Acknowledged geometry

Remarks on minimal rational curves on moduli spaces of stable bundles

Remarques sur les courbes rationnelles minimales sur les espaces des modules de faisceaux stables

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Abstract

Let \( C \) be a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field of characteristic zero, and \( M \) be the moduli space of stable bundles of rank \( 2 \) and with fixed determinant \( \mathcal{L} \) of degree \( d \) on the curve \( C \). When \( g = 3 \) and \( d \) is even, we prove that, for any point \([W]\) \( \in M \), there is a minimal rational curve passing through \([W]\), which is not a Hecke curve. This complements a theorem of Xiaotao Sun.

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Résumé

Soient \( C \) une courbe projective lisse de genre \( g \geq 2 \) et \( M \) l’espace des modules de faisceaux stables de rang 2 et de déterminant fixe \( \mathcal{L} \) de degré \( d \) sur \( C \). Nous prouvons que, lorsque \( g = 3 \) et \( d \) est pair, il existe, pour tout point \([W]\) \( \in M \), une courbe rationnelle minimale passant par \([W]\), qui n’est pas une courbe de Hecke. Cela complète un théorème de Xiaotao Sun.

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1. Introduction

Throughout this paper, we assume that \( C \) is a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field of characteristic zero. Let \( M := SU_C(r, \mathcal{L}) \) be the moduli space of stable vector bundles of rank \( r \geq 2 \) and with the fixed determinant \( \mathcal{L} \) of degree \( d \), which is a smooth quasi-projective Fano variety with \( \text{Pic}(M) = \mathbb{Z} \cdot \Theta \) and \( -K_M = 2(r, d)\Theta \), where \( \Theta \) is an ample divisor ([9,1]). By a rational curve of \( M \), we mean a nontrivial proper morphism \( \phi : \mathbb{P}^1 \to M \) and its degree is defined to be \( \text{deg}\phi^{\ast}(-K_M) \) (with respect to the ample anti-canonical line bundle \(-K_M\)).

In [10], Xiaotao Sun has determined all rational curves of minimal degree passing through generic points of \( M \) except in the case where \( g = 3, r = 2, \) and \( d \) is even.

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Theorem 1.1. (Theorem 1 of [10]) If \( g \geq 3 \), then any rational curve \( \phi : \mathbb{P}^1 \to M \) passing through the generic point has degree at least \( 2r \). It has degree \( 2r \) if and only if it is a Hecke curve unless \( g = 3 \), \( r = 2 \), and \( d \) is even.

This implies that all the rational curves of \((-K_M)\)-degree smaller than \( 2r \), called small rational curves, must lie in a proper closed subset \([3,4]\). In this note, we remark that the condition in Sun’s Theorem is necessary:

Theorem 1.2. If \( g = 2 \), \( r = 2 \) and \( d \) is odd, then, for any \([W] \in M\), there exists a rational curve passing through it, which has degree 2.
If \( g = 3 \), \( r = 2 \) and \( d \) is even, then, for any point \([W] \in M\), there exists a rational curve of degree 4 passing through it, which is not a Hecke curve.

Recall that, by Lemma 2.1 of [10], any rational curve \( \phi : \mathbb{P}^1 \to M \) is defined by a vector bundle \( E \) on \( f : X = C \times \mathbb{P}^1 \to C \).

If \( E \) is semi-stable on generic fiber \( X_p = f^{-1}(p) \) (tensoring a pullback of line bundle on \( \mathbb{P}^1 \)), we can assume the restriction of \( E \) to a generic fiber is of the form \( O_{X_p}^{\oplus 3} \), according to the arguments of section 2 in [10], there is a finite set \( S \subset C \) of points and a vector bundle \( V \) on \( C \) such that \( E \) just suits in the exact sequence

\[
0 \to f^*V \to E \to \bigoplus_{p \in S} Q_p \to 0
\]

where \( Q_p \) is a vector bundle on \( X_p = \{p\} \times \mathbb{P}^1 \). The curves defined by such \( E \) were said to be of Hecke type in \([8,11]\) (since a Hecke curve by definition is defined by a vector bundle \( E \) suited in \( 0 \to f^*V \to E \to \mathcal{O}_{X_p}(-1) \to 0 \)). If \( E \) is not semi-stable on the generic fiber \( X_p \) (curves defined by such \( E \) were said of split type in \([11]\)) and the curve has minimal degree \( 2(r,d) \), then \( E \) must suit in

\[
0 \to f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0
\]

where \( \pi : X \to \mathbb{P}^1 \) is the projection and \( V_1, V_2 \) are stable vector bundles on \( C \) of rank \( r_1, r_2 \), and degrees \( d_1, d_2 \) satisfy \( r_1d - d_1 = (r,d) \). Note that rational curves of degree \( 2(r,d) \) have degree 1 with respect to \( \Theta \) because of \(-K_M = 2(r,d)\Theta \), which will be called lines in \( M \).

The rational curves we constructed in Theorem 1.2 are of split type (thus they are not Hecke curves). We have in fact a more general result. Let \( M' = SU_C(2,L) \) be the moduli space of rank-two stable bundles with fixed determinant \( L \) on a smooth projective curve \( C \) of genus \( g \geq 3 \). Let \( M' \subset M \) be the locus of stable bundles \([W] \in M \) with the Segre invariant \( s(W) = s \) (refer to Section 3 for the definition of Segre invariant). Then we have the following theorem:

Theorem 1.3. When \( d \) is even, for any \([W] \in M_2 \), there is a rational curve of split type passing through it, which has degree 4. If \( d \) is odd, for any \([W] \in M_1 \), there is a rational curve of split type passing through it, which has degree 2.

When \( g = 3 \) and \( d \) is even, we have \( M_2 = M \) (see Lemma 3.1). Thus Theorem 1.2 is a corollary of Theorem 1.3.

2. Rational curves of split type

Let \( C \) be a smooth projective curve with genus \( g \geq 2 \) over an algebraically closed field of characteristic zero, \( W \) be a stable bundle of rank \( r \) and of degree \( d \) with determinant \( L \) over \( C \). Assume that there is a stable subbundle \( V_1 \) of \( W \) such that

\[
r_1d - d_1 = (r,d),
\]

where \( r_1 = \text{rank } V_1, d_1 = \text{deg } V_1 \) and \( d = \text{deg } W \). Let \( V_2 := V/V_1 \) be the quotient bundle, then \( W \) fits a non-trivial extension

\[
0 \to V_1 \to W \to V_2 \to 0.
\]

It is known that there is a family of vector bundles \( \{E_p\}_{p \in P} \) on \( C \) parametrized by \( P = P \text{Ext}^1(V_2,V_1) \) so that for each \( p \in P \), \( E_p \) is isomorphic to the bundle obtained as the extension of \( V_2 \) by \( V_1 \) given by \( p \) (see Lemma 2.3 of [9]). Let \( l \) be a line in \( P = P \text{Ext}^1(V_2,V_1) \) passing through the point \( p_0 \), where \( p_0 \) is the point in \( P \) given by \( (2) \). If it happens that \( E_p \) is stable for each \( p \in l \), then

\[
\{E_p\}_{p \in l}
\]

will define a rational curve of degree \( 2(r,d) \) (with respect to \(-K_M\)) passing through \([W] \in SU_C(r,L) \) ([10,4]). Such a rational curve in \( SU_C(r,L) \) will be called a rational curve of split type.

It is known that an extension \( 0 \to E \to W \to F \to 0 \), where \( E, W, F \) are vector bundles on \( C \), gives rise to an element \( \delta(W) \in H^1(C, \text{Hom}(F,E)) \), which is the image of the identity homomorphism in \( H^0(C, \text{Hom}(F,F)) \) by the connecting homomorphism \( H^0(C, \text{Hom}(F,F)) \to H^1(C, \text{Hom}(F,E)) \). This gives a one-to-one correspondence between the set of equivalent classes of extensions of \( F \) by \( E \) and \( H^1(C, \text{Hom}(F,F)) \) (refer to section 2 in [9]).
Lemma 2.1. Let \( d \) be an even number, and \( 0 \rightarrow L_1 \rightarrow W \rightarrow L_2 \rightarrow 0 \) be any non-trivial extension of \( L_2 \) by \( L_1 \), where \( L_1 \) (resp. \( L_2 \)) is a line bundle of degree \( \frac{d}{2} - 1 \) (resp. \( \frac{d}{2} + 1 \)). Then

(i) \( W \) is semi-stable;

(ii) \( W \) is non-stable if and only if the element \( \delta(W) \in H^1(C, L_2^{-1} \otimes L_1) \) corresponding to \( W \) is in the kernel of the map

\[
H^1(C, L_2^{-1} \otimes L_1) \rightarrow H^1(C, L_2^{-1} \otimes L_1 \otimes L_x),
\]

for some \( x \in C \), where \( L_x = O_C(x) \) is the line bundle defined by \( x \). In this case, \( W \) is S-equivalent to \( L_2 \otimes L_x^{-1} \otimes L_1 \otimes L_x \) (refer to section 2 of [7] for the definition of S-equivalent).

Proof. (i) See Lemma 2.2 in [4] and [5].

(ii) Let \( L' \) be a line bundle of degree \( \frac{d}{2} \). Then, since \( H^0(C, \text{Hom}(L', W)) = 0 \), it is easy to see that \( H^0(C, \text{Hom}(L', W)) \neq 0 \) if and only if \( L' \) is of the form \( L_2 \otimes L_x^{-1} \) for some \( x \in C \) and the natural map \( L_2 \otimes L_x^{-1} \rightarrow L_2 \) can be lifted into a map \( L_2 \otimes L_x^{-1} \rightarrow W \).

Consider the commutative diagram of vector bundles

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}(L_2, L_1) & \rightarrow & \text{Hom}(L_2, W) & \rightarrow & \text{Hom}(L_2, L_2) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Hom}(L_2 \otimes L_x^{-1}, L_1) & \rightarrow & \text{Hom}(L_2 \otimes L_x^{-1}, W) & \rightarrow & \text{Hom}(L_2 \otimes L_x^{-1}, L_2) & \rightarrow & 0,
\end{array}
\]

where the horizontal sequences are exact and the vertical maps are induced by the natural map \( L_2 \otimes L_x^{-1} \rightarrow L_2 \). From this, we deduce the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(C, \text{Hom}(L_2, W)) & \rightarrow & H^0(C, \text{Hom}(L_2, L_2)) & \rightarrow & H^1(C, \text{Hom}(L_2, L_1)) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^0(C, \text{Hom}(L_2 \otimes L_x^{-1}, W)) & \rightarrow & H^0(C, \text{Hom}(L_2 \otimes L_x^{-1}, L_2)) & \rightarrow & H^1(C, \text{Hom}(L_2 \otimes L_x^{-1}, L_1)) & \rightarrow & \cdots
\end{array}
\]

which implies the lemma. \( \square \)

Remark 2.2. Lemma 2.1 (ii) asserts that the non-stable bundles in \( \mathbb{P}H^1(L_2^{-1} \otimes L_1) \) correspond precisely to the image of \( C \) in \( \mathbb{P}H^1(L_2^{-1} \otimes L_1) \) under the map given by the linear system \( K_C \otimes L_1^{-1} \otimes L_2 \). Which implies that the dimension of the subset of non-stable bundles in \( \mathbb{P}H^1(L_2^{-1} \otimes L_1) \) is at most 1.

3. Proof of Theorem 1.3

Let \( C \) be a smooth irreducible curve over an algebraically closed field of characteristic zero, \( W \) a vector bundle of rank 2 over \( C \), set

\[
m(W) := \max\{\deg(L) | L \subset W \text{ is a sub line bundle of } W\}.
\]

A sub line bundle \( L \) of \( W \) of maximal degree \( m(W) \) is called a maximal sub line bundle. The Segre invariant is defined by

\[
s(W) := \deg(W) - 2m(W).
\]

Note that \( s(W) \equiv \deg(W) \pmod{2} \) and that \( W \) is stable (resp. semi-stable) if and only if \( s(W) \geq 1 \) (resp. \( s(W) \geq 0 \)). Nagata proved in [6] that

\[
s(W) \leq g.
\]

It is easy to see that

Lemma 3.1. If \( g = 3 \), then, for any stable bundle \( W \) over \( C \) of rank 2 and with even degree \( d \), we have \( s(W) = 2 \).

In general, the function \( s : M \rightarrow \mathbb{Z} \) defined by \([W] \mapsto s(W)\) is lower semicontinuous and gives a stratification of \( M \) into locally closed subsets \( M_s \) according to the value of \( s \). Then, by Proposition 3.1 in [2], we have

Proposition 3.2. ([2]) Suppose that \( 1 \leq s \leq g - 2 \) and \( s \equiv d \pmod{2} \). Then \( M_s \) is an irreducible algebraic variety of dimension \( 2g + s - 2 \).

The proof of Theorem 1.3 follows the following two propositions.
Proposition 3.3. Suppose that $g \geq 3$, $r = 2$, $d$ is even and $M_2$ is non-empty. Then, for any $[W] \in M_2$, there is a rational curve of split type passing through it, which has degree 4.

Proof. For any $[W] \in M_2$, there is a sub line bundle $L_1$ of $W$ with $\deg L_1 = \frac{d}{2} - 1$, where $d = \deg \mathcal{L}$. Let $L_2 := W / L_1$ be the quotient bundle, which has degree $\frac{d}{2} + 1$. It is easy to see that

$$1 \times d - (\frac{d}{2} - 1) \times 2 = 2 = (2, d).$$

Let $i : L_1 \to W$ be the natural injection, then

$$0 \to L_1 \xrightarrow{i} W \to L_2 \to 0$$

is a non-trivial extension (otherwise, we have $W \cong L_1 \oplus L_2$, which contradicts the stability of $W$).

It is known that there is a family of vector bundles $\mathcal{E}$ on $C$ parametrized by $P_{(L_1, L_2)} = \mathbb{P} \operatorname{Ext}^1(L_2, L_1)$ so that for each $p \in P_{(L_1, L_2)}$, the $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of $L_2$ by $L_1$ given by $p$ (see Lemma 2.3 of [9]). More precisely, there is a universal extension

$$0 \to f^*L_1 \otimes \mathbb{P}^n C_{P_{(L_1, L_2)}}(1) \to \mathcal{E} \to f^*L_2 \to 0$$

(5)

on $C \times P_{(L_1, L_2)}$, where $f : C \times P_{(L_1, L_2)} \to C$ and $\pi : C \times P_{(L_1, L_2)} \to P_{(L_1, L_2)}$ are projections. Then $\mathcal{E}$ is a family of semi-stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$ (Lemma 2.1). Thus, the universal extension (5) defines a morphism

$$\Phi_{(L_1, L_2)} : P_{(L_1, L_2)} \to UC(2, \mathcal{L}),$$

(6)

where $UC(2, \mathcal{L})$ denotes the moduli space of semi-stable bundles of rank 2 and with fixed determinant $\mathcal{L}$, which is a projective compactification of $M$.

It is easy to see that $P_{(L_1, L_2)}$ is a projective space of dimension $g \geq 3$. By Lemma 2.1 and Remark 2.2, there is a line $l$ in $P_{(L_1, L_2)}$ passing through

$$q = [0 \to L_1 \to i \to W \to L_2 \to 0]$$

such that $\mathcal{E}_p$ is stable for each $p \in l$. Thus, $\Phi_{(L_1, L_2)}(l) \subset M = SU_C(2, \mathcal{L})$ and

$$\Phi_{(L_1, L_2)}(l) = SU_C(2, \mathcal{L})$$

(7)

is a rational curve of split type passing through the point $[W] \in M$. □

Proposition 3.4. Suppose $g \geq 2$, $r = 2$, $d$ is odd and $M_1$ is non-empty. Then, for any $[W] \in M_1$, there is a rational curve of split type passing through it, which has degree 2.

Proof. Let $[W]$ be a point in $M_1$, then we have $s(W) = 1$ and there is a sub line bundle $L_1$ of $W$ with $\deg L_1 = \frac{d-1}{2}$, where $d = \deg \mathcal{L}$. Let $L_2 := W / L_1$, which is a line bundle of degree $\frac{d+1}{2}$. It is easy to see that

$$1 \times d - \frac{d-1}{2} \times 2 = 1 = (2, d).$$

Let $i : L_1 \to W$ be the natural injection, then

$$0 \to L_1 \xrightarrow{i} W \to L_2 \to 0$$

is a non-trivial extension because $W$ is a stable bundle.

It is known that there is a family of vector bundles $\mathcal{E}_p$ on $C$ parametrized by $P_{(L_1, L_2)} = \mathbb{P} \operatorname{Ext}^1(L_2, L_1)$ such that for each $p \in P_{(L_1, L_2)}$, $\mathcal{E}_p$ is isomorphic to the bundle obtained as the extension of $L_2$ by $L_1$ given by $p$ (Lemma 2.3 of [9]). By Lemma 3.1 of [10], $\mathcal{E}_p$ is a family of stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$, which defines a morphism

$$\Psi_{(L_1, L_2)} : P_{(L_1, L_2)} \to SU_C(2, \mathcal{L}) = M.$$

(8)

Let $l$ be a line in $P_{(L_1, L_2)}$ passing through

$$q = [0 \to L_1 \to i \to W \to L_2 \to 0],$$

then

$$\Psi_{(L_1, L_2)}(l) = SU_C(2, \mathcal{L})$$

(9)

is a rational curve of split type passing through the point $[W] \in M$, which has degree 2. □

When $g = 2$, the same as Lemma 3.1, we have:
Lemma 3.5. If $g = 2$, $r = 2$ and $d$ is odd, for any $[W] \in M$, $s(W) = 1$.

By Lemma 3.5 and Proposition 3.4, we have:

Proposition 3.6. If $g = 2$, $r = 2$ and $d$ is odd, then, for any $[W] \in M$, there exists a rational curve of split type passing through it, which has degree 2.

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