Cascade of Special Holonomy Manifolds and Heterotic String Theory

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Abstract

We investigate heterotic string theory on special holonomy manifolds including exceptional holonomy $G_2$ and $Spin(7)$ manifolds. The gauge symmetry is $F_4$ in a $G_2$ manifold compactification, and so(9) in a $Spin(7)$ manifold compactification. We also study the cascade of the holonomies: so(8) $⊃ Spin(7) ⊃ G_2 ⊃ su(3) ⊃ su(2)$. The differences of adjoining groups are described by Ising, tricritical Ising, 3-state Potts and u(1) models. These theories are essential for spacetime supersymmetries and gauge group enhancements. As concrete examples, we construct the modular invariant partition functions and analyze their massless spectra for $G_2$ and $Spin(7)$ orbifolds. We obtain the relation between topological numbers of the manifolds and multiplicities of matters in specific representations.
1 Introduction

It is a long time since the string theory attracted the attention of particle physicists as a candidate of the unified theory of elementary particles and their interactions. Extensive works have been devoted to the study of these theories, but it seems to be yet out of reach to gain fundamental understanding of them.

One of the most important things is the investigation of the properties of the manifolds on which the string should be compactified. Particularly the compactifications with minimum spacetime supersymmetries have received much attentions. From the point of view of the particle physics, geometrical properties of internal manifolds determine zero mass fields in the low energy effective theory and these manifolds play crucial roles in deciding the phenomenological features of the string theories.

If we require only one spacetime supersymmetry, we need only one covariantly constant spinor for a fixed chirality and this leads to the manifolds with minimal numbers of covariantly constant spinors. The condition of having an $N = 1$ spacetime supersymmetry for heterotic string leads to 4 distinct possibilities for compactifications namely compactifications down to 6,4,3,and 2 dimensions. Compactifications to 6 and 4 dimensions have been studied extensively before corresponding to $K3$ and a Calabi-Yau 3-fold respectively. The other two are special cases and correspond to compactification down to 3 on a 7 dimensional manifold of $G_2$ holonomy and compactification down to 2 on an 8 dimensional manifold with $Spin(7)$ holonomy. The possible existence of these two special cases had been known for a long time [1–4]. They have been investigated in the papers [3–10] and structures of their extended chiral algebras have been clarified [11]. The role the $U(1)$ current plays in the $N = 2$ superconformal theories, is played by tri-critical Ising model in the case of $G_2$ and Ising model in the case of $Spin(7)$ manifolds. It is mysterious that these statistical models appear unexpectedly in the cases of the exceptional holonomy manifolds. One might ask if these phenomena are restricted to these scattered exceptional manifolds. Also they yield a question that these manifolds could be related to the Calabi-Yau 3-folds or other special holonomy manifolds with different dimensions.

The aim of this article is to clarify these two questions based on analyses of compactifications of string theories. The study of spacetime $N = 1$ heterotic strings is the subject of the present paper. In the context of string theories the geometries of target manifolds can be studied by the worldsheet techniques and conformal field theories (CFTs) are powerful tools in the detailed description of the dynamics.

Motivated with these questions we intend to examine toroidal partition functions of the heterotic strings by means of CFT techniques. Particularly we elaborate branching
rules of gauge symmetries concentrating on the gauge sector of the partition function. We find that three 2 dimensional statistical models, “Ising”, “tricritical Ising”, “3-state Potts” models play important roles in connecting 3 special holonomy manifolds, “Spin(7)”, “G_2”, “CY_3”. At the same time spacetime gauge symmetries are respectively enhanced to SO(9), F_4, E_6 in associated heterotic cases for three manifolds. Then reductions of holonomies about these special manifolds are correlated to the enhancements of the spacetime gauge symmetries for the N = 1 susy theories. By studying branching rules of characters we make clear that extra degrees of freedom thrown away under the holonomy reductions are transferred to those of gauge symmetries and absorbed into them as necessary degrees of freedom in their enhancements. At the level of characters of affine Lie algebras it might be possible that the 7 dimensional G_2 manifold is related with a complex 3 dimensional Calabi-Yau manifold by transferring degrees of freedom of 3-state Potts model. Then a character of a U(1) current appears and we can obtain a state associated with a spectral flow operator of the CY_3. It leads us to enhancement of worldsheet currents from N = 1 susy to N = 2 susy of the CY_3.

The organization of this paper is as follows: In section 2 we will review some geometrical facts about manifolds with G_2 and Spin(7) holonomies and associated conformal theories that we will need in the rest of the paper. We also explain relations between ground states of the CFT and cohomology classes of the exceptional manifolds. In section 3 we discuss compactifications of heterotic strings on exceptional holonomy manifolds. We explain the gauge symmetry becomes F_4 in G_2 compactifications, and so(9) in Spin(7) compactifications. In section 4, we study compactifications on special holonomy manifolds from the point of view of coset CFT of level 1 affine Lie algebras. We concentrate on characters in the gauge sector of the model and study their detailed branching rules. We propose cascades of special holonomy manifolds with different dimensions and they are turned out to be controlled by statistical models. In section 5 we put a review of concrete orbifold examples of these exceptional manifolds constructed by Joyce [2-4]. By combining these left- and right-moving correspondings in the heterotic strings, we obtain partition functions of strings compactified on G_2 and Spin(7) holonomies. The resulting theories have spacetime N = 1 supersymmetries. For the G_2 holonomy case, the tricritical Ising model and SO(9) current algebra are combined so that the 3d spacetime gauge symmetry is enhanced to an exceptional group F_4. On the other side the 2d heterotic string on the Spin(7) manifold have an SO(9) spacetime gauge symmetry. We would like to point out that the tricritical (or Ising) parts are essential for enhancements of spacetime gauge symmetries in these N = 1 theories. Section 6 is devoted to conclusions and comments. In appendix, we collect several useful properties of theta functions.
2 Exceptional Holonomy Manifold

2.1 $G_2$ holonomy

Let us consider a seven manifold $M^{(7)}$ with a $G_2$ holonomy. The $G_2$ structure on $M^{(7)}$ is given by a closed $G_2$ invariant 3-form $\Phi$. By including this operator, an extended algebra of sigma model on $M^{(7)}$ has been constructed in the paper [5] based on analyses in the large volume limit. In addition to a set of stress tensor $T$ and its superpartner $G$, the conformal algebra contains sets of currents $(K, \Phi)$ with spins $(2, 3/2)$ and $(X, M)$ with spins $(2, 5/2)$. The $X$ is related with a dual 4-form $\ast \Phi$ and $(X, \Phi)$ is a set of currents $(T^{\text{Tr}i}, G^{\text{Tr}i})$ of $N = 1$ additional superconformal algebra. It is the conformal algebra of the tricritical Ising model with a Virasoro central charge $7/10$. Also the theory contains a spectral flow operator with the dimension $7/16$, in other words, the spin field of the statistical model. The appearance of the $N = 1$ minimal unitary model reflects a reduction of holonomy of seven manifold $M^{(7)}$ from $\text{SO}(7)$ to $G_2$ and we are left with the residual symmetry $\text{SO}(7)/G_2$. Its central charge is given as

$$\frac{7}{2} - \frac{14}{5} = \frac{7}{10},$$

and the correspondence has been proposed

$$\text{SO}(7)/G_2 \cong (\text{Tricritical Ising}).$$

Also the original stress tensor $T$ can be decomposed into a sum of two commutative Virasoro generators $T = T^{\text{Tr}i} + T^r$. The statistical model is a unitary minimal model with central charge $c = \frac{7}{10}$. There are 6 different scaling fields and the associated dimensions $h$’s are listed in table 1.

| field | $1$ | $\epsilon$ | $\epsilon'$ | $\epsilon''$ | $\sigma$ | $\sigma'$ |
|-------|----|------------|------------|------------|--------|---------|
| $h$   | 0  | $1/10$     | $3/5$      | $3/2$      | $3/80$  | $7/16$  |

Table 1: Conformal dimensions $h$’s of scaling fields in the tricritical Ising model.

The tricritical Ising model is also one of the relevant theories endowed with supersymmetry. The Neveu-Schwartz sector of the theory contains the fields $1, \epsilon, \epsilon'$, $\epsilon''$. In terms of superconformal representations, $\epsilon''$ is a descendant of the identity $1$ and $\epsilon$ and $\epsilon'$ are superpartners of each other. The fusion algebra of these 4 fields closes on itself. On the other hand the Ramond sector contains the spin fields $\sigma$ and $\sigma'$. We show the fields assignments in both sectors in table 2.
Table 2: Classification of scaling fields in $N = 1$ SCFT. The tricritical Ising model can be interpreted as an $N = 1$ susy model in the minimal unitary series.

| $h$         | field      | $(-1)^F$ sector |
|-------------|------------|-----------------|
| $[0, 3/2]$  | $(1, \epsilon)$ | $(+, (-))$      | $NS$            |
| $[1/10, 3/5]$ | $(\epsilon, \epsilon')$ | $(-), (+)$      | $NS$            |
| $3/80$      | $\sigma$   | $(\pm)$         | $R$             |
| $7/16$      | $\sigma'$  | $(\pm)$         | $R$             |

Here we put the $\mathbb{Z}_2$ assignments for the tricritical Ising model according to the paper [5]. In this assignment $(-1)^F = (-1)^{F_I}$ and one can use tricritical gradings for the whole theory. The Ramond ground states are coming in pairs and the $\pm$ sign reflects this degeneracy and we put two different $(-1)^F$ assignments in $R$-sectors.

Next we will classify the highest weight representations of the algebra by using a set of highest weights $(h^{\text{Tri}}, h')$ of $(T^{\text{Tri}}, T')$. These two Virasoro generators are commutative and the tricritical Ising part leads to unitary highest weight representations of the extended chiral algebra. Ramond vacua have dimension $\frac{7}{16}$ in this model and are classified as

$$R; \left| \frac{7}{16}, 0 \right> \left| \frac{3}{80}, \frac{2}{5} \right>.$$  

The operator corresponding to the ground state $| \frac{7}{16}, 0 \rangle = | \sigma', 0 \rangle$ plays the role of a spectral flow operator. By using fusion relations

$$\sigma' \cdot \sigma' = 1 + \epsilon'' ,$$

$$\sigma' \cdot \sigma = \epsilon + \epsilon' ,$$

one can show that the Ramond ground state $| \frac{7}{16}, 0 \rangle = | \sigma', 0 \rangle$ is mapped to an NS vacuum $| 0, 0 \rangle$ and the $| \frac{3}{80}, \frac{2}{5} \rangle$ is transformed into a primary state $| \frac{1}{10}, \frac{2}{5} \rangle$ with dimension $\frac{1}{2}$. This leads to construct the following states in NS sector

$$NS; | 0, 0 \rangle \left| \frac{1}{10}, \frac{2}{5} \right> .$$

Now we can describe the relation of Ramond ground states with the cohomology of the manifold $M^{(7)}$. The target manifold $M^{(7)}$ described by sigma model is characterized by its Betti numbers $b_\ell$ ($\ell = 0, 1, \cdots, 7$) with several relations

$$b_0 = b_7 = 1 , \ b_1 = b_6 = 0 ,$$

$$b_2 = b_5 , \ b_3 = b_4 ,$$
and its Euler number turns out to be 0. From the point of view of geometrical consideration, it is known that the moduli space $\mathcal{M}_{geom}$ of the $G_2$ manifold is related to the structure of the 3-form $\Phi$ and its dimension is given as

$$\dim \mathcal{M}_{geom} = b_3.$$ 

In the context of sigma model, the geometrical moduli space is extended to a string (CFT) moduli space $\mathcal{M}_{CFT}$ by an antisymmetric 2-form and its dimension is calculated as

$$\dim \mathcal{M}_{CFT} = b_2 + b_3.$$ 

In order to see the correspondence with the CFT, we glue left- and right-sectors of the CFT states and discuss the non-chiral states. The relevant states in $(R,R)$ sector are constructed as

$$\begin{align*}
\text{RR state} & \quad \text{number} \\
\left| \frac{7}{16},0 \right)_L \left| \frac{7}{16},0 \right)_R; + \right\rangle & \quad b_0 = 1 \\
\left| \frac{2}{5},2 \right)_L \left| \frac{2}{5},2 \right)_R; + \right\rangle & \quad b_2 + b_4 \\
\left| \frac{2}{5},2 \right)_L \left| \frac{2}{5},2 \right)_R; - \right\rangle & \quad b_3 + b_5 \\
\left| \frac{7}{16},0 \right)_L \left| \frac{7}{16},0 \right)_R; - \right\rangle & \quad b_0 = 1
\end{align*}$$

where the signs $\pm$ mean the values of $(-1)^F$. Let us consider specific counterparts in the NS sector. By acting on Ramond ground state with the operator associated with the state $\left| \left( \frac{7}{16},0 \right)_L \left( \frac{7}{16},0 \right)_R; + \right\rangle$, we obtain $(NS,NS)$ states

$$\begin{align*}
\text{NSNS state} & \quad \text{number} \\
\left| (0,0) \right)_L (0,0)_R; + \right\rangle & \quad 1 \\
\left| \frac{2}{5},2 \right)_L \left| \frac{2}{5},2 \right)_R; + \right\rangle & \quad b_2 + b_4
\end{align*}$$

As discussed in the paper [5], exactly marginal deformations are given by operators of the form

$$G_{-1/2} \tilde{G}_{-1/2} \left| \left( \frac{1}{10}, \frac{2}{5} \right)_L \left( \frac{1}{10}, \frac{2}{5} \right)_R; + \right\rangle \right\rangle \quad b_2 + b_4 = b_2 + b_3$$

which preserve the $G_2$ structure. These describe string moduli space $\mathcal{M}_{CFT}$. 

6
2.2 Spin(7) holonomy

In this subsection, we will review several properties of Spin(7) manifold $M^{(8)}$ and its associated conformal algebra. Let $M^{(8)}$ be an eight manifold with a Spin(7) holonomy. The structure is given by a closed self-dual Spin(7) invariant 4-form $\Phi$. The extended symmetry algebra of sigma model on $M^{(8)}$ has been found in paper \[5\]. In addition to a set of $N = 1$ superconformal currents $(T,G)$, it contains operators $(\tilde{X},\tilde{M})$ with spins $(2,3/2)$. The set is a pair of an extra $N = 1$ Virasoro conformal algebra $(T_1,G_1)$. The $\tilde{M}$ corresponds to the Cayley 4-form $\Phi$ and the $\tilde{X}$ is the energy momentum tensor for the $c = 1/2$ Majorana-Weyl fermion (Ising model). The latter is related to a spectral flow operator with the dimension 1/2 in the Ising model. The appearance of this statistical model can be explained by a reduction of holonomy for $M^{(8)}$ from SO(8) to Spin(7) by calculating central charge of SO(8)/Spin(7)

$$4 - \frac{7}{2} = \frac{1}{2}.$$ 

From this consideration, the correspondence has been proposed as

$$\text{SO}(8)/\text{Spin}(7) \cong (\text{Ising model}).$$

By using this Ising stress tensor $T_i$, the original stress tensor can be decomposed into a sum of two commutative Virasoro generators $T = T_i + T_r$. This statistical model is a unitary minimal model with central charge $c = \frac{1}{2}$. There are 3 local scaling operators in this model: the Ising spin $\sigma$ and the energy density $\epsilon$ and identity operator $1$. The associated dimensions $h$’s are listed in table \[3\]

| field $h$ | $h_{\text{Ising}}$ | $h_r$ |
|-------|----------------|---------|
| 1    | 0              | 0/16    |
| $(-1)^F$ | (+)          | (-)     |
| (+)          | (+)          | (+)     |

Table 3: Conformal dimensions $h$’s of scaling fields in the Ising model.

Here we put the $\mathbb{Z}_2$ assignments for the Ising model according to the paper \[3\]. In this assignment $(-1)^F = (-1)^{F_I}$ and one can use Ising gradings for the whole theory. It is the $\mathbb{Z}_2$ symmetry under spin flips $\sigma \rightarrow -\sigma$.

Next we will classify our state in the extended algebra by a set of two numbers $(h_{\text{Ising}}, h_r)$: Ising model highest weight $h_{\text{Ising}}$ and the highest weight $h_r$ of the $T_r$. In the Ramond sector we have ground states with dimension $\frac{1}{2}$ and they are classified as

$$R; \quad \left\{ \frac{1}{2}, 0 \right\}, \left\{ \frac{1}{2}, \frac{1}{2} \right\}, \left\{ \frac{1}{16}, \frac{7}{16} \right\}.$$
In this case the state $\left|\frac{1}{2}, 0\right\rangle = |\epsilon, 0\rangle$ plays the same role as a spectral flow operator. It is nothing but an energy operator of the Ising model. By using fusion relations

$$\epsilon \cdot \epsilon = 1, \epsilon \cdot \sigma = \sigma,$$

one can show that Ramond ground states $\left|\frac{1}{2}, 0\right\rangle, \left|0, \frac{1}{2}\right\rangle, \left|\frac{1}{16}, \frac{7}{16}\right\rangle$ are mapped to respectively NS vacua $|0, 0\rangle, \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{16}, \frac{7}{16}\right\rangle$.

Now we shall describe the relation of Ramond states with the cohomology of the manifold $M^{(8)}$. The $Spin(7)$ manifold $M^{(8)}$ associated with this CFT is characterized by geometrical data given by Betti numbers $b_\ell$ ($\ell = 0, 1, \cdots, 8$) together with relations

$$b_0 = b_8 = 1, \quad b_1 = b_7 = 0,$$

$$b_2 = b_6, \quad b_3 = b_5,$$

$$b_3 + b_4^+ - b_2 - 2b_4^- - 1 = 24,$$

where the $b_4^\pm$ mean (anti)self-dual parts of the $b_4$. The Euler number of the eight manifold $M^{(8)}$ is calculated as

$$\chi = 2(b_2 - b_3 + b_4 + 1).$$

From the point of view of geometrical consideration, it is known that the moduli space $\mathcal{M}_{geom}$ of the $Spin(7)$ manifold is related to the structure of the self-dual 4-form $\Phi$ and its dimension is given as

$$\dim \mathcal{M}_{geom} = b_4^- + 1.$$

In the context of string theory, the $\mathcal{M}_{geom}$ is extended to a CFT moduli space $\mathcal{M}_{CFT}$ by $B_{\mu \nu}$ and its dimension is evaluated as

$$\dim \mathcal{M}_{CFT} = b_2 + b_4^- + 1.$$

In order to see the correspondence with the CFT, we put left- and right-sectors together and discuss non-chiral states. The relevant states in $(R, R)$ sector and associated
\[(NS, NS)\] counterparts are given by the following form

\[
\begin{array}{ccc}
RR & NSNS & \text{number} \\
\langle \frac{1}{2}, 0 \rangle_L (\frac{1}{2}, 0)_R & \langle 0, 0 \rangle_L (0, 0)_R & b_0 = 1 \\
\langle 0, \frac{1}{2} \rangle_L (0, \frac{1}{2})_R & \langle \frac{1}{2}, \frac{1}{2} \rangle_L (\frac{1}{2}, \frac{1}{2})_R & b_6 + b_4^+ \\
\langle \frac{7}{16}, \frac{7}{16} \rangle_L (\frac{7}{16}, \frac{7}{16})_R & \langle \frac{7}{16}, \frac{7}{16} \rangle_L (\frac{7}{16}, \frac{7}{16})_R & 1 + b_2 + b_4^- \\
\langle 0, \frac{1}{2} \rangle_L (\frac{1}{2}, \frac{1}{2})_R & \langle 0, \frac{1}{2} \rangle_L (\frac{1}{2}, \frac{1}{2})_R & b_3 = b_5 \\
\langle \frac{7}{16}, \frac{1}{2} \rangle_L (0, \frac{1}{2})_R & \langle \frac{7}{16}, \frac{1}{2} \rangle_L (\frac{1}{2}, \frac{1}{2})_R & b_3 = b_5 \\
\end{array}
\]

The \((R, R)\) and \((NS, NS)\) states are exchanged by the operator corresponding to the state \(\langle \frac{7}{16}, 0 \rangle_L (\frac{7}{16}, 0) _R\). As discussed in the paper \([5]\), exactly marginal deformations are given by operators of the form

\[
G_{-1/2} G_{-1/2} \left| \left( \frac{7}{16}, \frac{7}{16} \right)_L (\frac{1}{2}, \frac{1}{2})_R \right> \quad 1 + b_2 + b_4^-
\]

which preserve the \(\text{Spin}(7)\) structure. These describe string moduli space \(\mathcal{M}_{\text{CFT}}\).

### 3 Compactifications of Heterotic String

We will consider compactifications of \(E_8 \times E_8\) heterotic string theory \([17, 19]\) on the (real) \(D\) dimensional special holonomy manifolds. The resulting theory compactified on \(M\) has \(d\) \((d = 10 - D)\) dimensional spacetime with an \(N = 1\) supersymmetry and spacetime gauge symmetries are \(G_0 \times E_8\). In order to construct consistent string theories, we have to impose several conditions on the gauge symmetries. First of all, let us use these from the point of view of worldsheet theories. In the original 10 dimensional string, the left-moving part has an \(N = 1\) spacetime supersymmetry with central charge \(c = 15\) and the right-moving counterpart is a bosonic theory with \(\bar{c} = 26\). In quantizing this model, we shall use a light-cone formula with transverse spacetime dimension \((d - 2)\) and the theory has total central charge \((c, \bar{c}) = (12, 24)\). A spacetime Lorentz group in the light-cone gauge is \(\text{SO}(d - 2)\) and is realized as level 1 affine Kac-Moody algebra \(\hat{\text{so}}(d - 2)\) by \((d - 2)\) free fermions on the worldsheet with central charge \((d - 2)\). Similarly the spacetime gauge symmetry \(G_0 \times E_8\) is represented by affine Lie algebras by worldsheet gauge fermions with central charge \(c_{G_0} + 8\). The \(D\) dimensional internal part can be described by an extended \(N = 1\) CFT associated with the manifold \(M\) in the previous section and has
central charge \((\frac{3}{2}D, \frac{3}{2}D)\). By collecting all these parts, we can write down conditions of balance of central charges on both left- and right-parts

right ; \ 24 = (d - 2) + \frac{3}{2}D + c_{G_0} + 8 ,
left ; \ 12 = \frac{3}{2}(d - 2) + \frac{3}{2}D . \ 
\rightarrow \ d + D = 10 , \ c_{G_0} + \frac{1}{2}D = 8 .

In this article, we will take \(M\) as exceptional holonomy manifolds \(M^{(D)}\) with (real) dimension \(D\). Concrete conditions can be written down for exceptional holonomy cases

\textbf{G}_2 \text{ case; } M^{(7)} \ (d = 3 \text{ theory})

\begin{align*}
12 &= \frac{3}{2} \times 1 + \frac{3}{2} \times 7 , \\
24 &= 1 + \frac{3}{2} \times 7 + (8 + \frac{9}{2}) , \\
1 &= d - 2 = (\text{spacetime transverse dimension}) , \\
7 &= (\text{dimension of } G_2 \text{ manifold}) , \\
8 &= (c \text{ of level 1 affine } E_8 \text{ algebra}) = (\text{rank of } E_8) , \\
\frac{9}{2} &= (c \text{ of level 1 affine } \text{SO}(9) \text{ algebra}) ,
\end{align*}

\textbf{Spin}(7) \text{ case; } M^{(8)} \ (d = 2 \text{ theory})

\begin{align*}
12 &= \frac{3}{2} \times 0 + \frac{3}{2} \times 8 , \\
24 &= 0 + \frac{3}{2} \times 8 + (8 + 4) , \\
0 &= d - 2 = (\text{spacetime transverse dimension}) , \\
8 &= (\text{dimension of } \text{Spin}(7) \text{ manifold}) , \\
8 &= (c \text{ of level 1 affine } E_8 \text{ algebra}) = (\text{rank of } E_8) , \\
4 &= (c \text{ of level 1 affine } \text{SO}(8) \text{ algebra}) = (\text{rank of } \text{SO}(8)) .
\end{align*}

The CFTs associated with \(M^{(D)}\) have extended algebras with spectral flow operators and naive gauge symmetries \(G_0 \times E_8\) are enhanced to \(G \times E_8\) by these special operators. These operators appear according to reductions of holonomies from \(\text{Spin}(7)\) (\(\text{SO}(8)\)) to \(G_2\) (\(\text{Spin}(7)\)) for respectively \(M^{(7)}\), \(M^{(8)}\). In other words, these are related to the degrees of freedom of quotient spaces \(\text{Spin}(7)/G_2\), \(\text{SO}(8)/\text{Spin}(7)\) and turn out to be associated with statistical models in the previous section

\begin{align*}
G_2 ; \ \ & \ \text{Spin}(7)/G_2 \cong \text{Tri-critical Ising} \ (c = \frac{7}{10}) , \\
\text{Spin}(7) ; \ & \ \text{SO}(8)/\text{Spin}(7) \cong \text{Ising} \ (c = \frac{1}{2}) .
\end{align*}
By taking account of these operators, we can propose enhancements of gauge symmetries as

\[ G_2 \ ; \ \frac{7}{10} + \frac{9}{2} = \frac{26}{5} \rightarrow \{ \text{tri-critical Ising} \} \times \text{SO}(9) \cong F_4 , \quad (3.1) \]

\[ \text{Spin}(7) \ ; \ \frac{1}{2} + 4 = \frac{9}{2} \rightarrow \{ \text{Ising} \} \times \text{SO}(8) \cong \text{SO}(9) , \quad (3.2) \]

where the left-hand sides of arrows represent consistency checks of central charges of enhanced currents. In fact, there are embeddings of these gauge groups in \( E_8 \)

\[ E_8 \supset G_2 \times F_4 , \quad (3.3) \]

\[ E_8 \supset \text{SO}(7) \times \text{SO}(9) , \quad (3.4) \]

and this also could be an evidence of the enhancements.

Under these embeddings, the representation 248 of a visible \( E_8 \) is decomposed by representations of their subgroups. For the \( G_2 \) case, this decomposition is expressed as

\[ E_8 \supset G_2 \times F_4 \]

\[ 248 = (1, 52) \oplus (7, 26) \oplus (14, 1) . \]

The 7 of \( G_2 \) is identified with an index of the tangent bundle of the 7 dimensional \( G_2 \) manifold. Also each representation of \( F_4 \) is decomposed into representations of \( \text{SO}(9) \subset F_4 \)

\[ 26 = 1 + 9 + 16 , \]

\[ 52 = 16 + 36 . \]

In this article, we consider standard embeddings and identify the spin connection of \( M^{(7)} \) directly with one of gauge \( F_4 \) singlet fields \( (14, 1) \).

Next we take the \( \text{Spin}(7) \) holonomy case. Through the embedding, the representation 248 of the \( E_8 \) is decomposed by representations of its subgroups

\[ E_8 \supset \text{SO}(7) \times \text{SO}(9) \]

\[ 248 = (1, 36) \oplus (7, 9) \oplus (8, 16) \oplus (21, 1) . \]

The 8 is identified with an index of the tangent space of the 8 dimensional \( \text{Spin}(7) \) manifold. 2nd rank antisymmetric tensors 28 on the 8 dimensional manifold (with a holonomy \( \text{SO}(8) \)) are decomposed into self-dual \( (\wedge^2_+) \) and anti self-dual parts \( (\wedge^2_-) \)

\[ 28 = 7 + 21 . \]
It corresponds to a decomposition into irreducible $Spin(7)$ modules

\[
\text{2 form } \Lambda^2(\mathbb{R}^8) \cong \text{so}(8),
\]

\[
\rightarrow \Lambda^2(\mathbb{R}^8) = \Lambda^2_+ \oplus \Lambda^2_-, \quad \text{dim } \Lambda^2_+ = 7, \quad \text{dim } \Lambda^2_- = 21,
\]

\[
\Lambda^2_- \cong Spin(7).
\]

In this decomposition, the Cayley 4-form $\Phi$ plays an important role. When we regard the $\Phi_{abcd}^\text{Cayley} (a, b, c, d = 1, 2, \cdots, 8)$ as a linear map $\hat{\Phi}$ of an so(8), eigenvalues of the operator $\frac{1}{2}\hat{\Phi}$ turn out to be $+1$ or $-3$. According to eigenvalues, we can construct projection operators $P_1, P_{-3}$

\[
P_1 = \frac{3}{4} \left( 1 + \frac{1}{6} \hat{\Phi} \right), \quad P_{-3} = \frac{1}{4} \left( 1 - \frac{1}{2} \hat{\Phi} \right),
\]

which project onto $\Lambda^2_-, \Lambda^2_+$ respectively. Especially $Spin(7)$ generators $\hat{G}_{ab}$’s are represented as

\[
\hat{G}_{ab} = \frac{3}{4} \left( \Gamma_{ab} + \frac{1}{6} \Phi_{abcd} \Gamma^{cd} \right) \in \Lambda^2_-,
\]

$\Gamma_{ab}$: SO(8) generator.

The anti self-dual parts are identified with adjoint 21 and one of them is set equal to the spin connection of the $Spin(7)$ manifold

\[(21, 1).\]

The remaining self-dual parts appear as matters of the vector representations 9 of SO(9)

\[(7, 9).\]

Also each representation of SO(9) is decomposed into representations of SO(8) $\subset$ SO(9)

\[
9 = 1 + 8_{\text{vec}},
\]

\[
16 = 8_{\text{spi}} + 8_{\text{cos}},
\]

\[
36 = 8_{\text{vec}} + 28.
\]

The subscripts $\text{vec}, \text{spi}, \text{cos}$ mean vector, spinor and cospinor representations of SO(8).

4 Special holonomy and character relations
4.1 Gauge symmetry enhancement from the viewpoint of characters

Next we study embeddings (3.3), (3.4) more precisely from branching relations of affine Lie algebras. The gauge symmetries of spacetime are realized by affine Kac-Moody algebras on the worldsheet and we will summarize the properties of several current algebras. As a first case, we take an affine $\mathfrak{so}(2r)_1$ algebra with level 1. It has central charge $c = r$ and its spectra (conformal dimension of the primary states) associated with integrable highest weight representations are evaluated as

- $\text{bas}: h = 0$, $\text{vec}: h = \frac{1}{2}$, $\text{spi}: h = \frac{2r}{16}$, $\text{cos}: h = \frac{2r}{16}$.

Here, $\text{bas, vec, spi, cos}$ mean basic, vector, spinor, cospinor representation respectively. Also the corresponding characters are evaluated by using Jacobi’s theta functions

\[
\chi_{\text{bas}}^{\mathfrak{so}(2r)} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^r + \left( \frac{\theta_4}{\eta} \right)^r \right),
\]
\[
\chi_{\text{vec}}^{\mathfrak{so}(2r)} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^r - \left( \frac{\theta_4}{\eta} \right)^r \right),
\]
\[
\chi_{\text{spi}}^{\mathfrak{so}(2r)} = \chi_{\text{cos}}^{\mathfrak{so}(2r)} = \frac{1}{2} \left( \frac{\theta_2}{\eta} \right)^r.
\]

For the affine $\mathfrak{so}(2r + 1)_1$ with level 1, there are three integrable highest weight representations and the associated conformal dimensions are calculated as

- $\text{bas}: h = 0$, $\text{vec}: h = \frac{1}{2}$, $\text{spi}: h = \frac{2r + 1}{16}$.

Their characters are constructed by combining theta functions

\[
\chi_{\text{bas}} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^{\frac{2r + 1}{2}} + \left( \frac{\theta_4}{\eta} \right)^{\frac{2r + 1}{2}} \right),
\]
\[
\chi_{\text{vec}} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^{\frac{2r + 1}{2}} - \left( \frac{\theta_4}{\eta} \right)^{\frac{2r + 1}{2}} \right),
\]
\[
\chi_{\text{spi}} = \frac{1}{\sqrt{2}} \left( \frac{\theta_2}{\eta} \right)^{\frac{2r + 1}{2}}.
\]

Similarly for level 1 affine $F_4, G_2, E_8$ cases, we will summarize integrable highest weight representations and their conformal dimensions in the following lists:

- level 1 affine $F_4$ ($c = \frac{26}{5}$)

  representations; $\text{bas}: h = 0$, $\text{fun}: h = \frac{3}{5}$,
level 1 affine \( G_2 \) \((c = \frac{14}{5})\)

representations; \( \text{bas}: h = 0 \), \( \text{fun}: h = \frac{2}{5} \),

level 1 affine \( E_8 \) \((c = 8)\)

representations; \( \text{bas}: h = 0 \),

where the “bas”, “fun” represent respectively the basic, fundamental representations of the corresponding algebras. Under these preparations we can obtain the tricritical Ising model by the coset construction \((\hat{F}_4)_1/\hat{so}(9)_1\). Then branching relation is expressed in the characters of each CFT algebra

\[
\lambda_{\Lambda}^{(F_4)} = \sum_{\lambda} \chi_{\text{Tri}}^{\Lambda,\lambda} \lambda_{\lambda}^{so(9)}. 
\]

The symbol \( \Lambda \) (= bas, fun) expresses each highest weight representation of \((\hat{F}_4)_1\) and \( \lambda \) (= bas, vec, spi) labels \( \hat{so}(9)_1 \) counterparts. The conformal dimensions of the Verma modules \((\Lambda, \lambda)\) are evaluated in the following table

| \((\Lambda, \lambda)\) | \text{(bas,bas)} | \text{(bas,vec)} | \text{(bas,spi)} | \text{(fun,bas)} | \text{(fun,vec)} | \text{(fun,spi)} |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( h \)                 | 0               | \( \frac{3}{2} \) | \( \frac{7}{16} \) | \( \frac{3}{5} \) | \( \frac{1}{10} \) | \( \frac{3}{80} \) |

That is to say, the \( \hat{F}_4 \) characters are decomposed according to the highest weights of the tricritical Ising model in the following way

\[
\begin{align*}
\chi_{\text{bas}}^{F_4} &= \chi_0^{so(9)} \lambda_{\text{bas}}^{G_2} + \chi_{3/2}^{so(9)} \lambda_{\text{vec}}^{G_2} + \chi_{7/10}^{so(9)} \lambda_{\text{spi}}^{G_2}, \\
\chi_{\text{fun}}^{F_4} &= \chi_{3/5}^{so(9)} \lambda_{\text{bas}}^{G_2} + \chi_{1/10}^{so(9)} \lambda_{\text{vec}}^{G_2} + \chi_{3/80}^{so(9)} \lambda_{\text{spi}}^{G_2}.
\end{align*}
\]

These are nothing but concrete realizations of the enhancement of gauge symmetry in Eq.(3.1) from \( \text{SO}(9) \) to \( \hat{F}_4 \). Also similar decompositions can be performed for sets \( \{\hat{so}(7)_1, \hat{G}_2\}_1, \text{(tricritical Ising)}\} \) and \( \{(\hat{E}_8)_1, \hat{so}(7)_1, \hat{so}(9)_1\} \) by applying the same technique as the \( \hat{F}_4 \) case

\[
\begin{align*}
\chi_{\text{bas}}^{so(7)} &= \chi_0^{so(7)} \lambda_{\text{bas}}^{so(7)} + \chi_{3/5}^{so(7)} \lambda_{\text{vec}}^{so(7)} + \chi_{3/80}^{so(7)} \lambda_{\text{spi}}^{so(7)}, \\
\chi_{\text{vec}}^{so(7)} &= \chi_{3/2}^{so(7)} \lambda_{\text{bas}}^{so(7)} + \chi_{1/10}^{so(7)} \lambda_{\text{vec}}^{so(7)}, \\
\chi_{\text{spi}}^{so(7)} &= \chi_{7/16}^{so(7)} \lambda_{\text{bas}}^{so(7)} + \chi_{3/80}^{so(7)} \lambda_{\text{vec}}^{so(7)}, \\
\chi_{\text{bas}}^{E_8} &= \chi_{\text{bas}}^{so(7)} \lambda_{\text{bas}}^{so(9)} + \chi_{\text{vec}}^{so(7)} \lambda_{\text{vec}}^{so(9)} + \chi_{\text{spi}}^{so(7)} \lambda_{\text{spi}}^{so(9)}.
\end{align*}
\]

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The first three equations mean that the holonomy group $G_2$ of our manifold $M$ is embedded in the $Spin(7)$. Collecting all these relations we can show an equation among characters for $(\hat{E}_8)_1$, $(\hat{F}_4)_1$ and $(\hat{G}_2)_1$

$$\chi_{bas}^{E_8} = \chi_{bas}^{F_4} G_2 + \chi_{fun}^{F_4} G_2.$$  

This describes embeddings of gauge groups $G_2$ and $F_4$ in $E_8$ considered in Eq.(3.3). The degrees of freedom in the tricritical Ising model are included in the symmetry algebra $\hat{so}(7)$. But they are transferred from this $\hat{so}(7)$ to $\hat{so}(9)$ and enhance the spacetime gauge symmetry from SO(9) to $F_4$.

Next we investigate the $Spin(7)$ holonomy case by taking account of the coset construction $\hat{so}(9)/\hat{so}(8)$ of the Ising model. The branching relation is expressed by characters of these CFTs

$$\chi_{\Lambda}^{so(9)} = \sum_{\lambda} \chi_{\Lambda,\lambda}^{Ising} \chi_{\lambda}^{so(8)}.$$  

Here the $\Lambda$ expresses highest weight representations of $\hat{so}(9)$ and $\lambda$ labels $\hat{so}(8)$ representations. The conformal dimensions of the Verma modules $(\Lambda, \lambda)$ are evaluated in the following table

| $(\Lambda, \lambda)$ | (bas,bas), (vec,vec) | (bas,vec), (vec,bas) | (spi,spi), (spi,cos) |
|----------------------|----------------------|----------------------|----------------------|
| $h$                  | 0                    | 1/2                  | 1/16                 |

and we can write down decompositions of characters of $\hat{so}(9)_1$ in terms of the weights of the Ising model concretely

$$\chi_{bas}^{so(9)} = \chi_0^{Ising} \chi_{bas}^{so(8)} + \chi_{1/2}^{Ising} \chi_{vec}^{so(8)};$$

$$\chi_{vec}^{so(9)} = \chi_{1/2}^{Ising} \chi_{bas}^{so(8)} + \chi_0^{Ising} \chi_{vec}^{so(8)};$$

$$\chi_{spi}^{so(9)} = \chi_{1/16}^{Ising} \chi_{spi}^{so(8)} + \chi_{1/16}^{Ising} \chi_{cos}^{so(8)}.$$  

(4.3)

These show the enhancement of gauge symmetry in Eq.(3.2) from SO(8) to SO(9). On the other hand the holonomy $Spin(7)$ is embedded in the SO(8) and this fact leads us to relations among characters of $\hat{so}(7)$ and $\hat{so}(8)$

$$\chi_{bas}^{so(8)} = \chi_0^{Ising} \chi_{bas}^{so(7)} + \chi_{1/2}^{Ising} \chi_{vec}^{so(7)};$$

$$\chi_{vec}^{so(8)} = \chi_{1/2}^{Ising} \chi_{bas}^{so(7)} + \chi_0^{Ising} \chi_{vec}^{so(7)};$$

$$\chi_{spi}^{so(8)} = \chi_{1/16}^{Ising} \chi_{spi}^{so(7)} + \chi_{1/16}^{Ising} \chi_{cos}^{so(7)}.$$  

(4.4)

By gathering these equations together with a decomposition of the $E_8$ character in terms of $so(8)$’s

$$\chi_{bas}^{E_8} = \chi_{bas}^{so(8)} \chi_{bas}^{so(8)} + \chi_{vec}^{so(8)} \chi_{vec}^{so(8)} + \chi_{spi}^{so(8)} \chi_{spi}^{so(8)} + \chi_{cos}^{so(8)} \chi_{cos}^{so(8)};$$

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we conclude the branching relation in terms of $so(7)$ and $so(9)$

$$
\chi_{\text{bas}}^{E_8} = \chi_{\text{bas}}^{so(7)} \chi_{\text{bas}}^{so(9)} + \chi_{\text{vec}}^{so(7)} \chi_{\text{vec}}^{so(9)} + \chi_{\text{spi}}^{so(7)} \chi_{\text{spi}}^{so(9)}.
$$

This describes embeddings of gauge groups $SO(7)$ and $SO(9)$ into $E_8$ in Eq.(3.4). In this case the degrees of freedom in the Ising model are transferred from one $\tilde{so}(8)$ to the other $\tilde{so}(8)$ and spacetime gauge symmetry is enhanced from $SO(8)$ to $SO(9)$. At the same time, the holonomy of $M^{(8)}$ is reduced from $SO(8)$ to $Spin(7)$. It is amazing that these phenomena about holonomies and gauge symmetries can be explained rigorously at the level of affine Lie algebras.

### 4.2 Relation to Calabi-Yau 3-fold and K3 compactification

Let us consider 8-dimensional space $M_0^{(8)}$ which is the whole transverse space of the string theory in light-cone gauge. $M_0^{(8)}$ might be a compact manifold, or a direct product $\mathbb{R}^{(D-2)} \times M^{(10-D)}$, where $M^{(10-D)}$ is a $(10-D)$-dimensional compact manifold.

Generally, the holonomy group $G_{\text{hol}}$ of $M_0^{(8)}$ is included in $so(8)$. In this case, the manifest gauge symmetry of the heterotic string theory on $M_0^{(8)}$ is $so(8)$, and there may be no supersymmetries. For this reason, let us denote this class of whole 8-dimensional (Ricci flat) manifolds or sigma models on these manifolds as $CFT(so(8))$.

As a subset of $CFT(so(8))$, we consider manifolds with holonomy group $G_{\text{hol}} \subset Spin(7) \subset so(8)$. We will name this class of manifolds or sigma model on them as $CFT(so(7))$. In such theories, the difference between $so(8)$ and $so(7)$ (in coset CFT meaning) — Ising model — is not broken by the holonomy. The relation $so(8)/so(7) = (\text{Ising})$ is shown in Eqs.(4.4). This extra “symmetry” causes the spacetime supersymmetry, and makes the naive gauge symmetry $so(8)$ enhanced to $so(9)$ as in Eqs.(4.3). This relation can be expressed by Eqs. (4.4).

There is a certain subset of $CFT(so(7))$ which has more spacetime supersymmetry (the number of supercharges is larger) and larger gauge symmetry. Its holonomy $G_{\text{hol}}$ is included in $G_2 \subset so(7)$. We call this class of manifolds $CFT(G_2)$. The prime example in this class of manifolds are a direct product of a flat line and a 7-dimensional $G_2$ holonomy manifold. A theory in $CFT(G_2)$ has more supercharges in spacetime and larger gauge symmetry than a general element in $CFT(so(7))$ because the theory in $CFT(G_2)$ has an extra symmetry expressed by the coset theory $so(7)/G_2 \cong (\text{tricritical Ising model})$. The relation $so(7)/G_2 \cong (\text{tricritical Ising model})$ is shown in Eqs.(4.2). This tricritical Ising model causes more supercharges, and larger gauge group than the general theory in $CFT(so(7))$ has. For example, the gauge symmetry enhancement $so(9)$ to $F_4$ occurs when we combine the $so(9)$ algebra and the tricritical Ising model as shown in Eqs.(4.1).
Moreover, as a subset of CFT\( (G_2) \), we can consider a class of manifolds whose holonomies are included in \( su(3) \subset G_2 \). We call this class of manifolds as CFT\( (su(3)) \). The prime example we mainly consider is a direct product of flat \( \mathbb{R}^2 \) and a Calabi-Yau 3-fold. A theory in CFT\( (su(3)) \) has more supercharges in spacetime and larger gauge symmetry than the general theory in CFT\( (G_2) \) because the theory in CFT\( (su(3)) \) has the extra symmetry expressed by the coset \( G_2/su(3) \). This coset \( G_2/su(3) \) turns out to be the 3-state Potts model from the following relations

\[
\begin{align*}
\chi_{G_2}^{bas} &= C_0^{3\text{-Potts}_{su(3)}} \chi_{bas}^{su(3)} + C_{2/3}^{3\text{-Potts}_{su(3)}} \chi_{fun}^{su(3)} + C_{2/3}^{3\text{-Potts}_{su(3)}} \chi_{fun}^{su(3)}, \\
\chi_{G_2}^{fun} &= C_{2/5}^{3\text{-Potts}_{su(3)}} \chi_{bas}^{su(3)} + C_{1/15}^{3\text{-Potts}_{su(3)}} \chi_{fun}^{su(3)} + C_{1/15}^{3\text{-Potts}_{su(3)}} \chi_{fun}^{su(3)}.
\end{align*}
\]

(4.5)

By the effect of this 3-state Potts model, a theory in CFT\( (su(3)) \) has the larger gauge symmetry \( E_6 \) than the gauge symmetry \( F_4 \) of a general theory in CFT\( (G_2) \). It can be shown as branching rules

\[
\begin{align*}
\chi_{E_6}^{bas} &= C_0^{3\text{-Potts}_{F_4}} \chi_{bas}^{F_4} + C_{2/5}^{3\text{-Potts}_{F_4}} \chi_{fun}^{F_4}, \\
\chi_{E_6}^{fun} &= \chi_{E_6}^{fun} = C_{2/3}^{3\text{-Potts}_{F_4}} \chi_{bas}^{F_4} + C_{1/15}^{3\text{-Potts}_{F_4}} \chi_{fun}^{F_4}.
\end{align*}
\]

(4.6)

A theory in CFT\( (su(3)) \) also has more supercharges in spacetime than a general theory in CFT\( (G_2) \). This theory also has a peculiar property. The \( N = 1 \) theory on the worldsheet has \( \mathbb{Z}_2 \) \( R \)-symmetry. But \( R \)-symmetry of the theory in this CFT\( (su(3)) \) is continuous \( U(1) \) and this theory has worldsheet \( N = 2 \) supersymmetry.

As a subset of CFT\( (su(3)) \), we can consider a class of manifolds (or CFT) CFT\( (su(2)) \subset CFT\( (su(3)) \). A manifold in this class has a holonomy included in \( su(2) \subset su(3) \). The prime example of the manifold in CFT\( (su(2)) \) is \( K3 \times \mathbb{R}^4 \), which we mainly consider. The difference between a theory in CFT\( (su(2)) \) and a general theory in CFT\( (su(3)) \) is evaluated by the coset \( su(3)/su(2) \cong u(1)_3 \) (see also appendix. [B.2]). That is seen from relations

\[
\begin{align*}
\chi_{bas}^{su(3)} &= \Theta_{0.3}^{su(2)} \chi_{bas}^{su(2)} + \Theta_{3.3}^{su(2)} \chi_{fun}^{su(2)}, \\
\chi_{fun}^{su(3)} &= \chi_{fun}^{su(3)} = \Theta_{2.3}^{su(2)} \chi_{bas}^{su(2)} + \Theta_{1.3}^{su(2)} \chi_{fun}^{su(2)}.
\end{align*}
\]

(4.7)

A theory in CFT\( (su(2)) \) has the larger gauge symmetry \( E_7 \) than the \( E_6 \) of a general theory in CFT\( (su(3)) \) by this \( u(1)_3 \). This is seen from equations about characters

\[
\begin{align*}
\chi_{bas}^{E_7} &= \Theta_{0.3}^{E_6} \chi_{bas}^{E_6} + \Theta_{2.3}^{E_6} \chi_{fun}^{E_6} + \Theta_{3.3}^{E_6} \chi_{fun}^{E_6}, \\
\chi_{fun}^{E_7} &= \chi_{fun}^{E_7} = \Theta_{1.3}^{E_6} \chi_{bas}^{E_6} + \Theta_{1.3}^{E_6} \chi_{fun}^{E_6}.
\end{align*}
\]

(4.8)

The number of spacetime supercharges of a theory in CFT\( (su(2)) \) is also larger than that of a general theory in CFT\( (su(3)) \).
Finally, there is a class of flat manifolds, such as $\mathbb{R}^8$. We name this class $\text{CFT}(1) \subset \text{CFT}(\text{su}(2))$. A flat CFT in $\text{CFT}(1)$ has more spacetime supercharges and the larger gauge symmetry than a general CFT in $\text{CFT}(\text{su}(2))$ because a flat CFT has the extra symmetry $\text{su}(2)$. A theory in $\text{CFT}(1)$ has the largest spacetime supersymmetry, and the largest gauge group $E_8$. The gauge symmetry enhancement from $E_7$ to $E_8$ can be seen from the relation about characters

$$\chi_E = \chi_{\text{bas}} E_7 + \chi_{\text{fun}} E_7.$$ 

Collecting these results, we find a sequence of inclusions of holonomy groups

$$\text{so}(8) \supset \text{so}(7) \supset G_2 \supset \text{su}(3) \supset \text{su}(2) \supset \{1\}. \quad (4.9)$$

This induces a sequence of classes of manifolds (or theories)

$$\text{CFT}(\text{so}(8)) \supset \text{CFT}(\text{so}(7)) \supset \text{CFT}(G_2) \supset \text{CFT}(\text{su}(3)) \supset \text{CFT}(\text{su}(2)) \supset \text{CFT}(1). \quad (4.10)$$

On the other side, there is also a sequence of gauge groups of theories

$$\text{so}(8) \subset \text{so}(9) \subset F_4 \subset E_6 \subset E_7 \subset E_8. \quad (4.11)$$

Each gauge group corresponds to each theory associated with a specific holonomy manifold. We can describe coset CFTs of subsequent two theories as rational CFTs

$$\text{so}(8)/\text{so}(7) \cong \text{so}(9)/\text{so}(8) \cong \text{Ising}, \quad \text{so}(7)/G_2 \cong F_4/\text{so}(9) \cong \text{tricritical Ising}, \quad (4.12)$$

$$G_2/\text{su}(3) \cong E_6/F_4 \cong \text{3-state Potts}, \quad \text{su}(3)/\text{su}(2) \cong E_7/E_6 \cong \text{u}(1)_3, \quad (4.13)$$

$$\text{su}(2)/\{1\} \cong E_8/E_7 \cong \text{su}(2).$$

These quotient theories play essential roles in gauge group enhancements and understanding spacetime supersymmetries.

Also, these sequences can be used to analyze special holonomy manifolds. For example, when one intends to study Calabi-Yau compactifications, he should consider the decomposition of $\text{so}(8)$

$$\text{so}(8) \cong (\text{Ising}) \times (\text{tricritical Ising}) \times (\text{3-state Potts}) \times \text{su}(3).$$

In this decomposition, the $\text{su}(3)$ part is absorbed as the holonomy, but the statistical part $(\text{Ising}) \times (\text{tricritical Ising}) \times (\text{3-state potts})$ remains unbroken and characterizes the universal structures of Calabi-Yau compactifications, such as spacetime supersymmetry and gauge group. We consider more about $\text{su}(3)$ holonomy and $\text{su}(2)$ holonomy cases in the following subsections.
4.2.1 $\text{su}(3)$ holonomy

Let us first consider the $\text{su}(3)$ holonomy case. The prime example of this case is the Calabi-Yau compactification. We expect there are $N = 2$ superconformal symmetry and a spectral flow operator. We explain how this symmetry can be seen from the cascade of holonomies.

The class $\text{CFT}(\text{su}(3))$ is characterized by the coset $\text{so}(8)/\text{su}(3)$, whose central charge is $c = 2$. We denote this $c = 2$ CFT as $\mathcal{X}$. One can construct this $\mathcal{X}$ from minimal models by using the sequence (4.11) and Eqs. (4.12), (4.13). We can define the characters of $\mathcal{X}$ by the set of equations

$$
\chi_{\Lambda}^{\text{so}(8)/\text{su}(3)} = \sum_{\Lambda} \chi_{(\lambda,\Lambda)}^{\text{so}(8)/\text{su}(3)} \chi_{\Lambda}^{\text{su}(3)}, \quad \lambda = \text{bas, vec, spi, cos}, \quad \Lambda = \text{bas, fun, \overline{fun}},
$$

(4.14)

where, $\text{bas, vec, spi, cos, fun, \overline{fun}}$ mean basic, vector, spinor, cospinor, fundamental, conjugate fundamental representation respectively. Then, from Eqs. (4.11), (4.12) and (4.13), the coset characters can be written by using characters of minimal models. The results are collected as

$$
\begin{align*}
\chi_{(\text{bas,bas})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(0,0,0)}^{\text{min}} + \chi_{(1/2,3/2,0)}^{\text{min}} + \chi_{(0,3,5/2,5)}^{\text{min}} + \chi_{(1/2,1/10,2/5)}^{\text{min}}, \\
\chi_{(\text{vec,bas})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(1/2,0,0)}^{\text{min}} + \chi_{(0,3/2,0)}^{\text{min}} + \chi_{(1/2,3/5,2/5)}^{\text{min}} + \chi_{(0,1/10,2/5)}^{\text{min}}, \\
\chi_{(\text{spi,bas})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(0,0,3)}^{\text{min}} + \chi_{(1/2,3/2,2/3)}^{\text{min}} + \chi_{(0,3,5,1/15)}^{\text{min}} + \chi_{(1/2,1/10,1/15)}^{\text{min}}, \\
\chi_{(\text{bas,fun})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(1/2,0,2/3)}^{\text{min}} + \chi_{(0,3,2,2/3)}^{\text{min}} + \chi_{(1/2,3,5,1/15)}^{\text{min}} + \chi_{(0,1/10,1/15)}^{\text{min}}, \\
\chi_{(\text{vec,fun})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(0,0,2)}^{\text{min}} + \chi_{(1/2,3,2,2)}^{\text{min}} + \chi_{(1/2,3,2,3)}^{\text{min}} + \chi_{(0,1/10,1/15)}^{\text{min}}, \\
\chi_{(\text{spi,fun})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(0,0,2)}^{\text{min}} + \chi_{(1/2,3,2,2)}^{\text{min}} + \chi_{(0,1/10,1/15)}^{\text{min}}, \\
\chi_{(\text{spi,fun})}^{\text{so}(8)/\text{su}(3)} &= \chi_{(0,0,2)}^{\text{min}} + \chi_{(1/2,3,2,2)}^{\text{min}} + \chi_{(0,1/10,1/15)}^{\text{min}},
\end{align*}
$$

(4.15)

where $\chi_{(a,b,c)}^{\text{min}}$ is the product of the characters of minimal models

$$
\chi_{(a,b,c)}^{\text{min}} = \chi_a^{\text{Ising}} \chi_b^{\text{Tri}} \chi_c^{3\text{-Potts}}.
$$

Also the symbol $(a,b,c)$ represents the set of conformal weights of each statistical model. Since the $\mathcal{X}$ causes the gauge symmetry enhancement $\text{so}(8) \to E_6$ (we are comparing $\text{CFT}(\text{so}(8))$ and $\text{CFT}(\text{su}(3))$), the coset $\text{CFT} E_6/\text{so}(8)$ is also identified with this $\mathcal{X}$. This fact can be seen from the sequence of gauge theory (4.11) and Eqs. (4.12), (4.13). We will also explain this fact from the point of view of characters later.

We can now obtain the characters of the coset CFT $E_6/\text{so}(8)$ by using the explicit forms of the characters shown in appendix 3.2. When we define the coset characters
\[ \chi_{(\Lambda, \lambda)}^{E_6/so(8)} \]’s by relations

\[ \chi_{\Lambda}^{E_6} = \sum_{\lambda} \chi_{(\Lambda, \lambda)}^{E_6/so(8)} \chi_{\lambda}^{so(8)}, \quad \Lambda = \text{bas, fun, fun}, \quad \lambda = \text{bas, vec, spi, cos}, \]

then we can obtain the results about characters

\[
\begin{align*}
\chi_{(\text{bas}, \text{bas})}^{E_6/so(8)} &= \frac{\Theta_{0,6}}{\eta} \chi_{\text{bas}}^{so(2)} + \frac{\Theta_{6,6}}{\eta} \chi_{\text{vec}}^{so(2)}, \\
\chi_{(\text{bas}, \text{vec})}^{E_6/so(8)} &= \frac{\Theta_{6,6}}{\eta} \chi_{\text{bas}}^{so(2)} + \frac{\Theta_{0,6}}{\eta} \chi_{\text{vec}}^{so(2)}, \\
\chi_{(\text{bas}, \text{spi})}^{E_6/so(8)} &= \chi_{(\text{bas}, \cos)}^{E_6/so(8)} = \frac{\Theta_{3,6}}{\eta} \chi_{\text{spi}}^{so(2)} + \frac{\Theta_{3,6}}{\eta} \chi_{\text{cos}}^{so(2)}, \\
\chi_{(\text{fun}, \text{bas})}^{E_6/so(8)} &= \frac{\Theta_{0,6}}{\eta} \chi_{\text{bas}}^{so(2)} + \frac{\Theta_{4,6}}{\eta} \chi_{\text{vec}}^{so(2)}, \\
\chi_{(\text{fun}, \text{vec})}^{E_6/so(8)} &= \frac{\Theta_{2,6}}{\eta} \chi_{\text{bas}}^{so(2)} + \frac{\Theta_{2,6}}{\eta} \chi_{\text{vec}}^{so(2)}, \\
\chi_{(\text{fun}, \text{spi})}^{E_6/so(8)} &= \chi_{(\text{fun}, \cos)}^{E_6/so(8)} = \chi_{(\text{fun}, \cos)}^{E_6/so(8)} = \frac{\Theta_{1,6}}{\eta} \chi_{\text{spi}}^{so(2)} + \frac{\Theta_{5,6}}{\eta} \chi_{\text{cos}}^{so(2)}. \tag{4.16}
\end{align*}
\]

We find the relation between the functions in Eqs. (4.17) and (4.16)

\[ \chi_{(\Lambda, \lambda)}^{E_6/so(8)} = \chi_{(\lambda, \Lambda)}^{so(8)/su(3)}, \tag{4.17} \]

must be satisfied for each set \((\Lambda, \lambda)\) from the following reason. By using the explicit forms of characters, the \(E_8\) character can be decomposed by two \(so(8)\) characters as

\[ \chi_{\text{bas}}^{E_8} = \sum_{\lambda=\text{bas, vec, spi, cos}} \chi_{\lambda}^{so(8)} \chi_{\lambda}^{so(8)}. \tag{4.18} \]

Using the definition of \(\chi^{so(8)/su(3)}\) and Eq. (4.18), we can decompose \(\chi_{\text{bas}}^{E_8}\) in the following formula

\[ \chi_{\text{bas}}^{E_8} = \sum_{\Lambda=\text{bas, fun, fun}} \sum_{\lambda=\text{bas, vec, spi, cos}} \chi_{\lambda}^{so(8)} \chi_{(\lambda, \Lambda)}^{so(8)/su(3)} \chi_{(\lambda, \Lambda)}^{su(3)}. \tag{4.19} \]

On the other hand, by using the explicit forms of the characters, the \(E_8\) character can be decomposed by those of \(E_6\) and \(su(3)\) as

\[ \chi_{\text{bas}}^{E_8} = \sum_{\Lambda=\text{bas, fun, fun}} \chi_{\Lambda}^{E_6} \chi_{\Lambda}^{su(3)}. \tag{4.20} \]

When we compare Eq. (4.19) and Eq. (4.20), the set of relations is obtained

\[ \chi_{\Lambda}^{E_6} = \sum_{\lambda=\text{bas, vec, spi, cos}} \chi_{\lambda}^{so(8)} \chi_{\lambda, \Lambda}^{so(8)/su(3)}. \tag{4.21} \]
By comparing the definition of $\chi_{E_6/so(8)}$ and Eq. (4.21), we can obtain the relation (4.17).

We also checked the relation (4.17) for several order in $q$-expansion by Mathematica. So we use the relation (4.17) in this paper and introduce a notation $\chi^{X}_{(\Lambda,\lambda)} := \chi_{E_6/so(8)}^{\Lambda,\lambda} = \chi_{so(8)/su(3)}^{\Lambda,\lambda}$.

Looking at the forms in Eqs. (4.16), we can show that the $X$ is decomposed into $so(2)$ and $u(1)$. This $so(2)$ corresponds to the rotation of flat space $\mathbb{R}^2$ in transverse directions of Calabi-Yau compactification $CY_3 \times \mathbb{R}^2$, and this $u(1)$ is the symmetry related to spacetime susy and gauge symmetry enhancement $so(10) \rightarrow E_6$. Actually, this $u(1)$ symmetry can be identified with the $u(1)$ symmetry in the $c=9$, $N=2$ superconformal algebra. In order to see this, we will concentrate on the gauge symmetry enhancement $so(10) \rightarrow E_6$. This phenomenon of the Calabi-Yau compactification can be realized as the relations about characters

$$
\chi^{E_6}_{\Lambda} = \sum_{\lambda} \frac{\Theta_{n(\Lambda)+3n(\lambda),6}}{\eta} \chi^{so(10)}_{\Lambda,\lambda},
$$

where $n(\Lambda)$ and $n(\lambda)$ are functions respectively depending on representations $\Lambda$’s and $\lambda$’s

$n(bas) = 0, n(fun) = n(spi) = 1, n(fun) = n(cos) = -1, n(vec) = 2$.

Now let us write an arbitrary state in the Calabi-Yau CFT as $|n, m\rangle \otimes |\text{other}\rangle$ where $|n, m\rangle$ is a state in the module “$m$” of the $u(1)_6$ theory. The integer “$m$” appears as an index of the character $\Theta_{m,6}/\eta$. The $|\text{other}\rangle$ is a state associated with other parts and has no contribution to $u(1)$ charge of the $N=2$ SCA. Only the part $|n, m\rangle$ has the relevant $u(1)$ charge. We can evaluate the $u(1)$ charge $Q_m$ of this state as $Q_m = m/2 \mod 6$. The $U(1)$ here serves as an R-symmetry of the $N=2$ theory and is used to construct $U(1)$ current of the $N=2$ algebra. Also a spectral flow operator of the $N=2$ CFT has conformal dimension 3/8 and is constructed by combining scaling operators of three statistical models. A candidate of a spectral flow operator appears in the character $\Theta_{3,6}/\eta$, more precisely in the sector

$$
\chi^{so(8)/su(3)}_{(spi,bas)} = \chi^{so(8)/su(3)}_{(cos,bas)} = \chi^{so(8)}_{(1/16,7/16,0)} + \chi^{so(8)}_{(1/16,3/80,2/5)},
$$

$$
= \frac{\Theta_{3,6}}{\eta} \chi^{so(2)}_{spi} + \frac{\Theta_{3,6}}{\eta} \chi^{so(2)}_{cos} = \chi^{E_6/so(8)}_{(bas,spi)} = \chi^{E_6/so(8)}_{(bas,cos)},
$$

(4.22)

This operator belongs to a sector with a $U(1)$ charge $Q = 3/2$. It implies that the state is related with a 3-form of the $CY_3$. Also the lowest term in this character is $q^{3/8-1/24}$. This represents a primary state with conformal weight 3/8 and its charge is 3/2. It is the same as the spectral flow operator $\Sigma$ has. We shall look at this more precisely. It is realized as a combination of states with $(h^{Ising}, h^{Tri}, h^{3-Potts}) = (1/16, 7/16, 0), (1/16, 3/80, 2/5)$. 

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The total weight of these states turns out to be 1/2. When we recall the identity $\chi^{\text{Ising}} \times \chi^{\text{Tri}} \times \chi^{3-\text{Potts}} = \chi^{\text{SO}(2)} \times \chi^{U(1)}$ in the above Eqs. (4.22), we can obtain a spin operator $\Sigma$ with $h = 3/8 (= 1/2 - 1/8)$ of the $\text{su}(3)$ holonomy model by subtracting contributions of a spin operator of $\text{SO}(2)$ with weight 1/8. This operator $\Sigma$ is nothing but a holomorphic 3-form of the $\text{CY}_3$ and confirms the validity of our discussions. (But we only look at the chiral part of the theory here).

It is remarkable that we can realize $N = 2$ CFT associated with $\text{CY}_3$ starting from $\text{SO}(8)$ theory by using three statistical models in 2 dimension.

### 4.2.2 $\text{su}(2)$ holonomy

Let us also consider the $\text{su}(2)$ holonomy case in the same way as the $\text{su}(3)$ case. The prime example of this case is the K3 compactification. In the $\text{su}(2)$ holonomy case, the coset $\text{so}(8)/\text{su}(2)$ is essential to explain spacetime supersymmetry and the gauge symmetry enhancement. We denote $\text{so}(8)/\text{su}(2)$ as $\mathcal{Y}$ and study how this determines the spacetime susy and gauge symmetry.

The characters of $\mathcal{Y}$ are defined by the branching relation

$$\chi_{\lambda}^{\text{so}(8)} = \sum_{\Lambda = \text{bas, fun}} \chi_{(\Lambda, \lambda)}^{\mathcal{Y}} \chi_{\Lambda}^{\text{su}(2)}.$$  

By using $\text{so}(8)/\text{su}(3) \cong \mathcal{X}$ in Eqs. (4.15), $\text{su}(3)/\text{su}(2) \cong u(1)_3$ in Eqs. (4.7), and the explicit forms of the $\chi_{(\Lambda, \lambda)}^{\mathcal{X}}$’s in Eqs. (4.16), the characters of $\mathcal{Y}$ can be written as

$$\chi_{(\text{bas, bas})}^{\mathcal{Y}} = \chi_{\text{bas}}^{\text{so}(4)} \chi_{\text{bas}}^{\text{su}(2)}, \quad \chi_{(\text{bas, vec})}^{\mathcal{Y}} = \chi_{\text{vec}}^{\text{so}(4)} \chi_{\text{bas}}^{\text{su}(2)},$$

$$\chi_{(\text{fun, bas})}^{\mathcal{Y}} = \chi_{\text{vec}}^{\text{so}(4)} \chi_{\text{fun}}^{\text{su}(2)}, \quad \chi_{(\text{fun, vec})}^{\mathcal{Y}} = \chi_{\text{bas}}^{\text{so}(4)} \chi_{\text{fun}}^{\text{su}(2)},$$

$$\chi_{(\text{bas, spi})}^{\mathcal{Y}} = \chi_{\text{bas, cos}}^{\text{so}(4)} \chi_{\text{spi}}^{\text{su}(2)}, \quad \chi_{(\text{fun, spi})}^{\mathcal{Y}} = \chi_{\text{fun, cos}}^{\text{so}(4)} \chi_{\text{spi}}^{\text{su}(2)}.$$  

These relations show that $\mathcal{Y}$ can be decomposed into $\text{so}(4)$ and $\text{su}(2)$. This $\text{so}(4)$ in $\mathcal{Y}$ is the rotation of flat $\mathbb{R}^4$ in $\mathbb{R}^4 \times \text{K3}$. On the other hand, this $\text{su}(2)$ in $\mathcal{Y}$ is the key symmetry for the spacetime supersymmetry and the gauge symmetry enhancement. Actually, the K3 CFT has $c = 6$, $N=4$ superconformal symmetry and the $\text{su}(2)$ in $\mathcal{Y}$ is identified with $R$-symmetry $\text{su}(2)$ in the $c = 6$, $N=4$ superconformal algebra.

The gauge symmetry is enhanced from $\text{so}(12)$ to $E_7$ with this $\text{su}(2)$ symmetry in $\mathcal{Y}$. This phenomenon can be explained by branching rules

$$\chi_{\text{bas}}^{E_7} = \chi_{\text{bas}}^{\text{su}(2)} \chi_{\text{bas}}^{\text{SO}(12)} + \chi_{\text{fun}}^{\text{su}(2)} \chi_{\text{spi}}^{\text{SO}(12)},$$

$$\chi_{\text{fun}}^{E_7} = \chi_{\text{bas}}^{\text{su}(2)} \chi_{\text{cos}}^{\text{SO}(12)} + \chi_{\text{fun}}^{\text{su}(2)} \chi_{\text{vec}}^{\text{SO}(12)}.$$
The spectral flow operators are the primary states in the fundamental representation of su(2) in \( \mathcal{Y} \). The conformal dimension of these states are both 1/4. This is the same property as spectral flow operators.

4.2.3 Comments on spacetime supersymmetry

Let us comment about the spacetime supersymmetry from the viewpoint of characters. In the flat case, the key ingredient for this susy is the Jacobi’s abstruse identity

\[ \theta_3^4 - \theta_4^4 = 0. \]

This can be rewritten by so(8) characters in the formula

\[ \chi^{so(8)}_{vec} - \chi^{so(8)}_{spi} = 0. \]

From this Jacobi’s abstruse identity, we can propose the key identities for the spacetime supersymmetries in compactifications on special holonomy manifolds

\[ \chi^{so(8)}/G_{hol, vec} - \chi^{so(8)}/G_{hol, spi} = 0, \tag{4.23} \]

where \( G_{hol} \)’s are the holonomy groups. Also \( \Lambda \) is the representation of \( G_{hol} \). We assume this identity is satisfied only for the cases \( \chi^{so(8)}/G_{hol, vec} \neq 0 \) and \( \chi^{so(8)}/G_{hol, spi} \neq 0 \). An evidence of our proposal is given by the following branching relation

\[ 0 = \chi^{so(8)}_{vec} - \chi^{so(8)}_{spi} = \sum_{\Lambda} (\chi^{so(8)}/G_{hol, vec} - \chi^{so(8)}/G_{hol, spi}) \chi^G_{hol, \Lambda}. \]

If we put the identity (4.23), we can show that the partition functions vanish in orbifold cases. In order to explain this, let us consider the orbifold group \( \Gamma \subset G_{hol} \subset so(8) \). The character of the \((g_1, g_2)\)-sector \((g_1, g_2 \in \Gamma)\) is defined as

\[ \chi^{so(8)}_{\Lambda, (g_1, g_2)} = \text{Tr}_{\lambda, g_1, \text{twisted}} [g_2 q^{L_0 - c/24}]. \]

This can be decomposed as

\[ \chi^{so(8)}_{\Lambda, (g_1, g_2)} = \sum_{\Lambda} \chi_{\Lambda, \Lambda}^{so(8)/G_{hol}} \chi^G_{hol, \Lambda}. \]

Note that \( \chi^{so(8)/G_{hol}}_{\Lambda, \Lambda} \) is independent of \((g_1, g_2)\) because \( g_1 \) and \( g_2 \) are elements of \( \Gamma \subset G_{hol} \). On the other hand, \( \chi^G_{hol, \Lambda, (g_1, g_2)} \) is defined in the same way as the so(8) case. By using these characters, the partition function of left-moving fermions in \((g_1, g_2)\)-sector can be written as

\[ Z^{(F)}_{g_1, g_2} = \chi^{so(8)}_{vec, (g_1, g_2)} - \chi^{so(8)}_{spi, (g_1, g_2)} = \sum_{\Lambda} (\chi^{so(8)/G_{hol}}_{vec, \Lambda} - \chi^{so(8)/G_{hol}}_{spi, \Lambda}) \chi^G_{hol, \Lambda, (g_1, g_2)}. \]
This $Z_{G_1,G_2}^{(F)}$ becomes 0 when we use the identities (4.23).

Let us see the explicit forms of these identities for each case of holonomies $G_2$, $\text{su}(3)$, $\text{su}(2)$.

First, we consider the explicit form of the identities of the $G_2$ case. The branching relation of the coset model $\text{so}(8)/G_2$ can be written as

$$\chi^{\text{so}(8)/G_2}_{\text{bas}} = (\chi^0_0, \chi^1_0 + \chi^1_{1/2} \chi^2_{3/2}) \chi^{G_2}_{\text{bas}} + (\chi^0_0, \chi^3_{1/2} + \chi^1_{3/2} \chi^0_{1/10}) \chi^{G_2}_{\text{fun}},$$

$$\chi^{\text{so}(8)/G_2}_{\text{vec}} = (\chi^1_{1/2} \chi^0_0 + \chi^0_0asectr) \chi^2_{3/2} \chi^{G_2}_{\text{bas}} + (\chi^1_{1/2} \chi^0_0asectr) \chi^3_{3/2} + \chi^0_0asectr) \chi^{G_2}_{\text{fun}},$$

$$\chi^{\text{so}(8)/G_2}_{\text{spi}} = \chi^{\text{so}(8)} = (\chi^0_0asectr) \chi^2_{1/16} \chi^2_{3/16} \chi^{G_2}_{\text{bas}} + (\chi^0_0asectr) \chi^3_{1/16} \chi^3_{3/80} \chi^{G_2}_{\text{fun}}.$$ 

The explicit susy identities in the $G_2$ holonomy case is as follows. From $\chi^{\text{so}(8)/G_2}_{\text{bas}} - \chi^{\text{so}(8)/G_2}_{\text{spi}, \text{bas}} = 0$, we obtain

$$\chi^0_0asectr) + \chi^1_0asectr) \chi^1_{3/2} - \chi^1_{1/2} \chi^3_{1/16} \chi^3_{7/16} = 0.$$ 

(4.24)

From $\chi^{\text{so}(8)/G_2}_{\text{bas}} - \chi^{\text{so}(8)/G_2}_{\text{spi}, \text{fun}} = 0$, we obtain

$$\chi^1_{1/2} \chi^3_{1/2} + \chi^0_0asectr) \chi^1_{1/10} - \chi^1_{1/16} \chi^3_{3/80} = 0.$$ 

(4.25)

These formulas are the same as the ones recently obtained in [15].

Next, let us go to the susy identities in the $\text{su}(3)$ holonomy case. The explicit form using the characters in (4.15) becomes

$$\chi^{\text{min}}_{(1/2,0,0)} + \chi^{\text{min}}_{(0,3/2,0)} + \chi^{\text{min}}_{(1/2,3/2,5/2)} + \chi^{\text{min}}_{(0,1,10,2/5)} - \chi^{\text{min}}_{(1/16,7,1/16)} - \chi^{\text{min}}_{(1/16,3,80/2)} = 0,$$

$$\chi^{\text{min}}_{(1/2,0,2/3)} + \chi^{\text{min}}_{(0,3/2,2/3)} + \chi^{\text{min}}_{(1/2,3/5,1/15)} + \chi^{\text{min}}_{(0,1,10,2/15)} - \chi^{\text{min}}_{(1/16,7,1/16,2)} - \chi^{\text{min}}_{(1/16,3,80,1/15)} = 0.$$ 

(4.26)

Since a Calabi-Yau compactification is a special case of $G_2$ compactifications, one may guess that the identities (4.24) can be derived from the identities (4.24) and (4.25). Actually, the following formulas show this guess is true

$$\chi^{\text{min}}_{(1/2,0,0)} + \chi^{\text{min}}_{(0,3/2,0)} + \chi^{\text{min}}_{(1/2,3/2,5/2)} + \chi^{\text{min}}_{(0,1,10,2/5)} - \chi^{\text{min}}_{(1/16,7,1/16)} - \chi^{\text{min}}_{(1/16,3,80/2)} = 0,$$

$$\chi^{\text{min}}_{(1/2,0,2/3)} + \chi^{\text{min}}_{(0,3/2,2/3)} + \chi^{\text{min}}_{(1/2,3/5,1/15)} + \chi^{\text{min}}_{(0,1,10,2/15)} - \chi^{\text{min}}_{(1/16,7,1/16,2)} - \chi^{\text{min}}_{(1/16,3,80,1/15)} = 0.$$ 

(4.26)

Besides the expression (4.26) of the susy identities, we can also write the explicit susy identities using the form of (4.25). These identities reduce to

$$\Theta_{6,6} \Theta_{0,2} + \Theta_{0,6} \Theta_{2,2} - 2 \Theta_{3,6} \Theta_{1,2} = 0,$$

$$\Theta_{2,6} \Theta_{0,2} + \Theta_{4,6} \Theta_{2,2} - \Theta_{1,6} \Theta_{1,2} - \Theta_{5,6} \Theta_{1,2} = 0.$$ 

(4.27)
These are the same identities obtained in [20] and [21]. If we use the identity (4.17), (4.26) and (4.27) are equivalent.

Finally, let us see the susy identities in the su(2) holonomy case. The explicit form of the identities are given by

\[
\lambda_{\text{vec}}^{\text{so}(4)} \lambda_{\text{bas}}^{\text{su}(2)} - \lambda_{\text{spi}}^{\text{so}(4)} \lambda_{\text{fun}}^{\text{su}(2)} = 0, \\
\lambda_{\text{bas}}^{\text{so}(4)} \lambda_{\text{fun}}^{\text{su}(2)} - \lambda_{\text{spi}}^{\text{so}(4)} \lambda_{\text{bas}}^{\text{su}(2)} = 0.
\]

These identities reduce to the ones obtained in [22].

5 Orbifold

In this section we investigate \(G_2\) and \(Spin(7)\) manifolds realized as orbifolds. These models have been discussed by Joyce [2–4] as concrete examples of compact manifolds with exceptional holonomies in mathematical contexts. We review his constructions in subsection 5.1. In subsection 5.2, we elaborate toroidal partition functions of heterotic strings on these orbifolds and study their modular properties. In subsection 5.3, we show our results about massless spectra of effective theories in our these heterotic models.

5.1 Examples of Special Holonomy Manifolds

In this subsection we study some of the examples constructed by Joyce [2–4]. A basic example of a compact seven manifold \(M^{(7)}\) with \(G_2\) holonomy is realized as a toroidal orbifold. Let \((x_1, x_2, \cdots, x_7)\) be a set of coordinates of \(T^7\) which is a product of seven circles of the radius \(R\). The \(M^{(7)}\) is defined as the desingularization of the \(T^7\) modded out by \(\Gamma \cong \mathbb{Z}_2^3\) group with generators

\[
T^7 \ni (x_1, x_2, x_3, x_4, x_5, x_6, x_7), \\
\Gamma \left\{ \begin{array}{l}
\alpha; \ ( -x_1, \ -x_2, \ -x_3, \ -x_4, \ x_5, \ x_6, \ x_7 ) \\
\beta; \ ( -x_1, \ 1/2 - x_2, \ x_3, \ x_4, \ -x_5, \ -x_6, \ x_7 ) \\
\gamma; \ ( 1/2 - x_1, \ x_2, \ 1/2 - x_3, \ x_4, \ -x_5, \ x_6, \ -x_7 ) 
\end{array} \right. ,
\]

where the generators of the \(\mathbb{Z}_2\)’s are denoted by \(\alpha, \beta, \gamma\). One can verify that \(\alpha^2 = \beta^2 = \gamma^2 = 1\) and \(\alpha, \beta, \gamma\) commute one another. Then discrete group \(\Gamma\) is isomorphic to \(\mathbb{Z}_2^3\). Also \(1/2\) means a shift \(1/2 \times 2\pi R\) around the circle in the case that each \(x_i\) of \(T^7\) has period \(2\pi R\). Then this holonomies preserve the flat \(G_2\) structure on \(T^7\) given by a \(\Phi\)

\[
\Phi = dx_{136} + dx_{145} + dx_{235} - dx_{127} - dx_{246} - dx_{347} - dx_{567} , \\
dx_{ijk} := dx_i \wedge dx_j \wedge dx_k.
\]
Next we review the cohomology classes on $M^{(7)}$. After this projection there remain a zero-form, one 7-form, seven 3-forms and seven 4-forms of $T^7$. But none of the two-forms are invariant under the action of the discrete group $\Gamma$. The elements $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ and $\alpha\beta\gamma$ have no fixed points on $T^7$. The fixed points of $\alpha$ in $T^7$ are $16$ $T^3$’s and the group $\langle \beta, \gamma \rangle$ acts freely on these 16 sets. It leaves us with 4 invariant combinations on the quotient $T^7/\Gamma$. Similarly one can see that the fixed $T^3$’s for each $\beta, \gamma$ are 4 copies of $T^3$. The local form of the singularities at the fixed $T^3$’s is $\mathbb{R}^4/\mathbb{Z}_2 \times T^3$ and resolving each of these yields one 2-form and three 3-forms. Since there are 12 fixed tori on $M^{(7)}$, one obtains Betti numbers after resolution by recalling $b_2(T^7/\Gamma) = 0$, $b_3(T^7/\Gamma) = 7$

$$b_2(M^{(7)}) = b_2(T^7/\Gamma) + 12 \cdot 1 = 12, \quad b_3(M^{(7)}) = b_3(T^7/\Gamma) + 12 \cdot 3 = 43.$$ 

Now we are able to write down all Betti numbers of this $G_2$ orbifold $M^{(7)}$

$$b_0 = b_7 = 1, \quad b_1 = b_6 = 0, \quad b_2 = b_5 = 12, \quad b_3 = b_4 = 43.$$ 

This is a compact, simply-connected seven manifold with holonomy $G_2$. The moduli space has dimension 43 and the associated CFT counterpart is a $b_2 + b_3 = 55$ dimensional space.

Next we shall explain a simple example of a compact 8 manifold $M^{(8)}$ with holonomy Spin$(7)$ constructed by Joyce [4]. This example proceeds similarly to the $G_2$ case. Let $(x_1, x_2, \cdots, x_7, x_8)$ be a set of coordinates of $T^8$ which is a product of eight circles of the radius $R$. The $M^{(8)}$ is constructed as the desingularization of the $T^8$ divided by the discreet group $\Gamma \cong \mathbb{Z}_2^4$ with generators

$$\Gamma \cong \left\{ \begin{array}{l}
\alpha; \quad ( -x_1, \quad -x_2, \quad -x_3, \quad -x_4, \quad x_5, \quad x_6, \quad x_7, \quad x_8 ) \\
\beta; \quad ( \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad -x_5, \quad -x_6, \quad -x_7, \quad -x_8 ) \\
\gamma; \quad ( \quad \frac{1}{2} - x_1, \quad \frac{1}{2} - x_2, \quad -x_3, \quad x_4, \quad \frac{1}{2} - x_5, \quad \frac{1}{2} - x_6, \quad x_7, \quad x_8 ) \\
\delta; \quad ( \quad -x_1, \quad x_2, \quad \frac{1}{2} - x_3, \quad -x_4, \quad x_5, \quad -x_6, \quad \frac{1}{2} - x_7, \quad x_8 )
\end{array} \right\}.$$ 

It is easy to see that $\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 1$ and $\alpha, \beta, \gamma, \delta$ all commute one another. Then the $\Gamma \cong \mathbb{Z}_2^4$ is a group of automorphisms of $T^8$ preserving the flat Spin$(7)$ structure given by a Cayley 4 form $\Phi$

$$\Phi = dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368}$$
$$-dx_{1458} - dx_{1467} - dx_{2358} - dx_{2367} - dx_{2457}$$
$$+ dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678},$$
$$dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l.$$
### Table 4: Fixed-point sets by \( \alpha, \beta, \gamma, \delta \) and actions of these generators on them.

|                | fixed points | \( \alpha \)-action | \( \beta \)-action | \( \gamma \)-action | \( \delta \)-action |
|----------------|-------------|----------------------|---------------------|---------------------|---------------------|
| \( \alpha \)-fixed points | \( 16 T^4 \) | *                     | trivial             | free                | free                |
| \( \beta \)-fixed points    | \( 16 T^4 \) | trivial              | *                   | free                | free                |
| \( \gamma \)-fixed points   | \( 16 T^4 \) | free                 | free                | *                   | free                |
| \( \delta \)-fixed points   | \( 16 T^4 \) | free                 | free                | free                | *                   |

Each of the fixed points of \( \alpha, \beta, \gamma, \delta \) are 16 copies of \( T^4 \). Also \( \beta \) acts trivially on the set of the 16 \( T^4 \) fixed by the \( \alpha \)-action and \( \langle \gamma, \delta \rangle \) acts freely on these \( T^4 \). It leaves us with 4 invariant combinations \( T^4/\{\pm 1\} \) on the quotient \( T^8/\Gamma \) from the \( \alpha \)-fixed points. We summarize similar properties about fixed \( T^4 \)'s by other generators in table 5.

The two sets of \( \alpha \)-fixed points and \( \beta \)-fixed points intersect in 64 points.

### Table 5: Fixed-point sets divided by actions of other generators.

|                | fixed points |
|----------------|-------------|
| \( \alpha \)-fixed points | \( 4 T^4/\{\pm 1\} \) |
| \( \beta \)-fixed points   | \( 4 T^4/\{\pm 1\} \) |
| \( \gamma \)-fixed points   | \( 2 T^4 \) |
| \( \delta \)-fixed points   | \( 2 T^4 \) |

As is the \( G_2 \) case, the Betti numbers \( b_i(T^8/\Gamma) \) are the dimension of the \( \Gamma \)-invariant subspaces of differential forms on \( T^8 \). After the projection there are no nonzero \( \Gamma \)-invariant 1-, 2-, and 3-forms. But one can show that there are four self-dual 4-forms and four anti self-dual 4-forms. Thus the Betti numbers of \( T^8/\Gamma \) are written down as

\[
b_1(T^8/\Gamma) = b_2(T^8/\Gamma) = b_3(T^8/\Gamma) = 0, \quad b_4(T^8/\Gamma) = 14, \quad b_4^+(T^8/\Gamma) = 7, \quad b_4^-(T^8/\Gamma) = 7.\]

Next we calculate the Betti numbers of \( M^{(8)} \). When one resolves each of the 4 fixed \( T^4/\{\pm 1\} \) by \( \alpha \)-action and 4 \( T^4/\{\pm 1\} \) fixed by \( \beta \) in \( T^8/\Gamma \), the \( b_3 \) is unchanged but 1 is added to the \( b_2 \). Also 3 is added to each of \( b_4^\pm \). For each of the 2 \( T^4 \) fixed by \( \gamma \)-action and 2 fixed \( T^4 \) by \( \delta \) in \( T^8/\Gamma \), there are contributions 1 to \( b_2 \) and 3 to each \( b_4^\pm \). When we resolve each of the 64 points in the intersection of the 4 \( \alpha \)-fixed sets \( T^4/\{\pm 1\} \) and the 4 \( \beta \)-fixed \( T^4/\{\pm 1\} \)'s, this operation does not change \( b_2, b_3 \) and \( b_4^- \) but adds 1 to \( b_4^+ \). By
collecting all the contributions, we obtain the Betti numbers of the $b_ℓ(M^{(8)})$

\begin{align*}
  b_0 = b_8 &= 1, \quad b_1 = b_7 = 0, \\
  b_2 = b_6 &= 12, \quad b_3 = b_5 = 16, \quad b_4 = 150, \\
  b^+_4 &= 107, \quad b^-_4 = 43.
\end{align*}

In this model, the moduli space of holonomy $\text{Spin}(7)$ metrics on $M^{(8)}$ is a smooth manifold of dimension $1 + b^-_4 = 44$.

## 5.2 Modular invariant partition function

In this subsection, we write down the partition functions of the orbifold string models explained in the previous subsection. In this paper, we work in light-cone gauge.

### 5.2.1 $G_2$ holonomy manifold case

First, we consider the $G_2$ compactification. In this case, we set $x^1, \ldots, x^7$ to be the coordinates of the $G_2$ manifold, and $x^8$ to be the transverse direction of the flat spacetime.

In our model, only one of the two $E_8$ has information about the holonomy group, and the other $E_8$ does not have any contribution of holonomy group of the internal manifold. We denote the $E_8$ including the holonomy group as $E_8^{(1)}$ and the other as $E_8^{(2)}$. We describe $E_8^{(1)}$ by 16 free fermions $\tilde{\lambda}^1, \ldots, \tilde{\lambda}^{16}$. Among them, $\tilde{\lambda}^1, \ldots, \tilde{\lambda}^7$ are orbifolded in the same way as the left-moving fermions $\psi^1, \ldots, \psi^7$, and others are not orbifolded. Therefore, the $\hat{s}_0(9)$ of $\lambda^1, \ldots, \lambda^{16}$ is manifestly realized.

The orbifold partition functions $Z(\tau, \bar{\tau})$ generally have the following form

\begin{equation}
  Z(\tau, \bar{\tau}) = \frac{1}{|Γ|} \sum_{g_1, g_2 \in Γ} Z_{g_1, g_2}(\tau, \bar{\tau}), \quad Z_{g_1, g_2}(\tau, \bar{\tau}) = \text{Tr} \left[ g_2 q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right],
\end{equation}

where $q = \exp(2\pi i \tau)$. Also $\tau$ is the modulus of the toroidal worldsheet. The subscript $g_1$ represents twisted boundary condition along the $σ_1$ (spatial) direction on the worldsheet. On the other side, the $g_2$ expresses the boundary condition along the temporal direction on the worldsheet. The $(L_0 - c/24, \bar{L}_0 - \bar{c}/24)$ is the set of Hamiltonians on the left- and right-moving parts in our heterotic string with $(c, \bar{c}) = (12, 24)$. To be modular invariant, the following modular properties should be satisfied

\begin{equation}
  Z_{g_1, g_2}(-1/τ) = Z_{g_2, g_1^{-1}}(τ), \quad Z_{g_1, g_2}(τ + 1) = Z_{g_1, g_2}(τ).
\end{equation}

In our case, each $Z_{g_1, g_2}$ can be decomposed into several blocks and can be written as a product of them

\begin{equation}
  Z_{g_1, g_2}(\tau, \bar{\tau}) = Z^{(\text{flat boson})}(\tau, \bar{\tau}) Z^{(B)}_{g_1, g_2}(\tau, \bar{\tau}) \times Z^{(F)}_{g_1, g_2}(\tau) \times (\lambda_{g_1, g_2}(τ) \lambda_{\text{bas}}(τ)).
\end{equation}
In this formula, $Z^{(\text{flat boson})}$ is the partition function of a single boson $x^8$; $Z^{(\text{flat boson})}(\tau, \bar{\tau}) = (\text{Im}\tau)^{-1/2}|\eta(\tau)|^{-2}$. $Z^{(B)}_{g_1,g_2}$ is the partition function of the bosons $x^1, \ldots, x^7$ in the $g_1$-twisted sector with $g_2$-insertion. This part describes the internal $G_2$ manifold. The $Z^{(B)}_{g_1,g_2}$‘s themselves satisfy the modular properties (5.2). Also, $Z^{(F)}_{g_1,g_2}(\tau)$ is the character of the left-moving fermions $\psi^1, \ldots, \psi^7$ of $g_1$-twisted sector with $g_2$-insertion. As a result of spacetime supersymmetry, each of $Z^{(F)}_{g_1,g_2}(\tau)$’s vanishes. Next we consider structures on the right-moving part. $\chi^{E_8}_{g_1,g_2}(\tau)$ is the character of $E_8$ in the $g_1$-twisted sector with $g_2$-insertion. $\chi^{E_8}_{\text{bas}}(\tau)$ is the $E_8^{(2)}$ character defined as

$$\chi^{E_8}_{\text{bas}}(\tau) = \frac{1}{2\eta(\tau)^8}\left\{\theta_3(\tau)^8 + \theta_4(\tau)^8 + \theta_2(\tau)^8\right\}.$$ 

The explicit formulae of $Z^{(B)}_{g_1,g_2}$, $Z^{(F)}_{g_1,g_2}(\tau)$ and $\chi^{E_8}_{g_1,g_2}(\tau)$ are concretely calculated in our model.

First, we consider the boson sector $Z^{(B)}_{g_1,g_2}$. Our orbifold group does not mix the coordinates one another, so we can concentrate on each $x_i$ separately. We have only to think the following 4 types of twistings

$$(0) : x \to x, \quad (1) : x \to x + \frac{1}{2}, \quad (2) : x \to -x, \quad (3) : x \to \frac{1}{2} - x$$

The (1) -twisted sector differs from untwisted sector by zero-modes. In (1) -twisted sector, the winding number becomes a half integer.

The (2) -twisted sector expresses an anti-periodic boson and it has half integral modes.

The (3) -twisted sector is the same as (2) -twisted sector: when we define $y = \frac{1}{4} - x$, then (3) is rewritten as $y \to -y$.

The (1) -operator insertion contributes $(-1)^n$ where $n$ is the momentum.

The (2) -operator insertion is represented on oscillators $\alpha_n \to -\alpha_n$. For zero-modes, only the zero momentum and zero winding part survives.

The (3) -operator insertion is the same as (2) -operator insertion.

As a result, we obtain the following partition function of a single boson $Z^{(B)}_{ab}(a, b = \ldots
We use the description by free fermions, and the result can be written by

\[
Z^{(B1)}_{(0)(0)} = |\eta(\tau)|^{-2} \sum_{n,w \in \mathbb{Z}} q^{\frac{1}{12}(\frac{n}{R\theta} + \frac{Rw}{2})^2} q^{\frac{1}{12}(\frac{n}{R\theta} - \frac{Rw}{2})^2},
\]

\[
Z^{(B1)}_{(0)(1)} = |\eta(\tau)|^{-2} \sum_{n,w \in \mathbb{Z}} (-1)^n q^{\frac{1}{12}(\frac{n}{\pi + R\theta})^2} q^{\frac{1}{12}(\frac{n}{\pi - R\theta})^2},
\]

\[
Z^{(B1)}_{(1)(0)} = |\eta(\tau)|^{-2} \sum_{n,w \in \mathbb{Z}} q^{\frac{1}{12}(\frac{n}{\theta + R(w+1/2)})^2} q^{\frac{1}{12}(\frac{n}{\theta - R(w+1/2)})^2},
\]

\[
Z^{(B1)}_{(1)(1)} = |\eta(\tau)|^{-2} \sum_{n,w \in \mathbb{Z}} (-1)^n q^{\frac{1}{12}(\frac{n}{\pi + R(w+1/2)})^2} q^{\frac{1}{12}(\frac{n}{\pi - R(w+1/2)})^2},
\]

\[
Z^{(B1)}_{(0)(2)} = Z^{(B1)}_{(0)(3)} = \left| q^{-\frac{1}{12}} \prod_{n=1}^{\infty} (1 + q^n)^{-1} \right|^2 = 2 \left| \frac{\eta(\tau)}{\theta_2(\tau)} \right|,
\]

\[
Z^{(B1)}_{(2)(0)} = Z^{(B1)}_{(3)(0)} = 2 \left| q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}})^{-1} \right|^2 = 2 \left| \frac{\eta(\tau)}{\theta_4(\tau)} \right|,
\]

\[
Z^{(B1)}_{(2)(2)} = Z^{(B1)}_{(3)(2)} = 2 \left| q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^{-1} \right|^2 = 2 \left| \frac{\eta(\tau)}{\theta_3(\tau)} \right|,
\]

\[
Z^{(B1)}_{(1)(1)} = Z^{(B1)}_{(2)(1)} = Z^{(B1)}_{(3)(1)} = Z^{(B1)}_{(2)(3)} = Z^{(B1)}_{(3)(3)} = 0.
\]

The \( \eta \) is the Dedekind’s eta function and \( \theta_i ’s \) (\( i = 2, 3, 4 \)) represent Jacobi’s theta functions. These \( Z^{(B1)}_{ab}’s \) satisfy the modular properties in Eqs.(5.2).

By using these results, the total bosonic part of the partition function \( Z^{(B)}_{g_1, g_2} \) can be obtained by multiplying these \( Z^{(B1)}_{ab}’s \). For example, we take the \( Z^{(B)}_{ab} \) concretely. Since \( \alpha \) and \( \beta \) have the following actions

\[
\alpha : \begin{pmatrix}
-1 & -x^1 \\
-2 & -x^2 \\
-3 & -x^3 \\
-4 & -x^4 \\
-5 & x^5 \\
-6 & x^6 \\
-7 & x^7
\end{pmatrix},
\]

\[
\beta : \begin{pmatrix}
-1 & -x^1 \\
-\frac{1}{2} & -x^2 \\
-3 & -x^3 \\
-4 & -x^4 \\
-5 & -x^5 \\
-6 & -x^6 \\
-7 & x^7
\end{pmatrix},
\]

the \( x^1 \)-sector produces \( Z^{(B)}_{(2)(2)} \) and the \( x^2 \)-sector produces \( Z^{(B1)}_{(2)(3)} \) and so on. Consequently, \( Z^{(B)}_{\alpha, \beta} \) becomes a product of \( Z^{(B1)}_{a,b}’s \) for each \( x^i, \ (i = 1, 2, \ldots, 7) \)

\[
Z^{(B)}_{\alpha, \beta} = Z^{(B1)}_{(2)(2)} Z^{(B1)}_{(2)(3)} Z^{(B1)}_{(2)(0)} Z^{(B1)}_{(0)(2)} Z^{(B1)}_{(0)(2)} Z^{(B1)}_{(0)(0)}.
\]

Next, we are going to write down \( \chi^{E_{g_1, g_2}} \). This part is essential for the spacetime gauge symmetry. We use the description by free fermions, and the result can be written by using five types of functions \( \chi^{E_{1,1}}, \chi^{E_{1,2}}, \chi^{E_{a,1}}, \chi^{E_{a,a}}, \chi^{E_{a,\gamma}} \). We can write down their explicit
This part is essential for the spacetime supersymmetry. We can construct this formulae by using theta functions where we use $\theta$ functions which are not defined in Eqs.(5.5) are determined by the following rules

$$
\chi^{Es}_{\alpha, \alpha} = \frac{1}{2\eta^8} \left\{ \theta_3^6 \theta_4^6 + \theta_4^6 \theta_3^6 - \theta_2^6 (\theta_1)^2 - (-i \theta_1)^6 \theta_2^2 \right\},
$$

$$
\chi^{Es}_{\alpha, \gamma} = \frac{1}{2\eta^8} \left\{ \theta_3^6 \theta_2^2 + \theta_2^6 \theta_3^2 + \theta_4^6 (\theta_1)^2 + (-i \theta_1)^6 \theta_2^2 \right\},
$$

$$
\chi^{Es}_{\gamma, \alpha} = \frac{1}{2\eta^8} \left\{ \theta_3^6 \theta_2^2 (\theta_1)^2 + \theta_2^6 (\theta_1)^2 \theta_3 + \theta_4^6 (\theta_1)^2 \theta_2 + (-i \theta_1)^5 \theta_2^2 \theta_3 \right\}.
$$

The general $\chi^{Es}_{g_1, g_2}$'s which are not defined in Eqs.(5.3) are determined by the following rules

$$
\chi^{Es}_{g_1, g_2} = \begin{cases} 
\chi^{Es}_{\alpha, \alpha} (g_1 = 1, g_2 \neq 1) \\
\chi^{Es}_{\alpha, \gamma} (g_1 \neq 1, g_2 = 1) \\
\chi^{Es}_{\gamma, \alpha} (g_1 = g_2 \neq 1) \\
\chi^{Es}_{\gamma, \gamma} (g_1 \neq g_2, g_1 \neq 1, g_2 \neq 1)
\end{cases},
$$

where the functions on the right-hand side are defined by Eqs.(5.5).

To check the modular invariance of the whole partition function, we need the modular transformation properties of the above functions. The modular properties of these functions are obtained by using the modular properties of theta functions in appendix A. For S transformation, these $\chi^{Es}$'s transform as

$$
\chi^{Es}_{1, 1}(-1/\tau) = \chi^{Es}_{1, 1}(\tau), \quad \chi^{Es}_{1, \alpha}(-1/\tau) = \chi^{Es}_{1, \alpha}(\tau),
$$

$$
\chi^{Es}_{\alpha, 1}(-1/\tau) = \chi^{Es}_{1, \alpha}(\tau), \quad \chi^{Es}_{\alpha, \alpha}(-1/\tau) = -\chi^{Es}_{\alpha, \alpha}(\tau), \quad \chi^{Es}_{\alpha, \gamma}(-1/\tau) = e \left[ \frac{1}{4} \right] \chi^{Es}_{\gamma, \alpha}(\tau),
$$

(5.6)

where we use $e[x] := \exp(2\pi i x)$. On the other hand, for the T transformation, they transform as

$$
\chi^{Es}_{1, 1}(\tau + 1) = e \left[ \frac{1}{3} \right] \chi^{Es}_{1, 1}(\tau), \quad \chi^{Es}_{1, \alpha}(\tau + 1) = e \left[ \frac{1}{3} \right] \chi^{Es}_{1, \alpha}(\tau),
$$

$$
\chi^{Es}_{\alpha, 1}(\tau + 1) = e \left[ \frac{1}{12} \right] \chi^{Es}_{\alpha, \alpha}(\tau), \quad \chi^{Es}_{\alpha, \alpha}(\tau + 1) = e \left[ \frac{1}{12} \right] \chi^{Es}_{\alpha, 1}(\tau), \quad \chi^{Es}_{\alpha, \gamma}(\tau + 1) = e \left[ \frac{1}{12} \right] \chi^{Es}_{\alpha, \gamma}(\tau).
$$

(5.7)

Finally, we construct the left-moving fermionic part of the partition function $Z^{(F)}_{g_1, g_2}$. This part is essential for the spacetime supersymmetry. We can construct this $Z^{(F)}_{g_1, g_2}$ in the same way as $\chi^{Es}_{g_1, g_2}$ case. As constituent blocks, we need to write five types of partition
functions $Z^{(F)}_{1,1}, Z^{(F)}_{1,\alpha}, Z^{(F)}_{\alpha,1}, Z^{(F)}_{\alpha,\alpha}, Z^{(F)}_{\alpha,\gamma}$. We can evaluate these functions concretely and the results are expressed as

\begin{align*}
Z^{(F)}_{1,1} &= \frac{1}{2\eta^4} \{\theta_3^4 - \theta_4^4 - \theta_2^4 + (-i\theta_1)^4\}, \\
Z^{(F)}_{1,\alpha} &= \frac{1}{2\eta^4} \{\theta_2^2\theta_4^2 - \theta_3^2\theta_3^2 + \theta_2^2(-i\theta_1)^2 - (-i\theta_1)^2\theta_2^2\}, \\
Z^{(F)}_{\alpha,1} &= \frac{1}{2\eta^4} \{\theta_2^2\theta_2^2 - \theta_3^2\theta_3^2 - \theta_1^2(-i\theta_1)^2 + (-i\theta_1)^2\theta_1^2\}, \\
Z^{(F)}_{\alpha,\alpha} &= \frac{1}{2\eta^4} \{\theta_2^2\theta_2^2 + \theta_3^2\theta_3^2 + \theta_3^2(-i\theta_1)^2 - (-i\theta_1)^2\theta_3^2\}, \\
Z^{(F)}_{\alpha,\gamma} &= \frac{i}{\eta^4} \{\theta_3\theta_4\theta_2(-i\theta_1) - \theta_2(-i\theta_1)\theta_3\theta_4 - \theta_4\theta_3(-i\theta_1)\theta_2 + (-i\theta_1)\theta_2\theta_4\theta_3\}. \quad (5.8)
\end{align*}

Each of these functions actually vanishes because of the spacetime supersymmetry.

The general $Z^{(F)}_{g_1,g_2}$'s which are not defined in Eqs. (5.8) can be written as the same way as the $\chi^{Es}_{g_1,g_2}$ case. They are determined by the following rules

\[
Z^{(F)}_{g_1,g_2} = \begin{cases}
Z^{(F)}_{1,\alpha} & (g_1 = 1, \ g_2 \neq 1) \\
Z^{(F)}_{\alpha,1} & (g_1 \neq 1, \ g_2 = 1) \\
Z^{(F)}_{\alpha,\alpha} & (g_1 = g_2 \neq 1) \\
Z^{(F)}_{\alpha,\gamma} & (g_1 \neq g_2, \ g_1 \neq 1, \ g_2 \neq 1)
\end{cases}.
\]

We also need the modular properties of these functions in Eqs. (5.8) to check the modular invariance of the whole partition function. For the S transformation, they transform as

\begin{align*}
Z^{(F)}_{1,1}(-1/\tau) &= Z^{(F)}_{1,1}(\tau), \quad Z^{(F)}_{1,\alpha}(-1/\tau) = Z^{(F)}_{\alpha,1}(\tau), \\
Z^{(F)}_{\alpha,1}(-1/\tau) &= Z^{(F)}_{1,\alpha}(\tau), \quad Z^{(F)}_{\alpha,\alpha}(-1/\tau) = -Z^{(F)}_{\alpha,\alpha}(\tau), \quad Z^{(F)}_{\alpha,\gamma}(-1/\tau) = e^{\frac{1}{4}} Z^{(F)}_{\gamma,\alpha}(\tau). \quad (5.9)
\end{align*}

On the other hand, for the T transformation, they transform as

\begin{align*}
Z^{(F)}_{1,1}(\tau + 1) &= e^{\frac{1}{3}} Z^{(F)}_{1,1}(\tau), \quad Z^{(F)}_{1,\alpha}(\tau + 1) = e^{\frac{1}{3}} Z^{(F)}_{1,\alpha}(\tau), \\
Z^{(F)}_{\alpha,1}(\tau + 1) &= e^{-\frac{5}{12}} Z^{(F)}_{\alpha,1}(\tau), \quad Z^{(F)}_{\alpha,\alpha}(\tau + 1) = e^{-\frac{5}{12}} Z^{(F)}_{\alpha,\alpha}(\tau), \quad Z^{(F)}_{\alpha,\gamma}(\tau + 1) = e^{-\frac{5}{12}} Z^{(F)}_{\alpha,\gamma}(\tau). \quad (5.10)
\end{align*}

Gathering these results, we can write down the $Z_{g_1,g_2}$ in Eq.(5.3). Also we can check that the $Z_{g_1,g_2}$ actually satisfy the modular properties (5.2) by using the modular properties (5.6), (5.7), (5.9), (5.10) and we can conclude the partition function is modular invariant.
5.2.2 Spin(7) holonomy manifold case

Now, we turn to construct the modular invariant partition function of the Spin(7) example. It is almost parallel to the case of $G_2$. In Spin(7) case, there are no transverse directions of the flat spacetime, and $Z_{g_1,g_2}$ can be decomposed as

$$Z_{g_1,g_2}(\tau, \bar{\tau}) = Z_{g_1,g_2}^{(B)}(\tau, \bar{\tau}) \times Z_{g_1,g_2}^{(F)}(\tau) \times (\chi_{g_1,g_2} \chi_{\text{bas}}(\tau)).$$  \hspace{1cm} (5.11)

The boson part $Z_{g_1,g_2}^{(B)}$ is constructed as in the $G_2$ case. For example, $Z_{\alpha,\gamma}^{(B)}$ becomes a product of each $Z^{(B1)}$'s

$$Z_{\alpha,\gamma}^{(B)} = Z^{(B1)}_{(2)(3)} Z^{(B1)}_{(2)(0)} Z^{(B1)}_{(2)(0)} Z^{(B1)}_{(0)(3)} Z^{(B1)}_{(0)(0)} Z^{(B1)}_{(0)(0)},$$

where $Z^{(B1)}$'s are single boson partition functions in Eqs.(5.4).

In order to write down the $Z_{g_1,g_2}^{(F)}$ and $\chi_{g_1,g_2}$, let us note $\alpha \beta = -1$. We also write $-1$. $g = -g \in \Gamma$. First, let us consider $\chi_{g_1,g_2}$. We need six new types of functions which do not appear in Eqs.(5.3). These functions are $\chi_{\alpha,-\alpha}, \chi_{\alpha,-1}, \chi_{-\alpha,\alpha}, \chi_{1,-1}, \chi_{-1,1}, \chi_{-1,-1}$. The explicit forms of them are expressed as

$$\chi_{\alpha,-\alpha}^{E_8} = \frac{1}{2\eta} \left\{ \theta_3^4 \theta_2^2 \theta_4 - \theta_2^4 \theta_3^2 (-i \theta_1)^2 + \theta_4^4 (-i \theta_1)^2 \theta_3^2 - (-i \theta_1)^4 \theta_2^2 \theta_3^2 \right\},$$

$$\chi_{\alpha,-1}^{E_8} = \frac{1}{2\eta} \left\{ \theta_2^4 \theta_3^2 \theta_4^2 + \theta_2^4 \theta_3^2 (-i \theta_1)^2 + \theta_4^4 (-i \theta_1)^2 \theta_2^2 + (-i \theta_1)^4 \theta_3^2 \theta_2^2 \right\},$$

$$\chi_{-1,\alpha}^{E_8} = \frac{1}{2\eta} \left\{ \theta_2^4 \theta_3^2 \theta_4^2 - \theta_2^4 \theta_3^2 (-i \theta_1)^2 - \theta_4^4 (-i \theta_1)^2 \theta_2^2 - (-i \theta_1)^4 \theta_3^2 \theta_2^2 \right\},$$

$$\chi_{1,-1}^{E_8} = \frac{1}{2\eta} \left\{ \theta_3^4 \theta_4^2 + \theta_4^4 \theta_3^2 + \theta_2^4 (-i \theta_1)^4 + (-i \theta_1)^4 \theta_2^4 \right\},$$

$$\chi_{-1,1}^{E_8} = \frac{1}{2\eta} \left\{ \theta_3^4 \theta_4^2 + \theta_4^4 \theta_3^2 + \theta_2^4 (-i \theta_1)^4 + (-i \theta_1)^4 \theta_2^4 \right\},$$

$$\chi_{-1,-1}^{E_8} = \frac{1}{2\eta} \left\{ \theta_4^4 \theta_2^4 + \theta_2^4 \theta_4^2 + \theta_3^4 (-i \theta_1)^4 + (-i \theta_1)^4 \theta_3^2 \right\}.$$  \hspace{1cm} (5.12)

The general $\chi_{g_1,g_2}^{E_8}$'s which are not in Eqs.(5.5), (5.12) are defined by using the functions in Eqs.(5.3) and (5.12). The general $\chi_{g_1,g_2}^{E_8}$ are defined as

$$\chi_{g_1,g_2}^{E_8} = \begin{cases} 
\chi_{1,\alpha}^{E_8} & (g_1 = 1, \ g_2 \neq 1) \\
\chi_{-1,\alpha}^{E_8} & (g_1 = -1, \ g_2 \neq 1) \\
\chi_{\alpha,1}^{E_8} & (g_1 \neq 1, \ g_2 = 1) \\
\chi_{\alpha,-1}^{E_8} & (g_1 \neq 1, \ g_2 = -1) \\
\chi_{\alpha,\alpha}^{E_8} & (g_1 = g_2 \neq 1) \\
\chi_{\alpha,-\alpha}^{E_8} & (g_1 = -g_2 \neq \pm 1) \\
\chi_{\alpha,\gamma}^{E_8} & (\text{others}). \end{cases}$$
We also need the modular properties of six functions introduced in Eq. (5.12) to check the modular invariance of the whole partition function. For the S transformation, they transform as

\[
\begin{align*}
\chi_{\alpha,-\alpha}^E(1/\tau) &= \chi_{\alpha,\alpha}^E(\tau), \\
\chi_{\alpha,-1}^E(1/\tau) &= \chi_{-1,\alpha}^E(\tau), \\
\chi_{1,-1}^E(1/\tau) &= \chi_{1,1}^E(\tau), \\
\chi_{\alpha,-\alpha}^E(1/\tau) &= \chi_{\alpha,-1}^E(\tau), \\
\chi_{-1,\alpha}^E(1/\tau) &= \chi_{\alpha,-1}^E(\tau), \\
\chi_{1,1}^E(1/\tau) &= \chi_{1,-1}^E(\tau).
\end{align*}
\]

(5.13)

On the other hand, for the T transformation they behave as

\[
\begin{align*}
\chi_{\alpha,-\alpha}^E(\tau + 1) &= e^{\frac{1}{12}} \chi_{\alpha,-\alpha}^E(\tau), \\
\chi_{\alpha,-1}^E(\tau + 1) &= e^{\frac{1}{6}} \chi_{\alpha,-1}^E(\tau), \\
\chi_{1,-1}^E(\tau + 1) &= e^{\frac{1}{3}} \chi_{1,-1}^E(\tau), \\
\chi_{\alpha,-\alpha}^E(\tau + 1) &= e^{\frac{1}{12}} \chi_{\alpha,-\alpha}^E(\tau), \\
\chi_{-1,\alpha}^E(\tau + 1) &= e^{\frac{1}{6}} \chi_{-1,\alpha}^E(\tau), \\
\chi_{1,1}^E(\tau + 1) &= e^{\frac{1}{3}} \chi_{1,1}^E(\tau).
\end{align*}
\]

(5.14)

As for the left-moving fermion part, we need six new types of the functions. We need explicit forms of \( Z_{\alpha,-\alpha}^{(F)}, Z_{\alpha,-1}^{(F)}, Z_{-1,\alpha}^{(F)}, Z_{1,-1}^{(F)}, Z_{-1,1}^{(F)}, Z_{1,-1}^{(F)} \), and they are written as

\[
\begin{align*}
Z_{\alpha,-\alpha}^{(F)} &= \frac{1}{2\eta^4} \{ \theta_3^2 \theta_2^2 + (-i\theta_1)^2 \theta_3^2 - \theta_3^2 (-i\theta_1)^2 - \theta_2^2 \theta_3^2 \}, \\
Z_{\alpha,-1}^{(F)} &= \frac{1}{2\eta^4} \{-\theta_3^2 \theta_2^2 + (-i\theta_1)^2 \theta_4^2 + \theta_2^2 (-i\theta_1)^2 + \theta_2^2 \theta_3^2 \}, \\
Z_{-1,\alpha}^{(F)} &= \frac{1}{2\eta^4} \{-\theta_3^2 \theta_4^2 + (-i\theta_1)^2 \theta_2^2 - \theta_2^2 (-i\theta_1)^2 + \theta_2^2 \theta_3^2 \}, \\
Z_{1,-1}^{(F)} &= Z_{-1,1}^{(F)} = \frac{1}{2\eta^4} \{ \theta_4^4 - \theta_3^4 - (-i\theta_1)^4 + \theta_2^4 \}, \\
Z_{1,-1}^{(F)} &= Z_{-1,1}^{(F)} = \frac{1}{2\eta^4} \{ \theta_3^4 - \theta_4^4 - (-i\theta_1)^4 + \theta_2^4 \}.
\end{align*}
\]

(5.15)

We introduce the general \( Z_{g_1,g_2}^{(F)} \)'s which are not in Eqs. (5.8) and (5.13). Each of these
are the same functions as in Eqs. (\ref{eq:5.8}), (\ref{eq:5.13}). They can be defined as

\[
Z^{(F)}_{g_1,g_2} = \begin{cases} 
Z^{(F)}_{1,\alpha} (g_1 = 1, g_2 \neq 1) \\
Z^{(F)}_{-1,\alpha} (g_1 = -1, g_2 \neq 1) \\
Z^{(F)}_{\alpha,1} (g_1 \neq 1, g_2 = 1) \\
Z^{(F)}_{\alpha,-1} (g_1 \neq 1, g_2 = -1) \\
Z^{(F)}_{\alpha,\alpha} (g_1 = g_2 \neq 1) \\
Z^{(F)}_{\alpha,\gamma} (g_1 = -g_2 \neq \pm 1) \\
\end{cases}
\]

Here, we write down the modular properties of the functions in Eqs. (\ref{eq:5.15}), which are needed to check the modular invariance of the partition function. For the S transformation, they transform as

\[
Z^{(F)}_{\alpha,-1}(-1/\tau) = Z^{(F)}_{-\alpha,\alpha}(\tau), \\
Z^{(F)}_{\alpha,-1}(-1/\tau) = Z^{(F)}_{-\alpha,1}(\tau), \\
Z^{(F)}_{\alpha,-1}(-1/\tau) = Z^{(F)}_{\alpha,-1}(\tau), \\
Z^{(F)}_{1,-1}(-1/\tau) = Z^{(F)}_{1,-1}(\tau), \\
Z^{(F)}_{-1,-1}(-1/\tau) = Z^{(F)}_{-1,-1}(\tau).
\]

\label{eq:5.16}

For the T transformation, they behave as

\[
Z^{(F)}_{\alpha,-1}(\tau + 1) = e^{\frac{-5}{12}} Z^{(F)}_{\alpha,-1}(\tau), \quad Z^{(F)}_{\alpha,-1}(\tau + 1) = e^{\frac{-5}{12}} Z^{(F)}_{\alpha,-1}(\tau), \\
Z^{(F)}_{-1,\alpha}(\tau + 1) = e^{\frac{-1}{6}} Z^{(F)}_{-1,\alpha}(\tau), \quad Z^{(F)}_{-1,\alpha}(\tau + 1) = e^{\frac{-1}{6}} Z^{(F)}_{-1,\alpha}(\tau), \\
Z^{(F)}_{1,-1}(\tau + 1) = e^{\frac{1}{3}} Z^{(F)}_{1,-1}(\tau), \quad Z^{(F)}_{1,-1}(\tau + 1) = e^{\frac{1}{3}} Z^{(F)}_{1,-1}(\tau), \\
Z^{(F)}_{-1,-1}(\tau + 1) = e^{\frac{-1}{6}} Z^{(F)}_{-1,-1}(\tau).
\]

\label{eq:5.17}

The partition function constructed from these constituent blocks satisfies the equations (\ref{eq:5.2}). It can be checked by using the modular properties (\ref{eq:5.6}), (\ref{eq:5.7}), (\ref{eq:5.9}), (\ref{eq:5.10}), (\ref{eq:5.13}), (\ref{eq:5.14}), (\ref{eq:5.16}), (\ref{eq:5.17}).

5.3 massless sector

In this subsection we will investigate massless spectra of the compactified models. The conformal dimension of a field in the whole theory is a sum of weights in each constituent CFT. The total weight on the theory is labelled by a set \((h^{\text{tot}}, \bar{h}^{\text{tot}})\) and can be written
down as

\[
\text{left } (N = 1) ; \ h^{\text{tot}} = h^M + h^{SO(d-2)} - \frac{12}{24} = 0 ,
\]

\[
\text{right } (N = 0) ; \ h^{\text{tot}} = \bar{h}^M + \bar{h}^G + \bar{h}^{E_8} - \frac{24}{24} = 0 ,
\]

\[
\rightarrow \bar{h}^M + \bar{h}^{G_0} + \bar{h}^{E_8} = 1 , \ h^M + h^{SO(d-2)} = \frac{1}{2},
\]

where \((h^M, \bar{h}^M)\) expresses a set of weights in the extended CFT for \(M\) and \(\bar{G}_0\), \(\bar{h}^{E_8}\), \(h^{SO(d-2)}\) are respectively conformal dimensions associated with affine Lie algebras \((\hat{G}_0)_1\), \((\hat{E}_8)_1\), \(\hat{so}(d - 2)_1\) \((d \geq 3)\). For the \(d = 3\) case we formally interpret the part “\(\hat{so}(d - 2)_1\)” as a current generated by a free fermion. In the case of \(d = 2\) this part does not appear and we set \(h^{SO(d-2)} = 0\).

As a first case we take gauge singlet states with conditions \((d \geq 3)\)

\[
\bar{h}^M = 1 , \ h^M + h^{SO(d-2)} = \frac{1}{2}.
\]

The \(h^\)'s are determined by representations of \(so(d-2)\) and can be classified in the following table \[3\]

In the table \[3\] we study models with spacetime transverse dimensions and the \(NS\) and \(R\)

\[d = \text{even} \] case

| rep. \(h^{so(d-2)}\) | bas | vec | spi | cos |
|----------------------|-----|-----|-----|-----|
| \(h^M\)             | \(\frac{1}{2}\) | \(0\) | \(\frac{d-2}{16}\) | \(\frac{d-2}{16}\) |
| sector              | \(NS\) | \(NS\) | \(R\) | \(R\) |

\[d = \text{odd} \] case

| rep. \(h^{so(d-2)}\) | bas | vec | spi |
|----------------------|-----|-----|-----|
| \(h^M\)             | \(\frac{1}{2}\) | \(0\) | \(\frac{10-d}{16}\) |
| sector              | \(NS\) | \(NS\) | \(R\) |

Table 6: Classifications of representations for \(SO(d-2)\) algebra.

distinguish sectors of susy states in the worldsheet theories. For the \(d = 2\) case a condition \(h^M = 1/2\) should be satisfied. By considering these conditions we can determine massless
fields \((d > 3)\) in this sector after GSO projections

\[
\psi^\nu_{-1/2} \tilde{\alpha}^\mu; \text{graviton, 2nd rank antisymmetric field, dilaton,} \\
S^\alpha \tilde{\alpha}^\mu_1; \text{gravitino, dilatino}.
\]

These represent an \(N = 1\) multiplet of supergravity. For the \(d = 3\) case, the excitations of the gravity and gravitino disappear after imposing on-shell conditions.

In the \(d = 2\) case transverse dimension of the spacetime vanishes and local excitations of graviton and \(B_{\mu\nu}\) do not exist. However a pair of dilaton and dilatino appears as its field content. For that case a set of weights is fixed to be \((h^M, \bar{h}^M) = (1, 1/2)\) and could be classified by states of the CFT associated with the internal manifold \(M^{(8)}\).

Secondly we consider the \(\bar{h}^{E_8} = 1\) part. The corresponding states are easily understood to be gauge fields and their superpartners with gauge symmetry in the hidden sector \(E_8\)

\[
\psi^\mu_{-1/2} \bar{J}_A^1, \ S^\alpha \bar{J}_A^1; \\
\bar{J}_A^1; E_8 \text{ current}.
\]

These fields are singlets with respects to the \(G_0\) group.

In the case of \(\bar{h}^{G_0} = 1\) the corresponding states are gauge fields with spacetime visible gauge symmetry \(G_0\). These transform as adjoint fields under this symmetry \(G_0\) and are identified with a set of an \(N = 1\) gauge multiplet

\[
\psi^\mu_{-1/2} \bar{J}_A^1, \ S^\alpha \bar{J}_A^1; \\
\bar{J}_A^1; G_0 \text{ current}.
\]

Next we shall study the \(E_8\) singlet matters with \(h^{E_8} = 0\). The right-moving part has an affine \(G_0\) current and the states are classified by its representations. On the other hand the left-movers have \(SO(d - 2)\) symmetry and its chiral states are labelled by representations of this group. We will concentrate on the \(d = 2, 3\) cases here. The \(d = 3\) case is realized through compactification on the \(G_2\) manifold with \(D = 7\). The right- and left-chiral states are respectively characterized by representations of \(G_0 = SO(9)\) and a free fermion \(\psi\). They are summarized in the table 7

Here the \("bas\", \("vec\", \("spi\) express respectively trivial, vector, spinor representations of \(SO(9)\) and \("tri-Ising\) means scaling operators of the associated tricritical Ising model. Also the \(\bar{h}^M\)’s can be decomposed as sums of pairs of weights \((\bar{h}^{Tri}, \bar{h}^r)\) of \((T^{Tri}, T^r)\). These states are collected into multiplets with a representation 26 of \(F_4\)

\[F_4 \supset SO(9) \times \text{ (tricritical Ising)} , \]

\[26 = 1_{bas} + 9_{vec} + 16_{spi} .\]
The left-part is classified in terms of the transverse fermion $\psi$ in the spacetime. The $h^M$'s are decomposed by the weights of the chiral fields of the tricritical Ising model as in table B.

$$
\begin{array}{|c|c|c|c|c|}
\hline
h^\psi & h^M & (h^{\text{Tri}}, h^r) & \text{tri-Ising sector} \\
\hline
0 & 1/2 & (1/10, 2/5) & \epsilon \\
1/2 & 0 & (0, 0) & 1 \\
1/16 & 7/16 & (3/80, 2/5) & \sigma \\
1/16 & 7/16 & (7/16, 0) & \sigma' \\
\hline
\end{array}
$$

Table 8: Left-moving part for $G_2$ case and its classification by $\psi$.

Now we are ready to write down spectra of the associated fields by gluing left- and right-parts together. We put them in the table C.

These states are $N = 1$ $F_4$ fundamental multiplets and transform as a representation

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{state} & \text{SO(9)} & h^\psi & F_4 & \#\text{multiplet} & (h, \bar{h}) & ((-1)^F, (-1)^{\bar{F}}) \\
\hline
(\frac{3}{5}, \frac{2}{5}) L (\frac{1}{10}, \frac{2}{5}) R & 1 & 0 & 26 & b_2 + b_4 & (1, \frac{7}{16}) & (+, -) \\
(\frac{1}{10}, \frac{2}{5}) L (\frac{1}{10}, \frac{2}{5}) R & 9 & 0 & & (\frac{1}{2}, \frac{1}{2}) & (-, -) \\
(\frac{3}{80}, \frac{2}{5}) L (\frac{1}{10}, \frac{2}{5}) R & 16 & 0 & & (\frac{7}{16}, \frac{1}{2}) & (-, -) \\
(\frac{3}{5}, \frac{2}{5}) L (\frac{3}{80}, \frac{2}{5}) R & 1 & 1/16 & b_2 + b_4 & (1, \frac{7}{16}) & (+, \pm) \\
(\frac{1}{10}, \frac{2}{5}) L (\frac{3}{80}, \frac{2}{5}) R & 9 & 1/16 & 26 & (\frac{1}{2}, \frac{7}{16}) & (-, \pm) \\
(\frac{3}{80}, \frac{2}{5}) L (\frac{3}{80}, \frac{2}{5}) R & 16 & 1/16 & & (\frac{7}{16}, \frac{7}{16}) & (+, \pm) \\
\hline
\end{array}
$$

Table 9: spectra ($d = 3$ heterotic theory on $G_2$ manifold)

26 of gauge group $F_4$. The number of these multiplets is evaluated by noticing the state $(\frac{3}{80}, \frac{2}{5}) L (\frac{3}{80}, \frac{2}{5}) R$. It is related with the string moduli space with $G_2$ manifold and its number is equal to the dimension of the $\mathcal{M}_{\text{CFT}}$, that is, $\dim \mathcal{M}_{\text{CFT}} = b_2 + b_3 = b_2 + b_4$. 

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In fact there are $b_2 + b_3 = 55$ fundamental 26-multiplets in our orbifold model. That illustrates the enhancement of the gauge symmetry from $G_0 = SO(9)$ to $G = F_4$. Next we shall recall there are adjoint fields with a representation 36 under $SO(9)$. They are combined into an adjoint 52-representation of $F_4$ together with 16-matter fields of $SO(9)$. Furthermore there are many gauge singlet states. We do not touch on details of these singlets here.

When one compactifies string theory on the $Spin(7)$ manifold, the transverse dimension is $d-2 = 0$ and there are no transverse excitations. In our light-cone formula it seems meaningless to discuss matter contents for this case. But we will explain associated left- and right-parts formally for mathematical interests. For simplicity we concentrate on the $\tilde{h}^{Es} = 0$ sector. Then formal massless sectors are classified according to the representation of the gauge symmetry $G_0 = SO(8)$ in the right-part. We show them in table 10.

| $SO(8)$ | $\tilde{h}^{SO(8)}$ | $\tilde{h}^{Spin(7)}$ | $(\tilde{h}^{Ising}, \tilde{h}^r)$ | Ising |
|---------|---------------------|-----------------------|-----------------------------|-------|
| bas (1) | 0                   | 1                     | (1/2, 1/2)                  | $\epsilon$ |
| vec (8vec) | 1/2               | 1/2                   | (0, 1/2)                    | 1     |
| spi (8spi) | 1/2               | 1/2                   | (1/16, 7/16)               | $\sigma$ |
| cos (8cos) | 1/2               | 1/2                   | (1/16, 7/16)               | $\sigma$ |

Table 10: Right-moving part for $Spin(7)$ case and its classification by $SO(8)$. The $\text{bas, vec, spi, cos}$ express respectively trivial, vector, spinor, cospinor representations of $SO(8)$ and “Ising” means scaling operators of the Ising model. The weights $\tilde{h}^M$’s are decomposed into sums of pairs of $(\tilde{h}^{Ising}, \tilde{h}^r)$. States here are collected into multiplets with representations 9 and 16 of an enhanced gauge symmetry $G = SO(9)$

$$SO(9) \supset SO(8),$$

$$9_{\text{vec}} = 1_{\text{bas}} + 8_{\text{vec}}, \quad 16_{\text{spi}} = 8_{\text{spi}} + 8_{\text{cos}}.$$  

The 9$_{\text{vec}}$ and 16$_{\text{spi}}$ represent transformation properties of matter contents under the SO(9) and express respectively the vector and spinor representations of SO(9).

On the other side the left-part always has weights $h^{Spin(7)} = 1/2$ and states are classified in terms of the chiral internal part in table 11.

By gluing left- and right-parts together we can write down non-chiral states in the table 12.

These states are $N = 1$ SO(9) gauge multiplets and transform as representations 9 (vector) and 16 (spinor). The number of these multiplet is evaluated by comparing the
### Table 11: Classification of chiral ground states for Spin(7) case.

| $h$   | 0   | $1/2$ | 1   |
|-------|-----|-------|-----|
| $NS$  | $|0,0\rangle$ | $|1/16,7/16\rangle$ | $|1/2,1/2\rangle$ |
| $R$   | $|0,1/2\rangle$, $|1/2,0\rangle$, $|1/16,7/16\rangle$ |

### Table 12: spectra ($d = 2$ heterotic theory on Spin(7) manifold)

| states | SO(8) | SO(9) | ≠multiplet | $(h, h)$ | $((-1)^F, (-1)^F)$ |
|--------|-------|-------|------------|----------|-------------------|
| $(\frac{1}{2}, \frac{1}{2})_{NS} (\frac{1}{16}, \frac{7}{16})_{NS}$ | 1 | 8 | $b_3 = b_5$ | $(\frac{1}{2}, \frac{1}{2})$ | (+, −) |
| $(0, \frac{1}{2})_{R} (\frac{1}{16}, \frac{7}{16})_{NS}$ | 8 | 9 | $b_3 = b_5$ | $(\frac{1}{2}, \frac{1}{2})$ | (+, −) |
| $(\frac{1}{16}, \frac{7}{16})_{NS, R} (\frac{1}{16}, \frac{7}{16})_{NS}$ | 8 + 8 | 16 | $1 + b_2 + b_4$ | $(\frac{1}{2}, \frac{1}{2})$ | (−, −) |
| $(\frac{1}{2}, \frac{1}{2})_{NS} (\frac{1}{16}, \frac{7}{16})_{R}$ | 1 | 8 | $b_3 = b_5$ | $(\frac{1}{2}, \frac{1}{2})$ | (+, −) |
| $(0, \frac{1}{2})_{R} (\frac{1}{16}, \frac{7}{16})_{R}$ | 8 | 9 | $b_3 = b_5$ | $(\frac{1}{2}, \frac{1}{2})$ | (+, −) |
| $(\frac{1}{16}, \frac{7}{16})_{NS, R} (\frac{1}{16}, \frac{7}{16})_{R}$ | 8 + 8 | 16 | $1 + b_2 + b_4$ | $(\frac{1}{2}, \frac{1}{2})$ | (−, −) |
| $(\frac{1}{2}, \frac{1}{2})_{NS} (\frac{1}{16}, \frac{7}{16})_{NS}$ | 1 | 8 | $b_6 + b_4^+$ | $(\frac{1}{2}, 1)$ | (+, +) |
| $(0, \frac{1}{2})_{R} (\frac{1}{16}, \frac{7}{16})_{NS}$ | 8 | 9 | $b_6 + b_4^+$ | $(\frac{1}{2}, 1)$ | (+, +) |
| $(\frac{1}{16}, \frac{7}{16})_{NS, R} (\frac{1}{16}, \frac{7}{16})_{NS}$ | 8 + 8 | 16 | $b_3 = b_5$ | $(\frac{1}{2}, 1)$ | (−, +) |
| $(\frac{1}{2}, \frac{1}{2})_{NS} (0, \frac{1}{2})_{R}$ | 1 | 8 | $b_6 + b_4^+$ | $(\frac{1}{2}, \frac{1}{2})$ | (+, +) |
| $(0, \frac{1}{2})_{R} (0, \frac{1}{2})_{R}$ | 8 | 9 | $b_6 + b_4^+$ | $(\frac{1}{2}, \frac{1}{2})$ | (+, +) |
| $(\frac{1}{16}, \frac{7}{16})_{NS, R} (0, \frac{1}{2})_{R}$ | 8 + 8 | 16 | $b_3 = b_5$ | $(\frac{1}{2}, \frac{1}{2})$ | (−, +) |
states in Eq. (2.1). It is related with the Betti numbers of $\text{Spin}(7)$ manifold and they are evaluated in our orbifold model as

\[
\begin{align*}
    b_0 &= b_8 = 1, \\
    b_1 &= b_7 = 0, \\
    b_2 &= b_6 = 12, \\
    b_3 &= b_5 = 16, \\
    b_4 &= 150, \\
    b_4^+ &= 107, \\
    b_4^- &= 43.
\end{align*}
\]

By using these data, we can calculate multiplicities of the $\text{SO}(9)$ matters

\[
\begin{align*}
    \{\text{multiplicity of 9}\} &= b_3 = b_5 = 16, \\
    \{\text{multiplicity of 9}\} &= b_6 + b_4^+ = 119, \\
    \{\text{multiplicity of 16}\} &= b_3 = b_5 = 16, \\
    \{\text{multiplicity of 16}\} &= 1 + b_2 + b_4^- = 56.
\end{align*}
\]

In particular the multiplet of representation 16 has multiplicity $1 + b_2 + b_4^- = 56$. It coincides with the dimension of the string moduli space $\mathcal{M}_{\text{CFT}}$.

6 Conclusions and Discussions

In this paper, we investigated heterotic strings on the exceptional holonomy manifolds by making use of the CFT techniques and found a cascade of special holonomy manifolds with different dimensions. In order to analyze these phenomena, we used standard CFT techniques of branching rules for characters.

We study partition functions of $E_8 \times E_8$ heterotic strings compactified on these manifolds and find that gauge symmetry enhancements are correlated with reductions of holonomies of the internal manifolds.

Gauge symmetry parts are exceptional groups $E_6$, $F_4$ respectively for $CY_3$ and $G_2$ theories and the $\text{Spin}(7)$ theory has an $\text{SO}(9)$ gauge symmetry in 2 dim spacetime.

The criticality condition on the left-moving side of this superstring is equivalent to a relation $d + D = 10$ for dimensions $d$, $D$ of spacetime and internal parts. In addition there are conditions for central charges $c_G$, $c_{\text{hol}}$, $c_{\text{spec}}$ in the gauge sector on the right-moving side $c_G + c_{\text{hol}} = 8$, $D = 2(c_{\text{hol}} + c_{\text{spec}})$. The $c_G$, $c_{\text{hol}}$ correspond to (enhanced) gauge group $G$, holonomy group $G_{\text{hol}}$ and the central charge $c_{\text{spec}}$ is associated with a CFT of a spectral flow operator. They also give us information on division of $E_8$ into the holonomy group $G_{\text{hol}}$ and (enhanced) gauge group $G$.

The essential part of our mechanism originates in two equations about characters

\[
\chi^{E_8} = \chi^{\text{SO}(8)} \times \chi^{\text{SO}(8)}, \quad \chi^{\text{SO}(8)/\text{su}(3)} = \chi^{\text{Ising}} \times \chi^{\text{Tri}} \times \chi^{\text{3-Potts}} = \chi^{\text{SO}(2)} \times \chi^{U(1)}.
\]

By multiplying
each character of $\chi^{\text{Ising}} \times \chi^{\text{Tri}} \times \chi^{\text{3-Potts}}$ one after another to the SO(8) part, we can obtain characters of visible enhanced gauge symmetries SO(9), $F_4$, $E_6$. At the same time holonomy parts are reduced to $\text{Spin}(7)$, $G_2$, $su(3)$ and associated manifolds could be changed. The first is the $\text{Spin}(7)$ holonomy case. The holonomy SO(8) part is decomposed into $\text{Spin}(7)$ in terms of Ising model because of an equation $\chi^{SO(8)} = \chi^{\text{Spin}(7)} \times \chi^{\text{Ising}}$. The second is a reduction from the $\text{Spin}(7)$ to $G_2$ holonomy by throwing away degrees of freedom of the tricritical Ising model. It can be explained by an equation $\chi^{G_2} = \chi^{\text{su}(3)} \times \chi^{\text{3-Potts}}$ including 3-state Potts model. It explains a reduction of holonomy from $G_2$ to $su(3)$, that is, a relation of $G_2$ manifolds and Calabi-Yau 3-folds.

It also changes gauge symmetries in spacetimes from $F_4$ to $E_6$ because we have a relation $\chi^{F_4} \times \chi^{\text{3-Potts}} = \chi^{E_6}$. By noticing the relation $\chi^{SO(9)/su(3)} = \chi^{\text{Ising}} \times \chi^{\text{Tri}} \times \chi^{\text{3-Potts}}$, we can understand the associated CFT has an affine $U(1)$ symmetry needed to enhance the worldsheet $N = 1$ CFT algebra to $N = 2$ conformal algebra of $CY_3$. It can be explained by an identity $\chi^{\text{Ising}} \times \chi^{\text{Tri}} \times \chi^{\text{3-Potts}} = \chi^{SO(2)} \times \chi^{U(1)}$. At the level of balance of central charges, this equation means relations $c = \frac{1}{2} + \frac{7}{10} + \frac{4}{5} = 2 = 1 + 1$.

This $U(1)$ serves as an R-symmetry of the $N = 2$ theory and is used to construct $U(1)$ current of the $N = 2$ algebra. Also a spectral flow operator of the $N = 2$ CFT has conformal dimension 3/8 and is constructed by combining scaling operators of three statistical models. It is realized as a combination of states with $(h^{\text{Ising}}, h^{\text{Tri}}, h^{\text{3-Potts}}) = (1/16, 7/16, 0), (1/16, 3/80, 2/5)$. This operator belongs to a sector with a $U(1)$ charge $Q = 3/2$. It implies that the state is related with a 3-form of the $CY_3$. Also the total weight of these states turns out to be 1/2 and we can obtain a spin operator $\Sigma$ with $h = 3/8(= 1/2 - 1/8)$ of the $su(3)$ holonomy model by subtracting contributions of a spin operator of SO(2) with weight 1/8. This operator $\Sigma$ is nothing but a holomorphic 3-form of the $CY_3$ and confirms the validity of our discussions.

By using this operator $\Sigma$ and combining the 4dim spacetime spin operator $S_\alpha$ together with contribution of ghost part, we can construct a spacetime supercharge $Q_\alpha = \int e^{-\frac{1}{2}\phi} S_\alpha \Sigma$. It guarantees spacetime $\mathcal{N} = 1$ supersymmetry.

It is amazing that we can realize $N = 2$ CFT associated with $CY_3$ starting from SO(8) theory by using three statistical models in 2 dimension. We will make several comments about these models here: The Ising model appears as the first entry (that is, with the lowest central charge) of minimal unitary models with $N = 0$. The tricritical Ising is a second model (with the lowest $c$ but one) in the $N = 0$ minimal series. But it is also a model in the $N = 1$ unitary minimal model with the lowest central charge. Furthermore
the 3-state Potts model is the third model in the $N = 0$ minimal series but it is the first model in a series of the $W_3$ algebra. It is a challenging task to analyze more precisely these structures. Particularly it is known that extended $N = 2$ algebras of $CY_3$’s have $W$-like symmetries, so-called $c = 9$ algebras. These structures with higher spin currents might be related with the $W_3$ algebra of the 3-state Potts model.

In our heterotic theory, the left-part has worldsheet $N = 1$ supersymmetry. This left-sector is composed of the internal manifold and transverse Lorentz group $SO(d - 2)$. Owing to the supersymmetry the left-part of the toroidal partition function vanishes by using identities about theta functions. We propose an identity that guarantees this symmetry in the context of CFTs for these minimal models (Ising, tricritical Ising, 3-state Potts). This left-part has the Lorentz group $SO(d - 2)$ and it contains information about spacetime dimension $d$. By changing holonomy groups there appear identities for characters associated with internal manifolds. They are some kinds of theta identities and could explain the dimension $d$ through some balance with the $SO(d - 2)$ part.

We would like to emphasize that our results are obtained under the completely general backgrounds. Especially it is remarkable that the forms of characters of statistical models are perfectly fitted to the holonomy parts of the manifolds in the gauge sector of the partition functions. Moreover identities in the worldsheet susy part are related with transverse Lorentz groups $SO(d - 2)$ combined with characters of CFTs for these special manifolds.

In section 5, we take concrete examples realized as orbifolds discussed by Joyce. We construct toroidal partition functions of heterotic strings compactified on these exceptional manifolds. We analyzed properties under modular transformations and studied consistencies of the strings on the orbifolds. Also we elaborate the spectra of massless sector of these models. For the $G_2$ case the matter parts are classified by representations of the gauge group $F_4$ and they are collected into 3 dim $N = 1$ multiplets of an $F_4$ gauge (supergravity) theory. The fundamental multiplets with 26-representation of $F_4$ are related with the (string) moduli space $\mathcal{M}_{CFT}$ of the internal $G_2$ manifold and its multiplicity is evaluated by a combination of topological numbers $b_2 + b_3 = 55$.

In the case of the $Spin(7)$ manifold, the matter parts transform as 9- and 16-representations under the enhanced gauge symmetry $SO(9)$. The associated fields of massless sectors are collected into 2 dim $N = 1$ multiplets of an $SO(9)$ gauge (supergravity) theory. The multiplets with spinor 16-representation of $SO(9)$ correspond to cohomology elements of the (string) moduli space $\mathcal{M}_{CFT}$ of the $Spin(7)$ manifold. Its multiplicity is calculated by using topological numbers as $1 + b_2 + b_4^- = 56$. 

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A Theta functions

We will review some properties of theta functions. The theta function is defined as

$$\theta(\nu|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi in^2\tau + 2\pi in\nu}.$$ 

The Jacobi's triple product identity is expressed in the following formula

$$\theta(\nu|\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + zq^n)(1 + z^{-1}q^n), \quad q = e^{2\pi i \tau}, \quad z = e^{2\pi i \nu}.$$ 

This function has periodicity 1 and its modular properties are summarized as

$$\theta(\nu + 1|\tau) = \theta(\nu|\tau), \quad \theta(\nu + \tau|\tau) = e^{-\pi i \tau - 2\pi i \nu} \theta(\nu|\tau),$$

$$\theta(\nu|\tau + 1) = \theta(\nu + \frac{1}{2}|\tau), \quad \theta(\nu|\tau - 1/\tau) = (-i\tau)^{1/2} e^{\pi i \frac{1}{4}} \theta(\nu|\tau).$$

Generalized theta functions are defined as

$$\theta^{[a]}_{[b]}(\nu|\tau) = \sum_{n\in\mathbb{Z}} \exp \left[ \pi i (n + a)^2\tau + 2\pi i (n + a)(\nu + b) \right]$$

$$= \exp \left[ \pi ia^2\tau + 2\pi ia(\nu + b) \right] \theta(\nu + a\tau + b, \tau).$$

Ordinary Jacobi’s theta functions are defined by using the generalized theta function

$$\theta_3(\nu|\tau) = \theta^{[0]}_{[0]}(\nu|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} z^n, \quad \theta_1(\nu|\tau) = -\theta^{[1/2]}_{[1/2]}(\nu|\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} z^{n-\frac{1}{2}},$$

$$\theta_2(\nu|\tau) = \theta^{[1/2]}_{[0]}(\nu|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} z^{-\frac{1}{2}}, \quad \theta_4(\nu|\tau) = \theta^{[0]}_{[1/2]}(\nu|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} z^n.$$
These theta functions are also expressed as infinite products

\[
\theta_3(\nu|\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + zq^{n-\frac{1}{2}})(1 + z^{-1}q^{n-\frac{1}{2}}),
\]

\[
\theta_4(\nu|\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - zq^{n-\frac{1}{2}})(1 - z^{-1}q^{n-\frac{1}{2}}),
\]

\[
\theta_2(\nu|\tau) = 2e^{\frac{\pi i}{4}\tau} \cos \pi \nu \prod_{n=1}^{\infty} (1 - q^n)(1 + zq^{n})(1 + z^{-1}q^{n}),
\]

\[
\theta_1(\nu|\tau) = -2e^{\frac{\pi i}{4}\tau} \sin \pi \nu \prod_{n=1}^{\infty} (1 - q^n)(1 - zq^{n})(1 - z^{-1}q^{n}).
\]

The Dedekind eta function is frequently used

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]

The modular properties of these functions are important and shown in the following equations

\[
\theta_3(\nu|\tau + 1) = \theta_4(\nu|\tau), \quad \theta_4(\nu|\tau + 1) = \theta_3(\nu|\tau),
\]

\[
\theta_2(\nu|\tau + 1) = e^{\frac{\pi i}{2}\nu} \theta_2(\nu|\tau), \quad \theta_1(\nu|\tau + 1) = e^{\frac{\pi i}{2}\nu} \theta_1(\nu|\tau), \quad \eta(\tau + 1) = e^{\frac{\pi i}{2}\nu} \eta(\tau),
\]

\[
\theta_3\left(\frac{\nu}{\tau} \bigg| \frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi i}{4}\nu^2} \theta_3(\nu|\tau), \quad \theta_4\left(\frac{\nu}{\tau} \bigg| \frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi i}{4}\nu^2} \theta_4(\nu|\tau),
\]

\[
\eta\left(\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} \eta(\tau).
\]

We also use the classical SU(2) theta function defined as

\[
\Theta_{m,k}(\nu|\tau) = \sum_{n \in \mathbb{Z}} q^{k(n + \frac{m}{2m})^2} z^{k(n + \frac{m}{2m})}. 
\]

We sometimes abbreviate arguments of theta functions: For example, \(\theta_3 = \theta_3(\tau)\) means \(\theta_3(\nu = 0|\tau)\).

\[\text{B} \quad \text{CFT and characters}\]

\[\text{B.1 Minimal models}\]

The unitary minimal models are labeled by an integer \(m\) (\(m = 3, 4, 5, \ldots\)). Its central charge is given by a formula

\[
c = 1 - \frac{6}{m(m + 1)}. 
\]
The Verma modules of each minimal model is classified by integers \( r, s \) in the regions
\[
r = 1, 2, \ldots, m - 1, \quad s = 1, 2, \ldots, m, \quad \text{with} \quad ms < (m + 1)r.
\]
The conformal dimension of the primary field is specified by the set \((r, s)\) and is evaluated as
\[
h_{r,s} = \frac{(m + 1)r - ms}{4m(m + 1)} - 1.
\]
The characters of these minimal models can be expressed for the primary field labelled by \((r, s)\)
\[
\chi_{r,s}^{(m)} = \frac{1}{\eta(\tau)} \{ \Theta_{(m+1)r-ms,m(m+1)}(\tau) - \Theta_{(m+1)r+ms,m(m+1)}(\tau) \}.
\]
We use \(m = 3, 4, 5\) minimal models in this paper. The details of properties of these models are listed in the following table:

- **Ising model \((c = \frac{1}{2})\)**
  \[
h_{1,1} = 0, \quad h_{2,1} = \frac{1}{2}, \quad h_{1,2} = \frac{1}{16}.
  \]
  We write the Virasoro characters for this model as \(\chi_{Ising}^{h_{r,s}}\).

- **Tricritical Ising model \((c = \frac{7}{10})\)**
  \[
h_{1,1} = 0, \quad h_{2,1} = \frac{7}{16}, \quad h_{1,2} = \frac{1}{10}, \quad h_{1,3} = \frac{3}{5}, \quad h_{2,2} = \frac{3}{80}, \quad h_{3,1} = \frac{3}{2}.
  \]
  We write the Virasoro characters of this model as \(\chi_{Tri}^{h_{r,s}}\).

- **3-state Potts model \((c = \frac{4}{5})\)**
  \[
h_{1,1} = 0, \quad h_{2,1} = \frac{2}{5}, \quad h_{3,1} = \frac{7}{5}, \quad h_{1,3} = \frac{2}{3}, \quad h_{4,1} = 3, \quad h_{2,3} = \frac{1}{15}.
  \]
  The notation \(\chi_{3-Potts}^{h_{r,s}}\) is used for Virasoro characters for this Potts model. But we mainly use \(W_3\) characters constructed from those of the Potts model
  \[
  C_{0 \cdot 3-Potts} = \chi_0 + \chi_3^{3-Potts}, \quad C_{2/5 \cdot 3-Potts} = \chi_{2/5} + \chi_{7/5}^{3-Potts},
  \]
  \[
  C_{2/3 \cdot 3-Potts} = \chi_{2/3}^{3-Potts}, \quad C_{1/15 \cdot 3-Potts} = \chi_{1/15}^{3-Potts}.
  \]
  The standard modular invariant partition function of the 3-state Potts model can be described by using these \(W_3\) characters \(C_{3-Potts}^{3-Potts}\)’s
  \[
  Z = |C_{0 \cdot 3-Potts}|^2 + |C_{2/5 \cdot 3-Potts}|^2 + 2|C_{2/3 \cdot 3-Potts}|^2 + 2|C_{1/15 \cdot 3-Potts}|^2.
  \]
The central charges and the conformal dimensions are summarized for representations of level 1 affine Lie algebras in Table 13.

Table 13: Properties of level 1 affine Lie algebras. The central charge and conformal dimension of each representation is shown here. The symbol “?” means there are no such representations.

### B.2 WZW models

The central charges and the conformal dimensions are summarized for representations of level 1 affine Lie algebras in Table 13.

Explicit forms of characters used in this paper are written down as follows:

\[
\chi_{\text{bas}}^{\text{so}(2r)} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^r + \left( \frac{\theta_4}{\eta} \right)^r \right), \quad \chi_{\text{vec}}^{\text{so}(2r)} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^r - \left( \frac{\theta_4}{\eta} \right)^r \right);
\]

\[
\chi_{\text{spi}}^{\text{so}(2r)} = \chi_{\text{cos}}^{\text{so}(2r)} = \frac{1}{2} \left( \frac{\theta_2}{\eta} \right)^r, \quad \chi_{\text{spi}}^{\text{so}(2r+1)} = \frac{1}{\sqrt{2}} \left( \frac{\theta_2}{\eta} \right)^{\frac{2r+1}{2}},
\]

\[
\chi_{\text{bas}}^{\text{so}(2r+1)} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^{\frac{2r+1}{2}} + \left( \frac{\theta_4}{\eta} \right)^{\frac{2r+1}{2}} \right), \quad \chi_{\text{vec}}^{\text{so}(2r+1)} = \frac{1}{2} \left( \left( \frac{\theta_3}{\eta} \right)^{\frac{2r+1}{2}} - \left( \frac{\theta_4}{\eta} \right)^{\frac{2r+1}{2}} \right);
\]

\[
\chi_{\text{bas}}^{\text{su}(2)} = \frac{\Theta_{0.1}}{\eta}, \quad \chi_{\text{fun}}^{\text{su}(2)} = \frac{\Theta_{1.1}}{\eta},
\]

\[
\chi_{\text{bas}}^{\text{su}(3)} = \frac{1}{\eta^2} \left( \Theta_{0.3} \Theta_{0.1} + \Theta_{3.3} \Theta_{1.1} \right), \quad \chi_{\text{fun}}^{\text{su}(3)} = \chi_{\text{fun}}^{\text{su}(3)} = \frac{1}{\eta^2} \left( \Theta_{2.3} \Theta_{0.1} + \Theta_{1.3} \Theta_{1.1} \right);
\]

\[
\chi_{\text{bas}}^{E_6} = \frac{1}{2 \eta^6 (\tau)} \left\{ \theta_3 (3 \tau) \cdot \theta_3 (\tau)^5 + \theta_4 (3 \tau) \cdot \theta_4 (\tau)^5 + \theta_2 (3 \tau) \cdot \theta_2 (\tau)^5 \right\},
\]

\[
\chi_{\text{fun}}^{E_6} = \frac{1}{2 \eta^6 (\tau)} \left\{ \theta_{[1/6]}_0 \right\} \cdot \theta_3 (3 \tau) \cdot \theta_2 (\tau)^5 + \theta_{[2/3]}_0 \right\} \cdot \theta_3 (3 \tau) \cdot \theta_4 (\tau)^5 + e^{-2 \pi i / 3} \theta_{[2/3]}_1 \right\} \cdot \theta_4 (3 \tau) \cdot \theta_2 (\tau)^5 \right\},
\]

\[
\chi_{\text{bas}}^{E_6} = \frac{1}{2 \eta^6 (\tau)} \left\{ \theta_{[5/6]}_0 \right\} \cdot \theta_3 (3 \tau) \cdot \theta_2 (\tau)^5 + \theta_{[1/3]}_0 \right\} \cdot \theta_3 (3 \tau) \cdot \theta_4 (\tau)^5 - e^{-2 \pi i / 3} \theta_{[1/3]}_1 \right\} \cdot \theta_4 (3 \tau) \cdot \theta_2 (\tau)^5 \right\},
\]

\[
\chi_{\text{fun}}^{E_7} = \frac{1}{2 \eta^7 (\tau)} \left\{ \theta_3 (2 \tau) \cdot \theta_2 (\tau)^6 + \theta_3 (2 \tau) \cdot \theta_4 (\tau)^6 \right\},
\]

\[
\chi_{\text{fun}}^{E_7} = \frac{1}{2 \eta^7 (\tau)} \left\{ \theta_3 (2 \tau) \cdot \theta_2 (\tau)^6 + \theta_2 (2 \tau) \cdot \theta_3 (\tau)^6 - \theta_4 (\tau)^6 \right\},
\]

\[
\chi_{\text{bas}}^{E_8} = \frac{1}{2 \eta^8 (\tau)} \left\{ \theta_2 (\tau)^8 + \theta_3 (\tau)^8 + \theta_4 (\tau)^8 \right\}.
\]
The algebra $u(1)_k$, $(k \in \mathbb{Z}, k > 0)$ also appears. Each module of the $u(1)_k$ is labeled by an integer $m \in \mathbb{Z}_{2k}$, and a character of a module $m$ is $\Theta_{m,k}/\eta$. The partition function of this CFT is written as

$$Z = \sum_{m \in \mathbb{Z}_{2k}} |\Theta_{m,k}/\eta|^2.$$ 

This theory describes a single free boson of radius $\sqrt{2k}$. We make a remark here: the $u(1)_1$ is the level 1 su(2) algebra, and $u(1)_2$ represents a level 1 affine so(2) algebra in our notation.

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