STRONG OPENNESS CONJECTURE FOR PLURISUBHARMONIC FUNCTIONS

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Abstract. In this article, we give a proof of the strong openness conjecture for plurisubharmonic functions posed by Demailly.

1. Introduction

Let $X$ be a complex manifold with dimension $n$, and $\varphi$ be a plurisubharmonic function on $X$. Let $I(\varphi)$ be the multiplier ideal sheaf associated to the plurisubharmonic function $\varphi$ on $X$. Denote by

$$I_+(\varphi) := \cup_{\epsilon > 0} I((1 + \epsilon)\varphi).$$

In [2], Berndtsson gave a proof of the openness conjecture of Demailly and Kollár in [7]:

Openness conjecture: Assume that $I(\varphi) = O_X$. Then

$$I_+(\varphi) = I(\varphi).$$

For the dimension two case it was proved by Favre and Jonsson in [10] (see also [9]). For arbitrary dimension it has been reduced to a purely algebraic statement in [14].

The strong openness conjecture is stated as follows, which is an open question posed by Demailly in [4], [5] (see also [8], [15], [13], [11], [3], [14], [16], [17], etc.):

Strong openness conjecture: For any plurisubharmonic function $\varphi$ on $X$, we have

$$I_+(\varphi) = I(\varphi).$$

The strong openness conjecture implies the openness conjecture.

In the present paper, we prove the strong openness conjecture by the following theorem:

Theorem 1.1. (solution of the strong openness conjecture) Let $\varphi$ be a negative plurisubharmonic function on the unit polydisc $\Delta^n \subset \mathbb{C}^n$, which satisfies

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty,$$

Date: 2013.10.06.

Key words and phrases. $L^2$ extension theorem, strong openness conjecture, plurisubharmonic function.

The second author was partially supported by NSFC.
where $F$ is a holomorphic function on $\Delta^n$ and $\lambda_n$ is the Lebesgue on $\mathbb{C}^n$. Then there exist $r > 0$ and $p > 1$, such that
\[
\int_{\Delta^n} |F|^2 e^{-p\varphi} d\lambda_n < +\infty,
\]
where $r < 1$.

For $n \leq 2$, the above theorem was proved in [13] by studying the asymptotic jumping numbers for graded sequences of ideals.

There are some immediate corollaries of the strong openness conjecture, we omit them here. Some further results are being in preparation (see our another paper), including a conjecture of Demailly and Kollár in [7] about the growth of the measure of the sublevel set of psh functions related to the complex singularity exponents.

2. SOME RESULTS USED IN THE PROOF OF THE THEOREM

In this section, we will show some preliminary results used in the proof of the main theorem.

2.1. A property of $L^1$ integrable function.

Let $G$ be a positive Lebesgue measurable and integrable function on a domain $\Omega \subset \subset \mathbb{C}^n$, i.e.,
\[
\int_{\Omega} G d\lambda_n < +\infty.
\]

Consider the function
\[
F_G(t) := \sup \{a | \lambda_n(\{G \geq a\}) \geq t\},
\]
$t \in (0, \lambda_n(\Omega)]$.

We first discuss the finiteness of $F_G(t)$.

If for some $t_0$, $F_G(t_0) = +\infty$, then $\lambda_n(\{G \geq A_j\}) \geq t_0$, where $A_j$ is a number sequence tending to $+\infty$ when $j \to +\infty$. Since $G$ is $L^1$ integrable, we have
\[
t_0 A_j \leq A_j \lambda_n(\{G \geq A_j\}) \leq \int_{\{G \geq A_j\}} G d\lambda_n \leq \int_{\Omega} G d\lambda_n < +\infty.
\]
Letting $A_j \to +\infty$, we thus obtain a contradiction. Therefore $F_G(t) < +\infty$ for any $t$.

Secondly, we discuss the decreasing property of $F_G(t)$.

Note that $\{A_j | \lambda_n(G \geq A_j) \geq t_1\} \supset \{A_j | \lambda_n(G \geq A_j) \geq t_2\}$, where $t_1 \leq t_2$. Then we have $F_G(t_1) \geq F_G(t_2)$, when $t_1 \leq t_2$.

The first lemma is about the sublevel set of $F_G$:

**Lemma 2.1.** We have
\[
\mu_{\mathbb{R}}(\{t | F_G(t) \geq a\}) = \lambda_n(\{G \geq a\}),
\]
for any $a > 0$, where $\mu_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$. Moreover, we have
\[
\mu_{\mathbb{R}}(\{t | F_G(t) > a\}) = \lambda_n(\{G > a\}).
\]

**Proof.** Since
\[
\mu_{\mathbb{R}}(\{t | F_G(t) > a\}) = \lim_{k \to +\infty} \mu_{\mathbb{R}}(\{t | F_G(t) \geq a + \frac{1}{k}\}),
\]
and

\[ \lambda_n(\{G > a\}) = \lim_{k \to +\infty} \lambda_n(\{G \geq a + \frac{1}{k}\}), \]

we only need to prove

\[ \mu(\{t \mid F_G(t) \geq a\}) = \lambda_n(\{G \geq a\}), \]

for any \( a > 0 \).

Note that

\[ \sup\{a_1 \mid \lambda_n(\{G \geq a_1\}) \geq \lambda_n(\{G \geq a\})\} \geq a. \]

Then we have

\[ \lambda_n(\{G \geq a\}) \in \{t \mid \sup\{a_1 \mid \lambda_n(\{G \geq a_1\}) \geq t\} \geq a\}, \]

therefore

\[ \{t \mid \sup\{a_1 \mid \lambda_n(\{G \geq a_1\}) \geq t\} \geq a\} \supseteq \{t \mid \lambda_n(\{G \geq a\}) \geq t\}, \]

where \( t > 0 \).

If " \( \supseteq \) " in the above relation is strictly " \( \geq \) " , then there exists \( t_0 \) such that

1). \( \sup\{a_1 \mid \lambda_n(\{G \geq a_1\}) \geq t_0\} \geq a; \)

2). \( t_0 > \lambda_n(\{G \geq a\}). \)

Let

\[ a_0 := \sup\{a_1 \mid \lambda_n(\{G \geq a_1\}) \geq t_0\} \geq a. \]

Note that

\[ \lambda_n(\cap_{a < a_0}(G \geq a_1)) = \inf_{a_1 < a_0} \lambda_n(\{G \geq a_1\}). \]

Then we have \( \lambda_n(\{G \geq a_0\}) \geq t_0. \)

As \( t_0 > \lambda_n(\{G \geq a\}), \) we have

\[ \lambda_n(\{G \geq a_0\}) > \lambda_n(\{G \geq a\}), \]

which is a contradiction to

\[ a_0 \geq a. \]

Then the following holds:

\[ \{t \mid \sup\{a_1 \mid \lambda_n(\{G \geq a_1\}) \geq t\} \geq a\} = \{t \mid \lambda_n(\{G \geq a\}) \geq t\}, \]

where \( t > 0 \).

According to the definition of \( F_G \) and equality \( \Box \), it follows that

\[ \{t \mid F_G(t) \geq a\} = \{t \mid \sup\{a_1 \mid \mu(\{G \geq a_1\}) \geq t\} \geq a\} = \{t \mid \lambda_n(\{G \geq a\}) \geq t\}, \]

where \( t > 0 \).

Note that

\[ \mu(\{t \mid F_G(t) \geq a\}) = \mu(\{t \mid \lambda_n(\{G \geq a\}) \geq t\}) = \lambda_n(\{G \geq a\}), \]

for \( t > 0. \) We have thus proved the present lemma. \( \Box \)

Denote by

\[ s(y) := y^{-1}(\log y)^{-1}, \]

where \( y \in (0, e^{-1}). \) It is clear that \( s \) is strictly decreasing on \((0, e^{-1}).\)

We define a function \( u \) by

\[ u(s(y)) = y^{-1}, \]

where \( u \in C^\infty((e, +\infty)). \) It is clear that \( u \) is strictly increasing on \((e, +\infty),\)

The second lemma is about the measure of the level set of \( G: \)
Lemma 2.2. For $A > e$, we have
\[ \liminf_{A \to +\infty} \lambda_n(\{G > A\})u(A) = 0, \]
where $\lambda_n$ is the Lebesgue measure of $\mathbb{C}^n$. Especially, $\lim_{A \to +\infty} \frac{A}{\mu(A)} = 0$.

Proof. According to the definition of Lebesgue integration and Lemma 2.1, it follows that
\[ \int_0^\mu(\Omega) F_G(t)dt = \int_\Omega Gd\lambda_n < +\infty. \]
Then we have
\[ \lim_{t \to 0} \frac{F_G(t)}{t^{-1}(-\log t)^{-1}} = 0, \]
which implies that there exists $t_j \to 0$ when $j \to +\infty$, such that
\[ \lim_{j \to +\infty} \frac{F_G(t_j)}{t_j^{-1}(-\log t_j)^{-1}} = 0. \]
(2.3)

Using Lemma 2.1 we have
\[ \lambda_n(\{G > F_G(t_j)\}) = \mu \{ t \mid F_G(t) > F_G(t_j) \} \leq \mu \{ 0, t_j \} = t_j. \]

We now want to prove that $u(F_G(t_j)) = o(t_j^{-1})$ by contradiction: if not, there exists $\varepsilon_0 > 0$, such that $u(F_G(t_j)) \leq \varepsilon_0 t_j^{-1}$. However,
\[ u(F_G(t_j)) \leq \varepsilon_0 t_j^{-1} = u(\frac{1}{\varepsilon_0}(-\log \frac{t_j}{\varepsilon_0})). \]

According to the strictly increasing property of $u$, it follows that $F_G(t_j) \leq \frac{1}{\varepsilon_0}(-\log \frac{t_j}{\varepsilon_0})$, which is contradict to equality 2.3 because $\lim_{t_j \to 0} \frac{t_j(-\log t_j)}{\varepsilon_0(-\log t_j)} = \varepsilon_0$.

Now we obtain $u(F_G(t_j)) = o(t_j^{-1})$.

Then we have
\[ \lim_{j \to +\infty} \mu(\{G > F_G(t_j)\})u(F_G(t_j)) \leq \lim_{j \to +\infty} t_j o(t_j^{-1}) = 0. \]

Note that if $F_G(t_j)$ is bounded above, when $t_j$ tends to 0, then $G$ has a positive upper bound. Therefore $\mu(\{G > A\}) = 0$ for $A$ large enough.

Then we have proved
\[ \liminf_{A \to +\infty} \lambda_n(\{G > A\})u(A) = 0. \]

As \( \frac{1}{t(-\log t)} \) is strictly decreasing on $(0, e^{-1})$, then for any $A > e$, there exists $t_A$ such that
\[ 1). \quad \frac{1}{t_A(-\log t_A)} = A; \]
\[ 2). \quad t_A \text{ goes to zero, when } A \text{ goes to } +\infty. \]

As
\[ \frac{A}{u(A)} = \frac{1}{u(\frac{1}{t_A(-\log t_A)})} = \frac{t_A(-\log t_A)}{1} = \frac{1}{-\log t_A}, \]
then we obtain
\[ \lim_{A \to +\infty} \frac{A}{u(A)} = 0, \]
by the above property 2) of $t_A$.

The present lemma is thus proved.
2.2. Estimation of integration of holomorphic functions on singular Riemann surfaces.

**Lemma 2.3.** Let \( h \not\equiv 0 \) be a holomorphic function on the disc \( \Delta_r \) in \( \mathbb{C} \). Let \( f_a \) be a holomorphic function on \( \Delta_r \), which satisfies \( f_a|_a = 0 \) and \( f_a(b) = 1 \) for any \( b^k = a(k \text{ is a positive integer) } \), then we have

\[
\int_{\Delta_r} |f_a|^2|h|^2 d\lambda_1 > C_1|a|^{-2},
\]

where \( a \in \Delta_r \) whose norm is small enough, \( C_1 \) is a positive constant independent of \( a \) and \( f_a \).

**Proof.** As \( h \not\equiv 0 \), we may write \( h = z^i h_1 \) near \( o \), where \( h_1|_o \not\equiv 0 \). Then there exists \( r' < r \), such that \( |h_1|_{\Delta_{r'}} \geq C_0 > 0 \). Therefore it suffices to consider the case that \( h = z^i \) on \( \Delta_{r'} \).

By Taylor expansion at \( o \), we have

\[
f(z) = \sum_{j=1}^{\infty} c_j z^j.
\]

As \( f(b) = 1 \), then

\[
\sum_{j=1}^{\infty} c_k a^j = \frac{1}{k} \sum_{1 \leq l \leq k} \sum_{j=1}^{\infty} c_j b^j_l = 1
\]

where \( b^k_l = a \), and \( \sum_{1 \leq l \leq k} b^j_l = 0 \) when \( 0 < j < k \).

It is clear that

\[
\int_{\Delta_{r'}} |f_a|^2|h|^2 d\lambda_1 = \int_{\Delta_{r'}} |f_a|^2|z^i|^2 d\lambda_1 = 2\pi \sum_{j=1}^{\infty} |c_j|^2 \frac{r^{2j+2i+2}}{2j+2i+2}
\]

(2.4)

Using Schwartz Lemma, we have

\[
\left( \sum_{j=1}^{\infty} |c_j|^2 \frac{r^{2j+2i+2}}{2j+2i+2} \right) \left( \sum_{j=1}^{\infty} \frac{2j+2i+2}{r^{2j+2i+2}} |a|^{2j} \right) \geq \left( \sum_{j=1}^{\infty} |c_k a^j|^2 \right) \geq \sum_{j=1}^{\infty} |c_k a^j|^2 = 1.
\]

(2.5)

Note that

\[
\sum_{j=1}^{\infty} \frac{2j+2i+2}{r^{2j+2i+2}} |a|^{2j} = \frac{a^{r^{-2i-4}} (2i+2) r^{r^2-2i-2}}{1-|\frac{a}{r^2}|^2} + 2 \frac{r^{r^2-2i-2}}{(1-|\frac{a}{r^2}|^2)^2},
\]

and \((2i+2)\frac{r^{r^2-2i-4}}{1-|\frac{a}{r^2}|^2} + 2 \frac{r^{r^2-2i-4}}{(1-|\frac{a}{r^2}|^2)^2}\) has a uniformly upper bound independent of \( a \) when \( |a| < \frac{r^2}{2} \). The Lemma thus follows.

\( \square \)

Let’s recall the local parametrization theorem:
Theorem 2.4. (see [6]) Let $\mathcal{I}$ be a prime ideal of $\mathcal{O}_n$ and let $C' = V(\mathcal{I})$ be an analytic curve at $o$. Then the ring $\mathcal{O}_n/\mathcal{I}$ is a finite integral extension of $\mathcal{O}_d$; let $q$ be the degree of the extension. There exists a local coordinates $(z'; z'') = (z_1; z_2, \cdots, z_n)$, such that if $\Delta'_o$ and $\Delta''_o$ are polydisks of sufficient small radii $r'$ and $r''$ and if $r' \leq \frac{r''}{2}$ with $C$ large, the projective map $\pi' : C' \cap (\Delta'_o \times \Delta''_o) \rightarrow \Delta'_o$ is a ramified covering with $q$ sheets, whose ramification locus is contained in $S = \{o'\} \subset \Delta'_o$.

This means that

a), the open set $C'_S := C' \cap ((\Delta'_o \setminus S) \times \Delta''_o)$ is a smooth 1-dimensional manifold, dense in $C' \cap (\Delta'_o \times \Delta''_o)$;

b), $\pi' : C'_S \rightarrow \Delta'_o \setminus S$ is an unramified covering;

c), the fibre $\pi'^{-1}(z')$ have exactly $q$ elements if $z' \in \Delta' \setminus S$ and at most $q$ if $z' \in S$.

Moreover, $C'_S$ is a connected covering of $\Delta'_o \setminus S$, and $C' \cap (\Delta'_o \times \Delta''_o)$ is contained in a cone $|z''| \leq \frac{1}{4}|z'|$.

Let $\Delta'$ and $\Delta''$ be unit disk with coordinates $(z_1)$ and unit polydisk with coordinates $(z_2, \cdots, z_n)$ respectively.

Let $\pi : \Delta' \times \Delta'' \rightarrow \Delta'$ be the projective map which is given by

$$\pi(z_1; z_2, \cdots, z_n) = z_1.$$ 

We have the following Remark of Theorem 2.4.

Remark 2.5. Let $\mathcal{I}$ be a prime ideal of $\mathcal{O}_n$ and let $C' = V(\mathcal{I})$ be an analytic curve at $o$. Then the ring $\mathcal{O}_n/\mathcal{I}$ is a finite integral extension of $\mathcal{O}_d$; let $q$ be the degree of the extension, there exists a biholomorphic map $j$ from a neighborhood of $\Delta' \times \Delta''$ to a neighborhood $U_o$ of $o$, such that the projective map $\pi|_{C \cap (\Delta' \times \Delta'')} \rightarrow \Delta'$ is a ramified covering with $q$ sheets, whose ramification locus is contained in $S = \{o'\} \subset \Delta'$ where $C := j^{-1}(C')$.

This means that

a), the open set $C_S := C \cap ((\Delta' \setminus S) \times \Delta'')$ is a smooth 1-dimensional manifold, dense in $C \cap (\Delta' \times \Delta'')$;

b), $\pi|_{C_S} : C_S \rightarrow \Delta' \setminus S$ is an unramified covering;

c), the fibre $\pi^{-1}(z')$ have exactly $q$ elements if $z' \in \Delta' \setminus S$ and at most $q$ if $z' \in S$.

Moreover, $C_S$ is a connected covering of $\Delta' \setminus S$, and $C \cap (\Delta' \times \Delta'')$ is contained in a cone $|z''| \leq \frac{1}{8}|z'|$.

Using Lemma 2.3 and Remark 2.5 we obtain the following singular version of Lemma 2.4.

Lemma 2.6. Let $h$ be a holomorphic function on an analytic curve $C$ as in Remark 2.5. Let $f_a$ be a holomorphic function on $C$, which satisfies $f(o) = 0$ and $f_a(\pi^{-1}(a) \cap C) = 1$, then we have

$$\int_{C_S} |f_a|^2 |h|^2 (\pi|_{C_S})^* d\lambda_{\Delta'} > C_2|a|^{-2}.$$
for any $b$ where $C$ then we have

\[
\text{Proof. } \quad \text{As } (C, o) \text{ is irreducible and locally irreducible, then there is a normalization } j_{\text{nor}} : (\Delta, 0) \to (C, o), \text{ denoted by }
\]

\[
j_{\text{nor}}(t) = (g_1(t), \ldots, g_n(t)),
\]

where $t$ is the coordinate of $\Delta$. As $\pi_C$ is a covering, then $g_1 \neq 0$. Without loss of generality, we may assume $g_1(t) = t^{i_1}$ on $\Delta_{r_0}$ for small enough $r_0 \in (0, 1)$.

There is a given $r > 0$, which is small enough, such that

\[
(C \cap \Delta^n_{r_0}) \ni \{(t^{i_1}, g_2(t), \ldots, g_n(t)) | t \in \Delta_r \},
\]

where $i_1 \geq 1$, and $g_i (i \geq 2)$ are holomorphic functions on $\Delta$, satisfying $|g_i| \leq \frac{1}{6} |t^{i_1}|$.

For given $r' < r$ small enough, we have

\[
\int_{C_S} |f_a|^2 |h|^2 (\pi|_{C_S})^* d\lambda_{\Delta'} \geq \int_{\Delta_{r'}} |j_{\text{nor}}^* f_a(t)|^2 |j_{\text{nor}}^* h(t)|^2 |t|^{2(i_1 - 1)} d\lambda_{\Delta}
\]

\[
= \int_{\Delta_{r'}} |j_{\text{nor}}^* f_a(t)(t^{i_1 - 1} j_{\text{nor}}^* h(t))|^2 d\lambda_{\Delta'},
\]

for any $a$ satisfying $|a|^{\frac{1}{6}} \in \Delta_{r'}$.

As

\[
j_{\text{nor}}^* f_a(b) = f_a(b^{i_1}, g_2(b), \ldots, g_n(b))
\]

and

\[
(b^{i_1}, g_2(b), \ldots, g_n(b)) \subset (\pi^{-1}(b^{i_1}) \cap C),
\]

then we have

\[
j_{\text{nor}}^* f_a(b) = 1,
\]

for any $b^{i_1} = a$.

Using Lemma 2.3, we have

\[
\int_{\Delta_{r'}} |j_{\text{nor}}^* f_a(t)(t^{i_1 - 1} j_{\text{nor}}^* h(t))|^2 d\lambda_{\Delta'} \geq C_1 |a|^{-2},
\]

where $C_1$ is independent of $a$ and $f_a$.

Combining with inequality 2.6, we thus obtain the present lemma. \qed

As $C \cap (\Delta' \times \Delta'')$ is contained in a cone $|z''| \leq \frac{1}{6} |z'|$, using the submean value property of plurisubharmonic function, we obtain the following lemma:

**Lemma 2.7.** For any holomorphic function $F$, which is a holomorphic on a $\Delta' \times \Delta''$, we obtain an approximation of the $L^2$ norm of $F$:

\[
\int_{\Delta' \times \Delta''} |F|^2 d\lambda_n \geq C_3 \int_{C_S} |F||_{C_S}^2 (\pi|_{C_S})^* d\lambda_{\Delta'},
\]

where $C_3$ is a positive constant independent of $F$. Here all symbols $C_S$, $\Delta'$ and $\pi$ are the same as in Remark 2.4.

**Proof.** Using the Fubini Theorem,

\[
\int_{\Delta' \times \Delta''} |F|^2 d\lambda_n = \int_{\Delta'} (\int_{\{z''\} \times \Delta''} |F|^2 d\lambda_{n-1}) d\lambda_{\Delta'},
\]
and the submean value inequality of plurisubharmonic function, we have
\[ \int_{\{z'\} \times \Delta^\nu} |F|^2 d\lambda_{n-1} \geq \left(\frac{\pi}{3}\right)^{n-1} |F(z', z'')|^2, \]
for \(|z''| \leq \frac{1}{6}\).

If \(w = (z', z'') \in (\pi^{-1}(z') \cap C_S)\), then \(|z''| \leq \frac{1}{6}\).
As
\[ \int_{\Delta^\nu \setminus \{0\}} \sum_{w \in (\pi^{-1}(z') \cap C_S)} |F(w)|^2 (z') d\lambda_{\Delta'} = \int_{C_S} |F|_{C_S}^2 (\pi|_{C_S})^* d\lambda_{\Delta'}, \]
it follows that
\[ q \int_{\Delta' \times \Delta''} |F|^2 d\lambda_n = q \int_{\Delta' \setminus \{0\}} \left( \int_{\{z'\} \times \Delta''} |F|^2 d\lambda_{n-1} \right) d\lambda_{\Delta'} \geq \left(\frac{\pi}{3}\right)^{n-1} \int_{\Delta' \setminus \{0\}} \left( \sum_{w \in (\pi^{-1}(z') \cap C_S)} |F(w)|^2 (z') d\lambda_{\Delta'} \right) \]
\[ = \left(\frac{\pi}{3}\right)^{n-1} \int_{C_S} |F|_{C_S}^2 (\pi|_{C_S})^* d\lambda_{\Delta'}, \]
where \(q\) is the degree of the covering map \(\pi|_{C_S}\).

2.3. \(L^2\) extension theorem with negligible weight.

We state optimal constant version of the Ohsawa’s \(L^2\) extension theorem with negligible weight (19) as follows:

**Theorem 2.8.** Let \(X\) be a Stein manifold of dimension \(n\). Let \(\varphi + \psi\) and \(\psi\) be plurisubharmonic functions on \(X\). Assume that \(w\) is a holomorphic function on \(X\) such that \(\text{sup}(\psi + 2 \log |w|) \leq 0\) and \(dw\) does not vanish identically on any branch of \(w^{-1}(0)\). Put \(H = w^{-1}(0)\) and \(H_0 = \{x \in H : dw(x) \neq 0\}\). Then there exists a uniform constant \(C = 1\) independent of \(X\), \(\varphi\), \(\psi\) and \(w\) such that, for any holomorphic \((n-1)\)-form \(f\) on \(H_0\) satisfying
\[ c_{n-1} \int_{H_0} e^{-\varphi - \psi} f \wedge \bar{f} < \infty, \]
where \(c_k = (-1)^{\frac{k(k-1)}{2}} (\sqrt{-1})^k\) for \(k \in \mathbb{Z}\), there exists a holomorphic \(n\)-form \(F\) on \(X\) satisfying \(F = dw \wedge \bar{f}\) on \(H_0\) with \(i^* \bar{f} = f\) and
\[ c_n \int_{X} e^{-\varphi} F \wedge \bar{F} \leq 2C\pi c_{n-1} \int_{H_0} e^{-\varphi - \psi} f \wedge \bar{f}, \]
where \(i : H_0 \to X\) is the inclusion map.

2.4. Curve selection lemma and Noetherian property of coherent sheaves.

We give the existence of some kind of germs of analytic curves, which will be used.

**Lemma 2.9.** Let \((Y, o)\) be a germ of irreducible analytic subvariety in \(\mathbb{C}^n\), and \((A, o)\) be a germ of analytic subvariety of \((Y, o)\), such that \(\text{dim}A < \text{dim}Y\). Then there exists a germ of holomorphic curve \((\gamma, o)\), such that \(\gamma \subset Y\), and \(\gamma \not\subset A\).
Proof. Note that \((Y,o)\) is locally Stein. Then using Cartan’s Theorem \(A\), we obtain the lemma. \(\square\)

Now we recall the curve selection lemma stated as follows:

**Lemma 2.10.** (see [5]) Let \(f, g_1, \cdots, g_s \in \mathcal{O}_n\) be germs of holomorphic functions vanishing at \(0\). Then we have \(|f| \leq C|g|\) for some constant \(C\) if and only if for every germ of analytic curve \(\gamma\) through \(0\) there exists a constant \(C_\gamma\) such that \(|f \circ \gamma| \leq C_\gamma|g \circ \gamma|\).

In order to obtain some uniform properties of \(\gamma\), we need to consider the following Lemma which was contained in the proof of Lemma 2.10 in [5].

**Lemma 2.11.** (see [5]) Let \(f, g_1, \cdots, g_s \in \mathcal{O}_n\) be germs of holomorphic functions vanishing at \(o\). Assume that for any given neighborhood of \(o\), \(|f| \leq C|g|\) doesn’t hold for any constant \(C\), where \(g = (g_1, \cdots, g_s)\). Then there exists a germ of analytic curve \(\gamma\) through \(o\) satisfying \(\gamma \cap \{f = 0\} = o\), such that \(\frac{\mathbf{g}_i}{\mathbf{f}}\) is holomorphic on \(\gamma \setminus o\) with

\[
\frac{\mathbf{g}_i}{\mathbf{f}|_\gamma}(o) = 0,
\]

for any \(i \in \{1, \cdots, s\}\), where \(\mathbf{g}_i\) is the holomorphic extension of \(\frac{\mathbf{g}_i}{\mathbf{f}}\) from \(\gamma \setminus o\) to \(\gamma\).

Proof. There exists \(\Delta^n, \Delta^n, \Delta^n, f \in \mathcal{O}(\Delta^n)\). We define a germ of analytic set \((Y,o) \subset (\Delta^n \times \mathbb{C}^s, o)\) by

\[
g_j(z) = f(z)z_{n+j}, \quad 1 \leq j \leq s.
\]

Let \(p\) be a projection \(p : \Delta^n \times \mathbb{C}^s \to \Delta^n\), such that

\[
p((z_1, \cdots, z_n), (z_{n+1}, \cdots, z_{n+s})) = (z_1, \cdots, z_n).
\]

Then \(Y \cap p^{-1}(\Delta^n \setminus \{f = 0\})\) is biholomorphic to \(\Delta^n \setminus \{f = 0\}\), which is irreducible. As every analytic variety has an irreducible decomposition, then \(Y\) contains an irreducible component \(Y_f\) which contains \(Y \cap p^{-1}(\Delta^n \setminus \{f = 0\})\).

Since \(Y_f\) is closed, then

\[
Y_f = Y \cap p^{-1}(\Delta^n \setminus \{f = 0\}).
\]

By assumption, for any given neighborhood of \(o\), \(|f| \leq C|g|\) doesn’t hold for any constant \(C\), then there exist a sequence of positive numbers \(C_\nu\) which goes to \(+\infty\) as \(\nu \to \infty\), and a sequence of points \(\{z_\nu\} \in \Delta^n\) which is convergent to \(o\) as \(\nu \to \infty\), such that \(|f(z_\nu)| > C_\nu|g(z_\nu)|\).

Then \((z_\nu, \frac{\mathbf{g}_i(z_\nu)}{\mathbf{f}(z_\nu)})\) converges to \(o\) as \(\nu\) tends to \(+\infty\), with \(f(z_\nu) \neq 0\).

As \((z_\nu, \frac{\mathbf{g}_i(z_\nu)}{\mathbf{f}(z_\nu)}) \in Y_f\), then \(Y_f\) contains \(o\).

It follows from Lemma 2.9 that there exists a germ of analytic curve \((\gamma, o) \subset Y_f\) through \(o\) satisfying \(\gamma \cap \{f = 0\} = o\), such that \(\frac{\mathbf{g}_i}{\mathbf{f}|_\gamma}\) is holomorphic on \(\gamma \setminus o\) for each \(i \in \{1, \cdots, s\}\).

By the Riemann removable singularity theorem, it follows that \(\frac{\mathbf{g}_i}{\mathbf{f}|_\gamma o}\) can be extended to \(\gamma\), and

\[
\frac{\mathbf{g}_i}{\mathbf{f}|_\gamma o}(o) = 0,
\]

for any \(i \in \{1, \cdots, s\}\). \(\square\)
Remark 2.12. Let \( g_1, \ldots, g_s \in O_n \) be germs of holomorphic functions vanishing at \( o \), and \( f(o) \neq 0 \). Then there exists a germ of analytic curve \( \gamma \) through \( o \) satisfying \( \gamma \cap \{ f = 0 \} = \emptyset \), such that \( \frac{g_i}{f} \gamma \) is holomorphic on \( \gamma \) with \( \frac{g_i}{f}(o) = 0 \), for any \( i \in \{1, \ldots, s\} \).

Let’s recall a strong Noetherian property of coherent sheaves as follows:

Lemma 2.13. (see [6]) Let \( \mathcal{F} \) be a coherent analytic sheaf on a complex manifold \( M \), and let \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \) be an increasing sequence of coherent subsheaves of \( \mathcal{F} \). Then the sequence \( (\mathcal{F}_k) \) is stationary on every compact subset of \( M \).

Remark 2.14. By Lemma 2.13 it is clear that \( \bigcup_{\varepsilon > 0} \mathcal{I}((1 + \varepsilon)\varphi) \) is a coherent subsheaf of \( \mathcal{I}(\varphi) \); actually for any open \( V_1 \subset M \), there exists \( \varepsilon_1 > 0 \), such that \( \bigcup_{\varepsilon > 0} \mathcal{I}((1 + \varepsilon)\varphi)|_{V_1} = \mathcal{I}((1 + \varepsilon_1)\varphi)|_{V_1} \).

By Remark 2.14 we derive the following proposition about the generators of the coherent sheaf \( \mathcal{I}_+(\varphi) \):

Proposition 2.15. Assume that \( F \in O_n \) is a holomorphic function on a neighborhood \( V_0 \) of \( o \), which is not a germ of \( \mathcal{I}_+(\varphi)_o \). Let \( g_1, \ldots, g_s \in \mathcal{I}_+(\varphi)_o \) be germs of holomorphic functions on a neighborhood \( V_1 \subset V_0 \) of \( o \), such that \( g_1, \ldots, g_s \) generate \( \mathcal{I}_+(\varphi)|_{V_1} \). Then there exists a germ of analytic curve \( \gamma \) on \( V_0 \) through \( o \) satisfying \( \gamma \cap \{ F = 0 \} = \emptyset \), such that \( \frac{g_i}{F} \gamma \) is holomorphic on \( \gamma \) for any \( i \), and

\[
\frac{g_i}{F} \gamma|_o = 0,
\]

where \( \frac{g_i}{F} \gamma \) is the holomorphic extension of \( \frac{g_i}{F} \gamma \) from \( \gamma \setminus o \) to \( \gamma \). Moreover, for any germ \( g \) of \( \mathcal{I}((1 + \varepsilon)\varphi)_o \), \( \frac{g}{F} \gamma \) is holomorphic on \( \gamma \setminus o \), and

\[
\frac{g}{F} \gamma|_o = 0,
\]

where \( \frac{g}{F} \gamma \) is the holomorphic extension of \( \frac{g}{F} \gamma \) from \( \gamma \setminus o \) to \( \gamma \).

Proof. By Remark 2.14 there exists \( \varepsilon_1 > 0 \), such that \( g_1, \ldots, g_s \in \mathcal{I}((1 + \varepsilon_1)\varphi)(V_1) \). As \( F \) is not a germ of \( \mathcal{I}_+(\varphi)_o \), then for any neighborhood of \( o \), \( |F| \leq C(\sum_{1 \leq j \leq s} |g_j|^2)^{1/2} \) doesn’t hold for any constant \( C \).

By Lemma 2.11 and Remark 2.12 there exists a germ of analytic curve \( \gamma \) on \( V_0 \) through \( o \) satisfying \( \gamma \cap \{ F = 0 \} = \emptyset \), such that \( \frac{g_i}{F} \gamma \) is holomorphic on \( \gamma \) for any \( i \), and

\[
\frac{g_i}{F} \gamma|_o = 0,
\]

where \( \frac{g_i}{F} \gamma \) is the holomorphic extension of \( \frac{g_i}{F} \gamma \) from \( \gamma \setminus o \) to \( \gamma \).

3. Proof Theorem 1.1

We will prove Theorem 1.1 by the methods of induction and contradiction, and by using dynamically \( L^2 \) extension theorem with negligible weight.
3.1. Step 1: Theorem 1.1 for dimension 1 case.

We first consider Theorem 1.1 for dimension 1 case, which is elementary but revealing.

We choose $r_0$ small enough, such that $\{F = 0\} \cap \Delta_{r_0} \subset \{o\}$.

As $\int_{\Delta} |F|^2 e^{-\varphi} d\lambda_1 < +\infty$, by Lemma 2.2 we have

$$\liminf_{A \to +\infty} \mu(\{|F|^2 e^{-\varphi} > A\}) u(A) = 0.$$  \(3.1\)

It is clear that, for any given $B > 0$, there exist $A > B$ and $z_A \in \Delta_{u(A)^{-1/2}}$, such that $e^{-\varphi(z_A)}|F(z_A)|^2 \leq A$. We assume that $A > 10$.

Let $\psi = -\log 2$, then $\log |z' - z_A| + \psi < 0$.

Using Theorem 2.8 on $\Delta$, we obtain a holomorphic function $F_A$ on $\Delta$ for each $A$ and $p_A > 1$, such that $F_A|_{z_A} = F(z_A)$, and

$$\int_{\Delta} |F_A|^2 e^{-p_A \varphi} d\lambda_1 < 8\pi A.$$  \(3.1\)

By the negativeness of $\varphi$, it follows that

$$\int_{\Delta} |F_A|^2 d\lambda_1 < 8\pi A.$$  \(3.2\)

Assume Theorem 1.1 for $n = 1$ is not true. Therefore

$$\int_{\Delta_r} |F|^2 e^{-p \varphi} d\lambda = +\infty,$$

for any $r > 0$ and $p > 1$.

Since $\{F = 0\} \cap \Delta_{r_0} \subset \{o\}$, then it follows from inequality 3.1 that there exists a holomorphic function $h_A$ on $\Delta_{r_0}$, such that

1). $F_A|_{\Delta_{r_0}} = F|_{\Delta_{r_0}} h_A$;
2). $h_A(o) = 0$;
3). $h_A(z_A) = 1$.

By Lemma 2.3 it follows that

$$\int_{\Delta_{r_0}} |F_A|^2 d\lambda_1 > C_1 |z_A|^{-2} > C_1 u(A),$$

where $C_1$ is independent of $A$.

It contradicts to

$$\int_{\Delta} |F_A|^2 d\lambda_1 < 8\pi A.$$  \(3.1\)

We have thus proved Theorem 1.1 for $n = 1$.

3.2. Step 2: Theorem 1.1 for $n = k$.

Assume Theorem 1.1 for $n = k$ is not true. Therefore,

$$\int_{\Delta_{r_k}} |F|^2 e^{-\varphi} d\lambda_k < +\infty,$$

for some $r > 0$, and

$$\int_{\Delta_{r_k}} |F|^2 e^{-p \varphi} d\lambda_k = +\infty,$$

for any $r > 0$ and any $p > 1$.

Then the germ of the holomorphic function $F$ is in $\mathcal{I}(\varphi)_o$, but is not in $\mathcal{I}_+(\varphi)_o$. 
Using Proposition 2.15, we have a germ of an analytic curve \( \gamma \) through \( o \) satisfying \( \{ F | \gamma = 0 \} \subseteq \{ o \} \), such that for any germ of the holomorphic function \( g \) in \( \mathcal{I}_+ ( \varphi ) \), and we also have a holomorphic function \( h_g \) on \( \gamma \) satisfying
\[
|F|_o = 0,
\]
such that
\[
g|_\gamma = F|_\gamma h_g. \tag{3.3}
\]

Then we can choose a biholomorphic map \( \iota \) from a neighborhood of \( \Delta_\prime \times \Delta_\prime' \) to a neighborhood \( V_0 \subset \Delta_k \) of \( o \), which is small enough, with origin keeping \( \iota(o) = o \), such that
1) \( \iota^{-1} (\gamma) \) is a closed analytic curve in the neighborhood of \( \Delta_\prime \times \Delta_\prime' \);
2) \( \iota^{-1} (\gamma) \) satisfies the parametrization property as the analytic curve \( \mathcal{C} \) in Remark 2.5.

Note that
\[
\int_{V_0} |F|^2 e^{-p\varphi} d\lambda_n = \int_{\Delta_\prime \times \Delta_\prime'} |\iota^* (F)|^2 e^{-p\iota^* (\varphi)} d\lambda_n.
\]

Then
\[
\int_{\Delta_k} |F|^2 e^{-p\varphi} d\lambda_n = +\infty,
\]
for any \( r > 0 \) and any \( p > 1 \), is equivalent to
\[
\int_{\Delta_\prime \times \Delta_\prime'} |\iota^* (F)|^2 e^{-p\iota^* (\varphi)} d\lambda_n = +\infty,
\]
for any \( r > 0 \) and any \( p > 1 \).

As \( \int_{\Delta_\prime \times \Delta_\prime'} |\iota^* (F)|^2 e^{-p\iota^* (\varphi)} d\lambda_k < +\infty \), it follows from Lemma 2.2 that
\[
\liminf_{A \to +\infty} \mu(\{ z | \int_{\pi^{-1}(z)} |\iota^* (F)|^2 e^{-p\iota^* (\varphi)} d\lambda_{k-1} > A \}) u(A) = 0,
\]
for any \( j \), where \( \pi \) is the projection in Remark 2.5.

It is clear that for any given \( B > 0 \), there exists \( A > B \), such that
\[
\{ z_k | \int_{\pi^{-1}(z_k)} |\iota^* (F)|^2 e^{-p\iota^* (\varphi)} d\lambda_{k-1} > A \}
\]
cannot contain \( \Delta_{u(A)^{-1/2}} \).

Then for any given \( B > 0 \), there exist \( A > B \) and \( z_A \in \Delta_{u(A)^{-1/2}} \), such that
\[
\int_{\pi^{-1}(z_A)} |\iota^* (F)|^2 e^{-p\iota^* (\varphi)} d\lambda_{k-1} \leq A.
\]

We assume that \( A > e^{10} \).

3.2.1. Using dynamically \( L^2 \) extension theorem with negligible weight.

As the conjecture for \( n = k - 1 \) holds, there exists a positive number \( p_A > 1 \), such that
\[
\int_{\pi^{-1}(z_A)} |\iota^* (F)|^2 e^{-p_A \iota^* (\varphi)} d\lambda_{k-1} < 2A.
\]
Let $\psi = -\log 2$, then $\log |z' - z_A| + \psi < 0$. Using Theorem 2.8 on $\Delta' \times \Delta''$, we obtain a holomorphic function $F_A$ on $\Delta' \times \Delta''$ for each $A$, such that $F_A|_{\pi^{-1}(z_A)} = \iota^*(F)|_{\pi^{-1}(z_A)}$, and

$$\int_{\Delta' \times \Delta''} |F_A|^2 e^{-p_A\psi} d\lambda_k < 8\pi A.$$ 

It follows from equality 3.3 that there exists a holomorphic function on $\gamma$, denoted by $h_A$, such that

$$\iota^*F_A|_{\gamma} = F|_{\gamma} h_A,$$

(3.4)

therefore,

$$F_A|_{\iota^{-1}(\gamma)} = \iota^*(F)|_{\iota^{-1}(\gamma)} \iota^*(h_A),$$

(3.5)

where $h_A(0) = 0$, $\iota^*(h_A)((\iota^{-1}(\gamma) \cap \pi^{-1}(z_A))) = 1$.

It follows from the negativeness of $\varphi$ that

$$\int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_1 < 8\pi A.$$ 

Using equality 3.5, the condition $|z_A| < u(A)^{-\frac{1}{2}}$, and Lemma 2.6 we have

$$\int_{\iota^{-1}(\gamma)} |F_A|^2 d\pi^*\lambda_{\Delta'} > C_1 u(A),$$

where $C_1 > 0$ is independent of $A$ and $F_A$. In our use of Lemma 2.6 the function $h_A$ corresponds to $f_a$ (where $a = z_A$) in Lemma 2.6, which does not correspond to $h$ in the Lemma (actually $F|_{\gamma}$ corresponds to $h$).

Using Lemma 2.7, we obtain

$$\int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_k \geq C_3 \int_{\iota^{-1}(\gamma)} |F_A|^2 d\pi^*\lambda_{\Delta'},$$

where $C_3 > 0$ is independent of $A$ and $F_A$.

Therefore

$$\int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_k \geq C_1 C_3 u(A),$$

which contradicts to

$$\int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_k < 8\pi A,$$

for $A$ large enough.

We have thus proved Theorem 1.1 for $n = k$.

The proof of Theorem 1.1 is thus complete.

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