NEW CHARACTERIZATIONS OF PLURISUBHARMONIC
FUNCTIONS AND POSITIVITY OF DIRECT IMAGE SHEAVES

FUSHENG DENG, ZHIWEI WANG, LIYOU ZHANG, AND XIANGYU ZHOU

ABSTRACT. We give new characterizations of plurisubharmonic functions and
Griffiths positivity of holomorphic vector bundles with singular Finsler met-
rics. As applications, we present a different method to prove plurisubharmonic
variation of generalized Bergman kernel metrics and Griffiths positivity of the
direct images of twisted relative canonical bundles associated to holomorphic
families of Kähler manifolds.

CONTENTS

1. Introduction
2. $m$-Bergman kernel metric and Narasimhan-Simha metric
   2.1. $m$-Bergman kernels associated to domains in $\mathbb{C}^n$
   2.2. $m$-Bergman kernel metrics on complex manifolds
   2.3. Relative Bergman kernel metrics and Narasimhan-Simha metrics
3. Singular Finsler metrics on coherent analytic sheaves
4. Extension theorems of Ohsawa-Takegoshi type
5. Regularity of Bergman kernel metrics and Hodge-type metrics
   5.1. For families of compact Kähler manifolds
   5.2. For families of pseudoconvex domains
6. New characterization of plurisubharmonic functions and positively
curved vector bundles
   6.1. Characterization of plurisubharmonic functions
   6.2. Characterization of positive vector bundles
7. Proof of Berndtsson’s minimum principle
8. Plurisubharmonic variation of relative $m$-Bergman kernel metrics
   8.1. For families of pseudoconvex domains
   8.2. For families of compact Kähler manifolds
9. Positivity of direct images of twisted relative canonical bundles
   9.1. For families of pseudoconvex domains
   9.2. For families of compact Kähler manifolds
References

1. INTRODUCTION

The aim of the present paper is to give a new characterization of plurisubhar-
monic (p.s.h. for abbreviation) functions and a new characterization of Griffiths
positivity of holomorphic vector bundles with singular Finsler metrics, and present
a different method to discuss plurisubharmonic variation of generalized Bergman kernel metrics and Griffiths positivity of the direct image of the twisted relative canonical bundle associated to a holomorphic family of complex Kähler manifolds, by using the characterizations. The work is inspired by Demailly’s method to the regularization of plurisubharmonic functions [10] and Berndtsson’s proof of the integral form of the Kiselman’s minimum principle for p.s.h. functions [20], [1].

To explain our observations, let us first take a look at Demailly’s method to the regularization of plurisubharmonic functions. Let \( \varphi \) be a plurisubharmonic function on a bounded pseudo-convex domain \( D \subset \mathbb{C}^n \). Let \( K_m \) be the weighted Bergman kernel of \( H^2(D, e^{-m \varphi}) \), the Hilbert space of holomorphic functions on \( D \) which are square-integrable with respect to the weight \( e^{-m \varphi} \). Applying the Ohsawa-Takegoshi extension theorem [26], Demailly showed that \( \frac{1}{m} \log K_m \) converges (in certain sense) to \( \varphi \) on \( D \) as \( m \to \infty \), and then got a regularization of the original plurisubharmonic function \( \varphi \).

The first observation in this paper is a new characterization of plurisubharmonic functions based on certain \( L^p \) extension property. We start from an upper semi-continuous function \( \varphi \) on \( D \) which is not assumed to be plurisubharmonic at the beginning, but we assume that the Ohsawa-Takegoshi extension theorem also holds on \( D \) with the weight \( e^{-m \varphi} \) for all \( m \geq 1 \). Then from the argument of Demailly, the convergence for \( \frac{1}{m} \log K_m \) to \( \varphi \) is also valid. On the other hand, it is clear that the logarithm of the Bergman kernels \( \frac{1}{m} \log K_m \) are always plurisubharmonic. Therefore at the end one can see that \( \varphi \) is plurisubharmonic. This implies that a rough converse of the Ohsawa-Takegoshi extension theorem holds. Precisely we have the following slightly stronger result.

**Theorem 1.1.** Let \( \varphi : D \to [-\infty, +\infty) \) be an upper semicontinuous function on \( D \subset \mathbb{C}^n \) that is not identically \( -\infty \). Let \( p > 0 \) be a fixed constant. If for any \( z_0 \in D \) with \( \varphi(z_0) > -\infty \) and any \( m > 0 \), there is \( f \in \mathcal{O}(D) \) such that \( f(z_0) = 1 \) and

\[
\int_D |f|^p e^{-m \varphi} \leq C_m e^{-m \varphi(z_0)},
\]

where \( C_m \) are constants independent of \( z_0 \) and satisfying the growth condition

\[
\lim_{m \to \infty} \frac{1}{m} \log C_m = 0,
\]

then \( \varphi \) is plurisubharmonic.

**Remark 1.1.** It is worth to mention that Berndtsson proved that a continuous function \( \varphi \) on a planar domain is subharmonic if \( e^{-m \varphi} \) can be used as a weight for Hörmander’s \( L^2 \)-estimate for \( \bar{\partial} \) [1]. For the argument in [1], dimension-one condition and continuity for \( \varphi \) are necessary assumptions. It seems interesting to generalize Berndtsson’s result to higher dimensions and upper semi-continuous functions.

Our second main observation is relating Theorem 1.1 with positivity of direct image sheaves of twisted relative canonical bundles, which is also inspired by Demailly’s regularization of p.s.h. functions. This observation comes from a new geometric interpretation of Theorem 1.1 as follows.

Let \( D \) and \( \varphi \) be as in Theorem 1.1. We view \( e^{-\varphi} \) as a (singular) hermitian metric on the trivial line bundle \( L = D \times \mathbb{C} \). Let \( \pi = \text{Id} : D \to D' = D \) be the trivial fibration with each fiber being one single point. It is obvious that the associated twisted relative canonical bundle \( K_{D/D'} \otimes L \) is isomorphic to \( L \) and its direct image \( L' := \pi_*(K_{D/D'} \otimes L) \) is also canonically isomorphic to \( L \). The Hodge-type metric...
NEW CHARACTERIZATIONS OF PLURISUBHARMONIC FUNCTIONS 3

on $L'$, denoted by $e^{-\varphi'}$, is given by integration along fibers. In the case under discussion, since all fibers are single points, the Hodge-type metric on $L'$ is given by valuations of the norms of the relative sections. It is obvious that $\varphi' = \varphi$. Therefore Theorem 1.1 implies that: if $(mL, e^{-m\varphi})$ satisfies the Ohsawa-Takegoshi extension theorem for all $m \geq 1$, then the direct image $\pi_* (K_{D/D'} \otimes L)$ is positively curved with respect to the Hodge-type metric. This simple observation leads to the expectation that certain extension property should imply positivity of direct image sheaves.

In connection to this direction, we give a characterization of Griffiths positivity of holomorphic vector bundles, as a generalization of Theorem 1.1. We first introduce the notion of multiple $L^p$-extension property for holomorphic vector bundles with singular Finsler metrics.

**Definition 1.1** (Multiple $L^p$-extension property). Let $(E, h)$ be a holomorphic vector bundle over a bounded domain $D \subset \mathbb{C}^n$ equipped with a singular Finsler metric $h$. Let $p > 0$ be a fixed constant. Assume that for any $z \in D$, any nonzero element $a \in E_z$ with finite norm $|a|$, and any $m \geq 1$, there is a holomorphic section $f_m$ of $E^\otimes m$ on $D$ such that $f_m(z) = a^\otimes m$ and satisfies the following estimate:

$$\int_D |f_m|^p \leq C_m |a^\otimes m|^p = C_m |a|^mp,$$

where $C_m$ are constants independent of $z$ and satisfying the growth condition $\frac{1}{m} \log C_m \to 0$ as $m \to \infty$. Then $(E, h)$ is said to have multiple $L^p$-extension property.

The following theorem says that multiple $L^p$-extension property for some $p > 0$ implies Griffiths positivity.

**Theorem 1.2.** Let $(E, h)$ be a holomorphic vector bundle over a bounded domain $D \subset \mathbb{C}^n$ equipped with a singular Finsler metric $h$, such that the norm of any local holomorphic section of $E^*$ is upper semicontinuous. If $(E, h)$ has multiple $L^p$-extension property for some $p > 0$, then $(E, h)$ is positively curved in the sense of Griffiths, namely $\log |u|$ is plurisubharmonic for any local holomorphic section $u$ of $E^*$.

The reader is referred to §3 for the definitions of singular Finsler metrics and dual Finsler metrics, and §6.2 for the definition of Finsler metrics on tensor products of vector bundles.

**Remark 1.2.** Some remarks about Theorem 1.2:

- Although this theorem is stated and proved for vector bundles of finite rank, the same argument also works for holomorphic vector bundles of infinite rank;
- In general a holomorphic vector bundle has certain positivity property if it has a lot of holomorphic sections. However the points here are different. Firstly the nature of this theorem is completely local, and secondly our aim here is not to show that $E$ admits some positively curved metric, instead our aim is to show that the metric $h$ itself is positively curved;
- It seems that multiple $L^p$-extension property is stronger than Griffiths positivity, and possibly it is more or less equivalent to Nakano positivity.
The third observation in the present paper is introducing fiber-product powers of holomorphic families. Motivated by Demailly’s method to regularization of p.s.h functions, it is natural to expect that considering the tensor product powers of a line bundle twisting the relative canonical bundle may enable us to prove the positivity of the direct image of the twisted relative canonical bundles. However, this is not powerful enough when we study families of complex manifolds with positive dimensional fibers.

In this paper, we consider fiber-product powers of holomorphic families, as well as tensor product powers of the involved line bundle. Recall that for a holomorphic map \( p : X \to Y \) between complex manifolds, the \( m \)-th power of fiber product is defined to be
\[
X \times_Y \cdots \times_Y X := \{(x_1, \cdots, x_m) \in X^m; p(x_1) = \cdots = p(x_m)\}.
\]
This observation is inspired by Berndtsson’s work on minimum principle for p.s.h functions [1].

We now discuss applications of the above observations to plurisubharmonic variation of Bergman kernels and Griffiths positivity of direct image sheaves associated to holomorphic families of complex manifolds. The main tool in the arguments is an Ohsawa-Takegoshi type \( L^2 \) extension theorem of the following form:

**Theorem 1.3.** Let \((X, \omega)\) be a weakly pseudoconvex Kähler manifold and \( L \) be a holomorphic line bundle over \( X \) with a (singular) hermitian metric \( h \). Let \( s : X \to \mathbb{C}^r \) be a holomorphic map such that \( 0 \in \mathbb{C}^r \) is not a critical value of \( s \). Assume that the curvature current of \((L, h)\) is semi-positive and \( |s(x)| \leq M \) for some constant \( M \). Let \( Y = s^{-1}(0) \) be the zero set of \( s \). Then for every holomorphic section \( f \) of \( K_X \otimes L \) over \( Y \) such that \( \int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_\omega < +\infty \), there exists a holomorphic section \( F \) of \( K_X \otimes L \) over \( X \) such that \( F|_Y = f \) and
\[
\int_X |F|^2 L dV_{X, \omega} \leq C_{r,M} \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y, \omega}
\]
where \( C_{r,M} \) is a constant depending only on \( r \) and \( M \).

**Remark 1.3.** For Theorem 1.3, the case that \( X \) is a pseudoconvex domain is proved by Ohsawa and Takegoshi in [26]. A geometric presentation in the case that \( X \) is Kähler and \( h \) is smooth was given by Manivel [24] and Demailly [11]. Recently, after the works by Blocki [8] and Guan-Zhou [15, 16, 17], the optimal \( L^2 \) extension in the setting of pseudoconvex Kähler manifolds presented as above was proved by Cao [9] and Zhou-Zhu [29] with an optimal estimate of the constant \( C_{r,M} \). In this paper, we also need an \( L^p(0 < p < 2) \) variant of Theorem 1.3 due to Berndtsson-Păun [7].

In [17] (see also [18]) Guan and Zhou observed a connection between positivity of the twisted relative canonical bundles and Ohsawa-Takegoshi type extension, by showing that plurisubharmonic variation of the relative Bergman kernels proved in [2] and [6] and Griffiths positivity of the twisted relative canonical bundles [9] can be deduced from Theorem 1.3 with optimal estimate. On the other hand, Berndtsson and Lempert [5] show that Theorem 1.3 with optimal estimate in the case of pseudoconvex domains can be deduced from Berndtsson’s result on positivity of direct image of the twisted relative canonical bundles [3].
Roughly speaking, by the works of Guan-Zhou and Berndtsson-Lempert, the Ohsawa-Takegoshi type extension theorems with optimal estimate and the positivity of the twisted relative canonical bundle are equivalent. Recently, based on Guan-Zhou’s above observation, Hacon, Popa and Schnell [19], and Păun and Takayama [27], and Zhou and Zhu [30] showed that, for a projective and Kähler family of manifolds, the positivity of direct images of the twisted relative pluricanonical bundles can be deduced from Theorem 1.3 with optimal estimate.

Note that the Ohsawa-Takegoshi extension theorem with optimal estimate and the positivity of direct image sheaves are equivalent, it seems to be interesting that the positivity of direct image sheaves could be deduced from Theorem 1.3 without optimal estimate. We should remark that the curvature positivity property involves the mean value inequality, which is an exact inequality, while the estimate in Theorem 1.3 contains a constant which is not exact.

In this paper, we’ll present a different approach to the Griffiths positivity of direct image sheaves based on the new characterizations of p.s.h. functions, together with Theorem 1.3. The key insight is Theorem 1.1 which says that the submean value inequality can be deduced from a multiple extension property, where nonexact constants are allowed provided they satisfy certain growth condition.

The first applications of the above observations is to plurisubharmonic variation of $m$-Bergman kernels associated to a family of pseudoconvex domains. Let $\Omega \subset \mathbb{C}^r \times \mathbb{C}^n$ be a pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection. Let $\varphi$ be a p.s.h. function on $\Omega$, which is assumed to be bounded. For $t \in U$, let $D_t = \{t\} \times D$ and $\varphi_t(z) = \varphi(t,z)$. Let $m$ be a positive integer, and let $K_{m,t}(z)$ be the $m$-Bergman kernel on $\Omega_t$ with weight $e^{-\varphi_t}$ (see §2.1 for definition). When $m = 1$, $K_{1,t}(z)$ is the ordinary relative weighted Bergman kernel. As $t$ varies, $K_{m,t}(z)$ gives a function on $\Omega$.

Theorem 1.4. The function $\log K_{m,t}(z)$ is plurisubharmonic on $\Omega$.

The method in the proof of Theorem 1.4 can also be applied to the case of families of pseudoconvex domains. Let $\Omega \subset \mathbb{C}^r \times \mathbb{C}^n$ be a pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$. Let $\varphi$ be a p.s.h. function on $\Omega$. Let $\Omega_t := p^{-1}(t) (t \in U)$ be the fibers. We also denote by $\varphi_t(z) = \varphi(t,z)$ the restriction of $\varphi$ on $\Omega_t$. Let $m$ be a positive integer, and let $K_{m,t}(z)$ be the $m$-Bergman kernel on $\Omega_t$ with weight $e^{-\varphi_t}$ (see §2.1 for definition). When $m = 1$, $K_{1,t}(z)$ is the ordinary relative weighted Bergman kernel. As $t$ varies, $K_{m,t}(z)$ gives a function on $\Omega$.

Theorem 1.4. The function $\log K_{m,t}(z)$ is plurisubharmonic on $\Omega$.

The second application of the observations is to prove positivity of certain direct image sheaves. We first consider the case of families of pseudoconvex domains.

Let $U, D$ be bounded pseudoconvex domains in $\mathbb{C}^r$ and $\mathbb{C}^n$ respectively, and let $\Omega = U \times D \subset \mathbb{C}^r \times \mathbb{C}^n$. Let $\varphi$ be a p.s.h function on $\Omega$, which is assumed to be bounded. For $t \in U$, let $D_t = \{t\} \times D$ and $\varphi_t(z) = \varphi(t,z)$. Let $E_t = H^2(D_t, e^{-\varphi_t})$ be the space of $L^2$ holomorphic functions on $D_t$ with respect to the weight $e^{-\varphi_t}$. Then $E_t$ are Hilbert spaces with the natural inner product. Since $\varphi$ is assumed to be bounded on $\Omega$, all $E_t$ for $t \in U$ are equal as vector spaces. However, the inner products on $E_t$ depend on $t$ if $\varphi(t,z)$ is not constant with $t$. So, under the natural projection, $E = \prod_{t \in U} E_t$ is a trivial holomorphic vector bundle (of infinite rank) over $U$ with varying Hermitian metric.

In [8], Berndtsson proved that $E$ is semipositive in the sense of Griffiths, namely, for any local holomorphic section $\xi$ of the dual bundle $E^*$, the function $\log |\xi|$ is plurisubharmonic (indeed Berndtsson proved a stronger result which says that $E$ is semipositive in the sense of Nakano). The aim here is to apply the above observations to present a new proof of Berndtsson’s result. The argument in [8]
involves taking derivatives and hence \( \varphi \) is assumed to be smooth up to the boundary of \( \Omega \). In the following Theorem 1.5, smoothness for \( \varphi \) is not necessary.

**Theorem 1.5.** The vector bundle \( E \) is semipositive in the sense of Griffiths.

**Remark 1.4.** We will see that Theorem 1.5 can be deduced from Theorem 1.3 for pseudoconvex domains. We have mentioned that Theorem 1.3 with optimal estimate for pseudoconvex domains can be deduced from Theorem 1.5 [5]. So logically one can think that Theorem 1.3 with optimal estimate is a consequence of Theorem 1.5 without optimal estimate. However, this is not a proper viewpoint since the work in [5] contains some essentially new input, which can be viewed as a new technique of localization. On the other hand, it is natural to expect that Theorem 1.3 with optimal estimate can be deduced from Theorem 1.3, by the method of raising powers of complex manifolds that is discussed above. To do this, it seems that one need to prove a modification of the \( L^2 \)-existence theorem in [11].

We now consider the positivity of the direct image sheaf of the twisted relative canonical bundle associated to a family of compact Kähler manifolds. Let \( X, Y \) be Kähler manifolds of dimension \( r + n \) and \( r \) respectively, and let \( p : X \to Y \) be a proper holomorphic map. For \( y \in Y \) let \( X_y = p^{-1}(y) \), which is a compact submanifold of \( X \) of dimension \( n \) if \( y \) is a regular value of \( p \). Let \( L \) be a holomorphic line bundle over \( X \), and \( h \) be a singular Hermitian metric on \( L \), whose curvature current is semi-positive. Let \( K_{X/Y} \) be the relative canonical bundle on \( X \). Let \( E = p^*(mK_{X/Y} \otimes L \otimes \mathcal{I}_m(h)) \) be the direct image sheaves on \( Y \) (see §4 for the definition of the idea sheaf \( \mathcal{I}_m(h) \)). One can choose a proper analytic subset \( A \subset Y \) such that:

1. \( p \) is submersive over \( Y \setminus A \),
2. \( E_m \) is locally free on \( X \setminus A \), and
3. \( E_{m,y} \) is naturally identified with \( H^0(X_y, mK_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}_m(h)|_{X_y}) \), for \( y \in Y \setminus A \).

where \( E_m \) is the vector bundle on \( Y \setminus A \) associated to \( E_m \). For \( u \in E_{m,y} \), the \( m \)-norm of \( u \) is defined to be

\[
H_m(u) := \|u\|_m = \left( \int_{X_y} |u|^2/m h^{1/m} \right)^{m/2} \leq +\infty.
\]

Then \( H_m \) is a Finsler metric on \( E_m \). In the case that \( m = 1 \), we denote \( E_1, H_1 \) by \( \mathcal{E}, H \) respectively. The following theorem says that \( H \) is a positively curved singular Hermitian metric in the coherent sheaf \( \mathcal{E} \) (see Definition 3.4 for definition).

**Theorem 1.6.** \( H \) is a positively curved singular Hermitian metric on \( \mathcal{E} \).

In the proof of Theorem 1.6, we first show that \( H \) is a positively curved Hermitian metric on \( p_*(K_{X/Y} \otimes L) \), and deduce form which the positivity of \( (\mathcal{E}, H) \) by Oka-Grauert principle.

By the plurisubharmonic variation of the \( m \)-Bergman kernel metrics and Theorem 1.6, one can see that the NS metric (see §2.3 for definition) on the direct image \( p_*(kK_{X/Y} \otimes L \otimes \mathcal{I}_k(h)) \) is positively curved in the sense of Griffiths. Plurisubharmonic variation of Bergman kernels and positivity of direct images of twisted relative canonical bundles have been extensively studied in recent years by many authors (see e.g. [2][3][6][7][17][14][27][9][19][29][30]), in different settings.
and different generality. The method in the present paper is quite different from those in the cited works.

Remark 1.5. The same argument can be used to show that Theorem 1.6 still holds if \( L \) is replaced by a holomorphic vector bundle with a Nakano semi-positive Hermitian metric (see also [22]).

Remark 1.6. It is possible to prove that the metric \( H_m \) (which is different from the NS metric defined in §2.3) is a positively curved singular Finsler metric on the coherent sheaf \( \mathcal{E}_m \) for all \( m \geq 1 \). Considering their significance in birational classification of algebraic varieties (see e.g. [28]), it seems that the study of the sheaves \( \mathcal{E}_m \) is more important than that of their descendent objects - \( m \)-Bergman kernel metrics. This topic will be discussed in a forthcoming work [12].

Acknowledgements. The authors are partially supported by NSFC grants.

2. \( m \)-Bergman kernel metric and Narasimhan-Simha metric

In this section, we recall the so-called (relative) \( m \)-Bergman kernel metrics and Narasimhan-Simha metrics (NS metric for short) on the twisted relative pluricanonical bundles associated to a family of complex manifolds. We also prove a product property of (relative) \( m \)-Bergman kernels which will be frequently used in the rest of this paper.

2.1. \( m \)-Bergman kernels associated to domains in \( \mathbb{C}^n \). We first recall the definition of \( m \)-Bergman kernels on domains (see [25] for more details). Let \( \Omega \subset \mathbb{C}^n \) be a domain. Let \( \varphi \) be an upper semicontinuous (u.s.c for short) function on \( \Omega \). Let \( m \geq 1 \) be an integer (indeed \( m \) can be a real number in domain case). The weighted \( m \)-Bergman space is defined as

\[
H_m(\Omega, \varphi) := \{ u \in \mathcal{O}(\Omega) : \|u\|_m := \left( \int_{\Omega} |u|^2/m \cdot e^{-\varphi/m} \right)^{m/2} < +\infty \}.
\]

Note that \( \|u\|_m \) is not a norm if \( m > 1 \) since the triangle inequality does not hold. The associated \( m \)-Bergman kernel is defined as

\[
K_m(x) := \sup_{u \in H_m(\Omega, \varphi) \setminus \{0\}} \frac{|u(x)|^2}{\|u\|_m^2},
\]

if \( u(x) \neq 0 \) for some \( u \in H_m(\Omega, \varphi) \), and \( K_m(x) = 0 \) if \( u(x) = 0 \) for all \( u \in H_m(\Omega, \varphi) \).

Equivalently, for any \( x \in \Omega \) such that \( K_m(x) \neq 0 \),

\[
K_m(x)^{-1} = \inf \{ \|u\|_m^2 : u(x) = 1, u \in H_m(\Omega, \varphi) \}.
\]

Remark 2.1. The 1-Bergman kernel is the usual weighted Bergman kernel with weight \( e^{-\varphi} \).

The \( m \)-Bergman kernels have the following product property.

Proposition 2.1. Let \( \Omega_1 \subset \mathbb{C}^n, \Omega_2 \subset \mathbb{C}^s \) be two domains, and \( \varphi_1, \varphi_2 \) be two u.s.c. functions on \( \Omega_1 \) and \( \Omega_2 \) respectively. Let \( K_{m,1}, K_{m,2}, K_m \) be the \( m \)-Bergman kernels of \( H_m(\Omega_1, \varphi_1) \), \( H_m(\Omega_2, \varphi_2) \) and \( H_m(\Omega_1 \times \Omega_2, \varphi_1 + \varphi_2) \) respectively. Then for any point \((x_1, x_2) \in \Omega_1 \times \Omega_2\),

\[
K_m(x_1, x_2) = K_{m,1}(x_1)K_{m,2}(x_2).
\]
Proof. It is obvious that $K_m(x_1, x_2) \geq K_m,1(x_1)K_m,2(x_2)$. It suffices to prove that $K_m(x_1, x_2) \leq K_m,1(x_1)K_m,2(x_2)$. If $K_m(x_1, x_2) = 0$, it is trivial, since one of $K_m(x_1)$ and $K_m(x_2)$ equals zero.

Now we assume that $K_m(x_1, x_2) \neq 0$. Let $u \in \mathcal{O}(\Omega_1 \times \Omega_2)$, such that $u(x_1, x_2) = 1$, and $u \in H_m(\Omega_1 \times \Omega_2, \varphi_1 + \varphi_2)$ Then from the definition of $m$-Bergman kernel, one can obtain that

$$\int_{\Omega_2} |u(z_1, z_2)|^{2/m} e^{-\varphi_2(z_2)/m} d\lambda_{z_2} \geq \frac{|u(z_1, z_2)|^{2/m}}{(K_m,2(x_2))^{1/m}}.$$ 

By Fubini theorem and the definition of $m$-Bergman kernel,

$$\left(\int_{\Omega_1 \times \Omega_2} |u(z_1, z_2)|^{2/m} e^{-(\varphi_1(z_1) + \varphi_2(z_2))/m} d\lambda_{z_1} d\lambda_{z_2}\right)^{1/m} \geq \frac{1}{K_m,2(x_2)} \int_{\Omega_1} |u(z_1, x_2)|^{2/m} e^{-\varphi_1(z_1)/m} d\lambda_{z_1} \geq \frac{1}{K_m,1(x_1)K_m,2(x_2)}.$$ 

Taking infimum on all $u \in H_m(\Omega_1 \times \Omega_2, \varphi_1 + \varphi_2)$ with $u(x_1, x_2) = 1$, we get

$$\frac{1}{K_m(x_1, x_2)} \geq \frac{1}{K_m,1(x_1)} \frac{1}{K_m,2(x_2)},$$

and hence $K_m(x_1, x_2) \leq K_m,1(x_1)K_m,2(x_2).$ $\square$

2.2. $m$-Bergman kernel metrics on complex manifolds. Let $X$ be a compact manifold of dimension $n$, and let $L$ be a holomorphic line bundle on $X$ with a (singular) Hermitian metric $h = e^{-\varphi}$. We assume that local weights $\varphi$ of $h$ takes values in $(-\infty, +\infty)$ and are upper semicontinuous. For a section $u \in H^0(X, mK_X \otimes L)$, we define its $m$-norm as

$$\|u\|_m := \left(\int_X |u|^{2/m} e^{\varphi/m}\right)^{m/2}.$$ 

Let $H^0_m(X, mK_X \otimes L) = \{u \in H^0(X, mK_X \otimes L); \|u\|_m < \infty\}$. If $H^0_m(X, mK_X \otimes L) \neq \{0\}$, it induces a metric on $mK_X \otimes L$ as follows. For $x \in X$ such that $u(x) \neq 0$ for some $u \in H^0_m(X, mK_X \otimes L)$, the evaluation map $H^0_m(X, mK_X \otimes L) \rightarrow (mK_X \otimes L)_x$ is surjective and hence induces a metric on $(mK_X \otimes L)_x$ given by

$$\|v\|_m := \inf\{\|u\|_m : u \in H^0_m(X, mK_X \otimes L), u(x) = v\},$$

$v \in (mK_X \otimes L)_x$; for $x \in X$ such that $u(x) = 0$ for all $u \in H^0_m(X, mK_X \otimes L)$, the metric on $(mK_X \otimes L)_x$ is defined to be $+\infty$ for all nonzero vectors.

The metric defined above is called the $m$-Bergman kernel metric on $mK_X \otimes L$ and will be denoted by $B^m_{(L,h)}$. The local weights of $B^m_{(L,h)}$ can be given as follows. Assume $(U, z)$ is a local coordinate on $X$ and $e$ is a local frame of $L$ on $U$. Then an element $u \in H^0(X, mK_X \otimes L)$ can be represented on $U$ as $u = \tilde{u}dz^m \otimes e$, where $\tilde{u}$ is a holomorphic function on $U$. Let $K_{(U,m)}(x) = \sup \{\|u\|_m : u \in H^0_m(X, mK_X \otimes L), \tilde{u}(x) = 1\}, x \in U$ if $u(x) \neq 0$ for some $u \in H^0_m(X, mK_X \otimes L)$, and set $K_{(U,m)}(x)$ otherwise. Then $\ln K_{(U,m)}$ is a local weight of $B^m_{(L,h)}$ on $U$ with respect to the frame $dz^m \otimes e$; namely, with respect to the $m$-Bergman metric,

$$|dz^m \otimes e|^2 = e^{-\ln K_{(U,m)}} = \frac{1}{K_{(U,m)}}.$$
The relation between the \( m \)-Bergman kernel defined in the previous subsection and the \( m \)-Bergman kernel metric is as follows. For a domain \( \Omega \subset \mathbb{C}^n \) and a p.s.h function \( \varphi \) on \( \Omega \), viewing \( e^{-\varphi} \) as a metric on the trivial line bundle \( L = \Omega \times \mathbb{C} \) over \( \Omega \), then the \( m \)-Bergman kernel metric on \( mK_\Omega \otimes L \equiv \Omega \times \mathbb{C} \) is given by \( \frac{1}{m} \), where \( K_m \) is the \( m \)-Bergman kernel on \( \Omega \) with weight \( \varphi \). Therefore, the \( m \)-Bergman kernel metric can be seen as an intrinsic definition of the \( m \)-Bergman kernel.

Similar to Proposition 2.2, we have the following product property for \( m \)-Bergman kernel metrics.

**Proposition 2.2.** Let \( X_1 \) and \( X_2 \) be two complex manifolds, and \( L_1 \rightarrow X_1 \) and \( L_2 \rightarrow X_2 \) be two holomorphic line bundles over \( X_1 \) and \( X_2 \) respectively. Let \( h_1 \) and \( h_2 \) be two Hermitian metrics on \( L_1 \) and \( L_2 \) respectively. Let \( p_i : X_1 \times X_2 \rightarrow X_i \) \((i = 1, 2)\) be the natural projections. Let \( L \) be the induced holomorphic line bundle \( p_1^*L_1 \otimes p_2^*L_2 \) over \( X_1 \times X_2 \) with the Hermitian metric \( h = p_1^*h_1 \cdot p_2^*h_2 \). Let \( B_m^{(L_1,h_1)}, B_m^{(L_2,h_2)} \) and \( B_m^{(L,h)} \) be the corresponding \( m \)-Bergman kernel metrics on \( mK_{X_1} \otimes L_1, mK_{X_2} \otimes L_2 \) and \( mK_X \otimes L \) respectively. Then

\[
B_m^{(L,h)} = B_m^{(L_1,h_1)} \cdot B_m^{(L_2,h_2)}.
\]

The statement in Proposition 2.2 is understood as follows. In a canonical way, we can identify \( K_X \) with \( p_1^*K_{X_1} \otimes p_2^*K_{X_2} \), and therefore identify \( mK_X \otimes L \) with \((mK_{X_1} \otimes L_1) \otimes (mK_{X_2} \otimes L_2)\). Then Proposition 2.2 says that the \( m \)-Bergman kernel metric \( B_m^{(L,h)} \) is induced by the \( m \)-Bergman kernel metrics \( B_m^{(L_1,h_1)}, B_m^{(L_2,h_2)} \) on \((mK_{X_1} \otimes L_1), (mK_{X_2} \otimes L_2)\).

**Proof.** The proof of Proposition 2.2 is similar to that of Proposition 2.1.

For simplicity, we denote \( B_m^{(L_1,h_1)}, B_m^{(L_2,h_2)}, B_m^{(L,h)} \) by \( B_1, B_2, B \) respectively.

By definition, it is obvious that \( B \leq B_1 \cdot B_2 \). We now prove that \( B \geq B_1 \otimes B_2 \).

Fix an arbitrary \((x_1, x_2) \in X_1 \times X_2\). If \( B(x_1, x_2) = +\infty \), there is nothing to prove. We assume that \( B(x_1, x_2) \neq +\infty \). Let \( a \in (mK_{X_1} \otimes L_1)|_{x_1}, b \in (mK_{X_2} \otimes L_2)|_{x_2} \) with \( a, b \neq 0 \). Let \( u \in B_m^{(L_1 \times L_2, mK_{X_1} \times X_2, L)} \) such that \( u(x_1, x_2) = a \otimes b \). From the definition of the Bergman kernel,

\[
\int_{X_2} |u(z, w)|^{2/m} h_2^{1/m} \geq |u(z, x_2)|^{2/m} B_2^{1/m}(b).
\]

By Fubini theorem and the definition of \( m \)-Bergman kernel,

\[
\left( \int_{X_1 \times X_2} |u(z, w)|^{2/m} h_1^{1/m} h_2^{1/m} \right)^m \geq B_2(b) \left( \int_{X_1} |u(z, x_2)|^{2/m} h_2^{1/m} \right)^m \geq B_1(a) B_2(b) = (B_1 \cdot B_2)(a \otimes b).
\]

Taking infimum on all such \( u \), we complete the proof. \( \square \)

2.3. Relative Bergman kernel metrics and Narasimhan-Simha metrics.

Let \( X \) be a compact complex manifold of dimension \( n \), and let \( L \) be a holomorphic line bundle on \( X \) with (singular) Hermitian metric \( h = e^{-\varphi} \). We assume that local weights \( \varphi \) of \( h \) take values in \([-\infty, +\infty)\) and are upper semicontinuous. In [2.2] we have defined the \( m \)-Bergman kernel metric \( B_m^{(L,h)} \) on \( mK_X \otimes L \). It is clear that \( h_m := (B_m^{(L,h)})^{-1} h^{1/m} \) defines a metric on \( L_m := (m-1)K_X \otimes L \), which is called
the Narasimhan-Simha metric (NS metric for short) on $L_m$. Then the Bergman kernel metric on $H^0(X, K_X \otimes L_m) = H^0(X, mK_X \otimes L)$ induced from $h_m$ is also called the Narasimhan-Simha metric on $H^0(X, mK_X \otimes L)$.

Let $X$ and $Y$ be complex manifolds of dimension $n + s$ and $s$ respectively. Let $p : X \to Y$ be a holomorphic surjective map. Denote by $X_y := p^{-1}(y)$. Note that if $p$ is a submersion, then all the fibers $X_y$ are complex submanifolds of the same dimension. Let $K_X, K_Y$ be the canonical bundles on $X, Y$ respectively, and $K_X/Y := K_X - p^*K_Y$ be the relative canonical bundle on $X$. Let $(L, h) \to X$ be a holomorphic line bundle over $X$ with a (singular) Hermitian metric $h$, whose weights are locally integrable functions.

Firstly, we assume that $p$ is a submersion. Let $t = (t_1, \ldots, t_s)$ be local coordinates on an open set $U \subset Y$. Then $dt := dt_1 \wedge \cdots \wedge dt_s$ is a trivialization of $K_Y$ on $U$, and gives a natural map from $(n, 0)$-forms on the fibers $X_y$ to sections $\tilde{u}$ of $K_X$ over $X_y$ for $y \in U$ by

$$\tilde{u} = u \wedge p^*(dt).$$

Conversely, given a local section $\tilde{u}$ of $K_X$, we can write $\tilde{u} = u \wedge dt$ locally. The restriction of $u$ to fibers is then uniquely defined and thus defines a section of $K_X$. From the correspondence between $u$ and $\tilde{u}$, we get an isomorphism

$$K_X \to K_X/Y|_{X_y}$$

$$u \mapsto u \wedge \frac{p^*(dt)}{dt}.$$ 

It is worth to mention that through the correspondence between $u$ and $\tilde{u}$ depends on the choice of $dt$, but the above isomorphism is independent of the choice of $dt$.

Similarly, we have the following isomorphism

$$mK_X \to mK_X/Y|_{X_y}$$

$$u \mapsto u \wedge \left(\frac{(p^*(dt))^{\otimes m}}{(dt)^{\otimes m}}\right),$$

which further induces a canonical isomorphism between $mK_X \times L|_{X_y}$ and $(mK_X/Y \otimes L)|_{X_y}$. Recall that we have defined the $m$-Bergman kernel metric on $mK_X \times L|_{X_y}$, which can also be viewed an Hermitian metric on $(mK_X/Y \otimes L)|_{X_y}$, according to the above isomorphism. As $y$ varies in $Y$, we get a (singular) Hermitian metric on $mK_X/Y \otimes L$, which will be called the relative $m$-Bergman kernel metric on $mK_X/Y \otimes L$. The relative NS metric on $(m - 1)K_X/Y \otimes L$ is defined in the same way from the NS metric on $(m - 1)K_X \times L|_{X_y}$.

If $p : X \to Y$ is not a submersion, the relative $m$-Bergman kernel metric given above is only defined on $(mK_X/Y \otimes L)|_{p^{-1}(U)}$ for some Zariski open set $U$ in $Y$. One of the main aims of the paper is to show that, for some cases, the $m$-Bergman kernel metric on $(mK_X/Y \otimes L)|_{p^{-1}(U)}$ is positively curved and can be extended to a positively curved metric on $mK_X/Y \otimes L$.

3. Singular Finsler metrics on coherent analytic sheaves

In this section, we recall the notions of singular Finsler metrics on holomorphic vector bundles and give a definition of positively curved singular Finsler metrics on coherent analytic sheaves.
Definition 3.1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold $X$. A (singular) Finsler metric $h$ on $E$ is a function $h : E \to [0, +\infty]$, such that $|v|^2_h := h(cv) = |c|^2 h(v)$ for any $v \in E$ and $c \in \mathbb{C}$.

In the above definition, we do not assume any regularity property of a singular Finsler metric. Only when considering Griffiths positivity certain regularity is required, as shown in the following Definition 3.1.

Definition 3.2. For a singular Finsler metric $h$ on $E$, its dual Finsler metric $h^*$ on the dual bundle $E^*$ of $E$ is defined as follows. For $f \in E_x^*$, the fiber of $E^*$ at $x \in X$, $|f|_{h^*}$ is defined to be 0 if $|v|_h = +\infty$ for all nonzero $v \in E_x$; otherwise,

$$|f|_{h^*} := \sup \{|f(v)|; v \in E_x, |v|_h \leq 1\} \leq +\infty.$$  

Definition 3.3. Let $E \to X$ be a holomorphic vector bundle over a complex manifold $X$. A singular Finsler metric $h$ on $E$ is called negatively curved (in the sense of Griffith) if for any local holomorphic section $s$ of $E$ the function $\log |s|^2_h$ is plurisubharmonic, and is called positively curved (in the sense of Griffiths) if its dual metric $h^*$ is negatively curved.

As far as our knowledge, there have not been natural definition of singular Finsler metric on a coherent analytic sheaf. In the present paper, we will propose a definition of positively curved Finsler metrics on coherent analytic sheaves. Let $\mathcal{F}$ be a coherent analytic sheaf on $X$, it is well known that $\mathcal{F}$ is locally free on some Zariski open subset $U$ of $X$. On $U$, we will identify $\mathcal{F}$ with the vector bundle associated to it.

Definition 3.4. Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold $X$. Let $Z \subset X$ be an analytic subset of $X$ such that $\mathcal{F}|_{X \setminus Z}$ is locally free. A positively curved singular Finsler metric $h$ on $\mathcal{F}$ is a singular Finsler metric on the holomorphic vector bundle $\mathcal{F}|_{X \setminus Z}$, such that for any local holomorphic section $g$ of the dual sheaf $\mathcal{F}^*$ on an open set $U \subset X$, the function $\log |g|^2_{h^*}$ is p.s.h. on $U \setminus Z$, and can be extended to a p.s.h. function on $U$.

Remark 3.1. Suppose that $\log |g|^2_{h^*}$ is p.s.h. on $U \setminus Z$. It is well-known that if $\text{codim}_C(Z) \geq 2$ or $\log |g|^2_{h^*}$ is locally bounded above near $Z$, then $\log |g|^2_{h^*}$ extends across $Z$ to $U$ uniquely as a p.s.h function. Definition 3.4 matches Definition 3.1 and Definition 3.3 if $\mathcal{F}$ is a vector bundle.

4. Extension theorems of Ohsawa-Takegoshi type

In this section, we prepare some extension theorems of Ohsawa-Takegoshi type which will be used as a central tool in this paper.

Theorem 4.1. Let $(X, \omega)$ be a weakly pseudoconvex Kähler manifold and $L$ be a holomorphic line bundle over $X$ with a (singular) hermitian metric $h$. Let $s : X \to \mathbb{C}$ be a holomorphic map such that $0 \in \mathbb{C}$ is not a critical value of $s$. Assume that the curvature current of $(L, h)$ is semi-positive and $|s(x)| \leq M$ for some constant $M$. Let $Y = s^{-1}(0)$ be the zero set of $s$. Then for every holomorphic section $f$ of $K_X \otimes L$ over $Y$ such that $\int_Y |f|^2 |\Lambda^*(ds)|^{-2}d\omega < +\infty$, there exists a holomorphic section $F$ of $K_X \otimes L$ over $X$ such that $F|_Y = f$ and

$$\int_X |F|^2_L dV_{X, \omega} \leq C_{r,M} \int_Y \frac{|f|^2_L}{|\Lambda^*(ds)|^2} dV_{Y, \omega},$$

where $C_{r,M}$ is a constant depending only on $r$ and $M$. 
Remark 4.1. For Theorem 1.3, the case that $X$ is a pseudoconvex domain is proved by Ohsawa and Takegoshi in [26]. A geometric presentation in the case that $X$ is Kähler and $h$ is smooth was given by Manivel [24] and Demailly [14]. Recently, after the works by Blocki ([8]) and Guan-Zhou ([15], [16], [17]), the optimal $L^2$ extension in the setting of pseudoconvex Kähler manifolds presented as above was proved by Cao [9] and Zhou-Zhu [29] with an optimal estimate of the constant $C_{r,M}$.

Combining Theorem 4.1 and the iteration method in [7], we get the following

**Theorem 4.2 ([7])**. Let $Ω ⊂ \mathbb{C}^{n+r}$ be a pseudoconvex domain, and $p : Ω → p(Ω) ⊂ \mathbb{C}^r$ be the natural projection. For $y ∈ p(Ω)$, we denote $Ω_y := p^{-1}(y) \cap Ω$. Let $ϕ$ be a p.s.h function on $Ω$. Assume that $|ϕ| ≤ M$ for all $y ∈ p(Ω)$. Let $m ≥ 1$ be an integer and $y_0 ∈ p(Ω)$ such that $ϕ$ is not identically $−∞$ on any branch of $Ω_{y_0}$. Then for any holomorphic function $u$ on $Ω_{y_0}$ such that

$$\int_{Ω_{y_0}} |u|^{2/m} e^{-ϕ} < +∞,$$

there exists a holomorphic function $U$ on $Ω$ such that $U|_{Ω_{y_0}} = u$ and

$$\int_{Ω} |U|^{2/m} e^{-ϕ} ≤ C_{r,M} \int_{Ω_{y_0}} |u|^{2/m} e^{-ϕ},$$

where $C_{r,M}$ is the constant as in Theorem 4.1.

**Proof.** We follow the idea as in the proof of Theorem in [7]. By standard regularization argument, without loss of generality, one may assume that $ϕ ∈ \text{Psh}(Ω) \cap C^\infty(Ω)$, $u$ is holomorphic on some neighborhood of $Ω_{y_0}$. We can also assume

$$\int_{Ω_{y_0}} |u|^{2/m} e^{-ϕ} dλ = 1.$$

From the pseudoconvexity of $Ω$, one can find a holomorphic function $F_1 ∈ \mathcal{O}(Ω)$ with $F_1|_{Ω_{y_0}} = u(x)$. Replacing $Ω$ by a relatively compact domain in it, we can assume

$$\int_{Ω} |F_1|^{2/m} e^{-ϕ} dλ ≤ A < +∞.$$

Let $ϕ_1 = ϕ + (1 − 1/m) \log |F_1|^2$, by Theorem 4.1 with the weight $ϕ_1$, there is a new extension of $F_2$ of $u$ satisfying

$$\int_{Ω} |F_2|^{2/m} e^{-ϕ} dλ ≤ C_{r,M} \int_{Ω_{y_0}} |u|^{2/m} e^{-ϕ} dλ = C_{r,M}.$$

By Hölder’s inequality,

$$\int_{Ω} |F_2|^{2/m} e^{-ϕ} dλ ≤ \left( \int_{Ω} |F_2|^{2/m} e^{-ϕ} dλ \right)^{1/m} \left( \int_{Ω} |F_1|^{2/m} e^{-ϕ} dλ \right)^{(m−1)/m} \leq C_{r,M}^{1/m} A^{(m−1)/m} = A(C_{r,M}/A)^{1/m} =: A_1.$$

We can assume $A > C_{r,M}$, then $A_1 < A$. Repeating the same argument with $F_1$ replaced by $F_2$, etc, we get a decreasing sequence of constants $A_k$, such that

$$A_{k+1} = A_k(C_{r,M}/A_k)^{1/m}.$$
for \( k \geq 1 \). It is easy to see that \( A_k \) tends to \( C_{r,M} \). Taking limit, we obtain a holomorphic function \( U \) on \( \Omega \) extending \( u \), such that

\[
\int_\Omega |U(x)|^{2/m} e^{-\varphi} d\lambda \leq C_{r,M}.
\]

\( \square \)

Theorem 4.2 can be extended to a family of compact Kähler manifolds. Let \( B \subset \mathbb{C}^r \) be the unit ball and let \( X \) be a Kähler manifold of dimension \( n + r \). Let \( p : X \to B \) be a holomorphic proper submersion. For \( t \in B \), denote by \( X_t \) the fiber \( p^{-1}(t) \). Let \( L \) be a holomorphic line bundle on \( X \) with a (singular) Hermitian metric \( h \) whose curvature current is positive.

Let \( k > 0 \) be a fixed integer. The multiplier ideal sheaf \( \mathcal{I}_k(h) \subset \mathcal{O}_X \) is defined as follows. If \( \varphi \) is a local weight of \( h \) on some open set \( U \subset X \), then the germ of \( \mathcal{I}_k(h) \) at a point \( p \in U \) consists of the germs of holomorphic functions \( f \) at \( p \) such that \( |f|^{2/k} e^{-\varphi/k} \) is integrable at \( p \). It is known that \( \mathcal{I}(h^{1/k}) \) is a coherent analytic sheaf on \( X \) [20].

Let \( \mathcal{E} = p_* (kK_X \otimes L \otimes \mathcal{I}_k(h)) \) be the direct image sheaf on \( B \). For any open subset \( U \subset B \) containing the origin \( 0 \) and any \( s \in H^0(U, \mathcal{E}_U) \), the restriction of \( s \) on \( X_0 \), denoted by \( s|_{X_0} \), gives a section in \( H^0(X_0, kK_X \otimes \mathcal{I}_k(h)|_{X_0}) \). By Cartan’s Theorem B, there exists a global section \( \tilde{s} \) of \( \mathcal{E} \) on \( B \) such that \( \tilde{s}|_{X_0} = s|_{X_0} \). For \( u \in H^0(X_0, kK_X \otimes \mathcal{I}_k(h)|_{X_0}) \), as in [22], the \( k \)-norm of \( u \) is defined to be

\[
\|u\|_k = \left( \int_{X_0} |u|^{2/k} h^{1/k} \right)^{k/2} \leq +\infty.
\]

Modifying the argument in the proof of Theorem 4.2, we can prove the following

**Theorem 4.3.** Let \( u \in H^0(X_0, kK_X \otimes \mathcal{I}_k(h)|_{X_0}) \) with \( \|u\|_k < \infty \). Assume there exist an open subset \( U \) containing the origin \( 0 \) and any \( s \in H^0(U, \mathcal{E}_U) \) such that \( s|_{X_0} \) = \( u \), then there exists \( s \in H^0(X, kK_X \otimes L \otimes \mathcal{I}_k(h)) \) such that \( s|_{X_0} = u \wedge dt^{\otimes k} \) and

\[
\int_X |s|^{2/k} h^{1/k} \leq C_{r,1} \int_{X_0} |u|^{2/k} h^{1/k},
\]

where \( t = (t_1, \ldots, t_r) \) is the standard coordinate on \( B \) and \( dt = dt_1 \wedge \cdots \wedge dt_r \), and \( C_{r,1} \) is the constant as in Theorem 4.2.

**Proof.** We assume that \( \|u\|_k = 1 \). As explained above, there is a \( \tilde{s}_1 \in H^0(B, \mathcal{E}) \) such that \( \tilde{s}_1|_{X_0} = u \). Let \( s_1 = \tilde{s}_1 \wedge dt \). By replacing \( B \) by a relatively smaller ball, we can assume that \( \int_X |s_1|^{2/k} h^{1/k} \leq A < \infty \) for some constant \( A \). The section \( s_1 \) induces a singular Hermitian metric

\[
h_1 = \left( \frac{1}{|s_1|^2} \right)^{k-1} h^{1/k}
\]

on \((k - 1)K_X \otimes L\), whose curvature current is positive. By Theorem 4.1, there is a section \( s_2 \in H^0(X, K_X \otimes ((k - 1)K_X \otimes L)) \) such that

\[
\int_X \frac{|s_2|^2}{|s_1|^{2-2/k}} h^{1/k} \leq C_{r,1} \int_{X_0} \frac{|u|^2}{|s_1|^{2-2/k}} h^{1/k} = C_{r,1}.
\]
By Hölder’s inequality,
\[
\int_X |s_2|^{2/k} h^{1/k} \leq \left( \int_X \frac{|s_2|^2}{|s_1|^{2-2/k}} h^{1/k} \right)^{1/k} \left( \int_X |s_1|^{2/k} h^{1/k} \right)^{(k-1)/k} \leq C_{r,1} 1/k A^{(k-1)/k} = A(C_{r,1}/A)^{1/k} =: A_1.
\]
We can assume \( A > C_{r,1} \), then \( A_1 < A \). Repeating the same argument with \( s_1 \) replaced by \( s_2 \), etc, we get a decreasing sequence of constants \( A_k \), such that
\[
A_{k+1} = A_k(C_{r,M}/A_k)^{1/m}
\]
for \( k \geq 1 \). It is easy to see that \( A_k \) tends to \( C_{r,M} \). Taking limit, we obtain a section \( s \) that satisfies the condition in the theorem. \( \Box \)

5. Regularity of Bergman kernel metrics and Hodge-type metrics

The aim of this section is to show certain continuity of relative \( m \)-Bergman kernel metrics and Hodge-type metrics on direct image sheaves.

5.1. For families of compact Kähler manifolds. Let \( X, Y \) be Kähler manifolds of dimension \( m + n \) and \( m \) respectively, let \( p : X \rightarrow Y \) be a proper holomorphic submersion. Let \( L \) be a holomorphic line bundle over \( X \), and \( h \) be a singular Hermitian metric on \( L \), whose curvature current is semi-positive. Let \( K_{X/Y} \) be the relative canonical bundle on \( X \).

Let \( \mathcal{E}_k = p_*(kK_{X/Y} \otimes L \otimes \mathcal{I}_k(h)) \) be the direct image sheaf on \( Y \). By Grauert’s theorem, \( \mathcal{E}_k \) is a coherent analytic sheaf on \( Y \). We assume that \( \mathcal{E}_k \) is locally free, then it is the sheaf of holomorphic sections of a holomorphic vector bundle, which will be denoted by \( E_k \). For any \( y \in Y \), we can identify the fiber \( E_{k,y} \) of \( E_k \) at \( y \) with \( H^0(X_y, (kK_{X/Y} \otimes L \otimes \mathcal{I}_k(h))|_{X_y}) \subset H^0(X_y, kK_{X_y} \otimes L|_{X_y}) \). For \( u \in E_{k,y} \), as in \([22]\) the \( k \)-norm of \( u \) is defined to be
\[
H_y(u) := \|u\|_k = \left( \int_{X_y} |u|^{2/k} h^{1/k} \right)^{k/2} \leq +\infty.
\]
Note that here we view \( u \) as an element in \( H^0(X_y, kK_{X_y} \otimes L|_{X_y}) \). Then \( H \) is a Finsler metric on \( E_k \). It is clear that \( H \) is locally bounded below by positive constants. The following proposition shows that \( H \) is lower semicontinuous.

**Proposition 5.1** \([19]\). Let \( s \) be a holomorphic section of \( E_k \). The function \( |s|_{k}(y) := \|s(y)\|_k : Y \rightarrow [0, +\infty] \) is lower semi-continuous.

**Proof.** We present the proof given in \([19]\). Without loss of generality, we assume that \( Y = B \), which is the unit ball in \( \mathbb{C}^m \) and prove that \( |s|_{k} \) is lower semicontinuous at the origin \( 0 \). Denote by \((t_1, \cdots, t_m)\) the standard coordinate system on \( B \), then the canonical bundle \( K_B \) is trivialized by the global section \( dt = dt_1 \wedge \cdots \wedge dt_m \), and the volume form on \( B \) is
\[
d\mu = c_m dt \wedge d\bar{t}.
\]
Denote by
\[
\beta = s \wedge (dt)^{\otimes k} \in H^0(B, kK_B \otimes E_k) \simeq H^0(X, kK_X \otimes L \otimes \mathcal{I}_k(h)).
\]
Since \( p : X \to B \) is a submersion, Ehresmann’s fibration theorem shows that \( X \) is diffeomorphic to the product \( B \times X_0 \). Choosing a Kähler metric \( \omega_0 \) on \( X_0 \), we can write

\[ |\beta|^{2/k} h^{1/k} = F \cdot d\mu \wedge \frac{\omega_0^n}{n!} \]

where \( F : B \times X_0 \to [0, +\infty) \) is lower semi-continuous and locally integrable; the reason is that the local weights for \((L, h)\) are upper semi-continuous. At every point \( y \in B \), we have that

\[ |s(y)|_{k, y} = \left( \int_{X_0} F(y, -) \frac{\omega_0^n}{n!} \right)^{k/2} \]

By Fubini’s theorem, the function \( |s|_k \) is measurable function on \( B \). Moreover, since \( F \) is locally integrable and \( X_0 \) is compact, \( |s(y)|_k \leq +\infty \) for almost every \( y \in B \).

We now need to show that

\[ |s(0)|_k \leq \liminf_{j \to +\infty} |s(y_k)|_k \]

holds for every sequence \( y_1, y_2, \ldots \in B \) which converges to the origin. By the lower semi-continuity of \( F \) and Fatou’s lemma, we obtain

\[
\int_{X_0} F(0, -) \frac{\omega_0^n}{n!} \leq \int_{X_0} \liminf_{k \to +\infty} F(y_k, -) \frac{\omega_0^n}{n!} \\
\leq \liminf_{k \to +\infty} \int_{X_0} F(y_k, -) \frac{\omega_0^n}{n!}.
\]

This completes the proof of this proposition. \( \square \)

The lower semicontinuity of the Finsler metric \( H \) on \( E_k \) does not imply automatically the upper semicontinuity of its dual metric \( H^* \) on \( E_k^* \). But in our case, \( H^* \) is indeed upper semicontinuous, as shown in the following proposition.

**Proposition 5.2.** With the same notations and assumptions as in Proposition 5.1. For every \( \xi \in H^0(Y, E_k^*) \), the function \( |\xi|_k := H^*(\xi(y)) : Y \to [0, +\infty] \) is upper semi-continuous.

**Proof.** Our proof here is based on the idea in \([19]\) and Theorem \([19]\). Without loss of generality, we assume that \( Y = B \), the unit ball in \( \mathbb{C}^m \). It suffices to prove \( |\xi|_k \) is upper semi-continuous at the origin of \( B \). We need to show that

\[ \limsup_{j \to +\infty} |\xi|(y_j) \leq |\xi|(0) \]

for every sequence \( y_1, y_2, \ldots \in B \) which converges to the origin. We may assume that \( |\xi|(y_j) \neq 0 \) for all \( k \in \mathbb{N} \), and that the sequence \( |\xi|(y_j) \) actually has a limit. By the lower semicontinuity of \( H \) on \( E_k \) as shown in Proposition 5.1, \( |\xi|(y_j) \) is upper semi-continuous for all \( j \). From the definition of the dual metric, for each \( j \in \mathbb{N} \), there is a holomorphic section \( u_j \in E_{k,y_j} \), such that \( \|u_j\|_k = 1 \) and \( |\xi(y_j), u_j| = |\xi|(0) \). By Theorem \([19]\), there are sections \( s_j \in H^0(X, \omega_X \otimes L) \) such that \( \int_X |s_j|^2/k h^{1/k} \leq C \) for some constant \( C > 0 \) independent of \( j \). By Montel’s theorem, we may assume \( s_j \) converges uniformly on compact sets of \( X \) to some \( s \in H^0(X, \omega_X \otimes L) \). Then \( \lim_{j \to \infty} <\xi(0), u_j> = <\xi(0), u := s(0)> \). It suffices to prove that \( \|u\|_k \leq 1 \). From
the proof of Proposition 5.1 each \( s_j \) determines a lower semi-continuous function 
\[ F_j : B \times X_0 \to [0, +\infty) \]
with
\[ 1 = \|u_j\|_k = \left( \int_{X_0} F_j(y_j, -e^{-\varphi_0}) \right) k/2. \]
In the same way, \( s \) determines a lower semi-continuous function 
\[ F : B \times X_0 \to [0, +\infty]. \]
Since the local weight \( e^{-\varphi} \) of \( h \) is lower semi-continuous, and \( s_j \) converges uniformly on compact subsets to \( s \), we get
\[ F(0, -) \leq \liminf_{j \to +\infty} F_j(y_j, -). \]
Then by Fatou’s lemma, we complete the proof of this proposition.

A direct consequence of Proposition 5.2 is the following

**Corollary 5.3.** For any \( m \geq 1 \), the relative \( m \)-Bergman kernel metric (see § 2.2 for definition) on \( mK_{X/Y} \otimes L \) is lower semi-continuous, namely, the norm of any local holomorphic section of \( mK_{X/Y} \otimes L \) with respect to the relative \( m \)-Bergman kernel metric is lower semi-continuous.

**5.2. For families of pseudoconvex domains.** Let \( \Omega \subset \mathbb{C}^{m+n} = \mathbb{C}^m \times \mathbb{C}^n \) be a pseudo-convex domain. Let \( p : \Omega \to \mathbb{C}^m \) be the natural projection. We denote \( p(\Omega) \) by \( D \) and denote \( p^{-1}(t) \) by \( \Omega_t \) for \( t \in D \). Let \( \varphi \) be a plurisubharmonic function on \( \Omega \) and let \( k \geq 1 \) be an fixed integer. For an open subset \( U \) of \( D \), we denote by \( F(U) \) the space of holomorphic functions \( F \) on \( p^{-1}(U) \) such that
\[ \int_{p^{-1}(K)} |F|^2/k e^{-\varphi} \leq \infty \]
for all compact subset \( K \) of \( D \). For \( t \in D \), let 
\[ E_{k,t} = \{ F |_{\Omega_t} : F \in F(U), U \subset D \text{ open and } t \in U \}. \]
\( E_{k,t} \) is a vector space and we define a norm on it as follows:
\[ H(f) := |f|_k = \left( \int_{D_t} |f|^2/k e^{-\varphi_t} \right) k/2 \]
where \( \varphi_t = \varphi|_{D_t} \). Let \( E_k = \bigsqcup_{t \in D} E_{k,t} \) be the disjoint union of all \( E_{k,t} \). Then we have a natural projection \( \pi : E_k \to D \) which maps elements in \( E_{k,t} \) to \( t \). We view \( H \) as a Finsler metric on \( E_k \).

In general \( E_k \) is not a genuine holomorphic vector bundle over \( D \). However, we can also talk about its holomorphic sections, which are the objects we are really interested in. By definition, a section \( s : D \to E_k \) is a **holomorphic section** if it varies holomorphically with \( t \), namely, the function \( s(t, z) : \Omega \to \mathbb{C} \) is holomorphic with respect to the variable \( t \). Note that \( s(t, z) \) is automatically holomorphic on \( z \) for \( t \) fixed, by Hartogs theorem, \( s(t, z) \) is holomorphic jointly on \( t \) and \( z \) and hence is a holomorphic function on \( \Omega \). In some sense, \( E_k \) can be viewed as an object similar to holomorphic vector fields studied in [21].

Let \( E^*_k \) be the dual space of \( E_{k,t} \), namely the space of all complex linear functions on \( E_{k,t} \). Let \( E^*_k = \bigsqcup_{t \in D} E^*_{k,t} \). The natural projection from \( E^*_k \) to \( D \) is denoted by \( \pi^* \). Note that we do not define any topology on \( E^*_{k,t} \) and \( E^*_k \). The only object we are interested in is holomorphic sections of \( E^*_k \) which we are going to define. Given a holomorphic section \( s \) of \( E_k \) on some open set \( U \) of \( D \), \( s \) induces a function \( |s|_k : U \to \mathbb{R} \) with \( |s|_k(t) \) given by \( |s(t)|_k \), which is lower semicontinuous and hence measurable, by the following Proposition 5.3.

**Definition 5.1.** A section \( \xi \) of \( E^*_k \) on \( D \) is holomorphic if:
(1) for any local holomorphic section \( s \) of \( E_k \), \( \langle \xi, s \rangle \) is a holomorphic function;
(2) for any sequence \( s_j \) of holomorphic sections of \( E_k \) on \( D \) such that \( \int_D |s_j|_k \leq 1 \), if \( s_j(t, z) \) converges uniformly on compact subsets of \( \Omega \) to \( s(t, z) \) for some holomorphic section \( s \) of \( E_k \), then \( \langle \xi, s_j \rangle \) converges uniformly to \( \langle \xi, s \rangle \) on compact subsets of \( D \).

In the same way we can define holomorphic section of \( E_k^* \) on open subsets of \( D \). The Finsler metric \( H \) on \( E_k \) induces a Finsler metric \( H^* \) on \( E_k^* \), as defined the Definition 5.2. We will show that \( H \) is lower semicontinuous and \( H^* \) is upper semicontinuous, as analogues of Proposition 5.1 and Proposition 5.2 in the case of families of pseudoconvex domains.

**Proposition 5.4.** With the above notations and assumptions. Assume \( s \) is a holomorphic section of \( E_k \), then the function \( |s|_k(t) := H(s(t)) : D \to [0, +\infty) \) is lower semicontinuous.

**Proof.** We assume \( 0 \in D \) and prove that \( |s|_k \) is lower semicontinuous for a point 0. Let \( K_1 \subset K_2 \subset \cdots \subset K_j \subset \cdots \subset \Omega_0 \) be an increasing sequence of compact subsets of \( \Omega_0 \), such that \( \cup_j K_j = \Omega_0 \). Since the set valued function \( t \to \Omega_t \) is lower semicontinuous, in the sense that if \( \Omega_t \) contains a compact set \( K \), then \( K \) is contained in all \( \Omega_{t'} \) for \( t' \) sufficiently close to \( t \). Thus for any \( j \), there is a small disk \( B_j \subset D \) centered at \( a \), such that \( B_j \times K_j \subset \Omega \). Note that \( e^{-\varphi} \) is lower semicontinuous, hence \( \liminf_{t \to 0} |s|_k(t) \geq \left( \int_{K_j} |s(0, z)|^{2/k} e^{-\varphi(t)} \right)^{k/2} \) for all \( j \). Letting \( j \) goes to \( \infty \), we get \( \liminf_{t \to 0} |s|_k(t) \geq |s|_k(0) \). 

The following lemma shows that \( |\xi|_k(t) \) can not take value \( +\infty \) anywhere.

**Lemma 5.5.** Let \( \xi \) be a holomorphic section of \( E_k^* \), then \( |\xi|_k(t) < +\infty \) for all \( t \in D \).

**Proof.** We argue by contradiction. Assume \( 0 \in D \) and \( |\xi(0)|_k = +\infty \). By definition, there is a sequence \( \{u_j\} \subset E_{k, 0} \) such that \( |u_j|_k = 1 \) and \( \lim_{j \to \infty} \langle \xi(0), u_j \rangle = +\infty \). By Theorem 4.2 there are holomorphic sections \( s_j \) of \( E_k \) such that \( s_j(0) = u_j \) and \( \int_D |s_j|_k \leq C \) for some constant \( C \) independent of \( j \). By Montel’s theorem there is a subsequence of \( \{s_j\} \), may assumed to be \( \{s_j\} \) itself, that converges uniformly on compact subsets of \( \Omega \) to some holomorphic section \( s \) of \( E_k \). By definition, \( \langle \xi, s_j \rangle \) converges uniformly on compact sets of \( D \) to \( \langle \xi, s \rangle \). In particular, \( \langle \xi(0), u_j \rangle \) converges to \( \langle \xi(0), s(0) \rangle \leq +\infty \), which is a contradiction.

**Proposition 5.6.** Let \( \xi : D \to E_k^* \) be a holomorphic section of \( E_k^* \). Then the function \( |\xi|_k(t) := H^*(\xi(t)) : D \to [0, +\infty) \) is upper semicontinuous.

**Proof.** We assume \( 0 \in D \) and prove that \( |\xi|_k \) is upper semicontinuous at 0. We need to show that
\[
\limsup_{j \to +\infty} |\xi|_k(t_j) \leq |\xi|_k(0).
\]
for every sequence \( t_1, t_2, \cdots \in D \) which converges to 0. We may assume that \( |\xi|_k(t_j) \neq -\infty \) for all \( j \in N \), and that the sequence \( |\xi|_k(t_j) \) actually has a limit. From the definition of the dual metric and Lemma 5.5 for each \( j \), there exists \( u_j \in E_{k, t_j} \), such that \( |u_j|_k = 1 \) and \( |\xi|_k(t_j) < |\langle \xi(t_j), u_j \rangle| + \epsilon \), where \( \epsilon > 0 \) is an
arbitrary constant. By Theorem \[\text{5.2} \] there are holomorphic sections \( s_j \) of \( E_k \) such that
\[
s_j(t_j) = u_j \quad \text{and} \quad \int_D |s_j(t)|_k \leq K
\]
for some constant \( K \) independent of \( j \). By Montel’s theorem, there is a subsequence of \( \{s_j\} \), may assumed to be \( \{s_j\} \) itself, that converges on compact subsets of \( \Omega \) uniformly to some holomorphic section \( s \) of \( E_k \). By definition, \( \langle \xi, s_j \rangle > \), as holomorphic functions on \( D \), converges uniformly on compact subsets of \( D \) to \( \langle \xi, s \rangle > \). In particular \( \lim \sup_{j \to \infty} |\xi|_k(t_j) \leq \left\langle \xi(t_j), u_j \right\rangle + \epsilon \to \left\langle \xi(0), s(0) \right\rangle \geq \epsilon \). If \( s(0) = 0 \), we are done. We assume \( s(0) \neq 0 \). Then it suffices to prove that \( |s(0)|_k \leq 1 \). But this is true since \( s_j \) converges to \( s \) uniformly on compact sets, \( |u_j|_k = 1 \), and \( e^{-\varphi} \) is lower semicontinuous. \( \square \)

A direct consequence of Proposition \[\text{5.6} \] is the following

**Corollary 5.7.** Let \( \Omega, p, \varphi \) as in the beginning of this subsection. For any positive integer \( k \), let \( K_k(t, z) \) be the \( k \)-Bergman kernel (see \[\text{2.1} \] for definition) on \( \Omega_t := p^{-1}(t), t \in D \), with \( e^{\varphi} \). Then the relative \( k \)-Bergman kernel \( K_k(t, z) \) is upper semi-continuous on \( \Omega \).

**Remark 5.1.** Let \( \xi \) be a holomorphic section of \( E_k^* \). By Lemma \[\text{5.5} \] and Theorem \[\text{5.6} \], \( |\xi|_k(t) \) is locally bounded above by positive constants. On the other hand, when \( k = 1 \), it is not difficult to show that a section of \( E_1^* \) is holomorphic if it satisfies condition (1) in Definition \[\text{5.1} \] and its norm is locally bounded above. It seems that the same result should be true for general \( k \), but we can not give a proof right off the bat.

**Remark 5.2.** For the case that \( \Omega = D \times D' \) is a product domain and \( \varphi \) is bounded on \( \Omega \), by the mean value inequality, \( |f|_k < \infty \) for any \( f \in E_{k,t} \). In the addition that \( k = 1 \), \( E_{1,t} \) consists of square integrable holomorphic functions on \( D' \) with respect to the weight \( e^{-\varphi} \), which is the setting considered by Berndtsson in \[\text{2} \].

### 6. New characterization of plurisubharmonic functions and positively curved vector bundles

**6.1. Characterization of plurisubharmonic functions.** It is known that a p.s.h function can be used as a weight in the Ohsawa-Takegoshi \( L^2 \) extension theorem. In this section, we prove a converse in some sense of this result, namely, if a function can be used as a weight in the Ohsawa-Takegoshi type \( \text{p.s.h functions} \) \[\text{10} \].

**Theorem 6.1.** Let \( \varphi : D \to [-\infty, +\infty) \) be a upper semicontinuous function on \( D \subset \mathbb{C}^n \) that is not identically \(-\infty \). Let \( p > 0 \) is a fixed constant. If for any \( z_0 \in D \) with \( \varphi(z_0) > -\infty \) and any \( m > 0 \), there is \( f \in \mathcal{O}(D) \) such that \( f(z_0) = 1 \) and
\[
\int_D |f|^p e^{-m\varphi} \leq C_m e^{-m\varphi(z_0)},
\]
where \( C_m \) are constants independent of \( z_0 \) and satisfying \( \log C_m/m \to 0 \), then \( \varphi \) is plurisubharmonic.
We need some preparation for the proof of Theorem 6.1. Let $D$, $\varphi$, and $p$ as in Theorem 6.1. Let

$$H^p(D, \varphi) = \{f \in \mathcal{O}(D); |f|_p := \int_D |f|^p e^{-\varphi} < \infty\}.$$  

For $z \in D$, define

$$K_{\varphi,p}(z) = (\inf \{|f|_p; f \in H^p(D, \varphi), f(z) = 1\})^{-1}$$

if there exist $f \in H^p(D, \varphi)$ with $f(z) \neq 0$, and otherwise $K_{\varphi,p}(z)$ is defined to be 0. It is also easy to see that $K_{\varphi,p}$ is continuous and log $K_{\varphi,p}$ is plurisubharmonic.

**Lemma 6.2.** With the above notations, $K_{\varphi,p}$ is a continuous function on $D$.

**Proof.** This is proved by an elementary normal family argument. By definition, it is clear that $K_{\varphi,p}$ is lower semicontinuous. Assume $a \in D$ and $z_j \in D$ which converge to $a$ as $j \to \infty$. Let $\epsilon > 0$ be arbitrary. There exist $f_j \in H^p(D, \varphi)$ such that $|f_j|_p = 1$ and $|f_j(z_j)|^p > K_{\varphi,p}(z_j) + \epsilon$. Since $\varphi$ is upper semicontinuous, $\{f_j\}$ is a normal family on $D$ and hence have a subsequence, may assumed to be $\{f_j\}$ itself, that converges uniformly to some $f \in \mathcal{O}(D)$ on compact sets of $D$. It is clear that $f \in H^p(D, \varphi)$ and $|f|_p \leq 1$. So

$$K_{\varphi,p}(a) \geq |f(a)|^p = \lim_{j \to \infty} |f_j(z_j)|^p \geq \lim_{j \to \infty} K_{\varphi,p}(z_j) - \epsilon.$$  

Letting $\epsilon$ goes to 0, we see $K_{\varphi,p}$ is upper semicontinuous. \hfill \Box

**Lemma 6.3.** $\log K_{\varphi,p}$ is a plurisubharmonic function on $D$.

**Proof.** Note that $\log K_{\varphi,p} = \sup \{p \log |f|; f \in H^p(D, \varphi), |f|_p = 1\}$ and log $K_{\varphi,p}$ is upper semicontinuous by Lemma 6.2. $\log K_{\varphi,p}$ is plurisubharmonic. \hfill \Box

We now give the proof of Theorem 6.1.

**Proof.** We will use the above notations and definitions. We denote $\frac{1}{m} \log K_{m\varphi,p}(z)$ by $\varphi_m(z)$. By Lemma 6.3, $\varphi_m$ is p.s.h on $D$. We want to show that $\varphi_m$ converges to $\varphi$ as $m \to \infty$.

By assumption, we know

$$-\varphi_m(z) \leq \frac{1}{m} \log \left(e^{-m\varphi(z)}\right) + \frac{\log C_m}{m} = -\varphi(z) + \frac{\log C_m}{m}.$$  

This is

$$\varphi_m(z) \geq \varphi(z) - \frac{\log C_m}{m}.$$  

On the other hand, if $d(z, \partial D) > r$, the mean value inequality for p.s.h functions implies

$$\int_D |f(\zeta)|^2 e^{-m\varphi(\zeta)} \geq \frac{\pi r^m}{m!} e^{-m \sup_{\zeta \in B(z,r)} \varphi(\zeta)},$$

where $f$ is the minimal solution in the definition of $K_{m\varphi,p}(z)$ and $B(z,r) = \{\zeta \in D; |\zeta - z| \leq r\}$. So we get

$$\varphi_m(z) \leq \sup_{\zeta \in B(z,r)} \varphi(\zeta) - \frac{1}{m} \log \left(\frac{\pi r^m}{m!}\right).$$
Definition 6.1. The metric \( h^m : (V^*)^m \to [0, +\infty] \) on \((V^*)^\otimes m\) is defined as:

\[
h^m(\xi) := \sup\{ |\xi(u_1, \cdots, u_m)| ; u_i \in V, h(u_i) \leq 1, 1 \leq i \leq m \}
\]

if \( h(u) < +\infty \) for some \( u \in V \), otherwise \( h^m(\xi) \) is defined to be 0. The metric \( h^m \) on \( V^\otimes m \) is defined in the same way by identifying \( V \) and \((V^*)^\otimes m\), the dual space of \( V^* \).

According to this definition, for \( \xi_1, \cdots, \xi_m \in V^* \), we have the product formula \( h^m(\xi_1 \cdots \xi_m) = h^*(\xi_1) \cdots h^*(\xi_m) \). Definition 6.1 can be applied to holomorphic vector bundles. If \( E \) is a holomorphic vector bundle over a complex manifold \( X \) and \( h \) is a Finsler metric on \( E \). The induced metrics \( h^m \) and \( h^* \) on \( E^\otimes m \) and \((E^*)^\otimes m\) is defined pointwise.

We now introduce a notion that may have independent interest.

**Definition 6.2** (Multiple \( L^p \)-extension property). Let \((E, h)\) be a holomorphic vector bundle over a bounded domain \( D \subset \mathbb{C}^n \) equipped with a singular Finsler metric \( h \). Let \( p > 0 \) be a fixed constant. Assume that for any \( z \in D \), any nonzero
element $a \in E_z$ with finite norm $|a|$, and any $m \geq 1$, there is a holomorphic section $f_m$ of $E^\otimes m$ on $D$ such that $f_m(z) = a^\otimes m$ and satisfies the following estimate:

$$\int_D |f_m|^p \leq C_m |a^\otimes m|^p = C_m |a|^m,$$

where $C_m$ are constants independent of $z$ and satisfying the growth condition

$$\frac{1}{m} \log C_m \to 0 \text{ as } m \to \infty.$$Then $(E, h)$ is said to have multiple $L^p$-extension property.

The following theorem says that multiple $L^p$-extension property for some $p > 0$ implies Griffiths positivity.

**Theorem 6.4.** Let $(E, h)$ be a holomorphic vector bundle over a bounded domain $D \subset \mathbb{C}^n$ equipped with a singular Finsler metric $h$, such that the norm of any local holomorphic section of $E^*$ is upper semicontinuous. If $(E, h)$ has multiple $L^p$-extension property for some $p > 0$, then $(E, h)$ is positively curved in the sense of Griffiths, namely $\log |u|$ is plurisubharmonic for any local holomorphic section $u$ of $E^*$.

**Proof.** Let $u$ be a local holomorphic section of $E^*$ on $U \subset D$. We need to show that the function $\varphi := \log |u|$ is plurisubharmonic on $U$. Our strategy is to prove that $\varphi$ satisfies the condition in Theorem 6.1.

Without loss of generality, we assume $U = D$. Let $z$ be a fixed point in $D$. We assume that $|u(z)| \neq 0$. Let $a \in E_z$ such that $|a| = 1$ and $<u(z), a> = |u(z)|$. By assumption, there is a holomorphic section $f$ of $E^\otimes m$ over $D$ such that $f(z) = a^\otimes m$ and $\int_D |f|^p \leq C_m |a^\otimes m|^p = C_m |a|^m = C_m$. We view $u^\otimes m$ as a holomorphic section of $(E^*)^\otimes m$. It is obvious that $|u^\otimes m| = |u|^m$ and $|u^\otimes m(z)| = <u^\otimes m, a^\otimes m>$. By definition,

$$|u(\zeta)|^m \geq <u^\otimes m(\zeta), f(\zeta)> |f(\zeta)|$$

for $\zeta \in D$, which is

$$e^{-m \varphi(\zeta)} \leq e^{-\log |<u^\otimes m(\zeta), f(\zeta)>||f(\zeta)|}.$$

Since $u^\otimes m, f$ are holomorphic section of $(E^*)^\otimes m$ and $E^\otimes m$ respectively, $<u^\otimes m, f>$ is a holomorphic function on $D$. By the Ohsawa-Takegoshi extension theorem, there is a holomorphic function $h$ on $D$ such that $h(z) = 1$ and

$$\int_D |h|^2 e^{-p \log |<u^\otimes m(\zeta), f(\zeta)>|} \leq C e^{-p \log |<u^\otimes m(z), f(z)>|} = C e^{-p \varphi(z)},$$

where $C$ is a constant independent of $m$ and $z$. By the above inequality, we have

$$\int_D |h| e^{-\frac{p}{2} \varphi} \leq \left( \int_D |h|^2 e^{-p \log |<u^\otimes m(\zeta), f(\zeta)>|} \right)^{1/2} \leq \left( \int_D |h|^2 e^{-p \log |<u^\otimes m(\zeta), f(\zeta)>|} \int_D |f|^p \right)^{1/2} \leq \left( C e^{-p \varphi(z)} C_m \right)^{1/2} = \sqrt{CC_m} e^{-\frac{p}{2} \varphi(z)}.$$

By Theorem 6.1, $\varphi = \log |u|$ is p.s.h on $D$. \qed
7. Proof of Berndtsson’s minimum principle

In this section we give a proof of Berndtsson’s integral form of minimum principle. The proof is motivated by Berndtsson’s original proof, but our starting point is Theorem 6.1 and the main ingredient is Ohsawa-Takegoshi extension theorem, while the main ingredient in Berndtsson’s original proof is Hörmander’s $L^2$ estimate of $\bar{\partial}$. Of course Theorem 7.1 is a corollary of the plurisubharmonic variation of relative Bergman kernels that will be discussed in §8, but we also present it here to show in this simple case how the main ideas work.

**Theorem 7.1 ([1]).** Let $\Omega \subset \mathbb{C}^r \times \mathbb{C}^m_w$ be a pseudoconvex domain and let $p : \Omega \to U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection. Let $\varphi(z, w)$ be a plurisubharmonic function on $\Omega$. If all fibers $\Omega_z := p^{-1}(z) \ (z \in p(\Omega))$ are Reinhardt domains in $\mathbb{C}^r$ and $\varphi(z, e^{i\theta}w) = \varphi(z, w)$ for all $\theta \in \mathbb{R}^n$. Then the function $\tilde{\varphi}$ defined by

$$e^{-\tilde{\varphi}(z)} = \int_{\Omega_z} e^{-\varphi(z, w)} d\lambda(w)$$

is a plurisubharmonic function on $p(\Omega)$, where $d\lambda(w)$ is the Lebesgue measure on $\mathbb{C}^m$.

**Proof.** It is clear that $\tilde{\varphi}$ is upper semicontinuous, so it suffices to prove that $\varphi$ satisfies the condition in Theorem 6.1. For $m \geq 1$, let

$$\Omega_m = \{(u_1, \ldots, u_m) \in \Omega^m; p(u_1) = \cdots = p(u_m)\}$$

be the fiber product of $\Omega$ and let $D^m_z = \{z\} \times \Omega^m_z \subset \Omega_m$ for $z \in U$. We can naturally identify $\Omega_m$ with the disjoint union $\bigsqcup_{z \in U} D^m_z$. For any $z \in U$ with $\int_{\Omega_z} e^{-\varphi(z, w)} d\lambda(w) < \infty$, by Theorem 7.1 there is $f \in \mathcal{O}(\Omega_m)$ such that $f|_{D^m_z} \equiv 1$ and

$$\int_{\Omega_m} |f(\zeta, w_1, \cdots, w_m)|^2 e^{-\sum_{i=1}^m \varphi(z, w_i)} \leq C \int_{D^m_z} e^{-\sum_{i=1}^m \varphi(z, w_i)} = e^{-m\tilde{\varphi}(z)},$$

where $C$ is a constant independent of $z$ and $m$. We choose $f$ such that the left hand side in the above inequality is minimal. By the uniqueness of the minimal element, $f(\zeta_1, e^{i\theta_1}w_1, \cdots, e^{i\theta_m}w_m) = f(\zeta, u_1, \cdots, w_m)$ for all $\theta_1 \in \mathbb{R}^n$. So $f$ is independent of $u_1, \cdots, w_m$. We denote $f(\zeta, u_1, \cdots, w_m)$ by $g(\zeta)$. From the above inequality, we have $g(\zeta) = 1$ and

$$\int_U |g(\zeta)|^2 e^{-m\tilde{\varphi}(\zeta)} \leq C e^{-m\tilde{\varphi}(z)}.$$

By Theorem 6.1 $\tilde{\varphi}$ is subharmonic on $U$. \hfill $\Box$

**Remark 7.1.** Theorem 7.1 can be generalized to holomorphically convex Kähler manifolds with more general group actions, which is the framework for minimum principle considered in [13, 14].

8. Plurisubharmonic variation of relative $m$-Bergman kernel metrics

The aim of this section is to prove that the relative $m$-Bergman kernel metric associated to a family of pseudoconvex domains or compact Kähler manifolds have semi-positive curvature current.
8.1. For families of pseudoconvex domains. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and let $p : \Omega \rightarrow U := p(\Omega) \subset \mathbb{C}^r$ be the natural projection. Let $\varphi(t, z)$ be a plurisubharmonic function on $\Omega$. Let $\Omega_t := p^{-1}(t) (t \in U)$ be the fibers. We also denote by $\varphi_t(z) = \varphi(t, z)$ the restriction of $\varphi$ on $\Omega_t$. Let $m$ be a positive integer, and let $K_{m,t}(z)$ be the $m$-Bergman kernel on $\Omega_t$ with respect to the weight $e^{-\varphi_t}$. As $t$ varies, we will view $K_{m,t}(z)$ as a function on $\Omega$, which will be called the relative $m$-Bergman kernel on $\Omega$. When $m = 1$, $K_{1,t}(z)$ is the ordinary relative Bergman kernel. Berndtsson proved that $K_{1,t}(z)$ is log-plurisubharmonic on $U$, based on regularity of $\partial$-Neumann and Hörmander’s $L^2$-estimate to $\partial$. In this section, we prove that the relative $m$-Bergman kernel is log-plurisubharmonic for general $m$. Our main techniques are Theorem 6.1 and raising powers of the domains.

**Theorem 8.1.** With the above assumptions and notations, for any $m \geq 1$ the function $\psi_m(t, z) := \log K_{m,t}(z)$ is a plurisubharmonic function on $\Omega$.

**Proof.** We may assume that $U$ is bounded. By Corollary 5.7, $\psi_m(t, z)$ is upper semicontinuous. It suffices to prove that $\psi_m(t, s(t))$ is subharmonic for all holomorphic sections $s : U \rightarrow \Omega$. We want to prove that $\psi_m(t, s(t))$ satisfies the condition in Theorem 6.1 on $U$.

For $k \geq 1$, let

$\Omega_k = \{ (u_1, \cdots, u_k) \in \Omega^k ; p(u_1) = \cdots = p(u_k) \}$

be the fiber product of $\Omega$ and let $D_t^k = \{ t \} \times \Omega_t^k \subset \Omega_k$ for $t \in U$. We can naturally identify $\Omega_k$ with the disjoint union $\bigsqcup_{t \in U} D_t^k$. Let $\varphi_k(t, z_1, \cdots, z_k) := \varphi(t, z_1) + \cdots + \varphi(t, z_k)$, then it is a plurisubharmonic function on $\Omega_k$.

Let $t_0 \in U$ and $s : U \rightarrow \Omega$ be a holomorphic section such that $\psi_m(t_0, s(t_0)) \neq -\infty$. By definition of the $m$-Bergman kernel and Proposition 2.1, there is $f \in \mathcal{O}(D_{t_0}^m)$ with $f(t_0, s(t_0), \cdots, s(t_0)) = 1$ and

$$\int_{D_{t_0}^k} |f|^{2/m} e^{-\varphi_k(t_0, z_1, \cdots, z_k)/m} = e^{-k\psi_m(t_0, s(t_0))/m}.$$ 

By Theorem 4.2 there is an $F \in \mathcal{O}(\Omega_k)$ such that $F|_{D_{t_0}^k} = f$ and

$$\int_{\Omega_k} |F|^{2/m} e^{-\varphi_k(t, z_1, \cdots, z_k)/m} \leq C \int_{D_{t_0}^k} |f|^{2/m} e^{-\varphi_k(t_0, z_1, \cdots, z_k)/m},$$

where $C$ is a constant independent of $t_0, s, k$ and $m$. The left hand side of the above inequality is

$$\int_U \int_{D_t^k} |F(t, z_1, \cdots, z_k)|^{2/m} e^{-\varphi_k(t, z_1, \cdots, z_k)/m}.$$ 

By definition of the $m$-Bergman kernel and Proposition 2.1 it is larger than

$$\int_U |F(t, s(t), \cdots, s(t))|^{2/m} e^{-k\psi_m(t, s(t))/m}.$$ 

Let $g(t) = F(t, s(t), \cdots, s(t))$, then $g$ is a holomorphic function on $U$ and $g(t_0) = 1$, and satisfies the following estimate

$$\int_U |g|^{2/m} e^{-k\psi_m(t, s(t))/m} \leq Ce^{-k\psi_m(t_0, s(t_0))/m},$$
where $C$ is a constant independent of $m$. By Theorem 6.1, $\psi_m(t, s(t))$ is a plurisubharmonic function on $U$. \hfill $\square$

8.2. For families of compact Kähler manifolds. Let $X, Y$ be Kähler manifolds of dimension $r+n$ and $r$ respectively, let $p : X \to Y$ be a proper holomorphic mapping. Let $L$ be a holomorphic line bundle over $X$, and $h$ be a singular Hermitian metric on $L$, whose curvature current is semi-positive. Let $K_{X/Y}$ be the relative canonical bundle on $X$.

Let $\mathcal{E}_m = p_* (mK_{X/Y} \otimes L \otimes T_m(h))$ be the direct image sheaf on $Y$. By Grauert’s theorem, $\mathcal{E}_m$ is a coherent analytic sheaf on $Y$. There is an Zariski open subset $U$ in $Y$ such that $\mathcal{E}_m|_U$ is locally free and the fiber $\mathcal{E}_{m,y}$ over $y \in U$ of the vector bundle associated to $\mathcal{E}_m|_U$, denoted by $E_m$, can be identified with $H^0(X_y, (kK_{X/Y} \otimes L \otimes T_k(h))|_{X_y}) \subset H^0(X_y, kK_{X/Y} \otimes L|_{X_y})$.

Recall that we have define the relative $m$ Bergman kernel metric, denoted by $B_{L,h}^m$, on $(mK_{X/Y} \otimes L)|_{p^{-1}(U)}$. The main result in this subsection is the following

**Theorem 8.2.** If $h$ is locally bounded, the relative $m$ Bergman kernel metric $B_{L,h}^m$ on $(mK_{X/Y} \otimes L)|_{p^{-1}(U)}$ has nonnegative curvature current; moreover, $B_{L,h}^m$ can be extended to a hermitian metric on $mK_{X/Y} \otimes L$ whose curvature current is nonnegative.

**Proof.** The proof of the first statement is similar to the proof of Theorem 8.1. We may assume that $U = \mathbb{B}^r$ is the unit ball in $C^r$. By Corollary 5.3, $B_{L,h}^m$ is lower semi-continuous.

For simplicity, we denote the line bundle $mK_{X/Y} \otimes L$ on $X$ by $F$ and denote $p^{-1}(U)$ by $\Omega$. Let $s : U \to \Omega$ be an arbitrary holomorphic map. Let $W$ be a neighborhood of the image of $h$ in $\Omega$ such that $F$ has a holomorphic frame $e$ on $W$. Let $\psi$ be the weight of the metric $B_{L,h}^m$ on $W$ with respect to the frame $e$. It suffices to prove that $\psi \circ s$ is a plurisubharmonic function on $U$ (or identically equal to $-\infty$). We will prove that $\psi \circ s$ satisfies the condition in Theorem 6.1.

For $k \geq 1$, let

$$\Omega_k = \{(x_1, \cdots, x_k) \in \Omega^k; p(x_1) = \cdots = p(x_k)\}$$

be the $k$-th fiber product power of $\Omega$ with respect to the map $p : \Omega \to U$. Let $p_k : \Omega_k \to U$ be the natural projection, and let $\Omega_k^t = p_k^{-1}(t)$ for $t \in U$. The line bundle $L$ on $\Omega$ induces a line bundle $L_k$ on $\Omega_k$ whose fiber at $(x_1, \cdots, x_k)$ is $L_{x_1} \otimes \cdots \otimes L_{x_k}$. The metric $h$ on $L$ naturally induces a metric $h_k$ on $L_k$ whose curvature current is nonnegative.

Let $t_0 \in U$ be an arbitrary point such that $\psi(s(t_0)) \neq -\infty$. We denote $s(t_0) \in \Omega$ by $z$ and denote $e(s(t_0))$ by $a$. Note that $(z, \cdots, z) \in \Omega_k$ and we can identify $(mK_{\Omega_k/U} \otimes L_m)|_{(z, \cdots, z)}$ with $(mK_{X/Y} \otimes L)|_{X_z}$. By Proposition 2.2, $a^\otimes_k := a \otimes \cdots \otimes a \in (mK_{\Omega_k/U} \otimes L_k)|_{(z, \cdots, z)}$ has square norm $e^{-k\psi(z)}$ with respect to the relative $m$-Bergman kernel metric on $mK_{\Omega_k/U} \otimes L_k$.

By definition of the $m$-Bergman kernel metric, there is $f \in H^0(\Omega_k, mK_{\Omega_k} \otimes L_m|_{\Omega_k})$ such that $f(z, \cdots, z) = a^\otimes_k$, and

$$\int_{\Omega_k^t} |f|^{2/m} h_k^{1/m} e^{-k\psi(z)/m}.$$
By Theorem 4.3, there is an $F \in H^0(\Omega_k, mK_{\Omega_k} \otimes L_k)$ such that $F|_{\Omega_0} = f \wedge dt_1 \wedge \cdots \wedge dt_r$ and

$$\int_{\Omega_k} |F|^{2/m} h_k^{1/m} \leq C \int_{\Omega_0} |f|^{2/m} h_k^{1/m},$$

where $(t_1, \cdots, t_r)$ are the coordinates on $U$ and $C$ is a constant independent of $t_0$ and $k$. The left hand side of the above inequality is

$$\int_{U} \int_{\Omega_k} |(F|_{\Omega_k^+})|^{2/m} h_k^{1/m}.$$  

Locally we write $F(x_1, \cdots, x_k) = \hat{F}(x_1, \cdots, x_k) e^{\psi}$. By definition of the $m$-Bergman kernel and Proposition 2.2, it is larger than

$$\int_{U} |\hat{F}(t, s(t), \cdots, s(t))|^{2/m} e^{-k\psi(t,s(t))}/m.$$  

Let $g(t) = \hat{F}(t, s(t), \cdots, s(t))$, then $g$ is a holomorphic function on $U$ and $g(t_0) = 1$, and satisfies the following estimate

$$\int_{U} |g|^{2/m} e^{-k\psi(t,s(t))}/m \leq Ce^{-k\psi(t_0, s(t_0))}/m,$$

where $C$ is a constant independent of $k$. By Theorem 6.1, $\psi(t,s(t))$ is a plurisubharmonic function on $U$.

For the proof of the second statement, it suffices to prove that the Bergman kernel metric is bounded locally below by positive constants. We follow the idea in [6]. Fix an arbitrary point $p \in X_y$. Let $u \in H^0(X_y, mK_{X_y} \otimes L|_{X_y})$ such that

$$\int_{X_y} |u|^{2/m} h|_{X_y}^{1/m} \leq 1.$$  

Let $\Omega \subset X$ be a bounded pseudoconvex neighborhood of $p$ in $X$. From Theorem 4.2, there is a local $m$-canonical form $\hat{U}$ on $\Omega$ which extends $u$ and satisfies

$$\int_{\Omega} |\hat{U}|^{2/m} h^{1/m} \leq C_0 \int_{\Omega_y} |u|^{2/m} h^{1/m} \leq C_0,$$

where $C_0$ is an absolute constant. We write as $\hat{U} = U'(dz)^{\otimes m}$. By the mean value inequality on a polydisk $D_{r_0}$ of poly-radius $(r_0, \cdots, r_0)$ centered at $p$, we obtain that

$$|u(p)|^{2/m} = |U'(q)|^{2/m} \leq \frac{1}{(\pi r_0)^{n+1}} \int_{D_{r_0}} |U'|^{2/m} \partial z \leq \frac{1}{(\pi r_0)^{n+1}} \sup_{D_{r_0}} h^{-1/m} \int_{D_{r_0}} |\hat{U}|^{2/m} h^{1/m} \leq C,$$

where $C$ is a constant which does not depend on the geometry of the fiber $X_y$, but on the ambient manifold $X$. We thus complete the proof of the theorem.

Remark 8.1. The boundedness assumption about $h$ in Theorem 8.2 is to ensure that the extension $F$ exists and satisfies the estimate in (5). When $m = 1$, this assumption is not necessary, since by Theorem 4.1, the extension with estimate always exists. On the other hand, the result in Theorem 8.2 still holds without the assumption that $h$ is locally bounded, because applying similar argument as in [6].
one can show that Theorem 5.2 is indeed a consequence of Theorem 7.3 that we will prove in §8. But we will not present the details of the argument of Berndtsson-P˘ aun in the present article.

9. Positivity of direct images of twisted relative canonical bundles

In this section, we generalize the idea in §8 to show that the direct image of relative canonical bundles twisted by pseudoeffective line bundles associated to certain families of pseudoconvex domains or compact Kähler manifolds are semipositive in the sense of Griffiths.

9.1. For families of pseudoconvex domains. Let \( U, D \) be bounded pseudoconvex domains in \( \mathbb{C}^r \) and \( \mathbb{C}^n \) respectively, and let \( \Omega = U \times D \subset \mathbb{C}^r \times \mathbb{C}^n \). Let \( \varphi \) be a p.s.h function on \( \Omega \), which is for simplicity assumed to be bounded. For \( t \in U \), let \( D_t = \{ t \} \times D \) and \( \varphi_t(z) = \varphi(t, z) \). Let \( E_t = H^2(D_t, e^{-\varphi_t}) \) be the space of \( L^2 \) holomorphic functions on \( D_t \) with respect to the weight \( e^{-\varphi_t} \). Then \( E_t \) are Hilbert spaces with the natural inner product. Since \( \varphi \) is assumed to be bounded on \( \Omega \), all \( E_t \) for \( t \in U \) are equal as vector spaces, however, the inner products on them depend on \( t \) if \( \varphi(t, z) \) is not constant with \( t \). So, under the natural projection, \( E = \bigcup_{t \in U} E_t \) is a trivial holomorphic vector bundle (of infinite rank) over \( U \) with varying Hermitian metric.

In [2], Berndtsson proved that \( E \) is semipositive in the sense of Griffiths, namely, for any local holomorphic section \( \xi \) of the dual bundle \( E^* \) of \( E \), the function \( \log | \xi | \) is plurisubharmonic (indeed Berndtsson proved a stronger result which says that \( E \) is semipositive in the sense of Nakano). The aim here is to provide a new proof of the positivity of \( E \), based on the new characterization of plurisubharmonic functions (Theorem 6.1) and the technique of rising powers of domains.

**Theorem 9.1.** The vector bundle \( E \) is semipositive in the sense of Griffiths.

Before giving the proof of Theorem 9.1, we first recall the notion of *Hilbert tensor product* of Hilbert spaces and prove a related lemma. Let \( V \) and \( W \) be two Hilbert spaces. For \( v \in V, w \in W \), the norm of \( v \otimes w \) is defined to be \( ||v|| ||w|| \). If \( \{ v_i \}_{i \in I} \) and \( \{ w_j \}_{j \in J} \) are orthonormal bases’ of \( V \) and \( W \) respectively, then the Hilbert tensor product \( V \otimes W \) of \( V \) and \( W \) is defined to be the Hilbert space with \( \{ v_i \otimes w_j \}_{(i,j) \in I \times J} \) as an orthonormal basis. It is easy to show that the definition of \( V \otimes W \) is independent of the choice of the orthonormal basis’ of \( V \) and \( W \). By definition one can check that \( (V \otimes W)^* = V^* \otimes W^* \). The definition can be naturally generalized to the tensor product of several Hilbert spaces. In particular, we can define the tensor powers \( V^\otimes k := V \otimes \cdots \otimes V \ ((k \geq 1)) \) of a Hilbert space \( V \). Let \( V \) be a Hilbert space. For \( v \in V \), it is obvious that the norm of \( v^\otimes k := v \otimes \cdots \otimes v \in V^\otimes k \) is \( ||v||^k \) for all \( k \geq 1 \).

**Lemma 9.2.** Let \( D_1, D_2 \) be bounded domains in \( \mathbb{C}^n_x \) and \( \mathbb{C}^m_w \) respectively. Let \( \phi_1 \) and \( \phi_2 \) be plurisubharmonic functions on \( D_1 \) and \( D_2 \). Then

\[
H^2(D_1 \times D_2, e^{-(\phi_1+\phi_2)}) = H^2(D_1, e^{-\phi_1}) \otimes H^2(D_2, e^{-\phi_2}).
\]

**Proof.** Let \( \{ f_i \}_{i=1}^\infty \) and \( \{ g_i \}_{i=1}^\infty \) be orthonormal bases’ of \( H^2(D_1, e^{-\phi_1}) \) and \( H^2(D_2, e^{-\phi_2}) \) respectively. Then \( K_1(z) = \sum_i |f_i(z)|^2 \) is the Bergman kernel of \( H^2(D_1, e^{-\phi_1}) \) and \( K_2(w) = \sum_j |g_j(w)|^2 \) is the Bergman kernel of \( H^2(D_2, e^{-\phi_2}) \). Let \( K(z, w) = \sum_{i,j} |f_i(z)g_j(w)|^2 \). It is clear that \( K(z, w) = K_1(z)K_2(w) \). By Fubini theorem.
\{f_i(z)g_j(w)\}_{i,j=1}^{\infty} is an orthonormal set of \( H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)}) \). By Proposition 2.1, the Bergman kernel of \( H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)}) \) equals to \( K_1(z)K_2(w) \). So \( \{f_i(z)g_j(w)\}_{i,j=1}^{\infty} \) is an orthonormal basis of \( H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)}) \) and hence
\[
H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)}) = H^2(D_1, e^{-\varphi_1}) \otimes H^2(D_2, e^{-\varphi_2}).
\]

\[\square\]

It is clear that Lemma 9.2 can be generalized to product of several domains. We now give the proof of Theorem 9.1.

**Proof.** Let \( u \) be a local holomorphic section of the dual bundle \( E^* \) of \( E \). We need to prove that \( \log |u(t)| \) is a plurisubharmonic function. Without loss of generality, we assume that \( u \) is a global holomorphic section, namely a holomorphic section of \( E^* \) on \( U \). The upper semi-continuity of \( \log |u(t)| \) follows from Proposition 5.6. We now prove that \( \log |u(t)| \) satisfies the condition in Theorem 6.1 for some \( p > 0 \).

For \( m \geq 1 \), let \( \Omega_m = U \times D^m \) and \( \varphi_m(t, z_1, \cdots, z_m) = \varphi(t, z_1) + \cdots + \varphi(t, z_m) \). For \( t \in U \), we denote \( t \times D^m \) by \( D^m_t \). Let \( E^{\otimes m} = H^2(D^m_t, e^{-\varphi_m}) \), and \( E^{\otimes m} = \bigoplus_{t \in U} E^{\otimes m}_t \). By Lemma 9.2, \( E^{\otimes m} \) is the \( m \)-th tensor power of \( E \) in the Hilbert space sense.

Let \( t_0 \in U \) be an arbitrary point such that \( u(t_0) \neq 0 \). By the definition of tensor powers of Hilbert spaces given as above, \( u^{\otimes m} \) is a nonvanishing holomorphic section of \( (E^{\otimes m})^* = (E^*)^{\otimes m} \), and \( |u^{\otimes m}(t)| = |u(t)|^m \). Let \( f \in E^{\otimes m}_{t_0} \) such that
\[
\int_{D^m_{t_0}} |f|^2 e^{-\varphi_m(t_0, z_1, \cdots, z_m)} = 1
\]
and \( \langle u^{\otimes m}(t_0), f \rangle = |u(t_0)|^m \).

By Theorem 4.1, there exists \( F \in \mathcal{O}(\Omega_m) \) such that \( F|_{D^m_{t_0}} = f \) and
\[
\int_{\Omega_m} |F(t, z_1, \cdots, z_m)|^2 e^{-\varphi_m(t, z_1, \cdots, z_m)} \leq C,
\]
where \( C \) is a constant independent of \( t_0 \) and \( m \). Let \( F_t(z_1, \cdots, z_m) = F(t, z_1, \cdots, z_m) \) and
\[
|F_t|^2 = \int_{D^m} |F_t|^2 e^{-\varphi_m(t, z_1, \cdots, z_m)}.
\]
Since \( \varphi \) is bounded, by the mean value inequality, \( |F_t| \leq +\infty \). This implies \( F_t \) lies in \( E^{\otimes m} \) for all \( t \in U \) and hence \( F \) can be seen as a holomorphic section of \( E^{\otimes m} \).

From the definition of \( \|u^{\otimes m}(t)\|_t \), it is clear that
\[
\|F_t\|_t |u(t)|^m \geq |\langle u^{\otimes m}(t), F_t \rangle|,
\]
and hence
\[
e^{-m \log |u(t)|} \leq e^{-\log |\langle u^{\otimes m}(t), F_t \rangle|} \|F_t\|_t.
\]

Note that \( \langle u^{\otimes m}(t), F_t \rangle \) is a holomorphic function on \( U \). By Theorem 4.1, there is a holomorphic function \( h \) on \( U \) such that \( h(t_0) = 1 \) and
\[
\int_U |h(t)|^2 e^{-2 \log |\langle u^{\otimes m}(t), F_t \rangle|} \leq C' e^{-2 \log |\langle u^{\otimes m}(t_0), F_{t_0} \rangle|} = C' e^{-2m \log |u(t_0)|},
\]
where \( C' \) is a constant independent of \( m \), \( t_0 \), and \( \Omega \).
where \( C' \) is a constant independent of \( m \) and \( t_0 \). So we have the estimate

\[
\int_U |h(t)| e^{-m \log |u(t)|} \leq \int_U |h(t)| e^{-\log <u^{\otimes m}(t), F_t>|} ||F_t||_t \\
\leq \left( \int_U |h(t)|^2 e^{-2 \log |<u^{\otimes m}(t), F_t>|} \int_U ||F_t||_t^2 \right)^{1/2} \\
\leq \sqrt{CC'} e^{-m \log |u(t)|},
\]

where the last inequality follows from \( \text{(3)} \), \( \text{(4)} \) and Fubini theorem. By Theorem \( \text{(6.1)} \) log \( |u(t)| \) is subharmonic. \( \square \)

9.2. For families of compact Kähler manifolds. In this subsection, we study the positivity of the direct image sheaf of the twisted relative canonical bundle associated to a family of compact Kähler manifolds.

Let \( X, Y \) be Kähler manifolds of dimension \( r + n \) and \( r \) respectively, and let \( p : X \to Y \) be a proper holomorphic map. For \( y \in Y \) let \( X_y = p^{-1}(y) \), which is a compact submanifold of \( X \) of dimension \( n \) if \( y \) is a regular value of \( p \). Let \( L \) be a holomorphic line bundle over \( X \), and \( h \) be a singular Hermitian metric on \( L \), whose curvature current is semi-positive. Let \( K_{X/Y} \) be the relative canonical bundle on \( X \).

Let \( \mathcal{E}_k = p_*(kK_{X/Y} \otimes L \otimes \mathcal{I}_k(h)) \) and \( \tilde{\mathcal{E}}_k = p_*(kK_{X/Y} \otimes L) \) be the direct image sheaves on \( Y \). We can choose a proper analytic subset \( A \subset Y \) such that:

1. \( p \) is submersive over \( Y \setminus A \),
2. both \( \mathcal{E}_k \) and \( \tilde{\mathcal{E}}_k \) are locally free on \( X \setminus A \),
3. for \( y \in Y \setminus A \), \( E_k,y \) and \( \tilde{E}_k,y \) are naturally identified with \( H^0(X_y, kK_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}_k(h)|_{X_y}) \) and \( H^0(X_y, kK_{X_y} \otimes L|_{X_y}) \) respectively,

where \( E_k \) and \( \tilde{E}_k \) are the vector bundles on \( Y \setminus A \) associated to \( \mathcal{E}_k \) and \( \tilde{\mathcal{E}}_k \) respectively. For \( u \in \tilde{E}_k,y \), as in \( \text{(2.2)} \) the \( k \)-norm of \( u \) is defined to be

\[
H_k(u) := ||u||_k = \left( \int_{X_y} |u|^{2/k} h^{1/k} \right)^{k/2} \leq +\infty.
\]

Then \( H_k \) is a Finsler metric on \( \tilde{E}_k \), whose restriction on \( E_k \) gives a Finsler metric on \( E_k \), which will be also denoted by \( H_k \). In the case that \( k = 1 \), we denote \( E_1, \tilde{E}_1, E_1, \tilde{E}_1, H_1 \) by \( \mathcal{E}, \tilde{\mathcal{E}}, E, \tilde{E}, H \) respectively. The following theorem says that \( H \) is a positively curved singular Hermitian metric in the coherent sheaf \( \mathcal{E} \) (see Definition \( \text{(3.4)} \) for definition).

**Theorem 9.3.** With the above assumptions and notations, \( H \) is a positively curved singular metric on \( \mathcal{E} \).

**Proof.** The proof splits into three steps.

*Step 1.* We prove that \( H \) is a positively curved singular Finsler metric on \( \tilde{E} \to U := Y \setminus A \). The argument is similar to that in the proof of Theorem 6.1. Let \( u \) be a local holomorphic section of \( \tilde{E}^* \). By definition, we need to show that log \( |u| \) is a plurisubharmonic function. Without loss of generality, in this step we can assume that \( U = \mathbb{B}^r \) is the unit ball and \( u \) is a holomorphic section of \( \tilde{E}^* \) on \( U \).


For $m \geq 1$, let $X_m = \{(y, z_1, \cdots, z_m) : y \in U, z_1, \cdots, z_m \in X_y\}$ be the $m$-th fiber-product power of $X$. The is a natural proper holomorphic submersion from $X_m$ to $U$, which is denoted by $p_m : X_m \to U$. Let $X^m_y = p_m^{-1}(y)$ be the fiber over $y \in U$.

For $1 \leq i \leq m$, we have a projection $\pi_i : X_m \to X$ which sends $(y, z_1, \cdots, z_m)$ to $(y, z_i)$. Let $L_m = \pi_1^*L \otimes \cdots \otimes \pi_m^*L$ and let $h_m$ be the singular Hermitian metric on $L_m$ induced from the metric $h$ on $L$. Then the curvature current of $h_m$ is nonnegative.

Note that $H^0(X^m_y, K_{X^m_y} \otimes L_m|X^m_y) = H^0(X_y, K_{X_y} \otimes L_y)^{\otimes m}$ for $y \in U$. Indeed, this follows from Proposition 2.2 by putting a smooth hermitian metric on $L$. In particular, the dimension of $H^0(X^m_y, K_{X^m_y} \otimes L_m)$ is independent of $y \in U$. Since $U$ is assumed to be the unit ball, we can identify $U$ can define a Finsler metric, say $|\log E_u|$. On $Y$, we have $E^{\otimes m}_u = E^{\otimes m}_u$. In the same way as defining $H$, we can define a Finsler metric, say $H^{m}_u$, on $E_y^{m}$. For $y \in U$, let $\hat{E}_y^{m}$ and $\hat{E}_y^{m}$ be subspaces of $\hat{E}_y^{m}$ consisting of vectors of finite norm. By Lemma 9.2, we have $\hat{E}_y^{m} = (\hat{E}_y^{m})^{\otimes m}$.

Recall that $u$ is a holomorphic section of $\hat{E}^*$ on $U$, and we need to prove that $\log |u|$ is a plurisubharmonic function on $U$. Note that the restriction $u|_E$ of $u$ on $E$ is a holomorphic section of $E^*$. The point is that, by definition of the dual norm in Definition 3.2, the norm of $u|_E$ and $u$ are equal. Therefore, by Proposition 5.2, $\log |u|$ is upper semicontinuous. Now it suffices to prove that $\log |u|$ satisfies the condition in Theorem 6.1.

Let $y_0 \in U$ be any given point such that $|u(y_0)| \neq 0$. $u^{\otimes m}$ is a holomorphic section of $\hat{E}^{\otimes m}_u = \hat{E}^{\otimes m}_u$. Note that the definition of the norm of $u^{\otimes m}$ only involves vectors in $\hat{E}^{\otimes m}$ of finite norm, by Lemma 9.2 we have $|u^{\otimes m}(y)| = |u(y)|^m$. There exists $f_{y_0} \in E^{m}_{y_0}$ such that $|f_{y_0}| := H^m(f_{y_0}) = 1$ and $\langle u^{\otimes m}(y_0), f_{y_0} \rangle = |u(y_0)|^m$. By Theorem 4.1 there is $F \in H^0(p_m^{-1}(U), (K_{X^m} \otimes L_m)|_{p_m^{-1}(U)})$ such that $F|_{X^m_{y_0}} = f_{y_0}$ and

$$\int_{X^m_{y_0}} |F(y, z_1, \cdots, z_m)|^2 e^{-\varphi_m(z_1, \cdots, z_m)} \leq C,$$

where $\varphi_m$ is the weight of $h_m$ and $C$ is an absolute constant independent of $y_0$ and $m$. For $y \in U$, let $F_y(z_1, \cdots, z_m) = F(y, z_1, \cdots, z_m)$, then $F_y \in E^{m}_{y}$ and

$$\|F_y\|^2 = \int_{X^m_{y}} |F_y|^2 e^{-\varphi_m(y, z_1, \cdots, z_m)}.$$

From the definition of $|u^{\otimes m}(y)|$, it is clear that

$$\|F_y\||u(y)|^m \geq |\langle u^{\otimes m}(y), F_y \rangle|,$$

and hence

$$e^{-m \log |u(y)|} \leq e^{-\log |\langle u^{\otimes m}(y), F_y \rangle||F_y||}.$$

Note that $F$ can be seen as a holomorphic section of $\hat{E}^{m}_u$ on $U$, so $\langle u^{\otimes m}(y), F_y \rangle$ is a holomorphic function on $U$. By Ohsawa-Takegoshi extension theorem (Theorem 4.1), there is a holomorphic function $h$ on $U$ such that $h(t_0) = 1$ and

$$\int_{U} |h(y)|^2 e^{-2 \log |\langle u^{\otimes m}(y), F_y \rangle|} \leq C e^{-2m \log |u(y_0)|},$$
where $C'$ is an absolute constant independent of $m$ and $y_0$. So we have the estimate
\begin{align}
\int_U |h(y)| e^{-m \log |u(y)|} &
\leq \int_U |h(y)| e^{-\log |<u^\otimes m(y),F_y>|} ||F_y|| \\
& \leq \left( \int_U |h(y)|^2 e^{-2 \log |<u^\otimes m(y),F_y>|} \int_U ||F_y||^2 \right)^{1/2} \\
& \leq \sqrt{CC'} e^{-m \log |u(y_0)|}.
\end{align}

By Theorem 5.1 log $|u(t)|$ is plurisubharmonic.

**Step 2.** We prove that $H$ is a positively curved singular Finsler metric on $E \to U := Y \setminus A$. We also assume that $U = B'$ be the unit ball. Note that $E$ is a holomorphic subbundle of $\tilde{E}$. Since $U$ is a Stein manifold, there is a holomorphic subbundle $E'$ of $\tilde{E}$ such that $\tilde{E}$ splits as $E \oplus E'$ (see e.g. Corollary 2.4.5 in [5]). So any holomorphic section $u$ of $E^*$ on $U$ can be extended to a holomorphic section $\tilde{u}$ of $\tilde{E}^*$ by setting $u(a) = 0$ for all $a \in E'$. Note that the norm of any vector in $\tilde{E} \setminus E$ is $+\infty$ (by Theorem 4.1), by definition, the norm of $u$ and $\tilde{u}$ are equal. By the result in Step 1, log $|\tilde{u}|$ is plurisubharmonic, so log $|u|$ is plurisubharmonic.

**Step 3.** We will complete the proof of Theorem 9.3 in this final step. Let $u$ be a holomorphic section of the dual sheaf $\mathcal{E}^*$ of $\mathcal{E}$ on some open set $V$ in $W$. We want to show that $|u|_{V \setminus A}$ is bounded above on all compact subsets of $V$. Once this is established, log $|u|$ can be extended uniquely to a plurisubharmonic function on $V$ and we are done.

The proof of the boundedness of log $|u|$ follows from the idea in the proof of Proposition 23.3 in [19]. Without loss of generality, we assume that $V = B'$ is the unit ball. For any $y \in V \setminus A$ such that $|u(y)| \neq 0$ (otherwise there is nothing to prove), there is $a \in E_y$ such that $|a| = 1$ and $<u(y),a> = |u(y)|$. By Theorem 4.1 there exists a holomorphic section $s$ of $\mathcal{E}$ on $V$ such that $s(y) = a$ and
\[ \int_V |s|^2_h \leq C, \]
where $C$ is a constant independent of $y$ and $a$. Let
\[ S = \{ f \in H^0(V,\mathcal{E}); \int_V |f|^2_h \leq C \}. \]
Since the metric $h$ on $L$ is lower semicontinuous, by the mean value inequality and Montel theorem, $S$ is a normal family, namely, any sequence in $S$ has a subsequence that converges uniformly on compact subsets of $V$. Note that if $s_j$ is a subsequence of $S$ that converges uniformly on compact subsets of $V$, then the sequence of holomorphic functions $<u,s_j>$ converges on compact subsets of $V$, and hence is uniformly bounded on compact sets of $V$. So $\{ <u,s>; s \in S \}$ is uniformly bounded on compact sets of $V$.

**Remark 9.1.** By Theorem 8.2 (and the remark following it) and Theorem 9.3, one can see that the NS metric (see §2.3 for definition) on the direct image $p_*(kK_{X/Y} \otimes L \otimes \Omega(h))$ is positively curved in the sense of Griffiths.
References

[1] B. Berndtsson. Prekopa’s theorem and Kiselman’s minimum principle for plurisubharmonic functions. *Math. Ann.*, 312(4):785–792, 1998.

[2] B. Berndtsson. Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains. *Ann. Inst. Fourier (Grenoble)*, 56(6):1633–1662, 2006.

[3] B. Berndtsson. Curvature of vector bundles associated to holomorphic fibrations. *Ann. of Math. (2)*, 169(2):531–560, 2009.

[4] B. Berndtsson. Complex Brunn-Minkowski theory and positivity of vector bundles. arXiv:1807.05844.

[5] B. Berndtsson and L. Lempert. A proof of the Ohsawa-Takegoshi theorem with sharp estimates. *J. Math. Soc. Japan*, 68(4):1461–1472, 2016.

[6] B. Berndtsson and M. Păun. Bergman kernels and the pseudoeffectivity of relative canonical bundles. *Duke Math. J.*, 145(2):341–378, 2008.

[7] B. Berndtsson and M. Păun. Bergman kernels and subadjunction. arXiv:1002.4145.

[8] Z. Băsă. Suita conjecture and the Ohsawa-Takegoshi extension theorem. *Invent. Math.*, 193(1):149–158, 2013.

[9] J. Cao. Ohsawa-Takegoshi extension theorem for compact Kähler manifolds and applications. In *Complex and symplectic geometry*, volume 21 of *Springer INdAM Ser.*, pages 19–38. Springer, Cham, 2017.

[10] J.-P. Demailly. Regularization of closed positive currents and intersection theory. *J. Algebraic Geom.*, 1(3):361–409, 1992.

[11] J.-P. Demailly. On the Ohsawa-Takegoshi-Manivel $L^2$ extension theorem. In *Complex analysis and geometry (Paris, 1997)*, volume 188 of *Progr. Math.*. Birkhäuser, Basel, 2000.

[12] F. Deng, Z. Wang, L. Zhang, and X. Zhou. Linear invariants of complex manifolds and their plurisubharmonic variation. Preprint.

[13] F. Deng, H. Zhang, and X. Zhou. Positivity of direct images of positively curved volume forms. *Math. Z.*, 278(1-2):347–362, 2014.

[14] F. Deng, H. Zhang, and X. Zhou. Positivity of character subbundles and minimum principle for noncompact group actions. *Math. Z.*, 286(1-2):431–442, 2017.

[15] Q. Guan and X. Zhou. Optimal constant problem in the $L^2$ extension theorem. *C. R. Math. Acad. Sci. Paris*, 350 (2012), no. 15-16, 753–756.

[16] Q. Guan and X. Zhou. Optimal constant in an $L^2$ extension problem and a proof of a conjecture of Ohsawa. *Sci. China Math.* 58 (2015), no. 1, 35–59.

[17] Q. Guan and X. Zhou. A solution of an $L^2$ extension problem with an optimal estimate and applications. *Ann. of Math. (2)*, 181(3):1139–1208, 2015.

[18] Q. Guan and X. Zhou. Strong openness of multiplier ideal sheaves and optimal $L^2$ extension. *Sci. China Math.* 60 (2017), no. 6, 967–976.

[19] C. Hacon, M. Popa, and C. Schnell. Algebraic fiber spaces over abelian varieties: Around a recent theorem by Cao and Păun. In *Local and global methods in algebraic geometry*, volume 712 of *Contemp. Math.*, pages 143–195. Amer. Math. Soc., Providence, RI, 2018.

[20] C. O. Kiselman. The partial Legendre transformation for plurisubharmonic functions. *Invent. Math.*, 49 (1978), no. 2, 137–148.

[21] L. Lempert and R. Szöke. Direct images, fields of Hilbert spaces, and geometric quantization. *Comm. Math. Phys.*, 327(1):49–99, 2014.

[22] K.-F. Liu and X.-K. Yang. DCurvature of direct image sheaves of vector bundles and applications. *J. Differ. Geom.*, 98, 117C145 (2014).

[23] F. Maitani and H. Yamaguchi. Variation of Bergman metrics on Riemann surfaces. *Math. Ann.*, 330(3):477–489, 2004.

[24] L. Manivel. Un théorème de prolongement $L^2$ de sections holomorphes d’un fibré hermitien. *Math. Z.*, 212(1):107–122, 1993.

[25] J. Ning, H. Zhang, and X. Zhou. On $p$-Bergman kernel for bounded domains in $\mathbb{C}^n$. *Comm. Anal. Geom.*, 24(4), 887-900 (2016).

[26] T. Ohsawa and K. Takegoshi. On the extension of $L^2$ holomorphic functions. *Math. Z.*, 195(2):197–204, 1987.

[27] M. Păun and S. Takayama. Positivity of twisted relative pluricanonical bundles and their direct images. *J. Algebraic Geom.*, 27(2):211–272, 2018.
[28] S.-T. Yau. On the pseudonorm project of birational classification of algebraic varieties. In *Geometry and analysis on manifolds*, volume 308 of *Progr. Math.*. Birkhäuser/Springer, Cham, 2015.

[29] X. Zhou and L. Zhu. An optimal $L^2$ extension theorem on weakly pseudoconvex kähler manifolds. *J. Differential Geom.*, Volume 110, Number 1 (2018), 135-186.

[30] X. Zhou and L. Zhu. Siu’s lemma, optimal $L^2$ extension, and applications to pluricanonical sheaves. Submitted, 2017.

FUSHENG DENG: SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, P. R. CHINA
E-mail address: fshdeng@ucas.ac.cn

ZHIWE WANG: SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, BEIJING, 100875, P. R. CHINA
E-mail address: zhiwei@bnu.edu.cn

LIYOU ZHANG: SCHOOL OF MATHEMATICAL SCIENCES, CAPITAL NORMAL UNIVERSITY, BEIJING, 100048, P. R. CHINA
E-mail address: zhangly@cnu.edu.cn

XIANGYU ZHOU: INSTITUTE OF MATHEMATICS, AMSS, AND HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA; SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, CHINA
E-mail address: xyzhou@math.ac.cn