HAUSDORFF MOMENTS, HARDY SPACES AND POWER SERIES

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Abstract. In this paper we consider power and trigonometric series whose coefficients are supposed to satisfy the Hausdorff conditions, which play a relevant role in the moment problem theory. We prove that these series converge to functions analytic in cut domains. We are then able to reconstruct the jump functions across the cuts from the coefficients of the series expansions by the use of the Pollaczek polynomials. We can thus furnish a solution for a class of Cauchy integral equations.

1. Introduction

The problem of characterizing the analytic properties of the functions, in terms of the coefficients of their power expansions, is very old and goes back to a classical result due to Le Roy [13]:

Theorem (Le Roy). If in the Taylor series \( \sum_{n=0}^{\infty} a_n z^n \) the coefficients \( a_n \) are the restriction to the integers of a function \( \tilde{a}(\lambda)(\lambda \in \mathbb{C}) \), holomorphic in the half–plane \( \text{Re}\lambda > -1/2 \), and moreover there exist two constants \( A \) and \( N \) such that

\[
|\tilde{a}(\lambda)| \leq A(1 + |\lambda|)^N \quad (\text{Re}\lambda > -\frac{1}{2}),
\]

then the series converges to a function \( f(z) \) analytic in the unit disk \( D = \{ z \mid |z| < 1 \} \), and furthermore \( f(z) \) admits a holomorphic extension to the cut plane \( \{ z \in \mathbb{C} \mid |1, +\infty) \} \).

Other similar results are due to Lindelöf [14] and Bieberbach [2]. More recently, Stein and Wainger [15] have reconsidered the problem in the framework of the Hardy space theory. More precisely, they assume that the coefficients \( a_n \) are the restriction to the integers of a function \( \tilde{a}(\lambda) \) holomorphic in the half–plane \( \text{Re}\lambda > -1/2 \), and, in addition, \( \tilde{a}(\lambda) \) is supposed to belong to the Hardy space \( H^2(\mathbb{C}_{-1/2}) \) with norm \( \|\tilde{a}\|_2 = \sup_{\sigma > -\frac{1}{2}} \left( \int_{-\infty}^{\infty} |\tilde{a}(\sigma + iv)|^2 dv \right)^{1/2} \). 

Correspondingly, they consider the class of functions \( f(z) \) analytic in the complex \( z \)–plane slit along the positive real axis from 1 to \(+\infty\) (this domain is denoted by \( \mathbb{S} \)). The space of functions analytic in \( \mathbb{S} \) for which \( \sup_{y \neq 0} \left( \int_{-\infty}^{\infty} |f(x + iy)|^2 dx \right)^{1/2} < \infty \) is denoted by \( H^2(\mathbb{S}) \). These authors have proved the following result:

Theorem (Stein–Wainger). Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). Then \( f \in H^2(\mathbb{S}) \) if and only if \( a_n = [\tilde{a}(\lambda)]_{(\lambda=n)} \), where \( \tilde{a} \in H^2(\mathbb{C}_{-1/2}) \). Moreover,

\[
\|F\|_2^2 = 2\pi\|\tilde{a}\|_2^2,
\]
where
\[ \|\tilde{a}\|_2^2 = \int_{-\infty}^{+\infty} \left| \tilde{a} \left( -\frac{1}{2} + i\nu \right) \right|^2 d\nu, \]
and
\[ \|F\|_2^2 = \int_{1}^{+\infty} |F(x)|^2 dx, \]
\( F(x) \) being the jump function, i.e., \( F(x) = -i(f_+(x) - f_-(x)) \); \( f_+(x) \) and \( f_-(x) \) are the boundary values of \( f(x + iy) \) and \( f(x - iy) \), respectively \( (f_{\pm}(x) = \lim_{y \to 0^\pm} f(x \pm iy)) \).

The main purpose of the present paper is the reconstruction of the jump function across the cut from the coefficients of the power expansion. This result is achieved by the use of the Pollaczek polynomials \([1, 16]\), and it will be illustrated in Section 4. This reconstruction allows us to solve the Cauchy integral equation of the following type:
\[ f(z) = \frac{1}{2\pi i} \int_{1}^{+\infty} \frac{F(x)}{x - z} dx, \]
when the Taylor coefficients \( a_n = f^{(n)}(0)/n! \), \( (n = 0, 1, 2, \ldots) \) are supposed to be known. Unfortunately, in the numerical analysis and in the applications to physical problems only a finite number of Taylor coefficients are known, and, moreover, they are affected by noise or, at least, by round-off errors. Furthermore, the integral equations of first kind, like equations (5), give rise to the so-called ill-posed problems in the sense of Hadamard \([8]\): The solution does not depend continuously on the data. We shall briefly return on this important point in Section 4. In a separate paper we shall discuss in detail how to manage numerically the method presented here.

In the theorems of Leroy and Stein–Wainger the coefficients \( a_n \) are required to be the restriction of a function \( \tilde{a}(\lambda)(\lambda \in \mathbb{C}) \) holomorphic in the half-plane \( \text{Re} \lambda > -1/2 \), and, in the case of the Stein–Wainger theorem, this function is also assumed to belong to the Hardy space \( H^2(\mathbb{C}_{-1/2}) \). We prefer to start by requiring that the coefficients \( a_n \) satisfy the so-called Hausdorff conditions \([20]\), which guarantee that they can be regarded as Hausdorff moments in a sense that will be explained in Section 2. This approach is convenient for several reasons:

i) From the Hausdorff conditions and by the use of the Carlson theorem \([3]\) we derive immediately the existence of a unique interpolation of the coefficients which is holomorphic in a half-plane.

ii) Following Watanabe \([18]\), we can give a suggestive probabilistic interpretation of the Hausdorff conditions. Accordingly, the coefficients \( a_n \) can be regarded as the values of an harmonic function associated with a Markov process in a sense that will be outlined in Section 2.

iii) By imposing to the coefficients \( a_n \) Hausdorff conditions of various types, we can, correspondingly, obtain more specific properties of smoothness of the function which gives the jump across the cut.

The last point is particularly relevant. In fact, in the Stein–Wainger approach one works essentially with the unitary equivalence between \( H^2(\mathbb{S}) \) and \( L^2(1, +\infty) \); accordingly, the jump function belongs to \( L^2(1, +\infty) \). On the other hand, more refined properties of continuity and differentiability of the jump function are relevant in the mathematical theory of the Cauchy integral equation, and particularly
in the physical applications [5]. In agreement with this approach, we shall prove in Section 3 theorems which are variations on the Stein–Wainger result. The main mathematical tool used in our approach is the Watson resummation method which leads, in a very natural way, to the Laplace transform of the jump function. It turns out that this Laplace transform coincides exactly with the Carlsonian interpolation of the coefficients. In this way all what is necessary for extending the methods and the results to expansions in terms of Legendre and ultraspherical polynomials is obtained. In this extension, in fact, the interpolating function \( \tilde{a}(\lambda) \) coincides with the spherical Laplace transform (in the sense of Faraut [4, 6, 7]), that can be regarded as a composition of the classical Laplace transform and the Abel–Radon transform.

This paper can be regarded as the completion and a large extension of a preliminary work by one of the authors (G.A.V.) [17], where the Hausdorff moment problem has been approached by the use of the Pollaczek polynomials.

2. Hausdorff Moments, Hardy Spaces and Markov Processes

2.1. Hausdorff Moments and Hardy Spaces. Given a sequence of (real) numbers \( \{f_n\}_{n=0}^\infty \), let \( \Delta \) denote the difference operator:

\[
\Delta f_n = f_{n+1} - f_n.
\] (6)

Then, we have:

\[
\Delta^k f_n = \Delta \times \Delta \times \cdots \times \Delta f_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^m f_{n+k-m},
\] (7)

(for every \( k \geq 0 \)); \( \Delta^0 \) is the identity operator by definition. Now, suppose that there exists a positive constant \( M \) such that:

\[
(n + 1) \sum_{i=0}^{n} \binom{n}{i}^2 |\Delta^i f_{(n-i)}|^2 < M \quad (n = 0, 1, 2, \ldots).
\] (8)

It can be proved [20] that condition (8) is necessary and sufficient in order to represent the sequence \( \{f_n\}_{n=0}^\infty \) as follows:

\[
f_n = \int_0^1 x^n u(x) \, dx \quad (n = 0, 1, 2, \ldots),
\] (9)

where \( u(x) \) belongs to \( L^2(0,1) \).

We can prove the following Proposition.

**Proposition 1.** If the sequence \( \{f_n\}_{n=0}^\infty \) satisfies condition (8), then there exists a unique interpolation of this sequence, denoted by \( \hat{F}(\lambda) \) (\( \lambda \in \mathbb{C} \), \( [\hat{F}(\lambda)](\lambda=n) = f_n \)) which belongs to the Hardy space \( H^2(\mathbb{C}_{-1/2}) \), and satisfies the following properties:

(i) \( \hat{F}(\lambda) \) is holomorphic in the half–plane \( \Re \lambda > -1/2 \);

(ii) \( \hat{F}(\sigma + iv) \) belongs to \( L^2(-\infty, +\infty) \) for any fixed value of \( \Re \lambda = \sigma \geq -1/2 \);

(iii) \( \hat{F}(\lambda) \) tends uniformly to zero as \( \lambda \) tends to infinity inside any fixed half–plane \( \Re \lambda \geq \delta > -1/2 \).
Proof. If the sequence \( \{ f_n \}_0^\infty \) satisfies condition (S), then representation (9) holds true. If in this representation we put \( x = e^{-t} \), then we obtain:

\[
f_n = \int_0^{+\infty} e^{-nt}e^{-t}u(e^{-t}) \, dt \quad (n = 0, 1, 2, \ldots).
\]

(10)

Therefore the numbers \( f_n \) can be regarded as the restriction to the integers of the following Laplace transform:

\[
\tilde{F}(\lambda) = \int_0^{+\infty} e^{-(\lambda+1/2)t}e^{-t/2}u(e^{-t}) \, dt.
\]

(11)

Indeed, one has \( [\tilde{F}(\lambda)](\lambda=n) = f_n \). Moreover, \( \int_0^{+\infty} |\exp(-t/2)u(\exp(-t))|^2 \, dt = \int_0^1 |u(x)|^2 \, dx < \infty \), and therefore the function \( \exp(-t/2)u(\exp(-t)) \) belongs to \( L^2(0, +\infty) \). Then, in view of the Paley–Wiener theorem \([9]\) and of formula (11), we can conclude that \( \tilde{F}(\lambda) \) belongs to the Hardy space \( H^2(\mathbb{C}_{-1/2}) \), and properties (i), (ii) and (iii) follow. Thus, we can make use of the Carlson theorem \([8]\), which guarantees that \( \tilde{F}(\lambda) \) represents the unique interpolation of the sequence \( \{ f_n \}_0^\infty \).

Now, we can prove the following Proposition.

**Proposition 2.** If the sequence \( \{ f_n \}_0^\infty \), where \( f_n = n^p a_n \) \( (p \geq 1) \), satisfies condition (S), then there exists a unique Carlsonian interpolation of the numbers \( \{ a_n \}_0^\infty \), denoted by \( \tilde{a}(\lambda) \) \( (\lambda \in \mathbb{C}) \), which satisfies the following properties:

(i) \( \tilde{a}(\lambda) \) is holomorphic in \( \text{Re} \lambda > -1/2 \);

(ii) \( \lambda^p \tilde{a}(\lambda) \) belongs to \( L^2(-\infty, +\infty) \) for any fixed value of \( \text{Re} \lambda = \sigma \geq -1/2 \);

(iii) \( \lambda^p \tilde{a}(\lambda) \) tends uniformly to zero as \( \lambda \) tends to infinity inside any fixed half-plane \( \text{Re} \lambda \geq \delta > -1/2 \);

(iv) \( \lambda^{(p-1)} \tilde{a}(\lambda) \) belongs to \( L^1(-\infty, +\infty) \) for any fixed value of \( \text{Re} \lambda = \sigma \geq -1/2 \).

**Proof.** Since the numbers \( f_n = n^p a_n \) \( (n = 0, 1, 2, \ldots; p \geq 1) \) satisfy condition (S), then there exists a unique Carlsonian interpolation of the sequence \( \{ f_n \}_0^\infty \), denoted by \( \tilde{F}(\lambda) \), \( (\lambda \in \mathbb{C}) \), which can be written as the product: \( \tilde{F}(\lambda) = \lambda^p \tilde{a}(\lambda) \), where \( \tilde{a}(\lambda) \) is the unique Carlsonian interpolation of the numbers \( \{ a_n \}_0^\infty \). In view of condition (S) and Proposition 1 it follows that \( \tilde{F}(\lambda) \) belongs to \( H^2(\mathbb{C}_{-1/2}) \). Therefore properties (i), (ii) and (iii) follow immediately.

By applying the Schwarz inequality, and recalling that \( \tilde{F}(\lambda) \in L^2(-\infty, +\infty) \) for any fixed \( \text{Re} \lambda = \sigma \geq -1/2 \), we have:

\[
\int_{-\infty}^{+\infty} \left| (\sigma + iv)^{(p-1)} \tilde{a}(\sigma + iv) \right| \, dv = \int_{-\infty}^{+\infty} \left| \frac{\tilde{F}(\sigma + iv)}{(\sigma + iv)} \right| \, dv 
\leq \left( \int_{-\infty}^{+\infty} \frac{1}{|\sigma + iv|^2} \, dv \right)^{1/2} \left( \int_{-\infty}^{+\infty} |\tilde{F}(\sigma + iv)|^2 \, dv \right)^{1/2} < \infty,
\]

(12)

if \( \sigma \geq -1/2, \sigma \neq 0, p \geq 1 \). Finally, from inequality (12), and in view of the regularity and integrability of the function \( \lambda^{(p-1)} \tilde{a}(\lambda) \) in the neighborhood of \( \text{Re} \lambda = 0 \) we can state in all generality that \( \lambda^{(p-1)} \tilde{a}(\lambda) \) belongs to \( L^1(-\infty, +\infty) \) for any \( p \geq 1 \). \( \square \)
2.2. Hausdorff Moments and Markov Processes. In this subsection we follow closely the paper of Watanabe [15]. Let \((\Omega, \mathcal{F}, P)\) be an abstract probability field. If \(\{y_n(\omega); n \geq 1\}\) is a sequence of random variables on \((\Omega, \mathcal{F}, P)\) which are mutually independent, and each one satisfies

\[
P\{y_n(\omega) = 1\} = p, \quad P\{y_n(\omega) = 0\} = 1 - p,
\]

then it is called a Bernoulli sequence and denoted by \(B(p)\). In the sequel we shall consider \(B(1/2)\). Let \(E\) be the set of all points \((n, i)\) such that \(n \geq i = 0, 1, 2, \ldots\)

Next, we consider the Markov process \(x_n\) attached to \(B(1/2)\). Let us note that:

\[
P_{(n,i)}(x_k = (m,j)) = \begin{cases} 
\left(\frac{1}{2}\right)^k \binom{k}{j-i} & \text{for } m = n + k, j \geq i, \\
0 & \text{otherwise,}
\end{cases}
\]

where \(k \geq 0, (n, i) \in E, (m, j) \in E\). The kernel \(K((n, i), (m, j))\) is given by [15]:

\[
K((n, i), (m, j)) = \frac{P_{(n,i)}(\sigma\{m,j\} < +\infty)}{P_{(0,0)}(\sigma\{m,j\} < +\infty)} = 2^n \frac{(m-n)! (j-m)!}{m! (m-n-j+i)! (j-i)!}.
\]

Now, consider an infinite sequence \((m_k, j_k)\) having no limit point in \(E\), and such that

\[
\lim_{k \to \infty} \frac{j_k}{m_k} = 1 - b,
\]

for a suitable \(0 \leq b \leq 1\). Then, using the Stirling formula, and taking into account equality (16), from (15) we obtain:

\[
K((n, i), b) = 2^n b^{(n-i)} (1-b)^i.
\]

Thus, one may consider that the Martin boundary \(M\) [10] induced by the process \(x_n\) coincides with the interval \([0, 1]\) as a set, and, accordingly, the generalized Poisson kernel \(K((n, i), b)\) is given by \(2^n b^{(n-i)} (1-b)^i\). Finally, we note that for a function \(u\) over \(E\) the expectation \(E_{(0,0)}\) is given by:

\[
E_{(0,0)}(|u(x_n)|) = 2^{-n} \sum_{i=0}^{n} |u(n, i)| \binom{n}{i}.
\]

Now, the following propositions due to Watanabe [15] can be stated.

**Proposition 3.** Let \(x_n\) be the Markov process attached to the Bernoulli sequence \(B(1/2)\).

(i) The Martin boundary induced by \(x_n\) is equivalent to the interval \([0, 1]\) with the ordinary topology;

(ii) The generalized Poisson kernel \(K((n, i), b)\) is:

\[
K((n, i), b) = 2^n b^{(n-i)} (1-b)^i.
\]

(iii) A function \(u\) (belonging to the set of all the finite real valued functions over \(E^* = E \cup \infty\), vanishing at \(\infty\)) can be represented by means of a bounded signed measure on \((0, 1), B_{[0,1]}\), (where \(B_{[0,1]}\) is the Borel field consisting of all the ordinary Borel subsets in \([0, 1]\)), as follows:

\[
u(n, i) = 2^n \int_0^1 b^{(n-i)} (1-b)^i d\mu(b),
\]
Therefore, from inequalities (8) and (25) we obtain:

\[
\sum_{i=0}^{n} |A^i f(n-i)| \left( \begin{array}{c} n \\ i \end{array} \right) < L \quad (n = 0, 1, 2, \ldots; L = \text{constant}),
\]

which can be compared with representation (9). Moreover, if the sequence following domains:

\[ I \]

that coincides with inequality (21), if we put

\[ y \]

In fact, if in inequality (24) we put:

\[ u \]

satisfies inequality (8), then it satisfies also inequality (21). This can be proved easily by the use of the Cauchy inequality:

\[
\sum_{i=0}^{n} |x_i y_i| \leq \left( \sum_{i=0}^{n} |x_i|^2 \right)^{1/2} \left( \sum_{i=0}^{n} |y_i|^2 \right)^{1/2}.
\]

Proof. See [18].

Proposition 4. Let \( x_n \) be the Markov process attached to the Bernoulli sequence \( B(1/2) \). Given a sequence of real numbers \( \{f_n; n \geq 0\} \) such that:

\[
\sum_{i=0}^{n} |A^i f(n-i)| \left( \begin{array}{c} n \\ i \end{array} \right) < L \quad (n = 0, 1, 2, \ldots; L = \text{constant}),
\]

then the function \( u(n, i) \) defined by:

\[
u(n, i) = 2^n (-1)^i A^i f(n-i)\]

is a \( x_n \)-harmonic function, and can be represented by formula (22).

Proof. See [18].

Notice that from representation (22) it follows:

\[
2^{-n} u(n, 0) = \int_{0}^{1} b^n d\mu(b),
\]

which can be compared with representation (9). Moreover, if the sequence \( \{f_n\}_{\geq 0} \) satisfies inequality (8), then it satisfies also inequality (21). This can be proved easily by the use of the Cauchy inequality:

\[
\sum_{i=0}^{n} |x_i y_i| \leq \left( \sum_{i=0}^{n} |x_i|^2 \right)^{1/2} \left( \sum_{i=0}^{n} |y_i|^2 \right)^{1/2}.
\]

In fact, if in inequality (24) we put: \( y_i = 1, \ \forall i \in (0, 1, 2, \ldots, n), \) \( x_i = |A^i f(n-i)| \left( \begin{array}{c} n \\ i \end{array} \right) \), we obtain:

\[
\sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) |A^i f(n-i)| \leq \left\{ \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) 2^i f(n-i) \right\}^{1/2} (n + 1)^{1/2}.
\]

Therefore, from inequalities (8) and (25) we obtain:

\[
\sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) |A^i f(n-i)| \leq \frac{\sqrt{M}}{(n + 1)^{1/2}} (n + 1)^{1/2} = \sqrt{M},
\]

that coincides with inequality (21), if we put \( L = \sqrt{M}. \)

3. A Double Analytic Structure for a Class of Trigonometric and Power Series

In the complex plane \( \mathbb{C} \) of the variable \( \theta = u + i v \) \( (u, v \in \mathbb{R}) \) we consider the following domains: \( \mathfrak{D}^{(\pm \xi_0)} = \{ \theta \in \mathbb{C} \mid \text{Im} \theta > \pm \xi_0, \xi_0 \geq 0 \} \), and \( \mathfrak{D}^{(\mp \xi_0)} = \{ \theta \in \mathbb{C} \mid \text{Im} \theta < \pm \xi_0, \xi_0 \geq 0 \} \). We introduce, correspondingly, the following cut domains:

\[ \mathfrak{D}^{(\pm \xi_0)} \setminus \Xi^{(\pm \xi_0)}, \text{ where } \Xi^{(\pm \xi_0)} = \{ \theta \in \mathbb{C} \mid \theta = 2k\pi + i v, v > \xi_0, \xi_0 \geq 0, k \in \mathbb{Z} \}, \text{ and } \mathfrak{D}^{(\pm \xi_0)} \setminus \Xi^{(\mp \xi_0)}, \text{ where } \Xi^{(\mp \xi_0)} = \{ \theta \in \mathbb{C} \mid \theta = 2k\pi + i v, v < -\xi_0, \xi_0 \geq 0, k \in \mathbb{Z} \}
\]

(for a detailed description of these cut domains see [4]). We will use the notation \( A = A \setminus 2\pi \mathbb{Z} \) for every subset \( A \) of \( \mathbb{C} \) which is invariant under the translation group \( 2\pi \mathbb{Z} \).

We can then prove the following theorem.
Theorem 1. Let us consider the following series:

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} a_n e^{-in\theta} \quad (\theta = u + iv, ~ u, v \in \mathbb{R}), \quad (27)$$

and suppose that the set of numbers \(\{f_n\}_{n=0}^{\infty}, \ f_n = n^p a_n \) \((p \geq 1, \ n = 0, 1, 2, \ldots)\) satisfies condition (B), then:

1. series (27) converges uniformly to a function \(f(\theta)\) analytic in \(\mathcal{J}^{(0)}_{-}\);
2. the function \(f(\theta)\) admits a holomorphic extension to the cut domain \(\mathcal{J}^{(0)}_{+} \setminus \Xi^{(0)}_{+}\) (see Fig. 14);
3. the jump function \(F(v)\) (which equals the discontinuity of \(i f(\theta)\) across the cuts \(\Xi^{(0)}_{+}\)) is a function of class \(C^{(p-1)}\) \((p \geq 1)\), and satisfies the following bound:

$$|F(v)| \leq \|\tilde{a}_\sigma\|_1 e^{\sigma v} \quad (\sigma \geq -\frac{1}{2}, \ v \in \mathbb{R}^+), \quad (28)$$

where \(\tilde{a}(\sigma + iv) ~ (v \in \mathbb{R})\) is the Carlsonian interpolation of the coefficients \(a_n\), and

$$\|\tilde{a}_\sigma\|_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\tilde{a}(\sigma + iv)| \ dv \quad (\sigma \geq -\frac{1}{2}); \quad (29)$$

4. \(\tilde{a}(\sigma + iv)\) is the Laplace transform of the jump function \(F(v)\): i.e.,

$$\tilde{a}(\sigma + iv) = \int_0^{+\infty} F(v)e^{-(\sigma + iv)} \ dv \quad (\sigma > -\frac{1}{2}); \quad (30)$$

5. the Plancherel equality holds true:

$$\int_{-\infty}^{+\infty} |\tilde{a}(\sigma + iv)|^2 \ dv = 2\pi \int_0^{+\infty} |F(v)e^{-\sigma v}|^2 \ dv \quad (\sigma \geq -\frac{1}{2}). \quad (31)$$

Proof. Since the set \(\{f_n\}_{n=0}^{\infty}\) satisfies condition (B), given an arbitrary number \(C\), there exists a real number \(m\) such that for \(n > m, \ |a_n| \leq C\). Therefore, we can write:

$$\left| \sum_{n=m}^{\infty} \left( \frac{1}{2\pi} a_n e^{-in\theta} \right) \right| \leq \frac{C}{2\pi} \sum_{n=m}^{\infty} e^{nv} \quad (\theta = u + iv). \quad (32)$$

The series at the r.h.s. of formula (32) is uniformly convergent for \(v \leq v_0 < 0\). Recalling the Weierstrass theorem on the uniformly convergent series of analytic functions, we can also conclude that the series \((1/2\pi) \sum_{n=m}^{\infty} a_n \exp(-in\theta) \ (\theta = u + iv)\) converges uniformly to a function analytic in \(\mathcal{J}^{(0)}_{-}\). On the other hand, series (27) can be rewritten as the following sum:

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} a_n e^{-in\theta} = \frac{1}{2\pi} \left\{ \sum_{n=m}^{\infty} a_n e^{-in\theta} + T_m(\theta) \right\}, \quad (33)$$

where \(T_m(\theta) = \sum_{n=0}^{[m]}(a_n e^{-in\theta})\) is a trigonometric polynomial analytic in \(\mathcal{J}^{(0)}_{-}\). Therefore the first statement is proved.

In order to prove the other statements, let us introduce the following integral:

$$f_\epsilon(u) = \frac{i}{4\pi} \int_{e^{\epsilon}}^{e^{-\epsilon}} \frac{\tilde{a}(\lambda)e^{-i\lambda(u-v)}}{\sin \pi \lambda} \ d\lambda \quad (\epsilon = \pm), \quad (34)$$
where $\tilde{a}(\lambda)$, $(\lambda = \sigma + i\nu)$ is the unique Carlsonian interpolation of the sequence $\{a_n\}_{n=0}^{\infty}$, which exists in view of the fact that the set $\{f_n\}_{n=0}^{\infty}$ satisfies condition $\mathcal{C}$, and the contour $\mathcal{C}$ is contained in the half-plane $\mathbb{C}_{-1/2}$ and encircles the positive real semi-axis of the $\lambda$-plane (or a part of it) as is illustrated in Fig. 2A.

Now, let us consider the following inequalities:

$$
\left| e^{-i(\sigma + i\nu)(u-\pi)} \right| \leq 2 \cosh \pi \nu \quad (u \in [0, 2\pi]),
$$

$$
\left| \sin \pi (\sigma + i\nu) \right| \geq \sinh \pi \nu,
$$

$$
\left| \frac{e^{-i(\sigma + i\nu)(u-\pi)}}{\sin \pi (\sigma + i\nu)} \right| \leq \frac{2 \cosh \pi \nu}{\sinh \pi \nu} \quad (u \in [0, 2\pi]).
$$

Let us recall that $\lambda^p \tilde{a}(\lambda)$ $(p \geq 1)$ tends uniformly to zero as $\lambda \to \infty$ inside any fixed half-plane $\text{Re} \lambda = \sigma \geq \delta > -1/2$, and $\lambda^{p-1} \tilde{a}(\lambda)$ belongs to $L^1(-\infty, +\infty)$ for $\text{Re} \lambda = \sigma \geq -1/2$ (see Proposition 2).

In view of these properties of $\tilde{a}(\lambda)$, and by the use of bound $\mathcal{C}$, we can guarantee that the integral $f_+(u)$ $(u \in [0, 2\pi])$ converges, and the contour $\mathcal{C}$ can be deformed and replaced by the line $L_\sigma = \{ \lambda = \sigma + i\nu, \nu \in \mathbb{R}, \sigma \geq -1/2 \}$, provided that the real variable $u$ is kept in $[0, 2\pi]$ (see Fig. 2A). Finally, by applying the Watson resummation method $\mathcal{W}$, we obtain for $u \in [0, 2\pi]$:

$$
f_+(u) = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\tilde{a}(\sigma + i\nu)e^{-i(\sigma + i\nu)(u-\pi)}}{\sin \pi (\sigma + i\nu)} d\nu = \frac{1}{2\pi} \sum_{n=l}^{\infty} a_n e^{-inu},
$$

$(l \geq 0$, integer; $-1/2 \leq \sigma < 0$ if $l = 0$, $l - 1 < \sigma < l$ if $l > 0)$.
Proceeding in an analogous fashion for the integral $f^-(u)$ (formula (34) with $\epsilon = -$), and distorting the contour integration in a similar way, we finally obtain for $u \in [-2\pi, 0]$

\[ f^-(u) = -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \hat{a}(\sigma + iv)e^{-i(\sigma + iv)(u + \pi)} \frac{d\nu}{\sin(\sigma + iv)} = \frac{1}{2\pi} \sum_{n=l}^{\infty} a_n e^{-inu}, \quad (39) \]

($l \geq 0$, integer; $-1/2 \leq \sigma < 0$ if $l = 0$, $l - 1 < \sigma < l$ if $l > 0$).

Now, in the integral (38) we substitute for $u$ the complex variable $\theta = u + iv$, and we see that the obtained integral provides an analytic continuation of $f_+$ in the strip $\{\theta = u + iv, 0 < u < 2\pi\}$, continuous in the closure of the latter. Indeed we have:

\[ e^{-\sigma v} f_+(u + iv) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iv\nu} H^\sigma_\nu(\nu) d\nu \quad (0 \leq u \leq 2\pi), \quad (40) \]
with
\[ H_\sigma^2(\nu) = -\frac{\hat{a}(\sigma + i\nu)e^{-i(\sigma + i\nu)(u-\pi)}}{2\sin\pi(\sigma + i\nu)}, \quad (41) \]
then, in view of bound (37), and since \( \lambda^{(p-1)}\hat{a}(\lambda) \in L^1(-\infty, +\infty) \) for any fixed value of Re \( \lambda = \sigma \geq -1/2 \) (see Proposition 2), the statement above is proved.

Similarly, the analytic continuation of the function \( f_- \) is defined in the strip \( \{ \theta = u + iv, -2\pi < u < 0 \} \). The discontinuity \( f_+(iv) - f_-(iv) \) can be computed by replacing \( u \) by \( iv \) in integrals (35) and (39), and subtracting Eq. (39) from Eq. (35). We then obtain
\[ i[f_+(iv) - f_-(iv)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{a}(\sigma + iv)e^{(\sigma + iv)v} \, dv \quad (v \in \mathbb{R}^+, \sigma \geq -\frac{1}{2}). \quad (42) \]
Thus, we have proved that the function \( f(\theta) (\theta = u + iv) \) admits a holomorphic extension to the cut domain \( \mathcal{J}_+^{(0)} \setminus \hat{\Xi}_+^{(0)} \).

From formula (42) we derive the following bound for the jump function \( F(v) \equiv i[f_+(iv) - f_-(iv)]: \]
\[ |F(v)| \leq \|\hat{a}_\sigma\|_1 e^{\sigma v} \quad (\sigma \geq -\frac{1}{2}, v \in \mathbb{R}^+), \quad (43) \]
where
\[ \|\hat{a}_\sigma\|_1 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{a}(\sigma + iv)| \, dv < \infty \quad (\sigma \geq -\frac{1}{2}). \quad (44) \]
Using again formula (42), and recalling the Riemann–Lebesgue theorem, we can prove that \( F(v) \) is a function of class \( C^{(p-1)} \) \( (p \geq 1) \) in view of the fact that \( \lambda^{(p-1)}\hat{a}(\lambda) \) belongs to \( L^1(-\infty, +\infty) \) for any Re \( \lambda = \sigma \geq -1/2 \) (see Proposition 2).

Inverting formula (42) we obtain:
\[ \hat{a}(\sigma + iv) = \int_{0}^{+\infty} F(v)e^{-(\sigma + iv)v} \, dv \quad (\sigma > -\frac{1}{2}), \quad (45) \]
which is, indeed, the Laplace transform of the jump function \( F(v) \), and it is holomorphic for Re \( \lambda = \sigma > -1/2 \).

Finally, recalling that \( \hat{a}(\sigma + iv) \) belongs to \( L^2(-\infty, +\infty) \) at any fixed value Re \( \lambda = \sigma \geq -1/2 \), we obtain the Plancherel equality (31) and, in particular:
\[ \int_{-\infty}^{+\infty} \left| \hat{a} \left( -\frac{1}{2} + iv \right) \right|^2 dv = 2\pi \int_{0}^{+\infty} |F(v)e^{v/2}|^2 \, dv. \quad (46) \]

Remarks. (i) Theorem 1 proves a double analytic structure, connected with series (27), in the following sense: To the functions \( \hat{a}(\lambda) \), that interpolate the coefficients \( a_n \), and are holomorphic in the half–plane Re \( \lambda > -1/2 \), there corresponds the class of functions \( f(\theta) \) holomorphic in the domain \( \mathcal{J}_+^{(0)} \cup (\mathcal{J}_+^{(0)} \setminus \hat{\Xi}_+^{(0)}) \), and moreover \( \hat{a}(\lambda) \) is the Laplace transform of the jump function \( F(v) \) across the cut.

(ii) Let us note that the Plancherel equality (formulae (31) and (46)), as well as the analyticity property of the function \( f(\theta) \), remain true under milder conditions on the coefficients \( a_n \). In fact, it is sufficient that the \( a_n \)'s form a sequence of numbers that satisfies condition (35). This is, indeed, the result contained in the theorem of Stein–Wainger [15] referred to series (27). In conclusion, the more restrictive conditions, assumed in Theorem 1, are reflected by the smoothness property of the
jump function, which is, however, a quite relevant property playing an important role in the applications to physical problems.

By substituting the complex plane of the variable \( z = \exp(-i\theta) \) to the \( 2\pi \)-periodic \( \theta \)-plane, we can now give an equivalent presentation of the results of Theorem 1 in terms of properties of Taylor series and of Mellin transformation. To the cut at \( \theta = iv \), it corresponds, in the \( \z \)-plane geometry, the cut located on the real axis \( (x \equiv \text{Re} z) \) from \( x = 1 \) up to \( +\infty \). To the jump function \( F(v) \) it corresponds the function \( F(\ln x) \), which shall still be denoted hereafter simply by \( F(x) \) with a small abuse of notation which avoids, however, an useless proliferation of symbols. Adopting the same convention, we shall always denote the jump function, in the various geometries, with the same symbol: i.e., \( F \).

**Theorem 2.** If in the Taylor series:

\[
\frac{1}{2\pi} \sum_{n=0}^{\infty} a_n z^n \quad (z = x + iy; \ x, y \in \mathbb{R}),
\]

the coefficients \( a_n \) satisfy the assumptions required by Theorem 1, then:

1. the series converges uniformly to a function \( f(z) \) analytic in the unit disk \( D_0 = \{ z \ | \ |z| < 1 \} \);
2. \( f(z) \) admits a holomorphic extension to the cut plane \( z \in \mathbb{C} \setminus (1, +\infty) \) (see Fig. 1C);
3. the jump function \( F(x) = -i(f_+(x) - f_-(x)) \), \( (f_\pm(x) = \lim_{\epsilon \to 0} f(x \pm i\epsilon)) \)
   is a function of class \( C^{(p-1)} \) \( (p \geq 1) \), and satisfies the following bound:

\[
|F(x)| \leq \|\tilde{a}_\sigma\|_1 x^\sigma \quad (\sigma \geq -\frac{1}{2}, \ x \in (1, +\infty)),
\]

where \( \tilde{a}(\sigma + iv) \) \( (v \in \mathbb{R}) \) is the Carlsonian interpolation of the coefficients \( a_n \), and \( \|\tilde{a}_\sigma\|_1 \) is given by formula (29):

\[
(4') \quad \tilde{a}(\sigma + iv) = \int_1^{+\infty} F(x) x^{-(\sigma + iv) - 1} dx \quad (\sigma > -\frac{1}{2});
\]

(5') the Plancherel formula associated with the Mellin transform gives

\[
\int_{-\infty}^{+\infty} |\tilde{a}(\sigma + iv)|^2 dv = 2\pi \int_1^{+\infty} |F(x)|^2 x^{-2\sigma - 1} dx \quad (\sigma \geq -\frac{1}{2}).
\]

**Remark.** Equality (51) that, in the particular case \( \sigma = -1/2 \), reads

\[
\int_{-\infty}^{+\infty} |\tilde{a}(-\frac{1}{2} + iv)|^2 dv = 2\pi \int_1^{+\infty} |F(x)|^2 dx,
\]

coincides with the result contained in the Stein–Wainger theorem \[15\]. In order to obtain this latter result it is sufficient to require that the set \( \{a_n\}_0^\infty \) satisfies condition \( \mathbf{8} \).

We now present, without giving the proof, two variants of Theorems 1 and 2, which are relevant in the physical applications, and specifically in the theory of the thermal Green functions \[5\]. The proofs can be easily obtained, with small variations, from those of Theorems 1 and 2.
Proposition 5. If in the following series
\[ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} a_n \cos nu \quad (u \in \mathbb{R}), \] (52)
the coefficients \( a_n \), in addition to the assumptions required by Theorem 1, satisfy also the following bound:
\[ |a_n| \leq C e^{-(n-m)\xi_0} \quad (n > m, m \in \mathbb{R}^+, \xi_0 > 0), \] (53)
then
(1) series (52) converges uniformly to a function \( f(\theta) \) analytic in the strip \( \{ \theta \in \mathbb{C} \mid |\text{Im}\theta| < \xi_0 \} \);
(2) the function \( f(\theta) \) admits a holomorphic extension to the cut domain \( \mathcal{J}^{(0)} \setminus \Xi_+^{(0)} \cup \mathcal{J}^{(0)} \setminus \Xi_-^{(0)} \) (see Fig. 1B);
(3) the jump function across the cuts \( \hat{\Xi}_\pm^{(\pm \xi_0)} \) satisfies all the properties proved in Theorem 1.

Proposition 6. If in the following Laurent series
\[ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} a_n z^n \quad (z = x + iy, x, y \in \mathbb{R}, a_n = a_{-n}), \] (54)
the coefficients \( a_n \) satisfy the assumption of Theorem 2 and condition (53), then:
(1) series (54) converges uniformly to a function \( F(z) \) analytic in the annulus \( \mathcal{A} = \{ z \in \mathbb{C} \mid \exp(-\xi_0) < |z| < \exp(\xi_0) \} \);
(2) \( F(z) \) admits a holomorphic extension to the cut domain \( z \in \mathbb{C} \setminus [(0, \exp(-\xi_0) \cup (\exp(\xi_0), +\infty)] \) (see Fig. 1D);
(3) the jump function \( \hat{F}(x) \) satisfies the following bounds:
\[ |\hat{F}(x)| \leq \|\tilde{a}_\sigma\|_1 x^\sigma \quad (\sigma \geq -\frac{1}{2}, x \in (\exp(\xi_0), +\infty)), \] (55)
and, accordingly,
\[ \left| \hat{F}\left(\frac{1}{x}\right) \right| \leq \|\tilde{a}_\sigma\|_1 x^{-\sigma} \quad (\sigma \geq -\frac{1}{2}, x \in (0, \exp(-\xi_0))), \] (56)
where \( \tilde{a}(\sigma + iv) \) is the Carlsonian interpolation of the coefficients \( a_n \), and \( \|\tilde{a}_\sigma\|_1 \) is given by formula (29);
(4) \( \hat{a}(\sigma + iv) \) is the Mellin transform of the jump function, and it is given by:
\[ \hat{a}(\sigma + iv) = \int_1^{+\infty} F(x)x^{-(\sigma+iv)-1} \ dx \quad (\sigma > -\frac{1}{2}), \] (57)
or, equivalently, by:
\[ \hat{a}(\sigma + iv) = \int_0^{1} F\left(\frac{1}{x}\right) x^{(\sigma+iv)-1} \ dx \quad (\sigma > -\frac{1}{2}). \] (58)
(5) The Plancherel formula associated with the Mellin transform gives:
\[ \int_{-\infty}^{+\infty} |\hat{a}(\sigma + iv)|^2 \ dv = 2\pi \int_1^{+\infty} |F(x)|^2 x^{-2\sigma-1} \ dx \quad (\sigma \geq -\frac{1}{2}), \] (59)
Analogous modifications should be considered in relation to the other expansions.

Therefore the inversion of the Fourier transform at \( \sigma \) in the mean order two, which reads:

\[
\{ \text{the sequence} \ a_n \ \text{of series} \ \{ f_n \}_{n=0}^{\infty} \ \text{satisfy the following condition: The sequence} \ \{ f_n \}_{n=0}^{\infty} \ \text{has been considered in Section} \ \mathbf{3} \}.
\]

Here we assume on the coefficients \( a_n \) the weakest possible condition: i.e., the sequence \( \{ a_n \}_{n=0}^{\infty} \) is supposed to satisfy condition \( \mathbf{8} \). Under this condition we can still guarantee that there exists a unique Carlsonian interpolation \( \tilde{a}(\lambda) \) of the coefficients \( a_n \), and also that \( \tilde{a}(\sigma + iv) \) belongs to \( L^2(\mathbb{R}^+, +\infty) \) for any fixed \( \Re \lambda = \sigma \geq -1/2 \); but \( \tilde{a}(-1/2 + iv) \) does not belong, in general, to \( L^1(\mathbb{R}^+, +\infty) \). Therefore the inversion of the Fourier transform at \( \sigma = -1/2 \) holds only as a limit in the mean order two, which reads:

\[
F(v)e^{v^2/2} = \lim_{\nu_0 \to +\infty} \left( \frac{1}{2\pi} e^{ivv} \right) (v \in \mathbb{R}^+).
\] (61)

We can prove the following theorem.

**Theorem 3.** If in series \( \{ f_n \}_{n=0}^{\infty} \) the coefficients \( a_n \) satisfy condition \( \mathbf{8} \), then the function \( \exp(v^2/2)F(v) \) can be represented by the following expansion, that converges in the sense of the \( L^2 \)-norm:

\[
e^{v^2/2}F(v) = \sum_{n=0}^{\infty} c_m \Phi_m(v) \quad (v \in \mathbb{R}^+),
\] (62)

where

\[
c_m = \frac{\sqrt{2}}{n!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} a_n \left( \begin{array}{c} n + 1 \ 2 \end{array} \right)
\] (63)

and

\[
\Phi_m(v) = i^m \sqrt{2} L_m(2e^{-v}) e^{-e^{-v}} e^{-v/2},
\] (64)

\( P_m \) and \( L_m \) being, respectively, the Pollaczek and the Laguerre polynomials.

**Proof.** The Pollaczek polynomials \( P_m(\alpha)(v) \ (v \in \mathbb{R}) \) are a set of polynomials orthogonal in \( (-\infty, +\infty) \) with the weight function (see \( \mathbf{11} \) \( \mathbf{14} \)):

\[
w(\nu) = \frac{1}{\pi} 2^{2(\alpha - 1)} |\Gamma(\alpha + iv)|^2 \quad (\alpha > 0),
\] (65)
(where $\Gamma(x)$ denotes the Euler gamma function).

We put $\alpha = 1/2$ (in the following we shall omit the index in the notation of the Pollaczek polynomials). Then the property of orthogonality reads as follows:

$$\int_{-\infty}^{+\infty} w(\nu) P_k(\nu) P_l(\nu) d\nu = \delta_{k,l},$$

where now $w(\nu) = (1/\pi)|\Gamma(1/2 + i\nu)|^2$.

Next, we introduce the following functions (that may be called Pollaczek functions):

$$\psi_m(\nu) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} + i\nu \right) P_m(\nu), \quad (P_0(\nu) = 1),$$

which form a complete basis in $L^2((0, \infty))$ (see [11]).

In view of the fact that the coefficients $\{a_n\}_0^\infty$ satisfy condition [13], then there exists a unique Carlsonian interpolation of the sequence $\{a_n\}_0^\infty$ denoted by $\tilde{a}(\lambda)$, such that $\tilde{a}(-1/2 + iv) \psi(\nu \in \mathbb{R})$ belongs to $L^2((0, \infty))$ (see Proposition 1).

Therefore the function $\tilde{a}(-1/2 + iv)$ can be expanded in terms of the Pollaczek functions as follows:

$$\tilde{a} \left( \frac{1}{2} + iv \right) = \sum_{m=0}^{\infty} d_m \psi_m(\nu) \quad (\nu \in \mathbb{R}),$$

and the convergence of this expansion is in the sense of the $L^2$-norm.

The coefficients $d_m$ are given by:

$$d_m = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \tilde{a} \left( \frac{1}{2} + iv \right) \Gamma \left( \frac{1}{2} - iv \right) P_m(\nu) d\nu.$$

Taking into account the asymptotic behavior of the gamma function, we may evaluate integral [60] by the contour integration method along the path shown in Fig. 2B. Note that the poles of the gamma function $\Gamma(1/2 - iv)$ are located at $\nu = -i(n + 1/2)$, and $\tilde{a}(-1/2 + iv))|_{\nu = -i(n+1/2)} = \tilde{a}(n) = a_n$. We obtain:

$$d_m = 2\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_n P_m \left[ -i \left( n + \frac{1}{2} \right) \right].$$

Next, we observe that:

$$\Gamma \left( \frac{1}{2} + iv \right) = \int_{-\infty}^{+\infty} e^{-iv\nu} e^{-\nu^2/2} d\nu = \mathcal{F} \left\{ e^{-v^2} e^{-\nu^2/2} \right\},$$

where $\mathcal{F}$ denotes the Fourier integral operator. Let us note that the function $\exp(-v)\exp(-v/2)$ belongs to the Schwartz space $S$ of the $C^\infty(\mathbb{R})$ functions which, together with all their derivatives, decrease, for $|v|$ tending to $+\infty$, faster than any negative power of $|v|$. Therefore we can write:

$$\psi_m(\nu) = \frac{1}{\sqrt{\pi}} \mathcal{F} \left\{ P_m \left( -i \frac{d}{dv} \right) e^{-v^2} e^{-v/2} \right\}.$$

Substituting in expansion [65] to the Pollaczek functions their representation [22], we obtain:

$$\tilde{a} \left( \frac{1}{2} + iv \right) = \sum_{m=0}^{\infty} d_m \left\{ \frac{1}{\sqrt{\pi}} \mathcal{F} \left\{ P_m \left( -i \frac{d}{dv} \right) e^{-v^2} e^{-v/2} \right\} \right\}. $$
Let us now apply the operator $\mathcal{F}^{-1}$ to the r.h.s. of formula (73). If we exchange the integral operator $\mathcal{F}^{-1}$ with the sum, and this is legitimate within the $L^2$-norm convergence, we obtain:

$$\mathcal{F}^{-1} \sum_{m=0}^{\infty} d_m \left\{ \frac{1}{\sqrt{\pi}} \mathcal{F} \left[ P_m \left(-i \frac{d}{dv} \right) \left[ e^{-v} e^{-v/2} \right] \right] \right\} = \sum_{m=0}^{\infty} d_m \left\{ \frac{1}{\sqrt{\pi}} \mathcal{F}^{-1} \mathcal{F} \left[ P_m \left(-i \frac{d}{dv} \right) \left[ e^{-v} e^{-v/2} \right] \right] \right\}. \quad (74)$$

Finally, recalling formula (61), we obtain the following expansion for the function $\exp(v/2)F(v)$:

$$e^{v/2}F(v) = \sum_{m=0}^{\infty} d_m \left\{ \frac{1}{\sqrt{\pi}} F \left[ P_m \left(-i \frac{d}{dv} \right) \left[ e^{-v} e^{-v/2} \right] \right] \right\}, \quad (75)$$

where the convergence is in the sense of the $L^2$-norm.

Then it can be verified easily that:

$$\sqrt{2} P_m \left(-i \frac{d}{dv} \right) \left\{ e^{-v} e^{-v/2} \right\} = i^m \sqrt{2} L_m(2e^{-v}) e^{-v} e^{-v/2}, \quad (76)$$

where $L_m$ denotes the Laguerre polynomials (see also [12, 17]).

It can be checked easily that the polynomials $L_m(v) = i^m \sqrt{2} L_m(2e^{-v})$ are a set of polynomials orthonormal on the real line with the weight function $w(v) = \exp(-v) \exp(-2 \exp(-v))$, and, consequently, the set of functions $\Phi_m(v)$, defined by formula (64) forms an orthonormal basis in $L^2(-\infty, +\infty)$. Finally, from formula (75) we obtain:

$$e^{v/2}F(v) = \sum_{m=0}^{\infty} c_m \left\{ i^m \sqrt{2} L_m(2e^{-v}) e^{-v} e^{-v/2} \right\} = \sum_{m=0}^{\infty} c_m \Phi_m(v) \quad (v \in \mathbb{R}^+), \quad (77)$$

where $c_m = d_m / \sqrt{2\pi}$, and the functions $\Phi_m(v)$ are given by formula (64). \hfill \Box

The results of Theorem 3 can be easily extended to all the cases considered in Section 3. For the sake of simplicity we limit ourselves to consider the Taylor series (47) treated in Theorem 2.

**Theorem 4.** If in the Taylor series (47) the coefficients $a_n$ satisfy condition (8), then the jump function $F(x)$ can be represented by the following series, which converges in the sense of the $L^2$-norm:

$$F(x) = \sum_{n=0}^{\infty} c_m \phi_m(x) \quad (x \in (1, +\infty)). \quad (78)$$

In the series (78) the coefficients $c_m$ are given by formula (63), while the functions $\phi_m(x)$ are given by:

$$\phi_m(x) = i^m \sqrt{2} L_m \left( \frac{2}{x} \right) e^{-1/x}, \quad (79)$$

$L_m$ being the Laguerre polynomials.

**Proof.** The proof of these results proceeds exactly as in the case of the previous theorem. It is, indeed, sufficient to observe that the Mellin transform (49) can be easily transformed in a Fourier–Laplace transform by putting: $x = e^v$. Let us
remind that we still denote (with a small abuse of language) the jump function by $F(x)$, as it has already been noted before the formulation of Theorem 2. Accordingly, the basis will be given by formula (49). Finally, let us note that the functions $\phi_m(x)$ form an orthonormal basis in $L^2(0, +\infty)$.

Let us note that expansion (78) furnishes a solution of the Cauchy integral equation of the type (5). However, let us observe that up to now we have supposed that the coefficients $a_n$ are infinite in number and noiseless. But this, in practice, is not the case. We have at our disposal only a finite number of coefficients and, in addition, they are affected by noise or by round–off errors. We are, therefore, forced to consider the following question: How to manage numerically expansions (77) and (78). Furthermore, let us note that the problem of reconstructing the jump function from the coefficients $a_n$ is a classical example of ill–posed problem in the sense of Hadamard [8]. It is, indeed, strictly connected to the problem of the analytic continuation up to the boundary of the analyticity domain. In fact, let us focus our attention on the cut $z$–plane geometry considered in Theorems 2 and 4. In view of the Riemann mapping theorem, this cut plane can be conformally mapped onto the unit disk in the $\zeta$–plane geometry (i.e., $|\zeta| < 1$) through a suitable transformation $\zeta = \zeta(z)$. In this map the upper (lower) lip of the cut is mapped in the upper (lower) half of the unit circle. Therefore, the problem of solving the Cauchy–type integral equations (5) corresponds to the analytic continuation up to the unit circle ($|\zeta| = 1$). It is, then, easy to exhibit Hadamard–like examples showing that the solution does not depend continuously on the data in various topologies, including uniform and $L^2$–topologies.

We shall treat all these questions in a separate paper devoted to the numerical analysis. The main result that will be proved there reads as follows: If we take as data a finite number $(N + 1)$ of coefficients perturbed by noise $a_n(\epsilon)$ (where $|a_n(\epsilon) - a_n| \leq \epsilon$, $n = 0, 1, 2, \ldots, N$, $\epsilon > 0$) we can still determine an approximation $F(\epsilon,N)(x)$ of the jump function $F(x)$, that asymptotically converges to $F(x)$, in the sense of the $L^2$–norm, as $\epsilon \to 0$ and $N \to \infty$.

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