ELEMENTARY DOUBLETS OF BOUND STATES OF THE RADIAL DIRAC EQUATION

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Abstract

For non-relativistic Schrödinger equations the lowering of their degree by substitution
\( \Psi(r) \rightarrow F(r) = \Psi'(r)/\Psi(r) \) is known to facilitate our understanding and use of their
(incomplete, so called quasi-exact) solvability. We show that and how the radial Dirac relativistic equation may quasi-exactly be solved in similar spirit.

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1 Introduction

The two-component radial Dirac equation with energy $E$, mass $M$ and centrifugal term $U(r) = \kappa/r$ reads

\[
\begin{bmatrix}
\partial_r - U(r) & M + W(r) - E - V(r) \\
M + W(r) + E + V(r) & \partial_r + U(r)
\end{bmatrix}
\begin{bmatrix}
f(r) \\
g(r)
\end{bmatrix} = 0. \quad (1)
\]

In a way similar to non-relativistic Schrödinger equation it proves exactly solvable for the electrostatic field of hydrogen atom $V(r) = \alpha/r$ accompanied, if necessary, by the auxiliary or external Lorentz scalar force $W(r) = \beta/r$ and by the possible central magnetic monopole charge $Q$ in $\kappa = \pm \sqrt{\ell + 1}(\ell + 1 + 2|Q|)$ where $\ell = 0, 1, \ldots$. Marginally, let us note that $\kappa = \ell + 1 = 0$ is also admitted whenever $\beta^2 > \alpha^2$ and $Q \neq 0$ (cf. ref. [2] for details).

Recently, Brihaye and Kosinski [3] conjectured that a formal parallel between relativistic and non-relativistic quantum mechanics may be extended to many other models. Explicitly, they have demonstrated that the perturbation of hydrogen atom by the linear relativistic force $W(r) \sim \omega r$ not only resembles its non-relativistic Coulomb plus linear plus quadratic analogue but also shares the incomplete, so called quasi-exact (QE) solvability with it. At certain exceptional energies and couplings, elementary bound states were obtained by non-numerical means in a way which complements the 25 years old non-relativistic result by A. Hautot [4].

In our present letter we intend to proceed one step further. Having in mind a deep non-relativistic connection between QE solvability and Riccati-Schrödinger equations [5], we shall formulate a parallel relativistic “order-lowering” idea and implement it in the technically slightly more difficult context of relativistic eq. (1). This will enable us to show that, in particular, the algebraic-equation approach of Brihaye and Kosinski just picks up a very specific portion of a much larger class of all the QE solvable Dirac equations.
2 New QE solutions: Explicit method

Once we represent a wavefunction in non-relativistic quantum mechanics as an integral \( \Psi_0(r) = \exp \int_{r_{ini}}^r F(\xi) d\xi \) we get \( \Psi'_0(r, \ell)/\Psi_0(r, \ell) = [F(r)]^2 + \partial_r F(r) \). This converts the radial differential Schrödinger equation to an equivalent first-order form \( V(r) = E_0 - \ell(\ell + 1)/r^2 + [F(r)]^2 + \partial_r F(r) \). Such a Riccati-type re-arranged equation is nonlinear but may be re-interpreted as an explicit closed definition of a partially solvable potential in terms of any of its “tentative” wavefunctions.

The latter point of view plays an important role in the understanding of non-relativistic QE systems \[5\]. *Mutatis mutandis*, the relativistic pair of equations (1) may also define the QE solvable pairs of potentials \( W(r) \) and \( V(r) \). Indeed, assuming that the latter forces are responsible for the existence of any particular elementary wavefunction with components \( f_0(r) \) and \( g_0(r) \) at a particular mass \( M_0 \) and energy \( E_0 \) we may write

\[
W(r) = -M_0 - \frac{1}{2} \left[ \frac{1}{g_0} (\partial_r - U) f_0 + \frac{1}{f_0} (\partial_r + U) g_0 \right], \tag{2}
\]

\[
V(r) = -E_0 + \frac{1}{2} \left[ \frac{1}{g_0} (\partial_r - U) f_0 - \frac{1}{f_0} (\partial_r + U) g_0 \right].
\]

On the basis of experience with non-relativistic models it is not too surprising that the mere correct threshold and asymptotic behaviour in tentative \( f_0(r) = r^\mu p \exp(-\lambda r) \) and \( g_0(r) = r^\mu q \exp(-\lambda r) \) with normalization constants \( p \) and \( q \) already leads to the exactly and, incidentally, completely solvable model as mentioned above (cf. also \[3\]). The next tentative elementary choice could mimick an unphysical singularity at negative \( r_u = -1/h < 0 \),

\[
f_0(r) = r^\mu (1 + h r) \exp(-\lambda r), \quad g_0(r) = r^\mu (1 + h r) \exp(-\lambda r). \tag{3}
\]

In terms of the same parameters \( \varepsilon = \pm 1 \) and \( t \in (-\infty, \infty) \) in the input ratio of norms \( p/q = \varepsilon \exp t \) we get the same formula for energy \( E = -\varepsilon \lambda \sinh t \) and mass \( M = \varepsilon \lambda \cosh t \) as above. Also both the Coulombic couplings remain the same,

\[
\begin{pmatrix}
\beta \\
\alpha
\end{pmatrix}
= \begin{pmatrix}
-\cosh t & \sinh t \\
\sinh t & -\cosh t
\end{pmatrix}
\begin{pmatrix}
\varepsilon \mu \\
\varepsilon \kappa
\end{pmatrix}. \tag{4}
\]
The only change emerges as a screening which enters the new and, by definition, QE solvable potentials

\[ V(r) = \frac{\alpha}{r} + \frac{\alpha_s}{1 + \hbar r}, \quad W(r) = \frac{\beta}{r} + \frac{\beta_s}{1 + \hbar r}, \quad \alpha_s = \varepsilon \hbar \sinh t, \quad \beta_s = -\varepsilon \hbar \cosh t. \quad (5) \]

Further states in such a model may be sought and studied, by the explicit algebraic method of ref. [3], in full analogy with semi-relativistic and non-relativistic QE solvable screened Coulomb potentials [6].

### 3 QE solutions: Implicit method

Our new QE model (5) looks particularly simple after transition to the elementary integral representation of wavefunctions

\[ f(r) = e^{\int_{r_{ini}}^{r} F(\xi) d\xi}, \quad g(r) = e^{\int_{r_{ini}}^{r} G(\xi) d\xi}. \quad (6) \]

In non-relativistic setting, similar re-arrangement proved useful in computations [7] as well as in the so called supersymmetric transformations of Hamiltonians [8]. Also here, our postulate (6) will lead to simplifications. Thus, with abbreviations \( Y(r) = [F(r) + G(r)]/2 \) and \( Z(r) = [F(r) - G(r)]/2 \), one of the components of Dirac equation lowers its order and becomes purely algebraic,

\[ [E + V(r)]^2 + [Y(r)]^2 = [M + W(r)]^2 + [U(r) - Z(r)]^2. \quad (7) \]

This enables us to parametrize, say,

\[ E + V(r) = R(r) \cos A(r), \quad M + W(r) = R(r) \cos B(r), \quad (8) \]

\[ Y(r) = R(r) \sin A(r), \quad U(r) - Z(r) = R(r) \sin B(r). \quad (9) \]

Such a transformation simplifies also the remaining Dirac equation

\[ M + W(r) - E - V(r) + [Y(r) + Z(r) - U(r)] \exp 2 \int_{r_{ini}}^{r} Z(\xi) d\xi = 0 \quad (10) \]

which, in terms of an abbreviation \( C(r) = -[A(r) + B(r)]/2 \), reads \( \tan C(r) = \exp 2 \int_{r_{ini}}^{r} Z(\xi) d\xi \) or, in differential form, \( Z(r) = \partial_r C(r) / \sin 2C(r) \). After an insertion
of such a definition of $Z(r)$ in the second item of eq. (4) we may eliminate the function $R(r)$ and are left with a set of the closed and compact simultaneous definitions of both the wavefunctions and potentials in terms of an arbitrary initial choice of the auxiliary but practically unrestricted pair of functions $A(r)$ and $B(r)$.

### 4 Re-parametrization

Our above construction of a QE solvable system exhibits still a certain similarity to its non-relativistic Riccati-like version. Unfortunately, the parallel is incomplete. In particular, we cannot derive potentials immediately from the wavefunctions since both of them enter our formulae together with their derivatives. At the same time, our implicit QE-type solution of Dirac eq. (1) still has to be made compatible with some overall physical requirements. Also its clearer physical interpretation is needed. For this purpose, let us make a further step. Recalling the standard Pauli matrices

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

let us re-write eq. (1) as a manifestly real two-component problem

$$\{ i \partial_r + [M + W(r)] \sigma_x - i[E + V(r)] \sigma_y - U(r) \sigma_z \} \psi(r) = 0 \quad (12)$$

and pre-multiply it by the transposed two-component real spinors $\sigma_x \psi(r)$ and $i\sigma_y \psi(r)$ from the left. The resulting pair of relations

$$
\begin{align*}
&\begin{bmatrix} Z(r) - U(r) \\ -Y(r) \end{bmatrix} = \begin{bmatrix} \cosh \Xi(r) & \sinh \Xi(r) \\ \sinh \Xi(r) & \cosh \Xi(r) \end{bmatrix} \begin{bmatrix} E + V(r) \\ M + W(r) \end{bmatrix} \\
&\Xi(r) = 2 \int_{r_{ini}}^{r} Z(\xi) d\xi
\end{align*}
$$

with $\Xi(r) = 2 \int_{r_{ini}}^{r} Z(\xi) d\xi$ represents another, integral representation of our original Dirac bound-state problem. In it, the two-by-two matrix is easily invertible,

$$
\begin{align*}
&\begin{bmatrix} E + V(r) \\ M + W(r) \end{bmatrix} = \begin{bmatrix} \cosh \Xi(r) & -\sinh \Xi(r) \\ -\sinh \Xi(r) & \cosh \Xi(r) \end{bmatrix} \begin{bmatrix} Z(r) - U(r) \\ -Y(r) \end{bmatrix}.
\end{align*}
$$

This induces the simplified ansatz

$$
E + V(r) = S(r) \sinh T(r), \quad M + W(r) = S(r) \cosh T(r), \quad (15)
$$
\[ Z(r) = U(r) + S(r) \sinh[T(r) + \Xi(r)], \quad Y(r) = -S(r) \cosh[T(r) + \Xi(r)] \] (16)

which, in particular, parametrizes all the QE solutions by the independent input functions \( S(r) \) and \( T(r) \). As already mentioned above, they must only be subject to the appropriate physical boundary conditions. We may conclude that the \textit{implicit} relativistic implementation of the idea of QE solvability is as straightforward as its \textit{explicit} nonrelativistic predecessor.

5 Relativistic QE doublets

After any change of our above “parametrization” point of view, technical complications may re-emerge immediately. Even for the elementary and popular Coulomb + polynomial form of forces as suggested for further study of the QE solvability in ref. [8] we immediately imagine that a seemingly trivial guarantee of their compatibility with our parameterizations leads in fact to a quite difficult algebraic problem.

Another, mathematically easier \textit{and} practically more important question is the possible existence and/or feasibility of constructions of the elementary QE multiplets. Indeed, in principle, after any choice of a QE wavefunction the resulting partially solvable potential may still remain compatible with another elementary bound state. Even in non-relativistic case such a physically useful requirement is mathematically non-trivial [8]. There, it may still be characterized by the comparatively transparent condition

\[ \partial_r[F_1(r) - F_2(r)] + [F_1(r) - F_2(r)][(F_1(r) + F_2(r)] = E_2 - E_1. \] (17)

For the \textit{same} QE potential (which was eliminated) this equation defines the superposition \( F_s(r) = [F_1(r) + F_2(r)]/2 \) in terms of the differences \( F_d(r) = [F_1(r) - F_2(r)]/2 \) and \( \delta = E_2 - E_1 \) [8]. This means that there exist very many nonrelativistic doublet partners \( F_1 = F_s + F_d \) and \( F_2 = F_s - F_d \) which are “numbered” by the choice of the virtually unconstrained functions \( F_d(r) \).

In relativistic case, we have to proceed in similar vain. The elimination of the two energy-independent QE solvable potentials \( V(r) \) and \( W(r) \) from the two versions
of eq. (14) will describe the difference between the two right hand sides as an $r-$independent spinor proportional to $\delta$. We get a relativistic differential-equation counterpart to eq. (17) in terms of the auxiliary integrals $\alpha = \alpha(r) = \int r(Z_2 - Z_1)$ and $\beta = \beta(r) = \int r(Z_2 + Z_1)$ and their derivatives $Z_2(r) = \partial_r \alpha/2 + \partial_r \beta/2$ and $Z_1(r) = -\partial_r \alpha/2 + \partial_r \beta/2$,

$$\begin{pmatrix} Z_2 - U \\ -Y_2 \end{pmatrix} = \delta \begin{pmatrix} \cosh(\alpha + \beta) \\ \sinh(\alpha + \beta) \end{pmatrix} + \begin{pmatrix} \cosh 2\alpha & \sinh 2\alpha \\ \sinh 2\alpha & \cosh 2\alpha \end{pmatrix} \begin{pmatrix} Z_1 - U \\ -Y_1 \end{pmatrix}. \quad (18)$$

This formula may be read as an algebraic linear set of definitions of the two sums of exponents $Y_1(r)$ and $Y_2(r)$. The symmetry of our new equation with respect to the simultaneous double reflection $\beta(r) \leftrightarrow -\beta(r)$, $U(r) \leftrightarrow -U(r)$ and permutation $Y_1(r) \leftrightarrow Y_2(r)$ is one of the reasons why its closed solution is still unexpectedly compact,

$$Y_1(r) + Y_2(r) = \delta \frac{\cosh \beta(r)}{\sinh \alpha(r)} - \alpha'(r) \frac{\cosh \alpha(r)}{\sinh \alpha(r)}, \quad (19)$$

$$Y_1(r) - Y_2(r) = \delta \frac{\sinh \beta(r)}{\cosh \alpha(r)} + [\beta'(r) - 2U(r)] \frac{\sinh \alpha(r)}{\cosh \alpha(r)}. \quad (20)$$

We may summarize that our “parameters” $\alpha(r)$ and $\beta(r)$ remain practically arbitrary functions. In the relativistic Dirac case there exist infinitely many QE bound-state doublets as well. Thus, the well known functional freedom of the implicit doublet solutions of the non-relativistic Schrödinger QE equations is shared by our present Dirac relativistic QE construction.

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References

[1] H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms (Academic Press, New York, 1957);
   J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964), p. 52;
   A. E. S. Green, in Antinucleon- and Nucleon-Nucleus Interactions, ed. by G. E. Walker et al (Plenum, New York, 1985), p. 143;
   C. J. Horowitz, in Relativistic Dynamics and Quark-Nuclear Physics, ed. by M. B. Johnson and A. Picklesimer (Wiley, New York, 1986), p. 221;
   C. Semay, R. Ceuleneer and B. Silvestre-Brac, J. Math. Phys. 34, 2215 (1993);
   V. Villalba, J. Math. Phys. 36, 3332 (1995).

[2] G. Torres del Castillo and L. Cortes-Curantli, J. Math. Phys. 38, 2996 (1997).

[3] Y. Brihaye and P. Kosinski, Mod. Phys. Lett. A 13, 1445 (1998).

[4] A. Hautot, Phys. Lett. A 38, 305 (1972).

[5] A. G. Ushveridze, Quasi-exactly solvable models in quantum mechanics (IOPP, Bristol, 1994).

[6] M. Znojil, Phys. Lett. A 94, 120 (1983) and J. Phys. A: Math. Gen. 29, 6443 (1996).

[7] F. M. Fernández, G. I. Frydman and E. A. Castro, J. Phys. A: Math. Gen. 22, 641 (1989);
   F. M. Fernández, Q. Ma and R. H. Tipping, Phys. Rev. A 39, 1605 (1989) and A 40, 6149 (1989);
   V. C. Aguilera-Navarro, F. M. Fernández, R. Guardiola and J. Ros, J. Phys. A: Math. Gen. 25, 6379 (1992);
   F. M. Fernández and R. Guardiola, J. Phys. A: Math. Gen. 26, 7169 (1993);
F. M. Fernández, R. Guardiola and M. Znojil, Phys. Rev. A 48, 4170 (1993);
M. Znojil, J. Phys. A: Math. Gen. 28, 6265 (1995).

[8] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).

[9] M. Znojil and P. G. L. Leach, J. Math. Phys. 33, 2785 (1992);
M. Znojil and R. Roychoudhury, Czech. J. Phys. 48, 1 (1998).