Total dominator chromatic number of $P_m \times C_n$

A. Vijayalekshmi$^1*$ and S. Anusha $^2$

Abstract
A total dominator coloring of a graph $G$ with $\delta(G) \geq 1$ is a proper coloring of $G$ with the extra property that every vertex in $G$ properly dominates a color class. The total dominator chromatic number of $G$ is the minimum number of colors needed in a total dominator coloring of $G$, denoted by $\chi_{td}(G)$. In this paper, we obtain total dominator chromatic number of $P_m \times C_n$.

Keywords
Total dominator chromatic number, ladder graph, grid graph and $P_m \times C_n$.

AMS Subject Classification
05C15, 05C69.

$^1$Department of Mathematics, S.T. Hindu College, Nagercoil-629002, Tamil Nadu, India.
$^2$Research Scholar [Reg. No:11506], Department of Mathematics, S.T. Hindu College, Nagercoil-629002, Tamil Nadu, India.
$^1$Corresponding author: vijitmath.a@gmail.com

1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definition of graph theory as found in [1]. Let $G=(V,E)$ be a graph of order $n$ with $\delta(G) \geq 1$. The open neighborhood $N(v) = \{u \in V(G) : uv \in E(G)\}$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The path and cycle of order $n$ are denoted by $P_n$ and $C_n$ respectively. For any two graphs $G$ and $H$, we define the cartesian product, denoted by $G \times H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices $(u_1, v_1)$ and $(u_2, v_2)$ iff either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$. $P_m \times C_n$ is defined as the cartesian product of path and cycle. A grid graphs can be defined as $P_m \times P_n$ where $m, n \geq 2$.

A subset $S$ of $V$ is called a total dominating set if every vertex in $V$ is adjacent to some vertex in $S$. The total dominating set is minimal total dominating set if no proper subset of $S$ is a total dominating set of $G$. The total domination number $\gamma_t$ is the minimum cardinality taken over all minimal total dominating set of $G$. A $\gamma_t$-set is any minimal total dominating set with cardinality $\gamma_t$.

A proper coloring of $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices have different colors. The chromatic number, $\chi(G)$, is the minimum number of colors in a proper coloring of $G$. A total dominator coloring of a graph $G$ is a proper coloring of $G$ with the extra property that every vertex in $G$ properly dominates a color class. The total dominator chromatic number of $G$ is the minimum number of colors needed in a total dominator coloring of $G$ denoted by $\chi_{td}(G)$. This concept was introduced by A. Vijayalekshmi in [2]. This notion is also referred as a Smarandachely $k$-dominator coloring of $G(k \geq 1)$ and was introduced by A. Vijayalekshmi in [4]. For an integer $k \geq 1$, a Smarandachely $k$-dominator coloring of $G$ is a proper coloring of $G$ such that every vertex in $G$ properly dominates a $k$ color class. The smallest number of colors for which there exist a Smarandachely $k$-dominator coloring of $G$ is called the Smarandachely $k$-dominator chromatic number of $G$, and is denoted by $\chi_{td}^k(G)$. For further details on this theory and on its applications, we advice the reader to refer [6–9].

In a proper coloring $\mathcal{C}$ of $G$, a color class of $\mathcal{C}$ is a set consisting of all those vertices assigned the same color. Let $\mathcal{C}^1$ be a minimal $td$-coloring of $G$. We say that a color class $c_1 \in \mathcal{C}^1$ is called a non-dominated color class ($n-d$ color class) if it is not dominated by any vertex of $G$. These color classes are also called repeated color classes.
2. Preliminaries

In this segment, we remember the critical [3] theorem which is quite helpful in our research. For the subsequent observation the total dominator chromatic number of a ladder graphs has been identified.

Theorem 2.1. [3] Let $G$ be $p_n$ or $C_n$. Then

$$\chi_{td}(G) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor + 2, & \text{if } n \equiv 0 \pmod{4} \\
\left\lfloor \frac{n}{2} \right\rfloor + 3, & \text{if } n \equiv 1 \pmod{4} \\
\left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor + 2, & \text{otherwise.} 
\end{cases}$$

Theorem 2.2. [3] For every $n \geq 2$, the total dominator chromatic number of a ladder graph $L_n$ is

$$\chi_{td}(L_n) = \begin{cases} 
2\left\lfloor \frac{n}{2} \right\rfloor + 2, & \text{if } n \equiv 0 \pmod{6} \\
2\left\lfloor \frac{n}{2} - \frac{1}{2} \right\rfloor + 4, & \text{if } n \equiv 1 \pmod{6} \\
2\left\lfloor \frac{n}{2} - \frac{1}{2} \right\rfloor + 4, & \text{otherwise.} 
\end{cases}$$

In this paper, we obtain the least value for total dominator chromatic number for $P_m \times C_n$.

3. Main Result

In this section, we present and establish the main results.

For our convenience, we denote $G_{m,n} = P_m \times C_n$ and let $D = \{v_{ij} | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$.

Lemma 3.1. For every $n$, $\chi_{td}(G_{2,n}) = 2\left\lfloor \frac{n}{2} \right\rfloor + 2$.

Proof. Since the td-colouring of $G_{2,n}$ is same as td-colouring of $L_n$, $\chi_{td}(G_{2,n}) = \chi_{td}(L_n)$. From Theorem 2.2, we get

$$\chi_{td}(G_{2,n}) = 2\left\lfloor \frac{m}{3} \right\rfloor + 2.$$

Illustration: Consider $G_{2,11}$

![Figure 1](image1)

Therefore

$$\chi_{td}(G_{2,11}) = 10.$$

Theorem 3.2. If $m,n \equiv 0 \pmod{3}$, then $\chi_{td}(G_{m,n}) = \frac{mn}{3} + 2$.

Proof. Let $D = \{v_{ij} | 1 \leq i \leq m \text{ and } j = 2,5,8,\ldots,(n-1)\}$ be a unique $\gamma$-set of $G_{m,n}$. We assign $\frac{mn}{3}$ distinct colors say $4,5,6,\ldots,\frac{mn}{3} + 1,\frac{mn}{3} + 2,\frac{mn}{3} + 3$ to vertices of $D$. Set $S = V(G_{m,n}) - D$, we assign two repeated colors say 1,2 to the vertices $v_{ij}$ and $v_{kl} \in S$ such that $|i-k| + |j-l| = 1$ and adjacent vertices in $S$ received different colors, we get a td-coloring of $G_{m,n}$.

So

$$\chi_{td}(G_{m,n}) = \frac{mn}{3} + 2.$$

Illustration: Consider $G_{6,10}$

![Figure 3](image3)

Therefore

$$\chi_{td}(G_{6,10}) = 22.$$

Illustration: Consider $G_{6,9}$

![Figure 2](image2)
Theorem 3.4. If \( m \equiv 1 \pmod{3} \) then

\[
\chi_{td}(G_{m,n}) = \begin{cases} 
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4} \\
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n}{4} \right\rceil + 4 & \text{if } n \equiv 1 \pmod{4} \\
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n+1}{4} \right\rceil + 2 & \text{if } n \equiv 2 \pmod{4} \\
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n+2}{4} \right\rceil + 3 & \text{otherwise.}
\end{cases}
\]

Proof. Since \( m-1 \equiv 0 \pmod{3} \), \( G_{m,n} \) is obtained by \( G_{m-1,n} \) followed by \( G_1 \). In a \( td \)-coloring of \( G_{m,n} \), \( \chi_{td}(G_{m,n}) = \chi_{td}(G_{m-1,n}) + \chi_{td}(G_1) \). Also the used repeated colors are the same the \( td \)-coloring of \( G_1 \). So \( \chi_{td}(G_{m,n}) = \chi_{td}(G_{m-1,n}) + \chi_{td}(G_1) - 2 \). By Theorem 2.1, we get

\[
\chi_{td}(G_{m,n}) = \begin{cases} 
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4} \\
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n}{4} \right\rceil + 4 & \text{if } n \equiv 1 \pmod{4} \\
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n+1}{4} \right\rceil + 2 & \text{if } n \equiv 2 \pmod{4} \\
\left(\frac{m-1}{3}\right)n + 2\left\lceil \frac{n+2}{4} \right\rceil + 3 & \text{otherwise.}
\end{cases}
\]

Illustration: Consider \( G_{6,7} \)

Therefore

\[ \chi_{td}(G_{6,7}) = 17. \]

Illustration: Consider \( G_{6,7} \)

Therefore

\[ \chi_{td}(G_{6,7}) = 17. \]

Theorem 3.5. If \( m \equiv 2 \pmod{3} \), then

\[
\chi_{td}(G_{m,n}) = \begin{cases} 
\left(\frac{m-2}{3}\right)n + 2\left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \text{ is even} \\
\left(\frac{m-2}{3}\right)n + 2\left\lceil \frac{n}{3} \right\rceil + 3 & \text{if } n \text{ is odd.}
\end{cases}
\]

Proof. Given \( m-2 \equiv 0 \pmod{3} \). We consider two cases.

Case (i): When \( n \) is even. We have \( G_{m,n} \) is obtained by \( G_{m-2,n} \) followed by \( G_2 \). From Theorem 3.4, \( \chi_{td}(G_{m,n}) = \chi_{td}(G_{m-2,n}) + \chi_{td}(G_2) - 2 \). By Theorem 3.3 and Lemma 3.1, we get

\[
\chi_{td}(G_{m,n}) = \left(\frac{m-2}{3}\right)n + 2\left\lceil \frac{n}{3} \right\rceil + 2.
\]

Case (ii): When \( n \) is odd. We have \( G_{m,n} \) is obtained by \( G_{m-2,n} \) followed by \( G_2 \). From Theorem 3.4, \( \chi_{td}(G_{m,n}) = \chi_{td}(G_{m-2,n}) + \chi_{td}(G_2) - 2 \).

By Theorem 3.3 and Lemma 3.1, we get

\[
\chi_{td}(G_{m,n}) = \left(\frac{m-2}{3}\right)n + 2\left\lceil \frac{n}{3} \right\rceil + 3.
\]

Thus

\[
\chi_{td}(G_{m,n}) = \begin{cases} 
\left(\frac{m-2}{3}\right)n + 2\left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \text{ is even} \\
\left(\frac{m-2}{3}\right)n + 2\left\lceil \frac{n}{3} \right\rceil + 3 & \text{if } n \text{ is odd.}
\end{cases}
\]

Illustration: Consider \( G_{4,10} \)

Therefore

\[ \chi_{td}(G_{4,10}) = 18. \]

Illustration: Consider \( G_{4,10} \)

Therefore

\[ \chi_{td}(G_{4,10}) = 18. \]

Illustration: Consider \( G_{4,11} \)

Therefore

\[ \chi_{td}(G_{5,8}) = 16. \]
Illustration: Consider $G_{5,7}$

![Figure 8](image)

Therefore

$$\chi_d(G_{5,7}) = 16.$$  

### 4. Conclusion

In this paper, we obtain total dominator chromatic number of $P_m \times C_n$.

### References

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