The Mabuchi geometry of low energy classes

Tamás Darvas

to the memory of Gabriela Kohr (1967-2020)

Abstract

Let $(X, \omega)$ be a Kähler manifold and $\psi : \mathbb{R} \to \mathbb{R}_+$ be a concave weight. We show that $\mathcal{H}_\omega$ admits a natural metric $d_\psi$ whose completion is the low energy space $\mathcal{E}_\psi$, introduced by Guedj-Zeriahi. As $d_\psi$ is not induced by a Finsler metric, the main difficulty is to show that the triangle inequality holds. We study properties of the resulting complete metric space $(\mathcal{E}_\psi, d_\psi)$.

1 Introduction

Let $(X, \omega)$ be a compact connected Kähler manifold. A basic problem in Kähler geometry is to find various canonical metrics among the Kähler metrics $\omega'$ that are in the same de Rham cohomology class as $\omega$ [Ca54, Ya78]. Due to Hodge theory, such $\omega'$ can be written as $\omega' = \omega + i\partial\bar{\partial}u$, where $u$ is a smooth function from the space of Kähler potentials:

$$\mathcal{H}_\omega := \{ v \in C^\infty(X) : \omega_v := \omega + i\partial\bar{\partial}v > 0 \}.$$

When studying weak notions of Kähler metrics, or degenerations of smooth ones, a natural space to consider is the space of $\omega$-plurisubharmonic ($\omega$-psh) functions $\text{PSH}(X, \omega)$. With slight abuse of precision, we say that $v : X \to [-\infty, \infty)$ is $\omega$-psh if it is usc, integrable and $\omega_v := \omega + i\partial\bar{\partial}v \geq 0$ in the sense of currents.

As pointed out in a series of works by Guedj-Zeriahi, and their collaborators [GZ07, BEGZ10, BBGZ13, EGZ09] the space of full mass potentials $\mathcal{E} = \{ v \in \text{PSH}(X, \omega) : \int_X \omega^n_v = \int_X \omega^n \}$ has a prominent role in the study of weak solutions to complex Monge-Ampère equations with measure theoretic right hand side. Here $\omega^n_v$ is the non-pluripolar complex Monge-Ampère measure of $v$, extending the interpretation of Bedford-Taylor and Cegrell from the local case [BT76, Ce98]. To study $\mathcal{E}$, it is feasible to consider various weights $\phi : \mathbb{R} \to \mathbb{R}_+$ and consider the subspaces with finite $\phi$-energy:

$$\mathcal{E}_\phi := \{ v \in \mathcal{E} \mid E_\phi(v) := \int_X \phi(v)\omega^n_v < \infty \}.$$

One important point is that $\mathcal{E}$ can be exhausted by a special class of finite energy subspaces [GZ07, Proposition 2.2]:

$$\mathcal{E} = \bigcup_{\psi \in W^-} \mathcal{E}_\psi. \quad (1)$$
Here $\mathcal{W}^-$ is the space of weights $\phi : [-\infty, \infty] \to [0, \infty]$ that are even and continuous on $\mathbb{R}$, in addition to being smooth, concave, and strictly increasing on $(0, \infty)$, normalized by $\psi(0) = 0$ and $\psi(\pm \infty) = \infty$. Following terminology of [GZ07], we will call elements of $\mathcal{W}^-$ *concave weights*, and the spaces $\mathcal{E}_\psi$, low energy classes (see Remark 2.1 for superficial differences in our discussion compared to [GZ07]).

As noticed in [BEGZ11, Section 2], the subspace $\mathcal{E}_1 = \{ v \in \mathcal{E} : \int_X |v| \omega^n_u < \infty \}$ has a complete metric topology. This was refined further in [Da15], where it was noticed that the more general high energy classes $\mathcal{E}_\chi$ are the metric completions of an appropriately defined Orlicz-Finsler metric structure on the smooth space $H_\omega$. Recall that high energy classes $\mathcal{E}_\chi$ are given by weights $\chi : \mathbb{R} \to \mathbb{R}^+$, that are even convex functions satisfying $\chi(0) = 0, \chi'(1) = 1$ and a growth estimate $t \chi'(t) \leq p \chi(t)$ (notation: $\chi \in \mathcal{W}_p^+$).

Moreover, in [Da15] it was also pointed out the the resulting metric spaces $(\mathcal{E}_\chi, d_\chi)$ admit geodesic segments connecting arbitrary points. This latter fact had a wide range of applications, including energy properness [CC1, CC2, BDL20, DR17], K-stability [BBJ21], convergence and more general high energy classes. However it is not difficult to see that this is not induced by the $\omega$-psh length metric of a Finsler structure, contrasting with [Da15]. Thus one has to work hard to prove the triangle inequality, this being the first main result of this paper:

**Theorem 1.1.** $(H_\omega, d_\psi)$ is a metric space.

Hoping for further analogies with the case of high energy classes [Da15], one might mistakenly expect that $(H_\omega, d_\psi)$ is at least a length space, and the weak geodesic $t \to u_t$ appearing in (2) is a metric $d_\psi$-geodesic connecting $u_0, u_1$. This is unfortunately not the case either. In fact, when $\psi(t) = |t|^{\alpha}, \alpha \in (0, 1)$, one can easily verify that the $d_\psi$-length of smooth curves inside $H_\omega$ is always zero. In addition, this also confirms that $d_\psi$ can not be induced by a Finsler metric on $H_\omega$.

In fact, the right analogy to follow here is the one coming from the case of toric Kähler manifolds $(X_T, \omega_T)$, and restricting $d_\psi$ to the torus invariant potentials $H^T_\omega$. As is well known,
the Legendre transform $\mathcal{L}$ transforms $\mathcal{H}_\omega^T$ bijectively into the space $\text{Conv}_\omega(P)$ of smooth convex functions on a Delzant polytope $P \subset \mathbb{R}^n$ with specific asymptotics near $\partial P$ (for details see [Gu14, Section 4] or [CGSZ19]). By the same calculations as in [Gu14, Proposition 4.3], we obtain that for $u_0, u_1 \in \mathcal{H}_\omega^T$ we have

$$d_\psi(u_0, u_1) = \int_P \psi(\mathcal{L}(u_0) - \mathcal{L}(u_1))d\mu,$$

with $\mu$ being the Lebesque measure on $P$. By Lemma 2.6 below we immediately see that in this case $d_\psi$ satisfies the triangle inequality trivially.

In addition, the $d_\psi$-completion of $\mathcal{H}_\omega^T$ will be $L^\psi(P) \cap \text{Conv}(P)$, the space of convex functions on $P$ that have finite $\psi$-integral. This space is exactly the Legendre dual of $\mathcal{E}_\psi^T$, the set of torus invariant potentials in $\mathcal{E}_\psi$ ([Gu14, Proposition 4.5]).

With the toric analogies in mind, our reader is perhaps less surprised by the statement of Theorem 1.1 above, and might also expect that the metric completion of $(\mathcal{H}_\omega, d_\psi)$ equals $\mathcal{E}_\psi$, even in the absence of toric symmetries. This is confirmed in our next main result.

**Theorem 1.2.** The metric $d_\psi$ extends to $\mathcal{E}_\psi$, making $(\mathcal{E}_\psi, d_\psi)$ a complete metric space, that is the metric completion of $(\mathcal{H}_\omega, d_\psi)$.

This result is analogous to [Da15, Theorem 2] that deals with the case of high energy classes. The similarities don’t stop here. Paralleling [Da15 Theorem 3], the $d_\psi$ metric is comparable to a concrete analytic expression:

**Theorem 1.3.** For any $u_0, u_1 \in \mathcal{E}_\psi$ we have

$$d_\psi(u_0, u_1) \leq \int_X \psi(u_0 - u_1)\omega_{u_0}^n + \int_X \psi(u_0 - u_1)\omega_{u_1}^n \leq 2^{2n+5}d_\psi(u_0, u_1). \tag{3}$$

This result implies that the expression $I_\psi(u_0, u) := \int_X \psi(u_0 - u_1)\omega_{u_0}^n + \int_X \psi(u_0 - u_1)\omega_{u_1}^n$ satisfies a quasi-triangle inequality, a result of independent interest. Previously this was obtained using analytic methods for the weights $\psi(t) = |t|^p$, $p \in (0, 1)$ in [GLZ19, Theorem 1.6].

As pointed about above, the absence of a background Finsler structure requires a new approach to the proof of Theorem 1.1. However once the triangle inequality is obtained, many pluripotential theoretic arguments can be used from [Da15], and this will be apparent in the proofs of Theorems 1.2 and 1.3.

Contrasting with the case of high energy classes explored in [Da15], our methods suggest that the metric $d_\psi$ is somehow positively curved (see Proposition 4.4 and Corollary 4.3). However it remains to be seen if such a notion can be defined for non-geodesic metric spaces, as it is the case here. Since weak geodesic segments are used to define $d_\psi$ in (2), it could be beneficial to understand what role these curves play from a metric/geometric point of view. In a different direction, it would be interesting to extend our results to more singular spaces. There has been a flurry of activity in this latter area recently, focusing mostly on the high energy case [DNL20, GT22, Xi19, Tr22, Tr23]. Lastly, we are curious if the quantization scheme of the high energy spaces from [DLR20] has an analogue in our low energy context. We hope to investigate these questions, as well as applications in future works.
Organization. In Section 2 we recall known results about finite energy classes, and obtain the second order variation of low energy weak quasi-norms. After some preliminary results on our candidate metric $d_\psi$ in Section 3, we prove the triangle inequality (and Theorem 1.1) in Section 4. Theorem 1.2 is proved in Section 5. Finally, Theorem 1.3 is proved in Section 6.

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2 Preliminaries

Most of our notation and terminology builds on that of [GZ07], [Da15] and the survey [Da19]. We refer our reader to these works for a detailed background. Below we only recall the basics, adapted to our specific context.

2.1 Finite energy classes

We recall here some basic facts about the class $E \subset \text{PSH}(X, \omega)$ and its subspaces. We refer to the original papers [GZ07] and [BEGZ10] and the recent book [GZ16] for a complete picture. For $v \in \text{PSH}(X, \omega)$, the canonical cutoffs $v_h$, $h \in \mathbb{R}$ are given by the formula $v_h := \max(-h, v)$.

By an application of the comparison principle of Bedford-Taylor theory, it follows that the Borel measures $\{v > -h\}(\omega + i\partial \bar{\partial} v_h)^n$ are increasing in $h$. As a result, one can make sense of $(\omega + i\partial \bar{\partial} v)^n$ as the limit of these increasing measures, even if $v$ is unbounded:

$$\omega^n_v := (\omega + i\partial \bar{\partial} v)^n = \lim_{h \to \infty} \{v > -h\}(\omega + i\partial \bar{\partial} v_h)^n.$$  \hfill (4)

With this definition, $\omega^n_v$ is called the non-pluripolar Monge-Ampère measure of $v$. It follows from (4) that

$$\int_X \omega^n_v \leq \int_X \omega^n =: V,$$

bringing us to the class of full mass potentials $E$. By definition, $v \in E$ if

$$\int_X \omega^n_v = \lim_{h \to \infty} \int_X \{v > -h\}(\omega + i\partial \bar{\partial} v_h)^n = V.$$  \hfill (5)

Suppose $\phi : [-\infty, \infty] \to [0, \infty]$ is a continuous even function, with $\phi(0) = 0$ and $\phi(\pm \infty) = \infty$. Such $\phi$ is referred to as a weight. The set of all weights is denoted by $\mathcal{W}$. By definition, for
If $v \in \mathcal{E}$ we have $v \in \mathcal{E}_\phi$ if

$$E_\phi(v) := \int_X \phi(v) \omega^n < \infty.$$ 

The two special classes of weights that are most interesting in the theory are:

$$\mathcal{W}^- = \{ \psi \in \mathcal{W} \mid \psi \text{ is concave, strictly increasing, and smooth on } (0, \infty) \},$$

$$\mathcal{W}_p^+ = \{ \chi \in \mathcal{W} \mid \chi \text{ is convex and } t\chi'(t) \leq p\chi(t), t \in \mathbb{R} \},$$

where $p \geq 1$. We note the sign difference between our convention for $\mathcal{W}^-$, $\mathcal{W}_p^+$, and the ones in [GZ07] and [Da17b, Section 2.3].

Of particular importance are the weights $\chi_p(t) = |t|^p$, $p > 0$, and the associated classes $\mathcal{E}_p := \mathcal{E}_{\chi_p}$. Note that $\chi_p \in \mathcal{W}_p^+$ for $p \geq 1$ and $\chi_p \in \mathcal{W}^-$ for $0 < p \leq 1$. The case $p = 1$ interpolates between convex and concave energy classes since

$$\mathcal{E}_\chi \subset \mathcal{E}_1 \subset \mathcal{E}_\psi,$$

for any $\chi \in \mathcal{W}_p^+$ and $\psi \in \mathcal{W}^-$. In this work we will be focusing on the concave weights $\mathcal{W}^-$. As mentioned in the introduction, the interest in them comes from the following fact [GZ07, Proposition 2.2]:

$$\mathcal{E} = \{ v \in \mathcal{E}_\psi \mid \psi \in \mathcal{W}^- \}. \quad (6)$$

**Remark 2.1.** To be precise, in [GZ07] the authors proved (6) for concave weights $\psi$ that are not necessarily smooth on $(0, \infty)$. However it is elementary to see that for a non-smooth concave weight $\psi$ we can find another concave weight $\tilde{\psi}$, smooth on $(0, \infty)$, such that $\mathcal{E}_\psi = \mathcal{E}_{\tilde{\psi}}$. Indeed, one can even make sure that $\psi - \tilde{\psi}$ is bounded. Because of this, very little is gained from working with more general concave weights. For sake of brevity we leave it to the interested reader to work out the details of our results in the case when the weights $\psi \in \mathcal{W}^-$ are not assumed to be smooth on $(0, \infty)$. This can be carried out using approximation, much in the same way as it is done in [Da15].

The following result is sometimes called the “fundamental estimate”:

**Proposition 2.2.** [GZ07, Lemma 2.3, Lemma 3.5] Let $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+$, $p \geq 1$. If $u, v \in \mathcal{E}_\phi$ with $u \leq v \leq 0$ then

$$E_\phi(v) \leq CE_\phi(u).$$

Here $C > 0$ depends only on $p$.

If $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+$, $p \geq 1$ then the $\phi$-energy has a very useful continuity property:

**Proposition 2.3.** [GZ07, Proposition 5.6] Let $\phi \in \mathcal{W}^- \cup \mathcal{W}_p^+$ and $\{u_j\}_{j \in \mathbb{N}} \subset \text{PSH}(X, \omega) \cap L^\infty$ is a sequence decreasing to $u \in \text{PSH}(X, \omega)$. If $\sup_j E_\phi(u_j) < \infty$ then $u \in \mathcal{E}_\phi$. Moreover we have

$$E_\phi(u) = \lim_{j \to \infty} E_\phi(u_j).$$

Using the canonical cutoffs, the last two results imply the very important “monotonicity property”:
Corollary 2.4. Let $\phi \in W^- \cup W^+_p$, $p \geq 1$. If $u \leq v$ and $u \in \mathcal{E}_\phi$ then $v \in \mathcal{E}_\phi$.

We note that the continuity property of the Monge-Ampère operator from Bedford-Taylor theory [BT76] is also preserved in this more general setting:

Proposition 2.5. [BEGZ10, Theorem 2.17] Suppose $\{v_k\}_{k \in \mathbb{N}} \subset \mathcal{E}(X, \omega)$ decreases (increases a.e.) to $v \in \mathcal{E}(X, \omega)$. Then $\omega^n_{v_k} \rightarrow \omega^n_v$ weakly.

A more general weak convergence result is proved in [Da19, Proposition 2.20], and the remark following it.

2.2 The $L^2$ metric and weak geodesics

As introduced by Mabuchi, and independently by Semmes and Donaldson, $\mathcal{H}_\omega$ can be endowed with a natural infinite dimensional $L^2$-type Riemannian metric:

$$\langle \alpha, \beta \rangle_u = \frac{1}{\int_X \omega^n} \int_X \alpha \beta \omega^n, \quad \alpha, \beta \in T_u \mathcal{H}_\omega = C^\infty(X).$$ (7)

One can compute the Levi-Civita connection $\nabla_i (\cdot)$ of this inner-product and the associated geodesic equation. For a thorough discussion of the $L^2$ Mabuchi–Semmes–Donaldson geometry, as well as its Levi-Civita connection, we refer to the surveys [Bl13, Section 4], [Da19, Section 3.1], as well as the original papers [Ma87, Se92, Do99, Ch00].

Unfortunately smooth geodesics connecting arbitrary $u_0, u_1 \in \mathcal{H}_\omega$ don’t exist, but a weak notion of geodesic was studied by Chen [Ch00]. His construction can be generalized to construct weak geodesic segments connecting points of $\text{PSH}(X, \omega) \cap L^\infty(X)$. Following Berndtsson, we recall how this argument works.

As before, let $S \subset \mathbb{C}$ be the strip $\{0 < \text{Re } s < 1\}$ and $\tilde{\omega}$ be the pullback of $\omega$ to the product $S \times X$. As argued in [Bern13a, Section 2.1], for $u_0, u_1 \in \text{PSH}(X, \omega) \cap L^\infty(X)$ the following Dirichlet problem has a unique solution:

$$u \in \text{PSH}(S \times X, \tilde{\omega}) \cap L^\infty(S \times X)$$

$$(\tilde{\omega} + i\partial \bar{\partial} u)^{n+1} = 0$$

$$u(t + ir, x) = u(t, x) \forall x \in X, t \in (0, 1), r \in \mathbb{R}$$

$$\lim_{t \rightarrow 0,1} u(t, x) = u_{0,1}(x), \forall x \in X.$$ (8)

Since the solution $u$ invariant in the imaginary direction, we denote it by $[0, 1] \ni t \rightarrow u_t \in \text{PSH}(X, \omega) \cap L^\infty$ and call it the weak geodesic joining $u_0$ and $u_1$.

In case $u_0, u_1 \in \mathcal{H}_\omega$ in [Ch00] it was proved that $\Delta u \in L^\infty(S \times X)$. Such a curve $[0, 1] \ni t \rightarrow u_t$ is called a $C^{1,1}$-geodesic.

A curve $[0, 1] \ni t \rightarrow v_t \in \text{PSH}(X, \omega)$ is called a subgeodesic if $v(s, x) := v_{\text{Re } s}(x) \in \text{PSH}(S \times X, \tilde{\omega})$. We recall that the solution $u$ of (8) is constructed as the upper envelope $u = \sup_{v \in S} v$, (9)

where $S$ is the following set of weak subgeodesics:

$$S = \{(0, 1) \ni t \rightarrow v_t \in \text{PSH}(X, \omega) \text{ is a subgeodesic with } \lim_{t \rightarrow 0,1} v_t \leq u_{0,1}\}.$$

For a thorough discussion of weak geodesics we refer to Section 3 in the survey [Da19].
2.3 First and second order variation of weak quasi-norms

To start, we observe that concave weights are subadditive:

**Lemma 2.6.** Let $\psi \in \mathcal{W}^-$. Then $\psi$ is subadditive, i.e., $\psi(a + b) \leq \psi(a) + \psi(b)$, $a, b \in \mathbb{R}$.

**Proof.** Since $\psi$ is increasing on $[0, \infty)$ we have that $\psi(a + b) = \psi(|a + b|) \leq \psi(|a| + |b|)$. On the other hand, since $\partial_+ \psi$ is decreasing on $(0, \infty)$, we can finish the proof in the following way:

\[
\psi(|a|) = \psi(|a|) - \psi(0) = \int_0^{|a|} \partial_+ \psi(t) dt \geq \int_{|b|}^{|a|+|b|} \partial_+ \psi(t) dt = \psi(|a| + |b|) - \psi(|b|).
\]

\[\square\]

To any concave weight $\psi \in \mathcal{W}^-$, and a finite measure space $(Y, \mu)$, one can associate the space $L^\psi_\mu$. These will be $\mu$-measurable functions $f$, such that $\int_Y \psi(f) d\mu < \infty$. For such functions $f$, we can associate the a weak quasi-norm that is only homogeneous, and typically does not satisfy (even weaker versions of) the triangle inequality:

\[\|f\|_{\psi, \mu} := \inf\{N > 0 \text{ s.t. } \int_Y \psi(f/N) d\mu \leq 1\}. \tag{10}\]

When $\psi \in \mathcal{W}_p^+$, the above quantity does define a bona-fide norm, and these are used in the Kähler geometry literature for approximation of $L^p$ Finsler metrics [Da15, DL20]. Despite the fact that in our case the triangle inequality fails, these weak quasi-norms will still be important in our discussion. To note, compared to [Da15, (13)], our definition in (10) is slightly different. There, to obtain the Hölder inequality [Da15, (14)], we needed a version of (10) that is invariant with respect to taking scalar multiples of $\psi$. Our definition here is intentionally not scale invariant, since we need exactly this property in the last step of the proof of Proposition 4.4 below.

Given $u \in \mathcal{H}_\omega$ and $f \in L^\psi_{\omega^0}$, we will denote $\|f\|_{\psi, \omega^0}$ simply as $\|f\|_{\psi, u}$. To start, we note the following elementary convergence result.

**Lemma 2.7.** Let $\mu$ and $\mu_k$ be finite Borel measures on $Y$. Let $\psi \in \mathcal{W}^-$, and $f_k, f$ be bounded functions that are $\mu_k$-measurable and $\mu$-measurable respectively. If $\int_Y \psi(cf_k) d\mu_k \to \int_Y \psi(cf) d\mu$ for all $c \in [0, \infty)$, then $\|f_k\|_{\psi, \mu_k} \to \|f\|_{\psi, \mu}$.

**Proof.** We can assume that $f \neq 0$ (a.e. with respect to $\mu$). In this case $[0, \infty) \ni c \to \int_Y \psi(cf) d\mu \in \mathbb{R}$ is strictly increasing and continuous (the latter by the dominated convergence theorem). As a result, for any $\varepsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that:

\[
1 + \frac{\varepsilon}{2} \leq \int_Y \psi\left(\frac{f}{\|f\|_{\psi, \mu} + \delta_1}\right) d\mu \leq 1 + \varepsilon \quad \text{and} \quad 1 - \varepsilon \leq \int_Y \psi\left(\frac{f}{\|f\|_{\psi, \mu} + \delta_2}\right) d\mu \leq 1 - \frac{\varepsilon}{2}.
\]

In addition, $\delta_1, \delta_2 \to 0$ as $\varepsilon \to 0$.

By our assumption we have that $\int_Y \psi(f_k/(\|f\|_{\psi, \mu} - \delta_1)) d\mu_k \to \int_Y \psi(f/(\|f\|_{\psi, \mu} - \delta_1)) d\mu$ and $\int_Y \psi(f_k/(\|f\|_{\psi, \mu} + \delta_2)) d\mu_k \to \int_Y \psi(f/(\|f\|_{\psi, \mu} + \delta_2)) d\mu$. By definition of our weak quasi-norm we conclude that $\|f\|_{\psi, \mu} - \delta_1 \leq \lim inf_k \|f_k\|_{\psi, \mu_k} \leq \lim sup_k \|f_k\|_{\psi, \mu_k} \leq \|f\|_{\psi, \mu} + \delta_2$, finishing the proof. Letting $\varepsilon \to 0$, the result follows. \[\square\]
Lemma 2.8. Let $f_k, f$ be continuous functions on a compact topological space $Y$ and $f_k \to f$ uniformly. Let $\mu$ and $\mu_k$ be Borel measures on $Y$ with finite mass such that $\mu_k \to \mu$ weakly. Then $\int_Y \psi(cf_k)d\mu_k \to \int_Y \psi(cf)d\mu$ for any $c \in [0, \infty)$ and $\|f_k\|_{\psi, \mu_k} \to \|f\|_{\psi, \mu}$.

Proof. For any $c > 0$ we have that $\psi(cf_k) \to \psi(cf)$ uniformly. Since we are dealing with finite measure spaces and $Y$ is compact, it follows that $\int_Y \psi(cf_k)d\mu_k \to \int_Y \psi(cf)d\mu$. The last claim follows from Lemma 2.7. \qed

In case we have smooth maps $[0, 1] \ni t \mapsto v_t \in \mathcal{H}_\omega$, $[0, 1] \ni t \mapsto f_t \in C^\infty$ with $f_t > 0$, it is easy to see that $t \mapsto \|f_t\|_{\psi, v_t}$ is smooth. Indeed, since $\psi|_{(0, \infty)}$ is smooth, the arguments of [Da15, Proposition 3.1] carry over without change, and we have the following precise formula for the first derivative:

Proposition 2.9. Suppose $\psi \in \mathcal{W}^-$. Given a smooth curve $(0, 1) \ni t \mapsto u_t \in \mathcal{H}$, i.e. $u(t, x) := u_t(x) \in C^\infty((0, 1) \times X)$, and a positive smooth vector field $(0, 1) \ni t \mapsto f_t \in C^\infty(X)$ along this curve, the following formula holds:

$$\partial_t \|f_t\|_{\psi, u_t} = \frac{\int_X \psi'(\|f_t\|_{\psi, u_t}) \nabla u_t f_t \omega_{u_t}^n}{\int_X \psi'(\frac{f_t}{\|f_t\|_{\psi, u_t}}) \frac{f_t}{\|f_t\|_{\psi, u_t}} \omega_{u_t}^n},$$

(11)

where $\nabla$ is the covariant derivative of the $L^2$ Mabuchi–Semmes–Donaldson metric [7].

Recall that Chen’s $\varepsilon$-geodesics are smooth curves $t \mapsto u_t$ that solve the following elliptic equation [Ch00]:

$$\nabla_{\partial_t u} \partial_t \omega_u^n = \varepsilon \omega_u^n.$$  

(12)

As pointed out in [Ch00], the advantage of $\varepsilon$-geodesics is that they are smooth, and approximate uniformly the weak $C^{1,1}$-geodesic connecting $u_0, u_1 \in \mathcal{H}_\omega$ that solves (8).

For this paper we need to compute the second order variation of the length of very special vector fields across $\varepsilon$-geodesics (c.f. [Le20, Section 4] and [CC3, Section 5]):

Proposition 2.10. Suppose $\psi \in \mathcal{W}^-$. Let $[0, 1]^2 \ni (s, t) \mapsto u(s, t) \in \mathcal{H}_\omega$ be smooth and an $\varepsilon$-geodesic in each $t$-direction, such that $\partial_s u > 0$. The following formula holds:

$$\partial^2_{t} \|\partial_s u\|_{\psi, u} =$$

$$= \int_X \psi''(\eta) \left( \|\partial_s u\|_{\psi, u} (\nabla_{\partial_s u} \eta)^2 + \frac{1}{\|\partial_s u\|_{\psi, u}} (\partial_s u, \partial_t u)^2 \omega_u^\varepsilon \right) + \frac{\varepsilon}{\|\partial_s u\|_{\psi, u}} (\nabla u, \nabla u, \nabla u \partial_s u, \nabla u \partial_s u \omega_u^n) \omega_u^n,$$

(13)

where $\{\cdot, \cdot\}_{\omega_u}$ is the Poisson bracket of $\omega_u$, and we introduced $\eta := \frac{\partial_u}{\|\partial_u\|_{\psi, u}}$, for simplicity. In particular, for fixed $s$, the map $t \mapsto \|\partial_s u\|_{\psi, u}$ is concave.

Proof. The proof is a careful calculation of the derivative of the right hand side of (13) in case $s \in [0, 1]$ is fixed and $f_t := \partial_s u(s, t), \ u_t := u(s, t)$.

We start with some side calculations, and put things together in the end. Since $\int_X \psi(\eta) \omega_u^n = 1$, the product rule for the Levi-Civita connection gives:

$$\int_X \psi'(\eta) \nabla_{\partial_s u} \eta \omega_u^n = \partial_t \int_X \psi(\eta) \omega_u^n = 0.$$  

(14)
Using this identity and the product rule of the Levi-Civita connection again, we can differentiate the denominator of the right hand side of (11) and obtain:

$$\partial_t \int_X \psi' (\eta) \eta \omega^u = \int_X \psi'' (\eta) \eta \nabla_{\partial_t u} (\eta) \omega^u. \quad (15)$$

Next we turn to the numerator of the right hand side of (11). The product rule again gives:

$$\partial_t \int_X \psi' (\eta) \nabla_{\partial_t u} \partial_s u \omega^u = \int_X \psi'' (\eta) \nabla_{\partial_t u} (\eta) \nabla_{\partial_t u} \partial_s u \omega^u + \int_X \psi' (\eta) \nabla_{\partial_t u} \nabla_{\partial_t u} \partial_s u$$

$$= \int_X \psi'' (\eta) \nabla_{\partial_t u} (\eta) \nabla_{\partial_t u} \partial_s u \omega^u + \int_X \psi' (\eta) \nabla_{\partial_t u} \nabla_{\partial_t u} \partial_t u \omega^u. \quad (16)$$

For the last term on the right of (16) we make the following side computation:

$$\int_X \psi' (\eta) \nabla_{\partial_t u} \nabla_{\partial_t u} \partial_t u \omega^u = \int_X \psi'' (\eta) R (\partial_t u, \partial_t u) \partial_t u \omega^u + \int_X \psi' (\eta) \nabla_{\partial_t u} \nabla_{\partial_t u} \partial_t u \omega^u$$

$$= \frac{1}{\| \partial_t u \|_{\psi, u}} \int_X \psi'' (\eta) \{ \partial_t u, \partial_t u \}^2 \omega^u + \varepsilon \int_X \psi' (\eta) \nabla_{\partial_t} (\omega^u) \omega^u$$

$$= \frac{1}{\| \partial_t u \|_{\psi, u}} \int_X \psi'' (\eta) \{ \partial_t u, \partial_t u \}^2 \omega^u -$$

$$\varepsilon \int_X \psi' (\eta) \left( \Delta_{\omega} \partial_t u \cdot \omega^u + \langle \nabla_{\omega} \left( \omega^u \right) \nabla_{\omega} \partial_t u \rangle \omega^u \right)$$

$$= \frac{1}{\| \partial_t u \|_{\psi, u}} \int_X \psi'' (\eta) \{ \partial_t u, \partial_t u \}^2 \omega^u + \varepsilon \int_X \psi'' (\eta) \langle \nabla_{\omega_t} \partial_t u, \nabla_{\omega_t} \partial_t u \rangle \omega^u, \quad (17)$$

where in the second we used the precise formula curvature $R (\cdot, \cdot) (\cdot)$ (computed in [CC3] (5.13)) or [B13, Theorem 5]) and (12), in the third line we used the formula for the Levi-Civita connection, and in the last line we used integration by parts.

For the first term of (16) we use that $\partial_t u = \| \partial_t u \|_{\psi, u}$, and the product rule for the Levi-Civita connection:

$$\int_X \psi'' (\eta) \nabla_{\partial_t u} (\eta) \nabla_{\partial_t u} \partial_t u \omega^u = \| \partial_t u \|_{\psi, u} \int_X \psi'' (\eta) \left( \nabla_{\partial_t u} (\eta) \right)^2 \omega^u + \partial_t \| \partial_t u \|_{\psi, u} \int_X \psi'' (\eta) \eta \nabla_{\partial_t u} (\eta) \omega^u$$

Substituting (17) and (18) into (16) we arrive at

$$\partial_t \int_X \psi' (\eta) \nabla_{\partial_t u} \partial_t u \omega^u = \int_X \psi'' (\eta) \left( \partial_t \| \partial_t u \|_{\psi, u} \eta \nabla_{\partial_t u} (\eta) + \| \partial_t u \|_{\psi, u} \left( \nabla_{\partial_t u} \eta \right)^2 \right) +$$

$$\frac{1}{\| \partial_t u \|_{\psi, u}} \{ \partial_t u, \partial_t u \}^2 + \frac{\varepsilon}{\| \partial_t u \|_{\psi, u}} \langle \nabla_{\omega_t} \partial_t u, \nabla_{\omega_t} \partial_t u \rangle \omega^u \quad (19)$$

Differentiating (11), we bring the above calculations together:
\begin{align}
\partial^2_t \| \partial_x u \|_{\psi,u} &= \frac{\partial_t \int_X \psi'(\eta) \nabla \partial_u \partial_x u \omega_u^n}{\int_X \psi'(\eta) \eta \omega_u^n} - \frac{\partial_t \| \partial_x u \|_{\psi,u} \partial_t \int_X \psi'(\eta) \eta \omega_u^n}{\int_X \psi'(\eta) \eta \omega_u^n} \\
&= \frac{\partial_t \int_X \psi'(\eta) \nabla \partial_u \partial_x u \omega_u^n}{\int_X \psi'(\eta) \eta \omega_u^n} - \frac{\partial_t \| \partial_x u \|_{\psi,u} \int_X \psi''(\eta) \nabla \partial_u (\eta \omega_u^n)}{\int_X \psi'(\eta) \eta \omega_u^n},
\end{align}

where in the last line we used (15). Next, in the numerator of the first fraction we now substitute (19) and notice that the last term on the right hand side of (20) will cancel with the first term on the right hand side of (19), ultimately yielding (13). \hfill \Box

### 3 The candidate metric \(d_\psi\)

We start with a preliminary discussion of our candidate metric \(d_\psi\), defined in (2). By He’s theorem [He15, Theorem 1.1], we know that for \(u_0, u_1 \in \mathcal{H}_\Delta := \text{PSH}(X, \omega) \cap \{\Delta \omega v \in L^\infty\} = \text{PSH}(X, \omega) \cap C^{1,1}\), we also have \(u_t \in \mathcal{H}_\Delta\), \(t \in [0,1]\), where \(t \to u_t\) is the weak geodesic connecting \(u_0, u_1\).

It is not yet known if \(t \to u_t\) is \(C^1\) in the \(t\)-direction when \(u_0, u_1 \in \mathcal{H}^{1,1}_\omega\). However, since \(t \to u_t\) is \(t\)-convex, it makes sense to define \(\dot{u}_0\) as the right derivative at \(t = 0\) and \(\dot{u}_1\) as the left derivative at \(t = 1\). As a result, it is possible to extend the definition (2) to potentials with bounded Laplacian:

\[
d_\psi(u_0, u_1) := \int_X \psi(\dot{u}_0) \omega_{u_0}^n.
\]

This will be helpful since many operations on potentials are stable in the class \(\mathcal{H}_\Delta\), and are not stable in \(\mathcal{H}_\omega\). For example, by [DR16, Theorem 2.5], we know that \(u, v \in \mathcal{H}_\Delta\) implies \(P(u, v) \in \mathcal{H}_\Delta\). The same property is not true for potentials of \(\mathcal{H}_\omega\).

In addition, we also introduce

\[
\dot{d}_\psi(u_0, u_1) = \| \dot{u}_0 \|_{\psi,u_0},
\]

where the term on the right hand side is the weak quasi-norm of \(\dot{u}_0\) with respect to the weight \(\psi\) and the measure \(\omega_{u_0}^n\) (11).

In case of \(u_0, u_1 \in \mathcal{H}_\omega\), by [Da15, Lemma 4.10] (slightly extending [Bern13a, Proposition 2.2]) we obtain that \(t \to \int_X \psi(c \dot{u}_t) \omega_u^x\) is constant for any \(c \in \mathbb{R}_+\). By definition of the weak quasi-norms, this immediately gives that \(t \to \| \dot{u}_t \|_{\psi,u_t}\) is constant as well, hence in this case:

\[
\dot{d}_\psi(u_0, u_1) = \| \dot{u}_t \|_{\psi,u_t}, \quad \text{for any } t \in [0,1].
\]

\[
d_\psi(u_0, u_1) = \int_X \psi(\dot{u}_t) \omega_u^x, \quad \text{for any } t \in [0,1].
\]

Though we will not need it, by [Da15, Lemma 4.10] the same formulas hold in case \(u_0, u_1 \in \mathcal{H}_\Delta\) as well. However in this case one has to clarify what \(\dot{u}_t\) means for \(t \in (0,1)\). As follows from [Da15, Lemma 4.10] (and its proof), \(\partial^2_r u_t = \partial^2_t u_t\) a.e with respect to \(\omega_u^x\). As a result, \(\dot{u}_t\) is a.e. well defined with respect to \(\omega_u^x\), allowing to make sense of the right hand side of (23) and (24) in this more general situation as well.

First we prove an approximation result for the above introduced notions:
Proof. First we show that $d_\psi(u_0^k, u_1^k) \to d_\psi(u_0, u_1)$. Let $[0, 1] \ni t \to u_t, u_t^k \in H^\Delta_\omega$ be the weak geodesic joining $u_0, u_1$ and $u_0^k, u_1^k$ respectively. We first claim that the push-forward measures $|\hat{u}_0^k|_*\omega^n_{u_0^k}$ weakly converge to $|\hat{u}_0|_*\omega^n_{u_0}$.

Assuming the claim, since $\hat{u}_0^k, \hat{u}_0$ are uniformly bounded [Da17a, Theorem 1], we can apply this to $\psi$ to arrive at the conclusion:

$$d_\psi(u_0^k, u_1^k) = \int_X \psi(\hat{u}_0^k)\omega^n_{u_0^k} = \int_X \psi(|\hat{u}_0^k|)\omega^n_{u_0^k} \to \int_X \psi(|\hat{u}_0|)\omega^n_{u_0} = d_\psi(u_0, u_1).$$

Now we prove the claim. From [Da15, Theorem 3] and the triangle inequality for $d_\rho$ we know that $d_\rho(u_0^k, u_1^k) \to d_\rho(u_0, u_1)$ for all $p \geq 1$. By [Da15, Lemma 4.11] this is equivalent with $\int_X |\hat{u}_0^k|_p\omega^n_{u_0^k} \to \int_X |\hat{u}_0|_p\omega^n_{u_0}$.

Since the global masses of the pushforward measures $|\hat{u}_0^k|_*\omega^n_{u_0^k}$, $|\hat{u}_0|_*\omega^n_{u_0}$ are finite, and $\hat{u}_0^k, \hat{u}_0$ are uniformly bounded, the Stone-Weierstrass theorem implies that $\int_X \alpha(|\hat{u}_0^k|)\omega^n_{u_0^k} \to \int_X \alpha(|\hat{u}_0|)\omega^n_{u_0}$ for any $\alpha \in C(\mathbb{R})$. This is equivalent with $|\hat{u}_0^k|_*\omega^n_{u_0^k} \to |\hat{u}_0|_*\omega^n_{u_0}$, as desired.

We can repeat the above for $\psi(ct)$ instead of $\psi(t)$ for any $c \in [0, \infty)$, and conclude that $d_\psi(u_0^k, u_1^k) \to d_\psi(u_0, u_1)$ via Lemma 2.7.

Next, we point out that an analogue of the Pythagorean identity holds for $d_\psi$:

**Lemma 3.2.** Let $\psi \in W^-$ and $u, v \in H^\Delta_\omega$. Then for $P(u, v) \in H^\Delta_\omega$ we have that

$$d_\psi(u, v) = d_\psi(u, P(u, v)) + d_\psi(v, P(u, v)).$$

**Proof.** This is a consequence of [Da15, Proposition 4.13] for $f := \psi$.

Next we point out that the operator $u \to P(u, w)$ is $d_\psi$-shrinking:

**Proposition 3.3.** Let $\psi \in W_-$ and $u, v, w \in H^\Delta_\omega$. Then we have

$$d_\psi(P(u, w), P(v, w)) \leq d_\psi(u, v).$$

**Proof.** The proof is exactly the same as that of [Da17b, Proposition 8.2], where one replaces the convex weight $|t|^2$ with our weight $\psi \in W^-$.

Using the argument of [Da15, Lemma 4.2] we note the following lemma:

**Lemma 3.4.** Let $\alpha, \beta, \gamma \in H^\Delta_\omega$ such that $\alpha \geq \beta \geq \gamma$. Then $d_\psi(\alpha, \beta) \leq d_\psi(\alpha, \gamma)$ and $\hat{d}_\psi(\alpha, \beta) \leq \hat{d}_\psi(\alpha, \gamma)$. Analogously, $d_\psi(\gamma, \beta) \leq d_\psi(\gamma, \alpha)$ and $\hat{d}_\psi(\gamma, \beta) \leq \hat{d}_\psi(\gamma, \alpha)$.

**Proof.** Let $[0, 1] \ni t \to u_t, v_t \in H^\Delta_\omega$ be the weak geodesics connecting $\alpha$, $\beta$ and $\alpha$, $\gamma$ respectively. We notice that they are both decreasing, satisfy $u_t \geq v_t$ by the comparison principle, and $u_0 = v_0 = \alpha$. From this it follows that $0 \geq \hat{u}_0 \geq \hat{v}_0$. Using this, (21) and (22) yield that $d_\psi(\alpha, \beta) \leq d_\psi(\alpha, \gamma)$ and $\hat{d}_\psi(\alpha, \beta) \leq \hat{d}_\psi(\alpha, \gamma)$. The last sentence is proved analogously, using two weak geodesics meeting at $\gamma$. 


4 The triangle inequality

First we obtain the triangle inequality for $d_\psi$ in a special case (Proposition 4.2), and then derive the general version from this using the Pythagorean identity for $d_\psi$.

We start with the analogue of [CC3 Lemma 5.2] in our setting, that will only hold in the particular case of increasing smooth curves (c.f. [Le20 Theorem 1.2]).

**Proposition 4.1.** Let $\varepsilon > 0$ and $[0, 1] \ni s \rightarrow u_{0,s}, u_{1,s} \in \mathcal{H}_\omega$ be smooth curves satisfying $\partial_s u_{0,s} > 0, \partial_s u_{1,s} > 0$. For fixed $s$, let $[0, 1] \ni t \rightarrow u_{t,s}^\varepsilon \in \mathcal{H}_\omega$ be the $\varepsilon$-geodesic connecting $u_{0,s}, u_{1,s}$. Then $t \rightarrow \int_0^1 \|\partial_s u_{t,s}^\varepsilon\|_\psi,\omega ds$ is concave.

**Proof.** By assumption, $\partial_s u_{0,s}, \partial_s u_{1,s} \geq \delta > 0$ for some constant $\delta$. By the proof of [Da19 Corollary 3.4] we get that $\partial_s u_{t,s}^\varepsilon \geq \delta > 0$ for any $t, s \in [0, 1]$ and $\varepsilon > 0$. In particular, $s \rightarrow \|\partial_s u\|_\psi,\omega$ is smooth and the results of Section 2.3 are applicable. In particular, since $\psi|_{[0,\infty)}$ is concave, Proposition 2.10 gives:

$$\frac{d^2}{dt^2} \int_0^1 \|\partial_s u_{t,s}^\varepsilon\|_\psi,\omega ds = \int_0^1 \frac{d^2}{dt^2}\|\partial_s u_{t,s}^\varepsilon\|_\psi,\omega ds \leq 0.$$  

This is equivalent to concavity of $t \rightarrow \int_0^1 \|\partial_s u_{t,s}^\varepsilon\|_\psi,\omega ds$. \hfill $\square$

**Proposition 4.2.** Let $u_0, u_1 \in \mathcal{H}_\omega$ with $u_0 < u_1$. Let $[0, 1] \ni s \rightarrow u_s \in \mathcal{H}_\omega$ be the (increasing) weak geodesic joining $u_0, u_1$. Then $\dot{\psi}(u_0, u_1) = \int_0^1 \|\dot{u}_s\|_\psi,\omega ds \geq \int_0^1 \|\dot{\zeta}_s\|_\psi,\zeta ds$, where $t \rightarrow \zeta_t$ is any smooth increasing curve ($\dot{\zeta}_s > 0$) joining $\zeta_0 := u_0$ and $\zeta_1 := u_1$.

The following argument is due to L. Lempert.

**Proof.** Let $\delta > 0$ such that $u_1 - u_0 > \delta$ and $\dot{\zeta}_s > \delta$ for all $s \in [0, 1]$. From (9) we obtain that $u_t \geq u_0 + \delta t$. Since $t \rightarrow u_t$ is $t$-convex, we obtain that $\dot{u}_t \geq u_0 \geq \delta$.

By (23) we know that $s \rightarrow \|\dot{u}_s\|_\psi,\omega$ is constant equal to $c > 0$. Since $\psi$ is concave and smooth on $(\delta/2c, \infty)$, it admits a concave extension $\tilde{\psi}$ to $(-\infty, \infty)$ such that $\tilde{\psi}|_{(\delta/2c, \infty)} = \psi|_{(\delta/2c, \infty)}$. Such extension of course is non-unique.

As $\dot{\zeta}_s > 0$, Proposition 2.9 implies that $s \rightarrow \|\dot{\zeta}_s\|_\psi,\zeta_s$ is smooth. Since weak quasi-norms are homogeneous, it is possible to reparametrize $[0, 1] \ni s \rightarrow \zeta_s \in \mathcal{H}_\omega$ to a smooth curve $[0, 1] \ni s \rightarrow \tilde{\zeta}_s \in \mathcal{H}_\omega$ such that $s \rightarrow \|\dot{\tilde{\zeta}}_s\|_\psi,\tilde{\zeta}_s$ is constant and the $\psi$-arclength does not change:

$$\int_0^1 \|\dot{\zeta}_s\|_\psi,\zeta_s ds = \int_0^1 \|\dot{\tilde{\zeta}}_s\|_\psi,\tilde{\zeta}_s ds = \|\dot{\tilde{\zeta}}_l\|_\psi,\tilde{\zeta}_l, \; l \in [0, 1].$$

Since $t \rightarrow -\tilde{\psi}(t/c)$ is convex, [Le20 Theorem 1.1] implies that

$$1 = \int_0^1 \int_X \tilde{\psi}\left(\frac{\dot{u}_s}{c}\right)\omega_{u_s}^n ds = \int_0^1 \int_X \tilde{\psi}\left(\frac{\dot{\bar{u}}_s}{c}\right)\omega_{\bar{u}_s}^n ds \geq \int_0^1 \int_X \tilde{\psi}\left(\frac{\dot{\zeta}_s}{c}\right)\omega_{\zeta_s}^n ds = \int_0^1 \int_X \tilde{\psi}\left(\frac{\dot{\tilde{\zeta}}_s}{c}\right)\omega_{\tilde{\zeta}_s}^n ds.$$  

In particular, by the mean value theorem, we obtain that $\int_X \tilde{\psi}\left(\frac{\dot{\tilde{\zeta}}_s}{c}\right)\omega_{\tilde{\zeta}_s}^n \leq 1$ for some $t \in [0, 1]$. By the definition of the weak quasi-norm, we get that $\|\dot{\tilde{\zeta}}_t\|_\psi,\tilde{\zeta}_t \leq c$. But since $s \rightarrow \|\dot{\tilde{\zeta}}_s\|_\psi,\tilde{\zeta}_s$ is
constant, we actually get that \( \|\hat{\zeta}_s\|_{\psi,\zeta_s} \leq c = \|\hat{u}_s\|_{\psi,\alpha_s} \) for all \( s \in [0,1] \). Integrating this inequality on \([0,1]\) yields the desired estimate.

As a corollary of the above two results, we obtain the following:

**Corollary 4.3.** Suppose we are given \( \alpha, \beta, \gamma \in \mathcal{H}_\omega^\Delta \) such that \( \alpha \geq \beta \geq \gamma \). Let \([0,1] \ni t \rightarrow \alpha_t, \gamma_t \in \mathcal{H}_\omega^\Delta \) be the weak geodesic joining \( \alpha_0 := \beta, \alpha_1 := \alpha \) and \( \gamma_0 := \beta, \gamma_1 := \gamma \) respectively. Then the function \( t \rightarrow \hat{d}_\psi(\alpha_t, \gamma_t) \) is concave.

**Proof.** Using Proposition 3.1 and [De94], we can assume that \( \alpha, \beta, \gamma \in \mathcal{H}_\omega \) and \( \alpha > \beta > \gamma \).

As in the first step of the proof of Proposition 4.2 there exists \( \delta > 0 \) such that \( \alpha_t > \delta \) and \( \gamma_t < -\delta \) for all \( t \in [0,1] \). Let \([0,1] \ni t \rightarrow \alpha_t, \gamma_t \in \mathcal{H}_\omega \) be the \( \varepsilon \)-geodesic joining \( \alpha_0 := \beta, \alpha_1 := \alpha \) and \( \gamma_0 := \beta, \gamma_1 := \gamma \) respectively. As \( \varepsilon \)-geodesics converge to weak geodesics in the \( C^{1,\alpha} \)-topology, for small enough \( \varepsilon \), we also have \( \alpha_t > \delta \) and \( \gamma_t < -\delta \). In particular, \( t \rightarrow \alpha_t \) is strictly increasing and \( t \rightarrow \gamma_t \) is strictly decreasing.

Let \( t, t' \in (0,1] \). Now let \( \varepsilon' > 0 \) be small enough so that both \( \varepsilon' \)-geodesics \( [0,1] \ni s \rightarrow v_s^{\varepsilon', t'}, u_s^{\varepsilon', t'} \in \mathcal{H}_\omega \) joining \( \gamma_0^{\varepsilon', t'} := \gamma_t \) and \( v_1^{\varepsilon', t'} := \alpha_t \), respectively \( v_0^{\varepsilon', t'} := \gamma_t \) and \( v_1^{\varepsilon', t'} := \alpha_t \), are strictly increasing (i.e. \( \partial_s v_s^{\varepsilon', t'}, \partial_s u_s^{\varepsilon', t'} > 0 \)).

For \( s \in [0,1] \) fixed, let \([0,1] \ni \lambda \rightarrow \eta^{\varepsilon', \varepsilon}(\lambda, s) \) be the \( \varepsilon'-\)geodesic joining \( \eta^{\varepsilon', \varepsilon}(0, s) := v_s^{\varepsilon', t'} \) and \( \eta^{\varepsilon', \varepsilon}(1, s) := v_s^{\varepsilon', t'} \). Notice that \( \eta^{\varepsilon', \varepsilon}(1, 0) = \eta^{\varepsilon', \varepsilon}(1 - \lambda) \) and \( \eta^{\varepsilon', \varepsilon}(0, 0) = \eta^{\varepsilon', \varepsilon}(1 - \lambda) \).

We fix \( \lambda \in [0,1] \). Combining previous results we can finish the proof:

\[
(1 - \lambda)d_\psi(\alpha_t, \gamma_t) + \lambda \hat{d}_\psi(\alpha_t, \gamma_t') = \lim_{\varepsilon \to 0} \left((1 - \lambda)d_\psi(\alpha_t, \gamma_t) + \lambda \hat{d}_\psi(\alpha_t, \gamma_t')\right)
= \lim_{\varepsilon \to 0} \int_0^1 \left((1 - \lambda)\|\partial_s v_s^{\varepsilon', t'} - \|\psi, v_s^{\varepsilon', t'}\|_{\psi, v_s^{\varepsilon', t'}} + \lambda\|\partial_s v_s^{\varepsilon', t'} - \|\psi, v_s^{\varepsilon', t'}\|_{\psi, v_s^{\varepsilon', t'}}ds\right)
\leq \lim_{\varepsilon \to 0} \int_0^1 \|\partial_s \eta^{\varepsilon', \varepsilon}((1 - \lambda)t + \lambda t', s)\|_{\psi, \eta^{\varepsilon', \varepsilon}((1 - \lambda)t + \lambda t', s)ds}
= \int_0^1 \hat{d}_\psi(\alpha(1 - \lambda)t + \lambda t', \gamma(1 - \lambda)t + \lambda t').
\]

where in the first line we have used Proposition 3.1, in the second line we have used Proposition 3.1, in the third line we have used Proposition 3.1, in the fourth line we have used Proposition 3.1, and in the last line we have used Proposition 3.1 again.

**Proposition 4.4.** Given \( \alpha, \beta, \gamma \in \mathcal{H}_\omega^\Delta \) such that \( \alpha \geq \beta \geq \gamma \), we have that

\[
d_\psi(\alpha, \gamma) \leq d_\psi(\alpha, \beta) + d_\psi(\beta, \gamma).
\] (25)

**Proof.** Using Proposition 3.1 we can assume that \( \alpha, \beta, \gamma \) are smooth Kähler potentials, moreover \( \alpha > \beta > \gamma \).

Let \([0,1] \ni t \rightarrow u_t, v_t \in \mathcal{H}_\omega^\Delta \) be the weak geodesics connecting \( u_0 := \beta \) and \( u_1 := \alpha \), respectively \( v_0 := \beta \) and \( v_1 := \gamma \). By Corollary 4.3 we get that \( t \rightarrow \hat{d}_\psi(u_t, v_t) \) is concave. Hence, since \( \hat{d}_\psi(u_0, v_0) = 0 \), \( t \rightarrow \hat{d}_\psi(u_t, v_t)/t \) is decreasing.
Let $[0, 1] \ni t \to \eta_t$ be the weak geodesic connecting $\eta_0 = \alpha$ and $\eta_1 = \gamma$. We can use the $\psi$-version of [Da15, Lemma 4.1] (whose proof is identical) to write:

$$\|\hat{\eta}_0\|_{\psi, \alpha} = \hat{d}_\psi(\alpha, \gamma) = \hat{d}_\psi(u_1, v_1) \leq \lim_{t \to 0} \frac{\hat{d}_\psi(u_t, v_t)}{t} \leq \limsup_{t \to 0} \frac{\|u_t - v_t\|_{\psi, v_t}}{t} = \limsup_{t \to 0} \left\| \frac{u_t - v_t}{t} \right\|_{\psi, v_t} = \|\hat{u}_0 - \hat{v}_0\|_{\psi, \beta},$$

where in the last step we have used that $\omega^n_\alpha \to \omega^n_\beta$ weakly, moreover $(u_t - v_t)/t \to \hat{u}_0 - \hat{v}_0$ uniformly, as $t \to 0$. Indeed, this allows an application of Lemma 2.8 to conclude.

Finally, if we replace $\psi(t)$ with $\tilde{\psi}(t) := \psi(t)/\int_X \psi(\tilde{\eta}_0)\omega^n_\alpha \in \mathcal{W}^-$, the same inequality as above implies that, $1 = \int_X \tilde{\psi}(\tilde{u}_0 - \tilde{v}_0)\omega^n_\beta \geq 1$, i.e., $\int_X \psi(\tilde{u}_0 - \tilde{v}_0)\omega^n_\beta \geq \int_X \tilde{\psi}(\tilde{\eta}_0)\omega^n_\alpha$. Using Lemma 2.6 we now conclude that

$$d_\psi(\alpha, \beta) + d_\psi(\beta, \gamma) = \int_X \tilde{\psi}(\tilde{u}_0)\omega^n_\beta + \int_X \tilde{\psi}(\tilde{v}_0)\omega^n_\beta \geq \int_X \psi(\tilde{u}_0 - \tilde{v}_0)\omega^n_\beta = \int_X \psi(\hat{u}_0)\omega^n_\alpha = d_\psi(\alpha, \gamma).$$

We are ready to prove the general case of the triangle inequality.

**Theorem 4.5.** Let $u, v, w \in \mathcal{H}_\omega^\Delta$. Then we have

$$d_\psi(u, w) \leq d_\psi(u, v) + d_\psi(v, w).$$

**Proof.** The triangle inequality follows from the following sequence of inequalities:

$$d_\psi(u, v) + d_\psi(v, w) = d_\psi(u, P(u, v)) + d_\psi(P(u, v), v) + d_\psi(v, P(v, w)) + d_\psi(P(v, w), w) \geq d_\psi(u, P(u, v)) + d_\psi(P(v, w), P(u, v)) + d_\psi(P(u, v), P(v, w)) \geq d_\psi(u, P(u, v)) + d_\psi(w, P(u, v)) \geq d_\psi(u, P(u, w)) + d_\psi(w, P(u, v)) \geq d_\psi(u, P(u, w)) + d_\psi(w, P(u, w)) = d_\psi(u, w),$$

where in the first and last line we have used the Pythagorean identity for $d_\psi$ (Lemma 3.2), in the second line we have used Proposition 3.3 for the second and third terms, in the fourth line we have used twice the particular case of the triangle inequality obtained in Proposition 4.4 and in the fifth line we used Lemma 3.4.

**Corollary 4.6.** $(\mathcal{H}_\omega^\Delta, d_\psi)$ is a metric space.

**Proof.** By the previous result, we only need to argue that $d_\psi(u_0, u_1) = 0$ implies $u_0 = u_1$.

If $d_\psi(u_0, u_1) = 0$, by Lemma 3.2 we have $d_\psi(u_0, P(u_0, u_1)) = 0$ and also $d_\psi(u_1, P(u_0, u_1)) = 0$. By the first estimate of Proposition 4.7 below, it follows that $u_0 = P(u_0, u_1)$ a.e. with respect to $\omega^n_{P(u_0, u_1)}$, and similarly, $u_1 = P(u_0, u_1)$ a.e. with respect to $\omega^n_{P(u_0, u_1)}$. We can now use the domination principle of full mass potentials due to Dinew [BL12, Proposition 5.9] to obtain that $u_0 \leq P(u_0, u_1)$ and $u_1 \leq P(u_0, u_1)$. As the reverse inequalities are trivial, we get that $u_0 = P(u_0, u_1) = u_1$. 


Proposition 4.7. Suppose \( u, v \in H^\Delta_\omega \) with \( u \leq v \). Then we have:

\[
\max \left( \frac{1}{2n+1} \int_X \psi(v-u)\omega_u^n, \int_X \psi(v-u)\omega_v^n \right) \leq d_\psi(u, v) \leq \int_X \psi(v-u)\omega_u^n. \tag{27}
\]

Proof. Using Proposition 3.1 we can assume that \( u \) and \( v \) are smooth. Suppose \([0, 1] \ni t \to w_t \in H^\Delta_\omega\) is the weak geodesic segment joining \( w_0 = u \) and \( w_1 = v \). By (24) we have

\[
d_\psi(u, v) = \int_X \psi(\dot{w}_0)\omega_u^n = \int_X \psi(\dot{w}_1)\omega_v^n.
\]

Since \( u \leq v \), we have that \( u \leq w_t \), as follows from the comparison principle. Since \((t, x) \to w_t(x)\) is convex in the \( t \) variable, we get \( 0 \leq \dot{w}_0 \leq v - u \leq \dot{w}_1 \), and together with the above identity we obtain part of (27):

\[
\int_X \psi(v-u)\omega_u^n \leq d_\psi(u, v) \leq \int_X \psi(v-u)\omega_v^n. \tag{28}
\]

Now we prove the rest of (27). Using \( \omega_u^n \leq 2^n \omega_{(u+v)/2}^n \) and concavity of \( \psi \) on \([0, \infty)\), we obtain that

\[
\frac{1}{2n+1} \int_X \psi(v-u)\omega_u^n \leq \int_X \psi \left( \frac{u+v}{2} - u \right)\omega_{(u+v)/2}^n.
\]

Since \( u \leq (u+v)/2 \), the first estimate of (28) allows to write:

\[
\frac{1}{2n+1} \int_X \psi(v-u)\omega_u^n \leq d_\psi \left( \frac{u+v}{2}, u \right).
\]

Finally, Lemma 3.4 implies that \( d_\psi((u+v)/2, u) \leq d_\psi(v, u) \), giving the remaining estimate in (27). \( \square \)

5 Extending \( d_\psi \) to \( \mathcal{E}_\psi \) and completeness

Given \( u_0, u_1 \in \mathcal{E}_\psi(X, \omega) \), by a classical result of Demailly [De94] (see [BK07] for a short argument) there exists decreasing sequences \( u_k^0, u_k^1 \in H_\omega \) such that \( u_0^0 \searrow u_0 \) and \( u_1^k \nearrow u_1 \). We propose to extend \( d_\psi \) to \( \mathcal{E}_\psi \) in the following way:

\[
d_\psi(u_0, u_1) = \lim_{k \to \infty} d_\psi(u_0^k, u_1^k). \tag{29}
\]

Very similar to the high energy case [Da15], we will show that the limit on the right hand side exists and is independent of the approximating sequences. For this, we first prove the next lemma:

Lemma 5.1. Suppose \( u \in \mathcal{E}_\psi \) and \( \{u_k\}_k \subset H^\Delta_\omega \) is a sequence decreasing to \( u \). Then \( d_\psi(u_l, u_k) \to 0 \) as \( l, k \to \infty \).

Proof. We can suppose that \( l \leq k \). Then \( u_k \leq u_l \), hence by Proposition 4.7 we have:

\[
d_\psi(u_l, u_k) \leq \int_X \psi(u_k - u_l)\omega_{u_k}^n.
\]
Let us fix $l$ momentarily, and let $\{v_j\}_j \in \mathcal{H}_\omega$ be such that $v_j \searrow u$. Then $u - v_j, u_k - v_j \in \mathcal{E}_\psi(X, \omega_{v_j})$ and $u - v_j \leq u_k - v_j \leq 0$. Hence, applying Proposition 2.2 for the class $\mathcal{E}_\psi(X, \omega_u)$ we obtain

$$d_\psi(u_l, u_k) \leq \int_X \psi(u_k - u_l)\omega^n_{u_k} \leq \lim_j \int_X \psi(u_k - v_j)\omega^n_{u_k}$$

$$\leq C \lim_j \int_X \psi(u - v_j)\omega^n_u = C \int_X \psi(u - u_l)\omega^n_u. \quad (30)$$

As $u_l$ decreases to $u \in \mathcal{E}_\psi$, by the dominated convergence theorem we have $d_\psi(u_l, u_k) \to 0$ as $l, k \to \infty$.

Our next lemma confirms that the way we proposed to extend the $d_\psi$ metric to $\mathcal{E}_\psi$ in (29) is consistent.

**Lemma 5.2.** Given $u_0, u_1 \in \mathcal{E}_\psi$, the limit in (29) is finite and independent of the approximating sequences $u^k_0, u^k_1 \in \mathcal{H}_\omega^A$.

**Proof.** By Proposition 3.1 we can assume that the approximating sequences are smooth. By the triangle inequality and Lemma 5.1 we can write:

$$|d_\psi(u^l_0, u^l_1) - d_\psi(u^k_0, u^k_1)| \leq d_\psi(u^l_0, u^k_0) + d_\psi(u^l_1, u^k_1) \to 0, \quad l, k \to \infty,$$

proving that $d_\psi(u^k_0, u^k_1)$ is indeed convergent.

Now we prove that the limit in (29) is independent of the choice of approximating sequences. Let $v^l_0, v^l_1 \in \mathcal{H}_\omega$ be different approximating sequences. By adding small constants we arrange that the sequences $u^k_0, u^k_1$, respectively $v^l_0, v^l_1$, are strictly decreasing to $u_0, u_1$.

Fixing $k$ for the moment, the sequence $\{\max\{u^{k+1}_0, v^l_0\}\}_j \in \mathbb{N}$ decreases pointwise to $u^{k+1}_0$. By Dini’s lemma there exists $j_k \in \mathbb{N}$ such that for any $j \geq j_k$ we have $v^l_0 < u^k_0$. By repeating the same argument we can also assume that $v^l_1 < u^k_1$ for any $j \geq j_k$. By the triangle inequality again

$$|d_\psi(u^k_0, u^k_1) - d_\psi(v^l_0, v^l_1)| \leq d_\psi(u^k_0, v^l_0) + d_\psi(u^k_1, v^l_1), \quad j \geq j_k.$$

From (30) it follows that for $k$ big enough $d_\psi(u^k_0, v^l_0), d_\psi(u^k_1, v^l_1), j \geq j_k$ are arbitrarily small. As a result, $d_\psi(u_0, u_1)$ is independent of the choice of approximating sequences.

When $u_0, u_1 \in \mathcal{H}_\omega^A$, one can approximate with the constant sequence, hence the restriction to $\mathcal{H}_\omega$ of the extended $d_\psi$ from (29) coincides with the original definition (21).

By the above result, [Da19, Proposition 2.20], and the remark following it, many properties of $d_\psi$ extend to $\mathcal{E}_\psi$, in particular the triangle inequality, the Pythagorean formula, etc. We list these in the proposition below and leave the standard proofs to the interested reader.

**Proposition 5.3.** Let $\psi \in \mathcal{W}^-$. Then the following hold:

(i) $d_\psi : \mathcal{E}_\psi \times \mathcal{E}_\psi \to \mathbb{R}$ satisfies the triangle inequality.

(ii) If $u, v \in \mathcal{E}_\psi$ then $P(u, v) \in \mathcal{E}_\psi$ and $d_\psi(u, v) = d_\psi(u, P(u, v)) + d_\psi(v, P(u, v))$.

(iii) Suppose $u, v \in \mathcal{E}_\psi$ with $u \leq v$. Then we have:

$$\max \left(\frac{1}{2n+1} \int_X \psi(v - u)\omega^n_u, \int_X \psi(v - u)\omega^n_v\right) \leq d_\psi(u, v) \leq \int_X \psi(v - u)\omega^n_u.$$

(iv) For $u, v, w \in \mathcal{E}_\psi$ we have $d_\psi(P(u, w), P(v, w)) \leq d_\psi(u, v)$. 

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We now argue that non-degeneracy of $d_\psi$ on $\mathcal{E}_\psi$ holds as well:

**Proposition 5.4.** Given $u_0, u_1 \in \mathcal{E}_\psi$ if $d_\psi(u_0, u_1) = 0$ then $u_0 = u_1$. In particular, $(\mathcal{E}_\psi, d_\psi)$ is a metric space.

**Proof.** We can repeat the argument of Corollary 4.6. By Proposition 5.3(ii) it follows that $d_\psi(u_0, P(u_0, u_1)) = 0$ and also $d_\psi(u_1, P(u_0, u_1)) = 0$. By Proposition 5.3(iii), it follows that $u_0 = P(u_0, u_1)$ a.e. with respect to $\omega^\nu_{P(u_0, u_1)}$, and similarly, $u_1 = P(u_0, u_1)$ a.e. with respect to $\omega^\nu_{P(u_0, u_1)}$. We can now use the domination principle of full mass potentials due to Dinew [BL12, Proposition 5.9] to obtain that $u_0 \leq P(u_0, u_1)$ and $u_1 \leq P(u_0, u_1)$. As the reverse inequalities are trivial, we get that $u_0 = P(u_0, u_1) = u_1$. \hfill \Box

**Corollary 5.5.** If $\{w_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\psi$ decreases or increases a.e. to $w \in \mathcal{E}_\psi$ then $d_\psi(w_k, w) \to 0$.

**Proof.** By Proposition 5.3(iii), we have $d_\psi(w, w_k) \leq \int_X \psi(w - w_k)(\omega^n_{w_k} + \omega^n_w)$. We can use [Da19, Proposition 2.20] (and the remark following it) to conclude that $d_\psi(w, w_k) \to 0$. \hfill \Box

**Lemma 5.6.** Suppose $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\psi$ be an increasing $d_\psi$-bounded sequence. Then sup$_X u_k$ is a bounded sequence.

**Proof.** Using Theorem 6.1 from below we have that $\int_X \psi(\max(u_k, 0))\omega^n \leq \int_X \psi(u_k)\omega^n \leq 2^{2n+5}d_\psi(u_k, 0) \leq C$, for some $C > 0$. Let $v := \lim_k \max(0, u_k)$, a measurable function on $X$. By the monotone convergence theorem we obtain that $\int_X \psi(v)\omega^n \leq C$. This implies that for some $d > 0$ the set $K := \{v \leq d\}$ has non-zero Lebesgue measure, hence $K$ is also non-pluripolar.

On $K$ we have that $u_k \leq d$. As a result, due to [GZ05, Corollary 4.3] we obtain that $\{u_k\}_{k \in \mathbb{N}} \subset \text{PSH}(X, \omega)$ is relatively $L^1$-compact, hence sup$_X u_k$ can not converge to $\infty$, finishing the proof. \hfill \Box

Next we argue that bounded monotone sequences in $\mathcal{E}_\psi$ have limits inside $\mathcal{E}_\psi$. Using the previous lemma, the proof of this result is very similar to [Da19, Lemma 3.34]:

**Lemma 5.7.** Suppose $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\psi$ is a decreasing/increasing $d_\psi$-bounded sequence. Then $u = \lim_k u_k \in \mathcal{E}_\psi$ and additionally $d_\psi(u, u_k) \to 0$.

**Proof.** Due to the previous lemma, after subtracting a constant, we can assume without loss of generality that $u_k \leq 0$.

Let us assume that $\{u_k\}_k$ is decreasing. From Proposition 5.3(iii) we have that $\int_X \psi(u_k)\omega^n_{u_k}$ is uniformly bounded. Due to [GZ07, Proposition 5.6] we get that $u = \lim_k u_k \in \mathcal{E}_\psi(X, \omega)$. Corollary 5.5 implies that $d_\psi(u_k, u) \to 0$.

Now assume that $\{u_k\}_k$ is increasing. Due to the previous lemma, there exists $u \in \mathcal{E}_\psi(X, \omega)$ such that $u_k \nearrow u$. By Corollary 5.5 again, $d_\psi(u_k, u) \to 0$. \hfill \Box

Finally, we argue completeness of $(\mathcal{E}_\psi, d_\psi)$:

**Theorem 5.8.** $(\mathcal{E}_\psi, d_\psi)$ is a complete metric space, that is the metric completion of $(\mathcal{H}_\omega, d_\psi)$. 

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Proof. By Corollary 5.5 and [De94] $\mathcal{H}_{\omega}$ is a $d_\psi$-dense subset of $E_\psi$. We need to argue completeness, which can be done identically as in [Da15], due to Proposition 3.3.

Indeed, suppose $\{u_k\}_{k \in \mathbb{N}} \subset E_\psi$ is a $d_\psi$-Cauchy sequence. We will prove that there exists $v \in E_\psi$ such that $d_\psi(u_k, v) \to 0$. After passing to a subsequence we can assume that

$$d_\psi(u_l, u_{l+1}) \leq 1/2^l, \quad l \in \mathbb{N}.$$ 

By [Da17b] Theorem 3.6] we can introduce $v^k_l = P(u_k, u_{k+1}, \ldots, u_{k+l}) \in E_\psi$, $l, k \in \mathbb{N}$. We argue first that each decreasing sequence $\{v^k_l\}_{l \in \mathbb{N}}$ is $d_\psi$-Cauchy. We observe that $v^k_{l+1} = P(v^k_l, u_{k+l+1})$ and $v^k_l = P(v^k_l, u_{k+l})$. Using this and Proposition 5.3(iv) we can write:

$$d_\psi(v^k_{l+1}, v^k_l) = d_\psi(P(v^k_l, u_{k+l+1}), P(v^k_l, u_{k+l})) \leq d_\psi(u_{k+l+1}, u_{k+l}) \leq \frac{1}{2^{k+l}}.$$ 

From Lemma 5.7 it follows now that each sequence $\{v^k_l\}_{l \in \mathbb{N}}$ is $d_\psi$-converging to some $v^k \in E_\psi$. By the same trick as above, we can write:

$$d_\psi(v^k, v^{k+1}) = \lim_{l \to \infty} d_\psi(v^k_{l+1}, v^k_l) = \lim_{l \to \infty} d_\psi(P(u_k, v^k_{l+1}), P(u_k, v^k_l)) \leq d_\psi(u_k, u_{k+1}) \leq \frac{1}{2^k},$$

$$d_\psi(v^k, u_k) = \lim_{l \to \infty} d_\psi(v^k_l, u_k) = \lim_{l \to \infty} d_\psi(P(u_k, v^k_{l+1}), P(u_k, u_k)) \leq \lim_{l \to \infty} d_\psi(v^k_{l-1}, u_k)$$

$$= \lim_{l \to \infty} d_\psi(P(u_{k+1}, v^k_{l-2}), u_k) \leq \lim_{l \to \infty} d_\psi(P(u_{k+1}, v^k_{l-2}), u_{k+1}) + d_\psi(u_{k+1}, u_k)$$

$$\leq \lim_{l \to \infty} \sum_{j=k}^{l+k} d_\psi(u_j, u_{j+1}) \leq \frac{1}{2^{k-1}}.$$ 

Consequently, $\{v^k\}_{k \in \mathbb{N}}$ is an increasing $d_\psi$-bounded $d_\psi$-Cauchy sequence that is equivalent to $\{u_k\}_{k \in \mathbb{N}}$. By Lemma 5.7 there exists $v \in E_\psi$ such that $d_\psi(v_k, v) \to 0$, which in turn implies that $d_\psi(u_k, v) \to 0$, finishing the proof. \[\square\]

6 An analytic expression governing the $d_\psi$ metric

As another application of the Pythagorean formula we will show that the $d_\psi$ metric is comparable to a concrete analytic expression, reminiscent of the analogous result for high energy classes [Da15, Theorem 3]:

**Theorem 6.1.** For any $u_0, u_1 \in E_\psi$ we have

$$d_\psi(u_0, u_1) \leq \int_X \psi(u_0 - u_1) \omega^n_{u_0} + \int_X \psi(u_0 - u_1) \omega^n_{u_1} \leq 2^{n+5} d_\psi(u_0, u_1). \quad (31)$$

*Proof. To obtain the first estimate we use the triangle inequality and Proposition 5.3(iii):

$$d_\psi(u_0, u_1) \leq d_\psi(u_0, \max(u_0, u_1)) + d_\psi(\max(u_0, u_1), u_1)$$

$$\leq \int_X \psi(u_0 - \max(u_0, u_1)) \omega^n_{u_0} + \int_X \psi(\max(u_0, u_1) - u_1) \omega^n_{u_1}$$

$$= \int_{\{u_1 > u_0\}} \psi(u_0 - u_1) \omega^n_{u_0} + \int_{\{u_0 > u_1\}} \psi(u_0 - u_1) \omega^n_{u_1}$$

$$\leq \int_X \psi(u_0 - u_1) \omega^n_{u_0} + \int_X \psi(u_0 - u_1) \omega^n_{u_1}.$$
Now we deal with the second estimate in (31). By the next result, Proposition 5.3(ii) and Proposition 5.3(iii) we can write

\[ 2^{n+2}d_\psi(u_0, u_1) \geq d_\psi\left(u_0, \frac{u_0 + u_1}{2}\right) \geq d_\psi\left(u_0, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \]

\[ \geq \int_X \psi\left(u_0 - P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \omega_{u_0}^n. \]

By a similar reasoning as above, and the fact that \(2^n \omega_{(u_0+u_1)/2}^n \geq \omega_{u_0}^n\) we can write:

\[ 2^{n+2}d_\psi(u_0, u_1) \geq d_\psi\left(u_0, \frac{u_0 + u_1}{2}\right) \geq d_\psi\left(\frac{u_0 + u_1}{2}, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \]

\[ \geq \int_X \psi\left(\frac{u_0 + u_1}{2} - P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \omega_{(u_0+u_1)/2}^n \]

\[ \geq \frac{1}{2^n} \int_X \psi\left(u_0 + u_1\right) - P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \omega_{u_0}^n. \]

Adding the last two estimates, and using sublinearity and concavity of \(\psi\) (Lemma 2.6) we obtain:

\[ 2^{2n+3}d_\psi(u_0, u_1) \geq \int_X \psi\left(u_0 - P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) + \psi\left(P\left(u_0, \frac{u_0 + u_1}{2}\right) - \frac{u_0 + u_1}{2}\right)\right) \omega_{u_0}^n \]

\[ \geq \int_X \psi\left(\frac{u_0 - u_1}{2}\right) \omega_{u_0}^n \geq \frac{1}{2} \int_X \psi(u_0 - u_1) \omega_{u_0}^n. \]

By symmetry we also have \(2^{n+4}d_\psi(u_0, u_1) \geq \int_X \psi(u_0 - u_1) \omega_{u_0}^n\), and adding these last two estimates together the second inequality in (31) follows. \(\square\)

**Lemma 6.2.** Suppose \(u_0, u_1 \in E_\psi\). Then we have

\[ d_\psi\left(u_0, \frac{u_0 + u_1}{2}\right) \leq 2^{n+2}d_\psi(u_0, u_1). \]

**Proof.** Using Proposition 5.3(ii) and (iii) we can start writing:

\[ d_\psi\left(u_0, \frac{u_0 + u_1}{2}\right) = d_\psi\left(u_0, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) + d_\psi\left(\frac{u_0 + u_1}{2}, P\left(u_0, \frac{u_0 + u_1}{2}\right)\right) \]

\[ \leq d_\psi(u_0, P(u_0, u_1)) + d_\psi\left(\frac{u_0 + u_1}{2}, P(u_0, u_1)\right) \]

\[ \leq \int_X \psi(u_0 - P(u_0, u_1)) \omega_{P(0, u_1)}^n + \int_X \psi\left(\frac{u_0 + u_1}{2} - P(u_0, u_1)\right) \omega_{P(0, u_1)}^n \]

\[ \leq \int_X \psi(u_0 - P(u_0, u_1)) \omega_{P(0, u_1)}^n + \int_X \psi(\max(u_1, u_0) - P(u_0, u_1)) \omega_{P(0, u_1)}^n \]

\[ \leq \int_X \left(1 + 1_{\{u_0 \geq u_1\}}\psi(u_0 - P(u_0, u_1)) + 1_{\{u_1 \geq u_0\}}\psi(u_1 - P(u_0, u_1))\right) \omega_{P(0, u_1)}^n \]

\[ \leq 2^{n+2}(d_\psi(u_0, P(u_0, u_1)) + d_\psi(u_1, P(u_0, u_1))) = 2^{n+2}d_\psi(u_0, u_1), \]

where in the second line we have used the first claim of Lemma 3.4 and the fact that \(P(u_0, u_1) \leq P(u_0, (u_0 + u_1)/2)\), in the third and sixth line Proposition 5.3(iii), and in the last equality we have used Proposition 5.3(ii). \(\square\)
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND

tdarvas@umd.edu