An Arithmetic Proof of John’s Ellipsoid Theorem

Peter M. Gruber and Franz E. Schuster

Abstract. Using an idea of Voronoi in the geometric theory of positive definite quadratic forms, we give a transparent proof of John’s characterization of the unique ellipsoid of maximum volume contained in a convex body. The same idea applies to the ‘hard part’ of a generalization of John’s theorem and shows the difficulties of the corresponding ‘easy part’.

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Introduction and Statement of Results

The following well-known characterizations of the unique ellipsoid of maximum volume in a convex body in Euclidean $d$-space are due to John [10] ((i)⇒(ii)) and Pelczyński [12] and Ball [1] ((ii)⇒(i)), respectively. For references to other proofs, a generalization and to the numerous applications see [2, 6, 11].

Theorem 1 Let $C \subset \mathbb{E}^d$ be compact, convex, symmetric in the origin $o$, and with $B^d \subset C$. Then the following claims are equivalent:

(i) $B^d$ is the unique ellipsoid of maximum volume in $C$.
(ii) There are $u_k \in B^d \cap \text{bd} \ C$ and $\lambda_k > 0$, $k = 1, \ldots, n$, where $d \leq n \leq \frac{1}{2}d(d+1)$, such that

$$I = \sum_k \lambda_k u_k \otimes u_k.$$ 

Here, $B^d$ is the solid unit ball in $\mathbb{E}^d$, $I$ the $d \times d$ unit matrix, and for $u, v \in \mathbb{E}^d$ the $d \times d$ matrix $u v^T$ is denoted by $u \otimes v$. bd stands for boundary.

Theorem 2 Let $C \subset \mathbb{E}^d$ be compact, convex, and with $B^d \subset C$. Then the following claims are equivalent:

(i) $B^d$ is the unique ellipsoid of maximum volume in $C$.
(ii) There are $u_k \in B^d \cap \text{bd} \ C$ and $\lambda_k > 0$, $k = 1, \ldots, n$, where $d + 1 \leq n \leq \frac{1}{2}d(d + 3)$, such that

$$I = \sum_k \lambda_k u_k \otimes u_k, \quad o = \sum_k \lambda_k u_k.$$
Our proof of Theorem 1 is based on the idea of Voronoi in the geometric theory of positive definite quadratic forms, to represent ellipsoids in $\mathbb{E}^d$ with center $o$ by points in $\mathbb{E}^{\frac{d}{2}(d+1)}$, see [4, 9, 14]. The problem on maximum volume ellipsoids in $\mathbb{E}^d$ is then transformed into a simple problem on normal cones in $\mathbb{E}^{\frac{1}{2}d(d+1)}$, which can be solved easily by Carathéodory’s theorem on convex hulls. This idea has been applied before by the first author [8]. The proof of Theorem 2 is a simple extension. The proof of the latter also gives Theorem 4 of Bastero and Romance [3], where $B^d$ is replaced by a compact connected set with positive measure.

In the context of John’s theorem, it is natural to ask whether ellipsoids can be replaced by more general convex or non-convex sets. The following is a slight refinement of results of Giannopoulos, Perissinaki and Tsolomitis [7] and Bastero and Romance [3] (Theorem 3). The result of Giannopoulos et. al. was first observed by Milman in the case, where both bodies are centrally symmetric, see [16].

**Theorem 3** Let $C \subseteq \mathbb{E}^d$ be compact and convex, and $B \subseteq C$ compact with positive measure. Then (i) implies (ii), where the claims (i) and (ii) are as follows:

(i) $B$ has maximum measure amongst all its affine images contained in $C$.

(ii) There are $u_k \in B \cap \text{bd}C, v_k \in N_C(u_k)$, and $\lambda_k > 0, k = 1, \ldots, n$, where $d + 1 \leq n \leq d(d + 1)$, such that

\[
I = \sum_{k} \lambda_k u_k \otimes v_k, \quad o = \sum_{k} \lambda_k v_k.
\]

Here $N_C(u), u \in \text{bd}C$, is the normal cone of $C$ at $u$. For this concept and other required notions and results of convex geometry we refer to [15].

Note that $B$ is not necessarily unique. A suitable modification of Voronoi’s idea applies in the present context and thus leads to a proof of Theorem 3, paralleling our proofs of Theorems 1 and 2. Incidentally, the proof of Theorem 3 shows, why it is not clear that property (ii) implies property (i), see the Final Remarks.

**Proof of Theorem 1**

For (real) $d \times d$-matrices $A = (a_{ij}), B = (b_{ij})$ define $A \cdot B = \sum a_{ij}b_{ij}$. The dot $\cdot$ denotes also the inner product in $\mathbb{E}^d$. Easy arguments yield the following:

(1) Let $M$ be a $d \times d$ matrix and $u, v, w \in \mathbb{E}^d$. Then $Mu \cdot v = M \cdot u \otimes v$ and $(u \otimes v)w = (v \cdot w)u$.

Next, we specify two tools:

(2) Each $d \times d$ matrix $M$ with $\det M \neq 0$ can be represented in the form $M = AR$, where $A$ is a symmetric, positive definite and $R$ is an orthogonal $d \times d$ matrix.

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(Put $A = (MM^T)^{\frac{1}{2}}, R = A^{-1}M$, see [5], p.112.) Identify a symmetric $d \times d$ matrix $A = (a_{ij})$ with the point $(a_{11}, \ldots, a_{1d}, a_{22}, \ldots, a_{2d}, \ldots, a_{dd})^T \in \mathbb{E}_+^{d(d+1)}$. The set of all symmetric, positive definite $d \times d$ matrices then is (represented by) an open convex cone $\mathcal{P} \subseteq \mathbb{E}_+^{d(d+1)}$ with apex at the origin. The set

\[
\mathcal{D} = \{ A \in \mathcal{P} : \det A \geq 1 \}
\]

is a closed, smooth, strictly convex set in $\mathcal{P}$ with non-empty interior.

(Use the implicit function theorem and Minkowski’s inequality for symmetric, positive definite $d \times d$ matrices, see [13], p.205.)

(i)$\Rightarrow$(ii): By (2), any ellipsoid in $\mathbb{E}^d$ can be represented in the form $AB^d$, where $A \in \mathcal{P}$. Thus the family of all ellipsoids in $C$ is represented by the set

\[
\mathcal{E} = \{ A \in \mathcal{P} : Au \cdot v = A \cdot u \otimes v \leq h_C(v) \text{ for } u, v \in S^{d-1} \},
\]

see (1). Here, $h_C(\cdot)$ is the support function of $C$. Clearly, $\mathcal{E}$ is the intersection of the closed halfspaces

\[
\{ A \in \mathbb{E}_+^{d(d+1)} : A \cdot u \otimes v \leq h_C(v) \} : u, v \in S^{d-1},
\]

with the set $\mathcal{P}$. Thus, in particular, $\mathcal{E}$ is convex. By (i), $\mathcal{E} \setminus \{I\} \subset \{ A \in \mathcal{P} : \det A < 1 \}$. This, together with (3), shows that

\[
\mathcal{D} \text{ and } \mathcal{E} \text{ are convex, } \mathcal{D} \cap \mathcal{E} = \{I\}, \text{ and } \mathcal{D} \text{ and } \mathcal{E} \text{ are separated by the unique support hyperplane } \mathcal{H} \text{ of } \mathcal{D} \text{ at } I \text{ in } \mathbb{E}_+^{d(d+1)}.
\]

$\mathcal{E}$ is the intersection of the closed halfspaces in (4) with the set $\mathcal{P}$, and these halfspaces vary continuously as $u, v$ range over $S^{d-1}$. Thus the support cone $\mathcal{K}$ of $\mathcal{E}$ at $I$ can be represented as the intersection of those halfspaces, which contain $I$ on their boundary hyperplanes, i.e. for which $I \cdot u \otimes v = u \cdot v = h_C(v)$. Since $u \cdot v \leq 1$ and $h_C(v) \geq 1$ and equality holds in both cases precisely when $u = v \in S^{d-1} \cap \text{bd } C$ (note that $B^d \subseteq C$), we see that

\[
\mathcal{K} = \bigcap_{u \in B^d \cap \text{bd } C} \{ A \in \mathbb{E}_+^{d(d+1)} : A \cdot u \otimes u \leq 1 \}.
\]

The normal cone $\mathcal{N}$ of ($\mathcal{E}$ or) $\mathcal{K}$ at $I$ is generated by the exterior normals of these halfspaces,

\[
\mathcal{N} = \text{pos} \{ u \otimes u : u \in B^d \cap \text{bd } C \}.
\]

The cone $\mathcal{K}$ has apex $I$ and, by (5), is separated from the convex set $\mathcal{D}$ by the hyperplane $\mathcal{H}$, where $\mathcal{H}$ is the unique support hyperplane of $\mathcal{D}$ at $I$. By considering the gradient of the function $A \rightarrow \det A$, we see that $I$ is an interior normal vector of $\mathcal{D}$ at $I$ and thus a normal vector of $\mathcal{H}$ pointing away from $\mathcal{K}$. Hence $I \in \mathcal{N}$. (7) and Carathéodory’s theorem for convex cones then yield the following: there are $u_k \otimes u_k \in \mathcal{N}$, i.e. $u_k \in B^d \cap \text{bd } C$, and $\lambda_k > 0$ for $k = 1, \ldots, n$, where $n \leq \frac{1}{2}d(d + 1)$, such that
\[(8) \quad I = \sum_k \lambda_k u_k \otimes u_k.\]

For the proof that \(n \geq d\), it is sufficient to show that \(\text{lin} \{u_1, \ldots, u_n\} = \mathbb{E}^d\). If this were not true, we could choose \(u \neq o, u \perp u_1, \ldots, u_n\), and then (1) yields the contradiction

\[0 \neq u^2 = I u \cdot u = \left( \sum_k \lambda_k (u_k \otimes u_k) u \right) \cdot u = \left( \sum_k \lambda_k (u_k \cdot u) u_k \right) \cdot u = 0.\]

(ii)\(\Rightarrow\)(i): Let \(E\) be as above. \(E\) is convex. \(B^d \subset C\) implies that \(I\) satisfies all defining inequalities of \(E\), in particular those corresponding to \(u = v = u_k, k = 1, \ldots, n\). Since \(h_C(u_k) = 1\), these inequalities are satisfied even with the equality sign. Thus \(I \in \text{bd} \, E\). Define \(K, N\) and \(H\) as before. (ii) implies that \(I \in N\). Hence \(K\) is contained in the closed halfspace with boundary hyperplane \(H\) through \(I\) and exterior normal vector \(I\). Clearly, \(H\) separates \(K\) and \(D\) and thus, a fortiori, \(E(\subset K)\) and \(D\). Since \(D\) is strictly convex by (3), \(D \cap E = \{I\}\). Hence \(B^d\) is the unique ellipsoid of maximum volume in \(C\).

**Outline of the Proof of Theorem 2**

The proof of Theorem 2 is almost identical with that of Theorem 1: an ellipsoid now has the form \(AB^d + a\) and is represented by \((A, a) \in P \times \mathbb{E}^d \subset E^{d(d+3)}\). \(E\) is the set

\[\{(A, a) \in P \times \mathbb{E}^d : A \cdot u \otimes v + a \cdot v \leq h_C(v) \text{ for } u, v \in S^{d-1}\}\]

and instead of (5) we have

\[D \times \mathbb{E}^d\] and \(E\) are convex, \((D \times \mathbb{E}^d) \cap E = \{(I, o)\}\) and \(D \times \mathbb{E}^d\) and \(E\) are separated by the hyperplane \(H \times \mathbb{E}^d\), where \(H\) is the unique support hyperplane of \(D\) at \(I\) (in \(E^{d(d+1)}\)).

\(K\) and \(N\) are the cones

\[K = \bigcap_{u \in B^d \cap \text{bd} \, C} (A, a) \in E^d : A \cdot u \otimes u + a \cdot u \leq 1,\]

\[N = \text{pos} \{(u \otimes u, u) : u \in B^d \cap \text{bd} \, C\}.\]

As before, \((I, o) \in N\). Carathéodory’s theorem for cones in \(E^{d(d+3)}\) then shows the following: there are \((u_k \otimes u_k, u_k) \in N\) or, equivalently, \(u_k \in B^d \cap \text{bd} \, C\) and \(\lambda_k > 0, k = 1, \ldots, n\), where \(n \leq 1/2 d(d+3)\), such that instead of (8) we have

\[(I, o) = \left( \sum_k \lambda_k u_k \otimes u_k, \sum_k \lambda_k u_k \right).\]

Since \(o = \sum \lambda_k u_k\) and \(\lambda_k > 0\), the proof that \(n \geq d + 1\) is the same as that for \(n \geq d\) above. This concludes the proof that (i)\(\Rightarrow\)(ii). The proof of (ii)\(\Rightarrow\)(i) is almost the same as that of the corresponding part of the proof of Theorem 1.
Proof of Theorem 3

Identify a $d \times d$ matrix $M = (m_{ij})$ with the point $(m_{11}, \ldots, m_{1d}, m_{21}, \ldots, m_{2d}, \ldots, m_{dd})^T \in \mathbb{E}^d$. The set $\mathcal{P}'$ of all non-singular $d \times d$ matrices then is (represented by) an open cone in $\mathbb{E}^d$ with apex at the origin. The set

$$\mathcal{D}' = \{ M \in \mathcal{P}' : |\det M| \geq 1 \}$$

is a closed body in $\mathcal{P}'$, i.e. it is the closure of its interior, with a smooth boundary surface.

The set of all affine images of $B$ in $C$ is represented by the set

$$\mathcal{E}' = \{(M, a) \subset \mathcal{P}' \times \mathbb{E}^d : Mu \cdot v + a \cdot v = M \cdot u \otimes v + a \cdot v \leq h_C(v) \text{ for } u \in B, v \in S^{d-1}\}.$$

This set is the intersection of the closed halfspaces

$$\{(M, a) \in \mathbb{E}^{d(d+1)} : M \cdot u \otimes v + a \cdot v \leq h_C(v)\} : u \in B, v \in S^{d-1},$$

and thus of a convex set, with the set $\mathcal{P}' \times \mathbb{E}^d$. Choose a convex neighborhood $\mathcal{U}'$ of $(I, o) \in (\mathcal{E}' \cap (\mathcal{P}' \times \mathbb{E}^d))$ which is so small that it is contained in the open set $\mathcal{P}' \times \mathbb{E}^d$. By (i),

the convex set $\mathcal{E}' \cap \mathcal{U}'$ and the smooth body $\mathcal{D}' \times \mathbb{E}^d$ only have boundary points in common, one being $(I, o)$.

Hence $\mathcal{E}' \cap \mathcal{U}'$ and thus the support cone $\mathcal{K}'$ of $\mathcal{E}' \cap \mathcal{U}'$ at $(I, o)$ is contained in the closed halfspace whose boundary hyperplane is the tangent hyperplane of the smooth body $\mathcal{D}' \times \mathbb{E}^d$ at $(I, o)$ and with exterior normal pointing into $\mathcal{D}' \times \mathbb{E}^d$. This normal is $(I, o)$. The normal cone $\mathcal{N}'$ of $\mathcal{K}'$ thus contains $(I, o)$.

The support cone $\mathcal{K}'$ is the intersection of those halfspaces in (9), which contain the apex $(I, o)$ on their boundary hyperplanes. Thus $I \cdot u \otimes v + o \cdot v = h_C(v)$, which is equivalent to $u \in B \cap \text{bd} C, v \in N_C(u)$. Hence, these halfspaces have the form

$$\{(M, a) \in \mathbb{E}^{d(d+1)} : M \cdot u \otimes v + a \cdot v \leq h_C(v)\} : u \in B \cap \text{bd} C, v \in N_C(u),$$

where $N_C(u)$ is the normal cone of $C$ at the boundary point $u$. Thus, being the normal cone of $\mathcal{K}'$,

$$\mathcal{N}' = \text{pos} \{(u \otimes v, v) : u \in B \cap \text{bd} C, v \in N_C(u)\}.$$

Since $(I, o) \in \mathcal{N}'$, Carathéodory’s theorem for convex cones in $\mathbb{E}^{d(d+1)}$ yields the following: there are $(u_k \otimes v_k, v_k) \in \mathcal{N}'$ or, equivalently, $u_k \in B \cap \text{bd} C, v_k \in N_C(u_k)$, and $\lambda_k > 0$, $k = 1, \ldots, n$, where $n \leq d(d + 1)$, such that

$$(I, o) = \left(\sum_k \lambda_k u_k \otimes v_k, \sum_k \lambda_k v_k\right).$$

For the proof that $n \geq d + 1$ we show by contradiction that $\text{lin}\{v_1, \ldots, v_n\} = \mathbb{E}^d$ as in the proof of Theorem 1.
Final Remarks

In different versions of the proofs of Theorems 1 and 2, which are closer to Voronoi’s idea, ellipsoids are represented in the form $x^T Ax \leq 1$ and $(x-a)^T A(x-a) \leq 1$, respectively.

If in Theorem 3 claim (ii) holds, then the support cone $K'$ of $E' \cap U'$ at $(I, o)$ is contained in the halfspace whose boundary is the tangent hyperplane of $D' \times E'^d$ at $(I, o)$ and with exterior normal pointing into $D' \times E'^d$. Since $D' \times E'^d$ is not convex, this does not guarantee that $D' \times E'^d$ and $E'$ do not overlap, i.e. that (i) holds.

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