Fibre derivatives: some applications to singular lagrangians

XAVIER GRÀCIA
Departament de Matemàtica Aplicada i Telemàtica
Universitat Politècnica de Catalunya
Campus Nord UPC, edifici C3
C. Jordi Girona, 1
08034 Barcelona
Catalonia, Spain
e-mail: xgracia@mat.upc.es
23 February 1999

Abstract

The fibre derivative of a bundle map is studied in detail. In the particular case of a real function, several constructions useful to study singular lagrangians are presented. Some applications are given; in particular, a geometric construction useful to solve the Euler-Lagrange equation of a singular lagrangian. A free particle in a curved space-time is studied as an example.

Keywords: fibre derivative, bundle map, vertical bundle, singular lagrangian, stabilisation algorithm.

MSC: 58C25, 58F05 PACS: 02.40.Vh, 03.20.+i

1 Introduction

During the last years the tools of Differential Geometry have been progressively applied to clarify the theory of singular lagrangians of Theoretical Physics. One of these tools is the fibre derivative. In the general form that will be considered in this article, the fibre derivative can be found in [GS73]; however, in the literature on lagrangian formalism the fibre derivative has been mostly limited to be the definition of the Legendre’s transformation between the velocity space and the phase space, whereas other geometric structures of the tangent and cotangent bundles have played a more important role. The purpose of this paper is twofold: to give a thorough presentation of the fibre derivative and its properties, and to apply these results to some problems about singular lagrangians.

To have a look at the question, let us consider a fibre bundle \( P \to M \) and a real function \( f: P \to \mathbb{R} \). If \( V(P) \) denotes the bundle of vertical tangent vectors of \( P \), we shall see that \( f \) defines a map \( \mathcal{V}f: P \to V^*(P) \). Indeed, if we write \( f_x: P_x \to \mathbb{R} \) for the restriction of \( f \) to the fibre \( P_x \), then \( \mathcal{V}f(p_x) = df_x(p_x) \), which is an element of \( T^*_{P_x}(P_x) = V^*_{P_x}(P) \). Its local expression is \( \mathcal{V}f(x,p) = (x,p; \partial f/\partial p) \).
The construction of $\nabla f$ cannot be extended to higher-order fibre derivatives. Let us consider again $f_x: P_x \to \mathbb{R}$; in general, it is not possible to define something like the hessian of $f_x$. Of course, several objects may be constructed involving (in coordinates) the second derivatives of $f_x$, but not as a symmetric bilinear form on a certain space. This is readily seen in coordinates: if a change of coordinates $p \mapsto \bar{p}$ is performed, with jacobian matrix $J = \partial p / \partial \bar{p}$, then the hessian matrix $\partial^2 f_x / \partial p \partial p$ becomes

$$\frac{\partial^2 f_x}{\partial \bar{p}^k \partial \bar{p}^l} = J^{-1}_i^k \frac{\partial^2 f_x}{\partial p^i \partial p^j} J^{-1}_j^l + J^{-1}_i^k \frac{\partial J_j^i}{\partial \bar{p}^j} \frac{\partial f_x}{\partial p^j}.$$

The annoying extra term disappears under two circumstances: when $p$ is a critical point of $f_x$, and when the manifold $P_x$ has an appropriate extra structure allowing for an atlas with constant jacobians. For instance, when $P_x$ is an affine space. Indeed, it will be shown that higher-order fibre derivatives can be defined for any bundle map $f: A \to B$, where $A$ and $B$ are affine bundles over a manifold $M$.

In the development of these concepts we pay special attention to the fibre derivative of a fixed real function $f: A \to \mathbb{R}$. Its fibre derivative is a map $\mathcal{F}: A \to E^*$, where $E$ is the vector bundle modelling the affine bundle $A$, and we shall study several constructions concerning the fibre derivatives of other functions defined either on $A$ or on $E^*$. Some of these constructions simplify other ones previously given in [CLR 87] [Gracia 91] [GPR 91] to study several structures of first- and higher-order singular lagrangians.

In the theory of singular lagrangians, and more particularly in the study of the relations between the lagrangian and the hamiltonian formalisms, several interesting ideas have been presented in [Kam 82] [BGPR 86] [Pon 88]. In particular, given a singular lagrangian $L$, the choice of a hamiltonian function $H$ and a set of primary hamiltonian constraints $\phi_\mu$ allows to construct a local expression of a vector field $D_0$ in the lagrangian formalism such that it satisfies the Euler-Lagrange equation on the primary lagrangian constraint submanifold; this vector field can be used as a departure point for an algorithm to find all the lagrangian constraints and the final dynamics. It turns out that $D_0$ is coordinate-dependent; however, in section 5 it will be shown that there is a geometric procedure —based on the fibre derivative— to construct a vector field that coincides with $D_0$ under a particular choice of coordinates.

The paper is organised as follows. Section 2 contains the definition and some properties of the fibre derivative, while section 3 is devoted to the particular case of a real function and several special constructions. In section 4 some relations between lagrangian and hamiltonian formalisms are presented and developed using the preceding constructions. Section 5 addresses the problem of solving the Euler-Lagrange equation for a vector field for a singular lagrangian. Finally, these results are applied in section 6 to the lagrangian of a free particle in a curved space-time.

Basic techniques about fibre bundles are needed in what follows, especially the pull-back of a bundle, the tangent bundle of a bundle, and particular properties of affine bundles. They may be found in several books, as for instance [AMR 88] [KMS 93] [LR 85] [Sau 89].


2 Fibre derivatives

As we have said in the introduction, the concept of fibre derivative can be found, for instance, in [GS73]; our presentation is more general and contains many details to be used later.

The fibre tangent map

Let $A \xrightarrow{\pi} M$ and $B \xrightarrow{\rho} M$ be fibre bundles over $M$, and $A \xrightarrow{f} B$ a fibre bundle morphism. Its tangent map $Tf: TA \to TB$ is a vector bundle morphism over $f$. Since $T(\pi) = T(\rho) \circ T(f)$, if a vector $v_a \in T_a(A)$ is vertical (i.e., tangent to the fibres, which amounts to saying that $T(\pi) \cdot v_a = 0$), so it is $T_a(f) \cdot v_a \in T_{f(a)}(B)$. Therefore $Tf$ restricts to a vector bundle morphism between the bundles of vertical vectors, $Vf: VA \to VB$, which can be called the fibre tangent map of $f$.

Recall that this vector bundle morphism over $f$ is equivalent to a vector $A$-bundle morphism —let us denote it by $\tilde{V}f$— with values in the pullback of $VB$, $f^*(VB) = A \times_f VB$, so we have:

$$\begin{array}{ccc}
VA & \xrightarrow{\tilde{V}f} & A \times_f VB \\
\downarrow \cong & & \downarrow \cong \\
A \times_M E & \xrightarrow{\hat{D}f} & B \times_M F
\end{array}$$

Moreover, this vector $A$-bundle morphism can also be identified with a section —let us denote it by $Vf$— of the vector $A$-bundle $\text{Hom}(VA, A \times_f VB) \to A$. If the local expression of $f$ is $(x^\mu, a^i) \mapsto (x^\mu, f^k(x, a))$, then the local expression of the section $Vf$ is

$$Vf(x^\mu, a^i) = \left( x^\mu, a^i, \frac{\partial f^k}{\partial a^i}(x, a) \right).$$

The fibre derivative

From now on our main concern is the following linear setting: $f: A \to B$ is a fibre $M$-bundle map between affine bundles $\pi: A \to M$ and $\rho: B \to M$ modelled respectively on vector bundles $E$ and $F$.

Recall that the vertical bundle of the affine bundle $A \to M$ can be identified with the pull-back of $E$ to $A$ through the vertical lift $\text{vl}_A: A \times_M E \xrightarrow{\cong} VA$; notice also for later use that a bundle map $\xi: A \to E$ is canonically identified with a vertical vector field $\xi^v$ on $A$: $\xi^v(a_x) = \text{vl}_A(a_x, \xi(a_x))$.

So, using also the isomorphism $A \times_f (B \times_M F) \cong A \times_M F$, we obtain a diagram

$$\begin{array}{ccc}
VA & \xrightarrow{\tilde{V}f} & A \times_f VB \\
\downarrow \cong & & \downarrow \cong \\
A \times_M E & \xrightarrow{\hat{D}f} & B \times_M F
\end{array}$$
From this diagram we identify the vertical tangent map with a vector $A$-bundle morphism $\tilde{D}f$. As before, this morphism can be identified with a section $Df$ of the vector bundle $\text{Hom}(A \times_M E, A \times_M F)$; indeed we have

$$\text{Hom}(VA, A \times fVB) \cong \text{Hom}(A \times_M E, A \times_M F) \cong A \times_M \text{Hom}(E, F) \rightarrow \text{Hom}(E, F)$$

From $Df$ and the maps in this diagram we obtain an $M$-bundle map

$$Df: A \rightarrow \text{Hom}(E, F) \cong F \otimes E^*,$$

which is called the fibre derivative of $f$. If the local expression of $f$ is $(x^\mu, a^i) \mapsto (x^\mu, f^k(x, a))$, then the local expression of $Df$ is

$$Df(x^\mu, a^i) = \left( x^\mu, \frac{\partial f^k}{\partial a^i}(x, a) \right).$$

Let us have a look at the fibres. In the general case of an $M$-bundle morphism $f: A \rightarrow B$ between fibre bundles, restriction to the fibres of $x \in M$ yields a manifold map $f_x: A_x \rightarrow B_x$. Its tangent map $T_{a_x}(f_x)$ at $a_x \in A_x$ is identified with a map $V_{a_x}(A) \rightarrow V_{f(a_x)}(B)$; this is the fibre map $V(f)_{a_x}$. In the linear case $A_x$ and $B_x$ are affine spaces, and then the ordinary derivative of $f_x$ at a point $a_x \in A_x$ is a linear map $Df_x(a_x): E_x \rightarrow F_x$ between the modelling vector spaces. This map is the fibre derivative $Df(a_x)$ at $a_x$ as defined above. See also [GS 73].

**Higher order fibre derivatives**

Since $Df$ is also a bundle map between affine bundles, the same procedure can be applied to compute its fibre derivative. The canonical isomorphism $\text{Hom}(E, \text{Hom}(E, F)) \cong \mathcal{L}^2(E; F)$ now yields the second fibre derivative, the fibre hessian, which is a map

$$D^2f: A \rightarrow \mathcal{L}^2(E; F) \cong \text{Hom}(E \otimes E, F) \cong F \otimes E^* \otimes E^*,$$

whose local expression is $D^2f(x^\mu, a^i) = \left( x^\mu, \frac{\partial^2 f^k}{\partial a^i \partial a^j}(x, a) \right)$.

More generally we obtain the $k$th order fibre derivative

$$D^k f: A \rightarrow \mathcal{L}^k(E; F) \cong \text{Hom}^k \otimes E, F) \cong F \otimes E^* \otimes \ldots \otimes E^*.$$  \hspace{1cm} (2.2)

The higher order fibre derivatives can also be considered as sections of certain bundles, and also as several vector $A$-bundle morphisms; for instance, the hessian $D^2f$ of $f: A \rightarrow B$ yields a section of $A \times_M \mathcal{L}^2(E; F)$ and vector bundle maps like $A \times_M (E \otimes E) \rightarrow A \times_M F$ and $A \times_M E \rightarrow A \times_M \text{Hom}(E, F)$. 

Rules of derivation

Some properties of the derivative apply to the fibre derivative. For instance, if we have as before a bundle map $f: A \to F$, and $\phi: A \to \mathbb{R}$ is a function, then

$$D(f\phi) = (Df)\phi + f \otimes D\phi,$$

(2.3)

where now the last term is a map $A \to F \otimes E^* \cong \text{Hom}(E, F)$. Now, if $\varphi: A \to F^*$ is another fibre bundle map, then the contraction $\langle \varphi, f \rangle$ is a function on $A$ whose fibre derivative is

$$D(\langle \varphi, f \rangle) = \varphi \bullet Df + D\varphi \bullet f,$$

(2.4)

where $\bullet$ denotes the composition between the images.

In the same way, if we consider three affine bundles bundles and two bundle maps, $A \overset{f}{\to} B \overset{g}{\to} C$, then $Df: A \to \text{Hom}(E, F)$ and we have $Dg \circ f: A \to \text{Hom}(F, G)$, and

$$D(g \circ f) = (Dg \circ f) \bullet Df.$$  

(2.5)

3 The fibre derivative of a real function

Let us consider a fibre bundle $A \xrightarrow{\pi} M$, and a real function $f: A \to \mathbb{R}$. This can be seen as a fibre bundle morphism from $A$ to the trivial line bundle $M \times \mathbb{R}$. Then the vertical tangent map is identified with a section $Vf$ of $V^*A$. Indeed, $Vf$ is the composition of $df$ with the canonical projection $T^*A \to V^*A$.

If $A$ is, as in the preceding section, an affine bundle modelled on a vector bundle $E$, then the fibre derivative of $f: A \to \mathbb{R}$ is an $M$-bundle map

$$\mathcal{F} = Df: A \to \text{Hom}(E, M \times \mathbb{R}) = E^*.$$  

(3.1)

In this section we shall study some properties of this map. We begin with the following remark:

**Proposition 1** Let $\mathcal{F}: A \to E^*$ be a fibre bundle map — for instance the fibre derivative of $f: A \to \mathbb{R}$. There exists an isomorphism $V^*(A) \cong A \times_\mathcal{F} V(E^*)$

**Proof.** The isomorphism is the composition of isomorphisms of vector $A$-bundles

$$b_\mathcal{F}: V^*(A) \xrightarrow{\text{can}} A \times M E^* \xrightarrow{\text{Id}_A \times_\mathcal{F} \eta_E} A \times_\mathcal{F} (E^* \times M E^*) \xrightarrow{\text{Id}_A \times_\mathcal{F} \eta_{E^*}} A \times_\mathcal{F} V(E^*),$$

(3.2)

where the second one is due to the contravariance of the pull-back. 

By projection to the second factor we obtain the vector bundle morphism over $\mathcal{F}$

$$b_\mathcal{F}: V^*(A) \to V(E^*),$$

(3.3)

which is an isomorphism at each fibre.

**Relation between $T(\mathcal{F})$ and the hessian**

Now we consider the fibre hessian $D^2f$ of $f$,

$$D^2f = D\mathcal{F}: A \to \text{Hom}(E, E^*),$$

(3.4)
whose associated vector $A$-bundle morphism is $\hat{D}\mathcal{F}: A \times_M E \to A \times_M E^*$.

To establish the relation between $D^2f$ and $T(\mathcal{F})$ it will be useful to consider the fibre hessian as a morphism between the vertical vectors, which is done through the vertical lift on $A$. So $\hat{D}\mathcal{F}$ is identified with a symmetric vector $A$-bundle morphism

$$\hat{W}: V(A) \longrightarrow V^*(A) \quad (3.5)$$

On the other hand, the isomorphism $V^*(A) \cong A \times_\mathcal{F} V(E^*)$ identifies the section $\mathcal{V}\mathcal{F}: A \to \text{Hom}(VA, A \times_\mathcal{F} V(E^*))$ —and therefore also $D\mathcal{F}$— with a section $\mathcal{W}: A \to \text{Hom}(V(A), V^*(A))$,

whose corresponding operator is indeed $\hat{W}$.

The following diagram shows the different morphisms involved:

\[
\begin{array}{ccc}
A \times_\mathcal{F} V(E^*) & \cong & A \times_\mathcal{F} (E^* \times_M E^*) \\
\mathcal{V}\mathcal{F} & \cong & \mathcal{V}\mathcal{F} \\
V(A) & \cong & A \times_M E \\
\hat{D}\mathcal{F} & \cong & A \times_M E^* (\mathcal{F}, \text{Id}_{E^*}) E^* \times_M E^* \cong V(E^*) \\
\hat{W} & \cong & b_F \\
V^*(A) & \cong & b_F \circ \hat{W}
\end{array}
\]

Since $\mathcal{F}$ is fibred over $M$, $\ker T(\mathcal{F}) \subset \ker T(\pi) = V(A)$; therefore the kernel of $T(\mathcal{F})$ coincides with the kernel of the fibre tangent map $V(\mathcal{F})$. Moreover,

$$V(\mathcal{F}) = b_F \circ \hat{W}$$

(where $b_F$ is the morphism defined by (3.3)), therefore this kernel is also the kernel of $\hat{W}$. So we have proved:

**Proposition 2** With the notations above,

$$\ker T(\mathcal{F}) = \ker \hat{W}. \quad (3.6)$$

In particular, $\mathcal{F}$ is a local diffeomorphism at $a_x \in A$ iff $\hat{W}_{a_x}$ is a linear isomorphism. ■

These statements follow also from the local expressions of the maps:

$$\mathcal{F}: \quad (x, a) \mapsto (x, \hat{\alpha}(x, a)), \quad \hat{\alpha}_i = \frac{\partial f}{\partial a_i},$$

$$T(\mathcal{F}): \quad (x, a; v, h) \mapsto \left( x, \hat{\alpha}(x, a); v, \frac{\partial^2 f}{\partial a \partial x} v + \frac{\partial^2 f}{\partial a \partial a} h \right),$$

$$V(\mathcal{F}): \quad (x, a; h) \mapsto \left( x, \hat{\alpha}(x, a); \frac{\partial^2 f}{\partial a \partial a} h \right),$$

$$\hat{W}: \quad (x, a; h) \mapsto \left( x, a; \frac{\partial^2 f}{\partial a \partial a} h \right),$$
The vector field $\Gamma_h$

Still let us consider the fibre derivative $F: A \to E^*$ of $f$. We are going to derive several properties of a function $h: E^* \to \mathbb{R}$ and its fibre derivatives.

We use the notation

$$\gamma_h = D_h \circ \mathcal{F}$$

for the composition $A \xrightarrow{\mathcal{F}} E^* \xrightarrow{Dh} E^{**} \cong E$. Recall that this map $\gamma_h: A \to E$ yields in a canonical way, through the vertical lift, a vertical vector field $\gamma_h^v$ on $A$:

$$\Gamma_h := \gamma_h^v = vl_A \circ (\text{Id}_A, Dh \circ \mathcal{F}): A \to A \times_M E \to VA \subset TA, \quad (3.8)$$

whose local expression is

$$\Gamma_h = F^* \left( \frac{\partial h}{\partial \alpha_i} \right) \frac{\partial}{\partial a^i}$$

if the natural coordinates of $A$ and $E^*$ are respectively $(x, a)$ and $(x, \alpha)$.

We also consider the composition

$$A \xrightarrow{\mathcal{F}} E^* \xrightarrow{D^2h} \text{Hom}(E^*, E^{**}) \cong \text{Hom}(E^*, E).$$

Then, application of the chain rule yields

$$D(h \circ \mathcal{F}) = D^2 f \bullet \gamma_h, \quad (3.9)$$

$$D(\gamma_h) = (D^2 h \circ \mathcal{F}) \bullet D^2 f. \quad (3.10)$$

(Here the composition is between the images of the maps $D^2 f: A \to \text{Hom}(E, E^*)$ and $D^2 h \circ \mathcal{F}: A \to \text{Hom}(E^*, E)$.) These properties will be used in the following section.

Notice from (3.3) that if $h$ vanishes on the image $\mathcal{F}(A) \subset E^*$ then $\Gamma_h$ is in the kernel of $\hat{W}$. Then, we obtain the following result:

**Proposition 3** Let $f: A \to \mathbb{R}$ with fibre derivative $\mathcal{F}: A \to E^*$. Suppose that $\mathcal{F}$ has connected fibres and is a submersion onto a closed submanifold—that is to say, $f$ is almost regular in the terminology of [GN 79]. Then the submanifold $\mathcal{F}(A) \subset E^*$ can be locally defined by the vanishing of a set of independent functions $\phi_\mu: E^* \to \mathbb{R}$, and the vertical fields $\Gamma_{\phi_\mu}$ constitute a frame for $\text{Ker} \hat{W} = \text{Ker} T(\mathcal{F})$. ■

So we have recovered and generalised earlier results from the theory of singular lagrangians—see for instance [BGPR 86].

The vector field along $\mathcal{F} \Upsilon^g$

Now we present a construction dual to $\Gamma_h$. Given a function $g: A \to \mathbb{R}$, the map

$$\Upsilon^g = vl_{E^*} \circ (\mathcal{F}, Dg): A \to E^* \times_M E^* \to VE^* \subset TE^*, \quad (3.11)$$

is a vector field along the fibre derivative $\mathcal{F}$ of $f$, with local expression

$$\Upsilon^g = \frac{\partial g}{\partial a^i} \left( \frac{\partial}{\partial \alpha_i} \circ \mathcal{F} \right).$$

Notice also that

$$\Upsilon^g = b_{\mathcal{F}} \circ (Dg)^v, \quad (3.12)$$
where \( b_F \) is defined by (3.3) and \((Dg)^\gamma\) is the section of \( V^*A \) constructed from \( Dg \). As differential operators, \( \Gamma_h \) and \( \Upsilon^g \) are related by

\[
\Upsilon^g \cdot h = \Gamma_h \cdot g.
\]

**The Liouville’s vector field**

Let \( A \) be an affine bundle modelled on a vector bundle \( E \). If \( \eta: A \to E \) is a bundle map with associated vertical field \( Y = \eta^\gamma \) on \( A \), and \( g: A \to \mathbb{R} \) is a function, then it is easily seen that

\[
Y \cdot g = \langle Dg, \eta \rangle. \tag{3.13}
\]

Recall that the Liouville’s vector field of a vector bundle \( E \) is the vertical field \( \Delta_E = \text{Id}_E^\gamma \). If \( g: E \to \mathbb{R} \) is a function, then (3.13) yields

\[
(\Delta_E \cdot g)(e_x) = \langle Dg(e_x), e_x \rangle, \tag{3.14}
\]

and then application of Leibniz’s rule gives

\[
D(\Delta_E \cdot g)(e_x) = Dg(e_x) + D^2 g(e_x) \cdot e_x. \tag{3.15}
\]

**4 Some structures relating lagrangian and hamiltonian formalisms**

The basic concepts about singular lagrangian and hamiltonian formalisms —Legendre’s map, energy, hamiltonian function, hamiltonian constraints . . . — are well-known and can be found in several papers, for instance [Car 90]. First we shall recall some of these concepts.

Let us consider a first-order autonomous lagrangian on a manifold \( Q \), that is to say, a map \( L: TQ \to \mathbb{R} \). Its fibre derivative (Legendre’s map) and fibre hessian are maps

\[
TQ \xrightarrow{F=L^\gamma} T^*Q,
\]

\[
TQ \xrightarrow{W=D^F} \text{Hom}(TQ, T^*Q).
\]

As said before, \( W \) can be identified with a vector bundle morphism \( \tilde{W}: V(TQ) \to V^*(TQ) \). If this is an isomorphism —equivalently, the Legendre’s map is a local diffeomorphism—the lagrangian \( L \) is called regular, otherwise it is called singular.

**Remark** A \( k \)th order lagrangian is a function \( L: T^kQ \to \mathbb{R} \). Since the \( k \)th order tangent bundle \( T^kQ \) is an affine bundle over \( T^{k-1}Q \times_Q TQ \), the fibre derivative and hessian of \( L \) can be studied in a similar way, and some of the following developments could be extended to this case. The case of a time-dependent lagrangian can also be dealt with in a similar way. Finally, some results could also be applied to field theory, where the lagrangian density may be considered as a function on a certain affine bundle [GS 73].
The Euler-Lagrange form

One of the basic objects of the variational problem associated to $L$ is the Euler-Lagrange form —see for instance [CLM91]—

$$\delta L: T^2Q \to T^*Q;$$

(4.1)

this is a 1-form along the second tangent bundle projection $\tau_2: T^2Q \to Q$ with local expression

$$\delta L(q, v, a) = \left(\frac{\partial L}{\partial q^i} - D_t \left(\frac{\partial L}{\partial v^i}\right)\right) dq^i.$$

A solution of the Euler-Lagrange equation is a path $\gamma: I \to Q$ such that $\delta L \circ \ddot{\gamma} = 0$, where $\ddot{\gamma}: I \to T^2Q$ is the second time-derivative of $\gamma$.

It is often convenient to consider second-order differential equations as first-order equations on the tangent bundle represented by vector fields satisfying the second-order condition. Let us denote by $N(Q) \subset T(TQ)$ the subset of tangent vectors satisfying the second-order condition; this is an affine subbundle modelled on the vector subbundle $V(TQ)$ of vertical tangent vectors. There is a canonical immersion $T^2Q \to T(TQ)$ that identifies $T^2Q$ with $N(TQ)$. Since $TQ \times_Q TQ$ is also identified with $V(TQ)$, we can regard the Euler-Lagrange form as a map

$$\hat{\delta}L: N(TQ) \to V^*(TQ),$$

(4.2)

which is indeed an affine bundle map; its associated vector bundle morphism is

$$\overline{\delta}L = -\hat{W}.$$

(4.3)

Let us remark that there are other geometric expressions of the Euler-Lagrange equation, using the presymplectic form $\omega_L$ defined by $L$ or the time-evolution operator $K$ —see for instance [Car90] [GP92]. Equations relating $\delta L$ with these objects can also be obtained.

Connection with the hamiltonian space

Let $h: T^*Q \to \mathbb{R}$ be a function. Recall the notation $\gamma_h = Dh \circ F: TQ \to TQ$, and that this map is canonically identified with a vertical vector field $\Gamma_h$ on $TQ$.

We assume that $L$ is an almost regular lagrangian [GN79]; this is the most basic technical requirement to develop a hamiltonian formulation from a singular lagrangian $L$. Then the image of the Legendre’s map is a submanifold $P_0 \subset T^*Q$, the primary hamiltonian constraint submanifold. Recall from Proposition 3 that, if $\phi_\mu$ constitute an independent set of primary hamiltonian constraints, then the vertical fields $\Gamma_\mu = \Gamma_{\phi_\mu}$ constitute a frame for $\text{Ker} \hat{W}$ and also for $\text{Ker} T(F)$.

Recall that the energy of $L$ is defined by $E = \Delta_{TQ} \cdot L - L$. Due to the properties of the Liouville’s vector field at the end of the preceding section,

$$E(v_q) = \langle DL(v_q), v_q \rangle - L(v_q),$$

(4.4)

$$DE(v_q) = W(v_q) \cdot v_q.$$
Finally, recall that a *hamiltonian* is a function $H: T^*Q \to \mathbb{R}$ such that $E = H \circ F$. It exists since $L$ is almost regular, and is unique on the primary hamiltonian constraint submanifold.

### A resolution of the identity

Given an almost regular lagrangian $L$, the choice of suitable data yields a (local) resolution of the identity map of $TQ$ as follows.

**Proposition 4** Let $L$ be an almost regular lagrangian, $\phi_\mu$ a set of independent primary hamiltonian constraints and $H$ a hamiltonian function (on an open set of $T^*Q$). Then there exist functions $\lambda^\mu$ (defined on an open set of $TQ$) such that, locally,

$$\text{Id}_{TQ} = \gamma_H + \sum_\mu \gamma_\mu \lambda^\mu. \quad (4.6)$$

Moreover,

$$\text{Id}_{\text{Hom}(TQ; TQ)} = M \circ W + \sum_\mu \gamma_\mu \otimes D\lambda^\mu, \quad (4.7)$$

where

$$M = (D^2 H \circ F) + \sum_\mu (D^2 \phi_\mu \circ F) \lambda^\mu. \quad (4.8)$$

(Notice that $W$ is a map $TQ \to \text{Hom}(TQ, T^*Q)$ and $M$ is a map $TQ \to \text{Hom}(T^*Q, TQ)$.)

**Proof.** Applying the chain rule (3.9) to the definition of $H$ yields

$$DE(v_q) = W(v_q) \cdot \gamma_H(v_q),$$

so using (4.5) we obtain

$$W(v_q) \cdot (v_q - \gamma_H(v_q)) = 0,$$

and since the terms in parentheses are in $\text{Ker} W(v_q)$, there exist numbers $\lambda^\mu(v_q)$ such that

$$v_q = \gamma_H(v_q) + \sum_\mu \gamma_\mu(v_q) \lambda^\mu(v_q),$$

which is (4.6).

Its fibre derivative is equation (4.7), which follows by applying equation (3.10) and the Leibniz’s rule to (4.6).

The above proposition can be given a slightly different form, using the identification of bundle maps $TQ \to TQ$ with vertical vector fields: equation (4.6) can be rewritten as

$$\Delta_{TQ} = \Gamma_H + \sum_\mu \Gamma_\mu \lambda^\mu. \quad (4.9)$$

In the same way, equation (4.7) can be expressed as an endomorphism of $V(TQ)$:

$$\text{Id}_{V(TQ)} = \hat{M} \circ \hat{W} + \sum_\mu \Gamma_\mu \otimes (D\lambda^\mu)^v, \quad (4.10)$$

where $\hat{M}: V^*(TQ) \to V(TQ)$ is the operator corresponding to the map $M$, and $(D\lambda^\mu)^v$ is the section of $V^*(TQ)$ deduced from the map $D\lambda^\mu: TQ \to T^*Q$.

Notice that application of (4.7) to $\gamma_\nu$ yields $\gamma_\nu = \sum_\mu \gamma_\mu(D\lambda^\mu, \gamma_\nu)$; then equation (3.13) shows that

$$\Gamma_\nu \cdot \lambda^\mu = \delta^\mu_\nu. \quad (4.11)$$
Now let us write the local expressions of (4.7) and (4.8):

\[ \delta^i_k = \sum_j M^{ij} W_{jk} + \sum_{\mu} F^* \left( \frac{\partial \phi_\mu}{\partial p_i} \right) \frac{\partial \lambda^\mu}{\partial v^k}, \]

\[ M^{ij} = F^* \left( \frac{\partial^2 H}{\partial p_i \partial p_j} \right) + \sum_{\mu} F^* \left( \frac{\partial^2 \phi_\mu}{\partial p_i \partial p_j} \right) \lambda^\mu. \]

These were deduced in [BGPR 86] by derivating the local expression of (4.6), which is

\[ v^i = F^* \left( \frac{\partial H}{\partial p_i} \right) + \sum_{\mu} F^* \left( \frac{\partial \phi_\mu}{\partial p_i} \right) \lambda^\mu. \]

See also [Kam 82].

**Remark** The manifold \( T^*(Q) \) has a canonical symplectic 2-form. Let \( X_h \) be the Hamiltonian vector field of a function \( h \). Then

\[ T(\tau^*_Q) \circ X_h \circ F : TQ \to TQ. \] (4.12)

coincides with the map \( \gamma_h \) (3.7); this yields the construction of the vertical fields \( \Gamma_h \) that appeared in [CLR 87]. A similar construction was given in [Grà 91] for the maps \( \Upsilon^g \) defined by (3.11).

## 5 The Euler-Lagrange equation for a vector field

**The equation of motion**

According to the preceding section, the Euler-Lagrange equation will be regarded as an first-order equation on the tangent bundle. If \( X : TQ \to N(TQ) \) is a second-order vector field, its integral curves are solutions of the Euler-Lagrange equation iff \( \hat{\delta}L \circ X = 0 \) —recall that \( \hat{\delta}L : N(TQ) \to V^*(TQ) \) is an affine bundle map.

If \( L \) is singular then \( \hat{\delta}L \) is not an isomorphism, therefore in general there is not a vector field \( X \) satisfying this equation everywhere. It is more correct to consider the Euler-Lagrange equation as an equation both for a second-order vector field \( X \) on \( TQ \) and a submanifold \( S \subset TQ \) such that

\[ \hat{\delta}L \circ X \simeq 0, \] (5.1)

supplemented with the condition that

\[ X \text{ is tangent to } S. \] (5.2)

(As a matter of notation, \( f \simeq g \) means that the maps \( f \) and \( g \) coincide on the subset \( S \).)

The *primary lagrangian constraint subset* \( V_1 \subset TQ \) is the set of points \( v \in TQ \) such that there exists a second-order tangent vector \( X_v \) such that \( \hat{\delta}L(X_v) = 0 \); it is assumed to be a closed submanifold. Then (5.1) has solutions on \( V_1 \); for simplicity, any of these solutions will be called a *primary dynamical field*. Notice that these solutions may not be unique: if \( X_0 \) is a fixed solution, then all the solutions are obtained by adding any section of \( \ker \hat{W} \) to \( X_0 \). On the other hand, a primary dynamical field is not necessarily
X. Gràcia, “Fibre derivatives: some applications to singular lagrangians” 12
tangent to $V_1$; we shall comment on this at the end of the section. See [GP92] for a careful
discussion on these problems.

**Construction of the primary lagrangian constraints**

The first step in order to solve the Euler-Lagrange equation for a vector field is to determine
the primary lagrangian constraint submanifold.

If $A_0$ is an affine space modelled on $E_0$, $F_0$ is a vector space, and $p_0 \in A_0$ is a fixed
point, the linear equation $f(p) = 0$ (for $p \in A_0$) is equivalent to $\vec{f}(u) = -f(p_0)$ (for
$u \in E_0$), therefore its consistency condition is $f(p_0) \in \text{Im} \vec{f}$. If $(s_\mu)$ is a frame for $\text{Ker} \vec{f}$,
the consistency condition is equivalent to the vanishing of the numbers $\langle s_\mu, f(p_0) \rangle$; of
course this does not depend on the point $p_0$ chosen.

We apply this remark to obtain the consistency condition of the Euler-Lagrange equa-
tion for a second-order vector field. If $X_0$ is any fixed second-order vector field in $TQ$, the
consistency condition is given by the vanishing of the functions

$$\chi_\mu = \langle \hat{\delta}L \circ X_0, \Gamma_\mu \rangle,$$

called the *primary lagrangian constraints*. Their vanishing defines the primary lagrangian
subset $V_1 \subset TQ$, assumed to be a submanifold. Notice that these functions are not
necessarily independent, and indeed may vanish identically.

**Remark** The primary lagrangian constraints can be also constructed without reference
to a concrete second-order vector field. Let

$$\check{\chi}_\mu = \langle \hat{\delta}L, \Gamma_\mu \circ \nu \rangle = \langle \delta L, \gamma_\mu \circ \tau_{21} \rangle \circ j_1^{-1},$$

where $\nu$ and $\tau_{21}$ are the projections of $N(TQ)$ and $T^2Q$ onto $TQ$, and $j_1$ is the diffeo-
morphism $T^2Q \to N(TQ)$; this defines functions $\check{\chi}_\mu$ on the affine bundle $N(TQ)$, and it is
readily seen that they are projectable to functions $\chi_\mu$ on $TQ$, since their fiber derivative
vanishes.

**Construction of primary fields**

Now we shall find a second-order vector field $X$ on $TQ$ such that $\hat{\delta}L \circ X \simeq 0$. If $X_0$ is
a fixed second-order vector field, giving $X$ amounts to giving a vertical vector field $Y$ on
$TQ$ such that $X = X_0 + Y$. Due to (4.10), the vertical field $Y$ can be written as

$$Y = \hat{\mathcal{M}} \circ (\hat{\mathcal{W}} \circ Y) + \sum_\mu (Y \cdot \lambda^\mu) \Gamma_\mu.$$

Moreover, by (4.3) $\hat{\delta}L \circ (X_0 + Y) = \hat{\delta}L \circ X_0 - \hat{\mathcal{W}} \circ Y$. Therefore we have:

$$X = X_0 + Y$$

$$= X_0 + Y + \hat{\mathcal{M}} \circ (\hat{\delta}L \circ X_0 - \hat{\mathcal{W}} \circ Y) - \hat{\mathcal{M}} \circ (\hat{\delta}L \circ X)$$

$$= X_0 + \hat{\mathcal{M}} \circ (\hat{\delta}L \circ X_0) + \sum_\mu (Y \cdot \lambda^\mu) \Gamma_\mu - \hat{\mathcal{M}} \circ (\hat{\delta}L \circ X).$$
If \( X \) is a primary field, then the last term is zero on \( V_1 \), and since the addition of vector fields of \( \text{Ker} \, W \) does not alter the set of primary fields, we conclude that \( X_0 + \hat{M} \circ (\hat{\delta}L \circ X_0) \) is also a primary field. Let us give a more precise statement of this result:

**Proposition 5** Under the same hypotheses of Proposition 4, let \( X_0 \) be any second-order vector field on \( TQ \). Then

\[
D_0 = X_0 + \hat{M} \circ (\hat{\delta}L \circ X_0)
\]

is a primary field. More precisely,

\[
\hat{\delta}L \circ D_0 = \sum_\mu \chi_\mu (D\lambda^\mu)^\nu \simeq 0,
\]

where \( \chi_\mu \) are the primary lagrangian constraints (5.3).

**Proof.** Just apply (4.3) and the transpose of (4.10) to \( D_0 \), bearing in mind that \( \hat{W} \) and \( \hat{M} \) are symmetric. Then

\[
\hat{\delta}L \circ D_0 = \hat{\delta}L \circ X_0 - \hat{W} \circ \hat{M} \circ \hat{\delta}L \circ X_0 = \hat{\delta}L \circ X_0 - \hat{\delta}L \circ X_0 + \sum_\mu (\hat{\delta}L \circ X_0, \Gamma_\mu) (D\lambda^\mu)^\nu = \sum_\mu \chi_\mu (D\lambda^\mu)^\nu.
\]

In coordinates, if \( X_0 = v \frac{\partial}{\partial q} + A(q, v) \frac{\partial}{\partial v} \), then

\[
D_0 = v \frac{\partial}{\partial q} + M \left( \frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial v \partial q} v \right) \frac{\partial}{\partial v} + \sum_\mu \left( \frac{\partial \lambda^\mu}{\partial v} A \right) \Gamma_\mu.
\]

The coordinate-dependent choice of \( X_0 = v^i \frac{\partial}{\partial q^i} (A = 0) \) yields a simpler expression for \( D_0 \), which is equation (4.19) in reference [BGPR 86]. The resulting vector field was also used in [Pon 88] to study some relations between lagrangian and hamiltonian formalisms.

Once a primary field \( D_0 \) has been obtained, all the primary fields are

\[
D_u \simeq D_0 + \sum_\mu u^\mu \Gamma_\mu,
\]

where \( u^\mu \) are arbitrary functions. The following step in the lagrangian stabilisation algorithm is to study the tangency of \( D_u \) to the submanifold \( V_1 \); this yields new constraints defining a submanifold \( V_2 \subset V_1 \) and determines some of the functions \( u^\mu \) and so on. The procedure follows the same lines as in [SNH 78, GP 92], and under some regularity assumptions finishes in a final constraint submanifold \( V_f \subset TQ \) and a family of second-order vector fields \( D^f + \sum_\mu u^\mu \Gamma_\mu \) tangent to \( V_f \) and solution of the Euler-Lagrange equation (5.1).

6  **An example: a free particle in a curved space-time**

In this simple example we will construct the geometric elements of the preceding sections, and show how they can be used to solve the equation of motion.
Let $Q$ be a $d$-dimensional manifold endowed with a metric tensor $g$ of signature $(1, d - 1)$. We write $v = \sqrt{g(v, v)}$, which is a function defined on the open subset
\[ V = \{ v \in TQ \mid g(v, v) > 0 \} \subset TQ. \]

The metric tensor defines an isomorphism $\hat{g}: TQ \to T^*Q$. We denote also by $g^*$ the 2-contravariant tensor deduced from $g$.

The lagrangian of a free particle of mass $m$ in $Q$ is $L = mv$; notice that it is defined only on $V$. Its fibre derivative (Legendre’s map) and fibre hessian are:
\[ V \xrightarrow{F} T^*Q, \quad F(v) = \frac{m}{v} \hat{g}(v), \]
\[ V \xrightarrow{W} \text{Hom}(TQ, T^*Q) \cong T^*Q \otimes T^*Q, \quad W(v) = \frac{m}{v} \left( g \circ \tau_Q - \frac{1}{v^2} \hat{g}(v) \otimes \hat{g}(v) \right). \]

The lagrangian is singular, since $\text{Ker} W(v)$ is spanned by $v$.

Using the vertical lift $TQ \times_Q TQ \xrightarrow{\tau} V(TQ)$ we can extend the product of $g$ to vertical vectors. So we obtain a section $g^V: TQ \to \text{Hom}(V(TQ), V^*(TQ)) \cong V^*(TQ) \otimes V^*(TQ)$, and the corresponding vector bundle isomorphism $\hat{g}^V: V(TQ) \to V^*(TQ)$. So we have, for instance, $g^V(\Delta, \Delta) = v^2$, since $\Delta$, the Liouville’s vector field of $TQ$, is the vertical lift of the identity.

In the same way, we know that the hessian yields a section $W: V \to \text{Hom}(V(TQ), V^*(TQ)) \cong V^*(TQ) \otimes V^*(TQ)$, and the corresponding operator $\hat{W}: V(TQ)|_V \to V^*(TQ)|_V$. Then, if $Y$ is a vertical vector field on $V$,
\[ \hat{W}(Y) = \frac{m}{v} \left( \hat{g}^V(\Delta, \Delta) \hat{g}(\Delta) \right). \]

Let $X$ be a second-order vector field. The Euler-Lagrange operator $\hat{\delta}L: \mathbb{N}(TQ)|_V \to V^*(TQ)|_V$ is
\[ \hat{\delta}L(X) = \hat{W}(S - X), \]
where $S$ is the geodesic vector field of $g$. This can be obtained by computing the local expression $\delta L(q, v, a) = \left( \frac{\partial L}{\partial q^i} - D_t \left( \frac{\partial L}{\partial v^i} \right) \right) dq^i$, and comparing with the local expression of the geodesic vector field, which is
\[ S = v^\mu \frac{\partial}{\partial q^\mu} - \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\mu}, \]
where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel’s symbols of the Levi-Civita connection of $g$.

Since $\text{Ker} \hat{W}$ is spanned by the Liouville’s vector field, from (6.4) the Euler-Lagrange equation $\hat{\delta}L(X) = 0$ is easily solved:
\[ X = S + \mu \Delta, \]
where $\mu: V \to \mathbb{R}$ is any function. The base integral curves of $X$ are the paths $\gamma$ in $Q$ satisfying $\nabla_t \dot{\gamma} = (\mu \circ \dot{\gamma}) \dot{\gamma}$, that is to say, they are reparametrised geodesics. The condition of being $\dot{\gamma}$ in $V$ is $g(\dot{\gamma}, \dot{\gamma}) > 0$, which means that $\gamma$ is a time-like curve.

Now let us see how the solution can be obtained from the hamiltonian formalism and the procedure of the preceding section.
The image of the Legendre’s map $\mathcal{F}$ is the submanifold $P_0 \subset T^*Q$ defined by the vanishing of the hamiltonian constraint
\begin{equation}
\phi(p) = \frac{1}{2}(g^*(p, p) - m^2).
\end{equation}
Since the lagrangian $L$ is homogeneous of degree 1, the lagrangian energy vanishes, and so does the hamiltonian $H$ on $P_0$.

It is readily seen that $D\phi(p) = \hat{g}^{-1}(p)$, therefore
\begin{equation}
\gamma_\phi(v) = \frac{m}{v}v.
\end{equation}
From the identity (4.3), $\text{Id}_{TQ} = \gamma_H + \gamma_\phi \lambda^\phi$, we obtain the function
\begin{equation}
\lambda^\phi(v) = \frac{v}{m},
\end{equation}
and thus $D\lambda^\phi(v) = \frac{1}{mv}\hat{g}(v)$. Finally, the map $M: V \to \text{Hom}(T^*Q, TQ)$ defined by (4.8) is
\begin{equation}
M(v) = \frac{v}{m}\hat{g}^{-1}.
\end{equation}
One can then check that (4.7) is satisfied.

Passing again to the vertical bundle, from $\gamma_\phi$ we obtain the vertical vector field
\begin{equation}
\Gamma_\phi = \frac{m}{v}\Delta,
\end{equation}
which spans the kernel of the fibre hessian and therefore the kernel of $T(\mathcal{F})$. We notice also that if $Y$ is a vertical vector field then $\langle (D\lambda^\phi)^V, Y \rangle = \frac{1}{mv}g^V(\Delta, Y)$. Finally, the operator $\hat{M}: V^*(TQ)|_V \to V(TQ)|_V$ is given by
\begin{equation}
\hat{M}(\Xi) = \frac{v}{m}(\hat{g}^V)^{-1}(\Xi).
\end{equation}

Now we are ready to compute the composition $\hat{M} \circ \hat{W}: V(TQ)|_V \to V(TQ)|_V$:
\begin{equation}
\hat{M} \circ \hat{W}(Y) = Y - \frac{g^V(\Delta, Y)\Delta}{v^2},
\end{equation}
that is to say, the orthogonal projection of $Y$ onto the subspace orthogonal to $\Delta$.

Let us also show that there are no lagrangian constraints. Indeed, from (5.3), if $X_0$ is any second-order vector field we obtain the primary lagrangian constraint as
\begin{equation}
\chi = \langle \delta L(X_0), \Gamma_\phi \rangle = \langle \hat{W}(S - X_0), \Gamma_\phi \rangle = \langle \hat{W}(\Gamma_\phi), S - X_0 \rangle = 0,
\end{equation}
where we have used that $\hat{W}$ is symmetric.

Now, if $X_0$ is any second-order vector field, application of (5.4) yields a lagrangian vector field
\begin{equation}
D_0 = X_0 + \hat{M} \circ (\delta L \circ X_0) = X_0 + \hat{M} \circ \hat{W}(S - X_0) = S - \frac{g^V(\Delta, S - X_0)\Delta}{v^2},
\end{equation}
which is one of the solutions (6.5) of the equation of motion.
Conclusions

In this paper we have studied the fibre derivative of a map between affine bundles. This permits a careful study of several structures constructed from the fibre derivative, for instance its tangent map and the fibre hessian.

In the particular case of a singular lagrangian we have applied these structures to obtain some geometric relations between lagrangian and hamiltonian formalisms; some of these relations were previously known in coordinates, but not as geometric objects. Since these developments work in any affine bundle, they may be useful also for higher-order lagrangians and field theory.

Acknowledgements

The author wishes to thank Profs. M.-C. Muñoz-Lecanda and J.M. Pons for some useful comments. He also acknowledges partial financial support from CICYT TAP 97–0969–C03–01.

References

[AMR 88] R. Abraham, J.E. Marsden and T. Ratiu, Manifolds, Tensor Analysis, and Applications (2nd ed.), (Springer, New York, 1988).

[BGPR 86] C. Batlle, J. Gomis, J.M. Pons and N. Román-Roy, “Equivalence between the lagrangian and hamiltonian formalism for constrained systems”, J. Math. Phys. 27 (1986) 2953–2962.

[Car 90] J.F. Cariñena, “Theory of singular lagrangians”, Fortschrit. Phys. 38 (1990) 641–679.

[CLM 91] J.F. Cariñena, C. López and E. Martínez, “Sections along a map applied to higher-order lagrangian mechanics. Noether’s theorem”, Acta Appl. Math. 25 (1991) 127–151.

[CLR 87] J.F. Cariñena, C. López and N. Román-Roy, “Geometric study of the connection between the lagrangian and the hamiltonian constraints”, J. Geom. Phys. 4 (1987) 315–334.

[GN 79] M.J. Gotay and J.M. Nester, “Presymplectic lagrangian systems I: the constraint algorithm and the equivalence theorem”, Ann. Inst. H. Poincaré A 30 (1979) 129–142.

[GNH 78] M.J. Gotay, J.M. Nester and G. Hinds, “Presymplectic manifolds and the Dirac-Bergmann theory of constraints”, J. Math. Phys. 19 (1978) 2388–2399.

[GP 92] X. Gràcia and J.M. Pons, “A generalized geometric framework for constrained systems”, Diff. Geom. Appl. 2 (1992) 223-247.
[GPR91] X. Gràcia, J.M. Pons and N. Román-Roy, “Higher-order lagrangian systems: geometric structures, dynamics, and constraints”, J. Math. Phys. 32 (1991) 2744–2763.

[Grà91] X. Gràcia, Sistemes lligats: estudi geomètric i transformacions de simetria, Ph. D. thesis, Universitat de Barcelona, 1991.

[GS73] H. Goldschmidt and S. Sternberg, “The Hamilton-Cartan formalism in the calculus of variations”, Ann. Inst. Fourier 23 (1973) 203–267.

[Kam82] K. Kamimura, “Singular lagrangian and constrained hamiltonian systems, generalized canonical formalism”, Nuovo Cim. B 68 (1982) 33–54.

[KMS93] I. Kolář, P.W. Michor and J. Slovák, Natural Operations in Differential Geometry, (Springer, Berlin, 1993).

[LR85] M. de León and P.R. Rodrigues, Generalized Classical Mechanics and Field Theory, North-Holland Math. Studies 112, (Elsevier, Amsterdam, 1985).

[Pon88] J.M. Pons, “New relations between hamiltonian and lagrangian constraints”, J. Phys. A: Math. Gen. 21 (1988) 2705–2715.

[Sau89] D.J. Saunders, The Geometry of Jet Bundles, London Math. Soc. Lecture Note Series 142, (Cambridge University Press, Cambridge, 1989).