Braid Group, Temperley–Lieb Algebra, and Quantum Information and Computation

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Abstract

In this paper, we explore algebraic structures and low dimensional topology underlying quantum information and computation. We revisit quantum teleportation from the perspective of the braid group, the symmetric group and the virtual braid group, and propose the braid teleportation, the teleportation swapping and the virtual braid teleportation, respectively. Besides, we present a physical interpretation for the braid teleportation and explain it as a sort of crossed measurement. On the other hand, we propose the extended Temperley–Lieb diagrammatical approach to various topics including quantum teleportation, entanglement swapping, universal quantum computation, quantum information flow, and etc. The extended Temperley–Lieb diagrammatical rules are devised to present a diagrammatical representation for the extended Temperley–Lieb category which is the collection of all the Temperley–Lieb algebras with local unitary transformations. In this approach, various descriptions of quantum teleportation are unified in a diagrammatical sense, universal quantum computation is performed with the help of topological-like features, and quantum information flow is recast in a correct formulation. In other words, we propose the extended Temperley–Lieb category as a mathematical framework to describe quantum information and computation involving maximally entangled states and local unitary transformations.

Key Words: Teleportation, Braid group, Temperley–Lieb algebra

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1 Introduction

Quantum entanglements [1] play key roles in quantum information and computation [2, 3] and are widely exploited in quantum algorithms [4, 5], quantum cryptography [6, 7] and quantum teleportation [8, 9]. On the other hand, topological entanglements [10] denote topological configurations like links or knots which are closures of braids. Aravind [11] observed that there are natural similarities between quantum entanglements and topological entanglements. As a unitary braid has the power of detecting knots or links, it can often transform a separate quantum state into an entangled one. Kauffman and Lomonaco [12, 13] identified a nontrivial unitary braid representation with a universal quantum gate [14]. Recently, a series of papers have been published on the application of the braid group [10] (or the Yang–Baxter equation [15, 16]) to quantum information and computation, see [13, 17, 18] for unitary solutions of the Yang–Baxter equation as universal quantum gates; see [19, 20, 21] for quantum topology and quantum computation. We focus on the project of setting up a bridge between low dimensional topology and quantum information, namely looking for low dimensional topology underlying quantum information and computation. Kauffman’s observation on the teleportation topology [13, 29] motivates our tour of revisiting in a diagrammatical approach all tight teleportation and dense coding schemes in Werner’s work [30]. In a joint article with Kauffman and Werner [31], we make a survey of diagrammatical tensor calculus and matrix representations, and explore topological and algebraic structures underlying multipartite entanglements. In this article as a further extension of our work [32, 33, 34], we describe quantum information and computation (especially quantum teleportation [8, 9]) in the language of the braid group and Temperley–Lieb (TL) algebra [35]. We propose the braid teleportation, teleportation swapping and virtual braid teleportation, and devise the extended TL diagrammatical rules to describe quantum teleportation, entanglement swapping, universal quantum computation and quantum information flow.

Quantum teleportation is a procedure of sending a message from Charlie to Bob with the help of Alice. She shares a maximally entangled state (for example, Bell states [36]) with Bob, and performs an entangling measurement on the composite system between Charlie and Alice. After Bob gets results of Alice’s measurement, he is able to obtain the message by exploiting the protocol between Alice and him. The transformation matrix between Bell states and product basis is found out to form a unitary braid representation [13], and it inspires us to reformulate the teleportation equation (which catches main features of quantum teleportation and is defined in the next section) in terms of the unitary braid representation $b$-matrix, and suggest the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$ with identity $Id$ to describe quantum teleportation. We present an interpretation for the braid teleportation in view of the crossed measurement [37, 38, 39], and explore the configuration for the braid teleportation in the state model [10] (which is
devised for the braid representation of the TL algebra). Furthermore, the virtual braid group \([40]\) is an extension of the braid group by the symmetric group, and it has virtual crossings acting like permutation \(P\). We suggest the teleportation swapping \((P \otimes Id)(Id \otimes P)\) as a special example of the braid teleportation. The virtual mixed relation for defining the virtual braid group is found to be a reformulation of the teleportation equation, which leads to our suggestion of the \textit{virtual braid teleportation}. Moreover, similar to the braid representation of the TL algebra \([10]\), the virtual braid representation can be constructed in terms of the Brauer algebra \([41]\) (or the virtual TL algebra \([31, 42]\)). The Temperley–Lieb configuration for quantum teleportation can be recognized as a fundamental configuration defining the diagrammatic Brauer algebra.

The maximally bipartite entangled pure state is a projector, and is able to form a representation of the TL algebra. Based on the diagrammatical representation for the TL algebra (i.e., configurations in terms of cups and caps \([10]\)), we devise the \textit{extended TL diagrammatical rules} to explore topological-like features in quantum circuits (or quantum information protocols) involving maximally bipartite entangled states and local unitary transformations, for examples, quantum teleportation, entanglement swapping, universal quantum computation, quantum information flow, and etc. A maximally entangled Dirac ket (bra) is represented by a configuration of a cup (cap), and a local unitary transformation (its adjoint) is denoted by a solid point (a small circle). In our extended TL diagrammatical framework, various approaches to quantum teleportation have a unified diagrammatical description, and they include its standard description \([8, 9]\), the transfer operator (or quantum information flow) \([43]\), measurement-based quantum teleportation \([38]\), and Werner’s tight teleportation schemes \([30]\). The transfer operator is described by a configuration involving a top cap and a bottom cup in which the teleportation appears to be a kind of the flow of quantum information. The measurement-based quantum teleportation has a typical TL configuration as a product of generators of the TL algebra, and this diagram is able to describe both discrete and continuous quantum teleportation schemes. The diagram describing the tight teleportation scheme is a closure of the configuration for measurement-based quantum teleportation, and it naturally derives a characteristic equation for quantum teleportation since a closed configuration in the extended TL diagrammatical rules corresponds to a trace of products of operators.

The extended TL diagrammatical rules present a diagrammatical representation of the \textit{extended TL algebra} as an extension of the TL algebra by local unitary transformations. The collection of all the extended TL algebras is called the \textit{extended TL category} \([32, 33, 34]\), and its diagrammatical representation includes all configurations made of cups, caps, solid points and small circles. Besides its application to quantum teleportation, it is able to describe entanglement swapping \([44]\), universal quantum computation \([45]\), and etc. Entanglement swapping is an approach to producing an entangled state between two independent systems via quantum measurements, and the closure of its diagrammatical description gives rise to the tight entanglement swapping scheme with a characteristic equation. Universal quantum computation is performed in the extended TL category, since we are able to construct unitary braid gates, the swap gate and CNOT gate, etc.,
with the extended TL diagrammatical rules. Furthermore, we recognize another equivalent description of quantum teleportation in terms of the swap gate and Bell measurements, after identifying the configuration for quantum teleportation with that for the axiom of the Brauer algebra [31, 42]. Moreover, we comment on multipartite entanglements in the extended TL diagrammatical approach.

Quantum teleportation can be viewed as a flow of quantum information from the sender to the receiver, and hence quantum information flow can be well described in the extended TL diagrammatical framework. In its configuration, the flow is only a part of the entire diagram, related to or even controlled by other parts. For example, it is zero due to the vanishing trace of a product of local unitary transformations which are not involved in the flow. Besides our diagrammatical approach, quantum information flow has been described by Kauffman’s teleportation topology [13, 29] and Abramsky and Coecke’s strongly compact closed categories [46]. There are essential physical and mathematical differences among them. Measurement-based quantum teleportation is chosen to present a full description of quantum teleportation in our study, whereas quantum information flow denoted by the transfer operator is regarded as the entire quantum teleportation in both [13, 29] and [46]. We show that the paradigm described by the transfer operator is a part of the picture by measurement-based quantum teleportation. Furthermore, only topological-like features [34] can be explored in the extended TL diagrammatical configuration instead of pure topology in [13, 29]. Moreover, we propose the extended TL category underlying quantum information protocols like quantum teleportation instead of strongly compact closed categories, see [34] for more details.

The plan of this paper is organized as follows. Section 2 introduces the teleportation equation and reformulates it respectively by the braid group, the symmetric group and the virtual braid group. Section 3-7 introduces the extended TL diagrammatical approach to quantum information and computation: Section 3 explains diagrammatical rules with examples; Section 4 unifies various descriptions of quantum teleportation at the diagrammatical level; Section 5 focuses on the TL algebra and the Brauer algebra; Section 6 deals with the entanglement swapping and universal quantum computation; Section 7 compares the quantum information flow in the extended TL category with other known approaches. Last section comments on our work in the project of setting up categorical foundations for quantum physics and information.

2 Braid teleportation, teleportation swapping and virtual braid teleportation

We describe quantum teleportation in the language of the braid group, the symmetric group, and the virtual braid group, respectively, and propose the braid teleportation, the teleportation swapping, and the virtual braid teleportation. First of all, we revisit the standard description of the teleportation [8, 9] and assign the name the teleportation equation to its most important equality. Secondly, we reformulate the teleportation equation in terms of the Bell matrix which forms a
unitary braid representation, and then realize that a braiding operator called the braid teleportation plays a key role in the formulation of the teleportation equation. Thirdly, we discuss the teleportation swapping as a simplest example of the braid teleportation, and recognize the teleportation equation as a reformulation of the virtual mixed relation defining the virtual braid group. Lastly, we look upon the braid teleportation as a kind of crossed measurement if it has a sort of physical correspondence, and expand it in the state model [10] which sheds an insight on the main topic in the following sections.

2.1 Quantum teleportation: the teleportation equation

The Pauli matrices $\sigma_1$, $\sigma_2$ and $\sigma_3$ have the conventional form,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

and quantum states $|0\rangle$ and $|1\rangle$ as a basis denoting a qubit have the coordinate presentation in the complex field $\mathbb{C}$,

$$
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

which give rise to useful formulas: $a, b \in \mathbb{C}$,

$$
\sigma_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}, \quad -i\sigma_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}, \quad \sigma_3 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}.
$$

The product basis of two-fold tensor products denoting two-qubit is chosen to be

$$
|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

which fixes our rule for calculating the tensor product of matrices, i.e., embedding the right matrix into the left one. With the product basis $|ij\rangle$, $i, j = 0, 1$, four mutually orthogonal Bell states have the form,

$$
|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),
$$

$$
|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),
$$

which derive the product basis $|ij\rangle$ in terms of Bell states,

$$
|00\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle), \quad |01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle),
$$

$$
|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle), \quad |11\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle).
$$

5
Figure 1: Diagrammatical description of quantum teleportation.

Bell states can be transformed to each other with local unitary transformations consisting of Pauli matrices and identity matrix $I_2$,

$$|\phi^+\rangle = (I_2 \otimes \sigma_3)|\phi^+\rangle = (\sigma_3 \otimes I_2)|\phi^+\rangle,$$

$$|\psi^+\rangle = (I_2 \otimes \sigma_1)|\phi^+\rangle = (\sigma_1 \otimes I_2)|\phi^+\rangle,$$

$$|\psi^-\rangle = (I_2 \otimes -i\sigma_2)|\phi^+\rangle = (i\sigma_2 \otimes I_2)|\phi^+\rangle,$$

where one-qubit unitary transformations are called local unitary transformations.

Quantum teleportation transports an unknown quantum state $|\psi\rangle_C = (a|0\rangle + b|1\rangle)_C$ from the sender, Charlie to the receiver, Bob, with the help of Alice, and it exploits properties of quantum entanglement and quantum measurement. Figure 1 is our diagrammatical interpretation for quantum teleportation. Let Alice and Bob share the Bell state $|\phi^+\rangle_{AB}$, a maximally bipartite entangled pure state. Do calculation:

$$|\psi\rangle_C|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(a|0\rangle + b|1\rangle)_C(|00\rangle + |11\rangle)_AB$$

$$= \frac{1}{2}b(|\phi^+\rangle + |\phi^-\rangle)_{CA}|0\rangle_B + \frac{1}{2}a(|\psi^+\rangle + |\psi^-\rangle)_{CA}|1\rangle_B$$

$$+ \frac{1}{2}b(|\psi^+\rangle - |\psi^-\rangle)_{CA}|0\rangle_B + \frac{1}{2}b(|\phi^+\rangle - |\phi^-\rangle)_{CA}|1\rangle_B$$

$$= \frac{1}{2}(|\phi^+\rangle_{CA}|\psi\rangle_B + |\phi^-\rangle_{CA}\sigma_3|\psi\rangle_B + |\psi^+\rangle_{CA}\sigma_1|\psi\rangle_B + |\psi^-\rangle_{CA}(-i\sigma_2)|\psi\rangle_B)$$

which is an identity but called the teleportation equation in our research for convenience. Alice performs Bell measurements in the composite system of Charlie and her, and obtains four kinds of outcomes. After Alice detects the Bell state
\( |\phi^+\rangle_{CA} \) and informs it to Bob through a classical channel, Bob knows that he has the quantum state \( |\psi\rangle_B \) which Charlie wants to send to him. As Alice gets Bell states \( |\phi^-\rangle_{CA} \) or \( |\psi^+\rangle_{CA} \) or \( |\psi^-\rangle_{CA} \), Bob applies local unitary transformations \( \sigma_3 \) or \( \sigma_1 \) or \( i\sigma_2 \) respectively, on the quantum state that he has to obtain \( |\psi\rangle_B \).

There exist beautiful mathematical structures underlying quantum teleportation, though only fundamental laws of quantum mechanics and a little linear algebra are involved in its standard description. In this paper, we make it clear that quantum teleportation can be described by the braid group, the symmetric group, the virtual braid group, the TL algebra and the Brauer algebra.

### 2.2 Teleportation equation in terms of Bell matrix

Let us introduce the Bell matrix \([13, 17, 12]\), denoted by 
\[
B = \begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{array},
\]
and its inverse 
\[
B^{-1} = B^T = \frac{1}{\sqrt{2}} \begin{array}{cccc}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{array}.
\]
(9)

It has an exponential formalism given by
\[
B = e^{i\frac{\pi}{4} (\sigma_1 \otimes \sigma_2)} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} (\sigma_1 \otimes \sigma_2)
\]
(10)
with interesting properties:
\[
B^2 = i\sigma_1 \otimes \sigma_2, \quad B^4 = -I_4, \quad B^8 = I_4, \quad B = \frac{1}{\sqrt{2}} (I_4 + B^2).
\]
(11)

In terms of the \( B \) matrix and product basis \( |ij\rangle \), Bell states is yielded in two ways:

\[
(I) : \quad |\phi^+\rangle = B|11\rangle, \quad |\phi^-\rangle = B|00\rangle,
|\psi^+\rangle = B|01\rangle, \quad |\psi^-\rangle = -B|10\rangle,
\]
(12)

and

\[
(II) : \quad |\phi^+\rangle = B^T|00\rangle, \quad |\phi^-\rangle = -B^T|11\rangle,
|\psi^+\rangle = B^T|10\rangle, \quad |\psi^-\rangle = B^T|01\rangle,
\]
(13)

where the Bell operator \( B \) acts on the basis \( |ij\rangle \) in the way
\[
B|ij\rangle = \sum_{k,l=0}^1 |kl\rangle B_{kl,ij} = \sum_{k,l=0}^1 |kl\rangle B^T_{ij,kl}.
\]
(14)
For simplicity, we exploit the first type of transformation law (I) between Bell states and the product basis, and rewrite the teleportation equation (8) into a new formalism,

$$\begin{align*}
(\mathbb{I}_2 \otimes B)(|\psi\rangle \otimes |11\rangle)_{CAB} &= \frac{1}{2}(B \otimes \mathbb{I}_2)(|00\rangle \otimes \sigma_3|\psi\rangle + |01\rangle \otimes \sigma_1|\psi\rangle + |10\rangle \otimes i\sigma_2|\psi\rangle + |11\rangle \otimes |\psi\rangle)_{CAB}, \\
&= (B \otimes \mathbb{I}_2)(\vec{v}^T \otimes \frac{1}{2}\vec{\sigma}_{11}|\psi\rangle)_{CAB}
\end{align*}$$

in which the vector \(\vec{v}\), its transpose given by

$$\vec{v}^T = (|00\rangle, |01\rangle, |10\rangle, |11\rangle), \quad v_{ij} = |i j\rangle, \quad i, j = 0, 1,$$

the vector \(\vec{\sigma}_{11}\), its transpose given by

$$\vec{\sigma}_{11}^T = (\sigma_3, \sigma_1, i\sigma_2, \mathbb{I}_2),$$

and the calculation of \(\vec{v}^T \otimes \vec{\sigma}_{11}\) follows the rule,

$$\vec{v}^T \otimes \vec{\sigma}_{11}|\psi\rangle \equiv (|00\rangle \otimes \sigma_3|\psi\rangle + |01\rangle \otimes \sigma_1|\psi\rangle + |10\rangle \otimes i\sigma_2|\psi\rangle + |11\rangle \otimes |\psi\rangle).$$

The remaining three teleportation equations are derived in a similar way with the help of local unitary transformations among Bell states (7). For example,

$$|\psi\rangle_C|\phi^-\rangle_{AB} = (\mathbb{I}_2 \otimes B)(|\psi\rangle \otimes |00\rangle)_{CAB}$$

$$= |\psi\rangle_C \otimes (\mathbb{I}_2 \otimes \sigma_3)|\phi^+\rangle_{AB} = (\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma_3)|\psi\rangle_C|\phi^+\rangle_{AB}$$

$$= (B \otimes \mathbb{I}_2)(\vec{v}^T \otimes \frac{1}{2}\sigma_3\vec{\sigma}_{11}|\psi\rangle)_{CAB}$$

where the local unitary transformation \(\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma_3\) commutes with \(B \otimes \mathbb{I}_2\), and the other two teleportation equations have the form,

$$(\mathbb{I}_2 \otimes B)(|\psi\rangle \otimes |01\rangle)_{CAB} = (B \otimes \mathbb{I}_2)(\vec{v}^T \otimes \frac{1}{2}\sigma_1\vec{\sigma}_{11}|\psi\rangle)_{CAB},$$

$$(\mathbb{I}_2 \otimes B)(|\psi\rangle \otimes |10\rangle)_{CAB} = (B \otimes \mathbb{I}_2)(\vec{v}^T \otimes -\frac{1}{2}i\sigma_2\vec{\sigma}_{11}|\psi\rangle)_{CAB}. \quad (20)$$

These four teleportation equations can be collected into an equation,

$$\begin{align*}
(\mathbb{I}_2 \otimes B)(|\psi\rangle \otimes \vec{v}^T)_{CAB} &= (B \otimes \mathbb{I}_2)(\vec{v}^T \otimes \frac{1}{2}\vec{\Sigma}|\psi\rangle)_{CAB}, \\
(\vec{v}^T \otimes \frac{1}{2}\vec{\Sigma})|\psi\rangle_{CAB} &= (B^{-1} \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B)(|\psi\rangle \otimes \vec{v}^T)_{CAB}, \quad (21)
\end{align*}$$

in terms of the new matrix \(\vec{\Sigma}\), a convenient notation given by

$$\vec{\Sigma} = (\sigma_3, \sigma_2, i\sigma_1, \mathbb{I}_2)\vec{\sigma}_{11}$$

(22)

where \(\vec{v}^T \otimes \vec{\Sigma}\) is defined as

$$\vec{v}^T \otimes \vec{\Sigma}|\psi\rangle \equiv (\vec{v}^T \otimes \sigma_3\vec{\sigma}_{11}|\psi\rangle, \vec{v}^T \otimes \sigma_2\vec{\sigma}_{11}|\psi\rangle, \vec{v}^T \otimes i\sigma_1\vec{\sigma}_{11}|\psi\rangle, \vec{v}^T \otimes \vec{\sigma}_{11}|\psi\rangle). \quad (23)$$
and $|\psi\rangle \otimes \vec{v}^T$ has the form

$$|\psi\rangle \otimes \vec{v}^T \equiv (|\psi\rangle \otimes |00\rangle, |\psi\rangle \otimes |01\rangle, |\psi\rangle \otimes |10\rangle, |\psi\rangle \otimes |11\rangle). \quad (24)$$

Obviously, the operator $(B^{-1} \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B)$ plays a fundamental role in the new formulations of the teleportation equation. In the following, we verify the Bell matrix $B$ a unitary braid representation, and then name the braiding operator of this kind as the braid teleportation.

### 2.3 Braid teleportation and teleportation swapping

A braid representation $b$-matrix is a $d \times d$ matrix acting on $V \otimes V$ where $V$ is an $d$-dimensional vector space over the complex field $\mathbb{C}$. The symbol $b_i$ denotes the braid $b$ acting on the tensor product $V_i \otimes V_{i+1}$. The classical braid group $B_n$ is generated by braids $b_i, b_2, \ldots, b_{n-1}$ satisfying the braid group relation,

$$b_i b_j = b_j b_i, \quad j \neq i \pm 1,$$

$$b_i b_{i+1} b_i = b_{i+1} b_{i+1} b_i, \quad i = 1, \cdots, n-2. \quad (25)$$

The virtual braid group $VB_n$ is an extension of the classical braid group $B_n$ by the symmetric group $S_n$ \cite{40}. It has two types of generators: braids $b_i$ and virtual crossings $v_i$ defined by the virtual crossing relation,

$$v_i^2 = Id, \quad v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}, \quad i = 1, \cdots, n-2,$$

$$v_i v_j = v_j v_i, \quad j \neq i \pm 1, \quad (26)$$

a presentation of the symmetric group $S_n$ with identity $Id$, and they satisfying the virtual mixed relation,

$$b_i v_j = v_j b_i, \quad j \neq i \pm 1,$$

$$b_i b_{i+1} v_i = v_i v_{i+1} b_i, \quad i = 1, \cdots, n-2. \quad (27)$$

The Bell matrix $B$ forms a unitary braid representation. After some algebra, the right handside of the braid group relation (25) has a form

$$(\mathbb{I}_2 \otimes B)(B \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B)$$

$$= \frac{1}{2\sqrt{2}}(\mathbb{I}_8 + \mathbb{I}_2 \otimes i\sigma_1 \otimes \sigma_2)(\mathbb{I}_8 + i\sigma_1 \otimes \sigma_2 \otimes \mathbb{I}_2)(\mathbb{I}_8 + \mathbb{I}_2 \otimes i\sigma_1 \otimes \sigma_2)$$

$$= \frac{i}{\sqrt{2}}(\mathbb{I}_2 \otimes \sigma_1 \otimes \sigma_2 + \sigma_1 \otimes \sigma_2 \otimes \mathbb{I}_2) = \frac{1}{\sqrt{2}}(\mathbb{I}_2 \otimes B^2 + B^2 \otimes \mathbb{I}_2) \quad (28)$$

and its left handside leads to the same result,

$$(B \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B)(B \otimes \mathbb{I}_2) = \frac{1}{\sqrt{2}}(\mathbb{I}_2 \otimes B^2 + B^2 \otimes \mathbb{I}_2). \quad (29)$$

Furthermore, the Bell matrix $B$ as a classical crossing and the permutation matrix $P$ as a virtual crossing,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P|ij\rangle = |ji\rangle, \quad i, j = 0, 1, \quad (30)$$
form a unitary virtual braid representation. The left handside of the virtual mixed relation (27) has a form
\[(1 \otimes 1 \otimes B)(P \otimes 1 \otimes 1)(1 \otimes 1 \otimes P)(|i\rangle \otimes |j\rangle \otimes |k\rangle) = (1 \otimes 1 \otimes B)(|k\rangle \otimes |i\rangle \otimes |j\rangle),\]
while its right handside leads to the form
\[\sum_{k,l=0}^{1} B_{ij,k,l}(P \otimes 1 \otimes 1)(1 \otimes 1 \otimes P)(|i\rangle \otimes |j\rangle \otimes |k\rangle) = (1 \otimes 1 \otimes B)(|k\rangle \otimes |i\rangle \otimes |j\rangle).\]

Here we suggest the unitary braiding operator \((b^{-1} \otimes 1 \otimes 1)(1 \otimes 1 \otimes b)\) as the braid teleportation underlying quantum teleportation. Its simplest example \((P \otimes 1 \otimes 1)(1 \otimes 1 \otimes P)\) is called the teleportation swapping in view of the fact that the braid group (25) is a generation of the symmetric group (26), and it has the other example \((B^{-1} \otimes 1 \otimes 1)(1 \otimes 1 \otimes B)\) in terms of the Bell matrix. There are two natural teleportation swapping operators \((P \otimes 1 \otimes 1)(1 \otimes 1 \otimes P)\) and \((1 \otimes 1 \otimes P)(P \otimes 1 \otimes 1)\) in terms of the permutation operator \(P\), satisfying the following teleportation swapping equalities,
\[|k\rangle \otimes |i\rangle = (P \otimes 1 \otimes 1)(1 \otimes 1 \otimes P)(|i\rangle \otimes |j\rangle \otimes |k\rangle),\]
\[|i\rangle \otimes |k\rangle = (1 \otimes 1 \otimes P)(P \otimes 1 \otimes 1)(|i\rangle \otimes |j\rangle \otimes |k\rangle),\]
which are shown in Figure 2, the virtual crossing \(P\) denoted by a cross with a small circle.

2.4 Virtual braid teleportation

Let us describe quantum teleportation in the language of the virtual braid group. The braid group relation (26) is a connection between topological entanglements and quantum entanglements, while the virtual mixed relation (27)
is a reformulation of the teleportation equation (38). A nontrivial unitary braid detecting knots or links can be often identified with a universal quantum gate transforming a separate state into an entangled one, see [13, 17, 18], and it acts as a device yielding an entangled source. On the other hand, the teleportation swapping in terms of a virtual crossings is responsible for quantum teleportation. In the following, the virtual mixed relation (27) is shown as a reformulation of the teleportation equation (38).

Note that the permutation matrix $P$ has a form

$$P = \frac{1}{2}(\mathbb{I}_4 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3),$$

(34)

and local unitary transformations (7) among Bell states are given by

$$|\phi^-\rangle = B|00\rangle = (\mathbb{I}_2 \otimes \sigma_3)B|11\rangle = (\sigma_3 \otimes \mathbb{I}_2)B|11\rangle,$$

$$|\psi^+\rangle = B|01\rangle = (\mathbb{I}_2 \otimes \sigma_1)B|11\rangle = (\sigma_1 \otimes \mathbb{I}_2)B|11\rangle,$$

$$|\psi^-\rangle = -B|10\rangle = (\mathbb{I}_2 \otimes -i\sigma_2)B|11\rangle = (i \sigma_2 \otimes \mathbb{I}_2)B|11\rangle.$$

(35)

In terms of the Bell matrix and teleportation swapping, the left handside of the teleportation equation (38) has a form,

$$|\psi\rangle_C \otimes |\phi^+\rangle_{AB} = (\mathbb{I}_2 \otimes B)(|\psi\rangle_C \otimes |11\rangle_{AB}) = (\mathbb{I}_2 \otimes B)(P \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes P)(|11\rangle_{CA} \otimes |\psi\rangle_B),$$

(36)

while its right handside leads to the other form,

$$\frac{1}{2}(|\phi^-\rangle_{CA} \sigma_3 |\psi\rangle_B + |\psi^-\rangle_{CA} (-i \sigma_2) |\psi\rangle_B + |\psi^+\rangle_{CA} \sigma_1 |\psi\rangle_B + |\phi^+\rangle_{CA} |\psi\rangle_B)$$

$$= \frac{1}{2} (\mathbb{I}_2 \otimes \sigma_3 \otimes \sigma_3 + \mathbb{I}_2 \otimes i \sigma_2 \otimes i \sigma_2 + \mathbb{I}_2 \otimes \sigma_2 \otimes \sigma_1 + \mathbb{I}_2)(B \otimes \mathbb{I}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B)$$

$$= (\mathbb{I}_2 \otimes P - \mathbb{I}_2 \otimes \sigma_2 \otimes \sigma_2)(B \otimes \mathbb{I}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B)$$

(37)

Hence the teleportation equation (38) can be recognized to be either a kind of the teleportation swapping,

$$|\psi\rangle_C \otimes |\phi^+\rangle_{AB} = (\mathbb{I}_2 \otimes P - \mathbb{I}_2 \otimes \sigma_2 \otimes \sigma_2)(|\phi^+\rangle_{CA} \otimes |\psi\rangle_B)$$

$$= (P \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes P)(|\phi^+\rangle_{CA} \otimes |\psi\rangle_B),$$

(38)

or a reformulation of the virtual mixed relation (27),

$$(\mathbb{I}_2 \otimes B)(P \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes P)(|11\rangle_{CA} \otimes |\psi\rangle_B)$$

$$= (\mathbb{I}_2 \otimes P - \mathbb{I}_2 \otimes \sigma_2 \otimes \sigma_2)(B \otimes \mathbb{I}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B)$$

$$= (P \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes P)(B \otimes \mathbb{I}_2)(|11\rangle_{CA} \otimes |\psi\rangle_B).$$

(39)

The remaining three teleportation equations can be reformulated by applying local unitary transformations to the teleportation equation (38). The teleportation equation for the Bell state $|\phi^+\rangle_{AB}$ is obtained to be

$$|\psi\rangle_C \otimes |\phi^-\rangle_{AB} = (\mathbb{I}_2 \otimes \sigma_3 \otimes \mathbb{I}_2)(|\psi\rangle_C \otimes |\phi^+\rangle_{AB})$$

$$= (\mathbb{I}_2 \otimes \sigma_3 \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes P - \mathbb{I} \otimes \sigma_2 \otimes \sigma_2)(\mathbb{I}_2 \otimes \sigma_3 \otimes \mathbb{I}_2)(|\phi^-\rangle_{CA} \otimes |\psi\rangle_B)$$

$$= (\mathbb{I}_2 \otimes P - \mathbb{I}_2 \otimes \sigma_1 \otimes \sigma_1)(|\phi^-\rangle_{CA} \otimes |\psi\rangle_B),$$

(40)
and the teleportation equations for Bell states $|\psi^\pm\rangle_{AB}$ have the form,

\[
|\psi\rangle_C \otimes |\psi^+\rangle_{AB} = (\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_3 \otimes \sigma_3)(|\psi^+\rangle_{CA} \otimes |\psi\rangle_B),
\]

\[
|\psi\rangle_C \otimes |\psi^-\rangle_{AB} = (\mathbb{1}_2 \otimes P - \mathbb{1}_8)(|\psi^-\rangle_{CA} \otimes |\psi\rangle_B),
\]

(41)

in which local unitary transformations of $(\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2)$ are exploited,

\[
(\mathbb{1}_2 \otimes \sigma_1 \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2)(\mathbb{1}_2 \otimes \sigma_1 \otimes \mathbb{1}_2) = \mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_3 \otimes \sigma_3,
\]

\[
(\mathbb{1}_2 \otimes i\sigma_2 \otimes \mathbb{1}_2)(\mathbb{1}_2 \otimes P - \mathbb{1}_2 \otimes \sigma_2 \otimes \sigma_2)(\mathbb{1}_2 \otimes i\sigma_2 \otimes \mathbb{1}_2) = \mathbb{1}_2 \otimes P - \mathbb{1}_8.
\]

(42)

As a remark, a unitary braid is a device of entangling separate states in the virtual braid teleportation, whereas a unitary braiding operator plays a role of quantum teleportation in the braid teleportation.

### 2.5 Braid teleportation, crossed measurement and state model

We explore the braid teleportation, i.e., a unitary braiding operator $(b^{-1} \otimes Id)(Id \otimes b)$, from both physical and mathematical perspectives. In view of Vaidman’s crossed measurement [37], a unitary braid or crossing acts as a device of non-local measurement in space-time. In Figure 3, two lines of the crossing $b$ represent two observable operations: the first one relating quantum measurement at the space-time point $(x_1, t_1)$ to that at the other point $(x_2, t_2)$ and the second one relating quantum measurement at $(x_2, t_2)$ to that at $(x_1, t_1)$. The crossed measurement $(Id \otimes b)$ plays the role of sending a qubit (with a possible local unitary transformation) from Charlie to Alice. Similarly, the crossed measurement $(b^{-1} \otimes Id)$ transfers the qubit from Alice to Bob with a possible local unitary transformation. Note that this kind of interpretation for a unitary braiding operator is not the same as the braid statistics of anyons [47].

Furthermore, the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$ is different from $(Id \otimes b)(b^{-1} \otimes Id)$, namely, two crossed measurements are not commutative with
Figure 4: Braid $b$ and its inverse $b^{-1}$ in the state model.

Figure 5: Braid teleportation in the state model.

each other. For example, we rewrite the Bell teleportation $(B^{-1} \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B)$ into the other formalism,

\[
(B^{-1} \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B) = \frac{1}{2}((\mathbb{I}_4 - B^2) \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes (\mathbb{I}_4 + B^2))
\]

\[
= \frac{1}{2}(\mathbb{I}_2 \otimes (\mathbb{I}_4 + B^2))((\mathbb{I}_4 - B^2) \otimes \mathbb{I}_2) + (\mathbb{I}_2 \otimes B^2)(B^2 \otimes \mathbb{I}_2)
\]

\[
= (\mathbb{I}_2 \otimes B)(B^{-1} \otimes \mathbb{I}_2) + (\mathbb{I}_2 \otimes B^2)(B^2 \otimes \mathbb{I}_2)
\]

where we exploit

\[
(\mathbb{I}_2 \otimes B^2)(B^2 \otimes \mathbb{I}_2) = -(B^2 \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B).
\]

and $B^2$ does not form a braid representation.

Moreover, we explore the configuration for the braid teleportation $(b^{-1} \otimes Id)(Id \otimes b)$ in the state model for knot theory [10]. It is an approach to a disentanglement of a knot or link, see Figure 4. A braid $b$ is denoted by a under-crossing and its inverse $b^{-1}$ is denoted by an over-crossing. Each crossing is identified with a linear combination of two types of configurations: the first given by two straight lines representing identity and the second given by a top cup together with a bottom cap representing a projector. The coefficients $A, A^{-1}$ are specified by which state model to be used [10]. In Figure 5, the braid teleportation has four diagrammatical terms. A part above a dashed line is contributed from $(b^{-1} \otimes Id)$ and a part under the dashed line is from $(Id \otimes b)$. Obviously, the first three are irrelevant with quantum teleportation but the fourth one takes charge for it.
As a remark, the state model for knot theory \cite{10} is a diagrammatical recipe for the braid representation of the TL algebra, and the teleportation term in Figure 5 is a typical configuration in the diagrammatical TL algebra. In the following, we focus on the topic how the TL algebra with local unitary transformations describes quantum information and computation.

3 Extended TL diagrammatical approach (I): diagrammatical rules and examples

In the following sections from Section 3 to Section 7, we propose the extended Temperley–Lieb diagrammatical approach and exploit it to study various topics in quantum information and computation. In this section, we devise the extended TL diagrammatical rules and explain them by examples.

3.1 Maximally entangled bipartite pure states

Maximally entangled bipartite pure states play key roles in quantum information and computation, and how to make a diagrammatical representation for them is a bone of the extended TL diagrammatical rules.

The vectors $|e_i\rangle$, $i = 0, 1, \cdots d - 1$ form an orthogonal basis in a $d$-dimension Hilbert space $H$ and the covectors $\langle e_i|$, are chosen in its dual Hilbert space $H^*$,

$$
\sum_{i=0}^{d-1} |e_i\rangle \langle e_i| = \mathbb{I}_d, \quad \langle e_j|e_i\rangle = \delta_{ij}, \quad i, j = 0, 1, \cdots d - 1,
$$

where $\delta_{ij}$ is the Kronecker symbol and $\mathbb{I}_d$ denotes a $d$-dimensional identity matrix.

The maximally entangled state $|\Omega\rangle$, a quantum state in the two-fold tensor product $H \otimes H$ of the Hilbert space $H$, and its dual state $\langle \Omega|$ are respectively given by

$$
|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i \otimes e_i\rangle, \quad \langle \Omega| = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle e_i \otimes e_i|,
$$

where the upper index $T$ denotes the transpose, and hence it is permitted to move a local action of the operator $M$ in the Hilbert space to the other Hilbert space.
if it acts on $|\Omega\rangle$. A trace of two operators $M^\dagger$ and $M'$ can be represented by an inner product between $|\psi\rangle$ and $|\psi'\rangle$,

$$\langle \psi | \psi' \rangle \equiv \langle \Omega | (M^\dagger \otimes \mathbb{1}_d)(M' \otimes \mathbb{1}_d)|\Omega\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \langle e_i | M^\dagger M' \langle e_j | e_i \rangle, \quad \langle \psi | \psi' \rangle = (M' \otimes \mathbb{1}_d)|\Omega\rangle,$$

which leads to an inner product with the operator $N_1 \otimes N_2$ given by a trace,

$$\langle \psi | N_1 \otimes N_2 \langle \psi' \rangle = \frac{1}{d} tr(M^\dagger N_1 M' N_2^T). \quad \langle 50 \rangle$$

The transfer operator $T_{BC}$, sending a quantum state from Charlie to Bob, is defined by

$$T_{BC}|\psi\rangle_C \equiv T_{BC} \sum_{k=0}^{d-1} a_k |e_k\rangle_C = \sum_{k=0}^{d-1} a_k |e_k\rangle_B = |\psi\rangle_B,$$

and has a form of an inner product between $C_A(\Omega)$ and $|\Omega\rangle_{AB}$,

$$C_A\langle \Omega | \Omega \rangle_{AB} = \frac{1}{d} \sum_{i,j=0}^{d-1} (C_\langle e_i | \otimes A(e_i | (|e_j\rangle_A \otimes |e_j\rangle_B) = \frac{1}{d} T_{BC} \equiv \frac{1}{d} \sum_{i=0}^{d-1} |e_i\rangle_B C_\langle e_i |, \quad \langle 51 \rangle$$

which is exploited by Braunstein et al. see [9].

A unitary transformation of $|\Omega\rangle$ is called local as the unitary operator $U_n$ only acts on the first (or second) Hilbert space of the two-fold Hilbert space $\mathcal{H} \otimes \mathcal{H}$. The local unitary transformation of $|\Omega\rangle$ denoted by $|\Omega_n\rangle = (U_n \otimes \mathbb{1}_d)|\Omega\rangle$, is still a maximally entangled vector. The set of unitary operators $U_n$ satisfying the orthogonal relation $tr(U_n^\dagger U_m) = d \delta_{nm}$, $n, m = 1 \cdots d^2$, forms a basis of $d \times d$ unitary matrices, where the upper index $\dagger$ denotes the adjoint. The collection of maximally entangled states $|\Omega_n\rangle$ has the properties,

$$\langle \Omega_n | \Omega_m \rangle = \delta_{nm}, \quad \sum_{n=1}^{d^2} |\Omega_n\rangle \langle \Omega_n | = \mathbb{1}_{d^2}, \quad n, m = 1, \cdots d^2. \quad \langle 52 \rangle$$

Introduce the symbol $\omega_n$ for the maximally entangled state $|\Omega_n\rangle \langle \Omega_n |$ and especially denote $|\Omega\rangle \langle \Omega |$ by $\omega$, namely,

$$\omega \equiv |\Omega\rangle \langle \Omega |, \quad \omega_n \equiv |\Omega_n\rangle \langle \Omega_n |, \quad U_1 = \mathbb{1}_d, \quad \langle 53 \rangle$$

and the set of $\omega_n$, $n = 1, 2, \cdots d^2$ forms a set of observables over an output parameter space.
3.2 Extended TL diagrammatical rules

Three pieces of extended TL diagrammatical rules are devised for assigning a diagram to a given algebraic object. The first is our convention; the second explains what straight lines and oblique lines represent; the third describes various configurations in terms of cups and caps.

**Rule 1.** Read an algebraic object such as an inner product from the left-hand side to the right-hand side, and draw a diagram from the top to the bottom. Represent the operator $M$ by a solid point, its adjoint operator $M^\dagger$ by a small circle, its transposed operator $M^T$ by a solid point with a cross line, and its complex conjugation operator $M^*$ by a small circle with a cross line. Denote the Dirac ket by the symbol $\nabla$ and the Dirac bra by the symbol $\triangle$.

**Rule 2.** See Figure 6. A straight line of type $A$ denotes the identity operator $\mathbb{1}_A$ in the system $A$. Straight lines of type $A$ with a bottom $\nabla$ or top $\triangle$ describe a vector $|\psi\rangle_A$, covector $A\langle\varphi|$, and an inner product $A\langle\varphi|\psi\rangle_A$ in the system $A$, respectively. Straight lines of type $A$ with a middle solid point or bottom $\nabla$ or top $\triangle$ describe an operator $M_A$, a vector $M_A|\psi\rangle_A$, a covector $A\langle\varphi|M_A|\psi\rangle_A$, and an inner product $A\langle\varphi|M_A|\psi\rangle_A$, respectively.

See Figure 7. An oblique line from the system $C$ to the system $B$ describes the transfer operator $T_{BC}$, and its solid point or bottom $\nabla$ or top $\triangle$ have the same interpretations as those on a straight line in Figure 6.

**Rule 3.** See Figure 8. A cup denotes the maximally bipartite entangled state
A vector $|\Omega\rangle$ and a cap does for its dual $\langle\Omega|$. A cup with a middle solid point on its one branch describes a local action of the operator $M$ on $|\Omega\rangle$, and this solid point can flow to its other branch and then is replaced by a solid point with a cross line representing $M^T$. The same happens for a cap except that a solid point is replaced by a small circle to distinguish the operator $M$ from its adjoint operator $M^\dagger$.

A cup and a cap can form different sorts of configurations. See Figure 9. As a cup is at the top and a cap is at the bottom for the same composite system, this configuration is assigned to the projector $|\Omega\rangle\langle\Omega|$. As a cap is at the top and a cup is at the bottom for the same composite system, this diagram describes an inner product $\langle\Omega|\Omega\rangle = 1$ by a closed circle. As a cup is at the bottom for the composite system $H_C \otimes H_A$ and a cap is at the top for the composite system $H_A \otimes H_B$, that is an oblique line representing the transfer operator $T_{CB}$ with the normalization factor $\frac{1}{d}$.

Additionally, as a cup has a local action of the operator $M$ and a cap has a local action of the operator $N^\dagger$, the resulted circle with a solid point for $M$ and a small circle for $N^\dagger$ represents the trace $\frac{1}{d} tr(MN^\dagger)$. As a convention, we describe a trace of operators by a closed circle with solid points or small circles, and assign each cap or cup a normalization factor $\frac{1}{\sqrt{d}}$ and a circle a normalization factor $d$.

Note that cups and caps are well known configurations in knot theory and statistical mechanics. They were used by Wu [48] in statistical mechanics, and exploited by Kauffman [10] for diagrammatically representing the Temperley-Lieb algebra soon after Jones’s work [49]. These configurations are nowadays called...
3.3 Examples for the extended TL diagrammatical rules

In the following, five examples are listed as well as their corresponding algebraic counterparts to explain the extended TL diagrammatical rules in detail.

Example 1: Figure 10 is a diagrammatical representation for the teleportation equation (8). The cup denotes the Bell state $|\phi^+\rangle$, and the cups with middle solid points $\sigma_3$, $-i\sigma_2$, $\sigma_1$ denote the Bell states $|\phi^-\rangle$, $|\psi^-\rangle$ and $|\psi^+\rangle$, respectively. The straight line with a bottom $\nabla$ denotes a unknown state $|\psi\rangle$ to be transported, and other straight lines with middle solid points respectively denote local unitary transformations of $|\psi\rangle$.

Example 2: Figure 11 presents how to compute the inner product between $|\phi\rangle$ and $|\psi\rangle$ in the extended TL diagrammatical approach. It consists of three terms: the covector $\langle \phi \otimes \Omega |$, the local operator $\mathbb{1}_d \otimes M \otimes \mathbb{1}_d$, and the vector $|\Omega \rangle \otimes |\psi\rangle$. The vector $|\psi\rangle$ is represented by a straight line with a bottom $\nabla$, and the local operator $\mathbb{1}_d \otimes M \otimes \mathbb{1}_d$ is denoted by a solid point on the cup $|\Omega\rangle$. Move the local operator $M$ from its beginning position to the other branch of the cap, change it to the local operator $M^T$ and then allow the bottom cup and the top cap to collapse into an oblique line denoting the transfer operator with a normalization factor $\frac{1}{d}$. Besides, it can be calculated in an algebraic way

\[
\langle \phi \otimes \Omega | (\mathbb{1}_d \otimes M \otimes \mathbb{1}_d) | \Omega \otimes \psi \rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \langle \phi | e_i \otimes e_i | (\mathbb{1}_d \otimes M \otimes \mathbb{1}_d) | e_j \otimes e_j \otimes \psi \rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \langle \phi | e_j \rangle \langle e_i | M | e_j \rangle \langle e_i | \psi \rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \phi^*_j M_{ji} \psi_i = \frac{1}{d} \langle \phi | M^T | \psi \rangle
\]

in which every step has a diagrammatical counterpart.

Example 3: Figure 12 provides a diagrammatical representation for the partial trace, which denotes the summation over a subsystem of a composite system,

\[
tr_A(|e^A_i \otimes e^B_j | (e^C_i \otimes e^C_m)) = |e^C_i \rangle \langle e^C_i | M_{ji} | \delta_{ji}, \quad tr_A(|e^A_j | (e^A_i \rangle \langle e^A_i |) = \delta_{ji}.
\]
\[ \langle \phi | M \rangle = \langle \phi | \psi \rangle M \langle \psi | \rangle \]

Figure 11: Inner product in terms of a cap and a cup.

\[ \frac{1}{d} \]

\[ \frac{1}{d} \]

\[ \frac{1}{d} \]

Figure 12: Three kinds of partial traces using a cup and a cap.

Note that the trace is a sort of partial trace, i.e., the summation over the entire composite system,

\[ tr_{CA}(|\rho_C \otimes 1\rangle \langle 1 | \rangle_{CA} = \frac{1}{d} \sum_{i,j=0}^{d-1} tr_A(|e_i^C \otimes e_i^A \rangle \langle e_j^C \otimes e_j^A |) = \frac{1}{d} (\mathbb{1}_d)_{C}, \quad 56 \]

The left diagrammatical term describes the one type of partial trace leading to a straight line for identity,

\[ tr_A(|\Omega \rangle_{CA} \langle 1 | \rangle_{CA} = 1, \quad 57 \]

and the other two diagrammatical terms represent the other type of partial trace leading to an oblique line for the transfer operator: the middle term denotes the transfer operator \( T_{CB} \) given by

\[ tr_A(|\Omega \rangle_{CA} AB \langle 1 | \rangle_{CB} = \frac{1}{d} T_{CB}, \quad 58 \]

and the right one gives rise to the transfer operator \( T_{CB} \) by

\[ tr_A(CA \langle \Omega | \Omega \rangle_{AB} = \frac{1}{d} T_{BC}. \quad 59 \]

Example 4: Figure 13 has two types of diagrams denoting the trace of operator products. In the first case, a top cup with a bottom cap forms a same closed circle as a top cap with a bottom cup, representing an algebraic equation given by

\[ tr_{CA}(|\rho_C \otimes \mathbb{1}_d \rangle \langle \mathbb{1}_d | \rangle_{CA} = tr_A(\mathbb{1}_d \otimes O^T) \langle \rho_C \otimes \mathbb{1}_d | \rangle_{CA}, \quad 60 \]
where $\rho_C$ and $\mathcal{O}_A$ are bounded linear operators in $d$-dimensional Hilbert space. In the second case, a closed circle formed by two oblique lines for transfer operators denotes a same trace as a top cap with a bottom cup, which is algebraically proved,

$$
\text{tr}_{CA}(\rho_C T_{CA})(\mathcal{O}_A T_{AC}) = \sum_{i,j=0}^{d-1} \text{tr}_{CA}(\rho_C |e_i\rangle_{CA} (\mathcal{O}_A |e_j\rangle_{AC} \langle e_j|)) \\
= \sum_{i,j=0}^{d-1} \text{tr}_{CA}(\rho_C |e_i\rangle_C \otimes \mathcal{O}_A |e_j\rangle_A (\mathcal{O}_A |e_j\rangle_A \langle e_j| \otimes A |e_i\rangle)) \\
= \sum_{i,j=0}^{d-1} (\rho_C)_{ij} (\mathcal{O}_A)_{ji} = \text{tr}(\rho_C \mathcal{O}_A) \\
= d \cdot \text{tr}_{CA}(\Omega)(\rho_C \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathcal{O}_A^T)|\Omega\rangle_{CA}.
$$

(61)

Example 5: Figure 14 recognizes the configuration of cup (cap) as compositions of cups and caps. In the left diagrammatical term, the cup $|\Omega\rangle_{AB}$ is a result of connecting the cap $|\Omega\rangle_{BC}$ with the cups $|\Omega\rangle_{AB}$, which is verified after some algebra

$$
(BC\langle\Omega|(|\Omega\rangle_{AB})(|\Omega\rangle_{CD}) \\
\equiv (\mathbb{1}_d \otimes BC\langle\Omega| \otimes \mathbb{1}_d)(|\Omega\rangle_{AB} \otimes \mathbb{1}_d \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathbb{1}_d \otimes |\Omega\rangle_{CD}) \\
= \frac{1}{d^2} \sum_{i,j,k=0}^{d-1} (|e_i^B \otimes e_i^C\rangle)(|e_j^A \otimes e_j^B\rangle)(|e_k^C \otimes e_k^D\rangle) \\
= \frac{1}{d^2} \sum_{i=0}^{d-1} |e_i^A \otimes e_i^D\rangle = \frac{1}{d} |\Omega\rangle_{AD}.
$$

(62)

The right diagrammatical term shows the cap $|\Omega\rangle_{AD}$ as a composition of the caps $AB\langle\Omega|$, $CD\langle\Omega|$ and the cup $|\Omega\rangle_{BC}$,

$$
(AB\langle\Omega|)(CD\langle\Omega|)(|\Omega\rangle_{BC}) \\
\equiv (AB\langle\Omega| \otimes \mathbb{1}_d \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \mathbb{1}_d \otimes CD\langle\Omega|)(\mathbb{1}_d \otimes |\Omega\rangle_{BC} \otimes \mathbb{1}_d) = \frac{1}{d} |\Omega\rangle_{AD}.
$$

(63)
As a remark, the above examples will be exploited in the following sections as topological-like diagrammatical tricks [34] in the extended TL diagrammatical configuration for quantum information and computation.

4 Extended TL diagrammatical approach (II): quantum teleportation

Three types of descriptions for quantum teleportation: quantum information flow denoted by the transfer operator, measurement-based quantum computation, and tight teleportation scheme, are unified in the extended TL diagrammatical approach.

4.1 Quantum teleportation using the transfer operator

Quantum teleportation can be formulated using the transfer operator $T_{BC}$ which sends the quantum state from Charlie to Bob in the way: $T_{BC}|\psi\rangle_C = |\psi\rangle_B$, besides its standard description [8, 9] using the teleportation equation (8). The entire teleportation process involves local unitary transformations which are not shown in the formalism (51) of the transfer operator $T_{BC}$. To be general, therefore, we recall the calculation [43] to reformulate the transfer operator in terms of maximally entangled states $|\Phi(U)\rangle_{CA} \text{ and } |\Phi(V^T)\rangle_{AB}$ labeled by local unitary actions of $U$ and $V^T$ on $|\Omega\rangle$,

$$
C_A\langle \Phi(U)|\Phi(V^T)\rangle_{AB} \equiv C_A\langle \Omega|(U^\dagger \otimes 1_d)(V^T \otimes 1_d)|\Omega\rangle_{AB} \\
\equiv C_{AB}\langle \Omega \otimes 1_d|(U^\dagger \otimes 1_d \otimes 1_d)(1_d \otimes V^T \otimes 1_d)|1_d \otimes \Omega\rangle_{CAB} \\
= C_{AB}\langle \Omega \otimes 1_d|(1_d \otimes 1_d \otimes VU^\dagger)1_d \otimes \Omega\rangle_{CAB} \\
= \frac{1}{d} \sum_{i=0}^{d-1} C_{B}\langle e_i \otimes 1_d|(1_d \otimes VU^\dagger)1_d \otimes e_i\rangle_{CB} \\
\equiv \frac{1}{d} \sum_{i=0}^{d-1} C\langle e_i |(VU^\dagger)B|e_i\rangle_{B} = \frac{1}{d} \sum_{i=0}^{d-1} (VU^\dagger)B|e_i\rangle_{B} C\langle e_i | \\
= \frac{1}{d} (VU^\dagger)B T_{BC}
$$

(64)
which has a special case of $U = V$ given by

$$\frac{1}{d} T_{BC} = C_A \langle \Phi(U) | \Phi(U^T) \rangle_{AB}. \quad (65)$$

In Figure 15, we repeat the above algebraic calculation at the diagrammatical level. From the left to the right, the inner product $C_A \langle \Phi(U) | \Phi(U^T) \rangle_{AB}$ has the $\langle \Omega |$, identity $\mathds{1}_d$, local unitary operators $U$ and $V^T$, identity $\mathds{1}_d$ and $| \Omega \rangle$ which are respectively drawn from the top to the bottom. Move local operators $U^\dagger$ and $V^T$ along the line from their positions to the line denoting the system $B$, and obtain the product $(VU^\dagger)_B$ of local unitary operators acting on the quantum state that Bob has. The transfer operator $T_{BC}$ has a normalization factor $\frac{1}{d}$ contributed by vanishing of a cup and a cap.

Hence in the extended TL diagrammatical approach, the quantum teleportation can be viewed as a kind of quantum information flow denoted by an oblique line from Charlie to Bob. The result $\frac{1}{d} (VU^\dagger)_B T_{BC}$ seems to argue that quantum measurement labeled by the unitary operator $U^\dagger$ plays a role before state preparation labeled by the unitary operator $V^T$. But it is not true. Let us read Figure 15 in the way where the $T$-axis denotes the time-arrow and the $X$-axis denotes the space-distance. Although the quantum information flow seems to start from the state preparation, pass through the quantum measurement, and come back to the state preparation again, and eventually return to the quantum measurement, it flows from the state preparation to the quantum measurement without violating the causality principle in its final form denoted by an oblique line.

Note that we have to add a rule on how to move operators in the extended TL diagrammatical approach: It is forbidden for an operator to cross over another operator. For example, we have the operator product $\frac{1}{d} (VU^\dagger)_B$ instead of $\frac{1}{d} (U^\dagger V)_B$. Obviously, a violation of this rule leads to a violation of the causality principle. In addition, there are another known approaches to the quantum information flow: the teleportation topology [13, 29] and categorical approach [46], which will be compared with the extended TL diagrammatical approach in Section 7.
4.2 Measurement-based quantum teleportation

Quantum teleportation can be described from the point of quantum measurement \[37, 38, 39, 50\]. The difference from its standard description \[8, 9\] is that the maximally entangled state \(|\Omega\rangle\rangle_{AB}\) between Alice and Bob is created via quantum measurement \[37\]. Here it is convenient to represent a quantum measurement in terms of a projector \(|\Omega\rangle\langle\Omega|\rangle_{AB}\). Therefore, quantum teleportation is determined by two quantum measurements: the one denoted by \(|\Omega\rangle\langle\Omega|\rangle_{AB}\) and the other denoted by \(|\Omega_n\rangle\langle\Omega_n|\rangle_{CA}\), which leads to a new formulation of the teleportation equation,

\[
\left(|\Omega_n\rangle\langle\Omega_n| \otimes 1_d\right)\left(|\psi\rangle \otimes |\Omega\rangle\langle\Omega|\right) = 1_d\left(|\Omega_n\rangle \otimes 1_d\right)\left(1_d \otimes U_n^\dagger |\psi\rangle\langle\Omega|\right),
\]

where lower indices \(A, B, C\) are omitted for convenience. Read this teleportation equation \([66]\) from the left to the right, and draw the diagram from the top to the bottom in view of the extended TL diagrammatical rules, i.e., Figure 16.

There is a natural connection between two formalisms \([8]\) and \([66]\) of the teleportation equation. Consider all unitary matrices \(U_n\) satisfying \([52]\) and \([53]\), and then make a summation of all teleportation equations of the type \([66]\) labeled by \(U_n\) to derive

\[
|\psi\rangle \otimes |\Omega\rangle = \frac{1}{d} \sum_{n=1}^{d^2} |\Omega_n\rangle \otimes U_n^\dagger |\psi\rangle
\]

which is a generalization of the teleportation equation of the type \([8]\) in the \(d\)-dimension Hilbert space. As \(d = 2\), the collection of the basis of unitary operators consist of unit matrix \(1_2\) and Pauli matrices \(\sigma_1, \sigma_2, \sigma_3\). Bell measurements are denoted by projectors in terms of Bell states \(|\phi^{\pm}\rangle\rangle\) and \(|\psi^{\pm}\rangle\rangle\). They lead to the
same kind of the teleportation equations of the type \((66)\),

\[
\begin{align*}
(|\phi^-\rangle\langle\phi^-| \otimes \mathbb{I}_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\phi^-\rangle \otimes \sigma_3|\psi\rangle), \\
(|\psi^+\rangle\langle\psi^+| \otimes \mathbb{I}_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\psi^+\rangle \otimes \sigma_1|\psi\rangle), \\
(|\psi^-\rangle\langle\psi^-| \otimes \mathbb{I}_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\psi^-\rangle \otimes -i\sigma_2|\psi\rangle), \\
(|\phi^+\rangle\langle\phi^+| \otimes \mathbb{I}_2)(|\psi\rangle \otimes |\phi^+\rangle) &= \frac{1}{2}(|\phi^+\rangle \otimes |\psi\rangle),
\end{align*}
\]

which has a summation as the teleportation equation \((8)\) since Bell states \(|\phi^\pm\rangle\)
and \(|\psi^\pm\rangle\) satisfy

\[
\mathbb{I}_2 = |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|. \tag{69}
\]

Furthermore, we comment on measurement-based quantum teleportation using continuous variables \([38]\),
which is a simple generalization of the discrete teleportation without essential conceptual changes.
A maximally entangled state \(|\Omega\rangle\) and teleportated state \(|\Psi\rangle\) in the continuous case have the form,

\[
|\Omega\rangle = \int dx \, |x, x\rangle, \quad |\Psi\rangle = \int dx \, \psi(x) \, |x\rangle, \tag{70}
\]

and other maximally entangled states \(|\Omega_{\alpha\beta}\rangle\) are formulated by the combined action
of the \(U(1)\) rotation with the translation \(T\) on \(|\Omega\rangle\),

\[
|\Omega_{\alpha\beta}\rangle = (U_\beta \otimes T_\alpha)|\Omega\rangle \equiv \int dx \exp(i\beta x)|x, x + \alpha\rangle, \quad \alpha, \beta \in \mathbb{R} \tag{71}
\]

where \(U_\beta|x\rangle = e^{i\beta x}|x\rangle\), \(T_\alpha|x\rangle = |x + \alpha\rangle\) and which is a common eigenvector of the
location operator \(X \otimes \mathbb{I} - \mathbb{I} \otimes X\) and conjugate momentum operator \(P \otimes \mathbb{I} + \mathbb{I} \otimes P\),

\[
(X \otimes \mathbb{I} - \mathbb{I} \otimes X)|\Omega_{\alpha\beta}\rangle = -\alpha|\Omega_{\alpha\beta}\rangle, \quad (P \otimes \mathbb{I} + \mathbb{I} \otimes P)|\Omega_{\alpha\beta}\rangle = 2\beta|\Omega_{\alpha\beta}\rangle. \tag{72}
\]

The teleportation equation of the type \((66)\) is obtained to be

\[
(|\Omega_{\alpha\beta}'\rangle\langle\Omega_{\alpha\beta}'| \otimes \mathbb{I})(|\Psi\rangle \otimes |\Omega'\rangle) = (|\Omega_{\alpha\beta}\rangle \otimes \mathbb{I})(\mathbb{I} \otimes \mathbb{I} \otimes U_{-\beta}T_\alpha|\Psi\rangle) \tag{73}
\]

which has a similar diagrammatical representation as Figure 16. The translation
operator \(T_\alpha\) is its own adjoint operator, and hence it is permitted to move along
a cup or cap without change.

As a remark, the difference between Figure 15 and 16 lies in there are an extra
cup and cap besides the quantum information flow in Figure 15, which become
crucial when we study the tight teleportation scheme in the next subsection.
In addition, Figure 16 is a typical configuration in the diagrammatical representation
for the TL algebra as involved local unitary operators are identity, see Section 5.
4.3 Tight teleportation and dense coding schemes

In the tight teleportation and dense coding schemes [30], all involved finite Hilbert spaces are $d$ dimensional and the classical channel distinguishes $d^2$ signals. All examples we treated in the above belong to the tight class. We exploit the same notations as in [30]. The density operator $\rho$ is a positive operator with a normalized trace. Charlie has his density operator $\rho_C = |\psi_1\rangle\langle\psi_2|_C$ which denotes the quantum state to be sent to Bob. Alice and Bob share the maximally entangled state $\omega_{AB}$. The set of observables $\omega_n$, $n = 1, 2, \cdots d^2$ over an output parameter space is a collection of bounded linear operators in the Hilbert space $\mathcal{H}$. Alice performs Bell measurements in the composite system between Charlie and her, and she chooses her observables $(\omega_n)_{CA}$ as local unitary transformations of the maximally entangled state $\omega$. After Bob gets the message denoted by $n$ from Alice, he applies a local unitary transformation $T_n$ on his observable $\mathcal{O}_B$, given by

$$T_n(\mathcal{O}_B) = U_n^\dagger\mathcal{O}_BU_n, \quad \mathcal{O}_B = |\psi_1\rangle\langle\psi_2|_B, \quad n = 1, 2, \cdots d^2. \quad (74)$$

The operator $T_n$ defined this way is called a channel, a complete positive linear operator with the normalization $T_n(\mathbb{I}_d) = \mathbb{I}_d$.

In terms of $\rho_C$, $\omega_{AB}$, $(\omega_n)_{CA}$ and $T_n(\mathcal{O}_B)$, the tight teleportation scheme is summarized in the equation

$$\sum_{n=1}^{d^2} \text{tr}((\rho \otimes \omega_n \otimes T_n(\mathcal{O}))) = \text{tr}(\rho \mathcal{O}), \quad (75)$$

where lower indices $A, B, C$ are neglected and which is called the characteristic equation for quantum teleportation in our previous work [34]. It catches the aim...
of a successful teleportation: Charlie performs quantum measurement in his system as he does in Bob’s system, though they are far away from each other. The term containing the message $n$ has a form denoted by $\text{term}_n$ in the following

$$\text{term}_n = \text{tr}(\langle \phi_1 | \phi_2 | \Omega \rangle \langle \Omega | \Omega_n | \otimes | U_n^\dagger | \psi_1 \rangle | \psi_2 | U_n)$$

$$= \langle \Omega_n \otimes \psi_2 U_n | \phi_1 \otimes \Omega \rangle | \phi_2 \otimes \Omega_n | \otimes | U_n^\dagger | \psi_1 \rangle$$

$$= \left( \frac{1}{d} | \psi_2 | \phi_1 \rangle \right) \left( \frac{1}{d} | \phi_2 | \psi_1 \rangle \right) = \frac{1}{d^2} \text{tr}(\rho_O),$$  \hspace{1cm} (76)

where the inner product (54) is applied twice. There are $d^2$ distinguished messages labeled by $n$, so we prove the characteristic equation (75). In the left term of Figure 17, we have two ways of deriving the characteristic equation (75) at the diagrammatical level. The first moves local operators $U_n^\dagger$, $O$ and $U_n$ from the branch of the cap to the other branch and then applies known diagrammatical tricks by the first term of Figure 12 and the first term of Figure 13. The second makes use of tricks by the second and third terms of Figure 12 and the second term of Figure 13. In addition, the number of classical channel, $d^2$ counts all possible teleportation diagrams of the same type.

In view of tight dense coding schemes have the same elements as the tight teleportation, all the tight dense coding schemes \[30\] are concluded in the characteristic equation,

$$\text{tr}(\omega(T_n \otimes \mathbb{I}_d)\omega_m)) = \delta_{nm}$$ \hspace{1cm} (77)

which is explained as follows. As Alice and Bob share the maximally entangled state $|\Omega\rangle_{AB}$, she transforms her state by the channel $T_n$ to encode the message $n$, and then Bob performs the measurement on observables $\omega_m$ in his system. At $n = m$, Bob gets the message. The entire process of dense coding is performed in the way

$$\text{tr}(|\Omega\rangle \langle \Omega | (U_n^\dagger \otimes \mathbb{I}_d)| \Omega_m \rangle \langle \Omega_m | (U_n \otimes \mathbb{I}_d))$$

$$= \langle \Omega | U_n^\dagger \otimes \mathbb{I}_d | \Omega_m \rangle \langle \Omega_m | U_n \otimes \mathbb{I}_d | \Omega \rangle = \frac{1}{d^2} (\text{tr}(U_n^\dagger U_m))^2 = \delta_{nm}$$ \hspace{1cm} (78)

which derives the characteristic equation for tight dense coding (77), also proved in the right term of Figure 17.

As a remark, Figure 16 denoting measurement-based quantum computation includes all elements of quantum teleportation: Figure 15 representing the quantum information flow is its part, and the left term of Figure 17 is its closure. Therefore, three approaches to quantum teleportation are unified in the extended TL diagrammatical approach.

### 5 Extended TL diagrammatical approach (III): TL algebra and Brauer algebra

We study algebraic structures in the extended TL diagrammatical approach: the TL algebra, the Brauer algebra, and the extended TL category. Note that symbols $\Omega$ labeling maximally entangled states are omitted for convenience without confusion in Figures 18, 19, 21.
5.1 Diagrammatical representation of TL algebra

The TL algebra $TL_n(\lambda)$ over the complex field $\mathbb{C}$ is generated by identity $Id$ and $n-1$ Hermitian projectors $e_i$ satisfying

$$
e_i^2 = e_i, \quad (e_j)^\dagger = e_i, \quad i = 1, \ldots, n-1,$$

$$e_ie_{i\pm1}e_i = \lambda^{-2}e_i, \quad e_ie_j = e_je_i, \quad |i-j| > 1,$$

(79)

with $\lambda$ called loop parameter. The diagrammatical representation of the TL algebra is called the Brauer diagram [41] or Kauffman diagram [10] in literature. It is a planar $(n,n)$ diagram including a “hidden” rectangle in the plane with $2n$ “hidden” distinct points: $n$ points on its top edge and $n$ points on its bottom edge, and they are connected by disjoint strings drawn in the rectangle. The identity is a diagram with all strings vertical, and $e_i$ has its $i$th and $i+1$th top (bottom) boundary points connected with all other strings vertical. The multiplication $e_ie_j$ identifies bottom points of $e_i$ with corresponding top points of $e_j$, removes a common boundary, and replaces each obtained loop with a factor $\lambda$. The adjoint of $e_i$ is its image under a mirror reflection on a horizontal line.

Let us set up a representation of the $TL_n(d)$ algebra in terms of the maximally entangled state $\omega$,

$$\omega = |\Omega\rangle \langle \Omega| = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle \langle jj|, \quad \omega^2 = \omega$$

(80)

by defining idempotents $e_i$ as

$$e_i = (Id)^{\otimes(i-1)} \otimes \omega \otimes (Id)^{\otimes(n-i-1)}, \quad i = 1, \ldots, n-1$$

(81)

The notation $e_i$ for an idempotent of the TL algebra is easily confused with a basis vector $|e_i\rangle$. As they both appear in the same subsection, we denote $|e_i\rangle$ by $|i\rangle$. 

Figure 18: Generator $e_i$, multiplication $e_1e_2$ and the axiom for the $TL_3(d)$ algebra.
with loop parameter $d$. For example, a representation of the $TL_3(d)$ algebra is generated by two idempotents $e_1$ and $e_2$, 

$$e_1 = \omega \otimes Id, \quad e_2 = Id \otimes \omega,$$

and they proved to satisfy the axiom $e_1 e_2 e_1 = \frac{1}{d^2} e_1$ in the way

$$e_1 e_2 e_1 |\alpha\beta\gamma\rangle = \frac{1}{d} \sum_{l=0}^{d-1} e_1 e_2 |ll\gamma\rangle \delta_{\alpha\beta} = \frac{1}{d^3} \sum_{n=0}^{d-1} |nn\gamma\rangle \delta_{\alpha\beta} = \frac{1}{d^2} e_1 |\alpha\beta\gamma\rangle$$

as well as the axiom $e_2 e_1 e_2 = \frac{1}{d^2} e_2$ using similar calculation. Figure 18 is a diagrammatical representation for $e_1$, $e_1 e_2$ and $e_1 e_2 e_1 = \frac{1}{d^2} e_1$ with loop parameter $d$. Therefore, a cup (cap) introduced in the extended TL diagrammatical approach is a connected line between top (bottom) boundary points. Each cup (cap) with a normalization factor $d^{-\frac{1}{2}}$ leads to an additional normalization factor $d^{-\frac{1}{2}N}$ as the number of vanishing cups and caps is $N$, and a closed circle yields a normalization factor $d = tr(\mathbb{I}_d)$. For examples, $e_1 e_2$ has a normalization factor $\frac{1}{d}$ from a vanishing cup and cap, and $e_1 e_2 e_1$ has a factor $\frac{1}{d^2}$ from four vanishing cups and caps.

In terms of the maximally entangled state $\omega_n$ as a local unitary transformation $U_n$ of $\omega$, we can set up a representation of the $TL_n(d)$ algebra, too. For example, a representation of the $TL_3(d)$ algebra is generated by $\tilde{e}_1$ and $\tilde{e}_2$, 

$$\tilde{e}_1 = \omega_n \otimes Id, \quad \tilde{e}_2 = Id \otimes \omega_n,$$

which are proved to satisfy the axioms of the TL algebra in the extended TL diagrammatical approach, see Figure 19. Hence the axioms of the TL algebra are invariant under local unitary transformations as idempotents $e_i$ are generated by the maximally entangled state $\omega$. 

Figure 19: The $TL_3(d)$ algebra in the extended TL diagrammatical approach.
Furthermore, a representation of the $TL_3(d)$ algebra is constructed in the way
\[ e'_1 = \rho \otimes \omega, \quad e'_2 = \omega \otimes \rho, \quad \rho = \ket{\phi} \bra{\psi}, \quad tr(\rho) = 1. \tag{85} \]
because $\rho$ and $\omega$ and the tensor products between them are all projectors. The axioms of the TL algebra are checked in both algebraic and diagrammatical approaches. Do calculation
\[
(\omega \otimes \rho)(\rho \otimes \omega) = \frac{1}{d^2} \sum_{i,j=0}^{d-1} (|ii\phi\rangle \langle jj\psi| \otimes |ii\phi\rangle \langle jj\psi|) = \frac{1}{d^2} \sum_{i,j=0}^{d-1} |ii\phi\rangle \langle jj\phi| = 1
\]
which leads to a proof for the axiom $e'_2 e'_1 e'_2 = \frac{1}{d^2} e'_2$.

\[
(\omega \otimes \rho)(\rho \otimes \omega)(\omega \otimes \rho) = \frac{1}{d^3} \sum_{i,j=0}^{d-1} \sum_{l,m=0}^{d-1} |ii\phi\rangle \langle jj\phi| |ll\phi\rangle \langle mm\phi| = \frac{1}{d^3} \sum_{i,j=0}^{d-1} |ii\phi\rangle \langle jj\phi| = 1
\]
which can be observed in Figure 19.

Similarly, to prove $e'_1 e'_2 e'_1 = \frac{1}{d^2} e'_1$. As a remark, $\rho = \ket{\phi} \bra{\psi}$ so that the transfer operator $T_{BC}$ sends a half of $\rho_C$, $\ket{\phi}_C$ from Charlie to Bob to form a unit inner product with $\ket{\psi}_B$, a half of $\rho_B$, see Figure 20. Moreover, in terms of $\rho$ and $\omega$, a representation of the $TL_3(d)$ algebra is also set up in another way
\[
e'_1 = \rho \otimes \omega_n, \quad e'_2 = \omega_n \otimes \rho, \quad n = 1, \ldots, d^2, \tag{88} \]
if and only if a local unitary transformation $U_n$ is a symmetric matrix:
\[
U_n^T U_n = U_n^* U_n = I_d \tag{89} \]
which can be observed in Figure 19.
5.2 Quantum Teleportation: the Brauer algebra

Quantum teleportation has a same diagrammatical representation as the element $e_1 e_2 (e_2 e_1)$ of the TL algebra, if involved local unitary transformations are identity (see Figure 16). There exists a natural question which can be possibly asked. Quantum teleportation plays important roles in quantum information, but the product $e_1 e_2 (e_2 e_1)$ is only an element of the TL algebra. It is meaningful to explore in which case the configuration of $e_1 e_2 (e_2 e_1)$ is crucial in the mathematical sense.

In the following, we present the axioms of the Brauer algebra [41], and explain that the quantum teleportation configuration forms a bone of this algebra.

The Brauer algebra $D_n(\lambda)$ is an extension of the TL algebra with virtual crossings, $\lambda$ called loop parameter. It has two types of generators: idempotents $e_i$ of the TL algebra $TL_n(\lambda)$ satisfying (79) and virtual crossings $v_i$ satisfying (26), $i = 1, \ldots, n - 1$. Both generators satisfy the mixed relations of the Brauer algebra, (ev/ve)

$e_i v_i = v_i e_i = e_i, \quad e_i v_j = v_j e_i, \quad j \neq i \pm 1,$

(vve): $v_{i\pm1} v_i e_{i\pm1} = \lambda e_i e_{i\pm1}, \quad (ev): \quad e_i v_{i\pm1} v_i = \lambda e_i e_{i\pm1}$

which has a diagrammatical representation in Figure 21. For example, the permutation $P$ as a virtual crossing and the maximally entangled state $\omega$ as an idempotent

$P = \sum_{i,j=0}^{d-1} |i \otimes j\rangle\langle j \otimes i|, \quad \omega = \frac{1}{d} \sum_{i=0}^{d-1} |i \otimes i\rangle\langle j \otimes j|$ (91)

form a representation of the Brauer algebra $D_2(d)$ with loop parameter $d$. The axiom (ev/ve) is verified in the way

$P \omega = \frac{1}{d} \sum_{i,j=0}^{d-1} \sum_{i',j'=0}^{d-1} |i \otimes j\rangle\langle j \otimes i' |i' \otimes j'\rangle\langle j' \otimes j'| = \omega = \omega P,$ (92)

Figure 21: Quantum teleportation: the Brauer algebra.
and the axioms \((vve)\) and \((eev)\) are proved after some algebra
\[
(\mathbb{1}_d \otimes P)(P \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes \omega) = d(\omega \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes P)(P \otimes \mathbb{1}_d),
\]
\[
(P \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes P)(\omega \otimes \mathbb{1}_d) = d(\mathbb{1}_d \otimes \omega)(\omega \otimes \mathbb{1}_d) = (\mathbb{1}_d \otimes \omega)(P \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes P),
\]
which are also proved in Figure 21 with \(\lambda = d\).

As a remark on Figure 21, it is clear that the configuration for quantum teleportation is fundamental for defining the Brauer algebra. The Brauer algebra presents an equivalent approach of performing the teleportation using the swap gate \(P\) and Bell measurement, where the teleportation swapping \((P \otimes \mathbb{1}_d)(\mathbb{1}_d \otimes P)\) or \((\mathbb{1}_d \otimes \omega)(P \otimes \mathbb{1}_d)\) are involved. As a summary of algebraic structures underlying quantum teleportation, we have proposed the braid teleportation, the teleportation swapping, the virtual braid teleportation, the Temperley–Lieb algebra, and the Brauer algebra.

### 5.3 Comment on the extended TL category

In this paper, we propose the extended TL diagrammatical approach to quantum information and computation involving maximally entangled states and local unitary transformations. Its diagrammatical configurations consist of cups, caps, solid points, empty circles, etc., as an extension of Kauffman diagrams or Brauer diagrams. An extension of the TL algebra with local unitary transformations is called the extended TL algebra, and the collection of all extended TL algebras is called the extended TL category which contains abundant mathematical objects such as braids, permutation, the TL algebra, the Brauer algebra, and others (see the next section). This category is a mathematical foundation of our diagrammatical approach.

Furthermore, interested readers are invited to refer our previous work [34], in which unitary Hermitian ribbon categories are suggested as a mathematical description of quantum information and physics as well as the extended TL diagrammatical approach is viewed as a diagrammatical representation for tensor categories.

### 6 Extended TL diagrammatical approach (IV): entanglement swapping and universal quantum computation

We study the entanglement swapping, universal quantum computing, and multipartite entanglements, in the extended TL diagrammatical approach which is a diagrammatical representation for the extended TL category.

#### 6.1 Entanglement swapping

Entanglement swapping [44] is an experimental technique realizing quantum entanglement between two independent systems as a consequence of quantum measurement instead of physical interaction. Let us make an example for its theoretical
interpretation in terms of a projector representing quantum measurement. Alice has a bipartite entangled state $|\Omega_l\rangle^A_{ab}$ for particles $a,b$, and Bob has $|\Omega_m\rangle^B_{cd}$ for particles $c,d$. They are independently created and do not share common history. Alice applies quantum measurement denoted by $I_d \otimes |\Omega_n\rangle \otimes I_d$ to the product state of $|\Omega_l\rangle^A_{ab}$ and $|\Omega_m\rangle^B_{cd}$ so that the output called the entanglement swapped state $|\Omega_{lnm}\rangle^A_{ad}$ is a bipartite entangled state shared by Alice and Bob for particles $a,d$,

$$
(I_d \otimes |\Omega_n\rangle \otimes I_d)(|\Omega_l\rangle^A_{ab} \otimes |\Omega_m\rangle^B_{cd})
$$

$$
= \frac{1}{d}(I_d \otimes |\Omega_n\rangle \otimes I_d)\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}(U_i U_n^* U_m|e_i\rangle^A_{a} \otimes I_d \otimes |e_i\rangle^B_{d})
$$

$$
\equiv \frac{1}{d}(I_d \otimes |\Omega_n\rangle \otimes I_d) |\Omega_{lnm}\rangle^A_{ad}.
$$

(93)

In other words, the entanglement swapping reduces a four-particle state $|\Omega_l\rangle^A_{ab} \otimes |\Omega_m\rangle^B_{cd}$ to a bipartite entangled state $|\Omega_{lnm}\rangle^A_{ad}$ using entangling measurement. As a remark, the entanglement swapped state $|\Omega_{lnm}\rangle^A_{ad}$ plays a role in the comparison of quantum mechanics with classical physics, because it is a quantum state but produced without any classical physical interactions.

Read the entanglement swapping equation (93) from the left to the right and draw a diagram from the top to the bottom according to the extended TL diagrammatical rules, i.e., the left term of Figure 22. It is a configuration for an element $e_2^1 e_3^1$ in the extended TL category, i.e.,

$$
\hat{e}_2 e_1^1 e_3^1 = (I_d \otimes \omega_n \otimes I_d)(\omega_l \otimes I_d \otimes I_d)(I_d \otimes I_d \otimes \omega_m).
$$

(94)

which changes the entangled state $|\Omega_l\rangle^A_{ab}$ in Alice’s system to the entangled state $|\Omega_{lnm}\rangle^A_{ad}$ in Alice and Bob’s composite system, see the right term of Figure 22.
Furthermore, the entanglement swapping is also called the teleportation using a cup state. Alice measures the Bell state $|\Omega\rangle_{AB}$ with a projector $|\psi\rangle_A \langle \psi|$ so that she transfers her quantum state $|\psi\rangle_A$ to Bob in the way

$$|\psi\rangle_A \langle \Omega|_{AB} = \frac{1}{\sqrt{d}} |\psi\rangle_A \langle \psi|_B,$$

which has a diagrammatical representation, the left term of Figure 23 with $\langle e_i | \rangle_T = |e_i\rangle$, similar to the two-way teleportation in the crossed measurement $^{38,50}$.

The characteristic equation for the tight entanglement swapping is derived by a similar procedure of obtaining characteristic equations for tight teleportation and dense coding schemes $^{30}$,

$$\sum_{n=1}^{d^2} tr((\rho \otimes \omega_n \otimes T_n(O))(\omega \otimes \omega)) = \frac{1}{d} tr(\rho O^T)$$

(96)

where the density operator $\rho$ for particle $a$, observable $O$ for particle $d$, and quantum channel $T_n(O)$ for particle $d$ are respectively given by

$$\rho = |\phi_1\rangle \langle \phi_2|, \quad O = |\psi_1\rangle \langle \psi_2|, \quad T_n(O) = U_n^\dagger O U_n$$

(97)

and the transpose of the density operator, $O^T$ is defined in the way

$$O^T = \sum_{i,j=0}^{d-1} \psi_{1i} \psi_{2j}^* (|e_i\rangle \langle e_j|)^T = \sum_{i,j=0}^{d-1} \psi_{1i} \psi_{2j}^* |e_i\rangle \langle e_j|, \quad (|e_i\rangle)^T = |e_i\rangle.$$

(98)
The tight entanglement swapping equation (96) is easily proved in the extended TL diagrammatical approach, see the right term of Figure 23 which is a closure of the left term of Figure 22. The diagrammatical trick by Figure 14 is exploited to derive the same configuration as the first term of Figure 13. It can be also verified in an algebraic way: the term \( n \) given by

\[
\text{term}_n = \text{tr}(\langle \phi_1 \rangle \langle \phi_2 \rangle \otimes | \Omega_n \rangle \langle \Omega_n | \otimes U_n^\dagger \psi_1 \langle \psi_2 | U_n^\dagger | \psi_1 \rangle = 1_d \sum_{i,j=0}^{d-1} \psi_i \psi_j^* \langle \phi_1 | \phi_2 \rangle | \phi_2 \rangle | \phi_1 \rangle = \text{term}_n,
\]

is found to be

\[
\frac{1}{d^3} \text{tr}(\rho O^T) = \frac{1}{d^3} \sum_{i,j=0}^{d-1} \psi_i \psi_j^* \langle \phi_1 | \phi_2 \rangle | \phi_2 \rangle | \phi_1 \rangle = \text{term}_n,
\]

and the characteristic equation (96) is proved due to there are \( d^2 \) \( \text{term}_n \) with each \( \text{term}_n \) independent of \( n \).

Therefore, the extended TL diagrammatical approach is not only to describe a quantum information protocol but also to assign it a characteristic equation.

### 6.2 Universal quantum computing

Quantum teleportation has been considered as a universal quantum computational primitive [45]. Under such a proposal, there are both theoretical observations and experimental motivations. The teleported state permits the action of local unitary transformations, and so quantum teleportation realizes single-qubit gates as local unitary transformations and two-qubit gates as linear combinations of products of single-qubit gates. Besides single-qubit transformations and Bell measurements can be performed in labs. Additionally, we have fault-tolerant quantum computation [51, 52], as single-qubit transformations are performed fault-tolerantly.

A fault-tolerant gate \( R \), an element of the Clifford group [51, 52], enters the teleportation via entangling measurement and then is transported in the form of

\[
B = \frac{1}{\sqrt{2}} (\mathbb{I} \otimes \mathbb{I} + i \sigma_1 \otimes \sigma_2)
\]
its conjugation $R^\dagger$, see the left term of Figure 24. A unitary braid gate \( \Omega \) has a form given by

$$B = \frac{1}{\sqrt{2}}(\mathbb{I}_2 \otimes \mathbb{I}_2 + i\sigma_1 \otimes \sigma_2)$$

which is performed in the way shown in the extended TL diagrammatical approach, see the right term of Figure 24. It has two diagrammatical terms and each one consists of two teleportation processes for sending a two-qubit. The swap gate $P$, denoted by \( \Omega \), is an element of the extended TL category, i.e., Figure 25 which has four diagrammatical terms and each one represents an algebraic term in \( \Omega \). The CNOT gate, a linear combination of products of Pauli matrices,

$$CNOT = (|0\rangle\langle 0| \otimes \mathbb{I}_2 + |1\rangle\langle 1| \otimes \sigma_1) = \frac{1}{2}(\mathbb{I}_2 + \sigma_3) \otimes \mathbb{I}_2 + \frac{1}{2}(\mathbb{I}_2 - \sigma_3) \otimes \sigma_1,$$

satisfying basic properties of the CNOT gate,

$$CNOT|00\rangle = |00\rangle, \quad CNOT|01\rangle = |01\rangle, \quad CNOT|10\rangle = |11\rangle, \quad CNOT|11\rangle = |10\rangle,$$

is an element of the extended TL category, Figure 26. Note that symbols $\Omega$ labeling a cup and cap are omitted in Figures 24-26 for convenience.

Knot polynomial in terms of a unitary braid gate \( \Omega \) in the extended TL category can be computed using quantum simulation of knot on quantum computer, which is different from an approximate quantum algorithm \[53\] for computing the Jones polynomial as well as topological quantum computing \[54, 55\] involving unitary braid representations as quantum gates acting on quasi-particles like anyons. Furthermore, virtual knots can be simulated via a quantum program with unitary braid gates and swap gates. Moreover, exactly solvable two dimensional quantum field theories or statistical models \[15, 16\] can be simulated on quantum computer, since unitary solutions of the Yang–Baxter equation with spectral parameters \[17, 18\] can be performed in the extended TL category.

As a remark, the extended TL category is a low-dimensional “topological” model for universal quantum computation, and “topological” or topological-like features \[54\] are expected to be helpful to look for new quantum algorithms.
6.3 Comment on multipartite entanglements

Bell measurements and local unitary transformations are crucial elements for the application of the extended TL diagrammatical rules. Hence, multipartite maximally entangled states like the GHZ state or the state $|\chi\rangle$ can be treated in the extended TL category if they have a form in terms of Bell measurements and local unitary transformations. For example, the GHZ state $|GHZ\rangle$ is a linear combination of local unitary transformations of Bell state,

$|GHZ\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$= \frac{1}{2}(|0\rangle + |1\rangle) \otimes |\phi^+\rangle + \frac{1}{2}(|0\rangle - |1\rangle) \otimes |\phi^-\rangle$

$= \frac{1}{2}(I_8 + \sigma_3 \otimes I_2 \otimes \sigma_3)(|\alpha\rangle \otimes |\phi^+\rangle)$

(103)

where $|\alpha\rangle = |0\rangle + |1\rangle$. Similarly, the four-particle state $|\chi\rangle$ [45], in the construction of the CNOT gate using quantum teleportation, has a form

$|\chi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|00\rangle + \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)|11\rangle$

$= \frac{1}{\sqrt{2}}|\phi^+\rangle(|\phi^+\rangle + |\phi^-\rangle) + \frac{1}{\sqrt{2}}|\psi^+\rangle(|\phi^+\rangle - |\phi^-\rangle)$

$= \frac{1}{\sqrt{2}}(I_{16} + I_8 \otimes \sigma_3 + I_2 \otimes \sigma_1 \otimes I_4 - I_2 \otimes \sigma_1 \otimes I_2 \otimes \sigma_3)|\phi^+\rangle|\phi^+\rangle$.

(104)

We will explore quantum teleportation using multipartite maximally entanglement states in the extended TL category in our further research. As a remark, the extended TL diagrammatical approach rules can be applied to topics like Bell inequalities, quantum cryptography and so on, in which Bell measurements and local unitary transformations play fundamental roles.
7 Extended TL diagrammatical approach (V): quantum information flow

We describe the quantum information flow in the extended TL category, and compare this description with another two known approaches: the teleportation topology \[13, 29\] and strongly compact closed categories \[46\].

7.1 Teleportation topology

Teleportation topology \[13, 29\] explains quantum teleportation as a kind of topological amplitude satisfying the topological condition. There are one-to-one correspondences between quantum amplitude and topological amplitude. The state preparation describes a creation of a two-particle quantum state from vacuum with a diagrammatical representation denoted by a cup state \(|\text{Cup}\rangle\), and quantum measurement denotes an annihilation of a two-particle quantum state with a diagrammatical representation by a cap state \(\langle\text{Cap}|\). See Figure 27. The cup and cap states are associated with the matrices \(M\) and \(N\) in the way

\[
|\text{Cup}\rangle = \sum_{i,j=0}^{d-1} M_{ij} |e_i \otimes e_j\rangle, \quad \langle\text{Cap}| = \sum_{i,j=0}^{d-1} \langle e_i \otimes e_j |N_{ij}
\]

which satisfy the topological condition, i.e., the concatenation of a cup and a cap is a straight line denoted by the identity matrix \(N_{ij} M_{jk} = \delta_{ki}\).

In the extended TL diagrammatical approach, the concatenation of a cup and a cap is formulated by the transfer operator which is not identity required by the topological condition, see Figure 15. Besides, the cup and cap states are normalized maximally entangled states given by

\[
|\text{Cup}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i \otimes e_i\rangle, \quad \langle\text{Cap}| = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \langle e_i \otimes e_i|
\]

which assigns a normalization factor \(\frac{1}{\sqrt{d}}\) to a straight line from the concatenation of a cup and a cap. Furthermore, a projector formed by a top cup and bottom cap
forms a representation of the TL algebra. Moreover, quantum teleportation is the quantum information flow denoted by the topological condition in the teleportation topology, whereas it is described in Figure 16 with the quantum information flow as its part.

### 7.2 Quantum information flow in terms of maps

Quantum teleportation is an information protocol transporting a unknown quantum state from Charlie to Bob with the help of Alice. To describe it in a unified mathematical formalism, we have to integrate standard quantum mechanics with classical features, since the outcomes of measurements are sent to Bob from Alice via classical channels and then Bob performs a required unitary operation. The categorical approach proposed by Abramsky and Coecke describes the quantum information flow by strongly compact closed categories, see [46, 56, 57].

To sketch the quantum information flow in the form of a composition of a series of maps which are central topics of the category theory, we study an example in detail. Set five Hilbert spaces $\mathcal{H}_i$ and its dual $\mathcal{H}_i^*$, $i = 1, \cdots, 5$, and define eight bipartite projectors $P_\alpha = |\Phi_\alpha\rangle\langle\Phi_\alpha|$, $\alpha = 1, \cdots, 8$ in which the bipartite vector $|\Phi_\alpha\rangle$ is an element of $\mathcal{H}_i \otimes \mathcal{H}_{i+1}$, $i = 1, \cdots, 4$. In the left diagram of Figure 28, every box represents a bipartite projector $P_\alpha$, and the vector $|\phi\rangle_C \in \mathcal{H}_1$ that Charlie owns is transported to Bob who obtains the vector $|\phi\rangle_B \in \mathcal{H}_5$ through the quantum information flow. The projectors $P_1$ and $P_2$ pick up an incoming vector in

$$|\phi\rangle_B = f_8^* \circ f_7 \circ f_6^* \circ f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1 |\phi\rangle_C$$
\(\mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5\), and the projectors \(P_7\) and \(P_8\) determine an outgoing vector in \(\mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5\). The right diagram in Figure 28 shows the quantum information flow from \(|\phi\rangle_C\) to \(|\phi\rangle_B\). It is drawn according to permitted and forbidden rules [46]; the flow is forbidden to go through a box from the one side to the other side, and is forbidden to be reflected at the incoming point, and has to change its direction from an incoming flow to an outgoing flow as it passes through a box. Obviously, if these rules are not imposed there will be many possible paths from \(|\phi\rangle_C\) to \(|\phi\rangle_B\).

Let us set up one-to-one correspondence between a bipartite vector and a map. There are a \(d_1\)-dimension Hilbert space \(\mathcal{H}_1\) and a \(d_2\)-dimension Hilbert space \(\mathcal{H}_2\). The bipartite vector \(|\Phi\rangle\) has a form in terms of the product basis \(|e_1^{(1)}\rangle \otimes |e_2^{(2)}\rangle\) in \(\mathcal{H}_1 \otimes \mathcal{H}_2\),

\[
|\Phi\rangle = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} m_{ij} |e_1^{(1)}\rangle \otimes |e_2^{(2)}\rangle, \quad \langle \Phi \rangle = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} m_{ij}^* \langle e_1^{(1)} | \otimes \langle e_2^{(2)} | \quad (107)
\]

where \(|\Phi\rangle\) denotes the dual vector of \(|\Phi\rangle\) in the dual product space \(\mathcal{H}_1^* \otimes \mathcal{H}_2^*\) with the basis \(|e_1^{(1)}\rangle \otimes |e_2^{(2)}\rangle\). Besides, once the product basis is fixed, bipartite vectors \(|\Phi\rangle\) or \(|\Phi\rangle\) are determined by a \(d_1 \times d_2\) matrix \(M_{d_1 \times d_2} = (m_{ij})\). Defining two types of maps \(f\) and \(f^*\) in the way

\[
f : \mathcal{H}_1 \rightarrow \mathcal{H}_2, \quad f(\cdot) = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} m_{ij} |e_1^{(1)}\rangle \langle e_2^{(2)}|,
\]

\[
f^* : \mathcal{H}_1^* \rightarrow \mathcal{H}_2^*, \quad f^*(\cdot) = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} m_{ij}^* \langle e_1^{(1)} | \langle e_2^{(2)} |, \quad (108)
\]

we have the bijective correspondences,

\[
|\Phi\rangle \approx \langle \Phi \rangle \approx M \approx f \approx f^* \quad (109)
\]

which suggests that the bipartite project box in Figure 28 can be labeled by the map \(f\) or \(f^*\) or matrix \(M\).

Now we work out the formalism of the quantum information flow in the categorical approach. Consider a projector \(P_7 = |\Phi_7\rangle \langle \Phi_7|\) and introduce a map \(f_1\) to represent the action of \(\langle \Phi_7|\), a half of \(P_7\),

\[
f_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2^*, \quad f_1(|\phi_C\rangle) = \langle \Phi_7|\phi_C\rangle. \quad (110)
\]

Similarly, the remaining seven boxes are respectively labeled by the maps \(f_2^*, f_3, f_4^*, f_5, f_6, f_7\) and \(f_8^*\), defined by

\[
f_2^* : \mathcal{H}_2^* \rightarrow \mathcal{H}_3, \quad f_2^* \circ f_1(|\phi_C\rangle) = \langle \Phi_7|\phi_C \otimes \Phi_6\rangle,
\]

\[
f_3 : \mathcal{H}_3 \rightarrow \mathcal{H}_4^*, \quad f_3 \circ f_2^* \circ f_1(|\phi_C\rangle) = \langle \Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4\rangle,
\]

\[
f_4^* : \mathcal{H}_4^* \rightarrow \mathcal{H}_3, \quad f_4^* \circ f_3 \circ f_2^* \circ f_1(|\phi_C\rangle) = \langle \Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4\rangle,
\]

\[
f_5 : \mathcal{H}_3 \rightarrow \mathcal{H}_2^*, \quad f_5 \circ f_4^* \circ f_3 \circ f_2^* \circ f_1(|\phi_C\rangle) = \langle \Phi_5 \otimes \Phi_7 \otimes \Phi_8|\phi_C \otimes \Phi_6 \otimes \Phi_4\rangle,
\]

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In the category theory, every step (or piece) is denoted by a specific map. For instance, consider the sequence of maps:

\[ f_6^* : \mathcal{H}_2^* \to \mathcal{H}_3, \quad f_7 : \mathcal{H}_3 \to \mathcal{H}_4^*, \quad f_8^* : \mathcal{H}_4^* \to \mathcal{H}_5, \]

where the composition of these maps, denoted by \( f_6^* \circ f_5 \circ f_3 \circ f_2 \circ f_1(\phi_C) \), is crucial for encoding the quantum information flow. This composition follows the rules of the teleportation topology [13, 29] to ensure that the quantum information is transferred correctly.

The quantum information flow in terms of a composition of maps naturally leads to its description in the category theory. Here we show one-to-one correspondences between the quantum information flow and strongly compact categories. To transport Charlie’s unknown quantum state \( |\psi\rangle_C \) to Bob, the teleportation has to complete all the operations: the preparation of \( |\psi\rangle_C \); the creation of \( |\Omega\rangle_{AB} \) in Alice and Bob’s system; the Bell-based measurement \( C_A(\Omega_n) \) in Charlie and Alice’s system; classical communications between Alice and Bob; Bob’s local unitary corrections. These steps divide the quantum information flow into six pieces, and they are shown in the left diagrammatical term of Figure 29 where the third piece represents a process bringing Alice and Charlie’s systems together for entangling measurement. In the category theory, every step (or piece) is denoted by a specific map.
map satisfying the axioms of strongly compact closed categories. A crucial point is to recognize a bijective correspondence between a Bell state and a map from the dual Hilbert space $\mathcal{H}^*$ to the Hilbert space $\mathcal{H}$,

$$\frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i\rangle_A \otimes |e_i\rangle_B \approx \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} A(e_i) \otimes |e_i\rangle_B, \quad \mathcal{H}_A \otimes \mathcal{H}_B \approx \mathcal{H}_A^* \otimes \mathcal{H}_B$$ (114)

so that the quantum information flow have a physical realization in strongly compact closed categories. See the right term of Figure 29: the symbol $\mathbb{C}$ denotes the complex field and

$$|\psi\rangle_C \approx |\psi\rangle_C \otimes \mathbb{C}, \quad |\psi\rangle_B \approx \mathbb{C} \otimes |\psi\rangle_B$$ (115)

which suggests a bipartite state is created from vacuum denoted by a complex number and is annihilated it into the vacuum.

### 7.3 Quantum information flow in the extended TL category

We revisit the example in Figure 28 and redraw a diagram, Figure 30, according to the extended TL diagrammatical rules. Every projector consists of a top cup and a bottom cap. Solid points $1, \ldots, 8$ on the left branches of cups respectively denote
local unitary transformations \(U_1, \ldots, U_8\), and small circles on the left branches of caps denote their adjoint operators \(U_1^\dagger, \ldots, U_8^\dagger\), respectively. The quantum information flow from \(|\phi\rangle_C\) to \(|\phi\rangle_B\) is determined by the transfer operator,

\[
|\phi\rangle_B = \frac{1}{d^6} \text{tr}(U_2^\dagger U_5) \text{tr}(U_4^\dagger U_7)(U_8^T U_4^T U_6^T U_5^T U_4 U_3^T U_2^T U_1^T) |\phi\rangle_C
\]

with the normalization factor \(\frac{1}{d^6}\) contributed from six vanishing cups and six vanishing caps as well as two traces from two closed circles.

Five remarks are made as Figure 28 is compared with Figure 30. 1) In the categorical approach, only the half of a projector is exploited to use the bijective correspondence between a bipartite vector and a map to represent the quantum information flow in terms of maps. In the extended TL diagrammatical approach, however, a projector is denoted as the combination of a top cup and a bottom cap instead of a single cup (or cap). Hence the quantum information flow from \(|\phi\rangle_C\) to \(|\phi\rangle_B\) in Figure 30 has a normalization factor contributed from closed circles which is crucial for the quantum formation flow. For examples, setting eight local unitary operators \(U_i\) to be identity leads to \(|\phi\rangle_B = \frac{1}{d^6} |\phi\rangle_C\), and assuming \(U_2\) and \(U_5\) (or \(U_4\) and \(U_7\)) orthogonal to each other causes a zero vector to be sent to Bob, \(|\phi\rangle_B = 0\), no flow! 2) The quantum information flow is only one part of the entire diagram in the extended TL diagrammatical approach. Hence the acausality problem of the quantum information flow in the categorical approach is not reasonable since the whole process is not considered from the global view. 3) In the extended TL diagrammatical approach, the bijective correspondence between a local unitary transformation and a bipartite vector is considered, which is different from the choice in the categorical approach. For example, we have \(|\psi(U)\rangle = (U \otimes \mathbb{I}_d) |\Omega\rangle\), \(|\psi(U)\rangle \approx U \approx |\psi(U)\rangle \langle \psi(U)|\),

\[
|\phi^+\rangle \approx \mathbb{I}_2, \quad |\phi^-\rangle \approx \sigma_3, \quad |\psi^+\rangle \approx \sigma_1, \quad |\psi^-\rangle \approx i \sigma_2.
\]

As a projector is labeled by a local unitary transformation, the equation (116) is called the quantum information flow in terms of local unitary transformations (instead of maps). 4) The quantum information flow in the categorical approach is created in view of additional permitted and forbidden rules [46], whereas it is derived in a natural way without imposed rules in the extended TL diagrammatical approach. 5) \(\mathcal{H}^* \otimes \mathcal{H}\) is imposed by the axioms of strongly compact closed categories, see Figure 29, but is not required by quantum teleportation, the quantum information flow, and the extended TL category.

8 Concluding remarks

In this paper, we study algebraic structures and low dimensional topology underlying quantum information and computation involving maximally entangled states and local unitary transformations. We describe quantum teleportation from the
points of the symmetric group, the braid group, the virtual braid group, the TL algebra and the Brauer algebra, and propose the teleportation swapping, the braid teleportation and the virtual braid teleportation. Especially, quantum teleportation can be performed using the teleportation swapping and Bell measurements, which is a description of quantum teleportation via the Brauer algebra. Besides, we propose the extended TL diagrammatical approach to study a series of topics: the transfer operator with the acausality problem; measurement-based quantum teleportation; tight teleportation and dense coding schemes; the diagrammatical representation of the TL algebra; quantum teleportation and the Brauer algebra; entanglement swapping; universal quantum computing; multipartite entanglements; and quantum information flow. All these examples show that the extended TL category is a mathematical framework describing quantum information and computation using maximally entangled states and local unitary transformations. For example, various descriptions to quantum teleportation can be unified in the extended TL diagrammatical approach.

As a further comment concluding this paper, we remark our previous work \[34\] on categorical foundation of quantum physics and information. We suggest *unitary Hermitian ribbon categories* as a natural extension of strongly compact closed categories with the extended TL category as its special example. All known diagrammatical approaches \[13 \ 29 \ 46 \ 56 \ 67 \ 68 \ 82 \ 83 \ 59 \ 32 \ 33 \ 59 \ 34\] to quantum information and computation including the extended TL diagrammatical approach can be viewed as different versions of the diagrammatical representation of tensor categories. Obviously, our future research is focused to look for new quantum information protocols or quantum algorithms with the help of mathematical structures presented in this paper, especially the extended TL category.

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