Abstract

We study the decay of unstable states by formulating quantum tunneling as a time-of-arrival problem: we determine the detection probability for particles at a detector located a distance $L$ from the tunneling region. For this purpose, we use a Positive-Operator-Valued-Measure (POVM) for the time-of-arrival determined in [1]. This only depends on the initial state, the Hamiltonian and the location of the detector. The POVM above provides a well-defined probability density and an unambiguous interpretation of all quantities involved. We demonstrate that the exponential decay only arises if three specific mathematical conditions are met. Their physical content is the following: (i) the decay time is much larger than any microscopic timescale, so that the fine details of the initial state can be ignored, (ii) there is no quantum coherence between the different 'attempts' of the particle to traverse the barrier, and (iii) the transmission probability varies little within the momentum spread of the initial state. We also determine the long time limits of the decay probability and we identify regimes, in which the decays have no exponential phase.

1 Introduction

This is a third in a series of papers [1] [2], which studies the properties and applications of a Positive-Operator-Valued-Measure for the time-of-arrival. In [1] this POVM was constructed and it was applied to the free particle case, where it coincided with the one of Kijowski [3]. In [2], this POVM was adapted to the tunneling case: it led the determination of tunneling time and through a generalization to sequential measurements it provided a probability distribution for the times of arrival. Here, we apply the formalism for the determination of the decay probability of unstable quantum states through tunneling.

Quantum tunneling refers to the escape of a particle from a region through a potential barrier, whose peak is larger than the particle’s energy. The question
we answer here is what is the law that determines the rate of the particle’s escape through the barrier?

The issue of the escape probability for particles through a barrier as a function of time has been important ever since the first days of quantum mechanics. It is an observed fact that in the vast majority of physical systems the escape rate is approximately constant: the decay is exponential. However, the constancy of the escape rate does not hold at very early times and at very late times. In the first regime, there is a behavior corresponding to quantum Zeno effect and in the latter the decay is governed by an inverse-power law (perhaps with oscillations). Moreover, it is not necessary that all systems decay exponentially. The method we develop in this paper allows a full characterization of the decay behavior for an unstable state that decays through tunneling.

1.1 Our approach

The basic feature of our approach to both problems is its operational character. We identify the escape probability by constructing probabilities for the outcome of specific measurements. We assume that the quantum system is prepared in an initial state $\psi_0$, which is localized in a region on one side of a potential barrier that extends in a microscopic region. At the other side of the barrier and a macroscopic distance $L$ away from it, we place a particle detector, which records the arrival of particles. Using an external clock to keep track of the time $t$ for the recorder’s clicks, we construct a probability distribution $p(t)$ for the time of arrival. The fact that the detector is a classical macroscopic object and that it lies at a macroscopic distance away from the barrier allows one to state (using classical language) that the detected particles must have passed through the barrier (quantum effects like a particle crossing the barrier and then backtracking are negligible). Hence, at the observational level, the probability $p(t)$ contains all information about the temporal behavior for the ensemble of particles.

For the purposes of this paper, we assume that each particle in the ensemble is located within a microscopic region, which is bounded by a potential barrier (e.g. nucleus). For the particle to escape this region and to be detected at a macroscopic distance away, it must tunnel through the barrier. In effect, the initial state is unstable and it decays through tunneling. It is evident that the detection probability $p(t)$ incorporates the decay probability for the unstable state: modulo a transient period (before the first particles arrive at the detector), its physical content is the same.

With the considerations above, the problem of determining the escape probability for a state that decays through tunneling is equivalent to the determination of probability for the time-of-arrival for an ensemble of particles described by the wave function $\psi_0$ at $t = 0$ and evolving under a Hamiltonian with a potential term. To solve this problem, we use the result of [1], namely the construction of a POVM for the time-of-arrival for particles for a generic Hamiltonian $\hat{H}$. This POVM provides a unique determination of the probability distribution $p(t)$ for the time-of-arrival. It is important to emphasize that by construction $p(t)$ is
linear with respect to the initial density matrix, positive-definite, normalized (when the alternative of non-detection is also taken into account) and a genuine density with respect to time. We refer the reader to [1] for the physical assumptions relevant to the definition of this POVM and to [2] for its detailed construction for the case relevant to tunneling.

Our approach involves the formulation of the decay probability as a time-of-arrival problem. There are other approaches that treat quantum tunneling as a time-of-arrival problem in the literature. In Ref. [6], a distribution for tunneling time is obtained by considering a detector model and defining a probability distribution for time as

\[ P_X(t) = \int_0^\infty ds |\psi(X,s)|^2, \]

where \( X \) is the position variable. The method yields positive definite probabilities. Unlike the present treatment, these probabilities are not linear with respect to the initial state (density matrix). Our approach is closer to the one developed in Ref. [7], in which a method is developed that models the measurement with an imaginary potential near the detection point—see also [8]. This leads to a class of POVMs for the time-of-arrival that provide a generalization of Kijowski’s and are similar in form to the one we employ in this paper.

### 1.2 Comparison to other approaches

The study of the decay probability of unstable states goes back to the first days of quantum mechanics, most notably in the work of Weisskopf and Wigner [9]. It is known that for the (overwhelming) majority of physical systems the decay laws of unstable states are exponential. It is also known that the exponential law cannot be valid at very short and at very long timescales. This implies that even though the exponential decay law is very common, it is not universal. This fact brings about many questions: why is the exponential decay observed valid in such a variety of systems? which are the physical conditions necessary for its appearance? are there systems, whose decay has no exponential phase? We shall see that the probability distribution \( p(t) \) allows us to provide an answer to these questions (at least for decays that can be formulated as a tunneling problem).

Most studies of the validity of the exponential decay law (see in particular [10, 11, 12, 13]) proceed through the determination of the properties of the survival amplitude \( \langle \phi_0 | e^{-iHt} | \phi_0 \rangle \), where \( \phi_0 \) is the initial (unstable) state and \( H \) the system’s Hamiltonian. The modulus square of the amplitude is the probability that the system lies at the state \( |\phi_0\rangle \) at time \( t \). The exponential decay then refers to the behavior of this probability.

The survival probability is a function of time: however, it is not a probability density with respect to time. One can immediately see this on dimensional grounds: \( \langle \phi_0 | e^{-iHt} | \phi_0 \rangle \) is a pure number, while a genuine probability density has dimensions of inverse time. However, the quantity \( w(t) := 1 - |\langle \phi_0 | e^{-iHt} | \phi_0 \rangle|^2 \) is the total probability that the state \( \phi_0 \) has decayed at time \( t \). If \( |\langle \phi_0 | e^{-iHt} | \phi_0 \rangle|^2 \to 0 \) as \( t \to \infty \), the first derivative of \( w(t) \) can be interpreted as a normalized probability density \( p(t) \) for the decay at time \( t \).
This argument would work in classical probability theory. However, in quantum theory
\[
\dot{\hat{w}}(t) = -iTr \left( \rho_0 [\hat{H}, \hat{Q}(t)] \right),
\]
(1.1)
where \( \hat{\rho} = |\phi_0\rangle\langle\phi_0| \) and \( \hat{Q}(t) = e^{i\hat{H}t}|\phi_0\rangle\langle\phi_0|e^{-i\hat{H}t} \). The quantity \(-i[\hat{H}, \hat{Q}(t)]\) is not a positive operator, hence there is no guarantee that \( \dot{\hat{w}}(t) \) will be positive at all times for a generic initial state \( t \). Its interpretation as probability density is therefore problematic.

Clearly, for exponential decay \( \dot{\hat{w}}(t) > 0 \). However, the exponential decay does not hold at all moments of time, and the survival probability is not a monotonously decreasing function of time. Hence, \( \dot{\hat{w}}(t) \) does not define a probability distribution for all \( t \in [0, \infty) \). For the times that the exponential decay holds, it is reasonable (if ultimately unjustified) to think of \( \dot{\hat{w}}(t) \) as a probability density, but outside the exponential regime this interpretation cannot be trusted. In fact, if one views \( \dot{\hat{w}}(t) \) as a functional of the initial state \( \rho \), it is not a linear functional; hence, convex combinations of initial density matrices do not lead to convex combinations of ‘densities’ \( \dot{\hat{w}}(t) \). This is a serious problem because according to the usual interpretations of the quantum state, a convex combination of initial states can be achieved by joining different statistical ensembles.

Another problem with the probabilities defined through the survival amplitude is that they lack clear interpretation in terms of quantum measurement theory. The ‘decay probability density’ \( \dot{\hat{w}}(t) \) is obtained formally as the expectation value of the operator \(-i[\hat{H}, \hat{Q}(t)]\); there is no clear procedure through which this can be directly measured. Moreover, the initial state (hence also \( \hat{Q} \)) is often unknown. In general, there is no immediate relation between \( \dot{\hat{w}}(t) \) and concrete measurement procedures performed on a quantum system.

### 1.3 Our results

In the operational approach we follow here, the problems characterizing the survival probability method do not arise. The probability density \( p(t) \) is genuine and the corresponding POVM refers to a concrete procedure for the measurement of the decay probability: we measure the time-of-arrival of the emitted particles, and we construct a probability density that refers to an ensemble of decaying quantum systems (e.g. nuclei). This method allows for no interpretational ambiguities: the probabilities we employ in this paper are soundly tied to the statistics of detection outcomes.

We find that exponential decay is generic, and we identify a primary physical reason for its validity. In effect, the exponential decay (at least in tunneling) arises when a specific condition holds (this was first assumed by Gamow \[14\] and by Gurney with Condon \[15\] in their classic studies of alpha decay). In semiclassical language, this condition is that the different attempts of the bound particle to cross the barrier are statistically independent, or in other words, that there is neither interference nor memory effects in the probability distribution.
The exponential decay is then a sign of a ‘quasi-classical’ (and Markovian) behavior of the bound particle.

A second condition is that the decay time should be substantially larger than the characteristic microscopic time scales associated to tunneling: if this condition does not hold, the decay probability exhibits a fine structure that is very sensitive to minor features of the initial state. In effect, the validity of the exponential decay law requires a separation of timescales, a condition similar to the one for the Markovian behavior in open systems. In fact, in absence of such a separation of time scales, it is questionable even if the word "decay" is suitable for the description of the phenomenon. A third condition is that there should be no coherence between decays characterized by substantially different characteristic timescales. We also show that the exponential phase of the decay has a finite duration, and we find the asymptotic behavior of the detection probability as $t \to \infty$ (inverse power law).

Our method provides a full characterization of the possible decay laws for tunneling systems, at least for the class of potentials we study here. (The choice of studied systems is guided by our desire to obtain analytic expressions for all relevant quantities. However, the POVM we employ is defined for a generic Hamiltonian: hence, the domain of applicability of the method is larger.) We find that there are specific regimes in many systems, in which the decays have no exponential phase. These regimes are identified by conditions on the transmission and reflection amplitude of the barrier at the relevant timescales.

The structure of this paper is the following. In Sec.2, we briefly review the basic object in the formalism, namely the POVM constructed in [1, 2]. In Sec.3, we present an informal argument about the assumptions involved in the derivation of the exponential decay law and the physical conditions that these necessitate. In Sec. 4, we construct explicitly the detection probability for a state decaying through tunneling: we use a simple model of a particle in the half-line, bounded close to $r = 0$ by a potential barrier $V(r)$. In Sec. 5, we identify the regime of exponential decay and we analyze the relevant conditions. In Sec.6, we study the deviations from exponential decay at long times and regimes for which the decays have no exponential phase. In Sec. 7, we compare our results to the ones obtained from the survival amplitude for the same systems and in Sec. 8, we conclude.

## 2 Summary of the formalism

The POVM defined in [1, 2] refers to the following situation. (We consider the case of a particle in one-dimension for concreteness.) An ensemble of particles is prepared in a state described by a wave function $\psi_0(x)$, which has support on values $x < L$. The Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x})$, where $M$ is the particle’s mass. At $x = L$ a detector is located, which register particles and records the time $t$ of this recording. The distribution of these times-of-arrival is given by a
probability distribution

\[ p(t) = Tr \left( \hat{\rho}_0 \hat{\Pi}(t) \right), \]  

(2.1)

where \( \hat{\rho}_0 = |\psi_0\rangle\langle \psi_0| \). The POVM is defined through the operators \( \hat{\Pi}(t) \), together with the operator \( \hat{\Pi}(N) = 1 - \int_0^\infty dt \hat{\Pi}(t) \), which corresponds to the event of no detection. The sample space for this POVM is therefore \([0, \infty) \times \{N\}\).

The POVM \( \hat{\Pi} \) involves in its definition a smearing function \( f^\tau(t) \), which determines the response of the detector; \( \tau \) is the characteristic response time.

For the physically relevant class of initial states that have support to values of energy \( E \) such that \( E \tau > 1 \), the POVM becomes \( \tau \)-independent. We proceed to describe its structure.

For the calculation of the probability density (2.1), it is necessary to find the spectrum of the Hamiltonian operator \( \hat{H} \) with Dirichlet boundary conditions at \( x = L \). We assume that the potential is short range, so that it vanishes around the neighborhood of \( x = L \). (In fact, it is only different from zero in a microscopic scale, while \( L \) is a macroscopic distance. We distinguish two cases: (i) if \( x \) takes value in the half-line, the spectrum of \( \hat{H}_D \) is expected to be discrete (this is the case relevant to this paper); (ii) if \( x \) takes values in the full real axis, at least the positive energy spectrum will be continuous. Either way, for \( x >> a \), \( V(x) = 0 \) and the solution of the Schrödinger equation \( \hat{H}_D \psi_E(x) = E \psi_E(x) \) with Dirichlet boundary conditions is proportional to \( \sin k(L - x) \), where \( k = (2ME)^{1/2} \). We choose to label the eigenstates of \( \hat{H}_D \) by \( k \), namely we write \( |k\rangle_D \) as a solution to the equation

\[ \hat{H} |k\rangle_D = \frac{k^2}{2M} |k\rangle_D, \]  

(2.2)

with Dirichlet boundary conditions.

Normalizing \( |k\rangle_D \) so that

\[ D \langle k|k'\rangle_D = \delta(k, k'), \]  

(2.3)

(and similarly in the discrete-spectrum case) we write

\[ \langle x|k\rangle_D = D_k \sin k(L - x), \]  

(2.4)

where the form of the normalization factor \( D_k \) is specified the Hamiltonian’s (generalized) eigenstates.

The probability distribution (2.1) is expressed as [see Sec.2 in Ref. [2]]

\[ p(t) = \frac{1}{2\sqrt{2M}} \sum_{kk'} D_k D_{k'} c_k c_{k'} \frac{k k'}{\sqrt{k^2 + k'^2}} e^{-i \frac{k^2 - k'^2}{2M} t}, \]  

(2.5)

where \( c_k = D \langle k|\psi_0\rangle \) and \( \sum_k \) denotes the integration with respect to the spectral measure of \( \hat{H}_D \). The probability for the time-of-arrival is expressed solely in terms of the system’s Hamiltonian, the initial state and the value of \( L \).
Eq. (2.5) is simplified if the spread $\Delta k$ of the initial state $|\psi_0\rangle$ ($\hat{k} = \sqrt{2M\hat{H}_D}$) is much smaller than the corresponding mean value $\bar{k}$: in this case, $k^2 + k'^2 \approx 2kk'$, hence

$$p(t) = \sum_k D_k c_k \sqrt{\frac{k}{4M}} e^{-ik^2 t/2M} \right|^2. \quad (2.6)$$

3 The origin of exponential decay

For the study of the decay probability undertaken in this paper, we will assume a particle in the half line, described by a wave function $\psi(r)$, $r \in (0, \infty)$. The particle is initially in region I ($0 < r < a$): the potential there can be in general attractive, even though for simplicity we consider the case that $V(r) = 0$. In region II ($a \leq r \leq b$), the potential $V(r)$ is repulsive and in region III ($r > b$), it vanishes.

Systems such as the above are described by an exponential law for the decay probability. However, it is well known that the exponential decay does not hold at all times: at short time we have a Zeno-type behavior and at very long times an inverse power fall-off. Our aim is to investigate the way the exponential decay law appears in such systems and the characteristic time-scales for its validity. Moreover, we would like to identify any conditions that lead to significant divergence from the exponential decay law.

For this purpose, before we explicitly construct the probability distribution (2.5) for this class of systems, we provide a simple argument (using the results of section 3) that the exponential behavior is generic\(^1\). We identify the main physical assumption underlying this argument, and we then examine whether it is consistent with the results arising from the evaluation of $p(t)$ in the present context.

We consider the potential described earlier with $V(r) = 0$ in the region I ($r \in [0, a]$). For a wave-function with momentum $k_0$ with momentum spread $\sigma$, it was shown in [2] that the probability of detection at distance $L$ at time $t$ is given by

$$p_0(t) \sim \sqrt{\frac{1}{1 + 4t^2\sigma^4/M^2}} \exp \left\{ -\frac{2k_0^2\sigma^2/M^2}{1 + 4t^2\sigma^4/M^2} \left[ t - \frac{M(L + \lambda_{k_0})}{k_0} \right]^2 \right\}, \quad (3.1)$$

where

$$\lambda_{k_0} = \frac{M}{k_0} \text{Im} \left( \frac{\partial \log T_k}{\partial k} \right)_{k=k_0}, \quad (3.2)$$

with $T_k$ the transmission amplitude of the potential for energy $\frac{k^2}{2M}$. (We ignored a small term in the exponential that corresponds to the center of the initial wave-function.)

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\(^1\)This is a variation of the classic argument that Gamow [14] and Gurney with Condon [15] put forward in their explanations of the alpha decay.
We assume that the initial state is localized in region I. Let \(|R_{k_0}|^2\) be the reflection probability on the barrier. A fraction \(|R_{k_0}|^2\) of the particles in the ensemble will be reflected. Since the particle is located in a bounded region (it is reflected at \(r = 0\)), it will attempt to cross the barrier: in a semi-classical approximation, this attempt will take place after time \(T = \frac{2Ma}{k_0}\), and the fraction of the particles that succeed will provide a contribution \(p_1\) to the probability of detection

\[
p_1(t) \sim |R_{k_0}|^2 \sqrt{\frac{1}{1 + 4t^2\sigma^4/M^2}} \exp \left\{-\frac{2k_0^2\sigma^2/M^2}{1 + 4t^2\sigma^4/M^2} \left[ t - \frac{M(L + \lambda_{k_0} + 2n\alpha)}{k_0} \right]^2 \right\} (3.3)
\]

where the coefficient of proportionality is the same for \(p_1\) and \(p_0\). Following the same argument for the multiply reflected particle, we obtain the total probability of detection

\[
p(t) \sim \sum_{n=0}^{\infty} |R_{k_0}|^{2n} \sqrt{\frac{1}{1 + 4t^2\sigma^4/M^2}} \exp \left\{-\frac{2k_0^2\sigma^2/M^2}{1 + 4t^2\sigma^4/M^2} \left[ t - \frac{M(L + \lambda_{k_0} + 2n\alpha)}{k_0} \right]^2 \right\} (3.4)
\]

For times \(t\) such that \(t\sigma^2/M << 1\), the expression above simplifies

\[
p(t) \sim \sum_{n=0}^{\infty} |R_{k_0}|^{2n} \exp \left\{-\frac{2k_0^2\sigma^2}{M^2} \left[ \frac{M(L + \lambda_{k_0} + 2n\alpha)}{k_0} \right]^2 \right\}. (3.5)
\]

The behavior of this function is the following: until the time of first detection \(t_0 = \frac{M(L + \lambda_{k_0})}{k_0}\), \(p(t)\) is practically zero; then it exhibits successive sharp peaks of width \(M/(\sigma k_0)\) separated by a time interval \(2Ma/k_0\). The height of the \(n\)-th peak is smaller than the height of the \((n-1)\)-th peak by a factor of \(|R_{k_0}|^2\). In effect, at time \(t = t_0 + 2nMa/k_0\) \((n\) integer) the height of the peak will be proportional to \(|R_{k_0}|^{2n}\). If the resolution of our time measurements is coarser than the interval \(2Ma/k_0\) between successive peaks, we can effectively substitute the probability by the curve connecting the peaks. Hence, for \(t > t_0\)

\[
p(t) \sim |R_{k_0}| \frac{k_0^2(t-t_0)}{2Ma} = e^{-\log |R_{k_0}|^2 \frac{k_0}{2Ma}(t-t_0)}. (3.6)
\]

We therefore have an exponential decay with decay coefficient \(\Gamma\)

\[
\Gamma = \frac{k_0}{2Ma} \log \left| |R_{k_0}|^2 \right|. (3.7)
\]

Note that \(|R_{k_0}|^2 = 1 - |T_{k_0}|^2\), where \(|T_{k_0}|^2\) is the transmission probability; if \(|T_{k_0}|^2 << 1\) then \(\log |R_{k_0}|^2 \simeq |T_{k_0}|^2\) and \(\Gamma = \frac{k_0}{2Ma}|T_{k_0}|^2\). However, as time increases and \(t\sigma^2/M\) becomes of order 1, there are no clear peaks in \(p(t)\) any more and the approximation by the exponential slowly worsens.

Clearly, the description above is oversimplified. Its key assumption is that the successive attempts of the particle to cross the barrier are statistically independent, i.e. the \(n\)-th attempt has no memory of the \((n-1)\)-th attempt:
the probability densities \( p_n \) can then be added. This is essentially an assumption of Markovian behavior, which is a fundamental property of the exponential decay law. However, in quantum theory one does not add probabilities, but *amplitudes*. Hence, one expects that in the most general case, there will be interference terms between the different attempts of the particle to cross the barrier: these may spoil the exponential decay law. A full treatment should focus on the size and contribution of these off-diagonal terms to the total probability. We next proceed to do this.

### 4 Evaluating the decay probability

We assume a potential \( V(r) \) as described in Sec. 3, for \( r \in [0, \infty) \): \( V(r) = 0 \) in the regions \([0, a] \) and \([b, \infty) \).

It is convenient to express the mode solutions \( u_k \) of Schrödinger’s equation

\[
-\frac{1}{2M}\partial^2_r u(r) + V(r)u(r) = \frac{k^2}{2M}u(r).
\]

in terms of the reflection and transmission amplitudes of the same potential, which are defined when the variable \( r \) extends from \(-\infty \) to \( \infty \). Let \( T_k \) and \( R_k \) be the transmission and reflection amplitude for particles coming from the left and \( T'_k \) and \( R'_k \) the same amplitudes for particles coming from the right. These coefficients satisfy the conditions

\[
T_k = T'_k, \quad |R_k| = |R'_k|, \quad T_k R_k = T'_k R'_k, \tag{4. 2}
\]

which arise from the property of the Schrödinger operator that the Wronskian of two eigenfunctions with the same energy must be a constant functions.

The mode functions \( u_k(r) \) corresponding to Dirichlet boundary conditions at \( r = 0 \) are then obtained (using a multiple scattering method)

\[
 u_k(r) = \begin{cases} 
 -2i \frac{T_k}{1+R_k} \sin kr & \text{in reg. I} \\
 e^{-ikr} + \left( R'_k - \frac{T^2_k}{1+R'_k} \right) e^{ikr} & \text{in reg. III} 
\end{cases} \tag{4. 3}
\]

Using Eqs. 4.2, we obtain \(|R'_k - \frac{T^2_k}{1+R'_k}| = 1\). Hence, we can write the mode function \( u_k(r) \) in region III as \( e^{-ikr} - e^{i\Theta_k} e^{ikr} \), where

\[
 e^{i\Theta_k} := -(R'_k - \frac{T^2_k}{1+R'_k}). \tag{4. 4}
\]

Imposing the Dirichlet boundary condition at \( r = L \) yields the eigenvalue equation

\[
e^{2ikL + \Theta_k} = 1, \tag{4. 5}
\]

with solutions \( k_n \) that satisfy the set of algebraic equations

\[
k_n = \frac{n\pi}{L} - \frac{\Theta_k}{2L}. \tag{4. 6}
\]
for all integers \( n \) that lead to positive value of \( k_n \). For \( V(r) = 0 \), we obtain \( k_n = n\pi/L, n = 1, 2, \ldots \).

The eigenstates \( u_{k_n}(x) \) of the Hamiltonian with Dirichlet boundary conditions are then

\[
u_{k_n}(x) = A_{k_n} \times \begin{cases} -2i \frac{T_{k_n}}{1 + R_{k_n}} \sin k_n r & \text{in reg. I} \\ 2i e^{-ik_n L} \sin k_n (L - r) & \text{in reg. III} \end{cases}
\]

(4.7)

where \( A_{k_n} \) is a normalization factor, chosen so that \( \int_0^L dx u^*_{k_n}(x) u_{k_n}(x) = \delta_{mn} \).

We immediately read the coefficient

\[
D_{k_n} = 2i A_{k_n} e^{-ik_n L} = 2i A_{k_n} e^{i\Theta_{k_n}/2}.
\]

(4.8)

We now choose an initial state: it should be concentrated in region I (i.e. we assume that no element of the ensemble has decayed at \( t = 0 \)) and it should have mean energy \( E = k_0^2/2M \), such that \( k_0 >> \sigma \), where \( \sigma \) is the momentum spread. For ease of calculation, we employ a Gaussian

\[
\psi_0(r) = \frac{1}{(2\pi \delta^2)^{1/4}} e^{-(r-a)^2/4\delta^2 + ik_0 r}.
\]

(4.9)

If \( e^{-\frac{a^2}{4\delta^2}} \ll 1 \) then this respects with good approximation the Dirichlet boundary condition at \( r = 0 \). We could have used a different state (exactly vanishing at \( r = 0 \)), but it turns out the precise form of the state makes only difference to fine details of the probability distribution and not to the basic features of the decay process\(^2\).

We then compute (with an error of order \( e^{-k_0^2/\sigma^2} \))

\[
c_{k_n} = -\bar{A}_{k_n} \frac{T_{k_n}}{1 + R_{k_n}} \left( \frac{4\pi}{\sigma^2} \right)^{1/4} e^{-\frac{(k_n-k_0)^2}{2\sigma^2} - i(k_0-k_0)\alpha/2},
\]

(4.10)

where \( \sigma = 1/(\sqrt{2}\delta) \). The overbar denotes complex conjugation.

Hence, the probability of detection is given by \( p(t) = |z(t)|^2 \), where

\[
z(t) = -\left( \frac{4\pi}{\sigma^2} \right)^{1/4} \sum_n \sqrt{\frac{k_n}{4M}} 2i|A_{k_n}|^2 e^{i\Theta_{k_n}/2} \frac{T_{k_n}}{1 + R_{k_n}} \times e^{-\frac{(k_n-k_0)^2}{2\sigma^2} - i(k_n-k_0)\alpha/2 - i k_0^2 t/2M}.
\]

(4.11)

Using the definition (4.4) and the identities (4.2) we find that

\[
e^{i\Theta_k} \frac{T_k}{1 + R_k} = \frac{T_{k_n}}{1 + R_{k_n}}.
\]

(4.12)

\(^2\)We have tried different initial states with the same values for \( k_0 \) and \( \sigma \) and the basic features of the decay remain unchanged.
Hence,

\[
z(t) = -2i \frac{\pi^{1/4}}{\sqrt{2M\sigma}} \sum_n |A_{k_n}|^2 e^{ik_n L} \sqrt{k_n} \frac{T_{k_n}}{1 + R_{k_n}} \times e^{-\frac{(k_n-k_0)^2}{2\sigma^2}} - i(k_n-k_0)a/2 - i\frac{k_n^2t}{2\sigma^2}. \tag{4.13}
\]

The distance \(\delta k_n\) between two neighboring eigenvalues \(k_n\) and \(k_{n-1}\) satisfies the equation

\[
\delta k_n = \frac{\pi}{L} - \Theta_{k_n} - \Theta_{k_{n-1}} \sim \frac{\pi}{L} + \frac{\Theta'_{k_n} \delta k_n}{2L}, \tag{4.14}
\]

where \(\Theta'_{k_n}\) is the derivative of \(\Theta_k\). We then obtain

\[
\delta k_n \sim \frac{\pi}{L}, \tag{4.15}
\]

Hence, we obtain the following expression for \(z(t)\)

\[
z(t) = -2i \frac{\pi^{1/4}}{\sqrt{2M\sigma}} \sum_n |A_{k_n}|^2 e^{ik_n L} \sqrt{k_n} \frac{T_{k_n}}{1 + R_{k_n}} \times e^{-\frac{(k_n-k_0)^2}{2\sigma^2}} - i(k_n-k_0)a/2 - i\frac{k_n^2t}{2\sigma^2}. \tag{4.16}
\]

Noting that \((1 + R_k)^{-1} = \sum_{n=0}^{\infty} (-R_k)^n\), we expand the corresponding terms around \(k = k_0\) and keep only the leading terms. Thus, we write

\[
\sqrt{kT_k} R_k^n \sim \sqrt{k_0 T_{k_0}} R_{k_0}^n e^{i(\lambda_{k_0} + n\beta_{k_0})(k-k_0) + (\xi_{k_0} + ns_{k_0})(k-k_0)}, \tag{4.17}
\]

where we defined

\[
\lambda_{k_0} := \text{Im} \left( \frac{\partial \log T_k}{\partial k} \right)_{k=k_0} \tag{4.18}
\]

\[
\xi_{k_0} := \frac{1}{2k_0} + \text{Re} \left( \frac{\partial \log T_k}{\partial k} \right)_{k=k_0} \tag{4.19}
\]

\[
\beta_{k_0} := \text{Im} \left( \frac{\partial \log R_k}{\partial k} \right)_{k=k_0} \tag{4.20}
\]

\[
s_{k_0} := \text{Re} \left( \frac{\partial \log R_k}{\partial k} \right)_{k=k_0}. \tag{4.21}
\]

Then \(z(t)\) takes the form

\[
z(t) = -2i \frac{\pi^{1/4}}{\sqrt{2M\sigma}} \sum_{n=0}^{\infty} T_{k_0} (-R_{k_0})^n \int_{-\infty}^{\infty} dk e^{ikL} e^{i(\lambda_{k_0} + n\beta_{k_0})(k-k_0) + (\xi_{k_0} + ns_{k_0})(k-k_0)} \times e^{-\frac{(k-k_0)^2}{2\sigma^2}} - i(k-k_0)a/2 - i\frac{k^2t}{2\sigma^2}. \tag{4.22}
\]
Evaluating the Gaussian integral we arrive at the following expression

\[ z(t) = -\frac{i}{\pi ^{1/4}}T_{k_0} \sqrt{\frac{k_0}{M\sigma(1/\sigma^2 + it/M)}} e^{ik_0 L - i\frac{k_0^2 t}{2\sigma^2}} \]

\[ \times \sum_{n=0}^{\infty} (-R_{k_0})^n \exp \left( \frac{[(\xi_{k_0} + ns_{k_0}) + i(L - a/2 + \lambda_{k_0} + n\beta_{k_0} - k_0 t/M)]^2}{2(1/\sigma^2 + it/M)} \right) \] (4.23)

The probability of detection is then

\[ p(t) = \frac{1}{\sqrt{\pi}} \frac{k_0}{M\sigma\sqrt{1/\sigma^2 + t^2/M^2}} |T_{k_0}|^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} R_{k_0}^n \bar{R}_{k_0}^m A_{nm}(t), (4.24) \]

where

\[ A_{nm}(t) = \exp \left[ \frac{[(\xi_{k_0} + ns_{k_0}) + i(L - a/2 + \lambda_{k_0} + n\beta_{k_0} - k_0 t/M)]^2}{2(1/\sigma^2 + it/M)} \right. \]

\[ + \frac{[(\xi_{k_0} + ns_{k_0}) - i(L - a/2 + \lambda_{k_0} + m\beta_{k_0} - k_0 t/M)]^2}{2(1/\sigma^2 - it/M)} \] (4.25)

5 The regime for exponential decay

5.1 Derivation

Eq. (4.24) was obtained for a physically reasonable choice of initial state with the additional assumption that its momentum spread is small enough so that the expansion (4.17) provides a good approximation. Recalling the discussion of Sec. 4.1, we see that the probability density (4.24) exhibits interference between the different attempts of the particle to cross the barrier (labeled by \( n \) and \( m \)), and for this reason it cannot give rise to an exponential decay law.

We next identify the regime in which Eq. (4.24) leads to exponential decay. We expect that exponential decay arises only for times such that \( t\sigma^2/M \ll 1 \), because at later times the peaks in the detection probability start deteriorating and the asymptotic long-time behavior sets-in. Note that in the present model we assumed that \( V(r) = 0 \) in region I: the more physical case of a strongly attracting potential would lead to a substantially smaller increase of the particle’s wave function spread for the time it spends in region I, and the relevant time scale could be significantly larger than \( M/\sigma^2 \).

Assuming that \( t \ll M/\sigma^2 \)

\[ A_{nm}(t) = \exp \left\{ -\sigma^2[L + \lambda_{k_0} + \frac{n+m}{2}\beta_{k_0} - k_0 t/M]^2 \right. \]

\[ + \sigma^2(\xi_{k_0} + \frac{n+m}{2}s_{k_0})^2 - \sigma^2(\beta_{k_0} - s_{k_0})^2 \]

\[ + i\sigma^2(n-m)[\xi_{k_0}\beta_{k_0} + s_{k_0}(L + \lambda_{k_0} - k_0 t/M) + (n+m)s_{k_0}\beta_{k_0}] \} \] (5.1)

\footnote{Note that in the context of probabilities for time in quantum theory, quantum coherence is essentially identical with memory effects.}
In the expressions above, we substituted $L - a/2$ (the distance of the detector from the center of the initial state) with $L$, since $L > a$. We see that $A_{nm}(t)$ has strong peaks at times $t = \frac{M(L + \lambda_{\text{km}} + \frac{a + m}{2} - \beta_{\text{km}})}{k_{\text{km}}}$, i.e. at multiples of $\beta_{\text{km}}$ after the time $t_0 = M(L + \lambda_{\text{km}})/k_{\text{km}}$ of first detection. There are also oscillating terms proportional to $(n - m)$, hence there are no sharp instants of detection. However, large differences in value between $n$ and $m$ are suppressed by a term $e^{-\sigma^2(n-m)^2(\beta_{\text{km}}^2 - s_{\text{km}}^2)}$. Hence, if
\[ e^{\sigma^2(\beta_{\text{km}}^2 - s_{\text{km}}^2)} \gg 1 \] the values of $A_{nm}$ for $n \neq m$ are very small for all times $t$. Effectively,
\[ A_{nm}(t) = \delta_{nm} e^{-\sigma^2[L + \lambda_{\text{km}} + n\beta - k_{\text{km}}t/M]^2 + \sigma^2(\xi_{\text{km}} + ns_{\text{km}})^2}. \] (5.3)
Substituting into Eq. (4.24), we obtain
\[ p(t) = \frac{k_{\text{km}}\sigma}{\sqrt{\pi}M}|T_{k_{\text{km}}}|^2 \sum_{n=0}^{\infty} |R_{k_{\text{km}}}|^2 e^{-\sigma^2[L + \lambda_{\text{km}} + n\beta - k_{\text{km}}t/M]^2 + \sigma^2(\xi_{\text{km}} + ns_{\text{km}})^2}. \] (5.4)
This probability distribution behaves as follows. For $t < t_0$, $p(t) \approx 0$. At $t = t_0$ there is a peak of width $M/k_{\text{km}}\sigma$ (corresponding to first detection), and then there are successive peaks of the same width centered around $t_n = t_0 + nM\beta_{\text{km}}/k_{\text{km}}$ with an amplitude differing by a factor of $|R_{k_{\text{km}}}|^2 e^{\sigma^2(\xi_{\text{km}} + ns_{\text{km}})^2}$.

This is similar to the semi-classical description of Sec. 3, namely that the peak at time $t_n$ corresponds to the $(n+1)$-th attempt of the particle to cross the barrier. The interpretation of the quantity $\beta_{\text{km}}$ is simpler if we assume that the potential $V(r)$ is parity-symmetric, or more precisely if $V'(x) := V(x - a - \frac{d}{2})$ for $d = b - a$ satisfies $V'(x) = V'(-x)$. For $V'(x)$, $R_k = R_k'$ and Eqs. (4.24) imply that
\[ \text{Arg}R_k = \frac{\pi}{2} + \text{Arg}T_k + 2ka + kd, \] (5.5)
so that $\beta_{\text{km}} = \lambda_{\text{km}} + 2a + d$. Hence, the distance between two successive peaks in $p(t)$ equals
\[ \frac{M\beta_{\text{km}}}{k_{\text{km}}} = t_{\text{cross}} + \frac{2M\alpha}{k_{\text{km}}}, \] (5.6)
where $t_{\text{cross}} = \frac{M}{k_{\text{km}}} \left( d + \left( \frac{\partial E}{\partial k_{\text{km}}} \right)_{k_{\text{km}} = \beta_{\text{km}}} \right)$ is the time it takes a particle to cross the barrier region. Hence, $M\beta_{\text{km}}/k_{\text{km}}$ is the sum of the time it takes a classical free particle between two attempts to cross the barrier plus the time for crossing the barrier.

\footnote{As shown in \cite{2}, $\frac{M\lambda_{\text{km}}}{k_{\text{km}}}$ is the delay time for the recording of the particles, which is due to the presence of the barrier.}
However, Eq. (5.4) does not correspond to exponential decay: the ratio of the amplitude of two successive peaks is not constant due to the presence of the term $e^{\sigma^2 (\xi_{k_0} + ns_{k_0})^2}$. We assume that the contribution of this term is much smaller than that of $|R_{k_0}|^{2n}$, i.e. we consider values of $n$ such that

$$\sigma^2 (\xi_{k_0} + ns_{k_0})^2 << n \log |R_{k_0}|^2. \quad (5.7)$$

In this regime,

$$p(t) = \frac{k_0 \sigma}{\sqrt{\pi M}} |T_{k_0}|^2 \sum_{n=0}^{\infty} |R_{k_0}|^{2n} e^{-\sigma^2 [L + \lambda_{k_0} + n\beta_{k_0} - k_0 t/M]^2} \quad (5.8)$$

The arguments of Sec. 4.1 can now be used in a straightforward manner. If the temporal resolution of measurements is larger than the distance between successive peaks, then for $t > t_0$, we can substitute $p(t)$ by the curve connecting these peaks. However, we have to preserve the normalization. The probability corresponding to each Gaussian in the sum (which equals $\sqrt{\pi M} \sigma/k_0$) must be spread within an interval of width $M\beta_{k_0}/k_0$ (i.e. the distance between two peaks.) This yields (for $t > t_0$)

$$p(t) \approx \frac{k_0}{M\beta_{k_0}} |T_{k_0}|^2 |R_{k_0}|^2 e^{-\sigma^2 |L + \lambda_{k_0} + n\beta_{k_0} - k_0 t/M|^2} \quad (5.9)$$

Since we assumed $|T_{k_0}|^2 << 1$, the expression above becomes

$$p(t) = \Gamma e^{-\Gamma(t-t_0)} = -\frac{d}{dt} e^{-\Gamma(t-t_0)} \quad (5.10)$$

where $\Gamma$ is the decay rate

$$\Gamma = \frac{k_0}{M\beta_{k_0}} |T_{k_0}|^2. \quad (5.11)$$

We thus obtained the standard expression for the exponential decay law.

Note that $t_d = M\beta_{k_0}/k_0$ is the distance between two successive peaks and that $\Gamma t_d = |T_{k_0}|^2 << 1$. The characteristic time-scale $\Gamma^{-1}$ associated to the decay must be much larger than the distance between successive peaks for the substitution of (5.8) with (5.9) to make sense: monitoring the detection at time scales of order $\Gamma^{-1}$ should not allow one to distinguish the fine structure of the probability distribution (5.8).

### 5.2 The conditions for exponential decay

We now examine in more detail the conditions that are necessary for the derivation of the exponential decay law. Most important amongst them is the fact that the transmission probability $|T_{k_0}|^2$ must be much smaller than unity. Otherwise, there is no regime for exponential decay. This is the case for example, if the particle’s energy is close to the peak of the potential barrier. In effect, this
Figure 1: The probability distribution of the time-of-arrival in the exponential decay regime. The exponential curve is the envelope of the distribution’s curve, and it has non-overlapping successive peaks at a scale of \( M \beta_{k_0} / k_0 \).

The condition involves a separation of time-scales: the typical decay time should be much larger than the characteristic time between two different attempts of the particle to cross the barrier. In open quantum systems, this separation of time-scales is a necessary condition for Markovian behavior—essentially implying that the memory effects can be ignored in the derivation of the evolution law. The context is different here, but the principle remains the same.

The condition (5.7) provides an upper time-limit for the validity of exponential decay. We first note that the definitions (4.19) and (4.21) together with the identities (4.2) we obtain

\[
s_{k_0} = - (\xi_{k_0} - \frac{1}{2k_0}) \frac{|T_{k_0}|^2}{|R_{k_0}|^2}. \tag{5.12}
\]

Since by (4.19) \( \xi_{k_0} > \frac{1}{2k_0} \) \((|T_k|\text{ increases with } k)\)

\[
\frac{|s_{k_0}|}{|\xi_{k_0}|} < \frac{|T_{k_0}|^2}{|R_{k_0}|^2}. \tag{5.13}
\]

Hence, for \( n \) less or of the order of \( |\xi_{k_0}/s_{k_0}| \), the left-hand-side of the inequality is \( n \)-independent and of order \( \sigma^2 \xi_{k_0}^2 \). Hence, the factor \( e^{\sigma^2 (\xi_{k_0} + ns_{k_0})^2} \) does not affect significantly the exponential decay law—at most by a multiplicative factor \( e^{\sigma^2 r_{k_0}} \). However, for larger values of \( n \) (time increasing) the term \( n^2 \sigma^2 s_{k_0}^2 \) dominates the left-hand side and the condition (5.7) becomes \( n << \frac{|T_{k_0}|^2}{\sigma^2 s_{k_0}^2} \), or
in terms of time $t$

$$t - t_0 << \frac{M\beta_{k_0}|T_{k_0}|^2}{k_0\sigma^2 s_{k_0}^2}. \quad (5.14)$$

Substituting $s_{k_0}$ from Eq. (5.12) and using Eq. (5.11) we obtain (for $|T_{k_0}|^2 << 1$)

$$\Gamma(t - t_0) << \left[\sigma^2 w_{k_0}^2\right]^{-1}. \quad (5.15)$$

where we wrote $w_{k_0} = \left(\frac{\partial \log |T_k|}{\partial k}\right)_{k=k_0}$.

Hence, for the exponential decay law to be valid for times much larger than the decay time $\Gamma^{-1}$, it is necessary that

$$\sigma \left(\frac{\partial \log |T_k|}{\partial k}\right)_{k=k_0} << 1, \quad (5.16)$$

or, defining the variation in transmission probability according to the momentum spread $\sigma$ of the initial state $\Delta|T_{k_0}| := \sigma \left(\frac{\partial |T_k|}{\partial k}\right)_{k=k_0}$,

$$\Delta|T_{k_0}| << |T_{k_0}|. \quad (5.17)$$

In effect, the interference between decay channels characterized by different decay coefficient must be negligible, in order to obtain an exponential decay law. Moreover, the time-scale $\Gamma^{-1}|T_{k_0}|^2/(\Delta|T_{k_0}|)^2$ characterizes the breakdown of the exponential decay law.

We finally examine condition (5.2). Since, for $|T_{k_0}| << 1, |s_{k_0}| \simeq |T_{k_0}|^2 |w_{k_0}|$, (5.16) implies that the condition (5.2) can only be satisfied if

$$\sigma \beta_{k_0} >> 1. \quad (5.18)$$

This means that the position spread $\sigma^{-1}$ of the initial state must be substantially smaller than the effective distance traversed by the particle between an unsuccessful attempt to cross the barrier and a successful one. If this does not hold, then there is quantum interference between the different attempts and the off-diagonal elements of $A_{mn}(t)$ will not be suppressed. An initial state with a spread equal to $a$ can only give rise to an exponential decay law if the effective tunneling length is very large.

We see therefore that the exponential decay law involves a specific intermediate regime for the characteristic features of the initial state. The position spread must be sufficiently small so that there is no interference between different attempts to cross the barrier, but it cannot be too small because the variation $\Delta|T_{k_0}|$ will become too large and create interference of decays with different characteristic time-scales.

We also note that the conditions (5.16) and (5.18) can be satisfied simultaneously only if

$$\Im \left(\frac{\partial \log R_k}{\partial k}\right)_{k=k_0} >> \Re \left(\frac{\partial \log T_k}{\partial k}\right)_{k=k_0}. \quad (5.19)$$
This is a necessary condition that any potential has to satisfy irrespective of the choice of the initial state, if the decay probability is to be characterized by an exponential regime.

To summarize, there are three conditions on the potential and on the initial state that have to be satisfied, for a regime of an exponential decay to exist.

1. \(|T_{k_0}|^2 << 1\). This is necessary for the clear identification of the decay behavior over the fine structure exhibited by \(p(t)\) at the time scale of \(M\beta_{k_0}/k_0\).

2. \(\sigma\beta_{k_0} >> 1\). This is sufficient for the suppression of interferences between different crossing attempts and the validity of the semiclassical picture of Sec. 4.1.

3. \(\sigma\omega_{k_0} << 1\). This guarantees that there is no interference between processes characterized by substantially different decay coefficients.

We also note that Eq. (4. 24), to which the conditions 1-3 above refer, is obtained by keeping only the first order in the expansion for the logarithm of the reflection and transmission amplitudes. We have therefore assumed the following conditions

\[
\sigma \text{Re} \left( \frac{\partial^2 \log T_k}{\partial k^2} \right)_{k=k_0} << w_{k_0} \quad (5. 20)
\]

\[
\sigma \text{Im} \left( \frac{\partial^2 \log T_k}{\partial k^2} \right)_{k=k_0} << \lambda_{k_0} \quad (5. 21)
\]

\[
\sigma \text{Re} \left( \frac{\partial^2 \log R_k}{\partial k^2} \right)_{k=k_0} << s_{k_0} \quad (5. 22)
\]

\[
\sigma \text{Im} \left( \frac{\partial^2 \log R_k}{\partial k^2} \right)_{k=k_0} << \beta_{k_0} \quad (5. 23)
\]

5.3 Special cases

We now examine the regime for exponential decay for the case of a square potential barrier, i.e. if \(V(x) = V_0\) for \(x \in [a, b]\) and \(V(x) = 0\) otherwise. Defining \(\gamma_k = \sqrt{2MV_0 - k^2}\), we obtain the following values for the coefficients \(T_k, R_k\)

\[
T_k = \frac{2k}{\gamma_k} e^{-ikd} \frac{2k\gamma_k [2k\gamma_k \cosh \gamma_k d - i(\gamma_k^2 - k^2) \sinh \gamma_k d]}{4k^2 \gamma_k^2 + (\gamma_k^2 + k^2) \sinh^2 \gamma_k d} \quad (5. 24)
\]

\[
R_k = -ie^{2ika} \frac{(\gamma_k^2 + k^2)[2k\gamma_k \cosh \gamma_k d - i(\gamma_k^2 - k^2) \sinh \gamma_k d]}{4k^2 \gamma_k^2 + (\gamma_k^2 + k^2) \sinh^2 \gamma_k d} \quad (5. 25)
\]

There are two limits, in which the expressions above simplify.

1. The limit of a long barrier \(\gamma_k d >> 1\), for which

\[
T_k \sim e^{-ikd} e^{-\gamma_k d} \frac{4k\gamma_k}{(\gamma_k^2 + k^2)^2} [2k\gamma_k - i(\gamma_k^2 - k^2)] \quad (5. 26)
\]
\[ R_k \approx e^{2ika} \frac{-\left(\frac{\gamma_k^2}{k} - k^2\right) + ik\gamma_k}{4\gamma_k^2} \] (5.27)

At this limit, \(|T_{k_0}| << 1\). We find,
\[ \beta_{k_0} = 2(a + \gamma_{k_0}^{-1}). \] (5.28)

The condition \(\sigma \beta_{k_0} >> 1\) follows by the assumption that \(\sigma a >> 1\).

We also obtain
\[ w_{k_0} \approx k_0^{-1} + 2ik_0 \frac{k_0}{\gamma_{k_0}}. \] (5.29)

Since \(\sigma/k_0 << 1\), the condition \(\sigma w_{k_0} << 1\) can only be satisfied if \(\sigma dk_0/\gamma_{k_0} << 1\). If \(d\) is of the order of \(a\) or smaller, then it is necessary that \(k_0/\gamma_{k_0} << 1\), i.e. that the particle’s energy \(k_0^2/2M\) is much smaller than the barrier’s height \(V_0\).

2. The limit of the delta function (very short) barrier. It is obtained by letting \(V_0 \to \infty\) and \(d \to 0\) such that \(V_0d\) is a constant (we denote this constant as \(\kappa/M\)). At this limit, \(\gamma_kd \approx \sqrt{\kappa d}\) and
\[ T_k = \frac{1}{1 + ik/\kappa} \] (5.30)
\[ R_k = e^{2ika} \frac{1}{1 + ik/\kappa}. \] (5.31)

At this limit, the condition \(|T_{k_0}| << 1\) is satisfied only if \(k_0/\kappa << 1\). In this regime,
\[ \beta_{k_0} = 2a + \frac{1}{\kappa}. \] (5.32)

Again, the condition \(\sigma \beta_{k_0} >> 1\) follows from the assumption that \(\sigma a >> 1\). We also compute
\[ w_{k_0} = k_0^{-1}. \] (5.33)

The condition \(\sigma w_{k_0} << 1\) is therefore satisfied for all values of \(\kappa\).

In both of the examples above, the validity of the conditions 1-3 implies the validity of the conditions (5.20–5.23) on the second derivatives of the transmission and reflection amplitudes.

### 6 Beyond exponential decay

#### 6.1 The long-time limit(s)

Eq. (5.15) implies that there is an upper limit to the duration of the exponential decay. In fact, we can find the effective probability distribution corresponding
to (5.4) by ignoring its fine structure without assuming the condition (5.7). Following the same steps as in the derivation of Eq. (5.10) we obtain (for $|T_{k_0}| << 1$) an equation for $p(t)$ which is valid even for times at which condition (5.15) breaks down

$$p(t) = \Gamma \exp \left[ -\Gamma(t-t_0) + \sigma^2 w_{k_0}^2 \right].$$  \hspace{1cm} (6.1)$$

However, the equation above cannot hold for all times, because it would not define a normalized probability. It is only valid for the regime that characterizes the breakdown of exponential decay. It shows an increase in the detection rate.

There is another limit forced by the eventual spread of the particle’s wave function in region I. In our model, we assumed that $V(r) = 0$ in region I: the time scale in which the wave-packet spread becomes significant is of the order $t \sim M/\sigma^2$. At this time-scale, the fine structure of $p(t)$ becomes blurred. Eq. (4.24) leads to the following asymptotic form for $p(t)$,

$$p(t) = \frac{2k_0|T_{k_0}|^2}{\sigma^2 |1 + R_{k_0}e^{2k_0(\beta_{k_0}-\alpha k_0)}|^2} t^{-1}. \hspace{1cm} (6.2)$$

Eq. (6.2) may be valid for times much larger than $M/\sigma^2$, but it is unacceptable as the limiting behavior for $t \to \infty$: the integral $\int_0^\infty dtp(t)$ diverges logarithmically, while by construction it should take a value less than unity. The reason is easy to identify: Eq. (4.24) involves an expansion of the logarithm of the transmission and reflection amplitudes, where only the first term is kept. However, the probability at the long time limit receives contributions from the *deep infrared* values of $k$, for which $T_k \simeq 0$; the expansion of $\log T_k$ is then inadequate as $\log T_k \to -\infty$. To find the behavior of $T_k$ at this limit we have to go back to a prior stage of our calculation, namely to Eq. (4.16) for $z(t)$. Substituting $k = y/\sqrt{t}$ and taking the dominant terms as $t \to \infty$, we obtain

$$z(t) = \frac{i}{\sqrt{\pi M\sigma}} \int_{-\infty}^{\infty} dy \frac{T_{y/\sqrt{t}}}{1 + R_{y/\sqrt{t}}} e^{-iy^2/2M}. \hspace{1cm} (6.3)$$

As $t \to \infty$, the dominant contribution comes from the values of $T_k$ around $k = 0$. Let $T_k \sim k^\alpha$ as $k \to 0$. Since $|R_{k=0}| = 1$, we see that Eq. (6.3) leads to a behavior $z(t) \sim 1/t^{3/4+\alpha/2}$. Hence, as $t \to \infty$

$$p(t) \sim \frac{1}{t^{3/2+\alpha}}. \hspace{1cm} (6.4)$$

For the square-well potential, $\alpha = 1$, hence $p(t)$ drops asymptotically as $t^{-5/2}$.

### 6.2 Non-exponential decays

We remind that in the derivation of the POVM (2.5) we took the limit that the ‘response time’ $\tau$ of the detector is much larger than energy spread of the
initial state \([2]\). As a result, there is no Zeno-like behavior at early times in
the POVM \((2.5)\): its predictions are only valid for times much larger than \(\tau\).
To study the detection probability at time scales relevant to the Zeno effect we
should employ the probability density \((2.19)\) of Ref. \([2]\) that has an explicit
\(\tau\)-dependence: the temporal resolution of the detector places limits on how
precisely one can identify the Zeno-like behavior.

We now examine the issue whether there exist physical systems with no
exponential decay regime. Clearly, the condition \(|T_{k_0}| << 1\) is crucial. If this
does not hold, the probability distribution exhibits a fine structure due to the
details of the initial state and it is not meaningful to talk about an exponential
regime. The ensemble will decay almost fully after the first few attempts of the
particles to cross the barrier: the relevant time-scale will be microscopic and of
the order of the \(M\beta_{k_0}\). Hence, the condition \(|T_{k_0}| << 1\) is necessary for a decay
characterized by a macroscopically distinguishable time-scale. If this condition
does not hold, the decay law will be highly irregular–such decays were identified
in \([16]\) for the special case of a delta-function barrier.

A question that arises in this context, is whether it is possible to predict
the fine structure (peaks in probability) at the scale of \(M\beta_{k_0}\), say in potentials
for which \(\beta_{k_0}\) takes very large values. We believe that in general it is not: the
reason is that in a realistic preparation of a decaying system we have no control
over the center of the initial wave packet. The only conditions we can safely
identify is its mean energy, the energy mean deviation and the fact that it is
localized in region I. This means that we cannot make a reasonable prediction
about the fine structure of \(p(t)\), which depends on other parameters of the state.
Moreover, our lack of control over the details of the initial state may imply that
the most adequate description may be in terms of a mixed state: we will then
obtain a convex combination of probability distributions of the type \([4, 24]\), in
which case the peaks in the probability distribution will be blurred. Hence, only
the behavior of \(p(t)\) at a sufficiently coarse-grained time scale can be predicted,
because it is insensitive to the details of the initial state.

The condition \(\sigma\beta_{k_0} >> 1\) essentially requires that the wave-function’s po-

tion spread is much smaller than the size \(a\) of region I. This depends on the
physical system under consideration. For example, in alpha-decay the spread
both \(\sigma\) and \(a\) are determined by the specifics of the interaction between the
alpha particle and the other nucleons: they are not free parameters that may be
varied by the experimentalist. However, in condensed matter systems, it might
be possible to control at least the value of \(a\). The violation of this condition
implies that the interference terms between different attempts to cross the bar-
rier become substantial. It should also be noted that \(\sigma/\beta\) is the ratio between
the distance of two successive peaks and the width of each peak. Hence, if this
quantity is of the order of unity the probability distribution is much smoother
at the scale of \(M\beta_{k_0}/k_0\) than it is in the case we considered earlier.

We noted that for the symmetric potentials \(\beta_{k_0}\) is larger than \(2a\). Hence,
\(\sigma\beta_{k_0}\) is in this case larger than \(2\sigma a\). Now \(1/2\sigma < a\) (since the state must be
localized in region I); hence, \(\sigma\beta_{k_0}\) cannot take values lower than unity in these
systems. This implies that to first approximation we can keep only the terms
with $|m-n|=0,1$ in Eq. (5.1). Dropping for simplicity, the $\lambda_k$ term that only changes slightly the value of $t_0$ and taking $|s_k|<<\beta_k$, we obtain the following expression

$$p(t) \simeq \frac{k_0}{\sqrt{\pi M}} |T_{k_0}|^2 \sum_{N=0}^{\infty} |R_{k_0}|^2 e^{-2N\sigma^2(L+N\beta_k-k_0 t/M)+\sigma^2(\xi_k+N s_k)^2} \times \sin^2 \left( \frac{\theta_k}{2} + \frac{\xi_k \beta_k}{2} + \frac{1}{2} s_k (L-k_0 t/M+2N\beta_k) \right), \quad (6.5)$$

where $\theta_k = \text{Arg}R_k$. The distance $M\beta_k/k_0$ between two peaks is now of the order of the peak’s width $\sigma$. At times $t_N = t_0 + N M \beta_k/k_0$, we obtain

$$p(t_N) \simeq \frac{k_0}{M \beta_k} |T_{k_0}|^2 \left( \sum_{N=0}^{\infty} |R_{k_0}|^2 \right) ^2 \frac{e^{-2N\sigma^2(L-N\beta_k-k_0 t_M)/M\beta_k} e^{-\sigma^2(\xi_k+N s_k)(t_N-t_0)/M\beta_k} \times \left( \frac{e^{-\sigma^2 \beta_k^2}}{|R_{k_0}|^2} + \frac{e^{-\sigma^2 \beta_k^2}}{|R_{k_0}|^2} + \frac{1}{|R_{k_0}|^2} \right)^2 + e^{-4\sigma^2 \beta_k^2} |R_{k_0}|^2 + e^{-4\sigma^2 \beta_k^2} |R_{k_0}|^2 + \ldots \right) \sin^2 \left( \frac{\theta_k}{2} + \frac{1}{2} \beta_k (r_{k_0} + M s_{k_0} (t_N-t_0)/k_0) \right).$$

At times, such that $M(t_N-t_0)\sigma_k/k_0 << r_{k_0}$, and assuming $|T_{k_0}|^2 << 1$ we obtain an exponential decay law. For $t > t_0$

$$p(t) \simeq \frac{k_0}{M \beta_k} (1+2e^{-\sigma^2 \beta_k^2}) |T_{k_0}|^2 \sin^2 \left( \frac{\theta_k}{2} + \frac{1}{2} r_{k_0} \beta_k \right) e^{-\frac{\lambda_{k_0}}{M \beta_k} |T_{k_0}|^2 (t-t_0)}, \quad (6.6)$$

where we only kept the leading order terms in $e^{-\sigma^2 \beta_k^2}$ and as in (5.10) we took $e^{-\sigma^2 \beta_k^2} \simeq 1$. Thus, we see that the relaxation of the condition $\sigma \beta_k << 1$ to $\sigma \beta_k \sim 1$ only affected the regime of exponential decay by a multiplicative constant factor. To obtain a qualitatively different behavior, one would have to assume $\sigma \beta_k << 1$, which is inadmissible at least for symmetric potentials. This implies that the suppression of the off-diagonal terms in $A_{nm}(t)$ of Eq. (5.1) is rather generic. Still, this result crucially depends on the fact that the potential vanishes in region I. A strongly attracting potential could lead to a value of $\beta_k$ substantially smaller than $2a$, whence the limit $\sigma \beta_k << 1$ could be applicable.

We next examine the consequences of violating the condition $\sigma \omega_{k_0} << 1$. Usually, the transmission probability is a bounded function of $k$ and for small values of $k_0$ (energy substantially lower than the barrier’s height), its rate of change is slow (unless the barrier varies rapidly at the scale of $k_0^{-1}$). However, even for the simple case of a long square barrier potential, there is a regime in which this condition is violated. If $\gamma_k d >> 1$, $|T_{k_0}|^2 << 1$, irrespective of the value of $k_0$; hence there is a distinguishable macroscopic decay time-scale. If the particle’s energy, is close to the barrier’s height, so that $\gamma_k/k_0$ is a very small number, it suffices that $\sigma/\gamma_k$ is of the same order as $\gamma_k/k_0$ to get a violation. However, in this regime we also have a violation of the conditions (5.20, 5.23), so Eq. 4.21 is not an adequate approximation. One would have to keep higher
order terms in the expansion for the logarithm of the reflection and transmission amplitudes. In this case, there are no meaningful detection peaks (even ones involving interference): the decay is going to be non-exponential. Note that in this regime, $k_0$ is near the potential’s threshold, and in this sense it is analogous to a regime of non-exponential decay identified in [17]—see also [18]. In general, there is going to be non-exponential decay in any potential characterized by a regime of energies, in which the transmission probability varies rapidly. This does not require that the energy be close to the potential’s peak. For example, in any barrier which can be approximated by the double-step potential

$$V(x) = \begin{cases} V_1, & a \leq x \leq b \\ V_2(> V_1), & b < x < c \end{cases} \quad (6.7)$$

the regime of energies around $k_0 = \sqrt{2MV_1}$ will give rise to non-exponential decay.

To obtain a picture for the behavior of such non-exponential decays without moving beyond the validity of the conditions $5.20–5.23$, we consider an initial state which is a superposition of two Gaussians of the same width $\sigma$, but with different values of momenta $k_1$ and $k_2$

$$\psi_0(r) = \frac{1}{\sqrt{2}} \left( \frac{1}{(2\pi\delta^2)^{1/4}} e^{i(r-a)/\sqrt{2\delta^2}} + \frac{1}{(2\pi\delta^2)^{1/4}} e^{i(r-a)/\sqrt{2\delta^2}} \right). \quad (6.8)$$

This initial state, being a superposition of two states with different mean energy, could be relevant for the description of quantum beats [19] in systems decaying through tunneling—for example [20]: we shall see that it leads to an oscillating behavior of the decay probability.

We assume that $|T_{k_1}| - |T_{k_2}|$ is of the order of unity. In cases such as the two step potential this does not necessitate that the absolute value of $q = k_1 - k_2$ is large—it suffices that it is substantially larger than $\sigma$. The conditions $5.20–5.23$ then need not be violated in the calculation. We neglect for simplicity all terms that are usually small in the exponential phase, i.e. we employ the same approximations involved in the derivation of Eq. $5.8$. We also drop the small $\lambda_k$ term in the exponential. Then, we obtain

$$p(t) = \frac{k_1\sigma}{2\sqrt{\pi M}} |T_{k_1}|^2 \sum_{n=0}^{\infty} |R_{k_1}|^2 e^{-\sigma^2(L+n\beta k_1-k_1 t)/M^2}$$

$$+ \frac{k_2\sigma}{2\sqrt{\pi M}} |T_{k_2}|^2 \sum_{n=0}^{\infty} |R_{k_2}|^2 e^{-\sigma^2(L+n\beta k_2-k_2 t)/M^2}$$

$$+ \sigma \sqrt{k_1 k_2} \sqrt{\pi M} \text{Re} \left[ e^{i(L-k_0 t)/M} T_{k_1} \bar{T}_{k_2} \sum_{n,m=0}^{\infty} (-R_{k_1})^n (-\bar{R}_{k_2})^m e^{-\sigma^2(L+n\beta k_1-k_1 t)^2/M^2} \right.$$}

$$\times e^{-\sigma^2(L-n\beta k_2-k_2 t)^2/M^2} \left] \right]. \quad (6.9)$$

$$22$$
where $k_0 = (k_1 + k_2)/2$. We assume that the coefficient $\beta_k$ does not vary much between $k_1$ and $k_2$—which is reasonable since the dominant contribution to $\beta_k$ is $2a$ (at least for the symmetric potentials). Then the off-diagonal elements in the last term are suppressed and this term becomes

$$\frac{\sigma \sqrt{k_1 k_2}}{\sqrt{\pi M}} e^{-\frac{2q^2 t^2}{M^2}} \Re \left( e^{i q (L - \frac{k_0 t}{M}) T_k} \sum_{n=0}^{\infty} (R_{k_1} R_{k_2})^n e^{-\sigma^2 (L + \frac{\beta_{k_1} + \beta_{k_2}}{2} - \frac{k_0 t}{M})^2} \right).$$

The same analysis as in Sec. 4 then yields for $t > t_0$

$$p(t) = \frac{\Gamma_1}{2} e^{-\Gamma_1 t} + \frac{\Gamma_2}{2} e^{-\Gamma_2 t} + \frac{2 \sqrt{k_1 k_2}}{M (\beta_{k_1} + \beta_{k_2})} e^{-\frac{2q^2 t^2}{M^2}} \Re \left( e^{-\frac{k_0 q}{M} (t-t_0) T_k} \sum_{n=0}^{\infty} (R_{k_1} R_{k_2})^n e^{-\sigma^2 (L + \frac{\beta_{k_1} + \beta_{k_2}}{2} - \frac{k_0 t}{M})^2} \right),$$

where $\Gamma_{1,2} = \frac{k_{1,2}}{M^{\beta_{k_{1,2}}}} |T_{k_{1,2}}|^2$.

We see then that $p(t)$ is a convex combination of two exponential decay terms together with an interference term that becomes negligible at times $t >> \frac{M}{q \sigma}$. Near a threshold the decay rates $\Gamma_{1,2}$ may be substantially different even for relatively small values of $q$. This implies that the oscillatory behavior arising from the interference term may persist long enough to be macroscopically distinguishable.

![Figure 2: The probability distribution (6.9) for $\frac{|T_{k_1}| - |T_{k_2}|}{|T_{k_1}| + |T_{k_2}|} = 0.7$ and $\sigma/q = 0.02$. The oscillations at long time arise from the interference in the time-of-arrival due to the different values of momentum.](image)

To summarize: the condition $|T_{k_0}|^2 << 1$ is a prerequisite for the existence of a meaningful decay law that does not depend on detailed knowledge of the
initial state. The condition $\sigma \beta_{k_0} >> 1$ can be relaxed substantially without loss of the exponential decay phase. The regimes at which the condition $\sigma w_{k_0} << 1$ is violated is expected to manifest the non-exponential decay behavior most clearly.

\section{Comparison with the survival probability}

We finally compare our results with the estimations obtained from the calculation of the survival probability for the same initial states and potentials. For the initial state \( \{4, 9\} \) we obtain the following expression for the survival amplitude

\[ (\psi_0 | e^{-iHt} | \psi_0) = \frac{1}{\sqrt{\pi \sigma}} \int_0^\infty dk \frac{|T_k|^2}{|1 + R_k|^2} e^{-\frac{(k - k_0)^2}{2\sigma^2}} e^{-i\frac{k^2}{2\sigma^2}}. \]

(7.1)

In the above equation, we set with a good approximation the mode normalization constant $|A_k|^2$ to $(2\pi)^{-1}$ (the value for the free particle on the half line). Expanding the logarithm of the transmission and reflection amplitudes and keeping the lower order terms, we obtain (still assuming that $\sigma/k_0 << 1$)

\[ \langle \psi_0 | e^{-iHt} | \psi_0 \rangle = \frac{1}{\sqrt{\pi \sigma}} |T_{k_0}|^2 \sum_{n, m=0}^\infty (-\bar{R}_{k_0})^n (-R_{k_0})^m \]

\[ \times \int_{-\infty}^{\infty} dk e^{-\frac{(k - k_0)^2}{2\sigma^2}} e^{i(m - n)\beta_{k_0} + 2(w_{k_0} + \frac{n + m}{2} s_{k_0})}, \]

(7.2)

where $\beta_{k_0}$, $w_{k_0}$, $s_{k_0}$ are defined as previously. The evaluation of the Gaussian integral yields

\[ \langle \psi_0 | e^{-iHt} | \psi_0 \rangle = |T_{k_0}|^2 e^{-\frac{k_0^2}{2\sigma^2}} \sum_{n, m=0}^\infty (-\bar{R}_{k_0})^n (-R_{k_0})^m \]

\[ \times \exp \left\{ \frac{i(k_0 - m - n)\beta_{k_0} + 2(w_{k_0} + \frac{n + m}{2} s_{k_0})^2}{4(\frac{1}{\sigma^2} + i \frac{2\sigma^2}{2M})} \right\}. \]

(7.3)

In the regime that we obtained the exponential decay previously $[t\sigma^2/M << 1, (w_{k_0} + \frac{n + m}{2} s_{k_0}) << 1]$ this expression yields

\[ \langle \psi_0 | e^{-iHt} | \psi_0 \rangle = |T_{k_0}|^2 e^{-\frac{k_0^2}{2\sigma^2}} \sum_{n, m=0}^\infty (-\bar{R}_{k_0})^n (-R_{k_0})^m e^{-\frac{\sigma^2\beta_{k_0}^2}{M^2}(m - n - \frac{k_0 s_{k_0}}{M})^2}. \]

(7.4)

If $\sigma \beta_{k_0} >> 1$, the Gaussian term has strong peaks at times $t_{m,n}$ such that $m - n = \frac{k_0 t_{m,n}}{M \beta_{k_0}}$. Ignoring the fine structure between such peaks, we drop the summation over $m$ substituting $m = n + \frac{k_0 t_{m,n}}{M \beta_{k_0}}$. We then obtain

\[ \langle \psi_0 | e^{-iHt} | \psi_0 \rangle \simeq e^{-\frac{k_0^2}{2\sigma^2}} (-R_{k_0})^{k_0 t_{k_0} / M \beta_{k_0}}. \]

(7.5)
Hence the survival probability $w(t) = |\langle \psi_0 | e^{-i\hat{H}t} | \psi_0 \rangle|^2$ equals

$$w(t) \simeq |R_{k_0}|^{\frac{2|k_0|^2}{M \beta_{k_0} / k_0}}.$$  

(7.6)

This expression describes exponential decay. Noting that our derivation employed implicitly the condition $|T_{k_0}|^2 << 1$, since we ignored the details of the distribution at the microscopic timescale $M \beta_{k_0} / k_0$, we obtain

$$w(t) \simeq e^{-\frac{k_0 |T_{k_0}|^2}{M \beta_{k_0}} t}.$$  

(7.7)

We note that $w(t)$ decays exponentially with the same decay constant $\Gamma$ that appears in Eq. (5.11). Moreover, the conditions for exponential decay are also the same. This agreement is quite remarkable given the very different characters of the two objects: $w(t)$ is quadratic with respect to the initial density matrix $\hat{\rho}$, while the probability $p(t)$ we obtained is linear. We believe that this coincidence is a consequence of the fact that the details of the initial state do not affect the coarser behavior of the object constructed: after all, $\Gamma$ only depends on the mean energy of the initial state and not on any of its moments.

The agreement between the results from the study of the survival probability and the detection probability only holds in the exponential regime. For example, in the asymptotic regime of $t \to \infty$, Eq. (7.1) yields

$$\langle \psi_0 | e^{-i\hat{H}t} | \psi_0 \rangle \sim t^{-\left(\frac{3}{2} + \alpha\right)},$$  

(7.8)

where, as previously, $\alpha$ is defined as $T_k \sim k^\alpha$ as $k \to 0$. Hence, $w(t) \sim t^{-(1+2\alpha)}$ and the ‘decay probability’ $-\dot{w}(t) \sim t^{-(2+2\alpha)}$, while as we showed in Sec. 4.4, $p(t) \sim t^{-(3/2+\alpha)}$.

More interesting is their divergence for the regimes that are not characterized by an exponential decay phase. For the case of a decay in which the condition $\sigma w_{k_0}$ is violated, we consider the same initial state as in Eq. (6.11). The assumption that $\sigma / (k_1 - k_2) << 1$ leads to a suppression of the interference terms in the evaluation of $\langle \psi_0 | e^{-i\hat{H}t} | \psi_0 \rangle$. We finally obtain

$$w(t) = \frac{1}{4} \left[ e^{-\Gamma_1 t} + e^{-\Gamma_2 t} + 2e^{-\sqrt{\Gamma_1 \Gamma_2} t} \cos \left( (\pi + \theta_{k_1}) \frac{k_1 t}{M \beta_{k_1}} - (\pi + \theta_{k_2}) \frac{k_2 t}{M \beta_{k_2}} - \frac{k_1^2 - k_2^2}{2M} t \right) \right],$$  

(7.9)

where $\theta_{k_1,2} = \text{Arg}R_{k_1,2}$.

The expression above for $w(t)$ is different from Eq (6.11) not only in the explicit form and the persistence of the oscillating term, but also in the coefficient $\frac{1}{4}$. This means that $w(t)$ is not a strictly decreasing function of $t$, and for this reason $-\dot{w}(t)$ is non-positive and it cannot be interpreted as a probability density for the time of decay. In other words, outside the exponential regime one cannot trust the results obtained from the survival amplitude to yield reliable statistics for the measurement outcomes.
8 Conclusions

We reformulated tunneling as a time-of-arrival problem, in order to study the decays of unstable states. We considered a general class of potentials for a particle in the half-line. We saw that the exponential regime is rather generic and we identified the specific conditions that are necessary for its validity. This allowed us to precisely identify the conditions necessary for the emergence of decays that have no exponential phase.

The key feature of our construction is that there is neither interpretational nor probabilistic ambiguity. The probabilities we derive are obtained through a POVM, hence (unlike other approaches to the problem) they are always positive and they respect the convexity of the space of quantum states. The interpretation of these objects is concretely operational, in the sense that it is tied to the statistics for the measurement of particles’ arrival times.

Another important point is that the POVM we used in the derivation of our results is defined in terms of the Hamiltonian, the initial state and the location of the detector. It can therefore be applied to much more general situations than the ones we considered here. In particular, it can be used for the study of relativistic tunneling, tunneling in open systems or even for the study of tunneling where the barrier is not an external potential but is caused by microscopic particle interaction (e.g. in nuclei).

Moreover, the analysis in Ref. [1, 2] need not apply only to particle systems. The measurement-theoretic context is specified in a choice of the projectors that represent the type of transition that is recorded by the measuring device [1]. These projectors can be completely general and they need not refer to particle positions. With a suitable choice for these objects, the results can be generalized to field theoretic systems (e.g. for the study of particle decays through field interactions) or even to a cosmological setting.

Finally, we note that the derivation of the POVM and consequently of these results depends crucially on concepts introduced by the histories approach to quantum theory, in particular on the algebraic (‘logical’) structure of the space of histories and on the decoherence functional. We have found impossible to rephrase the construction without explicitly referring the concepts above. For this reason, we believe that these results provide an argument that the distinctions and structures introduced by the histories approach provide an extension to the quantum mechanical formalism with a larger domain of applicability.

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