An amenable, radical Banach algebra

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Abstract

We give an example of an amenable, radical Banach algebra, relying on results from non-abelian harmonic analysis due to H. Leptin, D. Poguntke and J. Boidol.

Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-module. A bounded linear map $D: A \to E$ is called a a derivation if

$$D(ab) = a.Db + (Da).b \quad (a, b \in A).$$

A derivation $D: A \to E$ is said to be inner if

$$Da = x.a - a.x \quad (a \in A)$$

for some $x \in E$. For any Banach $A$-module $E$, its dual space $E^*$ is naturally equipped with a Banach $A$-module structure via

$$\langle x, a.\phi \rangle := \langle x.a, \phi \rangle \quad \text{and} \quad \langle x, \phi.a \rangle := \langle a.x, \phi \rangle \quad (a \in A, x \in E, \phi \in E^*).$$

We can now give the definition of an amenable Banach algebra:

**Definition 1** A Banach algebra $A$ is amenable if, for each Banach $A$-module $E$, every derivation $D: A \to E^*$ is inner.

The notion of amenability for Banach algebras was introduced by B. E. Johnson in [Job]. A locally compact group $G$ is called amenable if it possesses a translation-invariant mean, i.e. if there is a linear functional $\phi: L^\infty(G) \to \mathbb{C}$ satisfying

$$\phi(1) = \|\phi\| = 1 \quad \text{and} \quad \phi(\delta_x * f) = \phi(f) \quad (x \in G, f \in L^\infty(G)).$$

For instance, all abelian and all compact groups are amenable. For further information, see the monograph [Pat]. In [Job], B. E. Johnson proved the following fundamental theorem which provides the motivation for Definition 1:

**Theorem 2** Let $G$ be a locally compact group. Then $G$ is amenable if and only if $L^1(G)$ is amenable.
Since then, amenability has turned out to be an extremely fruitful concept in Banach algebra theory. We only would like to mention the following deep result due to A. Connes ([Con]) and U. Haagerup ([Haa]):

**Theorem 3** A $C^*$-algebra is amenable if and only if it is nuclear.

In [Loy], P. C. Curtis asked the following question:

**Question 1** Is there an amenable, radical Banach algebra?

In these notes, I would like to answer this question affirmatively.

Let $A$ be a Banach algebra, and let $\text{Prim}(A)$ denote the space of its primitive ideals endowed with the Jacobson topology. Recall that a closed subset $F$ of $\text{Prim}(A)$ is a set of synthesis for $A$ if $\ker(F)$ is the only closed ideal $I$ of $A$ such that $F = \text{hull}(I)$. Otherwise, $F$ is called a set of non-synthesis for $A$.

**Definition 4** A Banach algebra is said to be weakly Wiener if the empty set is a set of synthesis for $A$.

As is customary, we call a locally compact group $G$ weakly Wiener if $L^1(G)$ is weakly Wiener.

The following proposition is easily verified:

**Proposition 5** Suppose there is a locally compact group which is amenable, but not weakly Wiener. Then there is an amenable, radical Banach algebra.

**Proof** Since $G$ fails to be weakly Wiener, there is a proper, closed ideal $J$ of $L^1(G)$ whose hull in $\text{Prim}(L^1(G))$ is empty. Then $R := L^1(G)/J$ is a radical Banach algebra which, being the quotient of an amenable Banach algebra, has to be amenable. \[ \square \]

It thus makes sense to ask:

**Question 2** Is there a locally compact group which is amenable, but fails to be weakly Wiener?

By the proposition, an affirmative answer to Question 2 entails an affirmative one to Question 1. When I posed Question 2 to Jean Ludwig of Metz he was (to my surprise) not only able to answer the question immediately upon learning of it, but also claimed that Question 2 had already been settled in the eighties by J. Boidol. The example is the group

$$G_{4,9}(0) := \left\{ \begin{bmatrix} e^t & x & e^t z \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{bmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$
Another way of describing $G_{4,9}(0)$ is as follows. The Heisenberg group is defined as

$$H_1 := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$ 

Let $\mathfrak{A}(H_1)$ denote the group of automorphisms of $H_1$, and define $\phi : \mathbb{R} \to \mathfrak{A}(H_1)$ through

$$\phi(t) \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} := \begin{bmatrix} 1 & e^{-t}x & z \\ 0 & 1 & e^{t}y \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Then we may identify $G_{4,9}(0)$ with the semidirect product $H_1 \rtimes \phi \mathbb{R}$. In particular, $G_{4,9}(0)$ is an abelian extension of the nilpotent group $H_1$, thus solvable, and therefore amenable.

What remains to be shown is that $G_{4,9}(0)$ is not weakly Wiener. As Ludwig claims, this was proved by Boidol. Apparently, Boidol never published his finding. However, the proof of [L-P, Theorem 6] can be modified with the help of [Boi, Lemma 1] to yield the desired result.

We require two lemmas, the first of which is completely elementary:

**Lemma 6** Let $A$ be a Banach algebra which is weakly Wiener. Then every quotient of $A$ is weakly Wiener.

The second lemma is a variant of the lemma on [L-P, p. 130]. For a Banach algebra $A$ we write $M(A)$ to denote its double centralizer algebra.

**Lemma 7** Let $A$ be a Banach algebra which is weakly Wiener, and let $p \in M(A)$ be an idempotent such that $pAp$ is dense in $A$. Then $pAp$ is weakly Wiener.

**Proof** Let $J \subseteq pAp$ be a closed ideal, and let $I$ denote the closed ideal of $A$ generated by $J$. Obviously, $I = I_0^\perp$ with

$$I_0 := J + AJ + JA + JA,$$

and consequently, $pIp \subseteq J$. This means that $J \subseteq A$. Since, by assumption, $A$ is weakly Wiener, there is a primitive ideal $P$ of $A$ such that $I \subseteq P$. Let $Q := pAp \cap P$. Assume that $Q = pAp$. Then $(pAp)^2 \subseteq P$, and since $P$ is primitive, $pAp \subseteq P$. The density of $pAp$ in $A$ yields $A = P$, which is a contradiction. Let $\pi$ be an irreducible representation of $A$ on some linear space, say $E$, such that $\ker \pi = P$. Since $A$ is an ideal in $M(A)$, $\pi$ extends canonically to an irreducible representation of $M(A)$ on $E$, which we denote by $\pi$ as well. Since $pAp \not\subseteq P$, we have $\pi(p) \neq 0$. Let $x \in \pi(p)E \setminus \{0\}$. Then

$$\pi(pAp)x = \pi(pA)x = \pi(p)\pi(A)x = \pi(p)E,$$

and consequently $(\pi|_{pAp}, \pi(p)E)$ is an irreducible representation of $pAp$ (compare the proof of [B-D, Theorem 26.14]), and $Q$ is primitive. \qed
Furthermore, we require the theory of generalized $L^1$-algebras as given in [Lep]. Let $G$ be a locally compact group, and let $A$ be a Banach *-algebra with isometric involution such that $G$ acts on $A$ as a group of isometric *-automorphisms; for $x \in G$, we write $A \ni a \mapsto a^x$ for the automorphism implemented by $x$. The Banach space $L^1(G,A)$ becomes a Banach *-algebra with isometric involution via

\[ (f \ast g)(x) := \int_G f(xy)^{-1} g(y^{-1}) \, dy \quad \text{and} \quad f^*(x) := \Delta_G(x)^{-1}(f(x^{-1})^x)^* \]

\[ (f, g \in L^1(G,A), x \in G) \]

where $dx$ denotes left Haar measure, and $\Delta_G$ is the modular function on $G$.

Our main result will be that for a specific choice of $A$ the algebra $L^1(G,A)$ fails to be weakly Wiener and then use this to conclude that $G_{4,9}(0)$ is not weakly Wiener.

We shall be concerned with the following situation:

- $G$ and $H$ are locally compact groups, and
- $G$ acts continuously and automorphically on $H$, i.e. there is a continuous mapping $G \times H \to H$, $(g, x) \mapsto x^g$ such that $xy^g = x^g y^g$, $(x^g)^h = x^{gh}$, and $x^1 = x$.

For each $g \in G$, $dx^g$ is left Haar measure. Thus, there is a positive real number $\Delta_{G,H}(g)$ such that $dx^g = \Delta_{G,H}(g) \, dx$.

For $f \in C_0(H)$ and $h \in H$, we define $f_h$ and $f^h$ via

\[ f^h(x) := f(hx) \quad \text{and} \quad f_h(x) := f(xh) \quad (x \in H). \]

We shall also require a subalgebra $Q$ of $C_0(H)$ with the following properties:

(i) $Q$ is a *-subalgebra of $C_0(H)$ equipped with a Banach algebra $\cdot \mid \cdot$ such that $\mid q^* \mid = \mid q \mid \geq ||q||_\infty$.

(ii) For $q \in Q$ and $h$, we have $q^h \in Q$ and $\mid q^h \mid = \mid q \mid$.

(iii) For each $q \in Q$, the map $H \ni h \mapsto q^h$ is continuous.

(iv) The compactly supported functions in $Q$ form a dense subalgebra $Q_{00}$.

(v) For every neighborhood $U$ of $1_H$, there is $u \in Q$ such that

(a) $u \neq 0$ and $\text{supp}(u) \subset U$,

(b) $u_h \in Q$ for all $h \in H$, and

(c) the map $H \ni h \mapsto u_h$ is continuous.

Examples 1. Obviously, $C_0(H)$ satisfies all the requirements.
2. In case $H$ is abelian, we may choose $Q = A(H) \cong L^1(\hat{H})$.

Let $Q$ be as described above. For $u \in Q$ and $g \in G$, let
\[(u \circ g)(x) := u(x^g) \quad (x \in H).\]
We assume moreover:

(vi) For each $g \in G$, the map $Q \ni q \mapsto q \circ g$ is an isometric automorphism of $Q$.
(vii) For each $q \in Q$, the map $G \ni g \mapsto q \circ g$ is continuous.

These assumptions are certainly true for the two examples given above.

In this situation, we may speak of $L^1(H, Q)$. For $f \in L^1(H, Q)$ and $g \in G$, let
\[f^g(x) := \Delta_{G,H}(g)^{-1} f(x) \circ g \quad (x \in H).\]
Thus, for each $g \in G$, the mapping $L^1(H, Q) \ni f \mapsto f^g$ is an isometric \ast-isomorphism, and we may speak of $L^1(G, L^1(H, Q))$.

We are finally in a position to state the main theorem of these notes:

**Theorem 8** Let $G$, $H$ and $Q$ be given as above, let $A := L^1(H, Q)$, and suppose that $\Delta_{G,H} \not\equiv 1$. Then $L^1(G, A)$ is not weakly Wiener.

**Proof** For the sake of brevity, write $L := L^1(G, A)$. Let $\mathfrak{H} := L^2(H)$. We begin by defining a faithful \ast-representation $\rho$ of $A$ on $\mathfrak{H}$. For $f \in A$ and $\xi \in \mathfrak{H}$, let — note that we can view $f$ as a function on $H \times H$ —
\[(\rho(f)\xi)(x) = \int_H f(xy, y^{-1})\xi(y^{-1}) \, dy \quad (x \in H). \tag{1}\]

For $u, v \in Q_{00}$, define $u \circ v \in A$ through
\[(u \circ v)(x) := \Delta_H(x)^{1/2} u^x \bar{v}.\]
Then we have
\[(u \circ v)(x, y) = \Delta_H(x)^{1/2} u(xy) \bar{v(y^{-1})} \quad (x, y \in H). \tag{2}\]
Letting $u'(x) := \Delta_H(x)^{1/2} u(x)$, we obtain from (1) and (2) that
\[
(\rho(u \circ v)\xi)(x) := \int_H \Delta_H(xy^{-1})^{1/2} u(xy) \overline{v(y^{-1})} \xi(y^{-1}) \, dy
= u'(x) \int_H \xi(y^{-1}) \overline{v(y^{-1})} \Delta_H(y^{-1})^{1/2} \, dy
= \langle \xi, u'(x) \rangle.
\]
Thus \( \rho(u \circ v) \) is a rank one operator, which in case \( \|u'\|_2 = 1 \) is a projection.

Fix a real valued function \( u \in \mathcal{Q}_{00} \) such that \( u(1_H) > 0 \) and \( \|u'\|_2 = 1 \), and let \( p := u \circ u \). Since \( \rho \) is faithful and \( \rho(p) \) has rank one, it follows that \( p \) is a projection in \( \mathcal{A} \) such that \( p \mathcal{A} p = \mathbb{C} p \). Define a projection \( p^\# \in \mathcal{M}(\mathcal{L}) \) by letting

\[
(p^\# f)(g) := p^0 f(g) \quad \text{and} \quad (fp^\#)(g) := f(g)p \quad (g \in G).
\]

We wish to apply Lemma 7. As is shown in the proof of [L-P, Theorem 4], we see that \( \mathcal{A} \) are finished once we have established that \( \mathcal{L}_p := p^\# \mathcal{L} p^\# \) fails to be weakly Wiener.

Let \( \mathcal{K} \) denote the compact linear operators on \( \mathfrak{H} \). Again from the proof of [L-P, Theorem 4], we see that \( \rho(\mathcal{A}) \subset \mathcal{K} \). Define a unitary representation of \( G \) on \( \mathfrak{H} \) by letting

\[
(\pi(g)\xi)(x) := \Delta_{G,H}(g)^{1/2} \xi(x^g) \quad (g \in G, x \in H, \xi \in \mathfrak{H}).
\]

Then for \( g \in G, a \in \mathcal{A}, \) and \( \xi \in \mathfrak{H} \), we have

\[
(\pi(g)^x \rho(a) \pi(g)\xi)(x) = \Delta_{G,H}^{1/2}(g)(\rho(a)\pi(g)\xi(x^g))
\]

\[
= \Delta_{G,H}^{1/2}(g) \int_H a(x^g y, y^{-1})(\pi(g)\xi)(y^{-1}) \, dy
\]

\[
= \int_H a(x^g y, y^{-1})\xi((y^g)^{-1}) \, dy
\]

\[
= \int_H \Delta_{G,H}(g)^{-1} a((xy)^g, (y^{-1})^g)\xi(y^{-1}) \, dy
\]

\[
= (\rho(a^g)\xi)(x) \quad (x \in H),
\]

i.e. \( \pi \) implements the action of \( G \) on \( \rho(\mathcal{A}) \). In what follows, we shall suppress the symbol \( \rho \) and view \( \mathcal{A} \) as a subalgebra of \( \mathcal{K} \).

Let \( G \) act on \( \mathcal{K} \) in the trivial way, and consider the generalized group algebra \( L^1(G, \mathcal{K}) \). For \( f \in \mathcal{L} \), let

\[
f^\sharp(g) := \pi(g)f(g) \quad (g \in G).
\]

It is easy to see that \( f^\sharp \in L^1(G, \mathcal{K}) \), and that \( \sigma: \mathcal{L} \to L^1(G, \mathcal{K}), f \mapsto f^\sharp \) is a faithful *-homomorphism. We wish to compute \( \sigma(\mathcal{L}_p) \). For \( f \in \mathcal{L} \) and \( g \in G \), we have \( (p^\# fp^\#)(g) = p^0 f(g)p \), i.e. we have

\[
f \in \mathcal{L}_p \iff f(g) = p^0 f(g)p \text{ for almost all } g \in G.
\]

Thus, for \( f \in \mathcal{L} \) and for almost all \( g \in G \), we have

\[
f^\sharp(g) = \pi(g)f(g) = \pi(g)p^0 f(g)p = p\pi(g)f(g)p = \phi(g)p,
\]
for some \( \phi(g) \in \mathbb{C} \). Let \( | \cdot |_* \) denote the \( C^* \)-norm on \( \mathcal{K} \). Then we have

\[
|\phi(g)| = |f^2(g)|_* = |f(g)|_* \leq \|f(g)\|.
\]

Consequently, \( \sigma(\mathcal{L}_p) \subset L^1(G) \cong pL^1(G,\mathcal{K})p \). View \( \mathcal{Q}_{00} \) as a subspace of \( \mathfrak{E} \). Then clearly \( \pi(G)\mathcal{Q}_{00} \subset \mathcal{Q}_{00} \). In particular, we have

\[
(\pi(g)u)(x) = \Delta_{G,H}^{1/2}(g)u(x^g) \quad (g \in G, \ x \in H).
\]

Define

\[
w(g) := \pi(g^{-1})u \circ u \quad (g \in G).
\]

Is is easily seen (compare \([L-P\], p. 129\)) that

\[
p^qAp = \mathbb{C}w(g) \quad \text{and} \quad |\omega(g)|_* = 1.
\]

It follows hat each \( f \in \mathcal{L}_p \) has the form \( f(g) = \phi(g)w(g) \). Letting

\[
\omega(g) := \|w(g)\| \quad (g \in G),
\]

we conclude that \( \sigma(\mathcal{L}_p) \) is the Beurling algebra \( L^1(G,\omega) \) (it is shown on \([L-P\], p. 130\) that \( \omega \) is indeed a weight). For \( g \in G \), we have

\[
\begin{align*}
\omega(g) &= \|\pi(g^{-1})u \circ u\| \\
&= \int_H |(\pi(g^{-1})u)^x u| \Delta_H(x)^{1/2} \, dx \\
&\geq \int_H \| (\pi(g^{-1})u)^x u \|_{\infty} \Delta_H(x)^{1/2} \, dx \\
&= \int_H (\sup_{y \in H} |(\pi(g^{-1})u(x)u(y)|) \Delta_H(x)^{1/2} \, dx \\
&\geq \int_H |\Delta_{G,H}(g)^{1/2}u(1_H)u(x^{-1})| \Delta_H(x)^{1/2} \, dx \\
&= \Delta_{G,H}(g)^{1/2} \left[ u(1_H) \int_H \Delta_H(x)^{1/2} |u(x^{-1})| \, dx \right]_{:= \Omega > 0}.
\end{align*}
\]

Assume that \( \mathcal{L}_p \) is weakly Wiener. Then by Lemma \([L-P\] \( L^1(G,\omega) \) is weakly Wiener. Since \( \Delta_{G,H}^{1/2}(\cdot) \) is a homomorphism from \( G \) into the abelian group \( \mathbb{R} \setminus \{0\} \), there is no loss of generality if we assume that \( G \) is also abelian. By assumption, there is \( g \in G \), such that \( \Delta_{G,H}(g)^{1/2} > 1 \). We thus have

\[
\sum_{n=1}^{\infty} \frac{\log \omega(g^n)}{n^2} \geq \sum_{n=1}^{\infty} \frac{\log \Omega}{n^2} + \sum_{n=1}^{\infty} \frac{\log \Delta_{G,H}(g^n)^{1/2}}{n^2} = \sum_{n=1}^{\infty} \frac{\log \Omega}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\Delta_{G,H}(g)}{n} = \infty.
\]

Thus, \( L^1(G,\omega) \) does not satisfy the Beurling-Domar condition ([Ref. p. 132]) for a Beurling algebra to be weakly Wiener, and we have reached a contradiction (compare [Boi]). \( \Box \)
Corollary 9 $G_{4,9}(0)$ is not weakly Wiener.

Proof We identify $L^1(G_{4,9}(0))$ and $L^1(\mathbb{R}, L^1(H_1))$. For $f \in L^1(H_1)$, define

$$\hat{f}(x, s) := \int f(x, y, z) e^{-i(sy+z)} \, dy \, dz.$$  

The map $L^1(H_1) \ni f \mapsto \hat{f}$ is easily seen to be an epimorphism onto $A := L^1(\mathbb{R}, Q)$ with $Q := A(\mathbb{R})$, which in turn induces an epimorphism from $L^1(G_{4,9}(0))$ onto $L^1(\mathbb{R}, A)$. So, if $L^1(G_{4,9}(0))$ is weakly Wiener, the same must be true for $L^1(\mathbb{R}, A)$ by Lemma 6 (here, the action of $\mathbb{R}$ on $\mathbb{R}$ is given by $(t, x) \mapsto e^{-t}x$). Consequently, $\Delta_{\mathbb{R}, \mathbb{R}}(t) = e^{-t} \not\equiv 1$. From Theorem 8, we obtain that $L^1(\mathbb{R}, A)$ cannot be weakly Wiener.  

In view of the proposition, we finally obtain the promised answer to Question 1:

Corollary 10 There is an amenable, radical Banach algebra.

It is not clear at all (and in fact extremely unlikely) that the amenable, radical Banach algebra whose existence we have just proved is commutative. Hence, the following question remains open ([Hel, Problem 13]):

Question 3 Is there a commutative, amenable, radical Banach algebra?

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