NON-UNIFORM DEPENDENCE ON INITIAL DATA
FOR THE CH EQUATION ON THE LINE

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Abstract. For $s > 3/2$ two sequences of CH solutions living in a bounded set of the Sobolev space $H^s(\mathbb{R})$ are constructed, whose distance at the initial time is converging to zero while at any later time is bounded below by a positive constant. This implies that the solution map of the CH equation is not uniformly continuous in $H^s(\mathbb{R})$.

1. Introduction

We consider the Cauchy problem for the Camassa-Holm equation (CH)

$$
\partial_t u + u\partial_x u + \partial_x \left(1 - \partial_x^2 \right)^{-1} \left[u^2 + \frac{1}{2}(\partial_x u)^2 \right] = 0, \quad (1.1)
$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \quad (1.2)
$$

This equation appeared initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [FF]. However, it was written explicitly as a water wave equation by Camassa and Holm [CH], who showed that CH is biHamiltonian and studied its “peakon” solutions. Since then CH has been rederived in various ways by Misiolek [Mi], Johnson [J], Constantin and Lannes [CL], and Ionescu-Kruse [I]. Well-posedness on the line was first established by Li and Olver. In [LO] they showed that if $s > 3/2$ then CH is locally well-posed in $H^s(\mathbb{R})$ with solutions depending continuously on initial data. The proof was based on a regularization technique similar to the one used by Bona and Smith for the KdV equation [BS]. A similar result has also been proved by Rodriguez-Blanco [RB] by using Kato’s theory for quasilinear equations [K]. Moreover, global well-posedness in $H^1(\mathbb{R})$ for the CH equation has been studied by Bressan and Constantin in [BC]. However, well-posedness of CH in $H^s(\mathbb{R})$ for $s \in (1, 3/2]$ remains an open question.

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In this paper, we show that dependence of CH solutions on initial data in Sobolev spaces cannot be better than continuous. More precisely, we prove the following result.

**Theorem 1.** If \( s > 3/2 \) then the flow map \( u_0 \rightarrow u(t) \) for the CH equation is not uniformly continuous from any bounded set of \( H^s(\mathbb{R}) \) into \( C([-T,T]; H^s(\mathbb{R})) \). More precisely, there exist two sequences of CH solutions \( u_n(t) \) and \( v_n(t) \) in \( C([-T,T]; H^s(\mathbb{R})) \) such that

\[
\|u_n(t)\|_{H^s(\mathbb{R})} + \|v_n(t)\|_{H^s(\mathbb{R})} \lesssim 1, \tag{1.3}
\]

\[
\lim_{n \rightarrow \infty} \|u_n(0) - v_n(0)\|_{H^s(\mathbb{R})} = 0, \tag{1.4}
\]

and

\[
\liminf_{n \rightarrow \infty} \|u_n(t) - v_n(t)\|_{H^s(\mathbb{R})} \gtrsim \sin t, \quad |t| < T \leq 1. \tag{1.5}
\]

For \( s = 1 \) Theorem 1 has been already proved by Himonas, Misiolek and Ponce in [HMP] by using traveling wave solutions that are smooth except at finitely many points at which the slope is \( \pm \infty \) (cuspons). Also, in [HMP] the analogous result for the periodic CH was proved. For \( s \geq 2 \) non-uniform continuity of the CH solution map in the periodic case was established in [HM] using high frequency traveling wave solutions and following an approach similar to the one used in [KPV] by Kenig, Ponce and Vega. We mention that this method does not work in the non-periodic case because the traveling wave solutions do not live in \( H^s(\mathbb{R}) \).

Also, it is worth mentioning the following implication of Theorem 1 concerning ways for proving local well-posedness for CH. The fact that the data-to-solution map is not uniformly continuous from any bounded set of \( H^s(\mathbb{R}) \) into \( C([-T,T]; H^s(\mathbb{R})) \) tells us that local well-posedness of CH in \( H^s \) cannot be established by a solely contraction principle argument.

The proof of Theorem 1 is based on the method of approximate solutions used by Koch and Tzvetkov in [KT] and Christ, Colliander and Tao in [CCT]. The idea is to choose approximate solutions consisting of a low-frequency part and a high-frequency part, which satisfy the three conclusions of Theorem 1. Furthermore, solving the Cauchy problem with initial data given by evaluating the approximate solutions at \( t = 0 \) must yield actual solutions whose difference from the approximate solutions is negligible.

The literature about CH is extensive. For some other results about this equation we refer the reader to McKean [Mc], Constantin and Strauss [CS], Himonas, Misiolek, Ponce and Zhou [HMPZ], and Molinet’s survey article [Mo].

The paper is structured as follows. In section 2 we recall the well-posedness result of Li and Olver and use it to prove the basic energy estimate (see (2.3)) from which we derive a lower bound for the lifespan of the solution as well an estimate of the \( H^s \) norm.
of the solution \( u(t) \) in terms of the \( H^s \) norm of the initial data \( u_0 \) (see Proposition 1). In section 3 we construct approximate solutions consisting of a low-frequency part and a high-frequency part, and compute the error. In section 4 we estimate the \( H^1 \)-norm of this error. In section 5 we solve the Cauchy problem for the CH equation with initial data given by the approximate solutions evaluated at time zero, and estimate the \( H^1 \)-norm of the difference between actual and approximate solutions (see Lemma 6). Finally, in section 6 we conclude with the proof of Theorem 1.

2. LOCAL WELL-POSEDNESS

We shall need the following well-posedness result, proved in [LO] using a regularization technique.

**Theorem 2.** [Li-Olver] Suppose that the function \( u_0(x) \) belongs to the Sobolev space \( H^s(\mathbb{R}) \) for some \( s > 3/2 \). Then there is a \( T > 0 \), which depends only on \( \|u_0\|_{H^s} \), such that there exists a unique function \( u(x,t) \) solving the Cauchy problem (1.1)–(1.2) in the sense of distributions with \( u \in C([0,T];H^s) \). When \( s \geq 3 \), \( u \) is also a classical solution to (1.1)–(1.2). Moreover, the solution \( u \) depends continuously on the initial data \( u_0 \) in the sense that the mapping of the initial data to the solution is continuous from the Sobolev space \( H^s \) to the space \( C([0,T];H^s) \).

Using the information provided by Theorem 2, next we shall prove an explicit estimate for the time of existence \( T \) of the solution \( u(t) \). Also, we will show that at any time \( t \) in the time interval \([0,T]\) the \( H^s \) norm of the solution \( u(t) \) is dominated by the \( H^s \) norm of the initial data \( u_0 \).

**Proposition 1.** Let \( s > 3/2 \). If \( u \) is the solution of the Cauchy problem (1.1)–(1.2) described in Theorem 2 then its lifespan (the maximal existence time) is greater than

\[
T = \frac{1}{2c_s \|u_0\|_{H^s(\mathbb{R})}},
\]

where \( c_s \) is a constant depending only on \( s \). Also, we have that

\[
\|u(t)\|_{H^s(\mathbb{R})} \leq 2\|u_0\|_{H^s(\mathbb{R})}, \quad 0 \leq t \leq T.
\]

**Proof.** The derivation of the lower bound for the lifespan (2.1) and the solution size estimate (2.2) is based on the following differential inequality for the solution \( u \)

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^s(\mathbb{R})}^2 \leq c_s \|u(t)\|_{H^s(\mathbb{R})}^3, \quad 0 \leq t \leq T.
\]

This inequality can be extracted from the proof of Theorem 2 in [LO] using the energy estimate (3.6) proved for the following regularization

\[
\partial_t u - \partial_x^2 \partial_t u + \varepsilon \partial_x^4 \partial_t u + 3u \partial_x^2 u - \partial_x u \partial_x^2 u - u \partial_x^3 u = 0
\]
of the CH equation and letting \( \varepsilon \) go to zero. Here, we shall prove inequality (2.3) by following the approach used for quasilinear symmetric hyperbolic systems in Taylor [T1].

For any \( s \in \mathbb{R} \) let \( D^s = (1 - \partial_x^2)^{s/2} \) be the operator defined by

\[
\hat{D}^s f(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi),
\]

where \( \hat{f} \) is the Fourier transform

\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.
\]

Then for \( f \in H^s(\mathbb{R}) \) we have

\[
\|f\|_{H^s(\mathbb{R})} = \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \frac{d\xi}{2\pi} = \|D^s f\|_{L^2(\mathbb{R})}.
\]

Now let \( u \) be the solution to the Cauchy problem (1.1)–(1.2), which according to Theorem 2 belongs in \( C([0,T];H^s) \). Solving (1.1) for \( \partial_t u \) we obtain

\[
\partial_t u = -u \partial_x u - D^{-2} \partial_x \left[ u^2 + \frac{1}{2} (\partial_x u)^2 \right].
\]

Starting with (2.4) we want to derive the energy estimate in \( H^s \) expressed by inequality (2.3). We can form

\[
\frac{d}{dt} \|u\|_{H^s(\mathbb{R})}^2 = -\int_{\mathbb{R}} D^s J_\varepsilon (u \partial_x u) \cdot D^s J_\varepsilon u dx - \int_{\mathbb{R}} D^{s-2} \partial_x J_\varepsilon (u^2) \cdot D^s J_\varepsilon u dx
\]

where for each \( \varepsilon \in (0,1] \) the operator \( J_\varepsilon \) is the Friedrichs mollifier defined by

\[
J_\varepsilon f(x) = j_\varepsilon \ast f(x).
\]

Here \( j(x) \) is a \( C^\infty \) function supported in the interval \([-1,1]\) such that \( j(x) \geq 0, \int_{\mathbb{R}} j(x) dx = 1 \) and

\[
j_\varepsilon(x) = \frac{1}{\varepsilon} j\left(\frac{x}{\varepsilon}\right).
\]

Applying the operator \( D^s \) to both sides of (2.5), then multiplying the resulting equation by \( D^s J_\varepsilon u \) and integrating it for \( x \in \mathbb{R} \) gives

\[
\frac{1}{2} \frac{d}{dt} \|J_\varepsilon u\|_{H^s}^2 = -\int_{\mathbb{R}} D^s J_\varepsilon (u \partial_x u) \cdot D^s J_\varepsilon u dx - \int_{\mathbb{R}} D^{s-2} \partial_x J_\varepsilon (u^2) \cdot D^s J_\varepsilon u dx
\]

\[
- \frac{1}{2} \int_{\mathbb{R}} D^{s-2} \partial_x J_\varepsilon [(\partial_x u)^2] \cdot D^s J_\varepsilon u dx.
\]
In what follows next we use the fact that $D^s$ and $J_\varepsilon$ commute and that $J_\varepsilon$ satisfies the properties

\[(J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2}, \tag{2.8}\]

and

\[\|J_\varepsilon u\|_{H^s} \leq \|u\|_{H^s}. \tag{2.9}\]

**Estimating the Burgers term.** To estimate the first integral in the right-hand side of (2.7) we write it as follows

\[
\int_\mathbb{R} D^s J_\varepsilon (u \partial_x u) \cdot D^s J_\varepsilon u \, dx = \int_\mathbb{R} D^s (u \partial_x u) \cdot J_\varepsilon D^s J_\varepsilon u \, dx
\]

\[+ \int_\mathbb{R} u D^s (\partial_x u) \cdot J_\varepsilon D^s J_\varepsilon u \, dx. \tag{2.10}\]

Now, we estimate the first term in the right-hand side of (2.10). Applying the Cauchy-Schwarz inequality gives

\[
\left| \int_\mathbb{R} [D^s (u \partial_x u) - u D^s (\partial_x u)] J_\varepsilon D^s J_\varepsilon u \, dx \right| \leq \|D^s (u \partial_x u) - u D^s (\partial_x u)\|_{L^2} \|J_\varepsilon D^s J_\varepsilon u\|_{L^2}
\]

\[\leq \|D^s (u \partial_x u) - u D^s (\partial_x u)\|_{L^2} \|u\|_{H^s}
\]

\[\leq 2c_s \|\partial_x u\|_{L^\infty} \|u\|_{H^s}^2, \tag{2.11}\]

where the last step follows from the estimate

\[\|D^s (u \partial_x u) - u D^s (\partial_x u)\|_{L^2} \leq 2c_s \|\partial_x u\|_{L^\infty} \|u\|_{H^s}, \tag{2.12}\]

which we prove below by using the following Kato-Ponce commutator estimate [KP] (see also Ionescu and Kenig [IK]).

**Lemma 1.** [Kato-Ponce] If $s > 0$ then there is $c_s > 0$ such that for any $f, g \in H^s(\mathbb{R})$

\[\|D^s(fg) - f D^s g\|_{L^2} \leq c_s \left(\|D^s f\|_{L^2} \|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty} \|D^{s-1} g\|_{L^2}\right). \tag{2.13}\]

In fact, applying this estimate with $f = u$ and $g = \partial_x u$ gives

\[
\|D^s (u \partial_x u) - u D^s (\partial_x u)\|_{L^2} \leq c_s \left(\|D^s u\|_{L^2} \|\partial_x u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} \|D^{s-1} \partial_x u\|_{L^2}\right)
\]

\[\leq c_s \|\partial_x u\|_{L^\infty} \left(\|D^s u\|_{L^2} + \|D^s u\|_{L^2}\right)
\]

\[\leq 2c_s \|\partial_x u\|_{L^\infty} \|u\|_{H^s}, \tag{2.14}\]

which is the desired estimate (2.12).
Next, we estimate the second integral in the right-hand side of (2.10). Note if there were no $J_{\varepsilon}$’s involved then this would have been done in a straightforward manner as follows

$$\left| \int_{\mathbb{R}} u D^s (\partial_x u) \cdot D^s u \, dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}} u \partial_x [(D^s u)^2] \, dx \right|$$

$$= \left| - \frac{1}{2} \int_{\mathbb{R}} \partial_x u (D^s u)^2 \, dx \right|$$

$$\leq \frac{1}{2} \| \partial_x u \|_{L^\infty} \| u \|_{H^s}^2.$$  (2.15)

When the $J_{\varepsilon}$’s are involved the idea is the same. However, the implementation is more technical since we need to commute $J_{\varepsilon}$ so that is grouped correctly. We accomplish this as follows

$$\int_{\mathbb{R}} u D^s (\partial_x u) \cdot J_{\varepsilon} D^s J_{\varepsilon} u \, dx = \int_{\mathbb{R}} J_{\varepsilon} u D^s (\partial_x u) \cdot D^s J_{\varepsilon} u \, dx$$

$$= \int_{\mathbb{R}} \left( [J_{\varepsilon}, u] D^s (\partial_x u) + u J_{\varepsilon} D^s (\partial_x u) \right) \cdot D^s J_{\varepsilon} u \, dx$$

$$= \int_{\mathbb{R}} [J_{\varepsilon}, u] \partial_x D^s u \cdot D^s J_{\varepsilon} u \, dx$$

$$+ \int_{\mathbb{R}} u \partial_x D^s J_{\varepsilon} u \cdot D^s J_{\varepsilon} u \, dx.$$  (2.16)

Estimating the second integral of the right-hand side of (2.16) like we have done in (2.15) we get

$$\left| \int_{\mathbb{R}} u \partial_x D^s J_{\varepsilon} u \cdot D^s J_{\varepsilon} u \, dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}} u \partial_x [(D^s J_{\varepsilon} u)^2] \, dx \right|$$

$$= \left| - \frac{1}{2} \int_{\mathbb{R}} \partial_x u (D^s J_{\varepsilon} u)^2 \, dx \right|$$

$$\leq \frac{1}{2} \| \partial_x u \|_{L^\infty} \| J_{\varepsilon} u \|_{H^s}^2$$

$$\leq \frac{1}{2} \| \partial_x u \|_{L^\infty} \| u \|_{H^s}^2.$$  (2.17)

For estimating the first integral of the right-hand side of (2.16) we apply the Cauchy-Schwarz inequality and we have

$$\left| \int_{\mathbb{R}} [J_{\varepsilon}, u] \partial_x D^s u \cdot D^s J_{\varepsilon} u \, dx \right| \leq \|[J_{\varepsilon}, u] \partial_x D^s u\|_{L^2} \| D^s J_{\varepsilon} u\|_{L^2}$$

$$\leq \|[J_{\varepsilon}, u] \partial_x D^s u\|_{L^2} \| u\|_{H^s}$$

$$\leq c \| \partial_x u \|_{L^\infty} \| u \|_{H^s}^2,$$  (2.18)

where the last step of the above inequality is justified by the following result.
Lemma 2. Let $u(x)$ be a function such that $\|\partial_s u\|_{L^\infty} < \infty$. Then, there is $c > 0$ such that for any $f \in L^2(\mathbb{R})$ we have

$$\| [J_\varepsilon, u] \partial_x f \|_{L^2} \leq c \| \partial_x u \|_{L^\infty} \| f \|_{L^2}. \quad (2.19)$$

Proof. We have

$$[J_\varepsilon, u] \partial_x f(x) = J_\varepsilon(u \partial_x f)(x) - u J_\varepsilon(\partial_x f)(x)$$

$$= j_\varepsilon * (u \partial_x f)(x) - u(x)(j_\varepsilon * \partial_x f)(x)$$

$$= \int_\mathbb{R} j_\varepsilon(x-y)u(y)f'(y)\,dy - u(x) \int_\mathbb{R} j_\varepsilon(x-y)f'(y)\,dy \quad (2.20)$$

$$= \int_\mathbb{R} \frac{1}{\varepsilon} \left( \frac{x-y}{\varepsilon} \right) [u(y) - u(x)] f'(y)\,dy.$$

Integrating by parts and using the mean value theorem gives

$$[J_\varepsilon, u] \partial_x f(x) = - \int_\mathbb{R} \frac{1}{\varepsilon} j'(\frac{x-y}{\varepsilon}) u'(y) f(y)\,dy$$

$$+ \int_\mathbb{R} \frac{1}{\varepsilon^2} j'(\frac{x-y}{\varepsilon}) [u(y) - u(x)] f'(y)\,dy$$

$$= - \int_\mathbb{R} \frac{1}{\varepsilon^2} j'(\frac{x-y}{\varepsilon}) u'(y) f(y)\,dy$$

$$+ \int_\mathbb{R} \frac{1}{\varepsilon^2} j'(\frac{x-y}{\varepsilon}) u'(\xi(x,y))(y-x)f(y)\,dy. \quad (2.21)$$

Above we have used our assumption that $j(x)$ is supported on the interval $[-1, 1]$. So, using the bound $|(x-y)/\varepsilon| < 1$ and taking absolute values we obtain that

$$\| [J_\varepsilon, u] \partial_x f(x) \| \leq \| \partial_x u \|_{L^\infty} \left( \int_\mathbb{R} \frac{1}{\varepsilon} j'(\frac{x-y}{\varepsilon}) |f(y)|\,dy \right.$$

$$\left. + \int_\mathbb{R} \frac{1}{\varepsilon^2} |j'(\frac{x-y}{\varepsilon})| |f(y)|\,dy \right)$$

$$= \| \partial_x u \|_{L^\infty} \left( j_\varepsilon * |f|(x) + |j'_\varepsilon| * |f|(x) \right). \quad (2.22)$$

Finally, applying Young’s inequality we get

$$\| [J_\varepsilon, u] \partial_x f \|_{L^2} \leq \| \partial_x u \|_{L^\infty} \left( \| j_\varepsilon \|_{L^1} \| f \|_{L^2} + \| j'_\varepsilon \|_{L^1} \| f \|_{L^2} \right)$$

$$= \left( \| j \|_{L^1} + \| j' \|_{L^1} \right) \| \partial_x u \|_{L^\infty} \| f \|_{L^2}, \quad (2.23)$$

which gives the desired inequality (2.19) with constant $c = \| j \|_{L^1} + \| j' \|_{L^1}$. \hfill \Box

Combining the inequalities (2.10), (2.11), (2.17) and (2.17) we obtain the following estimate for the Burgers term of the CH equation

$$\left| \int_\mathbb{R} D^s J_\varepsilon(u \partial_x u) \cdot D^s J_\varepsilon u \,dx \right| \leq c_s \| \partial_x u \|_{L^\infty} \| u \|_{H^s}^2. \quad (2.24)$$
Estimating the nonlocal $D^{s-2} \partial_x J_\varepsilon(u^2)$. To estimate the second integral in the right-hand side of (2.7) we apply the Cauchy-Schwarz inequality and we get

$$\left| \int_\mathbb{R} D^{s-2} \partial_x J_\varepsilon(u^2) \cdot D^s J_\varepsilon u \, dx \right| \leq \| D^{s-2} \partial_x J_\varepsilon(u^2) \|_{L^2} \| D^s J_\varepsilon u \|_{L^2}$$

$$\leq \| u^2 \|_{H^{s-1}} \| u \|_{H^s} \quad \text{(2.25)}$$

Now, we use the following estimate for the Sobolev norm of a product, which can be found in Taylor [T2] (see Corollary 10.6). For any $s > 0$ and $1 < p < \infty$ there is $C = C_{s,p} > 0$ such that

$$\| fg \|_{H^{s,p}} \leq C \left[ \| f \|_{H^{s,p}} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{H^{s,p}} \right]. \quad \text{(2.26)}$$

Using this result with $s = 2$ and $f = g = u$ from (2.25) we obtain that

$$\left| \int_\mathbb{R} D^{s-2} \partial_x J_\varepsilon(u^2) \cdot D^s J_\varepsilon u \, dx \right| \leq 2c_s \| u \|_{L^\infty} \| u \|_{H^s}^2. \quad \text{(2.27)}$$

Estimating the nonlocal term $D^{s-2} \partial_x J_\varepsilon[(\partial_x u)^2]$. As before, applying the Cauchy-Schwarz inequality we have

$$\left| \int_\mathbb{R} D^{s-2} \partial_x J_\varepsilon[(\partial_x u)^2] \cdot D^s J_\varepsilon u \, dx \right| \leq \| D^{s-2} \partial_x J_\varepsilon[(\partial_x u)^2] \|_{L^2} \| D^s J_\varepsilon u \|_{L^2}$$

$$\leq \| (\partial_x u)^2 \|_{H^{s-1}} \| u \|_{H^s} \quad \text{(2.28)}$$

where in the last step we used estimate (2.26) applied with $s$ replace by $s - 1 > 0$ and $f = g = \partial_x u$.

Now, combining equation (2.6) and estimates (2.24), (2.27), (2.28) we obtain the differential inequality

$$\frac{1}{2} \frac{d}{dt} \| J_\varepsilon u(t) \|_{H^s}^2 \leq c_s \| u(t) \|_{C^1} \| u(t) \|_{H^s}^2, \quad 0 \leq t \leq T. \quad \text{(2.29)}$$

Next, integrating (2.29) from 0 to $t$, $t < T$, gives

$$\frac{1}{2} \| J_\varepsilon u(t) \|_{H^s}^2 - \frac{1}{2} \| J_\varepsilon u(0) \|_{H^s}^2 \leq c_s \int_0^t \| u(\tau) \|_{C^1} \| u(\tau) \|_{H^s}^2 \, d\tau. \quad \text{(2.30)}$$

Then, letting $\varepsilon$ go to 0 (2.30) gives

$$\frac{1}{2} \| u(t) \|_{H^s}^2 - \frac{1}{2} \| u(0) \|_{H^s}^2 \leq c_s \int_0^t \| u(\tau) \|_{C^1} \| u(\tau) \|_{H^s}^2 \, d\tau. \quad \text{(2.31)}$$

Finally, from (2.31) using Gronwall’s inequality we obtain the following lemma, which summarizes our estimates thus far.
Lemma 3. Let $s > 3/2$ and $u \in C([0,T]; H^s)$ be the solution of the Cauchy problem \((1.1)-(1.2)\). Then

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^s}^2 \leq c_s \|u(t)\|_{C^1} \|u(t)\|_{H^s}^2, \quad 0 \leq t \leq T. \tag{2.32}
\]

Since $s > 3/2$ using Sobolev’s inequality

\[
\|u(t)\|_{C^1} \leq c_s \|u(t)\|_{H^s}, \tag{2.33}
\]

from (2.32) we obtain the desired inequality (2.3).

Lifespan estimate. To derive an explicit formula for $T = T(\|v(0)\|_{H^s})$ we proceed as follows. Letting $y(t) = \|u(t)\|_{H^s}$ inequality (2.3) takes the form

\[
\frac{1}{2} y^{-3/2} \frac{dy}{dt} \leq c_s, \quad y(0) = y_0 = \|u_0\|_{H^s}^2. \tag{2.34}
\]

Integrating (2.34) from 0 to $t$ gives

\[
\frac{1}{\sqrt{y_0}} - \frac{1}{\sqrt{y(t)}} \leq c_s t. \tag{2.35}
\]

Replacing $y(t)$ with $\|u(t)\|_{H^s}$ and solving for $\|u(t)\|_{H^s}$ we obtain the formula

\[
\|u(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - c_s \|u_0\|_{H^s} t}. \tag{2.36}
\]

Now, from (2.36) we see that $\|u(t)\|_{H^s}$ is finite if

\[
c_s \|u_0\|_{H^s} t < 1,
\]

or

\[
t < \frac{1}{c_s \|u_0\|_{H^s}}. \tag{2.37}
\]

Therefore, the solution $u(t)$ to the CH Cauchy problem certainly exists for $0 \leq t < T_0$, where

\[
T_0 = \frac{1}{c_s \|u_0\|_{H^s}}. \tag{2.38}
\]

Size of the solution estimate. If we choose $T = 1/2T_0$, that is

\[
T = \frac{1}{2c_s \|u_0\|_{H^s}}, \tag{2.39}
\]

then for $0 \leq t \leq T$ inequality (2.36) gives

\[
\|u(t)\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{1 - (c_s \|u_0\|_{H^s})/(2c_s \|u_0\|_{H^s})},
\]

or

\[
\|u(t)\|_{H^s} \leq 2 \|u_0\|_{H^s}, \quad 0 \leq t \leq T. \tag{2.40}
\]

This completes the proof of Proposition 1. □
Remark. Inequality (2.32) can be used to show that if $u \in C([0, T]; H^s)$, $s > 3/2$, is a solution of Cauchy problem (1.1)–(1.2) such that $\sup_{0 \leq t < T} \|u(t)\|_{C^1} < \infty$ then $u(t)$ persists to be a solution beyond the time $T$. In particular, we can show that if the lifespan $T$ of $u$ is finite then $\sup_{0 \leq t < T} \|u(t)\|_{C^1} = \infty$ (see Theorem 6.2 in [LO]).

3. Construction of approximate solutions

Here we shall construct a two-parameter family of approximate solutions $u^{\omega, \lambda} = u^{\omega, \lambda}(x, t)$, each member of which consists of two parts, that is

$$u^{\omega, \lambda} = u_\ell + u^h. \quad (3.1)$$

The high frequency part $u^h$ is given by

$$u^h = u^{h, \omega, \lambda}(x, t) = \lambda^{-\delta/2-s} \varphi \left( \frac{x}{\lambda^\delta} \right) \cos(\lambda x - \omega t), \quad (3.2)$$

and is not a solution of CH. Here $\varphi$ is a $C^\infty$ function such that

$$\varphi(x) = \begin{cases} 
1, & \text{if } |x| < 1, \\
0, & \text{if } |x| \geq 2. 
\end{cases} \quad (3.3)$$

The low frequency part $u_\ell = u_{\ell, \omega, \lambda}(x, t)$ is the solution to the following Cauchy problem for CH:

$$\partial_t u_\ell + u_\ell \partial_x u_\ell + \Lambda^{-1} \left[ u_\ell^2 + \frac{1}{2} (\partial_x u_\ell)^2 \right] = 0, \quad (3.4)$$

$$u_\ell(x, 0) = \omega \lambda^{-1} \tilde{\varphi} \left( \frac{x}{\lambda^\delta} \right), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (3.5)$$

where $\tilde{\varphi}$ is a $C^\infty_0(\mathbb{R})$ function such that

$$\tilde{\varphi}(x) = 1, \quad \text{if } x \in \text{supp } \varphi. \quad (3.6)$$

Furthermore, $\Lambda^{-1}$ denotes the order $-1$ pseudodifferential operator

$$\Lambda^{-1} = \partial_x \left( 1 - \partial_x^2 \right)^{-1}. \quad (3.7)$$

As it is explained in Lemma 5 below, the initial value problem (3.4)–(3.5) has a unique smooth solution $u_\ell$ belonging in $H^s(\mathbb{R})$ for all $s$. Thus, the approximate solutions $u^{\omega, \lambda}$ belong in every Sobolev space.

Substituting the approximate solution $u^{\omega, \lambda} = u_\ell + u^h$ into CH equation we obtain the following expression

\begin{align*}
F &= \partial_t u^h + u_\ell \partial_x u^h + u^h \partial_x u_\ell + u^h \partial_x u_\ell + \Lambda^{-1} \left[ 2u_\ell u^h + (u^h)^2 + \partial_x u_\ell \partial_x u^h + \frac{1}{2} (\partial_x u^h)^2 \right] \\
&\quad + \partial_t u_\ell + u_\ell \partial_x u_\ell + \partial_x \left( 1 - \partial_x^2 \right)^{-1} \left[ u_\ell^2 + \frac{1}{2} (\partial_x u_\ell)^2 \right].
\end{align*}
Now, taking into consideration that $u_\ell$ solves CH we obtain the following error for the approximate solution

$$F = \partial_t u^h + u_\ell \partial_x u^h + u^h \partial_x u_\ell + u^h \partial_x u^h + \Lambda^{-1} \left[ 2u_\ell u^h + (u^h)^2 + \partial_x u_\ell \partial_x u^h + \frac{1}{2} (\partial_x u^h)^2 \right].$$

(3.8)

Computing $\partial_t u^h$ gives

$$\partial_t u^h(x, t) = \omega \lambda^{-\delta/2-s} \varphi(\frac{x}{\lambda^\delta}) \sin(\lambda x - \omega t).$$

(3.9)

Furthermore, since $\tilde{\varphi}$ is equal to 1 on the support of $\varphi$ we see that we can write $\partial_t u^h$ in the following form

$$\partial_t u^h(x, t) = \omega \lambda \varphi(x \lambda^\delta) \sin(\lambda x - \omega t).$$

(3.10)

Computing the spatial derivative of $u^h$ gives

$$\partial_x u^h(x, t) = -\lambda \lambda^{-\delta/2-s} \varphi(\frac{x}{\lambda^\delta}) \sin(\lambda x - \omega t),$$

$$+ \lambda^{-\frac{\delta}{2}-s} \partial_x \varphi(\frac{x}{\lambda^\delta}) \cos(\lambda x - \omega t).$$

(3.11)

Then, using (3.10) and (3.11) we find that

$$\partial_t u^h + u_\ell \partial_x u^h = \lambda \left[ u_\ell(x, 0) - u_\ell(x, t) \right] \lambda^{-\delta/2-s} \varphi(\frac{x}{\lambda^\delta}) \sin(\lambda x - \omega t)$$

$$+ u_\ell(x, t) \lambda^{-\frac{\delta}{2}-s} \partial_x \varphi(\frac{x}{\lambda^\delta}) \cos(\lambda x - \omega t).$$

(3.12)

Therefore, the error (3.8) of the approximate solution $u^{\omega, \lambda}$ is given by

$$F = F_1 + F_2 + \cdots + F_8,$$

(3.13)

where

$$F_1 = \lambda \left[ u_\ell(x, 0) - u_\ell(x, t) \right] \lambda^{-\delta/2-s} \varphi(\frac{x}{\lambda^\delta}) \sin(\lambda x - \omega t)$$

$$F_2 = u_\ell(x, t) \lambda^{-\frac{\delta}{2}-s} \partial_x \varphi(\frac{x}{\lambda^\delta}) \cos(\lambda x - \omega t)$$

$$F_3 = u^h \partial_x u_\ell$$

$$F_4 = u^h \partial_x u^h$$

$$F_5 = \Lambda^{-1} \left[ 2u_\ell u^h \right]$$

$$F_6 = \Lambda^{-1} \left[ (u^h)^2 \right]$$

$$F_7 = \Lambda^{-1} \left[ \partial_x u_\ell \partial_x u^h \right]$$

$$F_8 = \Lambda^{-1} \left[ \frac{1}{2} (\partial_x u^h)^2 \right].$$

(3.14)
Next we shall estimate the size of the error $F$.

4. Estimating the $H^1$ norm of the error

To estimate the $H^1$ norm of the error $F$ it suffices to estimate the $H^1$ norm of each term $F_j$. Observe that each $F_j$ is expressed in terms of $u_\ell$ and $u^h$. The high frequency part $u^h$ is defined by formula (3.2) and
\[
\|u^h(t)\|_{H^s(\mathbb{R})} \approx 1, \quad \text{for } \lambda \gg 1, \tag{4.1}
\]
because of the following result.

**Lemma 4.** Let $\psi \in S(\mathbb{R})$, $1 < \delta < 2$ and $\alpha \in \mathbb{R}$. Then for any $s \geq 0$ we have that
\[
\lim_{\lambda \to \infty} \lambda^{-\frac{1}{2}\delta-s}\|\psi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \alpha)\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}}\|\psi\|_{L^2(\mathbb{R})}. \tag{4.2}
\]
Relation (4.2) is also true if $\cos$ is replaced by $\sin$.

Although this lemma can be found in [KT], we include its proof here for the convenience of the reader.

**Proof.** Since
\[
\left(\psi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \alpha)\right)'(\xi) = \frac{1}{2} \lambda^\delta \left[ e^{-i\alpha} \hat{\psi}(\lambda^\delta (\xi - \lambda)) + e^{i\alpha} \hat{\psi}(\lambda^\delta (\xi + \lambda)) \right],
\]
we have that
\[
\begin{align*}
\lambda^{-\delta-2s}\|\psi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \alpha)\|_{H^s(\mathbb{R})}^2 &= \frac{\lambda^{-2s+\delta}}{8\pi} \int_{\mathbb{R}} (1 + \xi^2)^{s} \left| \hat{\psi}(\lambda^\delta (\xi - \lambda)) + \hat{\psi}(\lambda^\delta (\xi + \lambda)) \right|^2 d\xi \\
&= \frac{\lambda^{-2s+\delta}}{8\pi} \left[ \int_{\mathbb{R}} (1 + \xi^2)^{s} \left| \hat{\psi}(\lambda^\delta (\xi - \lambda)) \right|^2 d\xi \\
&\quad + \int_{\mathbb{R}} (1 + \xi^2)^{s} \left| \hat{\psi}(\lambda^\delta (\xi + \lambda)) \right|^2 d\xi \\
&\quad + 2 \int_{\mathbb{R}} (1 + \xi^2)^{s} \text{Re} \left[ e^{-2i\alpha} \hat{\psi}(\lambda^\delta (\xi - \lambda)) \hat{\psi}(\lambda^\delta (\xi + \lambda)) \right] d\xi \right].
\end{align*}
\]
Now, in the first and third integral we make the change of variables $\eta = \lambda^\delta (\xi - \lambda)$, while in the second we let $\eta = \lambda^\delta (\xi + \lambda)$. Thus, we have
\[
\begin{align*}
\lambda^{-\delta-2s}\|\psi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \alpha)\|_{H^s(\mathbb{R})}^2 &= \frac{\lambda^{-2s}}{8\pi} \int_{\mathbb{R}} \left[ 1 + \left( \frac{\eta}{\lambda^\delta} + \lambda \right)^2 \right]^s \left| \hat{\psi}(\eta) \right|^2 d\eta \\
&\quad + \int_{\mathbb{R}} \left[ 1 + \left( \frac{\eta}{\lambda^\delta} - \lambda \right)^2 \right]^s \left| \hat{\psi}(\eta) \right|^2 d\eta \\
&\quad + 2 \int_{\mathbb{R}} \left[ 1 + \left( \frac{\eta}{\lambda^\delta} + \lambda \right)^2 \right]^s \text{Re} \left[ e^{-2i\alpha}\hat{\psi}(\eta) \hat{\psi}(\eta + 2\lambda^{\delta+1}) \right] d\xi .
\end{align*}
\]
Moving the factor $\lambda^{-2s}$ inside the integrals gives

$$\lambda^{-\delta-2s}\|\psi\left(\frac{x}{\lambda}\right) \cos(\lambda x - \alpha)\|_{H^s(\mathbb{R})}^2 = \frac{1}{8\pi} \left[ \int_{\mathbb{R}} \left( \frac{1}{\lambda^2} + \left( \frac{\eta}{\lambda^{\delta+1}} + 1 \right)^2 \right)^s |\hat{\psi}(\eta)|^2 d\eta 
+ \int_{\mathbb{R}} \left( \frac{1}{\lambda^2} + \left( \frac{\eta}{\lambda^{\delta+1}} - 1 \right)^2 \right)^s |\hat{\psi}(\eta)|^2 d\eta 
+ 2 \int_{\mathbb{R}} \left( \frac{1}{\lambda^2} + \left( \frac{\eta}{\lambda^{\delta+1}} + 1 \right)^2 \right)^s \Re \left[ e^{-2i\alpha} \hat{\psi}(\eta) \hat{\psi}(\eta + 2\lambda^{\delta+1}) \right] d\xi \right].$$

Since $\psi \in S(\mathbb{R})$ we have that $\hat{\psi}(\eta + 2\lambda^{\delta+1}) \to 0$ as $\lambda \to \infty$. Therefore, applying the dominated convergence theorem we see that the third integral goes to zero while each of the other two goes to $\|\hat{\psi}\|_{L^2}^2$. Therefore, we obtain that

$$\lim_{\lambda \to \infty} \lambda^{-\delta-2s}\|\psi\left(\frac{x}{\lambda}\right) \cos(\lambda x - \alpha)\|_{H^s(\mathbb{R})}^2 = \frac{1}{4\pi} \|\hat{\psi}\|_{L^2}^2 = \frac{1}{2} \|\psi\|_{L^2}^2,$$

which proves the lemma. \(\square\)

As we have stated earlier, the low frequency part $u_\ell$ is the solution of the Cauchy problem (3.4)–(3.5). Next lemma summarizes the basic information about $u_\ell$.

**Lemma 5.** Let $\omega$ be bounded, $0 < \delta < 2$ and $\lambda \gg 1$. Then, the initial value problem (3.4)–(3.5) has a unique smooth solution $u_\ell \in C([0, 1]; H^s(\mathbb{R}))$, for all $s > 3/2$, and satisfying the estimate

$$\|u_\ell(t)\|_{H^s(\mathbb{R})} \leq c_s \lambda^{-1+\delta/2}, \quad 0 \leq t \leq 1. \quad (4.3)$$

**Proof.** Let $s \geq 0$. For any function $\psi \in S(\mathbb{R})$ we have

$$\|\psi\left(\frac{x}{\lambda}\right)\|_{H^s(\mathbb{R})} \leq \lambda^{\delta/2} \|\psi\|_{H^s(\mathbb{R})}. \quad (4.4)$$

In fact, using the relation $\hat{\psi}(\rho(\xi))(\xi) = \rho \hat{\psi}(\rho(\xi))$ and making the change of variables $\eta = \lambda^\delta \xi$ we obtain

$$\|\psi\left(\frac{x}{\lambda}\right)\|_{H^s(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \xi^2)^s |\lambda^\delta \hat{\psi}(\lambda^\delta \xi)|^2 d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \frac{\eta^2}{\lambda^{2\delta}})^s \lambda^{2\delta} |\hat{\psi}(\eta)|^2 \frac{d\eta}{\lambda^\delta}$$

$$= \lambda^\delta \cdot \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \frac{\eta^2}{\lambda^{2\delta}})^s |\hat{\psi}(\eta)|^2 d\eta \leq \lambda^\delta \cdot \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \eta^2)^s |\hat{\psi}(\eta)|^2 d\eta$$

$$= \lambda^\delta \|\hat{\psi}\|_{H^s(\mathbb{R})}^2.$$

Now, using inequality (4.4) we have that the initial data $u_\ell(0)$ satisfy the estimate

$$\|u_\ell(0)\|_{H^s(\mathbb{R})} \leq |\omega| \lambda^{-1+\delta/2} \|\hat{\psi}\|_{H^s(\mathbb{R})}, \quad (4.5)$$
which for $\omega$ bounded decays if

$$\delta < 2. \quad (4.6)$$

Next, using estimate (2.1) from Proposition 1 we have that the lifespan $T$ of the solution $u_\ell(t)$ satisfies

$$T \geq \frac{1}{2c_s\|u_\ell(0)\|_{H^s(\mathbb{R})}} \geq \frac{c_s^s}{\lambda^{1+\delta/2}} \geq 1, \quad \text{for } \lambda \gg 1,$$

since $\delta < 2$. Finally, if $s \geq 0$ then from estimate (2.2) of Proposition 1 we have

$$\|u_\ell(t)\|_{H^s(\mathbb{R})} \leq \|u_\ell(t)\|_{H^{s+2}(\mathbb{R})} \leq c_s\|u_\ell(0)\|_{H^{s+2}(\mathbb{R})} \leq c_s\lambda^{-1+\delta/2}. \quad \square$$

Now we are ready to estimate the $H^1$ norm of each error $F_j$.

**Estimating the $H^1$-norm of $F_1$**. We have

$$\|F_1\|_{H^1(\mathbb{R})} = \|\lambda\left[u_\ell(x,0) - u_\ell(x,t)\right]\lambda^{-\delta/2-s}\varphi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x - \omega t)\|_{H^1(\mathbb{R})}$$

$$= \lambda^{-\delta/2-s}\|\varphi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x - \omega t)\left[u_\ell(x,0) - u_\ell(x,t)\right]\|_{H^1(\mathbb{R})}. \quad (4.7)$$

Using the inequality

$$\|fg\|_{H^1(\mathbb{R})} \leq \sqrt{2}\|f\|_{C^1(\mathbb{R})}\|g\|_{H^1(\mathbb{R})}, \quad (4.8)$$

from (4.7) we get

$$\|F_1\|_{H^1(\mathbb{R})} \lesssim \lambda^{-\delta/2-s}\|\varphi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x - \omega t)\|_{C^1(\mathbb{R})}\|u_\ell(x,0) - u_\ell(x,t)\|_{H^1(\mathbb{R})}$$

And, since $\|\varphi\left(\frac{x}{\lambda^\delta}\right)\|_{C^1(\mathbb{R})} = \|\varphi\|_{L^\infty} \lambda$ the last inequality gives

$$\|F_1\|_{H^1(\mathbb{R})} \lesssim \lambda^{2-\delta/2-s}\|u_\ell(x,0) - u_\ell(x,t)\|_{H^1(\mathbb{R})}. \quad (4.9)$$

To estimate the $H^1$ norm of the difference $u_\ell(t) - u_\ell(0)$ we apply the fundamental theorem of calculus in the time variable to obtain

$$u_\ell(x, t) - u_\ell(x, 0) = \int_0^t \partial_\tau u_\ell(x, \tau) d\tau. \quad (4.10)$$

Then, taking the $H^1$ norm of the space variable to both sides of (4.10) and passing the norm inside the integral gives

$$\|u_\ell(x, 0) - u_\ell(x, t)\|_{H^1(\mathbb{R})} \leq \int_0^t \|\partial_\tau u_\ell(x, \tau)\|_{H^1(\mathbb{R})} d\tau, \quad t \in [0, 1]. \quad (4.11)$$

Next we estimate $\|\partial_\tau u_\ell(x, \tau)\|_{H^1(\mathbb{R})}$. For this we solve equation (3.4) for $\partial_\tau u_\ell$ to get

$$\partial_\tau u_\ell(x, \tau) = -u_\ell \partial_x u_\ell - \Lambda^{-1} \left[u_\ell^2 + \frac{1}{2} (\partial_x u_\ell)^2\right]. \quad (4.12)$$

Thus, at any time in $[0, T]$ we have

$$\|\partial_\tau u_\ell(x, \tau)\|_{H^1(\mathbb{R})} \leq \|u_\ell \partial_x u_\ell\|_{H^1(\mathbb{R})} + \|\Lambda^{-1} \left[u_\ell^2 + \frac{1}{2} (\partial_x u_\ell)^2\right]\|_{H^1(\mathbb{R})} \quad (4.13)$$
Now, using the inequality
\[ \|fg\|_{H^1(\mathbb{R})} \leq c\|f\|_{H^1(\mathbb{R})}\|g\|_{H^1(\mathbb{R})} \quad (4.14) \]
and the estimate
\[ \|\Lambda^{-1}f\|_{H^1(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})} \quad (4.15) \]
from (4.13) we obtain that
\[ \|\partial_t u_\ell(x,\tau)\|_{H^1(\mathbb{R})} \lesssim \|u_\ell\|_{H^1(\mathbb{R})}\|\partial_x u_\ell\|_{H^1(\mathbb{R})} + \|\partial_t^2 u_\ell\|_{L^2(\mathbb{R})} \]
\[ \lesssim \|u_\ell\|_{H^1(\mathbb{R})}\|u_\ell\|_{H^2(\mathbb{R})} + \|\partial_t u_\ell\|_{L^2(\mathbb{R})} + \|\partial_x u_\ell\|_{L^2(\mathbb{R})} \]
\[ \lesssim \|u_\ell\|_{H^2(\mathbb{R})} + \|u_\ell\|_{L^\infty(\mathbb{R})}\|\partial_x u_\ell\|_{L^2(\mathbb{R})} \]
\[ \lesssim \|u_\ell\|_{H^2(\mathbb{R})} + \|u_\ell\|_{H^1(\mathbb{R})}^2 + \|\partial_t u_\ell\|_{H^1(\mathbb{R})} \]
\[ \lesssim \|u_\ell\|_{H^2(\mathbb{R})}. \quad (4.16) \]

Using estimate (4.3), from the last inequality we get
\[ \|\partial_t u_\ell(x,\tau)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta}. \quad (4.17) \]
Substituting (4.17) into (4.11) we obtain
\[ \|u_\ell(x,0) - u_\ell(x,t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2+\delta}. \quad (4.18) \]
Finally, combining (4.18) and (4.9) gives
\[ \|F_1\|_{H^1(\mathbb{R})} \lesssim \lambda^{2-\delta/2-s} \cdot \lambda^{-2+\delta}, \quad (4.19) \]
which gives
\[ \|F_1\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s+\delta/2}, \quad \lambda > 1. \quad (4.20) \]

**Estimating the $H^1$-norm of $F_2$.** Reading $F_2$ from (3.14) we have
\[ \|F_2\|_{H^1(\mathbb{R})} = \|u_\ell(x,t) \cdot \lambda^{-\delta-s}\partial_x \varphi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \omega t)\|_{H^1(\mathbb{R})} \]
\[ \lesssim \lambda^{-\delta-s}\|\partial_x \varphi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \omega t)\|_{C^1(\mathbb{R})}\|u_\ell(x,t)\|_{H^1(\mathbb{R})} \]
\[ \lesssim \lambda^{-\delta-s}\|\partial_x \varphi\left(\frac{x}{\lambda^\delta}\right) \cos(\lambda x - \omega t)\|_{C^1(\mathbb{R})}\|u_\ell(x,t)\|_{H^2(\mathbb{R})} \]
\[ \lesssim \lambda^{-\delta-s} \cdot \lambda \cdot \lambda^{-1+\frac{s}{2}} \]
which gives
\[ \|F_2\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s-\delta}. \quad (4.21) \]

**Estimating the $H^1$-norm of $F_3$.** From (3.14) we have
\[ \|F_3(t)\|_{H^1(\mathbb{R})} = \|u_\ell(t)\|_{H^1(\mathbb{R})} \]
\[ \approx \|u_\ell(t)\|_{C^1(\mathbb{R})}\|\partial_x u_\ell(t)\|_{H^1(\mathbb{R})} \]
\[ \lesssim \|u_\ell(t)\|_{C^1(\mathbb{R})}\|u_\ell(t)\|_{H^2(\mathbb{R})}. \]
Using formula (3.2) for $u^h$ and estimate (4.3) for $u_\ell$, from the last inequality we obtain that
\[ \|F_3(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-\frac{1}{2}\delta-s+1} \cdot \lambda^{-1+\frac{1}{2}\delta}, \]
which gives
\[ \|F_3(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s}, \quad \lambda \gg 1. \quad (4.23) \]

**Estimating the $H^1$-norm of $F_4$.** Reading $F_4$ from (3.14) and using (2.26) we have
\[ \|F_4(t)\|_{H^1(\mathbb{R})} = \|u^h(t)\partial_x u^h(t)\|_{H^1(\mathbb{R})} \]
\[ \lesssim \|u^h(t)\|_{H^1(\mathbb{R})}\|\partial_x u^h(t)\|_{L^\infty(\mathbb{R})} + \|u^h(t)\|_{L^\infty(\mathbb{R})}\|\partial_x u^h(t)\|_{H^1(\mathbb{R})} \quad (4.24) \]
\[ \lesssim \|u^h(t)\|_{H^1(\mathbb{R})}\|\partial_x u^h(t)\|_{L^\infty(\mathbb{R})} + \|u^h(t)\|_{L^\infty(\mathbb{R})}\|u^h(t)\|_{H^2(\mathbb{R})}. \]

Since
\[ \|u^h(t)\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\frac{1}{2}\delta-s}, \quad \|\partial_x u^h(t)\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\frac{1}{2}\delta-s+1}, \]
and since, by Lemma 4, we have
\[ \|u^h(t)\|_{H^k(\mathbb{R})} = \lambda^{-\delta/2-s}\|\varphi\frac{\varphi}{\lambda^\delta}\cos(\lambda x - \omega t)\|_{H^k(\mathbb{R})} \]
\[ = \lambda^{-s+k} \cdot \lambda^{-\delta/2-k}\|\varphi\frac{\varphi}{\lambda^\delta}\cos(\lambda x - \omega t)\|_{H^k(\mathbb{R})} \]
\[ \lesssim \lambda^{-s+k}, \]
estimate (4.24) gives
\[ \|F_4(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s+1} \cdot \lambda^{-\frac{1}{2}\delta-s+1} + \lambda^{-\frac{1}{2}\delta-s} \cdot \lambda^{-s+2}. \]

Thus,
\[ \|F_4(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2s-\frac{1}{2}\delta+2}, \quad \lambda \gg 1. \quad (4.25) \]

**Estimating the $H^1$-norm of $F_5$.** We have
\[ \|F_5\|_{H^1(\mathbb{R})} = \|\Lambda^{-1}[2u_\ell u^h]\|_{H^1(\mathbb{R})} \]
\[ \leq 2\|u_\ell u^h\|_{L^2(\mathbb{R})} \]
\[ \lesssim \|u^h\|_{L^\infty(\mathbb{R})}\|u_\ell\|_{L^2(\mathbb{R})} \]
\[ \lesssim \|u^h\|_{L^\infty(\mathbb{R})}\|u_\ell\|_{H^2(\mathbb{R})} \]
\[ \lesssim \lambda^{-\frac{1}{2}\delta-s} \cdot \lambda^{-1+\frac{1}{2}\delta}, \]
which gives
\[ \|F_5(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s-1}, \quad \lambda \gg 1. \quad (4.26) \]
Estimating the $H^1$-norm of $F_6$. From (3.14) and Lemma 4 we have
\[
\|F_5\|_{H^1(\mathbb{R})} = \|\Lambda^{-1}[(u^h)^2]\|_{H^1(\mathbb{R})}
\leq \|[(u^h)^2]\|_{L^2(\mathbb{R})}
\lesssim \|u^h\|_{L^\infty(\mathbb{R})}\|u^h\|_{L^2(\mathbb{R})}
\lesssim \lambda^{-\frac{\delta}{2}} \cdot \lambda^{-s},
\]
which gives
\[
\|F_6(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2s-\frac{\delta}{2}}, \quad \lambda >> 1. \quad (4.27)
\]

Estimating the $H^1$-norm of $F_7$. Also, we have
\[
\|F_7\|_{H^1(\mathbb{R})} = \|\Lambda^{-1}[\partial_x u_t \partial_x u^h]\|_{H^1(\mathbb{R})}
\leq \|\partial_x u_t \partial_x u^h\|_{L^2(\mathbb{R})}
\lesssim \|\partial_x u^h\|_{L^\infty(\mathbb{R})}\|\partial_x u_t\|_{L^2(\mathbb{R})}
\lesssim \|\partial_x u^h\|_{L^\infty(\mathbb{R})}\|u_t\|_{H^2(\mathbb{R})}
\lesssim \lambda^{-\frac{\delta}{2}} \cdot \lambda^{-s+1},
\]
which gives
\[
\|F_6(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-s}, \quad \lambda >> 1. \quad (4.28)
\]

Estimating the $H^1$-norm of $F_8$. Finally, we have
\[
\|F_8\|_{H^1(\mathbb{R})} = \|\Lambda^{-1}[\frac{1}{2}(\partial_x u^h)^2]\]
\[
\leq \frac{1}{2}\|[(\partial_x u^h)^2]\|_{L^2(\mathbb{R})}
\lesssim \|\partial_x u^h\|_{L^\infty(\mathbb{R})}\|\partial_x u^h\|_{L^2(\mathbb{R})}
\lesssim \|\partial_x u^h\|_{L^\infty(\mathbb{R})}\|u^h\|_{H^1(\mathbb{R})}
\lesssim \lambda^{-\frac{\delta}{2}} \cdot \lambda^{-s+1},
\]
which gives
\[
\|F_6(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-2s-\frac{\delta}{2}}, \quad \lambda >> 1. \quad (4.29)
\]
Collecting all error estimates together gives the following proposition.

**Proposition 2.** Let $s > 1$ and $1 < \delta < 2$. Then, for $\omega$ bounded and $\lambda >> 1$ we have that
\[
\|F(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-r_s}, \quad \text{for } \lambda >> 1, \quad (4.30)
\]
with
\[
r_s \doteq (s - \frac{1}{2}\delta) > 0, \quad \text{if } s > \frac{1}{2}\delta. \quad (4.31)
\]
5. Estimating the difference between approximate and actual solutions

Let \( u_{\omega,\lambda}(x, t) \) be the solution to CH equation with initial data the value of the approximate solution \( u_{\omega,\lambda}(x, t) \) at time zero. That is, \( u_{\omega,\lambda}(x, t) \) solves the Cauchy problem

\[
\frac{\partial}{\partial t} u_{\omega,\lambda} + u_{\omega,\lambda} \frac{\partial}{\partial x} u_{\omega,\lambda} + \Lambda^{-1} \left[ u_{\omega,\lambda}^2 + \frac{1}{2} \left( \frac{\partial_x u_{\omega,\lambda}}{\Lambda} \right)^2 \right] = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{5.1}
\]

\[
u_{\omega,\lambda}(x, 0) = u_{\omega,\lambda}(x, 0) = \omega\lambda - 1 \tilde{\varphi}(\frac{x}{\lambda^\delta}) + \lambda^{-\delta/2-s} \varphi(\frac{x}{\lambda^\delta}) \cos(\lambda x). \tag{5.2}
\]

Note that \( u_{\omega,\lambda}(0) \) is in \( H^s(\mathbb{R}) \), \( s \geq 0 \), and

\[
\| u_{\omega,\lambda}(0) \|_{H^s} \leq \| u_\ell(0) \|_{H^s} + \| u_h(0) \|_{H^s} \lesssim \lambda^{-1} + \frac{1}{2} \lambda^{-\delta} + 1. \tag{5.3}
\]

Therefore, if \( s > 3/2 \) then using Theorem 2 and Proposition 1 we see that for any \( \omega \) in a bounded set and \( \lambda >> 1 \) the Cauchy problem (5.1)–(5.2) has a unique solution \( u_{\omega,\lambda} \) in \( C([0, T]; H^s(\mathbb{R})) \) with

\[
T \gtrsim \frac{1}{\| u_{\omega,\lambda}(0) \|_{H^s(\mathbb{R})}} \gtrsim \frac{1}{\lambda^{-1} + \delta/2 + 1} \gtrsim 1. \tag{5.4}
\]

In fact, \( u_{\omega,\lambda}(t) \) is in \( C^\infty \) for each \( t \in [0, T] \).

To estimate the difference between approximate and actual solutions we form the differential equation which it satisfies. So, if we let

\[
v = u_{\omega,\lambda} - u_{\omega,\lambda}, \tag{5.5}
\]

then a straightforward computation shows that \( v \) satisfies the Cauchy problem

\[
\frac{\partial}{\partial t} v - v \frac{\partial}{\partial x} v + u_{\omega,\lambda} \frac{\partial}{\partial x} v + \frac{\partial}{\partial x} u_{\omega,\lambda} v - \Lambda^{-1} \left[ v^2 + \frac{1}{2} (\frac{\partial_x v}{\Lambda})^2 - 2 u_{\omega,\lambda} v \right] = F(x, t), \tag{5.6}
\]

\[
v(x, 0) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{5.7}
\]

where \( F \) is defined by

\[
F \doteq \frac{\partial}{\partial t} u_{\omega,\lambda} + u_{\omega,\lambda} \frac{\partial}{\partial x} u_{\omega,\lambda} + \Lambda^{-1} \left[ (u_{\omega,\lambda})^2 + \frac{1}{2} (\frac{\partial_x u_{\omega,\lambda}}{\Lambda})^2 \right], \tag{5.8}
\]

and which it has been shown to satisfy the \( H^1 \)-estimate (4.30).

**Lemma 6.** Let \( 1 < \delta < 2 \). If \( s > 3/2 \) then

\[
\| v(t) \|_{H^1(\mathbb{R})} = \| u_{\omega,\lambda}(t) - u_{\omega,\lambda}(t) \|_{H^1(\mathbb{R})} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T, \tag{5.9}
\]

where \( r_s = s - \delta/2 > 0 \) (see (4.31)).

**Proof.** We have

\[
\frac{1}{2} \frac{d}{dt} \| v(t) \|_{H^1(\mathbb{R})}^2 = \int_\mathbb{R} \left[ v \frac{\partial}{\partial t} v + \frac{\partial}{\partial x} v \frac{\partial}{\partial x} v \right] dx \tag{5.10}
\]
Applying to both sides of (5.6) the operator \((1 - \partial_x^2)\) and solving for \(\partial_t v\) we obtain

\[
\partial_t v = (1 - \partial_x^2) F - (1 - \partial_x^2) [u^{\omega,\lambda} \partial_x v + \partial_x u^{\omega,\lambda} v] - \partial_x [2u^{\omega,\lambda} v + \partial_x u^{\omega,\lambda} \partial_x v] + 3v \partial_x v - 2\partial_x v \partial_x^2 v - v \partial_x^3 v + \partial_t \partial_x^2 v
\]

(5.11)

Substituting \(\partial_t v\) from (5.11) to (5.10) we get

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} v(1 - \partial_x^2) F dx - \int_{\mathbb{R}} v(1 - \partial_x^2) [u^{\omega,\lambda} \partial_x v + \partial_x u^{\omega,\lambda} v] dx - \int_{\mathbb{R}} v \partial_x [2u^{\omega,\lambda} v + \partial_x u^{\omega,\lambda} \partial_x v] dx + \int_{\mathbb{R}} [v(3\partial_x v - 2\partial_x v \partial_x^2 v - v \partial_x^3 v + \partial_t \partial_x^2 v) + \partial_x v \partial_x v] dx.
\]

(5.12)

Noting that the last integral can be rewritten as

\[
\int_{\mathbb{R}} [\partial_x (v^3) - \partial_x (v^2 \partial_x^2 v) + \partial_x (v \partial_x v)] dx = 0,
\]

which is a property special to CH, we see that equation (5.12) takes the form

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} v(1 - \partial_x^2) F dx - \int_{\mathbb{R}} v(1 - \partial_x^2) [u^{\omega,\lambda} \partial_x v + \partial_x u^{\omega,\lambda} v] dx - \int_{\mathbb{R}} v \partial_x [2u^{\omega,\lambda} v + \partial_x u^{\omega,\lambda} \partial_x v] dx.
\]

(5.13)

Integrating by parts and applying the Cauchy-Schwarz inequality, we estimate the three integrals in the right-hand side of (5.13) as follows. For the first integral we have

\[
\left| \int_{\mathbb{R}} v(1 - \partial_x^2) F dx \right| = \left| \int_{\mathbb{R}} [v F + \partial_x v \partial_x F] dx \right| \leq \|F(t)\|_{H^1(\mathbb{R})} \|v(t)\|_{H^1(\mathbb{R})}.
\]

(5.14)

Also, for the third integral we have

\[
\left| \int_{\mathbb{R}} v \partial_x [2u^{\omega,\lambda} v + \partial_x u^{\omega,\lambda} \partial_x v] dx \right| = \left| \int_{\mathbb{R}} \partial_x v [2u^{\omega,\lambda} v + \partial_x u^{\omega,\lambda} \partial_x v] dx \right| \leq 2 \left( \|u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} \right) \|v(t)\|_{H^1(\mathbb{R})}^2.
\]

(5.15)
Integrating by parts, we write the second integral in the form

$$\int_{\mathbb{R}} v(1 - \partial_x^2)[u^{\omega,\lambda} \partial_x v + \partial_x u^{\omega,\lambda}] dx = \int_{\mathbb{R}} v[u^{\omega,\lambda} \partial_x v + \partial_x u^{\omega,\lambda}] dx$$

$$+ \int_{\mathbb{R}} \partial_x v \partial_x [u^{\omega,\lambda} \partial_x v] dx + \int_{\mathbb{R}} \partial_x v \partial_x [\partial_x u^{\omega,\lambda}] dx$$

and estimate its first part by

$$\left| \int_{\mathbb{R}} v[u^{\omega,\lambda} \partial_x v + \partial_x u^{\omega,\lambda}] dx \right| \leq \left( \|u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} \right) \|v(t)\|_{H^1(\mathbb{R})}^2. \quad (5.17)$$

Its second part we can be written as

$$\int_{\mathbb{R}} \partial_x v \partial_x [u^{\omega,\lambda} \partial_x v] dx = \int_{\mathbb{R}} \left[ \frac{1}{2} u^{\omega,\lambda} \partial_x (\partial_x v)^2 + \partial_x u^{\omega,\lambda} (\partial_x v)^2 \right] dx = \frac{1}{2} \int_{\mathbb{R}} \partial_x u^{\omega,\lambda} (\partial_x v) dx,$$

which gives that

$$\left| \int_{\mathbb{R}} \partial_x v \partial_x [u^{\omega,\lambda} \partial_x v] dx \right| \leq \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} \|v(t)\|_{H^1(\mathbb{R})}^2. \quad (5.18)$$

Finally, writing the last part as follows

$$\int_{\mathbb{R}} \partial_x v \partial_x [\partial_x u^{\omega,\lambda}] dx = \int_{\mathbb{R}} \left[ \partial_x u^{\omega,\lambda} (\partial_x v)^2 + \partial_x^2 u^{\omega,\lambda} v \partial_x v \right] dx$$

we see that it can be estimated as follow

$$\left| \int_{\mathbb{R}} \partial_x v \partial_x [\partial_x u^{\omega,\lambda}] dx \right| \leq \left( \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} \right) \|v(t)\|_{H^1(\mathbb{R})}^2. \quad (5.19)$$

Combining the above estimates gives

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^1(\mathbb{R})}^2 \lesssim \|F(t)\|_{H^1(\mathbb{R})} \|v(t)\|_{H^1(\mathbb{R})} + \left( \|u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} \right) \|v(t)\|_{H^1(\mathbb{R})}^2. \quad (5.20)$$

From (3.2) we have

$$\partial_x^2 u^h = \lambda^{-\frac{1}{2} - \delta - s} \partial_x^2 \varphi \left( \frac{x}{\lambda^\delta} \right) \cos(\lambda x - \omega t)$$

$$- 2\lambda^{-\frac{1}{2} - \delta - s + 1} \partial_x \varphi \left( \frac{x}{\lambda^\delta} \right) \sin(\lambda x - \omega t) - 2\lambda^{-\frac{1}{2} - \delta - s + 2} \varphi \left( \frac{x}{\lambda^\delta} \right) \cos(\lambda x - \omega t). \quad (5.21)$$

so that

$$\|u^h(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^h(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^h(t)\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\frac{1}{2} - \delta + s - 2}. \quad (5.22)$$

For $u_\ell(t)$ we have

$$\|u_\ell(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u_\ell(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u_\ell(t)\|_{L^\infty(\mathbb{R})} \lesssim \|u_\ell(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-\frac{1}{2} - \delta}. \quad (5.23)$$

Therefore

$$\|u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 u^{\omega,\lambda}(t)\|_{L^\infty(\mathbb{R})} \lesssim \lambda^{-\rho}, \quad (5.24)$$
where
\[ \rho_s \doteq \min\{1 - \frac{1}{2} \delta, \frac{1}{2} \delta + s - 2\} > 0, \]
for any any \( s > 3/2 \) if \( \delta \) is chosen appropriately in the interval (1, 2).

Using (5.24) and the \( H^1 \)-estimate (4.30) for the error \( F \), from (5.20) we get
\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^1(\mathbb{R})}^2 \lesssim \lambda^{-\rho_s} \|v(t)\|_{H^1(\mathbb{R})}^2 + \lambda^{-r_s} \|v(t)\|_{H^1(\mathbb{R})}, \]
which gives the differential inequality
\[ \frac{d}{dt} \|v(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-\rho_s} \|v(t)\|_{H^1(\mathbb{R})} + \lambda^{-r_s}. \]
Since \( \|v(0)\|_{H^1(\mathbb{R})} = 0 \) and for \( s > 1 \) we can choose \( \delta \) such that \( \rho_s \geq 0 \) from (5.26) and Gronwall’s inequality we obtain that
\[ \|v(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T, \]
which concludes the proof of the lemma. \( \square \)

6. Non-uniform dependence in \( H^s(\mathbb{R}) \) for \( s > 3/2 \)

Next we shall prove non-uniform dependence for CH by taking advantage of the information provided by Theorem 2 and Proposition 1, and the \( H^1 \)-estimate (5.9) on the difference between approximate solutions and solutions with same initial data.

For this, let \( u_{1,\lambda}(x,t) \) and \( u_{-1,\lambda}(x,t) \) be the unique solutions to the the Cauchy problem (5.1)–(5.2) with initial data \( u^{1,\lambda}(x,0) \) and \( u^{-1,\lambda}(x,0) \) correspondingly. By Theorem 2 these solutions belong in \( C([0,T];H^s(\mathbb{R})) \). Recall, using Proposition 1 we proved estimate (5.4) which says that \( T \) is independent of \( \lambda >> 1 \). Also, for \( s > 3/2 \), using estimate (2.2), we have
\[ \|u_{\pm 1,\lambda}(t)\|_{H^s(\mathbb{R})} \lesssim \|u^{\pm 1,\lambda}(0)\|_{H^s(\mathbb{R})}, \quad 0 \leq t \leq T. \]
Furthermore, since our \( s \)-dependent initial data \( u^{\pm \lambda}(0) \) belong to every Sobolev space they do belong to \( H^{[s]+2}(\mathbb{R}) \). Since \( s > 3/2 \) by the argument in the last remark of section 2 we obtain a companion estimate to (6.1)
\[ \|u_{\pm 1,\lambda}(t)\|_{H^{[s]+2}(\mathbb{R})} \lesssim \|u^{\pm 1,\lambda}(0)\|_{H^{[s]+2}(\mathbb{R})}, \quad 0 \leq t \leq T. \]
Now let \( k = [s] + 2 \). If \( \lambda \) is large enough then from (4.2) and (4.3) we have
\[ \|u^{\pm 1,\lambda}(t)\|_{H^k(\mathbb{R})} \lesssim \|u_{\pm 1,\lambda}(t)\|_{H^k(\mathbb{R})} + \lambda^{\frac{1}{2} \delta - s} \|\varphi(\frac{x}{\lambda^2}) \cos(\lambda x - \lambda t)\|_{H^k(\mathbb{R})} \]
\[ \lesssim \lambda^{-1 + \frac{1}{2} \delta} + \lambda^{k-s} \cdot \lambda^{-\frac{1}{2} \delta - k} \|\varphi(\frac{x}{\lambda^2}) \cos(\lambda x - \lambda t)\|_{H^k(\mathbb{R})} \]
\[ \lesssim \lambda^{-1 + \frac{1}{2} \delta} + \lambda^{k-s} \|\varphi\|_{L^2(\mathbb{R})}, \]
where
\[ \|u_{\pm1,\lambda}(t)\|_{H^k(\mathbb{R})} \lesssim \lambda^{k-s}, \quad \text{hence by (6.2)} \]
\[ \|u_{\pm1,\lambda}(t)\|_{H^k(\mathbb{R})} \lesssim \lambda^{k-s}. \quad (6.3) \]

Therefore, from (6.3) we obtain the following estimate for the $H^k$-norm of the difference of $u_{\pm1,\lambda}$ and $u_{\pm1,\lambda}$
\[ \|u_{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^k(\mathbb{R})} \lesssim \lambda^{k-s}, \quad 0 \leq t \leq T. \quad (6.4) \]

Applying (5.9) with our particular choice of $\omega = \pm 1$ we have
\[ \|u_{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^1(\mathbb{R})} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T. \quad (6.5) \]

Now, applying the interpolation inequality
\[ \|\psi\|_{H^s(\mathbb{R})} \leq \|\psi\|_{H^1(\mathbb{R})}^{(s_2-s)/s_1} \|\psi\|_{H^{s_1}(\mathbb{R})}^{(s_1-s)/s_1} \]
with $s_1 = 1$ and $s_2 = \lfloor s \rfloor + 2 = k$ and using estimates (6.5) and (6.4) gives
\[ \|u_{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^s(\mathbb{R})} \leq \|u_{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^k(\mathbb{R})}^{(k-s)/(k-1)} \]
\[ \times \|u_{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^k(\mathbb{R})}^{(s-1)/(k-1)} \]
\[ \lesssim \lambda^{-r_s}[(k-s)/(k-1)]\lambda^{(k-s)/(k-1)} \]
\[ \lesssim \lambda^{-r_s}[(k-s)/(k-1)]. \quad (6.6) \]

From the last inequality we obtain that
\[ \|u_{\pm1,\lambda}(t) - u_{\pm1,\lambda}(t)\|_{H^s(\mathbb{R})} \lesssim \lambda^{-\varepsilon_s}, \quad 0 \leq t \leq T, \quad (6.7) \]
where $\varepsilon_s$ is given by
\[ \varepsilon_s = (1 - \frac{1}{2}\delta)/(s + 2). \quad (6.8) \]

Note that
\[ \varepsilon_s > 0, \quad \text{for} \quad s > 1. \quad (6.9) \]

Next, we shall use estimate (6.7) to prove non-uniform dependence when $s > 3/2$.

**Behavior at time zero.** Since $\delta < 2$, at $t = 0$ we have
\[ \|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s(\mathbb{R})} = \|2\lambda^{-1}\tilde{\phi}(\frac{x}{\lambda^s})\|_{H^s(\mathbb{R})} \]
\[ \leq 2\lambda^{-1+\frac{1}{2}\delta}\|\tilde{\phi}\|_{H^s(\mathbb{R})} \longrightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty. \quad (6.10) \]

**Behavior at time $t > 0$.** Then, we write
\[ \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \geq \|u_{1,\lambda}^{1,\lambda}(t) - u_{-1,\lambda}^{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \]
\[ - \|u_{1,\lambda}^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s(\mathbb{R})} \]
\[ - \|u_{-1,\lambda}^{-1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \quad (6.11) \]
Using estimate (6.7) for the last two terms in (6.11) we obtain
\[ \| u_1,\lambda(t) - u_{-1},\lambda(t) \|_{H^s(\mathbb{R})} \geq \| u^{1,\lambda}(t) - u^{-1,\lambda}(t) \|_{H^s(\mathbb{R})} - c\lambda^{-s}. \] (6.12)
In (6.12) letting \( \lambda \) go to \( \infty \) gives
\[ \lim_{\lambda \to \infty} \| u_1,\lambda(t) - u_{-1},\lambda(t) \|_{H^s(\mathbb{R})} \geq \lim_{\lambda \to \infty} \| u^{1,\lambda}(t) - u^{-1,\lambda}(t) \|_{H^s(\mathbb{R})}. \] (6.13)
Inequality (6.13) is a key estimate since it reduces finding a lower positive bound for the difference of the unknown solution sequences to finding a lower positive bound for the difference of the known approximate solution sequences. Using the identity
\[ \cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) \]
gives
\[ u^{1,\lambda}(t) - u^{-1,\lambda}(t) = u_{t,1,\lambda}(t) - u_{t,-1,\lambda}(t) + 2\lambda^{-\frac{1}{2}}s(\frac{x}{\lambda^s}) \sin(\lambda x) \sin t. \]
Therefore
\[ \| u^{1,\lambda}(t) - u^{-1,\lambda}(t) \|_{H^s(\mathbb{R})} \geq 2\lambda^{-\frac{1}{2}}s(\frac{x}{\lambda^s}) \sin(\lambda x) \|_{H^s(\mathbb{R})} \sin t| \]
\[ - \| u_{t,1,\lambda}(t) \|_{H^s(\mathbb{R})} - \| u_{t,-1,\lambda}(t) \|_{H^s(\mathbb{R})} \]
\[ \geq 2\lambda^{-\frac{1}{2}}s(\frac{x}{\lambda^s}) \sin(\lambda x) \|_{H^s(\mathbb{R})} \sin t| - \lambda^{-1+\frac{s}{2}}. \] (6.14)
Now letting \( \lambda \) go to \( \infty \), (6.14) gives
\[ \lim_{\lambda \to \infty} \| u^{1,\lambda}(t) - u^{-1,\lambda}(t) \|_{H^s(\mathbb{R})} \geq \| \varphi \|_{L^2(\mathbb{R})} \sin t|. \] (6.15)
Combining (6.13) and (6.15) gives
\[ \lim_{\lambda \to \infty} \| u_{1,\lambda}(t) - u_{-1,\lambda}(t) \|_{H^s(\mathbb{R})} \geq \| \varphi \|_{L^2(\mathbb{R})} \sin t|, \] (6.16)
which proves Theorem 1.

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