The heritage of S. Lie and F. Klein: 
Geometry via transformation groups

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Abstract

We outline a framework which generalizes Felix Klein’s Erlangen Programm which he announced in 1872 after exchanging ideas with Sophus Lie.

1 Introduction

Let $G$ be a Lie group which acts transitively on some space $M$. In this framework, Felix Klein defined geometry (later known as Erlangen Programm) as the study of properties which are invariant under the action of $G$. Realizing $M$ as the coset space $G/H$ where $H \subset G$ is the stabilizer of some point $p \in M$, we may thus speak of the Klein pair (or Klein geometry) $(G, G/H)$. In Euclidean geometry, the best known example, $G$ is the isometry group of $\mathbb{R}^n$ and is the semi-direct product of the orthogonal group $O(n)$ and $\mathbb{R}^n$, $H = O(n)$ and $G/H = \mathbb{R}^n$. Non-Euclidean geometries, which were known at Klein’s time, are other examples. Riemannian geometry can not be realized in this way unless the metric has constant curvature in which case the isometry group acts transitively and we have again a Klein geometry. We will refer the reader to [14], pg. 133 so that he/she can feel the philosophical disturbance created by this situation at that time. As an attempt to unify (among others) Riemannian geometry and Erlangen Programm, Elie Cartan introduced in 1922 his generalized spaces (principal bundles) which are curved analogs of the principal $H$-bundle $G \rightarrow G/H$. We will mention here the two outstanding books: [29] for Klein geometries, their generalizations called Cartan geometries in [29] and also for the notions of Cartan and Ehresmann connections, and [14] for more history of this subject (see in particular pg. 34-42 for Erlangen Programm). Cartan’s approach, which is later developed mainly by topologists from the point of view of fiber bundles, turned out to be extremely powerful. The spectacular achievements of the theory of principle bundles and connections in geometry, topology and physics are well known and it is therefore redundant to elaborate on them here. However, it is also a fact that this theory leaves us with the unpleasant question: What
happened to **Erlangen Programm**? The main reason for this question is that the total space $P$ of the principal bundle $P \to M$ does not have any group-like structure and therefore does not act on the base manifold $M$. Thus $P$ emerges in this framework as a new entity whose relation to the geometry of $M$ may not be immediate and we must deal now with $P$ as a separate problem. Consequently, it seems that the most essential feature of Klein’s realization of geometry is given up by this approach. Some mathematicians already expressed their dissatisfaction of this state of affairs in literature with varying tones, among which we will mention [29], [35], [26] and other controversial works of the author of [26] (see the references in [26]).

The purpose of this work is to present another such unification which we by no means claim to be the ultimate and correct one but believe that it is faithful to Klein’s original conception of geometry. This unification is based on the ideas which S. Lie and F. Klein communicated to each other before 1872 (see [14] for the extremely interesting history of this subject) and the works of D.C. Spencer and his co-workers around 1970 on the formal integrability of PDEs. The main idea is simple and seems to be the only possible one: We concentrate on the action of $G$ on $M = G/H$ and generalize this action rather than generalizing the action of $H$ on $G$ in the principal $H$-bundle $G \to G/H$. Now $G \subset \text{Diff}(M)$ and $G$ may be far from being a Lie group. As the natural generalization of **Erlangen Programm**, we may now deal directly with the group $G$ as in [2] (see also [23]), but this approach again does not incorporate Riemannian geometry unless the metric is homogeneous. We consider here the Lie pseudogroup $\tilde{G}$ determined by $G$ and filter the action of $\tilde{G}$ on $M$ via its jets, thus realizing $\tilde{G}$ as a projective limit $\text{Lim}_{\to k} S_k(M)$ of Lie equations $S_k(M)$. Lie equations (in finite form) are very special groupoids and are very concrete objects which are extensively studied by Spencer and his co-workers culminating in the difficult work [11]. We will refer to [11], [25], [26] for Lie equations and [19], [20] for differentiable groupoids and algebroids. On the infinitesimal level, we obtain the approximation $\text{Lim}_{\to k} s_k(M)$ where $s_k(M)$ is the infinitesimal Lie equation (or the algebroid) of $S_k(M)$. The idea is now to start with the expression $\text{Lim}_{\to k} S_k(M)$ as our definition of homogeneous geometry $S_\infty(M)$ (Section 3, Definition 2). Any transitive pseudogroup (in particular a complex, symplectic or contact structure) determines a homogeneous geometry and Klein geometries are special cases (Section 4). Some $S_k(M)$ may not prolong to a homogeneous geometry due to the lack of formal integrability and Riemannian geometry (almost complex, almost symplectic...structures) emerge as truncated geometries (Section 5). We associate various spectral sequences to a homogeneous geometry $S_\infty(M)$ (in particular to a truncated geometry $S_k(M)$) (Sections 2, 3). For a complex structure, we believe that one these spectral sequences is directly related to the Frölicher spectral sequence which converges to de Rham cohomology with $E_1$ term equal to Dolbeault cohomology. This unification is also a natural specialization of the standard principal bundle approach initiated by E. Cartan (Sections 6, 7, 8).

The idea of filtering an object via jets (for instance, the solution space of
some PDE called a diffiety in [32], [33]) is not new and is used by A.M.Vinogradov in 1978 in the construction of his C-spectral sequence and in the variational bicomplex approach to Euler-Lagrange equations (see [32], [33] and the references therein). In fact, this paper can also be considered as the first step towards the realization of a program stated in [33] for quasi-homogeneous geometries ([33], Section 6.4). Further, there is definitely a relation between the higher order de Rham complexes constructed here and those in [31]. We also believe that the main idea of this paper, though it may not have been stated as explicitly as in this paper, is contained in [26] and traces back to [11] and [25]. In particular, we would like to emphasize that all the ingredients of this unification are known and exist in the literature.

This paper consists of nine sections. Section 2 contains the technical core in terms of which the geometric concepts substantiate. This section may be somewhat demanding for the reader who is not much familiar with jets and the formalism of Spencer operator. However, as we proceed, technical points slowly evaporate and the main geometric concepts which we are all familiar with, start to take the center stage.

2 Universal homogeneous envelope

Let $M$ be a differentiable manifold and $Diff(M)$ be the group of diffeomorphisms of $M$. Consider the map $Diff(M) \times M \rightarrow M$ defined by $(g, x) \rightarrow g(x) = y$ and let $j_k(g)_y^x$ denote the $k$-jet of $g$ with source at $x$ and target at $y$. By choosing coordinates $(U, x^i)$ and $(V, y^i)$ around the points $x, y$, we may think $j_k(g)_y^x$ as the coefficients of the Taylor expansion of $g(x) = y$ up to order $k$. We will call $j_k(g)_y^x$ the $k$-arrow induced by $g$ with source at $x$ and target at $y$ and imagine $j_k(g)_y^x$ as an arrow starting at $x$ and ending at $y$. Let $(f_k)_y^x$ denote any $k$-arrow, i.e., $(f_k)_y^x$ is the $k$-jet induced by some arbitrary local diffeomorphism which maps $x$ to $y$. With some assumptions on orientability which involve only 1-jets (see [24] for details), there exists some $g \in Diff(M)$ with $j_k(g)_y^x = (f_k)_y^x$.

Therefore, with the assumption imposed by [24], the pseudogroup $Diff(M)$ of local diffeomorphisms on $M$ induces the same $k$-arrows as $Diff(M)$, but this fact will not be used in this paper. We can compose successive $k$-arrows and invert all $k$-arrows.

Now let $(G_k)_y^x$ denote the set of all $k$-arrows with source at $x$ and target at $y$. We define $G_k(M) = \cup_{x, y \in M}(G_k)_y^x$ and obtain the projections $\pi_{k,m} : G_k(M) \rightarrow G_m(M)$, $1 \leq m \leq k − 1$, and $\pi_{k,m}$ is compatible with composition and inversion of arrows. We will denote all projections arising from the projection of jets by the same notation $\pi_{k,m}$. Now $G_k(M)$ is a transitive Lie equation in finite form (TLEFF) on $M$ which is a very special groupoid (see [11], [25], [26] for Lie equations and [19], [20] for groupoids). We also have the locally trivial map $G_k(M) \rightarrow M \times M$ which maps $(G_k)_y^x$ to $(x, y)$. Note that $(G_k)_y^x$ is a Lie group and can be identified (not in a canonical way) with $k^{th}$-order jet group. Thus we obtain the sequence of homomorphisms
\[ \ldots \rightarrow \mathcal{G}_{k+1}(M) \rightarrow \mathcal{G}_k(M) \rightarrow \ldots \rightarrow \mathcal{G}_1(M) \rightarrow M \times M \rightarrow 1 \quad (1) \]

where the last arrow is used with no algebraic meaning but to express surjectivity. (1) gives the vague formula \( \text{Diff}(M) \times M = \text{Lim}_{k \rightarrow \infty} \mathcal{G}_k(M) \) or more precisely \( \text{Diff}(M) = \text{Lim}_{k \rightarrow \infty} \mathcal{G}_k(M) \). The ambiguity in this last formula is that a formal Taylor expansion may fail to recapture a local diffeomorphism. However, this ambiguity is immaterial for our purpose for the following reason:

Let \((j_\infty g)_x^p\) denote the \(\infty\)-jet of some local diffeomorphism \(g\) where \(g(x) = x\). Now \((j_\infty g)_x^p\) determines the \(\infty\)-jet of the identity diffeomorphism. This is a consequence of the following elementary but remarkable fact: For any sequence of real numbers \(a_0, a_1, \ldots\), there exists a real valued differentiable function \(f\) defined, say, near the origin \(o \in \mathbb{R}\), satisfying \(f^{(n)}(o) = a_n\). In particular, the same interpretation is valid for the \(\infty\)-jets of all objects to be defined below.

Since \(\mathcal{G}_k(M)\) is a differentiable groupoid (we will call the object called a Lie groupoid in [19], [20] a differentiable groupoid, reserving the term “Lie” for Lie equations), it has an algebroid \(\mathfrak{g}_k(M)\) which can be constructed using jets only. To do this, we start by letting \(J_k(T(M))_p\) denote the vector space of \(k\)-jets of vector fields at \(p \in M\) where \(T(M) \rightarrow M\) is the tangent bundle of \(M\). An element \(\xi \in J_k(T(M))_p\) is of the form \((p, \xi^1(p), \xi^2_j(p), \ldots, \xi^j_k, \ldots, \xi^j_{k_1k_2 \ldots j_1}(p))\) in some coordinates \((U, x^i)\) around \(p\). If \(X = (\xi^j(x)), Y = (\eta^j(x))\) are two vector fields on \(U\), differentiating the usual bracket formula \([X, Y](x) = \xi^a(x)\partial_a \eta^b(x) - \eta^a(x)\partial_a \xi^b(x)\) successively \(k\)-times and evaluating at \(p\), we obtain the algebraic bracket \(\{ , \}_{k,p} : J_k(T(M))_p \times J_k(T(M))_p \rightarrow J_{k-1}(T(M))_p\). Note that this bracket does not endow \(J_k(T(M))_p\) with a Lie algebra structure. However, for \(k = \infty\), \(J_\infty(T(M))_p\) is a graded Lie algebra with the bracket \(\{ , \}_{\infty,p}\), and is the well known Lie algebra of formal vector fields which is extensively studied in literature ([10]). However, let \(J_{k,0}(T(M))_p\) be the kernel of \(J_k(T(M))_p \rightarrow J_0(T(M))_p = T(M)_p\). Now \(J_{k,0}(T(M))_p\) is a Lie algebra with the bracket \(\{ , \}_{k,0,p}\) which is in fact the Lie algebra of \(\mathcal{G}_k(M)_p\).

We now define the vector bundle \(J_k(T(M)) \cong \bigcup_{\ell \in \mathbb{R}} J_{k,0}(T(M))\) by \(X\). To simplify our notation, we will use the same notation \(E\) for both the total space \(E\) of a vector bundle \(E \rightarrow M\) and also for the space \(\Gamma E\) of global sections of \(E \rightarrow M\). In a coordinate system \((U, x^i)\), \(X\) is of the form \(X(x) = (x, \xi^i_j(x), \xi^i_jj_1(x), \ldots, \xi^i_{j_1j_2 \ldots j_1}(x))\), but we may not have \(\xi^i_{j_1j_2 \ldots j_1}(x) = \frac{\partial \xi^i_{j_1j_2 \ldots j_1}}{\partial x^m}(x), 1 \leq m \leq k\). We can think \(X\) also as the function \(X(x, y) = \frac{\partial \xi^i_j(x)}{\partial x^m}(x, y-x)^a\) where \(\alpha\) is a multi-index with \(|\alpha| \leq k\) and we used summation convention. For some \(\tau \in U\), \(X(\tau, y)\) is some Taylor polynomial which is not necessarily the Taylor polynomial of \(\xi^i(x)\) at \(x = \tau\) since we may not have \(\xi^i_{\alpha+x}(\tau) = \frac{\partial \xi^i_{\alpha+x}}{\partial x}(\tau), |\alpha| \leq k\). Note that we have the bundle projections \(\pi_{k,m} : J_k(T(M)) \rightarrow J_m(T(M))\) for \(0 \leq m \leq k - 1\), where
\( J_0(T(M)) \cong T(M) \). We will denote \( J_k(T(M)) \) by \( \mathfrak{g}_k(M) \) for the reason which will be clear below.

We now have the Spencer bracket \([\ , \ ]\) defined on \( \mathfrak{g}_k(M) \) by the formula
\[
[k] (k+1) \quad [k] (k+1) \quad (0) (k+1) \quad (0) (k+1)
\]
\[ [X,Y] = [X,Y] + i(X)D Y - i(Y)D X \quad k \geq 0 \quad (2) \]

In (2), \( X \) and \( Y \) are arbitrary lifts of \( x \) and \( y \) to \( \mathfrak{g}_{k+1}(M) \). The bracket \([\ , \ ]\) is not a groupoid but rather a differentiable groupoid and its algebroid. However note that

\[
\tilde{\mathfrak{g}}_{k+1}(M) \rightarrow \mathfrak{g}_k(M) \]

is the algebraic bracket defined pointwise by \([X,Y](p) = [X(p), Y(p)] \). In fact, letting \( \mathfrak{X}(M) \) denote the Lie algebra of vector fields on \( M \), we have the prolongation map \( j_k : \mathfrak{X}(M) \rightarrow \mathfrak{g}_k(M) \) defined by \((x, \xi(x)) \rightarrow (x, \xi^i(x), \partial_i \xi^j(x), \partial_{j_1j_2} \xi^j(x), \ldots, \partial_{j_k \times \infty} \xi_j(x))\) which satisfies \( j_k[X,Y] = [j_k X, j_k Y] \). Thus (2) gives the usual bracket and its derivatives when restricted to \( j_k(\mathfrak{X}(M)) \).

Now \( \mathfrak{g}_k(M) \) is the transitive Lie equation in infinitesimal form (\( TLEIF \)) determined by \( \mathcal{G}_k(M) \). If we regard \( \mathcal{G}_k(M) \) as a differentiable groupoid and construct its algebroid as in [19], [20], we end up with \( \mathfrak{g}_k(M) \), justifying our notation \( \mathfrak{g}_k(M) \) for \( J_k(T(M)) \). The projection \( \pi_{k,m} : \mathfrak{g}_k(M) \rightarrow \mathfrak{g}_m(M) \) respects the bracket, i.e., it is a homomorphism of \( TLEIF \)'s.

In this way we arrive at the infinitesimal analog of (1):
\[
\ldots \rightarrow \mathfrak{g}_{k+1}(M) \rightarrow \mathfrak{g}_k(M) \rightarrow \ldots \rightarrow \mathfrak{g}_1(M) \rightarrow \mathfrak{g}_0(M) \rightarrow 0 \quad (3)
\]

Proceeding formally, the formula \( \widetilde{\text{Diff}}(M) = \lim_{k \rightarrow \infty} \mathcal{G}_k(M) \) now gives \( A\text{Diff}(M) = \lim_{k \rightarrow \infty} \mathfrak{g}_k(M) \) where \( A \) stands for the functor which assigns to a groupoid its algebroid. However note that \( \text{Diff}(M) \) is not a groupoid but rather a pseudogroup. Since a vector field integrates to some 1-parameter group of local diffeomorphisms (no condition on vector fields and diffeomorphisms since we have not imposed a geometric structure yet), we naturally expect \( A\text{Diff}(M) = \)
\( J_\infty(T(M)) \). As above, the vagueness in this formula is that a vector field need not be locally determined by the Taylor expansion of its coefficients at some point.

We now define the vector space \( J_k(M)_x \doteq \{ j_k(f)_x \mid f \in C^\infty(M) \} \) where \( C^\infty(M) \) denotes the set of smooth functions on \( M \) and \( j_k(f)_x \) denotes the \( k \)-jet of \( f \) at \( x \in M \). Now \( J_k(M)_x \) is a commutative algebra over \( \mathbb{R} \) with the multiplication \( \bullet \) defined by \( j_k(f)_x \bullet j_k(g)_x = j_k(fg)_x \). We define the vector bundle \( J_k(M) \doteq \bigcup_{x \in M} J_k(M)_x \rightarrow M \) with the obvious differentiable structure and projection map. The vector space of global sections of \( J_k(M) \rightarrow M \) is a commutative algebra with the fiberwise defined operations. We have the projection homomorphism \( \pi_{k,m} : J_k(M) \rightarrow J_m(M) \). We will denote an element of \( J_k(M) \) by \( f^{(k)} \) which is locally of the form \((x, f(x), f_1(x), f_{12}(x), \ldots, f_{1\ldots i_1}(x)) = (f_\alpha(x))\), \( |\alpha| \leq k \).

Now let \( X \in \mathfrak{g}_k(M) \) and \( f \in J_k(M) \). We define \( Xf \in J_k(M) \) by
\[
(0)(k)\quad Xf = X \ast f + i(X)D f
\]

In (4), \( \ast : \mathfrak{g}_k(M) \times J_{k+1}(M) \rightarrow J_k(M) \) is the algebraic action of \( \mathfrak{g}_k(M) \) on \( J_{k+1}(M) \) whose local formula is obtained by differentiating the standard formula \( Xf = \xi^a \partial_a f \) successively \( k \)-times and substituting jets, \( D : J_{k+1}(M) \rightarrow T^* \otimes J_k(M) \) is the Spencer operator defined by \( (x, f(x), f_1(x), f_{12}(x), \ldots, f_{1\ldots i_1}(x)) \rightarrow (x, \partial_{i_1} f(x) - f_1(x), \partial_{i_2} f_1(x) - f_{12}(x), \ldots, \partial_{i_{i_1}} f_{1\ldots i_1}(x) - f_{1\ldots i_1}(x)) \)

and \( f \) is some lift of \( f \). The RHS of (4) does not depend on the lift \( f \). It is easy to check that \( Xf = \xi^a \partial_a f \). Like (2), (4) is compatible with projection, i.e., we have \( \pi_{k,m}(Xf) = (\pi_{k,m}Xf) \{ \pi_{k,m}f \} \), \( 0 \leq m \leq k \). Since \( J_k(M)_x \) is a vector space over \( \mathbb{R} \), \( J_k(M) \) is a module over \( C^\infty(M) \). We will call some \( f^{(k)} = (f_\alpha) \in J_k(M) \) a smooth function if \( f_\alpha(x) = 0 \) for \( 1 \leq |\alpha| \leq k \). This definition does not depend on coordinates. Thus we have an injection \( C^\infty(M) \rightarrow J_k(M) \) of algebras. If \( f \in C^\infty(M) \), then \( f \bullet g = (\pi_{k,0} f \bullet g) = (f^{(0)}(k)) \). Similar considerations apply to \( J_k(T(M)) \).

We thus obtain the \( k \)-th-order analogs of the well known formulas:
\[
[\pi_{k,m}(X), \pi_{k,m}(Y)] = \pi_{k,m}(X(Yf) - Y(Xf)) \quad (5)
\]
\[
\pi_{k,m}(X(f \bullet g)) = \pi_{k,m}(Xf \bullet g) + f \pi_{k,m}(Xg) \quad (6)
\]

In particular, (6) gives
\[
\pi_{k,m}(X(f \bullet g)) = \pi_{k,m}(Xf \bullet g) + f \pi_{k,m}(Xg) \quad (7)
\]
where $X = \pi_{k,0}^{(k)}$. In the language of [19], [20], (5) and (7) define a representation of the algebroid $\mathfrak{g}_k(M)$ on the vector bundle $J_k(M) \to M$ (see also [25], pg.362 and [11], III, pg. 419). All constructions of this paper work also for other such “intrinsic” representations of $\mathfrak{g}_k(M)$. The passage from such intrinsic representations to general representations, we believe, is quite crucial for Erlangen Programm and will be touched at the end of Section 4 and in 5) of Section 9.

Now let $[k,r](M)_x$ denote the vector space of $r$-linear and alternating maps $\mathfrak{g}_k(M)_x \times \cdots \times \mathfrak{g}_k(M)_x \to J_k(M)_x$ where we assume $r \geq 1, k \geq 0$. We define the vector bundle $[k,r](M) = \cup_{x \in M} [k,r](M)_x \to M$. If $\omega \in \wedge^k(M) = \Gamma \wedge^k(M)$ and $X_1, \ldots, X_r \in \mathfrak{g}_k(M)$, then $\omega^r(X_1, \ldots, X_r) \in J_k(M)$ is defined by $(\omega^r(X_1, \ldots, X_r))(x) = \omega^r(x)(X_1(x), \ldots, X_r(x)) \in J_k(M)_x$. We define $d\omega^r$ by the standard formula: $(d\omega^r)(X_1, \ldots, X_{r+1}) = \frac{1}{r+1} \sum_{i \leq r+1} (-1)^{i+1} \omega^{r+1}(X_1, \ldots, \hat{X_i}, \ldots, X_r) - \frac{1}{r+1} \sum_{j \leq r+1} (-1)^{i+j} \omega^{r+1}([X_i, X_j], \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots, X_{r+1})$.

We also define $\wedge^k(M) = J_k(M)$ and $d : \wedge^{[k,1]}(M) \to \wedge^{[k,2]}(M)$ by $(d f)(X) = X \circ f$. We have $d : \wedge^{[k,r+1]}(M) : this follows from (6) or can be checked directly as in [9], pg. 489, using $f \circ g = f \circ g$ if $f \in C^\infty(M)$. In view of the Jacobi identity and the alternating character of $\omega^r$, the standard computation shows $d^2 = 0$. Thus we obtain the complex

\begin{equation}
\wedge^k(M) \to \wedge^{k+1}(M) \to \wedge^{k+2}(M) \to \cdots \to \wedge^n(M) \quad k \geq 0
\end{equation}

For $k = 0$, (9) gives de Rham complex.

We now assume $r \geq 1$ and define the subspace $\wedge^{[k,r]}(M)_x \subset [k,r](M)_x$ by the following condition: $\omega^r \in [k,r](M)_x$ belongs to $\wedge^{[k,r]}(M)_x$ iff $\pi_{k,m} \omega^r(X_1, \ldots, X_r)(x) \in J_m(M)_x$ depends on $X_1(x), \ldots, X_r(x), m \leq k$. This condition holds vacuously for $k = 0$. Thus we obtain the projection $\pi_{k,m} : \wedge^{[k,r]}(M)_x \to \wedge^{[m,r]}(M)_x$. We define the vector bundle $\wedge^{[k,r]}(M) = \cup_{x \in M} [k,r](M)_x \to M$ and set $\wedge = \wedge^0 = J_k(M).$
Definition 1 An exterior \((k,r)\)-form \(\omega\) on \(M\) is a smooth section of the vector bundle \(\wedge^k(M) \to M\).

An explicit description of \((k,r)\omega\) in local coordinates is not without interest but will not be done here. Applying \(\pi_k,m\) to (8), we deduce \(d_{\wedge} : \wedge^k(M) \to \wedge^{k+1}(M)\) and the commutative diagram:

\[
\begin{array}{ccc}
(k+1,r) & \wedge & (k+1,r+1) \\
\downarrow d & \downarrow \pi \\
(k,r) & \wedge & (k,r+1)
\end{array}
\]

Thus we obtain the array:

\[
\begin{array}{ccccccc}
(\infty,0) & \wedge & (\infty,1) & \wedge & (\infty,2) & \wedge & \ldots & \wedge & (\infty,n) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& (2,0) & \wedge & (2,1) & \wedge & (2,2) & \wedge & \ldots & (2,n) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& (1,0) & \wedge & (1,1) & \wedge & (1,2) & \wedge & \ldots & (1,n) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& (0,0) & \wedge & (0,1) & \wedge & (0,2) & \wedge & \ldots & (0,n) \\
\end{array}
\]

where all horizontal maps are given by \(d\) and all vertical maps are projections. The top sequence is defined algebraically by taking projective limits in each column.

Using \(\bullet\), we also define the wedge product of \((k,r)\)-forms in each row by the standard formula which turns \(\wedge^k(M) \oplus \oplus_{0 \leq r \leq n} (k,r)\wedge(M)\) into an algebra. This algebra structure descends to the cohomology. In particular, we obtain an algebra structure on the cohomology of the top row of (11), which we will denote by \(H^*(\Omega_{\infty}(M), J_{\infty}(M))\).

Now let \(C^{k+1,1}(M) \cong \text{Kernel}(\pi_{k,0} : \wedge^k(M) \to \wedge^k(M))\) and \(C^{k+1,*}(M) \cong \oplus_{0 \leq r \leq n} C^{k+1,r}(M), k \geq 0\). This gives the array
where all horizontal maps in (12) are restrictions of $d$ and all vertical maps are inclusions. The filtration in (12) preserves wedge product. In fact, we have

\[ C_{k,r}(M) \oplus C_{s,t}(M) \subset C_{k+s,r+t}(M) \]

which follows easily from the definition of $\bullet$. We will denote the spectral sequence of algebras determined by the filtration in (12) by $\mathcal{U}_M$ and call $\mathcal{U}_M$ the universal spectral sequence of $M$.

The above construction will be relevant in the next section. However, we remark here that (11) contains no information other than $H^\ast_{dR}(M, \mathbb{R})$. To see this, we observe that if $(k) \mathcal{X}(k) = 0$ for all $(k) \mathcal{X} \in \mathfrak{g}_k(M))$, then $(k) \mathcal{X} \in \mathbb{R} \subset C^\infty(M)$. Thus the kernel of the first operators in the horizontal rows of (11) define the constant sheaf $\mathbb{R}$ on $M$. Since $\wedge(M)$ is a module over $C^\infty(M)$, each row of (10) (which is easily shown to be locally exact) is a soft resolution of the constant sheaf $\mathbb{R}$ and thus computes $H^\ast_{dR}(M, \mathbb{R})$.

### 3 Homogeneous geometries

**Definition 2** A homogeneous geometry on a differentiable manifold $M$ is a diagram

\[
\begin{array}{ccccccc}
\quad \rightarrow & \mathcal{G}_2(M) & \rightarrow & \mathcal{G}_1(M) & \rightarrow & M \times M & \rightarrow & 1 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \| & \\
\quad \rightarrow & \mathcal{S}_2(M) & \rightarrow & \mathcal{S}_1(M) & \rightarrow & M \times M & \rightarrow & 1 \\
\end{array}
\]

where i) $\mathcal{S}_k(M)$ is a TLEFF for all $k \geq 1$ and therefore $\mathcal{S}_k(M) \subset \mathcal{G}_k(M)$ and the vertical maps are inclusions ii) The horizontal maps in the bottom sequence of (13) are restrictions of the projection maps in the top sequence and are surjective morphisms.

With an abuse of notation, we will denote (13) by $\mathcal{S}_\infty(M)$ and call $\mathcal{S}_\infty(M)$ a homogeneous geometry on $M$. We thus imagine that the lower sequence of (13) “converges” to some (pseudo)group $G \subset Diff(M)$ which acts transitively on $M$. However, $G$ may be far from being a Lie group and it may be intractible to deal with $G$ directly. The idea of Definition 2 is to work with the arrows of $G$ rather than to work with $G$ itself.
Definition 3 Let $\mathcal{S}_\infty(M)$ be a homogeneous geometry on $M$. $\mathcal{S}_\infty(M)$ is called a Klein geometry if there exists an integer $m \geq 1$ with the following property: If $(f_m)_y^z \in \mathcal{S}_m(M)_y$, then there exists a unique local diffeomorphism $g$ with $g(x) = y$ satisfying i) $j_k(g)_y^z = (f_m)_y^z$ ii) $j_k(g)_{g(x)}^z \in \mathcal{S}_m(M)_{g(z)}^z$ for all $z$ near $x$. The smallest such integer (uniquely determined if $M$ is connected) is called the order of the Klein geometry.

In short, a Klein geometry is a transitive pseudogroup whose local diffeomorphisms are uniquely determined by any of their $m$-arrows and we require $m$ to be the smallest such integer. Once we have a Klein geometry $\mathcal{S}_\infty(M)$ of order $m$, then all $\mathcal{S}_k(M)$, $k \geq m + 1$ are uniquely determined by $\mathcal{S}_m(M)$ and $\mathcal{S}_{k+1}(M) \to \mathcal{S}_k(M)$ is an isomorphism for all $k \geq m$, i.e., the Klein geometry $\mathcal{S}_\infty(M)$ prolongs in a unique way to a homogeneous geometry and all the information is contained in terms up to order $m$. We will thus denote a Klein geometry by $\mathcal{S}_{(m)}(M)$. We will take a closer look at these geometries in the next section. For instance, let $M^{2n}$ be a complex manifold. We define $\mathcal{S}_c(M)_x$ by the following condition: $(f_k)_y^z \in \mathcal{G}_k(M)_y$ belongs to $\mathcal{S}_k(M)_y$ if there exists a local holomorphic diffeomorphism $g$ with $g(x) = y$ and $j_k(g)_y^z = (f_k)_y^z$. We see that TLEIF $\mathcal{S}_k(M)$ defined by $\mathcal{S}_k(M) = \cup_{x,y \in M} \mathcal{S}_k(M)_y$ satisfies conditions of Definition 3 and therefore a complex structure determines a homogeneous geometry which is not necessarily a Klein geometry. Similarly, a symplectic or contact structure determines a homogeneous geometry (since these structures have no local invariants) which need not be Klein. More generally, any transitive pseudogroup determines a homogeneous geometry via its arrows.

Now given a homogeneous geometry $\mathcal{S}_\infty(M)$, we will sketch the construction of its infinitesimal geometry $\mathfrak{s}_\infty(M)$ referring to [25], [26] for further details. Let $x \in M$ and $X$ be a vector field defined near $x$. Let $f_t$ be the 1-parameter group of local diffeomorphisms generated by $X$. Suppose $X$ has the property that $j_k(f_t)_y^z$ belongs to $\mathcal{S}_k(M)_y$ for all small $t$ with $t \geq 0$ where $y_t = f_t(x)$. This is actually a condition only on the $k$-jet of $X$ at $x$. In this way we define the subspace $\mathfrak{s}_k(M)_x \subset \mathfrak{g}_k(M)_x$ which consists of those $X(x)$ satisfying this condition. We define the vector subbundle $\mathfrak{s}_k(M) = \cup_{x \in M} \mathfrak{s}_k(M)_x \to M$ of $\mathfrak{g}_k(M) \to M$ and the bracket (2) on $\mathfrak{g}_k(M)$ restricts to a bracket on $\mathfrak{s}_k(M)$ and $\mathfrak{s}_k(M)$ is the TLEIF determined by $\mathcal{S}_k(M)$.

In this way we arrive at the diagram

$$
\begin{array}{cccccc}
\vdots & \rightarrow & \mathfrak{g}_k(M) & \rightarrow & \mathfrak{g}_1(M) & \rightarrow & \mathfrak{g}_0(M) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\vdots & \rightarrow & \mathfrak{s}_k(M) & \rightarrow & \mathfrak{s}_1(M) & \rightarrow & \mathfrak{s}_0(M) & \rightarrow & 0 \\
\end{array}
$$

(14)

where the bottom horizontal maps are restrictions of the upper horizontal maps and are surjective morphisms. The vertical maps are injective morphisms induced by inclusion. Thus (14) is the infinitesimal analog of (13). We will denote (14) by $\mathfrak{s}_\infty(M)$ and call $\mathfrak{s}_\infty(M)$ the infinitesimal geometry of $\mathcal{S}_\infty(M)$.
Now we define \( L_{(k)} : \wedge \rightarrow \wedge \) and \( i_{(k)} : \wedge \rightarrow \wedge \) by the standard formulas: \( (L_{(k)} ω)(X_1, ..., X_r) = Y( ω (X_1, ..., X_r)) \) \( - Y( [Y, X_1], X_2, ..., X_r) \) \( - ... - Y( X_1, ..., [Y, X_r]) \) and \( (i_{(k)} ω)(X_1, ..., X_r) = (r+1) \) \( \omega (Y, X_1, ..., X_r) \). We also define \( \pi_{k,m} L_{(k)} : \wedge \rightarrow \wedge \) by \( \pi_{k,m} L_{(k)} = L \). We will call \( Θ(M) \) and similarly \( \pi_{k,m} i_{(k)} = i(M) \). With these definitions we obtain the well known formulas

\[
L_{(k)} = d \circ i_{(k)} + i_{(k)} \circ d \quad \text{(15)}
\]

\[
L_{(k)} \circ d = d \circ L_{(k)} \quad \text{(16)}
\]

We will now indicate briefly how a homogeneous geometry \( S_∞(M) \) gives rise to various spectral sequences.

1) We define \( \mathfrak{s}_k(M) \subset \mathfrak{s}(M) \) by the following condition: Some \( \mathfrak{s}_k(M) \) belongs to \( \mathfrak{s}_k(M) \) if \( L_{(k)} ω = 0, X \in \mathfrak{s}_k(M) \) and \( i_{(k)} ω = 0, X \in \mathfrak{s}_k(M) \). (15) and (16) show that \( d : \mathfrak{s}(M) \rightarrow \mathfrak{s}(M) \) and (10) holds. Thus we arrive at (11) and (12). Recall that if \( \mathfrak{g} \) is any Lie algebra with a representation on \( V \) and \( \mathfrak{s} \subset \mathfrak{g} \) is a subalgebra, then we can define the relative Lie algebra cohomology groups \( H^*(\mathfrak{g}, \mathfrak{s}, V) \). Since our construction is modelled on the definition of \( H^*(\mathfrak{g}, \mathfrak{s}, V) \), we will denote the cohomology of the top row of (11) by \( H^*(\mathfrak{s}_∞(M), \mathfrak{s}_∞(M), J∞(M)) \) in this case.

2) We will first make

**Definition 4** \( Θ(S_∞(M)) \equiv \{ f \in J_k(M) | \ X f = 0 \text{ for all } X \in \mathfrak{s}_k(M) \} \)

(6) shows that \( Θ(S_∞(M)) \) is a subalgebra of \( J_k(M) \). We will call \( Θ(S_∞(M)) \) the \( k^{th} \) order structure algebra of the homogeneous geometry \( S_∞(M) \). Note that \( Θ(G_k(M)) = \mathbb{R} \). For \( k \geq 1 \), we define \( \mathfrak{s}_k(M) \) as the space of alternating maps \( \mathfrak{s}_k(M) = \wedge s_k(M) \rightarrow \wedge s_k(M) \) and define \( \mathfrak{s}(M) \) as in Section 2. We have the restriction map \( θ_{(k)} : \wedge \mathfrak{s}(M) \rightarrow \wedge \mathfrak{s}(M) \) whose kernel will be denoted by \( \wedge \mathfrak{s}(M) \). Since \( Θ(S_∞(M)) = Ker(θ_{(k,1)} \circ d) \), we obtain the commutative diagram
Gmology can be localized, i.e., can be computed at any point of $M$. Cohomology groups in this case.

The reader may compare (17) to the diagram on page 183 in [25] which relates Spencer sequence to Janet sequence, called $P$-sequence in [25].

We will call (17) the horizontal crosssection of the representation triple $(G_\infty(M), S_\infty(M), J_\infty(M))$ at order $k$. Passing to the long exact sequence in (17), we see that local cohomology of the top and bottom sequences coincide (with a shift in order) in view of the local exactness of the middle row. Now defining $\Theta(\mathcal{S}_k(M))$ as in Section 2, (17) defines an exact sequence of three spectral sequences where the middle spectral sequence is (12). Note that local exactness of the top and bottom sequences would imply that their cohomologies coincide with the sheaf cohomology groups $H^*(\mathcal{S}_\infty(M), \Theta(S_k(M)))$ and $H^*(\mathcal{S}_\infty(M), 1_{\mathcal{S}_k(M)})$ respectively since the partition of unity applies to the spaces in these sequences. We will denote the limiting cohomology of the top and bottom sequences in (17) respectively by $H^*(_7(M), 0)$ and $H^*(_7(M), J_\infty(M))$. The reader may compare (17) to the diagram on page 183 in [25] which relates Spencer sequence to Janet sequence, called $P$-sequence in [25].

Before we end this section, note that $(f_{k+1})^r$ defines an isomorphism $(f_{k+1})^r : J_k(M)_x \rightarrow J_k(M)_y$ (in fact, $(f_k)^r$ does it) and also an isomorphism $(f_{k+1})^r : s_k(M)_x \rightarrow s_k(M)_y$. Let us assume that $S_\infty(M)$ is defined by some $G \subseteq Diff(M)$ as in the case of a symplectic structure, homogeneous complex structure or a Klein geometry (see Section 4). Now $G$ acts on $s_k(M)$ (defined as in 1) or 2 above) by $(g(\omega))(X_1, \ldots, X_r)(p) = j_{k+1}(g)^r(\omega)(g(j_{k+1}(g^{-1})^rX_1(p), \ldots, j_{k+1}(g^{-1})^pX_r(p))$ where $g(q) = p$. The cochains which are invariant under this action form a subcomplex whose cohomology can be localized, i.e., can be computed at any point of $M$. If $S_\infty(M) = G_\infty(M)$ and $G = Diff(M)$, then an invariant form must vanish but this need not be the case for a homogeneous geometry. We will not go into the precise description of this cohomology here though it is quite relevant for Klein geometries in Section 4 and can be expressed in terms of some relative Lie algebra cohomology groups in this case.
4 Klein geometries

Let $G$ be a Lie group and $H$ a closed subgroup. $G$ acts on the left coset space $G/H$ on the left. Let $o$ denote the coset of $H$. Now $H$ fixes $o$ and therefore acts on the tangent space $T(G/H)_{o}$ at $o$. However some elements of $H$ may act as identity on $T(G/H)_{o}$. The action of $h \in H$ (we regard $h$ as a transformation and use the same notation) on $T(G/H)_{o}$ depends only on 1-arrow of $h$ with source and target at $o$. Let $H_{1} \subset H$ be the normal subgroup of $H$ consisting of elements which act as identity on $T(G/H)_{o}$. To recover $H_{1}$, we consider 1-jets of vector fields at $o$ which we will denote by $J_{1}T(G(H)_{o})$. The action of $h$ on $J_{1}T(G(H)_{o})$ depends only on 2-arrow of $h$. Now some elements $h \in H_{1}$ may still act as identity at $J_{1}T(G(H)_{o})$ and we define the normal subgroup $H_{2} \subset H_{1}$ consisting of those elements. Iterating this procedure, we obtain a decreasing sequence $J$ of normal subgroups $\{1\} \subset \ldots \subset H_{k} \subset H_{k-1} \subset \ldots \subset H_{2} \subset H_{1} \subset H_{0} = H$ which stabilizes at some group $N$ which is the largest normal subgroup of $G$ contained in $H$ (see [29], pg. 161). We will call the smallest integer $m$ satisfying $N = H_{m}$ the order of the Klein pair $(G,H)$. In this case, it is easy to show that $g \in G$ is uniquely determined modulo $N$ by any of its $m$-arrows. $G$ acts effectively on $G/H$ iff $N = \{1\}$. If $(G,H)$ is a Klein pair of order $m$, then so is $(G/N,H/N)$ which is further effective. We will call $N$ the ghost of the Klein pair $(G,H)$ since it can not be detected from the action and therefore may have implications that fall outside the scope of Erlangen Programm. We will touch this issue again in 5) of Section 9. Thus we see that an effective Klein pair $(G,H)$ of order $m$ determines a Klein geometry $S_{(m)}(G/H)$ according to Definition 3 where the local diffeomorphisms required by Definition 3 are restrictions of global diffeomorphisms of $G/H$ which are induced by the elements of $G$. Conversely, let $S_{(m)}(M)$ be a Klein geometry according to Definition 3 and let $\tilde{M}$ be the universal covering space of $M$. We pull back the pseudogroup on $M$ to a pseudogroup on $\tilde{M}$ using the local diffeomorphism $\pi : \tilde{M} \rightarrow M$. Using simple connectedness of $\tilde{M}$ and a mild technical assumption which guarantees that the domains of the local diffeomorphisms do not become arbitrarily small, the standard monodromy argument shows that a local diffeomorphism defined on $\tilde{M}$ in this way uniquely extends to some global diffeomorphism on $\tilde{M}$. This construction is essentially the same as the one given in [30] on page 139-146. The global diffeomorphisms on $\tilde{M}$ obtained in this way form a Lie group $G$. If $H \subset G$ is the stabilizer of some $p \in \tilde{M}$, then $H$ is isomorphic to $S_{(m)}(M)_{q}$ where $\pi(p) = q$. To summarize, a Klein geometry $S_{(m)}(M)$ according to Definition 3 becomes an effective Klein pair $(G,H)$ of order $m$ when pulled back to $\tilde{M}$. Conversely, an effective Klein pair $(G,H)$ of order $m$ defines a Klein geometry $S_{(m)}(M)$ if we mode out by the action of a discrete subgroup. Keeping this relation in mind, we will consider an effective Klein pair $(G,H)$ as our main object below.

Now the above filtration of normal subgroups gives the diagram
where the spaces in the top sequence are jet groups in our universal situation in Section 2 and the vertical maps are injections. Since the kernels in the upper sequence are vector groups, we see that $H_1$ is solvable. As before, we now define $S_m(M)$ which consists of $m$-arrows of elements of $G$ and define $S_m(M) = \cup_{x,y \in M} S_m(M)^x_y$. As in the case $\text{Diff}(M) \times M \to M$ in Section 2, we obtain the map $G \times G/H \to S_m(M)$ defined by $(g, x) \to m$-arrow of $g$ starting at $x$ and ending at $g(x)$. This map is surjective by definition and this time also injective by the definition of $m$. Thus we obtain a concrete realization of $S_m(M)$ as $G \times G/H$. Note that $S_m(M)^o = H$. Going downwards in the filtration, we obtain the commutative diagram

$$
\begin{array}{cccccc}
G \times G/H & \rightarrow & S_m(M) & \\
\downarrow & & \downarrow & & \\
G/H_{m-1} \times G/H & \rightarrow & S_{m-1}(M) & \\
\downarrow & & \downarrow & & \\
\ldots & \rightarrow & \ldots & \\
\downarrow & & \downarrow & & \\
G/H_2 \times G/H & \rightarrow & S_2(M) & \\
\downarrow & & \downarrow & & \\
G/H_1 \times G/H & \rightarrow & S_1(M) & \\
\end{array}
$$

(19)

For instance, the bottom map in (19) is defined by $\{xH_1\} \times \{yH\} \to 1$-arrow of the diffeomorphism $\{xH_1\}$ starting at the coset $\{yH\}$ and ending at the coset $\{xyH\}$. Note that this is not a group action since $G/H_1$ is not a group but the composition and inversion of 1-arrows are well defined. This map is a bijection by the definition of $H_1$. Fixing one such 1-arrow, all other 1-arrows starting at $\{yH\}$ are generated by composing this arrow with elements of $S_1(M)^y_x = I_{xy^{-1}}H/I_{xy^{-1}}H_1$ where $I_{xy^{-1}}$ is the inner automorphism of $G$ determined by $xy^{-1} \in G$. The vertical projections on the right column of (19) are induced by projection of jets as in Sections 2, 3 and the projections on the left column are induced by projections on the first factor and identity map on the second factor.

A Lie group $G$ is clearly an effective Klein pair $(G, \{1\})$ with order $m = 0$. For many Klein geometries we have $m = 1$. This is the case, for instance, if $H$ is compact, in which case we have an invariant metric, $H$ is discrete or $(G, H)$ is a reductive pair which is extensively studied in literature from the point of view of principal bundles ([12]). If $G$ is semisimple, it is not difficult to show that the order of $(G, H)$ is at most two (see [18], pg. 131). For instance, let $M$ be a homogeneous complex manifold, i.e., $\text{Aut}(M)$ acts transitively on $M$. If $M$ is compact, then $M = G/H$ for some complex Lie group $G$ and a closed complex subgroup $H$ ([34]). If further $\pi_1(M)$ is finite, then $G/H = \overline{G/H}$ as
complex manifolds for some semisimple Lie group $\overline{G}$ ([34]). Thus it follows that jets of order greater than two do not play any role in the complex structure of $M$ in this case. If $G$ is reductive, it is stated in [35] that the order of $(G, H)$ is at most three. On the other hand, for any positive integer $m$, an effective Klein pair $(G; H)$ of order $m$ is constructed in [1] such that $G/H$ is open and dense in some weighted projective space. Other examples of Klein pairs of arbitrary order are communicated to us by the author of [35]. However, we do not know the answer to the following question

**Q1:** For some positive integer $m$, does there exist a Klein pair $(G, H)$ of order $m$ such that $G/H$ compact?

It is crucial to observe that $(G_1, H_1)$ and $(G_2, H_2)$ may be two Klein pairs with different orders with $G_1/H_1$ homeomorphic to $G_2/H_2$. For instance, let $H \subset G$ be complex Lie groups with $\pi_0(G) = \pi_1(G/H) = 0$ and $G/H$ compact. If $M \subset G$ is a maximal compact subgroup, then $(M, M \cap H)$ is a Klein pair of order one and $G/H = M/M \cap H$ as topological manifolds (in fact, $G/H$ is Kaehler iff its Euler characteristic is nonzero, see [34], [4]). The crucial fact here is that an abundance of Lie groups may act transitively on a homogeneous space $M$ with different orders and the topology (but not the analytic structure) of $M$ is determined by actions of order one only (the knowledge of a particular such action suffices, see Theorem VI in [34] where a detailed description of complex homogeneous spaces is given. It turns out that “they are many more than we expect” as stated there).

Now let $(G, H)$ be an effective Klein pair of order $m$ and let $L(G)$ be the Lie algebra of $G$. We have the map $\sigma : L(G) \to J_m(T(G/H))_p$ defined by $X \to j_m(X^*)_p$ where $X^*$ is the vector field on $G/H$ induced by $X$ and $p \in G/H$. $\sigma$ is a homomorphism of Lie algebras where the bracket on $J_m(T(G/H))_o$ is the algebraic bracket defined in Section 2. Note that the map $X \to X^*$ is injective due to effectiveness and also surjective due to transitivity. It is now easy to give a description of the infinitesimal analog of (19). We can thus express everything defined in Sections 2, 3 in concrete terms which will enable us to use the highly developed structure theory of (semisimple) Lie groups ([17]). A detailed description of $s_m(M)$ is given in [26] (Theorem 15 on pg. 199). The formula on pg. 200 in [26] is the same as the formula (15), Example 3.3.7, pg. 104 in the recent book [20], but higher order jets and Spencer operator remain hidden in (15) and also in (4), Example 3.2.9, pg. 98 in [20].

The following situation deserves special mention: Let $G$ be a complex Lie group, $H$ a closed complex subgroup with a holomorphic representation on the vector space $V$ and let $E \to G/H$ be the associated homogeneous vector bundle of $G \to G/H$. We now have the sheaf cohomology groups $H^*(G/H, E)$ where $E$ denotes the sheaf of holomorphic sections of $E \to G/H$. Borel-Weil theorem is derived in [4] from $H^0(G/H, E)$. If the Klein pair $(G, H)$ has order $m$ and is effective, the principal bundle $G \to G/H$ can be identified with the principal bundle $\mathcal{S}_m(M)^{(o)} \to G/H$ where $\mathcal{S}_m(M)^{(o)}$ consists of $m$-arrows in $G/H$ with source at the coset $o$ of $H$ (see Section 7). If $gx = y, g \in G, x, y \in G/H$, then the $m$-arrow of $g$ gives an isomorphism $E_x \to E_y$ between the fibers. Consequently, the action of $G$ on sections of $E \to G/H$ is equivalent to the representation
of the TLEFF $S_m(M) = G \times G/H$ on $E \to G/H$. On the infinitesimal level, this gives a representation of $s_m(M)$ on $E \to G/H$ and we can also define the cohomology groups $H^*(s_m(M), E)$ as in [19], [20]. Letting $\Theta$ denote the sections of $E$ killed by $s_m(M)$, we see that these two cohomology groups are related by a diagram similar to (17). It is crucial to observe here how the order of jets remains hidden in $H^*(G/H, E)$.

5 Truncated geometries

Definition 5 Some TLEFF $S_k(M)$ is called a truncated geometry on $M$ of order $k$.

We will view a truncated geometry $S_k(M)$ as a diagram (13) where all $S_m(M)$ for $m \leq k$ are defined as projections of $S_k(M)$ and $S_m(M)$ for $m \geq k+1$ do not exist. A homogeneous geometry defines a truncated geometry of any order. The question arises whether some truncated geometry always prolongs uniquely to some homogeneous geometry. The answer turns out to be negative. For instance, let $(M, g)$ be a Riemannian manifold and consider all 1-arrows on $M$ which preserve the metric $g$. Such 1-arrows define a TLEFF $S_1(M)$. We may fix some point $p \in M$ and fix some coordinates around $p$ once and for all so that $g_{ij}(p) = \delta_{ij}$, thus identifying $S_1(M)_p$ with the orthogonal group $O(n)$. Now any 1-arrow with source at $p$ defines an orthogonal frame at its target $q$ by mapping the fixed orthogonal coordinate frame at $p$ to $q$. The group $O(n)$ acts on all such 1-arrows by composing on the source. Now forgetting 1-arrows but keeping the orthogonal frames defined by them, we recover the orthogonal frame bundle of the metric $g$. However we will not adapt this point of view. In view of the existence of geodesic coordinates, we can now construct 2-arrows on $M$ which preserve the metric $g$, i.e., 1-jet of $g$ at all $x \in M$ can be identified (in various ways). Thus we obtain $S_2(M)$ and the projection $\pi_{2,1} : S_2(M) \to S_1(M)$. As a remarkable fact, $\pi_{2,1}$ turns out to be an isomorphism. This fact is equivalent to the well known Gauss trick of shifting the indices and showing the uniqueness of a metric connection which is symmetric (Levi-Civita connection). The Christoffel symbols are obtained now by twisting the 2-arrows of $S_2(M)$ by the 1-arrows of $S_1(M)$. Now we may not be able to identify 2-jet of $g$ over $M$ due to curvature of $g$ and thus we may fail to construct the surjection $\pi_{3,2} : S_3(M) \to S_2(M)$. If we achieve this (and $\pi_{3,2}$ will be again an isomorphism), the next obstruction comes from the 3-jet of $g$ which is essentially the covariant derivative of the curvature. However, if $g$ has constant curvature, then we can prolong $S_2(M)$ uniquely to a homogeneous geometry $S_\infty(M)$ which, as a remarkable fact, turns out to be a Klein geometry of order one since in this case $\lim_{k \to \infty} S_k(M)$ recaptures the isometry group of $(M, g)$ which acts transitively on $M$ and any isometry is uniquely determined by any of its 1-arrows. Thus we may view a truncated geometry $S_k(M)$ as a candidate for some homogeneous geometry $S_\infty(M)$ but $S_k(M)$ must overcome the obstructions, if any, put forward by $M$. Almost all geometric structures (Riemannian, almost complex, almost symplectic, almost
quaternionic, ..) may be viewed as truncated geometries of order at least one, each being a potential candidate for a homogeneous geometry.

**Definition 6** A truncated geometry $S_k(M)$ is called formally integrable if it prolongs to a homogeneous geometry.

However, we require the prolongation required by Definition 6 to be intrinsically determined by $S_k(M)$ in some sense and not be completely arbitrary. Given some $S_k(M)$, note that Definition 6 requires the surjectivity of $S_{j+1}(M) \rightarrow S_j(M)$, $j \geq k$. For instance, we may construct some $S_{k+1}(M)$ in an intrinsic way without $S_{k+1}(M) \rightarrow S_k(M)$ being surjective. We may now redefine all lower terms by $S_j(M) = \pi_{k+1,j}S_{k+1}(M)$ and start anew at order $k + 1$. This is not allowed by Definition 6. For instance, let $S_1(M)$ be defined by some almost symplectic form $\omega$ (or almost complex structure $J$). Then $\pi_{2,1} : S_2(M) \rightarrow S_1(M)$ will be surjective if $d\omega = 0$ (or $N(J) = 0$ where $N(J)$ is the Nijenhuis tensor of $J$).

This prolongation process which we tried to sketch above is centered around the concept of formal integrability which can be defined in full generality turning the ambiguous Definition 6 into a precise one. However, this fundamental concept turns out to be highly technical and is fully developed by D.C. Spencer and his co-workers from the point of view of PDEs, culminating in [11]. More geometric aspects of this concept are emphasized in [25], [26] and other books by this author.

## 6 Bundle maps

In this section we will briefly indicate the allowable bundle maps in the present framework. Consider the universal $\text{TLEFF } \mathcal{G}_k(M)$ of order $k$. We define the group bundle $\mathcal{AG}_k(M) = \cup_{x \in M} \mathcal{G}_k(M)^x$. The sections of this bundle form a group with the operation defined fiberwise. We will call such a section a universal bundle map (or a universal gauge transformation) of or der $k$. We will denote the group of universal bundle maps by $\Gamma \mathcal{AG}_k(M)$. We obtain the projection $\pi_{k+1,k} : \Gamma \mathcal{AG}_{k+1}(M) \rightarrow \Gamma \mathcal{AG}_k(M)$ which is a homomorphism. If $S_k(M) \subset \mathcal{G}_k(M)$, we similarly define $\mathcal{AS}_k(M) \subset \mathcal{AG}_k(M)$ and call elements of $\Gamma \mathcal{AS}_k(M)$ automorphisms (or gauge transformations) of $S_k(M)$. Now let $S_k(M) \subset \mathcal{G}_k(M)$ and $g_k \in \Gamma \mathcal{AG}_k(M)$. We will denote $g_k(x)$ by $(g_k)^x_x$, $x \in M$. We define the $\text{TLEFF } (Adg)S_k(M)$ by defining its $k$-arrows as $(Adg)S_k(M)^x = \{(g_k)^y_z(f_k)^z_y(g_k^{-1})^y_z | (f_k)^z_y \in S_k(M)^z_y\}$.

**Definition 7** $S_k(M)$ is called equivalent to $\mathcal{H}_k(M)$ if there exists some $g \in \Gamma \mathcal{AG}_k(M)$ satisfying $(Adg)S_k(M) = \mathcal{H}_k(M)$

A necessary condition for the equivalence of $S_k(M)$ and $\mathcal{H}_k(M)$ is that $S_k(M)^x_x$ and $\mathcal{H}_k(M)^x_x$ be conjugate in $\mathcal{G}_k(M)^x_x$, i.e., they must be compatible structures (like both Riemannian,...). Let $s_k(M)$ and $\mathcal{H}_k(M)$ be the corresponding $\text{TLEIF}$'s. The above action induces an action of $g$ on $s_k(M)$ which we will
denote also by \((Adg)s_k(M)\). This latter action uses 1-jet of \(g\) since the geometric order of \(s_k(M)\) is \(k + 1\), i.e., the transformation rule of the elements (sections) of \(s_k(M)\) uses derivatives up to order \(k + 1\). This construction is functorial. In particular, \((Adg)S_k(M) = H_k(M)\) implies \((Adg)s_k(M) = h_k(M)\). Clearly these actions commute with projections. In this way we define the moduli spaces of geometries.

Some \(g_k \in \Gamma A\mathcal{G}_k(M)\) acts on any \(k^{th}\) order object defined in Section 2. However, the action of \(g_k\) does not commute with \(d\) and therefore \(g_k\) does not act on the horizontal complexes in (10). To do this, we define \(A\mathcal{G}_{\infty}(M) = \cup_{x \in M} G_{\infty}(M)\) where an element \((g_{\infty})^x\) of \(G_{\infty}(M)\) is the \(\infty\)-jet of some local diffeomorphism with source and target at \(x\). As we noted above, \((g_{\infty})^x\) is far from being a formal object: it determines this diffeomorphism modulo the \(\infty\)-jet of identity diffeomorphism. Now some \(g_{\infty} \in \Gamma A\mathcal{G}_{\infty}(M)\) does act on the horizontal complexes in (10).

7 Principal bundles

Let \(S_k(M) \subset G_k(M)\). We fix some \(p \in M\) and define \(S_k(M)^{(p)} = \cup_{x \in M} S_k(M)^p_x\). The group \(S_k(M)^p_x\) acts on \(S_k(M)^{(p)}\) by composing with \(k\)-arrows of \(S_k(M)^p_x\) at the source as \((f_k)^x_p \to (f_k)^x_p h_k\) and the projection \(S_k(M)^{(p)} \to M\) with fiber \(S_k(M)^p_x\) over \(x\) is a principal bundle with group \(S_k(M)^p_x\). Considering the adjoint action of \(S_k(M)^p_x\) on itself, we construct the associated bundle whose sections are automorphisms (or gauge transformations) which we use in gauge theory. This associated bundle can be identified with \(AS_k(M)\) in Section 6 and therefore the two concepts of automorphisms coincide. In gauge theory, let \(h_k \in \Gamma AS_k(M)\) act on \(S_k(M)^{(p)}\) on the target as \((f_k)^x_p \to (h_k)^x_p(f_k)^x_p\) reserving the source for the group \(S_k(M)^p_x\). We will denote this transformation by \(f \to h \circ f\). We can regard the object \(h_k \circ S_1(g)^{(p)}\) as another principal \(S_k(M)^p_x\)-bundle: we imagine two copies of \(S_k(g)^p_x\), one belonging to principal bundle and one outside which is the group of the principal bundle and \(h_k\) acts only on the principal bundle without changing the group. To be consistent with \(\circ\), we now regard \(h_k \in \Gamma A\mathcal{G}_k(M)\) as a general bundle map and define the transform of \(S_k(M)^{(p)}\) by \(h_k\) using \(\circ\). Now \(\circ\) has a drawback from geometric point of view. To see this, let \((M, g)\) be a Riemannian manifold. We will denote the \(TLEF\) determined by \(g\) by \(S_1(g)\), identifying the principal \(S_1(g)^{(p)} \to M\) with the orthogonal frame bundle of \(g\) and the group \(S_1(g)^p_x\) with \(O(n)\) as in Section 5. Now the transformed object \(h \circ S_1(g)^{(p)}\), which is another \(O(n)\)-principal bundle, is not related to any metric in sight unless \(h = \text{identity}!!\). Thus we see that \(\circ\) dispenses with the concept of a metric but keeps the concept of an \(O(n)\)-principal bundle, carrying us from our geometric envelope outside into the topological world of general principal bundles. On the other hand, the action of \((G_1)^x_p\) on metrics at \(x\) gives an action of \(h\) on metrics on \(M\) which we will denote by \(g \to h \boxdot g\). Changing our notation \((Adh)S_1(g)\) defined in Section 6 to \(h \boxdot S_1(g)\) (using the same notation \(\boxdot\)), we see that \(h \boxdot S_1(g) = S_1(h \boxdot g)\). Thus \(\boxdot\) preserves both metrics and also 1-arrows determined by them.
Consider the naive inclusion
\[ \text{differential geometry} \subset \text{topology} \] (20)

If we drop the word differential in (20), we may adapt the point of view that the opposite inclusion holds now in (20). This is the point of view of A. Grothendieck who views geometry as the study of form, which contains topology as a special branch (see his Promenade #12, translated by Roy Lisker). This broad perspective is clearly a far reaching generalization of Erlangen Programm presented here.

In view of (20), we believe that any theorem in the framework whose main ingredients we attempted to outline here, however deep and far reaching, can be formulated and proved also using principal bundles. To summarize, we may say that differential geometry is the study smooth forms and the concepts of a form and continuous deformation of forms come afterwards as higher level of abstractions. We feel that it may be fruitful to start with differential geometry rather than starting at a higher level and then specializing to it.

8 Connection and curvature

Recall that a right principal $G$-bundle $P \to M$ determines the groupoid $\frac{P \times P}{G} \to M \times M$ where the action of $G$ on $P \times P$ is given by $(x, y)g = (xg, yg)$. Let $\mathcal{A}(P) \to M$ be the automorphism bundle obtained as the associated bundle of $P \to M$ using the adjoint action of $G$ on itself as in Section 7, whose sections are gauge transformations. We obtain in this way the groupoid extension
\[ 1 \to \mathcal{A}(P) \to \frac{P \times P}{G} \to M \times M \to 1 \] (21)

where we again use the first and last arrows to indicate injectivity and surjectivity without any algebraic meaning. On the infinitesimal level, (21) gives the Atiyah sequence of $P \to M$
\[ 0 \to \mathcal{L}\mathcal{A}(P) \to \frac{T P}{G} \xrightarrow{\pi} TM \to 0 \] (22)

(see [19], [20] for the details of the Atiyah sequence) where $\mathcal{L}\mathcal{A}(P) \to M$ is the Lie algebra bundle obtained as the associated bundle using the adjoint action of $G$ on its Lie algebra $\mathcal{L}(G)$. Connection forms $\omega$ on $P \to M$ are in 1-1 correspondence with transversals in (22), i.e., vector bundle maps $\omega : TM \to \frac{T P}{G}$ with $\pi \circ \omega = id$ and curvature $\kappa$ of $\omega$ is defined by $\kappa(X, Y) = \omega[X, Y] - [\omega X, \omega Y]$. The extension (22) splits iff (22) admits a flat transversal. Thus Atiyah sequence completely recovers the formalism of connection and curvature on $P \to M$ in the framework of algebroid extensions as long as we work over a fixed base manifold $M$.

Now let $\mathcal{S}_\infty(M)$ be a homogeneous geometry with infinitesimal geometry $s_\infty(M)$. The groupoid $\frac{\mathcal{S}_\infty(M)^{(p)} \times \mathcal{S}_\infty(M)^{(p)}}{\mathcal{S}_k(M)^{(p)}}$ in (21) determined by the principal
bundle $S_k(M)^{(p)} \to M$ as defined in Section 7 can be identified with $S_k(M)$ and the algebroid $\frac{T S_k(M)^{(p)}}{S_k(M)}$ in (22) can be identified with $s_k(M)$. We have $\mathcal{A}(S_k(M)^{(p)}) = \cup_{x \in M} S_k(M)_x^{\ast}$ as already indicated in Section 7 and $\mathcal{L}\mathcal{A}(S_k(M)^{(p)}) = \cup_{x \in M} \mathcal{L}(S_k(M)_x^{\ast})$ where the bracket of sections is defined fiberwise. Thus (21) becomes

$$1 \to \mathcal{A}S_k(M) \to S_k(M) \to M \times M \to 1$$

and the Atiyah sequence (22) is now

$$0 \to \mathcal{L}\mathcal{A}S_k(M) \to s_k(M) \to TM \to 0$$

It is easy to check exactness of (24) in local coordinates using (2).

Our purpose is now to indicate how the present framework captures information peculiar to jets by changing the base manifold, which Atiyah sequence does not detect. To see this, let $m \leq k + 1$ and consider the Lie group extension at $x \in M$:

$$1 \to S_{k,m}(M)_x^{\ast} \to S_k(M)_x^{\ast} \to S_m(M)_x^{\ast} \to 1$$

where the kernel $S_{k,m}(M)_x^{\ast}$ is nilpotent if $m \geq 1$ and is abelian if $k = m + 1 \geq 2$. Consider the $S_{k,m}(M)_x^{\ast}$-principal bundle $S_k(M)^{(p)} \to S_m(M)^{(p)}$. If $S_{k,m}(M)_x^{\ast}$ is contractible (this is the case in many examples for $m \geq 1$), this principal bundle is trivial and its Atiyah sequence admits flat transversals. Thus nothing is gained by considering higher order jets and all information is contained in the Atiyah sequence of $S_1(M)^{(p)} \to M$. On the other hand, we have the following extension of TLEIFS:

$$0 \to \mathcal{L}S_{k,m}(M) \to s_k(M) \to s_m(M) \to 0$$

where $\mathcal{L}S_{k,m}(M) = \cup_{x \in M} \mathcal{L}(S_{k,m}(M)_x^{\ast})$. Using (2), it is easy to check that the existence of a flat transversal in (26) a priori forces the splitting of the Lie algebra extension

$$0 \to \mathcal{L}(S_{k,m}(M)_x^{\ast}) \to \mathcal{L}(S_k(M)_x^{\ast}) \to \mathcal{L}(S_m(M)_x^{\ast}) \to 0$$

for all $x \in M$ where (27) is (25) in infinitesimal form. However (25) and (27) do not split in general. For instance, (27) does not split even in the universal situation $S_{\infty}(M) = G_{\infty}(M)$ when $m = 2$ and $k = 3$. In fact, the dimensions of the extension groups $H^2(\mathcal{L}(S_m(M)_x^{\ast})), S_{m+1,m}(M)_x^{\ast})$ are computed in [28] for all $m$ when $\text{dim} M = 1$.

Thus we see that the theory of principal bundles, which is essentially topological, concentrates on the maximal compact subgroup of the structure group of the principal bundle as it is this group which produces nontrivial characteristic classes as invariants of equivalence classes of principal bundles modulo bundle maps. Consequently, this theory concentrates on the contractibility of the kernel in (25) whereas it is the types of the extensions in (25), (27) which emerge as the new ingredient in the present framework.
The connections considered above are transversals and involve only $TLEIF'$s (algebroids). There is another notion of connection based on Maurer-Cartan form, which is incorporated by the nonlinear Spencer sequence (see Theorem 31 on page 224 in [25]), where the passage from $TLEFF$ (groupoid) to its $TLEIF$ (algebroid) is used in a crucial way. The passage from extensions of $TLEFF$ (torsionfree connections in finite form) to extensions of their $TLEIF$ (torsionfree connections in infinitesimal form) relates these two notions by means of a single diagram which we hope to discuss elsewhere. The approach to parabolic geometries adapted in [5] is a complicated mixture of these two notions whose intricacies, we believe, will be clearly depicted by this diagram.

However we view a connection, the main point here seems to be that it belongs to the group rather than to the space on which the group acts. Since there is an abundance of groups acting transitively on some given space, it seems meaningless to speak of the curvature of some space unless we specify the group. However, it turns out that the knowledge of the $k$-arrows of some ideal group is sufficient to define a connection but this connection will not be unique except in some special cases.

9 Some remarks

In this section (unfortunately somewhat long) we would like to make some remarks on the relations between the present framework and some other frameworks.

1) Let $E \to M$ be a differentiable fibered space. It turns out that we have an exterior calculus on $J_\infty(E) \to M$. Decomposing exterior derivative and forms into their horizontal and vertical components, we obtain a spectral sequence, called Vinogradov $C$-spectral sequence, which is fundamental in the study of calculus of variations (see [32], [33] and the references therein). The limit term of $C$-spectral sequence is $H^*_\text{deR}(J_\infty(E))$. In particular, if $E = T(M)$, we obtain $H^*_\text{deR}(g_\infty(M))$. The $C$-spectral sequence can be defined also with coefficients ([21]). On the other hand, we defined $H^*(g_\infty(M), J_\infty(M))$ and $H^*(g_\infty(M), g_\infty(\mathfrak{s}_\infty(M), J_\infty(M)))$ in Sections 2, 3. As we indicated in Section 2, we can consider representations of $g_\infty(M)$ other than $J_\infty(M)$ (for instance, see 2 below). These facts hint, we feel, at the existence of a very general Van Est type theorem which relates these cohomology groups.

2) Recall that (5) and (7) define a representation of the algebroid $g_k(M)$ on the vector bundle $J_k(M) \to M$. Thus we can consider the cohomology groups $H^*(g_k(M), J_k(M))$ as defined in [19], [20] (see also the references there for original sources) and $H^*(g_k(M), J_k(M))$ coincides with the cohomology of the bottom sequence of (17) by definition. Now $g_k(M)$ has other “intrinsic” representations, for instance $g_{k-1}(M)$. Lemma 8.32 and the formula on page 383 in [25] (which looks very similar to (4)) define this representation. In particular, the cohomology groups $H^*(g_k(M), g_{k-1}(M))$ are defined and are given by the bottom sequence of (17) for this case. Using this representation of $g_k(M)$ on $g_{k-1}(M)$, deformations of $TLEIF$’s are studied in [25] in detail using Janet
sequence (Chapter 7, Section 8 of [25]). Recently, some deformation cohomology groups are introduced in [8] in the general framework of algebroids. However, if the algebroid is a $TLEIF$, it seems to us that these cohomology groups coincide with those in [25] (and therefore also with the bottom sequence of (17)) and are not new (see also [20], pg. 309 for a similar claim). In view of the last paragraph of Section 4, the fact that deformation cohomology arises as sheaf cohomology in Kodaira-Spencer theory and as algebroid cohomology in [25], [8] is no coincidence.

3) Let $S_2(M)$ be a truncated geometry on $M$. We fix some $x \in M$ and consider the following diagram of Lie group extensions

$$
\begin{align*}
0 & \longrightarrow G_{2,1}(M)^x_x \longrightarrow G_2(M)^x_x \longrightarrow G_1(M)^x_x \longrightarrow 1 \\
0 & \longrightarrow S_{2,1}(M)^x_x \longrightarrow S_2(M)^x_x \longrightarrow S_1(M)^x_x \longrightarrow 1
\end{align*}
$$

(28)

where the vertical maps are inclusions. The top row of (28) splits and the components of these splittings are naturally interpreted as the Christoffel symbols of symmetric “point connections”. Some $k^x_x \in G_{2,1}(M)^x_x$, which is a particular bundle map defined in Section 6 when $M = \{x\}$, transforms $S_2(M)^x_x$ by conjugation but acts as identity on $S_{2,1}(M)^x_x$ and $S_1(M)^x_x$. Using $G_{2,1}(M)^x_x$ as allowable isomorphisms, we defined in [13] the group of restricted extensions $H^2_{res}(S_1(M)^x_x, S_{2,1}(M)^x_x)$. This group vanishes if the restriction of some splitting of the top row of (28) to the bottom row splits also the bottom row or equivalently, if the bottom row admits a symmetric point connection. The main point is that this group is sensitive to phenomena happening only inside the top row of (28) which is our universal envelope as in Section 2. Using the Lie algebra analog of (28), we defined also the group $H^2_{res}(\mathcal{L}(S_1(M)^x_x), S_{2,1}(M)^x_x)$ (note that $S_{2,1}(M)^x_x \subset G_{2,1}(M)^x_x$ are vector groups) obtaining the homomorphism $H^2_{res}(S_1(M)^x_x, S_{2,1}(M)^x_x) \rightarrow H^2_{res}(\mathcal{L}(S_1(M)^x_x), S_{2,1}(M)^x_x)$ ([13]).

On the other hand, regarding the bottom row of (28) as an arbitrary Lie group extension as in [15] without any reference to our universal envelope, we can define $H^2(S_1(M)^x_x, S_{2,1}(M)^x_x)$ and the homomorphism $H^2(S_1(M)^x_x, S_{2,1}(M)^x_x) \rightarrow H^2(\mathcal{L}(S_1(M)^x_x), S_{2,1}(M)^x_x)$. Thus we obtain the following commutative diagram

$$
\begin{align*}
H^2_{res}(S_1(M)^x_x, S_{2,1}(M)^x_x) & \quad \longrightarrow \quad H^2_{res}(\mathcal{L}(S_1(M)^x_x), S_{2,1}(M)^x_x) \\
\downarrow & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
We believe that we will lose information also in this global case. To summarize, even though the constructions in this paper can be formulated in the general framework of groupoids and algebroids as in [7], [8], [19], [20], we believe that this general framework will not be sensitive in general to certain phenomena peculiar to jets unless it takes the universal homogeneous envelop into account. In particular, we would like to express here our belief that Lie equations form the geometric core of groupoids which are the ultimate generalizations of (pseudo)group actions in which we dispense with the action but retain the symmetry that the action induces on the space (compare to the introduction of [36]).

4) Let $X(M)$ be the Lie algebra of smooth vector fields on $M$. Recalling that $j_k(X(M)) \subset g_k(M)$, (5) gives a representation of $X(M)$ on $j_k(M)$. Denoting the cochains computing this cohomology by $(k,r)^\wedge GF$, we obtain the chain map $(k,*)^\wedge \rightarrow (k,*)^\wedge GF$ which indicates that Gelfand-Fuks cohomology is involved in the present framework and plays a central role.

5) This remark can be considered as the continuation of Section 7 and the last paragraph of Section 4.

Let $Q$ be the subgroup of $G_1(n+1) = GL(n+1, \mathbb{R})$ consisting of matrices of the form

$$\begin{bmatrix} A & 0 \\ \xi & \lambda \end{bmatrix}$$

(30)

where $A$ is an invertible $n \times n$ matrix, $\xi = (\xi_1, ..., \xi_n)$ and $\lambda \neq 0$. We will denote (30) by $(A, \xi, \lambda)$. We have the homomorphism $Q/\lambda I \rightarrow G_1(n)$ defined by $(A, \xi, 1) \rightarrow A$, where $\lambda I$ denotes the subgroup $\{\lambda I | \lambda \in \mathbb{R}\}$, with the abelian kernel $K$ consisting of elements of the form $(I, \xi, 1)$. We also have the injective homomorphism $Q/\lambda I \rightarrow G_2(n)$ defined by $(A^i_j, \xi, 1) \rightarrow (A^i_j, \xi A^j_k + \xi_k A^j_i)$ which gives the diagram

$$
\begin{array}{cccc}
0 & \rightarrow & G_2,1(n) & \rightarrow G_2(n) \\
0 & \rightarrow & K & \rightarrow Q/\lambda I \\
& & \uparrow & \uparrow \\
& & G_1(n) & \rightarrow 1 \\
\end{array}
$$

(31)

Now $(G_1(n+1), Q)$ is a Klein pair of order two with ghost $N = \lambda I$. We also have the effective Klein pair $(G_1(n+1)/\lambda I, Q/\lambda I)$ which is of order two. Note that the standard action of $G_1(n+1)$ on $\mathbb{R}^{n+1}\setminus 0$ induces a transitive action of $G_1(n+1)$ on the the real projective space $\mathbb{R}P(n)$ and $Q$ is the stabilizer of the column vector $p = (0, 0, ..., 1)^T$ so that both Klein pairs $(G_1(n+1), Q)$ and $(G_1(n+1)/\lambda I, Q/\lambda I)$ define the same base $\mathbb{R}P(n)$. Clearly, $G_1(n+1)$ and $G_1(n+1)/\lambda I$ induce the same 2-arrows on $\mathbb{R}P(n)$.

At this stage, we have two relevant principle bundles.

i) The principle bundle $G_1(n+1)^{(p)} \rightarrow \mathbb{R}P(n)$ which consists of all 2-arrows emanating from $p$ and has the structure group $Q/\lambda I$. This is the same principle bundle as $(G_1(n+1)/\lambda I)^{(p)} \rightarrow \mathbb{R}P(n)$. In particular, (18) reduces to (31) in
this case (the reader may refer to Example 4.1 on pg. 132 of [18] and also to the diagram on pg.142).

ii) The principle bundle $G_1(n+1) \to \mathbb{R}P(n)$ with structure group $Q$. Note that we have the central extension

\[ 1 \to \lambda I \to Q \to Q/\lambda I \to 1 \]  

(32)

which splits: some $q = (A, \xi, \lambda) \in Q$ factors as $q = ab$ where $a = (\lambda^{-1}A, \lambda^{-1}\xi, 1) \in G_2(n)$ and $b = \lambda I$. Note that ii) is obtained from i) by lifting the structure group $Q/\lambda I$ to $Q$ in (31).

Now we have a representation of $Q$ on $\mathbb{R}$ defined by the homomorphism $(A, \xi, \lambda) \to \lambda^{-N}$ for some integer $N \geq 0$. Replacing $\mathbb{R}$ by $\mathbb{C}$ and working with complex groups and holomorphic actions, it is known that the holomorphic sections of the associated line bundle of $G_1(n+1, \mathbb{C}) \to \mathbb{C}P(n)$ realize all irreducible representations of the unitary group $U(n)$ as $N$ varies (see [16], pg. 138-152 and [17]). As a very crucial fact, we can repeat this construction by replacing the Klein pair $(G_1(n+1, \mathbb{C}), Q)$ of order two by the effective Klein pair $(U(n), U(n-1) \times U(1))$ of order one and this latter construction recovers the same line bundle (see [16] for details). The following question therefore arises naturally: Let $(G, H)$ be a Klein pair with ghost $N$. Let $\rho : H \to GL(V)$ be a representation and $E \to M = G/H$ be the associated homogeneous vector bundle of $G \to G/H$. Can we always find some effective Klein pair $(\overline{G}, \overline{H})$ (not necessarily of the same order) with $\overline{G}/\overline{H} = M$, a representation $\overline{\rho} : \overline{H} \to GL(V)$ such that $E \to M$ is associated with $(\overline{G})^{(p)} \to M$, or shortly

**Q2:** Can we always avoid ghosts in Klein geometry?

Replacing $\lambda I, Q, Q/\lambda I$ in (31) respectively by $U(1), Spin^c(4), SO(4)$ and recalling the construction of $Spin^c$-bundle on a 4-manifold ([22]), we see that $Q2$ is quite relevant as it asks essentially the scope and limitations of Erlangen Programm.

6) The assumption of transitivity, i.e., the surjectivity of the right arrows in (13), (14), is imposed upon us by Erlangen Programm. However, many of the constructions in this paper can be carried out without the assumption of transitivity. For instance, foliations give rise to intransitive Lie equations but they are studied in the literature mostly from the point of view of general groupoids and algebroids (see the references in [7], [8]).

Our main object of study in this paper has been a differentiable manifold $M$. The sole reason for this is that this author has been obsessed years ago by the question “What are Christoffel symbols?” and he could not learn algebraic geometry from books and he did not have the chance to learn it from experts (this last remark applies also to differential geometry) as he has always been at the wrong place at the right time. We feel (and sometimes almost see, for instance [27], [3]) that the present framework has also an algebraic counterpart valid for algebraic varieties.
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