Discontinuity growth of interval exchange maps

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Abstract

For an interval exchange map \( f \), the number of discontinuities \( d(f^n) \) either exhibits linear growth or is bounded independently of \( n \). This dichotomy is used to prove that the group \( \mathcal{E} \) of interval exchanges does not contain distortion elements, giving examples of groups that do not act faithfully via interval exchanges. As a further application of this dichotomy, a classification of centralizers in \( \mathcal{E} \) is given. This classification is used to show that \( \text{Aut}(\mathcal{E}) \cong \mathcal{E} \rtimes \mathbb{Z}/2\mathbb{Z} \).

1 Introduction

An interval exchange transformation is a map \( T^1 \to T^1 \) defined by a partition of the circle into a finite union of half-open intervals and a rearrangement of these intervals by translation. See Figure 1 for a graphical example.

The dynamics of interval exchanges were first studied in the late seventies by Keane [2], [3], Rauzy [9], Veech [10], and others. This initial stage of research culminated in the independent proofs by Masur [7] and Veech [11] that almost every interval exchange is uniquely ergodic. See the recent survey of Viana [12] for a unified presentation of these results. The current study of interval exchanges is closely related to dynamics on the moduli space of translation surfaces; an introduction to this topic and its connection to interval exchanges is found in a survey of Zorich [14].

To precisely define an interval exchange, let \( \pi \in \Sigma_n \) be a permutation and let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a vector in the simplex

\[
\Lambda_n = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0 \text{ and } \sum \lambda_i = 1 \}.
\]

The vector \( \lambda \) induces a partition of \( T^1 \cong [0,1) \) into intervals of length \( \lambda_j \):

\[
I_j = [\beta_{j-1}, \beta_j) \quad 1 \leq j \leq n,
\]

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where $\beta_0 = 0$, $\beta_j = \sum_{i=1}^{j} \lambda_i$ for $1 \leq j \leq n$.

The interval exchange $f_{(\pi,\lambda)}$ reorders the $I_j$ by translation, such that their indices are ordered by $\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n)$. Consequently, $f_{(\pi,\lambda)}$ is defined by the formula

$$f_{(\pi,\lambda)}(x) = x - \left( \sum_{i<j} \lambda_i \right) + \left( \sum_{\pi(i) < \pi(j)} \lambda_i \right) = x + \omega_j, \text{ for } x \in I_j.$$  

The vector $\omega(f) = (\omega_1, \ldots, \omega_n)$ is called the translation vector of $f_{(\pi,\lambda)}$.

| Before | I₁ | I₂ | I₃ | I₄ |
|--------|----|----|----|----|
| After  | I₃ | I₁ | I₄ | I₂ |

Figure 1: An interval exchange with $\pi = (2, 4, 1, 3)$

Let $d(f)$ denote the number of discontinuity points of $f : \mathbb{T}^1 \to \mathbb{T}^1$, where $\mathbb{T}^1$ is endowed with its standard topology. If $d(f) = k$, then it is easy to see that for iterates $f^n$, the discontinuity number $d(f^n)$ is bounded above by $k|n|$. It is possible for $d(f^n)$ to have linear growth at a rate which is strictly less than the maximum $d(f)$. For example, the map in Figure 2 has three discontinuities, but iteration will suggest that $d(f^n) \sim 2n$. Additionally, it is possible for $d(f^n)$ to be bounded independently of $n$. For example, a restricted rotation $r_{\alpha,\beta}$, as defined by Figure 3, satisfies $d(r_{n,\beta}^n) \leq 3$ for all $n \in \mathbb{Z}$. The key result of this paper is the observation that no intermediate growth rate may occur.

**Theorem 1.1.** For any interval exchange $f$, either $d(f^n)$ exhibits linear growth or $d(f^n)$ is bounded independently of $n$.

This theorem is a simplified statement of Proposition 2.3. The linear growth case is generic; for instance, given any irreducible permutation $\pi$ which permutes three or more intervals, $f_{(\pi,\lambda)}$ has linear discontinuity growth if the boundary points between intervals satisfy the infinite distinct orbit condition [2]. This condition is satisfied for the full measure set of $\lambda \in \Lambda_n$ that have rationally independent partition lengths $\{\lambda_i\}$. 

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This raises the question of what may be said about the interval exchanges $f$ for which $d(f^n)$ is bounded. A result of Li \cite{4} stated in Section 3 asserts that under certain additional conditions, the only such topologically minimal examples are maps conjugate to an irrational rotation.

By only assuming that $d(f^n)$ is bounded, it is still possible to give a complete description of the interval exchanges with bounded discontinuity growth. For $\gamma \in \mathbb{R}/\mathbb{Z} \cong [0, 1)$, let $r_\gamma$ denote the rotation $x \mapsto x + \gamma$, which is represented by the data $\pi = (2, 1), \lambda = (1 - \gamma, \gamma)$. An interval exchange is a restricted rotation if it is conjugate by some $r_\gamma$ to some $r_{\alpha, \beta}$. The support of an interval exchange $f$ is the complement of the set of fixed points $\text{Fix}(f)$. The following classification of interval exchanges with bounded discontinuity growth is proved in Section 3.

**Theorem 1.2.** Let $f$ be an infinite-order interval exchange transformation for which $d(f^n)$ is bounded. Then for some $k \geq 1$, $f^k$ is conjugate to a product of infinite-order restricted rotations with pairwise disjoint supports.

The discontinuity growth dichotomy and the subsequent classification of maps with bounded discontinuity growth may be applied to the study of interval exchange group actions. The set $\mathcal{E}$ of all interval exchange transformations
on $T^1$ forms a group under composition. For a group $G$, an \textit{interval exchange action} of $G$ is a group homomorphism $G \to \mathcal{E}$. Such an action is \textit{faithful} if this homomorphism is injective, in which case the image is a subgroup of $\mathcal{E}$ isomorphic to $G$.

Let $G$ be a finitely generated group, and let $S = \{g_1, \ldots, g_n\}$ be a set of generators. An element $f \in G$ is a \textit{distortion element} if $f$ has infinite order and

$$\liminf_{n \to \infty} \frac{|f^n|_S}{n} = 0,$$

where $|\cdot|_S$ denotes the minimal word length in terms of the generators and their inverses. For example, the central elements of the discrete Heisenberg group are distortion elements. In general, if $G$ is not finitely generated, an element $f \in G$ is said to be a distortion element in $G$ if it is a distortion element in some finitely generated subgroup of $G$.

\textbf{Theorem 1.3.} The group $\mathcal{E}$ contains no distortion elements.

A proof of this result is given in Section 4. The main consequence of this theorem is that any group $G$ containing a distortion element has no faithful interval exchange actions. A particularly interesting case is the following, which is analogous to a result of Witte \cite{13} for group actions $SL(n, \mathbb{Z}) \to \text{Homeo}_+(S^1)$ for $n \geq 3$.

\textbf{Corollary 1.4.} Suppose $\Gamma$ is a non-uniform irreducible lattice in a semisimple Lie group $G$ with $\mathbb{R}$-rank $\geq 2$. Suppose further that $G$ is connected, with finite center and no nontrivial compact factors. Then any interval exchange action $\Gamma \to \mathcal{E}$ has finite image.

For example, the lattices $SL(n, \mathbb{Z}), n \geq 3$, satisfy the above hypotheses; consequently, they do not act faithfully via interval exchange maps. This corollary follows from a theorem of Lubotzky, Moses, and Raghunathan \cite{5}.
which states that lattices satisfying the above conditions contain distortion elements (in fact, elements with logarithmic word growth) and a theorem of Margulis [6] which states that any irreducible lattice in a semisimple Lie group of \( \mathbb{R} \)-rank \( \geq 2 \) is almost simple; i.e., any normal subgroup of such a lattice is finite or has finite index.

A further application of the discontinuity growth dichotomy is the complete classification of centralizers in the group \( \mathcal{E} \), which is developed in Section 5. This classification relies on analyzing the centralizer \( C(f) \) in three cases that are distinguished by dynamical characteristics.

**Proposition 1.5.** Let \( f \) be an interval exchange transformation.

(i) \( f \) has periodic points if and only if \( C(f) \) contains a subgroup isomorphic to \( \mathcal{E} \).

(ii) If \( f \) is minimal and \( d(f^n) \) is bounded, then \( C(f) \) is virtually abelian and contains a subgroup isomorphic to \( \mathbb{R}/\mathbb{Z} \).

(iii) If \( f \) is minimal and \( d(f^n) \) has linear growth, then \( C(f) \) is virtually cyclic.

Minimality here refers to topological minimality: every orbit of \( f \) is dense in \( \mathbb{T}^1 \). The three parts of this result are restated and proved separately as Corollary 5.8, Proposition 5.2, and Proposition 5.3, respectively. These cases may be combined to give a general description of centralizers in \( \mathcal{E} \); this is stated and proved as Theorem 5.7.

The classification of centralizers in \( \mathcal{E} \) may be used to investigate the automorphism group \( \text{Aut}(\mathcal{E}) \). Since \( \mathcal{E} \) has trivial center, the inner automorphism group is isomorphic to \( \mathcal{E} \). A further automorphism is induced by switching the orientation of the circle \( \mathbb{T}^1 \). More precisely, let \( T: \mathbb{T}^1 \to \mathbb{T}^1 \) be defined by \( T(x) = -x \). For \( f \in \mathcal{E} \), \( T^{-1}fT \) is still an invertible piecewise translation, but it is now continuous from the left. Let \( \Psi_T \) be the automorphism of \( \mathcal{E} \) defined by conjugation with \( T \) followed by the natural isomorphism from the group of left-continuous interval exchanges to the right-continuous interval exchange group \( \mathcal{E} \).

The automorphism \( \Psi_T \) is of interest because it is not an inner automorphism. One way to see this is through the homomorphism \( \phi: \mathcal{E} \to \mathbb{R} \land \mathbb{Q} \mathbb{R} \), defined by

\[
\phi(f_{(\pi,\lambda)}) = \sum_{i=1}^{n} \lambda_i \land \mathbb{Q} \omega_i.
\]
See Arnoux [1] for a discussion of the properties of this map. The rotation $r_\alpha$ is defined by the data $\pi = (2,1), \lambda = (1-\alpha, \alpha)$, and it may be checked that $\phi(r_\alpha) = 1 \wedge \alpha$. Any inner automorphism preserves $\phi$, but the action of $\Psi_T$ changes the sign of the scissors invariant; for instance,

$$\phi(\Psi_T(r_\alpha)) = \phi(r_{-\alpha}) = -\phi(r_\alpha).$$

$\Psi_T$ is of further interest because it represents the only nontrivial class of outer automorphisms. Section 6 presents a proof of the following result.

**Theorem 1.6.** $Aut(\mathcal{E}) = Inn(\mathcal{E}) \rtimes \langle \Psi_T \rangle \cong \mathcal{E} \rtimes \mathbb{Z}/2\mathbb{Z}$.

Note that the inner automorphisms and the automorphism $\Psi_T$ act via conjugation by a transformation of $T^1$. Thus, all automorphisms of $\mathcal{E}$ are geometric, in the sense that they are induced by the action of $\mathcal{E}$ on $T^1$.

## 2 Discontinuity Growth

For a map $f \in \mathcal{E}$, let $D(f)$ denote the set of points at which $f$ is discontinuous as a map $T^1 \to T^1$. Let $D_{np}(f)$ be those discontinuities of $f$ which are not periodic:

$$D_{np}(f) = D(f) \setminus \text{Per}(f).$$

Note that if $f$ is an infinite-order map and $D(f)$ is nonempty, then $D_{np}(f)$ is also nonempty. If $x \in D_{np}(f)$, both the forward and backward orbits of $x$ eventually consist entirely of points at which $f$ is continuous, since $D_{np}(f)$ is a finite set of points with nonperiodic orbits. Moreover, for each $x \in D_{np}(f)$, there is some $k \geq 0$, such that $f^{-k}(x)$ is the last point of $D_{np}(f)$ encountered in the negative orbit of $x$. In particular, $f$ is continuous at all negative iterates $f^{-n}(x)$ for which $n > k$.

**Definition 2.1.** A nonperiodic discontinuity $x \in D_{np}(f)$ is a *fundamental discontinuity* if $f$ is continuous at all negative iterates of $x$:

$$\{ f^{-i}(x) \}_{i=1}^{\infty} \subseteq T^1 \setminus D(f).$$

The set of fundamental discontinuities of $f$ is denoted $D_F(f)$.

Thus, any point in $D_{np}(f)$ is either a fundamental discontinuity or a forward iterate of a fundamental discontinuity. In particular, the set of fundamental discontinuities is nonempty whenever $D(f)$ is nonempty and $f$ has infinite order.
Let $f_-$ denote the left-continuous form of $f$:

$$f_-(x) = \begin{cases} \lim_{y \to x^-} f(y), & \text{if } f \text{ is discontinuous at } x; \\ f(x), & \text{otherwise}. \end{cases}$$

Similarly, $f_+ = f$ may be used to denote the original right-continuous map. Observe that $(f_-)^n = (f^n)_-$ and $(f_+)^n = (f^n)_+$ for all integers $n$; such compositions are thus denoted $f^n_-$ and $f^n_+$ without ambiguity. It follows that an iterate $f^n$ is continuous at $x$ if and only if

$$f^n_-(x) = f^n_+(x).$$

The sets

$$\{f^n_+(x)\}_{n=0}^{\infty} \text{ and } \{f^n_-(x)\}_{n=0}^{\infty}$$

are called the right and left (forward) orbits of $x$, respectively.

Let $x \in D_{np}(f)$ be a fundamental discontinuity. By the definition of $D_{np}(f)$, the right orbit $\{f^n_+(x)\}$ is nonperiodic. Since $x$ is fundamental, $f$ is continuous at all points in the negative orbit of $x$. Thus, the left and right orbits coincide for negative iterates of $f$, and it follows that the left orbit of $x$ is also nonperiodic. Therefore, since the set $D(f)$ is finite and the left and right forward orbits of $x$ are nonperiodic, both of these forward orbits eventually consist entirely of points at which $f$ is continuous.

**Definition 2.2.** The stabilization time of an interval exchange $f$ is the smallest positive integer $n_0$, such that $f$ is continuous at $f^n_+(x)$ and $f^n_-(x)$ for all $n \geq n_0$ and for all fundamental discontinuities $x$. For a fundamental discontinuity $x$, if $f^n_+(x) = f^n_-(x)$, then $f^n_+(x) = f^n_-(x)$ for all $n \geq n_0$, since $f$ is continuous at all points in question. Such a fundamental discontinuity is said to be eventually resolving. Similarly, $f^n_+(x) \neq f^n_-(x)$ implies $f^n_+(x) \neq f^n_-(x)$ for all $n \geq n_0$; in this case, $x$ is said to be nonresolving.

Figure 5 gives examples of interval exchanges exhibiting the two types of fundamental discontinuity. The map $f$ has fundamental discontinuities at $\alpha$ and $\alpha + \beta$, and it may be checked that both of these discontinuities are nonresolving. The map $g$ is a product of two restricted irrational rotations. It has fundamental discontinuities at $\frac{1}{2} - \gamma$ and $1 - \delta$, both of which are eventually resolving.
Figure 5: Types of fundamental discontinuities; all parameters are irrational.

Fundamental discontinuities are so-named because they completely control the asymptotics of $d(f^n)$.

**Proposition 2.3.** For any infinite-order interval exchange $f$, exactly one of the following holds:

(a) All fundamental discontinuities of $f$ are eventually resolving, in which case $d(f^n)$ is bounded independently of $n$.

(b) The map $f$ has $k \geq 1$ nonresolving fundamental discontinuities, in which case $d(f^n)$ has linear growth on the order of $k|n|$.

**Proof.** If the map $f$ is continuous, then conclusion (a) holds vacuously. When $D(f)$ is nonempty, the discontinuities of $f^n$ are contained in the set

$$C_n = \bigcup_{i=0}^{n-1} f^{-i}(D(f)).$$

For $p \in C_n$, the left and right orbits (and hence the $f^n$-continuity status) of $p$ are determined by the left and right orbits of the first $f$-discontinuity $x$ that $p$ meets in its forward $f$-orbit. Suppose $x = f^j(p)$, for $j \geq 0$. If $n > j$, then

$$f^j_{\pm}(p) = f^{n-j}_{\pm}(f^j_{\pm}(p)) = f^{n-j}_{\pm}(x),$$

where $f^j_{\pm}(p) = x$ only because $f$ is continuous at the points

$$\{p, f(p), \ldots, f^{(j-1)}(p)\}.$$
It will first be shown that non-fundamental discontinuities of $f$ induce a uniformly bounded number of discontinuities for an iterate $f^n$. Let $x \in D_{np}(f)$ be a non-fundamental discontinuity. Then some negative iterate $y = f^{-j}(x)$, $j \geq 1$, is a fundamental discontinuity. Thus, for any $n \geq 1$, the discontinuity $x$ only determines the $f^n$-continuity status of points in the set

\[ \{x, f^{-1}(x), \ldots, f^{-(j-1)}(x)\} \]

The status of all other preimages $f^{-k}(x)$, such that $k \geq j$, is determined by the fundamental discontinuity $y$. Consequently, the number of points whose $f^n$-continuity status is determined by the non-fundamental discontinuity $x$ is bounded by $j$, independently of $n$. Similarly, there is a uniform bound to the number of points whose $f^n$-continuity status is determined by a periodic discontinuity of $f$.

Next, let $x$ be an eventually resolving fundamental discontinuity of $f$, and let $n_0$ be the stabilization time of $f$. Then,

\[ f^n(x) = f^n_-(x) \]

for all $n \geq n_0$. Suppose $n \geq n_0$ and $k$ is such that $0 \leq k \leq (n - n_0)$. The right and left orbits of $f^{-k}(x)$ are determined by the right and left orbits of $x$:

\[ f^n_+(f^{-k}x) = f^n_-(f^{-k}x) = f^n_{-k}(x) = f^n_-(f^{-k}x), \]

where the middle equality holds because $n - k \geq n_0$. Thus, $f^n$ is continuous at $f^{-k}(x)$, whenever $0 \leq k \leq (n - n_0)$. It follows that for all $n \geq n_0$, \[ \{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\} \] contains at most $n_0$ discontinuities of $f^n$. Therefore, $d(f^n)$ is bounded if all fundamental discontinuities of $f$ are eventually resolving.

Alternately, suppose that $x$ is a nonresolving fundamental discontinuity. Then

\[ f^n_+(x) \neq f^n_-(x) \]

for all $n \geq n_0$. By an argument similar to the one above, it follows that $f^n$ is discontinuous at $f^{-k}(x)$, for all $k$ such that $0 \leq k \leq (n - n_0)$. Thus, if $n \geq n_0$, $f^n$ has at least $n - n_0$ discontinuities in the set \[ \{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\} \]. Since $n_0$ is fixed relative to $n$, this implies that $d(f^n)$ has linear growth.

Consequently, the presence of at least one nonresolving fundamental discontinuity implies linear growth of $d(f^n)$, and the presence of $k$ nonresolving fundamental discontinuities implies $d(f^n) \sim kn$. 

\[ \square \]
3 Classification of maps with bounded discontinuity growth

The most simple example of an infinite-order interval exchange with bounded discontinuity growth is an irrational rotation \( r_\alpha \), for which \( d(r_\alpha^n) \) is always zero. The discontinuity growth rate of a map is invariant under conjugation, so we begin by stating a theorem of Li [4] which gives necessary and sufficient conditions for an interval exchange to be conjugate to an irrational rotation. For \( f \in \mathcal{E} \), let \( \delta(f) \) represent the number of intervals exchanged by \( f \) when viewed as a map \([0,1) \rightarrow [0,1)\):

\[
\delta(f) = \min \{ n : f = f(\pi, \lambda) \text{ for some } \pi \in \Sigma_n, \lambda \in \Lambda_n \}.
\]

**Theorem** (Li [4]). An interval exchange map \( f \) is conjugate to an irrational rotation if and only if the following hold:

(i) \( \delta(f^n) \) is bounded by some positive integer \( N \),

(ii) \( f^n \) is minimal for all \( n \in \mathbb{N} \), and

(iii) There are integers \( k > 0 \) and \( M \geq 2^{N^3 + 3N^2} \) such that \( \tilde{f} = f^k \) satisfies \( \delta(\tilde{f}) = \delta(\tilde{f}^2) = \cdots = \delta(\tilde{f}^M) \).

The quantities \( \delta(f) \) and \( d(f) \) are related, but they do not differ by a uniform constant for all \( f \in \mathcal{E} \). For a rotation \( r_\alpha \), \( \delta(r_\alpha) = 2 \) and \( d(r_\alpha) = 0 \), while \( \delta(f) = 3 \) for any map \( f = f(\pi, \lambda) \) with permutation \( \pi = (3,2,1) \). It may be checked that the continuity status of the points 0 and \( f^{-1}(0) \) account for any difference between \( \delta(f) \) and \( d(f) \); the function \( \delta(f) \) always counts these points as left endpoints of a partition interval of \( f \), but one or both of these points may fail to be a discontinuity of \( f \) when viewed as a map \( \mathbb{T}^1 \rightarrow \mathbb{T}^1 \). Consequently, some care must taken with condition (iii) in restating the above theorem in terms of the discontinuity number \( d \). Conceivably, one might observe \( d(f^k) \) to be constant over a large range of \( k \) while \( \delta(f^k) \) is changing frequently.

This difficulty may be overcome by a good choice of the base point on \( \mathbb{T}^1 \). Presenting an interval exchange as defined on \([0,1)\) amounts to specifying a base point 0 at which to cut the circle. Choosing a new base point amounts to conjugation by a rotation; since the conclusion of Li’s theorem is up to conjugacy, there is no loss in changing the base point. If \( f \) is replaced with a conjugate by a rotation, it may be assumed that \( f \) is continuous at all points of the orbit \( \mathcal{O}_f(0) \). Consequently, \( d(f^n) = \delta(f^n) - 2 \) for all integers \( n \), and observing \( d(f^n) \) to be constant is now equivalent to observing that \( \delta(f^n) \) is constant.
Theorem (Alternate Version of Li’s Theorem). An interval exchange map $f$ is conjugate to an irrational rotation if and only if the following hold:

(i) $d(f^n)$ is bounded by some integer $N$,

(ii) $f^n$ is minimal for all $n \in \mathbb{N}$, and

(iii) after redefining the base point (conjugating by a rotation) so that $f$ is continuous on the orbit of 0, there are integers $k > 0$ and $M \geq 2^{N^3 + 3N^2}$ such that $\tilde{f} = f^k$ satisfies $d(\tilde{f}) = d(\tilde{f}^2) = \cdots = d(\tilde{f}^M)$.

Given this version of the theorem, it may now be seen to what extent the conditions (ii) and (iii) hold when it is only assumed that $d(f^n)$ is bounded.

To introduce some terminology, a finite union $J$ of half-open intervals is a minimal component of $f$ if $J$ is $f$-invariant and the $f$-orbit of any $x \in J$ is dense in $J$. It is shown in [1] and [8] that for any interval exchange $f$, the set of non-periodic points of $f$ decomposes into finitely many minimal components.

Lemma 3.1. Suppose that $f$ is minimal and $d(f^n)$ is bounded. Then for some $k \in \mathbb{N}$, all nontrivial iterates $f^{nk}$ are minimal when restricted to each minimal component of $f^k$.

Proof. Suppose that no such integer $k$ exists. Then $f$ is minimal, but for some $k_1 = m_1 > 1$, $f^{m_1}$ has multiple minimal components. Suppose that this integer $k_1$ has been chosen to be as small as possible. Since $f$ and $f^{m_1}$ commute, $f$ permutes the minimal components of $f^{m_1}$. This permutation induced by $f$ is transitive since $f$ is minimal, and it must be of order $m_1$, by the choice of $m_1$. Thus $f^{m_1}$ has exactly $m_1$ minimal components, denoted by $J_{1,1}, \ldots, J_{1,m_1}$.

It has been assumed that no power $f^k$ is minimal for all iterates $f^{kn}$ when restricted to any of its minimal components. Thus, there exists a smallest integer $k_2 > 1$ such that $f^{m_2}$, where $m_2 = k_1 k_2$, is not minimal when restricted to some minimal component of $f^{m_1}$. Suppose this component is $J_{1,1}$. The map $f^{m_1}$ permutes the minimal components of $f^{m_2}$ which are contained in $J_{1,1}$; $f^{m_1}$ acts minimally on $J_{1,1}$, and so this permutation must be transitive and have order $k_2$. Additionally, the original map $f$ permutes the minimal components of $f^{m_2}$; since it also transitively permutes the minimal components of $f^{m_1}$, it follows that $f^{m_2}$ must have $k_2$ minimal components in each one of the $J_{1,j}$. Thus $f^{m_2}$ has exactly $k_2k_1 = m_2$ minimal components.

By the assumption that no $k$ satisfies the conclusion of the lemma, this process may continue indefinitely. In particular, there are sequences of integers $k_i > 1$ and $m_i = \prod_{j=1}^{i} k_j$, such that $f^{m_i}$ has exactly $m_i$ minimal components.
To arrive at a contradiction with the hypothesis that $d(f^n)$ is bounded, observe that if a map $g$ has $m > 1$ minimal components $J_1, \ldots, J_m$, then it must have at least $m$ discontinuities. To see this, consider a left-boundary point $x_i$ of $J_i$. Since some iterate of $x_i$ will eventually fall in the interior of $J_i$, it follows that the orbit of each $x_i$ must contain a discontinuity of $g$. Since these orbits are distinct, the map must have at least $m$ discontinuities. Thus, it is impossible for $f^n$ to have an arbitrarily large number of minimal components if $d(f^n)$ is bounded.

Remark 3.2. It seems plausible that the above lemma should hold in general; i.e., the condition that $d(f^n)$ is perhaps not necessary. However, the argument above strongly uses this assumption and breaks down without it.

Lemma 3.3. Suppose $f$ has infinite order and $d(f^n)$ is bounded. Then for some $N \in \mathbb{N}$, $d(f^{nN})$ is constant over all $n \in \mathbb{N}$.

Proof. By initially replacing $f$ with an iterate, it may be assumed that $\text{Per}(f) = \text{Fix}(f)$. Let $D_F = \{x_1, \ldots, x_k\}$ be the fundamental discontinuities of $f$. Since $d(f^n)$ is bounded, each $x_i$ is eventually resolving. All other non-fixed discontinuities are found in the forward orbits of the fundamental discontinuities. Choose an integer $N_1 > 0$ such that any point of $D_{np}(f)$ may be reached from $D_F$ by at most $N_1$ iterates of $f$. Such an $N_1$ exists since the set $D_{np}(f)$ is finite.

Choose $N_2$ such that the right and left orbits of all discontinuities in $D_{np}(f)$ are stabilized after $N_2$ iterates of $f$. In the situation where a non-fundamental discontinuity $x \in D_{np}(f)$ is fixed from the left (i.e., $f^-(x) = x$), it is the case that $f_n^+(x) \neq f^n(x)$ for all $n \geq 1$, since the right orbit of $x$ is nonperiodic. Otherwise, both the right and left forward orbits of any $x \in D_{np}(f)$ eventually consist entirely of continuity points of $f$. Thus, the notion of stabilization time is well-defined for all $x \in D_{np}(f)$.

Finally, choose $N > N_1 + N_2$. It will be shown that $d(f^{kN})$ is constant over all $k \in \mathbb{N}$. Since $\text{Per}(f) = \text{Fix}(f)$, the set of fixed discontinuities is identical for all nonzero iterates of $f$. Thus, it suffices to only consider the set $D_{np}(f^N)$ of non-fixed discontinuities of $f^N$; any such point must be of the form $f^{-i}(x)$, where $x \in D_{np}(f)$ and $0 \leq i < N$. The non-fixed discontinuities of $f$ are contained in the set

$$\bigcup_{i=0}^{N_1} f^i(D_F).$$
It follows that the non-fixed discontinuities of \( f^N \) are contained in the set
\[
\bigcup_{i=-N_1}^{i=N_1} f^i(D_F).
\]

Let
\[
P = D(f^N) \cap \left( \bigcup_{i=1}^{N_1} f^i(D_F) \right), \quad Q = D(f^N) \cap \left( \bigcup_{i=-(N-1)}^{0} f^i(D_F) \right).
\]

Consider a point \( x \in P \). Since this is a discontinuity of \( f^N \), the forward \( f \)-orbit of \( x \) must encounter a discontinuity of \( f \) whose right and left orbits control the continuity status of \( x \). Since \( x \) is in \( P \), this controlling discontinuity is non-fundamental, and it must be encountered within \( N_1 \) iterates of \( f \). Since \( N > N_1 + N_2 \), the inequality between \( f^N(x^+) \) and \( f^N(x^-) \) occurs at a place where the right and left orbits of the controlling discontinuity have already stabilized. Thus, the right and left orbits of the controlling discontinuity are nonresolving, and it follows that \( x \) is a discontinuity of \( f^n \), for all \( n \geq N \). In particular, \( x \) is a discontinuity for all \( f^{kN} \). Similarly, if a point in \( \bigcup_{i=1}^{N_1} f^i(D_F) \) is a point of continuity for \( f^N \), it must be a point of continuity for all \( f^{kN} \).

Next, consider a point \( x \in Q \). This point is a discontinuity of \( f^N \) whose \( f \)-orbit is controlled by a fundamental discontinuity \( x_i \) of \( f \). Observe that under \( f^{kN} \), the image of \( x \) is contained in
\[
\bigcup_{i=(k-1)N+1}^{kN} f^i(D_F).
\]
Consequently, if \( k \geq 2 \), the right and left orbits of \( x \) (which are controlled by the right and left orbits of the fundamental discontinuity \( x_i \)) have resolved once \( f^{kN} \) iterates have been applied to \( x \). Thus \( x \), as well as all other points in \( \bigcup_{i=-(N-1)}^{0} f^i(D_F) \), are continuity points for \( f^{kN} \), \( k \geq 2 \). In general, the \( f^{kN} \) continuity status of any point in \( \bigcup_{i=-(kN-1)}^{0} f^i(D_F) \) is controlled by the right and left orbits of a fundamental discontinuity. Since these orbits all resolve within \( N \) iterates, it follows that
\[
D(f^{kN}) \cap \left( \bigcup_{i=-(kN-1)}^{0} f^i(D_F) \right) = f^{-(k-1)N}(Q).
\]

The previous two paragraphs have shown that
\[
D(f^{kN}) = P \cup f^{-(k-1)N}(Q).
\]
This union is always disjoint, so the size of $D(f^{kN})$ is constant over all $k \in \mathbb{N}$. Since

$$d(f^{kN}) = |D(f^{kN})| + |\{\text{fixed discontinuities of } f^{kN}\}|,$$

and the second term in this sum is constant over all iterates of $f$, it follows that $d(f^{kN})$ is constant over all $k \in \mathbb{N}$, as desired.

\begin{proof}[Proof of Theorem 1.2] Since $f$ may be replaced with a power of itself, it may be assumed that $\text{Per}(f) = \text{Fix}(f)$. By applying Lemma 3.1 to the restriction of $f$ on each of its minimal components, there is some $k$ such that any $f^{nk}$ is minimal when restricted to any minimal component $J_1, \ldots, J_m$ of $f^k$. Since the result is up to conjugacy in $\mathcal{E}$, it may be assumed that the minimal components $J_i$ are all intervals.

Consider the restriction $f_j$ of $f^k$ to its minimal component $J_j$. It suffices to show that $f_j$ is conjugate to an irrational rotation. The function $d(f_n^j)$ is bounded, and by construction $f_n^j$ is minimal for all $n > 0$. If necessary, conjugate $f_j$ by a rotation to assure that $f_j$ is continuous at all points of the orbit of 0. By Lemma 3.3 there exists $N_j$ such that $d(f_n^{jN_j})$ is constant for all $n$. Consequently, the alternate version of Li’s theorem applies to the restricted map $f_j$, and so this map is conjugate to an irrational rotation.
\end{proof}

4 Proof of Theorem 1.3

We prove in this section that $\mathcal{E}$ does not contain distortion elements. To achieve this, it suffices to prove that an infinite-order interval exchange is not a distortion element in any finitely generated subgroup of $\mathcal{E}$ which contains it.

By Theorem 1.1, the iterates of $f$ have linear or bounded discontinuity growth. Suppose first that $f$ has linear discontinuity growth. Let $S = \{g_1, \ldots, g_k\}$ generate a subgroup of $G < \mathcal{E}$ which contains $f$, and let

$$M = \max_i \{d(g_i)\}.$$

Then

$$d(f^n) \leq M |f^n|_S,$$

since $f^n$ may be expressed as a composition of $|f^n|_S$ elements from the set of generators. Consequently, linear growth of $d(f^n)$ implies linear growth of $|f^n|_S$, and thus $f$ is not a distortion element of $G$.

Suppose now that $f$ has infinite order and bounded discontinuity growth, and again suppose $f \in G = \langle g_1, \ldots, g_n \rangle < \mathcal{E}$. By Theorem 1.2 after conjugation and replacing $f$ by an iterate it may be assumed that $f$ is a product of disjointly supported infinite-order restricted rotations.
Let $r_{\alpha,\beta}$ denote one of these rotations, and assume first that $\alpha \notin \mathbb{Q}$. Let $V$ be the $\mathbb{Q}$-vector subspace of $\mathbb{R}/\mathbb{Q}$ which is generated by the set of distances an element of $G$ may translate a point of $T^1$. The space $V$ is a finite-dimensional $\mathbb{Q}$-vector space, since it is generated by the components of the translation vectors $\omega(g_i), 1 \leq i \leq n$.

Fix a basis for $V$ which includes the class $[\alpha] \in \mathbb{R}/\mathbb{Q}$ and the class $[\beta]$ if $\beta \notin \mathbb{Q}$. Let

$$P_\alpha : V \to \mathbb{Q}$$

be the linear projection which returns the $[\alpha]$-coordinate of a vector with respect to this basis. For $p \in T^1$, define the function $\phi_{\alpha,p} : G \to \mathbb{Q}$ by

$$\phi_{\alpha,p}(g) = P_\alpha(g(p) - p).$$

Note that the maps $\phi_{\alpha,p}$ satisfy the cocycle relation

$$\phi_{\alpha,p}(fg) = \phi_{\alpha,p}(g) + \phi_{\alpha,g(p)}(f).$$

The map $f$ rotates by $\alpha \mod \beta$ on the interval $[0, \beta)$ and $P_\alpha(\beta) = 0$, so it follows that

$$\phi_{\alpha,0}(f^n) = n, \text{ for all } n \in \mathbb{Z}.$$ 

Now consider the generators $g_1, \ldots, g_n$. Each one of these maps induces only finitely many distinct translations, namely the components of $\omega(g_i)$. Consequently, there is a constant $M > 0$ such that

$$|\phi_{\alpha,p}(g_i)| \leq M, \text{ for all } p \in T^1, 1 \leq i \leq n.$$ 

Thus, for any $g \in G$,

$$|\phi_{\alpha,0}(g)| \leq M|g|_S.$$ 

In particular,

$$n = \phi_{\alpha,0}(f^n) \leq M|f^n|_S, \text{ for all } n \in \mathbb{Z},$$

which implies linear growth for $|f^n|_S$. Consequently, the map $f$ is not a distortion element in $G$.

Suppose we are in the case where $f$ is a product of infinite-order rotations, but all of these rotations are by some $\alpha_i \in \mathbb{Q}$ (mod $\beta_i \notin \mathbb{Q}$). The argument above fails in this case because the map $\phi_{\alpha,p}$ is not well-defined when $\alpha$ is rational. However, a similar argument can be made by tracking the contribution from the irrational number $\beta$. Choose a new basis for $V$ which contains $[\beta]$, and consider the map $\phi_{\beta,0}$. The rotation by $\alpha \mod \beta$ on $[0, \beta)$ contributes
(-1)β for every loop the iterated rotation makes around this interval. Thus, there exists some constant C > 0, (for instance, any C > β/α), such that

|φ_{β,0}(f^n)| \geq \frac{n}{C},

It is still the case that there is a constant M > 0 such that

φ_{β,0}(f^n) \leq M|f^n|_S,

which again implies linear growth for |f^n|_S. Thus, no infinite-order element of E is a distortion element.

5 Classification of Centralizers in E

5.1 The bounded growth case

For f ∈ E, let C(f) denote the centralizer of f in the group E:

C(f) = C_E(f) := \{g ∈ E : fg = gf\}.

If f is minimal, then the structure of C(f) is primarily determined by the discontinuity growth of f. In considering the situation where d(f^n) is bounded, the first case to consider is when f = r_α is an irrational rotation. Let R = \{r_α : α ∈ \mathbb{R}/\mathbb{Z}\} denote the subgroup of rotation maps.

Lemma 5.1. If α is irrational, then C(r_α) is the rotation group R.

Proof. (See [4] for an alternate proof.) Suppose that g ∈ E commutes with r_α, in which case g = r_α^{-1}gr_α. Since r_α is continuous as a map \( \mathbb{T} \rightarrow \mathbb{T} \), this conjugacy implies that the discontinuity set D(g) is r_α-invariant. Consequently, if D(g) is a nonempty set, it must be infinite, which is impossible. Thus, D(g) is empty, which implies g ∈ R, as rotations are the only continuous interval exchanges.

If f is minimal and d(f^n) is bounded, by Theorem 1.2 some power f^k is conjugate to a product of disjointly supported infinite-order restricted rotations. Suppose that k is chosen to be as small as possible, and let J_i, 1 ≤ i ≤ l, denote the minimal components of f^k. Replace f by a conjugate so that the J_i are intervals, and let r_i denote the restricted rotation supported on J_i induced by f^k. Since f is minimal and commutes with f^k, f transitively permutes the J_i and induces conjugacies between all of the r_i. Consequently, the J_i are
all intervals of length $1/l$, and each $r_i$ rotates by the same proportion of $1/l$. Let $R_i$ denote the rotation group supported on the interval $J_i$. Then $f^k$ is an element of the diagonal subgroup of

$$R_1 \times \cdots \times R_l,$$

and it follows from Lemma 5.1 that

$$C(f^k) = (R_1 \times \cdots \times R_l) \rtimes \Sigma_l,$$

where $\Sigma_l$ is the embedding of the symmetric group which permutes the $J_i$ by translation.

Since $C(f)$ is a subgroup of the virtually abelian group $C(f^k) \cong (\mathbb{R}/\mathbb{Z})^l \rtimes \Sigma_l$, it follows that $C(f)$ is also virtually abelian. In addition, $f \in C(f^k)$ implies that $f$ has the form

$$f = r_1 \cdots r_l \sigma,$$

where $r_i \in R_i$ and $\sigma$ is a permutation of the $J_i$ by translation. In particular, $f$ commutes with the diagonal subgroup in $R_1 \times \cdots \times R_l$, and we have proved the following.

**Proposition 5.2.** If $f$ is minimal and $d(f^k)$ is bounded, then $C(f)$ is virtually abelian and contains a subgroup isomorphic to $\mathbb{R}/\mathbb{Z}$.

### 5.2 The linear growth case

Next, suppose that $f$ is minimal and $d(f^k)$ exhibits linear growth. The discontinuity structure of $f$ and its iterates is significantly more complicated than the bounded case. Any map $g$ which commutes with $f$ must preserve this structure, and thus one would expect the centralizer of $f$ to be significantly smaller than in the bounded discontinuity situation.

**Proposition 5.3.** If $f$ is minimal and $d(f^k)$ has linear growth, then $C(f)$ is virtually cyclic.

To prove this, let $D = D(f)$ be the discontinuity set of $f$ and let $D_{NR} = \{x_1, \ldots, x_k\}$ be the set of nonresolving fundamental discontinuities of $f$, which is nonempty by Proposition 2.3. Let $n_0$ be the symmetric stabilization time for $f$: $n_0$ is the minimal positive integer such that $f$ is continuous at $f^i(x)$, for all $i$ such that $|i| \geq n_0$ and all $x \in D$. The following lemma states that if a sufficiently long piece of $f$-orbit contains enough discontinuity points of a large power of $f$, then the $f$-orbit must contain a nonresolving fundamental discontinuity of $f$. 

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Lemma 5.4. Suppose $f$ is minimal and has symmetric stabilization time $n_0$. Let $M > 3n_0$, and suppose that for some $y \in \mathbb{T}$ the set

$$B = \{ y, f^{-1}(y), \ldots, f^{-M+n_0}(y) \}$$

contains strictly more than $2n_0 + 1$ discontinuities of $f^M$. Then some $f^k(y)$, where $|k| \leq M$, is a nonresolving fundamental discontinuity of $f$.

Proof. Let $y_m$ denote $f^m(y)$, and let $j \in \mathbb{N}$ be the smallest positive integer such that $f^M$ is discontinuous at $y_j$. Since $f^M$ is discontinuous at $y_j$, this point has a controlling $f$-discontinuity at $y_k \in D(f)$, where $-j \leq k \leq -j + M - 1$. Consequently, there must be a fundamental discontinuity of $f$ at some $y_k$, where $-j - n_0 \leq k \leq -j + M - 1$.

Since $B$ contains more than $2n_0$ $f^M$-discontinuities at points $y_i$ with $i > j$, there are more than $n_0$ $f^M$-discontinuities whose status is controlled by $y_k$. In particular, at least one of the $f^M$-discontinuities in $B$ is induced by the stabilized behavior of $y_k$, which implies that $y_k$ is a nonresolving fundamental discontinuity of $f$.

Proof of Proposition 5.3. Suppose that $gf = fg$. Let $N \in \mathbb{N}$ be such that

$$N \gg n_0 \quad \text{and} \quad N \gg d(g).$$

Let $x \in D_{NR}$. Since $x$ is a nonresolving, the set

$$A = \{ x, f^{-1}x, f^{-2}x, \ldots, f^{-(N-n_0)}x \}$$

consists entirely of discontinuity points of $f^N$.

Since $f$ and $g$ commute, $g^{-1}f^Ng = f^N$ is discontinuous at all points of $A$. Consider how this composition acts upon the set $A$:

$$A = \{ x, f^{-1}x, \ldots, f^{-(N-n_0)}x \} \quad \downarrow g$$

$$g(A) = \{ gx, f^{-1}(gx), \ldots, f^{-(N-n_0)}(gx) \} \quad \downarrow f^N$$

$$f^N g(A) = \{ f^N(gx), f^{N-1}(gx), \ldots, f^{n_0}(gx) \} \quad \downarrow g^{-1}$$

$$f^N(A) = \{ f^N x, f^{N-1}x, \ldots, f^{n_0}x \}$$

The cardinality of $A$ is significantly larger than $d(g)$, which implies that $g$ acts continuously on most points of $A$ in the first stage of the above composition. Similarly, $g^{-1}$ acts continuously on most points of $f^N g(A)$ in the third
stage. However, since $g^{-1}f^Ng$ is discontinuous at all points of $A$, it follows that $f^N$ is discontinuous at most of the points in
\[ \{gx, f^{-1}(gx), \ldots, f^{-(N-n_0)}(gx)\}. \]

By Lemma 5.4 it follows that some $f$-iterate of $g(x)$ must be in $D_{NR}$.

The preceding paragraphs show that $g \in C(f)$ permutes the $f$-orbits of the points in $D_{NR} = \{x_1, \ldots, x_k\}$. In particular, there is some integer $i$ and some $x_j \in D_{NR}$ such that
\[ g(x_1) = f^i(x_j). \]

This relation determines $g$ on the entire $f$-orbit of $x_1$:
\[ g(f^nx_1) = f^n(gx_1) = f^{n+i}x_j. \]

Since the orbit $O_f(x_1)$ is dense, this relation fully determines $g$.

For each $j$ such that $1 \leq j \leq k$, let $h_j$ denote the unique interval exchange in $C(f)$ such that
\[ h_j(x_1) = x_j, \]
if such a map exists. Then, if $g \in C(f)$ satisfies $g(x_1) = f^k(x_j)$, it follows that $g = f^k h_j$. In particular, $\{h_i\}$ is a set of representatives for the finite quotient group $C(f)/\langle f \rangle$, and consequently $C(f)$ is virtually cyclic.

\[ \square \]

5.3 Centralizers of finite order maps

For $n \geq 2$, the rotation $r_{1/n}$ induces a cyclic permutation of the intervals
\[ I_i = \left[ \frac{i-1}{n}, \frac{i}{n} \right], \quad 1 \leq i \leq n. \]

Recall that the support of an interval exchange $f$ is the complement of its set of fixed points. For $1 \leq j \leq n$, let $E_{I_j}$ denote the subgroup of all interval exchanges whose support is contained in $I_j$. Note that the orientation-preserving affine bijection $I_j \to [0, 1)$ induces an isomorphism $E_{I_j} \cong \mathcal{E}$.

Consider the following subgroups in the centralizer $C(r_{1/n})$. Let $E^n_\Delta$ represent the maps in $C(r_{1/n})$ which preserve the intervals $I_i$:
\[ E^n_\Delta = \{ g \in C(r_{1/n}) : g(I_i) = I_i, \text{ for } 1 \leq i \leq n \}. \]

Note that $E^n_\Delta$ is the diagonal subgroup of the product
\[ \mathcal{E}_{I_1} \times \cdots \times \mathcal{E}_{I_n}, \]
as induced by the natural isomorphisms $\mathcal{E} \cong \mathcal{E}_I$. In short, a map in $E^n_\Delta$ acts on each of the $I_j$ in the same manner, and so $E^n_\Delta \cong I_j \cong \mathcal{E}$.

Next, let $P_n$ denote the subgroup of maps in $C(r_{1/n})$ which are invariant on $r_{1/n}$-orbits $\{x + k/n : k = 0, 1, \ldots, n - 1\}$:

$$P_n = \left\{ g \in C(r_{1/n}) : \forall x \in \mathbb{T}^1, \exists k \in \mathbb{Z}, \text{ such that } g(x) = x + \frac{k}{n} \ (\text{mod } 1) \right\}.$$  

Fix $g \in P_n$, and consider a point $x = x_1 \in I_1$. Let

$$x_k = r_{1/n}^{k-1}(x_1) = x_1 + \frac{k-1}{n}, \quad 2 \leq k \leq n,$$

denote the other points in the $r_{1/n}$-orbit of $x$, and let $\sigma_{g,x} \in \Sigma_n$ denote the permutation that $g$ induces on $\{x_i\}$:

$$g(x_i) = x_{\sigma_{g,x}(i)}.$$

The permutation $\sigma_{g,x}$ commutes with the permutation $r : i \mapsto i + 1 \ (\text{mod } n)$, which implies that $\sigma_{g,x}$ must be a power of $r$. Thus, the transformation $g$ is described by a right-continuous (and hence piecewise constant) map

$$\sigma_g : I_1 \to \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$  

Conversely, any such right-continuous map $I_1 \to \mathbb{Z}/n\mathbb{Z}$ with only finitely many discontinuities defines a map in $P_n$. Thus, $P_n$ is isomorphic to the abelian group of right-continuous functions $I_1 \to \mathbb{Z}/n\mathbb{Z}$ having finitely many discontinuities.

**Proposition 5.5.** $C(r_{1/n}) = P_n \rtimes E^n_\Delta$.

**Proof.** First, suppose $g \in P_n \cap E^n_\Delta$. Then $g$ preserves the intervals $I_i$, which implies that $\sigma_{g,x} = id$ for all $x \in I_1$. Thus $g = id$, and the subgroups $P_n$ and $E^n_\Delta$ have trivial intersection.

Next, suppose $g$ is an arbitrary element of $C(r_{1/n})$. Construct $h \in P_n$ as follows. For $x = x_1 \in I_1$, define $\{x_i\}$ as before and let $\sigma_{h,x}$ be the permutation such that

$$g(x_i) \in I_{\sigma_{h,x}(i)}.$$  

Observe that $\sigma_{h,x} \in \Sigma_n$ is well-defined since $g$ maps an $r_{1/n}$-orbit $\{x_i\}$ to another $r_{1/n}$-orbit. Since $g$ commutes with $r_{1/n}$, the permutation $\sigma_{h,x}$ is a power of the permutation $r$. Moreover, the function $x \mapsto \sigma_{h,x} \in \mathbb{Z}/n\mathbb{Z}[r]$ is right-continuous and has finitely many discontinuities, so it induces a map $h \in P_n$. From its construction, $gh^{-1}$ preserves each interval $I_i$, and so $gh^{-1} \in E^n_\Delta$. Thus $C(r_{1/n}) = P_n \cdot E^n_\Delta$.  

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It remains to show that \( P_n \) is a normal subgroup of \( C(r_{1/n}) \). Let \( g \in P_n \) and let \( h \in \mathcal{E}_n \Delta \). If \( \{x_i\} \) is an \( r_{1/n} \)-orbit, then \( h \) maps it to some other \( r_{1/n} \)-orbit \( \{y_i\} \), \( g \) permutes the orbit \( \{y_i\} \), and \( h^{-1} \) maps \( \{y_i\} \) back to \( \{x_i\} \). Thus \( h^{-1}gh \) is invariant on \( r_{1/n} \)-orbits, which implies that \( h^{-1}gh \in P_n \). Consequently, \( P_n \trianglelefteq C(r_{1/n}) \).

**Corollary 5.6.** For \( n \geq 2 \), let \( G_n = P_n \rtimes \mathcal{E}_n \) denote the centralizer of the rotation \( r_{1/n} \), and let \( G_1 = \mathcal{E} \). If \( f \) is any finite-order map, then \( C(f) \) is isomorphic to a finite direct product of the \( G_i \).

**Proof.** Decompose \( \mathbb{T}_1 \) into finitely many nonempty 

\[
I_j = \operatorname{Per}_j(f) = \{x \in \mathbb{T}_1 : |\mathcal{O}(x)| = j\}.
\]

This decomposition is finite because an interval exchange cannot have periodic points of arbitrarily large minimal period. After replacing \( f \) by a conjugate, it may be assumed that the \( I_j \) are intervals on which \( f \) acts by a finite-order rotation. The \( I_j \) are invariant under all \( g \in C(f) \), and \( C(f) \cap \mathcal{E}_I_j \) is isomorphic to \( G_j \). \[\Box\]

### 5.4 The general situation

Let \( f \) be any interval exchange. Let \( J_1, \ldots, J_k \) denote the minimal components of \( f \), let \( A = \operatorname{Per}(f) \setminus \operatorname{Fix}(f) \) and let \( B = \operatorname{Fix}(f) \). After replacing \( f \) by a conjugate, it may be assumed that each of these sets is an interval. Let \( f_i \) be the restriction of \( f \) to \( J_i \), here defined on all of \( \mathbb{T}_1 \) by

\[
f_i(x) = \begin{cases} 
  f(x), & \text{if } x \in J_i \\
  x, & \text{otherwise}.
\end{cases}
\]

Let \( g \in C(f) \). The sets \( A \) and \( B \) are both \( g \)-invariant, but \( g \) may permute the minimal components \( J_i \). However, if \( g \) maps \( J_i \) onto \( J_j \), then \( g \) induces a conjugacy between \( f_i \) and \( f_j \). After replacing \( f_j \) by a conjugate in \( \mathcal{E}_I_j \), it may be assumed that

\[
f_i = \tau_{ij}f_j\tau_{ij}
\]

where \( \tau_{ij} \) is the order-two map which interchanges \( J_i \) and \( J_j \) by translation and fixes all other points. Replace \( f \) by a further conjugate so that \( f_i = \tau_{ij}f_j\tau_{ij} \) holds for all pairs \( i \neq j \) such that \( f_i \) and \( f_j \) are conjugate, and let \( F \) be the group generated by all such \( \tau_{ij} \). Note that \( F \) is isomorphic to a direct product of symmetric groups, since the relation \( i \sim j \Leftrightarrow (f_i \text{ is conjugate to } f_j) \) is an equivalence relation on \( \{1, \ldots, k\} \). Let \( C_i = C_{\mathcal{E}_I_i}(f) = C(f) \cap \mathcal{E}_I_i \) denote the subgroup of maps in \( C(f) \) with support in \( J_i \), and let \( C_A = C(f) \cap \mathcal{E}_A \).
Theorem 5.7. For any \( f \in \mathcal{E} \), let \( A = \text{Per}(f) \setminus \text{Fix}(f) \) and let \( B = \text{Fix}(f) \). Then,
\[
C(f) \cong \left( \prod_{i=1}^{k} C_i \right) \rtimes F \times C_A \times \mathcal{E}_B,
\]
where \( F \) is a direct product of symmetric groups, where each \( C_i \) is either an infinite virtually cyclic group or a subgroup of \((\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n\) containing the diagonal in \((\mathbb{R}/\mathbb{Z})^n\), and where \( C_A \) is a direct product of finitely many factors \( G_n = P_n \rtimes \Sigma_n^\Delta \). The factors \( C_i \) and \( F \) are trivial if \( \text{Per}(f) = \emptyset \), and the factors \( C_A \) and \( \mathcal{E}_B \) are trivial if \( f \) has finite order.

Proof. It is clear that
\[
C(f) \cong C_{\cup J_i}(f) \times C_A \times \mathcal{E}_B,
\]
since these are disjoint and non-conjugate \( f \)-invariant sets which cover \( \mathbb{T}^1 \). The verification that
\[
C_{\cup J_i}(f) \cong \left( \prod_{i=1}^{k} C_i \right) \rtimes F
\]
is similar to the proof of Proposition 5.5.

A corollary to this result states that the existence of periodic points for an interval exchange \( f \) is characterized by its centralizer.

Corollary 5.8. For any \( f \in \mathcal{E} \), \( \text{Per}(f) \) is nonempty if and only if \( C(f) \) contains a subgroup isomorphic to \( \mathcal{E} \).

Proof. The factors \( C_A \) and \( \mathcal{E}_B \) both contain subgroups isomorphic to \( \mathcal{E} \) if they are nontrivial, and at least one of these factors is nontrivial when \( \text{Per}(f) \) is nonempty.

It remains to show that if \( \text{Per}(f) \) is empty, then no subgroup of \( C(f) \) is isomorphic to \( \mathcal{E} \). In this case,
\[
C(f) \cong \prod_{i=1}^{k} C_i \rtimes F,
\]
where each \( C_i \) is either virtually cyclic or isomorphic to a subgroup of \((\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n\) containing the diagonal. It may be seen that for any two infinite-order \( g, h \in C(f) \), there are nontrivial powers \( g^i \) and \( h^k \) of these maps which commute. This property does not hold for the group \( \mathcal{E} \). For instance, consider an irrational rotation \( r_\alpha \) and any infinite-order map \( f \in \mathcal{E} \) which is not a rotation; by Lemma 5.1, nontrivial powers of \( r_\alpha \) and \( f \) do not commute. Thus, it is not possible to embed \( \mathcal{E} \) as a subgroup of \( C(f) \) when \( f \) has no periodic points. \( \square \)
Corollary 5.9. For any \( f \in \mathcal{E} \) such that \( f \neq \text{id} \), the index \([\mathcal{E} : C(f)]\) is uncountable.

**Proof.** From the structure of \( C(f) \) given in the proposition, it suffices to consider the cases where \( f \) has finite order, where \( f \) is minimal with \( d(f^n) \) bounded, and where \( f \) is minimal with linear discontinuity growth.

If \( f \) has finite order, it suffices to consider the case \( f = r_{1/n} \). Fix an irrational \( \alpha \) in \((0, 1)\), and note that the product of restricted rotations \( r_{\alpha \epsilon}^{-1} r_{\alpha \epsilon'} \) is never an element of \( C(f) \) for any \( 0 < \epsilon < \epsilon' < \frac{1}{n} \). Consequently, the \( r_{\alpha \epsilon} \) provide an uncountable set of distinct coset representatives for \( \mathcal{E}/C(f) \).

If \( f \) is minimal and \( d(f^n) \) is bounded, consider a conjugate \( g \) of \( f^k \) which is a product of infinite-order restricted rotations on intervals of length \( 1/l \). Again, notice that for \( 0 < \epsilon < \epsilon' < 1/l \), the product \( r_{\alpha \epsilon}^{-1} r_{\alpha \epsilon'} \) is not an element of \( C(g) \). Consequently, the \( r_{\alpha \epsilon} \) also provide an uncountable set of coset representatives for \( \mathcal{E}/C(g) \), and it follows that \( C(f^k) \) and \( C(f) \) have uncountable index in \( \mathcal{E} \).

If \( f \) is minimal with linear discontinuity growth, then by proposition 5.3, \( C(f) \) is virtually cyclic. In particular, \( C(f) \) is countable, which implies that \( C(f) \) has uncountable index in \( \mathcal{E} \).

\[ \square \]

6 Computation of \( \text{Aut}(\mathcal{E}) \)

The proof of Theorem 1.6 is based on observing that an arbitrary \( \Psi \in \text{Aut}(\mathcal{E}) \) preserves the structure of centralizers, which implies that \( \Psi \) preserves various dynamical properties of individual maps and subgroups in \( \mathcal{E} \).

Lemma 6.1. An interval exchange \( f \) is conjugate to an irrational rotation \( r_{\alpha} \) if and only if the following conditions hold:

1. \( C(f) \cong \mathbb{R}/\mathbb{Z} \);
2. if \( g \in C(f) \) has infinite order, then \( C(g) = C(f) \).

**Proof.** By Lemma 5.1, conditions (1) and (2) hold if \( f = r_{\alpha} \) is an irrational rotation, and these conditions are both preserved under conjugation.

Conversely, assume that \( f \) satisfies (1) and (2). By Corollary 5.8, \( \text{Per}(f) \) is empty. Next, suppose that some \( f^n \) has at least two minimal components, and denote them by \( J_i \). Let \( g \) be the map which is equal to \( f \) on \( J_i \) and fixes all other points. Then \( g \) has infinite order and commutes with \( f \), so \( C(g) \cong \mathbb{R}/\mathbb{Z} \) by condition (2). However, \( g \) has fixed points, and so \( C(g) \) contains a subgroup
isomorphic to $E$, which is impossible by Corollary 5.8. Thus, $f^n$ is minimal for all $n \geq 1$.

Furthermore, $f$ has bounded discontinuity growth. If not, then $C(f)$ is virtually cyclic by Proposition 5.3, which is not the case for $\mathbb{R}/\mathbb{Z}$. Consequently, by Theorem 1.2 some power $f^k$ is conjugate to an irrational rotation. Since $C(f) = C(f^k)$, it follows that $f$ is also conjugate to an irrational rotation.

Let $R < E$ denote the group of circle rotations $\{r_\alpha : \alpha \in \mathbb{R}/\mathbb{Z}\}$. For any $f \in E$, let $\Phi_f$ denote conjugation by $f^{-1}$; i.e., $\Phi_f(g) = fgf^{-1}$.

**Corollary 6.2.** For any $\Psi \in Aut(E)$, $\Psi$ maps the rotation group $R$ to a conjugate. That is, there exists $g \in E$ such that $\Psi(R) = gRg^{-1}$.

**Proof.** Since conditions (1) and (2) in Lemma 6.1 are purely group theoretic, they are preserved by any automorphism $\Psi$. Fix an irrational rotation $r_\alpha$. By the Lemma, $\Psi(r_\alpha)$ is conjugate to an irrational rotation. In particular, there is some $g \in E$ and some irrational $\beta \in \mathbb{R}/\mathbb{Z}$ such that

$$\Psi(r_\alpha) = \Phi_g(r_\beta).$$

Then

$$\Psi(R) = \Psi(C(r_\alpha)) = C(\Psi(r_\alpha)) = C(\Phi_g(r_\beta)) = gRg^{-1}.$$

A similar result holds for maps that are conjugate to an infinite-order restricted rotation $r_{\alpha,\beta}$.

**Lemma 6.3.** An interval exchange $f$ is conjugate to an infinite-order restricted rotation $r_{\alpha,\beta}$ if and only if the following hold:

1. $C(f) = E_* \times H$, where $E_* \cong E$, $H \cong \mathbb{R}/\mathbb{Z}$, and $f \in H$;
2. if $g \in H$ has infinite order, then $C(g) = C(f)$;
3. for $h \in C(f)$, if the index $[C(f) : C(h) \cap C(f)]$ is finite and $C(h) \nsubseteq C(h) \cap C(f)$, then $h$ is a finite-order element of $H$.

**Proof.** Suppose that $f = r_{\alpha,\beta}$ with $\beta < 1$ and $\alpha/\beta$ irrational. Let $I = [\beta, 1)$. Then

$$C(r_{\alpha,\beta}) = E_I \times R_\beta,$$

where $R_\beta \cong \mathbb{R}/\mathbb{Z}$ is the group of all restricted rotations $r_{\gamma,\beta}$ on $[0, \beta)$. Any other infinite-order element of $R_\beta$ has the same centralizer as $r_{\alpha,\beta}$, and it follows that $r_{\alpha,\beta}$ satisfies conditions (1) and (2).
To verify condition (3) for \( r_{\alpha,\beta} \), take \( h \in C(r_{\alpha,\beta}) \) and write \( h = h_I r_{\gamma,\beta} \), where \( h_I \in \mathcal{E}_I \) and \( r_{\gamma,\beta} \in R_\beta \). Assume that \( C(h) \) satisfies the hypotheses of condition (3). Note that

\[
C(h) \cap C(r_{\alpha,\beta}) = C_{\mathcal{E}_I}(h_I) \times R_\beta,
\]

and consequently,

\[
[C(r_{\alpha,\beta}) : C(h) \cap C(r_{\alpha,\beta})] = [\mathcal{E}_I : C_{\mathcal{E}_I}(h_I)].
\]

Corollary 5.9 states that the index \([\mathcal{E}_I : C_{\mathcal{E}_I}(h_I)]\) is infinite if \( h_I \) is not the identity. However, it has been assumed that this index is finite; thus \( h_I = id \) and \( h = r_{\gamma,\beta} \) is a restricted rotation. It has also been assumed that

\[
C(h) \cap C(r_{\alpha,\beta}) = C_{r_{\alpha,\beta}},
\]

and this is possible only if the rotation \( h = r_{\gamma,\beta} \) has finite order.

Finally, observe that conditions (1)-(3) are all preserved under conjugation in \( \mathcal{E} \). Consequently, they hold for any conjugate of \( r_{\alpha,\beta} \).

Conversely, suppose that \( f \) is an interval exchange satisfying (1)-(3). Since \( C(f) \) contains a subgroup isomorphic to \( \mathcal{E} \), \( A = \text{Per}(f) \) is nonempty by Corollary 5.8. The map \( f \) does not have periodic points of arbitrarily large period, so \( \text{Fix}(f^k) = \text{Per}(f^k) = A \) for some \( k \geq 1 \). Since \( f^k \) fixes \( A \), \( \mathcal{E}_A \subset C(f^k) \). By condition (2), \( C(f^k) = C(f) \), and it follows that \( f \) fixes all points in \( A \). Similarly, all infinite-order \( g \in H \) must fix the set \( A \), and consequently all maps in \( H \) must fix \( A \). Thus, \( H \) is contained in \( \mathcal{E}_B \), where \( B = \mathbb{T}^1 \setminus A \).

Suppose now that \( f \) has \( k \geq 2 \) minimal components \( J_i \), and let \( h \) be the map which equals \( f \) on the component \( J_1 \) and fixes all other points. Then \( h \) has infinite order and commutes with \( f \). Thus, by Theorem 5.7,

\[
C(f) = \left( \prod_{i=1}^{k} C_i \right) \times F \times \mathcal{E}_A, \quad \text{and}
\]

\[
C(h) = C_1 \times C_{\mathcal{E}_{A \cup J_2 \cup \ldots \cup J_k}},
\]

where \( C_i = C(f) \cap \mathcal{E}_J \) and \( F \) is a finite group which permutes the \( J_i \). In particular, \( C(h) \cap C(f) \) contains \( \left( \prod C_i \right) \times \mathcal{E}_A \), which has finite index in \( C(f) \). In addition, \( C(h) \) strictly contains \( C(h) \cap C(f) \) since \( h \) has a larger fixed point set than \( f \). Thus, condition (3) implies that \( h \) must have finite order, which is a contradiction. A similar argument may be applied to any infinite-order \( g \in H \); thus, all such maps have a single minimal component, namely \( B \).
Consider the natural isomorphism 
\[ \mathcal{E} \to \mathcal{E}_B. \]

Let \( \tilde{f} \) denote the preimage of \( f \in \mathcal{E}_B \), and let \( \tilde{H} \) denote the preimage of \( H \). Then all infinite-order \( \tilde{g} \in \tilde{H} \) are minimal, and 
\[ C(\tilde{g}) = C(\tilde{f}) > \tilde{H}, \]
which implies that all infinite-order \( \tilde{g} \) have bounded discontinuity growth. As in the proof of Lemma 6.1, it follows that all \( \tilde{g} \in \tilde{H} \) are simultaneously conjugate to irrational rotations. Back in the group \( \mathcal{E}_B \), this implies that \( H \) is conjugate to a group of restricted rotations. \( \square \)

As in the earlier case, observe that the three conditions in the previous proposition are purely group-theoretic. Consequently, they are all preserved by any automorphism \( \Psi \), which implies the following corollary.

**Corollary 6.4.** For any \( \Psi \in Aut(\mathcal{E}) \) and any \( f \) which is conjugate to a restricted rotation, \( \Psi(f) \) is also conjugate to a restricted rotation.

Let \( \mathcal{P} \) denote the set algebra consisting of all finite unions of half-open intervals \( [a, b) \subseteq \mathbb{T}^1 \).

**Proposition 6.5.** For any \( \Psi \in Aut(\mathcal{E}) \) and any nonempty \( A \in \mathcal{P} \), there is a unique \( B \in \mathcal{P} \) such that \( \Psi(\mathcal{E}_A) = \mathcal{E}_B \).

**Proof.** It suffices to consider \( A \in \mathcal{P} \) to be a proper, nonempty subset of \( \mathbb{T}^1 \). Let \( g \in \mathcal{E} \) be a map that is conjugate to an infinite-order restricted rotation, such that \( \text{Fix}(g) = A \). By Corollary 6.4, \( \Psi(g) \) is also conjugate to a restricted rotation; let \( B = \text{Fix}(\Psi(g)) \).

Observe that two infinite-order restricted rotations \( g \) and \( h \) commute if and only if one of the following holds:

(a) their supports coincide and they are simultaneously conjugate to elements in some \( R_\beta \); or

(b) their supports are disjoint.

These conditions can be characterized in terms of centralizers: (a) implies that \( C(g) = C(h) \), while (b) implies \( C(g) \neq C(h) \). In particular, each condition is preserved by any automorphism of \( \mathcal{E} \).

Any restricted rotation with support contained in \( A = \text{Fix}(g) \) commutes with \( g \) and has support disjoint from that of \( g \). Consequently, all restricted rotations in \( \mathcal{E}_B \) are simultaneously conjugate to irrational rotations.
rotations in $\mathcal{E}_A$ must map under $\Psi$ to restricted rotations with support in $B = \text{Fix}(\Psi(g))$. The restricted rotations in $\mathcal{E}_A$ generate this subgroup; see [8] for a proof of this fact. Therefore, the image $\Psi(\mathcal{E}_A)$ is contained in $\mathcal{E}_B$.

Similarly, under $\Psi^{-1}$ all restricted rotations with support in $B$ are mapped to restricted rotations which commute with $g$ and have support disjoint from that of $g$. Therefore, $\Psi^{-1}(\mathcal{E}_B) \subseteq \mathcal{E}_A$, and it follows that $\Psi(\mathcal{E}_A) = \mathcal{E}_B$. □

6.1 Definition and properties of $\tilde{\Psi}$

Given an automorphism $\Psi \in \text{Aut}(\mathcal{E})$, Proposition 6.5 implies that there is a well-defined transformation

$$\tilde{\Psi}: \mathcal{P} \to \mathcal{P},$$

defined by the relation

$$\Psi(\mathcal{E}_A) = \mathcal{E}_{\Psi(A)}, \ A \in \mathcal{P}.$$  

In particular, $\tilde{\Psi}(\mathbb{T}^1) = \mathbb{T}^1$ and $\tilde{\Psi}(\emptyset) = \emptyset$, for all $\Psi \in \text{Aut}(\mathcal{E})$. An element $f \in \mathcal{E}$ also induces a transformation $\tilde{f}: \mathcal{P} \to \mathcal{P}$, defined by $\tilde{f}(A) = f(A)$.

**Proposition 6.6.** For all $\Psi \in \text{Aut}(\mathcal{E})$, the transformation $\tilde{\Psi}: \mathcal{P} \to \mathcal{P}$ has the following properties:

1. $\tilde{\Psi}$ is an automorphism of the set algebra $\mathcal{P}$.

2. For any $f \in \mathcal{E}$, $\tilde{\Psi}(f) = \tilde{\Psi} \tilde{f} \tilde{\Psi}^{-1}$.

3. The Lebesgue measure $\mu: \mathcal{P} \to [0,1]$ is $\tilde{\Psi}$-invariant: $\mu(\tilde{\Psi}(A)) = \mu(A)$.

**Proof.** To show that $\tilde{\Psi}$ is a set algebra automorphism, it suffices to show $\tilde{\Psi}$ preserves complements, inclusion, and unions in $\mathcal{P}$. If $A$ and $B$ are complements in $\mathcal{P}$, then the centralizer in $\mathcal{E}$ of $\mathcal{E}_A$ is $\mathcal{E}_B$, and vice versa. This same relation holds for $\Psi(\mathcal{E}_A) = \mathcal{E}_{\Psi(A)}$ and $\Psi(\mathcal{E}_B) = \mathcal{E}_{\Psi(B)}$, which implies $\tilde{\Psi}(A)$ and $\tilde{\Psi}(B)$ are complements.

For inclusion, notice that

$$A \subseteq B \Rightarrow \mathcal{E}_A \subseteq \mathcal{E}_B \Rightarrow \Psi(\mathcal{E}_A) \subseteq \Psi(\mathcal{E}_B) \Rightarrow$$

$$\mathcal{E}_{\Psi(A)} \subseteq \mathcal{E}_{\Psi(B)} \Rightarrow \tilde{\Psi}(A) \subseteq \tilde{\Psi}(B).$$
To verify that \( \tilde{\Psi} \) preserves unions, let \( A \) and \( B \) be elements of \( P \), and note that \( \tilde{\Psi}(A) \subseteq \tilde{\Psi}(A \cup B) \) and \( \tilde{\Psi}(B) \subseteq \tilde{\Psi}(A \cup B) \). Conversely, suppose that \( \tilde{\Psi}(A \cup B) \not\subseteq \tilde{\Psi}(A) \cup \tilde{\Psi}(B) \). To derive a contradiction, let

\[
C = \tilde{\Psi}(A \cup B) \setminus \left( \tilde{\Psi}(A) \cup \tilde{\Psi}(B) \right).
\]

Then \( C \in P \) is nonempty, and there exists a non-identity interval exchange \( f \in E \leq E_{\tilde{\Psi}(A \cup B)} \). The map \( f \) centralizes both \( E_{\tilde{\Psi}(A)} \) and \( E_{\tilde{\Psi}(B)} \), so the map \( \Psi^{-1}(f) \) centralizes \( E_A \) and \( E_B \). This implies \( \Psi^{-1}(f) \) has support disjoint from both \( A \) and \( B \). However, this is impossible since \( \Psi^{-1}(f) \) is in \( E_{A \cup B} \). Thus, \( \tilde{\Psi}(A \cup B) \subseteq \tilde{\Psi}(A) \cup \tilde{\Psi}(B) \), which completes the verification that \( \tilde{\Psi} \) is a set algebra automorphism.

To prove property (2), recall \( \Phi_f \in \text{Aut}(E) \) denotes conjugation by \( f^{-1} \). In particular, if \( \tilde{f} \) maps the set \( A \) to the set \( B \), then \( \Phi_f \) induces an isomorphism from \( E_A \) to \( E_B \). For any \( g \in E \),

\[
\Psi \Phi_f \Psi^{-1}(g) = \Psi(f(\Psi^{-1}g)f^{-1}) = \Psi(f) \circ g \circ \Psi(f)^{-1}.
\]

Thus \( \Psi \Phi_f \Psi^{-1} = \Phi_{(\Psi f)} \), and the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{E}_A & \xrightarrow{\Phi_f} & \mathcal{E}_B \\
\downarrow & & \downarrow \\
\mathcal{E}_{\tilde{\Psi}(A)} & \xrightarrow{\Phi_{(\Psi f)}} & \mathcal{E}_{\tilde{\Psi}(B)}
\end{array}
\]

Consequently, \( \tilde{\Psi}(f) = \tilde{\Psi} \tilde{f} \tilde{\Psi}^{-1} \).

To prove that \( \mu \) is invariant under \( \tilde{\Psi} \), it will first be shown that if \( A, B \in P \) are disjoint and \( \mu(A) = \mu(B) \), then \( \tilde{\Psi}(A) \) and \( \tilde{\Psi}(B) \) are also disjoint and have equal measure. For such \( A \) and \( B \), let \( f \in E \) be any interval exchange such that \( \tilde{f}(A) = B \). Then \( f \) induces a conjugacy between the subgroups \( E_A \) and \( E_B \), and \( \Psi(f) \) induces a conjugacy between \( E_{\tilde{\Psi}(A)} \) and \( E_{\tilde{\Psi}(B)} \). By (2), \( \tilde{\Psi}(f) \) maps \( \tilde{\Psi}(A) \) to \( \tilde{\Psi}(B) \), and as a result, \( \mu(\tilde{\Psi}(A)) = \mu(\tilde{\Psi}(B)) \), which proves the initial claim.

To prove that \( \mu(\tilde{\Psi}(A)) = \mu(A) \) for any \( A \in P \), assume first that \( \mu(A) \) is rational. Since any \( \tilde{\Psi} \) preserves finite disjoint unions, it may be further assumed that \( \mu(A) = 1/n \). Lebesgue measure is invariant under any conjugacy \( \Phi_f \), so it finally suffices to consider the case \( A = [0, 1/n) \). Each of the intervals

\[
A_i = \left[ \frac{i - 1}{n}, \frac{i}{n} \right), \quad 2 \leq i \leq n,
\]

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has the same measure as \( A = A_1 \) and is disjoint from it. Thus
\[
\mu(\tilde{\Psi}(A_i)) = \mu(\tilde{\Psi}(A)), \quad 2 \leq i \leq n.
\]
Since the sets \( \tilde{\Psi}(A_i) \) are also pairwise disjoint and cover \( \mathbb{T}^1 \), it follows that \( \mu(\tilde{\Psi}(A_i)) = 1/n \). Consequently, \( \tilde{\Psi} \) preserves the measure of sets with rational measure. In general, the set \( A \) may be approximated by an increasing family of sets in \( \mathcal{P} \) having rational measure.

\[\Box\]

### 6.2 Proof of Theorem 1.6

Let \( \Psi \) be an arbitrary automorphism of \( \mathcal{E} \). It will be shown that the identity automorphism may be reached by successively replacing \( \Psi \) with a composition of \( \Psi \) and an automorphism in \( \langle \text{Inn}(\mathcal{E}), \Psi_T \rangle \).

To begin, by Corollary 6.2, \( \Psi \) maps the rotation group \( R \) to a conjugate \( \Phi_g(R) \). Replacing \( \Psi \) by \( \Phi_g^{-1} \circ \Psi \), it may be assumed that \( R \) is invariant under \( \Psi \). Let \( \Psi_R : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) denote the restriction \( \Psi|_R \), where \( r_{\alpha} \mapsto \alpha \) is the natural identification of \( R \) and \( \mathbb{R}/\mathbb{Z} \).

**Lemma 6.7.** \( \Psi_R \) is continuous (w.r.t. the standard topology on \( \mathbb{R}/\mathbb{Z} \)).

**Proof.** Since \( \Psi_R \) is a group homomorphism, it suffices to show that \( \Psi_R \) is continuous at \( 0 \in \mathbb{R}/\mathbb{Z} \). Suppose that \( \alpha_n \to 0 \) in \( \mathbb{R}/\mathbb{Z} \). Then for any nonempty \( A \in \mathcal{P} \), there exists a constant \( M_A > 0 \) such that
\[
A \cap r_{\alpha_n}(A) \neq \emptyset, \quad \text{if } n \geq M_A.
\]
Conversely, this condition characterizes sequences in \( \mathbb{R}/\mathbb{Z} \) which converge to \( 0 \). In particular, given some sequence \( \alpha_n \), suppose that there exists a constant \( M_A \) as above for every nonempty \( A \in \mathcal{P} \). For any \( \epsilon > 0 \), let \( A_\epsilon = [0, \epsilon) \). Then
\[
A_\epsilon \cap r_{\alpha_n}(A_\epsilon) \neq \emptyset, \quad \text{if } n \geq M_{A_\epsilon},
\]
which implies that \( |\alpha_n| < \epsilon \) for all \( n \geq M_A \). Thus, \( \alpha_n \to 0 \).

Assuming again that \( \alpha_n \to 0 \), define \( \beta_n = \Psi_R(\alpha_n) \), so \( r_{\beta_n} = \Psi(r_{\alpha_n}) \). Let \( B \in \mathcal{P} \) be nonempty, and let \( A = \tilde{\Psi}^{-1}(B) \). Then by Proposition 6.6 part (2),
\[
r_{\beta_n}(B) = \tilde{\Psi}(r_{\alpha_n}(\tilde{\Psi}^{-1}(B))) = \tilde{\Psi}(r_{\alpha_n}(A)).
\]
Consequently, \( A \cap r_{\alpha_n}(A) \neq \emptyset \) if and only if \( B \cap r_{\beta_n}(B) \neq \emptyset \). Therefore, if \( \alpha_n \to 0 \), then there exists \( M_B \) (namely, the \( M_A \) associated with \( \alpha_n \)), such that
\[
B \cap r_{\beta_n}(B) \neq \emptyset, \quad \forall n \geq M_B.
\]
From the above characterization of sequences converging to zero, it follows that \( \Psi_R \) is continuous at zero. \[\Box\]
The only continuous automorphisms of \( \mathbb{R}/\mathbb{Z} \) are the identity and \( x \mapsto -x \). Note that the restriction of the orientation-reversing automorphism \( \Psi_T \) induces the second of these automorphisms. Subsequently, after replacing \( \Psi \) by \( \Psi \circ \Psi_T \) if \( \Psi_R \) is not the identity, it may be assumed that \( \Psi_R = \text{id} \).

**Lemma 6.8.** If \( \Psi \in \text{Aut}(\mathcal{E}) \) fixes the rotation group \( R \), then \( \tilde{\Psi} \) maps any interval in \( \mathcal{P} \) to another interval.

**Proof.** Since any rotation will preserve intervals in \( \mathcal{P} \), it suffices to consider \( I_a = [0, a) \). Then there exists some \( \epsilon > 0 \), such that for any \( \alpha \in (-\epsilon, \epsilon) \),

\[
\mu(I_a \cap r_{\alpha}(I_a)) = a - |\alpha|.
\]

By Proposition 6.6 part (2), and the hypothesis that \( \Psi(r_{\alpha}) = r_{\alpha} \),

\[
\tilde{\Psi} \circ \tilde{r}_{\alpha} = \tilde{\Psi}(r_{\alpha}) \circ \tilde{\Psi} = \tilde{r}_{\alpha} \circ \tilde{\Psi}.
\]

Therefore, it is also the case that

\[
\mu(\tilde{\Psi}(I_a) \cap r_{\alpha}(\tilde{\Psi}(I_a))) = a - |\alpha|,
\]

for \( \alpha \in (-\epsilon, \epsilon) \).

Suppose that \( \tilde{\Psi}(I_a) \) has \( k \geq 1 \) components:

\[
\tilde{\Psi}(I_a) = A_1 \cup \cdots \cup A_k,
\]

where the \( A_i \) are pairwise disjoint intervals. Since the \( A_i \) are disjoint, there is some \( \delta > 0 \) such that

\[
r_{\beta}(A_i) \cap A_j = \emptyset, \text{ for all } |\beta| < \delta \text{ and } i \neq j.
\]

Consequently, if \( |\beta| < \delta \), then

\[
\mu(\tilde{\Psi}(I_a) \cap r_{\beta}(\tilde{\Psi}(I_a))) = a - k|\beta|.
\]

It follows that \( k = 1 \), which implies that \( \tilde{\Psi}(I_a) \) must be an interval. \( \square \)

Continue with the assumption that \( \Psi_R \) is the identity. By the previous lemma, \( \tilde{\Psi} \) maps the interval \( I_a = [0, a) \) to some translate of \( I_a \). After composing \( \Psi \) with a suitable \( \Phi_{r_{\beta}} \), it may be assumed that \( \Psi_R \) is the identity and \( \tilde{\Psi}(I_a) = I_a \). Since

\[
\tilde{\Psi} \circ \tilde{r}_{\beta} = \tilde{r}_{\beta} \circ \tilde{\Psi},
\]

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for all $\beta \in \mathbb{R}/\mathbb{Z}$, it follows that $\tilde{\Psi}$ fixes any translate $r_\beta(I_a) = [\beta, a+\beta)$. Thus, for any $\beta$, $0 < \beta < a$, $\tilde{\Psi}$ fixes the intersection

$$I_a \cap r_\beta(I_a) = [\beta, a).$$

Thus $\tilde{\Psi}$ fixes all translates of arbitrarily small intervals, which implies that $\tilde{\Psi}$ is the identity on $\mathcal{P}$. Consequently, for any $f \in \mathcal{E}$, $\Psi(f)$ acts on the sets in $\mathcal{P}$ identically to the way $f$ does, which implies $\Psi$ is the identity. It has been shown that any $\Psi \in \text{Aut}(\mathcal{E})$ is in the group $\langle \text{Inn}(\mathcal{E}), \Psi_T \rangle$, and the proof of Theorem 1.6 is complete.

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