Non-dissipative space-time $hp$-discontinuous Galerkin method for the time-dependent Maxwell Equations

M. Lilienthal$^{a,c,*}$, S.M. Schnepp$^b$, T. Weiland$^c$

$^a$Graduate School of Computational Engineering, Technische Universitaet Darmstadt, Dolivostrasse 15, 64293 Darmstadt, Germany

$^b$ETH Zurich, Institute of Electromagnetic Fields (IFH), Gloriastrasse 35, 8092 Zurich, Switzerland

$^c$Institut fuer Theorie Elektromagnetischer Felder, Technische Universitaet Darmstadt, Schlooggartenstrasse 8, 64289 Darmstadt, Germany

Abstract

A finite element method for the solution of the time-dependent Maxwell equations in mixed form is presented. The method allows for local $hp$-refinement in space and in time. To this end, a space-time Galerkin approach is employed. In contrast to the space-time DG method introduced in [1] test and trial space do not coincide. This allows for obtaining a non-dissipative method. In order to obtain an efficient implementation, a hierarchical tensor product basis in space and time is proposed. In particular it allows to evaluate the local residual with a complexity of $O(p^4)$ and $O(p^5)$ for affine and non-affine elements, respectively.

1. Introduction

The accurate solution of large scale electromagnetic problems, where short wavelengths need to be resolved in large computational domains, remains a challenge. Examples include antenna design, broadband scattering problems or electrically large structures. Especially for problems, where dispersion errors dominate, high order methods have advantages. Furthermore, if $hp$-refinement is applied in a judicious way, it is possible to obtain exponential convergence, even for solutions, which are locally non-smooth [2]. This can lead to drastic savings in terms of degrees of freedom. In the past decade, there has been a lot of research on discontinuous Galerkin (DG) methods for Maxwell’s equations, see e.g. [3, 4, 5]. Due to the use of discontinuous finite element spaces, these methods allow in a natural way for $hp$-refinement without the use of special transition elements as it is the case for continuous finite element methods [6, 7]. With the traditional method of lines approach, DG methods are usually chosen

*Corresponding Author

Email addresses: lilienthal@gsc.tu-darmstadt.de (M. Lilienthal), sascha.schnepp@ifh.ee.ethz.ch (S.M. Schnepp), thomas.weiland@temf.tu-darmstadt.de (T. Weiland)

Preprint submitted to Elsevier

May 11, 2014
for the spatial part of the discretization, whereas time is discretized with explicit
time integrators. Due to the conditional stability of the resulting schemes, the
time step size in this case is determined by the smallest elements’ size and also
the degree of the approximating polynomials. Thus, the temporal resolution
is dictated by stability, not by actual accuracy requirements. In order to cure
this problem, local time-stepping schemes \cite{5, 8, 9} and locally implicit schemes
\cite{10, 11} have been proposed. For both approaches, good speedups on locally
refined meshes are reported. However, refinement in time is necessary for sta-

tility, rather than accuracy requirements.
Another approach are space-time DG methods \cite{1}. Since space and time are
discretized simultaneously, \(hp\)-refinement in space and time can be introduced
naturally, more or less out of the box. The space-time DG methods are uncondi-
tionally stable. Together with their high flexibility, these methods are well-
suited for space-time adaptivity. However, the previously introduced space-time
DG methods are dissipative. Dissipation may become an issue, especially for
low approximation orders. In this paper we propose a space-time finite element
method which is discontinuous with respect to the spatial directions and con-
tinuous in time. The resulting method allows for local \(hp\)-refinement in space
and time, is energy-conserving and unconditionally stable. In section 2 the
space-time finite element method for the time-dependent Maxwell equa-
tions
\begin{align}
\varepsilon \mathbf{E}_t - \nabla \times \mathbf{H} &= \mathbf{J} \\
\mu \mathbf{H}_t + \nabla \times \mathbf{E} &= 0 \text{ in } \Omega \\
\mathbf{n} \times \mathbf{E} &= \mathbf{n} \times \mathbf{g} \text{ on } \partial \Omega
\end{align}
is described. In section 3 we discuss the stability and energy conservation prop-
erty of the method. In section 4 we present a matrix-free implementation of the
space-time residual. It can be efficiently evaluated within an interactive solution
procedure, such that the method computationally behaves similarly to an ex-
plicit method. Section 5 is devoted to numerical experiments, including fully
space-time \(hp\)-adaptive simulations.

2. Description of the method

For the derivation of the method, we only consider spatial meshes consisting
of hexahedra. Nevertheless, most of the presented work will be applicable to
tetrahedral meshes, as well.

2.1. Partitioning of the space-time domain

We divide the time axis in intervals \(I_n = (t_{n-1}, t_n]\), and thus obtain a
partitioning of the space-time cylinder \(I \times \Omega\) in time slabs \(I_n \times \Omega\). For each
time slab, the spatial domain \(\Omega\) is partitioned in non-overlapping (hexahedral)
elements \(K\) resulting in a triangulation \(\mathcal{T}_n(\Omega)\). We require that \(\mathcal{T}_n(\Omega)\) can be
obtained by refinement of a coarse triangulation \(\overline{\mathcal{T}}(\Omega)\). The obtained space-
time volumes are further bisected in temporal direction \(I_n \times K = \bigcup_{k=1}^{N_K} I_k \times K\),
such that we obtain a partition of the time slab $I_n \times \Omega$ in space-time elements $I_k^K \times K \in S_n(I_n \times \Omega)$. Here $S_n(I_n \times \Omega)$ denotes the resulting triangulation of the time slab.

![Figure 1: Space-time partitioning of a time-slab](image)

### 2.2. Discrete spaces

Let $F_K$ denote the mapping from the unit cube $\tilde{K} = [0, 1]^3$ with axes $\hat{x}, \hat{y}, \hat{z}$ to the physical element $K = F_K(\tilde{K})$. Furthermore by $DF_K$ we denote the jacobian matrix of $F_K$. The electric and magnetic fields are transformed with the covariant transformation $v(x,t) = DF_K^{-1}(\hat{v}(\hat{x},t)) DF_K^{-1}$, as proposed in [5].

By $P_k(I)$ we denote polynomials of degree $k$ on interval $I$ and by $Q_{p_x,p_y,p_z}(\hat{K})$ tensor product polynomials of degrees $p_x, p_y$ and $p_z$ in the $\hat{x}, \hat{y}, \hat{z}$ directions.

Now we can introduce the following local discrete spaces

$$V_{h,\tilde{K}}^k := P_{p_t}(I_k^K) \otimes \left[ Q_{p_x,p_y,p_z}(\tilde{K}) \right]^3$$

$$W_{h,\tilde{K}}^k := P_{p_t-1}(I_k^K) \otimes \left[ Q_{p_x,p_y,p_z}(\tilde{K}) \right]^3.$$  \(2\)

Here $p^K_k = (p_t,p_x,p_y,p_z)$ denotes the local polynomial degree vector assigned to the space-time element $I_k^K \times \tilde{K}$. Note that the spatial part of the polynomial spaces in (2) is identical for all elements $I_k^K \times \tilde{K}$ in one patch of space-time elements $I_n \times \tilde{K}$, such that we have

$$V_{h,\tilde{K}} := S_K(I) \otimes \left[ Q_{p_x,p_y,p_z}(\tilde{K}) \right]^3, \quad S_K(I) := \{ u(t) \in H_1(I) : u|_{I_k^K} \in P_{p_t}(I_k^K) \}$$

$$W_{h,\tilde{K}} := T_K(I) \otimes \left[ Q_{p_x,p_y,p_z}(\tilde{K}) \right]^3, \quad T_K(I) := \{ u(t) \in L_2(I) : u|_{I_k^K} \in P_{p_t-1}(I_k^K) \}.$$  \(4\)
We collect the local polynomial degree vectors for all elements in a global vector \( \mathbf{p} \).

Now we can define the global spaces

\[
V_h(I_n \times \Omega; \mathbf{p}) := \{ v(t, x) \in H^1(I; L^2(\Omega)) : DF_T^T v \big|_{K \circ F} \in V_{h,K} \},
\]

and

\[
W_h(I_n \times \Omega; \mathbf{p}) := \{ v(t, x) \in L^2(I; L^2(\Omega)) : DF_T^T v \big|_{K \circ F} \in W_{h,K} \}.
\]

While both spaces are spatially discontinuous, the functions in \( V_h \) and \( W_h \) have different continuity properties in temporal direction: functions in \( V_h \) are time-continuous within each time slab whereas functions in \( W_h \) are allowed to be discontinuous at the interfaces in time direction. The situation is depicted for an example with three space-time elements in Fig. 2.

![Figure 2: Left: function belonging to trial space \( V_h \), right: function from the corresponding test space \( W_h \)](image)

### 2.3. Faces and trace operators

By \( \mathcal{F} \) we denote the set of all faces in the spatial triangulation \( T \), by \( \mathcal{F}_0 \) the set of all interior faces \( f := \partial K^1 \cap \partial K^2 : K^1, K^2 \in \mathcal{T}(\Omega) \) and by \( \mathcal{F}_b \) the set of all boundary faces \( f := \partial K \cap \partial \Omega : K \in \mathcal{T}(\Omega) \).

We define on interior faces \( f \in \mathcal{F}_0 \) the average and tangential jump operators as \( \{ v \} := (v^1 + v^2)/2 \) and \( [v]_T := n^1 \times v^1 + n^2 \times v^2 \) respectively. Here \( v^1 \) and \( v^2 \) denote the traces of \( v \) on \( f \) taken from within element \( K^1 \) and \( K^2 \) with unit normals \( n^1 \) and \( n^2 \). For a boundary face \( f \in \mathcal{F}_b \) we define averages and jumps as \( \{ v \} := v \) and \( [v]_T := n \times v \).

### 2.4. Weak formulation

In the following we consider the discretization of a single time-slab \( I \times \Omega \). Multiplying (1) by test functions \( \mathbf{v}, \mathbf{w} \in W_h \), integrating over \( I \times K \) and
performing integration by parts of the terms involving the curl operator with respect to the spatial variables yields:

\[
\int_{I} \int_{K} \varepsilon \partial_t \mathbf{E} \cdot \mathbf{v} \, dx \, dt - \int_{I} \int_{K} \mathbf{H} \cdot \nabla \times \mathbf{v} \, dx \, dt \\
- \int_{I} \int_{\partial K} \mathbf{n} \times \mathbf{H}^* \cdot \mathbf{v} \, dS \, dt = \int_{I} \int_{K} \mathbf{J} \cdot \mathbf{v} \, dx \, dt \\
\int_{I} \int_{K} \mu \partial_t \mathbf{H} \cdot \mathbf{w} \, dx \, dt + \int_{I} \int_{K} \nabla \times \mathbf{E} \cdot \mathbf{w} \, dx \, dt \\
+ \int_{I} \int_{\partial K} \mathbf{n} \times (\mathbf{E}^* - \mathbf{E}) \cdot \mathbf{w} \, dS \, dt = 0. \quad (7)
\]

Then by replacing \( \mathbf{E}, \mathbf{H} \) by the discrete fields \( \mathbf{E}_h, \mathbf{H}_h \in V_h \), summing over all \( I \times K \) and choosing centered fluxes

\[
\mathbf{E}^* = \{ \mathbf{E}_h \}, \ f \in F_0 \quad \mathbf{E}^* = \mathbf{g}, \ f \in F_D \quad \mathbf{H}^* = \{ \mathbf{H}_h \}, \ f \in F
\]

one obtains the bilinear form

\[
C_h(\mathbf{U}_h, \mathbf{V}) := \int_{I} \int_{\Omega} \varepsilon \partial_t \mathbf{E}_h \cdot \mathbf{v} \, dx \, dt + \int_{I} \int_{\Omega} \mu \partial_t \mathbf{H}_h \cdot \mathbf{w} \, dx \, dt + \\
- \int_{I} \int_{\Omega} \mathbf{H}_h \cdot \nabla_h \times \mathbf{v} \, dx \, dt + \int_{I} \int_{\Omega} \nabla_h \times \mathbf{E}_h \cdot \mathbf{w} \, dx \, dt \\
+ \int_{I} \int_{F_{0} \cup F_h} \{ \mathbf{H}_h \} \cdot [\mathbf{v}]_T \, dS \, dt - \int_{I} \int_{F_{0} \cup F_h} [\mathbf{E}_h]_T \cdot \{ \mathbf{w} \} \, dS \, dt \quad (8)
\]

and the functional

\[
L(\mathbf{V}) := \ell_E(\mathbf{v}) + \ell_H(\mathbf{w}) \\
\ell_E(\mathbf{v}) = \int_{I} \int_{\Omega} \mathbf{J} \cdot \mathbf{v} \, dx \, dt, \quad \ell_H(\mathbf{w}) = - \int_{I} \int_{F_h} \mathbf{n} \times \mathbf{g} \cdot \mathbf{w} \, dS \, dt. \quad (9)
\]

The discrete problem based on the bilinear form \( C_h \) does in general not yield stable solutions. To this end, an additional stabilization term

\[
S_h(\mathbf{U}_h, \mathbf{V}) := \frac{1}{2} \left( \int_{I} \int_{F_0} \{ \mathbf{H}_h \} \cdot [\pi_T \mathbf{v} - \mathbf{v}]_T \, dS \, dt + \int_{I} \int_{F_0} [\mathbf{H}_h]_T \cdot \{ \pi_T \mathbf{v} - \mathbf{v} \} \, dS \, dt \right) \\
- \frac{1}{2} \left( \int_{I} \int_{F_0} \{ \mathbf{E}_h \} \cdot [\pi_T \mathbf{w} - \mathbf{w}]_T \, dS \, dt + \int_{I} \int_{F_0} [\mathbf{E}_h]_T \cdot \{ \pi_T \mathbf{w} - \mathbf{w} \} \, dS \, dt \right). \quad (10)
\]

is introduced. For an interior face \( I \times f, f \in F_0 \) shared by \( I \times K^i, i = 1, 2 \), we denote by \( \pi_T \) the \( L_2 \)-orthogonal projection operator \( \pi_T : L_2(I) \to T_{K^i}(I) \cap T_{K^2}(I) =: \tilde{T}(I) \).

Finally the discrete problem reads

\[
B_h(\mathbf{U}_h, \mathbf{V}) := C_h(\mathbf{U}_h, \mathbf{V}) + S_h(\mathbf{U}_h, \mathbf{V}) = L(\mathbf{V}) \quad \forall \ \mathbf{V} \in W_h. \quad (11)
\]
3. Stability analysis

In this section we show stability by an energy argument.

3.1. Energy conservation

Denoting the electromagnetic energy by
\[ E(t) = \frac{1}{2} \int_\Omega (\varepsilon \mathbf{E}_h \cdot \mathbf{E}_h + \mu \mathbf{H}_h \cdot \mathbf{H}_h) \, dx, \] (12)
we demonstrate stability of the method by

**Theorem 3.1.** For element-wise constant material parameters \( \varepsilon, \mu \)
\[ E(t_{n+1}) - E(t_n) = \int_I \int_\Omega \mathbf{J} \cdot \pi_\tau \mathbf{E}_h \, dx \, dt - \int_I \int_{F_b} \mathbf{n} \times \mathbf{g} \cdot \pi_\tau \mathbf{H}_h \, dS \, dt. \] (13)

**Proof.** We denote by \( \pi_\tau \mathbf{E}_h, \pi_\tau \mathbf{H}_h \) the \( L_2 \)-projections of the discrete solution onto the test space. Choosing \( \mathbf{v} = \pi_\tau \mathbf{E}_h, \mathbf{w} = \pi_\tau \mathbf{H}_h \) in (11) yields
\[
\begin{align*}
\int_I \int_\Omega \varepsilon \partial_t \mathbf{E}_h \cdot \pi_\tau \mathbf{E}_h \, dx \, dt + \int_I \int_\Omega \mu \partial_t \mathbf{H}_h \cdot \pi_\tau \mathbf{H}_h \, dx \, dt \\
- \int_I \int_\Omega \mathbf{H}_h \cdot \nabla_h \times \pi_\tau \mathbf{E}_h \, dx \, dt + \int_I \int_\Omega \nabla_h \times \mathbf{E}_h \cdot \pi_\tau \mathbf{H}_h \, dx \, dt \\
+ \int_I \int_{F_{\partial \Omega} \cup F_b} \{ \mathbf{H}_h \} \cdot [\pi_\tau \mathbf{E}_h]_T \, dS \, dt - \int_I \int_{F_0 \cup F_b} [\mathbf{E}_h]_T \cdot \{ \pi_\tau \mathbf{H}_h \} \, dS \, dt \\
+ \frac{1}{2} \left( \int_I \int_\Omega \{ \mathbf{H}_h \} \cdot [\pi_\tau \pi_\tau \mathbf{E}_h - \pi_\tau \mathbf{E}_h]_T \, dS \, dt + \int_I \int_{F_0} [\mathbf{H}_h]_T \cdot \{ \pi_\tau \pi_\tau \mathbf{E}_h - \pi_\tau \mathbf{E}_h \} \, dS \, dt \right) \\
- \frac{1}{2} \left( \int_I \int_\Omega \{ \mathbf{E}_h \} \cdot [\pi_\tau \pi_\tau \mathbf{H}_h - \pi_\tau \mathbf{H}_h]_T \, dS \, dt + \int_I \int_{F_0} [\mathbf{E}_h]_T \cdot \{ \pi_\tau \pi_\tau \mathbf{H}_h - \pi_\tau \mathbf{H}_h \} \, dS \, dt \right) \\
= \int_I \int_\Omega \mathbf{J} \cdot \pi_\tau \mathbf{E}_h \, dx \, dt - \int_I \int_{F_b} \mathbf{n} \times \mathbf{g} \cdot \pi_\tau \mathbf{H}_h \, dS \, dt \tag{14}
\end{align*}
\]

For the first term in (14) we have since \( \partial_t \mathbf{E}_h \in W_h \) and integration by parts
\[
\begin{align*}
\int_I \int_\Omega \varepsilon \partial_t \mathbf{E}_h \cdot \pi_\tau \mathbf{E}_h \, dx \, dt &= \int_I \int_\Omega \varepsilon \partial_t \mathbf{E}_h \cdot \mathbf{E}_h \, dx \, dt \\
&= \frac{1}{2} \int_\Omega \varepsilon \mathbf{E}_h(t_k+1) \cdot \mathbf{E}_h(t_k+1) \, dx - \frac{1}{2} \int_\Omega \varepsilon \mathbf{E}_h(t_k) \cdot \mathbf{E}_h(t_k) \, dx.
\end{align*}
\]
Thus we obtain by treating the second term in (14) the same way
\[
\int_I \int_\Omega \varepsilon \partial_t \mathbf{E}_h \cdot \pi_\tau \mathbf{E}_h \, dx \, dt + \int_I \int_\Omega \mu \partial_t \mathbf{H}_h \cdot \pi_\tau \mathbf{H}_h \, dx \, dt = E(t_{n+1}) - E(t_n). \tag{15}
\]
The third term yields
\[
- \int_I \int_\Omega \mathbf{H}_h \cdot \nabla_h \times \pi_\tau \mathbf{E}_h \, dx \, dt = - \int_I \int_\Omega \pi_\tau \mathbf{H}_h \cdot \nabla \times \mathbf{E}_h \, dx \, dt
\]
Now, consider the mesh-dependent terms associated with an interior face \( f \in F_0 \) shared by two elements \( K^1, K^2 \). We have by a straightforward calculation

\[
- \frac{1}{2} \int_I \int_{\partial K^1} H^1_H \cdot n^1 \times \pi_T E^1_h \, dS \, dt - \frac{1}{2} \int_I \int_{\partial K^1} H^2_H \cdot n^1 \times \pi_T \pi_T E^1_h \, dS \, dt \\
- \frac{1}{2} \int_I \int_{\partial K^2} H^1_H \cdot n^2 \times \pi_T \pi_T E^2_h \, dS \, dt - \frac{1}{2} \int_I \int_{\partial K^2} H^2_H \cdot n^2 \times \pi_T \pi_T E^2_h \, dS \, dt \\
+ \frac{1}{2} \int_I \int_{\partial K^1} \pi_T H^1_H \cdot n^1 \times E^1_h \, dS \, dt + \frac{1}{2} \int_I \int_{\partial K^2} \pi_T H^1_H \cdot n^2 \times E^2_h \, dS \, dt \\
+ \frac{1}{2} \int_I \int_{\partial K^1} \pi_T \pi_T H^2_H \cdot n^1 \times E^1_h \, dS \, dt + \frac{1}{2} \int_I \int_{\partial K^2} \pi_T \pi_T H^2_H \cdot n^2 \times E^2_h \, dS \, dt. \tag{16}
\]

First, we inspect the terms, which do not couple to neighboring elements, for example the first and fifth term. Again, by the symmetry of \( \pi_T \), we have for the first term of (16)

\[
\frac{1}{2} \int_I \int_{\partial K^1} H^1_H \cdot n^1 \times \pi_T E^1_h \, dS \, dt = \frac{1}{2} \int_I \int_{\partial K^1} \pi_T H^1_H \cdot n^1 \times E^1_h \, dS \, dt,
\]

such that the sum of the first and the fifth term is zero.

For the terms involving neighbor-coupling we have for example for the second term

\[
\frac{1}{2} \int_I \int_{\partial K^1} H^2_H \cdot n^1 \times \pi_T \pi_T E^1_h \, dS \, dt = \frac{1}{2} \int_I \int_{\partial K^1} \pi_T \pi_T H^2_H \cdot n^1 \times E^1_h \, dS \, dt
\]

and \( \pi_T \pi_T H^2_H = \pi_T H^2_H \), since the temporal part of \( \pi_T H^2_H \) belongs to \( T_{K^1} \). Similarly we obtain for the seventh term in (16)

\[
\frac{1}{2} \int_I \int_{\partial K^2} \pi_T H^1_H \cdot n^2 \times E^2_h \, dS \, dt = \frac{1}{2} \int_I \int_{\partial K^2} \pi_T \pi_T H^2_H \cdot n^2 \times E^2_h \, dS \, dt.
\]

Proceeding similarly for the third and sixth term in (16) and noting that terms associated with a boundary face can be treated exactly the same way as the non-coupling terms, yields the desired result. \( \square \)

### 3.2. Stability in the space-time \( L_2 \)-Norm

For the special case of no local refinement in time we can also show the stability in the space-time \( L_2 \)-norm \( \| \cdot \|_{0,T \times \Omega} \). The general case is subject of ongoing work.

First we recall the recurrence relations for the Legendre-polynomials \( L_i(\xi) \)

\[
(i + 1)L_{i+1}(\xi) = (2i + 1)\xi L_i(\xi) - i L_{i-1}(\xi)
\]

\[
L'_i(\xi) = 2L_{i-1}(\xi)/\|L_{i-1}\|_{0,[0,1]}^2 + 2L_{i-3}(\xi)/\|L_{i-3}\|_{0,[0,1]}^2 + \ldots
\]

\[
\xi L'_{i+1} = (i + 1)L_{i+1}(\xi) + i L_{i-1}(\xi) + (i - 1)L_{i-1}(\xi) + (i - 2)L_{i-1}(\xi) \ldots \tag{17}
\]
Lemma 3.2. For element wise constant $\varepsilon, \mu$ there holds

$$
\frac{1}{2\Delta t} \|\varepsilon E_h\|_{0, I \times \Omega}^2 + \frac{1}{2\Delta t} \|\mu H_h\|_{0, I \times \Omega}^2 \leq \int_I \int_\Omega \varepsilon \partial_t E_h \cdot \pi_\tau \tau(t) \pi_\tau E_h \, dx \, dt
+ \int_I \int_\Omega \mu \partial_t H_h \cdot \pi_\tau \tau(t) \pi_\tau H_h \, dx \, dt + \varepsilon(t_k), \quad \tau(t) = \frac{(t_{k+1} - t)}{\Delta t}
$$

Proof. We have by integration by parts with respect to the temporal variable

$$
\int_I \int_K \varepsilon \partial_t E_h \cdot \pi_\tau \tau(t) E_h \, dx \, dt = \frac{1}{2\Delta t} \|\varepsilon E_h\|_{0, I \times K}^2 - \frac{1}{2} \|\varepsilon E_h(t_k)\|_K^2
$$

using that $\pi_\tau$ is an orthogonal projection and $\partial_t E_h \in W_h$ we can rewrite

$$
\int_I \int_K \varepsilon \partial_t E_h \cdot \pi_\tau \tau(t) E_h \, dx \, dt = \int_I \int_K \varepsilon \partial_t E_h \cdot \tau(t)(E_h - \pi_\tau E_h) \, dx \, dt
+ \int_I \int_K \varepsilon \partial_t E_h \cdot \tau(t) \pi_\tau E_h \, dx \, dt. \quad (18)
$$

We will now show that the first term on the right-hand side of (18) is non-positive. Due to the space-time tensor product construction of the local finite element space, we can expand the discrete solution and the projection error as

$$
E_h = \sum_{i=0}^{p_s} \sum_{k=1}^{N_s} L_i(\xi) \varphi_k(x, y, z) e_{i,k}
$$

$$
E_h - \pi_\tau E_h = \sum_{k=1}^{N_s} L_{p_s}(\xi) \varphi_k(x, y, z) e_{p_s,k} \quad (19)
$$

with $\xi = 2(t - t_k)/\Delta t - 1$ and $N_s = 3(p_s + 1)(p_y + 1)(p_z + 1)$. Using once more the projection property, inserting the expansions (19) we obtain

$$
\int_{t_k}^{t} \int_K \varepsilon \partial_t E_h \cdot \tau(t)(E_h - \pi_\tau E_h) \, dx \, dt = -\varepsilon \Delta t \int_{t_k}^{t} \int_K \partial_t E_h \cdot t(E_h - \pi_\tau E_h) \, dx \, dt
$$

$$
= -\varepsilon \Delta t \sum_{i=1}^{p_s} \sum_{j=1}^{N_s} \sum_{l=1}^{N_s} \int_{t_k + \Delta t(1 + \xi)/2}^{t_{k+1}} L_i'(\xi) L_{p_s}(\xi) \, d\xi
$$

$$
\times \int_K \varphi_j(x, y, z) \cdot \varphi_l(x, y, z) \, dx e_{i,j} e_{p_s,l}
$$

$$
= -\varepsilon \Delta t \sum_{i=1}^{p_s} \int_{t_k}^{t} \Delta t^2 \sum_{j=1}^{N_s} \int_K \varphi_j(x, y, z) \cdot \varphi_l(x, y, z) \, dx e_{i,j} e_{p_s,l}.
$$

In the last step we have used the recurrence relation for the derivatives of Legendre polynomials. Finally, using the third recurrence relation in (17), we obtain
by the orthogonality of the Legendre polynomials

\[
\int_{I_k} \int_{K} \varepsilon \partial_t \mathbf{E}_h \cdot \tau(t) (\mathbf{E}_h - \pi_T \mathbf{E}_h) \, dx \, dt
\]

\[= -\left( \varepsilon / 2 \right) \int_{-1}^{1} L_{p_k}(\xi)^2 \frac{d\xi}{\xi} \sum_{j=1}^{N_x} \sum_{l=1}^{N_x} \int_{K} \varphi_j(x, y, z) \cdot \varphi_l(x, y, z) \, dx \, e_{p_{n+1}j} e_{p_{l+1}l} \leq 0
\]

Denoting the dual norm on the discrete test space by

\[
\|\ell\|_{W_h^*} := \sup_{v \in W_h} \frac{|\ell(v)|}{\|v\|_{0, I \times \Omega}},
\]

we can show

**Lemma 3.3.** Provided, the temporal polynomial degree \( p_t \) is constant and no local \( h \)-refinement with respect to time is present in the discretization, for elementwise constant \( \varepsilon \geq \varnothing > 0 \) and \( \mu \geq \underline{\mu} > 0 \) there holds

\[
\|\varepsilon \mathbf{E}_h\|_{0, I \times \Omega}^2 + \|\mu \mathbf{H}_h\|_{0, I \times \Omega}^2 \leq 4 \Delta t^2 \left( \varnothing^{-1} \|\varepsilon\|_{W_h^*}^2 + \underline{\mu}^{-1} \|\varepsilon\|_{W_h^*}^2 \right) + 4 \Delta t \mathcal{E}(t_k)
\]

(20)

**Proof.** Choosing \( v = \pi_T(t) \pi_T \mathbf{E}_h, w = \pi_T(t) \pi_T \mathbf{H}_h \) in (11) yields

\[
\int_{I} \int_{\Omega} \varepsilon \partial_t \mathbf{E}_h \cdot \pi_T(t) \pi_T \mathbf{E}_h \, dx \, dt + \int_{I} \int_{\Omega} \mu \partial_t \mathbf{H}_h \cdot \pi_T(t) \pi_T \mathbf{H}_h \, dx \, dt
\]

\[- \int_{I} \int_{\Omega} \mathbf{H}_h \cdot \nabla_h \pi_T(t) \pi_T \mathbf{E}_h \, dx \, dt + \int_{I} \int_{\Omega} \nabla_h \times \mathbf{E}_h \cdot \pi_T(t) \pi_T \mathbf{H}_h \, dx \, dt
\]

\[+ \int_{I} \int_{\mathcal{F}_0 \cup \mathcal{F}_b} \{ \mathbf{H}_h \} \cdot [\pi_T(t) \pi_T \mathbf{E}_h]_T \, dS \, dt - \int_{I} \int_{\mathcal{F}_0 \cup \mathcal{F}_b} \{ \mathbf{E}_h \}_T \cdot [\pi_T(t) \pi_T \mathbf{H}_h] \, dS \, dt
\]

\[= \ell_E(\pi_T(t) \pi_T \mathbf{E}_h) + \ell_H(\pi_T(t) \pi_T \mathbf{H}_h).
\]

(21)

Note that the terms corresponding to the stabilization term are not present, since we do not consider local refinement with respect to time. Following the line of arguments of the proof of Theorem 3.1, all terms, except the first two terms on the left-hand side of (21) vanish.

Applying Lemma 3.2, the Cauchy-Schwarz inequality, the arithmetic-geometric-
mean inequality yields
\[
\frac{1}{2\Delta t} \left( \|\varepsilon E_h\|_{0,\mathcal{I} \times \Omega}^2 + \|\varepsilon H_h\|_{0,\mathcal{I} \times \Omega}^2 \right) \leq \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \mathcal{E}(t_k)
\]
\[
\leq \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \mathcal{E}(t_k)
\]
\[
\leq \Delta t \left( \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)}^2 + \mu^{-1} \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)}^2 \right) + \mathcal{E}(t_k)
\]
\[
+ \frac{\varepsilon}{4\Delta t} \|\pi_\tau(t) H_h\|_{0,\mathcal{I} \times \Omega}^2 + \frac{\mu}{4\Delta t} \|\pi_\tau(t) H_h\|_{0,\mathcal{I} \times \Omega}^2 + \mathcal{E}(t_k)
\]
\[
\leq \Delta t \left( \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)}^2 + \mu^{-1} \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)}^2 \right) + \mathcal{E}(t_k)
\]
\[
+ \frac{1}{4\Delta t} \left( \|\varepsilon \pi_\tau(t) E_h\|_{0,\mathcal{I} \times \Omega}^2 + \|\mu \pi_\tau(t) H_h\|_{0,\mathcal{I} \times \Omega}^2 \right).
\]
Since \( \|\pi_\tau\| \leq 1 \) and \( 1 \geq \tau(t) \geq 0 \) on \( \mathcal{I} \), we obtain the result. \( \square \)

**Theorem 3.4.** Provided the temporal polynomial degree \( p_t \) is constant and no local \( h \)-refinement with respect to time is present in the discretization there holds for elementwise constant \( \varepsilon \geq \varepsilon > 0 \) and \( \mu \geq \mu > 0 \)

\[
\mathcal{E}(t_n) \leq 2\mathcal{E}(t_1) + 2 \sum_{k=1}^{n} \left( \left( \frac{2t_n}{\varepsilon} \right)^{\frac{1}{2}} \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \left( \frac{2t_n}{\mu} \right)^{\frac{1}{2}} \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)} \right)
\]

\[
(22)
\]

**Proof.** The proof follows along the lines of [12] Corollary 1. Denoting \( n_{max} = \arg \max_k \mathcal{E}(t_k) \), we have by (13) and the Cauchy-Schwarz inequality, the arithmetic geometric mean inequality, (20)

\[
\max_k \mathcal{E}(t_k) \leq \mathcal{E}(t_1) + \sum_{k=1}^{n_{max}} \left( \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)} \right)
\]

\[
\leq \mathcal{E}(t_1) + \sum_{k=1}^{n_{max}} \left( \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \left( \frac{\delta}{\mu} \right)^{\frac{1}{2}} \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)} \right)
\]

\[
\leq \mathcal{E}(t_1) + \sum_{k=1}^{n_{max}} \left( \left( \frac{\delta}{\varepsilon} \right)^{\frac{1}{2}} \|E_E\|_{W_{h'}^1(\mathcal{I} \times \Omega)} + \left( \frac{\delta}{\mu} \right)^{\frac{1}{2}} \|E_H\|_{W_{h'}^1(\mathcal{I} \times \Omega)} \right)
\]

Choosing \( \delta = 2t_{n_{max}} \) we have

\[
\sum_{k=1}^{n_{max}} \frac{\Delta t}{2t_{n_{max}}} \mathcal{E}(t_k) \leq \frac{1}{2} \max_k \mathcal{E}(t_k),
\]

and thus the estimate for \( \mathcal{E}(t_n) \), which can be applied to the corresponding term in (20). \( \square \)

**4. Implementation**

For each time slab, (11) yields a linear system of equations. Since especially for three-dimensional problems, the direct solution becomes unfeasible
due to the large number of unknowns and high memory demands of sparse direct solvers, we resort to an iterative solution of (11). Rather than assembling a matrix we implement the evaluation of the residual
\[ R_h(V) = B_h(U_h, V) - L(V) \] (23)
directly. This is in particular advantageous in the context of adaptivity, where the discretization may change from time slab to time slab.
Furthermore under the assumptions of Theorem 3.4, we can derive a guaranteed error bound on the iteration error. This allows to balance discretization and iteration error, leading to greatly reduced computational costs. The resulting method will, from the computational point of view behave similarly as explicit methods.

4.1. Basis functions for trial- and testspace
In order to allow for the efficient evaluation of the residual, we chose the tensor product basis functions
\[ \hat{v}_{ijkl} = l_i(t_k) \varphi_{ijkl}(\hat{x}, \hat{y}, \hat{z}) \] (24)
for the local space \( V^k_{h,K} \). Here \( L_i(t) \) denote the orthonormal Legendre polynomials on \([0, 1]\) and \( l_i(t) \) the integrated Legendre polynomials on \([0, 1]\):
\[ l_0(\xi) = 1 - \xi, \quad l_1(\xi) = \xi, \quad l_i(\xi) = \int_0^\xi L_{i-1}(s)ds \quad i = 2, ..., p \]
Thus, the approximate solution in space-time element \( I^K_k \times K \) can be expanded as \( E_h \big| I^K_k \times K = D F T v_{ijkl} e_{ijkl} \), where \( e_{ijkl} \) denotes the coefficients.

For the local test-spaces \( W^k_{h,K} \), the basis is chosen to consist entirely of Legendre polynomials
\[ \hat{w}_{ijkl} = L_i(t_k) \varphi_{ijkl}(\hat{x}, \hat{y}, \hat{z}) \] (26)
\[ \varphi_{ijkl}(\hat{x}, \hat{y}, \hat{z}) = L_j(\hat{x})L_k(\hat{y})L_l(\hat{z}) e_c \]
i = 0, ..., \( p - 1 \), \( j = 0, ..., \( p_x \), \( k = 0, ..., \( p_y \), \( l = 0, ..., \( p_z \). (27)

4.2. Efficient evaluation of the space-time residual
In the following, we outline how the above choices of basis for the trial and test spaces lead to an efficient implementation of the space-time residual (23). In particular, for each space-time element, the residual can be evaluated with optimal complexity of \( O(p^4) \) operations in the case of affine elements and elementwise constant \( \varepsilon, \mu \) and \( O(p^5) \) operations for non-affine elements. In the following section we use index notation and summation convention.
Mass residual containing time derivatives

Because of the tensor product structure of the local basis functions we can factor the space-time integral as follows

\[
R^{\text{mass}}(v) = \int_{I^K} \int_K \varepsilon \partial_i \mathbf{E}_h \cdot v \, dx \, dt = \int_{I^K} \int_K \varepsilon \partial_i D F^{-T} \tilde{v}_{cijkl} \cdot D F^{-T} \tilde{v}_{dmnpq} \, dx \, dt \, e_{dmnpq}
\]

\[
= \int_{I^K} L_i(\xi) \partial_i l_m(\xi) d\xi \mathcal{M}_{cijkl, dmnpq} e_{dmnpq},
\]

\[
\mathcal{M}_{cijkl, dmnpq} = \int_K \varepsilon D F^{-T} \varphi_{cijkl}(\hat{x}, \hat{y}, \hat{z}) D F^{-T} \varphi_{dmnpq}(\hat{x}, \hat{y}, \hat{z}) |J| d\hat{x} d\hat{y} d\hat{z}
\]

Note, that we have \( \int_0^1 L_i(\xi) \partial_i l_m(\xi) d\xi = \delta_{im} \) for \( i > 1, m > 1 \) due to (26) and the orthogonality of the Legendre polynomials. Thus, the number of mass matrix \( \mathcal{M}_{cijkl, dmnpq} \) multiplications needed for evaluating the residual scales linearly with temporal polynomial degree \( p_t \). Each multiplication can be done with \( O(p^3) \) operations for affine and \( O(p^4) \) operations for non-affine elements using fast summation techniques [13].

Curl residual

\[
R^{\text{curl}}(v) = \int_{I^K} \int_K \nabla \times \mathbf{E}_h \cdot v \, dx \, dt \quad R^{\text{curl}}_{dmnpq} = \int_{I^K} L_i(\xi) l_m(\xi) d\xi C^{\text{vol}}_{cijkl, dmnpq}
\]

\[
C^{\text{vol}}_{cijkl, dmnpq} = \int_K \varphi_{cijkl}(\hat{x}, \hat{y}, \hat{z}) \cdot \nabla \times \varphi_{dmnpq}(\hat{x}, \hat{y}, \hat{z}) d\hat{x} d\hat{y} d\hat{z}
\]

Here, we also have linear complexity in \( p_t \) for the number of volume-curl evaluations, since for \( m > 2 \) there holds \( l_m(\xi) = (L_{m+1}(\xi) - L_{m-1}(\xi))/\sqrt{2m+1} \). Note that the curl can also be evaluated with \( O(p^4) \) operations using recurrence relations for the derivatives of the Legendre polynomials (see e.g. [14]), such that a total complexity of \( O(p^3) \) is obtained.

Flux terms

The flux terms are evaluated using fast summation techniques, such that in total \( O(p^3) \) operations are needed. For two space-time elements \( I^1 \times K^3, I^2 \times K^2 \) sharing the face \( f_t \times f_s, f_t = I^1 \cap I^2 \), \( f_s = \partial K^1 \cap \partial K^2 \), we consider the evaluation of the flux involving neighbor coupling. Note, that in particular we allow for nonconforming interfaces in space and time. Recalling that we have \( K^t = F_t([0, 1]^3), F^t = \tau_t([0, 1]) \) and we use a co-variant transform for the spatial variables, we can write

\[
R^{\text{flux}}(v) = \int_{f_t} \int_{f_s} \mathbf{E}_h \cdot (\mathbf{n}^1 \times \mathbf{\pi}^1) \, dS \, dt =
\]

\[
\int_{\tau_t^{-1}(f_t)} \int_{\tau_t^{-1}(f_s)} D F_1^T D F_2^{-T} \circ \psi_1^2 \hat{\mathbf{E}}_h \circ \psi_1^2 \cdot (\mathbf{n}^1 \times \mathbf{\pi}^1) \, d\hat{x} \, d\hat{y} \, dt.
\]
Here $\psi_1^2 : \tau_1^{-1}(I^1) \times F_1^{-1}(f_s) \to \tau_2^{-1}(I^2) \times F_2^{-1}(f_s)$ is the mapping from the reference coordinates of $I^1 \times K^1$ to those of $I^2 \times K^2$. The mappings $\psi_1^2$ and $DF_1^TDF_2^{-T}$ are constant [5] and in general affine linear with a scaled permutation matrix. Note, that the scaling part is different from the identity matrix for nonconforming interfaces only. Permutations in the spatial reference coordinates occur for general hexahedral meshes. However, for simplicity, we assume in the following that $\psi_2^1$ is of the form $(\hat{x}^2, \hat{y}^2, \hat{z}^2, \hat{t}^2) = (s_x\hat{x}^1 + b_x, s_y\hat{y}^1 + b_y, s_z\hat{z}^1 + b_z, s_t\hat{t}^1 + b_t)$, i.e. no coordinate permutations are present. We have for a face with (reference) normal $\hat{n}^1 = e_3$

$$\mathbf{E}_h = i(t)L_j(\hat{x})L_k(\hat{y})L_t(-1)c_{cijkl}e_c = i(t)L_j(\hat{x})L_k(\hat{y})e_{cijkl}e_c.$$  

(29)

Obviously, $c_{cijkl} = L_t(-1)c_{cijkl}$ can be computed with $O(p^4)$ operations. With $I_t = \tau_1^{-1}(f_t)$ and $I_x \times I_y = F_1^{-1}(f_s)$, using the tensor-product structure of the trial and testspace basis functions $\hat{v}_{cijkl}$ and $\hat{w}_{dmnop}$ respectively, (28) can be written as

$$R_{dmnop}^{flux} = L_p(1)[\pi_{t}]_{mq} \int_{I_t} l_i(s_i\hat{t}^1 + b_t)L_q(\hat{t}^1) d\hat{t}^1 \int_{I_x} L_j(s_x\hat{x}^1 + b_x)L_n(\hat{x}^1) d\hat{x}^1$$

$$\times \int_{I_y} L_k(s_y\hat{y}^1 + b_y)L_o(\hat{y}^1) d\hat{y}^1(\hat{n}^1 \times e_d) \cdot e_{cijkl}. $$

Thus, the flux-residual $R_{dmnop}^{flux}$ can be evaluated as

$$\text{Aux1}_{dno} = \int_{I_y} L_k(s_y\hat{y}^1 + b_y)L_o(\hat{y}^1) d\hat{y}^1(\hat{n}^1 \times e_d) \cdot e_{cijkl}$$

$$\text{Aux2}_{dno} = \int_{I_x} L_j(s_x\hat{x}^1 + b_x)L_n(\hat{x}^1) d\hat{x}^1 \text{Aux1}_{dno}$$

$$\text{Aux3}_{dno} = [\pi_{t}]_{mq} \int_{I_t} l_i(s_i\hat{t}^1 + b_t)L_q(\hat{t}^1) d\hat{t}^1 \text{Aux2}_{dno}$$

$$R_{dmnop}^{flux} = L_p(1)\text{Aux3}_{dno}.$$  

(30)

The complexity of each summation in (30) is $O(p^4)$, in the case of a nonconforming interface in the respective direction. If in contrast the interface is conforming, the summation can be skipped due to the orthogonality properties of the trial and test basis functions.

In total the flux-residuals can be evaluated with $O(p^4)$ operations. Note that the same applies to non-coupling terms also.

4.3. Inexact iterative solution - guaranteed iteration error bound

Solving the problem exactly can be very expensive. Instead we solve the problem inexactly and control the error introduced by the inexact solution.

Noting that the iteration error at solver iteration $n$ for timeslab $k$ fullfills

$$B_h(U^n_h - U_h, V) = R^n_{h,k}(V).$$

13
we can apply the stability estimate Theorem 3.4 in order to obtain the guaranteed bound on the errors $e^n = E^n_h - E_h$ and $h^n = H^n_h - H_h$ at time slab $I_N \times \Omega$

$$\| e^n(t_{N+1}) \|_{0, \Omega}^2 + \| h^n(t_{N+1}) \|_{0, \Omega}^2 \leq 4 \sum_{k=1}^{N} \left( \frac{2t_N}{\varepsilon} + \frac{\Delta t^2}{2t_N \varepsilon} \right) \| R^n_{k,E} \|_{W_h}^2 + \frac{2t_N}{\mu} + \frac{\Delta t^2}{2t_N \mu} \| R^n_{k,H} \|_{W_h}^2 \right).$$ (31)

The dual norm of the residual can be evaluated exactly by computing its Riesz-representor, which in this case, is just the application of a $L_2$-projection operator due to the entirely discontinuous test space.

5. Numerical Experiments

The first two numerical experiments feature basic academic examples for assessing convergence properties when local refinement, especially with respect to time, is present in the discretization.

5.1. Convergence tests on non-adaptive discretizations

5.1.1. TM Mode $hp$-discretization

The initial data is chosen such that the TM$_{mn}$ mode is approximated in $\Omega = [0,1] \times [0,1] \times [0,1/5]$. We choose $m = n = 1$ leading to frequency $\omega = \pi \sqrt{m^2 + n^2}$. The exact solution for the electric field is:

$$E = \sin(m \pi x) \sin(n \pi y) \cos(\omega t) e_z$$ (32)

The time traces of the spatial $L_2$-error are depicted for 200 periods of the solution in Fig. 4. Furthermore, in Fig. 4 one can observe exponential convergence in the norm $\| \cdot \|_{0,T \times \Omega}$.
5.1.2. Local $h$-refinement in time

In order to exploit the temporal accuracy of the method in the case of local $h$-refinement in time direction, the source term and boundary conditions are chosen such that the exact solution [15] is

\[
\begin{align*}
E &= e^t x(x-1)z(1-z)e_y, \\
H &= e^t x(x-1)(1-2z)e_x - e^t(2x-1)z(1-z)e_z.
\end{align*}
\]

The problem is solved on the space-time domain $I$ with $I = [0, 5], \Omega = [0, 1]^3$, which is subdivided in time slabs $\Delta t \times \Omega$. The spatial mesh and the temporal refinement level associated with each spatial cell is shown in Fig. 5. The solution is approximated with quadratic polynomials in the $x-z$ directions, such that the temporal error is expected to be dominating. In the temporal direction the polynomial degree is set to $p_t$. In Fig. 6 left, one can observe that the error in the norm $\| \cdot \|_{0, I \times \Omega}$ is of order $p_t + 1$. If we instead consider the quantity $\max_k \| U(t_k) - U_h(t_k) \|_{0, \Omega}$ evaluated at the end of each time slab, an order of $2p_t$ can be observed in Fig. 6 right. For cG time stepping schemes this nodal superconvergence behaviour is reported in [16].
5.1.3. Broadband pulse in a waveguide

We consider a broadband pulse in a coaxial waveguide. The exact solution is

\[
E = \frac{1}{r} e^{-(\pi(f_2-f_1)(z-t)/2)^2} \sin(\pi(f_1 + f_2)(z - t)) e_r
\]

\[
H = \frac{1}{r} e^{-(\pi(f_2-f_1)(z-t)/2)^2} \sin(\pi(f_1 + f_2)(z - t)) e_\varphi.
\]

Here \(r\), \(\varphi\) and \(z\) denote the radial, azimuthal and axial coordinates respectively and \(e_r\), \(e_\varphi\) and \(e_z\) the corresponding unit vectors. The mid-frequency \((f_1 + f_2)/2\) is chosen such that the corresponding wavelength is approximately 1/16 the length of the waveguide. The spatial mesh consists of 1600 hexahedra with an
edge-length of approximately $1/3\lambda$. The global time step is fixed, such that in total 400 time steps are necessary to propagate the pulse through the entire waveguide. In Fig. 7 we show the time trace of the spatial $L_2$-errors. The error exhibits an odd even pattern, the origin of which is yet unknown.

![Figure 7: Temporal evolution of the spatial $L_2$-error for $p = 2, 3, 4, 5, 6$](image)

5.1.4. Bi-static RCS, metallic sphere - global $p$-refinement

A metallic sphere of radius $a = 1$ is illuminated by a $e_z$-polarized plane wave traveling in $e_z$-direction with wave number $\omega = 2\pi$. We chose $\Omega$ as a spherical shell with outer radius $R = 5$. For the outer boundary a Silver-Müller boundary condition is applied. The spatial grid consists of only 576 hexahedral elements with about 2 elements per wavelength at the surface of the scatterer and one element per wavelength at the absorbing boundary. The polynomial degrees are chosen as $p = p_t = p_x = p_y = p_z = p_{geo}$, where $p_{geo}$ is the polynomial degree of the elemental mapping $F$. We compute the bi-static RCS

$$\sigma(\phi, \theta) = \lim_{r \to \infty} 4\pi r^2 \frac{|E_{sc}(r, \phi, \theta)|^2}{|E_{inc}(r, \phi, \theta)|^2}$$

Here $r$ denotes the distance from the center of the sphere to the point of observation $\mathbf{r}$, $\phi$ and $\theta$ the azimuthal and polar angle between the wave and $\mathbf{x}$. The scattered far field is evaluated at a closed surface $S$, one wavelength away from the scatterer by evaluating the near-to-far field transformation

$$E_{sc} = \frac{e^{-i\omega r}}{4\pi r} \int_{t_0}^{t_1} \int_S \left[ \mathbf{x} \times (\mathbf{x} \times (\mathbf{n} \times \mathbf{H}_h)) + \mathbf{x} \times (\mathbf{E}_h \times \mathbf{n}) \right] e^{i\omega\mathbf{x} \cdot \mathbf{y} - i\omega t} dS(y) dt$$

with Gauss quadrature of sufficiently high order. The computed RCS in Fig. 8 converge quickly to the analytical Mie-series solution.
5.2. Space-time hp-adaptive examples

In order to demonstrate the method’s capability for space-time hp-refinement, we propose an adaptive algorithm, which extends the concept of reference solutions [7, 17] to the space-time context.

- First a global time step $\Delta t$, resulting in equally sized time slabs, is chosen.

- Given a time slab $\Delta t \times \Omega$, an initial mesh $\mathcal{S}_0$ and polynomial-degree distribution $p_0$.

- Define a coarse discretization $(\mathcal{S}_H, p_H) = (\mathcal{S}_0, p_0)$ and a refined discretization $(\mathcal{S}_h, p_h)$ with finite element spaces $V_H$ and $V_h$ respectively. The refined discretization is obtained by isotropically refining all space-time elements and increasing the polynomial degrees by one. Note that $V_H$ is contained in $V_h$.

Then the following steps are performed iteratively

1. **SOLVE**: Solve the problem (11) on the coarse and fine discretizations.
2. **ESTIMATE**: Compute the error indicators $\eta(I \times K) = \|U_h - U_H\|_{I \times K}^2$ and the approximate error bound $\|U_h - U_H\|_{I \times \Omega}$, if $\|U_h - U_H\|_{I \times K} \leq TOL$ stop and proceed to the next time slab. Otherwise,
3. **MARK**: Apply a fixed fraction marking strategy [18] based on the indicators $\eta(I \times K)$
4. **REFINE**: For each marked $I \times K$, set up a list of refinement candidates $V_{H, I \times K}^{\text{co}}(I \times K, p_K)$. In particular, we allow for all combinations of the following modifications of the discretization parameters:
   - raise/decrease the polynomial degrees $p_x, p_y, p_z, p_t$
   - isotropically $h$-refine/derefine in the spatial directions
- increase/decrease the temporal refinement level

In case of a refinement, which yields new space-time elements, we restrict the number of candidates by choosing identical polynomial degrees for each new element.

Choose

\[
V_{H,I \times K} = \arg\min_{V_{H}^{\text{cand}}} \frac{\eta^{\text{cand}} - \eta}{\# \text{DOF}(V_{H}^{\text{cand}})}, \eta^{\text{cand}} = \|U_h - \Pi_{V_{H}^{\text{cand}}} U_h\|^2
\]

as the new local finite elements space, leading to a new global space \( V_H \). Build a new \( V_h \) from \( V_H \) and go back to SOLVE.

- Now, for the current time slab the final coarse and fine grid solutions have been obtained.

**Remark 1:** The initial data for coarse- and fine-grid solves for the current time slab is taken as the \( L_2 \)-projection of the refined solution \( U_h \) from the previous time slab. Note that in the case of spatial derefinement with respect to the previous time slab, i.e. if the spatial part of the finite element space is not contained in the current one, there will be dissipation introduced by the projection. See [19], for a more detailed discussion. However the amount of dissipation introduced seemed to be negligible compared to the total error for the examples we have considered.

**Remark 2:** If an iterative solver is applied, we can choose the coarse grid solution as starting point for solving the fine grid problem. We have observed, that this considerably cuts down the number of fine grid iterations.

### 5.2.1. Broadband pulse in coaxial waveguide

Figure 9: Visualization of tensor product polynomials with degrees \( p_x, p_y \)

We have solved the example from 5.1.3 with the \( hp \)-adaptive algorithm. In Fig. 10 we visualize the coarse grid \( hp \)-discretization for time slab 86, by showing from top to bottom the spatial polynomial discretization, using the
tensor product visualization depicted in Fig. 9, the temporal polynomial degree distribution and the magnitude of the electric field. In axial direction, where the pulse has greater variation, spatial polynomial degrees are chosen to be larger than in radial direction. The temporal polynomial degrees are raised in the area where the pulse is situated. The temporal refinement level was not raised in any of the time slabs. For the \( hp \)-adaptive solution we have obtained a relative fine-grid error in the norm \( \| \cdot \|_{0,\mathcal{I}\times\Omega} \) of \( 9.6e^{-3} \).

5.2.2. Scattering of a dielectric sphere

Now we consider the scattering of a Gaussian plane-wave by a dielectric sphere with \( \varepsilon = 4 \). The problem again was solved using the \( hp \)-adaptive algorithm. The discretization of time slabs 33, 65 and 124 (from top to bottom) is depicted in Fig. 11. The fact that the electric field has limited regularity at the material interface is reflected by the choice of moderate polynomial degrees in space and linear polynomial degrees in time and a comparatively small spatial
mesh size near the material interface. Refinement in the temporal direction was almost exclusively $h$-refinement, i.e. the temporal polynomial degree was almost exclusively equal to one. Thus we do not show the temporal polynomial degree distributions.

The bi-static RCS for $ka = 1$ is depicted in Fig. 12, where a relative $\ell_2$-error of $2.53 \cdot 10^{-2}$ was obtained.
Figure 11: $hp$-adaptive simulation of the scattering from a dielectric sphere. From left to right: distribution of $|E_h|$, spatial polynomial degrees $(p_x, p_y, p_z)$ and temporal refinement level at timeslab 34, 66, 125 (from top to bottom)
Figure 12: Bistatic RCS of a dielectric sphere with $\varepsilon = 4$ and $ka=1$
6. Conclusions

We have devised a space-time Galerkin method, which allows for local \(hp\)-refinement in space and time by treating the spatial part of the discretization with a DG approach, whereas the temporal part is treated with a continuous Galerkin approach. The resulting implicit method can be shown to be non-dissipative, as long as the spatial part of the discretization is kept constant from time-slab to time-slab. We have shown, that the method can be implemented, such that the complexity of the residual evaluation for an iterative solution is \(O(p^8)\) for affine elements and \(O(p^5)\) for non affine elements. Furthermore, for the case of no local refinement with respect to time, we have devised an \textit{a posteriori} bound on the iteration error. Thus a balancing of the iteration and discretization errors is possible, provided that an \textit{a posteriori} estimate for the discretization error is available.

We have presented numerical experiments indicating that the method can be used for fully space-time \(hp\)-adaptive simulations. Further work should be devoted to the \textit{a priori} and \textit{a posteriori} error analysis, which is subject of ongoing work.

References

References

[1] J. Van der Vegt, H. Van der Ven, Space–time discontinuous galerkin finite element method with dynamic grid motion for inviscid compressible flows: I. general formulation, Journal of Computational Physics 182 (2) (2002) 546–585.

[2] C. Schwab, p-and hp-Finite Element Methods: Theory and Applications to Solid and Fluid Mechanics, Oxford University Press, USA, 1999.

[3] J. Hesthaven, T. Warburton, Nodal high-order methods on unstructured grids, Journal of Computational Physics 181 (1) (2002) 186–221. doi:10.1006/jcph.2002.7118. URL http://linkinghub.elsevier.com/retrieve/pii/S0021999102971184

[4] L. Fezoui, S. Lanteri, S. Lohrengel, S. Piperno, Convergence and stability of a discontinuous galerkin time-domain method for the 3D heterogeneous maxwell equations on unstructured meshes, ESAIM: Mathematical Modelling and Numerical Analysis 39 (6) (2005) 1149–1176. doi:10.1051/m2an:2005049. URL http://www.esaim-m2an.org/10.1051/m2an:2005049

[5] G. Cohen, X. Ferrieres, S. Pernet, A spatial high-order hexahedral discontinuous galerkin method to solve maxwells equations in time domain, Journal of Computational Physics 217 (2) (2006) 340–363. doi:10.1016/j.jcp.2006.01.004. URL http://linkinghub.elsevier.com/retrieve/pii/S0021999106000131
[6] L. Demkowicz, Computing with Hp-adaptive Finite Elements: One and two dimensional elliptic and Maxwell problems, Vol. 1, CRC Press, 2006.

[7] L. Demkowicz, J. Kurtz, D. Pardo, M. Paszenski, W. Rachowicz, A. Zdunek, Computing with hp-ADAPTIVE FINITE ELEMENTS: Volume II Frontiers: Three Dimensional Elliptic and Maxwell Problems with Applications, Vol. 2, Chapman & Hall/CRC, 2007.

[8] S. Piperno, Symplectic local time-stepping in non-dissipative DGTD methods applied to wave propagation problems, ESAIM: Mathematical Modelling and Numerical Analysis 40 (05) (2006) 815–841.

[9] A. Taube, M. Dumbser, C.-D. Munz, R. Schneider, A high-order discontinuous galerkin method with time-accurate local time stepping for the maxwell equations, International Journal of Numerical Modelling: Electronic Networks, Devices and Fields 22 (1) (2009) 77–103.

[10] V. Dolean, H. Fals, L. Fezoui, S. Lanteri, Locally implicit discontinuous galerkin method for time domain electromagnetics, Journal of Computational Physics 229 (2) (2010) 512–526. doi:10.1016/j.jcp.2009.09.038. URL http://linkinghub.elsevier.com/retrieve/pii/S0021999109005300

[11] S. Descombes, S. Lanteri, L. Moya, Locally implicit time integration strategies in a discontinuous galerkin method for maxwells equations, Journal of Scientific Computing (2012) 1–29doi:10.1007/s10915-012-9669-5. URL http://dx.doi.org/10.1007/s10915-012-9669-5

[12] R. Griesmaier, P. Monk, Discretization of the wave equation using continuous elements in time and a hybridizable discontinuous galerkin method in space, Journal of Scientific Computing (2013) 1–27doi:10.1007/s10915-013-9741-9. URL http://dx.doi.org/10.1007/s10915-013-9741-9

[13] J. Melenk, K. Gerdes, C. Schwab, Fully discrete $hp$-finite elements: fast quadrature, Computer methods in applied mechanics and engineering 190 (32) (2001) 4339–4364.

[14] C. Koutschan, C. Lehrenfeld, J. Schöberl, Computer algebra meets finite elements: An efficient implementation for maxwells equations, Numerical and Symbolic Scientific Computing: Progress and Prospects 1 (2011) 105.

[15] J. G. Verwer, Composition methods, maxwell’s equations, and source terms, SIAM Journal on Numerical Analysis 50 (2) (2012) 439–457. doi:10.1137/100816122. URL http://epubs.siam.org/doi/abs/10.1137/100816122

[16] G. Akrivis, C. Makridakis, R. H. Nochetto, Galerkin and RungeKutta methods: unified formulation, a posteriori error estimates and nodal superconvergence, Numerische Mathematik 118 (3) (2011) 429–456.
Acknowledgements

The work of M. Lilienthal is supported by the ‘Excellence Initiative’ of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universität Darmstadt and the DFG under grant no. DFG WE 1239/27-2.

S.M. Schnepp acknowledges the support of the Alexander von Humboldt Foundation through a Feodor Lynen-Research Fellowship.

We thank Jens Niegemann for providing code for the computation of the Mie-series solutions.