Discrimination between nonorthogonal two-photon polarization states

Ulrike Herzog
Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstrasse 110, D-10115 Berlin, Germany

Abstract

It is shown that generalized measurements, required for optimally discriminating between nonorthogonal joint polarization states of two indistinguishable photons, can be realized with the help of polarization-dependent two-photon absorption and by means of sum-frequency generation. Optimization schemes are investigated with respect to minimizing the error probability in inferring the states, as well as with respect to maximizing the probability of success for unambiguous discrimination. Moreover, an implementation of error-minimizing discrimination between \( N \) symmetric single-photon states is studied. The latter can be used to extract information from the inconclusive results occurring in unambiguous discrimination between three symmetric two-photon polarization states.

03.65.Bz, 03.67-a, 42.50.Dv

*Contribution at the 8th Central-European Workshop on Quantum Optics, April 27 - 30, 2001

Typeset using REVTeX
I. INTRODUCTION

The problem of state discrimination consists in determining the actual state of a quantum system that is prepared in an unknown state belonging to a known finite set of given pure states. Nonorthogonal quantum states cannot be perfectly discriminated, and therefore optimization strategies have been developed that minimize either the probability of errors, or, when unambiguous discrimination is required, the probability of getting an inconclusive result [1]. For single-photon polarization states, experiments have been performed that realize optimum unambiguous discrimination between two states [2] and error-minimization for up to four states of a specific kind [3]. In this contribution we consider the discrimination between the states characterizing the joint polarization of two photons travelling in a single spatial mode. The Hilbert space of these states is spanned by a three-dimensional basis, corresponding to the three possibilities of distributing two indistinguishable photons among the two orthogonal polarization modes of a transverse field. Two-photon polarization states are a candidate for implementing a ternary quantum logic for quantum computation, or for applying the quantum cryptographic schemes that have been developed for three-state systems [4]. Recently three mutually orthogonal two-photon polarization states have been experimentally observed [5].

II. DISCRIMINATION WITH MINIMUM ERROR PROBABILITY

Given $N$ nonorthogonal quantum states $|\psi_k\rangle$ ($k = 1, 2, \ldots, N$) occurring with equal a priori probability, the problem of error minimization in state discrimination has been solved for arbitrary $N$ under the condition that the states are symmetric [3] which means that each state results from its predecessor by applying a unitary operator $\hat{V}$ in a cyclic way, i.e. $|\psi_k\rangle = \hat{V}^{k-1}|\psi_1\rangle$ and $|\psi_1\rangle = \hat{V}|\psi_N\rangle$. In this case the maximum achievable probability $P_C$ to infer the states correctly from a measurement, or the minimum error probability $P_E$, respectively, obeys the equation [3]

$$P_C = 1 - P_E = \frac{1}{N} \sum_{k=1}^{N} |\langle \mu_k|\psi_k\rangle|^2,$$

where $|\mu_k\rangle = \left(\sum_{k=1}^{N} |\psi_k\rangle\langle\psi_k|\right)^{-\frac{1}{2}}|\psi_k\rangle$. (1)

In the optimized measurement scheme that realizes Eq. (1), the quantum system is guessed to be in the state $|\psi_k\rangle$ provided that the state $|\mu_k\rangle$ is detected.

Let us consider $N$ specific symmetric quantum states given by

$$|\psi_k\rangle = \sum_{l=0}^{M} c_l e^{i\frac{2\pi}{N}lk} |u_l\rangle, \text{ with } k = 1, 2, \ldots, N \text{ and } N \geq M + 1,$$ (2)

where the coefficients $c_l$ are nonzero complex numbers and the state vectors $|u_l\rangle$ represent a set of $M + 1$ orthonormal basis states to be specified later. We find the optimum detection states to be

$$|\mu_k\rangle = \frac{1}{\sqrt{N}} \sum_{l=0}^{M} c_l e^{i\frac{2\pi}{N}lk} |u_l\rangle \text{ with } \langle \mu_j|\mu_k\rangle_{j \neq k} = \frac{1}{N} \frac{e^{i\frac{2\pi}{N}(k-j)(M+1)}}{e^{i\frac{2\pi}{N}(k-j)} - 1},$$ (3)
from which we obtain the probability of correct guesses $P_C = \frac{1}{N} \left( \sum_{l=0}^{M} |c_l|^2 \right)^2$. When $N = M + 1$, the $N$ states defined by Eq. (2) are linearly independent. In this case the detection states (3) are normalized and mutually orthogonal, the detection operators $|\mu_k\rangle\langle\mu_k|$ therefore being conventional projection operators. For $N > M + 1$ the states given by Eq. (2) form an overcomplete set of linearly dependent states, and the resulting nonorthogonal optimum detection states are nonnormalized since $\langle\mu_k|\mu_k\rangle = (M + 1)/N$. The positive Hermitian operators $|\mu_k\rangle\langle\mu_k|$ satisfy the resolution of the identity $\sum_{k=1}^{N}|\mu_k\rangle\langle\mu_k| = \hat{1}$ and can be interpreted to be quantum detection operators of a generalized measurement based on positive-operator valued measures (3).

In the following we restrict ourselves to the case $M = 2$ and study error-minimizing discrimination between the $N$ specific symmetric two-photon polarization states

$$|\psi_k\rangle = c_0|u_0\rangle + c_1e^{i\frac{\pi}{M}k}|u_1\rangle + c_2e^{i\frac{2\pi}{M}k}|u_2\rangle \quad \text{with} \quad |c_2| \leq |c_0|, |c_1|,$$

where $k = 1, \ldots N$. The basis states are now defined by

$$|u_0\rangle \equiv |2, 0\rangle = \frac{\hat{a}_1^2}{\sqrt{2}}|0\rangle, \quad |u_1\rangle = |1, 1\rangle = \hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle, \quad \text{and} \quad |u_2\rangle \equiv |0, 2\rangle = \frac{\hat{a}_2^2}{\sqrt{2}}|0\rangle. \quad (5)$$

Here the photon creation operators $\hat{a}_1$ and $\hat{a}_2$ refer to any two mutually orthogonal polarization modes of the field. An interesting special case arises when the $k$th state is created by the operator $\hat{b}_k^\dagger = (\hat{a}_1 + e^{i\frac{2\pi}{M}k}\hat{a}_2^\dagger)/\sqrt{2}$. The two-photon states $2^{-1/2}(\hat{b}_k^\dagger)^2|0\rangle$ are then given by Eq. (4) with $c_0 = c_2 = 1/2$ and $c_1 = 1/\sqrt{2}$, yielding the value $P_C = (3 + 2\sqrt{2})/(2N)$ for the maximum achievable probability of correct guesses. This value is larger than the result $P_C = 2/N$ following for the corresponding single-photon states $\hat{b}_k^\dagger|0\rangle = (|1, 0\rangle + e^{i\frac{2\pi}{M}k}|0, 1\rangle)/\sqrt{2}$.

For a physical implementation of state discrimination with minimum error probability, one could make use of two-photon absorption. Let us consider an atom that is pumped into a coherent superposition of three degenerate lower energy states $|g_0\rangle, |g_1\rangle,$ and $|g_2\rangle$, having the magnetic quantum numbers $m = -2, 0, \text{and} +2$, respectively. By two-photon transitions, the lower levels are assumed to be connected to an excited state $|e\rangle$ with $m = 0$. For simplicity, we write the interaction Hamiltonian as $H = \hbar \eta \left( \hat{a}_1^2|e\rangle\langle g_0| + \sqrt{2}\hat{a}_1\hat{a}_2|e\rangle\langle g_1| + \hat{a}_2^2|e\rangle\langle g_2| \right) + H.A.$, where $\eta$ is real and denotes the atom-field coupling constant. When the orientation of the atomic quantization axis coincides with the direction of wave propagation, $\hat{a}_1$ and $\hat{a}_2$ refer to right-handed and left-handed circular polarization, respectively. We start from an initial atomic superposition state $|\chi\rangle = \sum_{l=0}^{2} \alpha_l|g_l\rangle$ and from an initial two-photon polarization state of the field $|\psi\rangle = \sum_{l=0}^{2} \beta_l|u_l\rangle$, where the basis states are given by Eq. (3) and refer to circular polarization. By calculating the combined atom-field state $|\Phi_{tot}(t)\rangle = \exp(-\frac{i}{\hbar}Ht)|\chi\rangle|\psi\rangle$, we obtain after averaging with respect to the interaction time $t$ the time-independent single-atom excitation probability (4)

$$P_e = \Gamma \int_0^\infty dt \ e^{-\Gamma t} \text{Tr}\langle e|\Phi_{tot}(t)\rangle|^2 = \frac{2\eta^2}{\Gamma^2 + 12\eta^2} \left( \sum_{l=0}^{2} \left| \alpha_l \beta_l \right|^2 \right) = \frac{2\eta^2}{\Gamma^2 + 12\eta^2} \left( \langle \mu | \psi \rangle \right)^2. \quad (6)$$
Here the trace has been performed over the field and we introduced the notation $|\mu_l\rangle = \sum_{i=0}^2 \alpha_i^{(l)} |u_i\rangle$. Provided that $N$ kinds of atoms, labelled by the index $k$, are prepared in superposition states $|\chi_k\rangle = \sum_{i=0}^2 \alpha_i^{(k)} |g_i\rangle$ designed in such a way that $\alpha_i^{(k)} = \frac{1}{\sqrt{N}} c_i e^{-i 2\pi k i}$, the respective excitation probabilities are proportional to $|\langle \mu_k | \psi \rangle|^2$, where $|\mu_k\rangle$ is defined by Eq. (3) with $N = 2$. Therefore the atomic superposition states represent the detection states of the error-minimizing measurement scheme if it is possible to observe, e.g. by fluorescence detection, which kind of atom has been excited. For $N = 3$, when the optimum detection states are mutually orthogonal, one could use three separate gas cells being traversed one after the other by the unknown two-photon polarization states and being each filled with a sufficiently large number of atoms prepared in the realization of a different detection state.

For later use we still consider the specific symmetric single-photon states defined by $|\xi_k\rangle = c_0 |1,0\rangle + c_1 e^{i 2\pi k} |0,1\rangle$ with $k = 1, \ldots, N$. In this case error-minimizing state discrimination can be achieved by inferring the unknown state to be the state $|\xi_k\rangle$ provided that the photon is detected at the $k$th output port of a lossless linear optical network having $N$ input ports and $N$ output ports. For this purpose, the two modes have to be directed into separate input ports of the optical multiport. According to Eq. (4), the latter has to be constructed in such a way that, for any input state $|\xi_k\rangle$, the probability $|d_j^{(out)}|^2$ that the photon exits at the output port $j$ is given by

$$|d_j^{(out)}|^2 = |\langle \mu_j | \xi_k \rangle|^2 = \frac{1}{N} \left[ 1 + 2 |c_0||c_1| \cos \frac{2\pi (k - j)}{N} \right].$$

Here $|\mu_j\rangle$ has been determined from Eq. (3) with $M = 2$, using the basis states $|1,0\rangle$ and $|0,1\rangle$. In order to find the unitary transformation matrix $U$ characterizing a specific multiport that implements error minimization, we make use of the relation $d_j^{(out)} = \sum_{r=1}^N U_{jr} d_r^{(in)}$ that connects the single-photon input and output probability amplitudes, or the classical fields, respectively. With $d_1^{(in)} = c_0$, $d_2^{(in)} = c_1 e^{i 2\pi k}$, and $d_r^{(in)} = 0$ for $r \geq 3$ the required relation (4) is fulfilled provided that, for $j = 1, \ldots, N$,

$$U_{j1} = \frac{1}{\sqrt{N}} e^{i (\text{Arg} c_1 - \text{Arg} c_0)} \quad \text{and} \quad U_{jr} = \frac{1}{\sqrt{N}} e^{-i 2\pi j (r-1)} \quad \text{for} \quad r = 2, \ldots, N.$$ 

Once the transformation matrix is known, the desired linear optical multiport can be constructed using beam splitters and phase shifters. This has been used recently for proposing an implementation of optimum unambiguous discrimination between single-photon states in which the photon is divided among more than two input modes [8].

**III. OPTIMUM UNAMBIGUOUS DISCRIMINATION**

It has been proved that unambiguous state discrimination, with a certain probability of success, is possible if and only if the states are linearly independent [9], and that $N$ linearly independent and symmetric states can always be written in the form (2) with $M = N - 1$ and properly chosen basis states [10]. Provided that these states occur with equal a priori probability, the maximum probability of success has been derived to be $P_D = N \min |c_i|^2$ [10]. Here $\min |c_i|^2$ is the smallest square modulus arising from any of the coefficients $c_i$.
that occur in Eq. (2). We mention that \( P_D < P_C \) unless the states are orthogonal. The optimum value \( P_D \) can be achieved in a generalized measurement consisting of a two-step procedure [11]: First the given set of nonorthogonal states has to be transformed into a set of orthogonal ones by means of a suitable outcome-conditioned nonunitary transformation. By this operation, a quantum system prepared in one of the given nonorthogonal states \( |\psi_k\rangle \) will, with a certain probability of success, be transformed into the corresponding member of a set of orthogonal states \( |\tilde{\psi}_k\rangle \). In a second step the resulting orthogonal states can be perfectly discriminated by a conventional quantum mechanical projection measurement. Here we are interested in the physical mechanisms that enable optimum unambiguous discrimination between the three linearly independent two-photon polarization states given by Eq. (4) with \( N = 3 \), yielding the optimum value \( P_D = 3|c_2|^2 \).

First we discuss state orthogonalization via polarization-dependent two-photon absorption. We assume that atoms are prepared in such a way that they can perform two-photon absorbing transitions provided that the photons belong to prescribed polarization modes \( i \) and \( j \) with \( i,j = 1,2 \), and that single-photon absorption is negligible. With \( \rho \) being the reduced density operator of the radiation field, the master equation describing the absorption process reads \( \dot{\rho} = -\frac{2\gamma}{2}(\hat{a}_i^\dagger \hat{a}_j \rho + \rho \hat{a}_j^\dagger \hat{a}_i) + S \rho \). Here we used the abbreviation \( S \rho = \gamma_{ij} \hat{a}_i \hat{a}_j \rho \hat{a}_i \hat{a}_j^\dagger + \gamma_{11}, \gamma_{22}, \text{ and } \gamma_{12} = \gamma_{21} \) are the respective two-photon absorption constants. The superoperator \( S \) can be interpreted to be a jump operator responsible for jump-like changes of the density operator \( \rho \) due to the absorption of two photons. Under the condition that no photon is absorbed, the evolution of the radiation field is described by a nonnormalized conditioned density operator \( \tilde{\rho} \) obeying the evolution equation that ensues from the master equation when the jump-operator term is omitted [11]. Therefore, if the radiation field is initially in the pure state \( |\psi\rangle \), the conditioned state remains pure and is given by \( |\tilde{\psi}(t)\rangle = \exp(-\frac{2\gamma}{2} \hat{a}_i^\dagger \hat{a}_j t) |\psi\rangle \). The probability that no two-photon absorption process occurs is equal to \( \langle \tilde{\psi}(t)|\tilde{\psi}(t)\rangle \) [11]. We want to perform unambiguous state discrimination for the three two-photon polarization states \( |\psi_k\rangle \ (k = 1,2,3) \) that are defined by Eq. (4) with \( N = 3 \). Let us assume that the states interact during a time interval \( T_0 \) with a two-photon absorbing medium for which only \( \gamma_{11} \) is different from zero, and that after the end of this interaction (or before its beginning) the states interact during a time interval \( T_1 \) with another two-photon absorber for which only \( \gamma_{12} \) differs from zero. When both types of interaction are finished, and on the condition that no absorption has occurred, the incoming state \( |\psi_k\rangle \) is transformed into the nonnormalized state \( |\tilde{\psi}_k\rangle = c_0 e^{-\gamma_{11} T_0} |u_0\rangle + c_1 e^{-\gamma_{12} T_1} e^{i\Delta \omega k} |u_1\rangle + c_2 e^{i\Delta \omega k} |u_2\rangle \). To achieve orthogonalization, the interaction times have to be adjusted in such a way that \( \exp(-\gamma_{11} T_0) = |c_2|/|c_0| \) and \( \exp(-\gamma_{12} T_1/2) = |c_2|/|c_1| \). This yields the set of nonnormalized no-absorption-conditioned state vectors

\[
|\tilde{\psi}_k\rangle = |c_2| \left( \frac{c_0}{|c_0|} |u_0\rangle + \frac{c_1}{|c_1|} e^{i\Delta \omega k} |u_1\rangle + \frac{c_2}{|c_2|} e^{i\Delta \omega k} |u_2\rangle \right)
\]

with \( k = 1,2,3 \), which are mutually orthogonal and can be perfectly discriminated. The probability for successful discrimination, i. e. for the absence of absorption, is found to be \( \langle \tilde{\psi}_k|\tilde{\psi}_k\rangle = 3|c_2|^2 \) which is equal to the optimum achievable value \( P_D \). When the two incoming photons are absorbed, an inconclusive result is obtained.
If, instead of two-photon absorption, sum-frequency generation is used for state orthogonalization, the inconclusive results can be detected as well. The interaction Hamiltonian reads $H_{ij} = i\hbar \frac{\kappa_{ij}}{2} (\hat{a}_1^\dagger \hat{b}_i \hat{b}_j - \hat{a}_1 \hat{a}_i \hat{b}_j^\dagger)$ with $i, j = 1, 2$, where $\hat{b}_i$ is the creation operator for a photon in the corresponding up-converted mode. When $\hat{a}_1$ and $\hat{a}_2$ refer to horizontally and vertically linearly polarized light, respectively, the coupling constants $\kappa_{11}$ and $\kappa_{22}$, on the one hand, and $\kappa_{12} = \kappa_{21}$ on the other, correspond to type-I and type-II nonlinear crystals. For the purpose of state orthogonalization we assume that the two-photon polarization states are sufficiently efficient.

The nonorthogonal states have to be transformed for optimum unambiguous discrimination, conditioning the state orthogonalization to the set of detection states $\{|\psi_k\rangle\}$ with $N = 3$, interact during a time interval $T_0$ with a type-I crystal with $\kappa_{11} \neq 0$ and during a time interval $T_1$ with a type-II crystal. The interaction times are now required to obey the equations $\cos(\kappa_{11}T_0/\sqrt{2}) = |c_2|/|c_0|$ and $\cos(\kappa_{12}T_1/2) = |c_2|/|c_1|$. When the interaction is completed, the state vector of the enlarged system, including the up-converted modes labelled by $A$ and $B$ for the type-I and type-II crystal, respectively, takes the form

$$|\Psi_k^{tot}\rangle = e^{-i\hat{H}_{11}T_0}e^{-i\hat{H}_{12}T_1}|0\rangle_A|0\rangle_B|\psi_k\rangle = |0\rangle_A|0\rangle_B|\tilde{\psi}_k\rangle + |\xi_k\rangle |0\rangle,$$

where $|0\rangle$ is the vacuum state of the fundamental mode, $|\tilde{\psi}_k\rangle$ is given by Eq. (9), and

$$|\xi_k\rangle = \sqrt{|c_0|^2 - |c_2|^2} |1\rangle_A|0\rangle_B + \sqrt{|c_1|^2 - |c_2|^2} e^{i\frac{\pi}{4}k} |0\rangle_A|1\rangle_B.$$

Provided that no sum-frequency generation occurs, the state $|\psi_k\rangle$ is transformed into the conditioned state $\langle 0|_A|0|_B|\Psi_k^{tot}\rangle = |\tilde{\psi}_k\rangle$, belonging to the set of mutually orthogonal states that enable optimum unambiguous discrimination. On the other hand, when sum-frequency generation takes place, the up-converted photon is found to be in the conditioned single-photon superposition state $|\xi_k\rangle$. This happens with the $k$-independent probability of inconclusive results $\langle \xi_k|\xi_k\rangle = 1 - 3|c_2|^2$. The three states $|\xi_k\rangle$ refer to a two-dimensional basis and are therefore linearly dependent, rendering unambiguous discrimination impossible. However, if both $|c_0|$ and $|c_1|$ are larger than $|c_2|$, an inconclusive result still contains information about the original state which can be extracted in an optimum way by applying an error-minimizing discrimination scheme [4]. Upon normalization, the single-photon states (11) exactly correspond to the states that can be discriminated with minimum error probability using a linear multiport characterized by the transformation matrix given by Eq. (8) with $N = 3$. Hence, in order to infer the up-converted single-photon state and thus also the original state, the up-converted modes $A$ and $B$ have to be directed into two input ports of the specific multiport.

Interestingly, except for normalization, the set of orthogonal states $\{|\tilde{\psi}_k\rangle\}$, into which the nonorthogonal states have to be transformed for optimum unambiguous discrimination, is equivalent to the set of detection states $\{|\mu_k\rangle\}$ enabling error-minimizing discrimination, as can be seen by comparing Eq. (3) and Eq. (8) with $N = 3$ and $M = 2$. By generalization, we find this equivalence to be valid for linearly independent symmetric states in an arbitrary dimensional Hilbert space. Finally we note that it is planned to extend the investigations in order to study the discrimination between nonorthogonal joint polarization states of $M$ indistinguishable photons. In this case optimum unambiguous discrimination between up to $M + 1$ states could be implemented if the necessary $M$-photon interaction processes were sufficiently efficient.
REFERENCES

[1] For an overview, see A. Chefles, Contemp. Phys. 41, 401 (2000).
[2] R. B. M. Clarke et al., Phys. Rev. A 63, 040305(R) (2001).
[3] R. B. M. Clarke et al., quant-ph/0008028.
[4] H. Bechmann-Pasquinucci and A. Peres, Phys. Rev. Lett. 85, 3313 (2000).
[5] T. Tsegaye et al., Phys. Rev. Lett. 85, 5013 (2000).
[6] M. Ban, K. Kurokawa, R. Momose and O. Hirota, Int. J. Theor. Phys. 55, 22 (1997).
[7] U. Herzog, (to be published).
[8] Y. Sun, M. Hillery, and J. A. Bergou, quant-ph/0012131.
[9] A. Chefles, Phys. Lett. A 239, 339 (1998).
[10] A. Chefles and S. M. Barnett, Phys. Lett. A 250, 223 (1998).
[11] See, e. g., H. Carmichael, An Open Systems Approach to Quantum Optics (Berlin 1993).