The future of the universe in modified gravitational theories: approaching a finite-time future singularity

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Abstract. We investigate the future evolution of the dark energy universe in modified gravities, including $F(R)$ gravity, and string-inspired scalar Gauss–Bonnet and modified Gauss–Bonnet ones, and ideal fluid with an inhomogeneous equation of state (EoS). The modified Friedmann–Robertson–Walker dynamics for all of these theories may be presented in a universal form by using the effective ideal fluid with an inhomogeneous EoS without specifying its explicit form. We construct several examples of a modified gravity which produces accelerating cosmologies ending at the finite-time future singularities of all four known types by applying a reconstruction program. Some scenarios for resolving a finite-time future singularity are presented. Among these scenarios, the most natural one is related to additional modification of the gravitational action in the early universe. In addition, late-time cosmology in the non-minimal Maxwell–Einstein theory is considered. We investigate the forms of non-minimal gravitational coupling which generate finite-time future singularities and the general conditions...
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for this coupling such that the finite-time future singularities cannot emerge. Furthermore, it is shown that the non-minimal gravitational coupling can remove the finite-time future singularities or make the singularity stronger (or weaker) in modified gravity.

**Keywords:** magnetic fields, string theory and cosmology, gravity, physics of the early universe

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**1. Introduction**

Recent observational data strongly indicate the existence of dark energy, which generates the accelerating expansion of the present universe. In particular, the five-year Wilkinson Microwave Anisotropy Probe (WMAP) data [1] give the bounds for the value of the equation of state (EoS) parameter $w_{DE}$, which is the ratio of the pressure of the dark energy to its energy density, in the range of $-1.11 < w_{DE} < -0.86$. This could be consistent if the dark energy is a cosmological constant with $w_{DE} = -1$ and therefore our universe seems to approach asymptotically a de Sitter universe. It is also believed that there existed a period of another accelerating expansion of the universe, called inflation, in the early universe. In many models of inflation, the accelerating expansion could be
generated by an almost flat potential of scalar field(s), called the inflaton. Hence, in the period of inflation the universe could be described by (almost) de Sitter space. Thus, there is the striking similarity between the very early and very late universe scenarios.

Although both of the accelerating expansions seem to be of de Sitter type, the possibility that the current acceleration could be of quintessence type, in which \( w_{DE} > -1 \), or phantom type, in which \( w_{DE} < -1 \), is not completely excluded. Furthermore, even if the current accelerating universe is described by a \( \Lambda \)CDM epoch, it is quite possible that it may enter a quintessence/phantom phase in future. Similarly, while the early-time inflation may be de Sitter type, there may exist a pre-inflationary stage in which the evolution of the universe is different from the de Sitter type, e.g., it could be a quintessence/phantom epoch. It is often assumed that the early universe started from the singular point often called the big bang. However, if the current (or future) universe enters a quintessence/phantom stage, it may evolve to a finite-time future singularity, depending on the specific model under consideration and the value of the effective EoS parameter. This suggests that the same theory should describe the whole history of the expansion of the universe. To achieve this, two scalar fields, one field (inflaton) to describe the inflation and another field to provide the dark energy have often been introduced.

In this paper, we consider another approach, which might be more natural than introducing two scalar fields in order to unify the early-time inflation with late-time acceleration: that is modified gravity (for a review, see [2]). In this approach, one starts from some unknown fundamental gravity. At the very early universe in which the curvature is very large but quantum gravity effects may be neglected, the restricted specific sector of such a theory predicts inflation. In the course of the evolution, the curvature decreases and the next-to-leading terms become relevant to the intermediate universe (the radiation/matter-dominated stage). Note that the observational data seem to be consistent if the intermediate universe could have been governed by the standard general relativity. As the curvature becomes smaller, the universe enters the dark energy epoch controlled by a different sector of the unknown fundamental gravitational theory, different from general relativity. The corresponding gravitational terms are leading ones in comparison with the ones of general relativity at the current curvature. Hence, the evolution of the universe defines the modified gravitational theory, predicting its evolution at an each stage [3]. On the other hand, the effective evolution of modified gravity is responsible for the history of the expansion of the universe. This also indicates that the right approach to the understanding of the fundamental gravitational theory is to study the history of the expansion of the universe, which will give information about the leading sectors of modified gravity in each epoch. Moreover, consistent examples of modified gravitational theories passing the local tests and unifying the early-time inflation with late-time acceleration have already been constructed [4, 5]. To understand such theories more clearly, it is reasonable to investigate these theories in extreme situations, for instance, near to singular points, where some fundamental features of the theories may be discovered.

Besides dark energy, there is another unknown component in the universe, namely, dark matter. While dark energy has a large negative pressure, the pressure of dark matter is negligible. In many scenarios, dark matter could originate from particle physics, e.g., dark matter could be a lightest supersymmetric (SUSY) particle, or a particle coming from extra dimensional theories. There exists another kind of scenario in which dark matter
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would be explained by the modification of the gravity, as in the case of dark energy considered in this paper. There are many such scenarios like a modified Newtonian dynamics (MOND) [6], tensor–vector–scalar gravity (TeVeS) [7], and the generalized Einstein-aether theory [8] derived from the Einstein-aether theory [9] and the equivalent in the non-relativistic limit of MOND. Dark matter might be explained by $F(R)$ gravity [10], where $F(R)$ is an arbitrary function of the scalar curvature $R$. There is a scenario in which dark matter particles could be given by the scalar mode in $F(R)$ gravity (the first paper in [11]).

In the present paper, we study the future evolution of the dark energy epoch for various modified gravities: $F(R)$ gravity, scalar Gauss–Bonnet and modified Gauss–Bonnet ones, and effective ideal fluid with an inhomogeneous EoS, which includes the explicit dependence on the Hubble rate and curvature. In particular, we are interested in the behavior of the accelerating cosmological solutions in these theories when those solutions approach a finite-time future singularity. Of course, not all of these theories predict such singularities. It depends on the value of the effective EoS parameter and the structure of the theories. For instance, the expansion in the phantom phase which is not transient predicts a future big rip singularity.

The paper is organized as follows. In section 2, we present the modified Friedmann–Robertson–Walker (FRW) dynamics in a universal way. The following models are considered: $F(R)$ gravity, scalar Gauss–Bonnet and modified Gauss–Bonnet theories as well as an ideal fluid with an inhomogeneous EoS. It is indicated briefly how the history of the expansion of the universe can be reconstructed for such a universal formulation. Section 3 is devoted to the study of the finite-time future singularities in $F(R)$ gravity. Using the reconstruction technique, we present several examples which predict the accelerating FRW solutions ending at the finite-time future singularity. It is demonstrated that not only the big rip but also finite-time future singularities of three other types may appear. In section 4, we discuss various scenarios for resolving the finite-time future singularities. The most natural scenario is based on additional modification of the inhomogeneous EoS or the gravitational action by a term which is not relevant currently. However, such a term which may be relevant for the very early or very late universe may resolve the finite-time future singularities. It is interesting that the presence of such a next-to-leading order term does not conflict with the known local tests. Another scenario is described for accounting for quantum effects which become relevant near to the finite-time future singularities. Section 5 is devoted to the construction of scalar Gauss–Bonnet and modified Gauss–Bonnet gravities which predict the late-time acceleration ending in finite-time future singularities. Using the reconstruction technique, we present the corresponding effective potentials.

In sections 6–8, we study some cosmological effects in the non-minimal Maxwell–Einstein gravity with general gravitational coupling. In section 6, we describe our model and derive the effective energy density and pressure of the universe. In section 7, we consider finite-time future singularities in non-minimal Maxwell–Einstein gravity. We investigate the forms of the non-minimal gravitational coupling of the electromagnetic field generating the finite-time future singularities and the general conditions for the non-minimal gravitational coupling of the electromagnetic field such that finite-time future singularities cannot emerge. Furthermore, in section 8 we consider the influence of non-minimal gravitational coupling of the electromagnetic field on finite-time future
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It is shown that a non-minimal gravitational coupling of the electromagnetic field can remove the finite-time future singularities or make the singularity stronger (or weaker). A summary and outlook are given in section 9. We use units in which \( k_B = c = \hbar = 1 \) and denote the gravitational constant \( 8\pi G \) by \( \kappa^2 \), so that \( \kappa^2 \equiv 8\pi/M_{\text{Pl}}^2 \), where \( M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19} \text{ GeV} \) is the Planck mass. Moreover, in terms of electromagnetism we adopt Heaviside–Lorentz units.

2. Modified FRW dynamics and the reconstruction of the history of the expansion of the universe

In this section, we present the general point of view for the modification of the FRW equations which may be caused by alternative gravity or an ideal fluid with a complicated EoS. We consider \( F(R) \) gravity, scalar Gauss–Bonnet and modified Gauss–Bonnet theories, and an ideal fluid with an inhomogeneous EoS. In addition, we briefly indicate how the history of the expansion of the universe can be reconstructed through such a universal formulation.

The flat FRW space–time is described by the metric

\[
d s^2 = -dt^2 + a^2(t) \, d\mathbf{x}^2,
\]

where \( a(t) \) is the scale factor. In the Einstein gravity, the FRW equations are given by

\[
\rho = \frac{3}{\kappa^2} H^2, \quad p = -\frac{1}{\kappa^2} (2\dot{H} + 3H^2),
\]

where \( H = \dot{a}/a \) is the Hubble parameter, a dot denotes a time derivative, \( \dot{\cdot} = \partial/\partial t \), and \( \rho \) and \( p \) are the energy density and pressure of the universe, respectively. We have assumed a flat three-dimensional metric in accord with observational data. Let us consider any modified gravity (for a review, see [2]) like \( F(R) \) gravity (for reviews, see [2,11]), the scalar Gauss–Bonnet one, or the modified Gauss–Bonnet one \( (F(G) \text{ gravity, where } G \text{ is the Gauss–Bonnet invariant given by equation (2.16) below). In this case, the part of modified gravity may be formally included in the total effective energy density and the pressure as in [12], in which general inhomogeneous EoS fluid is introduced. In this case, the modified FRW equations have the well-known form

\[
\rho_{\text{eff}} = \frac{3}{\kappa^2} H^2, \quad p_{\text{eff}} = -\frac{1}{\kappa^2} (2\dot{H} + 3H^2),
\]

where \( \rho_{\text{eff}} \) and \( p_{\text{eff}} \) are the effective energy density and pressure of the universe, respectively. Note that the contribution of the (unusual) ideal fluid should also be included in the left-hand side (lhs) of the above modified FRW equations [12].

\( \rho_{\text{eff}} \) and \( p_{\text{eff}} \) satisfy a more general EoS:

\[
p_{\text{eff}} = -\rho_{\text{eff}} + f(\rho_{\text{eff}}) + G(H, \dot{H}, \ddot{H}, \ldots),
\]

or an even more complicated one. The other point of view is possible: one can only keep the contribution of matter in the energy density, while the gravitational modification should be parameterized by the above function \( G \). We should note that \( \rho_{\text{eff}} \) and \( p_{\text{eff}} \) defined in (2.3) satisfy the conservation law identically:

\[
\dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0.
\]
As an example, we may consider the $F(R)$ gravity [2, 11] whose action is given by

$$S_{F(R)} = \int d^4x, \sqrt{-g} \left\{ \frac{F(R)}{2\kappa^2} + \mathcal{L}_m \right\}, \quad (2.6)$$

where $F(R)$ is a proper function of the scalar curvature $R$ and $\mathcal{L}_m$ is the matter Lagrangian. One may separate the modified part in $F(R)$ from the Einstein–Hilbert one as

$$F(R) = R + f(R). \quad (2.7)$$

In the FRW background with flat spatial part, $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are given by

$$\rho_{\text{eff}} = \frac{1}{\kappa^2} \left( -\frac{1}{2} f(R) + 3 \left( H^2 + \dot{H} \right) f'(R) - 18 \left( 4H^2 \dot{H} + 6H\ddot{H} \right) f''(R) \right) + \rho_{\text{matter}}, \quad (2.8)$$

$$p_{\text{eff}} = \frac{1}{\kappa^2} \left( \frac{1}{2} f(R) - \left( 3H^2 + H \dot{H} \right) f'(R) + 6 \left( 8H^2 \dot{H} + 4H\ddot{H} + 6H\dddot{H} \right) f''(R) \right. \right. \left. \left. + 36 \left( 4H\ddot{H} + \dddot{H} \right)^2 f'''(R) \right) + p_{\text{matter}}. \quad (2.9)$$

Here, $\rho_{\text{matter}}$ and $p_{\text{matter}}$ are the energy density and pressure of the matter, respectively, and the scalar curvature $R$ is given by $R = 12H^2 + 6\dot{H}$. If the matter has a constant EoS parameter $w$, equation (2.4) has the following form:

$$p_{\text{eff}} = w\rho_{\text{eff}} + G(H, \dot{H}, \ddot{H}, \ldots),$$

$$G \left( H, \dot{H}, \ldots \right) = \frac{1}{\kappa^2} \left( \frac{1 + w}{2} f(R) - \left\{ 3 \left( 1 + w \right) H^2 + \left( 1 + 3w \right) \dot{H} \right\} f'(R) \right. \right. \left. \left. + 6 \left\{ (8 + 12w) H^2 \dot{H} + 4\dot{H}^2 + (6 + 3w) H \ddot{H} + \dddot{H} \right\} f''(R) \right. \right. \left. \right. \left. \left. + 36 \left( 4H\ddot{H} + \dddot{H} \right)^2 f'''(R) \right). \quad (2.10)$$

Let us consider several examples. In the case of the model [13] in which $f(R)$ is given by

$$f(R) = -\frac{\alpha}{R} + \beta R^n, \quad (2.11)$$

one finds

$$G \left( H, \dot{H}, \ldots \right) = \frac{1}{\kappa^2} \left( \frac{1 + w}{2} \left( -\frac{\alpha}{R} + \beta R^n \right) - \left\{ 3 \left( 1 + w \right) H^2 + \left( 1 + 3w \right) \dot{H} \right\} \right. \right. \left. \left. \right. \right. \times \left( \frac{\alpha}{R^2} + n\beta R^{n-1} \right) + 6 \left\{ (8 + 12w) H^2 \dot{H} + 4\dot{H}^2 + (6 + 3w) H \ddot{H} + \dddot{H} \right\} \right. \right. \times \left( -\frac{2}{R^3} + n(n - 1)\beta R^{n-2} \right) + 36 \left( 4H\ddot{H} + \dddot{H} \right)^2 \right. \right. \times \left( \frac{6}{R^4} + n(n - 1)(n - 2)R^{n-3} \right). \quad (2.12)$$

For the Hu–Sawicki model [14],

$$f_{\text{HS}}(R) = -\frac{m^2c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} = -\frac{m^2c_1}{c_2} + \frac{m^2c_1/c_2}{c_2 (R/m^2)^n + 1}. \quad (2.13)$$
we get
\[
G(H, \dot{H}, \ldots) = \frac{1}{\kappa^2} \left[ -\frac{1}{2} \frac{m^2 c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \right] + \left\{ 3 (1 + w) H^2 + (1 + 3w) \dot{H} \right\} \left( \frac{nc_1 (R/m^2)^{n-1}}{c_2 (R/m^2)^n + 1} \right) + 6 \left\{ (8 + 12w) H^2 \dot{H} + 4H^2 + (6 + 3w) H \ddot{H} + \dddot{H} \right\} \\
\times \left( \frac{\left( n(n-1)c_1 \right) (R/m^2)^{n-2}}{m^2} \frac{2n^2 c_1 c_2 (R/m^2)^{2n-2}}{m^4} \right) + 36 \left( 4H \dot{H} + \dddot{H} \right)^2 \\
\times \left( \frac{\left( n(n-1)(n-2)c_1 \right) (R/m^2)^{n-3}}{m^4} \right) - \frac{6n^2 (n-1)c_1 c_2 (R/m^2)^{2n-3}}{m^4} \left( c_2 (R/m^2)^n + 1 \right)^3 \\
+ \left\{ \frac{6n^4 c_1 c_2}{m^2} (R/m^2)^{3n-3} \right\} \\
\right]. \tag{2.14}
\]

Note that recently the observational bounds for $F(R)$ theories have been discussed in [15]. We may also consider the $F(\mathcal{G})$ gravity [16], whose action is given by
\[
S = \int d^4 x \sqrt{-g} \left\{ \frac{1}{2\kappa} \left( R + f_\mathcal{G}(\mathcal{G}) \right) + \mathcal{L}_m \right\}, \tag{2.15}
\]
where $\mathcal{G}$ is the Gauss–Bonnet invariant:
\[
\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}. \tag{2.16}
\]

In the model, the effective energy density and pressure are given by
\[
\rho_{\text{eff}} = \frac{1}{2\kappa^2} \left[ \mathcal{G} f'_\mathcal{G}(\mathcal{G}) - f_\mathcal{G}(\mathcal{G}) - 2A^2 H^4 \left( 2\dot{H}^2 + H\dddot{H} + 4H^2 \dot{H} \right) f''_\mathcal{G}(\mathcal{G}) \right] + \rho_{\text{matter}},
\]
\[
p_{\text{eff}} = \frac{1}{2\kappa^2} \left[ f_\mathcal{G}(\mathcal{G}) + 24^2 H^2 \left( 3H^4 + 20H^2 \dot{H}^2 + 6H^3 + 4H^3 \dot{H} + H^2 \dddot{H} \right) f''_\mathcal{G}(\mathcal{G}) \\
- 2A^2 H^5 \left( 2\dot{H}^2 + H\dddot{H} + 4H^2 \dot{H} \right)^2 f'''_\mathcal{G}(\mathcal{G}) \right] + p_{\text{matter}}. \tag{2.17}
\]

In the FRW background, we find $\mathcal{G} = 24(H^2 \dot{H} + H^4)$. If we assume that the matter has a constant EoS parameter $w$, again, equation (2.4) has the following form:
\[
p_{\text{eff}} = w\rho_{\text{eff}} + G_\mathcal{G}(H, \dot{H}, \dddot{H}, \ldots),
\]
\[
G_\mathcal{G} \left( H, \dot{H}, \ldots \right) = \frac{1}{2\kappa^2} \left[ (1 + w) f_\mathcal{G}(\mathcal{G}) - w\mathcal{G} f'_\mathcal{G}(\mathcal{G}) \\
+ 2A^2 H^2 \left( 3H^4 + 20H^2 \dot{H}^2 + 6H^3 + 4H^3 \dot{H} + H^2 \dddot{H} + w H^4 \right) \left( 2\dot{H}^2 + H\dddot{H} + 4H^2 \dot{H} \right)^2 f''_\mathcal{G}(\mathcal{G}) \right]. \tag{2.18}
\]
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An example is given by [16]:

$$f_G(G) = f_0 |G|^{1/2},$$

(2.19)

where $f_0$ is a constant. In this case, one gets

$$G_G(H, \dot{H}, \ldots) = \frac{f_0}{2\kappa^2} \left[ \left(1 + \frac{w}{2}\right) |G|^{1/2} - 144H^2 \left(3H^4 + 20H^2\dot{H}^2 + 6\dot{H}^3 + 4H^3\ddot{H} + H^2\dddot{H} + wH^4 \right) \right] \left(2\dot{H}^2 + H\ddot{H} + 2H^2\dot{H}\right) \frac{1}{|G|^{3/2}} + 9 \times 24^2H^5 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)^2 \frac{G}{2|G|^{7/2}}.$$

(2.20)

In the same way, one can obtain the function of modified gravity $G$ for other models including non-local gravity [17]. It is interesting to note that the perturbations of the above theory should be considered by using the analogy with effective field theory [18].

We now study the general reconstruction program in terms of $G$ in (2.4). For simplicity, we assume that the matter has a constant EoS parameter $w$. Using (2.3), we find

$$G \left(H, \dot{H}, \ldots\right) = -\frac{1}{\kappa^2} \left(2\dot{H} + 3(1 + w)H^2\right).$$

(2.21)

Let the cosmology be given by $H = H(t)$. The right-hand side (rhs) of (2.21) is given by a function of $t$. If the combination of $H, \dot{H}, \dddot{H}, \ldots$ in $G(H, \dot{H}, \ldots)$ reproduces such a function, this kind of cosmology could be realized. As an illustrative example, we may consider the case in which $H(t)$ is given by

$$H = h_0 + \frac{h_1}{t},$$

(2.22)

which gives

$$\dot{H} = -\frac{h_1}{t^2}, \quad \dddot{H} = \frac{2h_1}{t^3}, \quad \ldots,$$

(2.23)

and the rhs in (2.21) is given by

$$-\frac{1}{\kappa^2} \left(2\dot{H} + 3(1 + w)H^2\right) = -\frac{1}{\kappa^2} \left(3(1 + w)h_0^2 + \frac{6(1 + w)h_0h_1}{t} + \frac{-2h_1 + 3(1 + w)h_1^2}{t^2}\right).$$

(2.24)

For (trivial) example, if $G$ is given by

$$G \left(H, \dot{H}\right) = \frac{1}{\kappa^2} \left\{-3(1 + w)h_0^2 + 6(1 + w)h_0H + [2 - 3(1 + w)h_0] \dot{H}\right\},$$

(2.25)

(2.22) is a solution. Of course, there is large freedom of choice for $G(H, \dot{H}, \dddot{H}, \ldots)$, but the form could be determined by the kind of the modified gravitational theory which we are considering (for a detailed study of reconstruction for various modified gravities, see [19, 20]). The important point is that the realistic history of the expansion of the universe can be realized from the modified gravity reconstructed.
3. Finite-time future singularities in $F(R)$ gravity

In this section, we investigate $F(R)$ gravity models and show that some models generate several known types of the finite-time future singularities. This phenomenon is quite natural, as modified gravity represented as the Einstein gravity with the effective ideal fluid with the phantom or quintessence-like EoS (see the explicit transformation in [21]). In some cases, it is known that such ideal (quintessence or phantom) fluid induces the finite-time future singularity. We present several examples which predict the accelerating FRW solutions ending at the finite-time future singularity by using the reconstruction technique. We demonstrate that not only the big rip but also finite-time future singularities of three other types may appear.

As the first example, we consider the case of the big rip singularity [22], where $H$ behaves as

$$H = \frac{h_0}{t_0 - t},$$

where $h_0$ and $t_0$ are positive constants and $H$ diverges at $t = t_0$. To find the $F(R)$ gravity generating the big rip singularity, we use the method of reconstruction, namely, we construct the $F(R)$ model realizing any given cosmology using the technique proposed in [20] (for the related study of reconstruction in $F(R)$ gravity, see [23]). The action of $F(R)$ gravity with general matter is given as follows:

$$S = \int d^4x \sqrt{-g} \left\{ F(R) + L_{\text{matter}} \right\}.$$  \hspace{1cm} (3.2)

The action (3.2) can be rewritten by using proper functions $P(\phi)$ and $Q(\phi)$ for a scalar field $\phi$ [20]:

$$S = \int d^4x \sqrt{-g} \left\{ P(\phi) R + Q(\phi) + L_{\text{matter}} \right\}.$$  \hspace{1cm} (3.3)

One may regard the scalar field $\phi$ as an auxiliary scalar field because $\phi$ has no kinetic term. By the variation over $\phi$, we obtain

$$0 = P'(\phi) R + Q'(\phi),$$

which could be solved with respect to $\phi$ as $\phi = \phi(R)$. By substituting $\phi = \phi(R)$ into the action (3.3), we obtain the action of $F(R)$ gravity given by

$$F(R) = P(\phi(R)) R + Q(\phi(R)).$$

By assuming $\rho$, $p$ could be given by the corresponding sum of matter with constant EoS parameters $w_i$, and writing the scale factor $a(t)$ as $a = a_0 e^{g(t)}$, where $a_0$ is a constant, one gets the second-rank differential equation

$$0 = 2 \frac{d^2 P(\phi)}{d\phi^2} - 2g'(\phi) \frac{dP(\phi)}{d\phi} + 4g''(\phi) P(\phi) + \sum_i (1 + w_i) \rho_i a_0^{-3(1+w_i)} e^{-3(1+w_i)g(\phi)}.$$  \hspace{1cm} (3.6)
If one can solve equation (3.6) with respect to \( P(\phi) \), the form of \( Q(\phi) \) could be found as [20]

\[
Q(\phi) = -6 \left( g'(\phi) \right)^2 P(\phi) - 6g'(\phi) \frac{dP(\phi)}{d\phi} + \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3(1+w_i)g(\phi)}.
\] (3.7)

Thus, it follows that any given history of the expansion of the universe can be realized by some specific \( F(R) \) gravity. Specific models unifying the sequence: early-time acceleration, radiation/matter-dominated stage and dark energy epoch are constructed in [4, 5, 20].

In the case of (3.1), if we neglect the contribution from the matter, the general solution of (3.6) give

\[
P(\phi) = P_+ (t_0 - \phi)^{\alpha_+} + P_- (t_0 - \phi)^{\alpha_-}, \quad \alpha_\pm = \frac{-h_0 + 1 \pm \sqrt{h_0^2 - 10h_0 + 1}}{2},
\] (3.8)

when \( h_0 > 5 + 2\sqrt{6} \) or \( h_0 < 5 - 2\sqrt{6} \) and

\[
P(\phi) = (t_0 - \phi)^{-\left(h_0 + 1\right)/2} \left( \hat{A} \cos \left( (t_0 - \phi) \ln \frac{-h_0^2 + 10h_0 - 1}{2} \right) + \hat{B} \sin \left( (t_0 - \phi) \ln \frac{-h_0^2 + 10h_0 - 1}{2} \right) \right),
\] (3.9)

when \( 5 + 2\sqrt{6} > h_0 > 5 - 2\sqrt{6} \). Using (3.4), (3.5), and (3.7), we find that the form of \( F(R) \) when \( R \) is large is given by

\[
F(R) \propto R^{1-\alpha_-/2},
\] (3.10)

for the \( h_0 > 5 + 2\sqrt{6} \) or \( h_0 < 5 - 2\sqrt{6} \) case and

\[
F(R) \propto R^{(h_0 + 1)/4} \times (\text{oscillating parts}),
\] (3.11)

for the \( 5 + 2\sqrt{6} > h_0 > 5 - 2\sqrt{6} \) case.

Let us investigate a more general singularity [24]:

\[
H \sim h_0(t_0 - t)^{-\beta}.
\] (3.12)

Here, \( h_0 \) and \( \beta \) are constants, \( h_0 \) is assumed to be positive and \( t < t_0 \) because it should be for the expanding universe. Even for non-integer \( \beta < 0 \), some derivative of \( H \) and therefore the curvature becomes singular. We should also note that equation (3.12) tells us that the scale factor \( a \) (\( H = \dot{a}/a \)) behaves as

\[
a \sim \exp \left( h_0(t_0 - t) \frac{1}{1 - \beta} + \ldots \right).
\] (3.13)

Here, \( \ldots \) expresses the regular terms. From (3.13), we find that if \( \beta \) could not be any integer, the value of \( a \), and therefore the value of the metric tensor, would become complex numbers and include the imaginary part when \( t > t_0 \), which is unphysical. This could tell us that the universe could end at \( t = t_0 \) even if \( \beta \) could be negative or less than \(-1\).

Since the case \( \beta = 1 \) corresponds to the big rip singularity, which has been investigated, we assume \( \beta \neq 1 \). Furthermore, since \( \beta = 0 \) corresponds to de Sitter
space, which has no singularity, we assume $\beta \neq 0$. When $\beta > 1$, the scalar curvature $R$ behaves as

$$R \sim 12H^2 \sim 12h_0^2(t_0 - t)^{-2\beta}.$$  \hfill (3.14)

On the other hand, when $\beta < 1$, the scalar curvature $R$ behaves as

$$R \sim 6H \sim 6h_0\beta(t_0 - t)^{-\beta - 1}.$$  \hfill (3.15)

We may get the asymptotic solution for $P$ when $\phi \to t_0$.

- **$\beta > 1$ case:** We find the following asymptotic expression for $P(\phi)$:

$$P(\phi) \sim e^{(h_0/2(\beta-1))(t_0-\phi)^{-\beta+1}}(t_0 - \phi)^{3/2} \left(\tilde{A}\cos\left(\omega(t_0 - \phi)^{-\beta+1}\right) + \tilde{B}\sin\left(\omega(t_0 - \phi)^{-\beta+1}\right)\right), \quad \omega \equiv \frac{h_0}{2(\beta-1)}. \hfill (3.16)$$

When $\phi \to t_0$, $P(\phi)$ tends to vanish. Using (3.4), (3.5) and (3.7), we find that (at large $R$) $F(R)$ is given by

$$F(R) \propto e^{(h_0/2(\beta-1))R/(12h_0))^{(\beta-1)/2\beta} R^{-1/4} \times (\text{oscillating part}).$$ \hfill (3.17)

- **$1 > \beta > 0$ case:** We find the following asymptotic expression for $P(\phi)$:

$$P(\phi) \sim B e^{-(h_0/2(1-\beta))(t_0-\phi)^{1-\beta}}(t_0 - \phi)^{(\beta+1)/8}. \hfill (3.18)$$

Hence $F(R)$ is given by

$$F(R) \sim e^{-(h_0/2(1-\beta))(-6\beta h_0 R)^{(\beta-1)/(\beta+1)}} R^{7/8}. \hfill (3.19)$$

- **$\beta < 0$ case:** The asymptotic expression for $P(\phi)$ is as follows:

$$P(\phi) \sim A e^{-(h_0/2(1-\beta))(t_0-\phi)^{1-\beta}}(t_0 - \phi)^{-(\beta^2-6\beta+1)/8}. \hfill (3.20)$$

Thus $F(R)$ is given by

$$F(R) \sim (-6h_0\beta R)^{(\beta^2+2\beta+9)/8(\beta+1)} e^{-(h_0/2(1-\beta))(-6\beta h_0 R)^{(\beta-1)/(\beta+1)}}. \hfill (3.21)$$

Note that $-6h_0\beta R > 0$ when $h_0, R > 0$.

We found the behavior of the scalar curvature $R$ from that of $H$ in (3.12), namely, when $\beta > 1$, $R$ behaves as in (3.14), and when $\beta < 1$, $R$ behaves as in (3.15). Now, conversely we consider the behavior of $H$ from that of $R$. When $R$ behaves as

$$R \sim 6H \sim R_0(t_0 - t)^{-\gamma}, \hfill (3.22)$$

if $\gamma > 2$, which corresponds to $\beta = \gamma/2 > 1$, $H$ behaves as

$$H \sim \sqrt{\frac{R_0}{12}}(t_0 - t)^{-\gamma/2}, \hfill (3.23)$$

if $2 > \gamma > 1$, which corresponds to $1 > \beta = \gamma - 1 > 0$, $H$ is given by

$$H \sim \frac{R_0}{6(\gamma - 1)}(t_0 - t)^{-\gamma+1}, \hfill (3.24)$$
and if \( \gamma < 1 \), which corresponds to \( \beta = \gamma - 1 < 0 \), one obtains
\[
H \sim H_0 + \frac{R_0}{6(\gamma - 1)}(t_0 - t)^{-\gamma + 1}.
\] (3.25)

Here, \( H_0 \) is an arbitrary constant, and it does not affect the behavior of \( R \). \( H_0 \) is chosen to vanish in (3.12). From \( H = \dot{a}(t)/a(t) \), if \( \gamma > 2 \), we find
\[
a(t) \propto \exp \left( \frac{2}{\gamma - 1} \right) \sqrt{\frac{R_0}{12}} (t_0 - t)^{-\gamma/2 + 1},
\] (3.26)

when \( 2 > \gamma > 1 \), \( a(t) \) behaves as
\[
a(t) \propto \exp \left( \frac{R_0}{6\gamma (\gamma - 1)}(t_0 - t)^{-\gamma} \right),
\] (3.27)

and if \( \gamma < 1 \),
\[
a(t) \propto \exp \left( H_0 t + \frac{R_0}{6\gamma (\gamma - 1)}(t_0 - t)^{-\gamma} \right).
\] (3.28)

In any case, there appears a sudden future singularity [25] at \( t = t_0 \).

Since the second term in (3.25) is smaller than the first one, one may solve (3.6) asymptotically as follows:
\[
P \sim P_0 \left( 1 + \frac{2h_0}{1 - \beta} (t_0 - \phi)^{1-\beta} \right),
\] (3.29)

with a constant \( P_0 \), which gives
\[
F(R) \sim F_0 R + F_1 R^{2\beta/(\beta + 1)}.
\] (3.30)

Here, \( F_0 \) and \( F_1 \) are constants. When \( 0 > \beta > -1 \), we find \( 2\beta/(\beta + 1) < 0 \). On the other hand, when \( \beta < -1 \), we obtain \( 2\beta/(\beta + 1) > 2 \). As we saw in (3.10), for \( \beta < -1 \), \( H \) diverges when \( t \to t_0 \). Since we reconstruct \( F(R) \) so that the behavior of \( H \) could be recovered, \( F(R) \) generates the big rip singularity when \( R \) is large. Thus, even if \( R \) is small, \( F(R) \) generates a singularity where higher derivatives of \( H \) diverge.

Even for \( F(R) \) gravity, we may define the effective energy density \( \rho_{\text{eff}} \) and pressure \( p_{\text{eff}} \) as (2.3). We now assume that \( H \) behaves as (3.12). For \( \beta > 1 \), when \( t \to t_0 \),
\[
a \sim \exp(h_0(t_0 - t)^{1-\beta}/(\beta - 1)) \to \infty \quad \text{and} \quad \rho_{\text{eff}}, |p_{\text{eff}}| \to \infty.
\] If \( \beta = 1 \), we find
\[
a \sim (t_0 - t)^{-h_0} \to \infty \quad \text{and} \quad \rho_{\text{eff}}, |p_{\text{eff}}| \to \infty.
\] If \( 0 < \beta < 1 \), \( a \) goes to a constant but \( \rho, |p| \to \infty \). If \( -1 < \beta < 0 \), \( a \) and \( \rho \) vanish but \( |p_{\text{eff}}| \to \infty \). When \( \beta < 0 \), instead of (3.12), as in (3.24), one may assume
\[
H \sim H_0 + h_0(t_0 - t)^{-\beta}.
\] (3.31)

Hence, if \( -1 < \beta < 0 \), \( \rho_{\text{eff}} \) has a finite value \( 3H_0^2/\kappa^2 \) in the limit \( t \to t_0 \). If \( \beta < -1 \) but \( \beta \) is not any integer, \( a \) is finite, and \( \rho_{\text{eff}} \) and \( p_{\text{eff}} \) vanish if \( H_0 = 0 \) or \( \rho_{\text{eff}} \) and \( p_{\text{eff}} \) are finite if \( H_0 \neq 0 \) but higher derivatives of \( H \) diverge. We should note that the leading behavior of the scalar curvature \( R \) does not depend on \( H_0 \) in (3.31), and that the second term in (3.31) is relevant to the leading behavior of \( R \). We should note, however, that \( H_0 \) is relevant to the leading behavior of the effective energy density \( \rho_{\text{eff}} \) and the scale factor \( a \).
In [26], a classification of the finite-time future singularities was suggested, as follows.

- **Type I** (‘big rip’): For \( t \to t_s \), \( a \to \infty \), \( \rho \to \infty \) and \(|p| \to \infty\). This also includes the case of \( \rho \), \( p \) being finite at \( t_s \).
- **Type II** (‘sudden’): For \( t \to t_s \), \( a \to a_s \), \( \rho \to \rho_s \) and \(|p| \to \infty\).
- **Type III**: For \( t \to t_s \), \( a \to a_s \), \( \rho \to \infty \) and \(|p| \to \infty\).
- **Type IV**: For \( t \to t_s \), \( a \to a_s \), \( \rho \to 0 \), \(|p| \to 0\) and higher derivatives of \( H \) diverge.

This also includes the case in which \( p(\rho) \) or both of \( p \) and \( \rho \) tend to some finite values, while higher derivatives of \( H \) diverge.

Here, \( t_s \), \( a_s(\neq 0) \) and \( \rho_s \) are constants. We now identify \( t_s \) with \( t_0 \). Type I corresponds to the \( \beta > 1 \) or \( \beta = 1 \) case, type II to the \(-1 < \beta < 0 \) case, type III to the \( 0 < \beta < 1 \) case, and type IV to \( \beta < -1 \) but \( \beta \) is not any integer number. When the phantom dark energy was studied, it was found that the big rip type (type I) singularity could occur. After that, it was pointed out that there could be another kind of singularity corresponding to type II [25]. Then, it was found in [26] that there are other kinds of singularities corresponding to type III and IV. Note that if only higher derivatives of the Hubble rate diverge, then some combination of curvature invariants also diverges which leads to singularity. Thus, we have constructed several examples of \( F(R) \) gravity showing the above finite-time future singularities of any type. This is natural because it is known that modified gravity may lead to an effective phantom/quintessence phase [2], while the phantom/quintessence-dominated universe may end up with a finite-time future singularity.

The reconstruction method also tells us that there appear type I singularity for \( F(R) = R + \alpha R^n \) with \( n > 2 \) and type III singularity for \( F(R) = R - \beta R^{-n} \) with \( n > 0 \), where \( \alpha \) and \( \beta \) are constants. Note, however, that even if some specific model contains a finite-time future singularity, one can always effect a reconstruction of the model in the remote past in such a way that the finite-time future singularity disappears. Usually, positive powers of the curvature (polynomial structure) help to make the effective quintessence/phantom phase become transient and to avoid the finite-time future singularities. The corresponding examples are presented in [24,27].

### 4. Absence of singularity in modified gravity

In this section, we study various scenarios for resolving the finite-time future singularities. The most natural scenario is based on additional modification of the inhomogeneous EoS or the gravitational action by a term which is not relevant currently. In fact, however, such a term which may be relevant for the very early or very late universe may resolve the singularity. We note that the presence of such a next-to-leading order term does not conflict with the known local tests. Another scenario is related to accounting for quantum effects becoming relevant near to singularity.

Let us consider the conditions for \( G \) in (2.4), which prevent finite-time future singularities appearing. For simplicity, we only consider the case in which the matter has a constant EoS parameter \( w \). A key equation is equation (2.21). If equation (2.21) becomes inconsistent for any singularities, the singularity could not be realized. First, we put \( G = 0 \) in (2.21). In this case, a solution satisfying (2.21) is given by

\[
H = \frac{-2/(3(1 + w))}{t_0 - t}.
\]  

(4.1)
If \( w < -1 \), i.e., the phantom phase, there appears a big rip singularity. We may now assume \( w > -1 \). Even if \( w \) is not constant, in the case \(-\infty < w < -1\) there occurs the big rip type of singularity, where \( H \) diverges in the finite future. For example, we may consider the case in which \( w \) behaves as \( w = -1 - w_0(t_0 - t)\eta \) when \( t \sim t_0 \). Here, \( w_0 \) and \( \eta \) are constants. In order to make it that \( w \) could be less than \(-1\), \( w_0 \) should be positive. Hence the Hubble rate behaves as \( H \sim (t_0 - t)^{\eta} \). Thus, if \( \eta > -1 \), \( H \) diverges and the big rip type of singularity could occur. We should note that when \( \eta > 0 \), \( w \to -1 \) in the limit of \( t \to t_0 \).

If \( H \) evolves as (3.31), the rhs in (2.21) behaves as

\[
-\frac{1}{\kappa^2} \left( 2\dot{H} + 3(1 + w)H^2 \right) \sim \begin{cases} 
\frac{3(1 + w)h^2_0}{\kappa^2} (t_0 - t)^{-2\beta} & \text{when } \beta > 1 \\
-\frac{2\beta h_0 + 3(1 + w)h^2_0}{\kappa^2} (t_0 - t)^{-2} & 0 < \beta < 1 \\
-\frac{2\beta h_0}{\kappa^2} (t_0 - t)^{-\beta - 1} & 0 > \beta > -1.
\end{cases}
\]

(4.2)

If \( \beta > -1, \beta \neq 0 \), which correspond to type I, II and III singularities, the lhs on (4.2), i.e., \( G(H, \dot{H}, \ldots) \) in equation (2.21), diverges. One way to prevent such a singularity appearing could have \( G \) bounded. In this case, equation (2.21) becomes inconsistent with the behavior of the rhs of (4.2). An example is as follows:

\[
G = G_0 \frac{1 + aH^2}{1 + bH^2},
\]

where \( a \) and \( b \) are positive constants and \( G_0 \) is a constant.

We should also note that when \( \beta > 0 \), which corresponds to type I or III singularity, the rhs of (4.2) becomes negative. Hence, if \( G \) is positive for large \( \dot{H} \), the singularity could not be realized.

Another possibility is that if \( G \) contains the term like \( \sqrt{1 - a^2H^2} \), which becomes imaginary for large \( H \), (4.2) could become inconsistent. Thus, singularities where the curvature blows up (type I, II, III) could be prevented from appearing. This mechanism could be applied even if \( w < -1 \). One can add an extra term as

\[
G_1(H) = G_0 \left( \sqrt{1 - \frac{H^2}{H_0^2}} - 1 \right)
\]

(4.4)
to \( G \). Here, \( G_0 \) and \( H_0 \) are constants. If we choose \( H_0 \) to be large enough, \( G_1 \) is not relevant for the small curvature but is relevant for large scale and prohibits the curvature singularity from appearing. Moreover, it is easy to check the EoS of the effective ideal fluid induced by modified gravity. Some indications of the presence/absence of the singularities can be found here. Nevertheless, one should not forget that the description of the effective ideal fluid corresponds to the consideration of the theory in another (Einstein) frame. The transformation of the point where the singularity appears from one frame to another may be tricky in modified gravity [28].

As the universe approaches the singularity, the curvature may become large again. Hence, it is necessary to take into account quantum effects (or even those of modified gravity through the effective action approach [29]). One may include the massless quantum
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effects by taking account of the contribution of the conformal anomaly as the back-reaction near the singularity. The conformal anomaly $T_A$ has the following well-known form [26]:

$$T_A = b \left( F_W + \frac{2}{3} \Box R \right) + b' G + b'' \Box R. \quad (4.5)$$

Here, $F_W$ is the square of the 4D Weyl tensor, and it is given by

$$F_W = \frac{1}{3} R^2 - 2 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \quad (4.6)$$

In general, with $N$ scalar, $N_{1/2}$ spinor, $N_1$ vector fields, $N_2$ ($= 0$ or 1) gravitons and $N_{\text{HD}}$ higher derivative conformal scalars, $b$ and $b'$ are given by [26]

$$b = \frac{N + 6 N_{1/2} + 12 N_1 + 611 N_2 - 8 N_{\text{HD}}}{120(4\pi)^2},$$

$$b' = -\frac{N + 11 N_{1/2} + 62 N_1 + 1411 N_2 - 28 N_{\text{HD}}}{360(4\pi)^2}. \quad (4.7)$$

As is seen, $b > 0$ and $b' < 0$ for usual matter except for the higher derivative conformal scalars. Notice that $b''$ can be shifted by the finite renormalization of the local counter-term $R^2$, so $b''$ can be an arbitrary coefficient. For the FRW universe, we find

$$F_W = 0, \quad G = 24 \left( \dot{H} H^2 + H^4 \right). \quad (4.8)$$

In terms of the corresponding energy density $\rho_A$ and pressure $p_A$, $T_A$ is given by [26]

$$T_A = -\rho_A + 3 p_A, \quad \text{(4.9)}$$

and the energy density could be conserved:

$$\dot{\rho}_A + 3 H (\rho_A + p_A) = 0. \quad (4.10)$$

Hence we find

$$\rho_A = -a^{-4} \int dt a^4 H T_A, \quad p_A = \frac{T_A}{3} - \frac{a^{-4}}{3} \int dt a^4 H T_A. \quad (4.11)$$

By using the above expressions, equation (2.3) could be modified as

$$\rho_{\text{eff}} + \rho_A = \frac{3}{k^2} H^2, \quad p_{\text{eff}} + p_A = -\frac{1}{k^2} (2 \dot{H} + 3 H^2). \quad (4.12)$$

If we redefine the effective energy density and pressure as

$$\tilde{\rho}_{\text{eff}} = \rho_{\text{eff}} + \rho_A, \quad \tilde{p}_{\text{eff}} = p_{\text{eff}} + p_A, \quad (4.13)$$

equation (2.10) could be modified as

$$\tilde{p}_{\text{eff}} = w \tilde{\rho}_{\text{eff}} + \tilde{G}(H, \dot{H}, \ddot{H}, \ldots), \quad \tilde{G} \equiv G \left( H, \dot{H}, \ldots \right) + \frac{T_A}{3} - \left( \frac{1}{3} + w \right) a^{-4} \int dt a^4 H T_A. \quad (4.14)$$

It has been indicated [30] that an explicit account of such quantum effects tends to moderate the singularity, to make the space–time less singular or, at least, to delay the rip time. Clearly, these scenarios may be applied to any specific modified gravity under consideration. Furthermore, there remains the possibility that the action of modified gravity is changed by a term which is not relevant now and does not influence the local
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tests in such a theory currently. This kind of term, however, may be relevant for the very
early universe or the very late universe in such a way that the universe avoids approaching
the singularity (in other words, say, the phantom phase becomes transient as in the model
of [27]).

As an explicit realization of the above modification, one could consider an effective
ideal fluid with a complicated EoS depending on the Hubble rate:
\[ p_{\text{eff}} = w(H, \dot{H}, \ldots) \rho_{\text{eff}}, \]
namely, the EoS parameter could be a function of \( H, \dot{H}, \ddot{H}, \ldots \). The specific form of the
dependence on the Hubble rate is defined by the equivalent model of modified gravity. Using (2.3), we find
\[ 0 = 2\dot{H} + 3H^2 + 3w(H, \dot{H}, \ldots) H^2. \]
If we take \( H \) to be a constant, \( H_0 \), equation (4.16) is reduced to
\[ 0 = 1 + w(H_0, \dot{H} = 0, \ldots), \]
which is an algebraic equation and hence has a solution, e.g., there could be a de Sitter
space solution. For example, if \( w \) is given by
\[ w(H, \dot{H}) = \frac{H^2}{h_0^2} + f(\dot{H}), \]
it follows that
\[ H = h_0 \sqrt{1 - f(0)}. \]
As a special case, if \( w \) is given by
\[ w(H, \dot{H}) = -1 + \frac{2\dot{H}}{3H^2}, \]
equation (4.16) is trivially satisfied for any \( H \), and therefore any cosmology can be a
solution. This theory, however, has no predictive power. One non-trivial example is as follows:
\[ w(H) = -1 + \frac{2\beta}{3h_0^{1/\beta}} H^{-1 + 1/\beta}. \]
The solution of (4.16) is given by the exact form of (3.12), i.e., \( H = h_0(t_0 - t)^{-\beta} \). Hence,
any type of finite-time future singularity can be realized by the form of \( w \) in (4.20). Another non-trivial example is expressed by the following form:
\[ w(H) = -1 - \frac{2(h_i - h_l)}{3t_0 H^2} \left\{ 1 - \left( \frac{h_i + h_l}{h_i - h_l} - \frac{2H}{h_i - h_l} \right)^2 \right\}, \]
where \( t_0, h_i \) and \( h_l \) are constants satisfying \( h_i \gg h_l > 0 \). An exact solution of (4.16) is given by
\[ H = \frac{h_i + h_l}{2} - \frac{h_i - h_l}{2} \tanh \frac{t}{t_0}. \]
In the limit \( t \to -\infty \), we find \( H \to h_i \). On the other hand, in the limit \( t \to \infty \), \( H \to h_l \).
Hence the de Sitter universe could be realized in both limits \( t \to \pm\infty \). Thus, we may
identify the limit of \( t \to -\infty \) with inflation and the limit of \( t \to \infty \) with late-time
acceleration. Adding the effective ideal fluid with such a EoS to the model admitting
singularity, one can always check whether this addition resolves it.

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5. Finite-time future singularities in scalar Gauss–Bonnet and modified Gauss–Bonnet gravities

In this section, we discuss the scalar Gauss–Bonnet and modified Gauss–Bonnet gravities which predict the late-time acceleration ending at a finite-time future singularity. We present the corresponding effective potentials by using the reconstruction technique.

Let us consider the reconstruction [19,31,32] of the string-inspired scalar Gauss–Bonnet gravity proposed as dark energy in [33,34] for the investigation of the finite-time future singularities:

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) - \xi(\phi) G \right]. \]  

The theory depends on two scalar potentials. The explicit example motivated by string considerations is the following [33]:

\[ V(\phi) = V_0 e^{-2\phi/\phi_0}, \quad \xi(\phi) = \xi_0 e^{2\phi/\phi_0}, \]  

where \( V_0, \phi_0 \) and \( \xi_0 \) are constant parameters. By choosing the parameters properly, a dark energy universe ending at the big rip singularity emerges.

There exists another example corresponding to the following choice of the potentials [32]:

\[ V(\phi) = \frac{3}{\kappa^2} \left( g_0 + \frac{g_1}{t_0} e^{-\phi/\phi_0} \right)^2 - \frac{g_1}{\kappa^2 t_0^2} e^{-2\phi/\phi_0} - 3U_0 \left( g_0 + \frac{g_1}{t_0} e^{-\phi/\phi_0} \right) e^{g_1 / \phi_0} e^{g_1 t_0 e^{2\phi/\phi_0}}, \]

\[ \xi(\phi) = \frac{U_0}{8} \int_{t_0 e^{\phi/\phi_0}}^{t} dt_1 \left( g_0 + \frac{g_1}{t_1} \right)^{-2} \left( \frac{t}{t_0} \right) e^{g_1 t_1}. \]

Here \( g_0, g_1, \) and \( U_0 \) are constants. The Hubble rate is given by

\[ H = g_0 + \frac{g_1}{t}, \quad \phi = \phi_0 \ln \frac{t}{t_0}. \]  

Hence, when \( t \) is small, the second term in the expression for \( H \) in (5.4) dominates and the scale factor behaves as \( a \sim t^{g_1} \). Thus, if \( g_1 = 2/3 \), a matter-dominated period, in which a scalar may be identified with matter, could be realized. On the other hand, when \( t \) is large, the first term of \( H \) in (5.4) dominates and the Hubble rate \( H \) becomes constant. Consequently, the universe is asymptotically de Sitter space, which is an accelerating universe.

In the case without matter, the equations of motion in the FRW metric are given by

\[ 0 = -\frac{3}{\kappa^2} H^2 - \frac{1}{2} \dot{\phi}^2 + V(\phi) + 24H^3 \frac{d\xi(\phi(t))}{dt}, \]  

\[ 0 = \frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) + \frac{1}{2} \dot{\phi}^2 - V(\phi) - 8H^2 \frac{d^2\xi(\phi(t))}{dt^2} - 16H \dot{H} \frac{d\xi(\phi(t))}{dt} \]  

\[ - 16H^3 \frac{d\xi(\phi(t))}{dt}, \]  

\[ 0 = \ddot{\phi} + 3H \dot{\phi} + V'(\phi) + \xi(\phi) G. \]
The potentials are expressed as
\[ V(\phi(t)) = \frac{3}{\kappa^2}H(t)^2 - \frac{1}{2}\dot{\phi}(t)^2 - 3a(t)H(t) \int t \frac{dt_1}{a(t_1)} \left( \frac{2}{\kappa^2}\dot{H}(t_1) + \dot{\phi}(t_1)^2 \right), \]
\[ \xi(\phi(t)) = \frac{1}{8} \int \dot{\phi}(t) \frac{dt_1}{H(t_1)^2} \int \dot{\phi}_1 \frac{dt_2}{a(t_2)} \left( \frac{2}{\kappa^2}\dot{H}(t_2) + \dot{\phi}(t_2)^2 \right). \]

Equations (5.9) show that if we consider the theory including two functions \( g(t) \) and \( f(\phi) \) as
\[ V(\phi) = \frac{3}{\kappa^2}g'(f(\phi))^2 - \frac{1}{2}f'(f(\phi))^2 - 3g'(f(\phi))g^2(f(\phi))U(\phi), \]
\[ \xi(\phi) = \frac{1}{8} \int \frac{f''(f(\phi))e^{g(f(\phi))}}{f'(f(\phi))^2} U(\phi), \]
\[ U(\phi) = \int f\phi f'(f(\phi))e^{-g(f(\phi))} \left( \frac{2}{\kappa^2}g''(f(\phi)) + \frac{1}{f'(f(\phi))^2} \right), \]
the solution of the field equations is given by
\[ \phi = f^{-1}(t) \quad (t = f(\phi)), \quad a = a_0 e^{g(t)} (H = g'(t)). \]

It is easy to include matter with constant \( w = w_m \). In this case, it is enough to consider the theory where \( U(\phi) \) in \( V(\phi) \) and \( \xi(\phi) \) in (5.10) is replaced by
\[ U(\phi) = \int f\phi f'(f(\phi))e^{-g(f(\phi))} \left( \frac{2}{\kappa^2}g''(f(\phi)) + \frac{1}{f'(f(\phi))^2} + (1 + w_m)g_0 e^{-3(1+w_m)g(f(\phi))} \right). \]

We obtain the previous solution with \( a_0 = (g_0/\rho_0)^{-1/3(1+w_m)}. \)

As a cousin of the scalar Gauss–Bonnet theory, we may consider the modified Gauss–Bonnet theory [16, 35, 36], whose action is given by
\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + F(G) \right]. \]

By introducing an auxiliary scalar field \( \phi \), we can rewrite the action (5.13) in a form similar to the action of the scalar Gauss–Bonnet theory in (5.1):
\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - V(\phi) - \xi(\phi)G \right]. \]

In fact, by solving the equation for \( \phi \):
\[ 0 = V'(\phi) + \xi'(\phi)G, \]
with respect to \( \phi \) as \( \phi = \phi(\mathcal{G}) \), and substituting the expression for \( \phi \) into the action (5.14), we re-obtain the action (5.13) where \( F(\mathcal{G}) \) is given by

\[
F(\mathcal{G}) \equiv -V(\phi(\mathcal{G})) + \xi(\phi(\mathcal{G})) \mathcal{G}.
\]  

(5.16)

An example [16] is given by (2.19). When \( F_0^2 > 3/2\kappa^4 \), there are two solutions, which describe an effective phantom universe and admit the big rip singularity. When \( F_0^2 < 3/2\kappa^4 \), we have a solution which describes the effective quintessence, and another solution, which describes an effective phantom.

Another example with dust, whose energy density behaves as \( \rho = \rho_0 a^{-3} \), as follows [32]:

\[
V(\phi) = \frac{2C^2}{\kappa^2} \coth^2(C\phi) - 3CU_0 \coth(C\phi) \sinh^{2/3}(C\phi),
\]

\[
\xi(\phi) = \frac{U_0}{8} \int_0^\phi d\phi_1 \sinh^{-4/3}(C\phi) \cosh^2(C\phi).
\]

Here, \( C \) and \( U_0 \) are constants. The explicit solution is given by

\[
a(t) = a_0 e^{g(t)}, \quad g(t) = \frac{2}{3} \ln (\sinh (Ct)), \quad \rho_0 = \frac{27a_0^3 C}{4\kappa^2}.
\]

(5.18)

Equation (5.18) indicates that, when \( \phi = t \) is small, \( g(\phi) \) behaves as \( g(\phi) \sim (2/3) \ln \phi \) and therefore, the Hubble rate behaves as \( H(t) = g'(t) \sim (2/3)/t \), which certainly reproduces the matter-dominated phase. On the other hand, when \( \phi = t \) is large, \( g(\phi) \) behaves as \( g \sim (2/3)(C\phi) \), namely, \( H \sim 2C/3 \) and the universe asymptotically goes to de Sitter space. Hence, the model given by (5.17) with matter shows the transition from the matter-dominated phase to the accelerating universe, which is asymptotically de Sitter space.

We may consider the reconstruction of \( F(\mathcal{G}) \) gravity [19,32]. The field equations in the FRW background are given by

\[
0 = -\frac{3}{\kappa^2} H^2 + V(\phi) + 24H^3 \frac{d\xi(\phi(t))}{dt},
\]

\[
0 = \frac{1}{\kappa^2} \left( 2 \dot{H} + 3H^2 \right) - V(\phi) - 8H^2 \frac{d^2\xi(\phi(t))}{dt^2} - 16\dot{H} \frac{d\xi(\phi(t))}{dt} - 16H^3 \frac{d^2\xi(\phi(t))}{dt^2}.
\]

(5.19)

(5.20)

which could be rewritten as

\[
\xi(\phi(t)) = \frac{1}{8} \int_0^t dt_1 \frac{a(t_1)}{H(t_1)^2} W(t_1), \quad V(\phi(t)) = \frac{3}{\kappa^2} H(t)^2 - 3a(t)H(t)W(t),
\]

\[
W(t) \equiv \frac{2}{\kappa^2} \int_0^t dt_1 \frac{a(t_1)}{H(t_1)} H(t_1).
\]

(5.21)

Since there is no kinetic term of \( \phi \), there is a freedom to redefine \( \phi \) as \( \phi = \varphi = \varphi(\phi) \). By using the redefinition, we may choose the scalar field \( \phi \) as a time coordinate: \( \phi = t \). Thus one gets

\[
V(\phi) = \frac{3}{\kappa^2} g'(\phi)^2 - 3g'(\phi) e^{g(\phi)} U(\phi), \quad \xi(\phi) = \frac{1}{8} \int_0^\phi d\phi_1 \frac{e^{g(\phi_1)}}{g'(\phi_1)^2} U(\phi_1),
\]

\[
U(\phi) \equiv \frac{2}{\kappa^2} \int_0^\phi d\phi_1 e^{-g(\phi_1)} g''(\phi_1).
\]

(5.22)
The solution is given by
\[ a = a_0 e^{g(t)} (H = g'(t)). \quad (5.23) \]

As in the case of the scalar Gauss–Bonnet theory, we may include matter.

Let us apply the reconstruction program to the study of the finite-time future singularities, where the Hubble rate behaves as
\[ H \sim h_0 (t_s - t)^{-\beta}. \quad (5.24) \]

First, we consider the modified Gauss–Bonnet gravity. For the \( \beta = 1 \) and \( h_0 > 0 \) case, namely, the usual big rip singularity case, since \( g'(\phi) = H(\phi) \), we find
\[ g(\phi) = -h_0 \ln \frac{t_s - \phi}{t_0}, \quad (5.25) \]

where \( t_0 \) is a constant of the integration. \( U(\phi) \) in (5.22) has the following form:
\[
U(\phi) = \frac{2}{\kappa^2} \int d\phi_1 \left( \frac{t_s - \phi_1}{t_0} \right)^{h_0} \frac{h_0}{(t_s - \phi_1)^2} \\
= \begin{cases} 
U_0 - \frac{2h_0}{\kappa^2 t_0^{h_0} (h_0 - 1)} (t_s - \phi)^{h_0 - 1} & \text{when } h_0 \neq 1 \\
\frac{2}{\kappa^2 t_0} \ln \frac{t_s - \phi}{t_1} & \text{when } h_0 = 1.
\end{cases} \quad (5.26)
\]

Here, \( U_0 \) and \( t_1 \) are constants of the integration. At the next step, one gets
\[
V(\Phi) = \begin{cases} 
\frac{3h_0 t_0^{h_0} U_0}{\Phi^{h_0 + 1}} + \frac{3h_0^2 (h_0 + 1)}{\kappa^2 (h_0 - 1) \Phi^2} & \text{when } h_0 \neq 1 \\
\frac{3}{\kappa^2 \Phi^2} \left( 1 - 2 \ln \frac{\Phi}{t_0} \right) & \text{when } h_0 = 1,
\end{cases} \quad (5.27)
\]
\[
\xi(\Phi) = \begin{cases} 
\xi_0 - \frac{t_0^{h_0} U_0 \Phi^{3-h_0}}{8h_0^2 (3 - h_0)} + \frac{\Phi^2}{8\kappa^2 (h_0 - 1)} & \text{when } h_0 \neq 1, 3 \\
-\frac{t_0^3 U_0}{72} \ln \frac{\Phi}{t_2} + \frac{\Phi^2}{16\kappa^2} & \text{when } h_0 = 3 \\
\xi_0 - \left( \frac{1}{2} \ln \frac{\Phi}{t_1} + \frac{1}{4} \right) \Phi^2 & \text{when } h_0 = 1.
\end{cases} \quad (5.28)
\]

Here, \( \xi_0 \) and \( t_2 \) are integration constants but they are irrelevant to the action because the Gauss–Bonnet invariant is a total derivative and \( \xi_0 \) and \( t_2 \) correspond to the constant shift of the coefficient of the Gauss–Bonnet invariant. We further redefine the scalar field \( \phi \) as \( \Phi \equiv t_s - \phi \). The forms of \( V(\Phi) \) and \( \xi(\Phi) \) obtained do not contain the parameter \( t_s \) corresponding to the big rip time. Hence, the resulting \( F(G) \) does not contain \( t_s \) either and \( t_s \) could be determined dynamically by initial conditions.

When \( \beta \neq 0 \), \( g(\phi) \) is given by
\[ g(\phi) = \frac{h_0}{\beta - 1} (t_s - \phi)^{1-\beta} + g_0. \quad (5.29) \]
where \( g_0 \) is a constant of the integration but the constant is irrelevant and does not appear in the final expressions for \( V(\phi) \) and \( \xi(\phi) \). We therefore choose \( g_0 = 0 \). \( U(\phi) \) has the following form:

\[
U(\phi) = \frac{2h_0\beta}{\kappa^2} \int \phi_1(t_s - \phi_1)^{-1-\beta} e^{-(h_0/\beta-1)(t_s-\phi_1)^{1-\beta}}
\]

\[
= \frac{2h_0\beta}{\kappa^2 (\beta - 1)} \int x^1(\beta-1) e^{-(h_0/\beta-1)x}, \tag{5.30}
\]

where

\[
x \equiv (t_s - \phi)^{1-\beta}. \tag{5.31}
\]

We now consider the case \( \beta > 1 \), which corresponds to the type I singularity. In this case, \( x \to \infty \) when \( \phi \to t_s \). When \( x \) is large, the following expression can be used:

\[
\int dx x^\alpha e^{-ax} = -e^{-ax} \left( \frac{x^\alpha}{a} + \frac{\alpha}{a^2} x^{\alpha-1} + \frac{\alpha(\alpha-1)}{a^3} x^{\alpha-2} + \cdots \right),
\]

\[
(\alpha = \frac{1}{\beta}, a = \frac{h_0}{\beta - 1}). \tag{5.32}
\]

Keeping the leading order, one finds

\[
U(\phi) \sim -\frac{2\beta}{\kappa^2 (t_s - \phi)} e^{-(h_0/\beta-1)(t_s-\phi)^{1-\beta}}, \tag{5.33}
\]

and therefore

\[
V(\Phi) \sim \frac{3h_0^2}{\kappa^2} \Phi^{-2\beta} + \frac{6\beta}{\kappa^2} \Phi^{-\beta-1} \sim \frac{3h_0^2}{\kappa^2} \Phi^{-2\beta}, \quad \xi(\Phi) \sim \frac{\Phi^{2\beta}}{8\kappa^2 h_0^2} + \xi_0. \tag{5.34}
\]

Here, \( \Phi = t_s - \phi \) again and \( \xi_0 \) is an irrelevant constant of the integration.

In the case of \( \beta < 1 \), which corresponds to the type II, III, IV singularities, we find that \( x \to 0 \) when \( \phi \to t_s \). Using the expression

\[
\int dx x^\alpha e^{-ax} = \frac{1}{\alpha + 1} x^{\alpha+1} - \frac{a}{\alpha + 2} x^{\alpha+2} + \frac{a^2}{2!(\alpha + 3)} x^{\alpha+3} - \cdots, \tag{5.35}
\]

and only keeping the leading order, we find

\[
U(\phi) = \frac{2h_0}{\kappa^2} (t_s - \phi)^{-\beta} + U_0, \tag{5.36}
\]

where \( U_0 \) is a constant of the integration. Hence,

\[
V(\Phi) \sim -\frac{3h_0^2}{\kappa^2} \Phi^{-2\beta} - \frac{U_0}{h_0} \Phi^{-\beta},
\]

\[
\xi(\Phi) \sim \left\{ \begin{array}{ll}
-\frac{\Phi^{\beta+1}}{4\kappa^2 h_0 (\beta + 1)} - \frac{U_0 \Phi^{2\beta+1}}{8h_0^2 (2\beta + 1)} + \xi_0 & \text{when } \beta \neq -1, -\frac{1}{2} \\
- \frac{1}{4\kappa^2 h_0} \ln \frac{\Phi}{t_2} + \frac{U_0 \Phi^{-1}}{8h_0^2} & \text{when } \beta = -1 \\
- \frac{\Phi^{1/2}}{2\kappa^2 h_0} - \frac{U_0}{8h_0^2} \ln \frac{\Phi}{t_2} & \text{when } \beta = -\frac{1}{2}.
\end{array} \right. \tag{5.37}
\]

Here, \( \xi_0 \) and \( t_2 \) are (irrelevant) constants of the integration.
In the case of the scalar Gauss–Bonnet gravity, there is a freedom or ambiguity in the choice of \( f(\phi) \). For simplicity, we now choose
\[ t = f(\phi) = \kappa^2 \phi. \]
(5.38)

The equations (5.10) are rewritten as
\[
V(\varphi) = \frac{3}{\kappa^2} g'(\varphi)^2 - \frac{1}{2\kappa^4} - 3g'(\varphi) e^{\varphi} U(\varphi),
\]
\[
\xi(\varphi) = \frac{1}{8} \int \varphi g'(\varphi_1)^2 U(\varphi_1),
\]
\[
U(\varphi) \equiv \int d\varphi_1 e^{-g(\varphi_1)} \left( \frac{2}{\kappa^2} g''(\varphi_1) + \frac{1}{\kappa^4} \right),
\]
(5.39)

where \( \varphi \equiv \kappa^2 \phi \). By a calculation similar to that for \( F(\mathcal{G}) \) gravity, when \( \beta = 1 \) (the type I singularity, usual big rip), we find
\[
V(\Phi) = \begin{cases} 
U_0 - \frac{2h_0}{\kappa^2 t_0^h_0 (h_0 - 1)} (t_s - \varphi)^{h_0 - 1} - \frac{(t_s - \varphi)^{h_0 + 1}}{\kappa^4 t_0^h_0 (h_0 + 1)} & \text{when } h_0 \neq 1 \\
\frac{2}{\kappa^2 t_0} \ln \frac{t_s - \varphi}{t_1} - \frac{(t_s - \varphi)^{h_0 + 1}}{\kappa^4 t_0^h_0 (h_0 + 1)} & \text{when } h_0 = 1.
\end{cases}
\]
(5.40)

Here, \( U_0 \) and \( t_1 \) are constants of the integration again. Subsequently,
\[
V(\Phi) = \begin{cases} 
-3h_0 t_0 U_0 \frac{\Phi^{h_0 - 1}}{\Phi_0^{h_0 - 1}} + \frac{3h_0^2 (h_0 + 1)}{\kappa^2 (h_0 - 1)} \Phi^2 + \frac{2h_0 - 1}{\kappa^4 (h_0 + 1)} & \text{when } h_0 \neq 1 \\
\frac{3}{\kappa^2 \Phi^2} \left( 1 - 2 \ln \frac{\Phi}{\Phi_0} \right) + \frac{2h_0 - 1}{2\kappa^4 (h_0 + 1)} & \text{when } h_0 = 1.
\end{cases}
\]
(5.41)

\[
\xi(\Phi) = \begin{cases} 
\xi_0 - \frac{t_0^h U_0}{8h_0^2 (3 - h_0)} - \frac{3h_0^2}{8\kappa^2 (h_0 - 1)} \Phi^2 + \frac{\Phi^4}{32h_0^2 (h_0 + 1) \kappa^4} & \text{when } h_0 \neq 1, 3 \\
\frac{t_0^h U_0}{72} \ln \frac{\Phi}{t_2} + \frac{\Phi^2}{16\kappa^2} + \frac{\Phi^4}{32h_0^2 (h_0 + 1) \kappa^4} & \text{when } h_0 = 3
\end{cases}
\]
\[
\xi_0 - \frac{1}{2} \ln \frac{\Phi}{t_1} + \frac{1}{4} \Phi^2 + \frac{\Phi^4}{32h_0^2 (h_0 + 1) \kappa^4} & \text{when } h_0 = 1.
\]
(5.42)

Here, \( \Phi = t_s - \varphi \). In terms of \( \Phi \), the kinetic term of \( \phi \) is rewritten as
\[
\frac{1}{2} \partial_\mu \phi \partial^\mu \phi = -\frac{1}{2\kappa^4} \partial_\mu \Phi \partial^\mu \Phi.
\]
(5.43)

In the case of \( \beta > 1 \), one obtains
\[
U(\varphi) \sim -\frac{2\beta}{\kappa^2 (t_s - \varphi)} e^{-(h_0/(\beta - 1))(t_s - \varphi)^{1 - \beta}} + \frac{(t_s - \varphi)^\beta}{h_0 \kappa^4} e^{-(h_0/(\beta - 1))(t_s - \varphi)^{1 - \beta}},
\]
\[
V(\Phi) \sim \frac{3h_0^2}{\kappa^2} \Phi^{-2\beta} - \frac{3}{\kappa^4}, \quad \xi(\Phi) \sim \frac{\Phi^{2\beta}}{8\kappa^2 h_0^2} + \xi_0 - \frac{\Phi^{3\beta + 1}}{8h_0^3 (3\beta + 1) \kappa^4}.
\]
(5.44)
In the case of $\beta < 1$, we get
\[
U(\varphi) = \frac{2h_0}{\kappa^2} (t_s - \varphi)^{-\beta} + U_0 - \frac{t_s - \varphi}{\kappa^4},
\]
\[
V(\Phi) \sim -\frac{3h_0^2}{\kappa^2} \Phi^{-2\beta} - 3h_0 U_0 \Phi^{-\beta} + \frac{3h_0}{\kappa^4} \Phi^{1-\beta},
\]
\[
\xi(\Phi) \sim \begin{cases} 
\frac{\Phi^{\beta+1}}{4\kappa^2 h_0 (\beta + 1)} - \frac{U_0 \Phi^{2\beta+1}}{8h_0^2 (2\beta + 1)} + \xi_0 + \frac{\Phi^{2\beta+2}}{16h_0^2 \kappa^4 (\beta + 1)} & \text{when } \beta \neq -1, -\frac{1}{2} \\
- \frac{1}{4\kappa^2 h_0} \ln \frac{\Phi}{t_2} + \frac{U_0 \Phi^{-1}}{8h_0^2} + \frac{\Phi^{2\beta+2}}{16h_0^2 \kappa^4 (\beta + 1)} & \text{when } \beta = -1 \\
- \Phi^{1/2} - \frac{U_0}{8h_0^2} \ln \frac{\Phi}{t_2} + \frac{\Phi^{2\beta+2}}{16h_0^2 \kappa^4 (\beta + 1)} & \text{when } \beta = -\frac{1}{2}.
\end{cases}
\] (5.45)

Thus, any type of finite-time future singularity can be realized in $F(\mathcal{G})$ gravity and the scalar Gauss–Bonnet one. We have constructed specific examples of the above models containing the finite-time future singularity by using the reconstruction program. We should note, however, that the relation between the appearance of the singularity and the forms of $V(\phi)$ and $\xi(\varphi)$ in the scalar Gauss–Bonnet gravity is not so clear. In other words, even if the explicit forms of $V(\phi)$ and $\xi(\varphi)$ are given, it is difficult to understand whether the theory could generate the singularity or not. A corresponding investigation of the asymptotic behavior of the solution should be carried out in order to answer this question. In the case of $F(\mathcal{G})$ gravity, however, if $F(\mathcal{G})$ contains a term like $\sqrt{\mathcal{G}^2 - \mathcal{G}^2 (\mathcal{G}_0 > 0)}$, which becomes imaginary, and therefore inconsistent, if $|\mathcal{G}| > \mathcal{G}_0$, the singularity should not appear. In other words, even if the solution contains a finite-time future singularity, additional modification of the action may resolve it.

6. Non-minimal Maxwell–Einstein gravity

In this section, we consider some cosmological effects in the non-minimal Maxwell–Einstein gravity with general gravitational coupling. We account for our model and derive the effective energy density and pressure of the universe.

6.1. Model

We consider the following model action [37]:
\[
S_{\text{GR}} = \int d^4x \sqrt{-g} [\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{EM}}],
\] (6.1)
\[
\mathcal{L}_{\text{EH}} = \frac{1}{2\kappa^2} R,
\] (6.2)
\[
\mathcal{L}_{\text{EM}} = -\frac{1}{4} I(R) F_{\mu\nu} F^{\mu\nu},
\] (6.3)
\[
I(R) = 1 + \bar{I}(R),
\] (6.4)
The future of the universe in modified gravitational theories

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor. Here, $A_\mu$ is the $U(1)$ gauge field. Furthermore, $\tilde{I}(R)$ is an arbitrary function of $R$. It is known that the coupling between the scalar curvature and the Lagrangian of the electromagnetic field arises in curved space–time due to one-loop vacuum-polarization effects in quantum electrodynamics [38]. (In [39], a non-minimal gravitational Yang–Mills (YM) theory, in which the YM field couples to a function of the scalar curvature, has been discussed. Note that this study may be generalized also for $F(R)$ gravity coupled to non-linear electrodynamics [40].)

Taking variations of the action of equation (6.1) with respect to the metric $g_{\mu\nu}$ and the $U(1)$ gauge field $A_\mu$, we obtain the gravitational field equation and the equation of motion of $A_\mu$ as [37]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}^{(EM)},$$

(6.5)

with

$$T_{\mu\nu}^{(EM)} = I(R) \left( g^{\alpha\beta}F_{\mu\beta}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \right) + \frac{1}{2} \left\{ I'(R)F_{\alpha\beta}F^{\alpha\beta}R_{\mu\nu} + g_{\mu\nu}\Box \left[ I'(R)F_{\alpha\beta}F^{\alpha\beta} \right] - \nabla_\mu \nabla_\nu \left[ I'(R)F_{\alpha\beta}F^{\alpha\beta} \right] \right\},$$

(6.6)

and

$$-\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g}I(R)F^{\mu\nu} \right) = 0,$$

(6.7)

respectively, where $T_{\mu\nu}^{(EM)}$ is the contribution to the energy–momentum tensor from the electromagnetic field.

### 6.2. Effective energy density and pressure of the universe

We now assume the flat FRW space–time with the metric in equation (2.1). We here consider the case in which there exist only magnetic fields and hence electric fields are negligible. In addition, only one component of $B$ is non-zero and hence the other two components are zero. In this case, it follows from $\text{div} B = 0$ that the off-diagonal components of the last term on the rhs of equation (6.6) for $T_{\mu\nu}^{(EM)}$, i.e., $\nabla_\mu \nabla_\nu \left[ f'(R)F_{\alpha\beta}F^{\alpha\beta} \right]$, are zero. Thus, all of the off-diagonal components of $T_{\mu\nu}^{(EM)}$ are zero (for an argument about the problem of off-diagonal components of the electromagnetic energy–momentum tensor in non-minimal Maxwell gravity theory, see [41]). Moreover, because we assume that there exist magnetic fields as background quantities at the zeroth order, the magnetic fields do not have a dependence on the space components $x$.

In the FRW background, the equation of motion for the $U(1)$ gauge field in the Coulomb gauge, $\partial^j A_j(t, x) = 0$, for the case of $A_0(t, x) = 0$, becomes

$$\ddot{A}_i(t, x) + \left( H + \frac{i}{T} \right) \dot{A}(t, x) - \frac{1}{a^2} \Delta A_i(t, x) = 0,$$

(6.8)
where $\Delta = \partial^i \partial_i$ is the flat three-dimensional Laplacian. It follows from equation (6.8) that the Fourier mode $A_i(k, t)$ satisfies the equation

$$\ddot{A}_i(k, t) + \left( H + \frac{I}{I} \right) \dot{A}_i(k, t) + \frac{k^2}{a^2} A_i(k, t) = 0. \quad (6.9)$$

Replacing the independent variable $t$ by conformal time $\eta = \int dt/a(t)$, we find that equation (6.9) becomes

$$\frac{\partial^2 A_i(k, \eta)}{\partial \eta^2} + \frac{1}{I(\eta)} \frac{dI(\eta)}{d\eta} \frac{\partial A_i(k, \eta)}{\partial \eta} + k^2 A_i(k, \eta) = 0. \quad (6.10)$$

It is impossible to obtain the exact solution of equation (6.10) for the generic evolution of the coupling function $I$ at the inflationary stage. However, by using the WKB approximation on subhorizon scales and the long wavelength approximation on superhorizon scales, and matching these solutions at the horizon crossing [42], we find an approximate solution as

$$|A_i(k, \eta)|^2 = |\bar{C}(k)|^2 = \frac{1}{2kI(\eta_k)} \left| 1 - \frac{1}{2} \frac{1}{kI(\eta_k)} \frac{dI(\eta_k)}{d\eta} + i \int_{\eta_k}^{\eta_f} \frac{I(\eta_f)}{I(\eta_k)} d\eta_f \right|^2, \quad (6.11)$$

where $\eta_k$ and $\eta_f$ are the conformal time at the horizon crossing and that at the end of inflation, respectively. From equation (6.11), we obtain the amplitude of the proper magnetic fields in the position space

$$|B_i^{(\text{proper})}(t)|^2 = \frac{k|\bar{C}(k)|^2 k^4}{\pi^2 a^4}, \quad (6.12)$$

on a comoving scale $L = 2\pi/k$. Thus, from equation (6.12) we see that the proper magnetic fields evolves as $|B_i^{(\text{proper})}(t)|^2 = |B|^2/a^4$, where $|\bar{B}|$ is a constant. (The validity of this behavior of the proper magnetic fields, namely, that $|\bar{B}|$ is a constant, is shown in the appendix.) This means that the influence of the coupling function $I$ on the value of the proper magnetic fields exists only during inflation. (On the other hand, because the expression for the energy density of the magnetic fields is given using that for the magnetic fields multiplying $I$ due to the Lagrangian (6.3), the energy density of the magnetic fields depends on $I$ also after inflation. We can see this point from the first term on the rhs of equation (6.13) shown below.) The conductivity of the universe $\sigma_c$ is negligibly small during inflation, because there are few charged particles at that time. After the reheating stage, a number of charged particles are produced, so the conductivity immediately jumps to a large value: $\sigma_c \gg H$. Consequently, for a large enough conductivity at the reheating stage, the proper magnetic fields evolve in proportion to $a^{-2}(t)$ in the radiation-dominated stage and the subsequent matter-dominated stage [43].

In this case, it follows from equation (6.6) that the quantity corresponding to the effective energy density of the universe $\rho_{\text{eff}}$ and that corresponding to the effective pressure
The future of the universe in modified gravitational theories

\[ \rho_{\text{eff}} \text{ are given by} \]
\[ \rho_{\text{eff}} = \left\{ \frac{I(R)}{2} + 3 \left[ -5 H^2 + \dot{H} \right] I'(R) + 6H \left( 4H\dot{H} + \ddot{H} \right) I''(R) \right\} \frac{|\vec{B}|^2}{a^4}, \quad (6.13) \]
\[ p_{\text{eff}} = \left[ -\frac{I(R)}{6} + \left( -H^2 + 5\dot{H} \right) I'(R) - 6 \left( -20H^2\dot{H} + 4\ddot{H}^2 - H\dddot{H} + \ddot{H} \right) I''(R) \right. \]
\[ - 36 \left( 4H\dot{H} + \ddot{H} \right) \left( I''(R) \right) \frac{|\vec{B}|^2}{a^4}, \quad (6.14) \]

where we have used the following relations, taking into account that electric fields are negligible: \( g^{\alpha\beta} F_{0\beta} F_{0\alpha} - (1/4) g_{00} F_{\alpha\beta} F^{\alpha\beta} = |B_i^{\text{(proper)}(t)}|^2/2, \) and \( F_{\alpha\beta} F^{\alpha\beta} = 2|B_i^{\text{(proper)}(t)}|^2. \)

Finally, we remark on the following point. Suppose that \( I(R) \) is (almost) constant at the present time. We now assume that for the small curvature, \( I(R) \) behaves as
\[ I(R) \sim I_0 R^\alpha, \quad (6.15) \]
with constants \( I_0 \) and \( \alpha. \) Here, we consider the case \( \alpha < 0. \) The energy density of the magnetic fields is given by \( \rho_B = \left( 1/2 \right) |B_i^{\text{(proper)}(t)}|^2 I(R) = \left[ |B|^2/(2a^4) \right] I(R). \)

We investigate the form of \( I(R) \) which produces the big rip singularity,
\[ H \sim \frac{h_0}{t_0 - t}, \quad (7.3) \]
where \( h_0 \) is a positive constant, and \( H \) diverges at \( t = t_0. \) In this case,
\[ R \sim \frac{12h_0^2 + 6h_0}{(t_0 - t)^2}, \quad a \sim a_0(t_0 - t)^{-h_0}, \quad (7.4) \]
where \(a_0\) is a constant. We now assume that for the large curvature, \(I(R)\) behaves as equation (6.15). Hence \(\rho_{\text{eff}}\) in equation (6.13) behaves as \((t_0 - t)^{-2\alpha + 4h_0}\), but the lhs in the FRW equation \(3H^2/\kappa^2 = \rho_{\text{eff}}\) behaves as \((t_0 - t)^{-2}\). The consistency gives

\[-2 = -2\alpha + 4h_0,\]

namely

\[h_0 = \frac{\alpha - 1}{2} \quad \text{or} \quad \alpha = 1 + 2h_0.\]  

Equation (7.1) also shows

\[\frac{3h_0^2}{\kappa^2} = I_0 \left(12h_0^2 + 6h_0\right)^{\alpha - 2} \left\{ \frac{(12h_0^2 + 6h_0)^2}{2} + 3 \left[ -\alpha \left(12h_0^2 + 6h_0\right) (h_0 + 5h_0^2) + 6\alpha (\alpha - 1) h_0 (2h_0 + 4h_0^2) \right] \right\} \left| \tilde{B}\right|^2 \]

\[= -\frac{I_0 h_0 \left(12h_0^2 + 6h_0\right)^{\alpha} \left| \tilde{B}\right|^2}{2a_0^4},\]  

which requires that \(I_0\) should be negative. In the second line of (7.7), we have used (7.6) and deleted \(\alpha\).

As a result, it follows from equations (6.15) and (7.6) that the big rip singularity in equation (7.3) can emerge only when for the large curvature, \(I(R)\) behaves as \(R^{1 + 2h_0}\). If the form of \(I(R)\) is given by other terms, the big rip singularity cannot emerge. We here note that if exactly \(I(R) = I_0 R^\alpha\), \(H = h_0/(t_0 - t)\) is an exact solution.

Next, we study the form of \(I(R)\) which gives a more general singularity in equation (3.12). In this case,

\[R \sim 6h_0 \left[\beta + 2h_0(t_0 - t)^{-(\beta - 1)}\right] (t_0 - t)^{-(\beta + 1)}, \quad a \sim a_0 \exp \left[ \frac{h_0}{\beta - 1}(t_0 - t)^{-(\beta - 1)} \right].\]  

We also assume that for the large curvature, \(I(R)\) behaves as equation (6.15). If \(\beta < -1\), in the limit \(t \to t_0\), \(R \to 0\). Hence, we consider this case later. If \(\beta > 1\), \(a \to \infty\), and hence \(\rho_{\text{eff}} \to 0\) and \(p_{\text{eff}} \to 0\) because \(\rho_{\text{eff}} \propto a^{-4}\) and \(p_{\text{eff}} \propto a^{-4}\). On the other hand, \(H \to \infty\). Thus equations (7.1) and (7.2) cannot be satisfied.

If \(\alpha > 0\) and \(0 < \beta < 1\), \(\rho_{\text{eff}}\) in equation (6.13) evolves as \((t_0 - t)^{-\alpha(\beta + 1)}\), but the lhs of equation (7.1) evolves as \((t_0 - t)^{-2\beta}\). Thus, the consistency gives

\[-2\beta = -\alpha(\beta + 1),\]

namely,

\[\beta = \frac{\alpha}{2 - \alpha} \quad \text{or} \quad \alpha = \frac{2\beta}{\beta + 1}.\]  

From equation (7.1), we also find

\[\frac{3h_0^2}{\kappa^2} = -\frac{I_0 (6h_0\beta)^{\alpha} (1 - \beta)|\tilde{B}|^2}{2a_0^4(\beta + 1)},\]
where we have used equation (7.10), and on the lhs we have taken only the leading term. Equation (7.11) requires that $I_0$ should be negative. Consequently, if $\alpha > 0$ and $0 < \beta < 1$, in the limit $t \to t_0$, $a \to a_0$, $R \to \infty$, $\rho_{\text{eff}} \to \infty$, and $|\rho_{\text{eff}}| \to \infty$. Hence the type III singularity emerges. If $\alpha > 0$ and $-1 < \beta < 0$, $\rho_{\text{eff}} \to \infty$, but $H \to 0$. Hence equation (7.1) cannot be satisfied.

If $(\beta - 1)/(\beta + 1) < \alpha < 0$ and $-1 < \beta < 0$, in the limit $t \to t_0$, $a \to a_0$, $R \to \infty$, $\rho_{\text{eff}} \to 0$, and $|\rho_{\text{eff}}| \to \infty$. Although the final value of $\rho_{\text{eff}}$ is not a finite one but vanishes, this singularity can be considered to be type II. The reason is as follows. In this case, when $I$ and $H$ are given by $I = 1 + I_0 R^2$ and $H = H_0 + h_0 (t_0 - t)^{-\beta}$, where $H_0$ is a constant, respectively, in the above limit $\rho_{\text{eff}} \to \rho_0$. From equations (6.13) and (7.1), we find $\rho_0 = 3H_0^2/\kappa^2 = |\ddot{B}|^2/(2a_0^4)$. Hence, $\rho_0$ is a finite value.

If $\alpha \leq (\beta - 1)/(\beta + 1)$ and $-1 < \beta < 0$, in the limit $t \to t_0$, $a \to a_0$, $R \to \infty$, $\rho_{\text{eff}} \to 0$, and $|\rho_{\text{eff}}| \to 0$, but $H \to \infty$. Hence equation (7.2) cannot be satisfied. If $\alpha < 0$ and $0 < \beta < 1$, $\rho_{\text{eff}} \to 0$, but $H \to \infty$. Hence equation (7.1) cannot be satisfied.

In addition, we investigate the case in which $\beta < -1$. In this case, in the limit $t \to t_0$, $a \to a_0$ and $R \to 0$. We assume that for the small curvature, $R$ behaves as equation (6.15). If $\alpha \geq (\beta - 1)/(\beta + 1)$, in the limit $t \to t_0$, $\rho_{\text{eff}} \to 0$, $|\rho_{\text{eff}}| \to 0$, and higher derivatives of $H$ diverge. Hence the type IV singularity emerges. If $0 < \alpha < (\beta - 1)/(\beta + 1)$, $\rho_{\text{eff}} \to 0$ and $|\rho_{\text{eff}}| \to \infty$. However, $H \to 0$ and $\dot{H} \to 0$. Hence equation (7.2) cannot be satisfied.

We note that if $I(R)$ is a constant (the case in which $I(R) = 1$ corresponds to the ordinary Maxwell theory), no singularity can emerge.

Here we mention the case in which $I(R)$ is given by the Hu–Sawicki form [14]

$$I(R) = I_{\text{HS}}(R) \equiv \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1},$$

(7.12)

which satisfies the conditions $\lim_{R \to \infty} I_{HS}(R) = c_1/c_2 = \text{const}$ and $\lim_{R \to 0} I_{HS}(R) = 0$. Here, $c_1$ and $c_2$ are dimensionless constants, $n$ is a positive constant, and $m$ denotes a mass scale. The following form [5] also has the same features:

$$I(R) = I_{\text{NO}}(R) \equiv \frac{[(R/M^2) - (R_c/M^2)]^{2q+1} + (R_c/M^2)^{2q+1}}{c_3 + c_4 \left\{[(R/M^2) - (R_c/M^2)]^{2q+1} + (R_c/M^2)^{2q+1}\right\}},$$

(7.13)

which satisfies the conditions $\lim_{R \to \infty} I_{NO}(R) = 1/c_4 = \text{const}$ and $\lim_{R \to 0} I_{NO}(R) = 0$. Here, $c_3$ and $c_4$ are dimensionless constants, $q$ is a positive integer, $M$ denotes a mass scale, and $R_c$ is the current curvature. If $\beta < -1$ and $I(R)$ is given by $I_{HS}(R)$ in equation (7.12) or $I_{NO}(R)$ in equation (7.13), in the limit $t \to t_0$, $a \to a_0$, $R \to 0$, $\rho_{\text{eff}} \to 0$, and $|\rho_{\text{eff}}| \to 0$. In addition, higher derivatives of $H$ diverge. Thus the type IV singularity emerges.

Consequently, it is demonstrated that Maxwell theory which is coupled non-minimally with Einstein gravity may produce finite-time singularities in the future, depending on the form of the non-minimal gravitational coupling.

The general conditions for $I(R)$ such that the finite-time future singularities whose form is given by equations (7.3) or (3.12) cannot emerge are that in the limit $t \to t_0$, $I(R) \to I$, where $I(\neq 0)$ is a finite constant, $I'(R) \to 0$, $I''(R) \to 0$, and $I'''(R) \to 0$.
8. The influence of non-minimal gravitational coupling on the finite-time future singularities in modified gravity

In this section, we consider the case in which there exist finite-time future singularities in modified $F(R)$ gravity and investigate the influence of non-minimal gravitational coupling on them. We show that a non-minimal gravitational coupling of the electromagnetic field can remove the finite-time future singularities or make the singularity stronger (or weaker).

In this case, the total energy density and pressure of the universe are given by $\rho_{\text{tot}} = \rho_{\text{eff}} + \rho_{\text{MG}}$ and $p_{\text{tot}} = p_{\text{eff}} + p_{\text{MG}}$, respectively. Here, $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are given by equations (6.13) and (6.14), respectively. Moreover, it follows from equations (2.8) and (2.9) that $\rho_{\text{MG}}$ and $p_{\text{MG}}$ are given by

$$\rho_{\text{MG}} = \frac{1}{\kappa^2} \left[ -\frac{1}{2} f(R) + 3 \left( H^2 + \dot{H} \right) f'(R) - 18 \left( 4H^2\dot{H} + \dot{H}^2 \right) f''(R) \right],$$

$$p_{\text{MG}} = \frac{1}{\kappa^2} \left[ \frac{1}{2} f(R) - 3 \left( H^2 + \dot{H} \right) f'(R) + 6 \left( 8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H} \right) f''(R) \right. $$

$$\left. + 36 \left( 4H\dot{\dot{H}} + \dot{H} \right)^2 f'''(R) \right].$$

In this case, it follows from equations (2.2), (6.13), (6.14), (8.1), and (8.2) that the FRW equations are given by

$$\frac{3}{\kappa^2}H^2 = \rho_{\text{tot}} = \left\{ \frac{I(R)}{2} + 3 \left[ -\left( 5H^2 + \dot{H} \right) I'(R) + 6H \left( 4H\dot{H} + \dot{H} \right) I''(R) \right] \right\} \frac{|\dot{B}|^2}{a^4}$$

$$+ \frac{1}{\kappa^2} \left[ -\frac{1}{2} (F(R) - R) + 3 \left( H^2 + \dot{H} \right) (F'(R) - 1) \right.$$  

$$\left. - 18 \left( 4H^2\dot{H} + \dot{H}^2 \right) F''(R) \right],$$

$$\frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) = p_{\text{tot}} = \left[ -\frac{I(R)}{6} + \left( -H^2 + 5\dot{H} \right) I'(R) \right.$$  

$$\left. - 6 \left( -20H^2\dot{H} + 4\dot{H}^2 - H\ddot{H} + \dddot{H} \right) I''(R) \right.$$  

$$\left. - 36 \left( 4H\dot{\dot{H}} + \dot{H} \right)^2 I'''(R) \right] \frac{|\dot{B}|^2}{a^4} + \frac{1}{\kappa^2} \left[ \frac{1}{2} (F(R) - R) \right.$$  

$$\left. - \left( 3H^2 + \dot{H} \right) (F'(R) - 1) \right.$$  

$$\left. + 6 \left( 8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \dddot{H} \right) F''(R) + 36 \left( 4H\dot{H} + \dot{H} \right)^2 F'''(R) \right].$$

Using equations (8.3) and (8.4), we find

$$0 = \left[ \frac{I(R)}{3} + 2 \left( -8H^2 + \dot{H} \right) I'(R) + 6 \left( 32H^2\dot{H} - 4\dot{H}^2 + 4H\ddot{H} - \dddot{H} \right) I''(R) \right.$$  

$$\left. - 36 \left( 4H\dot{\dot{H}} + \dot{H} \right)^2 I'''(R) \right] \frac{|\dot{B}|^2}{a^4}.$$
respectively. When the non-minimal gravitational electromagnetic coupling and the modified gravity sectors, \( I(R) \) behaves as (6.15). The first and second terms on the rhs of equation (8.5) are given by

\[
R \sim \frac{R}{t_f} = \frac{R}{(t_0 - t)^u}, \quad q > 1,
\]

where in the expression for \( R \) we have taken only the leading term.

We investigate the form of the non-minimal gravitational coupling of the electromagnetic field \( I(R) \) which produces the solution (8.7). We here assume that \( I(R) \) behaves as (6.15). The first and second terms on the rhs of equation (8.5) are the non-minimal gravitational electromagnetic coupling and the modified gravity sectors, respectively. When \( t \) is close to \( t_0 \), the leading term of the non-minimal gravitational electromagnetic coupling sector evolves as \((t_0 - t)^{(u-1)(q-1)-2}\). On the other hand, if \( q \leq 1 \), or \( q > 1 \) and \( u < q/(q - 2) \), that of the modified gravity sector evolves as \((t_0 - t)^{(u-1)(q-1)-2}\). Hence the consistency gives \( \alpha = q \). Moreover, in order that the leading term of the non-minimal gravitational electromagnetic coupling sector should not diverge in the limit \( t \to t_0 \), \( \alpha \) must be \( \alpha \geq (u + 1)/(u - 1) \). Taking only the leading terms in equation (8.5) and using \( \alpha = q \), we find

\[
\frac{I_0}{B^2 \kappa^2} = \frac{a^2 F}{|B|^2 \kappa^2}.
\]

If \( q > 1 \) and \( u \geq q/(q - 2) \), the leading term of the modified gravity sector behaves as \((t_0 - t)^{u-1}\). Hence the consistency gives \( \alpha = 2u/(u - 1) \). In this case, taking only the leading terms in equation (8.5) and using \( \alpha = 2u/(u - 1) \), we obtain

\[
\frac{I_0}{B^2 \kappa^2} = \frac{a^2 F}{|B|^2 \kappa^2} \frac{u - 1}{6u^2 (u + 1)} (-6h_0 u)^{-2/(u-1)}.
\]

Consequently, we see that the non-minimal gravitational coupling of the electromagnetic field \( I(R) \) evolves with the specific values of \( I_0 \) and \( \alpha \) stated above can resolve the finite-time future singularities which occur in pure modified gravity.

Next, we study the case in which non-minimal gravitational coupling of the electromagnetic field does not remove the singularity but makes it stronger (or weaker). We also assume that for the large curvature, \( I(R) \) behaves as (6.15). Using the result in equation (3.10), we consider the case in which for the large curvature, \( F(R) \) behaves as \( F(R) \propto R^q \), where \( q \equiv 1 - \alpha \). In this case, the big rip singularity in equation (7.3) emerges. It follows from equation (7.6) that if \( \alpha = 1 + 2h_0 \), in the limit \( t \to t_0 \), \( \rho_{\text{eff}} \to \infty \).
and $|p_{\text{eff}}| \to \infty$. Hence, the non-minimal gravitational coupling of the electromagnetic field makes the singularity stronger.

When there exists a more general singularity in equation (3.12) with $0 < \beta < 1$, which is a type III singularity and can appear for the form of $F(R)$ in equation (3.19), and $\alpha > 0$, in the limit $t \to t_0$, $\rho_{\text{eff}} \to \infty$ and $|p_{\text{eff}}| \to \infty$. Thus the non-minimal gravitational coupling of the electromagnetic field makes the singularity stronger. If $-1 < \beta < 0$, namely, there exists a type II singularity, which can appear for the form of $F(R)$ in equation (3.21), and $(\beta - 1)/(\beta + 1) < \alpha < 0$, $\rho_{\text{eff}} \to 0$ and $|p_{\text{eff}}| \to \infty$. For $|p_{\text{tot}}| > |p_{\text{MG}}|$ ($|p_{\text{tot}}| < |p_{\text{MG}}|$), therefore, the non-minimal gravitational coupling of the electromagnetic field makes the singularity stronger (weaker). Thus, the non-minimal gravitational coupling in Maxwell theory may qualitatively influence future of the universe. For instance, for some forms of non-minimal gravitational coupling it resolves the finite-time future singularity or it may change its properties.

9. Conclusion

In the present paper, we have considered the finite-time future singularities in modified gravity: $F(R)$ gravity, the scalar Gauss–Bonnet or modified Gauss–Bonnet one, and an effective fluid with an inhomogeneous EoS. It has been demonstrated that depending on the specific form of the model under consideration, the universe may approach finite-time future singularities of all four known types. It is not easy to say from the very beginning whether the particular theory brings the accelerating universe to the singularity or not. As a rule, each theory should be checked to see whether the singularity appears in the FRW accelerating solutions. It is also interesting that the finite-time future singularity cannot be seen through the study of local tests: theory passing known local and cosmological tests may produce a future universe with (or without) a singularity. We have presented several examples of modified gravity having accelerating dark energy solutions with a finite-time future singularity by explicitly using the reconstruction program. Our results are shown in table 1, where the possibilities of four types of the finite-time future singularity are explained for the modified gravities under consideration.

Moreover, we have discussed several theoretical scenarios resolving finite-time future singularity. In particular, we have considered the additional modification of the fluid with the inhomogeneous EoS or that of the gravity action by a term which is relevant for the very early (or very late) universe. If the corresponding term is negligible at the current epoch, such modification is always possible, at least from the theoretical point of view. Nevertheless, in order to check whether the corresponding term is realistic, its role in the very early/late universe should be confirmed by using observational data. Another scenario for removing singularity is related to the accounting for quantum effects (or even quantum gravity). However, the corresponding consideration may give preliminary results, at best, due to the absence of a consistent theory of quantum gravity.

Furthermore, we have investigated the non-minimal Maxwell–Einstein (or Maxwell–$F(R)$) gravity in a similar fashion. We have studied the forms of the non-minimal gravitational coupling generating the finite-time future singularities and the general conditions for the non-minimal gravitational coupling such that the finite-time future singularities cannot emerge. In addition, we have considered the influence of the non-minimal gravitational coupling on the finite-time future singularities in modified gravity.
Table 1. Summary of the results: $\rho_0$, $c$, $c_1$, $c_2$, $c_3$, $\tilde{c}_1$, $\tilde{c}_2$, and $\tilde{c}_3$ are proper constants.

| Type I ($\beta \geq 1$) | Type II ($-1 < \beta < 0$) | Type III ($0 < \beta < 1$) | Type IV ($\beta < -1$, $\beta$: not integer) |
|--------------------------|---------------------------|---------------------------|------------------------------------------|
| Einstein gravity with inhomogeneous EoS | $p + \rho \propto \rho^{(\beta+1)/\beta}$ | $p + \rho \propto (\rho^{1/2} - \rho_0^{1/2})^{(\beta+1)/\beta}$ |
| $F(R)$ gravity | $F(R) \propto e^{cR^{\beta+1}/R_0}$ | $F(R) \propto e^{cR^{\beta+1}/R_0} R^{\beta_3+3/R_0}$ | $F(R) \propto e^{cR^{\beta+1}/R_0} R^{\beta^2+3/R_0}$ |
| Scalar | $V(\Phi) \propto \Phi^{-2\beta}$, $\xi(\Phi) \propto c_1 \Phi^{\beta+1} + c_2 \Phi^{2\beta+2}$ ($\beta \neq -1, -1/2$) | $c_1 \ln \Phi + c_2 \Phi^{-1}$ ($\beta = -1$) | $c_1 \Phi^{1/2} + c_2 \ln \Phi$ ($\beta = -1/2$) |
| Gauss–Bonnet gravity | $\xi(\Phi) \propto c_1 \Phi^{2\beta+1}$ | $\xi(\Phi) \propto c_1 \Phi^{2\beta+1} + c_2 \Phi^{3\beta+2}$ ($\beta \neq -1, -1/2$) | $\xi(\Phi) \propto c_1 \Phi^{2\beta+1} + c_2 \Phi^{2\beta+2}$ ($\beta = -1/2$) |
| Modified Gauss–Bonnet gravity | $V(\Phi) \propto c_1 \Phi^{-2\beta} + c_2 \Phi^{-\beta} + c_3 \Phi^{1-\beta}$, $\xi(\Phi) \propto c_1 \Phi^{3\beta+1} + c_2 \Phi^{2\beta+1} + c_3 \Phi^{2\beta+2}$ ($\beta \neq -1, -1/2$) | $\xi(\Phi) \propto c_1 \Phi^{1/2} + c_2 \ln \Phi + c_3 \Phi^{2\beta+2}$ ($\beta = -1/2$) |
| Non-minimal Maxwell–Einstein gravity | $I(R) \propto R^{1+2\beta_0}$ ($\beta = 1$), Not possible if $\beta > 1$ | Possible | Not possible | Possible |
As a result, it has been shown that the non-minimal gravitational coupling in Maxwell
modified gravity can remove finite-time future singularities or make the singularities
stronger (or weaker).

The observational data for the current dark energy epoch still cannot tell us what
its exact nature is: phantom, ΛCDM or quintessence type. It has been demonstrated
in this paper that in some modified gravities with the effective phantom or quintessence
EoS a finite-time future singularity can emerge. In this respect, the interpretation of
the observational data confirming (or excluding) the approach to a finite-time future
singularity is fundamentally important. On one hand, it may clarify the distant future of
our universe. On the other hand, it may help to define the evolution of the universe and
the current value of the effective EoS parameter: how close is \( w \) to \(-1\)?

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**Appendix**

We here consider equation (6.7). Let us now assume

\[
E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 = 0, \tag{A.1}
\]

where \( E_i \) is the electric field. Then

\[
A_i = \partial_i \int dt A_0 + C_i. \tag{A.2}
\]

Here \( C_i \) does not depend on time \( t \). Then we find

\[
F_{ij} = \partial_i A_j - \partial_j A_i = \partial_i C_j - \partial_j C_i. \tag{A.3}
\]

Hence, the \( F_{ij} \) and, therefore, the magnetic flux \( B_i \equiv \epsilon_{ijk} F_{jk} \) do not depend on time. Since
the metric \( g_{\mu\nu} \) and hence the scalar curvature \( R \) do not depend on the spatial coordinates,
equation (6.7) reduces to the following form:

\[
\partial^i F_{ij} = \Delta C_j - \partial_j \partial^i C_i = 0. \tag{A.4}
\]

The solution of (A.4) is given by the constant \( F_{ij} \) or constant magnetic flux \( B_i \).
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