Damage Spreading and Criticality in Finite Random Dynamical Networks

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We systematically study and compare damage spreading at the sparse percolation (SP) limit for random boolean and threshold networks with perturbations that are independent of the network size $N$. This limit is relevant to information and damage propagation in many technological and natural networks. Using finite size scaling, we identify a new characteristic connectivity $K_c$, at which the average number of damaged nodes $d$, after a large number of dynamical updates, is independent of $N$. Based on marginal damage spreading, we determine the critical connectivity $K_{c^{sparse}}(N)$ for finite $N$ at the SP limit and show that it systematically deviates from $K_c$, established by the annealed approximation, even for large system sizes. Our findings can potentially explain the results recently obtained for gene regulatory networks and have important implications for the evolution of dynamical networks that solve specific computational or functional tasks.

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Random boolean networks (RBN) were originally introduced as simplified models of gene regulation \cite{1, 2}, focusing on a system-wide perspective rather than on the often unknown details of regulatory interactions \cite{3}. In the thermodynamic limit, these disordered dynamical systems exhibit a dynamical order-disorder transition at a sparse critical connectivity $K_c$ \cite{4}; similar observations were made for sparsely connected random threshold (neural) networks (RTN) \cite{3, 4, 5}. For a finite system size $N$, the dynamics of both systems converge to periodic attractors after a finite number of updates. At $K_c$, the phase space structure in terms of attractor periods \cite{5}, the number of different attractors \cite{4} and the distribution of basins of attraction \cite{10} is complex, showing many properties reminiscent of biological networks \cite{2}.

Often, one is interested in the response of dynamical networks to external perturbations; because these signals can disrupt the generic dynamical state (fixed point or periodic attractor) of the network, they are usually referred to as “damage.” This type of study has numerous applications, e.g., the spreading of disease through a population \cite{11, 12}, the spreading of a computer virus on the internet \cite{13}, failure propagation in power grids \cite{14}, or the perturbation of gene expression patterns in a cell due to mutations \cite{13}. Mean-field approaches, e.g., the annealed approximation (AA) introduced by Derrida and Pomeau \cite{4}, allow for an analytical treatment of damage spreading and exact determination of the critical connectivity $K_c$ under various constraints \cite{14} \cite{17}. It has been shown that local, mean-field-like rewiring rules coupled to order parameters of the dynamics can drive both RBN and RTN to self-organized criticality \cite{18} \cite{19} \cite{20}.

Mean-field approximations of RBN/RTN dynamics rely on the assumption that $N \rightarrow \infty$ and study the rescaled damage $d(t)/N$ (where $d(t)$ is the number of damaged nodes at time $t$). For an application to real-world problems, these limits are often not very relevant. Going beyond the framework of AA, a number of recent studies focus on the finite-size scaling of (un-)frozen and/or relevant nodes in RBN with respect to $N$ \cite{21, 22}; only few studies, however, consider finite-size scaling of damage spreading in RBN \cite{15} \cite{23}. Here, of particular interest is the “sparse percolation (SP) limit” \cite{23}, where the initial perturbation size $d(0)$ does not scale up with network size $N$, i.e., the relative size of perturbations tends to zero for large $N$. This limit applies to many of the above-mentioned real-world networks (e.g., the spread of a new computer virus on the internet launched from a single computer). In this letter, we systematically study finite-size scaling of damage spreading in the SP limit for both RBN and RTN. We identify a new characteristic point $K_s$, where the expectation value of the number of damaged nodes after large number of dynamical updates is independent of $N$. By the definition of marginal damage spreading, we introduce a new approach to estimate the critical connectivity $K_c(N)$ for finite $N$, and present evidence that, even in the large $N$ limit, the critical connectivity for SP systematically deviates from the predictions of mean-field theory.

First, let us define the dynamics of RBN and RTN. A RBN is a discrete dynamical system composed of $N$ automata. Each automaton is a Boolean variable with two possible states: $\{0, 1\}$, and the dynamics is such that

$$\mathbf{F} : \{0, 1\}^N \rightarrow \{0, 1\}^N,$$

where $\mathbf{F} = (f_1, ..., f_i, ..., f_N)$, and each $f_i$ is represented by a look-up table of $K_i$ inputs randomly chosen from the set of $N$ automata. Initially, $K_i$ neighbors and a look-table are assigned to each automaton at random.

An automaton state $\sigma_i^t \in \{0, 1\}$ is updated using its
corresponding Boolean function:
\[ \sigma_i^{t+1} = f_i(x_i^t, x_{i_1}^t, ..., x_{i_K}^t). \]  

(2)

We randomly initialize the states of the automata (initial condition of the RBN). The automata are updated synchronously using their corresponding Boolean functions. The second type of discrete dynamical system we study is RTN. An RTN consists of \( N \) randomly interconnected binary sites (spins) with states \( \sigma_i = \pm 1 \). For each site \( i \), its state at time \( t+1 \) is a function of the inputs it receives from other spins at time \( t \):
\[ \sigma_i(t+1) = \text{sgn} (f_i(t)) \]  

(3)

with
\[ f_i(t) = \sum_{j=1}^{N} c_{ij} \sigma_j(t) + h. \]  

(4)

The \( N \) network sites are updated synchronously. In the following discussion the threshold parameter \( h \) is set to zero. The interaction weights \( c_{ij} \) take discrete values \( c_{ij} = +1 \) or \(-1\) with equal probability. If \( i \) does not receive signals from \( j \), one has \( c_{ij} = 0 \).

![FIG. 1: Average Hamming distance (damage) \( \bar{d} \) after 200 system updates, averaged over 10000 randomly generated networks for each value of \( \hat{K} \), with 100 different random initial conditions and one-bit perturbed neighbor configurations for each network. For both RBN and RTN, all curves for different \( N \) approximately intersect in a characteristic point \( K_s \).](image)

**Results.** We first study the expectation value \( \bar{d} \) of damage, quantified by the Hamming distance of two different system configurations, after a large number \( T \) of system updates. Let \( \mathcal{N} \) be a randomly sampled set (ensemble) of \( z_N \) networks with average degree \( \hat{K} \), \( \mathcal{I}_n \), a set of \( z_I \) random initial conditions tested on network \( n \), and \( \mathcal{I}^* \), a set of \( z_I \) random initial conditions differing in one randomly chosen bit from these initial conditions. Then we have
\[ \bar{d} = \frac{1}{z_N z_I} \sum_{n=1}^{z_N} \sum_{\sigma_i \in \mathcal{I}_n, \sigma_i^* \in \mathcal{I}^*} d^n_i(T), \]  

(5)

where \( d^n_i(T) \) is the measured Hamming distance after \( T \) system updates. Fig. 1 shows \( \bar{d} \) as a function of the average connectivity \( \hat{K} \) for different network sizes \( N \) by using a random ensemble for statistics. For both RBN and RTN, the observed functional behavior strongly suggests that the curves approximately intersect at a common point \((K_s, d_s)\), where the observed Hamming distance for large \( t \) is independent of the system size \( N \).

To verify this finding, let us now study the finite size scaling behavior of \( \bar{d} \) in this (SP) limit. For \( K \to 0 \) and for large \( \hat{K} \), it is straightforward to estimate the asymptotic scaling. In the case \( K \to 0 \), non-zero damage can only emerge if the initial perturbation hits a short loop of oscillating nodes (most likely a self-connection or a loop of length two, longer loops can be neglected). The a priori probability to generate these loops is \( \sim 1/N^2 \), and their number is proportional to the total number of links, \( K N \). Hence, we expect \( \bar{d} \sim KN/N^2 \propto 1/N \). For large \( \hat{K} \), damage percolates through the system, consequently, avalanche sizes are bounded only by the size of the system, and we expect \( \bar{d} \sim N \). At criticality, the frozen core of the network always remains undamaged; for this reason, \( \bar{d} \) should be limited by the number of unfrozen

![FIG. 2: Upper panels: \( \bar{d} \) as a function of \( N \) for different \( \hat{K} \): \( \hat{K} = 1.0 \) (+), \( \hat{K} = 1.5 \) (x), \( \hat{K} = 1.8 \) (RBN, *) and \( \hat{K} = 1.7 \) (RTN, *), \( \hat{K} = 2.1 \) (\( \square \), RBN) and \( \hat{K} = 1.9 \) (\( \square \), RTN), \( \hat{K} = 2.4 \) (\( \triangle \), RTN) and \( \hat{K} = 2.6 \) (\( \circ \)). The lines are fits of Eq. 6 to the data. Lower panel: Scaling exponents \( \gamma(\hat{K}) \) as a function of \( \hat{K} \), as obtained from fits of Eq. 6 for RBN (+) and RTN (x). The dashed/dotted lines mark the asymptotes as discussed in the text.](image)
of large damage events near $K_c$, measurements with finite $T$ can substantially underestimate $\bar{d}$ (in particular for $N \geq 512$). We performed high precision numerical experiments in the interval $1.6 \leq \bar{K} \leq 2.1$, waiting for $T = 5000$ update time steps to let the network dynamics relax after the initial one-bit perturbation; these simulations conclusively show an exponential dependence $\Delta \bar{d} \propto \exp(c(N) \cdot \bar{K})$ in this interval, with a constant $c(N)$ depending only on $N$ (Fig. 3 upper panels). This exponential dependence becomes apparent with the following assumptions: an increase $\Delta \bar{d}$ of the average damage is proportional to $\bar{d}$ itself (damage can generate new damage), to an increase $\Delta \bar{K}$ of the average connectivity, and to some function of the system size $N$. Actually, it cannot be directly proportional to $N$, because nodes that are part of the frozen core always remain undamaged asymptotically; hence, a rough upper limit is given by the number of nonfrozen nodes, which at $K_c$ scales as $N^{2/3}$. A lower bound can be derived by the number of relevant nodes, that almost certainly propagate damage, i.e. $N^{1/3}$ [21]. To summarize, we approximate

$$\Delta \bar{d} \approx c(N) N^\alpha \bar{d} \Delta \bar{K},$$  \hspace{1cm} (7)

with $1/3 \leq \alpha \leq 2/3$; replacing $\Delta \bar{d}$ and $\Delta \bar{K}$ with differentials and integrating yields

$$\bar{d}(\bar{K}, N) \approx c_1(N) \exp[c_2(N) N^\alpha \bar{K}].$$  \hspace{1cm} (8)

In simulations, we find $\alpha \approx 0.42$, which is well within the range we expect from our theoretical considerations as discussed above.

We now apply this dependence to obtain high-accuracy fits of Eq. 6 in the interval $1.6 \leq \bar{K} \leq 2.1$ (Fig. 3 lower panels); these fits yield

$$(K_s^{\text{RBN}}, \bar{d}_s^{\text{RBN}}) = (1.87 \pm 0.04, 0.65 \pm 0.05)$$  \hspace{1cm} (9)

for RBN and, correspondingly,

$$(K_s^{\text{RTN}}, \bar{d}_s^{\text{RTN}}) = (1.725 \pm 0.035, 0.52 \pm 0.04)$$  \hspace{1cm} (10)

for RTN.

Interestingly, $K_s$ is close to, but distinct from the critical connectivities $K_c^{\text{RBN}} = 2$ and $K_c^{\text{RTN}} = 1.845$, as predicted by mean-field theory. However, a natural comparison has to consider possible deviations of $K_c$ at the SP limit from these values. An intuitive definition of criticality for finite $N$ can be formulated in terms of marginal damage spreading. If at time $t$ one bit is flipped, one requires at time $t + 1$ [3, 7, 17]

$$\bar{d}(t + 1) = \langle p_s(K_c) \bar{K} \rangle = 1,$$  \hspace{1cm} (11)

where $\langle p_s(K_c) \rangle$ is the average damage propagation probability. Naturally, the iteration of this map implies $\bar{d} = 1$ for all $t$. Note that the relation: $\langle p_s(K_c) \bar{K} \rangle = 1$ is exact only in the framework of the AA. In the SP limit, we
instead have to set the right hand side of Eq. 8 to unity; inversion then leads to
\[ K_{c}^{\text{parsec}}(N) = -\frac{\ln c_1(N)}{c_2(N)N^\alpha}. \] (12)

Fig. 4 shows \( K_{c}^{\text{parsec}}(N) \), using the values \( c_1(N), c_2(N) \) obtained from numerical fits of Eq. 8 for both RBN and RTN. We find that both systems, in a very good approximation, obey the scaling relationship
\[ K_{c}^{\text{parsec}}(N) \approx b \cdot N^{-\delta} + K_{c}^\infty \] (13)
with \( b = 3.27 \pm 0.79, \delta = 0.85 \pm 0.07 \) and \( K_{c}^\infty = 1.90375 \pm 0.005 \) for RBN and \( b = 3.853 \pm 0.76, \delta = 0.736 \pm 0.05 \) and \( K_{c}^\infty = 1.75598 \pm 0.005 \) for RTN. Hence, in the limit \( N \to \infty \), we have
\[ K_{c}^{\infty,\text{RBN}} = 1.90375 \pm 0.005 \] (14)
for RBN, and for RTN
\[ K_{c}^{\infty,\text{RTN}} = 1.75598 \pm 0.005. \] (15)

Thus for both RBN and RTN, in the SP limit and for \( N \to \infty \), the critical connectivity is considerably below the value \( K_{c} \) predicted by the annealed approximation (notice that for small \( N < 128 \), however, \( K_{c}^{\text{parsec}}(N) > K_{c}^{\text{annealed}} \)).

It is beyond the scope of this letter to discuss possible causes for deviations from the annealed approximation in the limit of large networks, e.g., RBNs or RTNs, to robustly solve specific computational and functional tasks are required.

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