Global existence and estimates of the solutions to nonlinear integral equations

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Abstract

It is proved that a class of nonlinear integral equations of the Volterra-Hammerstein type has a global solution, that is, solutions defined for all \( t \geq 0 \), and estimates of these solutions as \( t \to \infty \) are obtained. The argument uses a nonlinear differential inequality which was proved by the author and has broad applications.

1 Introduction

Consider the equation:

\[
  u(t) = \int_0^t e^{-a(t-s)} h(u(s)) ds + f(t) := T(u), \quad t \geq 0; \quad a = \text{const} > 0.
\] (1)

that is, Volterra-Hammerstein equation. There is a large literature on nonlinear integral equations, [6], [1]. The usual methods to study such equations include fixed-point theorems such as contraction mapping principle and degree theory, (Schauder and Leray-Schauder theorems). The goal of this paper is to give a new approach to a study of equation (1). We give sufficient conditions for the global existence of solutions to (1) and their estimates as \( t \to \infty \).

Denote \( f' := \frac{df}{dt} \). By \( c > 0 \) various constants will be denoted.

Let us formulate our assumptions:

\[
  |h(u)| \leq c|u|^b, \quad |h'(u)| \leq c|u|^{b-1}, \quad b \geq 2,
\] (2)

\[
  |f(t)| + a|f'(t)| \leq ce^{-a_1t}, \quad a_1 = \text{const} > 0.
\] (3)

By \( c > 0 \) various constants are denoted.

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Our approach is based on the author's results on the nonlinear differential inequality formulated in Theorem 1 (see [2]–[5]). These results have been used by the author in a study of stability of solutions to abstract nonlinear evolution problems ([5]).

Denote \( R_+ = [0, \infty) \).

**Theorem 1.** Let \( g \geq 0 \) solve the inequality

\[
g'(t) \leq -ag(t) + \alpha(t, g) + \beta(t), \quad t \geq 0, \quad a = \text{const} > 0,
\]

where \( \alpha(t, g) \geq 0 \) and \( \beta(t) \geq 0 \) are continuous functions of \( t \), \( t \in \mathbb{R}_+ \) and \( \alpha(t, g) \) is locally Lipschitz with respect to \( g \). If there exists a function \( \mu(t) > 0 \), defined on \( \mathbb{R}_+ \), \( \mu \in C^1(\mathbb{R}_+) \), such that

\[
\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leq \frac{1}{\mu(t)} \left( a - \frac{\mu'(t)}{\mu(t)} \right), \quad \forall t \geq 0,
\]

and

\[
g(0)\mu(0) \leq 1,
\]

then \( g \) exists on \( \mathbb{R}_+ \) and

\[
0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0.
\]

A proof of Theorem 1 can be found in [5]. Its idea is described in Section 2.

The result of this paper is formulated in Theorem 2.

**Theorem 2.** Assume that \( (2) \) and \( (3) \) hold, \( a \geq 2, \ b \geq 2, \ c \in (0, 0.75), \ p \in (0, \min(0.75a, a_1)), \ R = (b-1)^{1/b} \). Then any solution to \( (1) \) exists on \( \mathbb{R}_+ \) and satisfies the estimate

\[
|u(t)| \leq R^{-1}e^{-pt}, \quad \forall t \geq 0, \quad p \in (0, \min(0.25a_1, a)).
\]

In Section 2 Theorem 2 is proved.

## 2 Proof of Theorem 2

Let us reduce equation \( (1) \) to the form suitable for an application of Theorem 1. Differentiate \( (1) \) and get

\[
u' = f' - a \int_0^t e^{-a(t-s)}h(u(s))ds + h(u(t)).
\]

Let \( g(t) := |u(t)| \) and take into account that \( |F(t)| \leq ce^{a_1t} \).

From \( (1) \) one gets \( \int_0^t e^{-a(t-s)}h(u(s))ds = u - f \). This and equation \( (9) \) imply \( u' = f' - a(u - f) + h(u(t)) \). Therefore, one gets

\[
 u' = -au + h(u) + F, \quad F := f' + af
\]

Multiply \( (10) \) by \( \overline{u} \), where \( \overline{u} \) stands for complex conjugate of \( u \), and get

\[
u'u = -ag^2 + h(u)\overline{u} + F\overline{u}.
\]
One has

\[ u'u + u(\overline{u})' = \frac{dg^2}{dt} = 2gg'. \]  

(12)

We define the derivative as \( g' = \lim_{h \to +0} \frac{g(t+h) - g(t)}{h} \). With this definition, \( g(t) \) is differentiable at every point if \( u(t) \) is continuously differentiable for all \( t \geq 0 \). Any solution \( u(t) \) to (11) is continuously differentiable under our assumptions. Take complex conjugate of (11), add the resulting equation to (11) and take into account (12). This yields

\[ 2gg' = -2ag^2 + 2Re(h(u)\overline{u}) + 2Re(F\overline{u}). \]  

(13)

Since \( g \geq 0 \), one derives from (13), using assumptions (2) and (3), that

\[ g'(t) \leq -ag(t) + cg^b + ce^{-a_1t}. \]  

(14)

Let

\[ \mu(t) = Re^{pt}, \quad R = const > 0, \quad p \in (0, \min(0.25a, a_1)). \]  

(15)

Condition (5) can be written as

\[ \frac{c}{Re^{pt}} + ce^{-a_1t} \leq \frac{1}{Re^{pt}}(a - p), \quad t \in \mathbb{R}_+. \]  

(16)

This inequality holds if

\[ \frac{c}{Re^{pt}}e^{(b-1)pt} + cRe^{-(a_1-p)t} \leq \frac{3a}{4}, \quad t \in \mathbb{R}_+. \]  

(17)

Inequality (17) holds if

\[ \frac{1}{Re^{pt}} + R \leq \frac{3a}{4c}. \]  

(18)

The minimum of the left side of (18) is attained at \( R = (b-1)^{1/b} \) and is equal to \( \frac{b}{(b-1)^{1/b}} \).

Thus, (18) holds if

\[ \frac{b}{(b-1)^{1/b}} \leq \frac{3a}{4c}. \]  

(19)

For example, assume that \( a \geq 2, \quad c \leq 0.75. \)

Then (19) holds if \( b \leq 2(b-1)^{(b-1)/b} \), that is, if

\[ b^b \leq 2^b(b-1)^{b-1}. \]  

(20)

Inequality (20) holds if \( b \geq 2 \). Thus, by Theorem1, any solution \( u(t) \) of (11) exists globally and

\[ |u(t)| \leq \frac{e^{-pt}}{R}, \]  

(21)

provided that

\[ |u(0)|R \leq 1, \quad R = (b-1)^{1/b}, \quad a \geq 2, \quad b \geq 2, \quad c = 0.75, \quad p \in (0, \min(0.25a, a_1)). \]  

(22)
Inequality $|u(0)|/R \leq 1$ holds if $f(0)/R \leq 1$. By assumption (3) this inequality holds if $c \leq 1/R$. Theorem 2 is proved. □

Let us prove existence of a solution to (1) using the contraction mapping principle and Theorem 2.

By estimate (21) one has $|u(t)| \leq 1/R$ for all $t \geq 0$. Therefore, using assumptions (2) and (3), one gets

$$\|Tu\| \leq c + \frac{c}{aR^b} \leq \frac{1}{R}, \quad (23)$$

provided that $cR \leq \frac{1}{1+aR^b}$. For $R = (b-1)^{1/b}$ this inequality holds if $c$ is sufficiently small. If $cR \leq \frac{1}{1+aR^b}$, then $T$ maps the ball $B_R := \{u : \|u\| \leq \frac{1}{R}\}$ into itself. Here $\|u\| = \max_{t \geq 0}|u(t)|$.

On the ball $B_R$ the operator $T$ is a contraction:

$$\|Tu - Tv\| \leq \| \int_0^t e^{-a(t-s)}c|\eta|^{b-1}|ds||u - v\| \leq \frac{c}{R^{b-1}a} \|u - v\|, \quad (24)$$

where the assumption (2) was used, and $\eta$ is the "middle" element between $u$ and $v$, $\|\eta\| \leq \frac{1}{R}$. The integral in (24) is estimated as follows:

$$\| \int_0^t e^{-a(t-s)}c|\eta|^{b-1}|ds\| \leq \frac{c}{R^{b-1}a} \max_{t \geq 0} \int_0^t e^{-a(t-s)}ds \leq \frac{c}{R^{b-1}a} \quad (25)$$

If

$$\frac{c}{R^{b-1}a} < 1, \quad (26)$$

then $T$ is a contraction on $B_R$. Condition (26) holds if $c$ is sufficiently small. Thus, if condition (26) and the assumptions of Theorem 2 hold, then, by the contraction mapping principle, there exists a unique solution to (1) in the ball $B_R$. □

For convenience of the reader we sketch the idea of the proof of Theorem 1 following [2]—[5].

Inequality (5) can be written for the function $w = \frac{1}{\mu}$ as follows:

$$-aw + \alpha(t, w) + \beta(t) \leq w'. \quad (27)$$

From (4) and (27) by a comparison lemma for ordinary differential equations it follows that

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad (28)$$

provided that $g(0) \leq w(0) = \frac{1}{\mu(0)}$. The last inequality is the assumption (3). Since $\mu(t) > 0$ and is assumed to be defined for all $t \geq 0$, the function $w = \frac{1}{\mu}$ is defined for all $t \geq 0$. Since $0 \leq g(t) \leq \frac{1}{\mu(t)}$, and $g(t) := |u(t)|$, the function $u$ is defined for all $t \geq 0$.

If $\lim_{t \to \infty} \mu(t) = 0$, then $\lim_{t \to \infty} |u(t)| = 0$ by estimate (28).
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