SOME NEW DESIGNS WITH PRESCRIBED AUTOMORPHISM GROUPS

VEDRAN KRČADINAC

ABSTRACT. We establish the existence of simple designs with parameters 2-(55, 10, 4), 3-(20, 5, 4), 3-(21, 7, 30), 4-(15, 5, 2), 4-(16, 8, 45), 5-(16, 7, 10), and 5-(17, 8, 40), which have previously been unknown. For the corresponding $t$, $v$, and $k$, we study the set of all $\lambda$ for which simple $t-(v, k, \lambda)$ designs exist.

1. Introduction

A $t-(v, k, \lambda)$ design $\mathcal{D}$ is a $v$-element set $\mathcal{P}$ of points together with a collection $\mathcal{B}$ of $k$-element subsets called blocks, such that every $t$-element subset of points is contained in exactly $\lambda$ blocks. The design is simple if $\mathcal{B}$ is a set, i.e. contains no repeated blocks. We refer to the Handbook of Combinatorial Designs [4] for definitions and results about designs. In this paper we are concerned with the existence problem for simple designs with small parameters.

An automorphism of $\mathcal{D}$ is a permutation of points leaving $\mathcal{B}$ invariant. The set of all automorphisms forms a group under composition, the full automorphism group $\text{Aut}(\mathcal{D})$. By prescribing suitable subgroups $G \leq \text{Aut}(\mathcal{D})$, we are able to construct simple designs with parameters 2-(55, 10, 4), 3-(20, 5, 4), 3-(21, 7, 30), 4-(15, 5, 2), 4-(16, 8, 45), 5-(16, 7, 10), and 5-(17, 8, 40). These designs are designated as unknown in [12, Table 1.35] and [7, Table 4.46]. Since the Handbook was published, new existence results about designs with small parameters have appeared in [1], [2], [15], [16], and [17]. To the best of our knowledge, existence of the constructed designs has previously been open.

The layout of our paper is as follows. In the next section we describe the construction method, the used computational tools, and some other preliminary matters. Sections 3 to 9 are dedicated to designs with particular $t$, $v$, and $k$. We present our new constructions and try to...
determine the set of all $\lambda$ such that simple $t-(v, k, \lambda)$ designs exist. This is accomplished for 4-$(15, 5, \lambda)$ designs in Section 6, and in the other sections up to three open cases remain. The prescribed groups $G$ and some computational details are laid out in the proofs of the theorems. Designs are presented by listing base blocks; $\mathcal{B}$ is the union of the corresponding $G$-orbits. The groups and the base blocks for the new designs are also available on our web page:

https://web.math.pmf.unizg.hr/~krcko/results/newdesigns.html

2. Preliminaries

Let $G$ be a group of permutations of $\mathcal{P} = \{1, \ldots, v\}$, and let $\mathcal{T}_1, \ldots, \mathcal{T}_m$ and $\mathcal{K}_1, \ldots, \mathcal{K}_n$ be the orbits of $t$-element subsets and $k$-element subsets of $\mathcal{P}$, respectively. Let $a_{ij}$ be the number of subsets $K \in \mathcal{K}_j$ containing a given $T \in \mathcal{T}_i$. This number does not depend on the choice of $T$. The matrix $A = [a_{ij}]$ is the Kramer-Mesner matrix. It is well known that simple $t-(v, k, \lambda)$ designs with $G$ as an automorphism group exist if and only if the system of linear equations $A \cdot x = \lambda j$ has 0-1 solutions $x \in \{0, 1\}^n$. Here, $j = (1, \ldots, 1)^T$ is the all-one vector of length $m$. This method of construction was pioneered by E. S. Kramer and D. M. Mesner [8] and has since been used to find many new designs with prescribed automorphism groups (see, e.g., [2], [3], [9], [10], [11], and [18]).

We use GAP [6] to compute the orbits $\mathcal{T}_i$, $\mathcal{K}_j$ and the matrix $A$. For most of our results this is a small and easy computation. Exceptions are Theorems 3.1 and 3.2 where we use an algorithm described in [10] to produce short orbits, and a program written in C for long orbits of the group $G_3 \cong M_{11}$. The second step of the computation is finding solutions of the Kramer-Mesner system $A \cdot x = \lambda j$. Solving systems of linear equations over $\{0, 1\}$ is a known NP complete problem. Our prescribed automorphism groups $G$ lead to systems of small to moderate size, that can be solved fairly quickly. We use the backtracking solver developed in [9] for 2-designs and A. Wasserman’s solver [18] based on the LLL algorithm for designs with $t \geq 3$. Running times varied from a few seconds to several days of CPU time. Finally, to decide whether the constructed designs are isomorphic and to compute their full automorphism groups, we use nauty/Traces by B. D. McKay and A. Piperno [13].

A $t-(v, k, \lambda)$ design $\mathcal{D}$ is also a $s-(v, k, \lambda_s)$ design, for $\lambda_s = \lambda \frac{(v-s)}{t-s}$; $0 \leq s \leq t$. The number of blocks of $\mathcal{D}$ is $b = |\mathcal{B}| = \lambda_0$. The parameters $t-(v, k, \lambda)$ are called admissible if all the $\lambda_s$ are integers, and realizable if simple designs with these parameters exists. Given $t$, $v$, and $k$, there
is a least integer $\lambda_{\text{min}}$ such that $t-(v, k, \lambda_{\text{min}})$ are admissible. Any $\lambda$ for which $t-(v, k, \lambda)$ are admissible is of the form $\lambda = m\lambda_{\text{min}}$, for $m \in \mathbb{N}$.

The largest $\lambda$ for which a simple $t-(v, k, \lambda)$ design exists is $\lambda_{\text{max}} = \binom{v-t}{k-t}$.

The corresponding complete $t-(v, k, \lambda_{\text{max}})$ designs contains all $k$-element subsets as blocks: $\mathcal{B} = \binom{P}{k}$.

For the parameters of our newly constructed designs, we try to determine all $\lambda$ between $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ such that $t-(v, k, \lambda)$ are realizable.

The supplement of a $t-(v, k, \lambda)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is the design $\mathcal{D}' = (\mathcal{P}, \binom{P}{k} \setminus \mathcal{B})$ with parameters $t-(v, k, \lambda_{\text{max}} - \lambda)$. Therefore it suffices to consider existence of $t-(v, k, \lambda)$ designs for $\lambda \leq \lambda_{\text{max}}/2$. We shall denote the largest integer $m$ such that $m\lambda_{\text{min}} \leq \lambda_{\text{max}}/2$ by $M$.

The complement of $\mathcal{D}$, obtained by taking the complement of each block in $\mathcal{P}$, is also a $t$-design and therefore it suffices to consider parameters with $k \leq v/2$.

3. Designs with parameters $2-(55, 10, \lambda)$

Let $t = 2$, $v = 55$, and $k = 10$. Then $\lambda_{\text{min}} = 1$, $\lambda_{\text{max}} = \binom{53}{8} = 886322710$, and $M = 443161355$. A $2-(55, 10, 1)$ design would have $b = 33$ blocks and cannot exist by Fisher's inequality [12, Theorem 1.9].

Designs $2-(55, 10, 2)$ are quasi-residuals of symmetric $2-(67, 12, 2)$ designs, which do not exist by the Bruck-Ryser-Chowla theorem [4, Theorem II.6.11]. By the Hall-Connor theorem [4, Theorem II.6.27], quasi-residual designs with $\lambda = 2$ are actually residual, hence $2-(55, 10, 2)$ designs also do not exist. According to [12, Table 1.35], $2-(55, 10, m)$ designs exist for $m = 5$, and are unknown for $m \in \{3, 4, 6\}$.

**Theorem 3.1.** Simple $2-(55, 10, 4)$ designs exist.

**Proof.** Let $G_1 \cong PSL(2, 11)$ be the group of order 660 generated by the permutations

\[(1, 2, \ldots, 11)(12, \ldots, 22)(23, \ldots, 33)(34, \ldots, 44)(45, \ldots, 55) \quad (1)\]

and

\[(2, 27)(3, 37)(4, 29)(5, 47)(6, 53)(7, 30)(8, 38)(9, 32)(11, 43)(13, 52) \]
\[(15, 48)(17, 28)(18, 42)(19, 49)(20, 51)(21, 44)(22, 31)(23, 46)(24, 50) \]
\[(25, 54)(26, 33)(34, 45)(36, 39)(41, 55).\]

The group $G_1$ has 6 orbits on 2-element subsets of $\mathcal{P} = \{1, \ldots, 55\}$. It suffices to consider orbits of 10-element subsets whose size does not exceed the number of blocks $b = 132$. This can be accomplished efficiently by an algorithm described in [10]; there are 97 such orbits. The $6 \times 97$ Kramer-Mesner system has 5 solutions for $\lambda = 4$, giving rise to three non-isomorphic designs. Two of them have $G_1$ as
their full automorphism group. They are generated by the base blocks \( \{1, 2, 5, 8, 10, 12, 23, 42, 45, 50\} \) and \( \{1, 2, 3, 9, 18, 27, 28, 33, 38, 43\} \), respectively. The third design has \( \text{Aut}(D) \cong G_1 \cdot Z_2 \cong PGL(2, 11) \) and is generated by the two base blocks \( \{1, 2, 4, 5, 16, 18, 24, 30, 51, 55\} \) and \( \{1, 2, 6, 8, 24, 27, 38, 40, 50, 53\} \). □

Designs 2-(55, 10, 5) can be constructed from the group \( G_2 \cong Z_{55} \cdot Z_{10} \).

There are five non-isomorphic designs with \( G_2 \) as their full automorphism group.

**Theorem 3.2.** Simple 2-(55, 10, m) designs exist for \( m \in \{4, 5, 8, \ldots, M\} \).

**Proof.** For \( m \in \{8, \ldots, 58\} \), designs can be constructed from the group \( G_1 \). For example, 2-(55, 10, 8) designs are obtained as unions of any two of the designs from Theorem 3.1 since they are disjoint. By the Kramer-Mesner method we found designs for all \( m \) in the specified range. Base blocks are available on the web page referred to in the Introduction.

For \( m \geq 59 \), we use the group \( G_3 \cong M_{11} \) of order 7920 generated by the permutations \( [11] \) and

\[
(3, 23)(4, 20)(5, 10)(6, 45)(7, 34)(8, 33)(11, 12)(13, 26)(14, 15)(16, 53)
(17, 43)(18, 28)(19, 52)(21, 49)(22, 47)(24, 46)(25, 32)(27, 51)(29, 42)
(31, 41)(36, 39)(38, 48)(40, 54)(44, 50).
\]

Orbits of size less than \( |G_3| \) were computed by the algorithm from [10]. There are 367 orbits of size less than 3960. Using these orbits and the Kramer-Mesner method, we found designs for \( m \in \{59, \ldots, 479\} \). Base blocks are available on our web page. There are 13753 orbits of size 3960, of which 1647 form the blocks of 2-(55, 10, 120) designs. These 1647 designs are mutually disjoint and disjoint from the previously constructed designs. Finally, we used a program written in C to find the 3686048 long orbits of size 7920. Among them, 555578 orbits form disjoint 2-(55, 10, 240) designs, and 1256273 pairs of long orbits can be combined into as many disjoint 2-(55, 10, 480) designs. It is clear that by taking unions of the so far constructed designs, simple 2-(55, 10, m) design with \( G_3 \) as an automorphism group can be constructed for any \( m \in \{59, \ldots, M\} \). □

Recently, D. Crnković and A. Švob [5] also found 2-(55, 10, m) designs for \( m \in \{4, 10\} \). Thus, the only remaining open cases are \( m \in \{3, 6, 7\} \). We tried to construct these designs using various prescribed automorphism groups, but did not find any examples.
4. Designs with parameters $3-(20, 5, \lambda)$

For $t = 3$, $v = 20$, and $k = 5$, we have $\lambda_{\min} = 2$, $\lambda_{\max} = \binom{17}{2} = 136$, and $M = 34$. According to [7, Table 4.46], $3-(20, 5, 2m)$ designs exist for $m \in \{3, \ldots, 6, 8, \ldots, 34\}$. We found designs for two of the three missing values of $m$.

**Theorem 4.1.** Simple $3-(20, 5, 2m)$ designs exist for $m \in \{2, 7\}$.

**Proof.** Let $G$ be the dihedral group of order 38, generated by the cycle $(1, \ldots, 19)$ and the involution

$$(2, 19)(3, 18)(4, 17)(5, 16)(6, 15)(7, 14)(8, 13)(9, 12)(10, 11).$$

There are 39 orbits of 3-element subsets and 444 orbits of 5-element subsets of $P = \{1, \ldots, 20\}$. One solution of the Kramer-Mesner system for $\lambda = 4$ is given by the first 12 base blocks in Table 1. The next 39 base blocks generate a $3-(20, 5, 10)$ design, disjoint from the $3-(20, 5, 4)$ design. The union of these two design is a simple $3-(20, 5, 14)$ design. $\square$

| $\lambda$ | Base blocks |
|-----------|-------------|
| 4         | $\{1, 2, 3, 5, 13\}$, $\{1, 2, 3, 8, 10\}$, $\{1, 2, 4, 6, 12\}$, $\{1, 2, 4, 8, 17\}$, $\{1, 2, 4, 10, 20\}$ |
| 10        | $\{1, 2, 5, 7, 15\}$, $\{1, 2, 5, 9, 16\}$, $\{1, 2, 6, 8, 20\}$, $\{1, 2, 6, 9, 11\}$, $\{1, 2, 7, 11, 14\}$ |
| 14        | $\{1, 3, 6, 9, 15\}$, $\{1, 4, 9, 13, 20\}$ |
| 12        | $\{1, 2, 3, 5, 12\}$, $\{1, 2, 3, 6, 16\}$, $\{1, 2, 3, 6, 17\}$, $\{1, 2, 3, 7, 12\}$ |
| 11        | $\{1, 2, 3, 7, 16\}$, $\{1, 2, 4, 5, 9\}$, $\{1, 2, 4, 9, 11\}$, $\{1, 2, 4, 9, 15\}$, $\{1, 2, 4, 9, 20\}$ |
| 13        | $\{1, 2, 4, 11, 16\}$, $\{1, 2, 4, 11, 18\}$, $\{1, 2, 4, 18, 20\}$, $\{1, 2, 5, 8, 9\}$, $\{1, 2, 5, 8, 12\}$ |
| 15        | $\{1, 2, 5, 8, 15\}$, $\{1, 2, 5, 8, 16\}$, $\{1, 2, 6, 14, 15\}$, $\{1, 2, 7, 12, 13\}$, $\{1, 2, 7, 12, 14\}$ |
| 16        | $\{1, 2, 8, 9, 20\}$, $\{1, 2, 8, 10, 12\}$, $\{1, 2, 8, 12, 20\}$, $\{1, 2, 9, 10, 20\}$, $\{1, 2, 9, 11, 13\}$ |
| 20        | $\{1, 2, 9, 13, 20\}$, $\{1, 3, 5, 9, 12\}$, $\{1, 3, 5, 11, 14\}$, $\{1, 3, 6, 8, 14\}$, $\{1, 3, 6, 8, 20\}$ |
| 17        | $\{1, 3, 6, 13, 16\}$, $\{1, 3, 6, 14, 16\}$, $\{1, 3, 6, 16, 20\}$, $\{1, 3, 6, 17, 20\}$, $\{1, 3, 7, 16, 20\}$ |
| 19        | $\{1, 3, 8, 11, 14\}$, $\{1, 4, 7, 11, 16\}$, $\{1, 4, 10, 14, 20\}$, $\{1, 5, 10, 14, 20\}$ |

**Table 1.** Base blocks for $3-(20, 5, \lambda)$ designs, $\lambda \in \{4, 10, 14\}$.

**Corollary 4.2.** Simple $3-(20, 5, 2m)$ designs exist for $m \in \{2, \ldots, 34\}$.

In fact, designs can be constructed from the same group $G$ for all $m \in \{2, \ldots, 34\}$. The only open case is now $m = 1$. We did not find any $3-(20, 5, 2)$ designs by prescribing automorphism groups. We examined the subgroups of $PGL(2,19)$ operating on 20 points and of $AGL(1,19)$ operating on 19 points, of orders greater than 19, systematically.
5. Designs with parameters 3-(21, 7, \(\lambda\))

For \(t = 3\), \(v = 21\), and \(k = 7\), we have \(\lambda_{\text{min}} = 15\), \(\lambda_{\text{max}} = \binom{18}{4} = 3060\), and \(M = 102\). According to [7 Table 4.46], 3-(21, 7, 15\(m\)) designs exist for 46 values of \(m\). Existence is unknown for 56 values of \(m\), starting with \(m \in \{1, 2, 5, 7, \ldots\}\). We found designs for all but the first of these unknown values.

**Theorem 5.1.** Simple 3-(21, 7, 15\(m\)) designs exist for \(m \in \{2, \ldots, 102\}\).

**Proof.** Let \(G_1 \cong A_6\) be the group of order 360 generated by the permutations

\[
(2, 3, 4, 5, 6)(7, 8, 9, 10, 11)(12, 13, 14, 15, 16)(17, 18, 19, 20, 21)
\]

and

\[
(1, 5, 2)(3, 4, 6)(9, 18, 16)(10, 13, 12)(11, 14, 19)(15, 17, 20).
\]

There are 12 orbits of 3-element subsets and 406 orbits of 7-element subsets of \(P = \{1, \ldots, 21\}\). The Kramer-Mesner system has 56 solutions for \(\lambda = 30\), giving rise to 28 non-isomorphic designs. All of them have \(G_1\) as their full automorphism group. Base blocks for one of the designs are the first 10 sets in Table 2.

The group \(G_2 \cong S_6\) of order 720 generated by the permutations

\[
(1, 4)(2, 6)(3, 5)(10, 12)(11, 14)(15, 20)(16, 18)
\]

and

\[
(1, 2, 3, 4, 7, 8, 20)
\]

can be used for \(m \geq 3\). The Kramer-Mesner system is of size \(11 \times 253\). We checked that solutions exist for all \(\lambda = 15m\), \(m \in \{3, \ldots, 102\}\). Base blocks for \(m = 5\) are the next 17 sets in Table 2. Base blocks for the other cases are available on our web page. \(\square\)

\begin{table}[h]
\centering
\begin{tabular}{lll}
\hline
\{1, 2, 3, 4, 5, 6, 7\} & \{1, 2, 3, 7, 8, 13, 17\} & \{1, 2, 3, 7, 9, 11, 14\} \\
\{1, 2, 3, 7, 9, 12, 21\} & \{1, 2, 7, 8, 10, 16, 21\} & \{1, 2, 7, 8, 11, 14, 20\} \\
\{1, 2, 7, 10, 16, 19, 20\} & \{1, 7, 8, 12, 13, 18, 19\} & \{1, 7, 8, 15, 18, 19, 21\} \\
\{1, 2, 3, 4, 7, 8, 20\} & \{1, 2, 3, 4, 7, 14, 15\} & \{1, 2, 3, 7, 8, 10, 12\} \\
\{1, 2, 3, 9, 13, 15, 20\} & \{1, 2, 7, 8, 9, 10, 16\} & \{1, 2, 7, 8, 9, 13, 19\} \\
\{1, 2, 7, 8, 9, 15, 18\} & \{1, 2, 7, 8, 11, 14, 17\} & \{1, 2, 7, 9, 12, 17, 20\} \\
\{1, 2, 7, 10, 11, 16, 17\} & \{1, 2, 7, 10, 11, 19, 20\} & \{1, 2, 7, 10, 16, 19, 20\} \\
\{1, 7, 8, 9, 10, 12, 15\} & \{1, 7, 8, 9, 17, 19, 21\} & \{7, 8, 9, 10, 12, 13, 21\} \\
\{7, 8, 9, 10, 17, 18, 20\} & \{7, 8, 9, 13, 17, 19, 21\} & \\
\hline
\end{tabular}
\caption{Base blocks for a 3-(21, 7, 30) design with \(G_1\) and a 3-(21, 7, 75) design with \(G_2\) as automorphism group.}
\end{table}
We did not find any 3-(21, 7, 15) designs, and the existence problem is still open.

6. Designs with parameters 4-(15, 5, \( \lambda \))

For \( t = 4 \), \( v = 15 \), and \( k = 5 \), we have \( \lambda_{\text{min}} = 1 \), \( \lambda_{\text{max}} = 11 \), and \( M = 5 \). It is known that 4-(15, 5, \( m \)) designs do not exist for \( m = 1 \) \[14\] and exist for \( m \in \{3, 4, 5\} \) \[7\ Table 4.46\]. We settle the remaining case \( m = 2 \).

**Theorem 6.1.** Simple 4-(15, 5, 2) designs exist.

**Proof.** Let \( G \cong \mathbb{Z}_3 \times S_3 \) be the group of order 18 generated by the permutations

\[
(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15),
\]

\[
(1, 4)(2, 5)(3, 6)(8, 13)(9, 10)(11, 15).
\]

The Kramer-Mesner system is of size \( 84 \times 178 \) and has 12 solutions for \( \lambda = 2 \), giving rise to two non-isomorphic designs with \( \text{Aut}(\mathcal{D}) = G \). Base blocks for one of them are given in Table 3. \( \square \)

| \{1, 2, 3, 4, 5\} | \{1, 2, 3, 7, 11\} | \{1, 2, 4, 7, 11\} | \{1, 2, 4, 7, 14\} |
|-------------------|-------------------|-------------------|-------------------|
| \{1, 2, 4, 8, 10\} | \{1, 2, 4, 8, 11\} | \{1, 2, 4, 9, 12\} | \{1, 2, 4, 9, 15\} |
| \{1, 2, 4, 10, 13\} | \{1, 2, 4, 12, 14\} | \{1, 2, 4, 13, 15\} | \{1, 2, 7, 8, 9\} |
| \{1, 2, 7, 8, 14\} | \{1, 2, 7, 12, 15\} | \{1, 2, 10, 11, 12\} | \{1, 2, 10, 11, 15\} |
| \{1, 2, 13, 14, 15\} | \{1, 4, 7, 8, 12\} | \{1, 4, 7, 9, 11\} | \{1, 4, 8, 9, 14\} |
| \{1, 4, 8, 12, 13\} | \{1, 4, 8, 13, 14\} | \{1, 4, 9, 10, 11\} | \{1, 4, 11, 12, 15\} |
| \{1, 4, 11, 14, 15\} | \{1, 7, 8, 10, 11\} | \{1, 7, 8, 10, 13\} | \{1, 7, 8, 11, 15\} |
| \{1, 7, 8, 14, 15\} | \{1, 7, 10, 11, 14\} | \{1, 7, 10, 13, 15\} | \{1, 7, 11, 12, 14\} |
| \{1, 7, 12, 13, 14\} | \{1, 10, 11, 13, 15\} | \{7, 8, 9, 10, 13\} | \{7, 8, 11, 12, 13\} |

**Table 3.** Base blocks for a 4-(15, 5, 2) design.

For \( m \in \{3, 5\} \) designs can be constructed from the same group \( G \), and for \( m = 4 \) from the group \( G_1 \cong A_5 \) of order 60.

7. Designs with parameters 4-(16, 8, \( \lambda \))

For \( t = 4 \), \( v = 16 \), and \( k = 8 \), we have \( \lambda_{\text{min}} = 15 \), \( \lambda_{\text{max}} = \binom{12}{2} = 495 \), and \( M = 16 \). By \[7\ Table 4.46\], 4-(16, 8, 15\( m \)) designs exist for \( m \in \{4, \ldots, 16\} \). We found designs for \( m = 3 \).

**Theorem 7.1.** Simple 4-(16, 8, 45) designs exist.
Proof. Let $G \cong \mathbb{Z}_{15} \times (\mathbb{Z}_4 \times \mathbb{Z}_2)$ be the group of order 120 generated by the permutations
\[
(2,3)(4,5,6,7)(8,9,10,11)(12,13,14,15), \\
(1,5)(2,13)(3,11)(6,15)(7,8)(9,14)(10,12).
\]
The Kramer-Mesner system is of size $25 \times 132$ and has four solutions for $\lambda = 45$. They correspond to four non-isomorphic designs with $\text{Aut}(\mathcal{D}) = G$. Base blocks for one of them are given in Table 4.

| Base blocks for a 4-(16, 8, 45) design. |
|-----------------------------------------|
| \{1,2,3,4,5,6,8,11\} | \{1,2,3,4,5,6,14,16\} | \{1,2,3,4,5,8,9,14\} |
| \{1,2,3,4,5,8,12,13\} | \{1,2,3,4,5,10,12,14\} | \{1,2,3,4,5,11,15,16\} |
| \{1,2,3,4,6,8,10,16\} | \{1,2,3,4,6,13,15,16\} | \{1,2,4,5,6,7,9,11\} |
| \{1,2,4,5,6,7,13,15\} | \{1,2,4,5,6,9,11,15\} | \{1,2,4,5,6,9,11,16\} |
| \{1,2,4,5,6,10,13,16\} | \{1,2,4,5,7,8,9,16\} | \{1,2,4,5,8,12,15,16\} |

The same group $G$ can be used to construct 4-(16, 8, 15$m$) designs for $m \in \{4, \ldots, 16\}$. We tried many groups for $m \in \{1, 2\}$, but did not find any designs.

8. Designs with parameters 5-(16, 7, $\lambda$)

For $t = 5$, $v = 16$, and $k = 7$, we have $\lambda_{\text{min}} = 5$, $\lambda_{\text{max}} = \binom{11}{2} = 55$, and $M = 5$. By [7, Table 4.46], 5-(16, 7, 5$m$) designs exist for $m \in \{3, 4, 5\}$. Here we settle the case $m = 2$.

**Theorem 8.1.** Simple 5-(16, 7, 10) designs exist.

**Proof.** Let $G \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2).A_4$ be the group of order 192 generated by the permutations
\[
(2,3,4)(5,6,7,8,9,10)(11,12,13,14,15,16), \\
(1,5)(2,12)(3,15)(4,8)(6,14)(7,16)(9,10)(11,13).
\]
The Kramer-Mesner system is of size $28 \times 71$ and has two solutions for $\lambda = 10$. The two designs are isomorphic and have $\text{Aut}(\mathcal{D}) = G$. Base blocks are listed in Table 5.

The same group $G$ gives designs for $m = 5$, and for $m \in \{3, 4\}$ a subgroup of index 2 can be used. We did not find any designs for $m = 1$. 

□
Theorem 9.1. Simple 5-(17, 8, 40) designs exist.

Proof. Let $G \cong \mathbb{Z}_{17}.\mathbb{Z}_{16}$ be the group of order 272 generated by the cycle $(1, 2, \ldots, 17)$ and the permutation

$$(2, 4, 10, 11, 14, 6, 16, 12, 17, 15, 9, 8, 5, 13, 7).$$

The Kramer-Mesner system is of size $25 \times 95$. It has 61 solutions for $\lambda = 40$, giving rise to 61 non-isomorphic designs with $\text{Aut}(D) = G$. Base blocks for one of the designs are given in Table 6. □

| $\{1, 2, 3, 4, 5, 6, 13\}$ | $\{1, 2, 3, 4, 5, 6, 14\}$ | $\{1, 2, 3, 5, 6, 7, 11\}$ | $\{1, 2, 3, 5, 6, 8, 9\}$ |
|----------------------------|----------------------------|-----------------------------|-----------------------------|
| $\{1, 2, 3, 5, 6, 9, 10\}$  | $\{1, 2, 3, 5, 6, 9, 12\}$ | $\{1, 2, 3, 5, 6, 10, 15\}$ | $\{1, 2, 3, 5, 6, 14, 16\}$ |
| $\{1, 2, 3, 5, 8, 11, 12\}$ | $\{1, 2, 5, 6, 7, 8, 16\}$ | $\{1, 2, 5, 6, 7, 9, 14\}$ | $\{1, 2, 5, 6, 7, 12, 13\}$ |
| $\{1, 2, 5, 6, 7, 14, 15\}$ |

Table 6. Base blocks for a 5-(17, 8, 40) design.

The same group can be used for $m \in \{3, 4, 5\}$. The existence of 5-(17, 8, 20) designs remains open.

References

[1] M. Araya and M. Harada, Mutually disjoint Steiner systems $S(5, 8, 24)$ and 5-(24, 12, 48) designs, Electron. J. Combin. 17 (2010), no. 1, Note 1, 6 pp.

[2] M. Araya, M. Harada, V. D. Tonchev, and A. Wassermann, Mutually disjoint designs and new 5-designs derived from groups and codes, J. Combin. Des. 18 (2010), no. 4, 305–317.

[3] A. Betten, R. Laue, and A. Wassermann, Simple 7-designs with small parameters, J. Combin. Des. 7 (1999), no. 2, 79–94.
[4] C. J. Colbourn and J. H. Dinitz (eds.), Handbook of Combinatorial Designs, Second Edition, Chapman & Hall/CRC, Boca Raton, 2007.

[5] D. Crnković and A. Švob, Transitive $t$-designs constructed from linear groups $L(2, q)$, $q \leq 23$, preprint.

[6] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.7, 2017, http://www.gap-system.org.

[7] G. B. Khosrovshahi and R. Laue, $t$-designs with $t \geq 3$, in: The Handbook of Combinatorial Designs, Second Edition (eds. C. J. Colbourn and J. H. Dinitz), Chapman & Hall/CRC, Boca Raton, 2007, pp. 79–101.

[8] E. S. Kramer and D. M. Mesner, $t$-designs on hypergraphs, Discrete Math. 15 (1976), no. 3, 263–296.

[9] V. Krčadinac, A. Nakić, and M. O. Pevčević, The Kramer-Mesner method with tactical decompositions: some new unitals on 65 points, J. Combin. Des. 19 (2011), no. 4, 290–303.

[10] V. Krčadinac and R. Vlahović, New quasi-symmetric designs by the Kramer-Mesner method, Discrete Math. 339 (2016), no. 12, 2884–2890.

[11] D. L. Kreher and S. P. Radziszowski, Constructing 6-$(v, k, \lambda)$ designs, Finite geometries and combinatorial designs (Lincoln, NE, 1987), 137-151, Contemp. Math. 111, Amer. Math. Soc., Providence, RI, 1990.

[12] R. Mathon and A. Rosa, 2-$(v, k, \lambda)$ designs of small order, in: The Handbook of Combinatorial Designs, Second Edition (eds. C. J. Colbourn and J. H. Dinitz), Chapman & Hall/CRC, Boca Raton, 2007, pp. 25–58.

[13] B. D. McKay and A. Piperno, Practical graph isomorphism, II, J. Symbolic Comput. 60 (2014), 94-112.

[14] N. S. Mendelsohn and S. H. Y. Hung, On the Steiner systems $S(3, 4, 14)$ and $S(4, 5, 15)$, Utilitas Math. 1 (1972), 5–95.

[15] P. R. J. Östergård and O. Pottonen, There exists no Steiner system $S(4, 5, 17)$, J. Combin. Theory Ser. A 115 (2008), no. 8, 1570–1573.

[16] D. R. Stinson, C. M. Swanson, and T. van Trung, A new look at an old construction: constructing (simple) 3-designs from resolvable 2-designs, Discrete Math. 325 (2014), 23–31.

[17] T. van Trung, Simple $t$-designs: a recursive construction for arbitrary $t$, Des. Codes Cryptogr. 83 (2017), no. 3, 493–502.

[18] A. Wassermann, Finding simple $t$-designs with enumeration techniques, J. Combin. Des. 6 (1998), no. 2, 79–90.

E-mail address: vedran.krcadinac@math.hr

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, HR-10000 Zagreb, Croatia