say from $T = 0$ to $T = 10^6$ K or more, as estimated for white dwarf interiors), electrons try to jump to a state higher in energy by an amount of the order of $k_B T$ but fail, as most of these transitions are Pauli blocked. Only those high-energy electrons that are within $k_B T \simeq 100$ eV from the Fermi surface can make the transition, but those represent a tiny fraction ($T/T_F \simeq 10^{-4}$) of the electrons in the star. Hence, for the purpose of computing the pressure of the system, it is extremely accurate — to 1 part in $10^4$ — to describe the electrons as a Fermi gas at zero temperature.

IV. NEWTONIAN STARS

Let us start by addressing Newtonian stars. For these stars we assume that corrections due to Einstein’s greatest triumph — the \textit{Theory of General Relativity} — may be safely ignored. White dwarf stars, with escape velocities of only 3% of the speed of light, fall into this category. Not so neutron stars — typical escape velocities of half the speed of light cause extreme sensitivity to these corrections.

We start by considering the radial force acting on a small mass element ($\Delta m = \rho(r) \Delta V$) located at a distance $r$ from the center of the star (see Fig. 1):

$$F_r = -\frac{G M(r) \Delta m}{r^2} - P(r + \Delta r) \Delta A + P(r) \Delta A = \Delta m \frac{d^2 r}{dt^2}. \quad (1)$$

Here $\rho(r)$ is the mass density of the star, $M(r)$ denotes the \textit{enclosed mass} within a radius $r$, and $P$ is the pressure. Expanding the above equation to lowest order in $\Delta r$ one obtains

$$-\frac{G M(r) \rho(r)}{r^2} - \frac{dP}{dr} = \rho(r) \frac{d^2 r}{dt^2}. \quad (2)$$

Assuming \textit{hydrostatic equilibrium} ($\ddot{r} = \dot{r} \equiv 0$), one arrives at the fundamental equations describing the structure of Newtonian stars. That is,

$$\frac{dP}{dr} = -\frac{G M(r) \rho(r)}{r^2}, \quad P(r=0) \equiv P_e; \quad (3a)$$

$$\frac{dM}{dr} = +4\pi r^2 \rho(r), \quad M(r=0) \equiv 0, \quad (3b)$$

where Eq. (3b) defines the enclosed mass.

It is simple to see that in hydrostatic equilibrium, the pressure of the star is a decreasing (or at least not increasing) function of $r$; otherwise the star collapses. Note that the radius of the star $R$ is defined as the value of $r$ at which the pressure goes to zero, \textit{i.e.}, $P(R) = 0$. Similarly, the mass of the star corresponds to the value of the enclosed mass at $r = R$, when $M = M(R)$.

A. Equation of State

The above set of equations, together with their associated boundary conditions, must be completed by an equation of state (EoS), namely a relation $P = P(\rho)$ between the density and pressure. For simplicity, we limit ourselves to the EoS of a zero-temperature Fermi
gas composed of constituents (e.g., electrons or neutrons) having a rest mass \( m \). The main assumption behind the Fermi gas hypothesis is that no correlations (or interactions) are relevant to the system other than those generated by the Pauli exclusion principle. For some standard references on the equation of state of a free Fermi gas — at both zero and finite temperatures — see Refs. [19, 20].

To start, the Fermi wavenumber \( k_F \) is defined; \( k_F \) represents the momentum of the fastest moving fermion and is solely determined by the number density \( n \equiv N/V \) of the system, where \( N \) is the total number of particles in our system, and \( V \) is the enclosed volume.

That is,

\[
N = 2 \sum_k \Theta(k_F - |k|) = 2 \int \frac{V}{(2\pi)^3} d^3k \Theta(k_F - |k|) = V \frac{k_F^3}{3\pi^2},
\]

or equivalently

\[
k_F = \left(\frac{3\pi^2 n}{2}\right)^{1/3}.
\]

In Eq. (1), \( \Theta(x) \) represents the Heaviside (or step) function. Having defined the Fermi wavenumber \( k_F \), the energy density of the system is obtained from a configuration in which all single-particle momentum states are progressively filled in accordance with the Pauli exclusion principle. For a degenerate (spin-1/2) Fermi gas at zero temperature, exactly two fermions occupy each single-particle state below the Fermi momentum \( p_F = \hbar k_F \); all remaining states above the Fermi momentum are empty. In this manner we obtain the following expression for the energy density:

\[
E \equiv E/V = 2 \int \frac{d^3k}{(2\pi)^3} \Theta(k_F - |k|) \epsilon(k),
\]

where \( \epsilon(k) \) is the single-particle energy of a fermion with momentum \( k \). In what follows, the most general free-particle dispersion (energy vs. momentum) relation is assumed, namely,
one consistent with the postulates of special relativity. That is,

$$\epsilon(k) = \sqrt{(\hbar c)^2 + (mc^2)^2} = mc^2 \sqrt{1 + x^2}, \quad \text{with } x \equiv \frac{\hbar c}{mc^2}. \quad (7)$$

In spite of its slightly intimidating form, the integral in Eq. (6) may be performed in closed form. We obtain

$$E = E_0 \mathcal{E}(x_F), \quad (8)$$

where $E_0$ is a dimensionful constant that may be written using dimensional analysis

$$E_0 \equiv \frac{(mc^2)^4}{(\hbar c)^3}, \quad (9)$$

and $\mathcal{E}(x_F)$ is a dimensionless function of the single variable $x_F = \hbar k_F c/mc^2$ given by

$$\mathcal{E}(x_F) \equiv \frac{1}{\pi^2} \int_0^{x_F} x^2 \sqrt{1 + x^2} \, dx = \frac{1}{8\pi^2} \left[ x_F \left( 1 + 2 x_F^2 \right) \sqrt{1 + x_F^2} - \ln \left( x_F + \sqrt{1 + x_F^2} \right) \right]. \quad (10)$$

The pressure of the system may now be directly obtained from the energy density by using the following thermodynamic relation — which is only valid at zero temperature:

$$P = - \left( \frac{\partial E}{\partial V} \right)_{N,T=0} = - \left( \frac{\partial (VE)}{\partial V} \right)_{N,T=0} \equiv P_0 \overline{P}. \quad (11)$$

In analogy to the energy density, dimensionful and dimensionless quantities for the pressure have been defined:

$$P_0 = \mathcal{E}_0 = \frac{(mc^2)^4}{(\hbar c)^3}, \quad (12a)$$

$$\overline{P}(x_F) \equiv \left[ \frac{x_F}{3} \mathcal{E}'(x_F) - \mathcal{E}(x_F) \right]. \quad (12b)$$

It may be surprising to find that a gas of particles at zero-temperature may still generate a non-zero pressure. It is quantum statistics, in the form of the Pauli exclusion principle — not temperature — that is responsible for generating the pressure. It is nevertheless surprising that quantum pressure, a purely microscopic phenomenon, should be ultimately responsible for supporting compact stars against gravitational collapse.

With an expression for the pressure in hand, we are finally in a position to compute its derivative with respect to $x_F$ (a quantity that we label as $\eta$). As we shall see in the next section, $\eta$ — a function closely related to the zero-temperature incompressibility — is the only property of the EoS that Newtonian stars are sensitive to [20]. We obtain

$$\eta \equiv \frac{dP}{dx_F} = P_0 \left[ \frac{x_F}{3} \overline{P}'(x_F) - \frac{2}{3} \overline{P}(x_F) \right] = \frac{P_0}{3\pi^2} \frac{x_F^4}{\sqrt{1 + x_F^2}}. \quad (13)$$

The above expression has a surprisingly simple form that depends on the energy density only through its derivatives. Alternatively, one could have bypassed the above derivation in favor of the following general relation valid for a zero-temperature Fermi gas:

$$\frac{dP}{dx_F} = n \frac{dx_F}{dx_F}. \quad (14)$$
In view of Eq. (14), the attentive reader may be asking why go through the trouble of computing the energy density and the corresponding pressure if all that is required is the dependence of the Fermi energy on $x_F$. The answer is general relativity. While Newtonian stars depend exclusively on $\eta$, the structure of relativistic stars (such as neutron stars) are highly sensitive to corrections from general relativity. These corrections depend on both the energy density and the pressure and will be treated in detail in a future publication.

B. Toy Model of White Dwarf Stars

Before attempting a numerical solution to the equations of hydrostatic equilibrium, we consider as a warm-up exercise a toy model of a white dwarf star [21]. Assume a white dwarf star with a uniform, spherically symmetric mass distribution of the form

$$\rho(r) = \begin{cases} \rho_0 = \frac{3M}{4\pi R^3}, & \text{if } r \leq R; \\ 0, & \text{if } r > R, \end{cases} \quad (15)$$

where $M$ and $R$ are the mass and radius of the star, respectively. For such a spherically symmetric star, the gravitational energy released during the process of “building” the star is given by

$$E_G = -4\pi G \int_0^R M(r)\rho(r) r \, dr, \quad (16a)$$

$$M(r) = \int_0^r 4\pi r'^2 \rho(r') \, dr'. \quad (16b)$$

For a uniform density star as assumed in Eq. (15), it is straightforward to perform the above two integrals. Thus, the gravitational energy released in “building” such a star is given by

$$E_G(M, R) = -\frac{3}{5} \frac{GM^2}{R}. \quad (17)$$

From Eq. (17), we conclude that without a source of gravitational support, a star with a fixed mass $M$ will minimize its energy by collapsing into an object of zero radius, namely, into a black hole. We know, however, that white dwarf stars are supported by the quantum-mechanical pressure from its degenerate electrons, which (at temperatures of about $10^6$ K) are fully ionized in the star (recall that 1 eV $\simeq 10^4$ K). In what follows, we assume that electrons provide all the pressure support of the star but none of its mass, while nuclei (e.g., $^4\text{He}$, $^{12}\text{C}$, ...) provide all the mass but none of the pressure. The electronic contribution to the mass of the star is inconsequential, as the ratio of electron to nucleon mass is approximately equal to $1 : 2000$.

The energy of a degenerate electron gas was computed in the previous section. Using Eqs. (5) and (8) we obtain,

$$E_F(M, R) = 3\pi^2 N m_e e^2 \frac{\mathcal{E}(x_F)}{x_F^3}, \quad (18)$$

where $m_e$ is the rest mass of the electron. Naturally, the above expression depends on the mass and the radius of the star, although this dependence is implicit in $x_F$. While the toy-problem at hand is instructive of the simple, yet subtle, physics that is displayed in compact stars, it also serves as a useful framework to illustrate how to scale the equations.
1. Scaling the Equations

One of the great challenges in astrophysics, and the physics of compact stars is certainly no exception, is the enormous range of scales that one must simultaneously address. For example, in the case of white dwarf stars it is the pressure generated by the degenerate electrons (constituents with a mass of $m_e = 9.110 \times 10^{-31}$ kg) that must support stars with masses comparable to that of the Sun ($M_\odot = 1.989 \times 10^{30}$ kg). This represents a disparity in masses of 60 orders of magnitude! Without properly scaling the equations, there is no hope of dealing with this problem with a computer.

We start by defining

$$f_F \equiv \frac{E_F}{N m_e c^2}$$

from Eq. (18), a quantity that is both dimensionless and intensive (i.e., independent of the size of the system). That is,

$$f_F(x_F) = 3 \pi^2 \frac{\overline{\mathcal{E}}(x_F)}{x_F^3},$$

where a closed-form expression for $\overline{\mathcal{E}}(x_F)$ has been displayed in Eq. (10). Note that the scaled Fermi momentum $x_F$ quantifies the importance of relativistic effects. At low density ($x_F \ll 1$) the corrections from special relativity are negligible and electrons behave as a non-relativistic Fermi gas. In the opposite high-density limit ($x_F \gg 1$) the system becomes ultra-relativistic and the “small” (relative to the Fermi momentum) electron mass may be neglected. We shall see that in the case of white dwarf stars, the most interesting physics occurs in the $x_F \sim 1$ regime.

The dynamics of the star consists of a tug-of-war between gravity that favors the collapse of the star and electron-degeneracy pressure that opposes the collapse. To efficiently compare these two contributions, the contribution from gravity to the energy must be scaled accordingly. Thus, in analogy to Eq. (19), we form the corresponding dimensionless and intensive quantity for the gravitational energy ($f_G \equiv E_G/N m_e c^2$):

$$f_G(M, R) = -\frac{3}{5} \left( \frac{GM}{Rc^2} \right) \left( \frac{M}{Nm_e} \right) = -\frac{3}{5} \left( \frac{GM}{Rc^2} \right) \left( \frac{m_n}{Y_e m_e} \right),$$

where we have assumed that the mass of the star, $M = A m_n$, may be written exclusively in terms of its baryon number $A$ and the nucleon mass $m_n$ (the small difference between proton and neutron masses is neglected). This is an accurate approximation as both nuclear and gravitational binding energies per nucleon are small relative to the nucleon mass. Further, $Y_e \equiv Z/A$ represents the electron-per-baryon fraction of the star (e.g., $Y_e = 1/2$ for $^4$He and $^{12}$C, and $Y_e = 26/56$ in the case of $^{56}$Fe).

The final step in the scaling procedure is to introduce dimensionful mass $M_0$ and radius $R_0$, quantities that, when chosen wisely, will embody the natural mass and length scales in the problem. To this effect we define

$$\overline{M} \equiv \frac{M}{M_0} \quad \text{and} \quad \overline{R} \equiv \frac{R}{R_0}.$$ 

In terms of these natural mass and length scales, the gravitational contribution to the energy of the system takes the following form:

$$f_G(\overline{M}, \overline{R}) = -\left[ \frac{3}{5} \left( \frac{GM_0}{R_0 c^2} \right) \left( \frac{m_n}{Y_e m_e} \right) \right] \frac{\overline{M}}{\overline{R}}.$$
While the dependence of the above equation on $M$ and $R$ is already explicit, the Fermi gas contribution to the energy depends implicitly on them through $x_F$. To expose explicitly the dependence of $f_F$ on $M$ and $R$ we perform the following manipulation aided by relations derived in Sec. IV A.

$$x_F^3 = \left( \frac{\hbar k_F c}{m_e c^2} \right)^3 = \left[ \left( \frac{9\pi}{4} Y_e \right) \left( \frac{M_0}{m_n} \right) \left( \frac{\hbar c / m_e c^2}{R_0} \right)^3 \right] \frac{M}{R^3}. \quad (23)$$

We have already referred earlier to $M_0$ and $R_0$ as the natural mass and length scales in the problem, but their values have yet to be determined. Thus, they are still at our disposal. Their values will be fixed by adopting the following choice: let the “complicated” expressions enclosed between square brackets in Eqs. (22) and (23) be set equal to one. That is,

$$\left[ \frac{3}{5} \left( \frac{G M_0}{R_0 c^2} \right) \left( \frac{m_n}{Y_e m_e} \right) \right] = \left[ \left( \frac{9\pi}{4} Y_e \right) \left( \frac{M_0}{m_n} \right) \left( \frac{\hbar c / m_e c^2}{R_0} \right)^3 \right] = 1. \quad (24)$$

This choice implies the following values for white dwarf stars with an electron-to-baryon ratio equal to $Y_e = 1/2$:

$$M_0 = \frac{5}{6} \sqrt{15\pi} \alpha_G^{-3/2} m_n Y_e^2 = 10.599 \, M_\odot \quad (25a)$$
$$R_0 = \frac{\sqrt{15\pi}}{2} \alpha_G^{-1/2} \left( \frac{\hbar c}{m_e c^2} \right) Y_e = 17\,250 \, \text{km} \quad (25b)$$

Here the minute dimensionless strength of the gravitational coupling between two nucleons has been introduced as

$$\alpha_G = \frac{G m_n^2}{\hbar c} = 5.906 \times 10^{-39} \quad (26)$$

The aim of this toy-model exercise is to find the minimum value of the total (gravitational plus Fermi gas) energy of the star as a function of its radius for a fixed value of its mass. Before doing so, however, a few comments are in order. First, from merely scaling the equations and with no recourse to any dynamical calculation we have established that white dwarf stars have masses comparable to that of our Sun but typical radii of only 10,000 km (recall that the radius of the Sun is $R_\odot \approx 700,000$ km). Further, we observe that while $R_0$ scales with the inverse electron mass, the mass scale $M_0$ is independent of it. This suggests that neutron stars, where the neutrons provide all the pressure and all the mass, will also have masses comparable to that of the Sun but typical radii of only about 10 km.

Now that the necessary “scaling” machinery has been developed, we return to our original toy-model problem. Taking advantage of the scaling relations, the energy per electron in units of the electron rest energy is given by:

$$f(M, x_F) = f_G(M, x_F) + f_F(x_F) = -\frac{M^{2/3}}{x_F} + 3\pi^2 \frac{\mathcal{E}(x_F)}{x_F^3}. \quad (27)$$

The mass-radius relation of the star may now be obtained by demanding hydrostatic equilibrium:

$$\left( \frac{\partial f(M, x_F)}{\partial x_F} \right)_{x_F} = 0. \quad (28)$$
While a closed-form expression has already been derived for the energy density $E(x_F)$ in Eq. (10), it is instructive to display explicit non-relativistic and ultra-relativistic limits. These are given by

$$f_F(x_F) = \begin{cases} 
1 + \frac{3}{10} x_F^2 & \text{if } x_F \ll 1; \\
\frac{3}{4} x_F & \text{if } x_F \gg 1.
\end{cases}$$  \hspace{1cm} (29)$$

To conclude this section an output from a Maple code has been included to illustrate how simple, within the present approximation, it is to compute the radius of an arbitrary mass star (see Fig. 2). For the present example, a 1 M$_\odot$ star has been used. First, scales for input quantities, such as the dimensionful mass ($M_0$) and length ($R_0$), are defined. Next, energies and their derivatives are computed and a plot displaying the latter is generated. The derivative of the gravitational energy (actually the negative of it) is constant and is displayed with a black horizontal line. Similarly, the derivative of the Fermi energy in the ultra-relativistic limit (horizontal green line) is also a constant equal to $3/4$, independent of the mass of the star. The blue line with a constant slope displays the derivative of the Fermi gas energy in the non-relativistic limit. Finally, the exact Fermi gas expression, which interpolates between the non-relativistic and the relativistic result, is displayed with the red line.

The equilibrium density of the star is obtained from the intersection of the red and blue lines with the gravitational line. In the non-relativistic case the solution may be computed analytically to be $x_{F0} = 5M_0^{2/3}/3$. However, this non-relativistic prediction overestimates the Fermi pressure and consequently also the radius of the star. The non-relativistic predictions for the radius of a 1 solar-mass white dwarf star is $R_{NR} = 7.162$ km. In contrast, the result with the correct relativistic dispersion relation is considerably smaller at an $R_{NR} = 4.968$ km. Yet an even more dramatic discrepancy emerges among the two models. While the non-relativistic result guarantees the existence of an equilibrium radius for any value of the star’s mass ($R_{NR} = 3/5M_0^{1/3}$), the correct dispersion relation predicts the existence of an upper limit beyond which the pressure from the degenerate electrons can no longer support the star against gravitational collapse. This upper mass limit, known as the Chandrasekhar mass, is predicted in the simple toy model to be equal to:

$$M_{Ch} = (3/4)^{3/2} M_0 = 1.72 \, M_\odot.$$ \hspace{1cm} (30)$$

As it will be shown later, accurate numerical results yield (for $Y_e = 1/2$) a Chandrasekhar mass of $M_{Ch} = 1.44 \, M_\odot$. Thus, not only does the toy model predict the existence of a maximum mass star, but it does so with an 80% accuracy.

C. Numerical Analysis

We now return to the exact (numerical) treatment of white dwarf stars. While the toy model problem developed earlier provides a particularly simple framework to understand the interplay between gravity, quantum mechanics, and special relativity, a quantitative description of the systems demands the numerical solution of the hydrostatic equations [Eq. (3)]. For the present treatment, however, we continue to assume that the equation of state is that of a simple degenerate Fermi gas. In this case the equation of state is known analytically and it is convenient to incorporate it directly into the differential equation. In
Toy Model of Compact Stars:

```maple
restart;
M0 := 2.651:     # dimensionfull mass scale (in solar masses)
R0 := 8625:      # dimensionfull length scale (in km)
M := 1.000:      # compute structure for a 1 solar-mass star
Mbar := M/M0:    # dimensionless mass

Compute energies:
fG := -Mbar^(2/3)*xF:              # gravitational energy
fF := (3/xF^3)*int(x^2*sqrt(1+x^2),x=0..xF): # Fermi energy
fFNR := 1+(3/10)*xF^2:             # Fermi energy (NR limit)
fFUR := (3/4)*xF:                  # Fermi energy (UR limit)

Compute derivative of the energies:
fG1 := diff(fG,xF):                # gravitational energy
fF1 := simplify(diff(fF,xF)):      # Fermi energy
fFNR1 := simplify(diff(fFNR,xF)):  # Fermi energy (NR limit)
fFUR1 := simplify(diff(fFUR,xF)):  # Fermi energy (UR limit)

Make a plot:
plot([-fG1,fF1,fFNR1,fFUR1],xF=0..2,
     color=[black,red,blue,green],thickness=3);

Compute the radius of a 1 solar mass star:
xF0 := fsolve(fG1+fF1,xF=1):       # solve for the density
Rbar0 := Mbar^(1/3)/xF0:           # solve for the Radius
Mass := Mbar*M0:                   # extract dimensionfull mass
Radius := Rbar0*R0:                # extract dimensionfull radius
printf("M=%4.2f MSolar; R=%4.0f km",Mass,Radius);
M=1.00 MSolar; R=4968 km
```

FIG. 2: Maple code displaying the interplay between various physical effects on the “toy-model” problem of a $M=1M_\odot$ star.
this way Eq. (3a) becomes
\[ \frac{dx_F}{dr} = -\frac{GM(r)\rho(r)}{r^2\eta}, \quad (31) \]
where the equation of state enters only through a quantity directly related to the zero-temperature incompressibility \[20\]. This quantity, \[\eta = dP/dx_F\], was defined and evaluated in Eq. (13). Moreover, the density of the system \[\rho(r)\] is easily expressed in terms of \[x_F\]. It is given by
\[ \rho = \left(\frac{m_e c^2}{\hbar c}\right)^3 \frac{m_n}{3\pi^2 Y_e} x_F^3. \quad (32) \]

At this point all necessary relations have been derived and the equations of hydrostatic equilibrium, first displayed in Eq. (3) but with an equation of state still missing, may now be written in the following form:
\[
\begin{align*}
\frac{dx_F}{d\tau} & = - \left[ \left( \frac{GM_0}{R_0 c^2} \right) \left( \frac{m_n}{Y_e m_e} \right) \right] \frac{M}{\tau^2} \sqrt{1 + \frac{x_F^2}{\tau^2}}, \quad x_F(\tau = 0) \equiv x_{Fc}; \quad (33a) \\
\frac{dM}{d\tau} & = + \left[ \left( \frac{3\pi Y_e}{4} \right) \left( \frac{M_0}{m_n} \right) \left( \frac{\hbar c}{m_e c^2} \right)^3 \right]^{-1} \tau^2 x_F^3, \quad M(\tau = 0) \equiv 0. \quad (33b)
\end{align*}
\]

Here the dimensionless distance \[\tau\] and the central (scaled) Fermi momentum \[x_{Fc}\] have been introduced. The structure of the above set of differential equations indicates that our goal of turning Eq. (3) into a well-posed problem, by directly incorporating the equation of state into the differential equations, has been accomplished. But we have done better. By defining the natural mass and length scales of the system \((M_0 \text{ and } R_0)\) according to Eq. (24), the two long expressions in brackets in the above equations reduce to the simple numerical values of \(5/3\) and \(1/3\), respectively. Finally, then, the equations of hydrostatic equilibrium describing the structure of white dwarf stars are given by the following expressions:
\[
\begin{align*}
\frac{dx_F}{d\tau} & = f(\tau; x_F, M), \quad x_F(\tau = 0) \equiv x_{Fc}; \quad (34a) \\
\frac{dM}{d\tau} & = g(\tau; x_F, M), \quad M(\tau = 0) \equiv 0, \quad (34b)
\end{align*}
\]
where the two functions on the right-hand side of the equations \((f \text{ and } g)\) are given by
\[ f(\tau; x_F, M) \equiv -\frac{5}{3} \frac{M}{\tau^2} \sqrt{1 + \frac{x_F^2}{\tau^2}} x_F \quad \text{and} \quad g(\tau; x_F, M) \equiv +\frac{3}{\tau^2} x_F^3. \quad (35) \]

These coupled set of first-order differential equations may now be solved using standard numerical techniques, such as the Runge-Kutta algorithm \[21\]. However, for those students not yet comfortable with writing their own source codes, the use of an “off-the-shelf” spreadsheet (here Microsoft Excel has been used), together with a crude low-order approximation for the derivatives has been shown to be adequate. As in the toy-model problem, solutions will be presented using the full relativistic dispersion relation as well as the non-relativistic approximation, where in the latter case the square-root term appearing in the function \(f(\tau; x_F, M)\) is set to one.
V. NUMERICAL TECHNIQUES FOR EVERYONE

In this section we present numerical solutions for the structure of Newtonian (white-dwarf) stars by employing a variety of numerical techniques and programming tools.

1. White Dwarf Stars with Excel

This section should be ideal for those students with a basic knowledge of calculus and with no programming skills. *High school* students that have learned the concept of derivatives in their introductory calculus class should be able to complete this part of the project with no problem. Indeed, one could use the definition of the derivative of a function $F(x)$

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h},$$

(36)

to approximate the value of the function at a neighboring point $x+h$. That is,

$$F(x+h) = F(x) + hF'(x) + \mathcal{O}(h^2).$$

(37)

The term $\mathcal{O}(h^2)$ indicates that the error that one makes in computing the value of the function at the neighboring point scales as the square of $h$. Thus, for this low-order approximation of the derivative, we selected a very small value of $h$ in order to ensure numerical accuracy. Other higher-order algorithms, such as the venerated Runge-Kutta method, contain errors that scale as $\mathcal{O}(h^4)$. These algorithms can attain the same degree of numerical accuracy as the one presented here with a dramatic reduction in computational time.

So how do we turn Eq. (37) to our advantage? By simply looking at the structure of the (scaled) equations of hydrostatic equilibrium Eq. (34) we can readily write

$$x_F(\tau + \Delta \tau) = x_F(\tau) + \Delta \tau f(\tau; x_F, \overline{M}),$$

(38a)

$$\overline{M}(\tau + \Delta \tau) = \overline{M}(\tau) + \Delta \tau g(\tau; x_F, \overline{M}).$$

(38b)

The resulting *difference equations* are recursion relations that enables one to “leapfrog” from point to point in a grid for which the various points are separated by a fixed distance $\Delta \tau$. Recursion relations such as this one are particularly well suited to be solved with a spreadsheet. Fig. 3 shows the results produced using Excel (lines).

To start the solution of the difference equations one notes that the right-hand side of the two recursion relations given above are completely known at $\tau = 0$ (recall that appropriate boundary conditions have already been specified). This enabled us to compute both the scaled Fermi momentum and the enclosed mass on the next grid point $\tau = \Delta \tau$. With this knowledge we could again evaluate the right-hand side of the recursion relations but now at $\tau = 2\Delta \tau$. Now the values of $x_F$ and $\overline{M}$ at the next grid point ($\tau = 2\Delta \tau$) may be computed. We continued in this manner until the Fermi momentum goes to zero. This point defines the radius of the star, while the mass of the star is the value of the enclosed mass at this last point (this point also corresponds to when the pressure goes to zero). Up to this point, both radius and mass are obtained in dimensionless units. To convert back to physical units we simply multiplied these dimensionless quantities by the dimensionful parameters ($R_0$ and $M_0$) defined in Eq. (25). Of course, there is no need to repeat the full calculation for a different value of $Y_e$ as we scaled $R_0$ and $M_0$ appropriately. In order to create the complete
mass-radius relation, we repeated the same procedure for a large number of central densities. Here a step size of $\Delta \tau = 0.0001$ was used throughout and scaled central densities ranged from 0.1 to 100, moving in steps of 0.1 at first, then to increments of 1. Doing so produces Fig. 3 and, in particular, a Chandrasekhar limit very close to $1.4 \, M_\odot$. However, as alluded in the toy-model problem and confirmed in this numerical calculation, there is no Chandrasekhar limit if one uses a non-relativistic dispersion relation.

![Mass-radius relation for white dwarf stars obtained using Excel (lines) and Maple (symbols).](image)

FIG. 3: Mass-vs-radius relation for white dwarf stars obtained using Excel (lines) and Maple (symbols).

2. White Dwarf Stars with Maple

The Maple code was created in a simple manner. It starts with a switch (or option) that queries the user for the type of Fermi gas equation of state (relativistic or non-relativistic) to be used. With this information one uses Eq. (25) to define appropriate dimensionful mass and radius parameters which are used after the scaled equations have been solved. Next, a Maple procedure was written to solve the scaled differential equations (consistent with the switch) for a given central (scaled) Fermi momentum using a classical numeric method. We noted that since in the structure equations the radius appears in the denominator of several expressions, we could not start calculating at a zero radius. Of course, as the limit is well defined and finite as the radius goes to zero, one could use an extremely small value to solve this problem (we used a scaled radius of $10^{-23}$ for the first point). For a given central Fermi momentum, the equations were numerically solved for $x_F$ and the enclosed mass $\overline{M}$ as a function of the scaled radius $\tau$ using a very small step size of $\Delta \tau = 0.00025$. Such a small value for $\Delta \tau$ is necessary in order to account for the rapidly-varying behavior of the density on the surface of the star. The program was instructed to stop once the scaled
Fermi momentum turned negative, with the previous point defining the radius of the star and the value of the enclosed mass at this point representing the mass of the star. Physical dimensions were restored by multiplying these scaled values by the dimensionful mass and radius parameters computed earlier. Having done this once, the procedure was repeated for a large range of central densities so one could accurately map the mass-vs-radius relation of the star. Once these values were stored, a plot was generated (Fig. 3). Fig. 3 illustrates the consistency of the results using either Excel or Maple; the data calculated using Excel are shown as lines, and the data from Maple are shown as symbols.

VI. SUMMARY - CONCLUDING REMARKS

Isaac Newton once said: If I have seen farther than others, it is because I was standing on the shoulders of giants. The foundations for the present experiment were laid by giants such as Chandrasekhar, Fermi, Dirac, and Einstein. Their seminal work placed the fascinating world of compact stars within the reach of the whole scientific community. Literally, they reduced the problem of compact stellar objects to quadratures. When these insights are combined with the remarkable advances in computer processing and algorithms, the problem becomes accessible to undergraduate students. (Indeed, highly motivated high-school students should also be able to tackle most of this problem.) The exercise reported in this manuscript is a testimony to this fact.

Our work benefited greatly from earlier contributions by Balian and Blaizot [1], and Silbar and Reddy [2] that pioneered the idea of bringing the physics of stars to the realm of the classroom. Here we have followed closely on their footsteps. For the present project a group involving two upper-level undergraduate physics students, a beginning physics graduate, and two advanced graduate students was assembled. It was demonstrated that with a limited knowledge of calculus and physics, undergraduate students can easily tackle the structure equations for white dwarf stars. Moreover, we are convinced that with a relative small amount of mentoring, the same will be true for motivated high school students [2].

Students learned several important lessons from this project. One of them relates to the usefulness of scaling the equations—without scaling, the problem would have been unsolvable. This is due to the tremendous range of scales encountered in this problem; there are more than 60 orders of magnitude between the minute electron mass and the immense solar mass. Another important lesson learned is that, contrary to what seems to happen in the classroom, most problems in physics have no analytic solution. Thus, numerical analysis is a necessary step towards a solution. It was shown that with a limited knowledge of calculus one can derive suitable recursion relations to arrive at accurate solutions to the differential equations by using a simple spreadsheet program like Microsoft Excel. For more advanced students, symbolic programs (such as Maple or Mathematica) provide a more efficient method for arriving at the solutions. Due to licensing agreements, Maple was used in this work.

In conclusion, this project successfully utilized common resources to solve structure equations for compact objects in the Newtonian regime. Building on this project, students are now in a position to study the fascinating physics of neutron stars. The structure of neutron stars, however, poses several additional challenges. First and foremost, Newtonian gravity must be replaced by general relativity. This implies that the structure equations must be replaced by the Tolman-Oppenheimer-Volkoff equations [13, 17, 18]. Second, at the higher densities encountered in the interior of neutron stars, the equation of state receives important corrections from the interactions among the neutrons. That is, Pauli correlations are
no longer sufficient to describe the equation of state and some realistic equations of state should be used \cite{2,22}. This topic of intense research activity is of relevance to the physics of neutron stars and to the structure of those exotic compact objects known as hybrid and quark stars.

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