Mathieu equation and Elliptic curve

Wei He\textsuperscript{1} \hspace{1cm} Yan-Gang Miao\textsuperscript{2}

\textit{Department of Physics, Nankai University, Tianjin 300071, China}

Abstract

We present a relation between the Mathieu equation and a particular elliptic curve. We find that the Floquet exponent of the Mathieu equation, for both $q \ll 1$ and $q \gg 1$, can be obtained from the integral of a differential one form along the two homology cycles of the elliptic curve. Certain higher order differential operators are needed to generate the WKB expansion. We provide a fifth order proof.

\textsuperscript{1}weihe@nankai.edu.cn
\textsuperscript{2}miaoyg@nankai.edu.cn
1 Introduction

Mathieu equation was first introduced by E. Mathieu when he studied vibrating elliptical membranes[1]. Its canonical form is

$$\frac{d^2u}{dz^2} + (\lambda - 2q\cos 2z)u = 0. \quad (1)$$

The related modified Mathieu equation is obtained by \(z \rightarrow iz\):

$$\frac{d^2u}{dz^2} - (\lambda - 2q\cosh 2z)u = 0. \quad (2)$$

The Mathieu equation is useful in various mathematics and physics problems. As an example, the separation of variables for the wave equation in the elliptical coordinates leads to the Mathieu equation.

According to the Floquet theory, the solution of the Mathieu equation can be written in the form:

$$u_\nu(z) = e^{i\nu z} f(z). \quad (3)$$

where \(f(z)\) is a function of period \(\pi\), and in general \(\nu\) is a constant independent of \(z\). \(\nu\) is called the Floquet characteristic exponent, it is a function of the constants \(\lambda\) and \(q\). A classical result is that the Floquet exponent can be obtained through the Hill’s determinant. Moreover, if \(\nu\) is an even integer, then the solution \(u(z)\) is a periodic function of period \(\pi\); if \(\nu\) is an odd integer, then the solution \(u(z)\) is a periodic function of period \(2\pi\). In our discussion in this paper, \(u(z)\) is not required to be periodic.

The Mathieu equation has been studied for a long time, for the collections of classical results see nice references [2, 3, 4, 5], and more recent studies in [6, 7].

Another object we study here is a particular elliptic curve. Geometrically the elliptic curve is topologically equivalent to a torus, it is a Riemann surface of genus \(g = 1\). The relation between the Mathieu equation and the elliptic curve naturally arises in the integrable theory. The (modified) Mathieu equation is the Shrödinger equation of the two body Toda system, while the elliptic curve is just the spectral curve of the classical Toda system. See [8] for relevant backgrounds. As an illustration, let us start from the Mathieu operator

$$\mathcal{L} = d_z^2 + \lambda - 2q\cos 2z = d_z^2 + \lambda - q(e^{2iz} + \frac{1}{e^{2iz}}). \quad (4)$$

Substituting \(d_z = x\) and \(q(e^{2iz} - e^{-2iz}) = y\), where \(x, y\) are complex coordinates. Then we have

$$\mathcal{L} = (x^2 + \lambda) \pm \sqrt{y^2 + 4q^2}. \quad (5)$$
The relation
\[ y^2 = (x^2 + \lambda)^2 - 4q^2 \] is nothing else but the elliptic curve we are interested in.

The curve (6) has two independent conjugate cycles \( \alpha \) and \( \beta \), they are canonical basis of the homology class of the torus. According to the general theory of Riemann surfaces, there is a holomorphic differential one form on the torus:
\[ \omega = \frac{dx}{y}, \] and we can construct two periods by integrating \( \omega \) along cycles \( \alpha \) and \( \beta \).
\[ A = \oint_\alpha \omega, \quad B = \oint_\beta \omega. \] Then \( \tau = \frac{B}{A}, \text{Im} \tau > 0 \) is the complex modulus of the elliptic curve.

However, we are interested in a meromorphic one form,
\[ \tilde{\omega} = \frac{x^2 dx}{y}. \] It is related to \( \omega \) by \( \omega = -2 \frac{\partial \tilde{\omega}}{\partial x} + \frac{\beta}{2} \frac{\partial^2 \tilde{\omega}}{\partial y} dx \), the total derivative term will not contribute to contour integrals. The reason for us to study \( \tilde{\omega} \), rather than \( \omega \), is that it is directly related to the Mathieu equation. As a first hint, let \( x^2 + \lambda = 2q \cos 2z \), then we have
\[ \tilde{\omega} = \sqrt{\lambda - 2q \cos 2z} dz. \] This is actually the leading WKB (Wentzel-Kramers-Brillouin) solution of the Mathieu equation. In the next section, we will see that they have an even deeper connection. In physics literature, the elliptic curve (6) is called Seiberg-Witten curve, and \( \tilde{\omega} \) is the Seiberg-Witten differential.\(^3\)

The elliptic curve (6) can be viewed as a double covering of the branched \( x \)-plane. There are four branch points at \( x = (i \sqrt{\lambda + 2q}, i \sqrt{\lambda - 2q}, -i \sqrt{\lambda - 2q}, -i \sqrt{\lambda + 2q}) \), and two branch cuts run between \( (i \sqrt{\lambda + 2q}, i \sqrt{\lambda - 2q}) \) and \( (-i \sqrt{\lambda - 2q}, -i \sqrt{\lambda + 2q}) \). The homology cycle \( \alpha \) of the elliptic curve corresponds to the contour encircling singularities \( (i \sqrt{\lambda + 2q}, i \sqrt{\lambda - 2q}) \), and the homology cycle \( \beta \) of the elliptic curve corresponds to the contour encircling singularities \( (i \sqrt{\lambda - 2q}, -i \sqrt{\lambda - 2q}) \). In the next two sections we will show that, for \( q \ll 1 \) the
\[^3\]In some literatures, as in [9], the elliptic curve is presented in a cubic form \( y^2 = \prod_{i=1}^{3} (x - e_i) \); while in some other literatures, as here, the elliptic curve is presented in a quartic form \( y^2 = \prod_{i=1}^{4} (x - e_i) \). They are equivalent forms of the same curve. The cubic form is obtained from the quartic one by proper coordinates transformations and parameter redefinitions, the result is that one of the zeros \( \tilde{e}_i \) is mapped to infinite. The cubic curve in [9] can be written as \( y^2 = (x^2 - \frac{2q}{16})(x - \frac{1}{2}) \), after identifying \( u = \frac{\hbar^2}{2} \lambda, \Lambda^2 = \frac{\hbar^2}{4} q \) according to [11] and rescaling \( x \to \hbar^2 x, y \to \hbar^2 y \).
Floquet exponent \( \nu \) is given by integrals of differential one forms along the \( \alpha \) cycle on the torus, for \( q >> 1 \) the \( \nu \) is given by integrals of the same differential forms along the \( \beta \) cycle.

The relation between Mathieu equation and elliptic curve we present here is found in our study in [10, 11], about a relation between gauge theories and quantization of integrable systems[12]. It suggests us to develop a WKB formalism to solve the Mathieu equation, as we explain in the next section. In this paper we try to present the problem as a differential equation problem, for relevant physics background, see [12] and [11, 13], and references therein.

## 2 Floquet characteristic exponents from elliptic curve

As the first step, we rewrite the Mathieu equation in a form convenient for WKB expansion. Suppose \( q >> 1 \), we rewrite it as

\[
\frac{\epsilon^2}{2} \frac{d^2 u}{dz^2} + (w - \cos 2z) u = 0,
\]

where \( \epsilon = \frac{1}{q} \), \( w = \frac{\lambda}{2q} \). Then \( \epsilon \) is a small expansion parameter. We expand \( u(z) \) as WKB series:

\[
u(z) = e^{i \int_{z_0}^z p(z')dz'} = e^{i \int_{z_0}^z (\text{ln} p_0 + \sum p_m(z'))dz'}.
\]

Substituting the series expansion (12) into the equation (11), we can solve \( p(z) \) order by order.

Of course, the requirement \( q >> 1 \) is not always satisfied. One may wonder if the results we get can be applied to the case \( q << 1 \). As we will see later, by suitably adjusting \( \lambda \), we actually obtain two convergent series. One series is convergent for \( q >> 1, \frac{\nu^2}{q} << 1 \), surprisingly it is still valid for the region \( q << 1, \nu >> 1 \). Another series is convergent for \( q >> 1, \frac{\nu^2}{q} << 1 \).

The first few recursive relations for \( p_m \) are:

\[
p_0 = \sqrt{2(w - \cos 2z)}, \quad p_1 = \frac{i}{2}(\ln p_0)', \quad p_2 = \frac{1}{8p_0} [2(\ln p_0)'' - (\ln p_0)'^2], \quad p_3 = \frac{i}{2} \left( \frac{p_2}{p_0} \right)',
\]

where the prime denotes \( \frac{2}{dz} \).

Then we extend the Mathieu equation and its periodic solution to the complex domain associated with the elliptic curve. Then \( p(z)dz \) is a differential one form associated to the elliptic curve. Actually, the leading order \( p_0(z)dz \) is proportional to the \( \tilde{\omega} \) we introduced.
above. We are interested in the integrals of $p(z)dz$ along the conjugate homology cycles $\alpha$ and $\beta$ on the elliptic curve, or equivalently, along the contours encircling $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(-\frac{1}{2}\cos^{-1} w, \frac{1}{2}\cos^{-1} w)$ on the $z$-plane. It is the monodromy of the Mathieu function along cycles $\alpha$ and $\beta$ on the torus.

The leading order integrals are related to the complete elliptic integrals of the first and second kind, the result is:

$$\oint_{\alpha} p_0(z)dz = \pi \sqrt{2(w+1)}F\left(-\frac{1}{2}, \frac{1}{2}; \frac{2}{w+1}\right),$$

$$\oint_{\beta} p_0(z)dz = \frac{i\pi}{2}(w-1)F\left(\frac{1}{2}, \frac{1}{2}; \frac{1-w}{2}\right).$$

(14)

As $p_1, p_3$ are total derivatives, the contour integrals of them are all zero

$$\oint_{\alpha,\beta} p_{2m+1}(z)dz = 0, \quad m = 0, 1,$$

(15)

and

$$\oint_{\alpha,\beta} p_2dz = \frac{1}{8\sqrt{2}}\oint_{\alpha,\beta} \frac{\sin^2 2z - 4w \cos 2z + 4}{(w - \cos 2z)^{3/2}}dz$$

$$= -\frac{1}{12\sqrt{2}}\oint_{A,B} \frac{\cos 2z}{(w - \cos 2z)^{3/2}}dz$$

$$= \frac{1}{12}(2wd_w^2 + d_w) \oint_{\alpha,\beta} \sqrt{2(w - \cos 2z)}dz,$$

(16)

where $d_w = \frac{d}{dw}$. We have simplified the integral by discarding some total derivative terms, this method was first used in [13]. In a similar way we find

$$\oint_{\alpha,\beta} p_4dz = \frac{1}{25}\left(\frac{28}{45}w^2d_w^4 + \frac{8}{3}wd_w^3 + \frac{5}{3}d_w^2\right)\oint_{\alpha,\beta} p_0dz.$$

(17)

We can proceed the same technique to obtain the differential operators for higher order $p_m$, by discarding total derivative terms and simplifying the expression as far as possible. We call these differential operators generating differential operators. Acting these differential operators on $\oint p_0dz$, we can get higher order contour integrals, they can be written as combinations of the hypergeometric functions by using the formula:

$$\frac{d}{dz} F(a, b, c; z) = \frac{ab}{c} F(a + 1, b + 1, c + 1; z).$$

(18)

As a demonstration, the expression for $\oint p_2dz$ can be found in [11], and $\oint p_4dz$ is even more lengthy. We can get series expansions near a suitable value of $w$ from these hypergeometric functions. However, it is much simpler to get the series expansion of $p_0$ first and then to act the generating differential operators on this series.
In principle, all higher order generating differential operators can be determined by WKB relations. However it turns out that the calculations become very involved and it is hard to determine whether the expressions can be simplified further by discarding total derivative terms. Based on some observation on $p_0, p_1, p_2, p_3, p_4$, we make a conjecture for higher order differential operators.

**Claim 1:** In general we have

$$\oint_{\alpha,\beta} p_{2m+1}dz = 0,$$

$$\oint_{\alpha,\beta} p_{2m}dz = (c_{m,m}w^m d_w^{2m} + c_{m,m-1}w^{m-1} d_w^{2m-1} + \cdots + c_{m,1}w d_w^{m+1} + c_{m,0} d_w^m) \oint_{\alpha,\beta} p_0dz, \quad (19)$$

where $m = 0, 1, 2, \cdots$, and $c_{m,i} (i = 0, 1, \cdots, m)$ are numerical coefficients.

Now we will state the relation between the monodromy of the Mathieu function along $\alpha, \beta$ and its Floquet exponent. The asymptotic expansions of hypergeometric function $F(a, b, c; z)$ are quite different for $z = 0, 1, \infty$. For example, let us look at the asymptotic behavior of the leading order results $\oint_{\alpha,\beta} p_0dz$. At $w = \infty$, we have

$$\sqrt{2(w+1)}F(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2}{w+1}) = \sqrt{2w}[1 - \frac{1}{4}(\frac{1}{2w})^2 - \frac{15}{64}(\frac{1}{2w})^4 - \frac{105}{256}(\frac{1}{2w})^6 + \cdots],$$

$$\frac{1}{2}(w-1)F(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}) = \frac{1}{\pi} \sqrt{2w}[(\ln 2w - 2 + 2\ln 2) + \frac{1}{4}(1 - 2\ln 2 - \ln w)(\frac{1}{2w})^2 + \frac{1}{128}(47 - 60\ln 2 - 30\ln 2w)(\frac{1}{2w})^4 + \cdots]. \quad (20)$$

While at $w \sim 1$, with $\sigma = w - 1$, we have

$$\sqrt{2(w+1)}F(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2}{w+1}) = \frac{4}{\pi} + \frac{(1 + 2\ln 2 - \ln \sigma)\sigma}{2\pi} + \frac{(3 - 4\ln 2 + 2\ln \sigma)\sigma^2}{64\pi} - \frac{3(2 - 2\ln 2 + \ln \sigma)\sigma^3}{512\pi} + \cdots,$$

$$\frac{1}{2}(w-1)F(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}) = \frac{1}{2}\sigma - \frac{1}{32}\sigma^2 + \frac{3}{512}\sigma^3 - \frac{25}{16384}\sigma^4 + \cdots. \quad (21)$$

It turns out that the asymptotic expansions of $\oint_{\alpha,\beta} p_0dz$ which are only powers of $w$ or $\sigma$ are related to the Floquet exponent of the Mathieu equation.

**Claim 2:** The contour integral of $p(z)$ along the $\alpha$-cycle gives the Floquet exponent

$$\nu = \frac{1}{\pi} \oint_{\alpha} p(z)dz, \quad (22)$$

for the case $q >> 1, \frac{\nu}{q} <<< 1 (or q << 1, \nu >> 1)$, the hypergeometric functions should be expanded near $\lambda >> q >> 1$, i.e. $w \sim \infty$. 

5
Claim 3: The contour integral of $p(z)$ along the $\beta$-cycle gives the Floquet exponent

$$\nu = \frac{1}{i\pi} \oint_{\beta} p(z)dz,$$

for the case $q >> 1, \frac{\nu^2}{q} << 1$, the hypergeometric functions should be expanded near $\lambda \sim 2q$, i.e. $w \sim 1$.

In this way, we can get the function $\nu = \nu(w, \epsilon)$ as series expansion of $\epsilon$ and $w$. In order to obtain the eigenvalue $\lambda$, we need to reverse the function $\nu = \nu(w, \epsilon)$ to get $w = w(\nu, \epsilon)$.

3 5th order proof

In order to prove the validity of our claims, we have to show that the asymptotic expansions of $\nu$ given by the contour integrals are indeed the same as results known in literatures. This has been successfully done in [11] for the first three orders $\epsilon^{-1}p_0 + \epsilon p_2 + \epsilon^3 p_4$. In this section, we will show how to determine the generating differential operators of $p_6$ and $p_8$, following the Claim 1,2,3, which would be very involved for manual calculation.

Let us start from a classical result of the asymptotic expansion for $\lambda_{\nu}$:

$$\lambda_{\nu} = \nu^2 + \frac{1}{2(\nu^2 - 1)}q^2 + \frac{5\nu^2 + 7}{32(\nu^2 - 1)^3(\nu^2 - 4)}q^4 + \frac{9\nu^4 + 58\nu^2 + 29}{64(\nu^2 - 1)^5(\nu^2 - 4)(\nu^2 - 9)}q^6 + \cdots$$

(24)

It often states in the literature that this asymptotic expansion is valid for $q << 1$ and $\nu \geq 4$. Actually, it is also valid in the parameter region $q >> 1$ and $\frac{\nu^2}{q} << 1$, this makes our WKB method applicable. Then we reverse the series (24) to obtain the series for $\nu$ as a function of $\lambda, q$. This can be easily achieved with the help of computer programs, for example the Mathematica software. We can trust the inverse results up to the order $q^6$.

The inverse series gives

$$\nu = \sqrt{\lambda} - \frac{q^2}{4} \lambda^{-3/2} - \frac{q^2}{4} \lambda^{-5/2} - \frac{15q^4}{64} \lambda^{-7/2} - \left(\frac{q^2}{4} + \frac{35q^4}{32}\right) \lambda^{-9/2} - \left(\frac{q^2}{4} + \frac{273q^4}{256}\right) \lambda^{-11/2} - \left(\frac{q^2}{4} + \frac{105q^6}{256}\right) \lambda^{-13/2} - \left(\frac{q^2}{4} + \frac{273q^4}{128}\right) \lambda^{-15/2} - \left(\frac{q^2}{4} + \frac{105q^6}{64}\right) \lambda^{-17/2} - \left(\frac{q^2}{4} + \frac{65637q^4}{256}\right) \lambda^{-19/2} - \left(\frac{q^2}{4} + \frac{722007q^6}{256}\right) \lambda^{-21/2} + O(\lambda^{-23/2}).$$

(25)
We have cut off the $\lambda$ expansion at $O(\lambda^{-23/2})$, and discarded all the $q$ expansion terms beyond the scope of the accuracy of (24). Rewrite the inverse series in $w$ and $\epsilon$:

$$\nu = \frac{1}{\epsilon} \left[ (2w)^{1/2} - \frac{1}{4} (2w)^{-3/2} - \frac{15}{64} (2w)^{-7/2} - \frac{105}{256} (2w)^{-11/2} \right] \epsilon^2 [\frac{1}{4} (2w)^{-5/2} - \frac{35}{32} (2w)^{-9/2} - \frac{1155}{256} (2w)^{-13/2}]$$

$$+ \epsilon^3 [\frac{1}{4} (2w)^{-7/2} - \frac{273}{64} (2w)^{-11/2} - \frac{5005}{128} (2w)^{-15/2}]$$

$$+ \epsilon^5 [\frac{1}{4} (2w)^{-9/2} - \frac{33}{2} (2w)^{-13/2} - \frac{42185}{128} (2w)^{-17/2}]$$

$$+ \epsilon^7 [\frac{1}{4} (2w)^{-11/2} - \frac{4147}{64} (2w)^{-15/2} - \frac{722007}{256} (2w)^{-19/2}] \right].$$

(26)

It is straightforward to expand the integral $\int \alpha_p (\epsilon^1 p_0 + \epsilon^2 p_2 + \epsilon^3 p_4) dz$ at $w = \infty$, and compare with (26). They indeed match [10, 11].

Interestingly, the expansion (26) is precise enough to further determine the differential operator for $\epsilon^5 p_6$. According to our first claim, we set

$$\int \alpha_p p_6 dz = (c_{3,3} w^3 d_w^6 + c_{3,2} w^2 d_w^5 + c_{3,1} w d_w^4 + c_{3,0} d_w^3) \int \alpha_p p_0 dz.$$

(27)

Expanding $\int \alpha_p p_0 dz = 2(w + 1) F(-1/2, 1/2, 1; 2/(w + 1))$ at $w = \infty$ as in (20), and acting the 6th order differential operator (27) on the series, we get

$$\int \alpha_p p_6 dz = -\frac{3}{8} (315 c_{3,3} - 70 c_{3,2} + 20 c_{3,1} - 8 c_{3,0})(2w)^{-5/2}$$

$$- \frac{105}{32} (1287 c_{3,3} - 198 c_{3,2} + 36 c_{3,1} - 8 c_{3,0})(2w)^{-9/2}$$

$$- \frac{10395}{512} (3315 c_{3,3} - 390 c_{3,2} + 52 c_{3,1} - 8 c_{3,0})(2w)^{-13/2}$$

$$- \frac{225225}{2048} (6783 c_{3,3} - 646 c_{3,2} + 68 c_{3,1} - 8 c_{3,0})(2w)^{-17/2}$$

$$- \frac{72747675}{131072} (12075 c_{3,3} - 966 c_{3,2} + 84 c_{3,1} - 8 c_{3,0})(2w)^{-21/2}$$

$$+ \ldots.$$  

(28)

In order to determine the four coefficients $c_{3,i}$, we have to match (28) with terms of order $\epsilon^5$ in formula (26). A crucial point is that although there are only three nonzero terms in (26), limited by the accuracy of (24) to the $q^6$ order, the leading term $w^{-5/2}$ is absent in (26), this fact enables us to determine the four coefficients in (28). We finally arrive at

$$\int \alpha_p p_6 dz = \frac{1}{26} \left( \frac{124}{945} w^3 d_w^6 + \frac{158}{105} w^2 d_w^5 + \frac{153}{35} w d_w^4 + \frac{41}{14} d_w^3 \right) \int \alpha_p p_0 dz.$$

(29)

Terms of order $w^{-21/2}$ and higher in (28) are superfluous for the determination of $c_{3,i}$. After $c_{3,i}$ are determined, these higher order terms can be subsequently determined, too.
The $\epsilon^7 p_8$ order contour integral has five unknown coefficients

$$\oint_{\alpha, \beta} p_8 dz = (c_{4,4} w^4 d^8_w + c_{4,3} w^3 d^7_w + c_{4,2} w^2 d^6_w + c_{4,1} w d^5_w + c_{4,0} d_4^1) \oint_{\alpha, \beta} p_0 dz. \quad (30)$$

In order to determine the coefficients we need at least four terms in the $\epsilon^7$ order in (26), as the coefficient for $\epsilon^7 (2w)^{-7/2}$ should vanish. Therefore, we need the $q^8$ order contribution for $\lambda_\nu$. Fortunately, it has been worked out in [7]:

$$\lambda^{(q^8)}_\nu = \frac{1469\nu^{10} + 9144\nu^8 - 140354\nu^6 + 64228\nu^4 + 827565\nu^2 + 274748}{8192(\nu^2 - 16)(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)^7} q^8. \quad (31)$$

It extends the $\epsilon^7$ order terms in (26) to

$$\epsilon^7[-\frac{1}{4} (2w)^{-11/2} - \frac{4147}{64} (2w)^{-15/2} - \frac{722007}{256} (2w)^{-19/2} - \frac{1000684685}{16384} (2w)^{-23/2}]. \quad (32)$$

This determines $\epsilon^7 \oint p_8 dz$ as

$$\oint_{\alpha, \beta} p_8 dz = \frac{1}{2^4 \lambda^{(q^8)}_\nu} \left( \frac{127}{4725 \times 2^3} w^4 d^8_w + \frac{13}{175} w^3 d^7_w + \frac{517}{63 \times 2^4} w^2 d^6_w + \frac{9539}{945 \times 2^3} w d^5_w + \frac{15229}{135 \times 2^7} d_4^1 \right) \oint_{\alpha, \beta} p_0 dz. \quad (33)$$

We have shown that the Claim 1 is correct up to the 5th order. Moreover, by using Claim 2 for $q << 1$, we have determined all the coefficients in the generating differential operators of $p_6$ and $p_8$. Then it is straightforward to expand $\oint_p p dz$ to the $\epsilon^7$ order, near $w \sim 1$, to obtain the Floquet index $\nu = \nu(w, \epsilon)$ for $q >> 1$. After reverse the series $\nu = \nu(w, \epsilon)$ to $w = w(\nu, \epsilon)$ and rewrite it in $\nu, \lambda, q$, we get

$$\lambda_\nu = 2q - 4\nu \sqrt{q} + \frac{4\nu^2 - 1}{2^3} + \frac{4\nu^3 - 3\nu}{2^6 \sqrt{q}} + \frac{80\nu^4 - 136\nu^2 + 9}{2^{12} q} + \frac{528\nu^5 - 1640\nu^3 + 405\nu}{2^{16} q^{3/2}} + \frac{2016\nu^6 - 10080\nu^4 + 5089\nu^2 - 243}{2^{19} q^2} + \frac{33728\nu^7 - 249872\nu^5 + 276004\nu^3 - 41607\nu}{2^{24} q^{5/2}} + \frac{2403072\nu^8 - 24881920\nu^6 + 45534368\nu^4 - 16087536\nu^2 + 506979}{2^{31} q^3} + \frac{44811520\nu^9 - 620967168\nu^7 + 1724770656\nu^5 - 1152647184\nu^3 + 130610637\nu}{2^{36} q^{7/2}}. \quad (34)$$

We keep only terms consistent with the accuracy limit $\epsilon^7 p_8$. This is the classical result of the Mathieu equation for $q >> 1$(See formula (20.2.30) in [5]). This finishes the proof of our Claim 3.
4 Concluding remarks

We show that the Mathieu equation is closely related to an elliptic curve, therefore there is a geometric structure for the Mathieu equation which is not captured by asymptotic analysis. The Floquet exponent of the Mathieu equation can be derived from integrals of certain differential forms along the homology cycles of the curve. These differential forms are determined by the WKB procedure. Integrals along each homology cycle give an asymptotic expression for the Floquet exponent expanded at a specific point, the inverse series gives the corresponding eigenvalue. Integrals along all homology cycles \( \alpha \) and \( \beta \) give the complete asymptotic expansions (24) and (34) for the eigenvalue.

The appearance of Riemann surfaces associated with differential equations is quite familiar in the theory of integrable models, the Riemann surfaces are the spectral curves of the classical integrable system while the differential equations are the Shrödinger equation of the same system. The (modified) Mathieu equation we discuss here is simply the two body Toda system, the eigenvalue formulae (24) and (34) are states with large quantum numbers(expanded at \( \lambda >> 1 \)) and states with small quantum numbers(expanded at \( \lambda \sim 2q \)), respectively[10] [11]. In fact, the Lax matrix \( L(z) \) of the classical two body periodic Toda system is[14]

\[
L(z) = \begin{pmatrix}
\begin{array}{cc}
p_1 e^{q_1-q_2} & e^{q_1-q_2} + z^{-1} e^{q_2-q_1} \\
pe^{q_2-q_1} & p_2
\end{array}
\end{pmatrix}
\]  

(35)

where \( p_1, p_2 \) are momentum and \( q_1, q_2 \) are coordinates of the particles, \( z \) is the complex spectral parameter. The spectral curve is obtained by the determinant

\[
det(\mu I - L(z)) = 0.
\]  

(36)

It’s of the form

\[
2t(\mu) - (z + z^{-1}) = 0,
\]  

(37)

where \( 2t(\mu) = \mu^2 - (p_1 + p_2)\mu + f(q_1 - q_2, p_1 p_2) \) is a polynomial of degree two. By a change of variables, \( x = \mu, y = z - t(\mu) \), the curve becomes

\[
y^2 = t^2(x) - 1.
\]  

(38)

This is the same curve as (6) modula a further coordinates change. Now let us substitute \( \mu \) by the differential \( \partial_z \), then formally we can get a second order differential equation through

\[
det(\partial_z \mu I - L(z)).
\]  

(39)

It’s easy to see that after some proper coordinates changes this is the Mathieu differential operator. So in some sense, the Mathieu equation defines a “quantum torus”. Actually,
Talalaev has constructed the “quantum spectral curve” for general integrable systems along this way [15].

It is possible that the relation we present here is just a particular case of a general picture. As an example, the spectral curve for the two body elliptic Calogero-Moser integrable system is an elliptic curve closely related to (6) (elliptic curves always can be written in the Weierstrass form), and its quantization leads to the Lamé equation. See a recent discussion in [16].

Another example is the direct generalization of the case we present here, the spectral curve for the $N$-body $A_N$ periodic Toda chain is a hyperelliptic curve of genus $g = N - 1$, with $N \geq 3$. The Gutzwiller’s quantization scheme of periodic Toda chain introduces a rather involved Bethe-like quantization condition which involves both the Floquet exponents and the integrals of motion [17]. Recently in [18] the quantization condition has been rewritten in a functional form that only involves the Floquet exponents of the associated Hill’s determinant. We wonder if the idea presented here can be generalized to higher genus curves and provide a solution to the eigenvalue problem of periodic Toda chain. Note that for a $N$-particle Toda system, there are $N - 1$ independent Floquet exponents $\nu_i, i = 1, 2, \cdots, N - 1$ (the condition $\sum_{i=1}^{N} \nu_i = 0$ just reduces the center of mass motion, or is the traceless condition for the SU($N$) group), and for the associated hyperelliptic curve there are $2(N - 1)$ independent homology cycles $\alpha_i$ and $\beta_i$. There are also a meromorphic one form and its WKB descendants on the curve. The differential forms involve exactly $N - 1$ coefficients $I_k$, with $k = 2, 3, \cdots, N$, which are the integrals of motion of Toda chain. Among the integrals of motions $I_2$ is interpreted as energy while $I_k$ for $k \geq 3$ have no physical interpretation. There is evidence that the Floquet exponents $\nu_i$ are given by integrals of these differential forms along the homology cycles on the hyperelliptic curve [19]. This is enough to determine the functional relations between the Floquet exponents and the integrals of motion $\nu_i = \nu_i(I_2, I_3, \cdots, I_N)$. There are $2(N - 1)$ asymptotic expansion points at $I_k \gg 1$ and at the dual points $I_k \sim I_k^{(0)}$. The critical values $I_k^{(0)}$ are determined by the Chebyshev polynomial [20]. If the $N - 1$ integral cycles are chosen as $(\alpha_{(i)}, \beta_{(j)})$, satisfying $\{i\} \subseteq \{1, 2, \cdots N - 1\}, \{j\} \subseteq \{1, 2, \cdots N - 1\}$ and $\{i\} \cup \{j\} = \{1, 2, \cdots, N - 1\}, \{i\} \cap \{j\} = \emptyset$, then functions $\nu_{(i)} = \nu_{(i)}(I_2, I_3, \cdots, I_N)$ have asymptotic expansions at $I_k \gg 1$, while $\nu_{(j)} = \nu_{(j)}(I_2, I_3, \cdots, I_N)$ have asymptotic expansions at $I_k - I_k^{(0)} \ll 1$. By reversing the Floquet exponents $\nu_i = \nu_i(I_2, I_3, \cdots, I_N)$ we obtain the eigenvalues $I_k = I_k(\nu_1, \nu_2, \cdots, \nu_{N-1})$. 
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