GLOBAL WELL-POSEDNESS FOR THE TWO DIMENSIONAL NAVIER-STOKES-VLASOV EQUATIONS

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ABSTRACT. The global well-posedness for the incompressible Navier-Stokes-Vlasov equations in two spatial dimensions is established by a priori estimates, the characteristic method and the semigroup analysis.

1. INTRODUCTION

The objective of this paper is to establish the global well-posedness for the two dimensional Navier-Stokes-Vlasov equations:

\[ \begin{align*}
\partial_t u + u \cdot \nabla u + \nabla P - \mu \Delta u &= -\int_{\mathbb{R}^2} (u - v) f \, dv, \\
\text{div} u &= 0, \\
\partial_t f + v \cdot \nabla_x f + \text{div}_v ((u - v)f) &= 0
\end{align*} \]  (1.1)

in \((0, T) \times \mathbb{R}^2 \times \mathbb{R}^2\), with the following initial data

\[ u(x, 0) = u_0(x), \quad f(x, v, 0) = f_0(x, v), \]  (1.2)

where \(u\) is the velocity of the fluid, \(P\) is the pressure, \(\mu\) is the kinematic viscosity of the fluid. Without loss of generality, we take \(\mu = 1\) throughout the paper. The distribution function \(f(t, x, v)\) depends on the time \(t \in [0, T]\), the physical position \(x \in \mathbb{R}^2\) and the velocity of particle \(v \in \mathbb{R}^2\). The number of particles enclosed at \(t \geq 0\) and location \(x \in \mathbb{R}^2\) in the volume element \(dv\) is given by \(f(t, x, v) \, dv\). We refer the readers to \([3, 6, 7, 8, 10]\) for more physical background and discussion of the Navier-Stokes-Vlasov equations and related problems.

There have been many mathematical studies on the Navier-Stokes-Vlasov equations and related problems. The global existence for the Stokes-Vlasov system in a bounded domain was established in \([6]\). The existence theorem for weak solutions has been extended in \([2]\), where the author did not neglect the convection term and considered the Navier-Stokes-Vlasov equations within a periodic domain. The weak solution of the Navier-Stokes-Vlasov-Poisson system with corresponding boundary value problem was obtained in \([1]\). The global existence of smooth solutions with small data for the Navier-Stokes-Vlasov-Fokker-Planck equations was obtained in \([5]\). More recently, the existence of global weak solutions with large data to the Navier-Stokes-Vlasov equations in a bounded domain was established in \([11]\).
However, there is no existence theory available for the Navier-Stokes-Vlasov equations with initial data in the whole space. Compared with \[2, 11\], the new difficulty is the loss of compactness of \( \int_{\mathbb{R}^2} f \, dv \) and \( \int_{\mathbb{R}^2} vf \, dv \) in the whole space. Since the methods in \[2, 11\] do not work here, mathematical analysis for this problem is challenging and requires new ideas and techniques. In this paper we shall study the initial value problem \((1.1)-(1.2)\) and establish the global well-posedness with large initial data. To achieve our goal, we will derive a priori estimates and use fixed point arguments. Partially motivated by the work of \[4\], we will use the semigroup analysis to establish the iteration of \((u, f)\) for using the fixed point theorem. To overcome the difficulty of the estimates of distribution function \(f\), we adopt the idea as in \[2\], and apply the characteristic method to the Vlasov equation, then the existence and uniqueness of the solutions for the Vlasov equation follows when \(u\) is continuous with respect to time \(t\). The continuous dependence of the solution \(f\) on \(u\) is also established. We also use Lemma 1.1 to deal with the distribution function and its coupling and interaction with fluid variables. With the above a priori estimates, continuous dependence, and semigroup analysis, we can apply the fixed point theorem to obtain the existence and uniqueness of strong solutions to problem \((1.1)-(1.2)\). Further regularity of \((u, f)\) can be deduced from the strong solution, thus the global well-posedness can be established.

In what follows, we denote
\[
m_k f = \int_{\mathbb{R}^2} |v|^k f \, dv, \quad \text{and} \quad M_k f = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v|^k f \, dv \, dx,
\]
\[
\rho = \int_{\mathbb{R}^2} f \, dv, \quad j = \int_{\mathbb{R}^2} vf \, dv.
\]
It is easy to see that
\[
M_k f = \int_{\mathbb{R}^2} m_k f \, dx.
\]
Here we state the following lemma due to \[6\]:

**Lemma 1.1.** Suppose that \((u, f)\) is a smooth solution to \((1.1)-(1.2)\). If \(f_0 \in L^p\) for some \(p > 1\), we have
\[
\|f(t, x; v)\|_{L^p} \leq C(T)\|f_0\|_{L^p}, \quad \text{for any } t \geq 0;
\]
and if \(|v|^k f_0 \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)\), then we have
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^k f \, dv \, dx \leq C(T) \left( \left( \int_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^k f_0 \, dv \, dx \right)^{\frac{1}{k}} + (\|f_0\|_{L^\infty} + 1)\|u\|_{L^r(0, T; L^{2+k})} \right)^{2+k}.
\]

Our first main result reads as follows.

**Theorem 1.1.** If \(u_0 \in W^{1,2}(\mathbb{R}^2)\) is a divergence-free vector, \(f_0 \in C^1(\mathbb{R}^2 \times \mathbb{R}^2)\), \(M_6 f_0 < \infty\), then there exists a unique strong solution \((u, f)\) to \((1.1)-(1.2)\) for any \(T > 0\).

The strong solution to system \((1.1)-(1.2)\) is defined as follows:

**Definition 1.1.** A pair \((u, f)\) is called a strong solution to the system \((1.1)-(1.2)\) if
\[
\bullet \quad u \in C(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2));
\]
\[
\bullet \quad f(t, x, v) \geq 0, \quad \text{for any } (t, x, v) \in (0, T) \times \mathbb{R}^2 \times \mathbb{R}^2;
\]
\[
\bullet \quad f \in C^1(0, T; W^{1,2}(\mathbb{R}^2 \times \mathbb{R}^2));
\]
• \( f|v|^2 \in L^\infty(0,T;L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \).

Based on Theorem 1.1, we can differentiate the system and apply similar arguments to obtain the further result:

**Theorem 1.2.** If \( u_0 \in W^{m+1,2}(\mathbb{R}^2) \) \( m > 0 \) integer, is a divergence-free vector, \( f_0 \in C^1(\mathbb{R}^2 \times \mathbb{R}^2) \), \( M_6 f_0 < \infty \), then the solution satisfies

\[
\begin{align*}
&u \in \mathcal{C}(0,T;W^{m+1,2}(\mathbb{R}^2)) \cap L^2(0,T;W^{m+2,2}(\mathbb{R}^2 \times \mathbb{R}^2)); \\
f(t,x,v) \geq 0, \text{ for any } (t,x,v) \in (0,T) \times \mathbb{R}^2 \times \mathbb{R}^2; \\
f \in \mathcal{C}^1(0,T;W^{m+1,2}(\mathbb{R}^2 \times \mathbb{R}^2)).
\end{align*}
\]

2. **a Priori Estimates**

The aim of this section is to obtain some a priori estimates. We start with deriving the energy inequality. Multiplying by \( u \) the both sides of the first equation in (1.1), integrating over \( \mathbb{R}^2 \) and by parts, we have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \frac{1}{2} |u|^2 \, dx + \int_{\mathbb{R}^2} |\nabla u|^2 \, dx = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(u - v)u \, dv \, dx. \tag{2.1}
\]

Multiplying by \((1 + \frac{1}{2}|v|^2)\) the both sides of the third equation in (1.1) and integrating over \( \mathbb{R}^2 \) and by parts, one obtains that

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(1 + |v|^2) \, dv \, dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f|u - v|^2 \, dv \, dx \tag{2.2}
\]

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(u - v)u \, dv \, dx.
\]

Using (2.1)-(2.2), one obtains

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} |u|^2 \, dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(1 + |v|^2) \, dv \, dx \right) + 2 \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f|u - v|^2 \, dv \, dx = 0. \tag{2.3}
\]

Taking the curl of the first equation in (1.1), we obtain

\[
\partial_t \omega + \nabla u - \Delta \omega = \nabla^T \cdot (-u \rho + j), \tag{2.4}
\]

where \( \omega = \text{curl} u \). Multiplying by \( \omega \) the both sides of (2.4) and integrating, we obtain, after integration by parts,

\[
\frac{d}{dt} \|\omega\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 \leq C \left( \|u\|_{L^4}^4 + \|\rho\|_{L^4}^4 + \|j\|_{L^4}^2 \right). \tag{2.5}
\]

On the other hand, by (2.3), we have

\[
u \in L^2(0,T;W^{1,2}(\mathbb{R}^2)),
\]

which implies that

\[
u \in L^2(0,T;L^p(\mathbb{R}^2)), \text{ for any } p \geq 1. \tag{2.6}
\]
Applying Lemma 2.1 the fact $M_0 f_0 < \infty$, and (2.6), we obtain

$$M_0 f < \infty.$$  

Applying Lemma 1 as in [2] in the two-dimensional space, we can control $\|\rho\|_{L^4}$ and $\|j\|_{L^2}$ by $M_0 f$. Thus, we obtain that

$$\sup_{0 \leq t \leq T} \|\omega\|_{L^2}^2 + \int_0^T \|\nabla\omega\|_{L^2}^2 dt \leq C(T),$$  

(2.7)

which implies that

$$u \in L^\infty(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2)).$$

Multiplying by $u_t$ the both sides of the first equation in (1.1), using integration by parts, we obtain

$$\begin{align*}
\frac{\partial}{\partial t} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |u_t|^2 dx \\
\leq C \left( \|u\|_{L^4}^4 + \|\rho\|_{L^4}^4 + \|j\|_{L^2}^2 \right) + \int_{\mathbb{R}^2} |\nabla u|^2 dx.
\end{align*}$$

(2.8)

Applying Gronwall’s inequality, one obtains that

$$u_t \in L^2(0, T; L^2(\mathbb{R}^2)), \quad \text{and} \quad u \in L^\infty(0, T; W^{1,2}(\mathbb{R}^2)).$$

Now, we rely on the following Lemma which is a very special case of interpolation theorem of Lions-Magenes. We refer the readers to [9] for the proof of this lemma.

**Lemma 2.1.** Let $V \subset H \subset V'$ be three Hilbert spaces, $V'$ is a dual space of $V$. If a function $u$ belong to $L^2(0, T; V)$ and its derivative $u'$ belong to $L^2(0, T; V')$ then $u$ is almost everywhere equal to a function continuous from $[0, T]$ into $H$.

Applying Lemma 2.1 with the following facts

$$\frac{\partial u}{\partial t} \in L^2(\Omega \times (0, T)), \quad \text{and} \quad u \in L^\infty(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2)),$$

we conclude that $u \in C(0, T; W^{1,2}(\mathbb{R}^2))$.

On the other hand, we can apply maximal principle to the Vlasov equation to obtain

$$\|f\|_{L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))} \leq \|f_0\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)}.$$

Thus we proved

**Proposition 2.1.** Let $(u, f)$ be a solution of (1.1)-(1.2) on $[0, T]$, with $u_0 \in W^{2,2}(\mathbb{R}^2)$ and $f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, $M_0 f_0 \leq C < \infty$, then we have the following regularity:

$$u \in C(0, T; W^{1,2}(\mathbb{R}^2)) \cap L^2(0, T; W^{2,2}(\mathbb{R}^2));$$

$$f \in L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)).$$
3. Local and Global Strong Solution

**Proposition 3.1.** Let \( u_0 \in W^{2,2}(\mathbb{R}^2) \) be a divergence-free vector, \( f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2) \), \( M_6 f_0 \leq C < \infty \), then there exists a time \( T_0 > 0 \) depending on the initial data and a unique strong solution

\[
\begin{align*}
    u & \in C(0, T_0, PW^{1,2}(\mathbb{R}^2)) \cap L^2(0, T_0, PW^{2,2}(\mathbb{R}^2)) ; \\
    f & \in L^\infty(0, T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))
\end{align*}
\]

of (\ref{eq:1.1}) with the initial data \( (u_0, f_0) \), where \( P \) is the Leray-Hodge projector on divergence-free vector. In addition, if \( f_0 \in C^1(\mathbb{R}^2 \times \mathbb{R}^2) \), then we have \( f \in C^1(0, T_0; W^{1,2}(\mathbb{R}^2 \times \mathbb{R}^2)) \).

**Proof.** We define \( \|(u, f)\|_B = \|u\|_X + \|f\|_Y \), where

\[
X = L^\infty(0, T_0, PW^{1,2}(\mathbb{R}^2)) \cap L^2(0, T_0, PW^{2,2}(\mathbb{R}^2)),
\]

\[
\|u\|_X = \|u\|_{L^\infty(0,T_0, PW^{1,2}(\mathbb{R}^2))} + \|u\|_{L^2(0,T_0, PW^{2,2}(\mathbb{R}^2))},
\]

and

\[
Y = L^\infty(0, T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)),
\]

\[
\|f\|_Y = \|f\|_{L^\infty(0,T_0, L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))}.
\]

Clearly the spaces \( X, Y \) are Banach spaces, and thus \( B \) is Banach space.

We let \( U = (u, f) \) in the Banach space \( B \), define the operator \( T(U) \) in \( B \), as \( T(U) = (\bar{u}, \bar{f}) \), where \( \bar{u}, \bar{f} \) are given by

\[
\bar{u} = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}P(u \cdot \nabla u)ds + \int_0^t e^{(t-s)\Delta}P(-\rho u + j)ds,
\]

\[
\bar{f} = N(u, v) \quad \text{where} \quad \partial_t \bar{f} + v \cdot \nabla \bar{f} + \text{div}_v((u - v)\bar{f}) = 0, \quad \bar{f}(x, v, 0) = f_0(x, v).
\]

We denote

\[
Q(u, w) = \int_0^t e^{(t-s)\Delta}P(u \cdot \nabla w),
\]

which solves

\[
\partial_t Q - \Delta Q = P(u \cdot \nabla w), \quad Q(x, 0) = 0.
\]

It is easy to obtain the following energy inequality,

\[
\frac{d}{dt} \|\Delta Q\|_{L^2}^2 + \|\nabla \Delta Q\|_{L^2}^2 \leq \|\nabla (u \cdot \nabla w)\|_{L^2} \|\nabla \Delta Q\|_{L^2}.
\]

Using Ladyzhenskaya inequality for the term involving \( \nabla u \cdot \nabla w \) and the interpolation inequality for the term involving \( u \cdot \nabla (\nabla w) \), one obtains that

\[
\sup_{0 \leq t \leq T} \|\Delta Q\|_{L^2}^2 + \int_0^T \|\nabla \Delta Q\|_{L^2}^2 dt \leq C T_0 \|u\|_X^2 \|w\|_X^2.
\]

We denote

\[
L := \int_0^t e^{(t-s)\Delta}P(-\rho u + j)ds,
\]

which solves

\[
\partial_t L - \Delta L = P(-\rho u + j), \quad L(x, 0) = 0.
\]
Multiplying $\Delta L$ the both sides of above equation, and using integration by parts, we get

\[
\sup_{0 \leq t \leq T} \|\nabla L\|_{L^2}^2 + 2 \int_0^{T_0} \|\Delta L\|_{L^2}^2 dt \leq \int_0^{T_0} \|\Delta L\|_{L^2}^2 ds
\]

\[
+ \int_0^{T_0} \|\rho u\|_{L^2}^2 ds + \int_0^{T_0} \|j\|_{L^2}^2 ds.
\]

(3.2)

By the Cauchy-Schwartz inequality, we have

\[
\sup_{0 \leq t \leq T} \|\nabla L\|_{L^2}^2 + \int_0^{T_0} \|\Delta L\|_{L^2}^2 dt \leq \int_0^{T_0} (\|\rho\|_{L^4}^2 + \|u\|_{L^4}^2 + \|j\|_{L^2}^2) ds.
\]

Using Lemma 1 as in [2] and Lemma 1.1, we can control $\|\rho\|_{L^4}$ as follows:

\[
\|\rho\|_{L^4} \leq M_6 f \leq C(M_6 f_0 + \|u\|_X)^8.
\]

Similarly, we can control the term $\|j\|_{L^2}$. Thus, we have the following estimate:

\[
\sup_{0 \leq t \leq T} \|\nabla L\|_{L^2}^2 + \int_0^{T_0} \|\Delta L\|_{L^2}^2 dt \leq C(1 + \|u\|_X)^8.
\]

Integrating the third equation in (1.1) with respect to $x$ and $v$, we have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} f dx dv = 0.
\]

(3.3)

Applying the maximum principle to the third equation in (1.1), one obtains that

\[
\|f\|_{L^\infty((\mathbb{R}^2 \times \mathbb{R}^2)} \leq C_T \|f_0\|_{L^\infty((\mathbb{R}^2 \times \mathbb{R}^2)}
\]

(3.4)

for all $t \in [0, T]$, if $f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. From (3.3) and (3.4), $f = N(u, v)$ satisfies the following estimate

\[
\|f\|_{L^\infty(0, T; L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)} \leq C_{T_0} \|f_0\|_{L^\infty((\mathbb{R}^2 \times \mathbb{R}^2)}
\]

which means

\[
\|N(u, v)\|_{L^\infty(0, T_0; L^\infty((\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))} \leq C_{T_0} \|f_0\|_{L^\infty((\mathbb{R}^2 \times \mathbb{R}^2)}.
\]

(3.5)

(3.6)

Following the same argument of [2], we let $g = e^{2t} f$ satisfy the following transport equation

\[
\partial_t g + v \cdot \nabla g + (u - v) \cdot \nabla_v g = 0.
\]

The above equation can be written by characteristics method as follows,

\[
\frac{dx}{dt} = v(t),
\]

\[
\frac{dv}{dt} = u(t, x(t)) - v(t),
\]

with the initial data

\[
x(0) = x \quad \text{and} \quad v(0) = v,
\]

and set $\chi(t, x, v) = (x(t), v(t))$ for any $(t, x, v)$.

Applying the fact $u \in C(0, T; W^{1,2}(\mathbb{R}^2))$ and the classical theory of Ordinary Differential Equations, we obtain the unique solution

\[
f(t, x, v) = e^{2t} f_0(\chi(t, x, v)), \quad \text{for any} \ (t, x, v).
\]

(3.7)
Thus, we conclude that
\[ f \in C^1(0, T; C^0(\mathbb{R}^2 \times \mathbb{R}^2)) \quad \text{if} \quad f_0 \in C^1(\mathbb{R}^2 \times \mathbb{R}^2). \]

Define
\[ f_1 = N(u_1, v), \quad f_2 = N(u_2, v), \]
and
\[ \chi_1 = (x(u_1), v), \quad \chi_2 = (x(u_2), v). \]

Notice that \( f_1 - f_2 \) can be controlled as follows
\[ \|f_1 - f_2\|_Y \leq C(T)\|\chi_1 - \chi_2\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}. \]

By the definition of \( \chi = (x,v) \), we have the following estimate
\[
\begin{aligned}
\|(\chi_1 - \chi_2)(t)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} &\leq C \left( \int_0^t \|(u_1 - u_2)(s)\|_{L^\infty(\mathbb{R}^2)} ds + \int_0^t (1 + \|u(s)\|_X) \|(\chi_1 - \chi_2)(s)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} ds \right) \\
&\leq C \left( \int_0^t \|(u_1 - u_2)(s)\|_{L^\infty(\mathbb{R}^2)} ds + \int_0^t \|(\chi_1 - \chi_2)(s)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} ds \right).
\end{aligned}
\]

Thus, for any \( t \), applying Gronwall’s inequality to (3.9), we have
\[ \|\chi_1 - \chi_2\|_Y \leq C\varepsilon \|u_1 - u_2\|_X. \]

This, with the help of (3.5), implies that
\[ \|f_1 - f_2\|_Y \leq C\varepsilon \|u_1 - u_2\|_X. \]

Thus
\[ \|N(u_1, v) - N(u_2, v)\|_Y \leq C\varepsilon \|u_1 - u_2\|_X. \]

Set \( U^0 = (u_0, f_0) \) and define the iteration \( U^{n+1} = T(U^n) \) for \( n = 0, 1, 2, \ldots \). It is easy to see that the sequence \( U^n \) is bounded in \( B \) and converges if we choose \( \varepsilon \) small enough. If there exist \( A, D \) and \( \varepsilon \) such that
\[ \|u^n\|_X \leq A, \quad \|f^n\|_Y \leq D, \]
then, by induction and (3.11), we have
\[ \|u^{n+1}\|_X \leq A_0 + \varepsilon A^2 + \varepsilon C(1 + A)^8, \quad \|f^{n+1}\|_Y \leq CT_0 D_0. \]

We can choose \( \varepsilon \) small enough, such that \( \varepsilon(1 + A)^8 + \varepsilon A^2 + A_0 \leq A \), and choose \( D \) such that \( CT_0 D_0 \leq D \). Thus, we conclude that the sequence is bounded in \( B \), then we can obtain the convergence of \( u^n \) in \( X, f^n \) in \( Y \).

By Proposition 3.1, there exists a strong solution on a short time interval \([0, T_0]\). For any given \( T > 0 \), we consider the maximal interval of the existence, \( T_1 = \sup T_0 \leq T \), such that the solution is strong on \([0, T_0]\). The main goal is to prove that \( T_1 \) can be taken to be equal to \(+\infty\). For any given \( T_0 > 0 \), there exists a constant \( K > 0 \) such that
\[ \|f\|_Y \leq \frac{K}{CT_0}; \]
which implies \( \|f(T_0, x, v)\|_Y \leq K \). Using a priori estimates in Section 2, and applying Proposition 3.1, the strong solution can be extended to \([0, T_0 + T^*]\) for a small number \(T^* > 0\). One can then repeat the argument many times and obtain the existence and uniqueness on the whole real line. Thus we proved Theorem 1.1.

To prove Theorem 1.2 the further regularity \((u, f)\) can be deduced from the regularity from Theorem 1.1. We can differentiate the equation (1.1) and apply similar arguments, Theorem 1.2 follows.

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