Yu Wang · Marjan Mirahmadi · Ahmed A. Elkamshishy · Jesús Pérez-Ríos

Hyperradial Distribution Function of Few-Body Problems: A New Arena for Extreme Value Theory

Abstract This work explores classical capture models for few-body systems via a Monte Carlo method in hyperspherical coordinates. In particular, we focus on van der Waals and charged-induced dipole interactions. As a result, we notice that, independently of the number of particles and interparticle interaction, the capture hyperradial distribution function follows a Fréchet distribution, a special type of the generalized extreme value distribution. Besides, we elaborate on the fundamentals of such universal feature using the general extreme value theory, thus, establishing a connection between extreme value theory and few-body physics.

1 Introduction

Classically, when the interaction potential energy is comparable to the colliding partners’ kinetic energy, the trajectory drawn by the particles deviates from the uniform rectilinear motion, leading to a collision [1,2]. The distance at which this occurs defines the range of the interaction, also known as the capture radius in the language of classical capture models. Then, using the capture radius, it is possible to define a threshold law for scattering observables such as the cross section and reaction rate.¹ A prime example of a classical capture is the well-known Langevin rate for charged-neutral collisions [3], although it includes the effects of the centrifugal barrier.

Recently, classical capture models have been introduced within the framework of direct three-body recombination processes $A + A + A \rightarrow A_2 + A$ [4–8], also known as ternary association reactions, i.e., the formation of a molecule from a collision of three free atoms. In this context, direct means that the reaction occurs without invoking the existence of an intermediate two-body complex that the third body may stabilize or dissociate. In particular, after using hyperspherical coordinates to characterize the configuration of the particles, in which

¹ This is strictly true in the case of zero impact parameter, or null angular momentum collision.
the hyperradius emerges as a natural generalization of the radius in spherical coordinates, a distribution of capture hyperradius is found for given collision energy. Moreover, and surprisingly enough, the distribution of capture hyperradius follows a generalized extreme value (GEV) distribution [9] independently of the collision energy. However, it has not yet explained why a GEV distribution appears. Also, classical capture models have not been introduced in the framework of four-body collisions and beyond.

In this work, we compute the capture hyperradius distribution for three- four- and five-body collisions using different interatomic interactions through a Monte Carlo approach sampling the different initial particle configurations of the N-body system. As a result, we find the GEV distribution in all studied scenarios independently of the collision energy, the number of particles, and interparticle interaction. Therefore, our results indicate that the shape of the hyperradial distribution function is a universal property of few-body systems. Additionally, we find that the extreme value theory applies to few-body systems and is key to understanding hyperradial distributions. The paper is organized as follows: Sect. 2 presents our methodology to extract the hyperradial distribution function; Sect. 3 shows our results that are further analyzed and discussed in Sect. 4. Finally, a summary of our chief findings and conclusions are discussed in Sect. 5.

2 Theoretical Framework

The dynamics of a system consisting of N particles (in a 3-dimensional space) with masses $m_i (i = 1, ..., N)$, interacting via potential $U(\vec{r}_1, ..., \vec{r}_N)$ is governed by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m_i} + U(\vec{r}_1, ..., \vec{r}_N),$$

(1)

where $\vec{r}_i$ and $\vec{p}_i$ are the position and momentum vectors of the $i$-th particle, respectively.

The classical capture radius is estimated as the distance for which the interaction potential energy is of the same order as the kinetic energy of a scattering system. Therefore, to find the classical capture radius for N-body processes we need to introduce the potential first.

2.1 Interaction Potential in N-Body Systems

Let us assume a system of N particles interacting through pair-wise potentials $U(r_{ij})$, wherein the relative distance $r_{ij}$ is

$$r_{ij} = ||\vec{r}_j - \vec{r}_i||,$$

(2)

such that the total interaction potential of the system reads as

$$U(\vec{r}_1, ..., \vec{r}_N) = \sum_{i=1}^{N} \sum_{j=i}^{N} U(r_{ij}).$$

(3)

Next, as customary on treating few-body systems from a chemical physics standpoint, we introduce the Jacobi coordinates given as

$$\vec{\rho}_1 = \vec{r}_2 - \vec{r}_1$$

$$\vec{\rho}_2 = \vec{r}_3 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vdots$$

$$\vec{\rho}_{N-1} = \vec{r}_N - \frac{\sum_{i=1}^{N-1} m_i \vec{r}_i}{\sum_{i=1}^{N-1} m_i}$$

$$\vec{\rho}_{CM} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}$$

(4)

which lead to the interaction potential $U(\vec{\rho}_1, ..., \vec{\rho}_{N-1})$. 
In the framework of classical capture models, one way to determine the characteristic interaction range at a given collision energy $E_{\text{col}}$ is to solve the equation $|U(\vec{\rho}_1, ..., \vec{\rho}_{N-1})| = E_{\text{col}}$. By doing so, we find different particle configurations associated with the same collision energy. This leads to a complicated distribution of interparticle distances, which is hard to analyze.

2.2 N-body Scattering Problem

It is well-known that the dynamics of an N-body problem in a 3-dimensional space can be mapped onto a single particle dynamics in a D-dimensional space, with $D = 3 \times N - 3$, i.e., three spatial degrees of freedom per particle minus the degrees of freedom associated with the trivial center of mass motion. In particular, following the pioneering work of Smith and Shui [10–13], it is possible to represent the position of N particles of the system via a single D-dimensional vector given by

$$
\vec{\rho} = \begin{pmatrix}
\vec{\rho}_1 \\
\vec{\rho}_2 \\
\vdots \\
\vec{\rho}_{N-1}
\end{pmatrix},
$$

(5)

as illustrated for a four-body system in Fig. 1. As a result, the interaction potential of the system is given by $U(\vec{\rho})$.

Similarly, it is possible to define the momentum vector in the same space as

$$
\vec{P} = \begin{pmatrix}
\vec{\pi}_1 \\
\vec{\pi}_2 \\
\vdots \\
\vec{\pi}_{N-1}
\end{pmatrix},
$$

(6)

where $\vec{\pi}_i$ stands for the conjugated momentum of $\vec{\rho}_i$ and $\kappa$ is a dimensionless parameter including different reduced masses according to the number of particles N. Finally, using Eqs. (5) and (6) the Hamiltonian (in the center-of-mass frame) in a D-dimensional space is written as

$$
H = \frac{\vec{P}^2}{2\mu} + U(\vec{\rho}) ,
$$

(7)

where $\mu = \left(\prod_{i=1}^{N} m_i / \sum_{i=1}^{N} m_i\right)^{1/(N-1)}$ is the reduced mass of the N-particle system.

Therefore, we can uniquely define the initial conditions of N-body collisions as single entities in a D-dimensional space. Furthermore, the cross section corresponding to the N-body collision is defined as the area of a (D-1)-dimensional hyperplane, described by the impact parameter, embedded in the D-dimensional space.
In this scenario, the cross section as a function of the magnitude of the momentum, $P$ (proportional to the collision energy), is given by [2,4]

$$\sigma(P) = \frac{\int \mathcal{P}(\vec{P}, \vec{b})b^{D-2}db \, d\Omega_b \, d\Omega_P}{\int d\Omega_P},$$  \hspace{1cm} (8)

where $\vec{b}$ is the impact parameter vector in the D-1-dimensional space, which is defined as the projection of the initial position vector $\vec{\rho}$ on the hyperplane perpendicular to the initial momentum vector. $d\Omega_b$ and $d\Omega_P$ denote the differential elements of the solid hyperangles associated with vectors $\vec{b}$ and $\vec{P}$, respectively. Finally, in Eq. (8), we find the so-called opacity function $\mathcal{P}$, which represents the probability of reaction toward the desired product state as a function of the impact parameter and the initial momentum. For instance, in the case of three-body recombination, the opacity function in Eq. (8) represents the probability of forming a molecule out of a three-body collision.

2.3 Hyperspherical Coordinates

Hyperspherical coordinates are a natural generalization of spherical coordinates into higher dimensional space. In particular, following Avery’s definition of hyperangles [14], the components of a D-dimensional vector are expressed as

$$\begin{align*}
x_1 &= \rho \sin \alpha_1 \sin \alpha_2 \cdots \sin \alpha_{D-3} \sin \alpha_{D-2} \\
x_2 &= \rho \cos \alpha_1 \sin \alpha_2 \cdots \sin \alpha_{D-3} \sin \alpha_{D-2} \\
x_3 &= \rho \cos \alpha_2 \cdots \sin \alpha_{D-3} \sin \alpha_{D-2} \\
\vdots \\
x_D &= \rho \cos \alpha_{D-1}
\end{align*}$$  \hspace{1cm} (9)

where $0 \leq \alpha_1 < 2\pi$ and $0 \leq \alpha_i \neq 1 \leq \pi$ stand for the D-1 hyperangles, whereas $\rho$ is the hyperradius. Similarly, the infinitesimal volume element is given by

$$d\tau = \rho^{D-1}d\rho \prod_{i=1}^{D-1} \sin^{i-1} \alpha_i d\alpha_i. \hspace{1cm} (10)$$

2.4 Capture Hyperradial Distribution for N-Body Systems

Via Eq. (4), it is possible to express the position of the particles ($\vec{r}_i$) as a function of the Jacobi vectors and with it, the interparticle distance. Next, using hyperspherical coordinates to describe the D-dimensional space (see Eq. (9)), we find

$$U(\vec{\rho}) = U(\rho, \vec{\alpha}) = \sum_{i=1}^{N} \sum_{j>i}^{N} U(r_{ij}(\rho, \vec{\alpha}))$$  \hspace{1cm} (11)

where $\vec{\alpha} = (\alpha_1, ..., \alpha_{D-1})$. Finally, solving

$$|U(\rho, \vec{\alpha})| = E_{\text{col}},$$  \hspace{1cm} (12)

leads to the capture hyperradius for a given set of hyperangles. Indeed, every configuration controlled by the hyperangles gives a different hyperradius, leading to a capture hyperradial distribution rather than a single capture hyperradius. However, it still is possible to assign a single capture hyperradius to each collision energy as the maximum likelihood of the distribution. Therefore, it is possible to calculate the long-range hyperradial potential as $U(\rho) \propto \rho^\gamma$ for each collision energy. Consequently, by choosing different $E_{\text{col}}$ in Eq. (12), we find $\gamma$. Note that $\gamma$ appears in capture models establishing threshold laws for the cross section and reaction rate. For instance, in the three-body recombination process, the threshold value of the cross section is given by $\propto E_{\text{col}}^{3/\gamma}$. For a more detailed discussion of threshold laws, we recommend looking into Ref. [8] for further details.
3 Methods

Herein, we explore $N$-body collisions in which one particle ($A$) is different from the rest ($B$). In particular, we envision a system of $N$ particles interacting via van der Waals (vdW) interactions described by

$$U(\rho, \vec{a}) = -\sum_{i \neq n}^{N-1} \frac{C_{A-B}^{A-B}}{r_{ij}^{6}(\rho, \vec{a})} - \sum_{i \neq n}^{N-2} \sum_{j \neq n > i}^{N-2} \frac{C_{B-B}^{B-B}}{r_{ij}^{6}(\rho, \vec{a})}, \quad (13)$$

where $C_{A-B}^{A-B}$ and $C_{B-B}^{B-B}$ stand for the vdW coefficient between $A$ and $B$ particles and $B–B$ particles, respectively. In Eq. (13) it is assumed that the $A$ particle is placed at $\vec{r}_n$. Indeed, the choice for labeling the particles in the system does not affect the results since we are interested in the energy landscape.

The second case under study is based on ion-neutral interactions, in which the $A$ particle is positive charged, whereas $B$ particles are neutral. Then, the interaction potential reads as

$$U(\rho, \vec{a}) = -\sum_{i \neq n}^{N-1} \frac{C_{A-B}^{A-B}}{r_{ij}^{6}(\rho, \vec{a})} - \sum_{i \neq n}^{N-2} \sum_{j \neq n > i}^{N-2} \frac{C_{B-B}^{B-B}}{r_{ij}^{6}(\rho, \vec{a})}, \quad (14)$$

wherein $C_{A-B}^{A-B}$ is the charged induced dipole long-range interaction coefficient that only depends on the polarizability of the atoms.

We use a Monte Carlo approach to generate the capture hyperradial distribution by generating random hyperangles. In each Monte Carlo step, we generate a set of hyperangles via the inverse transformation method to generate random numbers based on its probability density function [15,16]. This is obtained from the hyperangular part of the differential volume element given by Eq. (10) normalized via a constant $k_i$ associated with $\alpha_i$ as

$$k_i = \begin{cases} \frac{1}{2\pi}, & i = 1 \\ \frac{1}{\int_0^\pi \sin^{-1} \alpha_i d\alpha}, & i > 1 \end{cases} \quad (15)$$

Next, in virtue of Eq. (5) and Eq. (9) and using the generated $\vec{a}$, we find the Jacobi vectors associated with the generated configuration. Then, solving Eq. (4) for $\vec{r}_i$ ($i=1, \ldots, N$) allows us to obtain the interparticle vectors $\vec{r}_{ij}(\rho, \vec{a})$. With this, we have all the ingredients to solve $|U(\rho, \vec{a})| = E_{\text{col}}$ for $\rho$. In this way, for each Monte Carlo step, we obtain a capture hyperradius associated with $E_{\text{col}}$. Note that the interaction coefficients in Eqs. (13) and (14) are considered as constant parameters in the calculations.

4 Results and Discussion

The capture hyperradial distribution for $N^+–\text{Ar–Ar}$, $N^+–\text{He–He}$ and $\text{N–He–He}$ three-body collisions at $E_{\text{col}} = 1$ mK have been obtained, and the results are displayed in Fig. 2. Here, we use $C_{4}^{N–\text{He}} = 0.69$ a.u., $C_{4}^{\text{Ar–Ar}} = 5.54$ a.u., $C_{6}^{\text{He–He}} = 1.35$ a.u. [17], $C_{6}^{\text{Ar–Ar}} = 64.3$ a.u. [18], and $C_{6}^{N–\text{He}} = 5.7$ a.u. [19], to calculate the capture hyperradius. In Fig. 2, it is noticed that three-body processes involving charged-neutral interactions lead to a broader hyperradial distribution than solely vdW interactions. Similarly, the mode of the distributions involving a charged particle occurs at a larger hyperradius than in the case of vdW interactions. As expected, the mode and the width of the distribution are attributed to the stronger long-range nature of charge-neutral interactions ($1/r^4$) versus vdW interactions ($1/r^6$). Furthermore, it turns out that the hyperradial distribution is always described by a GEV distribution with the probability density function (PDF) given by (depicted by the red solid line in the figures)

$$f(\rho) = \frac{1}{\delta} \exp \left[ - \left( 1 + \xi \frac{\rho - \beta}{\delta} \right)^{-\frac{1}{\xi}} \right] \left( 1 + \xi \frac{\rho + \beta}{\delta} \right)^{-1-\frac{1}{\xi}}, \quad (16)$$

with $1 + \xi \frac{\rho - \beta}{\delta} > 0$. Here, $\xi$ is the shape parameter that is directly related to the assignment of different types of GEV distribution: Weibul, Fréchet and Gumbel, for $\xi = 0$, $\xi > 0$ and $\xi < 0$, respectively; $\delta$ is the scale
Fig. 2 Hyperradial distribution for a collision energy of 1 mK. Probability density function (PDF) associated with N$^+$–Ar–Ar [panel (a)], N$^+$–He–He [panel (b)] and N–He–He [panel (c)]. The solid lines represent the fitting to a GEV distribution, whereas the mode is indicated by the vertical lines: are 440 a$_0$, 203 a$_0$ and 52 a$_0$ for panels (a), (b) and (c), respectively.

To study the role of the number of particles on the hyperradial distribution, we have computed the hyperradial distribution for As–He–He, As–He–He–He and As–He–He–He–He. In particular, we include vdW-type interactions following Eq. (13), wherein A=As and B=He. Employing $C_{\text{As-He}}^6 = 17.16$ a.u [19] and $C_{\text{He-He}}^6 = 1.35$ a.u. [17], we obtain the results shown in Fig. 3. Analogously to three-body collisions shown in Fig. 2, the hyperradial distributions are well described via a GEV distribution. However, in this case, the width of the distribution shows a clear dependence on the number of particles, whereas the mode shows a more subtle dependence. By comparing the changes in panels (b) and (c) of Figs. 2 and 3, we notice that, even though the mode and width of the hyperradial distribution depend on both number of particles and interparticle interactions, they are more sensitive to the changes in the latter.

Similarly, Fig. 4 shows our results for the hyperradial distribution for three- four- and five-body collisions involving charged-neutral interactions following Eq. (14), wherein A=N$^+$ and B=He with $C_{\text{N-He}}^4 = 0.69$ a.u and $C_{\text{He-He}}^6 = 1.35$ a.u. [17]. As in previous cases, a GEV describes the computed distribution, in which the width and the mode depend on the number of colliding particles, although, here the relationship is weak in comparison with vdW-type interactions. In other words, the number of particles has a minimum effect on the hyperradial distribution when one of them is a charged particle in comparison with the scenario of vdW-type interactions.

Next, we analyze the parameters of the GEV as a function of the collision energy for a different number of particles and interparticle interactions. In particular, the collision energies spanned between 1 mK and 100 K, thus covering collisions between the cold and the thermal regimes. Fig. 5 displays the shape parameter, $\xi$, as a function of the collision energy for different few-body systems. In particular, panel (a) shows the three different three-body systems as in Fig. 2, whereas panel (b) covers the systems displayed in Fig. 3, i.e., N-body systems with $N = 3, 4, 5$. It is worth noting that, independently of the interparticle interaction and number of particles, $\xi$ is positive, thus, the GEV distribution is a Fréchet type ($\xi > 0$) in all cases. Furthermore, $\xi$ shows a constant value for energies below 10 K, i.e., the shape of the hyperradial distribution remains the same within a given interaction model (number of particles and interparticle interactions). However, for $E_{\text{col}} \gtrsim 10$ K, $\xi$ increases or decreases monotonically.

The study on the relationship between $\delta$ and $\beta$ parameters of the GEV distribution with the collision energy is shown in Fig. 6, for three-body systems presenting two different types of interparticle interactions. In this figure, it is noticed that, as opposed to the results for the parameter $\xi$, these parameters strongly depend on the collision energy. In particular, these parameters have a power-law relationship with the collision energy. To
Fig. 3 Hyperradial distribution for a collision energy of 1 mK as a function of the number of colliding particles. Panel (a) shows the Probability density function (PDF) for As-He-He, whereas panel (b) and (c) show the results for As-He-He-He and As-He-He-He-He, respectively. The solid lines represent the fitting to a GEV distribution, whereas the mode is indicated by the vertical lines: 64 \( a_0 \), 92 \( a_0 \) and 123 \( a_0 \) for panels (a), (b) and (c), respectively.

Fig. 4 Hyperradial distribution for a collision energy of 1 mK as a function of the number of colliding particles. Panel (a) shows the Probability density function (PDF) for N\(^+\)-He-He, whereas panel (b) and (c) show the results for N\(^+\)-He-He-He and N\(^+\)-He-He-He-He, respectively. The solid lines represent the fitting to a GEV distribution, whereas the mode is indicated by the vertical lines: 201 \( a_0 \), 273 \( a_0 \) and 311 \( a_0 \) for panels (a), (b) and (c), respectively.
study this in detail, we have extended our analysis to different few-body systems with different interactions, and the corresponding parameters of the GEV distribution ($\xi$, $\beta$ and $\delta$) are listed in Table 1. In accordance with the results depicted in Fig. 5, we notice that for each system under consideration, the parameter $\xi$ is almost independent of the collision energy (the maximum and minimum values of the variation interval are listed in the second and third columns of Table 1). On the other hand, we find that $\beta$ and $\delta$ follow a power-law relationship with the collision energy as $a \times E_{\text{col}}^b$, independently of the interparticle interaction. Intriguingly, we only observe two powers: $b = -\frac{1}{6}$ for vdW-type interactions and $b = -\frac{1}{4}$ for systems containing charged-neutral interactions, independently of the number of particles. Therefore, assuming that $\xi$ is energy independent, the mode of the distribution given by

$$\rho_m = \beta + \frac{\delta}{\xi} \left((1 + \xi)^{-\xi} - 1\right),$$

yields $\rho_m \propto E_{\text{col}}^{-\frac{1}{6}}$ and $\rho_m \propto E_{\text{col}}^{-\frac{1}{4}}$ for systems including vdW-type interactions and charged-neutral interactions, respectively. In other words, the most probable value of the capture hyperradius only depends on the long-range tail of the dominant inter-particle interactions, as it has been argued before [4,5,20], but never specifically proved.

4.1 Fréchet Distribution

All the simulations above show that capture hyperradial distribution follows the Fréchet distribution within the GEV family, i.e., $\xi > 0$. Throughout this section, we try to obtain these results independent of our observation and using the extreme value theory.

Each point in the D-dimensional space, indicated by a hyperradius and D-1 hyperangles, corresponds to a set of all possible configurations of the N particles in the 3-dimensional space. The hyperangles specify the shape of the configuration (e.g., in the 3-body case, it is a triangle). In contrast, the hyperradius describes the size (in a geometrical way) of the N-body system (the enlarging or shrinking of the atomic configuration by the same scale factor in all directions).
Fig. 6 \(\delta\) [panel (a)] and \(\beta\) [panel (b)] hyperradial distribution parameters as a function of the collision energy. The error bars represent the fitting uncertainty on this parameter. The dashed-lines represent the best fit of the data points to a power-law function, \(aE_{\text{col}}^b\).

Table 1 Relationship between parameters of the hyperradial distribution function and the collision energy for few-body systems including vdW-type and charged-neutral interactions

| System        | \(\xi\) | \(\beta\) | \(\delta\) |
|---------------|---------|-----------|------------|
| N-He-He       | 0.610   | 0.667     | 0.610      |
| \(N^7\)-He-He | 0.560   | 0.700     | 0.560      |
| N-3He         | 0.319   | 0.403     | 0.319      |
| N-4He         | 0.327   | 0.422     | 0.327      |
| N-Ar-Ar       | 0.487   | 0.583     | 0.487      |
| As-He-He      | 0.674   | 0.646     | 0.674      |
| As-3He        | 0.423   | 0.446     | 0.423      |
| As-4He        | 0.416   | 0.446     | 0.416      |

\(\xi\) only varies from system to system, but it is independent of the collision energy, and \(\beta\) and \(\delta\) follow a power-law relationship as a function of the collision energy as \(aE_{\text{col}}^b\).

As explained above, to find the capture hyperradius, a set of hyperangles (\(\vec{\alpha}\)) are randomly generated; those define the shape of the atomic configuration. Once the hyperangles are chosen, the hyperradius, as free of constraints, can be chosen randomly from a set of independent and identically distributed variables. In particular, by generating each set of \(\vec{\alpha}\), a sequence of hyperradius is available, and by solving Eq. (12), we choose the capture hyperradius. From a physical perspective, the “capture” hyperradius obtained from this equation is the maximum possible value of the hyperradius for a given reaction to occur. In other words, for a hyperradius larger than the capture hyperradius, the probability of the reaction event happening is zero. Finally, repeating this procedure for a large enough number of Monte Carlo steps, we generate a distribution of capture hyperradius.

According to the extreme value theorem, the only possible limit distribution of the capture radius distribution, i.e., the maxima of a sequence of independently and identically distributed random variables \(\rho\), is the GEV distribution. More precisely, based on the Fisher-Tippett-Gnedenko theorem [21–24], the limiting distribution of such variables (the extreme value distribution) after proper normalization is either Gumbel,
Fréchet, or Weibull distribution. These three distributions have different behavior over their range [9, 23]. The convergence towards the Weibull distribution requires the existence of an upper bound (finite number) on the support of the distribution. However, it is always possible to find configurations satisfying Eq. (12) in which \( \rho \) is very large. Therefore, the distribution can not be Weibull. The Gumbel distribution does not have finite upper or lower bounds, implying, in our case, that the size of the \( N \) particle configuration can be arbitrarily small and still satisfying Eq. (12), which is not possible. The range of the Fréchet distribution is bounded from below; thus, this is the only distribution that can accurately describe the capture hyperradial distribution.

5 Conclusion

The study of classical capture models relies on defining the capture radius, which in the case of few-body systems, translates into the distribution of hyperradius when the system is described in hyperspherical coordinates. This work has computed the hyperradial distribution for three- four- and five-body systems, including vdW and charged-neutral interactions in a wide range of collision energies between the cold and the thermal regimes. As a result, we conclude that the hyperradial distribution function follows a Fréchet distribution independently of the interparticle interaction nature and the number of interacting particles. In other words, classical capture models will lead to a Frechet distribution in N-body problems involving structureless particles. Moreover, by providing a standard measure in identifying the capture hyperradius, our finding helps to increase the accuracy and reduce the time and effort for developing classical capture models for N-body systems.

Finally, after using extreme value theory, we have found a plausible explanation behind the universality of the Fréchet distribution for the capture hyperradius, thus finding an application of extreme value theory in few-body physics.

Acknowledgements

M. M acknowledges the support of the Deutsche Forschungsgemeinschaft (DFG - German Research Foundation) under grant number PE 3477/2 - 493725479. J. P.-R. acknowledges the support of the Simons Foundations and the Deutsche Forschungsgemeinschaft (DFG - German Research Foundation) under grant number PE 3477/2 - 493725479. A A. E acknowledges the support of the AFOSR-MURI under grant number FA9550-20-1-0323.

Declarations

Conflict of interest

Not Applicable.

References

1. R. Levine, Molecular Reaction Dynamics (Cambridge University Press, Cambridge, 2005)
2. J. Pérez-Ríos, An Introduction to Cold and Ultracold Chemistry (Springer, Cham, 2020)
3. M.P. Langevin, Une formule fondamentale de théorie cinétique. Ann. Chim. Phys. 5, 245–288 (1905)
4. J. Pérez-Ríos, S. Ragole, J. Wang, C.H. Greene, Comparison of classical and quantal calculations of helium three-body recombination. J. Chem. Phys. 140(4), 044307 (2014). https://doi.org/10.1063/1.4861851
5. J. Pérez-Ríos, C.H. Greene, Universal temperature dependence of the ion-neutral-neutral three-body recombination rate. Phys. Rev. A 98(6), 23–26 (2018). https://doi.org/10.1103/PhysRevA.98.062707
6. J. Pérez-Ríos, Cold chemistry: a few-body perspective on impurity physics of a single ion in an ultracold bath. Mol. Phys. 119(8), 1881637 (2021). https://doi.org/10.1080/00268976.2021.1881637
7. M. Mirahmadi, J. Pérez-Ríos, On the formation of van der Waals complexes through three-body recombination. J. Chem. Phys. 154(3), 034305 (2021). https://doi.org/10.1063/5.0039610
8. M. Mirahmadi, J. Pérez-Ríos, Classical threshold law for the formation of van der waals molecules. J. Chem. Phys. 155(9), 094306 (2021). https://doi.org/10.1063/5.0062812
9. V.P. Singh, Entropy-Based Parameter Estimation in Hydrology (Springer, Dordrecht, 1998)
10. F.T. Smith, Generalized angular momentum in many-body collisions. Phys. Rev. 120(3), 1058–1069 (1960). https://doi.org/10.1103/PhysRev.120.1058
11. F.T. Smith, Three-body collision rates in atomic recombination reactions. Discuss. Faraday Soc. 33, 183–188 (1962). https://doi.org/10.1039/DF9623300183
12. V.H. Shui, Monte Carlo trajectory calculations of the three-body recombination and dissociation of diatomic molecules. J. Chem. Phys. 57(4), 1704–1717 (1972). https://doi.org/10.1063/1.1678459
13. V.H. Shui, Thermal dissociation and recombination of hydrogen according to the reactions \( h_2 + h \rightleftharpoons +h + h \). J. Chem. Phys. 58(11), 4868–4879 (1973). https://doi.org/10.1063/1.1679071
14. J.S. Avery, Hyperspherical Harmonics: Applications in Quantum Theory. Reidel Texts in the Mathematical Sciences (Springer, Dordrecht, 1986)
15. D.P. Landau, K. Binder, A Guide to Monte Carlo Simulations in Statistical Physics, 3rd edn. (Cambridge University Press, Cambridge, 2009)
16. Y.A. Shreider, *The Monte Carlo Method: the Method of Statistical Trials*, vol. 87 (Elsevier, Oxford, 2014)
17. R.A. Aziz, A.R. Janzen, M.R. Moldover, Ab initio calculations for helium; a standard for transport property measurements. Phys. Rev. Lett. 74, 1586–1589 (1995). https://doi.org/10.1103/PhysRevLett.74.1586
18. A. Kumar, W.J. Meath, Pseudo-spectral dipole oscillator strengths and dipole-dipole and triple-dipole dispersion energy coefficients for HF, HCl, HBr, He, Ne, Ar, Kr, and Xe. Mol. Phys. 54(4), 823–833 (1985). https://doi.org/10.1080/00268978500103191
19. H. Partridge, J.R. Stallcop, E. Levin, Potential energy curves and transport properties for the interaction of He with other ground-state atoms. J. Chem. Phys. 115(14), 6471–6488 (2001). https://doi.org/10.1063/1.1385372
20. J. Pérez-Ríos, C.H. Greene, Communication: Classical threshold law for ion-neutral-neutral three-body recombination. J. Chem. Phys. 143(4), 041105 (2015). https://doi.org/10.1063/1.4927702
21. B. Gnedenko, Sur la distribution limite du terme maximum d’une série aléatoire. Ann. Math. 44(3), 423–453 (1943)
22. R.A. Fisher, L.H.C. Tippett, Limiting forms of the frequency distribution of the largest or smallest member of a sample. Math. Proc. Cambridge Philos. Soc. 24(2), 180–190 (1928). https://doi.org/10.1017/S0305004100015681
23. M.R. Leadbetter, G. Lindgren, H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes* (Springer, New York, 2012)
24. A.F. Laurens de Haan, *Extreme Value Theory* (Springer, New York, 2007)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.