TOWARDS A CONSTRUCTIVE PROOF OF A THEOREM OF KOECK-LAU-SINGERMAN

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Abstract. In a couple of recent paper, Koeck-Lau-Singerman have proved that every symmetric Belyi curve $C$ is definable over $\mathbb{R} \cap \mathbb{Q}$. Their proof is based on Weil’s Galois descent theorem, so it asserts the existence of some isomorphism $R : C \to Z$, where $Z$ is a curve defined over $\mathbb{R} \cap \mathbb{Q}$. In this paper we work out an alternative proof which provides a method to obtain explicit equations for $R$ and $Z$. In fact, we are able to obtain the following stronger result. If both $C$ and the symmetry are defined over the number field $K$, then $Z$ is definable over $K \cap \mathbb{R}$.

1. Introduction

A complex algebraic variety $X$ is defined over a subfield $K$ of $\mathbb{C}$ if its ideal (which is finitely generated by Hilbert’s finiteness theorem) can be generated by polynomials with coefficients in $K$, equivalently, the variety is given as the common zeroes locus of a finite collection of polynomials with coefficients in $K$. We say that $X$ is definable over the subfield $K$ of $\mathbb{C}$ if there is a complex algebraic variety $Y$ defined over $K$ and there is a birational isomorphism $R : X \to Y$. Interesting types of algebraic varieties, used for instance in number theory, are those which are definable over $\mathbb{Q}$.

An irreducible non-singular complex algebraic variety $X$ is said to be symmetric if it admits a symmetry, that is, an antiholomorphic automorphism of order two. Weil’s Galois descent theorem [16] asserts that $X$ is symmetric if and only if it is definable over $\mathbb{R}$ (see also [15]).

Irreducible non-singular complex algebraic varieties of dimension one (that is, irreducible smooth complex algebraic curves) carry out natural structures of closed Riemann surfaces as a consequence of the Implicit Function Theorem. Conversely, by Torelli’s theorem, closed Riemann surfaces may be seen as irreducible smooth complex algebraic curves. In particular, these two categories are equivalent.

A closed Riemann surface $S$ is called a Belyi curve if there is a non-constant holomorphic map $\beta : S \to \mathbb{C}$ branched exactly at the values $\infty$, $0$ and $1$. Belyi’s theorem [2] states that $S$ is a Belyi curve if and only if $S$ can be described by an algebraic curve defined over $\mathbb{Q}$ (see also [17]).

The following result was obtained by Koeck-Singerman [9] in the case that the symmetry has fixed points and by Koeck-Lau [10] in the general situation.

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Theorem 1 (Koeck-Singerman [9], Koeck-Lau [10]). Let $X$ be an irreducible and non-singular complex algebraic variety with a finite group of holomorphic automorphisms. If $X$ is definable over $\mathbb{R}$ and it is also definable over $\overline{\mathbb{Q}}$, then it is definable over $\overline{\mathbb{Q}} \cap \mathbb{R}$. In particular, every symmetric Belyi curve of genus $g \geq 2$ can be described by an algebraic curve defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$.

Remark 2. Theorem 1 still valid if $X$ is an irreducible singular complex variety with a finite group of birational isomorphisms. In this case, one should define a symmetry of $X$ as an anti-holomorphic birational automorphism of order two. The constructive proof we provide in this paper also works in this more general situation.

The proof of Theorem 1 provided in [9, 10] is based on Weil’s Galois descent theorem [16] and it does not provide a method to obtain explicitly equations for an irreducible complex algebraic variety $Z$ defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$ and an explicit birational isomorphism $R : X \to Z$. In this paper, we provide an alternative constructive proof of Theorem 1 in the sense that, if we have explicit equations for $X$ over $\overline{\mathbb{Q}}$ and also an explicit symmetry $L : X \to X$, then it permits to compute $R$ and $Z$ explicitly.

The main idea is as follows. As $X$ is assumed to have a finite group of automorphisms, the symmetry $L$ is also defined over $\overline{\mathbb{Q}}$. We provide an explicitly birational isomorphism $R : X \to Z$, where $Z$ is some irreducible complex algebraic variety, maybe singular at some proper subvariety $W$ which is explicitly described. We notice that $Z$ and $W$ are defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$. As $R$ and $X$ are given explicitly, with the help of MAGMA [3], it is possible to compute explicitly equations for $Z$ (and also for $W$). In general, these provided equations by MAGMA are over $\overline{\mathbb{Q}}$ (but with many worked examples they are in most of the cases already given over $\overline{\mathbb{Q}} \cap \mathbb{R}$). Next, in order to obtain equations over $\overline{\mathbb{Q}} \cap \mathbb{R}$, we replace each polynomial (which is not already defined over the desired field) by its traces; which are then defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$.

Now, if $W = \emptyset$, then $R$ will be an holomorphic regular isomorphism and $Z$ an irreducible non-singular complex algebraic variety. If $W \neq \emptyset$, then we only have a birational isomorphism $R : X \to Z$ and $Z$ will have singularities at points in $W$. Anyway, as $W$ is also defined over $\overline{\mathbb{Q}} \cap \mathbb{R}$, one may desingularize it (by blowing-up process at $W$) to obtain an irreducible non-singular complex algebraic variety $\hat{Z}$ which is birationally equivalent to $X$.

An explicit example for which $X$ is a curve of genus 5 is worked out in the last section in order to explain how the computational method works.

Our proof (see Proposition 9) asserts a much stronger result than the one stated in Theorem 1.
Theorem 3. If both, $X$ and the symmetry $L : X \to X$, are defined over the field number $K$, then $X$ is definable over $K \cap \mathbb{R}$.

A direct consequence of Theorem 3 is the following fact concerning symmetric Belyi curves.

Corollary 4. Every symmetric Belyi curve $C$ admitting a symmetry $L : C \to C$ so that both, $C$ and $L$, are definable (at the same time) over $\mathbb{Q}(\sqrt{-d})$, for some positive integer $d$, then $C$ is definable over $\mathbb{Q}$.

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2. Generalities

In this section we set some notations and we recall some basic algebraic facts and definitions we will need in the rest of this paper.

2.1. Let us denote by $\text{Gal}(C/\mathbb{Q})$ the group of field automorphisms of $C$ and, for $K$ a subfield of $C$, we denote by $\text{Gal}(C/K)$ the subgroup of $\text{Gal}(C/\mathbb{Q})$ whose elements are those field automorphisms acting as the identity on $K$.

Notice that, since $C$ is algebraically closed and of characteristic zero, then for any subfield $K$ of $C$ it holds that the fixed field of $\text{Gal}(C/K)$ is $K$.

2.2. If $\eta \in \text{Gal}(C/\mathbb{Q})$ and $r \geq 1$ is an integer, then we have a natural bijective map

$$\widehat{\eta} : C^r \to C^r$$

$$\widehat{\eta}(x_1, \ldots, x_r) = (\eta(x_1), \ldots, \eta(x_r)).$$

Let $\eta \in \text{Gal}(C/\mathbb{Q})$. If $X \subset C^r$, then we set $X^{\eta} = \widehat{\eta}(X)$.

Let $H : A \subset C^n \to C^m$ be a rational map, that is, $H = (H_1, \ldots, H_m)$ where each $H_j$ is defined by a polynomial in $n$-variables and coefficients in $C$. If $\widehat{\eta}^{-1}(A) \subset A$, then we set $H^{\eta} = \widehat{\eta} \circ H \circ \widehat{\eta}^{-1} : A \subset C^n \to C^m$; so $H_{1}^{\eta} = (H_{1}^{\eta}, \ldots, H_{m}^{\eta})$.

2.3. Let $\mathbb{K}$ and $\mathbb{U}$ be subfields of $C$ so that $\mathbb{U}$ is a finite Galois extension of $\mathbb{K}$, say of index $m$, and $\text{Gal}(\mathbb{U}/\mathbb{K}) = \{\sigma_1, \ldots, \sigma_m\}$, where $\sigma_1 = e$ is the identity. Let $\{e_1, \ldots, e_m\}$ be a basis of $\mathbb{U}$ as a $\mathbb{K}$-vector space. The trace map

$$\text{Tr} : \mathbb{U} \to \mathbb{K} : a \mapsto \sum_{j=1}^{m} \sigma_j(a)$$

extends naturally to polynomial rings

$$\text{Tr} : \mathbb{U}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n] : P \mapsto \sum_{j=1}^{m} P^{\sigma_j}.$$ 

Next, we recall the following simple fact about invariant ideals.
Lemma 5.

1. If \( P \in \mathbb{U}[x_1, ..., x_n] \), then \( P \in \text{Span}_\mathbb{U}(\text{Tr}(e_1 P), ..., \text{Tr}(e_m P)) \).
2. If \( I \subset \mathbb{U}[x_1, ..., x_n] \) is an ideal so that \( \forall \sigma \in \Gamma \) and \( \forall P \in I \) it holds that \( P^\sigma \in I \), then \( I \) can be generated (as an ideal) by polynomials in \( I \cap \mathbb{K}[x_1, ..., x_n] \).

Proof. Set \( \Gamma_M = \{ \sigma_1, ..., \sigma_m \} \). If \( P \in \mathbb{U}[x_1, ..., x_n] \), then let us consider the following polynomials

\[ Q_1 = \text{Tr}(e_1 P), Q_2 = \text{Tr}(e_2 P), ..., Q_m = \text{Tr}(e_m P) \in \mathbb{K}[x_1, ..., x_n] \]

Now, as the matrix

\[
A = \begin{bmatrix}
e_1 & e_2 & \cdots & e_m \\
\sigma_2(e_1) & \sigma_2(e_2) & \cdots & \sigma_2(e_m) \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_m(e_1) & \sigma_m(e_2) & \cdots & \sigma_m(e_m)
\end{bmatrix}
\]

is invertible (see for instance [3]), the linear transformation \( A : \mathbb{U}^m \rightarrow \mathbb{U}^m \) (where \( \mathbb{U}^n \) is thought as the vector space of columns of length \( m \) and coefficients in \( \mathbb{U} \)) is invertible. This, in particular, ensures the existence of values \( \lambda_1, ..., \lambda_m \in \mathbb{U} \) so that \( A \lambda = E \), where \( \lambda = [\lambda_1 \lambda_2 \cdots \lambda_m] \) and \( E = [1 \ 0 \ \cdots \ 0] \). Then, \( P = \lambda_1 Q_1 + \lambda_2 Q_2 + \cdots + \lambda_m Q_m \) and we have proved 1.

Let us now assume that \( P \in I \) and that \( \forall \sigma \in \Gamma_M \) it holds that \( P^\sigma \in I \). It follows that \( Q_1, ..., Q_m \in I \). Moreover, by the construction, for each \( \tau \in \Gamma_M \) it holds that \( Q_1^\tau = Q_1, ..., Q_m^\tau = Q_m \), that is, \( Q_1, ..., Q_m \in \mathbb{K}[x_1, ..., x_n] \). In this way, \( Q_1, ..., Q_m \in I \cap \mathbb{K}[x_1, ..., x_n] \). This, together with 1, provides 2.

For instance, if \( \mathbb{U} = \mathbb{C} \) and \( \mathbb{K} = \mathbb{R} \), then \( \text{Tr}(P) = P + P^\sigma \in \mathbb{R}[x_1, ..., x_n] \) and, by taking \( e_1 = 1 \) and \( e_2 = i \), one has that \( P = \frac{1}{2} \text{Tr}(P) - \frac{i}{2} \text{Tr}(iP) \).

Let \( \rho : \text{Gal}(\mathbb{C}/\mathbb{K}) \rightarrow \text{Gal}(\mathbb{U}/\mathbb{K}) \) be the natural epimorphism defined by restriction. If \( P \in \mathbb{U}[x_1, ..., x_n] \), \( \eta \in \text{Gal}(\mathbb{C}/\mathbb{K}) \) and \( \sigma = \rho(\eta) \), then \( P^\sigma := P^\eta \) is the polynomial obtained from \( P \) by replacing its coefficients by their images under \( \sigma \). In particular, if \( Y \subset \mathbb{C}^n \) is a complex algebraic variety (it may be singular) defined over \( \mathbb{U} \), and if \( \eta_1, \eta_2 \in \text{Gal}(\mathbb{C}/\mathbb{K}) \) are so that \( \rho(\eta_1) = \rho(\eta_2) \), then \( Y^{\eta_1} = Y^{\eta_2} \). So, if \( \sigma \in \text{Gal}(\mathbb{U}/\mathbb{K}) \), then we may set \( Y^\sigma := Y^\eta \), where \( \eta \in \text{Gal}(\mathbb{C}/\mathbb{K}) \) is so that \( \rho(\eta) = \sigma \).

A direct consequence of Lemma [5] is the following.

Lemma 6. Let \( Y \) be an affine complex algebraic variety (it may be singular) defined over \( \mathbb{U} \). If \( Y^\sigma = Y \), for every \( \sigma \in \text{Gal}(\mathbb{U}/\mathbb{K}) \), then \( Y \) is defined over \( \mathbb{K} \). Moreover, if \( Y \) is defined by the polynomials \( P_1, ..., P_s \in \mathbb{U}[y_1, ..., y_n] \), then \( Y \) is also defined by the polynomials \( \text{Tr}(e_1 P_j), ..., \text{Tr}(e_m P_j) \in \mathbb{K}[y_1, ..., y_n] \), where \( j \in \{1, ..., s\} \).

Proof. Let \( I < \mathbb{C}[y_0, ..., y_n] \) be the ideal of \( Y \subset \mathbb{C}^n \). As \( Y \) is defined over \( \mathbb{U} \), the ideal \( I \) can be generated by polynomials with coefficients in \( \mathbb{U} \).
Let \( \sigma \in \text{Gal}(\mathbb{U}/\mathbb{K}) \) and let \( \eta \in \text{Gal}(\mathbb{C}/\mathbb{K}) \) be so that \( \rho(\eta) = \sigma \). If \( P \in I \) with coefficients in \( \mathbb{U} \) and \( (b_1, ..., b_n) \in Y \), then it holds that
\[
0 = \sigma(P(b_1, ..., b_n)) = \eta(P(b_1, ..., b_n)) = P^\sigma(\eta(b_1), ..., \eta(b_n)) = P^\sigma \circ \eta(b_1, ..., b_n).
\]

As \( \eta : Y \to Y \) is a bijection, by hypothesis, the above asserts that \( P^\sigma(c_1, ..., c_n) = 0 \) for each \( (c_1, ..., c_n) \in X \). This ensures that \( P^\sigma \in I \). Now Lemma 5 asserts the desired result. \( \square \)

2.4. Another ingredient in the computational method concerns with the algebra of invariants of a finite group of linear transformations. Let \( \mathcal{V} \) be a finite dimensional vector space over a field \( \mathcal{R} \), say of dimension \( n \geq 1 \). Let \( x_1, ..., x_n \) be a basis of \( \mathcal{V} \). The symmetric algebra of \( \mathcal{V} \) over \( \mathcal{R} \), say \( \mathcal{R}[[\mathcal{V}]] \), can be identified with the free unitary associative algebra generated by \( x_1, ..., x_n \) over \( \mathcal{R} \), that is, with the algebra of polynomials with variables \( x_1, ..., x_n \) and coefficients in \( \mathcal{R} \). If \( \Gamma \) is a group acting linearly over \( \mathcal{V} \), then that action extends naturally to the diagonal action on \( \mathcal{R}[[\mathcal{V}]] \).

**Theorem 7** (D. Hilbert - E. Noether [12, 13]). Let \( \mathcal{V} \) be a finite dimensional vector space over a field \( \mathcal{R} \). If \( \Gamma \) be a finite group acting linearly over \( \mathcal{V} \), then the algebra of \( \Gamma \)-invariants \( \mathcal{R}[[\mathcal{V}]]^\Gamma \) is finitely generated.

For a modern reference for the previous theorem see (Chap. 14 in [14]).

3. **Constructive proof of Theorem 1**

We set \( \mathcal{K} := \mathcal{Q} \cap \mathcal{R} \) and \( \Gamma := \text{Gal}(\mathbb{C}/\mathbb{K}) = \langle \sigma_2(z) = \overline{z} \rangle \cong \mathbb{Z}_2 \). We denote by \( \sigma_1 = e \) the identity element of \( \Gamma \). Clearly, \( \text{Gal}(\mathcal{Q}/\mathcal{K}) \) is also generated by the restriction of \( \sigma_2 \). Observe that \( \sigma_1 \) is the identity map and \( J := \sigma_2 \) is the conjugation map.

Let \( X \subset \mathbb{C}^n \) be an irreducible non-singular complex algebraic variety defined over \( \mathcal{Q} \), with a finite group of holomorphic automorphisms, admitting a symmetry \( L : X \to X \).

We notice that the symmetry \( L \) is defined over \( \mathcal{Q} \). In fact, if \( \rho \in \text{Gal}(\mathbb{C}/\mathcal{Q}) \), then we have the symmetry \( L^\rho : X \to X \). So, there is a holomorphic automorphism \( t \) of \( X \) so that \( L^\rho = L \circ t \). If we set \( K = \{ \rho \in \text{Gal}(\mathbb{C}/\mathcal{Q}) : L = L^\rho \} \) and \( \mathcal{U} \) is the fixed field of \( K \), then \( L \) is defined over \( \mathcal{U} \). Now, as \( X \) has a finite group of automorphisms, it follows that \( K \) is a subgroup of finite index of \( \text{Gal}(\mathbb{C}/\mathcal{Q}) \); so \( \mathcal{U} \) is a finite extension of \( \mathcal{Q} \). As \( \mathcal{Q} \) is algebraically closed, it now follows that \( \mathcal{U} = \mathcal{Q} \) and we are done.

3.1. **A non-constructive proof.** Just for completeness, we provide a short non-constructive proof of Theorem 1 as a consequence of Weil’s Galois descent theorem [16]. Let us consider the holomorphic isomorphism \( f_{\sigma_2} = J \circ L : X \to X^{\sigma_2} \), which is defined over \( \mathcal{Q} \) (since \( L \) and \( J \) are defined over \( \mathcal{Q} \)). If we set \( f_{\sigma_1} \) as the identity map of \( X \), then the collection \( \{ f_\sigma : \sigma \in \Gamma \} \) defines a Weil’s datum for \( X \) with respect to the Galois extension \( \mathcal{R} < \mathbb{C} \), that is, for every \( \tau, \eta \in \Gamma \), it holds that \( f_{\tau \eta} = f_\tau^\eta \circ f_\eta \). As the above isomorphisms are also defined over \( \mathcal{Q} \), it follows that \( \{ f_\sigma : \sigma \in \Gamma \} \)
also provides a Weil’s datum for \( X \) with respect to the Galois extension \( \mathcal{K} < \bar{\mathbb{Q}} \). It follows, from Weil’s Galois descent theorem, that \( X \) is definable over \( \mathcal{K} \).

3.2. A constructive proof. Next, we describe a method to obtain explicitly a birational isomorphism \( R \) between \( X \) and a suitable variety \( Z \) which is defined over \( \mathcal{K} \).

3.2.1. We assume that \( X \) is defined as the common zeroes of the polynomials

\[
P_1, \ldots, P_s \in \bar{\mathbb{Q}}[x_1, \ldots, x_n].
\]

We set \( f_j := f_{\sigma j} \), for each \( j = 1, 2 \) (as defined above). We already know that each \( f_j \) is defined over \( \mathbb{Q} \).

3.2.2. If \( X'_{\sigma^2} = X \), then it follows from Lemma 6 that \( X \) is defined over \( \mathcal{K} \). In fact, \( X \) is the common zeroes locus of the polynomials

\[
\left\{ \begin{array}{l}
P_1 + P'^{\sigma^2}_1, \quad i(P_1 - P'^{\sigma^2}_1) \\
\vdots \\
P_s + P'^{\sigma^2}_s, \quad i(P_s - P'^{\sigma^2}_s)
\end{array} \right.
\]

and we are done.

So, from now on, we assume that \( X'^{\sigma^2} \neq X \).

3.2.3. Let us consider the explicit holomorphic map (defined over \( \bar{\mathbb{Q}} \))

\[
\Phi : X \subset \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}^n = \mathbb{C}^{2n}
\]

\[
u \mapsto (f_1(\nu), f_2(\nu))
\]

As each \( f_j \) is a holomorphic isomorphism, \( \Phi : X \rightarrow \Phi(X) \) is a holomorphic isomorphism between \( X \) and \( \Phi(X) \). The inverse is just provided by the restriction to \( \Phi(X) \) of the projection

\[
\pi : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n : (x, z) \mapsto x.
\]

The complex algebraic variety \( \Phi(X) \) is defined by the equations

\[
\left\{ \begin{array}{l}
z = f_2(x), \\
P_1(x) = 0, \ldots, P_s(x) = 0
\end{array} \right.
\]

where \( (x, z) \in \mathbb{C}^n \times \mathbb{C}^n \), and it still an irreducible non-singular complex algebraic variety defined over \( \bar{\mathbb{Q}} \).

3.2.4. Each \( \sigma \in \Gamma \) produces a permutation \( \theta(\sigma) \in \mathfrak{S}_2 \) so that \( \sigma \sigma_j = \sigma \theta(\sigma_j) \), for every \( j = 1, 2 \). In fact, \( \theta(\sigma_1) = (1)(2) = e \) is the identity and \( \theta(\sigma_2) = (1, 2) \). We consider the natural isomorphism (Cayley representation)

\[
\theta : \Gamma \rightarrow \mathfrak{S}_2 : \sigma \mapsto \theta(\sigma).
\]

The symmetric group \( \mathfrak{S}_2 \) produces a natural permutation action \( \eta(\mathfrak{S}_2) \) on \( \mathbb{C}^n \times \mathbb{C}^n \) defined as follows. If \( x, z \in \mathbb{C}^n \), then

\[
\eta(e)(x, z) = (x, z), \quad \eta(1, 2)(x, z) = (z, x)
\]
If we set $\widehat{\theta}(\sigma) = \eta(\theta(\sigma))$, then we have a natural permutation action $\widehat{\theta}(\Gamma)$ on the 2 factors of the product $\mathbb{C}^n \times \mathbb{C}^n$, say

$$\widehat{\theta}(\sigma_1)(x, z) = (x, z)$$
$$\widehat{\theta}(\sigma_2)(x, z) = (z, x)$$

where $x, z \in \mathbb{C}^n$. We have obtained a representation $\widehat{\theta}(\Gamma) \cong \mathbb{Z}_2$ of $\Gamma$ as a subgroup of holomorphic automorphisms of $\mathbb{C}^n \times \mathbb{C}^n$.

3.2.5. If $\sigma \in \Gamma$, then we have the following equalities

$$\sigma(f_j(x)) = f_j^\sigma(\sigma(x)) = f_j^\sigma(\sigma(x)) = f_{\sigma(j)}(f_\sigma^{-1}(\sigma(x))) =$$

$$= f_{\sigma\sigma(j)}(f_\sigma^{-1}(\sigma(x))) = f_{\sigma(j)}(f_\sigma^{-1}(\sigma(x))).$$

It follows from the above that

(*) $\sigma(\Phi(x)) = \sigma(\eta) \circ \Phi \circ f_\sigma^{-1}(\sigma(x))$;

(**) if $y \in \Phi(X)$ and $\sigma \in \Gamma$, then $\sigma(y) \in \sigma(\Phi(X))$.

In this way, we have the following commutative diagram

\[
\begin{array}{ccc}
X & \overset{\Phi}{\rightarrow} & \Phi(X) \\
\downarrow \sigma & & \downarrow \sigma \\
X^\sigma & \overset{\Phi^\sigma}{\rightarrow} & \Phi^\sigma(X^\sigma) = \Phi(X)^\sigma = \widehat{\theta}(\sigma)(\Phi(X)) \\
\downarrow f_\sigma^{-1} & & \downarrow \widehat{\theta}(\sigma)^{-1} \\
X & \overset{\Phi}{\rightarrow} & \Phi(X)
\end{array}
\]

(3)

3.2.6. Set

$$G = \{ \tau \in \Gamma : \widehat{\theta}(\tau)(\Phi(X)) = \Phi(X) \} \subset \Gamma.$$ 

Clearly, from the definition, the elements $\tau \in G$ are exactly those for which $\widehat{\theta}(\tau)$ is a holomorphic automorphism of $\Phi(X)$; in particular, $X^{\tau} = X$. As either $G = \Gamma$ or $G = \{ e \}$ and we are assuming that $X^{\sigma_2} \neq X$, it follows that $G = \{ e \}$. This in particular asserts that the algebraic subvariety $W^* = \sigma_2(\Phi(X)) \cap \Phi(X)$ has positive co-dimension in $\Phi(X)$.

3.2.7. The algebra of $\Theta(\Gamma)$-invariant polynomials is generated by the following ones

$$t_1 = x_1 + z_1, \ldots, t_n = x_n + z_n,$$

$$t_{n+1} = x_1z_1, \ldots, t_{2n} = x_nz_n,$$

$$t_{2n+1} = x_1x_2 + z_1z_2, \ldots, t_{3n-1} = x_1x_n + z_1z_n.$$

Let us consider the holomorphic map

$$\Psi : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^{3n-1}$$

$$\Psi(x, z) = (t_1, \ldots, t_{3n-1})$$
Proposition 9. \( W \) is a subvariety of positive codimension of \( Z \) and the holomorphic map \( R \) provides a birational isomorphism between \( X \) and \( Z \). Moreover, both \( Z \) and \( W \) are defined over \( \mathbb{K} \). In fact, if both \( X \) and \( L \) are defined over the number field \( \mathbb{K} \), then \( Z \) is defined over \( \mathbb{K} \cap \mathbb{R} \).

Proof. As \( \Phi(X) \) is not \( \hat{\theta}(\Gamma) \)-invariant, the algebraic subvariety \( W^* = \hat{\theta}(\sigma_2)(\Phi(X)) \cap \Phi(X) \) has positive co-dimension in \( \Phi(X) \) and \( W^* = \Psi^{-1}(W) \). If \( \Omega = \Phi(X) - W^* \), then \( \Omega \) is a non-empty open dense subset over which \( \Psi \) is one-to-one. We notice, by the definition of \( W \), that \( Z \) may be singular only at points in \( W \). As \( \Psi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^{3n-1} \) is a holomorphic branched regular covering, with the finite group \( \hat{\theta}(\Gamma) \) as its deck group, it follows that \( \Psi : \Omega \to \Psi(\Omega) \) is a holomorphic isomorphism. In this way, \( \Psi : \Phi(X) \to Z \) is a holomorphic map which is also a birational isomorphism. In particular, \( R = \Psi \circ \Phi : X \to Z \) is a birational isomorphism and that \( W \) is a subvariety of positive codimension of \( Z \).

Let us now assume that both \( X \) and \( L \) are defined over the subfield \( \mathbb{K} \) of \( \mathbb{Q} \). We proceed to see that \( Z \) and \( W \) are defined over \( \mathbb{K} \cap \mathbb{R} \). Let \( \tau \in \text{Gal}(\mathbb{C}/\mathbb{K}) \). As \( X^* = X, \Phi^* = \Phi \) and \( \Psi^* = \Psi \), one has that \( R^* = R \) and, in particular, that \( Z^* = R(X)^* = R'(X^*) = R(X) = Z \). We have, by Lemma 6, that \( Z \) is definable over \( \mathbb{K} \). If \( \mathbb{K} \) is a subfield of \( \mathbb{R} \), then there is nothing to prove. Assume that this is not the case, so \( \mathbb{K} \) is a degree two extension of \( \mathbb{K} \cap \mathbb{R} \) and \( \text{Gal}(\mathbb{K}/\mathbb{K} \cap \mathbb{R}) \) is generated by the (restriction of) reflection \( \sigma_2 \). It follows, from (**) in Section 3.2.5 and (3) in

Remark 8. Observe that, for \( k = 2, \ldots, n, \)
\[
t_{n+k} = \frac{-t_1^2 t_{n+1} + t_1 t_{n-1+k+2n} - t_1^2 + 2n}{t_1^2 - 4t_{n+1}}.
\]

In particular, the image \( V = \Psi(\mathbb{C}^n \times \mathbb{C}^n) \subset \mathbb{C}^{3n-1} \) is the algebraic variety defined by these \( n - 1 \) previous equations.

3.2.8. Clearly, the map \( \Psi \) satisfies the following properties:

1. \( \Psi^\sigma = \Psi \), for every \( \sigma \in \Gamma \);
2. for every \( \sigma \in \Gamma \) it holds that \( \Psi \circ \tilde{\theta}(\sigma) = \Psi \); and
3. if \( \Psi(w) = \Psi(z) \), then there is some \( \gamma \in \Gamma \) so that \( w = \tilde{\theta}(\gamma)(z) \).

Conditions (1) and (2) are trivial to see. Condition (3) is consequence of the results in [5, 11] and the fact that finite groups are reductibles.

Conditions (2) and (3) assert that the map \( \Psi : \mathbb{C}^n \times \mathbb{C}^n \to V \) is a branched regular holomorphic covering with \( \hat{\theta}(\Gamma) \) as its deck group.

3.2.9. Set \( Z = \Psi(\Phi(X)) \subset V \), let us consider the restriction \( \Psi : \Phi(X) \to Z \) and set \( R = \Psi \circ \Phi : X \to Z \).

By the construction, the map \( R \) is holomorphic and explicitly given. Moreover, each point of \( Z \) either has one or two pre-images under \( R \).

Let \( W \) the subset of \( Z \) consisting of the points with two different pre-images on \( \Phi(X) \).
Section [3.2.7] that $\sigma_2 : Z \to Z$ is a bijection, that is, $Z^{\sigma_2} = Z$. Also, it can be seen that $\sigma_2(\Psi(W)) = \Psi(W)$ and so $W^{\sigma_2} = W$. Again, from Lemma 6, one obtains that $Z$ and $W$ are defined over $\mathbb{K} \cap \mathbb{R}$. □

Remark 10. If $W = \emptyset$, then $Z$ is non-singular and $R : X \to Z$ is a holomorphic isomorphism. If $W \neq \emptyset$, then, as a consequence of Proposition 9, $X$ is a normalization of $Z$ and $W$ is singular at $W$ which is also defined over $\mathbb{K}$. We may desingularize it along $W$ to obtain a non-singular model $\tilde{Z}$ over $\mathbb{K}$. Clearly, $\tilde{Z}$ will be defined over $\mathbb{K}$ and it will be holomorphically equivalent to $X$.

3.2.10. In the above process, we have provided an explicit birational map $R : X \to Z$, where it is known that $Z$ is defined over $\mathbb{K}$. As $R$ is explicitly given, with the help of MAGMA we may search for explicit polynomials $Q_1, \ldots, Q_m \in \mathbb{Q}[z_1, \ldots, z_N]$ defining $Z$. If one of these polynomials, say $F \in \{Q_1, \ldots, Q_m\}$, is not defined over $\mathbb{K}$, then it can be changed by the new polynomials (which are now defined over $\mathbb{K}$) $\text{Tr}(F), \text{Tr}(iF)$. Then Lemma 6 asserts that the new polynomials also define $Z$. This finishes the process.

4. An Example in genus 5

Closed Riemann surfaces of genus 5 admitting a group $H \equiv \mathbb{Z}_4$ are called classical Humbert’s curves. These curves appear in [7] by Humbert while investigating a net of conics and much later the same curves were encountered by Baker [1], related to a Weddle surface. In [4] a new set of quadrics defining those classical Humbert’s curves were obtained. We consider one particular of such curves in this section.

Let us consider the complex algebraic curve defined over $\overline{\mathbb{Q}}$

$$X : \left\{ \begin{array}{l}
1 + x_1^2 + x_2^2 = 0 \\
-1 + x_3^2 + x_4^2 = 0 \\
i + x_1^2 + x_4^2 = 0
\end{array} \right\} \subset \mathbb{C}^4.$$

It can be seen, by direct inspection or by using MAGMA, that $X$ is a non-singular irreducible algebraic curve of genus 5. The antiholomorphic map

$$L : \mathbb{C}^4 \to \mathbb{C}^4 : (x_1, x_2, x_3, x_4) \mapsto (-i \overline{x_1}, -i \overline{x_3}, -i \overline{x_2}, -i \overline{x_4})$$

keeps invariant $X$, so it defines an antiholomorphic automorphism of order two of $X$. It follows that $X$ is also definable over $\mathbb{R}$ and, by Theorem [1] that $X$ must also be definable over $\mathbb{Q} \cap \mathbb{R}$. In fact, as both $X$ and $L$ are defined over $\mathbb{Q}(i)$, we have seen that $X$ is definable over $\mathbb{Q}(i) \cap \mathbb{R} = \mathbb{Q}$. Another way to see this is as follows. As $X$ is defined over $\mathbb{Q}(i)$ and $X$ and $X^{\sigma_2}$ are holomorphically equivalent by $f : X \to X = X^{\sigma_2}$, given by

$$f : \mathbb{C}^4 \to \mathbb{C}^4; \quad f(x_1, x_2, x_3, x_4) = (i \overline{x_1}, i \overline{x_3}, i \overline{x_2}, i \overline{x_4}),$$

furthermore, we can restrict $f$ to $X$ and get the desired result.
it follows from Weil’s Galois descent theorem that $X$ has field of moduli $\mathbb{Q}$; therefore, by the results in [6], $X$ is definable over $\mathbb{Q}$.

Next, we follow the constructive proof, provided in the previous sections, for this particular example to obtain an explicit model of $X$ defined over $\mathbb{Q}$. In this example, $f_1$ is the identity and $f_2 = f$, so the holomorphic isomorphism $\Phi$ is given by

$$
\Phi : X \subset \mathbb{C}^4 \to \Phi(X) \subset \mathbb{C}^8, \quad (x_1, x_2, x_3, x_4) \mapsto (x, z)
$$

where

$$
x = (x_1, x_2, x_3, x_4), \quad z = (z_1, z_2, z_3, z_4) = (i x_1, i x_3, i x_2, i x_4).
$$

The equations defining $\Phi(X)$ are given by

$$
\begin{cases}
    z_1 - ix_1 = 0, & z_2 - ix_3 = 0, & z_3 - ix_2 = 0, & z_4 - ix_4 = 0 \\
    1 + x_1^2 + x_2^2 = 0, & -1 + x_1^2 + x_3^2 = 0, & i + x_1^2 + x_4^2 = 0
\end{cases}
$$

Notice that in this example $W = \emptyset$, so the two-to-one map $\Psi : \mathbb{C}^8 \to \mathbb{C}^{11}$ (as it was defined in Section [3.2.4] but with $n = 4$) provides a holomorphic isomorphism between $\Phi(X)$ and its image $Z = \Psi(\Phi(X))$; that is,

$$
R : X \to Z
$$

$$
R(x_1, x_2, x_3, x_4) \mapsto ((1+i)x_1, x_2+i x_3, x_3+i x_2, (1+i)x_4, ix_1^2, ix_2 x_3, ix_2 x_3, ix_3 x_3, ix_3^2, x_1 x_2 - x_1 x_3, x_1 x_3 - x_1 x_2, 0)
$$

is a holomorphic isomorphism.

Using MAGMA, we may see that $Z$ is the smooth curve of genus 5 given by the following equations, already defined over $\mathbb{Q}$ as desired,

$$
Z = \left\{ \begin{array}{l}
    t_{11} = 0, \quad t_9 + t_{10} = 0, \quad t_6 - t_7 = 0, \quad t_5 + t_8 - 1 = 0, \\
    t_2^2 - 2 t_8 = 0, \quad t_3^2 - 2 t_7 - 2 = 0, \quad t_2 - 2 t_7 + 2 = 0, \\
    t_8^2 t_{10}^2 + t_8^2 - 2 t_8^2 t_{10}^2 - 2 t_8 - 1 = 0, \quad t_7^2 t_{10}^2 + t_8 t_{10}^2 + 2 t_8 - t_{10}^2 - 2 = 0, \\
    t_7 t_8 - t_7 - t_{10}^2 + 2 t_8 + t_{10}^2 - 1 = 0, \quad t_2^2 - t_8^2 + 2 t_8 - 2 = 0, \\
    t_2 t_7 + t_2 - t_3 t_8 + t_3 = 0, \quad t_2 t_3 - 2 t_8 + 2 = 0, \\
    t_1 - 1/2 t_2 t_{10} - 1/2 t_3 t_{10} = 0, \quad t_2^2 - 2 t_3 t_7 + 2 t_3 t_8 + t_3 t_{10}^2 - 2 = 0, \\
    t_2 t_{10}^2 - 2 t_3 t_7 + 2 t_3 t_8 + t_3 t_{10}^2 - 2 = 0, \quad t_2 t_8 - t_2 - t_3 t_7 + t_3 = 0.
\end{array} \right\}
$$

**Remark 11.** Let us observe (as it can be seen from the form of $R$ or the above equations) that we may eliminate some of the coordinates in the above. We have, for $k = 2, 3, 4$, the following equality

$$
t_{5+k} = \frac{-t_2^2 t_5 + t_1 t_k t_{k+7} - t_{k+7}^2}{t_1^2 - 4 t_5}.
$$
So, after some elimination process, we may see that the only variables we need to use are \( w_1 = t_1, w_2 = t_2, w_3 = t_3 \) and \( w_4 = t_4 \), so

\[
Z \cong Y = \left\{ \begin{array}{l}
4 + w_2^2 - w_3^2 = 0 \\
w_1^2 + w_2w_3 = 0 \\
w_1^2 + w_4^2 - 2 = 0
\end{array} \right\} \subset \mathbb{C}^4.
\]

The curve \( X \) admits the group \( \langle A_1, A_2, A_3, A_4 \rangle \cong \mathbb{Z}_2^4 \) as subgroup of conformal automorphisms, where

\[
\begin{align*}
A_1(x_1, x_2, x_3, x_4) &= (-x_1, x_2, x_3, x_4) \\
A_2(x_1, x_2, x_3, x_4) &= (-x_1, -x_2, x_3, x_4) \\
A_3(x_1, x_2, x_3, x_4) &= (x_1, x_2, -x_3, x_4) \\
A_4(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, -x_4)
\end{align*}
\]

We see that all of them are defined over \( \mathbb{Q} \). The above automorphisms correspond to the following automorphisms in the model \( Y \):

\[
\begin{align*}
A_1(w_1, w_2, w_3, w_4) &= (-w_1, w_2, w_3, w_4) \\
A_2(w_1, w_2, w_3, w_4) &= (w_1, iw_3, -iw_2, w_4) \\
A_3(w_1, w_2, w_3, w_4) &= (w_1, -iw_3, iw_2, w_4) \\
A_4(w_1, w_2, w_3, w_4) &= (w_1, w_2, w_3, -w_4)
\end{align*}
\]

which are defined over \( \mathbb{Q}(i) \).

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