On the Sum and Product of Distinct Prime Factors of an Odd Perfect Number

Anirudh Prabhu
West Lafayette Jr/Sr High School
West Lafayette, IN 47906
aprabhu@purdue.edu

Abstract

We present lower bounds on the sum and product of the distinct prime factors of an odd perfect number, which provide a lower bound on the size of the odd perfect number as a function of the number of its distinct prime factors.

The study of perfect numbers dates back to Book IX of The Elements by Euclid circa 300 B.C. [5]. To date, 47 perfect numbers have been discovered and all of them are even [14]. The quest to find odd perfect numbers started more than 350 years ago, as evidenced by the 1638 communications between Descartes and Mersenne [5], and has remained unsuccessful to this day; at present, the non-existence of odd perfect numbers has not been proved either although it is known that if an odd perfect number exists it has to exceed $10^{300}$ [2]. Over the last 350 years a significant body of work by some of the most eminent mathematicians has focused on the conditions that must be satisfied by odd perfect numbers. Euler showed that an odd perfect number $N$ must have the form $N = P^n Q^2$ where $P$ is a prime number and $P \equiv n \equiv 1 \pmod{4}$ and a similar result was also derived by Frenicle in 1657 [5]. Sylvester’s 1887 conjecture that an odd perfect number must have at least six distinct prime factors was proved by Gradshtein in 1925 [1]; the result was subsequently improved by Nielsen who showed that every odd perfect number must have at least nine distinct prime factors [16]. In 1888 Catalan showed that an odd perfect number $N$ that is not divisible by 3, 5 or 7 must have at least 26 distinct prime factors; subsequently Norton improved the result by showing that $N$ must have at least 27 factors [17]. Norton also showed that an odd perfect number that is not divisible by 3 or 5 must have at least 15 distinct prime factors [17] while Nielsen [16] showed that an odd perfect number that is not divisible by 3 must have at least 12 distinct prime factors. Hare showed that the number of prime factors of an odd perfect number, counting multiplicity, is at least 75 [8]. In 1896 Stuyvaert observed that an odd perfect number must be a sum of two squares [5], while Touchard [22] showed that if an odd perfect number $N$ exists then $N \equiv 1 \pmod{12}$ or $N \equiv 9 \pmod{36}$. Iannucci and Jenkins have showed that the largest three factors of an odd perfect number must exceed $10^8, 10^4$ and $10^2$ respectively [11,10,11]. In 1913 Dickson showed that for every positive integer $r$, there can only be finitely many odd perfect numbers with $r$ distinct prime factors [6]. It is known that if an odd perfect number $N$ has $r$ distinct prime factors then $N < 2^r$ [15], and $p_i < 2^{2^{i-1}}(r - i + 1)$, where $2 \leq i \leq 6$ and $p_i$ is the $i^{th}$ smallest distinct prime factor [12]. Periasstri [19] showed that the smallest prime factor $p_1$ of an odd perfect number with $r$ distinct prime factors must satisfy $p_1 \leq \frac{r}{2}r + 3$. Cohen showed that an odd perfect number must have a factor of the form $p^n > 10^{39}$, where $p$ is a prime number [4]. Many of the results mentioned above are improvements

2010 Mathematics Subject Classification: Primary 11A25, 11A41, 11A51, 11Y05, 11Y11.

Keywords and phrases: Perfect numbers, divisor function, prime numbers, factorization,

$^1$A perfect number is a number that is equal to the sum of its proper divisors (divisors less than the number) is $N$ itself.
of earlier work, the references to which can be found in the cited papers. Details of an ongoing search for the odd perfect numbers can be found at [18].

We derive below lower bounds on the sum and product of the distinct prime factors of odd perfect numbers. If

\[ N = \prod_{i=1}^{r} p_i^{n_i} \]  

is an odd perfect number, where \( p_1, \ldots, p_r \) are prime numbers, then define

\[ \sigma(N) := \sum_{d \mid N} d; \quad \alpha(N) := \prod_{i=1}^{r} p_i; \quad \beta(N) := \sum_{i=1}^{r} p_i \]  

\( \sigma(N) \), the sum of all divisors of \( N \), is called the divisor function in the literature. We begin by proving the following lemma.

**Lemma 1** If \( N > 1 \) is an odd perfect number then \( \sigma(\alpha(N)) < 2 \alpha(N) \).

**Proof:** Let \( M := \alpha(N) \). Clearly \( M \leq N \) and first we show that \( M < N \). If \( M = N \), then by Euler’s theorem \( N = M = A^n Q^2 \) where \( A \) is a prime number and \( A \equiv n \equiv 1 \pmod{4} \). Since \( M \) is a product of distinct primes, the uniqueness of prime factorization implies that it cannot contain a factor that is a perfect square, and therefore \( N = M = A^n \). Since \( A \) is prime and every prime occurring in \( M \) has a unit exponent, \( n = 1 \), showing that \( N \) must be a prime number. But if \( N \) is a prime number, \( \sigma(N) = N + 1 \neq 2N \) for \( N > 1 \), contradicting the assumption that \( N \) is a perfect number. Therefore we conclude that \( M < N \). The above argument also proves that if \( N \) is an odd perfect number, as described in the Lemma, then the condition \( n_1 = n_2 = \ldots = n_r = 1 \) cannot hold, a conclusion that is also implied by a result of Steuerwald’s [20].

Next consider an arbitrary number \( B \) whose prime factorization is of the form

\[ B = p \cdot \prod_{i=1}^{s} q_i^{m_i} \]  

where \( p, q_1, \ldots, q_s > 1 \) are prime numbers. For \( n > 1 \), let

\[ C = p^n \cdot \prod_{i=1}^{s} q_i^{m_i}. \]  

Then

\[
\frac{\sigma(C)}{2C} = \left( \frac{\sum_{j=0}^{n} p^j}{p^n} \right) \frac{\prod_{i=1}^{s} \left( \sum_{k=0}^{m_i} q_i^k \right)}{\prod_{i=1}^{s} q_i^{m_i}} = \left[ \frac{\left( \sum_{j=0}^{n} p^j \right) \prod_{i=1}^{s} \left( \sum_{k=0}^{m_i} q_i^k \right)}{2p \cdot \prod_{i=1}^{s} q_i^{m_i}} \right] \left[ \frac{\sum_{l=0}^{n} p^l}{\sum_{k=0}^{l} p^k} \right] \left[ \frac{1}{p^{n+1}} \right]
\]

\[
= \left[ \frac{\sigma(B)}{2B} \right] \left[ \left( \frac{p^{n+1} - 1}{p - 1} \right) \frac{1}{1 + p} \right] \frac{1}{p^{n+1}}
\]

\[
= \left[ \frac{\sigma(B)}{2B} \right] \left[ \frac{p^{n+1} - 1}{p^{n+1} - p^{n-1}} \right]
\]

Since \( p, n > 1 \), we have \( p^{n+1} - 1 > p^{n+1} - p^{n-1} \) and therefore we conclude that for numbers \( B \) and \( C \) defined as in (3) and (4),

\[
\frac{\sigma(C)}{2C} > \frac{\sigma(B)}{2B}
\]  

(5)
If we consider the sequence \( M = B_0, B_1, \ldots, B_r = N \), where for \( 1 \leq k \leq r \)
\[
B_k = \prod_{i=1}^{k} p_i^{n_i} \prod_{j=k+1}^{r} p_j
\]  
(6)
then repeated application of inequality (5) shows that
\[
\frac{\sigma(M)}{2M} < \frac{\sigma(B_1)}{2B_1} < \frac{\sigma(B_2)}{2B_2} < \ldots < \frac{\sigma(B_r)}{2B_r} = \frac{\sigma(N)}{2N}
\]  
(7)
Since \( M \neq N \), and \( N \) being a perfect number\( \sigma(N) = 2N \), using (7) we conclude that
\[
\frac{\sigma(M)}{2M} < \frac{\sigma(N)}{2N} = 1
\]

The following theorem is the main result.

**Theorem 1** If \( N > 1 \) is an odd perfect number with \( r \) distinct prime factors then
\[
\alpha(N) > \frac{1}{\left(2^r - 1\right)^{\frac{r}{k}}}; \quad \beta(N) > \frac{r}{2^r - 1}
\]

**Proof**: Let \( N \) be defined as in (1) with distinct prime factors \( p_1, \ldots, p_r \). For \( 1 \leq k \leq r \), define
\[
S_k := \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} \frac{1}{(p_{i_1} \cdot p_{i_2} \cdots p_{i_k})}
\]  
(8)
The sum in (8) is over all \( k \)-subsets of \( \{p_1, \ldots, p_r\} \). Since each of the \( p_1, \ldots, p_r \) occurs in exactly \( \binom{r-1}{k-1} \) terms in the sum, using the GM-HM inequality we have
\[
\frac{(r \choose k)}{S_k} < \left[ \prod_{i=1}^{r} p_i \right] \left( \frac{r - 1}{k - 1} \right)^{\frac{1}{k}} = \left[ \alpha(N) \right]^{\frac{k}{r}}
\]
The GM-HM inequality is strict since we are considering distinct prime numbers. The above inequality can be rewritten as
\[
S_k > \left( \frac{r \choose k} \right) \left[ \alpha(N) \right]^{-\frac{k}{r}} \]  
(9)
Using Lemma 1 and (9) we get
\[
1 > \frac{\sigma(\alpha(N))}{2 \alpha(N)} = \prod_{i=1}^{r} \left(1 + p_i\right) = \frac{1}{2} \left[ 1 + \sum_{k=1}^{r} S_k \right] > \frac{1}{2} \left[ 1 + \sum_{k=1}^{r} \left( \frac{r \choose k} \right) \left[ (\alpha(N))^{-\frac{1}{r}} \right]^k \right] = \frac{1}{2} \left[ 1 + (\alpha(N))^{-\frac{1}{r}} \right]^r
\]
which implies that
\[
1 > \frac{1}{2} \left[ 1 + (\alpha(N))^{-\frac{1}{r}} \right]^r
\]
or
\[
\alpha(N) > \frac{1}{\left(\frac{2^r}{r} - 1\right)^r}
\]
(10)
as claimed.

The bound on \( \beta(N) \) can be derived using the AM-GM inequality, applying which and using (10) we get
\[
\frac{\beta(N)}{r} = \frac{\sum_{i=1}^{r} p_i}{r} \geq \left(\prod_{i=1}^{r} p_i\right)^\frac{1}{r} \geq \frac{1}{\left(\frac{2^r}{r} - 1\right)}
\]
from which we obtain the inequality
\[
\beta(N) > \frac{r}{\left(\frac{2^r}{r} - 1\right)}
\]
(11)
as claimed.

For an odd perfect number \( N \), since \( N > \alpha(N) \) we immediately have the corollary

**Corollary 1** If \( N \) is an odd perfect number with \( r \) distinct prime factors then
\[
N > \frac{1}{\left(\frac{2^r}{r} - 1\right)}
\]

Together with Nielsen’s result [15] the current lower and upper bounds on an odd perfect number with \( r \) distinct prime factors can be summarized as
\[
\frac{1}{\left(\frac{2^r}{r} - 1\right)^r} < N < 2^{4^r}
\]

Lemma 1 also yields a simple derivation of an upper bound on the sum of reciprocals of the distinct prime factors of an odd perfect number. The upper bounds presented below are slightly weaker than those reported in [3, 21] but the derivations are considerably shorter.

**Theorem 2** If \( N > 1 \) is an odd perfect number with prime factorization \( N = \prod_{i=1}^{r} p_i^{n_i} \), where \( p_1, \ldots, p_r \) are distinct prime numbers, then
\[
\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r} < 1
\]

**Proof:** Let \( M := \alpha(N) \). From Lemma 1, we have
\[
\frac{\sigma(M)}{2M} = \frac{\prod_{i=1}^{r} (1 + p_i)}{2 \prod_{i=1}^{r} p_i} = \frac{1}{2} + \frac{1}{2} \left(\sum_{i=1}^{r} \frac{1}{p_i}\right) + \frac{1}{2} \left(\sum_{1 \leq i < j \leq r} \frac{1}{p_i p_j}\right) + \ldots + \frac{1}{2} \left(\frac{1}{p_1 \ldots p_r}\right) < 1 \quad (12)
\]
Since \( p_1, \ldots, p_r > 0 \),
\[
\frac{1}{2} + \frac{1}{2} \left(\sum_{i=1}^{r} \frac{1}{p_i}\right) < 1
\]
from which the claim in the theorem follows.

The bound in Theorem 1 can be improved as follows.
Theorem 3 If \( N > 1 \) is an odd perfect number that has the prime factorization \( N = \prod_{i=1}^{r} p_i^{\nu_i} \), where \( p_1 < p_2 < \ldots < p_r = P \) are distinct prime numbers, then

\[
\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r} < 1 - \left\{ \left[ 1 + \frac{1}{P} \right]^r - \left[ 1 + \frac{r}{P} \right] \right\}
\]

Proof: Let \( a = \{a_1, \ldots, a_r\} \) be a set of \( r \) distinct positive real numbers. Without loss of generality we will assume that \( 0 < a_1 < a_2 < \ldots < a_r \). For \( 1 \leq k \leq r \), define

\[
S_k(a) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} (a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_k})
\] (13)

That is, \( S_k(a) \) is the sum of the products of subsets of \( k \) numbers chosen from the \( r \) numbers in \( a \). Observe that each number, such as \( a_1 \), occurs in \( \frac{(r-1)(r-2)\ldots(r-k)}{k!} \) products in the sum (13). Therefore, using the AM-GM inequality we have

\[
\frac{S_k(a)}{ \binom{r}{k} } \geq \left( \frac{a_1 \cdot \ldots \cdot a_r}{\binom{r}{k}} \right)^k = (a_1 \cdot \ldots \cdot a_r)^{\frac{k}{r}} > (a_1)^k \Rightarrow S_k(a) \geq \left( \frac{r}{k} \right)^k (a_1)^k
\]

using which we get

\[
\sum_{k=2}^{r} S_k(a) \geq \sum_{k=2}^{r} \left( \frac{r}{k} \right)^k (a_1)^k = (1 + a_1)^r - (1 + ra_1)
\] (14)

Let \( M := \alpha(N) \) and set \( a_1 = \frac{1}{p_r} = \frac{1}{P}, \ldots, a_r = \frac{1}{p_1} \). Then from (12), we have

\[
\sigma(M) = \frac{1}{2M} \prod_{i=1}^{r} \frac{(1 + p_i)}{p_i} = \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{r} \frac{1}{p_i} + \frac{1}{2} \left( \sum_{1 \leq i < j \leq r} \frac{1}{p_ip_j} \right) + \ldots + \frac{1}{2} \left( \frac{1}{p_1 \ldots p_r} \right)
\]

\[
= \frac{1}{2} + \frac{1}{2} S_1(a) + \frac{1}{2} S_2(a) + \ldots + \frac{1}{2} S_r(a)
\]

\[
< 1
\]

(15)

Using (14) and (15) we obtain

\[
\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r} < 1 - \left\{ \left[ 1 + \frac{1}{P} \right]^r - \left[ 1 + \frac{r}{P} \right] \right\}
\]

as claimed.

\[\square\]

References

[1] Ball, W. W. R. and Coxeter, H. S. M. Mathematical Recreations and Essays, 13th ed. New York: Dover, 1987.
[2] Brent, R. P.; Cohen, G. L.; te Riele, H. J. J. Improved Techniques for Lower Bounds for Odd Perfect Numbers. Math. Comput. 57, 857-868, 1991.

[3] Cohen, G. L. On odd perfect numbers. II. Multiperfect numbers and quasiperfect numbers, Journal of the Australian Mathematical Society, Vol. 29 , No. 3, pp. 369-384, 1980.

[4] Cohen, G. L. On the largest component of an odd perfect number, Journal of the Australian Mathematical Society, Vol. 42, No. 2, pp. 280-286, 1987.

[5] Dickson, L. E. History of the Theory of Numbers, Vol. 1: Divisibility and Primality. New York: Dover, pp. 3-33, 2005.

[6] Dickson, L. E. Finiteness of odd perfect and primitive abundant numbers with n-distinct prime factors, Amer. J. Math. 35 (1913), 413-422.

[7] Hagis, P., Sketch of a proof that an odd perfect number relatively prime to 3 has at least eleven prime factors, Mathematics of Computation, Vol. 40, No. 161 pp. 399-404, 1983.

[8] Hare, K.G., New techniques for bounds on the total number of prime factors of an odd perfect number, Mathematics of Computation, to appear.

[9] Iannucci, D. E. The Second Largest Prime Divisor of an Odd Perfect Number Exceeds Ten Thousand. Math. Comput. 68, 1749-1760, 1999.

[10] Iannucci, D. E. The Third Largest Prime Divisor of an Odd Perfect Number Exceeds One Hundred. Math. Comput. 69, 867-879, 2000.

[11] Jenkins, P. M. Odd Perfect Numbers Have a Prime Factor Exceeding $10^7$. Math. Comput. 72, 1549-1554, 2003.

[12] Kishore, M. On odd perfect, quasiperfect, and odd almost perfect numbers. Mathematics of Computation, Vol. 36, No. 154 (Apr., 1981), pp. 583-586.

[13] Kishore, M. Odd perfect numbers not divisible by 3. II. Mathematics of Computation, Vol. 40, No. 161, pp. 405-411, 1983.

[14] http://www.mersenne.org/

[15] Nielsen, P. P. An upper bound for odd perfect numbers. INTEGERS: Electronic Journal of Combinatorial Number Theory, Vol. 3, #A14, 2003.

[16] Nielsen, P. P. Odd Perfect Numbers Have at Least Nine Distinct Prime Factors. 22 Feb 2006. http://arxiv.org/abs/math.NT/0602485

[17] Norton, K. K. Remarks on the Number of Factors of an Odd Perfect Number. Acta Arith. 6, 365-374, 1960.

[18] http://www.oddperfect.org

[19] Periasastri, M. A note on odd perfect numbers, The Mathematics Student, Vol. 26 , pp. 179–182, 1958. pp. 179-181,

[20] Steuerwald, R. Verscharfung einen notwendigen Bedingung fur die Existenz einen ungeraden vollkommenen Zahl. Sitzungsber. Bayer. Akad. Wiss., 69-72, 1937. See also, Yamada, T. On the Divisibility of Odd Perfect Numbers by a High Power of a Prime. 16 Nov 2005. http://arxiv.org/abs/math.NT/0511410

[21] Suryanarayana, D. and Hagis, P., A theorem concerning odd perfect numbers, Fibonacci Quarterly, Vol. 8, No. 4, pp. 337-346, 374, 1970.

[22] Touchard, J. On Prime Numbers and Perfect Numbers. Scripta Math. 19, 35-39, 1953.