MULTIPLY WARPED PRODUCTS
WITH NON-SMOOTH METRICS

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Abstract. In this article we study manifolds with \( C^0 \)-metrics and properties of Lorentzian multiply warped products. We represent the interior Schwarzschild space-time as a multiply warped product space-time with warping functions and we also investigate the curvature of a multiply warped product with \( C^0 \)-warping functions. We give the Ricci curvature in terms of \( f_1 \), \( f_2 \) for the multiply warped products of the form \( M = (0, 2m) \times f_1 R^1 \times f_2 S^2 \).

I. INTRODUCTION

The concept of a warped product manifold was introduced by Bishop and O’Neill\(^1\), where it served to provide a class of complete Riemannian manifolds with everywhere negative curvature. The connection with general relativity was first made by Beem, Ehrich, and Powell\(^2,3\), who pointed out that several of the well-known exact solutions to Einstein’s field equations are pseudo-Riemannian warped products. Beem and Ehrich\(^4\) further explored the extent to which certain causal and completeness properties of a space-time may be determined by the presence of a warped product structure. O’Neill’s book on Semi-Riemannian geometry\(^5\) took this line of development to a natural conclusion by elevating warped products to a central role. After first developing the general theory of warped products to spaces, O’Neill then applied the theory to discuss, in turn, the special cases of Robertson-Walker, Friedmann, static and Schwarzschild space-time. The role of warped products in the study of exact solutions to Einstein’s equations is now firmly established, and it appears that these structures are generating interest in other areas of geometry.

We study manifolds with \( C^0 \)-metrics and properties of Lorentzian multiply warped products. Many authors\(^6,7,8,9\) have dealt with Lorentzian manifolds with non-smooth metric tensors from various viewpoints. Of particular interest are space-times which have a metric tensor which is fails to be \( C^1 \) across a hypersurface, and is \( C^\infty \) off the hypersurface. A space-time which, in an admissible coordinate system, the metric tensor is continuous but has a jump in its first and second derivatives across a submanifold will have a curvature tensor containing a Dirac delta function\(^10,11\). The support of this distribution may be of three, two, or one dimensional or may even consist of a single event. Lichnerowicz’s formalism\(^12\) for dealing with such tensors is modified so as to obtain a formalism in which the Riemannian curvature tensor and Ricci curvature tensor exist in the sense of distributions. We note that Smoller and Temple\(^13\) have presented a general theory for matching two solutions of the Einstein field equations at arbitrary shock-wave interface across which the metric \( g \) is \( C^0 \)-Lorentzian, i.e., smooth surface across which the first derivatives of the metric suffer at worst a jump discontinuity. A
multiply warped products space-time with base \((B, -dt^2)\), fibers \((F_i, g_i)\) \(i = 1, \ldots, n\) and warping functions \(f_i > 0\) is the product manifold \((B \times F_1 \times \ldots \times F_n, g)\) endowed with the Lorentzian metric:

\[ g = -\pi_B^* dt^2 + \sum_{i=1}^n (f_i \circ \pi_B)^2 \pi_i^* g_i \equiv -dt^2 + \sum_{i=1}^n f_i^2 g_i \]

where \(\pi_B, \pi_i\) \(i = 1, \ldots, n\) are the natural projections of \(B \times F_1 \times \ldots \times F_n\) onto \(B\) and \(F_1, \ldots, F_n\), respectively. Thus, warped product spaces are extended to a richer class of spaces involving multiply products. Multiply warped product spaces were studied by Flores, J. L. and M. Sánchez\(^{14}\).

The conditions of spacelike boundaries in the multiply warped product space-times were studied by Steven G. Harris\(^{15}\). From a physical point of view, these space-time are interesting, first, because they include classical examples of space-time: when \(n = 1\) they are Generalized Robertson-Walker space-times, standard models of cosmology; when \(n = 2\), the intermediate zone of Reissner-Nordström space-time and interior of Schwarzschild space-time appear as particular cases\(^{16}\).

The conditions of spacelike boundaries in the multiply warped product space-times were studied by Steven G. Harris. The Kasner metric was studied as a cosmological model by Schücking and Heckmann\(^{17}\)(1958).

The Schwarzschild solution is interpreted as describing the gravitational field of a point particle with mass \(m\). Generally, this metric is taken to represent the outside metric for a star with \(r > r_0\) where \(r_0\) gives the boundary of the star. The metric inside \(r < r_0\) is a different interior metric determined by the matter distribution \(T_{ij}\) inside the star and is matched at the boundary \(r = r_0\) with the metric. We represent the interior Schwarzschild space-time as a multiply warped product space-time with warping functions and we also investigate the curvature of a multiply warped product with \(C^0\)-warping functions. We given the Ricci curvature in terms of \(f_1, f_2\) for the multiply warped products of the form \(M = (0, 2m) \times f_1, R^1 \times f_2, S^2\).

II. MULTIPLY WARPED PRODUCTS MANIFOLDS WITH NON-SMOOTH METRIC TENSOR

In this section, we state some definitions and standard results\(^2,5\) which will be needed below. A Lorentzian manifold \((M, g)\) is a connected smooth manifold of dimension \(\geq 2\) with a countable basis together with a Lorentzian metric \(g\) of signature \((-+, +, +, \ldots, +)\). Let \((F_i, g_i)\) be Riemannian manifolds, and let \((B, g_B)\) be either a spacetime, or let \(B\) be \(R^1\) with \(g_B = -dt^2\). Let \(f_i > 0, i = 1, \ldots, n\) be smooth functions on \(B\). A multiply warped products space-time with base \((B, g_B)\), fibers \((F_i, g_i)\) \(i = 1, \ldots, n\) and warping functions \(f_i > 0\) is the product manifold \((B \times F_1 \times \ldots \times F_n, g)\) endowed with the Lorentzian metric:

\[ g = \pi_B^* g_B + \sum_{i=1}^n (f_i \circ \pi_B)^2 \pi_i^* g_i \equiv -dt^2 + \sum_{i=1}^n f_i^2 g_i \]  \hspace{1cm} (2.1)

where \(\pi_B, \pi_i\) \(i = 1, \ldots, n\) are the natural projections of \(B \times F_1 \times \ldots \times F_n\) onto \(B\) and \(F_1, \ldots, F_n\), respectively.

**Definition 2.1** Suppose \(M = B \times f_1, F_1 \times \ldots \times f_n, F_n\) is a multiply warped product of semi-Riemannian manifolds. For any \((p, q_1, \ldots, q_n) \in M\), the submanifold \(B \times q_1 \times \ldots \times q_n\) is a leaf of \(M\). The submanifolds \(p \times F_1 \times \ldots \times q_n\) is called type
1 fibers and \( p \times q_1 \times \ldots \times F_n \) is called type \( n \) fibers, respectively. As in the case of warped products, a multiply warped metric satisfies:

1. For each \( q_i \in F_i \), \( \pi_B|_{(B \times q_1 \times \ldots \times q_n)} \) is an isometry onto \( B \).
2. For each \( p \in B \) and \( q_j \in F_j \), \( \sigma_i|_{(p \times q_1 \times \ldots \times F_i \times \ldots \times q_n)} \) is homothety onto \( F_i \), with scale factor \( 1/f_i(p) \).
3. For each \( (p, q_1, \ldots, q_n) \in M \), the set \( B \times q_1 \times \ldots \times q_n \) and the \( i \)-th type fiber, \( p \times q_1 \times \ldots \times F_i \times \ldots \times q_n \) are mutually orthogonal at \( (p, q_1, \ldots, q_n) \).

Let \( M \) be a smooth manifold of dimension \( n \). Then subset \( S \) of \( M \) is a regularly embedded hypersurface of \( M \) if for all \( p \in S \), there exists a coordinate neighborhood \( U(p) \) with coordinates \( (x_1, \ldots, x_n) \) such that \( S \cap U = \{(x_1, \ldots, x_n) \in U \mid x_n = p \} \). For convenience, we say that such a neighborhood \( U \) is partitioned by \( S \). We denote \( \{x \in U \mid x_n > p \} \) and \( \{x \in U \mid x_n < p \} \) by \( U^+_p \) and \( U^-_p \), respectively. Now let \( M \) be a smooth manifold with a regularly embedded hypersurface \( S \). Let \( S^c \) denote the complement of \( S \). We define the concept of a \( C^0 \)-Lorentzian metric on \( M \).

**Definition 2.2** A \( C^0 \)-Lorentzian metric on \( M \) is a nondegenerate \((0,2)\) tensor of Lorentzian signature such that:

1. \( g \in C^0 \) on \( S \)
2. \( g \in C^\infty \) on \( M \cap S^c \)
3. For all \( p \in S \), and \( U(p) \) partitioned by \( S \), \( g|_{U^+_p} \) and \( g|_{U^-_p} \) have smooth extensions to \( U \).

We call \( S \) a \( C^0 \)-singular hypersurface of \( (M, g) \).

**III. MULTIPLY WARPED PRODUCTS MANIFOLDS WITH \( C^0 \) WARPING FUNCTION**

Let \( M = B \times f_1, F_1 \times \ldots \times f_n, F_n \) be a multiply warped products with \( g_B = -dt^2 \). Let \( f_i > 0 \) for \( i = 1, \ldots, n \) smooth functions on \( B = (a, b) \). Recall \( f_i \in C^\infty \) for \( t \neq t_0 \) if \( f_i \in C^0 \) at \( t = p \). When \( f \in C^0 \) at a single point \( p \in R^1 \) and \( S = \{p\} \times f_i, F_1 \times \ldots \times f_n, F_n \), we use the notation \( f \in C^0(S) \).

**Proposition 3.1** Let \( M = B \times f_1, F_1 \times \ldots \times f_n, F_n \) be a multiply warped products with Riemannian curvature tensor \( R \). If \( X, Y \in \mathcal{L}(B), V_i, W_i \in \mathcal{L}(F_i) \) where \( 1 \leq i \leq n, f_i \in C^0(S) \), then

1. \( \nabla_X Y \in \mathcal{L}(B) \) is the lift of \( \nabla_X Y \) on \( \mathcal{L}(B) \).
2. \( \nabla_X V_i = \nabla_{V_i} X = (X f_i / f_i) V_i \equiv \left\{ X^1 \left( f_i^+ u(t-p) + f_i^- u(p-t) \right) / f_i \right\} V_i \).
3. \( \nabla_{V_i} V_j = \nabla_{V_j} V_i = 0 \) if \( i \neq j \)
4. \( \nabla_{V_i} W_i = II^i(V_i, W_i) = - (\langle V_i, W_i > / f_i \rangle \text{ grad } f_i \)
5. \( \nabla_{V_i} W_i \in \mathcal{L}(F_i) \) is the lift of \( \nabla_{V_i} W_i \) on \( F \).

**Proof**

Refer to O’Neill results and similar arguments yield the (1), (3), (5).

Clearly, it is only necessary to establish (2) and (4)

2. Since \( X(f_i) = X^1 \left( f_i^+ u(t-p) + f_i^- u(p-t) \right) / f_i \) where \( X = X^1 \frac{\partial}{\partial t} \)
we have $(X f_i / f_i)V_i = \left\{ X^1 \left( f_i^{t+} u(t - p) + f_i^{t-} u(p - t) \right) / f_i \right\} V_i.$

(4) Since \( \text{grad} \ f_i = \sum_{k=1}^n g^{k\ell} \frac{\partial f_i}{\partial x^\ell} \frac{\partial}{\partial t} \ = - \left( f_i^{t+} u(t - p) + f_i^{t-} u(p - t) \right) \frac{\partial}{\partial t} \)

nor \( \nabla_x W_i = II(V_i, W_i) = - (< V_i, W_i > / f_i \text{grad} \ f_i)

= ( < V_i, W_i > / f_i) \left( f_i^{t+} u(t - p) + f_i^{t-} u(p - t) \right) \frac{\partial}{\partial t} \).

\[ \square \]

**Proposition 3.2** Let \( M = B \times f_1 \times \ldots \times f_n \) be a multiply warped products with Riemannian curvature tensor \( R \). If \( X, Y \in \mathfrak{X}(B), U_i, V_i, W_i \in \mathfrak{X}(F_i) \) where \( 1 \leq i \leq n, f_i \in C^0(S) \), then

(1) \( R_{XY} Z \in \mathfrak{X}(B) \) is the lift of \( B R_{XY} Z(= 0) \) on \( \mathfrak{X}(B) \).

(2) \( R_{XY} U_i = R_{U_i Y} X = R_{U_i X} U_i = 0 \) for \( i \neq j \).

(3) \( R_{XY} Y = (H^f(X, Y)/f_i) U_i \)

where \( H^f \) is the Hessian of \( f_i \).

(4) \( R_{XY} U_i = R_{U_i Y} X = R_{U_i X} U_i = 0 \) for \( i \neq j \).

(5) \( R_{XY} U_i = 0 \) for each \( i = 1, \ldots, n \).

(6) \( R_{XY} V_i = \left( \frac{U_{i+1}}{f_i} \right) \left( f_i^{t+} u + f_i^{t-} \right) \left( f_i^{t+} u + f_i^{t-} \right) U_i \) for \( i \neq j \).

(7) \( R_{XY} V_i \)

\[ = f_i R_{U_i V_i} W_i \]

\[ = < \text{grad} \ f_i, \text{grad} \ f_i > / f_i^2 (V_i < f_i^r u(t - p) + f_i^l u(p - t) > / f_i^2) \]

\[ < U_i, W_i > V_i - < V_i, W_i > U_i. \]

**Proof**

We will establish (3) and (7)

For \( X = X^1 \frac{\partial}{\partial t}, Y = Y^1 \frac{\partial}{\partial t} \)

since \( \nabla_X \text{grad} \ f_i = \nabla_X (-\frac{\partial f_i}{\partial t} \frac{\partial}{\partial t} = - \left( \frac{\partial f_i}{\partial t} X^1 \nabla \frac{\partial}{\partial t} + X^1 \frac{\partial}{\partial t} \left( \frac{\partial f_i}{\partial t} \right) \frac{\partial}{\partial t} \right) \)

\[ = - X^1 \left( f_i^\ell (t) + \delta_p(t) \left( f_i^{t+} - f_i^{t-} \right) \right) \frac{\partial}{\partial t} \]

(3) \( H^f(X, Y) = \left\{ \nabla_X \text{grad} \ f_i, Y \right\} \)

\[ = \left\{ - X^1 \left( f_i^\ell (t) + \delta_p(t) \left( f_i^{t+} - f_i^{t-} \right) \right) \frac{\partial}{\partial t}, Y^1 \frac{\partial}{\partial t} \right\} \]

\[ = \left\{ X^1 Y^1 \left( f_i^\ell (t) + \delta_p(t) \left( f_i^{t+} - f_i^{t-} \right) \right) \frac{\partial}{\partial t} \right\} \]

(7) \( R_{XY} V_i = \left( \frac{U_{i+1}}{f_i} \right) V_i \)

\[ = f_i \left( f_i^{t+} u(t - p) + f_i^{t-} u(p - t) \right) \frac{\partial}{\partial t} \]

\[ < U_i, W_i > V_i - < V_i, W_i > U_i. \]

\[ \square \]
Corollary 3.3 Let $M = B \times f_1, F_1 \times \ldots \times f_n, F_n$ be a multiply warped products with Riemannian curvature tensor $R$. If $X, Y \in \mathfrak{L}(B)$, $U_i, V_i \in \mathfrak{L}(F_i)$ where $1 \leq i \leq n$, $d_i = \dim F_i f_i \in C^0(S)$, then

1. $\text{Ric}(X, Y) = -\sum_{i=1}^n (d_i/f_i) H^i(X, Y)$
2. $\text{Ric}(X, U_i) = 0$
3. $\text{Ric}(U_i, V_i) = f_i \text{Ric}(U_i, V_i)$

Proof
We will establish (1) and (3)

1. $\text{Ric}(X, Y) = \sum_{i=1}^n \langle \nabla_X \nabla_Y \rangle + \langle \nabla_Y \nabla_X \rangle$
2. $\text{Ric}(X, Y) = -\sum_{i=1}^n (d_i/f_i) H^i(X, Y)$
3. $\text{Ric}(X, Y) = -\sum_{i=1}^n (d_i/f_i) H^i(X, Y)$
4. $\text{Ric}(X, Y) = 0$ for $i \neq j$

Now consider the multiply warped products, $M = R^1 \times f_1, F_1 \times \ldots \times f_n, F_n$ with the warping function $f_i$ on the spacelike hypersurface $\Sigma = \{(x_1, \ldots, x_{n+1}) | x_1 \text{ is constant}\}$. If $M$ has a metric $g = -dt^2 + \sum_{i=1}^n f_i^2 g_{i}$, metric of $F_i$ ( $\dim F_i = d_i$ ), is $C^\infty$ and symmetric.

Proposition 3.4 Let $M = B \times f_1, F_1 \times \ldots \times f_n, F_n$ be the multiply warped product space of dimension $d = 1 + \sum_{i=1}^n d_i$ with the warping functions $f_i$ on the spacelike hypersurface $\Sigma = \{p\} \times F_1 \times \ldots \times F_n$. The metric components of $g = g^L \cup g^R$ are $C^1$
functions of the coordinate variables $i.e., f_i \in C^1$ if and only if $[L_\eta] = (L^R_\eta - L^L_\eta) = 0$ at each point on the spacelike hypersurface $\Sigma$. Here $\eta = \frac{\partial}{\partial t}$

**Proof**

We can get the second fundamental form as follow. For $M = B \times f_1 F_1 \times \ldots \times f_n F_n$, consider the basis of $T_pM$. $
\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial \mu_1}, \ldots, \frac{\partial}{\partial \mu_1 d_1}, \ldots, \frac{\partial}{\partial \nu_n}, \ldots, \frac{\partial}{\partial \nu_n d_n} \right\} \Rightarrow \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d} \right\}$

For $p \in \Sigma$, $X_p = a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} + \ldots + a_d \frac{\partial}{\partial x_d} \in T_\Sigma$, choose normal vector $\eta_p = \frac{\partial}{\partial x_1}$ such that $g(X_p, \eta_p) = 0$ for non-degenerate $g$ where $p = (x_1, x_2, \ldots, x_d)$. 

Since $L_\eta X = -\nabla X \eta = -\sum_{i=2}^{d+1} a_i \frac{\partial f_i}{f_i} \frac{\partial}{\partial x_1}$, therefore, we have

$$[L_\eta] = -\sum_{i=2}^{d+1} a_i \left[ \frac{\partial f_i}{f_i} \right] \frac{\partial}{\partial x_1}$$

$$[L_\eta] = 0 \iff \left[ \frac{\partial f_i}{\partial x_1} \right] = 0 \iff f_i \in C^1(\Sigma) \Box$$

**IV. SCHWARZSCHILD SPACE-TIME AS A MULTIPLY WARPED PRODUCT**

In this section we will briefly discuss the interior Schwarzschild solution. We show how the interior solution can be written as a multiply warped product.

A *Schwarzschild black hole* for the region $r < 2m$ have the metric.

$$ds^2 = -\left(\frac{2m}{r} - 1\right)^{-1} dt^2 + \left(\frac{2m}{r} - 1\right) dr^2 + r^2 d\Omega^2 \quad (4.1)$$

Replacing $r$ with $\nu$ and $t$ with $\mu$, we have $0 < \nu < 2m$ and

$$ds^2 = -\left(\frac{2m}{\nu} - 1\right)^{-1} d\nu^2 + \left(\frac{2m}{\nu} - 1\right) d\mu^2 + \nu^2 d\Omega^2 \quad (4.2)$$

Put

$$d\mu^2 = \left(\frac{2m}{\nu} - 1\right)^{-1} d\nu^2 \quad (4.3)$$

Integrating

$$d\mu = \sqrt{\frac{\nu}{2m - \nu}} d\nu$$

we obtain

$$\mu = 2m \cos^{-1}\left(\sqrt{\frac{2m - \nu}{2m}}\right) - \sqrt{\nu(2m - \nu)} + C = F(\nu) + C. \quad (4.4)$$

Setting $C = 0$ we obtain $F(\nu) = 2m \cos^{-1}\left(\sqrt{\frac{2m - \nu}{2m}}\right) - \sqrt{\nu(2m - \nu)}$

This yields

$$\lim_{\nu \to 2m} F(\nu) = m\pi, \quad \lim_{\nu \to 0} F(\nu) = 0$$
Notice \( \frac{d\nu}{d\mu} > 0 \) implies \( F^{-1} \) is a well-defined function.

By using (4.4) we rewrite (4.2) as

\[
ds^2 = -d\mu^2 + \left( \frac{2m}{F^{-1}(\mu)} - 1 \right) d\nu^2 + F^{-1}(\mu)^2 d\Omega^2
\]

Therefore we can write Schwarzschild space-time as a multiply warped products

\[
ds^2 = -d\mu^2 + f_1^2(\mu)d\nu^2 + f_2^2(\mu)d\Omega^2
\]

where

\[
f_1(\mu) = \sqrt{\left( \frac{2m}{F^{-1}(\mu)} - 1 \right)}
\]

and

\[
f_2(\mu) = F^{-1}(\mu)
\]

Clearly one may investigate the curvature of the interior metric (4.6) of Schwarzschild space-time as a multiply warped product. Furthermore, we have the following Ricci curvature on the multiply warped products.

**Corollary 4.1** Let \( M \) be a multiply warped product \( M = R^1 \times f_1 R^1 \times f_2 S^2 \) with metric \( ds^2 = -d\mu^2 + f_1^2(\mu)d\nu^2 + f_2^2(\mu)d\Omega^2 \) (where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \)) for warping functions \( f_1, f_2 \).

Then we have Ricci curvature

1. \( \text{Ric}(\frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}) \cdot \text{R}_{\mu \nu} = \text{R}_{11} = -\frac{f''}{f_1} - \frac{2f'''}{f_2} \)
2. \( \text{Ric}(\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \nu}) \cdot \text{R}_{\nu \nu} = f_1 f'' + \frac{f_1 f''' + f_2'''}{f_2} \)
3. \( \text{Ric}(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}) \cdot \text{R}_{\theta \phi} = 0 \)
4. \( \text{Ric}(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) \cdot \text{R}_{\phi \phi} = \left( \frac{f_1 f_2 f'_2}{f_1} + f_2^2 + f_2 f_2'' + 1 \right) \sin \theta \)
5. \( \text{R}_{mm} = 0 \) for \( m \neq n \)

**Remark** Let \( M = (0, 2m) \times f_1 R^1 \times f_2 S^2 \) be given the metric \( ds^2 = -d\mu^2 + f_1^2(\mu)d\nu^2 + f_2^2(\mu)d\Omega^2 \). Then \( M \) is the Ricci flat with interior Schwarzschild metric if \( \lim_{\mu \to 0} f_1(\mu)^2 + 1 = 2m \) and \( \lim_{\mu \to 0} f_1(\mu) = \lim_{\mu \to 0} f_2(\mu) \)

**Proof**

From corollary 4.1 we have \( \frac{d^2 f_1}{d\mu^2} = -\frac{2f'''}{f_2} \) and \( \frac{d^2 f_2}{d\mu^2} = -\frac{2f''}{f_1} \) from \( \text{R}_{11} = 0, \text{R}_{22} = 0 \) respectively.

Substitute \( \frac{f_1'}{f_1} = \frac{f_2'}{f_2} = \frac{f''}{f_2} + 1 \) to \( \frac{1}{f_2} \) \( \times \text{R}_{33} = 0 \), then we have \( \frac{f_1''}{f_1} + \frac{f_2'' + 1}{f_2} = 0 \)

so \( 2f_2'' + \frac{f_2'' + 1}{f_2} = 0 \) thus \( 2f_2'' + \frac{f_2'' + 1}{f_2} = 0 \) \( \text{R}_{33} = 0 \) after integrating from 0 to \( \mu \)
we have \( \ln (f_2(\mu)^2 + 1) f_2(\mu) = \lim_{\mu \to 0} \ln (f_2(\mu)^2 + 1) f_2(\mu) \).

Put \( \lim_{\mu \to 0} f_2(\mu)^2 + 1 = F(\mu)^2 + 1 = \sqrt{\frac{C}{f_2(\mu)^2}} - 1 \) \( f_2(\mu) = \sqrt{\frac{C}{f_2(\mu)^2}} - 1 \).

From \( \frac{f_1'}{f_1} = \frac{f_2'}{f_2} \) we have \( f_1(\mu) = \frac{f_2(\mu)}{f_2(\mu)} f_2(\mu) = f_2(\mu) \) with initial condition \( \lim_{\mu \to 0} f_2(\mu)^2 + 1 = 1 \) that is, \( f_1(\mu) = \sqrt{\frac{C}{f_2(\mu)^2}} - 1 \) therfore \( f_1(\mu) = \sqrt{\frac{2m}{f_2(\mu)^2}} - 1 \) if \( C = 2m \).

Since \( f_2(\mu) = F^{-1}(\mu) = \nu \) and \( f_1(\mu) = \sqrt{\frac{2m}{F^{-1}(\mu)^2}} - 1 \) \( \frac{d\nu}{d\mu} \) we have (4.3) \( \square \)
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