SUPERCONFORMAL DEFORMATIONS AND

SPACE-TIME SYMMETRIES

Ioannis Giannakis †

Department of Physics, New York University
4 Washington Pl., New York, NY 10003.

Abstract

In this paper we present a method of deforming to first order the stress-tensor and the supercurrent of the superstring corresponding to turning on NS-NS bosonic fields. Furthermore we discuss the difficulties associated with turning on spacetime fermions and R-R bosons. We also derive the gauge symmetries of the massless spacetime fields.

† e-mail: ig231@sciress.nyu.edu
1. Introduction.

Superconformal field theories (with appropriate central charge) are solutions to the classical equations of motion of string theory. Thus by studying infinitesimal deformations which preserve this superconformal structure [1], we are investigating the linearised classical equation of motion about the corresponding solution. This is an interesting problem in its own right, but it also provides us with insights into the symmetry structure of string theory since a spacetime symmetry transformation is a particular deformation. To study symmetries, we seek transformations of the space-time fields that take one solution of the classical equations of motion to another that is physically equivalent. Since, “Solutions of the classical equations of motion,” are two-dimensional superconformal field theories, we are thus interested in isomorphic superconformal field theories.

2. Superconformal Deformations.

We shall work in a Hamiltonian formalism. Any quantum mechanical theory (including a superconformal field theory) is defined by three elements: i) an algebra of operator valued distributions, usually called superfields, \( \mathcal{A} \) (determined by the degrees of freedom of the theory and their equal-time commutation relations), ii) a representation of that algebra and iii) two distinguished elements of \( \mathcal{A} \), \( T(\sigma, \theta) = T_F(\sigma) + \theta T(\sigma) \) and \( \mathcal{T}(\sigma, \theta) = T_F(\sigma) + \theta T(\sigma) \). Superconformal operator algebras include also spin fields. In terms of these fields the Hamiltonian \( H \) and the generator of translations \( P \) are given by

\[
H = \int d\sigma (T(\sigma) + \mathcal{T}(\sigma)), \quad P = \int d\sigma (T(\sigma) - \mathcal{T}(\sigma)).
\]

(1)

Also the components \( T_F(\sigma) \) and \( T(\sigma) \) must satisfy two mutually commuting copies of the SuperVirasoro algebra (we have omitted one copy of the SuperVirasoro for simplicity):

\[
[T(\sigma), T(\sigma')] = -\frac{ic}{24\pi} \delta'''(\sigma - \sigma') + 2iT(\sigma')\delta'(\sigma - \sigma') - iT'(\sigma')\delta(\sigma - \sigma') \quad (2.2a)
\]

\[
\{T(\sigma), T_F(\sigma')\} = -\frac{1}{2\sqrt{2}} T(\sigma')\delta(\sigma - \sigma') + \frac{c}{24\sqrt{2}\pi} \delta''(\sigma - \sigma') \quad (2.2b)
\]

\[
[T(\sigma), T_F(\sigma')] = \frac{3i}{2} T_F(\sigma')\delta'(\sigma - \sigma') - iT'_F(\sigma')\delta(\sigma - \sigma'). \quad (2.2c)
\]

The existence of the SuperVirasoro algebra means that the states of the theory can be organised into modules of that algebra. If the Hamiltonian, Eq. (1), is bounded below, these are highest weight representations. The highest weight states of these representations are created by the other important operators in the theory, the superconformal primary
fields which are constructed from elementary fields and momenta and are defined as any pair of fermionic $\Phi_F(\sigma)$ and bosonic $\Phi_B(\sigma)$ fields that satisfy

$$[T(\sigma), \Phi_F(\sigma')] = id\Phi_F(\sigma')\delta'(\sigma - \sigma') - \frac{i}{\sqrt{2}} \partial\Phi_F(\sigma')\delta(\sigma - \sigma')$$

(2.3a)

$$[T(\sigma), \Phi_B(\sigma')] = i(d + \frac{1}{2})\Phi_B(\sigma')\delta'(\sigma - \sigma') - \frac{i}{\sqrt{2}} \partial\Phi_B(\sigma')\delta(\sigma - \sigma')$$

(2.3b)

$$\{T_F(\sigma), \Phi_F(\sigma')\} = -\frac{1}{2\sqrt{2}} \Phi_B(\sigma')\delta(\sigma - \sigma')$$

(2.3c)

$$[T_F(\sigma), \Phi_B(\sigma')] = id\Phi_F(\sigma')\delta'(\sigma - \sigma') - \frac{i}{2\sqrt{2}} \partial\Phi_F(\sigma')\delta(\sigma - \sigma')$$

(2.3d)

Note that for the same $\mathcal{A}$ we may have many choices of Hamiltonian, so that $\mathcal{A}$ should more properly be associated with a deformation class of superconformal theories than with one particular theory. The superfields do not exhaust the set of all operators in the algebra $\mathcal{A}$. On the complex plane for example the fermionic components of the superfields as two-dimensional spinors are double-valued fields, $\Phi_F(e^{2\pi i z}) = \pm \Phi_F(z)$. This implies that the operator algebra $\mathcal{A}$ encompasess spin fields $S^\alpha(z)$ whose presence modifies the boundary conditions of the fermionic components of the superfields. As a result the operator algebra is not local since the OPE of the fermionic component of a superfield $\Phi_F(z)$ with a spin field $S^\alpha(w)$ includes half integral powers of $\frac{1}{(z-w)}$. Locality appears to be essential in order to have a well defined string theory. We can either restrict ourselves to one of the two boundary conditions for the fermionic components of the superfields or include both boundary conditions (NS and R) but by eliminating half of the operators of each we regain a local operator algebra. This will involve the projection (GSO projection) of the non-local operator algebra $\mathcal{A}$ onto a local one $\mathcal{A}_1$.

String theory requires the full structure of the SuperVirasoro algebras in order to decouple negative norm states from physical processes.

We are interested in not just one, but a family of superconformal field theories parametrized by the values of the spacetime fields. Changing these spacetime fields changes the superconformal field theory but preserves the SuperVirasoro algebra (including the value of the central charge). We will discuss deformations of the local operator algebra $\mathcal{A}_1$ which involve a change in our choice of $T, T_F, \overline{T}$ and $\overline{T_F}$ in a way that preserves the SuperVirasoro algebra. Thus under deformations

$$T(\sigma) \rightarrow T(\sigma) + \delta T(\sigma) \quad T_F(\sigma) \rightarrow T_F(\sigma) + \delta T_F(\sigma)$$

(4)
and the preservation of the SuperVirasoro algebra, to first order in variations, we require

\[ \left[ \delta T(\sigma), T(\sigma') \right] + \left[ T(\sigma), \delta T(\sigma') \right] = 2i\delta T(\sigma')\delta'(\sigma - \sigma') - i\delta T'(\sigma')\delta(\sigma - \sigma') \]

\[ \{\delta T_F(\sigma), T_F(\sigma')\} + \{T_F(\sigma), \delta T_F(\sigma')\} = -\frac{1}{2\sqrt{2}}\delta T(\sigma')\delta(\sigma - \sigma') \]  

(5)

\[ \left[ \delta T(\sigma), T_F(\sigma') \right] + \left[ T(\sigma), \delta T_F(\sigma') \right] = \frac{3i}{2}\delta T_F(\sigma')\delta'(\sigma - \sigma') - i\delta T_F'(\sigma')\delta(\sigma - \sigma'). \]

We now seek solutions to the deformation equations. Let’s make the ansatz

\[ \delta T(\sigma) = \Phi_B(\sigma), \quad \delta T_F(\sigma) = \Phi_F(\sigma) \]  

(6)

where \( \Phi_F(\Phi_B) \) is the fermionic (bosonic) component of a superfield of dimension \((d, \overline{d})\). Substituting into the deformation equations and using Eq. (5) we find that our ansatz of \( \delta T \) and \( \delta T_F \) satisfy the deformation equations if the conformal dimension of \( \Phi_F(\sigma) \) is \( d = \overline{d} = \frac{1}{2} \) and the dimension of \( \Phi_B(\sigma) \) is \( d = \overline{d} = 1 \). These are the supersymmetric generalizations of the so called canonical deformations found in references [2].

For simplicity, consider a superconformal field theory of free scalars and free two-dimensional fermions, defined by the stress-tensor and the supercurrent

\[ T(\sigma) = -\frac{1}{2}\eta^{\mu\nu}\partial X_\mu \partial X_\nu - \frac{1}{2}\eta^{\mu\nu}\psi_\mu \partial \psi_\nu \]

\[ T_F(\sigma) = -\frac{1}{2}\eta^{\mu\nu}\psi_\mu \partial X_\nu. \]

(7)

The canonical deformations which correspond to turning on a massless NS-NS field, for example sending a weak gravitational and two-form gauge wave through this background can be found by identifying the bosonic component \( \Phi_B(\sigma) \) of Eq. (6) with the appropriate vertex operator ,

\[ \delta T(\sigma) = \Phi_B(\sigma) = K^{\mu\nu}(X)\overline{\partial}X_\nu \partial X_\mu + \partial^\lambda K^{\mu\nu}(X)\overline{\partial}X_\nu \psi_\lambda \psi_\mu. \]

(8)

The right hand side of this equation is a \((1, 1)\) primary field only if the functions \( K^{\mu\nu}(X) \) satisfy

\[ \Box K^{\mu\nu}(X) = 0, \quad \partial_\mu K^{\mu\nu}(X) = 0. \]

(9)

Its superpartner \( \Phi_F(\sigma) \) is then found by calculating the commutator of \( \Phi_B(\sigma) \) with the supercurrent \( T_F(\sigma) \). We find that

\[ \delta T_F(\sigma) = K^{\mu\nu}(X)\overline{\partial}X_\nu \psi_\mu. \]

(10)

By a tedious but rather straightforward calculation we can now verify that \( \delta T \) and \( \delta T_F \) satisfy the deformation equations. These superconformal deformations have also been found in papers [3].
3. Spacetime Symmetries.

Let us recall the essential features of the approach to spacetime symmetries which was developed in references [2], [4]. Given any algebra of operators we can construct another algebra isomorphic to the first one by means of a similarity transformation, also called inner automorphism \( \rho_h(O(\sigma)) = e^{ih}O(\sigma)e^{-ih} \) or in infinitesimal form \( \rho_h(O(\sigma)) = O(\sigma) + i[h,O(\sigma)]. \) For any infinitesimal operator \( h \) then the superconformal field theories specified by \( T_{\Phi}, T_{F(\Phi)} \) and \( T_{\Phi} + i[h,T_{\Phi}], T_{F(\Phi)} + i[h,T_{F(\Phi)}] \) are isomorphic. Thus if

\[
\begin{align*}
\delta T &= T_{\Phi} + \delta \Phi - T_{\Phi} = i[h,T_{\Phi}] \\
\delta T_{F} &= T_{F(\Phi)} + \delta \Phi - T_{F(\Phi)} = i[h,T_{F(\Phi)}]
\end{align*}
\]

then it follows that \( \Phi \mapsto \Phi + \delta \Phi \) is a symmetry transformation of the spacetime fields. As it should be obvious by now any inner automorphism preserves the physics but not every automorphism can be interpreted as a change in the spacetime fields. We should then think of Eq. (11) as a restriction on the operators \( h. \)

We now know how the stress-tensor \( T \) and the supercurrent \( T_{F} \) deform as we change the spacetime fields, Eq. (6). We then must find operators \( h \) that when commuted with \( T \) and \( T_{F} \) yield Eq. (11). Let \( \Psi(\sigma) \) be a sum of superfields of dimension \( (\frac{1}{2}, 0) \) and \( (0, \frac{1}{2}) \) and \( h \) to be

\[
h = \int d\sigma d\theta d\bar{\theta} \Psi(\sigma, \theta, \bar{\theta}).
\]

It then follows from the definition of superprimary fields that \( i[h,T] \) and \( i[h,T_{F}] \) reproduces a \( \delta T \) and \( \delta T_{F} \) of the form of a canonical deformation Eq. (6) and thus can be interpreted as change in the spacetime fields. The obvious choice for the operator \( h \) is

\[
h = \int d\sigma d\theta d\bar{\theta} \xi^\mu(\chi) D\chi^\mu = \int d\sigma \left( \xi^\mu(X) \partial X^\mu + \partial^\mu \xi^\nu(X) \psi_\mu \psi_\nu \right)(\sigma),
\]

where \( \chi^\mu = \psi^\mu + \theta X^\mu \) and \( D \) is the covariant derivative. This is the supersymmetric generalization of the operator that generates coordinate and two form gauge transformations about flat spacetime in the bosonic string theory. The integrand is only superprimary and of dimension \( \frac{1}{2} \) if the parameter \( \xi \) satisfies

\[
\Box \xi^\mu(X) = 0, \quad \partial_\mu \xi^\mu(X) = 0.
\]

These conditions are required because of normal ordering. We then proceed to calculate the commutator of \( h \) with \( T \) and \( T_{F} \) and we find

\[
\begin{align*}
i[h,T(\sigma)] &= \partial^\mu \xi^\nu \bar{\partial} X_\nu \partial X_\mu + \partial^\lambda \partial^\mu \xi^\nu \bar{\partial} X_\nu \psi_\lambda \psi_\mu \\
i[h,T_{F}(\sigma)] &= \frac{1}{2} \partial^\mu \xi^\nu \bar{\partial} X_\nu \psi_\mu.
\end{align*}
\]
The result provides us with the transformation properties of the physical fields under coordinate and two form gauge transformations
\[ \delta K^{\mu \nu} = \partial^{\mu} \xi^{\nu}. \] (16)

It is not hard to generalize this construction to an infinite class of infinitesimal gauge symmetries [5] and to finite symmetry transformations (T-duality) [6]. It is also worth noting that this approach treats on equal footing exact and spontaneously broken spacetime symmetries [7].

Canonical deformations have a number of interesting features: superprimary fields of dimension \((\frac{1}{2}, \frac{1}{2})\) are in natural correspondence with the physical states of string theory, being the vertex operators. As such they have a nice spacetime interpretation in terms of turning on spacetime fields. Appealing though they are canonical deformations have also significant drawbacks. They correspond to turning on NS-NS fields in a particular gauge as we have seen while they do not appear to describe spacetime fermions and R-R bosonic fields which are described in terms of spin fields. Spin fields cannot be written as superfields. These string backgrounds have attracted interest recently due to the conjectured AdS/CFT equivalence [8]. We might attempt to identify the bosonic component \(\Phi_B(\sigma)\) of the canonical deformation with the appropriate spacetime fermionic vertex operator
\[ \delta T(\sigma) = \Phi_B(\sigma) = \Psi_\mu^\alpha(X)S^\alpha e^{-\frac{w}{2} \partial X^\mu} \] (17)

This is a \((1, 1)\) primary field only if the functions \(\Psi_\mu^\alpha(X)\) obey
\[ \Box \Psi_\mu^\alpha(X) = 0, \quad \gamma^\mu \partial_\mu \Psi_\nu^\alpha(X) = 0, \quad \partial_\mu \Psi_\mu^\alpha(X) = 0. \] (18)

In order then to find its superpartner we need to calculate the commutator of \(\Phi_B(\sigma)\) with the supercurrent \(T_F(\sigma)\). The commutator of the vertex operator which is written in terms of spin fields with the supercurrent \(T_F\) is not well-defined since the corresponding OPE in the complex plane involves branch cut singularities
\[ T_F(z)\Phi_B(w) = \frac{\gamma^\lambda_{\alpha \beta} \partial_\lambda \Psi_\mu^\alpha(X)S^\beta e^{-\frac{w}{2} \partial X^\mu}}{(z-w)^{\frac{1}{2}}} + \frac{\gamma^\lambda_{\alpha \beta} \Psi_\mu^\alpha(X)S^\beta e^{-\frac{w}{2} \partial X^\lambda \partial X^\mu}}{(z-w)^{\frac{1}{2}}} \] (19)

This suggests then that the deformations we have just constructed in terms of superfields are not the most general solution to the deformation equations. In a forthcoming publication we intend to present the resolution to these problems.

4. Acknowledgments.

This work was done in collaboration with Jonathan Bagger and Mark Evans. I would like to thank James T. Liu and M. Porrati for useful discussions. This work was supported by NSF grant PHY-9722083.
References.

[1] J. Freericks and M. Halpern, Ann. Phys. (NY) **188** 258 (1988).
[2] M. Evans and B. Ovrut, Phys. Rev. **D41** 3149 (1990), M. Campbell, P. Nelson and E. Wong, Int. Jour. Mod. Phys. **A6** 4909 (1991).
[3] B. Ovrut and S. Kalyana Rama, Phys. Rev. **D45** 550 (1992), J. C. Lee, Z. Phys. **C54** 283 (1992).
[4] M. Evans and I. Giannakis, Phys. Rev. **D44** 2473 (1991).
[5] M. Evans, I. Giannakis and D. Nanopoulos Phys. Rev. **D50** 4022 (1994).
[6] M. Evans and I. Giannakis, Nucl. Phys. **B472** 139 (1996), I. Giannakis, Phys. Lett. **388B** 543 (1996).
[7] J. Bagger and I. Giannakis, Phys. Rev. **D56** 2317 (1997).
[8] J. Maldacena, Adv. Theor. Math. Phys. **2** 231 (1998), S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **428B** 105 (1998), E. Witten, Adv. Theor. Math. Phys. **2** 253 (1998)