Improved Monochromatic Double Stars in Edge Colorings

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Received: 26 April 2021 / Revised: 5 March 2022 / Accepted: 15 March 2022 / Published online: 28 March 2022 © The Author(s) 2022

Abstract

We give a slight improvement on the size of a monochromatic double star we can guarantee in every r-coloring of the edges of Kn.

Keywords Ramsey theory · Monochromatic double stars

1 Introduction

Gyárfás [5] obtained the following bound on the size of the largest monochromatic subtree we can find in any r-coloring of the complete graph Kn:

Theorem 1 ([5]) If the edges of Kn are colored with r colors then there is a monochromatic subtree with at least \( \frac{n}{r+1} \) vertices.

This estimate is sharp if \( r - 1 \) is a prime power and \( (r - 1)^2 \) divides n. Füredi [2] improved on this bound for r-s for which there is no affine space of order \( r - 1 \). The following lemma was a key tool in the proof of Theorem 1 in [5].

Lemma 2 ([5]) In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic subtree with at least \( \frac{n}{r} \) vertices.

A double star is a special tree obtained by joining the centers of two disjoint stars. Mubayi strengthened Lemma 2 in [9] as follows (proved also independently by Liu, Morris and Prince [8]).

Lemma 3 ([8, 9]) In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic double star with at least \( \frac{n}{r} \) vertices.

Research supported in part by NKFIH Grants No. K132696, K117879.

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This implies that if the edges of $K_n$ are $r$-colored, then either all color classes are connected or there is a monochromatic double star with at least $\frac{n}{r-1}$ vertices. Let us denote by $f(n, r)$ the size of the largest monochromatic double star we can find in any $r$-coloring of $K_n$. The previous remark raises the following natural problem.

**Problem 4** ([4]) Is it true that $f(n, r) \geq \frac{n}{r-1}$ for all $r \geq 3$?

Note that for $r = 2$ this is not true as seen by a random 2-coloring (see [4] for details). In [4] we proved the following weaker lower bound on $f(n, r)$.

**Theorem 5** ([4]) For $r \geq 2$ there is a monochromatic double star with at least $\frac{n(r+1)+r-1}{r^2}$ vertices in any $r$-coloring of the edges of $K_n$.

For $r \geq 3$ this improved a result of Mubayi [9], who proved that there is a monochromatic subgraph of diameter at most three with at least $\frac{n}{r} + 1$ vertices in every $r$-coloring of $K_n$.

For $r = 2$ the bound in Theorem 5 is close to best possible (see [4]). However, it is possible that for $r \geq 3$, $f(n, r)$ is larger, maybe even as large as $\frac{n}{r-1}$ (Problem 4). Perhaps surprisingly, despite significant research efforts (see e.g. [7, 9, 10]), as far as we know there has been no improvement on the bound in Theorem 5. In this note we give a slight improvement on this bound.

**Theorem 6** Let $\delta = \frac{1}{1500r^2}$. For $r \geq 3$ there is a monochromatic double star with at least $\frac{n(r+1)+r-1}{r^2} + \delta n$ vertices in any $r$-coloring of the edges of $K_n$.

No effort is made to optimize the $\delta$ improvement since it is probably far from optimal, we just wanted to make the point that Theorem 5 does not give the right asymptotics for $f(n, r)$. The original question whether $f(n, r) \geq \frac{n}{r-1}$ still remains open. Note that Letzter [7] proved that there is a *triple star* of size at least $\frac{n}{r-1}$ improving a result of Ruszinkó [10] (a triple star is a tree obtained by joining the centers of three disjoint stars by a path of length 2).

**2 Proofs**

**Proof of Theorem 6** Our starting point is the argument from [4]. Let $r \geq 3$, $\delta = \frac{1}{1500r^2}$ and $M = \frac{n(r+1)+r-1}{r^2}$. Consider an $r$-colored $K_n$. We will show that there is a monochromatic double star with at least $M + \delta n$ vertices. Consider an arbitrary vertex $p$ of $K_n$ and we denote by $A_i$ the set of vertices which are connected to $p$ with an edge of color $i$. If a vertex $a \in A_i$ has more than $M + \delta n - |A_i| - 1$ edges of color $i$ going to other $A_j$-s, then we have a monochromatic double star (with centers $a$ and $p$) in color $i$ with $M + \delta n$ vertices. Thus we may assume that this is not the case. Let us denote by $H_i$ the graph consisting of the edges of color $i$ going out of $A_i$. Consider the $r$-partite graph $G$ with partite classes $A_i$ obtained by the removal of the edges in $H_i$ and inside the $A_i$, $1 \leq i \leq r$. Just as in [4] (see (1) in [4]) from the previous remark and from the Cauchy-Schwartz inequality we can get the following lower bound:
\[ 2|E(G)| > \sum_{i=1}^{r} |A_i|(n - 1 - |A_i|) - 2 \sum_{i=1}^{r} |A_i|(M + \delta n - |A_i| - 1) \]
\[ = \sum_{i=1}^{r} |A_i|(n + |A_i| + 1 - 2M - 2\delta n) = \sum_{i=1}^{r} |A_i|^2 + (n - 1)(n + 1 - 2M - 2\delta n) \]
\[ \geq \frac{(n - 1)^2}{r} + (n - 1)(n + 1 - 2M - 2\delta n) = (n - 1)((r - 2)M - 2\delta n). \]

Define \( G(i, j) \) as the bipartite subgraph of \( G \) spanned by \([A_i, A_j]\). Note that there are no edges of color \( i \) or \( j \) in \( G(i, j) \) by definition. Let \( d(v, H) \) denote the degree of \( v \) in the graph \( H \) and let \( d_k(v, H) \) denote the degree of \( v \) in color \( k \) in the graph \( H \). For any edge \( e = xy \) of color \( k \), \( x \in A_i, y \in A_j \), we define
\[ s_{ijk}(x, y) = d_k(x, G) + d_k(y, G(i, j)), \quad t_{ijk}(x, y) = d_k(x, G(i, j)) + d_k(y, G). \]
Thus we have two double stars, \( S_{ijk}(x, y) \) of size \( s_{ijk}(x, y) \) and \( T_{ijk}(x, y) \) of size \( t_{ijk}(x, y) \), of color \( k \) in \( G \). We will sum \( s_{ijk}(x, y) + t_{ijk}(x, y) \) and estimate this sum. Again just as in \([4]\) (see page 534) using the Cauchy–Schwarz inequality again and (1) we get:
\[ \sum_{1 \leq i < j \leq r, k \neq i, j} \sum_{xy \in E(G(i, j))} (s_{ijk}(x, y) + t_{ijk}(x, y)) \]
\[ = \sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d_k^2(x, G) + \sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, j} \sum_{x \in A_i} d_k^2(x, G(i, j)) \]
\[ \geq \frac{\left( \sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d_k(x, G) \right)^2}{(r - 1)\sum_{i=1}^{r} |A_i|} + \frac{\left( \sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, j} \sum_{x \in A_i} d_k(x, G(i, j)) \right)^2}{(r - 1)(r - 2)\sum_{i=1}^{r} |A_i|} \]
\[ = \frac{(2|E(G)|)^2}{(r - 2)(n - 1)} > 2|E(G)| \left( M - \frac{2\delta n}{r - 2} \right). \]

Since altogether we summed the cardinalities of the vertex sets of \( 2|E(G)| \) monochromatic double stars, (3) implies that at least one of them has size at least \( M - \frac{2\delta n}{r - 2} \).
To improve on this first note that we must have the following upper bound
\[ \frac{2|E(G)|}{(r - 2)(n - 1)} < M + \delta n, \]
since otherwise the same argument would lead to a monochromatic double star of size at least \( M + \delta n \).
To get an improvement in (3) we will need to add back some of the removed edges in \( \cup H_i \). We will make use of the “defect form” of the Cauchy-Schwarz inequality (as in \([11]\) or in \([6]\)): if \( 1 \leq l \leq m \) and
\[ \sum_{i=1}^{l} x_i = \frac{l}{m} \sum_{i=1}^{m} x_i + \Delta, \]

then
\[ \sum_{i=1}^{m} x_i^2 \geq \frac{1}{m} \left( \sum_{i=1}^{m} x_i \right)^2 + \Delta^2 \frac{m}{l(m-l)} \geq \frac{1}{m} \left( \sum_{i=1}^{m} x_i \right)^2 + \frac{\Delta^2}{1}. \]

First we show the following stability result: in \( G \) apart from a small number of exceptional vertices all vertices have degrees close to the average degree in every color otherwise we are done again. Let \( \bar{d}(G) \) denote the average degree in graph \( G \) and let \( \bar{d}_k(G) \) denote the average degree in graph \( G \) in color \( k \). A vertex \( x \in A_i, 1 \leq i \leq r \) is called \( \eta \)-exceptional for some \( \eta>0 \) if (at least) one of the following holds:

- There exists \( k \neq i \) for which we have
  \[ \left| d_k(x, G) - \frac{\bar{d}(G)}{r-1} \right| > \eta n, \] (5)

- There exist \( j \neq i \) and \( k \neq j, i \) for which we have
  \[ \left| d_k(x, G(i,j)) - \frac{\bar{d}(G)}{(r-1)(r-2)} \right| > \eta n. \] (6)

Then we have the following stability claim:

**Claim 7** Let \( \varepsilon = 12\delta. \) One of the following two cases holds:

(i) \( G \) has a monochromatic double star of order at least \( M + \delta n. \)

(ii) The number of \( x^{1/3} \)-exceptional vertices in \( G \) is less than \( x^{1/3} n. \)

**Proof of Claim 7** Assume that (ii) does not hold, so we have at least \( x^{1/3} n \) exceptional vertices for which either (5) or (6) holds (for some \( k \) or \( k, j \), respectively). Then there are at least \( l = x^{1/3} n \) (for simplicity assume that this is an integer) exceptional vertices \( x \) for which always the same, say (5), holds and always in the same direction, say larger (the proof will be similar in the other cases), so for some \( k \)

\[ d_k(x, G) > \frac{\bar{d}(G)}{r-1} + x^{1/3} n. \] (7)

For each of these \( l \) exceptional vertices \( x \) we pick a \( k_x \) for which (7) holds. Denote by \( E = \{ (x, k_x) \} \) the set of these \( l \) pairs, so \( |E| = l = x^{1/3} n. \)

We will lower bound the sum of squares in line (2) a bit differently, namely for the summation
\[ \sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d^2_k(x, G) \] (8)

we will use the defect form of the Cauchy–Schwarz inequality instead of the original. We rearrange the summation in (8) so that we start with the \( l \) terms
\[ \sum_{(x,k) \in E} d^2_k(x, G). \]

On one hand by the definition of \( E \) we have
\[ \sum_{(x,k) \in E} d_k(x, G) > l \frac{d(G)}{r-1} + l2^{1/3}n. \] (9)

On the other hand \( \Delta \) is defined by
\[ \sum_{(x,k) \in E} d_k(x, G) = l \left( \frac{\sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d_k(x, G)}{(n-1)(r-1)} \right) + \Delta = l \frac{d(G)}{r-1} + \Delta. \] (10)

Hence, by (9) and (10), we get
\[ \Delta > l2^{1/3}n. \] (11)

Applying the defect form of the Cauchy-Schwarz inequality with \( l = \frac{2^{1/3}}{4}n \), we get
\[ \sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d^2_k(x, G) \geq \left( \frac{\sum_{i=1}^{r} \sum_{x \in A_i} \sum_{k \neq i} d_k(x, G)}{(n-1)(r-1)} \right)^2 + \frac{\Delta^2}{l}. \]

Thus using (11) we get an extra
\[ \frac{\Delta^2}{l} > \frac{l^22^{2/3}n^2}{l} = l2^{2/3}n^2 = \frac{a}{4}n^3 \]

term when we lower bound the sum of squares in line (2), which leads to an extra
\[ \frac{zn^3}{8|E(G)|} \geq \frac{zn}{4} \]

(just using \( 2|E(G)| \leq n^2 \)) increment in the size of the double star in (3). Thus we get a monochromatic double star of size at least
\[ M - \frac{2\delta n}{r-2} + \frac{zn}{4} \geq M + \delta n, \]

so indeed (i) holds (here we used \( \frac{a}{4} \geq 3\delta. \))

If (5) holds for \( l \) vertices in the other direction (smaller), the proof is almost identical. Finally, assume that (6) holds for \( l \) vertices in the larger direction, i.e. for some \( k, j \)
\[ d_k(x, G(i, j)) > \frac{\overline{d}(G)}{(r-1)(r-2)} + \alpha^{1/3}n. \] (12)

We proceed similarly as above. For each of these \( l \) exceptional vertices \( x \) we pick a pair \((k_x, j_x)\) for which (12) holds. Denote by \( E = \{(x, k_x, j_x)\} \) the set of these \( l \) triples, so \(|E| = l = \frac{2^{1/3}}{4}n\).

Here we will use the defect form of the Cauchy–Schwarz inequality for the other part of the summation in (2), i.e.

\[ \sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, j} \sum_{x \in \Delta_i} d_k^2(x, G(i, j)). \] (13)

We rearrange the summation in (13) so that we start with the \( l \) terms

\[ \sum_{(x, k, j) \in E} d_k^2(x, G(i, j)). \]

On one hand by the definition of \( E \) we have

\[ \sum_{(x, k, j) \in E} d_k(x, G(i, j)) > l \frac{\overline{d}(G)}{(r-1)(r-2)} + \alpha^{1/3}n. \] (14)

On the other hand here \( \Delta \) is defined by

\[ \sum_{(x, k, j) \in E} d_k(x, G(i, j)) = l \left( \frac{\sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, j} \sum_{x \in \Delta_i} d_k(x, G(i, j))}{(n-1)(r-1)(r-2)} \right) + \Delta \] (15)

Hence, by (14) and (15), we get

\[ \Delta > l\alpha^{1/3}n. \] (16)

Applying the defect form of the Cauchy–Schwarz inequality with \( l = \frac{2^{1/3}}{4}n\), we get

\[ \sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, j} \sum_{x \in \Delta_i} d_k^2(x, G(i, j)) \geq \left( \frac{\sum_{i=1}^{r} \sum_{j \neq i} \sum_{k \neq i, j} \sum_{x \in \Delta_i} d_k(x, G(i, j))}{(n-1)(r-1)(r-2)} \right)^2 + \frac{\Delta^2}{l}. \]

Thus using (16) again we get an extra

\[ \frac{\Delta^2}{l} \geq \frac{l^2 \alpha^{2/3}n^2}{l} = lx^{2/3}n^2 = \frac{\alpha}{4}n^3 \]

term when we lower bound the sum of squares in line (2), which leads to an extra
increment in the size of the double star in (3) and we can finish as above. Finally, the case when (6) holds for \( l \) vertices in the other direction (smaller), the proof is almost identical.

Let us return to the proof of Theorem 6, and assume that we have no monochromatic double star of order at least \( M + \delta n \). Then we have (ii) in Claim 7. An \( \delta^{1/3} \)-exceptional vertex is simply called exceptional, so we have fewer than \( \delta^{1/3} n \) exceptional vertices. Let us take a non-exceptional vertex \( x \). This exists since we may assume wlog that \( A_1 \) is the largest \( A_i \) and then for our choice of \( \alpha \) (\( \alpha = 12\delta \))

\[
|A_1| \geq \frac{n - 1}{r} > \delta^{1/3} n.
\]

Since \( x \) is non-exceptional we have

\[
d(x, G) \leq (r - 1) \frac{\overline{d}(G)}{r - 1} + (r - 1) \delta^{1/3} n = \frac{2|E(G)|}{n - 1} + (r - 1) \delta^{1/3} n.
\]

Recall that \( H_1 \) is the graph consisting of the removed edges of color 1 going out of \( A_1 \) and that

\[
d_1(x, H_1) < M + \delta n - |A_1|.
\]

(17) and (18) imply that there exists a \( j \neq 1 \) (wlog \( j = 2 \)) such that

\[
d_2(x, H_2) \geq \frac{1}{r - 1} ((n - 1) - |A_1| - d(x, G) - d_1(x, H_1)) > \frac{1}{r - 1} \left( (n - 1) - |A_1| - \frac{2|E(G)|}{n - 1} - (r - 1) \delta^{1/3} n - M - \delta n + |A_1| \right)
\]
\[
> \frac{1}{r - 1} \left( (n - 1) - (r - 2) M - (r - 2) \delta n - (r - 1) \delta^{1/3} n - M - \delta n \right)
\]
\[
\geq \frac{1}{r - 1} \left( (n - 1) - (r - 1) M - (r - 1) \delta n - (r - 1) \delta^{1/3} n \right)
\]
\[
\geq \frac{1}{r - 1} \left( \frac{n}{(r - 1)^2} - 2(r - 1) \delta^{1/3} n \right) = \frac{n}{(r - 1) r^2} - 2 \delta^{1/3} n.
\]

(19)

Here in the third line we used (4), while in (19) the definition of \( M \).

Denote by \( N^G_k(x, U) \) the color-\( k \) neighbors of the vertex \( x \) in the subset \( U \subseteq V(G) \) in the graph \( G \). Since \( x \) is non-exceptional we have

\[
|N^G_2(x, A_3)| \geq \frac{\overline{d}(G)}{(r - 1)(r - 2)} - \delta^{1/3} n > \frac{n - 1}{r(r - 1)(r - 2)} - \delta^{1/3} n > \delta^{1/3} n
\]

(20)

for our choice of \( \alpha \). Indeed,
Let us take a non-exceptional vertex \( y \in N_2^G(x, A_3) \) (this is possible because of (20)). Since \( y \) is non-exceptional we have

\[
d_2(y, G(1, 3)) \geq \frac{\overline{d}(G)}{(r-1)(r-2)} - \alpha^{1/3} n.
\]  

Consider the color-2 double star with centers \( x \) and \( y \) and leaf-sets \( N_2^G(x, V(G) \setminus A_1) \) and \( N_2^G(y, A_1) \). Using the fact that \( x \) is non-exceptional, (19) and (21), the size of this double star is at least

\[
d_2(x, G) + d_2(x, H_2) + d_2(y, G(1, 3))
\]

\[
> \frac{\overline{d}(G)}{r-1} - \alpha^{1/3} n + \frac{n}{(r-1)r^2} - 2\alpha^{1/3} n + \frac{\overline{d}(G)}{(r-1)(r-2)} - \alpha^{1/3} n
\]

\[
> M - \frac{2\delta n}{r-2} + \frac{n}{(r-1)r^2} - 4\alpha^{1/3} n > M + \delta n,
\]

(again the estimate in (3) using our choices for \( \delta \) and \( \alpha \)). Indeed,

\[
4\alpha^{1/3} + 3\delta < 5\alpha^{1/3} \leq \frac{1}{r^3} < \frac{1}{(r-1)r^2}.
\]

\( \square \)

### 3 Conclusion

We gave a slight improvement on the size of a monochromatic double star we can guarantee in every \( r \)-coloring of the edges of \( K_n \). Let us summarize where the improvement came from. Stability is used to show that most of the vertices are a part of monochromatic double stars of approximately the same size that was obtained in [4]. This allows us to start with a typical vertex, get the double star as we had before, but also get the extra edges from \( x \in A_1 \) to \( A_2 \) in color 2 which we didn’t have before.

However, the original question whether there is a double star of size at least \( \frac{n}{r} \) still remains open. No effort was made to optimize the improvement, we just wanted to make the point that Theorem 5 does not give the right asymptotics for \( f(n, r) \). With a more careful analysis one might be able to improve this further; perhaps even to the right order of magnitude, \( \delta = c/r^3 \) for some constant \( c \). However, our approach, using only the \( r \)-partite graph on the \( A_i \)-s, is not sufficient to prove \( f(n, r) \geq \frac{n}{r^2} \); there is an example demonstrating that. Indeed, consider the following random construction (for simplicity we describe it only for \( r = 3 \)). We have \( |A_i| = \frac{n-1}{3}, 1 \leq i \leq 3 \) and we have random bipartite graphs between the \( A_i \)-s with the following densities: in color 1 between \( A_1 \) and \( A_2 \) and between \( A_1 \) and \( A_3 \) with
density \((1/4 - \epsilon)\) for some small \(\epsilon\) and between \(A_2\) and \(A_3\) with density \((1/2 + 2\epsilon)\). The other colors are similar. Then it is easy to verify that there is no monochromatic double star of size at least \(n/2\) in the tripartite graph; the point is that any \(x \in A_2\) and \(y \in A_3\) will have a large common neighborhood in \(A_1\) in color 1 with high probability which can only be counted once. Thus in order to prove \(f(n, r) \geq \frac{n}{r+1}\) (if it is true), one needs to use the edges inside the \(A_i\)-s or of course use totally new ideas.

Acknowledgements Thanks to András Gyárfás and Louis DeBiasio for helpful discussions and comments. Thanks also to the referees for valuable comments.

Funding Open access funding provided by ELKH Alfréd Rényi Institute of Mathematics. Research supported in part by NKFIH Grants No. K132696, K117879.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

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