Dynamics of a modified Leslie-Gower system with diffusion in heterogeneous environment

Shuang Chen\textsuperscript{a,b,*}, Leqi Chen\textsuperscript{c}

\textsuperscript{a}School of Mathematics and Statistics, Huazhong University of Sciences and Technology, Wuhan, Hubei 430074, P. R. China
\textsuperscript{b}Center for Mathematical Sciences, Huazhong University of Sciences and Technology, Wuhan, Hubei 430074, P. R. China
\textsuperscript{c}School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P. R. China

Abstract
We investigate the dynamics of a diffusive Lotka-Volterra predator-prey system with modified Leslie-Gower and Holling type II functional response in spatially heterogeneous environment. First, we prove that the trivial steady state and one of two semi-trivial steady states are always linearly unstable, and the linear stability of the other semi-trivial steady state changes as the parameters vary. Second, when the semi-trivial steady state is not linearly unstable, we further obtain that it is globally asymptotically stable. The proof is based on the method of upper and lower solutions and the global dynamics of a competition-diffusion system. Finally, we establish the existence and the stability of a unique positive steady state, and the persistence of the two species under some suitable conditions.

Keywords: Predator-prey system, modified Leslie-Gower model, spatial heterogeneity, diffusion, stability

2010 MSC: 35Q92, 92D25, 35K57, 35P05.

1. Introduction
The Lotka-Volterra models, which are originated by Lotka [23] and Volterra [35, 36], have been frequently used to describe population dynamics in the past couple of decades. Among the investigation to the interactions between movement and environmental heterogeneity in population dynamics, many efforts have been devoted to studying the dynamics of single species models [7, 23, 24], Lotka-Volterra competition systems [5, 13, 14, 15, 19, 20, 26, 34, 40], and Lotka-Volterra predator-prey systems [10, 11, 25, 30, 37, 38, 39] with diffusion in spatially heterogeneous environment. For more detail concerning this topic, we also refer the readers to excellent monographs [6, 27].

*Corresponding author: Shuang Chen
Email addresses: schen@hust.edu.cn (Shuang Chen), CLQmath96@163.com (Leqi Chen)
In this paper, we consider a $2 \times 2$ diffusive predator-prey system with modified Leslie-Gower and Holling type II functional response in heterogeneous environment, which is in the form

$$
\begin{align*}
U_t &= d_1 \Delta U + U(r_1(x) - b_1 U) - VP(U) \quad \text{in } \Omega \times \mathbb{R}_+, \\
V_t &= d_2 \Delta V + V \left( r_2(x) - \frac{a_2 V}{U + k_2} \right) \quad \text{in } \Omega \times \mathbb{R}_+, \\
\frac{\partial U}{\partial n} &= \frac{\partial V}{\partial n} = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+, \\
U(0, x) &= U_0(x), \quad V(0, x) = V_0(x) \quad \text{in } \Omega,
\end{align*}
$$

with a Holling type II functional response

$$
P(U) = \frac{a_1 U}{U + k_1},$$

where $U(t, x)$ and $V(t, x)$ respectively represent the population densities of the prey and the predator at location $x$ and time $t > 0$, the parameters $d_1$ and $d_2$ are the dispersal rates, the growth rates $r_1(x)$ and $r_2(x)$ of the prey and the predator are spatially heterogeneous, the constant $b_1$ measures the strength of the competition among individuals of the prey. The predator’s numerical response is modified Leslie-Gower form proposed by Aziz-Alaoui and Daher Okiye in [3], which extends the classical Leslie form originated by Leslie in [21]. Here all model parameters are positive, the Laplacian operator $\Delta$ is defined by $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$, the vector $n$ denotes the outward unit normal vector on $\partial \Omega$, the habitat $\Omega$ is a bounded region in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, and $\mathbb{R}_+ = (0, +\infty)$.

As one of important classes of Lotka-Volterra models, the predator-prey systems with modified Leslie-Gower and Holling type II functional response were widely investigated by many authors. For example, [3] studied the global stability of a unique positive equilibrium by the method of Lyapunov functions and [12, 41] considered the periodic solutions for these systems of ordinary differential equations. [28] obtained a unique globally asymptotically stable positive equilibrium under some conditions for this system with delays, [1, 4, 8, 17] investigated the dynamics of the corresponding reaction-diffusion models in homogeneous environment.

Taking into consideration that real environments are highly heterogeneous in the abiotic factors, here we assume that the growth rates $r_1(x)$ and $r_2(x)$ of the prey and the predator depend on location $x$. This implies the spatial heterogeneity. More precisely, throughout this paper we assume that the growth rates $r_i(x)$ satisfy the following hypothesis:

**H1** The growth rates $r_i(x)$ are non-negative functions in $C^0(\Omega)$ for $a \in (0, 1)$, $(r_1(x), r_2(x))$ is not a constant vector in $\Omega$, and $r_i$ satisfy $\overline{r}_i > 0$, where $\overline{r}_i, i = 1, 2$, are defined by

$$
\overline{r}_i = \frac{1}{|\Omega|} \int_{\Omega} r_i(x) \, dx,
$$

where $|\Omega|$ denotes the measure of the bounded region $\Omega$.

No confusion should arise, when we use a function with a bar without any qualification we mean the average of this function on $\Omega$.

Our goal of this paper is to investigate the dynamics of the modified Leslie-Gower system (1) in spatially heterogeneous environment. We obtain that system (1) always has the trivial steady
states \((0, 0)\), and precisely two semi-trivial steady states \((\theta_{d, r_1}, 0)\) and \((0, k_2 \theta_{d, r_2}/a_2)\). By employing the principal spectral theory, we further prove that the steady states \((0, 0)\) and \((\theta_{d, r_1}, 0)\) are always linearly unstable, that is, the associated principal eigenvalues are always negative (see, for instance, \([27]\)), and the linear stability of the semi-trivial steady state \((0, k_2 \theta_{d, r_2}/a_2)\) changes as the parameters vary in different regions. A steady state is called to be linearly stable if the associated principal eigenvalue is positive. If the associated principal eigenvalue is equal to zero, then it is referred to as the degenerate case. Otherwise, it is called the non-degenerate case.

As we know, it is highly difficult to give a detailed study of the global dynamics of diffusive Lotka-Volterra systems in heterogeneous environment. In the recent work of He and Ni \([15]\), they considered a Lotka-Volterra competition-diffusion system of the form

\[
\begin{align*}
U_t &= d_1 \Delta U + U (r_1(x) - U - cV) \quad \text{in } \Omega \times \mathbb{R}_+ , \\
V_t &= d_2 \Delta V + V (r_2(x) - bU - V) \quad \text{in } \Omega \times \mathbb{R}_+ , \\
\frac{\partial U}{\partial n} &= \frac{\partial V}{\partial n} = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+, \\
U(0, x) &= U_0(x), \quad V(0, x) = V_0(x) \quad \text{in } \Omega .
\end{align*}
\]  

By mainly employing the principal spectral theory and the theory of monotone dynamical systems, \([15]\) obtained a powerful conclusion that the global dynamics of the competition-diffusion system \((2)\) could be determined by its local dynamics. Stimulated by the conclusions obtained by He and Ni \([15]\), an interesting question arises:

- Is the semi-trivial steady state \((0, k_2 \theta_{d, r_2}/a_2)\) of the modified Leslie-Gower system \((1)\) globally asymptotically stable when it is not linearly unstable?

The answer to this question is affirmative. It is worth mentioning that in contrast with the Lotka-Volterra competition model (see, for instance, \([13, 14, 15, 18, 31]\)), it is not easy task to obtain the monotonicity of the semiflow for a diffusive Lotka-Volterra predator-prey system with a Holling-type functional response in heterogeneous environment. This causes a big obstacle to analyze the dynamics of Lotka-Volterra predator-prey systems. To overcome this obstacle, we give a complete answer to the above problem by applying the global dynamics of a special case of system \((2)\) and the method of upper and lower solutions (see, for instance, \([29]\)). By the similar method, we further establish the existence and the stability of positive coexistence steady states, and the persistence of the two species under some suitable conditions.

This paper is organized as follows. In section 2 we first introduce some results on the global dynamics of a single-species logistic model with diffusion in spatially heterogeneous environment. In section 3 we analyze the trivial and the semi-trivial steady states, and obtain their local dynamics. In section 4 we give some suitable conditions under which \((0, k_2 \theta_{d, r_1}/a_2)\) is globally asymptotically stable. Section 5 is devoted to the properties of positive coexistence steady states and the persistence of the two species. Some concluding remarks are given in the final section.

2. Preliminaries

To analyze the linear stability of the trivial and the semi-trivial steady states for system \((1)\), here we introduce some results on the eigenvalues problems associated with the following single-
species logistic model

\[
\begin{split}
U_t &= d \Delta U + U(h(x) - U) \quad \text{in } \Omega \times \mathbb{R}_+, \\
\frac{\partial U}{\partial n} &= 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+,
\end{split}
\]  
(3)

where \(d > 0\) and the function \(h : \Omega \to \mathbb{R}\) satisfies the following hypothesis:

(H2) the function \(h\) is non-constant, bounded and measurable.

2.1. Global dynamics of the single-species logistic model

Under the hypothesis (H2) the single-species logistic model (3) has at most a unique positive steady state denoted by \(\theta_{d,h}\) (see, for instance, [24, 27] or Lemma 2.3). Then this steady state \(\theta_{d,h}\) satisfies the following equation

\[d \Delta \theta_{d,h} + \theta_{d,h} (h(x) - \theta_{d,h}) = 0.\]

Let both sides of this equation be divided by \(\theta_{d,h}\). And then integrating over the bounded region \(\Omega\), we get

\[d \int_\Omega \frac{\nabla \theta_{d,h}(x)}{\theta_{d,h}(x)} dx + \int_\Omega (h(x) - \theta_{d,h}(x)) dx = 0.\]

Since the function \(h\) is non-constant, then the steady state \(\theta_{d,h}\) is also non-constant, which yields that for each \(d > 0\),

\[\int_\Omega h(x) dx < \int_\Omega \theta_{d,h} dx.\]

To analyze the stability of the steady state \(\theta_{d,h}\) for system (3), it is useful to study the eigenvalue problem with indefinite weight:

\[
\begin{split}
\Delta \phi + \lambda h(x) \phi &= 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{split}
\]

(5)

A constant \(\lambda\) is referred to as a principal eigenvalue of the eigenvalue problem (5) if the problem (5) has a positive solution associated with \(\lambda\). Clearly, \(\lambda = 0\) is always a principal eigenvalue for each function \(h\). The properties of the principal eigenvalues for (5) are stated in the next lemma.

Lemma 2.1. [27, Theorem 4.2, p.67] Assume that the function \(h\) satisfies the hypothesis (H2) and changes sign in \(\Omega\). Then the eigenvalue problem (5) has a nonzero principal eigenvalue \(\lambda_1 = \lambda_1(h)\) if and only if \(\int_\Omega h(x) dx \neq 0\). More precisely, the following statements hold:

(i) if \(\int_\Omega h(x) dx > 0\), then \(\lambda_1(h) < 0\).
(ii) if \(\int_\Omega h(x) dx = 0\), then \(\lambda_1(h) = 0\) is a unique principal eigenvalue.
(iii) if \(\int_\Omega h(x) dx < 0\), then \(\lambda_1(h) > 0\). Moreover, \(\lambda_1(h)\) is given by

\[\lambda_1(h) = \inf \left\{ \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega h \phi^2} : \phi \in H^1(\Omega) \text{ and } \int_\Omega h \phi^2 > 0 \right\},\]

(6)

which satisfies the following properties:
The stability of the steady states for system (1) is characterized (see [27, Formula (4.9), p.69])

Let the following statements hold:

(i) if \(\int_\Omega h(x)\,dx \geq 0\) and \(\Omega \neq 0\), then \(\mu_1(d,h) < 0\) for each \(d > 0\).
(ii) if \(\int_\Omega h(x)\,dx < 0\), then the sign of \(\mu_1(d,h)\) has the following trichotomies:

- (B1) \(\mu_1(d,h) < 0\) for \(d < 1/\lambda_1(h)\);
- (B2) \(\mu_1(d,h) = 0\) for \(d = 1/\lambda_1(h)\);
- (B3) \(\mu_1(d,h) > 0\) for \(d > 1/\lambda_1(h)\), where \(\lambda_1(h)\) is defined by \(\mathcal{H}_2\).

(iii) the first eigenvalue \(\mu_1(d,h)\) is strictly increasing and concave in \(d\). Furthermore,

\[
\lim_{d \to 0^+} \mu_1(d,h) = \min(-h), \quad \lim_{d \to +\infty} \mu_1(d,h) = \theta = -\frac{1}{\Omega} \int_\Omega h(x)\,dx.
\]

(iv) if \(h \geq k\) and \(h \neq k\) in \(\Omega\), then \(\mu_1(d,h) < \mu_1(d,k)\). In particular, if \(h \leq 0\) and \(\Omega \neq 0\), then \(\mu_1(d,h) > 0\).

The results on the global dynamics of system (3) are summarized in the next lemma.

**Lemma 2.3.** [16, 27] Assume that the function \(h\) satisfies the hypothesis (H2) in the single-species logistic model (3). Then the following statements hold:

(i) if \(\int_\Omega h(x)\,dx \geq 0\), then for each \(d > 0\) system (3) has a unique positive steady state \(\theta_{d,h}\), which is globally asymptotically stable. Moreover, this steady state \(\theta_{d,h}\) satisfies the limits:

\[
\lim_{d \to 0^+} \theta_{d,h}(x) = h^+(x) := \max\{h(x),0\}, \quad \lim_{d \to +\infty} \theta_{d,h} = \bar{h}.
\]

(ii) if \(\int_\Omega h(x)\,dx < 0\) and the function \(h\) changes sign in \(\Omega\), then system (3) has a unique positive steady state \(\theta_{d,h}\) if and only if \(0 < d < 1/\lambda_1(h)\), where the constant \(\lambda_1(h)\) is defined by \(\mathcal{H}_2\). Moreover, the steady state \(\theta_{d,h}\) is globally asymptotically stable and \(\theta_{d,h}(x) \to h^+(x)\) as \(d \to 0^+\). If \(d \geq 1/\lambda_1(h)\), then the trivial steady state 0 is a global attractor of system (3) in \([U \in \mathbb{R} : U \geq 0]\).

(iii) if the function \(h\) satisfies \(h(x) \leq 0\) for each \(x \in \Omega\), then the trivial steady state 0 is a global attractor of system (3) in \(\mathbb{R}\).

By applying the method of upper and lower solutions (see, for instance, [24, 33]), the existence and uniqueness of the steady state \(\theta_{d,h}\) can be established. The results on the limit behavior of the steady state \(\theta_{d,h}\) are obtained by [16, Lemmas 2.4 and 2.5]. An outline of the proof for this lemma is given in [27, Section 4.1].
3. Analysis of the trivial and the semi-trivial steady states

In this section, based on the results on the global dynamics of the single-species logistic model (3), we give a study of the trivial and the semi-trivial steady states of the modified Leslie-Gower system (1).

Assume that the modified Leslie-Gower system (1) has a steady state \((u, v)\) with \(u \geq 0\) and \(v \geq 0\). We linearize the corresponding elliptic system of (1) at \((u, v)\), which is in the form

\[
\begin{align*}
    d_1\Delta\Phi + \left(r_1(x) - 2b_1u - \frac{a_1k_1v}{(u + k_1)^2}\right)\Phi - \frac{a_1u}{u + k_1}\Psi + \mu\Phi &= 0 \quad \text{in } \Omega, \\
    d_2\Delta\Psi + \frac{a_2v^2}{(u + k_2)^2}\Phi + \left(r_2(x) - 2b_2v\right)\Psi + \mu\Psi &= 0 \quad \text{in } \Omega, \\
    \frac{\partial\Phi}{\partial n} - \frac{\partial\Psi}{\partial n} &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

Then the local dynamics of system (1) at this steady state \((u, v)\) are determined by the principal eigenvalue \(\mu_1 \in \mathbb{R}\) of the linearized system (2), that is, \(\mu_1\) is the least eigenvalue. The steady state \((u, v)\) is called linearly stable (resp. linearly unstable) if \(\mu_1\) is positive (resp. negative) (see, for instance, [27]).

To obtain the stability of the trivial and the semi-trivial steady states of the modified Leslie-Gower system (1), we first make some preparations.

Assume that the rate \(d\) and the function \(h\) in the single-species logistic model (3) are in the form \(d = d_2\) and \(h = r_2\), where \(r_2\) satisfy the conditions in (H1). Then by Lemma 2.3, the single-species logistic model (3) has a unique positive steady state \(\theta_{d_2,r_2}\). Let the constants \(\alpha\) and \(\beta\) be respectively defined by

\[
\alpha = \inf_{d_2 > 0} \frac{r_1}{\theta_{d_2,r_2}}, \quad \beta = \sup_{d_2 > 0} \frac{r_1(x)}{\theta_{d_2,r_2}(x)},
\]

Then we have the following.

**Lemma 3.1.** Assume that the functions \(r_1\) and \(r_2\) satisfy the hypothesis (H1). Let \(\theta_{d_2,r_2}\) denote the unique positive steady state of the single-species logistic model (3) with \(d = d_2\) and \(h = r_2\). Then the constants \(\alpha\) and \(\beta\) in (10) satisfy \(\alpha < \beta\).

**Proof.** If \(r_2\) is a constant, then \(r_2(x) \equiv \overline{r}_2 > 0\) for each \(x \in \Omega\) and \(\theta_{d_2,r_2}(x) \equiv \overline{\theta}_{d_2,r_2}\) for all \(x \in \Omega\) and \(d_2 > 0\). By the hypothesis (H1), the function \(r_1\) is not a constant. Hence, we have the following:

\[
\beta = \sup_{d_2 > 0} \frac{r_1(x)}{\theta_{d_2,r_2}(x)} = \frac{1}{\overline{\theta}_{d_2,r_2}} \sup_{d_2 > 0} r_1(x) > \frac{1}{\overline{\theta}_{d_2,r_2}} \frac{1}{\overline{\theta}_{d_2,r_2}} \frac{r_1}{\overline{\theta}_{d_2,r_2}} = \alpha.
\]

Thus, the estimate \(\alpha < \beta\) holds.

If \(r_2\) is not a constant, then \(\theta_{d_2,r_2}\) is positive and non-constant in \(\Omega\). Thus we have

\[
0 = \int_{\Omega} \left(r_1(x) - 2b_2v\right)\overline{\theta}_{d_2,r_2}(x) - \overline{r}_1(\theta_{d_2,r_2}(x)) dx = \int_{\Omega} \overline{\theta}_{d_2,r_2}(x)\overline{\theta}_{d_2,r_2}(x) \left(r_1(x)/(\theta_{d_2,r_2}(x) - \overline{r}_1/\overline{\theta}_{d_2,r_2}\right) dx,
\]

which implies

\[
\sup_{d_2 > 0} \frac{r_1(x)}{\theta_{d_2,r_2}(x)} \geq \frac{\overline{r}_1}{\overline{\theta}_{d_2,r_2}}.
\]
By (4) and (8), we obtain that \( \theta_{d_{i},r_{i}} \) reaches its minimum at infinity and \( \theta_{d_{i},r_{i}} \) is not a constant as \( d_{i} \) varies. This together with (11) yields
\[
\beta = \sup_{d_{i}>0} \inf_{\theta_{d_{i},r_{i}}} \frac{r_{1}(x)}{\theta_{d_{i},r_{i}}(x)} \geq \sup_{d_{i}>0} \frac{r_{1}}{\theta_{d_{i},r_{i}}} > \inf_{d_{i}>0} \frac{r_{1}}{\theta_{d_{i},r_{i}}}. \]
Therefore, the proof is now complete. \( \square \)

Let the sets \( I, I_{1} \) and \( I_{2} \) be denoted by
\[
I := \left\{ d_{2} \in \mathbb{R}_{+} : \left( r_{1}(x) - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}}(x) \right) dx < 0 \right\},
\]
\[
I_{1} := \left\{ d_{2} \in \mathbb{R}_{+} : r_{1} - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}} \leq 0 \quad \text{and} \quad r_{1} - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}} \neq 0 \right\},
\]
\[
I_{2} := \left\{ d_{2} \in I : \sup_{\Omega} \left( r_{1}(x) - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}}(x) \right) > 0 \right\},
\]
respectively. Then \( I = I_{1} \cup I_{2} \). Then the results on the existence and stability of the trivial and the semi-trivial steady states are stated as follows.

**Theorem 3.2.** Assume that the modified Leslie-Gower system (1) satisfies the hypothesis (H1). Then system (1) has a trivial steady state \((0, 0)\), and two semi-trivial steady states \((\theta_{d_{i},r_{i}}/b_{i}, 0)\) and \((0, k_{2}\theta_{d_{i},r_{i}}/a_{i})\), where \(\theta_{d_{i},r_{i}}\) denotes the unique positive steady state of system (3) with \(d = d_{i}\) and \(h = r_{i}\) for each \(i = 1, 2\). Furthermore, the following statements hold:

(i) the trivial steady state \((0, 0)\) and the semi-trivial steady state \((\theta_{d_{i},r_{i}}/b_{i}, 0)\) are linearly unstable.

(ii) the semi-trivial steady state \((0, k_{2}\theta_{d_{i},r_{i}}/a_{i})\) is linearly stable for \((d_{1}, d_{2})\) in \(D_{+}\) and linearly unstable for \((d_{1}, d_{2})\) in \(D_{-}\), where the sets \(D_{+}\) in \(D_{-}\), respectively defined by
\[
D_{+} := \left\{ (d_{1}, d_{2}) \in \mathbb{R}_{+}^{2} : \mu_{1} \left( d_{1}, r_{1} - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}} \right) > 0 \right\},
\]
\[
D_{-} := \left\{ (d_{1}, d_{2}) \in \mathbb{R}_{+}^{2} : \mu_{1} \left( d_{1}, r_{1} - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}} \right) < 0 \right\},
\]
and the set \(D_{0}\) has the following trichotomies:

(T1) if \(a_{1}k_{2}/(a_{2}k_{1}) \in (0, \alpha]\), then \(D_{+} = \varnothing\);

(T2) if \(a_{1}k_{2}/(a_{2}k_{1}) \in [\beta, +\infty)\), then \(D_{+} = \mathbb{R}_{+}^{2}\);

(T3) if \(a_{1}k_{2}/(a_{2}k_{1}) \in (\alpha, \beta]\), then \(D_{+} = \{ (d_{1}, d_{2}) \in \mathbb{R}_{+}^{2} : d_{2} \in I, \ d_{1} > \phi(d_{2}) \}\), where the function \(\phi\) is defined by
\[
\phi(d_{2}) := \begin{cases} 0 & \text{for } d_{2} \in I_{1}, \\
\left( d_{1} \left( r_{1} - \frac{a_{1}k_{2}}{a_{2}k_{1}} \theta_{d_{i},r_{i}} \right) \right)^{-1} & \text{for } d_{2} \in I_{2},
\end{cases}
\]
Proof. It is clear that system (1) has the trivial steady state \((0, 0)\). Under the hypothesis (H1), by Lemma 2.3 we obtain that system (1) has two semi-trivial steady states \((\theta_{d_{i},r_{i}}/b_{i}, 0)\) and \((0, k_{2}\theta_{d_{i},r_{i}}/a_{i})\). Thus, the first statement holds.
Consider the linearized system \((9)\) with \((u, v) = (0, 0)\), that is,

\[
\begin{aligned}
&d_1 \Delta \Phi + r_1(x) \Phi + \mu \Phi = 0 \quad \text{in } \Omega, \\
&d_2 \Delta \Psi + r_2(x) \Psi + \mu \Psi = 0 \quad \text{in } \Omega, \\
&\frac{\partial \Phi}{\partial n} = \frac{\partial \Psi}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then by Lemma 2.2 (i), we see that \(\mu_1(d_i, r_i) < 0\) for \(d_i > 0\). This proves that \((0, 0)\) is linearly unstable.

Consider the linearized system \((9)\) with \((u, v) = \left(\theta_{d_i, r_i} / b_1, 0\right)\), that is,

\[
\begin{aligned}
&d_1 \Delta \Phi + \left(\gamma(x) - 2\theta_{d_i, r_i}(x)\right) \Phi - \frac{a_1 \theta_{d_i, r_i}(x)}{\theta_{d_i, r_i}(x) + b_1 k_1} \Psi + \mu \Phi = 0 \quad \text{in } \Omega, \\
&d_2 \Delta \Psi + \gamma(x) \Psi + \mu \Psi = 0 \quad \text{in } \Omega, \\
&\frac{\partial \Phi}{\partial n} = \frac{\partial \Psi}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

By (i) and (iv) in Lemma 2.2, we obtain that \(\mu_1(d_2, r_2) < 0\) for each \(d_2 > 0\), and \(\mu_1(d_1, r_1 - 2\theta_{d_i, r_i}) > \mu_1(d_1, r_1 - \theta_{d_i, r_i}) = 0\) for each \(d_1 > 0\). Then for \(\mu = \mu_1(d_2, r_2) < 0\), the operator

\[
d_1 \Delta \cdot + \left(\gamma(x) - 2\theta_{d_i, r_i}(x)\right) \cdot + \mu
\]

is invertible, which implies that the eigenvalue problem \((15)\) has a negative eigenvalue \(\mu_1(d_2, r_2)\). This yields that \(\left(\theta_{d_i, r_i} / b_1, 0\right)\) is linearly unstable. Thus, (i) is proved.

Consider the linearized system \((9)\) with \((u, v) = \left(0, k_2 \theta_{d_i, r_i} / a_2\right)\), that is,

\[
\begin{aligned}
&d_1 \Delta \Phi + \left(\gamma(x) - \frac{a_1 k_2}{a_2 k_1} \theta_{d_i, r_i}(x)\right) \Phi + \mu \Phi = 0 \quad \text{in } \Omega, \\
&d_2 \Delta \Psi + \left(\frac{\theta_{d_i, r_i}(x)}{a_2}\right)^2 \Phi + \left(\gamma(x) - 2\theta_{d_i, r_i}(x)\right) \Psi + \mu \Psi = 0 \quad \text{in } \Omega, \\
&\frac{\partial \Phi}{\partial n} = \frac{\partial \Psi}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

By similar method used in the proof for the stability of the semi-trivial steady state \((\theta_{d_i, r_i} / b_1, 0)\), the eigenvalue problem \((16)\) with \((d_1, d_2) \in D^-\) has a negative eigenvalue

\[
\mu = \mu_1 \left(d_1, r_1 - \frac{a_1 k_2}{a_2 k_1} \theta_{d_i, r_i}\right) < 0.
\]

Thus, \((0, k_2 \theta_{d_i, r_i} / a_2)\) is unstable for \((d_1, d_2) \in D^-\).

We next consider the case that \((d_1, d_2)\) is in \(D^+\). Let \((\Phi_0, \Psi_0) \neq (0, 0)\) be a solution of the eigenvalue problem \((16)\) with some eigenvalue \(\mu\). If \(\Phi_0\) satisfies \(\Phi_0 \neq 0\), then

\[
\mu \geq \mu_1 \left(d_1, r_1 - \frac{a_1 k_2}{a_2 k_1} \theta_{d_i, r_i}\right) > 0.
\]

If \(\Phi_0 = 0\) and \(\Psi_0 \neq 0\), then

\[
\mu \geq \mu_1 (d_2, r_2 - 2\theta_{d_i, r_i}) > \mu_1 (d_2, r_2 - \theta_{d_i, r_i}) = 0.
\]
Thus, \((0, k_2\theta_{d_1, r_2}/a_2)\) is linearly stable for \((d_1, d_2) \in D_+\).

To obtain the explicit expression of the set \(D_+\), we first recall that the constants \(\alpha\) and \(\beta\) defined by (10) satisfies \(\alpha < \beta\). If \(a_1k_2/(a_2k_1) \in (0, a_1]\), then by (10) we have that for each \(d_2 > 0\),

\[
\int_{\Omega} r_1(x) - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}(x) \, dx \geq 0.
\]

Lemma 2.2 (i) yields that \(\mu_1 \left(d_1, r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}\right) \leq 0\) for all \((d_1, d_2) \in \mathbb{R}^2_+\). Then (T1) is proved.

If \(a_1k_2/(a_2k_1) \in (\beta, +\infty)\), then from (10) it follows that for each \(x \in \Omega\) and each \(d_2 > 0\),

\[
r_1(x) - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}(x) \leq 0.
\]

We further claim that if \(a_1k_2/(a_2k_1) \in (\beta, +\infty)\), then for all \(d_2 > 0\),

\[
r_1(x) - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2} \not= 0.
\]

In fact, in case \(a_1k_2/(a_2k_1) \in (\beta, +\infty)\), it is clear that the claim holds. In case \(a_1k_2/(a_2k_1) = \beta\), the proof is divided into three different cases: (E1) if \(r_1\) is not a constant and \(r_2\) is a constant, then \(\theta_{d_1, r_2}\) is a constant for all \(d_2 > 0\), and \(r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}\) is not a constant. Thus, the claim holds. (E2) if \(r_1\) is a constant and \(r_2\) is not a constant, then \(\theta_{d_1, r_2}\) is not a constant for all \(d_2 > 0\), which yields the claims. (E3) if both \(r_1\) are not constants, then \(\theta_{d_1, r_2}\) is not a constant for all \(d_2 > 0\). Suppose that there exists a \(\hat{d}_2\) with \(0 < \hat{d}_2 < +\infty\) such that \(r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2} \equiv 0\). Then for each \(x \in \Omega\),

\[
\frac{r_1(x)}{\theta_{d_1, r_2}(x)} = \frac{a_1k_2}{a_2k_1} = \beta = \sup_{d_2 > 0} \frac{r_1(x)}{\theta_{d_2, r_2}(x)},
\]

which yields that \(\theta_{d_1, r_2} \leq \theta_{d_2, r_2}\) and \(\theta_{d_1, r_2} \not= \theta_{d_2, r_2}\) for all \(d_2 > 0\). This is a contradiction with the fact that \(\hat{d}_2\) and \(\hat{d}_2\) hold if \(r_2\) is not a constant. Hence, the claim is proved. Thus, by (17), (18) and Lemma 2.2 (iv), we obtain (T2).

If \(a_1k_2/(a_2k_1) \in (\alpha, \beta)\), then by (4), (5) and (10), there exist \(d_2 > 0\) and \(x_0 \in \Omega\) such that

\[
r_1(x_0) - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}(x_0) > 0 \quad \text{and} \quad \int_{\Omega} \left( r_1(x) - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}(x) \right) \, dx < 0,
\]

which yields that \(I_2\) is not an empty set. By Lemma 2.2 (ii), then for either \(d_1 > 0\) and \(d_2 \in I_1\) or \(d_1 > 1/\lambda (r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2})\) and \(d_2 \in I_2\), we have \(\mu_1 (d_1, r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}) > 0\). Thus, (T3) holds. Therefore, the proof is now complete. \(\square\)

4. More on the stability of the semi-trivial steady state

In this section we further show that the semi-trivial steady state \((0, k_2\theta_{d_1, r_2}/a_2)\) is globally asymptotically stable when it is not linear unstable.

Recall that the set \(D_+\) is in the form (14) and let the set \(D_0\) be defined by

\[
D_0 = \left\{ (d_1, d_2) \in \mathbb{R}^2_+ : \mu_1 \left(d_1, r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2}\right) = 0 \right\}.
\]

The main results on global stability of the semi-trivial steady state \((0, k_2\theta_{d_1, r_2}/a_2)\) are summarized as follows.
**Theorem 4.1.** Assume that the modified Leslie-Gower system (1) satisfies the hypothesis (H1), and the parameters $a_1$, $a_2$, $k_1$ and $k_2$ satisfy one of the following conditions: (C1) $a_1 k_2 / (a_2 k_1) \in [\beta, +\infty)$ and $k_1 \geq k_2$; (C2) $a_1 k_2 / (a_2 k_1) \in (\alpha, \beta)$ and $k_1 = k_2$; (C3) $a_1 k_2 / (a_2 k_1) \in (0, \alpha]$ and $k_2 \geq k_1$, where the constants $\alpha$ and $\beta$ are given by (17). Then for each $(d_1, d_2) \in \mathcal{D}_1 \cup \mathcal{D}_0$, the semi-trivial steady state $(0, 0)$ is globally asymptotically stable, and system (1) has no positive steady states.

The proof of this theorem is given in Section 4.3. This theorem gives a positive answer to the question in the introduction, that is, the semi-trivial steady state $(0, 0)$ is globally asymptotically stable when it is not linearly unstable. We also remark that the case that the principal eigenvalue $\mu_1$ is equal to zero, which is always called the degenerate case. Otherwise, it is called the non-degenerate case (see, for instance, [15, 40]). Then the case $(d_1, d_2) \in \mathcal{D}_e$ is non-degenerate and the case $(d_1, d_2) \in \mathcal{D}_0$ is degenerate. Here we obtain the global stability of the semi-trivial steady state $(0, k_2 \theta_{d_1, r_2} / a_2)$ based on the dynamics of the competition-diffusion system (28).

**4.1. The attraction of system (1) and the properties of $\mathcal{D}_0$**

To complete the proof of Theorem 4.1, we first make some preparations and start by the invariant regions of system (1).

**Lemma 4.2.** Assume that the modified Leslie-Gower system (1) satisfies the hypothesis (H1). Let the constants $M_1$ and $M_2$ be defined by $M_1 = \sup_{\mathbb{R}^+} r_1(x)$ and $M_2 = \sup_{\mathbb{R}^+} r_2(x)$, respectively. Then the set $\mathcal{A}$ of the form

$$
\mathcal{A} := \left\{ (U, V) : 0 \leq U \leq \frac{M_1}{b_1}, \ 0 \leq V \leq \frac{(M_1 + b_1 k_2) M_2}{a_2 b_1} \right\}
$$

is an invariant region of system (1). Furthermore, the set $\mathcal{A}$ is a global attractor of system (1) in the set $\{(U, V) \in \mathbb{R}^2 : U \geq 0, V \geq 0\}$.

**Proof.** Similarly to [3, Lemma 5], we obtain that the unique solution of system (1) is nonnegative and defined in the set $[0, +\infty) \times \Omega$. Then by the Comparison Principle, we obtain that the set $\mathcal{A}$ is an invariant region of system (1), and $U(t, x) \leq \bar{U}(t)$ for $t \geq 0$ and $x \in \Omega$, where $\bar{U}$ is the solution of the initial value problem

$$
\frac{d\bar{U}}{dt} = \bar{U}(M_1 - b_1 \bar{U}), \quad \bar{U}(0) = \max_{\mathbb{R}^+} U(x, 0).
$$

Hence, for $t \geq 0$ and $x \in \Omega$,

$$
U(t, x) \leq \bar{U}(t) = \frac{\bar{U}(0) M_1}{(M_1 - b_1 U(0)) e^{-b_1 t} + b_1 U(0)},
$$

which yields that for each $U_0$ with $0 \leq U_0 \leq M_1 / b_1$, the solution $(U, V)$ of system (1) satisfies $0 \leq U \leq M_1 / b_1$. Further, consider the following equation

$$
\frac{d\bar{V}}{dt} = \bar{V} \left( M_2 - \frac{a_2 b_1}{M_1 + k_2 b_1} \bar{V} \right), \quad \bar{V}(0) = \max_{\mathbb{R}^+} V(x, 0).
$$
Then for each \((U_0, V_0) \in \mathcal{A}\), we obtain that for each \(t \geq 0\) and each \(x \in \Omega\),
\[
V(t, x) \leq V(t) = \frac{\sqrt{V(0)M_2}}{(M_1 - K_V V(0))e^{-M_1 t} + K_V V(0)}.
\]
(21)

where \(K_V = a_2b_1/(M_1 + b_1k_2)\). The attraction of the set \(\mathcal{A}\) is obtained by letting \(t \to +\infty\) in (20) and (21). Therefore, the proof is now complete.

In order to study the degenerate case, we next give a description of the set \(D_0\).

**Lemma 4.3.** Assume that the functions \(r_1\) and \(r_2\) satisfy the hypothesis (H1). Let the set \(D_0\) be defined by (19) and a family of sets \(E_s\), \(s > 0\), be in the form
\[
E_s = \{ (d_1, d_2) \in \mathbb{R}^2 : r_1 \equiv s\theta_{d_1, r_2} \}, \quad s > 0.
\]
(22)

Then for each \(s > 0\), the set \(E_s\) has the following dichotomies:

- **(D1)** \(E_s = \emptyset\);
- **(D2)** \(E_s = \{ (d_1, d_2^+) \in \mathbb{R}^2_+ : d_1^+ > 0\}\), where \(d_2^+\) is the only \(d_2\) satisfying \(r_1 \equiv s\theta_{d_1, r_2}\),

and the set \(D_0\) can be rewritten as the form
\[
D_0 = \begin{cases} \emptyset & \text{if } \frac{a_1k_2}{a_2k_1} \in (0, \alpha) \cup [\beta, +\infty), \\ \{ (d_1, d_2) \in \mathbb{R}^2_+ : r_1 \equiv s\theta_{d_1, r_2} \} & \text{if } \frac{a_1k_2}{a_2k_1} = \alpha, \\ \partial D_+ \cup \{ (d_1, d_2) \in \mathbb{R}^2_+ : r_1 \equiv \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2} \} & \text{if } \frac{a_1k_2}{a_2k_1} \in (\alpha, \beta), \end{cases}
\]
(23)

where \(D_+\) is in the form (14), the boundary \(\partial D_+\) of \(D_+\) is given by
\[
\partial D_+ = \overline{D_+} \setminus D_+ = \begin{cases} \emptyset & \text{if } d_2 \in I_1, \\ \{ (d_1, d_2) \in \mathbb{R}^2_+ : d_1 = \left( a_1 \left( r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_1, r_2} \right) \right)^{-1} \} & \text{if } d_2 \in I_2, \end{cases}
\]
(24)

and the sets \(I_1\) and \(I_2\) are defined by (12) and (13), respectively.

**Proof.** The proof for the dichotomies (D1) and (D2) is divided into three different cases:

- **Case (E1):** If \(r_1\) is a constant and \(r_2\) is non-constant; (E2) \(r_1\) is non-constant and \(r_2\) is a constant; (E3) both \(r_1\) and \(r_2\) are non-constant.

  - **Case (E1):** If \(r_1\) is a constant and \(r_2\) is non-constant, then \(\theta_{d_1, r_2}\) is non-constant, which yields that no constants \(d_2\) satisfy \(r_1 \equiv s\theta_{d_1, r_2}\) for each \(s > 0\). Then \(E_s = \emptyset\).

  - **Case (E2):** If \(r_1\) is non-constant and \(r_2\) is a constant, then \(\theta_{d_1, r_2}\) is a constant. This implies that for each \(s > 0\), there are also no constants \(d_2\) such that \(r_1 \equiv s\theta_{d_1, r_2}\) holds. Then \(E_s = \emptyset\).

  - **Case (E3):** If both \(r_1\) and \(r_2\) are non-constant, then \(\theta_{d_1, r_2}\) is also non-constant and satisfies
\[
d_2 \Delta \theta_{d_1, r_2} + \theta_{d_1, r_2} (r_2(x) - \theta_{d_1, r_2}) = 0.
\]
(25)
For each \( s > 0 \), assume that \((d_1, d_2)\) is in the set \( E_s \). Then \( d_2 \) satisfies \( r_1 = s \theta_{d_1, r_2} \). This together with (25) yields that
\[
d_2 = \frac{\int_\Omega \left( \theta_{d_2, r_2}(x) - r_2(x) \right) dx}{\int_\Omega \left( \theta_{d_2, r_2}(x) \right)^2 dx} = \frac{\int_\Omega \left( \theta_{d_2, r_2}(x) - r_2(x) \right) dx}{\int_\Omega \left( \theta_{d_2, r_2}(x) \right)^2 dx}.
\]
By (24) we have \( d_2 > 0 \). Hence, the positive constant \( d_2 \) with the property \( r_1 = s \theta_{d_1, r_2} \) is well-defined and unique. This yields that \( E_s \) satisfies (D2). Thus, (D1) and (D2) are obtained.

Next we prove (23). If \( a_1 k_2 / (a_2 k_1) \in (0, \alpha) \), then by (10) we have that
\[
\int_\Omega \left( r_1(x) - \frac{a_1 k_2}{a_2 k_1} \theta_{d_1, r_2}(x) \right) dx > 0 \quad \text{for each } d_2 > 0.
\]
This together with Lemma 2.2(i) yields that for each \( d_1 > 0 \),
\[
\mu_1 \left( d_1, r_1 - \frac{a_1 k_2}{a_2 k_1} \theta_{d_1, r_2} \right) < 0.
\]
Thus, \( D_0 = \emptyset \) for \( a_1 k_2 / (a_2 k_1) \in (0, \alpha) \).

If \( a_1 k_2 / (a_2 k_1) \in [\beta, +\infty) \), then Theorem 3.2 yields \( D_+ = \mathbb{R}^+ \). This implies that \( D_0 = \emptyset \) for \( a_1 k_2 / (a_2 k_1) \in [\beta, +\infty) \).

If \( a_1 k_2 / (a_2 k_1) = \alpha \), then (10) yields that
\[
\int_\Omega \left( r_1(x) - \alpha \theta_{d_1, r_2}(x) \right) dx \geq 0, \quad \text{for each } d_2 > 0.
\]
Assume that \( d_2 \) satisfies \( r_1 = \alpha \theta_{d_1, r_2} \). Then \( \mu_1(d_1, 0) = 0 \) is the first eigenvalue of the Neumann boundary problem (7) with \( h \equiv 0 \). This implies
\[
\left\{(d_1, d_2) \in \mathbb{R}^2 : r_1 = \alpha \theta_{d_1, r_2} \right\} \subset D_0.
\]
Assume that \( d_2 \) satisfies \( r_1 = \alpha \theta_{d_1, r_2} \). Then by (26) and Lemma 2.2(i), we obtain that \( \mu_1(d_1, r_1 - \alpha \theta_{d_1, r_2}) < 0 \) for each \( d_1 > 0 \). This together with (27) yields that \( D_0 = \left\{(d_1, d_2) \in \mathbb{R}^2 : r_1 = \alpha \theta_{d_1, r_2} \right\} \) for \( a_1 k_2 / (a_2 k_1) = \alpha \).

If \( a_1 k_2 / (a_2 k_1) \in (\alpha, \beta) \), then Theorem 3.2(ii) we have
\[
D_+ = \left\{(d_1, d_2) \in \mathbb{R}^2 : d_2 \in I, \ d_1 > \varphi(d_2) \right\}.
\]
Thus \( \partial D_+ \) is in the form
\[
\partial D_+ = \left\{(d_1, d_2) \in \mathbb{R}^2 : d_2 \in I, \ d_1 = \varphi(d_2) \right\}.
\]
If \( d_2 \in I_1 \), then \( \varphi(d_2) = 0 \), which yields \( \partial D_+ = \emptyset \). If \( d_2 \in I_2 \), then by Lemma 2.2(ii),
\[
d_1 = \varphi(d_2) = \left( a_1 \left( r_1 - \frac{a_1 k_2}{a_2 k_1} \theta_{d_1, r_2} \right) \right)^{-1}, \quad \mu_1 \left( d_1, r_1 - \frac{a_1 k_2}{a_2 k_1} \theta_{d_1, r_2} \right) = 0
\]
Thus, we obtain (24) and \( \partial D_+ \subset D_0 \).

Similarly to (24), we obtain
\[
E_{a_1 k_2 / (a_2 k_1)} = \left\{(d_1, d_2) \in \mathbb{R}^2 : r_1 = \frac{a_1 k_2}{a_2 k_1} \theta_{d_1, r_2} \right\} \subset D_0.
\]
This together with \( \partial D_+ \subset D_0 \) yields \( (\partial D_+ \cup E_{a_1/k_2/(a_2k_1)}) \subset D_0 \) for \( a_1k_2/(a_2k_1) \in (\alpha, \beta) \). Thus, to finish the proof, it is only necessary to obtain \( D_0 \subset (\partial D_+ \cup E_{a_1/k_2/(a_2k_1)}) \) for \( a_1k_2/(a_2k_1) \in (\alpha, \beta) \).

If \((d_1, d_2) \in D_0 \) with \( r_1 \equiv a_1k_2\theta_{d_2,r_2}/(a_2k_1) \), then \((d_1, d_2) \in E_{a_1/k_2/(a_2k_1)}\).

If \((d_1, d_2) \in D_0 \) with \( r_1 \neq a_1k_2\theta_{d_2,r_2}/(a_2k_1) \), then by Lemma \( \ref{lem:global_dynamics} \) we have

\[
\int_{\Omega} \left( r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_2,r_2} \right) dx < 0
\]

which yields \( d_2 \in I = I_1 \cup I_2 \). For \( d_2 \in I_1 \), Lemma \( \ref{lem:global_dynamics} \) yields

\[
\mu_1 \left( d_1, r_1 - \frac{a_1k_2}{a_2k_1} \theta_{d_2,r_2} \right) > 0.
\]

Recall that \( \mu_1 (d_1, r_1 - a_1k_2\theta_{d_2,r_2}/(a_2k_1)) = 0 \) for \((d_1, d_2) \in D_0 \), then \( d_2 \in I_2 \). Following this assertion, by Lemma \( \ref{lem:global_dynamics}(ii) \) we have \( d_1 = (\Lambda r_1 - a_1k_2\theta_{d_2,r_2}/(a_2k_1))^{-1} \). Therefore, the proof is now complete.

4.2. Global dynamics of the competition-diffusion system

To obtain the globally asymptotical stability of \((0, k_2\theta_{d_2,r_2}/a_2)\), we give a detailed study of the global dynamics of the following competition-diffusion system

\[
\begin{aligned}
U_t &= d_1 \Delta U + U (r_1(x) - b_1 U - c_1 V) & \text{in } \Omega \times \mathbb{R}^+,
V_t &= d_2 \Delta V + V (r_2(x) - c_2 V) & \text{in } \Omega \times \mathbb{R}^+,
\frac{\partial U}{\partial n} &= \frac{\partial V}{\partial n} = 0 & \text{on } \partial \Omega \times \mathbb{R}^+,
U(0, x) &= U_0(x), V(0, x) = V_0(x) & \text{in } \Omega.
\end{aligned}
\]

(28)

By taking a variable transformation of the form \((U, V) \to (U/b_1, V/c_2)\), system (28) is changed into the form (3) with \( c = c_1/c_2 \) and \( b = 0 \). Thus system (28) can be viewed as a special case of system (3). The global dynamics of system (28) given in this section is a complement to the results obtained in [15]. Further, we observe that the second equation in system (28) is a single-species logistic system in heterogeneous environment and the evolution of the density \( V \) of the predator is independent of the density \( U \) of the prey.

Assume that the competition-diffusion system (28) has a steady state \((u, v)\) with \( u \geq 0 \) and \( v \geq 0 \). We linearize the corresponding elliptic system of (28) at \((u, v)\) and obtain

\[
\begin{aligned}
d_1 \Delta \Phi + \left( r_1(x) - 2b_1 u - c_1 v \right) \Phi - c_1 u \Phi + \mu \Phi &= 0 & \text{in } \Omega, \\
d_2 \Delta \Psi + \left( r_2(x) - 2c_2 V \right) \Psi - c_2 V \Psi + \mu \Psi &= 0 & \text{in } \Omega, \\
\frac{\partial \Phi}{\partial n} &= \frac{\partial \Psi}{\partial n} = 0 & \text{on } \partial \Omega.
\end{aligned}
\]

(29)

By Lemma \( \ref{lem:global_dynamics} \) the second equation in system (28) satisfying (H1) has a unique positive steady state \( \theta_{d_2,r_2}/c_2 \), where \( \theta_{d_2,r_2} \) is defined as in the section 2.1. Let the sets \( \tilde{I}, \tilde{I}_1 \) and \( \tilde{I}_2 \) be defined by

\[
\tilde{I} = \left\{ d_2 \in \mathbb{R}_+ : \int_{\Omega} \left( r_1(x) - \frac{c_1}{c_2} \theta_{d_2,r_2}(x) \right) dx < 0 \right\},
\]

(30)

\[
\tilde{I}_1 = \left\{ d_2 \in \mathbb{R}_+ : r_1 - \frac{c_1}{c_2} \theta_{d_2,r_2} \leq 0 \text{ and } r_1 - \frac{c_1}{c_2} \theta_{d_2,r_2} \neq 0 \right\},
\]

(31)

\[
\tilde{I}_2 = \left\{ d_2 \in \tilde{I} : \sup_{\tilde{I}} \left( r_1(x) - \frac{c_1}{c_2} \theta_{d_2,r_2}(x) \right) > 0 \right\}.
\]

(32)
It is clearly defined as in Lemma 4.4, and \( \tilde{\mu} \) is equal to \( \tilde{\theta} \). Then system (28) has a trivial steady state (28) with 
\[
d = d_i
\]
and \( h = r_i \), for each \( i = 1, 2 \). Furthermore, the following statements hold:

(i) the trivial steady state (0, 0) and the semi-trivial steady state \((\theta_{d_1,r_1}/b_1, 0)\) are both linearly unstable.

(ii) the semi-trivial steady state \((0, \theta_{d_2,r_2}/c_2)\) is linearly stable for \((d_1, d_2)\) in \( \tilde{D}_\alpha \) and linearly unstable for \((d_1, d_2)\) in \( \tilde{D}_\alpha \), where the sets \( \tilde{D}_\alpha \) are respectively given by

\[
\tilde{D}_\alpha = \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : \mu_1 \left( d_1, r_1 - \frac{c_1}{c_2} \theta_{d_2,r_2} \right) > 0 \right\},
\]

\[
\tilde{D}_\alpha = \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : \mu_1 \left( d_1, r_1 - \frac{c_1}{c_2} \theta_{d_2,r_2} \right) < 0 \right\},
\]

and the set \( \tilde{D}_\alpha \) has the following trichotomies:

(T1) if \( c_1/c_2 \in (0, \alpha] \), then \( \tilde{D}_\alpha = \emptyset \);

(T2) if \( c_1/c_2 \in [\beta, +\infty) \), then \( \tilde{D}_\alpha = \mathbb{R}_+^2 \);

(T3) if \( c_1/c_2 \in (\alpha, \beta) \), then \( \tilde{D}_\alpha = \{(d_1, d_2) \in \mathbb{R}_+^2 : d_2 \in \tilde{D}_1, d_1 > \tilde{\varphi}(d_2)\} \), where the function \( \tilde{\varphi} \) is defined by

\[
\tilde{\varphi}(d_2) := \begin{cases} 0 & \text{for } d_2 \in \tilde{D}_1, \\ (\lambda_1 (r_1 - c_1 \theta_{d_2,r_2}/c_2))^{-1} & \text{for } d_2 \in \tilde{D}_2. \end{cases}
\]  

We observe that the non-trivial steady state \((u, v)\) of the competition-diffusion system (28) satisfies that \( v = \theta_{d_2,r_2}/c_2 \) and \( u \) is the solution of the following single-species logistic model

\[
\begin{cases} U_t = d_1 \Delta U + U (r_1(x) - b_1 U) & \text{in } \Omega \times \mathbb{R}_+, \\ \frac{\partial U}{\partial n} = 0 & \text{on } \partial \Omega \times \mathbb{R}_+, \end{cases}
\]  

where the function \( \tilde{r}_1 \) is in the form \( \tilde{r}_1(x) = r_1(x) - c_1 \theta_{d_2,r_2}(x)/c_2 \) for \( x \in \Omega \). Let the sets \( \tilde{D}_\alpha \) be defined as in Lemma 4.4 and \( \tilde{D}_b \) are given by

\[
\tilde{D}_b = \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : \mu_1 \left( d_1, r_1 - \frac{c_1}{c_2} \theta_{d_2,r_2} \right) = 0 \right\}. 
\]  

It is clearly that \( \tilde{D}_b \) is equal to \( D_b \) when \( d_1 k_2/(a_2 k_1) = c_1/c_2 \), where the set \( D_b \) has the form as in Lemma 4.3. Finally, we summarize the global dynamics of the competition-diffusion system (28) with \((H1)\) in the following.

**Lemma 4.5.** Assume that the competition-diffusion system (28) satisfies the hypothesis \((H1)\). Then the following statements hold:

(i) if \((d_1, d_2) \in (\tilde{D}_\alpha \cup \tilde{D}_b)\), then the semi-trivial steady state \((0, \theta_{d_2,r_2}/c_2)\) is globally asymptotically stable, and system (28) has no positive coexistence steady states.

(ii) if \((d_1, d_2) \in (\mathbb{R}_+^2 \setminus (\tilde{D}_\alpha \cup \tilde{D}_b))\), then the semi-trivial steady state \((0, \theta_{d_2,r_2}/c_2)\) is unstable, and system (28) has a unique positive coexistence steady state \((\theta_{d_1,r_1}/b_1, \theta_{d_2,r_2}/c_2)\), which is globally asymptotically stable.
Then we obtain that \( (U, V) \) where

By applying the method of upper and lower solutions again, we obtain that

\[ \varepsilon > 0 \]

we have that if the competition-diffusion system (28) has a positive coexistence steady state, then it is globally asymptotically stable. We also observe that the semi-trivial steady state \((0, c_2)\) is not linearly unstable, then it is globally asymptotically stable.

### 4.3. Proof of Theorem 4.1

Now we are in the position to prove Theorem 4.1 by applying the method of upper and lower solutions, Theorem 4.2 and Lemma 4.5.

**Proof of Theorem 4.1**  

By Lemma 4.2, it suffices to consider the asymptotical behaviors of all solutions of system (11) with \( 0 \leq U_0(x) \leq M_1/b_1 \) and \( 0 \leq V_0(x) \leq (M_1 + b_1k_2)M_2/(a_2b_1) \) for each \( x \in \Omega \). Thus in the following we always assume that \((U_0, V_0) \in \mathcal{A}\). The detailed proof is divided into three different cases: (E1) \( a_1, k_2/(a_2k_1) \in [\beta, +\infty) \) and \( k_1 \leq k_2 \); (E2) \( a_1k_2/(a_2k_1) \in (0, \beta] \), \( k_1 \leq k_2 \) and \( (d_1, d_2) \in D_1 \cup D_2 \); (E3) \( a_1k_2/(a_2k_1) \in (0, \alpha), k_2 \geq k_1 \) and \( (d_1, d_2) \in D_1 \cup D_2 \).

**Case (E1):** If \( a_1k_2/(a_2k_1) \in [\beta, +\infty) \) and \( k_1 \leq k_2 \), then by Theorem 4.1, we have \( D_1, D_2 = \mathbb{R}^2 \), and \( \partial \Omega \times \mathbb{R}^+ \).

Consider the following systems:

\[
\begin{align*}
\frac{\partial U_{ij}}{\partial t} &= d_1 \Delta U_{ij} + U_{ij} \left( r_1(x) - b_1 U_{ij} - \gamma_{1j} V_{ij} \right) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\
\frac{\partial V_{ij}}{\partial t} &= d_2 \Delta V_{ij} + V_{ij} \left( r_2(x) - \eta_{1j} V_{ij} \right) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\
\frac{\partial U_{ij}}{\partial n} &= 0 \quad \text{on} \quad \partial \Omega \times \mathbb{R}^+, \\
U_{ij}(0, x) = U_0(x), \quad V_{ij}(0, x) = V_0(x) \quad \text{in} \quad \Omega.
\end{align*}
\]

where \( j = 1, 2 \). Define the constants \( \gamma_{1j} \) and \( \eta_{1j}, j = 1, 2 \), by

\[ \gamma_{11} = \frac{a_1}{k_1}, \quad \eta_{11} = \frac{a_2}{k_2}, \quad \gamma_{12} = \frac{a_1b_1}{M_1 + b_1k_1}, \quad \eta_{12} = \frac{a_2b_1}{M_1 + b_1k_2}. \]

Then by \( a_1k_2/(a_2k_1) \geq \beta \) and \( k_2 \leq k_1 \), we have that \( \gamma_{11}/\gamma_{11} \geq \beta \) and \( \gamma_{12}/\gamma_{12} \geq \beta \). This together with Lemma 4.4 (ii) and Lemma 4.5 (i) yields that for each \( j = 1 \), system (36) has a globally asymptotically stable steady state \((0, c_2)\) and has no positive coexistence steady states. Clearly, the solution \((U, V)\) of system (11) is an upper solution of system (36) with \( j = 1 \) and a lower solution of system (36) with \( j = 2 \), respectively. Then we obtain that

\[ U_{11}(t, x) \leq U(t, x) \leq U_{12}(t, x), \quad V_{11}(t, x) \leq V(t, x) \leq V_{12}(t, x), \quad t > 0, \quad x \in \Omega. \]

Thus for each small \( \varepsilon > 0 \), there exists a time \( t_1 > 0 \) such that \( 0 \leq U(t, x) < \varepsilon \) for each \( t \geq t_1 \) and \( x \in \Omega \), which implies that for \( t \geq t_1 \),

\[ V_{11}(r_2(x) - \eta_{11} V_{11}) \leq V \left( r_2(x) - \frac{a_2 V}{U + k_2} \right) \leq V \left( r_2(x) - \frac{a_1 V}{\varepsilon + k_2} \right), \quad t \geq t_1, \quad x \in \Omega. \]

By applying the method of upper and lower solutions again, we obtain that

\[ \frac{k_2}{a_2} \theta_{d_2r_2}(x) \leq \liminf_{t \to +\infty} V(t, x) \leq \limsup_{t \to +\infty} V(t, x) \leq \frac{k_2 + \varepsilon}{a_2} \theta_{d_2r_2}(x), \quad x \in \Omega. \]

Then we obtain that \((U(t, x), V(t, x)) \to (0, k_2 \theta_{d_2r_2}/a_2)\) as \( t \to +\infty \). Then this case is proved.
Case (E2'): If \( a_1 k_2 / (a_2 k_1) \in (\alpha, \beta) \), \( k_1 = k_2 \) and \((d_1, d_2) \in \mathcal{D}_+ \cup \mathcal{D}_0\), then

\[
\frac{\gamma_{11}}{\eta_{11}} = \frac{\gamma_{12}}{\eta_{12}} = \frac{a_1}{a_2} \in (\alpha, \beta),
\]

which together with Lemmas 4.4 and 4.5 yields that for each \( j = 1 \), system (36) has a globally asymptotically stable steady state \((0, \theta_{1,2}, \eta_{1,1})\) and has no positive coexistence steady states. Thus similarly to the case (E1'), we can prove (E2').

Case (E3'): If \( a_1 k_2 / (a_2 k_1) \in (0, a], k_1 \geq k_2 \) and \((d_1, d_2) \in \mathcal{D}_+ \cup \mathcal{D}_0\), then

\[
0 < \frac{\gamma_{11}}{\eta_{11}} = \frac{a_1 k_2}{a_2 k_1} \leq \alpha, \quad 0 < \frac{\gamma_{12}}{\eta_{12}} = \frac{a_1 k_2 (M_1 k_2 + b_1)}{a_2 k_1 (M_1 k_1 + b_1)} \leq \alpha. \tag{37}
\]

By the similar way used in the case (E2'), we obtain that this theorem holds in the case (E3'). Therefore, the proof is now complete.

\(\square\)

5. Analysis of the persistence and coexistence states

In this section, we always assume that the modified Leslie-Gower system (1) satisfies one of the following assumptions:

(A1) \( a_1 k_2 / (a_2 k_1) \in (0, \alpha], k_1 \leq k_2, (d_1, d_2) \in \mathbb{R}_+^2 \setminus \left\{ (d_1, d_2) \in \mathbb{R}_+^2 \mid r_1(x) = \alpha \theta_{d_1, d_2} \right\} \).

(A2) \( a_1 k_2 / (a_2 k_1) \in (0, \alpha], k_1 = k_2, (d_1, d_2) \in \mathbb{R}_+^2 \setminus \left\{ (d_1, d_2) \in \mathbb{R}_+^2 \mid r_1(x) = \alpha \theta_{d_1, d_2} \right\} \).

(A3) \( a_1 k_2 / (a_2 k_1) \in (\alpha, \beta), k_1 = k_2, (d_1, d_2) \in \left\{ (d_1, d_2) \in \mathbb{R}_+^2 \mid d_2 \in I_2, d_1 < \phi(d_2) \right\} \).

where the function \( \phi \) is defined as in Theorem 3.2. By applying the method of upper and lower solutions and the global dynamics of the competition-diffusion system (28), we prove the existence and the stability of a unique positive steady state, and establish the uniform persistence of the two species under some suitable conditions. The modified Leslie-Gower system (1) is said to be uniformly persistent [32, p.61] if there exists a positive constant \( \delta > 0 \) such that for each non-negative initial value \((U_0, V_0)\) with \( U_0(x) \neq 0 \) and \( V_0(x) \neq 0 \), the solution \((U, V)\) satisfies

\[
\liminf_{t \to +\infty} \min_{x \in \Omega} U(t, x) \geq \delta, \quad \liminf_{t \to +\infty} \min_{x \in \Omega} V(t, x) \geq \delta.
\]

The main results are stated as follows.

**Theorem 5.1.** Assume that that the modified Leslie-Gower system (1) satisfies the hypothesis (H1). Then the following statements hold:

(i) if either (A1) or (A3) holds, then system (1) is uniformly persistent.

(ii) if either (A2) or (A3) holds, then system (1) has at most one positive coexistence steady state. Further, if there exists a positive coexistence steady state, then it is a unique positive coexistence steady state, and is globally asymptotically stable.

To complete the proof for Theorem 5.1 we first consider the relation between the solutions of systems (1) and (36).

**Lemma 5.2.** Assume that either (A1) or (A3) holds. Then for each \((U_0, V_0) \in \mathcal{A}\), the solutions \((U, V)\) of system (1) and \((U_{1,j}, V_{1,j})\) of systems (36) with \( j = 1, 2 \), satisfy the following statements:
(i) $U_{11}(t, x) \leq U(t, x) \leq U_{12}(t, x), \ V_{11}(t, x) \leq V(t, x) \leq V_{12}(t, x), \ t > 0, \ x \in \Omega.$

(ii) Let $\tilde{r}_1$ be defined by $\tilde{r}_1(x) = r_1(x) - \gamma_{1j} / \eta_{1j}$ for $x \in \Omega$, then

$$\lim_{t \to \infty} (U_{11}(t, x), V_{11}(t, x)) = (\theta_{d_1, \tilde{r}_1}(x)/b_1, \theta_{d_2, \tilde{r}_1}(x)/\eta_{11}), \ j = 1, 2,$$ \hspace{1cm} (38)

and the following inequalities hold:

$$\theta_{d_1, \tilde{r}_1}(x)/b_1 \leq \theta_{d_1, \tilde{r}_1}(x)/b_1, \quad \theta_{d_2, \tilde{r}_1}(x)/\eta_{11} \leq \theta_{d_2, \tilde{r}_1}(x)/\eta_{11},$$ \hspace{1cm} (39)

Proof. Similarly to the case $\text{(E1')}$ in the proof of Theorem 4.1 we obtain that (i) holds.

To prove (ii), we only assume that the conditions stated in (A1) hold, the other case can be similarly discussed. By the similar method used in the proof of Theorem 4.1 we obtain that $\gamma_{1j} / \eta_{1j} \in (0, \alpha]$. This together with Lemma 4.3 the fact that $\mathcal{D}_0 = \mathcal{D}_0$ for $a_1k_2 + (a_2k_1) = c_1/c_2$ and Lemma 4.4 yields that $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \emptyset$ and $\mathcal{D}_4 = \{ (d_1, d_2) \in \mathbb{R}_+^2 \mid r_1(x) \equiv \alpha \theta_{d_2, \tilde{r}_1} \}$. Since $(d_1, d_2) \in \mathbb{R}_+^2 \setminus \{ (d_1, d_2) \in \mathbb{R}_+^2 \mid r_1(x) \equiv \alpha \theta_{d_2, \tilde{r}_1} \}$ in (A1), then $(d_1, d_2) \in (\mathbb{R}_+^2 \setminus ( \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 ))$. Thus by Lemma 4.5 (ii) the limits in (38) hold. The left statements in (ii) can be obtained by applying (37) and (i) in this lemma. Therefore, the proof is now complete.

We next show the limit behavior of $U(t, x)$ as $t$ tends to infinity under the assumption that either (A2) or (A3) holds. More precisely, we have the following lemma.

Lemma 5.3. Assume that either (A2) or (A3) holds. Then there exists a unique positive function $U^*(x)$ for $x \in \Omega$, which is independent of $(U_0, V_0)$, such that the solution $(U, V)$ of the modified Leslie-Gower system (7) satisfies that

$$\lim_{t \to \infty} U(t, x) = U^*(x), \ x \in \Omega.$$ \hspace{1cm} (40)

Proof. Since either (A2) or (A3) holds, then by Lemma 5.2 we have that

$$\theta_{d_1, \tilde{r}_1}(x)/b_1 \leq \liminf_{t \to \infty} U(t, x) \leq \limsup_{t \to \infty} U(t, x) \leq \theta_{d_1, \tilde{r}_1}(x)/b_1,$$ \hspace{1cm} (40)

Since $k_1 = k_2$, then

$$\gamma_{11} / \eta_{11} = \gamma_{12} / \eta_{12} = a_1 / a_2,$$

which yields that

$$\tilde{r}_1(x) = r_1(x) - \gamma_{11} / \eta_{11} \theta_{d_2, \tilde{r}_1}(x) \equiv r_1(x) - \gamma_{12} / \eta_{12} \theta_{d_2, \tilde{r}_1}(x) \equiv \tilde{r}_1(x).$$

Hence, by the above equalities and (40) the proof is finished.

In the end of this section we give the proof for Theorem 5.1

Proof of Theorem 5.1. If either (A1) or (A3) holds, then by Lemma 5.2 we have that

$$\liminf_{t \to \infty} \min_{x \in \mathbb{R}} U(t, x) > \delta > 0, \quad \liminf_{t \to \infty} \min_{x \in \mathbb{R}} V(t, x) > \delta > 0,$$
where \( \delta \) satisfies that

\[
2\delta = \min_{x \in \Omega} \left\{ \min_{x \in \partial \Omega} \frac{\theta_{d_1,r_1}(x)}{b_1}, \min_{x \in \partial \Omega} \frac{\theta_{d_2,r_2}(x)}{\eta_{11}} \right\}.
\]

Then (i) is proved.

Assume that system (1) has a positive coexistence steady state \((u^*, v^*)\), then \(u^* = U^*\), where \(U^*\) is a positive function obtained as in Lemma 5.3. To obtain \(v^*\) we consider the elliptic equation in the form

\[
\begin{aligned}
&d_2 \Delta V + V \left( r_2(x) - \frac{a_2 V}{U^*(x) + k_2} \right) = 0 \quad \text{in } \Omega \times \mathbb{R}_+,
&\frac{\partial V}{\partial n} = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+,
\end{aligned}
\]

which is equivalent to

\[
\begin{aligned}
&d_2 \Delta V + \frac{V}{U^*(x) + k_2} \left( r_2(x)(U^*(x) + k_2) - a_2 V \right) = 0 \quad \text{in } \Omega \times \mathbb{R}_+,
&\frac{\partial V}{\partial n} = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+.
\end{aligned}
\]

(41)

Let the constants \(W_0\) and \(W^*\) be given by

\[
W_0 = \min_{x \in \Omega} \frac{r_2(x)(U^*(x) + k_2)}{a_2}, \quad W^* = \max_{x \in \Omega} \frac{r_2(x)(U^*(x) + k_2)}{a_2}.
\]

Since \(U^*\) is a positive function and \(r_2\) is a non-negative function with \(\tau_2 > 0\), then \(W_0 \geq 0\) and \(W^* > 0\), which implies that \(W_0\) is a lower solution of (41) and \(W^* > 0\) is an upper solution. Consequently, by the method of upper and lower solutions, we have that (41) has a unique positive solution \(v^*\), which is globally asymptotically stable. This together with Lemma 5.3 yields that (ii) holds. Therefore, the proof is finished. \(\square\)

6. Concluding remarks

In this paper we have investigated the dynamics of a diffusive predator-prey system with modified Leslie-Gower and Holling-type II schemes in spatially heterogeneous environment. By applying the principal spectral theory, we obtain that the trivial steady state \((0, 0)\) and the semi-trivial steady state \((\theta_{d_1,r_1}, 0)\) are always linearly unstable. Similarly to the conclusion obtained by He and Ni in [13], we also prove that the semi-trivial steady state \((0, k_2\theta_{d_2,r_2}/a_2)\) is globally asymptotically stable when it is not linearly unstable. These results show that the prey \(U\) could die out and the predator \(V\) is always persistent. When \((0, k_2\theta_{d_2,r_2}/a_2)\) is linear unstable, we give some suitable conditions under which two species could be persistent or coexistent. It is also interesting to give a more detailed study of the positive coexistence steady states. For example, if the modified Leslie-Gower system (1) satisfies the following conditions:

\[
\frac{a_1 k_2}{a_2 k_1} \in (0, \alpha], \quad k_2 > k_1, \quad (d_1, d_2) \in \mathbb{R}_+^2 \setminus \{(d_1, d_2) \in \mathbb{R}_+^2 | r_1(x) \equiv a\theta_{d_2,r_2}\}.
\]

By Theorem 3.2, we see that \((0, k_2\theta_{d_2,r_2}/a_2)\) is linearly unstable. We conjecture that under the above conditions the modified Leslie-Gower system (1) has at most one positive coexistence steady state. Further, if it exists, we conjecture that it is globally asymptotically stable.
References

References

[1] W. Abid, R. Yafia, M. A. Aziz-Alaoui, H. Bouhafa, A. Abichou, Diffusion driven in stability and Hopf bifurcation in spatial predator-prey model on a circular domain, *Appl. Math. Comput.* **260** (2015), 292–313.

[2] E. J. Avila-Vales, R. S. Cantrell, Permanence of three competitors in seasonal ecological models with spatial heterogeneity, *Canad. Appl. Math. Quart.* **5** (1997), 145–169.

[3] M. A. Aziz-Alaoui, M. Daher Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes. *Appl. Math. Lett.* **16** (2003), 1069–1075.

[4] B. I. Camara, M. A. Aziz-Alaoui, Dynamics of predator-prey model with diffusion, *Dyn. Cont. Discret. Impul. Syst.* **15** (2008), 897–906.

[5] R. S. Cantrell, C. Cosner, The effects of spatial heterogeneity in population dynamics, *J. Math. Biol.* **29** (1991), 315–338.

[6] R. S. Cantrell, C. Cosner, *Spatial ecology via reaction-diffusion equations*, Wiley Ser. Math. Comput. Biol., John Wiley and Sons, Ltd., Chichester, 2003.

[7] D. L. DeAngelis, W.-M. Ni, B. Zhang, Dispersal and spatial heterogeneity: single species. *J. Math. Biol.* **72** (2016), 239–254.

[8] M. Daher Okiye, *Study and Asymptotic Analysis of Some NonLinear Dynamical Systems: Application to Predator-prey Problems* (Ph.D.thesis), Le Havre University, France, 2004 (in French).

[9] J. Dockery, V. Hutson, K. Mischaikow, M. Pernarowski, The evolution of slow dispersal rates: A reaction-diffusion model, *J. Math. Biol.* **37** (1998), 61–83.

[10] Y. H. Du, S. B. Hsu, A diffusive predator-prey model in heterogeneous environment, *J. Differential Equations* **203** (2004), 331–364.

[11] Y. H. Du, J. P. Shi, Allee effect and bistability in a spatially heterogeneous predator-prey model, *Trans. Am. Math. Soc.* **359** (2007), 4557–4593.

[12] J. Giné, C. Valls, Nonlinear oscillations in the modified Leslie-Gower model, *Nonlinear Anal. Real World Appl.* **51** (2020), 103010.

[13] X. He, W.-M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system I: Heterogeneity vs. homogeneity, *J. Differential Equations* **254** (2013), 528–546.

[14] X. He, W.-M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system II: The general case, *J. Differential Equations* **254** (2013), 4088–4108.

[15] X. He, W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system: diffusion and spatial heterogeneity I, *Comm. Pure Appl. Math.* **69** (2016), 981–1014.

[16] V. Hutson, J. López-Gómez, K. Mischaikow, G. Vickers, Limit behaviour for a competing species problem with diffusion, *Dynamical systems and applications*, 343–358, World Sci. Ser. Appl. Anal. **4**, World Sci. Publ., River Edge, NJ, 1995.

[17] J. Huang, G. Lu, S. Ruan, Existence of traveling wave solutions in a diffusive predator prey model, *J. Math. Biol.* **46** (2003), 132–152.

[18] D. Jiang, K.-Y. Lam, Y. Lou, Z. Wang, Monotonicity and global dynamics of a nonlocal two-species phytoplankton model, *SIAM J. Appl. Math.* **79** (2019), 716–742.

[19] K.-Y. Lam, W.-M. Ni, Uniqueness and complete dynamics in heterogeneous competition-diffusion systems. *SIAM J. Appl. Math.* **72** (2012), 1695–1712.

[20] K.-Y. Lam, Y. Lou, F. Lutscher, Evolution of dispersal in closed advective environments. *J. Biol. Dyn.* **9** (2015), 188–212.

[21] P. H. Leslie, Some further notes on the use of matrices in population mathematics, *Biometrika* **35** (1948), 213–245.

[22] S. Liang, Y. Lou, On the dependence of population size upon random dispersal rate, *Discr. Cont. Dyn. Syst. Ser. B.* **17** (2012), 2771–2788.

[23] A. J. Lotka, Contribution to the theory of periodic reaction, *J. Phys. Chem.* **14** (1910), 271–274.

[24] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differential Equations* **233** (2006), 400–426.

[25] Y. Lou, B. Wang, Local dynamics of a diffusive predator-prey model in spatially heterogeneous environment. *J. Fixed Point Theory Appl.* **19** (2017), 755–772.

[26] Y. Lou, X. Zhao, P. Zhou, Global dynamics of a Lotka-Volterra competition-diffusion-advection system in heterogeneous environments, *J. Math. Pures Appl.* **121** (2019), 47–82.

[27] W.-M. Ni, *The mathematics of diffusion*, CBMS-NSF Regional Conf. Ser. in Appl. Math., **82**, SIAM, Philadelphia, 2011.

[28] A. F. Nindjin, M. A. Aziz-Alaoui, M. Cadivel, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with delay, *Nonlinear Anal. Real World Appl.* **7** (2006), 1104–1118.
[29] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Springer, New York, 1992.

[30] R. Peng, J. Shi, Non-existence of non-constant positive steady states of two Holling type-II predator-prey systems: strong interaction case. *J. Differential Equations* **247** (2009), 866–886.

[31] H. Smith, *Monotone dynamical systems. An introduction to the theory of competitive and cooperative systems*, Math. Sur. and Monog. **41**, Amer. Math. Soc., Providence, RI, 1995.

[32] H. Smith, H. Thieme, *Dynamical Systems and Population Persistence*, Graduate Studies in Math. **118**, Amer. Math. Soc., Providence, RI, 2011.

[33] D. H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.** 21** (1972), 979–1000.

[34] D. Tang, P. Zhou, On a Lotka-Volterra competition-diffusion-advection system: Homogeneity vs heterogeneity, *J. Differential Equations* **268** (2020), 1570–1599.

[35] V. Volterra, Variazioni e fluttuazioni del numero d’individui in specie animali conviventi, *Mem. Acad. Lincei Roma*. 2 (1926), 31–113.

[36] V. Volterra, Variations and fluctuations of the number of individuals in animal species living together, in: Chapman, R. N. *Animal Ecology*, McGraw-Hill, 1931.

[37] Y. Wang, W. T. Li, Effects of cross-diffusion and heterogeneous environment on positive steady states of a prey-predator system, *Nonlinear Anal. Real World Appl.* **14** (2013), 1235–1246.

[38] J. Zhang, R. Cui, Qualitative analysis on a diffusive SIS epidemic system with logistic source and spontaneous infection in a heterogeneous environment, *Nonlinear Anal. Real World Appl.* **55** (2020), 103115.

[39] J. Zhao, M. Wang, A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, *Nonlinear Anal. Real World Appl.* **16** (2014), 250-263.

[40] P. Zhou, D. Xiao, Global dynamics of a classical Lotka-Volterra competition-diffusion-advection system, *J. Funct. Anal.* **275** (2018), 356–380.

[41] Y. Zhu, K. Wang, Existence and global attractivity of positive periodic solutions for a predator-prey model with modified Leslie-Gower Holling-type II schemes, *J. Math. Anal. Appl.* **384** (2011), 400–408.