Vortical and Self-similar Flows of 2D Compressible Euler Equations

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Abstract

This paper presents the vortical and self-similar solutions for 2D compressible Euler equations using the separation method. These solutions complement Makino’s solutions in radial symmetry without rotation. The rotational solutions provide new information that furthers our understanding of ocean vortices and reference examples for numerical methods. In addition, the corresponding blowup, time-periodic or global existence conditions are classified through an analysis of the new Emden equation. A conjecture regarding rotational solutions in 3D is also made.

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Key Words: Vortex, Compressible Euler Equations, Vortical Solution, Self-similar Solution, Time-periodic Solution

1 Introduction

In fluid mechanics, the N-dimensional isentropic compressible Euler and Navier-Stokes equations are expressed as follows:

\[ \begin{aligned}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0 \\
\rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + K \nabla \rho^\gamma &= 0,
\end{aligned} \]  \hspace{1cm} (1)

where \( \rho = \rho(t, \vec{x}) \) denotes the density of the fluid, \( \vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, \cdots, u_N) \in \mathbb{R}^N \) is the velocity, \( \vec{x} = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N \) is the space variable, and \( K > 0, \gamma > 1 \) are constants.

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The Euler equations have been studied in great detail by numerous scholars because of their significance in a variety of physical fields, such as fluids, plasmas, condensed matter, astrophysics, oceanography, and atmospheric dynamics. These equations are also important in physics and are widely used in different areas of study. For instance, the Euler system is the basic model for shallow water flows [3]. It also provides a good model of the superfluids produced by the Bose-Einstein condensates in the dilute gases of alkali metal atoms, in which identical gases do not interact at very low temperatures [4]. At the microscopic level, fluids or gases are formed by many tiny discrete molecules or particles that collide with one another. As the cost of directly calculating the particle-to-particle or molecule-to-molecule evolution of fluids on a large scale is expensive, approximation methods are needed to considerably simplify the process. Therefore, at the macroscopic scale, the continuum assumption, which considers fluids as continuous, is applied to the modeling. Here, the Euler equations provide a good description of the fluids at the statistical limit of a large number of small ideal molecules or particles by ignoring the less influential effects, such as the self-gravitational forces and relativistic effect [1]. For a mathematical introduction of the Euler equations, readers are referred to [8] and [2].

The construction of analytical or exact solutions is an important area in mathematical physics and applied mathematics, as it can further classify their nonlinear phenomena. For non-rotational flows, Makino first obtained the radial symmetry solutions for the Euler equations (1) in $\mathbb{R}^N$ in 1993 [9]. A number of special solutions for these equations [6], [7], [12], [13], and [14] were subsequently obtained. For rotational flows, Zhang and Zheng [15] constructed explicitly rotational solutions for the Euler equations with $\gamma = 2$ in 1997:

$$\rho = \frac{r^2}{8Kt^2}, \quad u_1 = \frac{1}{2t}(x+y), \quad u_2 = \frac{1}{2t}(x-y),$$  \hspace{1cm} (2)

where $x = r \cos \theta$ and $y = r \sin \theta$.

Very recently, Kwong and Yuen [5] constructed a family of rotational solutions for the Euler-Poisson equations

$$\begin{cases}
\rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\
\rho(\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u}) + K\nabla \rho = -\rho \nabla \Phi \\
\Delta \Phi(t, \vec{x}) = 2\pi \rho,
\end{cases}$$  \hspace{1cm} (3)

with $N = 2$ and $\gamma = 1$:

$$\begin{cases}
\rho(t, x) = \frac{1}{\alpha(t)}e^{f(r/a(t))}, \quad \vec{u}(t, \vec{x}) = \frac{\hat{a}(t)(x, y)}{a(t)} + \frac{x}{a(t)}(-y, x) \\
\hat{a}(t) = \frac{-\lambda}{\alpha(t)} + \frac{\xi^2}{a(t)}, \quad a(0) = a_0 > 0, \quad \hat{a}(0) = a_1 \\
f(s) + \frac{1}{2}f(s) + \frac{\xi^2}{2a^2}f(s) = \frac{\lambda}{\alpha}, \quad f(0) = \alpha, \quad f(0) = 0,
\end{cases}$$  \hspace{1cm} (4)

with arbitrary constants $\xi \neq 0$, $a_0$, $a_1$ and $\alpha$.

The rotational case complements Yuen’s solutions without rotation ($\xi = 0$) [11]. In this paper,
based on the foregoing development, we provide the corresponding vortical flows for 2D compressible Euler equations \(^\text{(1)}\) with \(\gamma > 1\) in the following result.

**Theorem 1** For \(\gamma > 1\), there exists a family of vortical flows in radial symmetry for the compressible Euler equations \(^\text{(1)}\) in 2D,

\[
\begin{align*}
\rho(t, \vec{x}) &= \max \left(\frac{(-\frac{\lambda(\gamma-1)}{2a^2} s + \alpha)^{\frac{1}{\gamma-1}}}{a(t)^2}, 0\right) \\
\vec{u}(t, \vec{x}) &= \frac{\hat{a}(t)}{a(t)}(x, y) + \frac{\xi}{a(t)^2}(-y, x) \\
\ddot{a}(t) - \frac{\lambda}{a(t)^2} = 0, \ a(0) = a_0 > 0, \ \dot{a}(0) = a_1, \\
\end{align*}
\]

with the self-similar variable \(s = \frac{x^2 + y^2}{a(t)^2}\) and arbitrary constants \(\xi \neq 0, a_0, a_1,\) and \(\alpha\).

**Remark 2** This result complements Makino’s solutions in radial symmetry without rotation \((\xi = 0)\). The vortical solutions \(^\text{(5)}\) provide new information that may further our understanding of oceans vortices and reference examples for numerical methods in computational physics.

**Remark 3** Zhang and Zheng’s solution \(^\text{(1)}\) for \(\gamma = 2\) is a special case in our solutions \(^\text{(5)}\).

### 2D Vortical and Self-similar Flows

Here, we provide a lemma for the vortical flows in 2D for the mass equation \(^\text{(1)}\). This lemma originates in Kwong and Yuen’s paper \(^\text{[5]}\), which constructs periodic and spiral solutions for 2D Euler-Poisson equations \(^\text{[3]}\) with \(\gamma = 1\).

**Lemma 4** \(^\text{[5]}\) For the equation of the conservation of mass,

\[
\rho_t + \nabla \cdot (\rho \vec{u}) = 0,
\]

there exist the following solutions:

\[
\rho(t, \vec{x}) = f\left(\frac{r}{a(t)}\right), \quad \vec{u}(t, \vec{x}) = \frac{\hat{a}(t)}{a(t)}(x, y) + \frac{G(t, r)}{r}(-y, x)
\]

with the radial \(r = \sqrt{x^2 + y^2}\) and arbitrary functions \(f \geq 0; G\) and \(a(t) > 0 \in C^1\).

**Proof.** The functional structure

\[
\rho(t, \vec{x}) = f\left(\frac{r}{a(t)}\right), \quad \vec{u}(t, \vec{x}) = \frac{F(t, r)}{r}(x, y) + \frac{G(t, r)}{r}(-y, x)
\]

with an arbitrary \(C^1\) function \(F(t, r)\), can be plugged into the mass equation \(^\text{(6)}\) to verify the result:

\[
\rho_t + \nabla \cdot (\rho \vec{u}) = \rho_t + \frac{\partial}{\partial x} \left[ \rho \frac{F_x}{r} - \rho \frac{G_y}{r} \right] + \frac{\partial}{\partial y} \left[ \rho \frac{F_y}{r} + \rho \frac{G_x}{r} \right]
\]
Then, the self-similar structure is taken for the density function, 

\[ \rho(t, x) = \rho(t, r) = \frac{f(r/a(t))}{a(t)^2} \]  

and \( F(t, r) = \frac{\dot{a}(t)}{a(t)^3} \) for velocity \( \vec{u} \) to balance equation (14) \( [11] \):

\[
\begin{align*}
\frac{\partial}{\partial t} f(\frac{r}{a(t)}) &+ \frac{\partial}{\partial r} f(\frac{r}{a(t)}) \dot{a}(t)r + f(\frac{r}{a(t)}) \ddot{a}(t) - f(\frac{r}{a(t)}) \dot{a}(t) \dot{a}(t) \\
&= \frac{-2a(t)f(r/a(t))}{a(t)^3} - \frac{\dot{a}(t)r f(r/a(t))}{a(t)^4} \\
&+ \frac{f(r/a(t)) a(t) r}{a(t)^3} + \frac{f(r/a(t)) a(t)}{a(t)^2} + \frac{f(r/a(t)) a(t)}{a(t)^2} + \frac{f(r/a(t)) a(t)}{a(t)^2} \\
&= 0.
\end{align*}
\]

The proof is completed.

The computational proof for Theorem 1 is as follows.

**Proof of Theorem 1.** The procedure for the proof for vortical fluids is similar to that for non-vortical fluids \([9], [14] \). It is clear that the following function

\[
\rho(t, x) = \frac{f(s)}{a(t)^2}, \quad \vec{u}(t, x) = \frac{a(t)}{a(t)^2}(x, y) + \frac{\xi}{a(t)^2}(-y, x),
\]

with an arbitrary \( C^1 \) function \( f \geq 0 \) of the self-similar variable \( s = \frac{x^2+y^2}{a(t)^2} \) and a undetermined \( C^2 \) time-function \( a(t) > 0 \), satisfies Lemma 4 for the mass equation (14). For the first momentum equation (16), we obtain:

\[
= \rho \left( \frac{\partial}{\partial t} u_1 + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \right) + K \gamma \rho^{\gamma-1} \frac{\partial}{\partial x} \rho
\]
with arbitrary constants \( \xi \) to ensure that equation (26) is zero, the following ordinary differential equation is required:

\[
\frac{\partial}{\partial t} u_1 + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + K\gamma f(s)^{\gamma-1} \frac{1}{a(t)^{2(\gamma-1)}} \frac{\partial}{\partial x} f(s) = 0 
\]

(21)

\[
\frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} = 0 
\]

(22)

\[
= \rho \left[ \frac{\partial}{\partial t} u_1 + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(23)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(24)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(25)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(26)

with the Emden equation

\[
\begin{align*}
\ddot{a}(t) - \frac{\xi^2}{a(t)^3} &= \frac{\lambda}{a(t)^{2\gamma}} \\
a(0) &= a_0 > 0, \quad \dot{a}(0) = a_1,
\end{align*}
\]

(27)

with arbitrary constants \( \xi \) and \( \lambda > 1 \).

To ensure that equation (26) is zero, the following ordinary differential equation is required:

\[
\begin{align*}
\lambda + 2K\gamma f(s)^{\gamma-2} \dot{f}(s) &= 0 \\
f(0) &= a > 0, \quad \dot{f}(0) = 0.
\end{align*}
\]

(28)

The function \( f(s) \) in the foregoing ordinary differential equation can be solved exactly:

\[
f(s) = \left( -\frac{\lambda(\gamma - 1)}{2K\gamma} s + a \right)^{\frac{1}{\gamma}}.
\]

(29)

For the second momentum equation (122), we also have:

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(30)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(31)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(32)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(33)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(34)

\[
= \rho \left[ \frac{\partial}{\partial t} u_2 + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + K\gamma f(s)^{\gamma-2} \frac{2x}{a(t)^{2(\gamma-1)}} \frac{\partial f}{\partial y} \right] 
\]

(35)
with the Emden equation (27) and function (29).

Therefore, for \( \xi \neq 0 \), we have the vortical and self-similar flows for the compressible Euler equations (1) in 2D [9].

To ensure the non-negative density function, we require that

\[
\rho(t, \vec{x}) = \max \left( \frac{\left( \frac{\lambda(\gamma-1)s + \alpha}{2K\gamma} \right)^{\frac{1}{\gamma-1}}}{a(t)^2}, 0 \right) \tag{37}
\]

This completes the proof.  

**Lemma 5** For the Emden equation,

\[
\begin{aligned}
\ddot{a}(t) - \frac{\xi^2}{a(t)^2} - \frac{\lambda}{a(t)^{\gamma-2}} &= 0 \\
a(0) &= a_0 > 0, \dot{a}(0) = a_1,
\end{aligned} \tag{38}
\]

with arbitrary constants \( \xi \neq 0 \), \( \lambda \) and \( \gamma > 1 \), the total energy is

\[
E(t) := \frac{\dot{a}(t)^2}{2} + \frac{\xi^2}{2a(t)^2} + \frac{\lambda}{(2\gamma - 2)a(t)^{2\gamma-2}}. \tag{39}
\]

We have the following.

(1) For \( 1 < \gamma < 2 \), if \( E(0) < 0 \), the solution is time-periodic; otherwise, the solution is global.

(2) For \( \gamma = 2 \),

(2a) with \( \xi^2 \geq -\lambda \), the solution is global;

(2b) with \( \xi^2 < -\lambda \), the solution blows up at a finite time if

\[
a_1 < \frac{\sqrt{-\lambda - \xi^2}}{a_0}; \tag{40}
\]

otherwise, the solution is global.

(3) For \( \gamma > 2 \),

(3a) with \( \lambda \geq 0 \), the solution is global;

(3b) with \( \lambda < 0 \) and a constant \( a_{\text{Max}} = \left( \frac{-\lambda}{\xi^2} \right)^{\frac{1}{\gamma-2}} \),

(3bI) and \( a_0 \geq a_{\text{Max}} \),

if \( E(0) \leq F_{\text{pot}}(a_{\text{Max}}) \) or \( E(0) > F_{\text{pot}}(a_{\text{Max}}) \) with \( a_1 \geq 0 \), the solution is global; otherwise, the solution blows up at a finite time.

(3bII) and \( a_0 < a_{\text{Max}} \),

if \( E(0) \geq F_{\text{pot}}(a_{\text{Max}}) \) with \( a_1 > 0 \), the solution is global; otherwise, the solution blows up at a finite time.

**Proof.** For equation (38), we multiply \( \dot{a}(t) \) and then integrate it, as follows:

\[
\frac{\dot{a}(t)^2}{2} + \frac{\xi^2}{2a(t)^2} + \frac{\lambda}{(2\gamma - 2)a(t)^{2\gamma-2}} = E(t), \tag{41}
\]
with a constant $E(0) = \frac{a_1^2}{2} + \frac{\xi^2}{2a_0^2} + \frac{\lambda}{(2\gamma-2)a_0^2}$. 

We define the kinetic energy as:

$$F_{\text{kin}} := \frac{\dot{a}(t)^2}{2},$$

and the potential energy as:

$$F_{\text{pot}} = \frac{\xi^2}{2a(t)^2} + \frac{\lambda}{(2\gamma-2)a(t)^{2\gamma-2}}.$$ (43)

The total energy is conserved thusly:

$$\frac{dE(t)}{dt} = \frac{d}{dt}(F_{\text{kin}} + F_{\text{pot}}) = 0.$$ (44)

The classical energy method for second-order autonomous ordinary differential equations (which readers can refer to pages 793–798 in [10]), can be applied to analyze the corresponding qualitative properties of the Emden equation (38).

(1) For $1 < \gamma < 2$, there exists a unique global minimum for the potential function (43) for $a(t) > 0$, and

$$\lim_{a(t) \to 0^+} F_{\text{pot}}(a(t)) = +\infty \quad \text{and} \quad \lim_{a(t) \to +\infty} F_{\text{pot}}(a(t)) = 0.$$ If

$$E(0) = \frac{a_1^2}{2} + \frac{\xi^2}{2a_0^2} + \frac{\lambda}{(2\gamma-2)a_0^{2\gamma-2}} < 0,$$ (45)

the solution is time-periodic; otherwise, it is global.

(2) For $\gamma = 2$, the ordinary differential equation (38) degenerates into the classical Emden equation

$$\ddot{a}(t) = \frac{\xi^2 + \lambda}{a(t)^3}.$$ (46)

(2a) For $\xi^2 \geq -\lambda$, the solution is global.

(2b) For $\xi^2 < -\lambda$, the potential function (43) is strictly increasing with $\lim_{a(t) \to +\infty} F_{\text{pot}}(a(t)) = 0$. The solution blows up at a finite time, if

$$a_1 < \sqrt{-\lambda - \frac{\xi^2}{a_0}};$$ (47)

otherwise, the solution is global.

(3) For $\gamma > 2$, 

(3a) with $\lambda \geq 0$, the potential function (43) is a decreasing function of $a(t) > 0$. Thus, the solution is global.

(3b) With $\lambda < 0$, the potential function (43) achieves a unique global maximum, $F_{\text{pot}}(a_{\text{Max}})$, with

$$a_{\text{Max}} = \left(\frac{\xi}{-\lambda}\right)^{\frac{1}{\gamma-1}}$$ for $a(t) > 0$.

(3bI) And $a_0 \geq a_{\text{Max}},$

if $E(0) \leq F_{\text{pot}}(a_{\text{Max}})$ or $E(0) > F_{\text{pot}}(a_{\text{Max}})$ with $a_1 \geq 0$, the solution is global; otherwise, the solution blows up at a finite time.

(3bII) and $a_0 < a_{\text{Max}},$


if \( E(0) \geq F_{\text{pot}}(a_{\text{Max}}) \) with \( a_1 > 0 \), the solution is global; otherwise the solution blows up at a finite time.

The proof is now complete. ■

The foregoing lemma makes it easy to determine the blowup or global existence of the corresponding solutions (59) for the compressible Euler equations (1) in 2D.

**Corollary 6** The total energy is defined

\[
E(t) := \frac{\dot{a}(t)^2}{2} + \frac{\xi^2}{2a(t)^2} + \frac{\lambda}{(2\gamma - 2)a(t)^{\gamma-2}},
\]

for the Emden equation (53).

(1) For \( 1 < \gamma < 2 \), if \( E(0) < 0 \), solution (53) is time-periodic; otherwise, solution (53) is global.

(2) For \( \gamma = 2 \),

(2aI) with \( \xi^2 > -\lambda \) or \( \xi^2 = -\lambda \) and \( a_1 \geq 0 \), solution (53) is global;

(2aII) with \( \xi^2 = -\lambda \) and \( a_1 < 0 \), solution (53) blows up at \( T = -a_1/a_0 \).

(2b) with \( \xi^2 < -\lambda \), solution (53) blows up at a finite time if

\[
a_1 < \frac{\sqrt{-\lambda - \xi^2}}{a_0};
\]

otherwise, solution (53) is global.

(3) For \( \gamma > 2 \),

(3a) with \( \lambda \geq 0 \), solution (53) is global;

(3b) with \( \lambda < 0 \) and a constant \( a_{\text{Max}} = \left(\frac{-\lambda}{\xi^2}\right)^{\frac{1}{\gamma-2}} \),

(3bI) and \( a_0 \geq a_{\text{Max}} \),

if \( E(0) \leq F_{\text{pot}}(a_{\text{Max}}) \) or \( E(0) > F_{\text{pot}}(a_{\text{Max}}) \) with \( a_1 \geq 0 \), solution (53) is global; otherwise, solution (53) blows up at a finite time.

(3bII) and \( a_0 < a_{\text{Max}} \),

if \( E(0) \geq F_{\text{pot}}(a_{\text{Max}}) \) with \( a_1 > 0 \), solution (53) is global; otherwise, solution (53) blows up at a finite time.

### 3 Conclusion and Discussion

This paper provides a class of self-similar vortical flows for the 2D compressible Euler equations. The result presented herein complements Makino’s solutions in radial symmetry without rotation. In addition, the corresponding blowup or global existence conditions are classified by analyzing the new Emden equation (53). Solution (53) is also the solution of the compressible Navier-Stokes equations in 2D:

\[
\begin{cases}
\rho \dot{u} + \nabla \cdot (\rho \vec{u}) = 0 \\
\rho [\dot{\vec{u}} + (\vec{u} \cdot \nabla)\vec{u}] + K\nabla \rho^{\gamma} = \mu \Delta \vec{u},
\end{cases}
\]

(50)
with a positive constant $\mu$.

Based on the existence of the rotational and self-similar solutions in 2D, it is natural to conjecture that there exists a class of rotational solutions for the compressible Euler equations in 3D, that complements those in radial symmetry, [9], [14]:

$$\rho = \frac{f(s)}{\prod_{k=1}^{3} a_k},$$

$$u_i = \frac{a_i}{a} (x_i - d_i^*) + d_i^*, \quad \text{for } i = 1, 2, 3,$$

(51)

where

$$f(s) = \max \left( \left( -\frac{\xi(\gamma - 1)}{2K\gamma} s + \alpha \right)^{\frac{1}{\gamma - 1}}, 0 \right),$$

(52)

with

$$d_i^* = d_i^{*0} + t d_i^{*1}, \quad s = \sum_{k=1}^{3} (x_k - d_k^*)^2.$$

(53)

Here, $\xi, d_i^{*0}, d_i^{*1}, \alpha \geq 0$ are constants, and functions $a_i = a_i(t)$:

$$\begin{cases}
\ddot{a}_i = \frac{\xi}{a_i \left( \prod_{k=1}^{3} a_k \right)}, & \text{for } i = 1, 2, 3 \\
a_i(0) = a_{i0} > 0, \quad \dot{a}_i(0) = a_{i1},
\end{cases}$$

(54)

with arbitrary constants $a_{i0}$ and $a_{i1}$.

Further research will be carried out to investigate the possibility of these vortical solutions in 3D.

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