THE DISPERSIONLESS MULTI-DIMENSIONAL INTEGRABLE SYSTEMS AND RELATED CONFORMAL STRUCTURE
GENERATING EQUATIONS OF MATHEMATICAL PHYSICS

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Abstract. Based on the diffeomorphism group vector fields on the complexified torus and the related Lie-algebraic structures, we study multi-dimensional dispersionless integrable systems, describing conformal structure generating equations of mathematical physics. An interesting modification of the devised Lie-algebraic approach subject to the spatial dimensional invariance and meromorphicity of the related differential-geometric structures is described and applied to proving complete integrability of some conformal structure generating equations. As examples, we analyzed the Einstein–Weyl metric equation, the modified Einstein–Weyl metric equation, the Dunajski heavenly equation system, the first and second conformal structure generating equations, the inverse first Shabat reduction heavenly equation, the first and modified Plebański heavenly equations and its multi-dimensional generalizations, the Husain heavenly equation and its multi-dimensional generalizations, the general Monge equation and its multi-dimensional generalizations.

1. Vector fields on the complexified torus and the related
Lie-algebraic properties

Consider the loop Lie group \( \tilde{G} := \text{Diff}(\mathbb{T}^n) \), consisting of the set of smooth mappings \( \{ C^1 \supset S^1 \to G : = \text{Diff}(\mathbb{T}^n) \} \), extended, respectively, holomorphically from the circle \( S^1 \subset C^1 \) on the set \( D_1^\perp \) of the internal points of the circle \( S^1 \), and on the set \( D_1^\perp \) of the external points \( \lambda \in \mathbb{C}\setminus(D_1^\perp) \). The corresponding diffeomorphism Lie algebra splitting \( \tilde{G} =: \tilde{G}_+ \oplus \tilde{G}_- \), where \( \tilde{G}_+ := \text{Diff}(\mathbb{T}^n)_+ \subset \Gamma(\mathbb{T}^n; \mathbb{R}(\mathbb{T}^n)) \) is a Lie subalgebra, consisting of vector fields on the complexified torus \( \mathbb{T}^n \subset \mathbb{T}^n \times \mathbb{C} \), suitably holomorphic on the disc \( D_1^\perp \), \( \tilde{G}_- := \text{Diff}(\mathbb{T}^n)_- \subset \Gamma(\mathbb{T}^n; \mathbb{R}(\mathbb{T}^n)) \) is a Lie subalgebra, consisting of vector fields on the complexified torus \( \mathbb{T}^n \subset \mathbb{T}^n \times \mathbb{C} \), suitably holomorphic on the set \( D_1^\perp \). The adjoint space \( \tilde{G}^* := \tilde{G}_+^* \oplus \tilde{G}_-^* \), where the space \( \tilde{G}_+^* \subset \Gamma(\mathbb{T}^n; \mathbb{T}^*(\mathbb{T}^n)) \) consists, respectively, from the differential forms on the complexified torus \( \mathbb{T}^n \), suitably holomorphic on the set \( \mathbb{T}^n \setminus(D_1^\perp) \), and the adjoint space \( \tilde{G}_-^* \subset \Gamma(\mathbb{T}^n; \mathbb{T}^*(\mathbb{T}^n)) \) consists, respectively, from the differential forms on the complexified torus \( \mathbb{T}^n \), suitably holomorphic on the set \( D_1^\perp \), so that the space \( \tilde{G}_+^* \) is dual to \( \tilde{G}_-^* \) and \( \tilde{G}_-^* \) is dual to \( \tilde{G}_{-}^* \) with respect to the following convolution form on the product \( \tilde{G}_+^* \times \tilde{G}_-^* \):

\[
(\tilde{l}, \tilde{a}) := \text{res}_{\lambda} \int_{\mathbb{T}^n} < \tilde{l}, \tilde{a} > dx
\]

for any vector field \( \tilde{a} := < a(x), \frac{\partial}{\partial x} > \in \tilde{G}_-^* \) and differential form \( \tilde{l} := < l(x), dx > \in \tilde{G}_+^* \) on \( \mathbb{T}^n \), depending on the coordinate \( x := (\lambda; x) \in \mathbb{T}^n \), where, by definition, \( < \cdot, \cdot > \) is the usual scalar product on the Euclidean space \( \mathbb{E}^{n+1} \) and \( \frac{\partial}{\partial x} := \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots , \frac{\partial}{\partial x_n} \right)^T \) is the usual gradient vector. The Lie algebra \( \tilde{G}_+^* \) allows the

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direct sum splitting $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, causing with respect to the convolution $(1.1)$ the direct sum splitting $\tilde{\mathcal{G}}^* = \tilde{\mathcal{G}}_+^* \oplus \tilde{\mathcal{G}}_-^*$. If to define now the set $I(\tilde{\mathcal{G}}^*)$ of Casimir invariant smooth functionals $h : \tilde{\mathcal{G}}^* \rightarrow \mathbb{R}$ on the adjoint space $\tilde{\mathcal{G}}^*$ via the coadjoint Lie algebra $\tilde{\mathcal{G}}$ action
\begin{equation}
\ad_{\nabla h(\tilde{l})} \tilde{l} = 0
\end{equation}
at a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, by means of the classical Adler-Kostant-Symes scheme one can generate a wide class of multi-dimensional completely integrable dispersionless (heavenly type) commuting to each other Hamiltonian systems
\begin{equation}
\frac{d\tilde{l}}{dt} := -\ad_{\nabla h(\tilde{l})} \tilde{l} ,
\end{equation}
for all $\tilde{h} \in I(\tilde{\mathcal{G}}^*)$, $\nabla h(\tilde{l}) := \nabla h_+(\tilde{l}) \oplus \nabla h_-(\tilde{l}) \in \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, on suitable functional manifolds. Moreover, these commuting to each other flows can be equivalently represented as a commuting system of Lax-Sato type vector field equations on the functional space $C^2(\mathbb{T}_C^1; \mathbb{C})$, generating an complete set of first integrals for them. As it was appeared, amongst them there are present important equations for modern studies in physics, hydrodynamics and, in particular, in Riemannian geometry, being related with such interesting conformal structures on Riemannian metric spaces as Einstein and Einstein-Weyl metrics equations, the first and second Plebański conformal metric equations, Dunajski metric equations etc. What was observed, some of them were generated by seed elements $\tilde{l} \in \tilde{\mathcal{G}}^*$, meromorphic at some points of the complex plane $\mathbb{C}$, whose analysis needed some modification of the theoretical backgrounds. Moreover, the general differential-geometric structure of seed elements, related with some conformal metric equations, proved to be invariant subject to the spatial dimension of the Riemannian spaces under regard, that made it possible to describe them analytically. We analyzed the Einstein–Weyl metric equation, the modified Einstein–Weyl metric equation, the Dunajski heavenly equation system, the first and second conformal structure generating equations, the inverse first Shabat reduction heavenly equation, the first and modified Plebański heavenly equations and its multi-dimensional generalizations, the Husain heavenly equation and its multi-dimensional generalizations, the general Monge equation and its multi-dimensional generalizations. Namely these and related aspects of the integrable multi-dimensional conformal metric equations, mentioned above, are studied and presented in the work.

2. The Lie-algebraic structures and integrable Hamiltonian systems

Consider the loop Lie algebra $\tilde{\mathcal{G}}$, determined above. This Lie algebra has elements representable as $a(x; \lambda) := < a(x; \lambda), \frac{\partial}{\partial x} > = \sum_{j=1}^{n} a_j(x; \lambda) \frac{\partial}{\partial x_j} + a_0(x; \lambda) \frac{\partial}{\partial x} \in \tilde{\mathcal{G}}$ for some holomorphic in $\lambda \in \mathbb{D}_0^+$ vectors $a(x; \lambda) \in \mathbb{E} \times \mathbb{E}^n$ for all $x \in \mathbb{T}^n$, where $\frac{\partial}{\partial x} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})^T$ is the generalized Euclidean vector gradient with respect to the vector variable $x := (\lambda, x) \in \mathbb{T}_C^n$. As it was mentioned above, the Lie algebra $\tilde{\mathcal{G}}$ naturally splits into the direct sum of two subalgebras:
\begin{equation}
\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-,
\end{equation}
allowing to introduce on it the classical $\mathcal{R}$-structure:
\begin{equation}
[\tilde{a}, \tilde{b}]_{\mathcal{R}} := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}]
\end{equation}
for any $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$, where
\begin{equation}
\mathcal{R} := (P_+ - P_-)/2,
\end{equation}
and
\begin{equation}
P_{\pm} \tilde{\mathcal{G}} := \tilde{\mathcal{G}}_\pm \subset \tilde{\mathcal{G}}.
\end{equation}
The space $\tilde{G}^*$, adjoint to the Lie algebra $\tilde{G}$, of vector fields on $T^*_C$, is functionally identified with $\tilde{G}$ subject to the metric (1.1). Now for arbitrary $f, g \in D(\tilde{G}^*)$, one can determine two Lie–Poisson type brackets
\begin{equation}
\{f, g\} := \langle \tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})] \rangle
\end{equation}
and
\begin{equation}
\{f, g\}_R := \langle \tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]_R \rangle,
\end{equation}
where at any seed element $\tilde{l} \in \tilde{G}^*$ the gradient element $\nabla f(\tilde{l})$ and $\nabla g(\tilde{l}) \in \tilde{G}$ are calculated with respect to the metric (1.1).

Now let us assume that a smooth function $\gamma \in I(\tilde{G}^*)$ is a Casimir invariant, that is
\begin{equation}
ad_{\gamma}(\tilde{l}) = 0
\end{equation}
for a chosen seed element $\tilde{l} \in \tilde{G}^*$. As the coadjoint mapping $ad_{\gamma}(\tilde{l}) : \tilde{G}^* \to \tilde{G}^*$ for any $f \in D(\tilde{G}^*)$ can be rewritten in the reduced form as
\begin{equation}
ad_{\gamma}(\tilde{l}) = \sum_{j=1}^{n} \left( l \frac{\partial}{\partial x} \nabla \gamma(l) \right),
\end{equation}
where, by definition, $\nabla f(l) := \langle \nabla f(l), \frac{\partial}{\partial x} \rangle$. For the Casimir function $\gamma \in D(\tilde{G}^*)$ the condition (2.7) is then equivalent to the equation
\begin{equation}
l \left( \frac{\partial}{\partial x}, \nabla \gamma(l) \right) + \left( \nabla \gamma(l), \frac{\partial}{\partial x} \right) l + \left( l, \nabla \gamma(l) \right) = 0,
\end{equation}
which should be solved analytically. In the case when an element $\tilde{l} \in \tilde{G}^*$ is singular as $|\lambda| \to \infty$, one can consider the general asymptotic expansion
\begin{equation}
\nabla \gamma := \nabla \gamma^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla \gamma^{(p)} \lambda^{-j}
\end{equation}
for some suitably chosen $p \in \mathbb{Z}_+$, and upon substituting (2.10) into the equation (2.9), one can proceed to solving it recursively.

Now let $h^{(y)}, h^{(t)} \in I(\tilde{G}^*)$ be such Casimir functions for which the Hamiltonian vector field generators
\begin{equation}
\nabla h^{(y)}(l) := (\nabla \gamma^{(p_y)}(l))_{y+}, \quad \nabla h^{(t)}(l) := (\nabla \gamma^{(p_t)}(l))_{t+}
\end{equation}
are, respectively, defined for special integers $p_y, p_t \in \mathbb{Z}_+$. These invariants generate, owing to the Lie–Poisson bracket (2.6), the following commuting flows:
\begin{equation}
\frac{\partial l}{\partial t} = - \left( l \frac{\partial}{\partial x}, \nabla h^{(t)}(l) \right) l - \left( l, \nabla h^{(t)}(l) \right),
\end{equation}
and
\begin{equation}
\frac{\partial l}{\partial y} = - \left( l \frac{\partial}{\partial x}, \nabla h^{(y)}(l) \right) l - \left( l, \nabla h^{(y)}(l) \right),
\end{equation}
where $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since the invariants $h^{(y)}, h^{(t)} \in I(\tilde{G}^*)$ commute with respect to the Lie–Poisson bracket (2.6), the flows (2.12) and (2.13) also commute, implying that the corresponding Hamiltonian vector field generators
\begin{equation}
A_{\nabla h^{(t)}} := \left( \nabla h^{(t)}(l), \frac{\partial}{\partial x} \right), \quad A_{\nabla h^{(y)}} := \left( \nabla h^{(y)}(l), \frac{\partial}{\partial x} \right)
\end{equation}
satisfy the Lax compatibility condition
\begin{equation}
\frac{\partial}{\partial y} A_{\nabla h^{(t)}} - \frac{\partial}{\partial t} A_{\nabla h^{(y)}} = [A_{\nabla h^{(t)}}, A_{\nabla h^{(y)}}, A_{\nabla h^{(y)}}]
\end{equation}
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for all $y, t \in \mathbb{R}$. On the other hand, the condition (2.15) is equivalent to the compatibility condition of two linear equations
\begin{equation}
(\frac{\partial}{\partial t} + A_{\psi}^h) \psi = 0, \quad \left(\frac{\partial}{\partial y} + A_{\psi}^h\right) \psi = 0
\end{equation}
for a function $\psi \in C^2(\mathbb{R}^2 \times T^o_C; \mathbb{C})$ for all $y, t \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$. The above can be formulated as the following key result:

**Proposition 2.1.** Let a seed vector field be $\tilde{l} \in \tilde{G}^*$ and $h^{(y)}_+, h^{(t)}_+ \in I(\tilde{G}^*)$ be Casimir functions subject to the metric $\langle \cdot, \cdot \rangle$ on the loop Lie algebra $\tilde{G}$ and the natural coadjoint action on the loop co-algebra $\tilde{\mathcal{G}}^*$. Then the following dynamical systems
\begin{equation}
\frac{\partial \tilde{l}}{\partial y} = -ad^*_{\psi h^{(y)}_+}(\tilde{l}), \quad \frac{\partial \tilde{l}}{\partial t} = -ad^*_{\psi h^{(t)}_+}(\tilde{l})
\end{equation}
are commuting Hamiltonian flows for all $y, t \in \mathbb{R}$. Moreover, the compatibility condition of these flows is equivalent to the vector fields representation (2.15), where $\psi \in C^2(\mathbb{R}^2 \times T^o_C; \mathbb{C})$ and the vector fields $A_{\psi h^{(y)}_+}, A_{\psi h^{(t)}_+} \in \tilde{G}$ are given by the expressions (2.14) and (2.17).

**Remark 2.2.** As mentioned above, the expansion (2.10) is effective if a chosen seed element $\tilde{l} \in \tilde{G}^*$ is singular as $|\lambda| \to \infty$. In the case when it is singular as $|\lambda| \to 0$, the expression (2.10) should be replaced by the expansion
\begin{equation}
\nabla \gamma^{(p)}_j(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla \gamma^{(p)}_j(l) \lambda^j
\end{equation}
for suitably chosen integers $p \in \mathbb{Z}_+$, and the reduced Casimir function gradients then are given by the Hamiltonian vector field generators
\begin{align}
\nabla h^{(y)}_+(l) &:= \lambda(\lambda^{-p_u} - 1) \nabla \gamma^{(p_u)}_j(l) \,, \\
\nabla h^{(t)}_+(l) &:= \lambda(\lambda^{-p_t} - 1) \nabla \gamma^{(p_t)}_j(l) \,.
\end{align}
for suitably chosen positive integers $p_u, p_t \in \mathbb{Z}_+$ and the corresponding Hamiltonian flows are, respectively, written as $\frac{\partial \tilde{l}}{\partial y} = ad^*_{\psi h^{(y)}_+}(\tilde{l})$, $\frac{\partial \tilde{l}}{\partial t} = ad^*_{\psi h^{(t)}_+}(\tilde{l})$.

It is also worth of mentioning that, following Ovsienko’s scheme [11] [12], one can consider a slightly wider class of integrable heavenly equations, realized as compatible Hamiltonian flows on the semidirect product of the holomorphic loop Lie algebra $\tilde{G}$ of vector fields on the torus $T^o_C$ and its regular co-adjoint space $\tilde{G}^*$, supplemented with naturally related cocycles.

**3. THE LAX–SATO TYPE INTEGRABLE MULTI-DIMENSIONAL HEAVENLY SYSTEMS AND RELATED CONFORMAL STRUCTURE GENERATING EQUATIONS**

**3.1. Einstein–Weyl metric equation.** Define $\tilde{G}^* = \tilde{df} \tilde{f}(T^o_C)^*$ and take the seed element
\[ \tilde{l} = (u_x \lambda - 2u_x v_x - u_y) \, dx + (\lambda^2 - v_x \lambda + v_y + v_x^2) \, d\lambda, \]
which generates with respect to the metric (1.1) the gradient of the Casimir invariants $h^{(p_u)}_+, h^{(p_t)}_+ \in I(\tilde{G}^*)$ in the form
\begin{align}
\nabla h^{(p_u)}_+(l) &\sim \lambda^2(0,1)^T + (-u_x, v_x)^T \lambda + (u_y, u - v_y)^T + O(\lambda^{-1}) , \\
\nabla h^{(p_t)}_+(l) &\sim \lambda(0,1)^T + (-u_x, v_x)^T - (u_y, -v_y)^T \lambda^{-1} + O(\lambda^{-2})
\end{align}
as $|\lambda| \to \infty$ at $p_u = 2, p_t = 1$. For the gradients of the Casimir functions $h^{(t)}_+, h^{(y)}_+ \in I(\tilde{G}^*)$, determined by (2.11) one can easily obtain the corresponding Hamiltonian vector field generators
\begin{align}
A_{h^{(t)}_+} &= \left(\nabla h^{(t)}_+(l), \frac{\partial}{\partial x}\right) = (\lambda^2 + \lambda v_x + u - v_y, \frac{\partial}{\partial x} - (\lambda u_x + u_y) \frac{\partial}{\partial \lambda}) , \\
A_{h^{(y)}_+} &= \left(\nabla h^{(y)}_+(l), \frac{\partial}{\partial x}\right) = (\lambda + v_x, \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \lambda}).
\end{align}
satisfying the compatibility condition (2.15), which is equivalent to the set of equations

\[ \begin{align*}
    u_{xt} + u_{yy} + (u_{ux})_x + v_x u_{xy} - v_y u_{xx} &= 0, \\
    v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} &= 0,
\end{align*} \]

(3.3)

describing general integrable Einstein–Weyl metric equations [6].

As is well known [10], the invariant reduction of (3.3) at \( \nu = 0 \) gives rise to the famous dispersionless Kadomtsev–Petviashvili equation

\[ (u_t + u u_x)_x + u_y = 0, \]

(3.4)

for which the reduced vector field representation (2.16) follows from (3.2) and is given by the vector fields

\[ A_{\nabla h^{(t)}} = (\lambda^2 + u) \frac{\partial}{\partial x} + (-\lambda u_x + u_y) \frac{\partial}{\partial \lambda}, \]

(3.5)

satisfying the compatibility condition (2.15), equivalent to the equation (3.4). In particular, one derives from (2.16) and (3.5) the vector field compatibility relationships

\[ \begin{align*}
    \frac{\partial \psi}{\partial t} + (\lambda^2 + u) \frac{\partial \psi}{\partial x} + (-\lambda u_x + u_y) \frac{\partial \psi}{\partial \lambda} &= 0, \\
    \frac{\partial \psi}{\partial y} + \lambda \frac{\partial \psi}{\partial x} - u_x \frac{\partial \psi}{\partial \lambda} &= 0,
\end{align*} \]

(3.6)

satisfied for \( \psi \in C^2(\mathbb{R}^2 \times T^1_\mathbb{C}; \mathbb{C}) \) and any \( y, t \in \mathbb{R}, (x, \lambda) \in T^1_\mathbb{C}. \)

### 3.2. The modified Einstein–Weyl metric equation

This equation system is

\[ \begin{align*}
    u_{xt} &= u_{yy} + u_x u_y + u_x^2 w_{xx} + u u_{xy} + u_{xy} w_x + u_{xx} a, \\
    w_{xt} &= u w_{xx} + u_y w_x + w_x w_{xy} + a w_{xx} - a_y,
\end{align*} \]

(3.7)

where \( a_x := u_x w_x - w_{xy}, \) and was recently derived in [17]. In this case we take also \( \hat{G}^* = \hat{d} \hat{f}(T^1_\mathbb{C}), \) yet for a seed element \( \tilde{l} \in \hat{G} \) we choose the form

\[ \begin{align*}
    \tilde{l} &= [\lambda^2 u_x + (2 u_x w_x + u_y + 3 u u_x) \lambda + 2 u_x \partial_x^{-1} u_x w_x + 2 u_x \partial_x^{-1} u_y + \\
    &+ 3 u_x w_x^2 + 2 u_y w_x + 6 u u_x w_x + 2 u u_y + 3 u^2 u_x - 2 a u_x] dx + \\
    &\quad + [\lambda^2 + (w_x + 3 u) \lambda + 2 \partial_x^{-1} u_x w_x + 2 \partial_x^{-1} u_y + w_x^2 + 3 u w_x + 3 u^2 - a] d\lambda,
\end{align*} \]

(3.8)

which with respect to the metric (1.1) generates two Casimir invariants \( \gamma^{(j)} \in I(\hat{G}^*), \) \( j = 1, 2, \) whose gradients are

\[ \begin{align*}
    \nabla \gamma^{(2)}(l) &\sim \lambda^2 [(u_x, -1)^\top + (u u_x + u_y, -u + w_x)^\top \lambda^{-1} + \\
    &+ (0, u w_x - a)^\top \lambda^{-2}] + O(\lambda^{-1}), \\
    \nabla \gamma^{(1)}(l) &\sim \lambda [(u_x, -1)^\top + (0, w_x)^\top \lambda^{-1}] + O(\lambda^{-1}),
\end{align*} \]

(3.9)

as \( |\lambda| \rightarrow \infty \) at \( p_y = 1, p_l = 2. \) The corresponding gradients of the Casimir functions \( h^{(t)}, h^{(y)} \in I(\hat{G}^*), \) determined by (2.11), generate the Hamiltonian vector field expressions

\[ \begin{align*}
    \nabla h^{(y)}_+ := \nabla \gamma^{(1)}(l)_+ &\sim (u_x \lambda, -\lambda + w_x)^\top, \\
    \nabla h^{(t)}_+ = \nabla \gamma^{(2)}(l)_+ &\sim (u_x \lambda^2 + (u u_x + u_y) \lambda, -\lambda^2 + (w_x - u) \lambda + u w_x - a)^\top.
\end{align*} \]

(3.10)

Now one easily obtains from (3.10) the compatible Lax system of linear equations

\[ \begin{align*}
    \frac{\partial \psi}{\partial y} + (-\lambda + w_x) \frac{\partial \psi}{\partial x} + u_x \lambda \frac{\partial \psi}{\partial \lambda} &= 0, \\
    \frac{\partial \psi}{\partial t} + (-\lambda^2 + (w_x - u) \lambda + u w_x - a) \frac{\partial \psi}{\partial x} + (u_x \lambda^2 + (u u_x + u_y) \lambda) \frac{\partial \psi}{\partial \lambda} &= 0,
\end{align*} \]

(3.11)

satisfied for \( \psi \in C^2(\mathbb{R}^2 \times T^1_\mathbb{C}; \mathbb{C}) \) and any \( y, t \in \mathbb{R}, (\lambda, x) \in T^1_\mathbb{C}. \)
3.3. The Dunajski heavenly equation system. This equation, suggested in [5], generalizes the corresponding anti-self-dual vacuum Einstein equation, which is related to the Plebański metric and the celebrated Plebański [13] second heavenly equation. To study the integrability of the Dunajski equations

\[ u_{x,t} + u_{y,x} + u_{x,y}u_{x} - u_{x}^{2} - v = 0, \]

\[ v_{x,t} + u_{x,y} + u_{x,y}u_{x} - 2u_{x}v_{x} = 0, \]

where \((u, v) \in C^{\infty}(\mathbb{R}^{2} \times \mathbb{T}^{2}; \mathbb{R}^{2})\), \((y, t; x_1, x_2) \in \mathbb{R}^{2} \times \mathbb{T}^{2}\), we define \(\hat{G}^{*} = \text{diff}(\mathbb{T}^{2})^{*}\) and take the following as a seed element \(\hat{l} \in \hat{G}^{*}\)

\[ \hat{l} = (\lambda + v_{x_{1}} - u_{x_{1},x_{1}} + u_{x_{1},x_{2}})dx_{1} + (\lambda + v_{x_{2}} + u_{x_{2},x_{2}} - u_{x_{1},x_{2}})dx_{2} + (\lambda - x_{1} - x_{2})d\lambda. \]

With respect to the metric \((1.1)\), the gradients of two functionally independent Casimir invariants \(h^{(p_{y})}, h^{(p_{t})} \in I(\hat{G}^{*})\) can be obtained as \(|\lambda| \to \infty\) in the asymptotic form as

\[ \nabla h^{(p_{y})} (l) \sim \lambda(1,0,0)^{T} + (-u_{x_{1},x_{2}}, u_{x_{1},x_{1}} - v_{x_{1}})^{T} + O(\lambda^{-1}), \]

\[ \nabla h^{(p_{t})} (l) \sim \lambda(0,1,0)^{T} + (u_{x_{2},x_{2}} - u_{x_{1},x_{2}}, v_{x_{2}})^{T} + O(\lambda^{-1}), \]

at \(p_{t} = 1 = p_{y} \). Upon calculating the Hamiltonian vector field generators

\[ \nabla h^{(y)} := \nabla h^{(p_{y})} (l)|_{+} = (\lambda - u_{x_{1},x_{2}}, u_{x_{1},x_{1}} - v_{x_{1}})^{T}, \]

\[ \nabla h^{(t)} := \nabla h^{(p_{t})} (l)|_{+} = (u_{x_{2},x_{2}} - \lambda - u_{x_{1},x_{2}}, v_{x_{2}})^{T}, \]

following from the Casimir functions gradients \((3.14)\), one easily obtains the following vector fields

\[ A_{\nabla h^{(y)}} = \langle \nabla h^{(y)}, \partial / \partial x \rangle >= u_{x_{2},x_{2}} \partial / \partial x_{1} - (\lambda + u_{x_{1},x_{2}}) \partial / \partial x_{2} + v_{x_{2}} \partial / \partial \lambda, \]

\[ A_{\nabla h^{(t)}} = \langle \nabla h^{(t)}, \partial / \partial x \rangle >= (\lambda - u_{x_{1},x_{2}}) \partial / \partial x_{1} + u_{x_{1},x_{1}} \partial / \partial x_{2} - v_{x_{1}} \partial / \partial \lambda, \]

satisfying the Lax compatibility condition \((2.15)\), which is equivalent to the vector field compatibility relationships

\[ \frac{\partial \psi}{\partial t} + u_{x_{2},x_{2}} \frac{\partial \psi}{\partial x_{1}} - (\lambda + u_{x_{1},x_{2}}) \frac{\partial \psi}{\partial x_{2}} + v_{x_{2}} \frac{\partial \psi}{\partial \lambda} = 0, \]

\[ \frac{\partial \psi}{\partial y} + (\lambda - u_{x_{1},x_{2}}) \frac{\partial \psi}{\partial x_{1}} + u_{x_{1},x_{1}} \frac{\partial \psi}{\partial x_{2}} - v_{x_{1}} \frac{\partial \psi}{\partial \lambda} = 0, \]

satisfied for \(\psi \in C^{2}(\mathbb{R}^{2} \times \mathbb{T}^{2}; \mathbb{C})\), any \((y, t) \in \mathbb{R}^{2}\) and all \((\lambda; x_{1}, x_{2}) \in \mathbb{T}^{2}_{\mathbb{C}}\). As was mentioned in [3], the Dunajski equations \((3.12)\) generalize both the dispersionless Kadomtsev–Petviashvili and Plebański second heavenly equations, and is also a Lax integrable Hamiltonian system.

3.4. First conformal structure generating equation: \(u_{yt} + u_{xt}u_{y} - u_{t}u_{xy} = 0\). The seed element \(\hat{l} \in \hat{G}^{*} = \text{diff}(\mathbb{T}^{2}_{\mathbb{C}})^{*}\) in the form

\[ \hat{l} = (u_{y}^{2} - \lambda - 1) \lambda^{-1} + u_{y}^{2} \lambda(\lambda - 1)^{-1} \]
where
\[(3.21)\quad \nabla h^{(y)}(\tilde{l}) := -\frac{1}{\lambda} \nabla \gamma^{(1)}(\tilde{l})|_\gamma = \frac{1}{\lambda \partial x},\]
\[\nabla h^{(t)}(\tilde{l}) := -\frac{1}{\lambda} \nabla \gamma^{(2)}(\tilde{l})|_\gamma = \frac{1}{\lambda \partial x},\]
leads to the heavenly type equation
\[u_{yt} + u_{xt}u_y - u_{xy}u_t = 0.\]
Its Lax-Sato representation is the compatibility condition for the first order partial differential equations
\[(3.22)\quad \frac{\partial \psi}{\partial y} - \frac{u_y}{\lambda - 1} \frac{\partial \psi}{\partial x} = 0,\]
\[\frac{\partial \psi}{\partial t} - \frac{u_t}{\lambda} \frac{\partial \psi}{\partial x} = 0,\]
where \(\psi \in C^2(\mathbb{R}^2 \times T^1_\mathbb{C}; \mathbb{C}).\)

### 3.5. Second conformal structure generating equation

For a seed element \(\tilde{l} \in \tilde{G}^* = \text{diff}(T^1_\mathbb{C})^*)\) in the form
\[(3.23)\quad \tilde{l} = [u_x^2 + 2u^2_y(u_y + \alpha)\lambda^{-1} + u^2_y(3u^2_y + 4\alpha u_y + \beta)\lambda^{-2}]dx,\]
where \(u \in C^2(T^1 \times \mathbb{R}^2; \mathbb{R}), \ x \in T^1, \ \lambda \in \mathbb{C} \setminus \{0\}, \text{and} \ \alpha, \beta \in \mathbb{R}, \) there is one independent Casimir functional \(\gamma^{(1)} \in I(\tilde{G}^*)\) with the following asymptotic as \(|\lambda| \to 0\) expansion of its functional gradient:
\[\nabla \gamma^{(1)}(l) \sim c_0 u_x^{-1} + (-c_0 u_y + c_1) u_x^{-1} \lambda + (-c_1 u_y + c_2) u_x^{-1} \lambda^2 + O(\lambda^3),\]
where \(c_r \in \mathbb{R}, \ r = 1, 2.\) If one assumes that \(c_0 = 1, \ c_1 = 0\) and \(c_2 = 0,\) then we obtain two functionally independent gradient elements
\[(3.24)\quad \nabla h^{(y)}(\tilde{l}) := -\nabla \gamma^{(1)}(\tilde{l})|_\gamma = \frac{1}{\lambda u_x} \frac{\partial}{\partial x},\]
\[\nabla h^{(t)}(\tilde{l}) := -\nabla \gamma^{(2)}(\tilde{l})|_\gamma = \left(\frac{1}{\lambda^2 u_x} - \frac{u_y}{\lambda u_x}\right) \frac{\partial}{\partial x}.\]
The corresponding commutativity condition \((3.19)\) of the vector fields \((3.20)\) give rise to the following heavenly type equation:
\[(3.25)\quad u_{xt} + u_x u_{yy} - u_y u_{xy} = 0,\]
whose linearized Lax-Sato representation is given by the first order system
\[(3.26)\quad \frac{\partial \psi}{\partial y} - \frac{1}{\lambda u_x} \frac{\partial \psi}{\partial x} = 0,\]
\[\frac{\partial \psi}{\partial t} + \left(\frac{1}{\lambda^2 u_x} - \frac{u_y}{\lambda u_x}\right) \frac{\partial \psi}{\partial x} = 0\]
of linear vector field equations on a function \(\psi \in C^2(\mathbb{R}^2 \times T^1_\mathbb{C}; \mathbb{R}).\)

### 3.6. Inverse first Shabat reduction heavenly equation

A seed element \(\tilde{l} \in \tilde{G}^* = \text{diff}(T^1_\mathbb{C})^*)\) in the form
\[(3.27)\quad \tilde{l} = (a_0 u_y^2 u_x^2 \lambda + 1)^{-1} + a_1 u_x^2 + a_1 u_x^2 \lambda)dx,\]
where \(u \in C^2(T^1 \times \mathbb{R}^2; \mathbb{R}), \ x \in T^1, \ \lambda \in \mathbb{C} \setminus \{-1\}, \text{and} \ a_0, a_1 \in \mathbb{R}, \) generates two independent Casimir functionals \(\gamma^{(1)}\) and \(\gamma^{(2)} \in I(\tilde{G}^*),\) whose gradients have the following asymptotic expansions:
\[(3.28)\quad \nabla \gamma^{(1)}(l) \sim u_y u_x^{-1} - u_y u_x^{-1} \mu + O(\mu^2),\]
as \(|\mu| \to 0, \ \mu := \lambda + 1,\) and
\[(3.29)\quad \nabla \gamma^{(2)}(l) \sim u_x^{-1} + O(\lambda^{-2}),\]
as $|\lambda| \to \infty$. If we put, by definition,

\begin{align}
\nabla h^{(\nu)}(\tilde{l}) := (\mu^{-1} \nabla \gamma^{(1)}(\tilde{l}))|_{-} &= -\frac{\lambda}{\lambda + 1} \frac{u_y}{u_x} \frac{\partial}{\partial x}, \\
\nabla h^{(l)}(\tilde{l}) := (\lambda \nabla \gamma^{(2)}(\tilde{l}))|_{+} &= \frac{\lambda}{u_x} \frac{\partial}{\partial x},
\end{align}

the commutativity condition (3.19) of the vector fields (3.20) leads to the heavenly equation

\begin{equation}
uxy + uy utx - u_{tx} ux = 0,
\end{equation}

which can be obtained as a result of the simultaneous changing of independent variables $\mathbb{R} \ni x \to t \in \mathbb{R}$, $\mathbb{R} \ni y \to x \in \mathbb{R}$ and $\mathbb{R} \ni t \to y \in \mathbb{R}$ in the first Shabat reduction heavenly equation. The corresponding Lax-Sato representation is given by the compatibility condition for the first order vector field equations

\begin{align}
\frac{\partial \psi}{\partial y} - \frac{\lambda}{\lambda + 1} \frac{u_y}{u_x} \frac{\partial \psi}{\partial x} = 0, \\
\frac{\partial \psi}{\partial t} + \frac{\lambda}{u_x} \frac{\partial \psi}{\partial x} = 0,
\end{align}

where $\psi \in C^2(\mathbb{R}^2 \times T_C^1; \mathbb{R})$.

### 3.7. First Plebański heavenly equation and its generalizations.

The seed element $\tilde{l} \in \mathcal{G}^* = \text{diff}(T_C^2)^*$ in the form

\begin{equation}
\tilde{l} = \lambda^{-1}(uyx_1, uyx_2) = \lambda^{-1}du_y,
\end{equation}

where $u \in C^2(T^2 \times \mathbb{R}^2; \mathbb{R})$, $(x_1, x_2) \in T^2$, $\lambda \in \mathbb{C}\{0\}$ and ”$d$” designates a full differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathcal{I}(\mathcal{G}^*)$, whose gradients have the following asymptotic expansions:

\begin{align}
\nabla \gamma^{(1)}(l) &\sim (-uyx_2, uyx_1)^T + O(\lambda), \\
\nabla \gamma^{(2)}(l) &\sim (-u_{tx_2}, u_{tx_1})^T + O(\lambda),
\end{align}

as $|\lambda| \to 0$. The commutativity condition (3.19) of the vector fields (3.20), where

\begin{align}
\nabla h^{(\nu)}(\tilde{l}) := (\lambda^{-1} \nabla \gamma^{(1)}(\tilde{l}))|_{-} &= -\frac{uyx_2}{\lambda} \frac{\partial}{\partial x_1} + \frac{uyx_1}{\lambda} \frac{\partial}{\partial x_2}, \\
\nabla h^{(l)}(\tilde{l}) := (\lambda \nabla \gamma^{(2)}(\tilde{l}))|_{+} &= -\frac{u_{tx_2}}{\lambda} \frac{\partial}{\partial x_1} + \frac{u_{tx_1}}{\lambda} \frac{\partial}{\partial x_2},
\end{align}

leads to the first Plebański heavenly equation [4]:

\begin{equation}
uyx_1 u_{tx_2} - uyx_2 u_{tx_1} = 1.
\end{equation}

Its Lax-Sato representation entails the compatibility condition for the first order partial differential equations

\begin{align}
\frac{\partial \psi}{\partial y} - \frac{uyx_2}{\lambda} \frac{\partial \psi}{\partial x_1} + \frac{uyx_1}{\lambda} \frac{\partial \psi}{\partial x_2} = 0, \\
\frac{\partial \psi}{\partial t} - \frac{u_{tx_2}}{\lambda} \frac{\partial \psi}{\partial x_1} + \frac{u_{tx_1}}{\lambda} \frac{\partial \psi}{\partial x_2} = 0,
\end{align}

where $\psi \in C^\infty(\mathbb{R}^2 \times T_C^1; \mathbb{C})$.

**Remark 3.1.** Taking into account that the determining condition for Casimir invariants is symmetric and equivalent to the system of nonhomogeneous linear first order partial differential equations for the covector function $l = (l_1, l_2)^T$, the corresponding seed element can be also chosen in another forms. Moreover, the form (3.33) is invariant subject to the spatial dimension of the underlying torus $T^n$, what makes it possible to describe the related generalized conformal metric equations for arbitrary dimension.
In particular, one easily observes that the asymptotic expansions are also true for such invariant seed elements as
\[ \hat{l} = \lambda^{-1} du_l, \]
and
\[ \hat{l} = \lambda^{-1} (du_y + du_l). \]

The above described Lie-algebraic scheme can be easily generalized for any dimension \( n = 2k \), where \( k \in \mathbb{N} \), and \( n > 2 \). In this case one has \( 2k \) independent Casimir functionals \( \gamma^{(j)} \in \mathcal{I(G^*)} \), where \( \mathcal{G^*} = \mathcal{Diff}(T^2)^* \), \( j = 1, 2k \), with the following asymptotic expansions for their gradients:
\[
\nabla \gamma^{(1)}(l) \sim (-u_{yx_2}, u_{yx_1}, 0, \ldots, 0)^\top + O(\lambda), \\
\nabla \gamma^{(2)}(l) \sim (-u_{tx_2}, u_{tx_1}, 0, \ldots, 0)^\top + O(\lambda), \\
\nabla \gamma^{(3)}(l) \sim (0, 0, -u_{yx_3}, u_{yx_3}, 0, \ldots, 0)^\top + O(\lambda), \\
\nabla \gamma^{(4)}(l) \sim (0, 0, -u_{tx_3}, u_{tx_3}, 0, \ldots, 0)^\top + O(\lambda), \\
\ldots , \\
\nabla \gamma^{(2k-1)}(l) \sim (0, \ldots, 0, -u_{yx_{2k}}, u_{yx_{2k-1}})^\top + O(\lambda), \\
\nabla \gamma^{(2k)}(l) \sim (0, \ldots, 0, -u_{tx_{2k}}, u_{tx_{2k-1}})^\top + O(\lambda).
\]

If we put
\[
\nabla h^{(y)}(\hat{l}) := (\lambda^{-1}(\nabla \gamma^{(1)}(\hat{l}) + \ldots + \nabla \gamma^{(2k-1)}(\hat{l}))) = -
\]
\[
= - \sum_{m=1}^{k} \left( \frac{u_{yx_{2m}}}{\lambda} \frac{\partial}{\partial x_{2m-1}} - \frac{u_{yx_{2m-1}}}{\lambda} \frac{\partial}{\partial x_{2m}} \right),
\]
\[
\nabla h^{(x)}(\hat{l}) := (\lambda^{-1}(\nabla \gamma^{(2)}(\hat{l}) + \ldots + \nabla \gamma^{(2k)}(\hat{l}))) = -
\]
\[
= - \sum_{m=1}^{k} \left( \frac{u_{tx_{2m}}}{\lambda} \frac{\partial}{\partial x_{2m-1}} - \frac{u_{tx_{2m-1}}}{\lambda} \frac{\partial}{\partial x_{2m}} \right),
\]
the commutativity condition (3.19) of the vector fields (3.20) leads to the following multi-dimensional analogs of the first Plebański heavenly equation:
\[
\sum_{m=1}^{k} (u_{yx_{2m-1}} u_{tx_{2m}} - u_{yx_{2m}} u_{tx_{2m-1}}) = 1.
\]

### 3.8. Modified Plebański heavenly equation and its generalizations

For the seed element \( \hat{l} \in \mathcal{G^*} = \mathcal{Diff}(T^2)^* \) in the form
\[
\hat{l} = (\lambda^{-1} u_{x_1 y} + u_{x_1 x_1} - u_{x_1 x_2} + \lambda) dx_1 + \\
+ (\lambda^{-1} u_{x_2 y} + u_{x_2 x_1} - u_{x_2 x_2} + \lambda) dx_2 = \\
= d(\lambda^{-1} u_y + u_x - u_x + \lambda x_1 + \lambda x_2),
\]
where \( d\lambda = 0, u \in C^2(T^2 \times \mathbb{R}^2; \mathbb{R}) \), \( (x_1, x_2) \in T^2 \), \( \lambda \in \mathbb{C} \setminus \{0\} \), there exist two independent Casimir functionals \( \gamma^{(1)} \) and \( \gamma^{(2)} \in \mathcal{I(G^*)} \) with the following gradient asymptotic expansions:
\[
\nabla \gamma^{(1)}(l) \sim (u_{yx_2}, -u_{yx_1})^\top + O(\lambda),
\]
as \( |\lambda| \to 0 \), and
\[
\nabla \gamma^{(2)}(l) \sim (0, -1)^\top + (-u_{x_2 x_2}, u_{x_1 x_2})^\top \lambda^{-1} + O(\lambda^{-2}),
\]
as $|\lambda| \to \infty$. In the case, when
\[
\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1} \nabla \gamma^{(1)}(\tilde{l}))|_{-} = \frac{u_{yx_2}}{\lambda} \frac{\partial}{\partial x_1} - \frac{u_{yx_1}}{\lambda} \frac{\partial}{\partial x_2},
\]
\[
\nabla h^{(t)}(\tilde{l}) := (\lambda \nabla \gamma^{(2)}(\tilde{l}))|_{+} = -u_{x_2x_2} \frac{\partial}{\partial x_1} + (u_{x_1x_2} - \lambda) \frac{\partial}{\partial x_2},
\]
the commutativity condition (3.19) of the vector fields (3.20) leads to the modified Plebański heavenly equation [4]:
\[
(3.38) \quad u_{yt} - u_{y_2} u_{x_2x_2} + u_{yx_2} u_{x_1x_2} = 0,
\]
with the Lax-Sato representation given by the first order partial differential equations
\[
\begin{align*}
\frac{\partial \psi}{\partial y} - & \frac{u_{yx_2}}{\lambda} \frac{\partial \psi}{\partial x_1} + \frac{u_{yx_1}}{\lambda} \frac{\partial \psi}{\partial x_2} = 0, \\
\frac{\partial \psi}{\partial t} - & \frac{u_{x_2x_2}}{\lambda} \frac{\partial \psi}{\partial x_1} + (u_{x_1x_2} - \lambda) \frac{\partial \psi}{\partial x_2} = 0
\end{align*}
\]
for functions $\psi \in C^2(\mathbb{R}^2 \times T^2; \mathbb{C})$.

Remark 3.2. The differential-geometric form of the seed element (3.37) is also dimension invariant subject to additional spatial variables of the torus $T^n$, $n > 2$, what poses a natural question of finding the corresponding multi-dimensional generalizations of the modified Plebański heavenly equation (3.38).

3.9. Husain heavenly equation. A seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \text{diff}(T^2)^*$ in the form
\[
(3.39) \quad \tilde{l} = \frac{d(u_y + i u_1)}{\lambda - i} + \frac{d(u_y - i u_1)}{\lambda + i} = \frac{2(\lambda d u_y - d u_1)}{\lambda^2 + 1},
\]
where $i^2 = -1$, $d \lambda = 0$, $u \in C^2(T^2 \times \mathbb{R}^2; \mathbb{R})$, $(x_1, x_2) \in T^2$, $\lambda \in \mathbb{C} \setminus \{-i, i\}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathcal{I}(\tilde{\mathcal{G}}^*)$, with the following gradient asymptotic expansions:
\[
\nabla \gamma^{(1)}(\tilde{l}) \sim \frac{1}{2}(u_{x_2} - i u_{tx_2}, u_{yx_1} + i u_{tx_1})^\top + O(\mu), \quad \mu := \lambda - i,
\]
as $|\mu| \to 0$, and
\[
\nabla \gamma^{(2)}(\tilde{l}) \sim \frac{1}{2}(u_{x_2} + i u_{tx_2}, u_{yx_1} - i u_{tx_1})^\top + O(\xi), \quad \xi := \lambda + i,
\]
as $|\xi| \to 0$. In the case, when
\[
\nabla h^{(y)}(\tilde{l}) := (\mu^{-1} \nabla \gamma^{(1)}(\tilde{l}) + \xi^{-1} \nabla \gamma^{(2)}(\tilde{l}))|_{-} = \\
= \frac{1}{2\mu} \left( -u_{yx_2} - i u_{tx_2} \right) \frac{\partial}{\partial x_1} + \left( u_{yx_1} + i u_{tx_1} \right) \frac{\partial}{\partial x_2} + \\
+ \frac{1}{2\xi} \left( u_{yx_2} + i u_{tx_2} \right) \frac{\partial}{\partial x_1} + \left( u_{yx_1} - i u_{tx_1} \right) \frac{\partial}{\partial x_2} = \\
= \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2 + 1} \frac{\partial}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2 + 1} \frac{\partial}{\partial x_2},
\]
\[
\nabla h^{(t)}(\tilde{l}) := (-\mu^{-1} \nabla \gamma^{(1)}(\tilde{l}) + \xi^{-1} \nabla \gamma^{(2)}(\tilde{l}))|_{-} = \\
= \frac{1}{2\mu} \left( -u_{tx_2} + i u_{yx_2} \right) \frac{\partial}{\partial x_1} + \left( u_{tx_1} - i u_{yx_1} \right) \frac{\partial}{\partial x_2} + \\
+ \frac{1}{2\xi} \left( u_{tx_2} + i u_{yx_2} \right) \frac{\partial}{\partial x_1} + \left( u_{tx_1} + i u_{yx_1} \right) \frac{\partial}{\partial x_2} = \\
= \frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2 + 1} \frac{\partial}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2 + 1} \frac{\partial}{\partial x_2},
\]
the commutativity condition (3.19) of the vector fields (3.20) leads to the Husain heavenly equation [4]:
\[
(3.40) \quad u_{yy} + u_{tt} + u_{yx_1} u_{tx_2} - u_{yx_2} u_{tx_1} = 0,
\]
with the Lax-Sato representation given by the first order partial differential equations
\[
\frac{\partial \psi}{\partial y} + \frac{u_{txz}}{\lambda} \frac{\partial \psi}{\partial x_2} + \frac{\lambda u_{x_2} - u_{txz}}{\lambda^2 + 1} \frac{\partial \psi}{\partial x_1} = 0,
\]
\[
\frac{\partial \psi}{\partial t} - \frac{u_{x_2} + \lambda u_{txz}}{\lambda^2 + 1} \frac{\partial \psi}{\partial x_1} + \frac{u_{x_1} + \lambda u_{txz}}{\lambda^2 + 1} \frac{\partial \psi}{\partial x_2} = 0,
\]
where \( \psi \in C^2(\mathbb{R}^2 \times T^2_n; \mathbb{C}) \).

Remark 3.3. The differential-geometric form of the seed element (3.39) is also dimension invariant subject to additional spatial variables of the torus \( T^4 \), \( n > 2 \), what poses a natural question of finding the corresponding multi-dimensional generalizations of the Husain heavenly equation (3.40).

3.10. The general Monge heavenly equation and its generalizations. A seed element \( \tilde{l} \in \tilde{G}^* = \text{diff}(T_n^4)^* \), taken in the form
\[
(3.41) \quad \tilde{l} = du_y + \lambda^{-1}(dx_1 + dx_2),
\]
where \( u \in C^2(T^4 \times \mathbb{R}^2; \mathbb{R}) \), \( (x_1, x_2, x_3, x_4) \in T^4 \), \( \lambda \in \mathbb{C}\setminus\{0\} \), generates four independent Casimir functionals \( \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \) and \( \gamma^{(4)} \), whose gradients have the following asymptotic expansions:
\[
(3.42) \quad \nabla \gamma^{(1)}(\tilde{l}) \sim (0, 1, 0, 0)^T + \nabla \gamma^{(2)}(\tilde{l}) \sim (1, 0, 0, 0)^T + \nabla \gamma^{(3)}(\tilde{l}) \sim (0, 0, -u_{x_4}, u_{x_3})^T + O(\lambda^2),
\]
\[
\nabla \gamma^{(4)}(\tilde{l}) \sim (0, 0, -u_{x_4}, u_{x_3})^T + (u_{x_3} u_{x_4} - u_{x_4} u_{x_3}, 0, 0)^T \lambda + O(\lambda^2),
\]
as \( |\lambda| \to 0 \). In the case, when
\[
(3.43) \quad \nabla h^{(\psi)}(\tilde{l}) := (\lambda^{-1}(\nabla \gamma^{(1)}(\tilde{l}) + \nabla \gamma^{(3)}(\tilde{l})) |_{\lambda = 0},
\]
\[
= \frac{\partial}{\partial x_1} + \frac{\partial}{\lambda \partial x_2} - \frac{u_{x_4}}{\lambda} \frac{\partial}{\partial x_3} + \frac{u_{x_3}}{\lambda} \frac{\partial}{\partial x_4},
\]
\[
\nabla h^{(l)}(\tilde{l}) := (\lambda^{-1}(\nabla \gamma^{(2)}(\tilde{l}) + \nabla \gamma^{(4)}(\tilde{l})) |_{\lambda = 0},
\]
\[
= -\frac{1}{\lambda} \frac{\partial}{\partial x_1} + 0 \frac{\partial}{\partial x_2} - \frac{u_{x_4}}{\lambda} \frac{\partial}{\partial x_3} + \frac{u_{x_3}}{\lambda} \frac{\partial}{\partial x_4},
\]
the commutability condition (3.19) of the vector fields (3.20) leads to the general Monge heavenly equation (3.20):
\[
(3.44) \quad u_{x_1} + u_{txz} + u_{x_3} u_{tx} - u_{x_4} u_{txz} = 0,
\]
with the Lax-Sato representation given by the first order partial differential equations
\[
\frac{\partial \psi}{\partial y} + \frac{1}{\lambda} \frac{\partial \psi}{\partial x_2} - \frac{u_{x_4}}{\lambda} \frac{\partial \psi}{\partial x_3} + \frac{u_{x_3}}{\lambda} \frac{\partial \psi}{\partial x_4} = 0,
\]
\[
\frac{\partial \psi}{\partial t} - \frac{1}{\lambda} \frac{\partial \psi}{\partial x_1} - \frac{u_{x_4}}{\lambda} \frac{\partial \psi}{\partial x_3} + \frac{u_{x_3}}{\lambda} \frac{\partial \psi}{\partial x_4} = 0,
\]
where \( \psi \in C^2(\mathbb{R}^2 \times T^2_n; \mathbb{R}) \) and \( \lambda \in \mathbb{C}\setminus\{0\} \).

Remark 3.4. Taking into account that the condition for Casimir invariants is equivalent to a system of homogeneous linear first order partial differential equations for a covector function \( l = (l_1, l_2, l_3, l_4)^T \), the corresponding seed element can be chosen in different forms.

If the expression
\[
\tilde{l} = du_t + \lambda^{-1}(dx_1 + dx_2)
\]
is considered as a seed element, one obtains that it generates four independent Casimir functionals \( \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \) and \( \gamma^{(4)} \in I(\hat{G}^*) \), whose gradients have the following asymptotic expansions:

\[
\nabla \gamma^{(1)}(l) \sim (0, 1, 0, 0)^\top +
\n\begin{pmatrix}
-u_{tx_2} - (\partial_{x_2} - \partial_{x_1})^{-1} u_{tx_2 x_1},
(\partial_{x_2} - \partial_{x_1})^{-1} u_{tx_2 x_1},
0, 0
\end{pmatrix}^\top \lambda + O(\lambda^2),
\]

\[
\nabla \gamma^{(2)}(l) \sim (1, 0, 0, 0)^\top +
\n\begin{pmatrix}
(\partial_{x_1} - \partial_{x_2})^{-1} u_{tx_1 x_2},
-u_{tx_1} - (\partial_{x_2} - \partial_{x_1})^{-1} u_{tx_1 x_2},
0, 0
\end{pmatrix}^\top \lambda + O(\lambda^2),
\]

\[
\nabla \gamma^{(3)}(l) \sim (0, 0, -u_{tx_4}, u_{tx_3})^\top + (0, u_{tx_3} u_{yx_4} - u_{tx_4} u_{yx_3}),
\]

\[
\nabla \gamma^{(4)}(l) \sim (0, 0, -u_{yx_4}, u_{yx_3})^\top + O(\lambda^2),
\]

as \( |\lambda| \to 0 \). If a seed element has the form

\[
\tilde{l} = du_y + du_t + \lambda^{-1} (dx_1 + dx_2),
\]

the asymptotic expansions for gradients of four independent Casimir functionals \( \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \) and \( \gamma^{(4)} \in I(\hat{G}^*) \) are written as

\[
\nabla \gamma^{(1)}(l) \sim (0, 1, 0, 0)^\top +
\n\begin{pmatrix}
-u_{yx_2} + u_{tx_2}
\end{pmatrix}^\top \lambda + O(\lambda^2),
\]

\[
\nabla \gamma^{(2)}(l) \sim (1, 0, 0, 0)^\top +
\n\begin{pmatrix}
(\partial_{x_1} - \partial_{x_2})^{-1} u_{yx_1 x_2},
-u_{yx_1} - (\partial_{x_2} - \partial_{x_1})^{-1} u_{yx_1 x_2},
0, 0
\end{pmatrix}^\top \lambda + O(\lambda^2),
\]

\[
\nabla \gamma^{(3)}(l) \sim (0, 0, -u_{yx_4}, u_{yx_3})^\top + (0, u_{yx_3} u_{tx_4} - u_{tx_4} u_{yx_3}),
\]

\[
\nabla \gamma^{(4)}(l) \sim (0, 0, -u_{yx_4}, u_{yx_3})^\top + O(\lambda^2),
\]

as \( |\lambda| \to 0 \).

The above described scheme is generalized for all \( n = 2k \), where \( k \in \mathbb{N} \), and \( n > 2 \). In this case one has \( 2k \) independent Casimir functionals \( \gamma^{(j)} \in I(\hat{G}^*), \) where \( \hat{G}^* = df \hat{f}(T^2) \), \( j = 1, 2k \), whose gradient asymptotic expansions are equal to the following expressions:

\[
\nabla \gamma^{(1)}(l) \sim (0, 1, 0, \ldots, 0)^\top +
\begin{pmatrix}
-u_{yx_2} + u_{tx_2}
\end{pmatrix}^\top \lambda + O(\lambda^2),
\]

\[
\nabla \gamma^{(2)}(l) \sim (1, 0, 0, \ldots, 0)^\top +
\begin{pmatrix}
(\partial_{x_1} - \partial_{x_2})^{-1} u_{yx_1 x_2},
-u_{yx_1} - (\partial_{x_2} - \partial_{x_1})^{-1} u_{yx_1 x_2},
0, 0
\end{pmatrix}^\top \lambda + O(\lambda^2),
\]
\[ \nabla \gamma^{(3)}(l) \sim (0, 0, -u_{yx4}, u_{yx3}, 0, \ldots, 0)^\top + (0, u_{tx3}u_{yx4} - u_{tx4}u_{yx3}, 0) \]

\[ u_{tx4}u_{yx2} - u_{tx2}u_{yx4}, u_{tx2}u_{yx3} - u_{tx3}u_{yx2}, 0, \ldots, 0)^\top \lambda + O(\lambda^2), \]

\[ \nabla \gamma^{(4)}(l) \sim (0, 0, -u_{tx4}, u_{tx3}, 0, \ldots, 0)^\top + (u_{yx3}u_{tx4} - u_{yx4}u_{tx3}, 0) \]

\[ u_{yx4}u_{tx1} - u_{yx1}u_{tx4}, u_{yx1}u_{tx3} - u_{yx3}u_{tx1}, 0, \ldots, 0)^\top \lambda + O(\lambda^2), \]

\[ \nabla \gamma^{(2k-1)}(l) \sim (0, \ldots, 0, 0, -u_{yx2k}, u_{yx2k-1})^\top + \]

\[ (0, \ldots, 0, 0, u_{tx2k-1}u_{yx2k} - u_{tx2k}u_{yx2k-1}, 0) \]

\[ u_{tx2k}u_{txk} - u_{txk}u_{tx2k}, u_{yx2k}u_{tx2k-1} - u_{tx2k}u_{yx2k-1} \lambda + O(\lambda^2), \]

\[ u_{tx2k}u_{txk} - u_{txk}u_{tx2k}, u_{yx2k}u_{tx2k-1} - u_{tx2k}u_{yx2k-1}, 0) \lambda + O(\lambda^2), \]

when a seed element \( \tilde{l} \in \tilde{G} \) is chosen as in (3.45). If

\[ \nabla \gamma^{(y)}(\tilde{l}) := (\lambda^{-1}(\nabla \gamma^{(1)}(\tilde{l}) + \nabla \gamma^{(3)}(\tilde{l}) + \ldots + \nabla \gamma^{(2k-1)}(\tilde{l}))) \right|_{\tilde{l}} = \]

\[ = 0 \frac{\partial}{\partial x_1} + \frac{1}{\lambda} \frac{\partial}{\partial x_2} - \sum_{m=2}^{k} \left( \frac{u_{yx2m}}{\lambda} \frac{\partial}{\partial x_{2m-1}} - \frac{u_{yx2m-1}}{\lambda} \frac{\partial}{\partial x_{2m}} \right), \]

\[ \nabla h^{(1)}(\tilde{l}) := (\lambda^{-1}(-\nabla \gamma^{(2)}(\tilde{l}) + \nabla \gamma^{(4)}(\tilde{l}) + \ldots + \nabla \gamma^{(2k)}(\tilde{l}))) \right|_{\tilde{l}} = \]

\[ = -\frac{1}{\lambda} \frac{\partial}{\partial x_1} + \frac{1}{\lambda} \frac{\partial}{\partial x_2} - \sum_{m=2}^{k} \left( \frac{u_{tx2m}}{\lambda} \frac{\partial}{\partial x_{2m-1}} - \frac{u_{tx2m-1}}{\lambda} \frac{\partial}{\partial x_{2m}} \right), \]

the commutability condition (3.19) of the vector fields (3.20) leads to the following multi-dimensional analogs of the general Monge heavenly equation:

\[ u_{yx1} + u_{tx2} + \sum_{j=2}^{k} (u_{yx2m-1}u_{tx2m} - u_{yx2m}u_{tx2m-1}) = 0. \]

4. CONCLUSION

We succeeded in applying the Lie-algebraic approach to studying vector fields on the complexified \( n \)-dimensional torus and the related Lie-algebraic structures, which made it possible to construct a wide class of multi-dimensional dispersionless integrable systems, describing conformal structure generating equations of modern mathematical physics. There was described a modification of the approach subject to the spatial dimensional invariance and meromorphy of the related differential-geometric structures, giving rise to new generalized multi-dimensional conformal metric equations. There have been analyzed in detail the related differential-geometric structures of the Einstein–Weyl conformal metric equation, the modified Einstein–Weyl metric equation, the Dunajski heavenly equation system, the first and second conformal structure generating equations, the inverse first Shabat reduction heavenly equation, the first and its multi-dimensional generalizations, the modified Plebański and Husain heavenly equations, the general Monge equation and its multi-dimensional generalizations.
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References

[1] D. Blackmore, A.K. Prykarpatsky and V.H. Samoylenko, Nonlinear dynamical systems of mathematical physics, World Scientific Publisher, NJ, USA, 2011
[2] M. Blaszak, Classical R-matrices on Poisson algebras and related dispersionless systems, Lett. A 297(3-4) (2002) 191–195
[3] L.V. Bogdanov, V.S. Dryuma, S.V. Manakov, Dunajski generalization of the second heavenly equation: dressing method and the hierarchy, J. Phys. A: Math. Theor. 40 (2007), 14383-14393
[4] B. Doubrov, E.V. Ferapontov, On the integrability of symplectic Monge-Ampère equations, J. Geom. Phys., 60 (2010), 10, 1604-1616, arXiv:0910.3407v2 [math.DG] 13 Apr 2010
[5] M. Dunajski, Anti-self-dual four-manifolds with a parallel real spinor, Proc. Roy. Soc. A. 458 (2002), 1205
[6] M. Dunajski, L.J. Mason, P. Tod, Einstein–Weyl geometry, the dKP equation and twistor theory, J. Geom. Phys. 37 (2001), no.1-2, 63-93
[7] L.A. Takhtajan, L.D. Faddeev, Hamiltonian Approach in Soliton Theory, Springer, Berlin-Heidelberg, 1987
[8] E.Ferapontov and B.S. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, J. Differential Geometry, 97 (2014) 215-254
[9] O.E. Hentosh, Y.A. Prykarpatsky, D. Blackmore and A.K. Prykarpatski, Lie-algebraic structure of Lax–ato integrable heavenly equations and the Lagrange-d’Alembert principle , Journal of Geometry and Physics 120 (2017) 208–227
[10] S.V. Manakov, P.M. Santini, On the solutions of the second heavenly and Pavlov equations, J. Phys. A: Math. Theor. 42 (2009), 404013 (11pp)
[11] V. Ovsienko, Bi-Hamilton nature of the equation $u_{tx} = u_{xy}u_y - u_{yy}u_x$, arXiv:0802.1818v1 [math-ph] 13 Feb 2008
[12] V. Ovsienko, C. Roger, Looped Cotangent Virasoro Algebra and Non-Linear Integrable Systems in Dimension 2 + 1, Commun. Math. Phys. 273 (2007), 357–378
[13] J.F. Plebański, Some solutions of complex Einstein equations, J. Math. Phys. 16 (1975), Issue 12, 2395-2402
[14] A. Pressley and G. Segal, Loop groups, Clarendon Press, London, 1986
[15] A.G. Reyman, M.A. Semenov-Tian-Shansky, Integrable Systems, The Computer Research Institute Publ., Moscow-Izhvek, 2003 (in Russian)
[16] A. Sergeyev and B. M. Szablikowski, Central extensions of cotangent universal hierarchy: (2+1)-dimensional bi-Hamiltonian systems, Phys. Lett. A, 372 (2008) 7016-7023
[17] B. Szablikowski, Hierarchies of Manakov–Santini Type by Means of Rota-Baxter and Other Identities, SIGMA 12 (2016), 022, 14 pp.
[18] M.B. Sheftel and D. Yazıcı, Bi-Hamiltonian representation, symmetries and integrals of mixed heavenly and Husain systems, arXiv:0904.3981v4 [math-ph] 4 May 2010
[19] M.B. Sheftel and D. Yazıcı, Evolutionary Hirota type (2+1)-dimensional equations: Lax pairs, recursion operators and bi-Hamiltonian structures, SIGMA 14 (2018), 017, 19 pp., arXiv:1712.01549v1 [math-ph] 5 Dec 2017
[20] B. Doubrov, E.V. Ferapontov, B. Kruglikov, V.S. Novikov, On a class of integrable systems of Monge-Ampère type, arXiv:1701.02270v1 [nlin.SI] 9 Jan 2017
[21] Manas M., Medina E., Martinez-Alonso L., On the Whitham hierarchy: dressing scheme, string equations and additional symmetries, J. Phys. A: Math. Gen. 39, 2349–2381, nlin.SI/0509017 (2006)
[22] Morozov O.I., A two-Component generalization of the integrable rd-Dym equation, SIGMA, 8, 051, 5 pp. (2012)
THE DISPERSIONLESS MULTI-DIMENSIONAL INTEGRABLE SYSTEMS AND RELATED CONFORMAL STRUCTURE GENERATING EQUATIONS OF MATHEMATICAL PHYSICS

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