Distributed Wasserstein Barycenters via Displacement Interpolation

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Abstract

Consider a multi-agent system whereby each agent has an initial probability measure. In this paper, we propose a distributed algorithm based upon stochastic, asynchronous and pairwise exchange of information and displacement interpolation in the Wasserstein space. We characterize the evolution of this algorithm and prove it computes the Wasserstein barycenter of the initial measures under various conditions. One version of the algorithm computes a standard Wasserstein barycenter, i.e., a barycenter based upon equal weights; and the other version computes a randomized Wasserstein barycenter, i.e., a barycenter based upon random weights for the initial measures. Finally, we specialize our algorithm to Gaussian distributions and draw a connection with opinion dynamics.

1 Introduction

Problem statement and motivation There has been strong interest in the theoretical study and practical application of Wasserstein barycenters over the last decade. In this paper, we characterize the evolution of a distributed system where all the computing units or agents hold a probability measure, interact through pairwise communication by performing displacement interpolations in the Wasserstein space. These pairwise interactions are asynchronous and stochastic. We study the conditions under which the agents’ measures will asymptotically achieve consensus and, additionally, consensus on a Wasserstein barycenter of the agents’ initial measures. We are interested in computing both the standard Wasserstein barycenter and randomized weighted versions of it – as a result of the stochastic interactions. We consider both undirected and directed communication graphs. To the best of our knowledge, these problems have not been studied in the literature on the distributed computation of Wasserstein barycenters.

Asynchronous pairwise algorithms are inherently robust to communication failures and do not require synchronization of the whole multi-agent system. Pairwise interactions may potentially reduce the local computation complexity of each agent. Indeed, displacement interpolations have the practical advantage that they may have a closed form expression, e.g., in the Gaussian case.

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Wasserstein barycenters and their applications  The Wasserstein barycenter of a set of measures can be interpreted as an interpolation or weighted Fréchet mean of multiple measures in the Wasserstein space [29]. In this interpolation, each measure has an associated positive weight that indicates its importance in the computation of the barycenter. When all weights are equal, we obtain the standard Wasserstein barycenter; otherwise, we obtain a weighted one.

There has been a strong interest in the theoretical study of Wasserstein barycenters over the last decade; e.g., uniqueness results and connections to multi-marginal optimal transport [2]; interpolation of discrete measures with finite support and connections with linear programming [15, 4]; the characterization of the barycenter as a fixed point of an operator and its computation [3]; the study of consistency and other statistical properties [25]. For further information, we refer to [29].

Along with the theoretical progress, many applications of Wasserstein barycenters have emerged, as well as numerical approaches for computing them. Wasserstein barycenters have found applications in economics [14], image processing [30, 27], computer graphics [10], physics [13], statistics [35, 33], machine learning [18, 32], signal processing [7, 5], and biology [24]. Examples of computational approaches include: exact algorithms [15, 17], algorithms that use entropic regularization [19, 20], and algorithms based on approximations of Wasserstein distances [11]. Finally, the particular case of interpolating two measures, i.e., the displacement interpolation, has applications in partial differential equations and geometry [39, 31], and fluid mechanics [6].

Moreover, the Wasserstein barycenter has been interpreted as a denoised version of an original signal whose sensor measurements are each of the noisy probability distributions that are being interpolated; thus, it has found applications as an information fusion algorithm [24, 7, 10, 17]. In this setting, a randomized Wasserstein barycenter, i.e., one which randomly weight each sensor, could be used to provide different estimates of the true signal when interpolating measurements of sensors of unknown accuracy or noise level.

Finally, we mention that the problem studied in this paper directly contributes to the literature on consensus, a research area which has attracted great interest from the systems and controls community. Specifically, we contribute to the fields of randomized consensus algorithms – e.g., see [12, Chapter 13] and references therein – and of consensus in spaces other than the Euclidean space – e.g., see [34, 26, 9]. Moreover, our work contributes to the field of opinion dynamics, where classic opinion models also use stochastic asynchronous pairwise interactions [21, 1]. Indeed, in our paper, we argue that displacement interpolation is a more suitable modeling approach for the non-Bayesian updating of individual’s beliefs in a social network, than classic averaging approaches in the literature.

Distributed algorithms for Wasserstein barycenters  To the best of our knowledge, there is only a recent and growing literature on distributed algorithms for Wasserstein barycenters. The idea of computing Wasserstein barycenters in a distributed way was first pioneered by Bishop and Doucet in their work [8] and its very recent extension [9]. Their work formally shows consensus towards the Wasserstein barycenter of the agents’ initial measures. In order to compute such consensus, each agent needs to fully compute the Wasserstein barycenter resulting from its own measure and the measures from all its neighbors at each iteration according to some time-varying graph. The work [9] focuses on the case of probability measures on the real line, with the communication between agents being deterministic, but flexible enough to consider both synchronous and asynchronous deterministic updating. It assumes agents are connected by an undirected graph.

The recent work [37] focuses on the design and distributed implementation of a numerical solver that approximates the standard Wasserstein barycenter when all the measures are discrete, through the use of entropic regularization. Moreover, the recent work [22] from the same authors
proposes another distributed solver for an approximate Wasserstein barycenter with the difference that the agents’ measures may correspond to continuous distributions. Indeed, its framework is semi-discrete, in that the measures to be interpolated can be continuous, but the sought measure that serves as a proxy for the barycenter is restricted to be a discrete measure with finite support. Therefore, we observe that the distributed algorithms from both works [37, 22] compute an approximate or a proxy of the true barycenter. Both works require synchronous updating and all the computations are performed over an undirected graph.

Our paper is more in line with the spirit of [8, 9], in the sense that we propose a theoretical formulation and analysis that prove how to generate Wasserstein barycenters from distributed computations. We do not propose specific designs of numerical solvers for the local computations of the agents, as it is instead performed in [37, 22]. Indeed, since the local computations in our algorithm are displacement interpolations at every time step, any numerical method that can solve optimal transport problems can be used, including for example any of the numerical algorithms mentioned above.

Contributions In this paper we propose the algorithm PaWBar (Pairwise distributed algorithm for Wasserstein Barycenters), where the agents update their measures via pairwise stochastic and asynchronous interactions implementing displacement interpolations. The algorithm has a directed and a symmetric version. As main contribution of this paper, we establish conditions under which both versions compute randomized and standard Wasserstein barycenters respectively. In the directed case, we prove that every time the algorithm is run, a barycenter with random convex weights is asymptotically generated as a result of the stochastic selection of the pairwise interactions. It is easy to characterize the first two moments of these random weights. On the other hand, in the symmetric case, although the interactions are stochastic, we prove that the asymptotically computed Wasserstein barycenter is the standard one (with probability one). In contrast to the works [37, 22], our algorithm does not require all the agents to synchronously update their measures at every time step. Moreover, our framework provides convergence guarantees towards the computation of the barycenter independently from the numerical implementation of the local computations. We also remark that work [9] is different from ours because: (i) it only focuses on measures on the real line \( \mathbb{R} \), while we consider \( \mathbb{R}^d, d \geq 1 \); (ii) its underlying communication graph is undirected at all time-steps and its changes are deterministic; (iii) it considers local computations of the full Wasserstein barycenter between agent and its neighbors; and (iv) we present sufficient conditions for the computation of the standard Wasserstein barycenter.

We now elaborate on the convergence results. We first prove convergence to a randomized or standard Wasserstein barycenter for a class of discrete measures on \( \mathbb{R}^d, d \geq 1 \). In particular, we show that the obtained barycenter interpolates the agents’ measures attained at some random finite time. However, if the initial measures are sufficiently close in the Wasserstein space, then such time is zero with probability one, i.e., there is an interpolation of the initial measures. For the case where these discrete measures are on \( \mathbb{R} \), the interpolation of the initial measures occurs with probability one irrespective of how distant they initially are from each other.

We then prove convergence to a randomized or standard Wasserstein barycenter for a class of measures that are absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^d \). As corollaries, we prove convergence of continuous probability distributions on the real line, and of a class of multivariate Gaussian distributions. In the Gaussian case, we also provide simpler closed form expressions for the computations of the PaWBar algorithm, and a simplified expression of the converged barycenter. We also conjecture that the convergence to Wasserstein barycenters holds for general absolutely continuous measures, and present supporting numerical evidence for the general
multivariate Gaussian case.

Moreover, in all the cases mentioned above, the convergence results are proved under general communication graphs: a strongly connected digraph for the directed PaWBar algorithm and a connected undirected graph for the symmetric algorithm. For randomized barycenters, we characterize their random convex coefficients by the limit product of random stochastic matrices.

Finally, we prove a general consensus result for the case of arbitrary initial measures on $\mathbb{R}^d$ by making strong use of general geodesic properties of the Wasserstein space. The results are proved over a cycle graph for the directed PaWBar algorithm and a line graph for the symmetric case. On $\mathbb{R}$, using a result from [9], our algorithms achieve consensus under general graphs.

**Paper organization**  Section 2 has notation and preliminary concepts. Section 3 has the proposed PaWBar algorithm and its theoretical analysis. Section 4 presents the proofs for Section 3. Section 5 presents the connection between our algorithm and opinion dynamics. Section 6 is the conclusion.

### 2 Notation and preliminary concepts

Let $z = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$ denote a vector. Let $\| \cdot \|_2$ denote the Euclidean distance. The standard unit vector $e_i \in \{0, 1\}^n$ has one in its $i$th entry. Let $\mathbb{I}_n, 0_n \in \mathbb{R}^n$ be the all-ones and all-zeros vectors respectively, and $I_n$ be the $n \times n$ identity matrix. Nonnegative matrix $A \in \mathbb{R}^{n \times n}$ is row-stochastic if $A \mathbb{I}_n = \mathbb{I}_n$, and doubly-stochastic if additionally $A^\top \mathbb{I}_n = \mathbb{I}_n$. The operator $\circ$ the composition of functions, and $\otimes$ the Kronecker product.

The numbers $\lambda_1, \ldots, \lambda_n$ are called convex coefficients if $\lambda_i \geq 0$, $i \in \{1, \ldots, n\}$, and $\sum_{i=1}^n \lambda_i = 1$. The vector $\lambda := (\lambda_1, \ldots, \lambda_n)^\top$ is called a convex vector.

The set of agents is $V = \{1, \ldots, n\}$, $n \geq 2$. The agents are connected according to the graph $G = (V, E)$; with set of nodes $V$ and set of edges $E$. When $E$ only has ordered pairs, i.e., $(i, j) \in E$ with $i, j \in V$, $G$ is a directed graph or digraph. Thus, $(i, j) \in E$ is a directed edge going from $i$ to $j$. When $E$ only has unordered pairs, i.e., $(i, j) \in E$ with $i, j \in V$, $G$ is an undirected graph, and its edges have no sense of direction. $G$ is weighted when a scalar value is assigned to every edge. An undirected graph $G$ is a line graph when its nodes can be labeled as $E = \{(1, 2), \ldots, (n-1, n)\}$. A digraph $G$ is a cycle when its nodes can be labeled as $E = \{(1, 2), \ldots, (n-1, n), (1, n)\}$. Given any $i, j \in V$, a digraph is strongly connected when it is possible to go from $i$ to $j$ by traversing the edges according to their direction (e.g., a cycle graph) and an undirected graph is connected when it is possible to go from $i$ to $j$ by traversing the edges in any direction (e.g., a line graph).

We denote the set of all probability measures on $\Omega \subseteq \mathbb{R}^d$ by $\mathcal{P}(\Omega)$, and define $\mathcal{P}^2(\Omega) = \{\mu \in \mathcal{P}(\Omega) \mid \int_{\Omega} \|x\|^2_2 \, d\mu(x) < \infty\}$. Consider $\mu \in \mathcal{P}(\Omega)$. For $\Omega = \mathbb{R}$, let $F_\mu$ be the cumulative distribution function, i.e., $F_\mu(x) = \mu((\infty, x])$. Let $\#$ be the push-forward operator, which, for any Borel measurable map $\mathcal{M} : \Omega \to \Omega$, defines the linear operator $\mathcal{M}_\# : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ by $(\mathcal{M}_\# \mu)(B) = \mu(\mathcal{M}^{-1}(B))$ for any Borel set $B \subseteq \Omega$. We denote the support of $\mu$ by $\text{supp}(\mu)$.

Given $\mu, \nu \in \mathcal{P}^2(\Omega)$, the 2-Wasserstein distance between $\mu$ and $\nu$ is

$$W_2(\mu, \nu) = \left( \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\Omega \times \Omega} \|x - y\|^2_2 \, d\gamma(x, y) \right)^{1/2} \tag{1}$$

where $\Pi(\mu, \nu)$ is the set of probability measures on $\Omega \times \Omega$ with marginals $\mu$ and $\nu$, i.e., if $\gamma \in \Pi(\mu, \nu)$, then $(\pi_1)_\# \gamma = \mu$ and $(\pi_2)_\# \gamma = \nu$ with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. The optimization problem that defines the Wasserstein distance is an optimal transport problem, and any of its solutions is an optimal transport plan. Let $\gamma^{\text{opt}}(\mu, \nu)$ denote an optimal transport plan between measures $\mu$ and $\nu$. 

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Given $\gamma^{\text{opt}}(\mu, \nu)$ such that $\nu = \mathcal{T}_{\#} \mu$, we say that $\gamma^{\text{opt}}$ solves the Monge optimal transport problem and the map $\mathcal{T}$ is called the optimal transport map from $\mu$ to $\nu$. The Wasserstein space of order 2 is the space $\mathcal{P}^2(\Omega)$ endowed with the distance $W_2$. In this paper, we will consider $\Omega = \mathbb{R}^d$, $d \geq 1$.

Given convex coefficients $\lambda_1, \ldots, \lambda_n$ – also called weights – and measures to interpolate $\mu_1, \ldots, \mu_n \in \mathcal{P}^2(\Omega)$, $\Omega$ convex, $n \geq 2$, a Wasserstein barycenter is defined by any solution to the convex problem

$$\min_{\nu \in \mathcal{P}^2(\Omega)} \sum_{i=1}^{n} \lambda_i W_2^2(\nu, \mu_i).$$

The displacement interpolation between measures $\mu, \nu \in \mathcal{P}^2(\Omega)$ is the curve $\mu_\lambda = (\pi_\lambda)_{\#} \gamma^{\text{opt}}(\mu, \nu)$, $\lambda \in [0, 1]$, where $\pi_\lambda : \Omega \times \Omega \to \Omega$ is defined by $\pi_\lambda(x, y) = (1 - \lambda)x + \lambda y$. The curve $\pi_\lambda$ is known to be a constant-speed geodesic curve in the Wasserstein space connecting $\mu_0 = \mu$ to $\mu_1 = \nu$ [31]. Moreover, for a fixed $\lambda \in [0, 1]$, it is known to be the solution to the Wasserstein barycenter problem

$$\min_{\rho \in \mathcal{P}^2(\Omega)} ((1 - \lambda)W_2^2(\rho, \mu_1) + \lambda W_2^2(\rho, \mu_2)).$$

When there exists an optimal transport map $\mathcal{T}$, then $(\pi_\lambda)_{\#} \mu = ((1 - \lambda)\text{Id} + \lambda \mathcal{T})_{\#} \mu$, $\lambda \in [0, 1]$, where $\text{Id}$ is the identity operator.

3 Proposed algorithm and analysis

3.1 The PaWBar algorithm

Let $\mu_i(t) \in \mathcal{P}^2(\mathbb{R}^d)$, $i \in V$, represent the measure of agent $i$ at time $t \in \{0, 1, \ldots\}$. Our proposed PaWBar (Pairwise distributed algorithm for Wasserstein Barycenters) algorithm has two versions.

**Definition 3.1** (Directed PaWBar algorithm). Let $G$ be a weighted directed graph with weight $a_{ij} \in (0, 1)$ for $(i, j) \in E$. Assume $\mu_i(0) := \mu_{i,0} \in \mathcal{P}^2(\mathbb{R}^d)$ for every $i \in V$. At each time $t$, execute:

(i) select a random edge $(i, j) \in E$ of $G$, independently according to some time-invariant probability distribution, with all edges having a positive selection probability;

(ii) update the measure of agent $i$ by

$$\mu_i(t + 1) := (\pi_{a_{ij}})_{\#} \gamma^{\text{opt}}(\mu_i(t), \mu_j(t))$$

where $\pi_{a_{ij}} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by $\pi_{a_{ij}}(x, y) = (1 - a_{ij})x + a_{ij}y$.

**Definition 3.2** (Symmetric PaWBar algorithm). Let $G$ be an undirected graph. Assume $\mu_i(0) := \mu_{i,0} \in \mathcal{P}^2(\mathbb{R}^d)$ for every $i \in V$. At each time $t$, execute:

(i) select a random edge $\{i, j\} \in E$ of $G$, independently according to some time-invariant probability distribution, with all edges having a positive selection probability;

(ii) update the measures of agents $i$ and $j$ by

$$\mu_i(t + 1) = \mu_j(t + 1) := (\pi_{1/2})_{\#} \gamma^{\text{opt}}(\mu_i(t), \mu_j(t))$$

where $\pi_{1/2} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by $\pi_{1/2}(x, y) = \frac{1}{2}(x + y)$.

**Remark 3.1** (Well-posedness). The PaWBar algorithm is well-posed since the displacement interpolation provides measures in $\mathcal{P}^2(\mathbb{R}^d)$ [31, Theorem 5.27].

**Remark 3.2** (Symmetry in the interpolated measure). Since $\pi_{1/2}(x, y) = \pi_{1/2}(y, x)$ for any $x, y \in \mathbb{R}^d$, the update rule (3) of the symmetric PaWBar algorithm is equivalent to $\mu_i(t + 1) := (\pi_{1/2})_{\#} \gamma^{\text{opt}}(\mu_i(t), \mu_j(t))$ and $\mu_j(t + 1) := (\pi_{1/2})_{\#} \gamma^{\text{opt}}(\mu_j(t), \mu_i(t))$.  

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For simplicity, we call edge selection process the underlying stochastic process of edge selection by the PaWBar algorithm, whose realizations are the infinite sequence of selected edges chosen every time the PaWBar algorithm is run. When a result is stated with probability one, it is to be understood with respect to the induced measure by the edge selection process. The following concept and proposition are useful for our results.

**Definition 3.3** (Evolution random matrix). Consider the edge selection process from the PaWBar algorithm. Define the evolution random matrix $A(t)$ by:

$$A(t) = \begin{cases} 
I_n - a_{ij} e_i^T e_j, & \text{if } (i, j) \in E \text{ is chosen,} \\
I_n - \frac{1}{2} (e_i e_i^T + e_j e_j^T + e_i e_j^T + e_j e_i^T), & \text{if } \{i, j\} \in E \text{ is chosen.}
\end{cases}$$

The following result is a direct application of [12, Theorem 13.1, Corollary 13.2].

**Proposition 3.3** (Convergence of products of evolution random matrices). Consider the PaWBar algorithm. For the directed case with a strongly connected digraph:

$$\lim_{t \to \infty} \prod_{\tau=0}^{t} A(\tau) = \frac{1}{n} \lambda^\top$$

for some random convex vector $\lambda$ with probability one. For the symmetric case with a connected undirected graph,

$$\lim_{t \to \infty} \prod_{\tau=0}^{t} A(\tau) = \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^\top$$

with probability one.

### 3.2 Analysis of discrete measures

**Theorem 3.4** (Wasserstein barycenters for discrete measures in $\mathcal{P}^2(\mathbb{R}^d)$). Consider initial measures $\{\mu_{i,0}\}_{i \in V}$, such that $\mu_{i,0} = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_i^j}$, with $x_1^i, \ldots, x_N^i \in \mathbb{R}^d$ being distinct points; i.e., $\mu_{i,0}$ is a discrete uniform measure.

(i) Consider the directed PaWBar algorithm with an underlying strongly connected digraph $G$; then, with probability one, for any $i \in V$,

$$W_2(\mu_i(t), \mu_\infty) \to 0 \text{ as } t \to \infty,$$

where the discrete uniform measure $\mu_\infty$ is a barycenter

$$\mu_\infty \in \arg \min_{\nu \in \mathcal{P}^2(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^{n} \lambda_i W_2(\nu, \mu_i(T))^2$$

with $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$ being a random convex vector satisfying $\prod_{\tau=1}^{\infty} A(\tau) = \mathbb{1}_n \lambda^\top$ with probability one, and $T \geq 0$ being some finite random time.

(ii) Consider the symmetric PaWBar algorithm with an underlying connected undirected graph $G$; then, with probability one, for any $i \in V$,

$$W_2(\mu_i(t), \mu_\infty) \to 0 \text{ as } t \to \infty,$$

where the discrete uniform measure $\mu_\infty$ is a barycenter

$$\mu_\infty \in \arg \min_{\nu \in \mathcal{P}^2(\mathbb{R}^d)} \frac{1}{n} \sum_{i=1}^{n} W_2(\nu, \mu_i(T))^2$$

with $T \geq 0$ being some finite random time.
In either case (i) or (ii), there exists $\epsilon > 0$ such that if $\max_{i,j \in V} W_2(\mu_{i,0}, \mu_{j,0}) < \epsilon$, then $T = 0$ with probability one.

Corollary 3.5 (Wasserstein barycenters for discrete measures in $\mathcal{P}^2(\mathbb{R})$). Consider initial measures $\{\mu_{i,0}\}_{i \in V}$, such that $\mu_{i,0} = \frac{1}{N} \sum_{j=1}^{N} \delta_{x^i_j}$, with $x^i_1, \ldots, x^i_N \in \mathbb{R}$ such that $x^i_1 < \cdots < x^i_N$. Then, the directed, resp. symmetric, PaWBar algorithm computes a randomized, resp. standard, Wasserstein barycenter of the initial measures under a strongly connected digraph, resp. connected undirected graph.

Remark 3.6 (Discussion of our results). (i) The setting of Theorems 3.4 and Corollary 3.5 has found applications in computational geometry, computer graphics and digital image processing; e.g., see [30, 19, 11]. (ii) In Theorem 3.4, if all initial measures are sufficiently close in the Wasserstein space, then the PaWBar algorithm computes a barycenter. This sufficient condition is not a problem in practical applications where the barycenter is used as an interpolation among measures that are known to be similar (e.g., measurements of the same object under noise). On the other hand, Corollary 3.5 tells us that the initial measures in $\mathcal{P}(\mathbb{R})$ can be arbitrarily distant from each other.

3.3 Analysis of absolutely continuous measures

We consider measures that are absolutely continuous with respect to the Lebesgue measure. For any such measures $\mu, \nu \in \mathcal{P}^2(\mathbb{R}^d)$, there exists a unique optimal transport map from $\mu$ to $\nu$, which we denote by $T^\nu_{\mu}$; we also denote $T^\mu_{\mu} = Id$. It is also known that there exists a unique Wasserstein barycenter when all the interpolated measures are of this class [2].

We focus on the following class of measures that form a compatible collection (based on [29, Definition 2.3.1]): a collection of absolutely continuous measures $\mathcal{C} \subset \mathcal{P}^2(\mathbb{R}^d)$ where for all $\nu, \mu, \gamma \in \mathcal{C}$, we have $T^\mu _\nu \circ T^\nu _\gamma = T^\mu _\gamma$. It is known that a displacement interpolation between any two absolutely continuous measures results in a curve of absolutely continuous measures [31]. This motivates the following definition: we say a compatible collection is closed under interpolation whenever the union of this set and any measure resulting from the displacement interpolation between any of its elements results in another compatible collection with the same property.

Theorem 3.7 (Wasserstein barycenters for absolutely continuous measures in $\mathcal{P}^2(\mathbb{R}^d)$). Consider initial measures $\{\mu_{i,0}\}_{i \in V}$ that are absolutely continuous with respect to the Lebesgue measure and that form a compatible collection closed under interpolation. Let $\gamma \in \{\mu_{i,0}\}_{i \in V}$.

(i) Consider the directed PaWBar algorithm with an underlying strongly connected digraph $G$; then, with probability one, for any $i \in V$,

$$W_2(\mu_i(t), \mu_\infty) \to 0 \quad \text{as} \quad t \to \infty,$$

where the absolutely continuous measure $\mu_\infty = \left( \sum_{j=1}^{n} \lambda_j T^\nu_{\mu_j} \right) \# \gamma$ is the barycenter

$$\mu_\infty = \arg \min_{\nu \in \mathcal{P}^2(\mathbb{R}^d)} \sum_{i=1}^{n} \lambda_i W_2(\nu, \mu_{i,0})^2 \quad \text{(7)}$$

with $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$ being a random convex vector satisfying $\prod_{\tau=1}^{\infty} A(\tau) = 1_n \lambda^\top$ with probability one.
(ii) Consider the symmetric PaWBar algorithm with an underlying connected undirected graph $G$; then, with probability one, for any $i \in V$,

$$W_2(\mu_i(t), \mu_\infty) \to 0 \text{ as } t \to \infty,$$

where the absolutely continuous measure $\mu_\infty = \left( \frac{1}{n} \sum_{j=1}^{n} T^{\mu_j,0} \right) \# \gamma$ is the barycenter

$$\mu_\infty = \arg \min_{\nu \in P^2(\mathbb{R}^d)} \sum_{t=1}^{n} W_2(\nu, \mu_i,0)^2.$$

The following corollary considers examples of measures relevant to our previous theorem. We use the term distribution and measure interchangeably for well-known probability measures with continuous distributions.

**Corollary 3.8** (Wasserstein barycenters for classes of absolutely continuous measures). Consider that initially either

(i) all agents have a probability measure in $\mathcal{P}^2(\mathbb{R})$ with continuous distribution; or

(ii) one agent has the standard Gaussian distribution on $\mathcal{P}^2(\mathbb{R}^d)$ and any other agent $i \in V$ has a Gaussian distribution $\mu_{i,0} = \mathcal{N}(0_d, \Sigma_{i,0})$, positive definite matrix $\Sigma_{i,0} \in \mathbb{R}^{d \times d}$; and there exists an orthogonal matrix $U$ such that $D_{k,0} = US_{k,0}U^\top$ is diagonal for all $k$.

Consider the directed PaWBar algorithm with an underlying strongly connected digraph $G$. Then, with probability one, $W_2(\mu_i(t), \mu_\infty) \to 0$ as $t \to \infty$ for any $i \in V$, where $\mu_\infty$ is the Wasserstein barycenter of the initial measures. In particular,

- for case (i), $\mu_\infty = \left( \sum_{j=1}^{n} \lambda_j F_{\mu_j,0} \odot F_{\mu_i,0} \right) \# \mu_{i,0} = \left( \sum_{j=1}^{n} \lambda_j F_{\mu_j,0}^{-1} \right) \# \mathcal{L}$, with $\mathcal{L}$ being the Lebesgue measure on $[0,1]$; and

- for case (ii), $\mu_\infty = \mathcal{N}(0_d, \Sigma_\infty)$ with $\Sigma_\infty \in \mathbb{R}^{d \times d}$ being a positive definite matrix that satisfies $\Sigma_\infty = \sum_{j=1}^{n} \lambda_j (\Sigma_{0,0}^{1/2} \Sigma_{j,0}^{1/2})^{1/2}$; with $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$ being a random convex vector such that $\prod_{\tau=1}^{\infty} A(\tau) = 1_n \lambda^\top$ with probability one.

Moreover, all the previous results also hold for the symmetric PaWBar algorithm when $G$ is a connected undirected graph, with the difference that the barycenters are now the standard one, i.e., with $\lambda = (\frac{1}{n} \ldots, \frac{1}{n})^\top$ in the previous bullet points.

For Gaussian distributions, since any displacement interpolation results in Gaussian distributions with a closed-form expression [16], the PaWBar algorithm can be written as follows.

**Definition 3.4** (PaWBar algorithm for Gaussian distributions). Assume any agent $i \in V$ has an initial distribution $\mu_{i,0} = \mathcal{N}(m_{i,0}, \Sigma_{i,0})$ with $m_{i,0} \in \mathbb{R}^d$ and $\Sigma_{i,0} \in \mathbb{R}^{d \times d}$ being a positive definite matrix. At any time $t$, let $m_i(t)$ and $\Sigma_i(t)$ be the mean and covariance matrix associated to agent $i \in V$.

(i) For the directed PaWBar algorithm, if $(i,j) \in E$ is selected at time $t$, update the Gaussian distribution of agent $i$ according to:

$$m_i(t + 1) := (1 - a_{ij})m_i(t) + a_{ij}m_j(t),$$

$$\Sigma_i(t + 1) = \Sigma_i(t)^{-1/2}((1 - a_{ij})\Sigma_i(t)) + a_{ij}(\Sigma_i(t)^{1/2} \Sigma_j(t)^{1/2})^{1/2} \Sigma_i(t)^{-1/2}.$$

(ii) For the symmetric PaWBar algorithm, if $(i,j) \in E$ is selected at time $t$, update the Gaussian distributions of agents $i$ and $j$ by using (8) with $a_{ij} = 1/2$ and $m_i(t + 1) = m_j(t + 1)$, $\Sigma_i(t + 1) = \Sigma_j(t + 1)$.
We remark that the work [3] proposes a non-distributed iterative algorithm tailored to compute the Wasserstein barycenter of Gaussian distributions. However, to the best of our knowledge, the PaWBar algorithm is the first one proposing a distributed computation of randomized and standard Gaussian barycenters.

**Remark 3.9** (Further characterizations of the randomized Wasserstein barycenter). The results in [36] can be applied to characterize the mean and covariance matrix associated to the random convex vector present in the randomized Wasserstein barycenter in Theorem 3.4, Corollary 3.5, Theorem 3.7, and Corollary 3.8, which numerically depends on the values of the time-invariant probabilities associated with the edge selection process.

We propose the following conjecture.

**Conjecture 1** (Computation under more general absolutely continuous measures). The convergence results of the PaWBar algorithm in Theorem 3.7 also hold for general absolutely continuous measures.

We provide some numerical evidence that Conjecture 1 is true at least for the case where all agents initially have multivariate Gaussian distributions that do not form a compatible collection. In the numerical evidence presented in Figure 1 and Figure 2, we use the update rules from Definition 3.4 and only focus on the evolution of the agents’ covariance matrices (the mean vectors evolve linearly and are easy to verify they converge to the mean of the Wasserstein barycenter). Although we only present results for the directed PaWBar algorithm, similar results supporting our conjecture were obtained for the symmetric case too.

### 3.4 Analysis of general measures

So far, we presented convergence results to a Wasserstein barycenter for classes of discrete (Theorem 3.4 and Corollary 3.5) and absolutely continuous (Theorem 3.7 and Corollary 3.8) measures. In these cases an optimal transport map exists between any two agents’ measures at every time. Now we analyze the case for general measures in \( \mathcal{P}^2(\mathbb{R}^d) \), which includes cases where there may not exist an optimal transport map between two initial measures or where there could exist a mix of discrete and absolutely continuous initial measures.

**Theorem 3.10** (Consensus result for general measures in \( \mathcal{P}^2(\mathbb{R}^d) \)). Consider the PaWBar algorithm with an underlying graph \( G \) which is either

(i) a cycle graph for the directed case, or

(ii) a line graph for the symmetric case;

and with the agents having initial measures \( \mu_{i,0} \in \mathcal{P}^2(\mathbb{R}^d), i \in V \). Then, with probability one, for any \( i \in V \),

\[
W_2(\mu_i(t), \mu_\infty) \to 0 \text{ as } t \to \infty,
\]

where \( \mu_\infty \in \mathcal{P}^2(\mathbb{R}^d) \) is a random measure whose possible values may depend on the realization of the edge selection process. If \( \mu_{i,0} = \mu_{j,0} \) for any \( i, j \in V \), then \( \mu_\infty = \mu_{i,0} \) with probability one.

**Remark 3.11** (Open problem: Theorem 3.10 and Wasserstein barycenters). The characterization of the consensus value in our theorem does not state any sufficient condition under which the converged consensus random measure is a Wasserstein barycenter or not: this is an open problem for further research.

For the particular case of general measures in \( \mathcal{P}^2(\mathbb{R}) \), consensus is guaranteed under the same general conditions for the underlying communication graph as in the previous results of our paper. This follows from a direct application of [9, Theorem 1].
Figure 1: Consider five agents that initially have multivariate Gaussian distributions on \( \mathbb{R}^5 \), with their covariance matrices being randomly generated. On the left, we present the underlying digraph over which the PaWBar algorithm is run. The weight associated to all edges is 0.75. We first fix a realization of the edge selection process. Then, we compute the covariance matrix \( \Sigma_\infty \) of the Wasserstein barycenter following the centralized numerical scheme proposed in [3]. On the right, each of the five plotted curves corresponds to the evolution of the error quantity \( \| \Sigma_i(t) - \Sigma_\infty \|_F \) for each agent \( i \in \{1, \ldots, 5\} \), where \( \Sigma_i(t) \) is the value of agent \( i \)'s covariance matrix at iteration \( t \), and \( \| \cdot \|_F \) is the Frobenius norm. All agents asymptotically reach consensus and converge to the randomized Wasserstein barycenter, thus giving evidence for the veracity of Conjecture 1 at least for the Gaussian case.

**Theorem 3.12** (Consensus result for general measures in \( \mathcal{P}^2(\mathbb{R}) \)). Consider the agents having initial measures \( \mu_{i,0} \in \mathcal{P}^2(\mathbb{R}) \), \( i \in V \). Then, for the directed, resp. symmetric, PaWBar algorithm under a strongly connected digraph, resp. connected undirected graph,

\[
W_2(\mu_{i}(t), \mu_\infty) \to 0 \text{ as } t \to \infty,
\]

(10)

where \( \mu_\infty \in \mathcal{P}^2(\mathbb{R}) \) is a random measure whose possible values may depend on the realization of the edge selection process.

4 Proofs of results in Section 3

4.1 Proofs of results in Subsection 3.2

Proof sketch of Theorem 3.4. Since measures \( \mu_{i,0} \) and \( \mu_{j,0}, i, j \in V \), are discrete uniform, we have \( W_2^2(\mu_{i,0}, \mu_{j,0}) = \min_{\sigma \in \Sigma_N} \sum_{k=0}^{N} \left\| x_{\sigma(k)}^i - x_{\sigma(k)}^j \right\|_2^2 \) [38, 30], with \( \Sigma_N \) being the set of all possible permutations of the elements in \( \{1, \ldots, N\} \); i.e., any permutation map \( \sigma \in \Sigma_N \) is a bijective function \( \sigma : \{1, \ldots, N\} \to \{1, \ldots, N\} \). Then, the displacement interpolation between any of the initial measures provides discrete measures. Now, consider \( \sigma \in \Sigma_N \) from solving the optimal
Figure 2: Consider five agents that initially have multivariate Gaussian distributions on $\mathbb{R}^5$, with their covariance matrices being randomly generated. The setting and methodology for computing the plot on the right is similar to the one described in Figure 1, with the difference that now the underlying digraph is a cycle (as seen on the left). All agents asymptotically compute the randomized Wasserstein barycenter.

transport problem from $\mu_{i,0}$ to $\mu_{j,0}$, i.e., $\gamma^{\text{opt}}(\mu_{i,0}, \mu_{j,0}) = \frac{1}{N} \sum_{k=0}^{N} \delta_{(x_i^k, x_j^k)}$. Consider two arbitrary points $(x_{k_1}^i, x_{\sigma(k_1)}^j), (x_{k_2}^i, x_{\sigma(k_2)}^j) \in \text{supp}(\gamma^{\text{opt}}(\mu_{i,0}, \mu_{j,0}))$. Then, for any $a_{ij} \in (0,1)$, the displacement interpolation implies the existence of some $z_{k_1}(a_{ij}), z_{k_2}(a_{ij}) \in \text{supp}((\pi_{a_{ij}})_{\#} \mu_{i,0})$, such that $z_{k_1}(a_{ij}) = (1 - a_{ij}) x_{k_1}^i + a_{ij} x_{\sigma(k_1)}^j$ and $z_{k_2}(a_{ij}) = (1 - a_{ij}) x_{k_2}^i + a_{ij} x_{\sigma(k_2)}^j$. Now, since the optimal transport plan $\gamma^{\text{opt}}(\mu_{i,0}, \mu_{j,0})$ has cyclically monotone support [38, Section 2.3], we can follow the treatment in [39, Chapter 8] and conclude that there exists no $a_{ij} \in (0,1)$ such that $z_{k_1}(a_{ij}) = z_{k_2}(a_{ij})$. As a consequence, $\text{supp}((\pi_{a_{ij}})_{\#} \mu_{i,0})$ has $N$ (different) elements for any possible edge weight $a_{ij}$, i.e., $(\pi_{a_{ij}})_{\#} \mu_{i,0}$ is a discrete uniform measure. It is easy to show by induction that in either the directed or symmetric PaWBar algorithm, $\mu_i(t)$ is a discrete uniform distribution for any $i \in V$ and time $t$ with probability one.

We now introduce some notation. Given $A \in \mathbb{R}^{m \times m}$, let $\text{diag}^{i,k}(A) \in \mathbb{R}^{km \times km}$ be the $k \times k$ block-diagonal matrix such that its $i$th block has the matrix $A$ and the rest of blocks are $I_m$. Given $A, B \in \mathbb{R}^{m \times m}$, let $\text{diag}^{i,j,k}(A, B) \in \mathbb{R}^{km \times km}$ be the $k \times k$ block-diagonal matrix such that its $i$th and $j$th blocks are the matrices $A$ and $B$ respectively, and the rest of blocks are $I_m$. Let $x^i(t) \in \mathbb{R}^d$ be a vector stacking the elements of $\text{supp}(\mu_i(t))$, which we call the support vector. Note that since the measures are discrete uniform at every time $t$ (with probability one), the order of the elements $x^i_k(t) \in \mathbb{R}^d$, $k \in \{1, \ldots, N\}$, in the vector $x^i(t)$ can be arbitrary; but for convenience we denote it as $x^i(t) = (x^i_1(t), \cdots, x^i_N(t))^\top$. For any $i, j \in V$ and time $t$, let $\sigma_{i,j,t} \in \Sigma_N$ be an optimal transport map from $\mu_i(t)$ to $\mu_j(t)$; and let $\sigma_{t,i,t} = \sigma_{i,j,t}^{-1}$.

We now focus on proving statement (i). Assume $(i, j) \in E$ is selected at time $t$. Then, $x^i_k(t+1) = \begin{cases} 1 & \text{if } x^i_k(t) = 1, \\ 0 & \text{otherwise,} \end{cases}$ and $x^j_0(t+1) = 1$. For any $i, j \in V$, let $\sigma_{t,i,t} = \sigma_{i,j,t}^{-1}$.
Now, set $k$ positive numbers $σ_i$, $i ∈ \{1, \ldots, N\}$, i.e.,
\[
x(t + 1) = (1 - a_{ij})x^i(t) + a_{ij}(P(t) ⊗ I_d)x^j(t)
\]
with the permutation matrix $P(t) ⊗ I_d$ defined by the permutation matrix $P(t) ∈ \{0, 1\}^{N × N}$ whose $k$th row is $e_{σ_i(k)}^T$. Indeed, with $Q_{ij,t} = P(t) ⊗ I_d$,
\[
W_2^2(μ_i(t), μ_j(t)) = \frac{1}{N} \|x^i(t) - Q_{ij,t}x^j(t)\|^2_2.
\]

Now, set $x(t) = (x^1(t), \ldots, x^n(t))^T ∈ \mathbb{R}^{Nd}$. Then, (11) can also be expressed as
\[
x(t + 1) = B(t)x(t)
\]
with the row-stochastic matrix $B(t) = diag\{P(t) ⊗ I_d(A(t) ⊗ I_{Nd}) diag\{P(t) ⊗ I_d\} \}$.

Consider now an initial vector $x(0)$ and a fixed realization of the edge selection process. Now consider $x'(0) = diag(P_1 ⊗ I_d, \ldots, P_n ⊗ I_d)x(0)$ with arbitrary permutation matrices $P_1, \ldots, P_n ∈ \{0, 1\}^{N × N}$. Notice that both $x(0)$ and $x'(0)$ represent the supports of the same group of measures $\{μ_i, 0\}_{i ∈ V}$ but may be the case that $x(0) ≠ x'(0)$. We claim that
\[
x'(t) = diag(P_1 ⊗ I_d, \ldots, P_n ⊗ I_d)x(t)
\]
for any time $t$.

To prove this claim, first recall that we have a fixed realization of the edge selection process. Assume $(i, j) ∈ E$ is selected at time $t = 0$ and obtain $x(1) = B(0)x(0)$. Likewise, $x'(1) = B'(0)x'(0)$, with
\[
B'(0) = diag\{P'(0) ⊗ I_d(A(0) ⊗ I_{Nd}) diag\{P'(0) ⊗ I_d\}\},
\]
is the update that results if the algorithm starts with initial vector $x'(0)$. Then,
\[
P'(0) ⊗ I_d = (P_1 ⊗ I_d)(P(0) ⊗ I_d)(P'_j ⊗ I_d) = (P, P(0)P'_j ⊗ I_d) ⊗ I_d.
\]
After some algebraic work, we obtain $B'(0) = diag\{P_1, P_j\}B(0) diag\{P'_j, P_j\}$, and so
\[
x'(1) = diag\{P_1, P_j\}B(0) diag\{P'_j, P_j\}x'(0)
\]
\[
\quad \quad \quad = diag\{P_1, P_j\}B(0) diag\{P'_j, P_j\}\cdot (P_1, \ldots, P_n)x(0)
\]
\[
\quad \quad \quad = diag(P_1, \ldots, P_n)B(0)x(0) = diag(P_1, \ldots, P_n)x(1).
\]

Finally (13) is easily proved by induction, and the claim is proved.

Now, let us make the following claim:
(i.a) for any $i^*, j^* ∈ V$, $i^* ≠ j^*$, $ε > 0$ and time $t$, the event \(W_2(μ_{i^*}(t + T), μ_{j^*}(t + T)) < ε\) for some finite $T > 0$ has positive probability.

We prove the claim. Define $d(i, j, σ_i, τ_i, t_1) := \left(\sum_{k=1}^N \frac{1}{N}\|x^i(σ_{k}(t_i)) - x^j_{σ_{k}(t_j)}\|^2\right)^{1/2}$, for $i, j ∈ V$, $σ_i, σ_j ∈ Σ_N$. Consider any $i^*, j^* ∈ V$ and $ε > 0$. Since $G$ is strongly connected, there exists a shortest directed path $P_{i^* → j^*} = ((i^*, t_1), \ldots, (t_{L-1}, j^*))$ from $i^*$ to $j^*$ of some length $L$. Now, pick positive numbers $ε_1, \ldots, ε_L$ such that $\sum_{i=1}^L ε_i < ε$. Consider any time $t$. Then, we can first select $T_1$ times the edge $(\ell_{L-1}, j^*)$ so that
\[
d(ℓ_{L-1}, j^*, σ_{ℓ_{L-1}j^*, t+T_1}, Id, t + T_1, t) = (1 - a_{ℓ_{L-1}j^*})T_1 \left(\sum_{k=1}^N \frac{1}{N}\|x^i(σ_{k}(t_i)) - x^j_{σ_{k}(t_j)}\|^2\right)^{1/2} < ε_L.
\]

Then, we can select $T_2$ times the edge $(\ell_{L-2}, ℓ_{L-1})$ so that
\[
d(ℓ_{L-2}, ℓ_{L-1}, σ_{ℓ_{L-2}ℓ_{L-1}, t+T_1+T_2}^{-1} ⊂ σ_{ℓ_{L-1}j^*, t+T_1}, σ_{ℓ_{L-1}j^*, t+T_1}, t + T_1 + T_2, t + T_1) < ε_{L-1},
\]
and we can continue like this until finally selecting \( T_k \) times the edge \((i^*, \ell_1)\) such that \\
\( d(i^*, \ell_1, \sigma, \sigma_{i_1}^{-1} \ell_1 \ell_2, t + \sum_{i=0}^{i_{L-1}} \sigma_{i_{L-1}j^*, t} \ell_{i_{L-1}} t + T, t + \sum_{i=1}^{L-1} T_i) < \epsilon_1 \) with \( \sigma = \sigma_{i_1}^{-1} \sigma_{i_2}^{-1} \cdots \sigma_{i_{L-1}j^*, t} \) and \( T = \sum_{i=1}^{L} T_i \). Then, \\
\[
W_2(\mu(t + T), \mu_j(t + T)) \leq \left( \sum_{k=1}^{N} \frac{1}{N} \left\| x_{\sigma(k)}^r(t + T) - x_j^r(t + T) \right\|_2 \right)^{\frac{3}{2}} < \sum_{i=1}^{L} \epsilon_i < \epsilon,
\]
where the first inequality follows by definition of the Wasserstein distance, and the second inequality from both the triangle inequality and the fact that \( x_j^r(t + T) = x_j^{L-1}(t + T), \ldots, x_j(t + T) = x_j^{i_{L}}(t + \sum_{i=1}^{L-1} T_i) \). Moreover, our construction implies, for any \( p \in V \) in the path \( \mathcal{P}_{i^*j^*} \), \\
\[
W_2(\mu_p(t + T), \mu_{j^*}(t + T)) < \epsilon.
\]

Now, consider any \( m \in V \) and construct a directed acyclic subgraph \( G' = (V, E') \), \( E' \subset E \), of \( G \) as follows: \( m \) is the unique node with zero out-degree (i.e, \( (m, i) \notin E' \) for any \( i \in V \)) and there exists a unique directed path from any node \( i \in V \setminus \{m\} \) to \( m \). Such subgraph \( G' \) exists because \( G \) is strongly connected. Consider any \( \epsilon > 0 \) and time \( t \). Then, the selection process just described above can make all nodes \( m \) with zero in-degree in \( G' \) (i.e., any \( m \in V \) such that \( (i, m) \notin E' \) for any \( i \in V \setminus \{m\} \)) satisfy \( W_2(\mu_m(t + T), \mu_m(t + T)) = W_2(\mu_{m}(t + T), \mu_{m}(T)) < \frac{\epsilon}{2} \) for some \( T \). Then, as a consequence of \((14)\), \( W_2(\mu_1(t + T), \mu_{j^*}(t + T)) < \frac{\epsilon}{2} \) for any \( i \in V \), and the triangle inequality then implies \( W_2(\mu_i(t + T), \mu_{j^*}(t + T)) < \epsilon \) for any \( j \in V \). Finally, for any \( i, j \in V \), the event \( "W_2(\mu_i(t + T), \mu_j(t + T)) < \epsilon \) for some \( T > 0" \) has a positive probability to occur at any time \( t \) because any selection of a finite sequence of edges has positive probability to occur at any time \( t \).

This finishes the proof of claim \((i.a)\).

Now, the event in \((i.a)\), due to its persistence, will eventually happen with probability one. Assume it happens at time \( t \). Then we claim that \( \epsilon \) in this event in this event could have been chosen sufficiently small so that, for any time \( t' \geq t \) and any \( i, j, p \in V \), \n
\begin{itemize}
  \item[(i.b)] \( \sigma_{ij,t'} = \sigma_{ij,t} \), and
  \item[(i.c)] \( \sigma_{ij,t'} = \sigma_{ip,t} \circ \sigma_{pj,t'} \).
\end{itemize}

Now we prove the claim. Firstly, note that from \((i.a)\) and the fact that the measures are discrete uniform at every time with probability one, we can consider a small enough \( \epsilon \) such that for any \( i, j \in V \) and any permutation map \( \sigma \neq \sigma_{ij,t} \),

\[
W_2(\mu_i(t), \mu_j(t)) = \left( \frac{1}{N} \sum_{k=1}^{N} \left\| x_i^j(t) - x_{\sigma_{ij,t}(k)}^j(t) \right\|_2 \right)^{\frac{1}{2}} < \epsilon \quad \text{and}
\]
\[
2\epsilon < \left( \frac{1}{N} \sum_{k=1}^{N} \left\| x_i^j(t) - x_{\sigma(t)}^j(t) \right\|_2 \right)^{\frac{1}{2}}.
\]

Such choice of \( \epsilon \) implies that \( \sigma_{ip,t} \circ \sigma_{pj,t} = \sigma_{ij,t} \) for any \( i, j, p \in V \); otherwise, if \( \sigma_{ip,t} \circ \sigma_{pj,t} \neq \sigma_{ij,t} \), then we obtain a contradiction:

\[
2\epsilon < \left( \frac{1}{N} \sum_{k=1}^{N} \left\| x_i^j(t) - x_{\sigma_{ip,t} \circ \sigma_{pj,t}(k)}^j(t) \right\|_2 \right)^{\frac{1}{2}} = \left( \frac{1}{N} \sum_{k=1}^{N} \left\| x_{\sigma_{ip,t}(k)}^j(t) - x_{\sigma_{pj,t}(k)}^j(t) \right\|_2 \right)^{\frac{1}{2}}
\leq \left( \frac{1}{N} \sum_{k=1}^{N} \left\| x_{\sigma_{ip,t}(k)}^j(t) - x_{\sigma_{ij,t}(k)}^j(t) \right\|_2 \right)^{\frac{1}{2}} + \left( \frac{1}{N} \sum_{k=1}^{N} \left\| x_{\sigma_{pj,t}(k)}^j(t) - x_{\sigma_{ij,t}(k)}^j(t) \right\|_2 \right)^{\frac{1}{2}} < 2\epsilon.
\]
We just proved that (i.c) holds for \( t' = t \). Note that (i.b) for \( t' = t \) is trivial. Now, assume any \((i^*, j^*) \in E \) is selected at time \( t \). Then, for any \( j \in V \setminus \{i^*, j^*\} \), using the identity \( Q_{j^*j,t} = Q_{j^*i^*,t}Q_{i^*j,t} \) from (i.c) for \( t' = t \) implies

\[
\left\| x^{i^*}(t+1) - Q_{i^*j,t}x^{j^*}(t+1) \right\|_2
\]

\[
\leq (1 - a_{i^*j^*}) \left\| x^{i^*}(t) - Q_{i^*j,t}x^{j^*}(t) \right\|_2 + a_{i^*j^*} \left\| Q_{j^*j^*,t}x^{j^*}(t) - Q_{i^*j,t}x^{j^*}(t) \right\|_2
\]

\[
= (1 - a_{i^*j^*}) \left\| x^{i^*}(t) - Q_{i^*j,t}x^{j^*}(t) \right\|_2 + a_{i^*j^*} \left\| x^{j^*}(t) - Q_{j^*j,t}x^{j^*}(t) \right\|_2
\]

\[
< (1 - a_{i^*j^*})\sqrt{\frac{1}{N}} + a_{i^*j^*}\epsilon \sqrt{\frac{1}{N}} = \epsilon \sqrt{\frac{1}{N}};
\]

likewise, we immediately obtain \( \frac{1}{N} \left\| x^{i^*}(t+1) - Q_{i^*j,t}x^{j^*}(t+1) \right\|_2 < \epsilon \) for any \( i \in V \setminus \{i^*, j\} \), and \( \frac{1}{N} \left\| x^{i^*}(t+1) - Q_{i^*j,t}x^{j^*}(t+1) \right\|_2 < \epsilon \) for any \( i, j \in V \) which implies \( \sigma_{i,j,t+1} = \sigma_{i,j,t} \) for any \( i, j \in V \); i.e., (i.b) holds for \( t' = t + 1 \). Now, to prove claim (i.c) holds for \( t' = t + 1 \), we must first prove that (15) holds for time \( t + 1 \).

Set \( y^1(t) := x^i(t), y^2(t) = Q_{i2,i}x^2(t), \ldots, y^n(t) = Q_{in,i}x^n(t) \) (this labeling is arbitrary and any other \( i \in V \setminus \{1\} \) could have been chosen to define \( Q_{1i}, \ldots, Q_{in} \) and \( y^i(t) = (y_1^i(t), \ldots, y_n^i(t))^\top \), \( y_1^i(t) \in \mathbb{R}^d, i \in \{1, \ldots, n\} \). For any \( k \in \{1, \ldots, N\} \), let \( L_k(t) \) be the convex hull of the set \( \{y_k^1(t), \ldots, y_k^n(t)\} \). For any \( p, q \in \{1, \ldots, N\} \), define the distance between \( L_p(t) \) and \( L_q(t) \) as \( d_{pq}(t) = \inf_{w_1 \in L_p(t), w_2 \in L_q(t)} \|w_1 - w_2\|_2 \). Assuming that \((i^*, j^*) \in E \) is selected at time \( t \), our result (i.b) for \( t' = t + 1 \) and (11) imply that \( y_k^{i^*}(t+1) = (1 - a_{i^*j^*})y_k^{i^*}(t) + a_{i^*j^*}y_k^{j^*} \in L_k(t), k \in \{1, \ldots, N\} \). Obviously, for any \( j \in V \setminus \{i^*\} \), \( y_k^j(t+1) \in L_k(t), k \in \{1, \ldots, N\} \). Then, \( L_i(t+1) \subseteq L_i(t) \) for any \( i \in \{1, \ldots, N\} \), and thus \( d_{pq}(t) \leq d_{pq}(t+1) \) for any \( p, q \in \{1, \ldots, N\} \). Now, for any \( i, j \in V \), time \( \tau \geq t \), and permutation map \( \sigma \neq Id \), we have

\[
(\sum_{k=1}^{N} d_{k\sigma(k)}(\tau))^\frac{1}{2} \leq (\sum_{k=1}^{N} \|y_k^i(\tau) - y_j^{\sigma(k)}(\tau)\|_2^2)^\frac{1}{2} .
\]

Now, we need to consider two cases. In the first case we consider

\[
2\epsilon < \min_{\sigma \in \Sigma_N, \sigma \neq Id}(\frac{1}{N} \sum_{k=1}^{N} d_{k\sigma(k)}(t+1))^\frac{1}{2} .
\]

Then,

\[
2\epsilon < (\frac{1}{N} \sum_{k=1}^{N} d_{k\sigma(k)}(t+1))^\frac{1}{2} \leq (\frac{1}{N} \sum_{k=1}^{N} \|y_k^i(t+1) - y_j^{\sigma(k)}(t+1)\|_2^2)^\frac{1}{2} .
\]

for any permutation map \( \sigma \neq Id \); and (i.b) for \( t' = t + 1 \) and (i.c) for \( t' = t \) imply \( 2\epsilon < (\frac{1}{N} \sum_{k=1}^{N} \|x_k^i(t) - x_j^{\sigma(k)}(t)\|_2^2)^\frac{1}{2} \) with \( \sigma' = \sigma_{i1,t+1} \circ \sigma \circ \sigma_{ij,t+1} \neq \sigma_{ij,t+1} \). Thus, (15) holds for time \( t+1 \) in this first case. Now, we consider the second case \( 2\epsilon \geq \min_{\sigma \in \Sigma_N, \sigma \neq Id}(\frac{1}{N} \sum_{k=1}^{N} d_{k\sigma(k)}(t))^\frac{1}{2} \).

Then, due to \( G \) being strongly connected and \( \{d_{pq}(\tau)\}_{\tau \geq t} \) being a nondecreasing sequence for any \( p, q \in \{1, \ldots, N\} \), we can follow the proof of result (i.a) and arbitrarily reduce the diameter of the set \( L_k \) for any \( k \in \{1, \ldots, N\} \) at some future time \( t \), i.e., \( L(\bar{t}) \subset L(t) \). This diameter reduction can be chosen such that \( d_{ij}(\bar{t}) > d_{ij}(t) \) for any \( i, j \in \{1, \ldots, N\} \), and this increase on the distances between sets can be done so that \( 2\epsilon' < \min_{\sigma \in \Sigma_N, \sigma \neq Id}(\frac{1}{N} \sum_{k=1}^{N} d_{k\sigma(k)}(\bar{t}))^\frac{1}{2} \) for some \( 0 < \epsilon' < \epsilon \). In other words, we are in the first case at time \( \bar{t} \). After this change, we will never be in the second case again for any time after \( \bar{t} \) with probability one. In summary, we just proved the conditions in equation (15) can be made to hold for time \( t+1 \), and so (i.c) holds for \( t' = t+1 \).

Now, assume results (i.b) and (i.c) hold for time \( t' = \tau \geq t \), and (15) holds for time \( \tau \). Following the proof just presented above, we easily establish that (i.b) and (i.c) hold for \( t' = \tau + 1 \) and that (15) holds for time \( \tau + 1 \). Then, by induction, we proved our initial claim about (i.b) and (i.c).
We now, notice that \( \epsilon \) in \( \max_{i,j \in V} W_2(\mu_i,0,\mu_j,0) < \epsilon \) can be made sufficiently small so that (i.b) and (i.c) hold, in which case (i.a) is satisfied at the beginning of time, i.e., with \( \lambda = 0 \). Therefore, in general, from results (i.a), (i.b) and (i.c), there exists some (possibly) random time \( \bar{T} \geq 0 \) such that, with probability one: for any time \( t \geq \bar{T} \) and any \( i,j \in V \), \( \sigma_{ij,t} = \sigma_{ij,\bar{T}} \) and \( \sigma_{ij,t} = \sigma_{ip,t} \circ \sigma_{pj,t} \).

Let us consider a fixed realization of the edge selection process, and then consider such time \( \bar{T} \), which is now a deterministic function of \( \mathbf{x}(0) \). Without loss of generality, as a consequence of (13), we can assume we started the algorithm with the initial support vectors \( \{Q_{1i,T}\mathbf{x}(0)\}_{i \in V} \) at time \( t = 0 \). Then, it is easy to prove that \( B(t) = A(t) \otimes I_{N_d} \) (as in (12)) for any \( t \geq \bar{T} \), i.e., \( B(t) \) has an associated permutation matrix \( P(t) = I_N \). Then, Proposition 3.3 let us conclude that \( \lim_{t \to \infty} \prod_{\tau=\bar{T}}^{t} B(\tau) = (I_n \lambda_\top) \otimes I_{N_d} \) for some convex vector \( \lambda \). Thus, \( \mathbf{x}(\infty) = \sum_{j=1}^{n} \lambda_j \mathbf{x}(\bar{T}), \quad i \in V \), which is the support vector of the final consensus measure \( \mu_\infty \).

It remains to prove that \( \mu_\infty \) corresponds to a Wasserstein barycenter. Let us formulate the Wasserstein barycenter problem \( \min_{\nu \in \mathcal{P}_2(\mathbb{R}^d)} \sum_{i=1}^{n} \lambda_i W_2(\nu, \mu_i(\bar{T}))^2 \). Since the measures \( \{\mu_i(\bar{T})\}_{i \in V} \) have finite support, any barycenter is a discrete measure with finite support [4]. Moreover, since all the measures are uniform, we can consider a minimizer with a discrete uniform distribution. We now prove that \( \mu_\infty \) is such a minimizer. Firstly, by construction and the fact that (15) holds for \( t \geq \bar{T} \), we have

\[
W_2(\mu_\infty, \mu_i(\bar{T}))^2 = \frac{1}{N} \sum_{j=1}^{N} \left\| \sum_{k=1}^{n} \lambda_k x_j^k(\bar{T}) - x_j^i(\bar{T}) \right\|_2^2 < 2\epsilon \frac{1}{N} \sum_{j=1}^{N} \left\| \sum_{k=1}^{n} \lambda_k x_j^{k}(\sigma_{i,j}(\bar{T})) - x_{\sigma_{i,j}(\bar{T})}^i(\bar{T}) \right\|_2^2
\]

for any \( \sigma^i \in \Sigma_N, \sigma_i \neq Id, \quad i \in V \). Then,

\[
\sum_{i=1}^{n} \lambda_i W_2(\mu_\infty, \mu_i(\bar{T}))^2 < \frac{1}{N} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{N} \left\| \sum_{k=1}^{n} \lambda_k x_{\sigma_{i,j}(\bar{T})}^k(\bar{T}) - x_{\sigma_{i,j}(\bar{T})}^i(\bar{T}) \right\|_2^2 \leq \frac{1}{N} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{N} \left\| y_i - x_{\sigma_{i,j}(\bar{T})}^i(\bar{T}) \right\|_2^2
\]

for any \( y = (y_1, \ldots, y_n)^\top \in \mathbb{R}^{N_d} \). The last inequality of the previous expression is proved by treating the last term as an objective function to minimize with respect to \( y \) using first optimality conditions to minimize such differential and strictly convex function (i.e., by setting the gradient with respect to \( y \) equal to the zero vector and solving for \( y \)). Now, take \( y_i \neq y_j \in \mathbb{R}^d \) for any \( i,j \in \{1, \ldots, N\} \), define the discrete uniform measure \( \nu = \frac{1}{N} \sum_{j=1}^{N} \delta_{y_j}, \quad y_j \in \mathbb{R}^d \); and let \( \sigma_{i,j}, \quad i \in V \), be such that

\[
W_2(\nu, \mu_i(\bar{T}))^2 = \frac{1}{N} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{N} \left\| y_i - x_{\sigma_{i,j}(\bar{T})}^i(\bar{T}) \right\|_2^2.
\]

Then, our recent analysis implies that:

1. If there exists \( i \in V \) such that \( \sigma_i \neq Id \), then \( \sum_{i=1}^{n} \lambda_i W_2(\mu_\infty, \mu_i(\bar{T}))^2 < \sum_{i=1}^{n} \lambda_i W_2(\nu, \mu_i(\bar{T}))^2 \); (2) if \( \sigma_i = Id \) for all \( i \in V \), then \( \sum_{i=1}^{n} \lambda_i W_2(\mu_\infty, \mu_i(\bar{T}))^2 \leq \sum_{i=1}^{n} \lambda_i W_2(\nu, \mu_i(\bar{T}))^2 \).

Given the generality of \( \nu \), cases (1) and (2) together imply that \( \mu_\infty \) is a Wasserstein barycenter.

Finally, all of our previous results hold with probability one because we considered an arbitrary realization of the edge selection process for our analysis (note that \( \lambda \) now becomes a random convex vector). This concludes the proof of statement (i).

We now focus on proving statement (ii). Assume \( \{i, j\} \in E \) is selected at time \( t \). Without loss of generality, the update of the PaWBar algorithm can be set as \( x_k^i(t+1) = \frac{1}{2} x_k^i(t) + \frac{1}{2} x_{\sigma_{ij,t}(k)}^j(t) \).
and $x_k^j(t+1) = \frac{1}{2} x_k^i(t) + \frac{1}{2} x_i^j(\sigma_{i,t}(k))(t)$, $k \in \{1, \ldots, N\}$; i.e.,

$$
x_i^i(t+1) = \frac{1}{2} x_i^i(t) + \frac{1}{2} (P(t) \otimes I_d)x_i^i(t),$$

$$
x_i^j(t+1) = \frac{1}{2} x_i^j(t) + \frac{1}{2} (P(t)^\top \otimes I_d)x_i^j(t)$$

(17)

recalling that the permutation matrix $P(t) \in \{0,1\}^{N \times N}$ has $e^\top_{\sigma_{i,t}(k)}$ as its $k$th row. Note that (17) can also be expressed as $x(t+1) = C(t)x(t)$, with matrix $C(t) = \text{diag}^i_{n}(P(t) \otimes I_d)(A(t) \otimes I_{Nd})\text{diag}^i_{n}(P^\top(t) \otimes I_d)$.

We make the following claim:

(ii.a) for any $i^*, j^* \in V$, $i^* \neq j^*$, $\epsilon > 0$ and time $t$, the event "$W_2(\mu_{i^*}(t + T), \mu_{j^*}(t + T)) < \epsilon$ for some finite $T > 0$" has positive probability.

Now, we prove the claim. Let us fix a spanning tree $G'$ of $G$. For any $i, j \in V$, let $P_{i-j}$ denote the unique path between $i$ and $j$ in $G'$. Let

$$U(t) = \max_{i,j \in V} \sum_{\{p,q\} \in P_{i-j}} W_2(\mu_p(t), \mu_q(t)).$$

Let $\{k, \ell\} \in \arg U(t)$ and $P_{k-\ell} = (\{k,p_1\}, \ldots, \{p_{L-1}, \ell\})$, i.e., edge $\{k, p_1\}$ is followed by $\{p_1, p_2\}$ and so on until $\{p_{L-1}, \ell\}$. Case 1 $W_2(\mu_k(t), \mu_p(t)) \neq 0$. For simplicity we also assume $W_2(\mu_i(t), \mu_j(t)) \neq 0$ for any $\{i, j\} \in P_{k-\ell}$; otherwise, if there exists $\{i^*, j^*\} \in P_{k-\ell}$ such that $W_2(\mu_{i^*}(t), \mu_{j^*}(t)) = 0$, we would need to use a similar analysis to Case 2 which will be treated later. Select $\{k, p_1\}$ at time $t$. If $P_{k-\ell}$ contains only one element, then $p_1 = \ell$ and $W_2(\mu_k(t + 1), \mu_{\ell}(t + 1)) = 0 < U(t) = W_2(\mu_k(t), \mu_\ell(t))$. Now, consider $P_{k-\ell}$ contains two or more elements. Set $U(t) = \sum_{\{p,q\} \in P_{k-\ell}\backslash \{(k,p_1)\}} W_2(\mu_p(t), \mu_q(t))$ (with $p_2 = \ell$ and $U(t) = 0$ if $P_{k-\ell}$ only has two elements). Then

$$\sum_{\{p,q\} \in P_{k-\ell}} W_2(\mu_p(t + 1), \mu_q(t + 1)) = U(t) + \frac{1}{\sqrt{N}} \|x^{p_1}(t + 1) - Q_{p_1(p_2,t+1)x^{p_2}(t)}\|_2$$

$$\leq U(t) + \frac{1}{\sqrt{N}} \|x^{p_1}(t + 1) - Q_{p_1(p_2,t)x^{p_2}(t)}\|_2$$

$$\leq U(t) + \frac{1}{2} W_2(\mu_{p_1}(t), \mu_{p_2}(t)) + \frac{1}{2\sqrt{N}} \|Q_{p_1(t)x^{k}(t)} - Q_{p_2(t)x^{p_2}(t)}\|_2$$

$$\leq U(t) + \frac{1}{2} W_2(\mu_{p_1}(t), \mu_{p_2}(t)) + \frac{1}{2\sqrt{N}} \|Q_{p_1(t)x^{k}(t)} - x^{p_1}(t)\|_2$$

$$+ \frac{1}{2\sqrt{N}} \|x^{p_1}(t) - Q_{p_1(p_2,t)x^{p_2}(t)}\|_2$$

$$= U(t) + W_2(\mu_{p_1}(t), \mu_{p_2}(t)) + \frac{1}{2} W_2(\mu_k(t), \mu_{p_1}(t)),$$

and so $\sum_{\{p,q\} \in P_{k-\ell}} W_2(\mu_p(t + 1), \mu_q(t + 1)) < U(t)$. Therefore, for any length of $P_{k-\ell}$, if $U(t + 1) \leq \sum_{\{p,q\} \in P_{k-\ell}} W_2(\mu_p(t + 1), \mu_q(t + 1))$, then $U(t + 1) < U(t)$. If $U(t + 1) > \sum_{\{p,q\} \in P_{k-\ell}} W_2(\mu_p(t + 1), \mu_q(t + 1))$, then we can choose $\{k, \ell\} \in \arg U(t + 1)$ and, using the analysis just presented, obtain $\sum_{\{p,q\} \in P_{k-\ell}} W_2(\mu_p(t + 2), \mu_q(t + 2)) < U(t + 1)$. If this does not imply $U(t + 2) < U(t)$, we can keep iterating this procedure until, eventually, obtain $U(t + T) < U(t)$ for some $T > 0$.

Case 2) $W_2(\mu_k(t), \mu_{p_1}(t)) = 0$. In this case, we do not select the edge $\{k, p_1\}$, but we consecutively check the edges along $P_{k-\ell}$ starting from $\{k, p_1\}$ and look for the first $\{i^*, j^*\} \in
sequence of edges has a positive probability of being consecutively selected at any time. We select this edge and a similar analysis to Case 1 implies that $\sum_{(p,q) \in P_{k-\ell}} W_2(\mu_p(t+1), \mu_q(t+1)) \leq U(t)$. Then, we select the edge previous to $\{i^*, j^*\}$ and continue to successively select the preceding edges until reaching the first edge $\{k, p_1\}$. Once this edge is selected, say at time $t$, the proof of case Case 1) let us conclude that $\sum_{(p,q) \in P_{k-\ell}} W_2(\mu_p(t+1), \mu_q(t+1)) < U(t)$, and we can continue the analysis of Case 1) until we have that $U(t+T) < U(t) \leq U(t)$ for some $T > 0$. In conclusion, we proved the existence of some finite sequence of selected edges such that $U(t+T) < U(t)$ for some $T > 0$. Moreover, we can iterate selections of such sequences to arbitrarily reduce the value of $U(t)$ after some finite time. Finally, claim (ii.a) follows from the fact that $\max_{i,j \in V} W_2(\mu_i(t), \mu_j(t)) \leq U(t)$ and that any finite sequence of edges has a positive probability of being consecutively selected at any time $t$.

We can now follow the same analysis as in the proof of statement (i) of the theorem – using result (ii.a) and its proof instead of (i.a) – to conclude that results (i.b) and (i.c) also hold for the symmetric PaWBar algorithm, after which the proof follows closely the one for statement (i) again.

**Proof of Corollary 3.5.** We follow the notation and proof of Theorem 3.4. Note that the entries of $x^i(0) = (x^i_1, ..., x^i_N)^\top$, $i \in V$, are sorted in ascending order. Then, $W_2^2(\mu_{i,0}, \mu_{j,0}) = \frac{1}{N} \sum_{k=0}^N (x^i_k - x^j_k)^2$ for $i, j \in V$. Now, consider the directed PaWBar algorithm and that $(i, j) \in E$ is selected at time $t = 0$. Then $x^i(1) = (1 - a_{ij})x^i(0) + a_{ij}x^j(0)$ and $x^i(1)$ has its entries sorted in ascending order. Then, it is easy to prove by induction that, at every time $t$, $x^i(t)$ for any $i \in V$ is sorted in ascending order with probability one. Considering $x(t) = (x^1(t), ..., x^n(t))^\top \in \mathbb{R}^{nN}$, we have that $x(t+1) = (A(t) \otimes I_N)x(t)$ and so $x(t) = (\prod_{t=0}^t A(i) \otimes \mathbb{1}_N)x(0)$. Then, we conclude the proof for the directed PaWBar algorithm by using Proposition 3.3 and the fact that $\sum_{i=1}^n \lambda_i x^i(0) \in \arg \min_{y \in \mathbb{R}^d} \frac{1}{N} \sum_{i=1}^n \lambda_i \sum_{k=0}^N (y_k - x^i_k)^2$ for any convex vector $\lambda \in \mathbb{R}^n$. The symmetric case is proved similarly.

**4.2 Proofs of results in Subsection 3.3**

**Proof of Theorem 3.7.** Fix any $\gamma \in \{\mu_{i,0}\}_{i \in V}$. Let $\mu(t) := (\mu_1(t), ..., \mu_n(t))^\top$ and $T_\gamma := (T^{\mu_{1,0}}_{\#}, ..., T^{\mu_{n,0}}_{\#})^\top$. We first claim that $\mu(t) = (\prod_{\tau=0}^t A(\tau) T_\gamma)_{\# \gamma}$, where the push-forward notation $(\cdot)_{\# \gamma}$ is applied element-wise. We will prove this claim by induction.

Assume any $(i, j) \in E$ is selected at $t = 0$. Then,

$$\mu_i(1) = ((1 - a_{ij})Id + a_{ij}T^{\mu_{i,0}}_{\#})_{\mu_{i,0}}$$

$$= ((1 - a_{ij})Id + a_{ij}T^{\mu_{i,0}}_{\#}(T^{\mu_{i,0}}_{\#})_{\# \gamma})$$

$$= ((1 - a_{ij})T^{\mu_{i,0}}_{\#} + a_{ij}T^{\mu_{i,0}}_{\#} \circ T^{\mu_{i,0}}_{\#})_{\# \gamma}$$

$$= ((1 - a_{ij})T^{\mu_{i,0}}_{\#} + a_{ij}T^{\mu_{i,0}}_{\#})_{\# \gamma},$$

where the third equality follows from the property that

$$(A)_{\# \mu} = ((B)_{\# \mu})(A^{-1}(\cdot)) = \mu(B^{-1}A^{-1}(\cdot)) = \mu((A \circ B)^{-1}(\cdot)) = (A \circ B)_{\# \mu}$$

for any measure $\mu$ and appropriate measurable maps $A, B$; and the last equality follows from the compatible collection. Thus, we have that, with probability one, $\mu(1) = (A(0)T_\gamma)_{\# \gamma}$. Again, without loss of generality, let us consider that $(i, j) \in E$ was chosen at $t = 0$ and analyze all possible updates at $t = 1$. Assume some edge $(p, q) \in E$ is chosen. Then, $\mu_p(2) = ((1 - a_{pq})Id + a_{pq}T^{\mu_{p,0}}_{\#})_{\mu_p(1)}$. Now, observe that $\mu_i(1) = ((1 - a_{ij})T^{\mu_{i,0}}_{\#} + a_{ij}T^{\mu_{i,0}}_{\#})_{\# \gamma}$ and $\mu_k(1) =$
for some random convex vector $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$ with probability one. This gives the consensus result $\lim_{t \to \infty} \mu_k(t) = (\sum_{j=1}^n \lambda_j T_{\gamma}^{\mu_{j,0}})^\# \gamma$ for any $k \in V$, and $T_{\mu_k(t)}^{\mu_{j,0}} \circ (\sum_{\ell=1}^n \xi_{\ell}^\mu T_{\gamma}^{\mu_{j,0}}) = \sum_{\ell=1}^n \xi_{\ell}^\mu T_{\gamma}^{\mu_{j,0}}$ with appropriate nonnegative constants $\{\xi_{\ell}^\mu\}_{\ell,k}$ such that $\mu(t) = (\prod_{j=0}^{t-1} \lambda_j T_{\gamma}^{\mu_{j,0}})^\# \gamma$. Finally, we conclude from [29, Theorem 3.1.9] that the measure $\mu_\infty := (\sum_{j=1}^n \lambda_j T_{\gamma}^{\mu_{j,0}})^\# \gamma$ is the unique solution to the barycenter problem with convex vector $\lambda$, i.e., equation (7) is proved. This concludes the proof of statement (i). Statement (ii) is proved with a similar analysis.

Proof of Corollary 3.8. We only focus on proving the results for the directed PaWBar algorithm, since the proofs for the symmetric PaWBar algorithm are very similar and thus omitted. Consider any two absolutely continuous measures $\alpha, \beta \in \mathcal{P}^2(\mathbb{R})$. Then, we have 1) $\alpha = (F_{\alpha}^{-1})_\# \mathcal{L}$, with $\mathcal{L}$ being the Lebesgue measure on $[0, 1]$; and 2) the optimal transport map from $\alpha$ to $\beta$ is the so-called Brenier’s map $T^{\beta}_{\alpha} = F_{\beta}^{-1} \circ F_{\alpha}$ [31, Theorem 2.5]. Thus, since any measure obtained from a displacement interpolation is another absolutely continuous measure in $\mathcal{P}^2(\mathbb{R})$, it is straightforward to conclude that the set of all absolutely continuous measures forms a compatible collection which is closed under interpolation. Since Theorem 3.7’s assumption is satisfied, we can fix any $\gamma \in \{\mu_i, 0\}_{i \in V}$ and replace the Brenier’s maps $F^{-1}_{\mu_{i,0}} \circ F_{\gamma}$, $i \in V$, in the Wasserstein barycenter $\mu_\infty$ expression in statement (i) of Theorem 3.7 to conclude the proof.

Now we consider case (ii). We first remark that a displacement interpolation between any two initial measures will result in zero-mean multivariate Gaussian variables with a closed form expression for their covariance matrices [16]. Thus, we consider a measure $\gamma$ with covariance matrix $\Sigma_\gamma$ resulting from the displacement interpolation with fixed parameter $\lambda \in (0, 1)$ between two arbitrary measures $\mu_{i,0}$ and $\mu_{j,0}$ from the initial set of measures. Then, $\Sigma_\gamma = (\Sigma_{i,0} - \lambda(\Sigma_{i,0}^1 + \lambda(\Sigma_{\gamma}(1/2)^2 + 1/2)^2)^2 \Sigma_{i,0}^1$ (see [16]), and some algebraic work using the fact that both $\Sigma_{i,0}$ and
Directed path the nontrivial case then implies $U$ probability one, (using our notation of the initial set of measures) [29, Section 2.3], we just proved that the initial set of measures is closed under interpolation. Thus, we can use Theorem 3.7 to imply the convergence to the Wasserstein barycenter and [16, Theorem 2.4] provides the shown characterization of the barycenter.

4.3 Proofs of results in Subsection 3.4

Proof sketch of Theorem 3.10. We first consider the directed PaWBar algorithm in case (i). Consider any $(i, j) \in E$ is selected at time $t$. From the definition of constant-speed geodesics [31], it follows that,

$$W_2(\mu_i(t + 1), \mu_j(t)) = (1 - a_{ij})W_2(\mu_i(t), \mu_j(t)),$$
$$W_2(\mu_i(t + 1), \mu_i(t)) = a_{ij}W_2(\mu_i(t), \mu_j(t)).$$  \hfill (18)

If $(i, j)$ is chosen $\tau$ times consecutively starting at time $t$, then $W_2(\mu_i(t + \tau), \mu_j(t)) = (1 - a_{ij})^\tau W_2(\mu_i(t), \mu_j(t))$.

Now, set

$$U(t) = \sum_{(i,j) \in E} W_2(\mu_i(t), \mu_j(t)).$$

Assume any $(i^*, j^*) \in E$ is selected at time $t$, and let $(k^*, i^*) \in E$ (since $G$ is a cycle). Then, setting $U(t) = \sum_{(i,j) \in E \setminus \{(i^*, j^*), (k^*, i^*)\}} W_2(\mu_i(t), \mu_j(t))$,

$$U(t + 1) = W_2(\mu_i(t + 1), \mu_j(t)) + W_2(\mu_{i^*}(t + 1), \mu_{k^*}(t)) + U(t)$$
$$\leq W_2(\mu_i(t + 1), \mu_{j^*}(t)) + W_2(\mu_{i^*}(t + 1), \mu_{j^*}(t)) + W_2(\mu_{i^*}(t), \mu_{k^*}(t)) + U(t)$$
$$= (1 - a_{i^*, j^*})W_2(\mu_{i^*}(t), \mu_{j^*}(t)) + a_{i^*, j^*}W_2(\mu_{i^*}(t), \mu_{j^*}(t))$$
$$+ W_2(\mu_{i^*}(t), \mu_{k^*}(t)) + U(t)$$
$$= W_2(\mu_{i^*}(t), \mu_{j^*}(t)) + W_2(\mu_{i^*}(t), \mu_{k^*}(t)) + U(t) = U(t)$$

where we used the triangle inequality, and then equation (18) for the last equality. Thus, with probability one, $(U(t))_{t \geq 0}$ is a non-increasing sequence uniformly lower bounded by zero, which then implies $U(t)$ converges to some lower bound which we need to prove to be zero. Consider the nontrivial case $U(t) \neq 0$ and again any $(i^*, j^*) \in E$. Since $G$ is a cycle, there is a unique directed path $P_{j^* \rightarrow i^*}$ from $j^*$ to $i^*$ of length $n - 1$. Let $P_{j^* \rightarrow i^*} = ((j^*, \ell_1), \ldots, (\ell_{n-2}, i^*))$. Consider $(i^*, j^*)$ was selected at any time $t$. Now, pick positive numbers $\epsilon_1, \ldots, \epsilon_{n-1}$ such that $\sum_{k=1}^{n-1} \epsilon_k < \frac{U(t)}{2}$. Then, from the sentence below (18), we can first select $T_1$ times the edge $(\ell_{n-2}, i^*)$ such that $W_2(\mu_{\ell_{n-2}}(t + T_1), \mu_{i^*}(t)) < \epsilon_L$; then, we can select $T_2$ times the edge $(\ell_{n-3}, \ell_{n-2})$ such that $W_2(\mu_{\ell_{n-3}}(t + T_1 + T_2), \mu_{\ell_{n-2}}(t + T_1)) < \epsilon_{n-2}$; and we can continue like this until finally selecting $T_{n-1}$ times the edge $(j^*, \ell_1)$ such that $W_2(\mu_{j^*}(t + T), \mu_{\ell_1}(t + \sum_{k=1}^{n-1} T_k)) < \epsilon_1$, with $T = \sum_{k=1}^{n-1} T_k$. 

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Then,

\[
\sum_{(i,j) \in P_{j \to i}} W_2(\mu_i(t + T), \mu_j(t + T)) = W_2(\mu_i^*(t + T), \mu_{e_1} \left( \sum_{k=1}^{n-2} T_k \right)) + \sum_{m=1}^{n-3} W_2(\mu_{e_m} (t + \sum_{k=1}^{n-1-m} T_k), \mu_{e_{m+1}} (t + \sum_{k=1}^{n-1-(m+1)} T_k)) + W_2(\mu_{e_{n-2}} (t + T_1), \mu_i^*(t)) < \sum_{i=1}^{n-1} \epsilon_i < \frac{U(t)}{2}.
\]

Moreover, this result and the triangle inequality imply

which along the triangle inequality implies

\[
W_2(\mu_i^*(t + T), \mu_j^*(t + T)) \leq \sum_{(i,j) \in P_{j \to i}} W_2(\mu_i(t + T), \mu_j(t + T)) < \frac{U(t)}{2},
\]

and thus \(U(t + T) = W_2(\mu_i^*(t + T), \mu_j^*(t + T)) + \sum_{(i,j) \in P_{j \to i}} \sum_{k=1}^{n-3} W_2(\mu_{e_m} (t + \sum_{k=1}^{n-1-m} T_k), \mu_{e_{m+1}} (t + \sum_{k=1}^{n-1-(m+1)} T_k)) + W_2(\mu_{e_{n-2}} (t + T_1), \mu_i^*(t)) < \frac{U(t)}{2} + \frac{U(t)}{2} = U(t)\). This implies the event “\(U(t + T) < U(t)\) for some finite \(T > 0\)” has positive probability of happening at any time \(t\) (because the finite sequence of edges described above has a positive probability of being selected sequentially at any time \(t\)), and so it can happen infinitely often with probability one. Therefore, we conclude that \(U(t) \to 0\) as \(t \to \infty\) with probability one. Then, \(G\) being a cycle implies \(U(t) = 0\) iff \(\mu_i(t) = \mu_j(t)\) for any \(i, j \in V\), and the consensus result (9) follows.

The particular value of the consensus measure \(\mu_\infty\) is random since it may depend on the specific realization of the edge selection process. This finishes the proof for case (i).

Finally, for the symmetric PaWBar algorithm in case (ii), let \(E = \{(1, 2), \ldots, (n-1, n)\}\) without loss of generality and set

\[
U(t) = \sum_{i=1}^{n-1} W_2(\mu_i(t), \mu_{i+1}(t)). \tag{19}
\]

Consider any \(\{i, i+1\} \in E\) is selected at time \(t\). In the following, consider this notation: for any \(a, b \in \{t, t+1\}\) and \(k \geq 1\), set \(W_2(\mu_{i-k}(a), \mu_{i+2}(b)) = 0\) and \(W_2(\mu_{i-k}(a), \mu_{n+2}(b)) = 0\). Then, setting \(U(t) = \sum_{j=1}^{\infty} W_2(\mu_j(t), \mu_{j+1}(t))\),

\[
U(t + 1) = W_2(\mu_{i-1}(t), \mu_i(t + 1)) + W_2(\mu_{i+1}(t + 1), \mu_{i+2}(t)) + U(t) \\
\leq W_2(\mu_{i-1}(t), \mu_i(t)) + W_2(\mu_i(t), \mu_i(t + 1)) + W_2(\mu_{i+1}(t + 1), \mu_{i+1}(t)) \\
+ W_2(\mu_{i+1}(t), \mu_{i+2}(t)) + U(t) \\
= \frac{1}{2} W_2(\mu_i(t), \mu_{i+1}(t)) + \frac{1}{2} W_2(\mu_i(t), \mu_{i+1}(t)) + \sum_{j=1}^{\infty} W_2(\mu_j(t), \mu_{j+1}(t)) \\
= U(t),
\]

where we used the triangle inequality and equation (18). Then \(U(t + 1) \leq U(t)\) with probability one. Following a similar analysis to case (i), assume the nontrivial case \(U(t) \neq 0\). If \(W_2(\mu_1(t), \mu_2(t)) \neq 0\) or \(W_2(\mu_{n-1}(t), \mu_n(t)) \neq 0\), then it follows from our previous derivation that choosing the edge \(\{1, 2\}\)
or \(\{n-1,n\}\) at time \(t\) implies \(U(t+1) < U(t)\). Now if \(W_2(\mu_1(t), \mu_2(t)) = W_2(\mu_{n-1}(t), \mu_n(t)) = 0\) (obviously we consider \(n \geq 4\) since for \(n = 2,3\) there is nothing to prove), then, it is easy to prove that we can select a finite sequence of edges, say of some length \(T'\), such that \(W_2(\mu_1(t + T'), \mu_2(t + T')) \neq 0\) or \(W_2(\mu_{n-1}(t + T'), \mu_n(t + T')) \neq 0\). After such sequence is selected, we can select \(\{1,2\}\) or \(\{n-1,n\}\) so that \(U(t + T' + 1) < U(t + T') \leq U(t)\). Therefore, at any time \(t\), the event \("U(t+T) < U(t)"\) for some finite \(T > 0\)" has positive probability. Finally, following a similar analysis to case (i), we conclude that \(U(t) \to 0\) as \(t \to \infty\) with probability one and conclude the convergence proof of case (ii).

5 The relevance of the PaWBar algorithm in opinion dynamics

In this section we discuss how the directed PaWBar algorithm generalizes a well-known opinion dynamics model with real-valued beliefs to a model with probability distributions as beliefs. Assume the strongly-connected weighted digraph \(G = (V,E,A)\) describes a social network, whereby each agent is an individual and the weight \(a_{ij} \in (0,1)\), for each \((i,j) \in E\), indicates how much influence individual \(i\) accords to individual \(j\). Traditionally in the field of opinion dynamics, the opinion or belief of any \(i \in V\) at time \(t\) is modeled as a scalar \(x_i(t) \in \mathbb{R}\). In the popular asynchronous averaging model (e.g., see [21, 1]) beliefs evolve as follows: if \((i,j) \in E\) is selected at time \(t\), then \(x_{i}(t+1) = (1 - a_{ij})x_{i}(t) + a_{ij}x_{j}(t)\). Note that the PaWBar algorithm specializes to the asynchronous averaging model (as a consequence of Theorem 3.4) when each agent has a degenerate initial distribution with unit mass at a single scalar value.

It is easy to formulate a second generalization of the asynchronous averaging model. Let \(\mu_i(t)\) and \(\mu_j(t)\) denote the beliefs of individuals \(i\) and \(j\), assume \((i,j) \in E\) is selected at time \(t\), and consider the update \(\mu_i(t+1) = (1 - a_{ij})\mu_i(t) + a_{ij}\mu_j(t)\). This second model is a simple (weighted) averaging of the beliefs; we call it the AoB model. To understand the similarities and difference between the PaWBar and AoB models, assume the beliefs of individuals \(i\) and \(j\) at time \(t\) are Gaussian distributions \(N(x_i(t), \sigma)\) and \(N(x_j(t), \sigma)\) with equal variance. Under this assumption, one can see that both models predict that \(i\)'s mean opinion evolves according to \(x_{i}(t+1) = (1 - a_{ij})x_{i}(t) + a_{ij}x_{j}(t)\). However, the two models differ in the predicted overall belief and, specifically:

\[
\text{PaWBar model: } \quad \mu_i(t+1) := N((1 - a_{ij})x_i(t) + a_{ij}x_j(t), \sigma), \quad \text{(20)}
\]

\[
\text{AoB model: } \quad \mu_i(t+1) := (1 - a_{ij})N(x_i(t), \sigma) + a_{ij}N(x_j(t), \sigma). \quad \text{(21)}
\]

In other words, the PaWBar model predicts a Gaussian belief and the AoB model predicts a Gaussian mixture belief. Even though both resulting beliefs have the same mean, they overall differ substantially.

Finally, we argue that the PaWBar algorithm is preferable over the AoB model for opinion evolution from a cognitive psychology viewpoint. In the case of initial Gaussian beliefs, the PaWBar algorithm dictates that \(i\)'s belief is simply Gaussian at every time. Thus, as \(i\) continues her interactions in the social network, the memory cost associated to her belief at all times is constant: \(i\) remembers only two scalars, i.e., the mean opinion and its variance. Instead, if \(i\) updates her belief according to the AoB model, then her belief is a Gaussian mixture at every time and \(i\) is required to remember a more complicated belief structure. Thus, the AoB model implies that \(i\) requires more cognitive power and memory to process the information she gathers from her interactions. The problem with the AoB approach is that arguably individuals tend to simplify beliefs in order to both remember and process thoughts more economically. This simplification of beliefs has attributed humans the metaphor of being *cognitive misers* in cognitive psychology [23, 28]. There-
fore, a model with more economic belief memory requirements, such as our PaWBar algorithm, is arguably more adequate.

6 Conclusion

We propose the PaWBar algorithm based on stochastic asynchronous pairwise interactions. For specific classes of discrete and absolutely continuous measures, we characterize the computation of both randomized and standard Wasserstein barycenters under arbitrary graphs. For the case of general measures, we prove a consensus result and leave the existence of a barycenter as an open problem. We also specialize our algorithm to the Gaussian case and establish a relationship with models of opinion dynamics.

We hope our paper elicits research on efficient numerical solvers for the distributed computation of Wasserstein barycenters based on pairwise computations. As future work, given the importance of Gaussian distributions, we envision theoretical progress in proving the conjecture proposed in our paper. Another open problem is to design consensus algorithms that guarantee the exact computation of a desired weighted Wasserstein barycenter through asynchronous pairwise computations, an unsolved problem presented in [8].

References

[1] D. Acemoglu and A. Ozdaglar. Opinion dynamics and learning in social networks. Dynamic Games and Applications, 1(1):3–49, 2011. doi:10.1007/s13235-010-0004-1.

[2] M. Agueh and G. Carlier. Barycenters in the Wasserstein space. SIAM Journal on Mathematical Analysis, 43:904–924, 2011. doi:10.1137/100805741.

[3] P. C. Álvarez Esteban, E. del Barrio, J. A. Cuesta-Albertos, and C. Matrán. A fixed-point approach to barycenters in Wasserstein space. Journal of Mathematical Analysis and Applications, 441(2):744–762, 2016. doi:10.1016/j.jmaa.2016.04.045.

[4] E. Anderes, S. Borgwardt, and J. Miller. Discrete Wasserstein barycenters: Optimal transport for discrete data. Mathematical Methods of Operations Research, 85:389–409, 2016. doi:10.1007/s00186-016-0549-x.

[5] M. Baum, P. K. Willett, and U. D. Hanebeck. On Wasserstein barycenters and MMOSPA estimation. IEEE Signal Processing Letters, 22(10):1511–1515, 2015. doi:10.1109/LSP.2015.2410217.

[6] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. Numerische Mathematik, 84:375–393, 2000. doi:10.1007/s002110050002.

[7] A. N. Bishop. Information fusion via the Wasserstein barycenter in the space of probability measures: Direct fusion of empirical measures and Gaussian fusion with unknown correlation. In International Conference on Information Fusion, pages 1–7, 2014.

[8] A. N. Bishop and A. Doucet. Distributed nonlinear consensus in the space of probability measures. IFAC Proceedings Volumes, 47(3):8662–8668, 2014. doi:10.3182/20140824-6-ZA-1003.00341.
[9] A. N. Bishop and A. Doucet. Network consensus in the Wasserstein metric space of probability measures. *SIAM Journal on Control and Optimization*, 59(5):3261–3277, 2021. doi:10.1137/19M1268252.

[10] N. Bonneel, G. Peyré, and M. Cuturi. Wasserstein barycentric coordinates: Histogram regression using optimal transport. *ACM Transactions on Graphics*, 35(4), 2016. doi:10.1145/2897824.2925918.

[11] N. Bonneel, J. Rabin, G. Peyré, and H. Pfister. Sliced and radon Wasserstein barycenters of measures. *Journal of Mathematical Imaging and Vision*, 51(1):22–45, 2015. doi:10.1007/s10851-014-0506-3.

[12] F. Bullo. *Lectures on Network Systems*. Kindle Direct Publishing, 1.6 edition, January 2022. URL: http://motion.me.ucsb.edu/book-lns.

[13] G. Buttazzo, L. De Pascale, and P. Gori-Giorgi. Optimal-transport formulation of electronic density-functional theory. *Physical Review A*, 85, 2012. doi:10.1103/PhysRevA.85.062502.

[14] G. Carlier and I. Ekeland. Matching for teams. *Economic Theory*, 42:397–418, 2010. doi:10.1007/s00199-008-0415-z.

[15] G. Carlier, A. Oberman, and E. Oudet. Numerical methods for matching for teams and Wasserstein barycenters. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49:1621–1642, 2015. doi:10.1051/m2an/2015033.

[16] Y. Chen, T. T. Georgiou, and A. Tannenbaum. Optimal transport for Gaussian mixture models. *IEEE Access*, 7:6269–6278, 2019. doi:10.1109/ACCESS.2018.2889838.

[17] S. Claici, E. Chien, and J. Solomon. Stochastic Wasserstein barycenters. In *International Conference on Machine Learning*, volume 80, pages 999–1008, 2018.

[18] N. Courty, R. Flamary, D. Tuia, and A. Rakotomamonjy. Optimal transport for domain adaptation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 39(9):1853–1865, 2016. doi:10.1109/TPAMI.2016.2615921.

[19] M. Cuturi and A. Doucet. Fast computation of Wasserstein barycenters. In *International Conference on Machine Learning*, volume 32, pages 685–693, 2014.

[20] M. Cuturi and G. Peyré. A smoothed dual approach for variational Wasserstein problems. *SIAM Journal on Imaging Sciences*, 9:320–343, 2016. doi:10.1137/15M1032600.

[21] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 3(1/4):87–98, 2000. doi:10.1142/S0219525900000078.

[22] P. Dvurechenskii, D. Dvinskikh, A. Gasnikov, C. Uribe, and A. Nedich. Decentralize and randomize: Faster algorithm for Wasserstein barycenters. 2018.

[23] S. Fiske and S. E. Taylor. *Social Cognition: From Brains to Culture*. Sage, 3 edition, 2003.

[24] S. Gallón, J.-M. Loubes, and E. Maza. Statistical properties of the quantile normalization method for density curve alignment. *Mathematical Biosciences*, 242(2):129–142, 2013. doi:10.1016/j.mbs.2012.12.007.
[25] T. Le Gouic and J.-M. Loubes. Existence and consistency of Wasserstein barycenters. *Probability Theory and Related Fields*, 168:901–917, 2017. doi:10.1007/s00440-016-0727-z.

[26] I. Matei and J. S. Baras. The asymptotic consensus problem on convex metric spaces. *IEEE Transactions on Automatic Control*, 60(4):907–921, 2015. doi:10.1109/TAC.2014.2362988.

[27] Y. Mroueh. Wasserstein style transfer. In *International Conference on Artificial Intelligence and Statistics*, volume 108, pages 842–852, 2020.

[28] P. J. Oakes and J. C. Turner. Is limited information processing capacity the cause of social stereotyping? *European Review of Social Psychology*, 1(1):111–135, 1990. doi:10.1080/14792779108401859.

[29] V. M. Panaretos and Y. Zemel. *An Invitation to Statistics in Wasserstein Space*. SpringerBriefs in Probability and Mathematical Statistics. Springer, 2020. doi:10.1007/978-3-030-38438-8.

[30] J. Rabin, G. Peyré, J. Delon, and M. Bernot. Wasserstein barycenter and its application to texture mixing. In *Scale Space and Variational Methods in Computer Vision*, pages 435–446. Springer, 2012. doi:10.1007/978-3-642-24785-9_37.

[31] F. Santambrogio. *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser, 2015. doi:10.1007/978-3-319-20828-2.

[32] M. A. Schmitz, M. Heitz, N. Bonneel, F. Ngolé, D. Coeurjolly, M. Cuturi, G. Peyré, and J.-L. Starck. Wasserstein dictionary learning: Optimal transport-based unsupervised nonlinear dictionary learning. *SIAM Journal on Imaging Sciences*, 11:643–678, 2018. doi:10.1137/17M1140431.

[33] V. Seguy and M. Cuturi. Principal geodesic analysis for probability measures under the optimal transport metric. 2015.

[34] R. Sepulchre. Consensus on nonlinear spaces. *Annual Reviews in Control*, 35(1):56–64, 2011.

[35] S. Srivastava, V. Cevher, Q. Dinh, and D. Dunson. WASP: scalable Bayes via barycenters of subset posteriors. In *International Conference on Artificial Intelligence and Statistics*, volume 38, pages 912–920, 2015.

[36] A. Tahbaz-Salehi and A. Jadbabaie. Consensus over ergodic stationary graph processes. *IEEE Transactions on Automatic Control*, 55(1):225–230, 2010. doi:10.1109/TAC.2009.2034054.

[37] C. A. Uribe, D. Dvinskikh, P. Dvurechensky, A. Gasnikov, and A. Nedić. Distributed computation of Wasserstein barycenters over networks. In *IEEE Conf. on Decision and Control*, pages 6544–6549, 2018. doi:10.1109/CDC.2018.8619160.

[38] C. Villani. *Topics in Optimal Transportation*. Graduate Studies in Mathematics. American Mathematical Society, 2003.

[39] C. Villani. *Optimal Transport: Old and New*. Springer, 2009. doi:10.1007/978-3-540-71050-9.