Transient Oscillations in Mechanical Systems of Automatic Control with Random Parameters

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Abstract. Transient oscillations in mechanical systems of automatic control with random parameters is a relevant but insufficiently studied issue. In this paper, a modified spectral method was applied to investigate the problem. The nature of dynamic processes and the phase portraits are analyzed depending on the amplitude and frequency of external influence. It is evident from the obtained results, that the dynamic phenomena occurring in the systems with random parameters under external influence are complex, and their study requires further investigation.

1. Introduction

Recently, in various fields of science and technology statistical methods have been widely used for solving the diverse challenges arising in machine building, control theory, transport etc. [1,4,5]. Among the application tasks solved with the methods of statistical dynamics a large place is occupied by the study of the behavior of mechanical systems with random parameters [1, 5].

The study of these systems presents serious mathematical difficulties [1, 5]. To solve this problem a modified spectral method was applied for the first time. [2, 3]. For other methods, such as method of moment functions, it is quite challenging to obtain a closed system of equations for moment functions [1]. The processes occurring in real systems are treated as extended Markov processes. To describe them, along with the equations of motion, additional relations (equation filters) are made in which the system is put into the Markov form. However, the empirical finding of the necessary filters is quite a complex and not always feasible task. The modified spectral method is free of most of these difficulties, which makes it possible to solve application problems [2, 3].

2. Statement of the problem

In this paper, a modified spectral method is used to study the transient oscillations in the mechanical automatic control systems with random parameters. The equation for the oscillations of such systems is as follows:

\[
\ddot{y} + 2\dot{\varepsilon}\dot{y} + \omega_0^2(1-f(t))y = g(t)
\]  

(1)

Here \(y(t)\) is the displacement, \(\omega_0^2\) is the squared eigenfrequency, \(\varepsilon\) is the damping ratio, \(f(t)\) and \(g(t)\) are parametrical and external effects. Using the modified method of spectral representations [3, 4], we introduce the following integral representations for \(y(t)\) and \(f(t)\):
\[ y(t) = y_0(t) + \int_{-\infty}^{\infty} \psi(\omega, t) Y(\omega) e^{i\omega t} d\omega \quad f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \]  

where \( y_0(t) \) is the mathematical expectation of the process \( y(t) \); \( \psi(\omega, t) \) is an unknown deterministic function; \( Y(\omega), F(\omega) \) are random spectra that satisfy the conditions of orthogonality of the stochastic. The external influence is determined and given in the form \( q(t) = q_0 \cos \Omega t \) (\( q_0, \Omega \) is the amplitude and frequency of the external influence).

3. Theory

After substituting (1) into (2) and compiling the torque ratios we obtain the system of equations for the functions \( y_0(t) \) and \( \psi(\omega, t) \)

\[
\begin{align*}
\dot{y}_0 + 2\dot{y}_0 + \omega_0^2 y_0 - \omega_0^2 \int y(\omega, t) S_{yf}(\omega) d\omega &= q_0 \cos \Omega t, \\
\left[ \ddot{y} + 2(\varepsilon + i\omega) \dot{y} + (\omega_0^2 - \omega^2 + 2i\omega \Omega) y \right] S_{yf}(\omega) - \omega_0^2 y_0 S_f(\omega) &= 0
\end{align*}
\]

Here \( S_{yf}(\omega), S_f(\omega) \) are the spectral densities of the processes \( y(t) \) and \( f(t) \). The solution to this system of equations is found under the following initial conditions

\[
\begin{align*}
\langle y \rangle &= s_0, \quad \langle \dot{y} \rangle = \dot{s}_0, \\
\langle y y \rangle &= 0, \quad \langle \dot{y} y \rangle = 0, t = 0 \\
\langle \dot{y} \dot{y} \rangle &= 0, \quad \langle \dot{y} \dot{y} \rangle = 0
\end{align*}
\]

Here \( s_0 \) and \( \dot{s}_0 \) are the initial values of the mathematical expectation of displacement and velocity. First we shall find the general solution to the homogeneous system of equations

\[
\begin{align*}
\dot{y}_0 + 2\dot{y}_0 + \omega_0^2 y_0 - \omega_0^2 \int y(\omega, t) S_{yf}(\omega) d\omega &= 0, \\
\left[ \ddot{y} + 2(\varepsilon + i\omega) \dot{y} + (\omega_0^2 - \omega^2 + 2i\omega \Omega) y \right] S_{yf}(\omega) - \omega_0^2 y_0 S_f(\omega) &= 0
\end{align*}
\]

The general solution to the mathematical expectation \( y_0(t) \) and to the function \( \psi(\omega, t) \) will be sought in the form

\[
y_0(t) = C e^{\lambda t} \quad \psi(\omega, t) = D(\omega) e^{i\omega t}
\]

where \( \lambda \) is characteristic indicators, \( C \) is constants, \( D(\omega) \) is non-random spectra. The characteristic equation relative to the values of \( \lambda \) has the form

\[
(\lambda + \varepsilon)^2 + \omega^2 = \omega_0^2 \int \frac{S_f(\omega) d\omega}{(\lambda + \varepsilon + i\omega)^2 + \omega_1^2} = 0
\]

Suppose the parametric effect is an ideal narrowband process with the spectral density

\[
S_f(\omega) = \frac{\sigma_f^2}{2} \delta [\omega - \omega_0]
\]

Here, \( \sigma_f^2 \) is the dispersion, \( \omega_0 \) is the carrier frequency of the process. In this case, the characteristic equation is a sextic polynomial.
the coefficients of which have the form:

\[ a_0 = 1, a_1 = 6\delta; a_2 = 12\delta^2 + 2\gamma^2 + 3; a_3 = 4\delta^2(2\gamma^2 + 2\delta^2 + 3) ; a_4 = 12\delta^2(1 + \gamma^2) + \gamma^4 + 3 - \sigma_f^2; \]

\[ a_5 = 2\delta(3 + 4\gamma^2)\delta^2 + \gamma^4 - \sigma_f^2 \]

Then we shall find a particular solution to the system (3) which describes the forced oscillations. Since the right part of the system is \( q(t) = q_0 \cos \Omega t \), the solution will be sought in the following form

\[ y_*(t) = A \cos \Omega t + B \sin \Omega t, \]

\[ \psi_*(\omega, t) = Q(\omega) \cos \Omega t + G(\omega) \sin \Omega t \] (10)

We shall find a particular solution to the displacement \( y_*(t) \). After substituting (10) into (3) and excluding spectra \( D(\omega) \) and \( G(\omega) \) from the obtained ratio we shall derive a system of algebraic equations for constant \( A \) and \( B \)

\[ \alpha_{11} A + \alpha_{12} B = q_0, \]

\[ \alpha_{21} A + \alpha_{22} B = 0 \] (11)

The solution to this system of equations has the form

\[ A = \frac{\alpha_{22} q_0}{\Delta_1}, B = -\frac{\alpha_{21} q_0}{\Delta_1}, \Delta_1 = \alpha_{11} - \alpha_{12} \cdot \alpha_{21}, \] (12)

where

\[ \alpha_{11} = \alpha_{22} = \omega_0^2 - \Omega^2 - \omega_0^2 \cdot I_1, \alpha_{12} = 2\delta \Omega + \omega_0^2 \cdot I_2, \alpha_{21} = -\alpha_{12}, \Delta_1 = \omega_0^2 - \omega^2, \]

\[ I_1 = \int_{-\infty}^{\infty} \left[ (\varepsilon + i\omega)^2 + \omega_0^2 \cdot I_1(\omega) \right] d\omega, I_2 = \int_{-\infty}^{\infty} \frac{2\Omega \cdot (\varepsilon + i\omega)}{\Delta_1} S_2(\omega) d\omega \]

After substituting (12) into the first expression (10) we obtain the following the expression for the displacement and velocity

\[ y_*(\tau) = -\frac{q_0}{\omega_0^2 \cdot \Delta_1} \cdot (\alpha_{11} \cos \gamma \tau + \alpha_{12} \sin \gamma \tau), \]

\[ \dot{y}_*(\tau) = \frac{q_0 \cdot \gamma}{\omega_0^2 \cdot \Delta_1} \cdot (-\alpha_{11} \sin \gamma \tau + \alpha_{12} \cos \gamma \tau), \] (13)

where

\[ \alpha_{11} = \alpha_{22} = \omega_0^2 \cdot (1 - \gamma^2 - I_1), \alpha_{12} = \omega_0^2 \cdot (2\delta \gamma + I_2), \alpha_{21} = -\alpha_{12}, I_1 = \alpha_{12} A \cdot \left( \frac{A_1}{A_1^2 + B_1^2} \right), \]

\[ I_2 = \frac{\sigma_f^2}{A_1^2 + B_1^2} \cdot a_1 = 1 - \gamma^2 - \gamma_1^2, b_1 = 2\delta \gamma, c_1 = 2\delta \gamma, d_1 = 2\gamma, \]

\[ A_1 = (1 - \gamma^2 - \gamma_1^2)^2 + 4 \cdot \left( \sigma_f^2 \cdot 2\gamma_1^2 - \sigma_f^2 \gamma_1^2 \right), B_1 = 2\delta \gamma_1 \cdot (1 + 3\gamma^2 - \gamma_1^2), \]

\[ \gamma = \frac{\Omega}{\omega_0}, \gamma_1 = \frac{\omega_0}{\omega_0}, \delta = \frac{\varepsilon}{\omega_0}, \tau = \omega_0 t. \]
For the constants $C_j$ and spectra $D_j(\omega)$ we substitute the general solution of the system of equations (3) into the initial conditions (4).

4. Discussion of results
The analysis of the processes occurring in systems with random parameters was carried out numerically and is shown in Figure 1-6. All the results are presented in the dimensionless form. In Figure 1-3 the time variation of the mathematical expectation of the displacement is shown, as well as of

![Figure 1](image1)
Figure 1. The expected displacement $y_0(t)$ with the external influence rating $q_0/\omega_0^2 = 2$ and $\gamma = 2$.

![Figure 2](image2)
Figure 2. The expected velocity $y_1(t) = \dot{y}_0(t)$ with the external influence rating $q_0/\omega_0^2 = 2$ and $\gamma = 2$.

![Figure 3](image3)
Figure 3. Phase portrait of the mechanical system with random parameters $y_1(t) = \dot{y}_0(t)$ with the external influence rating $q_0/\omega_0^2 = 2$ and $\gamma = 2$. 
The velocity with the carrier frequency $\gamma_1 = 1.5$, the parametric effect dispersion $\sigma^2 = 0.5$ and the damping parameter $\delta = 0.5$, which were chosen in accordance with the stability terms of the solution to the system of equations (5). The amplitude of external effect $q_0/\omega_0^2$ was taken equal to 2.0 and the frequency $\gamma = 2$. From Figure 1-2 it is evident that the process of forced oscillations establishment is quite fast. In Figure 3 the phase portrait is shown that is characterized by a change in displacement and velocity along damped trajectories, with jumps from one trajectory to another. Next we shall consider the response of a mechanical system to a change in the frequency of the external influence $\gamma$ with

![Figure 4](image-url)  
**Figure 4.** The expected displacement $y_0(t)$ with the external influence rating $q_0/\omega_0^2 = 2$ and $\gamma = 1.5$.

![Figure 5](image-url)  
**Figure 5.** The expected velocity $y_1(t) = \dot{y}_0(t)$ with the external influence rating $q_0/\omega_0^2 = 2$ and $\gamma = 1.5$.

![Figure 6](image-url)  
**Figure 6.** Phase portrait of the mechanical system with random parameters with the external influence rating $q_0/\omega_0^2 = 2$ and $\gamma = 1.5$. 


the carrier frequency of the parametric effects $\gamma_1 = 1.0$ being in the area of the side parametric resonance, the dispersion $\sigma_f^2 = 0.5$, the damping parameter $\delta = 0.5$ and the amplitude of external influence $q_0/\omega_0^2 = 2$. Firstly, the frequency of the external influence changes significantly the nature of dynamic processes (Figure 4-5). Secondly, there is a change in the nature of the trajectories on the phase portraits (Figure 6). It is seen that when the carrier frequency $\gamma_1 = 1.0$ is in the area of the side parametric resonance, another group of damped trajectories occurs on the phase portraits.

5. Conclusion
Based on the obtained results, the following conclusions can be made. First, the mechanical systems with random parameters are very sensitive to their change. Second, the nature of the processes depends essentially on the external influences, which reduce the effect of parametric oscillations. Dynamic phenomena occurring in these systems under external influence are complex, and their analysis requires more detailed discussion and further study.

References
[1] Bolotin V V 1979 Slochasticheskije kolebaniya uprugikh sistem (Random Oscillations of Elastic Systems), Moscow: Nauka p 336
[2] Roev B A 2013 Forced and parametric oscillations. Technical application. LAP LAMBERT Academic Publishing p 128
[3] Roev B and Vinokur A 2016 On the analysis of forced oscillations of systems with two random sources of parametric effects. JVE International LTD Vibroengineering PROCEDIA. OCT 8 pp 147-152
[4] Whitney C 1990 Random Processes in Physical Systems. New York: John Willey p 320
[5] Srinamachchivaya N, Pavlyukevich I and Wedig W 2016 Random Perturbations of Periodically Driven Nonlinear Oscillators Procedia IUTAM 19 pp 91-100