ON $\Lambda$-CONVERGENCE ALMOST EVERYWHERE OF MULTIPLE TRIGONOMETRIC FOURIER SERIES$^1$

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Abstract: We consider one type of convergence of multiple trigonometric Fourier series intermediate between the convergence over cubes and the $\lambda$-convergence for $\lambda > 1$. The well-known result on the almost everywhere convergence over cubes of Fourier series of functions from the class $LLn^{+}$\(\ln^{+}\ln\ln Ln^{+}\) has been generalized to the case of the $\Lambda$-convergence for some sequences $\Lambda$.

Key words: Trigonometric Fourier series, Rectangular partial sums, Convergence almost everywhere.

Suppose that $d$ is a natural number, $T^d = [-\pi, \pi]^d$ is a $d$-dimensional torus, and $\varphi$: $[0, +\infty) \to [0, +\infty)$ is a nondecreasing function. Let $\varphi(L)(T^d)$ be the set of all Lebesgue measurable real-valued functions $f$ on the torus $T^d$ such that

$$\int_{T^d} \varphi(|f(t)|) dt < \infty.$$  

Let $f \in L(T^d)$, $k = (k^1, k^2, \ldots, k^d) \in \mathbb{Z}^d$, $x = (x^1, x^2, \ldots, x^d) \in \mathbb{R}^d$, and $kx = k^1x^1 + k^2x^2 + \ldots + k^dx^d$. Denote by

$$c_k = \frac{1}{(2\pi)^d} \int_{T^d} f(t) e^{-ikt} dt$$

the $k$th Fourier coefficient of the function $f$ and by

$$\sum_{k \in \mathbb{Z}^d} c_k e^{ikx}$$  \hspace{1cm} (1)

the multiple trigonometric Fourier series of the function $f$.

Let $n = (n^1, n^2, \ldots, n^d)$ be a vector with nonnegative integer coordinates, and let $S_n(f, x)$ be the $n$th rectangular partial sum of series (1):

$$S_n(f, x) = \sum_{k = (k^1, \ldots, k^d): |k^j| \leq n^j, 1 \leq j \leq d} c_k e^{ikx}.$$  

Denote by $\text{mes}E$ the Lebesgue measure of a set $E$ and let $\ln^+ u = \ln(u + e)$, $u \geq 0$.

In 1915, in the case $d = 1$, N.N. Luzin (see [1]) suggested that the trigonometric Fourier series of any function from $L^2(T)$ converges almost everywhere. A.N. Kolmogorov [2] constructed an example of a function $F \in L(T)$ whose trigonometric series diverges almost everywhere and, later on [3], of a function from $L(T)$ with the Fourier series divergent everywhere on $T$. L. Carleson [4] proved that Luzin’s conjecture is true: if $f \in L^2(T)$, then the Fourier series of the function $f$ converges almost everywhere.

$^1$This work was supported by the Russian Science Foundation (project no. 14-11-00702).
everywhere. R. Hunt [5] generalized the statement about the almost everywhere convergence of the Fourier series to the class \( L(\ln^+ L)^2(T) \), particularly, to \( L^p(T) \) with \( p > 1 \). P. Sjölin [6] generalized it to the wider class \( L(\ln^+ L)(\ln^+ \ln^+ L)(T) \). In [7], the author showed that the condition \( f \in L(\ln^+ L)(\ln^+ \ln^+ L)(T) \) is also sufficient for the almost everywhere convergence of the Fourier series of the function \( f \). At present, the best negative result in this direction belongs to S.V. Konyagin [8]: if a function \( \varphi(u) \) satisfies the condition \( \varphi(u) = o(u\sqrt{\ln u / \ln \ln u}) \) as \( u \to +\infty \), then, in the class \( \varphi(L)(T) \), there exists a function with the Fourier series divergent everywhere on \( T \).

Let us now consider the case \( d \geq 2 \), i.e., the case of multiple Fourier series. Let \( \lambda \geq 1 \). A multiple Fourier series of a function \( f \) is called \( \lambda \)-convergent at a point \( x \in T^d \) if there exists a limit

\[
\lim_{\min\{n^i: 1 \leq i \leq d\} \to +\infty} S_n(f, x)
\]

considered only for vectors \( n = (n^1, n^2, \ldots, n^d) \) such that \( 1/\lambda \leq n^i/n^j \leq \lambda, \ 1 \leq i, j \leq d \). The \( \lambda \)-convergence is called the convergence over cubes (the convergence over squares for \( d = 2 \)) in the case \( \lambda = 1 \) and the Pringsheim convergence in the case \( \lambda = +\infty \), i.e., in the case without any restrictions on the relation between coordinates of vectors \( n \).

N.R. Tevzadze [9] proved that, if \( f \in L^2(T^2) \), then the Fourier series of the function \( f \) converges over cubes almost everywhere. Ch. Fefferman [10] generalized this result to functions from \( L^p(T^d) \), \( p > 1, d \geq 2 \). P. Sjölin [11] showed that, if a function \( f \) is from the class \( L(\ln^+ L)^d(\ln^+ \ln^+ L)(T^d) \), \( d \geq 2 \), then its Fourier series converges over cubes almost everywhere. The author [12] (see also [13]) proved the almost everywhere convergence over cubes of Fourier series of functions from the class \( L(\ln^+ L)^d(\ln^+ \ln^+ L)(T^d) \). The best current result concerning the divergence over cubes on a set of positive measure of multiple Fourier series of functions from \( \varphi(L)(T^d) \), \( d \geq 2 \), belongs to S.V. Konyagin [14]: for any function \( \varphi(u) = o(u(\ln u)^{d-1}\ln \ln u) \) as \( u \to +\infty \), there exists a function \( F \in \varphi(L)(T^d) \) with the Fourier series divergent over cubes everywhere.

On the other hand, Ch. Fefferman [15] constructed an example of a continuous function of two variables, i.e., a function from \( C(T^2) \) whose Fourier series diverges in the Pringsheim sense everywhere on \( T^2 \). M. Bakhbukh and E.M. Nikishin [16] proved that there exists \( F \in C(T^2) \) such that its modulus of continuity satisfies the condition \( \omega(F, \delta) = O(\ln^{-1}(1/\delta)) \) as \( \delta \to +0 \) and its Fourier series diverges in the Pringsheim sense almost everywhere. A.N. Bakhvalov [17] established that, for \( m \in \mathbb{N} \) and any \( \lambda > 1 \), there is a function \( F \in C(T^{2m}) \) such that the Fourier series of \( F \) is \( \lambda \)-divergent everywhere and the modulus of continuity of \( F \) satisfies the condition

\[
\omega(F, \delta) = O(\ln^{-m}(1/\delta)), \quad \delta \to +0.
\]

Later on, Bakhvalov [18] proved the existence of a function \( F \in C(T^{2m}) \) satisfying condition (2) and such that its Fourier series is \( \lambda \)-divergent for all \( \lambda > 1 \) simultaneously.

Let \( \Lambda = \{\lambda_\nu\}_{\nu \geq 1}^\infty \) be a nonincreasing sequence of positive numbers. Assume that

\[
\Omega_\Lambda = \left\{ n = (n^1, n^2, \ldots, n^d) \in \mathbb{N}^d : \frac{1}{1 + \lambda_{n^i}} \leq n^i/n^j \leq 1 + \lambda_{n^j}, \ 1 \leq i, j \leq d \right\}.
\]

We will say that a multiple Fourier series of a function \( f \in L(T^d) \) is \( \Lambda \)-convergent at a point \( x \in T^d \) if there exists a limit

\[
\lim_{n \in \Omega_\Lambda, \min\{n^i: 1 \leq j \leq d\} \to +\infty} S_n(f, x).
\]

Let us note that, if \( \lambda_\nu \equiv \lambda - 1 \) for some \( \lambda > 1 \), then the condition of \( \Lambda \)-convergence turns into the condition of \( \lambda \)-convergence defined above. And if \( \lambda_\nu \to 0 \) as \( \nu \to \infty \), then the condition of \( \Lambda \)-convergence is weaker than the condition of \( \lambda \)-convergence for any \( \lambda > 1 \).
The author proved [19] that, if a sequence $\Lambda = \{\lambda_{\nu}\}_{\nu=1}^{\infty}$ satisfies the condition $\ln^2 \lambda_{\nu} = o(\ln \nu)$ as $\nu \rightarrow \infty$, then there exists a function $F \in C(T^2)$ such that its Fourier series is $\Lambda$-divergent almost everywhere on $T^2$.

In the present paper, we obtain the following statement that strengthens the result of [12].

**Theorem 1.** Assume that a nonincreasing sequence of positive numbers $\Lambda = \{\lambda_{\nu}\}_{\nu=1}^{\infty}$ satisfies the condition

$$\lambda_{\nu} = O \left( \frac{1}{\nu} \right)$$

and a function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is convex on $[0, +\infty)$ and such that $\varphi(0) = 0$, $\varphi(u)u^{-1}$ increases on $[u_0, +\infty)$, and $\varphi(u)u^{-1-\delta}$ decreases on $[u_0, +\infty)$ for some $u_0 \geq 0$ and any $\delta > 0$. Assume that the trigonometric Fourier series of any function $g \in \varphi(L)(T)$ converges almost everywhere on $T$. Then, for any $d \geq 2$, the Fourier series of any function $f$ from the class $\varphi(L)(\ln^+ L)^{d-1}(T^d)$ is $\Lambda$-convergent almost everywhere on $T^d$.

Theorem 1 and the result of paper [7] imply the following statement.

**Theorem 2.** Let a nonincreasing sequence of positive numbers $\Lambda = \{\lambda_{\nu}\}_{\nu=1}^{\infty}$ satisfy condition (3), $d \geq 2$. Then the Fourier series of any function $f$ from the class $L(\ln^+ L)^d(\ln^+ \ln^+ L)(T^d)$ is $\Lambda$-convergent almost everywhere on $T^d$.

**Proof** of Theorem 1. Let a sequence $\Lambda = \{\lambda_{\nu}\}_{\nu=1}^{\infty}$ and a function $\varphi$ satisfy the conditions of the theorem. Let $\varphi_d(u) = \varphi(u)(\ln^+ u)^{d-1}$ for short. Without loss of generality, we can consider only functions $\varphi_d$ such that the functions $\varphi_d(\sqrt{\nu})$ are concave on $[0, +\infty)$. Otherwise, we can consider the functions $\varphi_d(u + a_d) - b_d$ (with appropriate constants $a_d$ and $b_d$) instead of $\varphi_d$. The corresponding class $\varphi_d(L)(T^d)$ will be the same in this case.

Denote by $S_n(f, x)$ the $n$th cubic partial sum of the Fourier series of the function $f$:

$$S_n(f, x) = S_n(f, x), \quad \text{where} \quad n = (n, \ldots, n).$$

Suppose that

$$M(f, x) = \sup_{n \in \mathbb{N}} |S_n(f, x)|,$$

$$M_\Lambda(f, x) = \sup_{n \in \mathbb{N} \in \Lambda} |S_n(f, x)|.$$

Under the conditions of the theorem (see [12, formula (3.1) and Lemma 3]), there are constants $K_d > 0$ and $y_d \geq 0$ such that

$$\text{mes} \left\{ x \in T^d : M(f, x) > y \right\} \leq \frac{K_d}{y} \left( \int_{T^d} \varphi_d(\|f(x)\|) \, dx + 1 \right), \quad y > y_d, \quad f \in \varphi_d(L)(T^d). \quad (4)$$

Using (4), we will prove that, for every $y > y_d$ and $f \in \varphi_d(L)(T^d)$,

$$\text{mes} \left\{ x \in T^d : M_\Lambda(f, x) > y \right\} \leq \frac{A_d}{y} \left( \int_{T^d} \varphi_d(\|f(x)\|) \, dx + 1 \right)$$

(5)
and, for every \( f \in \varphi_{d+1}(L)(\mathbb{T}^d) \),
\[
\int_{\mathbb{T}^d} M_\Lambda(f, x) \, dx \leq B_d \left( \int_{\mathbb{T}^d} \varphi_{d+1}(|f(x)|) \, dx + 1 \right),
\]
(6)
where \( A_d \) is independent of \( f \) and \( y \); \( B_d \) is independent of \( f \).

The proof is by induction on \( d \). Consider the base case, i.e., \( d = 1 \): statement (5) immediately follows from (4) because \( M(f, x) = M_\Lambda(f, x) \) in the one-dimensional case. Similarly, (6) is a consequence of [5, Theorem 2].

Let \( d \geq 2 \). Suppose that statements (5) and (6) hold for \( d - 1 \) and let us show that the same is true for \( d \).

First, let us prove the validity of (5). Let \( n = (n^1, n^2, \ldots, n^d) \in \Omega_\Lambda \). According to (3), there is an absolute constant \( C > 0 \) such that \( \lambda_{\nu\nu} \leq C \) for all natural numbers \( \nu \). Combining this with the definition of \( \Omega_\Lambda \), we obtain that, for all \( i, j \in \{1, 2, \ldots, d\} \),
\[
|n^i - n^j| \leq C.
\]
(7)

Recall that, if \( n = (n^1, n^2, \ldots, n^d) \), then the following representation holds for the \( n \)th rectangular partial sum of the Fourier series of the function \( f \):
\[
S_n(f, x) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{j=1}^{d} D_{n^j}(t^j) f(x^1 + t^1, \ldots, x^d + t^d) \, dt^1 \ldots dt^d,
\]
(8)
where \( D_n(t) = \sin((n + 1/2)t)/(2\sin(t/2)) \) is the one-dimensional Dirichlet kernel of order \( n \). Let us add to and subtract from the \( d \)-dimensional Dirichlet kernel \( \prod_{j=1}^{d} D_{n^j}(t^j) \) of order \( n \) the sum
\[
\sum_{k=2}^{d} \left( \prod_{j=1}^{k} D_{n^j}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \right)
\]
(here and in what follows, we suppose that all products \( \prod \) with an upper index less than a lower one are equal to 1). Rearranging the terms, we obtain
\[
\prod_{j=1}^{d} D_{n^j}(t^j) = \sum_{k=1}^{d-1} \left( \prod_{j=1}^{k} D_{n^j}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) - \prod_{j=1}^{k+1} D_{n^j}(t^j) \prod_{j=k+2}^{d} D_{n^j}(t^j) \right) + \prod_{j=1}^{d} D_{n^j}(t^j) =
\sum_{k=2}^{d} \left( \prod_{j=1}^{k-1} D_{n^j}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \left( D_{n^k}(t^k) - D_{n^1}(t^k) \right) \right) + \prod_{j=1}^{d} D_{n^j}(t^j).
From this and (8), it follows that
\[ S_n(f, x) = \sum_{k=2}^{d} \frac{1}{\pi} \int_{\mathbb{T}^d} \left( \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \right) \times \]
\[ \times f(x^1 + t^1, \ldots, x^d + t^d) \, dt^1 \ldots dt^d + \frac{1}{\pi} \int_{\mathbb{T}^d} \prod_{j=1}^{d} D_{n^1}(t^j) f(x^1 + t^1, \ldots, x^d + t^d) \, dt^1 \ldots dt^d = \]
\[ = \sum_{k=2}^{d} \frac{1}{\pi} \int_{\mathbb{T}} \left( D_{n^k}(t^k) - D_{n^1}(t^k) \right) \times \]
\[ \times \left( \int_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \right) f(x^1 + t^1, \ldots, x^d + t^d) \, dt^1 \ldots dt^{k-1} dt^{k+1} \ldots dt^d \, dt^k + S_n(f, x). \]
\[ (9) \]

Note that the latter term on the right hand side of (9) is the \( n^1 \)th cubic partial sum of the Fourier series of the function \( f \). By (7), for all \( k \in \{2, 3, \ldots, d\} \) and \( t \in \mathbb{T} \), we have \( |D_{n^k}(t) - D_{n^1}(t)| \leq C \). Combining this with (9), we obtain
\[ |S_n(f, x)| \leq \sum_{k=2}^{d} \frac{C}{\pi} \int_{\mathbb{T}^d} \int_{\mathbb{T}^{d-1}} \left( \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \times \right) \]
\[ \times f(x^1 + t^1, \ldots, x^d + t^d) \, dt^1 \ldots dt^{k-1} dt^{k+1} \ldots dt^d \, dt^k + |S_n(f, x)|. \]

Applying the definitions of \( M_\Lambda(f, x) \) and \( M(f, x) \), from the latter estimate, we obtain
\[ M_\Lambda(f, x) \leq M(f, x) + \frac{C}{\pi} \sum_{k=2}^{d} M_k(f, x), \]

where \( M_k(f, x) \) denotes the \( k \)th term of the sum on the left hand side of the equality in (10). Let \( k \in \{2, 3, \ldots, d\} \). Consider \( M_k(f, x) \). Denote by \( g_{k,t^k} \) the function of \( d-1 \) variables that can be obtained from the function \( f \) by fixing the \( k \)th variable \( t^k \):
\[ g_{k,t^k}(t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^d) = f(t^1, \ldots, t^{k-1}, t^k, t^{k+1}, \ldots, t^d), \quad (t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^d) \in \mathbb{T}^{d-1}. \]
Define \( \Omega'_k \) as the set of \( m_k = (m^1, \ldots, m^{k-1}, m^{k+1}, \ldots, m^d) \in \mathbb{N}^{d-1} \) such that \( m = (m^1, \ldots, m^d) \in \Omega_\Lambda \). Note that, in view of the invariance of \( \Omega_\Lambda \) with respect to a rearrangement of variables, the set \( \Omega'_k \) is independent of \( k \). Suppose that \( n'_k = (n^1, \ldots, n^{k-1}, n^{k+1}, \ldots, n^d) \in \mathbb{N}^{d-1} \). Then
\[ \frac{1}{\pi^{d-1}} \int_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^{d} D_{n^j}(t^j) \times \]
\[ \times f(x^1 + t^1, \ldots, x^{k-1} + t^{k-1}, x^k + t^k, x^{k+1} + t^{k+1}, \ldots, x^d + t^d) \, dt^1 \ldots dt^{k-1} dt^{k+1} \ldots dt^d = \]
to the inner integral on the right hand part of (11), we obtain

Similarly, from \([20, \text{Theorem 3}]\), we can obtain the following refinement of statement (5): there is

proved above.

According to (10),

\[ M_k(f, x) = \int \sup_{n'_k \in \Omega'_A} \left| S_{n'_k} \left( g_{k,z^k}, (x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^d) \right) \right| dx^k. \]

Further,

\[
\begin{align*}
\text{mes} \left\{ x \in \mathbb{T}^d : M_k(f, x) > y \right\} &= 2\pi \text{mes} \left\{ (x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^d) \in \mathbb{T}^{d-1} : M_k(f, x) > y \right\} \\
&\leq \frac{2\pi}{y} \int_{\mathbb{T}^{d-1}} M_k(f, x) \, dx^1 \ldots dx^{k-1} dx^{k+1} \ldots dx^d = \\
&= \frac{2\pi}{y} \int_{\mathbb{T}^d} \sup_{n'_k \in \Omega'_A} \left| S_{n'_k} \left( g_{k,\pi^k}, (x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^d) \right) \right| dx^k = \\
&= \frac{2\pi}{y} \int_{\mathbb{T}^d} \left( \int_{\mathbb{T}^{d-1}} \sup_{n'_k \in \Omega'_A} \left| S_{n'_k} \left( g_{k,\pi^k}, (x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^d) \right) \right| dx^1 \ldots dx^{k-1} dx^{k+1} \ldots dx^d \right) dx^k. \\
\end{align*}
\] (11)

From this, applying the induction hypothesis (more precisely, statement (6) for the dimension \(d-1\)) to the inner integral on the right hand part of (11), we obtain

\[
\begin{align*}
\text{mes} \left\{ x \in \mathbb{T}^d : M_k(f, x) > y \right\} &\leq \frac{2\pi}{y} \int_{\mathbb{T}^d} \left( B_{d-1} \int_{\mathbb{T}^{d-1}} \varphi_d(|f(x)|) \, dx^1 \ldots dx^{k-1} dx^{k+1} \ldots dx^d + 1 \right) \, dx^k \\
&\leq \frac{(2\pi)^2 B_{d-1}}{y} \left( \int_{\mathbb{T}^d} \varphi_d(|f(x)|) \, dx + 1 \right). \\
\end{align*}
\] (12)

According to (10),

\[
\left\{ x \in \mathbb{T}^d : M_A(f, x) > y \right\} \subset \left\{ x \in \mathbb{T}^d : M(f, x) > \frac{y}{2} \right\} \cup \left( \bigcup_{k=2}^{d} \left\{ x \in \mathbb{T}^d : M_k(f, x) > \frac{\pi y}{2(d-1)C} \right\} \right). \\
\] (13)

Combining (13), (4) and (12), we obtain (5) with the constant \(A_d = 2K_d + 8\pi(d-1)^2 B_{d-1} C\).

Now, we only need to prove the validity of statement (6). To this end, let us use statement (5) proved above.

From (5), it follows that the majorant \(M_A(f, x)\) is finite almost everywhere on \(\mathbb{T}^d\) for all \(f \in \varphi_d(L)(\mathbb{T}^d)\), in particular, for all \(f \in L^2(\mathbb{T}^d)\). Applying Stein’s theorem on limits of sequences of operators [20, Theorem 1], we see that the operator \(M_A(f, \cdot)\) is of weak type \((2, 2)\), i.e., there is a constant \(A^2_d > 0\) such that, for all \(y > 0\) and \(f \in L^2(\mathbb{T}^d)\),

\[
\text{mes} \left\{ x \in \mathbb{T}^d : M_A(f, x) > y \right\} \leq \frac{A^2_d}{y^2} \int_{\mathbb{T}^d} |f(x)|^2 \, dx. \\
\] (14)

Similarly, from [20, Theorem 3], we can obtain the following refinement of statement (5): there is a constant \(\bar{A}_d > 0\) such that, for all \(y \geq \frac{\bar{y}_d}{2} = \bar{A}_d\) and \(f \in \varphi_d(L)(\mathbb{T}^d)\),

\[
\text{mes} \left\{ x \in \mathbb{T}^d : M_A(f, x) > y \right\} \leq \int_{\mathbb{T}^d} \varphi_d \left( \frac{\bar{A}_d |f(x)|}{y} \right) \, dx \leq \frac{\bar{A}_d y}{y} \int_{\mathbb{T}^d} \varphi_d(|f(x)|) \, dx. \\
\] (15)
Further, let \( f \in \varphi_d(L)(\mathbb{T}^d) \) and \( y > 0 \). Suppose that
\[
g(x) = g_y(x) = \begin{cases} f(x), & |f(x)| > y, \\ 0, & |f(x)| \leq y; \end{cases} \quad h(x) = h_y(x) = f(x) - g(x).
\]
Define \( \lambda_f(y) = \text{mes} \{ x \in \mathbb{T}^d : M_A(f, x) > y \} \). Then
\[
\lambda_f(y) \leq \lambda \{ x \in \mathbb{T}^d : M_A(g, x) > y/2 \} + \lambda \{ x \in \mathbb{T}^d : M_A(h, x) > y/2 \} = \lambda_{g}(y/2) + \lambda_{h}(y/2).
\]
From this, using the equality
\[
\int_{\mathbb{T}^d} M_A(f, x) \, dx = -\int_{y}^{\infty} y \, d\lambda_f(y) = \int_{0}^{\infty} \lambda_f(y) \, dy
\]
(see, for example, [21, Chapter 1, § 13, formula (13.6)]), we obtain
\[
\int_{\mathbb{T}^d} M_A(f, x) \, dx \leq \bar{y}_d(2\pi)^d + \bar{\lambda} \int_{y}^{\infty} \lambda_f(y) \, dy \leq \bar{y}_d(2\pi)^d + \bar{\lambda} \int_{y}^{\infty} \lambda_g(y/2) \, dy + \bar{\lambda} \int_{y}^{\infty} \lambda_h(y/2) \, dy. \tag{16}
\]
Taking into account that \( g \in \varphi_d(L)(\mathbb{T}^d) \) and \( h \in L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \) and applying estimate (15) to \( \lambda_{g}(y/2) \) and estimate (14) to \( \lambda_{h}(y/2) \), from (16), we obtain
\[
\int_{\mathbb{T}^d} M_A(f, x) \, dx \leq \bar{y}_d(2\pi)^d + 2\bar{\lambda} \int_{y}^{\infty} \frac{1}{y} \varphi_d(|g(t)|) \, dt \, dy + 4\bar{\lambda}^2 \int_{y}^{\infty} \left( \frac{1}{y^2} \int \varphi_d(|h(t)|) \, dt \right) \, dy = \bar{y}_d(2\pi)^d + 2\bar{\lambda} \int_{y}^{\infty} \left( \frac{1}{y} \int \varphi_d(|f(t)|) \, dt \right) \, dy + 4\bar{\lambda}^2 \int_{y}^{\infty} \left( \frac{1}{y^2} \int \varphi_d(|f(t)|) \, dt \right) \, dy. \tag{17}
\]
Applying Fubibi’s theorem to the integrals on the right hand side of (17), we conclude that
\[
\int_{\mathbb{T}^d} M_A(f, x) \, dx \leq 2\bar{\lambda} \int_{\{ |f(t)| > \bar{y}_d \}} \varphi_d(|f(t)|) \left( \int_{\bar{y}_d}^{\infty} \frac{dy}{y} \right) \, dt + 4\bar{\lambda}^2 \int_{\mathbb{T}^d} |f(t)|^2 \left( \int_{\{ |f(t)| \}}^{\infty} \frac{dy}{y^2} \right) \, dt + \bar{y}_d(2\pi)^d,
\]
hence, statement (6) follows easily.

Finally, the Λ-convergence of the Fourier series of an arbitrary function from the class \( \varphi_d(L)(\mathbb{T}^d) \) can be obtained from (5) by means of standard arguments (see, for example, [12, Lemma 3]). Theorem 1 is proved. \( \square \)
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