QUIVER VARIETIES AND YANGIANS

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Abstract. We prove a conjecture of Nakajima ([5], for type $A$ it was announced in [2]) giving a geometric realization, via quiver varieties, of the Yangian of type $ADE$ (and more in general of the Yangian associated to every symmetric Kac-Moody Lie algebra). As a corollary we get that the finite dimensional representation theory of the quantized affine algebra and that of the Yangian coincide.

1. The algebra $Y_\hbar(Lg)$.

Let $g$ be a simple, simply laced, complex Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{kl})_{k,l \in I}$. Denote by $Lg = g[t, t^{-1}]$ the loop Lie algebra of $g$. The Yangian $Y_\hbar(Lg)$ is the associative algebra, free over $\mathbb{C}[h]$, generated by $x_{k,r}^\pm, h_{k,r} (k \in I, r \in \mathbb{N})$ with the following defining relations

\begin{align}
(1.1) & \quad [h_{k,r}, h_{l,s}] = 0, \quad [h_{k,0}, x_{l,s}^\pm] = \pm a_{kl} x_{l,s}^\pm, \\
(1.2) & \quad 2[h_{k,r+1}, x_{l,s}^+] - 2[h_{k,r}, x_{l,s+1}^+] = \pm a_{kl} (h_{k,r}x_{l,s}^+ + x_{l,s}^+ h_{k,r}), \\
(1.3) & \quad [x_{k,r}^+, x_{l,s}^-] = \delta_{kl} h_{k,r+s}, \\
(1.4) & \quad 2[x_{k,r+1}^+, x_{l,s}^+] - 2[x_{k,r}^+, x_{l,s+1}^+] = \pm a_{kl} (x_{k,r}^+ x_{l,s}^+ + x_{l,s}^+ x_{k,r}^+), \\
(1.5) & \quad \sum_{w \in S_m} [x_{k,r_w(1)}^+, [x_{k,r_w(2)}^+, \ldots, [x_{k,r_w(m)}^+, [x_{l,s1]^+, \ldots]]]] = 0, \quad k \neq l
\end{align}

for all sequences of non-negative integers $r_1, \ldots, r_m$, where $m = 1 - a_{kl}$.

Set

\[ [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \forall n \in \mathbb{Z}. \]

2. Quiver varieties.

Let $I$ (resp. $E$) be the set of vertices (resp. edges) of a finite graph $(I, E)$ with no edge loops. For $k, l \in I$ let $n_{kl}$ be the number of edges joining $k$ and $l$. Put $a_{kl} = 2\delta_{kl} - n_{kl}$. The map $(I, E) \mapsto A = (a_{kl})_{k,l \in I}$ is a bijection from the set of finite graphs with no loops onto the set of symmetric generalized Cartan matrices. Let $\alpha_k$ and $\omega_k, k \in I$, be the simple roots and fundamental weights of the symmetric Kac-Moody algebra corresponding to $A$. Let $H$ be the set of edges of $(I, E)$

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together with an orientation. For \( h \in H \) let \( h' \in I \) (resp. \( h'' \in I \)) the incoming (resp. the outgoing) vertex of \( h \). If \( h \in H \) we denote by \( \tilde{h} \in H \) the same edge with opposite orientation. Take two collection of finite dimensional complex vector spaces \( V = (V_k)_{k \in I}, \ W = (W_k)_{k \in I} \). Let us fix once for all the following convention: the dimension of the graded vector space \( V \) is identified with the element \( v = \sum_{k \in I} v_k \alpha_k \) in the root lattice (where \( v_k \) is the dimension of \( V_k \)). Similarly the dimension of \( W \) is identified with the weight \( w = \sum_k w_k \omega_k \) (where \( w_k \) is the dimension of \( W_k \)). Set

\[
M(v, w) = \bigoplus_{h \in H} \text{Hom}(V_{h''}, V_{h'}) \oplus \bigoplus_{k \in I} \text{Hom}(W_k, V_k) \oplus \bigoplus_{k \in I} \text{Hom}(V_k, W_k).
\]

The group \( G_v = \prod_k \text{GL}(V_k) \) acts on \( M(v, w) \) by \( g \cdot (B, i, j) = (gB g^{-1}, gi, jg^{-1}) \). We denote by \( B_h \) the component of the element \( B \) in \( \text{Hom}(V_{h''}, V_{h'}) \). Let us consider the map

\[
\mu_{v,w} : M(v, w) \to \bigoplus_{k \in I} \text{Hom}(V_k, V_k), \quad (B, i, j) \mapsto \sum_h \varepsilon(h) B_h B_{\tilde{h}} + ij,
\]

where \( \varepsilon \) is any function \( \varepsilon : H \to \mathbb{C}^\times \) such that \( \varepsilon(h) + \varepsilon(\tilde{h}) = 0 \). We say that a triple \( (B, i, j) \in \mu_{v,w}^{-1}(0) \) is stable if there is no nontrivial \( B \)-invariant subspace of \( \text{Ker} j \). Let \( \mu_{v,w}^{-1}(0)^s \) be the set of stable triples. The group \( G_v \) acts freely on \( \mu_{v,w}^{-1}(0)^s \). Put

\[
T(v, w) = \mu_{v,w}^{-1}(0)^s / G_v, \quad N(v, w) = \mu_{v,w}^{-1}(0) / G_v
\]

and let \( \pi : T(v, w) \to N(v, w) \) be the affinization map (it sends \( G_v \cdot (B, i, j) \) to the only closed \( G_v \)-orbit contained in \( G_v \cdot (B, i, j) \)). It is proved in [4, 3.10(2)] that \( T(v, w) \) is a smooth quasi-projective variety. Given \( v^1, v^2 \in \mathbb{N}[I] \) consider the fiber product \( Z(v^1, v^2; w) = T(v^1, w) \times_{\pi} T(v^2, w) \). Take \( v^2 = v^1 + \alpha_k \) where \( \alpha_k \) is a simple root and assume that \( V^1 \subseteq V^2 \) have dimension \( v^1, v^2 \), respectively. Consider the closed subvariety \( C^+_k(v^2, w) \) of \( Z(v^1, v^2; w) \) consisting of the pairs of triples \( (B^1, i^1, j^1), (B^2, i^2, j^2) \) such that \( B^2_{|v^1} = B^1, i^2 = i^1, j^2_{|v^1} = j^1 \). Put \( C^-_k(v^2, w) = \varphi(C^+_k(v^2, w)) \subset Z(v^2, v^1; w) \) where \( \varphi : T(v^1, w) \times T(v^2, w) \to T(v^2, w) \times T(v^1, w) \) permutes the components. The varieties \( C^+_k(v^2, w) \) are nonsingular [4, 5.7]. The group \( \tilde{G}_w = G_w \times \mathbb{C}^\times \) acts on \( T(v, w) \) by

\[
(g, t) \cdot (B, i, j) = (tB, t^2 i g^{-1}, gj), \quad \forall g \in G_w, \forall t \in \mathbb{C}^\times.
\]

Let \( V_k = \mu_{v,w}^{-1}(0)^s \times_{G_v} V_k \) and \( W_k \) be respectively the \( k \)-th tautological bundle and the trivial \( W_k \)-bundle on \( T(v, w) \). The bundles \( V_k, W_k \), are \( \tilde{G}_w \)-equivariant. The group \( \tilde{G}_w \) acts also on \( N(v, w), C^+_k(v^2, w), \) and \( Z(v^1, v^2; w) \). Let \( q \) be the trivial line bundle on \( T(v, w) \) with the degree one action of \( \mathbb{C}^\times \). For any complex \( G \)-variety \( X \) let \( K^G(X) \) be the Grothendieck ring of \( G \)-equivariant coherent sheaves on \( X \). Put

\[
F_k(v, w) = q^{-2} W_k - (1 + q^{-2}) V_k + q^{-1} \sum_{h'' = k} V_{h''} \in K^{\tilde{G}_w}(T(v, w)).
\]

The rank of \( F_k(v, w) \) is \( (w - v | \alpha_k) \), where \( (|) \) is the standard metric on the weight lattice of \( g \). We fix a pair of linear maps \( w \mapsto w_\pm \) on the weight lattice which are adjoin with respect to \( (\ | ) \), and such that \( w_+ + w_- = w \) for all \( w \).
3. Equivariant homology and convolution product.
Let \( G \) be a complex, connected, linear algebraic group. For any complex \( G \)-variety \( X \), let \( H^G_i(X) \) (resp. \( H^G_r(X) \)) be the \( i \)-th space of \( G \)-equivariant complex Borel-Moore homology (resp. of \( G \)-equivariant complex cohomology). Put

\[
H^G(X) = \bigoplus_i H^G_i(X), \quad H_G(X) = \bigoplus_i H_G^i(X).
\]

See [3] for details on equivariant Borel-Moore homology. Let us only recall the following well known facts.

- If \( Y \) is a closed \( G \)-subvariety of \( X \) and \( X \) is smooth, then \( H^G(Y) = H^G(X, X \setminus Y) \). Moreover there is a natural map \( H^G(X) \to H^G(Y) \). Call \( \alpha^o \in H^G(X) \) the image of \( \alpha \in H^G(Y) \). The \( \cup \)-product in equivariant cohomology induce, via the Poincaré duality, a product, noted \( . \), in equivariant homology. We will denote also by a dot the product \( H^G(X) \otimes H^G(X) \to H^G(X) \).

- Any \( G \)-equivariant vector bundle \( E \) on \( X \) admits an equivariant Chern polynomial \( \lambda_z(E) \in H^G(X)([z]) \). The coefficient of \( z \) in \( \lambda_z(E) \) is the equivariant first Chern class \( c_1(E) \in H^G(X) \). The coefficient of \( z^{rk(E)} \) in \( \lambda_z(E) \) is the equivariant Euler class \( \lambda(E) \in H^G(X) \). If \( E \) is invertible, then \( \lambda_z(E) = 1 + c_1(E)z \). Moreover, for any \( E \) and \( F \) we have \( \lambda_z(E \oplus F) = \lambda_z(E) \cup \lambda_z(F) \). The class \( \lambda_z(E) \) depends only on the class of \( E \) in \( K^G(X) \).

- If \( T \subset G \) is a maximal torus, put \( t = \text{Lie}(T) \). Then \( H^G_{2i}(pt) = S^{2i}(t^*)^W \), where \( S^i \) is the \( i \)-symmetric product and \( W \) is the Weyl group.

We will use the following (see [1, Proposition 2.6.47]):

**Lemma.** Let \( X \) be a smooth \( G \)-variety and let \( C_i \ (i = 1, 2) \) be two smooth closed \( G \)-subvarieties. Set \( C_3 = C_1 \cap C_2 \) and let \( \gamma_i : C_i \hookrightarrow X \ (i = 1, 2, 3) \) be the natural embedding. Suppose that \( C_1 \) and \( C_2 \) are transversal. Then, for all \( \alpha \in H^G(C_1) \) and \( \beta \in H^G(C_2) \),

\[
\gamma_{1*}(\alpha^o) \cdot \gamma_{2*}(\beta^o) = \gamma_{3*}((\alpha|_{C_3} \cup \beta|_{C_3})^o),
\]

where \( \alpha|_{C_3} \) (resp. \( \beta|_{C_3} \)) is the restriction of \( \alpha \) (resp. \( \beta \)) to \( C_3 \). \( \square \)

Let us recall the definition of the convolution product. Given quasi-projective \( G \)-varieties \( X_1, X_2, X_3 \), consider the projection \( p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j \) for all \( 1 \leq i < j \leq 3 \). Consider subvarieties \( Z_{ij} \subset X_i \times X_j \) such that the restriction of \( p_{13} \) to \( p_{12}^{-1}Z_{12} \cap p_{23}^{-1}Z_{23} \) is proper and maps to \( Z_{13} \). The convolution product is the map

\[
* : \ H^G(Z_{12}) \otimes H^G(Z_{23}) \to H^G(Z_{13}), \quad \alpha \otimes \beta \mapsto p_{13*}(p_{12}^*\alpha) \cdot (p_{23}^*\beta).
\]

See [1, 2.7 and the remark (iii), page 113] for more details on convolution product.

We will essentially consider the case \( X_i = T(v^i, w) \) and \( Z_{ij} = Z(v^i, v^j; w) \), where \( v^1, v^2, v^3, w \in \mathbb{N}[I] \) and \( 1 \leq i < j \leq 3 \).

4. Statement of the Result.
Let \( (I, E) \) be a graph of type \( ADE \). Fix \( w, v^1, v^2 \in \mathbb{N}[I] \), with \( v^2 = v^1 + \alpha_k \). For any \( k \), denote by \( \cal{V}_k^1 \) (resp. \( \cal{V}_k^2 \)) the vector bundle \( \cal{V}_k \otimes O_{T(v^2, w)} \) (resp. \( O_{T(v^1, w)} \otimes \cal{V}_k \)) over \( T(v^1, w) \times T(v^2, w) \). The restriction to \( C_k^+(v^2, w) \) of the sheaf \( \cal{V}_k^1 \) is a subsheaf
Let $\Delta : T^k \rightarrow T^k$ and (1.4) in the case $k = 1$. If $v, v', v''$ take all the possible values in $\mathbb{N}[I]$. Let $\Delta^\pm$ be the two natural embeddings

$$\Delta^+ : C^+_k(v^2, w) \hookrightarrow Z(v^1, v^2; w) \quad \text{and} \quad \Delta^- : C^-_k(v^2, w) \hookrightarrow Z(v^2, v^1; w).$$

If $r \geq 0$, put

$$x^\pm_{k,r} = \sum_{v^2} (-1)^{(\alpha_k | v^2)} \Delta^\pm_2(c_1(L^\pm_k)^\gamma)^r \in H^G_w(Z(w)).$$

Let $\Delta : T(v, w) \rightarrow T(v, w) \times T(v, w)$ be the diagonal embedding and set $h = c_1(q^2)^\circ$. Define $h_{k,r}$ as the coefficient of $hz^{-r-1}$ in

$$-1 + \sum_v \Delta_v \left( \frac{\lambda_{-1/z}(F_k(v, w))}{\lambda_{-1/z}(q^2 F_k(v, w))} \right)^-,$$

where $- \Delta$ stands for the expansion at $z = \infty$. The following result was conjectured by Nakajima ([5, Introduction], in [2] the result was announced for type $A$).

**Theorem.** For all $w \in \mathbb{N}[I]$, the map $x^\pm_{k,r} \mapsto x^\pm_{k,r}$, $h_{k,r} \mapsto h_{k,r}$ extends uniquely to an algebra homomorphism $\Phi_w : Y_h(Lq) \rightarrow H^G_w(Z(w))$. \[\square\]

**Remark.** We can prove a similar result for any symmetric Kac-Moody algebra. Let $A = (a_{kl})_{k,l \in I}$ be a symmetric generalized Cartan matrix. In the definition of the Yangian, the relation (1.4) becomes

$$\left\{ \begin{array}{l}
[x^\pm_{k,r+1}, x^\pm_{k,s}] - [x^\pm_{k,r}, x^\pm_{k,s+1}] = \pm h(x^\pm_{k,r} x^\pm_{k,s} + x^\pm_{k,s} x^\pm_{k,r}) \\
\eta_{-a_{kl}}(z, w)x^\pm_{k,r}(z)x^\pm_{k,s}(w) = \eta_{-a_{kl}}(z, w = \frac{h}{2})x^\pm_{k,s}(w)x^\pm_{k,r}(z) \quad \text{if } k \neq l
\end{array} \right.$$

where

$$x^\pm_{k}(z) = \sum_{r \geq 0} x^\pm_{k,r} z^{-r}, \quad \text{and} \quad \eta_a(z, w) = \prod_{j=1}^{n} (z - w + (1 + a - 2j)h/2).$$

In this case the action of $\mathbb{C}^\times$ on $T(v, w)$ and the complex $F_k(v, w)$ has to be changed as in [5]. In the proof of the theorem there are only minor and evident changes to do.

**5. Proof of the Result.**

The proof is as in [5, sections 10 and 11]: we check relations (1.1), (1.2), (1.5) and relations (1.3) and (1.4) in the case $k \neq l$ by direct computation. Relations (1.3) and (1.4) in the case $k = l$ are proved by reduction to the $\mathfrak{sl}_2$-case. We insist here only on the parts which need different calculations.
Relation (1.1). It is an immediate consequence of the definition, since for all \( x \in H^{G_w}(Z(v, v'; w)) \) we have

\[
h_{k, 0} \ast x = \text{rk} \mathcal{F}_k(v, w)x = (w - v|\alpha_k)x,
\]

\[
x \ast h_{k, 0} = \text{rk} \mathcal{F}_k(v', w)x = (w - v'|\alpha_k)x.
\]

Relation (1.2). We prove only the plus case, the minus being similar. Fix \( v^2 = v^1 + a_l \). We identify \( \mathcal{F}_k(v^1, w) \) and \( \mathcal{F}_k(v^2, w) \) with their pull-back to \( C^+_l(v^2, w) \) via the 1-st and the 2-nd projection. Then, in \( K^{G_w}(C^+_l(v^2, w)) \), we have

\[
\mathcal{F}_k(v^1, w) - q^2 \mathcal{F}_k(v^1, w) = \mathcal{F}_k(v^2, w) - q^2 \mathcal{F}_k(v^2, w) + [a_{kl}](q^{-1} - q)L^+_l.
\]

It follows that \([h_{k, r}, x^{+}_{l, s}] \in H^{G_w}(C^+_l(v^2, w))\) is the coefficient of \( h_z^{-r-1} \) in

\[
(\lambda_{-1/z}(\mathcal{F}_k(v^1, w) - q^2 \mathcal{F}_k(v^1, w)) - \lambda_{-1/z}(\mathcal{F}_k(v^2, w) - q^2 \mathcal{F}_k(v^2, w)))^{- x^{+}_{l, s}} =
\]

\[
= \left( (\lambda_{-1/z}([a_{kl}](q^{-1} - q)L^+_l) - 1)\lambda_{-1/z}(\mathcal{F}_k(v^2, w) - q^2 \mathcal{F}_k(v^2, w)) \right)^{- x^{+}_{l, s}}.
\]

Set

\[
A_s = \lambda_{-1/z}(\mathcal{F}_k(v^2, w) - q^2 \mathcal{F}_k(v^2, w))x^{+}_{l, s},
\]

\[
X = \lambda_{-1/z}([a_{kl}](q^{-1} - q)L^+_l) = \frac{1 - (c^+_l - a_{kl}h/2)z^{-1}}{1 - (c^+_l + a_{kl}h/2)z^{-1}}.
\]

Then the LHS and the RHS of the relation (1.2) are respectively equal to the coefficient of \( h_z^{-r-1} \) in

\[
(2z(X - 1)A_s - 2(X - 1)A_{s+1})^{-} = (2(X - 1)(z - c^+_l)A_s)^{-} \quad \text{and} \quad (ha_{kl}(X + 1)A_s)^{-}.
\]

We are then reduced to the identity, easily checked, in \( H^{G_w}(C^+_l(v^2, w)) \):

\[
2(X - 1)(z - c^+_l) = ha_{kl}(X + 1).
\]

Relation (1.3) with \( k \neq l \). Fix \( v^1, v^2, \tilde{v}^2, v^3 \), such that

\[
\tilde{v}^2 = v^1 - \alpha_l = v^3 - \alpha_k = v^2 - \alpha_k - \alpha_l.
\]

If \( 1 \leq i < j \leq 3 \), consider the projections

\[
p_{ij} : T(v^1, w) \times T(v^2, w) \times T(v^3, w) \rightarrow T(v^i, w) \times T(v^j, w),
\]

\[
\tilde{p}_{ij} : T(v^1, w) \times T(\tilde{v}^2, w) \times T(v^3, w) \rightarrow T(\tilde{v}^i, w) \times T(\tilde{v}^j, w),
\]

where we set \( \tilde{v}^1 = v^1, \tilde{v}^3 = v^3 \). We have

\[
x^{+}_{k, r} \ast x^{-}_{l, s} = (-1)^{(\alpha_k|v^3) + (\alpha_l|v^3)} p_{13*} (p_{12*}(c_1(V^2_k/V^1_k)^{or}) \cdot p_{23*}(c_1(V^3_l/V^1_l)^{os}))
\]

\[
x^{-}_{l, s} \ast x^{+}_{k, r} = (-1)^{(\alpha_l|\tilde{v}^2) + (\alpha_k|\tilde{v}^2)} \tilde{p}_{13*} (\tilde{p}_{12*}(c_1(V^2_l/V^1_l)^{os}) \cdot \tilde{p}_{23*}(c_1(V^3_k/V^1_k)^{or})).
\]
It is proved in [5, Lemma 10.2.1] that the intersections

\begin{align*}
p_{12}^{-1}C_k^+(v^2, w) \cap p_{23}^{-1}C_l^-(v^3, w) \quad \text{and} \quad \tilde{p}_{12}^{-1}C_l^-(\tilde{v}^2, w) \cap \tilde{p}_{23}^{-1}C_k^+(\tilde{v}^3, w)
\end{align*}

are transversal in \( T(v^1, w) \times T(v^2, w) \times T(v^3, w) \) and \( T(v^1, w) \times T(\tilde{v}^2, w) \times T(v^3, w) \) and that there exists a \( \mathcal{G}_w \)-equivariant isomorphisms between them which induces the isomorphisms:

\begin{align*}
V_k^2/V_k^1 \simeq V_l^2/V_l^1 \quad \text{and} \quad \tilde{V}_l^2/\tilde{V}_l^1 \simeq V_l^3/V_l^2.
\end{align*}

The result follows from the lemma in section 3.

Relation (1.4) with \( k \neq l \). We prove only the plus case. Fix \( v^1, v^2, \tilde{v}^2, v^3 \), such that

\begin{align*}
v^3 = \tilde{v}^2 + \alpha_k = v^2 + \alpha_l = v^1 + \alpha_k + \alpha_l.
\end{align*}

Consider the projections \( p_{ij} \) and \( \tilde{p}_{ij} \) \((1 \leq i < j \leq 3)\) as before. The intersections

\begin{align*}
Z_{kl} = p_{12}^{-1}C_k^+(v^2, w) \cap p_{23}^{-1}C_l^-(v^3, w) \quad \text{and} \quad Z_{lk} = \tilde{p}_{12}^{-1}C_l^-(\tilde{v}^2, w) \cap \tilde{p}_{23}^{-1}C_k^+(\tilde{v}^3, w)
\end{align*}

are transversal in \( T(v^1, w) \times T(v^2, w) \times T(v^3, w) \) and \( T(v^1, w) \times T(\tilde{v}^2, w) \times T(v^3, w) \) (see [5, Lemma 10.3.1]). Since \( k \neq l \), the restriction of \( p_{13} \) and \( \tilde{p}_{13} \) to \( Z_{kl} \) and \( Z_{lk} \) is an embedding into \( Z(v^1, v^3; w) \). Call it \( \iota_{kl} \) and \( \iota_{lk} \) respectively. Put \( b_k = c_1(V_k^3 - V_k^1), b_l = c_1(V_l^3 - V_l^1) \). We have (see the lemma in section 3)

\begin{align*}
x_{k,r}^+ \ast x_{l,s}^+ &\equiv (-1)^{(\alpha_k|v_k^2) + (\alpha_l|v_l^2) + \iota_{kl}}(p_{12}(b_k^r)_{|Z_{kl}} \cup p_{23}(b_l^s)_{|Z_{kl}})^o, \\
x_{l,s}^+ \ast x_{k,r}^+ &\equiv (-1)^{(\alpha_l|v_l^2) + (\alpha_k|v_k^2) + \iota_{lk}}(p_{12}(b_l^s)_{|Z_{lk}} \cup p_{23}(b_k^r)_{|Z_{kl}})^o.
\end{align*}

Take \( h \in H \) such that \( h' = l \) and \( h'' = k \). The map \( B_h \) may be viewed as a section of the \( \mathcal{G}_w \)-bundle \( E_{kl} = q(V_k^3/\mathbb{V}_k^1)^* \otimes (V_k^3/\mathbb{V}_k^1) \) on \( p_{13}(Z_{kl}) \) (where we set \( E_{kl} = 0 \) if \( a_{kl} = 0 \)). Similarly \( B_h \) is a section of the \( \mathcal{G}_w \)-bundle \( E_{lk} = q(V_l^3/\mathbb{V}_l^1)^* \otimes (V_l^3/\mathbb{V}_l^1) \) on \( \tilde{p}_{13}(Z_{lk}) \) (where again we set \( E_{lk} = 0 \) if \( a_{kl} = 0 \)). In [5, 10.3.9] it is proved that \( B_h \) and \( B_h \) are transversal to the zero section respectively. Moreover

\begin{align*}
p_{13}(Z_{kl}) \cap B_h^{-1}(0) &\equiv \tilde{p}_{13}(Z_{lk}) \cap B_h^{-1}(0).
\end{align*}

Then,

\begin{align*}
\iota_{kl} \ast (c_1(E_{kl})^o) x_{k,r}^+ \ast x_{l,s}^+ &\equiv (-1)^{\alpha_k \iota_{kl}}(c_1(E_{lk})^o) x_{l,s}^+ \ast x_{k,r}^+,
\end{align*}

i.e.

\begin{align*}
\iota_{kl}(b_k^r - b_l^s + h/2) x_{k,r}^+ \ast x_{l,s}^+ = \iota_{lk}(b_l^s - b_k^r + h/2) x_{l,s}^+ \ast x_{k,r}^+.
\end{align*}

The relation (1.4) follows immediately from this.

Relations (1.3) and (1.4) with \( k = l \). Thank to the same argument than in [5, 11.3] we are reduce to the case of \((I,E)\) of type \( A_1 \). In this case \( v \) and \( w \) are identified with natural numbers, so let us call them \( v \) and \( w \). Moreover we will omit everywhere the subindex 1. Let \( Gr_v(w) \) be the variety of \( v \)-dimensional subspaces in \( W \). It is easy to see that \( T(v,w) \simeq T^*Gr_v(w) \). The group \( G_w \) acts in the obvious way on
\(T(v, w)\). The group \(\mathbb{C}^\times\) acts by scalar multiplication on the fibers of the cotangent bundle. Fix \(T_1, ..., T_w\) such that
\[
K^G_\chi(\text{Gr}_v(w)) = \mathbb{C}[q^{\pm 1}, T_1^{\pm 1}, ..., T_w^{\pm 1}]^{S_v \times S_w - v},
\]
where \(e_i\) is the \(i\)-th elementary symmetric function. We get
\[
\mathcal{F}(v, w) = q^{-2}\mathcal{W} - (1 + q^{-2})\mathcal{V} = q^{-2}(T_{v+1} + \cdots + T_w) - (T_1 + \cdots + T_v).
\]
Put \(t_k = c_1(T_k)^\circ\). Then \(H^{G_\chi}(T(w)) = \bigoplus_{v=0}^w \mathbb{C}[h, t_1, \ldots, t_w]^{S_v \times S_w - v}\). The following lemma is proved as in [1, Claim 7.6.7].

**Lemma.** The space \(H^{G_\chi}(T(w))\) is a faithful module over \(H^{G_\chi}(Z(w))\). \(\square\)

The operators \(x_+^\pm\) on \(H^{G_\chi}(T(w))\) can be written down explicitly. Put
\[
O(v, w) = \{(V^1, V^2) \in \text{Gr}_{v-1}(w) \times \text{Gr}_v(w) | V^1 \subset V^2\}.
\]
The Hecke correspondence \(C^+(v, w)\) is the conormal bundle to \(O(v, w)\). Consider the projections \(p_1, p_2\) from \(O(v, w)\) to the first and the second component and let \(\pi : T(v, w) \to \text{Gr}_v(w)\) be the projection. We can prove as in [6, Lemme 5] that if \(\alpha \in H^{G_\chi}(O(v, w))\) and \(\beta \in H^{G_\chi}(\text{Gr}_v(w))\), then
\[
\pi^*(\alpha) \star \pi^*(\beta) = p_{1*}(\lambda(q^2T^*p_1) \cdot \alpha \cdot T^*\beta),
\]
where \(T^*p_1\) is the relative cotangent bundle to \(p_1\). The map \(p_1\) is a \(\mathbb{P}^{w-v}\)-fibration, then we have
\[
\lambda(q^2T^*p_1) = \prod_{m=v+1}^w (t_m - t_v + h) \in H^{G_\chi}(O(v, w)).
\]
Let us introduce the following notation. Fix \(z \in [1, w] = \{1, 2, ..., w\}\) and let \(I = (I_1, I_2)\) be a partition of \([1, w]\) into two subset of cardinality \(z\) and \(w - z\) respectively, say \(I_1 = \{a_1, a_2, ..., a_z\}\), \(I_2 = \{b_1, b_2, ..., b_{w-z}\}\). Then put
\[
f(t_{I_1}; t_{I_2}) = f(t_{a_1}, t_{a_2}, ..., t_{a_z}, t_{b_1}, t_{b_2}, ..., t_{b_{w-z}}).
\]
Thus (see [6, Lemme 1]), for any \(f \in \mathbb{C}[h, t_1, ..., t_w]^{S_v \times S_w - v}\),

(5.1)
\[
x_+^\pm(f)(t_{[1, v-1]}; t_{[v, w]}) = \sum_{k=v}^w f(t_{[1, v-1] \cup \{k\}}; t_{[v, w] \setminus \{k\}})\prod_{m \in [v, w] \setminus \{k\}} \left(1 + \frac{h}{t_k - t_m}\right),
\]

(5.2)
\[
x_-^\pm(f)(t_{[1, v+1]}; t_{[v+2, w]}) = \sum_{k=1}^{v+1} f(t_{[1, v+1] \setminus \{k\}}; t_{[v+2, w] \cup \{k\}})\prod_{m \in [1, v+1] \setminus \{k\}} \left(1 + \frac{h}{t_m - t_k}\right).
\]
We have
\[ \lambda_z F(v, w) = \frac{\prod_{m=v+1}^w (1 - z(t_m - h))}{\prod_{m=1}^v (1 - z(t_m + 1))}. \]

Thus \( h_r(f) \) is the coefficient of \( h z^{-r-1} \) in
\[ f \left( \prod_{m=1}^v \frac{z - t_m - h}{z - t_m} \prod_{m=v+1}^w \frac{z - t_m + h}{z - t_m} \right)^{-1}. \]

**Proposition.** Relations (1.3) and (1.4) hold in the \( \mathfrak{sl}_2 \)-case.

**Proof.** Let us prove relation (1.3). Fix \( f \in \mathbb{C}[h, t_1, ..., t_w] S_v \times S_{w-v} \). Using formulas (5.1) and (5.2), we have
\[
(x^-_r x^+_s(f))(t_{[1,v]}; t_{[v+1,w]}) = \sum_{l=1}^v \sum_{t \in [1,v] \cup \{l\}} f(t_{|[1,v] \cup \{l\}}; t_{|[v+1,w] \cup \{l\}}) t^r_t \cdot X_{kl},
\]
\[
(x^+_r x^-_s(f))(t_{[1,v]}; t_{[v+1,w]}) = \sum_{l \in [1,v] \cup \{k\}} \sum_{k=v+1}^w f(t_{|[1,v] \cup \{k\}}; t_{|[v+1,w] \cup \{k\}}) t^r_t Y_{kl},
\]
where
\[
X_{kl} = \prod_{m \in [1,v] \setminus \{l\}} \left( 1 + \frac{h}{t_m - t_l} \right) \prod_{n \in [v+1,w] \cup \{l\} \setminus \{k\}} \left( 1 + \frac{h}{t_k - t_n} \right),
\]
\[
Y_{kl} = \prod_{m \in [1,v] \cup \{k\} \setminus \{l\}} \left( 1 + \frac{h}{t_m - t_l} \right) \prod_{n \in [v+1,w] \setminus \{k\}} \left( 1 + \frac{h}{t_k - t_n} \right).
\]
The terms with \( k \neq l \) cancel out in the bracket. We get
\[
[x^+_r, x^-_s](f) = f \sum_{k=v+1}^w t^r_k t^-_k \prod_{m \in [1,v]} \left( 1 + \frac{h}{t_m - t_l} \right) \prod_{n \in [v+1,w] \setminus \{k\}} \left( 1 + \frac{h}{t_k - t_n} \right) -
\]
\[
-f \sum_{l=1}^v \prod_{m \in [1,v] \setminus \{l\}} \left( 1 + \frac{h}{t_m - t_l} \right) \prod_{n \in [v+1,w]} \left( 1 + \frac{h}{t_l - t_n} \right).
\]
Put
\[
A(z) = \prod_{m=1}^v (z - t_m), \quad B(z) = \prod_{m=1}^v (z - t_m - h) \prod_{m=v+1}^w (z - t_m + h).
\]
Then it is easy to check that
\[
h[x^+_r, x^-_s](f) = f \sum_{k=1}^w t^r_k B(t_k) A'(t_k) = f \text{res}_\infty z^{r+s} \frac{B(z)}{A(z)}.
\]
This is the definition of \( h_{r+s}(f) \) given in (5.3). As for the relation (1.4), note that, using (5.1), we get
\begin{align*}
(x^+_n x^+_m (f))(t_{[1,v-2]}; t_{[v-1,w]} &= \\
= & \sum_{l=v-1}^{w} \sum_{k \in [v-1,w] \setminus \{l\}} f(t_{[1,v-2] \cup \{k,l\}}; t_{[v-1,w] \setminus \{k,l\}}) t^+_l t^+_k Z_{kl},
\end{align*}

where

\[ Z_{kl} = \prod_{n \in [v-1,w] \setminus \{l\}} \left( 1 + \frac{\hbar}{t_l - t_n} \right) \prod_{m \in [v-1,w] \setminus \{k,l\}} \left( 1 + \frac{\hbar}{t_k - t_m} \right). \]

The relation in the plus case follows now by a direct computation. \qed

\textbf{Relation (1.5).} The proof is exactly as in [5, 10.4], so we omit it.

\textbf{Remark.} Nakajima [5, Theorem 9.4.1] has proved that there exists an algebra morphism

\[ \Psi_w : U_q(Lg) \to K^G_w(Z(w)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q), \]

where the algebra to the left is the quantized enveloping algebra of \( Lg \) and the algebra to the right is equipped with the convolution product. Using \( \Phi_w \) and \( \Psi_w \) we can construct the finite dimensional simple modules of \( Y_h(Lg) \) and \( U_q(Lg) \) respectively (see [5, section 14]). In particular \( Y_h(Lg) \) and \( U_q(Lg) \) have the same finite dimensional representation theory. More precisely let \( \mathcal{C} \) (resp. \( \mathcal{D} \)) be the abelian category of finite dimensional \( U_q(Lg) \)-modules such that the Drinfeld polynomials of the simple factors have roots in \( q^{-1} \mathbb{Z} \) (resp. \( \mathbb{Z} \)).

\textbf{Proposition.} The characters (as \( U_q(Lg) \)-modules and \( U(Lg) \)-modules resp.) of the simple finite dimensional modules in \( \mathcal{C} \) and in \( \mathcal{D} \) are the same. \qed

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