Reconstructing manifolds from truncated spectral triples

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Abstract

We explore the geometric implications of introducing a spectral cut-off on Riemannian manifolds. This is naturally phrased in the framework of non-commutative geometry, where we work with spectral triples that are truncated by spectral projections of Dirac-type operators. We prove that the underlying Riemannian manifold is the Gromov-Hausdorff limit of the metric spaces we associate to its truncations. This leads us to propose a computational algorithm that allows us to recover these metric spaces from the finite-dimensional truncated spectral data. We subsequently develop a technique for embedding the resulting metric graphs in Euclidean space to asymptotically recover an isometric embedding of the limit. We test these algorithms on the truncated sphere and a recently investigated perturbation thereof.

1 Introduction

A natural notion of scale is a major asset to any geometric theory with ties to the physical world. After all, our geometric knowledge of objects appearing around us is finite, being limited by our observational power which fails us at high energy scales. Additionally, the appearance of divergences in e.g. quantum field theory, especially when combined with gravity, is closely tied to bridging the gap between finite and infinite, or discrete and continuum, models. Moreover, computational representation and analysis of geometric models strongly relies on our ability to extract from our model what is relevant and computationally feasible.

The field of noncommutative geometry has had close ties to physics ever since its inception, and yet lacks a consistent treatment of scale in the sense imagined. The aim of this paper is to ameliorate this situation, by constructing
a natural metric counterpart to the finite objects that are here referred to as truncated (commutative) spectral triples, and showing that these do indeed carry enough information to describe their continuum limit in arbitrary detail.

Admittedly, finite-dimensional objects in noncommutative geometry have enjoyed enduring attention. General finite spectral triples have been classified [1, 2, 3] and parametrized [4], and the Connes metric on these spaces has been studied in depth [5]. However, this framework seems to lack simultaneous presence of 1) a natural link to the continuum in terms of metric spaces and 2) a natural link to the continuum in terms of spectral triples.

When representation-theoretic knowledge of the continuum is available, the framework of fuzzy spaces such as those of [6, 7, 8] seems to provide at least a natural link to the continuum in terms of spectral triples, and even some metric knowledge is available there [9]. However, it might be said that the construction of a ‘fuzzy’ version of a manifold is somewhat ad hoc from a Riemannian viewpoint, and at least the framework has not yet seen successful extension to a reasonably large class of manifolds.

Truncated (or ‘operator system’) spectral triples (see Section 2.1 and beyond) provide the advantage of a natural scale parameter and a natural correspondence between different scales. An interesting study of their topological and metric properties (with slightly different aims than pursued here) can be found in [10]. The example of the torus was examined in depth in [11], and the relation to the spectral action principle is being analyzed in [12]. Our companion paper [13] investigates the relevance of the higher Heisenberg equation of [14] in this framework (see Section 2.2).

Arguably the main mathematical result of this paper is Corollary 3.4.2, which shows that the notion of ‘localization’ in Section 3 allows one to associate metric spaces to truncated spectral triples in such a way that their Gromov-Hausdorff limit is isometric to the Riemannian manifold one started with.

In order to make the result more concrete and link back to the ‘computational representation’ alluded to above, Section 4 is devoted to the description of (a prototype of) an algorithm to recover (finite subsets of) these metric spaces from the raw datum of a truncated spectral triple. This allows us to test some of the results of Section 3 on the example of the sphere in Section 5.

Finite non-Euclidean metric spaces, as obtained by our algorithm, do not necessarily lend themselves to easy visualization or comparison by standard computational techniques. In order to gain traction in this direction, we propose to look for new (asymptotically, locally isometric) embedding techniques and present a candidate approach in Section 6. This allows us to visualize the metric results of Section 5 and – as originally inspired this visualization compare the truncated spectral triple for $S^2$ and its perturbation as in [13].
2 Truncated spectral geometries and point reconstruction

In noncommutative geometry one describes a spin manifold $M$ in terms of the associated spectral triple $(\mathcal{C}^\infty(M), L^2(M, \mathbb{S}), D_\mathbb{S})$, where $\mathbb{S}$ is a spinor bundle over $M$ and $D_\mathbb{S}$ is the corresponding Dirac operator.

Connes’ reconstruction theorem [15] shows that this association is a bijection: one can fully reconstruct the underlying manifold $M$ and the chosen spin structure from the spectral triple alone.

Of particular interest for the present paper is the way one can recover the Kantorovich-Rubinstein distance between probability measures on $M$ from the interplay of the algebra $\mathcal{A} = \mathcal{C}^\infty(M)$ and the Dirac operator $D = D_\mathbb{S}$, acting on the Hilbert space of spinors. Probability measures correspond to states on the C$^*$-algebra $C(M)$, and a function $f \in \mathcal{A}$ has Lipschitz constant $k$ if and only if $\| [D, f] \| \leq k$ (as operators on $H = L^2(M, \mathbb{S}_M)$). Thus, Kantorovich-Rubinstein duality allows us to write

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{ |\omega_1(a) - \omega_2(a)| \ | \|[D, a]\| \leq 1 \} .$$

When the states $\omega$ of $C(M)$ are pure, they correspond to the atomic measures on single points, and we can thus recover $M$ with its metric. This paper answers the question as to how we can understand this recovery of $M$ from the perspective of finite-dimensional parts of the representation of $\mathcal{A}$ and $D$ on $H$. That is, we will construct natural counterparts to the ingredients above in the setting of truncated spectral triples, and show that the metric space $M$ can be recovered as an asymptotic limit thereof.

2.1 Truncated spectral geometries

From a mathematical perspective, it is desirable to be able to describe the infinite-dimensional datum of a spectral triple as a limit of finite-dimensional data of increasing precision, just like one can describe a Riemannian manifold as a Gromov-Hausdorff limit of finite metric spaces. From a physical perspective, the same desire results from the view that one should be able to gain at least some information about the geometry by probing it at finite energies.

A natural way to introduce such a ‘finite-energy cutoff’, that is, truncation, of the geometric data $(\mathcal{A}, H, D)$ is to pick a scale $\Lambda$, and define the corresponding spectral projection of $D$,

$$P_\Lambda \overset{\text{def}}{=} \chi_{[-\Lambda, \Lambda]}(D)$$
to generate the finite-dimensional data

\[(A_\Lambda, H_\Lambda, D_\Lambda) \overset{\text{def}}{=} (P_\Lambda AP_\Lambda, P_\Lambda H, P_\Lambda D).\]  

(2)

Truncated spectral triples are further explored in as of yet unpublished work by Connes and van Suijlekom [12] and in [10].

This is the setting in which we wonder what counterpart to the duality \((A, H, D) \leftrightarrow M\), provided by the reconstruction theorem, can be found at the level of \((A_\Lambda, H_\Lambda, D_\Lambda)\).

### 2.2 The \(D_c\) operator on the sphere

All spin manifolds of dimension \(\leq 4\) satisfy (the two-sided version of) the higher Heisenberg equation introduced in [16]. The companion paper [13] explores the constraint that existence of solutions to the one-sided higher Heisenberg equation,

\[
\frac{1}{n!} \langle Y [Y, D] \ldots [Y, D] \rangle = \gamma,
\]

places on a truncated spectral triple. There we found that (3), with \(Y\) and \(\gamma\) obtained from the Dirac spectral triple of \(S^2\), is solved by a one-parameter class of operators \(\{D_c \mid c \in \mathbb{R}\} \subset \mathcal{D}\), where

\[
D_c = D_{S^2} + cB.
\]

Here \(B\) is a bounded, self-adjoint operator \(B = \text{sign}(D) \cos(\pi D_{S^2})\). This class of solutions does not strictly describe spectral triples, since the pseudo differential operator \(D_c\) does not satisfy the first order condition. As discussed there, however, failure of this condition is not detectable by standard methods at the level of truncated spectral triples. We will apply the POINTFORGE algorithm to this modified triple \((A, H, D_c)\) for \(S^2\) as well as to the standard triple for \(S^2\) itself, and view the embedded results in Section 6.

### 2.3 Point reconstruction

We aim to refine the reconstruction of a spin manifold \(M\) from its spectral triple \((A, H, D)\) through an understanding of the metric information contained in the truncations \((A_\Lambda, H_\Lambda, D_\Lambda)\). The full manifold should then emerge asymptotically, such as through a Gromov-Hausdorff limit of the objects corresponding to the truncations.
The vector states of \( C(M) \) that are induced by elements of \( H_\Lambda \) appear naturally as vector states of \( A_\Lambda \) as well, so that we have access to probability measures on \( M \) directly in the truncated setting. This, together with the distance formula (1), is the main ingredient of our approach: we will identify states that correspond to points of \( M \) in a suitable (asymptotic) sense, such that (1) asymptotically recovers the corresponding geodesic distance.

Our ‘proxy’ approach to state localization, as discussed in section 3.2, was inspired by the notion of quasi-coherent states on fuzzy spaces defined in [9]. However, as we aim just for the induced metric geometry on \( M \) and view \((C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)\) rather as a finite observation of a spectral triple than as a quantization thereof, we will not construct coherent states in any quantum-mechanical sense but rather aim for localization only. Moreover, as discussed below, this ‘proxy’ approach is merely introduced to gain computational feasibility, instead of as an essential ingredient to the definition.

After we define localized states, we prove existence of the desired objects: a sequence of metric spaces associated to the truncations \((A_\Lambda, H_\Lambda, D_\Lambda)\) that do indeed converge to \( M \) in the Gromov-Hausdorff sense. Then, Section 4 proposes an algorithm to construct these metric spaces computationally, in order to make actual examples amenable to computer simulation.

In Section 6 we propose a simple algorithm to obtain approximately locally isometric embedding of the resulting finite metric spaces into Euclidean space. These should asymptotically converge to an isometric embedding of \( M \) itself, and allow us to view the resulting finite metric spaces and investigate them more easily.

### 3 The metric space of localized states

Given a truncated commutative spectral triple \((C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)\), we aim to construct a finite metric space that describes \( M \) to the level of accuracy that the truncated spectrum will allow.

This implies we will first construct a reasonable set of ‘points’ (localized states), and then use the Connes metric on states to construct a finite metric space. The guiding demand for this construction will be that the finite metric spaces should asymptotically (as \( \Lambda \to \infty \) so that \( P_\Lambda \to 1 \)) converge, in the Gromov-Hausdorff sense, to the space \( M \) of pure states of \( C(M) \) equipped with the Connes metric.

Now, the pure states of \( C(M) \) – that is, actual points in the metric space \( M \) we are approximating – do not necessarily extend to \( C(M)_\Lambda \). We do, however, have access to vector states induced by \( v \in H_\Lambda \), which can be applied to either because both \( C(M) \) and \( C(M)_\Lambda \) are subsets of \( B(H) \). Let
$S_\Lambda$ denote the set of such states.

When applied to $C(M)$, elements of $S_\Lambda$ correspond to probability measures $(v, v)\, \text{vol}_M$ on $M$. When such probability measures are sufficiently concentrated near a single point in $M$ (‘localized’), they may well be said to resemble that point. In order to quantify the localization of such states, we will introduce the dispersion functional $\eta$ on $H_\Lambda$, below. The definition of $\eta$ requires identifying an element $Y$ of $C(M)$ that is bi-Lipschitz when viewed as a map from $M$ to $\mathbb{R}^n$. Then, $\eta(v)$ is just the variance of the random variable $Y$ under the probability measure $(v, v)\, \text{vol}_M$.

Lemma 3.3.3 will show that there is a map $b: S_\Lambda \to M$ with error $|d(\omega_1, \omega_2) - d(b(\omega_1), b(\omega_2))| = O(\sqrt{\eta(\omega_1)} + \sqrt{\eta(\omega_2)})$; that is, our localized states can be identified with points while controlling the induced distortion of distances. Lemma 3.4.1 will then show that there is a corresponding asymptotically inverse map $\phi_\Lambda: M \to S_\Lambda(C(M))$ such that $d(x, b(\phi_\Lambda(x))) = O(\Lambda^{-1})$ and $\eta(\phi_\Lambda(x)) = O(\Lambda^{-2} \log \Lambda)$ uniformly in $x$.

3.1 The metric on $S_\Lambda$

Connes’ generalization \cite{Connes} of the Kantorovich-Rubinstein distance between states of the algebra $C^\infty(M)$ extends verbatim to the setting $(C^\infty(M)_\Lambda, H_\Lambda, D_\Lambda)$. Because $D_\Lambda$ commutes with the projection on $H_\Lambda$ by design, it does not matter whether we treat elements of $S_\Lambda$ (i.e. vector states induced by $H_\Lambda$) as states of $C^\infty(M)_\Lambda$ or of $C^\infty(M)$, and thus we will use the metric

$$d(\omega, \omega') \overset{\text{def}}{=} \sup_{a \in C^\infty(M)_\Lambda} \{ |\omega(a) - \omega'(a)| \mid \|[D_\Lambda, a]\| \leq 1 \}$$

on the space $S_\Lambda$, with the reassuring knowledge that this turns the inclusion $S_\Lambda \hookrightarrow S(C(M))$ into an isometric embedding.

3.2 Localization: $Y$ and the dispersion functional

Since vector states in $S_\Lambda$ correspond uniquely to probability measures on $M$, a natural way to measure the localization of such a state would be to take e.g. the variance of (isometrically embedded) position under this measure; that is, one would naturally define the dispersion of a state $v$ to be $\inf_{x \in M} E_{\mu_v} [d(x, \cdot)^2]$, where $E_{\mu_v}$ denotes expectation values under $\mu_v$. In terms of the algebra, this quantity can be estimated as $\sup_{a \in C^\infty(M)} \{ \langle v, a^* av \rangle - |\langle v, av \rangle|^2 \mid \|[D, a]\| \leq 1 \}$. However, the relevant non-convex double optimization problem – to find minima $v$ of this dispersion in high-dimensional $H$ – is computationally extremely challenging except in the simplest cases.
Therefore, we will require a proxy, \( Y \), for the extremizing element \( a \) above. Thus, let \( Y : M \to \mathbb{R}^n \) be a (not necessarily Riemannian) embedding. It is interesting to note that natural candidates for such \( Y \) can algebraically be found as solutions of the higher Heisenberg equation from \([14]\).

With this identification, our measure of 'localization' is just the (trace of the) variance of \( Y \), viewed as a vector-valued random variable under the probability measure \( \mu_v \).

That is, \( \eta(v) = E_{\mu_v} [\|Y - E_{\mu_v} [Y]\|^2] \), where \( \|\cdot\| \) is the Euclidean norm. In local terms, this is by definition \( \int_M \|Y(y) - y_0\|^2 \, d\mu_v(y) \), with \( y_0 = E_{\mu_v}(Y) = \int_M y \, d\mu_v(y) \). In terms of the vector \( v \) itself,

**Definition 3.2.1.** For \( v \in H \) with \( \|v\| = 1 \), the dispersion \( \eta(v) \) of \( v \) equals

\[
\eta(v) \overset{\text{def}}{=} \sum_i \langle v, Y_i^2 v \rangle - \langle v, Y_i v \rangle^2.
\]

#### 3.3 The barycenter of a localized state

A state \( \omega \) that is considered to be localized should be localized somewhere, that is, around some 'barycenter' \( b(\omega) \). The distortion induced by the replacement of \( d_M(\cdot,\cdot) \) by \( \|Y(\cdot) - Y(\cdot)\| \) in the definition of the dispersion functional induces us to replace the usual measure-theoretic notion of barycenter by something more suited to the present construction. Hence,

**Definition 3.3.1.** Let \( \omega \in S(C(M)) \). Then a barycenter of \( \omega \) is any point \( x \in M \) that minimizes \( \|Y(x) - E_{\mu_v} [Y]\| \).

Localized states are indeed concentrated near their barycenters, as the following proposition shows. That is, the dispersion \( \eta(v) \), as measured using \( Y \), is a good proxy for the squared second Wasserstein distance \( W_2(\mu_{\omega}, \delta_x)^2 \) between the measure \( \omega_v \) and any given barycenter \( x \) thereof.

**Proposition 3.3.2.** Any barycenter \( x \) of a state \( \omega \) satisfies

\[
W_2(\mu_{\omega}, \delta_x)^2 \overset{\text{def}}{=} \int_M d_M(z, x)^2 \, d\mu_{\omega}(z) = O(\eta(\omega)),
\]

uniformly\(^1\) in \( \omega \), where \( \delta_x \) denotes the Dirac measure centered on \( x \). Moreover, any two barycenters of \( \omega \) are within distance \( O(\sqrt{\eta(\omega)}) \), uniformly in \( \omega \), of each other.

\(^1\)See e.g. \([17]\) for an introduction to the measure-theoretic notions that are applied (without any hint of sophistication) in this section.

\(^2\)That is, the relevant constant depends only on \( Y \) and \( M \), not on \( \omega \).
Proof. By Chebyshev’s inequality, $E_{\mu_\omega} \left[ \|Y - E_{\mu_\omega} [Y]\|^2 \right] \leq \frac{\eta(\omega)}{t^2}$, so there must exist a point of $Y(M)$ within Euclidean distance $\sqrt{\eta(\omega)}$ of $E_{\mu_\omega} [Y]$.

As a smooth embedding $Y$ is automatically bilipschitz, so there are $\alpha, \beta$ such that $\alpha \|Y(x) - Y(y)\| \leq d(x,y) \leq \beta \|Y(x) - Y(y)\|$, uniformly in $x,y$. That is, $d_M(z,x)^2 \leq \beta^2 \|Y(z) - Y(x)\|^2$ and so its integral under $d\mu_\omega$ is bounded by $\beta^2 E_{\mu_\omega} \left[ \|Y - Y(x)\|^2 \right] \leq 2\beta^2 \eta(\omega)$.

The final statement now follows by the triangle inequality and the bilipschitz property of $Y$. \qed

Lemma 3.3.3. There exists a map $b : S(C(M)) \to M$ such that

$$|d(\omega_1, \omega_2) - d(b(\omega_1), b(\omega_2))| = O(\sqrt{\eta(\omega_1)} + \sqrt{\eta(\omega_2)}),$$

as $\eta(\omega_i) \to 0$, uniformly in $\omega_i$.

Proof. Let $b$ assign a choice of barycenter to each $\omega \in S(C(M))$, and let $\delta_{x_i}$ be the Dirac measures centered on $b(\omega_i)$.

On a commutative spectral triple, Kantorovich-Rubinstein duality shows that the Connes distance $[1]$ between states $\omega_1$ and $\omega_2$ equals the Wasserstein distance $W_1(\mu_1, \mu_2)$ between the corresponding positive Radon measures $\mu_1$ and $\mu_2$. Now, by Jensen’s inequality $W_1(\mu_i, \delta_{x_i}) \leq W_2(\mu_i, \delta_{x_i})$, and so the triangle inequality for the metric $d$ shows that

$$|d(\omega_1, \omega_2) - d(x_1, x_2)| \leq W_1(\mu_1, \delta_{x_1}) + W_1(\mu_2, \delta_{x_2}) \leq W_2(\mu_1, \delta_{x_1}) + W_2(\mu_2, \delta_{x_2}).$$

\qed

3.4 Existence of localized states near any point

Lemma 3.3.3 tells us that states $\omega \in S_\Lambda$ of sufficiently small dispersion correspond well to their ‘barycenters’ $b(\omega)$. We would like to estimate the converse, i.e. to show that each point $x$ corresponds to a small-dispersion state $\phi_\Lambda(x)$. Then, with $\phi_\Lambda$ asymptotically an isometric embedding and $b \circ \phi_\Lambda$ asymptotically the identity on $M$, we can rightly say there is a picture of $M$ inside $(C(M)_\Lambda, H_\Lambda, D_\Lambda)$.

Lemma 3.4.1. For all $\Lambda$, there exists a map $\phi_\Lambda : M \to S_\Lambda$ such that

- The dispersion $\eta(\phi_\Lambda(x))$ is $O(\Lambda^{-2} \log \Lambda)$ as $\Lambda \to \infty$.
- $\phi_\Lambda$ asymptotically inverts $b$, in the sense that $d(x, b(\phi_\Lambda(x))) = O(\Lambda^{-1})$ and $d(\phi_\Lambda(b(\omega)), \omega) = O(\sqrt{\eta(\omega)} + \Lambda^{-2})$. 

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Corollary 3.4.2. The map $\phi_\Lambda$ is an $\epsilon$-isometry with $\epsilon = O(\Lambda^{-1}\sqrt{\log \Lambda})$.

Now let us prove lemma [3.4.1]. One natural approach to obtaining states ‘localized around $x$’ is to use the small-time heat asymptotics: one starts out with a fully localized spinor $v_x \in S_x$ and then lets heat flow for a short time in order to obtain a smooth section $y \mapsto p_t(x, y)(v_x)$ of the spinor bundle $S$. This section then corresponds to a vector in $H$ with a fully localized spinor $v$. The following proposition formalizes this intuition.

Proposition 3.4.3. Let $p_t(x, \cdot)$ be the heat kernel of the positive Laplace-type operator $D^2$ and let $0 \neq v_x \in S_x$. Let $\mu_t = \|p_t(x, \cdot)(v_x)\|^2 \text{vol}$ be the measure induced by the section $p_t(v_x): y \mapsto p_t(x, y)(v_x)$. Then, the state

$$\Phi_t(v_x): a \mapsto \frac{1}{\mu_t(M)} \int_M a(y)d\mu_t(y)$$

of the algebra $A = C(M)$ has dispersion $\Theta(t) + O(t^2)$.[3]

Proof. If we set $S(t, x, y) \overset{\text{def}}{=} \frac{e^{-d(x, y)^2/4t}}{(4\pi t)^{n/2}}$, the heat asymptotics tell us that

$$p_t(x, y)(v_x) = S(t, x, y)(\Psi(x, y)(v_x) + O(t)),$$

as $t \to 0$, where $\Psi$ is the parallel transport map along the spinor connection. Denote the metric $d_{\mathbb{R}^n}(Y(x), Y(y))$ on $M$ by $d_Y$. We will first use the heat asymptotics to approximate the moments $m_{t,k} \overset{\text{def}}{=} \int_M d_Y(x, y)^{2k}d\mu_t(y)$ by localized Gaussian integrals.

Let $\nu$ be a multivariate normal distribution of covariance matrix $\Sigma = 2t \text{Id}$ on the tangent space of $Y(M)$ at $Y(x)$. By Chebyshev’s inequality and elementary calculation, the conditional expectation $E_{\nu}(d_{\mathbb{R}^n}(X, Y(x))^{2k} | d_{\mathbb{R}^n}(X, Y(x)) \leq s) \leq \Theta(t^k) + O(s^{-4}t^k)$.

By smoothness of the embedding there exists a local diffeomorphism $\phi : U_x \to T_{Y(x)}Y(M)$ on a neighbourhood $U_x \subset M$ of $x$. Since $\phi$ and $Y$ are bi-Lipschitz by compactness, the ‘conditional expectation’ $m'_{t,s,k} \overset{\text{def}}{=} \int_{d_{\mathbb{R}^n}(x,y) \leq s} d_Y(x, y)^{2k}S(t, x, y)d\text{vol}(y)$ will be $\Theta(t^k) + O(s^{-4}t^k)$ as well.

Now, with $\alpha$ the Lipschitz constant of $Y^{-1}$, clearly

$$m_{t,k} = \int d_Y(x, y)^{2k}d\mu_t(y) + O\left(\frac{e^{-\alpha s^2/2t}}{t^n}\right).$$

[3] We use Bachmann-Landau notation throughout, so that $f(t) = \Theta(g(t))$ means there exist $k_1 > 0, k_2 > 0$ and $T$ such that $k_1g(t) \leq f(t) \leq k_2g(t)$ for all $t \geq T$. 

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By finiteness of diameter and the heat asymptotics, the integral on the right is $\Theta(t^{-n/2}m_{t,s,k}^{\prime}) + O(t^{1-n/2}m_{t,s,k}^{\prime})$ so we can scale $s$ as $t^{1/4}$ to conclude that $m_{t,k} = \Theta(t^{k-n/2}) + O(t^{k+1-n/2})$. Since the dispersion of $\Phi_t(v_x)$ is just $m_{t,1}/m_{t,0}$, the proposition follows. \hfill \Box

We want to map our smooth, localized section $p_t(v_x)$ of the spinor bundle, to an element of $H_\Lambda$ that is similarly localized. The natural thing to do is just to consider the element $P_\Lambda(p_t(v_x))$, and the following proposition shows this choice preserves the localization of $p_t(v_x)$ asymptotically, provided we balance the choices of $t$ and $\Lambda$ correctly.

**Proposition 3.4.4.** For $0 \neq v_x \in S_x$, let $\Phi_t(v_x)$ be the state induced by the normalization of $p_t(v_x) \in H$ and let $\Phi_{t,\Lambda}(v_x): a \mapsto \Phi_t(v_x)(a_\Lambda)$. Then, with $t_{\Lambda} \overset{\text{def}}{=} \frac{n}{2} c \Lambda^2 \log \Lambda$, the state $\Phi_{t,\Lambda}(v_x)$ is localized in the sense that

$$\left| \Phi_{t,\Lambda}(v_x)(a) - \Phi_t(v_x)(a) \right| = O\left(\|a\| \cdot \Lambda^{-k}\right)$$

uniformly in $a \in C^\infty(M)$, whenever $c > (6n + 4 + k)$.

**Proof.** By the spectral mapping theorem, the integral transform associated to the kernel $p_t(x,y)$ equals the bounded linear operator $w \mapsto e^{-tD^2}w$ on $H$. Therefore, $\langle p_t(v_x), w \rangle = \sum_{\lambda \in \sigma(D)} e^{-t\lambda^2} \langle v_x, e_\lambda(x) \rangle \langle e_\lambda, w \rangle$ for $w \in H$, where $(\cdot, \cdot)$ denotes the inner product on $S_x$.

Now, let us first prove that $|\langle p_t(v_x), ap_t(v_x) - p_t(v_x), ap_t(v_x) \rangle|$ decays exponentially in $t \Lambda^2$. Using the expression for $\langle p_t(v_x), \cdot \rangle$, the sum over the missing terms is clearly bounded by $2 \sum_{|\lambda| > \Lambda} \sum_{\kappa} Z_{\lambda,\kappa}$, where we write $Z_{\lambda,\kappa}$ for the term $e^{-t(\lambda^2 + \kappa^2)} |\langle v_x, e_\lambda(x) \rangle | |\langle e_\lambda, v_x \rangle | |\langle e_\lambda, a e_\kappa \rangle |$.

By the Sobolev inequality and (repeated) elliptic estimates, for all $s > n/2$ there exists a constant $C_s$ such that $\|e_\lambda\|_\infty \leq C_s(1 + \lambda^2)^s$, so we have $|\langle v_x, e_\lambda(x) \rangle | \leq C_s(1 + \lambda^2)^s \|v_x\|$ and thus $Z_{\lambda,\kappa} \leq C_s^2 e^{-t(\lambda^2 + \kappa^2)}(1 + \lambda^2)^s(1 + \kappa^2)^s \|v_x\|^2 \|a\|$. 

Now, $\sum_{|\lambda| > \Lambda} e^{-t(\lambda^2 + \kappa^2)}(1 + \lambda^2)^s(1 + \kappa^2)^s$ is, for $s \leq 1$, bounded by $e^{-t(1-z)\Lambda^2}$ times the total sum $\sum_{\lambda,\kappa} e^{-t(\lambda^2 + \kappa^2)}(1 + \lambda^2)^s(1 + \kappa^2)^s$. When $2s \leq n + 1$ (so that we are left with $n/2 < s \leq (n + 1)/2$), we may apply the heat asymptotics to estimate the latter as $O((t\zeta)^{-2n-1})$.

It remains to include the effect of the normalization of $\phi$. Note now that the case $a = 1$ shows $\|p_t,\Lambda(v_x)\|^2 = \|p_t(v_x)\|^2 + O(e^{-t(1-z)\Lambda^2})$, and since $\|p_t(v_x)\|^2 = \Theta(t^{-n/2}) + O(t^{1-n/2})$ by 3.4.3, the proposition follows when we choose $z = \frac{1}{2}$.

Since we have bounded the dispersion of $\Phi_t$ in $t$, the previous proposition gives us an estimate of the dispersion of $\Phi_{t,\Lambda}(v_x)$, and that is how we prove lemma 3.4.1.
Proof of Lemma 3.4.1. To pick our states continuously in $x$, choose a set $(\xi_0, \ldots, \xi_N)$ of spinors such that the corresponding fiber-wise norms add to an everywhere non-vanishing function on $M$. Let $\phi'_\Lambda : x \mapsto \sum_{\xi_i \neq 0} \Phi_{t\Lambda}(\xi_i(x))$ and define $\phi_{\Lambda} : M \mapsto S(B(H))$ by $\phi_{\Lambda} = \phi'_{\Lambda} / \|\phi'_{\Lambda}\|$.

By proposition 3.4.4, the dispersion $\eta_{\Lambda} \overset{\text{def}}{=} \sum_i \phi_{\Lambda}(x)(Y_i^2) - \phi_{\Lambda}(x)(Y_i)^2$ of $\phi_{\Lambda}(x)$ equals that of $\sum_{\xi_i \neq 0} \Phi_{t\Lambda}(\xi_i(x))/\|\phi'_{\Lambda}\|$ plus $O(\Lambda^{-k})$. That variance is, by proposition 3.4.3, itself $O(t)$. We conclude that $\eta_{\Lambda} = O(\Lambda^{-2\log \Lambda})$ as $\Lambda \to \infty$.

Now that we know there exists a set $\phi_{\Lambda}(M) \subset S_\Lambda$ of states that is $\epsilon$-isometric to $M$, we will propose an algorithm to find this set, and its metric, computationally.

4 The PointForge algorithm: associating a finite metric space

Once a set of localized vector states is found, the Connes (Kantorovich-Rubinstein) distance between them will serve as an estimate for the geodesic distance between the points in $M$ near which they are concentrated. The corresponding formula, (1), translates verbatim to the truncated setting because, by design, $D$ commutes with the truncating projection.

Localized vector states can be found by minimizing the dispersion functional in $H$.

Since nonzero dispersion induces a distortion of estimated distances (see section 3.3 below), there is a lower bound on the Gromov-Hausdorff distance between any graph of localized states and the manifold $M$. Computationally speaking, then, it would be desirable to minimize the number of states (and, hence, computational resources) required to approach this bound.

The main other factor, besides correctness of distances, influencing the Gromov-Hausdorff distance is the density (in the Hausdorff sense) of our set of points inside $M$. Optimally, therefore, the states would be equidistributed on $M$. 

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In order to construct a potential whose minima are both localized and roughly (that is, under the map $Y$) equidistributed, we add an electrostatic repulsion term to the dispersion. Given a set $V$ of states, the next state is then generated as the minimum of the energy functional

$$e(v; V) \equiv -\eta(v)^{-1} + g_e \sum_{w \in V} \left( \sum_i \left( \langle v, Y_i v \rangle - \langle w, Y_i w \rangle \right)^2 \right)^{-1}.$$  \hspace{1cm} (4)

The value of the coupling constant $g_e$ should ideally be sufficiently large to overcome local variation in minimal dispersion but is otherwise not expected to influence the generated states much – this is consistent with our observations for $M = S^1, S^2$.

### 4.1 The PointForge algorithm

Using the functional (4) we propose the following algorithm to construct states and thus a finite metric space $M_\Lambda$ that models the metric information about $M$ contained at cutoff $\Lambda$.

As preparation, we must estimate the number $N$ of states to generate.

- Estimate vol $M_\Lambda$ and dim $M_\Lambda$, e.g. using the asymptotic formulas of [18].
- Estimate the Euclidean dispersion $\eta_0 = E_\nu [\|X\|^2]$ under the multivariate normal distribution $\nu$ of covariance matrix $2\Lambda^{-2} \log \Lambda \text{id}$ on $\mathbb{R}^{\text{dim} M_\Lambda}$.
- Set $N = \text{vol} M_\Lambda / (\text{vol}(B_{\text{dim} M_\Lambda}) \eta_0^{\text{dim} M_\Lambda/2})$, where $B_{\text{dim} M_\Lambda}$ is the Euclidean unit ball of dimension $\text{dim} M_\Lambda$.

For cases where $Y$ is a perfect embedding for $M$, any $g_e$ will suffice to lead to equidistributed states in $M$, while for $g_e = 0$ the states generated numerically would mostly lie very close together. However in the cases where $Y$ might not match ideally with $M$, we need to chose $g_e$ to be sufficiently large to overcome local variations in minimal dispersion, and assume this to mean that $-\alpha^2 \eta_0^{-1} + g_e \alpha^2 \eta_0^{-1} \geq -\beta^2 \eta_0^{-1}$, where $\alpha$ and $\beta$ are the optimal local Lipschitz constants of $Y$ and $Y^{-1}$, respectively. This ensures that states in regions of $M$ where the dispersion is over-reported (due to stretching by $Y$) will be generated once the regions where the dispersion is under-reported are saturated with states, instead of being skipped.

Then, simply generate $N$ states by minimizing the iterative energy functional and calculate the Connes distance between them:

1: \textbf{while} $|V| \leq N$ \textbf{do}
2: Find a vector $w$ (locally) minimizing $e(w; V)$.
3: Append $w$ to $V$.
4: for $v \in V$, do
5: Set $d(v, w) = \max \{ |\langle v, av \rangle - \langle w, aw \rangle|: \| [D, a] \| \leq 1 \}$.
6: end for
7: end while

The algorithm, including the distance calculation and the examples $S^1$ and $S^2$, has been implemented in Python and is publicly available at [19].

4.2 Implementation: calculating the metric on $S_{\Lambda}$

When $v, w \in H_{\Lambda}$, the distance between the vector states $\langle v \mid \cdot \mid v \rangle$ and $\langle w \mid \cdot \mid w \rangle$ of the algebra $A = C^\infty(M)$ equals

$$\max_{a \in A_{\Lambda}} \{ |\langle v, av \rangle - \langle w, aw \rangle|: \| [D_\Lambda, a] \| \leq 1 \},$$

as in discussed in section 3.1.

The functional $a \mapsto \langle v \mid a \mid v \rangle - \langle w \mid a \mid w \rangle$ is linear and the space $\{ a a \mid \| [D_\Lambda, a] \| \leq 1 \}$ is convex, which ensures that computing the minimum is computationally feasible.

Indeed, if $a_0, \ldots, a_n$ is a basis for $(A_\Lambda)_{sa}$, we can reformulate the problem as:

**Problem.** Minimize $\sum_i c_i (\langle v, a_i v \rangle - \langle w, a_i w \rangle)$ over $c \in \mathbb{R}^{n+1}$, subject to the constraint

$$\left[ \sum_i c_i [D_\Lambda, a_i] \right] > 0$$

With the constraints formulated as a linear matrix inequality, we have put the problem in a form directly amenable to techniques from semi-definite programming. A reasonably effective algorithm, given the scale of the problem, is then provided by the Splitting Cone Solver of [20].

4.3 Complexity and the dimension of $C^\infty(M)_{\Lambda}$

Step 2 of the **PointForge** algorithm amounts to finding a local minimum of a quadratic function under quadratic constraints in a vector space of dimension $\dim H_{\Lambda}$, which can be done in $O(\dim H_{\Lambda})$, e.g. with the BFGS algorithm.

The convex problem in step 5 is convex, of dimension $\dim C_\Lambda(M)$. This factor is what limits the computational feasibility of high $\Lambda$ in our experiments, so it would be informative to analyze the scaling of $\dim C_\Lambda(M)$ with $\Lambda$. 

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As a simple example, one can represent the generator $e^{id}$ of $C^\infty(S^1)$ as the shift operator on $H = l^2$, with basis indexed by $\mathbb{Z}$, where the corresponding Dirac operator acts diagonally as $De_n = ne_n$. It is then easy to see that the dimension of $C^\infty(S^1)_\Lambda$ is equal to $\dim H_\Lambda = 2[\Lambda] + 1$.

For $M = S^2$, if we choose an orthonormal basis $e_{lm}$ of eigenvectors of $D$ and introduce the spherical harmonics $\alpha Y_{lm}$ then we can express $\langle (e_{l_1m_1} \cdot e_{l_2m_2})\cdot \alpha Y_{l_3m_3})_{L^2(M)}$ in terms of $3j$-symbols. In particular, these vanish unless $|l_1| - |l_2| \leq l_3 \leq |l_1| + |l_2|$, which tells us that $C^\infty(S^2)_\Lambda$ is spanned by $(\alpha Y_{lm})_\Lambda$ for $l \leq 2\Lambda$ and is thus of dimension bounded by $(2\Lambda + 1)^2$.

The general situation is not entirely clear. However, our Lemma 3.4.1, as noted there, provides a lower bound of $\Theta(\Lambda \dim M)$ on the scaling of $\dim C^\infty(M)_\Lambda$ with $\Lambda$.

5 Example: $S^2$

The simplest interesting example of a commutative spectral triple that allows for an isometric embedding in $\mathbb{R}^3$ is probably the sphere $S^2$. This section will cover the application of the PointForge algorithm to truncations of $(C^\infty(S^2), L^2(S^2, S_{S^2}), D_{S^2})$, and thereby illustrate (and test the optimality of) the analytic results of Section 3.

5.1 Implementation

The main ingredients are the vector space $C^\infty(S^2)$, the spectrum of $D_{S^2}$, the element $Y$, and their representation on $L^2(S^2, S_{S^2})_\Lambda$.

The vector space $C^\infty(S^2)$ is spanned by the spherical harmonic functions $Y_{lm}$ up to $l = 2\Lambda$, as in section 4.3. An eigenbasis $e_{lm}$ of $D$ can be expressed in terms of the spin-weighted spherical harmonics $sY_{lm}$, with $s = \pm \frac{1}{2}$, as discussed e.g. in [21], section 9.A. The matrix coefficients of the representation of $C^\infty(S^2)$ can then be expressed in terms of triple integrals of spin-weighted spherical harmonics. Note that a brute-force approach of calculating the inner products $\langle e_{lm}, Y_{l'm'}e_{l''m''} \rangle$ in order to obviate knowledge of the representation-theoretic machinery attached to $S^2$ would have been possible, however it would have introduced the additional complexity of calculating $(\dim H_\Lambda)^2 \cdot \dim C^\infty(S^2) \cdot \text{rank } S$ integrals numerically.

The element $Y$ is just the idempotent associated to the Bott projection:

$$Y = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix},$$

where $x, y, z$ are the standard coordinates on the embedding $S^2 \hookrightarrow \mathbb{R}^3$. Note that this embedding $Y$ is isometric, although that is not necessary for the algorithm or the theory in Section 3 to work.
The source code to this implementation is publicly available as part of the full Python implementation of the algorithm at [19].

5.2 The localized states $\phi_\Lambda(x)$

Because the densities associated to states in $S_\Lambda$ are of the form $(v, v) \text{vol}_M$, with $v$ in the finite-dimensional vector space $P_\Lambda H$, one can easily plot the corresponding function $(v, v)$ on $M$. This allows us to test them, by simply plotting them directly, using the spherical harmonics in the continuum. We can then compare these with the numerical states generated through the POINTFORGE algorithm for different $\Lambda$. The expectation is that the numerical states will be comparable but slightly less localized.

Figure 1 shows plot for $\phi_\Lambda(x)$, where $x \in S^2$ is fixed, and plots of numerical states for $\Lambda = 4, 10$. It is evident that the states are indeed peaked neatly near $x$, in both cases. We thus find that the states are well localized and become more localized the larger the cutoff is.

Other than this qualitative comparison we also have analytic control. Lemma 3.3.3 gives the functional form of the dispersion as a function of the cut-off $\Lambda$ as $\log \Lambda / \Lambda^2$. We can check this relation explicitly by plotting the size of the dispersion against the cutoff value, as done in Figure 2 for the cutoff up to $\Lambda = 16$.

5.3 Distribution of states over the sphere

Plotting several states simultaneously allows us to show how the repulsion term distributes them over the sphere. Figure 3 shows 17 states for $\Lambda = 11$. The distribution of states in Figure 3 has some inhomogeneities, some gaps between states are very large. This is because we only generated 17 states instead of the 110 we would expect to generate in the POINTFORGE algorithm. Restricting the number of states reduced computation time, and allowed for a clearer visualization of the independent states. In the right hand Figure we see the states as densities on the sphere, while the left hand plot shows the densities in the $\theta, \phi$ plane.

To test how the repulsive potential acts we can generate states on the sphere and just plot the coordinates for their center of mass associated with the embedding maps $Y_i$. We show this in Figure 4 for a maximal eigenvalue of $\Lambda = 10$, it is clear that without potential all states generated cluster at one point, while even a weak repulsive potential leads to points that are evenly distributed.
Figure 1: Plot of analytic and truncated localized states.
Figure 2: Plot of the dispersion of states versus the value of the cutoff for the states. The dashed line is a fit of the analytic result that the dispersion should scale like $\log \Lambda / \Lambda^2$.

Figure 3: Localized states on the sphere, the left hand image shows the states projected on the two dimensional plane using a sinusoidal projection, while the right hand image shows the states on the sphere.
Figure 4: This shows how the states are distributed dependent on the repulsive potential. We can see that even a weak repulsive potential suffices to lead to well distributed points. This figure shows the point distribution in the Sinusoidal projection, flattening the sphere onto the plane.
6 Embedding a distance graph in $\mathbb{R}^n$

Let $M$ be a Riemannian manifold, and assume that $M$ embeds isometrically into $\mathbb{R}^n$. Now take a finite set $V$ of points in $M$ and their geodesic distances $d(\cdot, \cdot)|_{V \times V}$, e.g. by generating states as above and calculating their distances using (1). Optimally, we would ask for a way to embed $V$ in $\mathbb{R}^n$ such that its image under this embedding equals its image under some Riemannian isometry $M \rightarrow \mathbb{R}^n$.

Of course, without knowledge of $M$ such a problem is unsolvable for any given $V$. Instead, we hope for our embedding procedure to satisfy such a property asymptotically, i.e. that for sequences of $V$ of increasing density their embeddings converge to an embedding of $M$, under some suitable notions of density and convergence. This is an open problem, and since our primary purpose at this point is one of visualization, we will only take it as a guiding principle.

6.1 Stress and local isometry of embeddings

The field of optimal graph embedding is well-established and provides many approaches to questions similar to the above. A particular model of interest is *metric multidimensional scaling*\(^4\) where one looks for an embedding $X : V \rightarrow \mathbb{R}^n$ that minimizes the stress function,

$$
\sigma(X) = \frac{\sum_{p \neq q \in V} w(p, q)(d(p, q) - ||X(p) - X(q)||)^2}{\sum_{p \neq q \in V} w(p, q)},
$$

where $w$ is a positive weight function: this is just a weighted version of the second Gromov-Wasserstein distance between $V \subset M$ and $X(V) \subset \mathbb{R}^n$.

Because our $M$ is not assumed to be Euclidean, the usual choice $w = 1$ would be quite unnatural here. In particular, an isometric embedding of $M$ in the Riemannian sense would not necessarily have minimal stress, because the model instead asks for isometry in the sense of maps of *metric spaces*, not Riemannian manifolds. Since all tangent space information is lost when discretizing like this, the Riemannian notion of isometry does not translate immediately and we must replace it using a measure of locality.

By the smoothness of an isometric embedding $\phi$ of $M$, the relative defect $|d(p, q) - ||\phi(p) - \phi(q)|||/d(p, q)$ must converge to 0 as $p \rightarrow q$. That is to say, as long as we only worry about pairs of points that are close in $M$, the stress function above places the correct restriction on $X$ - the further they are apart, the less sense the corresponding contribution to $\sigma$ makes. This

\(^4\)See e.g. [22]
motivates us to pick a positive weight function \( w(p, q) \) of \( d(p, q) \) that decays monotonically and sufficiently quickly to suppress those lengths that cannot be approximated well by an Euclidean embedding.

For example, imagine two points connected by a shortest geodesic (a great circle arc) of length \( l \leq \pi \) on the unit sphere and let that sphere be embedded isometrically in \( \mathbb{R}^3 \). In \( \mathbb{R}^3 \), the shortest geodesic connecting the points is a chord of length \( c(l) = 2 \sin(l/2) \). The defect for small geodesic distances \( l \) is thus quite small, being \( O(l^3) \). It reaches its maximum when the points are antipodal, with a relative error of \((\pi - 2)/\pi\). The weight function should suppress the contribution of the larger distances to the stress \( \sigma \), in order to still recognize when an embedding of the distance graph is locally isometric.

Let \( \phi : M \to \mathbb{R}^n \) be isometric and let \( w_k \) be a sequence of weight functions depending on the cardinality \( k \) of \( V \subset M \). If \( w_k(l) = o(1) \) for fixed \( l \) and the marginal defect, which is bounded by

\[
\frac{k \sup_{p,q \in M} w_k(p,q) (d(p,q) - \|\phi(p) - \phi(q)\|)^2}{\inf_{|V| = k} \sum_{p,q \in V} w_k(p,q)},
\]

is summable in \( k \), we can at least be sure that the stress function \( \sigma \) converges to 0 for embeddings \( X = \phi|_V \).

The optimal choice of \( w \) then depends (at least somewhat) on the geometry of \( M \) itself; the curvature \( \inf_{\phi : M \to \mathbb{R}^n} \sup\{d(p,q) - \|\phi(p) - \phi(q)\| \mid p, q \in M, d(p,q) \leq \epsilon\} \), as function of \( \epsilon \), together with the Hausdorff distance between \( V \) and \( M \), determines the optimal behaviour of \( w \).

### 6.2 Implementation

For \( \dim M = 2 \), we expect the length of the smallest edges to scale roughly as \( k^{-1/2} \). For \( w_k(l) = \exp\left(-\sqrt{k}l\right) \) the infimum in the denominator of the marginal defect, above, is roughly bounded from below by its value for an equidistributed \( V \), which is of order \( k^2 \int_0^\pi \sin(l)w_k(l)dl \sim k \) as \( k \to \infty \). The supremum in its numerator is \( O(k^{-3/2}) \), so this sequence \( w_k \) will do in the narrow sense that it will asymptotically detect when a sequence \( \{\phi_k : V_k \to \mathbb{R}^n\} \) corresponds asymptotically to an isometric embedding of \( M \), assuming the \( V_k \) are roughly equidistributed.

Given the choice of weights, minima of the resulting stress function can be found efficiently using the weighted SMACOF algorithm for stress majorization. A simple Python implementation of the weighted SMACOF algorithm is part of [19], but for more intensive use we recommend the more efficient FORTRAN version with Python bindings [23].
6.3 Result

The PointForge algorithm returns a metric graph, given an operator system spectral triple \((A, H, D)\) and a designated element \(Y \in A^n\). We apply the locally isometric embedding above not only to the example from Section 5 but also (tentatively) to the triple \((C^\infty(S^2)_\Lambda, H_\Lambda, D_{c,\Lambda})\) of [13], in order to investigate the metric properties of the latter. Here \(\Lambda = 5\), corresponding \(\text{dim } H_\Lambda = 84\), which leads to 35 states.

This leads to the results in Figure 5. The maximal distance between two points of the sphere is 2.58, which would correspond to a sphere with radius 0.82 compared to the expected radius of 1. However it is highly unlikely that the algorithm will generate two states that lie exactly antipodal so finding a smaller radius estimate is expected. The operator \(D_{S^2} - \frac{1}{2}B\) on the other hand has a maximal distance of 1.88, leading to a radius estimate of 0.60.

The embedding of \(D_{S^2}\) in Figure 5(a) is bigger than the unit sphere, while the embedding of \(D_{S^2} - \frac{1}{2}B\) in (b) is smaller, this leads to the impression that our SMACOF algorithm might lead to slightly large embeddings. This can be understood considering that the algorithm disfavours large distances in weighting the stress to avoid curvature effects at large scales. Both embeddings look quite good, however the embedding for \(D_{S^2} - \frac{1}{2}B\) is much better than for \(D_{S^2}\), which we can see comparing the stress function \(\sigma\) which is 0.0093 for \(D_{S^2}\) but 0.0005 for \(D_{S^2} - \frac{1}{2}B\).
7 Final remarks

The PointForge algorithm we introduced in section 4 was designed to reconstruct metric spaces from their truncated commutative (Dirac) spectral triples. However, the ingredients of the algorithm need not originate as truncations of a commutative spectral triple at all; the steps apply verbatim to arbitrary operator system spectral triples, provided a special 'embedding' element $Y$ is given. Obtaining such $Y$ could either be related to the higher Heisenberg equation of [16], or, computational resources allowing, be disposed of entirely as discussed in section 3.2. This would provide one with the means to construct finite metric spaces associated to an arbitrary noncommutative spectral triple.

It would be interesting to elaborate on this and relate it to quantization and e.g. fuzzy spaces, to get a geometric sense of the relation between a commutative spectral triple and its noncommutative deformations. This could be particularly useful in connection to more physically inspired explorations of spectral triples and fuzzy spaces, such as [24]. The ensemble of finite, random spectral triples defined there has shown signs of a phase transition [24, 25] and can be characterized through spectral dimension measures [26], which can be taken as an indication of possibly emergent geometric properties. The PointForge algorithm might then be an interesting tool to further explore some exemplary spectral triples from this class to gather further insights. It would also be instructive to test how the PointForge algorithm works for spectral triples of different topologies, e.g. the non-commutative torus [27] or a fuzzy torus [28].

Another possible application in this direction would be to exploit the explicit scale-dependence of the present formalism in order to obtain a better understanding of the gravitational properties of noncommutative approaches (such as [29]) to quantum field theory.

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