Equating the area sums of alternative sectors in a circle

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Abstract

We determine the conditions resulting from equating the area sums of alternative sectors in a circle generated by four, two, and three straight lines, respectively, that connect opposite points on its circumference while passing through a point that is arbitrarily placed within the circle.
I. INTRODUCTION

Circles are used in the modelling of physical phenomena in an amazing number and variety of situations in engineering, science, and other fields. The planetary orbits and their moons, the movement of electrons around nuclei, the path of a turning vehicle on road, the motion of galaxies etc are all represented, in their first approximations, as circles.

Problems in circle geometry [1, 2] are quite often not intuitively obvious and can appear to be surprising. One such problem is equating the area sums of alternative sectors in a circle and the required conditions. We consider sectors that are generated by four, two, and three straight lines, respectively, that connect opposing points on its circumference while passing through a point that is arbitrarily placed within the circle. We then determine the conditions that result from equating the area sums of these alternative sectors.

II. THE EIGHT SECTOR CASE

Using polar coordinates, the equation of a circle of radius $a$ and centre $C$ at $(r_0, \theta_0)$ is given by

$$r(\theta) = r_0 \cos(\theta - \theta_0) + \sqrt{a^2 - r_0^2 \sin^2(\theta - \theta_0)}$$  \hspace{1cm} (1)

Consider four lines passing through the pole O that join opposing points (relative to the pole O) on the circumference of the circle. The lines divide the circle into eight sectors that are denoted by $S_1, S_2...S_8$. 

Using polar integration the areas of these sectors can be calculated as

\[ S_i = \frac{1}{2} \int_{\theta_i}^{\theta_{i+1}} r^2(\theta) \, d\theta, \quad 1 \leq i \leq 8. \]

Note that

\[ \theta_5 = \theta_1 + \pi, \quad \theta_6 = \theta_2 + \pi, \quad \theta_7 = \theta_3 + \pi, \quad \theta_8 = \theta_4 + \pi. \] (2)

These can be evaluated as

\[ S_1 = \frac{r_0^2}{2} \int_{\theta_1}^{\theta_2} \left[ 2 \cos^2(\theta - \theta_0) - 1 + \left( \frac{a}{r_0} \right)^2 + 2 \sqrt{\left( \frac{a}{r_0} \right)^2 - \sin^2(\theta - \theta_0) \cos(\theta - \theta_0)} \right] \, d\theta \] (3)

or

\[ S_1 = r_0^2 \int_{\theta_1}^{\theta_2} \cos^2(\theta - \theta_0) \, d\theta + \frac{1}{2} \left( a^2 - r_0^2 \right) \int_{\theta_1}^{\theta_2} d\theta + \frac{2a}{r_0} \int_{\theta_1}^{\theta_2} \sqrt{1 - \left( \frac{r_0}{a} \right)^2 \sin^2(\theta - \theta_0) \cos(\theta - \theta_0)} \, d\theta \] (4)

Consider now the integral
\[ I = \int_{\theta_1}^{\theta_2} \sqrt{1 - \left(\frac{r_0}{a}\right)^2 \sin^2(\theta - \theta_0) \cos(\theta - \theta_0)} d\theta \]  

(5)

As \( a \geq r_0 \geq 0 \) we use the following substitution

\[ \frac{r_0}{a} \sin(\theta - \theta_0) = \sin x \]  

(6)

and the integral \( I \) becomes

\[ I = \int_{x_1}^{x_2} \sqrt{1 - \sin^2 x} \left(\frac{a}{r_0}\right) \cos x dx \]  

(7)

where

\[ x_1 = \sin^{-1} \left(\frac{r_0}{a} \sin(\theta_1 - \theta_0)\right) \quad \text{and} \quad x_2 = \sin^{-1} \left(\frac{r_0}{a} \sin(\theta_2 - \theta_0)\right) \]  

(8)

and we obtain

\[ I = \frac{a}{r_0} \int_{x_1}^{x_2} \frac{1 + \cos 2x}{2} \left(\frac{a}{r_0}\right) \cos x dx = \frac{a}{2r_0} \left[ (x_2 - x_1) + \frac{1}{2} (\sin 2x_2 - \sin 2x_1) \right] \]  

(9)

Now the first integral in (1) is

\[ r_0^2 \int_{\theta_1}^{\theta_2} \cos^2(\theta - \theta_0) d\theta = \frac{r_0^2}{2} \left[ (\theta_2 - \theta_1) + \frac{1}{2} (\sin 2(\theta_2 - \theta_0) - \sin 2(\theta_1 - \theta_0)) \right] \]  

(10)

and \( S_1 \) can then be re-expressed as

\[ S_1 = \frac{r_0^2}{2} [(\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0)] + \]  

\[ \frac{1}{2} (a^2 - r_0^2) (\theta_2 - \theta_1) + \]  

\[ \left(\frac{a}{r_0}\right)^2 [(x_2 - x_1) + \cos(x_1 + x_2) \sin(x_2 - x_1)] \]  

(11)

where \( x_1 \) and \( x_2 \) are defined in (8).

Similarly, we can find
\[ S_3 = \frac{r_0^2}{2} \left[ (\theta_4 - \theta_3) + \sin(\theta_4 - \theta_3) \cos(\theta_3 + \theta_4 - 2\theta_0) \right] + \]
\[ \frac{1}{2} (a^2 - r_0^2)(\theta_4 - \theta_3) + \]
\[ \left( \frac{a}{r_0} \right)^2 \left[ (x_4 - x_3) + \cos(x_3 + x_4) \sin(x_4 - x_3) \right] \tag{12} \]

where

\[ x_3 = \sin^{-1} \left( \frac{r_0}{a} \sin(\theta_3 - \theta_0) \right) \quad \text{and} \quad x_4 = \sin^{-1} \left( \frac{r_0}{a} \sin(\theta_4 - \theta_0) \right), \tag{13} \]

and

\[ S_5 = \frac{r_0^2}{2} \left[ (\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) \right] + \]
\[ \frac{1}{2} (a^2 - r_0^2)(\theta_2 - \theta_1) + \]
\[ \left( \frac{a}{r_0} \right)^2 \left[ (x_6 - x_5) + \cos(x_5 + x_6) \sin(x_6 - x_5) \right] \tag{14} \]

where

\[ x_5 = \sin^{-1} \left( \frac{r_0}{a} \sin(\theta_5 - \theta_0) \right) \quad \text{and} \quad x_6 = \sin^{-1} \left( \frac{r_0}{a} \sin(\theta_6 - \theta_0) \right). \tag{15} \]

Using Eqs. (2) and that \( \sin(\pi + \theta) = -\sin \theta, \sin^{-1}(-x) = -\sin(x) \), we obtain

\[ x_5 = -\sin^{-1} \left( \frac{r_0}{a} \sin(\theta_1 - \theta_0) \right) = -x_1, \]
\[ x_6 = -\sin^{-1} \left( \frac{r_0}{a} \sin(\theta_2 - \theta_0) \right) = -x_2, \tag{16} \]

and we have

\[ S_5 = \frac{r_0^2}{2} \left[ (\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) \right] + \]
\[ \frac{1}{2} (a^2 - r_0^2)(\theta_2 - \theta_1) - \]
\[ \left( \frac{a}{r_0} \right)^2 \left[ (x_2 - x_1) + \cos(x_1 + x_2) \sin(x_2 - x_1) \right]. \tag{17} \]
We therefore obtain

\[ S_1 + S_5 = r_0^2 \left[ (\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) \right] + \]
\[ (a^2 - r_0^2)(\theta_2 - \theta_1), \]

\[ S_3 + S_7 = r_0^2 \left[ (\theta_4 - \theta_3) + \sin(\theta_4 - \theta_3) \cos(\theta_3 + \theta_4 - 2\theta_0) \right] + \]
\[ (a^2 - r_0^2)(\theta_4 - \theta_3). \] (18)

The area sums of the two sets of alternative sectors are

\[ S_1 + S_3 + S_5 + S_7 = \]
\[ r_0^2 \left[ \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) + \sin(\theta_4 - \theta_3) \cos(\theta_3 + \theta_4 - 2\theta_0) \right] + \]
\[ a^2(\theta_2 - \theta_1 + \theta_4 - \theta_3), \]

\[ S_2 + S_4 + S_6 + S_8 = \]
\[ r_0^2 \left[ (\theta_3 - \theta_2 + \theta_5 - \theta_4) + \sin(\theta_3 - \theta_2) \cos(\theta_2 + \theta_3 - 2\theta_0) + \right. \]
\[ \sin(\theta_5 - \theta_4) \cos(\theta_4 + \theta_5 - 2\theta_0) \right] + \]
\[ (a^2 - r_0^2)(\theta_3 - \theta_2 + \theta_5 - \theta_4). \] (19)

As \( \theta_5 = \theta_1 + \pi \) the above sum can be written as

\[ S_2 + S_4 + S_6 + S_8 = \]
\[ r_0^2 \left[ \sin(\theta_3 - \theta_2) \cos(\theta_2 + \theta_3 - 2\theta_0) + \right. \]
\[ \sin(\theta_1 - \theta_4) \cos(\theta_4 + \theta_1 - 2\theta_0) \right] + \]
\[ a^2(\theta_3 - \theta_2 + \theta_1 + \pi - \theta_4), \] (20)

and the area sum of all eight sectors is
\[(S_1 + S_3 + S_5 + S_7) + (S_2 + S_4 + S_6 + S_8) =
πa^2 + r_0^2 \left[ \sin(θ_2 - θ_1) \cos(θ_2 + θ_2 - 2θ_0) +
\sin(θ_3 - θ_2) \cos(θ_3 + θ_2 - 2θ_0) +
\sin(θ_4 - θ_3) \cos(θ_4 + θ_2 - 2θ_0) +
\sin(θ_1 - θ_4) \cos(θ_1 + θ_1 - 2θ_0) \right].\] (21)

The area sums of the alternative sectors can be expressed as

\[S_1 + S_3 + S_5 + S_7 = \frac{r_0^2}{2} \left[ \sin 2(θ_2 - θ_0) - \sin 2(θ_1 - θ_0) +
\sin 2(θ_4 - θ_0) - \sin 2(θ_3 - θ_0) \right] +
a^2(θ_2 - θ_1 + θ_4 - θ_3),\]

\[S_2 + S_4 + S_6 + S_8 = \frac{r_0^2}{2} \left[ \sin 2(θ_3 - θ_0) - \sin 2(θ_2 - θ_0) +
\sin 2(θ_1 - θ_0) - \sin 2(θ_4 - θ_0) \right] +
a^2(θ_3 - θ_2 + θ_1 + π - θ_4),\] (22)

where \(\sum_{i=1}^{8} S_i = πa^2\). Also, when \(r_0 = 0\) i.e. when the centre \(C(r_0, θ_0)\) of the circle coincides with the pole O, we have the area sums of the alternative sectors given as

\[S_1 + S_3 + S_5 + S_7 = a^2(θ_2 - θ_1 + θ_4 - θ_3),\]

\[S_2 + S_4 + S_6 + S_8 = a^2(θ_3 - θ_2 + θ_1 + π - θ_4),\] (23)

which become equal for

\[(θ_2 - θ_1) + (θ_4 - θ_3) = \frac{π}{2}.\] (24)

When the pole O and the centre \(C(r_0, θ_0)\) coincide, the area sums of the alternative sectors corresponding to angles \((θ_2 - θ_1), (θ_4 - θ_3)\) are given by \(a^2(θ_2 - θ_1)\) and \(a^2(θ_4 - θ_3)\), respectively. The total areas of the alternative sectors i.e. \(a^2[(θ_2 - θ_1) + (θ_4 - θ_3)]\) becomes \(\frac{πa^2}{2}\) which is half the total area of the circle.

The area sums of the alternative sectors become equal when
\[
\frac{r_0^2}{2} [\sin 2(\theta_2 - \theta_0) - \sin 2(\theta_1 - \theta_0) + \sin 2(\theta_4 - \theta_0) - \sin 2(\theta_3 - \theta_0)] + \\
a^2(\theta_2 - \theta_1 + \theta_4 - \theta_3 - \frac{\pi}{2}) = 0
\]  
(25)
describing the general condition for the area sums of the alternative sectors to become equal.

A. A special case

As a special case of (25) is obtained when

\[
(\theta_2 - \theta_1) + (\theta_4 - \theta_3) = \frac{\pi}{2}, \text{ and } \\
\sin 2(\theta_4 - \theta_0) + \sin 2(\theta_2 - \theta_0) = \sin 2(\theta_3 - \theta_0) + \sin 2(\theta_1 - \theta_0),
\]  
(26)

for which we obtain

\[
S_1 + S_3 + S_5 + S_7 = \frac{\pi}{2} a^2 = S_2 + S_4 + S_6 + S_8,
\]  
(27)
i.e. when Eqs. (26) are true, the area sums of areas of the alternative sectors—corresponding to the case when the pole O does not coincide with the centre C—become equal.

We note that the second equation in (26) can be written as

\[
\sin(\theta_1 + \theta_3 - 2\theta_0) \cos(\theta_4 - \theta_2) = \sin(\theta_1 + \theta_3 - 2\theta_0) \cos(\theta_3 - \theta_1),
\]  
(28)
and substituting from the first equation of (26) into (28) to obtain

\[
\sin(\theta_1 + \theta_3 + \frac{\pi}{2} - 2\theta_0) \cos(\theta_4 - \theta_2) = \sin(\theta_1 + \theta_3 - 2\theta_0) \cos(\theta_3 - \theta_1).
\]  
(29)
As \(\sin(\theta + \frac{\pi}{2}) = \cos(\theta)\), the above equation can be written as

\[
\cos(\theta_1 + \theta_3 - 2\theta_0) \cos(\theta_4 - \theta_2) = \sin(\theta_1 + \theta_3 - 2\theta_0) \cos(\theta_3 - \theta_1),
\]  
(30)
and the conditions for equal area sums of the alternative sectors can then be written as

\[
(\theta_2 - \theta_1) + (\theta_4 - \theta_3) = \frac{\pi}{2}, \text{ and } \\
\cos(\theta_4 - \theta_2) = \tan(\theta_1 + \theta_3 - 2\theta_0) \cos(\theta_3 - \theta_1),
\]  
(31)
representing the requirement with which the area sums of the alternative sectors become equal with the centre \( C \) not coinciding with the pole O. For instance, the areas of the alternative sectors would be equal when

\[
\theta_1 + \theta_3 = 2\theta_0 \text{ and } \theta_4 = \frac{\pi}{2} + \theta_2
\] (32)

III. THE FOUR SECTOR CASE

With only two lines passing through the pole O result in four sectors \( S_1, S_2, S_3 \) and \( S_4 \), the area sum \( S_1 + S_3 \) is obtained as

\[
S_1 + S_3 = \frac{r_0^2}{2} \left[ (\theta_2 + \theta_4 - \theta_1 - \theta_3) + \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) + \sin(\theta_3 - \theta_4) \cos(\theta_3 + \theta_4 - 2\theta_0) \right] + \\
\frac{1}{2} (a^2 - r_0^2)(\theta_2 + \theta_4 - \theta_1 - \theta_3) + \\
\left( \frac{a}{r_0} \right)^2 \left[ (x_2 + x_4) - (x_1 + x_3) + \cos(x_1 + x_2) \sin(x_2 - x_1) + \cos(x_3 + x_4) \sin(x_4 - x_3) \right]
\] (33)

where
\[ x_i = \sin^{-1}\left[ \frac{r_0}{a} \sin(\theta_i - \theta_0) \right], \quad 1 \leq i \leq 4. \] (34)

As \( \theta_3 = \theta_1 + \pi \) and \( \theta_4 = \theta_2 + \pi \), we have \( x_3 = -x_1 \) and \( x_4 = -x_2 \) and we therefore obtain

\[ S_1 + S_4 = a^2(\theta_2 - \theta_1) + r_0^2 \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0), \] (35)

and the area sum \( S_2 + S_4 \) is

\[ S_2 + S_4 = a^2(\pi - \theta_2 + \theta_1) - r_0^2 \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0). \] (36)

For the case when \( S_1 + S_3 \) and \( S_2 + S_4 \) become equal, we obtain

\[ r_0^2 \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) + a^2(\theta_2 - \theta_1 - \frac{\pi}{2}) = 0, \] (37)

that can be expressed as

\[ \frac{r_0^2}{2} \left[ \sin 2(\theta_2 - \theta_0) - \sin 2(\theta_1 - \theta_0) \right] + a^2(\theta_2 - \theta_1 - \frac{\pi}{2}) = 0, \] (38)

describing the conditions for area sums of alternative sectors becoming equal with the centre \( C \) not coinciding with the pole \( O \). Note that with \( r_0 = 0 \), the Eq. (38) gives \( \theta_2 - \theta_1 = \frac{\pi}{2} \).

Comparing Eq. (38) with Eq. (25), reproduced below for reference, show the same structure of the conditions

\[ \frac{r_0^2}{2} \left[ \sin 2(\theta_2 - \theta_0) - \sin 2(\theta_1 - \theta_0) \right] + a^2(\theta_2 - \theta_1 - \theta_4 - \theta_3 - \frac{\pi}{2}) = 0, \] (39)

i.e. in the absence of two from four straight lines in the case of eight sectors, Eq. (25) can be reduced to Eq. (38). A special case of the conditions (38) is when

\[ \theta_2 - \theta_1 = \frac{\pi}{2}, \]

\[ \sin 2(\theta_2 - \theta_0) = \sin 2(\theta_1 - \theta_0), \] (40)

that has the same structure as the special case of the conditions for the eight sector case

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\[(\theta_2 - \theta_1) + (\theta_4 - \theta_3) = \frac{\pi}{2}, \text{ and} \]
\[
\sin 2(\theta_4 - \theta_0) + \sin 2(\theta_2 - \theta_0) = \sin 2(\theta_3 - \theta_0) + \sin 2(\theta_1 - \theta_0), \quad (41)
\]

IV. THE SIX SECTOR CASE

In this case, three lines passing through the pole O and result in six sectors $S_1, S_2, ... S_6$ and

\[
\theta_4 = \theta_1 + \pi, \; \theta_5 = \theta_2 + \pi, \; \theta_6 = \theta_3 + \pi. \quad (42)
\]

The area sum $S_1 + S_3 + S_5$ of alternative sectors is obtained as
\[ S_1 + S_3 + S_5 = \frac{r_0^2}{2} \left[ \pi + \sin(\theta_3 - \theta_2) \cos(\theta_2 + \theta_3 - 2\theta_0) + \sin(\theta_2 - \theta_1) \cos(\theta_1 + \theta_2 - 2\theta_0) + \sin(\theta_3 - \theta_1) \cos(\theta_1 + \theta_3 - 2\theta_0) \right] + \frac{\pi}{2} (a^2 - r_0^2) + \left( \frac{a}{r_0} \right)^2 \left[ 2(x_2 - x_3 - x_1) - \cos(x_2 + x_3) \sin(x_3 - x_2) + \cos(x_1 + x_2) \sin(x_2 - x_1) - \cos(x_3 - x_1) \sin(x_1 + x_3) \right], \] 

where \( x_1, x_2, x_3 \) are defined in (34). This sum can also be expressed as

\[ S_1 + S_3 + S_5 = \frac{\pi}{2} a^2 + \frac{r_0^2}{2} \left[ \sin 2(\theta_3 - \theta_0) - \sin 2(\theta_1 - \theta_0) \right] + \left( \frac{a}{r_0} \right)^2 \left[ 2(x_2 - x_3 - x_1) - \sin 2 x_3 + \sin 2 x_2 - \sin 2 x_1 \right], \] 

that can be expressed as

\[ S_1 + S_3 + S_5 = \frac{\pi}{2} a^2 + \frac{r_0^2}{2} \left[ \sin 2(\theta_3 - \theta_0) - \sin 2(\theta_1 - \theta_0) \right] + \left( \frac{a}{r_0} \right)^2 \left[ 2(x_2 - x_3 - x_1) - \sin 2 x_3 + \sin 2 x_2 - \sin 2 x_1 \right]. \] 

Note that in this case, when the pole coincides with the centre, the area sums of the alternative sectors become equal when

\[ \lim_{r_0 \to 0} \left( \frac{a}{r_0} \right)^2 \left[ 2(x_2 - x_3 - x_1) - \sin 2 x_3 + \sin 2 x_2 - \sin 2 x_1 \right] = 0 \] 

A. A special case

A special case of the conditions for equal area sums of the alternative sectors i.e. \( S_1 + S_3 + S_5 = S_2 + S_4 + S_6 \) are given by

\[ \frac{\pi}{2} a^2 + \frac{r_0^2}{2} \left[ \sin 2(\theta_3 - \theta_0) - \sin 2(\theta_1 - \theta_0) \right] + \left( \frac{a}{r_0} \right)^2 \left[ 2(x_2 - x_3 - x_1) - \sin 2 x_3 + \sin 2 x_2 - \sin 2 x_1 \right] = 0 \]
\[
\sin 2(\theta_3 - \theta_0) = \sin 2(\theta_1 - \theta_0),
\]
\[
2(x_2 - x_3 - x_1) - \sin 2x_3 + \sin 2x_2 - \sin 2x_1 = 0, \quad (47)
\]
where \( x_i = \sin^{-1} \left( \frac{a}{r} \sin (\theta_i - \theta_0) \right) \).

Note that although the conditions for the special cases of equal area sums of the alternative sectors have similar structure for the eight and the four sector case, the corresponding conditions for the six sector case do not have a similar structure.

V. CONCLUSIONS

When the point through which four, two, or three straight lines pass, is the centre of the circle, the conditions resulting from equating the area sums of alternative sectors are given, respectively by

\[
(\theta_2 - \theta_1) + (\theta_4 - \theta_3) = \frac{\pi}{2},
\]
\[
\theta_2 - \theta_1 = \frac{\pi}{2}, \quad (48)
\]
\[
\lim_{r_0 \to 0} \left( \frac{a}{r_0} \right)^2 [2(x_2 - x_3 - x_1) - \sin 2x_3 + \sin 2x_2 - \sin 2x_1] = 0. \quad (50)
\]

In the general case of the pole not coinciding with the center, however, these conditions are given, respectively, as

\[
\frac{r_0^2}{2} [\sin 2(\theta_2 - \theta_0) - \sin 2(\theta_1 - \theta_0) + \sin 2(\theta_4 - \theta_0) - \sin 2(\theta_3 - \theta_0)] + \]
\[
a^2(\theta_2 - \theta_1 + \theta_4 - \theta_3 - \frac{\pi}{2}) = 0, \quad (51)
\]
\[
\frac{r_0^2}{2} [\sin 2(\theta_2 - \theta_0) - \sin 2(\theta_1 - \theta_0)] + a^2(\theta_2 - \theta_1 - \frac{\pi}{2}) = 0, \quad (52)
\]
\[
\frac{r_0^2}{2} [\sin 2(\theta_3 - \theta_0) - \sin 2(\theta_1 - \theta_0)] + \]
\[
\left( \frac{a}{r_0} \right)^2 [2(x_2 - x_3 - x_1) - \sin 2x_3 + \sin 2x_2 - \sin 2x_1] = 0. \quad (53)
\]
From the symmetries in the above equations, one can directly obtain a generalization of the cases of four, eight, and six sectors to the ten, twelve and fourteen sectors etc.

[1] Ogilvy, C. Stanley, Excursions in Geometry, Dover, 1969, 14–17.

[2] Altshiller-Court, Nathan, College Geometry, Dover, 2007.