Casimir effect for arbitrary materials: contributions within and beyond the light cone

W L Mochán¹,³ and C Villarreal²

¹ Centro de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, 62251 Cuernavaca, Morelos, México
² Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 Distrito Federal, México
E-mail: mochan@fis.unam.mx and carlos@fisica.unam.mx

New Journal of Physics 8 (2006) 242
Received 3 June 2006
Published 20 October 2006
Online at http://www.njp.org/
doi:10.1088/1367-2630/8/10/242

Abstract. We obtain an expression for the Casimir force within a planar cavity without making particular models or assumptions about the nature of its walls. We obtain the energy and stress tensor in a closed ancillary system that has the same optical response as the target system, but that allows a fully quantum-mechanical treatment of the electromagnetic degrees of freedom. Our results constitute a generalization of the Lifshitz formula, applicable to a wide class of materials, which could be semi-infinite or finite, local or spatially dispersive, homogeneous or layered, dissipative or dissipationless. We discuss the incorporation of evanescent fields in the formalism, the electromagnetic normal modes of the system, its thermodynamic properties, and the generalization of our results to arbitrary geometries.

³ Author to whom any correspondence should be addressed.
1. Introduction

Half a century after Casimir [1] predicted that the fluctuations of the electromagnetic field within a cavity would produce an attractive force on its boundaries, experimental studies finally reached the necessary accuracy to test in detail the theoretical predictions [2]. Micro and nanoscopic devices for which the Casimir forces play a fundamental role have already been proposed and constructed; experimental precisions of the order of 1% have now become common [3]–[6] and distances down to $\approx 60$ nm have been explored [7]. Therefore, realistic theories of the Casimir force have become indispensable.

The original formulation assumed that the field was confined to a cavity by perfectly conducting mirrors, which could be simply replaced by homogeneous boundary conditions. To go beyond the perfect mirror approximation and introduce the temporal dispersion of the response of realistic materials, Lifshitz proposed in 1956 a macroscopic theory for two semi-infinite homogeneous dielectric slabs [8] characterized by complex frequency-dependent dielectric functions $\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$ and separated by a given distance $L$. Many corrections have been incorporated into the theoretical calculations, such as finite conductivity [9]–[12], surface roughness [13]–[15], finite temperatures [16, 17], grain structure [7], etc.

In his original derivation [8], Lifshitz calculated the stress tensor from the self-correlation of the fluctuating electromagnetic field. The source of this field consists of fluctuating charge $\rho(\vec{r}, t)$ and current $\vec{j}(\vec{r}, t)$ densities within each slab whose autocorrelations $\langle \rho(\vec{r}, t)\rho(\vec{r}', t) \rangle$ and $\langle \vec{j}(\vec{r}, t)\vec{j}(\vec{r}', t) \rangle$ are related through Kubo’s formalism, causality and the fluctuation–dissipation theorem to the complex dielectric response. In this calculation, only homogeneous and isotropic media were considered and it was assumed that fluctuating sources at a given position were completely uncorrelated with sources at nearby positions, i.e., the autocorrelation functions were presumed to be given by a function of the time delay $t - t'$ multiplied by a Dirac’s delta function of the separation $\delta(\vec{r} - \vec{r}')$. Under this assumption the Lifshitz derivation became applicable only to semi-infinite homogeneous local media, and could not cope with more complex systems, such as thin films, layered systems, superlattices, photonic crystals and metamaterials [18]. For the same reason, the Lifshitz theory seemed incapable of dealing with the spatial dispersion or non-locality of the electromagnetic response, although in principle it could be generalized by incorporating more complex correlation functions. By non-locality, we mean the dependence of the perturbation induced at a given position $\vec{r}$ within a material medium due to the action of
a polarizing field at nearby positions $\vec{r}'$ and not only at $\vec{r}$ [19]. Spatial dispersion arises from the spatial autocorrelation among the fluctuations of the equilibrium polarization within a solid. Non-locality is important only when the electromagnetic field has a rapid spatial variation within a lengthscale commensurate with the range of non-locality. As the wavelength of light is typically much larger than any characteristic microscopic scale of the system, non-local effects are not expected to be strong within the bulk of homogeneous systems. However, the electromagnetic field may have abrupt variations close to interfaces, so that non-local effects have to be taken into account for a realistic description of surface screening [20]–[22].

In order to circumvent the limitations of the Lifshitz theory, several alternatives have been proposed. Barash and Ginsburg [23] determined the allowed frequencies $\omega$ of the cavity fields from the dispersion relation obtained from Maxwell’s equations supplemented with planar dielectric boundary conditions. From the resulting density of states, the free energy and the force could be obtained employing a procedure that is straightforward in the absence of dissipation. However, when dissipation is included, the solution of the dispersion relation becomes complex, the corresponding fields decay in time, and their interpretation as normal modes loses meaning. To get around this problem, Barash and Ginsburg [23] introduced an auxiliary non-dissipative system in which the frequency dependence of the dielectric function is only parametric. This procedure allowed the expansion of the solutions of Maxwell equations for the field in inhomogeneous and absorbing media, in terms of the orthogonal solutions of Maxwell equations for the auxiliary system.

The use of an auxiliary system was further developed in physical terms by Kupiszewska [24] in a calculation of Casimir forces for lossy dielectrics in one dimension. In that study, the problem of quantizing a dissipative system is attacked by accounting both for the dynamics of the vacuum modes and of the atomic dipoles to which it is coupled and which make up the material, together with a thermal reservoir in which the atomic radiators dissipate the absorbed energy [24]. A disadvantage of this approach is that it requires an explicit microscopic model for the walls of the cavity, thus appearing to restrict its generality. A similar approach [25] was based on a Green’s function method and the Kubo theorem. In both cases, the stress tensor is obtained from the vacuum modes with an explicit dependence on $\epsilon(\omega)$. The final expression for the Casimir force in lossy media turns out to display the same structure as that for the non-absorptive situation.

An alternative treatment of the Casimir force was introduced by Jaekel and Reynaud [26], which calculated the radiation pressure within a cavity bordered by partially transmitting but lossless mirrors. Each mirror was replaced by an infinitesimally thin scatterer whose scattering matrix was taken to be unitary, so that energy was conserved. Their calculation was later generalized to the case of lossy optical cavities by Genet et al [27], which complemented the cavity modes with noise modes in such a way that the total scattering matrix was unitary. The scattering matrix corresponding to the cavity modes is then obtained through the optical theorem.

Mochán et al [28, 29] have obtained an expression for the Casimir force using both the scattering approach and a dissipationless ancilliary system. They have argued that in thermal equilibrium, all of the properties of the radiation field within a cavity are completely determined by the optical reflection amplitudes of the walls. Thus, the Casimir force may be obtained from the stress tensor of a fictitious system whose reflection amplitudes are identical to those of the real system. By choosing a fictitious system that has infinitesimal thin walls, and selecting their transmission amplitudes in such a way that their scattering matrix obeys the unitarity condition, it is ensured that the electromagnetic energy is conserved in the fictitious system and that no
degrees of freedom beyond those of the field itself are excited. This permits a full quantum mechanical calculation of the fields, even when the real system is dissipative. The field modes are quantized by adding perfect mirrors far away from the walls of the real cavity, so that all of the electromagnetic energy that leaves the cavity eventually returns but with a large phase shift that is very strongly dependent on the frequency and that becomes infinitely large in the limit in which the perfect mirrors are moved infinitely far away. Thus, the quantizing mirrors in the fictitious system produce a field that mimics the incoherent radiation back into the cavity that is responsible for maintaining the thermodynamic equilibrium in the real system when there are absorption processes.

The main result from the work mentioned above is that if Lifshitz formula is written in terms of the reflection coefficients of the walls of the cavity, or equivalently, in terms of their exact surface impedance [19, 30], it becomes applicable to any system with translational invariance along the surfaces and isotropy around their normal. Thus, it may be employed to calculate the Casimir force between semi-infinite or finite, homogeneous or layered, local or spatially dispersive, transparent or opaque, finite or semi-infinite systems. For example, through a simple substitution of the appropriate optical coefficients, the formalism has allowed the calculation of the Casimir force between photonic structures [31, 32], non local excitonic semiconductors [33], non local plasmon-supporting metals with sharp boundaries [29, 34], and between realistic spatially dispersive metals with a smooth self-consistent electronic density profile [28, 35].

The relative simplicity of the formalism has allowed its generalization to non-isotropic systems and the calculation of Casimir torques [36]. With a few modifications, it has also been employed for the calculation of other macroscopic forces, such as those due to electronic tunnelling across an insulating gap separating two conductors [37].

Nevertheless, there is a serious shortcoming in the derivation of the Casimir force presented in [28, 29]; namely, it fails to account properly for the fluctuating evanescent modes. The problem arises due to the fact, discussed in detail below, that a single evanescent wave excited at a surface is unable to transport energy along the normal to the surface. Therefore, for the ancillary system employed in [28, 29] it is impossible to construct a unitary scattering matrix in the evanescent region $Q > \omega / c$, where $\vec{Q}$ is projection of the wavevector on to the cavity walls. The contribution of evanescent waves to the Casimir force may be obtained as an analytic continuation from the region of propagating waves, but it is not obvious a priori that this extrapolation yields the correct result.

The purpose of the present paper is to develop a new derivation of the Casimir force that deals explicitly with the problem of energy transport in the evanescent region. Our results turn out to be consistent with Lifshitz formula when written in terms of the optical coefficients of the walls, showing its full generality within and beyond the light cone. Our derivation turns out to be very simple, so that we believe it can be readily generalized to other geometries, allowing the calculation of the dispersion forces in cavities of varied shapes whose walls are made up of realistic materials.

The structure of the paper is the following: in section 2, we review the previous scattering approach to the calculation of the Casimir force between two material slabs with flat surfaces through the stress tensor in terms of their optical properties. In section 3, we discuss the difficulties of the theory when dealing with evanescent waves and we introduce a modification which allows their incorporation. We believe that the simplicity of this derivation will readily permit its generalization to other geometries such as non-planar. In section 4, we discuss the structure of
Figure 1. Two slabs (1 and 2) with surfaces at $z_1$ and $z_2$ separated by an empty cavity $V$ of width $L$. A photon propagating within the cavity might be either reflected, absorbed, or transmitted (arrows). The coordinate axes are indicated.

the normal modes of the system, whose density is obtained in section 5 and employed in section 6 to calculate the thermodynamic properties. Finally, section 7 is devoted to conclusions.

2. Scattering approach

Consider the system $S$ shown in figure 1. The slabs represent rather arbitrary media. They could be made up of one or several components, they could be homogeneous or inhomogeneous, local or spatially dispersive, transparent or opaque, finite or semi-infinite, etc. We only impose translational symmetry along the $x$-$y$ plane, non-chirality and isotropy around the $z$-axis. Following [28, 29], we note that a photon within the cavity $V$ with a polarization $\alpha$ ($s$ or $p$) that arrives at a slab $a$ (1 or 2) may be coherently reflected with a probability amplitude $r^\alpha_a$. Otherwise, it may be transmitted with probability $T^\alpha_a = 1 - |r^\alpha_a|^2$ into the slab, where it may be absorbed, exciting electronic or vibrational material degrees of freedom, or it may be transmitted towards vacuum beyond the slab. In thermodynamic equilibrium, there would be radiation beyond the slab which could be transmitted back into $V$ and the slab would also be a source of radiation, thus replenishing the photons lost from the cavity. Detailed balance implies that photons are incoherently injected into the cavity with a probability $\propto T^\alpha_a$. Thus, any photon within the cavity that reaches the $a$th surface is either coherently reflected with amplitude $r^\alpha_a$ or incoherently replaced by a similar photon with probability $1 - |r^\alpha_a|^2$. As a consequence, the properties of the radiation field within $V$ are completely determined in equilibrium by the geometry of the cavity, characterized by $L$ and by the reflection amplitudes $r^\alpha_a$; any relevant material property is necessarily accounted for through the optical coefficients. In particular, it should be possible to calculate the Casimir force with no information about the material response beyond $r^\alpha_a$.

To this end, we consider as a first step a fictitious system $S'$ made up of two infinitely thin sheets at $z = z_a$ whose reflection amplitudes are postulated to coincide exactly with those of the real slabs (figure 2). Therefore, the radiation field within the fictitious cavity $V'$ corresponds to the real radiation field within the real cavity $V$. We introduce scattering matrices $S^\alpha_a$ for each sheet, some of whose components $r^\alpha_a$ are equal to those of the real system, and the rest of which are chosen so that $S^\alpha_a$ is unitary, i.e., conserves energy. This way, we can study the eigenstates of the electromagnetic field in the closed non-dissipative system $S'$, without mixing in electronic or vibrational degrees of freedom. The field obeys the vacuum Maxwell equations everywhere, and boundary conditions determined by $S^\alpha_a$ at $z = z_a$. In order to count modes, we discretize
Figure 2. Fictitious system made up of infinitely thin sheets at $z = z_a$ ($a = 1, 2$) characterized by reflection and transmission amplitudes $r_a$ and $t_a$. We indicate the empty regions I, II and III of widths $L_I$, $L_{II} = L$ and $L_{III}$ respectively, and the boundaries of the system at $z_0$ and $z_3$.

them by giving the whole system $S'$ a finite size $W = L_I + L_{II} + L_{III}$ (figure 2) and we impose homogeneous boundary conditions on the external boundaries $z = z_0, z_3$, taking the limit

$$L_I, L_{III}, W \to \infty.$$  \hspace{1cm} (1)

at the end of the calculation, where $L_I$ and $L_{III}$ are the widths of the empty regions of $S'$ to the left of $z_1$ and to the right of $z_2$ respectively, and $L_{II} \equiv L$. For any given large value of $L_I$ and of $L_{III}$, a photon that abandons the cavity $V'$ will come back after traversing one or several times the regions I or III, accumulating a large phase which varies very rapidly with frequency, mimicking the photons radiated into the cavity by the walls of the real system.

Consider now an $s$-polarized wave of frequency $\omega$ with wavevector $\vec{Q}$ along the interface. Without loss of generality, we choose $x-z$ as the plane of incidence, so that the field is given by

$$\vec{E}(t, \vec{r}) = \text{Re}(0, E_y(t, \vec{r}), 0) = \text{Re} \vec{E}_0 e^{i(Qx-\omega t)} \phi(z),$$  \hspace{1cm} (2)

with amplitude $\vec{E}_0 = (0, E_0, 0)$. The electromagnetic wave equation implies

$$\phi(z) = \alpha_r e^{ikz} + \beta_r e^{-ikz}, \quad (r = I, II, III),$$  \hspace{1cm} (3)

where

$$k_I = k_{II} = k_{III} \equiv k = \sqrt{q^2 - Q^2},$$  \hspace{1cm} (4)

$q = \omega/c$ is the free wavenumber, and the coefficients $\alpha_r, \beta_r$ for each of the regions $r = 1, II$ and III are to be obtained from the boundary conditions and the normalization condition

$$\| \phi \|^2 = \int_{z_0}^{z_3} dz |\phi(z)|^2 = \sum_r L_r \left( |\alpha_r|^2 + |\beta_r|^2 + 2 \text{Re} \frac{kL_r}{kL_r} \alpha_r \beta_r^* e^{2ik\zeta_r} \right) = 1.$$  \hspace{1cm} (5)

$\zeta_r$ denotes the midpoint of region $r$ and $(\cdots)^*$ is the complex conjugate of $(\cdots)$. In the limit (1), the normalization condition simplifies to

$$L_I(|\alpha_I|^2 + |\beta_I|^2) + L_{III}(|\alpha_{III}|^2 + |\beta_{III}|^2) = 1.$$  \hspace{1cm} (6)
Introducing the magnetic field $\mathbf{B} = (B_x, 0, B_z)$ with

$$B_x(t, \mathbf{r}) = \frac{i}{q} E_0 e^{i(Qx - \omega t)} \partial_z \phi(z),$$  \hfill (7)$$
$$B_z(t, \mathbf{r}) = \frac{Q}{q} E_0 e^{i(Qx - \omega t)} \phi(z),$$ \hfill (8)

we may integrate the energy density,

$$u = (|E|^2 + |B|^2)/(16\pi)$$ \hfill (9)

to obtain the electromagnetic energy

$$U = \frac{A}{8\pi} \sum_r L_r \left( |\alpha_r|^2 + |\beta_r|^2 + 2 \frac{Q^2}{q^2} \sin kL_r \Re \alpha_r \beta_r^* e^{2ikz} \right) |E_0|^2,$$ \hfill (10)

which simplifies to

$$U = \frac{A}{8\pi} |E_0|^2,$$ \hfill (11)

in the limit (1), when we use (6). Here, we introduced the area $A$ of the slabs, that will be taken to infinity at the end of the calculation. Notice that most of the energy lies within regions I and III, which play the role of a thermal bath for the cavity.

To obtain the force over slab 2, we calculate the stress tensor $T_{ij} = 1/(8\pi) \Re [E_i E_j^* + B_i B_j^* - (|E|^2 + |B|^2) \delta_{ij}/2]$ at an arbitrary position $z$ within the cavity,

$$-T_{zz}(z) = \left( \frac{k^2}{q^2} |\phi(z)|^2 + \frac{1}{q^2} |\partial_z \phi(z)|^2 \right) \frac{U}{2A},$$ \hfill (12)

where we used equation (11) to eliminate the amplitude in terms of the energy. By applying boundary conditions at $z_0$ and $z_3$, we obtain a discrete set of normal mode frequencies $\omega_n$ and corresponding perpendicular components $k_n$ of the wavevector. The total stress tensor is a sum of contributions similar to (12), one for each mode,

$$-T_{zz} = \frac{1}{2A} \sum_n \left( \frac{k_n^2}{q_n^2} |\phi_n(z)|^2 + \frac{1}{q_n^2} |\partial_z \phi_n(z)|^2 \right) (N_n + \frac{1}{2}) \hbar \omega_n, \quad \text{(fixed $\mathbf{Q}$)}$$ \hfill (13)

where we substituted the energy $U_n$ in terms of the average number of photons $N_n$ in state $n$. The sum over states may be rewritten as

$$-T_{zz} = \frac{\hbar c}{A} \int \frac{d(k^2)}{q} \rho_{k^2}, \quad \text{(fixed $\mathbf{Q}$)}$$ \hfill (14)
where

\[ f = \coth(\beta \hbar \omega / 2) / 2, \]  

(15)

is the equilibrium occupation number of a state with energy \( \hbar \omega \) at temperature \( T = 1/(k_B \beta) \), \( k_B \) is Boltzmann’s constant,

\[ \rho_k = \rho_k(z) = -\frac{1}{2\pi} \left( \text{Im}G_{k^2}(z, z') + \frac{1}{k^2} \partial_z \partial_{z'} \text{Im}G_{k^2}(z, z') \right) \bigg|_{z' \to z}, \]  

(16)

plays the role of a local density of states at \( z \) (in \( k^2 \) space), and

\[ G_{k^2}(z, z') = \sum_n \frac{\phi_n(z)\phi_n^*(z')}{k^2 - k_n^2}, \]  

(17)

is Green’s function for the 1D Helmholtz equation,

\[ (\partial_z^2 + k^2)G_{k^2}(z, z') = \delta(z - z'). \]  

(18)

Here, we introduced the complex wavenumber

\[ \tilde{k} = k + i\eta, \quad (\eta > 0), \]  

(19)

with the understanding that the limit

\[ \eta \to 0^+, \]  

(20)

is to be taken at the end of the calculation, and we have used the identity \( \text{Im}[1/(\tilde{k}^2 - k_n^2)] = -\pi \delta(k^2 - k_n^2) \) to replace the sum (13) by the integral (14).

The solution of equation (18) within \( \mathcal{V} \) is

\[ G_{\tilde{k}^2} = \frac{\phi^>(z)\phi^< (z_>)}{W}, \]  

(21)

where \( z_< \) and \( z_> \) denote the minimum and maximum values among \( z \) and \( z' \) respectively,

\[ \phi^<(z) = e^{-i\tilde{k}(z-z_1)} + r_1^* e^{i\tilde{k}(z-z_1)}, \]  

(22)

and

\[ \phi^>(z) = e^{i\tilde{k}(z-z_2)} + r_2^* e^{-i\tilde{k}(z-z_2)}, \]  

(23)

are the solutions to the Helmholtz equation with wavenumber \( \tilde{k} \) obeying the boundary conditions at \( z = z_1 \) and \( z_2 \) respectively and \( W \) is their Wronskian. The reflection amplitudes are in general functions of \( Q \) and \( \tilde{k} \).

Substitution of (22) and (23) in (21), (16) and (14), yields

\[ -T_{zz} = \frac{1}{\mathcal{A}} \frac{\hbar c}{\pi} \int d\tilde{k} f \frac{k^3}{q} \text{Re} \left( \frac{1 + r_1^* r_2^* e^{2i\tilde{k}L}}{k(1 - r_1^* r_2^* e^{2i\tilde{k}L})} \right). \]  

(24)
For $p$ polarization a similar result may be obtained by substituting $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$, and $r_a^p \to -r_a^p$. Summing equation (24) over $\vec{Q}$ and incorporating the contribution of $p$ polarized waves, we finally obtain

$$-T_{zz} = \frac{\hbar c}{4\pi^3} \int d^2 \vec{Q} \int dk f k^3 \frac{1}{q} \text{Re} \left( \frac{1 + r_1^p r_2^p e^{2i\vec{k}\cdot \vec{L}}}{k(1 - r_1^p r_2^p e^{2i\vec{k}\cdot \vec{L}})} + \frac{1 + r_1^p r_2^p e^{2i\vec{k}\cdot \vec{L}}}{k(1 - r_1^p r_2^p e^{2i\vec{k}\cdot \vec{L}})} \right),$$

(25)

where we assumed Born-von Karman periodic boundary conditions along the surface of area $A \to \infty$.

The flux of linear momentum $-T_{zz}$ in the fictitious cavity is the same as in the real cavity. To obtain the force on slab 2, we have to subtract the flux in the real system between the slab and infinity, which can be obtained following the derivation given above but replacing slab 1 by both slabs and replacing slab 2 by empty space. The result is identical to equation (25), but substituting $r_2^p \to 0$. Thus, the total force per unit area of slab 2 is

$$\frac{F_z}{A} = \frac{\hbar c}{2\pi^2} \text{Re} \int_0^\infty QdQ \int dk f k^3 \frac{1}{q} \left( \frac{1}{k(\xi - 1)} + \frac{1}{k(\xi' - 1)} \right),$$

(26)

where

$$\xi^a \equiv (r_1^p r_2^{-p} e^{2i\vec{k}\cdot \vec{L}})^{-1}.$$  

(27)

The integration in (26) is over all the allowed values of the parallel wavevector $\vec{Q}$ and of the normal wavevector $k$, or equivalently, of the frequency $\omega$, namely, $\vec{Q}$ ranges over all points in the 2D reciprocal space and $\omega$ ranges over all positive frequencies. Thus, in (26), $k$ goes from $iQ$ towards zero along the imaginary axis and then towards infinity along the real axis. We may remove the tilde from $\tilde{k}$ if we displace the integration trajectory infinitesimally into the upper complex half plane. We may then change integration variables from $Q, k$ to $Q, \omega$ and perform the usual Euclidean rotation on the frequency integral so that it becomes an integral over positive imaginary frequencies at zero temperature, or a sum over Matsubara frequencies at finite temperature.

Equation (26) generalizes the Lifshitz formula [8] to arbitrary slabs in terms of their optical reflection amplitudes. We remark that we have made no assumption about their constitution nor about their internal structure, and our results are only restricted by the translational symmetry imposed along $x$-$y$, by non-chirality, and by isotropy around $z$. An alternative derivation of equation (26) has been presented in [29]. Our approach is very similar to that proposed by Jaekel and Reynaud [26] and by Genet et al [27], who obtained the Casimir force in terms of the optical coefficients of the cavity through a unitary scattering matrix that may be employed even for dissipative systems. However, we have employed a fictitious system which has no degrees of freedom beyond the ordinary electromagnetic field, while they added a set of noise modes which interchange energy with the field accounting thus for absorption and radiation. Within individual local, homogeneous and isotropic layers of a multilayered system, the fields and noise modes are coupled through optical networks described by an enlarged unitary scattering matrix to which the ordinary electromagnetic non-unitary scattering matrix is related through an optical theorem. The optical networks of cavities and multilayered systems may be obtained by simple multiplication of transfer matrices which are simply related to the scattering matrices of each layer.
3. Evanescent waves

Although very general, the derivation presented in the previous section is not free from problems. The integration region in (26) contains the sector $Q < q$, for which all waves are propagating as assumed in the derivation, but it also contains a sector with $Q > q$, for which $k$ is imaginary and the waves are evanescent. For evanescent waves, the normalization condition cannot be simplified as in (5) and (6), nor can the energy be simplified as in (10) and (11). In particular, it is not true that the total energy is dominated by the empty fictitious regions I and III outside of the cavity $V'$, as the energy is concentrated close to the surfaces regardless of the lengths $L_1$ and $L_{III}$.

Even more serious is the fact that for evanescent waves it is impossible to construct a unitary scattering matrix that has the same reflection amplitude of the original dissipative system (figure 1), while conserving energy at the interface of the fictitious system (figure 2). The reason for this is that the $z$-component of the Poynting vector for a single transmitted evanescent wave is null; the transmitted wave cannot transport energy away from the interface in the fictitious cavities I and III (figure 2). On the other hand, the interaction of the incident and reflected waves allows energy to be carried from the cavity $V$ into its interfaces, namely, the power that is actually absorbed or transmitted through the slabs in the real system (figure 1).

At this point it is necessary to remark that in the presence of evanescent waves the energy flux is not an additive quantity [38]. An incident evanescent wave carries by itself no energy towards the interface, nor does the reflected wave carries energy away from the interface. We may convince ourselves of this fact by considering a single stationary evanescent wave in vacuum; if it were to transport energy, the energy flux would decay exponentially in space at twice the decay rate of the corresponding field, and thus energy would have to be deposited in vacuum, violating either the condition of stationarity or the law of conservation of energy, thus proving our statement by reductio ad absurdum. Alternatively, we may simply apply Poynting’s expression. Nevertheless, there is an energy flux in vacuum in the normal direction whenever the underlying system is dissipative. As a single evanescent wave carries no energy, the energy flux is a property of both the incident and the reflected wave (figure 3) taken together, i.e., it is a synergistic property of both the incident and reflected wave. Thus, the concept of reflectance, defined as the ratio between the reflected and the incident intensities becomes meaningless. It is for this reason that the reflection amplitude is no longer constrained to values smaller than unity [26], and can actually attain any magnitude outside of the light cone. For example, in a system that supports surface plasmons, whenever $\omega$ and $Q$ coincide with the frequency of a surface mode the reflection amplitude becomes infinitely large; surface modes may actually be defined through the poles of the reflection amplitudes [39].

We can recover the unitarity of the scattering matrix within the evanescent sector if we modify our fictitious system so that it does allow energy transport outside of the light cone. This can be done by filling regions I and III of figure 2 with a fictitious dielectric with a real dispersionless and dissipationless dielectric constant $\epsilon_f$, so that all waves propagate in regions I and III as long as $Q < \sqrt{\epsilon_f \omega / c}$ (figure 4), although they are evanescent within the cavity if $Q > \omega / c$.

We can now follow the steps of the derivation shown in section 2. We will only indicate the modifications required by the presence of the fictitious dielectric. Equations (2) and (3) would...
Figure 3. Left panel: an evanescent wave (i) is incident on a flat scatterer (vertical line) within vacuum, where it produces reflected \( (r) \) and transmitted \( (t) \) evanescent waves. The propagation directions are indicated by thin arrowheads and the dependence on position of the corresponding field strengths are schematically displayed. There is an incoming energy flux (double arrow) due to the interaction of the incident and the reflected wave, but there can be no transmitted energy (crossed double arrow) as the normal projection \( S_z \) of Poynting vector for a single evanescent wave is null, so that energy cannot be conserved. Right panel: if the vacuum beyond the scatterer is replaced by a fictitious medium with a large enough dielectric constant \( \epsilon_f \), the transmitted wave becomes propagating (wiggly arrow) and capable of transporting energy, restoring energy conservation.

Figure 4. Light cone edge in vacuum and within a dispersionless fictitious insulator with dielectric constant \( \epsilon_f \). Within the vacuum light cone (VLC, \( Q < \omega/c \)) waves may propagate in both vacuum and the dielectric. In the extended light cone (ELC, \( Q < \sqrt{\epsilon_f \omega/c} \)) they propagate in the dielectric but may be evanescent in vacuum. In the evanescent region (Ev, \( Q > \sqrt{\epsilon_f \omega/c} \)), they are evanescent in both vacuum and the dielectric.

remain valid, although the normal wavenumbers \( k_r \) would no longer obey (4) but

\[
k_r = \sqrt{\epsilon_r q^2 - Q^2} \quad (r = I, II, III), \quad k_I = k_{II} \equiv k_f, \quad k_{II} = k,
\]

where \( \epsilon_I = \epsilon_{III} = \epsilon_f \) and \( \epsilon_{II} = 1 \). A more subtle change appears when we consider the normalization (5) of the wavefunction. The field (2) ought to obey the Helmholtz equation

\[
\nabla^2 E_y(t, \vec{r}) = -\epsilon_r \frac{\omega^2}{c^2} \vec{E}_y(t, \vec{r}),
\]

but this is not properly an eigenvalue equation, due to the region-dependent factor \( \epsilon_r \) on its right-hand side. By dividing both sides by \( \epsilon_r \), it does become an eigenvalue equation, but the
corresponding wave operator $\mathcal{L} = \nabla^2 / \epsilon_r$, together with our boundary conditions turns out not to be Hermitian when we use the usual norm (5), as

$$2i \text{Im} \int_{S'} dz E^*_y(t, \mathbf{r}) [\mathcal{L} E_y(t, \mathbf{r})] = \frac{W(z_1^-)}{\epsilon_f} - W(z_1^+) + W(z_2^-) - \frac{W(z_2^+)}{\epsilon_f} \neq 0,$$

(30)

where we introduced the Wronskian $W(z) = E^*_y (\partial E_y / \partial z) - (\partial E^*_y / \partial z) E_y$ evaluated at a given $z$. Nevertheless, equation (7) implies that $W(z)$ is proportional to the normal component of Poynting vector $S_n(z)$ which we require to be continuous at both $z_1$ and $z_2$ in the fictitious system. Thus, the wave operator becomes Hermitian if we eliminate $\epsilon_f$ from the right-hand side of (30) by defining the norm as

$$\| \phi \|^2 = \int_{z_0}^{z_1} \epsilon_f |\phi(z)|^2 + \int_{z_1}^{z_2} |\phi(z)|^2 + \int_{z_2}^{z_3} \epsilon_f |\phi(z)|^2,$$

(31)

instead of (5). The normalization condition then becomes

$$\epsilon_f [L_I(|\alpha_I|^2 + |\beta_I|^2) + L_{III}(|\alpha_{III}|^2 + |\beta_{III}|^2)] = 1,$$

(32)

in the limit (1), instead of (6).

As usual, the energy density has to be written as

$$u = \text{Re}(\mathbf{E} \cdot \mathbf{D}^* + |\mathbf{B}|^2)/(16\pi),$$

(33)

instead of (9) to account for the dielectric response of the fictitious system, so that (10) has also to be modified, becoming

$$\mathcal{U} = \frac{\mathcal{A}}{16\pi q^2} \sum_r L_r \left[ (\epsilon_r q^2 + Q^2 + |k_r|^2)(|\alpha_r|^2 + |\beta_r|^2) + (\epsilon_r q^2 + Q^2 - |k_r|^2) \right] \times \left( \frac{2 Q^2 \sin k_r L_r}{q^2 k_r L_r \text{Re} \alpha_r \beta_r^* e^{ik_r z_r}} \right) |\mathcal{E}_0|^2.$$

(34)

Notice the presence of the absolute value in $|k_r|^2$ which might differ from $k_r^2$. Nevertheless, within the extended light cone $k_1 = k_{III} = k_f$ are real and $Q^2 + k_f^2 = \epsilon_f q^2$, so that in the limit (1) equation (34) simplifies to

$$\mathcal{U} = \frac{\mathcal{A}}{8\pi} \epsilon_f [L_I(|\alpha_I|^2 + |\beta_I|^2) + L_{III}(|\alpha_{III}|^2 + |\beta_{III}|^2)] |\mathcal{E}_0|^2.$$

(35)

Using the normalization condition (32), we finally obtain equation (11), showing that it is valid for both evanescent and propagating cavity waves, as long as $Q^2 < \epsilon_f \omega^2 / c^2$.

From equation (11) onwards, the derivation of the Casimir force can proceed in identical fashion as in the case of propagating waves since the fictitious regions play no further role in the calculation. Thus, we may arrive at (26) proving that Lifshitz expression for the Casimir force is valid in both the propagating and the evanescent sectors. However, we should point out that the integration in (26) should be restricted to the extended light cone $Q < \sqrt{\epsilon_f q}$ (figure 4).

Of course, the limit

$$\epsilon_f \to \infty$$

(36)
has to be taken at the end of the calculation in order to incorporate the contributions of all the evanescent modes to the force. Notice that the contributions to the force from evanescent waves could have been obtained from an analytical continuation of the contribution from propagating waves, but we did not have to assume that continuation nor discuss the analytical properties of the integrand [26, 27].

The procedure stated above for the calculation of the Casimir force, namely, the use of the Lifshitz formula within an extended light cone in the limit where its slope becomes zero, leads to an intriguing consequence, as the largest contributing wavevector $Q$ at a given frequency goes to zero as $\omega \to 0$ for any finite value of $\epsilon_f$, no matter how large. This procedure yields an unambiguous value for the zero frequency reflection amplitude of a conductor corresponding to $s$ polarization. For instance, for a local, semi-infinite, homogeneous conductor, $r_s \to -1$.

We remark that equations (32) and (35) may also be obtained from (31) and (34) without taking the limit (1), as we also take the limit (36). In fact, we can even take the limit $L_I, L_{III} \to 0$ as long as $\sqrt{\epsilon_f}L_I \to \infty$, $\sqrt{\epsilon_f}L_{III} \to \infty$.

By using fictitious regions that are optically thick, while they remain geometrically thin compared to the radius of curvature of the interfaces, we can easily extend the present calculation to other, non-planar geometries. Thus, our scheme may be applied to the calculation of the Casimir forces within cavities of arbitrary shape made of arbitrary materials (work in progress).

4. Normal modes

As emphasized above, all of the properties of the electromagnetic field within a cavity bordered by walls made up of arbitrary materials are determined solely in terms of their reflection amplitudes, and thus coincide with the field properties of fictitious cavities that have the same geometry and the same optical coefficients. In particular, as the electromagnetic stress tensor within the cavity is a local property of the field, we could obtain the Casimir force from the momentum flux in the auxiliary system. On the other hand, the energy is a global property and there is no reason to expect its value in the fictitious system to agree with its value in the real system. However, as the force in both systems coincides, then the dependence of the fictitious energy on the separation $L$ must agree with that of the real system. For this reason, it is interesting to calculate the energy of the fictitious system.

In order to calculate its energy, we require the electromagnetic normal modes of the fictitious system. We recall that in the real system energy may be lost from the cavity by being absorbed or transmitted away, but that in equilibrium this energy is replenished through re-radiation from the walls or through transmission of thermal radiation from the environment back into the cavity. In the fictitious system, we mimic this process by postulating that the electromagnetic energy that leaves the cavity propagates freely through the optically wide fictitious regions before being reflected back by perfect mirrors into the cavity with a large accumulated phase. Thus, the fictitious system is completely lossless and its electromagnetic normal modes are characterized by well defined, discrete, real frequencies $\omega_n$.

As a first step towards the calculation of $\omega_n$, we require the scattering matrix $S_a$ of each of the two surfaces $a = 1, 2$. For example, we consider interface $a = 2$. According to figure 5 and equation (3), there are two incoming waves with amplitudes $\alpha_{II}$ and $\beta_{III}$ and phases $e^{ikz}$.
and $e^{-ikz}$ respectively. We will assume that both are propagating waves. At the interface, two outgoing waves are generated with amplitudes $\beta_{II}$ and $\alpha_{III}$ and phases $e^{-ikz}$ and $e^{ikfz}$ respectively. Thus, we define $S_2$ through

$$
\left( \begin{array}{c}
\beta_{II} e^{-ikz} \\
\alpha_{III} e^{ikfz}
\end{array} \right) = S_2 
\left( \begin{array}{c}
\alpha_{II} e^{ikfz} \\
\beta_{III} e^{-ikz}
\end{array} \right).
$$

(38)

We can write the scattering matrix in terms of the optical coefficients of the interface as

$$
S_2 = \left( \begin{array}{cc}
 r_2 & \tilde{t}_2 \\
t_2 & \tilde{r}_2
\end{array} \right),
$$

(39)

where $r_2$ and $t_2$ are respectively the reflection and transmission amplitudes corresponding to a wave incident on the interface towards the right, and $\tilde{r}_2$ and $\tilde{t}_2$ are the corresponding quantities for incidence towards the left, and for notational simplicity, we have removed the polarization index (either $s$ or $p$). As discussed in section 2, we take $r_2$ to be identical to the reflection amplitude of the real system.

For propagating waves, the energy flux on the left- and right-hand sides is proportional to

$$
s^-_z = k(|\alpha_{II}|^2 - |\beta_{II}|^2),
$$

(40)

$$
s^+_z = k_f(|\alpha_{III}|^2 - |\beta_{III}|^2),
$$

(41)

so that energy continuity requires

$$
k|\beta_{II}|^2 + k_f|\alpha_{III}|^2 = k|\alpha_{II}|^2 + k_f|\beta_{III}|^2
$$

(42)

for arbitrary values of the incoming complex amplitudes $\alpha_{II}$ and $\beta_{III}$. Substituting (38) in (42) we obtain

$$
|r_2|^2 + \frac{k_f}{k} |t_2|^2 = 1,
$$

(43)

$$
\tilde{r}_2 \sqrt{k/k_f} + \tilde{t}_2 \sqrt{k_f/k} = 0,
$$

(44)
\(|\tilde{r}_2|^2 + \frac{k}{k_f}|\tilde{r}_2|^2 = 1, \tag{45}\)

which imply

\begin{align*}
R_2 &= \tilde{R}_2, \\
T_2 &= \tilde{T}_2, \\
R_2 + T_2 &= \tilde{R}_2 + \tilde{T}_2 = 1. \tag{46}
\end{align*}

where we identify the reflectance and transmittance as usual, \(R_2 = |r_2|^2\), \(T_2 = (k_f/k)|t_2|^2\), \(\tilde{R}_2 = |\tilde{r}_2|^2\), \(\tilde{T}_2 = (k/k_f)|\tilde{t}_2|^2\). We remark that although it is well known that Fresnel’s equations lead to (44), they need not hold for our fictitious system. The optical coefficients are determined not only by \(\epsilon_f\), but also by fictitious surface charges and currents induced at the interface which we have not fully specified; we have only demanded that the resulting reflection amplitude coincides with that of the real system and that energy be conserved. Equations (43)–(46) may also be obtained by using an alternative definition for the scattering matrix,

\[
\begin{pmatrix} \sqrt{k \beta_{\text{II}} \, e^{-ikz_2}} \\ \sqrt{k \beta_{\text{III}} \, e^{ikz_2}} \end{pmatrix} = S'_2 \begin{pmatrix} \sqrt{k \alpha_{\text{II}} \, e^{ikz_2}} \\ \sqrt{k \alpha_{\text{III}} \, e^{-ikz_2}} \end{pmatrix} \tag{47} \]

instead of (38), and then demanding the usual unitarity condition on \(S'_2\), namely, \((S'_2)^\dagger S'_2 = S'_2 (S'_2)^\dagger = 1\) where \(1\) represents the unit matrix.

If the field in region II is evanescent, the energy flux is no longer given by (40) expression, but by [38]

\[ s_{-}^- = 2 \kappa \, \text{Im} \, (\alpha_{\text{II}}^* \beta_{\text{II}}), \tag{48} \]

where we wrote \(k = i \kappa\) in terms of the decay constant \(\kappa\). Notice that \(s_{-}^-\) cannot be separated into an incident and a reflected contribution, as in the case of propagating waves [38].Demanding energy continuity now leads to

\[
|\tilde{r}_2|^2 = 1, \tag{49}
\]

\[
|t_2|^2 = 2 \frac{K}{k_f} r_{z}''', \tag{50}
\]

\[
\tilde{t}_2 = \frac{k_f}{k} t_{z}'' \tilde{t}_2. \tag{51}
\]

We remark that if we go beyond the extended light cone, we would also have to modify equation (41), which would become proportional to \(\text{Im}(\alpha_{\text{III}}^* \beta_{\text{III}})\). Thus it would be impossible to satisfy energy continuity for arbitrary values of \(\alpha_{\text{II}}\) and \(\beta_{\text{III}}\). It is for this reason that our derivation is restricted to the extended light cone which, nevertheless, becomes arbitrarily wide in the \(\epsilon_f \to \infty\) limit.

The perfect mirror at \(z_3\) represents a node of the field \(\phi(z_3) = 0\), so that \(\alpha_{\text{III}} e^{ikz_3} + \beta_{\text{III}} e^{-ikz_3} = 0\). This condition, together with (38) and (39) yields

\[ \beta_{\text{II}} e^{-ikz_2} = r_2 \alpha_{\text{II}} e^{ikz_2}, \tag{52} \]
where the total reflection amplitude is given by

\[ r_{2t} = r_2 - \frac{\tilde{t}_2 t_2 e^{2ikL_{III}}}{1 + \tilde{r}_2 e^{2ikL_{III}}}. \]  

(53)

The first term corresponds to the coherent reflection from the plate. The second represents the contribution of a wave that traverses the fictitious region III, acquiring in the process a large phase \(2k_f L_{III}\). After one transversal, the field might be reflected back instead of being immediately transmitted into the cavity, leading to multiple traversals that correspond to a geometrical series whose sum yields the denominator of (53).

For propagating waves, we may simplify (53) using (43)–(46) to obtain

\[ r_{2t} = \frac{(1 + r_2 e^{-i(k_f L_{III} + \delta_2)})^2 e^{i(k_f L_{III} + \delta_2)}}{|1 + r_2 e^{-i(k_f L_{III} + \delta_2)}|^2}, \]  

(54)

where the phase \(\delta_2\) is defined through \(e^{i\delta_2} = \tilde{r}_2 r_2^*\). Similarly, for evanescent waves, we may simplify (53) using equations (49)–(51) to obtain

\[ r_{2t} = \frac{\text{Re} \left[ r_2 (1 + e^{-i(k_f L_{III} + \delta_2)}) \right]}{1 + \cos(k_f L_{III} + \delta_2)}, \]  

(55)

where the phase \(\delta_2\) is now defined through \(e^{i\delta_2} = \tilde{r}_2\). Notice that although \(r_2\) may correspond to a partially transparent or absorptive plate, the total reflectance \(R_{2t} = |r_{2t}|^2 = 1\) for propagating waves, and \(r_{2t}'' = 0\) for evanescent waves, so in neither case is there a net energy flux out of the cavity, as expected in thermodynamic equilibrium.

Following an entirely similar procedure for interface \(a = 1\), we obtain

\[ \alpha_{1t} e^{ikz_1} = r_1 \beta_{1t} e^{-ikz_1}, \]  

(56)

where the total reflection amplitude is given by

\[ r_{1t} = r_1 - \frac{\tilde{t}_1 t_1 e^{2ikL_I}}{1 + \tilde{r}_1 e^{2ikL_I}}, \]  

(57)

which may be simplified to

\[ r_{1t} = \frac{(1 + r_1 e^{-i(k_f L_I + \delta_1)})^2 e^{i(k_f L_I + \delta_1)}}{|1 + r_1 e^{-i(k_f L_I + \delta_1)}|^2}, \]  

(58)

with \(e^{i\delta_1} = r_1/\tilde{r}_1^*\) for the case of propagating waves, and to

\[ r_{1t} = \frac{\text{Re} \left[ r_1 (1 + e^{-i(k_f L_I + \delta_1)}) \right]}{1 + \cos(k_f L_I + \delta_1)}, \]  

(59)

with \(e^{i\delta_1} = \tilde{r}_1\) for the case of evanescent waves.

The condition for having non-trivial simultaneous solutions of both equations (52) and (56) is

\[ D = 1 - r_{1t} r_{2t} e^{2ikL} = 0. \]  

(60)

After substituting either (54) and (58), or (55) and (59), the set of allowed frequencies \(\omega_n\) may be obtained for each value of \(Q\) and for each polarization. We remark that the resulting frequencies are real as energy is explicitly conserved in the fictitious system, corresponding to thermodynamic equilibrium in the real system.

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5. Density of states

The thermodynamic properties of a system may be obtained from appropriate sums over its allowed states. Before attempting those sums, we analyse the structure of the normal modes. We consider first the case of propagating waves for which $|r_1| = |r_2| = 1$, so that (60) may be recast as $\text{arg}(r_1 r_2 e^{2ikL}) = 2\pi n$, where $\text{arg}(\zeta) = \text{Im} \log(\zeta)$ denotes the argument of the complex number $\zeta$ and $n$ is an integer. Substituting (58) and (54), we obtain

$$2kL + \delta_1 + \delta_2 + k_i (L_1 + L_{III}) + 2 \text{Im} \log [(1 + r_1 e^{-i(k_i L_1 + \delta_1)})(1 + r_2 e^{-i(k_i L_{III} + \delta_2)})] = 2\pi n.$$

Notice that the first three terms of (61) are slowly varying functions of frequency, i.e., they vary with the characteristic frequency scales corresponding to the real separation $L$ and the real materials that make up the two slabs. On the other hand, in the limit (37), the fourth term becomes an extremely fast increasing function of the frequency, while the fifth term oscillates but does not contribute to the advance of the phase as the reflection amplitudes of real materials in equilibrium obey $|r_1|, |r_2| < 1$ and $-\pi < \text{arg} (1 + r_1 e^{-i(k_i L_1 + \delta_1)}), \text{arg} (1 + r_2 e^{-i(k_i L_{III} + \delta_2)}) < \pi$. Therefore, the separation of two consecutive modes is

$$\Delta \omega = \frac{\Delta \omega}{\Delta k_i} \frac{2\pi}{L_1 + L_{III}} \approx \frac{c}{\epsilon_i} \frac{2\pi}{(L_1 + L_{III})},$$

of the order of the inverse of the traversal time through the fictitious regions. Thus, $\Delta \omega \to 0$ in the limit (37) and the modes form a quasi-continuum in frequency space, regardless of the values of $r_1$ and $r_2$.

Consider now a finite frequency range $\Omega = (\omega_1, \omega_2)$ that is very small compared to all the characteristic frequencies of the real system, but that is large compared to $\Delta \omega$. Then, we may approximate

$$\sum_{\omega_n \in \Omega} f(\omega_n) \approx f(\bar{\omega}) N(\Omega)$$

for any slowly varying function $f$ of frequency, where $\bar{\omega}$ is some frequency within $\Omega$. The number $N(\Omega)$ of modes within the interval may be obtained from

$$N(\Omega) = \frac{1}{2\pi i} \oint_{\gamma} d\omega \frac{d}{d\omega} \log [g(\omega)],$$

where the integral is taken along a counterclockwise closed path $\gamma$ that contains all the zeros within $\Omega$ and no zero outside of $\Omega$, and $g(\omega)$ is an analytical function that has the same zeros as $D$ (equation (60)) and has no poles within $\gamma$. Notice that $D$ is not an analytical function due to the presence of absolute values in (58) and (54). We choose an integration path that goes from $\omega_1$ to $\omega_2$ a small distance $\eta$ below the real axis and returns to $\omega_1$ a distance $\eta$ above the real axis, and

$$g(\omega) = (1 + r_1^* e^{i(k_i L_1 + \delta_1)})(1 + r_2^* e^{i(k_i L_{III} + \delta_2)}) - (r_1 + e^{i(k_i L_1 + \delta_1)})(r_2 + e^{i(k_i L_{III} + \delta_2)}) e^{2ikL},$$

where we linearize all slowly varying functions around $\bar{\omega}$, so that we may treat them as analytic, recalling that $\Omega$ is a small region, and we take the limit

$$\eta \to 0^+.$$

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Notice that the naive choice \( g = D \) would be inappropriate even if all slow terms were replaced by constants, as it has poles arbitrarily close to the real axis in the limit (37). Thus,

\[
N(\Omega) = \frac{1}{2\pi i} \log \frac{g(w - i\eta)}{g(w + i\eta)} \bigg|_{\omega_1}^{\omega_2}.
\]

We can approach the limits (37) and (66) in such a way that \( \eta \gg \Delta \omega, \eta \sqrt{\varepsilon_1 L_1/c} \gg 1 \), \( \eta \sqrt{\varepsilon_1 L_III} \gg 1 \) although \( \eta \) is made much smaller than any other relevant frequency. Therefore, we simplify (67) to

\[
N(\Omega) = \frac{1}{2\pi i} \left[ \log \left( \frac{1 - r_1 r_2 e^{2ikL}}{r_1^* r_2^* e^{-2ikL} - 1} \right) e^{i[k L_1 + (L_1 + L_2 + L_III) + i]} \right] \bigg|_{\omega_1}^{\omega_2}.
\]

The number \( N_0(\Omega) \) of states within \( \Omega \) in the absence of the slabs, i.e., in vacuum, may be obtained by setting \( r_1 = r_2 = 0 \) in (68), so that the change in the number of states \( \Delta N(\Omega) \equiv N(\Omega) - N_0(\Omega) \) due to the presence of the plates is simply given by

\[
\Delta N(\Omega) = \frac{1}{2\pi i} \left[ \log \left( \frac{1 - r_1 r_2 e^{2ikL}}{1 - r_1^* r_2^* e^{-2ikL}} \right) \right] \bigg|_{\omega_1}^{\omega_2} = \frac{1}{\pi} \Im \log(1 - r_1 r_2 e^{2ikL}) \bigg|_{\omega_1}^{\omega_2}.
\]

As all quantities remaining in (69) are slowly varying we may simply divide by the small interval \( \omega_2 - \omega_1 \) to obtain a density of states

\[
\Delta \rho(\omega) = \frac{1}{\pi} \Im \frac{d}{d\omega} \log(1 - r_1 r_2 e^{2ikL}), \quad \Delta N(\Omega) = \Delta \rho(\bar{\omega})(\omega_2 - \omega_1).
\]

The case of evanescent waves may be analysed in a similar fashion as above, but substituting (59) and (55) instead of (58) and (54) in (60). Although the intermediate results differ, the final result is identical to (70) (work in progress), which is therefore valid within and beyond the vacuum light cone.

Notice that in the limit (37) both \( N(\Omega) \) and \( N_0(\Omega) \) diverge, but their difference remains finite. The approximation (63) may be made as precise as desired by increasing \( \sqrt{\varepsilon_1 L_1} \) and \( \sqrt{\varepsilon_1 L_III} \), and decreasing \( \omega_2 - \omega_1 \).

6. Thermodynamic properties

Expressions for the thermodynamic properties of the fictitious system may now be easily found by performing the corresponding sums over modes. By dividing the full frequency range \((0, \infty)\) into small subintervals \( \Omega \) and using (70) within each of them, we obtain

\[
\frac{1}{A} \sum_{\alpha, Q, \omega_n} \Delta f^\alpha(\omega_n, Q) = \frac{1}{2\pi^2} \Im \sum_{\alpha} \left( Q dQ \int d\omega f^{\alpha} \frac{\partial}{\partial \omega} \log(1 - r_1^* r_2^* e^{2ikL}), \right.
\]

where \( f^{\alpha}(Q, \omega) \) represents an arbitrary real function and we incorporated a sum over parallel wavevectors \( Q \) and over polarizations \( \alpha \). The \( \Delta \) in (71) denotes the difference between \( f^{\alpha} \) in
the presence of the material slabs of area $A$ and in their absence. For example, the ground state energy may be obtained by replacing $f^\alpha \rightarrow \hbar \omega / 2$, yielding

$$\Delta \frac{U}{A} = -\frac{\hbar}{4\pi^2} \text{Im} \sum_\alpha \int Q \, dQ \int d\omega \, \log(1 - r_1^\alpha r_2^\alpha e^{2i k L}) \quad (T = 0)$$

(72)

after an integration by parts. Similarly, the internal energy is obtained by replacing $f^\alpha \rightarrow \hbar \omega \coth(\beta \hbar \omega / 2)/2$,

$$\Delta \frac{U}{A} = -\frac{\hbar}{4\pi^2} \text{Im} \sum_\alpha \int Q \, dQ \int d\omega \, \log(1 - r_1^\alpha r_2^\alpha e^{2i k L}) \coth(\beta \hbar \omega / 2) - (\beta \hbar \omega / 2) \text{csch}^2(\beta \hbar \omega / 2),$$

(73)

the free energy through $f^\alpha \rightarrow -\log \left[ \text{csch}(\beta \hbar \omega / 2) / 2 \right] / \beta$,

$$\Delta \frac{F}{A} = -\frac{\hbar}{4\pi^2} \text{Im} \sum_\alpha \int Q \, dQ \int d\omega \, \log(1 - r_1^\alpha r_2^\alpha e^{2i k L}) \coth(\beta \hbar \omega / 2),$$

(74)

the entropy through $f^\alpha \rightarrow \partial \left[ \log \left[ \text{csch}(\beta \hbar \omega / 2) / 2 \right] / \beta \right] / \partial T$,

$$\Delta \frac{S}{A} = \frac{\hbar k_B \beta}{4\pi^2} \text{Im} \sum_\alpha \int Q \, dQ \int d\omega \, \log(1 - r_1^\alpha r_2^\alpha e^{2i k L}) \left[ (\beta \hbar \omega / 2) \text{csch}^2(\beta \hbar \omega / 2) \right],$$

(75)

etc, where $\coth$ denotes the hyperbolic cotangent and $\text{csch}$ denotes the hyperbolic cosecant. In the results above, we have omitted the $\omega = 0$ contributions that appear after the integration by parts and we remark that as in section 3, all the integrals ought to be done within an extended light cone which nevertheless becomes arbitrarily wide in the limit (36). As usual, the frequency integrals may be rotated into the imaginary axis and be converted into a discrete sum over Matsubara frequencies. The details will be discussed elsewhere (work in progress).

7. Conclusions

In this paper, we have presented several derivations of the Casimir force between two parallel plates relying on the fact that the properties of the radiation field within the cavity are completely determined in equilibrium by the width $L$ of the cavity and the optical coefficients of its boundaries. Thus, we calculated the force acting on a closed dissipationless fictitious system which has the same optical reflection amplitudes but for which there are no degrees of freedom beyond those of the electromagnetic field. For our first derivation, we considered a very large empty container in which we embedded two flat, parallel, infinitesimally thin scatterers characterized by appropriate, energy conserving scattering matrices, and we calculated the stress tensor between the plates. Although seemingly simple and applicable to a large class of materials, we found that our derivation did not account correctly for the coupling of the plates to fluctuating evanescent fields, namely it proved impossible to construct a scattering matrix with the desired properties for evanescent waves. Thus, we made a second derivation in which we added a fictitious dispersionless, dissipationless dielectric which allowed the evanescent field within the cavity to couple with propagating waves beneath the cavity, thus restoring energy conservation within an
extended light cone. In order to make quantum mechanical calculations for this system, it proved necessary to modify the norm in order to assure the Hermiticity of the wave operator.

The main result of these calculations is the confirmation that the Lifshitz formula is valid for a wide class of materials: semi-infinite or finite, local or spatially dispersive, homogeneous or layered, dissipative or dissipationless, etc, as long as it is written in terms the optical reflection amplitudes of the walls. The only assumption we made about the materials was non-chirality, translational invariance along their interfaces and rotational invariance around their normals.

We made a third derivation of the Casimir force from a calculation of the Casimir energy. Being a global property of the system, the Casimir energy of the fictitious system most certainly disagrees with the energy of the real system; the energy is not localized within the cavity. However, its changes under modifications of the separation $L$ ought to agree with the changes of the real energy and therefore it should lead to the correct force. To this end, we calculated the exact normal modes of the cavity, obtaining a dense quasi-continuous spectrum of real frequencies. The frequencies are real as each mode has a constant energy in thermodynamic equilibrium; detailed balance implies that energy lost through absorption is replenished through re-radiation. By subtracting the vacuum density of states from the cavity density of states, we obtained a finite difference given by an analytical expression which may be immediately employed for the calculation of any thermodynamic quantity.

Besides being quite simple and applicable to a very large class of materials, the schemes developed in this paper may be easily extended to other, non-planar geometries. This is the subject of ongoing research.

Acknowledgments

We acknowledge useful discussions with R G Barrera. This study was partially supported by DGAPA-UNAM under grant no. IN111306 and IN118605.

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