STRUCTURE OF OCTONIONIC HILBERT SPACES WITH APPLICATIONS IN THE PARSEVAL EQUALITY AND CAYLEY-DICKSON ALGEBRAS

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Abstract. Contrary to the simple structure of the tensor product of the quaternionic Hilbert space, the octonionic situation becomes more involved. It turns out that an octonionic Hilbert space can be decomposed as an orthogonal direct sum of two subspaces, each of them isomorphic to a tensor product of an irreducible octonionic Hilbert space with a real Hilbert space. As an application, we find that for a given orthogonal basis the octonionic Parseval equality holds if and only if the basis is weak associative. Fortunately, there always exists a weak associative orthogonal basis in an octonionic Hilbert space. This completely removes the obstacles caused by the failure of the octonionic Parseval equality. As another application, we provide a new approach to study the Cayley-Dickson algebras, which turn out to be specific examples of octonionic Hilbert spaces. An explicit weak associative orthonormal basis is constructed in each Cayley-Dickson algebra.

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1. Introduction

The theory of octonionic Hilbert spaces is initiated by Goldstine and Horwitz [5] in 1964. It has many developments in the spectral theory [15], operator theory [14], and mathematical physics [1, 8, 7, 17].

However, for the normed division algebras, many results are restricted to the quaternionic case and remain open in the octonionic case [21].

A quaternionic Hilbert space is of a simple structure:

• A Hilbert quaternionic bimodule is the tensor product of the quaternion algebra \(\mathbb{H}\) with a real Hilbert space [16].

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A Hilbert left quaternionic module is also a tensor product up to an isomorphism. This is because the compatible right multiplication always exits which depends on a choice of a Hilbert basis as shown in [4, Section 3.1].

A natural question for the octonionic case is whether a Hilbert left octonionic module is the tensor product of the octonionic algebra \( \mathbb{O} \) with some real Hilbert space. If the answer was affirmative, then it would extremely simplify the theory of Hilbert left octonionic modules. We refer to [14] for such a development of the theory of octonionic Hilbert spaces under the assumption of such a tensor product decomposition.

In this article, we give a negative answer to the question above. It turns out that a Hilbert left octonionic module can be decomposed into an orthogonal sum of two subspaces, each with a tensor product decomposition.

To state our results in detail, we first recall the definition of Hilbert left \( \mathbb{O} \)-modules due to Goldstine and Horwitz [5].

**Definition 1.1** ([5]). A Hilbert left \( \mathbb{O} \)-module \( H \) is a left \( \mathbb{O} \)-module with an \( \mathbb{O} \)-inner product \( \langle \cdot, \cdot \rangle : H \times H \to \mathbb{O} \) such that \((H, \text{Re} \langle \cdot, \cdot \rangle)\) is a real Hilbert space. Here the \( \mathbb{O} \)-inner product satisfies the following axioms for all \( u, v \in H \) and \( p \in \mathbb{O} \):

(a) \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \);
(b) \( \langle u, v \rangle = \langle v, u \rangle \);
(c) \( \langle u, u \rangle \in \mathbb{R}^+ \); and \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \);
(d) \( \langle tu, v \rangle = t \langle u, v \rangle \) for \( t \in \mathbb{R} \);
(e) \( \text{Re} \langle pu, v \rangle = \text{Re}(p \langle u, v \rangle) \);
(f) \( \langle pu, u \rangle = p \langle u, u \rangle \).

Recently, we find that axiom (f) is non-independent [11]. Since the functionals induced from the \( \mathbb{O} \)-inner product are not \( \mathbb{O} \)-linear, this motivates us to introduce the para-linearity instead [11]. It turns out that para-linearity is the main object of octonionic functional analysis, other than octonionic linearity.

We first consider the underline module structure of a Hilbert left \( \mathbb{O} \)-module \( H \). We refer to [20, 22] for the structure of alternative modules, which take \( \mathbb{O} \)-modules as specific examples.

It is known that there are only two irreducible left \( \mathbb{O} \)-modules, i.e., \( \mathbb{O} \) and \( \mathbb{O}^* \). Here the conjugate regular module \( \mathbb{O}^* \) coincides with \( \mathbb{O} \) as a set and its left module structure is defined by

\[
p^*x := \overline{px},
\]

for any \( p \in \mathbb{O} \) and \( x \in \mathbb{O} \). Moreover, any left \( \mathbb{O} \)-module \( H \) admits a decomposition [10]

\[
H = \mathbb{O}\mathcal{A}(H) \oplus \mathbb{O}\mathcal{A}^-(H),
\]

(1.1)

where

\[
\mathcal{A}(H) := \{ m \in H \mid \langle pq \rangle m - p(qm) = 0 \text{ for all } p, q \in \mathbb{O} \}
\]

is the set of **associative elements**, called the **nucleus** of \( M \), and

\[
\mathcal{A}^- (H) := \{ m \in H \mid \langle pq \rangle m - q(pm) = 0 \text{ for all } p, q \in \mathbb{O} \}
\]

is the set of **conjugate associative elements**. Here, we denote

\[
\mathbb{O}U := \left\{ \sum_{i=1}^{N} p_i u_i \mid p_i \in \mathbb{O}, \ u_i \in U, \ N \in \mathbb{N} \right\}
\]
for any $U \subseteq H$.

With the $\mathbb{O}$-module structure of $\mathbb{O}$ and $\overline{\mathbb{O}}$ at hands, we can endow a natural $\mathbb{O}$-module structure on the set

\begin{equation}
(\mathbb{O} \otimes_\mathbb{R} \mathcal{A}(H)) \oplus (\overline{\mathbb{O}} \otimes_\mathbb{R} \mathcal{A}^{-}(H))
\end{equation}

The decomposition (1.1) can be modified as

\begin{equation}
H \cong (\mathbb{O} \otimes_\mathbb{R} \mathcal{A}(H)) \oplus (\overline{\mathbb{O}} \otimes_\mathbb{R} \mathcal{A}^{-}(H)),
\end{equation}

which is a natural isomorphism of $\mathbb{O}$-modules.

Notice that this isomorphism is only at the level of left $\mathbb{O}$-modules. We need to seek further an isomorphism at the level of Hilbert left $\mathbb{O}$-modules. To this end, we will endow the left $\mathbb{O}$-module in (1.2) with a canonical $\mathbb{O}$-inner product. It is shown that the submodules in the two summands in (1.2) are orthogonal to each other, and the $\mathbb{O}$-inner products restricted to both $\mathcal{A}(H)$ and $\mathcal{A}^{-}(H)$ are all real-valued, respectively. This leads to the tensor product decomposition of Hilbert left $\mathbb{O}$-modules.

**Theorem 1.2 (Tensor product decomposition).** Let $H$ be a Hilbert left $\mathbb{O}$-module. Then there exists an $\mathbb{O}$-isomorphism of $\mathbb{O}$-Hilbert spaces

\begin{equation}
H \cong (\mathbb{O} \otimes_\mathbb{R} \mathcal{A}(H)) \oplus (\overline{\mathbb{O}} \otimes_\mathbb{R} \mathcal{A}^{-}(H)).
\end{equation}

Theorem 1.2 can be applied to study the Parseval theorem in $\mathbb{O}$-Hilbert spaces. As observed by Goldstine and Horwitz in the appendix of [6], the Parseval equality may fail for arbitrary orthonormal basis in an $\mathbb{O}$-Hilbert space. We find for a given orthonormal basis, the related Parseval equality holds if and only if the orthonormal basis is weak associative (see Definition 4.7). Fortunately, a weak associative orthonormal basis always exists in an $\mathbb{O}$-Hilbert space. This makes the Parseval equality applicable in the octonionic setting.

Theorem 1.2 can also be applied to study the Cayley-Dickson algebra $\mathbb{A}_n$. We refer to [2, 9] for the theory of Cayley-Dickson algebras. In this article, we introduce a new approach to study the Cayley-Dickson algebra $\mathbb{A}_n$. That is, we regard $\mathbb{A}_n$ as a Hilbert left $\mathbb{O}$-module when $n \geq 4$. It leads to an isomorphism of Hilbert left $\mathbb{O}$-modules

\[ \mathbb{A}_n \cong \mathbb{O}^{2^{n-4}} \oplus \overline{\mathbb{O}}^{2^{n-4}}. \]

In particular, for the algebra of sedenions $\mathbb{S}$, we have

\[ \mathbb{S} = \mathbb{O} \oplus \overline{\mathbb{O}}. \]

We can construct an explicit weak associative orthonormal basis in $\mathbb{A}_n$. Note that a Cayley-Dickson algebra is generally not a quaternionic vector space. The fact that $\mathbb{A}_n$ is a Hilbert left $\mathbb{O}$-module provides a new method to study Cayley-Dickson algebras.

2. Preliminaries

In this section we review some definitions and basic properties about the algebra $\mathbb{O}$ of octonions, $\mathbb{O}$-modules and $\mathbb{O}$-Hilbert spaces.
2.1. Octonions. The algebra \( \mathbb{O} \) of octonions is the 8-dimensional non-associative, non-commutative, normed division algebra over \( \mathbb{R} \). For convenience, we denote \( e_0 = 1 \). Let \( e_0, e_1, \ldots, e_7 \) be a basis of \( \mathbb{O} \) as a real vector space subject to the multiplication rule

\[
e_i e_j = \epsilon_{ijk} e_k - \delta_{ij}, \quad i, j = 1, \ldots, 7,
\]

where \( \delta_{ij} \) is the Kronecker delta and \( \epsilon_{ijk} \) is a completely skew-symmetric with value 1 precisely when

\[i j k = 123, 145, 176, 246, 257, 347, 365.\]

An octonion can be written as

\[x = x_0 + \sum_{i=1}^{7} x_i e_i,\]

where \( x_i \in \mathbb{R} \) for \( i = 0, \ldots, 7 \). Its conjugate is defined by

\[\overline{x} := x_0 - \sum_{i=1}^{7} x_i e_i,\]

its norm equals \( |x| = \sqrt{x \overline{x}} \in \mathbb{R} \), and its real part is \( \text{Re} x = x_0 = \frac{1}{2}(x + \overline{x}) \). We denote by \( \mathbb{S}_6 \) the set of imaginary units in \( \mathbb{O} \), i.e.,

\[\mathbb{S}_6 := \{J \in \mathbb{O} \mid J^2 = -1\}.
\]

Then there is a book structure on octonions (see [13]):

\[\mathbb{O} = \bigcup_{I,J} \mathbb{H}_{I,J},\]

where the sum runs over all \( I, J \in \mathbb{S}_6 \) which are mutually orthogonal and \( \mathbb{H}_{I,J} \) denotes the quaternionic algebra spanned by \( \{1, I, J, IJ\} \).

2.2. \( \mathbb{O} \)-modules. An octonionic module is a specific alternative module since the octonionic algebra is an alternative algebra. The general representation theory over alternative algebras has been fully studied; see [18, 12, 20, 19, 10]. We recall some basic notations and results on \( \mathbb{O} \)-modules in this subsection.

**Definition 2.1.** An \( \mathbb{R} \)-vector space \( M \) is called a left \( \mathbb{O} \)-module, if there is an \( \mathbb{R} \)-linear map

\[L : \mathbb{O} \rightarrow \text{End}_\mathbb{R}M, \quad p \mapsto L_p\]

satisfying \( L_1 = \text{id}_M \) and

\[(p, q, x) = -[q, p, x], \quad (2.2)\]

for all \( p, q \in \mathbb{O} \) and \( x \in M \). Here

\[ [p, q, x] := (pq)x - p(qx) = L_{pq}(x) - L_pL_q(x) \]

is called the left associator of \( M \). The definition of right \( \mathbb{O} \)-module is similar.

It is a useful fact that (2.2) is equivalent to the following statement

\[r(rm) = r^2 m\]

for all \( r \in \mathbb{O} \) and all \( m \in M \).
The structure of a left $\mathbb{O}$-module is characterized by its associative elements and conjugate associative elements as follows. The **associative elements** of a left $\mathbb{O}$-module $M$ form a set

$$\mathcal{A}(M) := \{ m \in M \mid [p, q, m] = 0 \text{ for all } p, q \in \mathbb{O} \},$$

called the **nucleus** of $\mathbb{O}$-module $M$. The **conjugate associative elements** of a left $\mathbb{O}$-module $M$ form a set denoted by

$$\mathcal{A}^{-}(M) := \{ m \in M \mid (pq)m = q(pm) \text{ for all } p, q \in \mathbb{O} \}.$$

**Theorem 2.2** ([10]). Let $M$ be a left $\mathbb{O}$-module. Then

$$M = \mathbb{O}\mathcal{A}(M) \oplus \mathbb{O}\mathcal{A}^{-}(M).$$

The category of left $\mathbb{O}$-modules is shown to be isomorphic to the category of left $\mathbb{C}\ell_7$-modules [10]. For a left $\mathbb{O}$-module $M$, we can endow it with a natural $\mathbb{C}\ell_7$-module structure, denoted by $M_{\mathbb{C}\ell_7}$. Let $M, M'$ be two left $\mathbb{O}$-module. We denote

$$\text{Hom}_{\mathbb{O}}(M, M') := \{ f \in \text{Hom}_R(M, M') \mid f(px) = pf(x) \text{ for all } p \in \mathbb{O} \}.$$

Then it follows from Schur’s lemma that

$$\text{Hom}_{\mathbb{O}}(\mathbb{O}, \mathbb{O}) \cong \text{Hom}_{\mathbb{C}\ell_7}(\mathbb{O}_{\mathbb{C}\ell_7}, \mathbb{O}_{\mathbb{C}\ell_7}) = \{0\}.$$

We restate this fact in a more precise way as follows.

**Lemma 2.3.** If $f \in \text{End}_R(\mathbb{O})$ satisfies $f(px) = pf(x)$ for all $p, x \in \mathbb{O}$, then $f = 0$.

A simple variant of Lemma 2.3 is as follows.

**Lemma 2.4.** Let $f \in \text{End}_R(\mathbb{O})$. If it holds $f(xq) = qf(x)$ for all $q, x \in \mathbb{O}$, then $f = 0$.

**Proof.** We define $g(x) := f(\overline{x})$. Then we obtain

$$g(px) = f(\overline{px}) = \overline{pf(x)} = \overline{pg(x)}.$$

It thus follows from Lemma 2.3 that $g = 0$, i.e., $f = 0$. \hfill $\square$

### 2.3. $\mathbb{O}$-Hilbert spaces

Octonionic Hilbert spaces are introduced by Goldstine and Horwitz [5]. Recently, we [11] develop this theory by using a new notion of $\mathbb{O}$-para-linearity.

**Definition 2.5** ([11]). Let $M$ be a left $\mathbb{O}$-module. A real linear map $f : M \rightarrow \mathbb{O}$ is called an **$\mathbb{O}$-para-linear function** if it satisfies

$$\text{Re}[p, x, f] = 0$$

for any $p \in \mathbb{O}$ and $x \in M$. Here

$$[p, x, f] := f(px) - pf(x),$$

which is called the **second associator**.

The definition of $\mathbb{O}$-Hilbert spaces can be restated in terms of para-linearity.

**Definition 2.6** ([11]). A left $\mathbb{O}$-module $H$ is called a **pre-Hilbert $\mathbb{O}$-module** if there exists an $\mathbb{R}$-bilinear map $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{O}$, which is referred to as an **$\mathbb{O}$-inner product**, satisfying:

(i) **($\mathbb{O}$-para-linearity)** $\langle \cdot, u \rangle$ is (left) $\mathbb{O}$-para-linear for all $u \in H$.

(ii) **(Octonion hermiticity)** $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in H$.

(iii) **(Positivity)** $\langle u, u \rangle \in \mathbb{R}^+$ and $\langle u, u \rangle = 0$ if and only if $u = 0$. 

The definition of $\mathbb{O}$-Hilbert spaces can be restated in terms of para-linearity.
We can define the norm \( ||\cdot|| : H \to \mathbb{R}^+ \) by setting
\[
||u|| = \sqrt{\langle u, u \rangle}.
\]
(2.4)

A pre-Hilbert left \( O \)-module \( H \) is said to be a Hilbert left \( O \)-module if it is complete with respect to its natural distance induced by the norm.

**Example 2.7.** We give two typical examples of Hilbert left \( O \)-modules.

(i) \( O \) is a Hilbert left \( O \)-module with the \( O \)-inner product \( \langle \cdot, \cdot \rangle_O \) defined by
\[
\langle x, y \rangle_O = xy
\]
for all \( x, y \in O \).

(ii) The conjugate regular module \( \overline{O} \) is a Hilbert left \( O \)-module with the \( O \)-inner product \( \langle \cdot, \cdot \rangle_{\overline{O}} \) defined by
\[
\langle x, y \rangle_{\overline{O}} = yx
\]
for all \( x, y \in O \). We remark that its left module structure is defined by
\[
p \cdot x := \overline{p}x,
\]
for any \( p \in O \) and \( x \in \overline{O} \).

In an \( O \)-Hilbert space \( H \), we can regard \( v \in H \) as an \( O \)-para-linear map induced by the \( O \)-inner product. This motivates us to introduce the second associator of \( H \) as
\[
[p, u, v] := \langle [p, q, v], u \rangle - [p, q, \langle v, u \rangle] + p[q, v, u] + \langle pq, u, v \rangle
\]
for all \( u, v \in H \) and \( p \in O \).

From now on, we shall denote
\[
\Re \langle u, v \rangle := \langle u, v \rangle_{\mathbb{R}}
\]
for all \( u, v \in H \). We assemble some identities concerning second associators for later use.

**Lemma 2.8 ([11]).** For all \( u, v \in H \) and all \( p, q \in O \), the following hold
\[
\langle [p, q, u], v \rangle_{\mathbb{R}} = -\langle u, [p, q, v] \rangle_{\mathbb{R}};
\]
(2.5)
\[
[p, v, u] = -[p, u, v];
\]
(2.6)
\[
[ pq, v, u ] = \langle [ p, q, v ], u \rangle - \langle p, q, [ v, u ] \rangle + p[q, v, u] + [ p, qv, u ].
\]
(2.7)

The following characterization of associative elements is very useful in the sequel.

**Lemma 2.9 ([11]).** An element \( x \in \mathcal{A}(H) \) if and only if
\[
[p, x, y] = 0
\]
(2.8)
for all \( p \in O \) and \( y \in H \).

In contrast to the complex and quaternionic cases, the \( O \)-inner products are \( O \)-para-linear, rather than \( O \)-linear. The following lemma reveals the close relations among \( O \)-inner product, \( O \)-scalar product, and the second associator.

**Lemma 2.10 ([11]).** Let \( H \) be a pre-Hilbert left \( O \)-module. Then for all \( u, v \in H \) and all \( p, q \in O \), the following hold:
\[
\langle u, pv \rangle = \langle u, v \rangle \overline{p} + [p, u, v];
\]
(2.9)
\[
\langle pq, u, v \rangle = \langle p \langle u, v \rangle \overline{q} + [pq, u, v] + \langle [p, q, v], u \rangle.
\]
(2.10)
3. Tensor product decompositions

In this section, we provide a Hilbert left $\mathbb{O}$-module with a tensor product decomposition structure. We first define the tensor product of a Hilbert left $\mathbb{O}$-module with a real Hilbert space.

**Definition 3.1.** Let $(H_1, \langle \cdot , \cdot \rangle_1)$ be a Hilbert left $\mathbb{O}$-module and $(H_2, \langle \cdot , \cdot \rangle_2)$ be a real Hilbert space. Then we endow the tensor product $H_1 \otimes_{\mathbb{R}} H_2$ with a canonical left $\mathbb{O}$-module structure

\[ \mathbb{O} \times (H_1 \otimes_{\mathbb{R}} H_2) \to H_1 \otimes_{\mathbb{R}} H_2 \]

\[ (p, u \otimes_{\mathbb{R}} x) \mapsto (pu) \otimes_{\mathbb{R}} x \]

and an $\mathbb{O}$-inner product

\[ \langle u \otimes_{\mathbb{R}} x, v \otimes_{\mathbb{R}} y \rangle := \langle u, v \rangle_1 \langle x, y \rangle_2 \] (3.1)

for all $u, v \in H_1$ and all $x, y \in H_2$. Then one can check that the tensor product $(H_1 \otimes_{\mathbb{R}} H_2, \langle \cdot , \cdot \rangle)$ is a Hilbert left $\mathbb{O}$-module.

**Definition 3.2.** Let $(H_1, \langle \cdot , \cdot \rangle_1)$ and $(H_2, \langle \cdot , \cdot \rangle_2)$ be two Hilbert left $\mathbb{O}$-modules. We endow the direct sum $H_1 \oplus H_2$ with the canonical left $\mathbb{O}$-module structure and the $\mathbb{O}$-inner product

\[ \langle (x, u), (y, v) \rangle := \langle x, y \rangle_1 + \langle u, v \rangle_2 \] (3.2)

for all $x, y \in H_1$ and $u, v \in H_2$. One can check that the direct sum $(H_1 \oplus H_2, \langle \cdot , \cdot \rangle)$ is a Hilbert left $\mathbb{O}$-module. By definition we know that $H_1$ is orthogonal to $H_2$ if we regard $H_1$ and $H_2$ as canonical subsets of $H_1 \oplus H_2$.

**Definition 3.3.** Let $(H_1, \langle \cdot , \cdot \rangle_1)$ and $(H_2, \langle \cdot , \cdot \rangle_2)$ be two Hilbert left $\mathbb{O}$-modules. A map $f : H_1 \to H_2$

is called an isomorphism of Hilbert left $\mathbb{O}$-modules if

(i) for all $p \in \mathbb{O}$ and $x \in H_1$, $f(px) = pf(x)$;

(ii) for all $x, y \in H_1$, $\langle f(x), f(y) \rangle_2 = \langle x, y \rangle_1$.

It is useful to know the behavior of the $\mathbb{O}$-inner product on associative and conjugate associative elements in an $\mathbb{O}$-Hilbert space.

**Lemma 3.4.** Let $H$ be a Hilbert left $\mathbb{O}$-module. Then we have

(i) for all $u, v \in \mathcal{A}(H)$,

\[ \langle u, v \rangle \in \mathbb{R}; \] (3.2)

(ii) for all $u \in \mathcal{A}(H)$ and $v \in \mathcal{A}^{-}(H)$,

\[ \langle u, v \rangle = 0. \] (3.3)

**Proof.** We first prove assertion (i). Recall identity (2.7):

\[ [pq, v, u] = \langle [p, q, v], u \rangle - [p, q, \langle v, u \rangle] + \langle q, v, u \rangle + [q, pq, u] \]

for all $p, q \in \mathbb{O}$ and $u, v \in H$. If $u, v \in \mathcal{A}(H)$, then it follows from Lemma 2.9 that

\[ [p, q, \langle v, u \rangle] = 0 \]

for all $p, q \in \mathbb{O}$. This implies $\langle v, u \rangle \in \mathbb{R}$ as desired.
Now we prove assertion (ii). Let \( u \in \mathcal{A}(H) \) and \( v \in \mathcal{A}^{-}(H) \). By definition, we have
\[
[p, q, v] = (pq)v - p(qv) = (pq)v - (qp)v = [p, q]v.
\]
According to identity (2.7) and Lemma 2.9, we obtain
\[
0 = \langle [p, q, v], u \rangle = \langle [p, q], \langle v, u \rangle \rangle - \langle [p, q], \langle v, u \rangle \rangle = \langle p(q \langle v, u \rangle) - (qp) \langle v, u \rangle \rangle.
\]
This means the map
\[
f : \mathcal{O} \rightarrow \mathcal{O}, \quad x \mapsto x \langle v, u \rangle
\]
satisfies
\[
pf(q) = f(qp)
\]
for all \( p, q \in \mathcal{O} \). By Lemma 2.4, we obtain that \( f = 0 \) and hence \( \langle v, u \rangle = 0 \). \( \square \)

Lemma 3.5. Let \( H \) be a Hilbert \( \mathcal{O} \)-module. If \( \mathcal{A}^{-}(H) = 0 \), then there exists an isomorphism of Hilbert \( \mathcal{O} \)-modules
\[
H \cong \mathcal{O} \otimes_{\mathcal{O}} \mathcal{A}(H).
\]
Here \( \mathcal{O} \) is a Hilbert \( \mathcal{O} \)-module as defined in Example 2.7, and \( \mathcal{A}(H) \) is a real Hilbert space with the inner product induced from \( H \) by restriction. More precisely, for any \( p, q \in \mathcal{O} \) and \( x, y \in \mathcal{A}(H) \), we have
\[
\langle px, qy \rangle = \langle p, q \rangle_{\mathcal{O}} \langle x, y \rangle.
\]

Proof. The \( \mathcal{O} \)-Hilbert space structure of \( \mathcal{O} \otimes_{\mathcal{O}} \mathcal{A}(H) \) is given as in Definition 3.1. That is,
\[
\left< \sum_{i=0}^{7} e_i \otimes_{\mathcal{O}} x_i, \sum_{i=0}^{7} e_i \otimes_{\mathcal{O}} y_i \right>_{\mathcal{O} \otimes_{\mathcal{O}} \mathcal{A}(H)} = \sum_{i,j=0}^{7} \left< e_i e_j, x_i, y_j \right>_{\mathcal{A}(H)}
\]
for any \( x, y \in \mathcal{A}(H) \).

Since \( \mathcal{A}^{-}(H) = 0 \), it follows from Theorem 2.2 that
\[
H = \mathcal{O} \mathcal{A}(H).
\]
We consider the canonical map
\[
\varphi : \mathcal{O} \otimes_{\mathcal{O}} \mathcal{A}(H) \rightarrow H
\]
\[
\sum_{i=0}^{7} e_i \otimes_{\mathcal{O}} x_i \mapsto \sum_{i=0}^{7} e_i x_i.
\]
One can easily check that this is an \( \mathcal{O} \)-isomorphism of left \( \mathcal{O} \)-modules. It follows from identity (2.10) and Lemma 3.4 that

\[
\sum_{i=0}^{7} e_i x_i, \sum_{i=0}^{7} e_i y_i \right) \mathcal{O} = \sum_{i,j=0}^{7} (e_i e_j (x_i, y_j) + [e_i e_j, x_i, y_j] + \langle [e_i, e_j, y_j], x_i \rangle) \\
= \sum_{i,j=0}^{7} (e_i \overline{e_j}) (x_i, y_j).
\]

This means that

\[
\sum_{i=0}^{7} e_i x_i, \sum_{i=0}^{7} e_i y_i \right) \mathcal{O} \otimes \mathcal{O} A(H).
\]

Hence \( \varphi \) is an isomorphism of \( \mathcal{O} \)-Hilbert spaces.

Every Hilbert left \( \mathcal{O} \)-module can be associated with another Hilbert left \( \mathcal{O} \)-module structure.

**Definition 3.6.** Let \( (H, \langle \cdot, \cdot \rangle) \) be a Hilbert left \( \mathcal{O} \)-module. We endow \( H \) with another left \( \mathcal{O} \)-multiplication

\[
\mathcal{O} \times H \rightarrow H \quad (p, x) \rightarrow p \hat{\cdot} x := \overline{p} x.
\]

This is also a left \( \mathcal{O} \)-module and we denote it by \( H^- \). We define

\[
\langle x, y \rangle_{H^-} := \langle y, x \rangle
\]
for any \( x, y \in H \).

We now check that \( (H^-, \langle \cdot, \cdot \rangle_{H^-}) \) is indeed a Hilbert left \( \mathcal{O} \)-module.

**Lemma 3.7.** Let \( (H, \langle \cdot, \cdot \rangle) \) be a Hilbert left \( \mathcal{O} \)-module. Then \( (H^-, \langle \cdot, \cdot \rangle_{H^-}) \) is a Hilbert left \( \mathcal{O} \)-module. Moreover, it holds

\[
\mathcal{A}(H) = \mathcal{A}^-(H^-),
\]

\[
\mathcal{A}(H^-) = \mathcal{A}^-(H).
\]

The two Hilbert left \( \mathcal{O} \)-modules have a close relation between their second associators

\[
[p, x, y]_{H^-} = [p, x, y] + \langle [x, y], p \rangle
\]

for all \( p \in \mathcal{O} \) and \( x, y \in H \).

**Proof.** By direct calculations, we have

\[
\text{Re} \left( p \hat{\cdot} (p^{-1} x, y)_{H^-} - p \langle x, y \rangle_{H^-} \right) = \text{Re} \left( \langle y, \overline{p} x \rangle - p \langle y, x \rangle \right) \\
= \text{Re} \left( \langle y, x \rangle p - [\overline{p}, y, x] - p \langle y, x \rangle \right) \\
= \text{Re} \left( \langle [y, x], p \rangle - [\overline{p}, y, x] \right) \\
= 0.
\]

This shows the para-linearity. The proof of hermiticity and positivity is trivial.
We come to prove (3.7). By definition,
\[ A^{-}(H^{-}) = \{ x \mid (pq)x = q(p^*x), \forall p, q \in O \} \]
\[ = \{ x \mid (pq)x = q(x^p), \forall p, q \in O \} \]
\[ = \{ x \mid [p, q, x] = 0, \forall p, q \in O \} \]
\[ = A(H). \]
This shows
\[ A^{-}(H^{-}) = A(H). \]
It is easy to see that
\[ (H^{-})^{-} = H. \]
Hence we get
\[ A(H^{-}) = A^{-}((H^{-})^{-}) = A^{-}(H). \]
Finally, we compute the second associators of \( H^{-} \). For any \( p \in O \) and \( x, y \in H \), we have
\[ [p, x, y]_{H^{-}} = \langle p^*x, y \rangle_{H^{-}} - p \langle x, y \rangle_{H^{-}} \]
\[ = \langle y, px \rangle - p \langle y, x \rangle \]
\[ = \langle y, x \rangle p - [p, y, x] - p \langle y, x \rangle \]
\[ = [p, x, y] + [\langle y, x \rangle, p]. \]
This proves the lemma.

By the duality in Lemma 3.7, we can extend the result in Lemma 3.4 to \( A^{-}(H) \).

**Lemma 3.8.** Let \( H \) be a Hilbert left \( O \)-module. Then
\[ \langle x, y \rangle \in \mathbb{R} \]
for all \( x, y \in A^{-}(H) \).

**Proof.** It is a direct consequence of Lemma 3.7 with \( H \) replaced by \( H^{-} \).

**Lemma 3.9.** Let \( H \) be a Hilbert left \( O \)-module. If \( A(H) = 0 \), then there exists an \( O \)-isomorphism of \( O \)-Hilbert spaces
\[ H \cong \mathcal{O} \otimes_{\mathbb{R}} A^{-}(H). \]

Here \( \mathcal{O} \) is a Hilbert left \( O \)-module as defined in Example 2.7, and \( A^{-}(H) \) is a real Hilbert space with the inner product induced from that of \( H \). More precisely, for any \( p, q \in O \) and \( x, y \in A^{-}(H) \), we have
\[ \langle px, qy \rangle = \langle p, q \rangle \langle x, y \rangle. \]

**Proof.** We consider the \( O \)-isomorphism
\[ \phi : \mathcal{O} \otimes_{\mathbb{R}} A^{-}(H) \rightarrow H \]
\[ \sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} x_i \mapsto \sum_{i=0}^{7} e_i x_i. \]
Note that the left \( O \)-module structure of \( \mathcal{O} \otimes_{\mathbb{R}} A^{-}(H) \) is determined by the \( O \)-multiplication of \( \mathcal{O} \) as follows:
\[ p \left( \sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} x_i \right) := \sum_{i=0}^{7} (pe_i) \otimes_{\mathbb{R}} x_i. \]
Hence it follows from $x_i \in \mathcal{A}^-(H)$ that
\[
\phi\left(p\left(\sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} x_i\right)\right) = \sum_{i=0}^{7} (\overline{e_i})p x_i = p \sum_{i=0}^{7} e_i x_i = p \phi\left(\sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} x_i\right).
\]

This proves that $\phi$ is indeed an $\mathbb{O}$-isomorphism.

We next consider the induced module $H^-$. By (3.7) and (3.8), we have
\[
\mathcal{A}^-(H^-) = \mathcal{A}(H) = 0
\]
and
\[
\mathcal{A}(H^-) = \mathcal{A}^-(H).
\]

Using Lemma 3.5, we obtain
\[
H^- \cong \mathbb{O} \otimes_{\mathbb{R}} \mathcal{A}^-(H).
\]
That is, for any $p, q \in \mathbb{O}$ and $x, y \in \mathcal{A}^-(H)$,
\[
\langle p \hat{x}, q \hat{y} \rangle_{H^-} = \langle p, q \rangle_{\mathbb{O}} \langle x, y \rangle_{H^-},
\]
or equivalently,
\[
\langle px, qy \rangle = \langle \overline{q}p \rangle \langle x, y \rangle = \langle \overline{q}, \overline{p} \rangle \langle x, y \rangle.
\]

Therefore,
\[
\left\langle \phi\left(\sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} x_i\right), \phi\left(\sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} y_i\right)\right\rangle = \sum_{i,j=0}^{7} \langle e_i e_j \rangle \langle x_i, y_j \rangle = \sum_{i,j=0}^{7} \langle e_i, e_j \rangle \langle x_i, y_j \rangle = \langle \sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} x_i, \sum_{i=0}^{7} e_i \otimes_{\mathbb{R}} y_i \rangle_{\mathbb{O} \otimes_{\mathbb{R}} \mathcal{A}^-}(H).
\]

This proves that $\phi$ is an isomorphism of $\mathbb{O}$-Hilbert spaces. □
Theorem 3.10 (Tensor product decomposition). Let $H$ be a Hilbert left $\mathcal{O}$-module. Then there exists an $\mathcal{O}$-isomorphism of $\mathcal{O}$-Hilbert spaces

$$H \cong (\mathcal{O} \otimes_{\mathbb{R}} \mathcal{A}(H)) \oplus (\overline{\mathcal{O}} \otimes_{\mathbb{R}} \mathcal{A}^-(H)).$$

(3.13)

Proof. By Theorem 2.2, we have

$$H = \mathcal{O}\mathcal{A}(H) \oplus \mathcal{O}\mathcal{A}^-(H).$$

Denote $H_1 = \mathcal{O}\mathcal{A}(H)$ and $H_2 = \mathcal{O}\mathcal{A}^-(H)$. Then by Lemmas 3.5 and 3.9,

$$H_1 \cong \mathcal{O} \otimes_{\mathbb{R}} \mathcal{A}(H), \quad H_2 \cong \overline{\mathcal{O}} \otimes_{\mathbb{R}} \mathcal{A}^-(H).$$

It remains to show that $H_1$ is orthogonal to $H_2$. Recall identity (2.10):

$$\langle pu, qv \rangle = (p \langle u, v \rangle)q + [pq, u, v] + \langle [p, q, v], u \rangle.$$

If $u \in \mathcal{A}^-(H)$ and $v \in \mathcal{A}(H)$, then it follows from (3.3) and Lemma 2.9 that

$$\langle pu, qv \rangle = 0.$$

This shows that $H_1$ is orthogonal to $H_2$ as desired. □

Remark 3.11. Note that the right-hand side of equality (3.13) with $\mathcal{A}(H)$ and $\mathcal{A}^-(H)$ replaced by any two real Hilbert spaces is always an $\mathcal{O}$-Hilbert space. Hence this theorem gives a complete classification of Hilbert left $\mathcal{O}$-modules.

4. Weak associative orthonormal basis and Parseval’s theorem

As observed by Goldstine and Horwitz in the appendix of [6], the Parseval theorem may fail for any orthonormal basis in an $\mathcal{O}$-Hilbert space. We introduce a new notion of weak associative orthonormal basis and show that the Parseval equality holds for an orthonormal basis if and only if it is weak associative.

Definition 4.1. A subset $S = \{x_\alpha\}_{\alpha \in \Lambda}$ of an $\mathcal{O}$-Hilbert space is said to be an orthonormal system if

$$\langle x_\alpha, x_\beta \rangle = \delta_{\alpha\beta}.$$

An orthonormal system $S = \{x_\alpha\}_{\alpha \in \Lambda}$ is said to be an orthonormal basis if there does not exist other orthonormal system $S'$ such that $S \subseteq S'$.

As usual, we have the following characterizations of an orthonormal basis. The proof runs in the same manner as in the classical case and is omitted here.

Lemma 4.2. Let $S$ be an orthonormal system. Then $S$ is an orthonormal basis if and only if there does not exist non-zero element which is orthogonal to every element of $S$.

In contrast to the classical cases, it is meaningless to define the dimension of an $\mathcal{O}$-Hilbert space as the cardinality of an orthonormal basis as shown in the following special case of $\mathcal{O}^2$.

Example 4.3. The $\mathcal{O}$-Hilbert space $\mathcal{O}^2$ admits two sets of basis. One has cardinality 2 and the other has cardinality 4.

(1). The $\mathcal{O}$-Hilbert space $\mathcal{O}^2$ admits an orthonormal basis given by

$$(0, 1), \quad (1, 0).$$

(2). The $\mathcal{O}$-Hilbert space $\mathcal{O}^2$ admits an alternative orthonormal basis given by
(4.1) \[ x_1 = \frac{1}{\sqrt{2}} (e_1, e_2), \quad x_2 = \frac{1}{\sqrt{2}} (e_4, e_7), \quad x_3 = \frac{1}{\sqrt{2}} (e_6, e_5), \quad x_4 = \frac{1}{\sqrt{2}} (1, e_3). \]

To show that (4.1) is an orthonormal basis, we first observe that \( \{x_n\}_{n=1}^4 \) is an orthonormal system of \( \mathbb{O}^2 \). If there exists an element \( x = (a, b) \in \mathbb{O}^2 \) orthogonal to \( x_1, \ldots, x_4 \), then from the assumption that \( \langle x, x_4 \rangle = 0 \) we have

\[ a = be_3. \]

Since \( \langle x, x_1 \rangle = 0 \), we get

\[ (be_3)e_1 + be_2 = 2be_2 + [b, e_3, e_1] = 0. \]

Using identity (2.5) in the \( \mathbb{O}\)-Hilbert space \( \mathbb{O} \), we obtain

\[ [I, J, x] \in \mathbb{H}_{I,J} \]

for any mutually orthogonal pair \( (I, J) \) of \( \mathbb{O} \) and all \( x \in \mathbb{O} \). Here the notation \( \mathbb{H}_{I,J} \) stands for the subset spanned by \( \{1, I, J, IJ\} \), which was introduced in Subsection 2.1. Combining (4.3) with (4.2), we get

\[ be_2 \in \mathbb{H}_{e_1, e_3}. \]

This yields that \( b_0 = b_1 = b_2 = b_3 = 0 \) if we write

\[ b = b_0 + \sum_{i=1}^7 e_i b_i. \]

The assumptions that \( \langle x, x_2 \rangle = \langle x, x_3 \rangle = 0 \) imply that

\[ \begin{cases} 2be_7 + [b, e_3, e_4] = 0, \\ 2be_5 + [b, e_3, e_6] = 0. \end{cases} \]

By similar argument, we conclude that \( b_4 = b_5 = b_6 = b_7 = 0 \) and hence \( b = 0 \). This means that \( x = 0 \). We thus conclude from Lemma 4.2 that \( \{x_n\}_{n=1}^4 \) is an orthonormal basis.

The Parseval equality may not hold for the orthonormal basis (4.1). Indeed, we take \( y = (1, 0) \in \mathbb{O}^2 \). By direct calculation, we have

\[ \langle y, x_1 \rangle = \frac{1}{\sqrt{2}} e_1, \quad \langle y, x_2 \rangle = \frac{1}{\sqrt{2}} e_4, \quad \langle y, x_3 \rangle = \frac{1}{\sqrt{2}} e_6, \quad \langle y, x_4 \rangle = \frac{1}{\sqrt{2}}. \]

It follows that

\[ y - \sum_{n=1}^4 \langle y, x_n \rangle x_n = (-1, e_3) \]

and

\[ \sum_{n=1}^4 ||\langle y, x_n \rangle||^2 \neq ||y||^2. \]
Denote $p_i = \langle y, x_i \rangle$ for $i = 1, \ldots, 4$. Using identity (2.5) and the left alternativity of associators, we obtain

$$
\sum_{m,n=1}^{4} \langle \langle y, x_n \rangle, \langle y, x_m \rangle, x_m \rangle, x_n \rangle_R = 2 \sum_{m<n} \langle \langle p_1, p_2, x_1 \rangle \rangle_R + \langle p_1, p_3, x_1 \rangle \rangle_R + \langle p_2, p_3, x_2 \rangle \rangle_R \\
= \frac{1}{2} \text{Re} \left( [e_1, e_4, e_7] + [e_1, e_6, e_5] + [e_4, e_6, e_5] \right) \\
= 3.
$$

Therefore, we have

$$
\sum_{n=1}^{4} |\langle y, x_n \rangle|^2 + \left\| y - \sum_{n=1}^{4} \langle y, x_n \rangle x_n \right\|^2 - \sum_{m,n=1}^{2} \langle \langle y, x_n \rangle, \langle y, x_m \rangle, x_m \rangle, x_n \rangle_R = \| y \|^2.
$$

In fact, the above equality even holds generally.

**Lemma 4.4.** Let $\{x_n\}_{n=1}^{N}$ be an orthonormal system of an $\mathbb{O}$-Hilbert space $H$. Then for all $x \in H$ we have

$$
\sum_{n=1}^{N} |\langle x, x_n \rangle|^2 + \left\| x - \sum_{n=1}^{N} \langle x, x_n \rangle x_n \right\|^2 - \sum_{m,n=1}^{N} \langle \langle x, x_n \rangle, \langle x, x_m \rangle, x_m \rangle, x_n \rangle_R \leq \| x \|^2.
$$

In particular, there holds the octonionic version of Bessel’s inequality:

$$
\sum_{n=1}^{N} |\langle x, x_n \rangle|^2 - \sum_{m,n=1}^{N} \langle \langle x, x_n \rangle, \langle x, x_m \rangle, x_m \rangle, x_n \rangle_R \leq \| x \|^2.
$$

**Proof.** Set

$$
p_n = \langle x, x_n \rangle, \quad u = \sum_{n=1}^{N} p_n x_n, \quad v = x - u.
$$

By calculation, we have

$$
\| x \|^2 = \langle u + v, u + v \rangle \\
= \| u \|^2 + \| v \|^2 + 2 \text{Re} \langle u, x - u \rangle \\
= \| v \|^2 - \| u \|^2 + 2 \text{Re} \langle u, x \rangle.
$$

Since

$$
\langle u, x \rangle = \langle \sum_{i=1}^{N} p_i x_n, x \rangle \\
= \sum_{n=1}^{N} p_n \langle x_n, x \rangle + [p_n, x_n, x] \\
= \sum_{n=1}^{N} \| x_n \|^2 + \sum_{n=1}^{N} [p_n, x_n, x],
$$

we obtain

$$
\sum_{n=1}^{N} |\langle x, x_n \rangle|^2 - \sum_{m,n=1}^{N} \langle \langle x, x_n \rangle, \langle x, x_m \rangle, x_m \rangle, x_n \rangle_R \leq \| x \|^2.
$$
we conclude that

\[ \text{Re} \langle u, x \rangle = \sum_{n=1}^{N} |\langle x, x_n \rangle|^2. \]

By (2.10), we obtain

\[
||u||^2 = \left( \sum_{n=1}^{N} p_n x_n, \sum_{m=1}^{N} p_m x_m \right) \\
= \sum_{m,n=1}^{N} (p_n \langle x_n, x_m \rangle) p_m + [p_n p_m, x_n, x_m] + [p_n, p_m, x_m, x_n] \\
= \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 + \sum_{m,n=1}^{N} [p_n p_m, x_n, x_m] + \{[\langle x, x_n \rangle, \langle x, x_m \rangle, x_m], x_n\},
\]

so that

\[
||u||^2 = \text{Re} \ ||u||^2 = \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 + \sum_{m,n=1}^{N} \{[\langle x, x_n \rangle, \langle x, x_m \rangle, x_m], x_n\}_\mathbb{R}.
\]

Hence (4.6) becomes

\[
||x||^2 = ||u||^2 - \left( \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 + \sum_{m,n=1}^{N} \{[\langle x, x_n \rangle, \langle x, x_m \rangle, x_m], x_n\}_\mathbb{R} \right) + 2 \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 \\
= \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 + \left\| x - \sum_{n=1}^{N} \langle x, x_n \rangle x_n \right\|^2 - \sum_{m,n=1}^{N} \{[\langle x, x_n \rangle, \langle x, x_m \rangle, x_m], x_n\}_\mathbb{R}.
\]

This proves the lemma. \(\square\)

**Remark 4.5.** In Example 4.3, it is worth pointing out that \(\{x_n\}_{n=1}^{4}\) is not \(\mathbb{O}\)-linearly independent. Indeed, if we denote

\[ x_5 = \frac{1}{\sqrt{2}} (1, -e_3), \]

then we have

\[ x_1 = e_1 x_5, \quad x_2 = e_4 x_5, \quad x_3 = e_6 x_5. \]

Moreover, it is easy to check that \(\{x_n\}_{n=4}^{5}\) is also an orthonormal basis. By direct calculation, for \(y = (1, 0)\) we have

\[ y - \sum_{i=4}^{5} \langle y, x_i \rangle x_i = 0 \]

and

\[ ||y||^2 = \sum_{i=4}^{5} |\langle y, x_i \rangle|^2. \]

In fact, this is not accidental in terms of the corollary below. The point is that the orthonormal basis \(\{x_n\}_{n=4}^{5}\) satisfies

\[ [p, x_4, x_5] = 0 \]

for all \(p \in \mathbb{O}\).
Corollary 4.6. Under the assumption of Lemma 4.4 and suppose \( \{x_n\}_{n=1}^N \) satisfies
\[
[p, x_n, x_m] = 0
\]
for all \( p \in \mathbb{O} \) and for each \( m, n = 1, \ldots, N \). Then we have
\[
|x|^2 = \sum_{n=1}^N |\langle x, x_n \rangle|^2 + \left\| x - \sum_{n=1}^N \langle x, x_n \rangle x_n \right\|^2
\]
and in this case we obtain the classical Bessel’s inequality
\[
\sum_{n=1}^N |\langle x, x_n \rangle|^2 \leq ||x||^2.
\]

Proof. If \( [p, x_n, x_m] = 0 \), then by taking the real part on both sides of (2.7), we have
\[
\sum_{m,n=1}^N \langle [\langle x, x_n \rangle, \langle x, x_m \rangle, x_m]_R, x_n \rangle = - \sum_{m,n=1}^N \text{Re} \left( \langle x, x_n \rangle \langle x, x_m \rangle, x_n, x_m \rangle \right) = 0.
\]
Therefore, (4.8) follows from (4.4) directly. This completes the proof. \( \square \)

This inspires us to introduce the following notion.

Definition 4.7. An orthonormal basis \( S = \{x_\alpha\}_{\alpha \in \Lambda} \) is said to be a weak associative orthonormal basis if \( S \) satisfies condition (4.7) in Corollary 4.6.

Lemma 4.8. If \( S = \{x_\alpha\}_{\alpha \in \Lambda} \) is a weak associative orthonormal basis in a Hilbert left \( \mathbb{O} \)-module \( H \), then
\[
\tilde{S} := \{e_i x_\alpha \mid i = 0, \ldots, 7, \alpha \in \Lambda\}
\]
is a real orthonormal basis of the real Hilbert space \( (H, \langle \cdot, \cdot \rangle_R) \).

Proof. According to identities (2.10) and (2.7), we have
\[
\langle e_i x_\alpha, e_j x_\beta \rangle_R = \text{Re} \left( e_i \langle x_\alpha, x_\beta \rangle e_j^* + [e_i e_j, x_\alpha, x_\beta] + \langle [e_i, e_j, x_\alpha], x_\beta \rangle_R \right)
\]
\[
= \text{Re} \left( e_i e_j \delta_{\alpha \beta} \right) - \text{Re} \left( e_i [e_j, x_\alpha, x_\beta] \right)
\]
\[
= \delta_{ij} \delta_{\alpha \beta}
\]
for each \( i, j = 0, \ldots, 7 \) and each \( \alpha, \beta \in \Lambda \). Hence \( \tilde{S} \) is a real orthonormal system of \( (H, \langle \cdot, \cdot \rangle_R) \). We next show that \( \tilde{S} \) is a real orthonormal basis. If not, then there exists a non-zero element \( x \in H \) such that \( x \) is orthogonal to \( \tilde{S} \) with respect to the real inner product \( \langle \cdot, \cdot \rangle_R \). Thus we have
\[
\langle x_\alpha, x \rangle = \langle x_\alpha, x \rangle_R - \sum_{i=0}^7 \langle e_i x_\alpha, x \rangle_R e_i = 0.
\]
In view of Lemma 4.2, this is impossible. \( \square \)

Now we are in position to establish the Parseval theorem for weak associative orthonormal bases.

Theorem 4.9 (Parseval theorem). Let \( H \) be an \( \mathbb{O} \)-Hilbert space and \( S = \{x_\alpha\}_{\alpha \in \Lambda} \) be a weak associative orthonormal basis. Then any \( x \in H \) can be uniquely expressed as
\[
x = \sum_{\alpha \in \Lambda} \langle x, x_\alpha \rangle x_\alpha
\]
and there holds

$$||x||^2 = \sum_{\alpha \in \Lambda} |\langle x, x_\alpha \rangle|^2.$$  

**Proof.** In view of Cauchy-Schwarz inequality, we get

$$|\langle x, x_\alpha \rangle| \leq ||x||$$

for each $\alpha \in \Lambda$. Note that

$$\bigcup_{k=1}^{\infty} \left[ \frac{1}{k+1} ||x||, \frac{1}{k} ||x|| \right] = (0, ||x||].$$

Thanks to Corollary 4.6, there are only countable $\alpha$, say $(\alpha_j)_{j=1}^{\infty}$, such that $\langle x, x_\alpha \rangle \neq 0$. Then by Bessel’s inequality (4.9) in Corollary 4.6, we have

(4.10) $$\sum_{j=1}^{\infty} |\langle x, x_\alpha \rangle|^2 < +\infty.$$  

Set

$$y_n = \sum_{j=1}^{n} \langle x, x_\alpha \rangle x_\alpha.$$  

For any $m < n$, we have

$$y_n - y_m = \sum_{j=m+1}^{n} \langle x, x_\alpha \rangle x_\alpha.$$  

Since $S$ is a weak associative orthonormal basis, we get

$$\langle y_n - y_m, x_\alpha \rangle = \sum_{j=m+1}^{n} \langle \langle x, x_\alpha \rangle x_\alpha, x_\alpha \rangle$$

$$= \sum_{j=m+1}^{n} \langle x, x_\alpha \rangle \langle x_\alpha, x_\alpha \rangle + \langle x, x_\alpha \rangle \langle x_\alpha, x_\alpha \rangle + \langle x, x_\alpha \rangle \langle x_\alpha, x_\alpha \rangle = \langle x, x_\alpha \rangle.$$  

It then follows from Corollary 4.6 again that

$$||y_n - y_m||^2 = \sum_{k=m+1}^{n} |\langle y_n - y_m, x_\alpha \rangle|^2 + \left\| \sum_{k=m+1}^{n} \langle y_n - y_m, x_\alpha \rangle x_\alpha \right\|^2$$

$$= \sum_{k=m+1}^{n} |\langle x, x_\alpha \rangle|^2 + \left\| \sum_{j=m+1}^{n} \langle x, x_\alpha \rangle x_\alpha - \sum_{k=m+1}^{n} \langle x, x_\alpha \rangle x_\alpha \right\|^2$$

$$= \sum_{k=m+1}^{n} |\langle x, x_\alpha \rangle|^2.$$  

Thus we conclude from (4.10) that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence. From the completeness of $H$, there exists an element $x' \in H$ such that

$$\lim_{n \to +\infty} y_n = x'.$$

We next prove that $x' = x$. In view of Lemma 4.8, it suffices to prove

(4.11) $$\langle x - x', e_i x_\alpha \rangle_{\mathbb{R}} = 0$$
for each \( \alpha \in \Lambda \) and \( i = 1, \ldots, 7 \). For any \( x_{\alpha k} \), we get

\[
\langle x - x', e_i x_{\alpha k} \rangle_R = \text{Re} \left( \langle x - x', x_{\alpha k} \rangle e_i + [e_i, x - x', x_{\alpha k}] \right) = \text{Re} \left( \langle x, x_{\alpha k} \rangle e_i - \lim_{n \to +\infty} \sum_{j=1}^{n} \langle \langle x, x_{\alpha j} \rangle, x_{\alpha j} \rangle, x_{\alpha k} \rangle e_i \right) = \text{Re} \left( - \lim_{n \to +\infty} \sum_{j=1}^{n} \left( \langle x, x_{\alpha j} \rangle \langle x_{\alpha j}, x_{\alpha k} \rangle \rangle e_i \right) \right) = 0.
\]

For any \( x_{\alpha} \) with \( \alpha \neq \alpha_j \), \( j = 1, 2, \ldots \), we also have

\[
\langle x - x', e_i x_{\alpha} \rangle_R = \text{Re} \left( \langle x - x', x_{\alpha} \rangle e_i + [e_i, x - x', x_{\alpha}] \right) = \text{Re} \left( \langle x, x_{\alpha} \rangle e_i - \lim_{n \to +\infty} \sum_{j=1}^{n} \langle \langle x, x_{\alpha j} \rangle, x_{\alpha j} \rangle, x_{\alpha} \rangle e_i \right) = \text{Re} \left( - \lim_{n \to +\infty} \sum_{j=1}^{n} \left( \langle x, x_{\alpha j} \rangle \langle x_{\alpha j}, x_{\alpha} \rangle \rangle e_i \right) \right) = 0.
\]

This proves \( x = x' \). Hence we have

\[
x = \sum_{j=1}^{\infty} \langle x, x_{\alpha j} \rangle x_{\alpha j} = \sum_{\alpha \in \Lambda} \langle x, x_{\alpha} \rangle x_{\alpha}.
\]

By Corollary 4.6 we obtain

\[
||x||^2 = \sum_{j=1}^{n} |\langle x, x_{\alpha j} \rangle|^2 + \left\| x - \sum_{j=1}^{n} \langle x, x_{\alpha j} \rangle x_{\alpha j} \right\|^2
\]

for any \( n \in \mathbb{N} \). Letting \( n \to \infty \), we get

\[
||x||^2 = \sum_{j=1}^{\infty} |\langle x, x_{\alpha j} \rangle|^2 = \sum_{\alpha \in \Lambda} |\langle x, x_{\alpha} \rangle|^2.
\]

The uniqueness of the decomposition follows from the condition (4.7). This completes the proof. \( \square \)

**Remark 4.10.** For any \( \mathcal{O} \)-Hilbert space, there always exists a weak associative orthonormal basis. This is because there always exists an orthonormal basis in a real Hilbert space. Note that by Theorem 3.10, an \( \mathcal{O} \)-Hilbert space \( H \) can be decomposed into an orthogonal sum. Let \( \{x_{\alpha}\}_{\alpha \in \Lambda} \) be an orthonormal basis of the real Hilbert space \( \mathcal{A}(H), \langle \cdot, \cdot \rangle_{\mathbb{R}} \) and \( \{x_{\beta}\}_{\beta \in \Lambda'} \) be an orthonormal basis of the real Hilbert space \( \mathcal{A}^-(H), \langle \cdot, \cdot \rangle_{\mathbb{R}} \). Then

\[
S := \{x_{\gamma}\}_{\gamma \in \Lambda \cup \Lambda'}
\]

is a weak associative orthonormal basis.

At last, we show that every orthonormal basis for which the Parseval theorem holds is also weak associative.
Definition 4.11. An orthonormal basis $S = \{x_\alpha\}_{\alpha \in \Lambda}$ is said to be a Hilbert basis, if for any $x \in H$ it holds
\[ ||x||^2 = \sum_{\alpha \in \Lambda} |\langle x, x_\alpha \rangle|^2. \]

Theorem 4.12. Let $S$ be an orthonormal basis. Then $S$ is a Hilbert basis if and only if $S$ is weak associative.

Proof. In view of Theorem 4.9, it suffices to show that every Hilbert basis is weak associative. Let $S = \{x_\alpha\}_{\alpha \in \Lambda}$ be a Hilbert basis. Then for $x = px_\beta$, we have
\[ ||x||^2 = \sum_{\alpha \in \Lambda} |\langle px_\beta, x_\alpha \rangle|^2 \]
\[ = \sum_{\alpha \in \Lambda} |(p\delta_{\alpha\beta} + [p, x_\beta, x_\alpha])|^2 \]
\[ = |p|^2 + \sum_{\alpha \neq \beta} ||[p, x_\beta, x_\alpha]||^2. \]

It follows from $||x|| = |p|$ that
\[ \sum_{\alpha \neq \beta} ||[p, x_\beta, x_\alpha]||^2 = 0. \]

This implies that $[p, x_\beta, x_\alpha] = 0$ for all $x_\beta, x_\alpha \in S$. This proves that $S$ is weak associative. \qed

5. Cayley-Dickson algebras as $\mathbb{O}$-Hilbert spaces

Any Cayley–Dickson algebra can be endowed with a weak $\mathbb{O}$-bimodule structure and an $\mathbb{O}$-inner product.

At first, we introduce the notion of weak $\mathbb{O}$-bimodules. Let $M$ be a left $\mathbb{O}$-module. There associates an involution
\[ C : M \rightarrow M, \]
called conjugate map. Since any $x \in M$ can be written as
\[ x = \sum_{i=0}^7 e_i x_i + \sum_{i=0}^7 e_i^{-} x_i, \]
where $x_i \in \mathcal{A}(M)$ and $x_i^{-} \in \mathcal{A}^{-}(M)$, we define the conjugate of $x$ as
\[ C(x) = \overline{x} := \sum_{i=0}^7 e_i x_i - \sum_{i=0}^7 e_i^{-} x_i. \]

Then by definition, we have
\[ \overline{x} = \begin{cases} x, & x \in \mathcal{A}(M), \\ -x, & x \in \mathcal{A}^{-}(M). \end{cases} \]

This means
\[ C(\mathcal{A}(M)) = \mathcal{A}(M), \quad C(\mathcal{A}^{-}(M)) = \mathcal{A}^{-}(M). \]
Definition 5.1. We call $M$ a weak $\mathcal{O}$-bimodule if $M$ is both a left $\mathcal{O}$-module and a right $\mathcal{O}$-module satisfying the compatibility conditions
\begin{equation}
px = \overline{p x}
\end{equation}
for all $x \in M$ and $p \in \mathcal{O}$.

Remark 5.2. The notion of a bimodule for a class of algebras has been introduced by Eilenberg [3]. Following [12, 18], we call $M$ an $\mathcal{O}$-bimodule if $M$ is both a left $\mathcal{O}$-module and a right $\mathcal{O}$-module satisfying the compatibility conditions
\[ [p, q, x] = [x, p, q] = [q, x, p] \]
for all $p, q \in \mathcal{O}$ and $x \in M$. The regular bimodule $\mathcal{O}$ is known to be the only irreducible $\mathcal{O}$-bimodule [12, 18]. We remark that a weak $\mathcal{O}$-bimodule may not be an $\mathcal{O}$-bimodule in general.

Remark 5.3. Let $M$ be a weak $\mathcal{O}$-bimodule.

(i) For all $p \in \mathcal{O}$ and $x \in M$, we have
\begin{equation}
\overline{p x} = \overline{x} \overline{p}.
\end{equation}
Indeed, it follows from (5.2) that
\[ px = \overline{px} = \overline{x} \overline{p} \]
which becomes (5.3) after replacing $p, x$ by $\overline{p}, \overline{x}$, respectively.

(ii) The right multiplication of $M$ is determined by its left multiplication due to (5.3). Conversely, every left $\mathcal{O}$-module can be viewed as a weak $\mathcal{O}$-bimodule by endowed with such right multiplication. That is, a weak $\mathcal{O}$-bimodule is nothing but a left $\mathcal{O}$-module endowed with a certain right multiplication.

(iii) We denote the left nucleus by
\[ \mathcal{A}_L(M) := \{ x \in M \mid [p, q, x] = 0, \forall p, q \in \mathcal{O} \} \]
and the right nucleus by
\[ \mathcal{A}_R(M) := \{ x \in M \mid [x, p, q] = 0, \forall p, q \in \mathcal{O} \}. \]
Similar notations $\mathcal{A}_L^-(M)$, $\mathcal{A}_R^-(M)$ can be also defined. One can check that
\[ \mathcal{A}_R(M) = C(\mathcal{A}_L(M)), \quad \mathcal{A}_R^-(M) = C(\mathcal{A}_L^-(M)). \]
Combining this with (5.1), we have
\begin{equation}
\mathcal{A}_R(M) = \mathcal{A}_L(M), \quad \mathcal{A}_R^-(M) = \mathcal{A}_L^-(M).
\end{equation}
Henceforth we shall denote
\[ \mathcal{A}(M) := \mathcal{A}_R(M) = \mathcal{A}_L(M), \quad \mathcal{A}^-(M) := \mathcal{A}_R^-(M) = \mathcal{A}_L^-(M). \]

We next consider the algebra of sedenions. The algebra of sedenions $\mathbb{S}$ is constructed from the algebra of octonions through the Cayley–Dickson construction [2]. The elements in the algebra of sedenions $\mathbb{S}$ take the form
\[ s = \sum_{i=0}^{15} x_i e_i, \]
where $x_i$ are reals, $e_0 = 1$ and $e_1, e_2, \ldots, e_{15}$ are imaginary units, i.e.,
\[ e_i^2 = -1 \]
for all $i = 1, \ldots, 15$. The conjugate of $s$ is defined as
\[
\bar{s} = x_0 - \sum_{i=1}^{15} x_i e_i.
\]
We regard $\mathcal{O}$ as the subalgebra of $S$ generated by $e_0, e_1, \ldots, e_7$. By the Cayley-Dickson construction, any $s \in S$ can be expressed uniquely as
\[
s = a + be_8
\]
with $a, b \in \mathcal{O}$ so that the conjugate of $s$ becomes
\[
\bar{s} = a - be_8
\]
and the multiplication in $S$ is defined as
\[
(a + be_8)(c + de_8) = (ac - \overline{db}) + (da + bc)e_8.
\]
We want to show that $S$ is a weak $\mathcal{O}$-bimodule. At first, we show that $S$ is a left $\mathcal{O}$-module with multiplication
\[
\mathcal{O} \times S \rightarrow S
\]
\[
(p, x) \mapsto px.
\]
Indeed, it follows from (5.5) that
\[
a(des) = (da)e_8
\]
for all $a, d \in \mathcal{O}$. Hence for all $p \in \mathcal{O}$ and $x = a + be_8 \in S$, we have
\[
p(px) = p(pa + (bp)e_8) = p^2 a + (bp^2)e_8 = p^2 x.
\]
This shows that $S$ is a left $\mathcal{O}$-module.

One can easily check that the conjugate of $s$ in sedenions coincides with the conjugate of $s$ when $S$ is regarded as a left $\mathcal{O}$-module. Hence the right multiplication induced by (5.2) is
\[
S \times \mathcal{O} \rightarrow S
\]
\[
(x, p) \mapsto \overline{p x} = xp.
\]
It can be seen from (5.6) that
\[
e_8 \in \mathcal{A}^- (S).
\]
Therefore we get
\[
S \cong \mathcal{O} \oplus \bar{\mathcal{O}}
\]
as left $\mathcal{O}$-modules.

We now define
\[
\langle a + be_8, c + de_8 \rangle_S := a\overline{c} + \overline{db}.
\]
In other words, if we denote by
\[
\pi_{\mathcal{O}} : S \rightarrow \mathcal{O}
\]
the projection from $S$ to $\mathcal{O}$, then for any $s, w \in S$,
\[
\langle s, w \rangle_S := \pi_{\mathcal{O}}(sw).
\]
One can check that this is an $\mathcal{O}$-inner product.

We move on our discussion to arbitrary Cayley–Dickson algebra $A_n$. For $n = 1, 2, 3$,
\[
A_1 = \mathbb{C}, \quad A_2 = \mathbb{H}, \quad A_3 = \mathcal{O}.
\]
We shall apply the induction to endow $\mathcal{A}_n$ with a Hilbert $\mathcal{O}$-module structure.

The Cayley-Dickson algebra $\mathcal{A}_n$ is defined by induction. An element $s$ in $\mathcal{A}_n$ can be written as

$$s = a + be_{2^{n-1}},$$

where $a, b \in \mathcal{A}_{n-1}$. Its conjugate is defined as

$$\overline{a + be_{2^{n-1}}} = \overline{a} - be_{2^{n-1}}.$$ 

the multiplication of $\mathcal{A}_n$ is defined as

$$\tag{5.7} (a + be_{2^{n-1}})(c + de_{2^{n-1}}) = (ac - db) + (da + b\overline{c})e_{2^{n-1}},$$

where $a, b, c, d \in \mathcal{A}_{n-1}$.

As a real vector space, $\mathcal{A}_n$ admits a canonical basis

$$\{e_0, e_1, \ldots, e_{2^n-1}\},$$

where

$$\tag{5.8} e_{2^{n-1}+j} := e_j e_{2^{n-1}}$$

for any $j = 1, \ldots, 2^{n-1} - 1$. The octonions $\mathcal{O}$ can be viewed as a subset of $\mathcal{A}_n$ when $n \geq 4$ generated by $e_0, e_1, \ldots, e_7$.

The left $\mathcal{O}$-module structure on $\mathcal{A}_n$ is induced by the multiplication of $\mathcal{A}_n$:

$$\mathcal{O} \times \mathcal{A}_n \rightarrow \mathcal{A}_n$$

$$(p, x) \mapsto px.$$

It turns out that $\mathcal{A}_n$ is a weak $\mathcal{O}$-bimodule for $n \geq 4$.

**Lemma 5.4.** If $n \geq 4$, then $\mathcal{A}_n$ is a weak $\mathcal{O}$-bimodule and the two conjugates coincide when $\mathcal{A}_n$ is regarded as either a weak $\mathcal{O}$-bimodule or a Cayley-Dickson algebra. Moreover, we have an isomorphism of left $\mathcal{O}$-modules

$$\mathcal{A}_n \cong \mathcal{O}^{2^{n-1}} \oplus \mathcal{O}^{2^{n-1}}.$$

**Proof.** When $n = 4$, we have $\mathcal{A}_4 = \mathbb{S}$ so the result has been proved. Suppose it holds for $n$, we come to prove the case of $n + 1$.

It follows from (5.7) that

$$p(qe_{2^n}) = (qp)e_{2^n}$$

for all $p, q \in \mathcal{O}$ and $n \geq 4$.

Now we check that the subset $\mathcal{A}_n e_{2^n}$ of $\mathcal{A}_{n+1}$ is also a left $\mathcal{O}$-module as well as a right $\mathcal{O}$-module. For any $p \in \mathcal{O}$ and $x \in \mathcal{A}_n$,

$$\tag{5.9} p(p(xe_{2^n})) = p((xp)e_{2^n}) = ((xp)p)e_{2^n}.$$ 

By induction, $\mathcal{A}_n$ is a right $\mathcal{O}$-module so that $(xp)p = xp^2$. Therefore (5.9) becomes

$$p(p(xe_{2^n})) = (xp^2)e_{2^n} = p^2(xe_{2^n}).$$

This shows that $\mathcal{A}_n e_{2^n}$ is a left $\mathcal{O}$-module. It can be proved that $\mathcal{A}_n e_{2^n}$ is a right $\mathcal{O}$-module similarly.

Moreover, it is easy to check that

$$\mathcal{A}_L(\mathcal{A}_n e_{2^n}) = \mathcal{A}_R(\mathcal{A}_n e_{2^n}), \quad \mathcal{A}_L^-(\mathcal{A}_n e_{2^n}) = \mathcal{A}_R^- (\mathcal{A}_n e_{2^n}).$$

Combining (5.10) and (5.4), we conclude that

$$\mathcal{A} (\mathcal{A}_n e_{2^n}) = \mathcal{A}^- (\mathcal{A}_n e_{2^n}), \quad \mathcal{A}^- (\mathcal{A}_n e_{2^n}) = \mathcal{A} (\mathcal{A}_n e_{2^n}).$$
By induction, as left modules
\[ \mathbb{A}_n \cong O^{2n-4} \oplus O^{2n-4}. \]
This implies that
\[ \mathbb{A}_n e_{2n} \cong O^{2n-4} \oplus O^{2n-4}. \]
Hence we obtain
\[ \mathbb{A}_{n+1} = \mathbb{A}_n \oplus \mathbb{A}_n e_{2n} \]
\[ \cong O^{2n+1-4} \oplus O^{2n+1-4}. \]
This completes the proof. \(\square\)

We finally define the \(O\)-inner product on \(\mathbb{A}_n\). We denote by \(\pi_O : \mathbb{A}_n \to O\) the projection. For any \(s, w \in \mathbb{A}_n\), we define
\[ \langle s, w \rangle_n := \pi_O(s \overline{w}) \] (5.12)

**Theorem 5.5.** \((\mathbb{A}_n, \langle \cdot, \cdot \rangle_n)\) is a Hilbert left \(O\)-module for any \(n \geq 4\).

**Proof.** We have shown that \(\langle s, w \rangle_4\) is an \(O\)-inner product. Suppose \(\langle s, w \rangle_{n-1}\) is also an \(O\)-inner product on \(\mathbb{A}_{n-1}\). We denote by \(\pi_{n-1} : \mathbb{A}_n \to \mathbb{A}_{n-1}\) the orthogonal projection. Then for any \(s = a + be_{2n-1}, w = c + de_{2n-1} \in \mathbb{A}_n\), we have
\[ \langle s, w \rangle_n = \pi_O(\overline{s w}) = \pi_3 \pi_4 \ldots \pi_{n-1}(\overline{s w}) = \langle a, c \rangle_{n-1} + \langle d, b \rangle_{n-1}. \]

Note that for any \(p \in O\),
\[ ps = pa + (bp)e_{2n-1}. \]
Hence we have
\[ \langle ps, w \rangle_n - p \langle s, w \rangle_n = \langle pa, c \rangle_{n-1} + \langle d, (bp)_{n-1} \rangle - p(\langle a, c \rangle_{n-1} + \langle d, b \rangle_{n-1}) \]
\[ = [p, a, c]_{n-1} + [\langle d, b \rangle_{n-1}, p] - [p, d, b]_{n-1}. \]
Here \([p, a, c]_{n-1}\) and \([p, d, b]_{n-1}\) are second associators in the \(O\)-Hilbert space \(\mathbb{A}_{n-1}\). Hence by induction we obtain
\[ \text{Re} \left( \langle ps, w \rangle_n - p \langle s, w \rangle_n \right) = 0. \]
This proves the para-linearity.

To prove the hermiticity, we apply the inductive assumption to get
\[ \langle w, s \rangle_n = \langle c, a \rangle_{n-1} + \overline{\langle b, d \rangle}_{n-1} = \langle a, c \rangle_{n-1} + \overline{\langle d, b \rangle}_{n-1} = \langle s, w \rangle_n. \]
At last, we come to prove the positivity. According to the inductive assumption, we have
\[ \langle s, s \rangle_n = \langle a, a \rangle_{n-1} + \overline{\langle b, b \rangle}_{n-1} \geq 0 \]
and
\[ \langle s, s \rangle_n = 0 \iff a = b = 0 \iff s = 0. \]
This shows the positivity.

Theorem 5.5 provides us a new approach to study the Cayley–Dickson algebras. We point out that (5.11) gives a complete description of the associative and conjugate associative elements of \( A_n \) by induction. More precisely,

\[
\mathcal{A}(A_{n+1}) = \mathcal{A}(A_n) \oplus \mathcal{A}^-(A_n)e_{2^n},
\]

\[
\mathcal{A}^-(A_{n+1}) = \mathcal{A}^-(A_n) \oplus \mathcal{A}(A_n)e_{2^n}.
\]

For example, we have

\[
\begin{array}{c|c|c}
A_n & \text{a basis of } \mathcal{A}(A_n) & \text{a basis of } \mathcal{A}^-(A_n) \\
\hline
A_4 & e_0 & e_8 \\
A_5 & e_0, e_8e_{16} & e_8, e_{16} \\
A_6 & e_0, e_8e_{16}, e_8e_{32}, e_{16}e_{32} & e_8, e_{16}, e_{32}, (e_8e_{16})e_{32}
\end{array}
\]

Let us look at a weak associative orthonormal basis in the Cayley-Dickson algebra as a Hilbert left \( O \)-module.

In the specific case \( n = 5 \), the algebra \( A_n = A_5 \) admits a weak associative orthonormal basis

\[
S := \{e_0, e_{24}, e_8, e_{16}\}.
\]

This means for any \( x \in A_5 \),

\[
x = \pi_O(x) + \pi_O(xe_8)e_8 + \pi_O(xe_{16})e_{16} + \pi_O(xe_{24})e_{24}.
\]

In the general \( A_n \), due to (5.8) we can take a weak associative orthonormal basis as

\[
S := \{e_{8i} : i = 0, \ldots, 2^{n-3} - 1\}.
\]

Any \( x \in A_n \) can be expressed in terms of this weak associative orthonormal basis as

\[
x = \sum_{i=0}^{2^{n-3}-1} \pi_O(xe_{8i})e_{8i}.
\]

(5.13)

This can be regarded as an octonionic version of the decomposition

\[
x = \sum_{i=0}^{2^n-1} \text{Re} (xe_{7^i})e_i.
\]

(5.14)

Remark 5.6. We have shown that the Cayley-Dickson algebra \( A_n \) is a Hilbert left \( O \)-module. However the corresponding result does not hold in the quaternionic case. That is, \( A_n \) is not a \( \mathbb{H} \)-vector space in general. This is because the definition of \( \mathbb{H} \)-vector space (see [16]) requires an associative condition

\[
p(qx) = (pq)x
\]

for all \( p, q \in \mathbb{H} \) and any element \( x \) of that \( \mathbb{H} \)-vector space.
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