Abstract

In this paper we compare two classical one-factor diffusion models which are used to model the term structure of interest rates. One of them is based on the Wiener-Bachelier process while the second one is based on the Ornstein-Uhlenbeck process. We show essential differences between the prices of European call options on a zero-coupon bond in these models.

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**Introduction**

The bond market forms a very important segment of financial markets. However, modelling of it is difficult because the bond prices depend on both the present situation of the market and the time to maturity. The prices of bonds are expressed in terms of various interest rates and yields, so understanding bonds pricing is equivalent to understanding interest rate behaviour. The term structure of interest rates is defined as the dependence between interest rates and maturities. We propose two models:

1) the Merton model based on the Wiener-Bachelier process,
2) the Vasicek model based on the quantum Ornstein-Uhlenbeck process.\(^1\)

In the above models we find the formula for the zero-coupon bond price and we price the option on this bond. In the first section we propose general zero-coupon bond pricing model and we give some information about forward contracts with the help of which we price options on zero-coupon bonds. Then, we discuss the Merton model while in the third one we discuss the Vasicek model. Finally, we compare both models.

We assume that the market is effective without an opportunity of arbitrage and the face value of any bond equals one. Let \( B_t^T \) be the price of the zero-coupon bond at some date \( t \) with the time to maturity \( T \geq t \). Let \( y_t^T \) denotes the discount rate called zero-coupon rate or spot rate for date \( T \) in continuous capitalisation \(^3\). Then \( B_t^T \) is related to the continuously compounded zero-coupon rate by

\[
B_t^T = e^{-(T-t)y_t^T},
\]

hence

\[
y_t^T = -\frac{1}{T-t} \ln B_t^T.
\]

We call the function \( T \rightarrow B_t^T \) the discount function and the transformation \( T \rightarrow y_t^T \) the yield curve.

The short-term interest rate is given by the formula

\[
r_t = \lim_{\tau \to 0^+} y_t^{t+\tau}.
\]

\(^1\) In the quantum game theory Ornstein-Uhlenbeck process has the interpretation of non-unitary tactics resulting in a new strategy. These strategies are the Hilbert’s spaces vectors. Quantum strategies create unique opportunities for making profits during intervals shorter than the characteristic thresholds for the Brown particle, see \(^3\).
We introduce few notions which are connected with the derivatives. First, we consider options on the zero-coupon bonds. Let us consider European options. We must clearly distinguish between the call option and the put option. The time of maturity of the option is denoted by $T$. Let $C_t$ denotes the time $t$ price of an European call option on the zero-coupon bond, which gives a payment of 1 at time $S$, where $S \geq T$, with the exercise price of $K$. Similarly, let $\pi_t$ denotes the price of the corresponding put option. The prices of these options at maturity are equal to $C_T = \max(B_T^S - K, 0)$ and $\pi_T = \max(K - B_T^S, 0)$. The prices of European call and put option on the zero-coupon bonds fulfil the put-call parity relation \cite{4}:

$$C_t + KB_t^T = \pi_t + B_t^S. \quad (3)$$

As the result, we can restrict ourself to pricing of European call options.

Another derivatives are forward contracts on the zero-coupon bonds. If the underlying variable is the price of the zero-coupon bond, the essential part by the pricing of the forward contracts plays the present prices of the zero-coupon bonds only. If there is no arbitrage, the time $t$ value $V_{t}^{T,S}$ of the forward contract on the zero-coupon bond, with the delivery date $T$ and the delivery price $K$, is given by the formula \cite{4}:

$$V_{t}^{T,S} = B_t^S - KB_t^T. \quad (4)$$

If $S > T$ is the maturity date of the underlying bond. For forwards contracted upon at time $t$, the delivery price $K$ is set so that the value of the forward at time $t$ is zero ($V_{t}^{T,S} = 0$). This value of $K$ is called the forward price $F_{t}^{T,S}$. Solving (4), we obtain that

$$F_{t}^{T,S} = \frac{B_t^S}{B_t^T}. \quad (5)$$

## 1 The pricing in affine models

We assume that for considered market model, there is a risk-neutral probability measure (or equivalent martingale measure) $P^*$ and a one-dimensional Wiener process $W^*$ under measure $P^*$. If the risk-neutral probability measure exists, then there is also the $T$-forward martingale measure $P^T$ and then $W^T$ is the Wiener process under measure $P^T$. If there exists a spot martingale measure, then the market is arbitrage free. Part of the market characteristic fulfils the short rate under the risk-neutral probability measure (i.e. the
spot martingale measure). The model in which this rate is represented by the process \((r_t)_{t>0}\), that fulfils the stochastic equation
\[
dr_t = \mu(r_t, t) dt + \sigma(r_t, t) dW_t^*,
\]
where \(\mu : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}\) and \(\sigma : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^d\), we call the affine model. Let us consider that \(\mu_t = \alpha_1 - \alpha_2 \cdot r_t\) and \(\sigma_t = \sqrt{\beta_1 + \beta_2 \cdot r_t}\), where \(\alpha_1, \alpha_2, \beta_1, \beta_2\) are constant. We know that the discount price of the zero-coupon bond is given by the formula
\[
B^T_t = e^{-\int_t^T r_s ds} \mathbb{E}^P \left[ e^{-R_{T-t} r_s ds} | \mathcal{F}_t \right]
\]
(7)
With the help of this assumption, we can calculate the formula for the discount price of the zero-coupon bond, see [6]. This price can be written as a function of time and the current short rate \(r\).

**Theorem 1.1.**

In an affine model, in which the short rate \(r\) is described by (6), the time \(t\) price of the zero-coupon with the time to maturity \(T\) and face value 1 is equal to
\[
B^T_t = e^{-a(T-t) - rb(T-t)}.
\]
(8)
The functions \(a, b : [0, T] \rightarrow \mathbb{R}\) fulfil the following system of ordinary differential equations:
\[
\frac{1}{2} \beta_2 b(t)^2 + \alpha_2 b(t) + b'(t) - 1 = 0,
\]
\[
a'(t) - \alpha_1 b(t) + \frac{1}{2} \beta_1 b(t)^2 = 0,
\]
for \(t \in (0, T)\) and \(a(0) = b(0) = 0\).

Proof.
The proof is based on the Feynman-Kac theorem, see [5,8] in Appendix. From theorem (5.8) follows that \(B^T_t\), given by the formula (8), fulfils the partial differential equation
\[
\frac{\partial B^T_t}{\partial t} + (\alpha_1 - \alpha_2 r) \frac{\partial B^T_t}{\partial r} + \frac{1}{2} (\beta_1 + \beta_2 r) \frac{\partial^2 B^T_t}{\partial r^2} - r B^T_t = 0.
\]
The relevant derivatives are
\[
\frac{\partial B^T_t}{\partial t} = B^T_t (a'(T-t) + rb'(T-t)),
\]
\[
\frac{\partial B^T_t}{\partial r} = -B^T_t b(T-t),
\]
\[
\frac{\partial^2 B^T_t}{\partial r^2} = B^T_t b(T-t)^2.
\]
After substituting these formulas into (5.8), we obtain
\[ a'(T - t) + rb'(T - t) - (\alpha_1 - \alpha_2 r)b(T - t) + \frac{1}{2}b(T - t)^2(\beta_1 + \beta_2 r)^2 = 0. \]

After gathering terms involving \( r \), we obtain
\[ (a'(T - t) - \alpha_1 b(T - t) + \frac{1}{2}\beta_1 b(T - t)^2) + \left( \frac{1}{2}\beta_2 b(T - t)^2 + \alpha_2 b(T - t) + b'(T - t) - 1 \right) r = 0. \]

The last equation must be fulfilled for arbitrary \( r \), so the expressions in the brackets must be identically 0. Hence, we find equations which the function \( a \) and \( b \) must fulfil. Because \( B_T^T \equiv 1 \), so \( a(0) + b(0) r \equiv 0 \) and hence \( a(0) = b(0) = 0 \). We see that the function (8) fulfils the Feynman-Kac equation.

Let us observe that, if we have the function \( b \) from the above formula, we can calculate the function \( a \) from the formula
\[ a(t) = \alpha_1 \int_0^t b(s)ds - \frac{1}{2}\beta_1 \int_0^t b(s)^2ds. \]  
(9)

If we want to determine the price of the bond, we have to find the solution of the differential equation given by the formula
\[ \frac{1}{2}\beta_2 b(t)^2 + \alpha_2 b(t) + b'(t) - 1 = 0, \quad b(0) = 0. \]

Now, let us observe that \( b(t) > 0 \) for \( t > 0 \). Above all, from the continuity of \( b \), the conditions \( b(0) = 0 \) and \( b'(t) = 1 - \frac{1}{2}\beta_2 b(t)^2 - \alpha_2 b(t) \) follow that \( b'(t) > 0 \) in the vicinity of 0. From this, \( b \) grows up in the vicinity of 0, that is \( b(t) > 0 \) for small \( t > 0 \). As if \( b(t) \leq 0 \) for certain \( t > 0 \), there is a point \( t_1 > 0 \) from the Intermediate Value Theorem \[7\], such that \( b(t_1) = 0 \). From the Rolle theorem \[7\], we obtain \( b(0) = b(t_1) = 0 \), so there must exist \( t_0 > 0 \) such that \( b'(t_0) = 0 \). If we substitute \( t_0 \) to the differential equation fulfilled by \( b \), we obtain the equality \( \frac{1}{2}\beta_2 y_0^2 + \alpha_2 y_0 - 1 = 0 \) for \( y_0 = b(t_0) \). Let us consider constant function \( b_0(t) = y_0 \). It fulfils the same differential equation, so \( b \). Furthermore, \( b(t_0) = b_0(t_0) \), that means that both functions fulfil the same initial-value problem. From the uniqueness of solution we obtain that \( b \equiv b_0 \)

\(*3\) If \( b(t) = \alpha_1 b(t) - \frac{1}{2}\beta_1 b(t)^2 \), so \( a(t) = a(0) + \alpha_1 \int_0^t b(s)ds - \frac{1}{2}\beta_1 \int_0^t b(s)^2ds \). Let us take into consideration that \( a(0) = 0 \).
and \( b_0(0) = b(0) \), that is, \( y_0 = 0 \), which is impossible because 0 does not fulfil the quadratic equation which is satisfied by \( y_0 \). We obtain contradiction, so \( b(t) > 0 \) for \( t > 0 \). It means that the function \( r \rightarrow B^T_t \) is strictly decreasing.

In the further parts of our paper we will use the formula for the volatility of the process \( B^T \) under \( T \)-forward martingale measure. It looks the same as for measure \( P^* \). From the Feynman-Kac theorem, it is given by the formula

\[
\sigma^T_t = \frac{\partial B^T_t}{\partial r} \beta_t
\]

and in our specific situation \( \beta_t = \sqrt{\beta_1 + \beta_2 r} \). Using the formula (8), we obtain

\[
\sigma^T_t = \frac{-b(T-t)B^T_t}{B^2_t} \beta_t = -b(T-t)\beta_t. \tag{10}
\]

Let \( \mu^T \) denotes the relative drift of the process \( B^T \) under measure \( P^T \).

2 The Merton model

The first dynamic, affine and continuous time model of the term structure of the interest rate was described by Merton [8]. The short rate follows a generalised Brownian motion under the spot martingale measure:

\[
\begin{align*}
&dr_t = \varphi dt + \sigma dW^*_t,
&\text{where } \sigma \text{ and } \varphi \text{ are constant. We begin by finding the price of the zero-coupon bond and then we calculate the price of the option. We see that the price of the bond in this model is given by the formula (see (5))}
&B^T_t = e^{-a(T-t)-rb(T-t)},
&\text{where the function } b \text{ fulfils the simple ordinary differential equation } b'(t) = 1
\end{align*}
\]

with \( b(0) = 0 \). So, \( b(t) = t \) and the function \( a \) is given by the formula (see (9))

\[
a(t) = \varphi \int_0^t s ds - \frac{1}{2} \sigma^2 \int_0^t s^2 ds = \frac{1}{2} \varphi t^2 - \frac{1}{6} \sigma^2 t^3,
\]

therefore

\[
B^T_t = e^{-\frac{1}{2} \varphi (T-t)^2 + \frac{1}{2} \sigma^2 (T-t)^3 - (T-t)r}. \tag{11}
\]

In order to price the option, we use the theorem (5.7), see Appendix. The price of an European call option on the zero-coupon bond with the expiration
date $S$, the exercise price $K$, and maturing at the time $T$ is given by, see (14) in Appendix,

$$C_t = B^T_t \left( E^{P_T} \left[ \max(B^S_T - K, 0) \right] | F_t \right) \right) (r).$$

From (5) we know that the forward price fulfills the formula:

$$F^{T,S}_t = \frac{B^S_t}{B^T_t}.$$

hence, in particular $F^{T,S}_t = B^S_t$. The forward price $(F^{T,S}_t)_{t \geq 0}$ is a martingale under the $T$-forward martingale measure, what means that the drift rate equals 0 under this probability measure. Simultaneously, this process is a quotient of the prices of two zero coupon bonds. In the diffusion equations for $B^T$ and $B^S$, the relative volatilities of the bond are equal to $\sigma^T_t = -\sigma \cdot (T-t)$ and $\sigma^S_t = -\sigma \cdot (S-t)$ respectively, so that

$$dB^T_t = \mu^T_t B^T_t dt - \sigma (T-t) B^T_t dW^T_t,$$

$$dB^S_t = \mu^S_t B^S_t dt - \sigma (S-t) B^S_t dW^T_t.$$

By an application of the Itô lemma, see Appendix (5.3), for functions of multiple stochastic processes and because of that, the drift of the process $F^{T,S}$ equals 0 under the $T$-forward martingale measure, we obtain

$$dF^{T,S}_t = \left[ \frac{\partial F^{T,S}_t}{\partial B^T_t} \sigma^T_t B^T_t + \frac{\partial F^{T,S}_t}{\partial B^S_t} \sigma^S_t B^S_t \right] dW^T_t$$

$$= \left[ -\frac{B^S_t}{(B^T_t)^2} \sigma^T_t B^T_t + \frac{1}{B^T_t} \sigma^S_t B^S_t \right] dW^T_t$$

$$= (\sigma^S_t - \sigma^T_t) F^{T,S}_t dW^T_t = -\sigma (S-T) F^{T,S}_t dW^T_t.$$

This means that, $F^{T,S}$ follows a geometric Brownian motion, see Appendix (5.4), with $\sigma^{T,S}(t) = -\sigma \cdot (S-T)$. Hence, the random variable $X := \ln \frac{F^{T,S}_T}{F^{T,S}_t}$ is normally distributed and

$$X = -\frac{1}{2} \int^T_t \sigma^{T,S}(u)^2 du + \int^T_t \sigma^{T,S}(u) dW^T_u,$$

what result from (5.5), see Appendix. Moreover,

$$E(X) = -\frac{1}{2} \int^T_t \sigma^{T,S}(u)^2 du = -\frac{1}{2} \sigma^2 (S-T)^2 (T-t),$$

$$\nu(t,T,S) := \sqrt{Var(X)} = \sqrt{\int^T_t \sigma^{T,S}(u)^2 du} = \sigma (S-T) \sqrt{T-t}.$$
Let us consider the function $g : \mathbb{R} \times \mathbb{R}_+ \ni (x, y) \mapsto \max(e^x y - K, 0)$ and a random variable $Z = e^X$. From the formula (12) and with the help of that $B_t^S = F_t^{T,S}$, we obtain

$$C_t = B_t^T \left( E^{PT} \left[ \max(F_t^{T,S} - K, 0) | \mathcal{F}_t \right] \right)(r) = B_t^T \left( E^{PT} \left[ g(X, F_t^{T,S}) | \mathcal{F}_t \right] \right)(r).$$

From the equation (13) we have that $X$ is independent of $\mathcal{F}_t$ and $F_t^{T,S}$ is $\mathcal{F}_t$-measurable, because the process $F_t^{T,S}$ is a martingale. Hence, we can use the lemma (5.2), see Appendix. By the using the lemma (5.6), see Appendix, we can calculate the expected value $[y > 0 !]$

$$E(g(X, y)) = E(\max(Z \cdot y - K, 0)) = yE \left( \max \left( \frac{Z - K}{y}, 0 \right) \right) =$$

$$= y \left[ e^{E(X)} + \frac{1}{2} v(t, T, S)^2 N \left( \frac{E(X) - \ln \frac{K}{y} + v(t, T, S)}{v(t, T, S)} \right) - \frac{K}{y} N \left( \frac{E(X) - \ln \frac{K}{y}}{v(t, T, S)} \right) \right] =$$

$$= ye^0 N \left( -\frac{1}{2} v(t, T, S) + \frac{1}{v(t, T, S)} \ln \frac{y}{K} + v(t, T, S) \right) +$$

$$- KN \left( -\frac{1}{2} v(t, T, S) + \frac{1}{v(t, T, S)} \ln \frac{y}{K} \right) =$$

$$= yN \left( \frac{1}{v(t, T, S)} \ln \frac{y}{K} + \frac{1}{2} v(t, T, S) \right) - KN \left( \frac{1}{v(t, T, S)} \ln \frac{y}{K} - \frac{1}{2} v(t, T, S) \right).$$

If we substitute to the above equation $y = F_t^{T,S}$, we obtain

$$\left( E^{PT} \left[ g(X, F_t^{T,S}) | \mathcal{F}_t \right] \right)(r) =$$

$$= F_t^{T,S} N \left( \frac{1}{v(t, T, S)} \ln \frac{F_t^{T,S}}{K} + \frac{1}{2} v(t, T, S) \right) +$$

$$- KN \left( \frac{1}{v(t, T, S)} \ln \frac{F_t^{T,S}}{K} - \frac{1}{2} v(t, T, S) \right)$$

and finally

$$C_t = B_t^T \left[ F_t^{T,S} N(d_1) - KN(d_2) \right] = B_t^S N(d_1) - KB_t^T N(d_2).$$

*4) Let us observe that $g(X, F_t^{T,S}) = \max(F_t^{T,S} - K, 0)$, so that $g(X, F_t^{T,S})$ is bounded, because $0 \leq F_t^{T,S} \leq 1.$
where
\[
\begin{align*}
    d_1 &= \frac{1}{v(t, T, S)} \ln \left( \frac{B_t^S}{KB_t^T} \right) + \frac{1}{2} v(t, T, S), \\
    d_2 &= \frac{1}{v(t, T, S)} \ln \left( \frac{B_t^S}{KB_t^T} \right) - \frac{1}{2} v(t, T, S),
\end{align*}
\]

\[v(t, T, S) = \sigma (S - T) \sqrt{T - t}.
\]

### 3 The Vasicek Model

In the paper [9] Vasicek proposed a mean-reverting version of the Ornstein-Uhlenbeck process for the short term rate. Specifically, under the risk-neutral measure \( P^* \), \( r_t \) is given by

\[
    dr_t = \kappa (\theta - r_t) dt + \sigma dW_t^*,
\]

where \( \kappa, \theta \) and \( \sigma \) are positive constants. The parameter \( \theta \) denotes *long-term level of the short rate*, because the rate \( r_t \) pull towards a long-term level of \( \theta \), \( \kappa \) determines the speed of adjustment, and \( \sigma \) is the average deviation of the rate of return.

Similarly to the Merton model, at the first we calculate the formula for the zero-coupon bond price. From the theorem 1.1 we know that

\[
    B_T^T = e^{-a(T-t) - rb(T-t)},
\]

but in the described model we have

\[
    \kappa b(t) + b'(t) - 1 = 0, \quad b(0) = 0.
\]

Hence,

\[
    b(t) = \frac{1}{\kappa} (1 - e^{-\kappa t}).
\]

From (9) we obtain that

\[
    a(t) = \kappa \theta \int_0^t b(s) ds - \frac{1}{2} \sigma^2 \int_0^t b(s)^2 ds =
    = \theta \int_0^t (1 - e^{-\kappa s}) ds - \frac{\sigma^2}{2\kappa^2} \int_0^t (1 - 2e^{-\kappa s} + e^{-2\kappa s}) ds =
    = (\theta - \frac{\sigma^2}{2\kappa^2})(t - b(t)) + \frac{\sigma^2}{4\kappa} b(t)^2.
\]
If we substitute $a(t)$ and $b(t)$ to the formula (8), we obtain analogous formula for the bond price to the Merton model, see (11).

We calculate the formula for the call option price similar to the Merton model. Let us use the basic formula

$$C_t = B_t^T E^{PT} \left[ \max(B_s^S - K, 0) | \mathcal{F}_t \right].$$

With the help of the Itô lemma (5.3) and because of that the drift of process $F_t^{T,S}$ equals 0 under the $T$-forward martingale measure, we obtain the diffusion equation for the forward price:

$$dF_t^{T,S} = \left((\sigma_t^S - \sigma_t^T) F_t^{T,S} dW_t^T - \sigma [b(S - t) - b(T - t)] F_t^{T,S} dW_t^T.\right.$$

Therefore, the process of forward prices is a geometric Brownian motion with $\sigma_t^{T,S}(t) = -\sigma [b(S - t) - b(T - t)]$. The random variable $X := \ln F_{S,T}^{T,S} F_{S,T}^{T,S}$ has the Gaussian distribution. Similarly, we obtain the formulas:

$${X} = -\frac{1}{2} \int_t^T \sigma_t^{T,S}(u)^2 du + \int_t^T \sigma_t^{T,S}(u) dW_u^T,$$

$$E(X) = -\frac{1}{2} \int_t^T \sigma_t^{T,S}(u)^2 du = -\frac{1}{2} \frac{\sigma^2}{\kappa^3} (1 - e^{-\kappa[S-t]})^2 (1 - e^{-2\kappa[T-t]}),$$

$$v(t, T, S) := \sqrt{\int_t^T \sigma_t^{T,S}(u)^2 du} = \frac{\sigma}{\kappa^{3/2}} (1 - e^{-\kappa[S-t]}) \sqrt{1 - e^{-2\kappa[T-t]}}.$$

Our further reasoning is the same like in the Merton model. Let the function $g : \mathbb{R} \times \mathbb{R}_+ \ni (x, y) \mapsto \max(e^x y - K, 0)$ and the random variable $Z = e^X$. We obtain the formula

$$C_t = B_t^T \left( E^{PT} \left[ \max(F_t^{T,S} - K, 0) | \mathcal{F}_t \right] \right)(r)$$

$$= B_t^T \left( E^{PT} \left[ g(X, F_t^{T,S}) | \mathcal{F}_t \right] \right)(r).$$

With the help of independence $X$ of $\mathcal{F}_t$ and $\mathcal{F}_t$-measurability of $F_t^{T,S}$, we can use the lemma (5.2). Then, if we use the lemma (5.6), we calculate that the expected value $E(g(X, y))$ equals

$$E(g(X, y)) = yN \left( \frac{1}{v(t, T, S)} \ln \frac{y}{K} + \frac{1}{2} v(t, T, S) \right) +$$

$$- KN \left( \frac{1}{v(t, T, S)} \ln \frac{y}{K} - \frac{1}{2} v(t, T, S) \right).$$
If we substitute to the above equation \( y = F_t^{T,S} \) we obtain

\[
\left( E^{PT} \left[ g(X, F_t^{T,S}|\mathcal{F}_t) \right] \right) (r) = 
\]

\[
= F_t^{T,S} N \left( \frac{1}{v(t, T, S)} \ln \frac{F_t^{T,S}}{K} + \frac{1}{2} v(t, T, S) \right) + 
\]

\[
- KN \left( \frac{1}{v(t, T, S)} \ln \frac{F_t^{T,S}}{K} - \frac{1}{2} v(t, T, S) \right) 
\]

and finally

\[
C_t = B_t^S N(d_1) - KB_t^T N(d_2), 
\]

where

\[
d_1 = \frac{1}{v(t, T, S)} \ln \left( \frac{B_t^S}{KB_t^T} \right) + \frac{1}{2} v(t, T, S), 
\]

\[
d_2 = \frac{1}{v(t, T, S)} \ln \left( \frac{B_t^S}{KB_t^T} \right) - \frac{1}{2} v(t, T, S), 
\]

\[
v(t, T, S) = \frac{\sigma}{\kappa^{3/2}} (1 - e^{-\kappa[S-t]}) \sqrt{1 - e^{-2\kappa[T-t]}}. 
\]

4 Final remarks

The Fig. 1 shows the prices of European call options on the zero-coupon bonds with the expiration date 5 years, the face value 1 and the real values of the market parameters based on the Merton model and the Vasicek model. In the above figures, we consider the European call option on the

![Figure 1: The prices of European call options on the zero-coupon bonds based on the: 1.1–Merton model and 1.2–Vasicek model. We assume that: \( \kappa = 0.4, \sigma = 0.03, r = 0 \).](image-url)
zero-coupon bond with the exercise price $K = 0.80$, time to maturity $S = 5$ years and the face value $B_S = 1$. As it follows from the general affine model pricing, see (6), we can compare this two models under the assumption that $\varphi = \kappa\theta$. The results are qualitatively different. Let us notice that, the price of the option described by Vasicek is higher than the price described by Merton and faster drawn towards to 0.2, where 0.2 is the expected price of the option with fixed expiration date $T = 5$, because the market is effective without an opportunity of arbitrage. Let us observe that, both models give equal prices for the long interest rate $\theta = 0$.

The difference between the logarithmic price of the European call option described by the Ornstein-Uhlenbeck process and the logarithmic price of the call option described by the Wiener-Bachelier process is illustrated in the Fig. 2. The difference between these logarithmic prices increases with the growth of $\theta$ and fades away in the limit of the expiration date ($T = 5$) of the zero-coupon bond. One of the inappropriate properties of the Merton model is the constant drift of the short rate. With the constant positive drift the short rate is expected to increase in the future. But this is not realistic. Many empirical studies show that, the interest rates exhibit mean reversion, in the sense that, if an interest rate is high by historical standards, it typically will be falling in the near future (analogically if the current interest rate is low).

Figure 2: We assume that: $\kappa = 0.4$, $\sigma = 0.03$, $r = 0$, and $C_M$ is the option price based on the Merton model, and $C_V$ is the option price based on the Vasicek model.
In paper \[2\] we considered the prices of options on company’s stock supported by Wiener-Bachelier model and Ornstein-Uhlenbeck model and we interpreted the Ornstein-Uhlenbeck process in terms of quantum market games theory as a non-unitary thermal tactic. We compared the probability density functions of both distributions and showed that the Ornstein-Uhlenbeck process would give the true reflection of the market in time scale where the approximation by a Wiener-Bachelier process is not valid, but such that market expectations and the asymptotic equilibrium state have not been changed. We called this time interval mezzo-scale. The differences between a classical look and the pricing which is supported by the quantum model; they are visible for very liquid financial instruments. The quantum game theory takes advantage the Ornstein-Uhlenbeck process for the financial modelling. Quantum strategies create unique opportunities for making profits during intervals shorter than characteristic thresholds for an effective market (Brownian motion). The additional possibilities offered by quantum strategies can lead to more successful outcomes than purely classical ones, see \[14\].
5 Appendix

In this section we introduce definitions and theorems (without proofs) which are used in our paper. They can be found in [3],[4],[10],[11],[12].

Definition 5.1.
Let \( X \) be an integrable random variable on a probability space \((\Omega, \mathcal{F}, P)\), and let \( \mathcal{G} \) be a \( \sigma \)-field contained in \( \mathcal{F} \). Then the conditional expectation value of \( X \), for given \( \mathcal{G} \), is defined to be a random variable \( E(X|\mathcal{G}) \) such that:

- \( E(X|\mathcal{G}) \) is \( \mathcal{G} \)-measurable,
- for any \( A \in \mathcal{G} \)
  \[ \int_A X dP = \int_A E(X|\mathcal{G}) dP. \]

Lemma 5.2.
If \( \mathcal{G} \subset \mathcal{F} \) is \( \sigma \)-field and \( f : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) is a Borel and bounded function, \( X \) is a random variable which is independent of \( \mathcal{G} \), and \( Y \geq 0 \) is \( \mathcal{G} \)-measurable and \( g(y) = E(f(X,y)) \), then

\[ E(f(X,Y)|\mathcal{G}) = g(Y). \]

Theorem 5.3 (Itô lemma).

(i) Let \( X = (X_t)_{t \geq 0} \) be a real-valued process with dynamics

\[ dX_t = a_t dt + b_t dW_t, \]

where \( a, b \) are real-valued processes, and \( W \) is a one-dimensional standard Brownian motion. Then, for any function \( F : [0, \infty) \times \mathbb{R} \ni (t, x) \mapsto F(t, x) \in \mathbb{R} \) which is two times continuously differentiable in \( x \) and continuously differentiable at \( t \), the process defined by \( Y = (F(t, X_t))_{t \geq 0} \) is an Itô process with dynamics

\[ dY_t = \left( \frac{\partial F}{\partial t}(t, X_t) + a_t \frac{\partial F}{\partial x}(t, X_t) + \frac{1}{2} b_t^2 \frac{\partial^2 F}{\partial x^2}(t, X_t) \right) dt + b_t \frac{\partial F}{\partial x}(t, X_t) dW_t. \]

(ii) Let \( W \) be a one-dimensional standard Brownian motion, \( a = (\alpha, \beta) \), \( b = (\gamma, \delta) \) and \( (X,Y) \) be two-dimensional stochastic processes with dynamics:

\[ dX_t = \alpha_t dt + \beta_t dW_t, \]
\[ dY_t = \gamma_t dt + \delta_t dW_t. \]
Then, for any function $F : [0, +\infty) \times \mathbb{R}^2 \ni (t, x, y) \to F(t, x, y) \in \mathbb{R}$ which is two times continuously differentiable in $x$ and $y$ and continuously differentiable at $t$, the process $Z = (F(t, X_t, Y_t))_{t \geq 0}$ is an Itô process with dynamics

$$dZ_t = \left[ \frac{\partial F}{\partial t}(t, X_t, Y_t) + \alpha_t \frac{\partial F}{\partial x}(t, X_t, Y_t) + \gamma_t \frac{\partial F}{\partial y}(t, X_t, Y_t) + \frac{1}{2} \left( \beta_t^2 \frac{\partial^2 F}{\partial x^2}(t, X_t, Y_t) + 2\beta_t \delta_t \frac{\partial^2 F}{\partial x \partial y}(t, X_t, Y_t) + \delta_t^2 \frac{\partial^2 F}{\partial y^2}(t, X_t, Y_t) \right) \right] dt + \left( \beta_t \frac{\partial F}{\partial x}(t, X_t, Y_t) + \delta_t \frac{\partial F}{\partial y}(t, X_t, Y_t) \right) dW_t.$$

**Definition 5.4.**
A stochastic process $X = (X_t)_{t \geq 0}$ is said to be the geometric Brownian motion if it is a solution of the stochastic differential equation

$$dX_t = \mu(t)X_t dt + \sigma(t)X_t dW_t,$$

where $\mu, \sigma : [0, +\infty) \to \mathbb{R}$ are deterministic functions of time. The initial value for process is assumed to be positive, $X_0 > 0$.

The function $\mu$ is the growing rate (drift) and $\sigma$ is a volatility rate of the process $X$.

**Theorem 5.5.**
Let $X$ be the geometric Brownian motion with $\mu \equiv 0$, then for any $t > 0$ the following conditions are fulfilled:

(i) $X_t > 0$,

(ii) a random variable $\ln \frac{X_t}{X_s}$ has a normal distribution,

(iii) if $\sigma$ is a volatility rate of the process $X$, then

$$X_t = X_s \exp \left( -\frac{1}{2} \int_s^t \sigma(u)^2 du + \int_s^t \sigma(u) dW_u \right),$$

$$E \left( \ln \frac{X_t}{X_s} \right) = -\frac{1}{2} \int_s^t \sigma(u)^2 du,$$

$$\text{Var} \left( \ln \frac{X_t}{X_s} \right) = \int_s^t \sigma(u)^2 du.$$
Lemma 5.6. If $Y = e^X$ and $X \sim N(m, s^2)$, then for any constant $K > 0$

$$E[\max(Y - K, 0)] = e^{m + \frac{1}{2} s^2} N\left(\frac{m - \ln K}{s} + s\right) - K N\left(\frac{m - \ln K}{s}\right) =$$

$$= E(Y) N\left(\frac{m - \ln K}{s} + s\right) - KN\left(\frac{m - \ln K}{s}\right).$$

Theorem 5.7. Let $T$ be the expiration date and $K$ the exercise price of the European option, and $h_T$ be the price of the underlying assets at the expiration date, then the price of the option at the time $t < T$ is equal to

$$C_t = B_t^T E^{P^*}[\max(h_T - K, 0)|\mathcal{F}_t].$$

(14)

The basic tool for pricing the financial models is the **Feynman-Kac theorem**. We assume that $x = (x_t)_{t \geq 0}$ is a diffusion process with dynamics given by the stochastic differential equation

$$dx_t = \alpha(x_t, t) dt + \beta(x_t, t)dW^*_t.\quad (15)$$

Moreover, we assume that a process $r_t = r(x_t, t)$ describes the short interest rate and the risk-neutral probability measure exists.

Let us consider a security with a single payment of $H_T = H(x_T, T)$ and the expiration date $T$. Let $V_t = V(x_t, t)$ denotes the price of the security at the moment $t$. It is proved that

$$V_t = E^{P^*}\left[ e^{-\int_t^T r(x_s, s) ds} H(x_T, T)|\mathcal{F}_t \right].$$

(16)

Theorem 5.8 (**Feynman-Kac theorem**).

The function $V$ defined by

$$V(x, t) = \left( E^{P^*}\left[ e^{-\int_t^T r(x_s, s) ds} H(x_T, T)|\mathcal{F}_t \right] \right)(x)$$

satisfies the partial differential equation

$$\frac{\partial V}{\partial t}(x, t) + \alpha(x, t) \frac{\partial V}{\partial x}(x, t) + \frac{1}{2} \beta(x, t)^2 \frac{\partial^2 V}{\partial x^2}(x, t) - r(x, t)V(x, t) = 0, \quad (17)$$

together with the terminal condition $V(x, T) = H(x, T)$. The process $V_t = V(x_t, t)$ describes the price of the considered security. The process $(V_t)_{t > 0}$ is a diffusion process under $P^*$ and $W^*$ with the drift

$$\mu(x, t) = r(x, t)V(x, t)$$

(18)

and the volatility rate

$$\sigma(x, t) = \frac{\partial V}{\partial x}(x, t)\beta(x, t).$$

(19)
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