A non-trivial spatial topology of the Universe is a potentially observable attribute, which can be probed through the circles-in-the-sky for all locally homogeneous and isotropic universes with no assumptions on the cosmological parameters. We show how one can use a possible circles-in-the-sky detection of the spatial topology of globally homogeneous universes to set constraints on the dark energy equation of state parameters.

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1. Introduction

In the context of general relativity, the observable Universe seems to be well described by a 4-manifold $\mathcal{M} = \mathbb{R} \times M$ with locally homogeneous and isotropic spatial sections $M$, and therefore endowed with a Robertson–Walker metric

$$ds^2 = -dt^2 + a^2(t) \left[d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)\right],$$

where $a(t)$ is the scale factor and $S_k(\chi) = \chi, \sin \chi, \sinh \chi$ depending upon whether the geometry of the spatial sections is Euclidean, spherical or hyperbolic with constant spatial curvature $k = 0, 1, \text{ or } -1$. The spatial geometry or the corresponding spatial curvature is an observable property, which can be determined by finding out whether the total energy-matter density of the Universe, $\Omega_{\text{tot}}$, is equal to, greater than or smaller than 1. In consequence, a key point in the search for the spatial geometry of the Universe is to use observations to constrain the density $\Omega_{\text{tot}}$. Often the homogeneous and isotropic spatial sections $M$ are assumed to be the simply connected 3-manifolds: Euclidean $\mathbb{R}^3$, spherical $S^3$, or hyperbolic $H^3$. However, the 3-space $M$ can also be one of the possible quotient (multiply-connected) manifolds $\mathbb{R}^3/\Gamma$, $S^3/\Gamma$, and $H^3/\Gamma$, where $\Gamma$ is a fixed-point free discrete group of isometries of the corresponding covering space $\mathbb{E}^3$, $S^3$, or $H^3$. The local geometry of the spatial sections $M$ thus constrains, but does not dictate, its topology (see, e.g., the review Refs. [1]).

The immediate observational consequence of a detectable multiply-connected spatial section $M$ is that an observer could potentially detect multiple images of
radiating sources. In this way, in a universe with a detectable\textsuperscript{2} non-trivial topology the last scattering surface (LSS) intersects some of its topological images in the so called circles-in-the-sky\textsuperscript{3}, i.e., pairs of matching circles of equal radii, centered at different points of the LSS with the same distribution of temperature fluctuations (up to a phase) along the circles of each pair. Therefore, to observationally probe a non-trivial spatial topology on the largest available scales, one needs to scrutinize the cosmic microwave background (CMB) sky-maps in order to extract such correlated circles, and use their angular radii, the relative phase and position of their centers to determine the spatial topology of the Universe. Hence, a detectable non-trivial cosmic topology is an observable attribute, which can be probed through the circles-in-the-sky for all locally homogeneous and isotropic universes with no assumptions on the cosmological parameters.

The question as to whether one can use the knowledge of the topology to either determine the geometry or set constraints on the density parameters naturally arises here. Regarding the geometry it is well-known that the topology of $M$ determines the sign of its curvature (see, e.g., Ref.\textsuperscript{4}) and therefore the $3$-geometry. Thus, the topology of the spatial section of the Universe dictates its geometry. In recent works\textsuperscript{5–8} it has been shown that the knowledge of a specific spatial topology through the circles-in-the-sky offers an effective way of setting constraints on the density parameters associated with matter ($\Omega_m$) and dark energy ($\Omega_\Lambda$) in the context of $\Lambda$CDM model. In other words, it has been shown in Refs.\textsuperscript{5 and 6} that a circles-in-the-sky detection of specific spatial topology can be used to reduce the degeneracies in the density parameter plane $\Omega_m - \Omega_\Lambda$, which arise from statistical analyses with data from current observations.

Our main aim here, which are complementary to our previous works\textsuperscript{5,6} is to show how one can use a possible circles-in-the-sky detection of the spatial topology of globally homogeneous universes to set constraints on the dark energy equation of state (EOS) parameters.

2. Topological Constraints and Concluding Remarks

To investigate how a possible detection of a nontrivial spatial topology can be used to place constraints on the dark energy equation of state parameters, we shall focus on the globally homogeneous spherical manifolds and indicate how a similar procedure can be used in the case of globally homogeneous flat topologies\textsuperscript{6}.

The topological constraints on dark energy EOS can be looked upon as having two main ingredients, namely one of observational nature, and another of theoretical character. Regarding the former, an important point about the globally homogeneous universes is that the pairs of topologically correlated circles on the LSS will be antipodal, as shown in Figure\textsuperscript{11}. A straightforward use of trigonometric relations

\textsuperscript{a}Since there are no Clifford translations in the hyperbolic geometry, there are no globally homogeneous hyperbolic manifolds.
Fig. 1. A schematic illustration of two antipodal matching circles in the sphere of last scattering. These pairs of circles occur in all globally homogeneous universes with a detectable nontrivial topology.

for the right-angled spherical triangle shown in Fig. 1 yields

\[ \cos \alpha = \frac{\tan r_{inj}}{\tan \chi_{lss}} \quad \text{or} \quad \chi_{lss} = \tan^{-1} \left[ \frac{\tan r_{inj}}{\cos \alpha} \right], \quad (2) \]

where \( r_{inj} \) is a topological invariant, whose values are given in Table 1 and the distance \( \chi_{lss} \) is the comoving distance to the LSS in units of the present-day curvature radius, \( a_0 = a(t_0) = \left( H_0 \sqrt{|1 - \Omega_{tot}|} \right)^{-1} \) for \( k \neq 0 \).

Now, a circles-in-the-sky detection of a given topology would give a value for the radius \( \alpha \) along with an observational uncertainty \( \sigma_\alpha \). These observational data along with Eq. (2) and the usual error propagation formula, give the observational distance \( \chi_{lss} \) to the LSS and the associated uncertainty \( \sigma_{lss} \).

The second important ingredient of the above mentioned topological constraints is related to the theoretical dark energy model. Indeed, the comoving distance to
the last scattering surface in units of the curvature radius is given by
\[ \chi_{lss}^{th} = \frac{d_{lss}}{a_0} = \sqrt{|\Omega_k|} \int_1^{1+z_{lss}} \frac{H_0}{H(x)} \, dx , \tag{3} \]
where \( d_{lss} \) is the radius of the LSS, \( x = 1+z \) is an integration variable, \( H \) is the Hubble parameter, \( \Omega_k = 1 - \Omega_{tot} \) is the curvature density parameter, and \( z_{lss} = 1089 \).

Clearly, different parametrizations of the equation of state \( \omega_x = p_x/\rho_x \) give rise to different Friedmann equations, i.e., different ratios \( H(z)/H_0 \). Thus, for example, assuming that the current matter content of the Universe is well approximated by a dust of density \( \rho_m \) (baryonic plus dark matter) along with a dark energy perfect fluid component of density \( \rho_x \) and pressure \( p_x \), for the parametrizations \( \omega_x = \text{const.} = \omega_0 \), \( \omega_x = \omega_0 + \omega_1 \) (Refs. [10] and [11]), and \( \omega_x = \omega_0 + \omega_1 z / (1 + z) \) (Refs. [12] and [13]) the Friedmann equation takes, respectively, the following forms:
\[ \left( \frac{H}{H_0} \right)^2 = \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_{x0}(1+z)^{3(1+\omega_0)} , \tag{4} \]
\[ \left( \frac{H}{H_0} \right)^2 = \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_{x0}(1+z)^{3(\omega_0-\omega_1+1)} \exp(3\omega_1 z) , \tag{5} \]
\[ \left( \frac{H}{H_0} \right)^2 = \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_{x0}(1+z)^{3(\omega_0+\omega_1)} \exp\left(-\frac{3\omega_1 z}{1+z}\right) . \tag{6} \]

In order to show how one can use the topology to set constraints on the dark energy equation of state parameters we combine the two above-mentioned ingredients by comparing the observational topological value \( \chi_{lss} \) for a given topology with the theoretical values \( \chi_{lss}^{th} \) for any dark energy equation of state parametrization [assuming a Gaussian distribution with mean given by Eq. (3)]. Thus, the constraints from a detectable spatial topology are taken into account in a \( \chi^2 \) statistical analysis of any parameter plane (as, for example, \( \omega_0 - \Omega_k \), \( \omega_1 - \Omega_k \), \( \omega_0 - \omega_1 \)) by adding a new term of the form
\[ \chi^2_{top} = \left( \frac{\chi_{lss} - \chi_{lss}^{th}}{\sigma_{\chi_{lss}}} \right)^2 \tag{7} \]
to the remaining \( \chi^2 \) terms that account for other observational data sets. A concrete application of this result can be found in Ref. [14] where by assuming the Poincaré dodecahedral space (PDS) as the circles-in-the-sky observable spatial topology, the current constraints on the equation of state parameters for the parametrizations Eq. (4) and Eq. (6) have been reanalyzed by using Type Ia supernovae data from the Legacy sample [15] along with the baryon acoustic oscillations (BAO) peak in the large-scale correlation function of the Sloan Digital Sky Survey (SDSS) [16] and CMB shift parameter [17] with and without the topological statistical term. It is shown that the PDS topology provides relevant additional constraints on the dark energy EOS parameters for the two-parameter Chevallier-Polarski-Linder parametrization [Eq. (6)], but negligible further constraints on \( \omega_0 \) of Eq. (4). The authors also show
that a suitable Gaussian prior on the curvature density parameter $\Omega_k$ can mimic the role of the topology in such statistical analyses.

Regarding the flat manifolds, we first note that there are three classes of multiply-connected manifolds of the form $\mathbb{R}^3/\Gamma$ which are globally homogeneous, namely the 3-torus class (compact in three directions), the class of chimney spaces (compact in two directions) and the slap space class (compact in one direction). Second, Eq. (2) clearly reduces to

$$\chi_{\text{lss}} = \frac{r_{\text{inj}}}{\cos \alpha},$$

but now since the curvature radius of Euclidean 3-space is infinite, one cannot identify $a_0 = a(t = t_0)$ with the curvature radius. Therefore, in this case there is no natural unit of length, and one has to use, for example, megaparsecs (Mpc) as unit of length. Hence, the comoving distance to the LSS (for $c = 1$) is given by

$$\chi_{\text{lss}}^{\text{th}} = d_{\text{lss}} = \int_{1}^{1+z_{\text{lss}}} \frac{1}{H(x)} \, dx.$$  

Finally, we note that Euclidean manifolds are not rigid: even though topologically equivalent the manifolds of a specific class (fixed group $\Gamma$) of quotient flat manifolds may have different size. In this way, the injectivity radius $r_{\text{inj}}$ for a class of flat multiply-connected manifolds, is not a topological invariant (constant). Thus, one should estimate $r_{\text{inj}}$ on physical grounds, as for example by fitting the CMB data.

To conclude, we emphasize that the above procedure may be employed for any specific globally homogeneous detectable topology and dark energy EOS parametrizations along with an arbitrary combination of data sets.

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b Although diffeomorphic they are not necessarily globally isometric. For example, a manifold in the 3-torus class can be constructed by taking a parallelepiped (or particularly a cube) of any size and identifying the opposite faces by translations.
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