Generalized thermoelastic interaction in an isotropic solid cylinder without energy dissipation

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Abstract. In this paper, we constructed the generalized thermoelastic equations of an isotropic solid cylinder. The formulation is applied in the context of Green and Naghdi theory of types II (without energy dissipation). The material of the cylinder is supposed to be homogeneous isotropic both mechanically and thermally. The governing equations have been written in the form of a vector-matrix differential equation in the Laplace transform domain, which is then solved by an eigenvalue approach. Numerical results for the temperature distribution, displacement and radial stress are represented graphically.

1. Introduction

The classical theory of thermoelasticity was studied by Carlson [1] in terms of generalizations and modifications into various thermoelastic models that can run under label hyperbolic thermoelasticity, see the survey of Chandrasekhar and Ignazack [2]. The notation hyperbolic reflects the fact that thermal waves are modelled, avoiding the physical paradox of infinite propagation speed of the classical models. Lately, Green and Naghdi (GN) [4-5] reinvestigated the principal hypotheses of thermo-mechanics that established a distinct entropy equality rather than the familiar entropy inequality. Model II of (GN) theory anticipates a limited speed for the propagation of heat and includes no energy dissipation which is known as thermoelasticity without energy dissipation.

During the second half of twentieth century, non-isothermal problems of the theory of elasticity become increasingly important. This is due to their many applications in widely diverse fields. First, the high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduces the strength of the aircraft structures. Second, in the nuclear field, the extremely high temperature and temperature gradients originating inside nuclear reactors influence their design and operations.

The counterparts of our problem in the contexts of the uncoupled thermoelasticity theory, the coupled thermoelasticity theory, the Green-Lindsay theory (GL-theory) [6] and the Lord-Shulman theory (LS-theory) [7] have been considered by Othman et al. [8-10]. Furthermore, related generalized Magneto-thermoelasticity problems in different hypotheses have been studied by Abbas et al. [11-13]. The effective method of eigenvalue approach is used to get the solutions of variously coupled thermoelasticity in an isotropic or an anisotropic media associated with additional fields that has been studied in [14-18].

In the present paper, we consider a problem of generalized thermoelasticity of an isotropic solid cylinder based on Green-Naghdi theory of type II. The non-dimensional equations are handled by employing an analytical-numerical techniques based on Laplace transform and eigenvalues approach.
Numerical results for the temperature distribution, displacement and radial stress are shown graphically.

2. Basic equations

Following Green, and Naghdi [4, 5], the system of equations that include the displacement, the stress, the strain and the temperature for a linear, homogeneous and isotropic thermoelastic continuum take the following form

The equation of heat conduction

\[ K T_{ii} + K T_{ii} = \rho C T + \gamma T_{ii}, \]  

(1)

The equations of motion

\[ (\lambda + \mu) u_{j,ij} + \mu u_{i,ij} - \gamma T_{j,i} = \rho \ddot{u}_i, \]  

(2)

The constitutive equations are given by

\[ \tau_{ij} = \lambda u_{i,i} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) - \gamma T \delta_{ij}, \]  

(3)

where \( C \) is the specific heat at constant strain, \( \lambda, \mu \) are Lame's constants, \( \gamma = (3\lambda + 2\mu) \alpha_t, \alpha_t \) is the coefficient of linear thermal expansion, \( T \) is the temperature above reference temperature \( T_0 \), \( K^* \) are material constant characteristic of the theory. In a cylindrical coordinate system \((r, \theta, z)\), for the axially symmetric problem, \( u_r = u_z, u_\theta = 0 \). Furthermore, if only axisymmetric plane strain problem is considered, we have \( u_r = u(r, t) \) and \( u_\theta = u_z = 0 \).

The strain-displacement relations are

\[ e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = e_{rz} = e_{\theta r} = e_{\theta z} = 0. \]  

(4)

The stress-strain relations are

\[ \tau_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma T, \]  

(5)

\[ \tau_{\theta\theta} = 2\mu \frac{u}{r} + \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma T, \]  

(6)

It is assumed that there are no body forces and heat sources in the medium, the equation of motion and energy equation has the form:

\[ \frac{\partial^2 \tau_{rr}}{\partial r^2} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2}, \]  

(7)

\[ K^* \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tau_r}{\partial r} \right) = \rho C \frac{\partial^2 T}{\partial t^2} + \gamma T \frac{\partial}{\partial r} \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right). \]  

(8)

It is convenient to change the preceding equations into the dimensionless forms. To do this, the dimensionless parameters are introduced as

\[ (r^o, u^o) = \left( \frac{r, u}{c_i \omega_i} \right), \quad t^o = \frac{t}{\omega_i}, \quad \left( \tau_{rr}^o, \tau_{\theta\theta}^o \right) = \frac{1}{\mu} \left( \tau_{rr}, \tau_{\theta\theta} \right), \quad \theta^o = \frac{\gamma T}{\rho c_i^2}. \]  

(9)
where, 
\[ c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad \omega_1 = \frac{K}{\rho C E c_1^2}. \]

From (9) into equations (5)-(8), it is easy to obtain (after dropping the superscript \( \circ \) for convenience) the following:

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{\partial \theta}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \]

(10)

\[ \varepsilon_2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial r}{\partial r} \right) = \frac{\partial^2 r}{\partial t^2} + \varepsilon_1 \frac{\partial^2 u}{\partial t^2} \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right), \]

(11)

\[ \tau_\pi = (2 + \beta) \frac{\partial u}{\partial r} + \beta \frac{u}{r} - (2 + \beta) \theta, \]

(12)

\[ \tau_{\theta0} = \beta \frac{\partial u}{\partial r} + (2 + \beta) \frac{u}{r} - (2 + \beta) \theta, \]

(13)

\[ \beta = \frac{\lambda}{\mu}, \quad \varepsilon_1 = \frac{\gamma^2 T_0}{\rho^2 c_1^2 C E}, \quad \varepsilon_2 = \frac{K^*}{\rho c_1^2 C E}. \]

From preceding description, the initial and boundary conditions may be expressed as

\[ u(r, 0) = \frac{\partial u(r, 0)}{\partial t} = 0, \theta(r, 0) = \frac{\partial \theta(r, 0)}{\partial t} = 0, \]

(14)

\[ -K \frac{\partial \theta(r, t)}{\partial r} = q_o \frac{1}{16\rho^2} \sigma_{rr}(r, t) = 0, \]

(15)

where \( b \) is an outer radius of the hollow cylinder, \( q_o \) is a constant, \( t_p \) is a characteristic time of the pulse heat flux.

### 3. Governing equations in the laplace transform domain

Applying the Laplace transform for equations (10)-(15) defined by the formula

\[ \bar{f}(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt. \]

(16)

Hence, we obtain the following system of differential equations

\[ \frac{d^2 \bar{u}}{dr^2} + \frac{1}{r} \frac{d\bar{u}}{dr} - \frac{\bar{u}}{r^2} - \frac{d\bar{\theta}}{dr} = s^2 \bar{u}, \]

(17)

\[ (\varepsilon_2 + s) \left( \frac{d^2 \bar{\theta}}{dr^2} + \frac{1}{r} \frac{d\bar{\theta}}{dr} \right) - s^2 \bar{\theta} = \varepsilon_1 s^2 \left( \frac{d\bar{u}}{dr} + \frac{\bar{u}}{r} \right), \]

(18)

\[ \tau_\pi = (2 + \beta) \frac{d\bar{u}}{dr} + \beta \frac{\bar{u}}{r} - (2 + \beta) \bar{\theta}, \]

(19)

\[ \tau_{\theta0} = \beta \frac{d\bar{u}}{dr} + (2 + \beta) \frac{\bar{u}}{r} - (2 + \beta) \bar{\theta}, \]

(20)
\[
\frac{\partial \bar{\sigma}}{\partial r} = \frac{-q_{o\sigma_p}}{\theta(s_{e\sigma+1})}, \quad \bar{\sigma}_{rr} = 0, \tag{21}
\]

Differentiating equation (18) with respect to \( r \) and using equation (17), it is easy and interesting to obtain
\[
\frac{d^2}{dr^2} \left( \frac{d\bar{\sigma}}{dr} \right) + \frac{1}{r} \frac{d}{dr} \left( \frac{d\bar{\sigma}}{dr} \right) - \frac{1}{r^2} \left( \frac{d\bar{\sigma}}{dr} \right) = \frac{\varepsilon s^4}{\varepsilon_2} + \frac{s^2 (1+\varepsilon_1)}{\varepsilon_2} \left( \frac{d\bar{\sigma}}{dr} \right). \tag{22}
\]

Equations (17) and (22) can be written in a vector-matrix differential equation as follows
\[
LV = AV, \tag{23}
\]
where
\[
L = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2}, \quad V = \begin{bmatrix} \bar{u} \\ \frac{d\bar{u}}{dr} \end{bmatrix}^T \quad \text{and} \quad A = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},
\]

\( m_{11} = s^2, \quad m_{12} = 1, \quad m_{21} = \frac{s^4 \varepsilon_1}{\varepsilon_2}, \quad m_{22} = \frac{s^2 (1+\varepsilon_1)}{\varepsilon_2} \).

4. Solution of the vector-matrix differential equation

Let us now proceed to solve equation (23) by the eigenvalue approach proposed by [29].

The characteristic equation of the matrix \( A \) takes the form
\[
m_{11}m_{22} - m_{12}m_{21} - (m_{22} + m_{11}) \lambda^2 + \lambda^4 = 0. \tag{24}
\]

The roots of the characteristic equation (24) which are also the eigenvalues of matrix \( A \) are of the form \( \lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2 \). The eigenvector \( \bar{X} = [x_1, x_2]^T \), corresponding to eigenvalue \( \lambda_1^2 \) can be calculated as:
\[
x_1 = m_{12}, \quad x_2 = \lambda_1^2 - m_{11}. \tag{25}
\]

From equations (24), we can easily calculate the eigenvector \( \bar{X}_1 \), corresponding to eigenvalue \( \lambda_j^2, j = 1, 2 \). For further reference, we shall use the following notations:
\[
\bar{X}_1 = \begin{bmatrix} x_1 \\ \lambda_1^2 - \lambda_1^2 \end{bmatrix}, \quad \bar{X}_2 = \begin{bmatrix} x_2 \\ \lambda_2^2 - \lambda_2^2 \end{bmatrix}. \tag{26}
\]

The general solutions of equation (23) must be finite when the radius \( r \to 0 \) and this leads to:
\[
\bar{V} = \bar{X}_1 B_1 I_1 (\lambda_1 r) + \bar{X}_2 B_1 I_1 (\lambda_2 r), \tag{27}
\]
where \( B_1 \) and \( B_2 \) are constants to be determined from the boundary condition of the problem. Thus, the field variables can be written for \( r \) and \( s \) as:
\[
\bar{u}(r,s) = x_1^1 B_1 I_1 (\lambda_1 r) + x_2^1 B_1 I_1 (\lambda_2 r), \tag{28}
\]
\[
\bar{\theta}(r,s) = -\frac{x_1^2}{\lambda_1} B_1 I_0 (\lambda_1 r) - \frac{x_2^2}{\lambda_2} B_2 I_0 (\lambda_2 r), \tag{29}
\]
\[
\tau_{r} = B_1 \left[ (2 + \beta) \left( \lambda_1 x_1^1 + \frac{x_1^1}{\lambda_1} \right) I_0 (\lambda_1 r) - \frac{2x_1^1}{r} I_1 (\lambda_1 r) \right] + B_2 \left[ (2 + \beta) \left( \lambda_2 x_2^1 + \frac{x_2^1}{\lambda_2} \right) I_0 (\lambda_2 r) - \frac{2x_2^1}{r} I_1 (\lambda_2 r) \right], \tag{30}
\]
\[
\tau_{\theta} = B_1 \left[ (\beta - 2) \left( \lambda_1 x_1^1 + \frac{x_1^1}{\lambda_1} \right) I_0 (\lambda_1 r) + \frac{2x_1^1}{r} I_1 (\lambda_1 r) \right] + B_2 \left[ (\beta - 2) \left( \lambda_2 x_2^1 + \frac{x_2^1}{\lambda_2} \right) I_0 (\lambda_2 r) + \frac{2x_2^1}{r} I_1 (\lambda_2 r) \right]. \tag{31}
\]
To complete the solution, constants $B_1$ and $B_2$ are required by using the boundary conditions (21) can be obtained

\[
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix} = \begin{pmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{pmatrix}^{-1} \begin{pmatrix}
0 \\
1/s
\end{pmatrix},
\]

(32)

where

\[
\begin{align*}
R_{11} &= (2 + \beta) \left( \lambda_1 x_1^1 + \frac{x_1^2}{\lambda_1} \right) I_0(\lambda_1 b) - \frac{2x_1^1}{a} I_1(\lambda_1 b), \\
R_{12} &= (2 + \beta) \left( \lambda_2 x_1^2 + \frac{x_2^2}{\lambda_2} \right) I_0(\lambda_2 b) - \frac{2x_2^2}{a} I_1(\lambda_2 b), \\
R_{21} &= -\frac{x_1^2}{\lambda_1} I_0(\lambda_1 b), \\
R_{22} &= -\frac{x_2^2}{\lambda_2} I_0(\lambda_2 b),
\end{align*}
\]

5. Numerical inversion of the Laplace transforms

For the final solution of displacement, temperature and stress distributions, a numerical inversion method was adopted based on the Riemann-sum approximation method which is used to obtain the numerical results. In this method, a function in the Laplace domain can be transformed into the time domain as:

\[
H(x, t) = \frac{\alpha t}{\tau} Re[H(x, \alpha)] + Re \sum_{q=0}^{N^*} (-1)^q H(x, m + \frac{i\pi q}{\tau}).
\]

(33)

where $Re$ is the real part and $i$ is the imaginary number unit. For faster convergence, numerical experiments stated that $m = \frac{4.7}{\tau}$ which satisfies the above relation.

6. Numerical results and discussion

The copper material was chosen for purpose of numerical evaluations and the constants of the problem were taken as follows [12]:

- $\lambda = 7.76 \times 10^{10} \text{(kg)(m)}^{-1} \text{(s)}^{-2}$, $\mu = 3.86 \times 10^{10} \text{(kg)(m)}^{-1} \text{(s)}^{-2}$,
- $K = 3.68 \times 10^5 \text{(kg)(m)(K)}^{-1} \text{(s)}^{-3}$, $c_e = 3.831 \times 10^5 \text{(m)}^2 \text{(K)}^{-1} \text{(s)}^{-2}$,
- $\rho = 8.954 \times 10^3 \text{(kg)(m)}^{-3}$, $T_0 = 293 \text{(K)}$, $\alpha_1 = 17.8 \times 10^{-6} \text{(K)}^{-1}$, $b = 3$.

Here all the variables are taken in non-dimensional forms. The outcomes of the field of physical quantities such as temperature distribution, displacement component, and radial stress component versus the distance $r$ for different values of $t = 0.2, 0.3, 0.4, 0.5$ based on Green-Naghdi theory of type II have been illustrated in Figures 1-3. The following observations can be summarized. From Fig. 1, it is noticed that the time has a great effect on the variation of temperature distribution $\theta$ as function of $r$. For example, the values of $\theta$ remain stable and almost equal zero until the value of $r = 0.225$ for all values of $t$. Next, the values of $\theta$ are increasing with direct proportional to going up of both $r$ and $t$. Fig. 2 shows a similar attitude as the previous case in Fig. 1 but for the variations of radial displacement versus $r$ for several values of time. Fig. 3 considers the variations of radial stress against $r$ for various values of $t$. It is observed that the radial stress starts from zero and vary with respect to...
the length to a maximum value then reduces with growing the distance to vanish once again at the centre of the cylinder which agrees with the boundary conditions.

7. Conclusions
The solution for a generalized thermoelastic interaction in an isotropic solid cylinder without energy dissipation has been found utilizing the eigenvalue approach in the context of Green and Naghdi theory of types II. The eigenvalue approach is applied successfully to get an explicit, totally analytic, and uniformly valid solution for the current problem. Numerical outcomes for the temperature, displacement and radial stress fields are illustrated and the graph for them is given. The methods which are used in this paper may be utilized in some recent usefulness workable problems and future particle research trends (see [19] -[28]). Furthermore, the solutions of these kinds of problems may open the scope of further studies in science, mathematics and engineering disciplines.

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Figure 1. The temperature distribution with radial distance.

Figure 2. The displacement distribution with radial distance.

Figure 3. The radial stress distribution with radial distance.