U_q(sl_2) AT FOURTH ROOT OF UNITY

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1 Introduction

This paper originated in a double motivation. In physics, we have strong reasons to believe that $SL_q(2, \mathbb{C})$ for a primitive third root of unity is fundamentally related with the fermion structure \[1\]. A precise implementation of this idea would open extremely interesting perspectives. In the noncommutative geometry version of the standard model of elementary particles \[2\] \[3\], replacement of the present phenomenological "spectral triple" by a fundamental Ansatz is expected to yield a scenario for a new "supersymmetric" standard model and a calculation of the fermion masses.

In mathematics, after Alain Connes axiomatic construction of spin-manifold \[1\], one is tempted by the project of an analogous theory of "generalized supermanifolds" (which we call "medusae"). This area, inevitable inasmuch as these objects exist (classical supermanifolds, gauge degrees of freedom of field theories) is mysterious and difficult, because the corresponding algebras are no longer subalgebras of $C^*$-algebras. In fact these are non semi-simple algebras, semi-simplicity arising only after quotienting by their nilradical. The projected survey of "medusae" lacks the two guiding features which have led Connes to the axiomatics of spin-manifold: the guidance through physics (hardly procured by the present supersymmetric standard model - in our opinion too ugly to be fundamental!) and (non semi-simplicity pending) the absence of Hilbert space techniques (presumably leading to abysses like "Hilbert spaces with indefinite metric", a situation already found but poor! ly investigated in quantum field theory).

We believe that the (finite dimensional) $U_q(sl_2)$ at roots of unity are models which hopefully yield features suggesting axioms for "medusae". A feature found in the two examples of $U_q(sl_2)$ for $q^3 = 1$ \[4\] and the present $H_1$ is the striking (apparently new) fact that the trace of the adjoint representation (in the quantum group sense \[3\]) has the nilradical in its kernel. This fact, in combination with an appropriate $*$-operation entails that semi-simplicity is synonymous with "positivity" (as is the case for the transversal degrees of freedom of quantum electrodynamics in the Lorentz gauge, eliminated by the requirement of a "strictly positive" Hilbert space, see e.g. \[5\])

We conjecture that semi-simplicity and positivity are synonymous for all roots of unity.

Apart from this feature, our paper displays various aspects of the quantum groups $H_N^1$, including a complete description of their algebra automorphisms and Hopf $*$-structures.

2 $U_q(sl_2)$ at fourth root of unity

$U_q(sl_2)$ is the algebra defined by the symbols $K, K^{-1}, E, F$ and the relations

\[
\begin{align*}
KE &= q^2EK, \\
KF &= q^{-2}FK, \\
[E,F] &= K - K^{-1}q^{-1}.
\end{align*}
\]

It has a Hopf algebra structure defined by

\[
\begin{align*}
\text{Comultiplication} & \quad \Delta(1) = 1 \otimes 1, \\
& \quad \Delta(K) = K \otimes K, \\
& \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \\
& \quad \Delta(E) = E \otimes 1 + K \otimes E, \\
& \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \\
\text{Counity} & \quad \varepsilon(1) = \varepsilon(K) = \varepsilon(K^{-1}) = 1, \\
& \quad \varepsilon(E) = \varepsilon(F) = 0,
\end{align*}
\]

(1)
Proof. \( S(1) = 1 \otimes 1, \]
\( S(K) = K^{-1}, \]
\( S(K^{-1}) = K, \]
\( S(E) = -K^{-1}E, \]
\( S(F) = -FK. \) \( \tag{4} \)

When we consider the special case \( q = i \), the relations (4) become
\[
\begin{align*}
KE &= -EK, \\
KF &= -FK, \\
[E,F] &= \frac{K-K^{-1}}{2i}.
\end{align*}
\] \( \tag{5} \)

The Casimir operator is defined as follows.
\[
C = FE + \frac{K-K^{-1}}{4i}.
\] \( \tag{6} \)

**Lemma 2.1** \( E^2, F^2 \) and \( K^2 \) belong to the center of \( U_q(sl_2) \).

**Proof:**

\( K^2 \) commutes with \( K, E \) and \( F \) since it anticommutes with \( E, F \). \( E^2 \) commutes with \( K \) since \( E \) and \( K \) anticommute. \( E^2 \) also commutes with \( F \),
\[
[E^2,F] = E[E,F] + [E,F]E = E\frac{K-K^{-1}}{2i} + \frac{K-K^{-1}}{2i}E = 0.
\]

Similarly, \( F^2 \) commutes with \( K \) since \( E \) and \( K \) anticommute. \( F^2 \) commutes with \( E \),
\[
[E,F^2] = F[E,F] + [E,F]F = F\frac{K-K^{-1}}{2i} + \frac{K-K^{-1}}{2i}F = 0.
\]

\( \square \)

**Definition 2.1** We define \( H^i_N = U_q(sl_2)/I_N \), with \( I_N \) the ideal of \( U_q(sl_2) \) generated by \( E^2, F^2, K^{2N} - 1 \).

**Proposition 2.1**

1) \( H^i_N \) is a Hopf algebra with the Hopf structure \( \Delta, \varepsilon \) and the Casimir operator \( C \).

2) \( H^i_N \) has a PWB-base \( \{ F^pK^nE^q \}_{p,q=0,1; n=0, \ldots, N} \). Thus it has dimension \( 8N \).

**Proof:**

In fact, we have in \( U_q(sl_2) \)
\[
\Delta(E^2) = E^2 \otimes 1 + K^2 \otimes E^2, \]
\[
\Delta(F^2) = F^2 \otimes K^{-2} + 1 \otimes F^2, \]
\[
\Delta(K^{2N} - 1) = K^{2N} \otimes (K^{2N} - 1) + (K^{2N} - 1) \otimes 1, \]
\[
\varepsilon(E^2) = 0, \]
\[
S(E^2) = -K^{-2}E^2, \]
\[
\varepsilon(F^2) = 0, \]
\[
S(F^2) = -F^2K^2, \]
\[
\varepsilon(K^{2N} - 1) = 0, \]
\[
S(K^{2N} - 1) = K^{-2N}(K^{2N} - 1). \] \( \tag{7} \)

1) The relations are immediately checked using multiplicativity of \( \Delta \) and \( \varepsilon \), as well as antipropagation of \( S \). These relations imply that \( I_N \) is a Hopf ideal. Indeed one has from (3) the inclusion \( \Delta(I_N) \subset I_N \otimes H^i_N + H^i_N \otimes I_N \), and \( I_N \) contains the elements \( \varepsilon(E^2), \varepsilon(F^2), \varepsilon(K^{2N} - 1), S(E^2), S(F^2), S(K^{2N} - 1) \) owing to (3).

2) Follows from the multiplication table below. \( \square \)

We now introduce a convenient alternative parametrization of the \( N \)-dimensional algebra \( K \) generated by \( K \). Owing to \( K^{2N} - 1, K \) is the group algebra of the finite abelian group \( \mathbb{Z}/2N\mathbb{Z} \). Using harmonic analysis on this group, we replace the basis \( \{ K^n \}_{n=0, \ldots, 2N-1} \) by the Fourier transformed basis \( \{ e_n \}_{n=0, \ldots, 2N-1} \), leading to a simpler description.
Lemma 2.2 Setting a complex number \( u = e^{\frac{2\pi i}{N}} \), the \(*\)-symmetric elements:

\[
e_k = e_k^* = \frac{1}{2N} \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{kj} K^j \quad k \in \mathbb{Z}/2N \mathbb{Z},
\]

(9)

with reversion formulae

\[
K^j = \sum_{k \in \mathbb{Z}/2N \mathbb{Z}} u^{-kj} e_k \quad j \in \mathbb{Z}/2N \mathbb{Z},
\]

(10)

yield a basis of \( K \). It has the following properties,

1) \( \sum_{k \in \mathbb{Z}/2N \mathbb{Z}} e_k = 1 \),
2) \( e_k e_m = \delta_{km} e_k \),
3) \( Ke_k = u^{-k} e_k \),
4) \( K^{-1} e_k = u^k e_k \), \( k \in \mathbb{Z}/2N \mathbb{Z} \)
5) \( \bar{e}_k = e_{k+N} \),
6) \( e_k F = F e_{k+N} \),
7) \( C = E F + \frac{1}{4N} \sum_{k \in \mathbb{Z}/2N \mathbb{Z}} (u^k - u^{-k}) e_k \).

(11)

Proof: The equivalence of (9) and (10) stems from the obvious fact that \( \sum_{k \in \mathbb{Z}/2N \mathbb{Z}} u^{(k-m)j} = \delta_{km} \). Check of the other claims: We have

2) \( e_k e_m = \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{kj} K^j e_m = \left( \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{(k-m)j} \right) e_m = \delta_{km} e_k \)

3) \( Ke_k = \frac{1}{2N} K \left( \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{kj} K^j \right) \)
\[= \frac{u^{-k}}{2N} \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{k(j+1)} K^{j+1} = u^{-k} e_k \]

5) \( E e_k = \frac{1}{2N} E \left( \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{kj} K^j \right) = \frac{1}{2N} \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{(N+k)j} K^j E = e_{k+N} E \)

6) \( e_k F = \frac{1}{2N} \left( \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{kj} K^j \right) F = \frac{1}{2N} \sum_{j \in \mathbb{Z}/2N \mathbb{Z}} u^{(N+k)j} K^j F = F e_{k+N} \)

Let us now describe in detail the two cases \( N = 1 \) and \( N = 2 \).

2.1 Case \( N = 1 \)

For \( N = 1 \) we have \( K^2 = 1 \), hence \( K = K^{-1} \).

Definition 2.2 The algebra \( H_1^1 \) is defined by the symbols \( K, E, F \) and the relations

\[
\begin{align*}
KE &= -EK, \\
KF &= -FK, \\
\{E,F\} &= 0, \\
E^2 &= 0, \\
F^2 &= 0, \\
K^2 &= 1.
\end{align*}
\]

(12)
Lemma 2.3 1) With
\begin{align*}
\begin{cases}
    e_0 = \frac{1+iK}{2}, \\
    e_1 = \frac{1-iK}{2}.
\end{cases}
\end{align*}

$H_1^1$ is equivalently defined by the symbols $e_0, e_1, E, F, K$ and the relations
\begin{align*}
\begin{cases}
    e_0^2 = e_0, \\
    e_1^2 = e_1, \\
    e_1 e_0 = 0, \\
    e_0 + e_1 = 1,
\end{cases}
\begin{cases}
    E^2 = 0, \\
    F^2 = 0, \\
    [E, F] = 0,
\end{cases}
\begin{cases}
    e_0 E = E e_1, \\
    e_1 E = E e_0, \\
    e_0 F = F e_1, \\
    e_1 F = F e_0.
\end{cases}
\end{align*}
\hspace{1cm} (13)

yielding the basis
\begin{align*}
\begin{cases}
    e_0, \\
    e_1, \\
    E_0 = e_0 E, \\
    E_1 = e_1 E, \\
    F_0 = e_0 F, \\
    F_1 = e_1 F, \\
    C_0 = e_0 EF, \\
    C_1 = e_1 EF.
\end{cases}
\end{align*}
\hspace{1cm} (14)

2) $H_1^1$ is a $*$-algebra under the $*$-operation specified by
\begin{align*}
\begin{cases}
    E^* = F, \\
    F^* = E, \\
    K^* = K,
\end{cases}
\begin{cases}
    e_0^*, e_1^* = e_0, \\
    e_1^* = e_1.
\end{cases}
\end{align*}
\hspace{1cm} (15)

3) The Casimir element is
\begin{align*}
    C = C^* = EF = FE.
\end{align*}

4) The basis $[5]$ is acted upon as follows by multiplications by $E, F$ and $C$:
\begin{align*}
\begin{cases}
    E e_0 = E_1, \\
    e_0 F = F_1, \\
    F e_0 = F_0, \\
    e_1 E = E_1, \\
    E E_0 = 0, \\
    F E_0 = C_1, \\
    E E_1 = 0, \\
    F E_1 = C_0, \\
    E F_0 = C_1, \\
    F F_0 = 0, \\
    F F_1 = 0, \\
    E F_1 = C_0, \\
    F C_0 = 0, \\
    E C_0 = 0, \\
    F C_0 = 0, \\
    E C_1 = 0, \\
    F C_1 = 0.
\end{cases}
\end{align*}
\hspace{1cm} (16)

5) The multiplication table of $H_1^1$ is as follows (we plugged the product $XY$ at the intersection of line $X$ and column $Y$, the latter so as to have $*$-symmetry w.r.t. the diagonal).

\begin{align*}
\begin{array}{cccccc}
    | & e_0 & e_1 & E_0 & F_0 & F_1 & C_0 \\
    \hline
    e_0 & e_0 & 0 & E_0 & 0 & 0 & C_0 \\
    e_1 & 0 & e_1 & E_1 & 0 & F_1 & 0 \\
    E_0 & 0 & E_0 & 0 & 0 & 0 & C_0 \\
    E_1 & E_1 & 0 & 0 & 0 & C_1 & 0 \\
    F_0 & 0 & F_0 & 0 & 0 & 0 & C_0 \\
    F_1 & F_1 & 0 & C_1 & 0 & 0 & 0 \\
    C_0 & C_0 & 0 & 0 & 0 & 0 & 0 \\
    C_1 & C_1 & 0 & 0 & 0 & 0 & 0
\end{array}
\end{align*}
\hspace{1cm} (17)

Proof:
Immediate from relations [4] and [5].
2.2 Structure of the algebra $H_1^1$

**Proposition 2.2** 1) Let $\textbf{M}_2$ be algebra of the $2 \times 2$ complex matrices, equipped with its natural grading (diagonal entries are even and off-diagonal entries odd) and $\Lambda = \Lambda_1 \otimes \Lambda_1$ with $\Lambda_1$ the Grassmann algebra over $\mathbb{C}$ (ordinary-not skew-tensor product equipped with the tensor product $\mathbb{Z}/2\mathbb{Z}$-grading).

As an algebra $H_1^1$ is isomorphic to the even part $(\textbf{M}_2 \otimes \Lambda)^+$ (again ordinary-not skew-tensor product equipped with the tensor product $\mathbb{Z}/2\mathbb{Z}$-grading). This isomorphism is specified as follows. With $\textbf{M}_2$ spanned by its matrix units $\{e_{ik}\}_{i,k=0,1}$, the first $\Lambda_1$-factor by $\textbf{1}_i \textbf{e}$; the second factor by $\textbf{1}_f \textbf{f}$; and the tensor product $\Lambda_1 \otimes \Lambda_1$ by $\textbf{1} \otimes \textbf{1}, \textbf{E} = \textbf{e} \otimes \textbf{1}, \textbf{F} = \textbf{1} \otimes \textbf{f}, EF = \textbf{e} \otimes \textbf{f}$, one has,

$$\begin{cases} e_0 = e_{00} \otimes \textbf{1}, & E_0 = e_{01} \otimes \textbf{E}, \quad F_0 = e_{00} \otimes \textbf{F}, \quad C_0 = e_{00} \otimes EF, \\ e_1 = e_{11} \otimes \textbf{1}, & E_1 = e_{10} \otimes \textbf{E}, \quad F_1 = e_{11} \otimes \textbf{F}, \quad C_1 = e_{11} \otimes EF. \end{cases}$$

2) The subspace $N_1^1$ of $H_1^1$ spanned by $E_0, E_1, F_0, F_1, C_0$ and $C_1$ is the latter's nilradical, giving rise to the quotient $H_1^1/N_1^1 \cong \mathbb{C} \oplus \mathbb{C}$.

3) We define as follows the scalar product $<.,.>$,

$$<a,b> = Tr(\lambda(a^*b)) \quad a, b \in H_1^1$$

where $\lambda$ denotes the left-regular representation of $H_1^1$. $<.,.>$ is positive semi-definite with null-space the nilradical $N_1^1$. It is positive definite on the span of $e_0, e_1$, where it coincides with the usual trace of $\mathbb{C} \oplus \mathbb{C}$ ($<e_i, e_k> = \delta_{ik}$, $i, k = 0, 1$).

**Proof:**

1) One immediately checks that the elements $\{\}$ fulfill the multiplication rules $\{\}.$

2) These multiplication rules imply that $N_1^1$ is a subalgebra fulfilling the inclusions $e_0 N_1^1, N_1^1 e_0, e_1 N_1^1, N_1^1 e_1 \subset N_1^1$, $N_1^1$ is thus an ideal of $H_1^1$ which moreover consists of nilpotent elements and yields a quotient generated by $e_0, e_1$ fulfilling $e_0^2 = e_0, e_1^2 = e_1, e_0 e_1 = 0$, thus isomorphic to the semi-simple $\mathbb{C} \oplus \mathbb{C}$. Accordingly, $N_1^1$ is the nilradical of $H_1^1$.

3) Clear by inspection of $\{\}$ implying first that $Tr(\rho)$ vanishes on $N_1^1$ and that $Tr(\rho(e_0)) = Tr(\rho(e_1)) = 1$. $\square$

2.3 Case $N = 2$

We now describe the case corresponding to $N = 2$, one has now $K^4 = 1$ and $K - K^{-1}$ no longer vanishes.

**Definition 2.3** The algebra $H_2^1$ is defined by the symbols $K, E$ and $F$ together with the relations

$$\begin{cases} KE = -EK, \\
KF = -FK, \\
[E, F] = \frac{1}{2}(K - K^{-1}), \end{cases} \quad \begin{cases} E^2 = 0, \\
F^2 = 0, \\
K^4 = 1. \end{cases}$$

**Lemma 2.4** 1) Let $e_m = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} i^{mk} K^k, i.e$

$$\begin{cases} e_0 = \frac{1}{4}(1 + K + K^2 + K^3), \\
e_1 = \frac{1}{4}(1 + iK - K^2 - iK^3), \\
e_2 = \frac{1}{4}(1 - K + K^2 - K^3), \\
e_3 = \frac{1}{4}(1 - K - K^2 + iK^3). \end{cases}$$

It implies the relations

$$\begin{cases} K = e_0 - ie_1 - e_2 + ie_3, \\
K^{-1} = e_0 + ie_1 - e_2 - ie_3, \\
K - K^{-1} = 2i(e_3 - e_1), \end{cases}$$
yielding the basis

\[
\begin{align*}
\{ e_0, & \quad e_2, & \\
E_0 = e_0 E, & \quad E_2 = e_2 E, & \\
F_0 = e_0 F, & \quad F_2 = e_2 F, & \\
P_0 = C_0 = e_0 EF, & \quad P_2 = C_2 = e_2 EF, &
\end{align*}
\]
(18)

Accordingly, \( H_2^1 \) is equivalently defined by the symbols \( E, F \) and \( e_j \) \( j = 0, 1, 2, 3 \) and the relations

\[
\begin{align*}
\begin{cases}
    e_0 + e_1 + e_2 + e_3 = 1, \\
e_i e_k = \delta_{ik},
\end{cases} & \quad \begin{cases}
    E^2 = 0, \\
F^2 = 0,
\end{cases} & \quad \begin{cases}
    [E,F] = e_3 - e_1, \\
e_j E = E e_{j+2}, \\
e_j F = e_{j+2} F, j \in \mathbb{Z}/4\mathbb{Z}.
\end{cases}
\end{align*}
\]

2) \( H_2^1 \) is a \(*\)-algebra under the \(*\)-operation specified by

\[
\begin{align*}
\begin{cases}
    E^* = F, \\
F^* = E,
\end{cases} & \quad \begin{cases}
    K^* = K, \\
e_j^* = e_j, j = 0, 1, 2, 3.
\end{cases}
\end{align*}
\]

3) The Casimir element is

\[
C = C^* = FE - \frac{1}{2}(e_1 - e_3) = EF + \frac{1}{2}(e_1 - e_3).
\]

4) Let

\[
\begin{align*}
\begin{cases}
    \pi_0 = e_0 + e_2, \\
\pi_1 = e_1 + e_3,
\end{cases} & \quad \begin{cases}
    H_2^{(0)} = \pi_0 H_2^1, \\
H_2^{(1)} = \pi_1 H_2^1.
\end{cases}
\end{align*}
\]

Then \( \pi_0 \) and \( \pi_1 \) are supplementary central idempotents yielding supplementary ideals \( H_2^{(0)} \) and \( H_2^{(1)} \) of \( H_2^1 \) spanned respectively by the left and right basis \[3\].

5) Multiplication by \( E, F \) and \( C \) act as follows on the basis \[3\],

\[
\begin{align*}
\begin{cases}
    E e_0 = E_2, \\
E e_2 = E_0, \\
E E_0 = 0, \\
E E_2 = 0,
\end{cases} & \quad \begin{cases}
    F e_0 = F_2, \\
F e_2 = F_0, \\
F E_0 = P_2, \\
F E_2 = P_0,
\end{cases} & \quad \begin{cases}
    e_0 E = E_0, \\
E_0 E = 0, \\
E_2 E = 0, \\
E_2 F = P_0,
\end{cases} & \quad \begin{cases}
    e_0 F = F_0, \\
F_0 E = 0, \\
F_2 E = P_2, \\
F_2 F = P_0,
\end{cases} & \quad \begin{cases}
    P e_0 = P_0, \\
P e_2 = P_2,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
    E e_1 = E_3, \\
E e_3 = E_1, \\
E E_1 = 0, \\
E E_3 = 0,
\end{cases} & \quad \begin{cases}
    F e_1 = F_3, \\
F e_3 = F_1, \\
F E_1 = P_3, \\
F E_3 = P_0,
\end{cases} & \quad \begin{cases}
    e_1 E = E_1, \\
e_3 E = E_3, \\
e_1 E = 0, \\
e_3 E = 0,
\end{cases} & \quad \begin{cases}
    e_1 F = F_1, \\
e_3 F = F_3, \\
E_1 F = P_1, \\
E_3 F = P_3,
\end{cases} & \quad \begin{cases}
    C e_1 = C_1, \\
C e_3 = C_3, \\
C E_1 = -\frac{1}{2} E_1, \\
C E_3 = -\frac{1}{2} E_3.
\end{cases}
\end{align*}
\]

6) One has the following multiplication tables (the product \( XY \) is at the intersection of line \( X \) and column \( Y \), ordering the latter so as to have \(*\) symmetry w.r.t the diagonal) for \( H_2^{(0)} \) in the basis \( \{e_0, e_2, E_0, E_2, F_0, F_2, P_0, P_2\} \).
and shows that the restriction of $H_2^{(1)}$ to the basis $\{e_1,e_3,E_1,E_3,F_1,F_3,P_1,P_3\}$

and for $H_2^{(1)}$ in the basis $\{e_1,e_3,E_1,E_3,F_1,F_3,P_1,P_3\}$

This yields the following action of the Casimir operator

and shows that the restriction of $C$ to $H_2^{(1)}$ has the eigenspaces $CE_1 + CF_3 + CP_1$ to the eigenvalue $-\frac{1}{2}$ and $CE_3 + CF_1 + CP_3$ to the eigenvalue $\frac{1}{2}$.

Proof:

The products $21$ and $22$ not involving the $C_j$ are immediate from $11$ and $14$. Check of the products $21$ involving $P_1,P_3$ is made using

so that,

$\{ EFE = -(e_1 - e_3)E, \\
F E F = (e_1 - e_3)F, \}$

□
2.4 Structure of the algebra $H_2^2$

Proposition 2.3 1) The algebra $H_2^2$ splits into the direct sum of the ideals $H_2^{(0)}$ and $H_2^{(1)}$.

2) The algebra $H_2^{(0)}$ is isomorphic to $H_1^1$ with the isomorphism given by

$$
\left\{ \begin{array}{l}
  e_0 \rightarrow e_0,
  E_0 \rightarrow E_0, \\
  e_2 \rightarrow e_1, \\
  F_0 \rightarrow F_0, \\
  C_0 \rightarrow C_0.
\end{array} \right.
$$

Consequently, $H_2^{(0)}$ is an algebra isomorphic to the even part $(M_2 \otimes \Lambda_1 \otimes \Lambda_1)^+$, the isomorphism being specified as follows: with $M_2$ spanned by its matrix units $\{e_{ik}\}_{i,k=0,1}$, the first $\Lambda_1$-factor by $1$, $e$, the second $\Lambda_1$-factor by $1$, $f$; and the tensor product $\Lambda_1 \otimes \Lambda_1$ by $1 \otimes 1$, $E = e \otimes 1$, $F = 1 \otimes f$, $EF = e \otimes f$, one has

$$
\left\{ \begin{array}{l}
  e_1 = e_{00} \otimes 1, \\
  e_2 = e_{11} \otimes 1, \\
  F_0 = e_{01} \otimes F, \\
  F_2 = e_{10} \otimes F.
\end{array} \right.
$$

3) The nilradical $N_1^{(0)}$ of $H_2^{(0)}$ is the eigenspace of the Casimir element $C$ to the eigenvalue $0$, it is spanned by the elements $E_0, E_2, F_0, F_2, P_0, P_2$. The quotient algebra $H_2^{(0)}/N_1^{(0)}$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$.

4) $H_2^{(1)}$ is an algebra isomorphic to the semi-simple algebra $M(2, \mathbb{C}) \oplus M(2, \mathbb{C})$ with the isomorphism given by

$$
K = i \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad E = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad F = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

5) We define the scalar product $<.,.>$,

$$
< a, b > = Tr(\lambda(a^*b)),
$$

where $\lambda$ denotes the left regular representation of $H_2^2$. One has that

a) the even and odd parts $H_2^{(0)}$ and $H_2^{(1)}$ are mutually $<.,.>$-orthogonal,

b) its restriction to $H_2^{(0)}$, $<.,.>$ is positive semi-definite with null space $N_1^{(0)}$, and is positive definite on the span of $e_0, e_2$, where it coincides with the usual trace of $\mathbb{C} \oplus \mathbb{C}$,

c) its restriction to $H_2^{(1)}$, $<.,.>$ behaves as follows with

$$
H_2^{(1,+)} = \text{span of } \left\{ \frac{1}{2}(e_1 - P_1), \frac{1}{2}(e_3 + P_3), F_1 \right\},
$$

$$
H_2^{(1,0)} = \text{span of } \left\{ \frac{1}{2}(e_1 + P_1), \frac{1}{2}(e_3 - P_3), E_1, E_3 \right\},
$$

$$
H_2^{(1,-)} = \text{span of } \{F_3\},
$$

$<.,.>$ is positive definite on $H_2^{(1,+)}$, with $\frac{1}{2}(e_1 - P_1), \frac{1}{2}(e_3 + P_3), F_1$ orthonormal, $<.,.>$ has $H_2^{(1,0)}$ as its null space (space of vectors orthogonal to all vectors), $<.,.>$ is negative definite on $H_2^{(1,-)}$, with $<F_3, F_3> = -1$.

Proof:

1) Recalls a former result, cf lemma 11.2.

2) The changes $\text{23}$ turn $\text{20}$ into $\text{17}$.

3) The eigenspaces of $C$ acting on $H_2^{(0)}$ are immediately found computing $\text{19}$ via $\text{20}$. The (a priori known) fact that these subspaces are ideals and the corresponding quotient are patent from $\text{20}$ and $\text{21}$.

4) Follows from the fact that these matrices satisfy the same relations and generate an 8-dimensional algebra.

5) Inspection of $\text{21}$ yields the following table of values of $Tr(\lambda(\cdot))$ on $H_2^{(1)}$.  

$^1$algebraic (not Hopf) ideals.
The coproduct is given by
\[
\Delta(C) = C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_0 - e_0 \otimes C_1 + E_0 \otimes F_0 + E_1 \otimes F_1 - F_0 \otimes E_1 - F_1 \otimes E_0.
\]

The antipode and the counit are
\[
\begin{aligned}
\varepsilon(C_0) &= 1, \\
\varepsilon(C_1) &= \varepsilon(E_0) = \varepsilon(E_1) = \varepsilon(F_0) = \varepsilon(F_1) = \varepsilon(C_0) = \varepsilon(C_1) = 0.
\end{aligned}
\]
Proof:

Check of (23): Using (3), whence $K = e_0 - e_1$, we have

$$\Delta(e_0) = \frac{1}{2} \Delta(1 + K) = \frac{1}{2} [1 \otimes 1 + K \otimes K] = \frac{1}{2} [(e_0 + e_1) \otimes (e_0 + e_1) + (e_0 - e_1) \otimes (e_0 - e_1)] = e_0 \otimes e_0 + e_1 \otimes e_1,$$

$$\Delta(e_1) = \frac{1}{2} \Delta(1 - K) = \frac{1}{2} [1 \otimes 1 - K \otimes K] = \frac{1}{2} [(e_0 + e_1) \otimes (e_0 + e_1) - (e_0 - e_1) \otimes (e_0 - e_1)] = e_0 \otimes e_1 + e_1 \otimes e_0.$$

$$\Delta(E_0) = \Delta(e_0 E) = \Delta(e_0) \Delta(E) = (e_0 \otimes e_0 + e_1 \otimes e_1) (E \otimes (e_0 + e_1) + (e_0 - e_1) \otimes E) = E_0 \otimes e_0 + E_1 \otimes e_1 + e_0 \otimes E_0 - e_1 \otimes E_1,$$

$$\Delta(E_1) = \Delta(e_1 E) = \Delta(e_1) \Delta(E) = (e_0 \otimes e_1 + e_1 \otimes e_0) (E \otimes (e_0 + e_1) + (e_0 - e_1) \otimes E) = E_0 \otimes e_1 + E_1 \otimes e_0 + e_0 \otimes E_1 - e_1 \otimes E_0.$$

Further, taking account of the fact that $K^{-1} = K = e_0 - e_1$

$$\Delta(F_0) = \Delta(e_0 F) = \Delta(e_0) \Delta(F) = (e_0 \otimes e_0 + e_1 \otimes e_1) (F \otimes (e_0 - e_1) + (e_0 + e_1) \otimes F) = F_0 \otimes e_0 - F_1 \otimes e_1 + e_0 \otimes F_0 + e_1 \otimes F_1,$$

$$\Delta(F_1) = \Delta(e_1 F) = \Delta(e_1) \Delta(F) = (e_0 \otimes e_1 + e_1 \otimes e_0) (F \otimes (e_0 - e_1) + (e_0 + e_1) \otimes F) = -F_0 \otimes e_1 + F_1 \otimes e_0 + e_0 \otimes F_1 + e_1 \otimes F_0,$$

$$\Delta(C_0) = \Delta(E_0) \Delta(F) = \Delta(e_0) \Delta(E) \Delta(F) = (e_0 \otimes e_0 + E_1 \otimes e_1 + e_0 \otimes E_0 - e_1 \otimes E_1) (F \otimes (e_0 - e_1) + (e_0 + e_1) \otimes F) = C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_0 - e_1 \otimes C_1 + E_0 \otimes F_0 + E_1 \otimes F_1 - F_0 \otimes E_0 - F_1 \otimes E_1,$$

$$\Delta(C_1) = \Delta(E_1) \Delta(F) = \Delta(e_1) \Delta(E) \Delta(F) = (e_0 \otimes e_1 + E_1 \otimes e_0 + e_0 \otimes E_1 - e_1 \otimes E_0) (F \otimes (e_0 - e_1) + (e_0 + e_1) \otimes F) = -C_0 \otimes e_1 + C_1 \otimes e_0 + e_0 \otimes C_1 - e_1 \otimes C_0 + F_0 \otimes E_1 + F_1 \otimes E_0 + E_0 \otimes F_1 + E_1 \otimes F_0.$$

Check of (24): Using again $K = e_0 - e_1$, we have

$$S(e_0) = \frac{1}{2} S(1 + K) = \frac{1}{2} (1 + K^{-1}) = \frac{1}{2} (1 + (e_0 - e_1)^{-1}) = e_0,$$

$$S(e_1) = \frac{1}{2} S(1 - K) = \frac{1}{2} (1 - K^{-1}) = \frac{1}{2} (1 - (e_0 - e_1)^{-1}) = e_1.$$
Since $S$ is an antiisomorphism, we also have

\[ S(E_0) = S(e_0 E) = S(E) S(e_0) = (-K^{-1} E)e_0 = -(e_0 - e_1) Ee_0 = E_1, \]
\[ S(E_1) = S(e_1 E) = S(E) S(e_1) = (-K^{-1} E)e_1 = -(e_0 - e_1) Ee_1 = -E_0, \]
\[ S(F_0) = S(e_0 F) = S(F) S(e_0) = (-FK)e_0 = -F(e_0 - e_1)e_0 = -F_1, \]
\[ S(F_1) = S(e_1 F) = S(F) S(e_1) = (-FK)e_1 = -F(e_0 - e_1)e_1 = F_0, \]
\[ S(C_0) = S(e_0 E) = S(F) S(E) S(e_0) = F(e_0 - e_1)(e_0 - e_1) Ee_0 = F e_1 E = e_0 EF = C_0, \]
\[ S(C_1) = S(e_1 EF) = S(F) S(E) S(e_1) = F(e_0 - e_1)(e_0 - e_1) Ee_1 = F e_0 E = e_1 EF = C_1. \]

Check of \( e \): using again \( K = e_0 - e_1 \), we have

\[ \varepsilon(e_0) = \frac{1}{2} \varepsilon(1 + K) = \frac{1}{2} (1 + 1) = 1, \]
\[ \varepsilon(e_1) = \frac{1}{2} \varepsilon(1 - K) = \frac{1}{2} (1 - 1) = 0. \]

Since \( \varepsilon \) is a morphism, we get, typically:

\[ \varepsilon(E_0) = \varepsilon(e_0) \varepsilon(E) = 1 \times 0 = 0. \]

\[ \square \]

**Remark 3.1** The eigenvalues of \( S \) in \( H^1_1 \) are 1, \( i \) and \( -i \) with the eigenspaces

\[ V_1 \text{ spanned by } \begin{cases} e_0 \\ e_1 \\ C_0 \\ C_1 \end{cases}, V_i \text{ spanned by } \begin{cases} E_0 - i E_1 \\ F_0 + i F_1 \end{cases} \text{ and } V_{-i} \text{ spanned by } \begin{cases} E_0 + i E_1 \\ F_0 - i F_1 \end{cases}. \]

### 3.2 Hopf structure of \( H^1_2 \)

Recall that the algebra \( H^1_2 \) is defined by the symbols \( K, E, F \) and the relations

\[
\begin{align*}
K E &= -E K, \\
K F &= -F K, \\
[E, F] &= \frac{1}{2}(K - K^{-1}), \\
E^2 &= 0, \\
F^2 &= 0, \\
K^4 &= 1,
\end{align*}
\]

or else symbols \( E, F \) and \( e_m = \frac{1}{4} \sum_{k \in \mathbb{Z}/4 \mathbb{Z}} \iota^{mk} K^k, \) \( i = 0, 1, 2, 3 \), and relations

\[
\begin{align*}
\sum_{m=0}^{3} e_m &= 1, \\
E e_m &= \delta_{lm}, \quad l, m = 0, 1, 2, 3 \\
E^2 &= 0, \\
F^2 &= 0, \\
[E, F] &= e_3 - e_1, \\
e_j E &= E e_{j+2}, \\
F e_j &= e_{j+2} F, \quad j = 0, 1, 2, 3.
\end{align*}
\]

It is spanned by \( e_0, e_2, E_0, E_2, E_0, F_0, F_2, P_0, P_2, e_1, e_3, E_1, E_3, F_1, F_3, P_1, P_3 \).

**Proposition 3.2** We have, for \( m \in \mathbb{Z}/4 \mathbb{Z},

\[
\begin{align*}
\Delta(e_m) &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4 \mathbb{Z}} e_k \otimes e_{m-k}, \\
\Delta(E_m) &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4 \mathbb{Z}} [E_k \otimes e_{m-k} + (i)^{-k} e_k \otimes E_{m-k}], \\
\Delta(F_m) &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4 \mathbb{Z}} [(i)^{m-k} F_k \otimes e_{m-k} + e_k \otimes F_{m-k}], \\
\Delta(P_m) &= \frac{1}{4} \sum_{k \in \mathbb{Z}/4 \mathbb{Z}} [(-1)^k (i)^{m-k} F_k \otimes e_{m-k} + (i)^{-k} e_k \otimes P_{m-k}],
\end{align*}
\]

(26)
Proof:

Using $K = e_0 - ie_1 - e_2 + ie_3$, we have $K^{-1}e_k = e_kK = (i)^{-k}e_k$, $K^{-1}e_k = e_k = K^{-1}(i)^ke_k$, we have

\[
\Delta(e_m) = \frac{1}{4} \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (i)^m \Delta(K)^r = \frac{1}{4} \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (i)^m K^r \otimes K^r = \frac{1}{4} \sum_{r,k,l \in \mathbb{Z}/4\mathbb{Z}} (i)^m (i)^{-kr}e_k \otimes (i)^{-lr}e_l
\]

\[
\Delta(E_m) = \Delta(e_m)\Delta(E) = \frac{1}{4} \left( \sum_{k \in \mathbb{Z}/4\mathbb{Z}} e_k \otimes e_{m-k} \right) (E \otimes 1 + K \otimes E)
\]

\[
\Delta(F_m) = \Delta(e_m)\Delta(F) = \frac{1}{4} \left( \sum_{k \in \mathbb{Z}/4\mathbb{Z}} e_k \otimes e_{m-k} \right) (F \otimes K^{-1} + 1 \otimes F)
\]

\[
\Delta(P_m) = \Delta(E_m)\Delta(F) = \frac{1}{4} \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (E_k \otimes e_{m-k} + (-i)^ke_k \otimes E_{m-k}) (F \otimes K^{-1} + 1 \otimes F)
\]

Check of $26$. We have by the antimultiplicativity of $S$,

\[
S(e_m) = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^mk S(K)^k = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^mk K^{-k} = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^{-mk} K^k = e_{-m},
\]

$S(E_m) = S(E)S(e_m) = (-K^{-1}E)e_{-m} = -K^{-1}e_{2-m}E = -(i)^{2-m} E_{2-m} = (-i)^{m} E_{2-m}$,

$S(F_m) = S(F)S(e_m) = (-FK)e_{-m} = -(i)^{m} Fe_{-m} = -(i)^{m} e_{2-m}F = -(i)^{m} F_{2-m}$,

$S(P_m) = S(F)S(E)S(e_m) = (-FK)(-K^{-1}E)e_{-m} = FE e_{-m} = (EF - e_1 + e_3)e_{-m} = P_{-m} - \delta_{1,-m} e_1 + \delta_{3,-m} e_3 = P_{-m} - \delta_{3,m} e_1 + \delta_{1,m} e_3$. 



Check of 28. Owing to the multiplicativity of \( \varepsilon \), we have

\[
\varepsilon(e_m) = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^m \varepsilon(K)^k = \sum_{k \in \mathbb{Z}/4\mathbb{Z}} (i)^m \delta_{0,m},
\]

whilst \( \varepsilon(E_m), \varepsilon(F_m) \) and \( \varepsilon(P_m) \) all vanish because they all contain the factor \( \varepsilon(E) \) or \( \varepsilon(F) \), \( \Box \)

4 Adjoint representations of \( H_1^i \) and \( H_2^i \)

4.1 Adjoint representation and adjoint trace of \( H_1^i \)

Proposition 4.1 1) We have the following values of the biregular (or adjoint ) representation \( \mu \)

| \( a \) | \( \mu(a)e_0 \) | \( \mu(a)e_1 \) | \( \mu(a)E_0 \) | \( \mu(a)E_1 \) | \( \mu(a)F_0 \) | \( \mu(a)F_1 \) | \( \mu(a)C_0 \) | \( \mu(a)C_1 \) |
|---|---|---|---|---|---|---|---|---|
| \( e_0 \) | \( e_0 \) | \( e_1 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( C_0 \) | \( C_1 \) |
| \( e_1 \) | \( 0 \) | \( 0 \) | \( E_0 \) | \( E_1 \) | \( F_0 \) | \( F_1 \) | \( 0 \) | \( 0 \) |
| \( E_0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( C_0 + C_1 \) | \( C_0 + C_1 \) | \( 0 \) | \( 0 \) |
| \( E_1 \) | \( -E_0 + E_1 \) | \( E_0 - E_1 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( F_0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( F_1 \) | \( F_0 + F_1 \) | \( -F_0 + F_1 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( C_0 \) | \( 2(C_0 + C_1) \) | \( 2(C_0 - C_1) \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( C_1 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

2) We have the following values of the adjoint trace \( Tr_\mu \) of \( H_1^i \)

| \( a \) | \( e_0 \) | \( e_1 \) | \( E_0 \) | \( E_1 \) | \( F_0 \) | \( F_1 \) | \( C_0 \) | \( C_1 \) |
|---|---|---|---|---|---|---|---|---|
| \( tr_\mu(a) \) | 4 | 4 | 0 | 0 | 0 | 0 | 0 |

Thus, \( Tr_\mu \) has the nilradical as its kernel and passes to the semi-simple quotient as its trace.

3) The scalar product determined by the adjoint trace and the *-operation

\[
< a, b > = \frac{1}{4} Tr_\mu(a^*b) \quad a, b \in H_1^i
\]

has the only non-vanishing values \( < e_0, e_0 > = 1 \) and \( < e_1, e_1 > = 1 \). In other terms the scalar product \( < ., . > \) is positive semi-definite with null-space the nilradical \( N_1^i \).

Proof:

1) From 23, 24, and 25, we deduce the following table of elements \((id \otimes S)\Delta(a)\) of \( End_C(H_1^i) \otimes End_C(H_1^i)\), \( a \in H_1^i \)

\[
(id \otimes S)\Delta(e_0) = e_0 \otimes e_0 + e_1 \otimes e_1, \quad (29)
(id \otimes S)\Delta(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0,
(id \otimes S)\Delta(E_0) = E_0 \otimes e_0 + E_1 \otimes e_1 + e_0 \otimes E_1 + e_1 \otimes E_0,
(id \otimes S)\Delta(E_1) = E_0 \otimes e_1 + E_1 \otimes e_0 - e_0 \otimes E_0 - e_1 \otimes E_1,
(id \otimes S)\Delta(F_0) = F_0 \otimes e_0 - F_1 \otimes e_1 - e_0 \otimes F_1 + e_1 \otimes F_0,
(id \otimes S)\Delta(F_1) = -F_0 \otimes e_1 + F_1 \otimes e_0 + e_0 \otimes F_0 - e_1 \otimes F_1,
(id \otimes S)\Delta(C_0) = C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_0 - e_1 \otimes C_1,
(id \otimes S)\Delta(C_1) = -C_0 \otimes e_1 + C_1 \otimes e_0 + e_0 \otimes C_1 - e_1 \otimes C_0,
(id \otimes S)\Delta(E_0) = -F_0 \otimes e_0 + F_1 \otimes e_1 + e_0 \otimes F_1 - e_1 \otimes F_0,
(id \otimes S)\Delta(E_1) = C_0 \otimes e_1 - C_1 \otimes e_0 + e_1 \otimes C_0 - e_0 \otimes C_1,
(id \otimes S)\Delta(F_0) = -C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_0 + e_1 \otimes C_1,
(id \otimes S)\Delta(F_1) = -C_0 \otimes e_1 + C_1 \otimes e_0 - e_0 \otimes C_1 + e_1 \otimes C_0,
(id \otimes S)\Delta(E_0) = -F_0 \otimes e_0 - F_1 \otimes e_1 + e_0 \otimes F_1 - e_1 \otimes F_0,
(id \otimes S)\Delta(E_1) = -C_0 \otimes e_1 - C_1 \otimes e_0 + e_1 \otimes C_0 - e_0 \otimes C_1,
(id \otimes S)\Delta(F_0) = -C_0 \otimes e_0 + C_1 \otimes e_1 + e_0 \otimes C_1 + e_1 \otimes C_0,
(id \otimes S)\Delta(F_1) = -C_0 \otimes e_1 - C_1 \otimes e_0 - e_0 \otimes C_1 - e_1 \otimes C_0,
(id \otimes S)\Delta(E_0) = -F_0 \otimes e_0 + F_1 \otimes e_1 - e_0 \otimes F_1 + e_1 \otimes F_0,
(id \otimes S)\Delta(E_1) = -C_0 \otimes e_1 + C_1 \otimes e_0 - e_0 \otimes C_1 + e_1 \otimes C_0,
(id \otimes S)\Delta(F_0) = -C_0 \otimes e_0 - C_1 \otimes e_1 + e_0 \otimes C_1 - e_1 \otimes C_0
\]
From these relations, the corresponding $\mu(a)x, x \in H^1_1$, are obtained by making $\otimes \to x$ (observe that $\mu(e_0)$ and $\mu(e_1)$ are the projections onto the even and odd part of $\text{End}_C(H^1_1)$ for the $\mathbb{Z}/2\mathbb{Z}$-grading).

2) Results by inspection conferring [29] with [17].

3) The scalar product $<.,.>$ is given by the following table (where $<a,b>$ is plotted at the intersection of the line $a$ and the column $b$), obtained by replacing the entries of [17] by their quantum traces.

|   | $e_0$ | $e_1$ | $E_0$ | $E_1$ | $F_0$ | $F_1$ | $C_0$ | $C_1$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $e_1$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $F_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $F_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

4.2 Adjoint representation and adjoint trace of $H^2_2$

**Proposition 4.2** 1) The biregular representation $\mu = \lambda \ast \rho$ of $H^2_2$ vanishes on $H^1_2$ and is given on $H^2_2$ by following table.

| $a$ | $\mu(a)e_0$ | $\mu(a)e_2$ | $\mu(a)E_0$ | $\mu(a)E_1$ | $\mu(a)F_0$ | $\mu(a)F_1$ | $\mu(a)P_0$ | $\mu(a)P_2$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | $\frac{1}{2}e_0$ | $\frac{1}{2}e_2$ | 0 | 0 | 0 | 0 | $\frac{1}{2}P_0$ | $\frac{1}{2}P_2$ |
| $e_2$ | 0 | 0 | $\frac{1}{2}E_0$ | $\frac{1}{2}E_2$ | $\frac{1}{2}F_0$ | $\frac{1}{2}F_2$ | 0 | 0 |
| $E_0$ | $\frac{1}{2}(E_0 + E_2)$ | $\frac{1}{2}(E_0 + E_2)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $E_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $F_0$ | 0 | 0 | $\frac{1}{2}P_0$ | $-\frac{1}{2}P_2$ | 0 | 0 | $\frac{1}{2}P_0$ | $-\frac{1}{2}P_2$ |
| $F_2$ | $\frac{1}{2}(F_0 + F_2)$ | $\frac{1}{2}(F_0 + F_2)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

and

| $a$ | $\mu(a)e_1$ | $\mu(a)e_3$ | $\mu(a)E_1$ | $\mu(a)E_3$ | $\mu(a)F_1$ | $\mu(a)F_3$ | $\mu(a)P_1$ | $\mu(a)P_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | $\frac{1}{2}e_1$ | $\frac{1}{2}e_3$ | 0 | 0 | $\frac{1}{2}e_1$ | $\frac{1}{2}e_3$ | $-\frac{1}{2}E_1$ | $-\frac{1}{2}E_3$ |
| $e_2$ | 0 | 0 | $\frac{1}{2}E_1$ | $\frac{1}{2}E_3$ | $\frac{1}{2}F_1$ | $\frac{1}{2}F_3$ | 0 | 0 |
| $E_0$ | $\frac{1}{2}(E_1 + E_3)$ | $\frac{1}{2}(E_1 + E_3)$ | 0 | 0 | $\frac{1}{2}e_1$ | $\frac{1}{2}e_3$ | $-\frac{1}{2}E_1$ | $-\frac{1}{2}E_3$ |
| $E_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $F_0$ | 0 | 0 | $\frac{1}{2}P_1$ | $\frac{1}{2}P_3$ | 0 | 0 | $\frac{1}{2}P_1$ | $\frac{1}{2}P_3$ |
| $F_2$ | $\frac{1}{2}(F_1 - F_3)$ | $\frac{1}{2}(F_1 - F_3)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_0$ | $\frac{1}{2}(e_1 + e_3)$ | $\frac{1}{2}(e_1 + e_3)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

We have the generic formulae

\[
\begin{align*}
\mu(e_1)e_j &= \frac{1}{2}\delta_{0,m}e_j, \\
\mu(e_2)e_j &= \frac{1}{2}\delta_{0,m}e_j, \\
\mu(e_0)E_j &= \frac{1}{2}\delta_{0,m}E_j, \\
\mu(e_0)F_j &= \frac{1}{2}\delta_{2,m}F_j, \\
\mu(e_0)P_j &= \frac{1}{2}\delta_{0,m}P_j, \\
\mu(e_1)e_j &= \frac{1}{2}\delta_{0,m}(E_{j+2} + E_j), \\
\mu(e_2)e_j &= \frac{1}{2}\delta_{0,m}(E_{j-2} + E_j), \\
\mu(e_0)E_j &= \frac{1}{2}\delta_{0,m}(E_{j+2} + E_j), \\
\mu(e_0)F_j &= \frac{1}{2}\delta_{2,m}F_j, \\
\mu(e_0)P_j &= \frac{1}{2}\delta_{0,m}P_j, \\
\mu(e_1)e_j &= \frac{1}{2}\delta_{0,m}(E_{j+2} + E_j), \\
\mu(e_2)e_j &= \frac{1}{2}\delta_{0,m}(E_{j-2} + E_j), \quad (30)
\end{align*}
\]
3) The scalar product determined by the trace and the ∗-operation

\[ <a, b> = \frac{1}{4} Tr_\mu(a^*b), \quad a, b \in H_2^1 \]

has the only non vanishing values \(<e_0, e_0> = 1\) and \(<e_2, e_2> = 2\). In other terms, the scalar product \(<\ldots>\) is positive semi definite with null-space \(N_2(0) \oplus H_2^{1(1)}\).

Proof:
1) From [26] and [27] we deduce the following elements \((id \otimes S) \Delta(a)\) of \(\text{End}_C(H_2^1) \otimes \text{End}_C(H_2^1), a \in H_2^1\), \((id \otimes S) \Delta(e_m) = \frac{1}{4} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} e_k \otimes e_{k-m}\), from which the corresponding \(4\mu(a)x, x \in H_2^1\), are obtained by making \(\otimes \rightarrow x\), yielding [30]

\[
\begin{align*}
\mu(e_m)e_j &= \frac{1}{4} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} e_k e_j e_{k-m} = \frac{1}{4} \delta_{0,m} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \delta_{k,j} e_j = \frac{1}{4} \delta_{0,m} e_j, \\
\mu(e_m)E_j &= \frac{1}{4} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} e_k E_j e_{k-m} = \frac{1}{4} \delta_{0,m} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \delta_{k,j} E_j = \frac{1}{4} \delta_{0,m} E_j, \\
\mu(e_m)F_j &= \frac{1}{4} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} e_k F_j e_{k-m} = \frac{1}{4} \delta_{k+2,k-m} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \delta_{k,j} F_j = \frac{1}{4} \delta_{2,m} F_j, \\
\mu(e_m)P_j &= \frac{1}{4} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} e_k P_j e_{k-m} = \frac{1}{4} \delta_{0,m} \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \delta_{k,j} P_j = \frac{1}{4} \delta_{0,m} P_j.
\end{align*}
\]

The derivation of the other formulae is obtained via multiplicativity of \(\mu\). Combining the latter with

\[
\begin{align*}
\mu(E)(e_j) &= E_{j+2} + E_j, \\
\mu(E)(E_j) &= 0, \\
\mu(E)(F_j) &= -i (\delta_{1,j} e_1 - \delta_{3,j} e_3), \\
\mu(E)(P_j) &= -i (\delta_{1,j} E_1 - \delta_{3,j} E_3),
\end{align*}
\]

\[
\begin{align*}
\mu(F)(e_j) &= (i)^j F_{j+2} + (i)^j F_j, \\
\mu(F)(E_j) &= - \left( (i)^j + (i)^j \right) E_j - (i)^j (\delta_{1,j} e_1 - \delta_{3,j} e_3), \\
\mu(F)(F_j) &= 0, \\
\mu(F)(P_j) &= -i (E_1 + E_3),
\end{align*}
\]

(31)

\[
\begin{align*}
\mu(P)(e_j) &= i (\delta_{1,j} + \delta_{3,j}) (e_1 + e_3), \\
\mu(P)(E_j) &= 2i (E_1 + E_3), \\
\mu(P)(F_j) &= 0, \\
\mu(P)(P_j) &= 0.
\end{align*}
\]

(32)

they are checked as follows: from [2] we have \((id \otimes S) \Delta(E) = E \otimes 1 + K \otimes K^{-1}E\) and \((id \otimes S) \Delta(F) = F \otimes K^{-1} - 1 \otimes FK\). Hence we have for \(x \in H_2^1\)

\[
\begin{align*}
\mu(E)(x) &= Ex + KxK^{-1}E, \\
\mu(F)(x) &= FxK^{-1} + xKF,
\end{align*}
\]

and thus

\[
\begin{align*}
\mu(E)e_j &= Ee_j + Ke_jK^{-1}E = E_{j+2} + E_j, \\
\mu(E)E_j &= EE_j + KE_jK^{-1}E = 0, \\
\mu(E)F_j &= EF_j + KFjK^{-1}E = P_j - e_j (EF + e_1 - e_3) = - (\delta_{1,j} e_1 - \delta_{3,j} e_3), \\
\mu(E)P_j &= EP_j + KP_jK^{-1}E = e_j EFE = -e_j (e_1 - e_3) E = - (\delta_{1,j} E_1 - \delta_{3,j} E_3),
\end{align*}
\]
and

\[
\mu(F) e_j = F e_j K^{-1} + e_j K F = (i)^j F_{j+2} + (i)^{-j} F_j,
\]

\[
\mu(F) E_j = F E_j K^{-1} + E_j K F = (i)^{j+2} e_j (EF + e_1 - e_3) - (i)^{-j} P_j,
\]

\[
\mu(F) F_j = F F_j K^{-1} + F_j K F = 0,
\]

\[
\mu(F) P_j = F P_j K^{-1} + P_j K F = (i)^j e_{j+2} F E F = (i)^j e_{j+2} (e_1 - e_3) F,
\]

further

\[
\mu(P) e_j = \mu(E) \mu(F) e_j = \mu(E) \left( (i)^j F_{j+2} + (i)^{-j} F_j \right),
\]

\[
= - (i)^j (\delta_{1,j+2} e_1 - \delta_{3,j+2} e_3) - (i)^{-j} (\delta_{1,j} e_1 - \delta_{3,j} e_3),
\]

\[
= - (i)^j (\delta_{3,j} e_1 - \delta_{1,j} e_3) - (i)^{-j} (\delta_{1,j} e_1 - \delta_{3,j} e_3),
\]

\[
= i \delta_{3,j} e_1 + i \delta_{1,j} e_3 + i \delta_{1,j} e_1 + i \delta_{3,j} e_3 = i (\delta_{1,j} + \delta_{3,j}) (e_1 + e_3),
\]

\[
\mu(P) E_j = \mu(E) \mu(F) E_j = \mu(E) \left( - \left( (i)^j + (i)^{-j} \right) P_j - (i)^j (\delta_{1,j} e_1 - \delta_{3,j} e_3) \right),
\]

\[
= \left( (i)^j + (i)^{-j} \right) (\delta_{1,j} E_1 - \delta_{3,j} E_3) + i \mu(E) (e_1 + e_3),
\]

\[
= 2 i (E_1 + E_3).
\]

\[
\mu(P) F_j = \mu(E) \mu(F) F_j = 0
\]

We now compute successively

\[
\mu(E_m) e_j = \mu(e_m) \mu(E) e_j = \mu(e_m) (E_{j+2} + E_j) = \frac{1}{4} \delta_{0,m} (E_{j+2} + E_j),
\]

\[
\mu(E_m) E_j = \mu(E) \mu(e_{m+2}) E_j = \frac{1}{4} \delta_{0,m+2} \mu(E)(E_j) = 0,
\]

\[
\mu(E_m) F_j = \mu(E) \mu(e_{m+2}) F_j = \frac{1}{4} \delta_{2,m+2} \mu(E)(F_j) = \frac{1}{4} \delta_{0,m} (\delta_{1,j} e_1 - \delta_{3,j} e_3),
\]

\[
\mu(E_m) P_j = \mu(e_m) \mu(E) P_j = \mu(E),
\]

\[
= - \mu(e_m) (\delta_{1,j} E_1 - \delta_{3,j} E_3) = - \frac{1}{4} \delta_{0,m} (\delta_{1,j} E_1 - \delta_{3,j} E_3),
\]

and

\[
\mu(F_m) e_j = \mu(e_m) \mu(F) e_j = \mu(e_m) \left( (i)^j F_{j+2} + (i)^{-j} F_j \right) = \frac{1}{4} \delta_{2,m} \left( (i)^j F_{j+2} + (i)^{-j} F_j \right),
\]

\[
\mu(F_m) E_j = \mu(e_m) \mu(F) E_j = \mu(e_m) \left( - \left( (i)^j + (i)^{-j} \right) P_j - (i)^j (\delta_{1,j} e_1 - \delta_{3,j} e_3) \right),
\]

\[
= - \frac{1}{4} \delta_{0,m} \left( (i)^j + (i)^{-j} \right) P_j + (i)^j (\delta_{1,j} e_1 - \delta_{3,j} e_3),
\]

\[
\mu(F_m) F_j = \mu(e_m) \mu(F) F_j = 0,
\]

\[
\mu(F_m) P_j = \mu(e_m) \mu(F) P_j = (i)^j \mu(e_m) P_j = \frac{1}{4} \delta_{0,m} (i)^j P_j,
\]

and

\[
\mu(P_m) e_j = \mu(e_m) \mu(P) e_j = i \mu(e_m) (\delta_{1,j} + \delta_{3,j}) (e_1 + e_3),
\]
5  Idempotents, automorphisms and real forms of $H_1^i$

Lemma 5.1 1) There is a unique $\kappa \in \text{Aut}(H_1^i)$ (the flip) such that
\[
\begin{aligned}
\kappa(K) &= -K, \\
\kappa(E) &= E, \\
\kappa(F) &= F.
\end{aligned}
\]
2) $\kappa$ is an involution performing the exchanges $e_0 \leftrightarrow e_1, E_0 \leftrightarrow E_1, F_0 \leftrightarrow F_1$ and $C_0 \leftrightarrow C_1$.
3) One has $\kappa \circ S \circ \kappa = S^{-1}$.

Proof:
1) and 2) : immediate from [17].
3) straightforward from $S^{-1}(K) = K^{-1}, S^{-1}(E) = K^{-1}E$ and $S^{-1}(K) = FK$. \qed

5.1  Idempotents of $H_1^i$

Proposition 5.1 1) The idempotents of $H_1^i$ are
-the element 0 of rank 0,
-the unit 1 of rank 8,
-a continous family of rank 4 idempotents
\[
\begin{aligned}
\{ e_{\alpha,\beta,\gamma,\delta,\eta} &= e_0 + \beta E_0 + \gamma F_0 + \delta E_1 + \eta F_1 + (\beta \eta + \delta \gamma) (C_1 - C_0), \\
e_{1,\beta,\gamma,\delta,\eta} &= e_1 + \beta E_0 + \gamma F_0 + \delta E_1 + \eta F_1 + (\beta \eta + \delta \gamma) (C_1 - C_0),
\end{aligned}
\]
indexed by the parameters $\beta, \gamma, \delta, \eta \in \mathbb{C}$.
2) Consequently the automorphisms $\phi$ of $H_1^i$ are of either the two types:
   a) fulfilling $\phi(e_0) = e_{0,\beta,\gamma,\delta,\eta}, \phi(e_1) = e_{1,-\beta,-\gamma,-\delta,-\eta},$
   b) fulfilling $\phi(e_0) = e_{1,\beta,\gamma,\delta,\eta}, \phi(e_1) = e_{0,-\beta,-\gamma,-\delta,-\eta}.$

Proof:
1) Every idempotent $e$ of $H_1^i = F_1^i \oplus N_1^i$ ($F_1^i$ a subalgebra, $N_1^i$ an ideal) decomposes as $e = e' + e''$ with $e'^2 = e', e''^2 + e'e' = e''$. We search $e'$
   \[ e' = e_0 + \beta e_1 = e'^2 = \alpha^2 e_0 + \beta^2 e_1 \] yields $\alpha (\alpha - 1) = \beta (\beta - 1) = 0$, hence $e'$ either $e_0, e_1$, or $e_0 + e_1$.
   We search $e'' = x_0 E_0 + y_0 F_0 + v_0 C_0 + x_1 E_1 + y_1 F_1 + v_1 C_1$ with $e'' = (x_0 y_1 + x_1 y_0)(C_0 + C_1)$
   - To $e' = 0, e''^2 = e''$ yields $x_0 = y_0 = x_1 = y_1 = 0, v_0 = v_1 = x_0 y_1 + x_1 y_0 = 0$ thus $e'' = 0$
   - To $e' = 1, 2e'' + e'^2 = e''$, $e'^2 = -e''$ yields $x_0 = y_0 = x_1 = y_1 = 0, v_0 = v_1 = -(x_0 y_1 + x_1 y_0) x_0 y_1 + x_1 y_0 = 0$ thus $e'' = 0$
   - To $e' = e_0$ since $e_0 e'' + e_0 e'' = x_0 E_0 + y_0 F_0 + v_0 C_0 + x_1 E_1 + y_1 F_1 + v_1 C_1 = e'' + v_0 C_0 - v_1 C_1$
   \[ e_0 e'' + e_0 e'' = e'' \] yields $v_1 = -v_0 = x_0 y_1 + x_1 y_0, x_0, y_0, x_1, y_1$ remaining arbitrary whence
\( c = e_0 + x_0 E_0 + y_0 F_0 + x_1 E_1 + y_1 F_1 + (x_0 y_1 + x_1 y_0) (C_0 - C_1) \). The flip then yields by exchange symmetry 0 \( \mapsto \) 1 the idempotent \( e_1 = x_0 E_0 + y_0 F_0 + x_1 E_1 + y_1 F_1 + x_0 y_1 + x_1 y_0 \) \((C_0 - C_1)\). 

Rank of \( c_{0, \beta, \gamma, \delta, \eta} \) \( \mapsto h = a_0 e_0 + X_0 E_0 + Y_0 F_0 + c_0 C_0 + a_1 e_1 + X_1 E_1 + Y_1 F_1 + c_1 C_1 \) fulfills \( c_{0, \beta, \gamma, \delta, \eta} h = h \) \( \iff \) one has the relations \( A_1 = 0, \eta X_0 + \delta Y_0 = (\beta \eta + \gamma) a_0 \) (automatic), \( X_1 = \delta a_0, Y_1 = \eta a_0 \). Thus in the two occurring cases \( \delta = 0 \) and \( \delta \neq 0, \eta \neq 0 \), one has rank 4.

2) Obvious from (1) since automorphisms turn an idempotent into an idempotent of the same rank, Moreover, \( \varphi \) must be bijective and satisfy \( \varphi(e_1) + \varphi(e_0) = 1 \).

\[ \square \]

### 5.2 Automorphisms of \( H_1^1 \)

**Proposition 5.2** 1) The set \( \text{Aut}(H_1^1) \) of automorphisms of \( H_1^1 \) coinciding with \( \text{Aut}(N_1^1) \) consists of elements of the two types (whose respective sets will be denoted \( \text{Aut}_1(H_1^1) \) and \( \text{Aut}_{11}(H_1^1) \)).

**Type I** :

\[
\varphi(e_0) = e_0 + \beta E_0 + \gamma F_0 + \delta E_1 + \eta F_1 + (\beta \eta + \gamma \delta) (C_1 - C_0), \\
\varphi(E_0) = \mu_0 E_0 + \nu_0 F_0 + (\delta \nu_0 + \eta \mu_0) (C_1 - C_0), \\
\varphi(F_0) = \sigma_0 E_0 + \tau_0 F_0 + (\delta \tau_0 + \eta \sigma_0) (C_1 - C_0), \\
\varphi(C_0) = \lambda C_0, \\
\]

and

\[
\varphi(e_1) = e_1 - \beta E_0 - \gamma F_0 - \delta E_1 - \eta F_1 - (\beta \eta + \gamma \delta) (C_1 - C_0), \\
\varphi(E_1) = \mu_1 E_1 + \nu_1 F_1 + (\beta \nu_1 + \gamma \mu_1) (C_1 - C_0), \\
\varphi(F_1) = \sigma_1 E_1 + \tau_1 F_1 + (\beta \tau_1 + \gamma \sigma_1) (C_1 - C_0), \\
\varphi(C_1) = \lambda C_1, \\
\]

for constants \( \beta, \gamma, \delta, \eta, \lambda, \mu_0, \nu_0, \sigma_0, \tau_0, \mu_1, \nu_1, \sigma_1, \tau_1 \in \mathbb{C} \) constrained by

\[
\begin{aligned}
1) & \mu_0 \nu_1 + \nu_0 \mu_1 = 0, \\
2) & \sigma_0 \tau_1 + \tau_0 \sigma_1 = 0, \\
3) & \lambda = \mu_0 \tau_1 + \nu_0 \sigma_1 = \sigma_0 \nu_1 + \tau_0 \mu_1, \\
4) & \mu_0 \tau_0 - \nu_0 \sigma_0 \neq 0, \\
5) & \mu_1 \tau_1 - \nu_0 \sigma_0 \neq 0, \\
6) & \lambda \neq 0.
\end{aligned}
\]

**Type II** : Product \( \varphi \kappa \) (or for that matter \( \kappa \varphi \)) with \( \varphi \) of the preceding type I

\[
\varphi(e_0) = e_1 + \beta E_1 + \gamma F_1 + \delta E_0 + \eta F_0 + (\beta \eta + \gamma \delta) (C_0 - C_1), \\
\varphi(E_0) = \mu_0 E_1 + \nu_0 F_1 + (\delta \nu_0 + \eta \mu_1) (C_0 - C_1), \\
\varphi(F_0) = \sigma_0 E_1 + \tau_0 F_1 + (\delta \tau_0 + \eta \sigma_0) (C_0 - C_1), \\
\varphi(C_0) = \lambda C_1, \\
\]

and

\[
\varphi(e_1) = e_0 - \beta E_1 - \gamma F_1 - \delta E_0 - \eta F_0 - (\beta \eta + \gamma \delta) (C_0 - C_1), \\
\varphi(E_1) = \mu_1 E_0 + \nu_1 F_0 + (\beta \nu_1 + \gamma \mu_1) (C_0 - C_1), \\
\varphi(F_1) = \sigma_1 E_0 + \tau_1 F_0 + (\beta \tau_1 + \gamma \sigma_1) (C_0 - C_1), \\
\varphi(C_1) = \lambda C_0, \\
\]

with constants constrained as in (35). Observe that these constraints, as well as the whole structure of \( H_1^1 \), is invariant under the flip.

2) \( \text{Aut}_1(H_1^1) \) is a normal subgroup of \( \text{Aut}(H_1^1) \).

3) The subgroup \( \text{Aut}(H_1^1) = \text{Aut}_1(H_1^1) \oplus \text{Aut}_{11}(H_1^1) \) of \( \text{Aut}_1(H_1^1) \) specified by \( \beta = \gamma = \delta = \eta = 0 \), consists of the \( 4 \times 4 \) matrices

\[
M = \begin{pmatrix}
\mu_0 & \nu_0 & 0 & 0 \\
\sigma_0 & \tau_0 & 0 & 0 \\
0 & 0 & \mu_1 & \nu_1 \\
0 & 0 & \sigma_1 & \tau_1 \\
\end{pmatrix} \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}),
\]
leaving stable the set of bilinear forms with vanishing sum of elements of their second diagonal. Specifically, for each matrix of the form

$$G = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & d \\ a & b & 0 & 0 \\ b & d & 0 & 0 \end{pmatrix} \quad a, b, c, d \in \mathbb{C},$$

$M^tGM$ is a matrix of the same type.

4) The constraints \[\text{entail the relations} \]

$$\begin{cases} \nu_0 \tau_1 + \tau_0 \nu_1 = 0, \\ \mu_0 \sigma_1 + \sigma_0 \mu_1 = 0. \end{cases} \quad (36)$$

Note that (2) answers the natural question why the constraints \[\text{propagate through matrix products. Indeed it is not clear a priori that } \text{Aut}_I(H_1) \text{ characterized as in (1) is multiplicative. Note also how two symmetries permute (either respecting or exchanging) formula, the flip } (0 \leftrightarrow 1) \text{ and the symmetry } (E \leftrightarrow F, \mu \leftrightarrow \tau, \nu \leftrightarrow \sigma). \]

Proof:

1) Exhausting the constraints stemming from multiplicativity of $\varphi$, in view of the multiplication table \[\text{we have the implications} \]

$$\varphi(e_0E_0) = \varphi(e_0)\varphi(E_0) \Rightarrow \varphi' = \nu_1^1 = \nu_1^1 = 0, \rho_1^1 = \delta \nu_0 + \eta \mu_0,$$

$$\varphi(e_0F_0) = \varphi(e_0)\varphi(F_0) \Rightarrow \sigma_1^1 = \tau_1^1 = \omega_1^1 = \delta \tau_0 + \eta \sigma_0, \quad (\nu_0 \tau_1 

\varphi' = \nu_1^1 = \nu_1^1 = 0, \rho_1^1 = \delta \tau_0 + \eta \sigma_0,$$

$$\varphi(F_0e_1) = \varphi(F_0)\varphi(e_1) \Rightarrow \mu_1^0 = \nu_1^0 = 0, \rho_0^0 = - (\delta \tau_0 + \eta \sigma_0),$$

$$\varphi(F_0^1e_1) = \varphi(F_0)\varphi(e_1) \Rightarrow \sigma_1^1 = \tau_1^1 = \omega_1^1 = - (\delta \tau_0 + \eta \sigma_0),$$

$$\varphi(e_1^1F_1) = \varphi(e_1)\varphi(F_1) \Rightarrow \sigma_1^0 = \tau_1^0 = 0, \omega_1^0 = - (\beta \tau_1 + \gamma \sigma_1),$$

$$\varphi(E_1e_0) = \varphi(E_1)\varphi(e_0) \Rightarrow \mu_0^0 = \nu_0^0 = 0, \rho_1 = \beta \nu_1 + \gamma \mu_1,$$

$$\varphi(F_1e_0) = \varphi(F_1)\varphi(e_0) \Rightarrow \sigma_0^0 = \tau_1^0 = 0, \omega_1 = \beta \tau_1 + \gamma \sigma_1,$$

$$\varphi(F_0e_1) = \varphi(F_0)\varphi(e_1) \Rightarrow \sigma_0^1 = \tau_0^1 = 0, \omega_1 = \beta \tau_1 + \gamma \sigma_1,$$

$$\varphi(E_0^1F_0) = \varphi(E_0)\varphi(F_0) \Rightarrow \sigma_1^0 = \tau_0^1 = 0, \omega_1 = \beta \tau_1 + \gamma \sigma_1,$$

$$\varphi(F_0^1e_1) = \varphi(F_0)\varphi(e_1) \Rightarrow \sigma_0^0 = \tau_0^0 = 0, \omega_1 = \beta \tau_1 + \gamma \sigma_1,$$

$$\varphi(F_0^1e_1) = \varphi(F_0)\varphi(e_1) \Rightarrow \sigma_0^0 = \tau_0^0 = 0, \omega_1 = \beta \tau_1 + \gamma \sigma_1,$$

$$\varphi(F_0^1e_1) = \varphi(F_0)\varphi(e_1) \Rightarrow \sigma_0^0 = \tau_0^0 = 0, \omega_1 = \beta \tau_1 + \gamma \sigma_1.$$

2) Let $\varphi \in \text{Aut}_I(H_1), \psi = \varphi \kappa \in \text{Aut}_I(H_1^t), \varphi_1 \in \text{Aut}_I(H_1^t)$, since $\psi^{-1} \varphi \psi = \kappa^{-1} \varphi^{-1}_1 \varphi_1 \kappa$, it suffices to prove that $\text{ad}x$ leaves $\text{Aut}_I(H_1^t)$ stable, now with $\varphi$ as in \[33\] one has $\kappa^{-1} \varphi \kappa = \varphi'$, $\varphi'$ of the form \[33\] and \[34\] with $\beta' = - \delta, \gamma' = - \eta, \delta' = - \beta, \eta' = - \gamma, \mu'_0 = \mu_1, \nu'_0 = \nu_1, \sigma'_0 = \sigma_1, \tau'_0 = \tau_1, \mu'_1 = \mu_0, \nu'_1 = \nu_0, \sigma'_1 = \sigma_0, \tau'_1 = \tau_0.$

3) Follows from

$$M^tGM = \begin{pmatrix} \mu_0 & \sigma_0 & 0 & 0 \\ \nu_0 & \tau_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \sigma_1 \\ 0 & 0 & \nu_1 & \tau_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & d \\ a & b & 0 & 0 \\ b & d & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_0 & 0 & 0 & 0 \\ 0 & \nu_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \nu_1 \\ 0 & 0 & \sigma_1 & \tau_1 \end{pmatrix}$$

$$= \begin{pmatrix} a \mu_0 - b \sigma_0 & b \mu_0 + d \sigma_0 \\ a \nu_0 - b \tau_0 & b \nu_0 + d \tau_0 \end{pmatrix} \begin{pmatrix} \mu_0 & 0 & 0 & 0 \\ 0 & \nu_0 & 0 & 0 \\ 0 & 0 & \mu_1 & \nu_1 \\ 0 & 0 & \sigma_1 & \tau_1 \end{pmatrix}.$$
Thus, the symmetric matrix $M'GM$ will be of the form
\[
\begin{pmatrix}
0 & 0 & a' & b' \\
0 & 0 & -b' & d \\
a' & -b' & 0 & 0 \\
b' & d' & 0 & 0
\end{pmatrix}
\]
for $a', b', c', d' \in \mathbb{C}$, whenever the sum $X + Y$ for its entries $X, Y$ located as
\[
\begin{pmatrix}
\ldots & \ldots & X \\
\ldots & Y & \ldots \\
\ldots & \ldots & \ldots
\end{pmatrix}
\]
vanishes. The latter are
\[
\begin{cases}
X = (a\mu_0 - b\sigma_0) \mu_1 + (b\mu_0 + d\sigma_0) \nu_1, \\
Y = (a\nu_0 - b\tau_0) \mu_1 + (b\nu_0 + d\tau_0) \nu_1,
\end{cases}
\]
whenever the vanishing of
\[
X + Y = a (\nu_1 \mu_0 + \mu_1 \nu_0) + b (\tau_1 \mu_0 + \sigma_1 \nu_0 - \nu_1 \sigma_0 - \mu_1 \tau_0) + d (\tau_1 \sigma_0 + \sigma_1 \tau_0)
\]
expresses the constraints in (35).

4) Multiplying both sides of (35) by $\tau_1 \nu_1$ yield using (35) (1) and (35) (2) yields
\[
-\nu_0 \tau_1 D_1 = \tau_0 \nu_1 D_1,
\]
with $D_1 = \mu_1 \tau_1 - \nu_1 \sigma_1$ assumed not to vanish thus implying relation (36).

5.2.1 The group $\text{Int}(H^1_1)$ of inner automorphisms of $H^1_1$

Proposition 5.3 1) The element $h = a_0 e_0 + X_0 E_0 + Y_0 F_0 + c_0 C_0 + a_1 e_1 + X_1 E_1 + Y_1 F_1 + c_1 C_1 \in H^1_1$ indexed by $a_0, X_0, Y_0, c_0, a_1, X_1, Y_1, c_1 \in \mathbb{C}$, is invertible iff $a_0 a_1 \neq 0$. Its inverse is then
\[
h^{-1} = \frac{1}{a_0} e_0 - a_0 a_1 (X_0 E_0 + Y_0 F_0) + \frac{1}{a_1} \left( \frac{P}{a_0 a_1} - \frac{1}{a_1} \right) C_0,
\]
\[
+ \frac{1}{a_0} e_1 - a_0 a_1 (X_1 E_1 + Y_1 F_1) + \frac{1}{a_1} \left( \frac{P}{a_0 a_1} - \frac{1}{a_1} \right) C_1,
\]
where $P = Y_0 Y_1 + X_1 Y_0$.

2) The matrix of $\text{ad}(h)$ then reads
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{X_0}{a_0} & \frac{a_0}{a_1} & 0 & 0 & X_0 & 0 & 0 & 0 \\
-\frac{Y_0}{a_0} & \frac{a_0}{a_1} & 0 & 0 & Y_0 & 0 & 0 & 0 \\
\frac{P}{a_0 a_1} & -\frac{X_1}{a_1} & \frac{a_0}{a_1} & 0 & -\frac{P}{a_0 a_1} & \frac{X_0}{a_0} & \frac{X_0}{a_0} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{X_1}{a_1} & \frac{a_0}{a_1} & 0 & 0 & -\frac{X_0}{a_0} & \frac{a_0}{a_1} & 0 & 0 \\
\frac{Y_1}{a_1} & \frac{a_0}{a_1} & 0 & 0 & -\frac{Y_0}{a_0} & \frac{a_0}{a_1} & 0 & 0 \\
\frac{Y_1}{a_1} & \frac{X_1}{a_1} & 0 & \frac{-P}{a_0 a_1} & -\frac{X_0}{a_0} & -\frac{Y_0}{a_0} & 0 & 1
\end{pmatrix}
\]
in the basis $(e_0, E_0, F_0, C_0, e_1, E_1, F_1, C_1)$.

This displays the four-parametric group $\text{Int}(H^1_1) \subset \text{Aut}_1(H^1_1)$ of inner automorphisms of $H^1_1$. The relationship with the parametrization of the full $\text{Aut}_1(H^1_1)$ is as follows.
\[
\begin{pmatrix}
\beta = -\frac{X_0}{a_0}, \\
\gamma = -\frac{Y_0}{a_0}, \\
\delta = \frac{X_1}{a_1}, \\
\eta = \frac{Y_1}{a_0}, \\
\mu_0 = \frac{a_0}{a_1}, \\
\nu_0 = 0, \\
\sigma_0 = 0, \\
\tau_0 = \frac{a_0}{a_1}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\beta = -\frac{X_0}{a_0}, \\
\gamma = -\frac{Y_0}{a_0}, \\
\delta = \frac{X_1}{a_1}, \\
\eta = \frac{Y_1}{a_0}, \\
\mu_0 = \frac{a_0}{a_1}, \\
\nu_0 = 0, \\
\sigma_0 = 0, \\
\tau_0 = \frac{a_0}{a_1}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\mu_1 = \frac{a_1}{a_0}, \\
\nu_1 = 0, \\
\sigma_1 = 0, \\
\tau_1 = \frac{a_1}{a_0}
\end{pmatrix}
\]
Proof:
Let $h' = a_0' e_0 + X_0' E_0 + Y_0' F_0 + c_0' C_0 + a_1' e_1 + X_1' E_1 + Y_1' F_1 + c_1' C_1 \in H_1$, we have $hh' = 1$ iff

\[
\begin{cases}
    a_0 a_0' = 1, \\
    a_0 X_0' + a_1' X_0 = 0, \\
    a_0 Y_0' + a_1' Y_0 = 0, \\
    a_0 c_0' + c_0 a_0' + X_0 Y_0' + Y_0 X_0' = 0,
\end{cases}
\]

\[
\begin{cases}
    a_1 a_1' = 1, \\
    a_1 X_1 + a_1 X_1' = 0, \\
    a_1 Y_1 + a_1 Y_1' = 0, \\
    a_1 c_1' + c_1 a_1' + X_1 Y_1' + Y_1 X_1' = 0,
\end{cases}
\]

hence \[37\] is proved. With $\lambda$ (resp. $\rho$) the regular representation (resp. antirepresentation), we have

\[
\text{matrix of } \lambda(h) = \begin{pmatrix}
    e_0 & E_0 & F_0 & C_0 & e_1 & E_1 & F_1 & C_1 \\
    e_0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    E_0 & 0 & a_0 & 0 & 0 & X_0 & 0 & 0 \\
    F_0 & 0 & 0 & a_0 & 0 & Y_0 & 0 & 0 \\
    C_0 & c_0 & 0 & 0 & a_0 & 0 & Y_0 & X_0 \\
    e_1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 \\
    E_1 & X_1 & 0 & 0 & 0 & a_1 & 0 & 0 \\
    F_1 & Y_1 & 0 & 0 & 0 & 0 & a_1 & 0 \\
    C_1 & 0 & Y_1 & X_1 & 0 & c_1 & 0 & a_1
\end{pmatrix},
\]

and

\[
\text{matrix of } \rho(h') = \begin{pmatrix}
    e_0 & E_0 & F_0 & C_0 & e_1 & E_1 & F_1 & C_1 \\
    e_0 & a_0' & 0 & 0 & 0 & 0 & 0 & 0 \\
    E_0 & X_0' & a_0' & 0 & 0 & 0 & 0 & 0 \\
    F_0 & 0 & a_0' & 0 & 0 & 0 & 0 & 0 \\
    C_0 & c_0' & Y_1' & a_1' & 0 & 0 & 0 & 0 \\
    e_1 & 0 & 0 & 0 & 0 & a_1' & 0 & 0 \\
    E_1 & 0 & 0 & 0 & 0 & X_1' & a_1' & 0 \\
    F_1 & 0 & 0 & 0 & 0 & Y_1' & 0 & a_1' \\
    C_1 & 0 & 0 & 0 & 0 & c_1' & Y_0' & X_0' & a_1'
\end{pmatrix}.
\]

The matrix product $\lambda(h)\rho(h')$ then yields relation \[38\] \qed

5.2.2 The subgroup $Aut^S(H_1^i)$ of automorphisms of $H_1^i$ commuting with the antipode $S$

Proposition 5.4 The $\varphi \in Aut^S(H_1^i)$ are of the following two types, corresponding to type $Aut_1(H_1^i)$ and type $Aut_{11}(H_1^i)$ with $\mu, \nu, \sigma, \tau \in \mathbb{C}$ such that $\mu \nu - \sigma \tau \neq 0$.

1) The $\varphi \in Aut^S_1(H_1^i)$ are as follows

\[
\begin{cases}
    \varphi(e_0) = e_0, \\
    \varphi(E_0) = \mu E_0 + \nu F_0, \\
    \varphi(F_0) = \sigma E_0 + \tau F_0, \\
    \varphi(C_0) = (\mu \sigma - \nu \tau) C_0,
\end{cases}
\]

Their action on the generators is given by

\[
\begin{cases}
    \varphi(K) = K, \\
    \varphi(E) = \mu E + \nu K F, \\
    \varphi(F) = \sigma K E + \tau F.
\end{cases} \tag{40}
\]

2) The $\varphi \in Aut^S_{11}(H_1^i)$ are as follows

\[
\begin{cases}
    \varphi(e_0) = e_1, \\
    \varphi(E_0) = -\mu E_0 + \nu F_0, \\
    \varphi(F_0) = \sigma E_1 + \tau F_1, \\
    \varphi(C_0) = (\mu \tau - \sigma \nu) C_1,
\end{cases}
\]

\[
\begin{cases}
    \varphi(e_1) = e_0, \\
    \varphi(E_1) = -\mu E_1 + \nu F_1, \\
    \varphi(F_1) = \sigma E_0 - \tau F_0, \\
    \varphi(C_1) = (\mu \tau - \sigma \nu) C_0.
\end{cases}
\]
Their action on the generators is given by

\[
\begin{align*}
\varphi(K) &= -K, \\
\varphi(E) &= -\mu KE + \nu F, \\
\varphi(F) &= \sigma E - \tauKF.
\end{align*}
\] (41)

Proof:
We confer [4] with the action of \(S\) which we recall.

\[
\begin{array}{c|cccccccc}
  S(a) & e_0 & E_0 & F_0 & C_0 & e_1 & E_1 & F_1 & C_1 \\
  \hline
  a & e_0 & E_1 & -F_1 & C_0 & e_1 & -E_0 & F_0 & C_1 \\
\end{array}
\]

1) The requirement that the action of \(\varphi \circ S\) and \(S \circ \varphi\) be the same,
- on \(e_0\) yields \(\beta = \gamma = \delta = \eta = 0\),
- on \(F_0\) or \(F_1\) yields \(\sigma_0 = -\sigma_1\) and \(\tau_0 = \tau_1\),
- on \(C_0\) is automatic.

The expression 38 immediately follows from 38 and from the facts that \(E_0 + E_1 = E, F_0 + F_1 = F, E_0 - E_1 = KE, F_0 - F_1 = KF\).

2) The requirement that the action of \(\varphi \circ S\) and \(S \circ \varphi\) be the same,
- on \(e_1\) yields \(\beta = \gamma = \delta = \eta = 0\),
- on \(F_1\) or \(F_0\) yields \(\sigma_0 = -\sigma_1\) and \(\tau_0 = \tau_1\),
- on \(C_1\) is automatic.

\(\square\)

5.2.3 The Hopf automorphisms of \(H_i^1\)

Corollary 5.1 Each element of \(\text{Aut}^S(H_i^1)\) and none of \(\text{Aut}^{II}(H_i^1)\) is a Hopf automorphism.

Proof:
We check that \(\varphi \in \text{Aut}^S(H_i^1)\) is coalgebra morphism. By the multiplicativity of \(\Delta\), it suffices to address the generators. We have, by [4]

\[
\begin{align*}
\Delta(\varphi(K)) &= \Delta(K) = K \otimes K = \varphi(K) \otimes \varphi(K), \\
\Delta(\varphi(E)) &= \Delta(\mu E + \nu KF) = \mu (E \otimes 1 + K \otimes E) + \nu (K \otimes K)(F \otimes K^{-1} + 1 \otimes F), \\
&= (\mu E + \nu KF) \otimes 1 + K \otimes (\mu E + \nu KF) = \varphi(E) \otimes \varphi(1) + \varphi(K) \otimes \varphi(F), \\
\Delta(\varphi(F)) &= \Delta(\sigma KE + \tau F) = \sigma (K \otimes K)(E \otimes 1 + K \otimes E) + \tau (F \otimes K^{-1} + 1 \otimes F), \\
&= (\sigma KE + \tau F) \otimes K^{-1} + 1 \otimes (\sigma KE + \tau F) = \varphi(F) \otimes \varphi(K^{-1}) + \varphi(1) \otimes \varphi(F),
\end{align*}
\]

whereas, for \(\varphi \in \text{Aut}^{II}(H_i^1)\), we have by [4]

\[
\Delta(\varphi(K)) = -\Delta(K) = -K \otimes K \neq \varphi(K) \otimes \varphi(K) = K \otimes K.
\]

\(\square\)

Lemma 5.2 The identity on the generators \(K, E, F\) extends uniquely to a Hopf \(\ast\)-operation \(I\) of \(H_i^1\), whose action is given as follows.

\[
\begin{array}{c|cccccccc}
  I(a) & e_0 & E_0 & F_0 & C_0 & e_1 & E_1 & F_1 & C_1 \\
  \hline
  a & e_0 & E_1 & F_1 & C_0 & e_1 & E_0 & F_0 & C_1 \\
\end{array}
\]
Proof
The defining \( \lambda \) and \( \mu \) are obviously respected, as well as the definition relations of the Hopf structure stated on the generators. \( \square \)

**Proposition 5.5**

1) The semi-Hopf \(*\)-operations of \( H_1^1 \) are of the following two types, corresponding to the above type I and type II automorphisms.

**Type I:** \( \Gamma = I \circ \varphi, \varphi \in \text{Aut}_I^S(H_1^1) \) is given by

\[
\begin{align*}
\Gamma(e_0) &= e_0, \\
\Gamma(E_0) &= \alpha E_1 + \beta F_1, \\
\Gamma(F_0) &= \gamma E_1 + \delta F_1, \\
\Gamma(C_0) &= \lambda C_0,
\end{align*}
\]

where

\[
\begin{align*}
\alpha &= ae^{i\phi}, \\
\beta &= \pm be^{i(\psi + \phi)/2}, \\
\gamma &= \pm ce^{i(\psi + \phi)/2}, \\
\delta &= ae^{i\psi}, \\
\lambda &= (\alpha \delta - \beta \gamma) = e^{i(\phi + \psi)},
\end{align*}
\]

with \( a, b, c \geq 0 \) fulfilling \( a^2 - bc = 1 \).

**Type II:** \( \Gamma = I \circ \varphi, \varphi \in \text{Aut}_I^S(H_1^1) \) is given by

\[
\begin{align*}
\Gamma(e_0) &= e_1, \\
\Gamma(E_0) &= \alpha E_1 + \beta F_0, \\
\Gamma(F_0) &= \gamma E_0 + \delta F_0, \\
\Gamma(C_0) &= \lambda C_1,
\end{align*}
\]

where

\[
\begin{align*}
\alpha &= ae^{i\phi}, \\
\beta &= \pm be^{i(\psi + \phi)/2}, \\
\gamma &= \pm ce^{i(\psi + \phi)/2}, \\
\delta &= ae^{i\psi}, \\
\lambda &= - (\alpha \delta - \beta \gamma) = -e^{i(\phi + \psi)},
\end{align*}
\]

with \( a > 0, b, c \geq 0 \) fulfilling \( a^2 + bc = 1 \).

2) The \(*\)-operations characterized under (I) above are in fact the Hopf \(*\)-operations of \( H_1^1 \). Indeed they all fulfill \( \Delta(\Gamma(a)) = \Gamma \otimes \Gamma(\Delta(a)), a \in H_1^1 \) whilst this is the case for none of the \(*\)-operations (II). One has thus a four-parameter family of Hopf \(*\)-operations belonging to the same orbit of right action of Hopf homomorphisms.

Observe that the \(*\)-operation I is obtained by making in (I) the choice \( \alpha = \delta = 1, \beta = \gamma = 0 \) whilst the choice \( \alpha = \delta = 0, \beta = -i, \gamma = i \) yields the \(*\)-operation

\[
\begin{align*}
e_0 &\rightarrow e_0, \\
E_0 &\rightarrow -iF_1, \\
F_0 &\rightarrow iE_1, \\
C_0 &\rightarrow -C_0,
\end{align*}
\]

\[
\begin{align*}
e_1 &\rightarrow e_1, \\
E_1 &\rightarrow iF_0, \\
F_1 &\rightarrow -iE_0, \\
C_1 &\rightarrow -C_1,
\end{align*}
\]

\[
\begin{align*}
K &\rightarrow K, \\
E &\rightarrow iKF, \\
F &\rightarrow iEK^{-1}.
\end{align*}
\]

Proof

1) We seek the semi-Hopf \(*\)-operations as the composition products \( I \circ \varphi, \varphi \in \text{Aut}_I^S(H_1^1) \), which are involutions. With \(*\) indicating complex conjugation, we have

- for \( \varphi \in \text{Aut}_I^S(H_1^1) \) iteration of \( \lambda_2 \) will yield the identity operation iff \( M \bar{M} = 1 \), \( M \) the matrix \( \begin{pmatrix} \mu & \nu \\ \sigma & \tau \end{pmatrix} \), and \( \lambda \bar{\lambda} = 1 \). Setting \( \mu^* = \alpha, \nu^* = \beta, \sigma^* = \gamma, \tau^* = \delta \), the first condition yields

\[
\begin{align*}
1) &\alpha \alpha^* - \beta \gamma^* = 1, \\
2) &\delta \delta^* - \beta \gamma^* = 1,
\end{align*}
\]

\[
\begin{align*}
3) &\alpha \beta^* = \beta \delta^*, \\
4) &\gamma \alpha^* = \delta \gamma^*.
\end{align*}
\]
expressed by \[43\] (observe that \(a \neq 0\) and that in the case \(b = c = 0\) the phases of \(\beta\) and \(\gamma\) are arbitrary); the second condition is then automatic.

- for \(\varphi \in Aut^2_1(H_1)\) iteration of \[43\] will yield the identity operation iff \(M \bar{M} = 1\), \(M\) the matrix \(\begin{pmatrix} \mu & -\nu \\ -\sigma & \tau \end{pmatrix}\), and \(\lambda \bar{\lambda} = 1\). Setting \(\mu^* = \alpha, \nu^* = \beta, \sigma^* = \gamma, \tau^* = \delta\), the first condition yields

\[
\begin{align*}
1) & \alpha \alpha^* + \beta \gamma^* = 1, \\
2) & \delta \delta^* + \beta \gamma^* = 1, \\
3) & \alpha \beta^* + \beta \delta^* = 0, \\
4) & \gamma \alpha^* + \delta \gamma^* = 0,
\end{align*}
\]

expressed by \[45\] (observe that \(a \neq 0\) and that in the case \(b = c = 0\) the phases of \(\beta\) and \(\gamma\) are arbitrary); the second condition is then automatic.

2) Follows from 5.2.2. or can be checked analogously. \(\square\)

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