Hypersurface family with a common geodesic

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Abstract

In this paper, we study the problem of finding a hypersurface family from a given spatial geodesic curve in $\mathbb{R}^4$. We obtain the parametric representation for a hypersurface family whose members have the same curve as a given geodesic curve. Using the Frenet frame of the given geodesic curve, we present the hypersurface as a linear combination of this frame and analyze the necessary and sufficient condition for that curve to be geodesic. We illustrate this method by presenting some examples.

Keywords: Hypersurface, Frenet frame, geodesic.

MSC: 53A04, 53A07
1. Introduction

Geodesic is a well-known notion in differential geometry. A geodesic on a surface can be defined in many equivalent ways. Geometrically, the shortest path joining any two points of a surface is a geodesic. Geodesics are curves in surfaces that play a role analogous that of straight lines in the plane. A straight line doesn’t bend to left or right as we travel along it [6].

In recent years, there have been various researches on geodesics. Kumar et al., [20] presented a study on geodesic curves computed directly on NURBS surfaces and discrete geodesics computed on the equivalent tessellated surfaces. Wang et al., [26] studied the problem of constructing a family of surfaces from a given spatial geodesic curve and derived a parametric representation for a surface pencil whose members share the same geodesic curve as an isoparametric curve. A practical method was presented by Sanchez and Dorado, [21] to construct polynomial surfaces from a polynomial geodesic or a family of geodesics, by prescribing tangent ribbons. Sprynski et al., [22] dealt with reconstruction of numerical or real surfaces based on the knowledge of some geodesic curves on the surface. Paluszny, [19] considered patches that contain any given 3D polynomial curve as a pregeodesic (i.e. geodesic up to reparametrization). Given two pairs of regular space curves $r_1(u), r_3(u)$ and $r_2(v), r_4(v)$ that define a curvilinear rectangle, Farouki et al., [10] handled the problem of constructing a $C^2$ surface patch $R(u,v)$ for which these four boundary curves correspond to geodesics of the surface. Farouki et al., [11] considered the problem of constructing polynomial or rational tensor-product Bézier patches bounded by given four polynomial or rational Bézier curves defining a curvilinear rectangle, such that they are geodesics of the constructed surface.

On the other hand, Wang et al., [26] tackled the problem of finding surfaces passing through a given geodesic. In 2011, given curve was changed to a line of curvature and Li et
al., [18] constructed a surface family from a given line of curvature. Bayram et al., [5] gave the necessary and sufficient conditions for a given curve to be an asymptotic on a surface.

However, while differential geometry of a parametric surface in $\mathbb{R}^3$ can be found in textbooks such as in Struik [24], Willmore [28], Stoker [23], do Carmo [7], differential geometry of a parametric surface in $\mathbb{R}^n$ can be found in textbook such as in the contemporary literature on Geometric Modeling [9, 16]. Also, there is little literature on differential geometry of parametric surface family in $\mathbb{R}^3$ [2, 8, 17, 26], but not in $\mathbb{R}^4$. Besides, there is an ascending interest on fourth dimension [1, 2, 8].

Furthermore, various visualization techniques about objects in Euclidean n-space ($n \geq 4$) are presented [3, 4, 14]. The fundamental step to visualize a 4D object is projecting first in to the 3-space and then into the plane. In many real world applications the problem of visualizing three-dimensional data, commonly referred to as scalar fields arouses. The graph of a function $f(x,y,z) : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$, $U$ is open, is a special type of parametric hypersurface with the parametrization $(x, y, z, f(x,y,z))$ in 4-space. There is an existing method for rendering such a 3-surface based on known methods for visualizing functions of two variables [13].

In this paper, we consider the four dimensional analogue of the problem of constructing a parametric representation of a surface family from a given spatial geodesic in Wang et al. [26], who derived the necessary and sufficient conditions on the marching-scale functions for which the curve $C$ is an isogeodesic, i.e., both a geodesic and a parameter curve, on a given surface. We express the hypersurface pencil parametrically with the help of the Frenet frame $\{T, N, B_1, B_2\}$ of the given curve. We find the necessary and sufficient constraints on the marching-scale functions, namely, coefficients of Frenet vectors, so that both the geodesic and parametric requirements met. Finally, as an application of our method one example for each type of marching-scale functions is given.
2. Preliminaries

Let us first introduce some notations and definitions. Bold letters such as \( \mathbf{a}, \mathbf{R} \) will be used for vectors and vector functions. We assume that they are smooth enough so that all the (partial) derivatives given in the paper are meaningful. Let \( \alpha : [1, \infty) \subset \mathbb{R} \to \mathbb{R}^4 \) be an arc-length curve. If \( \{ \mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2 \} \) is the moving Frenet frame along \( \alpha \), then the Frenet formulas are given by

\[
\begin{align*}
\mathbf{T}' &= k_1 \mathbf{N}, \\
\mathbf{N}' &= -k_1 \mathbf{T} + k_2 \mathbf{B}_1, \\
\mathbf{B}_1' &= -k_2 \mathbf{N} + k_3 \mathbf{B}_2, \\
\mathbf{B}_2' &= -k_3 \mathbf{B}_1,
\end{align*}
\]

where \( \mathbf{T}, \mathbf{N}, \mathbf{B}_1 \) and \( \mathbf{B}_2 \) denote the tangent, principal normal, first binormal and second binormal vector fields, respectively, \( k_i (i=1,2,3) \) the i-th curvature functions of the curve \( \alpha \) [14].

From elementary differential geometry we have

\[
\begin{align*}
\alpha'(s) &= \mathbf{T}(s), \\
\alpha''(s) &= k_1(s) \mathbf{N}(s), \\
k_1(s) &= \|\alpha'(s)\|.
\end{align*}
\]

By using Frenet formulas one can obtain the followings

\[
\begin{align*}
\alpha'''(s) &= -k_1^2 \mathbf{T}(s) + k_1' \mathbf{N}(s) + k_1 k_2 \mathbf{B}_1(s), \\
\alpha^{(iv)}(s) &= -3k_1 k_1' \mathbf{T}(s) + \left( -k_1^3 + k_1' + k_1 k_2^2 \right) \mathbf{N}(s) + \left( 2k_1' k_2 + k_1 k_2' \right) \mathbf{B}_1(s) + k_1 k_2 k_3 \mathbf{B}_2(s).
\end{align*}
\]

The unit vectors \( \mathbf{B}_2 \) and \( \mathbf{B}_1 \) are given by

\[
\begin{align*}
\mathbf{B}_2(s) &= \frac{\alpha'(s) \otimes \alpha''(s) \otimes \alpha'''(s)}{\|\alpha'(s) \otimes \alpha''(s) \otimes \alpha'''(s)\|}, \\
\mathbf{B}_1(s) &= \mathbf{B}_2(s) \otimes \mathbf{T}(s) \otimes \mathbf{N}(s),
\end{align*}
\]

where \( \otimes \) is the vector product of vectors in \( \mathbb{R}^4 \).

Since the vectors \( \mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2 \) are orthonormal, the second curvature \( k_2 \) and the third curvature \( k_3 \) can be obtained from (3) as
\[
\begin{cases}
    k_2(s) = \frac{B_1(s)\alpha''(s)}{k_1(s)}, \\
    k_3(s) = \frac{B_2(s)\alpha'''(s)}{k_1(s)k_2(s)},
\end{cases}
\]

where ‘\*’ denotes the standard inner product.

Let \{e_1, e_2, e_3, e_4\} be the standard basis for four-dimensional Euclidean space \(\mathbb{R}^4\). The vector product of the vectors \(u = \sum_{i=1}^{4} u_ie_i, v = \sum_{i=1}^{4} v_ie_i, w = \sum_{i=1}^{4} w_ie_i\) is defined by

\[
u \otimes v \otimes w = \begin{vmatrix}
e_1 & e_2 & e_3 & e_4 \\
u_1 & u_2 & u_3 & u_4 \\
v_1 & v_2 & v_3 & v_4 \\
w_1 & w_2 & w_3 & w_4
\end{vmatrix}
\]

[15, 27].

If \(u, v\) and \(w\) are linearly independent then \(u \otimes v \otimes w\) is orthogonal to each of these vectors.

3. Hypersurface family with a common geodesic

A curve \(r(s)\) on a hypersurface \(P = P(s,t,q) \subset \mathbb{R}^4\) is called an isoparametric curve if it is a parameter curve, that is, there exists a pair of parameters \(t_0\) and \(q_0\) such that \(r(s) = P(s,t_0,q_0)\). Given a parametric curve \(r(s)\), it is called an isogeodesic of a hypersurface \(P\) if it is both a geodesic and an isoparametric curve on \(P\).

Let \(C: r = r(s), L_i \leq s \leq L_2\), be a \(C^3\) curve, where \(s\) is the arc-length. To have a well-defined principal normal, assume that \(r''(s) \neq 0, L_i \leq s \leq L_2\).

Let \(T(s), N(s), B_1(s), B_2(s)\) be the tangent, principal normal, first binormal, second binormal, respectively; and let \(k_1(s), k_2(s)\) and \(k_3(s)\) be the first, second and the third curvature, respectively. Since \(\{T(s), N(s), B_1(s), B_2(s)\}\) is an orthogonal coordinate frame on \(r(s)\) the parametric hypersurface \(P(s,t,q) = [L_i,L_2] \times [Q_j, Q_k] \rightarrow \mathbb{R}^4\) passing through \(r(s)\) can be defined as follows:
\( P(s,t,q) = r(s) + u(s,t,q) \frac{\partial}{\partial s} v(s,t,q) \frac{\partial}{\partial t} + w(s,t,q) \frac{\partial}{\partial q} + x(s,t,q) \)

\( L_1 \leq s \leq L_2, T_1 \leq t \leq T_2, Q_1 \leq q \leq Q_2, \)

where \( u(s,t,q), v(s,t,q), w(s,t,q) \) and \( x(s,t,q) \) are all \( C^1 \) functions. These functions are called the marching scale functions.

We try to find out the necessary and sufficient conditions for a hypersurface \( P = P(s,t,q) \) having the curve \( C \) as an isogeodesic.

First to satisfy the isoparametricity condition there should exist \( t_0 \in [T_1, T_2] \) and \( q_0 \in [Q_1, Q_2] \) such that \( P(s,t_0,q_0) = r(s), L_1 \leq s \leq L_2, \) that is,

\[
\begin{align*}
\{ u(s,t_0,q_0) &= v(s,t_0,q_0) = w(s,t_0,q_0) = x(s,t_0,q_0) = 0, \\
\{ t_0 &\in [T_1, T_2], q_0 \in [Q_1, Q_2], L_1 \leq s \leq L_2. 
\end{align*}
\]

Secondly, the curve \( C \) is a geodesic on the hypersurface \( P(s,t,q) \) if and only if the principal normal \( N(s) \) of the curve and the normal \( \hat{a}(s,t_0,q_0) \) of the hypersurface \( P(s,t,q) \) are linearly dependent, that is, parallel along the curve \( C \). [25]. The normal \( \hat{a}(s,t_0,q_0) \) of the hypersurface can be obtained by calculating the vector product of the partial derivatives and using the Frenet formula as follows

\[
\left( \frac{\partial P(s,t,q)}{\partial s} \right) = 1 + \frac{\partial u(s,t,q)}{\partial s} - v(s,t,q) \kappa_1(s) T(s) + u(s,t,q) \kappa_1(s) - \frac{\partial v(s,t,q)}{\partial s} - w(s,t,q) \kappa_2(s) N(s) + v(s,t,q) \kappa_2(s) - \frac{\partial w(s,t,q)}{\partial s} - x(s,t,q) \kappa_3(s) B_1(s) + w(s,t,q) \kappa_3(s) + \frac{\partial x(s,t,q)}{\partial s} B_2(s),
\]

\[
\left( \frac{\partial P(s,t,q)}{\partial t} \right) = \frac{\partial u(s,t,q)}{\partial t} T(s) + \frac{\partial v(s,t,q)}{\partial t} N(s) + \frac{\partial w(s,t,q)}{\partial t} B_1(s) + \frac{\partial x(s,t,q)}{\partial t} B_2(s).
\]
and

\[
\frac{\partial \mathbf{P}(s,t,q)}{\partial q} = \frac{\partial \mathbf{u}(s,t,q)}{\partial q} \mathbf{T}(s) + \frac{\partial \mathbf{v}(s,t,q)}{\partial q} \mathbf{N}(s) \\
+ \frac{\partial \mathbf{w}(s,t,q)}{\partial q} \mathbf{B}_1(s) + \frac{\partial \mathbf{x}(s,t,q)}{\partial q} \mathbf{B}_2(s).
\]

Remark: Because,

\[
\begin{cases}
\mathbf{u}(s,t_0,q_0) = \mathbf{v}(s,t_0,q_0) = \mathbf{w}(s,t_0,q_0) = \mathbf{x}(s,t_0,q_0) = 0, \\
t_0 \in [T_1,T_2], q_0 \in [Q_1,Q_2], L_1 \leq s \leq L_2.
\end{cases}
\]

along the curve \( C \), by the definition of partial differentiation we have

\[
\begin{cases}
\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial s} = \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial s} = \frac{\partial \mathbf{w}(s,t_0,q_0)}{\partial s} = \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial s} = 0, \\
t_0 \in [T_1,T_2], q_0 \in [Q_1,Q_2], L_1 \leq s \leq L_2.
\end{cases}
\]

By using (7) we have

\[
\mathbf{u}(s,t_0,q_0) = \frac{\partial \mathbf{P}(s,t_0,q_0)}{\partial s} \otimes \frac{\partial \mathbf{P}(s,t_0,q_0)}{\partial t} \otimes \frac{\partial \mathbf{P}(s,t_0,q_0)}{\partial q}
\]

\[
= \mathbf{\phi}(s,t_0,q_0) \mathbf{T}(s) - \mathbf{\phi}(s,t_0,q_0) \mathbf{N}(s) + \mathbf{\phi}(s,t_0,q_0) \mathbf{B}_1(s) - \mathbf{\phi}(s,t_0,q_0) \mathbf{B}_2(s),
\]

where

\[
\mathbf{\phi}(s,t_0,q_0) = \begin{vmatrix}
\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial s} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial s} & \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial s} \\
\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial t} & \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial t} \\
\frac{\partial \mathbf{u}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{v}(s,t_0,q_0)}{\partial q} & \frac{\partial \mathbf{x}(s,t_0,q_0)}{\partial q}
\end{vmatrix} = 0
\]
\[
\phi(s_{t_0}, q_0) = \begin{vmatrix}
1 + \frac{\partial u(s_{t_0}, q_0)}{\partial s} & \frac{\partial w(s_{t_0}, q_0)}{\partial s} & \frac{\partial x(s_{t_0}, q_0)}{\partial s} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial t} & \frac{\partial w(s_{t_0}, q_0)}{\partial t} & \frac{\partial x(s_{t_0}, q_0)}{\partial t} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial q} & \frac{\partial w(s_{t_0}, q_0)}{\partial q} & \frac{\partial x(s_{t_0}, q_0)}{\partial q} \\
\end{vmatrix}
\]

\[
\phi(s_{t_0}, q_0) = \begin{vmatrix}
1 + \frac{\partial u(s_{t_0}, q_0)}{\partial s} & \frac{\partial v(s_{t_0}, q_0)}{\partial s} & \frac{\partial x(s_{t_0}, q_0)}{\partial s} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial t} & \frac{\partial v(s_{t_0}, q_0)}{\partial t} & \frac{\partial x(s_{t_0}, q_0)}{\partial t} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial q} & \frac{\partial v(s_{t_0}, q_0)}{\partial q} & \frac{\partial x(s_{t_0}, q_0)}{\partial q} \\
\end{vmatrix}
\]

\[
\phi(s_{t_0}, q_0) = \begin{vmatrix}
1 + \frac{\partial u(s_{t_0}, q_0)}{\partial s} & \frac{\partial v(s_{t_0}, q_0)}{\partial s} & \frac{\partial w(s_{t_0}, q_0)}{\partial s} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial t} & \frac{\partial v(s_{t_0}, q_0)}{\partial t} & \frac{\partial w(s_{t_0}, q_0)}{\partial t} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial q} & \frac{\partial v(s_{t_0}, q_0)}{\partial q} & \frac{\partial w(s_{t_0}, q_0)}{\partial q} \\
\end{vmatrix}
\]

\[
\phi(s_{t_0}, q_0) = \begin{vmatrix}
1 + \frac{\partial u(s_{t_0}, q_0)}{\partial s} & \frac{\partial v(s_{t_0}, q_0)}{\partial s} & \frac{\partial w(s_{t_0}, q_0)}{\partial s} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial t} & \frac{\partial v(s_{t_0}, q_0)}{\partial t} & \frac{\partial w(s_{t_0}, q_0)}{\partial t} \\
\frac{\partial u(s_{t_0}, q_0)}{\partial q} & \frac{\partial v(s_{t_0}, q_0)}{\partial q} & \frac{\partial w(s_{t_0}, q_0)}{\partial q} \\
\end{vmatrix}
\]
So, \( \mathbf{a}(s, t_0, q_0) \parallel \mathbf{N}(s) \) if and only if

\[
\mathbf{a}(s, t_0, q_0) = \mathbf{a}(s, t_0, q_0) = 0, \quad t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2.
\]

(8)

Thus, any hypersurface defined by (6) has the curve \( C \) as an isogeodesic if and only if

\[
\begin{align*}
\mathbf{u}(s, t_0, q_0) &= \mathbf{v}(s, t_0, q_0) = \mathbf{w}(s, t_0, q_0) = \mathbf{x}(s, t_0, q_0) = 0, \\
\mathbf{a}(s, t_0, q_0) &= \mathbf{a}(s, t_0, q_0) = 0, \quad \mathbf{a}(s, t_0, q_0) \neq 0,
\end{align*}
\]

(9)

\[ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2. \]

is satisfied. We call the set of hypersurfaces defined by (6) and satisfying (9) an isogeodesic hypersurface family.

To develop the method further and for simplification purposes, we analyze some types of marching-scale functions.

3.1. Marching-scale functions of type I

Let marching-scale functions be

\[
\begin{align*}
\mathbf{u}(s, t, q) &= \mathbf{l}(s) \mathbf{U}(t, q), \\
\mathbf{v}(s, t, q) &= \mathbf{m}(s) \mathbf{V}(t, q), \\
\mathbf{w}(s, t, q) &= \mathbf{n}(s) \mathbf{W}(t, q), \\
\mathbf{x}(s, t, q) &= \mathbf{p}(s) \mathbf{X}(t, q),
\end{align*}
\]

where \( \mathbf{l}(s), \mathbf{m}(s), \mathbf{n}(s), \mathbf{p}(s), \mathbf{U}(t, q), \mathbf{V}(t, q), \mathbf{W}(t, q), \mathbf{X}(t, q) \in C^1 \) and \( \mathbf{l}(s) \neq 0 \neq \mathbf{m}(s), \mathbf{n}(s) \neq 0 \neq \mathbf{p}(s), \forall s \in [L_1, L_2] \). By using (9) the necessary and sufficient condition for which the curve \( C \) is an isogeodesic on the hypersurface \( \mathbf{P}(s, t, q) \) can be given as

\[
\begin{align*}
\mathbf{U}(t_0, q_0) &= \mathbf{V}(t_0, q_0) = \mathbf{W}(t_0, q_0) = \mathbf{X}(t_0, q_0) = 0, \\
\frac{\partial \mathbf{V}(t_0, q_0)}{\partial t} \frac{\partial \mathbf{X}(t_0, q_0)}{\partial q} - \frac{\partial \mathbf{V}(t_0, q_0)}{\partial q} \frac{\partial \mathbf{X}(t_0, q_0)}{\partial t} &= 0, \\
\frac{\partial \mathbf{W}(t_0, q_0)}{\partial t} \frac{\partial \mathbf{X}(t_0, q_0)}{\partial q} - \frac{\partial \mathbf{W}(t_0, q_0)}{\partial q} \frac{\partial \mathbf{X}(t_0, q_0)}{\partial t} &= 0, \\
\frac{\partial \mathbf{W}(t_0, q_0)}{\partial t} \frac{\partial \mathbf{X}(t_0, q_0)}{\partial q} - \frac{\partial \mathbf{W}(t_0, q_0)}{\partial q} \frac{\partial \mathbf{X}(t_0, q_0)}{\partial t} &\neq 0,
\end{align*}
\]

(10)

\[ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2]. \]
With a closer investigation of (10), we should have \( \frac{\partial V(t_0, q_0)}{\partial t} = 0 \) and \( \frac{\partial V(t_0, q_0)}{\partial q} = 0 \).

So, (10) can be simplified to

\[
\begin{align*}
U(t_0, q_0) &= V(t_0, q_0) = W(t_0, q_0) = X(t_0, q_0) = 0, \\
\frac{\partial V(t_0, q_0)}{\partial t} &= \frac{\partial V(t_0, q_0)}{\partial q} = 0, \\
\frac{\partial W(t_0, q_0)}{\partial t} \cdot \frac{\partial X(t_0, q_0)}{\partial q} - \frac{\partial W(t_0, q_0)}{\partial q} \cdot \frac{\partial X(t_0, q_0)}{\partial t} &\neq 0,
\end{align*}
\]

\( t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2]. \)

### 3.2. Marching-scale functions of type II

Let marching-scale functions be

\[
\begin{align*}
u(s, t, q) &= l(s, t) U(q), \\
v(s, t, q) &= m(s, t) V(q), \\
w(s, t, q) &= n(s, t) W(q), \\
x(s, t, q) &= p(s, t) X(q),
\end{align*}
\]

where \( l(s, q), m(s, q), n(s, q), p(s, t, q) \in C^1 \) and \( U(t, q), V(t, q), W(t, q), X(t, q) \in C^1 \). Also let us choose

\[
V(q_0) = \frac{dV(q_0)}{dq} = U(q_0) = \frac{dU(q_0)}{dq} = 0.
\]

By using (9), the curve \( C \) is an isogeodesic on the hypersurface \( P(s, t, q) \) if and only if the followings are satisfied

\[
\begin{align*}
\begin{cases}
\frac{n(s, t_0)}{c t} W(q_0) + p(s, t_0) X(q_0) = 0, \\
\frac{n(s, t_0)}{c t} W(q_0) + p(s, t_0) X(q_0) = 0,
\end{cases}
\end{align*}
\]

\( t_0 \in [T_1, T_2], q_0 \in [Q_1, Q_2], 1 \leq s \leq L_2, T_1 \leq t \leq T_2, Q_1 \leq q \leq Q_2. \)

### 3.3. Marching-scale functions of type III

Let marching-scale functions be
\[
\begin{align*}
\mathbf{u}(s,t,q) &= 1(s,q) \mathbf{U}(t), \\
\mathbf{v}(s,t,q) &= \mathbf{m}(s,q) \mathbf{V}(t), \\
\mathbf{w}(s,t,q) &= \mathbf{n}(s,q) \mathbf{W}(t), \\
\mathbf{x}(s,t,q) &= \mathbf{p}(s,q) \mathbf{X}(t),
\end{align*}
\]
L_1 \leq s \leq L_2, T_1 \leq t \leq T_2, Q_1 \leq q \leq Q_2,

where \(1(s,q), \mathbf{m}(s,q), \mathbf{n}(s,q), \mathbf{p}(s,q), \mathbf{U}(t), \mathbf{V}(t), \mathbf{W}(t), \mathbf{X}(t) \in C^1\). Also let

\[
\mathbf{v}(t_0) = \frac{d\mathbf{V}(t_0)}{dt} = \mathbf{U}(t_0) = \frac{d\mathbf{U}(t_0)}{dt} = 0.
\]

By using (9) we derive the necessary and sufficient condition for which the curve \(C\) is an isogedeic on the hypersurface \(p(s,t,q)\) as

\[
\begin{align*}
\mathbf{n}(s,q_0) \mathbf{W}(t_0) &= \mathbf{p}(s,q_0) \mathbf{X}(t_0) = 0, \\
\mathbf{n}(s,t_0) \frac{d\mathbf{W}(t_0)}{dt} = \partial_q \mathbf{p}(s,q_0) \mathbf{X}(t_0) &= \mathbf{n}(s,q_0) \mathbf{W}(t_0) \frac{d\mathbf{p}(s,q_0)}{dt} = 0,
\end{align*}
\]
\(t_0 \in [T_1,T_2], q_0 \in [Q_1,Q_2], L_1 \leq s \leq L_2\).

4. Examples

**Example 1.** Let \(r(s) = \left(\frac{1}{2} \cos(s), \frac{1}{2} \sin(s), \frac{1}{2} s, \frac{\sqrt{2}}{2} s\right)\), \(0 \leq s \leq 2\pi\), be a curve parametrized by arc-length. For this curve,

\[
\begin{align*}
\mathbf{T}(s) &= r'(s) = \left(-\frac{1}{2} \sin(s), \frac{1}{2} \cos(s), \frac{1}{2} \frac{\sqrt{2}}{2} s\right), \\
\mathbf{N}(s) &= \left(-\cos(s), \sin(s), 0, 0\right), \\
\mathbf{B}_2(s) &= \frac{r'(s) \otimes r''(s) \otimes r'''(s)}{\|r'(s) \otimes r''(s) \otimes r'''(s)\|} = \left(\frac{1}{3}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{3}\right), \\
\mathbf{B}_1(s) &= \mathbf{B}_2 \otimes \mathbf{T} \otimes \mathbf{N} = \left(-\frac{\sqrt{3}}{2} \sin(s), \frac{\sqrt{3}}{2} \cos(s), \frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right).
\end{align*}
\]

Let us choose the marching-scale functions of type 1, where

\[
1(s) = \mathbf{m}(s) = \mathbf{n}(s) = \mathbf{p}(s) = 1
\]

and

\[
\mathbf{U}(t,q) = (t-t_0)(q-q_0), \mathbf{V}(t,q) = 0, \mathbf{W}(t,q) = t-t_0, \mathbf{X}(t,q) = q-q_0,
\]
\(t_0 \in [0,1], q_0 \in [0,1], 0 \leq s \leq 2\pi\).
So, we have

\[ u(s,t,q) = (t-t_0)(q-q_0), \]
\[ v(s,t,q) = 0, \]
\[ w(s,t,q) = t-t_0, \]
\[ x(s,t,q) = q-q_0. \]

The hypersurface

\[
P(s,t,q) = r(s) + u(s,t,q)T(s) + v(s,t,q)N(s) + w(s,t,q)B_1(s) + x(s,t,q)B_2(s)
\]

\[
= \left( \frac{1}{2} \cos(s) - \frac{1}{2}(t-t_0)(q-q_0) \sin(s) - \frac{\sqrt{3}}{2}(t-t_0) \sin(s), \right.
\]
\[
\frac{1}{2} \sin(s) + \frac{1}{2}(t-t_0)(q-q_0) \cos(s) + \frac{\sqrt{3}}{2}(t-t_0) \cos(s),
\]
\[
\frac{1}{2}s + \frac{1}{2}(t-t_0)(q-q_0) - \frac{\sqrt{3}}{6}(t-t_0) + \frac{\sqrt{6}}{3}(q-q_0),
\]
\[
\frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}(t-t_0)(q-q_0) - \frac{\sqrt{6}}{3}(t-t_0) - \frac{\sqrt{3}}{3}(q-q_0) \right)
\]

\[ 0 \leq s \leq 2\pi, 0 \leq t \leq 1, 0 \leq q \leq 1, t_0 \in [0,1], q_0 \in [0,1], \] is a member of the isogeodesic hypersurface family, since it satisfies (11).

By changing the parameters \( t_0 \) and \( q_0 \) we can adjust the position of the curve \( r(s) \) on the hypersurface. Let us choose \( t_0 = \frac{1}{2} \) and \( q_0 = 0 \). Now the curve \( r(s) \) is again an isogeodesic on the hypersurface \( P(s,t,q) \) and the equation of the hypersurface is

\[
P(s,t,q) = \left( \frac{1}{2} \cos(s) - \frac{1}{2}\left( t - \frac{1}{2} \right)(q+\sqrt{3}) \sin(s), \right.
\]
\[
\frac{1}{2} \sin(s) + \frac{1}{2}\left( t - \frac{1}{2} \right)(q+\sqrt{3}) \cos(s),
\]
\[
\frac{1}{2}s + \frac{1}{2}\left( t - \frac{1}{2} \right)q - \frac{\sqrt{3}}{6}\left( t - \frac{1}{2} \right) + \frac{\sqrt{6}}{3}q,
\]
\[
\frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}\left( t - \frac{1}{2} \right)q - \frac{\sqrt{6}}{6}\left( t - \frac{1}{2} \right) - \frac{\sqrt{3}}{3}q \right).
\]

The projection of a hypersurface into 3-space generally yields a three-dimensional volume. If we fix each of the three parameters, one at a time, we obtain three distinct families of 2-spaces in 4-space. The projections of these 2-surfaces into 3-space are surfaces in 3-space. Thus, they can be displayed by 3D rendering methods.
So, if we (parallel) project the hypersurface $P(s,t,q)$ into the $w = 0$ subspace and fix $q = \frac{1}{8}$ we obtain the surface

$$P_w(s,t,\frac{1}{8}) = \left( \frac{1}{2} \cos(s) - \frac{1 + 8\sqrt{3}}{16} \left( t - \frac{1}{2} \right) \sin(s), \right. $n
$$\left. \frac{1}{2} \sin(s) + \frac{1 + 8\sqrt{3}}{8} \left( t - \frac{1}{2} \right) \cos(s), \right. $n
$$\left. \frac{1}{2} + \frac{1}{16} \left( t - \frac{1}{2} \right) - \frac{\sqrt{3}}{6} \left( t - \frac{1}{2} \right) + \frac{\sqrt{6}}{24}, \right) \quad 0 \leq s \leq 2\pi, \ 0 \leq t \leq 1$$

in 3-space illustrated in Fig. 1.

![Image](image.png)

**Fig. 1.** Projection of a member of the hypersurface family with marching-scale functions of type I and its isogeodesic.

**Example 2.** Given the curve parameterized by arc-length

$$r(s) = \left( \frac{1}{2} \sin(s), \frac{1}{2} \cos(s), 0, \frac{\sqrt{3}}{2} s \right), \ 0 \leq s \leq 2\pi, \text{ it is easy to show that}$$
\[ T(s) = r'(s) = \left( \frac{1}{2} \cos(s), -\frac{1}{2} \sin(s), 0, \frac{\sqrt{3}}{2} \right), \]
\[ N(s) = (-\sin(s), -\cos(s), 0, 0), \]
\[ B_2(s) = \frac{r'(s) \otimes r''(s) \otimes r'''(s)}{\|r'(s) \otimes r''(s) \otimes r'''(s)\|} = (0, 0, -1, 0), \]
\[ B_1(s) = B_2 \otimes T \otimes N = \left( \frac{\sqrt{3}}{2} \cos(s), -\frac{\sqrt{3}}{2} \sin(s), 0, -\frac{1}{2} \right). \]

Let us choose the marching-scale functions of type II, where
\[ n(s, t) = s + t + 1, \quad p(s, t) = (s + 1)(t - t_0), \]
and
\[ U(q) = V(q) = 0, \quad W(q) = q - q_0, \quad X(q) = 1. \]

So, we get
\[ u(s, t, q) = 0, \]
\[ v(s, t, q) = 0, \]
\[ w(s, t, q) = (s + t + 1)(q - q_0), \]
\[ x(s, t, q) = (s + 1)(t - t_0). \]

From (12) the hypersurface
\[ P(s, t, q) = r(s) + u(s, t, q)T(s) + v(s, t, q)N(s) + w(s, t, q)B_1(s) + x(s, t, q)B_2(s) \]
\[ = \left( \frac{1}{2} \sin(s) + \frac{\sqrt{3}}{2} (s + t + 1)(q - q_0) \cos(s), \right. \]
\[ \left. \frac{1}{2} \cos(s) - \frac{\sqrt{3}}{2} (s + t + 1)(q - q_0) \sin(s), \right. \]
\[ -(s + 1)(t - t_0), \]
\[ \left. \frac{\sqrt{3}}{2} s - \frac{1}{2} (s + t + 1)(q - q_0) \right) \]

0 ≤ s ≤ 2\pi, 0 ≤ t ≤ 1, 0 ≤ q ≤ 1, is a member of the hypersurface family having the curve \( r(s) \) as an isogeodesic.

Setting \( t_0 = \frac{1}{2} \) and \( q_0 = 0 \) yields the hypersurface
By (parallel) projecting the hypersurface \( \mathbf{p}(s,t,q) \) into the \( w=0 \) subspace and fixing \( q = \frac{1}{500} \) we get the surface

\[
\mathbf{p}_w(s,t,\frac{1}{500}) = \left( \frac{1}{2} \sin(s) + \frac{\sqrt{3}}{2} (s+t+1) \cos(s), \right.
\]

\[
\frac{1}{2} \cos(s) - \frac{\sqrt{3}}{2} (s+t+1) \sin(s),
\]

\[-(s+1) \left(t - \frac{1}{2}\right),
\]

\[
\frac{\sqrt{3}}{2} s - \frac{1}{2} (s+t+1)q \right).\]

where, \( 0 \leq s \leq 2\pi, 0 \leq t \leq 1 \) in 3-space demonstrated in Fig. 2.

**Fig. 2.** Projection of a member of the hypersurface family with marching-scale functions of type II and its isogeodesic.
Example 3. Let \( r(s) = \left( \frac{1}{2} \sin(s), \frac{1}{2} \cos(s), 0, \frac{\sqrt{3}}{2} s \right), \) \( 0 \leq s \leq 2\pi, \) be an arc-length curve. One can easily show that

\[
T(s) = r'(s) = \left( \frac{1}{2} \cos(s), -\frac{1}{2} \sin(s), 0, \frac{\sqrt{3}}{2} \right),
\]

\[
N(s) = (-\sin(s), -\cos(s), 0, 0),
\]

\[
B_2(s) = \frac{r'(s) \otimes r''(s) \otimes r'''(s)}{\|r'(s) \otimes r''(s) \otimes r'''(s)\|} = (0, 0, -1, 0),
\]

\[
B_1(s) = B_2 \otimes T \otimes N = \left( \frac{\sqrt{3}}{2} \cos(s), -\frac{\sqrt{3}}{2} \sin(s), 0, -\frac{1}{2} \right),
\]

for this curve.

If we choose the marching-scale functions of type III, where

\[
n(s, q) = \sin(s(q - q_0)), \quad p(s, q) = sq^2
\]

and

\[
U(t) = V(t) = 0, \quad W(t) = 1, \quad X(t) = t - t_0
\]

then

\[
u(s, t, q) = 0,
\]

\[
v(s, t, q) = 0,
\]

\[
w(s, t, q) = \sin(s(q - q_0)),
\]

\[
x(s, t, q) = sq^2(t - t_0).
\]

Thus, from (13) if we take \( q_0 \neq 0 \) then the curve \( r(s) \) is an isogeodesic on the hypersurface

\[
P(s, t, q) = r(s) + u(s, t, q)T(s) + v(s, t, q)N(s) + w(s, t, q)B_1(s) + x(s, t, q)B_2(s)
\]

\[
= \left( \frac{1}{2} \sin(s) + \frac{\sqrt{3}}{2} \cos(s) \sin(s(q - q_0)), \right.
\]

\[
\frac{1}{2} \cos(s) - \frac{\sqrt{3}}{2} \sin(s) \sin(s(q - q_0)),
\]

\[
- sq^2(t - t_0),
\]

\[
\frac{\sqrt{3}}{2} s - \frac{1}{2} \sin(s(q - q_0)) \left], \right.
\]

where \( \pi \leq s \leq 3\pi, \ 0 \leq t \leq 1, \ 0 \leq q \leq 1. \)
By taking \( t_0 = 1 \) and \( q_0 = 1 \) we have the following hypersurface:

\[
\mathbf{P}(s,t,q) = \left( \frac{1}{2} \sin(s) + \frac{\sqrt{3}}{2} \cos(s) \sin(s(q - 1)), \frac{1}{2} \cos(s) - \frac{\sqrt{3}}{2} \sin(s) \sin(s(q - 1)), -sq^2(t-1), \frac{\sqrt{3}}{2}s - \frac{1}{2} \sin(s(q - 1)) \right).
\]

Hence, if we (parallel) project the hypersurface \( \mathbf{P}(s,t,q) \) into the \( z = 0 \) subspace we get the surface

\[
\mathbf{P}_z(s,q) = \left( \frac{1}{2} \sin(s) + \frac{\sqrt{3}}{2} \cos(s) \sin(s(q - 1)), \frac{1}{2} \cos(s) - \frac{\sqrt{3}}{2} \sin(s) \sin(s(q - 1)), \frac{\sqrt{3}}{2}s - \frac{1}{2} \sin(s(q - 1)) \right),
\]

where \( \pi \leq s \leq 3\pi, \ 0 \leq q \leq 1 \),

in 3-space shown in Fig. 3.

Fig. 3. Projection of a member of the hypersurface family with marching-scale functions of type III and its isogeodesic.
5. Conclusion

We have introduced a method for finding a hypersurface family passing through the same given geodesic as an isoparametric curve. The members of the hypersurface family are obtained by choosing suitable marching-scale functions. For a better analysis of the method we investigate three types of marching-scale functions. Also, by giving an example for each type the method is verified. Furthermore, with the help of the projecting methods a member of the family is visualized in 3-space with its isogeodesic.

However, there is more work waiting to study. For 3-space, one possible alternative is to consider the realm of implicit surfaces $F(x,y,z,t)=0$ and try to find out the constraints for a given curve $r(s)$ is an isogeodesic on $F(x,y,z,t)=0$. Also, the analogue of the problem dealt in this paper may be considered for 2-surfaces in 4-space or another types of marching-scale functions may be investigated.

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