Weighted composition operator on quaternionic Fock space

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Abstract
This paper is concerned with several important properties of weighted composition operator acting on the quaternionic Fock space $F^2(\mathbb{H})$. Complete equivalent characterizations for its boundedness and compactness are established. As corollaries, the descriptions for composition operator and multiplication operator on $F^2(\mathbb{H})$ are presented, which can indicate some well-known existing theories in complex Fock space. Finally, as an appendix the closed expression for the kernel function of $F^2(\mathbb{H})$ is exhibited, which can deepen the understanding of $F^2(\mathbb{H})$.

Keywords weighted composition operator · quaternionic Fock space · boundedness · compactness · closed expression

Mathematics Subject Classification 30G35 · 47B38 · 30H20

1 Introduction

It is known that many mathematicians have been in creating a theory of slice regular functions of a quaternionic variable, which would somehow resemble the classical theory of holomorphic functions of one complex variable. Indeed the beginning of this theory is due to two important work by Gentili and Struppa [9, 10] and equally it gets a lot of interests. And then several significant theories have been developed systematically and found a wide range of applications, especially promising for the study of quaternionic quantum mechanics, see e.g. [4]. For more about slice regular

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function theory, we refer the readers into the detailed monographs [5, 11] and their reference therein. In Sect. 2, some preliminaries for quaternions and slice regular functions will be recalled for our further use.

Up to now, there appear various slice regular function spaces along with the well-development of the theory of regular functions. We refer the readers to [1, 8] for Fock space, [18] for Bloch, Besov and Dirichlet spaces and [3] for Bergman space, in the slice regular settings. These quaternionic spaces have attracted interest in the past decade for their various applications especially in operator theory. Among them, Fock spaces play important role in quantum mechanics and quaternionic formulation, and a full account of properties for the Fock spaces in the holomorphic setting have been formulated in [19]. Fortunately, the recent study of slice regular Fock spaces over quaternions is carried out in [1] and it also presents the Fock space of slice monogenic functions with values in a Clifford algebra, which generalize the corresponding case in the excellent book [19]. The flavor of these results gives a great impulse to the theory of quaternionic function spaces.

For a long time, one of the fundamental problems is to describe the behavior of linear operators acting on various holomorphic function spaces over the complex field. The classical linear operators including composition operator [6], weighted composition operator [13, 15] and so on. The systematical research of composition operators on holomorphic functions with complex variables is a fairly active field, see the two well-known books Cowen and MacCluer [7] and Shapiro [17]. As regards to the slice regular composition operators, Ren and Wang [16] study their properties acting on the quaternionic counterpart of complex Hardy spaces mentioned in [2]. As far as we are concerned, there has been no systematical investigation on weighted composition operators acting on slice regular function spaces. Considering Fock space is a convenient setting for many problems in functional analysis and mathematical physics, we will explore some classical properties of weighted composition operators acting on quaternionic Fock space in this paper, which can degenerate into the cases in complex Fock space.

It is well-known that the combination of a composition operator with multiplication operator can generate a weighted composition operator. However, there exist different ways to define the slice regular composition, which can further induce several definitions for weighted composition operators. Besides, the pointwise product of slice regular functions does not preserve slice regularity, a new multiplication operation, the ∗-product, is introduced in [4, 5]. Based on the above foundations, we will investigate the weighted composition operator $W_{f,\varphi}$ induced by slice regular functions $f$ and $\varphi$, which is defined as

$$ (W_{f,\varphi}h)_I(z) = f_I \star (h_I \circ \varphi_I)(z), $$

where $\varphi(I)$ satisfies the slice condition: $\varphi(I) \subset I$ for some $I \in \mathbb{S}$ (unit sphere of purely imaginary quaternions), and the operation ∗-product is the slice regular product (see Sect. 2). The ∗-product will make the weighted composition operator theory more complicated than the complex cases. In this paper, we focus our attention on some classical and challenging problems to investigate and characterize the boundedness and compactness of weighted composition operators acting on
quaternionic Fock spaces. Our main results are firstly devoted to equivalently characterizing quaternionic weighted composition operators, which will inspire others to turn their attention to a series of problems concerning linear operators on non-commutative setting.

The paper consists of 5 sections and its outline is as follows. In Sect. 2 we recall some basic facts concerning the slice regular functions, slice regular quaternionic Fock space (for short quaternionic Fock space) \( \mathcal{F}^2(\mathbb{H}) \), and weighted composition operator. Sect. 3 is devoted to presenting some equivalent characterizations for the boundedness and compactness of weighted composition operator acting on quaternionic Fock space. As corollaries, we obtain all forms of bounded and compact composition operators on \( \mathcal{F}^2(\mathbb{H}) \) in Sect. 4. Especially, we show there is also only a trivial bounded multiplication operator on \( \mathcal{F}^2(\mathbb{H}) \). As an appendix, the closed expression for the kernel function of \( \mathcal{F}^2(\mathbb{H}) \) is exhibited for our subsequent applications.

2 Preliminaries

2.1 Quaternions and slice regular function

Throughout this paper, the symbol \( \mathbb{H} \) denotes the noncommutative, associative, real algebra of quaternions with standard basis \( \{1, i, j, k\} \), subject to the multiplication rules

\[ i^2 = j^2 = k^2 = ijk = -1. \]

That is to say \( \mathbb{H} \) is the set of the quaternions

\[ q = x_0 + x_1 i + x_2 j + x_3 k = Re(q) + Im(q), \]

with \( Re(q) = x_0 \) and \( Im(q) = x_1 i + x_2 j + x_3 k \), where \( x_j \in \mathbb{R} \) for \( j = 0, 1, 2, 3 \). The Euclidean norm of a quaternion \( q \) is given by

\[ |q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \]

where \( \bar{q} = Re(q) - Im(q) = x_0 - (x_1 i + x_2 j + x_3 k) \) representing the conjugate of \( q \). By the notation \( \mathbb{S} \), it is the two-dimensional unit sphere of purely imaginary quaternions, i.e.

\[ \mathbb{S} = \{q = x_1 i + x_2 j + x_3 k : x_1^2 + x_2^2 + x_3^2 = 1\}. \]

That is, \( I^2 = -1 \) for \( I \in \mathbb{S} \). For any fixed \( I \in \mathbb{S} \) we define

\[ \mathbb{C}_I := \{x + Iy : x, y \in \mathbb{R}\}, \]

which can be identified with a complex plane. In the sequel, an element in the complex plane \( \mathbb{C}_I = \mathbb{R} + I\mathbb{R} \) is denoted by \( x + Iy \). Moreover, it holds that

\[ \mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I. \]
Interestingly, the real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real number can be associated with any imaginary unit $I$. However any non real quaternion $q$ is uniquely associated to the element $I_q \in \mathbb{S}$ given by

$$I_q := (ix_1 + jx_2 + kx_3)/|ix_1 + jx_2 + kx_3|,$$

and then $q$ belongs to the complex plane $\mathbb{C}_{I_q}$.

We are now in a position to give a key concept–slice regular function, which will be mentioned many times in the context.

**Definition 2.1** [5, Definition 2.1.1] Let $U$ be an open set in $\mathbb{H}$ and a function $f : U \to \mathbb{H}$ be real differentiable. The function $f$ is called (left) *slice regular* if, for every $I \in \mathbb{S}$, its restriction

$$f_I(x + yI) = f(x + yI)$$

is holomorphic, i.e. it has continuous partial derivatives and satisfies

$$\overline{\partial} f_I(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all $x + yI \in U \cap \mathbb{C}_I$. The class of all (left) slice regular functions on $U$ is denoted by $\mathcal{R}(U)$.

Besides, the function $f$ is right slice regular if, for every $I \in \mathbb{S}$, it satisfies

$$(f_I \overline{\partial} f_I)(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + yI) + \frac{\partial}{\partial y} f_I(x + yI)I \right) = 0$$

for all $x + yI \in U \cap \mathbb{C}_I$.

It is easy to check that $\mathcal{R}(U)$ is a right linear space on $\mathbb{H}$, that is, $fa + gb \in \mathcal{R}(U)$ for $f, g \in \mathcal{R}(U)$ and $a, b \in \mathbb{H}$. Especially, a function slice regular on $\mathbb{H}$ will be called *entire slice regular*, which are all collected as $\mathcal{R}(\mathbb{H})$.

Let $I, J \in \mathbb{S}$ be such that $I$ and $J$ are orthogonal, so that $\{1, I, J, IJ\}$ is an orthogonal basis of $\mathbb{H}$ and write the restriction $f_I(x + yI) = f(x + yI)$ of $f$ to the complex plane $\mathbb{C}_I$ as

$$f = f_0 + If_1 + Jf_2 + IJf_3.$$ 

It can also be written as $f_I = F + GJ$, where $f_0 + If_1 = F$ and $f_2 + IJf_3 = G$. Hence we have the following Splitting Lemma, which relates slice regularity with classical holomorphy.

**Lemma 2.1** [5, Lemma 2.1.4] *(Splitting Lemma)* If $f$ is a slice regular function on the domain $U$, then for every $I, J \in \mathbb{S}$, with $I \perp J$, there are two holomorphic functions $F, G : U_I = U \cap \mathbb{C}_I \to \mathbb{C}_J$ such that

$$f_I(z) = F(z) + G(z)J \quad \text{for any } z = x + yI \in U_I.$$
It is proved that slice regular functions possess good properties on specific open sets called axially symmetric slice domains.

**Definition 2.2** [5, Definition 2.2.1] Let $U \subset \mathbb{H}$ be a domain.

1. $U$ is called a slice domain (or s-domain for short) if it intersects the real axis and if, for any $I \in \mathbb{S}, U_I := \mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$.

2. $U$ is axially symmetric if for every $x + yI \in U$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, all the elements $x + y\mathbb{S} = \{x + yJ : J \in \mathbb{S}\}$ is contained in $U$.

The Representation Formula of a slice regular function on an axially symmetric domain allows to recover all its values from its values on a single slice $\mathbb{C}_I$.

**Proposition 2.1** [5, Theorem 2.2.4] (Representation Formula) Let $f$ be a slice regular function on an axially symmetric s-domain $U \subset \mathbb{H}$. Let $J \in \mathbb{S}$ and let $x \pm yJ \in U \cap \mathbb{C}_J$, then the following equality holds for all $q = x + yI \in U$,

$$f(x + yI) = \frac{1}{2}[(1 + IJ)f(x - yJ) + (1 - IJ)f(x + yJ)].$$

For more details on the entire slice regular functions, we refer the readers into the excellent books [4, 5] and their references therein.

### 2.2 Quaternionic Fock space

In this subsection, we recall the definition for slice regular quaternionic Fock space (for short quaternionic Fock space) introduced in [1].

**Definition 2.3** [1, Definition 3.6] Let $I$ be any elements in $\mathbb{S}$ and $q|_{\mathbb{C}_I} = z$, consider

$$\mathcal{F}^2(\mathbb{H}) = \{f \in \mathcal{R}(\mathbb{H}) : \int_{\mathbb{C}_I} e^{-|z|^2} |f_I(z)|^2 d\sigma(x, y) < \infty\},$$

where $d\sigma(x, y) := \frac{1}{\pi} dxdy$. We call $\mathcal{F}^2(\mathbb{H})$ (slice regular) quaternionic Fock space.

And then $\mathcal{F}^2(\mathbb{H})$ is an inner space endowed with the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}_I} e^{-|z|^2} \overline{g_I(z)}f_I(z) d\sigma(x, y).$$

It has been proven that the definition of quaternionic Fock space does not depend on the imaginary unit $I \in \mathbb{S}$ (see, e.g. [1, Proposition 3.8]). The norm induced by the inner product is
\[ \|f\| = \left( \int_{C_t} e^{-|z|^2} |f(z)|^2 d\sigma(x, y) \right)^{1/2}. \]

It has been shown that \( \mathcal{F}^2(\mathbb{H}) \) contains the monomials \( q^n \) \((n \in \mathbb{N})\), which form an orthogonal basis and \( \langle q^m, q^n \rangle = n! \). Moreover, a function \( f(q) = \sum_{m=0}^{\infty} q^n a_m \in \mathcal{F}^2(\mathbb{H}) \)

if and only if \( \sum_{m=0}^{\infty} |a_m|^2 m! < \infty \) (see, e.g. [1, Proposition 3.11]).

### 2.3 Weighted composition operators

In this subsection, we introduce the weighted composition operator \( W_{f, \varphi} \) acting on \( \mathcal{F}^2(\mathbb{H}) \). Since the pointwise product of slice regular functions does not preserve slice regularity, the \( \ast \)-product was introduced in [4, 5]. For the case of power series, the regular product (or \( \star \)-product) is defined as below. Let \( U \) be a ball with center at a real point, \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) with \( a_n \in \mathbb{H} \) and \( g(q) = \sum_{n=0}^{\infty} q^n b_n \) with \( b_n \in \mathbb{H} \), the \( \star \)-product of \( f \) and \( g \) is

\[ (f \star g)(q) := \sum_{n=0}^{\infty} q^n \left( \sum_{r=0}^{n} a_r b_{n-r} \right). \]

In this case, the notion \( \star \)-product coincides with the classical notion of product of series with coefficients in a ring. It is easy to see the function \( f \star g \) is slice regular. The \( \star \)-product can be further generalized to the case of regular functions defined on axially symmetric \( s \)-domains. Now let \( U \subset \mathbb{H} \) be an axially symmetric \( s \)-domain and let \( f, g : U \to \mathbb{H} \) be slice regular functions. For any \( I, J \in \mathbb{S} \) with \( I \perp J \), the Splitting Lemma guarantees the existence of four holomorphic functions \( F, G, H, K : U \cap \mathbb{C}_I \to \mathbb{C}_I \) such that for all \( z = x + yI \in U \cap \mathbb{C}_I \),

\[ f_I(z) = F(z) + G(z)J, \quad g_I(z) = H(z) + K(z)J. \]

Then the \( \star \)-product \( f_I \star g_I : U \cap \mathbb{C}_I \to \mathbb{H} \) is defined as

\[ f_I \star g_I(z) = [F(z)H(z) - G(z)K(\bar{z})] + [F(z)K(z) + G(z)\overline{H(\bar{z})}]J. \]

The new function \( f_I \star g_I \) is holomorphic and then it admits an unique slice regular extension to \( U \) defined by \( \text{ext}(f_I \star g_I)(q) \). Hence the \( \star \)-product of two generally slice regular functions is given as follows.

**Definition 2.4** Let \( U \subset \mathbb{H} \) be an axially symmetric \( s \)-domain and let \( f, g : U \to \mathbb{H} \) be slice regular functions. The function

\[ (f \star g)(q) = \text{ext}(f_I \star g_I)(q) \]
defined as the extension of (1) (using Proposition 2.1) is called the \(\star\)-product of \(f\) and \(g\).

**Remark 2.1** By the definition of \(\star\)-product, we obtain that

\[ J \star H(z) = 
\overline{H(\bar{z})}J. \]

**Remark 2.2** By (1), the \(\star\)-product could be formally defined on any two quaternionic functions. So this notation in Sect. 3 will be employed to make the similarity between the slice regular setting and classical holomorphic case.

Furthermore, the \(\star\)-product has a relation with the pointwise product as shown in the following proposition.

**Proposition 2.2** \([5, \text{Theorem 2.3.10}]\) Let \(U \subset \mathbb{H}\) be an axially symmetric \(s\)-domain, \(f, g : U \to \mathbb{H}\) be slice regular functions. Then

\[
(f \star g)(p) = f(p)g(f(p)^{-1}pf(p)),
\]

for all \(p \in U, f(p) \neq 0\), while \((f \star g)(p) = 0\) when \(p \in U, f(p) = 0\).

**Remark 2.3** Let \(p \in \mathbb{R}\), then

\[
(f \star g)(p) = f(p)g(f(p)^{-1}pf(p)) = f(p)g(p),
\]

which implies the \(\star\)-product of two slice regular functions reduces to the usual product when the variable takes real values.

**Remark 2.4** After the \(\star\)-product, we recall the reproducing kernel of \(\mathcal{F}^2(\mathbb{H})\). Given a variable \(q \in \mathbb{H}\) and a parameter \(p \in \mathbb{H}\), set

\[
e^{-|q|^2} = \sum_{n=0}^{+\infty} \frac{(qp)^n}{n!} = \sum_{n=0}^{+\infty} \frac{p^n q^n}{n!},
\]

it is immediate that \(e^{-|q|^2}\) is a function slice regular in \(q\) and right slice regular in \(p\). The reproducing kernel of \(\mathcal{F}^2(\mathbb{H})\) is given by \(K_p(q) := e^{-|q|^2}\), and it holds that \((f, K_p) = f(p)\) for any \(f \in \mathcal{F}^2(\mathbb{H})\) (see, e.g. [1, Theorem 3.10]). Besides,

\[
\|K_p\|^2 = (K_p, K_p) = K_p(p) = e^{-|p|^2}.
\]

Furthermore, denote \(k_p(q) = K_p(q)/\|K_p\| = e^{-|q|^2}/||K_p||K_p(q)\) a unit-vector in \(\mathcal{F}^2(\mathbb{H})\).

So far, we can formulate the definition for weighted composition operator step by step. Let \(\varphi : \mathbb{H} \to \mathbb{H}\) be a slice regular function such that \(\varphi(C_I) \subset C_I\) for some \(I \in \mathbb{S}\). The composition operator \(C_\varphi : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H})\) is defined on \(C_I\) by

\[
(C_\varphi h)_I(z) = (h_I \circ \varphi_I)(z) = U \circ \varphi_I(z) + V \circ \varphi_I(z)J
\]
for all \( h \in \mathcal{F}^2(\mathbb{H}) \) with \( h f(z) = U(z) + V(z)J \). Using the Representation Formula, we can get the extension \( C_f h \) to the whole \( \mathbb{H} \).

Let \( f \in \mathcal{R}(\mathbb{H}) \) and \( \varphi \in \mathcal{R}(\mathbb{H}) \) satisfy \( \varphi(\mathbb{C}_I) \subset \mathbb{C}_I \) for some \( I \in \mathbb{S} \), the weighted composition operator \( W_{f,\varphi} : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H}) \), with the domain consisting of all \( h \in \mathcal{F}^2(\mathbb{H}) \) such that \( W_{f,\varphi} h \) also belongs to \( \mathcal{F}^2(\mathbb{H}) \), is defined on \( \mathbb{C}_I \) as

\[
(W_{f,\varphi} h)_I(z) = f_I(z) \ast (h_I \circ \varphi_I)(z)
\]

for all \( h \in \mathcal{F}^2(\mathbb{H}) \). The extension \( W_{f,\varphi} h \) on \( \mathbb{H} \) is obtained by the Representation Formula. It is easy to check that

\[
[W_{f,\varphi}(ga + hb)]_I(z) = (W_{f,\varphi} g)_I(z)a + (W_{f,\varphi} h)_I(z)b
\]

for any \( g, h \in \mathcal{F}^2(\mathbb{H}) \) and \( a, b \in \mathbb{H} \), which means \( W_{f,\varphi} \) is a right linear operator on \( \mathcal{F}^2(\mathbb{H}) \). The left linear quaternionic weighted composition operator has been partly investigated in [14], which contains the sufficient and necessary conditions for bounded and compact weighted composition operators under the assumption \( \lambda^{-1} \varphi(0) \in \mathbb{R} \) with \( \lambda \neq 0 \). Fortunately, we can obtain complete equivalent formulas for the right linear weighted composition operators on \( \mathcal{F}^2(\mathbb{H}) \) in this paper.

At the end of this section, we include a proposition concerning the representations of some special entire functions on \( \mathbb{C} \) for proving our main theorems.

**Proposition 2.3** [15, Proposition 2.1] Let \( f \) and \( \varphi \) be two entire functions on \( \mathbb{C} \) such that \( f \) is not identically zero. Suppose there is a positive constant \( M \) such that

\[
|f(z)|^2 e^{\varphi(z)^2 - |z|^2} \leq M \quad \text{for all} \quad z \in \mathbb{C}.
\]

Then \( \varphi(z) = \varphi(0) + az \) for some \( |a| \leq 1 \). If \( |a| = 1 \), then \( f(z) = f(0) e^{-\bar{\beta}z} \), where \( \beta = \bar{a} \varphi(0) \). Furthermore, if

\[
\lim_{|z| \to \infty} |f(z)|^2 e^{\varphi(z)^2 - |z|^2} = 0,
\]

then \( \varphi(z) = az + b \) with \( |a| < 1 \).

### 3 Bounded or compact weighted composition operators

During the last decades, many authors have contributed to develop a rich operator theory on the classical Fock spaces, see, e.g. [12]. Recently, the study of quaternionic Fock space has been gathered much attention. So we concentrate on investigating the properties of weighted composition operators \( W_{f,\varphi} \) on \( \mathcal{F}^2(\mathbb{H}) \). Its boundedness and compactness can be viewed as the starting points in this direction. In this section, we always let \( f \) and \( \varphi \) be two-slice regular functions on \( \mathcal{F}^2(\mathbb{H}) \) such that \( f \) is not identically zero and \( \varphi(\mathbb{C}_I) \subset \mathbb{C}_I \) for some \( I \in \mathbb{S} \). Furthermore, we always choose \( J \in \mathbb{S} \) with \( I \perp J \) and express the restriction \( f_I \) on \( \mathbb{C}_I \) into
\[ f_I(z) = F(z) + G(z)I \]

with two entire functions \( F \) and \( G \). As usual the adjoint of the weighted composition operator \( W_{f,\varphi} \) is denoted by \( W^*_{f,\varphi} \).

For the reproducing kernel \( K_p \in \mathcal{F}^2(\mathbb{H}) \), we have

\[
\langle W^*_{f,\varphi} K_p, K_q \rangle = \langle K_p, W_{f,\varphi} K_q \rangle = \langle W_{f,\varphi} K_q, K_p \rangle,
\]

which entails

\[
W^*_{f,\varphi} K_p(q) = \overline{W_{f,\varphi} K_q(p)}.
\]

(2)

Now, we are ready to prove an equivalent characterization for bounded weighted composition operator on \( \mathcal{F}^2(\mathbb{H}) \).

**Theorem 3.1** Let \( f \) and \( \varphi \) be two slice regular functions on \( \mathbb{H} \), such that \( f \) is not identically zero and \( \varphi(C_1) \subset C_1 \) for some \( 1 \in \mathbb{S} \). Denote \( f_I(z) = F(z) + G(z)I \) with two entire functions \( F \) and \( G \), then \( W_{f,\varphi} : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H}) \) is bounded if and only if \( f \in \mathcal{F}^2(\mathbb{H}) \), \( \varphi(q) = \varphi(0) + q\lambda \) with some \( |\lambda| \leq 1 \), \( \lambda \in C_1 \) and

\[
\tilde{M}(f, \varphi) := \sup_{w \in C_1} \left( |F(w)|^2 e^{\varphi(w)^2 - |w|^2} + |G(w)|^2 e^{\varphi(\overline{w})^2 - |w|^2} \right) < \infty.
\]

(3)

**Proof** Necessity. Suppose the operator \( W_{f,\varphi} \) is bounded on \( \mathcal{F}^2(\mathbb{H}) \), then

\[
f = f \star 1 = W_{f,\varphi} 1 \in \mathcal{F}^2(\mathbb{H}).
\]

On the other hand, we know

\[
\|W_{f,\varphi}\|^2 = \|W^*_{f,\varphi}\|^2 \geq \sup_{p \in \mathbb{H}} \frac{\|W^*_{f,\varphi} K_p\|^2}{\|K_p\|^2}.
\]

(4)

With the help of (2), the norm \( \|W^*_{f,\varphi} K_p\|^2 \) becomes

\[
\|W^*_{f,\varphi} K_p\|^2 = \|W^*_{f,\varphi} K_p(q)\|^2 = \|W_{f,\varphi} K_q(p)\|^2 = \|W_{f,\varphi} K_q(p)\|^2 = \|f(p) \star K_q \circ \varphi(p)\|^2
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{(\varphi(p))^n z^n}{n!} \right)^2 e^{-|q|^2} d\sigma(x,y)
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{(\varphi(p))^n z^n}{n!} \right)^2 e^{-|z|^2} d\sigma(x,y)
\]

where \( q|_{C_1} = z = x + yI \in C_I \). Denoting \( p|_{C_I} = w \in C_I \) and using (4), we deduce
\[ \|W_{f,\varphi}\|^2 \geq \sup_{w \in C_I} e^{-|w|^2} \left( \int_{C_I} \left| f_I(w) \star \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2 e^{-|z|^2} d\sigma(x,y) \right). \] (5)

Since \( \varphi : \mathbb{C}_I \rightarrow \mathbb{C}_I \), it follows that

\[ \varphi(p)^n = \text{ext}(\varphi_I(w)^n) = \text{ext}(\varphi_I(w)^n). \]

Hence, on \( \mathbb{C}_I \), by the operation rule (1) it turns out

\[
\begin{align*}
  f_I(w) \star \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} &= (F(w) + G(w)J) \star \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \\
  &= F(w) \cdot \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} + G(w) \cdot \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \cdot J.
\end{align*}
\]

The above formulas further imply

\[
\begin{align*}
  \left| f_I(w) \star \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2 &= \left| F(w) \cdot \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2 + \left| G(w) \cdot \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2 \\
  &= |F(w)|^2 \left| \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2 + |G(w)|^2 \left| \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2.
\end{align*}
\]

Therefore, the display (5) becomes

\[
\begin{align*}
  \|W_{f,\varphi}\|^2 &\geq \sup_{w \in C_I} e^{-|w|^2} \left( \int_{C_I} \left| f_I(w) \right|^2 \left| \sum_{n=0}^{\infty} \frac{(\varphi_I(w))^n z^n}{n!} \right|^2 e^{-|z|^2} d\sigma(x,y) \right) \\
  &+ \int_{C_I} |G(w)|^2 \left( \sum_{n=0}^{\infty} \frac{|(\varphi_I(w))^n|^2 n!}{(n!)^2} + \sum_{n=0}^{\infty} \frac{|\varphi_I(w)|^2 n!}{(n!)^2} \right) e^{-|z|^2} d\sigma(x,y) \\
  &= \sup_{w \in C_I} e^{-|w|^2} \left( |F(w)|^2 \sum_{n=0}^{\infty} \frac{|(\varphi_I(w))^n|^2 n!}{(n!)^2} + |G(w)|^2 \sum_{n=0}^{\infty} \frac{|\varphi_I(w)|^2 n!}{(n!)^2} \right) \\
  &= \sup_{w \in C_I} e^{-|w|^2} \left( |F(w)|^2 e^{||\varphi_I(w)||^2} + |G(w)|^2 e^{||\varphi_I(w)||^2} \right) \\
  &= \sup_{w \in C_I} \left( |F(w)|^2 e^{||\varphi_I(w)||^2 - |w|^2} + |G(w)|^2 e^{||\varphi_I(w)||^2 - |w|^2} \right).
\end{align*}
\] (6)

Since the operator \( W_{f,\varphi} : \mathcal{F}^2(\mathbb{H}) \rightarrow \mathcal{F}^2(\mathbb{H}) \) is bounded, it follows that
\[
\sup_{w \in \mathbb{C}_t} (|F(w)|^2 e^{2|\varphi_I(w)|^2 - |w|^2} + |G(w)|^2 e^{2|\varphi_I(w)|^2 - |w|^2}) < \infty,
\]

which implies (3) is true. The above inequality contains the following two facts,

\[
\sup_{w \in \mathbb{C}_t} |F(w)|^2 e^{2|\varphi_I(w)|^2 - |w|^2} < +\infty, \tag{7}
\]

\[
\sup_{w \in \mathbb{C}_t} |G(w)|^2 e^{2|\varphi_I(w)|^2 - |w|^2} < +\infty. \tag{8}
\]

Either of (7) and (8) together with Proposition 2.3 imply that the restriction of \( \varphi \) on \( \mathbb{C}_t \) is

\[ \varphi_I(w) = \varphi_I(0) + w\lambda \]

with some \( |\lambda| \leq 1 \) for \( w, \lambda \in \mathbb{C}_t \).

For any variable \( q = x + yI \in \mathbb{H} \), taking \( p = x + yI \in \mathbb{C}_t \) and using Proposition 2.1, we deduce that

\[
\varphi(q) = \frac{1}{2}(1 - 2I\sigma)\varphi(p) + \frac{1}{2}(1 + 2I\sigma)\varphi(-p) \\
= \frac{1}{2}(1 - 2I)(\varphi(0) + p\lambda) + \frac{1}{2}(1 + 2I)(\varphi(0) + \bar{\lambda}) \\
= \varphi(0) + \frac{1}{2}[(p + \bar{\lambda}) + 2I\lambda - p\lambda] \\
= \varphi(0) + \frac{1}{2}[2x + J(-2I\lambda)] \\
= \varphi(0) + (x + J\lambda) \\
= \varphi(0) + q\lambda, \quad \text{with } \lambda \in \mathbb{C}_t \text{ and } |\lambda| \leq 1.
\]

This means the extension of \( \varphi_I \) into the whole \( \mathbb{H} \) is

\[ \varphi(q) = \varphi(0) + q\lambda, \quad q \in \mathbb{H}, \]

with some \( \lambda \in \mathbb{C}_t \) and \( |\lambda| \leq 1 \).

**Sufficiency.** Under the assumptions, we will show the operator

\[ W_{f, \varphi} : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H}) \]

is bounded. Firstly, we suppose that \( \lambda \neq 0 \). For any \( h \in \mathcal{F}^2(\mathbb{H}) \), it follows

\[
\| W_{f, \varphi}h \|^2 = \int_{\mathbb{C}_t} |[f \ast (h \circ \varphi)]_t(z)|^2 e^{-|z|^2} d\sigma(x, y) \]

\[
= \int_{\mathbb{C}_t} |f_t \ast (h \circ \varphi)_t(z)|^2 e^{-|z|^2} d\sigma(x, y), \tag{9}
\]

with \( q|_{\mathbb{C}_t} = z = x + yI \in \mathbb{C}_t \). Denoting \( (h \circ \varphi)_t = H(z) + K(z)J \) with two entire functions \( H \) and \( K \), the display (1) implies
\[ f_t(z) \star (h \circ \varphi)_t(z) \]
\[ = (F(z)H(z) - G(z)K(z)) + (F(z)K(z) + G(z)H(z))J. \]

Thus the square of its module length is estimated as
\[
|f_t(z) \star (h \circ \varphi)_t(z)|^2
\leq 2|F(z)|^2 + |G(z)K(z)|^2 + |F(z)K(z)|^2 + |G(z)H(z)|^2 \tag{10}
= 2|F(z)|^2|h(z)|^2 + 2|G(z)|^2|\varphi(z)|^2.
\]

Putting (10) into (9), we obtain
\[
\|W_{f, \varphi}h\|^2
\leq 2 \int_{\mathbb{C}_z} |F(z)|^2|G(z)|^2 e^{-|z|^2} d\sigma(x, y)
+ 2 \int_{\mathbb{C}_z} |G(z)|^2|F(z)|^2 e^{-|z|^2} d\sigma(x, y)
\leq 2 \sup_{z \in \mathbb{C}_z} |F(z)|^2 e^{\varphi(z)^2} \int_{\mathbb{C}_z} |h(z)|^2 e^{-|\varphi(z)|^2} d\sigma(x, y)
+ 2 \sup_{z \in \mathbb{C}_z} |G(z)|^2 e^{\varphi(z)^2} \int_{\mathbb{C}_z} |h(z)|^2 e^{-|\varphi(z)|^2} d\sigma(x, y)
= 2 \hat{M}(f, \varphi) \int_{\mathbb{C}_z} |(h \circ \varphi)_t(z)|^2 e^{-|\varphi(z)|^2} d\sigma(x, y)
= 2 \hat{M}(f, \varphi) |\lambda|^{-2} \|h\|^2,
\]
where the change of variable \( w = \varphi(z) \) was used in the last line. Therefore, the operator \( W_{f, \varphi} \) is bounded on \( \mathcal{F}^2(\mathbb{H}) \) for \( \lambda \neq 0 \).

For the case \( \lambda = 0 \), it holds that \( \varphi(z) = \varphi(0) \). Taking any \( h \in \mathcal{F}^2(\mathbb{H}) \), we have
\[ W_{f, \varphi}h = f \star (h \circ \varphi(0)) = f \star \langle h, K_{\varphi(0)} \rangle \in \mathcal{F}^2(\mathbb{H}) \]
due to \( f \in \mathcal{F}^2(\mathbb{H}) \). As a matter of fact, similar to (10), it yields that
\[
|f_t(z) \star \langle h, K_{\varphi(0)} \rangle|^2
\leq 2|F(z)|^2|\langle h, K_{\varphi(0)} \rangle_t(z)|^2 + 2|G(z)|^2|\langle h, K_{\varphi(0)} \rangle_t(z)|^2
\leq 2|f_t(z)|^2 \|h\|^2 \|K_{\varphi(0)}\|^2.
\]
It follows that
\[ \|W_{f,\varphi}h\|^2 \]
\[ = \int_{\mathbb{C}_r} |f_I(z) \star \langle h, K_{\varphi(0)} \rangle|^2 e^{\|z\|^2} d\sigma(x, y) \]
\[ \leq 2\|f\|^2 \|K_{\varphi(0)}\|^2 \|h\|^2. \]

In a summary, the operator \( W_{f,\varphi} : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H}) \) is bounded for \( \varphi(q) = \varphi(0) + q\lambda \) with given \( |\lambda| \leq 1 \) and \( \lambda \in \mathbb{C}_r \), ending the proof. \( \square \)

We proceed to give the explicit form of \( f \) under the condition \( W_{f,\varphi} \) is bounded on \( \mathcal{F}^2(\mathbb{H}) \) with \( \varphi(q) = qa + b \), \( |a| = 1 \), \( a \in \mathbb{C}_r \).

**Proposition 3.1** Let \( f \) and \( \varphi \) be two slice regular functions on \( \mathbb{H} \), such that \( f \) is not identically zero and \( \varphi(\mathbb{C}_r) \subset \mathbb{C}_r \) for some \( I \in \mathbb{S} \). Suppose \( W_{f,\varphi} \) is bounded on \( \mathcal{F}^2(\mathbb{H}) \) with \( \varphi(q) = qa + b \), \( |a| \leq 1 \), \( a, b \in \mathbb{C}_r \) and \( f_I(z) = F(z) + G(z)J \) with two entire functions \( F \) and \( G \). If \( |a| = 1 \), then

\[ f_I(z) = F(0)e^{-a\bar{b}z} + G(0)e^{-\bar{a}b\bar{z}} = (F(0) + G(0)J) \star e^{-a\bar{b}z} \]

with at least one of \( F(0) \) and \( G(0) \) is not zero.

**Proof** Suppose the operator \( W_{f,\varphi} \) is bounded on \( \mathcal{F}^2(\mathbb{H}) \), Theorem 3.1 ensures the display (3) holds. Furthermore, the inequalities (7) and (8) are valid.

Under the case \( \varphi_I(z) = za + b \) with \( |a| = 1 \), denoting \( \beta = \bar{a}b \), it yields that

\[ |\varphi_I(z)|^2 - |z|^2 = \bar{\beta}z + \beta\bar{z} + |b|^2 \]
\[ |\varphi_I(\bar{z})|^2 - |z|^2 = \bar{\beta}z + \beta\bar{z} + |b|^2. \]

Furthermore, (7) and (8) entail that

\[ \sup_{z \in \mathbb{C}_r} |F(z)|^2 e^{\|\varphi_I(z)^2 - |z|^2} = \sup_{z \in \mathbb{C}_r} |F(z)e^{\bar{\beta}z}|^2 e^{\|b|^2} \leq \hat{M}(f, \varphi), \]
\[ \sup_{z \in \mathbb{C}_r} |G(z)|^2 e^{\|\varphi_I(\bar{z})^2 - |z|^2} = \sup_{z \in \mathbb{C}_r} |G(z)e^{\beta\bar{z}}|^2 e^{\|b|^2} \leq \hat{M}(f, \varphi). \]

These two inequalities together with Proposition 2.3 (\( |a| = 1 \)) imply

\[ F(z) = F(0)e^{-\bar{\beta}z} \quad \text{and} \quad G(z) = G(0)e^{-\beta\bar{z}}. \]  \( (11) \)

It follows that

\[ f_I(z) = F(0)e^{-a\bar{b}z} + G(0)e^{-\bar{a}b\bar{z}}J. \]

Since \( f \) is not identically zero, hence at least one of \( F(0) \) and \( G(0) \) is not zero, which completes the proof. \( \square \)
Theorem 3.1 together with Proposition 3.1 validate the following proposition, which can provide an interesting example for bounded $W_{f, \varphi}$ when $f, \varphi$ are specific slice functions.

**Proposition 3.2** Let $\varphi(p) = pA + B$ and $f_j(z) = C_1 e^{D_1 z} + C_2 e^{D_2 z} J$, where $A, B, C_i$ and $D_i$ are complex constants for $i = 1, 2$. Then the operator $W_{f, \varphi}$ is bounded on $\mathcal{F}^2(\mathbb{H})$ if and only if

(a) either $|A| < 1$,

(b) or $|A| = 1, D_1 + AB = 0$ and $D_2 + AB = 0$.

Next we explore the equivalent characterizations for compact $W_{f, \varphi}$ on $\mathcal{F}^2(\mathbb{H})$.

**Theorem 3.2** Let $f$ and $\varphi$ be two slice regular functions on $\mathbb{H}$, such that $f$ is not identically zero and $\varphi(C_i) \subset C_i$ for some $i \in \mathbb{S}$. Denote $f_j(z) = F(z) + G(z) J$ with two entire functions $F$ and $G$, then $W_{f, \varphi} : \mathcal{F}^2(\mathbb{H}) \rightarrow \mathcal{F}^2(\mathbb{H})$ is compact if and only if $\varphi(q) = q \lambda + \varphi(0)$ with some $|\lambda| < 1, \lambda \in \mathbb{C}_I$ and

$$
\lim_{|w| \to \infty} \left( |F(w)|^2 e^{2|\varphi(w)|^2 - |w|^2} + |G(w)|^2 e^{2|\varphi(w)|^2 - |w|^2} \right) = 0. 
$$

**(Proof)** **Necessity.** Suppose $W_{f, \varphi} : \mathcal{F}^2(\mathbb{H}) \rightarrow \mathcal{F}^2(\mathbb{H})$ is compact, it must be bounded and then Theorem 3.1 implies

$$
\varphi(q) = q \lambda + \varphi(0)
$$

with some $|\lambda| \leq 1, \lambda \in \mathbb{C}_I$. Besides, we know the adjoint operator $W_{f, \varphi}^* is also compact on $\mathcal{F}^2(\mathbb{H})$. Since

$$
k_p = \|K_p\|^{-1} K_p \to 0
$$

weakly as $|p| \to \infty$, we have

$$
\|K_p\|^{-2} \|W_{f, \varphi}^* K_p\|^2 = e^{-|p|^2} \|W_{f, \varphi}^* K_p\|^2 \to 0
$$

as $|p| \to \infty$. Employing the calculations in (4) and (6), it yields

$$
|F(w)|^2 e^{2|\varphi(w)|^2 - |w|^2} + |G(w)|^2 e^{2|\varphi(w)|^2 - |w|^2} \to 0,
$$

as $|w| \to \infty$. That means the display (12) holds. For the case $|\lambda| = 1$, Proposition 3.1 implies (11) is true for $\beta = \lambda \varphi(0)$. Then

$$
|F(w)|^2 e^{2|\varphi(w)|^2 - |w|^2} + |G(w)|^2 e^{2|\varphi(w)|^2 - |w|^2} = (|F(0)|^2 + |G(0)|^2) e^{2|\varphi(0)|^2} = |f_j(0)|^2 e^{2|\varphi(0)|^2}.
$$

So it yields (12) does not converge to zero for the case $|\lambda| = 1.$
Sufficiency. Assume that

\[ \varphi(q) = q\lambda + \varphi(0) \]

with some \( |\lambda| < 1, \lambda \in \mathbb{C}_I \) and (12) holds. For the case \( \lambda = 0 \), it is true that \( W_{f,\varphi} h = f \star h(\varphi(0)) \), which illustrates \( W_{f,\varphi} \) has finite rank, thus it is compact.

Now assume that \( \lambda \neq 0 \), we proceed to prove \( W_{f,\varphi} : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H}) \) is compact. Let \( \{h_m\}_{m=1}^\infty \) be a bounded sequence in \( \mathcal{F}^2(\mathbb{H}) \) with \( C := \sup \|h_m\| < +\infty \) and converge weakly to 0 as \( m \to \infty \). Then the sequence \( \{h_m\}_{m=1}^\infty \) converges to zero uniformly on compact subsets of \( \mathbb{H} \). In the sequel, we will show that

\[ \|W_{f,\varphi} h_m\|^2 \to 0 \]

as \( m \to \infty \). Similarly following the calculations in (9) and (10), we have

\[
\begin{align*}
\|W_{f,\varphi} h_m\|^2 &= \int_{C_I} |f_I(z) \star (h_m \circ \varphi_I)(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&\leq 2 \int_{C_I} |F(z)|^2 |(h_m \circ \varphi_I)(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&\quad + 2 \int_{C_I} |G(\overline{z})|^2 |(h_m \circ \varphi_I)(z)|^2 e^{-|z|^2} d\sigma(x,y). \tag{13}
\end{align*}
\]

Denote

\[
I_1 := \int_{C_I} |F(z)|^2 |(h_m \circ \varphi_I)(z)|^2 e^{-|z|^2} d\sigma(x,y), \tag{14}
\]

\[
I_2 := \int_{C_I} |G(\overline{z})|^2 |(h_m \circ \varphi_I)(z)|^2 e^{-|z|^2} d\sigma(x,y). \tag{15}
\]

Next we need to show \( I_1 \to 0 \) and \( I_2 \to 0 \) as \( m \to \infty \). For \( w = u + Iv \in C_I \), define the function

\[
\hat{F}(w) = |\lambda|^{-2} |F(\varphi_I^{-1}(w))|^2 e^{\|w|^2 - |\varphi_I^{-1}(w)|^2},
\]

and then

\[
\hat{F}(\varphi_I(w)) = |\lambda|^{-2} |F(w)|^2 e^{\|\varphi(w)|^2 - |w|^2}.
\]

Since \( \lim_{|w| \to \infty} |\varphi_I^{-1}(w)| = \infty \), the first term of (12) yields \( \lim_{|w| \to \infty} \hat{F}(w) = 0 \), hence \( \hat{F} \) is a bounded function on \( C_I \) with \( \|\hat{F}\|_\infty < \infty \). For any \( \epsilon > 0 \), there exists a \( R > 0 \) such that
\[
\sup_{|w|>R} \hat{F}(w) < \varepsilon. \tag{16}
\]

Alternatively, it is easy to check that
\[
|F(w)|^2 = |\lambda|^2 \hat{F}(\varphi_I(w))e^{|w|^2-|\varphi_I(w)|^2}.
\]

Putting it into (14) and employing (16), we deduce that
\[
I_1 = \int_{C_I} |\lambda|^2 \hat{F}(\varphi_I(z))e^{[z]^2-|\varphi_I(z)|^2}(h_m)_I \circ \varphi_I(z)|^2 e^{-|z|^2} \, d\sigma(x, y)
\]
\[
= \int_{C_I} \hat{F}(w)e^{-|w|^2}|(h_m)_I(w)|^2 \, d\sigma(u, v)
\]
\[
= \int_{|w|\leq R} \hat{F}(w)e^{-|w|^2}|(h_m)_I(w)|^2 \, d\sigma(u, v)
\]
\[
+ \int_{|w|> R} \hat{F}(w)e^{-|w|^2}|(h_m)_I(w)|^2 \, d\sigma(u, v)
\]
\[
\leq \|\hat{F}\|_\infty \int_{|w|\leq R} |(h_m)_I(w)|^2 \, d\sigma(u, v)
\]
\[
+ \sup_{|w|> R} \hat{F}(w) \int_{|w|> R} e^{-|w|^2}|(h_m)_I(w)|^2 \, d\sigma(u, v)
\]
\[
\leq C^2 \varepsilon, \text{ as } m \to \infty.
\]

Due to \( \varepsilon \) is arbitrary, then \( I_1 \to 0 \) as \( m \to \infty \). We note that in the second line, the variable substitution \( w = \varphi_I(z) \) was used.

Analogously, define
\[
\hat{G}(w) = |\lambda|^{-2}|\varphi_I^{-1}(w)|^2 e^{[w]^2-|\varphi_I^{-1}(w)|^2},
\]
and it also holds \( \lim_{|w|\to \infty} \hat{G}(w) = 0 \). Moreover,
\[
|G(\overline{w})|^2 = |\lambda|^2 \hat{G}(\varphi_I(w))e^{[w]^2-|\varphi_I(w)|^2}.
\]

Now for the term \( I_2 \) in (15), it follows that
\[
I_2 := \int_{C_I} |\lambda|^2 \hat{G}(\varphi_I(z))e^{[z]^2-|\varphi_I(z)|^2}(h_m)_I \circ \varphi_I(z)|^2 e^{-|z|^2} \, d\sigma(x, y)
\]
\[
= \int_{C_I} |\lambda|^2 \hat{G}(\varphi_I(z))e^{-|z|^2}(h_m)_I \circ \varphi_I(z)|^2 \, d\sigma(x, y)
\]
\[
= \int_{C_I} \hat{G}(w)e^{-|w|^2}|(h_m)_I(w)|^2 \, d\sigma(u, v)
\]
\[
\to 0, \text{ as } m \to \infty.
\]

Taking the above calculations into (13), we conclude that \( \|W_{f, \varphi} h_m\| \to 0 \) as \( m \to \infty \). This means \( W_{f, \varphi} : \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H}) \) is compact. \( \square \)
4 Composition operator and multiplication operator

4.1 Composition operator

As defined earlier, take a slice regular function \( \varphi : \mathbb{H} \rightarrow \mathbb{H} \) such that \( \varphi(C_I) \subset C_I \) for some \( I \in \mathbb{S} \), then the composition operator \( C_\varphi \) on \( C_I \) is

\[
(C_\varphi h)_I(z) = (h_I \circ \varphi)(z) = U \circ \varphi_I(z) + V \circ \varphi_I(z)J
\]

for all \( h \in F^2(\mathbb{H}) \) with \( h_I(z) = U(z) + V(z)J \). And then using the Representation Formula (Proposition 2.1), we can get all values of \( C_\varphi h \) on the whole \( \mathbb{H} \). Applying the fact \( C_\varphi \) is a special weighted composition operator \( W_{1,\varphi} \), all types of bounded and compact composition operators on \( F^2(\mathbb{H}) \) are presented as below. The following results reveal our quaternionic-valued composition operator is a natural generalization of complex cases ([12, Propositions 3.1 and 3.2]).

**Corollary 4.1** Let \( \varphi \) be a slice regular function on \( \mathbb{H} \) such that \( \varphi(C_I) \subset C_I \) for some \( I \in \mathbb{S} \). Then \( C_\varphi : F^2(\mathbb{H}) \rightarrow F^2(\mathbb{H}) \) is bounded if and only if \( \varphi(q) = q\lambda + \varphi(0) \) with some \( |\lambda| \leq 1 \) and \( \lambda \in C_I \).

**Corollary 4.2** Let \( \varphi \) be a slice regular function on \( \mathbb{H} \) such that \( \varphi(C_I) \subset C_I \) for some \( I \in \mathbb{S} \). Then \( C_\varphi : F^2(\mathbb{H}) \rightarrow F^2(\mathbb{H}) \) is compact if and only if \( \varphi(q) = q\lambda + \varphi(0) \) with some \( |\lambda| < 1 \) and \( \lambda \in C_I \).

4.2 Multiplication operator

By the operation \( \star \)-product, multiplication operator \( M_f : F^2(\mathbb{H}) \rightarrow F^2(\mathbb{H}) \), with \( f \in R(\mathbb{H}) \), is defined as

\[
(M_fh)_I(z) = f_I(z) \star h_I(z)
\]

for all \( h \in F^2(\mathbb{H}) \), which is a special case of \( W_{f,\varphi} \) with \( \varphi(q) = q \) for all \( q \in \mathbb{H} \). The corollary below implies that there is only trivial bounded multiplication operator \( M_f \) on \( F^2(\mathbb{H}) \) with constant function \( f \).

**Corollary 4.3** Given \( f \in R(\mathbb{H}) \) not identically zero, then \( M_f : F^2(\mathbb{H}) \rightarrow F^2(\mathbb{H}) \) is bounded if and only if \( f \) is a constant function.

**Proof** The sufficiency is a trivial implication. We only need to prove the necessity. From the proof in Theorem 3.1 with \( \varphi(q) = q \), it holds that

\[
\sup_{w \in C_I} \left( |F(w)|^2 + |G(w)|^2 J \right) < +\infty
\]
for the function \( f_I(w) = F(w) + G(w)J \) with two entire functions \( F \) and \( G \). Combining Liouville Theorem, it turns out both \( F(w) \) and \( G(w) \) are constant on \( \mathbb{C}_I \), and then \( f \) is a constant function on \( \mathbb{C}_I \). By Proposition 2.1, we can deduce \( f \) is a constant function on \( \mathbb{H} \), ending the proof. ⧫

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Appendix

It is well-known that the kernel function always plays significant role in the investigations on properties of Hilbert spaces. In this appendix, we show that the quaternionic exponential function \( e^{pq} = \sum_{n=0}^{\infty} \frac{p^n q^n}{n!} \) admits a closed expression.

**Theorem 5.1** For any \( p = a + \omega b, q = c + \eta d \), where \( \omega, \eta \) belong to the 2-sphere \( \mathbb{S} \). Then function \( e^{pq} \) can be expressed as

\[
\begin{align*}
\frac{1}{2} & (\cos(bd - ac) + \omega \sin(bd - ac))e^{ac+bd}(1 + \omega \eta) \\
&+ \frac{1}{2} (\cos(ac + bd) - \omega \sin(ac + bd))e^{ac-bd}(1 - \omega \eta).
\end{align*}
\]

**Proof** Denoting

\[
p = a + \omega b = r_1 (\cos x + \omega \sin x)
\]

and

\[
q = c + \eta b = r_2 (\cos y + \eta \sin y),
\]

by direct computation, we have that

\[
p^n q^n = r_1^n r_2^n (\cos nx + \omega \sin nx)(\cos ny + \eta \sin ny)
\]

\[
= r_1^n r_2^n \cos nx \cos ny + r_1^n r_2^n \omega \sin nx \cos ny
\]

\[
+ r_1^n r_2^n \eta \cos nx \sin ny + r_1^n r_2^n \omega \eta \sin nx \sin ny.
\]

Thus, the quaternionic exponential can be split into four series,

\[
e^{pq} = K_1 + K_2 + K_3 + K_4
\]

where
In the sequel, we compute the closed expression of each \( K_i, \ i = 1, 2, 3, 4 \).

\[
K_1 := \sum_{n=0}^{\infty} \frac{r_1^n r_2^n \cos nx \cos ny}{n!},
\]
\[
K_2 := \omega \sum_{n=0}^{\infty} \frac{r_1^n r_2^n \sin nx \cos ny}{n!},
\]
\[
K_3 := \eta \sum_{n=0}^{\infty} \frac{r_1^n r_2^n \cos nx \sin ny}{n!},
\]
\[
K_4 := \omega \eta \sum_{n=0}^{\infty} \frac{r_1^n r_2^n \sin nx \sin ny}{n!}.
\]

Putting the terms \( K_i, \ i = 1, 2, 3, 4 \) into (18), eliminating the angular variables and grouping corresponding terms, we obtain the desired closed expression (17). \( \square \)

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