Entropy of black holes in topologically massive gravity

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Abstract

We study the issue of black hole entropy in the topologically massive gravity. Assuming that the presence of gravitational Chern-Simons term with the coupling $1/\mu$ does modify the horizon radius $\tilde{r}_+$, we propose $\tilde{S}_{BH} = \pi \tilde{r}_+/2G_3$ as the Bekenstein-Hawking entropy. This entropy of CS-BTZ black hole satisfies the first-law of thermodynamics and the area-law but it is slightly different from the shifted-entropy $S_c = \pi r_+/2G_3 + (1/\mu l)\pi r_-/2G_3$ based on the BTZ black hole with outer $r_+$ and inner horizon $r_-$. In the case of $r_- = 0$, $\tilde{S}_{BH}$ represents the entropy of non-rotating BTZ black hole with the Chern-Simons term (NBTZ-CS), while $S_c$ reduces to the entropy of NBTZ black hole. It shows that $\tilde{S}_{BH}$ may be a candidate for the entropy of the CS-BTZ black hole.
1 Introduction

The gravitational Chern-Simons term in three dimensional Einstein gravity produces a propagating massive graviton \cite{1}. This theory with a negative cosmological constant $\Lambda = -1/l^2$ gives us the BTZ solution with mass $m$ and angular momentum $j$ as a trivial solution \cite{2,3}. However, there exists also a mixed case such that \cite{4,5}

$$M = m + \frac{j}{\mu l^2}, \quad J = j + \frac{m}{\mu},$$

which shows that two conserved quantities are shifted due to the presence of Chern-Simons term without affecting the solution of Einstein equation. There is a chiral point at $\mu l = 1$ which is a solution to the extremal condition of $Ml = J$ \cite{6}.

Also, its entropy is shifted to

$$S_c = \frac{\pi r_+}{2G_3} + \frac{1}{\mu l} \frac{\pi r_-}{2G_3}.$$

We observe that there is an unusual coupling between $\frac{1}{\mu l}$ and the inner horizon $r_-$. This shifted-entropy puts a difficulty on defining its thermodynamic relations of BTZ-CS black holes. Importantly, it implies that the entropy could see the region inside the outer horizon. That is, there should be some degrees of freedom associated with the inner horizon which would be responsible for the black hole entropy. However, it is hard to accept this view because a black hole entropy relates with the outer horizon. If an observer at infinity describes entropy and thermodynamics of the black hole, the region inside the outer horizon is a forbidden region.

In this Letter, we address this issue again and propose a new entropy $\tilde{S}_{BH}$ which satisfies the first-law of thermodynamics as well as the area-law of the entropy.

2 Thermodynamics and entropy for BTZ-CS black hole

We start with the action for the topologically mass gravity in anti-de Sitter spacetimes \cite{1}

$$I_{CTMG} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left[ R + \frac{2}{l^2} - \frac{1}{2\mu} \varepsilon^{\lambda\mu\nu} \Gamma^\rho_{\lambda\sigma} \left( \partial_\mu \Gamma^\sigma_{\nu\rho} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right) \right],$$

where $\varepsilon$ is the tensor density defined by $\varepsilon/\sqrt{-g}$ with $\varepsilon^{012} = 1$. The $1/\mu$-term is the first higher derivative correction in three dimensions because it is the third-order derivative.
Varying this action leads to the Einstein equation

\[ G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \]  

where the Einstein tensor including the cosmological constant is given by

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} \]  

and the Cotton tensor is

\[ C_{\mu\nu} = \varepsilon^{\alpha\beta} \nabla_\alpha \left( R_{\beta\nu} - \frac{1}{4} g_{\beta\nu} R \right). \]  

In the absence of Chern-Simons term, the BTZ solution to Eq.(4) is given by \[ ds^2_{\text{BTZ}} = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left( N^\phi dt + d\phi \right)^2, \]  

where \( f \) and \( N^\phi \) are the metric and shift functions defined by

\[ f(r) = -8G_3 m + \frac{r^2}{l^2} + \frac{16G_3^2 l^2}{r^2}, \quad N^\phi = -\frac{4G_3 j}{r^2}. \]  

Here \( m \) and \( j \) are the mass and the angular momentum of the BTZ black hole, respectively. From the condition of \( f = 0 \), we find two horizons located at

\[ r_{\pm} = l \left[ \sqrt{2G_3 \left( m + \frac{j}{l} \right)} \pm \sqrt{2G_3 \left( m - \frac{j}{l} \right)} \right]. \]  

We observe that in order to have appropriate horizon radii, it requires to have a bound of \( ml \geq j \). Its thermodynamic quantities of mass \( m \), temperature \( T_H = f'(r_+)/4\pi \), entropy \( S_{BH} \), angular momentum \( j \), and angular velocity \( \Omega_+ = -N^\phi(r_+) \) as function of \( r_+ \) and \( r_- \) are given by

\[ m = \frac{r_{+}^2 + r_{-}^2}{8G_3 l^2}, \quad T_H = \frac{r_{+}^2 - r_{-}^2}{2\pi l^2 r_{+}}, \quad S_{BH} = \frac{\pi r_{+}}{2G_3}, \quad j = \frac{r_{+} r_{-}}{4G_3 l}, \quad \Omega_+ = \frac{r_{-}}{l r_{+}}. \]  

We check that the first-law of

\[ dm = T_H dS_{BH} + \Omega_+ dj \]  

is satisfied and its integral form (Smarr formula) of \( m = \frac{T_H S_{BH}}{2} + \Omega_+ j \) holds for the BTZ black hole. We observe the area-form of the entropy \( S_{BH} \). Here we define the on-shell condition such that if the first-law of thermodynamics is satisfied with thermodynamic quantities which are obtained from using equations of motion. In this case, the on-shell condition implies the thermal equilibrium configuration of the BTZ black holes without
any conical singularity, while the off-shell condition means the off-equilibrium of the BTZ black hole with conical singularity [8, 9, 10].

However, it is known that the Chern-Simons term does not affect the BTZ solution of Einstein equation because the Cotton tensor is trivially satisfied with the BTZ-type metric. Hence the temperature $T_H$ and angular velocity $\Omega_+$ are fixed, because these are determined by solution of $f$ and $N^\phi$.

On the other hand, the presence of Chern-Simons term has an effect on the mass $M$ and angular momentum $J$ as Eq.(1) is shown. These shifted-quantities were obtained using the off-shell method, holographic gravitational anomalies [4]. Further we note that the off-shell approach of conical singularity was used to calculate the shifted-entropy $S_c$ [5]. In addition, this entropy was derived using other off-shell methods, Wald’s Noether formula [14] and Brown-Henneaux’s approach [17].

Using the entropy $S_c$, it is apparent that the first-law of black hole thermodynamics

$$dM = T_H dS_c + \Omega_+ dJ$$

is satisfied, but the area-law of black hole entropy violates. This implies that in order to have the first-law, the shift of entropy $(S_{BH} \rightarrow S_c)$ compensates changes of mass and angular momentum $(m \rightarrow M, j \rightarrow J)$ obtained from the off-shell approach. It is noted that $M, S_c, J$ are off-shell quantities but $T_H$ and $\Omega_+$ are on-shell quantities because they are derived from the on-shell approach of metric function $f$ and $N^\phi$. This picture gives rise to a difficulty in deriving thermodynamic relations if one considers $M(S_c, J)$ as function of $S_c$ and $J$. Actually, we find

$$\left( \frac{\partial M}{\partial S_c} \right)_J \neq T_H, \quad \left( \frac{\partial M}{\partial J} \right)_{S_c} \neq \Omega_+.$$  \hspace{1cm} (13)

Furthermore, we have a difficulty in defining the heat capacity, which is an important thermodynamic quantity to be used for testing the thermodynamic stability. Its usual definition

$$C_J = \left( \frac{\partial M}{\partial T_H} \right)_J$$

does not work because it is defined by $M$ (off-shell quantity) and $T_H$ (on-shell quantity). In order to cure this problem, we reexpress Eq.(12) as

$$dM = T_H dS_{BH} + \left( \Omega_+ + \frac{1}{\mu l^2} \right) dj$$

which shows that the change of angular velocity compensates the change of mass. Here we can check the thermodynamic relations

$$\left( \frac{\partial M}{\partial S_{BH}} \right)_j = T_H, \quad \left( \frac{\partial M}{\partial j} \right)_{S_{BH}} = \Omega_+ + \frac{1}{\mu l^2}$$  \hspace{1cm} (16)
since we can express the shifted-mass as function of $S_{BH}$ and $j$

$$M(S_{BH}, j) = \frac{G_3 S_{BH}^2}{2\pi^2 l^2} + \frac{\pi^2 j^2}{2G_3 S_{BH}^2} + \frac{j}{\mu l^2}. \quad (17)$$

In this case, we can define the heat capacity as that of the BTZ black hole

$$C_j = \frac{\partial M}{\partial T_H} = \frac{\pi r_+ \Delta}{2G_3(2 - \Delta)} = \frac{\partial m}{\partial T_H}, \quad (18)$$

with $\Delta = \sqrt{1 - \frac{j^2}{m^2 l^2}}$. This is independent of the CS-coupling $\mu$. It implies that the shifted-quantities of $M, J, S_c$ are not suitable for computing thermodynamic relations because all they are mixed as functions of $(r_+, r_-)$.

Let us discuss what happens for the extremal condition of $T_H = 0 = C_j$ when using the shifted-quantities. In the absence of the Chern-Simons term, the extremal black hole is located at $r_+ = r_- = r_e$ for $ml = j$. In the presence of the Chern-Simons term, this condition is $Ml = J$, which implies

$$\mu l = 1, \; ml = j, \quad (19)$$

where the first one is new for the BTZ-CS black hole and is called a chiral point. However, there is no way to implement this chiral condition (extremal black hole) without changing the temperature $T_H$ and heat capacity $C_j$ (metric function $f$).

## 3 Thermodynamics and entropy for CS-BTZ black hole

We are in a position to seek other solution by implementing the shifted condition of Eq. (1). We suggest that this condition changes the form of metric function. Plugging this into Eq. (8) ($m \rightarrow M, j \rightarrow J$), we find two shifted-horizons for the CS-BTZ located at

$$\tilde{r}_\pm = l \left[ \sqrt{1 + \frac{1}{\mu l} \sqrt{2G_3(m + \frac{j}{l})}} \pm \sqrt{1 - \frac{1}{\mu l} \sqrt{2G_3(m - \frac{j}{l})}} \right] \quad (20)$$

which shows clearly that the presence of Chern-Simons term modifies the black hole horizons. In this process, the Einstein equation (5) is trivially satisfied. This is our main result which differs from the BTZ-CS. We mention the difference between CS-BTZ and BTZ-CS. We mean by CS-BTZ that the Chern-Simons term shifts the horizon radii through Eq. (1). On the other hand, the BTZ-CS implies that the horizon radii remain
unchanged, while the mass and angular momentum are shifted so that the shifted-entropy does appear.

Fortunately, we find that the degenerate horizon of

\[ \tilde{r}_+ = \tilde{r}_- \equiv \tilde{r}_e \] (21)

which provides Eq. (19) of \( \mu l = 1 \) and \( lm = j \). The chiral point of \( \mu l = 1 \) \([6, 11]\) corresponds to the extremal CS-BTZ at \( \tilde{r}_e = 2l \sqrt{G_3(m + j/l)} \). From the condition of \( \tilde{r}_\pm \geq 0 \), we have the bound for \( \mu \), in addition to \( lm \geq j \):

\[ \mu l \geq 1. \] (22)

For the CS-BTZ black holes, we could define the thermodynamic quantities of mass \( M \), new temperature \( \tilde{T}_H \), angular momentum \( J \), and new angular velocity \( \tilde{\Omega}_+ \) as function of \( \tilde{r}_+ \) and \( \tilde{r}_- \),

\[ M = \frac{\tilde{r}_+^2 + \tilde{r}_-^2}{8G_3l^2}, \quad \tilde{T}_H = \frac{\tilde{r}_+^2 - \tilde{r}_-^2}{2\pi l^2 \tilde{r}_+}, \quad J = \frac{\tilde{r}_+ \tilde{r}_-}{4G_3l} \quad \tilde{\Omega}_+ = \frac{\tilde{r}_-}{l \tilde{r}_+}. \] (23)

Here we note that the shifted-mass \( M \) and angular momentum \( J \) are expressed in a compact way when using \( \tilde{r}_+ \) and \( \tilde{r}_- \). For \( \tilde{r}_+ = \tilde{r}_- \), we find the thermal condition for the extremal black hole

\[ \tilde{T}_H = 0, \quad \tilde{C}_J = 0, \] (24)

where the new heat capacity is given by

\[ \tilde{C}_J = \frac{\pi \tilde{r}_+ \tilde{\Delta}}{2G_3(2 - \tilde{\Delta})} \] (25)

with

\[ \tilde{\Delta} = \sqrt{1 - \frac{J^2}{M^2l^2}} = \sqrt{1 - \frac{1}{\mu l^2}} \sqrt{1 - \frac{j^2}{m^2l^2}}. \] (26)

Importantly, the new entropy is given by the Bekenstein-Hawking entropy as

\[ \tilde{S}_{BH} = \frac{\pi \tilde{r}_+}{2G_3} = \frac{\pi l}{2G_3} \left[ \sqrt{1 + \frac{1}{\mu l}} \sqrt{2G_3(m + j/l)} + \sqrt{1 - \frac{1}{\mu l}} \sqrt{2G_3(m - j/l)} \right] \] (27)

which is slightly different from the shifted-entropy \( S_c \) of Eq. (2) expressed as \([12, 13, 14]\)

\[ S_c = \frac{\pi l}{2G_3} \left[ (1 + \frac{1}{\mu l}) \sqrt{2G_3(m + j/l)} + (1 - \frac{1}{\mu l}) \sqrt{2G_3(m - j/l)} \right]. \] (28)

Here we check easily that the first-law of thermodynamics

\[ dM = \tilde{T}_H d\tilde{S}_{BH} + \tilde{\Omega}_+ dJ \] (29)
is satisfied for the CS-BTZ black hole. This implies that new thermodynamic quantities (tilde variables) compensate the shifts of mass and angular momentum to have the first-law and the area-law form of the entropy. Furthermore, we confirm that thermodynamic relations hold for the CS-BTZ black hole:

\[
\left( \frac{\partial M}{\partial S_{BH}} \right)_J = \bar{T}_H, \quad \left( \frac{\partial M}{\partial J} \right)_{\bar{S}_{BH}} = \bar{\Omega}_+. \tag{30}
\]

Especially, we have the Smarr formula of

\[
M = \frac{\bar{T}_H \bar{S}_{BH}}{2} + \bar{\Omega}_+ J.
\]

In order for \( \bar{S}_{BH} \) to compare with \( S_c \), we make a series expansion for \( \mu l \gg 1 \) as

\[
\bar{S}_{BH} \simeq \frac{\pi r_+}{2G_3} + \frac{1}{2\mu l} \frac{\pi r_-}{2G_3} + \frac{1}{2G_3} \left( \frac{\pi}{l} \right)^{d-2} v^2.
\]

At this stage, we introduce two similar cases in higher dimensions. Let us observe what happens for the Reissner-Nordström-AdS black holes when the Gauss-Bonnet term with coupling constant \( a \) is added [15, 16]. Its action \( S_d \) is given by

\[
S_d = \int d^d x \left( R - 2\Lambda - F^2/4 + a \mathcal{L}_{GB} \right).
\]

For \( d = 4 \), using the entropy function approach, one finds that

\[
S_4 = (\pi G_4)(v_2 + 4a),
\]

which means that the Gauss-Bonnet term does not modify the horizon radius (\( v_2 \) is independent of \( a \)) but it shifts the black hole entropy. Here \( v_2 \) is the radius square appeared in

\[
ds^2 = v_1(-r^2 dt^2 + dr^2/r^2) + v_2 d\Omega_{d-2} \text{ of } AdS_2 \times S^{d-2}.
\]

In this case, there is no change of equations of motion even for the presence of Gauss-Bonnet term because the Gauss-Bonnet term plays the role of a topological term. This is the same situation as the Chern-Simons term is added in three dimensions. However, for \( d = 5 \), we have

\[
S_5 = (\pi^2 \sqrt{v_2/2G_5})(v_2 + 12a).
\]

Here \( v_2 \) is a function of \( a \). The \( a \)-dependent term represents the deviation from the area-law due to the Gauss-Bonnet term. In this case, there is change of equations of motion in the presence of Gauss-Bonnet term. Hence, our entropy \( \bar{S}_{BH} \) is between \( S_4 \) and \( S_5 \) because the horizon radius was changed from \( r_+ \) to \( \bar{r}_+ \).

On the other hand, from the CFT on the boundary at infinity, we can calculate the entropy using the Cardy-formula. Taking into account the shifted mass and angular momentum in Eq.(1), the shifted Virasoro generators are given by [6, 17]

\[
\bar{L}_0^\pm = \frac{l}{2} \left( M \pm \frac{J}{l} \right) = \frac{l}{2} \left( 1 \pm \frac{1}{\mu l} \right) (m \pm \frac{J}{l}). \tag{32}
\]

Introducing the Cardy formula

\[
S^{CFT} = 2\pi \sqrt{\frac{c_L \bar{L}_0^+}{6}} + 2\pi \sqrt{\frac{c_R \bar{L}_0^-}{6}}, \tag{33}
\]

7
we recover $\tilde{S}_{BH}$ by choosing
\[
\tilde{c}_{L/R} = \frac{3l}{2G_3},
\] (34)
while $S_c$ is obtained by choosing the shifted central charges
\[
c_{L/R} = \frac{3l}{2G_3} \left(1 \pm \frac{1}{\mu l}\right).
\] (35)

4 Thermodynamics and entropy for CS-NBTZ black holes

We are in a position to consider the mixed case with $j = 0$ in Eq.(1) which is equivalent to the BTZ black hole with $m = m$ and angular momentum $j = m/\mu$. In this case, we denote
\[
M^{j=0} = m, \quad J^{j=0} = \frac{m}{\mu},
\] (36)
which indicates that the presence of Chern-Simons term shifts the angular momentum but does not change the mass if we start with $j = 0$ solution. Hence, the $j = 0$ case is more attractive to show the effect of the Chern-Simons term than the $j \neq 0$ case. Plugging this into Eq.(9), we find two shifted-horizons for the CS-NBTZ located at
\[
\tilde{r}^{j=0}_\pm = l\sqrt{2G_3m} \left[\sqrt{1 + \frac{1}{\mu l}} \pm \sqrt{1 - \frac{1}{\mu l}}\right]
\] (37)
which shows clearly that the presence of Chern-Simons terms modifies the black hole horizons. Explicitly, the presence of Chern-Simons term creates a new inner horizon $\tilde{r}_-$, in compared with the single horizon of NBTZ black hole. Furthermore, at the chiral point of $\mu l = 1$, we have a new degenerate horizon at $\tilde{r}^{j=0}_e = 2l\sqrt{G_3m}$.

For the CS-NBTZ, we define the thermodynamic quantities of mass, temperature, angular momentum, and angular velocity as function of $\tilde{r}_+$ and $\tilde{r}_-$ in Eq.(37),
\[
M^{j=0} = \frac{\tilde{r}_+^2 + \tilde{r}_-^2}{8G_3l^2}, \quad T^{j=0}_H = \frac{\tilde{r}_+^2 - \tilde{r}_-^2}{2\pi l^2\tilde{r}_+}, \quad J^{j=0} = \frac{2\tilde{r}_+\tilde{r}_-}{8G_3l}, \quad \Omega^{j=0} = \frac{\tilde{r}_-}{l\tilde{r}_+}.
\] (38)
Crucially, the entropy is given by the Bekenstein-Hawking formula as
\[
\tilde{S}_{BH}^{j=0} = \frac{\pi \tilde{r}_+^{j=0}}{2G_3} = \pi l\sqrt{m/2G_3} \left[\sqrt{1 + \frac{1}{\mu l}} + \sqrt{1 - \frac{1}{\mu l}}\right]
\] (39)
which contains the effect of Chern-Simons term. For the critical point of $\mu l = 1$, one has the extremal entropy $\tilde{S}_{BH}^{j=0} = \pi l\sqrt{m/G_3}$, while for $\mu l \to \infty$, one finds $\tilde{S}_{BH}^{j=0} = \pi l\sqrt{2m/G_3}$ which is just the entropy of the NBTZ black hole.
Table 1: Summary of thermodynamic picture for BTZ, BTZ-CS and CS-BTZ. Here First-L, TR, Heat-C, and Smarr-F represent the first-law of thermodynamics, thermodynamic relations, heat capacity, and Smarr formula, respectively.

|               | BTZ          | BTZ-CS       | CS-BTZ       |
|---------------|--------------|--------------|--------------|
| First-L       | $dm = T_H dS_{BH} + \Omega_+ dj$ | $dM = T_H dS_c + \Omega_+ dJ$ | $dM = \tilde{T}_H d\tilde{S}_{BH} + \tilde{\Omega}_+ dJ$ |
| TR            | $(\frac{dM}{dS_{BH}}) = T_H, (\frac{dM}{dj}) = \Omega_+$ | N/A | $(\frac{dM}{dS_{BH}}) = T_H, (\frac{dM}{dJ}) = \Omega_+$ |
| Entropy       | $S_{BH} = \frac{\pi r_+}{2G_3}$ | $S_c = \frac{\pi r_+}{2G_3} + \frac{1}{\mu l} \frac{\pi r_-}{2G_3}$ | $S_{BH} = \frac{\pi \tilde{r}_+}{2G_3}$ |
| Heat-C        | $C_J = \frac{\pi r_+ \Delta}{2G_3(2-\Delta)}$ | N/A | $\tilde{C}_J = \frac{\pi \tilde{r}_+ \Delta}{2G_3(2-\Delta)}$ |
| Smarr-F       | $m = T_H S_{BH}/2 + \Omega_+ j$ | $M = T_H S_c/2 + \Omega_+ J$ | $M = \tilde{T}_H \tilde{S}_{BH}/2 + \tilde{\Omega}_+ J$ |

On the other hand, the shifted-entropy $S_c^{j=0}$ is just the entropy of NTBZ black hole as

$$S_c^{j=0} = \pi l \sqrt{\frac{2m}{G_3}} = S^{NTBZ},$$

where shows clearly that there is no effect of Chern-Simons terms on the black hole entropy for $j = 0$ case, as is implied from Eq.(2) with $r_- = 0$. Furthermore, there is no chiral point of $\mu l = 1$. This means that there is no shift of Chern-Simons term to the entropy when using the NBTZ background.

5 Discussions

First of all, we summarize our thermodynamic results for BTZ, BTZ-CS, and CS-BTZ black holes in Table 1 for comparison.

We recover the shifted-mass $M$, angular momentum $J$ and the chiral condition of $\mu l = 1$ from shifted-horizons of the CS-BTZ black hole. In this case, we obtain a new entropy $\tilde{S}_{BH}$. Also the consistent thermodynamic relations are derived. This is a good achievement obtained by plugging the shifted-mass and angular momentum into the metric function $f$ and $N^\phi$.

Although the shifted-entropy $S_c$ is well-known as the landmark of the topologically massive gravity, it puts a difficulty on defining thermodynamic relations of BTZ-CS black holes. Importantly, it implies that the entropy could see the region inside the outer horizon because of the coupling between $1/\mu$ and the inner horizon $r_-$. However, it is hard to accept this view because a black hole entropy relates with the outer horizon.

The entropy of $\tilde{S}_{BH}$ seems to be a candidate for the entropy for the CS-BTZ black hole since it satisfies the first-law of thermodynamics and the area-law of the black hole entropy.
The effect of the Chern-Simons term shifts the horizons and creates a new degenerate horizon at the chiral point of \( \mu l = 1 \) even for the NBTZ background of \( m \neq 0, j = 0 \). The point to note is the equivalence between the CS-NBTZ and the BTZ black hole, which shows the role of the Chern-Simons term in the black hole clearly.

However, one thing to be understood is why the entropy of \( \tilde{S}_{BH} \) could be recovered from the CFT on the boundary at infinity with the central charges of \( \tilde{c}_R/L = 3l/2G_3 \). Assuming the AdS$_3$/CFT$_2$ correspondence, the bulk theory of topologically massive gravity is equivalent to a conformal field theory with different right and left central charges \( c_R/L \) in Eq. (35). In order to recover the shifted-entropy \( S_c \), one has to take into account the shifted-Virasoro generators in Eq. (32) as well as the shifted-central charges \( c_R/L \). This may correspond to doubly counting the Chern-Simons terms for computation of the entropy. In deriving the entropy \( \tilde{S}_{BH} \), we did not use the central charges \( c_R/L \).

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