Asymptotic behavior and orbital stability of galactic dynamics in relativistic scalar gravity

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\textit{Dedicated to Juan Luis Vázquez on his 60th birthday}

\textbf{Abstract}

The Nordström-Vlasov system is a relativistic Lorentz invariant generalization of the Vlasov-Poisson system in the gravitational case. The asymptotic behavior of solutions and the non-linear stability of steady states are investigated. It is shown that solutions of the Nordström-Vlasov system with energy greater or equal to the mass satisfy a dispersion estimate in terms of the conformal energy. When the energy is smaller than the mass, we prove existence and non-linear (orbital) stability of a class of static solutions (isotropic polytropes) against general perturbations. The proof of orbital stability is based on a variational problem associated to the minimization of the energy functional under suitable constraints.

\section{1 Introduction}

A classical problem in theoretical astrophysics is to establish the non-linear stability of galaxies in equilibrium. Neglecting relativistic effects and collisions among the stars of the galaxy, these equilibrium states can be described as...
stationary solutions of the Vlasov-Poisson system:
\[
\begin{align*}
\partial_t f + p \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_p f &= 0, \\
\Delta_x \phi &= 4\pi \rho, \\
\lim_{|x| \to \infty} \phi(t,x) &= 0, \\
\rho(t,x) &= \int_{\mathbb{R}^3} f(t,x,p) \, dp.
\end{align*}
\]

Here, \( f(t,x,p) \geq 0 \) denotes the distribution function in phase space of the stars, which are assumed to have all the same mass, with \( t \in \mathbb{R}, \, x \in \mathbb{R}^3, \, p \in \mathbb{R}^3 \) denoting time, position and momentum, respectively; \( \phi(t,x) \) stands for the mean gravitational field generated by the stars altogether. In our units, the gravitational constant and the mass of each star equal one.

A general method to approach the stability problem for an infinite dimensional dynamical system is to construct stationary solutions as minimizers of a suitable functional which is preserved by the evolution. The specific choice of the functional to minimize and of the constraints in the variational problem selects the type of steady states to be constructed, as well as the notion of distance which appears in the non-linear stability theorem. This approach was successfully applied to establish non-linear stability for a large class of steady states to the Vlasov-Poisson system, see [8, 11, 13, 14, 20, 21, 22, 24, 26]. An important pre-requisite for this method to yield a rigorous stability theorem is that sufficiently regular solutions of the dynamical system should exist for general initial data of the Cauchy problem (or at least for data close to the steady state). In the case of Vlasov-Poisson, the existence of global classical solutions for general initial data has been known for some time, see [17, 18, 22, 23].

In this paper we study the non-linear stability for a class of stationary solutions as well as the asymptotic behaviour of general solutions to the Nordström-Vlasov system [4, 6]. The latter provides a genuine relativistic generalization of the Vlasov-Poisson system in the following sense: It is invariant under Lorentz transformations and its solutions converge to solutions of Vlasov-Poisson as the speed of light tends to infinity. In units such that the speed of light equals one, the Nordström-Vlasov system is given by
\[
\begin{align*}
\partial^2_t \phi - \Delta_x \phi &= -\int_{\mathbb{R}^3} f(t,x,p) \frac{dp}{\sqrt{1 + |p|^2}}, \\
S f - \left[(S\phi) p + (1 + |p|^2)^{-1/2} \nabla_x \phi\right] \cdot \nabla_p f &= 4 f S \phi,
\end{align*}
\]

where
\[
\hat{p} = \frac{p}{\sqrt{1 + |p|^2}}, \quad S = \partial_t + \hat{p} \cdot \nabla_x
\]
are the relativistic velocity and the relativistic free transport operator. As compared to the Vlasov-Poisson case, the stability analysis for the Nordström-Vlasov system presents new difficulties related to the strongly nonlinear and hyperbolic character of the field equations. These new features make the problem under study relevant from a mathematical point of view, besides its original astrophysical motivation.
A fundamental role in our analysis is played by the conserved quantities for the system (1)-(2). Let us mention here the energy functional, or Hamiltonian,

\[ H(f, \phi, \partial_t \phi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f \, dp \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t \phi)^2 \, dx \]

and the Casimir functional

\[ C_Q(f, \phi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{3\phi} Q(f e^{-4\phi}) \, dp \, dx, \]

where \( Q : \mathbb{R} \to \mathbb{R} \) is any sufficiently regular function. In particular, for \( Q(z) = z^q, \ q \geq 1 \), we infer that \( \|e^{(3/q-4)\phi} f\|_{L^q} \) is constant, the case \( q = 1 \) being the conservation law of mass. For the purpose of the present investigation, it is convenient to introduce the new dynamical variable

\[ \tilde{f}(t, x, p) = e^{-4\phi} f(t, x, e^{-\phi} p), \]

in terms of which the Nordström-Vlasov system takes the form

\begin{align*}
\partial_t \tilde{f} + \frac{p}{\sqrt{e^{2\phi} + |p|^2}} \cdot \nabla_x \tilde{f} - \nabla_x \left( \sqrt{e^{2\phi} + |p|^2} \right) \cdot \nabla_p \tilde{f} &= 0, \\
\partial_t^2 \phi - \Delta_x \phi &= -e^{2\phi} \int_{\mathbb{R}^3} \tilde{f}(t, x, p) \frac{dp}{\sqrt{e^{2\phi} + |p|^2}}.
\end{align*}

(3)

(4)

The energy functional becomes

\[ H(\tilde{f}, \phi, \partial_t \phi) = E_{\text{kin}}(\tilde{f}, \phi) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t \phi)^2 \, dx, \]

(5)

where

\[ E_{\text{kin}}(\tilde{f}, \phi) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{2\phi} + |p|^2} \tilde{f} \, dp \, dx, \]

while the Casimir functional reads

\[ C_Q(\tilde{f}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Q(\tilde{f}) \, dp \, dx. \]

In particular \( \|\tilde{f}(t)\|_{L^q} \) is constant, for all \( q \in [1, \infty) \). The main reason to adopt this new formulation of the Nordström-Vlasov system is that the field \( \phi \) does not appear explicitly in the definition of the Casimir functional. Let us also notice that the particles distribution \( \tilde{f} \) and the gravitational potential \( \phi \) in the energy functional of the Nordström-Vlasov system are independent variables, whereas for Vlasov-Poisson they are related by the Poisson formula \( \phi = -\rho * \frac{1}{|x|} \). In the following we shall denote \( \tilde{f} \) simply by \( f \) to lighten the notation.

One of the objectives of this paper is to underline the differences between the Vlasov-Poisson system and the Nordström-Vlasov system related to the relativistic character of the latter model. We will then focus our attention on
the stability analysis to a simple class of steady states, namely the isotropic polytropes, which in the case of the Nordström-Vlasov system are defined as

\[ f_0(x, p) = \left(\frac{E_0 - E}{c}\right)^k_+, \quad E = \sqrt{e^{2\phi_0} + |p|^2}, \]

(6)

Here \( k > -1, c > 0 \) and \( E_0 > 0 \) are constants, \( E \) is the particles energy, \((\cdot)_+\) denotes the positive part. The existence of a maximum \( E_0 \) for the particles energy is necessary for the steady state to have finite energy [4]. Moreover, \( \phi_0 = \phi_0(x) \) is the gravitational potential induced by the distribution \( f_0 \) and is a solution of the non-linear Poisson equation

\[ \triangle \phi_0 = e^{2\phi_0} \int_{\mathbb{R}^3} \frac{f_0}{\sqrt{e^{2\phi_0} + |p|^2}} dp. \]

(7)

Our proof of stability for solutions of the form (6)-(7) is grounded on a variational argument similar to the one introduced in [24] to study the orbital stability of polytropic spheres for the Vlasov-Poisson system and which consists in minimizing the energy functional subject to the constraints of given mass and \( L^q \) norm of \( f \), for some \( q > 1 \).

This paper is organized as follows. We begin in Section 2 by proving that solutions of the Nordström-Vlasov system with energy greater or equal to the mass satisfy a dispersion estimate in terms of the conformal energy. This estimate does not imply that steady states solutions must have energy smaller than the mass, because for these solutions the conformal energy is unbounded. In Section 3 we state our main results on the stability of isotropic polytropes for the Nordström-Vlasov system and reduce the problem to that of minimizing the energy functional under suitable constraints. Some preliminary estimates necessary to solve the latter problem are given in Section 4. In particular it is shown that the energy of minimizers is bounded above by their mass, thus a connection with the result of Section 2 is established. In Section 5 we prove the existence of minimizers to the energy functional and show that they arise as the strong limit of suitable minimizing sequences. Our proof requires the minimizers to have energy strictly less than the mass, a property which is shown to be verified if the mass is sufficiently large. Finally, in Section 6 we establish uniqueness and spherical symmetry of the minimizer (up to a translation in space) and show that it is an isotropic polytrope solution of the Nordström-Vlasov system with finite radius.

To conclude this introduction we remark that relativistic theories of gravity, although not physically correct, are often used as simplified models for General Relativity [25]. Moreover, scalar fields play a central role in modern theories of classical and quantum gravity [7]. The physically correct relativistic model for self-gravitating collisionless matter is the Einstein-Vlasov system, which is discussed for instance in [11]. Existence and finite radius property of steady states to the Einstein-Vlasov system have been studied in [19], see also [10], but the question of their stability is currently open (see however [27]). We hope that the present work on the stability of steady states to the Nordström-Vlasov system may contribute to a better understanding of this important problem.
2 A dispersion estimate

The aim of this section is to prove a dispersion estimate for solutions of the Nordström-Vlasov system. Although this estimate will not be used in the following sections, it reveals an interesting link with the assumptions in our stability theorem, see Section 3. Define the local energy of a solution \((f, \phi)\) of \((3)-(4)\) as

\[
e(t, x) = \int_{\mathbb{R}^3} \sqrt{e^{2\phi} + |p|^2} f \, dp + \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\nabla_x \phi)^2,
\]

the local momentum

\[
q_i(t, x) = \int_{\mathbb{R}^3} p_i f \, dp - \partial_i \phi \partial_i \phi,
\]

and the local stress tensor

\[
\tau_{ij} = \int_{\mathbb{R}^3} \frac{p_i p_j}{\sqrt{e^{2\phi} + |p|^2}} f \, dp + \partial_i \phi \partial_j \phi + \frac{1}{2} \delta_{ij} \left[ (\partial_i \phi)^2 - (\nabla_x \phi)^2 \right],
\]

where \(\partial_i = \partial_{x_i}\). These quantities are related by the conservation laws

\[
\partial_t e + \nabla_x \cdot q = 0, \quad \partial_t q_i + \partial_j \tau_{ij} = 0, \quad (8)
\]

the sum over repeated indexes being understood. Upon integration, the previous identities lead to the conservation of the total energy and of the total momentum:

\[
H(t) = \int_{\mathbb{R}^3} e \, dx = \text{constant}, \quad Q(t) = \int_{\mathbb{R}^3} q \, dx = \text{constant}.
\]

Moreover, solutions of \((3)-(4)\) satisfy the conservation of the total rest mass:

\[
M = \int_{\mathbb{R}^3} f \, dp \, dx = \text{constant},
\]

which is obtained by integrating the local rest mass conservation law

\[
\partial_t \rho + \nabla_x \cdot j = 0, \quad \rho = \int_{\mathbb{R}^3} f \, dp, \quad j = e^\phi \int_{\mathbb{R}^3} \frac{p}{\sqrt{e^{2\phi} + |p|^2}} f \, dp.
\]

We define the conformal energy as

\[
\mathcal{E}_C(t) = \int_{\mathbb{R}^3} |x|^2 e(t, x) \, dx.
\]

The Nordström-Vlasov system is supplied with a set of initial data \((f_0, \phi_0, \phi_1)\), where

\[
f_0(x, p) = f(0, x, p), \quad \phi_0(x) = \phi(0, x), \quad \phi_1(x) = \partial_t \phi(0, x).
\]
In \cite{5} it is proved that \( f_0 \in C^1_c, \phi_0 \in C^3_b \cap H^1, \phi_1 \in C^2_b \cap L^2 \) launch a unique global classical solution of (3)-(4). This solution preserves energy and mass and has finite conformal energy for all times (provided it is bounded at time zero, e.g., by giving compactly supported initial data for the field). We prove the following

**Theorem 1** There exists a constant \( t_0 > 0 \), depending only on bounds on the initial data, such that the following holds: If the initial data satisfy \( H > M \) then

\[
E_C(t) \geq (H - M)t^2,
\]

for all \( t > t_0 \); if \( H = M \), then

\[
E_C(t) \geq 2Q_0 t,
\]

again for \( t > t_0 \), provided \( Q_0 > 0 \), where

\[
Q_0 = \int_{\mathbb{R}^3} (x \cdot q(0, x) - \phi_0 \phi_1) \, dx.
\]

**Proof:** By the first of (8) we have

\[
\frac{d}{dt} E_C = 2 \int_{\mathbb{R}^3} x \cdot q \, dx,
\]

whence, using the second equation in (8),

\[
\frac{d^2}{dt^2} E_C = 2 \int_{\mathbb{R}^3} \text{Tr}(\tau_{ij}) \, dx.
\]

Here \( \text{Tr}(\tau_{ij}) \) denotes the trace of the tensor \( \tau_{ij} \), which is given by

\[
\text{Tr}(\tau_{ij}) = \int_{\mathbb{R}^3} \frac{|p|^2}{\sqrt{e^{2\phi} + |p|^2}} f dp - \frac{1}{2}(\nabla_x \phi)^2 + \frac{3}{2}(\partial_t \phi)^2.
\]

It follows that one can rewrite (10) as

\[
\frac{d^2}{dt^2} E_C = 2H + 2Q(\partial_t \phi, \nabla_x \phi) - 2 \int_{\mathbb{R}^3} \mu(t, x) \, dx,
\]

where \( \mu \) is minus the right hand side of (4) and \( Q \) is the quadratic operator

\[
Q(\partial_t \phi, \nabla_x \phi) = \int_{\mathbb{R}^3} ((\partial_t \phi)^2 - (\nabla_x \phi)^2) \, dx.
\]

By means of the identity \((\partial_t \phi)^2 = \partial_t (\phi \partial_t \phi) - \phi \partial_t^2 \phi \) and using (4) we have

\[
\int_0^t Q(\partial_t \phi, \nabla_x \phi) \, ds = \int_0^t \int_{\mathbb{R}^3} \mu \phi \, dx \, ds + \frac{1}{2} \partial_t \int_{\mathbb{R}^3} \phi^2 \, dx \, ds - \int_{\mathbb{R}^3} \phi_0 \phi_1 \, dx.
\]
From (9), (11) and (12) we obtain

\[ E_C(t) = E_C(0) - \int_{\mathbb{R}^3} \phi_0^2 \, dx + \int_{\mathbb{R}^3} \phi^2 \, dx + 2Q_0 t + H t^2 \\
+ 2 \int_0^t \int_0^s \int_{\mathbb{R}^3} \mu (\phi - 1) \, dx \, d\tau \, ds. \]  

(13)

Using the simple lower bound \( \xi - 1 \geq -e^{-\xi} \), which holds for all \( \xi \in \mathbb{R} \), the last term in (13) is bounded from below by

\[ -2 \int_0^t \int_0^s \int_{\mathbb{R}^3} \frac{f e^\phi}{e^{2\phi} + |p|^2} \, dp \, dx \, d\tau \, ds \geq -Mt^2. \]

Substituting into (13) we finally obtain

\[ E_C(t) \geq E_C(0) - \|\phi_0\|_{L^2}^2 + 2Q_0 t + (H - M)t^2, \]

which yields the claim.

\[ \square \]

Remark 1 The estimate of Theorem 1 does not apply to steady states solutions. To see this, consider \( f_0 \) with compact support and \( \phi_0 \) a solution of (7). The fastest decay at infinity for \( \nabla \phi_0 \) is \( O(|x|^{-2}) \), which is too weak to bound the integral \( \int_{\mathbb{R}^3} |x|^2 |\nabla \phi_0|^2 \, dx \) in the conformal energy. Nevertheless the bound \( H < M \) will also appear as a crucial ingredient in our proof of stability for isotropic polytropes.

3 Orbital stability

In this section we state and comment our main results on the stability of isotropic polytropes of the Nordström-Vlasov system. In order to be precise with the formulation of our problem, we will start by introducing some notation. For \( J, M, k \) positive real numbers, we denote by \( \Gamma_{M,J}^k \) the space of functions \( f : \mathbb{R}^6 \to \mathbb{R} \) given by

\[ \Gamma_{M,J}^k = \{ f \in L^1 \cap L^{1+1/k}, \ f \geq 0 \text{ a.e.}, \ \|f\|_{L^1} = M, \ \|f\|_{L^{1+1/k}} \leq J \}. \]

Moreover we denote

\[ D^1(\mathbb{R}^3) = \{ \phi \in L^1_{\text{loc}}(\mathbb{R}^3) : \nabla \phi \in L^2 \text{ and } \phi \text{ vanishes at infinity} \}, \]

where the condition of \( \phi \) vanishing at infinity means that the set \( \{ x \in \mathbb{R}^3 : |\phi(x)| > a \} \) has finite (Lebesgue) measure, for all \( a > 0 \). Functions in the space \( D^1(\mathbb{R}^3) \) satisfy the Sobolev inequality

\[ \|\phi\|_{L^6} \leq \eta \|\nabla \phi\|_{L^2}, \quad \eta = \frac{2}{\sqrt{3}} \pi^{-2/3}, \]  

(14)

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see [15] Thm. 8.3]. The space $D^1(\mathbb{R}^3)$ has been extensively used in fluid mechanics, see for example [10].

Our first result is the existence of a minimizer to the variational problem

$$\inf \{ H(f, \phi, \psi), \ f \in \Gamma_{M,J}^k, \ \phi \in D^1, \ \psi \in L^2, \ E_{\text{kin}}(f, 0) < \infty \}, \ k \in (0, 2),$$

provided the mass $M$ is sufficiently large (depending on $J$ and $k$), where $H$ is the energy functional defined in [5]. Obviously, the above variational problem is equivalent to the following one:

$$I_{M,J}^k = \inf \{ E(f, \phi), \ f \in \Gamma_{M,J}^k, \ \phi \in D^1, \ E_{\text{kin}}(f, 0) < \infty \},$$

where

$$E(f, \phi) = H(f, \phi, 0) = E_{\text{kin}}(f, \phi) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx. \quad (15)$$

By abuse of terminology, we shall continue to refer to $E$ as the energy functional.

**Theorem 2** For all $0 < k < 2$ and $J > 0$, there exists $M_0 \in [0, \infty)$ such that $I_{M,J}^k < M$, for all $M > M_0$. Moreover, for all $M > M_0$ and any minimizing sequence $(f_n, \phi_n) \subset \Gamma_{M,J}^k \times D^1$ of the functional $E$, there exists a subsequence, still denoted $(f_n, \phi_n)$, a sequence of spatial translations $T_n f_n(x, p) = f_n(x + y_n, p)$, $T_n \phi_n(x) = \phi_n(x + y_n)$, with $y_n \in \mathbb{R}^3$, and a minimizer $(f_0, \phi_0) \in \Gamma_{M,J}^k \times D^1$ such that

$$\|T_n f_n - f_0\|_{L^1} \to 0, \ |T_n f_n - f_0|_{L^{1+1/k}} \to 0, \ |\nabla T_n \phi_n - \nabla \phi_0|_{L^2} \to 0,$$

as $n \to \infty$. Moreover, $(f_0, \phi_0)$ satisfies (7) in the sense of distributions.

Let us comment some aspects concerning the statements of Theorem 2.

i) As in the case of Vlasov-Poisson, the use of translations in the space variable is necessary, otherwise starting from a minimizer $(f_0, \phi_0)$ and a sequence of shift vectors $y_n \in \mathbb{R}^3$ such that $\lim_{n \to \infty} |y_n| = \infty$, the sequence $(T_n f_0, T_n \phi_0)$—which is still minimizing—converges weakly to zero, which is not in $\Gamma_{M,J}^k \times D^1$.

ii) With regard to the large mass condition, we notice first that if $M_0 = 0$, then our results apply to any $M > 0$. If $M_0 > 0$, then $I_{M,J}^k = M$, for all $0 < M < M_0$ and in this case it is possible to construct a minimizing sequence of the functional $E$ with vanishing weak limit, see Remark 2 in Section II.

iii) In view of Theorem 1 and the previous remark, it is reasonable to conjecture that the condition $I_{M,J}^k < M$ in Theorem 2 is optimal.

As an application of the Euler-Lagrange multipliers method we show that minimizers are isotropic polytropes solutions to the Nordström-Vlasov system.
Theorem 3 Let $0 < k < 2$, $J > 0$, $M > M_0$ and $(f_0, \phi_0)$ be a minimizer of $T_{f_0, \phi_0}$ over the space $\Gamma_{k, M, J}^k \times D^1$. Then $f_0$ has the form (6), with
\[
E_0 \in \left( \frac{k + 4 I_{M, J}^k}{6 M}, \frac{I_{M, J}^k}{M} \right), \quad c > 0
\]
given by
\[
E_0 = \frac{1}{M} \left( I_{M, J}^k - \frac{2 - k}{6} \int_{\mathbb{R}^3} |\nabla \phi_0|^2 \, dx \right), \quad c = \frac{k + 1}{2 - k} \left( \frac{I_{M, J}^k - E_0 M}{j^{1+1/k}} \right).
\]
Moreover different minimizers differ only by a spatial translation and for any representative in this class the following holds: $\phi_0(x) \to 0$, as $|x| \to \infty$, $f_0(x, p)$ is compactly supported, $(f_0, \phi_0)$ is spherically symmetric with respect to some point in $\mathbb{R}^3$ and is a time-independent mild solution of the Nordström-Vlasov system.

Uniqueness of minimizers in Theorem 3 (up to translations in space) is proved by showing that the Lagrange multiplier $E_0$ is the same for all minimizers. In the case of Vlasov-Poisson, this follows by the scaling properties of the Emden-Fowler equation, which is the ordinary differential equation (ODE) satisfied by the (spherically symmetric) gravitational potential of isotropic polytropes, see [13]. Here, this argument does not apply due to the strongly nonlinear and nonlocal character of the ODE which is the equivalent counterpart of the Emden-Fowler equation in the Nordström-Vlasov case. In fact, our strategy to prove uniqueness of minimizers for the Nordström-Vlasov system must combine analytical and numerical tools because some integrals involved in the associated ODE cannot be explicitly calculated. Our numerical/analytical computations reveal a monotonicity property of the set of minimizers with respect to the mass, which yields uniqueness by the mass constraint.

To conclude this section, we show how the previous results yield the orbital stability of the minimizer solution with respect to perturbed time-dependent solutions of the Nordström-Vlasov system. We adopt the expression orbital stability to invoke any criterium for which the orbit, described by $\{(T_y f_0, T_y \phi_0) \mid y \in \mathbb{R}^3\}$, of a stationary solution $(f_0, \phi_0)$ is the set of functions which remains close to a perturbed Nordström-Vlasov solution. This concept has been widely used in the literature, see [21] and the references therein.

Theorem 4 Let $(f_0, \phi_0)$ be the minimizer associated to $0 < k < 2$, $J > 0$ and $M > M_0$. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that, for all initial data $(f^\text{in}, \phi_0^\text{in}, \phi_1^\text{in}) = (f, \phi, \partial_t \phi)|_{t=0}$ of the Nordström-Vlasov system in the class
\[
0 \leq f^\text{in} \in \Gamma_{M, J}^k \cap C^1, \quad \phi_0^\text{in} \in C^3 \cap D^1, \quad \phi_1^\text{in} \in C^2 \cap L^2
\]
and
\[
|H(f^\text{in}, \phi_0^\text{in}, \phi_1^\text{in}) - H(f_0, \phi_0, 0)| \leq \delta,
\]
the associated solution \((f, \phi) \in C^1 \times C^2\) of \(\ref{3} - \ref{4}\) satisfies, for all \(t > 0\),

\[
\inf_{y \in \mathbb{R}^3} \| f - T_y f_0 \|_{L^1} + \inf_{y \in \mathbb{R}^3} \| f - T_y f_0 \|_{L^{1+1/k}} \leq \varepsilon,
\]

\[
\inf_{y \in \mathbb{R}^3} \| \nabla \phi - T_y \nabla \phi_0 \|_{L^2} + \| \partial_t \phi \|_{L^2} \leq \varepsilon.
\]

**Proof:** If the thesis of Theorem 4 were false, there would exist \(\varepsilon_0 > 0\), a sequence \((f_n^{in}, \phi_{0,n}^{in}, \phi_{1,n}^{in})\) and \(t_n > 0\) such that

\[
0 \leq f_n^{in} \in \Gamma_{M,J}^k \cap C^1_c, \quad \phi_{0,n}^{in} \in C^3 \cap D^1, \quad \phi_{1,n}^{in} \in C^2 \cap L^2,
\]

\[
| H(f_n^{in}, \phi_{0,n}^{in}, \phi_{1,n}^{in}) - H(f_0, \phi_0, 0) | < \frac{1}{n}
\]

and each of \(\| f_n(t_n) - T_y f_0 \|_{L^1}, \quad \| f_n(t_n) - T_y f_0 \|_{L^{1+1/k}}, \quad \| \nabla \phi_n(t_n) - T_y \nabla \phi_0 \|_{L^2}\) and \(\| \partial_t \phi_n(t_n) \|_{L^2}\) is greater than \(\varepsilon_0\) for all \(y \in \mathbb{R}^3\) and \(n \in \mathbb{N}\). Here \((f_n, \phi_n)\) is the solution of \(\ref{3} - \ref{4}\) associated to the initial data set \((f_n^{in}, \phi_{0,n}^{in}, \phi_{1,n}^{in})\). On the other hand, by conservation of energy and \(L^2\) norm, \((f_n(t_n), \phi_n(t_n))\) is a minimizing sequence. Hence the above conclusion contradicts the thesis of Theorem 2 \(\square\)

### 4 Bounds on the energy infimum

Throughout the paper we denote by \(C\) any positive constant that depends only on \(M, J\) and \(k\). Moreover, any subsequence of a minimizing sequence \((f_n, \phi_n)\) is still denoted \((f_n, \phi_n)\). We start by collecting some important estimates on the functions

\[
\rho_f = \int_{\mathbb{R}^3} f \, dp, \quad \mu_f = \int_{\mathbb{R}^3} f \, \frac{dp}{|p|}
\]

induced by an element \(f \in \Gamma_{M,J}^k\).

**Lemma 1** For \(0 < k < 2\), let \(\omega = 3 + k\), so that \(1 + 1/\omega \in (6/5, 4/3)\), and

\[
j = \frac{3}{2} + \frac{5 - \omega}{2(\omega - 1)}.
\]

Then for any \(f \in \Gamma_{M,J}^k\), there exists a constant \(C > 0\) such that

\[
\int_{\mathbb{R}^3} \rho_f^{1+1/\omega} \, dx \leq C \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{1+1/k} \, dp \, dx \right)^{k/\omega} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| \, f \, dp \, dx \right)^{3/\omega},
\]

\[
\int_{\mathbb{R}^3} \mu_f^2 \, dx \leq C \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{1+1/k} \, dp \, dx \right)^{2k} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| \, f \, dp \, dx \right)^{5-\omega},
\]

\[
\| \rho_f \|_{L^{6/5}} \leq C \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{1+1/k} \, dp \, dx \right)^{k/6} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| \, f \, dp \, dx \right)^{1/2}.
\]
Choosing a Proof: For all \(2\) interpolation with the finite mass constraint.

\[
\int_{\mathbb{R}^3} \rho_f^r \, dx = \int_{\mathbb{R}^3} \left( \int_{|p| \leq R} f \, dp + \int_{|p| > R} f \, dp \right)^r \, dx \\
\leq C \int_{\mathbb{R}^3} \left[ \left( \int_{\mathbb{R}^3} f^q \, dp \right)^{1/q} R^{3(q-1)/q} + \frac{1}{R} \int_{\mathbb{R}^3} |p| f \, dp \right]^r \, dx.
\]

Optimizing in \(R\) and applying Hölder’s inequality we obtain, for all \(\alpha > 1\),

\[
\int_{\mathbb{R}^3} \rho_f^r \, dx \leq C \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} f^q \, dp \right)^{\frac{r}{q}} \, dx \right)^{\frac{1}{\alpha}} \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |p| f \, dp \right)^{\frac{3r\alpha(1-q)}{(q-1)(3q-1)}} \, dx \right)^{\frac{\alpha-1}{\alpha}}.
\]

Choosing \(q = 1 + 1/k\), \(\alpha = \omega/k\) and \(r = 1 + 1/\omega\) yields the first estimate on \(\rho_f\) and the one on \(\mu_f\) is proved likewise. The bound on \(\|\rho_f\|_{L^{6/5}}\) follows by interpolation with the finite mass constraint.

We apply the preceding estimates to show that the energy infimum is strictly positive.

**Lemma 2** For all \(0 < k < 2\), \(J > 0\) and \(M > 0\), \(I_{M,J}^k > 0\).

**Proof:** Let \((f_n, \phi_n) \subset \Gamma_{M,J}^k \times D^1\) be a minimizing sequence. As \(e^x \geq 1 + x\), for all \(\xi \in \mathbb{R}\), we have the lower bound

\[
E_{\text{kin}}(f_n, \phi_n) \geq a \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{\phi_n} f_n \, dp \, dx + (1-a) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| f_n \, dp \, dx \\
\geq aM + a \int_{\mathbb{R}^3} \phi_n \int_{\mathbb{R}^3} f_n \, dp \, dx + (1-a) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| f_n \, dp \, dx,
\]

for all \(a \in [0,1]\). Using Hölder’s inequality, (14) and Lemma 1, we find

\[
E_{\text{kin}}(f_n, \phi_n) \geq aM - a\|\phi_n\|_{L^6} \|\rho_n\|_{L^{6/5}} + C(1-a)\|\rho_n\|_{L^{6/5}}^2 \\
\geq aM - a\eta \|\nabla \phi_n\|_{L^2} \|\rho_n\|_{L^{6/5}} + C(1-a)\|\rho_n\|_{L^{6/5}}^2,
\]

where the constant \(C\) is independent from \(n\). Letting \(X_n = \|\nabla \phi_n\|_{L^2}\) and \(Y_n = \|\rho_n\|_{L^{6/5}}\) we obtain

\[
E(f_n, \phi_n) \geq aM - a\eta X_n Y_n + C(1-a)Y_n^2 + \frac{1}{2} X_n^2 \\
\geq aM + \frac{1}{2}(1-a\eta)X_n^2 + \left[ C(1-a) - \frac{1}{2}a\eta \right] Y_n^2.
\]

Choosing \(a\) sufficiently small, precisely

\[
a \leq \min \left( \frac{1}{\eta}, \frac{C}{\frac{1}{2}\eta} \right),
\]

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we obtain $E(\tilde{f}_n, \tilde{\phi}_n) \geq aM > 0$. Note that $a < 1$. Letting $n \to \infty$ concludes the proof.

The proof of Theorem 2 requires precise bounds on $I^k_{M,J}$. We start with a simple scaling argument which relates the energy infimum for different masses.

**Lemma 3** For all $0 < k < 2$, $J > 0$ and $0 < M_1 \leq M_2$,

$$ I^k_{M_2,J} \leq \frac{M_2}{M_1} I^k_{M_1,J}. $$

**Proof:** Let $(f_n, \phi_n) \subset \Gamma^k_{M_1,J} \times D^1$ be a minimizing sequence and define

$$ \tilde{f}_n(x,p) = \alpha f_n(\beta x,p) \quad \tilde{\phi}_n(x) = \phi_n(\beta x), $$

where $\beta = M_1/M_2$ and $\alpha = \beta^2$. By direct computation,

$$ \|\tilde{f}_n\|_{L^1} = M_2, \quad \|\tilde{f}_n\|_{L^{1+1/k}} = \left( \frac{M_1}{M_2} \right)^{\frac{2}{1+k}} \|f_n\|_{L^{1+1/k}} \leq J, $$

so that $(\tilde{f}_n, \tilde{\phi}_n) \subset \Gamma^k_{M_2,J} \times D^1$. Moreover

$$ E(\tilde{f}_n, \tilde{\phi}_n) = \frac{M_2}{M_1} E(f_n, \phi_n), $$

whence

$$ I^k_{M_2,J} \leq \lim_{n \to \infty} E(\tilde{f}_n, \tilde{\phi}_n) = \frac{M_2}{M_1} I^k_{M_1,J}, $$

which concludes the proof.

**Proposition 1** The inequality $I^k_{M,J} \leq M$ holds, for all $0 < k < 2$, $J > 0$ and $M > 0$. Moreover, the strict inequality

$$ I^k_{M,J} < M $$

holds if the mass $M$ is sufficiently large.

**Proof:** Let $q = 1 + 1/k > 3/2$ and consider $f_\gamma \in \Gamma^k_{M,J}$ given by

$$ f_\gamma(x,p) = \left( \frac{J^q}{M} \right)^{\frac{1}{q+1}} \chi_{\{|x| \leq \beta\}}(x) \chi_{\{|p| \leq \gamma\}}(p), $$

where $\chi_A$ denotes the characteristic function of the set $A$ and

$$ \beta = \frac{1}{\gamma} \left[ \left( \frac{M}{J} \right)^{\frac{2}{q+1}} \left( \frac{3}{4\pi} \right)^{2} \right]^{1/3}. $$

Define $\phi_\alpha \in D^1$ as

$$ \phi_\alpha(x) = -\alpha \psi \left( \frac{x}{\beta} \right). $$
where \( \psi \in C_c^\infty (\mathbb{R}^3) \), \( 0 \leq \psi \leq 1 \), \( \psi(y) \equiv 1 \), for \( |y| \leq 1 \), \( \psi(y) \equiv 0 \), for \( |y| \geq 2 \). We estimate the energy of \((f_\gamma, \phi_\alpha)\) as

\[
E(f_\gamma, \phi_\alpha) \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{\phi_\alpha} f_\gamma \, dp \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| f_\gamma \, dp \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_\alpha|^2 \, dx \leq e^{-\alpha} M + \frac{3}{4} M \gamma + \alpha^2 \beta K,
\]

where, by (14),

\[
K = \int_{\mathbb{R}^3} |\nabla \psi|^2 \, dx \geq \eta^{-2} \|\psi\|^2_2 \geq \left( \frac{3}{4} \right)^{2/3} \pi^{5/3}.
\]

In particular, for \( \alpha = 0 \),

\[
I_{k,M,J}^k \leq \liminf_{\gamma \to 0} E(f_\gamma, 0) \leq M,
\]

which proves the first claim of the proposition. To prove the strict inequality (17) for large masses, we optimize the estimate (19) by choosing \( \gamma = \gamma(\alpha) \) as

\[
\gamma = \sqrt{\frac{4K}{3M}} \left[ \left( \frac{M}{J} \right)^{\frac{1}{q-1}} \left( \frac{3}{4\pi} \right)^2 \right]^{1/6} \alpha.
\]

So doing we obtain the inequality

\[
E(f_\alpha, \phi_\alpha) \leq M \left[ e^{-\alpha} + A^{-1} \alpha \right],
\]

where \( f_\alpha = f_{\gamma(\alpha)} \) and

\[
A = \frac{1}{\sqrt{K}} \left( \frac{4}{3} \right)^{5/6} \left( \frac{\pi}{8} \right)^{1/3} J^{-\frac{q}{2q-3}} M^{\frac{2q-3}{2q-6}}.
\]

We choose \( M \) so large so that \( A^{-1} < 1 \). Precisely,

\[
M > \left[ \left( \frac{3}{4} \right)^{5(q-1)} \left( \frac{8}{\pi} \right)^{2(q-1)} K^{3(q-1)} J^{-q} \right]^{1/(2q-3)}.
\]

Since \( A^{-1} < 1 \), we can optimize (20) by choosing \( \alpha = \log A \). The inequality (20) becomes

\[
E(f_\alpha, \phi_\alpha) \leq A^{-1} (1 + \log A) M < M,
\]

which concludes the proof of (17).

\[\square\]

**Remark 2** The constant \( M_0 \) in the statement of Theorem 2 is defined as

\[
M_0 = \inf \{ M' > 0 : I_{k,M,J}^k < M, \text{ for all } M > M' \}.
\]

By Proposition 1, \( M_0 < \infty \). We were not able to show that \( M_0 = 0 \), i.e., that the strict inequality (17) is always satisfied. If \( M_0 > 0 \), then it follows by Lemma 3 that \( I_{k,M,J}^k = M \), for all \( 0 < M < M_0 \). In this case, the sequence \((f_\gamma, 0)\), with \( f_\gamma \) given by (18), is minimizing. Since this sequence converges weakly to zero, then Theorem 2 is false when \( I_{k,M,J}^k = M \).

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Further information on $I_{M,J}$ can be obtained by assuming that the infimum is achieved. For any function $g$, we denote by $g^*$ the non-decreasing symmetric rearrangement of $g$ with respect to the $x$ variable (see [13]).

**Lemma 4** Let $(f_0, \phi_0)$ be a minimizer of the functional $E$ over the space $\Gamma_{M,J}^k \times D^1$. Then

(i) $\|f_0\|_{L^{1+k}} = J$ and $\phi_0 \leq 0$ almost everywhere;
(ii) $(\phi_0, f_0)$ satisfy (7) in the sense of distributions;
(iii) The following identity is satisfied:

\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_0|^2 dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|p|^2}{\sqrt{e^{2\phi_0} + |p|^2}} f_0 \, dp \, dx,
\]

which is the relativistic analogue of the Virial Theorem;
(iv) $(f_0^*, -\phi_0^*)$ is also a minimizer.

**Proof:** Let $q = 1 + 1/k > 3/2$. Assume $\|f_0\|_{L^q} = K < J$ and let $\bar{K} \in (K, J)$. Define $\tilde{f}(x, p) = \alpha^3 f_0(\alpha x, p)$, $\tilde{\phi}(x) = \phi_0(\alpha x)$, where $\alpha = (\bar{K}/K)^{q/(3q-3)}$. In this way, $\|\tilde{f}\|_{L^1} = M$, $\|\tilde{f}\|_{L^q} = \bar{K}$ and

\[
E(\tilde{f}, \tilde{\phi}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_0} + |p|^2} f_0 \, dp \, dx + \frac{1}{2} \left( \frac{\bar{K}}{K} \right)^{\frac{q}{3q-3}} \int_{\mathbb{R}^3} |\nabla \phi_0|^2 dx < E(f_0, \phi_0),
\]

which is a contradiction to $(f_0, \phi_0)$ being a minimizer. If $\phi_0 > 0$ in a set of non-zero measure, then the pair $(f_0, -|\phi_0|) \in \Gamma_{M,J}^k \times D^1$ would have energy strictly less than $E(f_0, \phi_0)$, which is again a contradiction. For the proof of (ii), let $\zeta \in C^\infty_c(\mathbb{R}^3)$. As $\phi_0 + t\zeta \in D^1$, for all $t \in \mathbb{R}$, it has to be

\[
0 = \left[ \frac{d}{dt} E(f_0, \phi_0 + t\zeta) \right]_{t=0} = \int_{\mathbb{R}^3} \frac{e^{2\phi_0}}{\sqrt{e^{2\phi_0} + |p|^2}} f_0 \zeta \, dp \, dx + \int_{\mathbb{R}^3} \nabla \phi_0 \cdot \nabla \zeta \, dx,
\]

which is the claim. Note that differentiation inside the integral in the kinetic energy is justified, since $\exp \phi_0$ is bounded (by one). For the proof of the Virial Theorem, consider the uniparametric family of functions $(f_\alpha, \phi_\alpha) \subset \Gamma_{M,J}^k \times D^1$ given by

$f_\alpha(x, p) = f_0(\alpha x, \alpha^{-1} p)$, $\phi_\alpha(x) = \phi_0(\alpha x)$.

Since $(f_0, \phi_0)$ is a minimizer, the equation

\[
\left[ \frac{d}{d\alpha} E(f_\alpha, \phi_\alpha) \right]_{\alpha=1} = 0
\]

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is to be satisfied, which is equivalent to identity claimed in the lemma. It remains to prove (iv). It follows by the general properties of symmetric rearrangements that \((f^\ast_0, -\phi^\ast_0) \in \Gamma_{M,J}^k \times D^1\). Moreover,

\[
\int_{\mathbb{R}^3} |\nabla \phi^*_0|^2 \, dx \leq \int_{\mathbb{R}^3} |\nabla \phi_0|^2 \, dx,
\]

see [15, Lemma 7.17]. To reach our goal it is therefore enough to prove that

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{-2\phi_0} + |p|^2} f_0 \, dp \, dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_0} + |p|^2} f_0 \, dp \, dx.
\]

To this purpose we use the layer cake representation of \(\sqrt{e^{2x} + |p|^2}\). For \(y > 0\) we write

\[
\sqrt{e^{-2y} + |p|^2} = \int_y^{\infty} \frac{e^{-2s}}{\sqrt{e^{-2s} + |p|^2}} \, ds + |p|.
\]

Applying this to \(y = -\phi_0\) we obtain

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_0} + |p|^2} f_0 \, dp \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[ \int_0^\infty \frac{e^{-2s}}{\sqrt{e^{-2s} + |p|^2}} \chi(-\phi_0 \leq s) \, ds \right] f_0 \, dp \, dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |p| f_0 \, dp \, dx.
\]

Next we use that

\[
\int_{\mathbb{R}^6} \frac{e^{-2s}}{\sqrt{e^{-2s} + |p|^2}} \chi(-\phi_0 \leq s) f_0 \, d(p,x) \geq \int_{\mathbb{R}^6} \frac{e^{-2s}}{\sqrt{e^{-2s} + |p|^2}} \chi(\phi_0^* \leq s) f_0^* \, d(p,x),
\]

see [15], eq. (3), pag. 83, and then

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_0} + |p|^2} f_0 \, dp \, dx \geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_0^*} + |p|^2} f_0^* \, dp \, dx \geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{e^{-2\phi_0^*} + |p|^2} f_0^* \, dp \, dx.
\]

This concludes the proof that \((f^\ast_0, -\phi^\ast_0)\) is a minimizer. \(\square\)

**Remark 3** We shall prove in Section 6 that \((f_0, \phi_0)\) coincides with \((f^\ast_0, -\phi^\ast_0)\), up to a translation in space. In particular, minimizers are spherically symmetric with respect to some point in \(\mathbb{R}^3\).

## 5 Existence of a minimizer

In this section we prove Theorem 2. We split the proof in several lemmas. In view of Lemma 4(i), it is necessary to show that the positive part of \(\phi_n\) vanishes in the limit, which is done in the next lemma. Denote by \(g^+ = \max(g,0)\) and \(g^- = \min(g,0)\), the positive and negative part of a real valued function \(g\), respectively.
Lemma 5 Let \((f_n, \phi_n)\) be a minimizing sequence. Then \((f_n, \phi_n^-)\) is also a minimizing sequence and

\[
\|\nabla \phi_n - \nabla \phi_n^-\|_{L^2} \to 0, \quad n \to \infty. \tag{24}
\]

In particular, after possibly extracting a subsequence, \(\phi_n^+ \to 0\) a.e.

Proof: It is clear that \((f_n, \phi_n^-) \subset \Gamma_{k,M,J}^k \times D^1\). Moreover, since \(|\nabla \phi_n|^2 = |\nabla \phi_n^-|^2 + |\nabla \phi_n^+|^2\), we have \(E(f_n, \phi_n^-) \leq E(f_n, \phi_n)\), whence \((f_n, \phi_n^-)\) is also a minimizing sequence. Now assume that (24) is false. Then there exists a subsequence \(\phi_n\) and \(\lambda > 0\) such that \(\|\nabla \phi_n - \nabla \phi_n^-\|_{L^2} > \lambda\). Thus

\[
E(f_n, \phi_n) = E_{\text{kin}}(f_n, \phi_n) + \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla \phi_n^-|^2 + |\nabla (\phi_n - \phi_n^-)|^2 \right) dx \\
\geq E(f_n, \phi_n^-) + \frac{1}{2} \|\nabla \phi_n - \nabla \phi_n^-\|_{L^2} \geq E(f_n, \phi_n^-) + \frac{\lambda^2}{2},
\]

which contradicts \(\lim_{n \to \infty} E(f_n, \phi_n) = \lim_{n \to \infty} E(f_n, \phi_n^-) = I_{k,M,J}^k\). \(\square\)

Let \((f_n, \phi_n)\) be a minimizing sequence. Since \((f_n, \phi_n)\) is bounded in \(L^{1+1/k} \times D^1\), there exist \(f_0 \in L^{1+1/k}, \phi_0 \in D^1\) and a subsequence \((f_n, \phi_n)\) such that

\[
f_n \to f_0 \text{ in } L^{1+1/k}, \quad \phi_n \to \phi_0 \text{ in } L^6, \quad \nabla \phi_n \to \nabla \phi_0 \text{ in } L^2
\]

and so, by [13, Cor. 8.7],

\[
\phi_n \to \phi_0, \text{ pointwise almost everywhere.}
\]

By Lemma 5, \(\phi_0(x) \leq 0\), for almost all \(x \in \mathbb{R}^3\) and by weak convergence, \(f_0 \geq 0\) a.e. It is clear that \((f_0, \phi_0)\) is our candidate for being a minimizer of the functional \(E\). In the next lemma we show that the energy functional is weakly lower semicontinuous.

Lemma 6 For all \(0 < k < 2, J > 0\) and \(M > 0\),

\[
I_{k,M,J}^k \geq E(f_0, \phi_0).
\]

Proof: Clearly,

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^3} |\nabla \phi_n|^2 dx \geq \int_{\mathbb{R}^3} |\nabla \phi_0|^2 dx.
\]

Moreover

\[
\sqrt{e^{2\phi_n} + |p|^2} \to \sqrt{e^{2\phi_0} + |p|^2}, \quad \text{pointwise a.e. (up to subsequences)}
\]

and so, for all \(R > 0\) and since \(\phi_0\) is non-positive,

\[
\sqrt{e^{2\phi_n} + |p|^2} f_n \to \sqrt{e^{2\phi_0} + |p|^2} f_0 \text{ in } L^{1+1/k}(B_R),
\]

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where $B_R = \{(x,p) \in \mathbb{R}^6 : |x|^2 + |p|^2 \leq R^2\}$. It follows that

$$\int_{\mathbb{R}^3} \int_{B_R} \sqrt{e^{2\phi} + |p|^2} f_0 \, dp \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} \int_{B_R} \sqrt{e^{2\phi_n} + |p|^2} f_n \, dp \, dx.$$  

We then have

$$\mathcal{E}(f_0, \phi_0) = \lim_{R \to \infty} \int_{\mathbb{R}^3} \int_{B_R} \sqrt{e^{2\phi} + |p|^2} f_0 \, dp \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_0|^2 \, dx \leq \lim_{n \to \infty} \mathcal{E}(f_n, \phi_n) = I_{k,M,J}^k,$$

which completes the proof. \(\square\)

In the next lemma we provide a sufficient condition for the strong convergence of $f_n$ in $L^1$.

**Lemma 7** Let $(f_n, \phi_n)$ be a minimizing sequence and assume

$$\|\nabla \phi_n - \nabla \phi_0\|_{L^2} \to 0, \quad n \to \infty. \tag{25}$$

Let $M > M_0$. Then

$$\|f_n - f_0\|_{L^1} \to 0, \quad as \ n \to \infty.$$

Moreover, $(f_0, \phi_0)$ is a minimizer of the functional $\mathcal{E}$ over the space $\Gamma_{M,J}^k \times D^1$ and

$$\|f_n - f_0\|_{L^{1+1/k}} \to 0, \quad as \ n \to \infty.$$

**Proof:** Let $q = 1 + 1/k > 3/2$. To prove weak convergence in $L^1$ of $f_n$, it is clearly enough to show that no mass is lost at infinity, i.e., that for all $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^3} \int_{A_R} f_n \, dp \, dx \leq \varepsilon,$$

where $A_R = \{(x,p) \in \mathbb{R}^6 : |x| > R \text{ or } |p| > R\}$. Since

$$\int_{\mathbb{R}^3} \int_{A_R} f_n \, dp \, dx \leq \int_{\mathbb{R}^3} \int_{|p| > R} f_n \, dp \, dx + \int_{|x| > R} \int_{\mathbb{R}^3} f_n \, dp \, dx \leq \frac{1}{R} E_{\text{kin}}(f_n, \phi_n) + \int_{|x| > R} \int_{\mathbb{R}^3} f_n \, dp \, dx$$

and since the term $R^{-1} E_{\text{kin}}$ can be made arbitrarily small—by taking $R$ sufficiently large—it is enough to prove that

$$\forall \varepsilon > 0, \ \exists R(\varepsilon) > 0 : \int_{|x| > R} \rho f_n \, dp \, dx \leq \varepsilon. \tag{26}$$

If (26) were false, then we could find $0 < Q < M$, a subsequence $f_n$ and a sequence $R(n)$ (depending also on $Q$) such that

$$\int_{|x| > R(n)} \int_{\mathbb{R}^3} f_n \, dp \, dx = Q, \quad \lim_{n \to \infty} R(n) = \infty.$$
We write

\[ Q = \int_{|x| > R(n)} \int_{\mathbb{R}^3} f_n \, dp \, dx \]

\[ \leq \int_{|x| > R(n)} \int_{\mathbb{R}^3} e^{\phi_n} f_n \, dp \, dx - \int_{|x| > R(n)} \int_{\mathbb{R}^3} \phi_n f_n \, dp \, dx \]

\[ \leq \int_{|x| > R(n)} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_n} + |p|^2} f_n \, dp \, dx + \int_{|x| > R(n)} |\phi_n| \int_{\mathbb{R}^3} f_n \, dp \, dx, \]

where the simple bound \( \exp(\phi_n) - \phi_n \geq 1 \) has been used. By (14) and (25), \( \|\phi_n - \phi_0\|_{L^6} \to 0, n \to \infty \); whence, for all \( \varepsilon > 0 \) and sufficiently large \( n \),

\[ \int_{|x| > R(n)} |\phi_n|^6 \, dx \leq \varepsilon. \]

Thus, by Lemma 1 and Hölder’s inequality,

\[ \int_{|x| > R(n)} |\phi_n| \int_{\mathbb{R}^3} f_n \, dp \, dx \leq \varepsilon(n), \]

where \( \varepsilon(n) \to 0 \), as \( n \to \infty \). In conclusion

\[ Q \leq \int_{|x| > R(n)} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_n} + |p|^2} f_n \, dp \, dx + \varepsilon(n). \]

We shall prove that the above inequality leads to the contradiction that \( I^k_{M,J} \) is not the energy infimum in the space \( \Gamma^k_{M,J} \). To this purpose, we define the sequence

\[ \bar{f}_n = \alpha f_n(\beta x, p) \chi_{\{|x| \leq \beta - 1 R(n)\}}, \quad \bar{\phi}_n = \phi_n(\beta x), \]

where

\[ \alpha = \beta^2, \quad \beta = \frac{M - Q}{M} < 1. \]

It is easy to check that the sequence \((\bar{f}_n, \bar{\phi}_n)\) is contained in \( \Gamma^k_{M,J} \times D^1 \). Moreover

\[ \mathcal{E}(\bar{f}_n, \bar{\phi}_n) = \beta^{-1} \left( \int_{|x| \leq R(n)} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_n} + |p|^2} f_n \, dp \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_n|^2 \right) \]

\[ = \beta^{-1} \left( \mathcal{E}(f_n, \phi_n) - \int_{|x| > R(n)} \int_{\mathbb{R}^3} \sqrt{e^{2\phi_n} + |p|^2} f_n \, dp \, dx \right) \]

\[ \leq \left( \frac{M}{M - Q} \right) \left( \mathcal{E}(f_n, \phi_n) - Q + \varepsilon(n) \right). \]

Letting \( n \to \infty \) we obtain

\[ \liminf_{n \to \infty} \mathcal{E}(\bar{f}_n, \bar{\phi}_n) \leq \left( \frac{M}{M - Q} \right) \left( I^k_{M,J} - Q \right). \]
Let us denote by $F(Q)$ the function in the right hand side of the latter inequality. It satisfies \( \lim_{Q \to 0^+} F(Q) = I_{k,M,J}^k \), \( \lim_{Q \to M^-} F(Q) = -\infty \) and

\[
F'(Q) = \left( \frac{M}{M - Q} \right) \left( \frac{I_{k,M,J}^k - M}{M - Q} \right) < 0,
\]

for all $Q \in (0, M)$ (because of the assumption $M > M_0$), which leads to the contradiction $\lim_{n \to \infty} E(\tilde{f}_n, \tilde{\phi}_n) < I_{k,M,J}^k$. This concludes the proof that $f_n \to f_0$ in $L^1$. By weak convergence, $\|f_0\|_{L^1} = M$ and $\|f_0\|_{L^q} \leq J$. Thus $(f_0, \phi_0)$ is a minimizer by Lemma 6. By Proposition 4, $\|f_0\|_{L^{1+1/k}} = J$. Hence $\liminf_{n \to \infty} \|f_n\|_{L^{1+1/k}} = \|f_0\|_{L^{1+1/k}}$ and so, after possibly extracting a subsequence, $f_n \to f$, strongly in $L^{1+1/k}$ (see for instance [15, Thm. 2.11]). Extracting a subsequence, the convergence also holds pointwise almost everywhere. A standard application of Egoroff’s theorem shows that weak convergence in $L^1$ and almost everywhere pointwise convergence of a sequence of functions imply strong convergence in $L^1$. This concludes the proof of the proposition. \( \square \)

By virtue of Lemma 7, Theorem 2 will follow if we prove that minimizing sequences, after properly translated in space, satisfy (25). Our strategy is to show that it suffices to prove this for a special class of minimizing sequences, which enjoy some additional regularity and to which one can apply arguments similar to those valid for the Vlasov-Poisson system.

**Lemma 8** Let $f \in \Gamma_{k,M,J}^k$ being fixed such that $E_{km}(f, 0) < \infty$.

(i) The variational problem

\[
I_f = \inf_{\psi \in D^1} J_f(\psi), \quad J_f(\psi) = \mathcal{E}(f, \psi),
\]

has a unique minimizer $\psi_f \in D^1$. Moreover, $\psi_f$ is a non-positive continuous function and satisfies

\[
\Delta \psi = e^{2\psi} \int_{\mathbb{R}^3} \frac{f}{\sqrt{e^{2\psi} + |p|^2}} dp, \tag{27}
\]

in the sense of distributions.

(ii) For all $\phi \in D^1$ we have

\[
J_f(\phi) - I_f \geq \frac{1}{2} \|\nabla \phi - \nabla \psi_f\|_{L^2}^2.
\]

**Proof:** Let $\psi_n \in D^1$ be a minimizing sequence for the functional $J_f$; there exists $\psi_f \in D^1$ and a subsequence—still denoted $\psi_n$—such that $\psi_n \to \psi_f$ in $L^6$, $\nabla \psi_n \to \nabla \psi_f$ in $L^2$ and $\psi_n \to \psi_f$ pointwise almost everywhere. Moreover, by Lemma 5 (with $f_n$ kept fixed), $\psi_f \leq 0$ a.e. We prove that $\psi_f$ is a minimizer by showing that the functional $J_f$ is weakly lower semicontinuous. Clearly

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^3} |\nabla \psi_n|^2 dx \geq \int_{\mathbb{R}^3} |\nabla \psi_f|^2 dx.
\]
Moreover, by convexity of the function $x \to \sqrt{e^{2x} + |p|^2}$,

$$E_{\text{kin}}(\psi, f) - E_{\text{kin}}(\psi_n, f) \leq \int_{\mathbb{R}^3} (\psi - \psi_n) \int_{\mathbb{R}^3} \frac{e^{2\psi}}{\sqrt{e^{2\psi} + |p|^2}} f \, dp \, dx.$$  

The function $e^{2\psi} \int_{\mathbb{R}^3} (e^{2\psi} + |p|^2)^{-1/2} f \, dp$ is dominated by $\rho_f$ and so it belongs to $L^{6/5}$ by Lemma [1]. By weak convergence, the right hand side of the last inequality converges to zero as $n \to \infty$, whence

$$\liminf_{n \to \infty} E_{\text{kin}}(f, \psi_n) \geq E_{\text{kin}}(f, \psi_f)$$

and the proof that $\psi_f$ is a minimizer is complete. Moreover, $\psi_f$ is a weak solution of the Euler-Lagrange equation for the functional $J_f$, which is (27). By Lemma [1] the right hand side of (27) is in $L^j$, where $j > 3/2$ is given by [16]. Whence $\psi_f \in W^{2,j}_\text{loc}(\mathbb{R}^3)$, which is continuously embedded in $C(\mathbb{R}^3)$. In particular, $\psi_f$ is bounded. Since the functional $J_f$ is uniformly convex in $D^1 \cap L^\infty$, uniqueness of minimizers follows by standard theory of calculus of variations (see [9], for instance). It remains to prove (ii). For this purpose we write

$$J_f(\phi) - J_f = J_f(\phi) - J_f(\psi_f) \geq \int_{\mathbb{R}^3} (\phi - \psi_f) \int_{\mathbb{R}^3} \frac{e^{2\psi}}{\sqrt{e^{2\psi} + |p|^2}} f \, dp \, dx$$

$$+ \int_{\mathbb{R}^3} \nabla \psi_f \cdot (\nabla \phi - \nabla \psi_f) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi - \nabla \psi_f|^2 \, dx$$

$$= \frac{1}{2} ||\nabla \phi - \nabla \psi_f||_{L^2}^2,$$

where the convexity of $x \to \sqrt{e^{2x} + |p|^2}$ and the Euler-Lagrange equation (27) have been used. This completes the proof of the lemma.

**Lemma 9** Let $f \in \Gamma_{M, f}$ being fixed such that $E_{\text{kin}}(f, 0) < \infty$. The equation (27) has a unique solution $\psi_f \in D^1$. Moreover, $\psi_f$ is a non-positive continuous function and satisfies

$$\psi_f(x) = - \int_{\mathbb{R}^3} \frac{e^{2\psi_f(y)}}{|x - y|} \int_{\mathbb{R}^3} \frac{f(y, p)}{\sqrt{e^{2\psi_f(y)} + |p|^2}} \, dp \, dy,$$

$$\nabla \psi_f = \int_{\mathbb{R}^3} \frac{(x - y)}{|x - y|^3} e^{2\psi_f(y)} \int_{\mathbb{R}^3} \frac{f(y, p)}{\sqrt{e^{2\psi_f(y)} + |p|^2}} \, dp \, dy,$$

$$\int_{\mathbb{R}^3} |\nabla \psi_f|^2 \, dx = \int_{\mathbb{R}^3} \frac{e^{2(\psi_f(x) + \psi_f(y))}}{|x - y|} \left( \int_{\mathbb{R}^3} \frac{f(x, p)}{\sqrt{e^{2\psi_f(x)} + |p|^2}} \, dp \right)$$

$$\times \left( \int_{\mathbb{R}^3} \frac{f(y, p')}{\sqrt{e^{2\psi_f(y)} + |p'|^2}} \, dp' \right) \, dx \, dy.$$
Proof: The existence claim follows directly from Lemma 8. To prove uniqueness, it suffices to show that any solution of (27) is a minimizer of the functional $\mathcal{J}_f$. This follows by the inequality

$$\mathcal{I}_f - \mathcal{J}_f(\psi) \geq \frac{1}{2} \left\| \nabla \psi - \nabla \psi_f \right\|^2_{L^2},$$

which is valid for all solutions $\psi \in D^1$ of (27) and which is derived by using the convexity of $x \rightarrow \sqrt{e^{2x} + |p|^2}$. Now set $f(x, p) = e^{4\psi} f(x, e^{\psi} p)$. So doing, (27) becomes the linear Poisson equation

$$\triangle \psi = \int_{\mathbb{R}^3} \tilde{f}(1 + |p|^2)^{-1/2} dp.$$

Note that $\tilde{f}$ has the same $L^q$ regularity of $f$. Moreover, $\int_{\mathbb{R}^3} \tilde{f} |p| dp = \int_{\mathbb{R}^3} f |p| dp < \infty$, whence, by Lemma 1, $\int_{\mathbb{R}^3} \tilde{f}(1 + |p|^2)^{-1/2} dp \in L^j$. The rest of the lemma follows by standard potential theory, see for instance [15, Thm. 6.21].

Corollary 1 Let $(f_n, \phi_n) \subset \Gamma_{M, f} \times D^1$ be a minimizing sequence for the functional $E$ and let $\psi_{f_n} \in D^1$ be the unique weak solution of

$$\triangle \psi = e^{2\psi} \int_{\mathbb{R}^3} \frac{f_n}{\sqrt{e^{2\psi} + |p|^2}} dp.$$  \hspace{1cm} (28)

Then $(f_n, \psi_{f_n})$ is still a minimizing sequence and

$$\| \nabla \psi_{f_n} - \nabla \phi_n \|_{L^2} \to 0, \quad n \to \infty. \hspace{1cm} (29)$$

In particular, $\psi_{f_n} \to \phi_0$, as $n \to \infty$, pointwise almost everywhere.

Proof: As $E(f_n, \phi_n) \geq \inf \{ \mathcal{J}_f(\psi), \psi \in D^1 \} = E(f_n, \psi_{f_n})$, then $(f_n, \psi_{f_n})$ is still a minimizing sequence. Moreover, by Proposition 8(ii) we have

$$E(f_n, \phi_n) - E(f_n, \psi_{f_n}) \geq \frac{1}{2} \left\| \nabla \phi_n - \nabla \psi_{f_n} \right\|^2_{L^2}

and since the left hand side converges to zero, (29) is proved. Thus, after possibly extracting a subsequence, $(\psi_{f_n} - \phi_n) \to 0$, $n \to \infty$, pointwise a.e. and since $\phi_n \to \phi_0$ in the same sense, the proof of the corollary is complete. \hfill \Box

Thus the problem of proving strong convergence for $\nabla \phi_n$ in $L^2$ up to spatial translations has now been reduced to that of proving the same claim for the sequence $\psi_{f_n}$. This problem will be addressed next, using the deep result by Burchard and Guo [2].

Lemma 10 Let $(f_n, \psi_{f_n})$ be a minimizing sequence to the functional $E$ given by Corollary 1. Then, there exits $\phi_0$ in $D^1$ such that

i) $\lim_{n \to \infty} \| \nabla \psi_{f_n} - \nabla \phi_0 \|^2_{L^2} = 0,$

ii) $\lim_{n \to \infty} \| \nabla \psi_{f_n} \|^2_{L^2} = \| \nabla \phi_0 \|^2_{L^2}.$
Proof: Thanks to Lemma 4 we deduce that \((f^*_n, -\psi^*_n)\) is also a minimizing sequence and verifies the inequalities (22) and (23). By uniqueness of solutions to (27), see Lemma 9, \(\psi^*_n\) is also spherically symmetric. Moreover, \(\psi^*_n = -\psi^*_n\) by uniqueness of minimizers to the variational problem in Lemma 8.

We now prove the first assertion of the Lemma. We notice first that since \(\nabla \psi f_n \in W^{1,j}(B_R)\), for any ball \(B_R = \{x \in \mathbb{R}^3 : |x| \leq R\}\) and the inclusion \(W^{1,j}(B_R) \subset L^2(B_R)\) is compact, then \(\nabla \psi_{f_n}\) converges strongly in \(L^2(B_R)\) (up to subsequences). Thus the only property that we need to prove is that

\[
\int_{|x|>R} |\nabla \psi^*_n| \to 0 \quad \text{as } R \to \infty,
\]

where we denote \(\psi^*_n = \psi^*_n\). Integrating (27) we obtain

\[
\psi_{n}^*(r) = \frac{1}{r^2} \int_0^r \frac{f_{n}}{s^2 e^{2\psi_{n}(s)}} dp ds,
\]

from which it follows that

\[
|\psi_{n}^*(r)| \leq \frac{M}{r^2}
\]

and the property (30) is then satisfied by the sequence \(\psi^*_n\). In conclusion, the sequence \(\nabla \psi^*_n\) converges strongly in \(L^2\).

The second assertion can be easily deduced from the above arguments and using (22) and (23).

Under the framework of Lemma 10, the hypotheses of Theorem 2 in [2] hold and as a consequence there is a sequence of spatial translations \(T_n\) such that

\[
\lim_{n \to \infty} \|T_n \nabla \psi_{f_n} - \nabla \phi_0\|^2_{L^2} = 0.
\]

This result coincides precisely with the hypothesis of Lemma 7 and, hence, allows to conclude the proof of Theorem 2.

6 Properties of minimizers

The aim of this section is to prove Theorem 3. We start by showing that the minimizers constructed in the previous section are isotropic polytropes solutions of the Nordström-Vlasov system. For this purpose we use the method of the Lagrange multipliers. Note that we are combining some previous established techniques, see for example [13, 14, 21, 22], together with precise change of scales that preserves both constraints in our minimizing problem.

Let \((f_0, \phi_0) \in \Gamma^k_{M,J} \times D^1\) be a minimizer and for any \(\varepsilon > 0\) fixed define \(S_\varepsilon = \{(x, p) \in \mathbb{R}^6 : \varepsilon \leq f_0(x, p) \leq \varepsilon^{-1}\}\). Let \(\eta \in L^\infty(\mathbb{R}^6)\) be a real valued function with compact support such that \(\eta \geq 0\), a.e. for \((x, p) \in \mathbb{R}^6 \setminus \text{supp } f_0\) and \(\text{supp } \eta \subseteq (\mathbb{R}^6 \setminus \text{supp } f_0) \cup S_\varepsilon\). For \(t \in [0, T]\) and \(T = (\|\eta\|_1 + \|\eta\|_q + \|\eta\|_\infty)^{-1} \varepsilon/2\) we define

\[
f_t(x, p) = \alpha(t)^3 M \frac{f_0 + t\eta}{\|f_0 + t\eta\|_1} (\alpha(t)x, p), \quad \phi_t(x) = \phi_0(\alpha(t)x),
\]

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where
\[
\alpha(t) = \left( \frac{J}{M} \frac{\|f_0 + t\eta\|_1}{\|f_0 + t\eta\|_q} \right)^{\frac{q}{3q-3}}, \quad q = 1 + 1/k > 3/2.
\]

Note that \(f_0 + t\eta \geq 0\) a.e. and, for a \(\varepsilon\) small enough,
\[
M/2 \leq \|f_0 + t\eta\|_1 \leq M + \varepsilon/2, \quad J/2 \leq \|f_0 + t\eta\|_q \leq J + \varepsilon/2.
\]

From this we infer that \(\alpha\) is a smooth function on \([0, T]\) and
\[
\alpha'(t) = \frac{q}{3q-3} \alpha(t) \left[ \frac{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta \, dp \, dx}{\|f_0 + t\eta\|_1} - \frac{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (f_0 + t\eta)^{q-1} \eta \, dp \, dx}{\|f_0 + t\eta\|_q} \right].
\]

Moreover \(\sup_{[0, T]} \alpha''(t)\) is bounded. By inspection, \((f_t, \phi_t) \in \Gamma_{M, J}^k \times D^1\), for all \(t \in [0, T]\) and
\[
\mathcal{E}(f_t, \phi_t) - \mathcal{E}(f_0, \phi_0) = \left( \frac{M}{\|f_0 + t\eta\|_1} - 1 \right) E_{\text{kin}}(f_0, \phi_0) + \frac{Mt}{\|f_0 + t\eta\|_1} E_{\text{kin}}(\eta, \phi_0) + \left( \frac{1}{\alpha(t)} - 1 \right) \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi_0|^2 \, dx.
\]

By a Taylor expansion at \(t = 0^+\) and straightforward estimates we obtain
\[
\frac{M}{\|f_0 + t\eta\|_1} - 1 = -\frac{t}{M} \int_{\mathbb{R}^3} \eta \, dp \, dx + O(t^2), \quad \frac{Mt}{\|f_0 + t\eta\|_1} = t + O(t^2),
\]
\[
\frac{1}{\alpha(t)} - 1 = -t \frac{q}{3q-3} \left[ \frac{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta}{M} - \frac{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0^{q-1} \eta}{f_0^q} \right] + O(t^2),
\]

where the notation \(O(t^2)\), as usual, means that the rest terms are bounded by \(Ct^2\), for a positive constant \(C\) depending on \(\varepsilon, f_0\) and \(\eta\), but not on \(t\). Substituting into (31) we get
\[
\mathcal{E}(f_t, \phi_t) - \mathcal{E}(f_0, \phi_0) = t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( E - E_0 + c f_0^{q-1} \right) \eta \, dp \, dx + O(t^2),
\]
where \(E_0\) and \(c\) are given in Theorem 3. Recalling the class of admissible test functions \(\eta\) and the fact that \(\varepsilon > 0\) is arbitrary, we conclude that \(E - E_0 \geq 0\) a.e. on \(\mathbb{R}^6 \backslash \text{supp} f_0\) and \(f_0 = (E_0 - E)^{1/(q-1)}\) a.e. on \(\text{supp} f_0\), which proves the first part of Theorem 3. As to the admissible range for \(E_0\) claimed in the theorem, we notice that, by the Virial Theorem and the definition of \(E_0\),
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\phi_0} \frac{f_0}{E} \, dp \, dx = \left( T_{M, J}^k - \int_{\mathbb{R}^3} |\nabla \phi_0|^2 \, dx \right)
\]
\[
= \left( \frac{6}{2-k} - \frac{k}{2-k} T_{M, J}^k \right).
\]

The above quantity and the constant \(c\) are both positive, from which the admissible range of \(E_0\) is derived.
We proceed proving some properties of the minimizers. By Lemma 9

\[ |\phi_0(x)| \leq \int_{\mathbb{R}^3} e^{2\phi_0(y)} \int_{\mathbb{R}^3} \frac{f_0(y, p)}{\sqrt{e^{2\phi_0(y)} + |p|^2}} dp \, dy \, \frac{dy}{|x - y|}. \]

For all \( R > 1 \) we split the integral in the right hand side according to \(|x - y| \leq 1/R, 1/R \leq |x - y| \leq R, \ |x - y| \geq R\). Straightforward estimates lead to

\[ |\phi_0(x)| \leq C||\mu_0||_2 \frac{R^{3-2j}}{J} + R \int_{|y| \geq |x| - R} \rho_0(y) dy + \frac{M}{R}, \]

which yields \( \lim_{|x| \to \infty} \phi_0(x) = 0 \). From this and the fact that \( E_0 < 1 \), it follows that \( \sqrt{e^{2\phi_0} + |p|^2} > E_0 \), for \( |p| > 1 \) or for \( |x| \) large enough, which proves the compact support property of \( f_0 \). Using \( f_0 = ((E - E_0)/c)_+ \) and a change of variable entail

\[ e^{2\phi_0} \int_{\mathbb{R}^3} \frac{f_0}{\sqrt{e^{2\phi_0} + |p|^2}} dp = 4\pi e^{-k} \int_{0}^{E_0} \sqrt{E^2 - e^{2\phi_0}(E_0 - E)^k} dE \leq C. \]

This result together with Lemma 9 implies that \( \phi_0 \in W^{2,\infty}_{loc} \), whence the characteristics of the time independent Vlasov equation are well defined \( C^1 \) curves. Since \( f_0 \) depends only on the particles energy \( E = \sqrt{e^{2\phi_0} + |p|^2} \), then it is constant along characteristics, which is the definition of mild solution. Next we show that \((f_0, \phi_0)\) is spherically symmetric with respect to some point in \( \mathbb{R}^3 \). By using Lemma 4, \((f_0^*, -\phi_0^*)\) is also a minimizer. Then

\[ \int_{\mathbb{R}^3} |\nabla \phi_0^*|^2 dx = \int_{\mathbb{R}^3} |\nabla \phi_0|^2 dx \]

and \((f_0^*, -\phi_0^*)\) is also a solution of (7). Writing \( \phi_0^*(x) = \psi(r), \ r = |x| \), and integrating (7) we obtain

\[ \psi'(r) = \frac{1}{r^2} \int_{0}^{r} s^2 e^{2\psi(s)} \int_{\mathbb{R}^3} \frac{f_0^*}{\sqrt{e^{2\psi(s)} + |p|^2}} dp \, ds, \]

where \( \psi' = d\psi/dr \). From the previous equation we deduce that \( \nabla \phi_0^* = (x/r) \psi' \neq 0 \) a.e., whence \( \phi_0(x) \) coincides with a spatial translation of \( \phi_0^*(x) \), see [3]. Since \( E_0 \) is the same for \( f_0 \) and \( f_0^* \), we conclude by using (6) that \( f_0 \) equals \( f_0^* \), up to a translation in space.

To conclude the proof of Theorem 3 it remains to establish the uniqueness statement. Without loss of generality, we assume that the potential function is spherically symmetric with respect to the origin, i.e., \( \psi(r) = \phi_0(x), \ r = |x| \); recall that uniqueness is up to a spatial translation. The non-linear Poisson equation (7) for \( \psi(r) \) is

\[ (r^2 \psi'(r))' = r^2 e^{2\psi(r)} \int_{\mathbb{R}^3} \frac{f_0}{E} dp. \]
On the other hand, $\psi$ is strictly increasing and vanishes at infinity which assures the existence of $r_0 \in \mathbb{R}^+$ such that

$$\psi(r_0) = \log E_0.$$  \hfill (33)

Let us note that if $\psi(0) > \log E_0$, then $\psi$ is constant in time since $f_0$ vanishes when $e^{\psi(r)} > E_0$. Now, we prove that $r_0$, $\psi(r_0)$ and $\psi'(r_0)$ are uniquely determined by $E_0$ for any minimizer. Let $r \geq r_0$; using the Virial Theorem and the definition of $E_0$ we find

$$\int_0^r s^2 e^{2\psi(s)} s \int_{\mathbb{R}^3} \frac{f_0}{E} dp ds = \int_0^\infty s^2 e^{2\psi(s)} s \int_{\mathbb{R}^3} \frac{f_0}{E} dp ds$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2\phi_0(x)} \frac{f_0}{E} dp dx$$

$$= \frac{1}{4\pi} \left( \frac{6}{2-k} E_0 M - \frac{k+4}{2-k} I_{M,J}^k \right), \quad \forall r \geq r_0.$$  \hfill (34)

Then, integrating (32) we have

$$\psi'(r) = \frac{1}{4\pi r^2} \left( \frac{6}{2-k} E_0 M - \frac{k+4}{2-k} I_{M,J}^k \right), \quad \forall r \geq r_0.$$  \hfill (35)

Using this expression we obtain

$$\int_{r_0}^{\infty} 4\pi s^2 (\psi'(s))^2 ds = \frac{1}{4\pi} \left( \frac{6}{2-k} E_0 M - \frac{k+4}{2-k} I_{M,J}^k \right)^2 \frac{1}{r_0}$$

or

$$\int_{r_0}^{\infty} 4\pi s^2 (\psi'(s))^2 ds = -\psi(r_0) \int_0^{r_0} \int_{\mathbb{R}^3} e^{2\psi(\xi)} \frac{f_0}{E} dp d\xi$$

which allow to prove that

$$r_0 = -\frac{1}{4\pi \log E_0} \left( \frac{6}{2-k} E_0 M - \frac{k+4}{2-k} I_{M,J}^k \right)$$  \hfill (35)

and

$$\psi'(r_0) = 4\pi (\log E_0)^2 \left( \frac{6}{2-k} E_0 M - \frac{k+4}{2-k} I_{M,J}^k \right)^{-1}.$$  \hfill (36)

The existence and uniqueness theory developed in [4] for (32) implies that any minimizer can be seen as the unique solution to (32) with initial conditions $\psi(r_0)$ and $\psi'(r_0)$. Since these quantities are uniquely parametrized by $E_0$ (see [33], [35] and [36]) the question about the uniqueness of the minimizer (up to spatial translations) is now equivalent to prove that the spherically symmetric minimizer is determined by an unique $E_0$.

For any $E_0 \in \left( \frac{k+4}{6} I_{M,J}^k \right)$ we have a solution $\psi = \psi_{E_0}$ to (32) with initial conditions given by (33), (35) and (36). Associated to every $\psi_{E_0}$ there is a density function $f = f_{E_0}$ defined by (6). At this point we check that the mass
associated with the functions $f_{E_0}$ is monotone as function of $E_0$, which entails uniqueness of the minimizer by the condition $\|f_0\|_1 = M$. Unfortunately, these quantities cannot be computed directly, since the initial conditions depend on several parameters, $M$, $J$, and $k$ and unknown quantities like $I_{M,J}^k$.

In order to skip the explicit dependence of equation (32) on $E_0$ we scale the potential as

$$
\bar{\psi}(r) = \psi(br) - \log E_0, \quad \text{where} \quad b = \frac{c^{k/2}}{E_0^{2+k/2}}.
$$

(37)

Now $\bar{\psi}(r)$ verifies

$$
(r^2 \bar{\psi}'(r))^\prime = r^2 e^{2\bar{\psi}(\bar{r})} \int_{\mathbb{R}^3} \frac{\bar{f}}{E} \, dp,
$$

(38)

with $\bar{E} = \sqrt{e^{2\bar{\psi}(\bar{r})} + |p|^2}$, $\bar{f} = (1 - \bar{E})^k_r$ and initial conditions

$$
\bar{\psi}(\bar{r}_0) = 0, \quad \bar{\psi}'(\bar{r}_0) = b \psi'(r_0), \quad \bar{r}_0 = r_0/b.
$$

(39)

Since the initial conditions (39) cannot be explicitly calculated, we consider numerical simulations with initial conditions at $r = 0$,

$$
\bar{\psi}(0) = \psi(0) - \ln E_0 := a \leq 0, \quad \bar{\psi}'(0) = b \psi'(0) = 0
$$

(40)

parametrized in terms of the new parameter $a$ whose relation with $E_0$ is at this moment unknown. Figure [1] shows these solutions for some values of $a$ in the case $k = 1$. Two different regimes have been detected numerically for several values of $k$. For small values of $a$ ($a \leq a^* \approx 0.723$ when $k = 1$) the solutions intersect themselves; in particular, for initial conditions $a^1 < a^2 (< a^*)$, the corresponding solutions $\bar{\psi}_1$ and $\bar{\psi}_2$ satisfy

$$
\bar{r}_0^1 < \bar{r}_0^2, \quad \bar{\psi}'_1(\bar{r}_0^1) > \bar{\psi}'_2(\bar{r}_0^2),
$$

(38)
where $r_0^1$, $r_0^2$ are the points such that $\tilde{\psi}_1(r_0^1) = \tilde{\psi}_2(r_0^2) = 0$. However, for large values of $a$ ($a \geq 0.723$ when $k = 1$) the solutions maintain the order meanwhile they reach negative values. In particular for initial conditions $a^1 < a^2$ then $\tilde{\psi}_1(r) < \tilde{\psi}_2(r)$ for $r < \tilde{r}_0^1$ and $\tilde{\psi}_2'(\tilde{r}_0^2) < \tilde{\psi}_1'(\tilde{r}_0^1)$ holds. It is not hard to see that the second type of solutions is not compatible with conditions (39). In fact, by the relations (35), (36) and (39) one can easily deduce that $\tilde{\psi}'(r_0)$ must be a decreasing function of $r_0$. The solutions with small values of $a$ corresponding to conditions (39) are thus the correct ones. We infer that the initial conditions in $r = 0$ verify a strictly increasing relation with the values of $E_0$, that is,

$$E_0^1 < E_0^2 \in \left( \frac{k + 4 \frac{I_{M,J}}{M}}{6}, \frac{I_{M,J}}{M} \right) \implies \tilde{\psi}_1(0) < \tilde{\psi}_2(0)$$

where $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are the solutions to (38) with initial conditions (39) (determined by $E_0^1$ and $E_0^2$).

Figure 2 shows the relation of the mass in terms of the $\tilde{\psi}_1$. We observe that

$$\tilde{\psi}_1(0) < \tilde{\psi}_2(0) \implies \|\tilde{f}_1\| > \|\tilde{f}_2\|.$$

Combining both estimates we conclude that the mass of the functions $\tilde{f}$ is a decreasing function of $E_0$. The masses of the original $f$ and the scaled $\tilde{f}$ are related by

$$\|f\|_1 = \frac{b}{E_0} \|\tilde{f}\|_1.$$

27
Since \( b/E_0 \) is also a strictly decreasing function of \( E_0 \), we deduce that the mass is a decreasing function of \( E_0 \), and in consequence this proves the uniqueness of minimizers.

In the remainder of this section we wish to comment on the validity of our numerical study. Let us note that the integral term in (38) can be rewritten equivalently as

\[
\int_{\mathbb{R}^3} \frac{\tilde{f}}{E} \, dp = \frac{4\pi}{k+1} \int_0^{\sqrt{1 - e^{\psi(r)}}} \left( 1 - \sqrt{e^{2\tilde{\psi}(r)} + \xi^2} \right)^{k+1} \, d\xi
\]  

(41)

by using radial coordinates in the variable \( p \) and integrating by parts. In general these integrals cannot be computed explicitly, but this is possible for \( k = 1 \), which avoids undesirable approximation errors in the equation. However, our numerical simulations show that the solutions to equation (38) with initial conditions (40) typically have the same behavior as for \( k = 1 \), i.e., two different qualitative shapes but only one of them is compatible with the constraints of the system (39). For \( k = 1 \), the integral (41) reads

\[
\int_{\mathbb{R}^3} \frac{\tilde{f}}{E} \, dp = \frac{2\pi}{3} \left( \sqrt{1 - e^{2\tilde{\psi}(r)}} (1 - 2e^{2\psi(r)}) - 3e^{2\psi(r)} \log \left( \frac{1 + \sqrt{1 - e^{2\psi(r)}}}{e^{\psi(r)}} \right) \right).
\]

In order to avoid the singularity in \( r = 0 \) exhibited by equation (38), we have performed numerical simulations with initial conditions

\[
\tilde{\psi}(\epsilon) = a \leq 0, \quad \tilde{\psi}'(\epsilon) = 0, \quad \text{where } \epsilon = 10^{-5}
\]  

(42)

which is a reasonable approximation of initial conditions (40), due to the vanishing of \( \tilde{\psi}' \) at \( r = 0 \). The mass of the functions \( \tilde{f} \) can be easily computed once \( \tilde{\psi}' \) is known.

A final interesting remark about these simulations is the validation of the virial relation for any computed pair \( \tilde{f}, \tilde{\psi} \). Let us observe that the scaling deriving \( \tilde{f}, \tilde{\psi} \) from \( f, \psi \) affects in the same way both terms in the virial relation. Since these functions verify

\[
\int_{\mathbb{R}^3} \frac{|p|^2}{E} \tilde{f}(x, p) \, dx = \frac{3}{k+1} \int_{\mathbb{R}^3} \tilde{f}^{1+1/k}(x, p) \, dx \quad \forall p \in \mathbb{R}^3
\]  

(43)

and

\[
\int_{\mathbb{R}^3} |\nabla \tilde{\phi}|^2 \, dx = \int_{\mathbb{R}^3} -\tilde{\phi} \int_{\mathbb{R}^3} \frac{\tilde{f}}{E} \, dx
\]  

(44)

where \( -\tilde{\phi}(x) = \tilde{\psi}(|x|) \), the radial versions of these expressions allow to compute (by means of integrals over finite intervals) both terms. The results obtained in our simulations show a nearly total agreement (errors of order \( 10^{-2} \)) between
both terms for each solution, indicating that these functions verify in general the Virial Theorem.

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