Gravitational and electromagnetic signatures of accretion into a charged black hole

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Abstract We present the derivation and the solutions to the coupled electromagnetic and gravitational perturbations with sources in a charged black hole background. We work in the so called \textit{ghost gauge} and consider as source of the perturbations the infall of radial currents. In this way, we study a system in which it is provoked a response involving both, gravitational and electromagnetic waves, which allows us to analyze the dependence between them. We solve numerically the wave equations that describe both signals, characterize the waveforms and study the relation between the input parameters of the infalling matter with those of the gravitational and electromagnetic responses.

Keywords Perturbation theory · Black holes · Gravitational-waves
1 Introduction

Gravitational waves emitted by distorted black holes carry information about the corresponding space-time. In some cases this signal may have an electromagnetic counterpart and one can either use electromagnetic or gravitational wave observations to probe strong-field gravity with black holes [1–3]. Regarding the former, one can use X-ray continuum spectra and Fe Kα line emissions to infer black hole spins. Also, such spectra can be used to probe deviations from the black hole solutions of General Relativity since the spectra depends on the position of the inner edge of the accretion disk around the black hole. Gravitational waves on the other hand may be used to follow the dynamics of the collision of two black holes as was done in the recent detection GW150914 and GW151226 by the LIGO-Virgo collaboration [4–6]. Furthermore, the recent gravitational-wave detection by aLIGO can be used to impose observational bounds on the black holes charge regardless of its nature as shown in [7].

There are scenarios in which is expected an electromagnetic counterpart associated with a large emission of gravitational wave, for instance the short duration gamma ray burst related with the double black hole merger GW150914 as described in [8]. Electromagnetic signals associated with gravitational wave events are indeed an interesting case of study. The interaction between electromagnetic radiation and gravitational waves gave rise to multimessenger astrophysics. An extensive study involving perturbations of neutron stars and black holes have been done in [9–11].

In General Relativity, black holes are uniquely described by three parameters, mass \(M\), angular momentum \(J\), and charge \(Q\) [12]. Whereas the first two parameters have been estimated with various observations, it has been widely believed that the charge must be very small or null. Due to the interaction with the interstellar media, a charged black hole should quickly discharge. Nevertheless charged spherical black holes allow us to set up scenarios in which electromagnetic waves are produced in conjunction with gravitational waves such as the collision of binary charged black holes [13–15].

One of the preferred models of charged black holes is the Reissner–Nördstrom spacetime, which is a stationary spherically symmetric solution of the coupled Einstein–Maxwell system. It represents a black hole when \(Q < M\) and it has been shown that is stable under linear perturbations [16,17]. The importance of this space-time relies in the fact that, when it is perturbed, one may expect a significant amount of energy released in form of gravitational and electromagnetic waves.

There are several studies involving perturbations of Reissner–Nördstrom black holes. Some of them have been made considering perturbations of the components of the metric, as in the pioneering work of Zerilli and Moncrief [18–20], or considering perturbations of the scalars of curvature given by projections of the Weyl tensor. The later is know as the Newman–Penrose formalism [21–23] and was used by Teukolsky to derive a master perturbation equation for rotating black holes [24,25] including electromagnetic radiation (see also [26,27] and references therein). Notice that the external perturbation are of first order respect the background spacetime, and the corresponding metric and electromagnetic responses are of the same order.

One of the most remarkable results of perturbation theory applied to the Reissner–Nördstrom solution is that the gravitational and electromagnetic perturbations can not be decoupled [28,29]. By virtue of this coupling the energy of an incident purely
gravitational wave, is reflected in part as electromagnetic waves and conversely. This transformation of gravitational energy into electromagnetic energy is encoded in a coupled set of equations for both perturbations.

In this paper we find the expressions for the perturbations of the Reissner–Nördstrom black hole in the Newman–Penrose formalism including the matter sources that caused the perturbations. We use the expansion of the perturbed fields in terms of spherical harmonics. We focus on a model in which the source of perturbation is a set of charged particles falling into the black hole. Then, we solve numerically the equations of gravitation-electromagnetic perturbations coupled to the equations of the dynamics of the particles. We study the dependence of the coupled electromagnetic and gravitational waveforms on the parameters of the accreting fluid, such as its charge and density profile.

The paper is organized as follows: In Sect. 2 we introduce the concepts of the null tetrad formalism. Next, in Sect. 3 we derive the perturbed Maxwell equations and in the following Sect. 4 we explicitly derive the outgoing coupled gravitational-electromagnetic equations. In Sect. 5 we introduce the tetrad and geometric quantities of relevance in the Reissner–Nördstrom background described in horizon penetrating coordinates, and we write the coupled system of equations for the perturbed Weyl scalars $\Psi_4^{(1)}, \Psi_3^{(1)}$ with sources. In Sect. 6 we describe the dynamics of the matter that cause the perturbations. In Sects. 7 and 8 we introduce the numerical procedure to solve the equations and analyze the properties of the waveforms. Finally, in Sect. 9 we give some concluding remarks.

In this work we adopt geometric units $c = G = 4\pi \epsilon_0 = 1$ where $\epsilon_0$ is the vacuum permittivity and the metric signature $(-, +, +, +)$. In the derivations, however, we will include the signature $(+, -, -, -)$ in order to be able to compare with the results obtained in classical works such as [22].

2 Foundations: Newman–Penrose formalism

The Newman–Penrose formalism is particularly suited for dealing with radiation in asymptotically flat space-times since ingoing and outgoing radiation are well represented by the Weyl scalars. We introduce a Newman–Penrose null tetrad $l^\mu, k^\mu, m^\mu$, choosing it so that $l^\mu$ points outward and $k^\mu$ points inward in the asymptotic region of a hyper-surface of constant time. The directional operators are defined as in Chandrasekhar’s book [22]:

$$D = l^\mu \partial_\mu, \quad \Delta = k^\mu \partial_\mu, \quad \delta = m^\mu \partial_\mu, \quad \text{and} \quad \bar{\delta} = \bar{m}^\mu \partial_\mu,$$

(2.1)
as well as the the spinor coefficients:

$$\kappa_s = \gamma_{311} = m^\mu l_{\mu;v} l^v, \quad \tau_s = \gamma_{312} = m^\mu l_{\mu;v} k^v,$$

$$\sigma_s = \gamma_{313} = m^\mu l_{\mu;v} m^v, \quad \rho_s = \gamma_{314} = m^\mu l_{\mu;v} \bar{m}^v,$$

$$\pi_s = \gamma_{241} = k^\mu \bar{m}_{\mu;v} l^v, \quad \nu_s = \gamma_{242} = k^\mu \bar{m}_{\mu;v} k^v,$$

$$\mu_s = \gamma_{243} = k^\mu \bar{m}_{\mu;v} m^v, \quad \lambda_s = \gamma_{244} = k^\mu \bar{m}_{\mu;v} \bar{m}^v,$$
\begin{align*}
\epsilon_s &= \frac{1}{2} (\gamma_{211} + \gamma_{341}) = \frac{1}{2} \left( k^\mu l_{\mu;v} + m^\mu \overline{m}_{\mu;v} \right) l^v; \\
\gamma_s &= \frac{1}{2} (\gamma_{212} + \gamma_{342}) = \frac{1}{2} \left( k^\mu l_{\mu;v} + m^\mu \overline{m}_{\mu;v} \right) k^v; \\
\beta_s &= \frac{1}{2} (\gamma_{213} + \gamma_{343}) = \frac{1}{2} \left( k^\mu l_{\mu;v} + m^\mu \overline{m}_{\mu;v} \right) m^v; \\
\alpha_s &= \frac{1}{2} (\gamma_{214} + \gamma_{344}) = \frac{1}{2} \left( k^\mu l_{\mu;v} + m^\mu \overline{m}_{\mu;v} \right) \overline{m}^v.
\end{align*}
(2.2)

The Weyl’s scalars \( \Psi_s \)’s, the Ricci projections \( \Phi_s \)’s and the electromagnetic scalars, \( \varphi \), are defined as

\begin{align*}
\Psi_4 &= -C_{\mu \nu \lambda \tau} k^\mu \overline{m}^v k^\lambda \overline{m}^\tau; \\
\Psi_3 &= -C_{\mu \nu \lambda \tau} l^\mu k^v \overline{m}^\lambda \overline{m}^\tau; \\
\Phi_{00} &= \overline{\Phi}_{00} = \frac{1}{2} R_{\mu \nu \tau} l^\mu l^\nu = 4 \pi T_{\mu \nu \tau} l^\mu l^\nu = 4 \pi T_{II}, \\
\Phi_{01} &= \overline{\Phi}_{10} = \frac{1}{2} R_{\mu \nu \tau} m^\mu m^\nu = 4 \pi T_{\mu \nu \tau} m^\mu m^\nu = 4 \pi T_{lm}, \\
\Phi_{02} &= \overline{\Phi}_{20} = \frac{1}{2} R_{\mu \nu \tau} m^\mu m^\nu = 4 \pi T_{\mu \nu \tau} m^\mu m^\nu = 4 \pi T_{nm}, \\
\Phi_{22} &= \overline{\Phi}_{22} = \frac{1}{2} R_{\mu \nu \tau} n^\mu n^\nu = 4 \pi T_{\mu \nu \tau} n^\mu n^\nu = 4 \pi T_{nn}, \\
\varphi_0 &= F_{\mu \nu} l^\mu m^\nu; \\
\varphi_1 &= \frac{1}{2} F_{\mu \nu} (l^\mu n^\nu + \overline{m}^\mu m^\nu); \\
\varphi_2 &= F_{\mu \nu} \overline{m}^\mu k^\nu.
\end{align*}
(2.3)

with \( C_{\mu \nu \lambda \tau} \) the Weyl tensor, \( R_{\mu \nu} \) the Ricci tensor, \( F_{\mu \nu} \) the Faraday tensor and \( T_{\mu \nu} \) is the stress energy tensor of the matter content. These definitions are given in the Newman–Penrose original paper [21]. In general, the null tetrad vectors \( Z_\mu^a \) satisfy the following normalization equations

\begin{align*}
Z_\mu^a Z_\nu^b = \eta_{ab}, \\
g_{\mu \nu} = 2 \eta^{ab} Z_\mu(Z_\nu^b).
\end{align*}
(2.6)

where \( \eta_{ab} \) has the form:

\[ \eta_{ab} = \begin{pmatrix}
0 & \eta & 0 & 0 \\
\eta & 0 & 0 & 0 \\
0 & 0 & 0 & -\eta \\
0 & 0 & -\eta & 0
\end{pmatrix}, \]
(2.7)

with \( \eta \) a constant related to the signature. For the signature \((+, -, -, -)\), \( \eta = 1 \) and for the signature \((-+, +, +, -)\), \( \eta = -1 \). In this work, we derive the equations for the electromagnetic and for the gravitational perturbations considering both signatures, displaying \( \eta \) explicitly in all the derivations, until we specify the components of the metric, from where we set \( \eta = -1 \).

With this set-up, the next step is to project Maxwell equations, the Riemann and Weyl curvature tensors, the Bianchi identities and the Einstein equations on the tetrad
to obtain the so called Newman–Penrose equations. Such procedure is described in detail for instance in [22,30].

3 Perturbed Maxwell equations

Let us consider first, the electromagnetic scalars, Eq. (2.5). These scalars are related with the ingoing and outgoing electromagnetic radiation at infinity and its dynamics is dictated by the Maxwell equations with sources, which are:

\[(D - 2\eta\rho_s)\varphi_1 - (\delta + (\pi_s - 2\alpha_s))\varphi_0 + \eta\kappa_s\varphi_2 = 2\eta\pi J_l,\]
\[(\delta - 2\eta\tau_s)\varphi_1 - (\Delta + \eta(\mu_s - 2\gamma_s))\varphi_0 + \eta\sigma_s\varphi_2 = 2\eta\pi J_m,\]
\[(D - \eta(\rho_s - 2\epsilon_s))\varphi_2 - (\delta + 2\eta\pi_s)\varphi_1 + \eta\lambda_s\varphi_0 = 2\eta\pi J_\mu,\]
\[(\delta - \eta(\tau_s + 2\beta_s))\varphi_2 - (\Delta + 2\eta\mu_s)\varphi_1 + \eta\nu_s\varphi_0 = 2\eta\pi J_\mu,\]

where \(J_l = J_\mu l^\mu, \quad J_m = J_\mu m^\mu\) and \(J_k = J_\mu k^\mu\). \(J_\mu\) is the 4-electric current [21].

To obtain an equation to describe the electromagnetic perturbations, we operate with \((\Delta + \eta(\overline{\mu}_s - \overline{\gamma}_s + \gamma_s + 2\mu_s))\) on Eq. (3.3) and with \((\delta - \eta(\overline{\tau}_s - \alpha_s - \overline{\beta}_s - 2\pi_s))\) on Eq. (3.4). After subtracting one equation from the other we get

\[\begin{align*}
[(\Delta + \eta(\overline{\mu}_s - \overline{\gamma}_s + \gamma_s + 2\mu_s))(D - \eta(\rho_s - 2\epsilon_s)) - (\delta - \eta(\overline{\tau}_s - \alpha_s - \overline{\beta}_s - 2\pi_s))]\varphi_2 & \\
- ((\Delta + \eta(\overline{\mu}_s - \overline{\gamma}_s + \gamma_s + 2\mu_s))(\delta + 2\eta\pi_s) - (\overline{\delta} - \eta(\overline{\tau}_s))
- \alpha_s - \overline{\beta}_s - 2\pi_s)))(\Delta + 2\eta\mu_s)\varphi_1 & \\
+ ((\Delta + \eta(\overline{\mu}_s - \overline{\gamma}_s + \gamma_s + 2\mu_s))\eta\lambda_s - (\delta - \eta(\overline{\tau}_s - \alpha_s - \overline{\beta}_s - 2\pi_s))]\eta\nu_s\varphi_0 & = 2\eta\pi J_2,
\end{align*}\]

where

\[J_2 = (\Delta + \eta(\gamma_s - \overline{\gamma}_s + \mu_s + \overline{\mu}_s))J_\mu - (\delta + \eta(\alpha_s + \overline{\beta}_s + 2\pi_s - \overline{\tau}_s))J_k.\]

Expanding the term acting on \(\varphi_1\) in Eq. (3.5) we have

\[\begin{align*}
[(\Delta + \eta(\overline{\mu}_s - \overline{\gamma}_s + \gamma_s + 2\mu_s))(D - \eta(\rho_s - 2\epsilon_s)) & \\
- (\delta - \eta(\overline{\tau}_s - \alpha_s - \overline{\beta}_s - 2\pi_s))(\delta - \eta(\tau_s - 2\beta_s)))]\varphi_2 & \\
- 2\eta(\delta\mu_s)\varphi_1 + 2\eta\pi_s(\Delta\varphi_1) - 2\eta\mu_s(\delta\varphi_1) + \eta(\gamma_s - \overline{\gamma}_s + \mu_s + \overline{\mu}_s)(\delta + 2\eta\pi_s)\varphi_1 & \\
- \eta(\alpha_s + \overline{\beta}_s + 2\pi_s - \overline{\tau}_s)\Delta + 2\eta\mu_s)\varphi_1 & \\
+ \varphi_0 [((\Delta + \eta(\overline{\mu}_s - \overline{\gamma}_s + \gamma_s + 2\mu_s))\eta\lambda_s - (\delta - \eta(\overline{\tau}_s - \alpha_s - \overline{\beta}_s - 2\pi_s))]\eta\nu_s & \\
+ \eta\lambda_s\Delta\varphi_0 - \eta\nu_s\overline{\delta}\varphi_0 = 2\eta\pi J_2.
\end{align*}\]

As is a common practice [22], we use the commutation properties of the differential operators, for instance, replacing the commutator \([\Delta, \overline{\delta}]\),
\[ [\Delta, \delta] = \eta \nu_s D + \eta (\alpha_s + \beta_s - \bar{\tau}_s) \Delta - \eta (\bar{\mu}_s - \bar{\nu}_s + \gamma_s) \delta - \eta \lambda_s \delta, \quad (3.7) \]

and also use the Ricci background identities \( \Delta \pi_s, \delta \mu_s, \)

\[
\eta \Delta \pi_s = \eta D \nu_s - \mu_s (\pi_s + \bar{\tau}_s) - \lambda_s (\bar{\sigma}_s + \tau_s) - \pi_s (\gamma_s - \bar{\nu}_s) + \nu_s (3 \epsilon_s + \bar{\epsilon}) - \eta \Psi_3 - \Phi_{21},
\]

\[
\eta \delta \mu_s = \eta \delta \lambda_s - \nu_s (\rho_s - \bar{\rho}_s) - \pi_s (\mu_s - \bar{\mu}) - \mu_s (\alpha_s + \beta_s)
\]

with such substitutions, Eq. (3.6) becomes

\[
\frac{[(\Delta + \eta (\gamma_s - \bar{\nu}_s + 2 \mu_s + \bar{\tau}_s)) (D - \eta (\rho_s + 2 \epsilon_s)] - (\bar{\delta} + \eta (\alpha_s + \beta_s + 2 \pi_s - \bar{\tau}_s)) (\delta - \eta (\tau_s + 2 \beta_s))}{\varphi_2 (D \varphi_1 - \eta \lambda_s \delta \varphi_1 + 2 \varphi_1)^2 (D + \eta (3 \epsilon_s + \bar{\epsilon} + \rho_s - \bar{\rho}_s)) \eta \nu_s - 2(\delta + \eta (\bar{\pi}_s + \tau_s - \bar{\sigma}_s + 3 \beta_s)) \eta \lambda_s - 4 \eta \Psi_3]}
\]

\[
+ \varphi_0 [(\Delta + \eta (\bar{\mu}_s - \bar{\nu}_s + \gamma_s + 2 \mu_s)) \eta \lambda_s - (\bar{\delta} - \eta (\bar{\tau}_s - \alpha_s - \beta_s + 2 \pi_s)) \eta \nu_s]
\]

\[
= 2 \eta \pi J_2.
\]

We perform a first order perturbation of the previous equations of the form \( f \Rightarrow f + f^{(1)} \). We restrict our analysis to background spacetimes where the quantities \( \nu_s, \lambda_s, \kappa_s, \sigma_s, \varphi_0, \varphi_2 \) are zero (as in the black hole family of spacetimes) to get

\[
\frac{[(\Delta + \eta (\gamma_s - \bar{\nu}_s + 2 \mu_s + \bar{\tau}_s)) (D - \eta (\rho_s + 2 \epsilon_s)] - (\bar{\delta} + \eta (\alpha_s + \beta_s + 2 \pi_s - \bar{\tau}_s)) (\delta - \eta (\tau_s + 2 \beta_s))}{\varphi_2 (D \varphi_1 - \eta \lambda_s \delta \varphi_1 + 2 \varphi_1)^2 (D + \eta (3 \epsilon_s + \bar{\epsilon} + \rho_s - \bar{\rho}_s)) \eta \nu_s - 2(\delta + \eta (\bar{\pi}_s + \tau_s - \bar{\sigma}_s + 3 \beta_s)) \eta \lambda_s - 4 \eta \Psi_3]}
\]

\[
= 2 \eta \pi J_2^{(1)},
\]

where we have kept only first order and background quantities. This expression can be further simplified by using Maxwell equations Eqs. (3.1): \( D \varphi_1 = 2 \eta \rho_s \varphi_1 + 2 \pi J_1 \), and (3.2): \( \delta \varphi_1 = 2 \eta \delta \varphi_1 + 2 \delta \varphi_2 \), to obtain

\[
\frac{[(\Delta + \eta (\bar{\mu}_s - \bar{\nu}_s + \gamma_s + 2 \mu_s)) (D - \eta (\rho_s - 2 \epsilon_s)] - (\bar{\delta} - \eta (\bar{\tau}_s - \alpha_s - \beta_s - 2 \pi_s)) (\delta - \eta (\tau_s - 2 \beta_s))}{\varphi_2 (D + \eta (3 \epsilon_s + \bar{\epsilon} + 2 \rho_s - \bar{\rho}_s)) \eta \nu_s - 2(\delta + \eta (\bar{\pi}_s - \bar{\sigma}_s + 3 \beta_s + 2 \tau_s)) \eta \lambda_s - 2 \eta \Psi_3}]
\]

\[
= 2 \eta \pi J_2^{(1)}.
\]
with

\[ J_2^{(1)} = (\Delta + \eta(\gamma_s - \tau_s + 2\mu_s + \nu_s))J^{(1)}_{\mu} \]
\[ - (\tilde{\delta} + \eta(\alpha_s + \tilde{\beta}_s + 2\pi_s - \tau_s))J^{(1)}_k. \]  
(3.12)

Equation (3.11) will be used later to eliminate the terms involving \( v_s^{(1)} \) and \( \lambda_s^{(1)} \) in favor of the Weyl and Maxwell scalars. It the next steps of the derivation it will be useful to commute the operators acting on \( \varphi_2^{(1)} \), using

\[ \left[ \Delta, D \right] = \eta(\gamma_s + \tau_s)D + \eta(\epsilon_s + \bar{\epsilon}_s)\Delta - \eta(\tau_s + \pi_s)\delta - \eta(\tau_s + \pi_s)\tilde{\delta}, \]  
(3.13)

\[ \left[ \tilde{\delta}, \delta \right] = \eta(\bar{\nu}_s - \mu_s)D + \eta(\bar{\rho}_s - \rho_s)\Delta + \eta(\bar{\alpha}_s - \bar{\beta}_s)\delta + \eta(\beta_s - \bar{\alpha}_s)\tilde{\delta}, \]  
(3.14)

taking into account that

\[ \eta D\mu_s = \eta^2\pi_s + (\bar{\pi}_s\mu_s + \sigma_s\lambda_s) + \pi_s(\bar{\pi}_s - \bar{\sigma}_s + \beta_s) \]
\[ - \mu_s(\bar{\epsilon}_s + \epsilon_s) - v_s\kappa_s + \eta\Psi_2 + 2R, \]  
(3.15)

\[ \eta\bar{\delta} \beta_s = - \eta\delta\alpha_s - (\mu_s\rho_s - \lambda_s\sigma_s) - \alpha_s\bar{\sigma}_s \]
\[ - \beta_s(\beta_s - 2\alpha_s) - \gamma_s(\rho_s - \bar{\rho}_s) - \epsilon_s(\mu_s - \bar{\mu}_s) + \eta\Psi_2 - \Phi_{11} - R, \]  
(3.16)

where \( R \) is the Ricci scalar. After some algebra we arrive to the following expression relating \( \varphi_2^{(1)}, v_s^{(1)}, \lambda_s^{(1)}, \Psi_3^{(1)} \), and the current \( J_2^{(1)} \) (recalling that we are considering a spacetime where the background quantities \( v_s, \lambda_s, \kappa_s, \sigma_s, \varphi_0, \varphi_2 \) are zero):

\[ \left[ (D - \eta (\rho_s - 3\epsilon_s - \bar{\epsilon}_s))(\Delta + \eta (2\mu_s + \bar{\mu}_s + 2\gamma_s)) \]
\[ - (\delta + \eta (3\beta_s - \bar{\alpha}_s - \bar{\tau}_s))\tilde{\delta} + \eta (2\alpha_s + 2\pi_s - \tau_s)) - 3\eta\Psi_2 \right] \varphi_2^{(1)} \]
\[ = 2\eta\varphi_1 \left[ (D + \eta (2\rho_s - \bar{\rho}_s + 3\epsilon_s + \bar{\epsilon}_s))v_s^{(1)} \right] \]
\[ - (\delta - \eta (\bar{\alpha}_s - 3\beta_s - \bar{\pi}_s - 2\tau_s))\lambda_s^{(1)} - 2\Psi_3^{(1)} \]
\[ + 2\eta \pi J_2^{(1)}. \]  
(3.17)

It is important to notice that there is a typo in [22] in the equivalent to this last equation.

4 Perturbed Bianchi identities

The perturbed Bianchi identitites determine an equation for the function \( \tilde{\chi} = 3\varphi_1 \Psi_3^{(1)} - 2\Psi_2 \varphi_2^{(1)} \). Remarkably it can be shown that \( \tilde{\chi} \) is invariant under tetrad transformations [22]. This is generally interpreted as a gauge freedom. As a result, the information about the perturbations of the electromagnetic field is contained in \( \tilde{\chi} \). As we will show, a suitable choice of gauge, allows one to simplify the equations. One the
commonest choices is to take \( \varphi_2^{(1)} \) equal to zero, which is called the *phantom gauge*. By using this gauge the electromagnetic perturbations are described by \( \Psi_3^{(1)} \).

In order to derive the equations for the perturbations of \( \Psi_4 \) and \( \Psi_3 \), we start with the next identities:

\[
(D + \eta (4\epsilon_s - \rho_s)) \Psi_4 - (\delta + 2\eta (2\pi_s + \alpha_s)) \Psi_3 + (3\eta \Psi_2 + 2\Phi_{11}) \lambda_s = \eta (\delta + 2\eta (\alpha_s - \tau_s)) \Phi_{21} - \eta (\Delta + \eta (\mu_s + 2\gamma_s - 2\varphi_s)) \Phi_{20},
\]

\[
- (\delta + \eta (4\beta_s - \tau_s)) \Psi_4 + (\Delta + 2\eta (\gamma_s + 2\mu_s)) \Psi_3 - (3\eta \Psi_2 - 2\Phi_{11}) \nu_s = \eta (\Delta + 2\eta (\mu_s + \tau_s)) \Phi_{21} - \eta (\delta + \eta (-\tau_s + 2\alpha_s + 2\beta_s)) \Phi_{22},
\]

and with the Ricci identity:

\[
\Psi_4 + (\Delta + \eta (\mu_s + \mu_s + 3\gamma_s - \varphi_s)) \lambda_s - (\delta + \eta (3\alpha_s + \beta_s + \tau_s - \tau_s)) \nu_s = 0.
\]

Performing a first order perturbation and considering spacetimes where the unperturbed quantities \( \Psi_4, \Psi_3, \nu_s, \lambda_s, \sigma_s \) are zero, and the unperturbed sources \( \Phi_{21}, \Phi_{20}, \Phi_{22} \) also vanish, one obtains:

\[
(D + \eta (4\epsilon_s - \rho_s)) \Psi_4^{(1)} - (\delta + 2\eta (2\pi_s + \alpha_s)) \Psi_3^{(1)} + (3\eta \Psi_2 + 2\Phi_{11}) \lambda_s^{(1)} = \eta (\delta + 2\eta (\alpha_s - \tau_s)) \Phi_{21}^{(1)} - \eta (\Delta + \eta (\mu_s + 2\gamma_s - 2\varphi_s)) \Phi_{20}^{(1)},
\]

\[
- (\delta + \eta (4\beta_s - \tau_s)) \Psi_4^{(1)} + (\Delta + 2\eta (\gamma_s + 2\mu_s)) \Psi_3^{(1)} - (3\eta \Psi_2 - 2\Phi_{11}) \nu_s^{(1)} = \eta (\Delta + 2\eta (\mu_s + \tau_s)) \Phi_{21}^{(1)} - \eta (\delta + \eta (-\tau_s + 2\alpha_s + 2\beta_s)) \Phi_{22}^{(1)},
\]

and the perturbation of the Ricci identity (4.3) gives,

\[
\Psi_4^{(1)} = (\Delta + \eta (\mu_s + \mu_s + 3\gamma_s - \varphi_s)) \lambda_s^{(1)} - (\delta + \eta (3\alpha_s + \beta_s + \tau_s - \tau_s)) \nu_s^{(1)} = 0.
\]

Eqs. (4.4)–(4.6) can be rewritten as

\[
O_{1a} \Psi_4^{(1)} + O_{1b} \Psi_3^{(1)} + (3\eta \Psi_2 + 2\Phi_{11}) \lambda_s^{(1)} = T_{1a} \Phi_{21}^{(1)} + T_{1b} \Phi_{20}^{(1)},
\]

\[
O_{2a} \Psi_4^{(1)} + O_{2b} \Psi_3^{(1)} - (3\eta \Psi_2 - 2\Phi_{11}) \nu_s^{(1)} = T_{2a} \Phi_{21}^{(1)} + T_{2b} \Phi_{20}^{(1)},
\]

\[
\Psi_4^{(1)} + O_{4a} \lambda_s^{(1)} - O_{4b} \nu_s^{(1)} = 0,
\]

where we have defined

\[
O_{3a} = (\Delta + \eta (3\gamma_s - \varphi_s + 4\mu_s + \mu_s)), \quad O_{3b} = (\delta + \eta (-\tau_s + \beta_s + 3\alpha_s + 4\pi_s)),
\]

\[
O_{4a} = (\Delta + \eta (\mu_s + \mu_s + 3\gamma_s - \varphi_s)), \quad O_{4b} = (\delta + \eta (3\alpha_s + \beta_s + \tau_s - \tau_s)),
\]

\[
O_{1a} = (D + \eta (4\epsilon_s - \rho_s)), \quad O_{2a} = -(\delta + \eta (4\beta_s - \tau_s)),
\]

\[
O_{1b} = -(\delta + 2\eta (2\pi_s + \alpha_s)), \quad O_{2b} = (\Delta + 2\eta (\gamma_s + 2\mu_s)),
\]
The next commutation relation, obtained from a generalization of the Eq. (3.7), see [21,22], will be useful to rewrite the Maxwell equations and the equation for the perturbations as second order differential equations,

\[
[\Delta - q \mu_s + \mu_s + (p + 1)\gamma_s - \gamma_s](\bar{\delta} + p\alpha_s - q\pi_s) \\
- [\bar{\delta} + \beta_s + (p + 1)\alpha_s - q\pi_s - \bar{\tau}_s](\Delta - q \mu_s + p\gamma_s) = 0,
\]

where \( p \) and \( q \) are real numbers. At this point it is possible to get a coupled equation for \( \Psi_4^{(1)} \), \( \Psi_3^{(1)} \) and \( \varphi_2^{(1)} \) using the anti-commutation relation Eq. (4.12). Applying the operator \( O_{3a} \) on Eq. (4.7) and \( O_{3b} \) on Eq. (4.8) and adding the resulting expressions we get:

\[
[O_{3a}O_{1a} + O_{3b}O_{2a}]\Psi_4^{(1)} + [O_{3a}O_{1b} + O_{3b}O_{2b}]\Psi_3^{(1)} + O_{3a} (3\eta \Psi_2 + 2\Phi_{11}) \lambda_s^{(1)} \\
- O_{3b} (3\eta \Psi_2 - 2\Phi_{11}) \nu_s^{(1)} \\
= [O_{3a}T_{1a} + O_{3b}T_{2b}]\Phi_2^{(1)} + [O_{3a}T_{1b}]\Phi_1^{(1)} + [O_{3b}T_{2a}]\Phi_2^{(1)},
\]

Furthermore, by using the anti-commutation relation \( [O_{3a}O_{1b} + O_{3b}O_{2b}] = 0 \) we eliminate the term multiplying \( \Psi_3^{(1)} \). We can further simplify this equation by means of the following relations: \( O_{3a} = O_{4a} + 3\eta \mu_s \), \( O_{3b} = O_{4b} + 3\eta \pi_s \), and

\[
O_{4a}(A \lambda_s^{(1)}) = A O_{4a}(\lambda_s^{(1)}) + \lambda_s^{(1)}(\Delta A), \\
O_{4b}(A \nu_s^{(1)}) = A O_{4b}(\nu_s^{(1)}) + \nu_s^{(1)}(\bar{\delta} A),
\]

for any function \( A \). As a result, we can rewrite Eq. (4.13) as

\[
[O_{3a}O_{1a} + O_{3b}O_{2a}]\Psi_4^{(1)} + 3\eta \Psi_2 \left[ (O_{4a} + 3\eta \mu_s) \lambda_s^{(1)} - (O_{4b} + 3\eta \pi_s) \nu_s^{(1)} \right] \\
+ \lambda_s^{(1)} \left[ 3\eta (\Delta \Psi_2) + 2(\Delta \Phi_{11}) \right] \\
+ 2\Phi_{11} \left[ (O_{4a} + 3\eta \mu_s) \lambda_s^{(1)} - (O_{4b} + 3\eta \pi_s) \nu_s^{(1)} \right] \\
- \nu_s^{(1)} \left[ 3\eta (\bar{\delta} \Psi_2) - 2(\bar{\delta} \Phi_{11}) \right] \\
= [O_{3a}T_{1a} + O_{3b}T_{2b}]\Phi_2^{(1)} + [O_{3a}T_{1b}]\Phi_1^{(1)} + [O_{3b}T_{2a}]\Phi_2^{(1)}.
\]

After collecting the terms that multiply at \( \lambda_s^{(1)} \) and \( \nu_s^{(1)} \) we get

\[
[O_{3a}O_{1a} + O_{3b}O_{2a}]\Psi_4^{(1)} + 3\eta \Psi_2 \left[ O_{4a} \lambda_s^{(1)} - O_{4b} \nu_s^{(1)} \right] \\
+ \lambda_s^{(1)} \left[ 3\eta (\Delta \Psi_2) + 9\mu_s \Psi_2 + 2(\Delta \Phi_{11}) + 6\eta \mu_s \Phi_{11} \right] \\
+ 2\Phi_{11} \left[ O_{4a} \lambda_s^{(1)} + O_{4b} \nu_s^{(1)} \right] 
\]
Eqs. (4.7), (4.8) to eliminate the operator acting on \( \phi \) response, obtain:

\[
- \nu_s^{(1)} \left[ 3 \eta (\bar{\delta} \Psi_2) + 9 \pi_s \Psi_2 - 2 (\bar{\delta} \Phi_{11}) - 6 \eta \pi_s \Phi_{11} \right] \\
= [O_{3a} T_{1a} + O_{3b} T_{2b}] \Phi_{21}^{(1)} + [O_{3a} T_{1b}] \Phi_{20}^{(1)} + [O_{3b} T_{2a}] \Phi_{22}^{(1)}. \tag{4.16}
\]

Using Eq. (4.9), \( O_{4a} \lambda_s^{(1)} - O_{4b} \nu_s^{(1)} = -\Psi_4^{(1)} \), in the second and fourth terms, we obtain:

\[
[O_{3a} O_{1a} + O_{3b} O_{2a} - 3 \eta \Psi_2 + 2 \Phi_{11}] \Psi_4^{(1)} + \lambda_s^{(1)} \\
[3 \eta (\Delta \Psi_2) + 9 \mu_s \Psi_2 + 2 (\Delta \Phi_{11}) + 6 \eta \mu_s \Phi_{11}] + 4 \Phi_{11} O_{4a} \lambda_s^{(1)} \\
+ \nu_s^{(1)} \left[ 3 \eta (\bar{\delta} \Psi_2) + 9 \pi_s \Psi_2 - 2 (\bar{\delta} \Phi_{11}) - 6 \eta \pi_s \Phi_{11} \right] \\
= [O_{3a} T_{1a} + O_{3b} T_{2b}] \Phi_{21}^{(1)} + [O_{3a} T_{1b}] \Phi_{20}^{(1)} + [O_{3b} T_{2a}] \Phi_{22}^{(1)}. \tag{4.17}
\]

Using the unperturbed Einstein and Maxwell equations:

\[
\Delta \Psi_2 = -3 \eta \mu_s \Psi_2 - 2 \mu_s \Phi_{11}, \quad \Delta \Phi_{11} = -4 \mu_s \eta \Phi_{11}, \tag{4.18}
\]
\[
\bar{\delta} \Psi_2 = -3 \eta \pi_s \Psi_2 + 2 \pi_s \Phi_{11}, \quad \bar{\delta} \Phi_{11} = -4 \eta \pi_s \Phi_{11}, \tag{4.19}
\]

Eq. (4.17) becomes:

\[
[O_{3a} O_{1a} + O_{3b} O_{2a} - 3 \eta \Psi_2 + 2 \Phi_{11}] \Psi_4^{(1)} \\
+ 4 \Phi_{11} (O_{4a} - 2 \eta \mu_s) \lambda_s^{(1)} - 8 \eta \pi_s \Phi_{11} \nu_s^{(1)} \\
= [O_{3a} T_{1a} + O_{3b} T_{2b}] \Phi_{21}^{(1)} + [O_{3a} T_{1b}] \Phi_{20}^{(1)} + [O_{3b} T_{2a}] \Phi_{22}^{(1)}. \tag{4.20}
\]

It is straightforward to see that in vacuum, the usual perturbation equation for \( \Psi_4^{(1)} \) is recovered [31].

In the present case, we will have three unknowns, \( \Psi_4^{(1)} \), the electromagnetic response, \( \varphi_2^{(1)} \), encoded in \( \Phi_{21}^{(1)} \) (see below), and the Weyl component \( \Psi_3^{(1)} \). The perturbed spinor coefficients \( \nu^{(1)} \), and \( \lambda^{(1)} \), can be expressed in terms of these three quantities by means of Eqs. (4.1) and (4.2).

Thus, it would seem that we need two more equations to close the system. However, there are no enough equations to solve for the three unknowns. This shortage comes from the fact that in the Newman–Penrose formulation of the Ricci equations is not possible to isolate \( \Psi_3 \) in order to get an equation similar to Eq. (4.3) for \( \Psi_4 \). Here, we will choose the phantom gauge and obtain an equation for \( \Psi_3^{(1)} \).

One can derive an equation for the combination of the two unknowns, \( \varphi_2^{(1)} \) and \( \Psi_3^{(1)} \), the gauge invariant \( \tilde{\chi} \) mentioned above.

This can be done by performing a similar procedure as before, but now acting on Eqs. (4.7), (4.8) to eliminate the operator acting on \( \Psi_4^{(1)} \). The operators needed to achieve that goal are:

\[
O_{5a} = (D + \eta (3 \varepsilon_s + \bar{\varepsilon}_s - \rho_s - \bar{\rho}_s)), \quad O_{5b} = (\delta + \eta (3 \beta_s - \bar{\alpha}_s - \tau_s + \bar{\pi}_s)). \tag{4.21}
\]

We also need to consider the action of the operators \( D \) and \( \delta \) on \( \Psi_2 \) and on \( \Phi_{11} \):
\[ D \Psi_2 = 3 \eta \rho_s \Psi_2 + 2 \rho_s \Phi_{11}, \quad D \Phi_{11} = 4 \eta \rho_s \Phi_{11}, \]
\[ \delta \Psi_2 = 3 \eta \tau_s \Psi_2 - 2 \tau_s \Phi_{11}, \quad \delta \Phi_{11} = 2 \eta \tau_s \Phi_{11}. \]  
(4.22)

Applying \( O_{5b} \) on Eq. (4.7), and \( O_{5a} \) on Eq. (4.8), adding the resulting equations and using the commuting operator (4.12), we obtain:

\[ [O_{5b} O_{1b} + O_{5a} O_{2b}] \Psi_3^{(1)} + (3 \eta \Psi_2 + 2 \Phi_{11}) O_{5b} \lambda_s^{(1)} \]
\[- O_{5a} (3 \eta \Psi_2 - 2 \Phi_{11}) \nu_s^{(1)} - [O_{5b} T_{1a} + O_{5a} T_{2b}] \Phi_{21}^{(1)} \]
\[ = [O_{5b} T_{1b}] \Phi_{20}^{(1)} + [O_{5a} T_{2a}] \Phi_{22}^{(1)}, \]  
(4.23)

In order to take into account the contribution of the different components of the matter content we will consider that the perturbation for the matter fields can be expressed as

\[ \Phi_{21}^{(1)} = \Phi_{21}^{(1) \, \text{elec}} + T_{21}^{(1)}, \quad \Phi_{20}^{(1)} = \Phi_{20}^{(1) \, \text{elec}} + T_{20}^{(1)}, \quad \Phi_{22}^{(1)} = \Phi_{22}^{(1) \, \text{elec}} + T_{22}^{(1)}, \]  
(4.24)

where \( T_{ab}^{(1)} \) is the term for external matter that perturbs the background. Thus, Eq. (4.23) gives:

\[ [O_{5b} O_{1b} + O_{5a} O_{2b}] \Psi_3^{(1)} + 3 \eta \Psi_2 [O_{5b} \lambda_s^{(1)} - (O_{5a} + \eta \rho_s) \nu_s^{(1)}] \]
\[ + 2 \Phi_{11} [O_{5b} \lambda_s^{(1)} + (O_{5a} + \eta \rho_s) \nu_s^{(1)}] \]
\[- 6 \rho_s \Psi_2 \nu_s^{(1)} - [O_{5b} T_{1a} + O_{5a} T_{2b}] \Phi_{21}^{(1) \, \text{elec}} \]
\[ = [O_{5b} T_{1b}] T_{20}^{(1)} + [O_{5a} T_{2a}] T_{22}^{(1)}. \]  
(4.25)

In order to get rid of some terms involving the perturbed spinor coefficients, we will now make use of the Maxwell relation, Eq. (3.17). Additionally, we will make two simplifying assumptions. The first one is that the spacetime is Reissner–Nördstrom, and the second, following Chandrasekhar [22], we will use the phantom gauge where \( \varphi_2^{(1)} = 0 \). Using this gauge we will be able to find an equation for \( \Psi_3^{(1)} \).

Since in the Reissner–Nördstrom background \( \varphi_2 = 0 \) and \( \varphi_0 = 0 \), we have that the three perturbed components of the Ricci tensor due to the perturbed electromagnetic field, are zero:

\[ \Phi_{21}^{(1) \, \text{elec}} = \varphi_2^{(1)} \varphi_1 + \varphi_2 \varphi_1^{(1)} = \varphi_2^{(1)} \varphi_1, \quad \text{and} \quad \Phi_{20}^{(1) \, \text{elec}} = \varphi_2^{(1)} \varphi_0 + \varphi_2 \varphi_0^{(1)} = 0, \]  
(4.26)

and also \( \Phi_{22}^{(1) \, \text{elec}} = 0. \)

In the next section we will show that, in the Reissner–Nördstrom spacetime, the spin coefficients are real and \( \pi_s = 0 = \tau_s = \gamma_s \) and \( \alpha_s = - \beta_s \). Furthermore, the following relations are valid \( O_{4a} - 2 \eta \mu_s = \Delta \), and, \( O_{3a} T_{1a} + O_{3b} T_{2b} = 2 \eta (\Delta + 4 \eta \mu_s) (\delta + 2 \eta \alpha_s) \). Finally, in the forthcoming analysis we will use the signature \((-++,+,+,-)\) so that \( \eta = -1. \)

Using the phantom gauge, the Maxwell relation Eq. (3.17) takes the form

\[ O_{5b} \lambda_s^{(1)} - (O_{5a} - 3 \rho) \nu_s^{(1)} = -2 \Psi_3^{(1)} + \frac{\pi}{\varphi_1} J_2^{(1)} \]  
(4.27)
Using (4.3) within the present assumptions, we obtain a coupled equation for $\Psi_4^{(1)}$ and $\lambda_s^{(1)}$ so that Eq. (4.20) takes the form:

$$
[O_{3a} O_{1a} + O_{3b} O_{2a} + 3 \Psi_2 + 2 \Phi_{11}] \Psi_4^{(1)} + 4 \Phi_{11} \left( O_{4a} + 2 \mu_s \right) \lambda_s^{(1)}
$$

$$
= [O_{3a} T_{1a} + O_{3b} T_{2a}] T_{21}^{(1)} + [O_{3a} T_{1b}] T_{20}^{(1)} + [O_{3b} T_{2b}] T_{22}^{(1)},
$$

(4.28)

that is, writing explicitly some of the coefficients we get

$$
[O_{3a} O_{1a} + O_{3b} O_{2a} + 3 \Psi_2 + 2 \Phi_{11}] \Psi_4^{(1)} + 4 \Phi_{11} \Delta \lambda^{(1)}
$$

$$
= -2 (\Delta - 4 \mu_s) \left( \delta + 2 \beta_s \right) T_{21}^{(1)}
$$

$$
+ [O_{3a} T_{1b}] T_{20}^{(1)} + [O_{3b} T_{2a}] T_{22}^{(1)}. \tag{4.29}
$$

Substituting Eq. (4.27) in Eq. (4.25), we obtain a second coupled equation for $\Psi_3^{(1)}$ and the perturbed spinor coefficients $\nu_s^{(1)}$, and $\lambda_s^{(1)}$:

$$
[O_{5b} O_{1b} + O_{5a} O_{2b} + 2 (3 \Psi_2 + 2 \Phi_{11})] \Psi_3^{(1)} + 4 \Phi_{11} O_{5b} \lambda_s^{(1)} + 4 \rho_s \Phi_{11} \nu_s^{(1)}
$$

$$
= [O_{5b} T_{1a} + O_{5a} T_{2b}] T_{21}^{(1)} + [O_{5b} T_{1b}] T_{20}^{(1)} + [O_{5a} T_{2a}] T_{22}^{(1)}
$$

$$
+ \frac{\pi}{\varphi_1} (3 \Psi_2 + 2 \Phi_{11}) J_2^{(1)}. \tag{4.30}
$$

The task now is to get rid of the perturbed spinor coefficients. This is done from Eq. (4.7), solving for $\lambda_s^{(1)}$ and from Eq. (4.8) solving for $\nu_s^{(1)}$

$$
\lambda_s^{(1)} = \frac{1}{2 \Phi_{11} - 3 \Psi_2} \left[ -O_{1a} \Psi_4^{(1)} - O_{1b} \Psi_3^{(1)} + T_{1a} T_{21}^{(1)} + T_{1b} T_{20}^{(1)} \right]. \tag{4.31}
$$

$$
\nu_s^{(1)} = -\frac{1}{(3 \Psi_2 + 2 \Phi_{11})} \left[ O_{2a} \Psi_4^{(1)} + O_{2b} \Psi_3^{(1)} - T_{2a} T_{22}^{(1)} - T_{2b} T_{21}^{(1)} \right]. \tag{4.32}
$$

Thus, we can express the term $4 \Phi_{11} \Delta \lambda_s^{(1)}$ in (4.30) as:

$$
4 \Phi_{11} \Delta \lambda_s^{(1)} = \frac{4 \Phi_{11}}{2 \Phi_{11} - 3 \Psi_2} \left( \Delta - \frac{14 \Phi_{11} - 9 \Psi_2}{2 \Phi_{11} - 3 \Psi_2} \mu_s \right) \left[ -O_{1a} \Psi_4^{(1)} - O_{1b} \Psi_3^{(1)} + T_{1a} T_{21}^{(1)} + T_{1b} T_{20}^{(1)} \right], \tag{4.33}
$$

where we have used Eq. (4.18).

After substituting expression (4.33) on Eq. (4.29) we get:

$$
\left\{ \left[ O_{3a} - \frac{4 \Phi_{11}}{2 \Phi_{11} - 3 \Psi_2} \left( \Delta - \frac{14 \Phi_{11} - 9 \Psi_2}{2 \Phi_{11} - 3 \Psi_2} \mu_s \right) \right] O_{1a}
$$

$$
+ O_{3b} O_{2a} + 3 \Psi_2 + 2 \Phi_{11} \right\} \Psi_4^{(1)} -
$$
\[
\frac{4 \Phi_{11}}{2\Phi_{11} - 3\Psi_2} \left( \Delta - \frac{14 \Phi_{11} - 9\Psi_2}{2\Phi_{11} - 3\Psi_2} \mu_s \right) O_{1b} \Psi_3^{(1)} \\
= - \left[ \frac{4 \Phi_{11}}{2\Phi_{11} - 3\Psi_2} \left( \Delta - \frac{14 \Phi_{11} - 9\Psi_2}{2\Phi_{11} - 3\Psi_2} \mu_s \right) T_{1a} \right. \\
+ 2 \left( \Delta - 4 \eta \mu_s \right) \left( \tilde{\delta} + 2 \beta_s \right) \left( T_{21}^{(1)} \right) \\
+ \left[ O_{3a} - \frac{4 \Phi_{11}}{2\Phi_{11} - 3\Psi_2} \left( \Delta - \frac{14 \Phi_{11} - 9\Psi_2}{2\Phi_{11} - 3\Psi_2} \right) \mu_s \right] T_{1b} \left. T_{20}^{(1)} \right] \\
+ \left[ O_{3b} T_{2a} \right] T_{22}^{(1)}. \quad (4.34)
\]

Writing explicitly the radial-temporal operators and after some algebraic steps we obtain

\[
\left\{ \Delta + \chi \mu_s \left( 5 - \frac{4 \Phi_{11} \left( 14 \Phi_{11} - 9\Psi_2 \right)}{(2\Phi_{11} - 3\Psi_2)^2} \right) \right\} \\
(D + \rho_s - 4\epsilon_s) - \chi O_{3b} O_{2a} - 2 \Phi_{11} + 3\Psi_2 \quad \Psi_4^{(1)} \\
- \frac{4 \Phi_{11}}{2\Phi_{11} + 3\Psi_2} \left( \Delta - \frac{14 \Phi_{11} - 9\Psi_2}{2\Phi_{11} - 3\Psi_2} \mu_s \right) O_{1b} \Psi_3^{(1)} \\
= - \left[ \frac{6 \Psi_2}{2\Phi_{11} + 3\Psi_2} \Delta + 4 \chi \left( 2 - \frac{\Phi_{11} \mu_s \left( 14 \Phi_{11} - 9\Psi_2 \right)}{(2\Phi_{11} - 3\Psi_2)^2} \right) \right] (\tilde{\delta} + 2 \beta_s) T_{21}^{(1)} \\
+ \left[ \Delta + \chi \mu_s \left( 5 - \frac{4 \Phi_{11} \left( 14 \Phi_{11} - 9\Psi_2 \right)}{(2\Phi_{11} - 3\Psi_2)^2} \right) \right] (\Delta - \mu_s) T_{20}^{(1)} \\
- \chi \left[ O_{3b} T_{2a} \right] T_{22}^{(1)}, \quad (4.35)
\]

which is the first coupled equation for \( \Psi_4^{(1)} \) and \( \Psi_3^{(1)} \), with sources \( T_{21}^{(1)} \), \( T_{20}^{(1)} \) and \( T_{22}^{(1)} \) and we have defined:

\[
\chi := \frac{2 \Phi_{11} - 3\Psi_2}{2\Phi_{11} + 3\Psi_2}. \quad (4.36)
\]

Regarding the second equation needed to close the system we substitute \( \lambda_s^{(1)} \) given by Eq. (4.31) and \( \nu_s^{(1)} \) given by Eq. (4.32), in Eq. (4.30) to get:

\[
\left[ \left( 1 + \frac{4 \Phi_{11}}{3\Psi_2 - 2\Phi_{11}} \right) O_{5b} O_{1b} + \left( O_{5a} - 4\rho_s \frac{\Phi_{11}}{3\Psi_2 + 2\Phi_{11}} \right) O_{2b} \right. \\
+ 2 \left( 3\Psi_2 + 2\Phi_{11} \right) \Psi_3^{(1)} + \left( 4 \frac{\Phi_{11}}{3\Psi_2 - 2\Phi_{11}} \left[ O_{5b} O_{1a} + \rho_s \chi O_{2a} \right] \Psi_4^{(1)} \right) \\
= + \left[ \left( 1 + \frac{4 \Phi_{11}}{3\Psi_2 - 2\Phi_{11}} \right) O_{5b} T_{1a} + \left( O_{5a} - 4\rho_s \frac{\Phi_{11}}{3\Psi_2 + 2\Phi_{11}} \right) T_{2b} \right] T_{12}^{(1)} \\
+ \left( 1 + \frac{4 \Phi_{11}}{3\Psi_2 - 2\Phi_{11}} \right) O_{5b} T_{1b} T_{20}^{(1)} + \left( O_{5a} - 4\rho_s \frac{\Phi_{11}}{3\Psi_2 + 2\Phi_{11}} \right) T_{2a} T_{22}^{(1)} \\
\left. + \frac{\pi}{\varphi_1} (3\Psi_2 + 2\Phi_{11}) J_2^{(1)} \right]. \quad (4.37)
\]
In order to proceed further, we will specify the derivation for the Reissner–Nördstrom spacetime in a particular coordinate system.

**5 Coupled equations with sources in Reissner–Nördstrom, ghost gauge**

The element of line for the Reissner Nördstrom spacetime in Kerr Schild-type coordinates is

\[
ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + 2\left(\frac{2M}{r} - \frac{Q^2}{r^2}\right)dtdr + \left(1 + \frac{2M}{r} - \frac{Q^2}{r^2}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).
\] (5.1)

In these coordinates we choose a null tetrad:

\[
l^\mu = \frac{1}{2} \left(1 + \frac{2M}{r} - \frac{Q^2}{r^2}, \frac{1}{r}, 0, 0\right),
\]
\[
k^\mu = (1, -1, 0, 0),
\]
\[
m^\mu = \frac{1}{\sqrt{2}r} (0, 0, 1, i \csc \theta).
\] (5.2)

The only non-vanishing components of the Weyl and Ricci scalars are

\[
\Psi_2 = \frac{M}{r^3} - \frac{Q^2}{r^4}, \quad \Phi_{11} = \frac{Q^2}{2r^4},
\] (5.3)

whereas the non-zero spin coefficients are

\[
\mu_s = \frac{1}{r}, \quad \rho_s = \frac{r^2 - 2Mr + Q^2}{2r^3}, \quad \epsilon_s = -\frac{1}{2} \left(\frac{M}{r^2} - \frac{Q^2}{r^3}\right),
\]
\[
\beta_s = -\alpha_s = -\frac{1}{2\sqrt{2}} \cot \theta.
\] (5.4)

In this coordinate system we will write explicit equations for the perturbations \(\Psi_3^{(1)}\) and \(\Psi_4^{(1)}\). From Eq. (4.37), simplifying, and using that \(\mathcal{O}_{5b} = -\mathcal{O}_{2a}\) in Reissner–Nördstrom, and that \(\mathcal{O}_{5b} \mathcal{O}_{1a} = (\mathcal{O}_{1a} + \rho_s) \mathcal{O}_{5b} = - (\mathcal{O}_{1a} + \rho_s) \mathcal{O}_{2a}\), we obtain the second equation for \(\Psi_3^{(1)}\), and \(\Psi_4^{(1)}\) with sources:

\[
\left[\left(\mathcal{O}_{5a} - 4\rho_s \frac{\Phi_{11}}{3\Psi_2 + 2\Phi_{11}}\right)\mathcal{O}_{2b} + \frac{1}{\chi} \mathcal{O}_{5b} \mathcal{O}_{1b} + 2 (3\Psi_2 + 2\Phi_{11})\right] \Psi_3^{(1)} - \frac{4\Phi_{11}}{3\Psi_2 - 2\Phi_{11}} \left(\mathcal{O}_{1a} + 6\rho_s \frac{\Psi_2}{3\Psi_2 + 2\Phi_{11}}\right) \mathcal{O}_{2a} \Psi_4^{(1)} = \left[\left(\mathcal{O}_{5a} - 4\rho_s \frac{\Phi_{11}}{3\Psi_2 + 2\Phi_{11}}\right) \mathcal{T}_{2b} + \frac{1}{\chi} \mathcal{O}_{5b} \mathcal{T}_{1a}\right] \mathcal{T}_{12}^{(1)}
\]
Regarding the source of the perturbation, let us assume that such matter source is a cloud of charged particles with charge $e$ and mass $\mu$ that behave like dust (pressure less fluid). The stress energy tensor is

$$\mathcal{T}_{\mu\nu} = \rho u_\mu u_\nu,\quad (5.6)$$

$\rho$ is the rest mass density and $u^\mu$ is the four velocity. Furthermore, in our analysis we will consider that the fluid is falling radially into the black hole with four velocity:

$$u^\mu = [u^t(t, r), u^r(t, r), 0, 0].\quad (5.7)$$

Let us further assume that the particles have a constant charge-mass ratio $q = (e/\mu)$ throughout the cloud. Then, the electric current induced by the motion of the particles is

$$J_\mu^{el} = q \rho u_\mu.\quad (5.8)$$

We will assume that this current is the source for the electromagnetic field in Maxwell equations namely, $J^{(1)}_\mu = J_\mu^{el}$. With the previous assumptions, the non-vanishing projections of the stress energy tensor and electric current are $T_{22}^{(1)}$ and $J_2^{(1)}$. The perturbation equations, Eqs. (4.35) and (5.5) become

$$\left\{ \begin{array}{l}
\Delta + \mu_s \chi \left( 5 + \frac{4 \Phi_{11} (9 \Psi_2 - 14 \Phi_{11})}{(3 \Psi_2 - 2 \Phi_{11})^2} \right) \\
(D + \rho_s - 4 \epsilon_s) \chi \mathcal{O}_{3b} \mathcal{O}_{2a} - 2 \Phi_{11} + 3 \Psi_2 \end{array} \right\} \Psi_4^{(1)}$$

$$- \frac{4 \Phi_{11}}{2 \Phi_{11} + 3 \Psi_2} \left( \Delta - \frac{9 \Psi_2 - 14 \Phi_{11}}{3 \Psi_2 - 2 \Phi_{11}} \mu_s \right) \mathcal{O}_{1b} \Psi_3^{(1)}$$

$$= -4 \pi \chi \left[ \mathcal{O}_{3b} \mathcal{T}_{2a} \right] T_{22}^{(1)}.$$  

$$\left[ \left( D + 2 \rho_s - 4 \epsilon_s - 4 \rho_s \frac{\Phi_{11}}{3 \Psi_2 + 2 \Phi_{11}} \right) \mathcal{O}_{2b} + \frac{1}{\chi} \mathcal{O}_{5b} \mathcal{O}_{1b} + 6 \Psi_2 + 4 \Phi_{11} \right] \Psi_3^{(1)}$$

$$- \frac{4 \Phi_{11}}{3 \Psi_2 - 2 \Phi_{11}} \left[ D + \rho_s - 4 \epsilon_s + 6 \rho_s \frac{\Psi_2}{3 \Psi_2 + 2 \Phi_{11}} \right] \mathcal{O}_{2a} \Psi_4^{(1)}$$

$$= 4 \pi \left( D + \rho_s - 4 \epsilon_s - 4 \rho_s \frac{\Phi_{11}}{3 \Psi_2 + 2 \Phi_{11}} \right) \mathcal{T}_{2a} \mathcal{T}_{22}^{(1)}$$

$$+ \frac{\pi}{\varphi_1} (3 \Psi_2 + 2 \Phi_{11}) J_2^{(1)},\quad (5.9)$$

where we have expanded the radial-temporal operators. Under the hypothesis of radially infalling matter the equations simplify and it is possible to separate the angular dependence from the radial-temporal part. In order to get this decomposition we expand
each relevant function into a basis of spin weighted spherical harmonics as follows. The \( \Psi_4^{(1)} \) function has spin weight \(-2\), \( \Psi_3^{(1)} \) spin weight \(-1\), and the density of matter is a scalar with spin weight zero. Expanding each function in the corresponding basis one gets

\[
\Psi_4^{(1)} = \sum_{l,m} P_4(t, r) Y_{-2}^{(l,m)}(\theta, \varphi),
\]

\[
\Psi_3^{(1)} = \sum_{l,m} P_3(t, r) Y_{-1}^{(l,m)}(\theta, \varphi),
\]

\[
\rho = \sum_{l,m} \rho(t, r) Y_0^{(l,m)}(\theta, \varphi).
\]

These expansions are convenient because the angular operators of Eqs. (5.9), (5.10) can be written in terms of the raising and lowering spin operators: \( \delta_s = -(\partial_\theta + i \csc \theta \partial_\varphi - s \cot \theta) \) and \( \bar{\delta}_s = -(\partial_\theta - i \csc \theta \partial_\varphi + s \cot \theta) \) [32,33]:

\[
\delta + q \beta_s = -\frac{1}{r \sqrt{2}} \delta_q, \quad \bar{\delta} + q \beta_s = -\frac{1}{r \sqrt{2}} \bar{\delta}_q,
\]

from these last equations and Eq. (4.11) with \( \pi_s = \tau_s = \gamma_s = 0 \), \( \alpha_s = -\beta_s \) and \( \eta = -1 \) one gets

\[
O_{2a} = -(\delta - 4 \beta_s) = \frac{1}{\sqrt{2} r} \bar{\delta}_2, \quad O_{3b} = (\bar{\delta} + 2 \beta_s) = -\frac{1}{\sqrt{2} r} \bar{\delta}_1,
\]

\[
\bar{\delta} + 2 \eta \alpha = \bar{\delta} + 2 \beta_s = -\frac{1}{\sqrt{2} r} \bar{\delta}_1, \quad T_{2a} = \bar{\delta} = -\frac{1}{\sqrt{2} r} \bar{\delta}_0,
\]

\[
O_{5b} = \delta - 4 \beta_s = -\frac{1}{\sqrt{2} r} \delta_2, \quad O_{1b} = -(\bar{\delta} + 2 \beta_s) = \frac{1}{\sqrt{2} r} \bar{\delta}_1.
\]

When the \( \delta \) operator acts on the spin weighted spherical harmonic, it raises the spin weight:

\[
\delta_s Y_s^{l,m} = \sqrt{(l - s) (l + s + 1)} Y_{s+1}^{l,m},
\]

and, when \( \bar{\delta} \) acts on the spin weighted spherical harmonic, it lowers the spin weight:

\[
\bar{\delta}_s Y_s^{l,m} = -\sqrt{(l + s) (l - s + 1)} Y_{s-1}^{l,m}.
\]

For a detailed description of these harmonics and on how to use them to extract physical information carried by gravitational waves see for instance [34] and references therein. Notice that in terms of the harmonic coefficients for the density, we have that

\[
[O_{3b} T_{2a}] T_{22}^{(1)} = \frac{1}{2 r^2} \left( k_\mu u^\mu \right)^2 \sum_{l,m} \rho_{l,m} \bar{\delta}_{l-1} \delta_0 Y_0^{l,m} = \left( k_\mu u^\mu \right)^2 \sum_{l,m} \rho_{l,m} \frac{\sqrt{(l-1) l (l+1) (l+2)}}{2 r^2} Y_{-2}^{l,m}.
\]
$$T_{2a} T_{22}^{(1)} = \frac{1}{\sqrt{2} r} \tilde{\delta}_0 \rho \left( k_\mu u^\mu \right)^2 = \left( k_\mu u^\mu \right)^2 \frac{1}{\sqrt{2} r} \sum_{l,m} \rho_{lm} \tilde{\delta}_0 Y_0^{l,m}$$

$$= \left( k_\mu u^\mu \right)^2 \sum_{l,m} \frac{\sqrt{l(l+1)}}{\sqrt{2} r} \rho_{lm} Y_{l-1}^{l,m}.$$  \hspace{1cm} (5.18)

$$J_2^{(1)} = -\delta J_n^{(1)} = k_\mu u^\mu \frac{1}{\sqrt{2} r} \sum_{l,m} \rho_{lm} \tilde{\delta}_0 Y_0^{l,m}$$

$$= k_\mu u^\mu \sum_{l,m} \frac{\sqrt{l(l+1)}}{\sqrt{2} r} \rho_{lm} Y_{l-1}^{l,m}. \hspace{1cm} (5.19)$$

Substituting the expansions Eq. (5.13) into Eqs. (5.9), (5.10), taking in consideration the eigenvalues of the angular operators and integrating over the solid angle, we can obtain an equation for each mode $(l, m)$ for $P_4$:

$$\left\{ \begin{array}{l}
\left[ \Delta + \mu_s \chi \left( 5 + \frac{4 \Phi_{11} (9 \Psi_2 - 14 \Phi_{11})}{(3 \Psi_2 - 2 \Phi_{11})^2} \right) \right] \\
(D + \rho_s - 4 \epsilon_s - \chi \frac{(l-1)(l+2)}{2 r^2} + 3 \Psi_2 - 2 \Phi_{11}) P_4(t, r)
\end{array} \right\} \hspace{1cm} (5.20)$$

and for $P_3$:

$$\left[ \begin{array}{l}
\left( D + 2 \rho_s - 4 \epsilon_s - 4 \rho_s \frac{\Phi_{11}}{3 \Psi_2 + 2 \Phi_{11}} \right) \left( \Delta - 4 \mu_s \right) \\
- \frac{1}{\chi} \frac{(l-1)(l+2)}{2 r^2} + 6 \Psi_2 + 4 \Phi_{11}
\end{array} \right] P_3(t, r)$$

$$\left[ \begin{array}{l}
- 4 \frac{\Phi_{11}}{3 \Psi_2 - 2 \Phi_{11}} \left[ D + \rho_s - 4 \epsilon_s + 6 \rho_s \frac{\Psi_2}{3 \Psi_2 + 2 \Phi_{11}} \right]
\end{array} \right] \frac{\sqrt{2} (l-1)(l+2)}{2 r} P_4(t, r)$$

$$= 4 \hat{\pi} \left( D + 2 \rho_s - 4 \epsilon_s - 4 \rho_s \frac{\Phi_{11}}{3 \Psi_2 + 2 \Phi_{11}} \right) \frac{\sqrt{l(l+1)}}{\sqrt{2} r} \rho(t, r) (u^\mu k_\mu)^2$$

$$+ \frac{\pi}{\varphi_1} (3 \Psi_2 + 2 \Phi_{11}) \frac{\sqrt{l(l+1)}}{\sqrt{2} r} \rho(t, r) (u^\mu k_\mu). \hspace{1cm} (5.21)$$

Replacing directional derivatives and spin coefficients for their explicit form in the specified coordinates we obtain an explicit system for the radial and temporal part of each mode of the perturbations $\Psi_4^{(1)}$ and $\Psi_3^{(1)}$. In Sect. 7 we will write down the system explicitly but first we will focus on the matter that produces the perturbations.
6 Matter content

The dynamics of the particles is described by the conservation of the number of particles and the conservation equation for the stress energy tensor \( T^{\mu \nu} \). The continuity equation holds because we are assuming conservation of the number of particles, the particles belong to the same species and are not created or annihilated

\[
\nabla_\mu J^\mu = 0, \tag{6.1}
\]

where \( J^\mu = \rho u^\mu \). The conservation Eq. (6.1) for radially infalling particles yields

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \rho u^\mu) = 0, \quad \Rightarrow \quad \partial_t (r^2 \rho u^t) + \partial_r (r^2 \rho u^r) = 0, \tag{6.2}
\]

which can be rewritten, given the metric (5.1), as:

\[
\partial_t \rho + v^r \partial_r \rho + \frac{\rho v^r}{r(u^t)^2} \left[ \left( E - \frac{q Q}{r} \right) \left( 2E - \frac{q Q}{r} \right) - 2 + \frac{3M}{r} - \frac{Q^2}{r^2} \right] = 0, \tag{6.3}
\]

where we have defined \( v^r \equiv \frac{u^r}{u^t} \). Since this is an equation that involves only temporal and radial derivatives, each mode \( \rho_{l,m} \) in the decomposition (5.13) obeys Eq. (6.3) once we can provide the velocity of each particle. In order to get this velocity, we will use the conservation of the stress energy tensor. First, notice that the equations

\[
T^{\mu \nu;l\mu} = 0, \tag{6.4}
\]

can be integrated once, and the components of the four velocity can be expressed in terms of the constants of motion using the symmetries of the space-time and the normalization on the four-velocity. This can be achieved by noticing that equation (6.4) can also be obtained using the Euler-Lagrange equations with the Lagrangian

\[
\mathcal{L} = \mu \left( \frac{1}{2} g_{\mu \nu} u^\mu u^\nu + q A_\mu u^\mu \right), \tag{6.5}
\]

where \( A_\mu \) is the vector potential. The Euler-Lagrange equations become

\[
u^\mu \nabla_\mu u^\nu = q F^\nu_\alpha u^\alpha, \tag{6.6}
\]

where \( F_{\mu \nu} = A_{\mu ;\nu} - A_{\nu ;\mu} \). In the RN background the nonvanishing component of vector potential is \( A_t = -Q/r \).

Since the Lagrangian Eq. (6.5) is independent of time, we get a constant of motion

\[-\varepsilon \equiv \frac{\partial}{\partial u^t} \mathcal{L}, \quad E = \frac{\varepsilon}{\mu}, \tag{6.7}\]
and considering the four velocity as \( u^\mu = (u^t, u^r, 0, 0) \) we get from Eqs. (6.5) and (6.7)

\[
u^t = \frac{E r^2 + (2 M r - Q^2) u^r + q A_t r^2}{r^2 - 2 M r + Q^2}.
\tag{6.8}
\]

The normalization of the four velocity \( u^\mu u_\mu = -1 \) together with Eq. (6.8), allow us to compute the radial component of the velocity. After some algebraic steps we get

\[
(u^r)^2 = (E + q A_t)^2 - \left(1 - \frac{2 M}{r} + \frac{Q^2}{r^2}\right).
\tag{6.9}
\]

Equation (6.9) can be rewritten in terms of an effective potential

\[
\left(\frac{d r}{d \tau}\right)^2 = E^2 - V_{\text{eff}},
\tag{6.10}
\]

where

\[
V_{\text{eff}} = 1 - \frac{2 M}{r} - \frac{q^2 Q^2}{r^2} + 2 q Q E + \frac{Q^2}{r^2},
\]

\[
= 1 - \frac{2 M}{r} \left(1 - \frac{E q Q}{M}\right) + \frac{Q^2}{r^2} \left(1 - q^2\right).
\tag{6.11}
\]

Notice that taking \( E = 1 \), an extremal particle \( (q = 1) \) in an extreme black hole \( (Q = M) \) will be in equilibrium [35]. The previous decomposition allows us to study, with a 1D numerical code, any radially in-falling dust matter distribution and its gravitational reaction.

### 7 Numerical implementation

According to the Peeling theorem [12], the Weyl scalars behave as

\[
\Psi_i \equiv \frac{1}{r^{5-i}}.
\tag{7.1}
\]

As such, it probes convenient to perform the evolution of the quantities \( r \, \Psi_4^{(1)} \), and \( r^2 \, \Psi_3^{(1)} \), so that the evolved quantity maintains a constant amplitude during the evolution. The evolution equation for \( R_4 = r \, P_4 \) and for \( R_3 = r^2 \, P_3 \) are

\[
\begin{align*}
\left( r^2 + 2 M r - Q^2 \right) \frac{\partial^2}{\partial t^2} - \left( r^2 - 2 M r + Q^2 \right) \frac{\partial^2}{\partial r^2} - 2 \left(2 M r - Q^2\right) \frac{\partial^2}{\partial t \partial r} - 2 \left(6 M r^3 + (3 M^2 - 10 Q^2) r^2 - 5 M Q^2 r - 2 Q^4\right) \frac{\partial}{\partial r} \\
- 2 \left(6 M r^3 - (3 M^2 + 10 Q^2) r^2 + 5 M Q^2 r + 2 Q^4\right) \frac{\partial}{\partial r}
\end{align*}
\]
\begin{align*}
&+ (l - 1) (l + 2) \left\{ \frac{3 M r - 4 Q^2}{3 M r - 2 Q^2} - \frac{6 M (M r - 2 Q^2)}{r (3 M r - 4 Q^2)} \right\} R_4(t, r) \\
&- 2 \sqrt{2} \frac{(l - 1) (l + 2)}{3 M r - 2 Q^2} Q^2 \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial r} + \frac{4 Q^2}{r (3 M r - 4 Q^2)} \right) \\
R_3(t, r) &= +4 \pi \sqrt{(l - 1) l} (l + 1) \frac{3 M r - 4 Q^2}{3 M r - 2 Q^2} r \rho(t, r) (u_\mu k^\mu)^2, \\
&\text{(7.2)}
\end{align*}

and:

\begin{align*}
&\left\{ (r^2 + 2 M r - Q^2) \frac{\partial^2}{\partial t^2} - (r^2 - 2 M r + Q^2) \frac{\partial^2}{\partial r^2} - 2 (2 M r - Q^2) \frac{\partial^2}{\partial t \partial r} \\
&- 2 \frac{3 M r^3 - Q^2 r^2 + M Q^2 r - Q^4}{(3 M r - 2 Q^2) r} \frac{\partial}{\partial t} - 2 \frac{(3 r^2 - Q^2) (M r - Q^2)}{(3 M r - 2 Q^2) r} \frac{\partial}{\partial r} \\
+ (l - 1) (l + 2) \frac{3 M r - 2 Q^2}{3 M r - 4 Q^2} \\
&- \frac{\sqrt{2} (l - 1) (l + 2)}{3 M r - 4 Q^2} Q^2 \left[ (r^2 + 2 M r - Q^2) \frac{\partial}{\partial t} + (r^2 - 2 M r + Q^2) \frac{\partial}{\partial r} \\
&- 2 Q^2 \frac{r^2 + M r - Q^2}{(3 M r - 2 Q^2) r} \right] R_4(t, r) \\
&= \pi \sqrt{2} (l + 1) r \left[ 2 (u_\mu k^\mu)^2 \cdot (r^2 + 2 M r - Q^2) \frac{\partial \rho(t, r)}{\partial t} \\
&+ (r^2 - 2 M r + Q^2) \frac{\partial}{\partial r} \left( \rho(t, r) (u_\mu k^\mu)^2 \right) \\
&- u_\mu k^\mu \left( 4 u_\mu k^\mu \frac{2 M r^3 - 2 (2 M^2 + Q^2) r + 9 M Q^2 r - 4 Q^4}{(3 M r - 2 Q^2) r} - \sqrt{2} \frac{3 M r - Q^2}{Q} \right) \rho(t, r) \right]. \\
&\text{(7.3)}
\end{align*}

To obtain a first order system of equations of motion suitable for numerical integration, we introduce the auxiliary functions

\[ \pi_a = \frac{1}{\alpha^2} (\partial_t R_a - \beta^r \psi_a), \quad \psi_a = \partial_r R_a, \quad \text{with} \quad a = 3, 4, \quad (7.4) \]

where we have dropped the mode number subscripts to simplify the notation and used the lapse and shift vector for the metric Eq. (5.1)

\[ \alpha = \left( 1 + \frac{2 M}{r} - \frac{Q^2}{r^2} \right)^{-1/2}, \quad \beta_r = \frac{2 M}{r} - \frac{Q^2}{r^2}, \quad \beta^r = \alpha^2 \beta_r. \quad (7.5) \]

The following system of evolution equations for the functions \( \pi_4, \psi_4, R_4 \) is obtained,
\[ \partial_t R_a = \alpha^2 \pi_a + \beta^r \psi_a, \quad \text{with} \quad a = 3, 4, \quad (7.6) \]
\[ \partial_t \psi_a = \alpha^2 \partial_r \pi_a + \beta^r \partial_r \psi_a \]
\[ + \frac{2r(Mr - Q^2)}{(r^2 + 2Mr - Q^2)^2} (\pi_a - \psi_a), \quad a = 3, 4, \quad (7.7) \]
\[ \partial_t \pi_4 = \beta^r \partial_r \pi_4 + \alpha^2 \partial_r \psi_4 + \beta_r (q_4 \partial_r \pi_4 + \psi_4 \partial_r \beta^r) \]
\[ + 2C_4t (\alpha^2 \pi_4 + \beta^r \psi_4) - 2C_4t \psi_4 - C_{4in} R_4 \]
\[ + 2C_3 \left( \alpha^2 (\pi_3 - \psi_3) + \frac{4Q^2}{r(3Mr - 4Q^2)} R_3 \right) \]
\[ + \frac{3Mr - 4Q^2}{r(3Mr - 2Q^2)} \sqrt{(\ell + 2)(\ell + 1)(\ell - 1)T_2}, \quad (7.8) \]

where the coefficients are
\[
C_{4t} = \frac{3Mr^2(M + 2r) - Q^2(10r^2 + 2Q^2 + 5Mr)}{r^3(3Mr - 4Q^2)},
\]
\[
C_{4r} = \frac{3Mr^2(M - 2r) - Q^2(-10r^2 + 2Q^2 + 5Mr)}{r^3(3Mr - 4Q^2)},
\]
\[
C_{4in} = \frac{(3Mr - 4Q^2)(l + 2)(l - 1)}{r^2(3Mr - 2Q^2)} - \frac{6M((Mr - 2Q^2))}{r^3(3Mr - 4Q^2)},
\]
\[
C_3 = Q^2 \frac{\sqrt{2(l + 2)(l - 1)}}{r^2(3Mr - 2Q^2)}.
\]

For the remaining functions \( \pi_3, \psi_3, R_3 \) we get the following set of equations
\[
\partial_t \pi_3 = \beta^r \partial_r \pi_3 + \alpha^2 \partial_r \psi_3 + \beta_r (q_3 \partial_r \pi_3 + \psi_3 \partial_r \beta^r) + 2C_{3t} (\alpha^2 \pi_3 + \beta^r \psi_3) \]
\[ - 2C_{3r} \psi_3 - C_{3in} R_3 \]
\[ + \frac{\sqrt{2(l + 2)(l - 1)}}{(3Mr - 4Q^2)} \left[ Q^2 (\pi_4 + \psi_4) - 2 \frac{Q^2(r^2 + Mr - Q^2)}{r^3(3Mr - 2Q^2)} R_4 \right] \]
\[ + \frac{r}{2} \sqrt{2(l + 1)} \left( \frac{1}{\alpha^2} \frac{\partial}{\partial t} T_2 + \left( \alpha^2 - \beta_r \beta^r \right) \frac{\partial}{\partial r} T_2 \right) \]
\[ - \frac{(6M^2r - 5MQ^2 - 9Mr^2 + 8Q^2)}{r^2(3Mr - 2Q^2)} T_2 + \left( \frac{3Mr - 2Q^2}{r^2Q\sqrt{2}} \right) J_2, \quad (7.9) \]

where \( T_2 = \rho_{l,m} (u^t + u^t)^2, \quad J_2 = \rho_{l,m} (u^t + u^t) \) and the coefficients are
\[
C_{3t} = \frac{6M^2r^2 - Q^2(-6Mr + Q^2 + r^2)}{r^3(3Mr - 2Q^2)},
\]
\[
C_{3r} = \frac{6Mr^2(M - r) + Q^2(-6Mr + Q^2 + 5r^2)}{r^3(3Mr - 2Q^2)},
\]
\[ C_3^{in} = \frac{(3Mr - 2Q^2)(l + 2)(l - 1)}{r^2(3Mr - 4Q^2)} - \frac{(3Mr^3 - 18M^2r^2 - 6Q^2r^2 + 23MQ^2r - 6Q^4)}{r^4(3Mr - 2Q^2)}. \]

8 Results and discussion

We solve the equations for the gravitational-electromagnetic field by using the method of lines. Our numerical code evolves the first order variables Eqs. (7.6)–(7.8) with a third order Runge–Kutta integrator with a fourth order spatial stencil in a finite computational domain \( r \in [r_{\text{min}}, r_{\text{max}}] \). We also introduce a small sixth order dissipation to eliminate high frequency modes. As boundary conditions at the last grid point, we impose that all the incoming waves as given by the characteristic fields vanish. We set \( r_{\text{max}} \) sufficiently far out in order to avoid any kind of contamination, typically \( r_{\text{max}} > t_{\text{evol}} \) where \( t_{\text{evol}} \) is the total evolution time. Since we are using horizon penetrating coordinates, \( r_{\text{min}} \) lies inside the event horizon. We solve the equation for the rest mass density and look for the gravitational and electromagnetic responses. Coupled waveforms are then extracted at a fixed radius \( r = r_{\text{obs}} \). In what follows we consider for simplicity, a shell of matter described by a single spherical harmonic mode

\[ \rho(t, r) = \rho_{l,m} Y_{l}^{m}_{0}, \quad (8.1) \]

we consider the modes with \( l = 2 \) since these give the main contribution to the quadrupole gravitational radiation. We have used as initial data for the radial distribution for the density a Gaussian of the form

\[ \rho_{l,m}(t = 0, r) = \rho_0 e^{-\left(r - r_{cg}\right)^2/2\sigma^2}, \quad (8.2) \]

with \( \rho_0 = 5 \times 10^{-3}, r_{cg} = 10M \) and \( \sigma = 0.5M \).

The gravitational-electromagnetic functions \( R_4, R_3 \) are set to zero initially as well as their derivatives. Our results indicate that this choice on the waveforms has negligible effects because the signals are ruled by the perturbation caused by the infalling matter.

Given an initial distribution of the infalling density \( \rho_0 \), centered at \( r_{cg} \) Eq. (6.3) can be integrated from \( r_{cg} \) to \( r \) to give the envelope of the density as

\[ \rho = \rho_0 \frac{r_{cg}}{r} \left[ \frac{(q^2 - 1)Q^2 + 2r_{cg}(M - Eq Q) + r_{cg}^2(E^2 - 1)}{(q^2 - 1)Q^2 + 2r(M - Eq Q) + r^2(E^2 - 1)} \right]^{1/2}. \quad (8.3) \]

In order to integrate Eq. (6.3) we use the fact that

\[ \frac{d}{dt} \rho = (\partial_t + v^r \partial_r) \rho, \quad (8.4) \]
Fig. 1 Snapshots of the evolution of the density given by (6.3) taken every $t = 25M$. The dashed line represents the trajectory of the point with the highest density for the Gaussian initial data. The trajectory is modelled by (8.3). For this plot we used $r_c = 10M$, $\sigma = 0.5M$, $q = 0.2$ and $Q = 0.9M$.

Fig. 2 The potential $V_{\text{eff}}$ is shown in the first row for two representative cases of the charge of the particles, $q = 0.6$ (left) and $q = -1.2$ (right), the particles have $E = 1$ and the charge of the BH is $Q = 0.9M$. The black line is the envelope of the density, i.e. the trajectory of the point with higher density in the fluid. In the second row it is shown the waveforms $R_4$ caused by the infalling on the particles, in the left for $q = 0.6$ and in the right for $q = -1.2$. Although the dynamics of the particles is different, as one may infer from the potential, the waveforms are quiet similar in structure, the same happens for $R_3$.

and $v^r = \frac{dr}{dt}$. In Fig. 1 we show the density profile at different times with the exact solution for the envelope, Eq. (8.3). This procedure allows us to prove the accuracy of our numerical code.

In Fig. 2 we plot $V_{\text{eff}}$, the density Eq. (8.3) and the resulting waveforms $R_4$ for two representative values $q = 0.6$, $q = -1.2$. For $q = 0.6$ the potential has a minimum and the potential barrier lies outside the external horizon. For $q = -1.2$ the electro-
magnetic force contributes to the gravitational attraction and the particles fall faster onto the black hole. The effective potential thus presents very distinctive properties depending on the value of $q$. The situation with the gravitational-electromagnetic signals is different. Although quantitatively the waveforms $R_4$ are different for $q = 0.6$ and $q = -1.2$, they show qualitatively the same three characteristics phases of a gravitational wave emitted by a perturbed black hole: the initial burst, the quasi-normal ringing and the power-law decay [36–38] despite the difference in the potential. The main difference is in the amplitude of the wave and the time of response. The same follows for $R_3$.

In Fig. 3 we plot the signals $R_4$ and $R_3$ produced by the infalling matter Eq. (8.2) as measured by an observer located at $r_{\text{obs}}/M = 100$. For this plot, we use a value of $Q = 0.9M$ and $q = 0.2$. For this relatively small value of $q$ the $R_4$ signal is stronger and displays the three characteristic phases. In the inset the signals are plotted in a semilogarithmic scale to improve the visualization of the two last phases. Regarding the frequencies of the signals emitted, in order to check the accuracy of our results, we performed a Fourier transform of the $R_3$ and $R_4$ at a fixed $r_{\text{obs}}$, our results are shown in Table 1. We compare our findings with the frequencies $\tilde{M} \tilde{\omega}_e$ and $\tilde{M} \tilde{\omega}_g$ given by Chandrasekhar in [22], we found a very good agreement up to the decimal figures shown.

Figure 4 shows the absolute value of $R_4$ and $R_3$ in a semilogarithmic scale for some values of the ratio $q$. Given the Fourier transform of the signal, we found that the frequency of $R_4$ does not depend on $q$. The same result holds for $R_3$. This is a consequence of the fact that the frequencies are the quasi-normal modes and hence depend only on the parameters of the BH. The plot of $R_4$ however, displays a shift in
Table 1 Frequencies of the electromagnetic $\omega_e$ and gravitational $\omega_g$ waveforms obtained with a Fourier transform of $R_3$ and $R_4$ respectively. The values are in agreement with the quasi normal frequencies reported in [22] shown in fourth and fifth columns.

| $Q/M$ | $\omega_e$ | $\omega_g$ | $\tilde{\omega}_e$ | $\tilde{\omega}_g$ |
|-------|------------|------------|---------------------|---------------------|
| 0.0   | 0.457      | 0.373      | 0.45760             | 0.37367             |
| 0.4   | 0.479      | 0.378      | 0.47993             | 0.37844             |
| 0.6   | 0.512      | 0.386      | 0.51201             | 0.38622             |
| 0.8   | 0.570      | 0.400      | 0.57013             | 0.40122             |
| 0.9   | 0.618      | 0.412      | 0.61939             | 0.41357             |

Fig. 4 The frequency of the waveforms $R_4$ (top) and $R_3$ (bottom) is independent of the value of $q$, it depends only on the charge and mass of the BH. The frequencies are those of the quasinormal modes. The difference in phase observed in $R_4$ is due to the difference in the infalling time of the charged particles.

The time in which the signal is emitted. This result is consistent with the fact that the electromagnetic repulsion plays a role in the time of infall of the particles.

We also studied the dependence of the signals on the initial properties of the shell. We focused on the gravitational and electromagnetic response of the black hole when the compactness of the shell is changed. We varied the initial dispersion $\sigma$ in (8.2) and monitor $R_4$ and $R_3$. We have found that the ringing phase is lost when the shell is larger than a threshold value $\sigma \sim M$. This result in consistent with previous analysis presented for instance in [39,40]. The waveforms are shown in the top panel of Fig. 5 for the gravitational signal and in the top panel of Fig. 6 for the electromagnetic signal. For a shell width of $\sigma = 0.5M$ the quasinormal ringing is very well defined whereas for $\sigma = 5M$ the quasinormal modes are not excited as can be seen in the bottom panel of the figure.
Fig. 5 In the top panel it is plotted $R_4$ for an observer located at $r_{\text{obs}} = 100M$ for several values of the shell width. In the bottom panel we show only three representative cases. For the largest $\sigma$, the quasinormal modes are not excited.

Fig. 6 The same as Fig. 5 but for the electromagnetic waveforms.

The radiated gravitational energy flux in terms of the multipolar decomposition is:

$$\frac{dE_{gw}}{dt} = \lim_{r \to \infty} \frac{1}{16\pi} \sum_{l,m} \int_{t'}^{t} |R_4|^2 dt'.$$  \hspace{1cm} (8.5)

We compute the gravitational radiated energy, $E_{gw}$ by integrating Eq. (8.5). In order to perform the integral and to get rid of the radiation resulting from the initial data, we start the integration of the fluxes with a shift in the initial time. The choice of that initial time depends on the extraction radius, typically we take it to be $t_s \sim 140M$ for an extraction radius of $r_{\text{obs}} = 100M$. That value was chosen in order to make sure that the energy computed had converged. We study the dependence of the radiated energy varying $q$. We also found that, in agreement with previous results, the energy grows quadratically with $q$, $E_{gw} = kq^2$, with $k$ a constant that depends on the amplitude.
of the initial perturbation. This results holds only when the gravitational signal $R_4$, displays the quasinormal ringing. For large values of $\sigma > 1M$, it was not possible to find any particular dependence.

9 Conclusions

The prime concern of this study has been the investigation, within linear perturbation theory, of waveforms of coupled electromagnetic and gravitational signals produced by point particles falling in the vicinity of the Reissner–Nördstrom black hole. The electromagnetic counterpart of gravitational waves caused by infalling charged particles into a neutral black hole was discussed in [41]. In this work we extend that study to consider a charged black hole. As in [41], the infalling of charged matter triggers gravitational and electromagnetic signals. In that work, we did not find a direct coupling between the frequencies of both types of waves. Since the spacetime was neutral. In this case, we allow the particles to interact the spacetime through the charge of the black hole and the charge of the particles, such coupling is reflected in the Weyl scalar $\Psi_3$.

We have shown that the signals, described by perturbations of the scalars $\Psi_3$ and $\Psi_4$ are coupled and the waveforms have the known phases: initial burst, quasinormal ringing and tail decay.

Since the particles consider here are charged, the electric field of the black hole affects their motion and they do not follow geodesics. However, the trajectories still have an analytic description and we were able to follow the motion and provoke a gravitational and electromagnetic response, as the particles cross the horizon.

We have compared the electromagnetic and gravitational response for an uncharged BH with the charged case. The amplitude and shape of the waveforms are different from the neutral case. However, we estimate the quasinormal modes via a numerical fit of the signals and found that the frequencies do correspond to the quasinormal modes of the Reissner–Nordstrom black hole [23,42,43].

The linear dependence of the electromagnetic waveforms with the charge-mass ratio of the particles $q$ found in [41] is also reproduced in this case. This result comes from the linear dependence of the perturbations with respect to $q$.

In [41], we had also found that the gravitational and electromagnetic energies were related by the square of $q$, which pointed to ways of determining the energy of one type of wave if the other was measured. In the present case, we found a similar relation for the gravitational signal $\Psi_4$. However for the electromagnetic counterpart, new forms of measuring the effects must be defined. A very interesting analysis along this direction was performed by Cardoso et al. in [7], where it is shown that for an extremal charged BH the electromagnetic quasinormal modes could be excited to considerable amplitude in the gravitational-wave spectrum. The results were obtained using a similar scenario to the one presented here (plunging point particles into a charged BH). However, in our case the amplitude of both signals depend strongly on the initial data. This last conclusion exemplifies the richness implicit in the Reissner–Nördstrom spacetime, and the dynamics of matter around it. Some new relations and challenges are still open, for instance, the interplay between electromagnetic and
gravitational fields under the scope of different gauges. Some of these subjects will be developed in future investigations.

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