UNIFORM VERSION OF WEYL–VON NEUMANN THEOREM

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Abstract. We prove a “quantified” version of the Weyl–von Neumann theorem, more precisely, we estimate the ranks of approximants to compact operators appearing in the Voiculescu’s theorem applied to commutative algebras. This allows considerable simplifications in uniform $K$-homology theory, namely it shows that one can represent all the uniform $K$-homology classes on a fixed Hilbert space with a fixed $*$-representation of $C_0(X)$, for a large class of spaces $X$.

1. Introduction

Voiculescu’s theorem [7] states that whenever one has a non-degenerate representation $\pi: E \to \mathcal{B}(H_{\pi})$ of a separable unital C*-algebra $E$ and a completely positive map $\rho: E \to \mathcal{B}(H_\rho)$ with the property that $\pi(e) \in \mathcal{K}(H) \implies \rho(e) = 0$ for every $e \in E$, then there exists a unitary $V: H_\rho \to H_{\pi}$, such that $\rho(e) - V^* \pi(e) V \in \mathcal{K}(H_\rho)$. Loosely worded, when $\pi: E \to \mathcal{B}(H_{\pi})$ is sufficiently big, then any other representation of $E$ is a compression of $\pi$ modulo compacts. In this paper, we prove a “quantified” version of this statement for commutative algebras $E$ and for $*$-homomorphisms $\rho$. More precisely, if we consider an class $\mathcal{X}$ of compact spaces $X$ having jointly locally bounded geometry (defined below), we can obtain bounds on the ranks of approximants of $\rho(f) - V^* \pi(f) V$ which depend only on the type of the function $f \in C(X)$ and is independent of the space $X \in \mathcal{X}$ itself.

The special case of Voiculescu’s theorem, when the algebra $E$ in question is commutative, is usually called the Weyl–von Neumann theorem and stated in the form that any normal operator on a separable Hilbert space can be expressed as a diagonal operator plus a compact operator, and additionally the norm of the compact can be made arbitrarily small.

The motivation and an application of our theorem come from considerations in analytic $K$-homology and coarse geometry. The analytic $K$-homology theory is a generalized homology theory on the category of locally compact Hausdorff topological spaces (with proper maps); in its full generality even a contravariant functor on the category of separable C*-algebras [3, 2]. The connection to coarse geometry and index theory comes from the existence of a coarse index map from analytic $K$-homology of a locally compact space into the $K$-theory of its Roe C*-algebra [5]. The assertion that this map is an isomorphism has notable applications for instance when the space $X$ is question is a Rips complex of a Cayley graph of a finitely generated group $\Gamma$, namely it implies
the Novikov conjecture for $\Gamma$ [8]. Also, since analytic $K$-homology of spaces is “computable” (by means of exact sequences), this provides a way to compute $K$-theory of Roe C*-algebras.

The Voiculescu’s theorem provides a way to simplify the definition and proofs in analytic $K$-homology as follows: The cycles for a C*-algebra $A$ are represented by Fredholm modules $(H, \phi, F)$, where $H$ is a (graded) Hilbert space, $\phi : A \to \mathcal{B}(H)$ is a *-homomorphism (of degree 0), $F \in \mathcal{B}(H)$ (has degree 1) and $[F, \phi(a)] \in \mathcal{K}(H)$, $(F^2 - 1)\phi(a) \in \mathcal{K}(H)$ for all $a \in A$. By introducing a suitable equivalence relation on Fredholm modules one obtains the $K$-homology group. A consequence of the Voiculescu’s theorem is that we can fix any Hilbert space $H_0$ with a representation $\phi_0 : A \to \mathcal{B}(H_0)$ which misses $\mathcal{K}(H_0) \setminus \{0\}$ and then any $K$-homology class of $A$ can be represented as a Fredholm module of the form $(H_0, \phi_0, \cdot)$.

We present here an application of the main result of this paper to uniform $K$-homology: Uniform $K$-homology is a version of analytic $K$-homology for locally compact separable metric spaces $X$ with bounded geometry, defined in [6], from which there is an index map into the $K$-theory of the uniform Roe C*-algebra $C^*_u(X)$. Furthermore, one can characterize amenability of $X$ in terms of this theory. A consequence of the main result of this paper is that one can work on a fixed uniform Fredholm module in the uniform $K$-homology (as explained in the previous paragraph for the ordinary $K$-homology) for locally compact spaces $X$ which have bounded geometry not only on the large scale (i.e. in the coarse geometric understanding of bounded geometry), but also on the small scale.

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2. Result

The following definition formalizes the notion of having bounded geometry on the small scale. Naturally, any single compact metric space $X$ by itself has bounded geometry in any sense, but the point of our main Theorem 2.5 is to obtain estimates which are independent of the space $X$. The notion defined below allows us to specify classes of spaces for which this is possible.

**Definition 2.1.** We say that compact metric spaces in a class $\mathcal{X}$ have **jointly locally bounded geometry**, or just shortly that $\mathcal{X}$ is admissible, if the following property holds: Given $\varepsilon > 0$, there exists $N \geq 0$, such that for any $X \in \mathcal{X}$ there exists an $\varepsilon$-net in $X$ of cardinality at most $N$.

**Example 2.2.** If $Y$ is a complete Riemannian manifold with bounded geometry (in the sense of Roe [4]), or a uniformly locally finite simplicial complex of finite dimension (for instance a Rips complex of a finitely generated group), then for any $R > 0$, the collection closed of $R$-balls in $Y$ comprises an admissible class of spaces.
Remark 2.3. Note that bounding only the diameter and the (covering) dimension of a space does not yield an admissible class. Just consider 1-dimensional simplicial complexes with one “central” vertex and $n$ different edges attached to it. The size of $\varepsilon$-nets in such a space grows with $n$, but neither the diameter nor the dimension does not.

For a metric space $X$ and $L \geq 0$, we denote

$$C_L(X) = \{ f : X \to \mathbb{C} \mid f \text{ is } L\text{-Lipschitz, } \|f\|_\infty \leq 1 \}$$

and by $C(X)$ the C*-algebra of continuous complex-valued functions on $X$.

**Lemma 2.4.** If $\mathcal{X}$ is admissible, then the following also holds: Given $L \geq 0$ and $\varepsilon > 0$, there exists $N \geq 0$, such that for any $X \in \mathcal{X}$ there exists an $\varepsilon$-net of simple Borel functions $C_L(X)$ containing at most $N$ elements.

**Proof.** The fact that $C_L(X)$ is compact follows for instance from Arzela–Ascoli theorem, but the point is to have a bound on the number of elements in an $\varepsilon$-net.

Given $L \geq 0$ and $\varepsilon > 0$, we choose $\varepsilon_1 > 0$ small enough and $K$ large enough, so that $\frac{1}{K} + LE_1 < \varepsilon$. Now for $\varepsilon_1 > 0$, the admissibility of $\mathcal{X}$ provides us with an upper bound on the size of $\varepsilon_1$-nets: denote it by $M \geq 0$. Taking any $X \in \mathcal{X}$ and an $\varepsilon_1$-net $E$ in $X$ with $|E| \leq M$, we consider any Borel partition $\{D_x \mid x \in E\}$ of $X$, such that $x \in D_x$ and $\text{diam}(D_x) \leq \varepsilon_1$ for each $x \in E$. For instance, we can obtain such a partition by considering the “closest point in $E$” map.

For an integer $K \geq 1$, consider the collection of simple functions of the form $s = \sum_{x \in E} \frac{i_x}{K} \chi_{D_x}$, where each $i_x \in \{0, \ldots, K\}$. There is at most $N = M^{K+1}$ of them. Furthermore, taking any $f \in C_L(X)$, there is at least one of them that approximates $f$ within $\frac{1}{K}$ at each $x \in E$, and then by the Lipschitz property, it approximates $f$ within $\frac{1}{K} + LE_1 < \varepsilon$ at any $y \in X$. \hfill \square

**Theorem 2.5.** For an admissible class $\mathcal{X}$ of compact metric spaces there exists a function $M_{\mathcal{X}} : [0, \infty) \times (0, 1] \to \mathbb{N}$, which satisfies the following: For any $X \in \mathcal{X}$, any *-representations $\pi : C(X) \to \mathcal{B}(H_\pi)$, $\rho : C(X) \to \mathcal{B}(H_\rho)$, such that $\pi$ is injective and $\pi(C(X)) \cap \mathcal{X}(H_\pi) = \{0\}$, there exists an isometry $V : H_\rho \to H_\pi$, such that for any $L \geq 0$ and $\varepsilon > 0$, all the operators $V^* \pi(f)V - \rho(f)$, $f \in C_L(X)$, are within $\varepsilon$ from an operator with rank $M_{\mathcal{X}}(L, \varepsilon)$.

**Proof.** The strategy is to follow parts of the proof that any representation $\rho : C(X) \to \mathcal{B}(H_\rho)$ is diagonalizable modulo compacts (this is the Weyl–von Neumann theorem), with some estimates on the ranks of the finite-rank approximants to the compacts involved. The diagonalizations of two representations of $C(X)$ will then provide us with an isometry. We then need to analyze the diagonalizations to prove that this construction will give us the estimates we require.

For a representation $\rho : C(X) \to \mathcal{B}(H_\rho)$, we shall without mention use the fact that it extends to a representation of the algebra of bounded Borel functions on $X$ into $\mathcal{B}(H_\rho)$. For the sake of brevity, we shall write $F^\rho \in \mathcal{B}(H_\rho)$ for $\rho(F)$, where $F$ is a bounded Borel function on $X$.\hfill \square
We fix sequences \((L_k)_{k \in \mathbb{N}}, L_k \not\to \infty,\) and \((\varepsilon_k)_{k \in \mathbb{N}}, \varepsilon_k \searrow 0,\) of positive reals. From the admissibility of \(\mathcal{X}\) it follows that there exists a sequence \((S_k)_{k \in \mathbb{N}}\) of positive integers, such that the following construction can be executed: An inductive application of the previous lemma provides us with a sequence \((I_k)_{k \in \mathbb{N}}, I_k \leq S_k\) of non-negative integers, and for any \(X \in \mathcal{X}\) with a sequence of (automatically commuting) projections \(\mathcal{P}_X = \{P_1 = \chi_{X_1}, \ldots\},\) such that \(C(X) \subset A = \text{span}(\mathcal{P}_X)\) and that \(\mathcal{C}_{L_k}(X)\) is contained in the \(\varepsilon_k\)-neighborhood of \(\text{span}\{P_1, \ldots, P_k\}\) (in the sup-norm on bounded Borel functions); whence it is also true after mapping through any representation \(\rho : C(X) \to \mathcal{B}(H_\rho)\).

For the sake of simplifying the notation later on in this proof, we shall arrange that \(\rho\) is without loss of generality injective. Furthermore, we assume that \(\rho\) is unital.

The proof of diagonalizability of \(\rho\) modulo compacts is similar to the proofs of the Weyl–von Neumann theorem, e.g. [1, Theorem II.4.1] and [2, Theorem 2.2.5]. Fix an orthonormal basis \(\{e_1, e_2, \ldots\}\) for \(H_\rho\). Denote \(R_k = \{P_{l_{k-1}+1}, \ldots, P_k\}\). Note that the projections in \(R_k\) are mutually orthogonal, their sum is 1 and the projections in \(R_j, j \leq k,\) are sums of projections from \(R_k\). We define subspaces \(E^\rho_k\) of \(H_\rho\) by setting

\[
E^\rho_k = \text{span}\{P^\rho e_j \mid 1 \leq j \leq k, P \in R_k\}.
\]

Then \(E^\rho_k\) is an increasing sequence of finite-dimensional spaces \((\dim(E^\rho_k) \leq k(I_k - I_{k-1}))\). Clearly \(\text{span}\{e_1, \ldots, e_k\} \subset E^\rho_k\), hence \(\bigcup_k E^\rho_k\) is dense in \(H_\rho\).

Consider now the algebras \(A_k\) generated by \(R_k\) (which is also a linear basis of \(A_k\)). Since both \(E^\rho_k\) and \(E^\rho_{k+1}\) are invariant for \(\rho(A_k)\), the subspace \(E^\rho_{k+1} \ominus E^\rho_k\) decomposes as the orthogonal sum of subspaces \(P^\rho(E^\rho_{k+1} \ominus E^\rho_k), P \in R_k\). Thus we may choose an orthonormal basis of \(E^\rho_{k+1} \ominus E^\rho_k\) which respects this decomposition. This diagonalizes \(\rho(A_k)\) on \(E^\rho_{k+1} \ominus E^\rho_k\). Note that since \(A_j \subset A_k\) for \(j \leq k\), it also diagonalizes the \(\rho\)-images of all these algebras. In this manner, we obtain an orthonormal basis of the whole \(H_\rho\).

In particular, each \(T^\rho, T \in A_k\), is eventually diagonal in this basis and it differs from a diagonal operator by at most a rank-(\(\dim(E^\rho_k)\)) operator. Furthermore, if we express \(T = \sum_{P \in R_k} c_P P\), then the entries on the diagonal of \(T^\rho\) in this basis are eventually just appropriate coefficients \(c_P\).
Let us remark here (although we shall not need in the rest of the proof), that the
the operators \( D_n = P_n^\rho(1 - F_n), n \geq 1 \), generate a diagonal C*-algebra \( \mathcal{D} \), such that
\( \rho(C(X)) \subset \mathcal{D} + \mathcal{K}(H_\rho) \).

Note that in the case \( \rho = \pi \), i.e. when \( \rho(C(X)) \cap \mathcal{K}(H_\pi) = \{0\} \), we can choose the
orthonormal basis \( \{e_1, \ldots\} \) in such a way that each \( \dim(E_k^\pi) \) is maximal possible (for a
given \( X \) and \( \mathcal{P}_X \)). We choose it inductively. First note that without loss of generality
we can assume that all the projections \( P_j^\pi \in \pi(\mathcal{P}_X) \) are infinite (since \( P_j = \chi_{X_j} \), with \( X_j \)
having nonempty interior), and also \( 1 - P_j^\pi \) is infinite.

Let us proceed to the induction which results in choosing \( e_1 \): We choose \( e_1^{(1)} \) so that
all the vectors from the set \( V_1 = \{P_j^\pi e_1^{(1)} \mid 1 \leq j \leq I_1\} \) are nonzero. In the \( k \)-th step,
we assume that we have already picked vectors \( e_1^{(1)}, \ldots, e_1^{(k-1)} \), such that if we denote
\( V_{k-1} = \{P_\pi e_1^{(i)} + \cdots + e_1^{(k-1)} \mid P \in R_{k-1}\} \) and \( T_{k-1} = \min(\|v\|, v \in V_{k-1}) \), then the vectors in \( V_{k-1} \) are non-zero, i.e. \( T_{k-1} > 0 \). Since the projections involved are infinite, we
can choose \( e_1^{(k)} \) in such a way that all the vectors \( P_\pi e_1^{(i)} + \cdots + e_1^{(k-1)} + e_1^{(k)} \), \( P \in R_k \),
are again non-zero and \( \|e_1^{(k)}\| \leq \frac{1}{2} T_{k-1} \). Finally let \( e_1 \) the multiple of \( \sum_{k \geq 1} e_1^{(k)} \) with norm 1.
By the bounds on the norms of \( e_1^{(k)} \)'s, this sum converges, and moreover \( P_j^\pi e_1 \neq 0 \) for any
\( j \) (since \( \|\sum_{k \geq m+1} e_1^{(k)}\| \leq \frac{2}{3} T_m \), thus \( P_\pi e_1 \neq 0 \) follows from \( \| P_\pi (e_1^{(1)} + \cdots + e_1^{(m)}) \| \geq T_m \)
for \( P \in R_m \). Furthermore for each \( m \), the vectors \( P_\pi e_1, P \in R_m \), are automatically
linearly independent since the projections in \( R_m \) are orthogonal.

We repeat this process for choosing \( e_2, \ldots \). The only change is that when choosing
\( e_i^{(j)} \), we need to also ensure that it is itself orthogonal to \( e_1, \ldots, e_{i-1} \) and that \( P_\pi (e_i^{(1)} +
\cdots + e_i^{(j)}) \), \( P \in R_j \), are orthogonal to the so-far chosen \( P_\pi e_1, \ldots, P_\pi e_{i-1} \). This is possible
since we are always excluding only finite-dimensional subspaces and the projections
involved are infinite. The result of this process is an orthonormal basis of \( H_\pi \), such that
\( \dim(E_k^\pi) = k(I_k - I_{k-1}) \), which is maximal possible. Moreover, also the individual
\( \dim(P_\pi(H_\pi) \cap (E_{k+1}^\pi \ominus E_k^\pi)) \), \( P \in R_k \) are maximal possible among all \( \dim(P^\rho(H_\rho) \cap (E_{k+1}^\rho \ominus E_k^\rho)) \) for all the representations \( \rho : C(X) \rightarrow \mathcal{B}(H_\rho) \).

By the conclusion in the previous paragraph, there exists an isometry \( V : H_\rho \rightarrow H_\pi \),
constructed so that within each \( P^\rho(H_\rho) \cap (E_{k+1}^\rho \ominus E_k^\rho) \), we send the chosen basis (which
diagonalizes \( \rho(A_k) \)) to the (possibly part of) the chosen basis of \( P_\pi(H_\pi) \cap (E_{k+1}^\pi \ominus E_k^\pi) \).

It remains to be shown that this isometry will indeed satisfy the estimates we require.
It is clear from the construction that \( V * T^\pi V - T^\rho \) for \( T \in A_k \) is a finite-rank operator,
with rank at most \( \dim(E_k^\rho) \leq k(I_k - I_{k-1}) \leq kS_k \). By our choices, any \( f \in C_l(X) \)
is at most \( \varepsilon \)-far from some \( T \in A_k \), hence \( V * \pi(f)V - \rho(f) \) is at most \( 2\varepsilon \)-far from a
rank-2kS_k operator. So to finish the proof, we just define the function \( M_{\mathcal{D}}(L, \varepsilon) = 2kS_k \),
where \( k \) is large enough so that \( L \leq L_k \) and \( \varepsilon \geq \varepsilon_k \). \( \square \)
3. APPLICATION TO UNIFORM $K$-HOMOLOGY

We apply Theorem 2.5 to show that any for “nice” locally compact metric spaces $X$ (namely the ones with both locally and coarsely bounded geometry), we can fix a suitable Hilbert space $H$ and a representation $\phi : C_0(X) \to \mathcal{B}(H)$ and then represent any class in the uniform $K$-homology $K^u(X)$ of $X$ as a uniform Fredholm module of the form $(H, \phi, F)$. The class of “nice” spaces includes open manifolds with bounded geometry and Rips complexes of finitely generated discrete groups (see Example 2.2).

Definition 3.1. We say that a metric space $X$ has locally bounded geometry, if it admits a countable Borel decomposition $X = \bigcup_{i \in I} X_i$, such that $\mathcal{X} = \{X_i \mid i \in I\}$ is an admissible class of compact metric spaces and that each $X_i$ has nonempty interior.

We say that $X$ has coarsely bounded geometry, if it contains a uniformly discrete subset $Y$, such that $\sup_{x \in X} d(x, Y) < \infty$ and which has bounded geometry in a sense that for each $R \geq 0$, $\sup_{y \in Y} \#B(y, R) < \infty$.

The tool we use to show that Theorem 2.5 implies the above statement in uniform $K$-homology is the notion of a uniformly covering isometry. We recall its definition [6, Definition 4.7]; it is a “quantified version” of the notion of a covering unitary [2, Definition 5.2.2].

Definition 3.2. Let $X$ and $Z$ be metric spaces, let $\varphi : C_0(X) \to C_0(Z)$ be a *-homomorphism, $\phi_X : C_0(X) \to \mathcal{B}(H_X)$ and $\phi_Z : C_0(Z) \to \mathcal{B}(H_Z)$ be *-representations. We say that an isometry $V : H_Z \to H_X$ uniformly covers $\varphi$, if for every $\epsilon > 0$, $R, L \geq 0$ there exists $M \geq 0$, such that for every $f \in C_0(X)$ which is $L$-Lipschitz and has support of diameter at most $R$, there exists an operator with rank at most $M$, which is at most $\epsilon$-far from $V^* \phi_X(f)V - \phi_Z(\varphi(f))$. [This relation is denoted in [6] by $V^* \phi_X(\cdot)V \sim_{l.u.a} \phi_Z(\varphi(\cdot))$; and it is a sharpening of $\sim$ which expresses that the difference of the two operators is compact.]

Corollary 3.3. Let $X$ be a locally compact metric space which has both locally and coarsely bounded geometry. Let $\pi : C_0(X) \to \mathcal{B}(H)$ be a *-representation that misses $\mathcal{K}(H) \setminus \{0\}$. Then for any non-degenerate *-representation $\rho : C_0(X) \to \mathcal{B}(H_\rho)$ that misses the compacts, there exists an isometry $V : H_\rho \to H$ that uniformly covers the identity map $id : C_0(X) \to C_0(X)$.

Remark 3.4. For a compact space $X$ any covering isometry is automatically uniformly covering; this is because we can disregard $R$ in this case and (as in Lemma 2.4) each $C_L(X)$ is compact, hence it is sufficient to consider only finitely many compacts for approximation by finite-rank operators.

Proof. By assumption on $X$, there is a Borel decomposition $X = \bigcup_{i \in \mathbb{N}} X_i$, such that $\mathcal{X} = \{X_i \mid i \in \mathbb{N}\}$ is an admissible class and each $X_i$ has nonempty interior. Also note that it follows from the admissibility that $R_0 = \sup_{i \in \mathbb{N}} (\text{diam}(X_i)) < \infty$.

Notice that we can apply the whole proof of the Theorem 2.5 also to the family $\mathcal{X}' = \{X_i \mid i \in \mathbb{N}\}$, where the algebras of functions $C(X_i)$ on $X_i$’s that we consider are the
algebras of those functions that are restrictions of functions from $C_0(X)$ (the conclusion of Lemma 2.4 is still true for $\mathcal{B}^\infty$).

Denoting $P_i = \chi_{X_i}, H_i = \pi(P_i)H$ and $H_{\rho,i} = \rho(P_i)H_{\rho}$, we have decompositions $H = \bigoplus_{i \in \mathbb{N}} H_i$ and similarly $H_{\rho} = \bigoplus_{i \in \mathbb{N}} H_{\rho,i}$. We may assume that $\pi$ is non-degenerate. The representations $\pi$ and $\rho$ restrict to families of representations $\pi_i = \pi|_{C(X_i)} : C(X_i) \to \mathcal{B}(H_i)$ and $\rho_i = \rho|_{C(X_i)} : C(X_i) \to \mathcal{B}(H_{\rho,i})$. Applying Theorem 2.5 we obtain a function $M$ and isometries $V_i : H_{\rho,i} \to H_i$, such that for any $\varepsilon > 0$ and $L \geq 0$, taking any $f \in C_L(X_i)$ implies that $V_i^* \pi_i(f) V_i - \rho_i(f)$ is at most $\varepsilon$-far from a rank-$M(\varepsilon,L)$ operator. If we now denote $V = \bigoplus_{i \in \mathbb{N}} V_i : H_{\rho} \to H$, we obtain an isometry which satisfies the condition for uniform covering of $\text{id} : C_0(X) \to C_0(X)$ for functions that are supported within some $X_i$.

The final step is to realize that coarsely bounded geometry condition implies that given $R \geq 0$, there is an upper bound on how many $X_i$’s can meet any ball of radius $R$ in $X$. Consequently the isometry $V$ is as required.

Let us now come the application to uniform $K$-homology. For precise definitions of uniform Fredholm modules and uniform $K$-homology groups, see [6, Section 2]. Let us only note here that they mimic the usual definitions of Fredholm modules and analytic $K$-homology groups, but where the relations “is compact” is replaced by the “quantified version” of being compact, in the same spirit as in Definition 3.2. The following proposition explains the connection between uniformly covering isometries and uniform Fredholm modules.

**Proposition 3.5.** Let $X$ be a locally compact metric space and let $\phi : C_0(X) \to \mathcal{B}(H)$ and $\rho : C_0(X) \to \mathcal{B}(H_{\rho})$ be $*$-representations. Assume that there exists an isometry $V : H_{\rho} \to H$ which uniformly covers the identity map $\text{id} : C_0(X) \to C_0(X)$. Then any uniform Fredholm module $(H_{\rho}, \rho, F)$ is equivalent in the uniform $K$-homology group $K_i^u(X)$ to a uniform Fredholm module $(H, \phi, F')$.

**Proof.** This follows from the results proved in [6], namely Proposition 4.3 (which explains the description of uniform $K$-homology groups in terms of the $K$-theory of “dual” C*-algebras), Lemma 5.4 (which shows that uniformly covering isometry induces a map between the dual C*-algebras) and Proposition 4.9 (where the “partial” uniform $K$-homology groups with specified $*$-representation $\phi$ are put together). See also the discussion preceding Lemma 4.8.

Compare also [2, Proposition 8.3.14], which supplies a more direct proof in the case when the uniformly covering isometry is actually onto.

Putting together Corollary 3.3 and Proposition 3.5 we obtain the final result.

**Corollary 3.6.** Let $X$ be a locally compact metric space which has both locally and coarsely bounded geometry. Let $\pi : C_0(X) \to \mathcal{B}(H)$ be a $*$-representation that misses $\mathcal{K}(H) \setminus \{0\}$. Then any uniform $K$-homology class over $X$ can be represented by a uniform Fredholm module of the form $(H, \pi, T)$ for some $T \in \mathcal{B}(H)$. 
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