Splittings of mapping tori of free group automorphisms

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Abstract

We present necessary and sufficient conditions for the existence of a splitting over \(\mathbb{Z}\) of the mapping torus \(M_\phi = F \rtimes_\phi \mathbb{Z}\) of a free group automorphism \(\phi\).

Introduction

Let \(\phi\) be an automorphism of a finitely generated free group \(F\) of rank at least two, and let \(M_\phi = F \rtimes_\phi \mathbb{Z}\) be the mapping torus of \(\phi\). If \(F\) is generated by \(x_1, \ldots, x_b\), then \(M_\phi\) is presented by

\[
\langle x_1, \ldots, x_b, t \mid tx_it^{-1} = \phi(x_i) \rangle.
\]

An automorphism \(\phi\) is called hyperbolic if \(M_\phi\) is a word-hyperbolic group, and it is called atoroidal if no positive power of \(\phi\) preserves the conjugacy class of a nontrivial element of \(F\). The following result gives us a nice way of detecting hyperbolicity.

\textbf{Theorem 0.1 ([Bri00])}. An automorphism \(\phi : F \to F\) is hyperbolic if and only if it is atoroidal.

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I. Kapovich \cite{Kap00} has proved a similar result for mapping tori of certain injective endomorphisms of free groups.

Given a word-hyperbolic mapping torus $M_\phi$, it is natural to ask whether it has a nontrivial JSJ decomposition \cite{Sel97} (see also \cite{SS}). This question appeared as Problem 6.7 in \cite{Kap00}. In particular, it is natural to ask whether $M_\phi$ splits over $\mathbb{Z}$, i.e., whether $M_\phi$ can be expressed as an HNN extension or a nontrivial free product with amalgamation over $\mathbb{Z}$.

The main result of this paper determines exactly when $M_\phi$ splits in this fashion (Theorem 1.3). This answers a question of Lee Mosher. The actual result is rather technical, but its main interest lies in the fact that splittings over $\mathbb{Z}$ can arise in various different and unexpected ways (contrast this with mapping tori of automorphisms of surface groups, which never split over $\mathbb{Z}$).

Note that $M_\phi$ does not split as a nontrivial free product, i.e., it is one-ended. Moreover, a result of M. Kapovich and B. Kleiner \cite{KK00}, Theorem 14 implies that the Gromov boundary $\partial_\infty M_\phi$ is the Menger curve if $M_\phi$ is hyperbolic and does not split over $\mathbb{Z}$.

There exists an algorithm for finding the JSJ decomposition of a one-ended, torsion-free hyperbolic group \cite{Sel95}. The results of this paper are nonconstructive, but it is conceivable that there exists a simpler (and faster) algorithm for finding splittings of mapping tori of free group automorphisms. Another open problem is the extension of the results of this paper to the case of mapping tori of injective endomorphisms of free groups.

Section 1 contains the statement and proof of the main result. In Section 2, we give some examples of hyperbolic automorphisms whose mapping tori split over $\mathbb{Z}$. One interesting consequence of these examples is that hyperbolic automorphisms may have train track representatives \cite{BH92} with strata of polynomial growth.

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1 Splittings of $M_\phi$ over $\mathbb{Z}$

Unless stated otherwise, we assume throughout this section that $M_\phi$ splits over $\mathbb{Z}$, either as an HNN-extension or as a free product with amalgamation.
Hence, $M_\phi$ acts (on the left) on a tree $T$ with one orbit of edges and cyclic edge stabilizers [Ser77] (see [Bau93] or [SW79] for a brief review of the facts from Bass-Serre theory that we use here). Let $t$ denote the stable letter of $M_\phi$. For a vertex $V$ of $T$, let $F_V = F \cap \text{Stab } V$. Given an edge $E$ of $T$, we write $\tau(E)$ for the terminal vertex of $E$ and $\iota(E)$ for the initial vertex.

Let $\Gamma = F \setminus T$ be the quotient of $T$ under the action of $F$. $\Gamma$ expresses $F$ as the fundamental group of a finite graph of groups with trivial edge stabilizers. In order to see this, suppose that $\Gamma$ is not finite, i.e., $\Gamma$ expresses $F$ as the fundamental group of an infinite graph of groups with cyclic edge groups. As $F$ is normal in $M_\phi$, there is an induced action of the stable letter $t$ on $\Gamma$, and the quotient of $\Gamma$ under the action of $\langle t \rangle$ consists of one edge. This forces $\Gamma$ to be homeomorphic to the real line, all vertex groups are conjugates of each other, and all edge groups are conjugates of each other. Hence, if the vertex groups are strictly larger than the edge groups, then $F$ is infinitely generated, and if all the vertex groups are the same as the edge groups, then $F$ is cyclic. Since $F$ is finitely generated and has rank at least two, both cases are impossible.

A careful analysis of the action of the stable letter $t$ on $\Gamma$ will give us a description of all possible splittings of mapping tori over $\mathbb{Z}$. The computational techniques are similar in all cases. In order to avoid redundancy, we spell out the details of the computations in the first few cases and restrict the exposition to an outline once all the ideas have appeared.

### 1.1 HNN extension

We assume that $M_\phi$ splits as an HNN extension over $\mathbb{Z}$, i.e., $M_\phi = G \ast_{\mathbb{Z}}$. In particular, $t$ acts on $\Gamma$ with one orbit of vertices and one orbit of edges. We distinguish three subcases depending on the translation length of the action of $t$ on $T$.

#### 1.1.1 Elliptic $t$-action

We consider quotients $\Gamma$ with one vertex and one orbit of $k$ edges (Figure [ ]). In this case, $t$ acts on $T$ as an elliptic isometry, and it fixes exactly one vertex $V$ of $T$. We have

$$F = F_V \ast \langle a_0, \ldots, a_{k-1} \rangle$$
such that

\[
\phi(F_V) = F_V \\
\phi(a_i) = \begin{cases} 
  a_{i+1} & \text{if } 0 \leq i < k - 1; \\
  wa_0v & \text{if } i = k - 1, \text{ for some } v, w \in F_V.
\end{cases}
\]

Proof. Clearly, if \( x \in F_V \), then \( V = tV = \phi(x)tV = \phi(x)V \), hence \( \phi(x) \in F_V \).

Let \( E \) be some edge with \( t(E) = V \) (\( V \) is the vertex fixed by \( t \)). Choose \( a_0, \ldots, a_{k-1} \in F \) such that \( a_i \tau(t^iE) = V \), and choose \( v \in F_V \) such that \( vt^kE = E \) (Figure 1). With these definitions, we have

\[ F = F_V \ast \langle a_0, \ldots, a_{k-1} \rangle. \]

We have \( ta_i \tau(t^iE) = tV \), which implies \( \phi(a_i) \tau(t^{i+1}E) = V \). Hence, for \( i < k - 1 \), there is no loss in assuming that \( \phi(a_i) = a_{i+1} \) (otherwise, we could simply modify our choice of \( a_{i+1} \)). Finally, we have \( \phi(a_{k-1}) \tau(t^kE) = \phi(a_{k-1})v^{-1} \tau(E) = V \), which implies \( a_0^{-1}V = v\phi(a_{k-1})V \), which in turn implies \( \phi(a_{k-1}) = wa_0v \) for some \( w \in F_V \).

Conversely, any automorphism \( \phi \) of the form listed above will give rise to a mapping torus \( M_{\phi} \) that splits over \( \mathbb{Z} \). To show this, we will give a sequence of Tietze transformations that exhibits the splitting.
1.1.2 HNN extension: Hyperbolic $t$-action with translation distance 1

We consider the case where $\Gamma$ has one orbit of $m$ vertices and one orbit of $km$ edges (Figure 2), and we assume that $t$ acts on $T$ as a hyperbolic isometry with translation distance 1.

Then there exists some vertex $V \in T$ and some edge $E \in T$ such that $\iota(E) = V$ and $\tau(E) = tV$. Let $v(i)$ be the remainder of $i$ under division by $m$. We have

$$F = \left( \prod_{i=0}^{m-1} F_{v(V)} \right) * \langle a_{m-1}, \ldots, a_{km-1} \rangle$$
such that
\[ \phi(F_t V) = \begin{cases} 
F_{i+1} V & \text{if } 0 \leq i < m - 1; \\
 a_{m-1}^{-1} F_V a_{m-1} & \text{if } i = m - 1;
\end{cases} \]
and
\[ \phi(a_i) = \begin{cases} 
a_{i+1} & \text{if } v(i) < m - 2; \\
 a_{m-1}^{-1} a_{i+1} & \text{if } v(i) = m - 2; \\
 a_{i+1} a_{m-1} & \text{if } v(i) = m - 1, \; i < km - 1; \\
 v a_{m-1} & \text{if } i = km - 1, \text{for some } v \in F_V.
\end{cases} \]

Proof. We first construct a subtree $S \subset T$ with $km$ edges that projects to $\Gamma$. Choose $a \in F$ such that $a t^{m} V = V$. Let $E_0 = E$, and define $E_{i+1} = t E_i$ if $v(i) \neq m - 1$, and $E_{i+1} = a t E_i$ if $v(i) = m - 1$. Let $S$ be the union of the edges $E_0, \ldots, E_{km - 1}$ (Figure 3, note that the union $S'$ of $E_0, \ldots, E_{m-2}$ projects to a spanning tree of $\Gamma$).

Now, choose $a_{m-1}, \ldots, a_{km-1} \in F$ such that $a_i \tau(E_i) = t^{v(i+1)} V$ (we choose $a_{m-1} = a$). With these definitions, we have
\[ F = \left( \prod_{i=0}^{m-1} F_{t_i V} \right) * \langle a_{m-1}, \ldots, a_{km-1} \rangle, \]
and our claim concerning the images of vertex stabilizers follows immediately. We still need to compute the images of the generators $a_i$. 

Figure 3: A lift of $\Gamma$ (case 1.1.2).
We have \( a_i \tau(E_i) = t^{v(i+1)}V \), which implies \( \phi(a_i) \tau(tE_i) = t^{v(i+1)+1}V \). If \( v(i) < m - 2 \), we have \( tE_i = E_{i+1} \) and \( v(i+1) + 1 = v(i+2) \), so there is no loss in assuming that \( \phi(a_i) = a_{i+1} \).

If \( v(i) = m - 2 \), then \( t^{v(i+1)+1}V = t^mV = a^{-1}V \) and \( tE_i = E_{i+1} \), which implies \( \phi(a_i) \tau(E_{i+1}) = a^{-1}V \), so we can let \( \phi(a_i) = a^{-1}a_{i+1} \).

If \( v(i) = m - 1 \) and \( i \neq km - 1 \), then \( tE_i = a^{-1}E_{i+1} \) and \( t^{v(i+1)+1}V = t^{v(i+2)}V \), so we have \( \phi(a_i) t \tau(E_i) = \phi(a_i)(a^{-1}E_{i+1}) = t^{v(i+2)}V \), hence \( \phi(a_i)a^{-1} = a_{i+1} \). This implies \( \phi(a_i) = a_{i+1}a \).

Finally, there exists some \( v \in F \) such that \( vat^{km-1} = E_0 \), and an argument similar to the previous one shows that \( \phi(a_{km-1}) = va \).

\[ \square \]

Conversely, any automorphism \( \phi \) of the form listed above will give rise to a mapping torus \( M_\phi \) that splits over \( \mathbb{Z} \). To show this, we will first find a new set of generators that will simplify this verification. Let \( b_i = a_{m+i-1}a_{m+i-2} \cdots a_{m-1} \) for \( i = 0, \ldots, (k-1)m \). A simple induction shows that

\[
\phi(b_i) = \begin{cases} 
  b_{i+1} & \text{if } v(i) < m - 1 \text{ and } i \neq (k-1)m; \\
  b_0^{-1}b_{i+1} & \text{if } v(i) = m - 1; \\
  vb_{(k-1)m} & \text{if } i = (k-1)m.
\end{cases}
\]

Moreover, we have

\[
(a_{m-1}t^{km})^k = (b_0t^m)^k = b_0^{\phi^m(b_0)}\phi^{2m}(b_0) \cdots \phi^{(k-1)m}(b_0)t^{km} = b_0b_0^{-1}b_mb_2^{-1}b_2^{-1} \cdots b_{(k-2)m}b_{(k-2)m}^{-1}b_{(k-1)m}t^{km} = b_{(k-1)m}t^{km}.
\]

After these preliminary computations, we can list a sequence of Tietze moves that exhibits the splitting over \( M_\phi \) of \( \mathbb{Z} \).

\[
M_\phi = \langle F_V, F_{IV}, \ldots, F_{m-1}V, b_0, \ldots, b_{(k-1)m}, t \mid \ldots \rangle \\
\cong \langle F_V, b_0, s, t \mid s = b_0t^m, sFVs^{-1} = F_V, \ldots \rangle \\
\cong \langle F_V, s, t \mid sFVs^{-1} = F_V, ts^{k-1} = vs^k \rangle \\
\cong \langle F_V, s \mid sFVs^{-1} = F_V \rangle \ast_{(s \sim ts^k)}.
\]

Note, in particular, that the last expression allows us to see the splitting geometrically.
1.1.3 HNN extension: Hyperbolic $t$-action with translation distance greater than 1

The main idea of the previous argument was to find a lift $S \subset T$ of the graph $\Gamma$ with the property that a spanning tree $S' \subset S$ is contained in the axis of $t$. This particular lift of $\Gamma$ allowed us to read off the exact appearance of the automorphism $\phi$. We will follow the same approach here.

As in 1.1.2, $\Gamma$ has $m$ vertices and $km$ edges. Now, however, given a vertex $V \in T$ and an edge $E$ with $\iota(E) = V$, we have $\tau(E) = x^t r V$ for some $x \in F$ and $r > 1$. Note that $m$ and $r$ are coprime because $\Gamma$ is connected and has only one orbit of edges. Let $s$ be the smallest positive number such that $rs \equiv \pm 1 \mod m$. Then the translation distance of $t$ equals $d = \min\{s, m - s\}$. By inverting $t$ if necessary, we may assume that $d = s$.

Label the edges of $\Gamma$ by letting $E_i = t^i E$ for $i = 0, \ldots, km - 1$. Similarly, let $V_j = t^j V$ for $j = 0, \ldots, m - 1$. As before, let $v(i)$ be the remainder of $i$ under division by $m$. Note that $\iota(E_j) = V_{v(j)}$ and $\tau(E_j) = V_{v(r(j + 1))}$. Hence, the edges $E_{r-1}, E_{v(2r-1)}, \ldots, E_{v((m-1)r-1)}$ form an edge path $\rho'$ that contains all vertices of $\Gamma$. We can lift $\rho'$ to a path $\rho$ in $T$ that is contained in the axis of $t$. We have found a lift of the edges $E_0, \ldots, E_{m-2}$, i.e., we have found a lift of a spanning tree of $\Gamma$.

The subpath of $t\rho$ that does not overlap with $\rho$ consists of the edges $tE_{m-2}, tE_{r-2}, \ldots, tE_{v((s-1)r-2)} = E_{m-r-1}$. Let $E_{m-1} = tE_{m-2}$, and choose $a \in F$ such that $atE_{r-2} = E_{r-1}$. (Figure 3 illustrates these choices in the case $m = 8, r = s = 3$.) Note that there exists some $w_1 \in F_{v(E_{r-1})}$ such that $w_1 atE_{v(2r-2)} = E_{v(2r-1)}$ (Figure 3). Similarly, for $j = 1, \ldots, s - 2$, we can find some $w_j \in F_{v(E_{v(rj-1)})}$ such that $w_j \cdots w_1 atE_{v((j+1)r-2)} = E_{v((j+1)r-1)}$.

Using the words $a, w_1 a, \ldots, w_{s-2} \cdots w_1 a$, we now recursively construct a
lift of the remaining edges of $\Gamma$. Then we find generators of $F$ and their images as before. The details are left to the reader.

1.2 Amalgamated free product

We assume that $M_\phi$ splits as an amalgamated free product over $\mathbb{Z}$, i.e., $M_\phi = G_1 \ast_{\mathbb{Z}} G_2$. In particular, $t$ acts on $\Gamma$ with two orbits of vertices and one orbit of edges. As before, we distinguish three subcases depending on the translation length of the action of $t$ on $T$.

1.2.1 Elliptic $t$-action

We consider graphs $\Gamma$ with two orbits of $1$, $n$ vertices respectively (Figure 5), and one orbit of $kn$ edges. Then we can express $F$ as

$$F = F_V \ast \left( \prod_{i=0}^{n-1} F_{t_i W} \right) \ast \langle a_n, \ldots, a_{kn-1} \rangle$$

such that

$$\phi(F_V) = F_V$$

$$\phi(F_{t_i W}) = \begin{cases} F_{t_{i+1} W} & \text{if } 0 \leq i < n-1; \\ a^{-1}_n F_W a_n & \text{if } i = n-1; \end{cases}$$

$$\phi(a_i) = \begin{cases} a_{i+1} & \text{if } i < kn-1; \\ a_{(k-1)n}^{-1} a_{(k-2)n}^{-1} \ldots a_n^{-1} w v & \text{if } i = kn-1, \text{where } v \in F_V \text{ and } w \in F_W. \end{cases}$$

Note that if $k = 1$, then the factor $\langle a_n, \ldots, a_{kn-1} \rangle$ is trivial.

Proof. As usual, we begin by lifting $\Gamma$ to $T$. Let $V$ be the (unique) vertex with $tV = V$, and let $E$ be some edge with $i(E) = V$. Let $W = \tau(E)$. Let $E_0 = E$ and $E_{i+1} = tE_i$ for $i < kn$. Then the union $S$ of $E_0, \ldots, E_{kn-1}$ is the desired lift of $\Gamma$. There exists some $v \in F_V$ such that $vtE_{kn-1} = E_0$. Choose $a_i \in F$ such that $a_i \tau(E_i) = \tau(E_{i-n})$ for $n \leq i < kn$.

Using computations analogous to those in previous sections, we can immediately read off that $\phi(a_i) = a_{i+1}$ if $i < kn$. Moreover, we have $a_{kn-1} t^{kn-1} W = t^{(k-1)n-1} W$, which implies that

$$\phi(a_{kn-1}) t^{kn} W = t^{(k-1)n} W = a_{(k-1)n}^{-1} t^{(k-2)n} W = a_{(k-1)n}^{-1} a_{(k-2)n}^{-1} \ldots a_n^{-1} W.$$
Using the identity $vt^{kn}W = W$, we immediately see that

$$a_na_2\cdots a_{(k-1)n}\phi(a_{kn-1})v^{-1} = w$$

for some $w \in F_W$, hence

$$\phi(a_{kn-1}) = a_{(k-1)n}^{-1}a_{(k-2)n}^{-1}\cdots a_n^{-1}wv.$$

As before, we find a sequence of Tietze moves that shows that the mapping torus of an automorphism of this form splits over $\mathbb{Z}$.

$$M_\phi \cong \langle F_V, F_W, a_n, \ldots, a_{kn-1}, t \mid tF_\nu t^{-1} = F_V, \ldots \rangle \cong \langle F_V, F_W, a_n, \ldots, a_{kn-1}, s, t \mid tF_\nu t^{-1} = F_V, s = a_nt^n, sF_Ws^{-1} = F_W, \ldots \rangle.$$

The crucial observation at this point is that the relation $t^{kn}a_nt^{-kn} = \phi^{kn}(a_n)$ can be rewritten as $w^{-1}s^k = vt^{kn}$, using $s = a_nt^n$ and the structure of $\phi$. After a few more Tietze moves, we can explicitly see the splitting.

$$M_\phi \cong \langle F_V, F_W, s, t \mid tF_\nu t^{-1} = F_V, sF_Ws^{-1} = F_W, w^{-1}s^k = vt^{kn} \rangle \cong \langle F_V, t \mid tF_\nu t^{-1} \rangle \ast_{\langle vt^{nk} = w^{-1}s^k \rangle} \langle F_W, s \mid sF_Ws^{-1} \rangle.$$
Remark 1.1. This case includes the obvious splittings that arise from automorphisms that respect a free product decomposition of $F$.

Example 1.2. This case contains certain inessential splittings [Sel97], i.e., splittings with edge groups that are cyclic, but not maximally cyclic. Consider, for example, the automorphism $\phi : F(x, y) \to F(x, y), x \mapsto y, y \mapsto x$. Then $M_\phi$ admits an inessential splitting over $\mathbb{Z}$:

$$M_\phi = \langle x, y, t \mid txt^{-1} = y, tyt^{-1} = x \rangle$$

$$\cong \langle x, s, t \mid s = t^2, sx^s = x \rangle$$

$$\cong \langle x, s \mid sx^s = x \rangle \ast \langle s \rangle \langle t \rangle.$$

In this example, $F_V$ is trivial, and $\Gamma$ has two edges.

1.2.2 Amalgamated free product: Two orbits of edges, special case

We consider graphs $\Gamma$ with two orbits of $m, n$ vertices respectively, and $kmn$ edges (Figure 6). Note that $m$ and $n$ are necessarily coprime. We may assume $m < n$. As above, we define $v(i)$ to be the remainder of $i$ under division by $m$, and we let $w(i)$ equal the remainder of $i$ under division by $n$.

We first assume that $n \equiv 1 \mod m$. In this case, we have

$$F = \left( \prod_{i=0}^{m-1} F_{t_i V} \right) * \left( \prod_{i=0}^{n-1} F_{t_i W} \right) * \langle a_{n+m-1}, \ldots, a_{kmn-1} \rangle$$

where

$$\phi(F_{t_{i+1} V}) = \begin{cases} F_{t_{i+1} V} & \text{if } i < m - 1; \\ a_{n+m-1}^{-1} F_V a_{n+m-1} & \text{if } i = m - 1; \end{cases}$$

$$\phi(F_{t_{i+1} W}) = \begin{cases} F_{t_{i+1} W} & \text{if } i < n - 1; \\ x F_W x^{-1} & \text{if } i = n - 1; \end{cases}$$

for

$$x = \phi(a_{n+m-1}^{-1}) \phi^{m+1}(a_{n+m-1}^{-1}) \phi^{2m+1}(a_{n+m-1}^{-1}) \cdots \phi^{n-1}(a_{n+m-1}^{-1})$$
Figure 6: Amalgamated product, hyperbolic $t$-action with translation distance 2.

Moreover, we have

$$\phi(a_i) = \begin{cases} a_{i+1} & \text{if } v(i) \neq m - 1, v(i - n) \neq m - 1; \\ a_{n+m-1}a_{i+1} & \text{if } v(i - n) = m - 1; \\ a_{i+1}a_{n+m-1}^{-1} & \text{if } v(i) = m - 1, i \neq km - 1; \\ a_{(km-1)n}a_{(km-2)n}^{-1} & \text{if } i = km - 1, \\ a_{3n}a_{2n}wva_{n+m-1}^{-1} & \text{if } i = knm - 1, \\ \end{cases}$$

for some $v \in F_V$ and $w \in F_W$.

Proof. As in the previous cases, we first construct a lift of the graph $\Gamma$. Pick some vertex $V \in T$ and some edge $E$ such that $\iota(E) = V$. Let $W = \tau(E)$. Since $n \equiv 1 \mod m$, there exists some edge $E'$ such that $\iota(E') = W$ and $E'$ is in the $F$-orbit of $t^n E$. Let $E_0 = E$. There is no loss in assuming that $tV = \iota(E')$ (Figure 3).

There exists some $a \in F$ such that $t^n V = a V$. For $0 \leq i < kmn - 1$, we now recursively define $E_{i+1}$ by letting $E_{i+1} = a^{-1}t E_i$ if $v(i) = m - 1$, $E_{i+1} = t E_i$ otherwise. There exists some $v \in F_V$ such that $v(a^{-1}t^n)^{nk} E_0 = E_0$. Moreover, there is no loss in assuming that $t(a^{-1}t^n)^{\lfloor \frac{m}{n} \rfloor} W = W$, i.e., $E_n = E'$. (here $\lfloor \frac{m}{n} \rfloor$ denotes the integral part of $\frac{m}{n}$)

We now choose $a_i$ for $n+m-1 \leq i \leq kmn-1$ such that $a_i \tau(E_i) = \tau(E_{i-n})$. In particular, we can let $a_{n+m-1} = a$. The elements $a_{n+m-1}, \ldots, a_{km-1}$ and the vertex groups generate $F$. With these definitions, a careful analysis of the action of the $a_i$s on $T$ yields the above description of $\phi$. The computations are essentially the same as before. 

$$\Box$$
In order to see that an automorphism of this form gives rise to a mapping torus that splits over \( \mathbb{Z} \), we first rewrite the presentation of \( M_\phi \) as

\[
M_\phi \cong \langle F_V, F_W, a, t \mid (a^{-1}t^m)F_V(a^{-1}t^m)^{-1} = F_V, \\
(t(a^{-1}t^m)[n])F_W(t(a^{-1}t^m)[n])^{-1} = F_W, \\
t^{knm}at^{-knm} = \phi^{knm}(a) \rangle
\]

Now we introduce the letters \( r = a^{-1}t^m \) and \( s = t(a^{-1}t^m)[n] \), and we observe that the relation \( t^{knm}at^{-knm} = \phi^{knm}(a) \) can be rewritten as \( vr^kn = w^{s}km \). Moreover, since \( t = s^{-1}r \) and \( a = t^m \), we can eliminate \( a \) and \( t \) and see the splitting of \( M_\phi \):

\[
M_\phi \cong \langle F_V, r \mid rF_Vr^{-1} = F_V \rangle \ast_{\langle vr^{kkn}w_{-1} \rangle} \langle F_W, s \mid sFs^{-1} = F_W \rangle.
\]

### 1.2.3 Amalgamated free product: Two orbits of edges, general case

Finally, we need to deal with the case where \( n \not\equiv 1 \mod m \). To this end, choose \( s \) such that \( sn \equiv 1 \mod m \) and \( 1 < s < m \). We construct a lift of \( \Gamma \) to \( T \) as above, except we start with a pair of edges \( E \) and \( E' \) in the \( F \)-orbit of \( E, t^{sn}E \) respectively. A spanning tree in this case consists of the edges \( E_0, \ldots, E_{n-1} \) and \( E_{sn}, \ldots, E_{sn+m-2} \). Hence, we obtain generators \( a_n, \ldots, a_{sn-1} \) and \( a_{sn+m-1}, \ldots, a_{knn-1} \). Our usual computation of images now gives us the structure of the automorphism. The details are left to the reader.

### 1.3 Summary

The above discussion covers all possible cases. We have obtained the main result of this paper.

**Theorem 1.3.** \( M_\phi \) splits over \( \mathbb{Z} \) if and only if \( \phi \) fits into one of the cases listed above. \( \square \)

We conclude this section with a corollary that shows that there is no shortage of automorphisms whose mapping tori do not split over \( \mathbb{Z} \). This corollary was also proved in [KK00], without a general characterization of splittings over \( \mathbb{Z} \). Recall that an automorphism \( \phi : F \to F \) is called *irreducible with*
irreducible powers if no positive power of \( \phi \) preserves the conjugacy class of a proper free factor of \( F \).

**Corollary 1.4.** If \( \phi \) is irreducible with irreducible powers, then \( M_\phi \) does not split over \( \mathbb{Z} \).

## 2 Examples

We list some examples of hyperbolic mapping tori that split over \( \mathbb{Z} \). Note, in particular, that the following proposition implies that there exist reducible hyperbolic automorphisms. Moreover, there exist hyperbolic automorphisms that have train track representatives with polynomially growing strata.

**Proposition 2.1.** Let \( \phi_i, i = 1, 2 \), be a hyperbolic automorphism of a free group \( F_i \).

1. The automorphism \( \phi = \phi_1 \ast \phi_2 \) is hyperbolic.

2. Let \( w \) be an element of \( F_1 \) such that
   \[
   w\phi_1(w)\phi_2^2(w) \cdots \phi_1^{k-1}(w) \neq v\phi_1^k(v^{-1})
   \]
   for all \( k \geq 1 \) and \( v \in F_1 \). Then the automorphism \( \psi \) of \( F = F_1 \ast <a> \) defined by \( \psi|_{F_1} = \phi_1 \), \( \psi(a) = aw \), is hyperbolic.

**Proof.**

1. Suppose \( \phi = \phi_1 \ast \phi_2 \) is not hyperbolic. Since an automorphisms is hyperbolic if and only if it is atoroidal by Theorem \( \ref{atord} \), there exists some word \( 1 \neq w \in F_1 \ast F_2 \) such that \( \psi^M(w) \) is conjugate to \( w \) for some \( M \geq 1 \). There is no loss in assuming that \( w = w_0w_1 \cdots w_{k-1} \), where \( k \) is even, \( 1 \neq w_{2i} \in F_1 \) and \( 1 \neq w_{2i+1} \in F_2 \).

   Clearly, both \( w \) and \( \psi^M(w) \) are cyclically reduced, and there exists some even number \( m \) such that \( \phi^M(w_i) = w_{i+m} \) (indices modulo \( k \)). But this implies that \( \psi^{KM}(w_i) = w_i \), hence neither \( \phi_1 \) nor \( \phi_2 \) are atoroidal, so they are not hyperbolic, which contradicts our hypothesis.

2. Suppose that \( \phi \) is not hyperbolic. Then, by Theorem \( \ref{atord} \), there exists some word \( 1 \neq w \in F \) such that \( \phi^M(w) \) is conjugate to \( w \) for some \( M > 1 \). We can write \( w = w_0w_1 \cdots w_{k-1} \) where either \( w_i \in F_1 \), or \( w_i = ax \) for some \( x \in F_1 \), or \( w_i = xa^{-1} \), or \( w = axa^{-1} \). Moreover,
we may assume that \( w \) is cyclically reduced and that \( k \) is minimal. In particular, there is no cancellation between successive subwords \( w_i, w_{i+1} \) (indices modulo \( k \)).

Then for any \( m > 1 \), there is no cancellation between \( \phi^m(w_i) \) and \( \phi^m(w_{i+1}) \), and \( \phi^m(w) \) is cyclically reduced. Moreover, the image of \( w_i \) is of the same form as \( w_i \), e.g., a subword of the form \( ax \) is mapped to a subword of the form \( ax' \). This implies that \( \phi^kM \) maps each subword \( w_i \) to a conjugate of itself. Since \( \phi \) is hyperbolic, this rules out subwords \( 1 \neq w_i \in \mathbb{F}_1 \) as well as subwords of the form \( w_i = axa^{-1} \).

The choice of \( w \) also rules out subwords of the form \( ax \) and \( xa^{-1} \), which implies \( w = 1 \), a contradiction.

\[ \square \]

**Example 2.2.**

1. The class of PV-automorphisms introduced in [GS91] provides a rich source of atoroidal automorphisms. For example, for \( F_1 = \langle x, y, z \rangle \), the automorphism \( \alpha : F_1 \rightarrow F_1 \) defined by \( \alpha(x) = y, \alpha(y) = z, \) and \( \alpha(z) = xy \) is irreducible and atoroidal.

2. Let \( \phi = \alpha^3 \). By abelianizing, we can easily see that \( w = x \) satisfies the hypothesis of Proposition 2.1, Part 2, so we obtain an explicit example of a hyperbolic automorphism with a train track representative with a stratum of polynomial growth [BH92].

**References**

[Bau93] Gilbert Baumslag. *Topics in combinatorial group theory*. Birkhäuser Verlag, Basel, 1993.

[BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.

[Bri00] P. Brinkmann. Hyperbolic automorphisms of free groups. *Geom. Funct. Anal.*, 10(5):1071–1089, 2000.

[GS91] S. M. Gersten and J. R. Stallings. Irreducible outer automorphisms of a free group. *Proc. Amer. Math. Soc.*, 111(2):309–314, 1991.
[Kap00] Ilya Kapovich. Mapping tori of endomorphisms of free groups. *Comm. Algebra*, 28(6):2895–2917, 2000.

[KK00] M. Kapovich and B. Kleiner. Hyperbolic groups with low-dimensional boundary. Ann. de ENS Paris, to appear, 2000.

[Sel95] Z. Sela. The isomorphism problem for hyperbolic groups. I. *Ann. of Math. (2)*, 141(2):217–283, 1995.

[Sel97] Z. Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.*, 7(3):561–593, 1997.

[Ser77] Jean-Pierre Serre. *Arbres, amalgames, SL₂*. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[SS] Peter Scott and Gadde Swarup. Regular Neighbourhoods and Canonical Decompositions for Groups. preprint.

[SW79] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

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