FREE PLANES IN LATTICE SPHERE PACKINGS

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Abstract. We show that for every lattice packing of $n$-dimensional spheres there exists an $(n/\log_2(n))$-dimensional affine plane which does not meet any of the spheres in their interior, provided $n$ is large enough. Such an affine plane is called a free plane and our result improves on former bounds.

1. Introduction

The main purpose of this note is to show that the complement of every $n$-dimensional lattice sphere packing contains an $n/\log_2(n)$ affine plane, if the dimension $n$ is large enough. In order to show the relations of this result to other problems in packing theory we give a more general introduction.

Let $K_0^n$ be the set of all 0-symmetric convex bodies in the $n$-dimensional Euclidean space $\mathbb{R}^n$. We always assume that $\text{int}(K) \neq \emptyset$ for $K \in K_0^n$, i.e., $K$ has non-empty interior, and we denote the volume of $K$ ($n$-dimensional Lebesgue measure) by $\text{vol}(K)$. A lattice $\Lambda \subset \mathbb{R}^n$ is the image of the standard lattice $\mathbb{Z}^n$ under a regular linear map $B$, i.e., $\Lambda = B\mathbb{Z}^n$, with $B \in \mathbb{R}^{n \times n}$ and $\det B \neq 0$. $|\det B|$ is called the determinant of $\Lambda$ and it is denoted by $\det \Lambda$. $\mathcal{L}^n$ denotes the set of all lattices in $\mathbb{R}^n$.

For a given $K \in K_0^n$, a lattice $\Lambda \in \mathcal{L}^n$ is called a packing lattice of $K$ if for $b_1 \neq b_2 \in \Lambda$ the translates $b_1 + K$ and $b_2 + K$ do not overlap, i.e., $\text{int}(b_1 + K) \cap \text{int}(b_2 + K) = \emptyset$. The density of a densest lattice packing of $K$, denoted by $\delta(K)$, is the maximum proportion of space that can be occupied by an arrangement of the $\Lambda + K$, where $\Lambda$ is a packing lattice. Thus

$$\delta(K) = \max \left\{ \frac{\text{vol}(K)}{\det \Lambda} : \Lambda \text{ packing lattice of } K \right\}.$$  

For a thorough treatment of packings of convex bodies and lattices we refer to [GL87] and [EGH89].

For a given lattice $\Lambda$ and a body $K \in K_0^n$ the maximum dilation factor $\lambda$ with the property that $\Lambda$ becomes a packing lattice of $\lambda K$ is called the packing radius of $\Lambda$ with respect to $K$ and is denoted by $\lambda(K, \Lambda)$, i.e,

$$\lambda(K, \Lambda) = \max \{ \lambda \in \mathbb{R}_{>0} : \Lambda \text{ is a packing lattice of } \lambda K \}.$$  

In particular, $\Lambda/\lambda(K, \Lambda)$ is the “smallest” dilation of $\Lambda$ which is a packing lattice of $K$. Hence we can write

$$\delta(K) = \text{vol}(K) \max \left\{ \frac{\lambda(K, \Lambda)^n}{\det \Lambda} : \Lambda \in \mathcal{L}^n \right\},$$
and thus
\[(1.1) \quad \lambda(K, \Lambda)^n \leq \delta(K) \frac{\det \Lambda}{\text{vol}(K)}.\]

As a counterpart to the packing radius we have the covering radius \(\mu(K, \Lambda)\) of \(\Lambda\) with respect to \(K\), defined by
\[
\mu(K, \Lambda) = \min \left\{ \mu \in \mathbb{R}_{>0} : \Lambda + \mu K = \mathbb{R}^n \right\}.
\]

One can also say, that \(\mu(K, \Lambda)\) is the smallest positive number \(\mu\) such that \(\Lambda\) is a covering lattice of \(\mu K\). If \(\theta(K)\) denotes the density of a thinnest lattice covering of \(K\), we get analogously to (1.1)

\[(1.2) \quad \mu(K, \Lambda)^n \geq \theta(K) \frac{\det \Lambda}{\text{vol}(K)}.\]

Here we are interested in lattices which are simultaneously good packing and covering lattices. Therefore we define
\[
\gamma(K) = \inf \{ \mu(K, \Lambda) : \Lambda \text{ is a packing lattice of } K \}.
\]

Zong baptised this number the simultaneous lattice packing and covering constant of \(K\) (cf. [Zon02a]). By definition of the packing radius of \(K\) we may also write
\[
\gamma(K) = \inf \left\{ \frac{\mu(K, \Lambda)}{\lambda(K, \Lambda)} : \Lambda \in \mathcal{L}^n \right\}
\]

and by (1.1) and (1.2) we get
\[
\gamma(K) \geq \left( \frac{\theta(K)}{\delta(K)} \right)^{1/n}.
\]

A remarkable upper bound on \(\gamma(K)\) is due to Butler [But72], who showed
\[
\gamma(K) \leq 2 + o(1),
\]
as \(n\) tends to infinity. By (1.3) it gives asymptotically a lower bound on \(\delta(K)\) which is of the same order of magnitude as the classical Minkowski-Hlawka bound.

\(\gamma(K)\) is a quite interesting functional. If we were able to prove, for instance for the unit ball \(B^n\), the existence of an \(\epsilon > 0\) such that \(\gamma(B^n) \leq 2 - \epsilon\) then we would get by (1.3) an improvement on the best known lower bound on \(\delta(B^n)\) which is of order \(2^{-n}\) [Ba92].

On the other hand, if we could verify \(\gamma(K) \geq 2\) then we know that even for a densest packing lattice \(\Lambda_K\) of \(K\), i.e. \(\delta(K) = \text{vol}(K)/\det \Lambda_K\), there exists a point \(x \in \mathbb{R}^n\) with \((x + \text{int}(K)) \cap (\Lambda_K + \text{int}(K)) = \emptyset\). Thus \(\Lambda_K \cup (x + \Lambda)\) is a packing set, which means that the bodies \((\Lambda_K \cup (x + \Lambda_K)) + K\) do not overlap. Since the space is occupied by \((\Lambda_K \cup (x + \Lambda_K)) + K\) two times as good as by \(\Lambda_K + K\) we get
\[
\delta_T(K) \geq 2\delta(K),
\]

where \(\delta_T(K)\) denotes the density of a densest arbitrary (without the restriction to lattices) packing of \(K\). So far a body \(K \in \mathcal{K}_0^n\) with \(\delta_T(K) > \delta(K)\)
is not known. For results on $\gamma(K)$ in the planar and three dimensional case we refer to \cite{Zon02b} and \cite{Zon03}.

Next we are interested in a generalisation of $\gamma(K)$. To this end we consider the covering radii introduced by Kannan and Lovász \cite{KL88}. For $1 \leq i \leq n$,

$$\mu_i(K, \Lambda) = \min \{ \mu > 0 : \Lambda + \mu K \text{ meets every } (n - i)\text{-dimensional affine subspace of } \mathbb{R}^n \}$$

is called the $i$-th covering minimum. With the help of these functionals we define for $1 \leq i \leq n$

$$\gamma_i(K) = \inf \left\{ \frac{\mu_i(K, \Lambda)}{\lambda(K, \Lambda)} : \Lambda \in \mathcal{L}^n \right\}.$$

Of course, $\gamma_n(K) = \gamma(K)$, and by definition we have

$$\text{(1.4) } \gamma_i(K) \geq 1 \iff \text{for every lattice packing } \Lambda \text{ of } K \text{ there exists an } (n-i)\text{-dimensional affine plane which does not intersect any of the translates } \Lambda + K \text{ in their interior.}$$

A plane which does not intersect any of the translates of the packing is called a free plane. In general, we can not expect to find a free plane for an arbitrary $K \in \mathcal{K}_n^0$ and $\Lambda \in \mathcal{L}^n$. However, it was shown by Heppes \cite{Hep61} that in every 3-dimensional lattice sphere packing one can find a cylinder of infinite length which does not intersect any of the spheres of this packing. In other words, one can always find an 1-dimensional free plane. Later this was generalised to any dimension $\geq 3$ by Horváth and Ryškov \cite{HR75}. Concerning the maximum dimension of free planes in lattice sphere packings it was recently shown that it is at least $4$ ($n$ large) and at most $c n$ for an absolute constant $c < 1$ \cite{HZZ02}. For a detailed treatment of the history of free planes in sphere packings and related results we refer to the recent survey of Zong \cite{Zon02a}.

Here we improve on the upper bound on free planes in the following way.

**Theorem 1.1.** For every lattice packing of $n$-dimensional spheres, $n$ large, there exists an affine plane of dimension $\geq n/\log_2(n)$ which does not intersect any of the spheres of the packing in their interior.

By (1.4) the theorem will be an immediate consequence of the next lemma giving a lower bound on $\gamma_i(K)$. To this end, for $1 \leq i \leq n$ and $K \in \mathcal{K}_n^0$ let

$$\rho_i(K) = \sup_{A \in \text{GL}(n, \mathbb{R})} \inf \left\{ \frac{\text{vol}(AK)^{1/n}}{\text{vol}_i((AK)|L)^{1/i}} : L \in G(i, n) \right\}.$$

Here $\text{GL}(n, \mathbb{R})$ denotes the set of all regular $(n \times n)$-matrices, $G(i, n)$ denotes the family of all $i$-dimensional linear subspaces of $\mathbb{R}^n$, and for a set $S$ the orthogonal projection onto a linear subspace $L$ is denoted by $S|L$. The $i$-dimensional volume of an $i$-dimensional $S \subset \mathbb{R}^n$ is denoted by $\text{vol}_i(S)$. 
Lemma 1.1. Let $K \in K_0^n$. Then

$$\gamma_i(K) \geq \delta(K)^{-\frac{i}{n}} \cdot \rho_i(K) \cdot \max \left\{ \left( \frac{1}{\sqrt{i+1}} \right)^{\frac{n-i}{i}}, \frac{1}{\sqrt{n-i+1}} \right\}.$$ 

In particular, for the sphere and $n$ large we have

$$\gamma_i(B^n) > \frac{3}{2} \sqrt{\frac{i}{n}} \cdot \max \left\{ \left( \frac{1}{\sqrt{i+1}} \right)^{\frac{n-i}{i}}, \frac{1}{\sqrt{n-i+1}} \right\}.$$ 

Substituting $i = n - n/\log_2(n)$ in the last formula leads to

$$\gamma_{n - \frac{n}{\log_2(n)}}(B^n) > \frac{3}{2} \sqrt{1 - \frac{1}{\log_2(n)}} \cdot \left( \frac{\log_2(n)}{n \log_2(n) - n + \log_2(n)} \right)^{1/(\log_2(n) - 1)}.$$ 

Since the last factor tends to $1/\sqrt{2}$ as $n$ approaches infinity, Theorem 1.1 follows from the observation (1.4).

Before giving the proof of the lemma we want to remark that in every dimension there exists a lattice $\tilde{\Lambda} \in L_n$ such that $\mu_i(B^n, \tilde{\Lambda})/\lambda(B^n, \tilde{\Lambda}) \leq c_i n$, where $c > 2$ is an absolute constant (cf. [HZZ02]). Hence, at least for the sphere and $i = 1$ Lemma 1.1 is best possible up to a constant.

2. Proof

First we have to fix some notations. A linear $i$-dimensional subspace $L_i \in G(i, n)$ is called a lattice plane (subspace) of $\Lambda \in L_n$ if $\dim(\Lambda \cap L_i) = i$. The set of all $i$-dimensional lattice planes of a given lattice $\Lambda$ is denoted by $L(\Lambda, i)$. For a linear subspace $L \subset \mathbb{R}^n$ we denote by $L^\perp$ the orthogonal complement. We note that the orthogonal projection of a lattice $\Lambda \in L_n$ onto a $i$-dimensional linear subspace $L$, say, is a lattice again, if and only if $L \in L(\Lambda, i)$. The polar (dual) lattice of $\Lambda \in L_n$ is denoted by $\Lambda^\ast$ and it is $\det \Lambda^\ast = 1/\det \Lambda$. There are some simple and useful relations between a lattice $\Lambda$ and its dual $\Lambda^\ast$, which we collect in the following statements

\begin{align}
&\text{i)} \quad L \in L(\Lambda, i) \iff L^\perp \in L(\Lambda^\ast, n - i), \\
&\text{(2.1) ii)} \quad (\Lambda | L^\perp)^\ast = \Lambda^\ast \cap L^\perp, \quad L \in L(\Lambda, i), \\
&\text{iii)} \quad \det(\Lambda \cap L) = \det(\Lambda^\ast \cap L^\perp) \cdot \det \Lambda, \quad L \in L(\Lambda, i).
\end{align}

For $1 \leq i \leq n$ let

$$\tau(n, i) = \sup_{\Lambda \in L_n} \inf \left\{ \frac{\det(\Lambda \cap L)^{1/i}}{\det \Lambda^{1/n}} : L \in L(\Lambda, i) \right\}.$$ 

The functional $[\tau(n, i)]^{2i}$ is the so called $i$-th Rankin invariant (or generalised Hermite constant), introduced by Rankin [Ran53] (see also [Mul64]). For our purposes, however, the above normalisation is more suitable. Among
others Rankin proved two basic relations for these numbers, which read in our notation

\begin{align}
\text{i) } \tau(n, i)^i &= \tau(n, n - i)^{n - i}, \\
\text{ii) } \tau(n, i) &\leq \tau(n, m)\tau(m, i), \quad 1 \leq i \leq m \leq n. 
\end{align}

(2.2)

In fact, the identity i) is a consequence of (2.1) iii), and the second inequality follows immediately from the definition.

Of particular interest is the \(\tau(n, 1)\) which can be rewritten as

\begin{equation}
\tau(n, 1) = 2 \left( \frac{\delta(B^n)}{\kappa_n} \right)^{1/n},
\end{equation}

where \(\kappa_n = \pi^{n/2}/\Gamma(n/2 + 1)\) is the volume of the \(n\)-dimensional unit ball \(B^n\). It is well known that (cf. e.g. [Mar03, p. 53])

\begin{equation}
\tau(n, 1) < \sqrt{n}.
\end{equation}

As a simple consequence of properties (2.2) we get for \(1 \leq i \leq n\)

\begin{equation}
\tau(n, i) \leq \min \left\{ \sqrt{n - i + 1}, \left( \sqrt{i + 1} \right)^{\frac{n-i}{i}} \right\}.
\end{equation}

(2.5)

In fact, let \(i \leq n - 1\). From (2.2) ii) and (2.4) we obtain

\begin{align}
\tau(n, i) &\leq \prod_{m=i}^{n-1} \tau(m + 1, m) = \prod_{m=i}^{n-1} \tau(m + 1, 1)^{\frac{1}{m}} \leq \prod_{m=i}^{n-1} (\sqrt{m + 1})^\frac{1}{m} \\
&\leq \left( \sqrt{i + 1} \right)^{\frac{n-i}{i}}.
\end{align}

Next we apply this bound to \(\tau(n, n - i)\) and with (2.2) i) we find

\begin{equation}
\tau(n, i) = \tau(n, n - i)^{\frac{n-i}{i}} \leq \sqrt{n - i + 1}.
\end{equation}

and so we get (2.5).

\textbf{Proof of Lemma 1.1.} Since \(\gamma_i(K)\) is invariant with respect to linear transformations we may assume that \(K\) is in such a position that

\[
\sup_{A \in \text{GL}(n, \mathbb{R})} \inf_{L \in G(n,i)} \frac{\text{vol}(AK)^{1/n}}{\text{vol}_{(L \perp)}(AK) \text{vol}_{L}^{1/i}}
\]

is attained for the identity matrix. It was shown in [KL88] that the \(i\)-th covering minimum can also be described as

\[
\mu_i(K, \Lambda) = \max \left\{ \mu(K|L^\perp, \Lambda|L^\perp) : L \in \mathcal{L}(\Lambda, n - i) \right\}.
\]
Hence we have 
\[ \frac{\mu_i(K, \Lambda)}{\lambda(K, \Lambda)} \geq \frac{1}{\lambda(K, \Lambda)} \max \left\{ \frac{\det(\Lambda|L^\perp)^{1/i}}{\text{vol}_i(K|L^\perp)^{1/i}} : L \in \mathcal{L}(\Lambda, n - i) \right\} \]

and with (2.3) we obtain
\[ \frac{\mu_i(K, \Lambda)}{\lambda(K, \Lambda)} = \frac{1}{\lambda(K, \Lambda)} \max \left\{ \frac{\det(\Lambda)^{1/n}}{\text{vol}(K)^{1/n}} \frac{\det(\Lambda^*)^{1/n}}{\text{vol}(\Lambda^* \cap L)^{1/i}} : L \in \mathcal{L}(\Lambda^*, i) \right\} \]
\[ = \frac{\det(\Lambda)^{1/n}}{\text{vol}(K)^{1/n} \lambda(K, \Lambda)} \max \left\{ \frac{\det(\Lambda^*)^{1/n}}{\text{vol}(\Lambda^* \cap L)^{1/i}} : L \in \mathcal{L}(\Lambda^*, i) \right\} \]
\[ \geq \delta(K)^{-1/n} \rho_i(K) \tau(n, i)^{-1}. \]

Together with (2.3) we obtain Lemma 1.1 in the general case. Next we firstly observe that by the isoperimetric inequality \( \rho_i(K) \leq \kappa^{1/n} / \kappa^{1/i} \) with equality if and only if \( K \) is an ellipsoid. Hence for \( K = B^n \) we have
\[ \rho_i(B^n) = \frac{1}{\kappa^{1/i}} = \frac{\Gamma(n/2 + 1)^{1/n}}{\Gamma(i/2 + 1)^{1/i}} \geq \sqrt{\frac{i}{n}}, \]
where the last inequality follows form the fact that \( \Gamma(m/2 + 1)^{1/m}/\sqrt{m} \) is monotonously decreasing which can be shown by elementary considerations.

Finally, as an upper bound on \( \delta(B^n) \) we use the asymptotic bound of Kabatianski and Levenstein
\[ \delta(B^n) \leq 2 - 0.599 n + o(1) \] (cf. [Zon99, p. 137]). \( \square \)

3. Concluding Remarks

We remark that already Rankin determined the value \( \tau(4, 2) = (3/2)^{1/4} \).

For further bounds for special choices of \( i \) and \( n \) and the relation of Rankin’s invariants to minimal vectors of the \( i \)-th exterior power of a lattice we refer to [Cou97] and the references within. Thunder [Thu98] gave a lower bound on a generalised version of \( \tau(n, i) \) in the context of algebraic extensions \( F \) of \( \mathbb{Q} \) of finite degree. In the case \( F = \mathbb{Q} \) his bound becomes
\[ \tau(n, i) \geq \left( 2 n \prod_{j=1}^{i} \frac{\zeta(n-i+j)}{(n-i+j)\kappa_{n-i+j}} \right)^{1/n}, \]
where \( \zeta(k) = \sum_{i=1}^{\infty} i^{-k} \) denotes the \( \zeta \)-function. For \( i \) proportional to \( n \) and \( n \to \infty \) this lower bound is of order \( \sqrt{n^{(n-i)/n}} \). However, even if this lower bound were the right order of magnitude for an upper bound on \( \tau(n, i) \) this would not lead to an improvement on the lower bound on the maximum dimension of free planes in lattice sphere packings.

Finally, we note that free planes are lattice phenomena. In the non-lattice case one can even find sphere packings such that the length of a segment contained in the complement is bounded by a constant depending only on the dimension (see [BT] and [HZ00]).
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