MOBILE ICOSAPODS

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Abstract. Pods are mechanical devices constituted of two rigid bodies, the base and the platform, connected by a number of other rigid bodies, called legs, that are anchored via spherical joints. It is possible to prove that the maximal number of legs of a mobile pod, when finite, is 20. In 1904, Borel designed a technique to construct examples of such 20-pods, but could not constrain the legs to have base and platform points with real coordinates. We show that Borel’s construction yields all mobile 20-pods, and that it is possible to construct examples with all real coordinates.

Introduction

A multipod is a mechanical linkage consisting of two rigid bodies, called the base and the platform, and a number of rigid bodies, called legs, connecting them. Each leg is attached to base and platform with spherical joints (see Figure 1), so platform points are constrained to lie on spheres — the center of each sphere is then the base point connected to the respective leg. If the platform can move respecting the constraints imposed by the legs we say that the multipod is mobile.

Note that multipods are also studied within Rigidity Theory as so-called body-bar frameworks [WW87], as two rigid bodies (platform and base) are connected by multiple bars (legs). Mobile multipods correspond to flexible body-bar frameworks, whose study is of great practical interest for e.g. protein folding [SSW14].

We can model the possible configurations of a multipod using direct isometries of $\mathbb{R}^3$, by associating to every configuration the isometry mapping it to a fixed configuration. At this point one can consider a motion, namely a one-dimensional set of direct isometries, and try to construct a multipod moving according to the fixed motion. This approach has a long history; there are motions allowing multipods with infinitely many legs, but among those that allow only a finite number of legs, the maximal number is 20. This was proved by Schoenflies, see Remark 2.6. Borel proposed a construction for icosapods (namely multipods with 20 legs) leading to linkages whose motion is line-symmetric, i.e. whose elements are involutions, namely rotations by 180° around a line (see Figure 1).

This paper provides two results on icosapods. First, we show that all mobile icosapods (with very mild restrictions) are instances of Borel’s construction. Second, we exhibit a mobile icosapod that is an instance of Borel’s construction (Borel, in fact, obtained equations for the base and platform points of a mobile icosapod,
Figure 1. Base points (pink) and platform points (yellow) of an icosapod. Notice that for every leg there is a symmetric one obtained by rotating the first one by $180^\circ$ around the axis $\ell$. The symmetry reverses the role of base and platform points.

but could not prove the existence of solutions in $\mathbb{R}^3$). The second part is closely related to the theory of quartic spectrahedra, which has been studied in [ORSV15].

Section 1 provides an historical overview of line-symmetric motions. In Section 2 we set up the formalism and the objects that are needed for our approach to the problem; in particular we recall a compactification of the group of direct isometries that has already been used by the authors to deal with problems on multipods, and we show how the constraints imposed by legs can be interpreted as a duality between the space of legs and the space of direct isometries. In Section 3 we use these tools to prove that, under certain generality conditions, mobile icosapods admit line-symmetric motions. In Section 4 we show how it is possible to construct example of mobile icosapods employing results in the theory of quartic spectrahedra.

1. Review on line-symmetric motions

Krames [Kra37b] studied special one-parametric motions, obtained by reflecting the moving system $\varsigma$ in the generators of a ruled surface in the fixed system $\Sigma$. This ruled surface is called the base surface for the motion. He showed some remarkable properties of these motions (see [Kra37b]), which led him to name them Symmetrische Schrütung (in German). This name was translated to English as symmetric motion by Tölke [Tö175]. Krames motion or line-symmetric motion by
Bottema and Roth [BR79] page 319]. As each Symmetrische Schrotung has the additional property that it is equal to its inverse motion (cf. Krames [Kra37c] page 415), it could also be called involutory motion. In this paper we use the name line-symmetry as it is probably the most commonly used term in today’s kinematic community.

Further characterizations of line-symmetric motions (beside the cited one of Krames [Kra37b]) were given by Tölke [Töl75], Bottema and Roth [BR79, Chapter 9, § 7], Selig and Husty [SH11] and Hamann [Ham11]. If one uses the so-called Study parameters \((e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)\) to describe isometries, then it is possible to characterize line-symmetric motions algebraically in the following way. Given such a motion, there always exist a Cartesian system of coordinates, or frame \((o; x, y, z)\) for the moving system \(\varsigma\) and a Cartesian frame \((O; X, Y, Z)\) for the fixed system \(\Sigma\) so that \(e_0 = f_0 = 0\) holds for the elements of the motion. With this choice of coordinates, the latter are rotations by 180° around lines; the Study coordinates \((e_1 : e_2 : e_3 : f_1 : f_2 : f_3)\) of these isometries coincide with the Plücker coordinates of the lines.

1.1. Historical results on line-symmetric motions with spherical paths.
In this paper we study line-symmetric motions that are solutions to the still unsolved problem posed by the French Academy of Science for the Prix Vaillant of the year 1904 (cf. [Hus00]): "Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths." Borel and Bricard were awarded the prize for their papers [Bor08] and [Bri06] containing partial solutions, and therefore this is also known as the Borel Bricard (BB) problem.

1.1.1. Krames’s results. Krames [Kra37c, Kra37d, Kra37a] studied some special motions already known to Borel and Bricard in more detail and stated the following theorem [Kra37c, Satz 6]:

**Theorem 1.1.** For each line-symmetric motion, that contains discrete, 1 or 2-dimensional spherical paths, the set \(f\) of points with spherical trajectories is congruent (direct isometry) to the set \(\mathfrak{F}\) of corresponding sphere centers.

Moreover Krames noted in [Kra37c] page 409 that[^1] "...most of the solutions given by Borel and Bricard are line-symmetric motions. In each of these motions both geometers detected this circumstance by other means, without using the above mentioned result" (Theorem 1.1). In the following we will take a closer look at the papers [Bor08, Bri06], which shows that the latter statement is not entirely correct.

1.1.2. Bricard’s results. Bricard studied these motions in [Bri06, Chapitre VIII]. His first result in this context (see end of [Bri06] § 32, page 70) reads as follows (adapted to our notation):

Dans toutes les solutions auxquelles on sera conduit, les figures liées \(\mathfrak{F}\) et \(f\) seront évidemment égales et semblablement placées par rapport aux deux trièdres \((O; X, Y, Z)\) et \((o; x, y, z)\).

[^1]: The following extract as well as Theorem 1.1 has been translated from the original German by the authors.
In the remainder of [Bri06, Chapitre VIII] he discussed some special cases, which also yield remarkable results, but he did not give further information on the general case.

1.1.3. Borel’s results. Borel discussed in [Bor08, Case Fb] exactly the case $e_0 = f_0 = 0$ and he proved in [Bor08, Case Fb1] that in general a set of 20 points are located on spherical paths but without giving any result on the reality of the 20 points. Moreover he studied two special cases in Fb2 and Fb3.

Borel did not mention the geometric meaning of the assumption $e_0 = f_0 = 0$. He only stated at the beginning of case F [Bor08, page 95] that the moving frame $(\alpha; x, y, z)$ is parallel to the frame obtained by a reflection of the fixed frame $(O; X, Y, Z)$ in a line. This corresponds to the weaker assumption $e_0 = 0$. He added that this implies the same consequences as already mentioned in [Bor08, page 47, case C], which reads as follows (adapted to our notation):

\[\ldots \text{dans le cas où les trièdres sont symétriques par rapport à une droite, si deux courbes sont représentées par des équations identiques, l’une en } X, Y, Z, \text{ l’autre en } x, y, z, \text{ elles sont symétriques par rapport à cette droite.}\]

But Borel did not mention, neither in case Fb1 nor in his conclusion section, that $f$ with $\#f = 20$ is congruent to $\widehat{f}$ (contrary to other special cases e.g. Fb3, where the congruence property is mentioned explicitly.)

1.2. Review of line-symmetric self-motions of hexapods. We denote the platform points of the $i$-th leg in the moving system $\varsigma$ by $p_i$ and its corresponding base points in the fixed system $\Sigma$ by $P_i$.

For a generic choice of the geometry of the platform and the base as well as the leg lengths $d_i$ the hexapod can have up to 40 configurations. Under certain conditions it can also happen that the direct kinematic problem has no discrete solution set but an $n$-dimensional one with $n > 0$. Clearly these so-called self-motions of hexapods are solutions to the BB problem.

In practice hexapods appear in the form of Stewart-Gough manipulators, which are 6 degrees of freedom parallel robots. In these machines the leg lengths can be actively changed by prismatic joints and all spherical joints are passive.

Moreover a hexapod (resp. Stewart-Gough manipulator) is called planar if the points $p_1, \ldots, p_6$ are coplanar and also the points $P_1, \ldots, P_6$ are coplanar; otherwise it is called non-planar. In the following we review those papers where line-symmetric self-motions of hexapods are reported.

1.2.1. Non-planar hexapods with line-symmetric self-motions. Line-symmetric motions with spherical paths already known to Borel [Bor08] and Bricard [Bri06] (and also discussed by Krames in [Kra37c, Kra37a]) were used by Husty and Zsombor-Murray [HZM94] and Hartmann [Har95] to construct examples of (planar and non-planar) hexapods with line-symmetric self-motions.

Point-symmetric hexapods with congruent platform and base possessing line-symmetric self-motions were given in [Naw14b, Theorem 11]. Further non-planar
hexapods with line-symmetric self-motions can be constructed from overconstrained pentapods with a linear platform [Naw15].

1.2.2. Planar hexapods with line-symmetric self-motions. All self-motions of the original Stewart-Gough manipulator were classified by Karger and Husty [KH98]. Amongst others they reported a self-motion with the property $e_0 = 0$ (see [KH98, page 208, last paragraph]), "which has the property that all points of a cubic curve lying in the plane ... and six additional points out of this plane have spherical trajectories. This seems to be a new case of a BB motion, not known so far."

Based on this result, Karger [Kar08a, Kar08b] presented a procedure for computing further "new self-motions of parallel manipulators" of the type $e_0 = 0$, where the points of a planar cubic $c$ have spherical paths.

Another approach was taken by Nawratil in his series of papers [Naw13, Naw11a, Naw12, Naw11b], by determining the necessary and sufficient geometric conditions for the existence of a 2-dimensional motion such that three points in the $xy$-plane of $\Sigma$ move on three planes orthogonal to the $XY$-plane of $\Sigma$ (3-fold Darboux condition) and two planes orthogonal to the $xy$-plane of $\Sigma$ slide through two fixed points located in the $XY$-plane of $\Sigma$ (2-fold Mannheim condition). It turned out that all these so-called type II Darboux-Mannheim motions are line-symmetric. Moreover a geometric construction of a 12-parametric set of planar Stewart-Gough platforms (cf. [Naw11b, Corollary 5.4]) with line-symmetric self-motions was given. It was also shown that the algorithm proposed by Karger in [Kar08a, Kar08b] yields these solutions.

While studying the classic papers of Borel and Bricard for this historical review we noticed that the solution set of the BB problem mentioned in the last two paragraphs was already known to these two French geometers; cf. [Bor08, Case Fb3] and [Bri06, Chapter V] (already reported by Bricard in [Bri01, page 21]). But in contrast to the above listed approaches (of Karger and Nawratil) both of them assumed that the motion with spherical trajectories is line-symmetric. Each of them additionally discovered one more property:

- Borel pointed out that there exist a further 8 points (all 8 can be real) with spherical trajectories. This set of points splits in four pairs, which are symmetric with respect to the carrier plane of the cubic $c$.
- Bricard showed the following: If we identify the congruent planar cubics of the platform and the base, i.e. $c = C$, then the tangents in a corresponding point pair $P$ and $p$ with respect to $c = C$ intersect each other in a point of the cubic $c = C$ ($P$ and $p$ form a so-called Steinerian couple).

Bricard communicated his result (published in [Bri01, page 21]) to Duporcq, who gave an alternative reasoning in [Dup01], which sank into oblivion over the past 100 years. Only a footnote in the conclusion section of Borel’s work [Bor08] points to Duporcq’s proof (but not to the original work of Bricard [Bri01]), which is based on the following remarkable motion (see Figure 2):

Let $P_1, \ldots, P_6$ and $p_1, \ldots, p_6$ be the vertices of two complete quadrilaterals, which are congruent. Moreover the vertices are labelled in a way that $p_1$ is the opposite
vertex of \( P_i \) for \( i \in \{1, \ldots, 6\} \). Then there exist a two-parametric line-symmetric motion where each \( p_i \) moves on a sphere centered at \( P_i \).

It can be easily checked that this configuration of base and platform points corresponds to an architecturally singular hexapod (e.g. [RM98] or [Kar03]). As architecturally singular manipulators are redundant we can remove any leg — w.l.o.g. we suppose that this is the sixth leg — without changing the direct kinematics of the mechanism. Therefore the resulting pentapod \( P_1, \ldots, P_5 \) and \( p_1, \ldots, p_5 \) also has a two-parameter, line-symmetric, self-motion.

Note that this pentapod yields a counter-example to Theorem 4.2 of [GNS15c] and as a consequence also the work [NS15] is incomplete as it is based on this theorem. For the erratum to [GNS15c] please see [GNS15d] and the addendum to [NS15] is given in [NS16].

Remark 1.2. If we assume for this pentapod that the line \([P_1P_2]\) is the ideal line of the fixed plane (so \([p_4p_5]\) is the ideal line of the moving plane) then we get exactly the conditions found in [Naw13], in which the points and ideal points of the plane’s normals must fulfill in order to get a type II Darboux-Mannheim motion. Note that \( P_2, P_3 \) can also be complex conjugates (so \( p_5, p_6 \) are complex conjugate too).

1.2.3. Computer search for mobile hexapods. In [GS09], Geiss and Schreyer describe a rather non-standard way to find mobile hexapods. They set up an algebraic system of equations equivalent to mobility, and then try random candidates with coordinates in a finite field of small size with a computer. After collecting statistical data indicating the existence of a family of real mobile hexapods, they try a (computationally more expensive) lifting process in the most promising cases. The method is extremely powerful and could still be used for finding new families of mobile hexapods. However, the family reported in [GS09] can be seen as an instance of Borel’s family \( F_b1 \). The line symmetry is not apparent because the method starts by guessing 6 legs, which may not form a line symmetric configuration; only if one adds the remaining 14 legs, or at least all real legs among them, one obtains a symmetric configuration. The question posed in Problem 5 in [GS09] is easily answered from this viewpoint: two legs have the same lengths because they are conjugated by line symmetry.

2. ISOMETRIES, LEGS AND BOND THEORY

This section illustrates the concepts and techniques that will be needed to carry out our analysis. From now on, by an \( n \)-pod we mean a triple \((\vec{p}, \vec{P}, \vec{d})\) where \( \vec{p}, \vec{P} \) are \( n \)-tuples of vectors in \( \mathbb{R}^3 \) (respectively, platform and base points), and \( \vec{d} \) is an
\(n\)-tuple of positive real numbers (representing leg lengths). We describe the set of admissible configurations of a given pod \(\Pi\) as the set of all \(\sigma \in \text{SE}_3\) (where \(\text{SE}_3\) denotes the group of direct isometries from \(\mathbb{R}^3\) to \(\mathbb{R}^3\)) such that
\[
\|\sigma(p_i) - P_t\| = d_i \quad \text{for all } i \in \{1, \ldots, n\}.
\]

Multipods with 20 legs are called icosapods.

Remark 2.1. The description of admissible configurations that we adopt allows independent choices for two coordinate systems, one for the base and one for the platform: this is why also the leg lengths have to be included in the definition of a pod. Moreover, we do not require that the identity belongs to the admissible configurations, namely we do not fix any initial position of the pod.

2.1. Isometries and leg space. Throughout this paper we use a compactification of the group \(\text{SE}_3\) of direct isometries that was introduced in [Sel13] and in [GNS15a]. We briefly recall the construction. A direct isometry of \(\mathbb{R}^3\) can be described by a pair \((M, y)\), where \(M\) is an orthogonal matrix with \(\det(M) = 1\) and \(y\) is the image of the origin under the isometry. We define \(x := -M^t y = -M^{-1} y\) and \(r := \langle x, x \rangle = \langle y, y \rangle\), where \(\langle \cdot, \cdot \rangle\) is the Euclidean scalar product. In this way, if we take coordinates \(m_{11}, \ldots, m_{33}, x_1, x_2, x_3, y_1, y_2, y_3\) and \(r\), together with a homogenizing variable \(h\), in \(\mathbb{P}^{16}_c\), then a direct isometry defines a point in projective space satisfying \(h \neq 0\) and
\[
MM^t = M^t M = h^2 \cdot \text{id}_{\mathbb{R}^3}, \quad \det(M) = h^3, \\
M^t y + hx = 0, \quad M x + hy = 0, \\
\langle x, x \rangle = \langle y, y \rangle = rh.
\]

Equations (1) define a variety \(X\) in \(\mathbb{P}^{16}_c\), whose real points satisfying \(h \neq 0\) are in one to one correspondence with the elements of \(\text{SE}_3\). A direct (computer aided) calculation shows that \(X\) is a variety of dimension 6 and degree 40, and its saturated ideal is generated by quadrics. The main feature of this choice of coordinates is that, if \(a, b \in \mathbb{R}^3\) and \(d \geq 0\), then the condition \(\|\sigma(a) - b\| = d\) on a direct isometry \(\sigma\) becomes linear, and in particular has the following form:
\[
((a, a) + \langle b, b \rangle - d^2) h + r - 2 \langle a, x \rangle - 2 \langle y, b \rangle - 2 \langle Ma, b \rangle = 0.
\]

We will usually refer to Equation (2) as the sphere condition imposed by \((a, b, d)\). Given an \(n\)-pod \(\Pi = (\vec{p}, \vec{F}, \vec{d})\), we can consider its configuration space, namely the set of direct isometries \(\sigma\) in \(\text{SE}_3\) such that \(\|\sigma(a_i) - b_i\| = d_i\) for all \(i \in \{1, \ldots, n\}\).

The fact that Equation (2) is linear in the coordinates of \(\mathbb{P}^{16}_c\) means that the configuration space of \(\Pi\) can be compactified as the intersection of \(X\) with the linear space determined by the \(n\) conditions imposed by its legs; we denote such variety by \(K_\Pi\). We say that the pod \(\Pi\) is mobile if the intersection \(\text{SE}_3 \cap K_\Pi\) has (real) dimension greater than or equal to one. Notice that if \(\Pi\) is mobile, then \(K_\Pi\) has (complex) dimension greater than or equal to one.

Notation. From now on, by the expression mobile icosapod we mean a 20-pod with mobility one to which we cannot add a leg without losing mobility.
For our purposes, it is useful to introduce another 16-dimensional projective
space, playing the role of a dual space of the one where \( X \) lives, where the duality
is given by a bilinear version of Equation (2). We start by introducing a new
quantity, called \textit{corrected leg length}, defined as \( l := \langle a, a \rangle + \langle b, b \rangle - d^2 \), so that
Equation (2) becomes

\[
l h + r - 2 \langle a, x \rangle - 2 \langle y, b \rangle - 2 \langle Ma, b \rangle = 0.
\]

We think of the points \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) as points in \( \mathbb{P}^3 \) by
introducing two extra homogenization coordinates \( a_0 \) and \( b_0 \). In this way, the pair
\((a, b)\) can be considered as a point in the Segre variety \( \Sigma_{3,3} \cong \mathbb{P}^3 \times \mathbb{P}^3 \), the latter is
embedded in \( \mathbb{P}^{15} \), where we take coordinates \{\( z_{ij} \)\} so that the points of \( \Sigma_{3,3} \) satisfy
\( z_{ij} = a_i b_j \) for some \((a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \in \mathbb{P}^3 \). If we homogenize
Equation (2) with respect to the coordinates \{\( z_{ij} \)\} and \( l \), then we get

\[
l h + z_{00}^2 - 2 (z_{10} x_1 + z_{20} x_2 + z_{30} x_3) -
2 \sum_{i,j=1}^3 m_{ij} z_{ij} = 0.
\]

Notice that the left hand side of Equation (3) is a bilinear expression in the co-
ordinates \((h, M, x, y, r)\) and in the coordinates \((z, l)\). We denote this expression
by BSC (for \textit{bilinear sphere condition}). Hence, if we denote by \( \mathbb{P}_{16} \) the projective
space with coordinates \((z, l)\), then we obtain a duality between \( \mathbb{P}_{16} \) and \( \mathbb{P}_{16} \)
sending a point \((h_0, M_0, x_0, y_0, r_0) \in \mathbb{P}_{16} \) to the hyperplane in \( \mathbb{P}_{16} \) of equation
BSC\((h_0, M_0, x_0, y_0, r_0, z, l) = 0\), and a point \((z_0, l_0) \in \mathbb{P}_{16} \) to the hyperplane in \( \mathbb{P}_{16} \)
of equation BSC\((h, M, x, y, r, z_0, l_0) = 0\).

\textbf{Remark 2.2.} Suppose that \((z_0, l_0) \in \mathbb{P}_{16} \) belongs to the cone \( Y \) with vertex \((0 : \cdots : 0 : 1)\) over the Segre variety \( \Sigma_{3,3} \) — namely \((z_{0})_{ij} = a_i b_j \) for some \((a_0 : a_1 : a_2 : a_3)\) and \((b_0 : b_1 : b_2 : b_3)\). Suppose furthermore that \(a_0 = b_0 = 1\) and
\(a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - l_0 \geq 0\). Then from the definition of BSC we see
that the hyperplane BSC\((h, M, x, y, r, z_0, l_0) = 0\) in \( \mathbb{P}_{16} \) is the same hyperplane
defined by Equation (2), where we take \(a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)\) and \(d = \sqrt{a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - l} \).

Remark 2.2 indicates that the cone \( Y \) in \( \mathbb{P}_{16} \) plays a sort of dual role to the compactification \( X \) in \( \mathbb{P}_{16} \), and we will exploit this in our arguments. Since
the Segre variety \( \Sigma_{3,3} \) has dimension 6 and degree 20, we obtain that \( Y \) has dimension 7
and degree 20.

\textbf{Definition 2.3.} Let \( C \subseteq X \) be a curve. We define the \textit{leg set} \( L_C \) as the set of all
points \((z, l) \in Y\) such that the BSC — evaluated at \((z, l)\) — holds for all points
in \( C \).

\textbf{Remark 2.4.} The leg set is a compactification of the set of all triples \((a, b, d) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_{\geq 0}\) such that the image of \( a \) under any point in \( C \cap \text{SE}_3 \) (considered as
an isometry) has distance \( d \) from \( b \).
Proposition 2.5. Let $C \subseteq X$ be a curve. If $L_C$ has only finitely many complex points, then its cardinality is at most 20. If $L_C$ has exactly 20 points, then the linear span of $L_C$ in $\mathbb{P}^9_c$ is a projective subspace of dimension 9.

Proof. By construction $L_C$ is defined by linear equations as a subset of $Y$; in other words $L_C = Y \cap \text{span}(L_C)$. Then the statement follows from general properties of linear sections of projective varieties. In fact, any linear subspace of codimension less than 7 intersects $Y$ in a subvariety of positive dimension, hence $\dim(\text{span}(L_C)) \leq 9$. A general linear subspace of dimension 9 intersects $Y$ in $\deg(Y) = 20$ points. In order to prove that $\dim(\text{span}(L_C)) = 9$ when $L_C$ has 20 points we assume the contrary, that $\dim(\text{span}(L_C)) < 9$. Then we take a general linear superspace $\Lambda$ of $\text{span}(L_C)$ of dimension 9. It intersects $Y$ again in 20 points, which must coincide with $L_C$. On the other hand, if the Hilbert series of $Y$ is $P(t) = (1 - t)^{16}$, then the Hilbert series of $\Lambda \cap Y$ is $P(t) = (1 - t)^9$, but this contradicts the fact that $\Lambda \cap Y = L_C$ is contained in a linear space of dimension strictly smaller than 9. □

In order to show that the maximal number of 20 intersections, as described by Proposition 2.5, can be achieved, we need to find a curve $C \subseteq X$ such that the bilinear sphere conditions of its points define a linear subspace in $\mathbb{P}^9_c$ of dimension 9. This is equivalent to requiring $\dim(\text{span}(C)) = 15 - 9 = 6$. We will deal with this problem in Section 3.

Remark 2.6. The proof of Proposition 2.5 gives also a simple alternative proof for the number of solutions (over $\mathbb{C}$) of the spatial Burmester problem, which reads as follows: Given are seven poses $\varsigma_1, \ldots, \varsigma_7$ of a moving system $\varsigma$, determine all points of $\varsigma$, that are located on a sphere in the given seven poses.

The given poses correspond to seven points in $X$, which span in the general case a $P^6_c$. Therefore its dual space is of dimension 9, and so intersects $Y$ in exactly 20 points.

This number was firstly computed by Schoenflies in [Sch86, page 148] and confirmed by Pimrose (see [Rot67, footnote 3]). We conclude by observing that the first solution of the spatial Burmester problem using an approach based on polynomial systems was presented by Innocenti [Inn95], who also gave an example with 20 real solutions.

2.2. Bond theory. The second ingredient for our arguments is bond theory, a technique for analyzing mobile multipods that has been introduced and developed (in the form we need here) in [GNS15a]. If $C$ is the configuration curve of a multipod $\Pi$ of mobility one, then the bonds of $\Pi$ are defined as the intersections of $C$ with the hyperplane $H = \{h = 0\}$. The set $B = H \cap X$, called the boundary, has dimension 5 and is a real variety with only one real point; the latter can never occur as a bond, hence bonds always arise in complex conjugate pairs. The points of $B$ do not represent isometries but they have geometric meaning: the bonds of a pod $\Pi$ impose conditions on its legs. Five different types of boundary points can be distinguished:

vertex: the only real point in $B$; it is never contained in a configuration curve.
**inversion bonds:** these correspond to inversion maps of orthogonal projections of base and platform; the BSC of an inversion bond states that the projections of base point and platform point correspond via this inversion.

**similarity bonds:** these correspond to similarity maps of orthogonal projections of base and platform; the BSC of an inversion bond states that the projections of base point and platform point correspond via this similarity.

**butterfly bonds:** these correspond to pairs of lines, one in the base and one in the platform; any leg satisfying the BSC of a butterfly bond either has the base point on the base line or the platform point on the platform line.

**collinearity bonds:** these correspond to lines; if a multipod admits a collinearity bond, then either its platform points or its base points are collinear.

A mobile icosapod cannot have butterfly bonds (nor can it have collinearity bonds): this would imply that there are at least 10 legs with collinear base points or platform points, and then an infinite number of legs could be added without changing the mobility, which is disallowed by our definition of mobile icosapods.

The inversion bonds are smooth points of $X$, and their tangent spaces are contained in $H$. Consequently every motion passing through an inversion bond touches the hyperplane $H$ tangentially at this bond. In particular, if a multipod $\Pi$ of mobility one has no bonds other than the inversion bonds, the degree of its configuration curve $C$ is twice the number of inversion bonds: the degree can be computed by intersecting with $H$, and there we see only pairs of double intersections.

Consider the projection $\mathbb{P}^1_{16} \rightarrow \mathbb{P}^9$ keeping only the coordinates $h$ and $m_{ij}$ for $i, j \in \{1, 2, 3\}$. The open subset of the image of $X$ defined by $h \neq 0$ is the group variety $SO_3$, and the image itself is a subvariety $X_m$ of degree 8 that is isomorphic to the Veronese embedding of $\mathbb{P}^3$ by quadrics. The coordinates of $\mathbb{P}^3$ are called Euler coordinates and denoted by $e_0, e_1, e_2, e_3$. The center of the projection $\mathbb{P}^1_{16} \rightarrow \mathbb{P}^9$ intersects $X$ in the union of the sets of similarity bonds, collinearity bonds and vertex.

### 2.3. The subvariety of involutions.

We focus our attention on a particular subvariety of $X$, the compactification of the set of involutions in $SE_3$. Involutions in $SE_3$ are rotations of 180° around a fixed axis, so their compactification — which we will denote by $X_{inv}$ for reasons that will be clear later — is a 4-dimensional subvariety of $X$, because the family of lines in $\mathbb{R}^3$ is 4-dimensional. One reason why involutions are particularly useful in the creation of mobile pods is that if $a$ and $b$ are a base and a platform point of a pod $\Pi$, and $\sigma \in K_\Pi$ is an involution in the configuration space of $\Pi$, then this means that $||\sigma(a) - b|| = d$, where $d$ is the distance between $a$ and $b$; on the other hand, since $\sigma$ is an involution we have $||\sigma(b) - a|| = d$. This means that if all isometries in the configuration space of $\Pi$ are involutions, then we can swap the roles of base and platform points and obtain “for free” new legs not imposing any further restriction to the possible configurations of $\Pi$. 


If $\sigma = (M, y)$ is an involution, then $M = M^t$, that is to say $M$ is symmetric, and $y = x$. Hence we can consider the subvariety

$$\{(h : M : x : y : r) \in X : M = M^t \text{ and } x = y\}.$$  

One verifies that such subvariety has two irreducible components, namely the isolated point corresponding to the identity and another one of dimension 4, which is cut out by a further linear equation, namely $m_{11} + m_{22} + m_{33} + h = 0$.

**Definition 2.7.** The subvariety of $X$ defined by the equations $M = M^t$ and $x = y$ and $m_{11} + m_{22} + m_{33} + h = 0$ is denoted $X_{\text{inv}}$.

### 3. Mobile Icosapods are Line-Symmetric

In this section we will show that if an icosapod of mobility one admits an irreducible configuration curve, then its motion is line-symmetric (Theorem 3.10).

We start by translating this concept in our formalism. Recall from [GNS15a, Section 2.2] that the group $SE_3$ acts on its compactification $X$: every isometry in $SE_3$ determines a projective automorphism of $\mathbb{P}^1^6_\mathbb{C}$ leaving $X$ invariant.

**Definition 3.1.** Let $C \subseteq X$ be a curve. Then $C$ is called an involutory motion if $C \subseteq X_{\text{inv}}$. The curve $C$ is called a line-symmetric motion if there exists an isometry $\tau$ such that the automorphism associated to $\tau$ maps $C$ inside $X_{\text{inv}}$.

From Proposition 2.5 we know that the configuration curve of an icosapod spans a linear subspace of dimension 6. To get an overview of possible examples of irreducible curves $C \subseteq X$ with $\dim(\text{span}(C)) = 6$, we consider the projection $\mathbb{P}^1^6_\mathbb{C} \dashrightarrow \mathbb{P}^9_\mathbb{C}$ described at the end of Subsection 2.2. Let $C_m \subseteq X_m$ be the projection of $C$, which can be either a point or a curve. It is possible to prove (see [Naw14a]) that if $C_m$ is a point, then there exists a multipod with infinitely many legs admitting $C$ as configuration set. Since we are interested in pods with finitely many legs, from now we suppose that $C_m$ is a curve. Let $C_c \subseteq \mathbb{P}^3_\mathbb{C}$ be its isomorphic preimage under the Veronese map.

**Proposition 3.2.** If $C \subseteq X$ is an irreducible curve such that $\dim(\text{span}(C)) = 6$, then $C_c$ is either planar or a twisted cubic.

**Proof.** The projection of $\text{span}(C)$ in $\mathbb{P}^9_\mathbb{C}$ is a linear subspace of dimension at most 6. Hence the ideal of $C_m$ contains at least 3 linear independent linear forms. Hence the ideal of $C_c$ contains at least 3 linear independent quadratic forms. The proposition then follows.

From now on, since we aim for a result on mobile icosapods, we will consider curves $C$ allowing exactly 20 legs, that is satisfying the following condition:

\[(\dagger)\quad \begin{cases} 
C \text{ is an irreducible real curve with real points,} \\
L_C \text{ consists of exactly 20 real finite points.} 
\end{cases} \]

Here by real “finite” points we mean that their $z_{00}$-coordinates are not zero; in other terms, such points determine pairs of base and platform points in $\mathbb{R}^3$ (and not at infinity).
Remark 3.3. Notice that condition \[1\] does not comprise every mechanical device that one could name a “mobile icosapod”: there may exist a device admitting infinitely many complex points in its leg space, of which only 20 are real and finite. Still, we believe that condition \[1\] is a good compromise because it will allow a uniform treatment of the topic.

Remark 3.4. Notice that condition \[1\] implies that \(\dim(\text{span}(C)) = 6\). Moreover, from Section 2.2 it follows that \(C\) does not pass through any butterfly or collinearity point.

Lemma 3.5. Suppose that \(C \subseteq X\) satisfies condition \[1\]. Then \(C_e\) cannot be a cubic (neither a twisted cubic, nor a plane cubic).

Proof. Suppose that \(C\) satisfies condition \[1\] and \(C_e\) is a twisted cubic. Then the curve \(C_m\), isomorphic to \(C_e\) under the Veronese embedding, is a rational normal sextic, and therefore spans a linear space of dimension 6. This means that the projection \(C \rightarrow C_m\) is a projective isomorphism, which means that the center of the projection is disjoint from \(\text{span}(C)\). By Section 2.2 this implies that \(C\) does not admit similarity or collinearity bonds. Recall from Remark 3.4 that the curve \(C\) does not admit butterfly points, so the only ones left are the inversion points. However, since the degree of \(C\) is twice the number of inversion bonds, we would get 3 of them, and this is not possible, since they come in conjugate pairs.

Now, suppose that \(C_e\) is a planar cubic. Then the curve \(C_m\) spans a linear space of dimension 5. This means that the center of the projection \(C \rightarrow C_m\) intersects \(\text{span}(C)\) in a single point. This shows again that \(C\) does not admit similarity bonds, because the similarity bonds are contained in the center of projection and they occur pairwise. Hence we can conclude as in the above case. \[\Box\]

We focus therefore on curves \(C\) satisfying \[1\] such that \(C_e\) is planar of degree different from 3. First we rule out the case when \(C_e\) is a line.

Lemma 3.6. Suppose that \(C \subseteq X\) satisfies condition \[1\]. Then \(C_e\) cannot be a line.

Proof. If \(C_e\) is a line the corresponding motion can only be a Schoenflies motion\(^2\). These motions with points moving on spheres where previously studied by Husty and Karger in [HK02]. It is not difficult to see that no discrete solution can exist, as any leg can be translated along the axis of the Schoenflies motion without restricting the self-motion. Therefore we always end up with an \(\infty\)-pod. \[\Box\]

Lemma 3.7. Suppose that \(C \subseteq X\) satisfies condition \[1\]. Then \(C_e\) cannot be a conic.

Proof. Assume that \(C_e\) is a conic. Consider the linear projection \(P^{16}_e \rightarrow P^9\), under the bilinear spherical condition BSC it determines a subspace \(\hat{P}^9 \subseteq \hat{P}^{16}_e\). By a direct inspection of Equation (3) one notices that the subvariety \(Y_\infty \subseteq Y\) composed of those pairs \((a, b)\) of points with \(a_0 = b_0 = 0\) (namely legs for which both the base

\(^2\)These motions can be composed by a rotation about a fixed axis and an arbitrary translation.
and the platform point are at infinity) is contained in $\mathbb{P}_c^9$. The dimension of $Y_\infty$ is 5 and its degree is 6, since it is a cone over the Segre variety $\mathbb{P}_c^2 \times \mathbb{P}_c^2$.

Consider now the set of linear forms in $\mathbb{P}_c^5$ vanishing on $C_m$: this is a vector space of dimension 5, since it is isomorphic to the vector space of all quadratic forms on $\mathbb{P}_c^3$ that vanish along $C_e$. In this way we get a linear space $\mathbb{P}_c^4 \subseteq \mathbb{P}_c^9$. The intersection $\mathbb{P}_c^4 \cap Y_\infty \subseteq L_C$ is non-empty and is in general constituted of 6 points. Thus the number of real legs not at infinity is at most 14, and this contradicts the assumption that $L_C$ has 20 real finite points.

For the remaining cases we want to prove that there exists an isometry $\tau$ such that the image of $C$ under the corresponding projective automorphism is contained in $X_{\text{inv}}$. Recall from Section 2.2 that we denoted $e_0, \ldots, e_3$ the coordinates of the $\mathbb{P}_c^3$ where $C_e$ lives. We may assume without loss of generality that $e_0 = 0$ holds for the points of $C_e$; this can be achieved by a suitable rotation of the coordinate frame of the platform — specifically by acting on $C_e$ with a suitable rotation — since by assumption $C_e$ is planar. In terms of the coordinates of $X$, this means that we can apply an automorphism of $\mathbb{P}_c^{16}$ induced by an isometry so that the points of $C$ satisfy $m_{ij} = m_{ji}$ and $m_{11} + m_{22} + m_{33} + h = 0$.

We can use the assumption $e_0 = 0$ in order to simplify the embedding. Let $X_1$ be the intersection of $X$ with the linear space

$$\{(h : M : x : y : r) \in \mathbb{P}_c^{16} : m_{ij} = m_{ji} \text{ and } m_{11} + m_{22} + m_{33} + h = 0\}.$$ 

Notice that $X_1$ is the set-theoretical preimage of the locus $\{e_0 = 0\}$ under the projection $X \rightarrow \mathbb{P}_c^3$. We project $X_1$ from the point $(0 : \cdots : 1)$ in $\mathbb{P}_c^{16}$ — the only singular point of $X$ of order 20 — giving a subvariety $X_2$ of $\mathbb{P}_c^{11}$. Consider the map from $X_2$ to the weighted projective space $\mathbb{P}_c(\vec{1}, \vec{2})$ of dimension 5 (here $\vec{1} = (1, 1, 1)$ and $\vec{2} = (2, 2, 2, 2, 2)$), with coordinates $e_1, e_2, e_3$ of weight 1 and coordinates $p_1, p_2, p_3, q_1, q_2, q_3$ of weight 2 obtained in the following way: express the coordinates $h$ and $m_{ij}$ for $i, j \in \{1, 2, 3\}$ in terms of the Euler coordinates $e_1, e_2, e_3$, and apply the coordinate change

$$p_i = x_i + y_i, \quad q_i = x_i - y_i, \quad \text{for } i \in \{1, 2, 3\}.$$

The image of $X_2$ is a weighted projective variety $Z \subseteq \mathbb{P}_c(\vec{1}, \vec{2})$ of dimension 5 defined by the equations

$$e_1 p_1 + e_2 p_2 + e_3 q_3 = p_1 q_1 + p_2 q_2 + p_3 q_3 = e_1 q_2 - e_2 q_1 = e_1 q_3 - e_3 q_1 = e_2 q_3 - e_3 q_2 = 0.$$

Therefore, for a curve $C \subseteq \mathbb{P}_c^{16}$ for which $C_e$ is planar and satisfies $e_0 = 0$ we get a map $C \rightarrow C_z \subseteq Z$ that is the composition of a projection, a linear change of variables and a Veronese map.

Remark 3.8. At the beginning of the section we pointed out that isometries determine automorphisms of $\mathbb{P}_c^{16}$ leaving $X$ invariant. Notice that the automorphisms corresponding to translations leave $X_1$ invariant, since its equations comprise only the rotational part of isometries. Therefore translations act also on $Z$. 
Lemma 3.9. Suppose that $C \subseteq X$ satisfies condition \([1]\). Let $C_z \subseteq Z$ be the image of the curve $C$ under the previously defined maps. Then there exists a translation of the platform such that the corresponding automorphism maps $C_z$ to a curve $C_z'$ whose points satisfy $q_1 = q_2 = q_3 = 0$.

Proof. From the discussion so far we infer that $\deg(C_z) > 3$. We are especially interested in the $q$-vector, so let $W \subseteq \mathbb{P}_C(1,1,1,2,2,2)$ be the projection of $Z$ to the $e$ and $q$-coordinates and let $C_w$ be the image of $C_z$ under such projection. The set $W$ has dimension 4, and its equations are

$$e_1 q_2 - e_2 q_1 = e_1 q_3 - e_3 q_1 = e_2 q_3 - e_3 q_2 = 0.$$  

By a direct inspection of the map $C \rightarrow C_w$ one notices that forms of weighted degree 2 on $C_w$ correspond to linear form on $C$. It follows that the vector space of weighted degree 2 forms on $C_w$ has dimension at most 7. There are 9 forms of weighted degree 2 on $\mathbb{P}_C(1,1,1,2,2,2)$ and they are all linear independent as forms on $W$ because the latter is defined by equations of weighted degree 3. Hence $C_w$ satisfies at least 2 equations $E_1 = E_2 = 0$ of weighted degree 2.

By construction, the polynomials $E_i$ are of the form $E_i = L_i(\overline{q}) + Q_i(\overline{c})$, where $L_i$ is linear and $Q_i$ is quadratic. One notices that $L_1(\overline{q}) L_2(\overline{c}) - L_1(\overline{c}) L_2(\overline{q})$ vanishes on $W$, because it is a multiple of the polynomials in Equation (4). Therefore on $C_w$ we have

$$E_1(\overline{c}, \overline{q}) L_2(\overline{c}) - E_2(\overline{c}, \overline{q}) L_1(\overline{c}) = Q_1(\overline{c}) L_2(\overline{c}) - Q_2(\overline{c}) L_1(\overline{c}) = 0.$$  

The latter is a cubic equation only in the variables $\overline{c}$, thus it is satisfied by $C_z$. On the other hand, $C_z$ is a planar curve of degree greater than 3, so $C_z$ cannot satisfy a cubic nontrivial equation. Therefore we conclude that $Q_1(\overline{c}) L_2(\overline{c}) - Q_2(\overline{c}) L_1(\overline{c})$ is zero on $\mathbb{P}_C^2$. Since $L_1$ and $L_2$ cannot be proportional (otherwise we would be able to obtain from $E_1$ and $E_2$ a quadratic equation in $\overline{c}$ satisfied by $C_z$) we conclude by unique factorization that $L_2$ is a factor of $Q_1$ and $L_1$ is a factor of $Q_2$. Hence $Q_1(\overline{c}) = L(\overline{c}) L_1(\overline{c})$ for some linear polynomial $L$.

From Equation (4) we infer that $L_1(\overline{q}) e_j = L_1(\overline{c}) q_j$ for $j \in \{1, 2, 3\}$. Since $E_1$ is zero on $C_w$, we have $-L_1(\overline{q}) L(\overline{c}) = L_1(\overline{c})$ on $C_w$. Multiplying by $e_j$ the last equation yields:

$$-L_1(\overline{c}) q_j = -L_1(\overline{q}) e_j = L(\overline{c}) L_1(\overline{c}) e_j,$$

this implying that $q_j = L(\overline{c}) e_j$ holds on $C_w$ for $j \in \{1, 2, 3\}$.

One can verify that the automorphism corresponding to the translation by a vector $\overline{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ acts on the coordinates of $\mathbb{P}_C(1,1,1,2,2,2)$ by sending

$$(\overline{c}, \overline{q}) \mapsto (\overline{c}, \overline{q} + \ell_{\overline{a}} \overline{c}),$$

where $\ell_{\overline{a}} = a_1 e_1 + a_2 e_2 + a_3 e_3$. Hence, if $L(\overline{c}) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, it is enough to apply to $C_w$ the automorphism corresponding to the translation by the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ to get that $q_1 = q_2 = q_3 = 0$ holds on $C_w$. This proves the statement.

We are now ready to prove our main result.
Theorem 3.10. Let $\Pi$ be a mobile icosapod such that its configuration curve $K_\Pi$ is irreducible and satisfies $[\Pi]$. Then $K_\Pi$ is a line-symmetric motion.

Proof. From the discussion so far (Proposition 3.2, Lemma 3.5 and the paragraph following it) we know that it is possible to apply a projective automorphism to $K_\Pi$ so that the equations $m_{ij} = m_{ji}$ and $m_{11} + m_{22} + m_{33} + h = 0$ hold. Hence we only need to ensure $x = y$. However, in the new embedding in $\mathbb{P}_c(\vec{1}, \vec{2})$ those equations correspond to $q_1 = q_2 = q_3 = 0$, so Lemma 3.9, Lemma 3.7 and Lemma 3.6 show the claim. □

Remark 3.11. From Theorem 3.10 the set of base and platform points of the icosapod possesses a line-symmetry during the complete self-motion. But this property holds for any pod with a line-symmetric self-motion (see Theorem 1.1). As a consequence one could call these mechanical linkages "line-symmetric icosapods" by analogy to the "line-symmetric Bricard octahedra".

4. Construction of real icosapods

Borel proposed to construct line-symmetric icosapods simply by intersecting $X_{\text{inv}}$ with a general linear subspace $T$ of dimension 7 in $\mathbb{P}_{c}^{10}$. Since $X_{\text{inv}}$ is a variety of codimension 6 and degree 12 in $\mathbb{P}_{c}^{10}$, the intersection $C = X_{\text{inv}} \cap T$ is an irreducible curve of degree 12. Indeed, $C$ is a canonical curve of genus 7, as one can read off from the Hilbert series of $X_{\text{inv}}$. The projection $C_\varepsilon$ of $C$ to the Euler parameters is a planar sextic. Recall that the leg set $L_C$ is the intersection of $Y$ with the dual of $\text{span}(C)$, a linear subspace of codimension 7; this is, in general, a set of 20 complex points. It is not clear whether there are examples with 20 real legs, and the goal of this section is to show that there are instances of such curves $C$ for which this is the case. We reduce the problem to a question on spectrahedra whose answer is well-known.

4.1. Borel’s construction. We rephrase Borel’s construction using the terminology and concepts introduced in this paper. Let $S$ be the linear subspace defined by the equations $M = M^t$ and $x = y$. Notice that $S$ has dimension 10 and contains $X_{\text{inv}}$, although $\text{span}(X_{\text{inv}})$ is a linear subspace of dimension 9. The restriction of the bilinear sphere condition from Equation (3) can be written in a symmetric way as

\begin{equation}
lh + z_{00}r - 4 \sum_{i=1}^{3} s_{0i}x_{i} - 2 \sum_{i=1}^{3} z_{ii}m_{ii} - 4 \sum_{1 \leq i < j}^{3} s_{ij}m_{ij} = 0,
\end{equation}

where $s_{ij} = z_{ij} + z_{ji}$ for $1 \leq i < j \leq 3$. We denote this equation by SBSC, for symmetric bilinear sphere condition. It defines a duality between $S$ and a linear subspace $\mathbb{P}_{c}^{10} \subseteq \mathbb{P}_{c}^{16}$ whose projective coordinates are $l, z_{00}, \ldots, z_{33}, s_{01}, \ldots, s_{23}$. The intersection of the leg variety $Y$ with such $\mathbb{P}_{c}^{10}$ parametrizes pairs of legs obtained by swapping the roles of the base and the platform points. Denote by $\pi : \mathbb{P}_{c}^{10} \dashrightarrow \mathbb{P}_{c}^{9}$ the projection defined by removing the $l$-coordinate. We denote by $Y_{\text{inv}}$ the image
of the map $\alpha: \mathbb{P}_c^3 \times \mathbb{P}_c^3 \rightarrow \mathbb{P}_c^9$, 
$$((a_0 : \cdots : a_3), (b_0 : \cdots : b_3)) \mapsto (a_0 b_0 : \cdots : a_3 b_3 : a_0 b_1 + a_1 b_0 : \cdots : a_2 b_1 + a_3 b_2).$$

One can easily prove that $Y_{\text{inv}}$ is nothing but the projection under $\pi$ of the intersection $Y \cap \mathbb{P}_c^{10}$. Note that $\alpha$ is a $2:1$ map, since $\alpha(a, b) = \alpha(b, a)$ for all $a, b \in \mathbb{P}_c^3$. Because of this, it might happen that two pairs of complex points are sent by $\alpha$ to a real point of $Y_{\text{inv}}$.

For any curve $C \subseteq X_{\text{inv}}$, the leg set $L_C$ is equal to the intersection of the linear space $\tilde{\Gamma}$, dual to $\text{span}(C)$, with the cone over $Y_{\text{inv}}$ in $\mathbb{P}_c^{10}$, namely $\pi^{-1}(Y_{\text{inv}})$. If $\dim \text{span}(C) = 6$, then $\dim \tilde{\Gamma} = 3$. Since $X_{\text{inv}}$ is contained in the hyperplane $\{m_{11} + m_{22} + m_{33} + h = 0\}$, it follows that $\tilde{\Gamma}$ passes through the point $p_e$ with coordinates $l = -2, z_{11} = z_{22} = z_{33} = 1$ and all other coordinates being zero. Borel’s construction can be rephrased as simply choosing a 3-space passing through $p_e$ and intersecting with the cone over $Y_{\text{inv}}$. This cone has degree 10 and codimension 3 in $\mathbb{P}_c^9$, so generically there are 10 solutions (possibly complex), each corresponding to a pair of legs.

For a general 3-space $\tilde{\Gamma}$ passing through $p_e$, one can ask three questions on reality:

1. How many of the 10 points of $\tilde{\Gamma} \cap \pi^{-1}(Y_{\text{inv}})$ are real?
2. How many of the real points above have real preimages under $\alpha$? Namely, how many real legs does the curve $C$ admit?
3. Does the curve $X_{\text{inv}} \cap \Lambda$ have real components, where $\Lambda$ is the dual to $\tilde{\Gamma}$ under SBSC?

The answers to Question (1) and (2) only depend on the projection of $\tilde{\Gamma}$ to $\mathbb{P}_c^9$. In order to obtain positive answers for Question 3, it is also convenient to start with the projection to $\mathbb{P}_c^9$.

**Definition 4.1.** A Borel subspace $\Gamma$ is a 3-space in $\mathbb{P}_c^9$ passing through $\pi(p_e)$.

The following proposition settles Question (3).

**Proposition 4.2.** Let $\Gamma$ be a Borel subspace. Then there exists a 3-space $\tilde{\Gamma}$ passing through $p_e$ such that $\pi(\tilde{\Gamma}) = \Gamma$ and $X_{\text{inv}} \cap \Lambda$ has real components, where $\Lambda$ is the dual of $\tilde{\Gamma}$ under SBSC.

**Proof.** Let $f: S \rightarrow \mathbb{P}_c^4$ the projection from the linear subspace $U$ dual to $\pi^{-1}(\Gamma)$. Then $U$ is contained in the hyperplane $\{m_{11} + m_{22} + m_{33} + h = 0\}$. Hence the image of $X_{\text{inv}}$ under $f$ is contained in a linear 3-space, and $f|_{X_{\text{inv}}}$ has one-dimensional fibers. Since $X_{\text{inv}}$ has real components, it follows that there exist fibers $(f|_{X_{\text{inv}}})^{-1}(q)$ with real components, for some $q \in \mathbb{P}_c^4$. We just need to choose $\tilde{\Gamma}$ dual to $f^{-1}(q)$; then $X_{\text{inv}} \cap \Lambda$ coincides with $(f|_{X_{\text{inv}}})^{-1}(q)$ and therefore has real components. \[\square\]

In order to get some statistical data on the answers to Question (1) and (2), we tested 10000 random examples.\(^\text{3}\) The results are shown in Tables 1 and 2.

\(^\text{3}\)The Maple code used to perform such experiments can be downloaded from [http://matteogallet.altervista.org/main/papers/icosapods2015/icosapods.mpl](http://matteogallet.altervista.org/main/papers/icosapods2015/icosapods.mpl)
Table 1. Points in $\tilde{\Gamma} \cap \pi^{-1}(Y_{\text{inv}})$.

| no. of real points | 2   | 4   | 6   | 8   | 10  |
|--------------------|-----|-----|-----|-----|-----|
| frequency          | 22  | 1067| 3638| 4035| 1238|

Table 2. Points in $\alpha^{-1}(\tilde{\Gamma} \cap \pi^{-1}(Y_{\text{inv}}))$.

| no. of real points | 2   | 4   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  |
|--------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| frequency          | 0   | 4107| 0   | 5240| 0   | 650 | 0   | 3   | 0   | 0   |

As one can see, the experimental data seem to indicate that there are no pods with 20 real legs. This is, however, misleading; see the next section.

4.2. Icosapods via spectrahedra. We conclude our work by showing how it is possible to construct a mobile icosapod with 20 real legs using some result in convex algebraic geometry.

Consider a 4-dimensional vector space $A$ of symmetric $4 \times 4$-matrices over $\mathbb{R}$. Classically, the spectrahedron defined by $A$ is the subset of $A$ comprised of positive semidefinite matrices. One can also consider the spectrahedron as a subset of the projective space $\mathbb{P}(A) \cong \mathbb{P}^3$. The boundary of the spectrahedron consists of the semidefinite matrices with determinant 0, and hence its Zariski closure is a quartic surface in $\mathbb{P}^3$, called the symmetroid defined by $A$. In general, a symmetroid has 10 double points, corresponding to matrices of rank 2.

Given a spectrahedron whose symmetroid has 10 complex double points, its type is the pair of integers $(a, b)$ where $a$ is the number of real double points of the symmetroid and $b$ is the number of real double points of the symmetroid that are also contained in spectrahedron.

**Theorem 4.3.** There is a bijective correspondence between quartic spectrahedra containing the matrix $E := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and Borel subspaces. For a spectrahedron defined by a vector space $A$ and the corresponding Borel subspace $\Gamma$, the following statement holds: if the spectrahedron has type $(a, b)$, then $\Gamma$ intersects $Y_{\text{inv}}$ in $a$ real points, and $a - b$ of them have real preimages under $\alpha$.

**Proof.** We identify $\tilde{\mathbb{P}}^9_\mathbb{C}$ with the projectivization of the vector space of symmetric $4 \times 4$ matrices in the following way: a point with homogeneous coordinates $z_{00}, \ldots, s_{23}$ corresponds to the class of the matrix

\[
\begin{pmatrix}
2 & z_{00} & s_{01} & s_{02} & s_{03} \\
 s_{01} & 2 & z_{11} & s_{12} & s_{13} \\
 s_{02} & s_{12} & 2 & z_{22} & s_{23} \\
 s_{03} & s_{13} & s_{23} & 2 & z_{33}
\end{pmatrix}.
\]

A linear subspace $A$ of dimension 3 in the space of symmetric matrices containing the matrix $E$ corresponds then to a Borel subspace $\Gamma$. 
The subvariety $Y_{inv}$ corresponds to the subvariety of matrices of rank 2. A real matrix of rank 2 does not lie on the spectrahedron if and only if the quadratic form defined by it is a product of two distinct real linear forms, and this is true if and only if its preimage under $\alpha$ is real.

Degtyarev and Itenberg in [DI11] determined all possible types of quartic spectrahedra. In particular, spectrahedra of type $(10, 0)$ do exist, hence by Theorem 4.3 they provide Borel subspaces intersecting $Y_{inv}$ in 10 real points, each of them having two real preimages under the map $\alpha$. This implies that there exist Borel icosa-pods with 20 real legs.

In [ORSV15], the authors give explicit examples of spectrahedra for all possible types. The given example of type $(10, 0)$ does not contain the matrix $E$, but it is easy to adapt their example to one of the same type that contains $E$.

4.3. Example. Starting from [ORSV15, Section 2, Case (10, 0)] we computed the following example, which is suitable for graphical representation:

$$
P_1 = p_4 = \left( -\frac{1943}{142100}, -\frac{2088}{25525}, -\frac{24}{5525} \right), \quad p_1 = P_4 = \left( -\frac{36411}{3578417}, -\frac{1608}{217751}, -\frac{304}{27059} \right),
$$

$$
P_2 = p_5 = \left( -\frac{269}{5000}, -\frac{30}{1000}, -\frac{17}{9001} \right), \quad p_2 = P_5 = \left( -\frac{47}{308}, -\frac{12}{1771}, -\frac{21}{1265} \right),
$$

$$
P_3 = p_6 = \left( -\frac{1864}{14615}, -\frac{106851}{1555400}, -\frac{2509}{222200} \right), \quad p_3 = P_6 = \left( -\frac{15185}{112462}, -\frac{120}{149303}, -\frac{48}{3047} \right).
$$

We apply a half-turn about a line $\ell$ through the point $\left( -\frac{1}{10}, 0, 0 \right)$ in direction $(1, \frac{7003716944}{10000000000}, \frac{8}{10})$ to the platform. In the resulting initial position, which is illustrated in Figure 1, the squared leg lengths of the first six legs read as follows:

$$
\begin{align*}
&d_1^2 = d_4^2 = 108164317973691297230954389391375992, \\
&d_2^2 = d_5^2 = -219492193057810817428484809989961, \\
&d_3^2 = d_6^2 = 6185635506762812517890674782558830797, \\
&d_3 = d_5 = 33621674987061375644068384333371560000.
\end{align*}
$$

For this input data the self-motion consists of two components. The trajectories of the component which passes through the initial position are illustrated in Figures 3 and 4. In the latter figure also the associated basic surface is displayed. An animation of this line-symmetric self-motion can be downloaded from

[www.geometrie.tuwien.ac.at/nawratil/icosapod.gif](http://www.geometrie.tuwien.ac.at/nawratil/icosapod.gif).

We close the paper by mentioning two open questions that we find of interest:

- Starting from spectrahedra of type $(a, b)$ with $a - b \geq 4$ one may construct mobile pods with 16, 12 or 8 legs: is it true that a general mobile pod with 16, 12 or 8 legs is line-symmetric? (It is known that $a, b$ have to be even numbers.)

- Identify all cases where more than 20 points move on spheres during a line-symmetric motion; i.e. (a) 1-dim, (b) 2-dim or even (c) 3-dim set of points with spherical trajectories. Case (c) is completely known due to Bricard [Bri06], but cases (a) and (b) are still open. Examples for both cases are known (cf. Section 1.2.2 and [Bor08, Bri06, Kra37a, Naw15]).

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4The Maple code containing a similar computation can be downloaded from [http://matteogallet.altervista.org/main/papers/icosapods2015/Icosapods.mpl](http://matteogallet.altervista.org/main/papers/icosapods2015/Icosapods.mpl)
Figure 3. The 20 spherical trajectories passing through the initial position of the icosapod.

Acknowledgments

The first and third-named author’s research is supported by the Austrian Science Fund (FWF): W1214-N15/DK9 and P26607 - “Algebraic Methods in Kinematics: Motion Factorisation and Bond Theory”. The second-named author’s research is funded by the Austrian Science Fund (FWF): P24927-N25 - “Stewart-Gough platforms with self-motions”.

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Figure 4. The same set-up as in Figure 3 but from another perspective. In addition a strip of the basic surface of this line-symmetric self-motion is illustrated, where $\ell$ is highlighted. Note that one of the two displayed families of curves on the surface is composed of straight lines, i.e. the generators of the ruled surface.

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