Action principle and weak invariants

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Abstract A weak invariant associated with a master equation is characterized in such a way that its spectrum is not constant in time but its expectation value remains constant under time evolution generated by the master equation. Here, an intriguing relationship between the concept of weak invariants and the action principle for master equations based on the auxiliary operator formalism is revealed. It is shown that the auxiliary operator can be thought of as a weak invariant.
Invariants play fundamental roles in physics. In particular, the profound connection between continuous symmetries and conservation laws is widely known as Noether’s theorem [1] and serves modern physics with the indispensable basis. In fact, the symmetry principle is frequently used upon construction of an action integral. However, a situation becomes somewhat involved if a system is not autonomous. In this case, as long as the dynamics is unitary and is generated by a Hamiltonian with explicit time dependence, still there exists an invariant, the spectrum of which does not depend on time. Such an invariant is referred to here as a strong invariant. A celebrated example is the Lewis-Riesenfeld invariant of the time-dependent harmonic oscillator [2] (see also Ref. [3]). It is known that Lewis-Riesenfeld strong invariant can be obtained through quantization of the classical strong invariant derived from the action principle and Noether’s theorem [4] (see, also, Ref. [5] for a much simplified discussion).

Recently, the concept of weak invariants has been discussed for nonunitary subdynamics [6]. In particular, the Lewis-Riesenfeld strong invariant has been generalized to the case of the time-dependent quantum damped harmonic oscillator, and the weak invariant, which is different from the quantity studied in Ref. [7], has explicitly been constructed within the framework of the master equation of the Lindblad type [8,9]. In contrast to a strong invariant, the spectrum of a weak invariant depends on time, but its expectation value is conserved. An origin of its physical importance is concerned with finite-time quantum thermodynamics of the isoenergetic process interacting with the energy bath, along which the internal energy is conserved [10,11]. There, the relevant weak invariant is the time-dependent Hamiltonian of a subsystem
under consideration. It seems worth emphasizing that such a process should be distinguished from the isothermal process familiar in classical thermodynamics because of the quantum-mechanical violation of the law of equipartition of energy.

In this article, we reveal a hidden connection between and the action principle for master equations without time reversal invariance and weak invariants, which lies outside of Noether’s theorem.

Consider a master equation

\[ i \frac{\partial \hat{\rho}}{\partial t} = \mathcal{E}(\hat{\rho}), \tag{1} \]

where \( \hat{\rho} = \hat{\rho}(t) \) is a density operator describing an open quantum system, and \( \mathcal{E} \) is a certain linear superoperator that may depend explicitly on time, in general. Here and hereafter, \( \hbar \) is set equal to unity for the sake of simplicity. A weak invariant, \( \hat{I} = \hat{I}(t) \), associated with Eq. (1) is defined as a solution of the following equation:

\[ i \frac{\partial \hat{I}}{\partial t} + \mathcal{E}^*(\hat{I}) = 0, \tag{2} \]

where \( \mathcal{E}^* \) stands for the adjoint of \( \mathcal{E} \). Then, it is straightforward to see that the expectation value, \( \langle \hat{I} \rangle = \text{tr}(\hat{I} \hat{\rho}) \), is constant in time: \( d \langle \hat{I} \rangle/dt = 0 \).

In a special case of the Lindblad equation, the superoperators in Eqs. (1) and (2) read
\[ \mathcal{L}(\hat{\rho}) = [\hat{H}, \hat{\rho}] - i \sum_n \alpha_n \left( \hat{L}^+_n \hat{\rho} \hat{L}_n + \hat{L}_n \hat{\rho} \hat{L}^+_n - 2 \hat{L}_n \hat{\rho} \hat{L}_n \right), \]  \hspace{1cm} (3)  

\[ \mathcal{L}^*(\hat{I}) = -[\hat{H}, \hat{I}] - i \sum_n \alpha_n \left( \hat{L}^+_n \hat{I} \hat{L}_n + \hat{I} \hat{L}^+_n \hat{L}_n - 2 \hat{L}^+_n \hat{I} \hat{L}_n \right), \]  \hspace{1cm} (4)  

respectively, where the subsystem Hamiltonian \( \hat{H} \), the nonnegative \( c \)-numbers \( \alpha_n \)’s, and the Lindbladian operators \( \hat{L}_n \)’s may also depend explicitly on time. The first term on the right-hand side in Eq. (3) is the unitary part that appears in the Liouville-von Neumann equation, whereas the second term is called the dissipator responsible for nonunitarity of the dynamics. From Eq. (4), it can be shown [6] that the spectrum of \( \hat{I} \) is, in fact, time-dependent.

Here, we make a comment on the adjoint superoperator in Eq. (2). In general, it is desirable that \( \mathcal{L}^* \) has the following property:

\[ \mathcal{L}^*(\hat{A} + c) = \mathcal{L}^*(\hat{A}) \]  \hspace{1cm} (5)  

where \( \hat{A} \) is a certain operator and \( c \) is any \( c \)-number. In fact, Eq. (4) satisfies this condition. The case of time-independent \( c \) is special since a weak invariant shifted by such a constant is clearly a weak invariant, and Eq. (2) remains unchanged under the constant shift if Eq. (5) is satisfied.

Now, our purpose is to elucidate how the weak invariant satisfying Eq. (2) is connected to the action principle for the master equation in Eq. (1).

As in the case of the Lindblad equation, the master equation describing the
subdynamics does not possess time reversal invariance. Therefore, to construct the action integral for the equation, it is convenient to extend the space of variables. Thus, following the work in Ref. [12], we introduce an auxiliary operator \( \hat{\Lambda} = \hat{\Lambda}(t) \) and consider the following action integral:

\[
S[\hat{\rho}, \hat{\Lambda}] = -\int_{t_i}^{t_f} dt \left\{ \hat{\Lambda} - i \mathcal{L}^* \left( \hat{\Lambda} \right) \right\} - \left\langle \hat{\Lambda} \right\rangle_{t_i}^{t_f},
\]

where \( t_i \) (\( t_f \)) is the initial (final) time and \( \dot{\hat{A}} = \partial \hat{A}(t) / \partial t \). Unlike the density operator, the auxiliary operator has to be neither positive semi-definite nor normalized, in general.

We wish to make a comment on the action integral given above. To calculate its variation with respect to \( \hat{\rho} \), the normalization condition on it should be taken into account. However, it turns out not to be necessary to add such a constraint to the action integral. The reason is as follows. Let us redefine the auxiliary operator as follows:

\[
\hat{\Lambda}(t) \rightarrow \hat{\Lambda}(t) + \int_{t}^{t_f} ds \lambda(s),
\]

where \( \lambda \) is a c-number function. Then, the action integral in Eq. (6) is rewritten as

\[
S[\hat{\rho}, \hat{\Lambda}] \rightarrow S[\hat{\rho}, \hat{\Lambda}] + \int_{t_i}^{t_f} dt \lambda(t) \left( \text{tr} \hat{\rho}(t) - \text{tr} \hat{\rho}(t_i) \right).
\]
In this form, $\lambda$ is seen to play a role of the Lagrange multiplier associated with the normalization condition: that is, $\text{tr}\hat{\rho}(t_i)=1$ holds if the initial density operator is fixed to be normalized $\text{tr}\hat{\rho}(t_f)=1$. Therefore, it is actually not necessary to add the constraint on the normalization condition to the action integral in Eq. (6). Furthermore, it should be noted that the “transformation” in Eq. (7) keeps the final condition $\hat{\Lambda}(t_f)$ unchanged.

Now, variations with respect to $\hat{\rho}$ and $\hat{\Lambda}$ yield

\begin{align}
\delta_\rho S[\hat{\rho}, \hat{\Lambda}] &= -\int_{t_i}^{t_f} dt \text{ tr} \left\{ \left[ \hat{\Lambda} - i \mathcal{L}^*(\hat{\Lambda}) \right] \delta \hat{\rho} \right\} - \text{tr} \left( \hat{\Lambda}(t_i) \delta \hat{\rho}(t_i) \right),
\end{align}

(9)

\begin{align}
\delta_\Lambda S[\hat{\rho}, \hat{\Lambda}] &= \int_{t_i}^{t_f} dt \text{ tr} \left\{ \left[ \hat{\rho} + i \mathcal{L}\hat{\rho} \right] \delta \hat{\Lambda} \right\} - \text{tr} \left( \hat{\rho}(t_f) \delta \hat{\Lambda}(t_f) \right),
\end{align}

(10)

respectively. Thus, under the fixed initial and final conditions, $\delta \hat{\rho}(t_i)=0$ and $\delta \hat{\Lambda}(t_f)=0$, we obtain, from Eq. (9) and (10), that

\begin{align}
i \frac{\partial \hat{\Lambda}}{\partial t} + \mathcal{L}^*(\hat{\Lambda}) = 0,
\end{align}

(11)

as well as the master equation in Eq. (1).

In conclusion, comparing Eq. (11) with Eq. (2), we find that an auxiliary operator in the action principle for a master equation is a weak invariant.
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[1] E. Noether, Nachr. Ges. Wiss. Gottingen 235 (1918).
[2] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).
[3] V. V. Dodonov and V. I. Man’ko, Invariants and The Evolution of Nonstationary Quantum Systems, edited by M. A. Markov (Nova Science Publishers, New York, 1989).
[4] M. Lutzky, Phys. Lett. A 68, 3 (1978).
[5] S. Abe, Y. Itto, and M. Matsunaga, Eur. J. Phys. 30, 1337 (2009).
[6] S. Abe, Phys. Rev. A 94, 032116 (2016).
[7] V. V. Dodonov and V. I. Man’ko, Physica A 94, 403 (1978).
[8] G. Lindblad, Commun. Math. Phys. 48, 119 (1976).
[9] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. 17, 821 (1976).
[10] C. Ou, R. V. Chamberlin, and S. Abe, Physica A 466, 450 (2017).
[11] C. Ou and S. Abe, e-print arXiv:1808.04128.
[12] O. Éboli, R. Jackiw, and S.-Y. Pi, Phys. Rev. D 37, 3557 (1988).