1. Introduction

A nilpotent orbit of a reductive group $G$ over $\mathbb{C}$ is an orbit of the adjoint or the coadjoint action of $G$ on the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ or $\mathfrak{g}^*$, respectively. The set of nilpotent orbits $\mathcal{O} \subset \mathcal{N}$ forms a poset with a partial order $\mathcal{O}_1 < \mathcal{O}_2$ defined by the closure order $\overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2}$. For classical groups, there is a bijection between the nilpotent orbits and certain integer partitions. In particular, for $GL(n)$ and $SL(n)$, each partition $\alpha$ of the integer $n$ corresponds to a nilpotent orbit $\mathcal{O}_\alpha$. The counterpart of the closure order $\mathcal{O}_\alpha \subset \overline{\mathcal{O}_\beta}$ on partitions is given by

$$\alpha < \beta \text{ if and only if } \sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \text{ for all } k.$$ 

The notion of induced orbits provides a connection between the nilpotent orbits of a subgroup and the orbits of the ambient group. Following [CM93, Chapter 7], for any nilpotent orbit $\mathcal{O}$ of a Levi subgroup $L \subset G$, there is an unique nilpotent orbit of $G$ which intersects $\mathcal{O} + \mathfrak{n}$ in a Zariski-open subset. This orbit is called the induced orbit $\text{ind}^G_L(\mathcal{O})$ in $G$. At the other end, we can define the Bala-Carter inclusion $\text{inc}^p(\mathcal{O})$ of a nilpotent orbit $\mathcal{O}$ of $I$ as the $G$-saturation $G \cdot \mathcal{O}$ in $\mathfrak{g}$. These two orbits are the maximal and minimal elements in the poset which consists of all the $G$-orbits contained in $G \cdot (\mathcal{O} + \mathfrak{n})$. The first part of the main result of this paper describes this poset of the nilpotent orbits contained in $G \cdot (\mathcal{O} + \mathfrak{n})$.

Another degree of freedom for the set $G \cdot (\mathcal{O} + \mathfrak{n})$ is the choice of parabolic subgroups $P$ and the nilpotent radicals $N \subset P$. For each pair of orbits $\mathcal{O}_\gamma$ of $I$ and $\mathcal{O}_\alpha$ of $\mathfrak{g}$, we will also describe an algebraic variety which parametrizes all the parabolic subgroups $P = LN$ such that

$$\mathcal{O}_\alpha \subset G \cdot (\mathcal{O}_\gamma + \mathfrak{n}).$$

The collection of all such conjugate parabolic subgroups forms a closed subvariety of the partial flag manifold $G/P$, and some information of its topology is encoded in the Littlewood-Richardson coefficients and the Hall-Littlewood polynomials. This variety is closely related to the geometry of $\mathcal{O}_\gamma \cap \mathfrak{p}$, and we will provide this result as the second part of our main theorem.

In Section 5, we will also discuss the application of this geometric result to the wavefront set of the subrepresentations of an induced representation, following the result in [GS20].

Now we state the main theorem of this paper:

**Theorem A.** For $G = GL(n, \mathbb{C})$ and a parabolic subgroup $P = LN$ with a standard Levi subgroup $L = GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \subset G$ embedded block-diagonally, let $\alpha$ be a partition of $n_1$ and $\beta$ be a partition of $n_2$, consider the nilpotent orbit $\mathcal{O}_{\alpha, \beta} = \mathcal{O}_{\alpha} \times \mathcal{O}_{\beta}$ of the subgroup $L$ where $\mathcal{O}_\alpha, \mathcal{O}_\beta$ are the nilpotent orbits corresponding to the partitions $\alpha, \beta$ in $GL(n_1)$ and $GL(n_2)$, respectively, then:

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(1) A $G$-orbit $O_\gamma$ is contained in $G \cdot (O_{\alpha, \beta} + n)$ if and only if the Littlewood-Richardson coefficient $c_{\alpha, \beta}^\gamma \neq 0$.

(2) The Littlewood-Richardson coefficient is the number of irreducible components of the subvariety

$$\mathcal{P}_{\alpha, \beta}^\gamma = \{ x \in O_\gamma \cap p \mid p_1(x) \in O_\alpha \times O_\beta \} \subset O_\gamma \cap p$$

where the map $p_1$ is the projection $p_1: p \mapsto p/n \cong 1$ onto the Levi subgroup.

The first part of this theorem is a direct consequence of the theory of Hall polynomials which was discussed for abelian $p$-groups in [Mac98] Appendix Chapter II, Theorem AZ.1, and can be proven as a corollary of the Hall’s theorem in [Kle69] Theorem 2.2. The proof of Hall’s theorem is combinatorial and makes use of Schubert calculus. However, in order to find an interpretation of the Littlewood-Richardson coefficients in our context, we would like to give a geometric proof to the first part of the theorem as a corollary of the geometric Satake isomorphism. The geometric constructions used in the proof of the first part will shed light on the proof of the second part of Theorem A in Section 4.

It is worth mentioning that since we can induce the orbits by stages, the first part of Theorem A can be generalized to arbitrary standard parabolic subgroups:

**Corollary B.** For any partition $\underline{n} = (n_1, \ldots, n_r)$ of $n$ and partitions $\alpha, \beta$ of the integers $n_i$, we take a standard parabolic subgroup $P_{\underline{n}} = L_{\underline{n}} N_{\underline{n}}$ with a Levi subgroup isomorphic to $GL(n_1) \times \ldots \times GL(n_r)$. A nilpotent orbit $O_\gamma$ of $GL(n)$ is contained in $G \cdot (\prod O_{\alpha} + n_0)$ if and only if the number

$$N_{\alpha_1, \ldots, \alpha_r}^{\gamma} = \sum_{\beta_2, \ldots, \beta_{r-1}} c_{\alpha_2, \alpha_3}^{\beta_2} c_{\beta_2, \alpha_4}^{\beta_3} \cdots c_{\beta_{r-1}, \alpha_r}^{\gamma}$$

is nonzero.

In terms of Schur functions, the number $N_{\alpha_1, \ldots, \alpha_r}^{\gamma}$ is the coefficient of $s_\gamma$ in the product of the Schur polynomials $s_{\alpha_1} \cdots s_{\alpha_r}$.

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## 2. Preliminaries and Notation

In this section, we specify the notations which are used throughout this paper. Let $G$ be a semisimple algebraic group over $\mathbb{C}$, we denote by

- $T \subset B \subset G$ a choice of maximum torus $T$ and a Borel subgroup $B$,
- $X_*(T)$ the cocharacter lattice, and $X^*(T)$ the character lattice of $T$,
- $\Delta_G(T)$ the set of roots, $\Delta_{G,+}(T)$ the set of positive roots and $\Delta_{G,\ast}(T)$ the set of simple roots, and $\Delta^G(T)$ the set of coroots.

We can identify the character lattice as the integer lattice $\mathbb{Z}^n$ where $n$ is the rank of $G$. Each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ corresponds to the character

$$(t_1, \ldots, t_n) \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n}.$$  

In particular, under this correspondence, the half sum of positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_{G,+}} \alpha$$

corresponds to the partition $(n-1, n-2, \ldots, 1, 0)$. Fixing a maximal torus $T$, the quadruple $(X_*, \Delta^G, X_+, \Delta_G)$ is called the root datum of $G$. Its dual root datum $(X_+, \Delta_G, X^*, \Delta^G)$ is obtained by switching the character lattice/roots and the cocharacter lattice/coroots. The algebraic group corresponding to the dual root datum is called the Langlands dual group and is denoted by $\hat{G}$.
2.1. Young Diagrams. In this section we will summarize the basic concepts related to Young diagrams and define the Littlewood-Richardson coefficients. A partition \( \alpha = (\alpha^1, \ldots, \alpha^r) \) of an integer \( n \) corresponds to a Young diagram (whose \( i \)-th row has length \( \alpha^i \)) of the shape \( \alpha \). In a Young tableau of the shape \( \alpha \), integers, starting with 1, are filled into the boxes of the Young diagram \( \alpha \). We denote the set of all Young tableaux of the shape \( \alpha \) by \( T_\alpha \). A Young tableau is called standard if the entries of each box are strictly increasing from the left to right in each row, and increasing from top to bottom in each column. If the entries are weakly increasing along the rows, the tableau is called a semistandard tableau. It is worth noting that each standard Young tableau \( T \) of the shape \( \alpha \) corresponds to a sequence of Young diagrams

\[
0 = \alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_r = \alpha
\]

with each \( \alpha_i \) formed by the boxes in \( T \) with contents \( \leq i \). For example, the Young tableau

\[
T = \begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & \\
\hline
6 & \\
\hline
\end{array}
\]

corresponds to the following sequence of Young diagrams:

\[
\emptyset \subset \begin{array}{|c|}
\hline
1 \\
\hline
\end{array} \subset \begin{array}{|c|c|}
\hline
2 & 3 \\
\hline
\end{array} \subset \begin{array}{|c|c|c|}
\hline
4 & 5 & 6 \\
\hline
\end{array}
\]

Similar to the usual Young diagrams discussed above, for each pair of Young diagrams \( (\alpha, \gamma) \) such that \( \alpha \subset \gamma \), a skew Young diagram is the complement of \( \alpha \) in \( \gamma \), with the two diagrams aligned along their top and left edges. For example, for \( \alpha = (2, 1) \) and \( \gamma = (4, 3, 2) \), the skew Young diagram \( \gamma/\alpha \) looks like

\[
\begin{array}{|c|c|c|c|}
\hline
\hline
1 \\
\hline
\hline
2 \\
\hline
\hline
3 \\
\hline
\hline
4 \\
\hline
\end{array}
\]

We denote the set of all semistandard skew tableaux of the shape \( \gamma/\alpha \) by \( T_{\gamma/\alpha} \). Similar to the correspondence between Young tableaux and sequences of Young diagrams described above, any skew tableau \( T \in T_{\gamma/\alpha} \) corresponds to a sequence of Young diagrams from \( \alpha \) to \( \gamma \), whose \( i \)-th stage \( \alpha_i \) consists of the boxes in \( T \) such that the entries in each of the selected boxes are \( \leq i \). Assuming the maximal content of the tableaux is \( r \), there is a sequence

\[
\alpha = \alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_r = \gamma.
\]

The entries of such a skew tableau also gives us a new partition \( \beta = (|\alpha_1/\alpha_0|, \ldots, |\alpha_r/\alpha_{r-1}|) \).

For example, the skew tableau

\[
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & \\
6 & \\
\hline
\end{array}
\]

yields the partition \( \beta = (4, 3) \). We say a tableau \( T \), or a corresponding sequence of Young diagrams from \( \alpha \) to \( \gamma \) has type \( (\alpha, \beta; \gamma) \) if this new partition is \( \beta \).

2.2. The Littlewood-Richardson Rule. For each pairs of Young diagrams \( (\alpha, \gamma) \) such that \( \alpha \subset \gamma \), a semistandard skew Young tableau is Littlewood-Richardson if its reverse lattice words, i.e. the string of content of each box, read from top to bottom and from right to left like in Hebrew, satisfies the Yamanouchi condition, i.e. for each content \( i \), the number of \( i \)'s in the length \( k \) prefix of its reverse lattice word is great or equal to the number of \( i+1 \)'s in the length \( k \) prefix. For example, the reverse lattice word of the tableau

\[
\begin{array}{|c|c|c|}
\hline
5 & 4 & 3 \\
\hline
2 & 1 & \\
\hline
\end{array}
\]

is 112132. It satisfies the Yamanouchi condition.

For each triple of Young diagrams \( (\alpha, \beta; \gamma) \) such that \( \alpha, \beta \subset \gamma \), the Littlewood-Richardson coefficient \( c^\gamma_{\alpha, \beta} \) is defined as the number of Littlewood-Richardson tableaux of type \( (\alpha, \beta; \gamma) \).

The usual combinatorial description of the Littlewood-Richardson coefficient is given by the Hall-Littlewood polynomials \( G^\gamma_{\alpha, \beta}(q) \). For a pair \((\mathcal{O}, \mathfrak{p})\) consisting of a discrete valuation ring \( \mathcal{O} \) and its maximal ideal \( \mathfrak{p} \) with a residue field of size \( q \), and a triple of partitions \( (\alpha, \beta; \gamma) \), the Hall-Littlewood polynomial \( G^\gamma_{\alpha, \beta}(q) \) is defined
as the number of submodules $N \subset M$ such that $N \cong M_\alpha$ and $M_\gamma/N \cong M_\beta$. The Hall polynomial $G_{\alpha,\beta}(q)$ has degree $(\rho, \alpha + \beta - \gamma)$, and its top-degree coefficient is equal to the Littlewood-Richardson coefficient $c_{\alpha,\beta}^\gamma$. A detailed proof of this fact is in the appendix of [Mac98, Chapter 2].

However, it’s important to have a geometric model of the abelian group extensions so that we can get more information related to nilpotent orbits. In the following sections, we will introduce the theory of affine Grassmannian and a geometric interpretation of the Littlewood-Richardson coefficient.

3. Lattices and Affine Grassmannian

In this paper, we will introduce four major pictures for the Littlewood-Richardson coefficient: the first picture is the extensions of torsion $\mathbb{C}[t]$-modules and the top-degree coefficient of the Hall-Littlewood polynomial, which we have already introduced in the previous section. The second picture is the extension of sublattices in $\mathcal{O}^n$. This picture is closely related to the first picture. The third picture uses the geometric Satake isomorphism and the interpretation of Littlewood-Richardson coefficients as decomposition multiplicities for tensor products of finite dimensional representation of $GL_n$, which are directly related to the affine Grassmannian. The fourth picture is the number of irreducible components of certain subvarieties of the affine Grassmannian. The whole Section 3 is devoted to the discussion of the second and third picture. The fourth picture will be introduced in Section 4.

In this section we would like to discuss the link between module extensions and geometry. Throughout this paper, we fix a ring $\mathcal{O} = \mathbb{C}[t]$ with a maximal ideal $\mathfrak{p} = t\mathcal{O}$. We will establish a geometric model for the extensions of finite dimensional modules of the form

$$M_\lambda = \mathbb{C}[t]/(t^{\lambda_1}) \oplus \ldots \mathbb{C}[t]/(t^{\lambda_n})$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition. We can add zeros to the end of $\lambda$ without causing any change to $M_\lambda$. The geometric model of module extensions is given by the following lemma, which we will prove in the later sections of this paper:

**Lemma 3.0.1.** For three partitions $\alpha, \beta, \gamma$ and three modules $M_\alpha, M_\beta, M_\gamma$ as defined above, there exists an exact sequence

$$0 \to M_\alpha \to M_\gamma \to M_\beta \to 0$$

if and only if the partitions $\alpha, \beta, \gamma$ makes the Littlewood-Richardson coefficient $c_{\alpha,\beta}^\gamma$ non-vanishing. The Littlewood-Richardson coefficient $c_{\alpha,\beta}^\gamma$ is equal to the number of isomorphism classes of all such possible extensions.

The first part of Theorem A is a direct consequence of this lemma.

3.1. Preliminaries on Affine Grassmannian. The affine Grassmannian is a geometric object parametrizing the full-rank $\mathcal{O}$-sublattices in $\mathcal{O}^n$.

**Definition 3.1.1.** Setting $\mathcal{O} = \mathbb{C}[t]$ and $K = \mathbb{C}((t))$, which are the rings of formal power series and formal Laurent series, respectively, let $G$ be a semisimple algebraic group, the affine grassmannian of $G$ is defined as the quotient

$$\text{Gr}_G = G(K)/G(\mathcal{O}).$$

The affine grassmannian is an ind-scheme, which means that it is the direct limit of finite-dimensional closed subschemes. To see this, we need a stratification on the affine Grassmannian based on the Cartan decomposition (See [MV07, Section 2] and [BR18, Proposition 1.3.2]), in which the cells are parametrized by the dominant coweights

$$G(K) = \prod_{\lambda \in X_+(T)} G(\mathcal{O}) \cdot L_\lambda \cdot G(\mathcal{O}).$$

In the case that $G = GL_n$, $L_\lambda$ is the diagonal matrix $L_\lambda = \text{diag}(t^{\lambda_1}, \ldots, t^{\lambda_n})$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of length $n$. We define an affine Schubert cell as the quotient

$$\text{Gr}_G^\lambda = G(\mathcal{O}) \cdot L_\lambda \cdot G(\mathcal{O})/G(\mathcal{O}).$$
By \[\text{[BR18 Proposition 1.3.2]}\], the closure \(\overline{\text{Gr}}_G^\mu\) of this cell is the disjoint union of affine Schubert cells \(\text{Gr}_G^\mu\) with \(\mu \leq \lambda\):

\[
\overline{\text{Gr}}_G^\mu = \bigsqcup_{\mu \in \lambda_\ast(T)'} \text{Gr}_G^\mu.
\]

In the disjoint union above, the order \(\mu \leq \lambda\) on the partitions are given by the closure order described in the beginning of Section II.

3.2. Affine Schubert Varieties and Lattices in \(O^n\). We will use the affine Schubert cycles to parametrize full-rank \(O\)-sublattices in \(O^n\). The correspondence is established by the following lemma:

**Lemma 3.2.1.** For any dominant coweight \(\lambda \in X_\ast(T)^+\), there is a bijection between the elements \(x \in \text{Gr}_G^\lambda\) and the full-rank \(O\)-sublattices \(\Lambda_x \subset O^n\) such that

\[
O^n/\Lambda_x \cong \bigoplus_i \mathbb{C}[t]/(t^{\lambda_i}).
\]

**Proof.** Since each element \(x \in \text{Gr}_G^\lambda\) represents an element in \(G(O)L_\lambda G(O)\) of the form

\[
x = \alpha L_\lambda, \quad \alpha \in G(O),
\]

the column vectors of \(L_\lambda\) generates a sublattice

\[
\alpha^{-1}L_\lambda = t^{\lambda_1}O \oplus \ldots \oplus t^{\lambda_n}O
\]

of \(\alpha^{-1}O^n \cong O^n\). The quotient \(\alpha^{-1}(O^n/\Lambda_x) \cong O^n/\Lambda_x\) is a module which satisfies the requirement of the lemma. Conversely, if there is a lattice \(\Lambda_x = \bigoplus_i O f_i\) which satisfies the requirement of the lemma, the isomorphism will give an isomorphism between lattices \(p : \bigoplus_i O t^{\lambda_i} \rightarrow \Lambda_x = \bigoplus_i O f_i\) such that \(f_i = \sum_j r_{ij}t^{\lambda_j}\) where the matrix \((r_{ij})\) is an invertible matrix in \(G(O)\).

As in the second picture mentioned in the beginning of Section 3.1, in order to for us to describe the set of extensions of lattices, we need to introduce the *twisted product* \(\overline{\text{Gr}}_G \times \overline{\text{Gr}}_G\) of two copies of the affine Grassmannians. Letting \(G(O)\) act on \(G(O) \times G(O)\) by \(k^+ (a, b) = (ka, k^{-1}b)\), the scheme \(\overline{\text{Gr}}_G \times \overline{\text{Gr}}_G\) is defined as the quotient of \(G(K) \times \overline{\text{Gr}}_G\) by the action of \(G(O)\) on \(G(O) \times G(O)\), fitting into the diagram

\[
\begin{array}{ccc}
G(K) \times \overline{\text{Gr}}_G & \overset{p}{\longrightarrow} & \overline{\text{Gr}}_G \times \overline{\text{Gr}}_G \\
\overline{\text{Gr}}_G \times \overline{\text{Gr}}_G & \overset{q}{\longrightarrow} & \overline{\text{Gr}}_G \times \overline{\text{Gr}}_G
\end{array}
\]

where the morphism \(p\) is the quotient by \(G(O)\) from the first component. There is a *semi-small, ind-proper* multiplication map \(m : \overline{\text{Gr}}_G \times \overline{\text{Gr}}_G \rightarrow \overline{\text{Gr}}_G\) defined by

\[
m : (g, x) \mapsto gx.
\]

The reason why this map \(m\) is called the multiplication map will be explained in Lemma 3.2.4.

The twisted product \(\times\) can be defined similarly for the Schubert cells and Schubert varieties by simply taking the quotient of the direct product of the corresponding cells by \(G(O)\). When restricted to the twisted product of the Schubert varieties, the multiplication map \(m\) maps \(\overline{\text{Gr}}^\alpha_G \times \overline{\text{Gr}}^\beta_G\) surjectively onto \(\overline{\text{Gr}}^{\alpha+\beta}_G\).

We can also take the twisted product of multiple affine Grassmannians and Schubert varieties. The variety \(\overline{\text{Gr}}^\lambda_1 \times \ldots \times \overline{\text{Gr}}^\lambda_n\) is defined as the multiple twisted product

\[
\overline{\text{Gr}}^\lambda_1 \times \ldots \times \overline{\text{Gr}}^\lambda_n = \overline{\text{Gr}}^\lambda_1 \times \ldots \times \overline{\text{Gr}}^\lambda_n.
\]

We can stratify the space \(\overline{\text{Gr}}^\lambda_1 \times \ldots \times \overline{\text{Gr}}^\lambda_n\) by the twisted product of Schubert cells corresponding to smaller partitions:

\[
\overline{\text{Gr}}^\lambda_1 \times \ldots \times \overline{\text{Gr}}^\lambda_n = \bigcup_{\mu \leq \lambda} \overline{\text{Gr}}^\mu_1 \times \ldots \times \overline{\text{Gr}}^\mu_n.
\]

To count the dimension of the twisted product of these Schubert cells, we need the following lemma from \[\text{[Zim16]}\] and \[\text{[BR18]}\]:

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Lemma 3.2.2. The dimension of the twisted product of cells $\text{Gr}_G^{\lambda_1, \ldots, \lambda_n}$ is equal to $\langle 2\rho, \sum_{i=1}^n \lambda_i \rangle$. The dimension of the fiber $m^{-1}(x) \cap \text{Gr}_G^{\lambda_1, \ldots, \lambda_n}$ of $m$ over any generic point $x \in \text{Gr}_G^{\lambda_1, \ldots, \lambda_n}$ is less than or equal to $\langle \rho, \sum_{i=1}^n \lambda_i - \lambda \rangle$, hence $m$ is a small-semi-small map.

Proof. Since the stabilizer of the action of $G(O)$ at $L_\lambda$ is $G(O) \cap L_\lambda G(O)(L_\lambda)^{-1}$, the tangent space $	ext{Gr}_G^{\lambda_1, \ldots, \lambda_n}$ at $L_\lambda$ is

$$\mathfrak{g}(O)/\mathfrak{g}(O) \cap \text{Ad}_{L_\lambda}\mathfrak{g}(O) \cong \bigoplus_{\langle \alpha, \lambda \rangle \geq 0} \mathfrak{g}_\alpha(\langle \alpha, \lambda \rangle) g(O).$$

The dimension of this tangent space above $(2\rho, \lambda)$.

The dimension of the twisted product follows from a similar stabilizer argument. The inequality for the dimension of the fiber $m^{-1}(x) \cap \text{Gr}_G^{\lambda_1, \ldots, \lambda_n}$ follows from [MV07, Lemma 4.4], also see [BR18, Lemma 6.4].

Remark 3.2.3. For any partition $\lambda$, the number $\langle \rho, \lambda \rangle$ is closely related to $\frac{1}{2} \| \lambda' \|^2 = \frac{1}{2} \sum (\lambda'_i)^2$. Since we know

$$\lambda_i - \lambda_{i+1} = \# \{ j \mid \lambda'_j = i \},$$

we can express $\frac{1}{2} \| \lambda' \|^2$ in terms of $\lambda$: $\frac{1}{2} \| \lambda' \|^2 = \sum_{i=1}^n i^2(\lambda_i - \lambda_{i+1})$.

Through simple calculation, we see that

$$\frac{1}{2} \| \lambda' \|^2 + \langle \rho, \lambda \rangle = \left( n - \frac{1}{2} \right) \sum_{k=1}^n \lambda_k.$$

Therefore, we can express the dimension of the fiber $m^{-1}(x) \cap \text{Gr}_G^{\lambda_1, \ldots, \lambda_n}$ differently in terms of the conjugate partitions:

$$\text{dim} \left( m^{-1}(x) \cap \text{Gr}_G^{\lambda_1, \ldots, \lambda_n} \right) = \frac{1}{2} \left( \| \lambda' \|^2 - \sum_{i=1}^n \| \lambda'_i \|^2 \right).$$

We can use the variety $\text{Gr}_G^{\alpha} \times \text{Gr}_G^\beta$ to parametrize the extensions of lattices (as promised in our second picture). This fact is shown in the following lemma:

Lemma 3.2.4. For any lattice $\Lambda \subset O^n$, denote by $\bar{\Lambda} = \text{Hom}_O(\Lambda, O)$ its dual lattice. The variety $\text{Gr}_G^{\alpha} \times \text{Gr}_G^\beta$ parametrizes the collection of sublattices $\Lambda_\gamma \subset \Lambda_\alpha \subset O^n$, such that

1. They both span the same vector space $K^n$ over $K = \mathbb{C}(t)$,
2. $\bar{\Lambda}_\gamma/O^n \cong M_\gamma$, and $\bar{\Lambda}_\alpha/O^n \cong M_\alpha$,
3. $\Lambda_\alpha/\Lambda_\gamma \cong M_\beta$.

The multiplication map $m : \text{Gr}_G^{\alpha} \times \text{Gr}_G^\beta \rightarrow \text{Gr}_G^{\alpha + \beta}$ sends any pair of such lattices $(\Lambda_\gamma, \Lambda_\alpha)$ to $\Lambda_\gamma$.

Proof. The scheme $\text{Gr}_G^{\alpha} \times \text{Gr}_G^\beta$ is a $G(O)$-quotient of the product $G(O)L_\alpha G(O) \times G(O)\lambda L_\beta$, which parametrizes pairs of elements $(l_1, l_2)$ with $l_2 \in G(O)L_\beta$ and $l_1 \in G(O)\Lambda_\alpha G(O)$. Each equivalent class under the quotient by $G(O)$ has a unique representative of the form $(xL_\alpha, yL_\beta)$, and can be moved to a representative $(L_\alpha, x^{-1}yL_\beta)$. In the moved lattice $x^{-1}O^n$, one can consider the lattice generated by the column vectors of the matrix $L_\alpha x^{-1}yL_\beta$. We would like to prove that this lattice is the correct $x\Lambda_\gamma$,

$L_\alpha = (e_1, \ldots, e_n)$,

where each $e_i = (0, \ldots, t^{e_i}, \ldots, 0)^t$ is only nonzero at the $i$-th entry. We represent by a matrix $x^{-1}yL_\beta = (g_{i,j})$, and column vectors of the matrix $L_\alpha x^{-1}yL_\beta$ are

$v_j = (t^{e_i}g_{i,j}) = \sum_i g_{i,j}e_i$

which generates a sublattice $x\Lambda_\gamma$ of $\sum_i O e_i$ which satisfies $\Lambda_\alpha/\Lambda_\gamma \cong M_\beta$. □
By Lemma 3.2.4, the two projections $\pi_1, \pi_2$ can be described as maps from the twisted product $\text{Gr}_G \times \text{Gr}_G$ onto the first and the second factor, respectively. The projection $\pi_1$ sends each pair of $(\Lambda_\gamma, \Lambda_\alpha)$ to $\Lambda_\alpha$, while the projection $\pi_2$ sends the pair $(\Lambda_\gamma, \Lambda_\alpha)$ to the sublattice $\Lambda_\beta = L^{-1}_\alpha L^{-1}_\gamma \Lambda_\alpha$ of $\mathbb{O}^n$. These two projections can also be used to set up an isomorphism between the twisted product $\text{Gr}_G \times \text{Gr}_G$ and the direct product $\text{Gr}_G \times \text{Gr}_G$, which sends any element $y$ in the twisted product to $(\pi_1(y), \pi_2(y))$. In terms of group element representatives, the isomorphism can be described by the map

$$\text{Gr}_G \times \text{Gr}_G \rightarrow \text{Gr}_G \times \text{Gr}_G \quad (x_1, x_2) \mapsto (x_1, x_1^{-1} x_2).$$

3.3. Equivariant Perverse Sheaf and the Geometric Satake Equivalence. According to [BL06, Theorem 2.6.2], for any $G(\mathcal{O}) \times G(\mathcal{O})$-equivariant constructible sheaf $\mathcal{F}$ on $G(K) \times \text{Gr}_G$, there is an unique $G(\mathcal{O})$-equivariant constructible sheaf $\tilde{\mathcal{F}}$ on $\text{Gr}_G \times \text{Gr}_G$ such that $q^* \tilde{\mathcal{F}} = \mathcal{F}$. In particular, for two constructible sheaves $\mathcal{A}, \mathcal{B}$ on $\text{Gr}_G$, consider the pullback $p^*(\pi_1^* \mathcal{A} \boxtimes \pi_2^* \mathcal{B})$ as the sheaf $\mathcal{F}$, where $\pi_1, \pi_2$ are the projection to the first and second factors defined in the previous section, respectively. We denote the corresponding unique constructible sheaf $\tilde{\mathcal{F}}$ by $\pi_1^* \mathcal{A} \boxtimes \pi_2^* \mathcal{B}$. We can thus define a convolution product in the derived category of $G(\mathcal{O})$-equivariant constructible sheaves on $\text{Gr}_G$ as

$$\mathcal{A} \ast \mathcal{B} = R\text{rm}_1(\pi_1^* \mathcal{A} \boxtimes \pi_2^* \mathcal{B}).$$

Equipped with the convolution product, the category $\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$ of $G(\mathcal{O})$-equivariant perverse sheaves is a monoidal category with irreducible objects the intersection cohomology sheaves

$$\text{IC}^{\lambda} := \text{IC}(\text{Gr}_G, C) = \iota^* (\mathbb{C}[\dim \text{Gr}_G])$$

supported on each Schubert variety $\text{Gr}_G^\alpha$ which are the intermediate extensions of the locally trivial bundle of $\text{Gr}_G^\alpha$ along the inclusion $\iota : \text{Gr}_G^\alpha \rightarrow \text{Gr}_G$. The main takeaway in the proof of our main Lemma 3.0.1 is the calculation of the convolution product $\text{IC}^{\lambda_1} \ast \text{IC}^{\lambda_2}$ of the two intersection cohomology sheaves. The main reference of the equivariant perverse sheaves on the affine Grassmannian in general is [BR18, Section 7.5].

The geometric Satake equivalence (see [Gin95, MV07] or [BR18]) is an equivalence of monoidal categories between the category of $G(\mathcal{O})$-equivariant perverse sheaves $\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$ and the category of finite dimensional representations $\text{Rep}_\mathbb{C}(\hat{G})$ of the Langlands dual group $\hat{G}$:

$$\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G) \cong \text{Rep}_\mathbb{C}(\hat{G}).$$

The actual construction of the categorical equivalence is not trivial, and requires the Tannakaian reconstruction theorems for an actual group scheme $\hat{G}$ which makes the following diagram of functors commute:

$$\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G) \xrightarrow{S} \text{Rep}_\mathbb{C}(\hat{G}) \, ,$$

$$\text{Vect}_\mathbb{C} \xrightarrow{\text{forget}} \text{H}^* \xrightarrow{\text{H}^*} \text{Rep}_\mathbb{C}(\hat{G}) \, .$$

This commutative diagram of functors respects the tensor products and the units of the three tensor categories. The total cohomology functor $\text{H}^*$ is the graded cohomology which decomposes into weight components:

$$H^k(\text{Gr}_G, \mathcal{A}) \cong \bigoplus_{\mu \in X_+(T)} H^{(2p, \mu)}(T^*_{\mu}, \iota^* \mathcal{A})$$

where $T_\mu$ is the locally closed subvariety of $\text{Gr}_G$ consisting of the points whose limit at $\infty$ is $L_\mu$ under the adjoint action of a regular dominant coroot.

If we identify the cocharacter lattice $X_+(T)$, which parametrizes the irreducible objects of the category $\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$, with the character lattice $X^*(\hat{T})$ of the Cartan subgroup in the Langlands dual group, the
functor $S$ sends the intersection cohomology sheaf supported on $\text{Gr}_G$ for each dominant coweight $\lambda$ to the irreducible representation $V(\lambda)$ of $\hat{G}$ with highest weight $\lambda$: 

$$\text{IC}^\lambda \overset{S}{\rightarrow} V(\lambda),$$

and the convolution product of the intersection cohomology sheaves

$$\text{IC}^{\lambda_1} \ast \ldots \ast \text{IC}^{\lambda_n} = \text{Rm}_1 \left( \pi_1^* \text{IC}^{\lambda_1} \boxtimes \ldots \boxtimes \pi_n^* \text{IC}^{\lambda_n} \right)$$

is sent to the tensor product $V(\lambda_1) \otimes \ldots \otimes V(\lambda_n)$. The functor $\text{Rm}_1$, which can be identified as $\text{Rm}_*$ due to the properness of $m$, sends a complex of constructible sheaves $\pi_1^* \text{IC}^{\lambda_1} \boxtimes \ldots \boxtimes \pi_n^* \text{IC}^{\lambda_n}$ on $\text{Gr}_G^{\lambda_1,\ldots,\lambda_n}$ to a complex of constructible sheaves on $\text{Gr}_G^{\lambda_1+\ldots+\lambda_n}$. As a derived direct image, the constructible sheaf $\text{IC}^{\lambda_1} \ast \ldots \ast \text{IC}^{\lambda_n}$ is perverse due to the semi-smallness of the morphism $m$. We need a lemma for use in the next section:

**Lemma 3.3.1.** The twisted product $\pi_1^* \text{IC}^{\lambda_1} \boxtimes \pi_2^* \text{IC}^{\lambda_2}$ is isomorphic to the intersection cohomology sheaf $\text{IC}(\text{Gr}_G^{\lambda_1,\lambda_2})$.

**Proof.** Consider the following pull-back square

$$
\begin{array}{ccc}
\text{Gr}_G^{\lambda_1} \times \text{Gr}_G^{\lambda_2} & \overset{h}{\longrightarrow} & \text{Gr}_G^{\lambda_1,\lambda_2} \\
\downarrow{\pi_2} & & \downarrow{m} \\
\text{Gr}_G^{\lambda_1+\ldots+\lambda_n} & \underset{\tilde{\eta}}{\longrightarrow} & \text{Gr}_G^{\lambda_1+\ldots+\lambda_n}
\end{array}
$$

such that the top row isomorphism $h$ sends $(x_1, x_2)$ to $(x_1, x_1^{-1}x_2)$ as described in the final remarks of Section 3.2. The morphism $h$ is an isomorphism which preserves the strata of $\text{Gr}_G^{\lambda_1} \times \text{Gr}_G^{\lambda_2}$ and $\text{Gr}_G^{\lambda_1,\lambda_2}$, and hence sends the external tensor product of IC-sheaves

$$\pi_1^* \text{IC}^{\lambda_1} \boxtimes \pi_2^* \text{IC}^{\lambda_2} \cong \text{IC}(\text{Gr}_G^{\lambda_1} \times \text{Gr}_G^{\lambda_2})$$

to the twisted tensor product $\pi_1^* \text{IC}^{\lambda_1} \boxtimes \pi_2^* \text{IC}^{\lambda_2}$. Therefore, the twisted tensor product is also the IC-sheaf $\text{IC}(\text{Gr}_G^{\lambda_1,\lambda_2})$. \hfill \Box

The fact that $S$ is an equivalence of tensor categories identifies the hom-sets between the intersection cohomology sheaves and those between the finite dimensional representations:

$$\text{Hom}(\text{IC}^\gamma, \text{IC}^\alpha \ast \text{IC}^\beta) \cong \text{Hom}_G(V(\gamma), V(\alpha) \otimes V(\beta)).$$

The dimension of this space is equal to the Littlewood-Richardson coefficient $c_{\alpha,\beta}^{\gamma}$.  

### 3.4. The Decomposition Theorem and the Proof of the Lemma 3.0.1

The derived direct image $\text{Rm}_1 \text{IC}(\text{Gr}_G^{\lambda_1,\ldots,\lambda_n})$ of the intersection cohomology sheaf on the variety $\text{Gr}_G^{\lambda_1,\ldots,\lambda_n}$ along the semi-small map $m : \text{Gr}_G \times \text{Gr}_G \to \text{Gr}_G$ can be decomposed non-canonically as a direct sum of irreducible objects. See [dCM09] Theorem 1.6.1] for the following well-known decomposition theorem:

**Theorem 3.4.1** (Beĭlinson-Bernstein-Deligne-Gabber). Let $f : X \to Y$ be a proper map of algebraic varieties, the derived push-forward $\text{Rf}_* \text{IC}_X$ of an intersection cohomology sheaf $\text{IC}_X$ decomposes (non-canonically) into a direct sum of semisimple perverse sheaves

$$\text{Rf}_* \text{IC}_X \cong \bigoplus_{i \in \mathbb{Z}} \text{pH}^i \left( \text{Rf}_* \text{IC}_X \right)[-i].$$

There is a canonical decomposition of $\text{pH}^i \left( \text{Rf}_* \text{IC}_X \right)$ as a direct sum of intersection cohomology sheaves on $Y$, where $\text{pH}^i$ is the perverse cohomology functor $\text{pH}^i_{\leq 1} \circ \text{pH}^i \circ \text{pH}^i_{\leq 1} = \text{pH}^i_{\leq 1} \circ \text{pH}^i_{\leq 1}$.  

We will make use of the decomposition theorem in the form of its corollary (see also [Hai03] Lemma 3.2):
**Corollary 3.4.2.** The convolution product of IC-sheaves decomposes into a direct sum

\[ \text{IC}^\alpha \star \text{IC}^\beta = \bigoplus_{\gamma \leq \alpha + \beta} L^\gamma_{\alpha,\beta} \otimes \text{IC}^\gamma \]

where \( L^\gamma_{\alpha,\beta} = H^c(2\rho, \alpha + \beta - \gamma)(m^{-1}(x) \cap G^\alpha_G) \) for a generic \( x \in G^\gamma_G \).

**Proof.** We consider embeddings of the big cell \( G^\alpha_G \subseteq G^\beta_G \) and its complement \( Z^\alpha_G \) in \( G^\beta_G \):

\[ Z^\alpha_G \ni i \quad \text{Gr}_G^\alpha \quad \ni j \quad \text{Gr}_G^\beta \]

and the corresponding embeddings of their intersections with the fiber of \( m \):

\[ m^{-1}(x) \cap Z^\alpha_G \ni i \quad m^{-1}(x) \cap \text{Gr}_G^\alpha \quad \ni j \quad m^{-1}(x) \cap \text{Gr}_G^\beta . \]

The stalk of the cohomology sheaf \( ^pH^i \left( Rm_* \text{IC}(\overline{G^\beta_G}) \right) \) at \( x \in G^\gamma_G \) can be identified as the cohomology group \( H^i(m^{-1}(x), \text{IC}(\overline{G^\alpha_G}) |_{m^{-1}(x)}) \). To compute this cohomology group, we need to apply the cohomology functor to the distinguished triangle

\[ j \circ i^* \text{IC}(\overline{G^\alpha_G}) \to \text{IC}(\overline{G^\beta_G}) \to i_\gamma^* \text{IC}(\overline{G^\beta_G}) \to \]

which gives rise to a long exact sequence of cohomologies

\[ \ldots \to H^i_x(m^{-1}(x) \cap G^\alpha_G, \text{IC}(\overline{G^\beta_G})) \to H^i(m^{-1}(x), \text{IC}(\overline{G^\beta_G})) \to H^i_x(m^{-1}(x) \cap Z^\alpha_G, \text{IC}(\overline{G^\beta_G})) \to \ldots . \]

Since \( \text{IC}(\overline{G^\alpha_G}) \) is a perverse sheaf, the cohomology sheaf \( ^pH^i(i^*_\gamma \text{IC}(\overline{G^\alpha_G})) = 0 \) if \( i \geq -(2\rho, \alpha + \beta) \). Also since the hypercohomology spectral sequence converges:

\[ E^{pq}_2 : H^p(m^{-1}(x) \cap Z^\alpha_G, i^*_\gamma \text{IC}(\overline{G^\beta_G})) \implies H^{p+q}(m^{-1}(x) \cap Z^\alpha_G, i^*_\gamma \text{IC}(\overline{G^\beta_G})) , \]

all the nonzero terms in this spectral sequence appears when

\[ q < -(2\rho, \alpha + \beta), \quad p \leq \dim(m^{-1}(x) \cap Z^\alpha_G) \leq (2\rho, \alpha + \beta - \gamma) - 2 . \]

Therefore, \( H^i_x(m^{-1}(x) \cap Z^\alpha_G, i^*_\gamma \text{IC}(\overline{G^\beta_G})) \) can be possibly nonzero only when \( i \leq -(2\rho, \gamma) - 2 \). Again by the long exact sequence (5), the stalk of the cohomology sheaf

\[ H^{-2\rho, \gamma}_x(m^{-1}(x), \text{IC}(\overline{G^\beta_G})) \cong H^{-2\rho, \gamma}_c(m^{-1}(x) \cap G^\alpha_G, j^*_\gamma \text{IC}(\overline{G^\beta_G})) . \]

When restricted to the open cell, \( \text{IC}(\overline{G^\alpha_G}) \) is the constant local system \( \mathbb{C}[(2\rho, \alpha + \beta)] \). Therefore, the stalk of the cohomology sheaf at \( x \) is isomorphic to

\[ H^{-2\rho, \gamma}_c(m^{-1}(x) \cap G^\alpha_G, j^*_\gamma \text{IC}(\overline{G^\beta_G})) \cong H^{-2\rho, \alpha + \beta - \gamma}_c(m^{-1}(x) \cap G^\alpha_G) . \]

We denote this space by \( L^\gamma_{\alpha,\beta} \). This finishes the proof of the corollary.

Now we are ready to prove Lemma 3.0.1.

**Proof of Lemma 3.0.1.** By Lemma 3.2.1 any element \( x \in G^\gamma_G \) represents a lattice \( \Lambda_x \) such that the quotient \( O^n/\Lambda_x \) has Jordan type \( \gamma \). If the multiplication map

\[ m : G^\alpha_G \times G^\gamma_G \to G^\alpha_G \]

sends a pair \( y = (\Lambda_\gamma, \Lambda_\alpha) \) to \( \Lambda_x \), by Lemma 3.2.1 we must have \( \Lambda_\gamma = \Lambda_\alpha \). Therefore, in order to prove Lemma 3.0.1 we will have to show that for any \( x \in G^\gamma_G \), the fiber \( m^{-1}(x) \cap G^\alpha_G \) is nonempty if and only if \( c^\gamma_{\alpha,\beta} = \dim L^\gamma_{\alpha,\beta} \neq 0 \).
By the decomposition theorem and the semi-simplicity of the category of equivariant perverse sheaves, \( \text{Hom}(IC^\gamma, IC^\alpha \ast IC^\beta) \) can be decomposed non-canonically as a direct sum:

\[
\text{Hom}(IC^\gamma, IC^\alpha \ast IC^\beta) \cong \text{Hom}(IC^\gamma, Rm_* IC(Gr_G^\alpha,\beta))
\]

\[= \text{Hom} \left( IC^\gamma, \bigoplus_{Gr_G^\alpha \subseteq Gr_G^\gamma, m^{-1}(x) \cap Gr_G^\alpha \neq \emptyset} L^\sigma_{\alpha,\beta} \otimes IC^\sigma \right) \]

\[= L^\gamma_{\alpha,\beta}. \]

By the geometric Satake equivalence, the dimension of the space is equal to the multiplicity of \( V(\gamma) \) in \( V(\alpha) \otimes V(\beta) \), which is equal to the Littlewood–Richardson coefficient \( c^\gamma_{\alpha,\beta} \).

Since on each Schubert cycle \( Gr_G^\alpha \), the group \( G(K) \) acts transitively, the preimage of the Schubert cycle \( m^{-1}(Gr_G^\gamma) \) intersects \( Gr_G^\alpha,\beta \) if and only if the fiber \( m^{-1}(x) \) intersects \( Gr_G^\alpha,\beta \). By [Zhu16, Corollary 5.1.5], the dimension of the space

\[ L^\gamma_{\alpha,\beta} = H^e_{\langle 2p,\alpha+\beta-\gamma \rangle} (m^{-1}(x) \cap Gr_G^\alpha,\beta), \]

is the number of irreducible components of \( m^{-1}(x) \cap Gr_G^\alpha,\beta \) of complex dimension \( \langle p, \alpha + \beta - \gamma \rangle \). All the dominant coweights of \( GL_n \) are sums of coweights of the form \( \epsilon^i = (1^i, 0^{n-i}) \), which are all minuscule, i.e. \( \langle \epsilon_i, \alpha \rangle \in \{-1, 0, 1\} \) for any root \( \alpha \). By [Hai06, Theorem 1.3], the irreducible components of \( m^{-1}(x) \cap Gr_G^\alpha,\beta \), if not empty, are equidimensional and of dimension \( \langle p, \alpha + \beta - \gamma \rangle \). Since the Littlewood-Richardson coefficient \( c^\gamma_{\alpha,\beta} \) is equal to the number of irreducible components of \( m^{-1}(x) \cap Gr_G^\alpha,\beta \), we have now proven that \( m^{-1}(x) \cap Gr_G^\alpha,\beta \neq \emptyset \) if and only if \( c^\gamma_{\alpha,\beta} \neq 0 \).

\[
\begin{align*}
\text{4. Interpretations of the Littlewood-Richardson Coefficient} \\
\text{In this section, we consider the Young diagrams with shapes } \alpha, \beta \subset \gamma \text{ such that } c^\gamma_{\alpha,\beta} \neq 0. \text{ We will fix our notations to denote the length of } \gamma \text{ by } m, \text{ the length of } \alpha \text{ by } n. \text{ The goal of this section is to prove the second part of the Theorem[A] which follows from the geometric interpretation of the Littlewood-Richardson coefficients studies by Springer in [Spr78] and Marc van Leeuwen in [VL00]. This geometric interpretation of } c_{\alpha,\beta} \text{ is our fourth picture mentioned in the beginning of Section 3.1.}
\end{align*}
\]

\[
\begin{align*}
\text{4.1. The Satake Fiber. The fiber } S^\gamma_{\alpha,\beta} = m^{-1}(L_\gamma) \cap Gr_G^\alpha,\beta \text{ of the multiplication map } m \text{ above the point } L_\gamma \in Gr_\gamma \text{ is sometimes referred to as the Satake fiber. The following lemma identifies the Satake fiber as a subvariety of the partial flag variety } G/P_{[n,m-n]}: \text{ }
\end{align*}
\]

\[
\text{Lemma 4.1.1. The Satake fiber } S^\gamma_{\alpha,\beta} \text{ parametrizes the subspaces } V \text{ of dimension } |\alpha| \text{ inside a vector space } W \text{ of dimension } |\alpha| + |\beta|, \text{ such that there is a nilpotent element } x \text{ which stabilizes } V \subset W, \text{ and}
\]

\[\begin{align*}
(1) & \text{ } x |_V \text{ acts with Jordan type } \alpha, \\
(2) & \text{ } x |_W/V \text{ with Jordan type } \beta, \\
(3) & \text{ } x |_W \text{ with Jordan type } \gamma.
\end{align*}\]

\[
\text{Proof. The Satake fiber } m^{-1}(L_\gamma) \cap Gr_G^\alpha,\beta \text{ parametrizes the pairs of lattices } (\Lambda_\gamma, \Lambda_\alpha) \text{ satisfying the conditions in Lemma } 3.2.3. \text{ The quotients } M_\alpha, M_\beta, M_\gamma \text{ of } \mathcal{O}^n \text{ by the three lattices } \Lambda_\alpha, \Lambda_\beta \text{ and } \Lambda_\gamma, \text{ respectively, fit into the exact sequence}
\]

\[0 \rightarrow M_\alpha \rightarrow M_\gamma \rightarrow M_\beta \rightarrow 0.
\]

The multiplication by the variable \( t \) of the ring \( \mathcal{O}[t] \) is a nilpotent linear operator with Jordan type \( \alpha, \beta, \gamma \) on the three spaces, respectively. Conversely, if we have a pair of spaces \( V \subset W \) satisfying the conditions of this lemma, \( V \) and \( W \) can be viewed as the \( \mathcal{O} \)-modules \( M_\alpha, M_\gamma \), respectively. \qed
4.2. The Springer-type Fibers. The tangent bundle of the flag variety $G/B$ can be realized as a subset of the product of the nilpotent cone $\mathcal{N}$ and $B = G/B$:

$$T^*B = \{(x, gB) \mid x \in \text{Ad}(g)b\} \subset \mathcal{N} \times B.$$ 

For each nilpotent orbit $O_{\gamma} \subset \mathcal{N} \subset \mathfrak{g}$, the projection $p$ from the cotangent bundle $T^*B$ to $B$ is a resolution of singularities called the Springer resolution for the nilpotent cone $\mathcal{N}$:

$$T^*B \subset \mathcal{N} \times B$$

![Diagram](http://example.com/diagram.png)

where $\pi, p$ are the projections onto the first and the second factor, respectively. The Springer fiber $B_{\gamma}(x)$ above $x \in O_{\gamma}$ of $p$ is the collection of Borel subalgebras $b \subset \mathfrak{g}$ containing the nilpotent element $x$

$$B_{\gamma}(x) = \{(x, gB) \mid x \in \text{Ad}(g)b\}.$$ 

It is proven by Spaltenstein in [Spa76] and Steinberg in [Ste76] and [Ste88] that the irreducible components of the Springer fiber $B_{\gamma}(x)$ are equidimensional and are parametrized by the standard Young tableaux of shape $\gamma$. The number of standard Young tableaux can be calculated from the hook-length formula.

For $G = GL_m$, the Springer fiber $B_{\gamma}(x)$ can be interpreted as the collection of $x$-stable full flags

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_m = V$$
in a vector space $V_m$ of dimension $m$, on which $x$ acts as a nilpotent linear transformation. Since the Springer fibers are isomorphic for different choices of $x$ in the same orbit $O_{\mu}$, we can denote the Springer fiber by $B_{\gamma}$ if there is no ambiguity caused by different choices of $x$.

Apart from the case of the full flag variety $G/B$, we can also consider the Springer-type partial resolution from the cotangent bundle of a partial flag variety $G/P$ corresponding to the parabolic subgroup $P = LN$. Similar to the Springer resolution of the nilpotent cone by the cotangent bundles of a full flag variety, given a parabolic subgroup $P$, we can define a partial resolution $\overline{\mathcal{N}}_P$ of the nilpotent cone $\mathcal{N}$

$$\overline{\mathcal{N}}_P = \{(x, gP) \mid x \in \mathcal{N} \cap \text{Ad}(g)p\} \subset \mathcal{N} \times G/P \xrightarrow{\pi_P} \mathcal{N}$$

which factorizes the Springer resolution $\pi$. The generalized Springer fiber $\pi_P^{-1}(x)$ above any element $x$ is sometimes referred to as the Steinberg variety (cf. [BM] Chapter 3), some references also call it the Spaltenstein variety). When $P = B$, the partial resolution $\pi_P$ becomes the usual Springer resolution, and the fiber $\pi_P^{-1}(x)$ becomes the usual Springer fiber above $x$. Equivalently, if we consider the parabolic subgroup $P$ as the stabilizer of a partial flag

$$V_{k_1} \subset V_{k_2} \subset \ldots \subset V_{k_r} = V_m$$

where each $V_{k_i}$ is a subspace of dimension $k_i$. The fiber $\pi_P^{-1}(x)$ can be identified as the collection of the $x$-stable flags which are stabilized by a conjugate of $P$.

We are particularly interested in the standard parabolic subgroups $P_{[n,1^{m-n}]}$ with a Levi subgroup $L_{[n,1^{m-n}]} \cong GL(n) \times GL(1)^{m-n}$, and $P_{[n,m-n]}$ with Levi subgroup $L_{[n,m-n]} \cong GL(n) \times GL(m-n)$. For the parabolic subgroup $P_{[n,1^{m-n}]}$, the generalized Springer fiber above $x \in O_{\gamma}$, denoted by $P_{[n,1^{m-n}]}(x)$, is the set of $x$-stable partial flags $V_n \subset \ldots \subset V_{m-1} \subset V_m$ in an $m$-dimensional vector space $V_m$ whose lowest piece $V_n$ has dimension $n$, and for every two consecutive spaces, we have $\dim(V_{i+1}/V_i) = 1$. The fiber $P_{[n,1^{m-n}]}(x)$ can be considered as

$$P_{[n,1^{m-n}]}(x) = \{(x, gP_{[n,1^{m-n}]}b) \mid \text{Ad}(g)^{-1}x \in p_{[n,1^{m-n}]}\}.$$ 

Similarly, for the parabolic subgroup $P_{[n,m-n]}$, the generalized Springer fiber above $x \in O_{\gamma}$, denoted similarly by $P_{[n,m-n]}(x)$, is a subvariety of $G/P_{[n,m-n]}$ given by

$$P_{[n,m-n]}(x) = \{(x, gP_{[n,m-n]}b) \mid \text{Ad}(g)^{-1}x \in p_{[n,m-n]}\}.$$
4.3. Springer Fibers, Jordan Forms, and the Proof of Part 2 of Theorem \[A\]. First, let’s consider the case of $\mathcal{P}_{\gamma}(x)$ and interpret this fiber as the collection of flags between $V_n$ and $V_m$. Recalling the notations in Section 2.1, for any nilpotent element $x$, take any skew Young tableau $T$ of the shape $\gamma/\alpha$ with the corresponding sequence of Young diagrams

$$\alpha = \alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_{m-n} = \gamma$$

such that $\beta = (|\alpha_1/\alpha_0|, \ldots, |\alpha_{m-n}/\alpha_{m-n-1}|)$, we can define a closed subvariety $\mathcal{P}^T(x)$ as the collections of $x$-stable flags on which the nilpotent element $x$ acts on each stage $V_{n+i}$ by a Jordan form $\lambda_i$, i.e. the restriction of $x$ on $V_{n+i}$ has Jordan form given by a partition $\alpha_i$.

For any nilpotent element $x \in O_\gamma$, there is a subvariety $G^{\mu}_n(x)$ of the Grassmannian variety $G(n, m)$ which parametrizes the $x$-stable subspaces of $V_m$ of dimension $n$. By the arguments in the Section 4 of [LLO00], this subvariety of the Grassmannian variety $G(n, m)$ can be decomposed into cells $G_{\alpha, \beta}(x)$ such that the Jordan type of $x$ on $V_n$ is $\alpha$, and the Jordan type of $x'$, the nilpotent linear operator acting on the quotient space $V_m/V_n$ induced from $x$, is $\beta$:

$$G^{\mu}_n(x) = \bigcup_{|\alpha| = n, |\gamma| = m, \alpha, \beta \subset \gamma} G_{\alpha, \beta}(x).$$

By Lemma 4.1.1 each cell $G_{\alpha, \beta}(x)$ is isomorphic to the Satake fiber $S_{\alpha, \beta}^\gamma$.

Moreover, the action of the nilpotent element $x$ with Jordan type $\beta$ on the space $V_m/V_n$ corresponds to an $x$-stable filtration

$$V_n \subset V_{n+1} \subset \ldots \subset V_m$$

such that the action of $x$ on each quotient $V_{n+i}/V_i$ has Jordan form $\alpha_i$. The collection of all the flags which satisfy this property can be identified with the subspace $\mathcal{P}_{\alpha, \beta}(x)$ of the partial flag variety $G/P_{[n, m-n]}$ defined below:

$$\mathcal{P}_{\alpha, \beta}(x) = \{gP \mid \text{Ad}(g)^{-1} x \in p_{[n, m-n]}, \text{ and } p_1(\text{Ad}(g)^{-1} x) \in O_{\alpha, \beta}\}$$

where $p_1 : p_{[n, m-n]} \to p_{[n, m-n]}/u_{[n, m-n]} \cong p_{[n, m-n]}$ is the projection map. Letting $(\alpha, \beta)$ go over all the possible pairs of Young diagrams satisfying $|\alpha| + |\beta| = |\gamma|$, the union of the spaces $\mathcal{P}_{\alpha, \beta}(x)$ is exactly the generalized Springer fiber $\mathcal{P}_{[n, m-n]}(x)$ over a point $x \in O_\gamma$ of the partial flag variety $G/P_{[n, m-n]}$.

The correspondence between flags and parabolic subgroups establishes isomorphisms between $G^{\mu}_n(x)$ and $\mathcal{P}_{[n, m-n]}(x)$, and between $G_{\alpha, \beta}(x)$ and $\mathcal{P}_{\alpha, \beta}(x)$.

Now we prove the second part of the Theorem 3.0.1

**Proof.** If $x \in O_\gamma$ and $gP_{[n, m-n]} \in \mathcal{P}_{\alpha, \beta}(x)$, each $\text{Ad}(g)^{-1} x \in p_{[n, m-n]} \cap O_\gamma$, and the space $\mathcal{P}_{\alpha, \beta}(x)$ is a closed subvariety of $G/P_{[n, m-n]}$ which consists of those parabolic subalgebras $q = m + u$ conjugate to $p_{[n, m-n]}$ such that

$$x \in q \text{ and } p_n(x) \in O_{\alpha, \beta}.$$
The number of connected components of \( P_{\alpha,\beta}(x) \cdot P \) is still \( c_{\alpha,\beta}^* \), since multiplying by an element \( p \in P \) simply fixes a basis of the flag. By \([3]\) stated in Remark \([3]\), the dimension of the connected components of \( P_{\alpha,\beta}(x) \cdot P \) is equal to

\[
\dim P_{\alpha,\beta}(x) \cdot P = \frac{1}{2} \left( \sum_i (\gamma_i^t)^2 - \sum_i (\alpha_i^t)^2 - \sum_i (\beta_i^t)^2 \right) + \left( \sum \alpha_i^t \right)^2 + \left( \sum \beta_i^t \right)^2 + \left( \sum \alpha_i^t \right) \left( \sum \beta_i^t \right).
\]

Therefore, the dimension of \( P_{\alpha,\beta}^* \) satisfies

\[
\dim P_{\alpha,\beta}^* = \dim P_{\alpha,\beta}(x) \cdot P - \dim C_G(x) > \dim n - \dim P_{\alpha,\beta}(x) > 0.
\]

The union of the closed orbits of \( C_G(x) \) will not be Zariski-dense. Therefore, the number of irreducible components of \( P_{\alpha,\beta}^* \) is also \( c_{\alpha,\beta}^* \). Since we can interpret the image of the map as the elements in \( O_\tau \cap p_{[n,m-n]} \) whose projections along \( p_i \) lies in \( O_{\alpha,\beta} \), we have completed the proof of the second part of the Theorem \( \Box \)

5. Applications: Wave Front Sets and Associated Varieties

In this section, let \( G \) be a semisimple algebraic group over an archimedean field \( F \). We would like to introduce invariants to describe the “growth” of irreducible representations \( \pi \) of the group \( G \). We denote the Lie algebra of \( G \) by \( \mathfrak{g}_0 \), and its complexified Lie algebra by \( \mathfrak{g} \).

5.1. Associated Varieties and Annihilator Varieties. For any irreducible representations \( \pi \) of \( G \), the “growth” of \( \pi \) is recorded in the associated variety \( \text{As}(\pi) \) and its annihilator variety \( \text{An}(\pi) \). The algebra \( U(\mathfrak{g}) \) and its ideals are filtered by the degrees, and the annihilator variety \( \text{An}(M) \) of a module \( M \) are defined as the zero sets of the ideal \( \text{Gr}(\text{Ann}(M)) \) and \( \text{Ann}(\text{Gr}(M)) \), respectively. For groups over \( \mathbb{R} \) and \( \mathbb{C} \), it turns out that the annihilator variety \( \text{An}(\pi) \) of any irreducible representation \( \pi \) is the closure of an unique orbit \( O(\tau) \) (cf. \([10]\) 3.10)).

5.2. Induction, Jacquet Functors and Associated Varieties. In \([20]\), Gourevitch and Sayag discussed a family of Lagrangian submanifolds of the annihilator varieties of any irreducible \( U(\mathfrak{g}) \)-module. According to \([20]\) Corollary 6.6:

**Theorem 5.2.1.** For any irreducible representation \( \pi \) of \( G \) and a parabolic subgroup \( P = LN \), denoting by \( r_P \) the Jacquet functor along the nilpotent radical \( N \), then for any irreducible quotient \( \tau \) of \( r_P(\pi) \),

\[ O(\tau) \subset p_i(O(\pi) \cap n^+). \]

Under the duality given by the Killing form, \( \mathfrak{g}^* \) and \( \mathfrak{g} \) are isomorphic, and the following spaces correspond under such isomorphism:

| \( \mathfrak{g} \) | \( \mathfrak{g}^* \cong \mathfrak{g} \) |
|---|---|
| \( \mathfrak{n}^- \) | \( \mathfrak{p} \) |
| projection \( p_i : \mathfrak{p} \to \mathfrak{p}/n \) | projection \( q_i : \mathfrak{n}^+ \to \mathfrak{n}^+/\mathfrak{n}^+ \) |
| \( \mathcal{O}(\pi) \cap \mathfrak{n}^+ \) | \( \mathcal{O}(\pi) \cap \mathfrak{p} \) |
| \( \mathfrak{n}^+ \cap p_i^{-1} \mathcal{O}(\tau) \) | \( \mathcal{O}(\tau) + \mathfrak{n} \).

Therefore, under the Killing form duality, the following two inclusions are equivalent:

\[ O(\tau) \subset p_i(O(\pi) \cap n^+) \Rightarrow O(\pi) \subset G \cdot (O(\tau) + n). \]

5.3. Examples. In this section we will give some examples and applications to the Theorem \( \Box \)

5.3.1. \( n = 3 \). Consider the standard parabolic subgroup \( P_{[2]} = L_{[2]} N_{[2]} \) where the Levi subgroup \( L_{[2]} = GL(2) \times GL(1) \), and the unipotent radical

\[ N_{[2]} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}. \]
The generalized flag variety $G/P_{[21]}$ is isomorphic to $\mathbb{P}^2$, and the isomorphism can be written explicitly down with the Plücker coordinates

$$G/P_{[21]} \rightarrow \mathbb{P}(\wedge^2 \mathbb{C}^3)$$

$$[g] \mapsto (g(e_1 \wedge e_2), g(e_2 \wedge e_3), g(e_3 \wedge e_1)).$$

For an element $x = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in O_{[21]}$, the generalized Springer fiber $P_{[21]}(x)$ above $x$ is isomorphic to the projective line $\mathbb{P}^1$, and is given by the equation $X_1 = 0$ in the Plücker coordinate $[X_0 : X_1 : X_2]$. The generalized Springer fiber above an element $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ is the point $[1 : 0 : 0]$.

For the case $\alpha = [11]$ and $\beta = [1]$, the two possible choices of $\gamma$ are $[21]$ and $[111]$ with Littlewood-Richardson coefficients $c_{[11],[1][21]} = c_{[11],[1][111]} = 1$. The orbit $O_{[111]}$ is the zero orbit, hence for all parabolic subalgebras $q = m + u$ conjugate to $p_{[21]}$, the variety $p_m (O_{[111]} \cap q)$ contains the zero orbit $O_{[11]} \times O_{[1]}$.

The orbit $O_{[21]}$ can be explicitly described with matrices:

$$O_{[21]} = \{ A \in g \mid A^2 = 0 \text{ and } A \neq 0 \}.$$

We can represent any element in $p_{[21]}$ as a matrix of the shape $\left(\begin{smallmatrix} A & v \\ 0 & b \end{smallmatrix}\right)$ where $v \in \mathbb{C}^2$. The subvariety $O_{[21]} \cap p_{[21]}$ is the following subset

$$O_{[21]} \cap p_{[21]} = \left\{ \left(\begin{smallmatrix} A & v \\ 0 & 0 \end{smallmatrix}\right) \neq 0 \mid A^2 = 0, Av = 0 \right\}$$

of $p_{[21]}$. The elements $[g]$ in the generalized Springer fiber $P_{[21]}(x)$ are those who satisfy $\text{Ad}(g)^{-1}x \in O_{[21]} \cap p_{[21]}$. The point $[0 : 0 : 1] \in P_{[21]}(x)$ projects to the zero orbit $O_{[11]}$ and the affine chart $[1 : 0 : x]$ projects to $O_{[2]}$.

The orbit $O_{[3]}$ is the set of matrices satisfying

$$O_{[3]} = \{ A \in g \mid A^3 = 0, A^2 \neq 0 \text{ and } A \neq 0 \}.$$

Therefore, the subvariety $O_{[3]} \cap p_{[21]}$ is the following set of matrices:

$$O_{[3]} \cap p_{[21]} = \left\{ \left(\begin{smallmatrix} A & v \\ 0 & 0 \end{smallmatrix}\right) \neq 0 \mid A^3 = 0, Av = 0, A^2 \neq 0 \text{ or } Ab \neq 0 \right\}.$$

The generalized Springer fiber over a point of $O_{[3]}$ is the point $[1 : 0 : 0]$, and the whole set $O_{[3]} \cap p_{[21]}$ projects to the orbit $O_{[2]}$.

5.3.2. $n = 6$. Consider the Levi blocks $GL(4) \times GL(2) \subset GL(6)$, and consider the orbit $O_{[22]} \times O_{[1]}$. Between the induced orbit $[33]$ and $[2211]$, the poset of $G$-orbits contained in $G \cdot (O_{[22],[1]} + n)$ is displayed as below:

$$\begin{array}{ccc}
[33] & & \\
& [321] & \\
\{3111\} & \{222\} & \\
& [2211] & \\
\end{array}$$

The orbits $[3111]$ and $[222]$ are excluded from the poset, since $c_{33,2211}^{3111} = c_{33,2211}^{222} = 0$.

It is important to mention that the $n = 6$ case for $P_{[33]}$ is the first case in which we can expect a Littlewood-Richardson coefficient greater than 1. The partial flag variety $G/P_{[33]}$ is isomorphic to the Grassmannian $G(3,6)$ and can be embedded into the projective space $\mathbb{P}(\wedge^3 \mathbb{C}^6) \cong \mathbb{P}^{19}$ with the Plücker embeddings by choosing the basis of $\wedge^3 \mathbb{C}^6$ as $e_i \wedge e_j \wedge e_k$ for each triple $i < j < k$. 

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We will discuss a slightly different space $P_{\alpha,\beta}(x,y)$ by taking those elements $[g]$ in $P_{\alpha,\beta}(x)$ such that $p_1(\text{Ad}^{-1}(g)x)$ is a fixed element $y \in O_\alpha \times O_\beta$. There is a map

$$\sigma : P_{\alpha,\beta}(x,y) \cdot P \rightarrow \text{Spec}(\mathbb{C}[p]^L)$$

which takes every $g$ to $\text{Ad}(g)^{-1}x$, and the ring $\mathbb{C}[p]^L$ is the ring of $\text{Ad}L$-invariant regular functions on $p$ under the adjoint action of $L$. Any element of $\text{Spec}(\mathbb{C}[p]^L)$ lying in the image of $\sigma$ can be represented by a matrix

$$\begin{pmatrix}
0 & 1 & 0 & u_{1,4} & u_{1,5} & u_{1,6} \\
0 & 0 & 0 & u_{2,4} & u_{2,5} & u_{2,6} \\
0 & 0 & 0 & u_{3,4} & u_{3,5} & u_{3,6} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

The images of the map $\sigma$ for different choices of $\gamma$ are listed in the following table:

| $\gamma$ | $\dim P_{\alpha,\beta}(x)$ | $c_{[21],[21]}$ | Ideal |
|-----------|-----------------------------|----------------|-------|
| [2211]    | 5                           | 1              | $(u_{3,6}, u_{2,6}, u_{2,4}, u_{3,4}, u_{1,4} + u_{2,5})$ |
| [222]     | 4                           | 1              | $(u_{1,4} + u_{3,5}, u_{2,4}, u_{3,4}, u_{2,6})$ |
| [3111]    | 4                           | 1              | $(u_{3,6}, u_{2,6}, u_{2,4}, u_{3,4})$ |
| [3,2,1]   | 2                           | 2              | $(u_{3,4}, u_{2,4}), (u_{2,6}, u_{2,4})$ |
| [3,3]     | 1                           | 1              | $(u_{2,4})$ |
| [4,1,1]   | 1                           | 1              | $(u_{2,6}u_{3,4} - u_{2,4}u_{3,6})$ |
| [4,2]     | 0                           | 1              | $(0)$ |

5.3.3. $n = 8$. We look at the Levi blocks $GL(4) \times GL(4) \subset GL(8)$, and consider the orbit $O_{[22]} \times O_{[22]}$. Between the induced orbit [44] and [2222], the poset of $G$-orbit $G \cdot (O_{[22]} \times O_{[22]} + n)$ is displayed as below:

```
[44]  \\
|    \\
[431]  \\
|    \\
[422]  \\
|    \\
[4211]  \\
|    \\
[332]  \\
|    \\
[3221]  \\
|    \\
[3311]  \\
|    \\
[3222]  \\
```

The orbits [4211] and [332] are excluded from the poset, since $c_{22,22}^{4211} = c_{22,22}^{332} = 0$. 
5.3.4. \( n = 12 \). Consider the Levi blocks \( GL(6) \times GL(6) \subset GL(12) \) and the orbit \( O_{[321]} \times O_{[321]} \), if we represent any element of \( \text{Spec}(\mathbb{C}[p]^L) \) by the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & u_{1,7} & u_{1,8} & u_{1,9} & u_{1,10} & u_{1,11} & u_{1,12} \\
0 & 0 & 1 & 0 & 0 & 0 & u_{2,7} & u_{2,8} & u_{2,9} & u_{2,10} & u_{2,11} & u_{2,12} \\
0 & 0 & 0 & 1 & 0 & 0 & u_{3,7} & u_{3,8} & u_{3,9} & u_{3,10} & u_{3,11} & u_{3,12} \\
0 & 0 & 0 & 0 & 1 & 0 & u_{4,7} & u_{4,8} & u_{4,9} & u_{4,10} & u_{4,11} & u_{4,12} \\
0 & 0 & 0 & 0 & 0 & 1 & u_{5,7} & u_{5,8} & u_{5,9} & u_{5,10} & u_{5,11} & u_{5,12} \\
0 & 0 & 0 & 0 & 0 & 0 & u_{6,7} & u_{6,8} & u_{6,9} & u_{6,10} & u_{6,11} & u_{6,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

If we set \( \gamma = [53211] \), the ideals for the image of the projection map from the 6 dimensional Satake fiber with 4 irreducible components are the following four ideals:

\[
I_1 = (u_{6,10}, u_{6,7}, u_{5,10}, u_{5,7}, u_{3,10}, u_{3,7})
\]

\[
I_2 = (u_{5,7} - u_{5,10} u_{6,12} - u_{3,12} u_{6,10} - u_{3,10} u_{6,12} - u_{3,12} u_{5,10} - u_{5,10} u_{5,12})
\]

\[
I_3 = (u_{5,12} - u_{5,7} - u_{5,10} - u_{3,12} - u_{3,7} - u_{3,10} - u_{3,12})
\]

\[
I_4 = (u_{3,12} - u_{3,7} - u_{3,10} - u_{3,12})
\]

The Littlewood-Richardson coefficient \( c_{[53211][321][321]} \) of this case is equal to 4.

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