Ex-Post Equilibrium and VCG Mechanisms*

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May 5, 2014

*This work is partly based on Rozen’s M.Sc thesis done under the supervision of Ron Holzman
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Abstract

Consider an abstract social choice setting with incomplete information, where the number of alternatives is large. Albeit natural, implementing VCG mechanisms is infeasible due to the prohibitive communication constraints. However, if players restrict attention to a subset of the alternatives, feasibility may be recovered.

This paper characterizes the class of subsets which induce an ex-post equilibrium in the original game. It turns out that a crucial condition for such subsets to exist is the existence of a type-independent optimal social alternative, for each player. We further analyze the welfare implications of these restrictions.

This work follows work by Holzman, Kfir-Dahav, Monderer and Tennenholtz [7] and Holzman and Monderer [8] where similar analysis is done for combinatorial auctions.

Keywords: Ex-post Equilibrium; mechanism design; communication complexity

JEL classification: C72, D70
1 Introduction

A classical result in the theory of mechanism design, known as the Vickrey-Clarke-Groves (VCG) mechanism, asserts that if agents' utility is based on his valuations of goods and his monetary situations, and these two are quasi-linear, then it is possible to align all agents' interest such that all agents would want to maximize the social welfare. Consequently, for each agent, truthful reporting dominates any other strategy. This is true even if an agent does not know exactly how many other agents are participating.

This remarkable result has been a cornerstone of many design problems, such as taxation, public good, auctions and more. In recent years, two major practical difficulties have been identified with this approach. Both are related to the design of “large” problems. Namely, problems where the number of social alternatives, $M$, is big.

The first problem relates to the computational complexity of determining the social maxima, among all $M$ alternatives. The second problem refers to the communication complexity of each agent’s message. For an agent to fully report her valuation she must provide $M$ numbers, and if $M$ is large this may be quite prohibitive. Although, these problems are of different nature both stem from the large size of social alternatives.

The classical example which gives rise to such problems is that of combinatorial auctions:

**Example 1.** Assume $N$ agents bid for $G$ different goods and each agent cares only about the goods she receives. An agent’s type is a vector of valuations, one for each of the $2^G - 1$ nonempty subsets of $G$. The case $|G| = 50$ is already prohibitively large.

This example has been studied extensively but it is far from being the unique setting giving rise communication complexity. Some other examples are:

**Example 2.** Assume $N$ agents bid for $G$ different goods and each agent cares for the partition of the goods. Namely agent $i$ utility also depends on the good agent $j$ receives. The type of an agent is a valuation for each of $(N + 1)^G$ possible allocations. The case $|G| = |N| = 20$ is prohibitively large.

Another example is that of ordering:
Example 3. Assume agents have to decide on an allocation of $C$ candidates to $C$ positions (or an order of $C$ jobs to be completed serially). Any agent then assigns a value to any of the $C!$ possible allocations. Once again, for $C = 50$ this becomes too demanding.

Additional examples are location problems where $F$ facilities need to be located in $L$ possible locations (inducing $|L|^{|F|}$ possibilities), network building where a set $E$ edges needs to be built in order to connect $V$, and more.

The problem of finding the social optimum has been studied, from a computational complexity point of view by various authors. Some examples are Rothkopf et al [14], Fujishima et al [5], Anderson et al [1], Sandholm et al [15] and Hoos and Boutilier [9]. The communication complexity problem motivated researchers to characterize alternative mechanisms which involve less demanding strategies than truthfulness. Some examples for this line of research in an auction setting are Gul and Stacchetti [6], Parkes [12], Parkes and Ungar [13], Wellman et al [16] and Bikhchandani et al [4].

In this paper we follow the framework of Holzman, Kfir-Dahav, Monderer and Tennenholtz [7] and Holzman and Monderer [8], who do not look for alternative mechanisms but rather study an alternative solution concept for the class of VCG mechanisms. In particular these two papers study the class of ex post equilibria of VCG mechanisms which exhibit many of the properties of dominant strategies, yet allow for less demanding communication complexity. In Holzman, Kfir-Dahav, Monderer and Tennenholtz [7] a family of ex post equilibria, called bundling equilibria, is introduced and its efficiency is studied. A bundling strategy is one where agents report valuations on a subset of all alternatives, where the subset must contain the empty set, be close under union of mutually exclusive sets, and close under complementarities. Holzman and Monderer [8] show that this family exhausts all the ex-post equilibria in combinatorial auction settings.

Whereas these two papers focus on the setting of example 2, namely on combinatorial auction without externalities, we study similar issues in a general and more abstract setting, making our results relevant to all the aforementioned examples. Additionally, the results obtained by in the two papers are restricted to the case of $N > 2$ players while we our results encompass the 2 player case as well. A more detailed comparison of our results with the results in the combinatorial auction setting is provided in section 5.
This paper focuses on three research questions: existence, characterization and efficiency. In particular we show that for the most general case, where valuations are arbitrary, the unique ex post equilibrium is that of the weakly dominant strategies. However, by imposing one restriction on the valuation space, namely that each agents optimal choice is independent of his valuation, we can recover the positive result and provide a large set of ex post equilibria. In terms of efficiency loss, we show that in the general case the efficiency loss grows with the number of players, and cannot be bounded uniformly. In fact we show that the efficiency loss is in the order of magnitude of the number of players and this cannot be improved upon. However, we propose two types of restrictions on the set of valuations which induce a uniform bound on the efficiency loss.

A related strand of the literature is that on ex-post implementation, which is part of the mechanism design literature. The research goal of most papers on ex-post implementation is to characterize the conditions needed for obtaining a mechanism which implements some social choice functions under the ex-post equilibrium solution. Some recent contributions to this are Bergemann and Morris [2], Bikhchandani [3] and Jehiel et al [10]. Needless to mention, given the research goal of this literature, that these papers do not yield mechanisms which have low complexity. Nisan and Segal [11] study the tradeoff between communication complexity and efficiency in allocation problems.

In section 2 we provide the basic model and definitions. Section 3 discusses existence and structure of ex-post equilibria, whereas section 4 analyzes their efficiency. Section 5 is devoted to a discussion of the results for the combinatorial setting and a comparison with the results of Holzman, Kfir-Dahav, Monderer and Tennenholtz [7] and Holzman and Monderer [8]. The proofs are relegated to the appendix.

2 Model

Let $A$ be a finite abstract set of social alternatives and let $N$ be a finite set of $n = |N|$ agents. A valuation function for agent $i$ is a function $v_i : A \to \mathbb{R}$ and $v = (v_1, \ldots, v_n) \in (\mathbb{R}^A)^n$ denotes a vector of valuations, one for each agent.

An *allocation mechanism*, $M : (\mathbb{R}^A)^n \to A$, chooses a single alternative for each vector of valuations. We say that $M$ is a social welfare maximizer if $M(v) \in \arg\max_{a \in A} \sum_i v_i(a)$
for all $v \in (\mathbb{R}^A)^n$. Let $\mathcal{M}_N$ be the set of all social welfare maximizers for the set of agents $N$. Note that two elements in $\mathcal{M}_N$ differ only in the way they break ties.

A set of agents $N$, a social welfare maximizing mechanism, $M$, and a set of functions $h_i : (\mathbb{R}^A)^{N-\{i\}} \to \mathbb{R}$, $\forall i \in N$, and a set of valuations, $\mathcal{V}_i \subset \mathbb{R}^A$, for each $i$, defines a Vickrey-Clarke-Groves (VCG) game, denoted $\Gamma(N, M, h, \mathcal{V})$, where $h = (h_i)_{i=1}^n$ and $\mathcal{V} = \prod_{i \in N} \mathcal{V}_i$.

- Agent $i$’s strategy, $b_i : \mathcal{V}_i \to \mathbb{R}^A$, maps his true valuation to an announced valuation (possibly not in $\mathcal{V}_i$). $b_i(v)(a)$ is the valuation announced for alternative $a$, when the actual valuation is $v$ and the strategy is $b_i$. Let $b = (b_1, \ldots, b_n)$ denote the agents’ strategy profile and let $b_i(\mathcal{V}_i) \subset \mathbb{R}^A$ be the set of all possible announcements of $i$.

- For any $v \in \mathcal{V}$ and any strategy profile, $b$, the utility of agent $i$ is $U_i = v_i(M(b(v)) + \sum_{j \neq i} b_j(v_j)(M(b(v))) - h_i(b_{-i}(v_{-i}))$.

We refer to the set of all VCG games where $h_i = 0$ as simple VCG games.

We note two observations about such VCG games:

- The bid $b_i$ is a best response for agent $i$, against $b_{-i}$, in some VCG game if and only if it is a best response in all VCG games.

- The truth-telling strategy, $b_i(v_i) = v_i$, weakly dominates any other strategy in any VCG game. Furthermore, it is the unique strategy with that property (up to a constant).

Given the above comments it seems that there is no need for any additional game theoretic analysis of VCG games. However, recently there has been a growing interest in the literature in situations where agents face a large number of social alternatives, in which case the communication of an agent’s valuation is prohibitively long, and practical reasons render the truth-telling strategy as impossible. One example for such a situation is a combinatorial auction, where the number of social alternatives is exponential in the number of goods. Another example is an assignment problem, where ”jobs” are assigned to ”resources” (e.g., people to positions). In this case the number of rankings grows fast with the number of jobs and resources.
The communication complexity issue, discussed above, motivates an alternative analysis of the solution concepts for large VCG games. We follow on two recent papers, Holzman et al [7] and Holzman and Monderer [8]. We look for natural solution concepts which are less demanding in terms of the agents communication needs, yet are as almost convincing, in terms of the incentive compatibility requirements, as weakly dominant strategies (truth telling).

2.1 Ex-Post Equilibrium

The solution concept of dominant strategies has the following appealing properties:

- Agents act optimally no matter what other agents do.
- The solution concept makes no use of any probabilistic information on agents’ valuations, either by the agents themselves or by the mechanism designer.
- Agent’s strategies are robust to changes in the number of players.

In addition, the solution is attained at truth telling strategies which are quite simple (knowing \( v_i \), agent \( i \) does not need to do any computation). Consequently we deduce that the chosen alternative is the socially efficient alternative. We also note that by choosing functions \( h_i \) properly the game is individually rational and budget balanced. An alternative, yet weaker, solution concept is that of an ex-post equilibrium:

**Definition 1.** A tuple of strategies, \( b \), is an ex-post equilibrium, for the VCG game, \( \Gamma(N, M, h, V) \), if for any player \( i \in N \), any valuation \( v_i \in V_i \) and any alternative strategy \( \hat{b}_i \) of \( i \),

\[
U_i((b_i, b_{-i}), (v_i, v_{-i})) \geq U_i((\hat{b}_i, b_{-i}), (v_i, v_{-i})) \quad \forall v_{-i} \in V^{N-\{i\}}.
\]

**Definition 2.** Fix a set of agents, \( N \), a set of social alternatives \( A \), and a set of valuations, one for each agent, \( V \subset (\mathbb{R}^A)^n \). A tuple of strategies, \( b \), is an ex-post equilibrium for the class of VCG games, over \( (N, A, V) \), if for all \( N' \subset N \), \( (b_j)_{j \in N'} \) is an ex-post equilibrium for \( \Gamma(N', M, h, V) \), for all \( M \in \mathcal{M}_{N'} \) and \( h \in H \).

Note that an ex-post equilibrium for the class of VCG games has many of the properties of the solution concept of dominant strategies:
• Agents act optimally no matter what other agents’s valuations are, as long as all agents keep to their strategies. In other words, agents have no incentive to unilaterally deviate, even after all valuations have been realized.

• The solution concept makes no use of any probabilistic information on agents’ valuations.

• Agent’s strategies are robust to changes in the number of players.

Example 4. Consider a standard Vickrey auction auction of 2 goods, A and B, with 2 bidders. There are 9 possible allocations of the goods (one can choose to allocate goods to none of the agents). Let \( V_i \) be the set of i’s valuations that depend only on the goods allocated to i and are monotonically non-decreasing. Consider the following strategies - Agent 1 announces the true valuation of the grand bundle (A and B), and the bundle A, and zero for B. Agent 2 announces his true valuation for the grand bundle and for good B and zero for A. These 2 strategies form an ex-post equilibrium of the standard Vickrey auction. In fact, they form an ex-post equilibrium for the class of VCG games, over \( V_i \).

Example 5. Consider an auction of M goods and N agents. As before, let \( V_i \) be the set of i’s valuations that depend only on the goods allocated to i and are monotonically non-decreasing. Let \( b_i \) be the strategy that assigns the true value for the grand coalition and zero to all other allocations. Holzman et al show that this is an ex-post equilibrium, for the class of VCG games, over \( V \).

3 Results

Obviously any solution in weakly dominant strategies is also an ex post equilibrium for the class of VCG games. However, Holzman et al [7] have shown that there exist ex-post equilibria for the class of VCG games induced by a combinatorial auction setting, other than the dominant ones. In particular some of these strategies have low communication complexity. A natural question to pursue is whether this result is an artifact of the particular structure of valuations of a combinatorial auction or whether it can be extended to larger valuation sets.

In what follows we study the class of ex-post equilibria for various classes of valuations. Our first result is immediate:

1This example is taken from a comment in [8] page 11.
Proposition 1. Assume \( b \) is an ex-post equilibrium for the class of VCG games, over \((N,A,V)\). If \( V'_i \subset V_i \) then \( b \) is an ex-post equilibrium for the class of VCG games, over \((N,A,V')\).

Proof. Follows directly from the definition. Q.E.D

Unfortunately, if the set of valuation functions is large enough then there are no ex post equilibria other than (near) truth telling for any large class of VCG games:

Theorem 1. Assume that \( n \geq 2 \) and \(|A| > 2\), or alternatively that \( n \geq 3\). Then a strategy profile \( b \) is an ex-post equilibrium for the class of VCG mechanisms over \((N,A,(\mathbb{R}^A)^n)\) if and only if \( b_i(v_i)(a) = v_i(a) + f_i(v_i) \), for any arbitrary function \( f_i : V \to \mathbb{R} \).

In particular, note that the valuations reported by the agents may differ from the true valuations, however, the difference between the true valuation and the reported ones must be constant. We shall refer to such strategies as nearly truth telling over \( A \) (see definition 3 below).

The proof of theorem 1 is composed of two distinct proofs, one for the case \( n \geq 2 \) and \(|A| > 2\) and a different one for the case \( n \geq 3\).

One can hope that by adding more structure to the problem the positive result can be salvaged. A valuation function is called non-negative if \( v_i(a) \geq 0 \) for all \( a \in A \). Let \( \mathbb{R}^A_+ \) be the set of all non-negative valuation functions. Indeed, this is a natural property in the case of combinatorial auctions. Nevertheless:

Theorem 2. Assume that \( n \geq 2 \) and \(|A| > 2\), or alternatively that \( n \geq 3\). Then a strategy profile \( b \) is an ex-post equilibrium for the class of VCG games over \((N,A,(\mathbb{R}^A_+)^n)\) if and only if \( b_i(v_i)(a) = v_i(a) + f_i(v_i) \), for any arbitrary function \( f_i : V \to \mathbb{R} \).

It is relatively straightforward to show that indeed the specific strategies prescribed in theorems 1 and 2 are indeed an ex post equilibrium. The difficulty of the proof lies in the second direction.
3.1 Constant Maximum Valuations

We say that a set of valuations $V_i$ has a maximum if there exists some $a \in A$, called the maximum, such that $v_i(a) \geq v_i(a')$ for all $a' \in A$ and all $v_i \in V_i$. Note that the set of non-decreasing valuations, in the combinatorial auction setting, has a maximum. In particular, any allocation that gives all the goods to agent $i$ is a maximum.

Valuations sets that have a maximum prevail in other settings as well. In the context of ordering a set of tasks, consider the valuations with the property that any agent want her own task to be processed first, but otherwise cares about which other tasks precede her own task. In the context of network construction, consider valuations where each player (who is a vertex in a graph) always prefers the star shaped graph centered around him over any other graph. Finally, in the context of facility location, assume agent always prefers all the ‘good facilities (e.g., library) to neighbor her and all the bad facilities (e.g., waste disposal) to be as far as possible.

Let $R_i(a_i)$ be set of all non-negative valuation functions for which $a_i$ is a maximum, and let $\mathcal{R}(\vec{a}) = \times_{i=1}^{n} R_i(a_i)$.

The following family of strategies will play an important role for this family of valuations.

**Definition 3.** Let $A' \subset A$ be a subset of social alternatives. A strategy profile is called nearly truth telling over $A'$ if:

- If $a \in A'$ then $b_i(v_i)(a) = v_i(a) + f(v_i)$ for some arbitrary function $f_i : \mathcal{V} \rightarrow \mathbb{R}$.
- If $a \notin A'$, then $b_i(v_i)(a) = C$, for an arbitrary $C \leq \min_{a \in A'} b_i(v_i)(a)$.

A nearly truth telling strategy prescribes telling the truth, up to a shift in a constant, on some subset of pre-selected alternatives, and assigns a valuation of zero to all other alternatives. So players using the strategy need only communicate $|A'| + 1$ numbers, instead of $|A|$ numbers.

**Theorem 3.** Consider the class of VCG games over $(N, A, \mathcal{R}(\vec{a}))$, where $\vec{a} = (a_1, a_2, \ldots, a_n)$, and let $A' \subset A$ satisfy $a_i \in A' \ \forall a_i, \ i = 1, \ldots, n$ (all the $n$ maxima are in $A'$). Then any nearly truth telling strategy profile over $A'$ is an ex-post equilibrium for this class.

\footnote{Note that we do not exclude negative bids, namely $v_i(a) < -f(v_i)$.}
Unfortunately, not all ex post equilibria are nearly truth telling for some subset $A'$. Consider the following strategy profile - $b_i(v_i)(a_k) = v_i(a_k) + 10$ for all maxima, $a_k$, $k = 1, \ldots, n$, and for all $a \not\in \{a_k : k = 1, \ldots, n\}$, $b_i(v_i)(a)$ is chosen arbitrarily in the interval $[0, 9]$. We leave it to the reader to verify that $b$ is an ex post equilibrium for the class of VCG games over $(N, A, \mathcal{R}(a))$.

Although we are not able to characterize all ex-post equilibria for the class of VCG games over $(N, A, \mathcal{R}(a))$, we can provide some necessary conditions.

**Theorem 4.** If $n \geq 3$ and the strategy profile, $b$, is an ex-post equilibrium for the class of VCG games over $(N, A, \mathcal{R}(a))$ then $b_i(v_i)(a_k) = v_i(a_k) + f(v_i)$, for all $i$ and $k$, where $a_k$ is the maximum for player $k$.

In words, in any ex-post equilibrium players (almost) report their true valuations on the set of maxima.

## 4 Efficiency

It is quite obvious that even if ex post equilibria exist, as in the models discussed in theorems 5 and 6, the demand on communication may be much smaller, compared with the dominant strategy solution. In fact, agents may need as little as reporting the value for $N$ alternatives only (compare $N = 50$ with $2^{50}$ or $50^{50}$ alternatives in example 2).

In section 2.1 we provided arguments why, conceptually, the notion of ex post equilibrium is almost as robust as the dominant strategy solution. However, when it comes to efficiency and social welfare the two solution concepts differ. Whereas, the dominant strategy solution maximizes social welfare (the sum of agents utilities) this is not so for many ex post equilibria.

**Example 6.** Consider a complete information combinatorial auction setting with $N$ agents and $N$ goods. Assume agent $i$ values the bundle of goods, $K$, as follows: $v_i(K) = 0$ if $i \not\in K$, $v_i(K) = 1$ if $i \in K$ and $|K| < N$ and finally $v_i(K) = 1 + \epsilon$ if $K$ is the grand bundle. Consider a strategy profile where each agent bids zero over any bundle that is not the grand bundle and truthfully on the grand bundle. This is an ex post equilibrium of the combinatorial auction and the communication complexity is very low. However
this results assigning the grand bundle randomly to one of the players, achieving a social welfare of $1 + \epsilon$, as opposed to the maximal social welfare that is achievable in the dominant strategy solution. Thus, the efficiency ratio is $N$.

Let $S(m) = \sum_i v_i(m)$ denote the social welfare for the social alternative $m$. Let $r(m, m') = \frac{S(m)}{S(m')}$. For any VCG mechanism and any strategy profile $d$ we denote by $VCG(d(v))$ the resulting social alternative, at the valuation profile $v$. Recall that the dominant strategy profile $b$ maximizes $S$, namely $S(VCG(b(v))) \geq S(m) \forall m$. The following theorem extends a result of Holzman et al [7].

**Theorem 5.** Consider the class of VCG games over $(N, A, \mathcal{R}(\vec{a}))$, where $\vec{a} = (a_1, a_2, \ldots, a_n)$, and let $A' \subset A$ satisfy $a_i \in A' \forall a_i, i = 1, \ldots, n$ (all the $n$ maxima are in $A'$). Let $d$ be a nearly truth telling strategy profile over $A'$ that is an ex-post equilibrium for this class and let $b$ be the dominant strategy equilibrium. Then $r(VCG(b(v)), VCG(d(v))) \leq N$.

**Proof.** The proof is similar to that of Remark 1 in Holzman et al [7]:

$$s(VCG(b(v))) \leq N \max_i v_i(VCG(b(v))) \leq N \max_i (v_i(a_i)) \leq N \max_i (\sum_j v_j(a_i))$$

where $a_i$ is the alternative $i$ prefers. On the other hand, $a_i$ is one of the alternatives for which valuations are announced, therefore:

$$\max_i (\sum_j v_j(a_i)) \leq \max_i (v_i(VCG(d(v)))) \leq S(VCG(d(v)))$$

which completes the proof. Q.E.D

Example 6 shows that this bound is tight. In fact, we can use the principles to that example to show that the efficiency loss is not gradual and that one can get high efficiency loss even when the communication complexity is very high:

**Example 7.** Consider a setting with $N$ players and $M$ social alternatives. Let $m_i$ denote the optimal social alternative for $i$ and let $m_0 \notin \{m_1, \ldots, m_N\}$ denote an arbitrary alternative. Assume players play an ex post equilibrium with near truth telling strategies on $M - \{m_0\}$. Now consider the following valuation for player $i$ - $v_i(m_i) = 1 + \epsilon, v_i(m_0) = 1$ and $v_i(m) = 0$ for all other alternatives. For this vector of valuations the resulting alternative is $m_N$ and the social welfare is $1 + N\epsilon$, whereas in the dominant strategy equilibrium the resulting alternative is $m_0$ with a social welfare of $N$. $r(VCG(b(v)), VCG(d(v)))$ approaches $N$ as $\epsilon$ approaches zero.
The bound we have shown is not a satisfactory one as the number of players may be quite large. We consider two families of valuations for which the efficiency loss is independent of the number of the players.

The family of valuations \( V = (V_1, \ldots, V_N) \) is called **homogeneous of degree** \( p \) if for any \( v \in V \) and any \( m \in M \max_i(v_i(m)) < p \cdot \frac{\sum v_i(m)}{N} \). In words, for each alternative there cannot be too much difference of opinion. In many settings valuations are bounded, say \( a \leq v_i(m) \leq b \) \( \forall i, v_i \in V, m \in A \). In such settings, if \( a > 0 \) then valuations are homogeneous of degree \( p = \frac{b}{a} \). Another example for valuations of degree \( p \), is in correlated settings where a common strictly positive signal is drawn and agents valuations are generated via idiosyncratic adjustments of the common signal. More concretely think of a set of firms which compete for some public resource. The quality of the resource, and hence the potential revenues is common, yet the production costs, as a ratio of the revenues can be between \( 0 < a \) and \( b < 1 \). In this case homogeneity of degree \( p = \frac{1-a}{1-b} \) prevails.

**Theorem 6.** Consider the class of VCG games over \((N, A, V)\), where \( V \subset R(\vec{a}) \) is homogeneous of degree \( p \). Let \( A' \subset A \) satisfy \( a_i \in A' \ \forall a_i, i = 1, \ldots, n \) (all the \( n \) maxima are in \( A' \)). Let \( d \) be a nearly truth telling strategy profile over \( A' \) that is an ex-post equilibrium for this class and let \( b \) be the dominant strategy equilibrium. Then \( r(\text{VCG}(b(v)), \text{VCG}(d(v))) \leq p \).

**Proof.** Let \( i_0 \) denote the agent that values the alternative \( \text{VCG}(b(v)) \) the most and let \( a_{i_0} \) be the alternative \( i_0 \) prefers.

\[
\frac{VCG(b(v)) \leq Nv_{i_0}(VCG(b(v))) \leq Nv_{i_0}(a_{i_0})}
\]

By homogeneity \( v_{i_0}(a_{i_0}) < p \cdot \frac{\sum v_i(a_{i_0})}{N} \) which implies that

\[
\frac{VCG(b(v)) < p \cdot \sum v_j(a_{i_0})}
\]

Because \( a_{i_0} \) is one of the alternatives for which valuations are announced it holds that:

\[
\frac{\sum v_j(a_{i_0}) \leq \sum v_j(\text{VCG}(d(v)))}
\]

An alternative, and perhaps more appealing, definition would involve the inequality \( \max_i(v_i(m)) < p \cdot (v_i(m)) \forall i \). However, we do not use this alternative as we do not want to rule out the possibility for some agents to assign a valuation of zero, while others assign a positive valuation.
which completes the proof. Q.E.D

Another family of valuations which we study is one where players valuations differ significantly over each alternative. A family of valuations \( V = (V_1, \ldots, V_N) \) is called **compatible of degree** \( p \) if for any \( v \in V \) and any \( m \in M \) there are at most \( p \) players for which \( v_i(m) > 0 \). As an example consider a combinatorial auction with \( p \) goods.

**Theorem 7.** Consider the class of VCG games over \((N,A,V)\), where \( V \subset \mathbb{R}(\vec{a}) \) is compatible of degree \( p \). Let \( A' \subset A \) satisfy \( a_i \in A' \ \forall i, \ i = 1, \ldots, n \) (all the \( n \) maxima are in \( A' \)). Let \( d \) be a nearly truth telling strategy profile over \( A' \) that is an ex-post equilibrium for this class and let \( b \) be the dominant strategy equilibrium. Then \( r(VCG(b(v)), VCG(d(v))) \leq p \).

The proof of this theorem mimics the proof of Theorem 5 with \( p \) replacing \( N \), and is therefore omitted. Note that in combinatorial auctions the number of players that have a positive valuation for any alternative is at most the number of goods. Additionally, in any bundling equilibrium derived from a partition of the set of goods, the number of players that have a positive valuation for any alternative is at most the size of the partition.

## 5 Combinatorial Auctions

An analysis of ex post equilibria in VCG mechanisms for the setting of combinatorial auctions is provided in In Holzman, Kfir-Dahav, Monderer and Tennenholtz \[7\] and Holzman and Monderer \[8\]. These papers focus on combinatorial auctions with 3 bidders or more with monotonic valuations. Their main finding is that the ex post equilibria of such auctions are characterized by submitting bids on a subset of the possible bundles (this is referred to as a bundling equilibrium), which is a quasi-field. Namely, it is non-empty set of sets that is closed under complements and under disjoint unions.

Note that a social alternative in the auction setting is an assignment of the set of all goods to the set of players (the bidders and the seller). However, a player's valuation depends only on the goods allocated to her (there are no externalities). Therefore specifying agent \( i \)'s valuation for a bundle \( B \) induces valuations for all social
alternatives in which agent $i$ receives the bundle $B$. In addition, monotonic valuations imply that allocating the grand bundle to agent $i$ always maximizes $i$’s valuation over the possible social alternatives and so a set of maximizers is identified.

This unique structure allows for a full characterization of the ex-post equilibria, in contrast with our partial characterization for the general case. Comparing the results for the general case with those of the auction setting is not obvious. To see this consider the following example:

**Example 8.** Consider an auction with 3 goods, \{a, b, c\} and 3 players. Consider the subset of social alternatives:

$$S' = \{(\emptyset, \emptyset, \emptyset), (\emptyset, bc, \emptyset), (\emptyset, abc, \emptyset), (ab, \emptyset, \emptyset), (abc, \emptyset, \emptyset), (\emptyset, \emptyset, abc)\}.$$  

The set $S'$ includes all the three maximizers, $(abc, \emptyset, \emptyset), (\emptyset, abc, \emptyset)$ and $(\emptyset, \emptyset, abc)$. Therefore, by Theorem 3 this set induces an ex post equilibrium, where players bid truthfully over this set. On the other hand, players do not submit bids on a quasi-field. In fact, note that this ex post equilibrium is not a bundling equilibrium as players do not bid on the same bundles. This seems to contradict the findings of Holzman, Kfir-Dahav, Monderer and Tennenholtz [7] and Holzman and Monderer [8].

Is this a real contradiction? The answer is clearly no. To settle this note that when we cast our general model to the combinatorial auction setting we do not assume additional restrictions on valuation functions. In particular agents valuation may have externalities and need not be monotonic. Therefore, valuations and bids over $S'$ are silent about valuations outside of $S'$.

If, however, we adopt the two restrictions of monotonicity and no-externality then the valuations of $S'$ extend to additional social alternatives. For example, if player 1 bids $v$ on the social alternative $(ab, \emptyset, \emptyset)$ then this implies a bid of $v$ on the social alternative $(ab, c, \emptyset) \notin S'$. However, the valuation of player 2 for this social alternative cannot be deduced from her valuations over $S'$, making the bids asymmetric. This, in turn, makes Theorem 3 mute as the conditions do not hold.

### 6 Appendix - Proofs

We begin by some preliminary observations needed to prove our main results.
6.1 Preliminary Observations

We observe that in any ex post equilibrium the most valued alternative for a player must have the highest reported valuation.

**Lemma 1.** Let \( n \geq 1 \) and let \( b \) be an ex-post equilibrium for the class of VCG games over \((N,A,V)\). Then for all \( i \) and all \( v_i \in V_i \), if \( v_i(a) > v_i(a') \), for all \( a' \neq a \) then \( b_i(v_i)(a) > b_i(v_i)(a') \) for all \( a' \neq a \).

**Proof.** Assume \( v_i(a) > v_i(a') \), for all \( a' \neq a \), and consider the one player game with player \( i \). In this game, the chosen alternative must be optimal for \( i \), and so it must be that \( i \)'s valuation on it was the highest, namely \( b_i(v_i)(a) > b_i(v_i)(a') \) for all \( a' \neq a \). Q.E.D

**Lemma 2.** Let \( b \) be an ex-post equilibrium for the class of VCG games over \((N,A,V)\). If \( a \) is chosen at the profile \( v \), then for any \( i \),

\[
v_i(a) + \sum_{j \neq i} b_j(v_j)(a) \geq v_i(a') + \sum_{j \neq i} b_j(v_j)(a'), \quad \forall a' \in A.
\]

**Proof.** Assume the claim is wrong, and that for some \( i \) \( v_i(a) + \sum_{j \neq i} b_j(v_j)(a) < v_i(a') + \sum_{j \neq i} b_j(v_j)(a') \leq \max_{\hat{a} \in A} v_i(\hat{a}) + \sum_{j \neq i} b_j(v_j)(\hat{a}) \). Note that the left hand side is \( i \)'s utility, whereas the right hand side is \( i \)'s utility from reporting truthfully. Thus, contradicting the ex-post equilibrium assumption. Q.E.D

**Lemma 3.** Let \( n \geq 2 \) and let \( b \) be an ex-post equilibrium for the class of VCG games over \((N,A,(\mathbb{R}_+^A)^n)\). Then for all \( i \) and all \( v_i \in \mathbb{R}_+^A \), if \( v_i(a) = v_i(a') \), then \( b_i(v_i)(a) = b_i(v_i)(a') \).

**Proof.** Assume \( v_i(a) = v_i(a') \) but \( b_i(v_i)(a) > b_i(v_i)(a') \). Let \( \bar{x} = \max_{\hat{a} \in A} |b_i(v_i)(\hat{a})| \). Consider the following valuation function for some player \( j \neq i \), \( v_j(\hat{a}) = 0 \), for all \( \hat{a} \not\in \{a,a'\} \), \( v_j(a) = 3\bar{x} \) and \( v_j(a') = 3\bar{x} + \frac{b_i(v_i)(a) - b_i(v_i)(a')}{2} \). Note that \( v_j(a') > v_j(a) \).

Assume that some \( \hat{a} \not\in \{a,a'\} \) is chosen in the two player game with \( i \) and \( j \). Then player \( j \)'s utility does not exceed \( \bar{x} \). However, by bidding truthfully either \( a \) or \( a' \) would have been chosen and \( j \)'s utility would be at least \( 2\bar{x} \), leading to a contradiction. Therefore, either \( a \) or \( a' \) are chosen.
If \( a' \) is chosen then \( j's \) utility is \( 3x + \frac{b_j(v_j)(a) - b_j(v_j)(a')}{2} + b_i(v_i)(a') \). This is strictly less than \( 3x + b_i(v_i)(a) \), which is the utility \( j \) could have received by reporting truthfully on \( a \) and zero on all other alternatives. This contradicts the ex-post equilibrium assumption, and therefore it must be the case that \( a \) is chosen. In this case the utility of \( i \) is \( v_i(a) + b_j(v_j)(a) \). By our assumption \( v_i(a) = v_i(a') \). By lemma \( b_j(v_j)(a) < b_j(v_j)(a') \). Therefore the utility of \( i \) is strictly less than \( v_i(a') + b_j(v_j)(a') \), which is what \( i \) could have received by reporting truthfully on \( a' \) and zero on all other alternatives. Q.E.D

### 6.1.1 The mean value exclusion property and parallelograms

The main lemma we will use is a simple observation due to Monderer and Holzman [8], which they refer to as the “mean value exclusion” property.

**Definition 4.** The pair of functions \( f_1, f_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies the mean value exclusion condition if \( \forall s, t, y \geq 0 \)

\[
s < y \leq f_1(s) \text{ or } f_1(s) \leq y < s \Rightarrow y \neq f_2(t).
\]

and symmetrically

\[
s < y \leq f_2(s) \text{ or } f_2(s) \leq y < s \Rightarrow y \neq f_1(t).
\]

Let \( I \subset \mathbb{R} \) denote an open interval. We denote its closure by \( \bar{I} \), its supremum by \( I^+ = \sup_{x \in I} x \) and its infimum, by \( I^- = \inf_{x \in I} x \).

Let \( \Omega \) denote the union of disjoint open intervals in \( \mathbb{R}_+ \) such that for any \( I \in \Omega \) \( I^- \neq 0 \). Consider an arbitrary function \( G : \Omega \to \{-1, +1\} \) satisfying \( G(I_1) \times G(I_2) = -1 \) whenever \( I_1^+ = I_2^- \). We say that a pair of function \( h_i : \mathbb{R}_+ \to \mathbb{R}_+ \), \( i = 1, 2 \), is \( (\Omega, G) \)-compatible if it satisfies the following conditions:

1. If \( x \in \mathbb{R}_+ \setminus \bigcup_{I \in \Omega} I \), then \( h_1(x) = h_2(x) = x \)
2. If \( x \in I \in \Omega \) and \( G(I) = -1 \) then \( h_1(x) = I^- \) and \( h_2(x) = I^+ \)
3. If \( x \in I \in \Omega \) and \( G(I) = +1 \) then \( h_1(x) = I^+ \) and \( h_2(x) = I^- \)
4. If \( I \in \Omega \) and \( I^- \neq J^+ \) for all \( J \in \Omega \), then:
(a) if $G(I) = -1$ then $h_1(I^-) = I^-$ and $h_2(I^-) = I^+$.
(b) if $G(I) = +1$ then $h_1(I^-) = I^+$ and $h_2(I^-) = I^-$.

5. If $I \in \Omega$ and $I^+ \neq J^-$ for all $J \in \Omega$, then:

(a) if $G(I) = -1$ then $h_1(I^+) = I^-$ and $h_2(I^+) = I^+$.
(b) if $G(I) = +1$ then $h_1(I^+) = I^+$ and $h_2(I^+) = I^-$.

6. If $I^+ = J^-$ for some $I \neq J \in \Omega$ and $G(I) = +1$ then $h_1(I^+) = I^+$ and $h_2(I^+) \in \{I^-, J^+\}$.

7. If $I^+ = J^-$ for some $I \neq J \in \Omega$ and $G(I) = -1$ then $h_1(I^+) \in \{I^-, J^+\}$ and $h_2(I^+) = I^+$.

Note that a set of disjoint open intervals, $\Omega$, and a function $G : \Omega \to \{-1, +1\}$ almost determines the pair of functions which is $(\Omega, G)$-compatible. The different variants of such a pair of functions stems from conditions (6) and (7) which allow a choice between two possible values.

**Proposition 2** (Parallelogram). Suppose that $g_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, satisfy the mean value exclusion condition. Then there exists a set of disjoint open segments $\Omega$ and a function $G : \Omega \to \{-1, +1\}$, with the restriction that $I^+ = J^-$ implies $G(I) \times G(J) = -1$, such that the pair $(g_1, g_2)$ is $(\Omega, G)$-compatible function.

To construct this family of segments from the given $g_1, g_2$ we use the following definitions and lemmas:

**Definition 2.1.** $D_1$: A set of open segments such that
$\forall I \in D_1, g_1(I^-) = I^+$.

**Definition 2.2.** $D_2$: A set of open segments such that
$\forall I \in D_2, g_1(I^+) = I^-$.

**Definition 2.3.** $D_3$: A set of open segments such that
$\forall I \in D_3, g_1(I^-) = I^+ \text{ and there is no } x' < I^- \text{ such that } g_1(x') = I^+.$

**Definition 2.4.** $D_4$: A set of open segments such that
$\forall I \in D_4, g_1(I^+) = I^- \text{ and there is no } y' > I^+ \text{ such that } g_1(y') = I^-.$
Definition 2.5. \(D_5\): A set of open segments such that \(\forall I \in D_5\) there exists a monotone decreasing sequence \(\{x_k\}_{k=1}^\infty, x_k \to I^-\) and \(\forall k \in \mathbb{N} \ g_1(x_k) = I^+\), but \(g_1(x') \neq I^+\) for all \(x' \leq I^-\).

Definition 2.6. \(D_6\): A set of open segments such that \(\forall I \in D_5\) there exists a monotone increasing sequence \(\{y_k\}_{k=1}^\infty, y_k \to I^+\) and \(\forall k \in \mathbb{N} \ g_1(y_k) = I^-\), but \(g_1(y') \neq I^-\) for all \(y' \geq I^+\).

Definition 2.7. \(D = D_3 \cup D_4 \cup D_5 \cup D_6\).

Definition 2.8. We say that a segment \(I\) satisfies the ""+"" condition if for all \(t \in I, g_1(t) = I^+, g_2(t) = I^-\).

Definition 2.9. We say that a segment \(I\) satisfies the ""-"" condition if for all \(t \in I, g_1(t) = I^-, g_2(t) = I^+\).

When a family of segments will be created in the sequel the ""+"" and ""-"" conditions will be attached to a segment with a function \(G\), which will give a segment a +1 if it satisfies the ""+"" condition and a -1 if it satisfies the ""-"" condition.

Lemma 4. \(\forall I \in D_1 \cup D_2 \exists I' \in D\) such that \(I \subseteq I'\).

Proof of Lemma 4. Let \(I \in D_1 \cup D_2\). We first assume that \(g_1(I^-) = I^+\), and let \(x_0 = \inf \{x' \mid g_1(x') = I^+\}\), of course \(x_0 \leq I^-\). If \(g_1(x_0) = I^+\) then \(I \subseteq (x_0, I^+) \in D_3\). Else there exists a sequence \(\{x_k\}_{k=1}^\infty\) that converges to \(x_0\), such that \(\forall k > 0 \ g_1(x_k) = I^+, \forall x' \leq x_0 \ g_1(x') \neq I^+\). So we have \(I \subseteq (x_0, I^+) \in D_5\). Now we assume that \(g_1(I^+) = I^-\), and let \(y_0 = \sup \{y' \mid g_1(y') = I^-\}\), of course \(y_0 \geq I^+\). If \(g_1(y_0) = I^-\) then \(I \subseteq (I^-, y_0) \in D_4\). Else there exists a sequence \(\{y_k\}_{k=1}^\infty\) that by Lemma 4.1 converges to \(y_0 < \infty\), such that \(\forall k > 0 \ g_1(y_k) = I^-, \forall y' \geq y_0 \ g_1(y') \neq I^-\). So, we have \(I \subseteq (I^-, y_0) \in D_6\). Q.E.D

Lemma 4.1. Let \(\{y_k\}_{k=1}^\infty\) be a sequence such that \(\forall k \in \mathbb{N} \ x < y_k \) and \(g_1(y_k) = x\) and \(y_k \to y\). Then \(y < \infty\).

Proof of Lemma 4.1. Assume for the sake of contradiction that \(y_k \to \infty\) and choose \(t\) such that \(t > x\). Then we shall see where \(g_2(t)\) can be:

\(g_2(t) \notin [x, \infty)\), otherwise we can find a \(y_m\) such that \(x \leq g_2(t) < y_m, g_1(y_m) = x \Rightarrow g_1(y_m) \leq g_2(t) < y_m\), a contradiction to mean value exclusion.

\(g_2(t) \notin [0, x)\), otherwise \(g_2(t) < x < t, g_1(y_0) = x \Rightarrow g_2(t) \leq g_1(y_0) < t\), a contradiction. Q.E.D
Lemma 5. Let \( x \in \mathbb{R}_+ \setminus \bigcup_{I \in D} I \) where \( \bar{I} \) is the closure of \( I \). Then \( g_1(x) = g_2(x) = x \).

Proof of Lemma 5. Let \( x \in \mathbb{R}_+ \setminus \bigcup_{I \in D} I \). Then \( g_1(x) = x \), otherwise, if \( x < g_1(x) \) then \((x, g_1(x)) \in D_1\) and by Lemma 4 it follows that there exists \( I \in D \) such that \((x, g_1(x)) \subseteq I\), and hence \( x \in \bar{I} \), contradicting the assumption. The same goes for \( x > g_1(x) \).

If \( x < g_2(x) \) then let \( x < t < g_2(x) \). We shall see where \( g_1(t) \) can be:
- \( g_1(t) \notin [g_2(x), \infty) \), otherwise \( t < g_2(x) \leq g_1(t) \), a contradiction.
- \( g_1(t) \notin (x, g_2(x)) \), otherwise \( x < g_1(t) < g_2(x) \), a contradiction.

It follows that \( g_1(t) \leq x < t \) and hence \((g_1(t), t) \in D_2\). By Lemma 4 \( \exists I \in D \) such that \((g_1(t), t) \subseteq I \) and therefore \( x \in \bar{I} \), contradicting the assumption.

If \( g_2(x) < x \) then let \( g_2(x) < t < x \). We shall see where \( g_1(t) \) can be:
- \( g_1(t) \notin [0, g_2(x)] \), otherwise \( g_1(t) \leq g_2(x) < t \), a contradiction.
- \( g_1(t) \notin [g_2(x), x) \), otherwise \( g_2(x) \leq g_1(t) < x \), a contradiction.

It follows that \( t < x \leq g_1(t) \) and hence \((t, g_1(t)) \in D_1\). By Lemma 4 \( \exists I \in D \) such that \((t, g_1(t)) \subseteq I \) and therefore \( x \in \bar{I} \), contradicting the assumption.

Q.E.D

Lemma 6. Let \( I \in D \). Then \( I \) satisfies the ”+” condition or the ”-” condition.

If \( I \) satisfies the ”+” condition then \( g_2(I^-) = I^- \) and \( g_1(I^+) = I^+ \).

If \( I \) satisfies the ”-” condition then \( g_1(I^-) = I^- \) and \( g_2(I^+) = I^+ \).

Proof of Lemma 6. Let \( I \in D \). We will split the proof into two parts:

Part 1: if \( I \in D_3 \cup D_5 \) then \( I \) satisfies the ”+” condition and \( g_2(I^-) = I^- \), \( g_1(I^+) = I^+ \).

Part 2: if \( I \in D_4 \cup D_6 \) then \( I \) satisfies the ”-” condition and \( g_1(I^-) = I^- \), \( g_2(I^+) = I^+ \).

Proof of part 1:

1. \( \forall t, I^- \leq t < I^+ \), we have \( g_2(t) \leq I^- \): Suppose this is not true. If \( I \in D_3 \) then \( g_1(I^-) = I^+ \) so if \( I^- < g_2(t) \leq g_1(I^-) \) it will be a contradiction. If \( t < g_1(I^-) < g_2(t) \) it will also be a contradiction. If \( I \in D_5 \) and \( I^- < g_2(t) \leq I^+ \) then we shall look at \( x_k \) of the sequence (that is given with a \( I \in D_5 \)) such that \( I^- < x_k < g_2(t) \) and \( g_1(x_k) = I^+ \). Then \( x_k < g_2(t) \leq g_1(x_k) \), a contradiction. If \( t < I^+ < g_2(t) \) then again we shall look at the same \( x_k \) and we will get that \( t < g_1(x_k) < g_2(t) \), a contradiction.
2. \( \forall t, I^- \leq t < I^+, \) we have \( g_2(t) = I^- \): Indeed, suppose there exists \( I^- \leq t < I^+ \) such that \( g_2(t) < I^- \). Choose \( s \) such that \( g_2(t) < s < I^- \leq t \). We shall see where \( g_1(s) \) can be:

- \( g_1(s) \notin [0, g_2(t)] \): otherwise \( g_1(s) \leq g_2(t) < s \), a contradiction.
- \( g_1(s) \notin (g_2(t), t) \): otherwise \( g_2(t) < g_1(s) < t \), a contradiction.
- \( g_1(s) \notin [t, I^+] \): otherwise we will find \( t' \) such that \( I^- \leq g_1(s) < t' < I^+ \), then from (1.) it follows that \( g_2(t') \leq I^- \), so \( g_2(t') \leq I^- \leq g_1(s) < t' \), a contradiction.
- \( g_1(s) \notin (I^+, \infty) \): otherwise choose \( t', I^+ < t' < g_1(s) \). We shall see where \( g_2(t') \) can be:

If \( g_2(t') \leq I^+ \) we have that

\[
g_2(t') \leq I^+ = \begin{cases} 
  g_1(I^-) < t' & I \in D_3 \\
  g_1(x_k) < t' & I \in D_5
\end{cases}
\]

a contradiction.

If \( I^+ < g_2(t') \leq g_1(s) \) we have that \( s < g_2(t') \leq g_1(s) \), a contradiction.

If \( g_1(s) < g_2(t') \) we have that \( t' < g_1(s) < g_2(t') \), a contradiction.

The only remaining possibility is \( g_1(s) = I^+ \). But since \( s < I^- \) and \( I \in D_3 \cup D_5 \), this is impossible.

3. \( \forall t, I^- < t \leq I^+ \), we have \( g_1(t) \geq I^- \): Otherwise there exists \( x_0, I^- < x_0 \leq I^+ \) such that \( I^- \leq g_1(x_0) < I^+ \) or \( g_1(x_0) < I^- \).

If \( g_1(x_0) < I^- \) then let \( t_0 \) be a number that satisfies (2.). We have that \( g_1(x_0) < I^- = g_2(t_0) < x_0 \), a contradiction.

If \( I^- \leq g_1(x_0) < I^+ \) then choose \( t_0, g_1(x_0) < t_0 < I^+ \). It follows from (2.) that \( g_2(t_0) = I^- \leq g_1(x_0) < t_0 \), a contradiction.

4. \( \forall t, I^- < t \leq I^+ \), we have \( g_1(t) = I^+ \): Otherwise there exists \( t_0, I^- < t_0 \leq I^+ \) such that \( g_1(t_0) > I^+ \). Choose \( s, I^+ < s < g_1(t_0) \). We shall see where \( g_2(s) \) can be:

- \( g_2(s) \notin [g_1(t_0), \infty) \): otherwise \( s < g_1(t_0) \leq g_2(s) \), a contradiction.
- \( g_2(s) \notin (t_0, g_1(t_0)) \): otherwise \( t_0 < g_2(s) < g_1(t_0) \), a contradiction.
- \( g_2(s) \notin [0, t_0] \): otherwise

\[
g_2(s) \leq t_0 \leq I^+ = \begin{cases} 
  g_1(I^-) < s & I \in D_3 \\
  g_1(x_k) < s & I \in D_5
\end{cases}
\]
Proof of part 2:
As the mean value exclusion condition is symmetric, by reversing the order of \( \mathbb{R}_+ \) and exchanging \( I^- \) and \( I^+ \) the proof of part 1 yields a proof of part 2. Q.E.D

Lemma 7. Let \( I_1, I_2 \in D \) such that \( I_1 \neq I_2 \). Then \( I_1 \cap I_2 = \phi \).

Proof of Lemma 7. Let us assume for the sake of contradiction that \( t \in I_1 \cap I_2 \neq \phi \). Then if \( I_1 = (I_1^-, I_1^+) \), \( I_2 = (I_2^-, I_2^+) \) we have three cases:

1. \( I_1, I_2 \) both satisfy the "+" condition. Then:
   \[ g_1(t) = I_1^+ \quad \text{and} \quad g_1(t) = I_2^+ \]
   \[ g_2(t) = I_1^- \quad \text{and} \quad g_2(t) = I_2^- \]
   \[ \Rightarrow I_1^+ = I_2^+, I_1^- = I_2^- \Rightarrow I_1 = I_2, \] a contradiction.

2. \( I_1, I_2 \) both satisfy the "-" condition. Then:
   \[ g_1(t) = I_1^- \quad \text{and} \quad g_1(t) = I_2^- \]
   \[ g_2(t) = I_1^+ \quad \text{and} \quad g_2(t) = I_2^+ \]
   \[ \Rightarrow I_1^+ = I_2^+, I_1^- = I_2^- \Rightarrow I_1 = I_2, \] a contradiction.

3. \( I_1 \) satisfies the "-" condition, and \( I_2 \) satisfies the "+" condition then:
   \( I_1 \) satisfies the "-" condition \( \Rightarrow g_1(t) = I_1^- \) and \( g_2(t) = I_1^+ \)
   \( I_2 \) satisfies the "+" condition \( \Rightarrow g_1(t) = I_2^+ \) and \( g_2(t) = I_2^- \)
   \[ \Rightarrow I_1^- = I_2^+, I_1^+ = I_2^- \] Hence one of the segments is not defined as a legal segment, a contradiction.

The symmetric case to (3.) has a symmetric proof. Q.E.D

Lemma 8. Let \( t \) be an end point of a segment \( I \in D \). Then:

1. If \( t \) is an end point of \( I \) alone then \( \forall x \in I, g_1(t) = g_1(x), g_2(t) = g_2(x) \).
2. If $t$ is an end point of two segments $I, J \in D$ then:

(a) The two segments have opposite signs.

(b) $\forall x \in I, g_1(t) = g_1(x)$ or $\forall x \in J, g_1(t) = g_1(x)$ and $\forall x \in I, g_2(t) = g_2(x)$ or
$\forall x \in J, g_2(t) = g_2(x)$.

Proof of Lemma. If $t$ is an end point of $I$ alone then we shall split the proof to 4 parts:

1. $I$ satisfies the ”+” condition and $t = I^-$. Then by Lemma 6 it follows that
$g_2(I^-) = I^-$, we will show that $g_1(I^-) = I^+$:

$g_1(I^-) \notin [I^- , I^+)$: Otherwise, we will find a number $s, g_1(I^-) < s < I^+$, and then by Lemma 6 it follows that $g_2(s) = I^-$. This implies $g_2(s) = I^- \leq g_1(I^-) < s$, a contradiction.

$g_1(I^-) \notin [0 , I^-)$: Otherwise $(g_1(I^-), I^-) \in D_2$ and by Lemma 4 it follows that there exists a segment $I' \in D$ such that $(g_1(I^-), I^-) \subseteq I'$. But by Lemma 7 $I \cap I' = \phi$. Hence $I^-$ is the right end point of $I'$, a contradiction to the assumption of this case.

$g_1(I^-) \notin (I^-, \infty)$: Otherwise $(I^-, g_1(I^-)) \in D_1$ and therefore $\exists I' \in D$ such that $I \subset (I^-, g_1(I^-)) \subseteq I'$, a contradiction to Lemma 7. The only remaining possibility is $g_1(I^-) = I^-.$

2. $I$ satisfies the ”+” condition and $t = I^+$. Then by Lemma 6 it follows that
$g_1(I^+) = I^+$, we will show that $g_2(I^+) = I^-$:

$g_2(I^+) \notin (I^- , I^+]$: Otherwise, we will find a number $s, I^- < s < g_2(I^+)$, and then by Lemma 6 it follows that $g_1(s) = I^+$. This implies $s < g_2(I^+) \leq I^+ = g_1(s)$, a contradiction.

$g_2(I^+) \notin [0 , I^-)$: Otherwise, the segment $(g_2(I^+), I^+]$ is not contained in $I$. We will show that there exists another $J \in D$ such that $(g_2(I^+), I^+) \subset J$. This will be a contradiction to Lemma 7. To show the existence of such a segment we shall show that for $s$ such that $g_2(I^+) < s < I^-$, $g_1(s) \geq I^+$. This will imply by using Lemma 4 that there exists a segment $J \in D$ as desired. Let $s$ satisfy $g_2(I^+) < s < I^-$. We will show that all other possibilities cannot be true:

$g_1(s) \notin [0, g_2(I^+)]$: Otherwise $g_1(s) \leq g_2(I^+) < s$, a contradiction.

$g_1(s) \notin (g_2(I^+), I^+]$: Otherwise $g_2(I^+) < g_1(s) < I^+$, a contradiction.
\( g_2(I^+) \notin (I^+, \infty) \): Otherwise, we will show that there exists a second segment \( J \in D \) such that \( J \neq I \) but \( I^+ \) is a left end point of \( J \). Let \( s \in (I^+, g_2(I^+)) \). Then

- \( g_1(s) \notin [0, I^-] \): Otherwise \( g_1(s) \leq I^- = g_2(I^-) < s \), a contradiction.
- \( g_1(s) \notin I \): Otherwise, we can find a number \( d \in (g_1(s), I^+) \) and then by Lemma 6 \( g_2(d) = I^- < g_1(s) < d \), a contradiction.
- \( g_1(s) \notin (I^+, g_2(I^+)] \): Otherwise \( I^+ < g_1(s) \leq g_2(I^+) \), a contradiction.
- \( g_1(s) \notin (g_2(I^+), \infty) \): Otherwise \( s < g_2(I^+) < g_1(s) \), a contradiction.

So, by Lemma 4 there exists a segment \( J \in D \) such that \((I^+, g_2(I^+)) \subseteq J\).

By Lemma 7 \( J \cap I = \emptyset \), so \( I^+ \) is the left end point of \( J \) and \( I \), a contradiction to the assumption of this case.

The only remaining possibility is \( g_2(I^+) = I^- \).

3. \( I \) satisfies the ”-” condition and \( t = I^- \): the proof is similar to (2.).

4. \( I \) satisfies the ”-” condition and \( t = I^+ \): the proof is similar to (1.).

If \( t \) is an end point of two segments, we split the proof to two parts:

1. The two segments have opposite signs:
   Assume for the sake of contradiction that \( t \) is an end point of two segments \( I_1 = (x, t), I_2 = (t, y) \) that both satisfy the ”+” condition. Then, by Lemma 6 and the fact that \( t \) is the left end point of \( I_2 \), it follows that \( g_2(t) = t \). It also follows by Lemma 3 that \( \forall x' \in (x, t) \ g_1(x') = t \). Hence \( x' < t = g_2(t) = g_1(x') \), a contradiction.
   In the same way it can be shown that \( t \) can’t be an end point of 2 segments that satisfy the ”-” condition.

2. Now we shall show that
   \( \forall x \in I_1, g_1(t) = g_1(x) \) or \( \forall x \in I_2, g_1(t) = g_1(x) \) and
   \( \forall x \in I_1, g_2(t) = g_2(x) \) or \( \forall x \in I_2, g_2(t) = g_2(x) \):
   Let us say that \( t \) is a common end point of \( I_1 = (x, t) \) that satisfies the ”+” condition, and of \( I_2 = (t, y) \) that satisfies the ”-” condition.
   (The opposite case is handled in a similar way.)
   By Lemma 6 and the fact that \( I_1 \) satisfies the ”+” condition, it follows that \( g_1(t) = t \) as for any \( x' \in (x, t) \).
So we need to show that \( g_2(t) \in \{x, y\} \):
\[ g_2(t) \notin [0, x): \text{Otherwise, choose } s, g_2(t) < s < x. \text{ We shall see where } g_1(s) \text{ can be:} \]
\[ g_1(s) \notin [0, g_2(t)]: \text{Otherwise } g_1(s) \leq g_2(t) < s, \text{ a contradiction.} \]
\[ g_1(s) \notin (g_2(t), t): \text{Otherwise } g_2(t) < g_1(s) < t, \text{ a contradiction.} \]
\[ g_1(s) \notin [t, \infty): \text{Otherwise choose } y', x < y' < t. \]

By Lemma 6 it follows that
\[ g_2(y') = x \text{ and hence } s < g_2(y') < t \leq g_1(s), \text{ a contradiction.} \]
\[ g_2(t) \notin (x, t]: \text{Otherwise choose } y', x < y' < g_2(t). \]

By Lemma 6 it follows that
\[ g_1(y') = t \text{ and hence } y' < g_2(t) \leq t = g_1(y'), \text{ a contradiction.} \]
\[ g_2(t) \notin (t, y): \text{Otherwise choose } y', g_2(t) < y' < y. \]

By Lemma 6 it follows that
\[ g_1(y') = t \text{ and hence } g_1(y') = t < g_2(t) < y', \text{ a contradiction.} \]
\[ g_2(t) \notin (y, \infty): \text{Otherwise choose } s, y < s < g_2(t). \]

We shall see where \( g_1(s) \) can be:
\[ g_1(s) \notin [0, y): \text{Otherwise choose } y', t < y' < y. \]

By Lemma 6 it follows that
\[ g_2(y') = y \text{ and hence } g_1(s) \leq g_2(y') < s, \text{ a contradiction.} \]
\[ g_1(s) \notin (y, g_2(t)]: \text{Otherwise } t < y < g_1(s) \leq g_2(t), \]
\[ \text{a contradiction.} \]
\[ g_1(s) \notin [g_2(t), \infty): \text{Otherwise } s < g_2(t) \leq g_1(s), \]
\[ \text{a contradiction.} \]

We have shown that \( g_2(t) \in \{x, y\} \).

Q.E.D

Proof of Proposition 2. Suppose that \( g_1, g_2 \) satisfy the mean value exclusion condition. Then the set of segments \( D \) (see Definition 2.7) is the set of segments promised in the proposition. \( D \) satisfies all the demanded properties:

By Lemma 7 the segments are disjoint.

From the definition of \( D \) the segments are open.

By Lemma 6 all the segments have signs. So we can define a function \( G : D \to \{+1, -1\} \) as follows:
\[ \forall I \in D \]
\[ G(I) = \begin{cases} +1 & \text{if } I \text{ satisfies the } "+" \text{ condition} \\ -1 & \text{if } I \text{ satisfies the } "-" \text{ condition} \end{cases} \]
By Lemma 8 no two segments with a common end point have the same sign.

By Lemmas 5, 6 and 8, \( g_i \) is a \((D - i, G)\) compatible function, \( i = 1, 2 \). Q.E.D

### 6.1.2 Ex post equilibria and parallelograms

Throughout the proofs we make use the following valuation function for \( i \), \( Z_i^{(a, s)} \in \mathbb{R}_{+}, \) which assigns \( a \in A \) the value \( s > 0 \) and zero otherwise.

We denote \( g_i^{(a, a')} (s) = b_i(Z_i^{(a, s)} (a)) - b_i(Z_i^{(a, s)} (a')) \)

**Lemma 9.** Let \( n \geq 2 \) and \( |A| \geq 3 \). Let \( b \) be an ex-post equilibrium for the class of VCG games over \((N, A, V)\). Assume that for some \( i, j \in N \) and for any \( s \in \mathbb{R} \), \( Z_i^{(a, s)} \in V_i \) and \( Z_j^{(a', s)} \in V_j \). Then \( g_i^{(a, a')} \) and \( g_j^{(a', a)} \) satisfy the mean exclusion condition.

**Proof.** Assume, to the contrary of the claim, \( b_i(Z_i^{(a, s)} (a)) - b_i(Z_i^{(a, s)} (a')) > s \), and there exists a player \( j \) and some \( t \) such that \( Z_j^{(a', t)} \in V_j \) and

\[
s < b_j(Z_j^{(a', t)} (a')) - b_j(Z_j^{(a', t)} (a)) \leq b_i(Z_i^{(a, s)} (a)) - b_i(Z_i^{(a, s)} (a')).
\]

By Lemma 11, \( b_j(Z_j^{(a', t)} (a')) > b_j(Z_j^{(a', t)} (\hat{a})) \) for all \( \hat{a} \neq a' \).

Let's consider a simple VCG game with 2 players, \( i \) and \( j \), where the mechanism’s tie breaking rule, in case of a tie between \( a \) and \( a' \), is to choose \( a \).

Consider the instance where \( i \)'s valuation is \( Z_i^{(a, s)} \) and \( j \)'s valuation is \( Z_j^{(a', t)} \). Assume that some \( \hat{a} \not\in \{a, a'\} \) is chosen in this game. In this case \( i \)'s utility is \( j \)'s valuation of \( \hat{a} \), namely \( b_j(Z_j^{(a', t)} (\hat{a})) \). Compare this to \( i \)'s utility had he announced zero on all alternatives. In this case \( a' \) would have been the chosen alternative and \( i \) would have received a utility of \( b_j(Z_j^{(a', t)} (a')) \). As \( b_j(Z_j^{(a', t)} (a')) > b_j(Z_j^{(a', t)} (\hat{a})) \) we have a contradiction with the assumption that \( b \) forms an ex-post equilibrium.

We conclude that either \( a \) or \( a' \) must chosen.

By our assumption \( b_j(Z_j^{(a', t)} (a')) + b_i(Z_i^{(a, s)} (a')) \leq b_j(Z_j^{(a', t)} (a)) + b_i(Z_i^{(a, s)} (a)) \), and so \( a \) is actually chosen, and the utility of \( i \) is \( s + b_j(Z_j^{(a', t)} (a)) \).

On the other hand let's assume \( i \) would have announced truthfully. By the assumption \( s + b_j(Z_j^{(a', t)} (a)) < b_j(Z_j^{(a', t)} (a')) \), leading to \( a' \) being chosen, and consequently \( i \)'s utility would have been \( b_j(Z_j^{(a', t)} (a')) \).

By our assumption \( b_j(Z_j^{(a', t)} (a')) > s + b_j(Z_j^{(a', t)} (a)) \), which stands in contradiction to the fact the \( b \) is an ex-post equilibrium of the 2 player game. Q.E.D
Corollary 1. Let \( n \geq 2 \) and \( |A| \geq 3 \). Let \( b \) be an ex-post equilibrium for the class of VCG games over \((N,A,V)\). For any \( a,a' \in A \) there exists a set of disjoint open segments, denoted \( \Omega^{(a,a')} \) and a function \( G^{(a,a')} : \Omega^{(a,a')} \to \{-1, +1\} \), for which the pair of functions \( g_1^{(a,a')}(.) \) and \( g_2^{(a,a')}(.) \) are compatible.

Proof. Follows directly from Lemma 9 and Proposition 2. Q.E.D

Lemma 10. Let \( n \geq 3 \), \( |A| \geq 3 \), and let \( b \) be an ex-post equilibrium for the class of VCG games over \((N,A,(\mathbb{R}^+)^n)\), then
\[ b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) = s \quad \text{for all} \quad i \in N, \quad s \in \mathbb{R}^+ \quad \text{and} \quad a' \neq a \in A. \]

Proof. Assume the claim is not true and that \( b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) \neq s \) for some \( i \in N, \quad s \in \mathbb{R}^+ \) and \( a \in A \). We will assume that \( b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) > s \). The case that \( b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) < s \) is similar, and therefore omitted.

Choose \( t \) such that \( b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) > t > s \) and a player \( j \neq i \).

Case 1: Assume \( b_j(Z_j^{(a',t)}(a')) - b_j(Z_j^{(a',t)}(a)) \geq t \). If in addition \( b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) \geq b_j(Z_j^{(a',t)}(a')) - b_j(Z_j^{(a',t)}(a)) \) then we get a contradiction to lemma 9.
Otherwise, \( b_j(Z_j^{(a',t)}(a')) - b_j(Z_j^{(a',t)}(a)) > b_i(Z_i^{(a,s)}(a)) - b_i(Z_i^{(a,s)}(a')) \), which leads again to a contradiction of lemma 9 with the roles of \( i \) and \( j \) reversed.

Case 2: Assume \( b_j(Z_j^{(a',t)}(a')) - b_j(Z_j^{(a',t)}(a)) < t \) and consider a third alternative \( a'' \notin \{a,a'\} \). By lemma 3 \( b_j(Z_j^{(a',t)}(a'')) - b_j(Z_j^{(a',t)}(a)) \) and therefore \( b_j(Z_j^{(a',t)}(a')) - b_j(Z_j^{(a',t)}(a'')) < t \).

Consider a third player \( l \). Obviously, \( b_l(Z_l^{(a'',t)}(a'')) - b_l(Z_l^{(a'',t)}(a')) < t \) as well (otherwise we can replicate the arguments of case 1). By applying lemma 3 we conclude that \( b_l(Z_l^{(a'',t)}(a'')) - b_l(Z_l^{(a'',t)}(a')) < t \) as well. Lets assume, without loss of generality that \( b_j(Z_j^{(a',t)}(a')) - b_j(Z_j^{(a',t)}(a'')) \leq b_l(Z_l^{(a'',t)}(a'')) - b_l(Z_l^{(a'',t)}(a')) < t \). This conflicts lemma 9 where \( j \) is in the role of \( i \) and \( l \) in the role of \( j \). Q.E.D

Lemma 11. \( I \in \Omega^{(a,a')} \) implies \( I^- \neq 0 \).

Proof. Assume the claim is wrong. This implies that there exists a player, w.l.o.g player 1, and a valuation \( v_1 \) such that \( v_1(a) - v_1(a') > 0 \) where \( a \) is a maximizing alternative for \( v_1 \), but \( b_1(v_1(a)) - b_1(v_1(a')) = 0 \). Among all the VCG mechanisms for the single player game, there exist one that chooses the alternative \( a' \), in case of tie between \( a \) and \( a' \). This contradicts the fact that, in an ex post equilibrium, if player 1 is on his own that the maximizing alternative must always be chosen. Q.E.D
Proposition 3. Let \((b_1, b_2)\) be an ex post equilibrium in the VCG mechanisms. Let \(v_1, v_2 \in V\) be two valuations for players 1 and 2, such that \(a\) is a maximizing alternative for \(v_1\) and \(a'\) is a maximizing alternative for \(v_2\). For any \(s\) which is not an end point of two segments in \(\Omega^{(a,a')}\):

- If \(v_1(a) - v_1(a') = s\) then \(b_1(v_1)(a) - b_1(v)(a') = g_1^{(a,a')}(s)\).
- If \(v_2(a') - v_2(a) = s\) then \(b_2(v_2)(a') - b_2(v_2)(a) = g_2^{(a,a')}(s)\).

Proof of Proposition. Let \(s \in \mathbb{R}_+, v_1 \in V\) such that \(v_1(a) - v_1(a') = s\). \(a\) is a maximizing alternative for \(v_1\) and \(s\) is not an end point of two segments. Then we have:

1. \(g_1^{(a,a')}(s) = s\):
   - If \(s = g_1^{(a,a')}(s) < b_1(v_1)(a) - b_1(v_1)(a')\): Choose \(t, s = g_1^{(a,a')}(s) < t < b_1(v_1)(a) - b_1(v_1)(a')\). From the mean value exclusion condition it follows that \(s < g_2^{(a,a')}(t)\). Consider the profile \((b_1(v_1), b_2(Z_2^{(a',t)}))\). Let \(\gamma\) be a maximizing alternative of \((b_1(v_1), b_2(Z_2^{(a',t)}))\). It follows by Lemma 2 that \(\gamma\) is also a maximizing alternative of \((v_1, b_2(Z_2^{(a',t)}))\). Then \(\gamma = a'\), for otherwise \(g_2(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(\gamma) < v_1(\gamma) - v_1(a') \leq v_1(a) - v_1(a') = s < g_2(t)\), a contradiction.
     Again by Lemma 2 \(a'\) should be a maximizing alternative of \((b_1(v_1), Z_2^{(a',t)})\) as well. But \(b_1(v_1)(a) - b_1(v_1)(a') > t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(a)\), a contradiction.
   - If \(b_1(v)(a) - b_1(v_1)(a') < s = g_1^{(a,a')}(s)\): Choose \(t, b_1(v_1)(a) - b_1(v_1)(a') < t < s\). From the mean value exclusion condition it follows that \(g_2^{(a,a')}(t) < s\). Consider the profile \((b_1(v_1), b_2(Z_2^{(a',t)}))\). Let \(\gamma\) be a maximizing alternative of \((b_1(v_1), b_2(Z_2^{(a',t)}))\). It follows by Lemma 2 that \(\gamma\) is also a maximizing alternative of \((b_1(v_1), Z_2^{(a',t)})\). Then \(\gamma = a'\), for otherwise \(t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(\gamma) > b_1(v_1)(a) - b_1(v_1)(a') \leq b_1(v_1)(\gamma) - b_1(v_1)(a'), a contradiction.
     Again by Lemma 2 \(a'\) should be a maximizing alternative of \((v_1, b_2(Z_2^{(a',t)}))\) as well. But \(v_1(a) - v_1(a') = s > g_2^{(a,a')}(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(a), a contradiction.

2. \(g_1^{(a,a')}(s) < s\): Let \(I \in \Omega^{(a,a')}\) be a segment with \(G(I)) = -1\) such that \(g_1^{(a,a')}(s) = I^- < s \leq I^+\). Such a segment exists by Proposition. Now consider three cases:
\begin{itemize}
\item $b_1(v_1)(a) - b_1(v_1)(a') < g_1^{(a,a')}(s) < s$: Choose $t, b_1(v_1)(a) - b_1(v_1)(a') < t < g_1^{(a,a')}(s)$. It emerges from the mean value exclusion condition that $g_2^{(a,a')}(t) < g_1^{(a,a')}(s)$. Consider the profile $(b_1(v_1), b_2(Z_2^{(a',t)}))$. Let $\gamma$ be a maximizing alternative of $(b_1(v_1), b_2(Z_2^{(a',t)}))$. It follows by Lemma 2 that $\gamma$ is also a maximizing alternative of $(b_1(v_1), Z_2^{(a',t)})$. Then $\gamma = a'$, for otherwise $b_1(v_1)(\gamma) - b_1(v_1)(a') \leq b_1(v_1)(a) - b_1(v_1)(a') < t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(\gamma)$, a contradiction.

Again by Lemma 2 $a'$ should be a maximizing alternative of $(v_1, b_2(Z_2^{(a',t)}))$ as well. But $v_1(a) - v_1(a') = s > g_1^{(a,a')}(s) > g_2^{(a,a')}(t)) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(a)$, a contradiction.

\item $g_1^{(a,a')}(s) < b_1(v_1)(a) - b_1(v_1)(a') < s$: Choose $t, g_1^{(a,a')}(s) < t < b_1(v_1)(a) - b_1(v_1)(a')$. It emerges from the mean value exclusion condition that $g_2^{(a,a')}(t) \geq s$. Consider the profile $(b_1(v_1), b_2(Z_2^{(a',t)}))$. Let $\gamma$ be a maximizing alternative of $(b_1(v_1), b_2(Z_2^{(a',t)}))$. Then $\gamma = a'$, for otherwise $b_1(v_1)(\gamma) - b_1(v_1)(a') \leq b_1(v_1)(a) - b_1(v_1)(a') < s \leq g_2^{(a,a')}(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(\gamma)$, a contradiction.

Again by Lemma 2 $a'$ should be a maximizing alternative of $(b_1(v_1), Z_2^{(a',t)})$ as well. But $b_1(v_1)(a) - b_1(v_1)(a') > t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(a)$, a contradiction.

\item $g_1^{(a,a')}(s) < s \leq b_1(v_1)(a)$: There are three cases:

(a) $s = b_1(v_1)(a)$: Choose $t, g_1^{(a,a')}(s) < t < s$. It emerges from the mean value exclusion condition that $g_2^{(a,a')}(t) \geq s$. Consider the profile $(b_1(v_1), b_2(Z_2^{(a',t)}))$. Note that $a'$ is a maximizing alternative of $(b_1(v_1), b_2(Z_2^{(a',t)}))$ (not necessarily the only one). For otherwise, there exists an alternative $\gamma$ which gives a better social surplus. But, $b_1(v_1)(\gamma) - b_1(v_1)(a') \leq b_1(v_1)(a) - b_1(v_1)(a') < g_2^{(a,a')}(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(\gamma)$, a contradiction.

By Lemma 2 $a'$ should be a maximizing alternative of $(b_1(v_1), Z_2^{(a',t)})$ as well. But $b_1(v_1)(a) - b_1(v_1)(a') > t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(a)$, a contradiction.

(b) $s = I^+, s < b_1(v_1)(a) - b_1(v_1)(a')$: There are two cases induced when $s$ is not an end point of two segments:

i. $I^+$ is a limit point of segments $I_k \in (a,a')$ that lie to the right of $I^+$. Then we can find a segment $I_j$ such that $I^+ < I_j^- < I_j^+ < b_1(v_1)(a) - b_1(v_1)(a')$. Choosing a number $s_0 \in I_j$ we have $I^+$ <
\[ g_1^{(a,a')}(s_0), g_2^{(a,a')}(s_0) < b_1(v_1)(a) - b_1(v_1)(a'). \]

ii. \( I^+ \) is not a limit of segments. Then we can find a number \( s_0, s < s_0 < b_1(v_1)(a) \) such that \( s_0 \in \mathbb{R}_+ \setminus \bigcup_{I \in Q(a,a')} I \) where \( I \) is the closure of \( I \). For such \( s_0 \) we have \( I^+ < g_1^{(a,a')}(s_0) = g_2^{(a,a')}(s_0) = s_0 < b_1(v_1)(a). \)

In both cases we shall look at the profile \((b_1(v_1), b_2(Z_2^a, s_0))\). As \( g_2^{(a,a')}(s_0) < b_1(v_1)(a) - b_1(v_1)(a'), \) it follows that \( a \) is a maximizing alternative. By Lemma 2 it should be a maximizing alternative of \((v_1, b_2(Z_2^a, s_0))\) as well. But \( v_1(a) - v_1(a') = s = I^+ < g_2^{(a,a')}(s_0) = b_2(Z_2(v_1, s_0))(a') - b_2(Z_2^a, s_0)(a) \), a contradiction.

(c) \( s < b_1(v_1)(a) - b_1(v_1)(a'), s \neq I^+ \): Then \( s < I^+ = g_2^{(a,a')}(I^+) \) and \( \forall s', s < s' < I^+ \), we have \( g_2^{(a,a')}(s') = I^+ \). There are two cases:

i. \( b_1(v_1)(a) - b_1(v_1)(a') \leq I^+ \). Choose \( s', s < s' < b_1(v_1)(a) - b_1(v_1)(a') \leq I^+ \). Consider the profile \((b_1(v_1), b_2(Z_2^a, s'))\).

Note that \( a' \) is a maximizing alternative of \((b_1(v_1), b_2(Z_2^a, s'))\) (not necessarily the only one). For otherwise, there exists an alternative \( \gamma \) which gives a better social surplus. But, \( b_1(v_1)(\gamma) - b_1(v_1)(a') \leq \gamma = b_1(v_1)(a) - b_1(v_1)(a') \leq I^+ = b_2(Z_2^a, s') (a') - b_2(Z_2^a, s') (a) \), a contradiction. It follows by Lemma 2 that \( a' \) also maximizes \((b_1(v_1), Z_2^a, s')\).

But \( b_1(v_1)(a) - b_1(v_1)(a') > s = Z_2^a, s' (a') - Z_2^a, s' (a) \), a contradiction.

ii. \( I^+ < b_1(v_1)(a) - b_1(v_1)(a') \). Consider the profile \((b_1(v_1), b_2(Z_2^a, I^+))\).

Note that \( a \) is a maximizing alternative of \((b_1(v_1), b_2(Z_2^a, I^+))\) (not necessarily the only one). For otherwise, there exists an alternative \( \gamma \) which gives a better social surplus. But if \( \gamma = a' \) then, \( b_1(v_1)(a) - b_1(v_1)(a') > I^+ = b_2(Z_2^a, I^+)(a') - b_2(Z_2^a, I^+)(a) \), a contradiction. If \( \gamma \neq a, a' \) then, \( b_1(v_1)(a) - b_1(v_1)(\gamma) = b_2(Z_2^a, I^+)(\gamma) - b_2(Z_2^a, I^+)(a) \), a contradiction. It follows by Lemma 2 that it also maximizes \((v_1, b_2(Z_2^a, I^+))\). But \( v_1(a) - v_1(a') = s < I^+ = b_2(Z_2^a, I^+)(a') - b_2(Z_2^a, I^+)(a) \), a contradiction.

3. \( g_1^{(a,a')}(s) > s \): This case is handled in a similar way as the previous case.

For player 2 the proof is similar. Q.E.D
Proposition 4. Let \((b_1, b_2)\) be an ex post equilibrium in the VCG mechanisms. Let \(v_1, v_2 \in \mathcal{V}\) be two valuations for players 1 and 2, such that \(a\) is a maximizing alternative for \(v_1\) and \(a'\) is a maximizing alternative for \(v_2\). Let \(s\) be an end point of two segments \(I = (x, s), J = (s, y) \in \Omega^{(a,a')}\). If \(v_1(a) - v_1(a') = s\) and \(v_2(a') - v_2(a) = s\) one of the following must hold:

1. If \(G^{(a,a')} (I) = -1\) and \(G^{(a,a')} (J) = +1\) then:
   \[ b_1(v_1)(a) - b_1(v_1)(a') = x \text{ or } y \]
   \[ b_2(v_2)(a') - b_2(v_2)(a) = s \]

2. If \(G^{(a,a')} (I) = +1\) and \(G^{(a,a')} (J) = -1\) then:
   \[ b_1(v_1)(a) - b_1(v_1)(a') = s \]
   \[ b_2(v_2)(a') - b_2(v_2)(a) = x \text{ or } y \]

Proof of Proposition 4. Let \(s \in \mathbb{R}_+, v_1 \in \mathcal{V}\) such that \(v_1(a) - v_1(a') = s\). \(a\) is a maximizing alternative for \(v_1\) and \(s\) is an end point of two segments \(I = (x, s), J = (s, y) \in \Omega^{(a,a')}\). There are three cases to consider, \(g_1^{(a,a')}(s) = s\), \(g_1^{(a,a')}(s) < s\) and \(g_1^{(a,a')}(s) > s\):

1. \(g_1^{(a,a')}(s) = s\): In the proof of 3 where \(g_1^{(a,a')}(s) = s\), there was no use of the fact that \(s\) wasn’t an end point of two segments. Therefore the result is valid in this case as well.

2. \(g_1^{(a,a')}(s) < s\):

   Assume for the sake of contradiction that \(b_1(v_1)(a) - b_1(v_1)(a') \notin \{g_1^{(a,a')}(s), y\}\).

   There are four cases:

   (a) \(b_1(v_1)(a) - b_1(v_1)(a') < g_1^{(a,a')}(s) = x\): Choose \(t, b_1(v_1)(a) - b_1(v_1)(a') < t < g_1^{(a,a')}(s) = x\). Consider the profile \((b_1(v_1), b_2(Z_2^{(a', t)})\)). Let \(\gamma\) be a maximizing alternative for this profile. Then by Lemma 2 it maximizes \((b_1(v_1), Z_2^{(a', t)})\) as well. Hence \(\gamma = a'\), for otherwise \(b_1(v_1)(\gamma) - b_1(v_1)(a') \leq b_1(v_1)(a) - b_1(v_1)(a') < t = Z_2^{(a', t)}(a') - Z_2^{(a', t)}(\gamma)\). It also follows by Lemma 2 that \(a'\) maximizes \((v_1, b_2(Z_2^{(a', t)})\)). Therefore \(g_2^{(a,a')}(t) = b_2(Z_2^{(a', t)})(a') - b_2(Z_2^{(a', t)})(a) \geq v_1(a) - v_1(a') = s\). This contradicts the mean value exclusion condition.
(b) \( g_1^{(a,a')}(s) < b_1(v_1)(a) - b_1(v_1)(a') \leq s \): Choose \( t, g_1^{(a,a')}(s) < t < b_1(v_1)(a) - b_1(v_1)(a') \leq s \). Then \( g_2^{(a,a')}(t) = s \). Consider the profile \( (b_1(v_1), b_2(Z_2^{(a',t)})) \). Note that \( a' \) is a maximizing alternative of \( (b_1(v_1), b_2(Z_2^{(a',t)})) \) (not necessarily the only one). For otherwise, there exists an alternative \( \gamma \) which gives a better social surplus. But, \( b_1(v_1)(\gamma) - b_1(v_1)(a') \leq b_1(v_1)(a) - b_1(v_1)(a') \leq s = g_2^{(a,a')}(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(\gamma) \), a contradiction. By Lemma 2 it maximizes \( (b_1(v_1), Z_2^{(a',t)}) \) as well, but \( b_1(v_1)(a) - b_1(v_1)(a') > t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(a) \), a contradiction.

(c) \( s < b_1(v_1)(a) - b_1(v_1)(a') < y \): Choose \( t, s < b_1(v_1)(a) - b_1(v_1)(a') < t < b \). Then \( g_2^{(a,a')}(t) = s \). Consider the profile \( (b_1(v_1), b_2(Z_2^{(a',t)})) \). Note that \( a \) is a maximizing alternative of \( (b_1(v_1), b_2(Z_2^{(a',t)})) \) (not necessarily the only one). For otherwise, there exists an alternative \( \gamma \) which gives a better social surplus. But if \( \gamma = a' \) then, \( b_1(v_1)(a) - b_1(v_1)(a') > s = g_2^{(a,a')}(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(a) \), a contradiction. If \( \gamma \neq a, a' \) then, \( b_1(v_1)(a) - b_1(v_1)(\gamma) \geq 0 = b_2(Z_2^{(a',t)})(\gamma) - b_2(Z_2^{(a',t)})(a) \), a contradiction. By Lemma 2 it maximizes \( (b_1(v_1), Z_2^{(a',t)}) \) as well, but \( b_1(v_1)(a) - b_1(v_1)(a') > t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(a) \), a contradiction.

(d) \( y < b_1(v_1)(a) - b_1(v_1)(a') \): Choose \( t, b < t < b_1(v_1)(a) \). Consider the profile \( (b_1(v_1), b_2(Z_2^{(a',t)})) \). Let \( \gamma \) be a maximizing alternative for this profile. Then by Lemma 2 it maximizes \( (b_1(v_1), Z_2^{(a',t)}) \) as well. Hence \( \gamma \neq a' \), for otherwise \( b_1(v_1)(a) - b_1(v_1)(a') > t = Z_2^{(a',t)}(a') - Z_2^{(a',t)}(a) \), a contradiction. It also follows by Lemma 2 that \( \gamma \) maximizes \( (v_1, b_2(Z_2^{(a',t)})) \). Therefore \( g_2^{(a,a')}(t) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(\gamma) = b_2(Z_2^{(a',t)})(a') - b_2(Z_2^{(a',t)})(a) < v_1(\gamma) - v_1(a') < v_1(a) - v_1(a') = s \). This contradicts the mean value exclusion condition.

3. \( g_1^{(a,a')}(s) > s \): This case is handled in a similar way as the previous case.

For player 2 the proof is similar. Q.E.D

Lemma 12. Let \( a, a' \) be any two alternatives and let \( \Omega^{(a,a')} \) be the set of segments induced by Proposition \( \Box \) then for any \( \epsilon > 0 \) we can find a \( \delta, 0 < \delta < \epsilon \) such that \( 0 < g_1^{(a,a')}(\delta), g_2^{(a,a')}(\delta) < \epsilon \).

Proof of Lemma 12. \( \Box \) Let \( \epsilon > 0 \), if there exists a segment \( I = (x, y) \) in \( \Omega^{(a,a')} \) such that \( y < \epsilon \) then due to Lemma 11 \( x > 0 \), and from Proposition \( \Box \) it follows that for any \( \delta \)
such that \( x < \delta < y \), \( 0 < x \leq g_1(a,a')(\delta), g_2(a,a')(\delta) \leq y < \epsilon \). Other wise there are two cases left:

- **case 1:** there exists a segment \( I = (x, y) \) in \( \Omega(a,a') \) such that \( x < \epsilon < y \) again due to Lemma 410 \( 0 < x \). As for this cases conditions it follows that for any segment \( J \in \Omega(a,a'), J \cap (0, x) = \phi \). So, for any \( \delta \in (0, x) \), \( g_1(a,a')(\delta) = g_2(a,a')(\delta) = \delta \) where \( 0 < \delta < \epsilon \) as required.

- **case 2:** For any segment \( J \in \Omega(a,a'), J \cap (0, \epsilon) = \phi \). then for any \( \delta \in (0, \epsilon) \), \( g_1(a,a')(\delta) = g_2(a,a')(\delta) = \delta \) where \( 0 < \delta < \epsilon \) as required.

Q.E.D

### 6.2 Proof of Theorems 1 and 2

#### 6.2.1 The Easy Direction

*Proof. The easy direction of Theorem 1* We shall first show for an arbitrary set of functions \( f_i : \mathcal{V} \rightarrow \mathbb{R}_+ \), \( i = 1, \ldots, n \). The strategy tuple \( b_i(v_i)(a) = v_i(a) + f_i(v_i) \) forms an ex-post equilibrium for the class of VCG games over \((N, A, \mathbb{R}_+^A)^n\).

Consider a VC mechanism \( d \), a profile of valuations \( v = (v_1, \ldots, v_n) \in V^N \), \( n \) arbitrary functions \( f_i : \mathcal{V} \rightarrow \mathbb{R}_+ \) and a buyer \( i \). According to the strategies \( b_i(v_i)(a) = v_i(a) + f_i(v_i) \), the profile of announced valuations is \( \hat{v} = (v_1(a) + f_1(v_1), \ldots, v_n(a) + f_n(v_n)) \). Let

\[
    t = \max_{a \in A} \sum_{j \neq i} v_j(\hat{a}) + f_j(v_j).
\]

Let \( v' \) be the profile of announced valuations consisting of an arbitrary announcement \( v'_i \) of buyer \( i \) and the fixed announcements \( v_j(a) + f_j(v_j) \) of buyers \( j \in N \setminus \{i\} \). Suppose that the alternative \( d(v') \) is \( a' \). Then the utility of buyer \( i \) is

\[
    u_i^d(v_i, v') = v_i(a') - c_i^d(v) = v_i(a') + \sum_{j \neq i} v_j(a') + f_j(v_j) - t
\]

This is maximized when \( a' \) maximizes \( v_i(\hat{a}) + \sum_{j \neq i} v_j(\hat{a}) \). But this is exactly what the mechanism maximizes when it chooses an alternative. So, by announcing \( v_i \) the utility of \( i \) will be maximized. But if he announces \( v_i + f_i(v_i) \) where \( f_i(v_i) \) does not change on the different alternatives then the mechanism still maximizes \( i \)'s utility.

Note that the above arguments fully mimic the proof of the standards arguments for proving that VCG mechanisms are incentive compatible.
The easy direction of Theorem 2 follows as a corollary from the above arguments and Proposition 1.

6.2.2 The Difficult Direction: \( n \geq 2 \) and \( A > 2 \)

The difficult direction of Theorem 2: In fact, to prove this direction we may assume, without loss of generality, that there are exactly \( n = 2 \) players (recall the definition of an ex post equilibrium) and \( |A| \geq 3 \), or, alternatively that there are \( n = 3 \) players. Assume for the sake of contradiction that the claim is wrong and that for some ex-post equilibrium \( b \), there exists an agent \( i \), without loss of generality \( i = 1 \) and a valuation function \( v_1 \), and two alternatives, \( a, a' \in A \) such \( b_1(v_1)(a) - v_1(a) \neq b_1(v_1)(a') - v_1(a') \). Without loss of generality we may choose \( a \) such that \( v_1(a) = \text{argmax}_{a' \in A} v_1(a') \). There are two cases:

1. \( b_1(v_1)(a) - v_1(a) > b_1(v_1)(a') - v_1(a') \). In this case \( b_1(v_1)(a) - b_1(v_1)(a') > v_1(a) - v_1(a') \) by Proposition 2, Proposition 3 and Proposition 4. The corresponding \( \Omega^{(a,a')} \) is not empty and there exists a segment \( I \) such that \( G(I) = +1 \) in \( \Omega^{(a,a')} \). Denote \( I = (x_1, x_2) \) and \( h = x_2 - x_1 \).

Consider the following two valuations:

\[
u_2(\hat{a}) = \begin{cases} 
 x_2 - h_3 - x_1 & \text{if } \hat{a} = a \\
 x_2 & \text{if } \hat{a} = a' \\
 x_2 - h_2 - x_1 & \text{if } \hat{a} = a'' \\
 x_2 - x_1 - h_4 & \text{otherwise}
\end{cases}
\]

Where \( 0 < h_2 < h_3 < h_4 < h \).

\[
u_1(\hat{a}) = \begin{cases} 
 M & \text{if } \hat{a} = a \\
 M - x_1 - h_1 & \text{if } \hat{a} = a' \\
 M - x_1 + \delta & \text{if } \hat{a} = a'' \\
 0 & \text{otherwise}
\end{cases}
\]

Where \( a'' \neq a, a', \) \( x_2 < M, \) \( 0 < h_1 < h \) and \( \delta \) is chosen such that it is not a common end point of two segments in \( \Omega^{(a,a'')} \) and \( 0 < g_{1}^{(a,a'')}(x_1 - \delta) < h_3 - h_2, \) \( 0 < \delta < x_1 \) this is possible as for lemma 12.

\[\text{Indeed, suppose there more players and there is an ex post equilibrium which is not of the form stated in the Theorem. By definition, it must be an equilibrium for 2 players as well.}\]
Note that the following emerges from Proposition 2 and Proposition 3:

\[ b_1(u_1)(a) - b_1(u_1)(a') = x_2 \]
\[ b_1(u_1)(a) - b_1(u_1)(a'') = g_1^{a,a''}(x_1 - \delta) \]

\[ b_2(u_2)(a') - b_2(u_2)(a) = x_1 \]
\[ b_2(u_2)(a') - b_2(u_2)(a'') = x_1 \]

The following will show that for the profile of strategies \((b_1(u_1), b_2(u_2))\), the alternative \(a\) is the only maximizing alternative.

We show that the total announcements at \(a\) exceeds that of \(\hat{a} \in A\), where \(\hat{a} \neq a, a', a''\):

\[ b_1(u_1)(a) - b_1(u_1)(\hat{a}) = g_1^{a,\hat{a}}(M) = g_1^{a,a'}(M) \geq x_2 \]

the second equality follows from lemma 3
\[ b_2(u_2)(a') - b_2(u_2)(a) = x_1 \]
\[ b_2(u_2)(a') - b_2(u_2)(\hat{a}) = x_1 \] again from lemma 3
Then it follows that \(b_2(u_2)(\hat{a}) - b_2(u_2)(a) = 0\)
So we have that \(b_2(u_2)(\hat{a}) - b_2(u_2)(a) = 0 < x_2 \leq b_1(u_1)(a) - b_1(u_1)(\hat{a})\)
which means that \(b_1(u_1)(\hat{a}) + b_2(u_2)(\hat{a}) < b_1(u_1)(a) + b_2(u_2)(a)\).

We now show that the total announcements at \(a\) exceeds that of \(a'\):

\[ b_2(u_2)(a') - b_2(u_2)(a) = x_1 < x_2 = b_1(u_1)(a) - b_1(u_1)(a') \]

We now show that The total announcements at \(a\) exceeds that of \(a''\):

\[ b_1(u_1)(a) - b_1(u_1)(a'') = g_1^{a,a''}(x_1 - \delta) \]
\[ b_2(u_2)(a') - b_2(u_2)(a) = x_1 \]
\[ b_2(u_2)(a') - b_2(u_2)(a'') = x_1 \]
it follows that \(b_2(u_2)(a'') - b_2(u_2)(a) = 0\)
So we that \(b_2(u_2)(a'') - b_2(u_2)(a) = 0 < g_1^{a,a''}(x_1 - \delta) = b_1(u_1)(a) - b_1(u_1)(a'')\).

This proves that \(a\) is the only maximum of \((b_1(u_1), b_2(u_2))\). By lemma 2 \(a\) should be a maximum of the profile \((b_1(u_1), u_2)\), but \(u_2(a'') - u_2(a) = x_2 - h_2 - x_1 - x_2 + h_3 + x_1 = h_3 - h_2 > g_1^{a,a''}(x_1 - \delta) = b_1(u_1)(a) - b_1(u_1)(a'')\) which means
that $u_2(a'') + b_1(u_1)(a'') > b_1(u_1)(a) + u_2(a)$ a contradiction.

2. The proof for the case that $b_1(v_1)(a) - v_1(a) < b_1(v_1)(a') - v_1(a')$ is similar to the previous case, and is therefore omitted.

The difficult direction of Theorem 1 follows as a corollary from the proof of the difficult direction of Theorem 2 and Proposition 1.

Q.E.D

6.2.3 The Difficult Direction: $n \geq 3$

**Proof of Theorem 2:** Assume the claim is wrong and that for some ex-post equilibrium $b$, there exists an agent $i$ and a valuation function, $v_i$, and two alternatives, $a, a' \in A$ such that $b_1(v_i)(a) - v_i(a) \neq b_1(v_i)(a') - v_i(a')$. Without loss of generality we may choose $a$ such that $b_1(v_i)(a) = \arg\max_{a \in A} b_1(v_i)(a)$.

**Case 1:** $b_1(v_i)(a) - v_i(a) > b_1(v_i)(a') - v_i(a')$.

In this case $b_1(v_i)(a) + v_i(a') > b_1(v_i)(a') + v_i(a)$. Let $t$ satisfy $b_1(v_i)(a) - b_1(v_i)(a') > t > v_i(a) - v_i(a')$, and consider player $j$ and a valuation $v_j = Z_j^{(a',t)}$.

By lemma 3, $b_j(Z_j^{(a',t)}(a)) = b_j(Z_j^{(a',t)})(a'')$, which together with the choice of $a$ implies $b_1(v_i)(a) + b_j(Z_j^{(a',t)}(a)) \geq b_1(v_i)(a'') + b_j(Z_j^{(a',t)})(a'')$. By a proper choice of the mechanism we may find a simple VCG game such that $a''$ is not chosen, and therefore, either $a$ or $a'$ are chosen.

Assume $a$ is chosen. Then the utility of $i$ is $v_i(a) + b_j(Z_j^{(a',t)}(a))$, which, by lemma 10, is equal $v_i(a) + b_j(Z_j^{(a',t)}(a')) - t$. This in turn is less than $v_i(a) + b_j(Z_j^{(a',t)}(a') - (v_i(a) - v_i(a'))) = b_j(Z_j^{(a',t)})(a') + v_i(a')$, contradicting lemma 2.

Therefore, it must be the case that $a'$ is chosen. However, consider $j$’s utility, $t + b_i(v_j)(a') < b_i(v_j)(a') - b_i(v_i)(a') + b_i(v_i)(a) = b_i(v_i)(a)$, again contradicting lemma 2.

**Case 2:** $b_1(v_i)(a) - v_i(a) < b_i(v_i)(a') - v_i(a')$.

This case is repeated with analogous arguments with $v_j = Z_j^{(a',t)}$, where $b_1(v_i)(a) - b_i(v_i)(a') < t < v_i(a) - v_i(a')$.

Q.E.D

**Proof of Theorem 1:** Follows as a corollary from the proof of the difficult direction of Theorem 2 and Proposition 1.

Q.E.D
6.3 Proof of Theorems 3 and 4

Proof of Theorem 3: Assume that for some $N' \subset N$ and for some specific realization of valuations, there exists a player $i \in N'$ which can benefit from deviation. This means that deviating to the strategy $\hat{b}_i(v_i)(a) = v_i(a)$ is also strictly beneficial (recall that truth telling is a dominant strategy for all VCG games). However, truth telling cannot change the chosen alternative and therefore cannot change $i$’s utility. Q.E.D.

Throughout this subsection we fix the valuation sets, $\mathcal{R}_i(a_i)$. For each player $i$, let $a_i$ denote the optimal element with respect to $\mathcal{R}_i(a_i)$. The following lemma is in the spirit of lemma 3.

Lemma 13. Let $n \geq 3$ and let $b$ be an ex-post equilibrium for the class of VCG games over $(N, A, \mathcal{R}(a))$. Then for all $i$, $k$, $m$ and all $Z^{(a_i,t)}_i, b_i(Z^{(a_i,t)}_i)(a_k) = b_i(Z^{(a_i,t)}_i)(a_m)$ for all $a_k, a_m \neq a_i$.

Proof. Assume the claim is wrong, and that for some $i$, $k$, $m$ and $t$, $b_i(Z^{(a_i,t)}_i)(a_k) > b_i(Z^{(a_i,t)}_i)(a_m)$, where $a_i$, $a_k$ and $a_m$ are three distinct alternatives. Now we can mimic the arguments in the proof of lemma 3 and reach a contradiction. Q.E.D.

One can note that the proof above provides a slightly stronger result, for which we only need 2 players:

Lemma 14. Let $n \geq 2$ and let $b$ be an ex-post equilibrium for the class of VCG games over $(N, A, \mathcal{R}(a))$. Then for all $i$, $k$ and all $Z^{(a_i,t)}_i, b_i(Z^{(a_i,t)}_i)(a_k) \geq b_i(Z^{(a_i,t)}_i)(a)$ for all $a_i \neq a_i$.

The next lemma is quite similar to lemma 10 and its proof is identical and therefore omitted:

Lemma 15. Let $n \geq 3$, $|A| \geq 3$, and let $b$ be an ex-post equilibrium for the class of VCG games over $(N, A, \mathcal{R}(a))$, then $b_i(Z^{(a_i,s)}_i)(a_i) - b_i(Z^{(a_i,s)}_i)(a_k) = s$ for all $i \in N$, $s \in \mathbb{R}_+$ and $a_k \neq a_i \in A$.

Proof of Theorem 4: Follows the arguments in the proof of Theorem 2 for the case $n \geq 3$, where the reference to lemmas 3 and 10 are replaced with lemma 13 and 15. Q.E.D.

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