Probabilistic cloning with supplementary information

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We consider probabilistic cloning of a state chosen from a mutually nonorthogonal set of pure states, with the help of a party holding supplementary information in the form of pure states. When the number of states is 2, we show that the best efficiency of producing $m$ copies is always achieved by a two-step protocol in which the helping party first attempts to produce $m-1$ copies from the supplementary state, and if it fails, then the original state is used to produce $m$ copies. On the other hand, when the number of states exceeds two, the best efficiency is not always achieved by such a protocol. We give examples in which the best efficiency is not achieved even if we allow any amount of one-way classical communication from the helping party.

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I. INTRODUCTION

The impossibility of deterministic cloning of nonorthogonal pure states is well known as the no-cloning theorem [1, 2]. The best one can do is to carry out weaker tasks, such as allowing the copies to be inaccurate [3–10] or allowing a failure to occur with a nonzero probability (probabilistic cloning) [11]. Another way to enable the cloning is to provide some hints in the form of a quantum state. Jozsa has considered [12] how much or what kind of supplementary information $\hat{\rho}_i$ is required to make two copies $|\psi_i\rangle|\psi_i\rangle$ from the original information $|\psi_i\rangle$. He has shown that for any mutually nonorthogonal set of original states $\{|\psi_i\rangle\}$, whenever two copies $|\psi_i\rangle|\psi_i\rangle$ are generated with the help of the supplementary information $\hat{\rho}_i$, the state $|\psi_i\rangle$ can be generated from the supplementary information $\hat{\rho}_i$ alone, independently of the original state, i.e.,

$$|\psi_i\rangle \otimes \hat{\rho}_i \xrightarrow{\text{CPTP}} |\psi_i\rangle|\psi_i\rangle \implies \hat{\rho}_i \xrightarrow{\text{CPTP}} |\psi_i\rangle,$$

where CPTP stands for a completely positive trace-preserving map, implying that the transformation can be done deterministically. This result, dubbed the stronger no-cloning theorem, implies that the supplementary information must be provided in the form of the result $|\psi_i\rangle$ itself, rather than a help, thereby obliterating the necessity of the cloning task itself.

An interesting question occurring here is whether we can find a similar property in the case of probabilistic cloning when we ask how much increase in the success probability is obtained with the help of supplementary information. Suppose that the success probability of cloning the $i$th state $|\psi_i\rangle$ without any help is $\gamma_i$. If we are directly given a right copy of state $|\psi_i\rangle$ with probability $\gamma'$, the success probability would increase to $\gamma'_i = q_i + (1-q_i)\gamma_i$. Hence the counterpart of the stronger no-cloning theorem in probabilistic cloning will be the implication

$$|\psi_i\rangle \otimes \hat{\rho}_i \xrightarrow{\gamma_i} |\psi_i\rangle|\psi_i\rangle \implies \hat{\rho}_i \xrightarrow{\gamma_i} |\psi_i\rangle|\psi_i\rangle$$

with $\gamma'_i = q_i + (1-q_i)\gamma_i$. In other words, it implies that the best usage of the supplementary information is to probabilistically create a copy $|\psi_i\rangle$ from it, independently of the original state.

If there are cases where the above implication is not true, it follows that the supplementary information can help directly the process of the cloning task in those cases. Then, the next question will be to ask what kind of interaction should occur between the supplementary information and the original information.

In this paper, we consider probabilistic cloning of mutually nonorthogonal pure states when supplementary information is given as a pure state. We prove that when the number of the possible original states is 2, the above implication is true, namely, the supplementary information only serves to provide a copy with a nonzero probability and it does not directly help the process of the cloning. On the other hand, when we have more than two states to choose from, the above implication is not always true. To see this, it is convenient to assume two parties, Alice and Bob, respectively holding the original information and the supplementary information. We give examples in which there is a gap between the efficiency when Bob only communicates to Alice with a one-way classical channel and the efficiency when they fully cooperate through a quantum channel.
This paper is organized as follows. In Sec. II we provide definitions and basic theorems used in later sections. We discuss the two-state problem in Sec. III and prove that the property similar to the stronger no-cloning theorem holds in this case. In Sec. IV we give examples with three or more states and show that there is a gap between the success probabilities in the scenarios with classical communication and quantum communication. Section V concludes the paper.

II. PROBABILISTIC TRANSFORMATION THEOREM

Throughout this paper, we consider a class of machines that conducts probabilistic transformation of input pure states into output pure states. We denote by \( \{ |\Phi_i\rangle \rightarrow |\Psi_i\rangle \}_{i=1,...,n} \) a machine having the following properties. (i) It receives a quantum state as an input, and returns a quantum state as an output, together with one bit of classical output indicating whether the transformation has been successful or not. (ii) When the input quantum state is \(|\Phi_i\rangle\), the transformation succeeds with probability \( \gamma_i \), and the successful output state is \(|\Psi_i\rangle\). Note that if the output states \(|\Psi_i\rangle\) form an orthonormal set, namely, \( \langle \Psi_i | \Psi_j \rangle = \delta_{ij} \), the machine carries out unambiguous discrimination of the set \(|\Phi_i\rangle\) with success probabilities \( \{ \gamma_i \} \).

A necessary and sufficient condition for the existence of a machine \( \{ |\Phi_i\rangle \rightarrow |\Psi_i\rangle \}_{i=1,...,n} \) is given by the following theorem.

**Theorem 1.** There exists a machine \( \{ |\Phi_i\rangle \rightarrow |\Psi_i\rangle \}_{i=1,...,n} \) if and only if there are normalized states \(|P^{(i)}\rangle\) \( (i = 1, \ldots, n) \) such that the matrix \( X = \sqrt{\Gamma Y} \sqrt{T} \) is positive semidefinite, where \( X := [\langle \Phi_i | \Phi_j \rangle] \), \( Y := [\langle \Psi_i | \Psi_j \rangle(\langle P^{(i)} | P^{(j)} \rangle)] \) and \( \Gamma := \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \) are \( n \times n \) matrices.

This theorem can be proved by a similar way as in the probabilistic cloning theorem by Duan and Guo [11]. A general description of a machine \( \{ |\Phi_i\rangle_A \rightarrow |\Psi_i\rangle_A \}_{i=1,...,n} \) for system \( A \) is given by a unitary operation \( \hat{U} \) acting on system \( A \) and an ancillary system \( E \), which is initially prepared in a state \( |\Sigma\rangle_E \), followed by a projection measurement on \( E \) to determine whether the transformation is successful or not. Hence the machine \( \{ |\Phi_i\rangle_A \rightarrow |\Psi_i\rangle_A \}_{i=1,...,n} \) exists iff there are a unitary \( \hat{U} \), normalized states \(|P^{(i)}\rangle_E\), and unnormalized states \(|\Omega_i\rangle_{AE}\) such that

\[
\hat{U}(|\Phi_i\rangle_A |\Sigma\rangle_E) = \sqrt{\gamma_i} |\Psi_i\rangle_A |P^{(i)}\rangle_E + |\Omega_i\rangle_{AE}
\]

and

\[
E\langle P^{(j)} | \Omega_i \rangle_{AE} = 0
\]

for all \( i, j \). Taking the inner products between the equations with different values of \( i \), we obtain

\[
X = \sqrt{\Gamma Y} \sqrt{T} + \Omega,
\]

where matrix \( \Omega := [\langle \Omega_i | \Omega_j \rangle] \) is positive semidefinite [11]. Hence it is necessary that \( X = \sqrt{\Gamma Y} \sqrt{T} \) be positive semidefinite. Conversely, if \( X = \sqrt{\Gamma Y} \sqrt{T} \) is positive semidefinite for a given set of \(|P^{(i)}\rangle_E\), there exist a unitary \( \hat{U} \) and unnormalized states \(|\Omega_i\rangle_{AE}\) satisfying Eqs. 3 and 4, as shown in Ref. 11. Theorem 1 is thus proved.

When the initial state is chosen from the set \(|\Phi_i\rangle\) with a priori probability \( p_i \), we may define the overall success probability \( \gamma_{\text{tot}} \) of a machine as

\[
\gamma_{\text{tot}} := \sum_i p_i \gamma_i.
\]

In this case, we can define the maximum success probability \( \gamma_{\text{tot max}} \) as

\[
\gamma_{\text{tot max}} := \max_{\{ \gamma_i \}} \sum_i p_i \gamma_i.
\]

where the maximum is taken over all combinations \( \{ \gamma_i \} \) for which there exists a machine \( \{ |\Phi_i\rangle \rightarrow |\Psi_i\rangle \}_{i=1, \ldots, n} \).

When the number of possible input states is 2, we can explicitly determine the achievable region \((\gamma_1, \gamma_2)\) from theorem 1.

**Corollary 1.** Let \( \eta_{\text{in}} := \langle \Phi_1 | \Phi_2 \rangle \) and \( \eta_{\text{out}} := \langle \Psi_1 | \Psi_2 \rangle \). There exists a machine \( \{ |\Phi_i\rangle \rightarrow |\Psi_i\rangle \}_{i=1,2} \) if and only if \( \gamma_1 \geq 0, \gamma_2 \geq 0 \), and

\[
\sqrt{(1 - \gamma_1)(1 - \gamma_2)} - \eta_{\text{in}} + \eta_{\text{out}} \sqrt{\gamma_1 \gamma_2} \geq 0.
\]

**Proof.** Without loss of generality, we may assume \( \eta_{\text{in}} = \langle \Phi_1 | \Phi_2 \rangle \) and \( \eta_{\text{out}} = \langle \Psi_1 | \Psi_2 \rangle \). Since \( \text{Tr}[X - \sqrt{\Gamma Y} \sqrt{T}] \geq 0 \), \( X - \sqrt{\Gamma Y} \sqrt{T} \) is positive semidefinite iff \( \text{det}[X - \sqrt{\Gamma Y} \sqrt{T}] \geq 0 \), or equivalently,

\[
\sqrt{(1 - \gamma_1)(1 - \gamma_2)} - \eta_{\text{in}} + \eta_{\text{out}} \sqrt{\gamma_1 \gamma_2} \geq 0.
\]

Since the left-hand side (LHS) of Eq. 9 is no larger than the LHS of Eq. 8, Eq. 8 is necessary for the existence of a machine. Conversely, whenever Eq. 8 holds and \( \eta_{\text{in}} \geq \eta_{\text{out}} \sqrt{\gamma_1 \gamma_2} \), we can satisfy Eq. 9 by choosing \( \langle P^{(1)} | P^{(2)} \rangle = 1 \) and there exists a machine. When \( \eta_{\text{in}} < \eta_{\text{out}} \sqrt{\gamma_1 \gamma_2} \), we can satisfy Eq. 9 by choosing \( \langle P^{(1)} | P^{(2)} \rangle = \eta_{\text{in}} / (\eta_{\text{out}} \sqrt{\gamma_1 \gamma_2}) \), and hence there exists a machine also in this case, proving the corollary.

When \( \eta_{\text{in}} > \eta_{\text{out}} \), the region \((\gamma_1, \gamma_2)\) determined by Eq. 8 is convex, and is bounded by the line \( \gamma_1 = 0 \), the line \( \gamma_2 = 0 \), and the curve specified by the equality in Eq. 8, which connects the points \((\gamma_1, \gamma_2) = (0, 1 - \eta_{\text{in}}^2) \) and \((\gamma_1, \gamma_2) = (1 - \eta_{\text{in}}^2, 0) \) through the point \( \gamma_1 = \gamma_2 = (1 - \eta_{\text{in}}) / (1 - \eta_{\text{out}}) \). When \( \eta_{\text{in}} \leq \eta_{\text{out}} \), \( \gamma_1 = \gamma_2 = 1 \) satisfies Eq. 8, namely, a deterministic machine \( \{ |\Phi_i\rangle \rightarrow |\Psi_i\rangle \}_{i=1,2} \) exists. Note that Eq. 8 still forbids regions of \((\gamma_1, \gamma_2)\) close to \((1, 0)\) and \((0, 1)\), reflecting the indistinguishability of the two input states.
III. PROBABILISTIC CLONING OF TWO STATES WITH SUPPLEMENTARY INFORMATION

In this section, we consider the case where one makes \( m \) copies of states \( \{ | \psi_1 \rangle, | \psi_2 \rangle \} \) with the help of supplementary information in the form of pure states \( \{ | \phi_1 \rangle, | \phi_2 \rangle \} \). We show that it is always better to try first the production of \( m - 1 \) copies of the original information from the supplementary information alone, independently of the original state, which is implied by the following theorem.

**Theorem 2.** If there exists a machine

\[
\{ | \psi_i \rangle | \phi_i \rangle \xrightarrow{\gamma_i} | \psi_i \rangle^{\otimes m} \}_{i=1,2},
\]

then there exist a machine

\[
\{ | \psi_i \rangle | \gamma_i^A \rightarrow | \psi_i \rangle^{\otimes m} \}_{i=1,2}
\]

and a machine

\[
\{ | \phi_i \rangle | \gamma_i^B \rightarrow | \psi_i \rangle^{\otimes m-1} \}_{i=1,2}
\]

with

\[
\gamma_i^B + (1 - \gamma_i^B)\gamma_i^A \geq \gamma_i \ (i = 1, 2).
\] (10)

Before the proof of this theorem, several remarks may be in order. If the original information is held by Alice, and the supplementary information by Bob, theorem 2 implies that the optimal performance is always achieved just by one-bit classical communication from Bob to Alice as follows: Bob, who possesses the supplementary state \( | \phi_i \rangle \), first runs the machine \( \{ | \phi_i \rangle \xrightarrow{\gamma_i^A} | \psi_i \rangle^{\otimes m} \} \), tells Alice whether the trial was successful or not. In the successful case, Alice just leaves her state \( | \psi_i \rangle \) as it is, and hence she obtain \( m \) copies in total. If Bob’s attempt has failed, Alice runs the machine \( \{ | \psi_i \rangle \xrightarrow{\gamma_i^A} | \psi_i \rangle^{\otimes m} \} \). The total success probability for input state \( | \psi_i \rangle | \phi_i \rangle \) in this protocol is given by \( \gamma_i^B + (1 - \gamma_i^B)\gamma_i^A \). Hence, by theorem 2, we see that the above protocol is as good as any other protocol in which Alice and Bob communicate through quantum channels. Note that when \( \{ | \psi_i \rangle \} \) includes no pair of identical states, \( \lim_{m \to \infty} \langle \psi_i | \psi_j \rangle^m = \delta_{ij} \) holds for any \( i \neq j \). Hence in the limit \( m \to \infty \) the machine \( \{ | \psi_i \rangle | \phi_i \rangle \xrightarrow{\gamma_i} | \psi_i \rangle^{\otimes m} \} \) effectively carries out unambiguous discrimination of the set \( \{ | \psi_i \rangle | \phi_i \rangle \} \). Therefore, in this limit theorem 2 reproduces the results in Ref. [13], namely, local operations and classical communication achieves the global optimality of unambiguous discrimination of any two pure product states with arbitrary a priori probability \( p_i \).

When the initial state \( | \psi_i \rangle | \phi_i \rangle \) is chosen with probability \( p_i \), it follows from theorem 2 that the maximum overall success probability \( \gamma_{\text{tot max}} \) is achieved by the above two-step protocol. For a special case of \( p_1 = p_2 = 1/2 \), we can directly confirm this as follows. The maximum overall success probability \( \gamma_{\text{tot max}} \) can easily be calculated by optimizing \( (\gamma_1 + \gamma_2)/2 \) over the region in corollary 1, and it is found to be

\[
\gamma_{\text{tot max}} = \frac{1 - |\alpha \beta|}{1 - |\alpha|^m}.
\] (11)

where \( \alpha := \langle \psi_1 | \psi_2 \rangle \) and \( \beta := \langle \phi_1 | \phi_2 \rangle \). Corollary 1 also shows the existence of a machine \( \{ | \phi_i \rangle \xrightarrow{\gamma_i^B} | \psi_i \rangle^{\otimes m-1} \}_{i=1,2} \) with

\[
\gamma_1^B = \gamma_2^B = \frac{1 - |\beta|}{1 - |\alpha|^m-1}
\] (12)

and a machine \( \{ | \psi_i \rangle \xrightarrow{\gamma_i^A} | \psi_i \rangle^{\otimes m} \}_{i=1,2} \) with

\[
\gamma_1^A = \gamma_2^A = \frac{1 - |\alpha|}{1 - |\alpha|^m}.
\] (13)

Hence, using these machines in the two-step protocol, we obtain an overall success probability

\[
\gamma_1^B + (1 - \gamma_1^B)\gamma_1^A = \frac{1 - |\alpha \beta|}{1 - |\alpha|^m},
\] (14)

which coincides with \( \gamma_{\text{tot max}} \). For cases with general \((p_1, p_2)\), it is even difficult to represent \( \gamma_{\text{tot max}} \) in an explicit form, but theorem 2 states that \( \gamma_{\text{tot max}} \) is always achieved by the above two-step protocol.

**Proof of Theorem 2.** When \( |\langle \phi_1 | \phi_2 \rangle| \leq |\langle \psi_1 | \psi_2 \rangle|^{m-1} \), from corollary 1, there exists a machine \( \{ | \phi_i \rangle \xrightarrow{\gamma_i^B} | \psi_i \rangle^{\otimes m-1} \}_{i=1,2} \) with \( \gamma_1^B = \gamma_2^B = 1 \), and theorem 2 obviously holds. We thus assume \( |\langle \phi_1 | \phi_2 \rangle| > |\langle \psi_1 | \psi_2 \rangle|^{m-1} \) in the following.

Let \( \mathcal{R} \) be the region of points \( (\gamma_1, \gamma_2) \) for which a machine \( \{ | \psi_i \rangle | \phi_i \rangle \xrightarrow{\gamma_i} | \psi_i \rangle^{\otimes m} \}_{i=1,2} \) exists. We first show that it suffices to prove theorem 2 for the cases where \( \gamma_1 \geq \gamma_2 \) and \( (\gamma_1, \gamma_2) \) is on a boundary of the achievable region \( \mathcal{R} \), namely (see corollary 1),

\[
\sqrt{(1 - \gamma_1)(1 - \gamma_2) - |\alpha \beta|} + |\alpha|^m \sqrt{\gamma_1 \gamma_2} = 0.
\] (15)

For any other point \( (\gamma_1', \gamma_2') \) in the region \( \mathcal{R} \) with \( \gamma_1' \geq \gamma_2' \), we can find a point \( (\gamma_1, \gamma_2) \) on the boundary with \( \gamma_1 \geq \gamma_2 \) satisfying \( \gamma_i' = y \gamma_i \ (i = 1, 2) \) with \( y \leq 1 \). If theorem 2 holds for \( (\gamma_1, \gamma_2) \), there are machines with \( \gamma_i \) and \( \gamma_i' \) satisfying \( \gamma_i^B + (1 - \gamma_i^B)\gamma_i^A \geq \gamma_i \) for \( i = 1, 2 \). This implies that theorem 2 also holds for \( (\gamma_1', \gamma_2') \). The cases \( \gamma_1 < \gamma_2 \) follow from the symmetry.

Consider a point \( (\gamma_1, \gamma_2) \) on the boundary and satisfying \( \gamma_1 \geq \gamma_2 \). Let \( x := \gamma_2/\gamma_1 \). From Eq. (15) we have

\[
\sqrt{(1 - \gamma_1)(1 - \gamma_2 x) - |\alpha \beta|} + |\alpha|^m \sqrt{x \gamma_1 \gamma_2} = 0.
\] (16)

For the machine \( \{ | \phi_i \rangle \xrightarrow{\gamma_i^B} | \psi_i \rangle^{\otimes m-1} \}_{i=1,2} \), we choose \((\gamma_1^B, \gamma_2^B)\) as the point satisfying \( \gamma_2^B = x \gamma_1^B \) and being on the boundary (for this machine), namely, satisfying

\[
\sqrt{(1 - \gamma_1^B)(1 - \gamma_2^B) - |\beta|} + |\alpha|^{m-1} \sqrt{\gamma_1^B \gamma_2^B} = 0.
\] (17)
Let us define $(\gamma_1^A, \gamma_2^A)$ by
\[
1 - \gamma_1^A := \frac{1 - \gamma_1}{1 - \gamma_1^B}, \quad \text{(18)}
\]
\[
1 - \gamma_2^A := \frac{1 - \gamma_2}{1 - \gamma_2^B} = \frac{1 - x\gamma_1}{1 - x\gamma_1^B}. \quad \text{(19)}
\]

For this choice, $\gamma_1^B + (1 - \gamma_1^B)\gamma_i^A = \gamma_i$ holds for $i = 1, 2$. Hence we only have to show the existence of a machine $\{\psi_i^A|\psi_i^B\}^{i=1,2}_i$, with $(\gamma_1^A, \gamma_2^A)$ defined above.

Now, consider a protocol in which Bob runs machine $\{\phi_i\}^B_1 \rightarrow |\psi_i^B\}^{i=1,2}_i$ and Alice does nothing. This protocol can be viewed as a machine $\{\psi_i^A|\phi_j\}^B_1 \rightarrow |\psi_i^B\}^{i=1,2}_i$, and hence $(\gamma_1^B, \gamma_2^B)$ is in the region $R$. Then, the points $(0, 0), (\gamma_1^B, \gamma_2^B), (\gamma_1, \gamma_2)$ should be on a straight line in this order, and $\gamma_2 \leq \gamma_1$ holds. Hence, from Eqs. (18) and (19), we have
\[
\gamma_1^A \geq 0, \quad \gamma_2^A \geq 0. \quad \text{(20)}
\]

Using Eqs. (18) and (19), we obtain
\[
\sqrt{(1 - \gamma_1^A)(1 - \gamma_2^A)} = \frac{\sqrt{(1 - \gamma_1)(1 - x\gamma_1)}}{\sqrt{(1 - \gamma_1^B)(1 - x\gamma_1^B)}} = |\alpha| |\beta| - |\alpha|m^{-1} \sqrt{x\gamma_1} \quad \text{(21)}
\]
and
\[
\sqrt{\gamma_1^A \gamma_2^A} = \frac{\sqrt{x(\gamma_1 - \gamma_1^B)}}{\sqrt{(1 - \gamma_1^B)(1 - x\gamma_1^B)}} = \frac{|\beta| - |\alpha|m^{-1} \sqrt{x\gamma_1}}{\sqrt{x\gamma_1}}. \quad \text{(22)}
\]

Hence it is not difficult to show that
\[
\sqrt{(1 - \gamma_1^A)(1 - \gamma_2^A)} - |\alpha| + |\alpha|m \sqrt{\gamma_1^A \gamma_2^A} = 0. \quad \text{(23)}
\]

From corollary 1, there exists a machine $\{\psi_i^A|\gamma_i^A \rightarrow |\psi_i^B\}^{i=1,2}_i$ with $(\gamma_1^A, \gamma_2^A)$, and theorem 2 is proved.

IV. PROBABILISTIC CLONING WITH SUPPLEMENTARY INFORMATION FOR THREE OR MORE STATES

When the number of the possible states is 3 or more, Theorem 2 is not always true, and there may exist a better protocol than just running machines $\{\phi_i\}^B_1 \rightarrow |\psi_i^B\}^{i=1,2}_i$ and $\{\psi_i^A|\psi_i^B\}^{i=1,2}_i$. We will give such an example in this section, and also show that a somewhat stronger statement holds about how the supplementary and the original information should be combined to give the optimal performance. For this purpose, we assume that two separated parties, Alice and Bob, have the original information $|\psi_i\}$ and the supplementary information $|\phi_j\}$, respectively. We do not care which of the parties produces the copies, as long as they produce $m$ copies of $|\psi_i\}$ in total, namely, the task is successful when
\[
|\psi_i^A|\phi_i^B \rightarrow |\psi_i^B\}^{m-k}_i \rightarrow |\psi_i^B\}^{k}_i, (i = 1, 2, \ldots, n), \quad \text{(24)}
\]
for any integer $k$. We consider two scenarios depending on the allowed communication between Alice and Bob.

Scenario I. Alice and Bob can use a one-way quantum channel from Bob to Alice. Note that this scenario is equivalent to the case where a single party having both the original and the supplementary information runs a machine $\{\psi_i|\phi_i\} \rightarrow |\psi_i^B\}^{i=1,2}_i$, and its success probabilities are determined by theorem 1.

Scenario II. Alice and Bob can use only a one-way classical channel from Bob to Alice. Note that the two-step protocol in the last section is included in this scenario. In what follows, we construct an example showing a gap between the two scenarios.

Consider an $n$-dimensional Hilbert space, and choose an orthonormal basis $\{|j\}\}_j=1,\ldots,n$. Let us define $n$ normalized states $\{\mu_j\}_{j=1,\ldots,n}$ as follows:
\[
|\mu_j\} := \sqrt{1 - (n - 1)z^2} |j\} - z \sum_{i \not= j} |i\}, \quad \text{(25)}
\]
where $z \geq 0$. The inner product between any pair of the states is given by
\[
\langle \mu_i | \mu_j \rangle = z \left[ (n - 2)z - 2\sqrt{1 - (n - 1)z^2} \right]. \quad \text{(26)}
\]
for $i \neq j$. The right-hand side is zero for $z = 0$, and is $-1/(n - 1)$ for $z = 1/\sqrt{n(n - 1)}$. By continuity, we see that for any $\alpha \in [-1/(n - 1), 0]$, there exists a set of $n$ normalized states $\{\mu_j\}_{j=1,\ldots,n}$ satisfying $\langle \mu_i | \mu_j \rangle = \alpha$ for $i \neq j$.

Now we consider a problem of producing $m$ copies of a state chosen randomly ($p_j = 1/n$) from the set $\{|\psi_j\}_j=1,\ldots,n$ satisfying
\[
\langle \psi_i | \psi_j \rangle = \alpha = -|\alpha| \left( 0 < |\alpha| < \frac{1}{n - 1} \right). \quad \text{(27)}
\]
for any $i \neq j$, each accompanied by supplementary information $\{\phi_j\}_j=1,\ldots,n$, satisfying
\[
\langle \phi_i | \phi_j \rangle = \beta = -\frac{1}{n - 1}. \quad \text{(28)}
\]
for any $i \neq j$. Both sets of states, $\{|\psi_j\}_j=1,\ldots,n$ and $\{|\phi_j\}_j=1,\ldots,n$, exist because they are special cases of the set $\{\mu_j\}_{j=1,\ldots,n}$ above.

Let $\gamma_{\text{totmax}}^I$ and $\gamma_{\text{totmax}}^II$ be the maximum overall probabilities in scenarios I and II, respectively. We show that for any $n \geq 3$ and any $m \geq 2$, there is a gap between the two scenarios $(\gamma_{\text{totmax}}^I > \gamma_{\text{totmax}}^II)$ for sufficiently small (but nonzero) $|\alpha|$.
First, we derive a lower bound for \( \gamma_{\text{tot}} \), written as
\[
\gamma_{\text{tot}} = \frac{1 - |\beta| |\alpha|}{1 - |\beta| |\alpha|^m}.
\]
(29)
This relation can be proved via theorem 1 with \(| \Phi_i \rangle = | \psi_i \rangle | \phi_i \rangle\) and \(| \Psi_i \rangle = | \psi_i \rangle | \phi_i \rangle \otimes | \mu \rangle\) as follows. We separate the cases depending on the parity of \( m \).

When \( m \) is even, we take
\[
\gamma_i = \frac{1 - |\beta| |\alpha|}{1 - |\beta| |\alpha|^m}.
\]
and \( \langle P_i | P_j \rangle = 1 \) for any \( i, j \). Then we obtain
\[
X - \sqrt{Y} Y \sqrt{L} = \frac{|\beta| |\alpha| - |\alpha|^m}{1 - |\beta| |\alpha|^m} Z,
\]
where \( Z \) is an \( n \times n \) matrix defined by
\[
Z := \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}.
\]
(32)
Since the eigenvalues of \( Z \) are only 0 (\( n - 1 \) degeneracy) and \( n \) (no degeneracy), \( Z \) is positive semidefinite and so is \( X - \sqrt{Y} Y \sqrt{L} \). Hence, by theorem 1, there exists a machine satisfying Eq. (30), and Eq. (20) holds in this case.

When \( m \) is odd, we take
\[
\gamma_i = \frac{1 - |\beta| |\alpha|}{1 - |\beta| |\alpha|^m}
\]
and \( \langle P_i | P_j \rangle = \beta = -1/(n - 1) \) for any \( i \neq j \). Then we obtain
\[
X - \sqrt{Y} Y \sqrt{L} = \frac{|\alpha|^{m-1} |\beta| |\alpha|^m}{1 - |\beta| |\alpha|^m} Z,
\]
which is positive semidefinite. By theorem 1, there exists a machine satisfying Eq. (33), and Eq. (20) holds also in this case, namely, irrespective of \( m \).

Next, we derive an upper bound on \( \gamma_{\text{tot}} \). We start by proving the following lemma.

**Lemma 1.** Consider a linearly independent set of \( n \) states \( \{ | \Phi_1 \rangle, | \Phi_2 \rangle, \ldots, | \Phi_n \rangle \} \), and another set of \( n \) states \( \{ | \Psi_1 \rangle, | \Psi_2 \rangle, \ldots, | \Psi_n \rangle \} \) satisfying
\[
\sum_{i=1}^{n} b_i | \Phi_i \rangle = 0.
\]
(35)
If \( b_i \neq 0 \), there is no machine \( \{ | \Phi_i \rangle \rightarrow | \Psi_i \rangle \}_{i=1}^{n} \) with \( \gamma_j > 0 \).

**Proof.** Suppose that there exists a machine \( \{ | \Phi_i \rangle \rightarrow | \Psi_i \rangle \}_{i=1}^{n} \). From theorem 1, \( n \times n \) matrix \( X - \sqrt{Y} Y \sqrt{L} \) is positive semidefinite. Then, for the vector \( b = (b_1, b_2, \ldots, b_n)^T \) satisfying the Eq. (35), we have
\[
b^\dagger (X - \sqrt{Y} Y \sqrt{L}) b \geq 0.
\]
(36)
Since \( b^\dagger X b = 0 \) from Eq. (35), we have \( b^\dagger (\sqrt{Y} Y \sqrt{L}) b \leq 0 \). Since the linear independence of \( \{ | \Psi_i \rangle \} \) implies that \( Y \) is positive definite, it follows that \( \sqrt{L} b = 0 \), and hence \( b \gamma_i = 0 \) for all \( i \). Then, \( b_j \neq 0 \) implies \( \gamma_j = 0 \).

In the problem at hand, the set of states \( \{ | \psi_i \rangle \otimes | \mu \rangle \} \) is linearly independent for any integer \( k \geq 1 \) since the eigenvalues of the \( n \times n \) matrix \( \{ | \psi_i \rangle | \psi_j \rangle \rangle = (1 - \alpha^k)I + \alpha^k Z \) are only \( 1 - \alpha^k > 0 \) (\( n - 1 \) degeneracy) and \( 1 + (n - 1) \alpha^k > 0 \) (no degeneracy), where \( I \) is the \( n \times n \) identity matrix. The set \( \{ | \phi_i \rangle \} \) satisfies \( \sum \gamma_i | \phi_i \rangle = 0 \) since \( \sum_{j} (\phi_i | \phi_j \rangle) = n + n(n - 1) \beta = 0 \). Then, we see from lemma 1 that any machine \( \{ | \phi_i \rangle \rightarrow | \psi_j \rangle \} \) is linearly independent and \( 0 \) is the zero success probability, \( \gamma_i = \gamma_j = 0 \). In scenario II, this fact implies that all of the \( m \) copies must be produced by Alice, and Bob’s role is just to provide classical information to help Alice’s operation. Hence, we are allowed to limit Bob’s action to a POVM measurement \( \{ E_\mu \} \) applied to his initial state \( | \phi_i \rangle \), providing outcome \( \mu \) with probability \( p_{\mu|i} := | \langle \phi_i | E_\mu \rangle |^2 \). Depending on the outcome \( \mu \) received from Bob, Alice runs a machine \( \{ | \psi_i \rangle \rightarrow | \psi_j \rangle \} \) to produce \( m \) copies of state \( | \psi_i \rangle \). Since the initial state \( | \psi_i \rangle \) is randomly chosen \( (p_i = 1/n) \), the overall success probability is
\[
\gamma_{\text{tot}} = \sum_{\mu} p_{\mu} \left( \sum_{i=1}^{n} p_i | \gamma_i |^{\mu} \right),
\]
(37)
where \( p_{\mu} := \sum_i p_\mu | p_i \) and \( p_{\mu|i} := p_\mu | p_i / p_\mu | \).

From theorem 1, \( \Gamma^{(\mu)} := \text{diag} \langle \gamma_1^{(\mu)}, \gamma_2^{(\mu)}, \ldots, \gamma_n^{(\mu)} \rangle \) should satisfy
\[
b^\dagger \left( X - \sqrt{Y} Y \sqrt{L} \right) b \geq 0
\]
(38)
for any \( b \). Here the elements of matrices \( X \) and \( Y \) are given by \( X_{ij} = (1 - \alpha) \delta_{ij} + \alpha \) and \( Y_{ij} = (1 - \alpha^m) \delta_{ij} + \alpha^m \langle P^i | P^j \rangle \). If we choose \( b = (\sqrt{P^1}, \sqrt{P^2}, \ldots, \sqrt{P^n}) \), we have
\[
0 \leq 1 - \sum_{i} p_\mu | \gamma_i^{(\mu)} - | \alpha \rangle \sum_{i,j(\neq i)} \sqrt{P_\mu P_j^\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle} + \alpha^m \sum_{i,j(\neq i)} \sqrt{P_\mu P_j^\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle} \leq 1 - \sum_{i} p_\mu | \gamma_i^{(\mu)} - | \alpha \rangle \sum_{i,j(\neq i)} \sqrt{P_\mu P_j^\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle} + \alpha^m \sum_{i} p_\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle \]
(39)
where we have used 2 \( \sqrt{P_\mu P_j^\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle} \leq p_\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle + P_\mu | \gamma_j^{(\mu)} | P^j \rangle | P^j \rangle \) and \( \langle | P^j \rangle | P^j \rangle \leq 1 \). Using this relation, we obtain
\[
\gamma_{\text{tot}} \leq \frac{1 - | \alpha \rangle \sum_{i,j(\neq i)} \sum_\mu p_\mu \sqrt{P_\mu P_j^\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle} + \alpha^m \sum_{i} p_\mu | \gamma_i^{(\mu)} | P^j \rangle | P^j \rangle}{1 - (n - 1) | \alpha |^m}.
\]
(40)
We further bound the term in the numerator by using
\[ p_{i|\mu}p_{j|\mu} = p_{i|\mu}p_{j} = \langle \phi_{i} | E_{\mu} \rangle \langle \phi_{j} | E_{\mu} \rangle / n, \]
the completeness relation \( \sum_{\mu} E_{\mu} = 1 \), and the Cauchy-Schwarz inequality
\[
\sum_{\mu} p_{\mu} \sqrt{p_{i|\mu}p_{j|\mu}} = \frac{1}{n} \sum_{\mu} \sqrt{\langle \phi_{i} | E_{\mu} \rangle \langle \phi_{j} | E_{\mu} \rangle \langle \phi_{i} | \phi_{j} \rangle} \\
\geq \frac{1}{n} \sum_{\mu} | \langle \phi_{i} | E_{\mu} \rangle | \langle \phi_{j} | \phi_{j} \rangle | \\
\geq \frac{1}{n} | \langle \phi_{i} | \phi_{j} \rangle | = \frac{1}{n(n-1)}. \tag{41}
\]
From Eqs. (40) and (41), we obtain
\[
\gamma^H_{\text{totmax}} \leq \frac{1-|\alpha|}{1-(n-1)|\alpha|^m}. \tag{42}
\]
Combining Eqs. (28) and (42), we obtain
\[
\gamma_{\text{totmax}}^{I} - \gamma_{\text{totmax}}^{H} \geq \frac{(n-2)|\alpha|(1+|\alpha|^m-n|\alpha|^{m-1})}{(n-1-|\alpha|^m)(1-(n-1)|\alpha|^m)} \tag{43}
\]
which shows that \( \gamma_{\text{totmax}}^{I} > \gamma_{\text{totmax}}^{H} \) when \( |\alpha| \) is a small enough positive number.

V. SUMMARY

In this paper, we have discussed probabilistic cloning of a mutually nonorthogonal set of pure states \( \{|\psi_{1}\rangle, \ldots, |\psi_{n}\rangle\} \), with the help of supplementary information. It has turned out that the situation is quite different for \( n = 2 \) and for other cases. When \( n = 2 \), the role of the supplementary information is limited to just produce copies on its own, independently of the original state. This property is quite similar to the property in deterministic cloning, stated in the stronger no-cloning theorem. For \( n \geq 3 \), such a simple property does not hold any longer. We assumed that the original and the supplementary information are held by separated parties, and asked what kind of communication is required to achieve the optimal performance. We have found examples in which the optimum performance cannot be achieved even if we allow any amount of classical communication from the party with the supplementary information to the other. If we limit to the one-way communication scenarios, this result means that a nonclassical interaction between the supplementary and the original information helps to improve the performance. On the other hand, if we allow the flow of information in the other direction, we are not sure the gap still exists. Analysis of such two-way protocols will be an interesting problem. The cases where the set \( \{|\psi_{i}\rangle\} \) includes a mutually orthogonal pair, or the cases where supplementary information is provided as a mixed state are also worth investigating.

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