Two-loop $\text{AdS}_5 \times S^5$ superstring: testing asymptotic Bethe ansatz and finite size corrections

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Abstract

We continue the investigation of two-loop string corrections to the energy of a folded string with a spin $S$ in $\text{AdS}_5$ and an angular momentum $J$ in $S^5$, in the scaling limit of large $J$ and $S$ with $\ell = \frac{J}{\sqrt{\lambda \ln S}}$ fixed. We compute the generalized scaling function at two-loop order $f_2(\ell)$ both for small and large values of $\ell$ matching the predictions based on the asymptotic Bethe ansatz. In particular, in the small $\ell$ expansion, we derive an exact integral form for the $\ell$-dependent coefficient of Catalan’s constant term in $f_2(\ell)$. Also, by resumming a certain subclass of multi-loop Feynman diagrams we obtain an exact expression for the leading $\ln \ell$ part of $f(\ell, \sqrt{\lambda})$ which is valid to any order in the $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ expansion. At large $\ell$ the string energy has a BMN-like expansion and the first few leading coefficients are expected to be protected, i.e. to be the same at weak and strong coupling. We provide a new example of this non-renormalization for the term which is generated at two loops in string theory and at one-loop in gauge theory (sub-sub-leading in $1/J$). We also derive a simple algebraic formula for the term of maximal transcendentality in $f_2(\ell)$ expanded at large $\ell$. In the second part of the paper we initiate the study of 2-loop finite size corrections to the string energy by formally compactifying the spatial world-sheet direction in the string action expanded near long fast-spinning string. We observe that the leading finite-size corrections are of ‘Casimir’ type coming from terms containing at least one massless propagator. We consider in detail the one-loop order (reproducing the leading Landau–Lifshitz model prediction) and then focus on the two-loop contributions to the $\frac{1}{\ln S}$ term (for $J = 0$). We find that in a certain regularization scheme used to discard power divergences the two-loop coefficient of the $\frac{1}{\ln S}$ term appears to vanish.

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1. Introduction

The correspondence between fast-spinning folded closed strings in $\text{AdS}_5 \times S^5$ and twist operators in the $\mathcal{N} = 4$ SYM theory is a remarkable tool for uncovering and checking the detailed structure of the AdS/CFT correspondence. In particular, string perturbative computations of quantum corrections to the spinning string energy, which should correspond to strong-coupling corrections to dimensions of gauge-theory operators, provide important data for checking the integrability-based (Bethe ansatz) predictions for the string spectrum. Here we will continue the investigation of two-loop string corrections to the energy of the folded $(S, J)$ spinning string [1, 2] using and extending the techniques developed in our previous papers [3, 4].

To put the results of our investigation into perspective let us first review the general structure of the dependence of string energies or gauge-theory dimensions $E = \Delta(S, J; \lambda)$ on spins and string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ or 't Hooft coupling $\lambda$. In general, $E$ is a complicated function of several variables and even having a formal set of asymptotic Bethe ansatz/thermodynamic Bethe ansatz equations describing the string spectrum (see, e.g., [5] for a review) one should still understand in detail the various patterns of behavior of this function in various limits. We shall concentrate on gauge theory states from the ‘$\mathfrak{sl}(2)$ sector’ represented by the operators such as $\text{tr}(D^2 \Phi^J)$ dual to strings with large spin $S$ in AdS$_5$ and large orbital momentum $J$ in $S^5$.

In perturbative planar gauge theory one first expands in $\lambda \ll 1$ for fixed spins $(S, J)$

$$E \equiv \Delta = S + J + \gamma(S, J, \lambda), \quad \gamma = \lambda \gamma_1(S, J) + \lambda^2 \gamma_2(S, J) + \lambda^3 \gamma_3(S, J) + \ldots, \quad (1.1)$$

and may then expand $\gamma_n$ in large spins. In the semiclassical string theory limit one first expands in $\alpha'$, i.e. in $\sqrt{\lambda} \gg 1$, for fixed semiclassical parameters $S = \frac{S}{\sqrt{\lambda}}, J = \frac{J}{\sqrt{\lambda}}$ (implying that $S$ are $J$ are assumed to be as large as $\sqrt{\lambda}$)

$$E = S + J + e(S, J, \sqrt{\lambda}), \quad e = \sqrt{\lambda} e_0(S, J) + e_1(S, J) + \frac{1}{\sqrt{\lambda}} e_2(S, J) + \ldots, \quad (1.2)$$

and may then expand in large $S, J$. The two limits are obviously very different and cannot be in general compared directly. The gauge–string duality relation implies that summing up the expansion in (1.1) (which should have a finite radius of convergence) and then re-expanding the result at strong coupling in the semiclassical string theory limit one should reproduce the string-theory expansion (1.2).

There are several special sub-limits depending on the relative values of $S$ and $J$. The more familiar ‘fast-string’ limit, generalizing the BMN limit, corresponds, on the gauge-theory side, to taking $J \gg 1, S \gg 1$ with $\frac{S}{J} = \text{fixed}$ (this is a limit of long but ‘locally-BPS’ operators). In this case $\gamma_n$ in (1.1) happen to have the following structure ($n = 1, 2, 3, \ldots$):

$$\gamma_n = \frac{1}{2^n (n-1)!} \left( a_{n1} + \frac{a_{n2}}{J} + \frac{a_{n3}}{J^2} + \cdots \right), \quad a_{nm} \equiv a_{nm} \left( \frac{S}{J} \right), \quad (1.3)$$

where $a_{n2}, a_{n3}, \ldots$ are the coefficients of subleading finite-size corrections at $n$th loop order from the underlying spin chain point of view (see, e.g., [6, 7]). The corresponding limit on

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(Some figures in this article are in colour only in the electronic version)
the string side is \( J \gg 1, \frac{S}{J} \) is fixed when \( e_k \) in (1.2) have the following expansion [8, 9] \((k = 0, 1, 2, \ldots)\):

\[
e_0 = \frac{J}{3} \left( b_{00} + \frac{b_{02}}{J^2} + \frac{b_{04}}{J^4} + \cdots \right), \quad e_1 = \frac{1}{J^2} \left( b_{10} + \frac{b_{12}}{J^2} + \frac{b_{13}}{J^3} + \cdots \right),
\]

\[
e_2 = \frac{1}{J^3} \left( b_{20} + \frac{b_{22}}{J^2} + \cdots \right), \quad e_3 = \frac{1}{J^4} \left( b_{30} + \frac{b_{31}}{J} + \cdots \right), \ldots, \quad b_{ij} = b_{ij} \left( \frac{S}{J} \right).
\]

Remarkably, due to the underlying supersymmetry of the theory and the special nature of the states the two different expansions have the same formal dependence of the spins and can be described by the following interpolating formula:

\[
E = S + J + \frac{h_1}{J} + \frac{h_2}{J^2} + \frac{h_3}{J^3} + \frac{h_4}{J^4} + \frac{h_5}{J^5} + \cdots, \quad h_n = h_n \left( \frac{S}{J}, \lambda \right), \tag{1.5}
\]

Here in perturbative gauge theory (i.e. in (1.3))

\[
h_1 = \lambda a_{11}, \quad h_2 = \lambda a_{12}, \quad h_3 = \lambda a_{13} + \lambda^2 a_{21}, \quad h_4 = \lambda a_{14} + \lambda^2 a_{22}, \tag{1.6}
\]

\[
h_5 = \lambda a_{15} + \lambda^2 a_{23} + \lambda^3 a_{31} + \cdots, \quad \tag{1.7}
\]

while in perturbative string theory (i.e. in (1.4))

\[
h_1 = \lambda b_{00}, \quad h_2 = \lambda b_{10}, \quad h_3 = \lambda b_{20} + \lambda^2 b_{02}, \quad h_4 = \lambda b_{30} + \lambda^2 b_{12}, \tag{1.8}
\]

\[
h_5 = \lambda^3 b_{04} + \lambda^{\frac{5}{2}} b_{13} + \lambda^2 b_{22} + \lambda^{\frac{3}{2}} b_{31} + \cdots. \tag{1.9}
\]

The functions \( h_1, h_2, h_3, h_4 \) are thus linear or quadratic functions of \( \lambda \), i.e. the corresponding coefficients should be the same in the gauge-theory and the string-theory expansions—they should match as functions of \( \frac{S}{J} \)

\[
a_{11} = b_{00}, \quad a_{12} = b_{10}, \quad a_{13} = b_{20}, \quad a_{21} = b_{02}, \quad a_{14} = b_{30}, \quad a_{22} = b_{12}. \tag{1.10}
\]

At the same time, \( a_{31} \neq b_{04} \) since \( h_5 \) (and also \( h_6, \ldots \)) is a non-trivial function of \( \lambda \); this is related to the presence of the non-trivial phase in ABA, explaining [9, 10], in particular, the well-known ‘3-loop disagreement’ [11].

The matching (1.10) should be due to the underlying supersymmetry of the large \( J \) expansion and the structure of the asymptotic Bethe ansatz [10]. The equivalence between the one-loop gauge and tree-level string coefficient functions \( a_{11} = b_{00} \) was explicitly demonstrated in [6, 8, 17]; the matching of the one-loop gauge and the one-loop string coefficients \( a_{12} = b_{10} \) was seen in [7]. However, the equality of the one-loop gauge and two-loop string coefficients \( a_{13} = b_{20} \) was not previously checked directly as the relevant two-loop string computation is non-trivial to perform (the sub-sub leading finite size correction on one-loop gauge theory side is also non-trivial to extract).

One of our aims below will be to provide such a check in a setting similar to the one described above. We shall consider another scaling limit of ‘fast long strings’ [13–16], i.e. \( \lambda \ll 1, \quad S \gg J \gg 1, \quad j = \frac{J}{\ln S} = \text{fixed} \). \tag{1.11}

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6 It may be understood also as a consequence of the exactness of the coefficients of the leading low-derivative terms in the underlying effective Landau–Lifshitz-type action [12, 37].

7 In the following we will use the normalization of [16]: \( j = \frac{J}{\ln S} \).
on the gauge theory side and
\[
\sqrt{\lambda} \gg 1, \quad S \gg J \gg 1, \quad \ell \equiv \frac{\pi J}{\ln S} = \frac{\pi j}{\sqrt{\lambda}} = \text{fixed} \quad (1.12)
\]
on the string theory side. One can then study also subcases of small or large \( \ell \) and \( j \). If \( \ell \ll 1 \), i.e. \( \ln S \gg J \gg \ell \gg 1 \), one finds on the string theory side
\[
E = S + \sqrt{\lambda} f(\ell, \sqrt{\lambda}) \ln S + \cdots,
\]
\[
f(\ell, \sqrt{\lambda}) = f_0(\ell) + \frac{1}{\sqrt{\lambda}} f_1(\ell) + \frac{1}{(\sqrt{\lambda})^2} f_2(\ell) + \cdots, \quad (1.13)
\]
\[
f_{\ell \to 0} = f(\lambda) + \ell^2 \sum_{n=0}^{\infty} c_n (\ln \ell)^n + d_n (\ln \ell)^{n-1} + \cdots + O(\ell^4),
\]
where \( c_n, d_n \) are essentially fixed by the \( O(6) \) sigma model truncation of the string action [15].
At tree level \( f_0 = \sqrt{1 + \ell^2} \), while the one-loop coefficient \( f_1 \) is [14]
\[
f_1(\ell) = \frac{1}{\sqrt{1 + \ell^2}} \left[ \sqrt{1 + \ell^2} - 1 + 2(1 + \ell^2) \ln(1 + \ell^2) - \ell^2 \ln \ell^2 \right. \\
\left. - 2 \left( 1 + \frac{1}{2} \ell^2 \right) \ln[\sqrt{1 + \ell^2}(1 + \sqrt{1 + \ell^2})] \right]
\]
\[
= -3 \ln 2 - 2\ell^2 \left( \ln \ell - \frac{3}{4} \right) + \ell^4 \left( \ln \ell - \frac{3}{8} \ln 2 - \frac{1}{16} \right) + O(\ell^6). \quad (1.14)
\]
The two-loop coefficient was found in [4, 18, 19]
\[
f_2(\ell) = -K + \ell^2 \left( 8 \ln^2 \ell - 6 \ln \ell - \frac{1}{3} \ln 2 + \frac{11}{4} \right)
\]
\[
+ \ell^4 \left( -6 \ln^2 \ell - \frac{7}{6} \ln \ell + 3 \ln 2 \ln \ell - \frac{9}{8} \ln^2 2 + \frac{11}{8} \ln 2 \right.
\left. - \frac{3}{32} K - \frac{233}{576} \right) + O(\ell^6), \quad (1.15)
\]
where \( K \) is Catalan’s constant. The string expression for \( \ell^2 \) term (1.15) is finally in agreement [4] with the ABA results at strong coupling [20–22]. Similarly the \( \ell^4 \) term [4] also agrees with [20].

Extending the techniques used in [3, 4] based on the AdS light-cone gauge string action [38], we shall find additional higher order terms in \( f_2 \) extending the agreement with the ABA result of [20]. In particular, we shall determine the exact form of the function of \( \ell \) that multiplies the transcendental constant \( K \).

We shall also find that the leading powers of \( \ln \ell \) terms in the \( \ell \to 0 \) expansion at each order in \( \frac{1}{\sqrt{\lambda}} \) are generated by a special class of Feynman graphs that can be resummed to all orders in the worldsheet loop expansion. This then leads to the exact form of the leading \( \ln \ell \) part of \( f(\ell) \) in (1.13), which is, again, in agreement with the ABA prediction of [20].

\[
f(\ell, \sqrt{\lambda}) \bigg|_{\text{lead log}} = \sqrt{1 + \frac{\ell^2}{1 + \frac{4}{\sqrt{\lambda}} \ln \ell}}. \quad (1.16)
\]

In contrast to the fast string limit (1.5) where the spin dependence was the same at large and strong coupling with the coefficients being, in general, interpolating functions of \( \lambda \), here the dependence on \( \ell \) on the string side is not the same as dependence on \( j \) on the gauge side where the small \( j \) expansion is polynomial in \( j \) [13, 16] (\( q_n \) are power series in \( \lambda \))
\[
E = S = \left[ f(\lambda) + q_1(\lambda) j + q_2(\lambda) j^2 + \cdots \right] \ln S + \cdots. \quad (1.17)
\]
To relate the gauge and string theory expressions it is necessary to first resum the gauge theory expansion and then to re-expand it at strong coupling keeping $\ell = \ell_0^\text{A} = \sqrt{\lambda}$ fixed [19].

The analog of the `fast string' limit is found by expanding $f(\ell, \sqrt{\lambda})$ in (1.13) at large $\ell$ on the string side and at large $j$ on the gauge side. Namely, assuming $\ell \gg 1$, i.e. $J \gg \ln S$ and $j = \ell_0 2 = \frac{\sqrt{\lambda}}{S} \ell \gg 1$ we should be encountering again ‘locally-BPS’ states for which the $\ell$ or $j$ dependence should appear to be the same at weak and at strong coupling. We may write then the string result expressed in terms of $j = \frac{\sqrt{\lambda}}{S} \ell$ in a form similar to (1.5) with coefficients depending on $\frac{\ln S}{S}$ instead of $\frac{\ell_0}{S}$. To leading order in $\ln S$ we then get

$$E = S + f(\ell, \ell) \ln S + \cdots,$$

$$f(\ell, \ell)_{\ell > 1} = j + \frac{c_{10} \lambda}{j} + \frac{c_{11} \lambda}{j^2} + \frac{c_{12} \lambda + c_{20} \lambda^2}{j^3} + \frac{c_{13} \lambda + c_{21} \lambda^2}{j^4} + \frac{p_5(\lambda) + p_6(\lambda)}{j^5} + \cdots, \quad (1.18)$$

with $p_5, p_6, \ldots$ being non-trivial interpolating functions of the coupling. Here the protected coefficients $c_{nm}$ are $m$-loop string theory contributions which should also match $n$-loop gauge theory contributions for the same reason as discussed above for (1.5). Explicit results found in tree-level and one-loop string theory [13, 14] should match the one-loop and two-loop gauge theory results [13, 23].

Here we shall consider the $c_{12}$ term in $E-S$ (1.18):

$$c_{12} \frac{\ln^4 S}{J^3} = c_{12} \frac{\ln^4 S}{\sqrt{\lambda} J^3}, \quad (1.20)$$

which originates at one loop on the gauge theory side and at two loops on the string theory side. This is a sub-sub-leading finite-size term from the weak-coupling $\text{sl}(2)$ sector spin-chain perspective (with $J$ being the length of the spin chain). The ABA predictions for the value of $c_{12}$ both at weak [22] and at strong [24] coupling appears to be

$$c_{12} = \frac{1}{3\pi^2}, \quad (1.21)$$

suggesting its non-renormalisation. Here we shall compute this coefficient directly as a two-loop correction in string theory (still defined on $\mathbb{R}^{1,1}$), providing an intricate two-loop check of the ABA at strong coupling.

Another limit that is useful to consider is that of ‘slow long strings’ which corresponds to ‘long’ far-from-BPS operators $\text{tr}(D^2 \Phi^4)$ with $\ln S \gg J$, $J = \text{twist} = \text{fixed}$ on the gauge side and $\ln S \gg J$, $J = \text{fixed}$ on the string side. In this case

$$E = S + f(\lambda, J) \ln S + h(\lambda, J) + \frac{u(\lambda, J)}{\ln S} + \cdots + O\left(\frac{1}{S}\right), \quad (1.22)$$

$$f_{\lambda, J > 1} = c_0 \sqrt{\lambda} + c_1 + \cdots, \quad f_{\lambda, J < 1} = b_1 \lambda + b_2 \lambda^2 + \cdots. \quad (1.23)$$

On the gauge theory side the scaling functions $f$ and $h$ are not sensitive to wrapping effects, i.e. they should be captured by the ABA (or BES-type integral equations) [10, 25, 26] and by string theory on $\mathbb{R}^{1,1}$.\textsuperscript{8} Indeed, the wrapping contributions at weak coupling are suppressed

\textsuperscript{8} In the string theory calculation of $h(\lambda, J)$ the string end-points turn out to be important [32]. While indeed the calculation can be carried out in the large $\ln S$ limit (i.e. on a decompacified worldsheet) it requires use of the exact solution, valid on a cylindric worldsheet $\mathbb{R} \times S^1$. A two-loop calculation for the exact folded string solution on $\mathbb{R}^{1,1}$ should reproduce the two-loop term in the strong coupling expansion of the virtual scaling function $h(\lambda, J)$ obtained from ABA in [26]. A linear integral equation governing $h(\lambda, J)$ was first written down in [27], where an alternative approach to the generalized scaling function $f(\lambda, J)$ and its subleading correction was discussed. In the following we shall ignore $h(\lambda, J)$ and focus on the sub-sub-leading $1/\ln S$ term.
by powers of $S$ (starting at five loops with $\frac{\ln^2 S}{S}$ [28]). The $1/\ln S$ terms are not present in the usual perturbative expansion.

One may wonder if the function $u$ may also be determined by a linear integral equation following from the ABA; it appears that for fixed $J$ at weak coupling ABA predicts $u = 0$ [29]. At the same time, $u$ is certainly non-zero at strong coupling, i.e. in the semiclassical expansion in string theory as we shall review below. Then matching the weak-coupling and strong-coupling forms of (1.22) would require resummation.

For $J = 0$, considering the one-loop correction to the folded spinning string energy on $\mathbb{R} \times S^1$, i.e. by replacing the integral over the continuous spatial momentum by a discrete summation

$$\int dp \rightarrow \frac{2\pi}{L} \sum_n, \quad L \sim \ln S,$$

one finds a finite-size ‘Casimir effect’-type correction to the string one-loop energy coming from five massless modes:

$$\mathcal{J} = 0: \quad u = k_1 + \frac{k_2}{\sqrt{\lambda}} + \cdots, \quad k_1 = -\frac{5\pi}{12}. \quad \text{(1.25)}$$

This correction was first found in [30] (by formally extending to $\mathbb{R} \times S^1$ world sheet the sum over the characteristic frequencies found in [31] in the infinite spin limit) and then confirmed rigorously in [32] (by starting with the exact form of the folded string solution).

As we shall discuss below, if one starts with the one-loop expression in the case of $\mathcal{J} \neq 0$ [14] (that leads to (1.14) in the case of the string on $\mathbb{R}^{1,1}$) and extends it to $\mathbb{R} \times S^1$ world sheet one finds instead only one massless-mode contribution; taking the $\mathcal{J} \rightarrow 0$ limit would lead to the conclusion that $k_1 = -\frac{\pi}{12}$, in an apparent contradiction with (1.25). More precisely, the massless-mode contribution to the one-loop string energy producing the ‘Casimir’ $\frac{1}{L} \sim \frac{1}{\ln S}$ term can be written as

$$\Delta E_1 = (E_1)_{\text{massless}} = -\frac{\pi}{12} \frac{1}{\ell^2 + 1} \ln S = -\frac{1}{12} \frac{\ell}{\ell^2 + 1} \ln \mathcal{J} = -\frac{1}{12\pi} \frac{\lambda}{J^2 + \frac{\lambda}{12\pi} \ln S} \ln S. \quad \text{(1.26)}$$

The relation to the $\mathcal{J} = 0$ case (1.25) can be understood by also taking into account the contribution of four massive (mass $\sim \mathcal{J}$) modes producing exponential $\sim \exp(-c \mathcal{J})$ corrections that should be added to (1.26) and resummed before taking the $\mathcal{J} \rightarrow 0$ limit. From the point of view of comparison to the $sl(2)$ sector Bethe ansatz result, the massless-mode contribution should be a finite size effect captured by ABA while the exponential contributions of four light massive modes should be ‘Lüscher’ corrections corresponding to wrapping contributions at weak coupling. The full one-loop string semiclassical result must match the TBA results as was demonstrated in [33].

The last form of the expression in (1.26) which is formally analytic in $\lambda$ suggests that at least first two terms in its large $J$ expansion

$$\Delta E_1 = -\frac{1}{12\pi} \frac{\lambda}{J^2} \ln S + \frac{1}{12\pi^2 \mathcal{J}^2} \ln^3 S + \cdots \quad \text{(1.27)}$$

may not be renormalized, i.e. should appear also with the same coefficients at weak coupling, originating from finite-size corrections in the $sl(2)$ sector Bethe ansatz. Like the leading order $\lambda$ term in the classical string energy (i.e. the $c_{10}$ term in (1.18) or the analog of $b_{00}$ term in (1.8))\footnote{As in [13] here $\ln S$ may be replaced by $\ln \frac{J}{J_0}$ but we will ignore this detail. The $\ln^2 S$ term was found in [2] on the string side and reproduced from the one-loop gauge-theory BA in [17].}

$$E_0 - S = E_0 - S = \sqrt{J^2 + \frac{\lambda}{\pi^2} \ln^2 S} + \cdots = J + \frac{\lambda}{2\pi^2 J} \ln^2 S + \cdots, \quad \text{(1.28)}$$
which matches the one-loop gauge theory result, the leading terms in (1.27) should be similar
to the protected $b_{10}$ and $b_{12}$ terms in (1.8) and (1.9) and $c_{11}$ and $c_{21}$ in (1.18) and (1.19). Note,
however, that the terms in (1.18) are found in a different limit than (1.26): by first fixing $\ell$ and
extracting the coefficient of the $\ln S$ term and then expanding this coefficient in large $\ell$. Here
instead we are discussing the coefficient of the $\frac{1}{mS}$ term at fixed $\ell$ and then expand in large $\ell$.

As in other similar cases of ‘fast string’ states [7] the non-renormalization of the leading
term in (1.27), i.e. it can also be obtained directly at weak coupling from the one-loop $sl(2)$
sector spin chain Hamiltonian, suggests that it should be reproduced as finite-size correction
from the corresponding Landau–Lifshitz (LL) model. The LL model [34] describes, on the
one hand, fast strings moving in $AdS_5 \times S^5$ and, on the other hand, the corresponding coherent
states of the one-loop $sl(2)$ sector Hamiltonian [35, 36]. Indeed, from the string theory point
of view, the LL model keeps the contribution of the ‘massless’ mode in the $AdS_3$ part that
is responsible for the ‘Casimir’ term in (1.26). We shall discuss the details of the LL model
relation in appendix A. In contrast, the order $\lambda$ term in (1.20) appears to receive contributions
from several massive string modes in the two-loop string computation that we shall describe
below and is not captured by the one-loop LL model.

While the matching of the one-loop string and the strong-coupling TBA results for
semiclassical ($J = \frac{\lambda}{\sqrt{4}}$, etc. kept fixed) string states appears to be guaranteed [33], this still
remains to be verified at the two-loop string level. This should provide further non-trivial
tests of quantum integrability of $AdS_3 \times S^1$ superstring and of consistency of TBA. Below
we shall initiate the study of two-loop string finite size corrections by formally extending the
computation of the string partition function in the folded string background done in [3] on $\mathbb{R}^{1,1}$ to the $\mathbb{R} \times S^1$ world sheet. We shall discuss in detail the computation of the two-loop string
correction to the $\frac{1}{mS}$ term in (1.22) at $J = 0$ and show that it appears to be zero in a
natural regularization scheme, in contrast to the non-vanishing two-loop term (1.26).

The detailed plan for the rest of the paper is as follows. In section 2, we discuss the
AdS light-cone gauge for the $AdS_5 \times S^5$ superstring. This is the gauge we will use for
all our computations. We also introduce the generalized cusp background with vanishing
winding (which is related to the spinning folded string by a conformal transformation [40]). In
section 3, we review the relation between the partition function for the long $(S, J)$ spinning
string and the corresponding quantum corrected AdS energy. The various contributions to
the two-loop string partition function in the generalized cusp background are discussed in
section 4. Compared to our previous work [4] we try to compute the various Feynman
integrals exactly rather than perturbatively at small $\ell$. With these results at hand, in section 5
we discuss the calculation of the generalized scaling function $f(\ell)$ at two-loop order both for
small and large values of $\ell$. In particular, in the small $\ell$ expansion, we show that the coefficients
of $\ln 2$ and $\ln^2 2$ match all the values reported in [20]. We also present an exact formula for the
$\ell$ dependent coefficient of the Catalan constant $K$ in $E_2(\ell)$ in terms of an integral generating
function, see equation (5.7). Still in the context of the small $\ell$ expansion, we obtain, by
summing all ‘maximally non 1-PI’ diagrams, an exact expression for the leading logarithm
coefficient in $f(\ell)$ which is valid at any loop order, see equation (5.16). In the case of the
large $\ell$ expansion, we derive a simple algebraic formula for the maximal transcendentality
piece which turns out to be proportional to $\pi^2$, see equation (5.22) which we match against the
ABA prediction. This result is the large $\ell$ analog of the coefficient of the $K$ constant at small $\ell$.

The non-renormalization of the coefficient $c_{12}$ in (1.18) mentioned above is discussed in
section 5.5.

In the second part of the paper we study the finite size $1/\ln S$ corrections to the string
energy by placing the string sigma model on a cylinder. In section 7.1, after replacing the
integral over continuous spatial momentum with a discrete summation, we extract the finite
We begin by reviewing the superstring action in the AdS light-cone gauge [38]. A great advantage of this gauge is its simplicity; for example the AdS5
one-loop energy of the long folded string reproducing the result of section 7.1.

In section 8 we apply the same strategy to the two-loop computation of the folded string energy. We observe that the only relevant finite size contributions come from terms containing at least a massless propagator, the purely massive ones producing exponentially suppressed contributions. The final result is presented in section 8.3 where an ambiguity in the regularization prescription and its possible resolution are discussed. Finally, in appendix A we study the LL model and show that it captures the leading finite size correction to the one-loop energy of the long folded (S, J) string.

Other appendices contain technical details on various aspects of our calculations.

2. AdS light-cone action and the generalized cusp solution

We begin by reviewing the superstring action in the AdS light-cone gauge [38]. A great advantage of this gauge is its simplicity; for example the AdS5 × S5 gauge-fixed action is at most quartic in the fermions. Expanding the action around the folded spinning string in the scaling limit of the infinite spin, one finds that the bosonic propagator is almost diagonal [3, 4]. This renders this gauge more efficient than the conformal gauge for higher-loop computations.

The AdS light-cone gauge is defined by imposing

\[ \tau \rightarrow -\tau, \quad p^+ \rightarrow i p^+, \quad \text{and after setting} \quad p^+ = 1, \quad \text{the AdS}_5 \times S^5 \text{ superstring action can be written as} \]

\[ I = \frac{1}{2} T \int d\tau \int d\sigma \mathcal{L}_E, \quad T = \frac{R^2}{2\pi \alpha'} = \frac{\sqrt{g}}{2\pi}, \]

\[ \mathcal{L}_E = \dot{x}^i \dot{x}^j + (\zeta^M + i z^2 \xi \rho_{MN} \eta^N) \dot{\gamma} + i (\dot{\gamma} \dot{\eta} + \dot{\gamma} \dot{\eta} - \text{h.c.}) - z^{-2} (\eta^2)^2 \]

\[ + z^{-4} (\dot{x}^i \dot{x}^j + \zeta^M \dot{\gamma}) + 2i [z^{-3} \eta^M \zeta^M (\dot{\theta} - i z^{-1} \eta^M \dot{x}) + \text{h.c.}]. \]

This action has manifest SO(6) ∼ SU(4) symmetry. The fermions are complex \( \theta^i = (\theta_i)^t, \eta^j = (\eta_j)^t \) (\( i = 1, 2, 3, 4 \)) transforming in the fundamental representation of SU(4). \( \rho^M_{ij} \) are off-diagonal blocks of six-dimensional gamma matrices in the chiral representation and \( (\rho^{MN})^j_i = (\rho^{[M} \rho^{N]} \eta^i)^j \) and \( (\rho^{MN})^i_j = (\rho^{[M} \rho^{N]} \eta^i)^j \) are the SO(6) generators.

Since the AdS light-cone gauge is adapted to the Poincaré patch, we apply a conformal transformation to the spinning folded string in global AdS5 × S5 to work with coordinates...
as in (2.1). In the scaling limit (1.12) this conformal transformation gives us the so-called
generalized null cusp background, see [4, 39, 40]. This is a bosonic solution for which only
the radial coordinate $z$ and one isometric angle $\phi$ of $S^5$ are nontrivial (i.e. $x = x^0 = 0$, $y^\alpha = 0$):

$$
\begin{align*}
z &= \sqrt{\frac{\kappa}{\mu}} \sqrt{\frac{\tau}{\sigma}}, \\
x^+ &= \tau, \\
x^- &= -\frac{\kappa}{2\mu} \frac{1}{\sigma}, \\
\phi &= \frac{\nu_e}{2\kappa} \ln \tau ,
\end{align*}
$$

(2.6)

where the parameters entering in the solution satisfy the following constraint:

$$
\kappa^2 + \nu_e^2 = \mu^2 .
$$

(2.7)

Since $x^+x^- = -\frac{1}{2}z^2$ this bosonic solution ends on a null cusp at the boundary $z = 0$ of AdS$_5$.
A generalization of this background which includes a winding parameter $w$, $\phi \sim w \ln \sigma$, is
also possible [4], but for simplicity it will not be considered here.

It is useful to define the fluctuations around the $z$ solution with extra rescalings:

$$
\begin{align*}
z &= \sqrt{\kappa} \sqrt{\frac{\tau}{\sigma}} \tilde{z}, \\
z^M &= \sqrt{\kappa} \sqrt{\frac{\tau}{\sigma}} \tilde{z}^M, \\
x &= \sqrt{\kappa} \sqrt{\frac{\tau}{\sigma}} \tilde{x}, \\
\theta &= \frac{1}{\sqrt{\sigma}} \tilde{\theta}, \\
\eta &= \frac{1}{\sqrt{\sigma}} \tilde{\eta}, \\
\tilde{x} &= \tilde{x}_1 + i\tilde{x}_2, \\
\tilde{z} &= e^{\tilde{\phi}} = 1 + \tilde{\phi} + \cdots, \\
\tilde{z}^M &= \tilde{z}^M u^M, \\
u^M u^M &= 1,
\end{align*}
$$

(2.8)

where we introduced the rescaled parameters

$$
\begin{align*}
\tilde{\nu}_e &\equiv \frac{\nu_e}{\mu}, \\
\tilde{\kappa} &\equiv \frac{\kappa}{\mu} = \sqrt{1 - \tilde{\nu}_e^2}.
\end{align*}
$$

(2.9)

Defining the fields as in (2.8) renders constant the coefficients in the Lagrangian for the bosonic
fluctuations, i.e. independent of the worldsheet variables. To obtain the constant coefficients
also in front of the fermionic fluctuations one has to additionally shift the $S^5$ angle $\phi$ by a
quantity $\tilde{\phi}$

$$
\tilde{\phi} = \frac{\tilde{\nu}_e}{2\kappa} \ln \tau + \tilde{\phi},
$$

(2.10)

where $\tilde{\phi}$ represents the fluctuation around the classical background.

It is also useful to replace the worldsheet coordinates ($\sigma$, $\tau$) with

$$
\begin{align*}
t &= \ln \tau , \\
s &= \ln \sigma ,
\end{align*}
$$

(2.11)

which puts the induced worldsheet metric in the conformal gauge:

$$
dx^2 = \frac{1}{4}(d\tau^2 + ds^2).
$$

(2.12)

The fluctuation spectrum can be derived by expanding the Lagrangian to quadratic order in the
fluctuations. It consists of eight bosonic and eight fermionic massive fields. The precise form
of the spectrum can be read off the pole structure of the bosonic and fermionic propagators
given in appendix B.

3. Generalized scaling function from the partition function

Before moving to the discussion of the two-loop calculation we briefly review how the
knowledge of the string partition function can be used to compute the generalized scaling
function. Note that to compare to the gauge theory predictions we need to compute the
partition function for a worldsheet with Lorentzian signature. We will therefore present all
results as function of the Lorentzian parameter $\theta = -i\nu_e$. 

}
The effective action $W$ has the following expansion:

$$W = - \ln Z_{\text{string}} = \frac{\sqrt{\lambda}}{2\pi} V F(\hat{\nu}), \quad F = 1 + \frac{1}{\sqrt{\lambda}} F_1 + \frac{1}{(\sqrt{\lambda})^2} F_2 + \ldots ,$$  \hspace{1cm} (3.1)

where $Z_{\text{string}}$ is the string partition function. For a generic 2D sigma model the expectation values of 2D conserved quantities in semiclassical approximation can be found using a thermodynamical approach, for more details see [4, 19]. In this case the relevant conserved quantities are $E - S$ and $J$ for which it is possible to show that the following relations hold:

$$E - S = M \sqrt{1 + \hat{\nu}^2} \left[ F(\hat{\nu}) - \hat{\nu} \frac{dF(\hat{\nu})}{d\hat{\nu}} \right],$$  \hspace{1cm} (3.2)

$$J = M \left[ \hat{\nu} F(\hat{\nu}) - (1 + \hat{\nu}^2) \frac{dF(\hat{\nu})}{d\hat{\nu}} \right],$$  \hspace{1cm} (3.3)

where $M$ is the ‘string mass’ (tension $\times$ length):

$$M = \frac{\sqrt{\lambda}}{2\pi} L = \frac{\sqrt{\lambda}}{\pi} \ln S = \sqrt{\lambda} \mu .$$  \hspace{1cm} (3.4)

Defining

$$f(\ell) \equiv \frac{E - S}{M}, \quad \ell \equiv \frac{J}{M} = \hat{\nu} + \frac{1}{\sqrt{\lambda}} \hat{\nu}_1(\hat{\nu}) + \frac{1}{(\sqrt{\lambda})^2} \hat{\nu}_2(\hat{\nu}) + \ldots ,$$  \hspace{1cm} (3.5)

we find from (3.2) and (3.3)

$$f(\ell) = \sqrt{1 + \hat{\nu}^2} \left[ F(\hat{\nu}) - \hat{\nu} \frac{dF(\hat{\nu})}{d\hat{\nu}} \right],$$  \hspace{1cm} (3.6)

$$\ell = \hat{\nu} F(\hat{\nu}) - (1 + \hat{\nu}^2) \frac{dF(\hat{\nu})}{d\hat{\nu}},$$  \hspace{1cm} (3.7)

allowing one to compute $f(\ell)$, given $F(\hat{\nu})$, by solving for $\hat{\nu}$.

Expanding (3.6) and (3.7) perturbatively in $1/\sqrt{\lambda}$ leads to the following expressions for the quantum corrections to the generalized scaling function $f(\ell)$ in terms of $F$:

$$f(\ell) = f_0(\ell) + \frac{1}{\sqrt{\lambda}} f_1(\ell) + \frac{1}{(\sqrt{\lambda})^2} f_2(\ell) + \ldots ,$$  \hspace{1cm} (3.8)

$$f_0 = \sqrt{1 + \ell^2}, \quad f_1 = \frac{F_1(\ell)}{\sqrt{1 + \ell^2}},$$  \hspace{1cm} (3.9)

$$f_2 = \frac{F_2(\ell)}{\sqrt{1 + \ell^2}} + \frac{1}{2} (1 + \ell^2)^{3/2} \left( \frac{df_1}{d\ell} \right)^2 .$$  \hspace{1cm} (3.10)

Higher-loop corrections can be obtained analogously.

4. Two-loop partition function

In this section we evaluate the diagrams which contribute to the effective action $W = - \ln Z_{\text{string}}$ at two-loop order

$$W_2 = W_{2B \text{ sunset}} + W_{2B \text{ double--bubble}} + W_{2F \text{ sunset}} + W_{2F \text{ double--bubble}} + W_2 \text{ tadpoles}$$

$$\equiv \frac{V}{2\pi \sqrt{\lambda}} F_2(\hat{\nu}).$$  \hspace{1cm} (4.1)
Figure 1. The two-loop 1PI topologies: ‘sunset’ and ‘double-bubble’. The propagators here are either bosonic or fermionic.

Figure 2. The two-loop tadpole topology. The non-vanishing graphs have the internal line corresponding to a $\tilde{\phi}$-propagator while the propagators in the loops can either be bosonic or fermionic.

This will be later used to extract the two-loop term $f_2(\ell)$ in the generalized scaling function. The relevant connected vacuum diagrams are shown in figures 1 and 2. Note that both one-particle irreducible (1PI) topologies and the non-1PI tadpole topology contribute. As already discussed in [3, 4], the presence of non-1PI tadpole graphs is crucial in this gauge for the cancellation of UV divergencies and to extract the correct expression of $f_2(\ell)$.

For the computation of the various topologies we need to expand the action up to fourth order in the fluctuations. Details on this expansion can be found in [4]. All manipulations of tensor structures appearing in the evaluation of the Feynman diagrams are performed in $d = 2$, and the resulting scalar integrals are computed using an analytic regularization scheme in which power divergent contribution are set to zero:

$$\int \frac{d^2 p}{(2\pi)^2} (p^2)^n = 0, \quad n \geq 0.$$  \hspace{1cm} (4.2)

We also use the following notation:

$$I[m^2] = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + m^2}. \hspace{1cm} (4.3)$$

Note that the integral $I[m^2]$ is UV divergent when $m^2 > 0$ and both UV and IR divergent when the mass vanishes.

We now begin with the analysis of the 1PI diagrams and end this section with the calculation of the tadpole.

4.1. Bosonic and fermionic double-bubble

The double-bubble diagrams arise from the quartic terms in the Lagrangian. If we introduce the following one-loop integrals:

$$J_B(i, j) = \int \frac{d^2 p}{(2\pi)^2} \frac{p_0^i p_0^j}{p^4 + p^2 + \tilde{\nu}^2 p_0^2}, \quad J_F(i, j) = \int \frac{d^2 p}{(2\pi)^2} \frac{p_0^i p_0^j}{\frac{4 \times 337}{16} \frac{\frac{1}{2} \tilde{\nu} p_0^2}{16}} \hspace{1cm} (4.4)$$

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we can express the bosonic and fermionic contributions as follows:

$$W_{\text{2-double--bubble}} = -2(J_B(0, 2) + J_B(2, 0))(J_B(0, 2) + (1 + \hat{\nu}^2)J_B(2, 0))$$
$$+ \frac{1}{2}(3J_B(0, 2)^2 + 2(3 + 2\hat{\nu}^2)J_B(0, 2)J_B(2, 0) + 3J_B(0, 2)^2)$$
$$- \hat{\nu}^2(J_B(0, 2) + J_B(2, 0))I[m_x^2] - 2\hat{\nu}^2 J_B(2, 0)I[m_y^2] - \frac{1}{2} \hat{\nu}^2 I[m_f^2]^2.$$

$$W_{\text{2-double--bubble}} = -\hat{\nu}^2 J_F(0, 0)^2 + 16J_F(1, 0)^2 + 8i\hat{\nu} J_F(0, 0)J_F(1, 0)$$
$$- 4(J_B(0, 2) + J_B(2, 0))[J_F(0, 0) + 2(J_F(0, 2) + J_F(0, 2))]$$
$$+ [2(2 + \hat{\nu}^2)J_F(0, 0) + 8(J_F(0, 2) - i\hat{\nu} J_F(1, 0) + J_F(0, 2))]I[m_f^2].$$

Here we have introduced the notation

$$m_x^2 \equiv \frac{1}{2}(2 + \hat{\nu}^2), \quad m_y^2 \equiv \frac{1}{4} \hat{\nu}^2$$

for the masses of the $\tilde{x} = x_1 + ix_2$ AdS fluctuation and of the four $\nu^a$ fluctuations on $S^5$, respectively.

Using the expressions for the one-loop integrals (4.4) given in appendix C we can find exact expressions for the $\hat{\nu}$ dependence of all double-bubble diagrams.

### 4.2. Bosonic sunset

The contributions to the sunset topology arise from the cubic interactions in the Lagrangian. We begin with the bosonic case. We can arrange the various terms depending on their denominator structure. From the form of the interactions we have two possibilities

$$\int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \delta^{(2)}(p + q + r) \frac{N(p,q,r)}{D_B(p)(q^2 + m^2)(r^2 + m^2)}, \quad m^2 = m_x^2 \text{ or } m_y^2$$

$$\int \frac{d^2p}{(2\pi)^2} \frac{d^2r}{(2\pi)^2} \delta^{(2)}(p + q + r) \frac{N(p,q,r)}{D_B(p)D_B(q)D_B(r)},$$

where

$$D_B(p) \equiv p^2(p^2 + 1) + \hat{\nu}^2 p_0^2$$

is the denominator appearing in the propagator of the mixed fields $\hat{\phi}$ and $\hat{\varphi}$.

There are two contributions with a single $D_B(p)$ in the denominator, with $m^2 = m_x^2, m_y^2$. These may be evaluated exactly. A convenient strategy is, for example, to solve the momentum conservation constraint by expressing the momenta as $p = P, q = Q, r = \bar{P} - Q$, and to perform the integration over $Q$ first. The remaining expression, which contains the non-Lorentz invariant factor $D_B(P)$, can be evaluated at the end. The final results for this structure can be written as follows:

$$W_{\text{2-sunset}, m_x} = -\frac{1}{8\pi} \left[ \frac{2 + \hat{\nu}^2}{4} \right] \left( 2 + \hat{\nu}^2 - 2\sqrt{1 + \hat{\nu}^2} + 8\pi \hat{\nu}^2 \left[ \frac{1}{4}(1 + \sqrt{1 + \hat{\nu}^2})^2 \right] \right)$$
$$+ \int_0^1 du \frac{(1 + \hat{\nu}^2)\text{arctanh} u}{2\pi^2[\sqrt{1 + \hat{\nu}^2} + u^2 + \sqrt{1 + (1 + \hat{\nu}^2)u^2}]^2}.$$

$$W_{\text{2-sunset}, m_y} = \frac{\hat{\nu}^2}{2} \left[ \frac{\hat{\nu}^2}{4} \right] \left( \frac{1}{4} \left( \frac{\hat{\nu}^2}{4} \right)^2 + 2I \left[ \frac{1}{4}(1 + \sqrt{1 + \hat{\nu}^2})^2 \right] \right).$$

While the integral in the second line of (4.9) cannot be evaluated in closed form in terms of elementary functions, it can be computed to any desired order in the small or large $\hat{\nu}$ expansion.
In particular, this integral produces Catalan’s constant term at small $\nu$ and $\pi^2$ term at large $\nu$, see section 5 below.

The remaining bosonic sunset corresponds to the more complicated structure with three $\mathcal{D}_B$ denominator factors, as in the second line of (4.7), which arises from the three-vertices involving only $\phi$ and $\bar{\phi}$ fluctuations. The presence of three Lorentz-non-invariant denominators makes an exact evaluation of this contribution more involved. We have computed it in the small $\nu$ expansion up to sixth order:

$$W_{2\text{Boson.}(\mathcal{D}_B)^3} = \frac{1}{2} I[1]^2 + \left( \frac{1}{2} I[1] - \frac{1}{8\pi} I[1] \right) \nu^2 + \left( -\frac{7}{64\pi} I[1] + \frac{5}{512\pi^2} \right) \nu^4$$

$$+ \left( \frac{7}{192\pi} I[1] + \frac{3}{512\pi^2} \right) \nu^6 + \mathcal{O}(\nu^8).$$

(4.11)

Note that the result only contains the UV divergent integral $I[1]$ and a rational finite part. We expect this to remain true to all orders in the small $\nu$ expansion. In particular, we do not expect this term to contain irreducible two-loop sunset-type integrals such as $I[1, 1, 1]$ and its variations with higher powers of the denominators\(^{10}\); these would introduce new transcendental numbers in the small $\nu$ expansion besides $\ln 2$ and Catalan’s constant (which do not appear for $\nu = 0$ and in the Bethe ansatz solution of [20]). Likewise, at large $\ell$ we do not expect this term to contribute to the highest transcendentality part of the free energy, on which we wish to concentrate in this paper. Therefore, we will not attempt here an exact evaluation of this specific term. Its contribution is, of course, crucial for complete cancellation of UV divergences. We will assume that this cancellation occurs, as it was rigorously checked up to order $\nu^5$ in [4].

4.3. Fermionic sunset

The possible structures of the two-loops integrals in this case are

$$\int \frac{d^2 p \, d^2 q \, d^2 r \, \delta^{(2)}(p + q + r)}{(2\pi)^4} \frac{\mathcal{N}(p, q, r)}{\mathcal{D}_F(p) \mathcal{D}_F(q)(r^2 + m^2)} + \text{c.c.,} \quad m^2 = m_x^2 \text{ or } m_y^2$$

$$\int \frac{d^2 p \, d^2 q \, d^2 r \, \delta^{(2)}(p + q + r)}{(2\pi)^4} \frac{\mathcal{N}(p, q, r)}{\mathcal{D}_F(p) \mathcal{D}_F(q) \mathcal{D}_B(r)} + \text{c.c.,}$$

(4.12)

where $\mathcal{N}(p, q, r)$ is a sum of tensors of rank up to four, and $\mathcal{D}_F(p)$ is the characteristic denominator which appears (together with its complex conjugate) in the fermion propagator:

$$\mathcal{D}_F(p) = \left( p_0 - i\frac{\nu}{4} \right)^2 + p_r^2 + \frac{1}{4} (1 + \nu^2).$$

(4.13)

The presence of more than one non-Lorentz invariant term in the denominators makes the exact computation of the fermionic sunset difficult. A possible approach is to shift the momenta $p$ and $q$ associated with the non-Lorentz invariant factor $\mathcal{D}_F$. Indeed, performing the shift $p_0 \rightarrow p_0 + i\frac{\nu}{4}$, or its conjugate version if we have $\mathcal{D}_F^*(p)$, reduces $\mathcal{D}_F(p)$ to a Lorentz invariant expression:

$$p_0 \rightarrow p_0 + i\frac{\nu}{4}, \quad \mathcal{D}_F \rightarrow p^2 + \frac{1 + \nu^2}{4}.$$  

(4.14)

\(^{10}\) Here we use the notation

$$I[m_1^2, m_2^2, m_3^2] = \int \frac{d^2 p \, d^2 q}{(2\pi)^4} \frac{1}{(p^2 + m_1^2)(q^2 + m_2^2)((q + p)^2 + m_3^2)}.$$  

In particular, one finds that $I[1, 1, 1] = \frac{1}{288\pi^4} (\psi'(1/3) - \psi'(2/3) + \psi'(1/6) - \psi'(5/6))$, where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.
Of course, the remaining momentum $r$ in (4.12) should accordingly be shifted to preserve overall momentum conservation. After such shift, one is left with two out of three Lorentz invariant propagators, and one can proceed with an exact evaluation as described in the previous subsection for the bosonic terms. We should point out, however, that such a shift of momenta is potentially dangerous due to the divergent nature of the integrals involved. It effectively amounts to a ‘change of scheme’ in regularizing the integrals, which is not necessarily compatible with the way we have computed all other two-loop contributions. Nonetheless, we have observed that, in practice, such issue only affects the ‘rational part’ of the free energy, both at large and small $\hat{\nu}$ (see equations (5.1) and (5.2) below). Therefore, for the purpose of this paper, we will compute these fermionic integrals by shifting momenta, keeping in mind that in the following results only terms containing transcendental numbers should be trusted when summing up all diagrams.

As in the bosonic case, the result can be divided into three contributions, the first two corresponding to the first structure in (4.12) with $m^2 = m_x^2$ or $m^2 = m_y^2$ and the third to the second structure in (4.12). After performing the momentum shift and evaluating the integrals, we obtain

$$W_{2\text{Fsumet};m_x} = -\frac{5\hat{\nu}^2}{8\pi}I\left[\frac{3}{4}(1 + \hat{\nu}^2)\right] - 2I\left[\frac{3}{4}(1 + \hat{\nu}^2)\right] + (2\hat{\nu}^2 + 1)I\left[\frac{3}{4}(1 + \hat{\nu}^2)^2\right] + W_1,$$

$$W_{2\text{Fsumet};m_y} = -\frac{5\hat{\nu}^2}{4\pi}I\left[\frac{3}{4}(1 + \hat{\nu}^2)\right] + 2I\left[\frac{3}{4}(1 + \hat{\nu}^2)\right] + (2\hat{\nu}^2 + 1)I\left[\frac{3}{4}(1 + \hat{\nu}^2)^2\right] + W_2,$$

$$W_{2\text{Fsumet};D_S} = -\frac{1}{8\pi}I\left[\frac{3}{4}(1 + \hat{\nu}^2)\right]\left[12 + 5\hat{\nu}^2 - 12\sqrt{1 + \hat{\nu}^2} + 8\pi(\hat{\nu}^2 - 2)I\left[\frac{3}{4}(1 + \sqrt{1 + \hat{\nu}^2})\right]\right],$$

where $W_1$, $W_2$ and $W_3$ are finite and have integral representations which we include in appendix D.

### 4.4. Tadpole contribution

Let us finally take into account the contribution of the non-1PI diagrams. The only fluctuation that can acquire a non-trivial expectation value is $\tilde{\phi}$. Therefore, the relevant non-1PI two-loop diagrams are obtained by sewing together two one-loop tadpoles with a $\tilde{\phi}$ propagator at zero momentum.

Exact expressions for these tadpoles were already found in [4]. Here we summarize the final result:

$$A_{\text{tadpole}} = -\frac{1}{4\pi}\left[1 - \sqrt{1 + \hat{\nu}^2}\right] + \frac{\hat{\nu}^2}{2}\left[\ln(2 + \hat{\nu}^2) - 4\ln(1 + \hat{\nu}^2) + 2\ln(\sqrt{1 + \hat{\nu}^2} + 1)\right] + 2I\left[\frac{3}{4}(1 + \hat{\nu}^2)\right].$$

Then the total contribution of the non-1PI graphs is

$$W_{2\text{tadpoles}} = -\frac{1}{2}A_{\text{tadpole}}^2.$$

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5. Generalized scaling function

Collecting together all the partial results for $W_2$ we can extract the generalized scaling function as outlined in section 3. In particular, we will extract the piece of maximal transcendentality, which is proportional to $K$ in the small $\hat{\nu}$ expansion, and proportional to $\pi^2$ in the large $\hat{\nu}$ expansion. Furthermore, we will also be able to verify that the coefficient of the $\frac{1}{\hat{\nu}}$ term (1.20) in $f_2$ is not renormalized, i.e. is the same as at weak coupling.

At small $\hat{\nu}$, one may represent the structure of the two-loop free energy (and analogously the structure of the generalized scaling function) in the form

$$F_2(\hat{\nu}) = \alpha_2(\hat{\nu}) \ln^2 \hat{\nu} + \alpha_1(\hat{\nu}) \ln \hat{\nu} + \beta_2(\hat{\nu}) \ln^2 2 + \beta_1(\hat{\nu}) \ln 2 + \gamma(\hat{\nu}) K + \delta(\hat{\nu}),$$

where $\alpha_2(\hat{\nu}), \ldots, \delta(\hat{\nu})$ can be expressed as analytic Taylor series around $\hat{\nu} = 0$. We recall that the presence of $\ln^2 \hat{\nu}$ and $\ln \hat{\nu}$ is due to the massive $S^5$ fluctuations which become very light as $\hat{\nu} \to 0$. The functions $\beta_1(\hat{\nu}), \gamma(\hat{\nu}), \delta(\hat{\nu})$ were computed to order $\hat{\nu}^4$ in [4], while the functions $\alpha_2(\hat{\nu}), \alpha_1(\hat{\nu})$ multiplying the logarithmic terms could actually be extracted in closed form. The results of the present paper allow us to also obtain $\beta_1(\hat{\nu})$ and $\gamma(\hat{\nu})$ to arbitrary order, as discussed below. However, we will not be able to fix the purely rational term $\delta(\hat{\nu})$ due to the fact that we do not exactly compute the complicated bosonic contribution, second line of (4.6), and also due to limitations of our approach to computation of the fermionic sunset integrals, see the previous subsection.

At large $\hat{\nu}$, on the other hand, the results of this paper (which match the Bethe ansatz calculations of [20, 22]) indicate that the two-loop free energy may be written in the form

$$F_2(\hat{\nu}) = p(\hat{\nu}) \pi^2 + r(\hat{\nu}),$$

where $p(\hat{\nu})$ and $r(\hat{\nu})$ are analytic around $\hat{\nu} = \infty$ and contain only rational numbers as coefficients in their expansion at large $\hat{\nu}$. As in the small $\hat{\nu}$ expansion, our analysis allows us to extract exactly only the coefficient $p(\hat{\nu})$ of the highest transcendentality constant $\pi^2 = 6 \xi(2)$, but not of the purely rational part $r(\hat{\nu})$.

5.1. $\ln^2 2$ and $\ln 2$ terms at small $\hat{\nu}$

A curious ‘experimental’ observation, which we have explicitly verified up to order $\hat{\nu}^{10}$, is that after summing up all two-loop diagrams the $\ln 2$ terms appearing in the small $\hat{\nu}$ expansion can be eliminated from the free energy by a simple rescaling

$$\hat{\nu} \to 2^{-3/4} \hat{\nu}.$$

In other words, the pattern in which $\ln^2 2$ and $\ln 2$ can appear is completely determined by the coefficients of the $\ln^2 \hat{\nu}$ and $\ln \hat{\nu}$ terms. This observation applies only to the complete free energy and does not hold integral by integral or even diagram by diagram. The small $\hat{\nu}$ form (5.1) of the two-loop free energy can then be written more compactly as

$$F_2(\hat{\nu}) = \alpha_2(\hat{\nu}) \left( \ln \hat{\nu} + \frac{3}{4} \ln 2 \right)^2 + \alpha_1(\hat{\nu}) \left( \ln \hat{\nu} + \frac{3}{4} \ln 2 \right) + \gamma(\hat{\nu}) K + \delta(\hat{\nu}).$$

Explicitly, the coefficients of the logarithmic terms are

$$\alpha_2(\hat{\nu}) = -2 \hat{\nu}^4,$$

$$\alpha_1(\hat{\nu}) = -\frac{1}{\hat{\nu}^2} (12 + 14 \hat{\nu}^2 + 2 \hat{\nu}^4 - 2 \hat{\nu}^2 (2 + \hat{\nu}^2) \sqrt{1 + \hat{\nu}^2}) + \frac{4}{\hat{\nu}^4} (3 + 4 \hat{\nu}^2 + \hat{\nu}^4) \ln (1 + \hat{\nu}^2) - 2 \hat{\nu}^4 \ln \left[ \frac{1}{2} \sqrt{1 + \frac{\hat{\nu}^2}{2}} \right] = -2 \hat{\nu}^2 + \frac{17}{6} \hat{\nu}^4 + \frac{7}{5} \hat{\nu}^6 - \frac{83}{160} \hat{\nu}^8 + \cdots$$

11 Note that the function $\alpha_1(\hat{\nu})$ differs from the function $\alpha_1(\hat{\nu})$ in (5.1); they are related as $\alpha_1(\hat{\nu}) = \alpha_1(\hat{\nu})^\perp + \frac{1}{2} \alpha_2(\hat{\nu}) \ln 2$. 

Computing the generalized scaling function \( f_2(\ell) \) by plugging (5.4) with the values (5.5) in equation (3.10) we find that the coefficients of \( \ln^2 2 \) and \( \ln 2 \) indeed match the Bethe ansatz result of [20] through the order \( \ell^6 \) explicitly given there (the coefficients of \( \ln^2 \ell \) and \( \ln \ell \) were already shown in [4] to match the results of [20] to all orders in \( \ell \)).

5.2. The Catalan constant term in the small \( \ell \) expansion of \( f_2 \)

Here we present a closed-form expression for the coefficient of the term in \( f_2 \) proportional to the Catalan constant appearing in the small \( \ell \) expansion of the generalized scaling function \( f_2(\ell) \) (we will denote this term \( f_2;K(\ell) \)). It can be expanded to any order in \( \ell \) extending the \( O(\ell^4) \) result obtained in [4].

From the discussion in section 4 it follows that the only bosonic contribution to the coefficient of the Catalan constant \( K \) comes from the integral term in (4.9) whose small \( \nu \) expansion reads

\[
\int_0^1 du \frac{8(1 + \nu^2) \text{arctanh} u}{[\sqrt{1 + \nu^2 + u^2} + \sqrt{1 + (1 + \nu^2)u^2}]^2} = \left( 1 + \frac{1}{2} \nu^2 - \frac{7}{32} \nu^4 + \frac{7}{64} \nu^6 - \frac{61}{1024} \nu^8 + \ldots \right) K
\]

Similarly, the fermions contribute only through the integral \( W \) in (4.15) and we observe that their net effect is to simply change the sign of the coefficient of the bosonic contribution to \( K \) term in (5.6), see the appendix D. Using (3.10), to obtain \( f_{2;K} \) we need to divide (5.6) by \( \sqrt{1 + \ell^2} \), while replacing \( \nu \to \ell \), and change the overall sign to account for the fermion contribution. Therefore, we find the following integral representation which can be expanded to any order in \( \ell \)

\[
f_{2;K}(\ell) = -\int_0^1 du \frac{8 \sqrt{1 + \ell^2} \text{arctanh} u}{[\sqrt{1 + \ell^2 + u^2} + \sqrt{1 + (1 + \ell^2)u^2}]^2} \bigg|_{K} = \left( -1 + \frac{3}{32} \ell^4 - \frac{3}{32} \ell^6 + \frac{81}{1024} \ell^8 + \ldots \right) K.
\]

Similarly, the fermions contribute only through the integral \( W \) in (4.15) and we observe that their net effect is to simply change the sign of the coefficient of the bosonic contribution to \( K \) term in (5.6), see the appendix D. Using (3.10), to obtain \( f_{2;K} \) we need to divide (5.6) by \( \sqrt{1 + \ell^2} \), while replacing \( \nu \to \ell \), and change the overall sign to account for the fermion contribution. Therefore, we find the following integral representation which can be expanded to any order in \( \ell \)

\[
f_{2;K}(\ell) = -\int_0^1 du \frac{8 \sqrt{1 + \ell^2} \text{arctanh} u}{[\sqrt{1 + \ell^2 + u^2} + \sqrt{1 + (1 + \ell^2)u^2}]^2} \bigg|_{K} = \left( -1 + \frac{3}{32} \ell^4 - \frac{3}{32} \ell^6 + \frac{81}{1024} \ell^8 + \ldots \right) K.
\]

Up to order \( \ell^6 \) this precisely matches the ABA result of [20], while higher order terms were not explicitly given there.

The fact that the fermionic contribution simply changes the sign of the bosonic contribution to the coefficient of the Catalan constant was first observed for the ordinary cusp anomaly \( J = 0 \) in [18]. It is remarkable that the same applies to all orders in \( \nu \).

5.3. Leading logarithms at small \( \ell \): all-loop resummation

The \( n \)-loop term in the strong coupling expansion of the generalized scaling function \( f(\ell, \sqrt{\lambda}) \) at small \( \ell \) is expected to contain the leading logarithmic term

\[
f_n(\ell) = \hat{c}_n(\ell) \ln^n \ell + \cdots, \quad \hat{c}_n(\ell) = \ell^2 c_n + O(\ell^4).
\]

12 Here we restored the overall factor \( \frac{\pi}{\sqrt{\lambda}} V = \frac{\pi}{\sqrt{\lambda}} V \) in \( W_2 \).

13 Here we do not need to include the piece proportional to the one-loop partition function in (3.10), as it does not contain terms proportional to \( K \).

14 Higher order terms can be found from the general expressions given in appendix C of [20].
Figure 3. Multi-loop maximally reducible diagrams contributing to the leading logarithmic terms in the string-free energy. The loops are made of the light $S^5$ fluctuations and the propagators correspond to the constant mode of the AdS fluctuation $\tilde{\phi}$.

The leading coefficient $c_n = (-1)^n 2^{2n-1}$ is completely captured by the $O(6)$ sigma model [15], while the exact function $\hat{c}_n(\ell)$ was derived in [20] from the asymptotic Bethe ansatz. In this subsection, we show that the AdS light-cone gauge action can be used to obtain an all-loop string theory prediction for this coefficient function which exactly matches the Bethe ansatz result.

At one and two loops, the leading logarithmic terms in the string free energy are

$$ F_1 = -2\hat{\nu}^2 \ln \hat{\nu} + \cdots, \quad F_2 = -2\hat{\nu}^4 \ln^2 \hat{\nu} + \cdots. \quad (5.9) $$

As recalled earlier, the string theory calculation makes it clear that the appearance of logarithms in the small $\hat{\nu}$ expansion is due to the presence of the light $S^5$ fluctuations $y^a$ with mass $m \sim \hat{\nu}$. Moreover, it was observed in [4] that the leading logarithmic term at two loops comes solely from the one-particle reducible diagram in figure 2 with the $S^5$ fields running in the two loops and the $\tilde{\phi}$ propagator at zero momentum.

It is natural to expect (and not difficult to show) that the same will hold at all loops, namely that the leading logarithms at the $n$-loop order come from the ‘maximally non-1PI’ diagrams containing $n$ loops of the $S^5$ fields connected by trees of $\tilde{\phi}$ propagators, as depicted in figure 3. Note that due to momentum conservation, the $\tilde{\phi}$ propagators are always at zero momentum.

It is, in fact, not difficult to exactly resum such diagrams by directly computing the path integral for an appropriate truncation of the AdS light-cone action (2.5). Since the relevant diagrams only contain one-loop subdiagrams of the $S^5$ fields, it is sufficient to truncate the action to quadratic order in $y^a$, while keeping the exact dependence on the constant mode of $\tilde{\phi}$ (since its propagator should be at zero momentum). In the following we will denote this constant mode as $\tilde{\phi}_0$. The truncated action of interest is then

$$ S_{\text{lead. log.}} = \sqrt{\lambda} \int dt ds L_{\text{lead. log.}}. $$

$$ L_{\text{lead. log.}} = \frac{1}{2} \cosh 2\tilde{\phi}_0 + e^{2\tilde{\phi}} (\partial_t y^a)^2 + e^{-2\tilde{\phi}} (\partial_s y^a)^2 + \frac{1}{4} \hat{\nu}^2 e^{2\tilde{\phi}} y^a y^a. \quad (5.10) $$

We can now integrate out the $y^a$ fields exactly. The corresponding one-loop determinant is

$$ W_1^{(y^a)} = 2V \int \frac{d^2 \hat{p}}{(2\pi)^2} 4 \ln \left( e^{2\tilde{\phi}} p_0^2 + e^{-2\tilde{\phi}} p_1^2 + \frac{1}{4} \hat{\nu}^2 e^{2\tilde{\phi}} \right). \quad (5.11) $$

where $V = \frac{1}{2} \int dt ds$. Performing the momentum integral one finds (discarding quadratic divergences)

$$ W_1^{(y^a)} = 2V \left( \ln \left( \frac{\hat{\nu}^2}{4} e^{2\tilde{\phi}_0} \right) \right) \hat{\nu}^2 e^{2\tilde{\phi}_0}. \quad (5.12) $$

Note that this is of course UV divergent, due to the presence of \( I[1] \). This divergence is supposed to cancel once we include all (bosonic and fermionic) modes in the theory. For the present purpose, we can just retain the finite piece proportional to \( \ln \nu \), so that after integrating out \( y^a \) we end up with the following effective action for the constant mode \( \tilde{\phi}_0 \):

\[
S_{\text{eff}}(\tilde{\phi}_0) = V \left( \frac{\sqrt{\lambda}}{2\pi} \cosh 2\tilde{\phi}_0 - \frac{\tilde{\nu}^2}{2\pi} e^{2\tilde{\phi}_0} \ln \tilde{\nu}^2 \right).
\]  

(5.13)

Now the exact path-integral for this reduced model can be obtained by performing the integration over the constant mode. In fact, since the relevant diagrams of figure 3 only contain \( \tilde{\phi}_0 \) at tree level, all we have to do is solve the classical equation of motion for \( \tilde{\phi}_0 \):

\[
\delta S_{\text{eff}} \frac{\delta}{\delta \tilde{\phi}_0} = 0 \rightarrow e^{2\tilde{\phi}_0} = \frac{1}{\sqrt{1 - 2\frac{\tilde{\nu}^2}{\sqrt{\lambda}} \ln \tilde{\nu}^2}}.
\]  

(5.14)

Plugging back into the effective action (5.13), and recalling that in our normalizations we define \( S_{\text{eff}} = V \frac{\sqrt{\lambda}}{2\pi} F \), we arrive at the following all-loop free energy for leading logarithms:

\[
F_{\text{lead.log.}} = 1 + 2\frac{\tilde{\nu}^2}{\sqrt{\lambda}} \ln \tilde{\nu}^2 = 1 + 2\frac{\tilde{\nu}^2}{\sqrt{\lambda}} F_{\text{lead.log.}}.
\]  

(5.15)

In the second equality we have stressed that this expression can be written entirely in terms of the leading logarithmic part of the one-loop free energy.

To obtain the coefficient of the leading logarithms in the generalized scaling function, we need just to plug this result for the free energy into equations (3.6)–(3.7). Eliminating \( \tilde{\nu} \) in favor of \( \ell \) from (3.7) and restricting to leading logarithms yields the following answer:

\[
f(\ell, \sqrt{\lambda})|_{\text{lead.log.}} = \sqrt{1 + \frac{\ell^2}{1 + \frac{\ell^2}{\sqrt{\lambda}}} \ln \ell^2}.
\]  

(5.16)

This precisely agrees with the Bethe ansatz result of [20]. In particular, the first few terms are

\[
f_1 = -\frac{2\ell^2}{\sqrt{1 + \ell^2}} \ln \ell + \cdots, \quad f_2 = \frac{8\ell^2 + 6\ell^4}{(1 + \ell^2)^{3/2}} \ln^3 \ell + \cdots,
\]  

(5.17)

\[
f_3 = -\frac{32\ell^2 + 48\ell^4 + 20\ell^6}{(1 + \ell^2)^{5/2}} \ln^3 \ell + \cdots
\]

5.4. The \( \pi^2 \) term in the large \( \ell \) expansion of \( f_2 \)

Let us now consider the large \( \tilde{\nu} \) expansion and again focus on the part of \( f_2 \) of maximal transcendentality which in this case turns out to be proportional to \( \xi(2) \) or \( \pi^2 \) (we will denote this term \( f_{2,\pi^2} \)). The \( \pi^2 \) term in the partition function turns out to be rather simple:

\[
F_{2,\pi^2} = \pi^2 \left( \frac{1}{3\tilde{\nu}^2} + \frac{1}{4\tilde{\nu}^4} \right).
\]  

(5.18)

Note that the expansion stops at next to leading order. It is interesting to discuss various partial contributions to this coefficient. Being of maximal transcendentality at two loops in large \( \ell \) expansion, \( \pi^2 \) can arise only from the sunset diagrams. Specifically, it originates from the integral (5.6) in \( W_{2\text{Boson,sunset},m} \), and from \( \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \) in the fermionic sunsets, see appendix D. As an example, let us explain how to obtain the large \( \tilde{\nu} \) expansion of integral (5.6) in the bosonic sunset (a similar approach also works for the integrals \( \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \) in the fermionic sunsets). In contrast with the small \( \tilde{\nu} \) case, a direct expansion of the integrand
in equation (5.6) leads to divergent integrals at sufficiently high orders in $\hat{v}^{-1}$. This signals that the large $\hat{v}$ expansion is non-analytic as a function of $\hat{v}^2$. A consistent expansion can be constructed by first using the identity
\[
\frac{\arctanh u}{u} = \int_0^1 dy \frac{1}{1 - u^2y^2}
\] (5.19)
to evaluate in closed form the $u$ integral in equation (5.6). The integrand of the resulting $y$ integral can be expanded at large $\hat{v}$, the integral of each term being finite. The absence of divergences indicates the consistency of this procedure. In this way we obtain
\[
\int_0^1 du \frac{8(1 + \hat{v}^2)\arctanh u}{(\sqrt{1 + \hat{v}^2} + u^2 + \sqrt{1 + (1 + \hat{v}^2)u^2})^2} = 2 + (6 - \pi^2)\frac{1}{\hat{v}^2} + \frac{16}{3} \frac{1}{\hat{v}^4} + \left(4 - \frac{\pi^2}{2}\right) \frac{1}{\hat{v}^4} - \frac{104}{45} \frac{1}{\hat{v}^6} + \cdots.
\] (5.20)

The presence of odd powers of $1/\hat{v}$ exposes the expected non-analyticity in $\hat{v}^2$ of the large $\hat{v}$ expansion. Note also that the result contains $\pi^2$ as well as rational numbers; as was already mentioned above, this is a general feature of the large $\ell$ expansion of the generalized scaling function.

Combining this with the fermionic terms listed in appendix D, we then have the following partial contributions to the coefficient of $\pi^2$ in $F_2$
\[
W_{2\text{Bouset}},m_i \to \pi^2 \left(-\frac{1}{\hat{v}^2} - \frac{1}{2\hat{v}^4} - \frac{1}{4\hat{v}^6} + \frac{7}{16\hat{v}^8} - \frac{49}{64\hat{v}^{10}} + \cdots\right)
\]
\[
W_{2\text{Fusnet},m_i} \to \pi^2 \left(\frac{4}{3\hat{v}^2} + \frac{3}{4\hat{v}^4} + \frac{1}{4\hat{v}^6} - \frac{7}{16\hat{v}^8} + \frac{49}{64\hat{v}^{10}} + \cdots\right)
\]
\[
W_{2\text{Fusnet},m_i} \to \pi^2 \left(-\frac{4}{3\hat{v}^2} - \frac{1}{\hat{v}^4}\right), \quad W_{2\text{Fusnet},D_2} \to \pi^2 \left(\frac{4}{3\hat{v}^2} + \frac{1}{\hat{v}^4}\right),
\] (5.21)

while $W_{2\text{Bouset},m_i}$ and $W_{2\text{Bouset},(D_2)}$ do not yield terms proportional to $\pi^2$. Note that the contributions in the last two lines precisely cancel each other, while the contributions of $W_{2\text{Bouset},m_i}$ and $W_{2\text{Fusnet},m_i}$ cancel each other beyond order $\frac{1}{\hat{v}^4}$ and leave (5.18) as a net result. Note in particular that even the leading term $\frac{\pi^2}{\hat{v}^2}$ receives both bosonic and fermionic contributions.

To extract the $\pi^2$ coefficient in the generalized scaling function $f_2(\ell)$, we then simply need to compute
\[
f_{2,\pi^2}(\ell) = \frac{F_2}{\sqrt{1 + \ell^2}} = \pi^2(3 + 4\ell^2) \frac{1}{12\ell^3\sqrt{1 + \ell^2}} = \pi^2 \left(\frac{1}{3\ell^3} + \frac{1}{12\ell^5} - \frac{1}{96\ell^7} + \cdots\right).
\] (5.22)

Again, we did not include the terms induced by the one-loop partition function as they cannot contain contributions proportional to $\pi^2$.

This answer can be compared directly to the asymptotic Bethe ansatz expression for $f_2$ derived in [20] which can be written as ($g = \sqrt{\frac{2}{3\pi}}$):
\[
f_2^{\text{BBA}} = \frac{16\pi^2}{\sqrt{\ell^2 + 1}} \left(\frac{2g^2\partial_\ell \tilde{F}^2(a_0)}{\sqrt{\ell^2 + 1}} - \frac{2g^2\tilde{F}^2(a_0)}{\ell^2 + 1} + 2g^2\delta F - \frac{5}{256\ell^6} + \frac{3}{64\ell^4} + \frac{1}{32\ell^2}\right)
\] (5.23)

Here $a_0 = \sqrt{1 + \ell^2}$ and we refer the reader to [20] for more details on the definition of the functions $\tilde{F}$ and $\delta F$. All the pieces in this formula can be analytically computed at large $\ell$. 

19
The first terms in this expansion are

$$\ell_2^{ABA} = \frac{\pi^2}{3} \frac{1}{\ell^3} + \left( \frac{32}{9} + \frac{\pi^2}{12} \right) \frac{1}{\ell^5} - \frac{232}{45} \frac{1}{\ell^6} + \frac{16}{5} \frac{1}{\ell^7} + \frac{20416}{1575} \frac{1}{\ell^8} - \left( \frac{3614}{1575} + \frac{\pi^2}{96} \right) \frac{1}{\ell^9} + \ldots$$

(5.24)

It turns out that the only relevant contributions to the $\pi^2$ coefficient arise from the last term in parenthesis in (5.23), i.e.

$$\delta F = \ldots + \frac{1}{\ell^3} \left( \frac{5}{512\ell^6} + \frac{1}{32\ell^6} + \frac{5}{192\ell^7} \right) + \ldots$$

(5.25)

Plugging these two expressions in equation (5.23) we reproduce our string theory result (5.22).

5.5. Non-renormalization of the leading terms in the large $\ell$ expansion

While the small $\ell$ expansion of the string theory result for $f(\ell, \lambda)$ should be compared with results of the all-loop Bethe ansatz expanded at strong coupling, the large $\ell$ expansion (or large $J$ 'BMN-type' expansion) makes contact with perturbative gauge theory results: as discussed in introduction, coefficients of the leading terms in this expansion may be protected, i.e. the same at strong and weak coupling.

On general grounds, the string energy is expected to have the expansion given in (1.18) with $j = \frac{\sqrt{2}}{\pi} \ell$. Rewritten in terms of $\ell$, the generalized scaling function in (1.18) takes the form

$$f(\ell, \lambda) = \frac{\pi}{\sqrt{\lambda}} f(\lambda, \ell) = \left( \ell + \frac{\pi^2 c_{10}}{\ell} + \frac{\pi^4 c_{20}}{\ell^3} + \ldots \right) + \frac{1}{\sqrt{\lambda}} \left( \frac{\pi^3 c_{11}}{\ell^2} + \frac{\pi^5 c_{21}}{\ell^4} + \ldots \right) + \frac{1}{\lambda} \left( \frac{\pi^4 c_{12}}{\ell^3} + \ldots \right).$$

(5.26)

The protected coefficients appear at one ($c_{10}, c_{11}, c_{12}$) and two ($c_{20}, c_{21}$) loops in gauge theory, while in string theory they appear at tree level ($c_{10}, c_{20}$), one loop ($c_{11}, c_{21}$) and two loops ($c_{12}$).

From tree-level and one-loop string results [14] we find

$$f_0 = \frac{\sqrt{1 + \ell^2}}{\lambda} = \ell + \frac{1}{2\ell} - \frac{1}{8\ell^3} + \ldots \rightarrow c_{10} = \frac{1}{2\pi^2}, \quad c_{20} = -\frac{1}{8\pi^4}$$

and

$$f_1 = \frac{\mathcal{F}_1(\ell)}{\sqrt{1 + \ell^2}} = -\frac{4}{3\ell} + \frac{4}{5\ell^3} + \ldots \rightarrow c_{11} = -\frac{4}{3\pi^3}, \quad c_{21} = \frac{4}{5\pi^5}.$$ 

(5.27)

On the gauge-theory side, the coefficients $c_{10}$ and $c_{11}$ were obtained from finite size corrections to the one-loop $sl(2)$ spin chain in [13]; the coefficients $c_{20}$ and $c_{21}$ were found from the analysis of the integral equation [16] for the generalized scaling function in [23].

Our results allow us to extract the expression for the term $\frac{\pi^2}{\ell^3}$ or (1.20), which is the leading two-loop contribution in the string sigma model.

As it turns out to be proportional to $\pi^2$, its computation is unambiguous (as discussed above, the shift of momenta performed in the fermionic sunset diagram does not affect $\pi^2$ terms). This coefficient can then be read off equation (5.22):

$$f_2 = \frac{\pi^2}{3\ell^3} + \ldots \rightarrow c_{12} = \frac{1}{3\pi^2}.$$ 

(5.28)

15 To be precise, since we do not have a complete handle on rational terms, in this paper we have not proven that at order $\frac{1}{\ell^3}$ there are no rational contributions coming from two-loop worldsheet diagrams. However, the full agreement with the ABA seen in [4] at small $\ell$ up to order $\ell^3$ strongly suggests that no such terms should be present.
The same result was obtained on the weakly coupled gauge-theory side (as a finite-size \(s(2)\) spin chain correction) in [22]. This provides the first direct check that the non-renormalization theorem for the leading terms in (5.26) at two-loop level in string theory\(^{16}\).

6. String finite size corrections: computations on \(\mathbb{R} \times S^1\) worldsheet

In the previous sections we discussed properties of the generalized scaling function in various limits of its argument \(\ell = \frac{-\pi J}{\sqrt{\ln S}}\). An interesting question is about finite size corrections in the case of small \(J\) which are proportional to \(\frac{1}{\ln S}\). As we will see in the following at the one loop order, such finite size corrections provide a sharp distinction between the \(\ell = 0\) and \(\ell \neq 0\) cases. The limit \(\ell \to 0\) of the latter should involve a resummation of infinitely many exponential corrections which may yield polynomial terms in \(\frac{1}{\ln S}\).

As already discussed in the introduction, a calculation of finite size corrections should potentially require use of the exact finite spin solution on a finite size worldsheet. This is indeed the case for the virtual scaling function \(h(\lambda, J)\) whose string theory evaluation, while possible on an \(\mathbb{R}^{1,1}\) worldsheet, requires use of the exact folded string solution. It was noted in [32], at one-loop order and for the leading \(\frac{1}{\ln S}\) correction, that use of the exact finite spin solution is not actually necessary and the correct result may be obtained by considering the folded string solution in its simplified scaling-limit form and using it in the \(\mathbb{R} \times S^1\) world sheet computation. We will assume that this short-cut applies also at higher loop orders as well.

Following [3, 4], in the previous sections we used the AdS light-cone gauge and the equivalence between a minimal surface describing a null cusp Wilson loop and the fast-spinning folded string in \(\text{AdS}_5 \times S^5\). A simple inspection of the spectra of fluctuations around the folded spinning string [14] and around the generalized cusp surface reveals that they are the same only up to a rescaling of worldsheet coordinates by a numerical factor. While this rescaling is not relevant on \(\mathbb{R}^{1,1}\) worldsheet, it should be accounted for on \(\mathbb{R} \times S^1\). Since it is the folded spinning string (dual to gauge theory twist operator) we are interested in, we will normalize the calculation to the closed string spectrum, even though we will formally use the same fluctuation action as in the ‘open string’ (cusp Wilson loop) case.

In sections 7 and 8, we will evaluate the leading finite size corrections to the energy of the folded string. We will comment along the way on the differences with the folded spinning string in the scaling limit with i.e. \(\ell \neq 0\). It was mentioned in [4] that for finite size systems differences may appear between the thermodynamic reasoning that led to expressions (3.9) and (3.10) for the quantum corrections to the target space energy and the calculation of the expectation values of the energy and spin operators. Below in section 7 we will show that no differences appear at the one-loop order.

Let us first comment on the map between the open and closed string normalizations and introduce the two-dimensional momenta on a cylindrical worldsheet with spatial length \(L\). By inspecting the open and closed string worldsheet volumes it is easy to see that the relation between them is given by (\(\beta \) is time interval)

\[
V \equiv \frac{1}{4} \int_{-\beta/2}^{+\beta/2} dt_o \int_0^{L_{\text{open}}} ds_o = 2\beta \ln S = \frac{1}{4} \int_{-\beta/2}^{+\beta/2} dt_c \int_0^{L_{\text{closed}}} ds_c. \tag{6.1}
\]

\(^{16}\) It is interesting to compare the terms which contribute to the leading coefficients \(c_{11}\) and \(c_{12}\) in the string expansion (5.26). The only contributions to \(c_{11}\) turn out to be coming from the AdS fluctuations \(\tilde{\phi}\) and \(\tilde{\psi}\) while as observed before (cf. discussion after equation (5.21)) this is not the case for \(c_{12}\). This is not necessarily in disagreement with the expectation that an effective Landau–Lifshitz model based on the AdS fluctuations should capture the leading protected terms in the expansion. Indeed, a calculation of \(c_{12}\) would be a two-loop one in the ‘one-loop’ (in gauge-theory sense) LL model and thus would require counterterms which would effectively take into account the contributions of other fluctuation fields.
From here it follows that the relation between coordinates is just

\[ (t, s)_{\text{open}} = 2(t, s)_{\text{closed}}, \quad p_{\text{open}} = \frac{1}{2} p_{\text{closed}}, \]  

(6.2)

In particular, the length of the open string worldsheet is twice that of the closed string worldsheet:

\[ L \equiv L_{\text{closed}} = \frac{1}{2} L_{\text{open}} = 2\pi \mu = 2 \ln S. \]  

(6.3)

Transformation (6.2) simply rescales the open string spectrum by a factor 4 which then becomes the spectrum of the fluctuations around the closed string background [2]. For \( J = 0 \) this consists of one field (\( \phi \)) with \( m^2 = 4 \), two fields (\( \tilde{x}, \tilde{x}^* \)) with \( m^2 = 2 \), five massless fields (\( y^a \)) and eight fermionic degrees of freedom with \( m^2 = 1 \).

In the calculation of the leading finite size corrections at one and two loops we will label momenta as \( p, q, r \), subject to momentum conservation \( p + q + r = 0 \). On a cylindrical world sheet the two components of momenta (\( p_0, p_1 \)) should be treated independently; the first is continuous while the second is discrete, being labeled by an integer:

\[ p_1 = \frac{2\pi}{L} n, \quad n \in \mathbb{Z}, \]  

(6.4)

with \( L = 2 \ln S \) being the length of the worldsheet cylinder. The two-dimensional loop momentum integration is therefore replaced by a one-dimensional integral over \( p_0 \) and a summation over the discrete values of \( p_1 \):

\[ \int d^2 p \rightarrow \int d p_0 \sum_{p_1} = \frac{2\pi}{L} \int d p_0 \sum_n. \]  

(6.5)

7. Leading finite size correction to the folded string energy at one-loop order

One-loop finite size corrections may be computed either in terms of the partition function (by directly applying the discussion in section 3) or by evaluating directly the expectation values of the energy and spin operators. We discuss both approaches in some detail and identify the precise origin of the leading \( \frac{1}{m_S} \) terms. The resulting observations will simplify the two-loop calculation in the next section by allowing us to focus only on a small set of terms.

7.1. Partition function approach

As discussed in section 3, the one-loop correction to the energy of the folded string is simply given by

\[ (E - S)_1 = \frac{1}{\beta} W_1, \quad W_1 = \frac{1}{2\pi} V F_1 = \frac{1}{2\pi} V (F_{1L=\infty} + \Delta F_1), \]  

(7.1)

where \( W_1 = - (\ln Z)_1 \) is the one-loop effective action, \( F_1 \) is the one-loop free energy, \( V \) is the worldsheet volume and \( \beta \) is the length of the non-compact worldsheet direction. Generalizing the expression in [2] in the long string limit to \( \mathbb{R} \times S^1 \) the one-loop free energy is given by

\[ F_1 = \frac{1}{2} \times \frac{2\pi}{L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d p_0 \sum_{n=0}^{\infty} \left[ \ln \left( p_0^2 + \left( \frac{2\pi n}{L} \right)^2 + 4 \right) + 2 \ln \left( p_0^2 + \left( \frac{2\pi n}{L} \right)^2 + 2 \right) + 5 \ln \left( p_0^2 + \left( \frac{2\pi n}{L} \right)^2 \right) - 8 \ln \left( p_0^2 + \left( \frac{2\pi n}{L} \right)^2 + 1 \right) \right]. \]  

(7.2)
Integrating by parts and noting that the integrand vanishes as $p_0^{-4}$ at large values of $p_0$ leads to

$$
\mathcal{F}_1 = -\frac{1}{L} \int_{-\infty}^{\infty} dp_0 \sum_n \left[ \frac{p_0^2}{p_0^2 + \left(\frac{2\pi n}{L}\right)^2} + 2 \frac{p_0^3}{p_0^2 + \left(\frac{2\pi n}{L}\right)^2} + 5 \frac{p_0^2}{p_0^2 + \left(\frac{4\pi n}{L}\right)^2} - 8 \frac{p_0^3}{p_0^2 + \left(\frac{4\pi n}{L}\right)^2} \right].
$$

(7.3)

Sums of this type have been discussed previously in [30] and are reviewed in appendix F. The sums have been discussed previously in [30] and are reviewed in appendix F.

As mentioned above, we ignore the terms that are independent of $\ln L$. Indeed, partial fractioning so that the resulting integral over $p_0$ is convergent, the massless contribution to $\mathcal{F}_1$ can be written as

$$
(\mathcal{F}_1)_{\text{massless}} = \frac{5}{L} \sum_{n=-\infty}^{\infty} \int dp_0 \frac{\left(\frac{2\pi n}{L}\right)^2}{p_0^2 + \left(\frac{2\pi n}{L}\right)^2} = \frac{20\pi^2}{L^2} \sum_{n=1}^{\infty} |n| \rightarrow \frac{4}{L^2} 5\pi^2 \zeta(-1).
$$

(7.6)

The other terms in equation (7.3) provide the necessary regularization of this sum. In the following we will use this observation to simplify the evaluation of the finite size corrections by first evaluating the integral over the continuous parameters and regularizing the resulting sums over $n$ using the zeta-function technique.

As a result, we find that the above expression for the partition function implies that the leading finite size correction to the energy of the long folded string is

$$
(E - S)_{\text{F}} = (E - S)_{1_{\text{L}}} + \Delta(E - S)_{\text{F}} = \frac{1}{\pi} \left[ -3 \ln 2 \ln S - \frac{5\pi^2}{12 \ln S} + O\left(\frac{\ln S}{S}\right) \right].
$$

(7.7)

### 7.2. Expectation value approach

As a test of the validity of thermodynamic arguments on finite-size worldsheets it is instructive to compute the one-loop finite size correction to the energy of the folded string by directly evaluating the expectation value of the energy operator [4]:

$$
E - S = \frac{\sqrt{\lambda}}{2\pi} \int_0^L ds [1 + 2\phi + (2\delta^+ + |x|^2)].
$$

(8.8)

As mentioned above, we ignore the terms that are independent of $\ln S$, whose evaluation requires use of the exact folded string solution, valid on a cylindrical worldsheet.
At the tree and one-loop level the expectation value of $E - S$ is

$$\frac{2}{L}(E - S)_0 = \frac{\sqrt{\lambda}}{\pi}, \quad \frac{2}{L}(E - S)_1 = \mathcal{E}_1 + \mathcal{E}_2$$

where

$$\mathcal{E}_1 = 2\langle \tilde{\phi} \rangle = -\int \frac{d\rho_0}{(2\pi)^2} \sum_{p_1} \left[ 4p_0^2 - p_1^2 \right] \frac{P(p, 0)}{P(p, 2)} - \frac{4 + 4p_1^2}{P(p, 0)} P(p, 0) + 2 \left( \frac{p_0^2 - p_1^2}{P(p, 0)} (p^2 + 2) + 4 \right] \frac{P(p, 0)}{P(p, 1)}$$

$$\mathcal{E}_2 = \int \frac{d\rho_0}{(2\pi)^2} \sum_{p_1} \left[ 4 \frac{P(p, 2)}{P(p, 4)} + \frac{4}{P(p, 0)} \right] P(p, m^2) \equiv p_0^2 + p_1^2 + m^2,$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the contributions of the tadpole and the quadratic term in equation (7.8), respectively.

These sums and integrals are of the same type as those appearing in the evaluation of the one-loop partition function (see appendix F). Choosing to first carry out the summation over the discrete component of the momentum we find that the leading $\ln S$ corrections are given by an expression analogous to equation (7.5). The complete contribution arises from $\mathcal{E}_1$, in particular, from the first and the third terms in equation (7.10) as these are the only ones containing massless propagators. The final result reproduces equation (7.7), confirming the validity of the thermodynamic arguments at the one-loop level.

### 7.3. Finite size corrections to the energy of folded spinning string with $J \neq 0$

The calculation in the two previous subsections may be extended without difficulty to the folded spinning string with an angular momentum $J$ on the $S^5$. We will again find that thermodynamic arguments still hold on a finite size worldsheet. We will also find that the limit $J \to 0$ is subtle: if taken in the final answer, it leads to a correction different from the one found above. We will discuss the origin of this difference.

Let us begin with the partition function approach. As was argued above, only the massless modes contribute to the $\frac{1}{m^2}$ terms. Here the light mode arises from the mixed $\tilde{\phi}$ and $\tilde{\phi}$ fields. The relevant part of the partition function is then

$$\mathcal{F}_1 \text{massless} = \frac{1}{2} \int \frac{d\rho_0}{2\pi} \frac{2\pi}{L} \sum_n \ln \text{det} K_{\tilde{\phi} \tilde{\phi}} = \frac{1}{2L} \int \frac{d\rho_0}{2\pi} \sum_n \ln \left[ p^2(p^2 + 4) + 4\tilde{\nu}^2 p_0^2 \right].$$

(7.12)

where $p^2 = p_0^2 + \left( \frac{2\pi n}{L} \right)^2$. Performing the integration over $\rho_0$, expanding to leading order in $L$ and replacing $\tilde{\nu} \to \ell$ we obtain for the leading finite size term:

$$\Delta \mathcal{F}_1 = \frac{4 \pi^2 \xi (-1)}{L^2} \frac{1}{\sqrt{1 + \ell^2}}.$$

(7.13)

From equations (3.5) and (3.9) it follows then that

$$\Delta (E - S)_1 = \frac{1}{\pi} \frac{\Delta \mathcal{F}_1}{\sqrt{1 + \ell^2}} \ln S = -\frac{\pi}{12} \frac{1}{1 + \ell^2} \frac{1}{\ln S}.$$

(7.14)

Note that the limit $\ell \to 0$ of this expression is different from the corresponding term in equation (7.7). The difference may be traced to the fact that, as $\ell \to 0$, four more massless

18 In what follows we will use the notation $(\cdot \cdot)_1 = \langle \cdot \cdot \rangle_1$. 24
modes emerge. They have been included in equation (7.7) but they produce only exponential corrections to equation (7.14) at $\ell \neq 0$. A resummation of these corrections should yield, in the limit $\ell \to 0$, the missing $-\frac{4}{\pi^2}$ contributions.

Let us now turn to the calculation of the expectation values of energy (7.8) and the angular momentum operators. We will first compute the expectation value of $E-\mathcal{S}$. Since $x$ is a massive field, we only need to compute the contributions proportional to the tadpole $\langle \hat{\phi} \rangle$ and to $\langle \hat{\phi}^2 \rangle$ which we called $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. As in the computation of the partition function, the only relevant contributions arise from the mixed fields $\hat{\phi}$ and $\hat{\phi}$. They are

\begin{align}
\mathcal{E}_1^{\text{massless}} &= -\tilde{k} \int \frac{d\nu_0}{(2\pi)^2} \sum_{\nu_1} \frac{2(p_0^4 - p_1^4) + 4(1 + 2\tilde{v}^2)\nu_0^2 - 4\nu_1^2}{p_0^2(p^2 + 4) + 4\nu_1^2 p_0^2}, \\
\mathcal{E}_2^{\text{massless}} &= \tilde{k} \int \frac{d\nu_0}{(2\pi)^2} \sum_{\nu_1} \frac{4\nu_0^2}{p_0^2(p^2 + 4) + 4\nu_1^2 p_0^2}.
\end{align}

Note that the latter contribution was not important in the $J = 0$ case but becomes relevant here due to the mixing of $\hat{\phi}$ and $\hat{\phi}$ which introduces a small-momentum singularity whose nature is slightly different from that of a regular massless field. Summing $\mathcal{E}_1^{\text{massless}}$ and $\mathcal{E}_2^{\text{massless}}$ and proceeding as above leads to the following finite size correction:

\begin{equation}
\Delta(E-\mathcal{S})_1 = \frac{2\pi(1+2\tilde{v}^2)\zeta(-1)}{(1+\tilde{v}^2) L^2} \ln S.
\end{equation}

To express $(E-\mathcal{S})_1$ in terms of $J$ it is necessary to compute the expectation value of the angular momentum $J$. Focusing again only on the terms which are sensitive to $1/L$ corrections, we have

\begin{equation}
J = \sqrt{\lambda} \int \frac{d\nu}{2\pi} \nu \mathcal{J}, \quad \mathcal{J} = \nu + 2\tilde{v} \hat{\phi} + 2\tilde{v} \hat{\phi}^2,
\end{equation}

leading to

\begin{equation}
\ell = \frac{2\pi}{\sqrt{\lambda} \ln S} \mathcal{J} = \nu + \frac{2\pi}{\sqrt{\lambda}} \frac{4\pi \nu \zeta(-1)}{\sqrt{1+\tilde{v}^2}} \frac{1}{L^2}.
\end{equation}

This can be inverted to express $\nu$ as a function of $\ell$

\begin{equation}
\nu \simeq \ell - \frac{2\pi}{\sqrt{\lambda}} \frac{4\pi \ell \zeta(-1)}{\sqrt{1+\ell^2}} \frac{1}{L^2} + \mathcal{O}(L^{-4}).
\end{equation}

Then

\begin{equation}
E - \mathcal{S} = (E - \mathcal{S})_0(\nu(\ell)) + (E - \mathcal{S})_1(\nu(\ell)) + \cdots, \quad (E - \mathcal{S})_0(\nu) = \sqrt{1 + \tilde{v}^2}.
\end{equation}

Using (7.20) and expanding to leading order we obtain for the finite size correction\(^{19}\)

\begin{equation}
\Delta(E - \mathcal{S})_1(\ell) = \frac{4\pi \zeta(-1)}{(1+\ell^2) L^2} \ln S = -\frac{\pi}{12} \frac{1}{(1+\ell^2) \ln S}.
\end{equation}

We have thus reproduced result (7.14) obtained in the partition function approach, supporting the expectation that the thermodynamic arguments are still valid for the leading finite size correction. We shall therefore use the free energy based approach also in the two-loop computation below\(^{20}\).

\(^{19}\) Note that the classical contribution $(E - J)_0$ has an effect on the one-loop result because of the one-loop expansion (7.20).

\(^{20}\) While we have not discussed explicitly the renormalization of the spin $\mathcal{S}$, it is possible to argue\(^{19}\) that corrections to it are suppressed by $S^{-n}$ factors and thus are subleading to the $\frac{1}{\pi^2}$ corrections we are interested in.
8. Leading finite size correction to the folded string energy at two-loop order

As we have seen above, only massless fields can yield \( \frac{1}{\ln S} \) finite size contributions and to compute them it suffices to first evaluate the integral(s) over the continuous component of the loop momentum and then evaluate the sum over the discrete momentum using zeta function regularization. Here we will follow this strategy in the two-loop calculation.

The relation between the two-loop partition function and the energy of the folded string follows from the discussion in section 3; the contributions to the former follow from Feynman diagrams with topologies shown in figures 1 and 2:

\[
W_2 = \beta E_2 = -(\ln Z)_2 = \frac{V}{2\pi \sqrt{\kappa}} F_2 = \beta \frac{\ln S}{\pi \sqrt{\kappa}} F_2
\]

\[
F_2 = -4\pi^2 \left( A_{\text{sunrise}}^{BBB} + A_{\text{double--bubble}}^{BB} + A_{\text{sunrise}}^{BF} + A_{\text{double--bubble}}^{BF} + A_{\text{double--bubble}}^{FF} + A_{\text{non--1PI}} \right)
\]

As discussed in [3, 4], for a non-compact worldsheet the partition function receives nontrivial contributions both from 1PI (figure 1) and non-1PI (figure 2), the role of the latter being to render the result finite. We shall see that the non-1PI graphs contribute nontrivially to finite size corrections as well.

The explicit expressions of the six terms in equation (8.2) are rather lengthy and are collected in appendix E. Their structure is determined by the topology of the Feynman graphs:

\[
A_{\text{sunrise}} = \int \frac{dp_0 dq_0 dr_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} \delta^{(2)}(p + q + r) \frac{f(p, q)}{P(p, m_1^2)P(q, m_2^2)P(r, m_3^2)} + \cdots
\]

\[
A_{\text{double--bubble}} = \int \frac{dp_0 dq_0 dr_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} \delta^{(2)}(p + q + r) \frac{g(p, q)}{P(p, m_1^2)P(q, m_2^2)}
\]

\[
A_{\text{non--1PI}} = \int \frac{dp_0 dq_0}{(2\pi)^4} \sum_{p_1, q_1} \frac{1}{2} T(p)T(q)
\]

where \( f(p_0, q_0) \) and \( g(p_0) \) have a polynomial dependence of degree 4 in \( p_0 \) and \( q_0 \).

\[
T(p) = + \frac{1 + \frac{5}{4} p_0^2 - \frac{5}{4} p_1^2 - \frac{1}{4} p_0^2 - \frac{1}{4} p_0^2}{P(p, 4)} + \frac{5}{4} P(p, 0) - 4 \frac{p_0^2}{P(p, 1)} + \frac{1 + p_0^2}{P(p, 2)}
\]

is the integrand of the one-loop tadpole for the field \( \tilde{\phi} \) and the ellipsis in the sunset contribution stand for terms with one canceled propagator. Such terms combine naturally with those arising from the double-bubble topology.

In obtaining the contributions listed in appendix E we have discarded power-like divergences in the continuous momentum integral. The various terms have been organized such that the summation over the space-like momenta is manifestly finite. The integral over the continuous (Euclidian time-like) momenta produces all the divergences which should cancel out when all integrals are added up.

The one-loop calculation described above suggests that only diagrams with at least one massless field (i.e. at least one factor \( P(p, 0) = p^2 \)) can yield a polynomial dependence in \( \frac{1}{\ln S} \). To demonstrate that this is indeed the case let us briefly discuss the \( L \)-dependence of the integrals that can appear in the two-loop partition function.
8.1. On the $L$-dependence of two-loop integrals

From equations (8.3) and the explicit expressions in appendix E it is clear that the integrals that enter the calculation of the two-loop partition function fall into two classes: (a) iterated one-loop integrals, and (b) sunset-type integrals involving three propagators.

Integrals of the first type are, up to numerator factors, similar to the integrals that enter the one-loop partition function. As in that case, a polynomial dependence in the inverse length of the string can arise only if the integrand is generated by a massless field. The precise $L^{-1}$ dependence of the result depends strongly on the numerator factors; these factors determine whether only one or both integral factors yield such contributions. For example, using the summation formulae in appendix F it is easy to see that for $m \geq 1$

$$
\int_{-\infty}^{+\infty} dp_0 \sum_{p_1} \frac{p_0^m}{p_0^2 + p_1^2} = \pi \int_{-\infty}^{+\infty} dp_0 p_0^{m-1} \coth \left( \frac{1}{2} L p_0 \right) = (\text{divergent}) + \frac{1}{L^m} \times (\text{finite}).
$$

(8.5)

Thus, the leading finite size contribution of the product of two such integrals contains one of the integrals evaluated in the $L \to \infty$ limit.

In the sunset-type two-loop integrals, the sums over the discrete components of momenta are generically of the type

$$
S(a, b, c) = \sum_{m,n} \frac{1}{\left[ \left( \frac{2\pi n}{L} \right)^2 + a^2 \right] \left[ \left( \frac{2\pi m}{L} \right)^2 + b^2 \right] \left[ \left( \frac{2\pi (n+m)}{L} \right)^2 + c^2 \right]}
$$

(8.6)

for some typically different $a, b, c$ with $a^2 = p_0^2 + m_0^2, b^2 = q_0^2 + m_0^2$ and $a^2 = (p_0 + q_0)^2 + m_0^2$. If all masses are nonvanishing, $m_{a,b,c} \neq 0$, such sums may be evaluated by a repeated application of the contour integral trick of [30]. We choose two copies of a contour that runs parallel to the real axis above and below it and write $S(a, b, c)$ as

$$
S(a, b, c) = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_-} dz \int_{\mathcal{C}_+} dy \frac{\cot \pi z \cot \pi y}{\left[ \left( \frac{2\pi z}{L} \right)^2 + a^2 \right] \left[ \left( \frac{2\pi y}{L} \right)^2 + b^2 \right] \left[ \left( \frac{2\pi (z+y)}{L} \right)^2 + c^2 \right]}. \tag{8.7}
$$

An integrand with suitable properties a contour deformation argument implies that the sum is given by the residues of the purely imaginary poles given by the rational part of the integrand. In these residues, $a, b, c$ or some combination thereof will appear in the argument of the cot function; moreover, since these poles occur at purely imaginary values of $p_0$ and $q_0$, the cot function will in fact become coth.21 We conclude that, if all propagators are massive (i.e. if $m_{a,b,c} \neq 0$), the $L$ dependence is exponentially suppressed. A slightly more involved analysis is necessary if some masses are equal (but nonvanishing); the conclusion, however, is unchanged. In appendix F we illustrate this conclusion by numerically evaluating the integral $I[\{1, \frac{1}{2}, \frac{1}{2}\}]$ which, in the $L \to \infty$ limit, yields the complete two-loop energy, see figure 4.

It therefore follows that, among all sunset-type integrals, only those with at least one massless field can yield a polynomial dependence on $\frac{1}{\ln L}$. For the purpose of finding the leading finite size corrections to the energy of the folded string it suffices to focus our attention only on these contributions. The other terms are, of course, crucial to guarantee the cancellation of UV divergences.

Let us note also that the arguments above require first to evaluate the sum over the discrete momentum. The $L$-dependence, however, is not expected to change if we first carry out the integral over the continuous momentum components in the presence of a suitable regulator

21 The $L \to \infty$ limit should be taken with care if the numerator polynomial has a high degree, as this makes the integral divergent. Formally, this limit amounts to the formal replacement $\coth(\cdot) \mapsto \text{sgn}(\cdot)$.
Figure 4. The plots of the massive two-loop integral $I[1, 1/2, 1/2]$ (blue) and $\frac{1}{L^2} + \frac{K^4}{\pi^2}$ (red) as function of $L$. $I[1, 1/2, 1/2]$ quickly approaches the continuum limit $\frac{K^4}{\pi^2}$.

(i.e. a regulator which does not depend on $L$). For example, it is possible to verify that

$$\int_{-\Lambda_1}^{\Lambda_1} \frac{1}{p_0 + p_1^2 + m^2} = \int_{-\infty}^{\infty} \frac{1}{p_0 + p_1^2 + m^2}$$

for suitable cutoffs $\Lambda_1$ and $\Lambda_2$. Thus, the formal consideration about the exponential suppression of all-massive sunset-type integrals should also hold if the integration over the continuous variables is performed first. We adopt this technically simpler strategy in our two-loop calculation; as seen at the one-loop level, the zeta-function regularization of the resulting sums should then yield the correct result.

8.2. Contribution of massless integrals

The discussion in the previous subsection implies that the only terms from appendix E which potentially contribute to the finite size correction in the two-loop effective action involve at least one massless field. Here and in appendix E we denote by $B_{m_1^2, m_2^2}$ and $B_{m_1^2, m_2^2, m_3^2}$ the terms in the integrand of the bosonic sunset and double-bubble diagrams with masses as indicated; similarly $C_{m_1^2, m_2^2}$, $F_{m_1^2, m_2^2, m_3^2}$ and $A_{m_1^2}^{\text{non-1PI}}$ denote, respectively, the terms from the mixed bosonic-fermionic double-bubble, fermionic sunset and non-1PI diagrams 22.

With this notation and after accounting for the various cancellations discussed in appendix E, the only terms in the two-loop integrand which may yield $O(1/L^2)$ contributions upon integration over the continuous momenta and summation over the discrete ones are

$$B_{0,0,4} = -\frac{5}{4} \frac{1}{P(p, 0) P(q, 0)} + \frac{5}{2} \frac{1 - 2p_0^2}{P(p, 0) P(r, 4)} + \frac{5}{P(p, 0) P(q, 0) P(r, 4)} (1 + p_0 q_0)^2$$

$$F_{0,1,1} + C_{0,1} = \frac{10 p_0 q_0 p_0^2}{P(p, 1) P(q, 1) P(r, 0)}$$

$$A_{0}^{\text{non-1PI}} = \frac{5 p_0^2 - p_0^2}{4 P(p, 0)} T(q).$$

To identify the relevant massless contribution coming from the non-1PI term, which is proportional to $T(p)T(q)$, we used the observation following from the calculation in 22 Purely fermionic double-bubble diagrams are identically vanishing; even if they were not, they could contribute only exponentially suppressed terms, as all fermions are massive.
section 7.2 that the leading finite size correction to the tadpole term is already of the desired $O(L^{-2})$ order. Thus, in the product $T(p)T(q)$ we need to keep the massless contribution from only one of the two tadpole graph factors, while the other one can be treated in the $L \to \infty$ limit.

Except for the third term in $B_{0.0.4}$ and for $F_{0.1.1} + C_{0.1}$ all other terms factorize into a product of one-loop integrals which may easily be evaluated using $(p^2 = p_0^2 + p_1^2)$:

\begin{align}
\int_{-\infty}^{+\infty} dp_0 \frac{1}{p^2 + m^2} &= \frac{\pi}{\sqrt{p_0^2 + m^2}} \\
\int_{-\infty}^{+\infty} dp_0 \frac{p_0^2}{p^2 + m^2} &= \int dp_0 \left( 1 - \frac{p_0^2 + m^2}{p^2 + m^2} \right) \implies -\pi \sqrt{p_0^2 + m^2}.
\end{align}

In the second integral above we discarded a linearly-divergent term; such terms are analogous to quadratically divergent terms which are discarded in the $L \to \infty$ calculation\textsuperscript{23}. They should be canceled by contributions of the path integral measure (or discarded using analytic regularization).

To carry out the integrals of products of three propagators it is useful to first Fourier-transform the integrals over the zeroth component of momenta to position space:

\begin{align}
\int_{-\infty}^{+\infty} dp_0 \frac{e^{ip_0 x}}{p_0^2 + p_1^2 + m^2} &= \frac{\pi}{\sqrt{p_0^2 + m^2}} e^{-|x|\sqrt{p_0^2 + m^2}}.
\end{align}

In this form, the numerator factors depending on the 0th component of momenta are realized as derivatives with respect to the position variable. The three relevant two-loop integrals, with constant numerator and with a numerator bilinear in the integration variables, are evaluated in appendix G. Now we will discuss the evaluation of the integrals of $B_{0.0.4}$, $F_{0.1.1} + C_{0.1}$ and the non-1PI contribution.

8.2.1. $B_{0.0.4}$.

Applying the strategy described above to the integral of $B_{0.0.4}$ leads to

\begin{equation}
\int_{-\infty}^{+\infty} dp_0 dq_0 B_{0,0.4} = \frac{1}{4} \left[ - \frac{15}{2} \pi^2 + \frac{5}{2} \pi q_1 |q_1| + \frac{5 \pi^2 p_1 \text{sgn}(q_1)}{\sqrt{4 + (p_1 + q_1)^2}} - 5 \pi^2 |q_1| \left( \frac{p_0^2 + p_1 q_1 + 1}{\sqrt{4 + (p_1 + q_1)^2}} \right) \right].
\end{equation}

(8.15)

To obtain this expression we performed some convenient relabelling of the discrete momenta $p_1$ and $q_1$. We also dropped terms which are odd in the discrete momenta and therefore vanish after summation over a symmetric domain.

In the last two terms, the remaining sums over the discrete momenta $p_1$ and $q_1$ are coupled due to the presence of nontrivial denominators. They may be decoupled by shifting one of the summation variables, e.g. $p_1 \to p_1 - q_1$. Since these sums are clearly divergent, such manipulations should be treated with care. It is not hard to check that, with some regulator $R(p)$,

\begin{equation}
\sum_{p_1} R(p_1) \left[ \frac{(p_1 + q_1)^2}{(p_1 + q_1)^2 + 4} - \frac{p_1^2}{p_1^2 + 4} \right] = q_1^6 \mathcal{O}(L^{-1}) = \mathcal{O}(L^{-(n+1)})
\end{equation}

(8.16)

\textsuperscript{23} Indeed, in the $L \to \infty$ limit integral (8.13) is just $\int d^2 p \frac{\hat{p}_0^2}{\hat{p}^4 m^2} = \frac{1}{2} \int d^2 p \frac{\hat{p}^2}{\hat{p}^4 m^2} \iff -\frac{m^2}{2} \int \frac{d^2 p}{\hat{p}^4 m^2}.$
for all exponents $n \neq 0$. The result depends strongly on the regulator $R$; however, the $L$ dependence is such that this difference is of too high an order in $L^{-1}$ to contribute to the leading $L^{-2}$ correction. Analyzing separately a constant numerator factor of the last term in equation (8.15) (which corresponds to the $n = 0$ terms in equation (8.16)) shows that the shift $p_1 \to p_1 - q_1$ does not affect the value of the sum either.

Decoupling the sums in the last two terms in (8.15) by appropriately shifting the summation variables leads to

$$
\sum_{p_1 \cdot q_1} B_{0,0,4} = \frac{1}{4} \sum_{p_1 \cdot q_1} \left( -\frac{15}{2} \pi^2 + \frac{5}{2} \pi^2 |p_1||q_1| - 5 \pi^2 |q_1| \frac{p_1^2 + 2}{\sqrt{4 + p_1^2}} \right). \tag{8.17}
$$

In deriving this expression we further discarded terms which are odd in the discrete momenta and thus vanish when summed over a symmetric domain. An example illustrating the discarded terms is the following:

$$
\sum_{p_1 \cdot q_1} \frac{p_1}{\sqrt{4 + (p_1 + q_1)^2}} = \frac{1}{2} \sum_{p_1 \cdot q_1} \frac{p_1 + q_1}{\sqrt{4 + (p_1 + q_1)^2}} = \frac{1}{2} \sum_{p_1 \cdot r_1} \frac{r_1}{\sqrt{4 + r_1^2}}, \tag{8.18}
$$

where we discarded a term odd under the interchange of $p_1$ and $q_1$. The remaining sum over $r_1$ also vanishes since the summand is odd. Alternatively, we can use the zeta-function regularization, which we assume, to show that the sum over $p_1$ can be set to zero. Indeed,

$$
\sum_{p_1} 1 = \frac{2 \pi}{L} \sum_{n=-\infty}^{\infty} 1 = \frac{2 \pi}{L} \left( 1 + 2 \sum_{n=1}^{\infty} 1 \right) = \frac{2 \pi}{L} (1 + 2 \zeta(0)) = 0. \tag{8.19}
$$

As in the integral in equation (8.13), such manipulations are similar to discarding quadratic divergences in two-loop integrals in the $L \to \infty$ limit.

With this prescription the constant term in (8.17) vanishes. It is easy to see that if in the second term we take both sums to contribute $L^{-1}$ terms, then the result is of order $L^{-4}$ and thus too high an order. It follows therefore that one of the sums should be evaluated in the $L \to \infty$ limit:

$$
\sum_{p_1 \cdot q_1} |p_1||q_1| = 2 \left( \frac{2 \pi}{L} \right)^2 2 \zeta(-1) \int_{-\infty}^{+\infty} |q_1| \, dq_1. \tag{8.20}
$$

This integral is a pure quadratic divergence, similar to other quadratically divergent integrals which have been discarded; thus, it may be discarded as well.

For the last term in (8.17) we observe that carrying out the sum over $q_1$ already yields a term of order $1/L^2$:

$$
-5 \pi^2 \sum_{p_1 \cdot q_1} |q_1| \frac{p_1^2 + 2}{\sqrt{4 + p_1^2}} = -5 \pi^2 \sum_{p_1 \cdot q_1} \frac{2 \pi |m|}{L} \sum_{p_1} \frac{p_1^2 + 2}{\sqrt{4 + p_1^2}} = 10 \pi^2 \left( \frac{2 \pi}{L} \right)^2 \zeta(-1) \sum_{p_1} \frac{p_1^2 + 2}{\sqrt{4 + p_1^2}} \tag{8.21}
$$

It is therefore appropriate to approximate the remaining sum over $p_1$ with the corresponding integral $^24$. We can therefore write the contribution of $B_{0,0,4}$ as

$$
\int_{-\infty}^{+\infty} dp_0 \, dq_0 \sum_{p_1 \cdot q_1} B_{0,0,4} = -\frac{5 \pi^2}{2} \left( \frac{2 \pi}{L} \right)^2 \zeta(-1) I_B, \quad I_B \equiv \int_{-\infty}^{+\infty} dp_1 \frac{p_1^2 + 2}{\sqrt{4 + p_1^2}}. \tag{8.22}
$$

$^24$ Alternatively, one may argue that the difference between the sum and the integral is exponentially suppressed due to the mass-like constant term in the denominator.
By power-counting $I_B$ is quadratically divergent; a closer inspection reveals that it does not contain logarithmic divergences. We will postpone its discussion until we analyze other terms contributing to the leading finite size correction.

8.2.2. $F_{0,1,1} + C_{0,1}$.

The integrals over the continuous momentum components $p_0$ and $q_0$ for the term $F_{0,1,1} + C_{0,1}$ are very similar to those appearing in $B_{0,0,4}$; the result is

$$
\int_{-\infty}^{+\infty} dp_0 \, dq_0 (F_{0,1,1} + C_{0,1}) = 5 \pi^2 \left( 1 - p_1 q_1 + 2 q_1 \sqrt{1 + p_1^2 \text{sgn}(p_1 + q_1)} - \sqrt{1 + p_1^2 \sqrt{1 + q_1^2}} \right).
$$

(8.23)

The sum over $p_1$ and $q_1$ of the first two terms vanishes due to zeta-function regularization and summation over a symmetric domain while the third term can be argued to contain only exponential dependence on $L$ and may therefore be ignored. The remaining contribution, after shifting $q_1$ and dropping a term odd under $p_1 \to -p_1$, becomes

$$
\int_{-\infty}^{+\infty} dp_0 \, dq_0 \sum_{p_1, q_1} (F_{0,1,1} + C_{0,1}) = 10 \pi^2 \left( \frac{2 \pi}{L} \right)^2 \sum_m \sum_{p_1} \sqrt{1 + p_1^2} 
$$

$$
= 20 \pi^2 \left( \frac{2 \pi}{L} \right)^2 \zeta(-1) I_F,
$$

(8.24)

$$
I_F = \int_{-\infty}^{+\infty} dp_1 \sqrt{1 + p_1^2}.
$$

Similarly to $I_B$ in $B_{0,0,4}$, the integral $I_F$ is quadratically divergent; unlike $I_B$, however, $I_F$ also exhibits subleading logarithmic divergences which cannot be removed by, e.g., an analytic regularization scheme. As we shall see, these divergences cancel out once all finite-size contributions are combined.

8.2.3. Non-1PI.

The finite size contributions of this type arise entirely from the factor

$$
T_0 = \int_{-\infty}^{+\infty} dp_0 \sum_{p_1} \frac{5 p_0^2 - p_1^2}{4 p_0^2 + p_1^2}.
$$

(8.25)

In the continuum limit this term can be neglected since it clearly vanishes due to the $p_0 \leftrightarrow p_1$ antisymmetry of the integrand. As was noted in the calculation of the expectation value of the energy operator at one loop, this is no longer so once $p_1$ is discrete. Following the same steps as for the evaluation of the contributions of $B_{0,0,4}$ and $F_{0,1,1} + C_{0,1}$ and carrying out first the integral over $p_0$ we find

$$
T_0 = \frac{5}{4} \sum_{p_1} \int_{-\infty}^{+\infty} dp_0 \left( 1 - 2 \frac{p_1^2}{p_0^2 + p_1^2} \right) = -\frac{5}{2} \sum_{p_1} \int_{-\infty}^{+\infty} dp_0 \frac{p_1^2}{p_0^2 + p_1^2}
$$

$$
= -\frac{5}{2} \pi \sum_{p_1} |p_1| = -5 \pi \left( \frac{2 \pi}{L} \right)^2 \zeta(-1),
$$

(8.26)

where one factor $\frac{2 \pi}{L}$ arises from the definition of the summation over $p_1$ in equation (6.5) while the second one from the definition $p_1 = \frac{2 \pi}{L}$ in equation (6.4). The constant term on the first line was discarded due to zeta-function regularization (see equation (8.19)) and also because its $p_0$ integral is linearly divergent.
To complete the calculation we should evaluate the continuum analog of this tadpole contribution following the same steps as in the discrete version of the calculation. Carrying first the \( q_0 \) integral we find that
\[
\int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_0 T(q) = \pi \int_{-\infty}^{+\infty} dq_1 \left( -\frac{5}{2} |q_1| + 4\sqrt{q_1^2 + 1} - \frac{1}{2} \frac{q_1^2 + 2}{\sqrt{q_1^2 + 4}} - \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} \right). 
\]

The first term is a pure quadratic divergence similar to (8.20) and other quadratic divergent integrals which have been discarded; we will discard it as well. The contribution of the non-1PI graphs is therefore
\[
\int_{-\infty}^{+\infty} dp_0 dq_0 \sum_{p_1, q_1} A^{\text{non-1PI}} = -5\pi^2 \left( \frac{2\pi}{L} \right)^2 \zeta(-1) \left[ 4T_F - \frac{1}{2} I_B - \int_{-\infty}^{+\infty} dq_1 \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} \right]. 
\]

(8.28)

8.3. Summing up

We are now in position to assemble the leading \( \frac{1}{m^2} \) term in the two-loop correction to the energy of the folded string. Combining equations (8.22), (8.24) and (8.28) and reconstructing the finite size correction \( \Delta F_2 \) to the free energy as defined in equation (8.2) we find that the divergent integrals \( I_F \) and \( I_B \) cancel out and the leading finite size correction to the free energy is
\[
\Delta F_2 = -5 \left( \frac{2\pi}{L} \right)^2 \zeta(-1) \int_{-\infty}^{+\infty} dq_1 \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}}. 
\]

(8.29)

The remaining integral in \( \Delta F_2 \) is clearly divergent. It is, however, free of logarithmic divergences as these cancel out in a nontrivial way between various contributions to \( \Delta F_2 \). The result then depends on how we deal with the remaining quadratic divergences.

It is easy to see that the quadratic divergence in equation (8.29) is of the type \( \int dq_1 |q_1| \), i.e. it is of the same nature as quadratic divergences that have been discarded in the calculation in the \( L \to \infty \) limit; they are also similar to divergences that have been discarded in the process of reorganizing the integrands of the two-loop Feynman integrals. One option then is to discard them here as well by simply replacing
\[
\int_{-\infty}^{+\infty} dq_1 \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} \rightarrow \int_{-\infty}^{+\infty} dq_1 \left( \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} - |q_1| \right) = 1. 
\]

(8.30)

If we adopt this prescription\(^{26}\) we end up with\(^{27}\)
\[
\Delta F_2 = \frac{5\pi^2}{12 \ln^2 S}. \quad (E - S)_2 = \frac{1}{\pi} \left( -\ln S + \frac{5\pi^2}{12 \ln S} \right). 
\]

(8.31)

\(^{25}\) Here we have restored a factor of \( \frac{1}{12 \pi^2} \) coming from the loop momentum integration. An additional multiplicative factor of \( (-16\pi^2) = (-4\pi^2) \times (4) \) arises from the definition of \( F_2 \) in equation (8.2) and from the definition of \( A_{\text{BIB}}^{\text{BB}} \), \( A_{\text{double--bubble}}^{\text{BB}} \), \( A_{\text{double--bubble}}^{\text{BB}} \), \( A_{\text{double--bubble}}^{\text{BB}} \) and \( A_{\text{non-1PI}} \) in appendix E.

\(^{26}\) Note that subtracting the quadratic divergence as \( \int_{-\infty}^{+\infty} dq_1 \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} \rightarrow \int_{-\infty}^{+\infty} dq_1 \left( \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} - 1 \right) \) is not valid, as it artificially introduces a logarithmic divergence.

\(^{27}\) As in (3.1) here we define \( (E - S)_a \) without the explicit loop-counting factor \( \frac{1}{12 \pi^2} \). Also, at one loop (7.7), we ignore the \( \ln S \)-independent term.
This result, however, may seem strange: such an evaluation of the integral in (8.29) leads to a departure from the pattern of transcendentality of coefficients noticed at one loop order: while there the coefficient of the finite size correction had one additional unit of transcendentality compared to the leading term (i.e. \( \pi^2 \) versus \( \ln 2 \) in (7.7)), the corresponding coefficients in the candidate two-loop expression (8.31) have the same transcendentality (\( \pi^2 = 6 \zeta(2) = 6 \sum_{n=1}^{\infty} \frac{1}{n^2} \) versus \( K = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2} \)). This observation may be considered as a hint that a different definition of the integral in (8.29) may be more appropriate.

Note that the term surviving in (8.29) is the last term in equation (8.27) which is nothing but the \( q_0 \) integrals of the fourth term in \( T(q) \), see equation (8.4). Interpreting it this way and evaluating the integral using the two-dimensional Lorentz invariance of the denominator it is easy to see that, up to quadratic divergences, this integral vanishes when evaluated in the ‘decompactified’ (continuum spatial momentum) case:

\[
\int \frac{d^2q}{q^2_0 + 1} = \int \frac{d^2q}{q^2 + 2} = \frac{1}{2} \int d^2q \rightarrow 0.
\]  

This then suggests that the integral in (8.29) should not have a finite part after the quadratic divergences are subtracted out

\[
\Delta F_2 \propto \int dq_1 \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}} \rightarrow 0.
\]  

This prescription then implies the vanishing of the leading finite size two-loop correction to the energy of the folded string.

The values (8.30) and (8.33) may be interpreted as corresponding to different regularization schemes, each preserving different amount of symmetries. Carrying out the momentum integrals iteratively obscures the fact that in the \( \ln S \to \infty \) limit the quadratic part of the action is invariant under the 2D Lorentz transformations. The second prescription corresponds to insisting on that symmetry in the limit \( \ln S \to \infty \). A bonus is that, as a result, one avoids violation of the pattern of transcendentality of coefficients observed at one loop.

Let us finally comment on the case of \( J \neq 0 \). At one-loop order we saw in detail that turning on a non-zero value of angular momentum on \( S^5 \) exposes the fact that part of the leading finite size corrections at \( J = 0 \) arises from the resummation of infinitely many exponentially small corrections at \( J \neq 0 \) (with \( \ell \) held fixed). From the Bethe ansatz standpoint such exponential terms may be interpreted as ‘Lüscher’ corrections (or wrapping corrections in gauge theory). A similar picture is expected at two loops: it would be interesting to identify in the two-loop calculation the terms that become exponentially suppressed as \( J \) is switched on. The structure of the \( J = 0 \) result (8.29) and an analogy with the one-loop case (the overall coefficient 5, which is related to the number of massless fields, replaced by 1) suggests that the leading term in the small \( \ell \) expansion of \( \Delta F_2 \) for \( J \neq 0 \) should be

\[
\Delta F_2 = - \left( \frac{2\pi}{L} \right)^2 \zeta(-1) \int_{-\infty}^{+\infty} dq_1 \frac{q_1^2 + 1}{\sqrt{q_1^2 + 2}}.
\]  

The final numerical value depends again on a regularization prescription used to subtract quadratic divergences and is thus zero if we adopt the ‘2D Lorentz-invariant’ prescription in (8.33).

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Appendix A. Long folded spinning string: one-loop finite size corrections and Landau–Lifshitz model

Let us start with a review of the form of the one-loop correction to the energy of the long folded \((S, J)\) string \cite{14}. In this case \(\mu = \frac{1}{\pi} \ln S \to \infty\) with \(\ell \equiv \frac{J}{\mu} = \text{fixed}\) and

\[
E_1 = \frac{1}{\kappa} E_{2D} = \frac{1}{\mu \sqrt{1 + \ell^2}} E_{2D},
\]

\[
E_{2D} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \mathcal{O}_{n-} + \mathcal{O}_{n+} + 2\sqrt{n^2 + (\ell^2 + 2)\mu^2} + 4\sqrt{n^2 + \ell^2 \mu^2} - 8\sqrt{n^2 + (\ell^2 + 1)\mu^2} \right]
\]

where

\[
\mathcal{O}_{n \pm} = \sqrt{n^2 + 2\mu^2 (1 + \ell^2) \pm 2\mu \sqrt{n^2 \ell^2 + \mu^2 (1 + \ell^2)^2}} \tag{A.2}
\]

are the contributions of the two ‘mixed’ AdS3 modes. We would like to determine the leading contribution to \(\mu^{-n}\) corrections coming from this sum over characteristic frequencies. It is easy to see that for non-zero \(\ell\) the massive modes give sums of exponential corrections but there is one special mode that becomes light in the \(\mu \to \infty\) limit: this is the lighter of two AdS3 modes in (A.2), i.e.

\[
\mathcal{O}_{n-} = \frac{n}{\sqrt{1 + \ell^2}} \left[ 1 + \frac{n^2 \ell^4}{8\mu^2 (1 + \ell^2)^2} + O\left(\frac{1}{\mu^4}\right) \right]. \tag{A.3}
\]

As a result, the leading \(\frac{1}{\mu}\) contribution to the one-loop correction to the energy comes from the first term in (A.3):

\[
(E_1)_{\frac{1}{\mu}} = \frac{1}{2\mu (1 + \ell^2)} \sum_{n=-\infty}^{\infty} n = -\frac{1}{12} \frac{1}{\mu (1 + \ell^2)} = -\frac{1}{12\pi} \frac{\lambda \ln S}{J^2 + \frac{J^2}{\pi} \ln^2 S}, \tag{A.4}
\]

where we used that \(\frac{1}{\mu} \sum_{n=-\infty}^{\infty} n = \zeta(-1) = -\frac{1}{12}\). Since the sum in \(E_{2D}\) in A.1 is UV finite, one may interchange summation over \(n\) with taking the large \(\mu\) limit and the use of the \(\zeta\)-function regularization is just a short-cut to extract the relevant term in that finite sum.

If we take \(\ell \ll 1\) or \(J^2 \ll \frac{1}{\pi} \ln^2 S\) we get \((E_1)_{\frac{1}{\mu}} = -\frac{1}{2\ell} \frac{\pi}{\ln^2 S}\) which is the ‘non-wrapping’ (from the BA point of view \cite{41}) part of the total string coefficient \(-\frac{5}{12}\) \cite{30,32} found for the \(\frac{1}{\mu^2}\) coefficient in the limit when \(J\) can be ignored. The distinction between the ‘non-wrapping’ and ‘wrapping’ contributions becomes clear for nonzero \(J\): to recover the extra \(-\frac{4}{12}\) contribution from four \(S^5\) modes that become massless in the strict \(J = 0\) limit we need to resum the exponential (‘Lüscher’) contributions corresponding to them before taking the large \(\mu\) limit\(^{28}\).

\(^{28}\) It is only in the massless or conformal limit that the contribution of a 2D mode is given by the Casimir effect on a cylinder, i.e. is proportional to \(-\frac{1}{12} \frac{\pi}{L^2}\) where \(T\) is the time interval and \(L\) is the length of the spatial circle.
As discussed in the introduction, the analytic dependence of \((A.4)\) on \(\lambda\) suggests that the order \(\lambda\) term there is not renormalized, i.e. its value is the same also at weak coupling. Then it can be reproduced as a one-loop correction in the corresponding Landau–Lifshitz model. This is the aim of this appendix.

Here we shall follow [36] and [37] (appendix D there). The semiclassical states from \(sl(2)\) sector correspond to strings rotating in \(AdS_3\) part of \(AdS_5\) and whose center of mass is moving along the big circle of \(S^3\), i.e. their energy depends on the two spins \((S, J)\). The fast string limit is when \(J\) is large with \(\tilde{\lambda} = \frac{J}{L}\) being fixed. On the gauge theory side we assume \(J\) is large and consider only the one-loop (order \(\lambda\)) term in the dilatation operator. In the previous discussions it was assumed that \(S/J\) is fixed in this limit but as we shall see below the LL description captures also the case when \(\ell\) or \(\tilde{\lambda}\) \(\ln^2 S\) is fixed\(^{29}\). The corresponding LL action \([35]\) derived from \(sl(2)\) spin chain Hamiltonian (or from the bosonic string action in \(AdS_3 \times S^1\) by fixing an analog of the static gauge \([34, 36]\)) is

\[
I = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} L, \quad L = -2 \sinh^2 \rho \eta - \frac{1}{2}(\rho^2 + \sinh^2 2\rho \eta^2), \quad \tilde{\lambda} = \frac{\lambda}{J^2} = \frac{1}{J^2}.
\]  

\(A.5\)

Here \(\rho\) and \(\eta = \frac{1}{\sqrt{2}}(t - \phi)\) are combinations of the \(AdS_3\) coordinates: \(ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2\).

The folded string solution is given by \(t = \kappa \tau, \phi = w \tau, \rho = \rho(\sigma), \varphi = \nu \tau,\) to leading order in the \(1/J\) expansion, the corresponding solution of the LL equations is

\[
\eta = \omega \tau, \quad \omega = \frac{1}{2}(\kappa - w), \quad \rho^2 + 2w \sinh 2\rho = 0, \quad \rho \equiv \frac{w}{\kappa}. \quad (A.6)
\]

\[\rho^2 = 2w(\cosh 2\rho_0 - \cosh 2\rho), \quad 0 < \rho < \rho_0. \quad (A.7)\]

As discussed in [37], one may follow the same steps as in the \(SU(2)\) sector and derive the Lagrangian for small fluctuations of \(\rho\) and \(\eta\) near the given solution:

\[
\bar{L} = 2g \bar{f} - \frac{1}{2} \bar{\lambda}[g^2 + f'^2 + 4w(3 \cosh 2\rho - 2 \cosh 2\rho_0)g^2 + 4w \cosh 2\rho f^2].
\]  

\(A.8\)

Here \(f\) and \(g\) are properly redefined fluctuation fields, i.e. linear combinations of \(\rho\) and \(\eta\), and \(\rho(\sigma)\) is a solution of \((A.7)\). The short string limit when \(\rho_0 \to 0\) was discussed in [37]; here we consider instead the long string limit when \(\rho_0 \to \infty\). In this case \(w = \kappa\) so that \(\omega = 0\), i.e.

\[
\eta = 0, \quad \rho = \mu \sigma. \quad (A.9)
\]

To describe this case as a limit of equations \((A.7)\) and \((A.8)\) we may take the limit

\[
\omega \to 0, \quad \rho_0 \to \infty, \quad \mu^2 = 2w \cosh 2\rho_0 = \text{fixed}, \quad w \cosh 2\rho \to 0. \quad (A.10)
\]

\(\mu\) may be related to spin since to leading \(\tilde{\lambda}\)-order the expression for the \(AdS_3\) spin \(S\) is \([36]\) (this follows directly from the action in \((A.5)\))

\[
S = 4J \int_0^{\pi/2} \frac{d\sigma}{2\pi} \sinh^2 \rho, \quad (A.11)
\]

where we integrate over one stretch of the string and the factor 4 accounts for the whole \((0, 2\pi)\) interval. Using that \(\rho = \mu \sigma\) this gives

\[
\mu = \frac{1}{\pi} \ln \frac{S}{J} + \text{const.} \quad (A.12)
\]

\(^{29}\) We may assume that \(\ln S\) should be replaced by \(\ln(\frac{1}{2})\) with \(\frac{1}{2}\) fixed, see below.
Even if $J$ is large, we are still allowed to assume $\frac{g}{J} \gg 1$ and even $\mu \gg 1$. The classical energy of this asymptotic LL solution is then\(^{30}\)

$$E_0^{(LL)} = \frac{1}{2} J \tilde{\lambda} \mu^2 = \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J}.$$  \hspace{1cm} (A.13)

That agrees with the expansion of the original classical string energy in (1.28).

In the limit (A.10) the fluctuation Lagrangian (A.8) becomes

$$\tilde{L} = 2g \tilde{f} - \frac{1}{\kappa} \tilde{f} (g^2 + f'^2 - 4\mu^2 g^2),$$  \hspace{1cm} (A.14)

and so that the characteristic frequencies on $\mathbb{R} \times S^1$ are found to be $\pm \tilde{O}_n$ where

$$\tilde{O}_n = \frac{\lambda}{2J^n} n \sqrt{n^2 + 4\mu^2},$$  \hspace{1cm} (A.15)

and the correction to the energy is given by their sum over $n$,

$$\tilde{E}_1 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{O}_n.$$  \hspace{1cm} (A.16)

Not too surprisingly, this is the same expression that one finds by expanding the contribution of the mode (A.3) to $E_1$ (i.e. $\frac{1}{\kappa} \tilde{O}_n = \frac{1}{\mu \sqrt{1+\ell^2}} \tilde{O}_n$) in (A.1) first in large $\ell$ or large $J$ to isolate the leading term corresponding to the LL model\(^{31}\)

$$\left[ \frac{1}{\mu \sqrt{1+\ell^2}} \tilde{O}_n \right]_{\ell \rightarrow \infty} = \frac{\lambda J^3}{2} \mu \sqrt{n^2 + 4\mu^2} - \frac{\lambda^2 J^5}{8} (n^2 + 4\mu^2)^{3/2} + \cdots.$$  \hspace{1cm} (A.17)

This implies that the LL model should capture the leading finite size correction (A.4) discussed above: indeed, taking now $\mu$ large gives the leading term as $\frac{1}{\mu} n \mu$ which is the same as (A.4) after summing over $n$.

To compare this to the discussion in [7] let us look at the $n=0$ contribution to the full string result in (A.1):

$$E^{(0)}_1 = \frac{1}{2\mu \sqrt{\ell^2 + 1}} (0 + 2\mu \sqrt{\ell^2 + 1} + 2\mu \sqrt{\ell^2 + 2} + 4\mu \ell - 8\mu \sqrt{\ell^2 + 1})$$

$$= -3 + \frac{\ell^2 + 2}{\sqrt{\ell^2 + 1}} + 2 \mu \sqrt{\ell^2 + 1}. \hspace{1cm} (A.18)$$

Note that there is no zero-mode contribution from the lightest AdS$_3$ mode. The zero-mode contribution is thus not contributing to the $\frac{1}{\mu}$ expansion at fixed $\ell$. Expanding (A.18) at large $\ell$ gives

$$[E^{(0)}_1]_{\ell \rightarrow \infty} = -\frac{1}{8\ell^4} + \cdots = -\frac{\lambda \mu^2}{2J^2} + O\left(\frac{\lambda^2}{J^4}\right). \hspace{1cm} (A.19)$$

Expanding the non-zero mode part of (A.1) we get

$$[E_1]_{\ell \rightarrow \infty} = \frac{\lambda}{2J^2} \sum_{n=1}^{\infty} \{n \sqrt{n^2 + 4\mu^2} - n^2 - 2\mu^2\} + O\left(\frac{\lambda^2}{J^4}\right). \hspace{1cm} (A.20)$$

The sum in (A.20) is UV finite, with the ‘regulator’ $-n^2 - 2\mu^2$ terms coming from other modes not seen in the AdS$_3$ LL model\(^{32}\). If we expand (A.20) in large $\mu$ first we would get

\(^{30}\)Since on the string theory side the LL action is derived in the gauge $t = \tau$, the 2D energy corresponding to the action in (A.5) is the same as the target space energy.

\(^{31}\)To match the LL model we need to take the large $J$ limit first. The large $\ell$ limit of the second AdS$_3$ mode $\tilde{O}_n$ with sign $+$ in (A.2) is $\frac{1}{\mu \sqrt{1+\ell^2}} \tilde{O}_n = 2 + \frac{\ell^2}{\ell^2 + 1} + O\left(\frac{\ell^4}{\ell^2 + 1}\right)$ and is not seen in LL model (cf. the discussion in [36]).

\(^{32}\)This expression is essentially equivalent to the expressions in [7] found for a circular string solution.
order $\mu$ term with coefficient $\zeta(-1) = -\frac{1}{12}$ and order $\mu^2$ term with coefficient $\zeta(0) = -\frac{1}{72}$. The latter cancels against the zero-mode contribution in (A.19) so we reproduce again the result (A.4). This confirms that this contribution is correctly captured by the lightest AdS$_3$ mode accounted for in the LL model.

By analogy with the circular string in AdS$_3 \times S^1$ case discussed in section 3.1 in [7] we expect the full expression, i.e. the sum of (A.19) and (A.20),

$$\tilde{E}_1 = -\frac{\lambda}{2J^2} \mu^2 + \frac{\lambda}{2J^2} \sum_{n=1}^{\infty} (n\sqrt{n^2 + 4\mu^2} - n^2 - 2\mu^2) + O\left(\frac{\lambda^2}{J^4}\right),$$  

(A.21)

can be reproduced from the BA equations for $sl(2)$ sector spin chain model. Note that the expression in (A.21) starts, in fact, with a $\mu^3$ term. This leading $\mu^3$ term comes from replacing sum by integral. Expanding in large $\ell$ the one-loop string result in (1.14) (found [14] by replacing the summation over $n$ by integration) we get

$$\langle E_1 \rangle_{\text{asymp.}} = -\frac{4\lambda}{3\ell^2} + \ldots = -\frac{4\lambda}{3J^2} \mu^3 + \ldots.$$  

(A.22)

This term comes from usual one-loop ‘non-anomaly’ part of BA (see, e.g., [42]) while here we are interested in true finite size corrections.

The conclusion is that the ‘non-wrapping’ string result (A.4) should be captured by the ABA since the LL model follows from the spin chain description.

Appendix B. Propagators

We present here the expressions for the bosonic and fermionic propagators:

$$K^{-1}_B(p) = \begin{pmatrix} 0 & \frac{2}{\rho^2 + \lambda^2} & 0 & 0 & 0_{1 \times 4} \\ \frac{2}{\rho^2 + \lambda^2} & 0 & 0 & 0 & 0_{1 \times 4} \\ 0 & 0 & \frac{\rho^2}{2\rho(p)} & \frac{\rho(p)}{2\rho(p)} & 0_{1 \times 4} \\ 0 & 0 & \frac{-\rho(p)}{2\rho(p)} & \frac{1 + \rho^2}{2\rho(p)} & 0_{1 \times 4} \\ 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & \frac{1}{\rho^2 + \lambda^2} \end{pmatrix}$$  

(B.1)

$$\mathcal{D}_B(p) \equiv p^2(p^2 + 1) + \hat{\nu}^2 p_0^2,$$

(B.2)

$$K^{-1}_F(p) = \frac{N_+(p)}{\mathcal{D}_F(p)} + \frac{N_-(p)}{\mathcal{D}_F(p)}, \quad \mathcal{D}_F(p) = \left(p_0 - \frac{i\hat{\nu}}{4}\right)^2 + p_1^2 + \frac{1 + \hat{\nu}^2}{4}.$$  

(B.3)

The precise form of $N_+(p)$ and $N_-(p)$ appearing in the fermionic propagator is given in [4].

Appendix C. Useful one-loop integrals

We have used the notation

$$I[m^2] = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + m^2}.$$  

(C.1)

This integral is logarithmically UV divergent. For zero mass it is also IR divergent. Integrals with different masses can be related thanks to the following identity:

$$I[m^2_1] - I[m^2_2] = \int \frac{d^2 p}{(2\pi)^2} \frac{m^2_1 - m^2_2}{(p^2 + m^2_1)(p^2 + m^2_2)} = \frac{1}{4\pi} \left(\ln m^2_1 - \ln m^2_1\right).$$  

(C.2)
Other convenient integrals, appearing in the evaluation of the double bubble topology, are

\[ J_F(k, r) = \int \frac{d^2 p}{(2\pi)^2} \frac{p_0^4 p_1^4}{p^4 + p^2 + \tilde{v}^2 p_0^2}, \quad J_F(k, r) = \int \frac{d^2 p}{(2\pi)^2} \frac{p_0^4 p_1^4}{p^4 + 4\pi^2 \tilde{v}^2 + \frac{1}{2} \tilde{v} p_0}, \]  

(C.3)

Their evaluation yields

\[ J_F(0, 0) = -\frac{1}{\sqrt{1 + \tilde{v}^2}} \left( \ln \frac{\sqrt{1 + \tilde{v}^2} + 1}{4} - I[0] + I[1 + \tilde{v}^2] \right), \]  

(C.4)

\[ J_F(2, 0) = -\frac{1}{8\pi \sqrt{1 + \tilde{v}^2}} \left( \frac{\sqrt{1 + \tilde{v}^2} - 1}{4} + \frac{1}{2} \left[ \frac{1}{4 \pi} \right] \right), \]  

(C.5)

\[ J_F(0, 2) = \frac{1}{8\pi \sqrt{1 + \tilde{v}^2}} \left( \frac{1 + \tilde{v}^2}{4} \right), \]  

(C.6)

\[ J_F(1, 0) = J_F(0, 1) = 0, \quad J_F(2, 0) = \frac{7\tilde{v}^2}{256\pi}, \]  

(C.7)

\[ J_F(0, 2) = -\frac{2 + 2\tilde{v}^2}{16} \left[ \frac{1 + \tilde{v}^2}{4} \right] + \frac{\tilde{v}^2}{256\pi}, \quad J_F(1, 1) = 0. \]  

(C.8)

**Appendix D. \( \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \) in the fermionic sunset**

We list below the explicit expression for the integral quantities appearing in the various expressions (4.15), (4.16) and (4.17) contributing to the fermionic sunset

\[ \mathcal{W}_i = \int_0^1 du \arctanh u U_i, \quad i = 1, 2, 3 \]

\[ U_1 = \frac{A[(1+u^3)\tilde{v}^4 u^2(3+6u^2-u^4)+\tilde{v}^2(1+7(u^2+u^4)+u^6)-(1+\tilde{v}^2)(1+(2+4\tilde{v})u^2+u^4)^2]}{4\pi^2 \tilde{v}^3 u^2(u^2-1)^3 A} \]

\[ \tilde{v}^4(-1+6u^2+3u^4)+8u^3(u-\sqrt{1+\tilde{v}^2}\sqrt{\tilde{v}^2+u^4})+4\tilde{v}^2 u^2(1+3u^2-2u\sqrt{1+\tilde{v}^2}\sqrt{\tilde{v}^2+u^4}) \]

\[ 2\tilde{v}^2 \pi^2(u^2-1)^3 u \]

\[ U_3 = \frac{8 - 8\sqrt{1+\tilde{v}^2}(1+\tilde{v}^2)^{3/2}+4\tilde{v}^2(3+u^2)+\tilde{v}^4(3+6u^2-u^4)}{4\pi^2(-1+u^2)^3} \]  

(D.1)

The large \( \tilde{v} \) expansion of \( \mathcal{W}_1, \mathcal{W}_2 \) and \( \mathcal{W}_3 \) is

\[ \mathcal{W}_1 = \frac{9}{64\pi^2} + \left( \frac{-51 + 8\pi^2 + 24 \ln 2 - 48 \ln \tilde{v}}{64\pi^2} \right) \left( \frac{1}{\tilde{v}^2} + \frac{1}{\tilde{v}^4} \right) + \left( \frac{1}{16} \right) \left( \frac{1}{256} \right) \left( \frac{1}{1152\pi^2} \right) \left( \frac{1}{\tilde{v}^6} \right) \]

\[ + \left( \frac{49}{1024} \right) \left( \frac{1}{1440\pi^2} \right) \left( \frac{1}{\tilde{v}^{10}} \right) = \left( \frac{1}{2048} \right) \left( \frac{1}{74677} \right) \left( \frac{1}{\tilde{v}^{12}} \right) + \cdots \]

\[ \mathcal{W}_2 = \frac{7}{32\pi^2} + \left( \frac{-1}{12} + \frac{11}{16\pi^2} \right) \left( \frac{1}{\tilde{v}^2} \right) + \left( \frac{-1}{16} + \frac{3}{32\pi^2} \right) \left( \frac{1}{\tilde{v}^4} \right) + \left( \frac{1}{12\pi^2} \right) \left( \frac{1}{\tilde{v}^6} \right) + \left( \frac{41}{576\pi^2} \right) \left( \frac{1}{\tilde{v}^8} \right) \]
\[ \mathcal{W}_3 = - \frac{9}{64\pi^2} + \frac{1219}{32\pi^2} - \frac{2100\pi^2}{32\pi^2} \hat{v}_{10} + \frac{1152\pi^2}{32\pi^2} \hat{v}_{10} + \frac{2099}{1} \hat{v}_{10} \hat{v}_{9} + \frac{13970}{8800\pi^2} \hat{v}_{11} \hat{v}_{10} + \ldots \] (D.2)

Similarly, one can derive the following small \( \hat{v} \) expansions:

\[ \mathcal{W}_1 = \frac{K}{8\pi^2} - \frac{(-1 + 2K + 2\ln 2)\hat{v}^2}{32\pi^2} + \frac{7(3 + 6K + 8\ln 2)\hat{v}^4}{1536\pi^2} - \frac{(1147 + 630K + 1152\ln 2)\hat{v}^6}{46080\pi^2} + \ldots \]

\[ \mathcal{W}_2 = \frac{7 - 12\ln \hat{v}}{48\pi^2} \hat{v}^2 + \frac{(17 + 168\ln \hat{v})\hat{v}^4}{1152\pi^2} + \frac{(-37 - 80\ln \hat{v})\hat{v}^6}{800\pi^2} + \frac{(127 + 180\ln \hat{v})\hat{v}^8}{2400\pi^2} + \ldots \]

\[ \mathcal{W}_3 = \frac{-\ln 2}{8\pi^2} \hat{v}^2 + \frac{(5 + 28\ln 2)\hat{v}^4}{384\pi^2} + \frac{(-41 - 128\ln 2)\hat{v}^6}{2560\pi^2} + \frac{3(7 + 16\ln 2)\hat{v}^8}{1280\pi^2} + \ldots \] (D.3)

**Appendix E. Two-loop contributions to the string free energy on the cylinder**

Here we list all the terms entering the computation of the two-loop free energy on \( \mathbb{R} \times S^1 \), discarding purely power-like divergences in the continuous momentum integral. The overall factor of 4 appearing in the definition of \( A_{sunset}^{BB}, A_{double-bubble}^{BB}, A_{sunset}^{BF}, A_{double-bubble}^{BF} \) and \( A_{double-bubble}^{BB} \) is a consequence of the closed string normalization we are using.

- **Bosonic sunset:**
  \[ A_{sunset}^{BB} = 4 \int \frac{dp_0 dq_0 dr_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} \delta^{(2)}(p + q + r) [B_{0,0,4} + B_{2,2,4} + B_{4,4,4}] \] (E.1)

  \[ B_{0,0,4} = -5 \frac{1}{4} P[p, 0] P[q, 0] + \frac{1}{2} P(p, 0) P(r, 4) + \frac{1}{2} P(p, 0) P(q, 0) P(r, 4) \] (E.2)

  \[ B_{2,2,4} = -4 \frac{1 + p_0^2}{P(p, 2) P(r, 4)} + 2 \frac{(1 + p_0^2)(1 + q_0^2)}{P(p, 2) P(q, 2) P(r, 4)} \] (E.3)

  \[ B_{4,4,4} = \frac{7 + p_0^2}{P(p, 4) P(r, 4)} + \frac{3 - \frac{7}{2} p_0^2 + \frac{1}{4} p_0^2}{P(p, 4) P(q, 4) P(r, 4)} \] (E.4)

- **Bosonic double-bubble:**
  \[ A_{double-bubble}^{BB} = 4 \int \frac{dp_0 dq_0 dr_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} \delta^{(2)}(p + q + r) [B_{2,2,4} + B_{4,4,4}] \] (E.5)

  \[ B_{2,2} = 2 \frac{1 + p_0^2}{P(p, 2) P(r, 4)} \quad B_{4,4} = \frac{1}{2} \frac{1}{P(p, 4) P(r, 4)} \] (E.6)
\textbf{Fermionic sunset:}

\[ A_{\text{sunset}}^{\text{BF}} = 4 \int \frac{d^4 p_0}{(2\pi)^4} \frac{d^4 q_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} \delta^{(4)}(p + q + r) \left[ F_{0,1,1} + F_{2,1,1} + F_{4,1,1} \right] \]  

(E.7)

\[ F_{0,1,1} = -\frac{5p_0^2}{P(p, 1)P(r, 0)} + \frac{10p_0q_0r_0^2}{P(p, 1)P(q, 1)P(r, 0)} \]  

(E.8)

\[ F_{2,1,1} = \frac{4p_0q_0(1 + r_0^2)}{P(p, 1)P(q, 1)P(r, 2)} \]

\[ F_{4,1,1} = \frac{12p_0^2}{P(p, 1)P(r, 4)} + \frac{2p_0(p_0 - q_0)^2q_0}{P(p, 1)P(q, 1)P(r, 2)} \]  

(E.9)

\textbf{Fermionic double-bubble:}

\[ A_{\text{double--bubble}}^{\text{BF}} = 0. \]  

(E.10)

\textbf{Bosonic-fermionic double-bubble:}

\[ A_{\text{double--bubble}}^{\text{BF}} = 4 \int \frac{d^4 p_0}{(2\pi)^4} \frac{d^4 q_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} \delta^{(4)}(p + q + r) \left[ C_{0,1} + C_{4,1} \right] \]  

(E.11)

\[ C_{0,1} = \frac{5p_0^2}{P(p, 1)P(r, 0)}, \quad C_{4,1} = -\frac{4p_0^2}{P(p, 1)P(r, 4)}. \]  

(E.12)

\textbf{Non-1PI:}

this diagram arises due to the presence of a tadpole \( T(p) \) and has \( \tilde{\phi} \) as internal leg

\[ A_{\text{non-1PI}} = 4 \int \frac{d^4 p_0}{(2\pi)^4} \sum_{p_1, q_1, r_1} A_{\text{non-1PI}}^{\text{non-1PI}}, \quad A_{\text{non-1PI}}^{\text{non-1PI}} = \frac{1}{2} T(p, q), \]  

(E.13)

\[ T(p) = \frac{1}{P(p, 4)} + \frac{1}{P(p, 0)} + \frac{1}{P(p, 2)} - \frac{4p_0^2}{P(p, 1)}. \]  

(E.14)

The first three contributions in \( T(p) \) come respectively from the bosonic vertices \( \tilde{\phi} \chi^3, \tilde{\phi} \chi^2 \) and \( \tilde{\phi} \chi \tilde{\phi} \), while the last term is due to the cubic vertex of \( \tilde{\phi} \) and the fermions.

We note that \( C_{0,1} \) cancels against the first term of \( F_{0,1,1} \). We also note the partial cancellation of \( C_{4,1} \) against the first term of \( F_{4,1,1} \), of \( B_{2,4} \) against the first term in \( B_{2,2,4} \) and of \( B_{4,4} \) with the first term in \( B_{4,4,4} \).

\textbf{Appendix F. Sums and integrals}

Some simple sums which occur in the calculation of one-loop integrals are

\[ \frac{2\pi}{L} \sum_{n \in \mathbb{Z}} \frac{1}{(\frac{2\pi}{L})^2 + p_0^2} = \frac{\pi}{p_0} \coth \left( \frac{1}{2} L p_0 \right), \]  

(F.1)

\[ \frac{2\pi}{L} \sum_{n \in \mathbb{Z}} \frac{1}{(\frac{2\pi}{L})^2 + p_0^2 + m^2} = \frac{\pi}{\sqrt{p_0^2 + m^2}} \coth \left( \frac{1}{2} L \sqrt{p_0^2 + m^2} \right). \]  

(F.2)

As already observed in the main text these expressions imply that massive one loop integrals are exponentially suppressed. Let us now consider an example of a purely massive two-loop integral and show explicitly show that it decays faster than \( 1/L^2 \), i.e. does not contribute to
the leading finite size term. The prototype of a two-loop integral in a theory on a 2D cylinder is

\[ I[m_1^2, m_2^2, m_3^2] = \int \frac{dp_0}{(2\pi)^2} \frac{dq_0}{(2\pi)^2} \Sigma[m_1^2, m_2^2, m_3^2] \]

\[ \Sigma[m_1^2, m_2^2, m_3^2] = \left( \frac{2\pi}{L} \right)^2 \sum_{m,n=-\infty}^{\infty} \frac{1}{\left( \frac{2\pi n}{L} \right)^2 + q_0^2 + m_1^2} \]

\[ A = \sqrt{(p_0 + q_0)^2 + m_3^2} \]

When all the masses are non-vanishing the integral \( I[m_1^2, m_2^2, m_3^2] \) is convergent. To study its behavior as a function of \( L \) we can first compute the sum over \( n, m, \Sigma[m_1^2, m_2^2, m_3^2] \). This can be found analytically. If we define the function

\[ \Theta[a, b, c] = -\frac{\pi}{2a} \coth[\pi a] \left( \frac{1}{(a + b)^2 - c^2} + \frac{1}{(a - b)^2 - c^2} \right) + \frac{\pi}{2c} \coth[\pi (b - c)] \left( \frac{1}{(b - c)^2 - a^2} - \frac{\pi}{2c} \coth[\pi (b + c)] \right) \]

we then have the following explicit expression for the sum:

\[ \Sigma[m_1^2, m_2^2, m_3^2] = \left( \frac{L}{2\pi} \right)^4 \left( \frac{2\pi^2}{\sqrt{q_0^2 + m_1^2}} \right) \left( \frac{L}{2\pi} \right)^{2\pi^2} \ LEFT[ \left( \frac{L}{2\pi} \right)^{\frac{p_0^2 + m_1^2}{2\pi}}, \left( \frac{L}{2\pi} \right)^{\frac{q_0^2 + m_2^2}{2\pi}}, \left( \frac{L}{2\pi} \right)^{\frac{r_0^2 + m_3^2}{2\pi}} \ RIGHT] \]

\[ \times \Theta \left[ \frac{L}{2\pi} \right]^{\frac{p_0^2 + m_1^2}{2\pi}}, \left( \frac{L}{2\pi} \right)^{\frac{q_0^2 + m_2^2}{2\pi}}, \left( \frac{L}{2\pi} \right)^{\frac{r_0^2 + m_3^2}{2\pi}} \]

The remaining continuous integrals over \( p_0 \) and \( q_0 \) can be computed numerically for various values of the size \( L \). To exemplify the behavior of a purely massive integral we have plotted \( I[1,1/2,1/2] \) for the range \( 1 < L < 10 \) in figure 4. Note that the function quickly approaches the continuum limit \( (L \to \infty) \) value of \( \frac{1}{2\pi^2} \), and in doing so its decay is faster than \( 1/L^2 \). For this reason it cannot contribute to finite size effects.

**Appendix G. Some useful one-dimensional two-loop integrals**

\[ \int dp_0 dq_0 \frac{d^3 r_0}{P(p, m_p) P(q, m_q) P(r, m_r)} \delta(p_0 + q_0 + r_0) \]

\[ = \frac{1}{2\pi} \sqrt{p_1^2 + m_p^2} \sqrt{q_1^2 + m_q^2} \sqrt{r_1^2 + m_r^2} (\sqrt{p_1^2 + m_p^2} + \sqrt{q_1^2 + m_q^2} + \sqrt{r_1^2 + m_r^2}) \]

\[ (G.1) \]

\[ \int dp_0 dq_0 \frac{d^3 r_0}{P(p, m_p) P(q, m_q) P(r, m_r)} p_0 q_0 \delta(p_0 + q_0 + r_0) \]

\[ = -\frac{1}{2\pi} \sqrt{r_1^2 + m_r^2} (\sqrt{p_1^2 + m_p^2} + \sqrt{q_1^2 + m_q^2} + \sqrt{r_1^2 + m_r^2}) \]

\[ (G.2) \]
\[
\int d\rho d\phi d\rho_0 d\phi_0 d\rho_1 d\phi_1 d\rho_2 d\phi_2 \frac{p_0^2 \delta(p_0 + q_0 + r_0)}{P(p, m_p) P(q, m_q) P(r, m_r)} = -\frac{1}{2\pi^2} \frac{2\pi^3}{\sqrt{p_0^2 + m_p^2} \sqrt{q_0^2 + m_q^2} \sqrt{r_0^2 + m_r^2}} \sqrt{p_0^2 + m_p^2 + q_0^2 + m_q^2 + r_0^2 + m_r^2}. \tag{G.3}
\]

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