Asymptotically periodic points, bifurcations, and transition to chaos in fractional difference maps

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Abstract

In this paper we derive analytic expressions for coefficients of the equations that allow calculations of asymptotically periodic points in fractional difference maps. Numerical solution of these equations allows us to draw the bifurcation diagram for the fractional difference logistic map. Based on the numerically calculated bifurcation points, we make a conjecture that in fractional maps the value of the Feigenbaum constant $\delta$ is the same as in regular maps, $\delta = 4.669\ldots$.

Keywords: Fractional maps, Periodic points, Bifurcations, Feigenbaum constant

1. Introduction

It is known that continuous and discrete fractional systems may not have periodic solutions except fixed points (see, for example, \cite{1,2}). But the asymptotically periodic solutions do exist and the equations for finding these points in generalized fractional maps were derived in \cite{3,4}. These equations contain coefficients which are converging series. The numerical evaluation of these series, in the case fractional and fractional difference maps, can be reduced to the calculation of the Riemann $\zeta$-function. It is also known from the stability analysis of the discrete fractional systems (see \cite{5}) that in the case of fractional difference maps the corresponding series are summable (see, e.g., \cite{6,7,8,9}). In the following sections, after preliminaries in Section 2 we derive the analytic expressions for the coefficients (sums) of the equations defining periodic points.
in the case of fractional difference maps in Section 3. Then, in Section 4, we present the bifurcation diagram for the order \( \alpha = 0.5 \) fractional difference logistic map, which is based on the solutions of the equations defining the periodic points and compare this diagram with the bifurcation diagram obtained after 100000 iterations on a single trajectory. Based on the obtained numerically solutions for the bifurcation points, we draw a conjecture that the Feigenbaum number \( \delta = 4.669... \) has the same value for fractional difference maps.

2. Preliminaries

For \( 0 < \alpha < 1 \), the generalized universal \( \alpha \)-family of maps is defined as (see [3, 4]):

\[
x_n = x_0 - \sum_{k=0}^{n-1} G^0(x_k) U_{\alpha}(n - k),
\]

where \( G^0(x) = h^\alpha G_K(x)/\Gamma(\alpha) \), \( x_0 \) is the initial condition, \( h \) is the time step of the map, \( \alpha \) is the order of the map, \( G_K(x) \) is a nonlinear function depending on the parameter \( K \), \( U_{\alpha}(n) = 0 \) for \( n \leq 0 \), and \( U_{\alpha}(n) \in D^0(\mathbb{N}_1) \). The space \( D^i(\mathbb{N}_1) \) is defined as (see [4])

\[
D^i(\mathbb{N}_1) = \{ f : \sum_{k=1}^{\infty} |\Delta^i f(k)| > N, \ \forall N, \ N \in \mathbb{N}, \ \sum_{k=1}^{\infty} |\Delta^{i+1} f(k)| = C, \ C \in \mathbb{R}_+ \},
\]

where \( \Delta \) is a forward difference operator defined as

\[
\Delta f(n) = f(n+1) - f(n).
\]

In the case Caputo fractional difference maps, which are defined as solutions of the Caputo h-difference equation [10, 11, 12]

\[
(0\Delta^\alpha_{h,n} x)(t) = -G_K(x(t + (\alpha - 1)h)),
\]

where \( t \in (h\mathbb{N})_m \), with the initial conditions

\[
(0\Delta^\alpha_{h,n} x)(0) = c_k, \ k = 0, 1, \ldots, m - 1, \ m = \lceil \alpha \rceil,
\]

the kernel \( U_{\alpha}(n) \) is the falling factorial function:

\[
U_{\alpha}(n) = (n + \alpha - 2)^{\alpha-1}, \ U_{\alpha}(1) = (\alpha - 1)^{\alpha-1} = \Gamma(\alpha).
\]
The definition of the falling factorial \( t^{(\alpha)} \) is
\[
t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad t \neq -1, -2, -3, \ldots
\] (7)

The falling factorial is asymptotically a power function:
\[
\lim_{t \to \infty} \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)t^\alpha} = 1, \quad \alpha \in \mathbb{R}.
\] (8)

The \( h \)-falling factorial \( t^{(\alpha)}_h \) is defined as
\[
t^{(\alpha)}_h = h\alpha \frac{\Gamma(t+1)}{\Gamma(t_h+1-\alpha)} = h\alpha \left( \frac{t}{h} \right)^{(\alpha)} \quad t \neq -1, -2, -3, \ldots
\] (9)

The following equations define period-\( l \) points in generalized fractional maps [3]
\[
x_{lim,m+1} - x_{lim,m} = S_1 G^0(x_{lim,m}) + \sum_{j=1}^{m-1} S_{j+1} G^0(x_{lim,m-j})
\]
\[
+ \sum_{j=m}^{l-1} S_{j+1} G^0(x_{lim,m-j+l}), \quad 0 < m < l,
\] (10)
\[
\sum_{j=1}^{l} G^0(x_{lim,j}) = 0,
\] (11)

where
\[
S_{j+1} = \sum_{k=0}^{\infty} \left[ U_\alpha(lk+j) - U_\alpha(lk+j+1) \right], \quad 0 \leq j < l.
\] (12)

It is easy to see that
\[
\sum_{j=1}^{l} S_j = 0.
\] (13)

3. Sums \( S_p \) for \( l \)-cycles of fractional difference maps

We will call \( S_p \) sum for an \( l \)-cycle \( S_{p,l} \). The definition of \( S_{p,l} \) from [3] for fractional difference maps may be rewritten using the following chain of transformations:
\[
S_{p,l} = \sum_{k=0}^{\infty} \left[ (lk+p+\alpha-3)^{(\alpha-1)} - (lk+p+\alpha-2)^{(\alpha-1)} \right] = \sum_{k=0}^{\infty} \frac{\Gamma(lk+p+\alpha-2)}{\Gamma(lk+p-1)}
\]
\[
- \frac{\Gamma(lk+p+\alpha-1)}{\Gamma(lk+p)} = (1-\alpha) \sum_{k=0}^{\infty} \frac{\Gamma(lk+p+\alpha-2)}{\Gamma(lk+p)} = -\Gamma(\alpha) \sum_{k=0}^{\infty} \frac{\Gamma(lk+p+\alpha-2)}{\Gamma(\alpha-1)\Gamma(lk+p)}
\]
\[
= -\Gamma(\alpha) \sum_{k=0}^{\infty} \left( \frac{lk+p+\alpha-3}{lk+p-1} \right) = \Gamma(\alpha)(-1)^p \sum_{k=0}^{\infty} (-1)^k \left( \frac{1-\alpha}{lk+p-1} \right).
\] (14)
Using absolute convergence of series and the following identity (see [13])

$$
\sum_{k=0}^{\infty} \left( \frac{1}{t+ks} \right) = \frac{1}{s} \sum_{j=0}^{s-1} \omega^{-j} (1 + \omega^{j})^s,
$$

where \( \omega = e^{i\pi/s} \), for the even and odd periods we obtain

\[
S_{p,2n} = \Gamma(\alpha)(-1)^p \sum_{j=0}^{\infty} \left( \frac{1 - \alpha}{2n j + p - 1} \right) = \frac{\Gamma(\alpha)}{2n} (-1)^p \sum_{j=0}^{2n-1} e^{-i\pi j(2p-1)/n} (1 + e^{i\pi j/n})^{1-\alpha} \\
= \frac{\Gamma(\alpha)}{2n} (-1)^p \left[ 2^{1-\alpha} + \sum_{j=1}^{n-1} (e^{-i\pi j(2p-1)/n} e^{i\pi j(1-\alpha)/2n} (2\cos(\pi j/(2n)))^{1-\alpha} \\
+ e^{i\pi j(2p-1)/n} e^{-i\pi j(1-\alpha)/2n} (2\cos(\pi j/(2n)))^{1-\alpha}) \right] \\
= \frac{\Gamma(\alpha) 2^{-\alpha}}{n} (-1)^p \left[ 1 + 2 \sum_{j=1}^{n} (\cos(\pi j/(2n)))^{1-\alpha} \cos(\pi j(2p + \alpha - 3)/(2n)) \right], \quad (15)
\]

\[
S_{p,2n+1} = \Gamma(\alpha)(-1)^p \sum_{j=0}^{\infty} \left( \frac{1 - \alpha}{2n j + p - 1} \right) \\
= \Gamma(\alpha)(-1)^p \left\{ \sum_{j=0}^{\infty} \left( \frac{1 - \alpha}{2(2n+1) j + p - 1} \right) - \sum_{j=0}^{\infty} \left( \frac{1 - \alpha}{2(2n+1) j + p + 2n} \right) \right\} \\
= \frac{\Gamma(\alpha)}{2(2n+1)} (-1)^p \left\{ \sum_{j=0}^{4n+1} e^{-i\pi j(2p-1)/2n+1} (1 + e^{i\pi j/(2n+1)})^{1-\alpha} - \sum_{j=0}^{4n+1} e^{-i\pi j(2p+2n-3)/2n+1} (1 + e^{-i\pi j/(2n+1)})^{1-\alpha} \right\} \\
+ e^{i\pi j/(2n+1)} 1^{1-\alpha} + e^{-i\pi j/(2n+1)} (1 + e^{-i\pi j/(2n+1)})^{1-\alpha} \right\} \\
= \frac{\Gamma(\alpha)}{2n+1} (-1)^p \sum_{j=1}^{2n} \left( 2\cos \frac{\pi j}{2(2n+1)} \right)^{1-\alpha} \sin \frac{j\pi}{2} \times \left( e^{i\pi j(2p+2n-3)/2n+1} - e^{-i\pi j(2p+2n-3)/2n+1} \right) \\
\times \sin \frac{\pi j(2p+2n-2+\alpha)}{2(2n+1)} \\
= \frac{2^{2-\alpha} \Gamma(\alpha)}{2n+1} (-1)^p \sum_{j=0}^{n-1} \left( \cos \frac{\pi (2j+1)}{2(2n+1)} \right)^{1-\alpha} (-1)^j \times \sin \frac{\pi (2j+1)(2p+2n-2+\alpha)}{2(2n+1)} \quad (16)
\\]

\[
\times \sin \frac{\pi j(2p+2n-2+\alpha)}{2(2n+1)} \\
= \frac{2^{2-\alpha} \Gamma(\alpha)}{2n+1} (-1)^p \sum_{j=0}^{n-1} \left( \cos \frac{\pi (2j+1)}{2(2n+1)} \right)^{1-\alpha} (-1)^j \times \sin \frac{\pi (2j+1)(2p+2n-2+\alpha)}{2(2n+1)} \quad (17)
\]
In [13], Eq. (15) is proven for integer values of \( \gamma \in \mathbb{N}_0 \). The question is whether this identity can be extended to \( \gamma \in \mathbb{R} \). The answer is NO. The calculated values of \( S_{2,4} \) obtained when we used Eq. (16) for \( \alpha = 0.5, \alpha = 0.99, \) and \( \alpha = 0.999 \) are 1.309697, 0.257228, and 0.2507115. The corresponding values obtained using the exact expression (see Eq. (35) in [3]) are 1.029970, 0.2571808, and 0.2507111.

**4. Bifurcation diagrams for the fractional difference logistic map with \( 0 < \alpha < 1 \)**

In the generalized fractional logistic map

\[
G_K(x) = x - Kx(1-x).
\]

We solved Eqs. (10) and (11) numerically using Mathematica’s Newton’s Method (the exact solution) to draw bifurcation diagrams for various values of \( \alpha \in (0,1) \) (we assume \( h = 1 \)). Our results and Mathematica codes are available upon request. Here, as an example, we present only the results for the value of \( \alpha = 0.5 \), which is far from the integer values zero and one. Fig. 1 shows the results of our calculations for \( \alpha = 0.5 \). From Fig. 1b one may see that already for the period four (\( T = 4 \)) points there is a noticeable difference between the exact solutions and the results obtained after \( 10^5 \) iterations on a single trajectory. For \( T > 16 \) (Figs. 1b, e, f) the results obtained after \( 10^5 \) iterations on a single trajectory seem to be noticeably inaccurate. Multiple papers investigating various fractional difference maps contain bifurcation diagrams obtained by iterations on a single trajectory and the number of iterations in all these papers is much less than \( 10^5 \).

Table 1 shows the bifurcation points \( K(n) \) (the period \( 2^{n-1} \) – period \( 2^n \) bifurcations) obtained using Eqs. (10) and (11) and the corresponding values of the ratios \( \delta_1 = \Delta K(n-2)/\Delta K(n-1) = [K(n-1) - K(n-2)]/[K(n) - K(n-1)] \) for the order \( \alpha = 0.5 \) fractional difference logistic map. The corresponding values for the regular logistic map are taken from Wikipedia [14]. As in the case of the regular logistic map, the values of \( \delta \) oscillate around the Feigenbaum number but converge significantly slower. The slow convergence is expected because, in general, the convergence in fractional maps follows the power law while the convergence in regular maps is exponential.
Figure 1: A part of the bifurcation diagram for the Caputo fractional difference logistic map of the order \( \alpha = 0.5 \) from \( K = 2 \) (fixed point) to approximately \( K = 3.16 \) (T=128 periodic point). In figures a-c the steady line represents the solution of Eqs. (10) and (11) (the exact solution) and the dots represent numerical calculations on a single trajectory with \( x_0 = 0.3 \) after 10^5 iterations. Figure d represents the exact solution from \( T = 32 \) on the left to \( T = 128 \) on the right. Figures e and f represent the calculations on a single trajectory.

From the authors’ point of view, the data present sufficient evidence to make a conjecture that the Feigenbaum number \( \delta \) exists in fractional difference maps and has the same value.

5. Conclusion

In this paper we derived the analytic expressions for the coefficients of the equations that define periodic points in fractional difference maps of the orders \( 0 < \alpha < 1 \) (Eqs. (16) and (17)). We draw the bifurcation diagram for the fractional difference logistic map of the order \( \alpha = 0.5 \) which is based on the solution of Eqs. (10) and (11) and showed that bifurcation diagrams based on the iterations on a single trajectory contain significant errors. The data in Table I present sufficient evidence to make a conjecture that the Feigenbaum number \( \delta \) exists in fractional difference maps and has the same
Table 1: Approaching the Feigenbaum constant $\delta$ in the regular and fractional difference logistic maps. $K_1$ is the period $2^n - 1$ – period $2^n$ bifurcation point in the regular logistic map ($\alpha = 1$); $K_5$ is the same point in the case $\alpha = .5$; $\delta_1 = \Delta K_1(n - 2)/\Delta K_1(n - 1) = [K_1(n - 1) - K_1(n - 2)]/[K_1(n) - K_1(n - 1)]; \delta_5$ is the same value in the case $\alpha = .5; \delta_F = \delta = 4.6692016$.

| $n(T = 2^n)$ | $K_1(n)$  | $K_5(n)$  | $\Delta K_1(n - 2) \over \Delta K_1(n - 1)$ | $\Delta K_5(n - 2) \over \Delta K_5(n - 1)$ | $\delta_1 - \delta_F$ | $\delta_5 - \delta_F$ |
|--------------|------------|------------|---------------------------------|---------------------------------|-----------------|-----------------|
| 1            | 3          | 2.4142136  |                                 |                                 |                 |                 |
| 2            | 3.4494897  | 3.0315081  |                                 |                                 |                 |                 |
| 3            | 3.5440903  | 3.1288030  | 4.751447                        | 6.345                           | .082245         | 1.6754          |
| 4            | 3.5644073  | 3.1522546  | 4.656229                        | 4.149                           | -.012973        | -.5204          |
| 5            | 3.5687594  | 3.1569819  | 4.668321                        | 4.961                           | -.000881        | .2917           |
| 6            | 3.5696916  | 3.1580209  | 4.668633                        | 4.550                           | -.000569        | -.1193          |
| 7            | 3.5698913  | 3.1582410  | 4.668002                        | 4.721                           | -.001200        | .0514           |
| 8            | 3.5699340  | 3.1582884  | 4.676815                        | 4.6435                          | .007613         | -.0257          |
| 9            | 3.15829845 | 4.716      |                                 |                                 |                 |                 |
| 10           | 3.15830062 | 4.631      |                                 |                                 |                 |                 |

value as in regular maps. We believe that it is worthwhile to invest time and to apply efforts to prove this conjecture.

**Data availability**

Data will be made available on request.

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References

[1] J. Jagan Mohan: Quasi-periodic solutions of fractional nabla difference systems, Fractional Differential Calculus 7 (2017) 339–355.

[2] E. Kaslik, S. Sivasundaram, Nonexistence of periodic solutions in fractional order dynamical systems and a remarkable difference between integer and fractional order derivatives of periodic functions, Nonlinear Analysis, Real World Applications 13 (2012) 1489–1497.

[3] M. Edelman, Cycles in asymptotically stable and chaotic fractional maps, Nonlinear Dynamics 104 (2021) 2829–2841.

[4] M. Edelman, A.B. Helman, Asymptotic cycles in fractional maps of arbitrary positive orders, Fract. Calc. Appl. Anal. (2022) https://doi.org/10.1007/s13540-021-00008-w

[5] M. Edelman, Stability of Fixed Points in Generalized Fractional Maps of the Orders $0 < \alpha < 1$, https://arxiv.org/abs/2209.01719 (2022).

[6] R. Abu-Saris, Q. Al-Mdallal, On the asymptotic stability of linear system of fractional-order difference equations, Fract. Calc. Appl. Anal. 16 (2013) 613–629.

[7] J. Čermák, I. Győri, L. Nechvátal, On explicit stability conditions for a linear fractional difference system, Fract. Calc. and Appl. Anal. 18 (2015) 651–672.

[8] D. Mozyrska, M. Wyrwas, The z-transform method and delta type fractional difference operators, Discrete Dynamics in Nature and Society 2015 (2015) 852734.

[9] P.T. Anh, A. Babiarz, A. Czornik, M. Niezabitowski, S. Siegmund, Asymptotic properties of discrete linear fractional equations, Bulletin of the Polish Academy of Sciences, Technical Sciences, 67 (2019) 749–759.

[10] F. Chen, X. Luo, Y. Zhou, Existence Results for Nonlinear Fractional Difference Equation, Adv. Differ. Eq. 2011 (2011) 713201.
[11] G.-C. Wu, D. Baleanu, S.-D. Zeng, Discrete chaos in fractional sine and standard maps, Phys. Lett. A 378 (2014) 484–487.

[12] M. Edelman, Caputo standard $\alpha$-family of maps: Fractional difference vs. fractional, Chaos 24 (2014) 023137.

[13] A.T. Benjamin, B. Chen, K. Kindred, Sums of evenly spaced binomial coefficients, Maths. Mag. 83 (2010) 370–373.

[14] Feigenbaum constants, Wikipedia https://en.wikipedia.org/wiki/Feigenbaum_constants.