On the local adjacency metric dimension of split graph

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Abstract. The metric dimension is one of an interesting studied graph topics. The local adjacency metric dimension is combination of the local metric dimension and the adjacency metric dimension. The graph $G = (V, E)$ in this study is connected, simple, and finite graph. Let $u, v$ are in $G$. For an order set of vertices $X = \{x_1, x_2, \ldots, x_k\}$, the adjacency representation of $v$ with respect to $X$ is the ordered $k$-tuple $rA(v|X) = (dA(v, x_1), dA(v, x_2), \ldots, dA(v, x_k))$, where $dA(u, v)$ represents the adjacency distance $u - v$. $dA(u, v)$ is defined by 0 if $u = v$, 1 if $u$ adjacents with $v$, and 2 if $u$ does not adjacent with $v$. For every two distinct vertices $u, v$ and $u$ adjacents with $v$ such that $rA(u|X) \neq rA(v|X)$. Then, we call $X$ as local adjacency resolving set of $G$. The basis of $G$ is a minimum local adjacency resolving set in $G$. The vertex cardinality in the basis is a local adjacency metric dimension of $G$ ($dim_{A,l}(G)$). In this research, we initiate to study the existence of the local adjacency metric dimension of split graph $G$.

1. Introduction

This section shows some definitions which are used to introduce the theme of this study. This research is using simple, connected, and undirected graph. A graph $G$ is defined by set of $V(G)$ and $E(G)$, the set of vertices and the set of edges of $G$. It can be seen at [13], [8], [7] and [12]. The metric dimension is one of an interesting studied graph topics [11], [3], [9]. Local means that every neighbouring vertex or edge has distinct representation [1], [2]. Let’s say, there are three neighboring vertex in a path, $a, b, c$ where each has representation: $a \neq b$ and $a$ may equals $c$. Then, the path $a - c$ is called local [10], [5], [4]. The local adjacency metric dimension is combination of local metric dimension and adjacency metric dimension.

Let $G = (V, E)$ be a connected simple finite graph and $u, v$ in $G$. For an order set of vertices $X = \{x_1, x_2, \ldots, x_k\}$, the adjacency representation of $v$ with respect to $X$ is the ordered $k$-tuple $rA(v|X) = (dA(v, x_1), dA(v, x_2), \ldots, dA(v, x_k))$, where $dA(u, v)$ represents the adjacency distance $u - v$. $dA(u, v)$ defined by 0 if $u = v$, 1 if $u$ adjacents with $v$, and 2 if $u$ does not adjacent with $v$. $X$ is a local adjacency resolving set of $G$ if for every two distinct vertices $u, v$ and $u$ adjacents with $v$ then $rA(u|X) \neq rA(v|X)$. A local adjacency metric basis of $G$ is a minimum local adjacency resolving set in $G$. The cardinality of vertices in the basis is a local adjacency metric dimension of $G$ ($dim_{A,l}(G)$).

The research is begun with Rodriguez, et all [14] about local adjacency metric dimension of corona graphs. Then in 2018 Dafik, et all [11] researched about The non-isolated resolving number of k-corona product of graphs. Recently, Darmaji, et all [6] studied about Local...
Figure 1. \( _{1}\text{Spl}(C_4) \) Graph

adjacency metric dimension of sun graph and stacked book graph this year. Of course they really motivate us to present this study.

2. Result

This section shows some new results on the local adjacency metric dimension.

2.1. \( m \)–Splitting of Cycle Graph

A \( m \)–splitting of cycle graph \( (m\text{Spl}(C_n)) \) is a graph obtained from a cycle graph \( (C_n) \) by adding new vertex \( v' \) in every vertex \( v \) as \( n \) such that \( v' \) adjacent \( v \) in \( C_n \). \( m \)–splitting graph is graph which has the number of vertex \( v' \) as \( m \). Let \( G = m\text{Spl}(C_n) \) with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{v_1^1, v_2^1, \ldots, v_k^i\} \), where \( v_i \) is vertex of \( C_n \) and \( v_k^i \) is copy of vertex \( v_i \) around \( C_n \) for \( i \in \{1, 2, \ldots, n\} \) and \( k \in \{1, 2, \ldots, m\} \). We can see at Figure 1 as illustration.

Theorem 2.1 Let \( G \) be a \( m \)–splitting graph of cycle graph \( (m\text{Spl}(C_n)) \) with \( |V(G)| = 2n \). For \( n \geq 3 \) and \( m, n \in \mathbb{N} \), then

\[
\text{dim}_{A,l}(G) = \begin{cases} 
2 & \text{for } 3 \leq n \leq 8; \\
\lceil \frac{n}{4} \rceil & \text{for } n \geq 9.
\end{cases}
\]

Proof 2.1 We divide this proof till some case.

• Case 1. For \( 3 \leq n \leq 8 \),

Choose \( S = \{v_i, v_{i-1}\} \subseteq V(G) \) where \( v_i \) does not adjacent with \( v_{i-1} \) and distance between them is a maximum 3 vertices. We will show that \( S \) is a local adjacency resolving set of \( G \). Since we have \( \text{dim}_{A,l}(G) = 2 \), then we just have two digits of \( r \). For the combination of the biggest possibility in \( C_n \) is \( \{12, 22, 21\} \) because the distance between \( v_i \) and \( v_{i-1} \) is 3 vertices. For the vertex around \( C_n \) \((m\text{–split})\), since every vertex in \( (v_k^i) \) has neighbour as \( (v_i) \) then the \( r \) of \( (v_k^i) \) is same as \( (v_i) \) except vertex \( v_k^i \) and \( v_k^{i-1} \). Since \( v_k^i \) does not adjacent with \( v_{k-1}^i \) and \( v_{k-1}^{i} \), \( v_k^{i-1} \) do not have same \( r \) of \( S \) with their neighbour. Then it ensures that \( v_k^i \) and \( v_k^{i-1} \) have distinct \( r \) with their neighbours. For your illustration, see the local adjacency representations of vertices from \( V(G) - S \) are as follow:
\( (i) \) for \( n = 3, \)
\[
\begin{align*}
r_A(v_2|S) &= r_A(v_3^2|S) = (11) \\
r_A(v_3^1|S) &= (21) \\
r_A(v_3^2|S) &= (12)
\end{align*}
\]

\( (ii) \) for \( n = 4, \)
\[
\begin{align*}
r_A(v_2|S) &= r_A(v_4|S) = r_A(v_3^2|S) = r_A(v_3^1|S) = (11) \\
r_A(v_3^1|S) &= (21) \\
r_A(v_3^2|S) &= (12)
\end{align*}
\]

\( (iii) \) for \( n = 5, \)
\[
\begin{align*}
r_A(v_2|S) &= r_A(v_5^2|S) = (11) \\
r_A(v_4|S) &= r_A(v_5^1|S) = r_A(v_3^1|S) = (21) \\
r_A(v_5|S) &= r_A(v_3^3|S) = r_A(v_5^3|S) = r_A(v_5^3|S) = (12)
\end{align*}
\]

\( (iv) \) for \( n = 6, \)
\[
\begin{align*}
r_A(v_2|S) &= r_A(v_6^2|S) = (11) \\
r_A(v_4|S) &= r_A(v_6^1|S) = r_A(v_3^1|S) = (21) \\
r_A(v_5|S) &= r_A(v_3^5|S) = (22) \\
r_A(v_6|S) &= r_A(v_3^6|S) = r_A(v_5^6|S) = (12)
\end{align*}
\]

\( (v) \) for \( n = 7, \)
\[
\begin{align*}
r_A(v_2|S) &= r_A(v_7|S) = r_A(v_6^5|S) = r_A(v_5^5|S) = (12) \\
r_A(v_3|S) &= r_A(v_7|S) = r_A(v_5^4|S) = r_A(v_3^4|S) = r_A(v_6^5|S) = r_A(v_5^5|S) = (22) \\
r_A(v_4|S) &= r_A(v_6|S) = r_A(v_5^6|S) = r_A(v_5^6|S) = (21)
\end{align*}
\]

\( (vi) \) for \( n = 8, \)
\[
\begin{align*}
r_A(v_2|S) &= r_A(v_8|S) = r_A(v_6^5|S) = r_A(v_5^5|S) = (12) \\
r_A(v_3|S) &= r_A(v_7|S) = r_A(v_7|S) = r_A(v_5^4|S) = r_A(v_3^4|S) = r_A(v_6^5|S) = r_A(v_5^5|S) = (22) \\
r_A(v_4|S) &= r_A(v_6|S) = r_A(v_5^6|S) = r_A(v_5^6|S) = (21)
\end{align*}
\]

As we see that all of the adjacency representations of adjacent vertices are distinct. So, \( S = \{v_i, v_{i-1}\} \) is a local adjacency resolving set for \( G \). The cardinality of \( S \), \(|S| = 2\) is minimum, because if \(|S| < n\) certainly there are \( x \neq y \subset V(G) - S \) such that \( r(x|S) = r(y|S) \).

Suppose \( S_1 = \{v_i\} \) where \(|S| = 1 < 2\). Then, \( r(v_{i-1}|S) = (1) = r(v_{i-2}|S) \) and \( v_{i-1} \) adjacents with \( v_{i-2} \). Thus, \( \text{dim}_{\text{AJ}}(G) = 2 \) for \( 3 \leq n \leq 8 \).

\* Case 2. For \( n \geq 9. \)
Choose \( S = \{v_1, v_2, \ldots, v_{\frac{n}{2}}\} \subset V(G) \). We will show that \( S \) is a local adjacency resolving set of \( G \). We know that \( d \) is defined by

\[
d(u, w) = \begin{cases} 
0 & \text{if } v = w \\
1 & \text{if } v \text{ adjacents with } w; \\
2 & \text{if } v \text{ does not adjacent with } w.
\end{cases}
\]

Suppose we call the "inside" vertices of \( G \) is the set of vertices in \( C_n \) and the "outside"
vertices of $G$ is the set of vertices outside $C_n$ (or in $m-$split of $C_n$). Then, there are three cases to prove the theorem, such that:

(i) When the resolving vertices set are inside the $G$. Choose resolving vertices of $S$ as much as $\lceil \frac{n}{4} \rceil$. Suppose we put any vertices of $S$ inside $G$. Based on construction of $C_n$, suppose we have maximum $3$ vertices ($d = 2$) between two resolving inside vertices. Since every vertex in $(v_i^k)$ has neighbour as $(v_i)$ then the $r$ of $(v_i^k)$ is same as $(v_i)$ except vertex which is a copy of every vertex in $S$. Since distance of every vertex in $S$ is more than and equal $2$, so we can ensure that every $v_i^k$ which is a copy of every vertex in $S$ has $r = (22\ldots2)$. Then it ensures that all vertices in $S$ are distinct. Then every adjacency $v_i | S$ must has different $r$.

(ii) When the resolving vertices set are outside the $G$. Choose resolving vertices of $S$ as much as $\lceil \frac{n}{4} \rceil$. Suppose we put any vertices of $S$ outside $G$. Without loss the generality, let $j$ be even number of $N$. Let $v_i^k, v_{i+2}^k, \ldots v_{i+j}^k$ in $S$. Then there must be minimum an outside vertex $(v_{i+1}^k)$ adjacent an inside vertex $(v_i)$ which have same $r = (22\ldots2)$.

(iii) When some resolving vertices are inside the $G$ and the others are outside the $G$. Choose resolving vertices of $S$ as much as $\lceil \frac{n}{4} \rceil$. Suppose we put some vertices of $S$ outside $G$ and some vertices of $S$ inside $G$. Without loss the generality, let $j$ be multiples of four in $N$ and let $l$ be multiples of three in $N$. Let $v_i^k, v_{i+4}^k, \ldots v_{i+j}^k$ and $v_{i-3}, \ldots v_{i-l}$ in $S$. Then there must be minimum an outside vertex $(v_{i+1}^k)$ adjacent an inside vertex $(v_{i+2})$ which have same $r = (22\ldots2)$.

Based on three points above, we focus in the first point of case. As we see that all of the adjacency representation of adjacent vertices are distinct. So, $S = \{v_1, v_2, \ldots, v_{\lceil \frac{n}{4} \rceil}\}$ is a local adjacency resolving set for $G$. The cardinality of $S$, $|S| = \lceil \frac{n}{4} \rceil$ is minimum, because if $|S| < \lceil \frac{n}{4} \rceil$ certainly there are $x \neq y \in V(G) - S$ such that $r(x|S) = r(y|S)$. Suppose $S_1 = \{v_{\lceil \frac{n}{4} \rceil}\}$ where $|S_1| = \lceil \frac{n}{4} \rceil - 1 < \lceil \frac{n}{4} \rceil$. Then, $r(v_{i-1}|S) = (22\ldots2) = r(v_{i-2}|S)$ and $v_{i-1}$adjacents with $v_{i-2}$. Thus, $\dim_{A,L}(G) = \lceil \frac{n}{4} \rceil$ for $n \geq 9$.

2.2. $m-$Splitting of Path Graph

A $m-$splitting of path graph ($mSpl(P_n)$) is a graph obtained from a path graph ($P_n$) by adding new vertex $v'$ in every vertex $v$ as $n$ such that $v'$ adjacent $v$ in $P_n$. $m-$splitting graph is graph which has the number of vertex $v'$ as $m$. Let $G = mSpl(P_n)$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_t\} \cup \{u_{1}^{k}, u_{2}^{k}, \ldots, u_{t}^{k}\}$, where $u_i$ is vertex of $P_n$ and $u_i^k$ is copy of vertex $u_i$ outside $P_n$ for $i \in \{1,2,\ldots,n\}$ and $k \in \{1,2,\ldots,m\}$. We can see at Figure 2 as illustration.
Theorem 2.2 Let $G$ be $m-$splitting of path graph $(m, Spl(P_n))$ with $|V(G)| = 2n$. For $n \geq 2$ and $m, n \in N$, then
\[
\dim_{A,l}(G) = \left\lfloor \frac{n-1}{4} \right\rfloor
\]

Proof 2.2 Choose $S = \{u_1, u_2, \ldots, u_{\left\lfloor \frac{n-1}{4} \right\rfloor} \} \subseteq V(G)$. We will show that $S$ is a local adjacency resolving set of $G$. We know that $d$ is defined by
\[
d(u, w) = \begin{cases} 
0 & \text{if } v = w; \\
1 & \text{if } v \text{ is adjacent with } w; \\
2 & \text{if } v \text{ does not adjacent with } w.
\end{cases}
\]

Based on the construction of $P_n$, we must have a maximum 3 vertices ($d = 2$) between two resolving vertices in $P_n$. For the vertex outside $P_n$ ($m$-split), since every vertex in $(u_i^k)$ has neighbour as $(u_i)$ then the $r$ of $(u_i^k)$ is same as $(u_i)$ except vertex which is a copy of every vertex in $S$. Since distance of every vertex in $S$ is more than and equal 2, so we can ensure that every $u_i^k$ which is a copy of every vertex in $S$ has $r = (22 \ldots 2)$. Then it ensures that all vertices in $S$ are distinct.

We see that all of the adjacency representations of adjacent vertices are distinct. So, $S = \{u_1, u_2, \ldots, u_{\left\lfloor \frac{n-1}{4} \right\rfloor} \}$ is a local adjacency resolving set for $G$. The cardinality of $S$, $|S| = \left\lceil \frac{n-1}{4} \right\rceil$ is minimum, because if $|S| < \left\lceil \frac{n-1}{4} \right\rceil$ certainly there are $b \neq c \in V(G) - S$ such that $r(b) = r(c)(S)$. Suppose $S_1 = \{u_{\left\lceil \frac{n-1}{4} \right\rceil} \}$ where $|S| = \left\lceil \frac{n-1}{4} \right\rceil - 1 < \left\lceil \frac{n-1}{4} \right\rceil$. Then, $r(u_{i-1}|S) = (22 \ldots 2) = u(v_{i-2}|S)$ and $u_{i-1}$ adjacent with $u_{i-2}$. Thus, $\dim_{A,l}(G) = \left\lceil \frac{n-1}{4} \right\rceil$.

2.3. $m-$Splitting of Ladder Graph
A $m-$splitting of ladder graph $(m, Spl(L_n))$ is a graph obtained from a ladder graph ($L_n$) by adding new vertex $v'$ in every vertex $v$ as $n$ such that $v'$ adjacent $v$ in $L_n$. $m-$splitting graph is graph which has the number of vertex $v'$ as $m$. Let $G = m, Spl(L_n)$ with vertex set $V(G) = \{u_1, u_2, \ldots, u_{\left\lfloor \frac{n-1}{4} \right\rceil} \} \cup \{v_1, v_2, \ldots, v_{\left\lfloor \frac{n-1}{4} \right\rceil} \} \cup \{u_1^k, u_2^k, \ldots, u_{\left\lfloor \frac{n-1}{4} \right\rceil}^k \} \cup \{v_1^k, v_2^k, \ldots, v_{\left\lfloor \frac{n-1}{4} \right\rceil}^k \}$, where $u_i, v_j$ is vertex of $TL_n$ and $u_i^k, v_j^k$ is copy of vertex $u_i$ and $v_j$ around $TL_n$ for $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, m\}$. We can see at Figure 2.3 as illustration.

Theorem 2.3 Let $G$ be $m-$splitting of ladder graph $(m, Spl(L_n))$ with $|V(G)| = 2nm$. For $n \geq 3$ and $m, n \in N$, then
\[
\dim_{A,l}(G) = \begin{cases} 
2 & \text{for } 3 \leq n \leq 6; \\
\left\lceil \frac{n}{3} \right\rceil & \text{for } n \geq 7.
\end{cases}
\]

Proof 2.3 We divide this proof till some cases.

- Case 1. For $3 \leq n \leq 6$.
  Choose $S = \{u_i, v_j\} \subseteq V(G)$ where $u_i$ does not adjacent with $v_j$ and distance between $u_i - u_{i-1}$ or $v_j - v_{j-1}$ is a maximum 5 vertices. We will show that $S$ is a local adjacency resolving set of $G$. Since we have $\dim_{A,l}(G) = 2$, then we just have two digits of $r$. For the combination of the biggest possibility in $L_n$ is $\{12, 22, 21\}$ because the distance between $u_i$ and $v_j$ is 2 vertices but in the opposite side (based on construct of $L_n$). For the vertex around $L_n$ ($m-$split), since every vertex in $(u_j^k)$ has neighbour as $(v_{j-1}^k)$ or $(u_{i+1})$ then the $r$ of $(u_j^k)$ is same as $r$ of $(v_j)$. Next, the $r$ of $u_j^k$ is same even they are not adjacent (neighbors). Thus, it ensures that every adjacent vertex in $V(G)$ has distinct $r$. For your illustration, see the local adjacency representations of vertices from $V(G) - S$ are as follow:
(i) for $n = 3$,
\[ r_A(u_1 | S) = r_A(u_3 | S) = r_A(u_4 | S) = r_A(u_5 | S) = (12) \]
\[ r_A(v_1 | S) = r_A(v_2 | S) = r_A(v_3 | S) = r_A(v_4 | S) = (21) \]

(ii) for $n = 4$,
\[ r_A(u_1 | S) = r_A(u_4 | S) = (12) \]
\[ r_A(u_3 | S) = r_A(u_5 | S) = r_A(v_2 | S) = r_A(v_3 | S) = (11) \]
\[ r_A(u_4 | S) = r_A(u_6 | S) = r_A(v_1 | S) = r_A(v_5 | S) = (22) \]
\[ r_A(v_4 | S) = r_A(v_7 | S) = (21) \]

(iii) for $n = 5$,
\[ r_A(u_1 | S) = r_A(u_4 | S) = r_A(u_5 | S) = r_A(u_6 | S) = r_A(v_2 | S) = r_A(v_3 | S) = (12) \]
\[ r_A(u_4 | S) = r_A(u_7 | S) = r_A(v_2 | S) = r_A(v_3 | S) = r_A(v_5 | S) = r_A(v_6 | S) = (21) \]
\[ r_A(u_5 | S) = r_A(v_1 | S) = r_A(v_7 | S) = (22) \]

(iv) for $n = 6$,
\[ r_A(u_1 | S) = r_A(u_4 | S) = r_A(u_5 | S) = r_A(u_6 | S) = r_A(v_2 | S) = r_A(v_3 | S) = r_A(v_5 | S) = (12) \]
\[ r_A(u_5 | S) = r_A(v_4 | S) = r_A(v_6 | S) = r_A(v_7 | S) = r_A(v_8 | S) = r_A(v_5 | S) = r_A(v_6 | S) = (21) \]
\[ r_A(u_4 | S) = r_A(u_7 | S) = r_A(u_6 | S) = r_A(v_1 | S) = r_A(v_5 | S) = r_A(v_6 | S) = r_A(v_7 | S) = (22) \]

As we see that all of the adjacency representations of adjacent vertices are distinct. So, \( S = \{ u_1, v_4 \} \) is a local adjacency resolving set for \( G \). The cardinality of \( S \), \( |S| = 2 \) is minimum, because if \( |S| < n \) certainly there are \( x \neq y \subset V(G) - S \) such that \( r(x | S) = r(y | S) \).

Suppose \( S_1 = \{ u_i \} \) where \( |S| = 1 < 2 \). Then, \( r(u_{i-1} | S) = (1) = r(u_{i-2} | S) \) and \( u_{i-1} \) adjacents with \( u_{i-2} \). Thus, \( \dim_{A_i}(G) = 2 \) for \( 3 \leq n \leq 6 \).

- For \( n \geq 7 \). Choose \( S = \{ w_1, w_2, \ldots, w_{\lfloor \frac{n}{3} \rfloor} \} \subset V(G) \). We will show that \( S \) is a local
adjacency resolving set of \( G \). We know that \( d \) is defined by

\[
d(u, w) = \begin{cases} 
0 & \text{if } v = w; \\
1 & \text{if } v \text{ adjacents with } w; \\
2 & \text{if } v \text{ does not adjacent with } w.
\end{cases}
\]

Suppose we call the "inside" vertices of \( G \) is the set of vertices in \( L_n \) and the "outside" vertices of \( G \) is the set of vertices outside \( L_n \) (or in \( m \)-split of \( L_n \)). Then, there are three cases to prove the theorem, such that:

(i) When the resolving vertices set are inside the \( G \). Choose resolving vertices of \( S \) as much as \( \lceil \frac{n}{3} \rceil \). Suppose we put any vertices of \( S \) inside \( G \). Based on construction of \( L_n \), suppose we have maximum 5 vertices \( (d = 6) \) between two resolving inside vertices, where: if we put any \( u_i \) and \( u_{i+1} \), then we start put \( v_j \) in the opposite middle vertex between \( u_i - u_{i-1} \) and \( v_{j-1} \) is put with distance maximum 5 vertices after \( v_j \), and otherwise. Since every vertex in \( (u^k_i) \) has neighbour as \( (u_i-1 \) and \( u_{i+1} \) then the \( r \) of \( (u^k_i) \) is same as \( (u_i) \) except vertex which is a copy of every vertex in \( S \). Since distance of every vertex in \( S \) is more than and equal 6, so we can ensure that every \( u^k_i \) which is a copy of every vertex in \( S \) has \( r = (22\ldots2) \). Then there ensures that all vertices in \( S \) are distinct. Then every adjacency \( v_i|S \) must has different \( r \).

(ii) When the resolving vertices set are outside the \( G \). Choose resolving vertices of \( S \) as much as \( \lceil \frac{n}{3} \rceil \). Suppose we put any vertices of \( S \) outside \( G \). Without loss the generality, let \( t \) be even number of \( N \). Let \( u^k_i, u^k_i+2, \ldots, u^k_i + t \), in \( S \). Then there must be minimum an outside vertex \( (u^k_i+1) \) adjacent an inside vertex \( (u_i) \) which have same \( r = (22\ldots2) \).

(iii) When some resolving vertices are inside the \( G \) and the others are outside the \( G \). Choose resolving vertices of \( S \) as much as \( \lceil \frac{n}{3} \rceil \). Suppose we put some vertices of \( S \) outside \( G \) and some vertices of \( S \) inside \( G \). Without loss the generality, let we have maximum 5 vertices \( (d = 6) \) between two resolving vertices \( (u_i \) and \( u_{i-(n-1)} \)) inside \( G \). Then we put another resolving vertex outside \( G \) in the middle vertex between \( u_i \) and \( v_j \), then, there must be minimum an inside vertex \( (v_{j-1}) \) which adjacent with vertex \( (v_{j-2}) \) has same \( r = (22\ldots2) \).

Based on three points above, we focus in the first point of case. As we see that all of the adjacency representation of adjacent vertices are distinct. So, \( S = \{w_1, w_2, \ldots, w_{\lceil \frac{n}{3} \rceil} \} \) is a local adjacency resolving set for \( G \). The cardinality of \( S \), \( |S| = \lceil \frac{n}{3} \rceil \) is minimum, because if \( |S| < \lceil \frac{n}{3} \rceil \) certainly there are \( x \neq y \in V(G) - S \) such that \( r(x|S) = r(y|S) \). Suppose \( S_1 = \{w_{\lceil \frac{n}{3} \rceil} \} \) where \( |S| = \lceil \frac{n}{3} \rceil - 1 < \lceil \frac{n}{3} \rceil \). Then, \( r(v_{j-1}|S) = (22\ldots2) = r(v_{j-2}|S) \) and \( v_{j-1} \) adjacent with \( v_{j-2} \). Thus, \( \text{dim}_{A,t}(G) = \lceil \frac{n}{3} \rceil \) for \( \geq 7 \).

2.4. \( m \)-Splitting of Triangular Ladder Graph

A \( m \)-splitting of triangular ladder graph \( (m_{\text{Spl}}(TL_n)) \) is a graph obtained from a triangular ladder graph \( (TL_n) \) by adding new vertex \( v' \) in every vertex \( v \) as \( n \) such that \( v' \) adjacent \( v \) in \( TL_n \). \( m \)-splitting graph is graph which has the number of vertex \( v' \) as \( m \). Let \( G = m_{\text{Spl}}(TL_n) \) with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_k\} \cup \{v'_1, v'_2, \ldots, v'_k\} \), where \( u, v_i \) is vertex of \( TL_n \) and \( u^k_i, v^k_i \) is copy of vertex \( u_i \) and \( v_i \) around \( TL_n \) for \( i \in \{1, 2, \ldots, n\} \) and \( k \in \{1, 2, \ldots, m\} \). We can see at Figure 2.4 as illustration.

Theorem 2.4 Let \( G \) be \( m \)-splitting of triangular ladder graph \( (m_{\text{Spl}}(TL_n)) \) with \( |V(G)| = 2nm \). For \( n \geq 3 \) and \( m, n \in N \), then

\[
\text{dim}_{A,t}(G) = n - 1
\]
Figure 4. \( _1\text{Spl}(TL_4) \) Graph

**Proof 2.4** Choose \( S = \{u_2, u_4, \ldots, u_n\} \cup \{v_2, v_4, \ldots, v_n\} \subset V(G) \). We will show that \( S \) is a local adjacency resolving set of \( G \). The local adjacency representations of vertices from \( V(G) - S \) are as follow:

\[
\begin{align*}
    r_A(u_1|S) &= r_A(u_1^k|S) = (112\ldots2) \quad \text{for any } n \\
    r_A(u_2|S) &= r_A(u_3^k|S) = (12112\ldots2) \\
    r_A(u_3|S) &= r_A(u_4^k|S) = (212112\ldots2) \\
    r_A(u_4|S) &= r_A(u_5^k|S) = (2212112\ldots2) \\ 
    \vdots
\end{align*}
\]

\[
\begin{align*}
    r_A(v_1|S) &= r_A(v_1^k|S) = (2\ldots212112\ldots2) \\
    r_A(v_2|S) &= r_A(v_3^k|S) = (2\ldots2121\ldots2) \\
    r_A(v_3|S) &= r_A(v_4^k|S) = (2\ldots212\ldots2) \\
    r_A(v_4|S) &= r_A(v_5^k|S) = (2\ldots21\ldots2) \\
    \vdots
\end{align*}
\]

As we see that all of the adjacency representations of adjacent vertices are distinct. So, \( S = \{u_1, u_2, \ldots, u_{n-1}\} \) is a local adjacency resolving set for \( G \). The cardinality of \( S \), \( |S| = n-1 \) is minimum, because if \( |S| < n-1 \) certainly there are \( a \neq b \in V(G) - S \) such that \( r(a|S) = r(b|S) \).
Suppose \( S_1 = \{ u_1, u_2, \ldots, u_{n-2} \} \), \(|S| = n - 2 < n - 1\). Then, \( r_A(u_i|S) = (11 \ldots 1) = r_A(u_{i-1}|S) \) and \( u_i \) adjacent with \( u_{i-1} \). Thus, \( \dim_{A,l}(G) = n - 1 \).

3. Concluding Remark
We have discussed about the local adjacency metric of \( m\)-splitting graphs for several sets of value \((m, n)\) in this paper. Four basic theorems are about cycle graph, path graph, ladder graph and triangular ladder graph which have any solutions for being basic graphs of operation \( m\)-splitting.

Open Problem
Find local adjacency metric of \( m\text{Spl}(H_n) \) graph for any \( n \) and \( m \) where \( H \) is any graph.

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