Purely coclosed $G_2$-structures on nilmanifolds

Giovanni Bazzoni$^1$ | Antonio Garvín$^2$ | Vicente Muñoz$^3$

$^1$Dipartimento di Scienza ed Alta Tecnologia, Università degli Studi dell’Insubria, Como, Italy
$^2$Departamento de Matemática Aplicada, Escuela de Ingenierías Industriales, Campus de Teatinos, Universidad de Málaga, Málaga, Spain
$^3$Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Campus de Teatinos, Málaga, Spain

Correspondence
Vicente Muñoz, Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Campus de Teatinos, s/n, 29071 Málaga, Spain.
Email: vicente.munoz@ucm.es

Abstract
We classify seven-dimensional nilpotent Lie groups, decomposable or of nilpotency step at most 4, endowed with left-invariant purely coclosed $G_2$-structures. This is done by going through the list of all seven-dimensional nilpotent Lie algebras given by Gong, providing an example of a left-invariant 3-form $\varphi$ which is a pure coclosed $G_2$-structure (i.e., it satisfies $d + \varphi = 0, \varphi \wedge d \varphi = 0$) for those nilpotent Lie algebras that admit them; and by showing the impossibility of having a purely coclosed $G_2$-structure for the rest of them.

KEYWORDS
purely coclosed $G_2$-structures, $SU(3)$-structures, nilmanifolds

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1 | INTRODUCTION

A seven-dimensional smooth manifold $M$ admits a $G_2$-structure if the structure group of its frame bundle reduces to the exceptional Lie group $G_2 \subset SO(7)$. Equivalently (see [6]), $M$ admits a $G_2$-structure if and only if it is orientable and spin. Further, a $G_2$-structure is equivalent to the existence of a positive 3-form $\varphi$ (see Section 2 for details), which defines a unique Riemannian metric $g_\varphi$ and an orientation $\text{vol}_\varphi$ on $M$. When $\varphi$ is parallel with respect to the Levi–Civita connection of $g_\varphi$, then the identity component of its holonomy group is contained in $G_2$; Fernández and Gray proved that this happens if and only if $\varphi$ is closed and coclosed [12]. In this case, $g_\varphi$ is Ricci-flat. A $G_2$-structure is called closed if $d \varphi = 0$, and coclosed if $d + * \varphi = 0$, where $* \varphi$ is the Hodge star operator associated with $g_\varphi$ and $\text{vol}_\varphi$. These two classes of $G_2$-structures are very different in nature; for instance, the closed condition is quite restrictive (see the recent survey [14]), while coclosed $G_2$-structures exist on any closed, oriented spin manifold, since they satisfy an $h$-principle, as proved by Crowley and Nordström in [8, Theorem 1.8].

As it is the case for general $G$-structures, the non-integrability of a $G_2$-structure is governed by its intrinsic torsion $\tau$, see [22]. In this particular case, $\tau$ has four components $\tau_0, \tau_1, \tau_2,$ and $\tau_3$, with $\tau_i \in \Omega^i(M)$, determined by the equations

$$
\begin{align*}
    d \varphi &= \tau_0 * \varphi + 3 \tau_1 \wedge \varphi + * \tau_3 \\
    d + * \varphi &= 4 \tau_1 \wedge * \varphi + \tau_2 \wedge \varphi.
\end{align*}
$$

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see [6, Proposition 1]. According to the vanishing of the various torsion components, one obtains 16 classes of $G_2$-structures, see [12]. We recognize closed $G_2$-structures as those for which $\tau_0 = \tau_1 = \tau_3$; on the other hand, coclosed $G_2$-structures are characterized by $\tau_1 = \tau_2 = 0$. A $G_2$-structure is of pure type if all the torsion components vanish, but one. Thus, closed $G_2$-structures are of pure type, while coclosed $G_2$-structures are not. A $G_2$-structure is locally conformally parallel (see [19]) if $\tau_0 = \tau_2 = \tau_3 = 0$, that is, if $d\varphi = 3\tau_1 \wedge \varphi$ and $d \ast \varphi \neq 0$. In this case, locally there is a function $f$ such that $e^f \varphi$ is a parallel $G_2$-structure. Nearly parallel $G_2$-structures (see [16]) are another important pure class; they are characterized by $\tau_1 = \tau_2 = \tau_3 = 0$, that is, $d\varphi = \tau_0 \ast \varphi$, where $\tau_0$ is a constant. In this case, the induced metric $g_\varphi$ is Einstein with positive scalar curvature.

In this paper, we focus on the last pure class of $G_2$-structures, called purely coclosed $G_2$-structures; these are given by the conditions $\tau_0 = \tau_1 = \tau_2 = 0$, that is, $d\varphi = \ast \varphi$ and $d \ast \varphi = 0$, or, equivalently [9], by

$$d \ast \varphi = 0 \quad \text{and} \quad \varphi \wedge d\varphi = 0;$$

clearly, they are a subclass of coclosed $G_2$-structures. The second one is an equality of 7-forms, hence it imposes a single extra condition. It is not clear whether there is an $h$-principle for purely coclosed $G_2$-structures.

A nilmanifold is a compact quotient $M = \Gamma \backslash G$, where $G$ is a connected, simply connected, nilpotent Lie group, and $\Gamma \subset G$ is a lattice. By Mal’cev Theorem [20], a lattice $\Gamma \subset G$ exists if and only if the Lie algebra $\mathfrak{g}$ of $G$ has a basis with respect to which the structure constants are rational numbers. A nilmanifold is parallelizable, hence it is spin for any Riemannian metric. Thanks to the aforementioned results, every nilmanifold has a coclosed $G_2$-structure.

We are interested on nilmanifolds endowed with left invariant $G_2$-structures. Since left-invariant differential forms on $\Gamma \backslash G$ are uniquely determined by forms on $\mathfrak{g}$, one can restrict the attention to seven-dimensional real nilpotent Lie algebras, which have been classified by Gong in [17]. Conti and Fernández classified nilpotent Lie groups endowed with a left-invariant closed $G_2$-structure, see [7]. Nilmanifolds cannot carry locally conformally parallel $G_2$-structures. Indeed, by a result of Ivanov, Parton and Piccinni [19], a compact manifold $M$ endowed with a locally conformally parallel $G_2$-structure fibers over the circle with fiber a compact, simply connected 6-manifold, hence $b_1(M) = 1$, while the first Betti number of a nilmanifold is at least 2. Also, non-toral nilmanifolds cannot have left-invariant nearly parallel $G_2$-structures. In fact, as we noticed above, the induced metric is Einstein in this case, and this never happens for nilmanifolds, due to a result of Milnor [21, Theorem 2.4]. As for left-invariant coclosed $G_2$-structures, there is an unpublished classification by Bagaglini [3], which does not seem to be complete. In [4], Bagaglini, Fernández and Fino determined which nilpotent Lie groups admit left-invariant coclosed $G_2$-structures in two cases: when the Lie algebra is decomposable, and when it is 2-step. The authors also showed that one can always choose the $G_2$-structure in such a way that the induced metric is a nilsoliton. In [15], Freibert obtained all nilpotent almost-Abelian Lie algebras with a coclosed $G_2$-structure. In [9], del Barco, Moroianu and Raffer obtained 2-step nilpotent Lie groups admitting left-invariant purely coclosed invariant $G_2$-structures. Their approach has a theoretical flavor and does not rely on the classification of seven-dimensional nilpotent Lie algebras.

In this paper, we study the existence of left-invariant purely coclosed $G_2$-structures on seven-dimensional decomposable nilpotent Lie groups and on indecomposable nilpotent Lie groups with nilpotency step $\leq 4$. We determine those which admit a left-invariant purely coclosed $G_2$-structures. For this, we go by exhaustion through the list of nilpotent Lie algebras of [17]. For each Lie group, in the positive case we provide an explicit example of a left-invariant purely coclosed $G_2$-structure. In the negative case, we show that it is not possible to find such $G_2$-structure by showing that there are suitable obstructions that forbid this to happen; such obstructions are described in Section 5. The results are summarized in Theorems 6.2, 6.5–6.7. In particular, we have the following.

**Theorem.** Every seven-dimensional decomposable nilpotent Lie algebra admitting a coclosed $G_2$-structure also admits a purely coclosed one, except for $\mathfrak{h}_3 \oplus \mathbb{R}^4$, where $\mathfrak{h}_3$ is the Heisenberg Lie algebra. Every seven-dimensional indecomposable nilpotent Lie algebra of nilpotency step $\leq 4$ admitting a coclosed $G_2$-structure also admits a purely coclosed one.

In future work, we shall address the remaining nilpotency steps.

In order to present our computations, we provide a worksheet for each Lie algebra. When it admits a left-invariant purely coclosed $G_2$-structure, we exhibit explicitly the relevant forms, and list the commands needed in order to verify that they indeed define a purely coclosed $G_2$-structure. In the negative case, we comment all the steps needed to show that the existence of a coclosed $G_2$-structure is obstructed. The worksheets can be found at [5]. For the ease of the verification, we use SageMath [11]: it is a free software, based on Python, which allows a number of computations in Mathematics.
Propositions 3.1 and 3.3 are mostly interesting because they allow to use SageMath in order to check that the forms in question define an SU(3)-structure. Our routines are freely available to anyone who may be interested in such computations.

2 | GENERALITIES ON $G_2$-STRUCTURES

Let $V$ be a seven-dimensional vector space. Given a 3-form $\varphi \in \Lambda^3(V^*)$, we define a symmetric bilinear form $b_{\varphi} : V \times V \to \Lambda^7(V^*)$ by $b_{\varphi}(x, y) := \frac{1}{6} i_x \varphi \wedge i_y \varphi \wedge \varphi$. We have that $\varphi$ is non-degenerate if $\varepsilon(\varphi) := (\det(b_{\varphi}))^{1/9} \neq 0$. Then

$$g_{\varphi} := \varepsilon(\varphi)^{-1} b_{\varphi}$$

(2.1)

is also a symmetric bilinear form on $V$. If it is positive definite, then $\varphi$ is a positive 3-form. By definition this is called a $G_2$-form. Then, there is a $g_{\varphi}$-orthonormal frame $\{e_1, \ldots, e_7\}$ such that

$$\varphi = e_{127} + e_{347} + e_{567} + e_{135} - e_{146} - e_{236} - e_{245},$$

where $\{e^1, \ldots, e^7\}$ is the dual coframe and $e^{ij} := e^i \wedge e^j$, $e^{ijk} := e^i \wedge e^j \wedge e^k$, and so on.

Recall that a seven-dimensional smooth manifold $M$ is said to admit a $G_2$-structure if there is a reduction of the structure group of its frame bundle from $GL(7, \mathbb{R})$ to the exceptional Lie group $G_2$, which can actually be viewed naturally as a subgroup of $SO(7)$. Thus, a $G_2$-structure determines a Riemannian metric and an orientation on $M$. In fact, the presence of a $G_2$-structure is equivalent to the existence of a 3-form $\varphi$ (the $G_2$-form) on $M$, which is positive on each tangent space $T_pM, p \in M$.

By Equation (2.1), a $G_2$-form $\varphi$ induces both an orientation $\text{vol}_\varphi$ and a Riemannian metric $g_\varphi$ on $M$, given by

$$6 g_\varphi(X, Y) \text{vol}_\varphi = i_X \varphi \wedge i_Y \varphi \wedge \varphi,$$

(2.2)

for vector fields $X, Y$ on $M$. Let $\ast_{\varphi}$ be the Hodge star operator determined by $g_{\varphi}$ and $\text{vol}_\varphi$. We say that a manifold $M$ has a coclosed $G_2$-structure if there is a $G_2$-structure on $M$ such that the $G_2$-form $\varphi$ is coclosed, that is, $d \ast_{\varphi} \varphi = 0$. The $G_2$-structure is called purely coclosed if, in addition, $\varphi \wedge d \varphi = 0$.

3 | LINEAR $SU(3)$-STRUCTURES

One way to understand $G_2$-structures on 7-manifolds is in terms of $SU(3)$-structures on 6-manifolds. In fact, both structures can be described coherently using spinors, as it was shown in [1].

Let $V$ be a six-dimensional vector space. Define the set

$$\Lambda_0(V^*) = \{ \omega \in \Lambda^2(V^*) \mid \omega^3 \neq 0 \}.$$

Given $\omega \in \Lambda_0(V^*)$, we orient $V$ declaring $\omega^3 > 0$.

For every $\tau \in \Lambda^3(V^*)$, we have a map $k_\tau : V \to \Lambda^5(V^*)$ given by $k_\tau(x) = i_x \tau \wedge \tau$, where $i_x$ denotes the contraction. Recall the natural isomorphism $V \otimes \Lambda^6(V^*) \to \Lambda^5(V^*)$, given by $(v, \omega) \mapsto v \omega$. Its inverse is $\mu : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*)$; if we fix a basis $\{u_1, \ldots, u_6\}$ of $V$, with dual basis $\{v^1, \ldots, v^6\}$, then $\mu(\xi) = \sum u_\ell \otimes (v^\ell \wedge \xi)$, for $\xi \in \Lambda^5(V^*)$. Composing $k_\tau$ with $\mu$ we obtain a map

$$K_\tau := \mu k_\tau : V \to V \otimes \Lambda^6(V^*).$$

In turn, this determines a function $\lambda : \Lambda^3(V^*) \to (\Lambda^6(V^*))^{\otimes 2}$ by

$$\lambda(\tau) = \frac{1}{6} \text{tr} \left( (K_\tau \otimes 1_{\Lambda^6(V^*)}) \circ K_\tau \right) \in (\Lambda^6(V^*))^{\otimes 2}.$$
Set

\[ \Lambda_{\pm}(V^*) := \{ \tau \in \Lambda^3(V^*) \mid \pm \lambda(\tau) > 0 \} . \]

Note that this condition is independent of orientations.

Take \( V_e = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle \), and consider copies \( V_f = \langle f_1, f_2, f_3, f_4, f_5, f_6 \rangle \), \( V_g = \langle g_1, g_2, g_3, g_4, g_5, g_6 \rangle \), and \( V_h = \langle h_1, h_2, h_3, h_4, h_5, h_6 \rangle \). Given \( \tau \in \Lambda^3(V^*) \), denote by \( \tau_e \) (resp. \( \tau_f, \tau_g, \tau_h \)) the corresponding elements in \( \Lambda^3(V^e) \) (resp. \( \Lambda^3(V^f), \Lambda^3(V^g), \Lambda^3(V^h) \)). A \((1,2,3)\)-shuffle is a collection \( \sigma = \{ \{i\}, \{j, k\}, \{r, s, t\} \} \) which is a permutation of \( \{1, \ldots, 6\} \) with \( j < k \) and \( r < s < t \). The sign of a \((1,2,3)\)-shuffle \( \sigma \), \((-1)^\sigma\), is its sign as permutation. Consider the element

\[ C_{xyz} = \sum_{\sigma} (-1)^\sigma x^i \wedge y^j \wedge y^k \wedge z^r \wedge z^s \wedge z^t , \]

for \( x, y, z = e, f, g, h \), where the sum runs over all \((1,2,3)\)-shuffles. We have the following:

**Proposition 3.1.** Let \( \tau \in \Lambda^3(V^*) \). Then

\[ \tau_e \wedge C_{gfe} \wedge \tau_f \wedge \tau_h \wedge C_{fgh} \wedge \tau_g = -6\lambda(\tau) \text{vol}_{efgh} \]

**Proof.** There is an easy equality

\[ C_{gfe} \wedge \tau_e = -\sum_{\ell=1}^{6} g_{\ell}^{e} \wedge 1_{f_{\ell}} \tau_f \wedge \text{vol}_{e} , \]

from which one has

\[ \tau_e \wedge C_{gfe} \wedge \tau_f = -\sum_{\ell=1}^{6} g_{\ell}^{e} \wedge k_{\tau_f}(f_{\ell}) \wedge \text{vol}_{e} . \] (3.1)

This implies

\[ \tau_e \wedge C_{gfe} \wedge \tau_f \wedge \tau_h \wedge C_{fgh} \wedge \tau_g = \sum_{\ell, m=1}^{6} g_{\ell}^{e} \wedge k_{\tau_f}(f_{\ell}) \wedge f^m \wedge k_{\tau_g}(g_{m}) \wedge \text{vol}_{eh} \]

\[ = -\sum_{\ell, m=1}^{6} g_{\ell}^{e} \wedge k_{\tau_g}(g_{m}) \wedge f^m \wedge k_{\tau_f}(f_{\ell}) \wedge \text{vol}_{eh} . \]

Recall that \( K_{\ell} = \mu_{k_{\tau}} \), so that \( K_{\tau_g}(g_{m}) = \sum g_{\ell} \otimes g_{\ell}^{e} \wedge k_{\tau_g}(g_{m}) \). Hence, \( k_{\ell, m} \), the entry \((\ell, m)\) of the matrix of \( K_{\tau_g} \), is determined by \( k_{\ell, m} \text{vol}_{g} = g_{\ell}^{e} \wedge k_{\tau_g}(g_{m}) \). This implies that the above sum is

\[ -\sum_{\ell, m=1}^{6} g_{\ell}^{e} \wedge k_{\tau_g}(g_{m}) \wedge f^m \wedge k_{\tau_f}(f_{\ell}) = -\sum_{\ell, m=1}^{6} k_{\ell, m} k_{\tau_g}(g_{m}) \text{vol}_{gf} = -\text{tr}(K_{\tau}^2)\text{vol}_{gf} = -6\lambda(\tau)\text{vol}_{gf} , \]

and the result follows. \( \square \)

We implement this computation in SageMath as follows:

D.<e1,e2,e3,e4,e5,e6,f1,f2,f3,f4,f5,f6,g1,g2,g3,g4,g5,g6,h1,h2,h3,h4,h5,h6> = GradedCommutativeAlgebra(QQ)
N= D.cdg_algebra()
N.inject_variables()
psie=-e2*e4*e6+e1*e3*e6+e1*e4*e5+e2*e3*e5
psif=-f2*f4*f6+f1*f3*f6+f1*f4*f5+f2*f3*f5
The next result relates elements in $\Lambda_0(V^*)$ and $\Lambda_- (V^*)$ with SU(3)-structures on $V$ (see [18, 23]).

**Theorem 3.2.** Let $(\omega, \psi_-) \in \Lambda_0(V^*) \times \Lambda_- (V^*)$ such that

$$\omega \wedge \psi_- = 0.$$  \hspace{1cm} (3.2)

Let $I = |K(\psi_-)|^{-1/2} K \psi_-$. If the tensor $h(x, y) = \omega(x, J y)$ is positive definite, then $(J, \omega)$ defines an SU(3)-structure on $V$, and every SU(3)-structure is obtained in this way. Taking $\psi_+ = - J^* \psi_-$, we have that $\psi := \psi_+ + i \psi_-$ is the complex volume form.

To apply Theorem 3.2, we have to check that the quadratic form $h(x) = \omega(x, J(x))$ is positive definite. Equivalently, we look at $\hat{h}(x) = \omega(x, K(x))$, where $K = K \psi_-, \psi = \psi_-$. This is allowed since $J$ is a positive multiple of $K$. Associated with $\omega = \sum c_{ijkl} e^i e^j$, we define the tensor

$$\omega_{ef} = \sum c_{ijkl} (e^i \wedge f^j - e^j \wedge f^i) \in V^*_e \otimes V^*_f.$$  

**Proposition 3.3.** We have

$$\hat{h}(x) \text{vol}_{e,f} = - \psi_e \wedge C_{x,fe} \wedge \psi_f \wedge \omega_{fx},$$

where $e, f$ are odd-degree variables, and $x$ is of even degree.

**Proof.** It is enough to prove this for $\omega = x^{ab}$, since the statement is linear in $\omega$. Let $x = \sum \alpha_i x_i \in V_x$, and let $K(x) = \sum k_{ij} \alpha_j x_i$. So

$$\hat{h}(x) = \omega(x, K(x)) = \sum_{\ell} (\alpha_g \alpha_{\ell} \kappa_{b,\ell} - \alpha_b \alpha_{\ell} \kappa_{g,\ell}).$$  \hspace{1cm} (3.3)
By Equation (3.1), we have $\psi_e \wedge C_{xfe} \wedge \psi_f = -\sum x^f \wedge k(f_e) \wedge \text{vol}_e$. By formula (3.4), we have $k(f_e) \wedge f^a = -\kappa_{ae} \text{vol}_f$, where $K(g_m) = \sum \kappa_{\ell m} g_{\ell}$. Therefore

$$\psi_e \wedge C_{xfe} \wedge \psi_f \wedge f^a = \sum x^f \kappa_{ae} \text{vol}_{ef}.$$ 

Finally,

$$\psi_e \wedge C_{xfe} \wedge \psi_f \wedge \frac{1}{2}(f^a \wedge x^b - f^b \wedge x^a) = \sum (x^f b \kappa_{ae} - x^f a \kappa_{be}) \text{vol}_{ef}. $$

Looking at Equation (3.3), we get the result. Recall that $x^f b = \frac{1}{2}(x^f \otimes x^b + x^b \otimes x^f)$ by convention on symmetric tensors. □

We implement this in SageMath as follows. Note that for setting even degree, we set the degree equal to two (since it is not allowed to have zero-degree variables).

```sage
D.<e1,e2,e3,e4,e5,e6,f1,f2,f3,f4,f5,f6,x1,x2,x3,x4,x5,x6> =
GradedCommutativeAlgebra(QQ,degrees=(1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2))
N=D.cdg_algebra()
N.inject_variables()
psie=-e2*e4*e6-e1*e3*e6+e1*e4*e5-e2*e3*e5
psif=-f2*f4*f6-f1*f3*f6+f1*f4*f5-f2*f3*f5
omegafx=f1*x2-f3*x4+f5*x6-f2*x1+f4*x3-f6*x5
Cxfe=[...]
1/2*psie*Cxfe*psif*omegafx
```

We will also need to compute $\psi_+$ explicitly out of $\omega, \psi_-$. We do this as follows. Suppose that $V = \langle x_1, ..., x_6 \rangle$ is a six-dimensional vector space. The complex structure is $J = cK$, where $c = |\lambda(\psi_-)|^{-1/2}$, and $K = K_{\psi_-}$. So $\psi_+ = -J^* \psi_- = -c^3 K^* \psi_-$. 

By Proposition 3.1, $K(g_m) = \sum \kappa_{\ell m} g_{\ell}$, where

$$\kappa_{\ell m} \text{vol}_g = g_{\ell} \wedge k_{\psi_-}(g_m).$$  \hfill (3.4)

For the action on forms, which is the dual one, we have $K^*(g^f) = \sum \kappa_{\ell m} g^m$, so that $K^*(g^f) \text{vol}_g = \sum g^m (g^f \wedge k_{\psi_-}(g_m))$. This means that for a 1-form $\alpha$, and writing the map in different variables for source and target, $K^*: V_x \rightarrow V_h$, it is

$$K^*(\alpha) \text{vol}_x = \sum h^m (\alpha \wedge k_{\psi_-}(x_m)).$$

For a 3-form $\tau = \sum a_{ijk} x^{ijk}$, we write

$$\tau_{xyz} = \sum a_{ijk} x^i \wedge y^j \wedge z^k.$$  

Then,

$$(K^* \tau) \text{vol}_{xyz} = \sum h^{abc} \wedge k_{\psi_-}(x_a) \wedge k_{\psi_-}(y_b) \wedge k_{\psi_-}(z_c) \wedge \tau_{xyz}.$$  

Using this for $\tau = \psi_-$ and formula (3.1), we have

$$\psi_+ = -c^3 (K^* \psi_-) \text{vol}_{xyzefg}$$

$$= -c^3 \psi_e \wedge C_{hxe} \wedge \psi_x \wedge \psi_f \wedge C_{hyf} \wedge \psi_y \wedge \psi_g \wedge C_{hzg} \wedge \psi_z \wedge \psi_{xyz}$$

where we have abbreviated $\psi = \psi_-$. The normalization can be obtained by means of

$$\psi_- \wedge \psi_+ = \frac{2}{3} \omega^3.$$
We implement this computation in SageMath as follows:

\[
D. <e_1, \ldots, h_6, x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, y_6, z_1, z_2, z_3, z_4, z_5, z_6> = \text{GradedCommutativeAlgebra}(\mathbb{Q})
\]
\[
N = D.\text{cdg\_algebra}()
\]
\[
N.\text{inject\_variables}()
\]
\[
\psi_1 = -e_2^*e_4^*e_6^* + e_1^*e_3^*e_6^* + e_1^*e_4^*e_5^* + e_2^*e_3^*e_5^*
\]
\[
\psi_2 = -z_2^*z_4^*z_6^* + z_1^*z_3^*z_6^* + z_1^*z_4^*z_5^* + z_2^*z_3^*z_5^*
\]
\[
\psi_{xyz} = -x_2^*y_4^*z_6^* + x_1^*y_3^*z_6^* + x_1^*y_4^*z_5^* + x_2^*y_3^*z_5^*
\]
\[
\chi_x = \ldots
\]
\[
\chi_y = \ldots
\]
\[
\chi_z = \ldots
\]
\[
\psi_1^*\chi_x^*\psi_2^*\psi_{xyz}
\]

4 | LEFT-INvariant G₂-STRUCTURES AND SU(3)-STRUCTURES ON LIE GROUPS

Let \( G \) be a seven-dimensional simply connected Lie group with Lie algebra \( \mathfrak{g} \). Then, a \( G_2 \)-structure on \( G \) is left-invariant if the corresponding 3-form is left-invariant. According to the discussion of Section 2, a left-invariant \( G_2 \)-structure on \( G \) is defined by a positive 3-form \( \varphi \in \Lambda^3(\mathfrak{g}^*) \), which can be written, in some orthonormal coframe \( \{e^1, \ldots, e^7\} \) of \( \mathfrak{g}^* \), as

\[
\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.
\]

We call this a \( G_2 \)-structure on \( \mathfrak{g} \). In this coframe, then,

\[
*\varphi \varphi = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}.
\]

A \( G_2 \)-structure on \( \mathfrak{g} \) is coclosed if \( \varphi \) is coclosed, that is, if

\[
d *\varphi \varphi = 0,
\]

where \( d \) denotes the Chevalley–Eilenberg differential on \( \mathfrak{g}^* \). A \( G_2 \)-structure on \( \mathfrak{g} \) is said to be purely coclosed if

\[
d *\varphi \varphi = 0 \quad \text{and} \quad \varphi \wedge d\varphi = 0.
\]

We explain next how to construct a \( G_2 \)-structure on a seven-dimensional Lie algebra, starting with a certain type of \( SU(3) \)-structure on a codimension 1 subspace and some extra data.

Let \( \mathfrak{g} \) be a seven-dimensional Lie algebra with non-trivial center \( \mathfrak{z}(\mathfrak{g}) \). Let \( V \subset \mathfrak{g} \) be a codimension 1 subspace, cooriented by \( X \in \mathfrak{z}(\mathfrak{g}) \); thus, \( X \) is a central vector with non-zero projection to \( \mathfrak{g}/V \). Let \( \omega \in \Lambda^2\mathfrak{g}^* \) and \( \psi_- \in \Lambda^3\mathfrak{g}^* \) be such that

- \( i_X\omega = 0; \)
- \( i_X\psi_- = 0; \)
- they define an \( SU(3) \)-structure on \( V \).

Hence, denoting by \( \bar{\omega} \) and \( \bar{\psi}_- \) the pull-back to \( V \) of the above tensors, we have \( \bar{\omega} \in \Lambda_0(V^*), \bar{\psi}_- \in \Lambda_-(V^*), \) and \( \bar{\omega} \wedge \bar{\psi}_- = 0. \) This determines \( \bar{\psi}_+ \in \Lambda^3(V^*) \). Extend \( \bar{\psi}_+ \) to an element \( \psi_+ \in \Lambda^3\mathfrak{g}^* \) by declaring \( i_X\bar{\psi}_+ = 0. \) Finally, let \( \eta \in \mathfrak{g}^* \) be such that \( \eta(X) \neq 0. \)

It follows that \( \varphi = \omega \wedge \eta + \psi_+ \) is a \( G_2 \)-form on \( \mathfrak{g} \); moreover, if \( h \) denotes the induced \( SU(3) \)-metric on \( V \), then the \( G_2 \)-metric on \( \mathfrak{g} \) is \( g = g_\varphi = h + \eta \wedge \eta. \) Clearly, \( *\varphi \varphi = \frac{\omega^2}{2} + \psi_- \wedge \eta. \) We want to find sufficient conditions on \( (\omega, \psi_-, \eta) \) in order for the \( G_2 \)-structure to be (purely) coclosed.

**Theorem 4.1.** In the above setting, the \( G_2 \)-structure is coclosed if
1. \( d\psi_- = 0 \);
2. \( \omega \wedge d\omega = \psi_- \wedge d\eta \).

Furthermore, the coclosed \( G_2 \)-structure is pure if

3. \( \omega^2 \wedge d\eta = -2\psi_+ \wedge d\omega \).

**Proof.** We compute

\[
d^* \varphi = \omega \wedge d\omega + d\psi_- \wedge \eta - \psi_- \wedge d\eta = (\omega \wedge d\omega - \psi_- \wedge d\eta) + d\psi_- \wedge \eta = 0 .
\]

The \( G_2 \)-form is \( \varphi = \omega \wedge \eta + \psi_+ \), so that \( d\varphi = d\omega \wedge \eta + \omega \wedge d\eta + d\psi_+ \). Hence,

\[
\varphi \wedge d\varphi = (\omega \wedge \eta + \psi_+) \wedge (d\omega \wedge \eta + \omega \wedge d\eta + d\psi_+) =
\]
\[
(\omega^2 \wedge d\eta + \psi_+ \wedge d\omega + d\psi_+ \wedge \omega) \wedge \eta + \psi_+ \wedge \omega \wedge d\eta + \psi_+ \wedge d\psi_+ .
\]

Now \( \psi_+ \wedge \omega = 0 \) and hence \( d\psi_+ \wedge \omega = \psi_+ \wedge d\omega \). Also, since \( X \) is central, \( i_X d\psi_+ = 0 \), hence \( i_X (\psi_+ \wedge d\psi_+) = 0 \) and so \( \psi_+ \wedge d\psi_+ = 0 \) since it is a 7-form. All in all, this implies that

\[
\varphi \wedge d\varphi = (\omega^2 \wedge d\eta + 2\psi_+ \wedge d\omega) \wedge \eta = 0 ,
\]

and the result follows. \( \square \)

The next example shows how to apply Theorem 4.1 in practice.

**Example 4.2.** Let us consider the Lie algebra \( \mathfrak{g} = 37B = (0, 0, 0, 0, 12, 23, 34) \) in the notation of [17]; this means that \( \mathfrak{g} \) is seven-dimensional and that it admits a basis \( \{ e_1, \ldots, e_7 \} \) such that, in terms of the dual basis \( \{ e_1, \ldots, e_7 \} \), the Lie algebra structure is given by \( de_j = 0 \), \( j = 1, \ldots, 4 \), \( de_5 = e_12 \), \( de_6 = e_23 \) and \( de_7 = e_34 \). Consider the subspace \( V = \text{span}(e_1, e_2, e_3, e_4, e_6, e_7) \subset \mathfrak{g} \) and the central vector \( X = e_5 \). Set

\[
\begin{align*}
\omega &= e_1^{13} + e_2^{24} - e_6^{7} ; \\
\psi_- &= e_1^{127} - e_1^{146} + e_2^{36} - e_3^{47} ; \\
\eta &= e_5 + e_7 .
\end{align*}
\]

Then, \( (\omega, \psi_-) \) defines an SU(3)-structure on \( V \) with induced metric \( h = \sum_{i \leq 5} e^i \otimes e^i \), and \( \psi_+ = e_1^{126} + e_1^{147} - e_3^{46} - e_2^{37} \).

Hence, \( \varphi = \omega \wedge \eta + \psi_+ \) defines a \( G_2 \)-structure on \( \mathfrak{g} \) with \( G_2 \)-metric \( g_\varphi = \sum_{i=1}^6 e^i \otimes e^i + 2e^7 \otimes e^7 + e^5 \otimes e^7 + e^7 \otimes e^5 \).

Since conditions 1–3. of Theorem 4.1 are satisfied, the \( G_2 \)-structure is purely coclosed: \( d^* \varphi = 0 \) and \( \varphi \wedge d\varphi = 0 \).

### 4.1 | Nilpotent Lie algebras and nilmanifolds

A nilmanifold is a compact manifold of the form \( \Gamma \setminus G \), where \( G \) is a connected, simply connected, nilpotent Lie group and \( \Gamma \) is a cocompact discrete subgroup (a lattice). By a result of Mal’cev [20], we know that if \( \mathfrak{g} \) is nilpotent with rational structure constants, then the associated connected, simply connected nilpotent Lie group \( G \) admits a lattice \( \Gamma \). Therefore, a left-invariant \( G_2 \)-structure on \( G \) determines a \( G_2 \)-structure on the nilmanifold \( \Gamma \setminus G \). Moreover, if the left-invariant \( G_2 \)-structure on \( G \) is coclosed (resp. purely coclosed), the same holds for the induced \( G_2 \)-structure on \( \Gamma \setminus G \). The left-invariant forms on \( G \) are given by the corresponding forms on \( \Lambda^*(\mathfrak{g}^*) \), so the computations can be done on the Lie algebra. Note that a nilpotent Lie algebra (NLA) has non-trivial center.

We use Gong’s classification [17] of seven-dimensional nilpotent Lie algebras that appears in the Appendix. We translate the Lie algebra brackets into the differential graded algebra (DGA) structure for \( \Lambda(\mathfrak{g}^*) \) for each of them. Gong’s classification [17] of seven-dimensional nilpotent Lie algebras is done over the real numbers. For those nilpotent Lie algebras that have a single representative, the structure constants are actually rational numbers, so they define nilmanifolds. For
those nilpotent Lie algebras that appear in families depending on a real parameter \( \lambda \), it is not clear that \( \lambda \) must be rational in order to define a nilmanifold, since a different choice of generators may have rational structure constants. By [10], if a family of nilpotent Lie algebras \( \{ \mathfrak{g}_\lambda \} \), which depends on a parameter \( \lambda \), satisfies that \( \mathfrak{g}_\lambda \cong \mathfrak{g}_{\lambda'} \) only if \( \lambda = \lambda' \), then the Lie algebra \( \mathfrak{g}_\lambda \) has a basis with rational structure constants if and only if \( \lambda \) is rational. In any case, we do not care about this issue, since we produce \( G_2 \)-structures for all values of the parameter \( \lambda \). When there are special values to be treated separately, these are always rational.

Note that the forms \( \omega, \psi_, \eta \) can be defined with real coefficients, and that it is only the structure constants of the DGA that should be rational to have a nilmanifold with a (purely coclosed) \( G_2 \)-structure.

5 | Obstructions for Coclosed \( G_2 \)-Structures on NLAs

We use a converse of Theorem 4.1, which follows from [23, Chapter 1, Proposition 4.5] and [4, Proposition 3.1]. First of all, suppose \( \varphi \) is a \( G_2 \)-structure on a seven-dimensional Lie algebra and let \( g_\varphi \) be the \( G_2 \)-metric. Pick a vector \( X \) of length 1. It follows from that \( X^\perp \) has an SU(3)-structure \((\omega, \psi_-)\) given by \( \omega := i_X \varphi \) and \( \psi_- := -i_X \phi \), where \( \phi = \ast \varphi \). Let \( \eta \) be the metric dual of \( X \). Then \( \phi = \frac{1}{2} \omega^2 + \psi_- \wedge \eta \). We have the following.

**Proposition 5.1.** Let \( \mathfrak{g} \) be a seven-dimensional Lie algebra with non-trivial center and let \( X \) be a unit central vector. Let \( \mathfrak{h} = \mathfrak{g} / \langle X \rangle \) be the quotient six-dimensional Lie algebra. If \( \varphi \) is a 3-form defining a coclosed \( G_2 \)-structure on \( \mathfrak{g} \), then it determines an SU(3)-structure \((\omega, \psi_-)\) on \( \mathfrak{h} \) such that

\[
\begin{align*}
d\psi_- &= 0, \\
\omega \wedge d\omega &= \psi_- \wedge d\eta .
\end{align*}
\]

In this section, we obtain obstructions to the existence of coclosed \( G_2 \)-structures. In the next subsections, we shall provide three of them.

5.1 | First obstruction

For the first obstruction we use the following result, which appears in [4, Lemma 3.3].

**Lemma 5.2.** Let \( \mathfrak{g} \) be a seven-dimensional Lie algebra. If for every closed 4-form \( \kappa \) on \( \mathfrak{g} \) there are linearly independent vectors \( X \) and \( Y \) in \( \mathfrak{g} \) such that \((i_X i_Y \kappa)^2 = 0\), then \( \mathfrak{g} \) does not admit coclosed \( G_2 \)-structures.

This is used in the following form:

**Corollary 5.3.** Let \( \mathfrak{g} \) be a seven-dimensional Lie algebra. Take the cohomology \( H^4( \Lambda(\mathfrak{g}^*) ) = \langle \{ z_\alpha \} \rangle \). Suppose that there exist linearly independent vectors \( X, Y \in \mathfrak{g} \), with \( Y \in \mathfrak{z}(\mathfrak{g}) \), such that \( i_X i_Y z_\alpha \in U \) for a subspace \( U \subset \Lambda^2 \mathfrak{g}^* \) such that \( \Lambda^2 U = 0 \). Then, \( \mathfrak{g} \) does not admit any coclosed \( G_2 \)-structure.

**Proof.** Thanks to Lemma 5.2, it suffices to check that for any \( \kappa \in \Lambda^4 \mathfrak{g}^* \) closed we have \( i_Y i_Y \kappa \in U \), and hence \((i_Y i_Y \kappa)^2 = 0\). Any such \( \kappa \) is a linear combination of the \( z_\alpha \)‘s and an exact 4-form. For the former, the result holds by assumption. If \( \kappa \) is an exact 4-form, then \( i_Y \kappa = 0 \), because \( Y \in \mathfrak{z}(\mathfrak{g}) \).

5.2 | Second obstruction

The second obstruction uses the following result, which appears in [4, Lemma 3.4].

**Lemma 5.4.** Let \((h, J)\) be an almost Hermitian structure on a six-dimensional oriented vector space \( V \), with orthogonal complex structure \( J \), Hermitian metric \( h \) and fundamental 2-form \( \omega \). Then, for any \( J \)-invariant four-dimensional subspace \( W \subset V \), we have that \((\ast \omega)|_W \neq 0\).
We use it in the following form.

**Corollary 5.5.** Let \( \mathfrak{g} \) be a seven-dimensional nilpotent Lie algebra and let \( \{e_1, \ldots, e_7\} \) be a nilpotent basis, with dual basis \( \{e^1, \ldots, e^7\} \). Take a list of generators of the space of closed 4-forms \( z_\alpha \in \Lambda^4(\mathfrak{g}^*) \). Suppose that \( z_\alpha \in \langle e^1, e^2 \rangle \wedge \Lambda^3(\mathfrak{g}^*) \). Then, \( \mathfrak{g} \) does not admit coclosed \( G_2 \)-structures.

**Proof.** Suppose that \( \varphi \) is a coclosed \( G_2 \)-structure, and let \( \phi = \ast \varphi \varphi \), so that \( d\phi = 0 \). Then, \( \phi \in \Lambda^4(\mathfrak{g}^*) \) is a closed 4-form. We fix \( X = e_7 \), and consider the quotient algebra \( \mathfrak{h} = \mathfrak{g} / \langle X \rangle \). By Proposition 5.1, the forms \( \omega = i_X \varphi \) and \( \psi_- = -i_X \phi \) determine an SU(3)-structure on \( \mathfrak{h} = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle \). Then, \( \psi_- = -i_X \phi \in \langle e^1, e^2 \rangle \wedge \Lambda^2(\mathfrak{h}^*) \).

Let us see that \( W = \langle e_3, e_4, e_5, e_6 \rangle \) is a complex subspace of \( \mathfrak{h} \). It is enough to see that \( K = K \psi_- \) leaves \( W \) invariant, since \( J \) is a multiple of it. Recall that \( k(x) = \iota_x \psi_- \wedge \psi_- \) and \( K = \mu \circ k \). To check that \( W \) is invariant, we need to check that \( k(\phi) \wedge e^1 = 0 \), \( k(\phi) \wedge e^2 = 0 \), for \( j = 3, \ldots, 6 \). This is clear since both \( \iota_x \psi_- \) and \( \psi_- \) contain at least one \( e^1, e^2 \), and hence a product with \( e^1 \) or \( e^2 \) kills it.

By Lemma 5.2, it must be \( \omega^2|_W \neq 0 \). This means that \( \omega^2(e_3, e_4, e_5, e_6) \neq 0 \); in other words, \( \omega^2 \) contains the monomial \( e^{3456} \). By Proposition 5.1, we have \( \omega^2 = 2(\phi - \psi_- \wedge \eta) \), where \( \eta \) is a 1-form with a component \( e^7 \). By assumption, \( \phi \) cannot contain \( e^{3456} \) (it always contains either \( e^1 \) or \( e^2 \)), and \( \psi_- \) also contains \( e^1 \) or \( e^2 \). Thus, \( e^{3456} \) cannot appear in \( \omega^2 \). This is a contradiction. \( \square \)

### 5.3 Third obstruction

This method is based on the last equation of Proposition 5.1. We fix \( X = e_7 \). We compute a basis \( \{z_\alpha\} \) of the closed 3-forms in \( \mathfrak{h}^* = \langle e^1, \ldots, e^6 \rangle \). Therefore for a closed 3-form, we can write \( \tau = \sum a_\alpha z_\alpha \).

**Proposition 5.6.** Suppose we have elements \( w_1, \ldots, w_\ell \in \Lambda^3(\mathfrak{h}^*) \), and let \( W \) be a subspace such that \( \Lambda^3(\mathfrak{h}^*) = W \oplus \langle w_1, \ldots, w_\ell \rangle \). Suppose furthermore that for any closed 2-form \( \beta \) and closed 3-form \( \tau \) on \( \mathfrak{h}^* \), we have

\[
\beta \wedge d\beta, \tau \wedge de^j \in W, \quad j = 1, \ldots, 6.
\]

Then, define the linear subspace

\[
H = \left\{ (a_\alpha) \mid \sum a_\alpha z_\alpha \wedge de^7 \in W \right\}.
\]

If \( \lambda(\sum a_\alpha z_\alpha) \geq 0 \) for all \( (a_\alpha) \in H \), then there is no coclosed \( G_2 \)-structure on \( \mathfrak{g} \).

**Proof.** If \( \varphi \) is a coclosed \( G_2 \)-structure on \( \mathfrak{g} \), then \( \psi_- \) is closed and thus \( \psi_- = \sum a_\alpha z_\alpha \). Next, \( \eta \) must have the form \( \sum_{i=1}^7 b_i e^i \), with \( b_7 \neq 0 \). Hence,

\[
\psi_- \wedge de^7 = b_7^{-1} \left( \omega \wedge d\omega - \sum_{i=1}^6 b_i \psi_- \wedge de^i \right) \in W,
\]

by assumption. Hence, \( (a_\alpha) \in H \), and so \( \lambda(\psi_-) \geq 0 \), which is a contradiction. \( \square \)

### 6 SEVEN-DIMENSIONAL NILPOTENT LIE ALGEBRAS WITH PURELY COCLOSED \( G_2 \)-STRUCTURES

#### 6.1 Decomposable nilpotent Lie algebras

Seven-dimensional decomposable nilpotent Lie algebras are listed in Tables A1 and A2. In [4], the authors identify seven-dimensional decomposable nilpotent Lie algebras that admit a coclosed \( G_2 \)-structure. The following result summarizes Proposition 4.1, Proposition 4.2, and Theorem 4.3 in [4].
### Table 1

| NLA | Purely coclosed $G_2$-structures on decomposable NLAs. |
|-----|------------------------------------------------------|
| $n_1$ | $e^{12} + e^{34} + e^{56}$ |
| $n_2$ | $e^{12} + e^{34} + e^{56}$ |
| $n_3$ | $e^{12} + e^{34} + e^{56}$ |
| $n_4$ | $e^{12} + e^{14} - 2e^{34} - e^{56}$ |
| $n_5$ | $e^{11} + e^{24} + e^{56}$ |
| $n_6$ | $e^{12} + e^{34} + e^{56}$ |
| $n_7$ | $e^{12} + e^{24} - e^{56}$ |
| $n_8$ | $-e^{12} + e^{15} - e^{26} - 2e^{34} + 3e^{36} - e^{45}$ |
| $n_9$ | $-e^{14} + e^{15} + 14e^{26} + e^{34} - 7e^{35} - e^{45}$ |
| $n_{10}$ | $e^{12} - 2e^{21} - e^{45} + \frac{4}{5}(e^{27} - e^{13}) + \frac{3}{5}(e^{37} - e^{15})$ |
| $n_{11}$ | $e^{16} + e^{24} - e^{35} - e^{45}$ |
| $n_{12}$ | $5e^{12} + 10e^{14} - 3e^{16} - 3e^{21} + e^{37} - 4e^{35}$ |
| $n_{13}$ | $e^{12} + e^{13} + \frac{1}{2}e^{16} + \frac{1}{2}e^{26} + e^{34} + 2e^{36} - 2e^{45}$ |
| $n_{14}$ | $e^{12} + e^{15} - e^{26} - 2e^{34} + 3e^{36} - e^{45}$ |
| $n_{15}$ | $-e^{14} + e^{15} + 14e^{26} + e^{34} - 7e^{35} - e^{45}$ |
| $n_{16}$ | $3e^{13} + e^{14} + e^{16} - e^{25} + e^{34} - e^{36} - e^{45}$ |
| $n_{17}$ | $3e^{13} + e^{14} + e^{16} - e^{25} + e^{34} - e^{36} - e^{45}$ |
| $n_{18}$ | $-e^{13} + e^{24} - e^{35} - 3e^{36} - e^{45}$ |
| $n_{19}$ | $5e^{13} + e^{25} + e^{46} + \frac{1}{2}(e^{12} + e^{24} + e^{34} + e^{56})$ |
| $n_{20}$ | $5e^{12} + 5e^{25} + e^{36} + \frac{1}{2}(e^{16} + e^{26} + e^{34} + e^{45})$ |
| $n_{21}$ | $-e^{14} + e^{15} + e^{26} + e^{35} + e^{36} + e^{45}$ |
| $n_{22}$ | $e^{15} + e^{24} + e^{36}$ |
| $n_{23}$ | $e^{13} + e^{24} + e^{36}$ |
| $n_{24}$ | $-e^{14} + e^{25} + e^{36} - e^{56}$ |

### Theorem 6.1

Among the 35 decomposable nilpotent Lie algebras of dimension 7, those that have a coclosed $G_2$-structure are $n_i$, $i = 1, \ldots, 24$, and those who do not admit any coclosed $G_2$-structure are $g_i$, $i = 1, \ldots, 8$ and $l_i$, $i = 1, 2, 3$.

We refer to Table 1 for the structure equations of these Lie algebras. The 24 decomposable nilpotent Lie algebras which admit a coclosed $G_2$-structure have the form $n = h \oplus \mathbb{R}$, where $h$ is a six-dimensional Lie algebra, generated by $\{e_1, \ldots, e_6\}$, endowed with a half-flat SU(3)-structure $(\omega, \psi_\perp)$, that is, $d(\omega^2) = 0$ and $d\psi_\perp = 0$. The $G_2$-metric is $g_\psi = h + e^7 \otimes e^7$, where $h$ is the SU(3)-metric on $h$ and $e_7$ is a generator of the factor $\mathbb{R}$. Then, $\ast_\psi \varphi = \frac{1}{2} \omega^2 + \psi_\perp \wedge e^7$ is automatically coclosed; indeed, conditions 1. and 2. of Theorem 4.1 are satisfied with $\eta = e^7$, which is closed. Condition 3, however, need not hold. Nevertheless, we prove the following result.

### Theorem 6.2

Every seven-dimensional decomposable nilpotent Lie algebra admitting a coclosed $G_2$-structure also admits a purely coclosed one, except for $n_2$.

**Proof.** We apply Theorem 4.1 to each of the Lie algebras $n_i$, $i = 1, 3, \ldots, 24$, and exhibit explicitly the required tensors $\omega$, $\psi_\perp$, and $\eta$ satisfying conditions 1–3. The results are contained in Table 1. That $n_2 = h_3 \oplus \mathbb{R}^4$ admits no purely coclosed $G_2$-structure follows from [9, Corollary 4.3].

**Remark 6.3.** As we shall see, $n_2$ is the only seven-dimensional nilpotent Lie algebra of nilpotency step $\leq 4$ which admits a coclosed $G_2$-structure but no purely coclosed structures. It turns out that $n_2$ admits the following purely coclosed $G_2^*$-structure:

$$\ast_\psi \varphi = -e^{1234} + e^{2386} + e^{1436} + e^{1357} + e^{2457} - e^{1367} + e^{2467}.$$
TABLE 2  Purely coclosed $G_2$-structures on indecomposable 2-step NLAs.

| NLA | $\omega$ | $\psi$ | $\eta$ |
|-----|---------|-------|-------|
| 17  | $-e^{12} + \frac{1}{2}e^{34} - e^{56}$ | $e^{135} + e^{146} - e^{236} + e^{245}$ | $e^7$ |
| 37A | $-e^{12} + e^{34} - e^{56}$ | $-e^{136} + e^{145} - e^{235} - e^{246}$ | $-e^7 + e^5$ |
| 37B | $e^{13} + e^{24} - e^{67}$ | $e^{127} + e^{146} + e^{236} + e^{247}$ | $2e^5$ |
| 37B1 | $-e^{12} - 2e^{34} + e^{67}$ | $-e^{137} + e^{146} + e^{236} + e^{247}$ | $e^5$ |
| 37C | $e^{12} - e^{34} + e^{67}$ | $e^{136} - e^{147} + e^{237} + e^{246}$ | $e^5$ |
| 37D | $-e^{12} + e^{34} - e^{67}$ | $e^{136} - e^{147} + e^{237} - e^{246} + 2e^5$ | $e^5$ |
| 37D1 | $e^{12} + e^{34} + e^{67}$ | $e^{136} + e^{147} + e^{236} = e^{247}$ | $e^5$ |

The following routine computes $\omega \wedge \psi_-$ and $\psi_- \wedge \psi_+ - \frac{2}{3} \omega^3$ to ensure that $(\omega, \psi_-)$ defines a normalized SU(3)-structure via Theorem 3.2. It also computes $d\psi_-, \omega \wedge d\omega - \psi_- \wedge d\psi_+ + 2\psi_+ \wedge d\omega$ to check that the $G_2$-structure $\varphi = \omega \wedge \eta + \psi_+$ is purely coclosed, via Theorem 4.1.

To check that the forms in Tables 1 satisfy the required conditions, we use a SageMath worksheet [11]. The worksheets are available in [5]. We include the worksheet for the Lie algebra $\mathfrak{n}_{15}$. A. $\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \rangle = \text{GradedCommutativeAlgebra}(\mathbb{Q})$

M = A.cdg_algebra( x4:x1*x2, x5:x1*x3, x6: x1*x4+x3*x5 )

M.inject_variables()

omega = x1*x7+x2*x3+2*x3*x4-x1*x4+x3*x5+x2*x7-4*x1*x5

psi = x1*x2*x4-x1*x2*x5+x1*x4*x7+x1*x5*x7+x2*x3*x4+x2*x3*x5-x3*x5*x7

psiplus = -2*(3*x1*x2*x4-x2*x3*x4+5*x1*x2*x5+3*x2*x3*x5-x1*x4*x7+2*x3*x4*x7+3*x1*x5*x7)

eta = x6

omega*psi

psi*psiplus-(2/3)*omega^3

psi.differential()

omega*omega.differential()-psi*eta.differential()

omega^2*eta.differential()+2*psiplus*omega.differential()
TABLE 3  Purely coclosed $G_2$-structures on indecomposable 3-step NLAs – 1.

| NLA   | $\omega$            | $\psi_-$            | $\eta$ |
|-------|----------------------|----------------------|--------|
| 137A  | $e^{13} - e^{24} + e^{56}$ | $e^{126} - e^{145} - e^{235} - e^{345} + e^{346}$ | $e^7$  |
| 137A₁ | $e^{12} + e^{14} + e^{56}$ | $-e^{246} + e^{135} + e^{145} + e^{235}$ | $e^7$  |
| 137B  | $e^{11} - \frac{3}{4} e^{24} + e^{56} - \frac{1}{2} (e^{15} + e^{36})$ | $e^{126} - e^{145} - e^{235} - e^{345} + e^{346}$ | $e^7$  |
| 137B₁ | $e^{12} + e^{35} + e^{56}$ | $2e^{136} + e^{145} + e^{235} - e^{346}$ | $-\frac{1}{2} e^7$ |
| 137C  | $e^{12} + \frac{2}{7} e^{13} - e^{24} - e^{34} + e^{56}$ | $e^{126} - e^{145} - e^{235} + e^{346}$ | $e^7 - e^5$ |
| 137D  | $e^{13} - e^{25} + e^{56}$ | $-e^{123} - 2e^{126} + e^{145} - e^{234} + e^{246} - e^{356}$ | $-\frac{1}{2} (e^7 + e^5)$ |
| 147A  | $e^{13} + e^{26} + e^{34} - e^{23}$ | $e^{124} + e^{126} + e^{145} + e^{345} + e^{346} + e^{356}$ | $-\frac{1}{3} (e^7 + e^5)$ |
| 147A₁ | $e^{11} + e^{25} - e^{66}$ | $-e^{123} - 2e^{126} + e^{145} - e^{234} + e^{246} - e^{356}$ | $e^7 - 2e^4$ |
| 147B  | $-e^{11} + e^{14} + e^{16} - e^{23}$ | $e^{124} + e^{126} + e^{145} + e^{345} + e^{346} + e^{356}$ | $-\frac{1}{3} (e^7 + e^5)$ |
| 147D  | $-2e^{12} - 6e^{23} - 3e^{25} + e^{26} - e^{34} - 4e^{26} - e^{35} - 3e^{56}$ | $-e^{123} - 2e^{134} - e^{136} + e^{154} + e^{234} + e^{235} - e^{236} - e^{245}$ | $\frac{1}{7} (e^7 - 2e^5) - e^4$ |
| 157E(\(\lambda\)) | $e^{13} - e^{25} + e^{34} + e^{45}$ | $-e^{123} - 2e^{125} - e^{126} + e^{145} + e^{234} + e^{235} + e^{236} - e^{245}$ | $e^7 - 2e^3$ |
| 157E₁(\(\lambda\)) | $e^{13} + e^{26} + e^{34} - e^{35}$ | $e^{123} + 2e^{126} + e^{145} + e^{234} + e^{235} + e^{236}$ | $-\frac{1}{2} e^7 - e^6$ |
| 247A  | $e^{12} + e^{35} + e^{56}$ | $e^{123} - e^{124} - 2e^{125} - e^{126} + e^{134} + e^{135}$ | $e^7 + e^6$ |
| 247B  | $e^{12} + e^{26} + e^{35}$ | $-e^{123} - e^{124} - 2e^{125} - e^{126} + e^{134} + e^{135}$ | $e^7 + e^6$ |
| 247C  | $e^{13} + e^{25} + e^{37}$ | $e^{124} - e^{157} + e^{237} + e^{345}$ | $e^6 + e^4$ |
| 247D  | $e^{12} - e^{34} - e^{56}$ | $e^{136} + e^{145} + e^{235} + e^{246}$ | $e^7 + e^4$ |
| 247F  | $e^{15} + e^{16} - e^{23} + e^{24} + e^{56}$ | $e^{123} + e^{124} + e^{125} + e^{235} + e^{236} - e^{245}$ | $3e^7 - e^4$ |
| 247F₁ | $e^{13} - e^{16} + \frac{1}{2} e^{23} + e^{26} - e^{45}$ | $e^{125} + e^{126} - e^{134} + e^{135}$ | $e^7 - e^5$ |
| 247I  | $e^{15} + \frac{1}{2} e^{16} + e^{23} - \frac{1}{2} (e^{24} - e^{45})$ | $e^{126} + e^{134} + e^{245}$ | $e^7 + e^4$ |
| 247J  | $e^{13} - e^{24} + e^{57}$ | $-e^{124} + e^{126} - e^{134} + e^{135}$ | $e^6 + 6e^5$ |
| 247L  | $e^{13} - e^{24} + e^{57}$ | $e^{125} + e^{134} + e^{147} + e^{237}$ | $\frac{1}{2} e^6$ |

We have the following result (see Tables 3 and 4):

**Theorem 6.6.** All seven-dimensional indecomposable 3-step nilpotent Lie algebra admit a purely coclosed $G_2$-structure, except for: $247E$, $247E_1$, $247G$, $247H_1$, $247K$, $247L$, $247R$, $247R_1$, $357B$, $357C$.

For the families $147E(\lambda)$ and $147E_1(\lambda)$, we cannot use the usual Sage worksheets, since the DGA package does not admit parameters in the definition of the differential. Instead we work with the following worksheet, inspired by [2], that implements an older way to deal with DGAs, less user-friendly, but that allows variables in the definition of the differential. For instance, $147E(\lambda)$ is implemented as follows:

```
E = ExteriorAlgebra(SR, 'x', 8)
l=var('l')
str_eq = (1,2):E.gens()[4], (2,3):E.gens()[5], (1,3):-E.gens()[6], (1,5):-E.gens()[7],
(2,6):l*E.gens()[7], (3,4):(1-l)*E.gens()[7]
d=E.coboundary(str_eq); d
print([d(b) for b in E.gens()])
omega=E.gens()[1]*E.gens()[3]-E.gens()[2]*E.gens()[6]-E.gens()[3]*E.gens()[4]
+E.gens()[4]*E.gens()[5]
psi=-E.gens()[1]*E.gens()[2]*E.gens()[3]-2*E.gens()[1]*E.gens()[2]*E.gens()[5]
+E.gens()[1]*E.gens()[4]*E.gens()[6]+E.gens()[2]*E.gens()[4]*E.gens()[5]
+E.gens()[3]*E.gens()[5]*E.gens()[6]
eta=(1/(l-1))*E.gens()[7] - E.gens()[6]
```
TABLE 4 Purely coclosed $G_2$-structures on indecomposable 3-step NLAs $-2$.

| NLA    | $\omega$                                                          | $\psi_-$        | $\eta$            |
|--------|------------------------------------------------------------------|------------------|-------------------|
| 247M   | $e^{15} + e^{23} + e^{35} - 2e^{46}$                             | $-2e^{124} - 2e^{126} - e^{134} - 2e^{136} - e^{145}$ | $e^7 + 2e^5$      |
|        |                                                                  | $+e^{156} + e^{234} + e^{236} + e^{245} + e^{345}$   | $-2e^4$           |
| 247N   | $-e^{14} - e^{25} - e^{37} - \frac{1}{2}(e^{17} + e^{34})$       | $-e^{123} + e^{157} - e^{247} + e^{345}$            | $e^6$             |
| 247O   | $e^{12} + e^{13} + e^{15} - e^{29} - \frac{1}{2}(e^{27} - e^{45}) + e^{57}$ | $e^{125} + e^{145} + e^{147} + e^{237} + e^{245} - e^{357}$ | $\frac{5}{4}e^6$ |
| 247P   | $e^{17} + e^{23} + e^{45}$                                       | $e^{125} + e^{237} + e^{134} - e^{157}$             | $-2e^6$           |
| 247P1  | $\frac{1}{2}(e^{15} + e^{27}) + e^{17} + e^{23} + e^{35} - e^{34} + \frac{1}{8}e^{45}$ | $e^{125} + e^{237} + e^{134} - e^{157}$             | $e^6 - \frac{1}{8}e^5$ |
| 247Q   | $e^{14} + e^{31} + e^{37}$                                       | $e^{125} - e^{137} + e^{247} + e^{345}$             | $-e^6$            |
| 257A   | $-(e^{12} + e^{31} - e^{46})$                                    | $e^{136} - e^{145} - e^{235} - e^{246}$             | $e^7$             |
| 257C   | $e^{13} + e^{14} + e^{25} - e^{36} - e^{56} + 3e^{46}$            | $e^{124} + e^{156} + e^{236} + e^{345}$             | $2e^7 - e^6 - 2e^3$ |
| 257F   | $-2e^{15} - e^{16} + 3e^{24} + e^{35} + e^{36}$                  | $e^{134} + e^{236} - e^{456} - e^{246} + e^{235}$ | $-4e^7$           |
| 257I   | $e^{12} - e^{35} - e^{45} + e^{46}$                              | $e^{134} + e^{156} - e^{236} - e^{245} - e^{26}$   | $-e^7 - e^3$      |
| 257I1  | $e^{13} + e^{25} + e^{46}$                                       | $-e^{124} + e^{156} - e^{236} - e^{345}$            | $e^7 + e^3$       |
| 357A   | $-e^{15} + e^{26} + \frac{1}{2}(e^{123} + e^{135} + e^{346})$    | $e^{124} + e^{136} - e^{234} - e^{256} - e^{345}$ | $-\frac{1}{8}e^7 + \frac{1}{4}e^6 - \frac{1}{8}e^4$ |
|        |                                                                  | $-e^{12} + e^{246} + e^{156} - e^{345}$             | $e^7 + e^5$       |

\[
\omega \ast \psi \ni \\
\psi \ast \psi_{+} - (2/3) \ast \omega^3 \\
d(\psi) \\
\omega \ast d(\omega) - \psi \ast d(\eta) \\
\omega^2 \ast d(\eta) + 2 \ast \psi_{+} \ast d(\omega)
\]

### 6.4 Indecomposable 4-step nilpotent Lie algebras

We deal next with 7D indecomposable four-step nilpotent Lie algebras. According to Gong [17] there are 43 of them, listed in Tables A7, A8, A9. Four of them (1357\textit{M}(\lambda), 1357\textit{N}(\lambda), 1357\textit{QRS}(\lambda) and 1357\textit{S}(\lambda)) depend on a real parameter \(\lambda\); there are conditions on the value of the parameter, which are included in Appendix A. Table A8.

We have the following result (see Tables 5, 6, 7):

**Theorem 6.7.** All seven-dimensional indecomposable 4-step nilpotent Lie algebra admit a purely coclosed $G_2$-structure, except for: 1357\textit{E}, 1457\textit{A}, 1457\textit{B} and 1357\textit{N}(−2).

In the Lie algebras 1357\textit{M}(\lambda), 1357\textit{N}(\lambda) (\lambda \neq -2) and 1357\textit{QRS}(\lambda), \psi_-$ depends on the parameter \(\lambda\). To check that \(\psi_- \in \Lambda_-(\mathfrak{h}^*)\), we use the commands of Section 6.2, adding an extra variable. However, we are not allowed to set its degree as 0, so we set it equal to 2 (any even degree would do).

Summarizing, the proofs of Theorems 6.5, 6.6 and 6.7 have two parts:

- Using Theorem 4.1, we exhibit an explicit purely coclosed $G_2$-structure on each indecomposable NLA which is not mentioned in the statements of the theorems. These are given in Tables 3, 4, 5, 6 and 7. For the convenience of the reader, we provide SageMath worksheets [5] which can be used as in Section 6.2.
- We use the obstructions of Section 5 to prove that the Lie algebras mentioned in the statements of Theorems 6.5–6.7 do not admit any coclosed $G_2$-structure.

As a consequence of the results of this section, we get the following result:

**Corollary 6.8.** Every seven-dimensional indecomposable nilpotent Lie algebra of nilpotency step \(\leq 4\) admitting a coclosed $G_2$-structure also admits a purely coclosed one.
TABLE 5  Purely coclosed $G_2$-structures on indecomposable 4-step NLAs – 1.

| NLA  | $\omega$                                                                 | $\psi_-$                                                                 | $\eta$                                                                 |
|------|---------------------------------------------------------------------------|--------------------------------------------------------------------------|------------------------------------------------------------------------|
| 1357A | $-e^{12} + e^{34} - e^{56}$                                                | $e^{135} + e^{136} - e^{145} + e^{146} + e^{235} + e^{246}$              | $e^{7} - 2e^{4}$                                                       |
| 1357B | $-e^{13} - e^{24} + \frac{1}{4}e^{45} - e^{56}$                            | $2e^{126} + e^{145} + \frac{1}{2}e^{156} - e^{235} - \frac{1}{2}e^{346}$ | $e^{7} - 4e^{4}$                                                       |
| 1357C | $-\frac{1}{2}e^{12} + 2e^{13} - 4e^{25} - 6e^{35}$                         | $-2e^{124} + 2e^{134} + 4e^{145} + 2e^{156}$                            | $-2e^{7} + 4e^{5}$                                                     |
| 1357D | $e^{12} - e^{13} + e^{56}$                                                 | $e^{136} + e^{145} - e^{235} + e^{246} + e^{246}$                       | $e^{7} + 2e^{3}$                                                       |
| 1357F | $e^{12} + 2e^{13} + e^{35}$                                               | $e^{132} - e^{125} + 2e^{126} + e^{136} + e^{145}$                     | $e^{7} + 2e^{3}$                                                       |
| 1357F1 | $e^{12} - e^{13} + e^{35}$                                                | $-e^{134} + e^{145} + e^{246}$                                         | $e^{7} + e^{6}$                                                        |
| 1357G | $e^{12} - e^{34} - e^{56}$                                                 | $e^{136} + e^{145} - e^{235} - e^{246} + \frac{3}{4}e^{246}$            | $e^{7} + \frac{1}{2}e^{3}$                                             |
| 1357H | $-e^{12} + 2e^{34} - e^{56}$                                              | $e^{135} + e^{145} + e^{235} - e^{236} + 2e^{245}$                     | $e^{7} - \frac{1}{2}e^{3}$                                            |
| 1357I | $e^{12} + e^{23} + 4e^{24} + e^{36} + e^{46}$                             | $2e^{123} + 3e^{124} - 2e^{125} + e^{136} + e^{146}$                   | $e^{7} - 2e^{5}$                                                       |
| 1357J | $-e^{13} + 2e^{23} + e^{25}$                                              | $2e^{135} - \frac{3}{2}e^{136} - e^{146} - e^{235} - 3e^{245} + e^{246}$ | $e^{7} - 3e^{6} + \frac{1}{2}e^{3}$                                    |
| 1357M(\lambda) | $\lambda < -1$                                | $e^{12} - e^{13} + e^{34} - e^{56}$                                   | $e^{7} + \frac{3}{2}e^{3}$                                            |
| 1357M(\lambda) | $\lambda = -1$                                      | $e^{135} - e^{146} - e^{235} - \lambda e^{236} - (\lambda + 1)e^{245} + e^{246}$ | $e^{7} - \frac{3}{2}e^{3}$                                            |
| 1357M(\lambda) | $-1 < \lambda < 0$                                          | $e^{235} + 2e^{145} - e^{36} + e^{46}$                                 | $e^{7} - \frac{3}{2}e^{3}$                                            |
| 1357M(\lambda) | $\lambda > 0$                                             | $e^{135} - e^{146} - \lambda e^{236} - (\lambda + 1)e^{245} + e^{246}$ | $e^{7} - \frac{3}{2}e^{3}$                                            |

TABLE 6  Purely coclosed $G_2$-structures on indecomposable 4-step NLAs – 2.

| NLA  | $\omega$                                                                 | $\psi_-$                                                                 | $\eta$                                                                 |
|------|---------------------------------------------------------------------------|--------------------------------------------------------------------------|------------------------------------------------------------------------|
| 1357N(\lambda) | $\lambda + 2)e^{12} + (\lambda + 2)e^{14} + e^{34} - e^{56}$ | $-2\lambda(\lambda - 2)(e^{126} - e^{134} - e^{146} + e^{245}) + 4e^{135} + 2e^{156}$ | $-\frac{2}{\lambda + 2}e^{7} - \frac{3\lambda - 10}{3\lambda}e^{6} + \frac{7}{3}e^{3}$ |
| 1357N(\lambda) | $\lambda e^{12} + e^{14} + (\lambda + \frac{1}{2})e^{23} + e^{36}$ | $-\frac{5}{2}\lambda^{2}e^{123} - e^{135} + 2e^{136} + 6e^{135}$ | $-\frac{5}{2}\lambda^{2}(e^{7} - e^{3}) - 2\frac{\lambda^{2} + 8\lambda + 8}{\lambda^{2} + 2\lambda + 2}e^{6}$ |
| 1357N(\lambda) | $\lambda e^{12} + e^{14} + (\lambda + \frac{1}{2})e^{23} + e^{36}$ | $-\frac{5}{2}\lambda^{2}e^{123} - e^{135} + 2e^{136} + 6e^{135}$ | $-\frac{5}{2}\lambda^{2}(e^{7} - e^{3}) - 2\frac{\lambda^{2} + 8\lambda + 8}{\lambda^{2} + 2\lambda + 2}e^{6}$ |
| 1357N(0) | $e^{14} + e^{33} + e^{56}$                                               | $e^{132} - e^{135} - 2e^{126} + 2e^{135}$                               | $e^{7} + \frac{16}{25}e^{6} + e^{2} = 3e^{3}$                          |
| 1357N(\lambda) | $2\lambda e^{12} + (\lambda + 2)e^{14} + \lambda e^{34} + e^{36}$ | $-2\lambda(e^{126} - e^{134} - e^{146} - e^{246}) - 4e^{135} + 2e^{136} + 2(\lambda + 2)e^{245}$ | $\frac{4\lambda^{2}(1 + \lambda)(\lambda + 2)}{4\lambda^{2}(1 + \lambda)(\lambda + 2)}e^{6} + \frac{4\lambda^{2}(1 + \lambda)(\lambda + 2)}{4\lambda^{2}(1 + \lambda)(\lambda + 2)}e^{6}$ |
| 1357O | $e^{12} - e^{34} - e^{56}$                                               | $e^{136} + 5e^{145} + \frac{1}{2}e^{146} + e^{235} + e^{236} + e^{245} + e^{246}$ | $e^{7} = \frac{11}{4}e^{3}$                                           |
| 1357P | $e^{12} + e^{34} - e^{56}$                                               | $-e^{135} + e^{146} - e^{235} + e^{236} + 2e^{245} + e^{246}$            | $e^{7} = 2e^{3}$                                                       |
| 1357P1 | $e^{12} + 2e^{34} + 2e^{56}$                                             | $e^{145} + e^{136} + e^{235} + e^{246}$                                 | $-2e^{7} = e^{3}$                                                      |
| 1357Q | $e^{12} + e^{34} + e^{56}$                                               | $-e^{135} + e^{146} + e^{235} + e^{245} + e^{246}$                      | $e^{7} = 2e^{3}$                                                       |
| 1357Q1 | $-e^{12} + e^{34} + 2e^{56}$                                             | $-e^{135} + 2e^{145} - e^{235} + e^{246} + e^{245}$                      | $e^{7} + \frac{153}{8}e^{3}$                                         |
| 1357QRS(\lambda) | $\lambda > 0$                                                      | $e^{12} + \lambda e^{34} + \lambda e^{56}$ | $-e^{135} + 2e^{145} + \lambda e^{146} + \lambda e^{235} + 2e^{245} + e^{246} - \lambda e^{246}$ | $\lambda e^{7} + \frac{1}{2}(3\lambda + 1)e^{3}$ |
**Table 7** Purely coclosed $G_2$-structures on indecomposable 4-step NLAs $- 3$.

| NLA $\lambda$ | $\omega$ | $\psi_-$ | $\eta$ |
|----------------|----------|-----------|--------|
| $1357QRS\lambda$ | $-e^{12} + \lambda e^{34} - \lambda e^{56}$ | $-e^{135} + 2e^{145} + \lambda e^{146} + e^{235}$ | $-\lambda e^7 + \frac{1}{2}(3\lambda + 1)e^3$ |
| $\lambda < 0$ | $+e^{236} + e^{245} - \lambda e^{246}$ | |
| $1357R$ | $-e^{12} - e^{34} + e^{56}$ | $-e^{135} - e^{145} + e^{235} + e^{245} + e^{246}$ | $e^7 + e^3$ |
| $1357S(0)$ | $-e^{12} - 3e^{34} + e^{56} - e^{35} + 2e^{46}$ | $-e^{145} + e^{136} - e^{245} - e^{235} - 2e^{246}$ | |
| $1357S\lambda$ | $\lambda \neq 0$ | $e^{135} - e^{125} - e^{146} + e^{235} + e^{246}$ | $\frac{1}{\lambda} e^7 - \left(\frac{1}{\lambda} + \frac{4}{3}\right)e^3$ |
| $2357A$ | $e^{13} - e^{24} + e^{37}$ | $-e^{125} - e^{137} - e^{237} - e^{345}$ | $3e^6$ |
| $2357B$ | $e^{13} + e^{24} - e^{57}$ | $-e^{125} - e^{137} - e^{237} - e^{345}$ | $-3e^6$ |
| $2357C$ | $e^{15} + e^{24} + e^{37}$ | $-e^{125} + e^{127} - e^{237} - e^{345}$ | $3e^6$ |
| $2357D$ | $3e^{13} - e^{24} + e^{37} + e^{12} - e^{34} + e^{14} - e^{23}$ | $-e^{125} + e^{127} - e^{237} - e^{345}$ | $-e^7 + 3e^6 - 2e^4$ |
| $2357D_1$ | $e^{13} + e^{24} - e^{37} + \frac{1}{2}(e^{15} - e^{37})$ | $e^{125} + e^{147} - e^{237} - e^{345}$ | $3e^6 - \frac{1}{2}e^4$ |
| $2457A$ | $e^{15} + e^{24} + e^{36}$ | $e^{125} - e^{146} + e^{256} + e^{345}$ | $e^7$ |
| $2457B$ | $2e^{15} - e^{24} - e^{36} + e^{13} - e^{56}$ | $e^{125} - e^{146} - e^{256} - e^{345}$ | $e^7 - e^6 + e^4$ |
| $2457C$ | $e^{15} + e^{24} - e^{36}$ | $-e^{125} + e^{146} + e^{256} + 2e^{345}$ | $e^7$ |
| $2457D$ | $2e^{15} - e^{23} + 2e^{26}$ | $e^{124} + e^{136} + e^{256} + 2e^{235} + e^{345}$ | $10e^7 + 6e^3$ |
| $2457E$ | $e^{15} - e^{23} + e^{47}$ | $e^{124} + e^{137} + e^{257} + e^{345}$ | $-e^6 + e^4$ |
| $2457F$ | $-e^{14} - e^{23} - 2e^{46} - e^{56}$ | $e^{124} + e^{125} + e^{136} - e^{256} - e^{345}$ | $-e^7$ |
| $2457G$ | $2e^{15} - e^{23} + e^{47}$ | $e^{124} + e^{137} + e^{257} + e^{345}$ | $-2e^6$ |
| $2457H$ | $e^{12} - e^{13} + e^{46}$ | $e^{136} + e^{145} + e^{234} + e^{256}$ | $-e^7 + e^4$ |
| $2457I$ | $e^{15} + e^{23} - e^{46} - e^{56} + \frac{1}{2}(e^{13} + e^{25})$ | $e^{124} + e^{125} + e^{136} - e^{256} - e^{345}$ | $\frac{1}{2}e^7 + e^4$ |
| $2457J$ | $e^{15} + e^{23} - e^{46} - e^{56}$ | $e^{124} + e^{125} + e^{136} - e^{256} - e^{345}$ | $-e^7 - e^4$ |
| $2457K$ | $e^{15} + e^{23} - e^{46}$ | $e^{124} + e^{145} + e^{234} + e^{256}$ | $-e^7 + e^4 + e^3$ |
| $2457L$ | $e^{15} + e^{24} + e^{36}$ | $e^{123} + e^{134} - e^{146} + e^{235} + 2e^{256} + 2e^{345}$ | $18e^7$ |
| $2457L_1$ | $-e^{14} + 2e^{25} + e^{37}$ | $-e^{123} + e^{135} + e^{157} + e^{234} + e^{247} - 2e^{345}$ | $\frac{2}{3}e^6$ |
| $2457M$ | $2e^{15} + e^{24} + e^{37}$ | $e^{123} - e^{147} + e^{257} + e^{345}$ | $\frac{2}{3}e^6$ |

We expect that all indecomposable seven-dimensional nilpotent Lie algebras carrying a coclosed $G_2$-structure also admit a purely coclosed one. Indeed, besides the ad hoc argument which allows to rule out $\mathfrak{n}_2$ from the list of Lie algebras admitting a purely coclosed $G_2$-structure (see [9, Corollary 4.3]), no general obstruction to the existence of a purely coclosed $G_2$-structure is known. However, it is difficult to deduce the existence of a purely coclosed $G_2$-structure directly from the existence of a coclosed one, under the indecomposability assumption.

For the sake of completeness, we give an example of a seven-dimensional indecomposable nilpotent Lie algebra of nilpotency step 5 admitting a purely coclosed $G_2$-structure.

**Example 6.9.** Consider the Lie algebra $12357B = (0, 0, 0, 12, 14 + 23, 15 - 34, 16 + 23 - 35)$ from [17]. Set $V = \text{span}(e_1, \ldots, e_6)$ and

- $\omega = -2e^{12} - \frac{7}{4}e^{13} + e^{14} - e^{23} - e^{26} + e^{45} - e^{56};$
- $\psi_- = -e^{124} - 2e^{125} + e^{145} - e^{146} - 2e^{234} - e^{235} - e^{236} - e^{345};$
- $\eta = e^7 + e^6 + \frac{1}{2}e^5 + 2e^4.$

Then, $(\omega, \psi_-)$ defines an SU(3)-structure on $V$, hence $\varphi = \omega \wedge \eta + \psi_-$ defines a $G_2$-structure on $12357B$. Also, conditions 1–3 of Theorem 4.1 are satisfied, and the $G_2$-structure is purely coclosed.

A final remark concerns the existence of exact purely coclosed $G_2$-structures: these are purely coclosed $G_2$-structures whose 4-form $\varphi = \ast \varphi$ is exact. Note that there are examples of exact coclosed $G_2$-structures: it suffices to consider a nearly parallel $G_2$-structure, for which $d\varphi = \ast \varphi \varphi = \varphi$. All the purely coclosed $G_2$-structures we constructed in this paper turn...
out not to be exact. The same question has been asked in the context of closed $G_2$-structures. A $G_2$-structure is exact if its defining 3-form $\varphi$ is exact. It was recently proved that no compact quotient of a Lie group by a discrete subgroup admits an exact $G_2$-structure which is left-invariant (see [13, Theorem 1.1]).

**Conjecture 6.10.** No compact quotient of a Lie group by a discrete subgroup admits an exact purely coclosed $G_2$-structure.

## 7 7D NILPOTENT LIE ALGEBRAS WITH NO COCLOSED $G_2$-STRUCTURES

We prove that the following algebras do not have coclosed $G_2$-structures:

- $27A, 27B$
- $247E, 247E_1, 247G, 247H, 247H_1, 247K, 247R, 247R_1$
- $257B, 257D, 257E, 257G, 257H, 257K, 257L$
- $357B, 357C$
- $1357E, 1357N(−2)$
- $1457A, 1457B$

In order to prove this, we use three different types of obstructions. We list in order:

- We use Corollary 5.3 to cover the families 27, 247, 257, as well as 1357E. For instance, for 27A, we have the following worksheet:
  
  A. $<x_1, x_2, x_3, x_4, x_5, x_6, x_7> = \text{GradedCommutativeAlgebra}(\mathbb{Q})$
  
  $M=\text{A.cdg_algebra}(x_6: x_1*x_2, x_7: x_1*x_5 + x_2*x_3)$
  
  M.inject_variables()
  
  M.cohomology(4)
  
  Defining $x_1, x_2, x_3, x_4, x_5, x_6, x_7$
  
  Free module generated by $[x_1*x_2*x_3*x_6], [x_1*x_2*x_4*x_6], [x_1*x_3*x_4*x_6], [x_2*x_3*x_4*x_6],$
  
  $[x_1*x_2*x_5*x_6], [x_1*x_4*x_5*x_6], [x_2*x_4*x_5*x_6], [x_3*x_4*x_5*x_6-x_2*x_3*x_5*x_7],$
  
  $[x_1*x_3*x_4*x_7], [x_2*x_3*x_4*x_7], [x_1*x_3*x_5*x_7], [x_1*x_4*x_5*x_7], [x_2*x_4*x_5*x_7],$
  
  $[x_3*x_4*x_5*x_7], [x_1*x_3*x_6*x_7] \text{ over Rational Field}$

  Recall that in the worksheet we write $x_i$ for $e^i$. We use $X = e_6, Y = e_7$, and the space $U = \langle e^{13} \rangle$. Using the list of representatives of the cohomology classes, we see that $t_{i_6, i_7, z_{a_6}} \in U$. Also $\Lambda^2 U = 0$, as required.

- We use Corollary 5.5 for the algebras 357B and 357C. In the Sage worksheet, we find the closed 4-forms by computing the degree 4 cohomology and the exact 4-forms, explicitly.

- We use Proposition 5.6 for the algebras 1357N(−2), 1457A, and 1457B. The Sage worksheets are self-commented and constructive. Bases of the spaces of closed 2-forms $\beta$ and closed 3-forms $\tau$ are found. Then, the conditions $\beta \wedge d\beta, \tau \wedge d\tau$ are checked. The computation of $\lambda(\sum a_\alpha z_\alpha)$ is done using parameters and the implementation of the condition $\lambda(\sum a_\alpha z_\alpha) \in W$ gives linear equations among $(a_\alpha)$ that reduces the expression of $\lambda(\sum a_\alpha z_\alpha)$ to a square in each case.

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APPENDIX A: 7D NILPOTENT LIE ALGEBRAS OF NILPOTENCY STEP ≤ 4

In this Appendix, we list seven-dimensional nilpotent Lie algebras of nilpotency step ≤ 4 using differentials. The Chevalley–Eilenberg differential \( d : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^* \) is dual to the bracket \([\cdot,\cdot] : \Lambda^2 \mathfrak{g} \to \mathfrak{g} \). The list of Gong [17] is given in terms of a nilpotent frame \( \{e_1, \ldots, e_7\} \) of \( \mathfrak{g} \); we give the structure constants in terms of the nilpotent coframe \( \{-e^1, \ldots, -e^7\} \) of \( \mathfrak{g}^* \).

- In the family 147E(\( \lambda \)), two Lie algebras 147E(\( \lambda_1 \)) and 147E(\( \lambda_2 \)) are isomorphic if and only if \( \frac{(1-\lambda_1+\lambda_2^3)}{\lambda_2^3(\lambda_2-1)^2} = \frac{(1-\lambda_2+\lambda_1^3)}{\lambda_1^3(\lambda_1-1)^2} \);

- in the family 1357QRS(\( \lambda_1 \)), two Lie algebras 1357QRS(\( \lambda_1 \)) and 1357QRS(\( \lambda_2 \)) are isomorphic if and only if \( \lambda_1 + \lambda_2^{-1} = \lambda_2 + \lambda_1^{-1} \), that is, if and only if \( \lambda_2 = \lambda_1 \) or \( \lambda_2 = \frac{1}{\lambda_1} \).
### Table A1: Seven-dimensional decomposable nilpotent Lie algebras – 1.

| NLA   | Structure equations                  | NLA   | Structure equations                  |
|-------|-------------------------------------|-------|-------------------------------------|
| n₁    | (0,0,0,0,0,0,0)                     | n₂    | (0,0,0,0,12,0)                      |
| n₃    | (0,0,0,0,12,34,0)                   | n₄    | (0,0,0,12,13,0)                     |
| n₅    | (0,0,0,12,14,23,0)                  | n₆    | (0,0,0,12,14 + 23,0)                |
| n₇    | (0,0,0,12,15 + 34,0)                | n₈    | (0,0,0,12,15 + 25,0)                |
| n₉    | (0,0,0,12,23,14 + 35,0)             | n₁₀   | (0,0,0,12,14,15 + 34,0)             |
| n₁₁   | (0,0,12,13,14 + 35,0)               | n₁₂   | (0,0,0,12,12,13,14 + 35,0)          |
| n₁₃   | (0,0,0,12,13,14 + 35,0)             | n₁₄   | (0,0,0,12,14,15 + 24 + 23,0)        |
| n₁₅   | (0,0,0,12,13,14 + 35,0)             | n₁₆   | (0,0,0,12,13,14 + 23,0)             |
| n₁₇   | (0,0,12,13,14 + 23,0)               | n₁₈   | (0,0,0,12,12,13,14 + 23,0)          |

### Table A2: Seven-dimensional decomposable nilpotent Lie algebras – 2.

| NLA   | Structure equations                  | NLA   | Structure equations                  |
|-------|-------------------------------------|-------|-------------------------------------|
| n₁₉   | (0,0,12,14,15 + 23,0)                | n₂₀   | (0,0,0,12,14 − 23,15 + 34,0)        |
| n₂₁   | (0,0,12,13,23,14 + 25,0)             | n₂₂   | (0,0,12,13,23,14 − 25,0)            |
| n₂₃   | (0,0,12,13,23,14,0)                  | n₂₄   | (0,0,12,13,14 + 23,15 + 24,0)       |
| g₁    | (0,0,0,12,15,0)                      | g₂    | (0,0,0,23,34,36)                    |
| g₂    | (0,0,0,12,13,14,0)                   | g₄    | (0,0,0,12,14,24,0)                  |
| g₃    | (0,0,12,13,14,35 + 23,0)             | g₅    | (0,0,12,13,14 + 23,34 − 25,0)       |
| g₄    | (0,0,12,13,14,23,14 + 23,0)          | g₆    | (0,0,12,13,14,15 + 23,0)            |
| g₅    | (0,0,12,13,14,23,14 + 23,0)          | g₇    | (0,0,12,13,14,34 − 25,0)            |
| g₆    | (0,0,12,13,14,23,14 + 23,0)          | g₈    | (0,0,12,13,14 + 23,34 − 25,0)       |
| l₁    | (0,0,0,12,13 − 24,14 + 23,0)         | l₂    | (0,0,0,12,14,13 − 24,0)             |
| l₃    | (0,0,0,12,14,13 + 24,0)              |       |                                     |

### Table A3: Seven-dimensional indecomposable 2-step nilpotent Lie algebras.

| NLA   | Structure equations                  | Center | NLA   | Structure equations                  | Center |
|-------|-------------------------------------|--------|-------|-------------------------------------|--------|
| 17    | (0³, 12 + 34 + 56)                  | ⟨e₇⟩  | 27A   | (0⁵, 12, 14 + 35)                   | ⟨e₆, e₇⟩ |
| 27B   | (0³, 12 + 34, 15 + 23)              | ⟨e₆, e₇⟩ | 37A   | (0⁵, 12, 23,24)                    | ⟨e₅, e₆, e₇⟩ |
| 37B   | (0³, 12, 23,34)                     | ⟨e₅, e₆, e₇⟩ | 37B₁  | (0⁵, 12 − 34, 13 + 24,14)         | ⟨e₅, e₆, e₇⟩ |
| 37C   | (0³, 12 + 34, 23,24)                | ⟨e₅, e₆, e₇⟩ | 37D   | (0⁵, 12 + 34, 13,24)              | ⟨e₅, e₆, e₇⟩ |
| 37D₁  | (0³, 12 − 34, 13 + 24,14 + 23)      | ⟨e₅, e₆, e₇⟩ |

### Table A4: Seven-dimensional indecomposable 3-step nilpotent Lie algebras – 1.

| NLA   | Structure equations                  | Center |
|-------|-------------------------------------|--------|
| 137A  | (0⁵, 12, 34, 15 + 36)               | ⟨e₇⟩  |
| 137A₁ | (0⁵, 13 + 24, 14 − 23,15 + 26)      | ⟨e₇⟩  |
| 137B  | (0⁵, 12, 34, 15 + 24 + 36)          | ⟨e₇⟩  |
| 137B₁ | (0⁵, 13 + 24, 14 − 23,15 + 26 + 34) | ⟨e₇⟩  |
| 137C  | (0⁵, 12, 14 + 23,16 − 35)           | ⟨e₇⟩  |
**TABLE A5** Seven-dimensional indecomposable 3-step nilpotent Lie algebras – 2.

| NLA      | Structure equations       | Center     |
|----------|---------------------------|------------|
| 137D     | \((0^4, 12, 14 + 23, 16 + 24 - 35)\) | \((e_7)\)   |
| 147A     | \((0^4, 12, 13, 0, 16 + 25 + 34)\) | \((e_7)\)   |
| 147A₁    | \((0^4, 12, 13, 0, 16 + 24 + 35)\) | \((e_7)\)   |
| 147B     | \((0^4, 12, 13, 0, 14 + 26 + 35)\) | \((e_7)\)   |
| 147C     | \((0^4, 12, 23, -13, 15 + 16 + 26 + 2 \cdot 34)\) | \((e_7)\) |
| 147E(\(\lambda\)) | \((0^3, 12, 23, -13, -15 + \lambda \cdot 26 + (1 - \lambda) \cdot 34), \lambda \neq 0, 1\) | \((e_7)\) |
| 147E₁(\(\lambda\)) | \((0^3, 12, 23, -13, -\lambda \cdot 16 + \lambda \cdot 25 + 2 \cdot 26 - 2 \cdot 34), \lambda > 1\) | \((e_7)\) |
| 157      | \((0^2, 12, 0, 0, 0, 13 + 24 + 56)\) | \((e_7)\)   |
| 247A     | \((0^1, 12, 13, 14, 15)\) | \((e_6, e_7)\) |
| 247B     | \((0^1, 12, 13, 14, 35)\) | \((e_6, e_7)\) |
| 247C     | \((0^1, 12, 13, 14 + 35, 15)\) | \((e_6, e_7)\) |
| 247D     | \((0^1, 12, 13, 14, 25 + 34)\) | \((e_6, e_7)\) |
| 247E     | \((0^1, 12, 13, 14 + 15, 25 + 34)\) | \((e_6, e_7)\) |
| 247E₁    | \((0^1, 12, 13, 14, 24 + 35)\) | \((e_6, e_7)\) |
| 247F     | \((0^1, 12, 13, 24 + 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247F₁    | \((0^1, 12, 13, 24 - 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247G     | \((0^1, 12, 13, 14 + 15 + 24 + 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247H     | \((0^1, 12, 13, 14 + 24 + 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247H₁    | \((0^1, 12, 13, 14 + 24 - 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247I     | \((0^1, 12, 13, 25 + 34, 35)\) | \((e_6, e_7)\) |
| 247J     | \((0^1, 12, 13, 15 + 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247K     | \((0^1, 12, 13, 14 + 35, 25 + 34)\) | \((e_6, e_7)\) |
| 247L     | \((0^1, 12, 13, 14 + 23, 15)\) | \((e_6, e_7)\) |
| 247M     | \((0^1, 12, 13, 14 + 23, 35)\) | \((e_6, e_7)\) |
| 247N     | \((0^1, 12, 13, 15 + 24, 23)\) | \((e_6, e_7)\) |
| 247O     | \((0^1, 12, 13, 14 + 35, 15 + 23)\) | \((e_6, e_7)\) |
### TABLE A6  Seven-dimensional indecomposable 3-step nilpotent Lie algebras – 3.

| NLA | Structure equations | Center |
|-----|---------------------|--------|
| 247\(P\) | \((0^1, 12, 13, 23, 25 + 34)\) | \(\langle e_6, e_7 \rangle\) |
| 247\(P_1\) | \((0^1, 12, 13, 23, 24 + 35)\) | \(\langle e_6, e_7 \rangle\) |
| 247\(Q\) | \((0^1, 12, 13, 14 + 23, 25 + 34)\) | \(\langle e_6, e_7 \rangle\) |
| 247\(R\) | \((0^1, 12, 13, 14 + 15 + 23, 25 + 34)\) | \(\langle e_6, e_7 \rangle\) |
| 247\(R_1\) | \((0^1, 12, 13, 14 + 23, 24 + 35)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(A\) | \((0^1, 12, 0, 0, 13 + 24, 15)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(B\) | \((0^1, 12, 0, 13, 14 + 25)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(C\) | \((0^1, 12, 0, 0, 13 + 24, 25)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(D\) | \((0^1, 12, 0, 0, 13 + 24, 14 + 25)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(E\) | \((0^1, 12, 0, 0, 13 + 45, 24)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(F\) | \((0^1, 12, 0, 0, 23 + 45, 24)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(G\) | \((0^1, 12, 0, 0, 13 + 45, 15 + 24)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(H\) | \((0^1, 12, 0, 0, 13 + 24, 45)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(I\) | \((0^1, 12, 0, 0, 13 + 14, 15 + 23)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(J\) | \((0^1, 12, 0, 0, 13 + 24, 15 + 23)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(J_1\) | \((0^1, 12, 0, 0, 13 + 14 + 25, 15 + 23)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(K\) | \((0^1, 12, 0, 0, 13, 23, 14)\) | \(\langle e_6, e_7 \rangle\) |
| 257\(L\) | \((0^1, 12, 0, 0, 13 + 24, 23 + 45)\) | \(\langle e_6, e_7 \rangle\) |
| 357\(A\) | \((0^1, 12, 0, 13, 24, 14)\) | \(\langle e_5, e_6, e_7 \rangle\) |
| 357\(B\) | \((0^1, 12, 0, 13, 23, 14)\) | \(\langle e_5, e_6, e_7 \rangle\) |
| 357\(C\) | \((0^1, 12, 0, 13 + 24, 23, 14)\) | \(\langle e_5, e_6, e_7 \rangle\) |

### TABLE A7  Seven-dimensional indecomposable 4-step nilpotent Lie algebras – 1.

| NLA | Structure equations | Center |
|-----|---------------------|--------|
| 1357\(A\) | \((0^1, 12, 14 + 23, 0, 15 + 26 - 34)\) | \(\langle e_7 \rangle\) |
| 1357\(B\) | \((0^1, 12, 14 + 23, 0, 15 - 34 + 36)\) | \(\langle e_7 \rangle\) |
| 1357\(C\) | \((0^1, 12, 14 + 23, 0, 15 + 24 - 34 + 36)\) | \(\langle e_7 \rangle\) |
| 1357\(D\) | \((0^1, 12, 0, 23, 24, 16 + 25 + 34)\) | \(\langle e_7 \rangle\) |
| 1357\(E\) | \((0^1, 12, 0, 23, 24, 25 + 46)\) | \(\langle e_7 \rangle\) |
| 1357\(F\) | \((0^1, 12, 0, 23, 24, 13 + 25 - 46)\) | \(\langle e_7 \rangle\) |
### TABLE A8  Seven-dimensional indecomposable four-step nilpotent Lie algebras – 2.

| NLA     | Structure equations | Center |
|---------|---------------------|--------|
| 1357F₁ | \((0^5, 12, 0, 23, 24, 13 + 25 + 46)\) | \((e_7)\) |
| 1357G  | \((0^5, 12, 0, 23, 14, 16 + 25)\) | \((e_7)\) |
| 1357H  | \((0^5, 12, 0, 23, 14, 16 + 25 + 26 - 34)\) | \((e_7)\) |
| 1357I  | \((0^5, 12, 0, 23, 14, 25 + 46)\) | \((e_7)\) |
| 1357J  | \((0^5, 12, 0, 23, 14, 13 + 25 + 46)\) | \((e_7)\) |
| 1357L  | \((0^5, 12, 0, 13 + 24, 14, 15 + 23 + \frac{1}{2} \cdot 26 + \frac{1}{2} \cdot 34)\) | \((e_7)\) |
| 1357M(\(\lambda\)) | \((0^5, 12, 0, 13 + 24, 14, 15 + \lambda \cdot 26 + (1 - \lambda) \cdot 34), \lambda \neq 0\) | \((e_7)\) |
| 1357N(\(\lambda\)) | \((0^5, 12, 0, 13 + 24, 14, 15 + \lambda \cdot 23 + 34 + 46)\) | \((e_7)\) |
| 1357O  | \((0^5, 12, 0, 13 + 24, 23, 16 + 25)\) | \((e_7)\) |
| 1357P  | \((0^5, 12, 0, 13 + 24, 23, 15 + 26 + 34)\) | \((e_7)\) |
| 1357P₁ | \((0^5, 12, 0, 13 + 24, 23, 15 - 26 + 34)\) | \((e_7)\) |
| 1357Q  | \((0^5, 12, 0, 13, 23 + 24, 15 + 26)\) | \((e_7)\) |
| 1357Q₁ | \((0^5, 12, 0, 13, 23 + 24, 15 - 26)\) | \((e_7)\) |
| 1357QRS(\(\lambda\)) | \((0^5, 12, 0, 13 + 24, 14, 15 + 23 + 26 + (1 - \lambda) \cdot 34), \lambda \neq 0\) | \((e_7)\) |
| 1357R  | \((0^5, 12, 0, 13, 23 + 24, 16 + 25 + 34)\) | \((e_7)\) |
| 1357S(\(\lambda\)) | \((0^5, 12, 0, 13, 23 + 24, 15 + 26 + 34, \lambda \neq 1\) | \((e_7)\) |
| 1457A  | \((0^5, 12, 13, 0, 0, 14 + 56)\) | \((e_7)\) |
| 1457B  | \((0^5, 12, 13, 0, 0, 14 + 23 + 56)\) | \((e_7)\) |
| 2357A  | \((0^5, 12, 14 + 23, 23, 15 - 34)\) | \((e_6, e_7)\) |
| 2357B  | \((0^5, 12, 14 + 23, 13, 15 - 34)\) | \((e_6, e_7)\) |
| 2357C  | \((0^5, 12, 14 + 23, 24, 15 - 34)\) | \((e_6, e_7)\) |
| 2357D  | \((0^5, 12, 14 + 23, 13 + 24, 15 - 34)\) | \((e_6, e_7)\) |
| 2357D₁ | \((0^5, 12, 14 + 23, 13 - 24, 15 - 34)\) | \((e_6, e_7)\) |
| 2457A  | \((0^5, 12, 13, 0, 14, 15)\) | \((e_6, e_7)\) |
| 2457B  | \((0^5, 12, 13, 0, 25, 14)\) | \((e_6, e_7)\) |
| 2457C  | \((0^5, 12, 13, 0, 14 + 25, 15)\) | \((e_6, e_7)\) |
| 2457D  | \((0^5, 12, 13, 0, 14 + 23 + 25, 15)\) | \((e_6, e_7)\) |
| 2457E  | \((0^5, 12, 13, 0, 23 + 25, 14)\) | \((e_6, e_7)\) |
| 2457F  | \((0^5, 12, 13, 0, 14 + 23, 15)\) | \((e_6, e_7)\) |
| 2457G  | \((0^5, 12, 13, 0, 15 + 23, 14)\) | \((e_6, e_7)\) |
| 2457H  | \((0^5, 12, 13, 0, 23, 14 + 25)\) | \((e_6, e_7)\) |

### TABLE A9  Seven-dimensional indecomposable four-step nilpotent Lie algebras – 3.

| NLA     | Structure equations | Center |
|---------|---------------------|--------|
| 2457I  | \((0^5, 12, 13, 0, 14 + 23, 25)\) | \((e_6, e_7)\) |
| 2457J  | \((0^5, 12, 13, 0, 14 + 23, 23 + 25)\) | \((e_6, e_7)\) |
| 2457K  | \((0^5, 12, 13, 0, 15 + 23, 14 + 25)\) | \((e_6, e_7)\) |
| 2457L  | \((0^5, 12, 13, 23, 14 + 25, 15 + 24)\) | \((e_6, e_7)\) |
| 2457L₁ | \((0^5, 12, 13, 23, 14 - 25, 15 + 24)\) | \((e_6, e_7)\) |
| 2457M  | \((0^5, 12, 13, 23, 15 + 24, 14)\) | \((e_6, e_7)\) |