Capturing Corruption with Hybrid Auctions: Social Welfare Loss in Multi-Unit Auctions

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Abstract. Corruption in auctions is a phenomenon that is theoretically still poorly understood, despite the fact that it occurs rather frequently in practice. In this paper, we initiate the study of the social welfare loss caused by a corrupt auctioneer, both in the single-item and the multi-unit auction setting. In our model, the auctioneer may collude with the winners of the auction by letting them lower their bids in exchange for a fixed fraction $\gamma$ of the surplus. As it turns out, this setting is equivalent to a $\gamma$-hybrid auction in which the payments are a convex combination (parameterized by $\gamma$) of the first-price and the second-price payments. Our goal is thus to obtain a precise understanding of the (robust) price of anarchy of $\gamma$-hybrid auctions. If no further restrictions are imposed on the bids, we prove a bound on the robust POA which is tight (over the entire range of $\gamma$) for the single-item and the multi-unit auction setting. On the other hand, if the bids satisfy the no-overbidding assumption a more fine-grained landscape of the price of anarchy emerges, depending on the auction setting and the equilibrium notion. We derive tight bounds for single-item auctions up to the correlated price of anarchy and for the pure price of anarchy in multi-unit auctions. These results are complemented by nearly tight bounds on the coarse correlated price of anarchy in both settings.

1 Introduction

Motivation and Background. We consider auction settings where a seller wants to sell some items and for this purpose recruits an auctioneer to organize an auction on their behalf\textsuperscript{4}. Such settings are widely prevalent in practice as they emerge naturally whenever the seller lacks the expertise (or facilities, time, etc.) to host the auction themselves. For example, individual sellers usually involve dedicated auctioneers or auction houses when they want to sell particular objects (such as real estate, cars, artwork, etc.). In private companies, the responsible finance officers are typically in charge of handling the procurement auctions. Similarly, government procurement is usually executed by some entity that acts on behalf of the government. The dilemma in such settings is that the incentives of the seller and the auctioneer are rather diverse in general: while the seller is interested in extracting the highest payments for the objects (or getting service at the

\textsuperscript{4} Throughout this paper, we use “they” as the gender-neutral form for third-person singular pronouns.
lowest cost), the agent primarily cares about maximizing their own gains from hosting the auction. Although undesirably, this misalignment leads (unavoidably) to fraudulent schemes which might be used by the auctioneer to manipulate the auction to their own benefit.

Corruption in auctions, where an auctioneer engages in bid rigging with one (or several) of the bidders, occurs rather frequently in practice, especially in the public sector (e.g., in construction and procurement auctions). For example, in 1999 the procurement auction for the construction of the new Berlin Brandenburg airport had to be rerun after investigations revealed that the initial winner was able to change the bid after they had illegally acquired information about the application of one of their main competitors (see [20]). As another example, in 1993 the New York City School Construction Authority caused a scandal when investigations revealed that they used a simple (but effective) bid-rigging scheme (see [15]): “In what one investigator described as a nervy scheme, that worker would unseal envelopes at a public bid opening, saving for last the bid submitted by the contractor who had paid him off. At that point, knowing the previous bids, the authority worker would misstate the contractor’s bid, insuring that it was low enough to secure the contract but as close as possible to the next highest bid so that the contractor would get the largest possible price.” This kind of bid rigging, where the winning bid “magically” aligns with the highest losing bid, is also known as magic number cheating (see [8]). We refer the reader to [10,14] (and the references therein) for several other bid rigging examples.

Despite the fact that this form of corruption occurs frequently in practice, its negative impact is still poorly understood theoretically and only a few studies (mostly in the economics literature) exist; see related work section. Our goal here is to initiate the study of the social welfare loss caused by corrupt auctioneers in fundamental auction settings. We focus on a basic model that captures the magic number cheating mentioned above and hope that this will lead the study of more sophisticated models in the future.

Capturing Corruption with Hybrid Auctions. Consider the single-item auction setting and suppose the auctioneer runs a sealed bid first-price auction. After receipt of all bids, the auctioneer approaches the highest bidder with the offer that they can lower their bid to the second highest bid in exchange for a bribe. If the highest bidder agrees, they win the auction and pay the second-highest bid for the items plus the corresponding bribe to the auctioneer. If the highest bidder disagrees, they still win the auction but pay their bid for the item according to the first-price auction format. We assume that the bribe to be paid to the auctioneer is a pre-determined fraction \( \gamma \in [0, 1] \) of the savings of the highest bidder, i.e., the auctioneer’s bribe amounts to \( \gamma \) times the difference between the highest and second highest bid. In case of the multi-unit auction setting, the procedure described above is adapted accordingly by offering the winning bidders to lower their bids to the highest losing bid.

Note that implicitly we make the following assumptions (see also [10]): (1) The auction is “semi-open” in the sense that at some stage a sealed bid part as described above takes place. In fact, in practice many auctions have this characteristic. (2) The auctioneer is aware of the fact that approaching the bidders is a risky endeavor and tries to minimize the number of illegal contacts by getting in touch with the winning bidders.
only. (3) Bid rigging allows the highest bidders only to lower their bids in exchange for a fixed share of the surplus. However, the allocation of the items must remain the same.

Observe that the payment scheme described above essentially reduces to the winning bidders paying a convex combination of $\gamma$ times their bids and $(1 - \gamma)$ times the highest losing bid. As we will argue below, this setting is tantamount to studying a hybrid auction $\gamma$-hybrid, where the items are assigned to the highest bidders (according to the respective single-item or multi-unit auction scheme) and the payments are a convex combination of the first-price and the second-price payments. By varying the parameter $\gamma \in [0, 1]$, $\gamma$-hybrid thus interpolates between the respective second-price auction ($\gamma = 0$) and the first-price auction ($\gamma = 1$) schemes. Note that the study of $\gamma$-hybrid might be interesting in its own right, purely from an auction design perspective (i.e., notwithstanding the corruption viewpoint adopted in this paper).

Our Contributions. We study the inefficiency of equilibria of $\gamma$-hybrid, both in the single-item and the multi-unit auction setting. More specifically, our goal is to obtain a precise understanding of the (robust) price of anarchy (POA) \cite{POA1,POA2,POA3} of $\gamma$-hybrid (definition is given below). We opt for the price of anarchy notion here because it is one of the most appealing and widely accepted measures to assess the efficiency of equilib-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Overview of our upper bounds on the POA (y-axis) for $\gamma$-hybrid as a function of $\gamma$ (x-axis). (a) CCE-POA for multi-unit auctions with overbidding (Theorem 6). (b) CCE-POA for multi-unit auctions without overbidding (Theorems 9 & 10). (c) CCE-POA for single-item auctions without overbidding (Theorems 6, 8, & 10). (d) CCE-POA for single-item auctions without overbidding and $n = 2$ bidders (Theorems 10, 11, & 12).}
\end{figure}
ria, especially in the context of social welfare analysis. We focus on the analysis of the robust price of anarchy under the complete information setting, incorporating equilibrium notions ranging from pure Nash equilibria (PNE) to coarse correlated equilibria (CCE).

Generally, we analyze the price of anarchy of $\gamma$-Hybrid distinguishing between the case when bidders can overbid and when they cannot overbid their actual valuations for the items.

The main results that we obtain in this paper are summarized below (see Figure 1 for an overview). Without any restrictions on the bids, we obtain the following result:

1. We prove an upper bound of $(1/\gamma) \cdot e^{1/\gamma}/(e^{1/\gamma} - 1)$ on the coarse correlated POA (CCE-POA) of $\gamma$-Hybrid in the multi-unit auction setting when bidders can overbid; see Figure 1(a). Our upper bound follows from a suitable adaptation of the smoothness technique for multi-unit auctions [19,3]. Further, we prove a matching lower bound over the entire range $\gamma \in [0, 1]$ by adapting an example of a first-price single-item auction by Syrgkanis [18] to the $\gamma$-hybrid auction. As a result, our bound settles the CCE-POA of $\gamma$-Hybrid exactly for both the single-item and multi-unit auction setting over the entire range of $\gamma \in [0, 1]$.

Oftentimes, it is justifiable to assume that the bidders cannot overbid. Under the no-overbidding assumption, a more fine-grained landscape of the price of anarchy emerges:

2. We show that the pure POA (PNE-POA) of $\gamma$-Hybrid in the multi-unit auction setting is 1 for $\gamma \in (0, 1)$. This result is complemented by PNE-POA = 2.1885 for $\gamma = 0$ [1] and PNE-POA = 1 for $\gamma = 1$ [4]. Note that this reveals an interesting transition at $\gamma = 0$.

3. We prove that the CCE-POA of $\gamma$-Hybrid in the multi-unit auction setting is upper bounded by

$$-(1 - \gamma)^W_1 \left(\frac{1}{e^{(2-\gamma)/(1-\gamma)}}\right),$$

for $\gamma \leq 0.607$ where $W$ is the Lambert-$W$ function. Combined with the upper bound in (1) for $\gamma > 0.607$ (i.e. with overbidding), we obtain the combined bound depicted in Figure 1(b).

4. We prove that the correlated POA (CE-POA) of $\gamma$-Hybrid in the single-item auction setting is 1 for every $\gamma \in (0, 1)$. This result together with CE-POA = 1 for $\gamma = 1$ [5] and our next result, shows that CE-POA = 1 for the entire range $\gamma \in [0, 1]$.

5. We show that the CCE-POA of $\gamma$-Hybrid in the single-item auction setting with $n$ bidders is bounded as indicated in Figure 1(c). Concretely, we prove an upper bound of $1/(1 - \gamma)$ and combine it with the multi-unit bounds from Figure 1(b).

6. We show that the CCE-POA of $\gamma$-Hybrid in the single-item auction setting with $n = 2$ bidders is bounded as indicated in Figure 1(d). This bound is derived by combining three different upper bounds, one of which the $1/(1 - \gamma)$ bound from Figure 1(c). Technically, this is the most challenging part of the paper as we use the cumulative distribution functions of equilibrium bids directly to derive these bounds.

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5 Several bounds are based on an adapted smoothness approach and extend to the incomplete information setting; see the extensions section below for more details.
Implications of our Results. Altogether, our bounds provide a rather complete picture of the POA of $\gamma$-Hybrid for different equilibrium notions both in the single-item and the multi-unit auction setting and with and without overbidding. If the bidders can overbid then our (tight) bound on the CCE-POA (Figure 1(a)) shows that the POA increases from a small constant $e/(e - 1)$ to infinity as $\gamma$ decreases from 1 to 0. We feel that this makes sense intuitively: As $\gamma$ approaches 0, the auctioneer only withholds a small fraction of the surplus and the bidders are thus incentivized to exploit the corruption (as it comes at a low cost). In contrast, as $\gamma$ approaches 1, the auctioneer charges a significant fraction of the surplus and while the bidders still have good reasons to join the corruption (explained below) they exploit it less drastically as it comes at a large cost. Our bounds reveal that there is a substantial difference in the POA depending on whether or not bidders can overbid; e.g., compare the bounds depicted in (a) and (b) (multi-unit setting), or (c) and (d) (single-item setting) in Figure 1. In general, it is not well-understood how the no-overbidding assumption influences the POA of auctions; this question also relates to the price of undominated anarchy studied by Feldman et al. [5] (see related work below). Our bounds shed some light on this question for $\gamma$-Hybrid.

Extensions. Although we focus on the complete information setting in this paper, most of our bounds can be lifted to the incomplete information setting as introduced by Harsanyi [7], where players have private valuation functions drawn from a common prior. Several of our upper bounds are based on an adapted smoothness approach for multi-unit auctions which extends (basically) directly to this incomplete information setting and (mixed) Bayes-Nash equilibria. More specifically, all bounds displayed in Figure 1(a–c) remain valid for Bayes-Nash equilibria as well. These extensions are proven along the same line of arguments as in [3,4], where similar smoothness arguments are used to bound the Bayes-Nash POA of (standard) multi-unit auctions. Given that these extensions cause quite some notational overhead without adding much analytically, we defer further details to the full version of the paper.

In the multi-unit auction setting, our basic bid rigging model might seem less realistic because the magic number cheating boils down to the winning bidders all bidding the same (highest losing) bid. We remark that, by using suitable transformations, our bounds on the POA of $\gamma$-Hybrid also apply to more general, non-uniform bid rigging models (see Appendix A.2 for more details).

Related Work. There is a large body of research in economics studying collusion among bidders in auctions (see, e.g., [6,13] for some standard references). Collusion between the auctioneer and the bidders in the form of bid rigging (as considered in this paper) has also been studied in the literature, but less intensively. Most existing works study certain aspects of equilibrium outcomes (e.g., equilibrium structure, auctioneer surplus, seller revenue, optimal bribe schemes, etc.); for an overview of the existing works along these lines, see [11,12,14] and the references therein.

The specific bid rigging model that we consider here was first studied by Menezes and Monteiro [14] and a slight generalization thereof by Lengwiler and Wolfstetter [11], both for the single-item auction setting. These works consider a Bayesian setting where the valuations are independent draws from a common distribution function. Menezes and Monteiro [14] prove the existence of symmetric equilibrium bidding strategies and
derive an optimal bribe function for the auctioneer. The authors also study a fixed-price bribe scheme, where the auctioneer charges a fixed amount that is independent of the gained surplus.

Subsequently, Lengwiler and Wolfschterter [10] study a more complex bid rigging scheme for the single-item auction setting, where the auctioneer additionally offers the second highest bidder to increase their bid. To the best of our knowledge, none of the existing works studied the price of anarchy of corrupt auctions.

Studying the price of anarchy in auctions has recently received a lot of attention; we refer to the survey paper by Roughgarden et al. [17] for an overview. A lot of work has gone into deriving bounds on the price of anarchy for various auction formats, both in the complete and incomplete information setting. The smoothness notion, originally introduced by Roughgarden in [16] to analyze the robust price of anarchy of strategic games, turned out to be very useful in an auction context as well. Syrgkanis and Tardos [19] build upon this notion and provide a powerful (smoothness-based) toolbox for the analysis of a broad range of auctions that fall into their composition framework.

With respect to the multi-unit auction setting, de Keijzer et al. [3] use an adapted smoothness approach to derive bounds on the POA of Bayes-Nash equilibria for the first-price and the second-price multi-unit auction (mostly focussing on the setting with no overbidding). Our bounds coincide with theirs for the extreme points $\gamma = 0$ and $\gamma = 1$. For the more general class of subadditive valuations, the POA of Bayes-Nash equilibria for the first-price multi-unit auction is 2, which follows from [3] and [2]. Birmpas et al. [1] recently settled the PNE-POA of the second-price multi-unit auction and show that it is 2.1885.

Our bounds on the CCE-POA are also based on a smoothness approach. We use a slightly adapted smoothness notion (inspired by [4][9]) to derive our bounds, both in the overbidding and the no overbidding setting. Interestingly, our smoothness proofs crucially exploit that the payments computed by $\gamma$-Hybrid recover at least a fraction of $\gamma$ of the first-price payments (but never exceed them). As a side result, Syrgkanis and Tardos [19] also derive a first bound on the CE-POA for $\gamma$-Hybrid in the single-item auction setting; our bound improves on theirs.

The POA of the first-price and second-price auction has been investigated intensively for both the single-item and the multi-unit auction setting. An assumption that often needs to made to derive meaningful bounds is that the bidders cannot overbid. For example, it is folklore that the PNE-POA of the second-price single-item auction is unbounded if the bidders can overbid. On the other hand, it is bounded by a small constant if bidders cannot overbid. In the second-price single-item auction, overbidding is a dominated strategy for each bidder and the no-overbidding assumption thus emerges naturally. But this might not be true in general. For example, for the second-price multi-unit auction, this analogy breaks already.

In general, the impact that the no-overbidding assumption has on the price of anarchy is not well-understood. This aspect also relates to the price of undominated anarchy studied by Feldman et al. [5]. The authors prove a clear separation for the POA in single-item first-price auctions: While the CE-POA is 1 (even with overbidding), the CCE-POA increases to 1.229 (without overbidding) and $e/(e-1)$ (with overbidding). A similar separation holds for the multi-unit auction setting and the uniform price auction,
where the PNE-POA is \( (e - 1)/e \) (without overbidding) \([12]\) and 2.1885 (with overbidding) \([1]\). Our results contribute to this line of research also because we show that the POA might improve significantly under the no-overbidding assumption.

2 Preliminaries

**Standard Auction Formats.** We focus on the description of the multi-unit auction setting; the single-item auction setting follows as a special case (choosing \( k = 1 \) below). In the multi-unit auction setting, there are \( k \geq 1 \) identical items (or goods) that we want to sell to \( n \geq 2 \) bidders (or players). We identify the set of bidders \( N \) with \( [n] = \{1, \ldots, n\} \). Each bidder \( i \) has a non-negative and non-decreasing valuation function \( v_i : \{0, \ldots, k\} \rightarrow \mathbb{R}_{\geq 0} \) with \( v_i(0) = 0 \), where \( v_i(j) \) specifies \( i \)'s valuation for receiving \( j \) items. We assume that for each bidder \( i \in N \) the valuation function \( v_i \) is submodular or, equivalently, that the marginal valuations are non-increasing, i.e., for every \( j \in [k - 1] \), \( v_i(j) - v_i(j - 1) \geq v_i(j + 1) - v_i(j) \). The valuation function \( v_i \) is assumed to be private information, i.e., it is only known to bidder \( i \) themselves. We use \( v = (v_1, \ldots, v_n) \) to denote the profile (or vector) of the valuation functions of the bidders. We assume that the bidders submit their bids according to the following standard format: Each bidder \( i \) submits a bid vector \( b_i = (b_i(1), \ldots, b_i(k)) \) of \( k \) non-negative and non-increasing marginal bids, i.e., \( b_i(j) \) specifies the additional amount \( i \) is willing to pay for receiving \( j \) instead of \( j - 1 \) items. The overall amount that \( i \) bids for receiving \( q \) items is thus \( \sum_{j=1}^{q} b_i(j) \). For \( k = 1 \) we write \( b_i = b_i(1) \).

Consider a multi-unit auction setting and suppose the auctioneer uses an auction mechanism \( M \) to determine an assignment of the items and the respective payments of the bidders. Each bidder submits their bid vector \( b_i \) to the mechanism. Based on the bidding profile \( b = (b_1, \ldots, b_n) \), the mechanism \( M \) orders the submitted marginal bids non-increasingly (breaking ties in an arbitrary but consistent way) and assigns the \( k \) items to the bidders who submitted the \( k \) highest marginal bids (according to this order). We use \( x(b) = (x_1(b), \ldots, x_n(b)) \) to refer to the resulting allocation, where \( x_i(b) \) specifies the number of items that bidder \( i \) receives; \( x_i(b) = 0 \) if \( i \) does not receive any item. Each bidder \( i \) who receives at least one item is called a winner.

There are two standard payment schemes that determine for each winner \( i \) the respective payment \( p_i(b) \); we adopt the convention that \( p_i(b) = 0 \) for each bidder \( i \) who is not a winner.

- **First-price payment scheme:** Every bidder \( i \) pays their bid for the received items, i.e., \( p_i(b) = \sum_{j=1}^{x_i(b)} b_i(j) \)
- **Second-price payment scheme:** Every bidder \( i \) pays the highest losing bid \( \bar{p}(b) \) for each received item, i.e., \( p_i(b) = x_i(b)\bar{p}(b) \)

Suppose we fix the payment scheme of mechanism \( M \) according to one of these schemes. We refer to mechanism \( M \) with the first-price payment or the second-price payment scheme, respectively, as FP-AUCTION or SP-AUCTION\(^6\)

\( ^6 \) We remark that in the multi-unit auction setting these auctions are usually referred to as discriminatory price auction and uniform price auction; however, here we stick to the given naming convention to align it with the common terminology of the single-item auction setting.
The utility \( u_i^v(b) \) of bidder \( i \) is defined as the total valuation minus the payment for receiving \( x_i(b) \) items, i.e., \( u_i^v(b) = v_i(x_i(b)) - p_i(b) \); note that \( u_i^v(b) = 0 \) by definition if bidder \( i \) is not a winner. Whenever \( v_i \) is clear from the context, we simply denote the utility of bidder \( i \) by \( u_i(b) \). We assume that each bidder strives to maximize their utility.

Finally, we introduce some standard assumptions that we use throughout this paper; we adopt the convention that the first two must always be satisfied by a mechanism.

1. **No positive transfers (NPT):** The payment of each bidder \( i \) is non-negative, i.e., \( p_i(b) \geq 0 \).
2. **Individual rationality (IR):** The payment of each bidder \( i \) does not exceed their bid, i.e., \( p_i(b) \leq \sum_{j=1}^{v_i(b)} b_{ij}(j) \).
3. **No overbidding (NOB):** The bid vector of each bidder \( i \) does not exceed their valuations, i.e., for every \( q \in [k] \), \( \sum_{j=1}^{q} b_{ij}(j) \leq v_i(q) \).

**Equilibrium Notions and the Price of Anarchy.** We focus on the complete information setting here. Below, we briefly review the different equilibrium notions used in this paper. A bidding profile \( b = (b_1, \ldots, b_n) \) is a **pure Nash equilibrium (PNE)** if no bidder has an incentive to deviate unilaterally; more formally, \( b \) is a PNE if for every bidder \( i \) and every bidding profile \( b'_i \) of \( i \) it holds that \( u_i(b) \geq u_i(b'_i, b_{-i}) \). Here we use the standard notation \( b_{-i} \) to refer to the bid vector \( b \) with the \( i \)th component being removed; \((b'_i, b_{-i}) \) then refers to the bid vector \( b \) with the \( i \)th component being replaced by \( b'_i \).

We also consider randomized bid vectors. Suppose bidder \( i \) chooses their bid vectors randomly according to a probability distribution \( \sigma_i \), independently of the other bidders. Let \( \sigma = \prod_{i \in N} \sigma_i \) be the respective product distribution. Then \( \sigma \) is a **mixed Nash equilibrium (MNE)** if for every bidder \( i \) and every bid vector \( b'_i \) it holds that \( \mathbb{E}_{b\sim\sigma}[u_i(b)] \geq \mathbb{E}_{b\sim\sigma}[u_i(b'_i, b_{-i})] \). We may also allow correlation among the bidders. Let \( \sigma \) be a joint distribution over bidding profiles of the bidders. Then \( \sigma \) is a **correlated equilibrium (CE)** if for every bidder \( i \in N \) and for every deviation function \( m_i(b_i) \) it holds that \( \mathbb{E}_{b\sim\sigma}[u_i(b)] \geq \mathbb{E}_{b\sim\sigma}[u_i(m_i(b_i), b_{-i})] \). Intuitively, conditional on bid vector \( b \), being realized, \( i \) has no incentive to deviate to any other bid vector \( m_i(b_i) \). The most general equilibrium notion that we consider in this paper is defined as follows: Let \( \sigma \) be a joint distribution over bidding profiles of the bidders. Then \( \sigma \) is a **coarse correlated equilibrium (CCE)** if for every bidder \( i \) and every bid vector \( b'_i \) it holds that \( \mathbb{E}_{b\sim\sigma}[u_i(b)] \geq \mathbb{E}_{b\sim\sigma}[u_i(b'_i, b_{-i})] \). Below, we also use PNE\((\nu)\), MNE\((\nu)\), CE\((\nu)\) and CCE\((\nu)\) to refer to the sets of pure, mixed, correlated and coarse correlated equilibria with respect to a valuation profile \( \nu = (v_1, \ldots, v_n) \), respectively.

We define the **social welfare** of a bidding profile \( b = (b_1, \ldots, b_n) \) as the overall valuation obtained by the bidders, i.e., \( \text{SW}(b) = \sum_{i \in N} v_i(x_i(b)) \). The expected social welfare of a joint distribution \( \sigma \) over bidding profiles is then defined as \( \mathbb{E}[[\text{SW}(\sigma)]] = \mathbb{E}_{b\sim\sigma}[\text{SW}(b)] \). We use \( x^*(\nu) \) to refer to an assignment that maximizes the social welfare with respect to the valuation functions \( \nu = (v_1, \ldots, v_n) \); i.e., \( \text{SW}(x^*(\nu)) = \sum_{i \in N} v_i(x_i^*(\nu)) \) is the maximum social welfare achievable for the bidders. \( x^*(\nu) \) is also called a social optimum.

The **price of anarchy** is defined as the maximum ratio of the social welfare of the social optimum and the (expected) social welfare of an equilibrium. Let \( X \) be a placeholder that refers to one of the equilibrium notions above, i.e., \( X \in \{ \text{PNE, MNE, CE, CCE} \} \).
More formally, given a valuation profile \( v = (v_1, \ldots, v_n) \), the price of anarchy with respect to \( X \) (or \( X\text{-POA} \) for short) is defined as \( X\text{-POA}(v) = \sup_{\sigma \in X} \frac{\text{SW}(x^*(v))/\mathbb{E}[\text{SW}(\sigma)]}{\text{SW}(x^*(v))} \).

The price of anarchy of an auction format then refers to the worst-case price of anarchy over all possible valuation profiles, i.e., \( X\text{-POA} = \sup_v X\text{-POA}(v) \). We use PNE-POA, MNE-POA, CE-POA and CCE-POA to refer to the respective price of anarchy notions.

### 3 Capturing Corruption with \( \gamma \)-Hybrid Auctions

We give a formal description of the model that we consider and elaborate on its relation to the \( \gamma \)-hybrid auction. We also introduce the adapted smoothness approach.

**Corruption in Auctions.** Suppose the bidders submit their bid vectors \( b = (b_1, \ldots, b_n) \) in a “sealed manner”, i.e., at first only the auctioneer sees the bidding profile \( b \). After receipt of the bidding profile \( b \), the auctioneer runs a first-price multi-unit auction (see Section 2) to obtain the respective assignment \( x(b) = (x_1(b), \ldots, x_n(b)) \) and payments \( p(b) = (p_1(b), \ldots, p_n(b)) \) but does not reveal this outcome yet. The auctioneer then approaches each winning bidder \( i \) individually with the offer that they can lower all their \( x_i(b) \) winning bids to the highest losing bid \( \bar{p}(b) \) (while receiving the same number of items), in exchange for a fixed fraction \( \gamma \in [0, 1] \) of the surplus gained by \( i \). The bidder can either reject or accept this offer. If bidder \( i \) rejects the offer, the allocation \( x_i(b) \) and respective payment \( p_i(b) \) remain unmodified. If bidder \( i \) accepts the offer, they receive the \( x_i(b) \) items at a reduced price of \( \bar{p}(b) \) each, but additionally pay a fee \( f_{\gamma i}(b) \) of \( \gamma \) times the surplus to the auctioneer; more formally, the total payment of a winning bidder \( i \) who accepts the offer is

\[
p_{\gamma i}(b) = x_i(b)\bar{p}(b) + f_{\gamma i}(b) \quad \text{where} \quad f_{\gamma i}(b) = \gamma \sum_{j=1}^{x_i(b)} (b_i(j) - \bar{p}(b)).
\]

We also refer to this setting as the \( \gamma \text{-corrupt FP-Auction} \).

Note that the change in the bid vector of player \( i \) is conform with the imposed bidding format, i.e., the modified marginal bids of bidder \( i \) are still non-negative and non-increasing. Note that in the complete information setting that we consider here we implicitly assume that all bidders are aware of the fact that the auctioneer is corrupt and also know the fraction \( \gamma \) that is withheld from the surplus.

It is not hard to show that it is a dominant strategy for every winning bidder to accept the offer of the auctioneer (see Appendix A.1). Subsequently, we assume that each winning bidder always accepts the offer.

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7 It is important to realize though that the final bids, which might not necessarily correspond to the submitted ones, might have to be revealed eventually because the bidders might want to verify the “soundness” of the outcome of the auction.

8 As the final payments are dependent on \( \gamma \), we need that the bidders are aware of this parameter (much alike it is assumed that the bidders know the used payment scheme in other auction formats). However, as we argue below we can assume that each winning bidder accepts the offer of the auctioneer, independently of the parameter \( \gamma \).
Hybrid Auction Scheme. We introduce our novel hybrid auction scheme, which we term $\gamma$-hybrid auction (or $\gamma$-Hybrid for short): $\gamma$-Hybrid uses the same allocation rule as in the multi-unit auction setting (see Section 2), but uses a convex combination of the first-price and second-price payment scheme (parameterized by $\gamma$), i.e.,

$$
p_{\gamma}(b) = \gamma \sum_{j=1}^{x_i(b)} b_{i}(j) + (1 - \gamma) x_i(b) \bar{p}(b).
$$

(1)

Said differently, $\gamma$-Hybrid interpolates between SP-Auction ($\gamma = 0$) and FP-Auction ($\gamma = 1$) as $\gamma$ varies from 0 to 1. We also use $p_{\gamma}(b)$ to refer to the above payment in the single-item auction setting.

The following proposition follows immediately from the discussion above.

**Proposition 1.** Fix some $\gamma \in [0, 1]$. Then $\gamma$-corrupt FP-Auction and $\gamma$-Hybrid admit the same set of equilibria and have identical social welfare objectives. Therefore, the price of anarchy for both these settings is the same.

We thus focus on the POA of $\gamma$-Hybrid. We remark that although our basic bid rigging model is such that all winning bidders lower their bids to the highest losing bid, we can use $\gamma$-Hybrid to capture more general, non-uniform bid rigging models as well (see Appendix A.2 for details).

### 3.1 Adapted Smoothness Notion

We introduce our adapted smoothness notion (based on the ones given in [19,4]) to derive upper bounds on the coarse correlated price of anarchy of $\gamma$-hybrid auction.

Given a bidding profile $b$, we let $\beta_j(b)$ refer to the $j$th lowest winning bid under $b$.

**Definition 2.** A mechanism $M$ for the multi-unit auction setting is $(\lambda, \mu)$-smooth for some $\lambda > 0$ and $\mu \geq 0$ if for every valuation profile $v$ and for each bidder $i \in N$ there exists a (possibly randomized) deviation $\sigma'_i$ such that for every bidding profile $b$ we have

$$
\sum_{i \in N} \mathbb{E}_{\sigma'_i \sim \sigma_i} [u_i(b'_i, b_{-i})] \geq \lambda SW(x^*(v)) - \mu \sum_{j=1}^{k} \beta_j(b).
$$

In essence, this definition comes close to the weak smoothness definition in [19], but relates more directly to the winning bids in the multi-unit auction setting. A similar definition is also used in [4], but there it is imposed on a per-player basis and used for the Bayesian setting.

We say that a mechanism is **first-price $\gamma$-approximate** for some $\gamma \in [0, 1]$ if it always recovers at least a fraction of $\gamma$ of the first-price payments, i.e., for every bidding profile $b$, $\sum_{i \in N} p_i(b) \geq \gamma \sum_{j=1}^{k} \beta_j(b)$.

**Theorem 3.** Let $M$ be a mechanism which is $(\lambda, \mu)$-smooth and first-price $\gamma$-approximate. Then $CCE$-POA $\leq \max\{1, 1 + \mu - \gamma\}/\lambda$, where we need that the no-overbidding assumption holds if $\mu > \gamma$. 
Proof. Fix a valuation profile \( v \) and let \( \sigma \) be a coarse correlated equilibrium. Consider some player \( i \) and let \( \sigma'_i \) be the (randomized) deviation of bidder \( i \) as given by the smoothness definition. Exploiting the coarse correlated equilibrium condition for \( i \), we have for every (deterministic) bid vector \( b'_i \) that \( \mathbb{E}_{b \sim \sigma}[u_i(b)] \geq \mathbb{E}_{b \sim \sigma}[u_i(b'_i, b_{-i})] \) and thus also

\[
\mathbb{E}_{b \sim \sigma}[u_i(b)] \geq \mathbb{E}_{b \sim \sigma}[\mathbb{E}_{b'_i \sim \sigma'_i}[u_i(b'_i, b_{-i})]]. \tag{2}
\]

Using this, we obtain

\[
\mathbb{E}[\text{SW}(\sigma)] = \sum_{i \in N} \mathbb{E}_{b \sim \sigma}[u_i(b) + p_i(b)] \geq \sum_{i \in N} \mathbb{E}_{b \sim \sigma}[\mathbb{E}_{b'_i \sim \sigma'_i}[u_i(b'_i, b_{-i})] + p_i(b)]
\]

\[
\geq \lambda \text{SW}(x^*(v)) + (\gamma - \mu) \mathbb{E}_{b \sim \sigma}[\beta_j(b)] \tag{3},
\]

where the first inequality follows from (2) and the second inequality holds because of the smoothness definition and because \( \lambda \) is first-price \( \gamma \)-approximate.

We distinguish two cases:

Case 1: \( \mu \leq \gamma \). Using (3), we obtain \( \mathbb{E}[\text{SW}(\sigma)] \geq \lambda \text{SW}(x^*(v)) \) and thus \( \text{POA}(v) \leq 1/\lambda \).

Case 2: \( \mu > \gamma \). Exploiting that the no-overbidding assumption holds in this case, we get that \( \sum_{j=1}^k \beta_j(b) \leq \sum_{i \in N} v_i(x_i(b)) \). Using (3), we obtain \( \mathbb{E}[\text{SW}(\sigma)] \geq \lambda \text{SW}(x^*(v)) + (\gamma - \mu) \mathbb{E}[\text{SW}(\sigma)] \). Rearranging terms yields \( \text{POA}(v) \leq (1 + \mu - \gamma)/\lambda \). Combining both cases proves the claim.

We will use the above smoothness definition in combination with the following lemma, which we import from [4] (adapted to our setting). We say that a mechanism is first-price dominated if the computed payments never exceed the first-price payments, i.e., \( \sum_{i \in N} p_i(b) \leq \sum_{j=1}^k \beta_j(b) \).

Lemma 4 (Lemma 3 in [4]). Let \( M \) be a mechanism that is first-price dominated and let \( \alpha > 0 \) be fixed arbitrarily. Then for every valuation profile \( v \) and for every bidder \( i \) there exists a randomized deviation \( \sigma'_i \) such that for every bidding profile \( b \) we have

\[
\mathbb{E}_{b'_i \sim \sigma'_i}[u_i(b'_i, b_{-i})] \geq \alpha \left( 1 - \frac{1}{e^{1/\alpha}} \right) v_i(x^*(v)) - \alpha \sum_{j=1}^k \beta_j(b) \tag{4}
\]

Combining Theorem [3] with Lemma [4] enables us to derive a smoothness bound on the POA of the \( \gamma \)-hybrid auction.

Theorem 5. Let \( \alpha > 0 \) be fixed arbitrarily. The coarse correlated price of anarchy of \( \gamma \)-Hybrid is

\[
\text{CCE-POA} \leq \frac{\max[1, 1 + \alpha - \gamma]}{\alpha(1 - e^{1/\alpha})}, \tag{5}
\]

where we need that the no-overbidding assumption holds if \( \alpha > \gamma \).

\( ^9 \) Note that this condition holds if the mechanism satisfies individual rationality.
Proof. Observe that our mechanism $\gamma$-HYBRID is both first-price $\gamma$-approximate and first-price dominated. We can thus use both Lemma 4 and Theorem 3.

Note that $\sum_{i \in N} \sum_{x^*} (v_j)(b) \leq \sum_{j=1}^{k} \beta_j(b)$. Hence, by summing inequality (4) over all players, we obtain that the mechanism is $(\alpha(1 - e^{-1/\alpha}),\alpha)$-smooth. The claimed bound now follows from Theorem 3. □

4 Overbidding

We derive a tight bound on the coarse correlated price of anarchy of $\gamma$-HYBRID for $\gamma > 0$ in the multi-unit auction setting when bidders can overbid. It is known that the price of anarchy is unbounded for SP-Auction ($\gamma = 0$). The bound is displayed in Figure 1(a).

Theorem 6. Consider $\gamma$-HYBRID in the multi-unit auction setting and suppose that bidders can overbid. For $\gamma \in (0,1]$, the coarse correlated price of anarchy of $\gamma$-HYBRID is $CCE$-POA $\leq \frac{1}{\gamma(1 - e^{-1})}$. Further, this bound is tight even for the single-item auction setting.

Proof. Upper bound: This bound is based on Theorem 5. Since bidders can overbid in this setting, we restrict to the part of equation (5) that does not require the no-overbidding assumption, namely $\alpha \leq \gamma$, with $CCE$-POA $\leq \frac{1}{\alpha(1 - e^{-1/\alpha})}$.

To minimize this upper bound for any given $\gamma \in [0,1]$, consider its derivative with respect to $\alpha$,

$$-\frac{1}{\alpha^2(1 - e^{-1/\alpha})^2} \left(1 - e^{-1/\alpha} - \alpha \frac{1}{\alpha^2} e^{-1/\alpha}\right) = -\frac{1 - (1 + \frac{1}{\alpha})e^{-1/\alpha}}{\alpha^2(1 - e^{-1/\alpha})^2}.$$ 

As $(1 + \frac{1}{\alpha})e^{-1/\alpha} < 1$ for all $\alpha > 0$, the derivative is negative for all $\alpha > 0$. Therefore, the bound is minimized by maximizing $\alpha \in (0, \gamma]$. Substituting $\alpha = \gamma$ for any $\gamma \in (0,1]$ yields the upper bound.

Tight lower bound: This bound can be proven to be tight for all $\gamma \in (0,1]$ by generalizing an example used by Syrgkanis [18] to provide a lower bound on the CCE-POA for the first-price single-item auction: Consider a single-item auction with two bidders and using the $\gamma$-hybrid pricing rule as defined above. We have $v_1 = v$ for some $v > 0$ and $v_2 = 0$. If both bidders bid $0$, the tie is broken in favor of bidder 2, whereas bidder 1 wins the auction if bidders tie with any positive bid. We construct a coarse correlated equilibrium for any $\gamma \in (0,1]$, with a welfare loss that matches the upper bound.

Let $t$ be a random variable with support $[0, (1 - e^{-1/\gamma})v]$ whose cumulative distribution function (CDF) $F$ and density function $f$ (which is well-defined for any $t \in (0, (1 - e^{-1/\gamma})v]$), respectively, are given as

$$F(t) = (1 - \gamma) + \frac{v}{v - t} ye^{-1/\gamma} \quad \text{and} \quad f(t) = \frac{v}{(v - t)^2} ye^{-1/\gamma}.$$
Note that $F$ has an atom at 0 with mass $(1 - \gamma) + \gamma e^{-1/\gamma}$.

Consider a bidding profile $\sigma = (t, t)$. Since ties are broken in favor of bidder 2 for $t = 0$, they win with probability $(1 - \gamma) + \gamma e^{-1/\gamma}$, which yields

$$\frac{\text{SW}(x^*(\psi))}{\mathbb{E}[\text{SW}(\sigma)]} = \frac{t}{1 - F(0)v} = \frac{1}{1 - (1 - \gamma) - \gamma e^{-1/\gamma}} = \frac{1}{\gamma (1 - e^{-1/\gamma})}.$$ 

It remains to show that $\sigma$ is a CCE. For bidder 2, this is quite obvious, since they either win by bidding 0, or lose if $t > 0$. Given any positive bid from bidder 1, the payment would be strictly greater than $v_2$, meaning bidder 2 could never profitably deviate.

For bidder 1, we show that for any $\gamma \in (0, 1]$, any deviation to a fixed bid $b_1 = b$ with $b \in (0, (1 - e^{-1/\gamma})v]$ leads to an expected utility of at most $\mathbb{E}_{b - \sigma}[u_1(b)]$. To start with $\sigma$ itself, note that bidder 1 wins whenever $t > 0$, and since both bidders bid $t$, we have a payment of $\gamma t + (1 - \gamma)t = t$. Recalling that $v_1 = v$, we get

$$\mathbb{E}_{b - \sigma}[u_1(b)] = \int_0^1 (1 - e^{-1/\gamma})v (v - t) f(t) dt = \int_0^1 (1 - e^{-1/\gamma})v v e^{-1/\gamma} v e^{-1/\gamma} dt$$

$$= v y e^{-1/\gamma} [-\ln(v - t)]_0^t = v y e^{-1/\gamma} \left(\ln(v) - \ln(e^{-1/\gamma}/v)\right)$$

$$= v y e^{-1/\gamma} \frac{1}{\gamma} = v e^{-1/\gamma}.$$

If bidder 1 deviates to $b$, they win the item if $b \geq t$, and for each $t \in (0, b]$, bidder 1 pays $\gamma b + (1 - \gamma)t$. Hence, the expected utility of bidder 1 becomes

$$\mathbb{E}_{b - F(\cdot)}[u_1(b, t)] = \int_0^b (b - \gamma b - (1 - \gamma)t) f(t) dt$$

To facilitate the calculations, note that

$$\int_0^b t f(t) dt = y e^{-1/\gamma} \int_0^b \frac{t}{(v - t)^2} dt = y e^{-1/\gamma} \left(\frac{v}{v - t} + \ln(v - t)\right)_0^b$$

$$= \left(\frac{v}{v - b} - 1\right) y e^{-1/\gamma} + \ln \left(\frac{v - b}{v}\right) y e^{-1/\gamma}$$

and

$$\int_0^b f(t) dt = F(b) - F(0) = \left(\frac{v}{v - b} - 1\right) y e^{-1/\gamma}.$$

Using this, we get

$$\mathbb{E}_{b - F(\cdot)}[u_1(b, t)] = (v - \gamma b) \int_0^b f(t) dt - (1 - \gamma) \int_0^b t f(t) dt$$

$$= (v - \gamma b - (1 - \gamma)v) \left(\frac{v}{v - b} - 1\right) y e^{-1/\gamma} - (1 - \gamma) \ln \left(\frac{v - b}{v}\right) y e^{-1/\gamma}$$

$$= y(v - b) \left(\frac{v}{v - b} - 1\right) y e^{-1/\gamma} - (1 - \gamma) \ln \left(\frac{v - b}{v}\right) y e^{-1/\gamma}$$

$$= b^2 y e^{-1/\gamma} - (1 - \gamma) \ln \left(\frac{v - b}{v}\right) y e^{-1/\gamma}. $$
Since $0 < b < v$, note that $-\ln \left( \frac{e^b}{v} \right)$ is increasing in $b$. Since $\gamma \in (0, 1]$, this implies the entire function above is increasing in $b$. Hence, it can be upper bounded by substituting the upper bound of the support: $b = (1 - e^{-1/\gamma}) v$. This yields
\[
\mathbb{E}_{t \sim F(t)}[u_1(b, t)] \leq (1 - e^{-1/\gamma}) v^2 e^{-1/\gamma} - (1 - \gamma) \ln \left( e^{-1/\gamma} \right) \gamma v e^{-1/\gamma}
\]
\[
= \left( (1 - e^{-1/\gamma}) \gamma^2 + (1 - \gamma) \right) v e^{-1/\gamma}
\]
\[
= \left( (1 - e^{-1/\gamma}) \gamma^2 + (1 - \gamma) \right) \mathbb{E}_{b \sim \sigma}[u_1(b)].
\]
Therefore, $\mathbb{E}_{t \sim F(t)}[u_1(b, t)] \leq \mathbb{E}_{b \sim \sigma}[u_1(b)]$ for any $b \in (0, (1 - e^{-1/\gamma}) v]$ if
\[(1 - e^{-1/\gamma}) \gamma^2 + (1 - \gamma) \leq 1 \iff \gamma(1 - e^{-1/\gamma}) \leq 1\]
which holds for any $\gamma \in (0, 1]$ as required. This shows that bidder 1 does not have any profitable deviation in the interval $(0, (1 - e^{-1/\gamma}) v]$. Finally, since $b = (1 - e^{-1/\gamma}) v$ already gives $F(b) = 1$, any higher bid will only lead to a (strictly) higher payment (since $\gamma > 0$), thereby being (strictly) worse than bidding $b = (1 - e^{-1/\gamma}) v$. Hence, deviations to a bid higher than this upper bound of the support of $F(t)$ need not be considered.

Concluding, $\sigma$ is a CCE for which the ratio of the social welfare of the social optimum and the expected social welfare of $\sigma$ exactly coincides with the upper bound derived in the previous section. □

5 No Overbidding

5.1 Multi-Unit Auction

In the previous section, we have completely settled the coarse correlated price of anarchy for $\gamma$-Hybrid when overbidding is allowed. We see that especially when $\gamma$ gets small, i.e., we get closer to a second-price auction, this has an extremely negative effect on the price of anarchy. In this section, we will investigate how these bounds improve under the no-overbidding assumption (NOB as defined above). It is a natural assumption to make and we will see that it leads to a significant improvement of the price of anarchy bounds, most notably for lower values of $\gamma$.

We start off with bounding the pure price of anarchy.

Pure Price of Anarchy. The pure price of anarchy of $\gamma$-Hybrid without overbidding has been analyzed before for $\gamma = 0$ and $\gamma = 1$: Birmpas et al. [1] show that the PNE-POA is 2.1885 for the second-price multi-unit auction ($\gamma = 0$), while de Keijzer et al. [4] show that the PNE-POA is 1 for the first-price multi-unit auction ($\gamma = 1$). Interestingly, we do not find a smooth interpolation between these two boundary points when analyzing the PNE-POA for the range $\gamma \in [0, 1]$. As it turns out, for $\gamma$-Hybrid the PNE-POA stays at 1 almost over the entire range, the only exception being at $\gamma = 0$ where it is 2.1885 by the result of Birmpas et al. [1].

Theorem 7. Pure Nash equilibria of $\gamma$-Hybrid without overbidding are always efficient, i.e., PNE-POA = 1 for all $\gamma \in (0, 1)$.
This theorem follows from a minor adaptation of the result of de Keijzer et al. [4], who show that pure Nash equilibria are always efficient for $\gamma = 1$. Intuitively, the same result goes through for $\gamma > 0$, because when considering $\gamma > 0$ there is a first price component and thus a player always has an incentive to lower their winning bid to the highest losing bid as that would increase their utility. For more details we refer to Appendix B.

**Coarse Correlated Price of Anarchy**

**Theorem 8.** Consider $\gamma$-Hybrid in the multi-unit auction setting and suppose that bidders cannot overbid. For $\gamma \leq 0.607$, the coarse correlated price of anarchy of $\gamma$-Hybrid is

$$CCE-POA \leq \frac{-1 + \alpha - \gamma}{\alpha (1 - e^{-1/\alpha})}$$

(6)

Combining the improved bound of Theorem 8 with the bound of Theorem 6 yields the upper bound displayed in Figure 1(b) for all $\gamma \in [0, 1]$. In particular, we obtain $CCE-POA \leq -W_{-1}(-e^{-2}) \approx 3.146$ for $\gamma = 0$ and $CCE-POA \leq e/(e - 1) \approx 1.582$ for $\gamma = 1$.

**Proof.** Similar to the proof of Theorem 6, we choose some $\alpha > 0$ to optimize the upper bound on the price of anarchy in Theorem 5 for any given $\gamma \in [0, 1]$. As argued in the proof of Theorem 6, it is optimal to use $\alpha = \gamma$ when restricting to $\alpha \leq \gamma$. Using the no-overbidding assumption, we can also set $\alpha \geq \gamma$ and obtain

$$CCE-POA \leq \frac{1 + \alpha - \gamma}{\alpha (1 - e^{-1/\alpha})}$$

(7)

This upper bound is minimized for

$$\alpha = \frac{1}{W_{-1}(-e^{-2}/(1-\gamma)) + \frac{\gamma}{1-\gamma}}$$

(8)

where $W_{-1}$ is the lower branch of the Lambert $W$ function. Substituting this into (7), we obtain the upper bound in (6). Importantly, the optimized bound in (6) is only valid if we have $\alpha \geq \gamma$, which does not hold for the entire range $\gamma \in [0, 1]$ if we use (8). More concretely, we have $\alpha \geq \gamma$ for all $\gamma \leq 0.607$... only. Thus, for $\gamma \leq 0.607$... we can use (6) to bound the price of anarchy. For $\gamma \geq 0.607$... the best we can do is to choose $\alpha = \gamma$ and obtain the same CCE-POA bound as in Theorem 6. □

### 5.2 Single-Item Auction

By restricting to the single-item auction setting, we can further improve the price of anarchy bounds. We start with the general $n$-player setting, for which we show that the single-item $\gamma$-hybrid auction is fully efficient up to correlated equilibria. For coarse correlated equilibria, we then derive a strong bound for low values of $\gamma$, namely $CCE-POA \leq 1/(1 - \gamma)$. This bound can in turn be complemented by the bound we derived
for multi-unit auctions. Finally, to improve upon this multi-unit bound for the higher range of $\gamma$, we derive two technically more involved bounds that work specifically in a two-player setting.

We need some more notation. Given a bid vector $\mathbf{b}$, let $HB(\mathbf{b}) = \max_i b_i$ and $SB(\mathbf{b})$ denote the highest and second highest bid in $\mathbf{b}$, respectively, and let $HB_n(\mathbf{b}) = \max_{j \neq i} b_j$ be the highest bid excluding bid $b_i$. For a randomized bid vector $\sigma$, let $HB(\sigma)$ be the random variable equal to the highest bid when the bids are distributed according to $\sigma$.

We sometimes write $\mathbb{E}[HB(\sigma)]$ for $\mathbb{E}_{b \sim \sigma}[HB(\mathbf{b})]$ (similarly for $SB(\sigma)$ and $HB_n(\sigma)$).

**Correlated Price of Anarchy.** We prove that $\gamma$-Hybrid is fully efficient for all $\gamma \in [0, 1]$ up to correlated equilibria. In Theorem 10 we show that for $\gamma = 0$ even coarse correlated equilibria are always efficient, so that Theorem 9 in fact holds for all $\gamma \in [0, 1]$. Similar to the argument in [5] for $\gamma = 1$, we show that any player with a value strictly lower than the highest value can never be a winner in a correlated equilibrium.

**Theorem 9.** Consider $\gamma$-Hybrid in the single-item auction setting and suppose that bidders cannot overbid. Then, the correlated price of anarchy of $\gamma$-Hybrid is 1 for all $\gamma \in (0, 1]$.

**Proof.** Without loss of generality assume that player 1 has the highest valuation $v_1$. Assume towards contradiction that the CE-POA is not 1. Then, there must be a player $i$ for which $v_i < v_1$ who has a positive probability of winning. Let $b^* = \inf \{ b \mid \mathbb{P}(HB(\sigma) < b) > 0 \}$. Since we assume that players cannot overbid, we know that $b^* \leq v_i < v_1$.

First, suppose $b^* = v_i$. Then $\mathbb{P}(HB(\sigma) < v_1) = 0$. If player 1 bids $(v_i + v_1)/2$ whenever $\sigma$ suggests $b_1 < b^*$ (in case ties are broken in favor of player 1) or whenever $\sigma$ suggests $b_1 \leq b^*$ (in case ties are broken in favor of player $i$) then player 1 strictly increases their utility. This contradicts the correlated equilibrium assumption.

Thus we can assume $b^* < v_i$. Define $\tilde{b} = (b^* + v_i)/2$. Fix a bid $b$ such that $b^* < b < \tilde{b}$. By assumption we have $\mathbb{P}(HB(\sigma) < b) > 0$. Either player 1 or player $i$ must win by bidding at most $b$ with probability at most $\mathbb{P}(HB(\sigma) < b)/2$. Let it be player $i$ (otherwise just fill in 1 for $i$ in what follows).

Consider the following deviating strategy for player $i$: bid $(\tilde{b} + b)/2$ whenever $\sigma$ suggests $b_i \leq b$ and $b_i$ otherwise. For $b_i > b$ nothing changes and so the utility stays the same. Next, consider $b_i \leq b$. Player $i$ already won with probability at most $\mathbb{P}(HB(\sigma) < b)/2$. Now that they bid higher note that the second bid part of the price does not change while the highest bid part goes up by at most $\gamma((\tilde{b} + b)/2 - b^*)$. This decreases their utility by at most $\mathbb{P}(HB(\sigma) < b)/2 \cdot (\tilde{b} + b)/2 - b^*)$. On the other hand player $i$ will gain (lower bounding the second highest bid by the highest bid) at least $\mathbb{P}(HB(\sigma) < b)/2 \cdot (v_i - (\tilde{b} + b)/2)$. Net, the utility of player $j$ increases by at least

\[
\frac{\mathbb{P}(HB(\sigma) < b)}{2} \left( \frac{v_i - \tilde{b} + b}{2} - \frac{(\tilde{b} + b - b^*)}{2} \right) > \frac{\mathbb{P}(HB(\sigma) < b)}{2} \left( v_i + b^* - 2\tilde{b} \right) = 0,
\]

where we use that $b < \tilde{b}$. Again, we find a contradiction with the CE conditions. Hence, there cannot be a player $i$ with $v_i < v_1$ having a positive probability of winning implying that the price of anarchy must be 1. \qed
Coarse Correlated Price of Anarchy. It is known that the coarse correlated price of anarchy for the first-price auction is approximately 1.229 [5], which implies that the result of Theorem [4] does not extend to coarse correlated equilibria. We derive the following bound which is good for small values of γ.

**Theorem 10.** Consider γ-Hybrid in the single-item auction setting and suppose that bidders cannot overbid. Then, the coarse correlated price of anarchy of γ-Hybrid is at most 1/(1 − γ) for all γ ∈ [0, 1).

**Proof.** Let player 1 be the player with highest valuation \( v_1 \), and if there are multiple players with the highest valuation the player for whom ties are broken in favor when bidding \( v_1 \). Let \( \sigma \) be a coarse correlated equilibrium. We have

\[
\mathbb{E}_{b \sim \sigma}[SW(b)] = \mathbb{E}_{b \sim \sigma}[u_1(b)] + \mathbb{E}_{b \sim \sigma}\left[ \sum_{i \neq 1} u_i(b) + p^\gamma(b) \right].
\]

(9)

Define \( E \) as the event that player 1 wins the auction with respect to \( \sigma \), and let \( \bar{E} \) be the complement event that player 1 does not win the auction with respect to \( \sigma \).

Suppose player 1 deviates to \( v_1 \). Then player 1 wins under \( (v_1, b_{-1}) \) because either they are the single highest bid or ties are broken in their favor by assumption and no player overbids; note that this holds independently for \( E \) and \( \bar{E} \). By the CCE conditions, we thus have

\[
\mathbb{E}_{b \sim \sigma}[u_1(b)] \geq \mathbb{E}_{b \sim \sigma}[u_1(v_1, b_{-1})] = (1 - \gamma)v_1 - (1 - \gamma)\mathbb{E}_{b \sim \sigma}[HB_{-1}(b)].
\]

Substituting this inequality in (9), we obtain

\[
\mathbb{E}_{b \sim \sigma}[SW(b)] \geq (1 - \gamma)v_1 - (1 - \gamma)\mathbb{E}_{b \sim \sigma}[HB_{-1}(b)] + \mathbb{E}_{b \sim \sigma}\left[ \sum_{i \neq 1} u_i(b) + p^\gamma(b) \right].
\]

The proof thus follows if we can show that

\[
\mathbb{E}_{b \sim \sigma}\left[ \sum_{i \neq 1} u_i(b) + p^\gamma(b) \right] \geq (1 - \gamma)\mathbb{E}_{b \sim \sigma}[HB_{-1}(b)].
\]

(10)

**Case 1:** Suppose \( b \in E \). Then player 1 wins the auction with respect to \( b \) and we have

\[
\sum_{i \neq 1} u_i(b) + p^\gamma(b) = \gamma b_1 + (1 - \gamma)HB_{-1}(b) \geq HB_{-1}(b).
\]

**Case 2:** Suppose \( b \in \bar{E} \). Then some other player \( i' \neq 1 \) wins the auction with respect to \( b \) and we have

\[
u_{i'}(b) + p^\gamma(b) = v_{i'} - p^\gamma(b) + p^\gamma(b) = v_{i'} \geq b_{i'} = HB_{-1}(b),
\]

where last inequality holds because \( i' \) does not overbid and the last equality holds because \( i' \) being the highest bidder implies that \( b_{i'} = HB_{-1}(b) \). This concludes the proof.

Any upper bound for the multi-unit auction setting of course also holds for the single-item setting. By combining the bounds of Theorem [6] Theorem [8] and Theorem [10] we obtain the upper bound displayed in Figure (1c) for the coarse correlated price of anarchy in the single-item auction setting.
Coarse Correlated Price of Anarchy for 2-player Auctions. We now present a more fine-grained picture for the coarse correlated price of anarchy for the 2-player setting. Ultimately, the upper bound for CCE-POA for two players becomes a combination of three upper bounds, as represented by the three colors in Figure 1(d). We already derived the bound we use for small values of $\gamma$ in Theorem 10, corresponding to the green graph in the figure. To derive the two remaining bounds, we use an approach inspired by [5]. These bounds significantly improve on the bounds of Theorem 6 and Theorem 8.

First we tackle the interval $\gamma \in \left[\frac{1}{2}, 1\right]$. Note that for $\gamma = 1$ this bound coincides with the (tight) bound in [5].

**Theorem 11.** Consider $\gamma$-Hybrid in a 2-player single-item auction setting and suppose that bidders cannot overbid. For $\gamma \in \left[\frac{1}{2}, 1\right]$, the coarse correlated price of anarchy of $\gamma$-Hybrid is upper bounded by the blue graph in Figure 1(d) (with CCE-POA $\leq 1.295...$ for $\gamma = 0.5$ and CCE-POA $\leq 1.229...$ for $\gamma = 1$).

**Proof.** Without loss of generality we assume that player 1 has a valuation of 1 and player 2 has a valuation of $v \leq 1$. Fix $\gamma$ and consider some coarse correlated equilibrium $\sigma$. Let $\alpha = \mathbb{E}[u_1(\sigma)]$ be the utility of player 1 and $\beta = \mathbb{E}[u_2(\sigma)]$ be the utility of player 2 in $\sigma$. The maximum social welfare is clearly 1, namely when player 1 wins all the time. Lower bounding the expected welfare of an arbitrary $\sigma$ translates into an upper bound on the price of anarchy. We have

$$\mathbb{E}[SW(\sigma)] \geq \alpha + \beta + \mathbb{E}[\rho^+(\sigma)] = \alpha + \beta + \gamma \mathbb{E}[HB(\sigma)] + (1 - \gamma) \mathbb{E}[SB(\sigma)]$$

We try to find the $v$, $\alpha$ and $\beta$ that minimize this expression and this will then give a lower bound on the expected social welfare. Let $F_X$ be the cumulative distribution function of the random variable $X$ where $X \in \{HB, HB_{-1}, HB_{-2}, SB\}$. Then by the CCE conditions, and the fact that a CDF is always bounded by 1, we know that

$$F_{HB_{-1}(\sigma)}(x) \leq \min \left\{ \frac{\alpha}{1 - x}, 1 \right\}, \quad F_{HB_{-2}(\sigma)}(x) \leq \min \left\{ \frac{\beta}{v - x}, 1 \right\}$$  \hspace{1cm} (11)

$$F_{HB(\sigma)}(x) \leq \min \left\{ \frac{\alpha}{1 - x}, \frac{\beta}{v - x}, 1 \right\}$$  \hspace{1cm} (12)

For example, if $F_{HB_{-1}(\sigma)} > \frac{\alpha}{1 - x}$ and player 1 changes their bid to $x$ their utility will be strictly greater than $\frac{\alpha}{1 - x} (1 - x) = \alpha$ which is more than their current utility contradicting the CCE conditions.

Also note that $\alpha \geq 1 - v$ because player 1 bidding $v + \epsilon$ will yield a utility of at least $1 - v - \epsilon$ for any positive $\epsilon$. The other player is not allowed to bid above $v$, thus player 1 always wins when bidding $v + \epsilon$.

Observe that for $n = 2$ players the following chain of equalities holds

$$F_{SB(\sigma)}(x) = \mathbb{P}[SB(\sigma) \leq x]$$

$$= \mathbb{P}[\min(HB_{-1}(\sigma), HB_{-2}(\sigma)) \leq x]$$

$$= \mathbb{P}[HB_{-1}(\sigma) \leq x] + \mathbb{P}[HB_{-2}(\sigma) \leq x] - \mathbb{P}[HB(\sigma) \leq x]$$

$$= F_{HB_{-1}(\sigma)}(x) + F_{HB_{-2}(\sigma)}(x) - F_{HB(\sigma)}(x).$$  \hspace{1cm} (13)
Let us get a more explicit expression for the expected payment using (14)
\[ E[p^r(\sigma)] = \gamma E[HB(\sigma)] + (1 - \gamma) E[SB(\sigma)] \]
\[ = \gamma \int_0^1 1 - F_{HB(\sigma)}(x)dx + (1 - \gamma) \int_0^1 1 - F_{SB(\sigma)}(x)dx \]
\[ = \gamma \int_0^1 1 - F_{HB(\sigma)}(x)dx + (1 - \gamma) \int_0^1 1 - F_{HB_{-1}(\sigma)}(x) - F_{HB_{-2}(\sigma)}(x) + F_{HB(\sigma)}(x)dx \]
\[ = (2\gamma - 1) \int_0^1 1 - F_{HB(\sigma)}(x)dx + (1 - \gamma) \sum_{i=1}^2 \int_0^1 1 - F_{HB_{-i}(\sigma)}(x)dx \]

Using the two bounds in (11) we can lower bound the two integrals in the summation
\[ \int_0^1 1 - F_{HB_{-1}(\sigma)}(x)dx \geq \int_0^{1-\alpha} 1 - \frac{\alpha}{1-x}dx = 1 - \alpha + \alpha \ln(\alpha) \]
\[ \int_0^1 1 - F_{HB_{-2}(\sigma)}(x)dx \geq \int_0^{\alpha-\beta} 1 - \frac{\beta}{\nu - x}dx = v - \beta + \beta \ln(\beta/\nu) \]

If \( \gamma \geq \frac{1}{2} \) then \( 2\gamma - 1 \geq 0 \) and so we can use (12) to lower bound the integral on the left by
\[ \int_0^1 1 - F_{HB(\sigma)}(x)dx \geq \int_0^1 1 - \min\left\{ \frac{\alpha}{1-x}, \frac{\beta}{\nu - x}\right\}dx \]

We split up in two cases.

**Case 1:** \( \beta \geq \nu \alpha \). Then \( \frac{\beta}{\nu - x} \geq \frac{\alpha}{1-x} \) for all \( x \in [0, v] \) and so
\[ \int_0^1 1 - \min\left\{ \frac{\alpha}{1-x}, \frac{\beta}{\nu - x}\right\}dx = \int_0^{1-\alpha} 1 - \frac{\alpha}{1-x}dx = 1 - \alpha + \alpha \ln(\alpha) \]
giving a lower bound on expected welfare of
\[ E[SW(\sigma)] \geq \alpha + \beta + (2\gamma - 1)(1 - \alpha + \alpha \ln(\alpha)) \]
\[ + (1 - \gamma)(1 - \alpha + \alpha \ln(\alpha) + v - \beta + \beta \ln(\beta/\nu)) \]

Using that \( v \geq 1 - \alpha \) and \( \beta \geq \nu \alpha \geq \alpha(1 - \alpha) \) this is lower bounded by
\[ 1 + \alpha(1 - \alpha) \ln(\alpha) + \gamma \alpha(1 - \alpha + \alpha \ln(\alpha)) \]

**Case 2:** \( \beta < \nu \alpha \). First \( \frac{\beta}{\nu - x} \) is smaller than \( \frac{\alpha}{1-x} \) until \( x = \theta = \frac{\alpha - \beta}{\nu - \beta} \) when \( \frac{\beta}{\nu - x} \) takes over. In this case the integral is bounded from below by
\[ \int_0^1 1 - \min\left\{ \frac{\alpha}{1-x}, \frac{\beta}{\nu - x}\right\}dx \geq \int_0^{\theta} 1 - \frac{\beta}{\nu - x}dx + \int_0^{1-\alpha} 1 - \frac{\beta}{\nu - x}dx \]
\[ = \alpha \ln\left( \frac{\alpha - \beta}{1 - v}\right) + 1 - \alpha + \beta \ln\left( \frac{\beta(1 - v)}{v(\alpha - \beta)}\right) \]
which gives a lower bound on the expected social welfare of
\[
\mathbb{E}[SW(\sigma)] \geq \alpha + \beta + (2\gamma - 1) \left( \alpha \ln \left( \frac{\alpha - \beta}{\alpha} \right) + 1 - \alpha + \beta \ln \left( \frac{\beta(1 - \nu)}{1 - \nu} \right) \right)
+ (1 - \gamma) \left( 1 - \alpha + \alpha \ln(\alpha) + \nu - \beta + \beta \ln(\beta/\nu) \right)
\] (17)

The derivative with respect to \( \nu \) is
\[
(2\gamma - 1) \frac{\alpha \nu - \beta}{(1 - \nu) \nu} + (1 - \gamma)(1 - \beta/\nu)
\]

For \( \beta < \nu \alpha \) this is positive and thus the minimum is attained when \( \nu \) is smallest, i.e., \( \nu = 1 - \alpha \). Substituting that in (17) gives
\[
\mathbb{E}[SW(\sigma)] \geq \alpha + \beta + (2\gamma - 1) \left( \alpha \ln \left( \frac{\alpha - \beta}{\alpha} \right) + 1 - \alpha + \beta \ln \left( \frac{\beta \alpha}{(1 - \alpha)(\alpha - \beta)} \right) \right)
+ (1 - \gamma) \left( 1 - \alpha + \alpha \ln(\alpha) + 1 - \alpha - \beta + \beta \ln \left( \frac{\beta}{1 - \alpha} \right) \right)
= 1 + \gamma \beta + (2\gamma - 1)(\alpha - \beta) \ln(\alpha - \beta) + (2 - 3\gamma) \alpha \ln(\alpha) + (2\gamma - 1) \beta \ln(\alpha)
\] + \gamma \beta \ln(\beta) - \gamma \beta \ln(1 - \alpha)
\] (18)

Note that filling in \( \beta = \alpha(1 - \alpha) \) yields the same revenue as in (16). Changing \( \beta < \nu \alpha \) to \( \beta \leq \nu \alpha = \alpha(1 - \alpha) \) subsumes case 1. So we only have to find the minimum in case 2.

The derivative of (18) with respect to \( \beta \) is
\[
(2\gamma - 1) \ln \left( \frac{\alpha}{\alpha - \beta} \right) + (2 - 3\gamma) \alpha \ln(\alpha) + 1
\]
This becomes 0 when
\[
\ln \left( \frac{\alpha^{2\gamma - 1} \beta^\gamma}{(\alpha - \beta)^{2\gamma - 1} (1 - \alpha)^\gamma} \right) = -1 \iff \frac{\beta^\gamma}{(\alpha - \beta)^{2\gamma - 1} (1 - \alpha)^\gamma} = \frac{(1 - \alpha)^\gamma}{e^{2\gamma - 1}} = 0
\]

For fixed \( \alpha \) the expression on the left is negative for \( \beta \) close to 0, and positive for \( \beta \) close to \( \alpha \). Also the second derivative with respect to \( \beta \) is always positive on \( [0, \beta] \). Thus we can use binary search to quickly find \( \beta \) satisfying the equality. Call this \( \beta_\alpha \) \( ^{10} \)

Then we have
\[
\mathbb{E}[SW(\sigma)] \geq 1 + \gamma \beta_\alpha + (2\gamma - 1)(\alpha - \beta_\alpha) \ln(\alpha - \beta_\alpha) + (2 - 3\gamma) \alpha \ln(\alpha) + (2\gamma - 1) \beta_\alpha \ln(\alpha)
+ \gamma \beta_\alpha \ln(\beta_\alpha) - \gamma \beta_\alpha \ln(1 - \alpha)
\]

For \( \gamma = 1/2 \) we compute \( \beta_\alpha = (1 - \alpha)/e^2 \) and then the social welfare is minimized for \( \alpha = e^{-1/e^2} \approx 0.3213 \ldots \) with value 0.7716\ldots. While for \( \gamma = 1 \) we have \( \beta_\alpha =
\]

\( ^{10} \beta_\alpha \) may violate the case assumption that \( \beta \leq \nu \alpha \) but removing this restriction can only decrease the minimum value of the expected social welfare.
\[\alpha(1 - \alpha)/(1 - \alpha + e\alpha)\] where the social welfare is minimized for \(\alpha \approx 0.2743\) and with value 0.8135... In both cases (13) becomes a unimodal function. Plotting (13) for various values of \(\alpha\), when doing binary search to find \(\beta_\alpha\) as a subroutine, suggests that this is the case for all \(\gamma\). Making this assumption we can use a ternary search on \(\alpha\) with a binary search to find \(\beta_\alpha\) as a subroutine to quickly find the minimum. Finally, taking 1 over this value gives us an upper bound on the price of anarchy, presented as the blue graph in Figure [d]. \(\Box\)

The previous theorem holds for \(\gamma \in [\frac{1}{2}, 1]\). With a similar proof template, making use of an upper bound on the highest bid, we can derive an upper bound on the coarse correlated price of anarchy for the lower to mid range of \(\gamma\).

**Theorem 12.** Consider \(\gamma\)-HYBRID in a 2-player single-item auction setting and suppose that bidders cannot overbid. For \(\gamma \in (0.217..., \frac{1}{2})\), the coarse correlated price of anarchy of \(\gamma\)-HYBRID is upper bounded by the orange graph in Figure [d] (with CCE-POA \(\leq 1.515\) for intersection point \(\gamma = 0.339\) and CCE-POA \(\leq 1.295\) for \(\gamma = 0.5\)).

**Proof.** For \(\gamma \in [0, 1/2]\), note that in the final equality of (14), we have \((2\gamma - 1) \leq 0\). To lower bound the social welfare, we should therefore upper bound the expected highest bid. For this, note that due to the fact that players cannot overbid, player 2 never bids higher than \(v\). Therefore, for any \(\gamma > 0\), any bid of player 1 that is (strictly) above \(v\) is (strictly) dominated by bidding \(v\) instead.\footnote{Formally, player 1 should bid \(v + \epsilon\) for any \(\epsilon > 0\). Since \(\epsilon\) can be an arbitrarily small number, we ignore it in the remainder of the proof for notational convenience.} Using this, it is clear that \(E[HB(\sigma)] \leq v\). Again using (11) to lower bound the two rightmost integrals, we get

\[
yE[HB(\sigma)] + (1 - \gamma)E[SB(\sigma)] \\
\geq (2\gamma - 1) \int_0^1 1 - F_{HB(\sigma)}(x)dx + (1 - \gamma)\left(\int_0^1 1 - F_{HB,1(\sigma)}(x)dx + \int_0^1 1 - F_{HB,2(\sigma)}(x)dx\right) \\
\geq (2\gamma - 1)v + (1 - \gamma)(1 - \alpha + \alpha \ln(\alpha) + v - \beta + \beta \ln(\beta/v)),
\]

so that

\[
E[SW(\sigma)] \geq \alpha + \beta + (2\gamma - 1)v + (1 - \gamma)(1 - \alpha + \alpha \ln(\alpha) + v - \beta + \beta \ln(\beta/v)) \\
= \gamma(\alpha + \beta + v) + (1 - \gamma)(1 + \alpha \ln(\alpha) + \beta \ln(\beta) - \beta \ln(v)).
\]

The derivative of this bound with respect to \(\beta\) equals

\[
\gamma + (1 - \gamma)(1 + \ln(\beta) - \ln(v)) = 1 + (1 - \gamma) \ln(\beta/v).
\]

Note that this derivative is equal to zero for \(\beta = ve^{-1/(1-\gamma)}\), and that it is positive for greater \(\beta\) and negative for smaller \(\beta\). Therefore, the bound attains its minimum at \(\beta = ve^{-1/(1-\gamma)}\). Substituting this \(\beta\) in (19) yields

\[
E[SW(\sigma)] \geq \gamma(\alpha + (1 + e^{-\frac{1}{1-\gamma}})v) + (1 - \gamma)(1 + \alpha \ln(\alpha) + ve^{-\frac{1}{1-\gamma}} \ln(e^{-\frac{1}{1-\gamma}})) \\
= \gamma(\alpha + (1 + e^{-\frac{1}{1-\gamma}})v) + (1 - \gamma)(1 + \alpha \ln(\alpha)) - ve^{-\frac{1}{1-\gamma}}. \tag{20}
\]
Next, we take the derivative of (20) with respect to $v$, which gives
\[ \gamma (1 + e^{-\frac{1}{1 - \gamma}}) - e^{-\frac{1}{1 - \gamma}} = \gamma - (1 - \gamma) e^{-\frac{1}{1 - \gamma}}. \]

This derivative is positive for all $\gamma > 0.21781 \ldots$, so for all $\gamma \in (0, 21781 \ldots, 1/2]$, we minimize the upper bound by setting $v$ to its lowest admissible value, being $v = 1 - \alpha$ (due to the CCE condition that $\alpha \geq 1 - v$).

Substituting the optimal parameter settings $\beta = ve^{-\frac{1}{1 - \gamma}} = (1 - \alpha) e^{-\frac{1}{1 - \gamma}}$ gives the following social welfare bound
\[
\mathbb{E}[SW(\sigma)] \geq \gamma (\alpha + (1 - \alpha)(1 + e^{-\frac{1}{1 - \gamma}})) + (1 - \gamma)(1 + \alpha \ln(\alpha)) - (1 - \alpha)e^{-\frac{1}{1 - \gamma}}
\]
\[= 1 - (1 - \gamma)(1 - \alpha)e^{-\frac{1}{1 - \gamma}} + (1 - \gamma)\alpha \ln(\alpha), \quad (21)\]
as a function of $\alpha$ only, which we optimize by setting its derivative with respect to $\alpha$ equal to zero. This yields
\[
(1 - \gamma)e^{-\frac{1}{1 - \gamma}} + (1 - \gamma)(1 + \ln(\alpha)) = 0 \iff \ln(\alpha) = -1 - e^{-\frac{1}{1 - \gamma}} \iff \alpha = e^{-1-e^{-\frac{1}{1 - \gamma}}}. 
\]

To facilitate the simplification of the formula of the final bound, we first substitute only $\ln(\alpha)$ in (21), after which $\alpha$ itself is substituted in the final step. We get
\[
\mathbb{E}[SW(\sigma)] \geq 1 - (1 - \gamma)e^{-\frac{1}{1 - \gamma}} + (1 - \gamma)e^{-\frac{1}{1 - \gamma}} + (1 - \gamma)e^{-1-e^{-\frac{1}{1 - \gamma}}} 
\]
\[= 1 - (1 - \gamma)e^{-\frac{1}{1 - \gamma}} - (1 - \gamma)e^{-1-e^{-\frac{1}{1 - \gamma}}}. \quad (22)\]

We divide 1 by (22) to get the upper bound on the price of anarchy presented as the orange graph in Figure [1d].

\[\Box\]

6 Conclusion and Future Work

Our bound on the CCE-POA of $\gamma$-Hybrid is tight over the entire range of $\gamma \in [0, 1]$ if players can overbid, both in the single-item and multi-unit auction setting. Despite the fact that our bounds on the CCE-POA are rather low already if players cannot overbid, further improvements might still be possible. We consider this a challenging open problem for future work.

On a more conceptual level, in this paper we considered a basic bid rigging model where the auctioneer colludes with the winning bidders only. It will be very interesting to study the price of anarchy of more complex bid rigging models; for example, the model introduced in [10] (ideally generalized to the multi-unit auction setting) might be a natural next step.
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A Proofs for Section 3 (Capturing Corruption with $\gamma$-Hybrid Auctions)

A.1 Dominant Strategy to Accept Offer

We show that it is a dominant strategy for every winning bidder to always accept the offer of the auctioneer (independent of $\gamma$).

**Proposition 13.** Fix some $\gamma \in [0, 1]$ and consider $\gamma$-corrupt FP-Auction. We can assume without loss of generality that each winning bidder always accepts the offer of the auctioneer.

**Proof.** Observe that the total payment to be made by a winning bidder $i$ who accepts the offer becomes

$$p_i^\gamma(b) = x_i(b)\bar{p}(b) + f_i^\gamma(b) = \gamma \sum_{j=1}^{x_i(b)} b_i(j) + (1 - \gamma)x_i(b)\bar{p}(b).$$  \hspace{1cm} (23)

Clearly, each winning bid $j$ of $i$ satisfies $b_i(j) \geq \bar{p}(b)$. Thus, $p_i^\gamma(b) \leq \sum_{j=1}^{x_i(b)} b_i(j) = p_i(b)$, where $p_i(b)$ is the payment that $i$ would have to pay when rejecting the offer. In fact, this inequality is strict unless all winning bids of $i$ are equal to $\bar{p}(b)$ or $\gamma = 1$. In both these cases, the offer made by the auctioneer does not have any effect for $i$ (as there is no surplus generated in the former case, and no difference in the final payment of $i$ in the latter case). Said differently, each winning bidder can only benefit from accepting the offer. Observe also that the above arguments hold for every winning bidder independently of what the other bidders do. Further, the final allocation remains invariant (assuming a consistent tie breaking rule). We conclude that it is a dominant strategy for every winning bidder to accept the offer of the auctioneer. \hfill $\Box$

A.2 Equivalence for Non-Uniform Bid Rigging

We exemplify the implications of our bounds on the POA of $\gamma$-Hybrid for an alternative, non-uniform bid rigging model.

As before, the bidders submit their bid vectors $b = (b_1, \ldots, b_n)$ to the auctioneer who runs a first-price multi-unit auction. The auctioneer then approaches each winning bidder $i$ individually with the offer that they can lower their $x_i(b)$ winning bids. However, in contrast to the basic model, the auctioneer and bidder $i$ agree to “camouflage” their bid rigging by bidding the highest losing bid $\bar{p}(b)$ plus a fraction $\alpha \in [0, 1]$ of the surplus $b_i(j) - \bar{p}(b)$ for each $j \in [x_i(b)]$. Note that this maintains the relative order among the winning bids and the magic number cheating becomes less obvious (as the winning bids fluctuate more). The remaining surplus of $(1 - \alpha)(b_i(j) - \bar{p}(b))$ is then split, where the auctioneer withholds a fraction of $\beta \in [0, 1]$. As before, bidder $i$ can either reject or accept the offer. But, also here, it is not hard to see that accepting the offer is a
dominant strategy. The total payment of a winning bidder $i$ is then

$$p_i^{(\alpha, \beta)}(b) = \sum_{j=1}^{x_i(b)} (\bar{p}(b) + \alpha(b_i(j) - \bar{p}(b))) + f_i^{(\alpha, \beta)}(b),$$

where

$$f_i^{(\alpha, \beta)}(b) = \beta \sum_{j=1}^{x_i(b)} (1 - \alpha)(b_i(j) - \bar{p}(b)).$$

After simplifying, we obtain

$$p_i^{(\alpha, \beta)}(b) = (\alpha + \beta (1 - \alpha)) \sum_{j=1}^{x_i(b)} b_i(j) + (1 - \alpha - \beta (1 - \alpha)) x_i(b) \bar{p}(b).$$

If we define $\gamma = \alpha + \beta - \alpha \beta$, the above payments $p_i^{(\alpha, \beta)}$ are equivalent to $p_i^{\gamma}$ as defined in (1). Note also that this mapping satisfies $\gamma \in [0, 1]$ for every $\alpha, \beta \in [0, 1]$. Said differently, given $\alpha, \beta \in [0, 1]$ the price of anarchy of the above non-uniform bid rigging scheme is determined by the price of anarchy of $\gamma$-Hybrid with $\gamma = \alpha + \beta - \alpha \beta$.

**B Proofs for Section 5 (No Overbidding)**

De Keijzer et al. [4] prove the following theorem.

**Theorem 14.** [4] Pure Nash equilibria of the Discriminatory Auction are always efficient, even for bidders with arbitrary valuation functions.

This theorem follows almost immediately from this lemma.

**Lemma 15.** [4] Let $b$ be a pure Nash equilibrium in a given Discriminatory Auction where the bidders have general valuation functions. Let $d = \max(b_i(j) : i \in [n], j \in [k], j > x_i(b))$. Then

(i) For any bidder $i$ who wins at least one item under $b$, and for all $j \in [x_i(b)] : b_i(j) = d$,  
(ii) $\ell d \leq \sum_{j=x_i(b)-\ell+1}^{x_i(b)} m_i(j)$, for all $i \in [n]$ and $\ell \in [x_i(b)]$,  
(iii) $\sum_{j=x_i(b)+\ell}^{x_i(b)+\ell+1} m_i(j) \leq \ell d$, for all $i \in [n]$ and $\ell \in [k-x_i(b)]$.

We can proceed along exactly the same line of arguments as in the proof of the lemma for $\gamma = 1$ to show that it holds for all $\gamma > 0$. Because there is a first-price component, the reasoning that a player would increase their utility by lowering their winning bid to the highest losing bid holds in the same way.