Generalized Second Law of Thermodynamics of Evolving Wormhole with Entropy Corrections

Tanwi Bandyopadhyay · Ujjal Debnath · Mubasher Jamil · Faiz-ur-Rahman · Ratbay Myrzakulov

Received: 3 August 2014 / Accepted: 11 October 2014 / Published online: 31 October 2014
© Springer Science+Business Media New York 2014

Abstract In this work, we study the generalized second law of thermodynamics (GSL) at the apparent horizon of an evolving Lorentzian wormhole. We obtain the expressions of thermal variables at the apparent horizon. Choosing the two well-known entropy functions i.e. power-law and logarithmic, we obtain the expressions of GSL. We have analyzed the GSL using a power-law form of scale factor \( a(t) = a_0 t^n \) and the special form of shape function \( b(r) = b_0 r^\frac{1}{2} \). It is shown that GSL is valid in the evolving wormhole spacetime for both choices of entropies if the power-law exponent is small, but for large values of \( n \), the GSL is satisfied at initial stage and after certain stage of the evolution of the wormhole, it violates.

Keywords Wormhole · Thermodynamics

T. Bandyopadhyay (✉)
Department of Mathematics, Shri Shikshayatan College, 11, Lord Sinha Road, Kolkata-71, India
e-mail: tanwib@gmail.com

U. Debnath
Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah 711103, India
e-mail: ujjal@iucaa.ernet.in

M. Jamil · F.-u.-Rahman
Center for Advanced Mathematics and Physics (CAMP), National University of Sciences and Technology (NUST), H-12, Islamabad, Pakistan
M. Jamil
e-mail: mjamil@camp.nust.edu.pk

M. Jamil · R. Myrzakulov
Eurasian International Center for Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan
R. Myrzakulov
e-mail: rmyrzakulov@gmail.com
1 Introduction

Different theories of quantum gravity particularly string theory and loop quantum gravity predict the thermal nature of black hole horizons [1–3]. These theories confirm the semi-classical results of Bekenstein and Hawking about the evaporation of black holes as a result of emission of charged particles [4, 5]. In the early seventies, it was found that laws of thermodynamics are applicable to black holes horizons by defining new quantities like entropy \((S \sim A)\) and surface gravity \((\kappa \sim T)\) [6]. Hence the laws of thermodynamics for black holes arise from the geometry of spacetime. More recently in the last few years, several new connections between geometry/gravity and thermodynamics have been found. Jacobson found a link between the Einstein equation and the thermal relation \(\delta Q = TdS\), where left hand side represents the energy flux [7] (see excellent reviews on connections between gravity and thermodynamics [8–10]). Later it was proved that the thermodynamics of black holes is also valid for cosmological horizons [11] and the Friedmann equation can be written as a first law of thermodynamics [12–14]. While more recently it is proposed by Verlinde that gravity has an entropic origin and Newton’s law of gravitation arises naturally and unavoidably in a theory in which space is emergent through a holographic scenario [15].

Wormholes are tunnels in spacetime geometry that connect two or more regions of the same spacetime or two different spacetimes [16]. Wormholes are classified in two categories: Euclidean wormholes that arise in Euclidean quantum gravity and the Lorentzian wormholes which are static spherically symmetric solutions of Einstein’s general relativistic field equations [17]. Interest in Lorentzian wormholes arose in order to make them useful for time travel for human beings, hence termed Time Machines in popular literature [18]. In order to support such exotic wormhole geometries, the matter required to stabilize them has to be exotic i.e. violating the energy conditions (null, weak and strong), however it has been shown that averaged null energy condition is satisfied in wormhole geometries [19–21]. Also the violation of weak energy condition (WEC) can be avoided for small intervals of time [22]. More recently traversable wormhole and time machine solutions of the field equations of an alternative of gravity with non-minimally curvature-matter coupling have been obtained [23]. Wormholes have been constructed in some alternative theories of gravity including \(f(R)\), teleparallel, Kaluza-Klein, loop quantum gravity, Brans-Dicke gravity and Lovelock gravities, to name a few [24] and is found that modified gravity can minimize the amount of anisotropy and exotic matter required to support the existence of wormhole’s throat.

Kar & Sahdev had shown that if the wormhole spacetime is taken to be evolving (non-static) than the matter threading the wormhole geometry need not to be WEC violating [31]. In their model, the wormhole initially inflates and later evolves like a FRW spacetime thereby implying our Universe to be an evolving wormhole. Later Anchordoqui et al [32] extended the previous work [31] by choosing the non-zero redshift function and obtained evolving wormhole solutions with non-exotic matter. Arellano & Lobo [33] found contracting wormhole solutions within non-linear electrodynamics. Cataldo et al discussed the construction of evolving wormholes using phantom energy [34, 35] and more recently using the cosmological constant [36]. Farooq et al [37] showed that the field equations of a \((2+1)\)-dimensional evolving wormhole can be recast into first law of thermodynamics. Debnath et al [38] investigated the non-static Lorentzian wormhole model in presence of anisotropic pressure and showed that the Einstein’s field equations and unified first law are equivalent for the dynamical wormhole model. Inspired by the above works, we are interested to study the thermodynamics of an evolving wormhole at its apparent horizon using general entropy function and two special entropy functions, to be discussed later.
The paper is organized as follows: In section II, we write down the governing dynamical equations of an evolving wormhole. The previous work of Rahaman et al. [39] has some computational errors for wormhole thermodynamics in presence of isotropic pressure. In this work, we correct these assigning with anisotropic pressure in the field equations. In section-III, we study the wormhole thermodynamics by deriving the expressions of temperature, entropy and surface gravity at the apparent horizon of the wormhole. Further we obtain the general expression of the GSL at the apparent horizon valid for any type of entropy function chosen. In section IV, we study the expression of GSL by choosing two well-known forms of entropy functions i.e. log corrected and the power-law corrected in two forthcoming subsections. We analyze the behavior of GSL by choosing a power-law form of scale factor. We briefly discuss our results in section V.

2 Basic Equations

A simple generalization of Morris-Thorne (MT) wormhole to the time-dependent background is given by the evolving Lorentzian wormhole [34]

\[ ds^2 = -e^{2\Phi(t, r)} dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2 \right], \]

where \( d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and \( a(t) \) is the scale factor of the universe. The functions \( b(r) \) and \( \Phi(t, r) \) are called the shape function and redshift function respectively. Earlier the evolving wormholes have been studied with time-independent redshift function \( \Phi(r) \) [35]. If \( a(t) \rightarrow \text{constant and } \Phi(t, r) \rightarrow \Phi(r) \), then the static MT wormhole can be obtained whereas for \( b(r) \rightarrow kr^3 \) and \( \Phi(t, r) \rightarrow \text{constant} \), (1) reduces to standard FRW metric.

For anisotropic pressure the components of the energy-momentum tensor are [35, 36] assumed as

\[ T_{t}^{t} = -\rho(t, r), \quad T_{r}^{r} = -p_{r}(t, r), \quad T_{\theta}^{\theta} = T_{\phi}^{\phi} = -p_{t}(t, r), \]

where, the traditionally homogeneous and isotropic exotic matter source has been generalized to an inhomogeneous and anisotropic fluid, but still with a diagonal energy-momentum tensor (as is usually considered in phantom cosmologies). Here \( \rho(t, r), p_{r}(t, r) \) and \( p_{t}(t, r) \) are the energy density, radial and tangential pressures respectively. The Einstein’s field equations are then given by

\[ 3e^{-2\Phi} \dot{H}^2 + \frac{b'}{a^2 r^2} = 8\pi G \rho, \]

\[ -e^{-2\Phi} \left( 2\dot{H} + 3H^2 \right) + 2e^{-2\Phi} \dot{\Phi} H - \frac{b}{a^2 r^3} = 8\pi G p_{r}, \]

\[ -e^{-2\Phi} \left( 2\dot{H} + 3H^2 \right) + 2e^{-2\Phi} \dot{\Phi} H + \frac{b - rb'}{2a^2 r^3} = 8\pi G p_{t}, \]

\[ 2\dot{\Phi}' H = 0, \]

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. Here dot and dash stand for the differentiation with respect to \( t \) and \( r \) respectively. From (6) we have \( \Phi' = 0 \rightarrow \Phi(t, r) = \Phi(t) \). So without
any loss of generality, we can set $\Phi = 0$ by re-scaling the time coordinate [34]. Then the field eqns change to

$$3H^2 + \frac{b'}{a^2 r^2} = 8\pi G \rho,$$

(7)

$$2\dot{H} + 3H^2 + \frac{b}{a^2 r^3} = -8\pi G p_r,$$

(8)

$$2\dot{H} + 3H^2 - \frac{b - rb'}{2a^2 r^3} = -8\pi G p_t,$$

(9)

From (8) and (9), we have

$$\frac{3b - rb'}{2a^2 r^3} = -8\pi G (p_t - p_r).$$

(10)

Also, from the conservation of energy equation, we get

$$\dot{\rho} + H(3\rho + p_r + 2p_t) = 0,$$

(11)

$$2(p_t - p_r) = rp'_r,$$

(12)

For isotropic pressure, $p_t = p_r$ and hence from (12), $p'_r = 0$ i.e., $p_r$ becomes a function of time only. Also (10) shows $b(r) = kr^3$ and thus the metric (1) becomes FRW metric. To avoid this structure, we must require matter with anisotropic pressure which automatically leads to inhomogeneity.

3 Wormhole Thermodynamics

Some basic laws of wormhole dynamics have been derived in [42], which suggest a genuine connection with thermodynamics. Ever since, the thermal properties of wormholes have been studied in literature [43–45]. For studying the generalized second law of thermodynamics for an evolving wormhole, let us consider the metric (1) as in the following form

$$ds^2 = h_{ij} dx^i dx^j + \tilde{r}^2 d\Omega_2^2, \quad i, j = 0, 1$$

(13)

where \( h_{ij} = \begin{pmatrix} -1, a^2 \left(1 - \frac{b(r)}{\tilde{r}}\right)^{-1} \end{pmatrix} \) and the physical radius $\tilde{r} = a(t)r$.

The dynamical apparent horizon (AH) $\tilde{r}_A$ is given by

$$\left[h^i_j \partial_i \tilde{r} \partial_j \tilde{r}\right]_{\tilde{r} = \tilde{r}_A} = 0,$$

(14)

i.e.,

$$H^2 \tilde{r}_A^2 = 1 - \frac{ab\left(\frac{\tilde{r}_A}{a}\right)}{\tilde{r}_A}.$$

(15)

Differentiating (15), we have

$$\dot{\tilde{r}}_A = \frac{H\tilde{r}_A \left[\tilde{r}_A b' \left(\frac{\tilde{r}_A}{a}\right) - ab \left(\frac{\tilde{r}_A}{a}\right) - 2H\tilde{r}_A^3\right]}{\tilde{r}_A b' \left(\frac{\tilde{r}_A}{a}\right) - ab \left(\frac{\tilde{r}_A}{a}\right) + 2H^2\tilde{r}_A^3}.$$

(16)

The associated surface gravity is defined as

$$\kappa = \frac{1}{2\sqrt{-h}} \partial_i \left(\sqrt{-h} h^{ij} \partial_j \tilde{r}\right),$$

(17)
\[ \kappa = -\frac{\ddot{r}_A}{2} \left( \dot{H} + 2H^2 \right) + \frac{1}{4\bar{r}_A^2} \left[ ab \left( \frac{\ddot{r}_A}{a} \right) - \ddot{r}_A b' \left( \frac{\ddot{r}_A}{a} \right) \right]. \] (18)

Hence the associated temperature on the AH is

\[ T_A = \frac{\kappa}{2\pi} = -\frac{\ddot{r}_A}{4\pi} \left( \dot{H} + 2H^2 \right) + \frac{1}{8\pi\bar{r}_A^2} \left[ ab \left( \frac{\ddot{r}_A}{a} \right) - \ddot{r}_A b' \left( \frac{\ddot{r}_A}{a} \right) \right]. \] (19)

Now, we assume a functional form of the entropy on the AH as in the following form [46]

\[ S_A = f(A) \frac{4}{4G}, \] (20)

where \( A = 4\pi\bar{r}_A^2 \) and \( f(A) \) is the entropy function. Then

\[ \frac{dS_A}{dt} = \frac{2\pi}{G} f'(A) \ddot{r}_A \dot{r}_A. \] (21)

Then from (19) and (21), we may write

\[ T_A \frac{dS_A}{dt} = \frac{2\pi}{G} f'(A) \ddot{r}_A \dot{r}_A \left[ -\frac{1}{4\pi} \ddot{r}_A (\dot{H} + 2H^2) \right. \]

\[ \left. + \frac{1}{8\pi\bar{r}_A^2} \left\{ ab \left( \frac{\ddot{r}_A}{a} \right) - \ddot{r}_A b' \left( \frac{\ddot{r}_A}{a} \right) \right\} \right]. \] (22)

On the other hand, for anisotropic pressure the Gibb’s equation can be defined as

\[ T_A dS_I = \frac{1}{3} (p_r + 2p_t) dV + d(\rho V), \] (23)

where \( S_I \) is the entropy within the AH and \( V = \frac{4}{3}\pi\bar{r}^3 \). So, it can be shown that

\[ T_A \frac{dS_I}{dt} = \frac{4\pi\bar{r}_A^2}{3} \left( 3\rho + p_r + 2p_t + \frac{\ddot{r}_A \rho'}{a} \right) \left( \dot{r}_A - H\ddot{r}_A \right). \] (24)

Thus from (22) and (24), the variation of total entropy on the AH can be expressed as

\[ T_A \left( \frac{dS_A}{dt} + \frac{dS_I}{dt} \right) = \frac{2\pi}{G} f'(A) \ddot{r}_A \dot{r}_A \left[ -\frac{1}{4\pi} \ddot{r}_A (\dot{H} + 2H^2) \right. \]

\[ \left. + \frac{1}{8\pi\bar{r}_A^2} \left\{ ab \left( \frac{\ddot{r}_A}{a} \right) - \ddot{r}_A b' \left( \frac{\ddot{r}_A}{a} \right) \right\} \right] \]

\[ + \frac{4\pi\bar{r}_A^2}{3} \left( 3\rho + p_r + 2p_t + \frac{\ddot{r}_A \rho'}{a} \right) \times \left( \dot{r}_A - H\ddot{r}_A \right). \] (25)

If \( \left( \frac{dS_A}{dt} + \frac{dS_I}{dt} \right) > 0 \) i.e., if the expression of the r.h.s of above equation is non-negative, then we say that GSL is valid. For this purpose we need to know the entropy function \( f(A) \) and the shape function \( b(r) \).
4 Two Special Cases

Here we choose a particular form of the shape function as \( b(r) = \frac{b_0}{r}, b_0 = \text{constant} \), in order to study the GSLT for two different types of wormhole models by choosing two specific forms of the horizon entropy \( S_A \) i.e., logarithmic and power law correction entropies.

4.1 Logarithmic Correction

It is a well-known fact that in Einstein’s gravity, the entropy of the horizon is proportional to the area, but when the gravity theory is modified by adding extra curvature terms in the action, it also modifies the entropy-area relation. In the context of loop-quantum gravity, this relation can be expanded into an infinite series as \[ S_A = S_0 + \tilde{\alpha} \ln S_0 - \sum_{i=1}^{\infty} \tilde{\alpha}_i S_i^0. \] (26)

Here \( S_0 \) is the classical entropy and the higher order terms are quantum corrections. Here \( \alpha_i \)'s are finite constants, but their values are highly debatable. For example, some take the value of \( \tilde{\alpha} \) to be negative [48, 49], some positive [50], whereas some have taken it to be zero [51]. This correction has been used in literature for variety of purposes: to study the dark energy models in numerous gravities [52–54], to unify inflation and dark energy models [55], study of GSL with log corrections [56, 57].

From (26), we continue our study by considering the expression of the horizon entropy with the logarithmic correction only [58], i.e, for our study

\[ S_A = \frac{A}{4G} + \pi \beta \ln \left( \frac{A}{4G} \right), \] (27)

so that the entropy function can be written as

\[ f(A) = A + 4\pi G \beta \ln \left( \frac{A}{4G} \right). \] (28)

With this choice of entropy, (25) takes the form

\[ T_A \left( \frac{dS_A}{dt} \right) = -\frac{H\dot{r}_A}{2} \left( 1 + \frac{\beta}{r_A^2} \right) \left( \frac{a^2 b_0 + \dot{H} r_A^4}{H^2 r_A^4 - a^2 b_0} \right) \] 
\[ \times \left[ \frac{a^2 b_0}{r_A^2} - \frac{\dot{r}_A^2}{r_A^2} (\dot{H} + 2H^2) \right] \] 
\[ -\frac{H^2}{3G} \left( \frac{a^2 b_0}{r_A^4} - 3\dot{H} \right) \left( \frac{H^2 + \dot{H}}{H^2 r_A^4 - a^2 b_0} \right), \] (29)

where we have used the field equations and (16) for the expressions of \( \rho, p_r, p_t \) and \( \dot{r}_A \).

For a special choice of the scale factor in power law form which is assumed as \( a(t) = a_0 t^n \). Now, we plot the evolution of the total entropy on the AH against time in Fig. 1, where the constant \( n \) has been taken to be equal to 2 (solid line) and 5 (dashed line), we see that the GSL is valid for small \( n \) but as \( n \) increases, GSL is not obeyed in the later epoch for logarithmic corrected entropy.
4.2 Power Law Correction

Another significant correction term to the horizon entropy expression appeared while dealing with the entanglement of quantum fields inside and outside of the horizon where the wave function of the field is chosen to be a superposition of ground state and excited state [59]. So the associated power law corrected entropy expression becomes [60, 61]

\[ S_A = \frac{A}{4G} \left( 1 - K_\alpha A^{\frac{1-\alpha}{2}} \right), \]  

(30)

so that in this case we have

\[ f(A) = A \left( 1 - K_\alpha A^{\frac{1-\alpha}{2}} \right), \]  

(31)

where \( \alpha \) is a dimensionless constant and \( K_\alpha \) is given by

\[ K_\alpha = \frac{\alpha (4\pi)^{\frac{\alpha}{2} - 1}}{(4 - \alpha) r_c^{2-\alpha}}. \]  

(32)

Here \( r_c \) is called the cross-over scale. The entanglement entropy of the ground state obeys the usual area law whereas the excited state contributes to the correction. Thus the correction term is more significant for higher excitations. It is important to note that the correction term falls off rapidly as \( A \) increases and hence in the semi classical limit (large \( A \)), the area law can be recovered. This correction has been recently used extensively in dark energy literature: to study the holographic and new-agegraphic dark energy in various gravitational theories [62–68] and the study of GSL in FRW cosmology with power-law entropy correction [69, 70].

For this choice of the horizon entropy, (25) becomes

\[ T_A \left( \frac{dS_A}{dt} + \frac{dS_l}{dt} \right) = -\frac{H \tilde{r}_A^{(2-\alpha)}}{4G} \left[ 2\tilde{r}_A^{(\alpha-1)} + (\alpha - 3) K_\alpha (4\pi)^{\frac{1-\alpha}{2}} \right] \]

\[ \left( \frac{a^2 b_0 + \dot{H} \tilde{r}_A^4}{H^2 \tilde{r}_A^4 - a^2 b_0} \right) \left[ \frac{a^2 b_0}{\tilde{r}_A^5} - \tilde{r}_A^2 (\dot{H} + 2H^2) \right] \]

\[ -\frac{H \tilde{r}_A^4}{3G} \left( \frac{a^2 b_0}{\tilde{r}_A^5} - 3\dot{H} \right) \left( \frac{H^2 + \dot{H}}{H^2 \tilde{r}_A^4 - a^2 b_0} \right), \]  

(33)
Here also the field equations together with (16) have been used to write the expressions of $\rho$, $p_r$, $p_t$ and $\tilde{r}_A$. For a special choice of the scale factor in power law form which is assumed as $a(t) = a_0 t^n$. Now, we plot the evolution of the total entropy on the AH against time in Fig. 2, where the constant $n$ has been taken to be equal to 2 (solid line) and 5 (dashed line), we see that the GSL is valid for small $n$ but as $n$ increases, GSL is not obeyed in the later epoch for power law corrected entropy. Hence the behavior is analogous for both power-law and Log-corrected entropies.

5 Exact Solutions

Let us find some exact solutions of the system (7–9).

i) Let $a = a_0 t^n$, then $H = n t^{-1}$. Substituting these expressions into (7–9) we get (below we assume $8 \pi G = 1$)

$$\rho = \frac{3n^2}{t^2} + \frac{b'}{a_0^2 t^{2n} r^2}, \quad (34)$$

$$p_r = \frac{n(2 - 3n)}{t^2} - \frac{b}{a_0^2 t^{2n} r^3}, \quad (35)$$

$$p_T = \frac{n(2 - 3n)}{t^2} + \frac{b - r b'}{2a_0^2 t^{2n} r^3}. \quad (36)$$

Now let us introduce the EoS parameters as

$$\omega_r(r, t) = \frac{p_r}{\rho}, \quad \omega_T(r, t) = \frac{p_T}{\rho}. \quad (37)$$

For the solutions (34–36) we obtain

$$\omega_r = \frac{n(2 - 3n)a_0^2 r^3 t^{2n-2} - b}{r \left[3n^2 a_0^2 r^2 t^{2n-2} + b'\right]}, \quad (38)$$

$$\omega_T = \frac{2n(2 - 3n)a_0^2 r^3 t^{2n-2} + (b - rb')}{2r \left[3n^2 a_0^2 r^2 t^{2n-2} + b'\right]}.$$

Fig. 2 This figure represents the rate of change of total entropy for power law corrected form of the horizon entropy function against time for $n = 2$ (solid line) and $n = 5$ (dashed line).
In these formulas we have one arbitrary function \( b(r) \) and parameters \( n, a_0 \). Let us we assume that in the \( r \)-direction we have an accelerated expansion so that we can put \( \omega_r = -1 \). Then from (38) we determine the unknown function \( b(r) \) as

\[
b(r) = r \left[ C - n a_0^2 r^2 t^{2n-2} \right],
\]

where \( C = \text{constant} \). To get rid of the \( t \) dependence of \( b \) we put \( n = 1 \). Then finally from (40) we get

\[
b(r) = r \left[ C - a_0^2 r^2 \right],
\]

So for the EoS parameters we get

\[
\omega_r = -1, \\
\omega_T = 0.
\]

ii) Our next example is by taking \( \omega_r = \text{const} = \omega r_0 \). Then for the density of energy, pressures and EoS parameters we get the same expressions as in the previous case. We get \( b(r) \) from (38)

\[
b(r) = C r^{1 - \frac{1}{\omega r_0}} + \frac{2n - 3(1 + \omega r_0)n^2}{1 + 3\omega r_0} a_0^2 r^3 t^{2n-2},
\]

where \( C = \text{constant} \). To get rid of the \( t \) dependence of \( b \) we put \( n = 1 \). Then finally we get

\[
b(r) = C r^{1 - \frac{1}{\omega r_0}} - a_0^2 r^3,
\]

Following [21], we can check the stability conditions for this evolving wormhole. These conditions include certain restrictions on the form of shape function including (1) \( b'(r_0) < 1 \), at the wormhole’s throat (2) \( b(r) < r \) for \( r > r_0 \) (flare-out condition) and (3) \( b(r)/r \to 0 \), as \( |r| \to \infty \) (asymptotic flatness). Clearly the last condition for (45) is not satisfied, however the first two condition impose the following inequality

\[
C < r^{1 + \frac{1}{\omega r_0}} \left( 1 + a_0^2 r^2 \right).
\]

In our case the formulas (38–39) become

\[
\omega_r = -\frac{a_0^2 r^3 + b}{r \left[ 3a_0^2 r^2 + b' \right]},
\]

\[
\omega_T = \frac{b - rb' - 2a_0^2 r^3}{2r \left[ 3a_0^2 r^2 + b' \right]}.
\]

So from these formulas and (45) finally we get

\[
\omega_r = \omega r_0,
\]

\[
\omega_T = -\frac{1 + \omega r_0}{2r}.
\]

Consider particular cases. 1) Let \( \omega r_0 = 1/3 \) that is raditation. Then

\[
\omega_r = 1/3, \\
\omega_T = -\frac{4}{6r}.
\]

The solution suggests a radiation state parameter in the \( r \) direction while a phantom energy state parameter \( r < \frac{2}{3} \).
2) Let $\omega_{r0} < -1$ that is phantom matter. Then

$$\omega_r < -1, \quad \omega_T > 0.$$ (52) (53)

3) Let $\omega_{r0} > 1$ that is ekpyrotic matter. Then

$$\omega_r > 1, \quad \omega_T < -\frac{1}{r}.$$ (54) (55)

It is interesting to note that in this case we have the ekpyrotic matter in $r$-direction and phantom in $T$-direction if $r < 1$. So that $r = 1$ is a crucial value.

iii) Now let us consider the case

$$\omega_{r0} = \frac{2 - 3n}{3n}.$$ (56)

In this case $b(r)$ takes the form

$$b(r) = Cr^{\frac{3n}{3n-2}},$$ (57)

Then the expressions for the parameters of EoS (38–39) become

$$\omega_r = \frac{2 - 3n}{3n},$$ (58)

$$\omega_T = -\frac{1}{3n} \left[ 1 + \frac{3n(1-n)(2-3n)a_0^2r^2t^{2n-2}}{Cr^{\frac{3n}{3n-2}} + n(3n - 2)a_0^2r^2t^{2n-2}} \right].$$ (59)

From (58), it is easy to see that a cosmological constant state parameter cannot be obtained. In other words, an evolving wormhole cannot be constructed and supported from vacuum energy (Fig. 3).

6 Conclusions

In this work, we have studied the basic equations of evolving Lorentzian wormhole by assuming the anisotropic pressure. The generalized second law of thermodynamics (GSL) at the apparent horizon of an evolving Lorentzian wormhole has been analyzed in general way when we have considered the horizon entropy proportional to a function of the horizon area. We have obtained the expressions of thermal variables at the apparent horizon. Choosing
the two well-known entropy functions i.e. power-law and logarithmic, we have obtained the expressions of the variation of the total entropy using Gibb’s equation. We have analyzed the GSL using a simple well established power-law form of scale factor $a(t) = a_0 t^n$ and the special form of shape function $b(r) = b_0 r^2$. It is shown that GSL is valid in the evolving wormhole spacetime for both choices of entropies if the power-law exponent $n$ is small, but for large values of $n$, the GSL is satisfied at initial stage and after certain stage of the evolution of the wormhole, it violates.

**Acknowledgments**  One of the authors (TB) wants to thank UGC, Govt. of India for providing with a research project No. F.PSW-063/10-11 (ERO). The authors (TB, UD) are thankful to IUCAA, Pune, India for warm hospitality where part of the work was carried out. The author UD is also thankful to Institute of Theoretical Physics, Chinese Academy of Science, Beijing, China for providing TWAS Associateship Programme under which some part of the work was carried out. Also UD is thankful to CSIR, Govt. of India for providing research project grant (No. 03(1206)/12/EMR-II).

**References**

1. Rovelli, C.: Quantum Gravity. Cambridge University Press (2004)
2. Rovelli, C.: Living Rev. Rel. 1, 1 (1998)
3. Zwiebach, B.: A First Course in String Theory. Cambridge University Press (2009)
4. Hawking, S.W.: Commun. Math Phys. 43, 199 (1975)
5. Bekenstein, J.D.: Phys. Rev. D 7, 2333 (1973)
6. Bardeen, J.M., Carter, B., Hawking, S.W.: Commun. Math. Phys. 31, 161 (1973)
7. Jacobson, T., : Phys. Rev. Lett. 75, 1260 (1995)
8. T. Padmanabhan: J. Phys. Conf. Ser. 306, 012001 (2011)
9. T. Padmanabhan: Rep. Prog. Phys. 73, 046901 (2010)
10. Padmanabhan, T.: Phys. Rept. 406, 49 (2005)
11. Davis, T.M., Davies, P.C.W., Lineweaver, C.H.: Class. Quant. Grav. 20, 2753 (2003)
12. Cai, R.-G., Kim, S.P.: JHEP 0502, 050 (2005)
13. Akbar, M., Cai, R.-G.: Phys. Lett. B 648, 243 (2007)
14. Akbar, M., Cai, R.-G.: Phys. Rev. D 75, 084003 (2007)
15. Verlinde, E.P.: JHEP 1104, 029 (2011)
16. Morris, M.S., Thorne, K.S.: Am. J. Phys. 56, 395 (1988)
17. Visser, M.: Lorentzian Wormholes: From Einstein to Hawking. AIP, New York (1995)
18. Thorne, K., Holes Black, Warps Time: Einstein’s Outrageous Legacy. W. W. Norton & Company (1995)
19. Visser, M.: Phys. Rev. D 39, 3182 (1989)
20. Jamil, M., Kuhfittig, P.K.F., Rahaman, F., Rakib, Sk.A.: Eur. Phys. J. C 67, 513 (2010)
21. Jamil, M., Farooq, M.U., Rashid, M.A.: Eur. Phys. J. C 59, 907 (2009)
22. Kar, S.: Phys. Rev D 49, 862 (1994)
23. Bertolami, O., Ferreira, R.Z. arXiv:1203.0523v1[gr-qc]
24. Lobo, F.S.N. arXiv:1112.6333v1[gr-qc]
25. Dehghani, H., Mehdizadeh, M.R.: Phys. Rev. D 85, 024024 (2012)
26. DeBenedictis, A., Horvat, D. arXiv:1111.3704v1[gr-qc]
27. Boehmer, C.G., Harko, T., Lobo, F.S.N.: Phys. Rev. D 85, 044033 (2012)
28. Darabi, F. arXiv:1110.5487v1[gr-qc]
29. DeBenedictis, A.: Phys. Rev. D 84, 104030 (2011)
30. Garcia, N.M., Lobo, F.S.N.: Mod. Phys. Lett. A 40, 3067 (2011)
31. Kar, S., Sahdev, D.: Phys. Rev. D 53, 722 (1996)
32. Anchordoqui, L.A., Torres, D.F., Trobo, M.L., Bergliaffa, S.E.P.: Phys. Rev. D 57, 829 (1998)
33. Arewllano, A.V., Lobo, F.S.N.: Class. Quant. Grav. 23, 5811 (2006)
34. Cataldo, M., del Campo, S., Minning, P., Salgado, P.: Phys. Rev. D 79, 024005 (2009)
35. Cataldo, M., Labrana, P., del Campo, S., Crisostomo, J., Salgado, P.: Phys. Rev. D 78, 104006 (2008)
36. Cataldo, M., Meza, P., Minning, P.: Phys. Rev. D 83, 044050 (2011)
37. Farooq, M., Akbar, M., Jamil, M.: AIP Conf. Proc. 1295, 176 (2010)
38. Debnath, U., Jamil, M., Akbar, M. arXiv:1202.1706v1[physics.gen-ph]
39. Rahman, F., Salahuddin, Akbar, M.: Chin. Phys. Lett. 28, 070403 (2011)
40. Visser, M., Kar, S., Dadhich, N.: Phys. Rev. Lett. 90, 201102 (2003)
41. Dadhich, N., Kar, S., Mukherjee, S., Visser, M.: Phys. Rev. D 65, 064004 (2002)
42. Hayward, S.A.: Phys. Rev. D 79, 124001 (2009)
43. Hong, S.T., Kim, S.W.: Mod. Phys. Lett. A 21, 789 (2006)
44. Bokhari, A.H., Akbar, M.: Int. J. Mod. Phys. D 19, 565 (2010)
45. Martin-Moruno, P., Gonzalez-Diaz, P.F.: Phys. Rev. D 80, 024007 (2009)
46. Chakraborty, S., Mazumder, N., Biswas, R.: Europhys. Lett. 91, 40007 (2010)
47. Banerjee, R., Modak, S.K.: JHEP 0905, 063 (2009)
48. Kaul, R.K., Majumdar, P.: Phys. Rev. Lett. 84, 5255 (2000)
49. Ghosh, A., Mitra, P.: Phys. Rev. D 71, 027502 (2005)
50. Hod, S.: Class. Quant. Grav. 21, L97 (2004)
51. Medved, A.J.M.: Class. Quant. Grav. 22, 133 (2005)
52. Setare, M.R., Jamil, M.: Europhys. Lett. 92, 49003 (2010)
53. Farooq, M.U., Rashid, M.A., Jamil, M.: Int. J. Theor. Phys. 49, 2278 (2010)
54. Jamil, M., Farooq, M.U.: JCAP 03, 001 (2010)
55. Sadjadi, H.M., Jamil, M.: Gen. Rel. Grav. 43, 1759 (2011)
56. Sadjadi, H.M., Jamil, M.: Europhys. Lett. 92, 69001 (2010)
57. Jamil, M., Sheykhi, A., Farooq, M.U.: Int. J. Mod. Phys. D 19, 1831 (2010)
58. Wei, H.: Commun. Theor. Phys. 52, 743 (2009)
59. Das, S., Shankaranarayanan, S., Sur, S.: Phys. Rev. D 77, 064013 (2008)
60. Radicella, N., Pavon, D.: Phys. Lett. B 691, 121 (2010)
61. Farooq, M.U., Jamil, M.: Canadian J. Phys. 89, 1251 (2011)
62. Sheykhi, A., Karami, K., Jamil, M., Kazemi, E., Haddad, M. arXiv:1107.4598v3[astro-ph.CO]
63. Karami, K., Sheykhi, A., Jamil, M., Ghaffari, S., Abdolmaleki, A. arXiv:1106.2406v1[physics.gen-ph]
64. Sheykhi, A., Jamil, M.: Gen. Rel. Grav. 43, 2661 (2011)
65. Karami, K., Abdolmaleki, A. arXiv:1111.7269v1[gr-qc]
66. Khodam-Mohammadi, A.: Mod. Phys. Lett. A 26, 2487 (2011)
67. Ebrahimi, E., Sheykhi, A.: Phys. Scripta 04, 045016 (2011)
68. Karami, K., Khaledian, M.S. arXiv:1108.5926v1[physics.gen-ph]
69. Debnath, U., Chattopadhyay, S., Hussain, I., Jamil, M. arXiv:1111.3858v1[physics.gen-ph]
70. Karami, K., Abdolmaleki, A., Sahraei, N., Ghaffari, S.: JHEP 1108, 150 (2011)