How long does it take to implement a projective measurement?

Philipp Strasberg, Kavan Modi, and Michalis Skotiniotis

Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
School of Physics and Astronomy, Monash University, Clayton, Victoria 3800, Australia

(Dated: March 17, 2022)

According to the Schrödinger equation, a closed quantum system evolves continuously in time. If it is subject to a measurement however, its state changes randomly and discontinuously, which is mathematically described by the projection postulate. But how long does it take for this discontinuous change to occur? Based on simple estimates, whose validity rests solely on the fact that all fundamental forces in nature are finite-ranged, we show that the implementation of a quantum measurement requires a minimum time. This time scales proportionally with the diameter of the quantum mechanical object, on which the measured observable acts non-trivially, with the proportionality constant being around $10^{-5} \text{s/m}$. We confirm our bound by comparison with experimentally reported measurement times for different platforms. We give a pedagogical exposition of our argumentation introducing along the way modern concepts such as ancilla-based measurements, the quantum speed limit, and Lieb-Robinson velocity bounds.

I. INTRODUCTION

The process of measurement in quantum theory is radically different as compared to any other physical theory. Upon observing a quantum mechanical system its state randomly and discontinuously changes into an eigenstate of the corresponding observable. It is this unpredictable change that most students have trouble coming to terms with as there is no analogue in classical mechanics. Indeed, the measurement or projection postulate remains a highly active area of research and is at the heart of interpretational issues concerning quantum theory. For instance, within the commonly taught Copenhagen interpretation of quantum mechanics, this change is epitomized by the “collapse” of the wave function.

Since quantum mechanics was conceived for the microworld, a naturally arising question is which postulates can be carried over to the macroworld. Here, we explain that the time to implement a measurement is necessarily finite. For macroscopic observables, this time can be much longer than the time scales of intrinsic quantum mechanical processes of the object itself, posing severe practical challenges for the application of the projection postulate. As we will explain, the minimum measurement time for a quantum mechanical observable that acts non-trivially on a physical system of size $d$ is

$$t_{\text{min}} \geq \frac{d}{v} \quad \text{with} \quad v = 10^5 \frac{\text{m}}{\text{s}}, \quad (1)$$

in good agreement with state-of-the-art experiments. Thus, the larger the measured object the longer it takes to measure it, highlighting that the discontinuity in the measurement postulate has to be taken cum grano salis (with a grain of salt). We remark that Eq. (1) does not hold for photons but for systems composed of electrons, atoms or molecules. Importantly, our conclusions hold independently of the reader’s preferred interpretation of quantum theory.

We derive Eq. (1) armed with nothing more than quantum theory itself and the fact that all forces in nature are finite-ranged. Our line of reasoning makes use of three important tools which are absent from most standard quantum mechanics curricula: ancilla-assisted measurements, which will help us to mathematically formulate the process of measurement in terms of the more familiar unitary evolution postulate of quantum theory, quantum speed limits which tell us that evolution in a Hilbert space takes finite time, and Lieb-Robinson bounds which limit the speed of propagation of information in space. We start with a pedagogical exposition of these techniques in Sec. II, which find applications in a wide range of problems, such as quantum sensing, metrology, and quantum thermodynamics, and form an active area of current research. We then employ these results to establish a lower bound on the time it takes to measure a macroscopic object of size $d$ in Sec. III. There we also provide a back-of-the-envelop comparison of our bound with experimentally reported measurement times in quantum systems. We summarise and conclude in Sec. IV.

II. REVIEW OF PERTINENT CONCEPTS

We start in Sec. (IIA) with a brief review of the mathematical postulates of quantum theory followed in Sec. (IIB) with a discussion about an important mathematical tool referred to as ancilla-assisted measurements. Whereas these first two sections are free from any physical considerations, we introduce physical constraints in the remaining three sections. These include energy constraints on the minimal evolution time of a quantum system (quantum speed limits, Sec. IIC), constraints on the form of the admissible Hamiltonians, which have to be local due to the finite range of all fundamental forces (Sec. IID), and constraints on the speed at which information can propagate in locally interacting objects (Lieb-Robinson bounds, Sec. IIE).
call the framework that results from supplementing the mathematical postulates of quantum theory with physical constraints quantum physics.

A. The mathematical postulates of quantum theory

For our purposes it suffices to consider the mathematical formalism of quantum theory as it applies to closed systems. The state of such a system is described by a vector $|\psi\rangle \in \mathcal{H}$ in a Hilbert space $\mathcal{H}$ or, more generally, by a positive-definite operator of unit trace, $\rho$, the density matrix. The closed-system evolution of such a system is described by a unitary operator, $U_t : \mathcal{H} \to \mathcal{H}$, with $U_t U_t^\dagger = U_t^2 = I$, the identity matrix, and $t$ the time. Measurements of an observable $R = \sum_{r=0}^{n-1} \lambda_r \Pi(r)$ are described by sets of projection operators $\{\Pi(r)\}$, with $r$ labeling the possible measurement outcomes (assumed to be discrete for simplicity), satisfying the conditions $\Pi(r)\Pi(s) = \delta_{r,s} \Pi(r)$ (the usual Kronecker delta) and $\sum_r \Pi(r) = I$. The probability of obtaining a given outcome, $r$, is given by Born’s rule, $p(r) = \text{tr}(\Pi(r) \rho)$. Upon obtaining the outcome $r$, the post-measurement state of the system changes to

$$\rho(r) = \frac{\Pi(r) \rho \Pi(r)}{p(r)}.$$  \hspace{1cm} (2)

Finally, the state space of a composite system, made out of $N$ distinguishable constituents whose Hilbert spaces are $\mathcal{H}_i$, $i \in \{1, \ldots, N\}$, is given by the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N := \bigotimes_{i=1}^N \mathcal{H}_i$. We assume that the time evolution of all objects in our discussion, including macroscopic ones, are described by Schrödinger’s equation. Consequently, we also model the interaction between an isolated quantum system and any object which one could potentially regard as measurement apparatus (including a human brain) by a unitary operator. This unitary entangles the system with the apparatus without either experiencing wavefunction collapse as per Everett’s “many-worlds” interpretation. We now review these ancilla-based measurement schemes which form part of the standard toolkit of quantum measurement theory.

B. Ancilla-based measurements

To explain the idea of ancilla-based measurements, which goes back to von Neumann, it is useful to look at an example first (see also Fig. 1). Consider a two-level system in the state $|\psi\rangle_S = \alpha |0\rangle_S + \beta |1\rangle_S$ with $|\alpha|^2 + |\beta|^2 = 1$ and suppose we want to perform a measurement in the basis $\{ |0\rangle, |1\rangle \}$. To this end, we let it interact with an external probe or detector, called an ancilla in the following, which is itself a quantum system. Quite intuitively, the measurement of a two-level system requires not more than a two-level system, which we assume to be prepared in the state $|0\rangle_A$. Thus, the initial system-ancilla state is $|\Psi(0)\rangle_{SA} = |\psi\rangle_S \otimes |0\rangle_A$. Now, we let the system and ancilla interact in such a way that the unitary after time $t$ is

$$U_{SA}(t) = |0\rangle\langle 0| + |01\rangle\langle 01| + |10\rangle\langle 1| + |11\rangle\langle 1|,$$  \hspace{1cm} (3)

where we used the shorthand notation $|0\rangle \equiv |0\rangle_S \otimes |0\rangle_A$, etc. What is the state after the interaction? A quick calculation reveals

$$|\Psi(t)\rangle_{SA} = U_{SA}(t)|\Psi(0)\rangle = \alpha |00\rangle + \beta |11\rangle,$$  \hspace{1cm} (4)

which is a perfectly correlated state: the system is in state $|0\rangle (|1\rangle)$ if and only if the ancilla is in state $|0\rangle (|1\rangle)$. Measuring the state of the ancilla therefore reveals the state of the system and implements the desired projective measurement. It is interesting to note that the reduced state of the system (as well as of the ancilla) is mixed,

$$\text{tr}_A[|\Psi(t)\rangle\langle \Psi(t)|_{SA}] = |\alpha|^2 |0\rangle\langle 0|_S + |\beta|^2 |1\rangle\langle 1|_S,$$  \hspace{1cm} (5)

which is a consequence of the quantum entanglement between system and ancilla. Another quick calculation reveals that this state is identical to

$$|0\rangle\langle 0|_S |\psi\rangle |\psi\rangle_{SA} + |1\rangle\langle 1|_S |\psi\rangle |\psi\rangle_{SA},$$  \hspace{1cm} (6)

which corresponds to the average effect of a measurement in the basis $\{ |0\rangle, |1\rangle \}$. It is therefore also called the unconditional measurement state because it is not yet conditioned on receiving the outcome 0 or 1. In fact, it is impossible to implement Eq. (2) in a unitary way, which in the present case means a transformation of the form $|0\rangle\langle 0|_S |\psi\rangle_{SA} + |\beta|^2 |1\rangle\langle 1|_S$ if $r = 1$. The reason is that Eq. (2) cannot follow from unitary time-evolution alone as it is a non-linear transformation with respect to $\rho$ (recall that the probability $p(r)$ depends on $\rho$ too).

The main question we are asking in this manuscript is: What is the minimum measurement time $t$ required to implement the unitary in Eq. (3), i.e., what is the minimum time required to sufficiently correlate a system with an external detection apparatus or ancilla? We will address this question in particular for macroscopic systems by using physically motivated requirements. Before we come to them, we show that the approach above is general and can be used for any system and measurement.

We consider an arbitrary system state $|\psi\rangle_S$ and an arbitrary system observable $R_S$ with $n$ associated projectors $\Pi_S(r)$. As above, we assume the ancilla is prepared in the state $|0\rangle_A$ such that the initial system-ancilla state reads $\rho_{SA}(0) = \rho_S \otimes |0\rangle_A$. Next, we assume that the experimentalist can engineer an interaction between $S$ and $A$ resulting in the unitary operator

$$U_{SA}(t) = \sum_{r=0}^{n-1} \Pi_S(r) \otimes \sum_{x=0}^{n-1} |x+r\rangle\langle x|_A.$$  \hspace{1cm} (7)
transformation of the system initially in state $|\psi\rangle$ by a physical Hamiltonian $H$.

We shall define precisely what we mean by the term Hamiltonian and how the system evolves under $H$. The crucial distinction lies in the fact that we must consider the dynamics of the system in a joint system (or self-adjoint) operators. For simplicity in the presentation, we decide to focus on the general case of two-level systems.

In this, our conclusions will not rely on it. Moreover, we are interested in how fast we can implement unitary evolution in a joint system-ancilla space.

FIG. 1. An ancilla-based strategy for implementing the projective measurement $\{\Pi_S(0) = |0\rangle\langle 0|_S, \Pi_S(1) = |1\rangle\langle 1|_S\}$ (equivalently implementing a measurement of the observable $\sigma^z = |1\rangle\langle 1| - |0\rangle\langle 0|$). The system is a two-level system initially prepared in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. The ancilla is also a two-level system initially prepared in state $|0\rangle$. They interact via the Hamiltonian $H_{SA} = \hbar \omega |0\rangle\langle 0|_S \otimes 1 + |1\rangle\langle 1|_S \otimes (1 + \sigma^z)_A$ (with $\sigma^z = |0\rangle\langle 0| + |1\rangle\langle 1|$) and evolve unitarily for a time $t = \pi/\omega$. Up to a global phase this implements the transformation $U_{SA} \approx \exp\left(-i \frac{\hbar}{\omega} |1\rangle\langle 1|_S \otimes 1 + |0\rangle\langle 0|_S \otimes (1 + \sigma^z)_A \right)$. Here, the ancilla is prepared in state $|0\rangle$ and the system evolves to an orthogonal state, such as $|\psi\rangle = |\psi\rangle \otimes |\psi\rangle'$. Measurably, it turns out that the map $\rho' = \sum K_r \rho K_r^\dagger$ can be also implemented by a suitable unitary evolution in a joint system-ancilla space.

C. Quantum speed limits

So far, our exposition was based purely on mathematical considerations. Quantum physics starts to emerge by a set of correspondence rules that assign dynamical variables, such as position, momentum, etc., to Hermitian (or self-adjoint) operators.

For instance, unitary time evolution can always be written as (which formally follows from Stone’s theorem)

$$U_t = \exp \left(-i \frac{H}{\hbar} t \right),$$

where $H$ is some Hermitian operator and $\hbar$ the reduced Planck constant. In quantum theory, any Hermitian operator can generate unitary evolution. In quantum physics, on the other hand, unitary evolution is determined by the systems Hamiltonian through Schrödinger’s equation and Eq. (10) holds for time-independent Hamiltonians only. The crucial distinction between quantum theory and quantum physics is that not every Hermitian operator corresponds to a physical Hamiltonian. We shall define precisely what we mean by a physical Hamiltonian shortly. Presently, we are interested in how fast we can implement unitary evolution given a Hermitian operator $H$.

The quantum speed limit quantifies the minimum time required for a system, initially in state $|\psi\rangle$, to evolve to an orthogonal state, $|\psi\rangle'\perp$, under the unitary evolution of Eq. (10). Given a Hermitian operator $H$, the quantum speed limit provides a lower bound to

\begin{align*}
U_{SA}(t) \rho_{SA}(0) U_{SA}^\dagger(t) &= \sum_{r,r'} \Pi_S(r) \rho_S \Pi_S(t') \otimes |r\rangle\langle r'|_A, \\
\rho_S(t) &= \sum_r \Pi_S(r) \rho_S \Pi_S(r),
\end{align*}

which describes the unconditional measurement state as explained above. It is easy to check that this corresponds to the average of Eq. (2): $\rho_S(t) = \sum_r p(r) \rho_S(r)$. In reality, the general description above is minimal but somewhat simplistic. However, as we will see, we will not put any restrictions on the ancilla in the following. In our view, the use of an ancilla corresponds to a minimal theoretical description of the process, which contains all essential features to understand it. In present-day experiments, these abstract ancillae can indeed be realized and controlled using either physically distinct quantum systems (electrons, photons, etc.) or even additional degrees of freedom of the same physical system, such as the motional degrees of freedom of the quantum system, or higher electronic (energy) eigenstates. Whatever the physical realization of the ancilla is, our conclusions will not rely on it.

Before we proceed, we remark that measurements in quantum mechanics are not always described by the projection postulate, Eqs. (2) or (9). An example is the detection of particles or photons. This measurement certainly reveals information about the state of the system before the measurement, but afterwards the system no longer ‘exists’ in the sense considered here. Mathematically, it turns out that such a state change can be captured by generalizing Eq. (9) to $\rho' = \sum K_r \rho K_r^\dagger$.
the evolution time as
\[ t \geq \tau_{\text{qsl}} = \max \left\{ \frac{\hbar \pi}{2 \Delta E}, \frac{\hbar \pi}{2 (\langle H \rangle - E_0)} \right\} . \] (11)

Here, \( \Delta E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2} \) is the standard deviation of the energy with expectation value \( \langle H \rangle = \langle \psi | H | \psi \rangle \), and \( E_0 \) is the energy of the ground state. The bound \( \hbar \pi / 2 \Delta E \) is due to Mandelstam and Tamm\(^{[2]} \) whereas the bound \( \hbar \pi / 2 (\langle H \rangle - E_0) \) is due to Margolus and Levitin\(^{[12]} \).

It is worth pausing for a moment to reflect on the motivation behind the Mandelstam and Tamm bound discovered in 1945. Shortly after Heisenberg postulated his famous uncertainty principle of position and momentum, an analogous time-energy uncertainty relation was postulated based on dimensional arguments. While the position-momentum uncertainty relation was formally derived, the time-energy uncertainty relation was difficult to derive as time is not an observable. Mandelstam and Tamm resolved this issue with their bound giving the time-energy uncertainty relation firm mathematical footing. This simple relation between time and energy is now an integral part of a quantum theorist’s toolkit\(^{[13]} \).

By way of example, let us compute the quantum speed limit for any ancilla-based measurement of a binary observable, \( R_S \), on the system. In this case it suffices to consider a two-dimensional ancilla for its implementation. Denote by \( \{|0\}_A, |1\>_A \) the orthonormal basis states of the ancilla and assume it is initially prepared in the state \( |0\>_A \). The unitary of Eq. (7) can be implemented by the following system-ancilla Hamiltonian (see also Fig. 1).

\[ H_{SA} = \frac{\hbar g}{2} \left[ |0\>_S \langle 0| \otimes |1\>_A + |1\>_S \langle 1| \otimes (|1\>_A + \sigma^x)_A \right], \] (12)

with \( \sigma^x \), \( \sigma^y \) and \( \sigma^z \) the standard Pauli matrices. If \( H_{SA} \) is “switched on” for a time \( t = \pi / g \) we obtain \( U_{SA} = |0\>_S \langle 0| \otimes |1\>_A + |1\>_S \langle 1| \otimes |1\>_A \) as desired.

Now let us determine the minimum time according to the quantum speed limit. Note that we have \( E_0 = 0 \) for the Hamiltonian in Eq. (12). If the system-plus-ancilla are originally in the state \( |00\>_SA \), then \( U_{SA} |00\>_SA = |00\>_SA \) and there is no quantum speed limit to apply. However, if our system-plus-ancilla state is \( |10\>_SA \), then we find that both bounds in Eq. (11) coincide, giving rise to

\[ \tau_{\text{qsl}} = \frac{\hbar \pi}{2 \Delta E_{SA}} = \frac{\hbar \pi}{2 \langle 10 | H_{SA} | 10 \rangle} = \frac{\pi}{g}. \] (13)

Thus, the interaction Hamiltonian of Eq. (12) is the quickest way of implementing the ancilla-based measurement; any other interaction Hamiltonian would require equal or longer time.

### D. Locality of interactions

The fact that not every Hermitian operator corresponds to a physical Hamiltonian arises from the fact that physical systems are subject to further constraints. For instance, in closed systems energy, momentum, etc., are conserved quantities. Moreover, for macroscopic physical systems, composed out of a large number of particles with non-zero rest mass (such as electrons, nuclei, atoms, or molecules), the dominant interactions are mediated via the Coulomb force. The latter gives rise to a set of physical Hamiltonians which describe locally-interacting systems as we now explain.

Let \( H_S \) be the Hamiltonian associated to a physical system occupying a space-like volume, \( V \), of diameter \( d \) (following the spherical-cow tradition in theoretical physics we characterize the size of the object by its diameter only). To explain the notion of interacting and local we split \( V \) into an arbitrary set \( \{ V_i \} \) of non-overlapping volumes \( V_i \) (i.e., \( V_i \cap V_j = \emptyset \) if \( i \neq j \), see Fig. 2). To each such volume we assign a Hilbert space, \( \mathcal{H}_i \), such that the Hilbert space of the entire system is \( \mathcal{H}_S = \bigotimes \mathcal{H}_i \). We can now split the Hamiltonian into a sum involving a local and interacting part,

\[ H_S = H_{\text{loc}} + H_{\text{int}}, \] (14)

where \( H_{\text{loc}} = \sum_i H_i \) contains all the local Hamiltonians—\( H_i \) acting non-trivially only on \( \mathcal{H}_i \)—and \( H_{\text{int}} = \sum_{i \neq j} J_{ij} \) describes the pair-wise interactions between volumes \( V_i \) and \( V_j \). The assumption of pair-wise interactions is ultimately justified by the fact that the Coulomb force is a two-body interaction. We define a locally interacting physical system if the following two conditions are satisfied:

(i) Each \( V_i \) interacts at least with one other \( V_j \). Here, by interaction we mean that \( [H_i, J_{ij}] \neq 0 \).

(ii) For all \( i, j \) the strength of the interaction is inversely proportional to the distance, \( d_{ij} \), between the two locations. If \( V_i \) and \( V_j \) have themselves a non-negligible diameter, \( d_{ij} \) can be defined as an average or minimal distance.
The first condition allows the transfer of energy and information through the physical system, which we assume to avoid trivial situations (e.g., objects composed out of non-interacting parts). The second condition arises from the fact that all fundamental forces in nature are finite-ranged and decrease with increasing distance. Together, conditions (i) and (ii) above set the constraints on the set of physically admissible Hamiltonians that we consider here.

To give a specific example, we present a simple model of such a physical system made out of interacting spins on a one-dimensional lattice as shown in Fig. 3. The simplest interaction one can conceive of is that of nearest-neighbour interactions: each spin interacts only with the spins adjacent to it. Whilst being a strong simplification, we note that models restricted to nearest-neighbour interactions can qualitatively describe many phenomena in condensed matter physics. The reason is that nearest-neighbour interactions often play the most dominant role. In Fig. 3, for example, we depict the Ising model whose Hamiltonian

$$ H_{\text{Ising}} = h \sum_{i=1}^{N} \sigma_i^{(z)} + g \sum_{i=1}^{N-1} \sigma_i^{(x)} \sigma_{i+1}^{(x)} $$  \hspace{1cm} (15)$$
describes \( N \) interacting spins in an external magnetic field, \( h \), and with spin-spin interaction strength, \( g \). While it is good to keep the example of the 1D Ising model in mind for illustrative purposes, we remark that our conclusions below are general and not restricted to these kind of models.

E. Finite size systems and the Lieb-Robinson bound

The objects we are here interested in are composed out of a very large number of locally interacting particles with non-zero rest mass (this includes, e.g., electrons, nuclei, atoms, or molecules, but excludes the treatment of, e.g., noninteracting photons). To define a macroscopic object we estimate the size of each constituent by invoking the equipartition theorem. The latter states that the kinetic energy of a single particle at temperature \( T \) is of the order \( mv^2 \approx k_B T \), where \( m \) is the mass of the particle, \( v \) its velocity, and \( k_B \) is Boltzmann’s constant. By identifying the particle’s momenta via \( mv = h/\lambda_{\text{th}} \) with \( h = 2\pi h \), we obtain the thermal de Broglie wavelength \( \lambda_{\text{th}} = h/\sqrt{mk_B T} \) of a single particle. For example, a single hydrogen atom at room temperature has \( \lambda_{\text{th}} \approx 100 \text{ nm} \). According to this reasoning, a physical system is macroscopic if \( d \gg \lambda_{\text{th}} \).

Moreover, we are interested in observing global properties of such systems. For example, in Schrödinger’s cat gedanken experiment we measure the whole cat, not just its tail, in order to determine its state of wellbeing. The corresponding elements of such global dynamical variables correspond to observables, \( R \), that are highly non-local, i.e., they act non-trivially on the entire macroscopic object.

Following Sec. II B, the goal is now to engineer a suitable interaction between the system and a measuring device—the ancilla—such that the unitary of Eq. (7) ensues. After suitably choosing the initial ancilla state, this gives rise to the post measurement state of Eq. (9) as desired. Recalling Eq. (12), the most straightforward way seems to be a system-ancilla Hamiltonian of the form

$$ H_{SA} = \sum_r \Pi_S(r) \otimes H_A(r) $$  \hspace{1cm} (16)$$
with suitably chosen ancilla Hamiltonians \( \{ H_A(r) \} \). However, for highly non-local observables this implies that the projectors \( \Pi_S(r) \) act non-trivially on a large part of the system. It follows that the Hamiltonian of Eq. (16) is unphysical in that it does not belong to the class of Hermitian operators corresponding to locally-interacting physical systems.

Realistic Hamiltonians have to take into account that the measuring device can only interact locally with the system. Returning to the example of the Ising model from Eq. (15), a more physical system-ancilla interaction Hamiltonian is

$$ H_{SA} = H_{\text{Ising}} + h_A \sigma_A^{(z)} + g_A \sigma_A^{(x)} \sigma_A^{(x)} $$  \hspace{1cm} (17)$$
where we assumed—for illustrative purposes—that the ancilla is represented by a single spin coupled to the \( N \)th spin of the Ising chain from Eq. (15) (see Fig. 3).

One may wonder whether there is actually any chance at all to implement the unitary of Eq. (7) based on the limited amount of control offered by Eq. (17). Insights into this question are offered by Zassenhaus’s formula (see Appendix A), a special case of the more general Baker-Campbell-Hausdorff formula. Despite being a challenging theoretical and experimental task, we assume for the remainder of this work that local interactions are sufficient to generate the required unitary in Eq. (7).
As locally-interacting Hamiltonians satisfy conditions (i) and (ii), information cannot travel arbitrarily fast through the physical system, even if we neglect relativistic considerations. A local perturbation needs time before it can influence a different region in space. This idea can be formalized as follows. Let $A_i, A_j$ be two observables acting on the Hilbert spaces $\mathcal{H}_i, \mathcal{H}_j$ corresponding to two particles a distance $d_{ij}$ apart. If the interaction between sites $i$ and $j$ is finite ranged, then the Lieb-Robinson bound\(^{19}\) states

$$
||[A_j(t), A_i]|| \leq C \exp[-a(v_f t - d_{ij})],
$$

where $A_j(t) = e^{i t H/\hbar} A_j e^{-i t H/\hbar}$ with $H$ the locally interacting Hamiltonian and $a \geq 0$ and $C \geq 0$ are suitable constants. Here, $\| \cdot \|$ is a suitable operator norm measuring the “size” or “significance” of the term on the left hand side of Eq. (18). Furthermore, $v_f \geq 0$ is known as the Lieb-Robinson velocity, where we use the subscript $I$ to emphasize that this determines the speed by which information can travel through the system.

In our case, $H$ includes the system-ancilla interaction as well as the Hamiltonian of the isolated system itself. Due to the Lieb-Robinson bound, excitations can travel through a locally interacting object only with a finite (perhaps averaged in case of a non-homogenous object) velocity $v_f$ (see Fig. 4). In particular, for an excitation to travel through an object of diameter $d$, it takes a time

$$
\tau = \frac{d}{v_f}. \quad \text{(19)}
$$

Like the quantum speed limit, the Lieb-Robinson bound also has a rich history and plays an integral role as a toolkit in modern quantum science\(^ {19}\).

### III. PROJECTIVE MEASUREMENT TIMES OF FINITE-SIZE SYSTEMS

We now have everything we need in order to show that projective measurements on finite-size systems take a finite amount of time. To ensure that our arguments are fundamentally valid for all physical systems (irrespective of size), we do not impose any restrictions on the degrees of freedom of the measurement apparatus. In particular, we allow that some technologically advanced “alien civilization” is in possession of a fully controllable, fault-tolerant, large scale quantum computer allowing arbitrary pre- and post-processing of arbitrarily many ancillae. However, what has to be respected is that the unitary of Eq. (7) is implemented by a locally-interacting Hamiltonian.

Two strategies exist for measuring the nonlocal observable, $R_S$, of an object with diameter $d$; either we perform a single local measurement by coupling only one ancilla to some part of the system, or we perform many local measurements using several ancillae coupled to the macroscopic system at different locations.

![Diagram](image)

**FIG. 4.** Illustration of the Lieb-Robinson bound. Consider the Ising model from Eq. (15) prepared initially in its ground state (shaded balls). Suppose now that we locally perturb one end of the chain (dark ball and wiggly arrow). How long does it take until the other end of the chain “feels” an influence of this perturbation? Computing this time exactly is challenging, but it is easy to understand from the structure of Eq. (13) that this must take a finite time. For this purpose assume that we discretize time into small steps $\delta t$ such that the unitary time evolution operator can be approximated as $e^{-i t H_{\text{Ising}}/\hbar} \approx I - i t H_{\text{Ising}} \delta t / \hbar$. Now, recall that $H_{\text{Ising}}$ is local and contains only nearest-neighbor interactions. Thus, in the first time step the excited spin on the right of the chain can only influence its nearest neighbor to the left (now also sketched as a dark ball). In the next time step, this neighbor can again influence only its nearest left neighbor, and so on and so forth until the excitation reaches the leftmost spin. In our example the time taken is $t = 6 \delta t$. If one takes the ratio of the distance traveled by the excitation, which here equals the diameter of the object, and this time one obtains the finite velocity $v_f$.

For the first strategy it is sufficient to consider the one-dimensional situation depicted in Fig. 3. Computing the quantum speed limit, given in Eq. (11), now gives us a minimum time that varies inversely proportional to the system-ancilla coupling strength, $g_A$. If the interaction is electrostatic, then $g_A = u_{\text{Coulomb}}/\hbar$ with the Coulomb potential

$$
u_{\text{Coulomb}} = \frac{k_e q_S q_A}{r_{SA}}. \quad \text{(20)}
$$

Here, $k_e \approx 9 \cdot 10^9 \text{ Nm}^2/\text{C}^2$ is the Coulomb constant and $r_{SA}$ is the distance between the system with charge $q_S$ and the ancilla with charge $q_A$. Our simple model specific result, Eq. (13), suggests that the quantum speed limit scales like $\tau_{\text{sql}} \approx \pi / g_A$. Equating this with the minimum measurement time, $t_{\text{min}} = \tau_{\text{sql}}$, we can write the bound more conveniently as

$$
t_{\text{min}} = \frac{r_{SA}}{v_E} \quad \text{with} \quad v_E = \frac{k_e \pi}{k_e q_S q_A}. \quad \text{(21)}
$$

Above, we have added the subscript $E$ to emphasize that this speed comes from energetic considerations.
Assuming a single elementary charge, \( q_S = q_A = 1.6 \cdot 10^{-19} \) C (e.g., the ancilla is a single electron, which we let precisely interact with one electron of the system), Eq. (20) gives
\[
t_{\text{min}} = \frac{r_{SA}}{v_E} \quad \text{with} \quad v_E \approx 10^5 \frac{\text{m}}{\text{s}}. \quad (22)
\]

If the distance between the system and apparatus is \( r_{SA} = 10^{-10} \) m—of the order of the size of a hydrogen atom—then we obtain a minimum measurement time of 1 fs. Remarkably, this order of magnitude fits surprisingly well cutting-edge technological standards based on ultrashort laser pulses, used in femtochemistry, to study single chemical reactions in real time.\(^{20}\) Higher time resolutions are possible using, e.g., free electron lasers, but they destroy the system such that the post-measurement state can no longer be described by Eq. (2).

Let us now consider a macroscopic object of diameter \( d = 1 \) m with an ancilla attached to one end of it. Putting \( r_{SA} = 1 \) m in Eq. (22) results in a minimum measurement time of \( t_{\text{min}} \approx 10^{-5} \) s and corresponds to an ability to resolve processes in the low-frequency radio wave regime, much slower than the timescale many quantum mechanical processes evolve at (e.g., the timescale of chemical reactions in a cat). In turn, this also gives us an optimistic estimate of the Lieb-Robinson velocity, \( v_I \approx 100,000 \) m/s: one order of magnitude larger than the speed of sound in very stiff materials such as diamond. Finally, note that for a macroscopic object the distance \( r_{SA} \) between the ancilla and the farthest end of the object roughly equals the diameter of the object, i.e., \( d \approx r_{SA} \). Thus, Eq. (22) directly implies our central result (1).

Can we improve the situation by coupling several ancillae at different locations on the macroscopic system? Afterwards, one could imagine that clever post-processing of the ancillae gives rise to a faster measurement. For one- and two-dimensional systems this strategy may in fact be possible since one could couple ancillae to every part of this object. However, the world around us is three-dimensional and three-dimensional systems are typically dominated by their bulk or volume properties. If it is only possible to couple ancillae to the surface of the object, then the Lieb-Robinson velocity sets a minimum time needed for some information about the interior of the object to influence the ancilla. Taking our above estimate for the Lieb-Robinson velocity, the minimum time based on Eq. (19) is
\[
t_{\text{min}} = \frac{d}{v_I} \quad \text{with} \quad v_I \approx 10^3 \frac{\text{m}}{\text{s}}. \quad (23)
\]

In contrast to Eq. (22), the minimum measurement time is here bounded by the speed with which information can travel through the system. Even though this estimate is independent of the quantum speed limit, both estimates agree and are in unison with Eq. (1).

While we have focused on electromagnetic interactions, our results are general. Even if we could use strong nuclear forces for the system-ancilla interaction, which allow for the measurement of a small object on a yoctosecond time-scale \( (10^{-24}) \) s, the measurement time for a macroscopic object would still be determined by the Lieb-Robinson velocity since the strength of the nuclear forces decays very quickly to zero with increasing distance. Thus, our bound on the time it takes to measure a macroscopic quantum object remains unaffected even if it turns out that a quantum measurement of a single microscopic system can be implemented extremely fast. Furthermore, even if we consider the speed of light for the Lieb-Robinson velocity, i.e., \( v_I = c \), the time needed for an excitation to travel a distance of \( d = 1 \) m would be \( \frac{d}{v_S} \approx 3 \times 10^{-8} \) s. This implies an ability to resolve instantaneously processes involving energy differences of 300 MHz (in the low microwave regime). Once more, it is impossible to justify applying an instantaneous projective measurement postulate at this time-scale.

In Fig. 5 we compare our main result, the minimum measurement time given in Eq. (1) (blue line in Fig. 5), with various state-of-the-art experimental platforms\(^{21–27}\) (black dots with reference number in Fig. 5). We emphasize that the derivation of our bound is based on assuming that macroscopic systems are composed of many locally interacting particles, so we do not necessarily expect our bound to hold for a single electron or atom. Nevertheless, as we see in Fig. 5 our bound is never violated.

**IV. DISCUSSION**

We have illustrated how basic physical concepts (in particular, the locality of interactions), which are absent from the axioms of quantum theory, can be fruitfully used to learn more about those very axioms. In particular, we found that a basic postulate of quantum theory, the instantaneous implementation of a projective measurement, is not applicable to macroscopic objects as soon as one takes into account physical consid-
operations. Based on intuitive physical reasoning, we derived a lower bound, Eq. [1], in agreement with experimental reality. By shedding some light on the abstract measurement postulate from a physical perspective, we hope it appears less confusing for students. At least we introduce many modern and multi-faceted techniques of quantum theory.

The present insights might have also consequences on the philosophical questions raised by the ingenious thought experiments of Schrödinger, Wigner, and others. This is because in the mathematical description of these thought experiments cats, or human observers, are often modelled as two-level systems which may be subjected to instantaneous measurements in any basis. Whilst these assumptions are compatible in principle with quantum theory, our analysis shows that they are incompatible with our actual physical world.

It further remains an open question how tight our bound actually is. However, our bound is likely to be rather generous and real macroscopic quantum measurements are likely to take even longer. This is because we only took into consideration the time required for the information to be transferred from the system to the ancillae, and do not account for the time taken to process the information in the ancillae.

Finally, our results are of importance for the ongoing endeavor to realize macroscopic quantum states of matter in practice. Even if we can create such states the important question remains whether, and how, we can detect them. In this context it is worth pointing out the related studies, which raised similar doubts from a somewhat different perspective than ours. Finally, it is interesting to note that our results complement those of Ref. demonstrating that ideal projective measurements require infinite resources: if infinitely strong interactions are possible, the lower bound set by the quantum speed limit predicts $t_{\text{min}} = 0$.

Acknowledgements. This research has received financial support from the DFG (project STR 1505/2-1), the Spanish MINECO FIS2016-80681-P (AEI-FEDER, UE), Spanish MICINN PCI2019-111869-2, the Spanish Agencia Estatal de Investigación, project PID2019-107609GB-I00, the Generalitat de Catalunya CIRIT 2017-SGR-1127, the Secretaria d’Universitats i Recerca del Departament d’Empresa i Coneixement de la Generalitat de Catalunya, project ref. 001-P-001644(QuantumCat), the QuantERA grant C’MON-QSENS!, and the Australian Research Council Future Fellowship FT160100073.

1. E. P. Wigner, “The Problem of Measurement,” Am. J. Phys. 31, 6 (1963).
2. J. Bell, “Against ‘measurement’,” Phys. World 3, 33 (1990).
3. F. Pokorny, C. Zhang, G. Higgins, A. Cabello, M. Kleinmann, and M. Henrich, “Tracking the Dynamics of an Ideal Quantum Measurement,” Phys. Rev. Lett. 124, 080401 (2020).
4. A. Cabello, “What is Quantum Information?” (Cambridge University Press, Cambridge, 2017) Chap. Interpretations of Quantum Theory: A Map of Madness, pp. 138–144.
5. H. Everett, “‘Relative State’ Formulation of Quantum Mechanics,” Rev. Mod. Phys. 29, 454–462 (1957).
6. A. S. Holevo, Statistical Structure of Quantum Theory (Springer-Verlag, Berlin Heidelberg, 2001).
7. D. D’Alessandro, Introduction to Quantum Control and Dynamics (Chapman & Hall/CRC, London, 2007).
8. H. M. Wiseman and G. J. Milburn, Quantum Measurement and Control (Cambridge University Press, Cambridge, 2010).
9. K. Jacobs, Quantum Measurement Theory and its Applications (Cambridge University Press, Cambridge, 2014).
10. L. E. Ballentine, Quantum Mechanics: A Modern Development (World Scientific, Singapore, 1998).
11. L. Mandelstam and I. G. Tamm, “The uncertainty relation between energy and time in nonrelativistic quantum mechanics,” J. Phys. 9, 249 (1945).
12. N. Margolus and L. B. Levitin, “The maximum speed of dynamical evolution,” Physica D 120, 188–195 (1998).
13. S. Deffner and S. Campbell, “Quantum speed limits: from Heisenberg’s uncertainty principle to optimal quantum control,” J. Phys. A 50, 453001 (2017).
14. P. Fulde, Electron Correlations in Molecules and Solids (3rd ed., Springer-Verlag, Berlin, 1995).
15. H. Bruus and K. Flensberg, Many-body quantum theory in condensed matter physics: an introduction (Oxford University Press, Oxford, 2004).
16. S. Sachdev, Quantum Phase Transitions (John Wiley & Sons, 2007).
17. G. Jaeger, “What in the (quantum) world is macroscopic?” Am. J. Phys. 82, 896 (2014).
18. E. H. Lieb and D. W. Robinson, “The Finite Group Velocity of Quantum Spin Systems,” Commun. Math. Phys. 28, 251–257 (1972).
19. B. Nachtergaele and R. Sims, “Lieb-Robinson Bounds in Quantum Many-Body Physics,” arXiv: 1004.2086 (2010).
20. A. H. Zewail, “Femtochemistry: Atomic-Scale Dynamics of the Chemical Bond Using Ultrafast Lasers (Nobel Lecture),” Angew. Chem. Int. Ed. 39, 2586–2631 (2000).
21. J. M. Elzerman, R. Hanson, L. H. Willems van Beveren, B. Witkamp, L. M. K. Vandersypen, and L. P. Kouwenhoven, “Single-shot read-out of an individual electron spin in a quantum dot,” Nature 430, 431–435 (2004).
22. T. Nakajima, A. Noiri, J. Yoneda, M. R. Delbecq, P. Stano, T. Otsuka, K. Takeda, S. Amaha, G. Allson, K. Kawasaki, A. Ludwig, A. D. Wieck, D. Loss, and S. Tarucha, “Quantum non-demolition measurement of an electron spin qubit,” Nat. Nanotech. 14, 555–560 (2019).
23. F. Schäfer, I. Herrera, S. Cherukattil, C. Lovecchio, L. E. Ballentine, and A. Smerzi, “Experimental realization of quantum zeno dynamics,” Nat. Comm. 5, 3194 (2014).
24. D. Ristè, C. C. Bultink, K. W. Lehnhert, and L. DiCarlo,
“Feedback Control of a Solid-State Qubit Using High-Fidelity Projective Measurement,” Phys. Rev. Lett. 109, 240502 (2012).

D. B. Hume, T. Rosenband, and D. J. Wineland, “High-Fidelity Adaptive Qubit Detection through Repetitive Quantum Nondeletion Measurements,” Phys. Rev. Lett. 99, 120502 (2007).

T. L. Nicholson, M. J. Martin, J. R. Williams, B. J. Bloom, M. Bishof, M. D. Swallows, S. L. Campbell, and J. Ye, “Comparison of Two Independent Sr Optical Clocks with 1×10^{-17} Stability at 10^3 s,” Phys. Rev. Lett. 109, 230801 (2012).

J. T. Monroe, N. Yungar Halpern, T. Lee, and K. W. Murch, “Weak measurement of a superconducting qubit reconciles incompatible operators,” Phys. Rev. Lett. 126, 100403 (2021).

E. Schrödinger, “Die gegenwärtige Situation in der Quantenmechanik,” Naturwissenschaften 23, 807–812 (1935).

E. Schrödinger, “Die gegenwärtige Situation in der Quantenmechanik,” Naturwissenschaften 23, 823–828 (1935).

E. P. Wigner, “The Scientist Speculates,” (Heinemann, London, 1961) Chap. Remarks on the mind-body question, pp. 284–302.

D. Deutsch, “Quantum theory as a universal physical theory,” Int. J. Theor. Phys. 24, 1–41 (1985).

L. Hardy, “Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories,” Phys. Rev. Lett. 68, 2981–2984 (1992).

L. Hardy, “Nonlocality for two particles without inequalities for almost all entangled states,” Phys. Rev. Lett. 71, 1665–1668 (1993).

C. Brukner, “Quantum [Un]Speakables II: Half a Century of Bell’s Theorem,” (Springer, 2017) Chap. On the quantum measurement problem, pp. 95–117.

D. Frauchiger and R. Renner, “Quantum theory cannot consistently describe the use of itself,” Nat. Comm. 9, 3711 (2018).

M. Skotiniotis, W. Dürr, and P. Sekatski, “Macroscopic superpositions require tremendous measurement devices,” Quantum 1, 34 (2017).

F. Fröwis, P. Sekatski, W. Dürr, N. Gisin, and N. Sangouard, “Macroscopic quantum states: Measures, fragility, and implementations,” Rev. Mod. Phys. 90, 025004 (2018).

A. López-Incera, P. Sekatski, and W. Dürr, “All macroscopic quantum states are fragile and hard to prepare;” Quantum 3, 118 (2019).

Y. Guryanova, N. Friis, and M. Huber, “Ideal Projective Measurements Have Infinite Resource Costs,” Quantum 4, 222 (2020).

Hanhee Paik, D. I. Schuster, Lev S. Bishop, G. Kirchmair, G. Catelani, A. P. Sears, B. R. Johnson, M. J. Reagor, L. Frunzio, L. I. Glazman, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, “Observation of High Coherence in Josephson Junction Qubits Measured in a Three-Dimensional Circuit QED Architecture,” Phys. Rev. Lett. 107, 240501 (2011).

M. Naghlioo, Introduction to Experimental Quantum Measurement with Superconducting Qubits, Ph.D. thesis, Washington University (2019).

Appendix A: The Zassenhaus Formula

Let $X, Y$, be two Hermitian operators. Zassenhaus’ formula states

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}} [X,Y]$$

$$\times e^{t^2 \left( \frac{3}{2} [Y,[X,Y]] + [X,[X,Y]] \right)} \ldots,$$

where the terms that follow are exponentials of higher-order nested commutators. To derive Zassenhaus’ formula one simply Taylor expands the exponentials and collects terms.

To see how one can use Zassenhaus’ formula to engineer long-range interactions we return to the one-dimensional Ising interaction considered in the main text (Eq. (17)). Setting $X = H_{\text{Ising}}$ and $Y = h_A \sigma_A^{(z)} + g_A \sigma_N^{(z)} \sigma_A^{(z)}$, one sees that if all the high-order nested commutators in Eq. (A1) are non-zero, then by controlling $h_A, g_A$ in Eq. (17) one can engineer a large class of unitaries. It remains a challenging and interesting problem in quantum control theory whether with such limited “local” control one is able to generate highly non-trivial global interactions, but it is possible in principle.

Appendix B: Experimental data for measurement times

Ref.\cite{Ref} reports on the readout of the spin state of an electron confined in a quantum dot with a size of roughly 100 nm. The readout timescale is reported to be 8 µs.

In Ref.\cite{Ref2}, a quantum non-demolition measurement of a similar system (a quantum dot) was reported using an ancilla of two adjacent quantum dots. The smallest measurement time $\tau_k$ for $k = 1$ was reported to be 2.33 ns.

Ref.\cite{Ref3} reports on experimentally observed quantum Zeno dynamics in a Bose-Einstein condensate of Rubidium atoms. To estimate the size of the atomic cloud, we use the waist of the laser beam (70 µm). To estimate the measurement time, we add the 0.8 µs for the $\pi$-pulse created by a Raman beam to the 0.6 µs required for the time to illuminate the condensate with dissipative light (in total 1.4 µs).

Ref.\cite{Ref4} reports on projective measurements of a superconducting transmon qubit, whose size we estimate based on Ref.\cite{Ref5} to be 1 mm, using a microwave pulse of 400 ns, which we set as our measurement time.

Ref.\cite{Ref6} reports an ancilla-assisted read-out of a single Aluminium ion. The interaction time between the Aluminium ion and the ancilla (a Beryllium ion) is quoted with 25 µs. The diameter of the object is estimated using the radius of an Aluminium atom (143 pm).

Ref.\cite{Ref7} reports on the read-out of optical lattice clocks using Rabi spectroscopy with a probe time of 160 ns. To estimate the diameter of the sample, we multiply the
lattice constant of 813 nm with the number of atoms, which can be scaled up to 50,000 “under typical experimental conditions.” Thus, the diameter of the object is roughly 4 cm.

Ref[27] reports on projective measurements of a superconducting transmon qubit on a time-scale of 350 ns. As for Ref[24], the size of the transmon qubit in this experiment is estimated to be 1 mm[41]. Note that the 50 ns faster read-out in Ref[27] compared to Ref[24] is invisible in Fig. 5 due to the logarithmic scale covering many orders of magnitude.