Quantum recurrences versus stability

LOUIS E. LABUSCHAGNE
DEPARTMENT OF MATHS, APPLIED MATHS AND ASTRONOMY
UNIVERSITY OF SOUTH AFRICA
P.O. BOX 392
0003 PRETORIA, SOUTH AFRICA
E-mail address: labusle@unisa.ac.za

WŁADYSŁAW A. MAJEWSKI
INSTITUTE OF THEORETICAL PHYSICS AND ASTROPHYSICS
GDAŃSK UNIVERSITY
WITA STWOSZA 57
80-952 GDAŃSK, POLAND
E-mail address: fizwam@univ.gda.pl

ABSTRACT. Consequences of quantum recurrences on the stability of a broad class of dynamical systems is presented.

Key words and phrases: quantum recurrences, detailed balance condition, Kadison-Schwarz inequality, decoherence, return to equilibrium.

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1. Definitions, notations and stating the problem

For any \( C^* \)-algebra \( \mathcal{A} \) let \( \mathcal{A}^+ \) denote the set of all positive elements in \( \mathcal{A} \). A state on a unital \( C^* \)-algebra \( \mathcal{A} \) is a linear functional \( \omega : \mathcal{A} \to \mathbb{C} \) such that \( \omega(a) \geq 0 \) for every \( a \in \mathcal{A}^+ \) and \( \omega(1) = 1 \) where \( 1 \) is the unit of \( \mathcal{A} \). By \( \mathcal{S}(\mathcal{A}) \) we will denote the set of all states on \( \mathcal{A} \). For any Hilbert space \( \mathcal{H} \) we denote by \( \mathcal{B}(\mathcal{H}) \) the set of all bounded linear operators on \( \mathcal{H} \). Clearly, \( \mathcal{B}(\mathcal{H}) \) is an example of \( C^* \)-algebra.

A linear map \( \tau : \mathcal{A} \to \mathcal{B} \) between \( C^* \)-algebras is called positive if \( \tau(\mathcal{A}^+) \subset \mathcal{B}^+ \). For \( k \in \mathbb{N} \) we consider a map \( \tau_k : M_k(\mathcal{A}) \to M_k(\mathcal{B}) \) where \( M_k(\mathcal{A}) \) and \( M_k(\mathcal{B}) \) are the algebras of \( k \times k \) matrices with coefficients from \( \mathcal{A} \) and \( \mathcal{B} \) respectively, and \( \tau_k([a_{ij}]) = [\tau(a_{ij})] \). We say that \( \tau \) is \( k \)-positive if the map \( \tau_k \) is positive. The map \( \tau \) is said to be completely positive when it is \( k \)-positive for every \( k \in \mathbb{N} \).

The triple \( (\mathcal{A}, \tau, \omega) \) consisting of a unital \( C^* \)-algebra \( \mathcal{A} \), a linear positive unital map \( \tau \), and a state \( \omega \) will be called a (quantum) dynamical system. We will need:

**Definition 1.1.** We say that the quantum dynamical system \( (\mathcal{A}, \tau, \omega) \) satisfies detailed balance II if there exists another linear positive unital map \( \tau^\beta \) such that

\[
\omega(A^* \tau(B)) = \omega(\tau^\beta(A^*)B)
\]

for all \( A, B \in \mathcal{A} \).

There have been various versions of the detailed balance condition (cf discussion in \[9\]). Here we would mention only the detailed balance I given in \[8\] as that version is related to the existence of a form of time-reversal for the underlying dynamics. The relations between both conditions are described in \[9\].

We will assume that the dynamical system \( (\mathcal{A}, \tau, \omega) \) satisfies detailed balance condition (DB)II. We recall that DB II (the same will be true under DB I) implies: i) the state \( \omega \) is \( \tau \)-invariant, ii) in the GNS representation \((\mathcal{H}, \pi_\omega, \Omega)\) of \((\mathcal{A}, \omega)\), the definition \( T_\omega \pi_\omega(A) \Omega = \pi_\omega(\tau A) \Omega \) gives a contraction \( T_\omega \) on the Hilbert space \( \mathcal{H} \), iii) if additionally \( \omega \) is a faithful state then \( T_\omega \) commutes (strongly) with the associated modular operator.

Finally, to formulate the quantum Khintchin theorem we need the concept of a relatively dense subset. We say that \( \mathcal{N} \subset \mathbb{N} \) is relatively dense provided that there exists an \( L > 0 \) such that in any interval of natural numbers having length larger than \( L \) one can find a number \( n \in \mathcal{N} \). Recently, the following quantum generalization of Khintchin’s theorem was proved (see \[13, 3\])

**Theorem 1.2.** Let \( \mathcal{A} \) be a \( C^* \)-algebra, \( \varphi \) a state on \( \mathcal{A} \) and \( \tau : \mathcal{A} \to \mathcal{A} \) a positive linear map such that \( \varphi \circ \tau = \varphi \). Let us assume that

\[
\varphi(\tau(A)^* \tau(A)) \leq \varphi(A^* A)
\]

for every \( A \in \mathcal{A} \). Then, for every \( A \in \mathcal{A} \) and \( \epsilon > 0 \), there exists a relatively dense subset \( \mathcal{N} \) of \( \mathbb{N} \) such that

\[
Re(\varphi(A^* \tau^n(A))) \geq |\varphi(A)|^2 - \epsilon
\]

for all \( n \in \mathcal{N} \).

The aim of that note is to show that DB II combined with the quantum Khintchin theorem (so with quantum recurrences) compel a quantum dynamical system to pattern upon reversible evolution. Clearly, it can be considered as a “quantum reminiscence” of the famous controversy between Boltzmann and Poincaré; \( \tau \) in Theorem 1.2 represents a general stochastic map! Here, we will argue that our result
may be used in the study of decoherence and stability of a large class of quantum systems. Namely, defining decoherence to be an irreversible emergence of classical properties in a quantum system (so disappearing of macroscopic interferences) one can say that the essential character of decoherence appears to be irreversibility (cf. [14], [16], [4], and [15]). In other words, it seems that decoherence is not an intrinsic property of Nature but rather a dynamical effect. In that context, our next result says that DB together with quantum recurrences spoil the stability of dynamics thereby producing an obstacle for creation dynamical effects which could have lead to a decoherence phenomenon.

2. Stability

As mentioned, the DB II implies that $T_\omega$ is a contraction. Combining that result with the fact that (1.2) was used in the proof of Theorem 1.2 only to get a contraction in the GNS space one has

$$(2.1) \quad ||a\Omega||||T^a_\omega a\Omega|| \geq ||a\Omega, T^a_\omega a\Omega|| \geq |(\Omega, T^a_\omega a\Omega)|^2 - \epsilon$$

for any $a \in \pi_\omega(A) = A_0$, and $n \in \mathbb{N}$. Suppose, $\omega(A) \neq 0$ and $\lim_{n \to \infty} ||T^a_\omega a\Omega|| = 0$ for $\pi_\omega(A) \equiv a \in A_0$. Hence, $\forall_{\epsilon > 0} \exists_{N} \forall_{n > N} \quad ||T^a_\omega a\Omega|| < \epsilon$. This and (2.1) implies

$$(2.2) \quad \forall_{\epsilon > 0} \exists_{N} \forall_{n > N} \forall_{n' \in \mathbb{N}} \quad \epsilon ||a\Omega|| > ||T^a_\omega a\Omega|| ||a\Omega|| \geq |(\Omega, a\Omega)|^2 - \epsilon$$

which is a contradiction. Hence, for any $a(= \pi_\omega(A)$ such that $\omega(A) \neq 0$ the sequence $\{||T^a_\omega a\Omega||\}$ does not go to 0. The limit exists as the sequence being monotonic nonincreasing and bounded below is convergent. Thus we get a form of stability for the discrete evolution $\{T^a_\omega\}$.

Let us discuss the consequences of that result. Firstly, we recall that for positive semigroups on $C^*$-algebras with unit, weak stability and uniform stability coincide (see [12], Theorem 1.7 in Chapter D-IV). This means that we are not able to split our original algebra into two subalgebras in such a way that there would exist two $\tau$-invariant subalgebras, one of them such that expectation values for observables from that set are practically equal to zero after large time. The main obstacle to such a splitting would be the existence of the time invariant state that is guaranteed by the DB condition.

Therefore, one can expect a similar decomposition of observables associated with the considered system to that given in (cf. [2], [7]) but now only in the representation space $\mathcal{H}$. We emphasize that such a type of decomposition was the main ingredient of the discussion of decoherence in [2], [7]. Let us define the desired form of decomposition. The search for a decomposition of the full algebra which contains selected observables with a weak stability property can be justified by the phenomenon called the environment-induced decoherence (cf. [15] and the references given there). Assume $\mathcal{A}$ is a $W^*$-algebra, so $A_0$ is a von Neumann algebra on the Hilbert space $\mathcal{H}$. Further, $\omega$ is a faithful normal state given by a cyclic and separating vector $\Omega \in \mathcal{H}$. We wish to have:

$$(2.3) \quad \mathcal{A}_0 = \mathcal{A}_1 \oplus \mathcal{A}_2$$

where both subsets $\mathcal{A}_1, \mathcal{A}_2$ are $\pi_\omega(\tau)$-invariant and the following properties hold:

- $\mathcal{A}_1$ is a von Neumann subalgebra of $\mathcal{A}$ and the evolution $\pi_\omega(\tau)$ when restricted to $\mathcal{A}_1$ is reversible.
• \( \mathcal{A}_2 \) is a linear space (closed in the norm topology) such that for any observable \( B = B^* \in \mathcal{A}_2 \) and any normal state \( \phi \) of the system with the support \( s(\phi') \) of its extension \( \phi' \) on \( \mathcal{B}(\mathcal{H}) \) orthogonal to \( |\Omega><\Omega| \), there is

\[
\lim_{t \to \infty} \phi(\pi_\omega(t)(B)) = 0
\]

However we note that (2.4) implies

\[
\forall x \in A_0' \forall \epsilon > 0 \exists N = N(\epsilon, x) \forall n > N \quad |(x^* x \Omega, T^n_\omega B \Omega)| < \epsilon
\]

where \( B \in \mathcal{A}_2 \). Here, \( \mathcal{A}_0' \) stands for the commutant of \( \mathcal{A}_0 \). We have also used that \( \Omega \) is both a cyclic and separating vector. Thus, an \( f \in \mathcal{H} \) can be approximated by vectors of the form \( \{ y \Omega, \ y \in \mathcal{A}_0' \} \). On the other hand, for all \( B \in \mathcal{A}_2 \) such that \( \omega(B) \neq 0 \), the contractivity of \( T_\omega \) and (2.1) imply

\[
\forall \epsilon > 0 \exists N(\epsilon) \forall n > N \quad \text{const} - \epsilon < ||T^n_\omega B \Omega|| < \text{const} + \epsilon
\]

where \( \text{const} \) is a positive number (depending on \( B \)). This leads to

\[
\forall \epsilon > 0 \exists N(\epsilon) \forall n > N \quad \text{const} - 2\epsilon < |(f, T^n_\omega B \Omega)| < \text{const} + 2\epsilon
\]

which would contradict (2.7). We have used that

\[
\forall \epsilon > 0 \exists N(\epsilon) \forall n > N \quad ||T^n_\omega B \Omega|| - |(f, T^n_\omega B \Omega)| < \epsilon.
\]

Consequently, only a very specific decomposition of the von Neumann algebra associated to the set of observables of the system will be possible.

3. Discussion

Inequality (1.2) is nothing but a composition of the Kadison-Schwarz inequality with a state. To get it 2-positivity would be enough. However, to get the decomposition of the form (2.3) one needs some extra conditions (cf [7]). It is an easy observation that DB provides these conditions. In other words, a general dynamical semigroup consisting of completely positive maps does not fulfill the necessary requirements unless it possesses additional properties, eg. the DB condition.

Further, as we do not expect that the collective variables form a \( C^* \)-subalgebra we should distinguish between weak and uniform stability. Clearly, this makes sense if the considered system has an infinite number of degrees of freedom. At this point it is worth mentioning that it was Heisenberg who pointed out, on various occasions, the role of environment (so infinite systems) in the problem of suppression of macroscopic interferences. Therefore, our approach relying on the idea of infinite systems is well justified.

Our approach sheds some new light on relations between recurrences and stability of dynamical semigroups \( V_t \) on a Hilbert space \( (V_t \) is a one parameter strongly continuous semigroup of linear contractions on \( \mathcal{H} \)). Namely, strong stability of such semigroups is defined as

\[\text{Definition 3.1.} \quad \text{The semigroup } V_t \text{ on } \mathcal{H} \text{ is strongly stable if as } t \to \infty \| V_t f \| \to 0 \text{ for all } f \in \mathcal{H}.\]

Let us recall (cf. [6]).
Theorem 3.2. Let the semigroup $V_t$ be a contraction semigroup on $\mathcal{H}$. $\mathcal{H}$ has a maximal closed subspace $\mathcal{H}_1$ on which $V_t$ is (i.e. restricts to) a unitary semigroup. The restriction of $V_t$ on $\mathcal{H}_1$ is a completely non unitary semigroup. Moreover, both $V_t$ and $V_t^*$ are strongly stable on $\mathcal{H}_1$ if and only if $P = Q$ is a projection, where

$$P f = \lim_{t \to +\infty} V_t^* V_t f \quad \& \quad Q f = \lim_{t \to +\infty} V_t V_t^* f$$

for $f \in \mathcal{H}$. The range of $P = Q$ is then $\mathcal{H}_1$.

Remark 3.3.
(i) The limits in (3.1) exist (cf. [5]).
(ii) Conditions leading to semigroups being strongly stable on $\mathcal{H}_1$ was also studied in ([10] and [11]).

Stability of semigroups is also strongly related to peculiar properties of infinitesimal generators of semigroups. Namely,

Theorem 3.4. (see [1]) Let $V_t$ be a bounded $C_0$-semigroup with generator $A$. Assume that $\sigma_r(A) \cap i\mathbb{R} = \emptyset$, where $\sigma_r(A)$ denotes the residual part of the spectrum of $A$. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $V_t$ is a strongly stable $C_0$-semigroup.

Firstly, we recall that a dynamical semigroup $\tau_t$ (one parameter semigroup of linear unital positive maps on $\mathcal{A}$) satisfying DB II or DBI gives rise to a dynamical semigroup of type $V_t$ on the GNS Hilbert space $\mathcal{H}$ such that $\sigma_r(A) = \emptyset$ if DBI holds. Secondly, the results of Section 2 have shown that the quantum recurrences spoil stability. Consequently, the recurrence phenomenon leads to a peculiar form of the spectrum of the infinitesimal generator of $V_t$ (uncountable intersection of the spectrum with the imaginary axis) and gives special asymptotic behaviour of the semigroup ($P \neq Q$, cf (3.1)). Furthermore, for dynamics consisting of positive maps satisfying DB II there is only room for “scattered” collective observables, i.e. they do not form such a “rich” structure as that which was used to define $\mathcal{A}_2$. Consequently, one may conjecture that the interaction between the system and the environment singles out a subset of states with well defined stability for some but not all of the operators.

It may be worth noting that the obtained results could be viewed from another perspective. If one takes stability of certain observables in selected states as a given, then the contradiction obtained in (2.7) can also be interpreted as saying that recurrence of quantum phenomena in the sense of Stroh, Zsidó, et al. [13] can only occur in states which suppress stability in the sense of annihilating $\mathcal{A}_2$. Hence in the presence of the DB condition, stability limits recurrence.

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