ON THE DIMENSION OF THE STRATUM OF THE MODULI OF POINTED CURVES BY WEIERSTRASS GAPS

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Abstract. The dimension of the moduli space of smooth pointed curves with prescribed Weierstrass semigroup at the marked point is computed for three families of symmetric semigroups of multiplicity six. We also collect the dimensions of such moduli spaces for all semigroups of genus not greater than seven. A question related to an improvement of a Deligne–Pinkham’s bound is also formulated, suggesting that the positive graded part of the first module of the cotangent complex associated to a semigroup algebra is a missing invariant. The answer for this question is positive for all these moduli that we know their dimensions.

1. Introduction

Given a smooth pointed curve $(C, P) \in M_{g,1}^S$ of genus $g > 1$, its associated Weierstrass semigroup $S$ consists of the set of nonnegative integers $n$, called non-gaps, such that there is a rational function on $C$ with pole divisor $nP$. Equivalently, $n$ is a nongap if and only if $H^0(C, \mathcal{O}_C(n - 1)P) \subsetneq H^0(C, \mathcal{O}_C(nP))$. Follows from the Riemann–Roch theorem that the set of positive integers that are not in $S$ (the set of gaps) has magnitude exactly $g$.

Inverting the above considerations, given a numerical semigroup $S$ of genus $g > 1$, let $M^S_{g,1}$ be the moduli space parameterizing smooth pointed curves whose associated Weierstrass semigroup is $S$. It is very known that the moduli space $M^S_{g,1}$ can be empty. If it is nonempty, since the $i$-th gap of a Weierstrass semigroup is an upper semicontinuous function, the moduli space $M^S_{g,1}$ is a locally closed subspace of $M_{g,1}$. Hence, the moduli spaces $M^S_{g,1}$ give a stratification of $M_{g,1}$. Naturally, many general problems about this stratification arise. But, the simplest problems of determine when $M^S_{g,1}$ is nonempty, describe its irreducible components and their dimensions, remain unsolved.

In this paper we focus on the problem of find the dimension of a given $M^S_{g,1}$. There are two general and important bounds for the dimension of $M^S_{g,1}$. On the one hand a Deligne–Pinkham’s upper bound, cf. [6] [12], and on the other hand a lower bound given by Pflueger in [10]. The section two of the present paper details this two bounds, it is also showed that this two bounds can be not attained.

In section three of this paper we recall an important construction given by Stoehr [15] and Contiero–Stoehr [5] of a compactification of $M^S_{g,1}$, when $S$ is symmetric, by allowing canonical Gorenstein curves at its bordering.

In section four we compute the dimension of $M^S_{g,1}$ when $S$ runs over the following two families of symmetric semigroups, $S := < 6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau >$ and

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$\mathcal{S} := \langle 6, 7 + 6\tau, 8 + 6\tau, 9 + 6\tau, 10 + 6\tau \rangle$, with $\tau > 0$. The method to compute the dimension is to construct a compactification of $\mathcal{M}_{9,1}^S$ as a quotient of an affine quasi-cone $X$ by an action of the multiplicative group $\mathbb{G}_m(k)$, this approach gives a compactification of the moduli space $\mathcal{M}_{9,1}^S$ constructed by Pinkham in [12] using equivariant deformation theory. Then we realize a quadratic approximation of $X$ as a scheme over a suitable artinian algebra. Hence we get an upper bound for the dimension of $\mathcal{M}_{9,1}^S$ which is better than Deligne–Pinkham’s bound and it is equal to Pflueger’s lower bound.

Finally, in section five we collect the dimensions of $\mathcal{M}_{9,1}^S$ for all semigroups of genus not greater than seven, and also for three families of symmetric semigroups. Therefore, we can verify that for all these semigroups the dimension of $\mathcal{M}_{9,1}^S$ is equal to Pflueger’s lower bound and also it is equal to

\[ 2g - 2 + \lambda(S) - \dim T^{1,+}(k[S]), \]

where $\dim T^{1,+}(k[S])$ stands for the dimension of the positive graded part of the first module of the cotangent complex associated to the semigroup algebra $k[S]$ introduced by Lichtenbaum and Schlessinger in [8]. We also noted that there are examples where the Pflueger’s bound does not provide the exact dimension of $\mathcal{M}_{9,1}^S$, but the above possible bound does, for example $S = \langle 6, 7, 8 \rangle$ and $S = \langle 6, 7, 15 \rangle$. As a final result we show that Pflueger’s bound is not greater than $2g - 2 + \lambda(S) - \dim T^{1,+}(k[S])$ for any numerical semigroup $S$ of genus $g > 0$.

2. General bounds for $\dim \mathcal{M}_{9,1}^S$

Let $C$ be a projective reduced algebraic curve defined over an algebraically closed field $k$ and $P \in C$ a rational point. Set $\delta := \dim_k \mathcal{O}/\mathcal{O}$ the singularity degree of $P$ and $\mu := \dim(\text{Der}_k(\mathcal{O}, \mathcal{O})/\text{Der}_k(\mathcal{O}, \mathcal{O}))$, where $\text{Der}_k(R, M)$ is the module of $k$-derivations from $R$ to $M$. By analyzing three different parameters spaces, using a result of Rim [13, Cor. 2.10] and the Kodaira–Spencer correspondence, Deligne [6, Thm. 2.27] established the following formula.

**Theorem 2.1** (Deligne’s formula). *For any smoothing component $E$ of the formal versal space of deformations of $\mathcal{O}$, it follows*

\[ \dim E = 3\delta - \mu. \]

A smoothing component stands for an irreducible component whose fiber over its generic point does not intersect the singular locus of the total space.

An interesting consequence of Deligne’s theorem can be obtained by assuming that $C$ is a monomial curve. In this case, by computing the right hand-side of Deligne’s formula, do not assuming the smoothing condition, using Pinkham’s equivariant deformation theory [12, Thm. 13.9], Rim and Vitulli [14, §6] noted that:

**Theorem 2.2** (Deligne–Pinkham’s bound). *For any numerical semigroup $S$,*

\[ \dim \mathcal{M}_{9,1}^S \leq 2g - 2 + \lambda(S) = 3\delta - \mu \]

where $\lambda(S)$ is the number of gaps $\ell$ such that $\ell + n \in S$ whenever $n$ is a nongap.

Deligne–Pinkham’s upper bound is attained, Rim and Vitulli showed, cf. [14 Cor. 5.14], that if $S$ is negatively graded, then $\dim \mathcal{M}_{9,1}^S = 2g - 2 + \lambda(S)$. A numerical semigroup is negatively graded if the first cohomology module of the cotangent complex associated to the semigroup algebra $k[S]$ is negatively graded.
Despite this, there are families of semigroups where Deligne–Pinkham’s bound is far from being tight, see sections 4 and 5 below, specially the tables 1 and 2 in section 5 of this paper.

On the other hand, Pflueger in [10] produced an upper bound for the codimension of $M_{g,1}^S$ seen as a locally closed subset of $M_{g,1}$. His bound is an improvement of a bound given by Eisenbud and Harris in [7]. Pflueger introduced the effective weight of a numerical semigroup $S$,

$$\text{ewt}(S) := \sum_{\text{gaps } l_i}(\# \text{ generators } n_j < l_i),$$

as a substitute of the weight of a numerical semigroup which appears in Eisenbud–Harris bound. More specifically, Pflueger showed [10, Thm. 1.2]:

**Theorem 2.3. (Pflueger’s bound)** If $M_{g,1}^S$ is nonempty, and $X$ is any irreducible component of it, then

$$\dim X \geq 3g - 2 - \text{ewt}(S).$$

Although Pflueger’s bound is attained for a lot of classes of numerical semigroups, there are examples where his bound does not give the exact dimension of $M_{g,1}^S$, an example was given by Pflueger himself, cf. [10, pg. 12], taking the following symmetric semigroup $S := <6, 7, 8>$ of genus 9. The moduli variety for this particular semigroup can be completely described by classical results as follows. Let $S$ be a symmetric semigroup generated by less than 5 elements. Using Pinkham’s equivariant deformation theory [12], a quasi-homogeneous version of Buchsbaum-Eisenbud’s structure theorem for Gorenstein ideals of codimension 3 (see [2, p. 466]), one can deduce that the affine monomial curve Spec $k[S] = \text{Spec } k[t^{m_1}, t^{m_2}, t^{m_3}, t^{m_4}]$ can be negatively smoothed without any obstructions (see [3, 16, 17, Satz 7.1]), hence

$$\dim M_{g,1}^S = \dim \mathbb{P}(T^1_{k[S]|k}),$$

and therefore, by the Jacobian criterion and elimination theory,

$$\dim M_{g,1}^S = \dim \mathbb{P}(T^1_{k[S]|k}).$$

If $S = <6, 7, 8>$, then $\dim M_{g,1}^S = 14$, see theorem [10] in section 5 below. In contrast, Pflueger’s bound gives $3g - 2 - \text{ewt}(S) = 27 - 2 - 12 = 13$, while Deligne–Pinkham’s bound provides $2g - 2 + \lambda(S) = 18 - 2 + 1 = 17$. In the same way, if we take the following symmetric semigroup $S := <6, 7, 15>$ of genus 12, Pflueger’s bound gives 17, while Deligne–Pinkham’s bound 23. But one can verify using (2) and theorem 5.3 that $\dim M_{g,1}^S = \dim \mathbb{P}(T^1_{k[S]|k}) = 18$.

In a later paper, cf. [11], Pflueger made a detailed study of the moduli variety $M_{g,1}^S$ when $S$ is a Castelnuovo semigroup, ie. semigroups generated by consecutive suitable positive integers. In [10] Pflueger noted that this class of semigroups is one he is aware that his bound in 2.3 does not provide the exact dimension of $M_{g,1}^S$, see [10] pg. 2.

Summarizing, for any semigroup $S$ such that $M_{g,1}^S \neq \emptyset$ we have

$$3g - 2 - \text{ewt}(S) \leq \dim M_{g,1}^S \leq 2g - 2 + \lambda(S).$$

3. **Weierstrass points on Gorenstein curves**

Let $C$ be a complete integral Gorenstein curve of arithmetic genus $g > 1$ defined over an algebraically closed field $k$. For each smooth point $P$ of $C$, let $S$ be the
Weierstrass semigroup of $C$ at $P$. By the very definition, for each $n \in S$ there is a rational function $x_n$ on $C$ with pole divisor $nP$. Let us assume that the semigroup $S$ is symmetric, i.e., the last gap $\ell_g$ is the biggest possible, $\ell_g = 2g - 1$. Equivalently, $n \in S$ if, and only if, $\ell_g - n \notin S$. Let $\omega$ be the dualizing sheaf of $C$. A basis for the vector space $H^0(C, \omega)$ is \{x_n, x_{n_1}, \ldots, x_{n_{g-1}}\}, and thus $\omega \cong O_C((2g - 2)P)$. By assuming that $C$ is nonhyperelliptic, the canonical morphism

$$(x_{n_0} : x_{n_1} : \ldots : x_{n_{g-1}}) : C \hookrightarrow \mathbb{P}^{g-1}$$

is an embedding. Thus $C$ becomes a curve of degree $2g - 2$ in $\mathbb{P}^{g-1}$ and the integers $l_i - 1$ are the contact orders of the curve with the hyperplanes at $P = (0 : \ldots : 0 : 1)$. Conversely, any nonhyperelliptic symmetric semigroup $S$ can be realized as the Weierstrass semigroup of the Gorenstein canonical monomial curve

$$C_S := \left\{ (s^{n_0}t^{g-1}: s^{n_1}t^{g-1}: \ldots : s^{n_{g-2}}t^{g-1} : s^{n_{g-1}}t^{g-1}) \mid (s : t) \in \mathbb{F}^1 \right\} \subset \mathbb{P}^{g-1}$$

at its unique point $P$ at the infinity.

Now we recall a construction of a compactification of $M^{g,1}$ given by Stoehr [15] and Contiero-Stoehr [5]. This construction will be required for the next section.

Since $S$ is symmetric, each nongap $s \in S, s \leq 4g - 4$ can be written as a sum of two others nongaps (see [5, theorem 1.3]),

$$s = a_s + b_s, \quad a_s \leq b_s \leq 2g - 2.$$ 

By taking $a_s$ as the smallest possible, the $3g - 3$ rational functions $x_a, x_b$ form a $P$-hermitian basis of the space of global sections $H^0(C, \omega^2)$ of the bicanonical divisor. If $r \geq 3$, then a $P$-hermitian basis of the vector space $H^0(C, \omega^r)$ (cf. [5, Lemma 2.1]) is

$$
\begin{align*}
&x_{n_0}^{-1}x_{n_1}, \\
&x_{n_0}^{-2}x_{a_0}x_{b_0}x_{n_{g-1}}, \quad (i = 0, \ldots, g - 1), \\
&x_{n_0}^{-3}x_{n_1}x_{2g-n_1}x_{n_{g-2}}x_{n_{g-1}}, \quad (i = 0, \ldots, r - 3).
\end{align*}
$$

A consequence of the existence of a $P$-hermitian basis of $H^0(C, \omega^r)$ for any $r \geq 1$, is a Max-Noether’s theorem, namely the following homomorphism

$$k[X_{n_0}, \ldots, X_{n_{g-1}}]_r \longrightarrow H^0(C, \omega^r)$$

induced by the substitutions $X_{n_i} \longrightarrow x_{n_i}$ is surjective for each $r \geq 1$, where $k[X_{n_0}, \ldots, X_{n_{g-1}}]_r$ is the vector space of $r$-forms.

Let $I(C) = \bigoplus_{r=2}^{\infty} I_r(C) \subset k[X_{n_0}, \ldots, X_{n_{g-1}}]$ be the ideal of $C \subset \mathbb{P}^{g-1}$. The codimension of $I_r(C)$ in $k[X_{n_0}, \ldots, X_{n_{g-1}}]_r$ is equal to $(2r - 1)(g - 1)$, in particular

$$\dim I_2(C) = (g - 2)(g - 3)/2.$$ 

For $r \geq 2$, let $\Lambda_r$ be the vector space in $k[X_{n_0}, \ldots, X_{n_{g-1}}]_r$ spanned by the lifting of the $P$-hermitian basis of $H^0(C, \omega^r)$. Since $\Lambda_r \cap I_r(C) = 0$ and

$$\dim \Lambda_r = \dim H^0(C, \omega^r) = \text{codim} I_r(C),$$

it follows that

$$k[X_{n_0}, \ldots, X_{n_{g-1}}]_r = \Lambda_r \oplus I_r(C), \quad r \geq 2.$$ 

For each nongap $s \leq 4g - 4$, let us consider all the partitions of $s$ as sum of two nongaps not greater than $2g - 2$,

$$s = a_s + b_s, \quad \text{with } a_s \leq b_s, \quad (i = 1, \ldots, \nu_s), \quad \text{where } a_{s0} := a_s.$$
Hence, given a nongap $s \leq 4g - 4$ and $i = 1, \ldots , \nu_s$ we can write

$$x_{a_i}x_{b_{si}} = \sum_{n=0}^{s} c_{sin} x_{a_i} x_{b_{si}},$$

where $a_i$ and $b_{si}$ are nongaps of $S$ whose sum is equal to $s$, and $c_{sin}$ are suitable constants in $k$. By normalizing the coefficients $c_{sis} = 1$, it follows that the $(g+1) - (3g - 3) = \frac{(g-2)(g-3)}{2}$ quadratic forms

$$F_{si} = X_{a_i} x_{b_{si}} - X_{a_i} x_{b_i} - \sum_{n=0}^{s-1} c_{sin} x_{a_i} x_{b_{si}}$$

vanish identically on the canonical curve $C$, where the coefficients $c_{sin}$ are uniquely determined constants. They are linearly independent, hence they form a basis for the space of quadratic relations $I_2(C)$.

It is necessary to make some assumptions on the symmetric semigroup $S$ to assure that the ideal $I(C)$ is generated by quadratic relations. More precisely, we suppose that $S$ satisfies $3 < n_1 < g$ and $S \neq \langle 4, 5 \rangle$. According to [4, Lemma 3.1], both the conditions $n_1 \neq 3$ and $n_1 \neq g$ on $S$ are to avoid possible trigonal Gorenstein curves whose Weierstrass semigroup at $P$ equal to $S = \langle 3, g + 1 \rangle$ and $S = \langle g, g + 1, \ldots , 2g - 2 \rangle$, respectively. This two avoided cases are treated by Contiero and Fontes in [4]. By the assumptions on the semigroup $S$ it follows by the Enriques–Babbage theorem that $C$ is nontrigonal and it is not isomorphic to a plane quintic.

If $C$ is smooth, then Petri’s theorem [5] assure that the ideal of $C$ is generated by the quadratic relations. Given a canonical curve $C$, not necessarily smooth, an algorithmic proof that the ideal of $C$ is generated by the quadratic forms $F_{si}$ was done by Contiero and Stoehr in [5, Theorem 2.5].

On the other hand, for each symmetric semigroup $S$ with $3 < n_1 < g$ and $S \neq \langle 4, 5 \rangle$, we can take the following $(g - 2)(g - 3)/2$ quadratic forms

$$F_{si} = X_{a_i} x_{b_{si}} - X_{a_i} x_{b_i} - \sum_{n=0}^{s-1} c_{sin} x_{a_i} x_{b_{si}},$$

where $c_{sin}$ are constants to be determined in order that the intersection $\cap V(F_{si}) \subset \mathbb{P}^{g-1}$ is a canonical Gorenstein curve of genus $g$ whose Weierstrass semigroup at $P$ is $S$. Analogously, let

$$F_{si}^{(0)} := X_{a_i} x_{b_{si}} - X_{a_i} x_{b_i}$$

be the quadratic forms that generate the ideal of the canonical monomial curve $C_S$, cf. [5, Lemma 2.2]. One of the keys to construct a compactification of $\mathcal{M}_{g,1}^S$ is the following lemma.

**Syzygy Lemma** (cf. [4]). For each of the $\frac{1}{2}(g-2)(g-5)$ quadratic binomials $F_{si}^{(0)}$ different from $F_{n_1 + 2g - 1}^{(0)}$ ($i = 0, \ldots , g - 3$) there is a syzygy of the form

$$X_{2g-2} F_{s'i'}^{(0)} + \sum_{n \neq 1}^{c_{n's'i'}} X_{n} F_{si}^{(0)} = 0$$

where the coefficients $c_{n's'i'}$ are integers equal to $1$, $-1$ or $0$, and where the sum is taken over the nongaps $n < 2g - 2$ and the double indexes $si$ with $n + s = 2g - 2 + s'$. 
Let us described briefly the algorithmic construction of a compactification of $\mathcal{M}_{g,1}^S$ which was done by Stoehr [15] and Contiero–Stoehr [5]. First, we replace the binomials $F_{s'i'}^{(0)}$ and $F_{si}^{(0)}$ on the left hand side of the Syzygy Lemma by the corresponding quadratic forms $F_{s'i'}$ and $F_{si}$. Hence we obtain a linear combination of cubic monomials of weight $< s' + 2g - 2$. By virtue of [5] Lemma 2.4 this linear combination of cubic monomials admits the following decomposition.

$$X_{2g-2}F_{s'i'} + \sum_{n, s} e_{nsi}^{(s'i')} X_n F_{si} = \sum_{n, s} \eta_{nsi}^{(s'i')} X_n F_{si} + R_{s'i'}$$

where the sum on the right hand side is taken over the nongaps $n \leq 2g - 2$ and the double indexes $si$ with $n + s < s' + 2g - 2$, where the coefficients $\eta_{nsi}^{(s'i')}$ are constants, and where $R_{s'i'}$ is a linear combination of cubic monomials of pairwise different weights $< s' + 2g - 2$.

For each nongap $m < s' + 2g - 2$ we denote by $\varrho_{s'i'm}$ the unique coefficient of $R_{s'i'}$ of weight $m$. Finally, let us consider the following quasi-homogeneous polynomial in the constants $c_{s'in}$,

$$R_{s'i'}(t^{n_0}, t^{n_1}, \ldots, t^{n_g-1}) = \sum_{m=0}^{s' + 2g - 3} \varrho_{s'i'm} t^m.$$ 

Since that the coordinates functions $x_n$, $n \in S$ and $n \leq 2g - 2$, are not uniquely determined by their pole divisor $nP$ by assuming the characteristic of the field $k$ to be zero (or a prime not dividing any of the differences $m - n$ where $n, m$ are nongaps of $S$ such that $n, m \leq 2g - 2$), we transform

$$X_{n_i} \mapsto X_{n_i} + \sum_{j=0}^{i-1} c_{n_i n_{i-j}} X_{n_{i-j}},$$

for each $i = 1, \ldots, g - 1$, and so we can normalize $\frac{g}{2}g(g - 1)$ of the coefficients $c_{s'in}$ to be zero, see [15 Proposition 3.1]. Due to these normalizations and the normalization of the coefficients $c_{s'in} = 1$ with $n = s$, the only left to us is to transform $x_{n_i} \mapsto e^{n_i} x_{n_i}$ for $i = 1, \ldots, g - 1$. Summarizing, we get

**Theorem 3.1.** [5 Theorem 2.6] Let $S$ be a symmetric semigroup of genus $g$ satisfying $3 < n_1 < g$ and $S \neq \langle 4, 5 \rangle$. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $S$ correspond bijectively to the orbits of the $\mathbb{G}_m(k)$-action

$$(c_1, \ldots, c_{s'in}, \ldots) \mapsto (c_1, \ldots, c^{s-n} c_{s'in}, \ldots)$$

on the affine quasi-cone of the vectors whose coordinates are the coefficients $c_{s'in}$ of the normalized quadratic $F_{si}$ satisfying the quasi-homogeneous equations $\varrho_{s'i'm} = 0$.

### 4. Families of Symmetric Semigroups

In this section we apply the techniques developed in [5] and [15] (briefly described in the above section) to deal with families of symmetric semigroups. We note that if the symmetric semigroup is generated by less than five elements, the dimension of the moduli variety $\mathcal{M}_{g,1}^S$ is very known, as we noted in section 2 of this paper. So, we must consider symmetric semigroups of multiplicity greater than 5, just because a symmetric semigroup of multiplicity $m$ can be generated by $m - 1$ elements. The main idea is to adapt the techniques developed in [5] and [15] to handle with a
projection of the (affine) canonical monomial curve over an (affine) ambient space whose dimension does not depend on the genus \( g \), but only on the multiplicity of the semigroup. Thus we are able to handle with a family of symmetric semigroups of a given multiplicity. It is clear this approach is closely related to the equivariant deformation theory developed by Pinkham in [12].

4.1. A family of symmetric semigroups. For each positive integer \( \tau \) let
\[
S := (6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau) = \mathbb{N} \sqcup \bigcup_{j \in \{3, 4, 7, 8\}} (j + 6\tau + 6\mathbb{N}) \sqcup (11 + 12\tau + 6\mathbb{N}).
\]
be a semigroup of multiplicity 6 generated minimally by five elements. Counting the number of gaps of \( S \) and picking up the largest nongap, we have
\[
g = 3 + 6\tau \quad \text{and} \quad l_g = 12\tau + 5 = 2g - 1,
\]
and so \( S \) is a symmetric semigroup.

Let \( C \) be a complete integral Gorenstein curve and \( P \) be a smooth point of \( C \) whose Weierstrass semigroup is \( S \) at \( P \). For each \( n \in S \), let \( x_n \) be a rational function on \( C \) whose pole divisor \( nP \). We abbreviate
\[
x_0 := x_6 \quad \text{and} \quad y_j := x_{j+6\tau} \quad (j = 3, 4, 7, 8)
\]
and normalize
\[
x_{6i} = x^i \quad \text{and} \quad x_{j+6\tau+6i} = x^i y_j, \quad \forall i \geq 1.
\]
A \( P \)-hermitian basis for the vector space \( H^0(C, (2g - 2)P) \) consists of the functions
\[
\begin{align*}
x^0, & \ldots, x^{2\tau}, \\
x^0 y_j, & \ldots, x^\tau y_j \quad (j = 3, 4), \\
x^0 y_j, & \ldots, x^{\tau-1} y_j \quad (j = 7, 8).
\end{align*}
\]
Since the complete integral Gorenstein curve \( C \) is nonhyperelliptic, it can be identified with its image under the canonical embedding
\[
(x_{n_0} : x_{n_1} : \ldots : x_{n_{g-1}}) : C \hookrightarrow \mathbb{P}^{g-1}.
\]
By considering the above the normalizations, the projection map
\[
(1 : x : y_3 : y_4 : y_7 : y_8) : C \hookrightarrow \mathbb{P}^5
\]
defines an isomorphism of the canonical curve \( C \) onto a curve \( D \subset \mathbb{P}^5 \) which has degree \( 8 + 6\tau \). Instead of study the ideal of the canonical curve \( C \), which has \((g-2)(g-3)/2\) quadratic generators, we study the ideal (and the relations between its generators) of the projected curve \( D \subset \mathbb{P}^5 \). The advantage is that the number of generators of the ideal of \( D \) does not depends on the genus \( g \).

Let us consider a \( P \)-hermitian basis of the vector space \( H^0(C, 2(2g - 2)P) \) of the bicanonical divisor \((4g - 4)P = (24\tau + 8)P\) which consists of the \( 3g - 3 \) functions
\[
\begin{align*}
x^i \quad (i = 0, 1, \ldots, 4\tau + 1), \\
x^i y_j \quad (i = 0, 1, \ldots, 3\tau, \ j = 3, 4, 7, 8), \\
x^i y_3 y_8 \quad (i = 0, 1, \ldots, 2\tau - 1).
\end{align*}
\]
Let $X, Y_3, Y_4, Y_7, Y_8$ be indeterminate whose weight we attached $6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau$, respectively. For each $n \in \mathcal{S}$, we define a monomial $Z_n$ of weight $n$ as follows

$$Z_{6i} = X^i, \quad Z_{j+6\tau+6i} = Y_j X^i \quad \text{and} \quad Z_{11+12\tau+6i} = Y_3 Y_8 X^i.$$ 

By writing the nine products $y_i y_j, (i, j) \neq (3, 8)$ as linear combination of the basis elements we obtain polynomials in the indeterminates $X, Y_3, Y_4, Y_7, Y_8$ that vanish identically on the affine curve $D \cap \mathbb{A}^5$, say

\begin{equation}
\begin{align*}
F_i &= F_i^{(0)} + \sum_{j=0}^{12\tau+i} f_{ij} Z_{12\tau+i-j} \quad (i = 6, 7, 11, 12, 14, 15) \\
G_i &= G_i^{(0)} + \sum_{j=0}^{3\tau+i} g_{ij} Z_{3\tau+i-j} \quad (i = 8, 10, 16),
\end{align*}
\end{equation}

where

$$F_6^{(0)} = Y_3^2 - X^{2\tau+1}, \quad F_7^{(0)} = Y_3 Y_4 - X^\tau Y_7, \quad G_8^{(0)} = Y_4^2 - X^\tau Y_8,$$

$$G_{11}^{(0)} = Y_3^2 Y_7 - X^{2\tau+1} Y_4, \quad F_{14}^{(0)} = Y_7^2 - X^{\tau+1} Y_8, \quad G_{15}^{(0)} = Y_7 Y_8 - X^{\tau+2} Y_3, \quad G_{16}^{(0)} = Y_8^2 - X^{\tau+2} Y_4,$$

and the index $j$ only varies through integers with $12\tau + i - j \in \mathcal{S}$. The proof of the following lemma is very similar to [5, Lemma 4.1].

**Lemma 4.1.** The ideal of the affine curve $D \cap \mathbb{A}^5$ is equal to the ideal $\mathcal{I}$ generated by the forms $F_i$ ($i = 6, 7, 11, 12, 14, 15$) and $G_i$ ($i = 8, 10, 16$). In particular, if $C$ is the canonical monomial curve $\mathcal{C}_S$, then the ideal of the affine monomial curve

$$\mathcal{D}_S \cap \mathbb{A}^5 = \{(t^6, t^{3+6\tau}, t^{4+6\tau}, t^{7+6\tau}, t^{8+6\tau}) : t \in \mathbb{k}\}$$

is generated by the initial forms $F_i^{(0)}$ ($i = 6, 7, 11, 12, 14, 15$) and $G_i^{(0)}$ ($i = 8, 10, 16$).

**Proof.** It is clear that $\mathcal{I} \subseteq I(D \cap \mathbb{A}^5)$. Let $f$ be a polynomial in the variables $X, Y_3, Y_4, Y_7, Y_8$. By applying induction on the degree of $f$ in the indeterminates $Y_3, Y_4, Y_7, Y_8$ we note that, module the ideal generated by the nine forms $F_i, G_i$, the monomials of this polynomial $f$ are not divisible by the nine products $Y_i Y_j, (i, j) \neq (3, 8)$, hence the class of $f$ is a sum $\sum c_n Z_n$ of monomials $Z_n$ of pairwise different weights with $n \in \mathcal{S}$ and $c_n \in \mathbb{k}$. Thus the polynomial $f$ belongs to the ideal of the curve $D \cap \mathbb{A}^5$ if and only if the linear combination $\sum c_n Z_n$ vanishes identically on the curve $D \cap \mathbb{A}^5$ and by taking the corresponding linear combination $\sum c_n x_n$ of rational functions on $\mathbb{k}(C)$ we have $c_n = 0$ for each $n \in \mathcal{S}$, hence $f$ belongs to $\mathcal{I}$. \hfill \Box

Now let us invert the above situation. Taking the fixed above symmetric semigroup $\mathcal{S}$, let us consider the lifting to the polynomial ring $\mathbb{k}[X, Y_3, Y_4, Y_7, Y_8]$ the basis in [3] and [1]. We now introduce nine isobaric polynomials like in [3]. Note that the lifting of the above two basis and the nine isobaric polynomials just depend on the semigroup $\mathcal{S}$. We look for relations on the coefficients $f_{ij}$ and $g_{ij}$ in order that this nine polynomials gives rise a Gorenstein curve whose Weierstrass semigroup is $\mathcal{S}$ at the marked point. We also note that a $P$-hermitian basis of $H^0(C, \omega)$ is not uniquely determinate by the pole divisors $nP$ with $n \in \mathcal{S}$ and $n \leq 2g-2$. In
this way, we keep the $P$-hermitian property by transforming

\[
x \mapsto x + c_6 \\
y_3 \mapsto y_3 + \sum_{i=0}^{\tau} c_{3+6\tau} x^{\tau-i} \\
y_4 \mapsto y_4 + c_1 y_3 + \sum_{i=0}^{\tau} c_{4+6\tau} x^{\tau-i} \\
y_7 \mapsto y_7 + c_3 y_4 + c_4 y_3 + \sum_{i=0}^{\tau+1} c_{1+6\tau} x^{\tau+1-i} \\
y_8 \mapsto y_8 + c'_1 y_7 + c'_4 y_4 + c_5 y_3 + \sum_{i=0}^{\tau+1} c_{2+6\tau} x^{\tau+1-i},
\]

where $c_1, c'_1, c_3, c_4, c'_4, c_5$ and $c_6$ are constants of weight $1, 1, 3, 4, 4, 5$ and $6$, respectively. Assuming that the characteristic of $k$ is different from $2$, and making the lifting of above linear change of variables, we can normalize the following coefficients

\[
f_{7,3+6i} = f_{11,4+6i} = 0 \quad (i = 0, \ldots, \tau) \quad g_{10,1+6i} = f_{11,2+6i} = 0 \quad (i = 0, \ldots, \tau + 1),
\]

\[
f_{12,1} = f_{14,3} = f_{15,4} = g_{16,5} = 0 \quad g_{8,1} = g_{8,4} = g_{16,6} = 0.
\]

Due the above normalizations and those such that $c_{axr} = 1$, the only freedom left us is to transform $x_{n_i} \mapsto c^n x_{n_i} (i = 1, \ldots, g - 1)$, where $c \in k^* = \mathbb{G}_m (k)$. By virtue of theorem 5.1 the isomorphism classes of pointed Gorenstein curves $(C, P)$ determine uniquely the coefficients up to the $\mathbb{G}_m$-action

\[
g_{ij} \mapsto c^j g_{ij} \quad e \quad f_{ij} \mapsto c^j f_{ij},
\]

where $c \in k^*$. We attach to coefficients $f_{ij}, g_{ij}$ the weight $j$. Applying the Syzygy Lemma we get only six syzygies of the affine monomial curve $D_S \cap k^5$

\[
Y_2 F_6^{(0)} - Y_3 F_7^{(0)} + X^\tau G_{10}^{(0)} = 0 \\
XY_4 F_2^{(0)} - Y_7 G_{10}^{(0)} + Y_3 F_{14}^{(0)} - XY_3 G_8^{(0)} = 0 \\
Y_4 F_{11}^{(0)} - Y_7 G_8^{(0)} + Y_8 F_7^{(0)} = 0 \\
Y_4 F_{12}^{(0)} - Y_8 G_6^{(0)} - X^\tau G_{16}^{(0)} = 0 \\
Y_4 F_{14}^{(0)} - Y_8 G_{10}^{(0)} - Y_7 F_{11}^{(0)} = 0 \\
Y_4 F_{15}^{(0)} - Y_8 F_{11}^{(0)} - Y_3 G_{16}^{(0)} = 0.
\]

Replacing $F_i^{(0)}$ and $G_i^{(0)}$ by $F_i$ and $G_i$ on the above syzygies and applying the division algorithm to the cubic monomials that do not belong to lifting of the basis of the vector space $H^0(C, 2(2g-2)P)$, the six syzygies of the affine monomial curve $D_S \cap k^5$ give rise to the following six syzygies module $\Lambda_3$

\[
Y_4 F_6 - Y_3 F_7 + X^\tau G_{10} \equiv \\
- \sum_{i=0}^{\tau-1} X^\tau X^{\tau-1-i} (f_{6,4+6i} F_{12} + f_{6,5+6i} F_{11} - f_{7,6+6i} G_{10}) \\
- \sum_{i=0}^{\tau-1} X^\tau (f_{6,2+6i} G_8 + (f_{6,3+6i} - f_{7,3+6i}) F_7 - f_{7,4+6i} F_6),
\]
\[ XY_4F_7 - Y_7G_{10} + Y_8F_{14} - XY_3G_8 \equiv \\ + \sum_{i=0}^r X^{r-i}(g_{8,1+6i}XG_{10} + g_{10,2+6i}F_{15} + g_{10,3+6i}F_{14}) \\
+ \sum_{i=0}^{r+1} X^{r-i}(g_{8,5+6i}XF_6 + (g_{8,4+6i} - f_{7,4+6i})XF_7 + g_{10,6+6i}F_{11}), \\
- \sum_{i=0}^{r+1} X^{r+1-i}(f_{14,1+6i}G_{10} + f_{14,4+6i}F_7 + f_{14,5+6i}F_6) \\
- \sum_{i=0}^r X^{r-1-i}(f_{7,5+6i}F_{12} + f_{7,6+6i}F_{11}) \]

\[ Y_4F_{11} - Y_7G_{10} + Y_8F_7 \equiv \\ - \sum_{i=0}^r X^{r+1-i}(f_{11,1+6i}G_8 + f_{11,2+6i}F_7) \\
- \sum_{i=0}^{r-1} X^{r-i}(f_{11,3+6i}F_{12} - g_{8,1+6i}F_{14} + (f_{11,4+6i} - g_{8,4+6i})F_{11} - g_{8,5+6i}G_{10}), \\
- \sum_{i=0}^{r-1} X^{r-1-i}(f_{7,5+6i}G_{16} + (f_{7,6+6i} - g_{8,6+6i})F_{15}) \]

\[ Y_4F_{12} - Y_6G_8 - X^rG_{16} \equiv \\ - \sum_{i=0}^r X^{r-i}(-g_{8,1+6i}F_{15} + (f_{12,4+6i} - g_{8,4+6i})F_{12} + f_{12,5+6i}F_{11}) \\
+ \sum_{i=0}^{r+1} g_{8,6+6i}X^{r-1-i}G_{10} - \sum_{i=0}^{r+1} X^{r+1-i}(f_{12,2+6i}G_8 + f_{12,3+6i}F_7), \]

\[ Y_4F_{14} - Y_6G_{10} - Y_7F_{11} \equiv \\ - \sum_{i=0}^r X^{r+1-i}(f_{14,1+6i} - f_{11,1+6i})F_{11} - f_{11,2+6i}G_{10} + f_{14,4+6i}G_8 + f_{14,5+6i}F_7) \\
\sum_{i=0}^r X^{r-i}(g_{10,2+6i}G_{16} + g(10,6+6i - f_{14,6+6i})F_{12}) \\
\sum_{i=0}^r X^{r-i}(g_{10,3+6i} + f_{11,3+6i})F_{15} + f_{11,4+6i}F_{14} \]

\[ Y_4F_{15} - Y_6F_{11} - Y_3G_{16} \equiv \\ - \sum_{i=0}^{r+1} X^{r+1-i}(f_{15,5+6i}G_8 - g_{16,3+6i}G_{10} + f_{15,2+6i}F_{11}) \\
- \sum_{i=0}^r X^{r+1-i}(f_{15,1+6i} - f_{11,1+6i})F_{12} + (f_{15,6+6i} - g_{16,6+6i})F_{7} \]

\[ + \sum_{i=0}^{r+2} X^{r-i}(f_{11,3+6i}G_{16} + f_{11,4+6i}F_{15}) + \sum_{i=0}^{r+2} g_{16,1+6i}X^{r+2-i}F_6. \]

For each of the above six syzygies, the right-hand side differs from the corresponding left-hand side by a linear combination of basis elements of the vector space \( \Lambda_3 \) which are a lifting of the basis elements of \( H^0(\mathcal{C}, 3(2g - 2)P) \). The vanishing of the coefficients of the six linear combinations provides quasi-homogeneous equations.
between the coefficients $f_{ij}$ and $g_{ij}$. To express these equations in a concise manner we introduce polynomials in only one variable. For each $i = 6, 7, 11, 12, 14, 15$ let us consider the polynomial

$$f_i := \sum_{r=1}^{12\tau+i} F_i(t^{-6}, t^{-6-3\tau}, t^{-6-4\tau}, t^{-6-7\tau}, t^{-8-3\tau}) t^{i+12\tau}$$

and we write each one as the sum of its partial polynomials

$$f_i^{(j)} = \sum_{r \equiv j \mod 6} f_{ir} t^r, \ (j = 1, \ldots, 6),$$

which are defined by collecting every terms whose exponents are in the same residue class module 6. Analogously we define the polynomials $g_j$ ($j = 8, 10, 16$) and its partial polynomials $g_j^{(j)}$.

From our normalizations of the constants $f_{ij}, g_{ij}$, the eight partial polynomials $f_{12}, f_{14}, f_{16}, g_{16}, f_7, f_{11}, g_{10}$ and $f_{12}$ are equal to zero, and so we may express each $g_j$ and $f_j$ in terms of the remaining 41 partial polynomials.

The formal degree of the partial polynomials with $i = j$ and $i - j \equiv 6$ (namely: $f_6^{(6)}, f_7^{(1)}, g_8, g_{10}, f_1^{(1)}, f_{12}$ and $f_{12}$, $g_{16}$) is $i + 12\tau$. The partial polynomials $f_6^{(4)}, f_7^{(5)}$, and $g_{16}$ have formal degree $j + 6(\tau - 1)$, and $13 + 6\tau$, respectively. Among the remaining 26 polynomials, 13 have formal degree $j + 6\tau$ and the other ones have formal degree $j + 6(\tau + 1)$. Therefore, the number of the coefficients that are still involved is equal to

$$(2\tau + 1) + 5(2\tau + 2) + 3(2\tau + 3) + 6\tau + 3 + 13(\tau + 1) + 13(\tau + 2) - 3 = 50\tau + 59,$$

where the subtraction by three corresponds to the normalizations $g_{8,1} = g_{8,4} = g_{16,6} = 0$. By virtue of theorem 3.1 we get an explicit construction of a compactification of moduli space $M_{{\mathcal{MS}}_{g,1}}^S$ for each $\tau \geq 1$, as follows.

**Theorem 4.2.** Let $S$ be the semigroup generated by $6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau$ where $\tau$ is a positive integer. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $S$ correspond bijectively to the orbits of the $\mathbb{G}_m$-action on the quasi-cone of the vectors of length $50\tau + 59$ whose coordinates are the coefficients $g_{ij}, f_{ij}$ of the 41 partial polynomials that satisfy the six equations:
\[
f_6 - f_7 + g_{10} = -f_6^{(2)} g_8 - (f_6^{(3)} - f_7^{(3)}) f_7 + f_7^{(4)} f_6 - f_6^{(4)} f_{12} - f_6^{(5)} f_{11} + f_7^{(6)} g_{10},
\]
\[
f_7 - g_{10} + f_{14} - g_8 = (g_8^{(1)} - f_7^{(1)}) g_{10} + g_{10}^{(2)} f_{15} + g_{10}^{(3)} f_{14} + (g_8^{(6)} - f_7^{(6)}) f_{11} + (g_8^{(4)} - f_7^{(4)} - f_7^{(3)}) f_7 + (g_8^{(5)} - f_{14}^{(5)} - f_7^{(5)}) f_6 - f_7^{(6)} f_{12},
\]
\[
f_{11} - g_8 + f_7 = g_8^{(5)} g_{10} - f_{11}^{(1)} g_{10} - f_{11}^{(2)} f_7 - f_{11}^{(5)} g_{16} + (g_8^{(6)} - f_7^{(6)}) f_{15} + g_8^{(1)} f_{14} - (f_7^{(3)} + f_7^{(5)}) f_{12} - (f_7^{(4)} - g_8^{(4)}) f_{11},
\]
\[
f_{12} - g_8 - g_{16} = g_8^{(4)} f_{15} - (f_{12}^{(4)} - g_8^{(4)}) f_{12} - f_{12}^{(5)} f_{11} + g_8^{(6)} g_{10} - f_{12}^{(2)} g_8 - f_{12}^{(3)} f_7,
\]
\[
f_{14} - g_10 - f_{11} = (f_{11}^{(1)} - f_{11}^{(1)}) f_{11} + f_{11}^{(2)} g_{10} - f_{11}^{(4)} g_{8} - f_{14}^{(5)} f_7 + g_{10}^{(2)} f_{16} + (g_8^{(3)} + f_1^{(3)}) f_{15} + f_7^{(4)} f_{14} - (f_7^{(6)} - g_{10}^{(6)}) f_{12},
\]
\[
f_{15} - f_{11} - g_{16} = (f_{11}^{(1)} - f_{15}^{(1)}) f_{12} - (f_{15}^{(6)} - g_{16}^{(6)}) f_7 + g_{16}^{(3)} g_{10} - f_{15}^{(5)} g_8 - f_{15}^{(2)} f_{11} + g_8^{(1)} f_{16} + f_{11}^{(3)} g_{16} + f_{16}^{(3)} f_{15}.
\]

Note that the compactified moduli space \( \overline{M}_{g,1} \) can be embedded into a weighted projective space of dimension \( 50 \tau + 58 \). Now the key is diminish the dimension of the ambient space by projecting this space onto a space of lower dimension. Initially, we take the six equations of the moduli space given by the above theorem and rewritten this equations in terms of 36 polynomial equations between 41 partial polynomials. Among this equations, there are the following six linear equations between the partial polynomials

\[
f_7^{(5)} = f_6^{(5)}, f_{14}^{(5)} = g_8^{(5)} - f_6^{(5)}, g_8^{(4)} = f_7^{(4)}, f_{12}^{(5)} = g_8^{(5)} f_{14}^{(1)} = f_{11}^{(1)}, g_{16}^{(1)} = f_{15}^{(1)} - f_{11}^{(1)}.
\]

With this normalizations we diminish the dimension of the ambient space to \( 44 \tau + 50 \). By analyzing the formal degree in the remaining 30 equations we can eliminate more partial polynomials, until the remaining quasi-homogeneous equations do not admit linear terms. However, even with this procedure, the solution of the remaining polynomial equations are far from being practicable for every \( \tau \geq 1 \), even with a computer.

We avoid this technical difficulty by considering a much simpler algebraic space which contains the moduli variety \( M_{g,1}^S \), which consists of a space given by only the forms of degree 2 of the generators of the ideal of the moduli variety \( M_{g,1}^S \). First we determine the vector space \( T_{g,1}^{S} \) which is, up to an isomorphism, the locus of the linearizations of the 36 equations between the partial polynomials. Indeed the linearizations consist in substituting the right hand side of the equations in theorem by zeros and solving the linear systems in terms of the partial polynomials.

\[
f_7^{(5)} = f_6^{(5)}, f_{14}^{(5)} = g_8^{(5)} - f_6^{(5)}, g_8^{(4)} = f_7^{(4)}, f_{12}^{(5)} = g_8^{(5)} f_{14}^{(1)} = f_{11}^{(1)}, g_{16}^{(1)} = f_{15}^{(1)} - f_{11}^{(1)}.
\]
can solving this system as follows.

\[ f_1^{(1)} = f_1^{(1)} = 0, f_1^{(1)} = g_8^{(1)}, f_1^{(1)} = g_8^{(1)}, f_1^{(1)} = g_8^{(1)}; \\
\]
\[ g_2^{(2)} = 0, g_2^{(2)} = f_2^{(2)}, f_2^{(2)} = f_2^{(2)}, f_2^{(2)} = f_2^{(2)}, f_2^{(2)} = f_2^{(2)}; \\
\]
\[ f_3^{(3)} = f_3^{(3)} = g_{10}^{(3)} = 0, f_3^{(3)} = f_{12}^{(3)} = g_{16}^{(3)} = f_{12}^{(3)}; \\
\]
\[ g_4^{(4)} = 0, f_4^{(4)} = g_8^{(4)}, f_4^{(4)} = g_8^{(4)}, f_4^{(4)} = g_8^{(4)}, f_4^{(4)} = f_4^{(4)} - g_8^{(4)}; \\
\]
\[ f_5^{(5)} = f_6^{(5)}, f_5^{(5)} = f_6^{(5)}, f_5^{(5)} = f_6^{(5)}, f_5^{(5)} = f_6^{(5)} + g_8^{(5)}, f_5^{(5)} = f_6^{(5)} + g_8^{(5)}; \\
\]
\[ f_6^{(6)} = f_7^{(6)} + g_{10}^{(6)}, g_9^{(6)} = f_7^{(6)}, f_6^{(6)} = f_7^{(6)} + g_8^{(6)}, f_6^{(6)} = f_7^{(6)} + g_8^{(6)}; \\
\]

Thus we conclude that the vector space \( T_{k[S]}^{1,-} \) can be identified with the space whose entries are the coefficients of the remaining partial polynomials

\[ g_8^{(1)}, f_6^{(2)}, f_1^{(2)}, f_2^{(2)}, f_3^{(4)}, g_8^{(4)}, f_6^{(4)}, f_6^{(5)}, g_8^{(5)}, f_7^{(6)}, g_9^{(6)}, f_1^{(6)}, f_2^{(6)}, f_6^{(6)}, f_7^{(6)}, g_9^{(6)}; \\
\]

By counting the coefficients of this partial polynomials we have \( 11\tau + 11 \) coefficients, and discounting the conditions corresponding to the three normalizations

\[ g_{9,1} = g_{8,4} = g_{16,6} = 0, \]

we obtain

\[ \dim T_{k[S]}^{1,-} = 11\tau + 8. \]

Now, if we enter with the linearizations into the right hand side of the equations in theorem \( 3.1 \) then we can solve these equations in terms of the partial polynomials which appear only in the linearizations. We solve in a way that on the right hand-side only appear the partial polynomials in the linearizations and the corresponding left hand-side only the partial polynomials which are not in the linearizations. Collecting those equations whose formal degrees of the left hand-sides are less than the corresponding formal degrees of the right hand-sides, we obtain

\[ f_4^{(1)} = -f_6^{(4)} f_2^{(3)} - f_6^{(5)} f_2^{(5)} + f_6^{(6)} g_8^{(1)} + g_8^{(1)}; \\
\]
\[ g_9^{(1)} = f_6^{(1)} f_2^{(2)} - f_6^{(6)} g_8^{(6)} + f_6^{(2)} g_9^{(2)} + f_6^{(3)} g_8^{(1)} - g_8^{(1)}; \\
\]
\[ f_6^{(3)} = f_6^{(6)} g_8^{(6)} + f_6^{(4)} g_8^{(4)} - f_6^{(6)} g_8^{(4)}; \\
\]
\[ f_7^{(4)} = f_6^{(6)} f_1^{(2)} - f_6^{(6)} f_1^{(2)} + f_6^{(4)} f_7^{(2)} - g_6^{(6)} g_8^{(6)} - f_6^{(6)} g_8^{(4)} + f_6^{(4)} g_8^{(4)}; \\
\]
\[ f_8^{(5)} = f_6^{(6)} f_1^{(3)} + f_6^{(6)} g_8^{(4)} + f_6^{(5)} g_8^{(6)} + f_6^{(5)} g_8^{(6)} + f_6^{(5)} g_8^{(6)}; \\
\]

By comparing the formal degrees of the left with the right hand-sides of the above five equations, we introduce the following polynomials equations in the coefficients \( f_{ij} \) and \( g_{ij} \).

\[ \pi_{7+6\tau}(-f_6^{(4)} f_2^{(3)} - f_6^{(5)} f_2^{(5)} + g_8^{(6)} g_8^{(1)}) = 0; \\
\]
\[ \pi_{13+6\tau}(-g_8^{(1)} g_9^{(6)} + f_6^{(2)} g_8^{(5)} + f_6^{(2)} g_8^{(4)}) = 0; \\
\]
\[ \pi_{3+6\tau}(f_6^{(2)} g_8^{(6)} + f_6^{(4)} g_9^{(5)} - f_6^{(6)} g_8^{(4)}) = 0; \\
\]
\[ \pi_{10+6\tau}(f_6^{(2)} f_1^{(2)} + f_6^{(4)} g_9^{(6)} - g_9^{(6)} g_9^{(6)}) = 0; \\
\]
\[ \pi_{11+6\tau}(f_6^{(2)} f_1^{(3)} - f_6^{(5)} g_9^{(6)} + g_9^{(6)} g_9^{(6)}) = 0; \]

where \( g_9^{(6)} = g_9^{(6)} - f_7^{(6)}, g_9^{(6)} = g_9^{(6)} - f_7^{(6)} \) and \( \pi_i \) denotes the projection operator in \( t \) that annihilates the terms of degree not greater than \( i \). The above polynomials equations give rise to an affine quadratic quasi-cone \( \mathcal{Q} \subset \mathbb{A}^{11\tau+8} \), which contains the affine quasi-cone \( \mathcal{X} \subset \mathbb{A}^{11\tau+8} \), where this last one is such that \( \mathcal{M}_{g,1} \cong \mathcal{X} / \mathbb{G}_m \). Hence \( \dim \mathcal{Q} \geq \dim \mathcal{X} = \dim \mathcal{M}_{g,1} + 1 \). This is the method presented in \([5\, & \, 3]\).
We note that the congruences in \( j \) do not depend on the coefficients
\[
f_{6,2}, f_{12,2}, f_{12,3}, g_{8,5}, \tilde{g}_{10,6}, \tilde{g}_{16,6}, f_{12,8}, f_{12,9}, \tilde{g}_{16,12} \text{ and } f_{7,6i}, i = 1, \ldots, \tau - 1.
\]
These congruences depend only on \( 10\tau \) coefficients. They can be expressed in five equations between ten elements of the \( \tau \)-dimensional artinian algebra
\[
A := k[e] = \bigoplus_{j=0}^{\tau-1} ke^j, \text{ where } e^\tau = 0.
\]

**Theorem 4.3.** The quadratic quasi-cone \( Q \) is isomorphic to the direct product
\[
Q = M \times N,
\]
where \( M \) is the \( (\tau + 8) \)-dimensional weighted space of weights 2, 2, 3, 5, 6, 6, 8, 9, 12 and 6i, \( i = 1, \ldots, \tau - 1 \), and \( N \) is the quadratic quasi-cone consisting of vectors
\[
(\omega_1, \ldots, \omega_{10}) = \left( \sum_{j=0}^{\tau-1} \omega_{1j} e^j, \ldots, \sum_{j=0}^{\tau-1} \omega_{10j} e^j \right),
\]
such that satisfying the five equations
\[
\begin{align*}
\omega_4\omega_9 - \omega_3\omega_7 - \omega_2\omega_8 &= 0, \\
\omega_6\omega_7 - \omega_4\omega_{10} + \omega_5\omega_8 &= 0, \\
\omega_1\omega_4 + \omega_2\omega_6 - \omega_3\omega_5 &= 0, \\
\omega_1\omega_7 + \omega_2\omega_{10} - \omega_5\omega_9 &= 0, \\
\omega_6\omega_9 - \omega_3\omega_{10} + \omega_1\omega_8 &= 0,
\end{align*}
\]
in the artinian algebra \( A \).

**Proof.** Defining
\[
\begin{align*}
\omega_{1j} &= f_{6,6\tau+2-6j} & \omega_{2j} &= f_{6,6\tau-2-6j}, \\
\omega_{3j} &= f_{6,6\tau-1-6j} & \omega_{4j} &= g_{8,6\tau-5-6j}, \\
\omega_{5j} &= g_{8,6\tau-2-6j} & \omega_{6j} &= g_{8,6\tau+5-6j}, \\
\omega_{7j} &= f_{12,6\tau+8-6j} & \omega_{8j} &= f_{12,6\tau+9-6j}, \\
\omega_{9j} &= g_{10,6\tau+6-6j} & \omega_{10j} &= g_{16,6\tau+6-6j},
\end{align*}
\]
it is sufficient to observe that the conditions on the \( 10\tau \) coefficients are equivalents to the five quadratic equations in the artinian algebra \( A \). \( \square \)

By applying induction on \( \tau \) one can proof that

**Corollary 4.4.** \( \dim Q = 8\tau + 8 \) and hence
\[
\dim \mathcal{M}_{g,1}^S \leq 8\tau + 7.
\]

4.2. **A second family of symmetric semigroups.** We apply the same method above for the following particular family of symmetric semigroups. For each \( \tau \geq 1 \), let
\[
\mathcal{S} = \langle 6, 7 + 6\tau, 8 + 6\tau, 9 + 6\tau, 10 + 6\tau \rangle
\]
\[
= \mathbb{N} \cup \bigcup_{j \in \{7, 8, 9, 10\}} (j + 6\tau + 6\mathbb{N}) \sqcup (17 + 12\tau + 6\mathbb{N}),
\]
be a symmetric semigroup of genus \( g = 6 + 6\tau \). We note if \( \tau \) is equal to zero, \( \mathcal{S} \) is also a symmetric semigroup of multiplicity 6 generated minimally by five elements. However the ideal of the canonical monomial curve \( C_{\mathcal{S}} \) can not be generate by only
quadratic forms, this special cases were treated in a recent preprint by Contiero and Fontes in [4]. Since \( \tau \geq 1 \) and the method is the same of the preceding subsection, we make a lot of shortcuts and we do not explain the method again.

Let \( C \) be a complete integral Gorenstein curve and \( P \) be a nonsingular point of \( C \) whose Weierstrass semigroup at \( P \) is \( S \). For each \( n \in S \) let \( x_n \) be a rational function on \( C \) with pole divisor \( nP \). Let us consider

\[
x := x_6 \text{ and } y_j := x_{j+6\tau} \quad (j = 7, 8, 9, 10) \text{ with } x_{6i} = x^i, x_{j+6\tau+6i} = x^iy_j, \quad i \geq 1.
\]

Since \( C \) is a nonhyperelliptic curve, it can be canonically embedded in \( \mathbb{P}^{g-1} \), and the projection map

\[
(1 : x : y_7 : y_8 : y_9 : y_{10}) : C \hookrightarrow \mathbb{P}^5
\]

defines an isomorphism of the canonical curve \( C \) onto a curve \( D \subset \mathbb{P}^5 \) of degree \( 6\tau + 10 \). A \( P \)-hermitian basis of the vector space \( H^0(C, (4g - 4)P) \) is

\[
x^i \quad (i = 0, 1, \ldots, 4\tau + 3),
\]

\[
x^iy_j \quad (i = 0, 1, \ldots, 3\tau + 2, \quad j = 7, 8),
\]

\[
x^iy_j \quad (i = 0, 1, \ldots, 3\tau + 1, \quad j = 9, 10),
\]

\[
x^iy_7y_{10} \quad (i = 0, 1, \ldots, 2\tau).
\]

For a nongap \( n \in S \), the monomial \( Z_n \) of weight \( n \) is

\[
Z_{6i} = X^i, \quad Z_{j+6\tau+6i} = Y_jX^i \text{ and } Z_{11+12\tau+6i} = Y_7Y_{10}X^i.
\]

Writing the nine products \( y_iy_j, (i,j) \neq (7,10) \) as linear combination of the basis elements of the \( k \)-vector space \( H^0(C, 2(2g - 2)P) \) we obtain, in the indeterminates \( X, Y_7, Y_8, Y_9, Y_{10}, \) the polynomials

\[
F_i = F_i^{(0)} + \sum_{j=0}^{12\tau+1} f_{ij}Z_{12\tau+i-j} \quad (i = 14, 15, 16, 17, 18)
\]

\[
G_i = G_i^{(0)} + \sum_{j=0}^{12\tau+1} g_{ij}Z_{12\tau+i-j} \quad (i = 16, 18, 19, 20),
\]

that vanish identically on the affine curve \( D \cap \mathbb{A}^5 \), where

\[
F_1^{(0)} = Y_7^2 - X^{\tau+1}Y_8, \quad F_5^{(0)} = Y_7Y_8 - X^{\tau+1}Y_9, \quad F_6^{(0)} = Y_7Y_9 - X^{\tau+1}Y_{10},
\]

\[
G_6^{(0)} = Y_8^2 - X^{\tau+1}Y_{10}, \quad F_7^{(0)} = Y_8Y_9 - Y_7Y_{10}, \quad F_8^{(0)} = Y_8Y_{10} - X^{2\tau+3},
\]

\[
G_8^{(0)} = Y_9^2 - X^{2\tau+3}, \quad G_9^{(0)} = Y_9Y_{10} - X^{\tau+2}Y_7, \quad G_{10}^{(0)} = Y_7^2 - X^{\tau+2}Y_8.
\]

Lemma 4.5. The ideal of the affine curve \( D \cap \mathbb{A}^5 \) is equal to the ideal \( \mathcal{I} \) generated by the above forms \( F_i \) and \( G_i \). In particular, if \( C \) is the canonical monomial curve \( C_S \), then the ideal of the affine monomial curve

\[
D_S \cap \mathbb{A}^5 = \{ (t^6, t^7+6\tau, t^8+6\tau, t^9+6\tau, t^{10}+6\tau) : t \in k \}
\]

is generated by the initial forms \( F_i^{(0)} \) and \( G_i^{(0)} \).

Inverting the above situation and considering the polynomials \( F_i \) and \( G_i \), just induced by the semigroup \( S \), we normalize the coefficients

\[
f_{18,1} = g_{18,1} = g_{19,2} = g_{20,3} = 0, \quad f_{15,6} = f_{16,2} = g_{16,1} = 0
\]
Hence we obtain seven polynomial equations module $\Lambda_3$

\[
Y_{10} F_{14} - Y_8 F_{16} + Y_7 F_{17} \equiv \\
\quad - \sum_{i=0}^{\tau+1} X^{\tau+1-i}[(f_{17,3+6i} - f_{16,3+6i}) F_{15} + f_{17,2+6i} F_{16} - f_{16,2+6i} G_{16}]
\]
\[
\quad - \sum_{i=0}^{\tau} X^{\tau-i}[(f_{14,6+6i} - f_{16,6+6i}) F_{18} + f_{14,5+6i} G_{19} + f_{14,4+6i} G_{20}],
\]
\[
Y_{10} F_{15} - Y_9 G_{16} + Y_8 F_{17} \equiv \\
\quad \sum_{i=0}^{\tau} X^{\tau-i}[(g_{16,6+6i} - f_{15,6+6i}) G_{19} - f_{15,5+6i} G_{20}]
\]
\[
\quad + \sum_{i=0}^{\tau+1} X^{\tau+1-i}[(g_{16,3+6i} F_{16} - f_{17,3+6i} G_{16} - (f_{17,2+6i} - g_{16,2+6i}) F_{17}
\quad - (f_{15,1+6i} + f_{17,1+6i}) F_{18} + g_{16,1+6i} G_{18}],
\]
\[
Y_{10} G_{16} - Y_8 F_{18} + X^{\tau+1} G_{20} \equiv - \sum_{i=0}^{\tau} X^{\tau-i} g_{16,6+6i} G_{20} +
\]
\[
\quad \sum_{i=0}^{\tau+1} X^{\tau+1-i}[(f_{18,5+6i} F_{15} + f_{18,4+6i} G_{16} + f_{18,3+6i} F_{17} - g_{16,2+6i} F_{18} - g_{16,1+6i} G_{19}],
\]
\[
Y_{10} F_{17} - Y_8 G_{19} + Y_7 G_{20} \equiv \\
\quad \sum_{i=0}^{\tau+1} X^{\tau+1-i}[(g_{19,6+6i} - g_{20,6+6i}) F_{15} - g_{20,5+6i} F_{16} + g_{19,5+6i} G_{16} + g_{19,4+6i} F_{17}
\quad - f_{17,3+6i} F_{18} - f_{17,2+6i} G_{19} - f_{17,1+6i} G_{20}] - \sum_{i=0}^{\tau+2} X^{\tau+2-i} g_{20,1+6i} F_{14},
\]
\[
Y_{10} F_{18} - X^{\tau+2} G_{16} - Y_8 G_{20} \equiv \sum_{i=0}^{\tau+2} X^{\tau+2-i} g_{20,1+6i} F_{15}
\]
\[
\quad + \sum_{i=0}^{\tau+1} X^{\tau+1-i}[(g_{20,6+6i} G_{16} + g_{20,5+6i} F_{17} - (f_{18,4+6i} - g_{20,4+6i}) F_{18} - f_{18,3+6i} G_{19}],
\]
\[
Y_{10} G_{18} - X^{\tau+2} F_{16} - Y_9 G_{19} \equiv \sum_{i=0}^{\tau+1} X^{\tau+1-i}[(g_{19,6+6i} F_{16} + g_{19,5+6i} F_{17}
\quad + g_{19,4+6i} G_{18} - g_{18,4+6i} F_{18} - g_{18,3+6i} G_{19} - g_{18,2+6i} G_{20}],
\]
\[ Y_{10}G_{19} - X^{\tau+2}F_{17} - Y_9G_{20} = \sum_{i=0}^{\tau+2} X^{\tau+2-i}g_{20,1+6i}F_{16} + \sum_{i=0}^{\tau+1} X^{\tau+1-i}[g_{20,6+6i}F_{17} + g_{20,5+6i}G_{18} - g_{19,5+6i}F_{18} - (g_{19,4+6i} - g_{20,4+6i})G_{19}]. \]

Theorem 4.6. Let \( S \) be the semigroup generated by \( 6, 7+6\tau, 8+6\tau, 9+6\tau \) and \( 10+6\tau \) where \( \tau \) is a positive integer. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup \( S \) correspond bijectively to the orbits of the \( G_m \)-action on the quasi-cone of the vectors of length \( 50\tau + 84 \) whose coordinates are the coefficients \( g_{ij}, f_{ij} \) of the 41 partial polynomials that satisfy the seven equations:

\[
\begin{align*}
    f_{18} - g_{16} - g_{20} &= g_{20}g_{16} + \left( g_{20}g_{16} \right) f_{17} - \left( f_{18} - g_{20} \right) f_{18} - f_{19} + f_{20} + 2g_{15}, \\
    g_{18} - f_{16} - g_{19} &= g_{16}g_{19} + \left( g_{16}g_{19} \right) f_{17} + \left( g_{16}g_{19} \right) g_{18} - g_{18}f_{18} - g_{18}g_{19} \quad \text{or} \\
    g_{16} - f_{18} + g_{20} &= f_{16}f_{15} + f_{14}g_{16} + f_{13}g_{17} - g_{16}f_{18} - g_{16}g_{19} \quad \text{or} \\
    f_{14} - f_{16} + f_{17} &= f_{16}g_{16} - \left( f_{17} - f_{13} \right) f_{15} - f_{17}f_{16} - \left( f_{16} - f_{16} \right) f_{18} \quad \text{or} \\
    f_{15} - g_{16} + f_{17} &= \left( g_{16}g_{19} \right) f_{18} + g_{16}g_{19}f_{16} - f_{17}g_{16} - f_{15}g_{16} - f_{15}g_{20} \quad \text{or} \\
    g_{17} - g_{19} + g_{20} &= g_{19}f_{17} + \left( g_{19}f_{17} \right) f_{18} - g_{20}f_{16} - g_{19}f_{16}f_{15} \quad \text{or} \\
    &- f_{17}f_{18} - f_{17}g_{19} - f_{17}g_{20} - g_{20}f_{14}. 
\end{align*}
\]

The linearizations depend only on the 11 following partial polynomials

\[
\begin{align*}
    f_{14}^{(1)}, f_{17}^{(2)}, f_{16}^{(2)}, g_{16}^{(3)}, f_{14}^{(4)}, g_{20}^{(4)}, f_{14}^{(5)}, f_{15}^{(5)}, f_{14}^{(6)}, f_{15}^{(6)} \quad \text{and} \quad g_{20}^{(6)}. 
\end{align*}
\]

Counting its coefficients and discounting the three normalizations \( f_{15,6} = f_{16,2} = g_{16,1} = 0 \), we obtain \( 11\tau + 15 \) coefficients. Thus

\[
\dim T_{\mathbb{K}[S]\mathbb{K}} = 11\tau + 15.
\]

The equations of the affine quadratic quasicone \( Q \) are given by:

\[
\begin{align*}
\pi_{13+6\tau}(f_{14}^{(1)} - g_{20}^{(6)} + f_{14}^{(4)} + f_{14}^{(5)} + f_{14}^{(6)} - f_{17}^{(5)} + g_{16}^{(4)} - g_{20}^{(4)} &= 0, \\
\pi_{7+6\tau}(f_{14}^{(1)} - f_{14}^{(2)} + f_{14}^{(3)} - f_{14}^{(4)} + f_{14}^{(5)} + f_{14}^{(6)} &= 0, \\
\pi_{9+6\tau}(f_{14}^{(1)} - f_{14}^{(2)} + f_{14}^{(4)} - f_{14}^{(5)} + f_{14}^{(6)} &= 0, \\
\pi_{10+6\tau}(f_{14}^{(2)} - f_{14}^{(4)} - f_{14}^{(6)} &= 0, \\
\pi_{11+6\tau}(f_{14}^{(3)} - f_{14}^{(5)} - f_{14}^{(6)} &= 0, \\
\pi_{11+6\tau}(f_{14}^{(5)} - f_{14}^{(6)} &= 0, \\
\pi_{11+6\tau}(f_{14}^{(6)} &= 0, \\
\end{align*}
\]

where \( g_{20}^{(6)} = g_{20}^{(6)} - f_{14}^{(6)} - f_{15}^{(6)} - f_{16}^{(6)} \) and \( \pi_i \) denotes the projection operator in \( t \) that annihilates the terms of degree not greater than \( i \). We can observe that
these equations does not depend of the coefficients $f_{14,1}, f_{17,2}, g_{16,3}, g_{20,4}, f_{18,5}$ and $f_{14,6}, i = 2, \ldots, \tau + 1$. By considering the $(\tau + 1)$-dimensional artinian algebra
\[ A := k[\epsilon] = \bigoplus_{j=0}^{\tau} k\epsilon^j, \text{ where } \epsilon^{\tau+1} = 0, \]
we can write the equations in \((7)\) in terms of five polynomial equations between $\tau + 1$ elements of the $A$.

**Theorem 4.7.** The quadratic quasi-cone $Q$ is isomorphic to the direct product
\[ Q = M \times N, \]
where $M$ is the $(\tau + 5)$-dimensional weighted space of weights $1,2,3,4,5$ and $6i, i = 2, \ldots, \tau + 1$, and $N$ is the quadratic quasi-cone consisting of vectors
\[ (\omega_1, \ldots, \omega_{10}) = \left( \sum_{j=0}^{\tau} \omega_{1j} \epsilon^j, \ldots, \sum_{j=0}^{\tau} \omega_{10j} \epsilon^j \right), \]
such that satisfying the five equations
\begin{align*}
\omega_1\omega_{10} + \omega_2\omega_5 + \omega_3\omega_7 - \omega_4\omega_8 + \omega_6\omega_9 &= 0, \\
-\omega_1\omega_9 - \omega_2\omega_5 - \omega_3\omega_7 &= 0, \\
-\omega_1\omega_2 - \omega_5\omega_8 - \omega_6\omega_7 &= 0, \\
\omega_{13} + \omega_5(\omega_{10} - \omega_9) - \omega_6\omega_9 &= 0, \\
\omega_{19} - \omega_1\omega_{10} - \omega_5\omega_9 + \omega_6\omega_4 &= 0,
\end{align*}
in the artinian algebra $A$.

**Corollary 4.8.** We have $\dim Q_S = 8\tau + 12$. Thus
\[ \dim M^S_{g,1} \leq 8\tau + 11. \]

5. Collecting known dimensions

As noted in section 2 of this paper, if a numerical semigroup $S$ is negatively graded then the dimension of $M^S_{g,1}$ is equal to Deligne–Pinkham’s upper bound $2g - 2 + \lambda(S)$, which is also equal to Pfleuger’s bound $3g - 2 - \text{ewt}(S)$. Additionally, it is also know, cf. [10], that the dimension of $M^S_{g,1}$ is equal to Pfleuger’s bound for all numerical semigroups whose genus is not greater than $6$.

In the following table we collect all numerical semigroups of genus $g \leq 6$ and compare the bounds given by Deligne–Pinkham and Pfleuger. Of course we just consider non-negatively graded numerical semigroups. Notations, D–P stands for the Deligne–Pinkham’s bound, NP for Pfleuger’s bound, and finally $\dim T^1: = \sum_{s=1}^{\infty} \dim T^1(k[S])$, see theorem 5.3 below and [3].

Let us now compare, see table below, the lower bound given by Pfleuger with the upper bound obtained in section four of this paper, cf. corollaries 4.4 and 4.8. We also include the upper bound obtained by Contiero and Stoehr in [3] Cor. 4.5 for the symmetric semigroup $\langle 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau \rangle$ with $\tau \geq 1$.

We summarize the dimensions of $M^S_{g,1}$, which appears in table and table, and also the dimension of $M^S_{g,1}$ given by the theorem of Rim and Vitulli on negatively graded semigroups, in the next corollary.
ON THE DIMENSION OF $\mathcal{M}_{g,1}^S$

Table 1. non-negatively graded semigroups of genus $\leq 6$

| gaps       | NP | $\dim \mathcal{M}_{g,1}^S$ | D–P | $\dim T^{1,+}$ |
|------------|----|----------------------------|-----|----------------|
| 1, 2, 4, 5, 8 | 9  | 9                          | 10  | 1              |
| 1, 2, 3, 5, 7 | 10 | 10                         | 11  | 1              |
| 1, 2, 3, 6, 7 | 9  | 9                          | 10  | 1              |
| 1, 2, 4, 5, 7, 10 | 11 | 11                         | 12  | 1              |
| 1, 2, 4, 5, 8, 11 | 10 | 10                         | 11  | 1              |
| 1, 2, 3, 5, 6, 9 | 12 | 12                         | 13  | 1              |
| 1, 2, 3, 5, 6, 10 | 11 | 11                         | 12  | 1              |
| 1, 2, 3, 5, 7, 9 | 11 | 11                         | 13  | 2              |
| 1, 2, 3, 5, 7, 11 | 10 | 10                         | 11  | 1              |
| 1, 2, 3, 6, 7, 11 | 10 | 10                         | 11  | 1              |
| 1, 2, 3, 4, 6, 8 | 13 | 13                         | 14  | 1              |
| 1, 2, 3, 4, 6, 9 | 12 | 12                         | 13  | 1              |
| 1, 2, 3, 4, 7, 8 | 12 | 12                         | 13  | 1              |
| 1, 2, 3, 4, 7, 9 | 11 | 11                         | 12  | 1              |
| 1, 2, 3, 4, 8, 9 | 10 | 10                         | 12  | 2              |

Table 2. $\dim \mathcal{M}_{g,1}^S$ for three families of semigroups

| semigroup                                           | NP       | CFV-CS  | D–P  | $\dim T^{1,+}$ |
|----------------------------------------------------|----------|---------|------|----------------|
| $\langle 6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau \rangle$ | $8\tau + 7$ | $8\tau + 7$ | $12\tau + 5$ | $4\tau - 2$ |
| $\langle 6, 7 + 6\tau, 8 + 6\tau, 9 + 6\tau, 10 + 6\tau \rangle$ | $8\tau + 11$ | $8\tau + 11$ | $12\tau + 11$ | $4\tau$     |
| $\langle 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau \rangle$ | $8\tau + 5$ | $8\tau + 5$ | $12\tau + 1$ | $4\tau - 4$ |

Corollary 5.1. For each numerical semigroup $S$ of genus $g \leq 6$, or any non-negatively graded numerical semigroup $S$, or one of the following symmetric semigroups $\langle 6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau \rangle$, $\langle 6, 7 + 6\tau, 8 + 6\tau, 9 + 6\tau, 10 + 6\tau \rangle$ or $\langle 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau \rangle$, we get

$$3g - 2 - \text{ewt}(S) = \dim \mathcal{M}_{g,1}^S = 2g - 2 + \lambda(S) - \dim T^{1,+}(k[S]).$$

Remark 5.2. We also recall that for the following two symmetric semigroups $S = \langle 6, 7, 8 \rangle$ and $S = \langle 6, 7, 15 \rangle$, Pflueger’s bound does not provide the exact dimension of the $\mathcal{M}_{g,1}^S$, but one can see that $\dim \mathcal{M}_{g,1}^S = 2g - 2 + \lambda(S) - \dim T^{1,+}(k[S])$ for this two particular semigroups.

The dimension of the homogeneous part of degree $\ell$ of the cotangent complex $T^1(k[S])$ can be easily computed using a description of the cotangent complex given by Buchsweitz in [3]. Let $S := \langle a_1, \ldots, a_r \rangle$ be a numerical semigroup of genus $g > 1$. By a theorem due to Herzog, the ideal of $C_S := \{(t^{a_1}, \ldots, t^{a_r}) ; t \in k\} \subset k^r$ can be generated by isobaric polynomials $F_i$ which are differences of two monomials

$$F_i := X_1^{\alpha_{i1}} \cdots X_r^{\alpha_{ir}} - X_1^{\beta_{i1}} \cdots X_r^{\beta_{ir}}$$

with $\alpha_i \cdot \beta_i = 0$. As usual, the weight of $F_i$ is $d_i := \sum_j a_j \alpha_{ij} = \sum_j a_j \beta_{ij}$. For each $i$ let $v_i := (\alpha_{i1} - \beta_{i1}, \ldots, \alpha_{ir} - \beta_{ir})$ be a vector in $k^r$. 
Theorem 5.3 (cf. Thm. 2.2.1 of [3]). For each $\ell \notin \text{End}(S)$,
\[
\dim T^1(k[S])_\ell = \# \{i \in \{1, \ldots, r\}; a_i + \ell \notin S\} - \dim V_\ell - 1
\]
where $V_\ell$ is the subvector space of $k^r$ generated by the vectors $v_i$ such that $d_i + \ell \notin S$.

It also true that
\[
\dim T^1(k[S])_s = 0, \quad \forall s \in \text{End}(S).
\]

The following question was shared (in private communications) with a large number of specialist in the field of deformation theory and moduli of curves. We do not know a partial answer or even an example where the inequality fails.

Question 5.1. For which numerical semigroups it is true that
\[
\dim \mathcal{M}_{g,1}^S \leq 2g - 2 + \lambda - \dim T^{1,+}(k[S])?
\]

The last result of this paper shows that Pfueger’s lower bound can not be greater than $2g - 2 + \lambda(S) - \dim T^{1,+}(S)$.

Lemma 5.4. For any numerical semigroup $S$ of genus $g \geq 1$,
\[
3g - 2 - \text{ewt}(S) \leq 2g - 2 + \lambda(S) - \dim T^{1,+}(S).
\]

Proof. For each $\ell \in \mathbb{Z}$, set $A_\ell := \{i \in \{1, \ldots, r\}; i + \ell \notin S\}$. Using the theorem we obtain
\[
\dim T^{1,+}(S) = \sum_{\ell \notin \text{End}(S)} (\#A_\ell - \dim V_\ell) - g + \lambda(S).
\]

Hence, we just have to prove that $\text{ewt}(S) - \sum_{\ell \notin \text{End}(S)} \#A_\ell \geq 0$. We proceed by induction on the genus $g$ of $S$. The statement is trivial for $g = 1$. If $S$ is a numerical semigroup of genus $g > 1$, whose biggest gap is $\ell_g$, then consider the numerical semigroup $S' := S \cup \{\ell_g\}$, whose genus is $g - 1 \geq 1$. It is clear that $\{\ell \notin \text{End}(S)\} = \{\ell \notin \text{End}(S')\} \bigcup \{\ell \mid \ell + a_j = \ell_g \text{ and } \ell + a_j \in S, \forall j \neq i\}$. Now the result follows easily. \qed

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