Differentiation of Integrals

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Abstract

No functions class for general measurable sets classes are known whose functions have the property of differentiability of integrals associated to such sets classes.

In this paper, we give some subspaces of $L^s$ with $1 < s < \infty$, whose functions are proven to have the differentiability of integrals associated to measurable sets classes in $\mathbb{R}^n$, this gives an answer to a question stated by Stein in his book Harmonic Analysis. We give also a example of some functions in these classes on $\mathbb{R}^2$, which is continuous nowhere.

1. Introduction

It is considered that the question of the differentiability of integrals in $\mathbb{R}^n$ represents one of the main issues in real-variable theory, Stein formulated this question in [9] as follows:

**Problem A** for what collections $\mathcal{C}$ of sets $\{C\}$, is it true that for "all" $f$

$$\lim_{\text{diam}(C) \to 0, x \in C \in \mathcal{C}} \frac{1}{|C|} \int_C f(y) \, dy = f(x) \quad \text{a.e. } x \in \mathbb{R}^n? \quad (1)$$

**Problem B** given $\mathcal{C}$, for what classes of functions $f$ does (1) hold?

If $\mathcal{C} = \{\text{all cubes}\}$ or $\{\text{all balls}\}$, the Lebesgue’s differentiation theorem asserts that (1) holds for all $f$ in $L^1_{\text{loc}}$ (see [9]).

If $\mathcal{C} = \{\text{rectangles: whose major axes point in a fixed directions}\}$, then (1) holds for all $f$ in $L^p_{\text{loc}}$ with $p > 1$, but fails when $p = 1$, (see [9]).

If $\mathcal{C} = \{\text{all rectangles}\}$, for each $p$ with $1 \leq p < \infty$, there exists an $f \in L^p$ such that (1) does not hold, (see [9]).

If $\mathcal{C} = \{\text{tubes that point in the given directions}\}$, then (1) holds for all $f$ in $L^p_{\text{loc}}$ with $p > 1$, (see [6]).

If the set in $\mathcal{C}$ have bounded eccentricity, in the sense that the ratio between the smallest ball containing $C$ to the largest ball contained in $C$ is uniformly bounded over $C \in \mathcal{C}$, then (1) holds for all $f$ in $L^1_{\text{loc}}$, (see [9]).

Efforts on the issues above includes the works in a lot of literatures, for example, [7, 3, 8, 5, 1], etc.
Even if for the class of arbitrary rectangles, as we see, $L^p$ is not suitable for (1). So far, only the class $C_0$ of continuous functions with compact supports for the class of arbitrary rectangles, while no class for general measurable sets classes, have been seen in the literatures, whose functions $f$ satisfying (1).

Let $C_0 = \{\text{all measurable sets } C \subset \mathbb{R}^n \text{ with } |C| \neq 0\}$. In this paper, we give some functions classes on $\mathbb{R}^n$, which are subspaces of $L^p$ with $1 < p < \infty$, whose functions are proven to satisfy (1) for $C = C_0$. This gives an answer for Problem B above. We give also a example of some functions in these classes on $\mathbb{R}^2$, which is continuous nowhere.

Let $C_1 = \{\text{cubes } Q : |Q| \geq 1\}$. Let $0 < p < \infty, -n < \alpha < \infty$. Denote that $\bar{p} = \min\{p, 1\}$.

**Definition 1** (A) $a(x)$ is said to be a $(p, \alpha)$-block, if

(i) $\sup \, a \subseteq Q \in C_1$, (ii) $\|a\|_{L^\infty} \leq |Q|^{-\alpha/pn-1/p}$.

(B) A $(p, \alpha)$-block $a(x)$ is said to be a $(p, \alpha)$-continuous block, if $a(x) \in C_c$.

(C) $a(x)$ is said to be a $(p, \alpha)$-characteristic block, if $a(x) = |Q|^{-\alpha/pn-1/p} \chi_Q(x)$ with $Q \in C_1$.

**Definition 2** (A) $BL^{p, \alpha} = \{f : f = \sum_{k=-\infty}^{\infty} |\lambda_k|^p < +\infty\}$, here, the "convergence" means a.e. convergence. Moreover, define a quasinorm on $BL^{p, \alpha}$ by $\|f\|_{BL^{p, \alpha}} = \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right)^{1/p}$, where the infimum is taken over all the a.e.-equal forms of $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$.

(B) $BC^{p, \alpha}$ is defined if each $a_k$ in the definition above is a $(p, \alpha)$-continuous block.

(C) $B^{p, \alpha}$ is defined if each $a_k$ in the definition above is a $(p, \alpha)$-characteristic block.

2. The Maximal Theorem

Let

$$M_{C_0} f(x) = \sup_{x \in C \in C_0, \text{diam}(C) < 1} \frac{1}{|C|} \int_C |f(y)| dy. \tag{2}$$

**Theorem 1** Let $0 < p < \infty$ and $-n < \alpha < n(p-1)$. $M_{C_0}$ is bounded on $BL^{p, \alpha}$.

We will prove Theorem 1 with the aid of the idea of [2].

**Definition 3** Let $0 < p < +\infty, -n < \alpha < +\infty, -\infty < A_0, B_0 < \infty, A_0 - B_0 = \frac{1}{p} - \frac{\alpha}{np}$ and $-\infty < \varepsilon < A_0$. Set $a = A_0 - \varepsilon$, and $b = B_0 - \varepsilon$. A function $M(x) \in L^\infty$ is said to be a $(p, \alpha, \varepsilon)$-molecular (centered at $x_0$), if

(i) $M(x)|x - x_0|^{nb} \in L^\infty$, (ii) $\|M\|_{L^\infty}^a \|M(x)|x - x_0|^{nb}\|_{L^\infty}^{1-a/b} \equiv \Re(M) < \infty$.

**Theorem 2** Let $p, \alpha, \varepsilon, a, b$ as in Definition 3. Let $M(x)$ be a $(p, \alpha, \varepsilon)$-molecular centered at any point $x_0 \in \mathbb{R}^n$. If there exists $Q_0 \in C_1$ such that

$$\|M\|_{L^\infty} \leq |Q_0|^{-1/p-\alpha/np},$$

then $M \in BL^{p, \alpha}$ and $\|M\|_{BL^{p, \alpha}} \leq C \Re(M)$, where the constant $C$ is independent of $M$.

**Proof** Without loss of generality, we can assume $\Re(M) = 1$. In fact, assume $\|M\|_{BL^{p, \alpha}} \leq C$ holds whenever $\Re(M) = 1$. Then, for general $M$, let $M' = M/\Re(M)$. We have $\Re(M') = 1$ and hence $\|\Re(M)M'\|_{BL^{p, \alpha}} \leq \Re(M)\|M'\|_{BL^{p, \alpha}} \leq C \Re(M)$.

Define cube $Q_1$ centered at 0 whose sides parallel to the axes by setting

$$\|M\|_{L^\infty} = |Q_1|^{-1/p-\alpha/np}.$$
We see that $|Q_1| \geq |Q_0|$ since $|Q_1|^{-1/p-\alpha/np} \leq |Q_0|^{-1/p-\alpha/np}$ and $\alpha > -n$, so we can assume that $Q_1 \in C_1$. Let

$$E_0 = Q_1, E_k = 2^k Q_1 \setminus 2^{k-1} Q_1, k = 1, 2, \cdots.$$  

Set $M_k = M \chi_{E_k}$. By $\mathcal{R}(M) = 1$, we have

$$\left\| M(x) |x|^{nb} \right\|_{L^\infty} = \| M \|_{L^\infty}^{-\frac{b}{\| R \|}} = |Q_1|^{-\frac{n}{\| R \|}} = |Q_1|^a.$$  

For $x \in E_k$, $k = 1, 2, \cdots$, we have

$$|M_k(x)| = |M_k(x)||x|^{nb}|x|^{-nb} \leq |2^{k-1} Q_1|^{-b}|M_k(x)||x|^{nb} = 2^{nb} |2^k Q_1|^{-b}|M_k(x)||x|^{nb},$$  

since $b > 0$. It follows that

$$\| M_k \|_{L^\infty} \leq 2^{nb} |2^k Q_1|^{-b} \| M_k(x) |x|^{nb} \|_{L^\infty} \leq 2^{nb} |2^k Q_1|^{-b} |Q_1|^a = 2^{nb} 2^{-kna} |2^k Q_1|^{-1/p-\alpha/np},$$  

for $k = 1, 2, \ldots$, while

$$\| M_0 \|_{L^\infty} \leq \| M \|_{L^\infty} = |Q_1|^{-1/p-\alpha/np} \leq 2^{nb} |Q_1|^{-1/p-\alpha/np}.$$  

Let $a_k(x) = 2^{-nb} 2^{kna} M_k(x)$, $k = 0, 1, 2, \ldots$,

then

$$M(x) = \sum_{k=0}^\infty M_k(x) = \sum_{k=0}^\infty 2^{nb} 2^{-kna} 2^{-nb} 2^{kna} M_k(x) = 2^{nb} \sum_{k=0}^\infty 2^{-kna} a_k(x),$$  

and each $a_k$ is a $(p, \alpha)$-block centered at 0 with $\text{supp } a_k \subset 2^k Q_1$, and

$$2^{nb} \left\{ \sum_{k=0}^\infty 2^{-\rho nk} \right\}^{1/\rho} = C < \infty$$  

since $a > 0$. That is $\|M\|_{BL,p,\alpha} \leq C$. Thus, Theorem 2 is proved.

**Proposition 1** Let $0 < p < \infty$, $-n < \alpha < n(p - 1)$ . Then,

$$\| M_{C_0} h \|_{BL,p,\alpha} \leq C,$$  

for a $(p, \alpha)$-block $h$, where constant $C$ is independent of $h$ .

**Proof** By the molecular theorem, it suffices to check that $M_{C_0} h$ is a $(p, \alpha, \varepsilon)$-molecular centered at $x_0$ for every $(p, \alpha)$-block $h$ centered at $x_0$ and $\mathcal{R}(M_{C_0} h) \leq C$, where $C$ is independent of $h$. Given a $(p, \alpha)$-block $h$ with $\text{supp } h \subset Q_0$ in $C_1$ centered at $x_0$. We see that $\| M_{C_0} h \|_{L^\infty} \leq \| h \|_{L^\infty} \leq |Q_0|^{-1/p-\alpha/np}$. Since $1 - 1/p - \alpha/np > 0$, we can choose $\varepsilon$ such that

$$0 < \varepsilon < \min\{1 - 1/p - \alpha/np, 1\}.$$
and set

\[ a = 1 - 1/p - \alpha np - \varepsilon \quad \text{and} \quad b = 1 - \varepsilon. \]

Clearly, \(a > 0, b > 0\) and \(b - a = (n + \alpha)/np > 0\). To get \(\Re(M_{C_0}h) \leq C\), it suffices to prove that

\[ J =: \|M_{C_0}h(x)|x - x_0|^{nb}\|_{L^\infty} \leq C|Q_0|^b\|h\|_{L^\infty}. \quad (3) \]

Write

\[ J \leq \|M_{C_0}h(x)|x - x_0|^{nb}\chi_{2Q_0}(x)\|_{L^\infty} + \|M_{C_0}h(x)|x - x_0|^{nb}\chi(2Q_0)^c(x)\|_{L^\infty} =: J_1 + J_2. \]

For \(J_1\), we have that

\[ J_1 \leq \|M_{C_0}h\|_{L^\infty}\|x - x_0|^{nb}\chi_{2Q_0}(x)\|_{L^\infty} \leq C|Q_0|^b\|h\|_{L^\infty} \]

since \(b > 0\) and \(L^\infty\) boundedness of \(M_{C_0}\). For \(J_2\), we see that

\[ M_{C_0}h(x) = \sup_{x \in C, \text{diam}(C) < 1} \frac{1}{|C|} \int_C |h(y)| dy = \sup_{x \in C, \text{diam}(C) < 1} \frac{1}{|C|} \int_{C \cap Q_0} |h(y)| dy = 0 \]

for \(x \in (2Q_0)^c\), since \(C \cap Q_0 = \emptyset\). So, \(J_2 = 0\). Thus, \((3)\) holds. Proposition 1 follows.

Theorem 1 follows easily from Proposition 1.

Let \(0 < p < \infty\) and \(-n < \alpha < n(p - 1)\), it is known that \(B^{p,\alpha}_{BL} \subset L^{np/(n+\alpha)}\), see [4]. Then,

**Corollary 1** Let \(0 < p < \infty\) and \(-n < \alpha < n(p - 1)\). \(M_{C_0}\) is bounded from \(B^{p,\alpha}_{BL}\) to \(L^{np/(n+\alpha)}\).

3. The Differentiation Theorems By Corollary 1 and a standard arguments, we have that

**Theorem 3** Let \(0 < p < \infty\) and \(-n < \alpha < n(p - 1)\). Let \(B\) be a subspace of \(B^{p,\alpha}_{BL}\). If \((1)\) holds as \(C = C_0\) for the functions in a dense subspace \(D\) of \(B\), then \((1)\) holds for all functions in \(B\).

\(BC^{p,\alpha}_\bullet\) and \(B^{p,\alpha}_\bullet\) are in \(B^{p,\alpha}_{BL}\). At the same time, we see that

**Property 1** Let \(0 < p < \infty\), and \(-n < \alpha < \infty\). \(C_\bullet\) is dense in \(BC^{p,\alpha}_\bullet\).

**Proof** Let \(f \in BC^{p,\alpha}_\bullet\), i.e. \(f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)\) where each \(a_k\) is a \((p, \alpha)\)-continuous block with \(\text{supp } a_k \subseteq Q_k\) and \(\sum_{k=1}^{\infty} |\lambda_k|^p < \infty\). Then for any \(\varepsilon > 0\), there exists an \(i_0\) such that \(\sum_{k=i_0+1}^{\infty} |\lambda_k|^p < \varepsilon^p\). Let \(f_{i_0}(x) = \sum_{k=1}^{i_0} \lambda_k a_k(x)\), we see that \(f_{i_0} \in C_\bullet\), and

\[ \|f - f_{i_0}\|_{BC^{p,\alpha}_\bullet} = \| \sum_{k=i_0+1}^{\infty} \lambda_k a_k \|_{BC^{p,\alpha}_\bullet} \leq \sum_{k=i_0+1}^{\infty} |\lambda_k|^p < \varepsilon^p. \]

Thus, Property 1 holds.

Let \(C_{R}\) be the collection of arbitrary rectangles in \(\mathbb{R}^n\). it is known that \((1)\) holds as \(C = C_R\) for all functions in \(C_{\bullet}\). So, we have that

**Corollary 2** Let \(0 < p < \infty\) and \(-n < \alpha < n(p - 1)\). \((1)\) holds as \(C = C_R\) for all functions in \(BC^{p,\alpha}_\bullet\).

Let \(FB^{p,\alpha}_\bullet\) be the functions class consists of finite linear combinations of \((p, \alpha)\)-characteristic blocks. It is easy to see that \(FB^{p,\alpha}_\bullet\) is dense in \(B^{p,\alpha}_\bullet\). While \((1)\) holds as
\[ C = C_0 \] for all functions in \( FB^{p,\alpha} \), in fact, for each \((p, \alpha)\)-characteristic block \( a_k \) with support set \( Q \) (cube), the equation in (1) holds for \( x \in Q \setminus \partial Q \) (inner point of \( Q \)). So,

**Corollary 3** Let \( 0 < p < \infty \) and \(-n < \alpha < n(p - 1)\). (1) holds as \( C = C_0 \) for all functions in \( B^{p,\alpha} \).

4. A count example Denote \( Q \) the set of rational numbers. Denote \( I_i \) the closed interval \([i, i + 1], i \in \mathbb{Z}\). Let \( Q_{i,j} = \{(r, q) \in I_i \times I_j : r, q \in Q\} \). Let \( E_{r,q}^{i,j} \) be the closed cube with sides parallel to the axes, with side length 1 and with a vertex \((r, q)\) in the upper right corner. Let

\[
 f_{i,j}(x) = \sum_{(r,q) \in Q_{i,j}} \lambda_{r,q}^{i,j} \chi_{E_{r,q}^{i,j}}(x),
\]

with each \( \lambda_{r,q}^{i,j} > 0 \) for even \( j \) and all \( i \in \mathbb{Z} \) and \( \lambda_{r,q}^{i,j} < 0 \) for odd \( j \) and all \( i \in \mathbb{Z} \), and \( \sum_{(r,q) \in Q_{i,j}} |\lambda_{r,q}^{i,j}|^\beta < \infty \). Let

\[
 g(x) = \sum_{i,j} f_{i,j}(x).
\]

It is easy to see that \( g(x) \in B^{p,\alpha} \). So (1) holds for \( g \) as \( C = C_0 \).

**Proposition 2** There exists \( g(x) \) which is continuous nowhere on \( \mathbb{R}^2 \) such that for which (1) holds as \( C = C_0 \).

**Proof** Need only to prove that \( g(x) \) above is continuous nowhere on \( \mathbb{R}^2 \). Taking \((r_i, q_j) \in Q_{i,j}\) for each pair \((i, j)\), we see that

\[
 \begin{align*}
 (r_i, q_j) \in \cap_{q_j \leq q \leq q+1,q \in Q} E_{r,q}^{i,j}, & \quad (r_i, q_j) \in \cap_{q_j \leq q \leq q+1,q \in Q} E_{r,q}^{i,j+1}, \quad (r_i, q_j) \in \cap_{q_j \leq q \leq q+1,q \in Q} E_{r,q}^{i,j+1}, \\
 (r_i, q_j) \in \cap_{j+1 \leq q \leq q+1,q \in Q} E_{r,q}^{i,j+1}, & \quad (r_i, q_j) \in \cap_{j+1 \leq q \leq q+1,q \in Q} E_{r,q}^{i,j+1}, \quad (r_i, q_j) \in \cap_{j+1 \leq q \leq q+1,q \in Q} E_{r,q}^{i,j+1}, \quad (r_i, q_j) \in \cap_{j+1 \leq q \leq q+1,q \in Q} E_{r,q}^{i,j+1},
\end{align*}
\]

it follows that

\[
 f_{i,j}((r_i, q_j)) = \sum_{q_j \leq q \leq q+1,q \in Q} \lambda_{r,q}^{i,j},
\]

\[
 f_{i+1,j}((r_i, q_j)) = \sum_{q_j \leq q \leq q+1,q \in Q} \lambda_{r+1,q}^{i+1,j},
\]

\[
 f_{i,j+1}((r_i, q_j)) = \sum_{j \leq q \leq j+1,q \in Q} \lambda_{r,q}^{i,j+1},
\]

\[
 f_{i+1,j+1}((r_i, q_j)) = \sum_{j \leq q \leq j+1,q \in Q} \lambda_{r,q}^{i+1,j+1}.
\]

Noticing

\[
 g((r_i, q_j)) = f_{i,j}((r_i, q_j)) + f_{i+1,j}((r_i, q_j)) + f_{i,j+1}((r_i, q_j)) + f_{i+1,j+1}((r_i, q_j)),
\]

we have for \( \tilde{q}_j > q_j \) that

\[
 g((r_i, \tilde{q}_j)) - g((r_i, q_j)) = \sum_{q_j \leq q < \tilde{q}_j,q \in Q} \lambda_{r,q}^{i,j} - \sum_{q_j \leq q < \tilde{q}_j,q \in Q} \lambda_{r+1,q}^{i+1,j} + \sum_{q_j \leq q < \tilde{q}_j+1,q \in Q} \lambda_{r,q}^{i,j+1} + \sum_{q_j \leq q < \tilde{q}_j+1,q \in Q} \lambda_{r+1,q}^{i+1,j+1},
\]

it follows that

\[
 |g((r_i, \tilde{q}_j)) - g((r_i, q_j))| > |\lambda_{r,q}^{i,j}|
\]
It follows that \( f(x) \) is not continuous at \((r_i, q_j)\). Proposition 2 follows.

5. Two remarks

**Remark 1** Let \( C_l = \{ \text{cubes } Q : |Q| \geq l^n > 0 \} \). Let \( BL^{p,\alpha}_l, BC^{p,\alpha}_l \) and \( B^{p,\alpha}_l \) be the classes replacing \( C_1 \) in Definition 1 and 2 by \( C_l \). By a simple dilation transform, we see that Theorem 3, Corollary 2 and 3, and Proposition 1 hold for \( l > 0 \) if \( BL^{p,\alpha}, BC^{p,\alpha} \) and \( B^{p,\alpha} \) are replaced by \( BL^{p,\alpha}_l, BC^{p,\alpha}_l \) and \( B^{p,\alpha}_l \), respectively. Let \( M_{C_0,l} f(x) = \sup_{x \in C \in C_{l}, \text{diam}(C) < l} \frac{1}{|C|} \int_C |f(y)| dy \), it is easy to see from the proofs that Theorem 1 and 2, Proposition 1 and Corollary 1 hold for \( M_{C_0,l} \) and \( BL^{p,\alpha}_l \) when \( l > 0 \).

**Remark 2** Let \( BL^{p,\alpha}_R, BC^{p,\alpha}_R \) and \( B^{p,\alpha}_R \) be the classes replacing \( C_1 \) in Definition 1 and 2 by \( R_0 = \{ \text{rectangles } R \text{ with sides length } \geq 1 \text{ and fixed eccentricity} \} \), (the eccentricity of a rectangle is the ratio of its longest size over its shortest size). Then, Theorem 1-3, Corollary 1-3 and Proposition 1 hold still if \( BL^{p,\alpha}, BC^{p,\alpha} \) and \( B^{p,\alpha} \) are replaced by \( BL^{p,\alpha}_R, BC^{p,\alpha}_R \) and \( B^{p,\alpha}_R \), respectively. This can be seen by the facts:

\[
R \subset Q \quad \text{and} \quad |R|^{-\alpha/pn-1/p} \leq (d_{\min}^n)^{-\alpha/pn-1/p} = c^n(\alpha/pn+1/p) |Q|^{-\alpha/pn-1/p},
\]

since \(-\alpha/pn-1/p < 0\), where \( R \in R_0 \) with the shortest side length \( d_{\min} \) and eccentricity \( c \), \( Q \) is the cube with sides length \( cd_{\min} \).

**Acknowledgements** This work was supported in part by NSF of China grant 11171280

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