Unfamiliar Aspects of Bäcklund Transformations and an Associated Degasperis-Procesi Equation

Alexander G. Rasin
Department of Mathematics,
Ariel University, Ariel 40700, Israel
E-mail: rasin@ariel.ac.il

Jeremy Schiff
Department of Mathematics,
Bar-Ilan University, Ramat Gan, 52900, Israel
E-mail: schiff@math.biu.ac.il

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Abstract

We summarize the results of our recent work on Bäcklund transformations (BTs), particularly focusing on the relationship of BTs and infinitesimal symmetries. We present a BT for an associated Degasperis-Procesi (aDP) equation and its superposition principle, and investigate the solutions generated by application of this BT. Following our general methodology, we use the superposition principle of the BT to generate the infinitesimal symmetries of the aDP equation.

1 Introduction: Bäcklund Transformations and Symmetries

Bäcklund transformations (BTs) are a characteristic feature of integrable differential equations, which are typically regarded as curious, but not of fundamental importance. A
prototypical example of a BT is that of the potential KdV (pKdV) equation: If $u$ satisfies the pKdV equation

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u_x^2$$

then so does $u_\alpha = u + v_\alpha$, where $v_\alpha$ is a solution of the system

$$v_{\alpha,x} = \alpha - 2u_x - v_\alpha^2,$$

$$v_{\alpha,t} = u_{xx} v_\alpha - \frac{1}{2} u_{xxx} + (\alpha + u_x)(\alpha - 2u_x - v_\alpha^2),$$

and $\alpha$ is a parameter. Note that the equations of this system are ordinary differential equations for $v_\alpha$. (It is not immediately obvious that the two equations are consistent — but the consistency condition turns out to be that $u_x$ satisfies the KdV equation, which is the case if $u$ satisfies pKdV.) The general solution of the first equation will depend on an arbitrary function of $t$, and the second equation in the system will determine this up to an arbitrary constant. Thus $v_\alpha$ in fact depends on a second “hidden” parameter, as well as the parameter $\alpha$. In this sense the BT, for a fixed value of the parameter $\alpha$, generates a 1-parameter family of solutions $u_\alpha$ of pKdV for a given starting solution $u$.

A fundamental result on BTs is the Bianchi permutability theorem, that states that BTs commute. The solutions $u_{\alpha,\beta}$ generated from the starting solution $u$ and first applying the BT with parameter $\alpha$ and then the BT with parameter $\beta$ coincide with the solutions $u_{\beta,\alpha}$ obtained by applying the BTs in the reverse order. Furthermore we have the algebraic superposition principle

$$u_{\alpha,\beta} = u_{\beta,\alpha} = u + \frac{\beta - \alpha}{u_\beta - u_\alpha}.$$  

In [23] we realized the following simple fact: In the fraction in the last expression, if we let $\beta$ tend to $\alpha$, the numerator of the fraction will tend to zero, but the denominator need not tend to zero, because of the dependence on different values of the hidden parameters in $u_\alpha$ and $u_\beta$. Thus

$$Q(\alpha) = \frac{1}{u_\alpha^{(1)} - u_\alpha^{(2)}}$$

is an infinitesimal symmetry of pKdV. Here $u_\alpha^{(1)}$ and $u_\alpha^{(2)}$ denote 2 distinct solutions obtained from $u$ by application of the BT with parameter $\alpha$. Recall [21] that $\eta$ is an infinitesimal symmetry of pKdV if both $u$ and $u + \epsilon \eta$ solve pKdV to first order in $\epsilon$, i.e. if

$$\eta_t = \frac{1}{4} \eta_{xxx} + 3u_x \eta_x.$$  

\[\tag{2}\]

It is straightforward to verify directly that $\eta = Q(\alpha)$ given in \[1\] satisfies \[2\]. A general solution $\eta$ of \[2\] is called a local symmetry if it is a function of the coordinates $x, t$, the function $u$ and a finite number of derivatives of $u$. The symmetry of $Q(\alpha)$ is nonlocal; however
we showed in [23] that by expanding \( Q(\alpha) \) in an asymptotic series for large \( \alpha \) we recover the standard infinite hierarchy of local symmetries of pKdV. Furthermore we derived the standard recursion relation between the local symmetries [20] and proved their commutativity. We also showed that the symmetry \( Q(\alpha) \) includes known nonlocal symmetries of pKdV.

The fact that a BT can be used to derive local symmetries of an integrable equation was shown by Kumei in [14] for the Sine-Gordon equation, but this was thought to be an isolated case. In fact deriving local symmetries from BTs seems to be possible, using superposition principles, for many, if not all, integrable equations. In [24] we investigated the BT of the Camassa-Holm (CH) equation, following on the previous work [26, 23] for the associated CH equation (an equation related to CH by a change of coordinates). Once again, the superposition principle for two BTs was found to encode the infinitesimal symmetries. In [25] we investigated the BT of the Boussinesq equation. In this case the superposition principle that encodes the infinitesimal symmetries was found to be a superposition for 3 BTs. It seems natural to conjecture that this is related to the fact that the Lax pair for the Boussinesq equation is third order.

A variety of other unfamiliar aspects of BTs emerged in these works. In the case of the CH equation [24] the BT acts not only on the dependent variable, but also on (one of the) independent variables. In the case of the Boussinesq equation [25] a single application of the BT produces not only travelling wave solutions, but also what we called “merging soliton” solutions, in which two solitary waves merge into a single solitary wave. Using the superposition principle we found superpositions of merging solitons with solitons (thus, for example, giving a solution in which three solitary waves merge to two), but superpositions of merging solitons with themselves seemed to always produce finite time singularities (which in certain cases disappeared again at a later, but finite, time). The fact that BTs encode infinitesimal symmetries renews interest in BTs as a solution-generating technique. However, we note that we have still not succeeded to use BT techniques to generate multipeakon solutions of the CH equation.

The aim of this paper is to show another example of many of these interesting features of BTs. In the next section we introduce and explain the significance of an equation we called the associated Degasperis-Procesi (aDP) equation. In Section 3 we give the BT and the superposition principle. Sections 4 and 5 are devoted, respectively, to solutions generated by the application of 1 and 2 BTs. A single application of the BT gives rise to two kinds of solitons, as well as a number of types of singular solutions, and also “mergers” in which a singular soliton absorbs (either kind of) nonsingular soliton (mergers of nonsingular

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1We thank Peter Olver for bringing this reference to our attention.
solitons do not seem to be allowed). By application of a second BT these solutions can be superposed, in this case (it seems) without any new singularity emerging. In section 6 we use the superposition principle of 3 BTs to derive the infinitesimal symmetries of the aDP equation; the first nontrivial symmetry is the Kaup-Kupershmidt (KK) equation (this being in accord with the results of Degasperis, Holm and Hone [11], who related aDP to the KK hierarchy). In section 7 we conclude. In an appendix, we briefly give a direct derivation of the travelling wave solutions of aDP.

2 The associated DP equation

The Degasperis-Procesi equation [22] is the equation

\[ u_T - u_{XXT} + 4uu_X - uu_{XXX} - 3u_Xu_{XX} = 0 \]  

This equation is the case \( b = 3 \) of the so-called “\( b \)-family” of equations

\[ m_T + um_X + bmu_X = 0, \quad m = u - u_{XX} \]  

In [2] this family of equations (except for the case \( b = -1 \)) was shown to arise from a shallow water equation via a Kodama transformation. For an extensive study of the \( b \)-family see [9, 10]. For \( b > 1 \) there are stable peakon solutions \( u(X,T) = ce^{-|X-cT|} \), which can be superposed to give more general solutions of the form

\[ u(X,T) = \sum_{i=1}^{N} p_i(T)e^{-|X-q_i(T)|} \]

provided the positions \( q_i(T) \) and speeds \( p_i(T) \) satisfy the equations of motion of a certain finite dimensional Hamiltonian system. In [10] it is suggested that \( b = 3 \) is a possible bifurcation point of the \( b \)-family. Furthermore, it is known that there are only two integrable cases of the \( b \)-family, the case \( b = 2 \), for which it reduces to the CH equation, and the case \( b = 3 \) of DP.

Integrability in the case \( b = 3 \) was established in [11], where a Lax pair was given, and a bihamiltonian structure proposed, for an equation related to DP by a change of coordinates. The bihamiltonian structure was validated in [11], and further evidence presented singling out the cases \( b = 2 \) and \( b = 3 \). A version of the DP equation after the change of coordinates is what we call here the associated DP (aDP) equation, so we present its derivation in full detail. Equation [11] can be written in the form

\[ \left( m^{1/b} \right)_T + \left( m^{1/b} u \right)_X = 0 \].
Writing \( p = m^{1/b} \), this implies we can define new coordinates \( x, t \) via

\[
\frac{dx}{dt} = pdX - pudT, \quad \frac{dt}{dt} = dT.
\]

Moving to the new coordinates we find that \( u \) can be eliminated via

\[
u = p^b - px + \frac{p_x p_t}{p}.
\]

and there remains a single equation for \( p \), that can be written in either of the forms

\[
\left( \frac{1}{p_x} \right)_x + \left( p_{xt} - \frac{p_x p_t}{p} - p^b \right)_x = 0
\]

or

\[
\left( \frac{p_{xx}}{p} - \frac{p_x^2}{2p^2} + \frac{1}{2p^2} \right)_t + \left( \frac{bp^{b-1}}{b-1} \right)_x = 0.
\]

The second form suggests introducing a new unknown \( f(x,t) \) defined by \( p = f_1^{1/(b-1)} \). The equation for \( f \) is then

\[
\frac{f_{xxt}}{f_t} - \frac{(2b - 3)f_{xt}^2}{2(b - 1)f_t} + c_1 f_x f_t + c_2 f_t^{\frac{b+3}{2}} = 0
\]

where \( c_1, c_2 \) are constants, which can be chosen freely by rescalings of \( f, x \) and \( t \). Making a convenient choice we arrive, in the case \( b = 3 \), to what we call for the purposes of this paper the associated DP \( (aDP) \) equation:

\[
f_{xxt} - \frac{3f_{xt}^2}{4f_t} + 3(1 - f_x f_t) = 0.
\]

The reason we do not study DP directly, but rather work with \( aDP \), is that it turns out that formulas for the BT are simpler when expressed in terms of \( f \) than in terms of the other fields, \( u, m, p \). Specifically, in \([1]\) it was established that after the change of coordinates we have described, the DP equation can be considered as a negative flow in the Kaup-Kupershmidt (KK) hierarchy. The KK variable in the notation of \([1]\) is the field \( V \), which in our notation can be identified, modulo rescaling, with \( f_x \). Thus it will come as no surprise that in section 6 we find the first nontrivial symmetry of \((5)\) is the potential KK equation. The connection between the DP and KK equations was further explored in the recent paper \([13]\).

One motivation for the study of the BT of \( aDP \) \((5)\) is to advance the state of knowledge of solutions of DP \((3)\). A large number of solutions of DP are already known. Multi-peakon solutions are discussed in \([1]\) and \([16]\). Travelling wave solutions are discussed in \([29]\) and \([15]\), the former including the description of “loop-like” solitary waves and periodic solutions. In
the formidable papers [17, 18] Matsuno uses the relation of the DP with the KK hierarchy, the fact that the KK hierarchy is a reduction of the CKP hierarchy, and known results about CKP tau functions to derive multisoliton solutions of DP. A similar approach appears in works of Feng et al. [3, 4], and the paper [28] uses an extension of Matsuno’s results to describe superpositions of loop solitons and mixed superpositions of solitons and loop solitons. In the current paper, however, we focus on application of the BT of aDP and defer comparison with existing results on DP for a later work.

A second motivation for the study of the BT of aDP (5) is the link to the KK equation. As far as we know, a BT for the KK equation is not known, and we believe it should be possible to extend the BT given here for aDP to KK. Furthermore, KK is a piece of a larger picture: the KK and the Sawada-Kotera (SK) equations are both reductions of a general Lax equation describing the evolution of a 3rd order Lax operator [27]. KK and SK are related, both being generated by Miura maps from a common “modified” equation [3, 5–7]. In the same way as the DP equation is related to KK, the Novikov equation [19] is related to SK, as demonstrated in [12], by a change of coordinates similar to the one we have presented for DP. We note in passing that introducing the function $f$ as described above for the Novikov equation yields the extremely simple “associated Novikov” (aN) equation

$$f_{xxx} + 3(1 - f_x f_t) = 0 .$$

(6)

We believe there is an analog of our results for the aDP equation for the aN equation. But once again, we defer discussion of BTs for the full set of aDP, KK, aN and SK equations, and their interrelations, and the BT for the general Lax system from which they are obtained as a reduction, to a later work. Note that a BT for SK is known [27]. The interrelations of the DP, KK, Novikov and SK were studied in [13], with focus on Hamiltonian structures.

3 The BT and the Superposition Principle

A direct calculation verifies that the aDP equation (5) has a BT that can be written in either the form

$$f \rightarrow f - \frac{2\theta}{s^2 + 2s_x - 3f_x} \quad \text{or} \quad f \rightarrow f - 2q$$

(7)
where \( s \) satisfies

\[
\begin{align*}
    s_{xx} &= \theta - s^3 - 3ss_x + 3fxs + \frac{3}{2}f_{xx}, \\
    s_t &= f_t + \frac{1}{\theta} \left( \frac{1}{2}f_{xt}s_x - f_tss_x - f_ts^3 + f_{sx}s^2 + \left( \frac{3}{2} + \frac{3fxf_t}{2} - \frac{3f_{xt}^2}{8f_t} \right) s \
    &\quad - \frac{3fxf_{xt}}{4} + \frac{3f_{xt}}{4f_t} + \frac{f_{xt}^3}{16f_t^2} \right), 
\end{align*}
\]

(8)

and \( q \) satisfies

\[
\begin{align*}
    q_{xx} &= \theta - q^3 + \frac{3q^2}{4q} - 3qq_x + 3qf_x, \\
    q_t &= \frac{1}{2}f_t - \frac{1}{2\theta^2f_t} \left( q^3f_t + f_tqq_x - q^2fx + \frac{f_tq^2}{4q} - \frac{1}{2}qf_{xt} + \frac{qf_{xt}^2}{4f_t} - 3q \right)^2. 
\end{align*}
\]

(9)

The advantages of each of the formulations are clear: the action of the BT is simpler in the \( q \) formulation, but the equations satisfied by \( q \) are more complicated than those satisfied by \( s \) (see below, equation (15), for a simpler version of equation (9)). The relationships between \( q \) and \( s \) are

\[
q = \frac{\theta}{s^2 + 2ss_x - 3fx} \quad \text{and} \quad s = \frac{q_x}{2q} + q.
\]

Note that the equations for \( s \) (or for \( q \)) can be linearized via the substitution \( s = \frac{\psi_x}{\psi} \) yielding

\[
\begin{align*}
    \psi_{xxx} &= 3fx\psi_x + \left( \frac{3}{2}f_{xx} + \theta \right) \psi, \\
    \psi_t &= \frac{1}{\theta} \left( f_t\psi_{xx} - \frac{1}{2}f_{xx}\psi_x + \left( \frac{1}{8f_t} - \frac{3}{2}f_{xt} \right) \psi \right).
\end{align*}
\]

(12)

This is the Lax pair for the aDP equation (compare [II]). Note that equation (12) can also be written in the form

\[
\left( \psi\psi_{xx} - \frac{1}{2}\psi_x^2 - \frac{3}{2}f_x\psi^2 \right)_x = \theta\psi^2
\]

so

\[
\psi\psi_{xx} - \frac{1}{2}\psi_x^2 - \frac{3}{2}f_x\psi^2 = \theta \int \psi^2 \, dx.
\]

and thus

\[
q = \frac{\theta}{2\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} - 3fx} = \frac{\psi^2}{2\int \psi^2 \, dx}.
\]

(14)

We mention in passing that the BT can be used to find conservation laws for the aDP equation. From (8)-(9) it follows that

\[
s_t = \frac{1}{\theta} \left( f_t(s_x + s^2) - \frac{1}{2}fxs - \frac{3}{2}f_{fx} + \frac{f_{xt}^2}{8f_t} \right)_x
\]

(15)
which is a $\theta$ dependent conservation law for aDP. For large $|\theta|$, $s$ can be expanded in an asymptotic series
\[ s \sim \sum_{i=0}^{\infty} s_i(x, t) \theta^{(1-i)/3}, \quad s_0(x, t) = 1. \] (16)
Substituting this series into (8) gives a recursion relation for the coefficients $s_i(x, t)$ in terms of $f$ and its $x$-derivatives. Once the coefficients are determined, substituting the series into (15) and expanding in powers of $\theta^{-1/3}$ gives an infinite sequence of conservation laws. Many of these are trivial. The first nontrivial one (after a little cleaning up) is
\[ (f_{xx}^2 + 4f_x^3)_t = \left(3f_t f_x^2 - 6f_x + \frac{3f_x f_{xt}^2 - 18}{2f_t} + \frac{3f_{xt}^2}{2f_t} - \frac{f_{xt}^4}{16f_t^3}\right)_x \]

We note the BT for aDP has an intriguing superficial similarity to the BT for the Boussinesq equation (equations (8)-(9) in [25]). However the BT was actually discovered by trial and error searching for a BT for DP that was superficially similar to the BT for CH given in [24].

The BT has a superposition principle which we present in the $q$ formulation: If $q_1, q_2$ are (respectively) solutions of (10)-(11) with parameters $\theta_1, \theta_2$, then the solution obtained by applying the 2 BTs with parameters $\theta_1$ and $\theta_2$, in either order, is
\[ f_{12} = f - \frac{2(\theta_1 + \theta_2) \left(2Z + (\theta_1 - \theta_2) \left(\frac{1}{q_1} - \frac{1}{q_2}\right)\right)}{\left(Z + \left(\frac{q_1}{q_1} + \frac{q_2}{q_2}\right)^2 - \frac{(\theta_1 + \theta_2)^2}{q_1 q_2}\right)} \] (17)
where
\[ Z = \left(\frac{q_{1x}}{2q_1} - \frac{q_{2x}}{2q_2} + q_1 - q_2\right)^2 = (s_1 - s_2)^2. \]
This result was also found by trial and error, but once known, can easily be verified directly.

### 4 Solutions from a Single BT

We apply the BT to the “trivial” solution
\[ f = \beta x + \frac{t}{\beta} + \alpha \]
where $\alpha, \beta$ are constants with $\beta \neq 0$. The general solution of (12)-(13) is
\[ \psi = C_1 e^{\lambda_1 x + \frac{t}{\lambda_1}} + C_2 e^{\lambda_2 x + \frac{t}{\lambda_2}} + C_3 e^{\lambda_3 x + \frac{t}{\lambda_3}} \]
where $\lambda_1, \lambda_2, \lambda_3$ are the 3 roots of $\lambda^3 = 3\beta \lambda + \theta$, and $C_1, C_2, C_3$ are constants. Using (14) this gives us new solutions

$$ f = \beta x + \frac{t}{\beta} - \frac{2(\eta_1 + \eta_2 + \eta_3)^2}{\eta_1^2 + \eta_2^2 + \eta_3^2 + \frac{4\eta_1\eta_2}{\lambda_1} + \frac{4\eta_1\eta_3}{\lambda_2}} \tag{18} $$

where

$$ \eta_i = C_i e^{\lambda_i x + \beta \lambda_i t}, \quad i = 1, 2, 3. $$

Analysis of these solutions proceeds in a similar manner as in [25]. We restrict to the case $4\beta^3 > \theta^2$, in which case all the roots $\lambda_1, \lambda_2, \lambda_3$ are real. If two of the constants $C_i$ vanish, the solution is trivial. If exactly one of the constants $C_i$ vanishes then the solution is a travelling wave, with speed $\frac{\lambda_i}{\beta \theta}$ (where $\lambda_i$ is the root of the same index as the coefficient $C_i$ that vanishes). There are two kinds of (right moving) soliton solution and two kinds of (left moving) singular soliton solution; the wave profiles are displayed in Figure 1. One of the solitons has a regular soliton profile — we call this a “simple soliton”. The other one has “dimples”, at which the profile drops to zero, before and after the main mass of the soliton — we call this a “dimpled soliton”. The singular solitons all have a single dimple, but differ in whether it is on the left or the right of the main soliton mass. See the appendix for a direct derivation of the travelling wave solutions of aDP, and more useful formulae.

The solutions with all of the $C_i$ nonzero are not travelling wave solutions. They describe the absorption of one of the two types of soliton by one of the two types of singular soliton. When a simple soliton gets absorbed, by either of the types of singular soliton, the type of the singular soliton remains the same. But when a dimpled soliton gets absorbed, then the type of the singular soliton is switched. See Figures 2,3 for two examples.

The dimpled soliton solutions are to some extent an artefact of the fact we are plotting $f_t$ as the wave height. In terms of the variable $p$ satisfying $f_t = p^2$ the dimpled solitons appear as antisolitons. But then there is an asymmetry between solitons and antisolitons: whereas solitons can have arbitrarily small elevation over the background level, antisolitons have a minimum depth (corresponding to $f_t$ dropping from $\frac{1}{3}$ to 0).

5 Solutions from Double BTs

Experiments with the superposition formula (17) show that a wide range of different kinds of solution can be obtained using double BTs. It is not the case that the physical content of a 2 BT solution is the “sum” of the physical content of the solutions obtained by applying the 2 BTs individually — the superposition is highly nonlinear.
Figure 1: Profiles of travelling wave solutions of aDP \((5)\). Plots are of \(f_t\) as a function of \(x\), with \(f\) given by \((18)\). Parameters are \(\beta = 1, \theta = \frac{1}{2}\), so \(\lambda_1 \approx -1.64, \lambda_2 \approx -0.17, \lambda_3 \approx 1.81\).

Top left: \(C_1 = 1, C_2 = 1, C_3 = 0\), solution moves right, the “simple soliton”. Top right: \(C_1 = 1, C_2 = 0, C_3 = 1\), solution moves left. Bottom left: \(C_1 = 1, C_2 = -1, C_3 = 0\), solution moves right, the “dimpled soliton”. Bottom right: \(C_1 = 1, C_2 = 0, C_3 = -1\), solution moves left.
Figure 2: Absorption of simple soliton by singular soliton. Parameter values as in Figure 1, but $C_1 = C_2 = C_3 = 1$. Times shown are $t = -0.6, -0.3, -0.1, 0.1, 0.3, 0.6$. Note the type of the singular soliton is the same before and after absorption of the soliton — the dimple remains on the right.

Figure 3: Absorption of dimpled soliton by singular soliton. Parameter values as in Figure 1, but $C_1 = -1, C_2 = C_3 = 1$. Times shown are $t = -0.6, -0.3, -0.1, 0.1, 0.3, 0.6$. Note the type of the singular soliton changes as a result of absorption of the soliton — the dimple moves from the right to the left. Note also that at a certain time a pair of dimples disappears, by colliding with a singularity.
Figure 4: Superposition of fast dimpled soliton and slow simple soliton. The dimpled soliton absorbs the simple soliton and emits it on the other side. Parameters are $\beta = 1; \theta = 0.8$, $C_1 = C_2 = 1, C_3 = 0$ for $q_1$; $\theta = 1.0$, $C_1 = C_2 = 1, C_3 = 0$ for $q_2$.

It is possible to superpose simple solitons with simple solitons, dimpled solitons with dimpled solitons, and also simple with dimpled. The process by which a fast dimpled soliton passes a slow simple soliton is very straightforward: the dimpled soliton absorbs the simple soliton on its right, and later emits it on its left, as shown in Figure 4. The process by which a fast simple soliton passes a slow dimpled soliton is much more complicated and involves 3 stages (see Figure 5): (1) As the simple soliton approaches the dimpled soliton it starts to absorb it. Its height grows rapidly, while the height of the hump between the two dimples drops dramatically. (But, it should be emphasized, the dimples never disappear at any stage of the process, even though they get closer together, and the height of the hump between them becomes very low.) (2) When the hump between the two dimples has become sufficiently low, a new hump starts to grow on their right, while the hump on their left (the enlarged incoming simple soliton) decays. This stage is extremely rapid. (3) Once the hump on the left of the dimples has totally decayed, and the hump on their right is fully grown, the latter hump starts to decay again, while the hump between the dimples regrows, to assume its original size.

We give just one further example of a superposition. Recall the merger solution illus-
Figure 5: Superposition of fast simple soliton and slow dimpled soliton. Parameters are $\beta = 1; \theta = \frac{1}{2}, C_1 = 1, C_2 = -1, C_3 = 0$ for $q_1; \theta = \frac{4}{3}, C_1 = 1, C_2 = -1, C_3 = 0$ for $q_2$.

First row: reduction of the central hump. Second row: emergence of the humo on the right and annihilation of the hump on the left. Third row: reemergence of the central hump.
Figure 6: Superposition describing a simple soliton colliding with a singular soliton, and emerging as a dimpled soliton. Parameters are $\beta = 1; \theta = \frac{1}{2}, C_1 = 1, C_2 = -1, C_3 = 0$ for $q_1; \theta = \frac{4}{3}, C_1 = 0, C_2 = 1, C_3 = 1$ for $q_2$.

It is possible to verify that the double BT described by (17) does not include the identity transformation, for any choice of $\theta_1, \theta_2$ or the hidden parameters in $q_1, q_2$. Thus we are led to look at triple BTs. We have not, as of yet, succeeded to find algebraic formulas for triple superpositions, along the lines of those found for the Boussinesq equation in [25], though it is natural to conjecture there should be some relation with the discrete CKP equation, as described in [8]. However, it is possible to compute the effect of a triple BT in the limit that the 3 parameters $\theta_1, \theta_2, \theta_3$ tend to a common limit $\theta$. In this limit the triple BT gives
an infinitesimal symmetry

\[ Q(\theta) = \frac{s^{(1)} - s^{(2)}}{(s^{(1)} - s^{(2)})s^{(3)}_x + (s^{(2)} - s^{(3)})s^{(1)}_x + (s^{(3)} - s^{(1)})s^{(2)}_x - (s^{(1)} - s^{(2)})(s^{(2)} - s^{(3)})(s^{(3)} - s^{(1)})} \]  

(19)

where \( s^{(1)}, s^{(2)}, s^{(3)} \) are three distinct solutions of (8)-(9). Remarkably, this is exactly the form of generating function for symmetries found for the Boussinesq equation in [25], equation (32). Even though the calculation to arrive at the result (19) is enormous, once the result has been found it is straightforward to check directly that \( \eta = Q(\theta) \) solves the linearized aDP equation

\[ \eta_{xxt} - \frac{3}{2} \frac{f_x \eta_{xt}}{f_t} + \frac{3}{4} \frac{f_x^2 \eta_t}{f_t^2} - 3(f_x \eta_t + \eta_x f_t) = 0 . \]  

(20)

To generate local symmetries for aDP from (19) we take \( s^{(1)}, s^{(2)}, s^{(3)} \) to be given by the 3 possible asymptotic expansions of \( s \) for large \( |\theta| \). One expansion was given already in (16) and the others are obtained from it by replacing \( \theta^{1/3} \) in the expansion with \( \omega \theta^{1/3} \) and \( \omega^2 \theta^{1/3} \), where \( \omega = \frac{1}{2}(-1 + \sqrt{3}i) \) is a cube root of 1. Substituting these asymptotic expansions into \( Q(\theta) \) and expanding in inverse powers of \( \theta^{1/3} \) gives the first two nontrivial local symmetries of aDP to be

\[ \eta^{(1)} = f_{5x} - 15f_x f_{xxx} - \frac{45}{4} f_{xx}^2 + 15 f_x^3 , \]

\[ \eta^{(2)} = f_{7x} - 21f_x f_{5x} - \frac{105}{2} f_{4x} f_{xx} - \frac{147}{4} f_{xx}^2 + 126 f_x^2 f_{xxx} + \frac{315}{2} f_x f_{xx}^2 - 63 f_x^4 . \]

(Here \( f_{4x}, f_{5x}, \ldots \) denote 4th, 5th etc derivatives with respect to \( x \).) The first of these defines the potential KK flow, as expected.

7 Concluding Remarks

In this paper we have introduced a BT for the aDP equation and given a superposition principle for it. We have seen how this BT generates interesting solutions, and in particular that a single BT on the trivial solution generates not just traveling wave solutions, but also certain “mergers”. We have further seen that the superposition principle of 3 BTs allows us to compute the hierarchy of infinitesimal symmetries, this being a further illustration of the general scheme we have proposed.

A number of matters for further work have arisen, and we briefly recap these. We have not yet translated the solutions obtained for aDP to solutions for DP, which is of more interest because of its physical significance. We have also indicated that the DP and aDP
equations fit into a larger picture of integrable equations with a third order Lax operator, including the KK, SK, DP and Novikov equations, and it would be good to see the results presented here emerge as special cases or reductions of more general results. (The fact that the generator of infinitesimal symmetries \( Q(\theta) \) for aDP is at least superficially the same as the corresponding object for the Boussinesq equation would seem to be good evidence for this.) Finally, we have not yet found an explicit formula for the superposition of 3 BTs of aDP, and there is good reason to believe such a thing must exist. Although we succeeded in presenting the superposition formula in the reasonably compact form (17), this involves 4th powers of \( s_1 - s_2 \) and is difficult to handle. We suspect there is some underlying formalism or choice of fields in which everything becomes much more transparent.

**Appendix: Travelling wave solutions of aDP**

We look for travelling wave solutions of (5) in the form

\[
f = \beta x + \frac{t}{\beta} + U(\sqrt{\beta}(x - ct))
\]

where \( \beta \) is a positive constant. Writing \( z = \sqrt{\beta}(x - ct) \) and \( P(z) = 1 - c\beta^{3/2}U'(z) \), we find we need

\[
P'' = \frac{3P'^2}{4P} + 3(1 - P) \left(1 - \frac{P}{\beta^2c}\right).
\]

Equation (21) can be integrated to give

\[
P'^2 = -\frac{4}{\beta^2c}P^3 + 12 \left(1 + \frac{1}{\beta^2c}\right)P^2 + 4dP^{3/2} + 12P.
\]

Clearly it is advantageous to substitute \( P(z) = Q(z)^2 \), giving

\[
Q'^2 = -\frac{1}{\beta^2c}Q^4 + 3 \left(1 + \frac{1}{\beta^2c}\right)Q^2 + dQ + 3.
\]

For a soliton solution we require that the polynomial on the right hand side should have a double root. Choosing \( d \) so that there is a double root at \( Q = 1 \) (corresponding to a soliton with a background value of \( f_t = \frac{1}{\beta} \)) we obtain

\[
Q'^2 = -\frac{1}{\beta^2c}(Q - 1)^2(Q^2 + 2Q - 3\beta^2c).
\]
If $\beta^2 c > 1$, the RHS of (22) has 2 simple real roots, in addition to the double root at $Q = 1$, with one simple root on each side of the double root. The corresponding solutions are

$$Q(z) = 1 + \frac{3(\beta^2 c - 1)}{2 \pm \sqrt{3\beta^2 c + 1} \cosh \left( z \sqrt{3 \left( 1 - \frac{1}{\beta^2 c} \right)} \right)} .$$

The solution with a $+$ corresponds to the simple soliton of the Section 4 ($Q > 1$ for all $z$), while the solution with a $-$ corresponds to the dimpled soliton ($Q$ changes sign, but $P = Q^2$ remains positive). If $\beta^2 c < -\frac{1}{3}$, then the RHS (22) has no roots apart from the double root at $Q = 1$, and is nonnegative for all $Q$. The corresponding solutions are

$$Q(z) = 1 - \frac{3(\beta^2 |c| + 1)}{2 \pm \sqrt{3\beta^2 |c| - 1} \sinh \left( z \sqrt{3 \left( 1 + \frac{1}{\beta^2 |c|} \right)} \right)} .$$

These correspond to the singular solitons of Section 4. It is straightforward to verify that (23) and (24) are the only solitary wave solutions.

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