Massive two-column bosonic fields
in the frame-like formalism

Yu. M. Zinoviev *

Institute for High Energy Physics
of National Research Center ”Kurchatov Institute”
Protvino, Moscow Region, 142280, Russia

Abstract

In this paper we develop the frame-like gauge invariant formulation for the massive
two-column bosonic fields in (anti) de Sitter space-times. We begin with the partially
massless cases in $AdS$ and $dS$ and then we combine these results into the general
massive theory. Separate section is devoted to the special case where both columns
have equal number of indices.

*E-mail address: Yurii.Zinoviev@ihep.ru
Introduction

Two-column fields (tensors with two groups of completely anti-symmetric indices) are the special and interesting case of the general mixed symmetry fields that naturally arise in space-time dimensions greater than four. One of the reason for this interest is that such fields (in the appropriate dimensions) can be considered as the dual forms of the usual spin-2 graviton (see e.g. [1, 2] and references therein). In the frame-like formalism the general massless mixed symmetry fields (including two-column ones) in the flat Minkowski space have been constructed in [3]. One of the specific features of the massless mixed symmetry fields is that most of them do not admit straightforward deformation into (anti) de Sitter space and to keep all the gauge symmetries one has to introduce some additional fields [4] (see also [5, 6]). Thus the irreducible (A)dS representations for such fields become reducible ones in the flat limit. In what follows by massless fields we will always assume massless fields in the Minkowski space while all other cases that become reducible in the flat limit we will call partially massless or massive. Partially massless two-column fields in AdS space have been constructed in [7] as the special case of the general formalism in [8, 9, 10]. Note that approaches of [3] and [7] are quite different. In the first one to describe two-column fields $Y(k, l)$, $k > l$, one introduces physical $k$-form with $l$ anti-symmetric local indices $e^{a[k]}$ and auxiliary $l$-form with $k$ local ones $\omega^{a[k]}$. At the same time in the second approach both physical and auxiliary fields are $l$-forms with $k$ and $k + 1$ local indices correspondingly. Let us stress that the second approach keeps only half of the gauge symmetries that massless two-column fields possess in the flat case and in what follows we will see that the results of [7] can be reproduced by the partial gauge fixing.

In the metric-like formalism the general massive two-column fields in the (A)dS spaces with all possible (partially) massless limits have been constructed in [11] using so-called multi-forms (see e.g. [12] and references therein). Our aim here is to reconstruct the results of [11] in the frame-like gauge invariant formalism following the approach of [3]. As it could be expected the results of [11] appeared to be quite similar to the results for the simplest ”hook” and ”window” cases [20]. So to simplify calculations we begin with the partially massless cases in de Sitter and anti de Sitter spaces and than we combine them into the general massive one.

Our main objection here is that the frame-like formalism seemed to be not only much simpler but also very effective in the investigations of possible interactions. Till now the results on the mixed symmetry fields interactions are not numerous. There was a number of ”no-go” results (e.g. [21, 22, 23]). In [24] cubic interactions for the system containing not only completely symmetric fields but also the ”long hook” ones in AdS$_5$ have been constructed. Gravitational interactions for the simple hook both in Minkowski and in AdS space were considered in [25], while a complete non-trivial theory for the set of ”high hooks” have been constructed in [20]. For the massive hook electromagnetic and gravitational interactions have been considered in [27, 28].

The paper is organized as follows. In the first section we provide all necessary formulas

---

1BRST approach for the general massive mixed symmetry fields have been developed in [13, 14] (see also [15, 16] for the two-column case), while frame-like gauge invariant description for the massive two-row fields were given in [17, 18]. Weyl actions and their possible AdS and massive deformations for the two-column fields have been considered recently in [19].
for the frame-like description of massless two-column fields in Minkowski space following [3].

Then in sections 2 and 3 we consider partially massless cases in anti de Sitter and de Sitter spaces combining them in section 4 into general massive theory. Section 5 devoted to the special case $Y(k, k)$ where both columns have equal number of cells.

**Notation and conventions** We will work in the (anti) de Sitter space with the background frame $e^a$ and its inverse $\hat{e}_a$. We will heavily use short-hand notation for their wedge products:

$$E^{a(k)} = e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_k}$$

$$\hat{E}_{a(k)} = \hat{e}_{a_1} \wedge \hat{e}_{a_2} \wedge \ldots \wedge \hat{e}_{a_k}$$

A couple of useful relations:

$$\hat{E}_{a(k)} \wedge e^b = \delta^a_b \hat{E}_{a(k-1)}$$

$$\hat{E}_{a(k)} \wedge e^a = (d - k + 1) \hat{E}_{a(k-1)}$$

An (A)dS covariant derivative $D$ is defined so that

$$D \wedge D \xi^a = -\kappa E^a_b \xi^b$$

In what follows we will systematically omit the wedge product sign $\wedge$.

# 1 Massless fields in the Minkowski space

As we have already noted, massless fields in the Minkowski space play the role of minimal building blocks for the construction of general partially massless and massive fields in (A)dS spaces with arbitrary value of the cosmological constant. So in this section we give all necessary formulas for the massless two-column fields. We will follow the approach of Skvortsov [3] but we will use an anti-symmetric basis which is natural for the two-column fields.

## 1.1 General case $Y(k, l)$, $k > l$

Frame-like description requires physical $k$-form $R^{a(l)}_k$ as well as auxiliary $l$-form $\Sigma^{a(k+1)}_l$. The free Lagrangian can be written as follows:

$$\mathcal{L}_0(\Sigma^{k+1}_l, R^{a(l)}_k) = a_0(k, l) \hat{E}_{a(2l)} \Sigma^{a(l)b(l+1)}_l \Sigma^{a(l)}_b(k-l+1) + \hat{E}_{a(k+1+l)} \Sigma^{a(k+1)}_l dR^{a(l)}_k$$  \hspace{1cm} (1)

where $d$ is a usual external derivative. This Lagrangian is invariant under the following local gauge transformations:

$$\delta_0 \Sigma^{a(k+1)}_l = d\eta_{l-1}^{a(k+1)}, \quad \delta_0 R^{a(l)}_k = d\xi_{k-1}^{a(l)} + E_{b(k-l+1)} \eta_{l-1}^{a(l)b(k-l+1)}$$  \hspace{1cm} (2)

where $\xi^{a(l)}_{k-1}$ is a $(k - 1)$-form, while $\eta^{a(k+1)}_{l-1}$ is a $(l - 1)$-form.

Note that such massless field does not admit deformation into (A)dS space without introduction of some additional fields (see below). In this particular case (see general discussion in [4]) the reason is simple: we can not add to the Lagrangian any term quadratic in the physical field $R^{a(l)}_k$. 


1.2 Special case $Y(k, k)$

In this case both physical field $R_{k}^{a(k)}$ as well as auxiliary one $\Sigma_{k}^{a(k+1)}$ are $k$-forms. The free Lagrangian looks like (it corresponds to (1) for $l = k$):

$$\mathcal{L}_0(\Sigma_{k}^{a(k+1)}, R^k_{k}) = \frac{(k + 1)}{2} \hat{E}_{a(2k)} \Sigma_{k}^{a(k)b} \Sigma_{b}^{a(k)}_{k} + \hat{E}_{a(2k+1)} \Sigma_{k}^{a(k+1)} dR^a_{k}$$

and is invariant under the following gauge transformations:

$$\delta_0 \Sigma_{k}^{a(k+1)} = d\eta^a_{k-1}, \quad \delta_0 R_{k}^{a(k)} = d\xi^a_{k-1} + e_b \eta^a_{k-1}$$

where both gauge parameters are $(k-1)$-forms.

One of the reasons why this case is indeed special is that this massless field admits deformation into AdS space without introduction of any additional fields. Let us consider the following deformed Lagrangian:

$$\mathcal{L}_0 = \frac{(k + 1)}{2} \hat{E}_{a(2k)} \Sigma_{k}^{a(k)b} \Sigma_{b}^{a(k)}_{k} + \hat{E}_{a(2k+1)} \Sigma_{k}^{a(k+1)} D R^a_{k} + b_1 \hat{E}_{a(2k)} R^a_{k} R^a_{k}$$

where $D$ is AdS covariant derivative, and corrected gauge transformations:

$$\delta_0 \Sigma_{k}^{a(k+1)} = D\eta^a_{k-1} + \beta_1 e^a \xi^a_{k-1}, \quad \delta_0 R_{k}^{a(k)} = D\xi^a_{k-1} + e_b \eta^a_{k-1}$$

Variations under the $\eta$ transformations give

$$\delta_\eta \mathcal{L}_0 = k((k + 1)(d - 2k)\kappa - 2b_1) \hat{E}_{a(2k-1)} \eta^a_{k-1} b_{k-1} R^a_{k}$$

while variations under the $\xi$ transformations give

$$\delta_\xi \mathcal{L}_0 = \left[(-1)^{k}2b_1 - (k + 1)(d - 2k)\beta_1 \right] \hat{E}_{a(2k)} R^a_{k} b_{k-1} \xi^a_{k-1} + k(k + 1)(d - 2k) \left[(-1)^{k} \kappa - \beta_1 \right] \hat{E}_{a(2k-1)} \Sigma_{k}^{a(k)} b_{k-1} \xi^a_{k-1}$$

Thus we obtain

$$2b_1 = (k + 1)(d - 2k)\kappa, \quad \beta_1 = (-1)^{k} \kappa$$

2 Partially massless case in anti de Sitter space

In this section we will show that the combination of two fields $Y(k, l)$ and $Y(k - 1, l)$ can be deformed into anti de Sitter space with $\kappa < 0$ and such deformation keep all four (appropriately modified) gauge symmetries. Note that this case have already been considered in our paper [25], so we briefly reproduce corresponding results in our current notation and conventions because we will need them for the general massive case.
2.1 Lagrangian and gauge transformations

We introduce two pairs of physical and auxiliary fields \((\Sigma_l^{k+1}, R^l_k)\) and \((\omega_l^k, h_{k-1}^l)\). We begin with the sum of their kinetic terms:

\[
\mathcal{L}_0 = \mathcal{L}_0(\Sigma_l^{k+1}, R^l_k) + \mathcal{L}_0(\omega_l^k, h_{k-1}^l)
\]  

and their initial gauge transformations:

\[
\delta_0 \Sigma_l^{a(k+1)} = D\eta_l^{a(k+1)}, \quad \delta_0 R_k^{a(l)} = D\eta_{k-1}^{a(l)} + E_{b(k-l+1)}\eta_{k-1}^{a(l)b(k-l+1)} \\
\delta_0 \omega_l^{a(k)} = D\eta_l^{a(k)}, \quad \delta_0 h_{k-1}^{a(l)} = D\eta_{k-2}^{a(l)} + E_{b(k-l)}\eta_{k-1}^{a(l)b(k-l)}
\]  

where all derivatives are \(AdS\) covariant ones. Then we add all possible terms with one derivative:

\[
\mathcal{L}_1 = a_3 \hat{E}_{a(k+l-1)} \Sigma_l^{a(k)} R^l_k + a_4 \hat{E}_{a(k+l)} \omega_l^{a(l)} R_k^{a(l)}
\]

and corresponding corrections to the gauge transformations

\[
\delta_1 \Sigma_l^{a(k+1)} = \alpha_5 e^{a(k)} \eta_{l-1}^{a(k)}, \quad \delta_1 R_k^{a(l)} = \alpha_6 e^{a(l)b} \eta_{k-2}^{a(l)b(k-l)} \\
\delta_1 \omega_l^{a(k)} = \alpha_7 e^{a(k)} \eta_{l-1}^{a(k)}, \quad \delta_1 h_{k-1}^{a(l)} = \alpha_8 \eta_{k-1}^{a(l)}
\]

To cancel all variations with one derivative \(\delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0\) (see Appendix A for details) we have to put:

\[
\alpha_5 = -\alpha_6 = -\frac{(-1)^l a_4}{(k+1)(d-k-l)}, \quad \alpha_7 = (-1)^{k-l} a_4, \quad \alpha_8 = -a_4, \quad a_3 = (-1)^l a_4
\]

Note that in this case case there are no any terms quadratic in the physical fields \(R\) and \(h\) that can be added to the Lagrangian. But all variations without derivatives \(\delta_1 \mathcal{L}_1\) vanish provided

\[
a_4^2 = -(k+1)(d-k-l)\kappa
\]

2.2 Gauge invariant curvatures

One of the nice features of the frame-like formalism is that for each field (physical or auxiliary) one can construct gauge invariant object (that we will call curvature). It is clear that in any gauge theory such gauge invariant objects play important role both in the free theory as well as in any attempt to construct non-trivial interactions.

For the case at hands we can construct four such gauge invariant curvatures\(^2\):

\[
\mathcal{R}_l^{a(k+1)} = D\Sigma_l^{a(k+1)} + \alpha_5 e^{a(k)} \omega_l^{a(k)} \\
\mathcal{T}_{k+1}^{a(l)} = DR_k^{a(k+1)} + (-1)^{k-l} E_{b(k-l+1)} \Sigma_l^{a(l)b(k-l+1)} - \alpha_6 e^{a(l)b} h_{k-1}^{a(l)b(k-l+1)} \\
\mathcal{R}_l^{a(k)} = D\omega_l^{a(k)} + \alpha_7 e^{a(k)} \Sigma_l^{a(k)} \\
\mathcal{T}_k^{a(l)} = Dh_{k-1}^{a(l)} - (-1)^{k-l} E_{b(k-l)} \omega_l^{a(l)b(k-l)} - \alpha_8 R_k^{a(l)}
\]

\(^2\)These expressions can be easily read out from the ones for gauge transformations. Of course, after that one has to check (as we have done) that the resulting curvatures are indeed gauge invariant.
These curvatures satisfy the following differential identities:

\[ DR_{l+1}^{a(k+1)} = -\alpha_5 e^a R_{l}^{a(k)} \]
\[ DT_{k+1}^{a(l)} = -E_{b(k-l-1)} R_{l+1}^{a(l)b(k-l-1)} - \alpha_6 e^a b_i R_{l+1}^{b(k-l-1)} \]
\[ DR_{k+1}^{a(l)} = -\alpha_7 e^a R_{l+1}^{a(k)b} \]
\[ DT_{k}^{a(l)} = -E_{b(k-l)} R_{l+1}^{a(l)b(k-l)} - \alpha_8 T_{k+1}^{a(l)} \]

Using these curvatures one can rewrite the free Lagrangian in the explicitly gauge invariant form:

\[ \mathcal{L} = \hat{E}_{a(2l+2)} [b_1 R_{l+1}^{a(l+1)b(k-l)} R_{l+1}^{a(l+1)b(k-l)} + b_2 R_{l+1}^{a(l+1)b(k-l)} R_{l+1}^{a(l+1)b(k-l-1)}] + b_3 \hat{E}_{a(l+1)} R_{l+1}^{a(k+1)} T_{k}^{a(l)} \]

where

\[ \frac{(d-k-l-1)}{(k+1)(d-k-l)} b_1 + b_2 = -\frac{a_0(k, l)}{(l+1)^2 a_4^2}, \quad b_3 = -\frac{(-1)^l}{a_4} \]

Here we face an ambiguity in the choice of coefficients \( b_{1,2,3} \). It is related with the following identity:

\[ 0 = \hat{E}_{a(2l+3)} D [R_{l+1}^{a(l+2)b(k-l-1)} R_{l+1}^{a(l+1)b(k-l-1)}] = -(-1)^l (l + 2) a_4 \hat{E}_{a(2l+2)} [R_{l+1}^{a(l+1)b(k-l)} R_{l+1}^{a(l+1)b(k-l)}] \]
\[ \frac{(d-k-l-1)}{(k+1)(d-k-l)} R_{l+1}^{a(l+1)b(k-l-1)} R_{l+1}^{a(l+1)b(k-l-1)} \]

where we have used the differential identities given above.

### 2.3 Partial gauge fixing

Relatively large number of components and corresponding curvatures make investigations of possible interactions for such fields rather involved. To simplify these investigations one can try to use partial gauge fixing. In the case at hands we use \( \xi_{k-1}^{a(l)} \) transformations and put \( h_{a(l)} = 0 \). Then the on-shell constraint \( T_{k}^{a(l)} \approx 0 \) (which is algebraic in this gauge) can be solved and gives:

\[ R_{k}^{a(l)} = \frac{(-1)^{k-l}}{a_4} \hat{E}_{b(k-l)} T_{k}^{a(l)b(k-l)} \]

Taking into account that the field \( \omega^{a(k)} \) will play the role of physical (and not auxiliary) field now, we make a re-scaling:

\[ \omega^{a(k)} \Rightarrow (-1)^{k-l} a_4 \omega^{a(k)} \]

\(^3\)Of course, one can not be sure that these two procedures, namely switching on an interaction and partial gauge fixing, commute. For the simplest example of mixed symmetry field \( Y(2,1) \) (the so-called hook) it was shown in [25] that it is indeed the case.
After that the free Lagrangian takes the form:

\[
\mathcal{L} = \frac{(-1)^{k-l}(l+1)}{2} \hat{E}_{a(2l)} \Sigma_l a(l) b(k-l+1) \Sigma_l a(l) b(k-l+1) + \hat{E}_{a(2l+1)} \Sigma_l a(l) b(k-l) D \omega_l a(l) b(k-l)
\]

This Lagrangian is still invariant under the remaining gauge transformations:

\[
\delta_0 \Sigma_l a^{(k+1)} = D \eta_{k+1} a^{(k)} + (-1)^{k} \kappa e^{a} \eta_{l} a^{(k)} \\
\delta_0 \omega_l a^{(k)} = D \eta_{k} a^{(k)} + e_b \eta_{l} a^{(k)b}
\]

Such a procedure leaves us with just two gauge invariant curvatures:

\[
\mathcal{R}_{l+1}^{a(k+1)} = D \Sigma_l a^{(k+1)} + (-1)^{k} \kappa e^{a} \omega_l a^{(k)} \\
\mathcal{T}_{l+1}^{a(k)} = D \omega_l a^{(k)} + e_b \Sigma_l a^{(k)b}
\]

Using these curvatures the Lagrangian can be written as follows:

\[
\mathcal{L} = \hat{E}_{a(2l+2)} [c_1 \mathcal{R}_{l+1}^{a(l+1)b(k-l)} \mathcal{R}_{l+1}^{a(l+1)b(k-l)} + c_2 \mathcal{T}_{l+1}^{a(l+1)b(k-l-1)} \mathcal{T}_{l+1}^{a(l+1)b(k-l-1)}]
\]

where

\[
[c_2 - (d-k-l-1) \kappa C_1] = \frac{(-1)^{k-l}}{2(l+1)}
\]

Once again we face the ambiguity in the choice of the parameters. This time it is related with the identity:

\[
\hat{E}_{a(2l+2)} [\mathcal{R}^{a(l+1)b(k-l)} \mathcal{R}^{a(l+1)b(k-l)} + (d-k-l-1) \kappa \mathcal{T}^{a(l+1)b(k-l-1)} \mathcal{T}^{a(l+1)b(k-l-1)}] = 0
\]

Note that up to some difference in notation and conventions all the results in this subsection are in agreement with the results in [7].

3 Partially massless case in de Sitter space

In this section we will show that combination of two fields \(Y(k, l)\) and \(Y(k, l - 1)\) can be deformed into de Sitter space with \(\kappa > 0\). Moreover, such deformation will keep all four (appropriately modified) gauge symmetries of these two fields.

3.1 Lagrangian and gauge transformations

Thus we introduce two pairs of physical and auxiliary fields: \((\Sigma_{d}^{k+1}, R_{k}^{l})\) and \((\Omega_{l-1}^{k+1}, \Phi_{k}^{l-1})\). We begin our construction with the sum of their kinetic terms:

\[
\mathcal{L}_0 = \mathcal{L}_0(\Sigma_{d}^{k+1}, R_{k}^{l}) + \mathcal{L}_0(\Omega_{l-1}^{k+1}, \Phi_{k}^{l-1})
\]

as well as their initial gauge transformations:

\[
\delta_0 \Sigma_l a^{(k+1)} = D \eta_{k+1} a^{(k+1)} \\
\delta_0 R_k a^{(l)} = D \xi_{k-1} a^{(l)} + E_{b(k-l+1)} \eta_{l-1} a^{(l)b(k-l+1)} \\
\delta_0 \Omega_{l-2}^{a(k+1)} = D \eta_{k} a^{(k+1)} \\
\delta_0 \Phi_k a^{(l-1)} = D \xi_{k-1} a^{(l-1)} + E_{b(k-l+2)} \eta_{l-2} a^{(l-1)b(k-l+2)}
\]
where all derivatives are replaced by the the $dS$ covariant ones.

Now we add all possible terms with one derivative:
\[
L_1 = a_1 \hat{E}_{a(k+l)} \Sigma^{a(k+1)}_{l} \Phi^{a(l-1)} + a_2 \hat{E}_{a(k+l-1)} \Omega^{a(k)}_{l-1} R^{a(l-1)b}_{k} 
\]  
and corresponding corrections to the gauge transformations:
\[
\delta_1 \Sigma^{a(k+1)}_l = \alpha_1 e^a e_b \eta^{a(b)l-2}, \quad \delta_1 R^{a(l)}_k = \alpha_2 e^a \xi^{a(l-1)}_k, \\
\delta_1 \Omega^{a(k+1)}_{l-1} = \alpha_3 \eta^{a(k+1)}_l, \quad \delta_1 \Phi^{a(l-1)}_k = \alpha_4 e_b \xi^{a(l-1)b}_k. 
\]

Then all variations with one derivative $\delta_0 L_1 + \delta_1 L_0$ (see Appendix B for details) cancel provided we set
\[
\alpha_1 = \alpha_2 = -\frac{(-1)^{k-1} a_1}{l(d-k-l)}, \quad \alpha_3 = (-1)^{k+1} a_1, \quad \alpha_4 = (-1)^k a_1, \quad a_2 = (-1)^l (k+1) a_1
\]

As we have already noted, there is no possible terms quadratic in the physical fields $R$ and $\Phi$ that can be added to the Lagrangian. Nonetheless all variations without derivatives $\delta_1 L_1$ indeed cancel if we put
\[
a_1^2 = l(d-k-l) \kappa
\]

### 3.2 Gauge invariant curvatures

In this case we also obtain four gauge invariant curvatures:
\[
\begin{align*}
\mathcal{R}^{a(k+1)}_{l+1} &= D \Sigma^{a(k+1)}_l - \alpha_1 e^a e_b \Omega^{a(k)b}_{l-1} \\
\mathcal{T}^{a(l)}_{k+1} &= D R^{a(l)}_k + (-1)^{k-1} E_{b(k-l+1)} \Sigma^{a(l)b(k-l+1)}_l + \alpha_2 e^a \Phi^{a(l-1)}_k \\
\mathcal{R}^{a(k+1)}_l &= D \Omega^{a(k+1)}_l - \alpha_3 \Sigma^{a(k+1)}_l \\
\mathcal{T}^{a(l-1)}_{k+1} &= D \Phi^{a(l-1)}_k - (-1)^{k-1} E_{b(k-l+2)} \Omega^{a(l-1)b(k-l+2)}_l + \alpha_4 e_b R^{a(l-1)b}_k
\end{align*}
\]

By straightforward calculations one can check that these curvatures satisfy the following differential identities:
\[
\begin{align*}
D \mathcal{R}^{a(k+1)}_{l+1} &= -\alpha_1 e^a e_b \mathcal{R}^{a(k)b}_l \\
D \mathcal{T}^{a(l)}_{k+1} &= -E_{b(k-1+l)} \mathcal{R}^{a(l)b(k-1+l)}_{l+1} - \alpha_2 e^a \mathcal{T}^{a(l-1)}_{l+1} \\
D \mathcal{R}^{a(k+1)}_l &= -\alpha_3 \mathcal{R}^{a(k+1)}_{l+1} \\
D \mathcal{T}^{a(l-1)}_{l+1} &= -E_{b(k-1+l+1)} \mathcal{R}^{a(l-1)b(k-1+l+1)}_l - \alpha_4 e_b \mathcal{T}^{a(l-1)b}_k
\end{align*}
\]

Using these curvatures one can rewrite the free Lagrangian in the explicitly gauge invariant form:
\[
\mathcal{L} = b_1 \hat{E}_{a(2l+2)} \mathcal{R}^{a(l+1)}_{l+1} b(k-l) \mathcal{R}^{a(l+1)b(k-l)}_{l+1} + b_2 \hat{E}_{a(2l)} \mathcal{R}^{a(l)}_l b(k-l+1) \mathcal{R}^{a(l)b(k-l+1)}_l + b_3 \hat{E}_{a(k+l+1)} \mathcal{R}^{a(k+1)}_l \mathcal{T}^{a(l)}_{k+1}
\]

where
\[
\frac{(l+1)^2(d-k-l-1)}{l(d-k-l)} b_1 - b_2 = \frac{a_0(k,l)}{a_1^2}, \quad b_3 = \frac{(-1)^l}{a_1}
\]
One can see that there is an ambiguity in the choice of the coefficients $b_{1,2,3}$. This ambiguity is related with the fact that the three terms in the Lagrangian are not independent. Indeed, using the differential identities given above, one can show that:

\[
0 = \hat{E}_{a(2l+2)} D\left[\mathcal{R}_{l+1}^{a(l+1)} b_{k-l} \mathcal{R}_{l}^{a(l+1)b(k-l)}\right] = -a_1 \left(\frac{h-k-l-1}{2(l-k-l)}\right) \hat{E}_{a(2l)} \mathcal{R}_{l}^{a(l)} b_{k-l+1} \mathcal{R}_{l}^{a(l)b(k-l+1)}
\]

\[
+ \hat{E}_{a(2l+2)} \mathcal{R}_{l+1}^{a(l+1)} b_{k-l} \mathcal{R}_{l+1}^{a(l+1)b(k-l)}
\]

### 4 General massive case

In this section we consider general massive case for $Y(k,l)$. By analogy with the simple examples considered previously [20], we will use four massless fields $Y(k,l), Y(k,l-1), Y(k-1,l)$ and $Y(k-1,l-1)$.

#### 4.1 Lagrangian and gauge transformations

Thus we introduce four pairs of physical and auxiliary fields: $(R_k^I, \Sigma_{l+1}^k), (h_{k-1}^{l}, \omega_k^l), (\Phi_{k-1}^l, \Omega_{l-1}^{k+1}), (B_{k-1}^{l-1}, C_{l-1}^{k})$. We begin with the sum of their kinetic terms:

\[
\mathcal{L}_0 = \mathcal{L}_0(\Sigma_{l+1}^k, R_k^I) + \mathcal{L}_0(\Omega_{l-1}^{k+1}, \Phi_{l-1}^k) + \mathcal{L}_0(\omega_k^l, h_{k-1}^{l}) + \mathcal{L}_0(C_{l-1}^{k}, B_{k-1}^{l-1})
\]

(28)

as well as their initial gauge transformations:

\[
\begin{align*}
d_0 \Sigma_{l+1}^k & = Dn_{l+1}^k, & d_0 R_k^I & = D\Sigma_{l+1}^k + E_b(k-l+1)\eta_{l-1}^a b(k-l+1) \\
d_0 \Omega_{l-1}^{k+1} & = D\Omega_{l-1}^{k+1}, & d_0 \Phi_{l-1}^k & = D\Omega_{l-1}^{k+1} + E_b(k-l+2)\eta_{l-2}^a b(k-l+2) \\
d_0 \omega_k^l & = D\omega_k^l, & d_0 h_{k-1}^{l} & = D\omega_k^l + E_b(k-l)\eta_{l-1}^a b(k-l) \\
d_0 C_{l-1}^{k} & = Dc_{l-1}^{k}, & d_0 B_{k-1}^{l-1} & = Dc_{l-1}^{k} + E_b(k-l+1)\eta_{l-2}^a b(k-l+1)
\end{align*}
\]

(29)

where all derivatives are $(A)dS$ covariant now.

Now we add all possible terms with one derivative (note that the choice of the coefficients is compatible with the results in the two previous sections):

\[
\mathcal{L}_1 = a_1 \hat{E}_{a(k+l)} \Sigma_{l+1}^k \Phi_{l}^{a(l)} + a_2 \hat{E}_{a(k+l)} \mathcal{R}_{l-1}^{a(k)b} R_{l}^{a(l)b} \\
+ a_3 \hat{E}_{a(k+l-1)} \Sigma_{l}^{a(k)} h_{k-1}^{l} \eta_{l-1}^{a} b(k-l+1) + a_4 \hat{E}_{a(k+l)} \omega_{l}^{a(k)} R_{l}^{a(l)} \\
+ a_5 \hat{E}_{a(k+l+2)} \mathcal{R}_{l-1}^{a(k)b} B_{k-1}^{l} \eta_{l-2}^{a} b(k-l+1) + a_6 \hat{E}_{a(k+l-1)} \mathcal{R}_{l+1}^{a(k)b} \eta_{l-1}^{a} b(k-l+1) \\
+ a_7 \hat{E}_{a(k+l+2)} \omega_{l}^{a(k)} R_{k-1}^{a(l)} + a_8 \hat{E}_{a(k+l+1)} \omega_{l+1}^{a(k)} R_{k-1}^{a(l)}
\]

(30)

and corresponding corrections to the gauge transformations:

\[
\begin{align*}
d_1 \Sigma_{l+1}^k & = \alpha_1 \Sigma_{l+1}^k + \alpha_2 \omega_{l}^{a(k)} R_{l}^{a(l)} + \alpha_3 \omega_{l+1}^{a(k)} R_{k-1}^{a(l)} \\
d_1 R_k^I & = \alpha_4 \Sigma_{l+1}^k + \alpha_5 \omega_{l}^{a(k)} R_{l}^{a(l)} + \alpha_6 \omega_{l+1}^{a(k)} R_{k-1}^{a(l)} \\
d_1 \Omega_{l-1}^{k+1} & = \alpha_7 \Sigma_{l+1}^k + \alpha_8 \omega_{l}^{a(k)} R_{l}^{a(l)} + \alpha_9 \omega_{l+1}^{a(k)} R_{k-1}^{a(l)}
\end{align*}
\]
\[ \delta_1 \Phi^{a(l-1)}_k = \alpha_4 e_b \xi_k^{a(l-1)b}_k + \alpha_{10} e^a \eta_k^{a(l-2)b} \]
\[ \delta_1 \omega^{a(k)}_l = \alpha_7 e_b \eta_l^{a(k)b} + \alpha_{13} e^a \eta_l^{a(k-1)b} \]
\[ \delta_1 h^{a(l)}_{k-1} = \alpha_8 \xi_{k-1}^{a(l)} + \alpha_{14} e^a \xi_{k-2}^{a(l-1)} \]
\[ \delta_1 C^{a(k)}_{l-1} = \alpha_{11} e_b \eta_{l-2}^{a(k)b} + \alpha_{15} \eta_{l-1}^{a(k)} \]
\[ \delta_1 B^{a(l-1)}_{k-1} = \alpha_{12} \xi_{k-1}^{a(l-1)} + \alpha_{16} e_b \xi_{k-2}^{a(l-2)b} \]

Requirement that all the variations with one derivative \( \delta_0 \mathcal{L}_1 + \delta_1 \mathcal{L}_0 = 0 \) cancel gives the same expressions for \( \alpha_1 \ldots \alpha_8 \) as before and

\[
\begin{align*}
\alpha_9 &= -\alpha_{10} = -\frac{(-1)^l a_6}{(k+1)(d-k-l+1)}, & \alpha_{11} &= (-1)^{k-l} a_6, & \alpha_{12} &= a_6 \\
\alpha_{13} &= \alpha_{14} = \frac{(-1)^{k-l} a_7}{l(d-k-l+1)}, & \alpha_{15} &= (-1)^l a_7, & \alpha_{16} &= -(-1)^k a_7 \\
\end{align*}
\]

In this case also there are no any possible terms quadratic in physical fields that can be added to the Lagrangian. Nonetheless, all the remaining variations vanish provided we put:

\[
\begin{align*}
a_6^2 &= \frac{(d-k-l+1)}{(d-k-l)} a_4^2, & a_7^2 &= \frac{(d-k-l+1)}{(d-k-l)} a_1^2 \\
(k+1) a_4^2 - l a_4^2 &= l(k+1)(d-k-l) \kappa 
\end{align*}
\]

Thus all the coefficients for the terms in \( \mathcal{L}_1 \) gluing our four massless fields together are determined by the two main ones \( a_1 \) and \( a_4 \) (as schematically shown on Fig.1) that satisfy the relation (34). From the relation (34) it follows that in the de Sitter space \( (\kappa > 0) \) we can take a limit \( a_4 \to 0 \). In this case the whole system decomposes in two independent subsystems (see Fig.2) both of them corresponding to partially massless case considered in Section 3.

From the other side, in the anti de Sitter space \( (\kappa < 0) \) we can take a limit \( a_1 \to 0 \) where our system also decomposes into two independent subsystems (see Fig.3), each of them corresponding to the partially massless case considered in Section 2.

![Figure 1: General massive theory for \( Y(k, l) \) field](image)
we use these invariants satisfy. So as in the Section 2 we proceed with partial gauge fixing. Namely, here one can construct nine invariants quadratic in curvatures as well as four identities that

In this case there exist as many as eight gauge invariant curvatures:

\[ \mathcal{R}^{(k+1)}_l = D \sum_l \omega^{(k+1)}_l - \alpha_1 e^a e_b \Omega^{(k+1)}_{l-1} + \alpha_5 e^a \omega^{(k)}_l \]
\[ \mathcal{T}^{(l)}_{k+1} = DR^{(l)}_{k} + (-1)^{k-l} E_b(k-l+1) \sum_l \omega^{(l)b(k-l+1)} + \alpha_2 e^a \Phi^{(l-1)}_k - \alpha_6 e^a e_b h^{(l-1)b}_{k-1} \]
\[ \mathcal{F}^{(k+1)}_l = D \sum_l \Phi^{(k+1)}_l - \alpha_3 \sum_l \Phi^{(k+1)}_l + \alpha_9 e^a C^{(k)}_{l-1} \]
\[ \mathcal{F}^{(l-1)}_{k+1} = D \Phi^{(l-1)}_k - (-1)^{k-l} E_b(k-l+1) \sum_l \omega^{(l)b(k-l+1)} + \alpha_4 e_b R^{(l-1)}_{k} - \alpha_{10} e^a e_b B^{(l-2)b}_{k-1} \]
\[ \mathcal{G}^{(k)}_l = D \omega^{(k)}_l + \alpha_7 e_b \sum_l \omega^{(k)}_l - \alpha_{13} e^a e_b C^{(k-1)b}_l \]
\[ \mathcal{G}^{(l)}_{k+1} = D \Phi^{(l)}_k - (-1)^{k-l} E_b(k-l+1) \sum_l \omega^{(l)b(k-l)} - \alpha_8 R^{(l)}_k + \alpha_{14} e^a B^{(l-1)}_{k-1} \]
\[ \mathcal{H}^{(k+1)}_l = D \omega^{(k)}_l + \alpha_8 e_b \sum_l \omega^{(k)}_l - \alpha_{15} \omega^{(k)}_l \]
\[ \mathcal{H}^{(l-1)}_{k+1} = D B^{(l-1)}_{k-1} + (-1)^{k-l} E_b(k-l+1) C^{(l-1)b(k-l+1)} - \alpha_{12} \Phi^{(l-1)}_k + \alpha_{16} e_b h^{(l-1)b}_{k-1} \]

In principle, it is possible to rewrite the free Lagrangian in terms of these curvatures, but here one can construct nine invariants quadratic in curvatures as well as four identities that these invariants satisfy. So as in the Section 2 we proceed with partial gauge fixing. Namely, we use \( \xi_{k-1}^{(l)} \) and \( \zeta_{k-1}^{(l)} \) transformations to set \( h^{(l)}_{k-1} = 0 \) and \( B^{(l-1)}_{k-1} = 0 \). Then we solve the on-shell constraints \( \mathcal{T}^{(l)}_l \approx 0 \) and \( \mathcal{T}^{(l-1)}_k \approx 0 \) and get:

\[ R^{(l)}_k = \frac{(-1)^{k-l}}{a_4} E_b(k-l) \omega^{(l)b(k-l)}_l \]
\[ \Phi^{(l-1)}_k = \frac{(-1)^{k-l}}{a_6} E_b(k-l+1) C^{(l-1)b(k-l+1)}_{l-1} \]
Taking into account that the fields \( \omega \) and \( C \) will now play the roles of physical (and not the auxiliary) ones, we make appropriate re-scaling:

\[
\omega_l^{a(k)} \Rightarrow (-1)^{k-l} a_4 \omega_l^{a(k)}, \quad C_{l-1}^{a(k)} \Rightarrow (-1)^{k-l} a_6 C_{l-1}^{a(k)}
\]

These leaves with the set of four fields with the Lagrangian:

\[
\mathcal{L} = \frac{(-1)^{k-l}(l+1)}{2} \hat{\mathcal{E}}_a^{(2l)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + \hat{\mathcal{E}}_a^{(2l+1)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + \frac{(-1)^{k-l}(l+1)}{2} \hat{\mathcal{E}}_a^{(2l-2)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + (l+1) \hat{\mathcal{E}}_a^{(2l-2)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + (l+1) a_1 \hat{\mathcal{E}}_a^{(2l-1)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + (l+1) a_1 \hat{\mathcal{E}}_a^{(2l-1)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + (l+1) a_1 \hat{\mathcal{E}}_a^{(2l-1)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)} + (l+1) a_1 \hat{\mathcal{E}}_a^{(2l-1)} \sum_l \mathcal{O}_{l(k-l+1)} \mathcal{O}_{l(k-l+1)}
\]

which is still invariant under the remaining gauge transformations:

\[
\begin{align*}
\delta \Sigma_{l}^{a(k+1)} &= D_{l} \eta_{l-1}^{a(k+1)} + \alpha_{1} e_{a} e_{b} \eta_{l-2}^{a(k+1)} + \alpha_{2} e_{a} \eta_{l-1}^{a(k+1)} \\
\delta \omega_{l}^{a(k)} &= D_{l} \eta_{l-1}^{a(k)} + e_{b} \eta_{l-1}^{a(k)} - \alpha_{1} e_{a} e_{b} \eta_{l-2}^{a(k-1)} \\
\delta \mathcal{O}_{l-1}^{a(k+1)} &= D_{l} \eta_{l-2}^{a(k+1)} + \alpha_{3} \eta_{l-1}^{a(k+1)} + \alpha_{5} e_{a} \eta_{l-2}^{a(k)} \\
\delta \mathcal{C}_{l-1}^{a(k)} &= D_{l} \eta_{l-2}^{a(k)} + e_{b} \eta_{l-1}^{a(k)} + \alpha_{3} \eta_{l-1}^{a(k)}
\end{align*}
\]

where

\[
\alpha_{5} = \frac{(-1)^{k-2} a_{4}^{2}}{(k+1)(d - k - 1)}
\]

and the new set of curvatures

\[
\begin{align*}
\mathcal{R}_{l+1}^{a(k+1)} &= D_{l} \eta_{l-1}^{a(k+1)} - \alpha_{1} e_{a} e_{b} \mathcal{O}_{l-1}^{a(k+1)} + \alpha_{5} e_{a} \omega_{l}^{a(k)} \\
\mathcal{T}_{l+1}^{a(k)} &= D_{l} \eta_{l-1}^{a(k+1)} + e_{b} \eta_{l-1}^{a(k+1)} - \alpha_{1} e_{a} e_{b} \mathcal{C}_{l-1}^{a(k+1)} \\
\mathcal{R}_{l}^{a(k+1)} &= D_{l} \eta_{l-1}^{a(k+1)} - \alpha_{3} \mathcal{O}_{l}^{a(k+1)} + \alpha_{5} e_{a} \mathcal{C}_{l-1}^{a(k)} \\
\mathcal{T}_{l}^{a(k)} &= D_{l} \eta_{l-1}^{a(k+1)} - \alpha_{3} \mathcal{O}_{l}^{a(k)} - \alpha_{3} \omega_{l}^{a(k)}
\end{align*}
\]

Trying to rewrite this Lagrangian it terms of curvatures, we found that there are six invariants quadratic in them as well as four identities they satisfy. The simplest example we managed to find looks like:

\[
\mathcal{L} = c_{1} \hat{\mathcal{E}}_a^{(2l+2)} \mathcal{T}_{l+1}^{a(l+1)(b(k-l+1))} \mathcal{T}_{l+1}^{a(l+1)(b(k-l+1))} + c_{2} \hat{\mathcal{E}}_a^{(2l)} \mathcal{T}_{l}^{a(l+1)(b(k-l))} \mathcal{T}_{l}^{a(l+1)(b(k-l))}
\]

where

\[
c_{1} = \frac{(-1)^{k-l}}{2(l+1)}, \quad c_{2} = \frac{(-1)^{k-l}(l+1)}{2l}
\]
5  Special case \( Y(k, k) \)

In this section we consider massive version for the special case \( Y(k, k) \). Similarly to the example \( Y(2, 2) \) considered previously \cite{20} we will need three massless fields \( Y(k, k), Y(k, k - 1) \) and \( Y(k - 1, k - 1) \) only.

5.1  Lagrangian and gauge transformations

We introduce three pairs of physical and auxiliary fields: \((R_k^k, \Sigma_k^{k+1}), (\Phi_k^{k-1}, \Omega_k^{-1} k^{k+1}), (h_{k-1}^{k-1}, \omega_{k-1}^k)\) and begin with the sum of their kinetic terms:

\[
L_0 = L_0(\Sigma_k^{k+1}, R_k^k) + L_0(\Omega_k^{-1} k^{k+1}, \Phi_k^{k-1}) + L_0(\omega_{k-1}^k, h_{k-1}^{k-1})
\]

(39)

as well as their initial gauge transformations:

\[
\delta_0 \Sigma_k^{a(k+1)} = D_1 k_1^{a(k+1)}, \quad \delta_0 R_k^a = D_2 k_1^a + e_b^a \eta_{k-1}^b \\
\delta_0 \Omega_k^{-1} k^{a(k+1)} = D_3 k_2^{-1} k^{a(k+1)}, \quad \delta_0 \Phi_k^{a(k-1)} = D_4 k_1^{-1} k^{a(k-1)} + e_b^a \eta_{k-2}^b \\
\delta_0 \omega_{k-1}^a = D_5 k_1^{-1} k^{a(k-1)} + e_b^a \eta_{k-2}^b
\]

(40)

Now we add all possible terms with one derivative:

\[
L_1 = a_1 \hat{E}_a(2k) \Sigma_k^{a(k+1)} \Phi_k^{a(k-1)} + a_2 \hat{E}_a(2k-1) \Omega_k^{-1} k^{a(k-1)b} + a_3 \hat{E}_a(2k-2) \omega_{k-1}^a \Phi_k^{a(k-1)}
\]

(41)

and corresponding corrections to the gauge transformations:

\[
\delta_1 \Sigma_k^{a(k+1)} = a_1 e_a^b \eta_{k-2}^b, \quad \delta_1 R_k^a = a_2 e_a^b \xi_{k-1}^b \\
\delta_1 \Omega_k^{-1} k^{a(k+1)} = a_3 e_a^b \eta_{k-2}^b + a_4 e_a^b \xi_{k-1}^b \\
\delta_1 \Phi_k^{a(k-1)} = a_5 e_b^a \xi_{k-1}^b + a_6 e_b^a \eta_{k-2}^b \\
\delta_1 \omega_{k-1}^a = a_7 e_b^a \xi_{k-1}^b, \quad \delta_1 h_{k-1}^{a(k-1)} = a_8 e_a^b \xi_{k-1}^b
\]

(42)

First of all we calculate all variations with one derivative. We found that all of them vanish provided we set

\[
\alpha_1 = \alpha_2 = \frac{a_1}{k(d-2k)}, \quad \alpha_3 = \alpha_5 = -(-1)^k a_1 \\
\alpha_4 = -(-1)^k a_4, \quad \alpha_7 = \alpha_8 = -a_4 \\
\alpha_2 = (-1)^k (k+1) a_1, \quad \alpha_3 = (-1)^k (k-1) a_4
\]

(43)

But in this case it is not possible to cancel the remaining variations just by adjusting the two main parameters \( a_1 \) and \( a_4 \). Happily, we have two physical fields \( R_{k}^{a(k)} \) and \( h_{k-1}^{a(k-1)} \) with equal numbers of world and local indices, so we proceed and introduce terms without derivatives:

\[
L_2 = b_1 \hat{E}_a(2k) R_{k}^{a(k)} + b_2 \hat{E}_a(2k-1) R_{k}^{a(k)} + b_3 \hat{E}_a(2k-1) h_{k-1}^{a(k-1)}
\]

(44)
and corresponding corrections to the gauge transformations:

\[ \delta_2 \Sigma_k^{a(k+1)} = \beta_1 e^a \xi_{k-1}^{(k)} + \beta_2 e^a \xi_{k-2}^{(k-1)} \]
\[ \delta_2 \omega_{k-1}^{a(k)} = \beta_3 \xi_{k-1}^{(k)} + \beta_4 e^a \xi_{k-2}^{(k-1)} \]  \hspace{1cm} (45)

For this corrections to be consistent with \( L_2 \) we have to put:

\[ \beta_1 = \frac{(-1)^k b_1}{(k+1)(d-2k)}, \quad \beta_2 = -\frac{(-1)^k b_2}{k(k+1)(d-2k)(d-2k+1)} \]
\[ \beta_3 = -(1)^k b_2, \quad \beta_4 = \frac{(-1)^k b_3}{k(d-2k+2)} \]

Now we require that all variations vanish and obtain:

\[ 2k b_1 = - (k+1) a_1^2 + k(k+1)(d-2k) \kappa, \quad b_2 = -a_1 a_4 \]
\[ 2b_3 = - \frac{k(d-2k+2)}{(k+1)(d-2k+1)} a_4^2 - k(d-2k+2) \kappa \] \hspace{1cm} (46)

\[ (k+1)(d-2k+1) a_1^2 - k(d-2k) a_4^2 = k(k+1)(d-2k)(d-2k+1) \kappa \] \hspace{1cm} (47)

Thus all the coefficients that determine the mixing our three massless fields together are determined by the two main ones \( a_1 \) and \( a_4 \) (see Fig.4) that in turn satisfy the relation (47).

Figure 4: General massive case for the \( Y(k, k) \) field

The relation (47) shows that in the de Sitter space (\( \kappa > 0 \)) we can take a limit \( a_4 \to 0 \). In this limit our system decomposes into two independent subsystems (see Fig.5) where the first one corresponds to the partially massless case considered in Section 3.

Figure 5: Partially massless limit in de Sitter space

From the other hand, in the anti de Sitter space (\( \kappa < 0 \)) we can take a limit \( a_1 \to 0 \). Then the whole system also decomposes into two independent subsystems (see Fig.6) where the first one is just the massless \( Y(k, k) \) field as in Section 1, while the second one corresponds to partially massless case considered in Section 2.
5.2 Curvatures and partial gauge fixing

This time we have six gauge invariant curvatures:

\[ R_{k+1}^{a(k+1)} = D \Sigma_{k}^{a(k+1)} - \alpha_{1} e_{a} e_{b} \Omega_{k-1}^{a(k)b} + \beta_{1} e_{a} R_{k}^{a(k)} - \beta_{2} e_{a} e_{b} h_{k-1}^{a(k-1)b} \]

\[ T_{k+1}^{a(k)} = D R_{k}^{a(k)} + e_{b} \Sigma_{k}^{a(k)b} + \alpha_{2} e_{a} \Phi_{k}^{a(k-1)} \]

\[ R_{k+1}^{a(k+1)} = D \Omega_{k-1}^{a(k+1)} - \alpha_{3} \Sigma_{k}^{a(k+1)} + \alpha_{4} e_{a} \omega_{k-1}^{a(k)} \]

\[ T_{k+1}^{a(k)} = D \Phi_{k}^{a(k-1)} - e_{(b} \Omega_{k-1}^{a(k-1)b(2)} + \alpha_{5} e_{b} R_{k}^{a(k-1)b} - \alpha_{6} e_{a} e_{b} h_{k-1}^{a(k-1)b} \]

Again to simplify further investigations we proceed with the partial gauge fixing. Namely, we use \( \xi_{k-1}^{a(k-1)} \) gauge transformations to fix \( h_{k-1}^{a(k-1)b} = 0. \) Then solving the on-shell constraint \( T_{k}^{a(k-1)} = 0 \) we obtain:

\[ \Phi_{k}^{a(k-1)} = -\frac{1}{a_{4}} e_{b} \omega_{k-1}^{a(k-1)b} \]

Again we make the appropriate re-scaling

\[ \omega_{k}^{a(k)} \Rightarrow a_{4} \omega_{k}^{a(k)} \]

Then for the remaining four fields we obtain the following Lagrangian:

\[ \mathcal{L} = -\frac{(k + 1)}{2} \hat{E}_{a(2k)} \Sigma_{k}^{a(k)} \Sigma_{k}^{a(k)b} - \hat{E}_{a(2k+1)} \Sigma_{k}^{a(k+1)} D R_{k}^{a(k)} \]

\[ -\frac{k(k + 1)}{2} \hat{E}_{a(2k-2)} \Omega_{k-1}^{a(k-1)b(2)} \Omega_{k-1}^{a(k+1)b(2)} - (k + 1) \hat{E}_{a(2k-1)} \Omega_{k-1}^{a(k)b} D \omega_{k-1}^{a(k-1)b} \]

\[ + (k + 1) a_{1} \hat{E}_{a(2k-1)} \left[ -\Sigma_{k}^{a(k)} \omega_{k-1}^{a(k-1)b} + (-1)^{k} \Omega_{k-1}^{a(k)b} R_{k}^{a(k-1)b} \right] \]

\[ + b_{1} \hat{E}_{a(2k)} R_{k}^{a(k)} R_{k}^{a(k)} + \frac{k(d + 2k + 1)b_{1}}{(d - 2k)} \hat{E}_{a(2k-2)} \omega_{k-1}^{a(k-1)b} \omega_{k-1}^{a(k-1)b} \]  

which is still invariant under the following gauge transformations:

\[ \delta \Sigma_{k}^{a(k+1)} = D \eta_{k-1}^{a(k+1)} + \alpha_{1} e_{a} e_{b} \eta_{k-2}^{a(k)b} + \beta_{1} e_{a} \xi_{k-1}^{a(k)} \]

\[ \delta R_{k}^{a(k)} = D \xi_{k-1}^{a(k)} + e_{b} \eta_{k-1}^{a(k)b} + \alpha_{1} e_{a} e_{b} \eta_{k-2}^{a(k)b} \]

\[ \delta \Omega_{k-1}^{a(k-1)} = D \eta_{k-2}^{a(k-1)} - (-1)^{k} a_{1} \eta_{k-1}^{a(k+1)} - \beta_{1} e_{a} \eta_{k-2}^{a(k)} \]

\[ \delta \omega_{k-1}^{a(k)} = D \eta_{k-2}^{a(k)} - e_{b} \eta_{k-2}^{a(k)b} + (-1)^{k} a_{1} \xi_{k-1}^{a(k)} \]  

Figure 6: Massless limit in anti de Sitter space
Also we can construct a set of the new gauge invariant curvatures:

\[
R^{a(k+1)}_{k+1} = D\Sigma_k^{a(k+1)} - \alpha_1 e^a e^b \Omega^{a(k+1)}_{k-1} + \beta_1 e^a R^{a(k)}_k \\
T^{a(k)}_{k+1} = DR^{a(k)}_k + e^a \Sigma_k^{a(k)} + \alpha_1 e^a \omega^{a(k-1)}_{k-1} \\
R^{a(k+1)}_k = D\Omega^{a(k+1)}_{k-1} + (-1)^k a_1 \Sigma_k^{a(k+1)} - \beta_1 e^a \omega^{a(k)}_{k-1} \\
T^{a(k)}_k = D\omega^{a(k)}_{k-1} - e^a \Omega^{a(k)}_{k-1} - (-1)^k a_1 R^{a(k)}_k
\]

(50)

Trying to rewrite this Lagrangian in terms of curvatures we have found that there exist five invariants quadratic in them as well as two identities these invariants satisfy. The simplest solution solution we managed to find is:

\[
\mathcal{L} = c_1 \hat{E}_a(2k+2) R^{a(k+1)}_{k+1} \mathcal{R}^{a(k+1)}_{k+1} + c_2 \hat{E}_a(2k) \mathcal{R}^{a(k)}_k \mathcal{R}^{a(k)}_b
\]

(51)

where

\[
c_1 = \frac{(d - 2k)}{4(d - 2k - 1)b_1}, \quad c_2 = \frac{(k + 1)^2}{4kb_1}
\]

Acknowledgments

Work supported in parts by RFBR grant 14-02-01172.

A Partially massless case in anti de Sitter space

Non-invariance due to non-commutativity of AdS covariant derivatives:

\[
\delta_0 \mathcal{L}_0 = -l(k + 1)(d - k - l) \kappa \hat{E}_a(2k+1) \eta^{a(k+1)}_{l-1} R^{a(l-1)}_k \\
-(-1)^l l(k + 1)(d - k - l) \kappa \hat{E}_a(2k+1) \Sigma^{a(k)}_{l} e^{a(l-1)b}_{k-1} \\
-lk(d - k - l + 1) \kappa \hat{E}_a(2k+2) \eta^{a(k-1)}_{l-1} h^{a(l-1)}_{k-1} \\
-(-1)^l lk(d - k - l + 1) \kappa \hat{E}_a(2k+2) \omega^{a(k-1)}_{l} a^{a(l-1)b}_{k-2}
\]

Requirement the $\delta_1 \mathcal{L}_0 + \delta_0 \mathcal{L}_1 = 0$ gives

\[
\alpha_5 = -\alpha_6 = -\frac{a_3}{l(k + 1)(d - k - l)}, \quad \alpha_7 = (-1)^{k-l} a_4, \quad \alpha_8 = -a_4, \quad a_3 = (-1)^l l a_4
\]

Contributions from $\delta_1 \mathcal{L}_1$

\[
\delta_1 \mathcal{L}_1 = -(-1)^{k-l} l a_4 \alpha_7 \hat{E}_a(2k+1) \eta^{a(k+1)}_{l-1} R^{a(l-1)}_k \\
+a_3 \alpha_8 \hat{E}_a(2k+1) \Sigma^{a(k)}_{l} e^{a(l-1)b}_{k-1} \\
+k(d - k - l + 1) a_3 \alpha_5 \hat{E}_a(2k+2) \eta^{a(k-1)}_{l-1} h^{a(l-1)}_{k-1} \\
-(-1)^l lk(d - k - l + 1) a_4 \alpha_8 \hat{E}_a(2k+2) \omega^{a(k-1)}_{l} a^{a(l-1)b}_{k-2}
\]

Requirement $\delta_1 \mathcal{L}_1 + \delta_0 \mathcal{L}_0 = 0$ gives

\[
a_4^2 = -(k + 1)(d - k - l) \kappa
\]
B  Partially massless case in de Sitter space

Non-invariance due to non-commutativity of AdS covariant derivatives:

\[
\delta_0 \mathcal{L}_0 = -l(k+1)(d-k-l)\kappa \hat{E}_{a(k+l-1)} a^{(k)b} R_k^{a(l-1)} b \\
+ (l-1)(k+1)(d-k-l+1)\kappa \hat{E}_{a(k+l-2)} a^{(k)b} \Phi_k^{a(l-2)} b \\
-(-1)^l(l+1)(d-k-l)\kappa \hat{E}_{a(k+l-1)} \Sigma_l^{a(k)} b \xi_{k-1}^{a(l-1)} b \\
-(-1)^l(l-1)(k+1)(d-k-l+1)\kappa \hat{E}_{a(k+l-2)} \Omega_l^{a(k)} b \xi_{k-2}^{a(l-2)} b
\]

where \( \mathcal{L}_0 \) given by (20) and \( \delta_0 \) by (21). Requirement that \( \delta_1 \mathcal{L}_0 + \delta_0 \mathcal{L}_1 = 0 \) gives

\[
\alpha_1 = \alpha_2 = -\frac{(-1)^{l-1}a_1}{l(d-k-l)} , \quad \alpha_3 = (-1)^l a_1 , \quad \alpha_4 = (-1)^k a_1 , \quad a_2 = (-1)^l(k+1)a_1
\]

Contributions from \( \delta_1 \mathcal{L}_1 \):

\[
\delta_1 \mathcal{L}_1 = a_2 \alpha_3 \hat{E}_{a(k+l-1)} a^{(k)b} R_k^{a(l-1)} b \\
+ (-1)^l(l-1)(k+1)(d-k-l+1)\alpha_1 \alpha_1 \hat{E}_{a(k+l-2)} a^{(k)b} \Phi_k^{a(l-2)} b \\
+ (-1)^l(l+1)\alpha_1 \alpha_2 \hat{E}_{a(k+l-1)} \Sigma_l^{a(k)} b \xi_{k-1}^{a(l-1)} b \\
-(-1)^l(l-1)(d-k-l+1)\alpha_2 \alpha_2 \hat{E}_{a(k+l-2)} \Omega_l^{a(k)} b \xi_{k-1}^{a(l-2)} b
\]

Requirement \( \delta_0 \mathcal{L}_0 + \delta_1 \mathcal{L}_1 = 0 \) gives

\[
a_1^2 = l(d-k-l)\kappa
\]

References

[1] Nicolas Boulanger, Dmitry Ponomarev "Frame-like off-shell dualisation for mixed-symmetry gauge fields", J. Phys. A46 (2013) 214014, [arXiv:1206.2052]

[2] Thomas Basile, Xavier Bekaert, Nicolas Boulanger "A note about a pure spin-connection formulation of General Relativity and spin-2 duality in (A)dS", Phys. Rev. D93 (2016) 124047, [arXiv:1512.09060]

[3] E. D. Skvortsov "Frame-like Actions for Massless Mixed-Symmetry Fields in Minkowski space", Nucl. Phys. B808 (2009) 569, [arXiv:0807.0903]

[4] L. Brink, R. R. Metsaev, M. A. Vasiliev "How massless are massless fields in AdS\(_d\)?", Nucl. Phys. B586 (2000) 183, [arXiv:hep-th/0005136]

[5] N. Boulanger, C. Iazeolla, P. Sundell "Unfolding Mixed-Symmetry Fields in AdS and the BMV Conjecture: I. General Formalism", JHEP 0907 (2009) 013, [arXiv:0812.3615]

[6] N. Boulanger, C. Iazeolla, P. Sundell "Unfolding Mixed-Symmetry Fields in AdS and the BMV Conjecture: II. Oscillator Realization", JHEP 0907 (2009) 014, [arXiv:0812.4438]
K. B. Alkalaev "Two-column higher spin massless fields in AdS(d)", Theor. Math. Phys. 140 (2004) 1253, arXiv:hep-th/0311212.

K. B. Alkalaev, O. V. Shaynkman, M. A. Vasiliev "On the Frame-Like Formulation of Mixed-Symmetry Massless Fields in (A)dS(d)", Nucl. Phys. B692 (2004) 363, arXiv:hep-th/0311164.

K. B. Alkalaev, O. V. Shaynkman, M. A. Vasiliev "Lagrangian Formulation for Free Mixed-Symmetry Bosonic Gauge Fields in (A)dS(d)", JHEP 0508 (2005) 069, arXiv:hep-th/0501108.

K. B. Alkalaev, O. V. Shaynkman, M. A. Vasiliev "Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in AdS(d)", arXiv:hep-th/0601225.

P. de Medeiros "Massive gauge-invariant field theories on spaces of constant curvature", Class. Quant. Grav. 21 (2004) 2571, arXiv:hep-th/0311254.

P. de Medeiros, C. Hull "Geometric Second Order Field Equations for General Tensor Gauge Fields", JHEP 0305 (2003) 019, arXiv:hep-th/0303036.

I. L. Buchbinder, V. A. Krykhtin, H. Takata "Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields", Phys. Lett. B656 (2007) 253, arXiv:0707.2181.

I.L. Buchbinder, A. Reshetnyak "General Lagrangian Formulation for Higher Spin Fields with Arbitrary Index Symmetry. I. Bosonic fields", Nucl. Phys. B862 (2012) 270, arXiv:1110.5044.

Alexander A. Reshetnyak "BRST-BFV Lagrangian Formulations for Higher Spin Fields subject to two-column Young Tableaux", arXiv:1412.0200.

Alexander A. Reshetnyak "Gauge-invariant Lagrangians for mixed-antisymmetric higher spin fields", arXiv:1604.00620.

Yu. M. Zinoviev "Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields", Nucl. Phys. B812 (2009) 46, arXiv:0809.3287.

Yu. M. Zinoviev "Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields. II. General Young tableau with two rows", Nucl. Phys. B826 (2010) 490, arXiv:0907.2140.

Euihun Joung, Karapet Mkrtchyan "Weyl Action of Two-Column Mixed-Symmetry Field and Its Factorization Around (A)dS Space", JHEP 06 (2016) 135, arXiv:1604.05330.

Yu. M. Zinoviev "On Massive Mixed Symmetry Tensor Fields in Minkowski space and (A)dS", arXiv:hep-th/0211233.
[21] X. Bekaert, N. Boulanger, M. Henneaux "Consistent deformations of dual formulations of linearized gravity: A no-go result", Phys. Rev. D67 (2003) 044010, arXiv:hep-th/0210278.

[22] N. Boulanger, S. Cnockaert "Consistent deformations of [p,p]-type gauge field theories", JHEP 0403 (2004) 031, arXiv:hep-th/0402180.

[23] X. Bekaert, N. Boulanger, S. Cnockaert "No Self-Interaction for Two-Column Massless Fields", J. Math. Phys. 46 (2005) 012303, arXiv:hep-th/0407102.

[24] K.B. Alkalaev "FV-type action for AdS(5) mixed-symmetry fields", JHEP 1103 (2011) 031, arXiv:1011.6109.

[25] Nicolas Boulanger, E. D. Skvortsov, Yu. M. Zinoviev "Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds", J. Phys. A44 (2011) 415403, arXiv:1107.1872.

[26] Nicolas Boulanger, E. D. Skvortsov "Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime", JHEP 1109 (2011) 063, arXiv:1107.5028.

[27] Yu. M. Zinoviev "On electromagnetic interactions for massive mixed symmetry field", JHEP 03 (2011) 082, arXiv:1012.2706.

[28] Yu. M. Zinoviev "Gravitational cubic interactions for a massive mixed symmetry gauge field", Class. Quantum Grav. 29 (2012) 015013, arXiv:1107.3222.