Abstract

We consider the problem of minimizing a generalized relative entropy, with respect to a reference diffusion law, over the set of path-measures with fully prescribed marginal distributions. When dealing with the actual relative entropy, problems of this kind have appeared in the stochastic mechanics literature, and minimizers go under the name of Nelson Processes.

Through convex duality and stochastic control techniques, we obtain in our main result the full characterization of minimizers, containing the related results in the pioneering works of Cattiaux & Léonard [23] and Mikami [48] as particular cases. We also establish that minimizers need not be Markovian in general, and may depend on the form of the generalized relative entropy if the state space has dimension greater or equal than two. Finally, we illustrate how generalized relative entropy minimization problems of this kind may prove useful beyond stochastic mechanics, by means of two applications: the analysis of certain mean-field games, and the study of scaling limits for a class of backwards SDEs.

Keywords: Nelson processes, Schrödinger problem, entropy minimization, marginal constraints, convex duality, mean field games, generalized entropy, BSDE, minimal supersolution.

1 Introduction

Overview

Let \( t \mapsto \mu_t \) be a given weakly continuous flow of probability measures on \( \mathbb{R}^q \). In this work we consider the following variational problem:

\[
\inf \left\{ \mathbb{E}^{\mathcal{Q}} \left[ \int_0^T g^*(t, X_t, \sigma'(t, X_t) \beta_t) \, dt \right] : \mathcal{Q} \circ X_t^{-1} = \mu_t, \; \forall t \in [0, T] \right\},
\]

where the optimization is performed over all probability measures \( \mathcal{Q} \) solution of the martingale problem with coefficients \( (b + a \beta, a) \), where \( a = \sigma \sigma' \) and \( b \) are fixed functions (contrary to \( \beta \)). When \( g^*(t, x, \cdot) = \| \cdot \|^2 / 2 \) this corresponds to minimizing the relative entropy of \( \mathcal{Q} \) with respect to the law of the solution of the martingale problem \( (b, a) \), given the flow of marginals constraint. In such case a unique extremal solution to (1.1) is known to exist provided this problem is finite, and it is known to
be a Markovian measure. The construction of such trajectorial law goes under the name of “Nelson Processes” in the stochastic mechanics literature; see [24, 23] and references therein. For generalized entropy minimization as in (1.1), the problem has only been analysed in [48], to the best of our knowledge. Our aim is to:

- Obtain existence, duality, and characterization of the optimizers of (1.1).
- Establish the nature of the optimizer of (1.1) in terms of its Markovianity and robustness (i.e. interplay between $g^*$, its growth, and the spatial dimension $q$).
- Introduce novel applications for (1.1) beyond stochastic mechanics.

**Proper Setting and Assumptions**

Let

$$ b : [0, T] \times \mathbb{R}^q \to \mathbb{R}^q, \quad \sigma : [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q \times q}. $$

Throughout we use the apostrophe (’) to denote transposition, and we let

$$ a := \sigma \sigma', $$

which is then an $\mathbb{R}^{q \times q}$-valued function. We work under the assumption

(A) $a$ is bounded and $b, \sigma$ are once differentiable in time, twice differentiable in space, and satisfy the usual linear-growth and Lipschitz conditions of Itô theory. The matrix $\sigma$ is invertible.

Let us define the differential operators:

$$ L = b' \nabla_x + \frac{1}{2} \sum_{i,j} a^{i,j} \partial^2_{x_i, x_j}, \quad L_t = \partial_t + L. $$

Under the above assumption, the martingale problem with generator $L$ (one also says, with coefficients $(b, a)$) and domain $C^\infty_0((0, T) \times \mathbb{R}^q)$ admits for each starting point $x$ a unique solution, which we denote $P^x$. Equivalently, the diffusion SDE

$$ dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x, \quad (1.2) $$

has a unique weak solution.

From now on we fix a starting distribution $m_0(dx)$ and define

$$ \mathbb{P} := \int m_0(dx)\mathbb{P}_x. $$

We also denote throughout by $X$ the canonical process on $\Omega := C([0, T]; \mathbb{R}^q)$, and by $\{F_t\}$ the canonical filtration. Further, we define

$$ M_t := X_t - X_0 - \int_0^t b(s, X_s)ds = \int_0^t \sigma(s, X_s)dW_s. $$

For simplicity we write $\mathbb{E}$ for expectation under $\mathbb{P}$. With $\mu := \{\mu_t\}_t$ we also denote

$$ Q(\mu) := \left\{ Q \in \mathcal{P}(\Omega) : \begin{array}{l}
Q \circ X_t^{-1} = \mu_t \quad \forall t \in [0, T], \quad M - \int_0^t a(t, X_t)\beta_t^Q dt \\
\text{is a } \mathbb{Q}\text{-martingale and } \langle M \rangle_t = \int_0^t a(t, X_t)dt
\end{array} \right\}. $$
We stress that $Q(\mu)$ may contain measures singular with respect to $P$. We can now properly define \([1.1]\): the \textit{primal} optimization problem central to this article is:

$$\inf_{Q \in Q(\mu)} \mathbb{E}^Q \left[ \int_0^T g^* \left( t, X_t, \sigma'(t, X_t) \beta_t^Q \right) dt \right], \quad (P^\text{ext}[\mu])$$

where $g^* : [0, T] \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}_+$. It is implicitly assumed that $\beta^Q$ above is a predictable functional (of $X$) s.t. the $dt$-integral is well-defined.

Regarding the family $\mu := \{\mu_t\}_t$ we make the standing assumption:

\textbf{(B)} Each $\mu_t$ is a Borel probability measure on $\mathbb{R}^q$ and the function $t \mapsto \mu_t$ is continuous w.r.t. the usual weak topology of measures on the target space. We further assume $\mu_0 = m_0$.

We let $g$ be the convex conjugate of $g^*$ w.r.t. the last argument:

$$(t, x, y) \in [0, T] \times \mathbb{R}^q \times \mathbb{R}^q \mapsto g(t, x, y) := \sup_{z \in \mathbb{R}^q} \{\langle z, y \rangle - g^*(t, x, z)\}.$$

Outstanding assumption on $g^*$ is

\textbf{(C)} $g^*$ is measurable in the first two coordinates, whereas it is strictly convex, even and continuously differentiable in the last one. Moreover, we have

1. $g^*(t, x, y) = 0 \iff y = 0$,
2. $(t, x) \mapsto \sup_{|y| \leq n} g^*(t, x, y)$ is $\mu_t(dx)dt$-integrable for each $n \in \mathbb{N}$,
3. $\limsup_{|y| \to 0} \frac{g^*(t, x, y)}{|y|} = 0$ and for some $p > 1$ we have uniformly on $(t, x)$

$$\liminf_{|y| \to \infty} \frac{g^*(t, x, y)}{|y|^p} > 0,$$

4. $\exists C > 1$ and $h : [0, T] \times \mathbb{R}^q \to \mathbb{R}_+$ $\mu_t(dx)dt$-integrable s.t.

$$g^*(t, x, 2y) \leq C g^*(t, x, y) + h(t, x),$$

5. $\exists \ell > 1$ and $H : [0, T] \times \mathbb{R}^q \to \mathbb{R}_+$ $\mu_t(dx)dt$-integrable s.t.

$$g^*(t, x, y) \leq \frac{g^*(t, x, \ell y)}{2\ell^2} + H(t, x).$$

Note that in particular $g^*$ is jointly measurable, and necessarily $g$ is non-negative and finite-valued. Furthermore, $g$ is strictly convex, differentiable and even w.r.t. the last coordinate.

\textbf{Remark 1.1} The case $g^*(t, x, y) = \|y\|^2$ corresponds to the entropy criterion. Notice that $g^*(t, x, y) = R(t, x)\|y\|^p$, with $1 < p < \infty$ and $R(\cdot, \cdot)$ integrable and uniformly strictly positive, satisfies the above assumptions. More generally, $g^*(t, x, y) = R(t, x)\|y\|^p[1 + | \log \|y\| |]$ does it too, and so forth.

We shall occasionally refer to the property

$$\liminf_{|y| \to \infty} \frac{g^*(t, x, y)}{|y|^2} > 0, \text{ uniformly on } (t, x),$$

by saying that “$g^*$ has at least quadratic growth.” This is not assumed for most results in this article.
Main results

We introduce the space of test functions

\[ C := \{ w : [0, T] \times \mathbb{R}^q \to \mathbb{R} \in C^{1,2} : \text{supp}(w) \subset (0, T) \times \mathbb{R}^q \text{ compact} \} , \]

with an associated variational problem:

\[ \sup_{w \in C} \int \int \{ L_t w(t, x) - g(t, x, \sigma'(t, x) \nabla w(t, x)) \} \mu_t(dx)dt. \quad (D_0[\mu]) \]

Problem \( (D_0[\mu]) \) has to be supplemented with a suitable extension, namely

\[ \sup_{\psi \in L^2_{\mu}} L(\psi) - \int \int g(t, x, \sigma'(t, x) \psi(t, x)) \mu_t(dx)dt. \quad (D[\mu]) \]

We postpone the definition of \( L^2_{\mu} \) and the interpretation of the linear functional \( L(\psi) \) to Section 4. Problems \( (D_0[\mu]) \) and \( (D[\mu]) \) are referred to as the dual and the extended dual problems respectively. We can now state the main structural result of the article.

**Theorem 1.1** There is no duality gap:

\[ \text{value}(D_{\text{ext}}[\mu]) = \text{value}(D_0[\mu]) = \text{value}(D[\mu]). \quad (1.3) \]

If this common value is finite, then the primal problem is attained by a unique \( Q \in Q(\mu) \), and the extended dual problem is attained by a \( \mu_t(dx)dt \)-a.s. unique \( \Psi \in L^2_{\mu} \). These optimizers are related as follows: Under \( Q \) the canonical process satisfies

\[ dX_t = \{ b(t, X_t) - \sigma(t, X_t) \nabla g(t, X_t, \sigma'(t, X_t) \Psi(t, X_t)) \} dt + \sigma(t, X_t) dW_t, \quad (1.4) \]

and the common value in (1.3) equals

\[ \int \int g^*(t, x, \nabla g(t, x, \sigma'(t, x) \Psi(t, x))) \mu_t(dx)dt. \]

If furthermore \( g^* \) has at least quadratic growth, then \( Q \ll P \) and

\[ \frac{dQ}{dP} = \mathcal{E} \left( - \int \nabla g(t, X_t, \sigma'(t, X_t) \Psi(t, X_t))' \sigma^{-1}(t, X_t) dM_t \right). \quad (1.5) \]

We now provide two applications of the main result. First we ask whether the optimal measure for the primal problem has the Markov property. Recall that this does not simply follow from the coefficients being “Markovian,” and indeed we will show that the Markov property may fail for the optimal measure. This answers an open question in [48] to the negative.

**Corollary 1.1** There is \( P \) and \( \mu \) with value\( (P_{\text{ext}}[\mu]) < \infty \), for which the optimal solution \( Q \) does not have the Markov property. On the other hand, if we assume that \( (P_{\text{ext}}[\mu]) \) is attained by a probability measure absolutely continuous w.r.t. \( P \) (which is guaranteed if \( g^* \) has at least quadratic growth and the problem is finite), then the optimal \( Q \) must have the Markov property.
Second, we address the following question: is the optimizer of the primal problem universal, i.e. independent of the concrete $g^*$? Our insight is that the answer depends on the dimension $q$:

**Corollary 1.2** In dimension one ($q = 1$) the solution of the primal problem does not depend on the cost $g^*$, as long as (C) is fulfilled. In higher dimensions ($q \geq 2$) there is dependence on $g^*$.

The fact that the optimizer is universal for dimension one, and that otherwise the optimizer does depend on the cost criterion, attests to the richness of the problem.

**Comparison with the literature**

Problem $(P_{ext}[\mu])$ was first analyzed in Mikami’s [48]. Unlike in that article, we treat the subject directly, rather than as a limiting problem where only finitely many marginals are prescribed. This is the main methodological difference between the two works. In particular, this allows us to obtain duality directly with a continuum of prescribed marginals. The emphasis on duality theory allows us to relax the requirements on the cost function $g^*$, which in [48] is assumed to be rather smooth owing to the use of PDE theory (strong solutions thereof). We also cover the case where $\mathbb{P}$ is a diffusion law, rather than just Wiener measure; in particular, we make no use of uniform ellipticity. Other important differences are: the treatment of applications outside of the realm of stochastic mechanics (they will be given in Section 2 below), and a detailed study of the universality and Markovianity of the optimal primal solutions. In this last regard, we answer an open question in [48] to the negative.

Our duality approach is closest to Cattiaux & Léonard’s [23], where the entropic case is dealt with. Unlike in that article however, we do not use large deviations arguments but only duality and stochastic control techniques, and we cover generalized entropies rather than the relative entropy only. We also make use of backwards SDE techniques as in the works of Drapeau, Kupper, Tangpi and others [30, 31, 32].

A number of well studied problems in the literature share a similar nature with Problem $(P_{ext}[\mu])$. For instance in the works on Markovian projections of Semimartingales by Bentata, Brunick, Cont, Gyöngy, Shreve [34, 35, 18, 14] among others. On a similar note, this is close to the so-called Peacock problem explored by Kellerer [39], Lowther [46], Hirsch & Profeta & Yor [36], Beiglböck & Huesmann & Stebegg [10], Juillet [37], Källblad & Tan & Touzi [38], and many other authors: given a continuum of marginals in increasing convex order, does there exist a simple martingale (eg. Markovian) having them as marginals? Another close cousin of Problem $(P_{ext}[\mu])$ is the celebrated Schrödinger problem (also called entropic optimal transport), wherein only initial and final marginal distributions are prescribed: we refer to the survey by Léonard [45] for a detailed historical account and to the works by Backhoff, Benamou, Carlier, Chen, Conforti, Cuturi, Gentil, Georgiou, Léonard, Nenna, Pammer, Pavon, Peyré [28, 11, 25, 27, 7] for a sample of recent developments. By mixing the Schrödinger problem with $(P_{ext}[\mu])$ in the entropic case, one obtains the so-call Bredinger Problem, which can be seen as a regularized version of Brenier’s incompressible fluid model [16, 17]; See the works by Arnaudon, Baradat, Benamou, Carlier, Cruzeiro, Léonard, Monsaingeon, Nenna, Zambrini [11, 9, 12, 8].
Outline

First we provide in Section 2 applications for the results hitherto obtained, namely for Mean-Field games and non-exponential large deviations of empirical flows. The rest of the article is devoted to the proofs of the main result and its corollaries. In Section 3 we look in depth at the primal problem. In Section 4 we introduce the dual problem(s). In Section 5 we establish the absence of duality gap. In Section 6 we prove the main theorem. Finally in Section 7 we provide important (counter)examples and complete the proofs of the main corollaries.

2 Applications

So far we have worked with a fixed flow of marginals $\mu$, in this part we shall let $\mu$ vary. The notation so far has been set up to deal with this situation.

2.1 McKean-Vlasov control and Mean-Field games of potential type

Let us write

$$Q := \bigcup_{\mu \text{ satisfying (B)}} Q(\mu).$$

(2.1)

We consider the following McKean-Vlasov control problem in canonical space (i.e. in weak formulation):

$$\inf \left\{ \mathbb{E}^Q \left[ \int_0^T g^*(t, X_t, \sigma'(t, X_t) \beta^Q_t(X))dt \right] + \int_0^T R_t(Q \circ X_t^{-1})dt : Q \in Q \right\}.$$  

($MKV_0$)

Here

$$(t, m) \ni [0, T] \times \mathcal{P}(\mathbb{R}^d) \mapsto R_t[m] \in \mathbb{R}_+,$$

is assumed measurable. We have the following technical result whose straightforward proof we omit.

Lemma 2.1 Problem ($MKV_0$) is equivalent to

$$\inf \left\{ \text{value}[P_{\text{ext}}^\mu] + \int_0^T R_t[\mu_t]dt : \mu = \{\mu_t\}_t \text{ satisfies Assumption (B)} \right\}.$$  

($MKV$)

In particular: $\mu$ is an optimizer for ($MKV$) and $Q$ is an optimizer for ($P_{\text{ext}}[\mu]$) iff $Q$ is an optimizer for ($MKV_0$) and the marginals of $X$ under $Q$ are given by $\mu$.

The goal of this part of the article is to illustrate the use of Theorem 1.1 to obtain that the “optimal control” $\beta$ is of Markovian feedback form. The same will be true for associated Mean Field games that we will introduce shortly. We stress that this is then a purely variational argument for the existence of optimal Markov controls, as opposed to analytical arguments. We refer to [20, 19, 3, 17, 13, 21, 21, 1] for references on McKean-Vlasov control (also known as mean-field control), to [22] for extensive references on mean-field games, to the works of Lacker [33, 14] for the general question of existence of Markovian optimizers, and to [2] for dynamic potential games. We make all simplifying assumptions necessary to keep technicalities at a minimum.
Proposition 2.1 Assume that $(MKV_0)$ is finite, and that:

for all $t : m \mapsto R_t[m]$ is lower-semicontinuous.

Then Problem $(MKV_0)$ has an optimizer $Q$ for which the associated optimal control is Markov: $\beta^Q_t(X) = \beta^Q_t(X_t)$ for each $t$.

Proof. Let $Q^n$ be $(1/n)$-optimizers for $(MKV_0)$. It follows that

$$E^{Q^n} \left[ \int_0^T g^*(t, X_t, \sigma'(t, X_t) \beta^{Q^n}_t(X)) dt \right] \leq 1 + \text{value}((MKV_0)).$$

By Lemma 3.1, $\{Q^n\}_n$ is tight. We denote by $Q$ an accumulation point. Again by this lemma we deduce $Q \in Q$. Analogously, and due to the assumption on $R$ (plus Fatou’s lemma), we derive the lower-semicontinuity of the objective function. This implies the optimality of $Q$ for $(MKV_0)$. Denoting $\mu$ the flow of marginals of this measure, we clearly have that $\mu$ satisfies (B), and necessarily $Q$ is optimal for $(P^\text{ext}|\mu]$). By Theorem 1.1, the associated $\beta^Q$ is of the desired form.

From now on we assume that $R$ is differentiable, meaning that the following directional derivatives exist

$$\lim_{\epsilon \to 0^+} \frac{R_t[m + \epsilon(\bar{m} - m)] - R_t[m]}{\epsilon} = \int \nabla R_t[m](y)(\bar{m} - m)(dy),$$

along with a bounded measurable function $\nabla R_t[m] : \mathbb{R}^q \to \mathbb{R}$.

We consider the following Mean Field game (MFG) of potential form on canonical space (this is again a weak formulation): Find $(Q, \mu)$ such that

(1) $Q$ attains

$$\inf \left\{ E^Q \left[ \int_0^T g^*(t, X_t, \sigma'(t, X_t) \beta^Q_t(X)) + \nabla R_t[\mu_t](X_t) \right] dt : \hat{Q} \in Q \right\},$$

(2) $Q \circ X_t^{-1} = \mu_t$ for all $t \in [0, T]$.

Leveraging on Proposition 2.1, we prove the existence of a Mean Field equilibrium where the optimal control $\beta^Q$ is Markovian.

Proposition 2.2 Under the conditions in Proposition 2.1 and the differentiability assumption on $R$, let $Q$ be any optimizer for $(MKV_0)$ and $\mu_t := Q \circ X_t^{-1}$. Then $(Q, \mu)$ is a solution (i.e. an equilibrium) to the Mean Field game (1)-(2) above, and the associated control $\beta^Q$ is Markov.

Proof. Let $Q$ as in Proposition 2.1 with marginals $\mu$. By Lemma 2.1 we have

$$\text{value}(P^\text{ext}[\mu]) + \int_0^T R_t[\mu_t] dt$$

$$\leq \text{value}(P^\text{ext}[\mu + \epsilon(\bar{\mu} - \mu)]) + \int_0^T R_t[\mu_t + \epsilon(\bar{\mu}_t - \mu_t)] dt$$

$$\leq \epsilon \text{value}(P^\text{ext}[\mu]) + (1 - \epsilon) \text{value}(P^\text{ext}[\mu]) + \int_0^T R_t[\mu_t + \epsilon(\bar{\mu}_t - \mu_t)] dt.$$
Indeed, one can see the convexity of \( \text{value}(P^{ext}[:]) \) either directly or as a consequence of the absence of duality gap (Theorem 1.1) since the dual problem \( \text{value}(D[:]) \) is obviously convex. Rearranging we obtain

\[
\text{value}(P^{ext}[:]) \leq \text{value}(P^{ext}[\bar{\mu}]) + \lim_{\epsilon \to 0^+} \int_0^T R_\epsilon[\mu_t + \epsilon(\bar{\mu}_t - \mu_t)] - R_\epsilon[\mu_t] \, dt.
\]

By dominated convergence and the differentiability assumption, we deduce

\[
\text{value}(P^{ext}[:]) \leq \text{value}(P^{ext}[\bar{\mu}]) + \int_0^T \nabla R_\epsilon[\mu_t](y)(\bar{\mu}_t - \mu_t)(dy) \, dt,
\]

so

\[
\text{value}(P^{ext}[:]) \leq \text{value}(P^{ext}[\bar{\mu}]) + \int_0^T \nabla R_\epsilon[\mu_t](y)d\bar{\mu}_t(y) \, dt.
\]

Since \( \bar{\mu} \) is arbitrary, this is clearly equivalent to saying that \((Q, \mu)\) is a Mean Field game equilibrium. \(\blacksquare\)

### 2.2 A generalized Laplace principle for empirical flow of particles

We interpret here the value of our primal problem \((P^{ext}[:])\), seen as a function of the flow \(\mu\), as the rate function of a non-exponential Laplace principle for empirical flow of marginals. In this way we come full circle with the work \cite{23}, where the authors start from an exponential Laplace principle, and then study \((P^{ext}[:])\) in the entropic case. Indeed, we do the opposite here, starting from the study of \((P^{ext}[:])\) and then referring to a non-exponential Laplace principle. Furthermore, we cover situations vastly more general than the entropic case. Our starting point is the work \cite{12} by Lacker, and its Wiener space specialization \cite{6} by Lacker, Tangpi, and one of the authors. We let \(\gamma\) denote the Wiener measure in state space \(\mathbb{R}^q\) and start at the origin, and assume for simplicity that

\[
g^*(t, x, y) = g^*(y),
\]

and that \(m_0\) is concentrated on a point (w.l.o.g. the origin). We have

**Proposition 2.3** Let \(F\) be a real-valued, measurable and bounded functional over flows of probability measures, namely \(F \in B_b(C([0, T]; \mathcal{P}(\mathbb{R}^q)))\). Let \(\{W^i\}_{i \in \mathbb{N}}\) distributed like \(\gamma^\otimes\mathbb{N}\) and \(\{X^i\}_{i \in \mathbb{N}}\) be the associated i.i.d. sequence of solutions to (1.2). We consider the following backwards SDE under \(\gamma^\otimes\mathbb{N}\):

\[
dY^n_t = -g(\sqrt{n}Z_t)dt + Z_t dW^{(n)}_t, \quad Y^n_T = nF \left( t \mapsto \frac{1}{n} \sum_{i \leq n} \delta X^i_t \right),
\]

where \(W^{(n)}\) is the \(\gamma^\otimes\mathbb{N}\)-Brownian motion obtained by appropriate scaling and consecutive concatenation of \(W^1, \ldots, W^n\) over the time-index set \([0, T]\). Then

\[
\lim_{n \to \infty} \frac{1}{n} Y^n_0 = \sup_{\mu} \left\{ F \left( t \mapsto \mu_t \right) - \inf_{Q \in \mathcal{Q}(\mu)} E^Q \left[ \int_0^T g^*(\sigma'(t, X^i_t)\beta^Q_t) \, dt \right] \right\}
\]

\[
= \sup_{\mu} \left\{ F \left( t \mapsto \mu_t \right) - \text{value}(P^{ext}[:]) \right\}.
\]
Proof.

Step 1: We recall here the crucial result of [6]. Let $\tilde{F} \in B_b(P(C([0,T];\mathbb{R}^q)))$.

With the same ingredients as in the statement, we have

$$\lim_{n \to \infty} \frac{1}{n} \tilde{Y}^n_0 = \sup \left\{ \tilde{F}(\tilde{Q}) - E^{\tilde{Q}} \left[ \int_0^T g^*(q^\flat_t) \, dt \right] : \text{\tilde{Q} s.t. } X. - \int_0^T q^\flat_t \, dt \text{ is \tilde{Q}-B.m.} \right\},$$

where $\tilde{Y}^n$ solves the same BSDE as $Y^n$ but with the terminal condition

$$\tilde{Y}^n_T = n\tilde{F} \left( \frac{1}{n} \sum_{i \leq n} \delta_{W^i} \right),$$

under $\gamma \otimes N$.

Step 2: We now move from Wiener measure to the diffusion law $P$. Since (1.2) has a unique strong solution, there is a measurable map $H$ between path-spaces such that $X = H(W)$. For $\hat{F} \in B_b(P(C([0,T];\mathbb{R}^q)))$, we consider $\tilde{F}(Q) := \hat{F}(Q \circ H^{-1}).$

Observe that pointwise

$$\tilde{F} \left( \frac{1}{n} \sum_{i \leq n} \delta_{\omega^i} \right) = \hat{F} \left( \frac{1}{n} \sum_{i \leq n} \delta_{H(\omega^i)} \right).$$

Notice that $\tilde{Q}$ is associated to $q^\hat{Q}$ iff $Q = \tilde{Q} \circ H^{-1}$ is associated to $\sigma'(t,X_t)\beta^Q_t = q^\hat{Q}$. This and Step 1 show that

$$\lim_{n \to \infty} \frac{1}{n} \tilde{Y}^n_0 = \sup \left\{ \hat{F}(Q) - E^Q \left[ \int_0^T g^*(\sigma'(t,X_t)\beta^Q_t) \, dt \right] : Q \in \mathcal{Q} \right\},$$

where $\hat{Y}^n$ solves the same BSDE as $Y^n$ but with the terminal condition

$$\hat{Y}^n_T = n\hat{F} \left( \frac{1}{n} \sum_{i \leq n} \delta_{X^i} \right),$$

under $\gamma \otimes N$.

Step 3: We now change the state space from $P(C([0,T];\mathbb{R}^q))$ to $C([0,T];P(\mathbb{R}^q))$, much as in the contraction principle in large deviations theory. Let $F$ as in the statement. Then $F$ can be seen as belonging to $B_b(P(C([0,T];\mathbb{R}^q)))$ via the identification

$$\tilde{F}(Q) = F(t \mapsto Q \circ X_t^{-1}).$$

Applying Step 2 to this $F$ we easily obtain the desired result and finish the proof. ■

In the entropic case (i.e. when $g^*$ is quadratic), this Laplace principle is equivalent to a large deviations principle (LDP) for the same objects. It is unclear whether the above general result can be translated into a LDP of sorts. Nevertheless, we think it is a curious observation that generalized entropy minimization is so closely related to scaling limits of backwards SDEs.
3 The primal problem

Recall the notation $Q$ from (2.1). Let

$$
\tilde{I}(Q) := \mathbb{E}^Q \left[ \int_0^T g^* \left( t, X_t, \sigma'(t, X_t) \beta_t^Q \right) dt \right], \quad (3.1)
$$

if $Q \in \mathcal{Q}$, and otherwise we set $\tilde{I}(Q) = +\infty$. This is our primal objective function.

**Lemma 3.1** The function $\tilde{I}$ is strictly convex, lower-semicontinuous with respect to weak convergence, and has tight sub-level sets (i.e. $\tilde{I}$ is inf-compact).

**Proof.** This is folklore. It readily follows e.g. from [7, Theorem 8.3].

We now prove that $(P_{ext}\[\mu])$ is attained.

**Lemma 3.2** If value $(P_{ext}\[\mu]) < \infty$, this problem has a unique optimizer.

**Proof.** Immediate from Lemma 3.1 and the fact that the constraints are closed w.r.t. weak convergence.

We will need further properties of the functional $\tilde{I}$ when we establish the absence of duality gap in Section 5. First we must introduce some terminology from stochastic analysis. We follow [31], in the simpler so-called translation-invariant setting. By a supersolution of a Backward Stochastic Differential Equation (BSDE) with generator $g$ and terminal condition $F(X)$ we mean a couple of processes $(Y,Z)$, the first one càdlàg adapted and the second predictable and making

$$
\int Z_r \cdot dM_r \text{ supermartingale},
$$

such that

$$
\begin{cases}
Y_s - \int_s^t g(r, X_r, \sigma'(t, X_t) Z_r) dr + \int_s^t Z_r \cdot dM_r \geq Y_t, & \text{for all } 0 \leq s \leq t \leq T, \\
Y_T \geq F(X).
\end{cases}
$$

Observe that $Y_0$ is $X_0$-measurable. A supersolution $(\bar{Y}, \bar{Z})$ is said minimal if a.s. $\bar{Y}_t \leq Y_t$ for every $t$ and every supersolution $(Y,Z)$. Let us denote by $\mathcal{A}(F)$ the set of supersolutions. From our assumptions follows that $g(r, X_r, 0) = 0$, so if $F$ is essentially bounded we have that $(\|F\|_\infty, 0) \in \mathcal{A}(F)$. As proved originally in [30], and extended in [31, Theorem 2.1], we may define the minimal supersolution operator by $E^\mathcal{A}_t(F) = +\infty$ for all $t$ if $\mathcal{A}(F) = \emptyset$, and otherwise

$$
E^\mathcal{A}_t(F) := \text{ess inf } \{ Y_t : (Y,Z) \in \mathcal{A}(F) \},
$$

in which case the process $E^\mathcal{A}(F)$ is the minimal supersolution for the terminal condition $F$. Again, $E^\mathcal{A}_0(F)$ is $X_0$-measurable. When $g$ has at most quadratic growth in its last component then $E^\mathcal{A}(F)$ may reduce to the solution of the BSDE with generator $g$. In general, a BSDE may have no solutions (see [29]) and this is the reason one works with supersolutions.

**Lemma 3.3** Define

$$
Q \ll P \mapsto I(Q) := \mathbb{E}^Q \left[ \int_0^T g^* \left( t, X_t, \sigma'(t, X_t) \beta_t^Q \right) dt \right]. \quad (3.2)
$$

\(^4\)Strictly speaking, the stochastic integral term is often taken to be of the form $\int Z_r dW_r$ in the literature. Since our $\sigma$ is invertible we can and prefer to write $\int Z_r dM_r$, as $M$ is the most natural martingale for us.
if \( Q \circ X^{-1}_0 = m_0 \), and \(+\infty\) otherwise. The minimal supersolution operator (at time zero) is related to \( I \) via the following conjugate relationship:

\[
\int [E^Q_0(F)](x_0)dm_0(x_0) = \sup_{Q \ll P}\{E^Q[F] - I(Q)\}, \ F \in L^\infty(P).
\]

The converse is also true, namely

\[
I(Q) = \sup_{F \in L^\infty(P)} \left\{ E^Q[F] - \int [E^Q_0(F)](x_0)dm_0(x_0) \right\}, \ Q \ll P. \tag{3.3}
\]

Proof. By regular disintegration of \( Q \) w.r.t. its initial condition, and the fact that the space \( L^\infty \) is decomposable, it is elementary to see that proving the conjugate duality relations in this lemma can be reduced to the case when \( m_0 \) is concentrated in a singleton. We now assume this. Then the first statement is [31, Theorem 3.4], upon observing that what the authors call \( q \) is our \( \beta \) and that there is no “discounting factor” in our case since our \( E^P \) is translation-invariant. The proof of (3.3) can be found in [31, Theorem 3.10], more precisely in the part of the proof entitled Second equality therein (again, there is no discounting factor for us), if we assume that \( Q \sim P \). The case \( Q \ll P \) is obtained by convexity and elementary computations.

Lemma 3.4 We have \( \bar{I}(Q) \leq I(Q) \) with equality if \( Q \ll P \). Accordingly,

\[
\int [E^Q_0(F)](x_0)dm_0(x_0) \leq \sup_{Q}\{E^Q[F] - \bar{I}(Q)\}, \tag{3.4}
\]

for \( F \) Borel bounded. If \( F \) is lower semicontinuous and bounded from below, then there is equality in (3.4).

Proof. Given \( Q \), if \( \beta, \bar{\beta} \) satisfy the conditions on \( \beta^Q \) for (3.1), then \( a(t, X_t)(\beta - \bar{\beta})(t, X) = 0 \) holds \( dQ \times dt \)-a.s. and from here \( \sigma'(t, X_t)\beta(t, X) = \sigma'(t, X_t)\bar{\beta}(t, X) \) \( dQ \times dt \)-a.s. Ergo the value of \( \bar{I} \) is well-defined. If \( Q \) is not abs. continuous then \( \bar{I}(Q) \leq I(Q) \) is trivial. Otherwise, we obtain by Girsanov that \( M - \int a(t, X_t)\beta^Q dt \) is a \( Q \)-martingale with quadratic variation process \( \int a(t, X_t)dt \), where \( dQ/dP = E(\int \beta^Q dM) \). So there is equality in that case. As for (3.3), it follows from Lemma 3.3, whereas the equality case is contained in [6].

4 The dual problem and relevant function spaces

We start by motivating the relevance of \([D_0|\mu]\).

Lemma 4.1 Weak duality holds: \( \text{value}_P[D^{ext}|\mu] \geq \text{value}[D_0|\mu] \).

Proof. By definition of convex conjugates, and since \( g \) is even in the last argument, we have for any admissible \( Q, w \) that

\[
\mathbb{E}^Q \left[ \int_0^T g^* \left( t, X_t, \sigma'(t, X_t)\beta^Q \right) dt \right]
\]

\[
\geq \mathbb{E}^Q \left[ \int_0^T \left\{ - (\beta^Q)'a(t, X_t)\nabla w(t, X_t) - g(t, X_t, \sigma'(t, X_t)\nabla w(t, X_t)) \right\} dt \right]
\]

\[
= \mathbb{E}^Q \left[ \int_0^T \left\{ \nabla w(t, X_t) - g(t, X_t, \sigma'(t, X_t)\nabla w(t, X_t)) \right\} dt \right]
\]

\[
= \int \left[ \int L_t w(t, x) - g(t, x, \sigma'(t, x)\nabla w(t, x)) \right] \mu_t(dx) dt.
\]
Indeed, since $Q$ is a solution to the martingale problem $(b + a\beta^Q, a)$, and as $w \in C$ implies that $\nabla w$ is bounded, we have

$$
\mathbb{E}^Q \left[ \int_0^T \left\{ (\beta^Q_t)^' a(t, X_t) \nabla w(t, X_t) + L_t w(t, X_t) \right\} dt \right] = 0.
$$

Thus we are entitled to call $[D_\mu]$ the dual problem. We shall soon extend this problem, but first we need to introduce a few more elements. Let us define a semi-norm on functions $\psi : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}$ as follows

$$
\|\psi\|_g := \inf \left\{ \ell \geq 0 : \int\int g(t, x, \sigma'(t, x)\psi(t, x)/\ell) \mu_t(dx)dt \leq 1 \right\},
$$

as well as the following Orlicz-like space:

$$
L^g := \left\{ \psi : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q : \|\psi\|_g < \infty \right\}.
$$

Under Assumption (C) we actually have (see proof of Lemma 4.2 below)

$$
L^g = \left\{ \psi : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q : \forall \alpha > 0, \int\int g(t, x, \alpha\sigma'(t, x)\psi(t, x)) \mu_t(dx)dt < \infty \right\}.
$$

We cannot call $L^g$ an actual Orlicz space because of the presence of the time-space parameters $(t, x)$ in $g$. It is however an Orlicz-Musielak space (see [40, 41]). Similarly, we define

$$
L^{g^*} := \left\{ \psi : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q : \|\psi\|_{g^*} < \infty \right\}.
$$

$$
\|\psi\|_{g^*} := \inf \left\{ \ell \geq 0 : \int\int g^*(t, x, \sigma'(t, x)\psi(t, x)/\ell) \mu_t(dx)dt \leq 1 \right\}.
$$

**Lemma 4.2** Identifying $\mu_t(dx)dt$-a.s. equal functions, the semi-norm $\|\cdot\|_g$ (respect. $\|\cdot\|_{g^*}$) is an actual norm on $L^g$ (respect. $L^{g^*}$). The norm dual of $L^{g^*}$ is isometrically isomorphic to $L^g$, and both are reflexive Banach spaces. The duality pairing is

$$
(\beta, \psi) \mapsto L^{g^*} \times L^g \mapsto \int\int \beta(t, x)' a(t, x) \psi(t, x) \mu_t(dx)dt.
$$

**Proof.** Observe that the convex conjugate of $g(t, x, \sigma'(t, x)\cdot)$ is

$$
h(t, x, \cdot) := g^*(t, x, \sigma(t, x)^{-1}).
$$

Let us call $L^h$ the Orlicz-like space

$$
L^h := \left\{ \psi : [0, T] \times \mathbb{R}^q \rightarrow \mathbb{R}^q : \|\psi\|_h < \infty \right\}.
$$

$$
\|\psi\|_h := \inf \left\{ \ell \geq 0 : \int\int h(t, x, \psi(t, x)/\ell) \mu_t(dx)dt \leq 1 \right\}.
$$

Notice that [40] Conditions A and B, p. 109-110] are fulfilled. Indeed taking $F := L^h$ in the author’s notation, the first condition is a consequence of $F$ containing functions taking two values, whereas the second condition follows from Assumption (C) 2. Also [40] Definition 2.1.1, 2.1.2 and 2.1.3] hold for $\Phi := h$, thanks to Assumption (C). By [40] Theorem 2.4] $\|\cdot\|_h$ is a norm and $L^h$ is Banach. By Assumption (C) 4 and [41] Corollary 1.7.4] we have that the norm dual of $L^h$ is isometrically isomorphic to $L^g$. 

12
In particular \( \| \cdot \|_g \) is a norm and \( L^g \) is Banach. Observe that Assumption (C).5 on \( g^* \) implies that (C).4 holds but on \( g \). Thus the equivalent expression for \( L^g \) holds, and further applying [10, Proposition 4.5] and again [11, Corollary 1.7.4] we get that the norm dual of \( L^g \) is isometrically isomorphic to \( L^h \). Putting things together, this shows the reflexivity of both spaces. Now, the mapping
\[
L^g \ni \phi \mapsto a\phi \in L^h,
\]
is clearly an isometric isomorphism. It follows that we can identify \( L^g^* \) and \( L^h \), so the former is reflexive Banach and with dual isometrically isomorphic to \( L^g \). Since the duality product between \( L^g \) and \( L^h \) is given by
\[
(\beta, \psi) \ni L^g \times L^g \mapsto \int \int \beta(t, x)\psi(t, x)\mu_t(dx)dt,
\]
we obtain the desired duality product between \( L^g^* \) and \( L^g \).

Let us introduce the “space of gradient fields”
\[
L^g_\nabla := \{ \nabla w : w \in \mathcal{C} \}^{L^g}.
\]

**Lemma 4.3** Assume that value\([D_0[\mu]]\) < \( \infty \). Then there is a unique continuous linear functional
\[
L : L^g_\nabla \to \mathbb{R},
\]
for which
\[
\forall w \in \mathcal{C} : L(\nabla w) = \int \int L_t(t, x, \nabla w(t, x))\mu_t(dx)dt.
\]

**Proof.** From value\([D_0[\mu]]\) < \( \infty \) we easily get
\[
|L(\nabla w)| \leq \int \int g(t, x, \sigma'(t, x)\nabla w(t, x))\mu_t(dx)dt + \text{value}[D_0[\mu]],
\]
for all \( w \in \mathcal{C} \). Replacing \( w \) by \( w/\ell \), and choosing \( \ell \) appropriately, we also get
\[
|L(\nabla w)| \leq \{1 + \text{value}[D_0[\mu]]\} \|\nabla w\|_g.
\]
By linearity and density of gradients in \( L^g_\nabla \), we conclude. \( \blacksquare \)

We denote by \((L^g_\nabla)^*\) the norm dual of \( L^g_\nabla \) equipped with the \( \| \cdot \|_g\)-topology, i.e.
\[
(L^g_\nabla)^* := \left\{ \ell : L^g_\nabla \to \mathbb{R} \text{ linear and s.t. } \|\ell\|_{(L^g_\nabla)^*} := \sup_{w \in \mathcal{C}, \|\nabla w\|_g \leq 1} \ell(\nabla w) < \infty \right\}.
\]

**Lemma 4.4** \((L^g_\nabla)^*\) can be identified with (i.e. is isometrically isomorphic to) the quotient of \( L^g^* \) by the relation
\[
\beta \mathcal{R} \bar{\beta} \iff \forall w \in \mathcal{C} : \int \int [\bar{\beta}(t, x) - \beta(t, x)]'a(t, x)\nabla w(t, x)\mu_t(dx)dt = 0,
\]
when \( L^g^* \) is given the “operator norm” as the dual of \( L^g \), and the quotient space the derived norm topology.

In particular, if value\([D_0[\mu]]\) < \( \infty \), then there is a unique equivalence class \([\beta]_\mathcal{R}\) such that
\[
L\psi = \int \int \beta(t, x)'a(t, x)\psi(t, x)\mu_t(dx)dt, \text{ for all } \psi \in L^g_\nabla.
\]
Proof. The subspace

\[ M := \{ \beta \in L^g \text{ s.t. } \forall w \in C : \int \int \beta(t,x)^t a(t,x) \nabla w(t,x) \mu_t(dx)dt = 0 \}, \]

is clearly closed. Notice that \( \beta \mathcal{R} \tilde{\beta} \iff \beta - \tilde{\beta} \in M \). By classical results, the quotient space \( L^g / \mathcal{R} \) is Banach with the norm \( \| \beta \mathcal{R} \| = \inf \{ \| \beta + m \mathcal{R} \| : m \in M \} \). On the one hand, each equivalence class \( [\beta] \) defines an element of \( (L^g)\mathcal{R}^* \). On the other hand, if \( \ell \in (L^g)\mathcal{R}^* \), by Hahn-Banach theorem, \( \ell \) can be extended by an \( \tilde{\ell} \in (L^g)^* = L^g \) with \( \| \ell \|_{(L^g)^*} = \| \tilde{\ell} \|_{g^*} \). By definition the function \( \ell \mapsto |\tilde{\ell}| \mathcal{R} \) is well-defined, surjective and linear. This function is also an isometry. Indeed, we have already obtained \( \| [\tilde{\ell}] \mathcal{R} \| \leq \| \ell \|_{(L^g)^*} \), by the Hahn-Banach argument, whereas the converse inequality is trivial for the operator norm. The last statement is a consequence of the identification of \( (L^g)^* \) and Lemma 4.3. 

Owing to the previous lemmas, we can finally say that the expression of the extended dual problem \( \tilde{D}[\mu] \), given in Section 1, is now rigorously defined. We have

Lemma 4.5 The functional

\[ \psi \in L^g \mapsto G(\psi) := \int \int g(t,x,\sigma'(t,x)\psi(t,x)) \mu_t(dx)dt \]

is convex and norm-continuous. As a consequence, the values of \( \tilde{D}_0[\mu] \) and \( \tilde{D}[\mu] \) coincide.

Proof. Clearly \( G \) is convex and finite, so we only need to show its local boundedness. Let \( \psi \) given and take \( \phi \) s.t. \( \| \psi - \phi \|_g \leq 1/2 \). By convexity, we find

\[ G(\phi) \leq \frac{1}{2} G(2\psi) + \frac{1}{2} G(2|\phi - \psi|) \leq \frac{1}{2} G(2\psi) + 1/2, \]

since by assumption \( G(0) = 0 \) so by convexity again

\[ G([\phi - \psi]/(1/2)) \leq 1 \text{ if } \| \psi - \phi \|_g \leq 1/2. \]

The second statement follows from the first one and the continuity in Lemma 4.3.

Lemma 4.6 For any \( \psi \in L^g \), we have that \( (t,x) \mapsto \nabla g(t,x,\sigma(t,x)\psi(t,x)) \in L^g \).

Proof. Denote \( f(t,x) := \nabla g(t,x,\sigma(t,x)^t \psi(t,x)) \). By definition of convex conjugates, we have

\[ \int \int g^*(t,x,f(t,x)) \mu_t(dx)dt = - \int \int g(t,x,\sigma(t,x)^t \psi(t,x)) \mu_t(dx)dt + \int \psi(t,x)^t \sigma(t,x) f(t,x) \mu_t(dx)dt, \]

so finiteness of the l.h.s. is equivalent to that of the second term in the r.h.s., since \( \sigma' \) is bounded. By convexity, \( g(t,x,2y) \geq g(t,x,y) + y^t \nabla g(t,x,y) \). From Assumption (C).5 on \( g^* \) we can conclude that (C).4 holds for \( g \) instead. Thus, there is \( c > 0 \) and \( \alpha(\cdot,\cdot) \) non-negative and \( \mu_t(dx)dt \)-integrable such that

\[ y^t \nabla g(t,x,y) \leq cg(t,x,y) + \alpha(t,x). \tag{4.5} \]

In particular, \( \psi(t,x)^t \sigma(t,x) f(t,x) \leq cg(t,x,\sigma(t,x)^t \psi(t,x)) + \alpha(t,x) \), so we conclude that the expressions above are finite as desired.
Lemma 4.7 We have
\[
\lim_{\|\psi\|_g \to \infty} \frac{G(\psi)}{\|\psi\|_g} = +\infty, \tag{4.6}
\]
with \(G\) as in Lemma 4.5. Further, \(G\) is directionally Gâteaux differentiable, and for all \(\psi \in L^2_{\mathcal{P}}\), \(w \in C\) we have:
\[
DG(\psi)(\nabla w) = \iint \nabla g(t,x,\sigma'(t,x)\psi(t,x))'\sigma'(t,x)\nabla w(t,x)\mu_t(dx)dt. \tag{4.7}
\]

**Proof.** Assumption (C).4 on \(g^*\) implies Assumption (C).5 written on \(g\) instead (for some \(\ell > 1\) and some integrable \(H\)). Applying this inequality repeatedly, one finds for each \(p \in \mathbb{N}\) that \(g(t,x,y^{\ell p}) \geq 2^p \ell^p g(t,x,y) - k\ell^p H(t,x)\). Let \(\psi\) be s.t. \(\|\psi\|_g \geq \ell^p\), then
\[
g(t,x,\sigma'\psi(t,x)) = g(t,x,\sigma'\psi(t,x))\frac{\ell^p}{\|\psi\|_g} \geq 2^p \ell^p g(t,x,\sigma'\psi(t,x))/\ell^p - \frac{k\ell^p H(t,x)}{\|\psi\|_g}
\]
\[
\geq 2^p g(t,x,\frac{\ell^p}{\|\psi\|_g} \sigma'\psi(t,x)) - k\ell^p H(t,x) \|\psi\|_g,
\]
since \(g(t,x,\cdot)\) is convex and clearly null at 0. Since \(g\) is finite (by the superlinear growth of \(g^*\)) and convex in its last argument, it is a continuous function of it. By monotone convergence, this proves
\[
\iint g(t,x,\sigma'\psi(t,x))/\|\psi\|_g \mu_t(dx)dt = 1,
\]
so by the previous inequalities we find
\[
\frac{G(\psi)}{\|\psi\|_g} = \iint \frac{g(t,x,\sigma'\psi(t,x))}{\|\psi\|_g} \mu_t(dx)dt \geq 2^p - k\ell^p \iint \frac{H(t,x)}{\|\psi\|_g} \mu_t(dx)dt.
\]
Taking \(\|\psi\|_g \to \infty\) and then \(p \to \infty\) implies \(4.6\). As for the Gâteaux differentiability, we must compute \(\lim_{\epsilon \to 0} \frac{G(\psi + \epsilon \nabla w) - G(\psi)}{\epsilon}\), which is equal to
\[
\lim_{\epsilon \to 0} \iint_0^1 \nabla g(t,x,\sigma'(t,x)\psi(t,x) + \epsilon \theta \nabla w(t,x))'\sigma'(t,x)\nabla w(t,x) d\theta \mu_t(dx)dt.
\]
But the innermost integral, as a function of \((t,x)\) converges a.s. when \(\epsilon \to 0\) to \(\nabla g(t,x,\sigma'(t,x)\psi(t,x))'\sigma'(t,x)\nabla w(t,x)\). Applying the bound \(4.5\) and the integrability result in Lemma 4.6 we may use dominated convergence to conclude. ■

5 No duality gap

For our main results in Section we will be crucial to establish the equality between the Primal \(P^{\text{ext}}[\mu]\) and the Dual \(D[\mu]\) problems. We obtain this in the present section. So far we have kept the flow of marginals \(\mu := \{\mu_t\}_t\) fixed (see Assumption (B)). For this part of the article we shall vary this flow of marginals. Thus, we let
\[
\nu := \{\nu_t\}_t \in C([0,T]; \mathcal{P}^\nu(\mathbb{R}^q)),
\]
stand for a generic weakly continuous flow of measures with \(\nu_0 = m_0\), and use the notation \(P[\nu]\) and \(D[\nu]\) respectively for the Primal and Dual problem under such flow, in accordance to the notations used so far. For convenience, we write \(P[\nu]\) and \(D[\nu]\) for the value of these problems.
Let us define
\[
P^*[f] := \sup_{\nu \in C([0,T];P(\mathbb{R}^q)), \nu_0 = \mu_0} \left\{ \int \int f(t,x) \nu_t(dx) dt - P[\nu] \right\},
\]
\[
P^{**}[\nu] := \sup_{f \in \mathcal{C}} \left\{ \int \int f(t,x) \nu_t(dx) dt - P^*[f] \right\}.
\]

Lemma 5.1. We have:
1. \( \{\nabla f : f \in C_b^{1,2}([0,T] \times \mathbb{R}^q)\} \subset L_{\mathcal{C}}^2 \).
2. Problem \((D[\nu])\) is equal to
\[
\sup_{h \in C_b^{1,2}([0,T] \times \mathbb{R}^q)} \left\{ - \int h(T,x) \nu_T(dx) + \int h(0,x) m_0(dx) + \int \int [L_t h(t,x) - g(t,x,\sigma(t,x) \nabla h(t,x))] \nu_t(dx) dt \right\},
\]
3. Problem \((D[\nu])\) is equal to
\[
\sup_{f \in \mathcal{C}} \left\{ \int \int f(t,x) \nu_t(dx) dt - \int \left[ \frac{E^g_0}{0} \left( \int_0^T f(t,X_t) dt \right) \right] (x_0) m_0(dx_0) \right\},
\]
where \(E^g_0\) denotes the minimal supersolution operator.

Proof.
For Point 1 we follow the final part of the proof of \textsuperscript{[23]} Proposition 3.2. One first observes that \( \{\nabla w : w \in \mathcal{C}\} \) is dense in \( \{\nabla f : f \in C_b^{1,2}\} \) with respect to the weak topology \( \langle \nabla f, \nabla w \rangle = \int \int \nabla f a \nabla w \nu_t dt \). By Ascoli Theorem, this shows that \( \{\nabla f : f \in C_b^{1,2}\} \) is in the closure of \( \{\nabla w : w \in \mathcal{C}\} \) w.r.t. the weak topology \( \sigma(L^g, L^g^*) \). By Mazur’s Lemma this closure coincides with \( L_{\mathcal{C}}^2 \) and we conclude.

We prove Point 2. Clearly \( D[\nu] \geq D[\nu] \). For the converse, we may assume \( D[\nu] < \infty \). One verifies, for all \( h \in C_b^{1,2} \), that
\[
L(\nabla h) = - \int h(T,x) \nu_T(dx) + \int h(0,x) \nu_0(dx) + \int \int L_t h(t,x) \nu_t(dx) dt,
\]
by Point 1 and standard approximation arguments. We conclude by Lemma \textsuperscript{[4,5]}

Finally we prove Point 3. Let \( f \in \mathcal{C} \) and observe that
\[
E^g_0 \left( \int_0^T f(t,X_t) dt \right) = E^g_0 + g(0),
\]
both being functions of \( X_0 \). By \textsuperscript{[32]} Theorem 5.2, which is applicable thanks to \textsuperscript{[32]} Proposition 3.5.(iv) and our Assumption (C).3, the above values equal \( u(0,X_0) \), where \( u \) is the minimal viscosity supersolution of
\[
L_t u + g(t,x,\sigma(t,x) \nabla u(t,x)) + f(t,x) \leq 0, \quad u(T,\cdot) \geq 0.
\]
Let \( \Phi \) be a sufficiently smooth function\textsuperscript{[2]} such that \( L_t \Phi(t,x) + g(t,x,\sigma(t,x) \nabla \Phi(t,x)) + f(t,x) \leq 0 \) and \( \Phi(T,\cdot) \geq 0 \). Then
\[
\int \int f(t,x) \nu_t(dx) dt \leq \int \Phi(T,x) \nu_T(dx) - \int \int [L_t \Phi(t,x) + g(t,x,\sigma(t,x) \nabla \Phi(t,x))] \nu_t(dx) dt.
\]
\textsuperscript{[2]}For instance \( \Phi(t,x) := C - t \sup_{s,y} |f(s,y)| \) with \( C \geq T \sup_{s,y} |f(s,y)| \) fulfills this.
By [33, Theorem 5], we actually have \( u(0, X_0) = \inf \Phi(0, X_0) \text{ m}_0\text{-a.s.}, \) namely that the minimal viscosity supersolution is the infimum over classical supersolutions. From this and the previous considerations, we obtain for each \( \epsilon > 0 \) the existence of \( \Phi = \Phi^\epsilon \) such that

\[
\iint f(t, x)\nu_t(dx)dt - \int \mathbb{E}_0^\omega \left( \int_0^T f(t, X_t)dt \right) dm_0(X_0) = -\int u(0, x)m_0(dx) + \iint f(t, x)\nu_t(dx)dt \\
\leq \epsilon - \int \Phi(0, x)m_0(dx) + \int \Phi(T, x)\nu_T(dx) - \iint |L_t\Phi(t, x) + g(t, x, \sigma'\nabla \Phi(t, x))|\nu_t(dx)dt \\
\leq \epsilon + D3[\nu] \\
= \epsilon + D[\nu].
\]

The last inequality comes from taking \(-h\) instead of \( h \) in \([D3[\nu]]\). So \( D4[\nu] \leq D[\nu] \). The converse inequality follows by taking, for each \( w \in C \), \( f^w := L_tw - g(\sigma'\nabla w) \), and elementary approximation arguments.

**Lemma 5.2** We have

\[
P^*[f] = \int \mathbb{E}_0^\omega \left( \int_0^T f(t, X_t)dt \right) (x_0)dm_0(x_0), \text{ for } f \in C_b([0, T] \times \mathbb{R}^d),
\]

and

\[
D[\nu] = \sup_{f \in C} \left\{ \iint f(t, x)\nu_t(dx)dt - \int \mathbb{E}_0^\omega \left( \int_0^T f(t, X_t)dt \right) (x_0)dm_0(x_0) \right\},
\]

\[
= P^{**}[\nu].
\]

**Proof.** We start with (5.3). To wit

\[
P^*[f] = \sup_{\nu \in C([0, T] \times \mathbb{R}^d), \nu_0 = \text{m}_0} \left\{ \iint f(t, x)\nu_t(dx)dt - P[\nu] \right\}
\]

\[
= \sup_{\nu \in C([0, T] \times \mathbb{R}^d), \nu_0 = \text{m}_0, Q \in \mathcal{Q}(\nu)} \mathbb{E}^Q \left[ \int_0^T f(t, X_t)dt - \int_0^T g^*(t, X_t, \sigma'(t, X_t)\tilde{\nu}_t^Q)dt \right]
\]

\[
= \sup_{Q} \left\{ \mathbb{E}^Q \left[ \int_0^T f(t, X_t)dt \right] - \tilde{I}(Q) \right\}
\]

\[
= \int \mathbb{E}_0^\omega \left( \int_0^T f(t, X_t)dt \right) (x_0)dm_0(x_0),
\]

by Lemma 5.2. This shows that

\[
P^{**}[\nu] = \sup_{f \in C} \left\{ \iint f(t, x)\nu_t(dx)dt - \int \mathbb{E}_0^\omega \left( \int_0^T f(t, X_t)dt \right) (x_0)dm_0(x_0) \right\},
\]

which yields (5.5), whereas (5.4) follows by Lemma 5.1. 

**Proposition 5.1** We have \( P[\nu] = P^{**}[\nu] = D[\nu] \), i.e. there is no duality gap.
Proof. It is easy to see that \( P[\nu] \geq P^{**}[\nu] \) and that \( P[\cdot] \) is convex. In light of Lemma 3.2 to obtain no duality gap it suffices to prove \( P[\nu] = P^{**}[\nu] \). We now establish that \( P[\cdot] \) is lower-semicontinuous in an appropriate sense. Let \( \{\nu^\alpha\} \) be a net in \( C([0,T]; \mathcal{P}(\mathbb{R}^q)) \) for which \( P[\nu^\alpha] \leq k \) and

\[
\forall f \in \mathcal{C} : \int_0^T \int f(t,x)\nu^\alpha_t(dx)dt \rightarrow \int_0^T \int f(t,x)\nu_t(dx)dt,
\]

for some \( \nu \in C([0,T]; \mathcal{P}(\mathbb{R}^q)) \); we may assume all these functions start at \( m_0 \) at time zero. By Lemma 3.2 we have that \( P[\nu^\alpha] = \bar{I}(Q^\alpha) \leq k \) for unique probability measures \( Q^\alpha \in \mathcal{Q}(\nu^\alpha) \). By Lemma 3.1 the family \( \{Q^\alpha\} \) is tight. Let \( Q \) be any accumulation point. For ease of notation we still index the subnet accumulating into \( Q \) by the same indices. By the lower semicontinuity of \( \bar{I} \) given in Lemma 3.1, we obtain \( \bar{I}(Q) \leq k \). On the other hand, for each \( f \in \mathcal{C} \) we have

\[
\mathbb{E}^Q \left[ \int_0^T f(t,X_t)dt \right] = \lim \mathbb{E}^{Q^\alpha} \left[ \int_0^T f(t,X_t)dt \right] = \lim \int \int f(t,x)\nu^\alpha_t(dx)dt = \int \int f(t,x)\nu_t(dx)dt.
\]

Now take \( F \) a smooth function on \( \mathbb{R}^q \) with bounded support, \( \bar{t} \in (0,T) \) and \( m^\alpha \) a sequence of smooth functions of time converging monotonically (hence uniformly) to \( 1_{(\bar{t},\bar{t}+\epsilon)} \). Take \( f(t,x) = m^\alpha(t)F(x) \). By monotone convergence and the above equality, we get

\[
\mathbb{E}^Q \left[ \frac{1}{\epsilon} \int_{\bar{t}}^{\bar{t}+\epsilon} F(X_t)dt \right] = \frac{1}{\epsilon} \int_{\bar{t}}^{\bar{t}+\epsilon} \int_{\mathbb{R}^q} F(x)\nu_t(dx)dt.
\]

By dominated convergence we get as \( \epsilon \to 0 \) that

\[
\mathbb{E}^Q[F(X_t)] = \int F(x)\nu_t(dx),
\]

since \( t \to \nu_t \) is weakly continuous. This identity must also hold for \( F \) continuous bounded by further approximation arguments. The limiting cases of \( \bar{t} \in \{0,T\} \) follow taking limits, as \( t \to \nu_t \) is weakly continuous. Therefore \( Q \) has \( \nu \) as its marginal flow. Since \( \bar{I}(Q) < \infty \) we conclude that \( Q \in \mathcal{Q}(\nu) \), therefore is feasible for \( (P[\nu]) \), and we deduce \( P[\nu] \leq k \) as desired.

Wrapping up, we obtained that \( P[\cdot] \) is convex and lower semicontinuous w.r.t. pointwise convergence on \( \mathcal{C} \) (i.e. in the sense of (5.6)). By construction \( P^{**}[\cdot] \) is the greatest minorant of \( P[\cdot] \) having these properties, so we conclude \( P^{**}[\cdot] = P[\cdot] \). \( \blacksquare \)

6 Proof of the main result

The following is a crucial result for this part:

Proposition 6.1 \( \text{value}[D_0[\mu]] < \infty \) is equivalent to the existence of some \( \Psi \in L^2_N \) such that for all \( w \in \mathcal{C} \):

\[
\int \int [L_t w(t,x) - \nabla g(t,x,\sigma'(t,x)\Psi(t,x))'\sigma'(t,x)\nabla w(t,x)]\mu_t(dx)dt = 0 \quad (6.1)
\]

18
When this holds, then $\Psi$ is an optimizer for $(D_{\mu})$ and
\[
\text{value}(D_{\mu}) = \text{value}(D_{\mu}) = \int \int g^*(t, x, \nabla g(t, x, \sigma'(t, x)\Psi(t, x))) \mu_t(dx)dt. \quad (6.2)
\]
This $\Psi$ is $\mu_t(dx)dt$-a.s. unique, and we further have that
\[(t, x) \mapsto (\sigma')^{-1}(t, x)\nabla g(t, x, \sigma'(t, x)\Psi(t, x))\]
is the unique (up to equivalence class) representative of $L$ (cf. Lemma 4.4).

**Proof.** First we assume $\text{value}(D_{\mu}) < \infty$. Clearly (4.2) implies that $L(\psi) \leq \{1 + \text{value}(D_{\mu})\} \|\psi\|_g$. We thus find
\[
L(\phi) - \int \int g(t, x, \sigma'(t, x)\psi(t, x)), \mu_t(dx)dt \leq \|\psi\|_g \left[1 + \text{value}(D_{\mu}) - G(\psi)\right],
\]
in the notation of Lemma 4.5. Using (4.6) we find that the l.h.s. goes to $-\infty$ if we let $\|\psi\|_g \to \infty$. We deduce that computing $(D_{\mu})$ can be done over a fixed ball in $L^g_\psi$. But $L^g$ is reflexive by Lemma 4.2 so balls in $L^g_\psi$ are weakly compact. The objective function of the extended dual problem being concave continuous (see Lemmata 4.5, 4.3), it is also weakly upper semi-continuous. We conclude the existence of an optimizer for the extended dual problem. Let $\Psi$ denote any optimizer and $\nabla w$ any “direction”. The optimality of $\Psi$ easily yields
\[
\int \int L_t w(t, x) \mu_t(dx)dt - DG(\Psi)(\nabla w) = 0.
\]
Thanks to (4.7) this proves (6.1), which further implies for all $\psi \in L^g_\psi$:
\[
L(\psi) = \int \int \nabla g(t, x, \sigma'(t, x)\Psi(t, x))'\sigma'(t, x)\psi(t, x) \mu_t(dx)dt. \quad (6.3)
\]
For the converse direction, we observe that (6.1) combined with Lemma 4.6 allows to perform the continuous extension $L$ of Lemma 4.3. Thus one can define the extended dual problem anew. By (6.1) and continuity, the extended dual becomes
\[
\sup_{\psi \in L^g_\psi} \int \int [\nabla g(t, x, \sigma'(t, x)\Psi(t, x))'\sigma'(t, x)\psi(t, x) - g(t, x, \sigma'(t, x)\psi(t, x))] \mu_t(dx)dt,
\]
which is bounded above by the r.h.s. of (6.2) by convex conjugacy. This bound is finite by Lemma 4.6, so a fortiori the non-extendeed dual problem is finite as desired.

For (6.2), substitute (6.3) into the extended dual (evaluated at $\Psi$), obtaining
\[
\text{value}(D_{\mu}) = \int \int [\nabla g(t, x, \sigma'(t, x)\Psi(t, x))'\sigma'(t, x)\Psi(t, x) - g(t, x, \sigma'(t, x)\Psi(t, x))] \mu_t(dx)dt,
\]
which in effect yields (6.2) due to the conjugacy relationship. The remark on uniqueness of $\Psi$ follows from the differentiability of $g^*$, which implies the strict convexity of $g$. The last statement follows by (6.1) (equiv. (6.3)), which implies that the given element does represent $L$ acting on $L^g_\psi$, and Lemma 4.4 implying uniqueness of such representative up to equivalence class. 

19
We can now prove the main structural result of the article.

**Proof of Theorem 1.1.** Absence of duality gap was obtained in Proposition 5.1. From now on we assume \(P_{ext}[\mu] < \infty\). The existence of a unique optimal \(\Psi\) is given by Proposition 6.1. The existence of a (unique) primal optimizer \(Q\) was established in Lemma 3.2. We proceed to show that this \(Q\) must have the desired property.

Since \(Q \in Q(\mu)\), we have for some drift \(\beta\):

\[
\mathbb{E}^Q \left[ \int_0^T (L_t + \beta'_t a \nabla) w(t, X_t) dt \right] = 0, \forall w \in C.
\]

Let \(\bar{\beta}(t, x) = \mathbb{E}^Q[\beta_t | X_t = x]\), so that obviously

\[
-\int \int L_t w(t, x) \mu_t(dx) dt = \int \int \bar{\beta}(t, x) a(t, x) \nabla w(t, x) \mu_t(dx) dt, \forall w \in C. \quad (6.4)
\]

Plugging in this representation of the l.h.s. into the dual problem, and using the Young-Fenchel inequality we obtain

\[
\text{value}(D_0[\mu]) \leq \int \int g^*(t, x, \sigma'(t, x) \bar{\beta}(t, x)) \mu_t(dx) dt. \quad (6.5)
\]

By Jensen’s inequality, the fact that \(Q\) has marginals \(\{\mu_t\}_t\), the above equation and (6.2), we deduce

\[
\text{value}(P_{ext}[\mu]) = \int \int g^*(t, x, \sigma'(t, x) \bar{\beta}(t, x)) \mu_t(dx) dt \geq \text{value}(D_0[\mu]) \geq \int \int g^*(t, x, \nabla g(t, x, \sigma'(t, x) \Psi(t, x))) \mu_t(dx) dt.
\]

By no duality gap, the above inequalities are actual equalities. Since \(g^*\) is strictly convex, this shows that

\[
Q \times dt - a.s. \quad \beta_t(X) = \bar{\beta}(t, X_t). \quad (6.7)
\]

On the other hand (6.1) with (6.6) show that the problem

\[
\inf_{k(\cdot, \cdot), \forall w \in C} \int \int g^*(t, x, \sigma' k(t, x)) \mu_t(dx) dt,
\]

has \(-\bar{\beta}(\cdot, \cdot)\) and \((\sigma')^{-1}(\cdot, \cdot)\nabla g(\cdot, \cdot, \sigma'(\cdot, \cdot) \Psi(\cdot, \cdot))\) as feasible elements, where the latter is optimal. Indeed, (6.5) holds also for any \(k(\cdot, \cdot)\) participating in the infimum above.

Again by strict convexity of \(g^*\) and the equality in (6.6) we find that

\[
\mu_t(dx) \times dt - a.s. \quad \bar{\beta}(t, x) = -(\sigma')^{-1}(t, x) \nabla g(t, x, \sigma'(t, x) \Psi(t, x)). \quad (6.8)
\]
Calling \( \Lambda \subset \mathbb{R}^q \times [0, T] \) the set on which (6.8) fails, we have
\[
0 = \int \int \mathbf{1}_\Lambda(t, x) \mu_t(dx)dt = \int_0^T \mathbb{E}^Q[\mathbf{1}_\Lambda(t, X_t)]dt,
\]
showing that
\[
Q \times dt - a.s. \quad \bar{\beta}(t, X_t) = - (\sigma')^{-1}(t, X_t) \nabla g(t, X_t, \sigma'(t, X_t)\Psi(t, X_t)). \tag{6.9}
\]
Putting (6.7) and (6.9) together, we find (1.4). From this (1.5) is also clear.

7 Proofs of the main corollaries

We prove here Corollaries 1.1 and 1.2. Most of the effort is devoted to the construction of counterexamples.

**Proof of Corollary 1.1.** We show in Section 7.1 below an example of an optimizer without the Markov property. Let us now assume the sufficient condition in the statement, so we have
\[
\frac{dQ}{dP} := \mathbb{E} \left( - \int \nabla g(t, X_t, a(t, X_t)\Psi(t, X_t))'dM_t \right) .
\]
The argument is now as in [49, Theorem 12]. Let us call \( Z_t \) the associated density process, which is a true \( P \)-martingale. Let \( s \leq t \), \( F \) be an \( \mathcal{F}_s \)-measurable bounded function and \( f : \mathbb{R}^q \to \mathbb{R} \) Borel bounded. Then
\[
\mathbb{E}^Q[F f(X_t)] = \mathbb{E}^P[F Z_s \mathbb{E} \left( - \int_s^t \nabla g(t, X_t, a(t, X_t)\Psi(t, X_t))'dM_t \right) f(X_t)]
\]
\[
= \mathbb{E}^P[F Z_s \mathbb{E}^P \left[ \mathbb{E} \left( - \int_s^t \nabla g(t, X_t, a(t, X_t)\Psi(t, X_t))'dM_t \right) f(X_t) \bigg| \mathcal{F}_s \right]]
\]
\[
= \mathbb{E}^Q[F \mathbb{E}^P \left[ \mathbb{E} \left( - \int_s^t \nabla g(t, X_t, a(t, X_t)\Psi(t, X_t))'dM_t \right) f(X_t) \bigg| X_s \right]].
\]
The last equality by the Markov property under \( P \) and the fact that nothing in the stochastic exponential there depends on \( \{X_r : r \leq s\} \). This finishes the proof.

**Proof of Corollary 1.2.** The assertion in one-dimension is fully analogous to [24, Proposition 5.2 and Remark 5.5]. Indeed, there is actually at most one Markovian measure with the given marginals and with an integrable drift. For higher dimensions, see the example in Section 7.2 below.

7.1 Non-Markovian optimal solution

The (counter)example is based on the Bessel(\( \delta \)) process, with dimension parameter \( 1 < \delta < 2 \), equiv. index \( \nu = \delta/2 - 1 \in (-1/2, 0) \); see [15, Appendix I.21]. From the expression of the probability density function \( p^\nu \) of this process, and the asymptotics of Bessel functions, we have that
\[
p^\nu_t(x, y) y^{-2\nu - 1}
\]
is bounded away from zero and infinity, for each \( t > 0 \) fixed and \( y \) in a neighbourhood of the origin. Therefore

\[
\int y^{-p} \nu_t(x,y) \, dy < \infty,
\]
as soon as \( 1 < p < 2\nu + 2 \). Denoting by \( X \) the Bessel process described, it is an easy consequence of scaling and the finite integral above, that

\[
\mathbb{E}^{x_0} \left[ \int_0^1 \frac{1}{|X_t|^p} \, dt \right] < \infty.
\]

We recall that \( X \) started at \( x_0 > 0 \) satisfies

\[
dX_t = \frac{\delta - 1}{2X_t} \, dt + dW_t, \quad X_0 = x_0,
\]

and is in fact the unique positive solution of this SDE. Actually, the origin is instantaneously reflected by this process. Denote \( \tau \) the first time that \( X \) touches the origin. We now construct a second process, as in [26, Example 3.10], by

\[
Y_t = X_t \text{ if } t \leq \tau,
\]
and \( Y_t = \text{sign}(X_{\tau/2} - 1)X_t \), for \( t > \tau \). One can see that \( Y \) is a weak solution of the same SDE as \( X \), and has the same finite moment

\[
\mathbb{E} \left[ \int_0^1 \frac{1}{|Y_t|^p} \, dt \right] < \infty.
\]

On the other hand \( Y \) is clearly non-Markovian. Denoting \( \mu_t := \text{Law}(Y_t) \), and taking \( \mathbb{P} \) the Wiener measure, we have

**Lemma 7.1** *Law(Y) is the unique optimizer of our primal problem for the cost \( g^*(t, x, b) = |b|^p/p \) and the marginals \( \{\mu_t\}_t \), with finite optimal cost if \( 1 < p < 2\nu + 2 \). In particular, solutions to our primal problem can fail to have the Markov property even if the value of the problem is finite.*

**Proof.** We have \( g(z) = |z|^q/q \) with \( q \) the Hölder conjugate of \( p \). By the first order conditions of the dual problem problem, and the fact that \( \nabla g(z) = \text{sign}(z)|z|^{q-1} \), it is easy to guess that

\[
\Psi(t, x) = -\left(\frac{\delta - 1}{2} |x|^{-1}\right)^{\frac{q-1}{q-1}} \text{sign}(x),
\]
is the dual optimizer. Indeed, to see that \( \Psi \) is an \( L^q(d\mu_t \, dt) \)-limit of gradients, we just consider \( w_n(x) = -\frac{\delta - 1}{2} \times \frac{q-1}{q-2} \left( \frac{2}{\delta} |x| + n^{-1} \right)^{\frac{q-2}{q-1}} \), take gradients, and use dominated convergence. \( \blacksquare \)

**7.2 Non-universality of optimal solution**

We shall see that the optimizer can depend on the cost criterion. Let \( q = 2 \). For simplicity we shall consider a “stationary” case. We do so only to spare the reader with the heavier computations needed for the “non-stationary” analogue argument. The cost to pay is that the marginal distributions \( \{\mu_t\} \) must be \( \sigma \)-finite measures.
Let $B : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable with bounded support. We take

$$dX_t = \nabla B(X_t)dt + dW_t,$$

with initial condition $X_0$ distributed like two-dimensional Lebesgue measure, that is $\text{Law}(X_0) = \lambda^2$. We denote by $\mathbb{P}$ the law of the unique strong solution of this SDE. We denote by $Q^{ent}$ the law of stationary (i.e. reversible) Brownian motion, that is Brownian motion with initial (and stationary) distribution $\lambda^2$. Let us take $\mu_t = \lambda^2$ for all $t$, so the $t$-marginals of $Q^{ent}$ are precisely $\mu_t$. It is easy to see that

$$dQ^{ent}/d\mathbb{P} = \exp \left\{ - \int_0^T \nabla B(X_t)\cdot dW_t - \frac{1}{2} \int_0^T |\nabla B(X_t)|^2 dt \right\},$$

and that $Q^{ent}$ is optimal for the entropy minimization (primal) problem

$$\inf \left\{ \mathbb{E}_Q \left[ \int_0^T \|\beta Q\|^2 dt \right] : d\mathbb{Q}/d\mathbb{P} = \mathcal{E} \left(-\int \beta' dW\right)_T, \ Q \circ X_t^{-1} = \lambda^2 \text{ for all } t \right\}.$$ 

Indeed, taking $\beta^B(t,X) := \nabla B(X_t)$ ensures producing the correct marginals, provides finite entropy, and has to be an optimal choice being a gradient (for instance by first order conditions, or see previous sections with $g$ quadratic).

We now claim that different cost criteria than the above quadratic one may yield different optimizers. Consider

$$\inf \left\{ \mathbb{E}_Q \left[ \int_0^T \|\beta Q\|^3 dt \right] : d\mathbb{Q}/d\mathbb{P} = \mathcal{E} \left(-\int \beta' dW\right)_T, \ Q \circ X_t^{-1} = \lambda^2 \text{ for all } t \right\}.$$ 

Observe that the power cost $g^*(\cdot) := (\cdot)^3$ satisfies our assumptions and that $Q^{ent}$ is feasible and produces a finite value for this cost criterion. We also have $g(\cdot) = \frac{2}{3} (\cdot)^{3/2}$. The optimizer for this problem has the structure

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \mathcal{E} \left(-\int \nabla g(\Psi(t,X_t))'dW_t\right)_T = \mathcal{E} \left(-\int \frac{\Psi(t,X_t)'}{\sqrt{\|\Psi(t,X_t)\|^2}}dW_t\right)_T,$$

for $\Psi$ a solution to the dual problem, and so a limit of gradients. We want to give conditions so that $Q^{ent} \neq \mathcal{Q}$. For the sake of the argument let us assume now that $\Psi = \nabla w$ for $w$ suitable smooth. So we want to ensure the impossibility of

$$\nabla B = \frac{\nabla w(t, X_t)}{\sqrt{\|\nabla w(t, X_t)\|}}.$$ 

Taking norms on both sides we get $\|\nabla B\| = \sqrt{\|\nabla w\|}$, so we explore instead

$$\|\nabla B\| \nabla B = \nabla w.$$ 

(7.1)

The argument is simple now. For the r.h.s. we know, no matter who $w$ may be, that $\partial_x \partial_x w = \partial_y (1\text{st coordinate of the r.h.s.}) = \partial_x \partial_y w = \partial_x (2\text{nd coordinate of the r.h.s.})$. But by (7.1) one computes that this is possible only if

$$\partial_x B[\partial_x B \partial_{yy}^2 B + \partial_x B \partial_{xx}^2 B] = \partial_y B[\partial_x B \partial_{xx}^2 B + \partial_y B \partial_{yy}^2 B].$$

So choosing $B$ such that this does not occur (for instance take $B(x,y) = p(x)q(y)$ with $p,q$ non-trivial, smooth and with bounded support) we see that there is no smooth $w$ for which (7.1) may hold. The general case with $\Psi$ is similar, by integration by parts and from the fact that $\Psi$ is a limit of actual gradients. In such case, no matter who the dual optimizer is, the induced optimal measure will not have a stochastic logarithm equal to $\nabla B(t, X_t)$.

23
References

[1] Beatrice Acciaio, Julio Backhoff-Veraguas, and René Carmona. Extended mean field control problems: stochastic maximum principle and transport perspective. *SIAM Journal on Control and Optimization*, 57(6):3666–3693, 2019.

[2] Beatrice Acciaio, Julio Backhoff-Veraguas, and Junchao Jia. Cournot-Nash equilibrium and optimal transport in a dynamic setting. *arXiv preprint arXiv:2002.08786*, 2020.

[3] Daniel Andersson and Boualem Djehiche. A maximum principle for SDEs of mean-field type. *Applied Mathematics & Optimization*, 63(3):341–356, 2011.

[4] Marc Arnaudon, Ana Bela Cruzeiro, Christian Léonard, and Jean-Claude Zambrini. An entropic interpolation problem for incompressible viscous fluids. *arXiv preprint arXiv:1704.02126*, 2017.

[5] Julio Backhoff-Veraguas, Giovani Conforti, Ivan Gentil, and Christian Léonard. The mean field Schrödinger problem: ergodic behavior, entropy estimates and functional inequalities. *arXiv preprint arXiv:1905.02393*, 2019.

[6] Julio Backhoff-Veraguas, Daniel Lacker, and Ludovic Tangpi. Non-exponential Sanov and Schilder theorems on Wiener space: BSDEs, Schrödinger problems and Control. *Forthcoming at Annals of Applied Probability*, 2018.

[7] Julio Backhoff-Veraguas and Gudmund Pammer. Applications of weak transport theory. *arXiv preprint arXiv:2003.05338*, 2020.

[8] Aymeric Baradat. On the existence of a scalar pressure field in the Brézis–Gallouët problem. *SIAM Journal on Mathematical Analysis*, 52(1):370–401, 2020.

[9] Aymeric Baradat and Léonard Monsaingeon. Small noise limit and convexity for generalized incompressible flows, Schrödinger problems, and optimal transport. *Archive for Rational Mechanics and Analysis*, pages 1–47, 2019.

[10] M. Beiglböck, M. Huesmann, and F. Stebegg. Root to Kellerer. *Séminaire de Probabilités, to appear*, 2016.

[11] Jean-David Benamou, Guillaume Carlier, Marco Cuturi, Luca Nenna, and Gabriel Peyré. Iterative Bregman projections for regularized transportation problems. *SIAM Journal on Scientific Computing*, 37(2):A1111–A1138, 2015.

[12] Jean-David Benamou, Guillaume Carlier, and Luca Nenna. Generalized incompressible flows, multi-marginal transport and Sinkhorn algorithm. *Numerische Mathematik*, 142(1):33–54, 2019.

[13] Alain Bensoussan, Jens Frehse, and Phillip Yam. *Mean Field Games and Mean Field Type Control Theory*, volume 101. Springer, 2013.

[14] Amel Bentata and Rama Cont. Forward equations for option prices in semimartingale models. *Finance and Stochastics*, 19(3):617–651, 2015.

[15] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[16] Yann Brenier. The least action principle and the related concept of generalized flows for incompressible perfect fluids. *Journal of the American Mathematical Society*, 2(2):225–255, 1989.
[17] Yann Brenier. The dual least action problem for an ideal, incompressible fluid. Archive for rational mechanics and analysis, 122(4):323–351, 1993.

[18] G. Brunick and S. Shreve. Mimicking an Itô process by a solution of a stochastic differential equation. Ann. Appl. Probab., 23(4):1584–1628, 2013.

[19] Rainer Buckdahn, Boualem Djehiche, and Juan Li. A general stochastic maximum principle for SDEs of mean-field type. Applied Mathematics & Optimization, 64(2):197–216, 2011.

[20] Rainer Buckdahn, Boualem Djehiche, Juan Li, and Shige Peng. Mean-field backward stochastic differential equations: a limit approach. The Annals of Probability, 37(4):1524–1565, 2009.

[21] René Carmona and François Delarue. Forward–backward stochastic differential equations and controlled McKean–Vlasov dynamics. The Annals of Probability, 43(5):2647–2700, 2015.

[22] René Carmona and François Delarue. Probabilistic Theory of Mean Field Games with Applications. Volume I: Mean Field FBSDEs, Control and Games. Springer, 2017.

[23] P. Cattiaux and C. Léonard. Large deviations and Nelson processes. Forum Math., 7(1):95–115, 1995.

[24] Patrick Cattiaux and Christian Léonard. Minimization of the Kullback information of diffusion processes. Ann. Inst. H. Poincaré Probab. Statist., 30(1):83–132, 1994.

[25] Yongxin Chen, Tryphon T Georgiou, and Michele Pavon. On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint. Journal of Optimization Theory and Applications, 169(2):671–691, 2016.

[26] Alexander S. Cherny and Hans-Jürgen Engelbert. Singular stochastic differential equations, volume 1858 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2005.

[27] Giovanni Conforti. A second order equation for Schrödinger bridges with applications to the hot gas experiment and entropic transportation cost. Probability Theory and Related Fields, 174(1-2):1–47, 2019.

[28] Marco Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. In Advances in neural information processing systems, pages 2292–2300, 2013.

[29] Freddy Delbaen, Ying Hu, and Xiaobo Bao. Backward SDEs with superquadratic growth. Probab. Theory Related Fields, 150(1-2):145–192, 2011.

[30] Samuel Drapeau, Gregor Heyne, and Michael Kupper. Minimal supersolutions of convex BSDEs. Ann. Probab., 41(6):3973–4001, 2013.

[31] Samuel Drapeau, Michael Kupper, Emanuela Rosazza Gianin, and Ludovic Tangpi. Dual representation of minimal supersolutions of convex BSDEs. Ann. Inst. Henri Poincaré Probab. Stat., 52(2):868–887, 2016.

[32] Samuel Drapeau and Christoph Mainberger. Stability and Markov property of forward backward minimal supersolutions. Electron. J. Probab., 21:Paper No. 41, 15, 2016.

[33] Wendell H. Fleming and Domokos Vermes. Convex duality approach to the optimal control of diffusions. SIAM J. Control Optim., 27(5):1136–1155, 1989.
[34] I. Gyöngy. Mimicking the one-dimensional marginal distributions of processes having an Itô differential. Probab. Theory Relat. Fields, 71(4):501–516, 1986.

[35] I. Gyöngy. Mimicking complicated stochastic differential equations by simpler ones. In Probability theory and mathematical statistics with applications (Visegrád, 1985), pages 87–96. Reidel, Dordrecht, 1988.

[36] F. Hirsch, C. Profeta, B. Roynette, and M. Yor. Peacocks and associated martingales, with explicit constructions, volume 3 of Bocconi & Springer Series. Springer, Milan; Bocconi University Press, Milan, 2011.

[37] Nicolas Juillet. Peacocks parametrised by a partially ordered set. In Séminaire de probabilités XLVIII, pages 13–32. Springer, 2016.

[38] Sigrid Källblad, Xiaolu Tan, Nizar Touzi, et al. Optimal skorokhod embedding given full marginals and azéma–yor peacocks. The Annals of Applied Probability, 27(2):686–719, 2017.

[39] Hans G. Kellerer. Integraldarstellung von Dilationen. In Transactions of the Sixth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes (Tech. Univ., Prague, 1971; dedicated to the memory of Antonín Špaček), pages 341–374. Academia, Prague, 1973.

[40] A. Kozek. Orlicz spaces of functions with values in Banach spaces. Comment. Math. Prace Mat., 19(2):259–288, 1976/77.

[41] A. Kozek. Convex integral functionals on Orlicz spaces. Comment. Math. Prace Mat., 21(1):109–135, 1980.

[42] D. Lacker. A non-exponential extension of sanov’s theorem via convex duality. Preprint.

[43] Daniel Lacker. Mean field games via controlled martingale problems: existence of Markovian equilibria. Stochastic Processes and their Applications, 125(7):2856–2894, 2015.

[44] Daniel Lacker. Limit theory for controlled McKean–Vlasov dynamics. SIAM Journal on Control and Optimization, 55(3):1641–1672, 2017.

[45] Christian Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. Discrete Contin. Dyn. Syst., 34(4):1533–1574, 2014.

[46] G. Lowther. Fitting martingales to given marginals. ArXiv e-prints, August 2008.

[47] Thilo Meyer-Brandis, Bernt Øksendal, and Xun Yu Zhou. A mean-field stochastic maximum principle via Malliavin calculus. Stochastics An International Journal of Probability and Stochastic Processes, 84(5-6):643–666, 2012.

[48] Toshio Mikami. Semimartingales from the Fokker-Planck equation. Appl. Math. Optim., 53(2):209–219, 2006.

[49] W. A. Zheng. Tightness results for laws of diffusion processes application to stochastic mechanics. Ann. Inst. H. Poincaré Probab. Statist., 21(2):103–124, 1985.