Transformation Optics for the Modelling of Waves in a Universe with Nontrivial Topology

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Abstract

We consider how transformation optics and invisibility cloaking can be used to construct models, where the time-harmonic waves for a given angular wavenumber \(k\), are equivalent to the waves in some closed orientable manifold. The obtained models could in principle be physically implemented using a device built from metamaterials. In particular, the measurements, that is, the source-to-solution operator, in the metamaterial device are equivalent to measurements in a universe given by \((\mathbb{R}_+ \times M, -dt^2 + g)\), where \((M, g)\) is a closed, orientable, \(C^\infty\)-smooth, 3-manifold. Thus the obtained construction could be used to simulate cosmological models using metamaterial devices.

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1 Introduction

The study of cosmic topology seeks to answer a fundamental question: what is the shape of the universe? By shape, cosmologists mean to identify both the topological structure of the universe and its Lorentzian spacetime structure. It is assumed that the universe is given by a $C^\infty$-smooth differentiable 4-manifold of the form $\mathbb{R}_+ \times M$, where $M$ is a 3-manifold. The spacetime structure on $\mathbb{R}_+ \times M$ is proposed to be a Friedmann-Lemaître-Robertson-Walker metric

$$-dt^2 + A(t)g(x), \quad (1.1)$$

where $A : \mathbb{R}_+ \times M \to \mathbb{R}$ is a positive function and $g$ is a Riemannian metric on $M$. At the time of writing this paper, it is not yet known what are the topological and curvature properties of the spatial manifold $(M, g)$ which best fits gathered cosmological data. However, new cosmic microwave background radiation (CMBR) data obtained by the European Space Agency’s Planck satellite has implicated that $(M, g)$ has nonzero curvature [6, 28]. Further, using the new Planck measurements, it is argued in [6] and [32] that a closed manifold $(M, g)$ best fits the data. Earlier literature using CMBR data from NASA’s WMAP satellite have also suggested that $(M, g)$ might be a closed 3-manifold [31], [33], [23]. For example, Luminet et. al. contended that the correlations in the cosmic microwave background data available in 2003 were consistent of the Universe having the topology of the Poincaré dodecahedral space [33].

In this paper, we consider a spacetime model of the universe given by a Lorentzian manifold $(\mathbb{R}_+ \times M, dt^2 + g)$ (we have set $A(t) = 1$ in (1.1)), whose constant time slices are isometric to a closed, orientable, $C^\infty$-smooth Riemannian 3-manifold $(M, g)$. Given such a model, we propose a method to construct a laboratory device which represents the approximate optical properties of $(M, g)$. The existence of such a device would enable cosmologists to both simulate the optical properties of a candidate manifold $(M, g)$ for the universe structure and compare this information to experimental data.

The device we propose is a cloaking device based on the principle of transformation optics. An object is cloaked if it is rendered invisible to measurements. Devices which electromagnetically cloak an object have been built using the technology of metamaterials [1], [35], [4]. These so-called metamaterials are composite materials designed to manipulate the properties of electromagnetic radiation, so that the phase velocity of the waves with an (angular) wave number $k$ may be a very large or small. One way to prescribe the properties and arrangement of the metamaterials for a cloaking device is through the method of transformation optics. Here, the viewpoint is that the metamaterials comprising the cloaking device mimic the properties of waves on an abstract Riemannian manifold, which is called virtual space. To realize these waves in the laboratory, a piecewise smooth diffeomorphism, called the transformation map, from virtual space into $\mathbb{R}^3$ is constructed. The image of the transformation map is called...
physical space. The Riemannian metric induced on physical space from the pushforward of the Riemannian metric on virtual space by the transformation map dictates the metamaterial properties, whereas the transformation map itself describes their arrangement.

Transformation maps which blow up a point in an Euclidean ball were first proposed in [15], in the context of demonstrating nonuniqueness to Calderón’s problem. Extensions and perturbations of this type of transformation map construction have been widely promoted for the manufacturing of cloaking devices. In particular, the mathematical feasibility of such cloaks has been studied in various physical and theoretical settings. For optical waves governed by a Helmholtz equation, a non-exhaustive list of the literature wherein the stability and accuracy of such a cloak has been analyzed is [13], [22], [30], [14], [24], [10], [29], [26] and [5]. For a survey of some of these results, please see [12] and [37]. For electromagnetic waves described by Maxwell’s equations, the mathematical analysis of cloaks in this setting has been addressed in [10], [9], [11], and [25].

We highlight that the authors in [9] and [11] constructed a cloaking device which models the electromagnetic properties of a virtual space with the geometry of a wormhole. They achieved a cloak with this geometry by building a transformation map which “blows up a curve”. This construction led to the experimental implementation of a magnetic wormhole by physicists in 2015 [36]. In this experimental implementation, the required metamaterial was built using superconducting elements.

We take inspiration from [9] and [11] and define a transformation map which allows us to render a physical space model $\tilde{M} \subset \mathbb{R}^3$ for any virtual space given by a closed, orientable, $C^\infty$-smooth, 3-manifold $M$ as above.

To do this, we demonstrate and exploit the existence of a link in $M$. We call a union $L$ of finitely many, pair-wise disjoint, embedded, simple closed curves in $M$ a link in $M$. Assume for now that $M$ contains a link $L$, and that there is a smooth diffeomorphism

$$\Psi : M \setminus L \to \tilde{M} \subset \mathbb{R}^3.$$  

We equip $\tilde{M}$ with the pushforward metric $\tilde{g} := \Psi_\ast g$. For us, $(\tilde{M}, \tilde{g})$ is then the physical space candidate for a cloaking device which describes static optical waves in the universe model $(\mathbb{R} \times M, -dt^2 + g)$. That is, we use the map $\Psi$ to define the properties of a cloaking device which recreates a good approximation of the optics of time-harmonic waves on the spatial manifold $(M, g)$. Time-harmonic waves on $(M, g)$ are solutions $U(t, x) = e^{-ikt}u(x)$, to the wave equation

$$\partial_t^2 U(t, x) - \Delta_g U(t, x) = e^{-ikt}f(x)$$

on $\mathbb{R} \times M$, where $\Delta_g$ is the Laplace-Beltrami operator associated to $g$, $k \in \mathbb{C}$ is the wavenumber, $f : M \to \mathbb{C}$ is an electromagnetic source, and $u : M \to \mathbb{C}$ solves the Helmholtz equation

$$\Delta_g u + k^2 u = \frac{1}{\sqrt{|\det g|}} \sum_{a,b=1}^3 \partial_a \left( \sqrt{|\det g|} g^{ab} \partial_b u \right) + k^2 u = f$$  

(1.2)

on $M$. Here $g^{ab}$ are the components of the inverse metric $g^{-1} : T^*M \times T^*M \to [0, \infty)$. The symmetric tensor $\sigma$ with components

$$\sigma^{ab} = \sqrt{|\det g|} g^{ab}$$  

(1.3)
is called the *conductivity* (in the electrostatic case). We note that the conductivity tensor is in fact a product of a 2-contravariant tensor and a half density. If \( V \subset M \) is an open set and \( f \) has compact support in \( V \), we define the *source to solution map* as

\[
\Lambda_V : L^2(V, g) \to L^2(V, g), \quad \Lambda_V(f) = u|_V,
\]

where \( u \) solves (1.2). We note that in this paper we consider \( L^2(V, g) \) as the subspace of \( L^2(M, g) \) consisting of functions with support over \( V \).

To show our transformation map gives an appropriate prescription of \( \tilde{g} = \Psi^* g \), and hence for the conductivity \( \sigma \) of the materials used for the cloaking device, we establish that waves in the physical space \((\tilde{M}, \tilde{g})\) stably reproduces waves in the virtual space \((M, g)\).

That is, for \( \epsilon > 0 \) let \( T(\epsilon) \) be a tubular neighbourhood of \( L \) of radius \( \epsilon \) and consider the Helmholtz equation

\[
\Delta_{\tilde{g}} \tilde{u}_\epsilon + \kappa^2 \tilde{u}_\epsilon = \tilde{f} \quad \text{on} \quad \tilde{M} \setminus \tilde{T}(\epsilon)
\]

\[
\partial_{\tilde{n}} \tilde{u}_\epsilon = 0 \quad \text{on} \quad \partial \tilde{T}(\epsilon),
\]

now where \( \tilde{T}(\epsilon) \), \( \epsilon > 0 \), is a tubular neighbourhood of \( \Sigma = \partial \tilde{M} \) and \( \tilde{n}_\epsilon \) is the unit outward pointing normal vector field on \( \partial \tilde{T}(\epsilon) \). In this setting, if \( \tilde{V} \subset \tilde{M} \setminus \tilde{T}(\epsilon) \) is an open set and \( \tilde{f} \) has compact support in \( \tilde{V} \), the *source to solution map* associated to the problem (1.5) is defined as

\[
\tilde{\Lambda}_{\tilde{V}, \epsilon} : L^2(\tilde{V}, \tilde{g}) \to L^2(\tilde{V}, \tilde{g}), \quad \tilde{\Lambda}_{\tilde{V}, \epsilon}(\tilde{f}) = \tilde{u}_\epsilon|_{\tilde{V}},
\]

where \( \tilde{u}_\epsilon \) solves (1.5) with source \( \tilde{f} \).

The following theorem is our main result, which roughly speaking states that in the device, designed by transformation optics, it is possible to simulate with arbitrary precision the time-harmonic waves in a static universe that are produced by any compactly supported source.

**Theorem 1.1.** Let \( \tilde{V} \subset \tilde{M} \) be a relatively compact open set, \( \tilde{f} \in L^2(\tilde{V}, \tilde{g}) \), and \( -\kappa^2 \in \text{res}(\Delta_{\tilde{g}}) \). Then,

\[
\lim_{\epsilon \to 0} \tilde{\Lambda}_{\tilde{V}, \epsilon} \tilde{f} = \Psi^* \Lambda_V \Psi^* \tilde{f}
\]

strongly in \( L^2(\tilde{V}, \tilde{g}) \).

Moreover, if additionally \( \tilde{V} \subset \tilde{M} \) is a bounded domain with boundary \( \partial \tilde{V} \in C^m(\tilde{M}) \) for some \( m \geq 0 \), and also for some \( 1 > \alpha \geq 0 \) satisfying \( m + \alpha > 1/2 \), we have \( g_{ab}, \tilde{g}^{ab} \in C^{m+1,\alpha}(\tilde{V}) \), for \( a, b \in \{1, 2, 3\} \), and \( f \in C^{m,\alpha}_0(\tilde{V}) \), then

\[
\lim_{\epsilon \to 0} \tilde{\Lambda}_{\tilde{V}, \epsilon} \tilde{f} = \Psi^* \Lambda_V \Psi^* \tilde{f}
\]

in \( C^{m+2,\alpha}_0(\tilde{V}) \).

Now we return to the existence of the transformation map \( \Psi \) used in the Theorem (1.1). We show that we may obtain a link blow-up map \( \Psi \) as a consequence of the following extension of the classical Lickorish–Wallace theorem. This theorem states that the class of the homeomorphism performing the Lickorish–Wallace surgery contains a \( C^\infty \)-smooth map which differential satisfies also the bound (2.6) given below.
Proposition 1.2. Let $M$ be a smooth, closed, oriented, and connected Riemannian 3-manifold. Then, there exists a smooth link $L \subset M$ and a $C^\infty(M \setminus L)$-smooth embedding $\Psi : M \setminus L \to \mathbb{R}^3$.

1.1 Overview

In Section 2, we motivate our proof of Proposition 1.2 by considering a similar situation in two dimensions. We also prove that the analogous result to Proposition 1.2 does not hold in two dimensions. We defer the proof of Proposition 1.2 to Section 4. We also give soft estimates for the metric $\tilde{g}$ on our virtual space.

Section 3 contains the set up and preliminary work to prove Theorem 1.1. First, in Section 3.1 we define the function spaces we work with and establish equivalences between Sobolev spaces on our virtual space $(M, g)$ and those of the same order on $(M \setminus L, g)$ and our physical space $(\tilde{M}, \tilde{g})$. Then, in Section 3.2, we define Friedrichs extensions of the operators appearing in (1.2) and (1.5), as well as their corresponding sesquilinear forms. Section 3.3 establishes $\Gamma$-convergence results for the forms in Section 3.2. These convergence results allow us to prove Theorem 1.1 in the case where the wavenumber appearing in (1.2) and (1.5) is negative. As we are concerned with wave solutions, in Section 3.4, we establish convergence of the resolvents associated to the Friedrich's extensions of $\Delta g$. Lastly, in Section 3.5 we use the convergence results in Section 3.4 to prove Theorem 1.1.

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2 Virtual and physical spatial models

In this section, we describe our construction of a piecewise smooth embedding $M \setminus L \to \mathbb{R}^3$, where $L$ is a link in $M$. This embedding will be composition of a piecewise smooth embedding $M \setminus L \to \mathbb{S}^3$ and a stereographic projection map. We will give a heuristic overview of the construction of how we obtain the embedding $M \setminus L \to \mathbb{S}^3$, and place the details of the argument in Section 4. To begin, we recall the concept of a link in $M$:

Definition 2.1. A subset $L$ is a link in a manifold $M$ if $L$ is a union of finitely many, pairwise disjoint embedded circles $S_1, \ldots, S_J$. That is, $L = \bigcup_{j=1}^J S_j$ where $S_j \cap S_k = \emptyset$ for $j \neq k$, and for each $j = 1, \ldots, J$, there exists an embedding $\phi_j : \mathbb{S}^1 \to M$ for which $S_j = \phi_j(\mathbb{S}^1)$. Further, we say that a link $L$ in $M$ is smooth if we may take each embedding $\phi_j$ to be a smooth embedding.

If there exists a link $L \subset M$, we may define a compact tubular neighbourhood $T$ of this link which is diffeomorphic to a union of pairwise disjoint solid tori. Then, we perform a surgery of $M$ along the boundary $\partial T$. That is, we remove the interior of $T$ and transform $T$ to a homeomorphic copy $T'$. Then we glue $T'$ to $M \setminus \text{int}T$ along the boundary of $T'$. This is analogous to the situation in 2-dimensions, where a pair of disks on a surface is replaced with a cylinder to obtain a surface with higher genus. Below, we explain how to use surgery to construct a piecewise smooth homeomorphism in the 2-dimensional case to motivate the role surgery near the link $L \subset M$ plays in the construction of the embedding in the 3-dimensional case.
2.1 An excursion: 2-dimensional embedding theorem

We define a \textit{surface} as a 2-manifold with or without boundary. The boundary of a surface will always refer to its manifold boundary.

Closed and orientable surfaces are classified by their genus; we do not define the notion of genus here, please see [17] for the appropriate details. We use the genus to represent a closed and orientable surface as the boundary of a 3-dimensional manifold in $\mathbb{R}^3$ as follows.

Recall that the 2-sphere $S^2 \subset \mathbb{R}^3$, which is the boundary of the closed unit 3-ball $\bar{B}^3$, is a surface of genus 0. Let now $J \in \mathbb{Z}_+$. For $j = 1, \ldots, J$, we fix embeddings $\psi_j: \bar{B}^2 \times [0, 1] \to \mathbb{R}^3$ having mutually disjoint images which satisfy

$$\phi_j(\bar{B}^2 \times [0, 1]) \cap \bar{B}^3 = \phi_j(\bar{B}^2 \times \{0\}) \subset S^2$$

for each $j = 1, \ldots, J$. Then

$$M = \bar{B}^3 \cup \phi_1(\bar{B}^2 \times \{0\}) \cup \cdots \cup \phi_J(\bar{B}^2 \times \{0\})$$

is a 3-manifold with boundary in $\mathbb{R}^3$ and the boundary $\Sigma = \partial M$ is a surface of genus $J$. Typically, the images of embeddings $\phi_i$ are called \textit{handles} of $M$ and the manifold $M$ a handlebody.

Having the surface $\Sigma$ at our disposal, we may formulate a 2-dimensional analog of Proposition 1.2 as follows.

\textbf{Proposition 2.2.} Let $\Sigma$ be a surface of genus $J \in \mathbb{Z}_+$. Then there exists a link $L \subset \Sigma$ consisting of $J$ circles, pair-wise disjoint closed disks $D_1, \ldots, D_{2J} \subset S^2$, and a homeomorphism $\Sigma \setminus L \to S^2 \setminus (D_1 \cup \cdots \cup D_{2J})$.

A careful reader will recall that the embedding in Proposition 1.2 is smooth, whereas the embedding $\Sigma \setminus L \to S^2$ in Proposition 2.2, induced by the homeomorphism, is merely topological. The existence of a smooth embedding follows from the classical results of Radó on Riemann surfaces, which states that each topological surface carries a unique smooth structure which in turn implies that homeomorphic smooth surfaces are diffeomorphic.

\textit{Proof of Proposition 2.2.} We may assume that $\Sigma = \partial M$, where $M$ is the genus $J$ handlebody constructed above. Let also embeddings $\phi_i: \bar{B}^2 \times [0, 1] \to \mathbb{R}^3$ be as in the construction of $M$ for $j = 1, \ldots, J$.

For each $j = 1, \ldots, J$, we denote

$$E_{2j+k} = \phi_j(B^2 \times \{k\}), \quad D_{2j+k} = \phi_j(\bar{B}^2(1/2) \times \{k\}), \quad \text{and} \quad H_j = \phi_j(S^1 \times (0, 1)).$$

We observe first that

$$\Sigma \cap S^2 = S^2 \setminus (E_1 \cup \cdots \cup E_{2J})$$

and that

$$\Sigma = (\Sigma \cap S^2) \cup (H_1 \cup \cdots \cup H_J);$$

heuristically, $\Sigma$ is obtained by removing open disks $E_1, \ldots, E_{2J}$ from $S^2$ and attaching open cylinders $H_1, \ldots, H_J$. 
Let $L \subset \Sigma$ be the link with circles
$$S_j = \phi(S^1 \times \{1/2\}) \subset H_j$$
for $j = 1, \ldots, J$. Let also
$$H_j^- = \phi_j(B^2 \times (0, 1/2)) \quad \text{and} \quad H_j^+ = \phi_j(B^2 \times (1/2, 1))$$
be open cylinders for each $j = 1, \ldots, J$; note that $H_j \setminus S_j = H_j^- \cup H_j^+$.

We define now a homeomorphism $h: \Sigma \setminus L \to S^2 \setminus (D_1 \cup \cdots \cup D_J)$ as follows. On $\Sigma \cap S^2$, we set $h$ to be the identity. Then, for each $j = 1, \ldots, J$, we take, $h|_{H_j^-}$ to be the unique embedding satisfying
$$h(\phi_j(x,t)) = \phi_j((1-t)x,0)$$
for $(x,t) \in \bar{B}^2 \times (0,1/2)$. Similarly, we take $h|_{H_j^+}$ to be the unique embedding satisfying
$$h(\phi_j(x,t)) = \phi_j(tx,1)$$
for $(x,t) \in \bar{B}^2 \times (1/2,1)$. This completes the proof.

We remark that for two dimensional manifolds there are invisibility cloaking results where one removes the origin from a two dimensional disc $D^2$ of radius 2 and center 0 and uses the transformation optics approach with a blow-up diffeomorphism $F: \mathbb{D}^2 \setminus \{0\} \to \mathbb{D}^2 \setminus \mathbb{D}^1$, where $\mathbb{D}^1$ is a disc of radius 1. These cloaking results are based on the fact that the single point $\{0\}$ has the zero capacitance in $\mathbb{D}^2$. Thus it would be natural to ask, could one construct a two dimensional model where waves are equivalent to those on a two dimensional closed manifold $S$ by removing a finite number of points $p_1, p_2, \ldots, p_J$ from the manifold and then mapping the set $S \setminus \{p_1, p_2, \ldots, p_J\}$ to the two-dimensional plane. As the capacitance zero sets on $S$ (such as $\{p_1, p_2, \ldots, p_J\}$) have Hausdorff dimension zero, the following theorem shows that unlike in the 3-dimensional case of Theorem 1.2, in the 2-dimensional case we cannot achieve an embedding into $\mathbb{R}^2$ by removing a subset capacitance zero (or generally, a set which Hausdorff dimension is less than one) from a manifold that is not homeomorphic to a sphere.

**Theorem 2.3.** Let $S$ be a Riemann surface of positive genus $g > 0$ and $E \subset S$ a set of Hausdorff $1$-measure zero, that is, $\mathcal{H}^1(E) = 0$. Then there is no embedding $S \setminus E \to \mathbb{R}^2$.

**Remark 2.4.** Note that, although the Riemannian metric is not fixed on $S$, the null sets of Hausdorff $1$-measure are well-defined on $S$.

**Proof.** By the Uniformization theorem, there exists a covering map $\pi: X \to S$, from the universal cover $X$ to $S$, which is a conformal local diffeomorphism and where $X$ is the Euclidean space $\mathbb{C}$ in the case $g = 1$ or the hyperbolic disk $D$ for $g \geq 2$. Let also $\Gamma$ be the deck group of isometries of $\pi$ and let $\Omega$ be a fundamental domain for $\Gamma$. Then $\Omega \subset X$ is a $2g$-gon, whose opposite faces the covering map $\pi$ identifies.

Since $\pi$ is conformal local diffeomorphism, we have that the set $\tilde{E} = \pi^{-1}(E) \cap \Omega$ has Hausdorff $1$-measure zero.

Given that $g > 0$, we may fix two different pairs $\alpha, \alpha'$ and $\beta, \beta'$ of opposite faces of $\Omega$ for which the interiors of $\alpha \cup \alpha'$ and $\beta \cup \beta'$ on $\partial \Omega$ do not meet.
We fix now line segments \( \ell_\alpha \subset \Omega \) and \( \ell_\beta \subset \Omega \) as follows. First, consider all line segments \( \ell = [x, x'] \) in \( \Omega \) connecting faces \( \alpha \) and \( \alpha' \) for which points \( x \in \alpha \) and \( x' \in \alpha' \) are not corner points and \( \pi(x) = \pi(x') \).

From the fact that \( H^1(\tilde{E}) = 0 \), we may apply Fubini’s theorem to obtain that there exists among these line segments a line segment \( \ell_\alpha \) for which \( \ell_\alpha \cap \tilde{E} = \emptyset \). We fix a line segment \( \ell_\beta \) analogously. To this end, we also record the observation that, since \( \alpha \neq \beta \) and the end points of the line segments are not corner points, we have that \( \ell_\alpha \) and \( \ell_\beta \) meet exactly at one point.

Let now \( \gamma_\alpha = \pi(\ell_\alpha) \) and \( \gamma_\beta = \pi(\ell_\beta) \). Then \( \gamma_\alpha \) and \( \gamma_\beta \) are closed curves on \( S \). Furthermore, since \( \Omega \) is a fundamental domain, we have that \( \gamma_\alpha \) and \( \gamma_\beta \) are simple closed curves, that is, they are homeomorphic to \( S^1 \). Moreover, \( \gamma_\alpha \cap E = \gamma_\beta \cap E = \emptyset \) by construction.

Suppose now that there exists an embedding \( f : S \setminus E \to \mathbb{R}^2 \). Then \( f(\gamma_\alpha) \) is a Jordan curve in \( \mathbb{R}^2 \). By the Jordan curve theorem, \( \mathbb{R}^2 \setminus f(\gamma_\alpha) \) consists of two components \( D \) and \( D' \) and \( \partial D = \partial D' = f(\gamma_\alpha) \). Since the line segment \( \ell_\beta \) meets \( \gamma_\alpha \) in \( \Omega \) transversally in one point, the same holds for \( \gamma_\beta \) and \( \gamma_\alpha \) in \( S \), and further for \( f(\gamma_\beta) \) and \( f(\gamma_\alpha) \) in \( \mathbb{R}^2 \). Let \( x_0 \in \mathbb{R}^2 \) be the intersection point \( f(\gamma_\beta) \cap f(\gamma_\alpha) \). Thus \( f(\gamma_\beta) \) intersects non-trivially both sets \( D \) and \( D' \).

Let \( x_0 \in \mathbb{R}^2 \) be the intersection point \( f(\gamma_\beta) \cap f(\gamma_\alpha) \). Then \( \{D \cap f(\gamma_\beta), D' \cap f(\gamma_\beta)\} \) is a separation of \( f(\gamma_\beta) \setminus \{x_0\} \). This is a contradiction \( f(\gamma_\beta) \) is homeomorphic to \( S^1 \) and \( S^1 \) cannot be separated by one point. We conclude that such embedding \( f \) does not exist. \( \square \)

### 2.2 Estimates for the physical model metric

Given \( M \), by Proposition 1.2 or its refinement Theorem 4.6 (please see section 4 and the proof therein), there exists a link \( L \subset M \), a domain \( D \subset \mathbb{S}^3 \), and a diffeomorphism \( F : M \setminus L \to D \). Consider a point \( p_0 \in \mathbb{S}^3 \setminus [D \cup \partial D] \), and let \( \pi_{p_0} : \mathbb{S}^3 \setminus \{p_0\} \to \mathbb{R}^3 \) be stereographic projection through the point \( p_0 \). This gives a diffeomorphism

\[
\Psi : M \setminus L \to \tilde{M} \subset \mathbb{R}^3, \quad \Psi := \pi_{p_0} \circ F,
\]

where \( \tilde{M} := \pi_{p_0}(D) \) is a \( C^\infty \)-smooth manifold embedded in \( \mathbb{R}^3 \).

By setting

\[
\tilde{g} : T\tilde{M} \times T\tilde{M} \to [0, \infty), \quad \tilde{g} = \Psi_* g.
\]

we achieve a Riemannian isometry

\[
\Psi : (M \setminus L, g) \to (\tilde{M}, \tilde{g}).
\]

The submanifold \( \Sigma := \partial \tilde{M} \subset \mathbb{R}^3 \) is called the cloaking surface. We emphasize that the metric \( \tilde{g} \), represented in the Euclidean coordinates, is not assumed to be bounded from above or below near \( \Sigma \). From this point onwards, we will use a label with a tilde to denote an object associated to the physical space \( (\tilde{M}, \tilde{g}) \) and un-tilded labels to denote objects associated to the virtual space \( (M, g) \).

Recall that the optical and electromagnetic properties of the virtual space are captured by the Riemannian metric \( g \), via the conductivity tensor \( \sigma = \sqrt{|\det g|} g^{-1} \). In our physical space
model $(\tilde{M}, \tilde{g})$, the conductivity tensor is given by the pushforward of $\sigma$ by $\Psi$:

$$\tilde{\sigma}(\tilde{x}) := \Psi_*\sigma(x) = \frac{D\Psi(x)\sigma(x)D\Psi^T(x)}{|\det D\Psi(x)} = \sqrt{|\det \tilde{g}(\tilde{x})\tilde{g}^{-1}(\tilde{x})}|.$$  \tag{2.3}$$

Above, $D\Psi$ denotes the differential of $\Psi$ and $D\Psi^T$ denotes the transpose of the matrix $D\Psi$.

We shall see that the quantities $\tilde{g}(\tilde{x})$, $\sqrt{|\det \tilde{g}(\tilde{x})|}$, and $\tilde{\sigma}(\tilde{x})$ become singular as $\tilde{x} \in \tilde{M}$ approaches $\Sigma$. In the next section we will demonstrate this behaviour by employing an advantageous coordinate system near $\Sigma$.

About each curve $S_j \subset L$, we consider the (open) tubular neighbourhood of radius $r > 0$:

$$T_j(r) := \{x \in M : \text{dist}_g(x, S_j) < r\}.$$ 

Since the curves $S_j$ are disjoint, there exists an $0 < R \leq 1$ such that if $r < R$, the elements of the collection $\{T_j(r) : j = 1, 2, \ldots, J\}$ are pairwise disjoint. We further restrict the value $R$ to be smaller than the injectivity radius of $L$ in $M$.

Then, for any $0 < r \leq R$, let

$$T(r) := \bigcup_{j=1}^{J} T_j(r)$$ 

be a tubular neighbourhood about the link $L$ and let $V_\theta$ be the inward-pointing unit normal vector to $\partial T(R)$ at $\phi(R \times \{s\} \times \{j\})$. Now, by our choice of $R$, for all $r \leq R$ and for all $j = 1, \ldots, J$, the map

$$\phi_{\text{norm}} : [0, R] \times \mathbb{S} \times \mathbb{S} \times \{1, \ldots, J\} \to T(R), \quad \phi_{\text{norm}}(r, \theta, s, j) = \exp_{\phi(R, s, j)}((R - r)V_\theta),$$

is a diffeomorphism from $[0, R] \times \mathbb{S} \times \mathbb{S} \times \{1, \ldots, J\}$ to $T(R)$. We call the coordinate system given by $x = (r, \theta, s, j)$ normal coordinates adapted to $L$. Additionally, we denote by $\partial T(\epsilon)$ the boundary of $T(\epsilon)$ and by $\nu_\epsilon$ the outward-point unit normal vector field on $\partial T(\epsilon)$.

Next, we turn our attention to the model $(\tilde{M}, \tilde{g})$ and define an analogous coordinates system near the surface $\Sigma \subset \mathbb{R}^3$ via the map $\Psi = \pi_{p_0} \circ F$. For this purpose, consider once again the tori $T(r)$, $0 < r \leq R$ defined in (2.4). From the map $\Psi$, we obtain corresponding open tori

$$\tilde{T}(r) := \{\tilde{x} \in \tilde{M} : \tilde{x} = \Psi(x), x \in T(r) \setminus L\} \subset \tilde{M}.$$  \tag{2.5}$$

The union $\tilde{T}(r) = \bigcup_{j=1}^{J} \tilde{T}_j(r)$ give a tubular neighbourhood of $\Sigma$:

$$\Sigma \subset \tilde{T}(r).$$

Since $F$ is a diffeomorphism away from the link $L$ and $\tilde{g} = \Psi_*g$, for $R > 0$ as above the map

$$\tilde{\phi}_{\text{norm}} := \pi_{p_0} \circ \beta,$$
where
\[
\beta : \overline{B}^2(R) \setminus \overline{B}^2(1/2) \times S \times \{1, \ldots, J\} \to \tilde{T}(R),
\]
\[
\beta(\tilde{r}, \theta, s, j) = \exp_{F\circ\phi(R,s,j)}((2\tilde{r} - 1)F_V\theta),
\]
\[
\tilde{r} = \frac{1 + R - r}{2},
\]
is a diffeomorphism. We call the coordinates \((\tilde{r}, \theta, s, j)\) normal coordinates adapted to \(\Sigma\).

Similar as before, we write \(\partial \tilde{T}(\epsilon)\) for the boundary of \(\tilde{T}(\epsilon)\) and by \(\tilde{\nu}_\epsilon\) the outward-point unit normal vector field on \(\partial \tilde{T}(\epsilon)\).

In the above normal coordinates adapted to \(\Sigma\), we have that away from \(\tilde{M} \cup \Sigma\),
\[
|\tilde{g}^{-1}(\tilde{x})| \leq C \text{ for } \tilde{x} \in \mathbb{R}^3 \setminus (\tilde{M} \cup \Sigma),
\]
where \(C > 0\) is a uniform constant independent of \(\tilde{r}\) and depending on \(\Psi\) and its derivatives. Similarly, for \(x \in \tilde{M} \cup \Sigma\), there is a uniform constant \(C > 0\) such that
\[
|\tilde{g}^{aa}(\tilde{x})| \leq \frac{C}{(2\tilde{r} - 1)} \text{ for } a \in \{s, \theta\},
\]
\[
|\tilde{g}^{a\tilde{r}}(\tilde{x})| \leq C \text{ for } a \in \{\tilde{r}, s\},
\]
\[
|\tilde{g}^{\theta\theta}(\tilde{x})| \leq \frac{C}{(2\tilde{r} - 1)^2}.
\]
Additionally, we have the estimate
\[
\sqrt{\tilde{g}} \leq C(2\tilde{r} - 1). \tag{2.7}
\]

Combining the above inequalities, the conductivity \(\tilde{\sigma} := \sqrt{\tilde{g}}\tilde{g}^{-1}\) satisfies
\[
|\tilde{\sigma}^{ab}(\tilde{x})| \leq C(2\tilde{r} - 1) \text{ for } a \in \{\tilde{r}, s\}, \tag{2.8}
\]
\[
|\tilde{\sigma}^{a\theta}(\tilde{x})| \leq C \text{ for } a \in \{s, \theta\}, \tag{2.9}
\]
\[
|\tilde{\sigma}^{\theta\theta}(\tilde{x})| \leq \frac{C}{2\tilde{r} - 1}, \tag{2.10}
\]
and hence becomes singular as \(\tilde{r} \to \frac{1}{2}\).

### 3 The behaviour of optical waves in virtual and physical space

In this section, we present the proof of Theorem 1.1; using the map \(\Psi : M \setminus L \to \tilde{M}\), we will actually show that Theorem 1.1 is equivalent to the claim

**Theorem 3.1.** Let \(V \subset M \setminus L\) be a relatively compact open set, \(f \in L^2(V, g)\), and \(-k^2 \in \text{res}(\Delta_g)\). Then,
\[
\lim_{\epsilon \to 0} \Lambda_{V, \epsilon} f = \Lambda_{V, 0} f \tag{3.1}
\]
strongly in \(L^2(V, g)\).
Moreover, if additionally $V \subset M \setminus L$ is a bounded domain with $\partial V \in C^m(M,g)$ for some $m \geq 0$, and also for some $1 > \alpha \geq 0$ satisfying $m + \alpha > 1/2$, we have $g^{ab}, g_{ab} \in C^{m+1,\alpha}(V,g)$, for $a,b \in \{1,2,3\}$, and $f \in C_0^{m,\alpha}(V,g)$, then

$$\lim_{\epsilon \to 0} \Lambda_{V,\epsilon} f = \Lambda_{V,0} f$$

in $C_0^{m+2,\alpha}(V,g)$.

Above,

$$\Lambda_{V,\epsilon} : L^2(V,g) \to L^2(V,g), \quad \Lambda_{V,\epsilon}(f) = u_\epsilon|_V,$$

is the source to solution map associated to the Helmholtz problem

$$\Delta g u + k^2 u = f \quad \text{on } M \setminus T(\epsilon)$$

$$\partial_\nu u = 0 \quad \text{on } \partial T(\epsilon),$$

where $T(\epsilon), \epsilon > 0$ are the tubular neighbourhoods specified by (2.4) and $\nu_\epsilon$ is the outward pointing unit normal vector field on $\partial T(\epsilon)$. The map

$$\Lambda_{V,0} : L^2(V,g) \to L^2(V,g), \quad \Lambda_{V,0}(f) = u|_V,$$

is the source to solution map for the problem

$$\Delta g u + k^2 u = f \quad \text{on } M \setminus L.$$ 

Having demonstrated the equivalence between Theorem 1.1 and Theorem 3.1, we will then proceed to prove Theorem 3.1. The proof of Theorem 3.1 will be split into preliminary steps, which we group into subsections. First, to simplify the notation in this section and the following sections, when the context is clear, we let $g$ to denote the Riemannian metric on either $M$ or the induced metric $g|_{M \setminus L}$ on $M \setminus L$.

In Section 3.1, using the map $\Psi$, we will define and establish equivalences between Sobolev spaces on $(M,g)$, $(M \setminus L,g)$, and $(\tilde{M},\tilde{g})$.

Next, in Section 3.2, we will extend the operators in (3.3) and (1.2) to operators defined on $L^2(M \setminus L,g)$. Combined with the results in Section 3.1, we will then show that Theorem 1.1 is equivalent to 3.1.

Sections 3.3 and 3.4 contain the bulk of the work needed to establish (3.1). In Section 3.3, we use the machinery of $\Gamma$-convergence to prove (3.1) for $k^2 > 0$. In Section 3.4, we use classical linear perturbation theory to show (3.1) for other values of $k \in \mathbb{C}$, in particular for $k^2 < 0$, which corresponds to sinusoidal solutions of (3.3) and (1.2).

Before proceeding, we define convenient notation which will be used from this point onward. We use $\nabla$ for the covariant derivative associated to $g$, and similarly write $\nabla$ for the covariant derivative associated to $\tilde{g}$. Throughout this paper, $\mu$ denotes the Lebesgue measure on $\mathbb{R}^3$, and $\mu_g$ (respectively $\mu_{\tilde{g}}$) denotes the measure induced on $M$ (respectively $\tilde{M}$) by $g$ (respectively $\tilde{g}$).

We note here that given local coordinates $M$ or $\tilde{M}$, the volume forms satisfy $d\mu_g = \sqrt{|\det g|} d\mu$ and $\mu_{\tilde{g}} = \sqrt{|\det \tilde{g}|} d\mu$. We will abuse notation and write $d$ to denote exterior differentiation of forms either on $M$, $\tilde{M}$, or $\mathbb{R}^3$, and use the context to make the distinction.
3.1 Function spaces

For any $C^\infty$-smooth 3-manifold $\Omega$ and any smooth Riemannian metric $h$ on $\Omega$, we write $L^1_{\text{loc}}(\Omega, h)$ for the set of all locally $d\mu_h$-integrable functions $u : \Omega \to \mathbb{C}$.

We denote the space of square integrable functions on $(\Omega, h)$ by

$$L^2(\Omega, h) = \{ u \in L^1_{\text{loc}}(\Omega, h) : \| u \|_{L^2(\Omega, h)} < \infty \},$$

where the norm is given by

$$\| u \|_{L^2(\Omega, h)}^2 = \int_\Omega |u|^2 \, d\mu_h.$$

When $h = g_e$, we use the shorthand notion $L^2(\Omega) = L^2(\Omega, g_e)$. Sometimes, it will be convenient to write $H^0(\Omega, h) = L^2(\Omega, h)$. Also, we observe that if $\sqrt{\det h}$ is bounded and measurable with respect to $\mu$, then $L^2(\Omega) \subset L^2(\Omega, h)$.

Now, given a coordinate chart $(x^1, x^2, x^3) : W \subset \Omega \to \mathbb{R}$ and a multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we use $\partial^\alpha := \partial^{\alpha_1}_{x^1} \partial^{\alpha_2}_{x^2} \partial^{\alpha_3}_{x^3}$ to denote the distributional derivative. With this notation set, we next define the following Sobolev spaces. First,

$$H^1(\Omega, h) = \{ u \in L^1_{\text{loc}}(\Omega, h) : \forall |\alpha| \leq 1, \partial^\alpha u \in L^1_{\text{loc}}(\Omega, h), \| u \|_{H^1(\Omega, h)} < \infty \},$$

where

$$\| u \|_{H^1(\Omega, h)}^2 = \| u \|_{L^2(\Omega, h)}^2 + \int_\Omega h(\nabla u, \nabla u) \, d\mu_h,$$

and $(\nabla u)^a = \sum_{b=1}^3 h^{ab} \partial_b u$ is the local coordinate expression of the covariant derivative of $u$. We emphasize that $\partial_b u$, $b = 1, 2, 3$, denotes the distributional derivative, when it exists.

Second, if $\Omega$ has boundary $\partial \Omega \neq \emptyset$, we define the set of trace-free functions $u \in H^1(\Omega, h)$:

$$H^1_0(\Omega, h) = \text{cl}_{H^1(\Omega, h)}[C_0^\infty(\Omega)].$$

Third,

$$H^2(\Omega, h) = \{ u \in L^1_{\text{loc}}(\Omega, h) : \forall |\alpha| \leq 2, \partial^\alpha u \in L^1_{\text{loc}}(\Omega, h), \| u \|_{H^2(\Omega, h)} < \infty \},$$

with norm

$$\| u \|_{H^2(\Omega, h)}^2 = \| u \|_{H^1(\Omega, h)}^2 + \int_\Omega \| \nabla^2 u \|^2_h \, d\mu_h.$$

Here, in coordinates $(x^1, x^2, x^3)$ on $\Omega$, we have

$$\| \nabla^2 u \|^2_h = \sum_{a,b,c,d=1}^3 h^{ab}h^{cd}(\nabla^2 u)_{ac}(\nabla^2 u)_{bd},$$

and $\nabla^2 u$ is the Hessian of $u$; it has the local expression

$$(\nabla^2 u)_{ab} = \partial_a \partial_b u - \sum_{c,d=1}^3 \frac{1}{2} h^{cd}(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}) \partial_c u.$$
The Hessian is also related to the Laplace-Belrami operator on $\Omega$ through the trace over $h$:

$$\Delta_h u = \text{tr}_h \nabla^2 u.$$  

In local coordinates this becomes

$$\Delta_h u = \sum_{a,b=1}^3 \frac{1}{\sqrt{|\det h|}} \partial_a \left( \sqrt{|\det h|} h^{ab} \partial_b u \right).$$

**Lemma 3.2.** Let $p = 0, 1, 2$. Then, the restriction map

$$\mathfrak{B} : H^p(M, g) \to H^p(M \setminus L, g), \quad \mathfrak{B}(u) = u|_{M \setminus L}$$

is an isomorphism. Moreover,

$$H^1(M, g) = H^1_0(M \setminus L, g).$$

**Proof.** First consider the case when $p = 0$. Let $u \in L^2(M, g)$. As $L$ has measure zero with respect to the measure induced on $M$ by $g$, we have

$$||u||^2_{L^2(M, g)} = ||\mathfrak{B}(u)||^2_{L^2(M \setminus L, g)}.  \quad (3.6)$$

From this, we see that $\mathfrak{B}$ is well-defined, continuous, injective, and an isometry onto its image. If $v \in L^2(M \setminus L, g)$, by extending $v$ to be zero on $L$, we obtain a function $u^v \in L^2(M, g)$ with $\mathfrak{B}(u^v) = v$. This proves that $\mathfrak{B}$ is surjective, and hence $\mathfrak{B} : L^2(M, g) \to L^2(M \setminus L, g)$ is an isomorphism.

Next, let $p = 1$. To prove the claim in this case, we need only to show that the link $L \subset M$ has zero capacitance. Then, by [18, Theorem 2.44] we achieve the desired isometry.

For $\epsilon > 0$, let $T(\epsilon)$ be a tubular neighbourhood about the link $L \subset M$, as defined in (2.4), and let $r_0 > 0$ be such that the tubular neighborhood of radius $r_0$ is well defined. The 2-capacitance [18, Section 2.35] of $L$ in $T(r_0)$ is defined as

$$\text{Cap}(L) = \inf \{ ||\nabla u||^2_{L^2(M)} : u \in H^1(M, g), \ u \geq 1 \text{ on a neighbourhood of } L \text{ and supp } \subset \bar{T}(r_0) \}.$$  

As in [18, Example 2.12] and [18, Thm. 2.2] (see also [18, Cor 2.39]), we see that the 2-capacity of the set $L$ in $T(r_0)$ is zero.

Before proceeding with the $p = 2$ case, we prove the moreover part of the lemma. Since $L$ has vanishing capacity, by [18, Theorem 2.45], we have the isometry

$$H^1(M \setminus L, g) = H^1_0(M \setminus L, g).$$

Our above argument then gives

$$H^1_0(M \setminus L, g) = H^1(M, g),$$

as claimed.

Finally, suppose that $p = 2$. As in the previous cases, it is easy to see that $\mathfrak{B} : H^2(M, g) \to H^2(M \setminus L, g)$ is well-defined, continuous, injective, and an isometry onto its image.
Let \( v \in H^2(M \setminus L, g) \); by definition we also have \( v \in H^1(M \setminus L, g) \). From the proof for the \( p = 1 \) case, there exists a \( u^v \in H^1(M, g) \) with \( \mathfrak{B}(u^v) = v \) and \( \|u^v\|_{H^1(M, g)} = \|u\|_{H^1(M \setminus L, g)} \).

Let \( W \subset M \) be a coordinate neighbourhood and \( \psi \in C_0^\infty(W) \). As \( H^1_0(M \setminus L, g) = H^1(M, g) \) and \( \mathfrak{B} : H^1(M, g) \to H^1(M \setminus L, g) \) is an isomorphism, there exists functions \( \phi_j \in C_0^\infty(W \setminus L) \) such that

\[
\|\phi_j - \psi\|_{H^1(W, g)} \to 0
\]
as \( j \to \infty \). Additionally, set \( f = \Delta_g u \in L^2(M \setminus L, g) \); from the \( p = 0 \) case proved above, there exists an \( h^f \in L^2(M, g) \) with \( \mathfrak{B}(h^f) = f \). We thus compute

\[
\int_M g(\nabla u^v, \nabla \phi_j) \, d\mu_g = \int_{M \setminus L} g(\nabla v, \nabla \phi_j) \, d\mu_g = -\int_{M \setminus L} f \phi_j \, d\mu_g = -\int_M h^f \phi_j \, d\mu_g.
\]

Taking the limit as \( j \to \infty \) and integrating by parts gives us

\[
\int_M \Delta_g u^v \psi \, d\mu_g = \int_M h^f \psi \, d\mu_g.
\]

Since both \( \psi \) and the chart \( W \) were arbitrary, we deduce that \( u^v \in H^2(M, g) \). This proves that \( \mathfrak{B} \) is surjective, which is the last property we needed to confirm that \( \mathfrak{B} \) is an isomorphism.

\[\Box\]

**Lemma 3.3.** For \( p \geq 0 \), the diffeomorphism \( \Psi : M \setminus L \to \tilde{M} \) induces the unitary operators

\[
\mathfrak{S} : H^p(\tilde{M}, \tilde{g}) \to H^p(M \setminus L, g), \quad \mathfrak{S}(\tilde{u}) = \tilde{u} \circ \Psi.
\]

**Proof.** We prove the claim for \( p = 0 \). The proof for \( p > 0 \) is analogous.

Let \( \tilde{u} \in L^2(\tilde{M}, \tilde{g}) \) and set \( u := \mathfrak{S}(\tilde{u}) \). For \( x \in \tilde{M} \) we also write \( \tilde{x} \) for its local coordinate expression. Since \( \Psi \) is a diffeomorphism and \( \tilde{g} = \Psi_\ast g \), employing a change of coordinates \( x = \Psi^{-1}(\tilde{x}) \) gives

\[
||\tilde{u}||^2_{L^2(\tilde{M}, \tilde{g})} = \int_{M \setminus L} ||u \circ \Psi(x)||^2 \sqrt{|\det g(x)|} \, |\det D\Psi(x)| \, dx = ||u||^2_{L^2(M \setminus L, g)}.
\]  

By construction, \( \mathfrak{S} \) is a linear operator. From the above computation we achieve that \( \mathfrak{S} \) is well-defined, continuous, and an isometry onto its image. To prove \( \mathfrak{S} \) is unitary, it only remains to show that \( \mathfrak{S} \) is surjective.

Let \( u \in L^2(M \setminus L, g) \). Define \( \tilde{u} := u \circ \Psi^{-1} \). From (3.7), \( \tilde{u} \in L^2(\tilde{M}, \tilde{g}) \) and we have \( \mathfrak{S}(\tilde{u}) = u \). This concludes the proof.

\[\Box\]

We will also require use of the so-called Hölder function spaces. Once again, let \( (\Omega, h) \) be a \( C^\infty \)-smooth Riemannian manifold. Given \( m \geq 0 \) and \( 1 > \alpha > 0 \), we define the \((m, \alpha)\)-Hölder space as

\[
C^{m, \alpha}(\Omega, h) = \{ u \in C^m(\Omega, h) : ||u||_{C^{m, \alpha}(\Omega, h)} < \infty \},
\]
where
\[ \|u\|_{C^{m,\alpha}(\Omega,h)} = \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{\text{dist}_g(x,y)} + \sum_{a=1}^{m} \sup_{x \in \Omega} h(\nabla^a u, \nabla^a u). \]

Lastly, if \( \Omega \) has boundary \( \partial \Omega \), we set
\[ C_0^{m,\alpha}(\Omega,h) = C^{m,\alpha}(\Omega,h) \cap C_0(\Omega,h) \]
to be the \((m,\alpha)\)-Hölder space of functions which vanish on \( \partial \Omega \).

### 3.2 Sesquilinear forms and their Friedrichs extensions

In this section, we will construct the elliptic operators appearing in equation (1.2) and the Neumann boundary value problem (3.3) by using Friedrich’s method. We then will relate these operators which act on functions defined on virtual space to operators which act on functions on physical space.

We start with prescribing a set of forms which will eventually give rise to our desired operators. First, designate by \( a_0 \) the sesquilinear form
\[ a_0 : L^2(M,g) \times L^2(M,g) \rightarrow \mathbb{R} \cup \{\infty\} \]
\[ a_0[u,v] = \begin{cases} \int_M g(\nabla u, \nabla v) \, d\mu_g & \text{if } u,v \in H^1(M,g) \\ \infty & \text{otherwise} \end{cases} . \]

This form defines a quadratic form \( a_0[u] = a_0[u,u] \), with domain
\[ \text{dom}(a_0) = H^1(M,g) . \]

We also call \( \text{dom}(a_0) \) the domain of \( a_0[\cdot,\cdot] \). In general:

**Definition 3.4.** Let \((H, \|\cdot\|_H)\) be a Hilbert space, and let \( s[\cdot,\cdot] : H \times H \rightarrow \mathbb{C} \) be a sesquilinear form and \( s[\cdot] : H \rightarrow \mathbb{C} \) the associated quadratic form. The domain of \( s[\cdot,\cdot] \) (respectively \( s[\cdot] \)) is the subset \( D \subset H \) such that
\[ s[\cdot,\cdot] : D \times D \rightarrow \mathbb{C} \text{ and } s[\cdot] : D \rightarrow \mathbb{C} \]
are defined. In this case, we write \( \text{dom}(s) = D \).

Second, let
\[ q_0 : L^2(M \setminus L,g) \times L^2(M \setminus L,g) \rightarrow \mathbb{R} \cup \{\infty\} \]
\[ q_0[u,v] = \begin{cases} \int_{M \setminus L} g(\nabla u, \nabla v) \, d\mu_g & \text{if } u,v \in H^1(M \setminus L,g) \\ \infty & \text{otherwise} \end{cases} , \]
with domain \( \text{dom}(q_0) = H^1(M \setminus L,g) \).
Third and last, we define a 1-parameter family of forms which will be associated to the operators appearing in 3.3. Let $T(\epsilon)$, with $\epsilon > 0$, be a tubular neighbourhood of $L$ as in (2.4), and define a sesquilinear form $q_{\epsilon}$ as
\[
q_{\epsilon} : L^2(M \setminus L, g) \times L^2(M \setminus L, g) \to \mathbb{R} \cup \{\infty\}
\]
\[
q_{\epsilon}[u, v] = \begin{cases} 
\int_{M \setminus T(\epsilon)} \sigma(\partial u, \partial v) \, d\mu & \text{if } u|_{M \setminus T(\epsilon)}, v|_{M \setminus T(\epsilon)} \in H^1(M \setminus T(\epsilon), g) \\
\infty & \text{otherwise}
\end{cases}
\]
These forms have domain given by
\[\text{dom}(q_{\epsilon}) = \{u \in L^2(M \setminus L, g) : u|_{M \setminus T(\epsilon)} \in H^1(M \setminus T(\epsilon), g)\}.
\]
Foremost, we record some properties of the forms $q_{\epsilon}$ in the following lemma.

**Lemma 3.5.** Let $a_0$ and $q_{\epsilon}$, $\epsilon \geq 0$, be the forms defined above. Then,

1. the form $a_0$ is densely defined in $L^2(M, g)$ and the forms $q_{\epsilon}$, $\epsilon \geq 0$, are densely defined in $L^2(M \setminus L, g)$.

Additionally, these forms are

2. closed,
3. positive, and
4. symmetric.

**Proof.**

1. By definition, $H^1(M \setminus L, g) \subset \text{dom}(q_{\epsilon})$ and $H^1(M \setminus L, g) = \text{dom}(q_0)$. As $H^1(M \setminus L, g)$ is dense in $L^2(M \setminus L, g)$ and $\text{dom}(a_0) = H^1(M, g)$ is dense in $L^2(M, g)$, this proves the first claim.

2. We now prove that $q_{\epsilon}$ is closed for $\epsilon > 0$; the proof is similar for $q_0$ and $a_0$.

Let $v \in \text{dom}(q_{\epsilon})$; notice that
\[q_{\epsilon}[v, v] \leq \|v|_{M \setminus T(\epsilon)}\|_{H^1(M \setminus T(\epsilon), g)}^2.
\]
Suppose now that $u_j \in \text{dom}(q_{\epsilon})$ is a sequence such that $u_j \to u$ in $L^2(M \setminus L, g)$ and $q_{\epsilon}[u_j, u_j] \to q_{\epsilon}[u, u]$ as $j \to \infty$. We compute
\[
|q_{\epsilon}[u_j, u_j] - q_{\epsilon}[u, u]| \leq \|u_j\|_{H^1(M \setminus T(\epsilon), g)}^2 - \|u\|_{H^1(M \setminus T(\epsilon), g)}^2
\]
\[= (\|u_j\|_{H^1(M \setminus T(\epsilon), g)} + \|u\|_{H^1(M \setminus T(\epsilon), g)}) \cdot \left(\sqrt{q_{\epsilon}[u_j, u_j]} - \sqrt{q_{\epsilon}[u, u]}\right).
\]
As $q_{\epsilon}[u_j, u_j] \to q_{\epsilon}[u, u]$ as $j \to \infty$, we must have that $\|u_j\|_{H^1(M \setminus T(\epsilon), g)} + \|u\|_{H^1(M \setminus T(\epsilon), g)}$ is bounded. Thus, $u \in \text{dom}(q_{\epsilon})$ which implies that $q_{\epsilon}$ is closed.

3. and 4. The forms $a_0$ and $q_{\epsilon}$, $\epsilon \geq 0$ are symmetric and positive as the tensor $\sqrt{\det g} \cdot g$ is both symmetric and positive.
Next, we present a classical result built on the ideas of Friedrichs and Kato which demonstrates that the forms $a_0$ and $q_\epsilon$, $\epsilon \geq 0$, define self-adjoint operators on an appropriate $L^2$-space, and these operators extend the Laplace-Beltrami operator $\Delta_g$ to a subset of $L^2$-functions. This result holds due to the preceding Lemma 3.5.

**Theorem 3.6** (Friedrich’s Extension [7]). Let $a_0$ and $q_\epsilon$, $\epsilon \geq 0$, be the sesquilinear forms defined above. Then, there exists densely defined, closed, self-adjoint, linear operators

$$A_0 : L^2(M, g) \rightarrow L^2(M, g)$$

and

$$Q_\epsilon : L^2(M \setminus L, g) \rightarrow L^2(M \setminus L, g), \quad \epsilon \geq 0,$$

which satisfy

- $\text{dom}(A_0) = H^2(M);$  
- $\text{dom}(Q_\epsilon) = \{ u \in H^1(M \setminus L) : u|_{M \setminus T(\epsilon)} \in H^2(M \setminus T(\epsilon)), \partial_{\nu} u|_{\partial T(\epsilon)} = 0 \}, \epsilon > 0;$  
- $\text{dom}(Q_0) = H^2(M \setminus L);$  
- $\mathbf{a}_0[u, v] = \langle A_0 u, v \rangle_{L^2(M, g)};$  
- $\mathbf{q}_\epsilon[u, v] = \langle Q_\epsilon u, v \rangle_{L^2(M \setminus L, g)}, \epsilon \geq 0;$  
- for $u \in \text{dom}(Q_\epsilon)$, $\epsilon \geq 0$, $\Delta_g u = -Q_\epsilon u$ in the distributional sense on $M \setminus T(\epsilon)$ if $\epsilon > 0$ and on $M \setminus L$ if $\epsilon = 0;$  
- for $u \in \text{dom}(A_0)$, $\Delta_g u = -A_0 u$ in the distributional sense on $M$.

With the above operators defined, we conclude this section by demonstrating the equivalence of Theorem 1.1 and Theorem 3.1. To do so, we first define weak solutions $\tilde{u}$ to the Helmholtz equation on $(\tilde{M}, \tilde{g})$:

**Definition 3.7.** We say that $\tilde{u} \in H^1(\tilde{M}, \tilde{g})$ is a finite energy solution of the Helmholtz equation with source $\tilde{f} \in L^2(\tilde{M}, \tilde{g})$ and wavenumber $k \in \mathbb{C}$, $k \neq 0$, if for all $\tilde{\phi} \in H^1(\tilde{M}, \tilde{g})$

$$\int_{\tilde{M}} \left[ \tilde{g}(\tilde{d}\tilde{u}, d\tilde{\phi}) - k^2 \tilde{u}\tilde{\phi} \right] d\mu_{\tilde{g}} = -\int_{\tilde{M}} \tilde{f}\tilde{\phi} d\mu_{\tilde{g}}. \quad (3.8)$$

When (3.8) holds, we write

$$\Delta_{\tilde{g}} \tilde{u} + k^2 \tilde{u} = \tilde{f} \quad \text{on } \tilde{M}. \quad (3.9)$$

Then, we show:

**Lemma 3.8.** Let $k \in \mathbb{C}$, $f \in L^2(M, g)$, and $u \in H^1(M, g)$ solve $\Delta_g u + k^2 = f$ on $M$. Then, $u \circ \Psi^{-1}$ is a finite energy solution the Helmholtz equation with source $f \circ \Psi^{-1}$ and wavenumber $k$.

**Proof.** Set $\tilde{u} := u \circ \Psi^{-1}$ and $\tilde{f} = f \circ \Psi^{-1}$. By Lemma 3.3 $\tilde{u} \in H^1(\tilde{M}, \tilde{g})$ and $\tilde{f} \in L^2(\tilde{M}, \tilde{g})$.

It remains to show that $\tilde{u}$ is a finite energy solution of the Helmholtz equation on $\tilde{M}$.
Let $r, \epsilon > 0$. To evaluate integrals to follow, let $T(r)$ be a tubular neighbourhood about the link $L \subset M$, as defined in (2.4). Let $\tilde{T}(\epsilon)$ be a tubular neighbourhood about $\Sigma$, as defined in (2.5).

Given $\tilde{\phi} \in H^1(\tilde{M}, \tilde{g})$, set $\phi = \tilde{\phi} \circ \Psi$. Then, $\phi \in H^1(M \setminus L)$ and by the above considerations $\phi$ has an extension $\phi_\epsilon \in H^1(M)$ that satisfies $\phi_\epsilon|_{M \setminus L} = \phi$. Since $u$ solves $\Delta_g u + k^2 u = f$ on $M$ and the support of $f$ does not intersect $L$, we see that $u$ is $C^\infty$-smooth in some $M$-neighborhood of the set $L$. Moreover, $\Psi : M \setminus T(\epsilon) \to \tilde{M} \setminus \tilde{T}(\epsilon)$ is a Riemannian isometry,

$$
\int_{\tilde{M}} \tilde{g}^{-1}(\tilde{d}u, d\tilde{\phi}) - k^2 \tilde{u} \tilde{\phi} + \tilde{f} \tilde{\phi} \, d\tilde{\mu}_{\tilde{g}} = \lim_{\epsilon \to 0} \int_M \chi_\epsilon [g^{-1}(d\tilde{u}, d\tilde{\phi}) - k^2 \tilde{u} \tilde{\phi} + \tilde{f} \tilde{\phi}] \sqrt{|\det g|} d\mu \\
= \lim_{r \to 0} \int_{M \setminus T(r)} [g^{-1}(du, d\phi) - k^2 u \phi + f \phi] \sqrt{|\det g|} d\mu \\
= \int_{M \setminus L} [g^{-1}(du, d\phi) - k^2 u \phi + f \phi] \sqrt{|\det g|} d\mu \\
= \int_{M \setminus L} [g^{-1}(du, d\phi_e) - k^2 u \phi_e + f \phi_e] \sqrt{|\det g|} d\mu \\
= \int_M [g^{-1}(du, d\phi_e) - k^2 u \phi_e + f \phi_e] \sqrt{|\det g|} d\mu \\
= 0,
$$
as $L \subset M$ is zero-measurable. This proves that $\tilde{u}$ is a finite energy solution of the Helmholtz equation on $\tilde{M}$.

Now to show the equivalence of Theorem 1.1 and Theorem 3.1, we next use the diffeomorphism $\Psi : M \setminus L \to \tilde{M}$ to relate waves on $(\tilde{M}, \tilde{g})$ and waves on $(M \setminus L, g)$.

**Lemma 3.9.** Let $T(\epsilon) \subset M$ for $\epsilon > 0$ be the tubular neighbourhood of $L$ defined in (2.4), and let $V \subset M \setminus L$ be open. We have for all $f \in L^2(M \setminus T(\epsilon), g)$

$$
\Lambda_{V, \epsilon}(f) = \Psi^* \tilde{\Lambda}_{\Psi(V), \epsilon} \Psi_* f. \tag{3.10}
$$

Additionally, for all $f \in L^2(M \setminus L, g)$,

$$
\Lambda_{V, 0}(f) = \Psi^* \tilde{\Lambda}_{\Psi(V)} \Psi_* f. \tag{3.11}
$$

**Proof.** The first claim follows from Lemma 3.3. The second claim follows from the Lemma 3.3 and Lemma 3.8.

To obtain the equivalence of Theorem 1.1 and Theorem 3.1, it now only remains to prove an equivalence between measurements on the virtual space $(M, g)$ and measurements on the modified space $(M \setminus L, g)$.

**Lemma 3.10.** Let $V \subset M \setminus L$ be open. We have for all $f \in L^2(M, g)$

$$
\Lambda_{V, 0}(\mathcal{B} f) = \Lambda_V f,
$$

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where \( \mathcal{B} \) is the restriction map.

**Proof.** From Theorem 3.6, we have that \( \text{dom}(A_0) = H^2(M, g) \) and \( \text{dom}(Q_0) = H^2(M \setminus L, g) \). Then, Lemma 3.2 implies that

\[
\mathcal{B}(\text{dom}(A_0)) = \text{dom}(Q_0) \quad \text{and} \quad \text{dom}(A_0) = \mathcal{B}^{-1}(\text{dom}(Q_0)).
\]

Hence we have

\[
\mathcal{B} \circ A_0 = Q_0 \quad \text{and} \quad Q_0 \circ \mathcal{B}^{-1},
\]

which implies the claim. \( \square \)

Since now our aim is to prove that (3.1) holds, our next natural step is to demonstrate that \( q_\epsilon \) converges to \( q_0 \) in a suitable sense. This is the goal of the next Section.

### 3.3 \( \Gamma \)-convergence of the sesquilinear forms

We recall the definition of \( \Gamma \)-convergence:

**Definition 3.11 ([34] and [3]).** Let \((X, h)\) be a metric space and \( \{ F_\epsilon : X \to [-\infty, \infty], \epsilon > 0 \} \) be a 1-parameter family of functionals on \( X \). For \( x \in X \) let \( N(x) \) denote the set of all open neighbourhoods \( U \subset X \) of \( x \). We say the family \( F_\epsilon \) \( \Gamma \)-converges to a functional \( F : X \to [-\infty, \infty] \) if

\[
F = \sup_{U \in N(x)} \liminf_{\epsilon \to 0} \inf_{y \in U} F_\epsilon = \sup_{U \in N(x)} \limsup_{\epsilon \to 0} \inf_{y \in U} F_\epsilon.
\]

In this case, we say that \( F_\epsilon \) \( \Gamma \)-converges to \( F \) in \( X \) and write

\[
F_\epsilon \Gamma \rightharpoonup F.
\]

We now proceed to formulate the stability of solutions to the Helmholtz problems (3.3). This is done by considering the quadratic forms \( q_\lambda \) and \( q_{\lambda, \epsilon} \) associated to the Helmholtz operators \( Q_\lambda \) and \( Q_{\lambda, \epsilon} \) in the framework of \( \Gamma \)-convergence.

**Lemma 3.12.** The quadratic form

\[
q_\epsilon[u] = q_\epsilon[u, u], u \in L^2(M \setminus L, g)
\]

converges pointwisely in \( L^2(M \setminus L, g) \) to

\[
q^1[u] := q_0[u, u], u \in L^2(M \setminus L, g).
\]

Additionally,

\[
q_\epsilon^1 \Gamma \rightharpoonup q^1
\]

strongly in \( L^2(M \setminus L, g) \) and weakly in \( H^1(M \setminus L, g) \).
Proof. First we show that $q_\ell^1 \to q^1$ pointwisely in $L^2(M \setminus L, g)$. Let $u \in L^2(M \setminus L, g)$. If $u \notin H^1(M \setminus L, g)$, then $q_\ell^1[u] = \infty = q^1[\bar{u}]$ for all $\epsilon > 0$.

Thus, suppose that $u \in H^1(M \setminus L, g)$. For $m \in \mathbb{N}$, $x \in M \setminus L$, and $\xi \in \mathbb{R}^3$, let

$$w_m(x, \xi) := \chi_{1/m} \sqrt{|\det g| g^{ab}(x) \xi_a \xi_b},$$

where $a, b \in \{1, 2, 3\}$ and $\chi_{M \setminus T(1/m)}$ is a cutoff function which vanishes on $T(1/m)$. For $\xi$ fixed, each $w_m(\cdot, \xi)$ is measurable with respect to the Lebesgue measure on $\mathbb{R}^3$. Further, these functions satisfy

$$0 \leq w_m(\cdot, \xi) \leq w_{m+1}(\cdot, \xi)$$

for all $m \in \mathbb{N}$. As $m \to \infty$,

$$w_m(\cdot, \xi) \to w(\cdot, \xi) := \sqrt{|\det g| g^{ab}(\cdot) \xi_a \xi_b}.$$  

Using the above monotone pointwise convergence of $\{w_m(\cdot, \xi)\}_{m \in \mathbb{N}}$ in $M \setminus L$ and Lebesgue monotone convergence theorem,

$$\lim_{\epsilon \to 0} q_\ell^1[u] = \lim_{m \to \infty} \int_{M \setminus L} w_m(x) \, d\mu(x) = \int_{M \setminus L} \sigma^{ab}(x) \partial_a u(x) \partial_b u(x) \, d\mu(x) = q^1[u].$$

This demonstrates the claimed pointwise convergence $q_\ell^1 \to q^1$ as $\epsilon \to 0$. From (3.12), we see $q_\ell^1[u] \geq 0$ for all $u \in L^2(M \setminus L, g)$ and all $\epsilon \geq 0$.

Without loss of generality, let $\epsilon = \frac{1}{m}$ for $m \in \mathbb{N}$ and consider the sequence $\{q_m^1\}_{m \in \mathbb{N}}$. We will now prove that $q_m^1 \Gamma$-converges to $q^1$ as $m \to \infty$ strongly in $L^2(M \setminus L, g)$.

Since $q_m^1 \to q^1$ pointwisely, if additionally $\{q_m^1\}_{m \in \mathbb{N}}$ is an increasing sequence of lower semi-continuous functionals, then [34, Proposition 5.4] implies the desired $\Gamma$-convergence in the strong topology of $L^2(M \setminus L, g)$. From (3.12) we see that $\{q_m^1\}_{m \in \mathbb{N}}$ is increasing. Thus to conclude the proof we next show that each $q_m^1$ is a lower semi-continuous function on $L^2(M \setminus L, g)$.

Indeed, let $u \in L^2(M \setminus L, g)$ and let $(u_\ell)_{\ell \in \mathbb{N}} \subset L^2(M \setminus L, g)$ be a sequence converging to $u$ in $L^2(M \setminus L, g)$. Suppose that $\liminf_{\ell \to \infty} q_\ell^1[u_\ell] < \infty$; otherwise there is nothing to prove. By definition of the limit inferior, there exists a subsequence $(u_{\ell_j})_{j \in \mathbb{N}}$ of $(u_\ell)_{\ell \in \mathbb{N}}$ such that the sequence $\|u_{\ell_j}|_{M \setminus T(\ell)}\|_{H^1(M \setminus T(\ell))}$ is uniformly bounded. Again, by choosing a subsequence, we can assume that the functions $u_{\ell_j}|_{M \setminus T(\ell)}$ converge weakly in $H^1(M \setminus T(\ell))$. Then, by the weak lower semi-continuity of the norm in the weak topology of $H^1(M \setminus T(\ell)), g$, we see that

$$q^1_\ell[u] \leq \liminf_{\ell \to \infty} q_\ell^1[u_{\ell_j}].$$

This demonstrates that $q_\ell^1$ is lower semicontinuous in the strong topology of $L^2(M \setminus L, g)$ and completes the proof of $\Gamma$-convergence in the strong topology of $L^2(M \setminus L, g)$.

Now, as $g$ is a $C^\infty$-smooth Riemannian metric on $M$, and since we have the pointwise convergence (3.13), by [34, Proposition 5.14], $q_\ell^1 \Gamma$-converges to $q^1$ in the weak topology of $H^1(M \setminus L, g)$. 

$\square$
Lemma 3.13. For $\lambda \in (-\infty, 0)$, and every $u \in L^2(M \setminus L, g)$, the functional

$$q^2_{\lambda, \epsilon}[u] := \lambda \int_{M \setminus \Gamma(\epsilon)} |u|^2 d\mu_g,$$

converges pointwisely in $L^2(M \setminus L, g)$ to

$$q^2_{\lambda}[u] := \lambda \int_{M \setminus L} |u|^2 d\mu_g.$$

Moreover,

$$q^2_{\lambda, \epsilon}[u] \Gamma \rightarrow q^2_{\lambda}[u]$$

with respect to the strong topology of $L^2(M \setminus L, g)$.

Proof. We first note that $\text{dom}(q^2_{\lambda}) = L^2(M \setminus L, g)$ and is lower semicontinuous on $L^2(M \setminus L, g)$. In particular, it agrees with its lower semicontinuous envelope, $\text{sc}^{-} q^2_{\lambda}$.

Given that $\lambda < 0$, if $\epsilon_2 > \epsilon_1 > 0$, we have $q^2_{\lambda, \epsilon_2} > q^2_{\lambda, \epsilon_1}$. Thus $q^2_{\lambda, \epsilon}$ decreases as $\epsilon \rightarrow 0$.

Let $u \in L^2(M \setminus L, g)$. Then, using the Lebesgue dominated convergence theorem we compute

$$|q^2_{\lambda, \epsilon}[u] - q^2_{\lambda}[u]| = \left| -\lambda \int_{\Gamma(\epsilon)} |u|^2 \sqrt{\det g} d\mu_g \right| \rightarrow 0.$$

That is, $q^2_{\lambda, \epsilon} \rightarrow q^2_{\lambda}$ pointwisely in $L^2(M \setminus L, g)$.

Then, by [34, Proposition 5.7], $q^2_{\lambda, \epsilon} \Gamma$-converges to $\text{sc}^{-}(q^2_{\lambda}) = q^2_{\lambda}$ in $L^2(M \setminus L, g)$, as desired. \qed

Definition 3.14. Let $\epsilon \geq 0$. The resolvent set of $Q_{\epsilon}$, denoted by $\text{res}(Q_{\epsilon})$, is the set of $\lambda \in \mathbb{C}$ such that the associated resolvent operator

$$R_{\epsilon}(\lambda) : L^2(M \setminus L, g) \rightarrow L^2(M \setminus L, g), \quad R_{\epsilon}(\lambda)\tilde{f} := (Q_{\epsilon} - \lambda)^{-1}\tilde{f},$$

is bounded and satisfies

$$(Q_{\epsilon} - \lambda)R_{\epsilon}(\lambda) = R_{\epsilon}(\lambda)(Q_{\epsilon} - \lambda) = \text{Id}_{L^2}.$$

Then, the spectrum of $Q_{\epsilon}$ is the complement set $\text{spec}(Q_{\epsilon}) := \mathbb{C} \setminus \text{res}(Q_{\epsilon})$.

Corollary 3.15. Let $\lambda \in \mathbb{C}$. Then

1. $q^1_{\epsilon} + q^2_{\lambda, \epsilon} \rightarrow q^1 + q^2_{\lambda}$ pointwisely in $L^2(M \setminus L, g)$.
2. $q^1_{\epsilon} + q^2_{\lambda, \epsilon} \Gamma \rightarrow q^1 + q^2_{\lambda}$ in the strong topology of $L^2(M \setminus L, g)$.
3. for $\lambda > 0$, the operator $R_{\epsilon}(\lambda)$ converges in the strong topology of $L^2(M \setminus L, g)$ to the resolvent operator $R_0(\lambda)$ associated to $Q_0$. 21
Proof. First consider the case when \( \lambda \leq 0 \). From Lemmas 3.13 and 3.12, \( q^1 \overset{\Gamma}{\rightarrow} q^1 \) and \( \lambda q_1^2 \overset{\Gamma}{\rightarrow} \lambda q^2 \). Further, we have the pointwise convergence \( q^1(u) \rightarrow q^1(u) \) and \( q^2_\lambda(u) \rightarrow q^2_\lambda(u) \) for each \( u \in L^2(M \setminus L, g) \). Then, by [34, Proposition 6.25], (1) and (2) hold in this case.

Next consider \( \lambda > 0 \). By Lemma 3.5, the quadratic forms \( q^1 + q^2_\lambda \) and \( q^1 + q^2_\lambda \) are positive. In this setting we may apply [34, Theorem 13.6] to achieve claims (1)-(3).

\[ \square \]

### 3.4 Strong resolvent convergence

In the previous section, we concluded with Corollary 3.15 which gave strong convergence of the resolvent operators \( \mathcal{R}_\epsilon(\lambda) \) to \( \mathcal{R}_0(\lambda) \) as \( \epsilon \to 0 \) in the case when \( \lambda > 0 \). In this section, we prove that as \( \epsilon \to 0 \), the resolvents \( \mathcal{R}_\epsilon(\lambda) \) converge strongly in \( L^2(M \setminus L, g) \) to \( \mathcal{R}_0(\lambda) \), for an appropriate set of complex values \( \lambda \in \mathbb{C} \). In particular, we will show that the values of \( \lambda \) for which we have strong convergence include those of the form \( -k^2 < 0 \) for some \( k \in \mathbb{R} \), which correspond to sinusoidal wave solutions of the Helmholtz equations \( -Q_\epsilon u + k^2 u = f \).

First, we show that a compact set \( K \subset \mathbb{C} \) which avoids the the spectrum of \( Q_0 \) also avoids the spectrum of \( Q_\epsilon \) for sufficiently small \( \epsilon > 0 \):

**Lemma 3.16.** Let \( K \subset \mathbb{C} \) be compact and such that \( K \cap \text{spec}(Q_0) = \emptyset \). Then, there exists an \( \epsilon_K > 0 \) such that for \( \epsilon < \epsilon_K \)

\[ K \cap \text{spec}(Q_\epsilon) = \emptyset. \]

**Proof.** By [21, Chapter IV, Section 3.1, Theorem 3.1] and [21, Chapter IV, Section 2.6, Theorem 2.25] is is enough to show that there exists some \( \lambda_0 \in \text{res}(Q_0) \) and an \( \epsilon_0 > 0 \) such that for \( \epsilon < \epsilon_0 \),

\[ \| R_\epsilon(\lambda_0) - R_0(\lambda_0) \|_{L^2(M \setminus L, g) \to L^2(M \setminus L, g)} \to 0 \quad \text{as} \quad \epsilon \to 0. \]  

(3.14)

As \( Q_\epsilon \) are positive, self adjoint operators, if \( \lambda_0 > 0 \), then \( \lambda_0 \in \text{res}(Q_\epsilon) \) for all \( \epsilon > 0 \). From Corollary 3.15, we achieve (3.14) with \( \lambda_0 = 1 \).

\[ \square \]

Next, we introduce notation for subsets of \( \mathbb{C} \) for which the family of resolvents \( \mathcal{R}_\epsilon(\lambda) \) exhibit strong convergence or boundedness as \( \epsilon \to 0 \).

**Definition 3.17.** [21, Chapter VIII, Section 1.1] The region of boundedness for the family \( \{ \mathcal{R}_\epsilon(\lambda) : \epsilon > 0 \} \) is the subset of all \( \lambda \in \mathbb{C} \) with the property that there exists an \( \epsilon_0 > 0 \) such that the family \( \{ \| \mathcal{R}_\epsilon(\lambda) \|_{L^2} : \epsilon_0 > \epsilon > 0 \} \) is bounded. We denote this set by \( D_b \).

The set of values \( \lambda \in \mathbb{C} \) for which the strong convergence limit \( \lim_{\epsilon \to 0} \mathcal{R}_\epsilon(\lambda) \) exists in \( L^2(M \setminus L, g) \) is denoted by \( D_s \), and is called the region of strong convergence for the family \( \{ \mathcal{R}_\epsilon(\lambda) : \epsilon > 0 \} \).

Finally, we write \( D_u \) for the set of \( \lambda \in \mathbb{C} \) for which the family \( \{ \mathcal{R}_\epsilon(\lambda) : \epsilon > 0 \} \) convergences in norm to some limit in \( L^2(M \setminus L, g) \). We call \( D_u \) the region of convergence in norm. Observe that \( D_u \subset D_s \subset D_b \).
With these definitions in hand, we prove:

Lemma 3.18. Let \( \lambda \in \text{res}(Q_0) \). For \( f \in L^2(M \setminus L, g) \), we have as \( \epsilon \to 0 \),
\[
R_\epsilon(\lambda) f \to R_0(\lambda) f
\]
strongly in \( L^2(M \setminus L, g) \). Furthermore, if \( K \subset \mathbb{C} \) is compact and satisfies \( K \cap \text{spec}(Q_0) = \emptyset \), the convergence is uniform for \( \lambda \in K \).

Proof. The first part of the claim is to show that \( D_s = \mathbb{C} \setminus \text{spec}(Q) \). We prove it by using an open/closed argument which gives \( D_s = D_b = \mathbb{C} \setminus \text{spec}(Q) \).

To begin, observe that \( D_b \) is connected since \( Q_0 \) is self-adjoint and thus the spectrum of \( Q_0 \) is both discrete and countable [7, Chapter 6, Theorem 1.9]. From Corollary 3.15 we know the sets \( D_s \) and \( D_b \) are nonempty since \( \mathbb{R}_+ \subset D_s \subset D_b \).

Now, let \( K \subset \mathbb{C} \) be compact and such that \( K \cap \text{spec}(Q_0) = \emptyset \). From Lemma 3.16 there is an \( \epsilon_K \) and \( \delta > 0 \) such that the set
\[
K_\delta = \{ z \in \mathbb{C} : \text{dist}(z, K) < \delta \}
\]
satisfies \( K_\delta \cap \text{spec}(Q_\epsilon) = \emptyset \) for all \( \epsilon < \epsilon_K \).

Given that the operators \( Q_\epsilon : L^2(M \setminus L, g) \to L^2(M \setminus L, g), \epsilon \geq 0, \) are self-adjoint, by [21, Chapter V, Section 3.5], for all \( \lambda \in K \)
\[
\|R_\epsilon(\lambda)\|_{L^2 \to L^2} \leq \left[\text{dist}(\lambda, \text{spec}(Q_\epsilon))\right]^{-1} < \delta^{-1}.
\]
(3.15)

Thus we deduce that \( K \subset D_b \). Since \( K \subset \mathbb{C} \setminus \text{spec}(Q_0) \) was arbitrary and \( \mathbb{C} \setminus \text{spec}(Q_0) \) is connected, we have \( D_b = \mathbb{C} \setminus \text{spec}(Q_0) \).

Utilizing [21, Chapter VIII, Section 1.1, Theorem 1.2], we have that \( D_s \) is both relatively open and relatively closed in \( D_b \). Thus \( D_s = D_b = \mathbb{C} \setminus \text{spec}(Q_0) \).

The result [21, Chapter VIII, Section 1.1, Theorem 1.2] further states that for \( f \in L^2(M \setminus L, g) \), \( R_\epsilon(\lambda) f \to R_0(\lambda) f \) uniformly on all compact subsets of \( D_b \), which completes our proof.

\[\Box\]

3.5 Proof of Theorem 1.1

Now we are ready to prove our main result. As shown in Section 3.2, it is equivalent to prove:

Theorem 3.1. Let \( V \subset M \setminus L \) be a relatively compact open set, \( f \in L^2(V, g) \), and \( -k^2 \in \text{res}(Q_0) \). Then,
\[
\lim_{\epsilon \to 0} \Lambda_{V,\epsilon} f = \Lambda_{V,0} f
\]
strongly in \( L^2(V, g) \).
Moreover, if additionally $V \subset M \setminus L$ is a bounded domain with $\partial V \in C^m(M)$ for some $m \geq 0$, and also for some $1 > \alpha \geq 0$ satisfying $m + \alpha > 1/2$, we have $g^{ab}, g_{ab} \in C^{m+1,\alpha}(V, g)$, for $a, b \in \{1, 2, 3\}$, and $f \in C^{m,\alpha}_0(V, g)$, then
\[
\lim_{\epsilon \to 0} \Lambda_{V, \epsilon} f = \Lambda_{V, 0} f
\]
in $C^{m+2,\alpha}_0(V)$.

Proof. Write $\tilde{f} = \Psi \circ f$. By Lemma 3.3, $\tilde{f} \in L^2(\Psi(V), \tilde{g})$. As well, let $\tilde{u} \in H^1(\tilde{M}, \tilde{g})$ be the corresponding finite energy solution of the Helmholtz equation on $\tilde{M}$ with wavenumber $k$ and source $\tilde{f}$.

Denote by $u_\epsilon \in H^1(M \setminus T(\epsilon), g)$ the solution to (3.3) with wavenumber $k$ and source $f$. Then, by Lemma 3.18,
\[
\lim_{\epsilon \to 0} u_\epsilon = u
\]
strongly in $L^2(M \setminus L, g)$, which implies that
\[
\lim_{\epsilon \to 0} u_\epsilon|_V = u|_V
\]
strongly in $L^2(V, g)$. By definition we have $u_\epsilon|_V = \Lambda_{V, \epsilon} f$ and $u|_V = \Lambda_V f$; applying Lemma 3.9 we thus obtain the first part of the claim.

For the moreover part of the claim, suppose that $f \in C^{m,\alpha}_0(V, g)$ for some $m \geq 0$ and $\alpha \in (0, 1)$. Let $W \subset M \setminus L$ be a bounded open set such that $W \subset V$. Then, we have for for $w = u$ or $u_\epsilon$ that $w \in C^{m+2,\alpha}(V)$ and the the standard elliptic estimate [20, Theorem A.2.3]
\[
\|w\|_{C^{k,\alpha}(W)} \leq C(\|f\|_{C^{k,\alpha}(V)}) + \|w\|_{C^{0,\alpha}(V)}, \quad (3.17)
\]
holds for some $C = C(M \setminus L, V, g) > 0$.

Now, also there exists a $C = C(M \setminus L, V, g) > 0$ such that
\[
\|w\|_{H^2(V, g)} \leq C\|w\|_{L^2(M \setminus L, g)}. \quad (3.18)
\]
By applying Morrey’s inequality for $\text{dim}(V) = 3$ and $H^2(V, g)$, we have
\[
\|w\|_{C^{0,1/2}(V)} \leq C\|w\|_{H^2(V, g)} \quad (3.19)
\]
for some $C = C(V, m, \alpha, g) > 0$.

Since we assumed that $m + \alpha > 1/2$, putting together estimates (3.17) - (3.19) gives us the inequality
\[
\|u_\epsilon - u\|_{C^{m,\alpha}(W, g)} \leq C\|u_\epsilon - u\|_{L^2(V, g)},
\]
for $C = C(M \setminus L, V, m, \alpha, g) > 0$.

Using, (3.16), we obtain the desired convergence in $C^{m,\alpha}_0(V, g)$.

\[\square\]
4 Proof of Proposition 1.2

In this section, we prove a version of Proposition 1.2 satisfying also the necessary additional properties in (2.6). The proof of the existence of the embedding is based on a characterization theorem for closed 3-manifolds due to Lickorish [27] and Wallace [38]: Each closed and oriented 3-manifold is obtained by a surgery along a link in $S^3$. In particular, each closed and oriented 3-manifold admits an embedding into $\mathbb{R}^3$ after removal of a suitable link.

We do not discuss the proof of this topological statement in detail, but introduce the necessary terminology related to a precise statement and recall a smoothing result in 3-dimensions used to deduce Proposition 1.2 from the Lickorish–Wallace theorem.

Let $M$ be a 3-manifold. We call the product space $\bar{B}^2 \times S^1$ and the circle $\{0\} \times S^1$ the solid 3-torus and its core curve, respectively. Given an embedding $\phi: \bar{B}^2 \times S^1 \to M$, we call the images $\phi(\bar{B}^2 \times S^1)$ and $\phi(\{0\} \times S^1)$ an embedded solid 3-torus and its core curve, respectively.

Heuristically, a surgery over a circle $S$ on a 3-manifold $M$ is an operation which replaces an interior of an embedded solid 3-torus $T$ in $M$ by the interior of another solid 3-torus $T'$ which is not a priori contained in $M$. Formally, the replacement is obtained by gluing a copy of $\bar{B}^2 \times S^1$ to $M \setminus \text{int} T$. More precisely, we fix first a neighbourhood for $S$ by choosing an embedding $\phi: \bar{B}^2 \times S^1 \to M$, for which $S = \phi(\{0\} \times S^1)$. To formalize the gluing, let $h: (\partial \bar{B}^2) \times S^1 \to \phi((\partial \bar{B}^2) \times S^1)$ be a homeomorphism. We consider then a disjoint union

$$\left( M \setminus \phi(\bar{B}^2 \times S^1) \right) \bigcup \bar{B}^2 \times S^1$$

and form the quotient space

$$\widehat{M} = \left( \left( M \setminus \phi(\bar{B}^2 \times S^1) \right) \bigcup \bar{B}^2 \times S^1 \right) / \sim_h,$$

where the equivalence relation $\sim_h$ is the equivalence relation generated by the condition that $y \sim_h (x, e^{i\theta})$ if $y = h(x, e^{i\theta})$ for $y \in M$ and $(x, e^{i\theta}) \in (\partial \bar{B}^2) \times S^1$. In what follows, we use the common notation

$$\widehat{M} = \left( M \setminus \phi(\bar{B}^2 \times S^1) \right) \bigcup_h \bar{B}^2 \times S^1$$

for the space $\widehat{M}$.

A surgery over a link $L$ in $M$ is obtained similarly as follows. The additional condition we impose is that the solid 3-tori neighbourhoods of the circles in the link $L$ do not meet. For brevity, we introduce the following notation. Let $L = S_1 \cup \cdots \cup S_J$ be a link with $J$ circles in $M$, and let $T_1, \ldots, T_J$ be mutually disjoint solid 3-tori having circles $S_1, \ldots, S_J$ as their core curves, respectively. Let also $h: (\partial \bar{B}^2 \times S^1) \times \{1, \ldots, J\} \to (\partial T_1 \cup \cdots \cup \partial T_J)$ be a homeomorphism. Then

$$\widehat{M} = \left( \left( M \setminus \text{int}(T_1 \cup \cdots \cup T_J) \right) \bigcup \bar{B}^2 \times S^1 \times \{1, \ldots, J\} \right) / \sim_h,$$

is a space obtained from $M$ as a surgery over the link $L$. Here the equivalence relation with respect to $h$ is analogous to the case of one circle, that is, he equivalence relation $\sim_h$ is the
equivalence relation $\sim_h$ generated by the condition $y \sim_h (x,t,j)$, where $y = h(x,e^{i\theta},j)$ for $y \in M$, $(x,e^{i\theta}) \in \mathbb{B}^2 \times S^1$ and $j = 1,\ldots, J$. Again, we denote
\[
\widehat{M} = (M \setminus \text{int}(T_1 \cup \cdots \cup T_J)) \bigcup_h \mathbb{B}^2 \times S^1 \times \{1,\ldots, J\}.
\]
In what follows, let
\[
\pi_h: (M \setminus \text{int}(T_1 \cup \cdots \cup T_J)) \bigcup_h \mathbb{B}^2 \times S^1 \times \{1,\ldots, J\}
\to (M \setminus \text{int}(T_1 \cup \cdots \cup T_J)) \bigcup_h \mathbb{B}^2 \times S^1 \times \{1,\ldots, J\}
\]
be the canonical projection, i.e. the quotient map $x \mapsto [x]$ for the equivalence relation $\sim_h$.

For each $j = 1,\ldots, J$, we also denote $\widehat{T}_j = \pi_h(\mathbb{B}^2 \times S^1 \times \{j\})$ and $\widehat{S}_j = \pi_h(\{0\} \times S^1 \times \{j\})$.

Note that each $\widehat{T}_j$ is a solid 3-torus and each $\widehat{S}_j$ the core curve of $\widehat{T}_j$. In particular,
\[
\widehat{L} = \widehat{S}_1 \cup \cdots \cup \widehat{S}_J
\]
is a link in $\widehat{M}$.

Having this terminology at our disposal, we may now give a precise statement of the Lickorish–Wallace theorem.

**Theorem 4.1** (Lickorish [27], Wallace [38]). Let $M$ be a closed, connected, and orientable 3-manifold. Then there exists a link $L = S_1 \cup \cdots \cup S_J \subset S^3$, mutually disjoint solid 3-tori $T_1,\ldots, T_J$ in $S^3$ with core curves $S_1,\ldots, S_J$, respectively, and a homeomorphism
\[
h: (\partial \mathbb{B}^2 \times S^1) \times \{1,\ldots, J\} \to (\partial T_1 \cup \cdots \cup \partial T_J)
\]
for which
\[
M \approx \widehat{M} = (S^3 \setminus \text{int}(T_1 \cup \cdots \cup T_J)) \bigcup_h \mathbb{B}^2 \times S^1 \times \{1,\ldots, J\}.
\]

### 4.1 Embeddings satisfying (2.6)

Although our primary aim is to obtain a smooth embedding $\widehat{M} \setminus \widehat{L} \to \mathbb{R}^3$, in our applications we need the embedding to have controlled derivative close to the link $\widehat{L}$ as stated in (2.6). For the definition, we use polar coordinates in the $\mathbb{B}^2$ factor of $\mathbb{B}^2 \times S^1$. In practice, this means that we will use three coordinates $(r, \theta, s)$, $r \in [0, 1]$, $\theta, s \in \mathbb{R}$, to denote a point in $\mathbb{B}^2 \times S^1$ instead of just two $(x, e^{i\psi})$, $x \in \mathbb{B}^2$, $e^{i\psi} \in S^1$. As usual, these two coordinate systems are related to each other with the formulas $x = (r \cos \theta, r \sin \theta)$ and $\psi = s$.

Now we can give the precise definition for the controlled derivative.

**Definition 4.2.** Let $\widehat{M}$ and $M$ be closed 3-manifolds and $\widehat{L} \subset \widehat{M}$ and $L \subset M$ links. We say that the derivatives of a diffeomorphism $F: \widehat{M} \setminus \widehat{L} \to M \setminus L$ are controlled close to the link $\widehat{L}$, if there exist mutually disjoint solid 3-tori $\widehat{T}_1,\ldots, \widehat{T}_J \subset \widehat{M}$ and $T_1,\ldots, T_J \subset M$ with parametrizations $\phi: \mathbb{B}^2 \times S^1 \times \{1,\ldots, J\} \to \widehat{T}_1 \cup \cdots \cup \widehat{T}_J$ and $\phi : B^2 \times S^1 \times \{1,\ldots, J\} \to T_1 \cup \cdots \cup T_J$, for which $\widehat{L} = \phi(\{0\} \times S^1 \times \{1,\ldots, J\})$, $L = \phi(\{0\} \times S^1 \times \{1,\ldots, J\})$, $F((\widehat{T}_1 \cup \cdots \cup \widehat{T}_J) \setminus \widehat{L}) \subset T_1 \cup \cdots \cup T_J$ and the partial derivatives of the composite map
\[
\pi = \phi^{-1} \circ F \circ \phi|_{(B^2 \setminus \{0\}) \times S^1 \times \{j\}}: (\mathbb{B}^2 \setminus \{0\}) \times S^1 \to (\mathbb{B}^2 \setminus \{0\}) \times S^1
\]
are bounded for each \( j = 1, \ldots, J \), i.e. there are positive constants \( C_\pi, C_\theta, C_\rho, C_\theta, \ldots \) that satisfy
\[
\left| \frac{\partial \pi_r}{\partial r} \right| \leq C_\pi, \quad \left| \frac{\partial \pi_r}{\partial \theta} \right| \leq C_\theta, \quad \left| \frac{\partial \pi_r}{\partial s} \right| \leq C_\rho, \quad \left| \frac{\partial \pi_\theta}{\partial r} \right| \leq C_\theta, \quad \ldots
\]

To be able to construct a diffeomorphism with this property for our needs, we require also our gluing homeomorphisms to be diffeomorphisms. Our first preliminary result is the following lemma, which states that for every gluing homeomorphism there exists a gluing diffeomorphism that yields in the surgery the same 3-manifold up to a homeomorphism.

**Lemma 4.3.** Let \( M \) be a compact, smooth 3-manifold and \( \hat{M} \) a 3-manifold obtained from \( M \) by a surgery along mutually disjoint solid tori \( T_1, \ldots, T_J \subset M \) with gluing homeomorphism \( h: (\partial \hat{B}^2 \times S^1) \times \{1, \ldots, J\} \to (\partial T_1 \cup \cdots \cup \partial T_J) \). Then there exists another gluing homeomorphism \( h' \), which is a diffeomorphism and for which the manifold
\[
\hat{M}' = (M \setminus \text{int}(T_1 \cup \cdots \cup T_J)) \bigcup_{h'} \hat{B}^2 \times S^1 \times \{1, \ldots, J\}
\]
is homeomorphic to \( \hat{M} \). Furthermore, \( \hat{M}' \) has a smooth structure induced by the smooth structures of \( M \setminus \text{int}(T_1 \cup \cdots \cup T_J) \) and \( \hat{B}^2 \times S^1 \times \{1, \ldots, J\} \).

The proof is based on a classical two dimensional smoothing result that all homeomorphisms between surfaces are isotopic to diffeomorphisms; see Baer [2], Epstein [8], or an unpublished short proof due to Hatcher [16]. We formulate the needed result as follows:

**Theorem 4.4.** Let \( M \) and \( M' \) be closed, smooth 2-manifolds, and \( f: M \to M' \) a homeomorphism. Then there exists a diffeomorphism \( f': M \to M' \) that is isotopic to \( f \), i.e. there exists a map
\[
F: M \times [0, 1] \to M',
\]
for which \( F(x, 0) = f(x) \) and \( F(x, 1) = f'(x) \) for all \( x \in M \) and the map \( F_t: M \to M' \), \( x \mapsto F(x, t) \), is an embedding for each \( t \in [0, 1] \).

**Proof of Lemma 4.3.** Let \( h': (\partial \hat{B}^2 \times S^1) \times \{1, \ldots, J\} \to (\partial T_1 \cup \cdots \cup \partial T_J) \) be a diffeomorphism isotopic to \( h \) as in Theorem 4.4. It is a well-known fact that if the gluing map is a diffeomorphism, the smooth structures of the manifolds glued together induce a unique smooth structure on the resulting manifold (see for example Hirsch [19, Section 8.2]).

Now it remains to show that \( \hat{M}' \) is homeomorphic to \( \hat{M} \). We will define the homeomorphism \( \omega: \hat{M}' \to \hat{M} \) piecewise. Write \( \hat{B}^2(\frac{1}{2}) = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq \frac{1}{2}\} \). Let \( \omega|_{M \setminus \text{int}(T_1 \cup \cdots \cup T_J)} \) and \( \omega|_{\hat{B}^2(\frac{1}{2}) \times S^1 \times \{1, \ldots, J\}} \) be the identity maps. Let \( F: (\partial \hat{B}^2 \times S^1) \times \{1, \ldots, J\} \times [0, 1] \to (\partial T_1 \cup \cdots \cup \partial T_J) \) be the isotopy between \( h \) and \( h' \). Define \( \omega|_{(\hat{B}^2(\frac{1}{2}) \setminus \hat{B}^2(\frac{1}{2})) \times S^1 \times \{1, \ldots, J\}} \) to be
\[
(x, e^{i\theta}, j) \mapsto |x| \cdot (h^{-1} \circ F)(x/|x|, e^{i\theta}, j, 2(|x| - \frac{1}{2})),
\]
where we use the notation \( a \cdot (x, e^{i\theta}, j) = (ax, e^{i\theta}, j) \) with \( a \in [0, 1] \) and \( (x, e^{i\theta}, j) \in \partial \hat{B}^2 \times S^1 \times \{1, \ldots, J\} \).

Now fix \( t \in [0, 1] \). The map \( F_t \) is an embedding and hence injective. This implies with the domain invariance theorem that \( F_t \) is an open map. Now fix \( j \in \{1, \ldots, J\} \). The map \( F \) is
continuous and the set $\partial B^2 \times S^1 \times \{j\} \times [0,1]$ is connected, so its image $F(\partial B^2 \times S^1 \times \{j\} \times [0,1])$ is also connected and hence contained in some $\partial T_j'$. We may assume that $j' = j$. The set $\partial B^2 \times S^1 \times \{j\}$ is open and compact and hence also its image $F_i(\partial B^2 \times S^1 \times \{j\})$ is open and compact and thus open and closed. This implies that it covers the whole $\partial T_j$. Hence $F_i$ is surjective.

We have shown that $F_i$ is bijective for each $t \in [0,1]$. This together with the initial assumption that $h$ is a homeomorphism and $h^{-1}$ hence bijective implies that $\omega|_{(\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1,\ldots,J\}}$ is bijective.

We will show next that $\omega$ is well-defined. Let $(x, e^{\theta}, j) \in \partial B^2 \times S^1 \times \{1, \ldots, J\}$. Now we have

$$h(\omega|_{(\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1,\ldots,J\}}(x, e^{\theta}, j)) = h((|x|(h^{-1} \circ F)(x/|x|, e^{\theta}, j, 2(|x| - \frac{1}{2}))))
$$

$$= h(1 \cdot h^{-1} \circ F)(x/1, e^{\theta}, j, 2(1 - \frac{1}{2})) = h((h^{-1} \circ F)(x, e^{\theta}, j, 1))
$$

$$= F(x, e^{\theta}, j, 1) = h'(x, e^{\theta}, j) = \omega|_{M \setminus \text{int}(T_1 \cup \cdots \cup T_J)}(h'(x, e^{\theta}, j)).$$

Hence the definitions of $\omega|_{(\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1,\ldots,J\}}$ and $\omega|_{M \setminus \text{int}(T_1 \cup \cdots \cup T_J)}$ agree on the intersection $((\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1, \ldots, J\}) \cap (M \setminus \text{int}(T_1 \cup \cdots \cup T_J))$.

Let now $(x, e^{\theta}, j) \in (\tilde{B}^2(\frac{1}{2}) \setminus B^2(\frac{1}{2})) \times S^1 \times \{1, \ldots, J\}$. We have

$$\omega|_{(\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1,\ldots,J\}}(x, e^{\theta}, j) = \frac{1}{2} \cdot h^{-1} \circ F)(x/\frac{1}{2}, e^{\theta}, j, 2(\frac{1}{2} - \frac{1}{2}))
$$

$$= \frac{1}{2} \cdot (h^{-1} \circ F)(2x, e^{\theta}, j, 0) = \frac{1}{2} \cdot (h^{-1} \circ h)(2x, e^{\theta}, j)
$$

$$= \frac{1}{2} \cdot (2x, e^{\theta}, j) = (x, e^{\theta}, j) = \omega|_{\tilde{B}^2(\frac{1}{2}) \times S^1 \times \{1,\ldots,J\}}(x, e^{\theta}, j).$$

Hence the definitions of $\omega|_{(\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1,\ldots,J\}}$ and $\omega|_{\tilde{B}^2(\frac{1}{2}) \times S^1 \times \{1,\ldots,J\}}$ also agree on the intersection $((\tilde{B}^2 \setminus B^2(\frac{1}{2})) \times S^1 \times \{1, \ldots, J\}) \cap (\tilde{B}^2(\frac{1}{2}) \times S^1 \times \{1, \ldots, J\})$.

All of its pieces are continuous, so also the whole map $\omega$ is continuous. Furthermore, the manifold $\hat{M}$ is compact, so we conclude that $\omega$ is a homeomorphism. \hfill \Box

Having now all the terminology and this lemma at our disposal, we may formulate an embedding lemma:

**Lemma 4.5.** Let $M$ be a closed, smooth 3-manifold and $\hat{M}$ a 3-manifold obtained from $M$ by a surgery in link $L \subset M$. Then there exists a link $\hat{L}$ in $\hat{M}$ and a diffeomorphism $\hat{M} \setminus \hat{L} \rightarrow M \setminus L$. Furthermore, the derivatives of this diffeomorphism are controlled close to the link $\hat{L}$.

**Proof.** For the argument, we assume that link $L$ has $J$ circles $S_1, \ldots, S_J$ and these circles are core curves of mutually disjoint solid 3-tori $T_1, \ldots, T_J$ in $M$. Let $h: (\partial B^2 \times S^1) \times \{1, \ldots, J\} \rightarrow (\partial T_1 \cup \cdots \cup \partial T_J)$ be the gluing homeomorphism for which

$$\tilde{M} = (M \setminus \text{int}(T_1 \cup \cdots \cup T_J)) \bigcup h B^2 \times S^1 \times \{1, \ldots, J\}$$

and let $\hat{T}_1, \ldots, \hat{T}_J$ be solid 3-tori and $\hat{S}_1, \ldots, \hat{S}_J$ their core curves, respectively, as above.

We follow the idea of the proof of Proposition 2.2 and define the diffeomorphism $F: \hat{M} \setminus \hat{L} \rightarrow M \setminus L$ in parts. On $M \cap \hat{M}$, we set $F$ to be the identity. Thus it suffices to define $F$ on $\hat{T}_j \setminus \hat{S}_j$. 

for each \( j = 1, \ldots, J \). To simplify the notation, let \( \Omega_j = T_j \setminus S_j \) and \( \hat{\Omega}_j = \hat{T}_j \setminus \hat{S}_j \) for each \( j = 1, \ldots, J \). Let also \( \Omega = \Omega_1 \cup \cdots \cup \Omega_J \) and \( \hat{\Omega} = \hat{\Omega}_1 \cup \cdots \cup \hat{\Omega}_J \).

Let \( \phi : B^2 \times S^1 \times \{1, \ldots, J\} \to T_1 \cup \cdots \cup T_J \) be a diffeomorphism simultaneously parametrizing each solid torus \( T_1, \ldots, T_J \). By Lemma 4.3, we may assume the gluing homeomorphism \( h \) to be a diffeomorphism. Then it induces a diffeomorphism

\[
h' : (\partial B^2) \times S^1 \times \{1, \ldots, J\} \to (\partial B^2) \times S^1 \times \{1, \ldots, J\}
\]
satisfying

\[
\phi \circ h' = h.
\]

Let now

\[
H : (\hat{B}^2 \setminus \{0\}) \times S^1 \times \{1, \ldots, J\} \to (\hat{B}^2 \setminus \{0\}) \times S^1 \times \{1, \ldots, J\}
\]
be the homeomorphism

\[
(x, e^{i\theta}, j) \mapsto |x| : h'(x/|x|, e^{i\theta}, j)
\]
extending the homeomorphism \( h' \), where we are again using the notation \( a \cdot (x, e^{i\theta}, j) = (ax, e^{i\theta}, j) \).

Finally, let

\[
\psi : \hat{B}^2 \times S^1 \times \{1, \ldots, J\} \to \hat{T}_1 \cup \cdots \cup \hat{T}_J
\]
be the restriction of the quotient map \( \pi_h \) and define the restriction

\[
F|_{\hat{\Omega}} : \hat{\Omega} \to \Omega
\]
by the formula

\[
F|_{\hat{\Omega}} = \phi \circ H \circ \psi^{-1}|_{\hat{\Omega}}.
\]

Since \( \phi \circ H \circ \psi^{-1}|_{\hat{\Omega}} \) is the identity, the map \( F \) is well-defined. Since restrictions \( F|_{\hat{M} \setminus \text{int} \hat{\Omega}} \) and \( F|_{\hat{\Omega}} \) are continuous, we have that \( F \) is continuous. Clearly, \( F \) is also bijective. Since \( F \) extends to a continuous bijection over the link \( \hat{L} \) to a continuous bijection \( \hat{M} \to M \) and \( \hat{M} \) is compact, we conclude that \( F \) is a homeomorphism.

Recall that we defined the map \( F \) so that the restriction \( F|_{\hat{M} \setminus \text{int} \hat{\Omega}} \) is the identity. Now the domain and codomain of \( F|_{\hat{M} \setminus \text{int} \hat{\Omega}} \) have the same differential structure, so this restriction is furthermore a diffeomorphism. For the restriction \( F|_{\hat{\Omega}} \), we have \( F|_{\hat{\Omega}} = \phi \circ H \circ \psi^{-1}|_{\hat{\Omega}} \). Since the map \( h' \) is a diffeomorphism, also \( H \) is a diffeomorphism. In addition, the maps \( \phi \) and \( \psi \) are smooth embeddings, so as their composite map, the restriction \( F|_{\hat{\Omega}} \) is a diffeomorphism.

We have now shown that the both restrictions

\[
F|_{\hat{M} \setminus \text{int} \hat{\Omega}} : \hat{M} \setminus \text{int} \hat{\Omega} \to M \setminus \text{int} \Omega \quad \text{and} \quad F|_{\hat{\Omega}} : \hat{\Omega} \to \Omega
\]
are diffeomorphisms. It then follows that there exists a diffeomorphism \( \hat{M} \setminus \hat{L} \to M \setminus L \) that agrees with \( F \) on the subset \( \hat{\Omega} \) (see for example Hirsch [19, Theorem 8.1.9]). For the rest of this proof, we will use \( F \) to denote this diffeomorphism.
It remains to show that the derivatives of the diffeomorphism $F$ are controlled close to the link $\hat{L}$. We have
\[
\pi = \phi^{-1} \circ F \circ \psi|_{(B^2\setminus\{0\}) \times S^1 \times \{j\}} = \phi^{-1} \circ \phi \circ H \circ \psi^{-1} \circ \psi|_{(B^2\setminus\{0\}) \times S^1 \times \{j\}} = H|_{(B^2\setminus\{0\}) \times S^1 \times \{j\}}.
\]
We want to now use polar coordinates for $\hat{B}^2$, so we use the notation
\[
h'(\theta, s, j) = (h'_\theta(\theta, s, j), h'_s(\theta, s, j), h'_j(\theta, s, j)),
\]
where $h'_\theta$, $h'_s$ and $h'_j$ are the coordinate functions of $h'$. Now we have for the map $H$ the formula
\[
H(r, \theta, s, j) = (r, h'_\theta(\theta, s, j), h'_s(\theta, s, j), j).
\]
Then we may calculate the partial derivatives. For the derivative $\frac{\partial \pi}{\partial r}$, we obtain
\[
\frac{\partial \pi}{\partial r}(r, \theta, s) = \frac{\partial H}{\partial r}(r, \theta, s) = \frac{\partial r}{\partial r} = 1.
\]
For the derivative $\frac{\partial \pi_\theta}{\partial s}$, we obtain
\[
\frac{\partial \pi_\theta}{\partial s}(r, \theta, s) = \frac{\partial H_\theta}{\partial s}(r, \theta, s, j) = \frac{\partial h'_\theta}{\partial s}(r, \theta, s, j).
\]
The derivative $\frac{\partial h'_\theta}{\partial s}$ is continuous and its domain $\hat{B}^2 \times S^1 \times \{1, \ldots, J\}$ is compact, so also its image $\frac{\partial h'_\theta}{\partial s}(\hat{B}^2 \times S^1 \times \{1, \ldots, J\})$ is compact and hence bounded. The remaining partial derivatives can be shown to be bounded in a similar manner.

4.2 Proof of Proposition 1.2

The controlled version of Proposition 1.2 reads as follows:

**Theorem 4.6.** Let $M$ be a closed, connected, and orientable 3-manifold. Then there exist links $L \subset M$ and $\hat{L} \subset S^3$, and a diffeomorphism $M \setminus L \to S^3 \setminus \hat{L}$. Furthermore, the derivatives of this diffeomorphism are controlled close to the link $L$.

The last ingredient of the proof of Theorem 4.6 is the smoothing theorem due to Munkres, which states that homeomorphic smooth 3-manifolds are diffeomorphic.

**Theorem 4.7.** Let $M$ and $N$ be smooth 3-manifolds. If there exists a homeomorphism $M \to N$, then there exists a diffeomorphism $M \to N$.

**Proof of Theorem 4.6.** Let $\hat{L}$ be the link in $S^3$ and $\hat{M}$ the closed, connected, and orientable 3-manifold given by Theorem 4.1. Let also $\hat{L}$ be the link in $\hat{M}$ induced by the surgery of $S^3$ along the link $\hat{L}$. By Lemma 4.3, we may assume $\hat{M}$ to be a smooth manifold. Then, since $M$ and $\hat{M}$ are homeomorphic smooth 3-manifolds, there exists a diffeomorphism $\Phi: M \to \hat{M}$. Let $L = \Phi^{-1} \hat{L}$. Finally, there exists a diffeomorphism $\psi: \hat{M} \setminus \hat{L} \to S^3 \setminus \hat{L}$ by Lemma 4.5. Let now $F: M \setminus L \to S^3 \setminus \hat{L}$ be the diffeomorphism $F = \psi \circ \Phi$.

Let $\tilde{T}$ be the union of the solid tori in $S^3$ that the surgery is performed along and let $\tilde{p}: \hat{B}^2 \times S^1 \times \{1, \ldots, J\} \to \tilde{T}$ be its parametrization. Let $\tilde{T} \subset \hat{M}$ be the union of the solid tori that were
attached to $\mathbb{S}^3 \backslash \text{int} \hat{T}$ in the surgery and let $\phi: \bar{B}^2 \times \mathbb{S}^1 \times \{1, \ldots, J\} \to \hat{T}$ be its parametrization induced by the quotient map $\pi_h$. Let $\rho = \Phi^{-1} \circ \phi$. We now have

$$\pi = \tilde{\rho}^{-1} \circ F \circ \rho|_{\bar{B}^2 \backslash \{0\} \times \mathbb{S}^1 \times \{j\}} = \tilde{\rho}^{-1} \circ \psi \circ \Phi \circ \rho|_{\bar{B}^2 \backslash \{0\} \times \mathbb{S}^1 \times \{j\}}$$

Let $\rho = \Phi^{-1} \circ \phi$. We now have

$$\pi = \tilde{\rho}^{-1} \circ \psi \circ \phi^{-1} \circ \Phi \circ \rho|_{\bar{B}^2 \backslash \{0\} \times \mathbb{S}^1 \times \{j\}}.$$

Using the previous formula and the chain rule for derivatives, we have for the derivative $\frac{\partial \pi \rho}{\partial \theta}$(r, θ, s)

$$\frac{\partial \pi \rho}{\partial \theta}(r, \theta, s) = \frac{\partial (\tilde{\rho}^{-1} \circ \psi \circ \phi^{-1} \circ \Phi \circ \rho)}{\partial \theta}(r, \theta, s)$$

$$= \frac{\partial (\phi^{-1} \circ \Phi \circ \rho)}{\partial \theta}(r, \theta, s) \cdot \frac{\partial (\tilde{\rho}^{-1} \circ \psi \circ \phi)}{\partial \theta}(\phi^{-1} \circ \Phi \circ \rho(r, \theta, s))$$

$$+ \frac{\partial (\phi^{-1} \circ \Phi \circ \rho)}{\partial \theta}(r, \theta, s) \cdot \frac{\partial (\tilde{\rho}^{-1} \circ \psi \circ \phi)}{\partial \theta}(\phi^{-1} \circ \Phi \circ \rho(r, \theta, s))$$

$$+ \frac{\partial (\phi^{-1} \circ \Phi \circ \rho)}{\partial \theta}(r, \theta, s) \cdot \frac{\partial (\tilde{\rho}^{-1} \circ \psi \circ \phi)}{\partial \theta}(\phi^{-1} \circ \Phi \circ \rho(r, \theta, s)).$$

We know by Lemma 4.5 that the derivatives of the diffeomorphism $\psi$ are controlled close to the link $\hat{L}$. Hence the partial derivatives of the composite map $\tilde{\rho}^{-1} \circ \psi \circ \phi$ are bounded. The partial derivatives of the composite map $\phi^{-1} \circ \Phi \circ \rho$, on the other hand, are continuous and their domain $\bar{B}^2 \times \mathbb{S}^1 \times \{1, \ldots, J\}$ is compact, so also their images are compact and hence bounded. Since all these partial derivatives are bounded, it follows that also $\frac{\partial \pi \rho}{\partial \theta}$ is bounded. The remaining partial derivatives of the map $\pi$ can be shown to be bounded using similar argument.

\[
\text{References}
\]

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