Diffusive transport of light in two-dimensional disordered packing of disks: Analytical approach to transport-mean-free path

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We study photon diffusion in a two-dimensional random packing of monodisperse disks as a simple model of granular media and wet foams. We assume that the intensity reflectance of disks is a constant \( r \). We present an analytic expression for the transport-mean-free path \( l^* \) in terms of the velocity of light in the disks and host medium, radius \( R \) and packing fraction of the disks, and the intensity reflectance. For the glass beads immersed in the air or water, we estimate transport-mean-free paths about half the experimental ones. For the air bubbles immersed in the water, \( l^*/R \) is a linear function of \( 1/\varepsilon \), where \( \varepsilon \) is the liquid volume fraction of the model wet foam. This throws new light on the empirical law of Vera et al. [Applied Optics 40, 4210 (2001)], and promotes more realistic models.

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I. INTRODUCTION

There is a good reason to study wave propagation in turbid or random media: Multiply scattered waves can probe temporal changes in physical systems [1, 2, 3]. Thus, light transport through fog [4], milky liquids, nematic liquid crystals [5], granular media [6, 7, 8, 9], foams [10, 11, 12, 13, 14, 15, 16], and human tissue [17]; propagation of elastic waves in the Earth’s crust [18, 19]; acoustic waves in the fluidized or sedimenting suspensions [20]; etc., have attracted much attention.

In a turbid medium, light undergoes many scattering events before leaving the sample, and the transport of light energy is diffusive [2]. Therefore, the photon can be considered as a random walker. The transport-mean-free path \( l^* \), over which the photon direction becomes randomized, depends on the structural details of the opaque medium. Experimental techniques like diffuse-transmission spectroscopy (DTS) [21] and diffusing-wave spectroscopy (DWS) [22] can be used to measure \( l^* \). In DTS, the average fraction \( T \) of incident light transmitted through a slab of thickness \( L \) is measured. The transport-mean-free path is then deduced from \( T \propto l^*/L \). Utilizing the temporal intensity fluctuations in the speckle field of the multiply scattered light, DWS determines \( l^* \) and the mean-squared displacement of the scattering sites due to time evolution, thermal motion, or flow.

A plethora of light-scattering experiments show that light transport reaches its diffusive limit in granular media [6, 7, 8, 9] and foams [10, 11, 12, 13, 14, 15, 16], which means that photons perform a random walk. However, the mechanisms underlying this random walk are not elucidated. A wet foam is composed of spherical gas bubbles dispersed in liquid. A relatively dry foam consists of polyhedral cells separated by thin liquid films. Three of them meet in the so-called Plateau borders which then define tetrahedral vertices [23]. In their studies of foams with the liquid volume fraction \( \varepsilon \) in the range \( 0.008 < \varepsilon < 0.3 \), Vera, Saint-Jalmes, and Durian [11] observed the empirical law

\[ l^* \approx 2R \left( \frac{0.14}{\varepsilon} + 1.5 \right), \]

where \( R \) is the average bubble radius. Recent studies of scattering from Plateau borders [11, 24], vertices [25], and films [26, 27, 28, 29, 30, 31]; or transport effects such as total internal reflection of photons inside the Plateau borders [12, 32], have not yet clarified the empirical law of Vera et al.. For granular media, systematic measurements of the transport-mean-free path \( l^* \) as a function of the refractive indices of grains and the host medium (air, water,...), grain size, and packing fraction, have not been performed. Menon et al. [6] determined \( l^* \approx 15R \) for glass spheres of radius \( R = 47.5 \mu m \) dispersed in air. For glass beads dispersed in water, Leutz et al. [8] found \( l^* \approx 14R - 16R \) for 80\( \mu m \leq R \leq 200\mu m \). Their samples had a packing fraction \( \phi \approx 0.64 \). Crassous [3] performed numerical simulations to find \( l^* \) as a function of refractive indices of the grain and host medium, but only for packing fraction \( \phi \approx 0.64 \).

It is instructive to consider simple or even toy models of granular media and wet foams, which allow an analytic access to the transport-mean-free path \( l^* \). Apparently, such models pave the way for deeper understanding of fascinating DWS experiments. In this paper, we consider two-dimensional packing of monosize disks. The disks are much larger than the wavelength of light, thus one can employ ray optics to follow a light beam or photon as it is reflected by the disks with a probability \( r \) called the intensity reflectance. We assume that the intensity reflectance is constant, and the velocity of light inside and outside the disks are \( c/n_{in} \) and \( c/n_{out} \), respectively. We show that the photon’s random walk based on the above rules is a persistent random walk [26, 33, 34]. Writing a master equation to describe the photon transport, we

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find in Sec. III A the transport-mean-free path as

\[ l^* = \frac{\pi R}{4} \left( \frac{3}{2} - 1 \right) \left( \frac{r}{1-r} + \frac{1}{\phi} \right) \left( \frac{r}{1-r} + \frac{1}{\phi} \right) \],

(2)

where \( \phi \) and \( R \) denote the packing fraction and radius of disks, respectively. We further study our model by numerical simulation of the photon’s random walk. We observe the overall agreement between our numerical and analytical estimates of the transport-mean-free path.

For glass beads immersed in air or water, we find transport-mean-free paths about half the experimental ones \([6, 8]\). For the air bubbles immersed in the water, we use Eq. (2) to derive \( l^* \) as a function of the liquid volume fraction \( \varepsilon = 1 - \phi \). \( r \approx 0.20 \) is estimated as a weighted average of Fresnel’s intensity reflectance. We find that in the range \( 0.08 < \varepsilon < 0.15 \), our analytical result agrees well with the relation \( l^* = R(0.11/\varepsilon + 2.37) \). In other words, we find that \( l^*/R \) is a linear function of \( 1/\varepsilon \).

Using the hybrid lattice gas model for two-dimensional foams and Fresnel’s intensity reflectance, Sun and Hutzel performed numerical simulation of photon transport and found \( l^* \approx R(0.26/\varepsilon + 4.90) \) \([24]\). Quite remarkably, our analytic estimate of the transport-mean-free path throws new light on the empirical law of Vera et al. and the numerical simulation of Sun et al.

Our article is organized as follows. In Section II we introduce the two-dimensional packing of disks as a simple model for a granular medium or a wet foam. Photon transport in a random packing of disks using constant intensity reflectance is discussed in Sec. III. Discussions, conclusions, and an outlook are presented in Sec. IV.

II. MODEL

As a simple model for a two-dimensional disordered granular medium, wet foam, and bubbly liquid, we choose the random packing of circular disks. All non-overlapping disks have the same radius \( R \), and cover a fraction \( \phi \) of the plane. To address the photon transport in such medium, we have made the following assumptions: (i) Disks or grains, are much larger than the wavelength of light, thus one can employ ray optics to follow a light beam or photon as it is reflected by the disks with a probability \( r \) called the intensity reflectance. (ii) \( r \) is a constant, with no dependence on the incidence angle. (iii) Although disks of refractive index \( n_{in} \) are immersed in a medium of refractive index \( n_{out} \), the incident and the transmitted rays have the same direction. In other words, we assume that the angle of refraction equals the angle of incidence. (iv) The velocity of light inside and outside the disks are \( c/n_{in} \) and \( c/n_{out} \), respectively.

Our first assumption is inspired by the experiments \([6, 7, 8, 9]\). Our second and third assumptions do not agree with Fresnel’s formulas and Snell’s law, respectively. Consequently, our model does not consider total internal reflection of rays. However, we deliberately adopt a step-by-step approach to photon transport in granular media, and will consider more realistic models later.

As already mentioned, we model single photon paths in a packing of disks as a random walk with rules motivated by ray optics, i.e., an incoming light beam is reflected from a disk surface with a probability \( r \) or it traverses the disk surface with a probability \( t = 1 - r \). This naturally leads to a persistent random walk of the photons \([20]\), where the walker remembers its direction from the previous step \([33, 34]\). Persistent random walks are employed in biological problems \([35]\), turbulent diffusion \([36]\), polymers \([37]\), Landauer diffusion coefficient for a one-dimensional solid \([38]\), and in general transport mechanisms \([39, 40]\). More recent applications are reviewed in \([41]\). In the following section, we adopt the approach of \([28, 50]\) to study persistent random walk of the photons in a granular medium.

III. PHOTON TRANSPORT IN A TWO-DIMENSIONAL PACKING OF DISKS

A. Analytical treatment

The photon random walk in a packing of disks consists of steps inside and outside the disks. We denote the average length of steps inside and outside the grains by \( L_{in} \) and \( L_{out} \), respectively. We characterize each step by an angle relative to the \( x \)-axis. As Fig. I(a) demon-
strates, on hitting a disk with an incidence angle $\gamma$, a photon moving in the host medium along the direction $\theta + \pi + 2\gamma$ will be either reflected to the direction $\theta$, or enters the disk. The probability distribution of the random variable $\gamma$ ($0 < \gamma < \pi/2$) is $F(\gamma) = \cos \gamma$, see Appendix [A]. Similarly, a photon moving in a disk along the direction $\theta + \pi + 2\gamma'$ and hitting its surface with an angle $\gamma'$, will be either reflected to the direction $\theta$ or enters the host medium, see Fig. 4(b). Quite remarkably, the probability distribution of the incidence angles $\gamma$ and $\gamma'$ are the same, see Appendix [A]. We can therefore conclude that diffusion of the photons inside and outside the grains are not inherently different. As will be shown in the following, we write a master equation to describe the photon diffusion inside (outside) the grains, utilizing the step length $\bar{L}_{in}$ ($\bar{L}_{out}$) and velocity $c/n_{in}$ ($c/n_{out}$), and extract the diffusion constant $D_{in}$ ($D_{out}$). According to the two-state model of Lennard-Jones [34], the diffusion constant of photons in the granular medium is

$$D_m = f_{in}D_{in} + f_{out}D_{out},$$  (3)

where $f_{in}$ ($f_{out} = 1 - f_{in}$) is the fraction of time that the photons spend inside (outside) the disks.

We introduce the probability $P_n(x, y|\theta)dx dy$ that the photon after its $n$th step of length $\bar{L}$ along the direction $\theta$, arrives in the area $dx dy$ at position $x = (x, y)$. Then the following master equation expresses the evolution of $P_n(x, y|\theta)$:

$$P_{n+1}(x, y|\theta) = \frac{1}{2} r \int_{-\pi}^{\pi} P_n(x - \bar{L} \cos \theta, y - \bar{L} \sin \theta | \theta + \pi + 2\gamma) F(\gamma) d\gamma + t P_n(x - \bar{L} \cos \theta, y - \bar{L} \sin \theta). \quad (4)$$

The first term on the right-hand side describes the reflection of the photon with a probability $r$. The photon which has arrived at position $(x - \bar{L} \cos \theta, y - \bar{L} \sin \theta)$ along the direction $\theta + \pi + 2\gamma$, changes its direction by an angle $\pi + 2\gamma$ according to the probability function $F(\gamma)$ [12]. The second term describes the transmission with a probability $t = 1 - r$. The photon performs a ballistic motion with step length $\bar{L}$ along direction $\theta$ from position $(x - \bar{L} \cos \theta, y - \bar{L} \sin \theta)$ to $(x, y)$.

The diffusion constant follows from the evaluation of the second moment of $P_n(x, y|\theta)$ with respect to the spatial coordinates $x$ and $y$. The probability distribution as an exact solution of the master equation [41] is hard to obtain. However, there is a more direct method for the evaluation of the moments which employs the characteristic function $P_n(\omega_x, \omega_y|m)$ associated with $P_n(x, y|\theta)$ [34]:

$$\langle x^{k_1} y^{k_2} \rangle_n = \int \int \int x^{k_1} y^{k_2} P_n(x, y|\theta) dx dy d\theta$$

$$= (-i)^{k_1+k_2} \frac{\partial^{k_1+k_2} P_n(\bar{\omega}|m = 0)}{\partial \omega_x^{k_1} \partial \omega_y^{k_2}} \bigg|_{\omega = 0}, \quad (5)$$

where $k_1$ and $k_2$ are either zero or positive integers,

$$P_n(\bar{\omega}|m) = \frac{1}{2\pi} e^{im\theta} \int \int e^{i\bar{\omega} \cdot \bar{x}} P_n(x, y|\theta) dx dy d\theta, \quad (6)$$

and $\bar{\omega} = (\omega_x, \omega_y)$. We are interested in the first and second moments of $P_n(x, y|\theta)$, thus we focus on the Taylor expansion

$$P_n(\omega, \alpha|m) \approx Q_{0,n}(\alpha|m) + i\omega \bar{L} Q_{1,n}(\alpha|m) - \frac{\omega^2 \bar{L}^2}{2} Q_{2,n}(\alpha|m) + \ldots. \quad (7)$$

Fourier transforming Eq. (4), we obtain

$$P_{n+1}(\omega, \alpha|m) = \sum_{k=-\infty}^{\infty} e^{ik\alpha} J_k(\omega \bar{L}) P_n(\omega, \alpha|k + m) \times[r(1)^{m+k} F(2m + 2k) + l], \quad (9)$$

where $F(m) = \frac{1}{2} J_{-\pi/2} e^{im\gamma} F(\gamma) d\gamma$, and

$$J_k(z) = \frac{1}{2\pi i k} \int_{-\pi}^{\pi} e^{iz \cos \theta} e^{-ik\theta} d\theta \quad (10)$$

is the $k$th-order Bessel function. Since we are only interested in the Taylor coefficients $Q_{1,n}(\alpha|m)$ and $Q_{2,n}(\alpha|m)$, we insert Eq. (7) into Eq. (9). Using the Taylor expansion of the relevant Bessel functions $J_k(z)$ $(|k| \leq 2)$ and $J_k(0) = \delta_{0,k}$ [13] and collecting all terms with the same power in $\omega$, results in the following recursion relations for
the $Q_{i,n}(\alpha|m)$:

\[
Q_{0,n+1}(\alpha|m) = [t + r(1-m)F(2m)]Q_{0,n}(\alpha|m),
\]

\[
Q_{1,n+1}(\alpha|m) = [t + r(1-m)F(2m)]Q_{1,n}(\alpha|m) + e^{-i\alpha} [t + r(1-m)^{1+F}(2m+2)]Q_{0,n}(\alpha|m + 1) + \frac{e^{i\alpha}}{2} [t + r(1-m)^{-1}F(2m-2)]Q_{0,n}(\alpha|m - 1),
\]

\[
Q_{2,n+1}(\alpha|m) = [t + r(1-m)F(2m)]Q_{2,n}(\alpha|m) + e^{-i\alpha} [t + r(1-m)^{1+F}(2m+2)]Q_{1,n}(\alpha|m + 1) + e^{i\alpha} [t + r(1-m)^{-1}F(2m-2)]Q_{1,n}(\alpha|m - 1) + \frac{1}{2} [t + r(1-m)^F(2m)]Q_{0,n}(\alpha|m) + \frac{e^{-2i\alpha}}{4} [t + r(1-m)^{+2}F(2m+4)]Q_{0,n}(\alpha|m + 2) + \frac{e^{2i\alpha}}{4} [t + r(1-m)^{-2}F(2m-4)]Q_{0,n}(\alpha|m - 2).
\]

(11)

\[
Q_0(z,\alpha|m) = \frac{Q_{0,n=0}(\alpha|m)}{1 - z [t + r(1-m)F(2m)]},
\]

\[
Q_1(z,\alpha|m) = \frac{Q_{1,n=0}(\alpha|m)}{1 - z [t + r(1-m)F(2m)]} + \frac{z}{2(1 - z [t + r(1-m)F(2m)])} \left\{ e^{-i\alpha} [t + r(1-m)^{1+F}(2m+2)]Q_{0,n=0}(\alpha|m + 1) + e^{i\alpha} [t + r(1-m)^{-1}F(2m-2)]Q_{0,n=0}(\alpha|m - 1) \right\}.
\]

\[
Q_2(z,\alpha|m) = \frac{Q_{2,n=0}(\alpha|m)}{1 - z [t + r(1-m)F(2m)]} + \frac{zQ_{0,n=0}(\alpha|m)}{2(1 - z [t + r(1-m)F(2m)])^2} + \frac{z}{1 - z [t + r(1-m)F(2m)]} \left\{ e^{-i\alpha} [t + r(1-m)^{1+F}(2m+2)]Q_1(z,\alpha|m + 1) + e^{i\alpha} [t + r(1-m)^{-1}F(2m-2)]Q_1(z,\alpha|m - 1) + \frac{e^{-2i\alpha}}{4} [t + r(1-m)^{+2}F(2m+4)]Q_{0,n=0}(\alpha|m + 2) + \frac{e^{2i\alpha}}{4} [t + r(1-m)^{-2}F(2m-4)]Q_{0,n=0}(\alpha|m - 2) \right\}.
\]

(12)

where $F(m) = (1 - m^2)^{-1} \cos(m\pi/2)$, especially $F(0) = 1$. The above expressions contain the sum of several terms whose inverse $z$-transform are readily accessible:

\[
1 \leftrightarrow \frac{1}{1 - z},
\]

\[
n \leftrightarrow \frac{1}{(1 - z)^2},
\]

\[
a^n \leftrightarrow \frac{1}{1 - az},
\]

\[
n^{an} \leftrightarrow \frac{az}{(1 - az)^2},
\]

(13)

For an arbitrary initial distribution $P_0(x,y|\theta)$, the relevant function $Q_1,n(\alpha|0)$ contains terms which are either constant or behave as $a^n$ with $|a| < 1$. They are associated with the randomization of the initial distribution of the random walkers but are not essential for large $n$. According to Eq. (3),

\[
\langle x \rangle_n = \langle y \rangle_n = 0 \tag{14}
\]

The behavior of the mean-square displacements is associated with $Q_{2,n}(\alpha|0)$, see Eq. (3). We checked that for large $n$ or in the long-time limit it is purely diffusive, i.e.,

\[
\langle x^2 \rangle_n = 2D_x \tau, \quad \langle y^2 \rangle_n = 2D_y \tau, \tag{15}
\]
where we introduced the time \( \tau = n \tilde{L} / v \) which passes when the random walker makes \( n \) steps at a speed \( v \). We extract the diffusion constants from \( Q_{2,n}(0|0) \) and \( Q_{2,n}(\pi/2|0) \):

\[
D_x = D_y = \frac{1}{4} \tilde{L} v \left( \frac{3}{2r} - 1 \right)
\]

As already mentioned, we write the master equation [4] to describe photon diffusion in the grains and in the host medium. Equation (16) immediately leads to

\[
D_{\text{in}} = \frac{1}{4} \tilde{L}_{\text{in}} c \frac{3}{2n_{\text{in}}} \left( \frac{3}{2r} - 1 \right),
\]

\[
D_{\text{out}} = \frac{1}{4} \tilde{L}_{\text{out}} c \frac{3}{2n_{\text{out}}} \left( \frac{3}{2r} - 1 \right).
\]

The task is now expressing \( \tilde{L}_{\text{in}}, \tilde{L}_{\text{out}}, f_{\text{in}} \) and \( f_{\text{out}} \) in terms of the model parameters \( R, \phi, r \), and using the two-state model of Lennard-Jones to derive \( D_m \). First, we note that \( \tilde{L}_{\text{in}} = < 2R \cos \gamma > \), where \( \gamma \) is the incidence angle of photons moving in the disk, and here \( < > \) denotes averaging with respect to the probability distribution \( F'(\gamma) = \cos \gamma \), see Appendix A and Fig. 1(b). Second, \( \phi = L_{\text{in}} / (L_{\text{in}} + L_{\text{out}}) \). Hence we find

\[
\tilde{L}_{\text{in}} = \frac{\pi R}{2},
\]

\[
\tilde{L}_{\text{out}} = \frac{\pi R (1 - \phi)}{2}. \tag{18}
\]

The evaluation of \( f_{\text{in}} \) is more exacting. Each step length \( \tilde{L}_{\text{in}} \) inside a disk takes a time \( \tilde{\tau}_{\text{in}} = \tilde{L}_{\text{in}} n_{\text{in}} / c \). The probability of \( m \) internal steps before leaving the disk is \( t^m \). Hence the average time that a photon spends in the disk is \( \sum_{m=0}^{\infty} m \tilde{\tau}_{\text{in}} t^m = \tilde{\tau}_{\text{in}} r / t \). The photon spends a time \( \tilde{\tau}_{\text{out}} = L_{\text{out}} n_{\text{out}} / c \) before reaching a disk. Hence

\[
f_{\text{in}} = \frac{\tilde{\tau}_{\text{in}} r / t}{\tilde{\tau}_{\text{in}} + \tilde{\tau}_{\text{in}} r / t} = \frac{n_{\text{in}} \frac{r}{\tau}}{n_{\text{in}} \frac{r}{\tau} + n_{\text{out}} \frac{1 - \phi}{\phi}},
\]

\[
f_{\text{out}} = \frac{\tilde{\tau}_{\text{out}}}{\tilde{\tau}_{\text{out}} + \tilde{\tau}_{\text{out}} r / t} = \frac{n_{\text{out}} \frac{1 - \phi}{\phi}}{n_{\text{in}} \frac{r}{\tau} + n_{\text{out}} \frac{1 - \phi}{\phi}}. \tag{19}
\]

Now we utilize Eq. 3 to derive the diffusion constant of photons in the granular medium as

\[
D_m = \frac{\pi R c \left( \frac{3}{2r} - 1 \right) \left( \frac{1 - \phi}{\phi} \right)^2}{8 n_{\text{in}} \frac{r}{\tau} + n_{\text{out}} \frac{1 - \phi}{\phi}}. \tag{20}
\]

In two-dimensional space, the transport-mean-free path follows from

\[
l^* = 2D_m / c_m, \tag{21}
\]

where \( c_m \) is the transport velocity of light in the medium. To a first approximation

\[
c_m = \phi \frac{c}{n_{\text{in}}} + (1 - \phi) \frac{c}{n_{\text{out}}}. \tag{22}
\]

Note that the velocity of light in the disks (the host medium) covering a fraction \( \phi \) of the plane is \( c / n_{\text{in}} (c / n_{\text{out}}) \). From Eqs. 20, 22, we find the transport-

mean-free path \( l^* \) mentioned already in Eq. 2 in the introduction.

**B. Numerical simulations**

We presented an analytic theory to calculate the diffusion constant of photons. Now we carefully examine this analytic result by performing numerical simulations.

In order to generate random configurations of monodisperse disks with a desired packing fraction, we compress a dilute system of rigid disks into a smaller space. Simulation methods based on a confining box generate a packing whose properties in the vicinity of walls differ from those

![FIG. 2: Part of a packing of 10^4 disks, covering a fraction \( \phi = 0.35 \) of the plane.](image)

![FIG. 3: The diffusion constant \( D_m \) (in units of the disk radius \( R \) times the velocity of light \( c \)) as a function of the intensity reflectance \( r \), for various packing fractions. Here \( n_{\text{in}} = 1.5 \) and \( n_{\text{out}} = 1.0 \) are assumed. Monte Carlo simulation results and \( D_m(r) \) are denoted, respectively, by points and the line.](image)
in the bulk. Hence we utilize the compaction method of Ref. [45] which combines the contact dynamics algorithm [46, 47] with the concept of the Andersen dynamics [48]. This combined simulation method involves variable area of the simulation box with periodic boundary conditions in all directions. Due to the exclusion of side walls, the algorithm generates homogenous packings.

We let photons perform a random walk in our packing of disks by applying the rules introduced in Sec. [11]. For improving the speed of our ray tracing program, we adopt the cell index method commonly used in the molecular dynamics simulations [49]. The square simulation box is divided into a regular lattice of J × J cells. We maintain a list of disks in each of these cells. A photon moving in the cell j (1 ≤ j ≤ J²) probably hits the disks in the cell j or its neighbor cells. Thus it is not necessary to check collision between the photon and all disks of the medium.

Our computer program shrinks an initial dilute sample of 10⁴ non-overlapping disks randomly distributed in a two-dimensional simulation box. In the course of shrinking the packing, the program saves snapshots of the grain positions if the packing fraction φ ∈ [0.15, 0.25, ..., 0.75], see Fig. [2]. For each medium, the program takes 10⁴ photons at an initial position, and launches them in a direction specified by angle θ₀ relative to the x-axis. Then it generates the trajectory of each photon following a standard Monte Carlo procedure and evaluates the statistics of the photon cloud at times τ ∈ [7000, 7100, ..., 9900] (in units of R/c). As detailed in Ref. [20], we determine the diffusion constant D_m from the temporal evolution of the average mean-square displacement of the photons: 

\[ \langle x^2 + y^2 \rangle = 4D_m \tau. \]

For angles θ₀ ∈ [20°, 40°, ..., 320°], the simulation is repeated for each intensity reflectance r ∈ [0.1, 0.2, ..., 0.9]. As a reasonable result, no dependence on the starting point and the starting direction is observed. In Fig. [3] we plot the diffusion constant D_m in units of the disk radius R times the velocity of light c as a function of the intensity reflectance r, for the glass disks (n_{in} = 1.5) immersed in the air (n_{out} = 1.0). For this medium, Fig. [3] shows the diffusion constant D_m as a function of the packing fraction φ, for the intensity reflectances r = 0.1 and 0.4. The errorbars reflect the standard deviations when we average over all diffusion constants for different starting positions and angles. We also performed simulations for the other examples (n_{in} = 1.34, n_{out} = 1.0), (n_{in} = 1.0, n_{out} = 1.34), (n_{in} = 1.0, n_{out} = 1.5), etc., but the results are not shown here. We observed the overall agreement between the numerical results and our theoretical value for D_m. Quite remarkably, Eq. (20) involves no free parameters, but reasonably agrees with the numerical results in a wide range of φ, r, n_{in}, and n_{out}.

IV. DISCUSSION, CONCLUSIONS, AND OUTLOOK

Diffusing-wave spectroscopy provides invaluable information about the static and dynamic properties of granular media [6, 7, 8, 9] and foams [10, 11, 12, 13, 14, 15, 16]. The transport-mean-free path l∗ in terms of the microscopic structure, however, remains to be fully elucidated. In this paper, we consider a simple model for photon diffusion in a two-dimensional packing of disks. Our analytical result for l∗ provides new insights into the light transport, and promotes more realistic models.

We have studied the photon’s persistent random walk in a two-dimensional packing of monodisperse disks. We employed ray optics to follow a light beam or photon as it is reflected by the disks. We used a constant intensity reflectance r. Moreover we assumed that on hitting a disk, the incident and transmitted rays have the same direction. To achieve a better understanding of photon diffusion in granular medium and wet foams, we are extending our studies by considering Fresnel’s formulae for the intensity reflectance, Snell’s law of the refraction, and the three-dimensional packing of polydisperse spheres. We are also improving our estimate of l∗ by taking into account the distribution of photons’ step length, and the transport velocity of photons [50]. We discuss these points in the following.

Many two-dimensional systems offer a rich and unexpected behavior. For an experimental observation of photon diffusion in a two-dimensional packing, a set of parallel fibres can be used. Photons injected in a plane perpendicular to the axis of fibres perform a planar diffusion. One can also drill parallel cylinders in a host medium, and fill all the cylinders with a liquid: l∗ dependence on the refractive index n_{in} can be measured. Note that a ray maintains its polarization state on hitting a disk, and Fresnel’s intensity reflectance depends on the polarization state: Quite interesting, the transport-mean-free paths for the transverse electric and transverse magnetic polarizations, are different. Inspired by the rich optics of a two-dimensional packing and following a step-by-step approach to a real system, our attention is natu-
FIG. 5: The probability distribution \( G(\text{L}_{\text{out}}) \) as a function of \( \text{L}_{\text{out}}/R \) for various packing fraction \( \phi \). Inset: The same plot in the logarithmic scale. Note that \( G(\text{L}_{\text{out}}) \) decays exponentially.

...nonced maximum, the distribution in a medium composed of dielectric spheres comparable to light transport in dry foams, we studied the two- and three-dimensional Voronoi foams [27, 31]. The interest in three-dimensional V oronoi foams [27, 31]. The interest...
immersed in the water \((n_{\text{out}} = 1.34)\), we find \(r \approx 0.09\) and \(l^* \approx 9R\). Our transport-mean-free paths are smaller than the experimental values \([3, 8]\) by a factor of about 2. Now we consider a simple model for wet foams. For the air bubbles \((n_{\text{in}} = 1)\) immersed in the water \((n_{\text{out}} = 1.34)\), we estimate \(r \approx 0.20\). Figure 6 delineates \(l^*\) (in units of \(R\)) as a function of the liquid volume fraction \(\varepsilon = 1 - \phi\). From Fig. 6 we find that in the range \(0.08 < \varepsilon < 0.15\), our analytical result agrees well with the relation

\[
l^* \approx R(\frac{0.11}{\varepsilon} + 2.37).
\]

Using the hybrid lattice gas model for two-dimensional foams and Fresnel’s intensity reflectance, Sun and Hutzler performed numerical simulation of photon transport \([24]\). For \(0.02 < \varepsilon < 0.16\), their numerical results can be fitted to \(l^* \approx R(0.26/\varepsilon + 4.90)\). Fig. 6 compares our analytic prediction with the numerical result of Ref. [24]. Again our analytic estimate is smaller than the numerical simulations by a factor of about 2.

Quite remarkably, our analytic estimate of the transport-mean-free path \(l^*\) quoted in Eq. (2), sheds some light on the empirical law of Vera et al. and the numerical simulation of Sun and Hutzler: \(l^*/R\) is a linear function of \(1/\varepsilon\), see Eqs. (1) and (23). For a better understanding of the empirical law \([1]\), we aim at a more realistic model which not only considers Fresnel’s intensity reflectance with its significant dependence on the incidence angle, but also the broad distribution of photons’ step length. Also an extension to the three-dimensional packing of polydisperse spheres is envisaged.

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APPENDIX A: THE PROBABILITY DISTRIBUTIONS \(F(\gamma)\) AND \(F'(\gamma')\)

To find the probability distribution of the random variable \(\gamma\), we assume that the impact parameter \(s\) in Fig. 1(c) has a uniform distribution in the interval \([0, R]\). In other words, we assume that the number of incident rays with an impact parameter less than \(s\) is proportional to \(s\). The cumulative distribution function \(F_c(\gamma) = \int_0^\gamma F(\psi)d\psi\) is then \(F_c(\gamma) = \text{Prob}(s < R \sin \gamma) = \sin \gamma\). It follows that

\[
F(\gamma) = \frac{dF_c(\gamma)}{d\gamma} = \cos \gamma.
\]

Now we consider path of photons inside the disk, see Fig. 1(d). Each ray can be characterized by its distance \(s'\) from the center of the disk. We assume that the random variable \(s'\) has a uniform distribution in the interval \([0, R]\). Since \(s' = R \sin \gamma'\), the cumulative distribution function is \(F_c'(\gamma') = \sin \gamma'\), and

\[
F'(\gamma') = \frac{dF_c'(\gamma')}{d\gamma'} = \cos \gamma'.
\]

Further numerical simulations of our model confirm these analytical results for \(F(\gamma)\) and \(F'(\gamma')\).

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