Generalizations of the abstract boundary singularity theorem

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Abstract
The abstract boundary singularity theorem was first proven by Ashley and Scott. It links the existence of incomplete causal geodesics in strongly causal, maximally extended spacetimes to the existence of abstract boundary essential singularities, i.e., non-removable singular boundary points. We give two generalizations of this theorem: the first to continuous causal curves and the distinguishing condition, the second to locally Lipschitz curves in manifolds such that no inextendible locally Lipschitz curve is totally imprisoned. To do this we extend generalized affine parameters from \( C^1 \) curves to locally Lipschitz curves.

Keywords: general relativity, singularity theorem, abstract boundary, generalized affine parameter, locally Lipschitz curve, continuous causal curve

1. Introduction
The end goal of our program of research is to link the Penrose Hawking singularity theorems to curvature singularity results. Our three theorems, theorems 1.1, 1.2 and 1.3, are a further step towards this goal. They prove that the Penrose Hawking singularity theorems actually imply the existence of irremovable, also called essential, singularities and they provide a location for these singularities in terms of boundary points of an envelopment. It is our hope that this additional structure can be exploited to complete our program of research.

The abstract boundary singularity theorem, proven by Ashley and Scott [1, theorem 4.12], is:
Theorem 1.1. Let \((\mathcal{M}, g)\) be a strongly causal, \(C^l\) maximally extended, \(C^k\) spacetime \((1 \leq l \leq k)\). Let \(C\) be the set of affinely parametrized causal geodesics in \(\mathcal{M}\). There exists an incomplete curve in \(C\) if and only if the abstract boundary \(B(\mathcal{M})\) contains an abstract \(C^l\) essential singularity.

An abstract \(C^l\) essential singularity is an abstract boundary set which has a singleton \(\{p\}\) as a representative boundary set where \(p\) is a singular boundary point which cannot be removed by a change of coordinates and is approached by a curve in \(C\) with bounded parameter. This theorem does not prove the existence of an incomplete causal geodesic, but rather shows that the existence of an incomplete causal geodesic is equivalent to the existence of an endpoint for the incomplete geodesic: that is, a location for the singularity in the abstract boundary. Hence, the theorem extends the ‘standard’ singularity theorems, e.g., the Penrose and Hawking singularity theorems [2, section 8.2], by showing that they actually produce genuine singularities, at least according to the abstract boundary classification of boundary points [1, theorem 4.13]. For further details about the abstract boundary please refer to one of [1, 3–5].

Ideally, the use of geodesics and the assumption of strong causality could be relaxed to increase the generality of the theorem. Geroch [6] has shown that in order to identify all singular behaviour in a spacetime it is necessary to consider, at least, all causal curves. Hence, it is desirable that the singularity theorem above be generalized to include, at least, all causal curves.

The most general singularity theorems, like that given by Maeda and Ishibashi [7], use causality conditions much weaker than strong causality. Ashley and Scott have investigated weakening the strong causality condition in some cases. In particular, they have shown that the abstract boundary singularity theorem is false in chronological spacetimes and have indicated how a counter example may be provided in causal spacetimes [1, section 4.4.1]. In [1, theorem 4.22] Ashley and Scott also prove that in two-dimensional spacetimes the theorem holds for the distinguishing condition. From a theoretical point of view it is also pleasing to find the weakest conditions under which the theorem holds.

The main theorem of this paper is the following, which we suggest should be considered as the abstract boundary singularity theorem.

Theorem 1.2 (The abstract boundary singularity theorem). Let \((\mathcal{M}, g)\) be a future (past) distinguishing, \(C^l\) maximally extended, \(C^k\) spacetime \((1 \leq l \leq k)\) and let \(C\) be the family of generalized affinely parametrized continuous causal curves in \(\mathcal{M}\). There exists an incomplete curve in \(C\) if and only if \(B(\mathcal{M})\) contains an abstract \(C^l\) essential singularity.

There are two main difficulties with proving this theorem. The first is that we need a definition of a generalized affine parameter on a continuous causal curve (definition 4.2) as generalized affine parameters are usually defined on \(C^1\) curves [8, p 208]. The second is that we need the existence of an endpoint for a continuous causal curve in order to imply that the generalized affine parameter is bounded and the curve is extendible (proposition 4.7). To tackle these difficulties we work with a more general class of curves, locally Lipschitz curves, definition 2.2 (although some additional work is required to apply results about locally Lipschitz curves to continuous causal curves; see proposition 4.8). In proving the required results for locally Lipschitz curves we get, in addition to theorem 1.2, the following theorem.

Theorem 1.3. Let \((\mathcal{M}, g)\) be a \(C^1\) maximally extended, \(C^k\) spacetime \((1 \leq l \leq k)\) so that no inextendible locally Lipschitz curve is totally imprisoned. Let \(C\) be the family of generalized
affinely parametrized locally Lipschitz curves in \( \mathcal{M} \). There exists an incomplete curve in \( \mathcal{C} \) if and only if \( B(\mathcal{M}) \) contains an abstract \( C^l \) essential singularity.

While theorem 1.3 is more general than theorem 1.2 it lacks physical motivation for the condition on \( \mathcal{M} \).

Consider the three statements in theorem 1.3:

1. The spacetime does not contain an inextendible totally imprisoned locally Lipschitz curve,
2. The set of curves, \( \mathcal{C} \), contains an incomplete curve,
3. The abstract boundary contains an abstract \( C^l \) essential singularity.

The following implications are apparent from the definitions and the proof of theorem 1.3:

1. implies that (2) and (3) are equivalent,
2. implies that either (1) does not hold or (3) holds,
3. implies that (2) holds and that this incomplete curve is inextendible and not totally imprisoned.

Hence, given the use of locally Lipschitz curves, theorem 1.3 cannot be weakened further. Theorem 1.3 can, therefore, be considered a proof of the generally accepted statement that incompleteness of curves implies either the existence of incomplete trapped curves or of a singularity.

More generally, this paper fits into the wider field of research into singularities and the use and application of boundary constructions in general relativity. We will not provide further context here, but we refer the interested reader to Senovilla’s series of review articles on singularity theorems [9–11], recent work on the causal boundary [12] and Geroch’s g-boundary [13] as well as newer boundary constructions [5, 14, 15]. With regards to the abstract boundary we refer the reader to [5, 16, 17] for discussions of topological properties, [5, 18] for its relationship to boundaries induced by charts and distances, [19, 20] for homotopy and rigidity results, [1, 21, 22] for other abstract boundary singularity results and [1, 4] for general reviews of the abstract boundary.

2. Common definitions

Throughout this paper we restrict our attention to \( n \)-dimensional, paracompact, connected, Hausdorff, \( C^\infty \) manifolds \( \mathcal{M} \) which are Lorentzian and time orientable. Standard definitions, e.g., of strong causality, the maximal extension of a manifold and of the generalized affine parameter, are taken from [8]. We do, however, deviate from [8] by using the ‘total imprisonment’ of [2] rather than the ‘imprisonment’ of [8]. The two definitions are equivalent; we simply prefer the terminology of [2]. We do not review the abstract boundary, though we encourage the reader to refer to one of [1, 3, 4]. Because curves and their extensions play a central role in this paper, we remind the reader of the following definitions.

**Definition 2.1.** A \( C^k \), \( k \geq 1 \), (regular) curve \( \gamma : [a, b) \to \mathcal{M} \), \( a, b \in \mathbb{R} \sqcup \{ \infty \}, a < b \), is a \( C^k \) function with \( \gamma' \) everywhere non-zero. We say that \( \gamma \) is non-spacelike (timelike) and future (past) directed if \( \gamma' \) is everywhere non-spacelike (timelike) and future (past) directed.

We shall need the following non-standard definition.
Definition 2.2. A (regular) locally Lipschitz curve $\gamma: [a, b) \to \mathcal{M}$ is a function so that, for each chart $\phi: U \subset \mathcal{M} \to \mathbb{R}^n$ and each $t \in [a, b)$ such that $\gamma(t) \in U$, there exists $V$, a neighbourhood of $t$ in $[a, b)$, and $K \in \mathbb{R}^+$ such that $\gamma(V) \subset U$ and for all $t_1, t_2 \in V$,

$$d(\phi \circ \gamma(t_1), \phi \circ \gamma(t_2)) \leq K |t_1 - t_2|,$$

where $d$ is the Euclidean distance on $\mathbb{R}^n$ and so that $\gamma'$ is non-zero apart from a set of measure zero. Note that $K$ depends on the chart $\phi$ and the point $t$.

The definition is independent of the choice of chart since the set $V$ can be taken to be compact and changes of coordinates between charts are invertible and bounded on compact sets contained in the intersection of their domains.

Definition 2.3 ([8, section 3.2] or [2, p 184]). A continuous future directed, non-spacelike curve $\gamma: [a, b) \to \mathcal{M}$ is a continuous function so that for each $t_0 \in [a, b)$ there is a neighbourhood $N$ of $t_0$ in $[a, b)$ and a convex normal neighbourhood $U$ of $\gamma(t_0)$ so that for all $t \in N$, $t \neq t_0$, if $t > t_0$ then $\gamma(t) \in J^+(\gamma(t_0), U) \setminus \gamma(t_0)$ or if $t < t_0$ then $\gamma(t) \in J^-(\gamma(t_0), U) \setminus \gamma(t_0)$. We say that $\gamma$ is past directed if, for $t > t_0$, then $\gamma(t) \in J^+(\gamma(t_0), U) \setminus \gamma(t_0)$ and for $t < t_0$ then $\gamma(t) \in J^-(\gamma(t_0), U) \setminus \gamma(t_0)$. We say that $\gamma$ is timelike if the sets $I^+(\gamma(t_0), U)$ and $I^-(\gamma(t_0), U)$ are used instead of $J^+(\gamma(t_0), U) \setminus \gamma(t_0)$ and $J^-(\gamma(t_0), U) \setminus \gamma(t_0)$ respectively. We shall assume that every continuous causal (timelike) curve is equipped with a parametrization so that it is also a locally Lipschitz curve. That such a parametrization always exists is proven on page 75 of [8].

It is worth noting that the idea that every continuous causal curve is also locally Lipschitz makes up part of the folklore of general relativity. To the best of the authors’ knowledge the first mention of this result, where it is stated without proof, is by Penrose [27, remark 2.26] who credits it to Geroch but gives no reference. While it is certainly true that every continuous causal curve has a reparametrization so that it is locally Lipschitz, it is possible to give continuous causal curves parameters so that they are not locally Lipschitz (e.g. $t \mapsto t^{1/3}$).

The condition that $\gamma(t) \neq \gamma(t_0)$, for all $t \in N \setminus \{t_0\}$, is the continuous causal curve analogue of the non-zero tangent vector condition for $C^k$, $k > 0$, curves.

Definition 2.4 ([3, definition 2]). A curve $\gamma: [a, b) \to \mathcal{M}$ is a subcurve of a curve $\lambda: [a', b') \to \mathcal{M}$, if $a' \leq a < b \leq b'$ and $\lambda|_{[a,b)} = \gamma$. If $a = a'$ and $b < b'$ then $\lambda$ is an extension of $\gamma$.

Definition 2.5 (Compare to [8, definition 6.2]). An affinely parametrized causal geodesic $\gamma: [a, b) \to \mathcal{M}$ is incomplete if $b < \infty$ and $\gamma$ is not extendible by any affinely parametrized causal geodesic.

Definition 2.6. Two $C^k$, $k \geq 0$, (locally Lipschitz) curves $\gamma: [a, b) \to \mathcal{M}$ and $\lambda: [a', b') \to \mathcal{M}$ are related by a change of parameter if there exists a $C^k$ (locally Lipschitz) surjective strictly monotonically increasing function $f: [a', b') \to [a, b)$ so that $\lambda = \gamma \circ f$.

Note that, unlike $C^k$ changes of parameter, the inverse of a locally Lipschitz change of parameter is not necessarily a locally Lipschitz change of parameter. If we restricted to locally bi-Lipschitz (bi-locally Lipschitz) reparametrizations then there would be no difference between the $C^k$ and locally Lipschitz cases. We do not make that restriction here, however.
The next definition is non-standard, however it simplifies the discussion of many of the results in this paper.

**Definition 2.7.** Let \( \gamma: [a, b) \to \mathcal{M} \) be a \( C^k \), \( k \geq 0 \), curve. A full sequence in \( \gamma \) is a sequence \( \{x_i = \gamma(t_i)\}_{i \in \mathbb{N}}, \{t_i\} \subset [a, b) \), \( t_i < t_{i+1} \), so that \( t_i \to b \) as \( i \to \infty \). We say that \( x \in \mathcal{M} \) is a limit point of \( \gamma \) if there exists a full sequence \( \{x_i\} \) in \( \gamma \) so that \( x_i \to x \). We write \( \gamma \to x \) if and only if every full sequence in \( \gamma \) converges to \( x \), in which case \( x \) is the endpoint of \( \gamma \). We say that a curve \( \gamma \) is a winding curve if there exist two full sequences in \( \gamma \) with different limit points.

This terminology is inspired by the Misner spacetime [2, section 5.8] in which every curve with at least two limit points ‘winds’ around the cylinder.

**Definition 2.8.** A curve, \( \gamma \), is precompact if the closure of its image, \( \overline{\gamma} \), is compact.

### 3. The abstract boundary singularity theorem

Since details of the abstract boundary singularity theorem have only appeared in Ashley’s PhD Thesis [1, theorem 4.12] we present here Ashley and Scott’s proof of theorem 1.1. We do not review the end-point theorem [4, theorem 3.2.1] which plays an important part in the proof below (Ashley’s Thesis [1, section 4.2] contains a nice discussion of the end-point theorem but does not give its proof). Note that the \( \Leftarrow \) implication of this proof follows simply by definition and the Hausdorff property of manifolds. It does not require a restriction on the causality of the spacetime \( (\mathcal{M}, g) \).

**Proof of theorem 1.1.** \( \Leftarrow \) Let \( [p] \in B(\mathcal{M})[\mathcal{M}] \) be an abstract \( C^1 \) essential singularity. That is, there exists \( \mu_p: \mathcal{M} \to \mathcal{M}_p \) an envelopment so that \( p \in \partial\mu_p(\mathcal{M}) \) is a \( C^1 \) essential singularity. Thus there exists \( \gamma: [a, b) \to \mathcal{M} \subseteq \mathcal{C}, b < \infty \), so that \( p \) is a limit point of \( \mu_p(\gamma) \).

Let \( \{x_i = \gamma(t_i)\}_{i \in \mathbb{N}} \) be a full sequence in \( \gamma \) so that \( \mu_p(x_i) \) converges to \( p \).

Suppose that there exists \( \lambda: [a, c) \to \mathcal{M} \) an extension of \( \gamma \), where \( \lambda \in \mathcal{C} \). Consider the sequence \( \{y_i = \lambda(t_i)\} \). Since \( \lambda \) is an extension of \( \gamma \), \( x_i \to y_i \), so that \( \{\mu_p(y_i)\} \) converges to \( p \). Yet, \( t_i \to b \) and \( b < c \) so \( y_i \to \lambda(b) \). Since \( \mathcal{M}_p \) is Hausdorff and \( \mu_p \) is continuous we see that \( \mu_p(\lambda(b)) = p \) which is a contradiction since \( \mu_p(\lambda(b)) \in \mu_p(\mathcal{M}) \) and \( p \notin \partial\mu_p(\mathcal{M}) \). Therefore \( \gamma \) is an incomplete curve in \( \mathcal{C} \) as required.

\( \Rightarrow \) Let \( \gamma \in \mathcal{C} \) be incomplete. We have two cases:

**Case 1.** Suppose that there exists a full sequence \( \{x_i\}_{i \in \mathbb{N}} \) in \( \gamma \) so that \( \{x_i\} \) has no limit points in \( \mathcal{M} \).

By the end-point theorem [4, theorem 3.2.1] there exists an envelopment \( \phi: \mathcal{M} \to \mathcal{M}_g \) so that \( \{\phi(x_i)\} \) converges to \( x \in \partial\phi(\mathcal{M}) \). We will now proceed to classify \( x \) according to the classification of boundary points given in [3].

Since \( (\mathcal{M}, g) \) is \( C^1 \) maximally extended, \( x \) cannot be a \( C^1 \) regular boundary point. By assumption, \( x \) is approached by \( \phi(\gamma) \) (\( \gamma \) is incomplete). Hence we can conclude that \( x \) is a \( C^1 \) singularity. Suppose that \( x \) is a \( C^1 \) removable singularity, then by theorem 43 of [3] there exists a boundary set \( B \) of another envelopment, so that \( B \supseteq x \) and \( B \) contains at least one \( C^1 \) regular boundary point. This contradicts that \( (\mathcal{M}, g) \) is \( C^1 \) maximally extended. Thus \( x \) is a \( C^1 \) essential singularity. This implies that \( [x] \in B(\mathcal{M})[\mathcal{M}] \) is an abstract \( C^1 \) essential singularity.
Case 2. Suppose that every full sequence in $\gamma$ has a limit point in $M$. We distinguish two further cases.

Case 2.1. Suppose that every full sequence in $\gamma$ has the same limit point. Thus there exists $p \in M$ so that $\gamma \to p$.

Let $\gamma; [a, b) \to M$, where $b < \infty$, and let $N$ be a convex normal neighbourhood of $p$. Since $\gamma \to p$ there exists $t \in [a, b)$ so that $y_{\gamma}^{t} \subset N$.

By the convexity of $N$ there exists $v \in T_{p}M$ so that $\gamma(t) = \exp_{p}(v)$. Since $\gamma$ is a geodesic this implies that there exists $\lambda \in \mathbb{R}^{+}$ so that $\gamma(t) = \exp_{p}(\lambda \cdot v)$.

Thus there exists $\mu \in \mathbb{R}^{+}$ so that $\gamma(t) = \exp_{p}(\mu \cdot v)$. Let $\gamma = \mu_{[a, b)}$ and $\lambda(t) = \mu(b + t)$. Since $[a, b) \subset [a, b + c)$, the causal geodesic $\mu$ is an extension of $\gamma$. Hence $\gamma$ is not incomplete.

This is a contradiction and thus this case cannot occur.

Case 2.2. Suppose that there exist two full sequences in $\gamma$ with different limit points.

Let $\gamma; [a, b) \to M$, where $b < \infty$, and let $\{x_{i} = \gamma(t_{i}^{x})\}_{i \in \mathbb{N}}$ and $\{y_{i} = \gamma(t_{i}^{y})\}_{i \in \mathbb{N}}$ be the two full sequences with limit points $x, y \in M$, so that $x_{i} \to x$, $y_{i} \to y$ and $x \neq y$. Without loss of generality, we can assume that for all $i$, $t_{i}^{x} < t_{i}^{y} < t_{i+1}^{x}$.

Let $U, V \subset M$ be open sets so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $x_{i} \to x$ there exists $N_{x} \in \mathbb{N}$ so that for all $i \geq N_{x}$, $x_{i} \in U$. Similarly, as $y_{i} \to y$ there exists $N_{y} \in \mathbb{N}$ so that for all $i \geq N_{y}$, $y_{i} \in V$. Let $i \geq \max\{N_{x}, N_{y}\}$, then $\gamma(t_{i}^{x}) \in U$, $\gamma(t_{i}^{y}) \in V$ and $\gamma(t_{i+1}^{x}) \in U$. Since $t_{i}^{x} < t_{i}^{y} < t_{i+1}^{x}$, $x \neq y$ and as $U$ and $V$ are arbitrary the spacetime $(M, g)$ is not strongly causal.

This is a contradiction and therefore this case cannot occur.

As only case 1 may occur, we have proven our result.

Note that the proof of the abstract boundary singularity theorem is based on the division of causal geodesics into three classes:

(i) curves with a full sequence with no limit points, i.e. non-precompact curves,
(ii) curves with all full sequences having the same limit point, i.e. precompact curves with an endpoint,
(iii) curves with all full sequences having a limit point and with two full sequences with different limit points, i.e. precompact curves without an endpoint (these curves are necessarily winding).

The first class of curves correspond to abstract boundary essential singularities. The second class of curves correspond to curves that are not incomplete. The third class of curves correspond to curves that violate strong causality. Thus to generalize this theorem to all continuous causal curves and the distinguishing condition we need three things:

(i) an incomplete continuous causal curve must correspond to an abstract essential singularity,
(ii) a continuous causal curve with an endpoint must not be incomplete,
(iii) the existence of a precompact winding continuous causal curve must violate the distinguishing condition.

The first point requires that each continuous causal curve carries a particular parametrization so that the set of all continuous causal curves satisfies the bounded parameter property. For more background on this we refer the reader to a discussion of the classification of the abstract boundary; see one of [1, 3, 4].

**Definition 3.1** ([3, definition 4]). Let $C$ be a set of curves in $\mathcal{M}$. The set $C$ has the bounded parameter property if:

(i) for all $p \in \mathcal{M}$ there exists $\gamma \in C$ so that $p \in \gamma$,
(ii) if $\gamma \in C$ then every subcurve of $\gamma$ is in $C$,
(iii) for all $\gamma, \lambda \in C$ if there exists a change of parameter relating $\gamma$ and $\lambda$ then either both parameters are bounded or both are unbounded.

The set of all affinely parametrized causal geodesics satisfies the bounded parameter property. We will show that locally Lipschitz curves carry a generalization of the generalized affine parameter defined on $C^1$ curves [8, p 208] and that this parametrization ensures that the set of all locally Lipschitz curves satisfies the bounded parameter property. Since continuous causal curves are locally Lipschitz (definition 2.3) it will then be the case that the set of all continuous causal curves satisfies the bounded parameter property.

The second point requires a definition of an incomplete continuous causal curve. We use the new parameter to define an incomplete locally Lipschitz curve, in the same spirit as definition 2.5. We then show that a locally Lipschitz curve with endpoint is extendible, i.e., that the curve is not incomplete.

For the third point we rely on a result from Hawking and Ellis, proposition 6.4.8 of [2], to get the needed contradiction. Paraphrased to accommodate our definitions of extension and total imprisonment [8, definition 7.29] (note that we differ here from Beem et al as we use the ‘total imprisonment’ of Hawking and Ellis which is the ‘imprisonment’ of Beem et al) the result is as follows:

**Proposition 3.2** ([2, proposition 6.4.8]). If the future or past distinguishing condition holds on a compact set $S$, there can be no inextendible causal curve totally imprisoned in $S$.

Note that the definition of causal curve [2, page 184] used by Hawking and Ellis is the same as ours, definition 2.3; thus proposition 3.2 can be applied in our situation.

4. The bounded parameter property and incompleteness for locally Lipschitz curves

4.1. The bounded parameter property for locally Lipschitz curves

A generalized affine parameter is given as the arc length of a curve, with respect to a Riemannian metric, induced by an orthonormal frame which is parallelly propagated along the curve; see [8, p 208]. Because of the need for parallel propagation, generalized affine parameters are usually defined on $C^1$ curves. There are, however, existence and uniqueness results for ordinary differential equations involving functions with weaker regularity than $C^1$. We exploit these results to define parallel propagation along locally Lipschitz curves and thus to equip them with a generalized affine parameter. We go through the proof in detail.
Proposition 4.1. Let \( \gamma: [a, b] \to \mathcal{M} \) be a locally Lipschitz curve. Then for all \( \tau \in [a, b] \) (excluding a set of Lebesgue measure zero) and all \( v \in T_{\gamma(\tau)}\mathcal{M} \) there exists \( V: [a, b] \to TM \), a vector field on \( \gamma \), which is absolutely continuous on compact subsets of \([a, b]\) and so that \( V(\tau) = v \) and \( V_\tau V = 0 \) except on a set of Lebesgue measure zero.

Proof. Since \( \gamma \) is locally Lipschitz, \( \gamma'(t) \) exists for almost all \( t \in [a, b] \) ; see [8, p 75]. The equations for parallel propagation of a vector, \( X \), along \( \gamma \) in some coordinate chart \( \phi \) are

\[
\frac{d}{dt} X'(t) + (\gamma'(t)) X'(t) \Gamma^i_{jk}(t) = 0.
\]

We will apply Carathéodory’s theorem [23, theorem 2.1.1] to this equation to prove the existence of \( X' \) on some compact connected subinterval \( I_\theta \subset [a, b] \), such that \( \gamma'(I_\theta) \) is contained in the chart \( \phi \). As initial conditions we take \( \tau \in \text{interior}(I_\theta) \) so that \( \gamma'(\tau) \) exists and \( v \in T_{\gamma(\tau)}\mathcal{M} \).

With this purpose in mind, let \( \partial_i, i = 1, \ldots, n \), be the coordinate vectors for the chart \( \phi \). We can view \( \gamma'(t) \) and \( \Gamma^i_{jk} \) as functions from \( I_\theta \) to \( \mathbb{R} \). Define \( \hat{\gamma}: I_\theta \to \mathbb{R} \) by

\[
\hat{\gamma}'(t) = \begin{cases} 
(\gamma'(t)) & \text{if } (\gamma'(t)) \text{ is defined,} \\
0 & \text{otherwise.}
\end{cases}
\]

The vector \( v \in T_{\gamma(\tau)}\mathcal{M} \) can be written as \( v = v^i \partial_i \), where \( v^i \in \mathbb{R} \). We consider \((\tau, v^1, \ldots, v^n)\) as a point in \( \mathbb{R}^{n+1} \). Let \( |\cdot| \) be the \( L_1 \) norm on \( \mathbb{R}^n \) given by \( |(x^1, \ldots, x^n)| = \sum |x^i| \). We will write \( x \) for \((x^1, \ldots, x^n)\). Thus \( |x(t)| = |(x^1(t), \ldots, x^n(t))| \).

Choose \( c > \max \{|v|, 1\} \) an otherwise arbitrary positive constant. Let \( R = \text{interior}(I_\theta) \times \{x \in \mathbb{R}^n: |x| < c\} \). Thus \((\tau, v) \in R\). Letting \((t, x) \in \mathbb{R}^{n+1}\), we can define the functions \( f^i: R \to \mathbb{R}, i = 1, \ldots, n \), by

\[
f^i(t, x^1, \ldots, x^n) = \sum_{j,k} \hat{\gamma}^j(t)x^k \Gamma^i_{jk}(t).
\]

Since \( \gamma \) is Lipschitz on \( I_\theta \), we can see that \( f^i \) is continuous in \( x^k, k = 1, \ldots, n \), for all fixed \( t \) and that \( f^i \) is Lebesgue measurable in \( t \) for all fixed \( x^1, \ldots, x^n \). We will write \( f(t, x) \) for \((f^1(t, x), \ldots, f^n(t, x))\).

By the Lipschitz condition on \( \gamma \) and as each \( \Gamma^i_{jk} \) is continuous we know that the function \( m: I_\theta \to \mathbb{R} \) given by

\[
m(t) = c \sum_{i,j,k} \left| \hat{\gamma}^j(t) \Gamma^i_{jk}(t) \right|
\]

is Lebesgue integrable. In particular, noting that \( |x + y| \leq |x| + |y| \), some algebra shows that for all \((t, x) \in R\) we have that \( |f(t, x)| < m(t) \).

Carathéodory’s theorem [23, theorem 2.1.1], rephrased for our particular situation, states that:

Carathéodory’s theorem. Let \( f^i, i = 1, \ldots, n \), be defined on \( R \), and suppose each \( f^i \) is Lebesgue measurable in \( t \) for each fixed \( x \) and continuous in \( x \) for each fixed \( t \). If there exists a Lebesgue integrable function \( m: I_\theta \to \mathbb{R} \) so that \( |f(t, x)| < m(t) \), then there exists \( J \subset I_\theta \), a subinterval of \( I_\theta \), so that \( \tau \in J \) and for each \( i = 1, \ldots, n \) there exists an absolutely continuous function \( y^i(\tau) : J \to \mathbb{R} \) so that \((t, y^1(t), \ldots, y^n(t)) \in R, y^i(\tau) = v^i_\tau \), and so that
\[
\frac{d}{dt} y^i(t) + f^i\left( t, y^1(t), ..., y^n(t) \right) = 0
\]  
for almost all \( t \).

Since the functions \( f \) and \( m \) satisfy the conditions of the theorem we know that on some subinterval \( J \) of \( I_{\phi} \), with \( \tau \in J \), an absolutely continuous solution \( X^i: J \to \mathbb{R}, \ i = 1, ..., n \), exists except on a set of Lebesgue measure zero. Since \( J \subset I_{\phi} \) could be taken to be very small, we only know the local existence of a solution about \( \tau \).

To prove global existence on \( I_{\phi} \) it is necessary to find an upper bound on the length of solutions with initial condition \( v \) at \( \tau \). Since each \( \Gamma^j_{jk} \) is defined on all of \( \phi \), as \( \gamma \) is Lipschitz on \( \phi I \) and as \( \phi I \) is compact there exists \( C_0 > 0 \) so that
\[
\sum_{i,j,k} \gamma^i \Gamma^j_{jk} \leq C_0 \sum_{i,j} |\gamma^j(t)\Gamma^j_{jk}(t)| = C |X(t)|.
\]
Gronwall’s inequality [24, p 624] now implies that there exists \( K \in \mathbb{R}^+ \) such that for all \( t \in I_{\phi} \),
\[
|X(t)| \leq K \exp(Ct).
\]
Since \( I_{\phi} \) is a compact subinterval of \( I = [a, b] \), the function \( \exp(Ct) \) has a maximum value on \( I_{\phi} \). Hence, we know that if \( X(t) \) is a solution, then there exists \( b \in \mathbb{R}^+ \) so that \( |X(t)| < b \) on \( I_{\phi} \).

Since \( c \) was arbitrary we can choose \( c \) so that \( c > \max\{|v|, 1, b\} \).

Theorem 2.1.3 of [23] now allows us to conclude global existence; rephrased for our situation it states:

**Theorem 2.1.3 of [23].** Let \( R \) be an open, connected subset of \( \mathbb{R}^{n+1} \), with points \( (t, x) \), so that for each \( i = 1, ..., n \), the function \( f^i \) is defined on \( R \), Lebesgue measurable in \( t \) for each fixed \( x \) and continuous in \( x \) for each fixed \( t \). If there exists a Lebesgue integrable function \( m(t) \) so that \( |f(t, x)| < m(t) \) for all \( (t, x) \in R \), then any solution \( y^i \) of the system (1) in the sense of Carathéodory’s theorem can be extended to the boundary of \( R \).

Since \( f \) and \( m(t) \) satisfy the conditions of the theorem, we know that the solution, \( X(t) \), extends to the boundary of \( R \). The boundary of \( R \) is
\[
\text{interior}(I_{\phi}) \times \{ x \in \mathbb{R}^n: |x| = c \} \cup \partial I_{\phi} \times \{ x \in \mathbb{R}^n: |x| < c \},
\]
where \( \partial I_{\phi} \) is the boundary of \( I_{\phi} \) in \( \mathbb{R} \). By construction we know that \( |X(t)| < b \), hence, as \( c > \max\{|v|, 1, b\} \), the intersection of the solution (t, \( X^1(t), ..., X^n(t) \)) with the boundary of \( R \) cannot lie in interior \((I_{\phi}) \times \{ x \in \mathbb{R}^n: |x| = c \}\). Therefore the solution must extend to \( \partial I_{\phi} \times \{ x \in \mathbb{R}^n: |x| < c \} \). This implies that the solution is defined on all of \( I_{\phi} \).

To prove uniqueness we use theorem I.5.3 of [25]. This theorem is a special case of theorem 2.2.1 of [23] which better fits our situation.

**Theorem I.5.3 of [25].** Suppose that \( R \) is an open set in \( \mathbb{R}^{n+1} \), with points \((t, x)\), and for each \( i = 1, ..., n \), \( f^i \) is defined on \( R \), Lebesgue measurable in \( t \) for each fixed \( x \) and continuous in \( x \) for each fixed \( t \). Suppose that for each compact set \( U \) in \( R \), there is a Lebesgue integrable function \( m_U(t) \) such that for all \( (t, x), (t, y) \in U \).
Then for any \((t_0, x_0) \in U\) there exists a unique solution to the system \((1)\).

Since \(f\) and \(m\) satisfy the conditions of the theorem and as

\[
\sum_{i,j} |\hat{f}_{ij}(t)\hat{G}_{jk}^i(x^k - y^k)| \leq \sum_{i,j,k} |\hat{f}_{ij}(t)\hat{G}_{jk}^i| |x^k - y^k| \\
\leq \sum_{i,j,k} |\hat{f}_{ij}(t)\hat{G}_{jk}^i| |x^k - y^k| + \sum_{i,j,k,l \neq k} |\hat{f}_{ij}(t)\hat{G}_{jk}^i| |x^k - y^k| \\
= \sum_{i,j,k,l} |\hat{f}_{ij}(t)\hat{G}_{jk}^i| |x^l - y^l| \\
= \left( \sum_{i,j,k} |\hat{f}_{ij}(t)\hat{G}_{jk}^i| \right) \left( \sum_l |x^l - y^l| \right) \\
= \frac{m(t)}{c} |x - y| < m(t)|x - y|,
\]

holds on all of \(\mathbb{R}\), where we have used our requirement that \(c > 1\), our solution, \(X^i: I_\phi \rightarrow \mathbb{R}\), \(i = 1, \ldots, n\), is unique. Hence for \(i = 1, \ldots, n\) there exists a unique and absolutely continuous function \(X^i: I_\phi \rightarrow \mathbb{R}\) so that \(X^i(t) = v^i\) and

\[
\frac{d}{dt} X^i(t) + (\gamma')^i(t)X^k(t)\hat{G}_{jk}^i(t) = 0
\]

for almost all \(t\).

Since \(\mathcal{M}\) is paracompact and Hausdorff and as \([a, b] \subset \mathbb{R}\) we know that there exists a covering of \([a, b]\) by a countable collection of compact connected intervals \(\{I_i \subset [a, b]: i \in \mathbb{N}\}\) so that \(a \in I_1\), \(I_i \cap I_{i+1} \neq \emptyset\), \(I_{i-1} \cap I_i \neq \emptyset\) and \(I_i \cap I_j = \emptyset\) if \(j \neq i\), \(i \pm 1\) and so that for each \(i\) there exists a chart \(\phi_i\) so that \(\gamma(I_i)\) lies in \(\phi_i\). Without loss of generality we assume that \(\phi = \phi_i\) and that \(I_\phi = I_i\) for some \(l \in \mathbb{N}\).

Suppose that there exist \(X^k: I_i \rightarrow \mathbb{R}\), \(k = 1, \ldots, n\), absolutely continuous functions satisfying \((1)\) except on a set of Lebesgue measure zero. As \(I_i \cap I_{i+1}\) has non-zero Lebesgue measure there exists \(t_{i,i+1} \in I_i \cap I_{i+1}\) such that each \(X^k\) exists at \(t_{i,i+1}\). From above, there exists, except on a set of Lebesgue measure zero, an absolutely continuous solution \(Y^k: I_{i+1} \rightarrow \mathbb{R}\) of \((1)\) so that for each \(k = 1, \ldots, n\), we have \(Y^k(t_{i,i+1}) = X^k(t_{i,i+1})\). By uniqueness of the solution we must have \(Y^k = X^k\) on \(I_i \cap I_{i+1}\) and therefore we can extend our solution to \(I_i \cup I_{i+1}\) in an absolutely continuous way. The same argument can be used to show that the solution on \(I_i\) can be extended to \(I_{i-1} \cup I_i\). By induction, and as we have proven existence on \(I_\phi\), we can extend our solution to all of \([a, b]\). We denote the vector that results from this extension by \(V\).

Therefore, we have that \(V\) is a vector field on \(\gamma\), which is absolutely continuous on compact subsets of \([a, b]\) and so that \(V(t) = v\) and \(V_t V = 0\) except on a set of Lebesgue measure zero, as required.

Given two such parallelly propagated vector fields \(X, Y\) on a locally Lipschitz curve \(\gamma: [a, b] \rightarrow \mathcal{M}\) then \(g(X, Y): [a, b] \rightarrow \mathbb{R}\) is a function that is absolutely continuous on compact subsets of \([a, b]\) and \(g(X, Y) = 0\) almost everywhere. From the comments just before proposition 9.6.4, from proposition 9.6.4 itself and from proposition 9.6.6 of [26] the
function \(g(X, Y)\) is constant on \([a, b]\). Thus, if \(X\) and \(Y\) are chosen so that for some \(t \in [a, b]\), 
\[g(X, Y)(t) = c\] 
then \(g(X, Y)(\tau) = c\) for all \(\tau \in [a, b]\).

Using similar arguments it is possible to show that this definition of parallel propagation on locally Lipschitz curves satisfies all the expected properties.

**Definition 4.2.** Let \(\gamma: [a, b] \to \mathcal{M}\) be a locally Lipschitz curve, choose \(c \in \mathbb{R}\), \(t_0 \in [a, b]\) and let \(X_1, \ldots, X_n\) be a frame of linearly independent vector fields on \(\gamma\) which are absolutely continuous on compact subsets of \([a, b]\) and so that \(\nabla_{\dot{\gamma}} X_i = 0\) almost everywhere. Then we may write \(\gamma' = (\gamma')^i X_i\) for almost all \(t \in [a, b]\). Since \(\gamma\) is locally Lipschitz the function \(\tau: [a, b] \to \mathbb{R}\) given by

\[
\tau(t) = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} ((\gamma')^i(u))^2} \, du + c,
\]

is well defined. We shall call \(\tau\) a generalized affine parameter.

It is clear that definition 4.2 when applied to \(C^1\) curves reproduces the standard generalized affine parameter. Thus definition 4.2 is a generalization of the generalized affine parameter for locally Lipschitz curves.

Unlike general locally Lipschitz changes of parameter, any locally Lipschitz curve reparametrized with a generalized affine parameter remains locally Lipschitz. This is because, if \(\gamma\) is a locally Lipschitz curve and \(\mu\) is \(\gamma\) reparametrized with the generalized affine parameter \(\tau\), i.e. \(\mu(\tau(t)) = \gamma(t)\), then the length of \(\mu'(\tau)\) is 1 in the Riemannian metric induced by the frame used to define \(\tau\). Restricting to relatively compact subsets of the domain of \(\tau\) and translating this result into a coordinate frame demonstrates that \(\mu\) is locally Lipschitz.

Since, by assumption, every continuous causal curve is also locally Lipschitz, continuous causal curves can be given generalized affine parameters.

**Definition 4.3.** Let \(C^l(\mathcal{M})\) be the set of all locally Lipschitz curves in \(\mathcal{M}\) with generalized affine parameters. Let \(C_{cc}(\mathcal{M})\) be the set of all continuous causal curves in \(\mathcal{M}\) with generalized affine parameters.

By definition 2.3, \(C_{cc}(\mathcal{M}) \subset C^l(\mathcal{M})\).

**Proposition 4.4.** The sets \(C^l(\mathcal{M})\) and \(C_{cc}(\mathcal{M})\) satisfy the bounded parameter property.

**Proof.** Through any point \(p \in \mathcal{M}\) there exists a causal geodesic. Such a geodesic is a continuous causal curve and therefore an element of \(C_{cc}(\mathcal{M})\), hence also of \(C^l(\mathcal{M})\). It is clear that any subcurve of a generalized affinely parametrized locally Lipschitz curve is also a generalized affinely parametrized locally Lipschitz curve. Likewise, the definition of a continuous causal curve makes it clear that a subcurve of a generalized affinely parametrized continuous causal curve is also a generalized affinely parametrized continuous causal curve. Since the inner products of parallelly propagated vectors along locally Lipschitz curves are constant, the standard proof [2, p 259] that if one generalized affine parameter (on \(C^1\) curves) is bounded then every generalized affine parameter (on \(C^l\) curves) is bounded carries over to locally Lipschitz curves. Thus, if two generalized affinely parametrized locally Lipschitz curves are related by a change of parameter then either both parameters are bounded or both
are unbounded. Since generalized affinely parametrized continuous causal curves are generalized affinely parametrized locally Lipschitz curves this holds for generalized affinely parametrized continuous causal curves as well. Hence, the bounded parameter property holds on both sets.

Thus, we have identified a bounded parameter property satisfying set of curves that is larger than the set of all affinely parametrized causal geodesics and includes all continuous causal curves.

4.2. Incompleteness for locally Lipschitz curves

We now give a definition of incompleteness for locally Lipschitz curves and show that locally Lipschitz curves with endpoints are not incomplete. To do this we mirror definition 2.5.

**Definition 4.5.** A generalized affinely parametrized locally Lipschitz curve \( \gamma: [a, b) \to \mathcal{M} \) is incomplete if \( b < \infty \) and \( \gamma \) is not extendible by any generalized affinely parametrized locally Lipschitz curve.

The following technical lemma will be of use below.

**Lemma 4.6.** Let \( \gamma: [a, b) \to \mathcal{M} \) be a generalized affinely parametrized locally Lipschitz curve and let \( \lambda: [a', b') \to \mathcal{M} \) be a locally Lipschitz curve such that there exists a change of parameter \( f: [a, b) \to [a', c), \ c' \leq b', \) so that \( \lambda \circ f = \gamma. \) Then there exists \( s: [a', b') \to [a, d), \ b \leq d, \) a generalized affine parameter on \( \lambda, \) so that \( s \circ f \) is the identity on \( [a, b), \) and \( c' < b' \) if and only if \( b < d. \)

**Proof.** Since \( \gamma \) is generalized affinely parametrized there exists \( t_0 \in [a, b) \) and a parallelly propagated frame \( X_1, \ldots, X_n \) on \( \gamma \) so that the parameter \( t \in [a, b) \) is given by

\[
t = \int_{t_0}^{t} \sqrt{\sum_{i=1}^{n} \left( \gamma'(t) \right)^2} \, dt + t_0,
\]

where \( \gamma' = \gamma' X_i. \) Since \( \gamma(t) = \lambda \circ f(t) \) and \( \lambda \) is locally Lipschitz the equation \( \gamma'(t) = f'(t)\lambda'(f(t)) \) holds almost everywhere. Therefore,

\[
0 = \nabla_{\gamma'} X_i = f' \nabla_{\lambda'} X_i.
\]

Hence, \( \nabla_{\lambda'} X_i = 0 \) almost everywhere and so the frame \( X_1, \ldots, X_n \) can be extended to a parallelly propagated frame along \( \lambda. \) Let \( \lambda' = \lambda' X_i \) and let \( s_0 = f(t_0). \) This allows us to define a generalized affine parameter, \( s, \) on \( \lambda \) by

\[
s(\tau_s) = \int_{s_0}^{\tau_s} \sqrt{\sum_{i=1}^{n} \left( \lambda'(s) \right)^2} \, ds + t_0, \quad \tau_s \in [a', b').
\]

Choose \( c, \ d \in \mathbb{R} \cup \{\infty\}, \ c < d, \) so that \( s: [a', b') \to [c, d) \) is surjective.

Let \( \tau_s \in [a, b) \) and let \( \tau_s = f(\tau_s). \) Since \( f \) is locally Lipschitz we know that \( f \) is absolutely continuous on the interval \([t_0, \tau_s]. \) Hence, [26, theorem 9.7.5] implies that we can use \( f \) to change variables in the equation for \( s. \) From above \( (\gamma')'(t) = f'(t)(\lambda')'(f(t)) \) and therefore
\[
s(\tau) = \int_{0}^{\tau} \sqrt{\sum_{i} \left( \dot{\lambda}^{(i)}(\dot{\tau}) \right)^2} \, d\dot{\tau} + t_0 = \int_{0}^{\tau} \sqrt{\sum_{i} \left( \ddot{\lambda}^{(i)}(f(\dot{\tau})) \right)^2} f'(\dot{\tau}) \, d\dot{\tau} + t_0
\]
\[
= \int_{0}^{\tau} \sqrt{\sum_{i} \left( \dot{\gamma}^{(i)}(\dot{\tau}) \right)^2} \, d\dot{\tau} + t_0 = \tau.
\]

Hence, \( s \circ f(\tau) = \tau \). Since \( \tau \) was arbitrary this implies that \( s \circ f(t) = t \) for all \( t \in [a, b) \), as required.

As \( \dot{s} \) is strictly monotonically increasing and surjective we have that \( f(a) = a' \). Since \( \dot{\lambda} \) is locally Lipschitz the function
\[
s'(\tau) = \sqrt{\sum_{i} \left( \dot{\lambda}^{(i)}(\tau_i) \right)^2}
\]
is positive almost everywhere. Thus \( s \) is strictly monotonically increasing. Hence, \( s(a') = c \) and therefore, as \( s \circ f(a) = a, c = a \). So \( s: [a', b') \rightarrow [a, d) \) as required.

Let \( c' < b' \) and, again, let \( c > 0 \) be such that \( b - c \in [a, b) \). Then \( f(b - c) \in [a', c') \) and thus \( f(b - c) < c' \). Since \( c' < b' \), \( s(c) \) is defined. Hence \( b - c = s \circ f(b - c) < s(c') \). As this is true for all such \( c > 0 \) we have that \( b \leq s(c') \). Since \( s(c') \in [a, d) \) we see that \( b < d \) as required.

Suppose that \( b < d \). Let \( c > 0 \) be such that \( c' - c \in [a', c') \). Since \( f \) is surjective there exists \( x \in [a, b) \) so that \( f(x) = c' - c \). Since \( b < d, s^{-1}(b) \) is well defined. As \( x \in [a, b) \) we have \( x < b \) and hence \( s^{-1}(x) < s^{-1}(b) \). By construction
\[
s^{-1}(x) = s^{-1} \circ s \circ f(x) = s^{-1} \circ s(c' - c) = c' - c.
\]

Thus \( c' - c < s^{-1}(b) \). As this holds for all such \( c > 0 \), \( c' \leq s^{-1}(b) \). Since \( s^{-1}(b) \in [a', b') \) we see that \( c' \leq s^{-1}(b) < b' \) as required.

**Proposition 4.7.** Let \( \gamma: [a, b) \rightarrow \mathcal{M} \) be a generalized affinely parametrized locally Lipschitz curve. If \( \gamma \) has an endpoint then it is extendible and \( b < \infty \).

**Proof.** If \( b < \infty \), let \( f: [a, b) \rightarrow [0, 1) \) be defined by \( f(t) = \frac{1}{b-a} (t-a) \). If \( b = \infty \), let \( f: [a, b) \rightarrow [0, 1) \) be defined by \( f(t) = \frac{2}{\pi} \arctan(t-a) \). In either case both \( f \) and \( f^{-1} \) are changes of parameter.

Let \( p \in \mathcal{M} \) be such that \( \gamma \rightarrow p \). Let \( \lambda: [0, c) \rightarrow \mathcal{M}, c \in \mathbb{R}^+ \), be a geodesic from \( p \). Let \( \mu: [0, c+1) \rightarrow \mathcal{M} \) be defined by
\[
\mu(t) = \begin{cases} 
\gamma \circ f^{-1}(t) & t \in [0, 1), \\
\lambda(t-1) & t \in [1, c+1).
\end{cases}
\]

Since \( 1 < c + 1 \), by lemma 4.6 there exists \( d \in \mathbb{R} \cup \{\infty\}, b < d \), and a generalized affine parameter \( s: [0, c+1) \rightarrow [a, d) \) on \( \mu \) so that \( s \circ f(t) = t \) for all \( t \in [a, b) \). Let \( \delta: [a, d) \rightarrow \mathcal{M} \) be the generalized affinely parametrized curve defined by \( \delta \circ s = \mu \). Then \( \delta = \delta \circ s \circ f = \mu \circ f = \gamma \) on \([a, b)\). Since \( b < d, b < \infty \) and the curve \( \delta \) is an extension of \( \gamma \). □
It is possible to extend continuous causal curves with endpoints by continuous causal
curves.

**Proposition 4.8.** Let \( \gamma \colon [a, b) \to \mathcal{M} \) be a generalized affinely parametrized continuous
future directed non-spacelike (timelike) curve. If \( \gamma \) has an endpoint then \( \gamma \) is extendible by a
generalized affinely parametrized continuous future directed non-spacelike (timelike) curve\and \( b < \infty \).

**Proof.** From proposition 4.7 it is clear that \( b < \infty \) and there exists a generalized affinely parametrized locally Lipschitz curve \( \delta \) that is an extension of \( \gamma \), where \( \gamma \) is considered as a
locally Lipschitz curve. To prove the result it remains to show that \( \delta \), as defined in the proof of
proposition 4.7, is a continuous future directed non-spacelike (timelike) curve. Note that, by
construction \( \delta \colon [a, b) \to \mathcal{M}, b < d \) and \( \delta = \gamma \) on \( [a, b) \).

By construction \( \delta \) is continuous. Since \( \delta \big|_{(a,b)} \) is a continuous, future directed, non-
spacelike (timelike) curve and \( \delta \big|_{(b,d)} \) can be chosen to be a smooth, future directed, non-
spacelike (timelike) geodesic, we know that for \( \delta \) to be a non-spacelike, future directed, continuous curve we need only show that there exists a neighbourhood \( N \subset [a, d) \) of \( b \) and a
convex normal neighbourhood \( U \) of \( p = \delta(b) \) so that for all \( t \in N \), \( t \neq b \), if \( t > b \) then
\( \delta(t) \in J^+(p, U) \) and if \( t < b \) then \( \delta(t) \in J^-(p, U) \). In the timelike case we need \( N \)
and \( U \) as before so that for all \( t \in N \), \( t \neq b \), if \( t > b \) then \( \delta(t) \in I^+(p, U) \) and if \( t < b \) then
\( \delta(t) \in I^-(p, U) \).

Choose \( U \) a convex normal neighbourhood of \( p \) and let \( N \) be a neighbourhood of \( b \) in
\([a, d) \) so that \( \delta(N) \subset U \). Let \( t \in N \). Suppose that \( t > b \). Then the condition immediately
follows as \( \delta \big|_{[b,d]} \) is a non-spacelike (timelike), future directed geodesic.

Suppose that \( t < b \) and that \( \gamma \) is non-spacelike. As \( U \) is a convex normal neighbourhood
there exists a unique geodesic \( \alpha \colon [0,1] \to U \) so that \( \alpha(0) = \delta(t) \) and \( \alpha(1) = p \). If \( \alpha \) is future
directed non-spacelike then we are done, so suppose that \( \alpha \) is spacelike. Then by the
continuity of \( g \) on \( T_{\delta(t)} \mathcal{M} \) and proposition 4.5.1 of \([2]\) there exists a neighbourhood \( V \) of \( p \) in
\( U \) so that for all \( q \in V \) the unique geodesic in \( U \) from \( \delta(t) \) to \( q \) is spacelike. Since \( \delta(t) \to p \)
as \( t \to b \), we can consider all \( t' < b \), where \( t < t' \), such that \( \delta(t') \in V \). By construction, for
each such \( t' \), there is a unique spacelike geodesic \( \tilde{\alpha}_{t'} \colon [0,1] \to U \) so that \( \tilde{\alpha}_{t'}(0) = \delta(t) \) and \( \tilde{\alpha}_{t'}(1) = \delta(t') \). By proposition 4.5.1 of \([2]\), however, this implies that for each such \( t' \),
\( \delta(t') \not\in J^+(\delta(t'), U) \). This is a contradiction and so \( \alpha \) must be non-spacelike. Also each \( \tilde{\alpha}_{t'} \)
must be non-spacelike and future directed since \( \delta(t) \in J^-(\delta(t'), U) \). By continuity
\( \alpha(t) = \lim_{t' \to t} \tilde{\alpha}_{t'}(t) \) and so \( \alpha \) is future directed. Lastly since \( U \) is normal and as \( t < t' \),
\( \delta(t) \neq \delta(t') \).

Suppose that \( t < b \) and that \( \gamma \) is timelike. Let \( \alpha \) be as above. If \( \alpha \) is future directed,
timelike then we are done, so suppose that \( \alpha \) is non-timelike. If \( \alpha \) is spacelike the same
argument as above can be used to find a contradiction. Thus we can restrict here to the case
that \( \alpha \) is a null geodesic. We may choose \( t' \in (t, b) \), then by assumption \( \delta(t') \in I^+(\delta(t), U) \).
This implies that the unique geodesic from \( \delta(t') \) to \( p \) is spacelike (otherwise we would have a
timelike curve from \( \delta(t) \) to \( p \) in \( U \) which contradicts the assumption that \( \alpha \) is null \([2, \text{proposition 4.5.1}]\). The same argument as above can then be applied to find a contradiction.
Therefore \( \alpha \) must be timelike and future directed.

Thus \( \delta \) is a future directed continuous non-spacelike (timelike) curve as required. \( \square \)
5. Proofs of the generalizations

Proof of theorem 1.2. ⇐ Unchanged from theorem 1.1. ⇒ Let $\gamma \in C$ be incomplete. We have two cases:

Case 1. Unchanged from theorem 1.1.

Case 2. By assumption $\gamma$ is precompact. Then either $\gamma$ has an endpoint or $\gamma$ does not have an endpoint.

Case 2.1. Suppose that $\gamma$ has an endpoint. By proposition 4.8 we know that $\gamma$ is extendible. This is a contradiction and thus this case cannot occur.

Case 2.2. By assumption $\gamma$ is a precompact, winding curve. Since $\gamma$ is winding it has two full sequences with different limit points. As $\mathcal{M}$ is Hausdorff this implies that $\gamma$ is inextendible. Thus the inextendible curve $\gamma$ is totally imprisoned in the compact set $\mathcal{P}$. Proposition 3.2 implies that the future (past) distinguishing condition fails on $\mathcal{P}$. This is a contradiction and therefore this case cannot occur.

As only case 1 may occur, we have proven our result.

Proof of theorem 1.3. ⇐ Unchanged from theorem 1.1. ⇒ Let $\gamma \in C$ be incomplete. We have two cases:

Case 1. Unchanged from theorem 1.1.

Case 2. By assumption $\gamma$ is precompact. Then either $\gamma$ has an endpoint or $\gamma$ does not have an endpoint.

Case 2.1. Suppose that $\gamma$ has an endpoint. By proposition 4.7 we know that $\gamma$ is extendible. This is a contradiction and thus this case cannot occur.

Case 2.2. By assumption $\gamma$ is a precompact, winding curve. This implies that $\gamma$ is inextendible and totally imprisoned. By assumption such curves cannot exist. This is a contradiction and therefore this case cannot occur.

As only case 1 may occur, we have proven our result.

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