FIRST MILNOR COHOMOLOGY OF HYPERPLANE ARRANGEMENTS

NERO BUDUR, ALEXANDRU DIMCA, AND MORIHIKO SAITO

Dedicated to Professor Anatoly Libgober

Abstract. We show a combinatorial formula for a lower bound of the dimension of the non-unipotent monodromy part of the first Milnor cohomology of a hyperplane arrangement satisfying some combinatorial conditions. This gives exactly its dimension if a stronger combinatorial condition is satisfied. We also prove a non-combinatorial formula for the dimension of the non-unipotent part of the first Milnor cohomology, which apparently depends on the position of the singular points. The latter generalizes a formula previously obtained by the second named author.

Introduction

Let $D$ be a hyperplane arrangement in $X = \mathbb{C}^n$. We assume $D$ is reduced and central, i.e. $D$ is defined by a reduced homogeneous polynomial $f$. We also assume that $D$ does not come from an arrangement in $\mathbb{C}^n-1$, i.e. $f$ cannot be expressed as a polynomial of $n-1$ variables. Set $F_f = f^{-1}(1)$. This is the Milnor fiber of $f$. Let $T$ be the Milnor monodromy on $H^j(F_f, \mathbb{C})$. We have the monodromy decomposition

$$H^j(F_f, \mathbb{C}) = \bigoplus_{\lambda} H^j(F_f, \mathbb{C})_{\lambda},$$

with $H^j(F_f, \mathbb{C})_{\lambda}$ the $\lambda$-eigenspace. Let $d = \deg D$. Set $Z := \mathbb{P}(D) \subset Y := \mathbb{P}^{n-1} \setminus Z$. It is well-known (see e.g. [CS], [Di1]) that $H^j(F_f, \mathbb{C})_{\lambda} = 0$ if $\lambda^d \neq 1$, and there are local systems $L^{(k)}$ of rank 1 on $U$ for $k = 0, \ldots, d-1$ such that

$$H^j(F_f, \mathbb{C})_{\lambda} = H^j(U, L^{(k)}) \quad \text{for} \quad \lambda = \exp(2\pi i k/d),$$

and the monodromy around any irreducible component of $Z$ is the multiplication by $\exp(-2\pi i k/d)$. In particular, $L^{(0)} = \mathcal{C}_U$ so that $H^j(F_f, \mathbb{C})_{1} = H^j(U, \mathbb{C})$. It has been conjectured that the $H^j(F_f, \mathbb{C})_{\lambda}$ are combinatorial invariants. By [OS], the cohomology $H^*(U, \mathbb{C}) = H^*(F_f, \mathbb{C})_{1}$ is combinatorially described.

Let $Y' = \mathbb{P}^2$ be a sufficiently general linear subspace of dimension 2 in $Y = \mathbb{P}^{n-1}$. By the generalized weak Lefschetz theorem for perverse sheaves (i.e. Artin's...
have an isomorphism
\[(0.1) \quad H^1(U, L^{(k)}) = H^1(Y' \cap U, L^{(k)}|_{Y' \cap U}).\]
Thus we may assume \(n = 3\) since we are interested in \(b_1(F_f)_{\lambda} := \dim H^1(F_f)_{\lambda}\).

Set \(\Sigma = \{y \in Z \mid \text{mult}_y Z \geq 3\}\). Let \(Z_1, \ldots, Z_d\) be the irreducible components of \(Z\). For \(y \in \Sigma\) and \(I \subset \{1, \ldots, d\}\), set \(m_y := \text{mult}_y Z = \#\{i \mid y \in Z_i\}\), and
\[
m_{I,y} := \#\{i \in I \mid y \in Z_i\}, \quad \alpha_{I,y} := \frac{|I|}{d} m_y - m_{I,y}.
\]

The following would be known to specialists.

**Proposition 1.** Let \(\lambda = \exp(2\pi i k/d)\) with \(k \in \mathbb{Z} \cap [1, d/2]\). Then \(b_1(F_f, C)_{\lambda}\) is described combinatorially by using the Aomoto complex as in (1.2) with \(\omega\) given in (1.2.3), if there is \(I \subset \{1, \ldots, d\}\) such that \(|I| = k\) and
\[(0.2) \quad \text{either } \alpha_{I,y} \notin \mathbb{Z}_{>0} (\forall y \in \Sigma) \quad \text{or} \quad \alpha_{I,y} \notin \mathbb{Z}_{<0} (\forall y \in \Sigma).
\]

In case the second condition of (0.2) is satisfied, we actually consider \(k' := d - k\) and the complement \(I'\) of \(I\) instead of \(k\) and \(I\) by using the complex conjugation on the Milnor cohomology so that \(\alpha_{I,y}\) is replaced by \(-\alpha_{I,y}\). In fact, this is the reason for which we assume \(k \leq d/2\); the case \(k > d/2\) is treated in the second case of (0.2). The proof of Proposition 1 follows from a theorem of H. Esnault, V. Schechtman and E. Viehweg [ESV] about Aomoto’s conjecture (here [STV] is not needed since \(n = 3\) together with a combinatorial description of the cohomology of the complement due to P. Orlik and L. Solomon [OS], see (1.2) below. Proposition 1 was essentially used in a calculation of examples in [Sa2] although the combinatorial description of \(H^*(U)\) in [OS] was not mentioned there. It is possible to give a combinatorial description of \(H^*(U)\) in a more geometric way using the theory of perverse sheaves [BBD], see also [BS].

If \(k = d/3 \in \mathbb{N}\) and \(m_y = 3 (\forall y \in \Sigma)\), then condition (0.2) in Proposition 1 becomes
\[(0.3) \quad \text{either } m_{I,y} \geq 1 (\forall y \in \Sigma) \quad \text{or} \quad m_{I,y} \leq 1 (\forall y \in \Sigma).
\]
Note that the first condition of (0.3) is equivalent to that \(\Sigma\) is covered by \(\bigcup_{I \in I} Z_i\).

There are examples where neither condition of (0.3) is satisfied, see Example (3.1)(iii) below.

Returning to the general situation (with \(n = 3\)), it is well-known that \(b_1(F_f)_{\lambda} = 0\) unless there is \(y \in \Sigma\) with \(\lambda^{m_y} = 1\), see Remark (3.4)(i) below. (In this case we have actually a stronger assertion that \(b_1(F_f)_{\lambda} = 0\) unless each \(Z_i\) contains \(y \in \Sigma\) with \(\lambda^{m_y} = 1\), see Remark (3.4)(ii) below.) Using Proposition 1, we can show the following.

**Theorem 1.** Assume \(n = 3\). Let \(m, r \in \mathbb{Z} \cap [3, d - 1]\) with \(d/m \in \mathbb{Z}\). Assume there is a map \(\phi : \{1, \ldots, d\} \to \{1, \ldots, r\}\). Set \(I_f = \phi^{-1}(j)\), \(m_{j,y} = m_{I_{j,y}}\), \(\Sigma(d) = \Sigma \setminus Z_d\),
\[
\Sigma^\phi = \{y \in \Sigma \mid \{y\} = Z_i \cap Z_{\phi} \text{ with } \phi(i) \neq \phi(i')\}, \quad \Sigma^\phi(d) = \Sigma^\phi \cap \Sigma(d).
\]

(i) Assume \(\lambda\) is an \(m\)-th root of unity and the following conditions are satisfied.
\[(0.4) \quad m_{j,y} > 0 (\forall y \in \Sigma^\phi(d)), \quad \text{and the } m_{j,y}/m_{j',y} \text{ are independent of } y \in \Sigma^\phi(d).
\]
(0.5) \( \exists j_0 \text{ s.t. } |I_{j_0}| = d/m, \ m_{j_0,y} = \frac{1}{m} \sum_{j=1}^{r} m_{j,y} \ (\forall y \in \Sigma_{(d)}^\phi). \) Then \( b_1(F_f)_\lambda \geq r - 2. \)

(ii) Assume \( \lambda \) is a primitive \( m \)-th root of unity and the following condition is satisfied.

(0.6) \( m_{j,y} = 1 \ (\forall y \in \Sigma^\phi), \ m = r, \) and (0.2) is satisfied for some \( I_{j_0}. \) Then \( b_1(F_f)_\lambda = m - 2. \)

If the first condition of (0.6) is satisfied, i.e. if \( m_{j,y} = 1 \ (\forall y \in \Sigma^\phi), \) then we have \(|\Sigma^\phi| = |I_j||I_{j'}|\) for any \( j \neq j'\) so that \( |I_j| = d/r \) for any \( j, \) and conditions (0.4-5) with \( m = r \) are satisfied. Hence \( Z \) is an \((m, d/m)\)-net in the sense of [FY]. By the Nullstellensatz there is a pencil such that \( \bigcup_{\phi(i)=j} Z_i \) is a special fiber of the pencil for any \( j, \) see [Yu2], Lemma 3.1. This implies another proof of Theorem 1 (i) in this case, see [DP], Th. 3.1 (i) (or Remark (3.3)(ii) below). Note that we have \( m \leq 4 \) in this case by [St2], [Yu3]. For the moment any known examples are essentially of this type. For \( m = 3, \) there are lots of examples where (0.6) is satisfied, see [Yu2] (and Remark (3.2)(ii) below). For \( m = 4, \) however, only one example is known, see [FY] (and Remark (3.3)(iii) below). Note that the last example implies a rather artificial example where conditions (0.4–5) are satisfied for \( m = 4 \) and \( r = 3 \) by considering \( I_1, I_2 \) and \( I_3 \cup I_4 \) as a partition. Note also that a hyperplane arrangement has a structure of a multi-net in the sense of [FY] and \( r \leq 4, \) if the hypotheses (0.4–5) of Theorem 1(i) are satisfied, see Remark (3.3)(ii) below.

In this paper we also prove a non-combinatorial formula for the dimension of the non-unipotent monodromy part of the first Milnor cohomology generalizing [Di1], Ch.6, Th. 4.15. Let \( \Sigma \) be as in Proposition 1, and set

\[ \Sigma(k) = \{ y \in \Sigma \mid m_y k/d \in \mathbb{Z} \}. \]

For \( y \in \Sigma, \) let \( I_{\{y\}} \subset \mathcal{O}_Y \) be the reduced ideal of \( \{y\} \subset Y, \) and define

\[ \mathcal{J}^{(k)} := \bigcap_{y \in \Sigma}(\bigcup_{y \in \Sigma} I_{\{y\}}^{[m_y k/d]-2}), \quad \mathcal{J}^{(k')} := \bigcap_{y \in \Sigma}(\bigcup_{y \in \Sigma} I_{\{y\}}^{[m_y k/d]-1}). \]

Here \( [\alpha] := \min\{k \in \mathbb{Z} \mid k \geq \alpha \}, \ [\alpha] := \max\{k \in \mathbb{Z} \mid k \leq \alpha \}, \) and \( \mathcal{I}_{\{y\}} = \mathcal{O}_Y \) for \( j \leq 0. \) Let \( \mathcal{C}[X]_j \) denote the space of homogeneous polynomials of degree \( j. \) This is identified with \( \Gamma(Y, \mathcal{O}_Y(j)). \) Define

\[ J_j^{(k)} := \Gamma(Y, \mathcal{O}_Y(j) \otimes \mathcal{O}_Y) \mathcal{J}^{(k)} \subset \Gamma(Y, \mathcal{O}_Y(j)) = \mathcal{C}[X]_j. \]

**Theorem 2.** Assume \( n = 3. \) For \( k \in [1, d-1], \) let \( k' = d - k \) and \( \lambda = \exp(2\pi ik/d). \) Then

\[ \dim \text{Gr}_F^0H^1(F_f)_\lambda = \dim \text{Coker}(\rho^{(k)} : J_{k-3}^{(k)} \rightarrow \bigoplus_{y \in \Sigma(k)} \mathcal{J}^{y(k)} / \mathcal{J}^{(k)}>_{y(k)})) \]

\[ = \dim \text{Coker}(\tilde{\rho}^{(k)} : \mathcal{C}[X]_{k-3} \rightarrow \bigoplus_{y \in \Sigma} \mathcal{O}_Y / \mathcal{J}^{(k)}>_{y(k)}), \]

\[ \dim \text{Gr}_F^1H^1(F_f)_\lambda = \dim \text{Coker}(\rho^{(k')} : J_{k'-3}^{(k')} \rightarrow \bigoplus_{y \in \Sigma(k')} \mathcal{J}^{y(k')} / \mathcal{J}^{(k')}>, \)

\[ = \dim \text{Coker}(\tilde{\rho}^{(k')} : \mathcal{C}[X]_{k'-3} \rightarrow \bigoplus_{y \in \Sigma} \mathcal{O}_Y / \mathcal{J}^{(k')}>). \]
and \( b_1(F_f)_\lambda = \dim \text{Gr}_F^1 H^1(F_f)_\lambda + \dim \text{Gr}_F^0 H^1(F_f)_\lambda \). Here we choose local trivialization of \( \mathcal{O}_Y(k - 3) \), \( \mathcal{O}_Y(k' - 3) \) to define the restriction morphisms \( \bar{\rho}^{(k)} \), \( \bar{\rho}^{(k')} \), etc. at \( y \in \Sigma \).

Similar assertions in terms of superabundances (but without reference to the mixed Hodge structure) were obtained by A. Libgober, see [Li1], [Li2]. Note that for \( \alpha = k \frac{d}{3} \) and \( 0 < \varepsilon \ll \frac{1}{d} \), we have

\[
\mathcal{J}^{(k)} = \mathcal{J}(Y, (\alpha - \varepsilon)Z), \quad \mathcal{J}^{(>k)} = \mathcal{J}(Y, \alpha Z),
\]

where \( \mathcal{J}(Y, \alpha Z) \) is the multiplier ideal sheaf [La], see (2.3.4) below. Moreover, the target of the restriction morphism \( \rho^{(k)} \) is identified with

\[
\mathcal{G}(Y, \alpha Z) := \mathcal{J}(Y, (\alpha - \varepsilon)Z)/\mathcal{J}(Y, \alpha Z).
\]

For simplicity assume \( d/3 \in \mathbb{Z} \) and \( m_y = 3 (\forall y \in \Sigma) \). Set \( k = 2d/3 \), \( k' = d/3 \). Then the target of \( \rho^{(k')} \) vanishes, and \( \rho^{(k)} \) coincides with \( \bar{\rho}^{(k)} \) which is identified with the evaluation map (choosing points of \( C^3 \setminus \{0\} \) representing the points of \( \Sigma \))

\[
\bigoplus_{y \in \Sigma(k)} \text{ev}_y^{k-3} : C[X]_{k-3} \to \bigoplus_{y \in \Sigma(k)} C_y.
\]

So we get a partial generalization of [Di1], Ch.6, Th. 4.15 where \( d = 9 \). Note that \( \text{ev}_y^{k-3} \) can be defined and is surjective if \( 2d/3 < k < d \), although the surjectivity for \( k = 2d/3 \) does not hold in general. This follows from formulas for the spectrum, see [BS], Th. 3 and 5 (these are closely related to Mustaţă’s formula for the multiplier ideals [Mu], see also [Te]).

We would like to thank A. Libgober and S. Yuzvinsky for useful remarks, and the referee for valuable comments.

In Section 1 we recall theorems of Orlik, Solomon [OS] and Esnault, Schechtman, Viehweg [ESV] to show Proposition 1, and then prove Theorem 1. In Section 2 we prove Theorem 2 using the theory of multiplier ideals and Hodge theory. In Section 3 we give some examples and remarks.

1. Combinatorial description

1.1. Orlik-Solomon algebra. Let \( D \) be a central hyperplane arrangement in \( X = \mathbb{C}^n \). Set \( Y = \mathbb{P}^{n-1} \), and \( U = Y \setminus Z \), where \( Z = \mathbb{P}(D) := (D \setminus \{0\})/\mathbb{C}^* \). Let \( \mathcal{S}(Z) \) denote the intersection lattice consisting of any intersections of the irreducible components \( Z_i \) of \( Z \). For the definition of the Orlik-Solomon algebra, we allow the ambient space \( \mathbb{P}^{n-1} \) as a member, but the empty set corresponding to \( 0 \in \mathbb{C}^n \) is excluded. (This is different from the definition in [BS], 1.1.) In particular, \( \mathcal{S}(Z) \) is not a lattice in the strict sense, and is often called the “poset of intersections” in the literature. By [OS] there is an isomorphism of \( \mathbb{C} \)-algebras

\[
A^{\bullet}_{\mathcal{S}(Z)} \cong H^\bullet(U, \mathbb{C}),
\]

where \( A^{\bullet}_{\mathcal{S}(Z)} \) is the quotient of the exterior algebra \( \bigwedge^\bullet \left( \bigoplus_i \mathbb{C} e_i \right) \) by the ideal \( \mathcal{I}_{\mathcal{S}(Z)} \). Here the \( e_i \) correspond to \( Z_i \) for \( i = 1, \ldots, d - 1 \) since we use the induced affine
arrangement on $\mathbb{C}^{n-1} = \mathbb{P}^{n-1} \setminus Z_d$, see [Br]. Moreover, $\mathcal{I}_S(Z)$ is determined by the combinatorial data, see [OS].

1.2. Solution of Aomoto’s conjecture. Let $\alpha_i \in \mathbb{C}$ for $i \in \{1, \ldots, d\}$, and assume $\sum_{i=1}^d \alpha_i = 0$, and

\begin{equation}
\sum_{Z_i \supset V} \alpha_i \notin \mathbb{N} \setminus \{0\} \quad \text{for any dense edge } V \in S(Z).
\end{equation}

For the definition of “dense”, see [STV]. (In this paper the condition that $V$ is dense may be replaced with $V \subset \Sigma$ since we assume essentially $n = 3$.) Note that (1.2.1) should be satisfied for $S(Z)$, and not for $S(Z)_{(d)}$ in (1.3) below. In the notation of (1.1), set

$$\omega = \sum_{i=1}^{d-1} \alpha_i e_i \in \mathcal{A}^1_{S(Z)}.$$ 

This defines a complex $(\mathcal{A}^\bullet_{S(Z)}, \omega \wedge)$, called the Aomoto complex associated to $\omega$. Note that $e_i$ is identified with $dg_i/g_i$ where $g_i$ is a linear function with a constant term defining $Z_i \setminus Z_d \subset \mathbb{C}^{n-1}$. We also have a regular singular connection $\nabla^\omega$ on $\mathcal{O}_U$ such that

$$\nabla^\omega g = dg + g\omega \quad \text{for } g \in \mathcal{O}_U.$$ 

Note that (1.2.1) is the condition on the distribution of the residues of the connection in [STV]. The main theorem of [ESV], [STV] assures that if (1.2.1) is satisfied, then we have a quasi-isomorphism

\begin{equation}
H^\bullet_{DR}(U, (\mathcal{O}_U, \nabla^\omega)) \rightarrow (\mathcal{A}^\bullet_{S(Z)}, \omega \wedge).
\end{equation}

This implies Proposition 1 in Introduction since the first condition of (0.2) coincides with (1.2.1) by setting

\begin{equation}
\alpha_i = \frac{k}{d} - 1 \quad \text{if } i \in I, \quad \text{and } \alpha_i = \frac{k}{d} \quad \text{otherwise}.
\end{equation}

In case the second condition of (0.2) is satisfied, we replace $k$ with $d - k$, and $I$ with its complement, using the complex conjugation on the Milnor cohomology.

If condition (1.2.1) is not satisfied, then (1.2.2) may be false. However, we have always the following inequality (see [LY], Prop. 4.2)

\begin{equation}
\dim H^i_{DR}(U, (\mathcal{O}_U, \nabla^\omega)) \geq \dim H^j(\mathcal{A}^\bullet_{S(Z)}, \omega \wedge).
\end{equation}

So we get for any $a \in \mathbb{C}^*$

\begin{equation}
\dim H^j_{DR}(U, (\mathcal{O}_U, \nabla^\omega)) \geq \dim H^j(\mathcal{A}^\bullet_{S(Z)}, a\omega \wedge) = \dim H^j(\mathcal{A}^\bullet_{S(Z)}, \omega \wedge).
\end{equation}

1.3. Generalized residues. Set

$$S(Z)^{(k)} = \{ V \in S(Z) \mid \text{codim}_V V = k \}.$$ 

Let $j_U : U \hookrightarrow Y$ denote the inclusion. Since it is an affine morphism, $R(j_U)_* \mathcal{Q}_U[n-1]$ is a perverse sheaf [BBD] underlying naturally a mixed Hodge module [Sa1]. Let $W$ be the weight filtration. By [BS], 1.7, there are constant variations of Hodge structures $L_V$ of type $(0, 0)$ on $V$ for $V \in S(Z)$ such that we have for $i = 1, \ldots, n-1$

\begin{equation}
\text{Gr}^W_{n-1+i}(R(j_U)_* \mathcal{Q}_U[n-1]) = \bigoplus_{V \in S(Z)^{(i)}} L_V(-i)[n-1-i],
\end{equation}

where
where \( \text{Gr}_{n-1}^W (R(jU), Q_U[n-1]) = Q_Y[n-1] \). We may identify \( L_V \) with a Hodge structure since it is constant. Set
\[
S(Z)_{(d)} = \{ V \in S(Z) \mid V \not\subseteq Z_d \}, \quad S(Z)^{(k)}_{(d)} = S(Z)_{(d)} \cap S(Z)^{(k)}.
\]
Then, restricting (1.3.1) to \( Y \setminus Z_d = \mathbb{C}^{n-1} \), we get by [BS], 1.9
\[
(1.3.2) \quad H^i(U, \mathbb{Q}) = \bigoplus_{V \in S(Z)^{(i)}_{(d)}} L_V(-i) \quad (i = 1, \ldots, n-1).
\]
This is compatible with the results of Brieskorn [Br] and Orlik, Solomon [OS], since \( \dim L_V \) is given by the Möbius function in loc. cit. For each \( V \in S(Z)^{(i)}_{(d)} \), we have the projection
\[
\pi_V : H^i(U, \mathbb{Q}) \to L_V(-i).
\]
This may be called the generalized residue along \( V \).

In the notation of (1.2), set \( e_i = dg_i/g_i \) for \( 1 \leq i < d \), and
\[
e_{i_1, \ldots, i_j} := e_{i_1} \wedge \cdots \wedge e_{i_j} \in H^j(U, \mathbb{Q}).
\]
This is compatible with the decomposition (1.3.2), i.e.
\[
(1.3.3) \quad e_{i_1, \ldots, i_j} \in L_V(-j) \subset H^j(U, \mathbb{Q}) \quad \text{with} \quad V = Z_{i_1} \cap \cdots \cap Z_{i_j}.
\]
This means that its image by \( \pi_V \) for \( V' \neq V \) vanishes. The assertion is shown by taking a sub-arrangement \( Z' \subset Z \) consisting of \( Z_{i_1}, \ldots, Z_{i_j}, Z_d \), and using the compatibility of the exterior product with the restriction morphism by the inclusion \( U \hookrightarrow U' := \mathbb{P}^{n-1} \setminus Z' \), since the construction of \( L_V \) is functorial for \( Z \). Indeed, taking the graded pieces of the canonical morphism
\[
R(jU'), Q_U[n-1] \to R(jU), Q_U[n-1],
\]
the obtained morphism is compatible with the direct sum decomposition in (1.3.1) (since the direct factors in (1.3.1) are simple perverse sheaves with different supports). Note that \( H^j(U', \mathbb{Q}) = \mathbb{Q} \) where \( j \) is as above, and \( L_V = \mathbb{Q} \) if \( Z \) is a divisor with normal crossings at the generic point of \( V \).

The following lemma is well known to the specialists, see [LY], Lemma 3.1 (and also [Fa], [Li2], [Yu1]) where the situation is localized at \( V \) using the fact that the relations of the Orlik-Solomon algebra are of the form \( \partial(e_J) \) for certain \( J \) and are compatible with the decomposition by \( V \). We note here a short proof for the convenience of the reader.

Set \( I_V = \{ i \mid Z_i \supset V \} \), and \( \alpha_V = \sum_{i \in I_V} \alpha_i \), where \( \omega = \sum_{i=1}^{d-1} \alpha_i e_i \) as above.

**Lemma 1.4.** Let \( \eta = \sum_{i=1}^{d-1} \beta_i e_i \) with \( \beta_i \in \mathbb{C} \). For \( V \in S(Z)^{(2)}_{(d)} \) we have
\[
(1.4.1) \quad \alpha_V \beta_i = \beta_V \alpha_i \quad (\forall i \in I_V) \quad \text{if} \quad \pi_V(\omega \wedge \eta) = 0.
\]

**Proof.** For \( i, j, k \in I_V \), it is easy to show the relation (see e.g. [OS])
\[
e_{i,j} + e_{j,k} = e_{i,k}.
\]
In the case \( \text{codim}_Y V = 2 \), we have \( \dim L_V = |I_V| - 1 \) by the definition of \( L_V \) in [BS], 1.7. (This is related to the theory of Möbius function [OS]). Hence \( L_V \) has a basis consisting of \( e_{i,k} (i \neq k) \) for any fixed \( k \in I_V \).
By hypothesis we have for any \( k \in I_V \) the vanishing of
\[
\sum_{i,j \in I_V} \alpha_i \beta_j e_{i,j} = \sum_{i,j \in I_V} \alpha_i \beta_j (e_{i,k} - e_{j,k}) = \sum_{i \in I_V} (\alpha_i \beta_V - \beta_i \alpha_V) e_{i,k}.
\]
This implies \( \alpha \cdot \beta = \beta_V \alpha \) for any \( i \in I_V \setminus \{ k \} \). However, \( k \) can vary in \( I_V \). So the assertion follows.

1.5. Kernel of the differential \( \omega \wedge \). The subject of this subsection has been studied very well in [Fa], [LY], [FY]. Assume \( n = 3 \). Set \( J' = \{ 1, \ldots, d - 1 \} \), and let
\[
\omega = \sum_{i=1}^{d-1} \alpha_i e_i \quad \text{with} \quad \alpha_i \in C^*.
\]
With the notation of (1.4), set \( \alpha_y = \alpha_V := \sum_{i \in I_V} \alpha_i \) if \( V = \{ y \} \). Let \( Z' = Z \setminus Z_d \), and
\[
\Sigma_{(d)} = \{ y \in Z' \mid \operatorname{mult}_y Z' \geq 3 \}, \quad \Sigma_{(d)}^\alpha = \{ y \in \Sigma_{(d)} \mid \alpha_y = 0 \}.
\]
Let \( Z'^{\alpha} \) be the proper transform of \( Z' \) by the blow-up of \( Y' := Y \setminus Z_d \) along \( \Sigma_{(d)}^\alpha \).
We say that \( I \subset J' \) is \( \alpha \)-connected, if \( \bigcup_{i \in I} Z'_i \) is the image of a connected subvariety of \( Z'^{\alpha} \), where \( Z'_1, \ldots, Z'_{d-1} \) are the irreducible components of \( Z' \). Note that the set of \( \alpha \)-connected components is a “neighborly partition” in [Fa], [LY]. Let \( J'_1, \ldots, J'_r \) be the \( \alpha \)-connected components of \( J' \).

Let \( \gamma = \sum_{i=1}^{d-1} \beta_i e_i \) with \( \beta_i \in C \), and assume
\[
\omega \wedge \gamma = 0.
\]
By Lemma (1.4), there are \( c_k \in C \) \( (k = 1, \ldots, r') \) such that
\[
\beta_i = c_k \alpha_i \quad \text{for} \quad i \in J'_k.
\]
So we assume \( r' \geq 2 \) since \( H^1(A^*_{\mathcal{S}(Z)}, \omega \wedge) = 0 \) otherwise. In this case there are intersections \( Z'_i \cap Z'_j = \{ y \} \) such that \( i, j \) belong to different \( \alpha \)-connected components and \( y \in \Sigma_{(d)}^\alpha \). So \( \alpha_y = 0 \), and Lemma (1.4) implies a further condition
\[
\beta_y := \sum_{i \in I_y} \beta_i = 0 \quad (y \in \Sigma_{(d)}^\alpha).
\]
However, the relation between these conditions for different \( y \in \Sigma_{(d)}^\alpha \) is quite non-trivial, see [Fa], [FY]. Since the image of \( \omega \wedge \) in \( A^1_{\mathcal{S}(Z)} \) is 1-dimensional, we get at least
\[
\dim H^1(A^*_{\mathcal{S}(Z)}, \omega \wedge) \leq r' - 2.
\]

1.6. Proof of Theorem 1. Define the connection \( \nabla^\omega \) as in (1.2) setting \( I = I_{\tilde{J}_0} \) for the \( \tilde{J}_0 \) in (0.5), where \( k = d/m \) and the \( \alpha_i \) are defined as in (1.2.3) (although (1.2.1) is not necessarily satisfied). Then \( \alpha_y = 0 \) \( (y \in \Sigma_{(d)}^\phi) \) by the last condition of (0.5), and we get
\[
\Sigma_{(d)}^\alpha = \Sigma_{(d)}^\phi.
\]
since the inclusion \( \subset \) follows from the constantness of \( \alpha_i \) on each \( I_j \). Note that \( I_j \) is not necessary \( \alpha \)-connected in the sense of (1.5). However, we may consider the case
\[
\beta_i = c_i \phi(i) \quad (i = 1, \ldots, d - 1),
\]
where \( c_j \in \mathbb{C} \) \((j = 1, \ldots, r)\). Then the condition given by (1.5.2) is written as

\[
\sum_{j=1}^{r} m_{j,y} c_j = 0 \quad (y \in \Sigma_{(d)}),
\]

and it is independent of \( y \in \Sigma_{(d)} \) by the last condition of (0.4). We get thus

\[
\dim H^1(A_{S(Y)}^\bullet, \omega \wedge) \geq r - 2,
\]

since \( \dim \omega \wedge A_{S(Y)}^0 = 1 \). (Here \( \Sigma_{(d)} \) may be empty.) So the desired inequality for \( \lambda = \exp(2\pi i/a) \) \((a = 1, \ldots, m)\) follows from (1.2.5).

If (0.6) is satisfied, then (0.4) and (0.5) are also satisfied. In this case we have \( \Sigma_{(d)} \neq \emptyset \) since \( |I_j| = d/r \geq 2 \). Since the Milnor monodromy is defined over \( \mathbb{Q} \), it is enough to consider the case \( \lambda = \exp(2\pi i/m) \) using the Galois group of \( \overline{\mathbb{Q}}/\mathbb{Q} \) which acts transitively on the primitive \( m \)-th roots of unity. We define \( \alpha_i \) and \( \omega \) as above where (0.2) is satisfied for \( I = I_{j_0} \). In this case each \( I_j \) is \( \alpha \)-connected, since \( Z_i \cap Z_{i'} \) for \( i, i' \in I_j \) cannot belong to \( \Sigma_{(d)} \) by the condition \( m_{j,y} = 1 \) for \( y \in \Sigma_{(d)} \). So we get

\[
\dim H^1(A_{S(Y)}^\bullet, \omega \wedge) = m - 2,
\]

and the assertion follows from Proposition 1. This finishes the proof of Theorem 1.

### 2. Non-combinatorial description

**2.1. Mixed Hodge complexes.** Set \( \Sigma = \{ y \in Z \mid \text{mult}_y Z \geq 3 \} \) as in Introduction. Let \( \pi : \tilde{Y} \to Y \) be the blow-up of \( Y = \mathbb{P}^2 \) along \( \Sigma \). Set \( \tilde{Z} = \pi^{-1}(Z) \). Let \( L^{(k)} \) be the local system on \( U \) calculating \( H^\bullet(F_\lambda, C) \) for \( \lambda = \exp(2\pi ik/d) \) as in Introduction. Let \( L^{(k)} \) be the Deligne extensions over \( \tilde{Y} \) such that the eigenvalues of the residue of the connection are contained in \([0, 1)\), see [D1].

It is well-known (see e.g. [Es], [Ti]) that the Hodge filtration \( F^p \) on the Milnor cohomology \( H^\bullet(F_\lambda, C) = H^\bullet(U, L^{(k)}) \) is given by the logarithmic de Rham complex whose \( j \)-th component is

\[
DR^j_{\log} L^{(k)} := \Omega^j_{\tilde{Y}}(\log \tilde{Z}) \otimes \mathcal{L}^{(k)},
\]

where the Hodge filtration \( F^p \) is induced by the truncation \( \sigma_{\geq p} \) in [D2], 1.4.7. By the strict compatibility of the Hodge filtration \( F \), we get

\[
\text{Gr}^p_F H^{p+q}(F_\lambda, C) = H^q(\tilde{Y}, \Omega^p_{\tilde{Y}}(\log \tilde{Z}) \otimes \mathcal{L}^{(k)}).
\]

It is also well-known (see e.g. [BS], 1.4) that

\[
L^{(k)} = \mathcal{O}_{\tilde{Y}}((-k)\tilde{H} + \sum_{y \in \Sigma} \lfloor m_y k/d \rfloor E_y),
\]

where \( \lfloor \alpha \rfloor := \max\{ k \in \mathbb{Z} \mid k \leq \alpha \} \), and \( \tilde{H} \) is the pull-back of a general hyperplane \( H \subset Y \).

**2.2. Weight filtration.** Let \( \tilde{j} : U \to \tilde{Y} \) denote the inclusion. The perverse sheaf \( R\tilde{j}_*L^{(k)[2]} \) has the weight filtration \( W \) such that

\[
\text{Gr}^W_2 (R\tilde{j}_*L^{(k)[2]}) = IC_Y L^{(k)} = (\tilde{j}_*L^{(k)})[2],
\]
where the middle term is the intersection complex with coefficients in the local system $L^{(k)}$, see [BBD], [GM]. Here the local monodromy of $L^{(k)}$ is trivial only around the exceptional divisors $E_y$ for $y \in \Sigma(k)$, and these are smooth and disjoint. This implies
\[ (2.2.2) \quad \text{Gr}^W_k(\mathbf{R}j_*L^{(k)}[2]) = 0 \quad \text{for} \quad k \neq 2, 3. \]

Let $\tilde{Z}'$ be the proper transform of $Z$, and set $E'_y = E_y \setminus \tilde{Z}'$ with the inclusion $j_y : E'_y \to E_y$. For $y \in \Sigma(k)$, $L^{(k)}$ can be extended over $U \cup E'_y$, since the local monodromy around $E'_y$ is trivial. Let $L^{(k)}_{E'_y}$ be its restriction over $E'_y$. Then
\[ (2.2.3) \quad \text{Gr}^W_3(\mathbf{R}j_*L^{(k)}[2]) = \bigoplus_{y \in \Sigma(k)} (j_y)_*(L^{(k)}_{E'_y})[1], \]

2.3. Multiplier ideals. Let $Y$ be a smooth complex algebraic variety, and $Z$ be a divisor on it. Let $\pi : \tilde{Y} \to Y$ be an embedded resolution of $Z$. Set $\bar{Z} := \pi^*Z = \sum_i m_i \tilde{Z}_i$. The following is well-known (see e.g. [La]).

For $\alpha \in \mathbb{Q}_{>0}$, the multiplier ideal sheaves $\mathcal{J}(Y, \alpha Z)$ are defined by
\[ (2.3.1) \quad \mathcal{J}(Y, \alpha Z) = \pi_*(\omega_{\tilde{Y}/Y} \otimes \mathcal{O}_{\tilde{Y}}(-\sum_i [\alpha m_i] \tilde{Z}_i)), \]
and we have the local vanishing theorem (see loc. cit. 9.4.1)
\[ (2.3.2) \quad R^i\pi_*(\omega_{\tilde{Y}/Y} \otimes \mathcal{O}_{\tilde{Y}}(-\sum_i [\alpha m_i] \tilde{Z}_i)) = 0 \quad (i > 0). \]

Moreover, if $Y$ is proper and $Z'$ is another divisor on $Y$ such that $Z' - \alpha Z$ is nef and big, then we have the vanishing theorem of Nadel (see loc. cit. 9.4.8)
\[ (2.3.3) \quad H^i(Y, \omega_Y \otimes \mathcal{O}_{\tilde{Y}}(Z') \otimes \mathcal{J}(Y, \alpha Z)) = 0 \quad (i > 0). \]

If $Z$ is a projective hyperplane arrangement in $Y = \mathbb{P}^2$ with reduced structure, let $\mathcal{I}_{(y)}$ be the reduced ideal sheaf of $\{y\} \subset Y$. It is well-known that
\[ (2.3.4) \quad \mathcal{J}(Y, \alpha Z) = \bigcap_{y \in \Sigma} \mathcal{I}_{(y)}^{\lfloor \alpha m_y \rfloor - 1} \quad \text{for} \quad \alpha > 0. \]

This is a special case of Mustaţă’s formula [Mu].

Theorem 2.4. For $\lambda = \exp(2\pi ik/d)$, we have a canonical isomorphism
\[ (2.4.1) \quad \text{Gr}^k_\lambda H^1(F, \mathcal{C}) = H^1(Y, \mathcal{O}_Y(k - 3) \otimes \mathcal{J}(\frac{\lambda}{d} Z)). \]

Proof. By (2.1.1–2) together with Serre duality, we have
\[ \text{Gr}^0_\lambda H^1(F, \mathcal{C}) = H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}((-k)\tilde{H} + \sum_{y \in \Sigma} [m_y k/d] E_y)) \]
\[ = H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}((k - 3)\tilde{H} + \sum_{y \in \Sigma} (1 - [m_y k/d]) E_y)) \]
Here $\omega_{\tilde{Y}/Y} = \mathcal{O}_{\tilde{Y}}(\sum_{y \in \Sigma} E_y)$, and hence $\Omega^2_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-3\tilde{H} + \sum_{y \in \Sigma} E_y).$ So the assertion follows from (2.3.1–2).

2.5. Proof of Theorem 2. Set first $\alpha = \frac{k}{d}$, and $\alpha' = \alpha - \varepsilon$ with $0 < \varepsilon \ll 1/d$. With the notation of Introduction, we have the short exact sequence
\[ (2.3.1) \quad 0 \to \mathcal{J}(Y, \alpha Z) \to \mathcal{J}(Y, \alpha' Z) \to \mathcal{G}(Y, \alpha Z) \to 0. \]
Since $O_Y(Z) = O_Y(d)$ and $\alpha'd < k$, we have by (2.3.3)
\begin{equation}
(2.5.2) \quad H^1(Y, O_Y(k - 3) \otimes J(Y, \alpha'Z)) = 0.
\end{equation}
Then, after taking the tensor product of (2.3.1) with $O_Y(k - 3)$, we get the associated long exact sequence
\[ H^0(Y, J(\alpha'Z)(k - 3)) \xrightarrow{\gamma} H^0(Y, G(\alpha Z)(k - 3)) \rightarrow H^1(Y, J(\alpha Z)(k - 3)) \rightarrow 0, \]
where $J(\alpha Z)(k - 3) := J(Y, \alpha Z) \otimes O_Y(k - 3)$ and similarly for $G(\alpha Z)(k - 3)$. Moreover, $\gamma$ is identified with $\rho^{(k)}$ as explained after Theorem 2. So the first equality in Theorem 2 follows from Theorem (2.4).

For the proof of the second equality, we show, in the above notation, that the first equality holds for $\beta'$ system
\begin{equation}
\begin{aligned}
(2.2.3) & \quad \text{ (i) Let } n = 3, d = 6, \text{ and } \\
& \quad f = xyz(x - y)(x - z)(y - z).
\end{aligned}
\end{equation}
This is the simplest example with $b_1(F) \neq 0$ ($\lambda \neq 1$). Here $\lambda = \exp(2\pi i/3)$. In this case, (0.2) for $k = 2$ and (0.6) for $r = m = 3$ are both satisfied.

(ii) Let $n = 3, d = 9, \text{ and }$
\begin{equation}
\begin{aligned}
& \quad f = xyz(x - y)(y - z)(x - y - z)(2x + y + z)(2x + y - z)(2x - 5y + z).
\end{aligned}
\end{equation}

3. Examples and Remarks

Examples 3.1. The following examples are studied in [CS], [Di1], etc.

(i) Let $n = 3, d = 6, \text{ and }$
\begin{equation}
\begin{aligned}
& \quad f = xyz(x - y)(x - z)(y - z).
\end{aligned}
\end{equation}
This is the simplest example with $b_1(F) \neq 0$ ($\lambda \neq 1$). Here $\lambda = \exp(2\pi i/3)$. In this case, (0.2) for $k = 2$ and (0.6) for $r = m = 3$ are both satisfied.

(ii) Let $n = 3, d = 9, \text{ and }$
\begin{equation}
\begin{aligned}
& \quad f = xyz(x - y)(y - z)(x - y - z)(2x + y + z)(2x + y - z)(2x - 5y + z).
\end{aligned}
\end{equation}
This is the second simplest example with \( b_1(F_f)_{\lambda} \neq 0 \) (\( \lambda \neq 1 \)). Here \( \lambda = \exp(2\pi i/3) \). In this case, (0.2) for \( k = 3 \) and (0.6) for \( r = m = 3 \) are both satisfied. This arrangement is dual to the Pappus configuration.

(iii) Let \( n = 3 \), \( d = 9 \), and
\[
f = xyz(x + y)(y + z)(x + 3z)(x + 2y + z)(x + 2y + 3z)(2x + 3y + 3z).
\]
Then \( b_1(F_f)_{\lambda} = 0 \) for \( \lambda \neq 1 \). In this case neither (0.2) for \( k = 3 \) nor (0.6) for \( r = m = 3 \) is satisfied. So we need Theorem 2 or [Di1], Ch.6, Th. 4.15 to calculate \( b_1(F_f)_{\lambda} \) for \( \lambda = \exp(\pm 2\pi i/3) \), see also [CS]. The referee has pointed out that (1.2.2) holds for this example even though (0.2) is not satisfied.

Remarks 3.2. (i) According to A. Libgober, there is an example with \( n = 3 \), \( d = 9 \), \( |\Sigma| = 12 \), and \( Z := \mathbf{P}(D) \) has no ordinary double point. It is the dual of the nine inflection points of a smooth cubic curve \( E \) in the dual projective space \( \mathbf{P}^2 \), see [Li2], p. 243, Ex. 2. This is shown by choosing an inflection point as the origin \( O \) of the group law so that there is a line passing through three points \( P_1, P_2, P_3 \) on \( E \) if and only if
\[
P_1 + P_2 + P_3 = O.
\]
In this case conditions (0.4–5) are satisfied so that \( b_1(F_f)_{\lambda} \neq 0 \) for \( \lambda = \exp(\pm 2\pi i/3) \) although (0.2) cannot be satisfied. The referee has pointed out that the inequality (1.2.4) is strict in this example.

If the cubic curve is given by \( u^3 + v^3 + w^3 = 0 \), then
\[
f = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3),
\]
and setting \( \theta = \exp(\pm 2\pi i/3) \), we have
\[
\Sigma = \{(\theta^i : \theta^j : 1) \mid i, j = 0, 1, 2\} \cup \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} \subset \mathbf{P}^2.
\]
In this case the evaluation map \( \bigoplus_{y \in \Sigma} \text{ev}_y^3 \) explained after Theorem 2 is injective so that \( b_1(F_f)_{\lambda} = 2 \) for \( \lambda = \exp(\pm 2\pi i/3) \) using [Di1], Ch.6, Th. 4.15 or Theorem 2 in Introduction. (This follows from the assertion that \( a = b = c = 0 \) if \( a\theta^{2i} + b\theta^i + c = 0 \) for \( i = 0, 1, 2 \) where \( a, b, c \in \mathbf{C} \). We have a more geometric proof using the fact that any \( Z_i \) contains 4 points of \( \Sigma \) so that any cubic polynomial vanishing on \( \Sigma \) has to vanish on any \( Z_i \).)

(ii) The argument in Remark (i) can be used to construct further examples with \( b_1(F_f)_{\lambda} = 1 \) for \( \lambda = \exp(\pm 2\pi i/3) \). Indeed, take a finite subgroup \( G \) of \( E \) together with three points \( P_1, P_2, P_3 \) such that \( P_1 + P_2 + P_3 = O \) and \( P_i - P_j \notin G \) for any \( i \neq j \), see [Yu2]. Consider the dual line arrangement associated to \( P_i + Q \) for \( i = 1, 2, 3 \) and \( Q \in G \) where \( d = 3|G| \) and mult\( _y Z \leq 3 \) for any \( y \in \Sigma \) (since the degree of the curve is 3). Then (0.6) for \( r = m = 3 \) is satisfied if furthermore \( 3P_i \notin G \) for some \( i \) (since the last condition implies that the second condition of (0.3) holds). This gives examples with the same combinatorial data as Examples (3.1)(i) and (ii). Here “same combinatorial data” means that there is an isomorphism between the intersection lattices defined by the intersections of their irreducible components. If we set \( P_1 = 0 \) with \( P_2 = -P_3 \) generic and if \( |G| = 3 \), then we get an example where
\(d = 9, |\Sigma| = 10\) and \(b_1(F_f)_\lambda = 1\) for \(\lambda = \exp(\pm 2\pi i/3)\) by Theorem 1 (ii). This example is the specialization of the Pappus arrangement in Example (3.1)(ii), see [Fa].

(iii) As for Example (3.1)(iii), a line arrangement with the same combinatorial data can be constructed as follows. Choose a finite subgroup \(G = \mathbb{Z}/27\mathbb{Z}\) in an elliptic curve, and then take the subset \(J\) of \(G\) consisting of \(a \in G\) with \(a - 1 \in 3G\). The irreducible components of \(D\) defined by the linear factors of \(f\) in Example (3.1)(iii) correspond, for example, to \(7, 1, 4, 19, 22, 16, 13, 10, 25 \in \mathbb{Z}/27\mathbb{Z}\), respecting the order of the factors of \(f\) (and this is confirmed by the referee using Mathematica). It means that we have \(a + b + c = 0\) in \(G\) if and only if the corresponding three lines meet at a point. (We can associate \(a' := (a-1)/3 \in \mathbb{Z}/9\mathbb{Z}\) to \(a \in J\). Then the above condition is equivalent to that \(a' + b' + c' = -1\) in \(\mathbb{Z}/9\mathbb{Z}\), and the latter may be easier to handle.)

(iv) In the case \(\text{mult}_y Z \leq 3\ \forall y \in \Sigma\), it is sometimes possible to construct a finite subset of an elliptic curve defining a line arrangement which is combinatorially equivalent to a given line arrangement \(Z\) as follows. Let \(M\) be a matrix of size \((q,d)\) (where \(q = |\Sigma|\)) such that if \(y \in \Sigma\) is the intersection of three lines \(Z_{i_1}, Z_{i_2}, Z_{i_3}\), then it corresponds to a row vector \(v_y\) of \(M\) such that
\[
v_{y,i} = \begin{cases} 1 & \text{if } i \in \{i_1, i_2, i_3\}, \\ 0 & \text{if } i \notin \{i_1, i_2, i_3\}. \end{cases}
\]
Consider the equation
\[Mx = 0.\]
If it has a nontrivial solution \(x\) in a finite abelian group \(G\) generated by two elements, and if the components \(x_i\) of \(x\) are all different from each other, then the \(x_i\) may define the desired line arrangement after embedding \(G\) into an elliptic curve. Here one problem is that new triple points may appear, i.e. there may exist distinct elements \(x_{i_1}, x_{i_2}, x_{i_3}\) whose sum is \(O\) and which do not correspond to any row vector of \(M\). (In the case of Example (iii) the size of \(M\) is 9 \times 9, and its determinant is 27. So we can solve the equation mod 27. We can also count the number of the triple points, which is 9.)

**Remarks 3.3.** (i) If \(m_{ij,y} = 1\ \forall y \in \Sigma^d\) in the notation of Theorem 1, then \(|I_j|\) is independent of \(j\) as remarked after Theorem 1, and hence \(Z\) is an \((m,d/m)\)-net. So it is associated to a pencil on \(\mathbb{P}^2\) by the Nullstellensatz, see [Yu2], Lemma 3.1. Then Theorem 1 (i) follows as is shown in [DP], Th. 3.1 (i) in a more general case. Indeed, the existence of a pencil implies that there is a surjective morphism of algebraic varieties
\[h : U \to S := \mathbb{P}^1 \setminus \{m \text{ points}\},\]
and moreover the local system \(L^{(k)}\) on \(U\) is the pull-back of a local system \(L^{(k)}_S\) on \(S\). Since \(\dim H^1(S, L^{(k)}_S) = m - 2\), Theorem 1 (i) then follows from the injectivity
of the pull-back
\[ H^1(S, L_S^{(k)}) \to H^1(U, L^{(k)}). \]

Note that the condition \( m_{j,y} = 1 (\forall y \in \Sigma^0) \) implies that there is a natural compactification \( \tilde{h} : \tilde{U} \to S \) of \( h : U \to S \) such that \( \tilde{U} \setminus U = \Sigma^0 \times S \). Hence the base change holds for \( h : U \to S \) and \( \{s\} \hookrightarrow S \), and the Leray spectral sequence for \( h \) degenerates at \( E_2 \) since \( E_2^{p,q} = 0 \) unless \( q = 0, 1 \).

(ii) By [DPS], [FY], [Yu3], we have \( r \leq 4 \) if conditions (0.4–5) are satisfied in Theorem 1 (i). Indeed, if \( E \) denotes the irreducible component of the resonance variety \( R_1(Z) \) (see [Fal]) containing \( \omega \) in the proof of Theorem 1, then we have by [DPS], Prop. 7.8(i)
\[ \dim E = H^1(\mathcal{A}^*_S(D), \omega \wedge) + 1 \geq r - 1, \]
where the last inequality follows from the proof of Theorem 1(i). Moreover, there is a correspondence between the components \( E \) and the pencils, see [FY] (and also [Di3]). Then we get the inequality \( \dim E \leq 3 \) by using [Yu3]. So the desired inequality follows. This argument also shows that a hyperplane arrangement has a structure of a multi-net in the sense of [FY] if the hypotheses (0.4) and (0.5) of Theorem 1(i) are satisfied.

(iii) For the moment only one example is known where the hypotheses of Theorem 1 are satisfied with \( m = 4 \). This is induced by the Hesse pencil (see [FY], Ex. 3.5 and [St1])
\[ s_0(x^3 + y^3 + z^3) - s_1xyz \quad (s_0, s_1 \in \mathbb{C}). \]
Calculating the logarithmic differential of \( (x^3 + y^3 + z^3)/xy \), the singular members of the pencil are given by
\[ s_1/s_0 = 3, \quad 3\theta, \quad 3\bar{\theta}, \quad \infty, \]
where \( \theta = \exp(\pm 2\pi i/3) \). So the line arrangement is defined by
\[ f = xyz \prod_{j=0}^2(x^3 + y^3 + z^3 - 3\theta^j xyz) = xyz \prod_{i,j=0}^2(\theta^i x + \theta^j y + z). \]
In this case \( d = 12 \), and \( |\Sigma| = 9 \). More precisely
\[ \Sigma = \{x^3 + y^3 + z^3 = 0\} \cap \{xyz = 0\} \]
\[ = \{(-\theta^i : 1 : 0), (-\theta^i : 0 : 1), (0 : -\theta^i : 1) \mid i = 0, 1, 2\} \subset \mathbb{P}^2. \]
Then condition (0.6) is satisfied for \( m = r = 4 \) where \( \phi \) corresponds to the first factorization of \( f \). So we get \( \lambda_{(F_f)} \geq 2 \) by Theorem 1 (ii) and \( \lambda_{(F_f)} = 2 \) by Theorem 1 (ii) (see also [DP]). For \( \lambda = -1 \), we can calculate \( \lambda_{(F_f)} \) by using Theorem 2 where \( d = 12 \), \( k = k' = 6 \), and \( m_y = 4 \) for \( y \in \Sigma \). In this case, \( \dim \mathbb{C}[X]_3 = 10 \), and \( \lambda_{(F_f)} = \rho_{(6)} \) is identified with the evaluation map at \( \Sigma \). Its kernel is generated by \( x^3 + y^3 + z^3 \) and \( xyz \), and its cokernel is 1-dimensional. So we get \( \lambda_{(F_f)} = 2 \). It is also possible to calculate \( \lambda_{(F_f)} \) by using Theorem 2.

Remarks 3.4. (i) Let \( f \) be a holomorphic function on a complex manifold \( X \). Let \( \psi_{f,\lambda} \mathcal{C}_X \) denote the \( \lambda \)-eigenspace of the nearby cycle functor \( \psi_f \mathcal{C}_X \). It is well-known that \( \psi_{f,\lambda} \mathcal{C}_X \) is a perverse sheaf up to a shift of complex, and its stalk at \( y \in f^{-1}(0) \)
gives $H^\bullet(F_{f,y}, \mathbb{C})_\lambda$ where $F_{f,y}$ denotes the Milnor fiber of $f$ around $y$. These imply that $H^j(F_{f,y}, \mathbb{C})_\lambda = 0$ for $j \neq \dim X - 1$ if $H^j(F_{f,y^\prime}, \mathbb{C})_\lambda = 0$ for any $j \in \mathbb{Z}$ and any $y^\prime \neq y$ sufficiently near $y$.

(ii) In the case of a hyperplane arrangement, an assertion stronger than (i) is known as follows. We have $b_1(F_f)_\lambda = 0$ unless each $Z_i$ contains $y \in \Sigma$ with $\lambda^{m_y} = 1$ (see [Li3], Th. 3.1 or [Di2], Th. 6.4.13), where we may assume $n = 3$ by (0.1). Indeed, if there is $Z_i$ containing no point $y$ of $\Sigma$ with $\lambda^{m_y} = 1$, then letting $j : U \hookrightarrow Y$, $j_i : U_i := Y \setminus Z_i \hookrightarrow Y$ denote the inclusions, we have

$$R_j^*L^{(k)} = (j_i)_!j_i^*(R_j^*L^{(k)})$$

using the blow-up along $\Sigma$, where $\lambda = \exp(2\pi ik/d)$. Then the assertion follows from Artin’s theorem on the generalization of the weak Lefschetz theorem for perverse sheaves [BBD].

REFERENCES

[BBD] Beilinson, A., Bernstein, J. and Deligne, P., Faisceaux pervers, Astérisque, vol. 100, Soc. Math. France, Paris, 1982.

[Br] Brieskorn, E., Sur les groupes de tresses [d’après V.I. Arnold], Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, Lect. Notes in Math. Vol. 317, Springer, Berlin, 1973, pp. 21–44.

[BS] Budur, N. and Saito, M., Jumping coefficients and spectrum of a hyperplane arrangement, preprint (arXiv:0903.3839).

[CS] Cohen, D.C. and Suciu, A., On Milnor fibrations of arrangements, J. London Math. Soc. 51 (1995), 105–119.

[D1] Deligne, P., Equations Différentielles à Points Singuliers Réguliers, Lect. Notes in Math. vol. 163, Springer, Berlin, 1970.

[D2] Deligne, P., Théorie de Hodge II, Publ. Math. IHES, 40 (1971), 5–58.

[Di1] Dimca, A., Singularities and Topology of Hypersurfaces, Universitext, Springer, Berlin, 1992.

[Di2] Dimca, A., Sheaves in Topology, Universitext, Springer, Berlin, 2004.

[Di3] Dimca, A., Pencils of plane curves and characteristic varieties, preprint.

[DP] Dimca, A. and Papadima, S., Finite Galois covers, cohomology jump loci, formality properties, and multinet, preprint (arXiv:0906.1040).

[DPS] Dimca, A., Papadima, S. and Suciu A., Topology and geometry of cohomology jump loci, preprint (arXiv:0902.1250), to appear in Duke Math. J.

[Es] Esnault, H., Fibre de Milnor d’un cône sur une courbe plane singulière, Inv. Math. 68 (1982), 477–496.

[ESV] Esnault, H., Schechtman V. and Viehweg, E., Cohomology of local systems on the complement of hyperplanes, Inv. Math. 109 (1992), 557–561.

[Fa] Falk, M., Arrangements and cohomology, Ann. Combin. 1 (1997), 135–157.

[FY] Falk, M. and Yuzvinsky, S., Multinets, resonance varieties, and pencils of plane curves, Compos. Math. 143 (2007), 1069–1088.

[GM] Goresky, M. and MacPherson, R., Intersection homology theory, Topology 19 (1980), 135–162.

[La] Lazarsfeld, R., Positivity in algebraic geometry II, Springer, Berlin, 2004.

[Li1] Libgober, A., Alexander invariants of plane algebraic curves, in Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983, pp. 135–143.
[Li2] Libgober, A., Characteristic varieties of algebraic curves, in Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), NATO Sci. Ser. II Math. Phys. Chem., 36, Kluwer Acad. Publ., Dordrecht, 2001, pp. 215–254.

[Li3] Libgober, A., Eigenvalues for the monodromy of the Milnor fibers of arrangements, in: Trends in Singularities, Birkhäuser, Basel (2002).

[LY] Libgober, A. and Yuzvinsky, S., Cohomology of the Orlik-Solomon algebras and local systems, Compos. Math. 121 (2000), 337–361.

[Mu] Mustață, M., Multiplier ideals of hyperplane arrangements, Trans. Amer. Math. Soc. 358 (2006), 5015–5023.

[OS] Orlik, P. and Solomon, L., Combinatorics and topology of complements of hyperplanes, Inv. Math. 56 (1980), 167–189.

[Sa1] Saito, M., Mixed Hodge modules, Publ. RIMS, Kyoto Univ. 26 (1990), 221–333.

[Sa2] Saito, M., Bernstein-Sato polynomials of hyperplane arrangements (math.AG/0602527).

[STV] Schechtman, V., Terao, H. and Varchenko, A., Local systems over complements of hyperplanes and the Kac-Kazhdan conditions for singular vectors, J. Pure Appl. Algebra 100 (1995), 93–102.

[St1] Stipins, J., Old and new examples of $k$-nets in $\mathbb{P}^2$, preprint (arXiv:0701046).

[St2] Stipins, J., On finite $k$-nets in the complex projective plane, Ph. D. Thesis, The University of Michigan, 2007.

[Te] Teitler, Z., A note on Mustață’s computation of multiplier ideals of hyperplane arrangements, Proc. Amer. Math. Soc. 136 (2008), 1575–1579.

[Ti] Timmerscheidt, K., Mixed Hodge theory for unitary local systems, J. Reine Angew. Math. 379 (1987), 152–171.

[Yu1] Yuzvinsky, S., Cohomology of the Brieskorn-Orlik-Solomon algebras, Comm. Algebra 23 (1995), 5339–5354.

[Yu2] Yuzvinsky, S., Realization of finite abelian groups by nets in $\mathbb{P}^2$, Compos. Math. 140 (2004), 1614–1624.

[Yu3] Yuzvinsky, S., A new bound on the number of special fibers in a pencil of curves, Proc. Amer. Math. Soc. 137 (2009), 1641–1648.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF NOTRE DAME, IN 46556, USA
E-mail address: nbudur@nd.edu

LABORATOIRE J.A. DIEDONNÉ, UMR DU CNRS 6621, UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS, PARC VALROSE, 06108 NICE CEDEX 02, FRANCE
E-mail address: Alexandru.DIMCA@unice.fr

RIMS KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN
E-mail address: msaito@kurims.kyoto-u.ac.jp