GLOBAL WEAK SOLUTIONS FOR THE TWO-COMPONENT NOVIKOV EQUATION

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Abstract. The two-component Novikov equation is an integrable generaliza-
tion of the Novikov equation, which has the peaked solitons in the sense of
distribution as the Novikov and Camassa-Holm equations. In this paper, we
prove the existence of the $H^1$-weak solution for the two-component Novikov
equation by the regular approximation method due to the existence of three
conserved densities. The key elements in our approach are some a priori esti-
mates on the approximation solutions.

1. Introduction. This paper is devoted to the existence of weak solutions to the
Cauchy problem for the two-component Novikov equation [18]
\[
\begin{align*}
    m_t + uvm_x + (2uv_x + uw_x)m &= 0, \quad m = u - u_{xx}, \quad t > 0, \\
    n_t + un_x + (2uv_x + vu_x)n &= 0, \quad n = v - v_{xx}.
\end{align*}
\]

(1)

Note that this system reduces respectively to the Novikov equation [23]
\[
m_t + 3uu_x m + u^2 m_x = 0,
\]
when $v = u$, and the celebrated Camassa-Holm (CH) equation [1]
\[
m_t + 2u_x m + um_x = 0,
\]
when $v = 1$.

The CH equation was proposed as a nonlinear model describing the unidirec-
tional propagation of the shallow water waves over a flat bottom [1]. Based on the
Hamiltonian theory of integrable systems, it was found earlier by using the method of
recursion operator due to Fuchssteiner and Fokas [10]. It can also be obtained by
using the tri-Hamiltonian duality approach related to the bi-Hamiltonian represen-
tation of the Korteweg-de Vries (KdV) equation [9, 25]. The CH equation exhibits
several remarkable properties. One is the existence of the multi-peaked soli-
tons on the line $\mathbb{R}$ and unit circle $\mathbb{S}^1$ [1, 2], where the peaked solitons are the weak
solution in the sense of distribution. Second, it can describes wave breaking phe-
nomena [4], which is different from the classical integrable systems. The existence
of $H^1$-conservation law to the CH equation enables ones to define the $H^1$-weak
solution [28]. There have been a number of results concerning about integrability,

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well-posedness, blow up and wave breaking, orbital stability in the energy space and geometric formulations etc, see for instance [4, 5, 6, 8, 28] and references therein.

The Novikov equation (2) can be viewed as a cubic generalization of the CH equation, which was introduced by Novikov [23, 24] in the classification for a class of equations while they possesses higher-order generalized symmetries. Eq. (2) was proved to be integrable since it enjoys Lax-pair and bi-Hamiltonian structure [14], and is equivalent to the first equation in the negative flow of the Sawada-Kotera hierarchy via Liouville transformation [16]. The Novikov equation (2) also admits peaked solitons over the line \( \mathbb{R} \) and unit circle \( S^1 \) [14, 20], which can be derived by the inverse spectral method. Orbital stability of peaked solitons over the line \( \mathbb{R} \) and unit circle \( S^1 \) of (2) in the energy space were verified [20] based on the conservation laws and the structure of peaked solitons of the Novikov equation (2). The well-posedness and wave breaking of the Novikov equation have been discussed in a number of papers, and it reveals that the Cauchy problem of the Novikov equation (2) has global strong solutions when the initial data \( u_0 \in H^s, s > 3/2 \) [3, 15, 26, 27]. The existence of global weak solutions to the Cauchy problem of the Novikov equation (2) was also discussed in [17].

As the two-component generalization of Novikov equation (2), the so-called Geng-Xue system [11]

\[
\begin{align*}
m_t + 3vu_xm + uvm_x &= 0, \\
n_t + 3uv_xn + uvn_x &= 0,
\end{align*}
\]

has been studied extensively [11, 13]. The integrability [11, 19], dynamics and structure of the peaked solitons of (4) [21] were discussed. In [13], well-posedness and wave breaking phenomena of the Cauchy problem of (4) were discussed. The single peakons and multi-peakons of system (4) were constructed in [21] by using compatibility of Lax-pair, which are not the weak solutions in the sense of distribution. Furthermore, the Geng-Xue system does not have the \( H^1 \)-conserved density, this is different from the CH and Novikov equations. The weak solution in \( H^1 \) is not well-defined since it does not obey the \( H^1 \)-conservation law.

The main object in this work is to investigate the existence of weak solutions to system (1). It is of great interest to understand the effect from interactions among the two-components, nonlinear dispersion and various nonlinear terms. More specifically, we shall consider the Cauchy problem of (1) and aim to leverage ideas from previous works on CH and Novikov equations. The weak solution of the Cauchy problem associated with (1) is established in Theorem 3.1.

The remainder of this paper is organized as follows. In the next section 2, we review some basic results and lemmas as well as invariant properties of momentum densities \( m \) and \( n \). In Section 3, we establish the existence of weak solutions, our approach is the regular approximation method together with some a priori estimates.

2. Strong solutions and some a priori estimates. In this section, we recall the local well-posedness, some properties of strong and weak solutions to equation (1) and several approximation results.

First, we introduce some notations. Throughout the paper, we denote the convolution by \(*\). Let \( \| \cdot \|_X \) denote the norm of Banach space \( X \) and let \( \langle \cdot, \cdot \rangle \) denote the duality paring between \( H^1(\mathbb{R}) \) and \( H^{-1}(\mathbb{R}) \). Let \( \mathcal{M}(\mathbb{R}) \) be the space of Radon measures on \( \mathbb{R} \) with bounded total variation and \( \mathcal{M}^+(\mathbb{R}) \) be the subset of
positive Radon measures. Moreover, we write $BV(\mathbb{R})$ for the space of functions with bounded variation, $\mathcal{V}(f)$ being the total variation of $f \in BV(\mathbb{R})$. Furthermore, for $0 < p < \infty$, $s \geq 0$, let $\| \cdot \|_{L^p}$ and $\| \cdot \|_s$ denote the norm of $L^p(\mathbb{R})$ space and $H^s(\mathbb{R})$ space, respectively.

With $m = u - u_{xx}$ and $n = v - v_{xx}$, the Cauchy problem of equation (1) takes the form:

$$
\begin{cases}
m_t + um_{xx} + (2vu_x + uv_x)m = 0, & m = u - u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\
n_t + wn_{xx} + (2uw_x + vu_x)n = 0, & n = v - v_{xx}, \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}.
\end{cases}
$$

(5)

Note that if $P(x) = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$, we have $(1 - \partial_x^2)^{-1} f = P * f$ for all the $f \in L^2(\mathbb{R})$ and $P * m = u$, $P * n = v$. Then we can rewrite the equation (5) as follows:

$$
\begin{cases}
\frac{1}{2} \left( u_t + wu_x \right) + P_x \left( \frac{1}{2} u_x^2 + uu_x + u^2 v \right) + \frac{1}{2} P \left( u_x^2 v_x \right) = 0, & t > 0, \quad x \in \mathbb{R}, \\
\frac{1}{2} \left( v_t + wv_x \right) + P_x \left( \frac{1}{2} v_x^2 + vv_x + v^2 u \right) + \frac{1}{2} P \left( v_x^2 u_x \right) = 0, \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}.
\end{cases}
$$

(6)

Next we recall the local well-posedness and the conservation laws.

**Lemma 2.1.** [12] Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$. Assume that $T = T(u_0,v_0) > 0$ be the maximal existence time of the corresponding strong solution $(u, v)$. Then the initial value problem of system (1) possesses a strong solution

$$
u, v \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $(u_0, v_0) \mapsto (u(\cdot,u_0), v(\cdot,v_0)) : H^s(\mathbb{R}) \times H^s(\mathbb{R}) \rightarrow C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})) \times C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ is continuous.

**Lemma 2.2.** [12] Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $(u(t,x), v(t,x))$ be the corresponding solution to equation (1) with the initial data $(u_0,v_0)$. Then we have

$$
\int_{\mathbb{R}} (u^2(t,x) + u_x^2(t,x)) \, dx = \int_{\mathbb{R}} (u_0^2 + u_{0x}^2) \, dx,
$$

$$
\int_{\mathbb{R}} (v^2(t,x) + v_x^2(t,x)) \, dx = \int_{\mathbb{R}} (v_0^2 + v_{0x}^2) \, dx,
$$

$$
\int_{\mathbb{R}} (u(t,x)v(t,x) + u_x(t,x)v_x(t,x)) \, dx = \int_{\mathbb{R}} (u_0v_0 + u_{0x}v_{0x}) \, dx.
$$

Moreover, we have

$$
|u(t,x)| \leq \frac{\sqrt{2}}{2} \|u_0\|_1, \quad |v(t,x)| \leq \frac{\sqrt{2}}{2} \|v_0\|_1.
$$

Note that equation (1) has the solitary waves with corner at their peaks. Obviously, such solitons are not strong solutions to equation (6). In order to provide a mathematical framework for the study of these solitons, we define the notion of weak solutions to equation (6). Let

$$
F_u(u, v) = uvu_x + P_x \left( \frac{1}{2} u_x^2 + uu_x + u^2 v \right) + \frac{1}{2} P \left( u_x^2 v_x \right),
$$

$$
F_v(u, v) = uvv_x + P_x \left( \frac{1}{2} v_x^2 + vv_x + v^2 u \right) + \frac{1}{2} P \left( v_x^2 u_x \right).
$$
Then equation (6) can be written as
\[
\begin{align*}
\frac{du}{dt} + F_u(u,v) &= 0, \\
\frac{dv}{dt} + F_v(u,v) &= 0, \\
u(0,x) &= u_0(x), \quad v(0,x) = v_0(x).
\end{align*}
\tag{7}
\]

**Lemma 2.3.** [22] Let $T > 0$. If $f, g \in L^2((0,T);H^1(\mathbb{R}))$ and $\frac{df}{dt}, \frac{dg}{dt} \in L^2((0,T);H^{-1}(\mathbb{R}))$, then $f, g$ are a.e. equal to functions continuous from $[0,T]$ into $L^2(\mathbb{R})$ and
\[
\langle f(t), g(t) \rangle - \langle f(s), g(s) \rangle = \int_s^t \left\langle \frac{df(\tau)}{d\tau}, g(\tau) \right\rangle d\tau + \int_s^t \left\langle \frac{dg(\tau)}{d\tau}, f(\tau) \right\rangle d\tau
\]
for all $s, t \in [0,T]$.

Throughout this paper, let $\{\rho_n\}_{n \geq 1}$ denote the mollifiers
\[
\rho_n = \left( \int_{\mathbb{R}} \rho(\xi)d\xi \right)^{-1} n \rho(nx), \quad x \in \mathbb{R}, \quad n \geq 1,
\]
where $\rho \in C^\infty_c(\mathbb{R})$ is defined by
\[
\rho(x) = \begin{cases} 
e \frac{e^{x^2-1}}{x^2}, & \text{for } |x| < 1, \\
0, & \text{for } |x| \geq 1. 
\end{cases}
\]

Next, we recall two crucial approximation results and two identities.

**Lemma 2.4.** [7] Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. If $\mu \in M(\mathbb{R})$, then
\[
[\rho_n * (f \mu) - (\rho_n * f)(\rho_n * \mu)] \to 0, \quad \text{as } n \to \infty \quad \text{in } L^1(\mathbb{R}).
\]

**Lemma 2.5.** [7] Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and bounded. If $g \in L^\infty(\mathbb{R})$, then
\[
\rho_n * (f g) - (\rho_n * f)(\rho_n * g) \to 0, \quad \text{as } n \to \infty \quad \text{in } L^\infty(\mathbb{R}).
\]

**Lemma 2.6.** [7] Assume that $u(t, \cdot) \in W^{1,1}(\mathbb{R})$ is uniformly bounded in $W^{1,1}(\mathbb{R})$ for all $t \in \mathbb{R}_+$. Then for a.e. $t \in \mathbb{R}_+$, there hold
\[
\frac{d}{dt} \int_{\mathbb{R}} |\rho_n * u| dx = \int_{\mathbb{R}} (\rho_n * u_t) \text{sgn}(\rho_n * u) dx
\]
and
\[
\frac{d}{dt} \int_{\mathbb{R}} |\rho_n * u_x| dx = \int_{\mathbb{R}} (\rho_n * u_{xt}) \text{sgn}(\rho_n * u_x) dx.
\]

Consider the flow governed by $(uv)(t,x)$:
\[
\begin{align*}
\frac{dq(t,x)}{dt} &= (uv)(t,q), \quad t > 0, \quad x \in \mathbb{R}, \\
q(0,x) &= x, \quad x \in \mathbb{R}.
\end{align*}
\tag{8}
\]

Applying classical results in the theory of ODEs, one can obtain the following useful result on the above initial value problem.
Lemma 2.7. [12] Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$, and $T > 0$ be the life-span of the corresponding strong solution $(u, v)$ to equation (5) with the initial data $(u_0, v_0)$. Then equation (8) has a unique solution $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism over $\mathbb{R}$ with

$$q_x = \exp \left( \int_0^t (uv)_x(s, q(s, x))ds \right), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$ 

Furthermore, setting $m = u - u_{xx}$ and $n = v - v_{xx}$, we obtain

$$m(t, q) = \exp \left( -\int_0^t (2vu_x + uv_x)(s, q(s, x))ds \right) m_0,$$

$$n(t, q) = \exp \left( -\int_0^t (2uv_x + vu_x)(s, q(s, x))ds \right) n_0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$ 

Theorem 2.8. Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$. Assume that $m_0 = u_0 - \partial_x^2 u_0$ and $n_0 = v_0 - \partial_x^2 v_0$ are nonnegative, and $T > 0$ be the maximal existence time of the corresponding strong solution $(u, v)$. Then the initial value problem of system (1) possesses a pair of unique strong solution $(u, v)$, where

$$u, v \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})).$$

Set $m(t, \cdot) = u(t, \cdot) - u_{xx}(t, \cdot)$ and $n(t, \cdot) = v(t, \cdot) - v_{xx}(t, \cdot)$. Then, $E_u(u) = \int_\mathbb{R} (u^2 + u_x^2) dx$, $E_v(v) = \int_\mathbb{R} (v^2 + v_x^2) dx$, $H(u, v) = \int_\mathbb{R} (uw + u_xv_x) dx$ and $E_0(u, v) = \int_\mathbb{R} (mn)^{\frac{3}{2}} dx$ are four conservation laws and we have for all $t \in \mathbb{R}_+$

(i). $m(t, \cdot) \geq 0$, $n(t, \cdot) \geq 0$, $u(t, \cdot) \geq 0$, $v(t, \cdot) \geq 0$ and $|u_x(t, \cdot)| \leq u(t, \cdot)$, $|v_x(t, \cdot)| \leq v(t, \cdot)$ on $\mathbb{R}$;

(ii). $\|u(t, \cdot)\|_{L^1}, \|n(t, \cdot)\|_{L^1}, \|u(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} \|u(t, \cdot)\|_1 = \frac{\sqrt{2}}{2} \|u_0\|_1$, and $\|v(t, \cdot)\|_{L^1}, \|n(t, \cdot)\|_{L^1}, \|v(t, \cdot)\|_{L^\infty} \leq \frac{\sqrt{2}}{2} \|v(t, \cdot)\|_1 = \frac{\sqrt{2}}{2} \|v_0\|_1$;

(iii). $\|u_x(t, \cdot)\|_{L^1}, \|v_x(t, \cdot)\|_{L^1} \leq \|n(t, \cdot)\|_{L^1}$ and $\|u_x(t, \cdot)\|_{L^1}, \|v_x(t, \cdot)\|_{L^1} \leq \|n(t, \cdot)\|_{L^1}$.

Moreover, if $m_0, n_0 \in L^1(\mathbb{R})$, we obtain

$$\|m(t, \cdot)\|_{L^1} \leq e^{\|u_0\|_1 \|v_0\|_1} \|m_0\|_{L^1} \quad \text{and} \quad \|n(t, \cdot)\|_{L^1} \leq e^{\|u_0\|_1 \|v_0\|_1} \|n_0\|_{L^1}.$$ 

Proof. Let $u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the solution $(u, v)$ to equation (5) with the initial data $(u_0, v_0)$. If $m_0 \geq 0$ and $n_0 \geq 0$, then Lemma 2.7 ensures that $m(t, \cdot) \geq 0$ and $n(t, \cdot) \geq 0$ for all $t \in [0, \infty)$. By $u = P \ast m$, $v = P \ast n$ and the positivity of $P$, we infer that $u(t, \cdot) \geq 0$ and $v(t, \cdot) \geq 0$ for all $t \geq 0$. Note that $v$ is analogous as $u$ and

$$u(t, x) = \frac{e^x}{2} \int_{-\infty}^x e^y m(t, y)dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y)dy,$$ 

and

$$u_x(t, x) = -\frac{e^x}{2} \int_{-\infty}^x e^y m(t, y)dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y)dy.$$ 

From the above two relations and $m \geq 0$, we deduce that

$$|u_x(t, x)| \leq u(t, x) \leq \frac{\sqrt{2}}{2} \|u(t, x)\|_1.$$
In view of Lemma 2.2, we obtain that $E_u(u)$ and $E_v(v)$ are conserved and

$$u(t, x) \leq \frac{\sqrt{2}}{2} \|u_0\|_1, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$ 

Since $m(t, x) = u - u_{xx}$, it follows that $u = P \ast m$ and $u_x = P_x \ast m$. Note that $\|P\|_{L^1} = \|P_x\|_{L^1} = 1$. Applying Young’s inequality, one can easily obtain (i) – (iii).

Since equation (1) can be used to derive the following form

$$\left((mn)^{\frac{1}{2}}\right)_t + \left((mn)^{\frac{1}{2}}uv\right)_x = 0,$$

it immediately follows that $E_0(u, v)$ is a conserved density. On the other hand, by equation (5), we have

$$\frac{d}{dt} \int_{\mathbb{R}} m(t, x) dx = -\int_{-\infty}^{\infty} (uv m_x + (2vu_x + uv_x)m) dx$$

$$= \int_{-\infty}^{\infty} (vu_x m - (uv)m_x) dx \leq \|u\|_{L^\infty} \|v\|_{L^\infty} \int_{-\infty}^{\infty} m(t, x) dx$$

$$\leq \|u_0\|_1 \|v_0\|_1 \int_{-\infty}^{\infty} m(t, x) dx.$$

Since $m_0 \in L^1(\mathbb{R})$, in view of Gronwall’s inequality, we can get

$$\|m(t, \cdot)\|_{L^1} \leq e^{\|u_0\|_1 \|v_0\|_1 t} \|m_0\|_{L^1}.$$ 

Similarly, we find

$$\|n(t, \cdot)\|_{L^1} \leq e^{\|u_0\|_1 \|v_0\|_1 t} \|n_0\|_{L^1}.$$ 

This completes the proof of Theorem 2.8. \(\square\)

3. Global weak solutions. In this section, we will prove that there exists a unique global weak solution to equation (6), provided the initial data $(u_0, v_0)$ satisfy certain sign-invariant conditions.

**Theorem 3.1.** Let $u_0, v_0 \in H^1(\mathbb{R})$. Assume $m_0 = u_0 - u_{0xx}$ and $n_0 = v_0 - v_{0xx} \in \mathcal{M}^+(\mathbb{R})$. Then equation (6) has a pair of unique weak solution $(u, v)$, where

$$u, v \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$$

with the initial data $u(0, x) = u_0, v(0, x) = v_0$ and such that $m = u - u_{xx}, n = v - v_{xx} \in \mathcal{M}^+(\mathbb{R})$ are bounded on $[0, T)$, for any fixed $T > 0$. Moreover, $E_u(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx, E_v(v) = \int_{\mathbb{R}} (v^2 + v_x^2) dx$ and $H(u, v) = \int_{\mathbb{R}} (uv + uv_x) dx$ are conserved densities.

**Proof.** First, we shall prove $u, v \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$. Let $u_0, v_0 \in H^1(\mathbb{R})$ and assume that $m_0 = u_0 - u_{0xx}, n_0 = v_0 - v_{0xx} \in \mathcal{M}^+(\mathbb{R})$. Note that $u_0 = P \ast m_0$ and $v_0 = P \ast n_0$. Thus, we have for any $f \in L^\infty(\mathbb{R})$,

$$\|u_0\|_{L^1} = \|P \ast m_0\|_{L^1} = \sup_{\|f\|_{L^\infty} \leq 1} \int_{\mathbb{R}} f(x)(P \ast m_0)(x) dx$$

$$= \sup_{\|f\|_{L^\infty} \leq 1} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} P(x - y) dm_0(y) dx$$

$$= \sup_{\|f\|_{L^\infty} \leq 1} \int_{\mathbb{R}} (P \ast f)(y) dm_0(y)$$

$$\leq \sup_{\|f\|_{L^1} \|f\|_{L^\infty} \|m_0\|_{\mathcal{M}(\mathbb{R})}} \|P\|_{L^1} \|f\|_{L^\infty} \|m_0\|_{\mathcal{M}(\mathbb{R})} = \|m_0\|_{\mathcal{M}(\mathbb{R})}.$$ 

(11)
Similarly, we have
\[ \|u_0\|_{L^1} \leq \|n_0\|_{\mathcal{M}(\mathbb{R})}. \] (12)

We first prove that there exists a corresponding \((u, v)\) with the initial data \((u_0, v_0)\), which belongs to \(H^1_{loc}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R})) \times H^1_{loc}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R}))\), satisfying equation (6) in the sense of distributions.

Let us define \(u_0^0 = \rho_n * u_0 \in H^\infty(\mathbb{R})\) and \(v_0^0 = \rho_n * v_0 \in H^\infty(\mathbb{R})\) for \(n \geq 1\). Obviously, we have
\[ u_0^0 \to u_0 \quad H^1(\mathbb{R}), \quad n \to \infty, \]
\[ v_0^0 \to v_0 \quad H^1(\mathbb{R}), \quad n \to \infty, \] (13)
and for all \(n \geq 1\),
\[ \|u_0^n\|_1 = \|\rho_n * u_0\|_1 \leq \|u_0\|_1, \quad \|v_0^n\|_1 \leq \|v_0\|_1, \]
\[ \|u_0^n\|_{L^1} = \|\rho_n * u_0\|_{L^1} \leq \|u_0\|_{L^1}, \quad \|v_0^n\|_{L^1} \leq \|v_0\|_{L^1}, \] (14)
in view of Young’s inequality. Note that for all \(n \geq 1\),
\[ m_0^n = u_0^n - u_{0,xx} = \rho_n * m_0 \geq 0, \quad \text{and} \quad n_0^n = v_0^n - v_{0,xx} = \rho_n * v_0 \geq 0. \]
Comparing with the proof of relation (11) and (12), we get
\[ \|m_0^n\|_{L^1} \leq \|m_0\|_{\mathcal{M}(\mathbb{R})}, \quad \text{and} \quad \|n_0^n\|_{L^1} \leq \|n_0\|_{\mathcal{M}(\mathbb{R})}, \quad n \geq 1. \] (15)
By Theorem 2.8, we obtain that there exists a global strong solution
\[ u^n = u^n(\cdot, u_0^n), v^n = v^n(\cdot, v_0^n) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})) \]
for every \(s \geq 3\), and we have \(u^n(t, x) - u^n_{xx}(t, x) \geq 0\) and \(v^n(t, x) - v^n_{xx}(t, x) \geq 0\) for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\). In view of theorem 2.8 and (14), we obtain for \(n \geq 1\) and \(t \geq 0\),
\[ \|u^n_2(t, \cdot)\|_{L^{\infty}} \leq \|u^n(t, \cdot)\|_{L^{\infty}} \leq \|u^n(t, \cdot)\|_1 = \|u^n_2\|_1 \leq \|u_0\|_1, \]
\[ \|v^n_2(t, \cdot)\|_{L^{\infty}} \leq \|v^n(t, \cdot)\|_{L^{\infty}} \leq \|v^n(t, \cdot)\|_1 = \|v^n_2\|_1 \leq \|v_0\|_1. \] (16)
By the above inequality, we have
\[ \|u^n(t, \cdot)v^n(t, \cdot)u^n_{x}(t, \cdot)\|_{L^2} \leq \|u^n(t, \cdot)\|_{L^\infty} \|v^n(t, \cdot)\|_{L^\infty} \|u^n_{x}(t, \cdot)\|_{L^2} \]
\[ \leq \|u^n(t, \cdot)\|_{L^2}^2 \|v^n(t, \cdot)\|_1 \leq \|u_0\|_{L^2}^2 \|v_0\|_1. \] (17)
Similarly, we have
\[ \|v^n(t, \cdot)u^n(t, \cdot)v^n_{x}(t, \cdot)\|_{L^2} \leq \|v_0\|_{L^2}^2 \|u_0\|_1. \] (18)
By Young’s inequality and (16), for all \(t \geq 0\) and \(n \geq 1\), we obtain
\[ \left\| P_x \left( \frac{1}{2} (u^n_x)^2 v^n + u^n u^n_x v^n + (u^n)^2 v^n \right) + \frac{1}{2} P \ast \left( (u^n_x)^2 v^n \right) \right\|_{L^2} \leq \left\| P_x \right\|_{L^2} \left\| \frac{1}{2} (u^n_x)^2 v^n + u^n u^n_x v^n + (u^n)^2 v^n \right\|_{L^1} + \frac{1}{2} \left\| P \right\|_{L^2} \left\| (u^n_x)^2 v^n \right\|_{L^1} \]
\[ \leq \frac{1}{2} \left\| u^n_x \right\|_{L^2}^2 \|v^n\|_{L^{\infty}} \leq \frac{1}{2} \left\| u^n \right\|_{L^{\infty}} \|u^n_x\|_{L^1} \|v^n\|_{L^2} + \left\| u^n \right\|_{L^2} \|v^n\|_{L^2} + \left\| u^n \right\|_{L^2} \|v^n\|_{L^{\infty}} \leq \frac{1}{2} \|v^n\|_{L^{\infty}} \|u^n_x\|_{L^2} \leq \frac{5}{2} \|u^n\|_{L^2} \|v^n\|_{L^1}. \] (19)
Similarly, we get
\[ \left\| P_x \left( \frac{1}{2} (v^n_x)^2 u^n + v^n u^n_x v^n + (v^n)^2 u^n \right) + \frac{1}{2} P \ast \left( (v^n_x)^2 u^n \right) \right\|_{L^2} \leq \frac{5}{2} \|v^n\|_{L^2} \|u^n\|_{L^1}. \] (20)
Combining (17)-(20) with equation (6) for all \( t \geq 0 \) and \( n \geq 1 \), we find
\[
\left\| \frac{d}{dt} u^n(t, \cdot) \right\|_{L^2} \leq \frac{7}{2} \| u_0 \|_1^2 \| v_0 \|_1, \quad \text{and} \quad \left\| \frac{d}{dt} v^n(t, \cdot) \right\|_{L^2} \leq \frac{7}{2} \| v_0 \|_1^2 \| u_0 \|_1. 
\] (21)

For fixed \( T > 0 \), by Theorem 2.8 and (21), we have
\[
\int_0^T \int_{\mathbb{R}} \left( |u^n(t,x)|^2 + |u_0^n(t,x)|^2 + |u_0^n(t,x)|^2 \right) dx dt \leq \left( \| u_0 \|_1^2 + \frac{49}{4} \| v_0 \|_1^2 \| v_0 \|_1^2 \right) T, 
\]
\[
\int_0^T \int_{\mathbb{R}} \left( |v^n(t,x)|^2 + |v_0^n(t,x)|^2 + |v_0^n(t,x)|^2 \right) dx dt \leq \left( \| v_0 \|_1^2 + \frac{49}{4} \| v_0 \|_1^2 \| v_0 \|_1^2 \right) T. 
\] (22)

It follows that the sequence \( \{u^n\}_{n \geq 1} \) is uniformly bounded in the space \( H^1((0,T) \times \mathbb{R}) \). Thus we can extract a subsequence such that
\[
u_{n_k} \rightharpoonup u \quad \text{weakly in} \quad H^1((0,T) \times \mathbb{R}) \quad \text{for} \quad n_k \to \infty \] (23)
and
\[
u_{n_k} \to u, \quad \text{a.e. on} \quad (0,T) \times \mathbb{R} \quad \text{for} \quad n_k \to \infty, \] (24)
for some \( u \in H^1((0,T) \times \mathbb{R}) \). By Theorem 2.8, (11) and (14), we have that for fixed \( t \in (0,T) \), the sequence \( u_{n_k}^x(t, \cdot) \in BV(\mathbb{R}) \) satisfies
\[
\mathcal{V}[u_{n_k}^x(t, \cdot)] = \| u_{n_k}^x(t, \cdot) \|_{L^1} \leq \| u^n_k(t, \cdot) \|_{L^1} + \| m_{n_k}^x(t, \cdot) \|_{L^1} \\
\leq 2\| m_{n_k}^x(t, \cdot) \|_{L^1} \leq 2e^{\| u_n \|_1 \| u_0 \|_1 t} \| m_0 \|_{\mathcal{M}(\mathbb{R})} \\
\leq 2e^{\| u_0 \|_1 \| v_0 \|_1 t} \| m_0 \|_{\mathcal{M}(\mathbb{R})}
\]
and
\[
\| u_{n_k}^x(t, \cdot) \|_{L^\infty} \leq \| u_{n_k}^x(t, \cdot) \|_1 = \| u_0^n(t, \cdot) \|_1 \leq \| u_0 \|_1.
\]

Applying Helly’s theorem, we obtain that there exists a subsequence, denoted again by \( \{ u_{n_k}^x(t, \cdot) \} \), which converges at every point to some function \( \bar{u}(t, \cdot) \) of finite variation with
\[
\mathcal{V}[\bar{u}(t, \cdot)] \leq 2e^{\| u_0 \|_1 \| v_0 \|_1 t} \| m_0 \|_{\mathcal{M}(\mathbb{R})}.
\]
Since for almost all \( t \in (0,T) \), \( u_{n_k}^x(t, \cdot) \to u_x(t, \cdot) \) in \( D'(\mathbb{R}) \) in view of (24), it follows that \( \bar{u}(t, \cdot) = u_x(t, \cdot) \) for a.e. \( t \in (0,T) \). Therefore, we have
\[
u_{n_k}^x \rightharpoonup u_x \quad \text{a.e. on} \quad (0,T) \times \mathbb{R} \quad \text{for} \quad n_k \to \infty, \] (25)
and for a.e. \( t \in (0,T) \),
\[
\mathcal{V}[u_x(t, \cdot)] = \| u_x(t, \cdot) \|_{\mathcal{M}(\mathbb{R})} \leq 2e^{\| u_0 \|_1 \| v_0 \|_1 t} \| m_0 \|_{\mathcal{M}(\mathbb{R})}.
\]

We can analogously extract a subsequence of \( \{ v_{n_k}^x \} \), denoted again by \( \{ v_{n_k}^x \} \) such that
\[
v_{n_k}^x \to v \quad \text{a.e. on} \quad (0,T) \times \mathbb{R} \quad \text{for} \quad n_k \to \infty \] (26)
and \( v_{n_k}^x \to v_x \quad \text{a.e. on} \quad (0,T) \times \mathbb{R} \quad \text{for} \quad n_k \to \infty \).

By Theorem 2.8 (ii) – (iii) and (16), we have
\[
\left\| \frac{1}{2} (u_{n_k}^x)^2 v^n + u^n u_{n_k}^x v_{n_k}^x + (u^n)^2 v^n + \frac{1}{2} (u_{n_k}^x)^2 v_{n_k}^x \right\|_{L^1} \leq \| u_0 \|_1^2 \| v_0 \|_1.
\]
For fixed \( t \in (0,T) \), it follows that the sequence \( \left\{ \frac{1}{2} (u_{n_k}^x)^2 v^n + u^n u_{n_k}^x v_{n_k}^x + (u^n)^2 v^n + \frac{1}{2} (u_{n_k}^x)^2 v_{n_k}^x \right\} \) is uniformly bounded in \( L^1(\mathbb{R}) \). Therefore, it has a subsequence converging weakly in \( L^1(\mathbb{R}) \), denoted again by \( \left\{ \frac{1}{2} (u_{n_k}^x)^2 v^n + u^n u_{n_k}^x v_{n_k}^x + (u^n)^2 v^n + \frac{1}{2} (u_{n_k}^x)^2 v_{n_k}^x \right\} \).
By (24) and (25), we deduce that the weak $L^1(\mathbb{R})$-limit is \( \frac{1}{2}(u_x^2)^2v^n + uu_xv_x + u^2v + \frac{1}{2}(u_x^2)^2v_x \). Note that \( P, P_x \in L^\infty(\mathbb{R}) \). It follows that

\[
P_x \left[ \frac{1}{2}(u_x^n)^2v^n + uu_xv_x^n + (u^n)^2v^n \right] + P \left[ \frac{1}{2}(u_x^n)^2v^n \right] 
\rightarrow P_x \left[ \frac{1}{2}v_x^n + uu_xv_x + u^2v \right] + P \left[ \frac{1}{2}u_x^n v_x \right], \quad \text{as} \quad n \to \infty.
\]

We can analogously obtain that

\[
P_x \left[ \frac{1}{2}(v_x^n)^2u^n + v^n v_x^n - v^n u^n + (v^n)^2u^n \right] + P \left[ \frac{1}{2}(v_x^n)^2u^n \right] 
\rightarrow P_x \left[ \frac{1}{2}v^2 u^n + v^n v_x u_x + u^2v \right] + P \left[ \frac{1}{2}u^2 v_x \right], \quad \text{as} \quad n \to \infty.
\]

Combining (24)-(26) with (27) and (28), we deduce that \((u, v)\) satisfies equation (6) in \( D'((0, T) \times \mathbb{R}) \).

Since \( u^n(t, \cdot) \) is uniformly bounded in \( L^2(\mathbb{R}) \) for all \( t \in \mathbb{R}_+ \) and \( \|u^n(t, \cdot)\|_1 \) has a uniform bound as \( t \in \mathbb{R}_+ \) and all \( n \geq 1 \). Hence the family \( t \mapsto u^n(t, \cdot) \in H^1(\mathbb{R}) \) is weakly equicontinuous on \([0, T]\) for any \( T > 0 \). An application of the Arzela-Ascoli theorem yields that \( \{u^n\} \) contains a subsequence, denoted again by \( \{u^n\} \), which converges weakly in \( H^1(\mathbb{R}) \), uniformly in \( t \in [0, T] \). The limit function is \( u \). Because \( T \) is arbitrary, we have that \( u \) is locally and weakly continuous from \([0, \infty)\) into \( H^1(\mathbb{R}) \), i.e.

\[
u \in C_{w, loc}(\mathbb{R}_+; H^1(\mathbb{R})).
\]

For a.e. \( t \in \mathbb{R}_+ \), since \( u^n(t, \cdot) \rightharpoonup u(t, \cdot) \) weakly in \( H^1(\mathbb{R}) \), in view of (15) and (16), we obtain

\[
\|u(t, \cdot)\|_{L^\infty} \leq \|u(t, \cdot)\|_1 \leq \liminf_{n_k \to \infty} \|u^n(t, \cdot)\|_1 = \liminf_{n_k \to \infty} \|v^n_0(t, \cdot)\|_1 \leq \liminf_{n_k \to \infty} \|P_1\| \|m^n_0(t, \cdot)\|_{L^1} \leq \|m_0\|_{\mathcal{M}(\mathbb{R})}
\]

for a.e. \( t \in \mathbb{R}_+ \). The previous relation implies that

\[
u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})).
\]

Note that by Theorem 2.8 and (15), we have

\[
\|u^n_0(t, \cdot)\|_{L^\infty} \leq \|u^n(t, \cdot)\|_{L^\infty} \leq \|u^n(t, \cdot)\|_1 \leq \|P_1\| \|m^n_0(t, \cdot)\|_{L^1} \leq \|m_0(t, \cdot)\|_{\mathcal{M}(\mathbb{R})}.
\]

Combining this with (25), we deduce that

\[
u_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R}).
\]

This shows that

\[
u \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})).
\]

Taking the same way as \( u \), we get

\[
v \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})).
\]

Please note that we use the subsequence of \( \{v^n\} \) which is determined after using the Arzela-Ascoli theorem.
By differentiation of the first equation of (31), we obtain

\[
\begin{align*}
\rho_n \ast u_t + \rho_n \ast (uvu_x) + \rho_n \ast P_x \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \\
\quad + \frac{1}{2} \rho_n \ast P \ast (u_x^2 v_x) = 0, \\
\rho_n \ast v_t + \rho_n \ast (uvu_x) + \rho_n \ast P_x \ast \left( \frac{1}{2} v_x^2 v + vu_x v_x + v^2 u \right) \\
\quad + \frac{1}{2} \rho_n \ast P \ast (v_x^2 u_x) = 0.
\end{align*}
\]

By differentiation of the first equation of (31), we obtain

\[
\begin{align*}
\rho_n \ast u_{xt} + \rho_n \ast (uvu_x)_x + \rho_n \ast P_x \ast \left( \frac{1}{2} u_x^2 v_x \right) \\
\quad + \rho_n \ast P_{xx} \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) = 0. 
\end{align*}
\]

(32)

Note that $\partial^2 (P \ast f) = P \ast f - f$, $f \in L^2(\mathbb{R})$. We can rewrite (32) as

\[
\begin{align*}
\rho_n \ast u_{xt} + \rho_n \ast (uvu_x) + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \\
\quad - \rho_n \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v_x \right) = 0. 
\end{align*}
\]

(33)

Take these two equation (32) and (33) into the integration below, we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\rho_n \ast u)^2 + (\rho_n \ast u_x)^2 dx \\
= \int_{\mathbb{R}} (\rho_n \ast u)(\rho_n \ast u_t) + (\rho_n \ast u_x)(\rho_n \ast u_{xt}) dx \\
= - \int_{\mathbb{R}} (\rho_n \ast u) \left( \rho_n \ast (uvu_x) + \rho_n \ast P_x \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right) \\
\quad + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v_x \right) dx \\
\quad - \int_{\mathbb{R}} (\rho_n \ast u_x) \left( \rho_n \ast (uvu_x) + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right) \\
\quad - \rho_n \ast \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v_x \right) \right) dx.
\end{align*}
\]

(34)

Note that

$$
\lim_{n \to \infty} ||\rho_n \ast u - u||_{L^2} = \lim_{n \to \infty} ||\rho_n \ast (uvu_x) - uvu_x||_{L^2} = 0.
$$

Therefore, by using Hölder inequality, we have for a.e. $t \in \mathbb{R}^+$

$$
\int_{\mathbb{R}} (\rho_n \ast u)(\rho_n \ast (uvu_x)) dx \rightarrow \int_{\mathbb{R}} u^2 v u_x dx, \quad \text{as} \quad n \to \infty.
$$
Similarly, for a.e. $t \in \mathbb{R}$

$$
\int_{\mathbb{R}} (\rho_n * u) \left( \rho_n * P_x \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right) \, dx \longrightarrow \int_{\mathbb{R}} u P_x \left( \frac{1}{2} u_x^2 v \right) \, dx, \quad \text{as } n \to \infty,
$$

$$
\int_{\mathbb{R}} (\rho_n * u) \left( \rho_n * P \left( \frac{1}{2} u_x^2 v \right) \right) \, dx \longrightarrow \int_{\mathbb{R}} u P \left( \frac{1}{2} u_x^2 v \right) \, dx, \quad \text{as } n \to \infty,
$$

$$
\int_{\mathbb{R}} (\rho_n * u_x) \left( \rho_n * P \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right) \, dx \longrightarrow \int_{\mathbb{R}} u_x P \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \, dx, \quad \text{as } n \to \infty,
$$

$$
\int_{\mathbb{R}} (\rho_n * u_x) \left( \rho_n * \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right) \, dx \longrightarrow \int_{\mathbb{R}} u_x \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \, dx, \quad \text{as } n \to \infty,
$$

$$
\int_{\mathbb{R}} (\rho_n * u_x) \left( \rho_n * P_x \left( \frac{1}{2} u_x^2 v_x \right) \right) \, dx \longrightarrow \int_{\mathbb{R}} u_x P_x \left( \frac{1}{2} u_x^2 v_x \right) \, dx, \quad \text{as } n \to \infty,
$$

as $u(t, \cdot), v(t, \cdot) \in H^1(\mathbb{R})$ and $u_x, v_x \in L^{\infty}(\mathbb{R} \times \mathbb{R})$. Furthermore, note that

$$
\int_{\mathbb{R}} (\rho_n * u_x)(\rho_n * (uvu_x)) dx = - \int_{\mathbb{R}} (\rho_n * u)(\rho * (uvu_x)) dx
$$

$$
+ \int_{\mathbb{R}} (\rho_n * u_x)(\rho_n * uv)(\rho_n * u_x) dx + \frac{1}{2} \int_{\mathbb{R}} (\rho_n * u_x)^2 (\rho_n * (uv)_x) dx. \tag{35}
$$

Observe that

$$
\int_{\mathbb{R}} (\rho_n * u_x)^2 (\rho_n * (uv)_x) dx \longrightarrow \int_{\mathbb{R}} u_x^2 (uv)_x dx, \quad \text{as } n \to \infty.
$$

On the other hand

$$
\|\rho_{nxx} * u\|_{L^1} \leq \|u_{xx}\|_{M(\mathbb{R})} \leq 2e^{|u_0||\|v_0||_{t}||m_0|_{M(\mathbb{R})}}, \forall t \in [0, T).
$$

As $u(t, \cdot), v(t, \cdot) \in H^1(\mathbb{R})$ and $u_x, v_x \in L^{\infty}(\mathbb{R} \times \mathbb{R})$, by Lemma 2.5, it follows that

$$
\|(\rho_n * uv)(\rho_n * u_x) - (\rho_n * (uvu_x))\|_{L^\infty} \to 0, \quad \text{as } n \to \infty.
$$

Therefore,

$$
\int_{\mathbb{R}} (\rho_{n,xx} * u) ((\rho_n * uv)(\rho_n * u_x) - \rho_n * (uvu_x)) \, dx \to 0, \quad \text{as } n \to \infty.
$$

In view of the above relations and (35), we obtain

$$
\int_{\mathbb{R}} (\rho_n * u_x)(\rho_n * (uvu_x)) dx \to \frac{1}{2} \int_{\mathbb{R}} u_x^2 (uv)_x dx, \quad \text{as } n \to \infty. \tag{36}
$$

Let us define

$$
E_n^u(t) = \int_{\mathbb{R}} (\rho_n * u)^2 + (\rho_n * u_x)^2 dx, \tag{37}
$$
and

\[ G_n^u(t) = -2 \int_{\mathbb{R}} (\rho_n * u)(\rho_n * (uvu_x) + \rho_n * P_x * \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) + \rho_n * P * \left( \frac{1}{2} u_x^2 v_x \right)) \, dx \]

\[-2 \int_{\mathbb{R}} (\rho_n * u_x)(\rho_n * (uvu_x) + \rho_n * P * \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) - \rho_n * \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) + \rho_n * P_x * \left( \frac{1}{2} u_x^2 v_x \right)) \, dx.\]

We have proved that for fixed \( T > 0, \) for a.e. \( t \in [0, T], \)

\[
\begin{cases}
\frac{d}{dt} E_n^u(t) = G_n^u(t), & n \geq 1, \\
G_n^u(t) \to 0, & n \to \infty.
\end{cases}
\] (38)

Therefore, we get

\[ E_n^u(t) - E_n^u(0) = \int_0^t G_n^u(s) \, ds, \quad t \in [0, T), \quad n \geq 1. \] (39)

By Young’s inequality and Hölder’s inequality, it follows that there is a \( K^u(T) > 0 \) such that

\[ |G_n^u(t)| \leq K^u(T), \quad n \geq 1. \]

In view of (38) and (39), an application of Lebesgue’s dominated convergence theorem yields that for fixed a.e. \( t \in \mathbb{R}_+, \)

\[ \lim_{n \to \infty} (E_n^u(t) - E_n^u(0)) = 0. \]

By (24) and the above relation, for fixed \( t \in \mathbb{R}_+, \) we can get

\[ E_u(u) = \lim_{n \to \infty} E_n^u(t) = \lim_{n \to \infty} E_n^u(0) = E_u(u_0). \]

By Theorem 2.8, we infer that for all fixed \( t \in \mathbb{R}_+, \) \( E_u(u) \) is conserved. Similarly, we can show that \( E_v(v) \) is also conserved.

Next, we prove that \( H(u, v) \) is a conserved density.

By differentiation of the second equation of (31), we obtain this relation:

\[ \rho_n * v_{xt} + \rho_n * (uvu_x) + \rho_n * P * \left( \frac{1}{2} v_x^2 u + uu_x v_x + v^2 u \right) - \rho_n * \left( \frac{1}{2} v_x^2 u_x + uu_x v_x + v^2 u \right) + \rho_n * P_x * \left( \frac{1}{2} v_x^2 u_x \right) = 0. \] (40)

In view of (31), (33) and (40), we obtain

\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}} (\rho_n * u)(\rho_n * v) + (\rho_n * u_x)(\rho_n * v_x) \, dx \\
= \int_{\mathbb{R}} (\rho_n * u)(\rho_n * v_t) + (\rho_n * u_x)(\rho_n * v_{xt}) + (\rho_n * u_t)(\rho_n * v) + (\rho_n * u_{xt})(\rho_n * v_x) \, dx \\
= -\int_{\mathbb{R}} (\rho_n * u) \left( \rho_n * (uvu_x) + \rho_n * P_x * \left( \frac{1}{2} v_x^2 u + uu_x v_x + v^2 u \right) \right)
\end{align*}
\]
\[
+ \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u_x \right) \, dx \\
- \int_{\mathbb{R}} \left( \rho_n \ast u_x \right) \left( \rho_n \ast (u v u_x) + \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \right) \\
- \rho_n \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \, dx \\
- \int_{\mathbb{R}} \left( \rho_n \ast v \right) \left( \rho_n \ast (v u u_x) + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) \right) \\
+ \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v \right) \, dx \\
- \int_{\mathbb{R}} \left( \rho_n \ast v_x \right) \left( \rho_n \ast (u v v_x) + \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \right) \\
- \rho_n \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \, dx \\
- \int_{\mathbb{R}} \left( \rho_n \ast v \right) \left( \rho_n \ast (v u u_x) + \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v + u u_x v_x + u^2 v \right) \right) \\
+ \rho_n \ast P \ast \left( \frac{1}{2} u_x^2 v \right) \, dx \\
- \int_{\mathbb{R}} \left( \rho_n \ast v \right) \left( \rho_n \ast (v u v_x) + \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \right) \\
- \rho_n \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \, dx.
\]

We can analogously get the similar convergence like the case \( \frac{d}{dt} \int_{\mathbb{R}} (\rho_n \ast u)^2 + (\rho_n \ast u_x)^2 \, dx \) by using Lemma 2.5, \( u(t, \cdot), v(t, \cdot) \in H^1(\mathbb{R}) \) and \( u_x, v_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \).

It is nature to define

\[
H_n(t) = \int_{\mathbb{R}} \left( \rho_n \ast u \right) \left( \rho_n \ast v \right) + \left( \rho_n \ast u_x \right) \left( \rho_n \ast v_x \right) \, dx,
\]

and

\[
G_n^{u,v}(t) = - \int_{\mathbb{R}} \left( \rho_n \ast u \right) \left( \rho_n \ast (u v v_x) + \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \right) \\
+ \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u \right) \, dx \\
- \int_{\mathbb{R}} \left( \rho_n \ast v \right) \left( \rho_n \ast (v u v_x) + \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u + v u_x v_x + v^2 u \right) \right) \\
+ \rho_n \ast P \ast \left( \frac{1}{2} v_x^2 u \right) \, dx.
\]

And it is easy to get

\[
H_n(t) - H_n(0) = \int_0^t G_n^{u,v}(s) \, ds, \quad t \in [0, T), \quad n \geq 1.
\]

Similarly, we get this estimate by using Young’s inequality and Hölder’s inequality:

\[
|G_n^{u,v}(t)| \leq K^{u,v}(T), \quad n \geq 1.
\]
Similarly, we arrive at the conclusion:

\[
H(u, v) = \lim_{n \to \infty} H_n(t) = \lim_{n \to \infty} H_n(0) = H(u_0, v_0),
\]

which indicates that \( H(u, v) \) is a conserved density.

Since \( L^1(\mathbb{R}) \subset (L^\infty(\mathbb{R}))^\ast \subset (C_0(\mathbb{R}))^\ast = \mathcal{M}(\mathbb{R}) \). It is not too hard to show that for a.e. \( t \in [0, T) \),

\[
\|m(t, \cdot)\| \leq 3e^{\|u_0\|_1 \|v_0\|_1} t \|m_0\|_\mathcal{M}(\mathbb{R}).
\]

For any fixed \( T, \forall t \in [0, T) \), we have proved

\[
(u(t, \cdot) - u_{xx}(t, \cdot)) \in \mathcal{M}(\mathbb{R}).
\]

Therefore, in view of \((24) \text{ and } (25)\), we obtain that for all \( t \in [0, T) \), as \( n \to \infty \),

\[
u^{n_k}(t, \cdot) - u^{n_k}_{xx}(t, \cdot) \to u(t, \cdot) - u_{xx}(t, \cdot) \text{ in } D'(\mathbb{R}).
\]

Since \( u^n(t, \cdot) - u^{n_k}_{xx}(t, \cdot) \geq 0 \) for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \), we deduce that for a.e. \( t \in [0, T) \)

\[
u(t, \cdot) - u_{xx}(t, \cdot) \in \mathcal{M}^+(\mathbb{R}).
\]

Similarly, we arrive at the conclusion:

\[
u(t, \cdot) - v_{xx}(t, \cdot) \in \mathcal{M}^+(\mathbb{R}).
\]

Finally, we show the uniqueness of the weak solutions of equation \((6)\). Let \((u, v)\) and \((\bar{u}, \bar{v})\) be two weak solutions of equation \((6)\) in the class \((f, g) \in W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R})) \times W^{1, \infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))\)

Note that

\[
\|u(t, \cdot) - u_{xx}(t, \cdot)\|_{\mathcal{M}(\mathbb{R})} \leq 3e^{\|u_0\|_1 \|v_0\|_1} t \|m_0\|_\mathcal{M}(\mathbb{R}),
\]

\[
\|v(t, \cdot) - v_{xx}(t, \cdot)\|_{\mathcal{M}(\mathbb{R})} \leq 3e^{\|u_0\|_1 \|v_0\|_1} t \|n_0\|_\mathcal{M}(\mathbb{R}) \text{ for a.e. } t \in [0, T).
\]

Define

\[
M(T) = \sup_{t \in [0, T)} \left\{ \|u(t, \cdot) - u_{xx}(t, \cdot)\|_{\mathcal{M}(\mathbb{R})} + \|v(t, \cdot) - v_{xx}(t, \cdot)\|_{\mathcal{M}(\mathbb{R})} + \|\bar{u}(t, \cdot) - \bar{u}_{xx}(t, \cdot)\|_{\mathcal{M}(\mathbb{R})} + \|\bar{v}(t, \cdot) - \bar{v}_{xx}(t, \cdot)\|_{\mathcal{M}(\mathbb{R})} \right\}.
\]

Then for fixed \( T \), we obtain \( M(T) < \infty \). For all \((t, x) \in [0, T) \times \mathbb{R}\), in view of \((11)\), we find that

\[
\|u(t, \cdot)\|_{L^1} \leq \|P\|_{L^1} M(T) = M(T),
\]

\[
\|u_x(t, \cdot)\|_{L^1} \leq \|P_\ast\|_{L^1} M(T) = M(T),
\]

\[
\|v(t, \cdot)\|_{L^1}, \|v_x(t, \cdot)\|_{L^1}, \|\bar{u}(t, \cdot)\|_{L^1}, \|\bar{u}_x(t, \cdot)\|_{L^1}, \|\bar{v}(t, \cdot)\|_{L^1} \text{ and } \|\bar{v}_x(t, \cdot)\|_{L^1} \leq M(T).
\]

On the other hand, from \((29) \text{ and } (30)\), we have

\[
\|u(t, \cdot)\|_{L^\infty} \leq \|m_0\|_{\mathcal{M}(\mathbb{R})} \leq N, \quad \|u_x(t, \cdot)\|_{L^\infty} \leq \|m_0\|_{\mathcal{M}(\mathbb{R})} \leq N,
\]

\[
\|v(t, \cdot)\|_{L^\infty} \leq \|n_0\|_{\mathcal{M}(\mathbb{R})} \leq N, \quad \|v_x(t, \cdot)\|_{L^\infty} \leq \|n_0\|_{\mathcal{M}(\mathbb{R})} \leq N,
\]

\[
\|\bar{u}(t, \cdot)\|_{L^\infty}, \|\bar{u}_x(t, \cdot)\|_{L^\infty}, \|\bar{v}(t, \cdot)\|_{L^\infty} \text{ and } \|\bar{v}_x(t, \cdot)\|_{L^\infty} \leq N.
\]

Let us define

\[
\hat{u}(t, x) = u(t, x) - \bar{u}(t, x) \quad \text{and} \quad \hat{v}(t, x) = v(t, x) - \bar{v}(t, x), \quad (t, x) \in [0, T) \times \mathbb{R}.
\]
Convoluting equation (6) for \((u, v)\) and \((\bar{u}, \bar{v})\) with \(\rho_n\), we have that for a.e. \(t \in [0, T)\) and all \(n \geq 1,\)

\[
\frac{d}{dt} \int_R |\rho_n * \hat{u}| dx = \int_R \rho_n * \hat{u}_t \text{sgn}(\rho_n * \hat{u}) dx \\
= - \int_R \rho_n * (\hat{u}u_x + \bar{u}u_x v + \bar{u}\hat{v}u_x) \text{sgn}(\rho_n * \hat{u}) dx \\
- \int_R \rho_n * P_x * \left( \frac{1}{2} \hat{u}(u_x + \bar{u}x) + \frac{1}{2} \hat{u}^2 \hat{v} \hat{v} + \hat{u}u_x v_x + \bar{u}v_x \hat{u}_x \right) dx \\
+ \hat{u}u_x \hat{v}_x + \hat{u}(u + \bar{u})v + u^2 \hat{v}) \text{sgn}(\rho_n * \hat{u}) dx \\
- \int_R \rho_n * \frac{1}{2} P_x * (\hat{u}_x (u_x + \bar{u}), v_x + \bar{u}^2 \hat{v}x) \text{sgn}(\rho_n * \hat{u}) dx.
\]

Using (46) and Young’s inequality, we infer that for a.e. \(t \in [0, T)\) and all \(n \geq 1\)

\[
\frac{d}{dt} \int_R |\rho_n * \hat{u}| dx \\
\leq C \left( \int_R |\rho_n * \hat{u}| dx + \int_R |\rho_n * \hat{u}_x| dx + \int_R |\rho_n * \hat{v}| dx + \int_R |\rho_n * \hat{v}_x| dx \right).
\]

where \(C\) is a constant depending on \(N\). Similarly, convoluting equation (6) for \((u, v)\) and \((\bar{u}, \bar{v})\) with \(\rho_{nx}\), it follows that

\[
\frac{d}{dt} \int_R |\rho_n * \hat{u}_x| dx = \int_R \rho_n * \hat{u}_x \text{sgn}(\rho_{nx} * \hat{u}) dx \\
= - \int_R \rho_n * (\hat{u}u_x + \bar{u}u_x v + \bar{u}\hat{v}u_x)_x \text{sgn}(\rho_{nx} * \hat{u}) dx \\
- \int_R \rho_n * P_{xx} * \left( \frac{1}{2} \hat{u}(u_x + \bar{u}_x) + \frac{1}{2} \hat{u}^2 \hat{v} + \hat{u}u_x v_x + \bar{u}v_x \hat{u}_x \right) dx \\
+ \hat{u}_x \hat{v}_x + \hat{u}(u + \bar{u})v + u^2 \hat{v}) \text{sgn}(\rho_{nx} * \hat{u}) dx \\
- \int_R \rho_n * \frac{1}{2} P_x * (\hat{u}_x (u_x + \bar{u}), v_x + \bar{u}^2 \hat{v}x) \text{sgn}(\rho_{nx} * \hat{u}) dx.
\]

For the term \(I_1\), we have

\[
I_1 = - \int_R \rho_n * (\hat{u}_x v u_x + \bar{u}_x v_x + \bar{u}_x u_x \hat{v} + \bar{u}_x u_x \hat{v} + \bar{u}_x \hat{v}_x + \bar{u}\hat{v}_x) \text{sgn}(\rho_{nx} * \hat{u}) dx \\
\leq C \left( \int_R |\rho_n * \hat{u}| dx + \int_R |\rho_n * \hat{u}_x| dx + \int_R |\rho_n * \hat{v}| dx + \int_R |\rho_n * \hat{v}_x| dx \right) \\
- \int_R \rho_n * (\hat{u}_x v u_x + \bar{u}_x v_x + \bar{u}_x u_x \hat{v} + \bar{u}_x u_x \hat{v} + \bar{u}_x \hat{v}_x + \bar{u}\hat{v}_x) \text{sgn}(\rho_{nx} * \hat{u}) dx \\
\leq C \left( \int_R |\rho_n * \hat{u}| dx + \int_R |\rho_n * \hat{u}_x| dx + \int_R |\rho_n * \hat{v}| dx + \int_R |\rho_n * \hat{v}_x| dx \right) \\
- \int_R (\rho_n * \bar{u} v) (\rho_n * u_x) \text{sgn}(\rho_{nx} * \hat{u}) dx - \int_R (\rho_n * \bar{u} \hat{v}) (\rho_n * u_x) \text{sgn}(\rho_{nx} * \hat{u}) dx \\
- \int_R (\rho_n * \bar{u} \hat{v}) (\rho_n * \hat{u}_x) \text{sgn}(\rho_{nx} * \hat{u}) dx + R_n(t)
\]
\[
\begin{align*}
\leq C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right) \\
+ \int_R (\rho_n \ast (\hat{u}v)_x)(\rho_n \ast u_x) \text{sgn}(\rho_{nx} \ast \hat{u}) dx + \int_R (\rho_n \ast (\hat{u}v)_x)(\rho_n \ast u_x) \text{sgn}(\rho_{nx} \ast \hat{u}) dx \\
+ \int_R (\rho_n \ast (\hat{u}v)_x)(\rho_n \ast \hat{u}_x) \text{sgn}(\rho_{nx} \ast \hat{u}) dx + R_n(t)
\end{align*}
\]

(50)  

where \( C \) is a constant depending on \( M(T), N, \|u_0\|_1 \) and \( \|v_0\|_1 \) and \( R_n(t) \) satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
R_n(t) \rightarrow 0, \quad n \to \infty, \\
|R_n(t)| \leq \kappa(T), \quad n \geq 1, \quad t \in [0, T).
\end{array} \right.
\end{align*}
\]

(51)

For the second term \( I_2 \), we find

\[
\begin{align*}
I_2 &= - \int_R \rho_n \ast P_x \ast \left( \frac{1}{2} \hat{u}(u_x + \hat{u}_x)v + \frac{1}{2} \hat{u}_x^2 \hat{v} + \frac{1}{2} u^2 \hat{v} + \hat{u}_x v_x + \hat{u} v_x \hat{u}_x \right) + \hat{u}_x v_x + \hat{u} u_x \hat{v} + \hat{u} u \hat{v}_x + \hat{u} (u + \hat{u}) v + u^2 \hat{v} \text{sgn}(\rho_{nx} \ast \hat{u}) dx \\
\leq & 2 \int_R \rho_n \ast \left( \frac{1}{2} \hat{u}(u_x + \hat{u}_x)v + \frac{1}{2} \hat{u}_x^2 \hat{v} + \frac{1}{2} u^2 \hat{v} + \hat{u}_x v_x + \hat{u} v_x \hat{u}_x + \hat{u}_x \hat{v}_x + \hat{u} (u + \hat{u}) v + u^2 \hat{v} \right) \text{sgn}(\rho_{nx} \ast \hat{u}) dx \\
\leq & C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right).
\end{align*}
\]

(52)

For the final term \( I_3 \), we have

\[
\begin{align*}
I_3 &= - \int_R \rho_n \ast \frac{1}{2} P_x \ast \left( \hat{u}_x (u_x + \hat{u}_x) v_x + \hat{u}_x^2 \hat{v}_x \right) \text{sgn}(\rho_{nx} \ast \hat{u}) dx \\
\leq & C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right).
\end{align*}
\]

(53)

Adding these three terms, we obtain

\[
\frac{d}{dt} \int_R |\rho_n \ast \hat{u}_x| dx
\]

\[
\leq C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right) + R_n(t).
\]

(54)

For these terms \( \frac{d}{dt} \int_R |\rho_n \ast \hat{v}| dx \) and \( \frac{d}{dt} \int_R |\rho_n \ast \hat{v}_x| dx \), we have similar results:

\[
\frac{d}{dt} \int_R |\rho_n \ast \hat{v}| dx
\]

\[
\leq C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right),
\]

(55)

\[
\frac{d}{dt} \int_R |\rho_n \ast \hat{v}_x| dx
\]

\[
\leq C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right) + R_n(t).
\]

From (48), (54) and (55), we infer that

\[
\frac{d}{dt} \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right)
\]

\[
\leq C \left( \int_R |\rho_n \ast \hat{u}| dx + \int_R |\rho_n \ast \hat{u}_x| dx + \int_R |\rho_n \ast \hat{v}| dx + \int_R |\rho_n \ast \hat{v}_x| dx \right) + R_n(t).
\]

(56)
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If \( \int_{\mathbb{R}} |\rho_n * \hat{u}| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}| dx \neq 0 \), then by Gronwall’s inequality, we obtain

\[
\left( \int_{\mathbb{R}} |\rho_n * \hat{u}| dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x| dx \right) \leq e^{\int_0^t C \tilde{R}_n(\tau) d\tau} (|\rho_n * \hat{u}| + |\rho_n * \hat{u}_x| + |\rho_n * \hat{v}| + |\rho_n * \hat{v}_x|) (0, x),
\]

(57)

where \( \tilde{R}_n(t) = R_n(t) \left( \int_{\mathbb{R}} |\rho_n * \hat{u}| dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x| dx \right)^{-1} \).

From Lebesgue’s dominated convergence theorem, it follows that

\[
\left( \int_{\mathbb{R}} |\rho_n * \hat{u}| dx + \int_{\mathbb{R}} |\rho_n * \hat{u}_x| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}| dx + \int_{\mathbb{R}} |\rho_n * \hat{v}_x| dx \right) \leq e^{Ct} (|\rho_n * \hat{u}| + |\rho_n * \hat{u}_x| + |\rho_n * \hat{v}| + |\rho_n * \hat{v}_x|) (0, x),
\]

(58)

As \( T \) is arbitrary, \( \hat{u}_0 = \hat{u}_{0,x} = \hat{v}_0 = \hat{v}_{0,x} = 0 \), we obtain \( (u, v) = (\bar{u}, \bar{v}) \). This completes the proof of theorem 3.1. \( \Box \)

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