Abstract

The fitness-level method, also called the method of $f$-based partitions, is an intuitive and widely used technique for the running time analysis of randomized search heuristics. It was originally defined to prove upper and lower bounds on the expected running time. Recently, upper tail bounds were added to the technique; however, these tail bounds only apply to running times that are at least twice as large as the expectation.

We remove this restriction and supplement the fitness-level method with sharp tail bounds, including lower tails. As an exemplary application, we prove that the running time of randomized local search on OneMax is sharply concentrated around $n \ln n - 0.1159n$.

1 Introduction

The running time analysis of randomized search heuristics, including evolutionary algorithms, ant colony optimization and particle swarm optimization, is a vivid research area where many results have been obtained in the last 15 years. Different methods for the analysis were developed as the research area grew. For an overview of the state of the art in the area see the books by Auger and Doerr (2011), Neumann and Witt (2010) and Jansen (2013).

The fitness-level method, also called the method of fitness-based partitions, is a classical and intuitive method for running time analysis, first formalized by Wegener (2001). It applies to the case that the total running time of a search heuristic can be represented as (or bounded by) a sum of geometrically distributed waiting times, where the waiting times account for the number of steps spent on
certain levels of the search space. [Wegener, 2001] presented both upper and lower bounds on the running time of randomized search heuristics using the fitness-level method. The lower bounds relied on the assumption that no level was allowed to be skipped. [Sudholt, 2013] significantly relaxed this assumption and presented a very general lower-bound version of the fitness-level method that allows levels to be skipped with some probability.

Only recently, the focus in running time analysis turned to tail bounds, also called concentration inequalities. [Zhou, Luo, Lu, and Han, 2012] were the first to add tail bounds to the fitness-level method. Roughly speaking, they prove w.r.t. the running time $T$ that $\Pr(T > E(T) + 2\delta h) = e^{-\delta}$ holds, where $h$ is the worst-case expected waiting time over all fitness levels and $\delta > 0$ is arbitrary. An obvious open question was whether the factor 2 in front of the expected value could be “removed” from the tail bound, i.e., replaced with 1; [Zhou et al., 2012] only remark that the factor 2 can be replaced with 1.883.

In this article, we give a positive answer to this question and supplement the fitness-level method also with lower tail bounds. Roughly speaking, we prove in Section 2 that $\Pr(T < E(T) - \delta) \leq e^{-\delta^2/(2s)}$ and $\Pr(T > E(T) + \delta) \leq e^{-\min\{\frac{\delta^2}{16s},\frac{\delta h}{4}\}}$, where $s$ is the sum of the squares of the waiting times over all fitness levels. We apply the technique to a classical benchmark problem, more precisely to the running time analysis of randomized local search (RLS) on OneMax in Section 3, and prove a very sharp concentration of the running time around $n \ln n - 0.1159n$. We finish with some conclusions and a pointer to related work.

## 2 New Tail Bounds for Fitness Levels

[Miscellaneous authors, 2011] on the internet discussed tail bounds for a special case of our problem, namely the coupon collector problem [Motwani and Raghavan, 1995, Chapter 3.6]. Inspired by this discussion, we present our main result in Theorem 1 below. It applies to the scenario that a random variable (e.g., a running time) is given as a sum of geometrically distributed independent random variables (e.g., waiting times on fitness levels). A concrete application will be presented in Section 3.

**Theorem 1.** Let $X_i$, $1 \leq i \leq n$, be independent random variables following the geometric distribution with success probability $p_i$, and let $X := \sum_{i=1}^{n} X_i$. If $\sum_{i=1}^{n}(1/p_i^2) \leq s < \infty$ then for any $\delta > 0$

$$\Pr(X < E(X) - \delta) \leq e^{-\frac{\delta^2}{2s}}.$$  

For $h := \min\{p_i \mid i = 1, \ldots, n\}$,

$$\Pr(X > E(X) + \delta) \leq e^{-\frac{\delta}{4} \min\{\frac{\delta}{h}\}}.$$
For the proof, the following two simple inequalities will be used.

**Lemma 1.**

1. For \( x \geq 0 \) it holds \( \frac{e^x}{1+x} \leq e^{x^2/2} \).
2. For \( 0 \leq x \leq 1 \) it holds \( \frac{e^{-x}}{1-x} \leq e^{x^2/(2-2x)} \).

**Proof.** We start with the first inequality. The series representation of the exponential function yields

\[
\frac{e^x}{1+x} \leq \sum_{i=0}^{\infty} \frac{x^i}{i!} \leq \sum_{i=0}^{\infty} (1 + x)^{2i} \frac{x^i}{(2i)!}
\]

since \( x \geq 0 \). Hence,

\[
\frac{e^x}{1+x} \leq \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!}.
\]

Since \( (2i)! \geq 2^i i! \), we get

\[
\frac{e^x}{1+x} \leq \sum_{i=0}^{\infty} \frac{x^{2i}}{2^i i!} = e^{x^2/2}.
\]

To prove the second inequality, we omit all negative terms except for \(-x\) from the series representation of \( e^{-x} \) to get

\[
\frac{e^{-x}}{1-x} \leq \frac{1 - x + \sum_{i=1}^{\infty} \frac{x^{2i}}{(2i)!}}{1 - x} = 1 + \sum_{i=1}^{\infty} \frac{x^{2i}}{(1-x)(2i)!}.
\]

For comparison,

\[
e^{x^2/(2-2x)} = 1 + \sum_{i=1}^{\infty} \frac{x^{2i}}{2^i(1-x)^i i!},
\]

which, as \( x \leq 1 \), is clearly not less than our estimate for \( e^{-x}/(1-x) \). \( \square \)

**Proof of Theorem [7]** Both the lower and upper tail are analyzed similarly, using the exponential method (see, e. g., the proof of the Chernoff bound in Motwani and Raghavan [1995] Chapter 3.6). We start with the lower tail. Let \( d := E(X) - \delta = \sum_{i=1}^{n}(1/p_i) - \delta \). Since for any \( t \geq 0 \)

\[
X < d \iff -X > -d \iff e^{-tX} > e^{-td},
\]
Markov's inequality and the independence of the $X_i$ yield that
\[
\Pr(X < d) \leq \frac{E(e^{-tX})}{e^{-td}} = e^{td} \prod_{i=1}^{n} E(e^{-tX_i}).
\]
Note that the last product involves the moment-generating functions (mgf’s) of the $X_i$. Given a geometrically distributed random variable $Y$ with parameter $p$, its moment-generating function at $r \in \mathbb{R}$ equals $E(e^{rY}) = \frac{pe^r}{1 - e^{r(1-p)}} = \frac{1}{1 - (1-e^{-r})/p}$ for $r < -\ln(1-p)$. We will only use negative values for $r$, which guarantees existence of the mgf’s used in the following. Hence,
\[
\Pr(X < d) \leq e^{td} \prod_{i=1}^{n} \frac{1}{1 - (1-e^{-t/p_i})} \leq e^{td} \prod_{i=1}^{n} \frac{1}{1 + t/p_i},
\]
where we have used $e^x \geq 1 + x$ for $x \in \mathbb{R}$. Now, by writing the numerators as $e^{t/p_i} \cdot e^{-t/p_i}$, using $e^x \leq e^{x^2/2}$ for $x \geq 0$ (Lemma 1) and finally plugging in $d$, we get
\[
\Pr(X < d) \leq e^{td} \cdot \left( \prod_{i=1}^{n} e^{t^2/(2p_i^2)} e^{-t/p_i} \right) = e^{td} e^{(t^2/2) \sum_{i=1}^{n} (1/p_i)^2} e^{-tE(X)} \leq e^{-t\delta + (t^2/2)s}.
\]
The last exponent is minimized for $t = \delta/s$, which yields
\[
\Pr(X < d) \leq e^{-\frac{\delta^2}{2s}}
\]
and proves the lower tail inequality.

For the upper tail, we redefine $d := E(X) + \delta$ and obtain
\[
\Pr(X > d) \leq \frac{E(e^{tX})}{e^{td}} = e^{-td} \prod_{i=1}^{n} E(e^{tX_i}).
\]
Estimating the moment-generating functions similarly as above, we get
\[
\Pr(X > d) \leq e^{-td} \cdot \left( \prod_{i=1}^{n} e^{-t/p_i} \cdot e^{t/p_i} \right) = \left( \prod_{i=1}^{n} e^{-t\delta + t^2/p_i^2} \right) \leq e^{-t\delta + t^2}.
\]
Since now positive arguments are used for the moment-generating functions, we limit ourselves to $t \leq \min\{p_i \mid i = 1, \ldots, n\}/2 = h/2$ to ensure convergence. Using $\frac{x^2}{1-x^2} \leq e^{x^2/(2-2x)}$ for $0 \leq x \leq 1$ (Lemma 1), we get
\[
\Pr(X > d) \leq e^{-td} \cdot \left( \prod_{i=1}^{n} e^{t^2/(2-2t/p_i)} \cdot e^{t/p_i} \right) = \left( \prod_{i=1}^{n} e^{-t\delta + t^2/p_i^2} \right) \leq e^{-t\delta + t^2},
\]
which is minimized for \( t = \delta/(2s) \). If \( \delta \leq sh \), this choice satisfies \( t \leq h/2 \). Then 
\[-t\delta + t^2s = -\delta^2/(4s)\] 
and we get 
\[\Pr(X > d) \leq e^{-\delta^2/(4s)}\].

Otherwise, i.e. if \( \delta > sh \), we set \( t = h/2 \) to obtain 
\[-t\delta + t^2s = -\delta h/2 + s(h/2)^2 \leq -\delta h/2 + \delta h/4 = -\delta h/4.\] Then
\[\Pr(X > d) \leq e^{-\delta h/4}.\]

Joining the two cases in a minimum leads to the lower tail. \( \square \)

Based on Theorem 1, we formulate the fitness-level theorem with tail bounds for general optimization algorithms \( A \) instead of a specific randomized search heuristic (see also Sudholt, 2013, who uses a similar approach).

**Theorem 2** (Fitness Levels with Tail Bounds). Consider an algorithm \( A \) maximizing some function \( f \) and a partition of the search space into non-empty sets \( A_1, \ldots, A_m \). Assume that the sets form an \( f \)-based partition, i.e., for \( 1 \leq i < j \leq m \) and all \( x \in A_i, y \in A_j \) it holds \( f(x) < f(y) \). We say that \( A \) is in \( A_i \) or on level \( i \) if the best search point created so far is in \( A_i \).

1. If \( p_i \) is a lower bound on the probability that a step of \( A \) leads from level \( i \) to some higher level, independently of previous steps, then the first hitting time of \( A_m \), starting from level \( k \), is at most
\[
\sum_{i=k}^{m-1} \frac{1}{p_i} + \delta.
\]
with probability at least \( 1 - e^{-\frac{\delta}{4} \cdot \min\{ s, h \}} \), for any finite \( s \geq \sum_{i=k}^{m-1} \frac{1}{p_i} \) and
\( h = \min\{ p_i \mid i = k, \ldots, m - 1 \} \).

2. If \( p_i \) is an upper bound on the probability that a step of \( A \) leads from level \( i \) to level \( i + 1 \), independently of previous steps, and the algorithm cannot increase its level by more than \( 1 \), then the first hitting time of \( A_m \), starting from level \( k \), is at least
\[
\sum_{i=k}^{m-1} \frac{1}{p_i} - \delta
\]
with probability at least \( 1 - e^{-\frac{s^2}{\pi}} \).

**Proof.** By definition, the algorithm cannot go down on fitness levels. Estimate the time to leave level \( i \) (from above resp. from below) by a geometrically distributed random variable with parameter \( p_i \) and apply Theorem 1. \( \square \)
3 Application to RLS on OneMax

We apply Theorem 2 to a classical benchmark problem in the analysis of randomized search heuristics, more precisely the running time of RLS on OneMax. RLS is a well-studied randomized search heuristic, defined in Algorithm 1. The function OneMax: \{0, 1\}^n \to \mathbb{R} is defined by \( \text{OneMax}(x_1, \ldots, x_n) = x_1 + \cdots + x_n \), and the running time is understood as the first hitting time of the all-ones string (plus 1 to count the initialization step).

\[
t := 0, \quad \text{choose an initial bit string } x_0 \in \{0, 1\}^n \text{ uniformly at random.}
\]

\[
\text{repeat}
\begin{align*}
&\text{create } x' \text{ by flipping a uniformly chosen bit in } x_t. \\
&x_{t+1} := x' \text{ if } f(x') \geq f(x_t), \text{ and } x_{t+1} := x_t \text{ otherwise.}
\end{align*}
\]

\[
t := t + 1.
\]

forever.

Algorithm 1: RLS for the maximization of \( f: \{0, 1\}^n \to \mathbb{R} \)

**Theorem 3.** Let \( T \) be the running time of RLS on OneMax. Then

1. \( n \ln n - 0.11594n - o(n) \leq E(T) \leq n \ln n - 0.11593n + o(n) \).

2. \( \Pr(T \leq E(T) - rn) \leq e^{-\frac{3r^2}{\pi^2}} \) for any \( r > 0 \).

3. \( \Pr(T \geq E(T) + rn) \leq \begin{cases} e^{-\frac{3r^2}{\pi^2}} & \text{if } 0 < r \leq \frac{\pi^2}{6} \\ e^{-\frac{r^2}{4}} & \text{otherwise} \end{cases} \).

**Proof.** We start with Statement 1, i.e., the bounds on the expected running time. Let the fitness levels \( A_0, \ldots, A_n \) be defined by \( A_i = \{x \in \{0, 1\}^n \mid \text{OneMax}(x) = i\} \) for \( 0 \leq i \leq n \). By definition of RLS, the probability \( p_i \) of leaving level \( i \) equals \( p_i = n/(n-i) \) for \( 0 \leq i \leq n-1 \). Therefore, the expected running time from starting level \( k \) is

\[
\sum_{i=k}^{n-1} \frac{n}{n-i} = n \sum_{i=1}^{n-k} \frac{1}{i},
\]

which leads to the weak upper bound \( E(T) \leq n \ln n + n \) in the first place. Due to the uniform initialization in RLS, Chernoff bounds yield \( \Pr(n/2 - n^{2/3} \leq k \leq n/2 + n^{2/3}) = 1 - e^{-\Omega(n^{1/3})} \). We obtain

\[
E(T) \leq n \left( \sum_{i=1}^{n/2+n^{2/3}} \frac{1}{i} \right) + e^{-\Omega(n^{1/3})} \cdot (n \ln n + n) = n \left( \sum_{i=1}^{n/2+n^{2/3}} \frac{1}{i} \right) + o(n).
\]
We can now estimate the Harmonic number by \( \ln(n/2 + n^{2/3}) + \gamma + o(1) = \ln n + \gamma - \ln 2 + o(1) \), where \( \gamma = 0.57721 \ldots \) is the Euler-Mascheroni constant. Plugging in numerical values for \( \gamma - \ln 2 \) proves the upper bound on \( E(T) \). The lower one is proven symmetrically.

For Statement 2, the lower tail bound, we use Theorem 2. Now, \( \sum_{i=1}^{n-k} \frac{1}{p_i} \leq \sum_{i=1}^{n} \frac{n^2}{p_i^2} \leq \frac{n^2 s^2}{6} =: s \). Plugging \( \delta := rn \) and \( s \) in the second part of the theorem yields \( \Pr(T \leq E(T) - rn) \leq e^{-\frac{r^2 n^2}{2s}} = e^{-\frac{n^2}{\pi^2}} \).

For Statement 3, the upper tail bound, we argue similarly but have to determine when \( \frac{\delta}{s} \leq h \). Note that \( h = \min\{p_i\} = 1/n \). Hence, it suffices to determine when \( \frac{\delta n}{n^2 \pi^2} \leq 1/n \), which is equivalent to \( r \leq \frac{\pi^2}{6} \). Now the two cases of the lower bound follow by appropriately plugging \( \frac{\delta}{s} \) or \( h \) in the first part of Theorem 2.

The stochastic process induced by RLS on OneMax equals the classical and well-studied coupon collector problem (started with \( k \) full bins). Despite this fact, the lower tail bound from Theorem 3 could not be found in the literature (see also the comment introducing Theorem 1.24 in Doerr, 2011, which describes a simple but weaker lower tail). There is an easy-to-prove upper tail bound for the coupon collector of the kind \( \Pr(T \geq E(T) + rn) \leq e^{-r} \), which is stronger than our result but not obvious to generalize. Finally, Scheideler (2000, Theorem 3.38) suggests upper and lower tail bounds for sums of geometrically distributed random variables, which could also be tried out in our example; however, it then turns out that these bounds are only useful if \( r = \Omega(\sqrt{\ln n}) \).

4 Conclusions

We have supplemented upper and lower tail bounds to the fitness-level method. The lower tails are novel contributions and the upper tails improve an existing result from the literature significantly. As a proof of concept, we have applied the fitness levels with tail bounds to the analysis of RLS on OneMax and obtained a very sharp concentration result.

If the stochastic process under consideration is allowed to skip fitness levels, which is often the case with globally searching algorithms such as evolutionary algorithms, our upper tail bound may become arbitrarily loose and the lower tail is even unusable. To prove tail bounds in such cases, drift analysis may be used, which is another powerful and in fact somewhat related method for the running time analysis of randomized search heuristics. See, e.g., Lehre and Witt (2013) and references therein for further reading.

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