Classification of Cohomogeneity One Strings

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Abstract

We define the cohomogeneity one string, string with continuous symmetries, as its world surface is tangent to a Killing vector field of a target space. We classify the Killing vector fields by an equivalence relation using isometries of the target space. We find that the equivalence classes of Killing vectors in Minkowski spacetime are partitioned into seven families. It is clarified that there exist seven types of strings with spacelike symmetries and four types of strings with timelike symmetries, stationary strings.

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I. INTRODUCTION

Recently, extended objects, such as strings and membranes, gather much attention in the contexts of unified theories and cosmologies. A trajectory of the extended object, world hypersurface, is a submanifold embedded in a target space. The geometry induced on the world hypersurface has indeed much variety which depends on the possible solutions to the equations of motion of the object, but extended objects which have geometrical symmetries are especially interesting for their simplicity.

A matter of main interest in the brane universe scenario is the structure of symmetric extended objects, i.e., spatially homogeneous brane universe models embedded in a symmetric bulk space[1]. Another example is classical stationary solutions for Nambu-Goto string in stationary spacetimes [2, 3, 4, 5]. Equations of motion of an extended object governed by the Nambu-Goto action or its generalization has the form of partial differential equations with non-linear constraint equations in general. However, the equations reduce to ordinary differential equations when the object has enough symmetries. In the investigations noted above, rich properties of the extended objects are clarified by solving the ordinary differential equations explicitly.

Suppose that an $m$-dimensional hypersurface is embedded in an $n(> m)$-dimensional target space which has isometries, the embedded hypersurface can possess a part of the symmetries of the target space. If a subgroup of the isometries of target space acts on the hypersurface, the restriction on the hypersurface is isometries of it. In addition, the extrinsic geometry of the hypersurface is also symmetric with the isometries. The hypersurface is said to be a cohomogeneity one hypersurface if a subgroup of isometries of a target space acts on the hypersurface, and the orbits of the action on it span an $(m - 1)$-dimensional space. As is discussed later in the Nambu-Goto string case, the equations of motion for a cohomogeneity one object can be reduced to ordinary differential equations.

Among a variety of extended object’s dimension, we concentrate ourself especially on cohomogeneity one strings, as the simplest extended objects, in this letter. The one-parameter group of isometries acting on the world surface is generated by a Killing vector field on a target space, that is, the string’s world surface is tangent to the Killing vector field.

If a target space has only one Killing vector, there is one class of cohomogeneity one strings associated with the Killing vector. On the other hand, in a space with more than
two independent Killing vectors, there would be infinite classes of cohomogeneity one strings since infinite numbers of linear combinations of the Killing vectors are possible.

Let us consider the four-dimensional Minkowski spacetime, which has ten linearly independent Killing vectors, as a target space. Suppose two Killing vectors which denote translation symmetries along different directions. Though these Killing vectors are linearly independent, cohomogeneity one strings associated with these Killing vectors are members of the same class because the world surfaces of these strings are transformed each other by a suitable isometry of Minkowski spacetime which rotates one direction of translation symmetry into the other.

The issue treated in this letter is how many classes of cohomogeneity one strings are there in a symmetric spacetime. For clarifying this problem, we classify the Killing vectors in Minkowski spacetime, as a typical example of symmetric spacetime, introducing an equivalence relation by using isometries of the spacetime.

II. EQUATIONS OF MOTION OF COHOMOGENEITY ONE STRING

Let us consider a two-dimensional world surface $\Sigma$ is embedded in an $n$-dimensional target spacetime $\mathcal{M}$ with a metric $g_{ab}$ which possesses Killing vector fields. If the world surface $\Sigma$ is tangent to one of the Killing vector fields of $\mathcal{M}$, say $\xi^a$, we call the world surface $\Sigma$ a cohomogeneity one string associated with the Killing vector $\xi^a$. The stationary string is one of the example associated with a Killing vector which is timelike on the surface[4].

When we choose a Killing vector field in $\mathcal{M}$, there would be many cohomogeneity one world surfaces associated with the Killing vector. If the equations of motion governing the dynamics of the string is given, the possible cohomogeneity one string surfaces are selected out as solutions to the equations. Hereafter, we consider the Nambu-Goto string, as the simplest example, whose action is given by

$$S = \int_{\Sigma} \sqrt{-\gamma},$$  \hspace{1cm} (1)

where $\gamma$ is the determinant of induced metric on the world surface.

The orbit space of a Killing vector field $\xi^a$ of $\mathcal{M}$, say $\mathcal{N}$, is an $(n - 1)$-dimensional space on which the metric

$$h_{ab} = g_{ab} - \frac{\xi_a \xi_b}{f}$$  \hspace{1cm} (2)
is introduced naturally, where \( f := \xi^a \xi_a \). The metric \( h_{ab} \) has Euclidean signature in the region where \( f < 0 \) and Lorentzian signature in \( f > 0 \).

Suppose that a curve \( \mathcal{C} \) is given in \( \mathcal{N} \), orbits of the Killing vectors \( \xi^a \) starting from \( \mathcal{C} \) span a two-dimensional surface. Since \( h_{ab} \) measures the length along the direction perpendicular to \( \xi^a \), area of the surface element is simply given by

\[
dA = \sqrt{|f|} \sqrt{|h_{ab} dx^a dx^b|},
\]

where \( dx^a \) is an infinitesimal displacement along \( \mathcal{C} \). Therefore, the action (1) reduces to

\[
S = \int_{\mathcal{C}} \sqrt{-\tilde{h}_{ab} dx^a dx^b},
\]

where

\[
\tilde{h}_{ab} := fh_{ab}.
\]

The action (4) gives the length of \( \mathcal{C} \) with respect to the metric \( \tilde{h}_{ab} \) on \( \mathcal{N} \). Therefore, the problem for finding solutions of cohomogeneity one string associated with \( \xi^a \) reduces to the problem for solving \( (n-1) \)-dimensional geodesic equations with respect to the metric \( \tilde{h}_{ab} \).

When a world surface is associated with a spacelike Killing vector, i.e., \( f > 0 \) on the surface, \( \mathcal{C} \) should be a timelike geodesic with respect to the metric \( \tilde{h}_{ab} \) so that \( \Sigma \) is a timelike world surface.

### III. CLASSIFICATION IN MINKOWSKI SPACETIME

First, let us consider a pair of two-dimensional world surfaces, \( \Sigma_A \) and \( \Sigma_B \), embedded in a symmetric target space \( \mathcal{M} \) which admits isometries. The world surfaces \( \Sigma_A \) and \( \Sigma_B \) are geometrically equivalent if there is an isometry \( \phi \) of \( \mathcal{M} \) which maps \( \Sigma_A \) onto \( \Sigma_B \):

\[
\phi : \Sigma_A \rightarrow \Sigma_B.
\]

Next, let \( \Sigma_A \) be a cohomogeneity one string associated with a Killing vector field \( \xi_A \) and let \( \Sigma_B \) be a geometrically equivalent to \( \Sigma_A \). The isometry \( \phi \) which maps \( \Sigma_A \) to \( \Sigma_B \) pushes forward \( \xi_A \) to a vector field, say \( \xi_B \):

\[
\phi^* : \xi_A \rightarrow \xi_B.
\]
It is clear that $\xi_B$ is also a Killing vector field on $\mathcal{M}$, and $\Sigma_B$ is a cohomogeneity one string associated with it. If $\mathcal{M}$ has a large number of symmetries, it is possible that $\xi_A$ and $\xi_B$ are linearly independent vector fields which are connected by the isometry $\phi$. Therefore, it suggests the idea that we classify Killing vector fields by the equivalence relation with respect to the isometries.

We introduce an equivalence relation, $\sim$, as follows: Killing vectors $\xi_A$ and $\xi_B$ are equivalent if and only if there is an isometry $\phi$ which pushes forward $\xi_A$ to $\lambda \xi_B$ ($\lambda$ : constant):

$$
\phi^* : \xi_A \rightarrow \lambda \xi_B \quad \Leftrightarrow \quad \xi_A \sim \xi_B.
$$

(8)

The scalar multiplication comes from the fact that the Killing vector field is irrelevant to the constant scaling.

Here, we classify the Killing vectors in Minkowski spacetime based on the equivalence relation noted above. In Minkowski spacetime with the metric

$$
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (9)
$$

there are ten independent Killing vectors which generate Poincaré group:

- $P_a$ ($a = t, x, y, z$), Translations along $a$-direction;
- $K_i$ ($i = x, y, z$), Lorentz boosts along $i$-direction;
- $L_i$ ($i = x, y, z$), Space rotations around $i$-axis.

An arbitrary Killing vector $\xi$ is expressed as a linear combination of them:

$$
\xi = \alpha_a P_a + \beta_i K_i + \gamma_i L_i, \quad (10)
$$

where $\alpha_a$, $\beta_i$ and $\gamma_i$ are constant coefficients.

We will list up inequivalent representatives under the equivalence relation (8) using Poincaré group of isometries. The task we should do is logically straightforward but tedious. Then, we would like to show some typical procedure and final result.

As an example of the procedure of classification, we consider a Killing vector in the form

$$
\xi = a K_y + b L_z, \quad (11)
$$

5
where $a$ and $b$ are arbitrary constants. We use the Lorentz boost along $x$-direction denoted by $\phi = \text{Exp}[\varphi K_x]$, where $\varphi$ is a parameter. The Killing vector $\xi$ is pushed forward by $\phi$ as
\[ \phi^*: \xi \rightarrow (a \cosh \varphi - b \sinh \varphi) K_y + (-a \sinh \varphi + b \cosh \varphi) L_z. \] (12)

Choosing the parameter $\varphi$ suitable for the values of $a$ and $b$, we see that $\xi$ is equivalent to one of three Killing vectors:
\[ \xi = a K_y + b L_z \sim \begin{cases} 
K_y & \text{for } |a| > |b| \\
L_z & \text{for } |a| < |b| \\
K_y + L_z & \text{for } |a| = |b|
\end{cases}, \] (13)
where we also use space or time reflection.

In contrast, it is worthwhile to note that the Killing vector in the form
\[ \xi = a K_z + b L_z, \] (14)
for example, is not equivalent to
\[ \xi' = a' K_z + b' L_z \] (15)
except the case $|a/b| = |a'/b'|$. In the metric (9), the norm of $\xi$ is given by
\[ a^2(-z^2 + t^2) + b^2(x^2 + y^2), \] (16)
Then, equinorm surfaces of $\xi$ are in the form of squashed hyperboloids, where $a^2/b^2$ describes the amount of squashing. Since the pushing forward $\phi^*$ preserves the norm of vectors then the shape of the equinorm surfaces should be same for equivalent vector fields. If $|a/b| \neq |a'/b'|$ the amount of squashing of the hyperboloid is different. Thus, there is no isometry which maps $\xi$ to $\xi'$, i.e., $\xi$ is inequivalent to $\xi'$.

Starting from a general linear combination (10), we can complete the classification. We find that equivalence classes are partitioned into seven families listed in Table I. Each partition consists of infinite classes parameterized by one parameter. We introduce a notation, $\boxplus$, such that $P_t \boxplus L_z$, for example, means the set of all linear combinations $a P_t + b L_z$, where $a$ and $b$ are non-negative constants satisfying $a^2 + b^2 = 1$. The constraints on $a$ and $b$ are for eliminating total scaling of Killing vectors and redundancy under space and time reflections.
The number of partitions would crucially depends on the structure of the group of isometries. Since Poincaré group is the group of isometries preserving indefinite Minkowski metric then the first three families, I \sim III, should be distinguished. If the classification is done in Euclidean space, these families would fall into one family since the rotations in \( t - z \) plane connect members in these families. Similarly, families of III, IV and V would fall into one family in the flat Euclidean space. In addition, it seems that the existence of translation group makes the issue complicated. It would be interesting to classify the Killing vectors in other maximally symmetric spacetime: de Sitter and anti-de Sitter spacetimes[6].

The Killing vector fields in the families I, V, VI and VII can be timelike in some regions in Minkowski spacetime except edges of the families i.e., \( L_z, P_z, \) and \( K_y + L_z \). Then, there are four types of stationary strings which correspond to four families of timelike Killing vectors. The rigidly rotating strings are one of them; they are associated with the Killing vectors in the family \( P_t \ominus L_z \). All Killing vector fields except \( P_t \) and \( (P_t + P_z) \) have spacelike region then there are seven types of cohomogeneity one strings with spacelike symmetries.

From (2) and (5), we see that a one-parameter family of Killing vectors gives one-parameter family of metrics \( \tilde{h} \) on the orbit space \( \mathcal{N} \). Then, we should solve seven types of geodesic equations for finding all of cohomogeneity one Nambu-Goto string solutions in Minkowski spacetime. Some of them are easily solved [4] [7].

Generalizations of the present work to higher dimensional target spaces and higher dimensional cohomogeneity one objects are interesting issues. It is easy to get ordinary differential
equations for higher dimensional cohomogeneity one objects, but classification of them would require some labors.

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