UNIFORM CONVERGENCE OF MULTIGRID FINITE ELEMENT
METHOD FOR TIME-DEPENDENT RIESZ TEMPERED
FRACTIONAL PROBLEM *

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Abstract. In this article a theoretical framework for the Galerkin finite element approximation
to the time-dependent Riesz tempered fractional problem is provided without the fractional regularity
assumption. Because the time-dependent problems should become easier to solve as the time step
τ → 0, which correspond to the mass matrix dominant [R. E. Bank and T. Dupont, Math. Comp.,
153 (1981), pp. 35–51]. Based on the introduced and analysis of the fractional τ-norm, the uniform
convergence estimates of the V-cycle multigrid method with the time-dependent fractional problem
is strictly proved, which means that the convergence rate of the V-cycle MGM is independent of the
mesh size h and the time step τ. The numerical experiments are performed to verify the convergence
with only O(NlogN) complexity by the fast Fourier transform method, where N is the number of
the grid points. To the best of our knowledge, this is the first proof for the convergence rate of the
V-cycle multigrid finite element method with τ → 0.

Key words. uniform convergence of V-cycle multigrid method, finite element method, fractional
τ-norm, time-dependent Riesz tempered fractional problem, fast Fourier transform

AMS subject classifications. 65M55, 76M10, 35R11, 65T50

1. Introduction. In most practical problems, the real physical domain is bounded
and/or the observables involved in dynamical have finite moments. Then, based on
the continuous time random walk (CTRW) model [15], the Riesz tempered fractional
diffusion equation can be derived by tempering the probability of large jump length of
the Lévy flights [5, 15, 20, 26], which describes the probability density function (PDF)
of the truncated Lévy flights [42, 44]. In this article we investigate the multigrid fi-
nite element method (FEM) for solving the time-dependent Riesz tempered fractional
problem

\begin{equation}
\begin{aligned}
P_t(x, t) - \nabla_x^{\alpha,\lambda} P(x, t) &= \varphi(x, t) \quad \text{in } \Omega \times (0, T], \\
P(x, t) &= 0 \quad \text{on } (\mathbb{R} \setminus \Omega) \times (0, T], \\
P(x, 0) &= \psi(x) \quad \text{on } \Omega \times \{t = 0\},
\end{aligned}
\end{equation}

where we assume Ω = (a, b) to be an open, bounded subset of \( \mathbb{R} \) for some fixed time
T > 0. The Riesz tempered fractional derivative is defined by [15]

\begin{equation}
\nabla_x^{\alpha,\lambda} P(x, t) = \kappa_{\alpha} \left[ a D_x^{\alpha,\lambda} + x D_x^{\alpha,\lambda} \right] P(x, t)
\end{equation}

* This work was supported by NSFC 11601206 and NSFC 11601460.
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with $\lambda > 0$, $\kappa_\alpha = -\frac{1}{2\cos(\pi \alpha/2)} > 0$, $\alpha \in (1, 2)$. The left and right tempered Riemann-Liouville fractional derivatives are, respectively, defined by

\begin{equation}
\begin{aligned}
d_a^{\alpha,\lambda} u(x) &= d_a^{\alpha,\lambda} u(x) - \lambda^\alpha u(x) - \alpha \lambda^{\alpha-1} \frac{\partial u(x)}{\partial x} , \\
dx D_b^{\alpha,\lambda} u(x) &= x D_b^{\alpha,\lambda} u(x) - \lambda^\alpha u(x) + \alpha \lambda^{\alpha-3} \frac{\partial u(x)}{\partial x} .
\end{aligned}
\end{equation}

(1.3)

Here the main term of the left and right tempered Riemann-Liouville fractional derivatives, respectively, are given in [19, 20]

\begin{equation}
\begin{aligned}
a D_a^{\alpha,\lambda} u(x) &= e^{-\lambda x} a D_a^{\alpha} [e^{\lambda x} u(x)] = e^{-\lambda x} \frac{d^2}{dx^2} d_a^{\alpha} [e^{\lambda x} u(x)] = e^{-\lambda x} \frac{d^2}{dx^2} \int_a^x e^{\lambda \xi} u(\xi) \frac{d\xi}{(x-\xi)^{\alpha-1}} d\xi , \\
x D_b^{\alpha,\lambda} u(x) &= e^{\lambda x} x D_b^{\alpha} [e^{-\lambda x} u(x)] = e^{\lambda x} \frac{d^2}{dx^2} x D_b^{\alpha} [e^{-\lambda x} u(x)] = e^{\lambda x} \frac{d^2}{dx^2} \int_x^b e^{-\lambda \xi} u(\xi) \frac{d\xi}{(\xi-x)^{\alpha-1}} d\xi ,
\end{aligned}
\end{equation}

(1.4)

and the left and right tempered Riemann-Liouville fractional integrals, respectively, are defined by [35]

\begin{equation}
\begin{aligned}
a D_a^{-\alpha,\lambda} u(x) &= e^{-\lambda x} a D_a^{-\alpha} [e^{\lambda x} u(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x e^{-\lambda (x-\xi)} (x-\xi)^{\alpha-1} u(\xi) d\xi , \\
x D_b^{-\alpha,\lambda} u(x) &= e^{\lambda x} x D_b^{-\alpha} [e^{-\lambda x} u(x)] = \frac{1}{\Gamma(\alpha)} \int_x^b e^{-\lambda (\xi-x)} (\xi-x)^{\alpha-1} u(\xi) d\xi .
\end{aligned}
\end{equation}

(1.5)

To make a coercive bilinear form for the model (see section 2.2 for details), we can take $P = e^{\sigma t} u$ with $\sigma > 0$ [16, 30]. Then (1.1) can be rewrite as

\begin{equation}
\begin{aligned}
&\left\{ u_t(x,t) - \kappa_\alpha \left[ a D_a^{\alpha,\lambda} + x D_b^{\alpha,\lambda} \right] u(x,t) + \sigma u(x,t) = f(x,t) \quad \text{in} \quad \Omega \times (0,T), \\
u(x,t) = 0 \quad \text{on} \quad (\mathbb{R} \setminus \Omega) \times (0,T), \\
u(x,0) = \psi(x) \quad \text{on} \quad \Omega \times \{ t = 0 \}
\right. \\
\end{aligned}
\end{equation}

(1.6)

with $f(x,t) = e^{-\sigma t} \varphi(x,t)$.

There are already some important progress for numerically solving the FEM fractional problems [45]. For example, the time-fractional and/or space-fractional problems are discussed in [13, 14, 27, 29, 30, 43, 50, 58]. A theoretical framework with the fractional regularity assumption for the Galerkin finite element approximation to the steady-state fractional model is first presented in [30] and intensive studied in [1, 41, 54]. Ros-Oton and Serra study the regularity up to the boundary of solutions to the Dirichlet problem for the fractional Laplacian with Hölder estimates [51]; Jin et al. pointed out that there is still a lack of the regularity of weak solution in general but given the regularity results of strong solutions [17]; and Ervin et al. investigated the regularity of the solution to the two-side fractional diffusion equation [31]. Recently, the tempered fractional problems with the tempered fractional regularity assumption are discussed in [10, 25]. Here a theoretical framework for the Galerkin finite element approximation to the time-dependent Riesz tempered fractional problem is presented without the fractional regularity assumption.

When considering iterative solvers for the large-scale linear systems arising from the approximation of elliptic partial differential equations (PDEs), multigrid methods (MGM) (such as V-cycle and W-cycle) are often optimal order process [8, 10, 32]. The elegant theoretical framework and uniform convergence of the V-cycle MGM
for second order elliptic equation is well established in [7, 9, 55]. The convergence rate independent of the number of levels is presented by multigrid finite difference method for elliptic equations with variable coefficients [38]. In the case of multilevel matrix algebras, for special prolongation operators [35], the convergence rate of the V-cycle MGM is derived in [3, 4, 8] for the time-dependent second elliptic problems, the new convergence proofs for V-cycle MGM including multilevel linear systems are given in [22, 24]. For the time-independent fractional PDEs, based on the idea of [7, 9, 10], the convergence rate of the V-cycle MGM is discussed in [39, 57] and the nearly uniform convergence result is derived in [24]. For the time-dependent fractional PDEs, the convergence rate of the two-grid method has been performed in [21, 22, 47] by following the ideas in [17, 35]; and the convergence of the V-cycle MGM is investigated with \( \tau \) (time step) a positive constant [12]. Because the time-dependent PDEs should become easier to solve as the time step \( \tau \to 0 \), which correspond to the mass matrix dominant [6]. As far as we know, the convergence rate of the V-cycle multigrid finite element method has not been consider for a time-dependent PDEs with \( \tau \to 0 \). In this paper, based on the introduced and analysis of the fractional \( \tau \)-norm, the convergence rate of the V-cycle MGM is strictly proved, i.e., the uniform convergence of the V-cycle MGM is independent of the mesh size \( h \) and \( \tau \). Moreover, the fast Toeplitz matrix-vector multiplication is utilized to lower the computational cost with only \( O(N\log N) \) complexity by the fast Fourier transform (FFT) method, where \( N \) is the number of the grid points.

The outline of the paper is as follows. In the next section, we reivew the tempered fractional Sobolev space and prove a coercivity and a bondness for the bilinear form. A theoretical framework for the Galerkin finite element approximation to the time-dependent Riesz tempered fractional problem is presented without the fractional regularity assumption in Section 3. In Section 4, we first define the fractional \( \tau \)-norm and prove the convergence estimates of the V-cycle MGM with time-dependent fractional PDEs. Results of numerical experiments are reported and discussed in Section 5, in order to show the effectiveness of the presented schemes. Finally, we conclude the paper with some remarks.

\section{Preliminaries.}
Throughout this article, \( c, C \) and \( C_i, i \geq 0 \) will denote positive constants, not necessarily the same at different occurrences, which are independent of the mesh size \( h \) and the time step \( \tau \). Let \( C^0_0(\Omega) \) denote the set of all functions \( u \in C^\infty(\Omega) \) that vanish outside a compact subset of \( \Omega \). For \( \nu \geq 0 \), let \( H^\nu(\Omega) \) denote the Sobolev space of order \( \nu \) on the interval \( \Omega \) and \( H^\nu_0(\Omega) \) the set of functions in \( H^\nu(\Omega) \) whose extension by 0 are in \( H^\nu(\mathbb{R}) \). This section is devoted to some results on tempered fractional Sobolev spaces and some properties on bilinear form.

\subsection{Tempered fractional Sobolev spaces.}
Based on the idea of [16, 25, 30], we further develop the abstract setting for the analysis of the approximation to the tempered fractional diffusion equation.

\begin{definition}
Let \( \nu > 0, \lambda > 0 \). Define the semi-norm
\[
|u|_{J^\nu_\lambda(\mathbb{R})} := |\int_{-\infty}^{\infty} D^{\nu} x^\lambda u|_{L^2(\mathbb{R})},
\]
and norm
\[
||u||_{J^\nu_\lambda(\mathbb{R})} := \left( ||u||^2_{L^2(\mathbb{R})} + |u|^2_{J^\nu_\lambda(\mathbb{R})} \right)^{1/2},
\]
\end{definition}
where $J_{\nu,\lambda}^\infty(R)$ denotes the closure of $C_0^\infty(R)$ with respect to $\| \cdot \|_{J_{\nu,\lambda}^\infty(R)}$.

**Definition 2.2.** Let $\nu > 0$, $\lambda > 0$. Define the semi-norm

$$|u|_{J_{\nu,\lambda}^\infty(R)} := \|xD_{\nu,\lambda}^\infty u\|_{L^2(R)},$$

and norm

$$\|u\|_{J_{\nu,\lambda}^\infty(R)} := \left(\|u\|_{L^2(R)}^2 + |u|_{J_{\nu,\lambda}^\infty(R)}^2\right)^{1/2},$$

where $J_{\nu,\lambda}^\infty(R)$ denotes the closure of $C_0^\infty(R)$ with respect to $\| \cdot \|_{J_{\nu,\lambda}^\infty(R)}$.

In the following analysis, we define a semi-norm for functions in $H_{\nu,\lambda}^\infty(R)$ in terms of the Fourier transform. We shall depart from the constant coefficient $\sqrt{2\pi}$ of the inverse Fourier transform. This convention simplifies the appearance of results such as the following Parseval’s Formula \[52]\n
$$\int_R u(x)\overline{v(x)}dx = \int_R \hat{u}(\omega)\overline{\hat{v}(\omega)}d\omega,$$

where the bar denotes complex conjugation.

**Lemma 2.3.** (\[13,14,27\]) Let $\nu > 0$, $\lambda > 0$. Let $u$, $-\infty D_{x}^{\nu,\lambda} u$, $x D_{\nu}^{\nu,\lambda} u$ and their Fourier transform belong to $L^1(R)$. Then

$$\mathcal{F}\left(-\infty D_{x}^{\nu,\lambda} u(x)\right) = (\lambda - i\omega)^{\nu} \hat{u}(\omega),$$

and

$$\mathcal{F}\left(x D_{\nu}^{\nu,\lambda} u(x)\right) = (\lambda + i\omega)^{\nu} \hat{u}(\omega),$$

where $\mathcal{F}$ denotes Fourier transform operator and $\hat{u}(\omega) = \mathcal{F}(u)$, i.e.,

$$\hat{u}(\omega) = \int_R e^{i\omega x}u(x)dx.$$

**Definition 2.4.** Let $\nu > 0$, $\lambda > 0$. Define the semi-norm

$$|u|_{H_{\nu,\lambda}^\infty(R)} := \| (\lambda^2 + |\omega|^2)^{\nu/2} \hat{u}\|_{L^2(R)},$$

and norm

$$\|u\|_{H_{\nu,\lambda}^\infty(R)} := \left(\|u\|_{L^2(R)}^2 + |u|_{H_{\nu,\lambda}^\infty(R)}^2\right)^{1/2},$$

where $H_{\nu,\lambda}^\infty(R)$ denotes the closure of $C_0^\infty(R)$ with respect to $\| \cdot \|_{H_{\nu,\lambda}^\infty(R)}$.

**Definition 2.5.** Let $\nu > 0$, $\lambda = 0$. Define the semi-norm

$$|u|_{H_{\nu}^\infty(R)} := \| \lambda^\nu \hat{u}\|_{L^2(R)},$$

and norm

$$\|u\|_{H_{\nu}^\infty(R)} := \left(\|u\|_{L^2(R)}^2 + |u|_{H_{\nu}^\infty(R)}^2\right)^{1/2} = \left(1 + |\omega|^{2\nu}\right)^{1/2} \|\hat{u}\|_{L^2(R)},$$

where $H_{\nu}^\infty(R)$ denotes the closure of $C_0^\infty(R)$ with respect to $\| \cdot \|_{H_{\nu}^\infty(R)}$. 
Lemma 2.6. Let $\nu > 0$ and $x \geq 0$. Then
\[
(1 + x^n) \leq (1 + x)^\nu \leq 2^{\nu-1}(1 + x^n) \quad \forall \nu \geq 1;
\]
\[
2^{\nu-1}(1 + x^n) \leq (1 + x)^\nu \leq (1 + x^n) \quad \forall 0 < \nu \leq 1.
\]

Lemma 2.7. \textbf{[Fractional Poincaré-Friedrichs]} For $u \in J^\nu_L(\mathbb{R})$ or $u \in J^\nu_R(\mathbb{R})$ with a compact subset of $\Omega$, respectively, we have
\[
||u||_{L^2(\mathbb{R})} \leq C|u|_{J^\nu_L(\mathbb{R})} \quad \text{and} \quad ||u||_{L^2(\mathbb{R})} \leq C|u|_{J^\nu_R(\mathbb{R})}.
\]

Theorem 2.8. Let $\nu > 0$, $\lambda > 0$, $u \in J^\nu_L(\mathbb{R})$. Then the space $J^\nu_L(\mathbb{R})$, $J^\nu_R(\mathbb{R})$, $H^\nu(\mathbb{R})$ are equal with respect to the semi-norms and norms; and in fact
\[
|u|_{H^\nu(\mathbb{R})} = |u|_{J^\nu_L(\mathbb{R})} = |u|_{J^\nu_R(\mathbb{R})},
\]
and
\[
C_1|u|_{J^\nu_L(\mathbb{R})} \leq |u|_{H^\nu(\mathbb{R})} \leq C_2|u|_{J^\nu_R(\mathbb{R})}.
\]

Proof. According to Definition 2.1 and Lemma 2.7, there exists
\[
|u|_{J^\nu_L(\mathbb{R})} \leq \|u\|_{J^\nu_L(\mathbb{R})} = \left(\|u\|_{L^2(\mathbb{R})}^2 + |u|_{J^\nu_L(\mathbb{R})}^2\right)^{1/2} \leq (1 + C^2)^{1/2}|u|_{J^\nu_L(\mathbb{R})},
\]
which means that we have norm equivalence of $|u|_{J^\nu_L(\mathbb{R})}$ and $|u|_{J^\nu_R(\mathbb{R})}$. From Definitions 2.1, 2.2, 2.3, 2.4, 2.5 and Lemma 2.6 and Parseval’s Formula 2.1, we obtain
\[
|u|_{H^\nu(\mathbb{R})} = |u|_{J^\nu_L(\mathbb{R})} = |u|_{J^\nu_R(\mathbb{R})} = \|\left(\lambda^2 + |\omega|^2\right)^{\nu/2} \hat{u}\|_{L^2(\mathbb{R})},
\]
and $|u|_{H^\nu(\mathbb{R})} = |u|_{J^\nu_L(\mathbb{R})} = |u|_{J^\nu_R(\mathbb{R})}$.

Next we shall prove norm equivalence of $|u|_{H^\nu(\mathbb{R})}$ and $|u|_{J^\nu_L(\mathbb{R})}$ (or $|u|_{J^\nu_R(\mathbb{R})}$).

Using Lemma 2.7 with $\nu \geq 1$, we obtain
\[
(1 + |\omega|^{2\nu})^{\frac{1}{\nu}} \leq (1 + |\omega|^2)^{\frac{1}{2}} \leq 2^{\frac{\nu-1}{\nu}} (1 + |\omega|^{2\nu})^{\frac{1}{\nu}} \quad \forall \nu \geq 1,
\]
and it is easy to get
\[
\frac{1}{\lambda^\nu} \left(\lambda^2 + |\omega|^2\right)^{\nu/2} \leq (1 + |\omega|^2)^{\nu/2} \leq \left(\lambda^2 + |\omega|^2\right)^{\nu/2} \quad \forall \lambda \geq 1,
\]
\[
\left(\lambda^2 + |\omega|^2\right)^{\nu/2} \leq \frac{1}{\lambda^\nu} \left(\lambda^2 + |\omega|^2\right)^{\nu/2} \quad \forall 0 < \lambda \leq 1.
\]

Multiplying (2.2) by $||\hat{u}||_{L^2(\mathbb{R})}$, and using Definitions 2.1, 2.2, 2.3, 2.4, 2.5 and Lemma 2.6 there exists
\[
\frac{1}{\lambda^\nu} |u|_{J^\nu_L(\mathbb{R})} \leq (1 + |\omega|^2)^{\frac{1}{2}} ||\hat{u}||_{L^2(\mathbb{R})} \leq 2^{\frac{\nu-1}{\nu}} (1 + |\omega|^{2\nu})^{\frac{1}{\nu}} ||\hat{u}||_{L^2(\mathbb{R})}
\]
\[
= 2^{\frac{\nu-1}{\nu}} |u|_{H^\nu(\mathbb{R})} \leq 2^{\frac{\nu-1}{\nu}} (1 + |\omega|^2)^{\frac{1}{2}} ||\hat{u}||_{L^2(\mathbb{R})}
\]
\[
\leq 2^{\frac{\nu-1}{\nu}} \left(\lambda^2 + |\omega|^2\right)^{\nu/2} ||\hat{u}||_{L^2(\mathbb{R})} = 2^{\frac{\nu-1}{\nu}} |u|_{J^\nu_L(\mathbb{R})} \quad \forall \lambda \geq 1.
\]
Similarly, we have
\[ |u|_{J_{L}^{\nu,\lambda}(\mathbb{R})} \leq (1 + |\omega|^2)^{|\nu|/2} |\bar{u}|_{L^2(\mathbb{R})} \leq 2 \frac{|\nu|}{\lambda^\nu} ||u||_{H^\nu(\mathbb{R})} \leq \frac{1}{|\lambda|^\nu} 2 \frac{|\nu|}{\lambda^\nu} |u|_{J_{L}^{\nu,\lambda}(\mathbb{R})} \forall 0 < \lambda \leq 1. \]

Therefore, we obtain the norm equivalence of ||u||_{H^\nu(\mathbb{R})} and |u|_{J_{L}^{\nu,\lambda}(\mathbb{R})}, moreover
\[ |u|_{H^\nu(\mathbb{R})} = |u|_{J_{L}^{\nu,\lambda}(\mathbb{R})} \geq \min \{1, \lambda^\nu\} ||u||_{H^\nu(\mathbb{R})} \forall \lambda > 0. \]

From [30], we obtain \[ ||u||_{L^2(\mathbb{R})} \leq C||u||_{H^\nu(\mathbb{R})} \]
which means that we have norm equivalence of ||u||_{H^\nu(\mathbb{R})} and |u|_{J_{L}^{\nu,\lambda}(\mathbb{R})}. The similar arguments can be performed as stated above with 0 < \nu \leq 1, we omit it here. The proof is completed. \[ \square \]

2.2. Coercive and continuous of bilinear form. Assume that u is a sufficiently smooth function and v \in C_0^\infty(\Omega), we obtain
\[
\left(- \left(aD_x^{\alpha,\lambda} + zD_b^{\alpha,\lambda}\right) u, v \right) = \left(- \left(aD_x^{\alpha,\lambda} + zD_b^{\alpha,\lambda}\right) u, v \right) + 2\lambda^\alpha(u,v) \\
= -2 \left(aD_x^{\alpha/2,\lambda} u, zD_b^{\alpha/2,\lambda} v \right) + 2\lambda^\alpha(u,v).
\]

Thus, we can define the associated bilinear form \( b: H_0^{\nu/2}(\Omega) \times H_0^{\nu/2}(\Omega) \rightarrow \mathbb{R} \) as
\[ b(u, v) = -2\kappa_\alpha \left(aD_x^{\alpha/2,\lambda} u, zD_b^{\alpha/2,\lambda} v \right) + 2\kappa_\alpha\lambda^\alpha(u,v) + \sigma(u,v). \]

**Definition 2.9.** Define the spaces \( J_{L,0}^{\nu,\lambda}(\Omega), J_{R,0}^{\nu,\lambda}(\Omega), H_0^{\nu,\lambda}(\Omega), H_0^{\nu}(\Omega) \) as the closures of \( C_0^\infty(\Omega) \) under their respective norms.

For a simple and intuitive derivation, we denote
\[ a \circ |u|^2_{H^\nu(\mathbb{R})} := \int_\mathbb{R} a \cdot (\lambda^2 + \omega^2)^\nu |\bar{u}|^2 d\omega. \]

**Lemma 2.10.** Let \( \nu > 0, \lambda > 0, u \in H_0^{\nu/2}(\Omega) \). Then
\[ \left(aD_x^{\nu/2,\lambda} u, zD_b^{\nu/2,\lambda} v \right) = \cos(\nu \theta) \circ |u|^2_{H^{\nu/2,\lambda}(\mathbb{R})} \]
with \( \theta = \arctan \frac{|\omega|}{\lambda} \in \left[0, \frac{\pi}{2}\right) \).

**Proof.** Let overbar denotes complex conjugate. It is easy to get
\[ (\lambda + i\omega)^{\nu/2} = \begin{cases} (\lambda - i\omega)^{\nu/2} e^{-i\nu\theta}, & \omega \geq 0; \\ (\lambda - i\omega)^{\nu/2} e^{i\nu\theta}, & \omega < 0 \end{cases} \]
with \( \theta = \arctan \frac{|\omega|}{\lambda} \in \left[0, \frac{\pi}{2}\right) \). It should be noted that \( \theta = \frac{\pi}{2} \) if \( \lambda = 0 \), see in [30].
Using Lemma 2.3 and Parseval’s Formula (2.1), we have

\[
\left( s D^{\nu/2, \lambda} u, s D^{\nu/2, \lambda} u \right) = \int_{-\infty}^{0} (\lambda - i\omega)^{\nu/2} \hat{u}_{\nu/2, \lambda} \hat{x}^{\nu/2} \hat{u} d\omega + \int_{0}^{\infty} (\lambda - i\omega)^{\nu/2} \hat{u}_{\nu/2, \lambda} (\lambda + i\omega)^{\nu/2} \hat{u} d\omega
\]

\[
= \int_{\mathbb{R}} \cos(\nu \theta) \cdot (\lambda^2 + \omega^2)^{\nu/2} |\hat{u}|^2 d\omega = \cos(\nu \theta) \circ |u|_{H^{\nu/2, \lambda}(\mathbb{R})}^2.
\]

The proof is completed.

Lemma 2.11. Let \( \nu > 0, \lambda > 0, u \in H^{\nu/2, 0}_0(\Omega) \). Then

\[
\|u\|^2_{L^2(\Omega)} = \frac{\cos^\nu(\theta)}{\lambda^\nu} \circ |u|_{H^{\nu/2, \lambda}(\mathbb{R})}^2
\]

with \( \theta = \arctan \left| \frac{\lambda}{\lambda} \right| \in [0, \frac{\pi}{2}) \).

Proof. From Parseval’s Formula (2.1), it yields

\[
\|u\|^2_{L^2(\Omega)} = \int_{\mathbb{R}} |\hat{u}|^2 d\omega = \frac{1}{\lambda^\nu} \int_{\mathbb{R}} \left( \frac{\lambda^2}{\lambda^2 + \omega^2} \right)^{\nu/2} \cdot (\lambda^2 + \omega^2)^{\nu/2} |\hat{u}|^2 d\omega
\]

\[
= \frac{1}{\lambda^\nu} \cos^\nu(\theta) \circ |u|_{H^{\nu/2, \lambda}(\mathbb{R})}^2,
\]

where we use

\[
\frac{\lambda^2}{\lambda^2 + \omega^2} = \frac{1}{1 + \tan^2 \theta} = \cos^2(\theta).
\]

The proof is completed.

Lemma 2.12. Let \( 1 < \alpha < 2, \lambda > 0, u \in H^{\alpha/2, 0}_0(\Omega) \). Then

\[
-2 \left( s a D^{\alpha/2, \lambda} u, s D^{\alpha/2, \lambda} b \right) + 2\lambda^\alpha (u, u) = 2 \cos^\alpha(\theta) \circ |u|_{H^{\alpha/2, \lambda}(\mathbb{R})}^2 \geq 0
\]

with \( \theta = \arctan \left| \frac{\lambda}{\alpha} \right| \in [0, \frac{\pi}{2}) \).

Proof. According to Lemmas 2.10 and 2.11, there exists

\[
-2 \left( s a D^{\alpha/2, \lambda} u, s D^{\alpha/2, \lambda} b \right) + 2\lambda^\alpha (u, u) = 2 \cos^\alpha(\theta) \circ |u|_{H^{\alpha/2, \lambda}(\mathbb{R})}^2
\]

Let

\[(2.7) \quad f(\theta) := \cos^\alpha(\theta) - \cos(\alpha \theta). \]

Obviously, \( f(\theta) = 0 \) if \( \theta = 0 \). Next we prove \( f(\theta) \geq 0 \) if \( \theta \in (0, \frac{\pi}{2}) \). Since

\[
f'(\theta) = \alpha \left[ \sin(\alpha \theta) - \cos^{\alpha-1}(\theta) \sin(\theta) \right] \geq \alpha \sin(\theta) g(\theta)
\]

with

\[
g(\theta) = \cos((\alpha - 1)\theta) - \cos^{\alpha-1}(\theta),
\]

The proof is completed.
and
\[ g'(\theta) = (\alpha - 1) \sin(\theta) \left[ \cos^{\alpha - 2}(\theta) - \frac{\sin((\alpha - 1)\theta)}{\sin(\theta)} \right] > 0 \text{ with } 1 < \alpha < 2. \]

Hence, we know that \( f \) is strictly increasing in \([0, \frac{\pi}{2}]\). The proof is completed. \( \square \)

**Lemma 2.13.** The bilinear form \( b(\cdot, \cdot) \) is coercive over \( H_0^{\alpha/2}(\Omega) \), i.e., there exists a constant \( C_0 \) such that
\[ b(u, u) \geq C_0 \| u \|^2_{H_0^{\alpha/2}(\Omega)}, \]

where \( \lambda > 0, 1 < \alpha < 2 \) and
\[ C_0 = \min \{ 1, \lambda^\alpha \} \cdot \min \left\{ 2\kappa_\alpha \left[ \frac{1}{2} - \cos \left( \frac{\alpha \pi}{3} \right) \right], \frac{\sigma}{(2\lambda)^\alpha} \right\} > 0. \]

**Proof.** According to (2.5) and Lemmas 2.12, 2.11 we have
\[ b(u, u) = 2\kappa_\alpha \left[ \cos^\alpha(\theta) - \cos(\alpha \theta) + \frac{\sigma}{2\kappa_\alpha \lambda^\alpha} \cos^\alpha(\theta) \right] \cdot |u|^2_{H_0^{\alpha/2, \lambda}(\Omega)}. \]

From (2.4), we know that \( f(\theta) = \cos^\alpha(\theta) - \cos(\alpha \theta) \geq 0 \) is strictly increasing in \([0, \frac{\pi}{2}]\), which leads to
\[ 2\kappa_\alpha \left[ \cos^\alpha(\theta) - \cos(\alpha \theta) + \frac{\sigma}{2\kappa_\alpha \lambda^\alpha} \cos^\alpha(\theta) \right] \geq \frac{\sigma}{\lambda^\alpha} \cos^\alpha \left( \frac{\pi}{3} \right) = \frac{\sigma}{(2\lambda)^\alpha} > 0 \quad \forall \theta \in \left[ \frac{\pi}{3}, \frac{\pi}{2} \right), \]

and
\[ 2\kappa_\alpha \left[ \cos^\alpha(\theta) - \cos(\alpha \theta) + \frac{\sigma}{2\kappa_\alpha \lambda^\alpha} \cos^\alpha(\theta) \right] \geq 2\kappa_\alpha \left[ \frac{1}{2} - \cos \left( \frac{\alpha \pi}{3} \right) \right] > 0 \quad \forall \theta \in \left[ \frac{\pi}{3}, \frac{\pi}{2} \right). \]

According to (2.4), and the above equations, there exists
\[ b(u, u) \geq \min \left\{ 2\kappa_\alpha \left[ \frac{1}{2} - \cos \left( \frac{\alpha \pi}{3} \right) \right], \frac{\sigma}{(2\lambda)^\alpha} \right\} \cdot |u|^2_{H_0^{\alpha/2, \lambda}(\Omega)} \]

\[ \geq \min \left\{ 2\kappa_\alpha \left[ \frac{1}{2} - \cos \left( \frac{\alpha \pi}{3} \right) \right], \frac{\sigma}{(2\lambda)^\alpha} \right\} \min \{ 1, \lambda^\alpha \} \| u \|^2_{H_0^{\alpha/2}(\Omega)}. \]

The proof is completed. \( \square \)

**Lemma 2.14.** The bilinear form \( b(\cdot, \cdot) \) is continuous on \( H_0^{\alpha/2}(\Omega) \times H_0^{\alpha/2}(\Omega) \) with \( 1 < \alpha < 2 \), i.e., there exists a constant \( C_1 \) such that
\[ |b(u, v)| \leq C_1 \| u \|_{H_0^{\alpha/2}(\Omega)} \| v \|_{H_0^{\alpha/2}(\Omega)}. \]

**Proof.** From (1.3), (2.4) and Theorem 2.8 we have
\[ |b(u, v)| \leq 2\kappa_\alpha \left| a D^{\alpha/2}_x u \cdot x D^{\alpha/2}_v v \right| \]
\[ \leq 2\kappa_\alpha |u|_{L_{x, \lambda}^{\alpha/2}(\Omega)} |v|_{L_{x, \lambda}^{\alpha/2}(\Omega)} + 2\kappa_\alpha |\lambda^\alpha + \sigma| \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} \]
\[ \leq C_1 \| u \|_{H_0^{\alpha/2}(\Omega)} \| v \|_{H_0^{\alpha/2}(\Omega)}. \]

The proof is completed. \( \square \)
3. Finite element method for tempered fractional problem. In this section a theoretical framework for the Galerkin finite element approximation to the time-dependent tempered fractional problem is presented, which does not require for the fractional regularity assumption \[16, 25, 30\]. The proposed method is based on a Crank-Nicolson scheme on time and Galerkin finite element in space for (1.6). This section is devoted to the stability analysis of the time-stepping scheme and the detailed error analysis of semidiscretization on time and of full discretization.

3.1. Stability analysis and error estimates for the semi-discrete scheme. Let \( T > 0, \Omega = (a, b) \) and \( t_n = n\tau, \quad n = 0, 1, \ldots, N \), where \( \tau = \frac{T}{N} \) is the time steplength. We set \( u^n \) or \( u^n(x) \) as an approximation of \( u(x, t_n) \) and \( f^{n-1/2} \) as an approximation of \( f(x, t_{n-1/2}) \) and denote \( u^n = \frac{1}{2}(u^n + u^{n-1}) \). We now turn to the Crank-Nicolson scheme, which will produce a second order accurate method in time, i.e.,

\[
\left. \frac{\partial u(x, t)}{\partial t} \right|_{t = t_{n-1/2}} = \bar{\partial}u^n + r^n_{\tau}(x)
\]

with

\[
|r^n_{\tau}(x)| \leq C \tau^2 \quad \text{and} \quad \bar{\partial}u^n = \frac{u^n(x) - u^{n-1}(x)}{\tau},
\]

where \( C \) is a constant depending only on \( u \). Then we get the following variational formulation of (1.6): Find \( u^n \in H^{1/2}_0(\Omega) \) such that

\[
(\bar{\partial}u^n, \chi) + b(u^n, \chi) = (f^{n-1/2}, \chi) \quad \forall \chi \in H^{1/2}_0(\Omega).
\]

**Theorem 3.1.** The semi-discretized scheme (3.2) is unconditionally stable.

**Proof.** Let \( \tilde{u}^n \) be the approximate solution of \( u^n \), which is the exact solution of the variational formulation (3.2). Taking \( E^n = \tilde{u}^n - u^n \). Then we get the following perturbation equation

\[
(\bar{\partial}E^n, \chi) + b(E^n, \chi) = 0
\]

with \( E^n = \frac{1}{2}(E^n + E^{n-1}) \). Taking \( \chi = E^n \), we obtain

\[
||E^n||_{L^2(\Omega)} \leq ||E^{n-1}||_{L^2(\Omega)} \leq ||E^0||_{L^2(\Omega)}.
\]

The proof is completed. \( \square \)

**Theorem 3.2.** Let \( u \) be the exact solution of (1.6) and \( u^n \) be the solution of semi-discrete scheme (3.2). Then we have the following error estimates

\[
||u(x, t_n) - u^n(x)||_{L^2(\Omega)} \leq 2C u(b - a)^{1/2}T\tau^2,
\]

where \( C u \) is defined by (3.7) and \((x, t_n) \in (a, b) \times (0, T)\) with \( N\tau \leq T\).

**Proof.** Define \( e^n = u(x, t_n) - u^n(x) \). According to \( e^0 = 0 \) and (3.1), (3.2), (1.6), there exists

\[
(e^n - e^{n-1}, \chi) + \tau b(e^n, \chi) = (R^n_{\tau}, \chi)
\]

with \( e^n = \frac{1}{2}(e^n + e^{n-1}) \) and \( ||R^n_{\tau}|| \leq c||\tau r^n_{\tau}|| \leq C u\tau^3 \). Taking \( \chi = e^n \) in the above equation, we get

\[
||e^n||_{L^2(\Omega)}^2 - ||e^{n-1}||_{L^2(\Omega)}^2 \leq (R^n_{\tau}, e^n + e^{n-1})
\]
Replacing $n$ with $s$ and summing up for $s$ from 1 to $n$, there exists

\[(3.3) \quad \|e^n\|^2_{L^2(\Omega)} \leq \sum_{s=1}^{n} (R^s_n, e^s + e^{s-1}) \leq \sum_{s=1}^{n} \|R^s_n\|_{L^2(\Omega)} \cdot \left(\|e^s\|_{L^2(\Omega)} + \|e^{s-1}\|_{L^2(\Omega)}\right).\]

Suppose $m$ is chosen so that $\|e^m\|_{L^2(\Omega)} = \max_{0 \leq s \leq N} \|e^s\|_{L^2(\Omega)}$. Then

$$\|e^m\|^2_{L^2(\Omega)} \leq \sum_{s=1}^{m} \|R^s_n\|_{L^2(\Omega)} \cdot \left(\|e^m\|_{L^2(\Omega)} + \|e^m\|_{L^2(\Omega)}\right).$$

Hence

$$\|e^n\|_{L^2(\Omega)} \leq \max_{0 \leq s \leq N} \|e^s\|_{L^2(\Omega)} = \|e^m\|_{L^2(\Omega)} \leq 2 \sum_{s=1}^{m} \|R^s_n\|_{L^2(\Omega)} \leq 2C_u(b-a)^{1/2}T^2.$$ 

The proof is completed. \[\square\]

### 3.2. Finite element approximation and error estimates for full discretization.

Let $S_h$ be a partition of $\Omega$ such that $\Omega = \{\cup K : K \in S_h\}$ with the uniform mesh size $h$. Denote a finite dimensional subspace $V_h \subset H^{\alpha/2}_0(\Omega)$ with a basis of the piecewise polynomials of degree at most $r - 1$ as

$$V_h = \{v \in H^{\alpha/2}_0(\Omega) \cap C^0(\Omega) : v|_K \in P_{r-1}(K) \forall K \in S_h\}.$$ 

Let us define the energy norm associated with (2.5) as

$$\|u\|_E := \sqrt{b(u,u)}.$$ 

Form Lemmas 2.13 and 2.14 we have norm equivalence of $\| \cdot \|_{H^{\alpha/2}_0(\Omega)}$ and $\| \cdot \|_E$.

Before we start to discuss the time-dependent fractional problem we shall briefly review some basic relevant material about the finite element method for the corresponding stationary problem.

**Definition 3.3 (Variational Solution of Stationary Problem).** For $f \in H^{-\alpha/2}(\Omega)$, find $u \in H^{\alpha/2}_0(\Omega)$ such that

$$b(u, v) = (f, v) \quad \forall v \in H^{\alpha/2}_0(\Omega).$$

The approximate problem is then to find $u_h \in V_h$ such that

$$b(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

**Lemma 3.4.** \[\square\] Let $u$ be the exact solution to $b(u, v) = (f, v)$ and let $P_h$ be the orthogonal projection from $H^{\alpha/2}_0(\Omega)$ to its finite dimensional subspace $V_h$. Then

$$b(P_h u - u, v) = 0 \quad \forall v \in V_h.$$ 

Next we prove the following fractional regularity estimate of strong solutions, which is the intermediate situation of weak solution and classical solutions \[\square\].
which contradicts $1\prec \alpha\prec z$.

According to the above equation and Lemmas 2.10-2.12 and (2.4) with $\sigma$, where $f$, then there exists a positive constant $C_a$ such that

$$\|z\|_{H^s(\Omega)} \leq C_a\|g\|_{L^2(\Omega)}.$$ (3.4)

Proof. From Lemma 2.12, we have

$$-\left(\alpha D^\alpha_x + x D^\alpha_x \right)z + \sigma \tau = g(x), \quad x \in \Omega, \quad z(x) = 0, \quad x \in \mathbb{R} \setminus \Omega.$$

Then there exists the positive constant $C$ such that $C_a = C$.

According to the above equation and Lemmas 2.10-2.12 and 2.4 with $\sigma \geq 3\alpha$, we obtain

$$\|g\|_{L^2(\Omega)} = \| - \left(\alpha D^\alpha_x + x D^\alpha_x \right)z + \sigma \tau\|_{L^2(\Omega)}\leq \| - \left(\alpha D^\alpha_x + x D^\alpha_x \right)z\|_{L^2(\Omega)} + 2\|\sigma \tau\|_{L^2(\Omega)}$$

$$= \|\alpha D^\alpha_x z\|_{L^2(\Omega)} + \|x D^\alpha_x z\|_{L^2(\Omega)} + 2 \left(\alpha D^\alpha_x z, x D^\alpha_x z \right)$$

$$- 4\lambda^\alpha \left(\alpha D^\alpha_x z, x D^\alpha_x z \right) + (\sigma^2 - 4\lambda^2z) \|z\|_{L^2(\Omega)}$$

$$\geq \left[2 + 2\cos(2\alpha\tau)\right] + (\sigma^2 - 4\lambda^2z) \|z\|_{L^2(\Omega)}$$

$$\geq 4f(\theta)|z|_{H^{s,\tau}(\Omega)}^2 \leq C\|z\|_{H^s(\Omega)}^2,$$

where $f(\theta) = \cos^2(\alpha\theta) + \cos^2(\alpha\theta) > 0$. In fact, if $f(\theta) = 0$, it lead to $\theta = \frac{\pi}{2}$ and $\alpha = 1$, which contradicts $1 < \alpha < 2$. The proof is completed.

Remark 3.1. In recent years, the optimal error estimate was provided under the assumption that the weak solution has full regularity \cite{16, 25, 30}; Ros-Oton and Serra study the regularity up to the boundary of solutions to the Dirichlet problem for the fractional Laplacian with Hölder estimates \cite{51}; Jin et al. pointed out that there is still a lack of the regularity of weak solution in general but given the regularity estimate of strong solutions \cite{44}; and Ervin et al. investigated the regularity of the strong solution to the two-side fractional diffusion equation \cite{31}.

Lemma 3.6. (Approximation Property) Let $u \in H^s_0(\Omega) \cap H^r(\Omega)$ with $r > s/2$. Then there exists the positive constant $C$ such that

$$\|P_hu - u\|_{L^2(\Omega)} + h^{s/2}\|P_hu - u\|_{H^r_0(\Omega)} \leq C h^{s/2}||u||_{H^r(\Omega)}.$$ (3.6)

Proof. For any $v \in V_h \subset H^s_0(\Omega)$, from Lemmas 2.13 3.8 2.14 there exists

$$C_0\|P_hu - u\|^2_{H^s_0(\Omega)} \leq b(P_hu - u, P_hu - u)$$

$$= b(P_hu - u, v - u) + b(P_hu - u, P_hu - v)$$

$$\leq C_1\|P_hu - u\|_{H^s_0(\Omega)}\|v - u\|_{H^s_0(\Omega)}.$$
which leads to

\[ \| P_h u - u \|_{H_0^\alpha(\Omega)} \leq \frac{C_1}{C_0} \inf_{v \in V_h} \| v - u \|_{H_0^\alpha(\Omega)} \leq \frac{C_1}{C_0} \| I_h u - u \|_{H_0^\alpha(\Omega)} \]

\[ \leq \frac{C_2 C_0}{C_0} h^{\alpha/2} \| u \|_{H^\alpha(\Omega)} \leq C_3 h^{\alpha/2} \| u \|_{H^\alpha(\Omega)}, \]

where \( I_h \) is the interpolation operator \cite{10, 30}, i.e., for \( u \in H^r(\Omega) \) and \( 0 \leq s \leq r \), there exists

\[ \| I_h u - u \|_{H^s(\Omega)} \leq C \| u \|_{H^r(\Omega)}. \]

For the error bound in \( L_2 \)-norm we proceed by a Aubin-Nitsche technology. Let \( z \) be the solution to (3.4). Then \( z \) satisfies the variational formulation

\[ b(v, z) = (P_h u - u, v) \quad \forall v \in H_0^\alpha(\Omega) \]

with \( g = P_h u - u \). Using Lemmas 3.4 and 3.5, we obtain

\[ \| P_h u - u \|_{L^2(\Omega)}^2 = b(P_h u - u, z) = \frac{1}{2} \| P_h u - u \|_{H_0^\alpha(\Omega)}^2 \]

\[ \leq C_1 \| z - I_h z \|_{H_0^\alpha(\Omega)} \| P_h u - u \|_{H_0^\alpha(\Omega)} \]

\[ \leq C_1 C_2 h^{\alpha/2} \| z \|_{H^\alpha(\Omega)} \| P_h u - u \|_{H_0^\alpha(\Omega)} \]

\[ \leq C_1 C_2 C_3 h^{\alpha/2} \| P_h u - u \|_{L^2(\Omega)} \| P_h u - u \|_{H_0^\alpha(\Omega)}. \]

Using the above equation and (3.5), there exists

\[ \| P_h u - u \|_{L^2(\Omega)} \leq C_1 C_2 C_3 h^{\alpha/2} \| P_h u - u \|_{H_0^\alpha(\Omega)} \leq C_1 C_2 C_3 h^{\alpha/2} \| u \|_{H^\alpha(\Omega)}. \]

The proof is completed.

Now we give the finite element approximation of (3.2): Find \( u^n_h \in V_h \) such that

\[ (\overline{\partial u^n_h}, \chi) + b(u^n_h, \chi) = (f^n_{n - 1/2}, \chi). \]

Let

\[ \overline{\partial u^n} = \overline{\partial P_h u^n} + r^n_h(x) = \frac{P_h u^n(x) - P_h u^{n-1}(x)}{\tau} + r^n_h(x). \]

Combine (3.1) and (3.7), there exists

\[ \frac{\partial u(x, t)}{\partial t} \bigg|_{t = t_{n-1/2}} = \overline{\partial P_h u^n} + r^n_{\tau, h}(x) \]

with

\[ r^n_{\tau, h}(x) = r^n_{\tau}(x) + r^n_h(x). \]

**Lemma 3.7.** The truncation error \( r^n_{\tau, h}(x) \) is bounded by

\[ \| r^n_{\tau, h} \|_{L^2(\Omega)} \leq C (\tau^2 + h^r). \]
Proof. Here, \( r^n_h(x) \) is given in (3.6), i.e.,
\[
\| r^n_h \|_{L^2(\Omega)} \leq C_u \tau^2.
\]
From Lemma 3.6, there exists
\[
(3.9) \quad \| P_h u - u \|_{L^2(\Omega)} \leq C_1 h^r \| u \|_{H^r(\Omega)}.
\]
Then using (3.7) and (3.9), we have
\[
r^n_h(x) = (I - P_h)^\tau u^n = (I - P_h) \tau^{-1} \int_{t_n}^{t_{n+1}} u_t ds = \tau^{-1} \int_{t_n}^{t_{n+1}} (I - P_h) u_t ds,
\]
it yields
\[
\| r^n_h \|_{L^2(\Omega)} \leq \tau^{-1} \int_{t_n}^{t_{n+1}} \| (I - P_h) u_t \|_{L^2(\Omega)} ds \leq C_1 h^r \| u_t \|_{H^r(\Omega)}.
\]
Hence, from (3.8), there exists
\[
\| \Delta u_t \|_{L^2(\Omega)} \leq \| \Delta u \|_{L^2(\Omega)} + h |
\]
and
\[
\| \Delta u \|_{L^2(\Omega)} \leq C_2 \| \Delta u \|_{L^2(\Omega)} \leq C (\tau^2 + h^r).
\]
The proof is completed.

**Theorem 3.8.** Let \( u \) be the exact solution of (1.6) and \( u^n_h \) be the solution of the full discretization scheme (3.6). Then we have the following error estimates
\[
\| u(x,t_n) - u^n_h \|_{L^2(\Omega)} \leq C (\tau^2 + h^r).
\]

**Proof.** Let \( \varepsilon^n = u^n_h - P_h u(x,t_n) \) with \( \varepsilon^0 = 0 \). Using Lemmas 3.4 and 3.6, we get the following error equation
\[
(\varepsilon^n - \varepsilon^{n-1}, \chi) + \tau b(\varepsilon^n, \chi) = (R^n_{\tau,h}, \chi)
\]
with \( \varepsilon^n = \frac{1}{\tau}(\varepsilon^n + \varepsilon^{n-1}) \) and \( \| R^n_{\tau,h} \|_{L^2(\Omega)} \leq C_1 \| \tau r^n_{\tau,h} \|_{L^2(\Omega)} \leq C_7 (\tau^2 + h^r) \). Taking \( \chi = \varepsilon^n \) in the above equation, we get
\[
\| \varepsilon^n \|_{L^2(\Omega)}^2 - \| \varepsilon^{n-1} \|_{L^2(\Omega)}^2 \leq (R^n_{\tau,h}, \varepsilon^n + \varepsilon^{n-1}).
\]
Replacing \( n \) with \( s \) and summing up for \( s \) from 1 to \( n \), there exists
\[
\| \varepsilon^n \|_{L^2(\Omega)}^2 \leq \sum_{s=1}^{n} (R^n_{\tau,h}, \varepsilon^s + \varepsilon^{s-1}).
\]
It can be treated in the same way as (3.3), we have
\[
\| u^n_h - P_h u(x,t_n) \|_{L^2(\Omega)} = \| \varepsilon^n \|_{L^2(\Omega)} \leq \max_{0 \leq s \leq N} \| \varepsilon^s \|_{L^2(\Omega)} \leq C_2 (\tau^2 + h^r).
\]
According to the above equation and Lemma 3.6, which leads to
\[
\| u(x,t_n) - u^n_h \|_{L^2(\Omega)} \leq \| u(x,t_n) - P_h u(x,t_n) \|_{L^2(\Omega)} + \| P_h u(x,t_n) - u^n_h \|_{L^2(\Omega)} \leq C_1 h^r \| u \|_{H^r(\Omega)} + C_2 (\tau^2 + h^r) \leq C (\tau^2 + h^r).
\]
The proof is completed.
4. Multigrid method for time-dependent tempered fractional problem.
The time-dependent fractional MGM can be treated as the elliptic equations arising at a fixed time step [12]. However, the time-dependent PDEs should become easier to solve as the time step $\tau \to 0$, which correspond to the mass matrix dominant [6]. Therefore, we need to look for an estimate of other form in the fractional $\tau$-norm, which is independent of the time step $\tau$.

4.1. Multigrid method. Let $V_k$ denote $C^0$ piecewise linear functions with the uniform mesh size $h_k = \frac{1}{2} h_{k-1}$, i.e., $V_{k-1} \subset V_k$, $k \geq 1$.

**Definition 4.1.** The mesh-dependent inner product $(\cdot, \cdot)_k$ on $V_k$ is defined by

$$(v, w)_k = h_k \sum_{i=1}^{n_k} v(x_i)w(x_i),$$

where $\{x_i\}_{i=1}^{n_k}$ is the set of internal nodes of mesh grid with $n_k = \dim V_k$.

**Definition 4.2.** The coarse-to-fine operator $I_{k-1}^k : V_{k-1} \to V_k$ is taken to the natural injection, i.e.,

$$I_{k-1}^k v = v \ \forall v \in V_{k-1}.$$

The fine-to-coarse operator $I_k^{k-1} : V_k \to V_{k-1}$ is defined to be the transpose of $I_{k-1}^k$ with respect to the $(\cdot, \cdot)_{k-1}$ and $(\cdot, \cdot)_k$ inner products. In other words,

$$(I_k^{k-1} w, v)_{k-1} = (w, I_k^{k-1} v)_k = (w, v)_k \ \forall v \in V_{k-1}, w \in V_k.$$

We rewrite (1.0) as the following variational form

$$(u_t, v) + b(u, v) = (f, v) \ \forall v \in H_0^{\alpha/2}(\Omega), \ t \geq 0.$$

We use the Crank-Nicolson scheme for the above equation, there exists

$$\tau^{-1}(u^n, v) + \frac{1}{2} b(u^n, v) = (g^{n-1}, v) \ \forall v \in H_0^{\alpha/2}(\Omega)$$

with

$$(g^{n-1}, v) = \tau^{-1}(u^{n-1}, v) - \frac{1}{2} b(u^{n-1}, v) + (f^{n-1/2}, v).$$

For the simplicity of illustration, we rewrite (4.1) as

$$a_\tau(z, v) =: \tau^{-1}(z, v) + \frac{1}{2} b(z, v) = (g, v) \ \forall v \in H_0^{\alpha/2}(\Omega)$$

with $z = u^n$.

Then the discretized problems is: Find $w_k \in V_k$ such that

$$a_\tau(w_k, v) = (g, v) \ \forall v \in V_k.$$

The operator $A_{k, \tau} : V_k \to V_k$ is defined by

$$(A_{k, \tau} w, v)_k = a_\tau(w, v) \ \forall v, w \in V_k.$$

In terms of the operator $A_{k, \tau}$, the resulting systems (4.3) can be written as

$$A_{k, \tau} u_k = g_k.$$
Since $A_{k,\tau}$ is symmetric positive definite with respect to $(\cdot, \cdot)_k$, we can define a scale of mesh-dependent norms $|||\cdot|||_{s,k,\tau}$ in the following way

$$|||v|||_{s,k,\tau} := \sqrt{(A_{s,k,\tau}^* v, v)_k}.$$  

**Definition 4.3.** Let $P_k : H^{\alpha/2}_0(\Omega) \to V_k$ be the orthogonal projection with respect to $a_\tau(\cdot, \cdot)$. In other words, $P_k v \in V_k$ and

$$a_\tau(v, w) = a_\tau(P_k v, w) \quad \forall w \in V_k.$$  

Let $K_k$ be the iteration matrix of the smoothing operator. In this work, we take $K_k$ to be the weighted (damped) Jacobi iteration matrix

$$K_k = I - S_k A_{k,\tau}, \quad S_k := S_{k,\eta} = \eta D_{k,\tau}^{-1}$$  

with a weighting factor $\eta \in (0, 1/2]$, and $D_{k,\tau}$ is the diagonal of $A_{k,\tau}$. A multigrid process can be regarded as defining a sequence of operators $B_k : B_k \mapsto B_k$ which is an approximate inverse of $A_{k,\tau}$ in the sense that $|||I - B_k A_{k,\tau}|||$ is bounded away from one. The V-cycle multigrid algorithm [9] is provided in Algorithm 1.

**Algorithm 1** V-cycle Multigrid Algorithm: Define $B_1 = A_{1,\tau}^{-1}$. Assume that $B_{k-1} : B_{k-1} \mapsto B_{k-1}$ is defined. We shall now define $B_k : B_k \mapsto B_k$ as an approximate iterative solver for the equation $A_{k,\tau} z = g$.

1: Pre-smooth: Let $S_{k,\eta}$ be defined by (4.8), $z_0 = 0$, $l = 1 : m_1$, and

$$z_l = z_{l-1} + S_{k,\eta_{pre}} (g_k - A_{k,\tau} z_{l-1}).$$

2: Coarse grid correction: Denote $e^{k-1} \in B_{k-1}$ as the approximate solution of the residual equation $A_{k-1} e^{k-1} = I_{k-1}^k (g - A_{k,\tau} z_{m_1})$ with the iterator $B_{k-1}$

$$e^{k-1} = B_{k-1} I_{k-1}^k (g - A_{k,\tau} z_{m_1}).$$

3: Post-smooth: $z_{m_1+1} = z_{m_1} + I_{k-1}^k e^{k-1}, \quad l = m_1 + 2 : m_1 + m_2 + 1$, and

$$z_l = z_{l-1} + S_{k,\eta_{post}} (g - A_{k,\tau} z_{l-1}).$$

4: Define $\text{MG}(k, z_0, g) := B_k g = z_{m_1+m_2+1}.$

**4.2. Uniform convergence of the V-cycle MGM.** Based on the (4.2), we define the fractional $\tau$-norm

$$||v||_{\tau,\alpha}^2 = \tau^{-1} ||v||_{L^2(\Omega)}^2 + ||v||_{H^{\alpha}(\Omega)}^2 \quad \forall v \in H^{\alpha}(\Omega).$$

**Remark 4.1.** The fractional $\tau$-norm reduces to $\tau$-norm [6, 28] when $\alpha = 1$. In order to estimate the spectral radius, $\rho(A_{k,\tau})$, of $A_{k,\tau}$, we first introduce the following lemmas.
LEMMA 4.4. The bilinear form $a_\tau(u,v)$ is symmetrical, continuous and coercive. In other words, there exist the positive constants $C_2,C_3$ such that

$$a_\tau(u,u) \geq C_2\|u\|_{\tau,\alpha/2}^2 \quad \text{and} \quad |a_\tau(u,v)| \leq C_3\|u\|_{\tau,\alpha/2}\|v\|_{\tau,\alpha/2}.$$

Proof. According to (1.2) and Lemma 2.13 there exists

$$a_\tau(u,u) = \tau^{-1}(u,u) + \frac{1}{2} b(u,u) \geq \tau^{-1}(u,u) + \frac{C_0}{2}\|v\|_{H^{\alpha/2}(\Omega)} \geq C_2\|u\|_{\tau,\alpha/2}^2$$

with $C_2 = \min\{1,C_0/2\}$. On the other hand, using Lemma 2.14 we have

$$|a_\tau(u,v)| \leq \tau^{-1}|(u,v)| + \frac{1}{2}|b(u,v)|$$

$$\leq \left(1 + \frac{1}{2}C_1\right) \left(\tau^{-1}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|u\|_{H^{\alpha/2}(\Omega)}\|v\|_{H^{\alpha/2}(\Omega)}\right)$$

$$\leq \left(1 + \frac{1}{2}C_1\right) \left(\tau^{-2}\|u\|_{L^2(\Omega)}^2\|v\|_{L^2(\Omega)}^2 + \|u\|_{H^{\alpha/2}(\Omega)}^2\|v\|_{H^{\alpha/2}(\Omega)}^2\right)^{1/2}$$

$$= \left(1 + \frac{1}{2}C_1\right) \|u\|_{\tau,\alpha/2}\|v\|_{\tau,\alpha/2}.$$

The proof is completed.  \[\square\]

LEMMA 4.5. [Interpolation theorem] Let $\Omega$ be an open set of $\mathbb{R}$ with a Lipshitz continuous boundary. Let $s_1 < s_2$ be two real numbers, and $\mu = (1 - \theta)s_1 + \theta s_2$ with $0 \leq \theta \leq 1$. Then there exists a constant $C > 0$ such that

$$\|v\|_\mu \leq C\|v\|_{s_1}^{1-\theta}\|v\|_{s_2}^\theta \quad \forall v \in H^{s_2}(\Omega).$$

LEMMA 4.6 (Spectral radius theorem). Let $A_{k,\tau}$ be defined by (4.4). Then there exists a constant $C > 0$ such that

$$\rho(A_{k,\tau}) \leq C(1 + \tau^{-1}h^\alpha)h^{-\alpha}.$$  

Proof. Form Lemmas 2.14, 1.13 and inverse estimation of (4.4), there exists

$$b(v,v) \leq C_1\|v\|_{H^{\alpha/2}(\Omega)}^2 \leq C_1 \left(C_2\|v\|_{L^2(\Omega)}^{1-\alpha/2}\|v\|_{H^1(\Omega)}^{\alpha/2}\right)^2$$

$$\leq C_1 \left(C_2\|v\|_{L^2(\Omega)}^{1-\alpha/2}\cdot h^{-\alpha/2}\|v\|_{L^2(\Omega)}^{\alpha/2}\right)^2 \leq C_3 h^{-\alpha}\|v\|_{L^2(\Omega)}^2.$$

Let $\Lambda$ be an eigenvalues of $A_{k,\tau}$ with eigenvector $v$. From the above equation and Lemmas 4.4, 2.13 we have

$$\Lambda(A_{k,\tau}) = \frac{(A_{k,\tau}v,v)}{(v,v)} = \frac{a_\tau(v,v)}{(v,v)} \leq \frac{C_4\|v\|_{\tau,\alpha/2}^2}{\|v\|_{L^2(\Omega)}^2}$$

$$\leq \frac{C_5 \left(\tau^{-1}\|v\|_{L^2(\Omega)}^2 + b(v,v)\right)}{\|v\|_{L^2(\Omega)}^2} \leq C(1 + \tau^{-1}h^\alpha)h^{-\alpha}. $$
The proof is completed. □

**Lemma 4.7.** Let \( A_{k,\tau} = \{a_{i,j}\}^{n_k}_{i,j=1} \) be defined by (4.7) with \( \lambda = 0 \). Then

\[
\eta \rho(A_{k,\tau}) (v_k, v_k) \leq (S_k v_k, v_k) \leq (A_{k,\tau}^{-1} v_k, v_k) \quad \forall v_k \in V_k,
\]

where \( S_k = \eta D_k^{-1}, \eta \in (0, 1/2] \) and \( D_k,\tau \) is the diagonal of \( A_{k,\tau} \).

**Proof.** It is easy to check that \( A_{k,\tau} \) is a weakly diagonally dominant symmetric Toeplitz M-matrix [20, 56], i.e., \( A_{k,\tau} \) is a positive definite matrix with positive entries on the diagonal and nonpositive off-diagonal entries and the diagonal element of a matrix is at least as large as the sum of the off-diagonal elements in the same row or column [11]. Then the similar arguments can be performed as Lemma 2.4 of [23], the desired result is obtained. □

**Remark 4.2.** For \( \lambda = 0 \), the Riesz tempered fractional derivative (4.2) reduces to the fractional Laplace operator. We conclude that the stiffness matrix \( A_{k,\tau} \) of the linear finite element approximation on a uniform grid, after proper scaling \( h \), is the same as the stiffness matrix of the finite difference scheme, see [20, 56]. For \( \lambda > 0 \), the analytical solution of the stiffness matrix are special function 46, from our numerical experiences, we find that it is also a weakly diagonally dominant symmetric Toeplitz M-matrix.

According to (4.6), (4.9) and Lemma 4.4 it is easy to get

(4.10)

\[
\begin{align*}
&c||v||_{L^2(\Omega)} \leq |||v|||_{0,k,\tau} \leq C||v||_{L^2(\Omega)}, \\
&c||v||_{\tau,\alpha/2} \leq |||v|||_{1,k,\tau} \leq C||v||_{\tau,\alpha/2}, \\
&c||A_{k,\tau} v||_{L^2(\Omega)} \leq |||v|||_{2,k,\tau} \leq C||A_{k,\tau} v||_{L^2(\Omega)}.
\end{align*}
\]

**Lemma 4.8.** (Generalized Cauchy-Schwarz inequality with \( \tau \)-norm) For any real number \( \theta \), it holds

\[
|a_\tau(v, w)| \leq |||v|||_{1+\theta,k,\tau}|||w|||_{1-\theta,k,\tau} \quad \forall v, w \in V_k.
\]

**Proof.** Let \( \lambda_i \) with \( 1 \leq i \leq n_k \) be the eigenvalues of the operator \( A_{k,\tau} \) and \( \psi_i \) be the corresponding eigenfunction satisfying the orthogonal relation \( (\psi_i, \psi_j)_k = \delta_{i,j} \).
We can write \( v = \sum_{i=1}^{n_k} c_i \psi_i, w = \sum_{j=1}^{n_k} d_j \psi_j \). From (4.6) and (4.10), we obtain

\[
a_\tau(v, w) = (A_{k,\tau} v, w) = \left( \sum_{i=1}^{n_k} \lambda_i c_i \psi_i, \sum_{j=1}^{n_k} d_j \psi_j \right)
\]

\[
= \sum_{i=1}^{n_k} \lambda_i c_i d_i \leq \left( \sum_{i=1}^{n_k} c_i^2 \lambda_i^{1+\theta} \right)^{1/2} \left( \sum_{i=1}^{n_k} d_i^2 \lambda_i^{1-\theta} \right)^{1/2}
\]

\[
= \left( A_{k,\tau}^{1+\theta} v, v \right)^{1/2} \left( A_{k,\tau}^{1-\theta} w, w \right)^{1/2} = |||v|||_{1+\theta,k,\tau} |||w|||_{1-\theta,k,\tau}.
\]

The proof is completed. □

**Lemma 4.9.** Let \( u \in H^\alpha_{0,\Omega} \) and \( u_h \in V_h \) denote the solution of the variational problems \( a_\tau(u, v) = (g, v) \forall v \in H^\alpha_{0,\Omega} \) and \( a_\tau(u_h, v_h) = (g, v_h) \forall v_h \in V_h \), respectively. Then there exists a positive constant \( C \) such that

\[
||u - u_h||_{L^2(\Omega)} \leq C||u - u_h||_{\tau,\alpha/2} \left( \sup_{g \neq 0} \frac{1}{||g||_{L^2(\Omega)}} \inf_{v_h \in V_h} ||w_g - v_h||_{\tau,\alpha/2} \right),
\]
where \( w_g \in H^\alpha_0(\Omega) \) is the unique solution of \( a_\tau(v, w_g) = (g, v) \) \( \forall v \in H^\alpha_0(\Omega) \). In particular, if \( w_g \in H^\alpha(\Omega) \), we have

\[
\|u - u_h\|_{L^2(\Omega)} \leq Ch^\alpha/2 \left( 1 + \tau^{-1}h^\alpha \right)^{1/2} \|u - u_h\|_{\tau,\alpha/2}.
\]

**Proof.** Since

\[
\|u - u_h\|_{L^2(\Omega)} = \sup_{g \neq 0} \frac{|(g, u - u_h)|}{\|g\|_{L^2(\Omega)}} = \sup_{g \neq 0} \frac{|a_\tau(u - u_h, w_g)|}{\|g\|_{L^2(\Omega)}}
\]

\[
= \sup_{g \neq 0} \frac{|a_\tau(u - u_h, w_g - v_h)|}{\|g\|_{L^2(\Omega)}} \leq \sup_{g \neq 0} \frac{C\|w_g - v_h\|_{\tau,\alpha/2}}{\|g\|_{L^2(\Omega)}} \|u - u_h\|_{\tau,\alpha/2}
\]

\[
\leq C\|u - u_h\|_{\tau,\alpha/2} \left( \sup_{g \neq 0} \frac{1}{\|g\|_{L^2(\Omega)}} \inf_{v_h \in V_h} \|w_g - v_h\|_{\tau,\alpha/2} \right).
\]

Using the property of the interpolation operator \( I_h \) and \( (4.10) \), we have

\[
\|w_g - v_h\|_{\tau,\alpha/2} = \tau^{-1}\|w_g - v_h\|_{L^2(\Omega)} + \|w_g - v_h\|_{H^\alpha/2(\Omega)}^2
\]

\[
\leq C_1 h^\alpha \left( 1 + \tau^{-1}h^\alpha \right) \|w_g\|_{H^\alpha(\Omega)}^2,
\]

i.e.,

\[
\|w_g - v_h\|_{\tau,\alpha/2} \leq C_2 h^{\alpha/2} \left( 1 + \tau^{-1}h^\alpha \right)^{1/2} \|w_g\|_{H^\alpha(\Omega)}.
\]

According to the above equations and Lemma 3.5 there exists

\[
\|u - u_h\|_{L^2(\Omega)} \leq CC_2 h^{\alpha/2} \left( 1 + \tau^{-1}h^\alpha \right)^{1/2} \|u - u_h\|_{\tau,\alpha/2}.
\]

The proof is completed. \( \Box \)

**Lemma 4.10.** *(Approximation property with \( \tau \)-norm)* There exists a positive constant \( C \) such that

\[
\|(I - P_{k-1})v\|_{\tau,\alpha/2} \leq Ch^{\alpha/2} \left( 1 + \tau^{-1}h^\alpha \right)^{1/2} \|v\|_{2,k,\tau} \ \forall v \in V_k.
\]

**Proof.** According to \( (4.11) \), \( (4.6) \), \( (4.4) \), \( (4.7) \) and Lemmas 3.8, 4.10

\[
\|(I - P_{k-1})v\|_{\tau,\alpha/2}^2 \leq C_1 \|(I - P_{k-1})v\|_{1,k,\tau}^2 = C_1(A_{k,\tau}(I - P_{k-1})v, (I - P_{k-1})v)_k
\]

\[
= C_1 \alpha_\tau((I - P_{k-1})v, (I - P_{k-1})v) = C_1 \alpha_\tau((I - P_{k-1})v, v)
\]

\[
\leq C_1 \|(I - P_{k-1})v\|_{L^2(\Omega)} \|v\|_{2,k,\tau}
\]

\[
\leq C_1 Ch^{\alpha/2} \left( 1 + \tau^{-1}h^\alpha \right)^{1/2} \|(I - P_{k-1})v\|_{\tau,\alpha/2} \|v\|_{2,k,\tau}.
\]

The proof is completed. \( \Box \)

**Definition 4.11.** The error operator \( E_k : V_k \to V_k \) is defined recursively by

\[
E_1 = 0,
\]

\[
E_k = K_k^{\alpha}[I - (I - E_{k-1})P_{k-1}]K_k^{\alpha},
\]
where $K_k = I - S_k A_{k,\tau}$, $k \geq 1$ in (13).

**Lemma 4.12.** Let $z, g \in V_k$ satisfy $A_{k,\tau}z = g$ with the initial guess $z_0$. Then
\[ E_k(z - z_0) = z - \text{MG}(k, z_0, g) \quad \forall k \geq 1. \]

**Proof.** The similar arguments can be performed as [7, 9, 10], we omit it here. [16]

**Lemma 4.13.** (Smoothing Property for the $V$-cycle)
\[ a_\tau((I - K_k)K_k^{2m}v, v) \leq \frac{1}{2m}a_\tau((I - K_k^{2m})v, v). \]

**Proof.** The similar arguments can be performed as [7, 9, 10], we omit it here. [16]

**Lemma 4.14.** Let $m$ be the number of smoothing steps. Then
\[ a_\tau(E_kv, v) \leq \frac{C^*}{m + C^*}a_\tau(v, v) \quad \forall v \in V_k \]
where $C^*$ is a positive constant independent of $h$ and $\tau$.

**Proof.** Let $\gamma = \frac{C^*}{C + \tau^m}$. We prove (4.11) by the mathematical induction. It obviously holds for $k = 1$ by Definition 4.11. Assume that
\[ a_\tau(E_{k-1}v, v) \leq \gamma a_\tau(v, v). \]
Next we prove that (4.11) holds. From Definition 4.11 and the above equation, it yields
\[ a_\tau(E_kv, v) \leq C_2(1 - \gamma)(|I - P_{k-1})K_k^{m}v|_{\tau,\alpha/2}^2 + \gamma a_\tau(K_k^{m}v, K_k^{m}v). \]
According to Lemmas 4.10, 4.6 and 4.13 we get
\[ |I - P_{k-1})K_k^{m}v|_{\tau,\alpha/2}^2 \leq C(1 + \tau^{-1}h^\alpha)^2 \frac{1}{2m}(a_\tau(v, v) - a_\tau(K_k^{m}v, K_k^{m}v)). \]
Taking $C^* = \frac{C_2(1 + \tau^{-1}h^\alpha)^2}{2}$ and using the above equations, the desired results is obtained. [16]

**Theorem 4.15.** [Uniform convergence of $V$-cycle MGM with fractional $\tau$-norm]
Let $m$ be the number of smoothing steps. Then
\[ ||z - \text{MG}(k, z_0, g)||_{\tau, E} \leq \frac{C^*}{m + C^*}||z - z_0||_{\tau, E} \quad \forall z \in V_k, \]
where the time-dependent energy norm associated with (4.2) is defined by
\[ ||z||_{\tau, E} = \sqrt{a_\tau(z, z)}. \]

**Proof.** Let $\mu_i$ be the eigenvalues of the operator $E_k$ and $\varphi_i$ be the corresponding eigenfunction satisfying the orthogonal relation $a_\tau(\varphi_i, \varphi_j) = \delta_{ij}$. Using Lemma 4.14 we obtain $0 < \mu_1 \leq \mu_1 \cdots \mu_n \leq \gamma$, where $\gamma = \frac{C^*}{m + C^*}$ is given in (4.11). Let $v = \sum_{i=1}^{n_k} c_i \varphi_i$, we have
\[ ||E_kv||_{\tau, E}^2 = a_\tau(E_kv, E_kv) = \sum_{i=1}^{n_k} c_i^2 \mu_i^2 \leq \gamma^2 a_\tau(v, v). \]
From Lemma 4.12 and the above equation, the desired results is obtained. [16]

**Remark 4.3.** The time-dependent fractional MGM can be treated as the elliptic equations arising at a fixed time step [12]. Here, it is independent of the time step, i.e., $\tau \to 0$. [16]
5. Numerical Results. We employ the V-cycle MGM described in Algorithm 4 to solve the resulting system, where for operation counts, the cost of using FFT to would lead to \( O(N \log(N)) \) \cite{[18]}, where \( N \) denotes the number of the grid points. The stopping criterion is taken as \( \frac{\|r^{(i)}\|}{\|r^{(0)}\|} < 10^{-10} \), where \( r^{(i)} \) is the residual vector after \( i \) iterations; and the number of iterations \( (m_1, m_2) = (1, 2) \) and \( (\eta_{pre}, \eta_{post}) = (1/2, 1/2) \). In all tables, the numerical errors are measured by the \( L_2 \) norm, ‘Rate’ denotes the convergence orders. ‘CPU’ denotes the total CPU time in seconds (s) for solving the resulting discretized systems; and ‘Iter’ denotes the average number of iterations required to solve a general linear system \( A_h \tau u_h = g_h \) at each time level.

All numerical experiments are programmed in Matlab, and the computations are carried out on a PC with the configuration: Inter(R) Core (TM) i5-3470 CPU 3.20 GHz and 8 GB RAM and a Windows 7 operating system.

Example 5.1. Let us consider the time-dependent tempered fractional problem \( (1.6) \) with \( \sigma = 3\lambda^\alpha \kappa_\alpha \) and \( a < x < b \), \( 0 < t \leq T \). Take the exact solution of the equation as \( u(x, t) = e^{-t}x^2(1-x/b)^2 \), then the corresponding initial and boundary conditions are, respectively, \( u(x,0) = x^2(1-x/b)^2 \) and \( u(0, t) = u(b, t) = 0 \); and the forcing function
\[
f(x, t) = -e^{-t}x^2(1-x/b)^2(1-5\lambda^\alpha \kappa_\alpha) - e^{-t}\kappa_\alpha \left( e^{-\lambda x} D_x^\alpha \left[ e^{\lambda x} x^2(1-x/b)^2 \right] + e^{\lambda x} D_b^\alpha \left[ e^{-\lambda x} x^2(1-x/b)^2 \right] \right).
\]
Here the left and right fractional derivatives of the given functions are calculated by the following Algorithm 2

**Algorithm 2** Calculating the Left and Right Fractional Derivatives

**Input:**
Original function \( u(x) \in C^2(a, b) \cap C_0^1(a, b) \) and \( \alpha \in (1, 2) \)

**Output:**
Denote the values of numerically calculating \( a D_x^\alpha u(x) \) and \( b D_b^\alpha u(x) \) by \( v_l \) and \( v_r \).

The algorithm JacobiGL of generating the nodes and weights of Gauss-Labatto integral with the weighting function \((1-x)^{1-\alpha} \) or \((1+x)^{1-\alpha} \) can be seen in \([33]\).

1: \( z, w := \text{JacobiGL}(1-\alpha, 0, 100) \)
2: \( v_l := \frac{1}{\Gamma(2-\alpha)} \left( \frac{x-a}{z} \right)^{2-\alpha} \sum_{i=1}^{100} \frac{\partial^2}{\partial x^2} \left( \frac{x-a}{z} z_i + \frac{x+a}{2} \right) w_i \)
3: \( z, w := \text{JacobiGL}(0, 1 - \alpha, 100) \)
4: \( v_r := \frac{1}{\Gamma(2-\alpha)} \left( \frac{x-b}{z} \right)^{2-\alpha} \sum_{i=1}^{100} \frac{\partial^2}{\partial x^2} \left( \frac{b-x}{z} z_i + \frac{b+x}{2} \right) w_i \)

From Table 5.1, we numerically confirm that the numerical scheme has second-order accuracy and the computational cost is almost \( O(N \log N) \) operations.

Example 5.2. Consider the time-dependent tempered fractional problem \( (1.6) \) on a domain \( 0 < x < 1 \), \( 0 < t \leq 1 \). We take the initial condition \( u(x,0) = x(1-x) \) with the homogeneous boundary conditions, and the forcing function is \( f(x, t) = 0 \).

Since the analytic solutions is unknown for Example 5.2, the order of the convergence of the numerical results are computed by the following formula

\[
\text{Convergence Rate} = \frac{\ln \left( \frac{\|u^N_{2h} - u^N_h\|_{L_2} / \|u^N_h - u^N_{h/2}\|_{L_2}}{\ln 2} \right)}{\ln 2}
\]
Table 5.1: MGM to solve the resulting system \(^{(4.5)}\) with \(\lambda = 0.5, a = 0, b = 32, T = 1\) and \(\tau = T/N, h = b/M, N = M\).

| \(N\) | \(\alpha = 1.1\)       | Rate | Iter | CPU(s)     | \(\alpha = 1.8\)       | Rate | Iter | CPU(s)     |
|------|-------------------------|------|------|------------|-------------------------|------|------|------------|
| \(2^7\) | 4.7035e-03               | 21   | 1.63 | 1.5598e-02 | 17                      | 1.41 |
| \(2^8\) | 1.1469e-03               | 20   | 3.70 | 3.8736e-03 | 20.909               | 14  | 3.06 |
| \(2^9\) | 2.8262e-04               | 20   | 8.07 | 9.6415e-04 | 20.063               | 11  | 6.14 |
| \(2^{10}\) | 7.0031e-05               | 20   | 19.17| 2.4353e-04 | 19.852               | 13  | 18.20|

Table 5.2: The \(L_2\) errors and convergence orders for \(^{(4.5)}\) with \(\sigma = 0, \lambda = 0.5\) and \(\tau = 1/N, h = 1/M, N = M\).

| \(N\) | \(\alpha = 1.1\)       | Rate | \(\alpha = 1.5\)       | Rate | \(\alpha = 1.9\)       | Rate |
|------|-------------------------|------|-------------------------|------|-------------------------|------|
| \(2^7\) | 5.4283e-05               | 8.1597e-06 | 3.6689e-06               | 1.5277 |
| \(2^8\) | 2.7895e-05               | 0.9605 | 3.9436e-06               | 1.0490 |
| \(2^9\) | 1.4239e-05               | 0.9702 | 1.9427e-06               | 1.0214 |
| \(2^{10}\) | 7.2344e-06               | 0.9769 | 9.6532e-07               | 1.0090 |

Table 5.2 shows that the scheme \(^{(4.5)}\) preserves the desired first order convergence with nonhomogeneous initial conditions. And it is not possible to reach second-order convergence because of the weak regularity of the solution in the region close to the initial point and the boundaries.

6. Conclusions. In this work we have developed variational formulations for the time-dependent Riesz tempered fractional problem. The stability of the variational formulation, and the Sobolev regularity of the variational solutions were established. There are already some uniform convergence estimates for using the V-cycle MGM to solve the time-dependent PDEs, however, we notice that the proofs are mainly based on the fixed time step \(\tau\). We introduce and define the fractional \(\tau\)-norm, then the convergence rate of the V-cycle MGM is strictly proved with \(\tau \to 0\). The numerical experiments are performed to verify the convergence with only \(\mathcal{O}(N\log N)\) complexity by FFT. We remark that the corresponding theoretical can be extended to the time-fractional Feynmann-Kac equation \(^{(24)}\) including the classical parabolic PDEs. It should be noted that the real challenge is the verification of the weak solution regularity for the fractional operator and this will be the subject of future researches.

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