Non-gaussianity in the strong regime of warm inflation

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The bispectrum of scalar mode density perturbations is analysed for the strong regime of warm inflationary models. This analysis generalises previous results by allowing damping terms in the inflaton equation of motion that are dependent on temperature. A significant amount of non-gaussianity emerges with constant (or local) non-linearity parameter \( f_{NL} \sim 20 \), in addition to the terms with non-constant \( f_{NL} \) which are characteristic of warm inflation.

I. INTRODUCTION

Observations of the cosmic microwave background are consistent with the existence of gaussian, weakly scale dependent, density perturbations as predicted by most inflationary models [1–4]. The amount of non-gaussianity produced by the simplest inflationary models is small and unlikely to be to be observable by the next generation of experiments, but this still leaves open the possibility that a slightly more exotic inflationary model could produce a measurable effect.

One variation on inflation is the warm inflationary scenario [5] (see also [6] and the review [7]). The non-gaussianity produced by a warm inflationary scenario with constant friction coefficient has been presented elsewhere [8]. The bispectrum of the non-gaussianity is large, and it has a distinctive dependence on wave number. In this paper, we shall examine the bispectrum of the non-gaussianity in the more likely situation where the friction coefficient is not constant. We shall see that the bispectrum has two terms, one which is like the previous warm inflation bispectrum and a new term which is typical of density perturbations with local non-linearities.

Warm inflation is characterised by the rate of radiation production during the inflationary era. The radiation can affect both the homogeneous evolution of the inflaton field and the inhomogeneous fluctuations. If the radiation field has a strong damping effect on the inflaton dynamics, then we have what is known as the strong regime of warm inflation. If the damping effect is small, but the fluctuations are still influenced by radiation, then we have the weak regime of warm inflation. In both strong and weak regimes of warm inflation, fluctuations in the radiation are transferred to the inflaton [9–16] and become the primary source of density fluctuations. This is the most significant difference between warm inflation and traditional cold inflation.

The simplest warm inflationary scenarios assume that the radiation produced during the inflationary era thermalises at a rate faster than the expansion rate. This type of warm inflationary model is therefore rather restrictive. However, the possibility of thermalisation occurring is enhanced by inflaton decay channels, which are naturally present in many supersymmetric theories, where the inflaton decays into light radiation fields through heavy particle intermediaries [17–20]. In these models, the damping of the inflaton field is described by a friction coefficient \( \Gamma \propto T^c \), where \( T \) is the temperature. In particular, \( c \approx 3 \) at temperatures small compared to the heavy particle masses [21, 22]. The temperature dependence has been found to have a large effect on the size of the density fluctuations [23], and we shall examine now how the temperature dependence affects the non-gaussianity.

A feature of warm inflation is that a significant amount of non-gaussianity is produced whilst the density fluctuations are still on sub-horizon scales. In the previous analysis, this was due to the non-linearity caused by the bulk velocity of the radiation. In the new analysis, there are terms in the non-gaussianity which are proportional to the parameter \( c \). The only restriction on \( c \) for the consistency of warm inflation is that \( c < 4 \) [24]. This leads to effects which are large compared to slow-roll parameters. For comparison, the non-linearities produced by derivatives of the inflaton potential in warm inflationary models was looked at by Gupta et al. [25, 26], but their contribution to the non-gaussianity is governed by slow-roll parameters and it is tiny in comparison with the true non-gaussianity produced in warm inflation.

Fluctuations in the cosmic microwave background provide an observational link to the density fluctuations at the surface of last scattering. We know, in principle, how to evolve these fluctuations from early times using, for example, the Bardeen variable \( \zeta \) [27]. Observations can be compared to predictions for various moments of the probability

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distribution of $\zeta$. The most important of these is the primordial power spectrum of fluctuations $P_\zeta(k)$, defined by the stochastic average
\begin{equation}
\langle \zeta(k_1)\zeta(k_2) \rangle = (2\pi)^3 P_\zeta(k_1) \delta^3(k_1 + k_2).
\end{equation}
The bispectrum $B_\zeta(k_1, k_2, k_3)$, defined by
\begin{equation}
\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta^3(k_1 + k_2 + k_3),
\end{equation}
can be used to examine the non-gaussianity in the density fluctuations. The normalised amount of non-gaussianity in the bispectrum is described by a non-linearity function $f_{NL}$, defined by
\begin{equation}
 f_{NL}(k_1, k_2, k_3) = \frac{5}{6} \frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1)}.
\end{equation}
where the $5/6$ factor is convenient for cosmic microwave background comparisons [28]. Models with constant $f_{NL}$ are often called local models because this type of non-gaussianity can arise from local non-linearities in the density perturbations.

Non-linear evolution during the inflationary era can result in non-gaussianity appearing in the primordial density fluctuations. The amount of non-gaussianity produced by vacuum fluctuations in single-field inflationary models is typically around a few per cent [29, 31], and can be related to the standard set of inflationary slow-roll parameters [32]. This is small compared to non-inflationary effects. For example, the second order Sachs-Wolfe effect is expected to act as a source of non-gaussianity in the cosmic microwave background observations equivalent to $f_{NL} \sim 1$ [33, 34].

Models of inflation with multiple scalar fields, acting as sources of density fluctuations in the curvaton scenario [34, 35], or modifying the reheating phase of the universe [37], can produce a level of non-gaussianity above the foreground effects, even significantly above the foreground for particular parameter choices. Non-gaussianity can also be produced by modifications to the kinetic part of the inflaton Lagrangian in D-brane models [38, 39]. This type of non-gaussianity is concentrated on equilateral wave-vector triangles $k_1, k_2, k_3$, unlike the local form, which is concentrated on oblique triangles.

The best observational limit on the non-gaussianity at present is from the WMAP seven-year data release [40], which gives $-10 < f_{NL}^{local} < 74$ with 95% confidence for a constant (or local) component. The Planck satellite observations may have a sensitivity limit of around $|f_{NL}^{local}| \sim 5 - 10$, depending on how well the signal can be separated from the galactic foreground [28]. Our prediction for $f_{NL}^{local}$ from the strong regime of warm inflation lies well above the Planck detection threshold in most models, but the most significant feature is the presence of a term in the bispectrum which could eventually provide a means to distinguish warm inflation from other sources of non-gaussianity. The prospects for observing this term have been discussed in Refs. [11] and [32].

The paper is organised as follows. We begin in section II with a brief introduction to the notion of warm inflation. In section III, we introduce fluctuations of the inflaton field described by a Langevin equation and expand the fluctuations to second order. The limit of strong dissipation is introduced in section IV, and the bispectrum is calculated in section V. Some observational prospects are discussed in the conclusion. We use units with $\hbar = c = 1$.

II. WARM INFLATION

Warm inflation occurs when there is a significant amount of particle production during the inflationary era. We shall assume that the particle interactions are strong enough to produce a thermal gas of radiation with temperature $T$. In this case, warm inflation will occur when $T$ is larger than the energy scale set by the expansion rate $H$. The production of radiation is associated with a damping effect on the inflaton, whose equation of motion becomes
\begin{equation}
\ddot{\phi} + (3H + \Gamma)\dot{\phi} + V_\phi = 0
\end{equation}
where $\Gamma(\phi, T)$ is a friction coefficient, $H$ is the Hubble parameter and $V_\phi$ is the $\phi$ derivative of the inflaton potential $V(\phi, T)$.

The effectiveness of warm inflation can be parameterised by a parameter $r$, defined by
\begin{equation}
r = \frac{\Gamma}{3H}
\end{equation}
When $r \gg 1$ the warm inflation is described as being in the strong regime and when $r \ll 1$ the warm inflation is in the weak regime.
Consistent models of warm inflation \cite{24} require a suppression of thermal corrections to the inflaton potential, so that the effective potential separates into inflaton and radiation components
\[ V(\phi, T) = V(\phi) + \rho_r(T), \] (6)
where \( \rho_r \) is the radiation density
\[ \rho_r = \frac{\pi^2}{30} g_* T^4. \] (7)

In this case, the time evolution is described by the equations
\[ \ddot{\phi} + (3H + \Gamma) \dot{\phi} + V_\phi = 0, \] (8)
\[ \dot{\rho}_r + 4H \rho_r = \Gamma \dot{\phi}^2, \] (9)
\[ 3H^2 = 4\pi G \left( 2V + 2\rho_r + \dot{\phi}^2 \right) \] (10)

During inflation we apply a slow-roll approximation and drop the highest derivative terms in the equations of motion,
\[ 3H(1 + r) \dot{\phi} + V_\phi = 0, \] (11)
\[ 4H \rho_r = \Gamma \dot{\phi}^2, \] (12)
\[ 3H^2 = 8\pi G V \] (13)

The validity of the slow-roll approximation depends on the slow-roll parameters defined in \cite{16},
\[ \epsilon = \frac{1}{16\pi G} \left( \frac{V_\phi}{V} \right)^2, \quad \eta = \frac{1}{8\pi G} \left( \frac{V_{\phi\phi}}{V} \right), \quad \beta = \frac{1}{8\pi G} \left( \frac{\Gamma V_\phi}{V} \right) \] (14)

The slow-roll approximation holds when \( \epsilon \ll 1 + r, \eta \ll 1 + r \) and \( \beta \ll 1 + r \). Any quantity of order \( \epsilon/(1+r) \) will be described as being first order in the slow-roll approximation.

The temperature dependence of the friction coefficient \( \Gamma \) plays an important role in the present analysis. We parameterise this by a parameter \( c \),
\[ c = \frac{T \Gamma_T}{\Gamma}, \] (15)
where \( \Gamma_T \) denotes the \( T \) derivative of \( \Gamma \). This parameter is not necessarily small, but a stability analysis of warm inflation shows that \( c < 4 \) for a consistent model \cite{24}.

III. FLUCTUATIONS

Thermal fluctuations are the main source of density perturbations in warm inflation. Thermal noise is transferred to the inflaton field; mostly on small scales. As the comoving wavelength of a perturbation expands, the thermal effects decrease until the fluctuation amplitude freezes out \cite{13}. In the strong regime of warm inflation, this occurs when the wavelength of the fluctuation is still small in comparison with the size of the cosmological horizon.

A. Inflaton fluctuations

The behaviour of a scalar field interacting with radiation can be analysed using the Schwinger-Keldysh approach to non-equilibrium field theory \cite{43, 44}. In flat spacetime, when the small-scale behaviour of the fields is averaged out, a simple picture emerges in which the field can be described by a stochastic system whose evolution is determined by a Langevin equation \cite{45}. This takes the form
\[ -\nabla^2 \phi(x, t) + \Gamma \dot{\phi}(x, t) + V_\phi = (2\Gamma T)^{1/2} \xi(x, t), \] (16)
where \( \nabla^2 \) is the flat spacetime Laplacian and \( \xi \) is a stochastic source. For a weakly interacting radiation gas the probability distribution of the source term can be approximated by a localised gaussian distribution with correlation function \cite{46, 47}.
\[ \langle \xi(x, t) \xi(x', t') \rangle = \delta(3)(x - x') \delta(t - t'). \] (17)
We shall restrict ourselves to this gaussian noise approximation.

We can use the equivalence principle to adapt the flat spacetime Langevin equation to an expanding universe during a period of warm inflation by replacing ordinary derivatives with covariant derivatives in the cosmological metric with scale factor $a$ and co-moving coordinates $x^\alpha$. The Langevin equation will retain its local form as long as the microphysical and thermal scales in the problem are small compared to the cosmological scale \cite{[7, 47]}. However, the rest frame of the fluid will have a non-zero 3–velocity with respect to the cosmological frame and we must include an advection term. Another alteration is that the coefficient of the noise term is changed slightly by the expansion (for details see ref \cite{[23]}), to $K = (2\Gamma_{\text{eff}} T)^{1/2}$, where $\Gamma_{\text{eff}} = \Gamma + H$.

The Langevin equation becomes \cite{[8]}

$$\ddot{\phi}(x, t) + 3H\dot{\phi}(x, t) + \Gamma D\phi + V_{\phi} - a^{-2}\partial^2\phi(x, t) = K\xi(x, t)$$

(18)

where $\partial^2$ is the Laplacian in the expanding frame and $D\phi$ is the derivative along the radiation fluid. The correlation function for the noise, expressed in terms of the comoving cosmological coordinates, has the form

$$\langle \xi(x, t)\xi(x', t') \rangle = a^{-3}(2\pi)^2\delta^{(3)}(x - x')\delta(t - t').$$

(19)

The fluid velocity components in a coordinate frame are $u = (\gamma, u^\alpha)$, where $\gamma$ is the Lorentz factor. With this choice of components,

$$D\phi = \gamma\dot{\phi} + u^\alpha\partial_\alpha\phi.$$  

(20)

The inflaton will generate metric inhomogeneities, but with a suitable choice of gauge, these can be discarded on sub-horizon scales (see Sect. \\ref{sect:III:C}). We shall use a uniform expansion rate gauge. Eq. (18) applies on scales which are intermediate between the thermal averaging scale and the horizon scale. Later, we use a matching argument to extend the fluctuations to large scales.

The analysis of the Langevin equation can be simplified by introducing a new time coordinate $\tau = (aH)^{-1}$ and using the slow-roll approximation. We are led to the equation

$$\phi''(x, \tau) - (3\gamma r + 2)\tau^{-1}\phi'(x, \tau) - 3r\tau^{-1}a u^\alpha\partial_\alpha\phi(x, \tau) - \partial^2\phi(x, \tau) = K\hat{\xi}(x, \tau)$$

(21)

where a prime denotes a derivative with respect to $\tau$ and we have kept only the leading terms in the slow-roll approximation. The noise term has been rescaled so that its correlation function is now

$$\langle \hat{\xi}(x, \tau)\hat{\xi}(x', \tau') \rangle = (2\pi)^3\delta^{(3)}(x - x')\delta(\tau - \tau').$$

(22)

This equation is non-linear because $\Gamma$, $u^\alpha$ and $K$ depend on $\phi$ and $T$.

Now we treat the source term as a small perturbation and expand the inflaton field

$$\phi(x, \tau) = \phi(\tau) + \delta_1\phi(x, \tau) + \delta_2\phi(x, \tau) + \ldots.$$  

(23)

where $\delta_1\phi$ is the linear response due to the source $\hat{\xi}$. Similarly, for the fluid velocity,

$$u^\alpha(x, \tau) = u_1^\alpha(x, \tau) + u_2^\alpha(x, \tau) + \ldots.$$  

(24)

This expansion is substituted into the langevin equation. Only the zeroth order terms in the slow-roll approximation will be retained.

The first two perturbation equations are

$$\delta_1\phi'' - (3r + 2)\tau^{-1}\delta_1\phi' - \partial^2\delta_1\phi + 3H^{-1}\tau^{-2}\dot{\phi}\delta_1r = K\hat{\xi},$$

(25)

$$\delta_2\phi'' - (3r + 2)\tau^{-1}\delta_2\phi' - \partial^2\delta_2\phi + 3H^{-1}\tau^{-2}\dot{\phi}\delta_2r = \delta_1K\hat{\xi} + 3r^{-1}\delta_1r\delta_1\phi' - 3ar^{-1}u_1^\alpha\partial_\alpha\delta_1\phi - 3r\dot{\phi}H^{-1}\tau^{-2}\delta_2\gamma.$$  

(26)

To leading order in the slow-roll approximation, the perturbations of $r$ are determined entirely by the temperature dependence of the friction coefficient $\Gamma$. Since $r \propto T^c$ and $\rho_r \propto T^4$, we obtain

$$\delta_1r = cr\frac{\delta_1\rho_r}{4\rho_r}$$  

(27)

$$\delta_2r = cr\frac{\delta_2\rho_r}{4\rho_r} = c\frac{r(4 - c)}{2}\left(\frac{\delta_1\rho_r}{4\rho_r}\right)^2.$$  

(28)
Similarly,
\[ \delta K = K \frac{d \ln K}{d \ln T} \frac{\delta \rho_r}{4 \rho_r}. \]  
(29)

Before substituting these quantities back into the perturbation equations, it is convenient to replace perturbations by dimensionless parameters \( \zeta_n \) and \( \epsilon_n \),
\[ \zeta_n = \frac{H \delta_n \dot{\phi}}{\phi}, \]
(30)
\[ \epsilon_n = \frac{\delta_n \rho_r}{4 \rho_r}. \]
(31)

On large scales, the parameters \( \zeta_1 \) and \( \epsilon_1 \) become the Bardeen variables for an inflaton dominated and a radiation dominated universe respectively. Note that \( \rho_r \) and \( \dot{\phi} \) are related by the slow-roll equation (12).

After substituting Eqs. (27) and (28) into the perturbation equations and converting to dimensionless variables we have
\[ \zeta_1'' - (3r + 2) \tau^{-1} \zeta'_1 - \partial^2 \zeta_1 + 3cr \tau^{-2} \epsilon_1 = \hat{K} \hat{\xi}, \]
(32)
\[ \zeta_2'' - (3r + 2) \tau^{-1} \zeta'_2 - \partial^2 \zeta_2 + 3cr \tau^{-2} \epsilon_2 = \delta_1 \hat{K} \hat{\xi} + 3 \frac{2}{c} \epsilon(4 - c) \tau^{-2} \epsilon_1^2 + 3cr^{-1} \epsilon_1' - 3ar \tau^{-1} u_1^a \partial \alpha \zeta_1 - 3r \tau^{-2} \delta_2 \gamma. \]
(33)

where \( \hat{K} = HK/\dot{\phi} \).

**B. Radiation fluctuations**

The dominant source of fluctuations in the radiation field is an inhomogeneous energy-momentum flux from the inflaton field. This transfer of momentum and energy into the radiation is described by an energy-momentum four-vector \( Q_a \),
\[ Q_a = -\Gamma u^b \partial_b \partial_a \phi, \]
(34)

where \( \Gamma \) is the 4—velocity of the radiation fluid, \( u = (\gamma, u^a) \).

We shall model the radiation field by a perfect barotropic fluid with pressure \( p = w \rho_r \), and energy momentum tensor
\[ T_{ab} = (1 + w) \rho_r u_a u_b + pg_{ab}, \]
(35)
and field equations
\[ \nabla_a T^{ab} = Q^a. \]
(36)

The time and space components of the field equations are
\[ D \rho_r + (1 + w) \rho_r \nabla_a u^a = Q, \]
(37)
\[ w \partial^a \rho_r + (1 + w) \rho_r D u_a = Q^a \]
(38)

where \( \perp \) denotes components perpendicular to \( u \) and the source terms are
\[ Q = -u^a Q_a = \Gamma(D \phi)^2, \]
(39)
\[ Q_a = -\Gamma(D \phi) \partial_a \phi. \]
(40)

As before, metric perturbations are small on sub-horizon scales and we can use the cosmological background metric with flat spacial sections. The fluid divergence is given by
\[ \nabla_a u^a = \partial_a u^a + 3H \gamma + \dot{\gamma} \]
(41)

Indices are lowered with the background metric, so that \( u_\alpha = a^2 u^a \) and \( \gamma^2 = 1 + u_\alpha u^\alpha \).
1. **First order perturbations**

First order perturbations of Eqs. (37) and (38) give

\[ \delta_1 \rho_r + 3H(1 + w)\delta_1 \rho_r + (1 + w)\rho_r \partial_\alpha u_1^\alpha = \delta_1 Q, \]  

\[ w\partial_\alpha \delta_1 \rho_r + (1 + w)\rho_r u_{1\alpha} + 3H(1 + w)\rho_r u_{1\alpha} = \delta_1 Q\alpha. \]  

(42)  

(43)

Perturbations in the energy and momentum fluxes (39) and (40) are caused by perturbations in the scalar field and perturbations in the friction coefficient, Eq. (27),

\[ \delta_1 Q = cH\delta_1 \rho_r + 2\Gamma \delta_1 \phi \]  

\[ \delta_1 Q\alpha = -\Gamma \partial_\alpha \delta_1 \phi. \]  

(44)  

(45)

From this point on we shall take a radiation fluid with \( w = 1/3 \).

The dimensionless flux perturbation \( q_n = \delta_n Q/Q \) can be introduced, in addition to the perturbations \( \zeta_1 \) and \( \epsilon_1 \) used previously. Using the time coordinate \( \tau = 1/(aH) \),

\[ a\partial_\alpha u_1^\alpha = 3\epsilon_1 - 12\tau^{-1}\epsilon_1 + 3\tau^{-1}q_1, \]  

\[ u_1^\alpha' - 5\tau^{-1}u_1^\alpha = a\partial^\alpha \epsilon_1 - 3\tau^{-1}q_1^\alpha. \]  

(46)  

(47)

where

\[ q_1 = c\epsilon_1 - 2\tau \zeta_1', \]  

\[ q_{1\alpha} = -a\tau \partial_\alpha \zeta_1. \]  

(48)  

(49)

The velocity is given in terms of the inflaton and density fluctuations by

\[ a\partial_\alpha u_1^\alpha = 3\epsilon_1' - 6\zeta_1' - (12 - 3c)\tau^{-1}\epsilon_1 \]  

After eliminating the velocity,

\[ \epsilon_1'' - (8 - c)\tau^{-1}\epsilon_1' + (20 - 5c)\tau^{-2}\epsilon_1 - \frac{1}{3}\partial^2 \epsilon_1 \]  

\[ -2\zeta_1'' + 8\tau^{-1}\zeta_1' - \partial^2 \zeta_1 = 0. \]  

(50)  

(51)

This equation was derived previously in Ref. [23].

2. **Second order perturbations**

Second order perturbations of Eqs. (37) and (38) give

\[ \delta_2 \rho_r + 3H(1 + w)\delta_2 \rho_r + (1 + w)\rho_r \partial_\alpha u_2^\alpha \]  

\[ + D_1 \delta_1 \rho_r + (1 + w)\delta_1 \rho_r \partial_\alpha u_1^\alpha + (1 + w)H\rho_r \theta_2 = \delta_2 Q \]  

\[ w\partial_\alpha \delta_2 \rho_r + (1 + w)\rho_r \partial_\alpha u_2^\alpha + 3H(1 + w)\rho_r u_{2\alpha} \]  

\[ + u_{1\alpha} \delta_1 \rho_r + (1 + w)\delta_1 \rho_r u_{1\alpha} + (1 + w)\rho_r D_1 u_{1\alpha} = \delta_2 Q\alpha - u_{1\alpha} \delta_1 Q \]  

(52)  

(53)

where \( D_1 = u_1^\alpha \partial_\alpha \) and \( H\theta_2 = 3H\delta_2 \gamma + \delta_2 \gamma' \). In terms of dimensionless parameters and \( w = 1/3 \),

\[ a\partial_\alpha u_2^\alpha = 3\epsilon_2' - 12\tau^{-1}\epsilon_2 + 3\tau^{-1}q_2 \]  

\[ -4a\epsilon_1 \partial_\alpha u_1^\alpha - 3aD_1 \epsilon_1 - 3\tau^{-1}\delta_2 \gamma + \delta_2 \gamma' \]  

\[ u_2^\alpha' - 5\tau^{-1}u_2^\alpha - a\partial^\alpha \epsilon_2 + 3\tau^{-1}q_2^\alpha = \]  

\[ -u_1^\alpha \epsilon_1 - 4\epsilon_1 u_1^\alpha + 8\tau^{-1}\epsilon_1 v_1^\alpha + aD_1 u_1^\alpha + 3\tau^{-1}u_1^\alpha q_1. \]  

(54)  

(55)

The second order momentum flux is

\[ q_2^\alpha = -a\tau \partial^\alpha \zeta_2 - ca\tau \partial^\alpha \zeta_1 + a\tau^2 \zeta_1' \partial^\alpha \zeta_1, \]  

(56)

and the second order energy flux is

\[ q_2 = c\epsilon_2 - 2\tau \zeta_2' - \frac{1}{2}c(4 - c)\epsilon_2' - 2c\tau \epsilon_1 \zeta_1 + \tau^2 \zeta_1' + 2a\tau D_1 \zeta_1 + 2\delta_2 \gamma. \]  

(57)

An equation for the energy density fluctuations can be found by eliminating the velocity as before. We have done this using a computer algebra package, and the result is given in the summary section below.
We have been claiming up to now that the metric fluctuations play no role in the analysis. This can be checked using the gauge ready equations for metric perturbations from Ref. [18]. The new quantities we require at first order are the proper time perturbation $\alpha_1$, the scale factor perturbation $\varphi_1$, the perturbed expansion of the hypersurface normal vectors $\kappa_1$ and the perturbed shear of the hypersurface normal vectors $\chi_1$. For convenience we pick a uniform expansion-rate gauge $\kappa_1 = 0$, although the conclusions hold in for any reasonable gauge choice.

When the metric perturbations are included and we take the leading order in the slow-roll approximation, the scalar field equation [32] and the fluid equation (47) become

$$\zeta'' - (3r + 2)\tau^{-1}\zeta' - \partial^2 \zeta_1 + 3cr\tau^{-2}\epsilon_1 + \tau^{-1}\alpha' - 3\tau^{-2}\alpha_1 = \dot{K}\hat{\epsilon},$$

$$u_1'' - 5\tau^{-1}u_1' = a\partial^2(\epsilon_1 - \alpha_1) - 3\tau^{-1}\dot{q}_1.$$  \hfill (58)

The Raychaudhuri equation gives the metric perturbation $\alpha_1$ (see Eq. (14) in [48] or Eq. (A22) in [8]),

$$\alpha_1 = \frac{\epsilon}{(1 + r)^2} k^2 \tau^2 - 3\epsilon/(1 + r) + 4\epsilon/(1 + r)^2$$ \hfill (60)

Clearly $\alpha_1 << \epsilon_1$ and $\alpha_1 << \zeta_1$, not only for sub-horizon scales $k\tau >> 1$ but also for scales comparable to the horizon when $k\tau \approx 1$. The equations for the second order metric variation $\alpha_2$ follow a similar pattern, apart from the presence of a large number of extra quadratic terms in the first order perturbations. The effects of $\alpha_2$ are also suppressed by the slow-roll parameters.

D. Summary

We now have a full set of perturbation equations up to second order which can be used to determine the radiation and scalar field fluctuations $\epsilon$ and $\zeta$. The first order equations were solved numerically in Ref [23], and an encouraging feature of the numerical work was that the results agreed well with analytic approximations. We shall be making similar approximations to the second order equations in the next section.

The first order equations are

$$\zeta'' - (3r + 2)\tau^{-1}\zeta' - \partial^2 \zeta_1 + 3cr\tau^{-2}\epsilon_1 = \dot{K}\hat{\epsilon},$$

$$\epsilon'' - (8 - c)\tau^{-1}\epsilon' + (20 - 5c)\tau^{-2}\epsilon_1 - \frac{4}{\tau}\partial^2 \epsilon_1 - 2\zeta'' + 8\tau^{-1}\zeta' - \partial^2 \zeta_1 = 0.$$ \hfill (62)

The fluid velocity is irrotational at this order, and it can be determined by the equation,

$$a\partial_\omega u_1^a = 3\epsilon'_1 - 6\zeta'_1 - (12 - 3c)\tau^{-1}\epsilon_1.$$ \hfill (63)

The second order perturbation equations are

$$\zeta'' - (3r + 2)\tau^{-1}\zeta' - \partial^2 \zeta_1 + 3cr\tau^{-2}\epsilon_1 = \delta_1 \dot{K}\hat{\epsilon},$$

$$\epsilon'' - (8 - c)\tau^{-1}\epsilon' + (20 - 5c)\tau^{-2}\epsilon_1 - \frac{4}{\tau}\partial^2 \epsilon_2 - 2\delta_1'' + 8\tau^{-1}\delta_1' - \partial^2 \delta_1 = j^{(1)} + j^{(2)} + j^{(3)},$$ \hfill (65)

where the source terms $j^{(n)}$ are ordered so that increasing $n$ implies decreasing size for large $\tau$,

$$j^{(1)} = a^2 \partial^2 [(c\epsilon_1 - \tau\zeta'_1)\partial_\alpha \zeta_1] - \frac{1}{3}a^2 \partial^2 (3\zeta_1 + \epsilon_1)\partial_\omega (9\zeta_1 + 2\epsilon_1),$$

$$j^{(2)} = (2\epsilon_1 - 2\tau\zeta'_1)^2 - 27\epsilon_1^2 - (24 - 2c)\zeta'_1 + 2(27 - 3c)\tau^{-1}\epsilon_1 - \frac{1}{2}c\dot{\omega}_1^2 + 32(3c - 8)(c - 4)c\tau^{-2}\epsilon_1^2 + \frac{1}{2}(7c - 48)\dot{\omega}_1^2 + a\omega_1^2 \partial_\alpha \omega,$$

$$j^{(3)} = -11\tau^{-2}\delta_2,$$ \hfill (66)

where

$$\omega = -7\zeta'_1 + \frac{5}{2}\zeta'_1 + \frac{1}{3}(6c - 3)\epsilon_1 - 11\tau^{-1}\zeta_1.$$ \hfill (69)
IV. STRONG DISSIPATION AND SMALL SCALES

The perturbation equations simplify considerably in the strong regime of warm inflation. In this regime, the thermal fluctuations are generated on small scales, compared to the cosmological horizon, and their amplitude approaches a simple power-law behaviour. This contrasts with vacuum fluctuations, which oscillate on sub-horizon scales and only freeze-out when they grow larger than the horizon.

Fourier transforms are defined with respect to the comoving coordinates $x^\alpha$,

$$\zeta_n(k) = \int d^3 x \zeta_n(x)e^{i k \cdot x} \quad (70)$$

The fluid velocity decomposes into an irrotational scalar component $u_n$ along $k$ and a solenoidal component $u_n^{\alpha} \perp k$. Only the scalar component,

$$u_n = i a \hat{k}_n \zeta_n, \quad (71)$$

contributes to the scalar density perturbations at second order. In this section we shall take $r = \Gamma/3H \gg 1$ and concentrate on physical scales small compared to the horizon, where the wave number $k \gg aH$.

From Eq. (62), the first-order density and scalar fluctuations are related by

$$\epsilon_1 \approx -3 \zeta_1. \quad (72)$$

This approximation can be substituted into Eq. (25).

$$\zeta_1'' - (3r + 2) \tau^{-1} \zeta_1' + k^2 \tau^{-2} \zeta_1 = \hat{K} \hat{\zeta} \quad (73)$$

We solve this using a Green function,

$$\zeta_1 = \frac{1}{k} \int_\tau^\infty G(k\tau, k\tau')(k\tau')^{-1-2\nu} \hat{K} \hat{\zeta}(\tau') d\tau', \quad (74)$$

where the retarded Green function $G(z, z')$ is given by

$$G(z, z') = \frac{\pi}{2} z^{\nu} z'^{\nu} (J_{\nu+3c}(z)Y_{\nu+3c}(z') - J_{\nu+3c}(z')Y_{\nu+3c}(z)) \quad \text{for } z < z'. \quad (75)$$

and $\nu = 3r/2$. A useful feature of the solution is that in the range $1 << k\tau << \sqrt{\nu}$ the fluctuation has an approximate power law behaviour, $\zeta_1 \propto \tau^{-3c}$. This allows us to use

$$\zeta_1' \approx -3c \tau^{-1} \zeta_1. \quad (76)$$

We can use both the approximations (72) and (76) in the velocity equation (63), to get

$$k u_1 \approx 36(1 + c) \tau^{-1} \zeta_1. \quad (77)$$

At second order in the perturbation amplitude, the dominant terms for large $k\tau$ in Eq. (65) are

$$k^2 \epsilon_2 + 3k^2 \zeta_2 = j^{(1)}(k), \quad (78)$$

where $j^{(1)}$ is given by Eq. (66). The approximations (72) and (76) imply $j^{(1)} \approx 0$ and

$$\epsilon_2 \approx -3 \zeta_2. \quad (79)$$

We use this together with the approximations (72), (76) and (77) in Eq. (64) to get

$$\zeta_2'' - (3r + 2) \tau^{-1} \zeta_2' + k^2 \tau^{-2} \zeta_2 \approx A_s \tau^{-2} \zeta_1 \ast \zeta_1 + A_v \tau^{-2} (k^2 \zeta_1) \ast (k\alpha \zeta_1) + A_r \hat{K} \zeta_1 \ast \hat{\zeta}, \quad (80)$$

where $\ast$ denotes a convolution and

$$A_s = \frac{27}{2} c(4 + c)r, \quad (81)$$

$$A_v = -108(1 + c)r, \quad (82)$$

$$A_r = -\frac{3}{2}(1 + c). \quad (83)$$
Note that, when the friction coefficient is temperature independent and \( c = 0 \), then only the \( A_v \) and \( A_r \) terms survive. These terms agree with the \( c = 0 \) case investigated in Ref. [8], apart from a sign change in \( A_v \) caused by an error in the sign of the green function used in the earlier work.

The \( A_s \) term arises for the temperature dependence of the friction coefficient. This term is local in space, and we might therefore expect it to produce a local type of non-gaussianity. The \( A_v \) term arises from the fluid advection terms \( u^\alpha \partial_\alpha \) in the perturbation equations, and this term is associated closely with the bulk behaviour of the radiation field. It distinguishes between vacuum and non-vacuum fluctuations.

V. NON-GAUSSIANITY AND THE BISPECTRUM

The small scale inflaton fluctuations develop a simple power-law growth well in advance of the time when they cross the horizon. During this stage the metric fluctuations are relatively small (in the uniform expansion-rate gauge). Eventually, the wavelength of the perturbations crosses the effective cosmological horizon and the wavelength grows to a regime where the metric perturbations become important. On large scales it becomes possible to use a small-spatial-gradient expansion (first formalised by Salopeck and Bond [49] and later developed into the ‘delta-N’ formalism [50–52]). This approach allows us to define a perturbation \( \zeta \) which is constant on large scales, even in the full non-linear theory. The bispectrum and the non-linearity of the density fluctuations can be approximated, to a reasonable accuracy, by matching the small and large scale approximations at horizon crossing.

In a uniform spatial curvature gauge, the perturbation \( \zeta \) is defined by

\[
\zeta = \int_0^{\phi(x, \tau)} \frac{H(\phi')}{\phi(\phi')} d\phi',
\]

where \( H \equiv H(\phi) \) and \( \dot{\phi} \equiv \dot{\phi}(\phi) \) are given by solving the slow-roll equations [11,13]. When the inflaton perturbations are expanded as before in eq. (23), we have

\[
\zeta = \zeta_0 \delta_1 \phi + \zeta_\phi \delta_2 \phi + \frac{1}{2} \zeta_\phi \delta_3 \phi \ast \delta_1 \phi + \ldots
\]

(85)

where \( \phi \) subscripts denote derivatives with respect to \( \phi \) and \( \zeta_\phi = H/\dot{\phi} \). Note that, according to eqs. (11,13), \( \zeta_{\phi \phi} / \zeta_\phi^2 \) can be dropped because it is first order in the slow-roll expansion. We therefore have

\[
\zeta = \zeta_1 + \zeta_2 + \ldots,
\]

(86)

where the \( \zeta_n \) are identical to the parameters defined in Eq. (30) of the previous section.

The power spectrum and bispectrum of the density perturbations were defined in the introduction. The first order perturbations \( \zeta_1 \) are gaussian fields and their bispectrum vanishes. The leading order contribution to the bispectrum must therefore include a contribution to \( \zeta \) from the second order perturbation,

\[
\sum_{\text{cyclic}} \langle \zeta_1(k_1, \tau) \zeta_1(k_2, \tau) \zeta_2(k_3, \tau) \rangle \approx (2\pi)^3 B_1(k_1, k_2, k_3) \delta^3(k_1 + k_2 + k_3),
\]

(87)

where ‘cyclic’ denotes cyclic permutations of \( \{k_1, k_2, k_3\} \). The second order perturbation \( \zeta_2 \) can be obtained by solving Eq. (80) using Eq. (74).

We shall split the second order perturbation \( \zeta_2 \), into three parts \( \zeta_2 = \zeta_1 + \zeta_v + \zeta_r \), where each part is sourced by the corresponding terms with coefficients \( A_s \), \( A_v \) or \( A_r \) in Eq. (80). Beginning with the \( \zeta_s \) term, we have

\[
\zeta_s'' - (3r + 2) \tau^{-1} \zeta_s' + k^2 \zeta_s - 9c \tau^2 \zeta_s = A_s \tau^{-2} \zeta_s, \quad \zeta_s \equiv \zeta_1 + \zeta_v + \zeta_r.
\]

(88)

The solution can be obtained using the green function (75).

\[
\zeta_s = A_s \int_\tau^\infty k d\tau' G(k\tau, k\tau')(k\tau')^{-1 - 2\nu} \zeta_1 \ast \zeta_1
\]

(89)

The contribution \( B_s \) to the density fluctuation bispectrum from \( \zeta_s \) is

\[
(2\pi)^3 B_s(k_1, k_2, k_3) \delta^3(k_1 + k_2 + k_3) = A_s \sum_{\text{cyclic}} \int_\tau^\infty k_3 d\tau' G(k_3\tau, k_3\tau')(k_3\tau')^{-1 - 2\nu} \langle \zeta_1(k_1, \tau) \zeta_1(k_2, \tau) \zeta_1(k_3, \tau') \rangle
\]

(90)
Now we can use Eq. \[74\] for \(\zeta_1\) and decompose the stochastic average of the noise terms into products of the correlation function \[22\]. The result is that

\[
B_s = 2A_s \sum_{\text{cyclic}} \frac{\partial^4}{\partial k_1^4 k_2^4} \int_\tau^\infty k_3 d\tau' G(k_3, k_3\tau')(k_3\tau')^{-1-2\nu} F(k_1\tau, k_1\tau') F(k_2\tau, k_2\tau'),
\]

where

\[
F(k\tau_1, k\tau_2) = k \int_{\tau_2}^\infty d\tau' G(k\tau_1, k\tau') G(k\tau_2, k\tau')(k\tau')^{2-4\nu}.
\]

This integral can be evaluated analytically when \(\nu\) is large. The integral is expressed in terms the power spectrum \(P_\zeta(k, \tau)\) in appendix A and gives

\[
B_s = 2A_s \sum_{\text{cyclic}} P_\zeta(k_1, \tau) P_\zeta(k_2, \tau)(k_3\tau)^{6c} \int_\tau^\infty k_3 d\tau' G(k_3\tau, k_3\tau')(k_3\tau')^{-1-2\nu-6c}.
\]

This integral can also be found in appendix A leaving

\[
B_s = \frac{2A_s}{\nu^c} f(c) \sum_{\text{cyclic}} P_\zeta(k_1, \tau) P_\zeta(k_2, \tau),
\]

where we have used \(\nu = \Gamma/2H\). The factor \(f(c)\) is,

\[
f(c) = 1 - (2\Gamma/H)^{-3c/2}.
\]

We can usually take \(f(c) \approx 1\) unless \(c\) is very small. The value of \(A_s\) was found in the previous section \[81\], so that

\[
B_s = 3(4 + c)f(c) \sum_{\text{cyclic}} P_\zeta(k_1, \tau) P_\zeta(k_2, \tau).
\]

This part of the bispectrum has a local form, as expected, and the value of the non-linearity parameter for \(B_s\) alone is,

\[
f_{\text{local}} = (4 + c)f(c).
\]

Remarkably, this is only weakly dependent on \(\Gamma\). However, some caution needs to be exercised due to the rather crude nature of the matching procedure applied at the horizon scale, where the approximations used in Sect. \[IV\] break down.

In Ref. \[23\], an alternative approximation scheme was introduced to cover fluctuations at the horizon scale. This approximation implied that the density fluctuation power spectra approach constant values at the horizon scale, with \(P_\zeta(k_1, \tau) \propto k^{-3}\) (or \(k^{-4}\) at first order in the slow-roll parameters). In the present context, this suggests that the small scale approximation will extend to the horizon scale when the bispectrum is expressed in terms of the power spectrum, as we have done above.

The bispectrum reaches its largest values for oblique wave-vector triangles, with \(k_1 << k_2 \approx k_3\) (up to permutations of the sides). However, if the triangle becomes too oblique, then the result no longer applies because it is not possible to have \(k_1\tau\) and \(k_2\tau\) in the same range where the approximations are valid. The solution \[8\] is to cut off the bispectrum when \(k_2/k_1 > r = \Gamma/3H\) (or similarly for any other pairs of sides). There is some residual dependence of the bispectrum on \(\Gamma\) due to this effect. The bispectrum \(B_\rho\) with a cut-off for \(k_2/k_1 > r\) is plotted in figure 1.

The other contributions to the bispectrum are obtained by following the same steps. The result for \(B_v\) is

\[
B_v = -\frac{12}{c}(1 + c)f(c) \sum_{\text{cyclic}} (k_1^{-2} + k_2^{-2})k_1 \cdot k_2 P_\zeta(k_1) P_\zeta(k_2),
\]

Note that, in the limit \(c \to 0\), the function \(f(c) \sim (3c/2)\ln(\Gamma/2H)\) and the result agrees with Ref. \[8\], apart from the overall sign mentioned earlier. The bispectrum is concentrated on oblique triangle shapes as before, but this time the contribution to \(f_{\text{NL}}\) is strongly dependent of the shape of the wave vector triangle and even changes sign as the triangle becomes more oblique. The dependence of the bispectrum \(B_v\) on \(k_1\) and \(k_2\) for \(r = 20\) is plotted in Fig. 2. The contribution from the squeezed triangles is reduced relative to the local case, but the squeezed triangle contribution rises for larger values of \(r\).
FIG. 1: A contour plot of the truncated local bispectrum $B_s$ as a function of $k_1$ and $k_2$. The plot shows $B_s(x_1, x_2, 1)x_1^2x_2^2 \tanh^2(rx_1) \tanh^2(rx_2)/B_s(1, 1, 1)$, where $x_1 = k_1/k_3$, $x_2 = k_2/k_3$ and $r = 20$.

FIG. 2: A contour plot of the truncated local bispectrum $B_s$ as a function of $k_1$ and $k_2$. The plot shows $B_v(x_1, x_2, 1)x_1^2x_2^2 \tanh^2(rx_1) \tanh^2(rx_2)/B_v(1, 1, 1)$, where $x_1 = k_1/k_3$, $x_2 = k_2/k_3$ and $r = 20$.

The final contribution to the bispectrum $B_r$ depends on the integral (A18). This contribution is suppressed by factors of $\Gamma/3H$. It is only relevant when $c \approx 0$, and even then this contribution is dominated by $B_v$. The total bispectrum is effectively the sum of two terms,

$$B = \frac{6}{5} f_{NL}^{\text{local}} \sum_{\text{cyclic}} P_\zeta(k_1)P_\zeta(k_2) - \frac{6}{5} f_{NL}^{\text{adp}} \sum_{\text{cyclic}} (k_1^{-2} + k_2^{-2}) \mathbf{k}_1 \cdot \mathbf{k}_2 P_\zeta(k_1)P_\zeta(k_2),$$

(99)
where $f_{adv}^{nl}$ is named after fluid advection terms. The negative sign in front of $f_{adv}^{nl}$ is chosen so that, for equilateral triangles, $f_{NL} = f_{NL}^{local} + f_{adv}^{nl}$.

The values of $f_{NL}^{local}$ and $f_{adv}^{nl}$ are plotted in Fig. 3. The contribution to the bispectrum from $B_v$ dominates at small values of $c$, but then decreases, whilst the local component $B_s$ becomes larger as $c$ increases. The momentum dependence makes the contribution to $B$ from $B_v$ harder to detect, but since this component is characteristic of warm inflation it would be desirable to try to isolate this component in future CMB experiments.

FIG. 3: A plot of $f_{NL}$ for equilateral triangles against $c$. The curves represent two different contributions with different dependence on wave numbers. In this plot, $\Gamma/3H = 10$ and $\Gamma \propto T^c$. The contribution from $f_{adv}^{nl}$ should be multiplied by the values in Fig. 1 for other triangle shapes, and experiments which search for constant (or local) $f_{NL}^{local}$ will see predominantly $f_{NL}^{local}$.

VI. CONCLUSIONS AND OUTLOOK

The main results of this paper are given in Fig. 3 which shows the amplitudes of two contributions to the bispectrum for primordial fluctuations which originate in the strong regime of warm inflation, with friction coefficient $\Gamma \propto T^c$. Contemporary models of warm inflation typically have $c = 3$ [20]. The local component with amplitude $f_{NL}^{local}$ is easier to detect, and would most likely be seen first.

It would be very desirable to search for the component of the bispectrum with amplitude $f_{adv}^{nl}$, since this component is a characteristic feature of fluctuations which originate from a non-vacuum source. Some idea of the difficulty of measuring both components of the bispectrum can be gauged by considering an ideal experiment where the only source of noise is the cosmic variance. We can parameterise the bispectrum as we did in Eq. (99),

$$ B = \sum_i f_i B_i^j, $$

where $i = 1, 2$ for $f_{local}^{nl}$ of $f_{adv}^{nl}$ respectively. The observations give a set of spherical harmonic components $B_{l_1l_2l_3}^{obs}$, with an upper limit $l \leq l_{\text{max}}$. The estimator for the parameter $f_i$ is

$$ \hat{f}_i = \sum_j (F^{-1})_{ij} \sum_{l_1l_2l_3} B_{l_1l_2l_3}^{obs} \frac{B_{l_1l_2l_3}^j}{6C_{l_1}C_{l_2}C_{l_3}}, $$

where $C_l$ are obtained from the power spectrum, and the Fisher matrix

$$ F_{ij} = \sum_{l_1l_2l_3} \frac{B_{l_1l_2l_3}^i B_{l_1l_2l_3}^j}{6C_{l_1}C_{l_2}C_{l_3}}. $$
The Fisher matrix for the relevant models was evaluated in Ref. [41], where it was found that the standard deviation of the estimator $f_{NL}^{adv}$ is around 5 times larger than the standard deviation in the estimator $f_{NL}^{local}$. For Planck, the detection limit for $f_{NL}^{local}$ is expected to be around $5 - 10$, depending on how successfully the backgrounds can be removed. This would imply that Planck would only be able to detect the presence of $f_{NL}^{adv}$ if the value was at least 25. The problems arising from cosmic variance could be overcome by taking higher resolution surveys, but then the issue of backgrounds contributing to $f_{NL}^{adv}$ has to be addressed.

There is another regime of warm inflation, the weak regime, where the approximations we have used here all break down. Nevertheless, the dependence of the bispectrum on the wave-vector triangle will contain the same components that we found here, with the possibility of two extra components appearing from terms which we have been able to discard in the strong regime of warm inflation. In the weak regime we have to resort to numerical investigation of the second order perturbation equations to determine the bispectrum fully, and this is something we hope to report on at a later date.

**Appendix A: Integrals**

We begin with an approximation to the integral

$$I(k\tau) = k \int_{\tau}^{\infty} d\tau' G(k\tau, k\tau')(k\tau')^{1-2\nu-6c}$$  \hspace{1cm} (A1)

where $\nu = \Gamma/2H$ and the retarded green function $G$ is given in Eq. (75). The leading contributions to the integral for large $\nu$ and fixed $\tau$ come from

$$I(k\tau) \approx -\frac{\pi}{2}(k\tau)^{-3c}(k\tau)^{\nu} Y_{\nu'}(k\tau) \int_{k\tau}^{\infty} J_{\nu'}(z) z^{-1-\nu'-3c} dz,$$  \hspace{1cm} (A2)

where $\nu' = \nu + 3c$. We also have

$$Y_{\nu}(z) \sim -z^{-\nu} \frac{2^{\nu}}{\pi \Gamma_R(\nu)}, \quad J_{\nu}(z) \sim z^{\nu} \frac{2^{-\nu}}{\Gamma_R(\nu + 1)}.$$  \hspace{1cm} (A3)

which give

$$I(k\tau) \sim \frac{1}{6c\nu}(k\tau)^{-6c}.$$  \hspace{1cm} (A4)

The approximations used above fail when $c$ is small. In that case, there is a saddle point in the integrand which comes to dominate the result. The easiest way to obtain the contribution from this saddle point is to obtain the result for $c < 0$ and extend to $c > 0$. When $c < 0$,

$$I(k\tau) \approx -\frac{\pi}{2}(k\tau)^{-3c}(k\tau)^{\nu} Y_{\nu'}(k\tau) \int_{k\tau}^{\infty} J_{\nu'}(z) z^{-1-\nu'-3c} dz.$$  \hspace{1cm} (A5)

The integral,

$$\int_{0}^{\infty} J_{\nu'}(z) z^{-1-\nu'-3c} dz = 2^{-3c-1-\nu'} \frac{\Gamma_R(3c/2)}{\Gamma_R(\nu' + 1 + 3c/2)}.$$  \hspace{1cm} (A6)

For large $\nu$ and small $c$,

$$I(k\tau) \sim -\frac{1}{6c\nu}(k\tau)^{-3c(4\nu)^{-3c/2}}.$$  \hspace{1cm} (A7)

Combining the two asymptotic results together gives

$$I(k\tau) \sim \frac{1}{6c\nu}(k\tau)^{-6c} \left(1 - (4\nu)^{-3c/2}\right).$$  \hspace{1cm} (A8)

The next integral is

$$F(k\tau) = k \int_{\tau}^{\infty} d\tau' G(k\tau, k\tau')G(k\tau, k\tau')(k\tau')^{2-4\nu}$$  \hspace{1cm} (A9)
This integral arises when we calculate the power spectrum using Eq. (74),

\[ P_\zeta(k, \tau) = k^{-3} \hat{K}^2 F(k\tau). \]  

The leading contributions to the integral for large \( \nu \) and fixed \( \tau \) come from

\[ F(k\tau) \approx \frac{\pi^2}{4}(k\tau)^{2\nu} Y_\nu(k\tau)^2 \int_0^\infty J_\nu(z)^2 z^{2-2\nu} dz, \]  

where \( \nu' = \nu + 3c \). This gives a standard Schafheitlin integral,

\[ \int_0^\infty J_\nu(z)^2 z^{2-2\nu} dz = \frac{\Gamma_R(\nu - 1/2)\Gamma_R(2\nu + 3c - 1/2)}{2\sqrt{\pi} \Gamma_R(3c + 3/2)}, \]  

where \( \Gamma_R \) is the gamma function. Hence,

\[ F(k\tau) \sim \sqrt{\frac{\pi}{32\nu}} \frac{\Gamma_R(3c + 3/2)}{\Gamma_R(3/2)} \left( \frac{2\nu}{k^2 \tau^2} \right)^{3c} \left( 1 + \frac{k^2 \tau^2}{2\nu} + \ldots \right), \]  

for \( k\tau << \nu^{1/2} \). The \( \mathcal{O}(\nu^{-1}) \) terms come from the expansion of \( Y_\nu(k\tau) \). The next integral is

\[ F(k\tau_1, k\tau_2) = k \int_{\tau_2}^{\tau_1} d\tau' G(k\tau_1, k\tau')G(k\tau_2, k\tau')(k\tau')^{2-4\nu} \]  

This is evaluated in exactly the same way as \( F(k\tau) \),

\[ F(k\tau_1, k\tau_2) \sim \left( \frac{\tau_1}{\tau_2} \right)^{3c} F(k\tau_1). \]  

We can also use Eq. (A10) to express this in terms of the power spectrum,

\[ \hat{K}^2 F(k\tau_1, k\tau_2) \sim \left( \frac{\tau_1}{\tau_2} \right)^{3c} k^3 P_\zeta(k, \tau_1) \]  

Another integral of this type is

\[ \hat{F}(k_1, k_2) = (k_1 k_2)^{1/2} \int_{\tau}^{\infty} d\tau' G(k_1 \tau, k_1 \tau')G(k_2 \tau, k_2 \tau')(k_1 \tau')^{1-2\nu}(k_2 \tau')^{1-2\nu}. \]  

This time, for large \( \nu \),

\[ \hat{F}(k_1, k_2) \sim \left( \frac{2k_1 k_2}{k_1^2 + k_2^2} \right)^{(3c+3)/2} (2\nu)^{-3c/2} F(k_1 \tau, k_2 \tau). \]  

This integral is negligible compared to \( F(k_1 \tau, k_2 \tau) \).

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