Ferromagnetism of Weakly-Interacting Electrons in Disordered Systems

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It was realized two decades ago that the two-dimensional diffusive Fermi liquid phase is unstable against arbitrarily weak electron-electron interactions. Recently, using the nonlinear sigma model developed by Finkelstein, several authors have shown that the instability leads to a ferromagnetic state. In this paper, we consider diffusing electrons interacting through a ferromagnetic exchange interaction. Using the Hartree-Fock approximation to directly calculate the electron self energy, we find that the total energy is minimized by a finite ferromagnetic moment for arbitrarily weak interactions in two dimensions and for interaction strengths exceeding a critical value proportional to the conductivity in three dimensions. We discuss the relation between our results and previous ones.

I. INTRODUCTION

Since the discovery of Anderson localization, it has been well known that disorder can profoundly change the electronic properties of solids. For non-interacting electronic systems, the celebrated scaling theory of Abrahams et al.\textsuperscript{,}\textsuperscript{1} shows that electrons are always localized in one and two dimensions, while in three dimensions, there exists a metal-insulator transition at a critical value of the disorder strength. On the other hand, when interactions between electrons are taken into account, the results are much less clear. In a disordered system, at length scales beyond the mean free path, electrons move diffusively. Since diffusion is slow, electron-electron interactions are consequently enhanced. First, electrons’ ability to screen the long-ranged Coulomb interaction is hampered. Second, electrons stay in each other’s region longer and hence interact with each other more strongly.\textsuperscript{2} As shown by Finkelstein\textsuperscript{3} and Castellani et al.\textsuperscript{4,5}, the two-dimensional diffusive Fermi liquid phase is, as a result, unstable. However, it is not clear what is the stable fixed point controlling the low-energy behavior of weakly-interacting electrons in the presence of disorder.

Using the nonlinear sigma model developed by Finkelstein\textsuperscript{3}, several authors\textsuperscript{6,7} have shown that the enhancement of electron-electron interactions due to disorder leads to the formation of ferromagnetic moment. The non-linear sigma model takes as its starting point a repulsive density-density interaction which is customarily split into small-angle and large angle scattering with coupling constants $\Gamma_1$ and $\Gamma_2$. The latter coupling drives ferromagnetism. In this paper, we consider itinerant electrons interacting through a ferromagnetic exchange $H_{\text{int}} = -\int J(r-r')S(r) \cdot S(r')$. As in the case of a density-density interaction, this is an interaction which, in the clean, disorder-free case, does not lead to ferromagnetism for small $J$; if it occurs at all, it can only be at large $J$. This is because a kinetic energy penalty must be paid when the up- and down-spin Fermi surfaces are split in order generate a net ferromagnetic moment. The kinetic energy penalty for splitting the Fermi surfaces by $\delta\mu$ is $\sim \nu \delta\mu^2$, where $\nu$ is the density of states at the Fermi surface. On the other hand, the energy gain for non-interacting electrons is $\sim J\nu^2\delta\mu^2$. Hence, ferromagnetism can only occur for $J > \frac{1}{\nu}$, the Stoner criterion. However, as we shall see in the next section, in a two-dimensional disordered system, the interaction energy gain varies instead as $\sim J\nu^2\delta\mu^2 \ln(\tau\delta\mu)$; consequently, it is always favorable to develop a ferromagnetic moment. (This is similar to what occurs in a quantum dot.) We show this by directly calculating the disorder-averaged electron self-energy, using the trick of ref.\textsuperscript{8}, in which it is related to the diffusive pole in the density-density correlation function. This method is physically transparent since one can directly see how the seemingly innocent diffusive pole leads to a logarithmic correction to the self energy. We also avoid the need for the replica trick used in the sigma model treatment. For a three dimensional system, our calculation shows that ferromagnetic order develops when $J$ exceeds a critical value which is proportional to the conductivity of the system. We compare these results with previous ones.

II. THE MODEL

We consider the following Hamiltonian for a weakly-interacting electronic system in the presence of a random distribution of impurities:

$$H = H_0 + H_{\text{int}}$$

where

$$H_0 = \sum_{m, \alpha} E_m \hat{c}_{m\alpha}^\dagger \hat{c}_{m\alpha}$$

is the noninteracting Hamiltonian, which includes the kinetic energy of electrons and the random potential. $E_m$, $\varphi_m(r)$ are respectively the $m^{th}$ eigenvalue and eigenfunction of $H_0$, which we assume to be spin-independent. We will call $E_m$ the ‘kinetic’ energy of the electron to distinguish it from the interaction energy associated with $H_{\text{int}}$, even though $E_m$ includes the effects of the random potential. $\hat{c}_{m\alpha}$ is the annihilation operator for an electron of spin $\alpha = \uparrow, \downarrow$ in the eigenstate $\varphi_m(r)$. The annihilation operator $\psi_{\alpha}(r)$ for an electron of spin $\alpha$ at $r$ is:

$$\psi_{\alpha}(r) = \sum_m \varphi_m(r) \hat{c}_{m\alpha}$$
As mentioned in the introduction, the interaction Hamiltonian $H_{\text{int}}$ considered here is:

$$H_{\text{int}} = - \int d^3r \, d^3r' \, J(r - r') S(r) \cdot S(r')$$
$$= - \int d^3r \, d^3r' \, J(r - r') \left( S^z(r) S^z(r') + 2 \left( S^+(r) S^-(r') + S^-(r) S^+(r') \right) \right)$$

(4)

Here $J(r - r') > 0$ is the exchange interaction which is assumed to be weak, ferromagnetic, and short-ranged. $S(r) = \hat{\psi}_\alpha^\dagger(r) \sigma_{\alpha \beta} \hat{\psi}_\beta(r)$ is the spin density operator, and $\sigma$ are the Pauli matrices.

III. HARTREE AND FOCK ENERGIES

As a warmup, let us first consider a clean system, for which the $\varphi_m(r)$ are plane waves. We look for ferromagnetism in this model by assuming taking a trial wavefunction $|\Psi_0\rangle_{\text{clean}}$ with a ferromagnetic moment:

$$|\Psi_0\rangle_{\text{clean}} = \prod_{k < k_F} \prod_{k' < k_{F_\perp}} c_{k \uparrow}^\dagger c_{k' \downarrow}^\dagger |0\rangle$$

and minimizing $\langle \Psi_0 | H | \Psi_0 \rangle$ with respect to the magnetization $M = \nu \delta \mu$, where $\nu$ is the single-electron density of states and $2 \delta \mu = (k_{F_\perp}^2 - k_{F_\parallel}^2)/2m$ is the difference in the kinetic energies of up- and down-spin electrons at the Fermi level: $M = \nu \delta \mu$, where $\nu$ is the single-electron density of states. Now, we find that the disorder-averaged variational interaction energy is:

$$\langle \Psi_0 | H_{\text{int}} | \Psi_0 \rangle = - \int d^3r \, d^3r' \, J(r - r') \left( S^z(r) S^z(r') \right) + \int dE \, dE' \int d^3r \, d^3r' \, F(E, E', r, r') \, N_E N_{E'}$$

(10)

where

$$\langle S^z(r) \rangle = \sum_m \varphi_m^\dagger(r) \varphi_m(r) \langle N_{m\uparrow} - N_{m\downarrow} \rangle$$

(11)

$$\langle N_{m\uparrow, \downarrow} \rangle = \langle c_{m\uparrow}^\dagger c_{m\uparrow} \rangle = \theta(n_0 \pm M - m)$$

(12)

$N_E$ is the number of electrons in the state at energy $E$, namely $N_E = 2$ for $E < -\delta \mu$, $N_E = 1$ for $-\delta \mu < E < \delta \mu$, and $N_E = 0$ for $E > \delta \mu$. The function $F(E; E'; r, r')$ is defined as

$$F(E, E'; r, r') = \sum_{m,n} \delta(E - E_m) \delta(E' - E_n) \times \varphi_m^\dagger(r) \varphi_m(r') \varphi_n^\dagger(r') \varphi_n(r)$$

(13)

The upper bar indicates average over disorder. The first term in (10) is similar in behavior to the clean case, and will be neglected in the following. The second term is more interesting because the function $F$ is closely related to the density-density correlation function. The crucial difference with the clean case is that the latter has a diffusive form, as we now describe.

We assume that the density propagates diffusively, so that

$$\Pi(q, \omega) = \nu Dq^2 / -i\omega + Dq^2$$

(14)

where, again, $\nu$ is the density of states at the Fermi surface, and $D$ is the diffusion constant. Then

$$\Pi^*(q, \omega) = \int d^3x \, \langle [\rho(0, 0), \rho(x, t)] e^{i q \cdot x - i\omega t} \rangle = \nu Dq^2 e^{i q \cdot x + i\omega t}$$

(15)

On the second line, we have used the fact that after averaging over disorder, the density-density correlation function is translationally invariant. This assumption is valid in the high conductivity regime of interest here. From this, we can obtain the Fourier transform of the density-density correlation function,

$$A(q, \omega) = \int d^3x \, e^{-i q \cdot x + i\omega t} \langle [\rho(x, t), \rho(0, 0)] \rangle$$

(16)
by observing that
\[ A(\mathbf{q}, \omega) = \frac{1}{i} \langle \Pi(\mathbf{q}, \omega) - \Pi^*(\mathbf{q}, \omega) \rangle = \frac{1}{2} \text{Im}(\Pi(\mathbf{q}, \omega)) = \nu \frac{Dq^2 \omega}{2(\omega^2 + (Dq^2)^2)} \]

We can relate this to our function \( F \) by noting that
\[ \langle \rho(rt), \rho(r', 0) \rangle = \sum_{m, n} \varphi_m^*(r) \varphi_m^*(r') \varphi_n(r) \times (N_m - N_n) \tag{18} \]

So we get
\[ F(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} dE \int_{-\infty}^{\mu} dE' F(E, E' ; r, r') \times e^{i\mathbf{q}(r-r')} \delta(E - E' - \omega) \tag{19} \]

Where, on the last line, \( F(\mathbf{q}, \omega) \) is the Fourier transform of \( F(E, E' ; r, r') \), and \( \omega \) is the energy difference \( E - E' \). \( \mu \) is the chemical potential. So we get
\[ F(\mathbf{q}, \omega) = \nu \frac{Dq^2}{2(\omega^2 + (Dq^2)^2)} \tag{20} \]

and the second term in (10) becomes
\[ \langle \Psi_0 | H_{\text{int}} | \Psi_0 \rangle = \ldots + \nu \int dE \int_{-\infty}^{\mu} dE' \int \frac{d^d q}{(2\pi)^d} \frac{Dq^2}{(E - E')^2 + (Dq^2)^2} J(q) \tag{21} \]

The \( \ldots \) refers to the first term in (10), which is smaller because it does not have any infrared singularities in its integral. In the preceding derivation, we performed the disorder average of \( F \), by comparing it to the density-density correlation function, thereby avoiding the replica trick normally used in the field theoretical treatment.

**IV. FERROMAGNETIC GROUND STATE**

We now minimize \( \langle H \rangle \) with respect to the magnetization \( M \). We define the effective interaction \( V(E - E') \) between electrons in states at energies \( E, E' \) by
\[ \langle \Psi_0 | H_{\text{int}} | \Psi_0 \rangle = \int_{E > E'} dE dE' V(E - E') N_E N_{E'} \tag{22} \]

Evaluating the momentum integral in (21), we obtain
\[ V(E - E') = \frac{\nu J}{8\pi D} \ln \left[ \frac{(E - E')^2 + \Lambda^2}{(E - E')^2} \right] \tag{23} \]

Here, we have ignored the momentum dependence of \( J(q) \), and as usual, \( \Lambda \) is the ultraviolet cutoff which can be taken to be \( 1/\tau \), where \( \tau \) is the elastic scattering time.

Thus, the interaction energy is:
\[ \langle H_{\text{int}} \rangle_M = \int_{-\Lambda}^{\Lambda} dE \int_{-\Lambda}^{E} dE' V(E - E') \cdot 2 \cdot 2 + \int_{-\Lambda}^{\Lambda} dE \int_{-\Lambda}^{-\nu J} dE' V(E - E') \cdot 1 \cdot 1 + \int_{-\Lambda}^{\Lambda} dE \int_{-\nu J}^{\nu J} dE' V(E - E') \cdot 1 \cdot 1 \tag{24} \]

Subtracting the interaction energy in the paramagnetic state, \( M = 0 \), we have
\[ \langle H_{\text{int}} \rangle_M - \langle H_{\text{int}} \rangle_0 = \int_{-\nu J}^{\nu J} dE \int_{-\nu J}^{\nu J} dE' V(E - E') = \frac{\nu J}{8\pi D} \int_{-\nu J}^{\nu J} dE \int_{-\nu J}^{\nu J} dE' \ln \left[ \frac{(E - E')^2 + \Lambda^2}{(E - E')^2} \right] = \frac{\nu J}{8\pi D} \left( 6 \delta \mu^2 + 4 \delta \mu^2 \ln \frac{\Lambda}{2\delta \mu} \right) \tag{26} \]

Including the ‘kinetic energy’, we have
\[ \langle H \rangle_M - \langle H \rangle_0 = \left( \nu - \frac{3\nu J}{4\pi D} \right) \delta \mu^2 - \frac{\nu J}{2\pi D} \delta \mu^2 \ln \left( \frac{\Lambda}{2\delta \mu} \right) \tag{27} \]

Even at small \( J \), ferromagnetic order occurs because the total energy is minimal at a finite value of \( \delta \mu \) and, therefore, of \( M \):
\[ M = \frac{\nu \Lambda}{2} e^{-2\pi D/\ell} \tag{28} \]

So we have shown that the diffusive fermi liquid is always unstable against the ferromagnetic order in two dimensions.

In three dimensions, one can do the analogous calculations, leading to
\[ \langle H \rangle - E_0 = \left( \nu - \nu^2 J - \frac{J \nu \Lambda^{1/2}}{2\pi^2 D^{3/2}} \right) \delta \mu^2 + \frac{4J\nu}{15\pi D^{3/2}} \delta \mu^{5/2} \tag{29} \]

In the first term, we have included the Hartree contribution because the Fock part is non-divergent and can be viewed as a correction. Hence, the critical coupling \( J_c \), above which ferromagnetism occurs is modified from the Stoner value:
\[ J_c = \frac{1}{\nu} \left( 1 - \frac{3\sqrt{3}}{(k_F \ell)^2} \right) \tag{30} \]

where \( \ell = v_F \tau \) is the elastic mean-free path and we have used \( D = v_F^2 \tau/3 \) in three dimensions.
V. DISCUSSION

In this paper, we use a straightforward Fock approximation to show that in two dimensions an arbitrarily weak ferromagnetic interaction causes an instability of the diffusive Fermi liquid. In three dimensions, this instability occurs only when the interaction strength exceeds a critical value proportional to the conductivity. We worked directly with the electronic Hamiltonian, without recourse to Finkelstein’s non-linear $\sigma$ model, which was used at an intermediate stage of previous calculations. Our results are consistent with these earlier ones. The model and method of this paper has the advantage of physical transparency: it shows the direct connection between diffusive motion and the ferromagnetic instability.

By using the method in ref. 9 we avoided using the replica trick, and saw the crucial feature of diffusive motion: it implies a spatial correlation between single-particle states which are nearby in energy. As a result of this correlation, the effective interaction strength diverges as the energy of two electrons approach each other. This strongly favors ferromagnetism since electrons of the same spin cannot occupy the same energy level. This strong dependence on the energy separation of the electrons is completely absent in a clean system.

We note that once ferromagnetic order develops, the system goes into the universality class of electrons in a Zeeman field, which flows towards an insulating state in two dimensions.

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