PARALLEL TRANSPORT ON PRINCIPAL BUNDLES OVER STACKS

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Abstract. In this paper we introduce a notion of parallel transport for principal bundles with connections over differentiable stacks. We show that principal bundles with connections over stacks can be recovered from their parallel transport thereby extending the results of Barrett, Caetano and Picken, and Schreiber and Waldorf from manifolds to stacks.

In the process of proving our main result we simplify Schreiber and Waldorf’s original definition of a transport functor for principal bundles with connections over manifolds and provide a more direct proof of the correspondence between principal bundles with connections and transport functors.

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1. Introduction

Let $G$ be a Lie group and $M$ a $C^\infty$ manifold. Recall that a choice of a connection 1-form $A \in \Omega^1(P, g)^G$ on a principal $G$-bundle $P$ over the manifold $M$ and a choice of a base point $x \in M$ gives rise to the holonomy map

$$\Omega(M, x) \to \text{Aut(fiber of } P \text{ at } x) \cong G,$$

where $\Omega(M, x)$ is the set of smooth loops at $x$ in $M$. For a connected manifold $M$ holonomy map uniquely determines the connection $A$ and, in fact, the bundle $P$ itself [12]. If two loops in $\Omega(M, x)$
differ by a homotopy that sweeps no area, a so-called “thin homotopy,” then their holonomies are the same. Therefore the holonomy map descends to a well defined map on the quotient
\[ \mathcal{H} : \Omega(M, x)/\sim \to G, \]
where \( \sim \) means “identify thinly homotopic loops.” The quotient \( \pi_1^{\text{thin}}(M, x) := \Omega(M, x)/\sim \) is a group and \( \mathcal{H} \) is a homomorphism. Moreover \( \pi_1^{\text{thin}}(M, x) \) has a smooth structure — it is a diffeological group (see Appendix A and Remark 2.13) — and \( \mathcal{H} \) is smooth. Barrett [2], motivated by questions coming from general relativity and Yang-Mills theory, proved that a homomorphism
\[ T : \pi_1^{\text{thin}}(M, x) \to G \]
is defined by parallel transport on some principal \( G \)-bundle with connection if and only if \( T \) is smooth. More precisely, he proved that assigning parallel transport homomorphisms to a principal bundle with a connection induces a bijection of sets:
\[
(\text{principal bundles with connections over } M)/\text{isomorphisms} \quad \leftrightarrow \quad (\text{smooth homomorphisms } \pi_1^{\text{thin}}(M, x) \to G)/\text{conjugation}.
\]
Barrett’s proofs were simplified by Caetano and Picken [7]. Wood [23] reformulated Barrett’s theorem in terms of the groupoids of paths in \( M \); this obviates the need to choose a base point. Schreiber and Waldorf [20] categorified Wood’s version of Barrett’s theorem. They showed that assigning holonomy to a bundle defines an equivalence of categories
\[ \text{hol}_M : B^\nabla G(M) \to \text{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G-\text{tor}). \]
Here, and in the rest of the paper, \( B^\nabla G(M) \) denotes the category of principal \( G \)-bundles with connections over a manifold \( M \), \( \Pi^{\text{thin}}(M) \) is the thin fundamental groupoid of \( M \) (see Definition/Proposition 2.9), \( G-\text{tor} \) is the category of \( G \)-torsors (Definition 3.2) and \( \text{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G-\text{tor}) \) denotes a category of functors that are smooth in an appropriate sense (see Definition 3.5). Schreiber and Waldorf’s definition of \( \text{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G-\text{tor}) \) is fairly involved and the proof that \( \text{hol}_M \) is an equivalence of categories is indirect. Nor is it clear if the equivalence \( \text{hol}_M \) is natural in the manifold \( M \).

In this paper we propose a simple definition of what it means for a functor \( T : \Pi^{\text{thin}}(M) \to G-\text{tor} \) to be smooth. We refer to such smooth functors as transport functors. We shorten our notation by setting
\[ \text{Trans}_G(M) := \text{Hom}_{C^\infty}(\Pi^{\text{thin}}(M), G-\text{tor}); \]
\( \text{Trans}_G(M) \) is a category whose objects are parallel transport functors and morphisms are natural transformations (see Definition 3.16). We provide a sanity check by showing that the parallel transport functor \( \text{hol}_M(P, A) \) defined by a connection \( A \) on a principal \( G \)-bundle \( P \to M \) is smooth in the sense of this paper. We then prove that for a manifold \( M \) the functor \( \text{hol}_M \) is an equivalence of categories (Theorem 4.1). This part of the paper does not require any knowledge of stacks.

In the second part of the paper we assume that the reader is familiar with stacks over the site \( \text{Man} \) of manifolds. The standard references are Behrend and Xu [3], Heinloth [10] and Metzler [18].

We first prove that the assignment \( M \mapsto \text{Trans}_G(M) \) extends to a contravariant functor \( \text{Trans}_G : \text{Man}^{op} \to \text{Groupoid} \) (Lemma 5.1). By a Grothendieck construction, the presheaf \( \text{Trans}_G \) defines a category fibered in groupoids (CFG) \( \text{Trans}_G \to \text{Man} \). The collection of functors
\[ \{\text{hol}_M : B^\nabla G(M) \to \text{Trans}_G(M)\}_{M \in \text{Man}} \]
extends to a morphism of CFGs \( \text{hol} : B^\nabla G \to \text{Trans}_G \) (Lemma 5.3). Since each functor \( \text{hol}_M \) is an equivalence of categories, the functor \( \text{hol} \) is an equivalence (Theorem 5.4). Consequently, since
$B^\nabla G$ is a stack, so is $\text{Trans}_G$ (Corollary 5.5). Together the two results imply one of the main results of the paper:

$$\text{hol} : B^\nabla G \rightarrow \text{Trans}_G$$

is an isomorphism of stacks. In section 6 we work out some consequence of Theorem 5.4 for principal bundles with connections over stacks. We start by recalling a definition of a principal $G$-bundle over a stack $\mathcal{X}$: it is a functor $P : \mathcal{X} \rightarrow BG$, where $BG$ denotes the stack of principal $G$ bundles. By analogy we introduce the notion of a principal bundle with connection and of a transport functor over a stack $\mathcal{X}$. Note that we do not assume that $\mathcal{X}$ necessarily has an atlas. As an immediate consequence of Theorem 5.4, we obtain that for each stack $\mathcal{X}$ the functor $\text{hol}$ induces an equivalence of categories between the categories of principal bundles with connections over $\mathcal{X}$ and transport functors over $\mathcal{X}$ (Theorem 6.4). We then recall that for a CFG $\mathcal{X} \rightarrow \text{Man}$ and a Lie groupoid $\Gamma$, there is the category $\mathcal{X}(\Gamma)$ of cocycles with values in $\mathcal{X}$. We discuss the fact that the cocycle category $\mathcal{X}(\Gamma)$ is equivalent to the functor category $[\Gamma_0/\Gamma_1, \mathcal{X}]$ (Proposition 6.6). Here and elsewhere in the paper $[\Gamma_0/\Gamma_1]$ denotes the stack quotient of the Lie groupoid $\Gamma$. We end Part 2 of the paper by reformulating Theorem 6.4 in terms of the cocycle categories: for any Lie groupoid $\Gamma$ the isomorphism of stacks $\text{hol} : B^\nabla G \rightarrow \text{Trans}_G$ induces an equivalence $\text{hol}_\Gamma : B^\nabla G(\Gamma) \rightarrow \text{Trans}_G(\Gamma)$ of the cocycle categories (Theorem 6.7).

The paper has two appendices. In Appendix A we review the definition of a diffeological space both from a traditional point of view and as a concrete sheaf of sets. We prove the folklore result that the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ of a manifold $M$ is a diffeological groupoid. We also prove two technical results that are needed elsewhere in the paper. We show that the target map $t$ of the thin fundamental groupoid has local sections (Lemma A.26). We prove that the assignment $M \mapsto \Pi^{\text{thin}}(M)$ extends to a functor $\Pi^{\text{thin}} : \text{Man} \rightarrow \text{DiffGpd}$ from the category of manifolds to the category $\text{DiffGpd}$ of diffeological groupoids. In Appendix B we prove that for any Lie groupoid $\Gamma$ an equivalence of CFGs $F : \mathcal{X} \rightarrow \mathcal{Y}$ induces an equivalence $F_{\Gamma} : \mathcal{X}(\Gamma) \rightarrow \mathcal{Y}(\Gamma)$ of the corresponding cocycle categories (Proposition 6.8).

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Part 1. Parallel transport for bundles over manifolds

2. Thin Homotopy and the thin fundamental groupoid

In this section, following Schreiber and Waldorf, we define the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ of a manifold $M$. Nothing in this section is new. Our purpose for presenting this material is to keep the paper self-contained and to fix notation. To start we recall the notion of a path with sitting instances of Caetano and Picken [7].

**Definition 2.1** (A path with sitting instances). Let $[a, b] \subset \mathbb{R}$ be a closed interval and $M$ a manifold. A smooth map $\gamma : [a, b] \rightarrow M$ is a path with sitting instances if $\gamma$ is constant on neighborhoods of $a$ and $b$.

**Remark 2.2.** It will be useful for us to fix a smooth non-decreasing map

$$\beta : [0, 1] \rightarrow [0, 1]$$

which takes the value 0 on all points sufficiently close to 0 and the value 1 for all points sufficiently close to 1 (that is, there is $\epsilon > 0$ so that $\beta(t) = 0$ for $t \in [0, \epsilon)$ and $\beta(t) = 1$ for $t \in (1 - \epsilon, 1]$). Given
any path \( \gamma : [0, 1] \to M \), \( \gamma \circ \beta \) is a path with sitting instances. Therefore, up to “reparameterization,” all paths on \( M \) are paths with sitting instances.

**Notation 2.3.** We denote the set of all paths with sitting instances from the interval \([0, 1]\) to a manifold \( M \) by \( \mathcal{P}(M) \):

\[
\mathcal{P}(M) := \{ \gamma : [0, 1] \to M \mid \gamma \text{ is a path with sitting instances} \}.
\]

A useful notion of homotopy fixing end points between two paths with sitting instances is that of a thin homotopy:

**Definition 2.4 (Thin homotopy).** Two paths \( \gamma_0, \gamma_1 : [0, 1] \to M \) with sitting instances and the same endpoints in a manifold \( M \) are thinly homotopic relative the endpoints if there is a smooth map \( H : [0, 1]^2 \to M \) with the following properties:

1. \( H \) is a smooth homotopy from \( \gamma_0 \) to \( \gamma_1 \) relative the endpoints:
   \[
   H(s, 0) = \gamma_0(s), \quad H(s, 1) = \gamma_1(s) \quad \text{for all } s \in [0, 1]
   \]
   and
   \[
   H(0, t) = \gamma_0(0) = \gamma_1(0), \quad H(1, t) = \gamma_0(1) = \gamma_1(1) \quad \text{for all } t \in [0, 1];
   \]
2. \( H \) is “thin”:
   \[
   \text{rank}(dH)(s,t) \leq 1
   \]
   for all \((s, t) \in [0, 1]^2\);
3. \( H \) has sitting instances near the boundary of the square: \( H(s, t) \) is constant in \( s \) for all \((s, t) \) near \([0, 1] \times [0, 1]\) and is constant in \( t \) for all \((s, t) \) near \([0, 1] \times \{0, 1\}\).

We refer to \( H \) as a thin homotopy from \( \gamma_0 \) to \( \gamma_1 \).

**Notation 2.5.** We write \( H : \gamma_0 \Rightarrow \gamma_1 \) to indicate that \( H \) is a thin homotopy from a path \( \gamma_0 \) to a path \( \gamma_1 \).

We make several comments about the notion of thin homotopy.

**Remark 2.6.**

1. Two thin homotopies can be easily pasted together (vertically or horizontally) to give rise to a new thin homotopy. Consequently “being thin homotopic” is an equivalence relation \( \sim \) on the space \( \mathcal{P}(M) \) of paths with sitting instances. The relation \( \sim \) is also compatible with concatenations.
2. The pullback by a thin homotopy of any differential 2-form is zero. Consequently if \((P, A) \to M\) is a principal bundle with connection and \( H : [0, 1]^2 \to M \) is a thin homotopy, then the pullback bundle \( H^*(P, A) \to [0, 1]^2 \) is flat. This, in turn, implies that parallel transport maps defined by two thinly homotopic curves are equal (see Proposition 3.12 below). In particular, when studying parallel transport along loops based at some point \( x_0 \), we may safely divide out by thin homotopy.
3. The collection of all loops at a point \( x_0 \in M \) parameterized by \([0, 1]\) do not form a group under concatenation: for example the composition is not associative. It does become associative once we divide out by thin homotopy. Thus, if we want to think of parallel transport along loops as a representation of the “group” of loops, we need to pass to the thin fundamental group \( \pi_1^{\text{thin}}(M, x) \) (see Definition 2.12 below).

**Notation 2.7 (\( \mathcal{P}(M)/\sim \)).** Since being thinly homotopic is an equivalence relation, the corresponding equivalence classes make sense. We denote the equivalence class of a path \( \gamma \) by \([\gamma]\). We denote the set of equivalence classes of paths in a manifold \( M \) by \( \mathcal{P}(M)/\sim \).

\footnote{The map \( \beta \) is not a reparametrization in the strict sense of the word since it is not invertible.}
Remark 2.8. In [7] thin homotopy is called “intimacy” to distinguish it from the notion of thin homotopy between piece-wise smooth paths introduced by Barrett [2]. The terminology of Definition 2.4 is now standard.

Definition/Proposition 2.9 (Thin fundamental groupoid $\Pi^{\text{thin}}(M)$). The concatenation of paths with sitting instances in a manifold $M$ descends to an associative multiplication of their thin homotopy classes. This multiplication gives rise to a groupoid $\Pi^{\text{thin}}(M)$ with the points of $M$ as objects and thin homotopy classes of paths as morphisms.

While Definition/Proposition 2.9 is familiar to experts, cf. for example [20, Lemma 2.3], we will recall the argument to keep this paper self-contained. First, we fix our notation for groupoids.

Notation 2.10. Let $\Gamma$ be a groupoid, that is, a category with all morphisms invertible. We denote its collection of objects by $\Gamma_0$ and its collection of arrows/morphisms by $\Gamma_1$. If $\gamma \coloneqq \gamma_2 \gamma_1$ is an arrow in $\Gamma$, (i.e., a morphism from an object $x$ to an object $y$) we say that $x$ is the source of $\gamma$, $y$ is the target and write $x = s(\gamma)$, $y = t(\gamma)$.

The collection of pairs of composable arrows of $\Gamma$ is the fiber product

$$\Gamma_1 \times_{s, \Gamma_0, t} \Gamma_1 \equiv \Gamma_1 \times_{\Gamma_0} \Gamma_1 := \{ (\gamma_2, \gamma_1) \in \Gamma_1 \times \Gamma_1 \mid s(\gamma_2) = t(\gamma_1) \}.$$ 

We denote the composition/multiplication in $\Gamma$ by $m$ by:

$$m : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1, \quad m(\gamma_2, \gamma_1) = \gamma_2 \gamma_1.$$ 

In particular, we write the composition from right to left. This will matter when we define multiplication in the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ of a manifold $M$. The inversion map is denoted by $i$:

$$i : \Gamma_1 \to \Gamma_1, \quad i(\gamma) := \gamma^{-1},$$ 

and the unit map is denoted by $u : \Gamma_0 \to \Gamma_1$.

Finally, we will often write $\Gamma = \{ \Gamma_1 \Rightarrow \Gamma_0 \}$ to single out the collections of arrows and objects of the groupoid $\Gamma$ together with the associated source and target maps. This suppresses the other three structure maps from notation.

Proof of Definition/Proposition 2.9. We define the set of objects of the groupoid $\Pi^{\text{thin}}(M)$ to be the set of points of the manifold $M$:

$$\Pi^{\text{thin}}(M)_0 := M.$$ 

We define the set of arrows of the groupoid $\Pi^{\text{thin}}(M)$ to be the set of thin homotopy classes of paths:

$$\Pi^{\text{thin}}(M)_1 := \mathcal{P}(M)/\sim.$$ 

The source and target maps are defined by assigning endpoints to classes of paths:

$$s([\gamma]) := \gamma(0) \quad t([\gamma]) := \gamma(1);$$ 

these maps are well-defined since thin homotopies fix endpoints. The unit map $u : M \to \mathcal{P}(M)/\sim$ assigns to each point $x \in M$ the class of the constant path $1_x(t) \equiv x$:

$$u(x) := [1_x] \quad \text{for all } x \in M.$$ 

Recall that if $\gamma : [0, 1] \to M$ is a path, its reversal $\gamma^{-1}$ is defined by $\gamma^{-1}(t) := \gamma(1 - t)$. We define the inversion map $i : \Pi^{\text{thin}}(M)_1 \to \Pi^{\text{thin}}(M)_1$ by

$$i([\gamma]) := [\gamma^{-1}].$$ 

The map $i$ is well-defined.
The multiplication $m$ in the groupoid $\Pi^{\text{thin}}(M)$ is defined by concatenating representatives of the equivalence classes of paths. If $\gamma, \tau : [0, 1] \to M$ are two paths with $\gamma(0) = \tau(1)$, define $\gamma \tau$ by

$$
\gamma \tau(t) := \begin{cases} 
\tau(2t) & \text{if } t \in [0, 1/2] \\
\gamma(2t - 1) & \text{if } t \in [1/2, 1]
\end{cases}.
$$

Note that since both $\gamma$ and $\tau$ are paths with sitting instances, their concatenation $\gamma \tau$ is $C^\infty$. This is one of the reasons why working with paths with sitting instances is so convenient. We then set

$$
m([\gamma], [\tau]) := [\gamma \tau]
$$

for all composable classes of paths $([\gamma], [\tau]) \in (\mathcal{P}(M)/\sim) \times_M (\mathcal{P}(M)/\sim)$. The map $m$ is well-defined since thin homotopies can be concatenated.

Finally one needs to check that the five maps $s, t, u, i, m$ defined above do define the structure of a groupoid on $\Pi^{\text{thin}}(M)$. That is, one needs to check that $s \circ u = id_M = t \circ u, s \circ i = t, m$ is associative and so on. This is straightforward. We omit the verification. □

Remark 2.11. In Proposition A.25 in Appendix A we check that $\Pi^{\text{thin}}(M)$ is a diffeological groupoid. This fact is well-known to experts. We haven’t been able to find a proof in literature.

Definition 2.12. Let $M$ be a manifold. We define the thin fundamental group $\pi_1^{\text{thin}}(M, x)$ of $M$ at a point $x \in M$ to be the automorphism group of $x$ in the groupoid $\Pi^{\text{thin}}(M)$:

$$
\pi_1^{\text{thin}}(M, x) := s^{-1}(x) \cap t(x);
$$

it is the group of thin homotopy classes of loops at $x$ in $M$.

Remark 2.13. Since for any manifold $M$, the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ is a diffeological groupoid, the automorphism groups $\pi_1^{\text{thin}}(M, x)$ are diffeological groups.

3. TRANSPORT FUNCTORS OVER MANIFOLDS

In this section we axiomatize parallel transport in principal bundles with connections by introducing the notion of a transport functor (Definition 3.5). We check that each principal bundle with connection $(P, A)$ over a manifold $M$ gives rise to a transport functor $\text{hol}_M(P, A)$ (Theorem 3.9). Transport functors over a manifold $M$ form a groupoid; we denote it by $\text{Trans}_G(M)$. We end the section by showing (Proposition 3.18) that the assignment of a transport functor to a principal bundle with a connection extends to a functor

$$
\text{hol}_M : B\nabla G(M) \to \text{Trans}_G(M).
$$

Definition 3.1. Let $G$ be a Lie group. A $G$-torsor is a manifold $X$ with a free and transitive right action of a Lie group $G$. In particular the map

$$
X \times G \to X \times X, \quad (x, g) \mapsto (x, x \cdot g)
$$

is a diffeomorphism.

Definition 3.2 (The category $G$-tors of $G$-torsors). Fix a Lie group $G$. The collection of all $G$-torsors form a category: by definition a morphism from a torsor $X$ to a torsor $Y$ is a $G$-equivariant map $f : X \to Y$.

We denote the category of $G$-torsors by $G$-tors. Note that $G$-tors is a groupoid. We denote the set of morphisms in $G$-tors from a torsor $X$ to a torsor $Y$ by $\text{Hom}_{G\text{-tors}}(X, Y)$.

Remark 3.3. Let $X$ be a $G$-torsor. Since the map (3.1) is a bijection, for every pair of points $(x, y) \in X \times X$ there is a unique element $d = d(x, y) \in G$ so that

$$
x \cdot d(x, y) = y.
$$

Since (3.1) is a diffeomorphism, the map $d : X \times X \to G$ defined by (3.2) is smooth.
Lemma 3.4. Let $G$ be a Lie group. For any $G$-torsor $X$ the group of automorphisms

$$\text{Aut}(X) := \text{Hom}_{G\text{-tors}}(X, X)$$

is canonically a Lie group.

Proof. A choice of a point $x \in X$ gives rise to a map

$$\psi_x : \text{Aut}(X) \to G, \quad \psi_x(f) := d(x, f(x))$$

where $d : X \times X \to G$ is the smooth map implicitly defined by

$$x \cdot d(x, y) = y$$

(q.v. Remark 3.3 above). It is not hard to check that $\psi_x$ is a group isomorphism. Hence $\psi_x$ gives $\text{Aut}(X)$ the structure of a Lie group.

If $y \in X$ is another choice of base point then $y = x \cdot b$ for some $b \in G$. It is easy to check that

$$\psi_{x,b} = b^{-1} \psi_x b.$$ 

Hence the Lie group structure on $\text{Aut}(X)$ does not depend on a choice of $x \in X$. \qed

Definition 3.5 (Transport functor). Let $M$ be a manifold. A functor $F : \Pi^{\text{thin}}(M) \to G\text{-tors}$ is a (parallel) transport functor if, for every point $x \in M$, the map $F|_{\pi^{\text{thin}}_1(M,x)} : \pi^{\text{thin}}_1(M,x) \to \text{Aut}(F(x))$ is a map of diffeological spaces (see Definition A.4).

Remark 3.6. Recall that the group $\pi^{\text{thin}}_1(M,x)$ is a diffeological group (see Remark 2.13) and the group $\text{Aut}(F(x))$ of automorphisms of the torsor $F(x)$ is a Lie group (see Lemma 3.4). Therefore, it makes sense to require that $F|_{\pi^{\text{thin}}_1(M,x)}$ is a map of diffeological spaces, that is, smooth.

Remark 3.7. One would have liked to define a functor $F : \Pi^{\text{thin}}(M) \to G\text{-tors}$ to be a transport functor if and only if it were a morphism of diffeological groupoids. Unfortunately the groupoid $G\text{-tors}$ has no natural diffeological structure even though it is equivalent (as a category) to the one object action groupoid $\{G \times \{\ast\} \rightrightarrows \{\ast\}\}$. The action groupoid $\{G \times \{\ast\} \rightrightarrows \{\ast\}\}$ is a Lie groupoid, and Definition 3.5, in effect, exploits this fact. The definition in [20] also makes use of this fact, but in a more complicated way. As a result the two notions of transport functors (ours and Schreiber and Waldorf’s) are difficult to compare directly. None less the two notions are equivalent: see [20, Theorem 3.12]. We thank Konrad Waldorf for pointing out this fact.

We now argue that our definition is a conservative extension of the notion of parallel transport defined by Barrett and by Caetano and Picken (op. cit.).

Lemma 3.8. Let $M$ be a manifold and $F : \Pi^{\text{thin}}(M) \to G\text{-tors}$ a functor. Suppose that there is a point $x \in M$ so that $F|_{\pi^{\text{thin}}_1(M,x)} : \pi^{\text{thin}}_1(M,x) \to \text{Aut}(F(x))$ is smooth. Then, for any point $y$ in the path component of $x$, the map $F|_{\pi^{\text{thin}}_1(M,y)} : \pi^{\text{thin}}_1(M,y) \to \text{Aut}(F(y))$ is smooth. In particular, if $M$ is connected, $F : \Pi^{\text{thin}}(M) \to G\text{-tors}$ is a transport functor if and only there exists one point $x \in M$ so that $F|_{\pi^{\text{thin}}_1(M,x)}$ is smooth.

Proof. Choose a path $\gamma : [0,1] \to M$ with sitting instances such that $\gamma(0) = x$, $\gamma(1) = y$. Then, since $\Pi^{\text{thin}}(M)$ is a diffeological groupoid, the equivalence class $[\gamma] \in \mathcal{P}(M)/\sim$ defines a smooth map

$$c_{[\gamma]} : \pi^{\text{thin}}_1(M,x) \to \pi^{\text{thin}}_1(M,y), \quad [\tau] \mapsto [\gamma]^{-1}[\tau][\gamma].$$

Similarly, we have the smooth map

$$c_{F([\gamma])} : \text{Aut}(F(x)) \to \text{Aut}(F(y)), \quad \varphi \mapsto F([\gamma])^{-1} \circ \varphi \circ F([\gamma]).$$
Since $F$ is a functor, the diagram

\[
\begin{array}{ccc}
\pi_1^{\text{thin}}(M,x) & \xrightarrow{c[\gamma]} & \pi_1^{\text{thin}}(M,y) \\
F \downarrow & & \downarrow F \\
\text{Aut}(F(x)) & \xrightarrow{c_F(\gamma)} & \text{Aut}(F(y))
\end{array}
\]

commutes. That is,

\[
F|_{\pi_1^{\text{thin}}(M,y)} = (c_F(\gamma))^{-1} \circ F|_{\pi_1^{\text{thin}}(M,x)} \circ c[\gamma].
\]

Hence, $F|_{\pi_1^{\text{thin}}(M,y)}$ is smooth. \qed

For Definition 3.5 to be reasonable, parallel transport in principal bundles with connections have to define transport functors; we now show that it does.

**Theorem 3.9.** A principal bundle with a connection $(P \xrightarrow{\pi} M, A \in \Omega^1(P, g)^G)$ over a manifold $M$ gives rise to a parallel transport functor

\[
\text{hol}_M(P, A) : \Pi^{\text{thin}}(M) \to G\text{-tors}
\]

It is defined by

\[
\text{hol}_M(P, A)(x [\gamma] \to y) := (||^A : P_x \to P_y)
\]  

(3.3)

for any arrow $(x [\gamma] \to y) \in \Pi^{\text{thin}}(M)$. Here $P_x$ denotes the fiber of the bundle $P \to M$ above $x \in M$ and $||^A$ denotes parallel transport along a path $\gamma$ defined by the connection $A$ (see Definition 3.10 below).

To prove Theorem 3.9 we need a proposition and two lemmas. We start with a definition to fix our notation (cf. [13, Proposition 3.1]).

**Definition 3.10.** Let $\pi : P \to M$ be a principal $G$-bundle with a connection 1-form $A$ and $\gamma : [a, b] \to M$ a path (a smooth curve). The **horizontal lift** of $\gamma$ with respect to $A$ is a curve $\gamma^A_z : [a, b] \to P$ with the following three properties:

1. $\pi \circ \gamma^A_z = \gamma$;
2. $\gamma^A_z(a) = z$; and
3. $\frac{d}{dt} \gamma^A_z(t) \in \ker (A_{\gamma^A_z(t)})$ for all $t \in [a, b]$.

The **parallel transport** along the path $\gamma : [a, b] \to M$ defined by $A$ is a $G$-equivariant diffeomorphism (i.e., a map of $G$-torsors)

\[
||^A : P_{\gamma(a)} \to P_{\gamma(b)}.
\]

(3.4)

It is defined by

\[
||^A(z) := \gamma^A_z(b).
\]

for all $z \in P_{\gamma(a)}$.

**Lemma 3.11** (Parallel transport pulls back). Let $(P \to M, A)$, $(P' \to M', A')$ be two principal $G$-bundles with connections and $f : P \to P'$ a $G$-equivariant map with $A = f^* A'$. Suppose also that $f$ covers $f' : M \to M'$. Then for any horizontal lift $\gamma^A$ of a curve $\gamma : [a, b] \to M$ with respect to $A$, the curve $f \circ \gamma^A$ is a horizontal lift of the curve $f' \circ \gamma : [a, b] \to M'$ with respect to $A'$. Consequently

\[
(f|_{P_x}) \circ ||^A = ||^{A'}_{f|_{P_y}} \circ (f|_{P_x}) : P_x \to P'_{f(y)},
\]  

(3.4)

where $x = \gamma(a)$ and $y = \gamma(b)$.

**Proof.** Let $\gamma^A_z : [a, b] \to P$ denote the horizontal lift of $\gamma$ starting at $z \in P_x$. Then, since $f^* A' = A$, the curve $f \circ \gamma^A_z : [a, b] \to P'$ is a horizontal lift of $f \circ \gamma$ starting at $f(z) \in P'_{f(x)}$. \qed
Proposition 3.12. Let $P \to M$ be a principal $G$-bundle with a connection 1-form $A$. Let $\gamma_0, \gamma_1 : [0,1] \to M$ be two paths with sitting instances and $H : [0,1]^2 \to M$ a thin homotopy from $\gamma_0$ to $\gamma_1$. Let $p \in P$ be a point in the fiber above $\gamma_0(0) = \gamma_1(0)$ and let $\tilde{\gamma}_0, \tilde{\gamma}_1$ be the corresponding horizontal lifts that start at $p$. Then

\[ \tilde{\gamma}_0(1) = \tilde{\gamma}_1(1). \]

Hence the parallel transport maps $||_{\gamma_0}||_{\gamma_1} : P_{\gamma_0(0)} \to P_{\gamma_1(1)}$ are equal.

Proof. Our proof is essentially that of [7, Section 6]). Pull back the principal bundle $P \to M$ to $[0,1]^2$ by the thin homotopy $H$. The result is a principal $G$-bundle $H^*P \to [0,1]^2$ and a map of principal bundles

\[ \tilde{H} : H^*P \to P. \]

Let $z'$ denote the point in the fiber of $H^*P$ above $(0,0)$ with $\tilde{H}(z') = z$. Let $\Omega$ be the curvature of $A$. Since $H$ is a thin homotopy, $H^*\Omega = 0$. Hence, the pullback connection $A' := \tilde{H}^*A$ is flat, and $\ker A'$ defines an integrable distribution on $H^*P$. Since the square $[0,1]^2$ is contractible there is a global horizontal section $\sigma : [0,1]^2 \to H^*P$ with $\sigma(0,0) = z'$. Note that the restrictions of $\sigma$ to the four sides of the square $[0,1]^2$ are horizontal lifts of the corresponding curves. By Lemma 3.11, $\tilde{H}$ sends horizontal lifts to horizontal lifts. Hence,

\[ \tilde{H}(\sigma(0,t)) = \gamma_z^A(t). \]

It also follows that

\[ s \mapsto \tilde{H}(\sigma(s,0)) \quad \text{and} \quad s \mapsto \tilde{H}(\sigma(s,1)) \]

are the horizontal lifts of the constant curves $s \mapsto \gamma(0)$ and $s \mapsto \tau(1)$; hence, they are constant curves themselves. Since $\tilde{H}(\sigma(0,0)) = \tilde{H}(\sigma(1,0))$, the curve

\[ t \mapsto \tilde{H}(\sigma(1,t)) \]

is the horizontal lift of $\tau$ that starts at $z = \gamma_z^A(a)$, that is, $\tau_z^A$. Since $\tilde{H}(\sigma(0,1)) = \tilde{H}(\sigma(1,1))$,

\[ \gamma_z^A(1) = \tau_z^A(1). \]

\[ \square \]

Lemma 3.13. Suppose $(P \xrightarrow{\pi} M, A)$ is a principal $G$-bundle with connection, $F : U \times [0,1] \to M$ is a smooth map (so that the associated map $\tilde{F} : U \to \mathcal{P}(M)$ is a plot (q.v. Definition A.19)), and $F_0, F_1 : U \to M$ are the restrictions of $F$ to $U \times \{0\}$ and $U \times \{1\}$, respectively. Then the map

\[ \Psi : F_0^*P \to F_1^*P, \quad \Psi(u,z) := (u,||^A_{F(u)}(z)) \]

is smooth. Here as before $||^A$ denotes parallel transport on the principal bundle $P \to M$ defined by $A$.

Proof. Recall that parallel transport $||^A_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ is defined by sending $z \in P_{\gamma(0)}$ to $\gamma_z^A(1)$ where $\gamma_z^A : [0,1] \to P$ is the horizontal lift of $\gamma$ starting at $z$. Recall also that the curve

\[ t \mapsto (t, \gamma_z^A(t)) \]

is an integral curve of a vector field $X_\gamma$ on $\gamma^*P \to [0,1]$; $X_\gamma$ is the horizontal lift of $\frac{\partial}{\partial t}$ with respect to $\gamma^*A$. Similarly let $X$ be the horizontal lift of the vector field $(0, \frac{\partial}{\partial t})$ with respect to $F^*A$ to the bundle $F^*P \xrightarrow{F^*\pi} U \times [0,1]$. Then for any $(u,z) \in F_0^*P = \{(u,z) \in U \times P \mid u = \pi(z)\}$ the curve

\[ t \mapsto (u, F(u)_{\gamma_z^A(t)}) \]

is an integral curve of $X$. Let $\Phi_1$ denote the time-1 flow of the vector field $X$. Then

\[ (u,||^A_{F(u)}(z)) = \Phi_1(u,z) \]
for all \((u, z) \in F_0^*P\). It follows that
\[
\Psi = \Phi_1|_{F_0^*P},
\]
which is smooth. \(\square\)

**Proof of Theorem 3.9.** We need to check that (1) \(\text{hol}_M(P, A)\) is well-defined, (2) it is a functor and (3) it is a transport functor in the sense of Definition 3.5.

Suppose \(x \xrightarrow{[\gamma]} y\) is an arrow in \(\Pi^{\text{thin}}(M)\) and \(\tau \in [\gamma]\). Then there is a thin homotopy \(H : \gamma \Rightarrow \tau\).

By Proposition 3.12 \(\|\gamma\| = \|\tau\|\). Hence \(\text{hol}_M(P, A)\) is well-defined.

Let \(z \xleftarrow{\gamma} y \xleftarrow{\tau} x\) be a pair of composable paths in \(M\). Then by definition of multiplication in \(\Pi^{\text{thin}}(M)\)
\[
[\tau \gamma] = [\tau][\gamma].
\]
On the other hand, by a well-known property of parallel transport
\[
\|\tau \gamma\| = \|\tau\| \circ \|\gamma\|.
\]
Hence
\[
\text{hol}_M(P, A)([\tau \gamma]) = \text{hol}_M(P, A)([\tau]) \text{hol}_M(P, A)([\gamma]).
\]
Parallel transport along a constant path is the identity. We conclude that \(\text{hol}_M(P, A)\) is a functor.

Finally we check that for a point \(x \in M\)
\[
\text{hol}_M(P, A)(\tau_{\text{thin}}(M, x)) : \tau_{\text{thin}}(M, x) \to \text{Aut}(P_x)
\]
is smooth. By Lemma A.16 it is enough to check that for any plot \(p : U \to \Omega(M, x)\) on the space of loops at \(x\) the composite map
\[
\text{hol}_M(P, A) \circ q \circ p : U \to \text{Aut}(P_x)
\]
is \(C^\infty\). Here \(q : \Omega(M, x) \to \tau_{\text{thin}}(M, x)\) is the quotient map. By construction of the \(C^\infty\) structure on \(\text{Aut}(P_x)\) (see Lemma 3.4) a map \(L : U \to \text{Aut}(P_x)\) is smooth if and only if the map
\[
U \ni u \mapsto d(z, L(u)(z)) \in G
\]
is smooth for some \(z \in P_x\) (the map \(d\) is defined in Remark 3.3). Since the map \(d : P_x \times P_x \to G\) is smooth, the smoothness of \(L : U \to \text{Aut}(P_x)\) follows from the smoothness of the map
\[
U \to P_x, \quad u \mapsto L(u)(z)
\]
for some (any) choice of \(z \in P_x\). In the case we care about, this amounts to showing that the map
\[
U \to P_x, \quad u \mapsto \|\gamma\|_{[p(u)]}(z)
\]
is \(C^\infty\). This fact, in turn, easily follows from Lemma 3.13. \(\square\)

**Definition 3.14** (The category \(B^\nabla G(M)\) of principal bundles with connections). Principal \(G\)-bundles with connections over a manifold \(M\) form a category \(B^\nabla G\). The objects are principal bundles with connections, that is, pairs \((P \to M, A)\) where \(P \to M\) is a principal \(G\)-bundle and \(A \in \Omega^1(M, g)^G\) a connection. Morphisms are connection preserving gauge transformations. That is a morphism from \((P \xrightarrow{\pi} M, A)\) to \((P' \xrightarrow{\pi'} M, A')\) is a \(G\)-equivariant map \(f : P \to P'\) with \(\pi' \circ f = \pi\) and \(f^*A' = A\).

**Remark 3.15.** The category \(B^\nabla G(M)\) is a groupoid since every gauge transformation is automatically invertible.

**Definition 3.16** (The category \(\text{Trans}_G(M)\) of \(G\)-transport functors). Fix a Lie group \(G\) and a manifold \(M\). Transport functors on \(M\) form a category \(\text{Trans}_G(M)\): the objects are transport functors (q.v. Definition 3.5) and morphism are arbitrary natural transformations.
Remark 3.17. Since the category of $G$-torsors is a groupoid, a natural transformation between two transport functors is automatically a natural isomorphism. Hence, $\text{Trans}_G(M)$ is automatically a groupoid.

Proposition 3.18. For a manifold $M$ the assignment
\[(P, A) \mapsto \text{hol}_M(P, A)\]
of a transport functor to a principal bundle with a connection extends to a functor
\[
\text{hol}_M : B^\nabla G(M) \rightarrow \text{Trans}_G(M)
\]
from the category of principal $G$-bundles with connections over $M$ to the category of transport functors.

Proof. Let $f : (P, A) \rightarrow (P', A')$ be a morphism in $B^\nabla G(M)$. We want to define a natural transformation $\text{hol}_M(f) : \text{hol}_M(P, A) \Rightarrow \text{hol}_M(P', A')$. Let $x \xrightarrow{[\gamma]} y$ be an arrow in $\Pi^{\text{thin}}(M)$. By (3.4) the diagram
\[
\begin{array}{ccc}
P_x & \xrightarrow{||\gamma||} & P_y \\
\downarrow f|_{P_x} & & \downarrow f|_{P_y} \\
P'_x & \xrightarrow{||\gamma'||} & P'_y
\end{array}
\]
commutes. Since $||\gamma|| = \text{hol}_M(P, A) ([\gamma])$ and $||\gamma'|| = \text{hol}_M(P', A') ([\gamma])$ it follows that the assignment
\[
x \mapsto f|_{P_x}
\]
is a natural transformation from $\text{hol}_M(P, A)$ to $\text{hol}_M(P', A')$. We denote it by $\text{hol}_M(f)$. It is easy to check that the map
\[
\text{hol}_M : B^\nabla G(M) \rightarrow \text{Trans}_G(M), \quad (P, A) \xrightarrow{f} (P', A') \mapsto \left(\text{hol}_M(P, A) \xrightarrow{\text{hol}_M(f)} \text{hol}_M(P', A')\right)
\]
sends identity maps to identity natural transformations and preserves composition. In other words $\text{hol}_M$ is a functor. \qed

4. Equivalence of the categories of principal bundles with connections and of transport functors

The goal of this section is to prove that the functor $\text{hol}_M : B^\nabla G(M) \rightarrow \text{Trans}_G(M)$ constructed in Proposition 3.18 is an equivalence of categories. Our proof is in the same spirit as Barrett’s original proof [2]. The details are necessarily different since we are carefully keeping track of morphisms. We start by formally stating the theorem in question:

Theorem 4.1. For every manifold $M$ the holonomy functor
\[
\text{hol}_M : B^\nabla G(M) \rightarrow \text{Trans}_G(M)
\]
constructed in Proposition 3.18 is an equivalence of categories.

We first reduce the proof of Theorem 4.1 to the case where the manifold $M$ is connected. This is done to simplify the proof of Lemma 4.11 below. One can also prove Lemma 4.11 directly for arbitrary manifolds at a cost of additional fiddling.

Lemma 4.2. Suppose the functor $\text{hol}_M : B^\nabla G(M) \rightarrow \text{Trans}_G(M)$ is an equivalence of categories for all connected manifolds $M$. Then $\text{hol}_M$ is an equivalence of categories for any manifold $M$. 11
Proof. Fix a manifold $M$. We may assume that the set of connected components of $M$ is indexed by a set $I$. Then
\[ M = \bigsqcup_{\alpha \in I} M_\alpha, \]
where the $M_\alpha$'s are connected components of $M$. Observe that
\[ B^\nabla G(M) = \prod_{\alpha \in I} B^\nabla G(M_\alpha) \quad \text{and} \quad \Trans_G(M) = \prod_{\alpha \in I} \Trans_G(M_\alpha), \]
where $\prod$ denotes product of categories. It is not hard to check that the diagram
\[
\begin{array}{c}
\prod_{\alpha \in I} B^\nabla G(M_\alpha) \\
\downarrow \\
B^\nabla G(M_\beta)
\end{array}
\xrightarrow{\text{hol}_{M_\beta}}
\begin{array}{c}
\prod_{\alpha \in I} \Trans_G(M_\alpha) \\
\downarrow \\
\Trans_G(M_\beta)
\end{array}
\]
(4.1)
commutes for all $\beta \in I$. The result follows from these observations. \hfill $\square$

To prove Theorem 4.1 for connected manifolds we introduce a third category, $B^pG(M)$. To properly define this category we recall the notion of a (left) groupoid action (see [17], for example).

**Definition 4.3.** A (left) action of a groupoid $\Gamma = \{ \Gamma_1 \rightrightarrows \Gamma_0 \}$ on a set $M$ consists of a map $a: M \to \Gamma_0$ called the anchor and a map $a(\gamma, m) = \gamma \cdot m$ called the action so that
1. $a(\gamma \cdot m) = t(\gamma)$ for all $\gamma \in \Gamma_1$ and $m \in M$ with $s(\gamma) = a(m)$;
2. the identity arrows act trivially: $1_{a(m)} \cdot m = m$ for all $m \in M$; and
3. $\tau \cdot (\gamma \cdot m) = (\tau \gamma) \cdot m$ for any two composable arrows $\tau, \gamma$ and any point $m \in M$ with $s(\gamma) = a(m)$.

Here, as before, $t$ and $s$ are the target and the source maps of the groupoid $\Gamma$, respectively.

**Definition 4.4.** Fix a Lie group $G$ and a manifold $M$. The category of $B^pG(M)$ of principal $G$-bundles over $M$ with smooth actions of the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ is defined as follows: the objects are smooth (left) actions $a: \mathcal{P}(M)/\sim \times P \to P$ of $\Pi^{\text{thin}}(M)$ on principal $G$-bundles $P \to M$ that commute with the right actions of $G$. Morphisms are $\Pi^{\text{thin}}(M) \times G$-equivariant maps. That is, they are maps $f: P \to P'$ of principal $G$-bundles so that
\[ f([\gamma] \cdot z) = [\gamma] \cdot f(z) \]
for all $([\gamma], z) \in \mathcal{P}(M)/\sim \times P$.

The superscript $^p$ in $B^pG(M)$ refers to parallel transport. See Lemma 4.5 below. There are several reasons for introducing this category. Note first:

**Lemma 4.5.** A connection 1-form $A$ on a principal $G$-bundle $P \xrightarrow{\pi} M$ defines a smooth action $a$ of the thin fundamental groupoid $\Pi^{\text{thin}}(M) := \{ \mathcal{P}(M)/\sim \Rightarrow M \}$ on the manifold $P$ relative to the anchor map $\pi$.

**Proof.** We define the map $a: \mathcal{P}(M)/\sim \times_M P \to P$ by $a([\gamma], z) := ||_{\gamma}(z)$.
for all pairs \( ([\gamma], z) \in \mathcal{P}(M) / \sim_s \times_{s, M, \pi} P \), that is, for all pairs \( ([\gamma], z) \) with \( s([\gamma]) = \gamma(0) = \pi(z) \). Since \( ||_{s(z)}(z) = z \) and, of any pair of composable paths \( \gamma \) and \( \tau \), \( ||_\gamma \circ ||_\tau = ||_{\gamma \tau} \), \( a \) is indeed an action.

To check that the action \( a \) is smooth, pick a pair of plots \( p : U \to \mathcal{P}(M) / \sim_P \) and \( r : U \to P \) with \( s \circ p = \pi \circ r \) (q.v. Construction A.17, the construction of the fiber product diffeology). We need to show that \( a \circ (p, r) : U \to P \) is \( C^\infty \). We may assume that \( p \) has a global lift \( P : U \to \mathcal{P}(M) \). Then
\[
\begin{align*}
a \circ (p, r)(u) &= ||_{p(u)}(r(u)).
\end{align*}
\]

Let \( F : U \times [0, 1] \to M \) denote the map associated to the plot \( p : F(u, t) := p(u)(t) \). By Lemma 3.13, the map \( F_0^*P \to F_1^*P \) given by \( (u, z) \mapsto ||_{p(u)}z \) is smooth. Since \( r : U \to P \) is smooth, \( u \mapsto ||_{p(u)}(r(u)) \) is smooth as well.

In fact, much more is true. We will show:

- The map that sends a principal bundle with connection \((P, A)\) to an action of \( \Pi^{\text{thin}}(M) \) on \( P \) extends to an isomorphism of categories \( \text{tp} : B^G \Pi^{\text{thin}}(M) \to B^P \Pi^{\text{thin}}(M) \); see Lemma 4.10.
- There is a natural equivalence of categories \( \text{rep} : B^P \Pi^{\text{thin}}(M) \to \text{Trans}_G(M) \) so that the diagram
\[
\begin{array}{ccc}
B^G \Pi^{\text{thin}}(M) & \xrightarrow{\text{hol}} & \text{Trans}_G(M) \\
\text{tp} \downarrow & & \downarrow \text{rep} \\
B^P \Pi^{\text{thin}}(M) & & 
\end{array}
\]

commutes; see Lemma 4.11 and Remark 4.9.

Clearly these two facts imply that \( \text{hol}_M : B^G \Pi^{\text{thin}}(M) \to \text{Trans}_G(M) \) is an equivalence of categories and thereby Theorem 4.1. We now proceed to define the relevant functors.

**Definition 4.6** (The functor \( \text{rep} : B^P \Pi^{\text{thin}}(M) \to \text{Trans}_G(M) \)). Given an action \( a : \mathcal{P}(M) / \sim \times_M P \to P \) of the thin fundamental groupoid \( \Pi^{\text{thin}}(M) \) on a principal \( G \)-bundle \( P \to M \), define the associated transport functor \( \text{rep}(a) : \Pi^{\text{thin}}(M) \to G \)-tors by
\[
\text{rep}(a)(x \xrightarrow{[\gamma]} y) := P_x \xrightarrow{[\gamma]} P_y.
\]

Here \( \cdot - \) denotes the action of \( [\gamma] \) on the points of the fiber \( P_x \). Given two actions \( a \) and \( a' \) on principal \( G \)-bundles \( P \) and \( P' \) over \( M \) and a \((\Pi^{\text{thin}}(M) \times G)\)-equivariant map \( f : P \to P' \), define a natural isomorphism \( \text{rep}(f) : \text{rep}(a) \Rightarrow \text{rep}(a') \) by
\[
\text{rep}(f)_x := f|_P)_x : P_x \to P'_x
\]
for all \( x \in M \).

**Remark 4.7.** Note that, since \( a \) is an action, \( \text{rep}(a) \) is indeed a functor. Furthermore, since \( a \) is a smooth action, the map
\[
\pi^{\text{thin}}_1(M, x) \times P_x \to P_x, \quad ([\gamma], z) \mapsto [\gamma] \cdot z
\]
is smooth. Hence, since the category of diffeological spaces is Cartesian closed \([16, 1]\), the adjoint map \( \pi^{\text{thin}}_1(M, x) \to \text{Aut}(P_x) \) is smooth. Therefore \( \text{rep}(a) \) is a transport functor. We conclude that the functor \( \text{rep} \) is well-defined.

**Definition 4.8** (The functor \( \text{tp} : B^G \Pi^{\text{thin}}(M) \to B^P \Pi^{\text{thin}}(M) \)). Given a principal \( G \)-bundle with connection \((P, A)\) define an action \( \text{tp}(P, A) : \mathcal{P}(M) / \sim \times_M P \to P \) of \( \Pi^{\text{thin}}(M) \) on \( P \) by
\[
\text{tp}(P, A)([\gamma], z) := ||_{\gamma}^A(z)
\]
for all \( ([\gamma], z) \in \mathcal{P}(M) / \sim \times_M P \) (q.v. Lemma 4.5).
Equation (3.4) implies that a map $f : (P, A) \to (P', A')$ of principal $G$-bundles with connection intertwines the actions $tp(P, A)$ and $tp(P', A')$. Hence $f : P \to P'$ is also a morphism in $B^pG(M)$.

**Remark 4.9.** It follows easily from the definitions that $\text{rep} \circ tp = \text{hol}$, that is, the diagram (4.2) commutes.

We now state:

**Lemma 4.10.** For a manifold $M$ the functor $tp : B^\nabla G(M) \to B^pG(M)$ of Definition 4.8 is an isomorphism of categories.

**Lemma 4.11.** For a connected manifold $M$, the functor $\text{rep} : B^pG(M) \to \text{Trans}_G(M)$ of Definition 4.6 is an equivalence of categories.

The proofs of these two lemmas take up the rest of the section. Our proof of Lemma 4.10 is surprisingly fiddly. We first describe a procedure for building a smooth family of paths in $\mathcal{P}(M)/\sim$ from any path on $M$.

**Construction 4.12.** Let $(a, b)$ be an interval containing 0 and $\gamma : (a, b) \to M$ a smooth path in a manifold $M$. We construct a smooth path (a plot)

$$\Upsilon : (a, b) \to \mathcal{P}(M)$$

as follows. Let $\beta \in C^\infty([0, 1])$ be the function of Remark 2.2. Define

$$\Upsilon(s)(t) := \gamma(s\beta(t))$$

for all $t \in [0, 1]$, $s \in (a, b)$. Since the map $\hat{\Upsilon} : (a, b) \times [0, 1] \to M$ given by

$$\hat{\Upsilon}(s, t) := \gamma(s\beta(t))$$

is smooth, the map $\Upsilon$ is indeed a smooth path in the diffeological space $\mathcal{P}(M)$ of paths in $M$. Note that the path $\Upsilon$ satisfies $\Upsilon(0) = 1_x$, where $1_x$ denotes the constant path at $x = \gamma(0)$. Additionally $\Upsilon(s)(0) = x$ and $\Upsilon(s)(1) = \gamma(s)$ for all $s$.

Note also that the composite $q \circ \Upsilon$ is a smooth path in the space $\mathcal{P}(M)/\sim$ of arrows of the groupoid $\Pi^{\text{thin}}(M)$ (here, as before, $q : \mathcal{P}(M) \to \mathcal{P}(M)/\sim$ is the quotient map).

**Lemma 4.13.** For any manifold $M$, the functor $tp : B^\nabla G(M) \to B^pG(M)$ is bijective on morphisms and injective on objects.

**Proof.** We first argue that the functor $tp$ is bijective on morphisms. That is, for any two principal $G$-bundles with connection $(P, A)$ and $(P', A')$ over the manifold $M$, the map

$$tp : \text{Hom}_{B^\nabla G(M)}((P, A), (P', A')) \to \text{Hom}_{B^pG(M)}(tp(P, A), tp(P', A'))$$

is a bijection. A morphism in $\text{Hom}_{B^\nabla G(M)}(tp(P, A), tp(P', A'))$ is a map of principal bundles $f : P \to P'$ which is also $\Pi^{\text{thin}}(M)$ equivariant (with the two actions defined by the connections $A$ and $A'$, respectively). A morphism in $\text{Hom}_{B^pG(M)}((P, A), (P', A'))$ is a map of principal bundles $h : P \to P'$ covering the identity on $M$ with $h^*A' = A$. Thus the map (4.3) is a bijection if and only if any $\Pi^{\text{thin}}(M)$-equivariant map of principal bundles preserves the respective connections.

Given a path $\gamma : [a, b] \to M$, recall that $\gamma^A_z$ denotes the horizontal lift to $P$ with respect to the connection $A$ which starts at $z \in P_{\gamma(a)}$ (see Definition 3.10). To prove that a $\Pi^{\text{thin}}(M)$-equivariant map of principal bundles $f : P \to P'$ preserves connections, it is enough to show that

$$f \circ \gamma^A_z = \gamma^A_{f(z)}$$

(4.4)
for all curves $\gamma$ in $M$. This is because given a point $x \in M$ and a vector $v \in T_x M$, we can choose $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then $\dot{\gamma}^A(0)$ is the horizontal lift of $v$ to $T_x P$ with respect to $A$. So if (4.4) holds, then

$$Df_x(\dot{\gamma}^A(0)) = \dot{\gamma}^A_{f(z)}(0).$$

Consequently, since $f$ is $G$-equivariant, we have to have

$$f^* A' = A.$$

To prove that (4.4) holds, first observe that if $\tau : (-\epsilon, \epsilon) \to P$ is a curve with $\pi \circ \tau = \gamma$, $\tau(0) = z$, and

$$\tau(s) = ||_\gamma\gamma^A(z)$$

for all $s \in (-\epsilon, \epsilon)$, then $\tau$ is the horizontal lift $\gamma^A_z$ of $\gamma$. This is because $\gamma^A_z$ is the unique curve with $\gamma^A_z(0) = z$ and $\gamma^A_z(s) = ||_\gamma\gamma^A(z)$.

By Construction 4.12 a path $\gamma : (-\epsilon, \epsilon) \to M$ gives rise to a a smooth path $\Upsilon : (-\epsilon, \epsilon) \to P(M)$ with $\Upsilon(0) = 1_x$ (the constant path at $x = \gamma(0)$), $\Upsilon(s)(0) = x$, and $\Upsilon(s)(1) = \gamma(s)$ for every $s \in (-\epsilon, \epsilon)$. Consequently

$$\tau(s) = \Upsilon(s) \cdot z$$

is a smooth path in the principal bundle $\tau : P \to M$ with $\tau(0) = [1_x] \cdot z = z$ and $\pi(\tau(s)) = \gamma(s)$. Moreover, by definition of the action of $\Pi^{thin}(M)$ on $P$, the point $\tau(s) \in P$ is the image of $z$ under parallel transport along the curve $\nu(t) := \gamma(s)(\beta(t))$. Since $\nu(t)$ is a “reparameterization” of $\gamma|_{[0,s]}$, we have $\tau(s) = ||_\gamma\gamma^A(s)$ for all $s$. We conclude that the curve $\tau$ is the horizontal lift $\gamma^A_z$ of $\gamma$. Since $f$ is $\Pi^{thin}(M)$-equivariant,

$$f(\tau(s)) = f([\Upsilon(s)] \cdot z) = [\Upsilon(s)] \cdot f(z).$$

By the same argument, the curve $s \mapsto [\Upsilon(s)] \cdot f(z)$ is the horizontal lift $\gamma^A_{f(z)}$ of $\gamma$ to $P'$. Hence (4.4) holds and we conclude that the map (4.3) is a bijection.

The argument above also implies that if $A$ and $A'$ are two connections on a principal $G$-bundle $P \to M$ that define the same action of $\Pi^{thin}(M)$, then $(id_P)^* A' = A$, that is, $A = A'$. Therefore, the functor $\mathfrak{tp}$ is injective on objects.

It remains to show that the functor $\mathfrak{tp}$ is surjective on objects; that is, an action of the thin fundamental groupoid $\Pi^{thin}(M)$ on a principal $G$-bundle $P \to M$ defines a connection. We prove this in a series of lemmas and corollaries. The first is a variant of a lemma due to Barrett [2].

**Lemma 4.14.** Let $Q \to \mathbb{R}^n$ be a principal $G$-bundle with a smooth action of the thin fundamental groupoid $\Pi^{thin}(\mathbb{R}^n)$ and $p : (-\epsilon, \epsilon) \to \pi^1(A)(x) \subset \mathcal{P}(\mathbb{R}^n)/\sim$ a smooth family of (thin homotopy classes of) loops with $p(0) = [1_x]$, the class of the constant loop at $x \in \mathbb{R}^n$. Then

$$\frac{d}{ds} \bigg|_0 (p(s) \cdot z) = 0$$

for any point $z$ in the fiber of $Q$ above $x$.

**Proof.** We may assume that the path $p : (-\epsilon, \epsilon) \to \mathcal{P}(\mathbb{R}^n)/\sim$ has a global lift $\tilde{p} : (-\epsilon, \epsilon) \to \mathcal{P}(\mathbb{R}^n)$. Then

$$(\tilde{p}(s))(t) = (\tilde{p}_1(s,t), \ldots, \tilde{p}_n(s,t))$$

for some smooth functions $\tilde{p}_j : (-\epsilon, \epsilon) \times [0,1] \to \mathbb{R}$ with $\tilde{p}_j(0,t) = x_j$ for all $t$. Consider the map $P : (\epsilon, \epsilon)^n \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$P(s_1, \ldots, s_n)(t) = (\tilde{p}_1(s_1,t), \ldots, \tilde{p}_n(s_n,t)),$$

it is also a plot for $\mathcal{P}(\mathbb{R}^n)$. We then have

$$p(s) = q \circ P(s, \ldots, s) = [P(s, \ldots, s)],$$
where \( q : P(\mathbb{R}^n) \rightarrow P(\mathbb{R}^n)/\sim \) is the quotient map. Since the action of \( \Pi^{\text{thin}}(\mathbb{R}^n) \) on the bundle \( Q \) is smooth, the map

\[
F : (\epsilon, \epsilon)^n \rightarrow Q, \quad F(s_1, \ldots, s_n) := [P(s_1, \ldots, s_n)] \cdot z
\]

is smooth. For each \( s \in (\epsilon, \epsilon) \), the paths \( s \mapsto P(s, 0, \ldots, 0), s \mapsto P(0, s, \ldots, 0), \ldots, s \mapsto P(0, \ldots, 0, s) \) are all thinly homotopic to the constant path \( 1_x \).

It follows that

\[
\frac{\partial F}{\partial s_j}(0, \ldots, 0) = 0
\]

for all \( j \).

By the chain rule

\[
\frac{d}{ds}(p(s) \cdot z) \bigg|_0 = \frac{d}{ds} \bigg|_0 F(s, \ldots, s) = \sum_{j=0}^{n} \frac{\partial F}{\partial s_j}(0, \ldots, 0) \cdot 1 = 0.
\]

\[\square\]

**Corollary 4.15.** Let \( Q \rightarrow \mathbb{R}^n \) be a principal \( G \)-bundle with a smooth action of the thin fundamental groupoid \( \Pi^{\text{thin}}(\mathbb{R}^n) \) as above and \( \tau, \tau' : (\epsilon, \epsilon) \rightarrow \pi^{\text{thin}}_1(\mathbb{R}^n, x) \subset P(\mathbb{R}^n)/\sim \) be two smooth maps with \( \tau(0) = \tau'(0) = [1_x], x \in \mathbb{R}^n \), and \( t(\tau(s)) = t(\tau'(s)) \) for all \( s \in (\epsilon, \epsilon) \). Then

\[
\frac{d}{ds} \bigg|_0 (\tau(s) \cdot z) = \frac{d}{ds} \bigg|_0 (\tau'(s) \cdot z)
\]

for any point \( z \) in the fiber of \( Q \) above \( x \).

**Proof.** By assumption, \( p(s) = \tau(s)^{-1} \tau'(s) \) is a loop for each \( s \in (\epsilon, \epsilon) \) and \( p(0) \) is the constant loop \([1_x]\). Note that

\[
\tau'(s) = \tau(s) \cdot \tau(s)^{-1} \cdot \tau'(s) = \tau(s) \cdot p(s).
\]

Hence

\[
\frac{d}{ds} \bigg|_0 (\tau'(s) \cdot z) = \frac{d}{ds} \bigg|_0 (\tau(s) \cdot (p(s) \cdot z))
\]

Since the action of \( \Pi^{\text{thin}}(M) \) on \( Q \) is smooth the map \( F : (\epsilon, \epsilon)^2 \rightarrow Q \) defined by

\[
F(s_1, s_2) = \tau(s_1) \cdot (p(s_2) \cdot z)
\]

is smooth. By the chain rule

\[
\frac{dF}{ds}(s, s) \bigg|_0 = \frac{\partial F}{\partial s_1}(0, 0) + \frac{\partial F}{\partial s_2}(0, 0).
\]

Hence

\[
\frac{d}{ds} \bigg|_0 (\tau'(s) \cdot z) = \frac{d}{ds} \bigg|_0 (\tau(s) \cdot (p(0) \cdot z)) + \frac{d}{ds} \bigg|_0 (\tau(0) \cdot (p(s) \cdot z)).
\]

By Lemma 4.14 the second term is zero, and we are done. \[\square\]

**Lemma 4.16.** Let \( \pi : Q \rightarrow \mathbb{R}^n \) be a principal \( G \)-bundle. A smooth action \( a : (P(\mathbb{R}^n)/\sim) \times_s \mathbb{R}^n, sQ \rightarrow Q \) of the thin fundamental groupoid \( \Pi^{\text{thin}}(\mathbb{R}^n) \) on \( Q \) defines a connection \( A \) (which is necessarily unique by Lemma 4.13) so that for any path \( \gamma : [0, 1] \rightarrow \mathbb{R}^n \) and any point \( z \) in the fiber \( Q_x \) above \( x = \gamma(0) \)

\[
[\gamma] \cdot z = \gamma^A_z(1).
\]

Here, as before, \( \gamma^A_z : [0, 1] \rightarrow Q \) is a lift of \( \gamma \) to \( Q \) which is \( A \)-horizontal and starts at the point \( z \).
Proof. As we have seen in the proof of Lemma 4.13, an action of $\Pi^\text{thin}(\mathbb{R}^n)$ on a principal bundle $Q \to \mathbb{R}^n$ allows us to lift curves of the form $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ to $Q$. Thus, given a point $x \in \mathbb{R}^n$ and a point $z \in Q$ in the fiber above $x$, it is tempting to define the horizontal subspace $\mathcal{H}_Q z \subset T_z Q$ by
\[
\mathcal{H}_Q z := \left\{ \frac{d}{ds} \bigg|_{s=0} [\Upsilon(s)] \cdot z \bigg| \Upsilon(s)(t) = \gamma(s \beta(t)), \quad \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n, \gamma(0) = x \right\}.
\] (4.5)

Here as before $\Upsilon$ is the smooth path in $\mathcal{P}(M)$ constructed from the path $\gamma$ (q.v. Construction 4.12). However, it is not clear that (4.5) defines a vector space. Nor is it clear that curves of the form $s \mapsto [\Upsilon(s)] \cdot z$ are tangent to the purported distribution $\mathcal{H}$ when $s \neq 0$. We therefore proceed a little differently.

Consider the map
\[
\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n), \quad \sigma(x, y)(t) := x + \beta(t)(y - x),
\] (4.6)
for all $t \in [0, 1]$. Here as before $\beta \in C^\infty([0, 1])$ is the function of Remark 2.2. The map $\sigma$ is smooth since the map $\hat{\sigma}(x, y, t) := x + \beta(t)(y - x)$ is smooth. We denote the induced plot on the space of arrows of the thin fundamental groupoid $\Pi^\text{thin}(\mathbb{R}^n)$ by $[\sigma]$. By construction, $[\sigma(x, y)]$ is the thin homotopy class of the straight line paths from $x$ to $y$ with $[\sigma(x, x)] = [1_x]$, the class of the constant path.

Next consider the smooth map
\[
F : \mathbb{R}^n \times Q \to Q, \quad F(y, z) := [\sigma(\pi(z), y)] \cdot z.
\]
By construction
\[
\pi(F(y, z)) = y.
\]
Thus for each fixed $z \in P$ the map
\[
F(\cdot, z) : \mathbb{R}^n \to Q
\]
is a smooth section of $\pi : P \to \mathbb{R}^n$ with
\[
F(\pi(z), z) = [\sigma(\pi(z), \pi(z))] \cdot z = [1_{\pi(z)}] \cdot z = z.
\]
Denote the derivative of this section at $y \in \mathbb{R}^n$ by $\partial_1 F(y, z)$. By definition, the derivative $\partial_1 F(\pi(z), z)$ is a linear map from $T_{\pi(z)} \mathbb{R}^n$ to $T_z Q$. We define
\[
\mathcal{H}_Q z := \text{image } \left( \partial_1 F(\pi(z), z) : T_{\pi(z)} \mathbb{R}^n \to T_z Q \right).
\]
Since $y \mapsto F(y, z)$ is a section of $\pi$, the image of its differential at every point is a subspace complementary to the vertical bundle of $\pi : Q \to \mathbb{R}^n$. Since $F : \mathbb{R}^n \times Q \to Q$ is smooth, the map $z \mapsto \partial_1 F(\pi(z), z)$ is smooth. It follows that the subspace $\mathcal{H}_Q z$ depends smoothly on $z \in Q$. Since the action of $\Pi^\text{thin}(\mathbb{R}^n)$ on $Q$ commutes with the action of $G$,
\[
F(y, z \cdot g) = [\sigma(\pi(z \cdot g), y)] \cdot (z \cdot g) = ([\sigma(\pi(z), y)] \cdot z) \cdot g = F(y, z) \cdot g
\]
for any $y \in \mathbb{R}^n$, $z \in Q$ and $g \in G$. Consequently, $\mathcal{H}_Q \subset TP$ is a $G$-invariant distribution. Denote the corresponding connection 1-form by $A$. It remains to check that $[\gamma] \cdot z = \gamma^A_z(1)$ for a path $\gamma : [0, 1] \to \mathbb{R}^n$ and any point $z$ in the fiber $Q_{\gamma(0)}$ above $\gamma(0)$.

Since $\gamma$ is smooth, there is $\epsilon > 0$ and an extension of $\gamma$ to a smooth map from $(-\epsilon, 1 + \epsilon)$ to $\mathbb{R}^n$. We denote the extension by the same symbol $\gamma$. Consider the corresponding path $\Upsilon : (-\epsilon, 1 + \epsilon) \to \mathcal{P}(\mathbb{R}^n)$ satisfying $\Upsilon(s)(0) = \gamma(0)$ and $\Upsilon(s)(1) = \gamma(s)$ for any $s \in (-\epsilon, 1 + \epsilon)$ (Construction 4.12).

We want to show that
\[
\frac{d}{ds} \bigg|_{s=s_0} [\Upsilon(s)] \cdot z \in \mathcal{H}_{[\Upsilon(s)] \cdot z} Q
\]
for any $s_0 \in [0, 1]$, and any $z \in Q_{\gamma(0)}$. Note that, by Construction 4.12,
\[
\pi([\Upsilon(s)] \cdot z) = t([\Upsilon(s)]) = \Upsilon(s)(1) = \gamma(s).
\]
Set \( z_0 := [\Upsilon(s_0)] \cdot z \). Then:

\[
\frac{d}{ds} \bigg|_{s=s_0} \Upsilon(s) \cdot z = \frac{d}{ds} \bigg|_{s=0} [\Upsilon(s + s_0)] \cdot z
\]

\[
= \frac{d}{ds} \bigg|_{s=0} ([\Upsilon(s + s_0)][\Upsilon(s_0)]^{-1}) \cdot ([\Upsilon(s_0)] \cdot z)
\]

\[
= \frac{d}{ds} \bigg|_{s=0} ([\Upsilon(s + s_0)][\Upsilon(s_0)]^{-1}) \cdot z_0
\]

We next consider two paths in \( P(\mathbb{R}^n)/\sim \) defined on \((-\epsilon, \epsilon)\):

\[
\tau(s) = [\Upsilon(s + s_0)][\Upsilon(s_0)]^{-1}
\]

and

\[
\tau'(s) = [\sigma(\gamma(s_0), \gamma(s + s_0))],
\]

where \( \sigma \) is defined by (4.6). These two paths satisfy the hypotheses of Corollary 4.15. Hence

\[
\frac{d}{ds} \bigg|_{s=0} \tau(s) \cdot z_0 = \frac{d}{ds} \bigg|_{s=0} \tau'(s) \cdot z_0.
\]

By construction of the path \( \tau' \) and the distribution \( \mathcal{H} \), the right hand side of (4.7) is a vector in \( \mathcal{H}_{z_0} Q = \mathcal{H}_{[\Upsilon(s)]} \cdot z Q \). Hence \( \frac{d}{ds} \big|_{s=s_0} [\Upsilon(s)] \cdot z = \frac{d}{ds} \big|_{s=0} \tau(s) \cdot z_0 \in \mathcal{H}_{[\Upsilon(s)]} \cdot z Q \) as well. \( \square \)

**Lemma 4.17.** Let \( \pi : Q \to M \) be a principal \( G \)-bundle. An action \( a : (P(M)/\sim) \times \mathbb{R}^n, \pi Q \to Q \) of the thin fundamental groupoid \( \Pi^{\text{thin}}(M) \) on \( Q \) defines a unique connection \( A \) so that for any path \( \gamma : [0, 1] \to M \) and any point \( z \) in the fiber \( Q_x \) above \( x = \gamma(0) \)

\[
[\gamma] \cdot z = \gamma_z^A(1).
\]

**Proof.** For any point \( x \in M \) there is a coordinate chart \( \varphi : U \to \mathbb{R}^n \) on \( M \) with \( x \in U \) and \( \varphi \) a diffeomorphism. Since the desired connection \( A \) would have to be unique (see Lemma 4.13), it is enough to define it on restrictions \( Q|_U \) where \( U \subset M \) is a domain of a diffeomorphism \( \varphi : U \to \mathbb{R}^n \).

The action of \( \Pi^{\text{thin}}(M) \) on \( Q \) defines a smooth action of \( \Pi^{\text{thin}}(U) \) on \( Q|_U \). The diffeomorphism \( \varphi \) allows us to transfer this action to an action of \( \Pi^{\text{thin}}(\mathbb{R}^n) \) on the principal \( G \)-bundle \( (\varphi^{-1})^* Q \to \mathbb{R}^n \).

By Lemma 4.16, the action of \( \Pi^{\text{thin}}(\mathbb{R}^n) \) defines a connection on \( (\varphi^{-1})^* Q \to \mathbb{R}^n \). Its pullback to \( Q|_U \to U \) defines the restriction of the desired connection \( A \) to \( Q|_U \). \( \square \)

**Proof of Lemma 4.10.** By Lemma 4.13 the functor \( \text{tp} \) is bijective on morphisms and injective on objects. By Lemma 4.17 it is also surjective on objects. Therefore, \( \text{tp} \) is an isomorphism of categories. \( \square \)

To prove Lemma 4.11 it will be convenient for us to define principal diffeological bundles over manifolds. Note that our definition is different from the one in [11]; the definition we use is more convenient for our purposes.

**Definition 4.18.** Let \( \mathcal{G} \) be a diffeological group. A diffeological space \( \mathcal{P} \) is a principal \( \mathcal{G} \)-bundle over a manifold \( M \) if the following three conditions hold:

1. there is a surjective map \( \varpi : \mathcal{P} \to M \) which has local sections: for any point \( x \in M \), there is a neighborhood \( U \) of \( M \) and a smooth section \( \sigma : U \to \mathcal{P} \) of \( \varpi \);
2. there is smooth right action of \( \mathcal{G} \) on \( \mathcal{P} \);
3. the map \( \psi : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times \varpi, M, \varpi \mathcal{P} \) given by \( \psi(z, g) := (z, z \cdot g) \) is an isomorphism of diffeological spaces.
Lemma 4.19. Let $M$ be a connected manifold. Then for any point $x \in M$ the fiber $s^{-1}(x)$ of the source map $s : \mathcal{P}/\sim \to M$ of the groupoid $\Pi^{\text{thin}}(M)$ with the subspace diffeology is a principal $\pi^{\text{thin}}(M,x)$ bundle over $M$; the projection $\varpi : s^{-1}(x) \to M$ is the restriction of the target map $t : \mathcal{P}/\sim \to M$:

$$\varpi = t|_{s^{-1}(x)}.$$ 

Proof. Since the multiplication $m$ in the fundamental groupoid $\Pi^{\text{thin}}(M)$ is smooth, the map

$$\psi : s^{-1}(x) \times \pi^{\text{thin}}(M,x) \to s^{-1}(x) \times \varpi, \varpi s^{-1}(x), \quad \psi([\tau], [\gamma]) := ([\tau], [\tau][\gamma])$$

is smooth. If $[\tau_1], [\tau_2] \in s^{-1}(x)$ have the same target, then $[\tau_1]^{-1}[\tau_2]$ is a loop in $\pi^{\text{thin}}(M,x)$ and $[\tau_2] = [\tau_1]([\tau_1]^{-1}[\tau_2])$. Hence the map

$$([\tau_1], [\tau_2]) \mapsto ([\tau_1], [\tau_1]^{-1}[\tau_2])$$

is a smooth inverse of $\psi$.

By Lemma A.26, the map $\varpi = t|_{s^{-1}(x)} : s^{-1}(x) \to M$ has local sections.

Lemma 4.20. Let $\mathcal{G} \to \mathcal{P} \xrightarrow{\varpi} M$ be a diffeological principal bundle. Suppose the diffeological group $\mathcal{G}$ acts smoothly on the left on a manifold $T$. Then the associated bundle

$$\mathcal{P} \times^G T := (\mathcal{P} \times T)/\mathcal{G}$$

is a manifold; the left action of $\mathcal{G}$ on $\mathcal{P} \times T$ is given by $\gamma \cdot (z,t) = (z \cdot \gamma^{-1}, \gamma \cdot t)$. Moreover, $\pi : \mathcal{P} \times^G T \to M$, given by $\pi([z,t]) = \varpi(z)$, makes $\mathcal{P} \times^G T$ into a fiber bundle over $M$ with typical fiber $T$. Additionally, if $T$ is a $G$-torsor for a Lie group $G$ and $\mathcal{G}$ acts on $T$ by torsor automorphisms, then $\pi : \mathcal{P} \times^G T \to M$ is a principal $G$-bundle.

Proof. By Definition 4.18 the map $\psi : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times_{\varpi,M,\varpi} \mathcal{P}$ given by $\psi(z,g) := (z, z \cdot g)$ is an isomorphism of diffeological spaces. By composing its inverse $\psi^{-1}$ with the projection on the second factor, we obtain a smooth map

$$d : \mathcal{P} \times_M \mathcal{P} \to \mathcal{G}$$

characterized by

$$z_1 \cdot d(z_1, z_2) = z_2$$

for all points $(z_1, z_2)$ in the fiber product $\mathcal{P} \times_M \mathcal{P}$. Consequently, if $s_\alpha : U_\alpha \to \mathcal{P}$ and $s_\beta : U_\beta \to \mathcal{P}$ are two local sections, then

$$s_\alpha(x) \cdot d(s_\alpha(x), s_\beta(x)) = s_\beta(x)$$

for all $x \in U_\alpha \cap U_\beta$. A local section $s_\alpha : U_\alpha \to \mathcal{P}$ defines a smooth map

$$\sigma_\alpha : U_\alpha \times T \to (\mathcal{P} \times^G T)|_{U_\alpha}, \quad \sigma_\alpha(x,t) := [s_\alpha(x), t].$$

By construction, the diagram

$$\begin{tikzcd}
U_\alpha \times T \ar{r}{\sigma_\alpha} \ar[swap]{dr}{pr_1} & (\mathcal{P} \times^G T)|_{U_\alpha} \\
& U_\alpha \ar{u}{\pi}
\end{tikzcd}$$

commutes, where $pr_1$ is the projection on the first factor and $\pi([z,t]) := \varpi(z)$.

The map $\sigma_\alpha$ has a smooth inverse: given $x \in U_\alpha$ and $z \in \mathcal{P}_x := \varpi^{-1}(x)$, we have

$$s_\alpha(x) \cdot d(s_\alpha(x), z) = z.$$ 

Hence, for all $t \in T$,

$$[z,t] = [s_\alpha(x) \cdot d(s_\alpha(x), z), t] = [s_\alpha(x), d(s_\alpha(x), z) \cdot t].$$
Consequently
\[ \sigma^{-1}_\alpha([z,t]) = (\pi(z), d(s_\alpha(\pi(z))) \cdot t) \]
is smooth. We conclude that \( \sigma_\alpha : U_\alpha \times T \to (\mathcal{P} \times^G T)|_{U_\alpha} \) is an isomorphism of diffeological spaces.

Now choose an open cover \( \{U_\alpha\} \) of the manifold \( M \) so that for each index \( \alpha \), the restriction \( \mathcal{P}|_{U_\alpha} \to U_\alpha \) has a section \( s_\alpha \). Then the images of the corresponding trivializations \( \sigma_\alpha : U_\alpha \times T \to \mathcal{P} \times^G T \) cover \( \mathcal{P} \times^G T \). Any diffeological space that has an open cover consisting of manifolds is itself a manifold [11, Section 4.2, p. 78]. Therefore the diffeological space \( \mathcal{P} \times^G T \) is a manifold. Moreover, since diagram (4.8) commutes, \( \pi : \mathcal{P} \times^G T \to M \) is a locally trivial fiber bundle with fiber \( T \).

Additionally, if \( T \) is a \( G \)-torsor for a Lie group \( G \) and \( \mathcal{G} \) acts on \( T \) by torsor automorphisms, then \( \mathcal{P} \times^G T \) admits a right \( G \)-action. By construction, the local trivialization maps \( \sigma_\alpha : U_\alpha \times T \to \mathcal{P} \times^G T \) are \( G \)-equivariant. It follows that \( \pi : \mathcal{P} \times^G T \to M \) is a principal \( G \)-bundle. \( \square \)

**Proof of Lemma 4.11.** Let \( \alpha : F \to F' \) be a natural isomorphism between two transport functors. Fix a point \( x \in M \). Then \( \alpha_x : F(x) \to F'(x) \) is a map of \( G \)-torsors. Hence, it defines a smooth, \((\Pi^\text{thin}_1(M,x) \times G)\)-equivariant map:
\[
(id, \alpha_x) : s^{-1}(x) \times F(x) \to s^{-1}(x) \times F'(x).
\]

This map descends to a smooth \( G \)-equivariant map on the quotient
\[
[id, \alpha_x] : (s^{-1}(x) \times F(x))/\Pi^\text{thin}_1(M,x) \to (s^{-1}(x) \times F'(x))/\Pi^\text{thin}_1(M,x).
\]

There is also a natural action of \( \Pi^\text{thin}_1(M) \) on \( s^{-1}(x) \) that descends to an action on the quotient. Since \([id, \alpha_x] \) is \( \Pi^\text{thin}_1(M) \)-equivariant the procedure defines a functor
\[
\text{assoc} : \text{Trans}_G(M) \to B^pG(M).
\]

For any principal \( G \)-bundle \( P \to M \) with an action \( a \) of \( \Pi^\text{thin}_1(M) \), the principal \( G \)-bundle \( \text{assoc}(\text{rep}(a)) \) is naturally isomorphic to the bundle \( P \). The isomorphism
\[
\eta_P : (s^{-1}(x) \times P_x)/\Pi^\text{thin}_1(M,x) \to P
\]
is defined by
\[
\eta_P([[\gamma], z]) := [\gamma] \cdot z.
\]

Conversely, any transport functor \( F : \Pi^\text{thin}_1(M) \to \text{G-tors} \) is isomorphic to the transport functor \( \text{rep}(\text{assoc}(F)) \). To see this, note first that \( \text{rep}(\text{assoc}(F)) \) is defined on objects by sending \( y \in M \) to the fiber of the bundle \( \text{assoc}(F) \) above \( y \). This fiber is the torsor
\[
\{[[\gamma], z] \in s^{-1}(x) \times F(x)/\mathcal{G} \mid \gamma(1) = y\}.
\]

The natural isomorphism \( \varepsilon : \text{rep}(\text{assoc}(F)) \Rightarrow F \) is given by
\[
\varepsilon_y : [[\gamma], z] \mapsto F([\gamma])z.
\]

It is well-defined. It follows that the functor \( \text{rep} : B^pG(M) \to \text{Trans}_G(M) \) is an equivalence of categories. \( \square \)

**Part 2. Parallel transport and stacks**

From now on we assume that the reader is familiar with stacks over the site of differentiable manifolds. The standard references are Behrend and Xu [3], Heinloth [10] and Metzler [18]. We will primarily think of a stack \( \mathcal{X} \) over the site \( \text{Man} \) of manifolds as a category fibered in groupoids (CFG) that satisfies descent. One can also think of stacks over \( \text{Man} \) as lax presheaves of groupoids with descent. Grothendieck’s construction (see for example [22]) converts lax presheaves into CFGs. A choice of cleavage turns a CFG into a lax presheaf. Finally recall that any Lie groupoid \( \Gamma = \{\Gamma_1 \Rightarrow \Gamma_0\} \) has a stack quotient \([\Gamma_0/\Gamma_1] \): it is a category fibered in groupoids over \( \text{Man} \) whose
objects are principal \( \Gamma \)-bundles (see [15] for example). It is well known that stack quotients, as the name implies, are stacks.

5. Holonomy functor as an isomorphism of stacks

We start by constructing a presheaf of groupoids out of the assignment of transport functors to manifolds.

**Lemma 5.1.** The assignment

\[
M \mapsto \text{Trans}_G(M)
\]

extends to a contravariant functor, that is, a strict presheaf of groupoids

\[
\text{Trans}_G : \text{Man}^{op} \to \text{Groupoid}
\]

from the category of manifolds to the category of groupoids.

**Proof.** We need to check that for any smooth map \( f : N \to M \) between manifolds, we have a map \( f^* : \text{Trans}_G(M) \to \text{Trans}_G(N) \) such that, for a pair of composable maps \( Q \overset{h}{\to} N \overset{f}{\to} M \), we have \((f \circ h)^* = h^* \circ f^*\) and that \( id_M^* = id_{\text{Trans}_G(M)} \).

Since \( \Pi^{\text{thin}} \) is a functor from the category of manifolds to the category of diffeological groupoids, for any smooth map \( f : N \to M \) between manifolds, we have a morphism \( \Pi^{\text{thin}}(f) : \Pi^{\text{thin}}(N) \to \Pi^{\text{thin}}(M) \) between diffeological groupoids (see Proposition A.27). In particular \( \Pi^{\text{thin}}(f) : \mathcal{P}(N)/\sim \to \mathcal{P}(M)/\sim \) is a map of diffeological spaces and so is the restriction \( \Pi^{\text{thin}}(f)|_{\pi_1^{\text{thin}}(N,x)} \) for any point \( x \in N \). If \( T : \Pi^{\text{thin}}(M) \to G\text{-tors} \) is a transport functor then, by definition, for any \( x \in N \) the map \( T|_{\pi_1^{\text{thin}}(M,f(x))} : \pi_1^{\text{thin}}(M,f(x)) \to \text{Aut}(T(f(x))) \) is smooth. Consequently the map

\[
(T \circ \Pi^{\text{thin}}(f))|_{\pi_1^{\text{thin}}(N,x)} = T|_{\pi_1^{\text{thin}}(M,f(x))} \circ (\Pi^{\text{thin}}(f)|_{\pi_1^{\text{thin}}(N,x)})
\]

is smooth as well. Thus,

\[
f^*T := T \circ \Pi^{\text{thin}}(f)
\]

is a transport functor. Since \( \Pi^{\text{thin}} \) is a functor,

\[
(f \circ h)^*T = T \circ \Pi^{\text{thin}}(fh) = T \circ \Pi^{\text{thin}}(f) \circ \Pi^{\text{thin}}(h) = h^*(f^*T).
\]

Since \( \Pi^{\text{thin}}(id_M) = id_{\Pi^{\text{thin}}(M)} \),

\[
(id_M)^* = id_{\text{Trans}_G(M)}
\]

for all manifolds \( M \). \( \square \)

Grothendieck’s construction [22] applied to the presheaf of groupoids \( \text{Trans}_G \) produces a category \( \text{Trans}_G \) which is fibered in groupoids over the category of manifolds \( \text{Man} \). Explicitly we define \( \text{Trans}_G \) as follows.

**Definition 5.2** (the category \( \text{Trans}_G \) of transport functors over the category \( \text{Man} \) of manifolds). The objects of \( \text{Trans}_G \) are pairs \((M,F)\) where \( M \) is a manifold and \( F \in \text{Trans}_G(M) \) is a transport functor. A morphism of \( \text{Trans}_G \) from \((N,H)\) to \((M,F)\) is a pair \((f,\alpha)\) where \( f : N \to M \) is a smooth map of manifolds and \( \alpha : F \circ \Pi^{\text{thin}}(f) \Rightarrow H \) is a natural isomorphism.

The functor \( \varpi_T : \text{Trans}_G \to \text{Man} \) is given on arrows by

\[
\varpi_T((N,H) \overset{(f,\alpha)}{\to} (M,F)) = (N \overset{f}{\to} M).
\]

We would like to extend Theorem 4.1 to a statement about maps of stacks. As a first step we prove
Lemma 5.3. The collection of functors
\[ \{ \text{hol}_M : B^\nabla G(M) \to \text{Trans}_G(M) \}_{M \in \text{Man}} \]
extends to a functor
\[ \text{hol} : B^\nabla G \to \text{Trans}_G, \]
which is a morphism of categories fibered in groupoids over \( \text{Man} \).

Proof. Let \( f : (P \to M, A) \to (P' \to M', A') \) be a map of principal \( G \)-bundles with connections. The connections \( A \) and \( A' \) define transport functors \( \text{hol}(P, A) : \Pi^{\text{thin}}(M) \to G\text{-tors} \) and \( \text{hol}(P', A') : \Pi^{\text{thin}}(M') \to G\text{-tors} \), respectively. We need to define a morphism
\[ \text{hol}(f) : \text{hol}(P, A) \to \text{hol}(P', A') \]
in \( \text{Trans}_G \). Such a morphism is a pair of the form \( (\bar{f}, \eta) \) where \( \eta : \text{hol}(P, A) \Rightarrow \text{hol}(P', A') \circ \Pi^{\text{thin}}(\bar{f}) \) is a natural transformation (see Definition 5.2 above) and \( \bar{f} : M \to M' \) the induced map on the base.

Recall that the objects of the groupoid \( \Pi^{\text{thin}}(M) \) are points of \( M \). For each point \( x \in M \), define
\[ \eta_x := f|_{P_x} : P_x \to P'_{f(x)}. \]
We check that the collection \( \{\eta_x\}_{x \in M} \) is a natural transformation. Since (3.4) holds, the diagram
\[
\begin{array}{ccc}
P_x & \xrightarrow{\|_\gamma} & P_y \\
\downarrow f|_{P_x} & & \downarrow f|_{P_y} \\
P'_{f(x)} & \xrightarrow{\|_{f \circ \gamma}} & P'_{f(y)}
\end{array}
\]
commutes. By definition, \( \text{hol}(P, A)([\gamma]) = \|_\gamma \), the parallel transport along \( \gamma \) in \( P \) defined by the connection \( A \). On the other hand \( \Pi^{\text{thin}}(\bar{f})([\gamma]) = [\bar{f} \circ \gamma] \) and \( \text{hol}(P', A')([\bar{f} \circ \gamma]) = \|_{f \circ \gamma}. \) Hence the diagram
\[
\begin{array}{ccc}
P_x & \xrightarrow{\text{hol}(P, A)([\gamma])} & P_y \\
\downarrow \eta_x & & \downarrow \eta_y \\
P'_{f(x)} & \xrightarrow{\text{hol}(P', A')(\Pi^{\text{thin}}(\bar{f}))(\[\gamma\])} & P'_{f(y)}
\end{array}
\]
commutes for every arrow \( x \xrightarrow{[\gamma]} y \) in \( \Pi^{\text{thin}}(M) \). Therefore \( \eta \) is a natural transformation. Thus \( \text{hol}(f) \) is a morphism in \( \text{Trans}_G \).

It is not hard to check that \( \text{hol} \) is actually a functor. Finally the functor \( \text{hol} \) commutes with the projections \( \varpi_B : B^\nabla G \to \text{Man} \), \( \varpi_T : \text{Trans}_G \to \text{Man} \) to the category of manifolds since
\[
\varpi_B(f : (P \to M, A) \to (P' \to M', A')) = \bar{f}
\]
and
\[
\varpi_T(\text{hol}(f : (P \to M, A) \to (P' \to M', A'))) = \varpi(\bar{f}, \eta) = \bar{f}. \]

We are now in position to state and prove the main result of the paper. The proof is short since most of the work has already been done.

Theorem 5.4. The functor
\[ \text{hol} : B^\nabla G \to \text{Trans}_G; \]
is an equivalence of categories fibered in groupoids over \( \text{Man} \).
Definition 6.1. Let $X$ be a stack over the category (site) $\text{Man}$ of manifolds. Recall that $B^\nabla G \to \text{Man}$ is a stack. The proof is the same as the proof that the CFG $BG \to \text{Man}$ is a stack: principal $G$-bundles glue and connection 1-forms on the principal bundles glue as well. Since the CFG $\text{Trans}_{G} \to \text{Man}$ is equivalent to the CFG $B^\nabla G \to \text{Man}$ and since $B^\nabla G$ is a stack, $\text{Trans}_{G}$ is a stack.

Corollary 5.5. The category $\text{Trans}_{G}$ of transport functors is a stack over the category (site) $\text{Man}$ of manifolds.

Proof. Recall that $B^\nabla G \to \text{Man}$ is a stack. The proof is the same as the proof that the CFG $BG \to \text{Man}$ is a stack: principal $G$-bundles glue and connection 1-forms on the principal bundles glue as well. Since the CFG $\text{Trans}_{G} \to \text{Man}$ is equivalent to the CFG $B^\nabla G \to \text{Man}$ and since $B^\nabla G$ is a stack, $\text{Trans}_{G}$ is a stack.

6. PRINCIPAL BUNDLES OVER STACKS AND PARALLEL TRANSPORT

In this section we work out some consequences of Theorem 5.4 for principal bundles with connections over stacks. We start with a definition of a principal $G$-bundle over a stack (where as before $G$ is a Lie group) which is known to experts [4, 5]. We then define principal bundles with connections over stacks and the associated parallel transport. We prove that for each stack $\mathcal{X}$ the functor $\text{hol}$ induces an equivalence between the category of principal $G$-bundles with connections over $\mathcal{X}$ and the category of corresponding transport functors.

Definition 6.1. Let $G$ be a Lie group. A principal $G$-bundle over a stack $\mathcal{X} \to \text{Man}$ is a a 1-morphism of stacks $p : \mathcal{X} \to BG$, where $BG$ denotes the stack of principal $G$-bundles.

There are several reasons why this definition makes sense.

- If the stack $\mathcal{X}$ is a manifold $M$ then by the 2-Yoneda lemma (see [22], for example) there is an equivalence of categories $[M, BG] \xrightarrow{\cong} BG(M)$. Under this equivalence a functor $p \in [M, BG]$ corresponds to the principal bundle $p(id_M)$ over $M$. Thus functors $p : M \to BG$ “are” principal $G$-bundles over $M$.

- Suppose the stack $\mathcal{X}$ is a stack quotient $[\Gamma_0/\Gamma_1]$ of a Lie groupoid $\Gamma$. The bicategory $\text{Bi}$ of Lie groupoids, bibundles and bibundle isomorphism is 2-equivalent to the 2-category of geometric stacks over $\text{Man}$ (see [15] or [6]). Consequently the functor category $[[\Gamma_0/\Gamma_1], BG]$ is equivalent to the category of bibundles from $\Gamma$ to the action groupoid $\{G \equiv *\}$:

$$[[\Gamma_0/\Gamma_1], BG] \xrightarrow{\cong} \{P : \Gamma_1 \rightrightarrows \Gamma_0 \to \{G \equiv *\} | P \text{ is a right } G \text{ principal bibundle} \}.$$ 

Any bibundle $P : \Gamma_1 \rightrightarrows \Gamma_0 \to \{G \equiv *\}$ is a principal $G$ bundle over the Lie groupoid $\Gamma$ (see [14]).

- The functor category $[[\Gamma_0/\Gamma_1], BG]$ is also equivalent to the cocycle category $BG(\Gamma_1 \rightrightarrows \Gamma_0)$ (see Definition 6.5 and Proposition 6.6 below). Objects of the cocycle category $BG(\Gamma_1 \rightrightarrows \Gamma_0)$ are again principal $G$-bundles over the groupoid $\Gamma$. In particular, if the groupoid $\Gamma$ is a cover groupoid arising from a cover $\{U_\alpha\}$ of a manifold $M$ then the objects of the cocycle category are Čech cocycles with values in the Lie group $G$. Hence they “are” principal $G$-bundles over the manifold $M$.

By analogy with the notion of a principal $G$ bundle over a stack $\mathcal{X}$ we define principal bundles with connections and parallel transport functors over $\mathcal{X}$ as follows.

Definition 6.2. Let $\mathcal{X}$ be a stack over $\text{Man}$ and $G$ a Lie group. We define the category of principal $G$ bundles with connection over the stack $\mathcal{X}$ to be the functor category $[\mathcal{X}, B^\nabla G]$.

In particular a principal bundle with connection over a stack $\mathcal{X}$ is a 1-morphism of stacks $F : \mathcal{X} \to B^\nabla G$. 

Proof. Recall that a functor between two categories fibered in groupoids is an equivalence of categories if and only if its restriction to each fiber is an equivalence of categories; see for example [22, Proposition 3.36]. By Theorem 4.1 for each manifold $M$ the functor $\text{hol}_M : B^\nabla G \to \text{Trans}_{G}(M) = \text{Trans}_{G}(M)$ is an equivalence of categories.

As an immediate corollary of Theorem 5.4, we obtain:

6.2. Простые пучки над стакками и параллельный транспорт

В этом разделе мы разработаем некоторые следствия Теоремы 5.4 для простых пучков с соединениями над стакками. Мы начинаем с определения простого $G$-пучка над стакком (где $G$ является линейным группой) которое известно специалистам [4, 5]. Мы затем определяем простые пучки с соединениями над стакками и соответствующие транспортные функции. Мы доказываем, что для каждого стакка $\mathcal{X}$ функция $\text{hol}$ индуцирует эквивалентность между категорией простых $G$-пучков с соединениями над $\mathcal{X}$ и категорией соответствующих транспортных функций.

Определение 6.1. Пусть $G$ — линейная группа. Простым $G$-пучком над стакком $\mathcal{X} \to \text{Man}$ является 1-морфизм стакка $p : \mathcal{X} \to BG$, где $BG$ — стакк простых $G$-пучков.

Существуют несколько причин, по которым этот определение имеет смысл.

- Если стакк $\mathcal{X}$ является пространством $M$ тогда по 2-й цитоне лемме (см. [22] например) существует эквивалентность категорий $[M, BG] \xrightarrow{\cong} BG(M)$. Под этим эквивалентностью функция $p \in [M, BG]$ соответствует простому пучку $p(id_M)$ над $M$. Таким образом, функции $p : M \to BG$ “являются” простыми $G$-пучками над $M$.

- Предположим, что стакк $\mathcal{X}$ является стакком квотирования $[\Gamma_0/\Gamma_1]$ линейного группоида $\Gamma$. Бикатегория $\text{Bi}$ линейных группоидов, бибундлей и бибундлей изоморфизм является эквивалентной 2-категории геометрических стакков над $\text{Man}$ (см. [15] или [6]). Поэтому функция категория $[[\Gamma_0/\Gamma_1], BG]$ эквивалентна категории бибундлей от $\Gamma$ к действию группоиду $\{G \equiv *\}$:

$$[[\Gamma_0/\Gamma_1], BG] \xrightarrow{\cong} \{P : \Gamma_1 \rightrightarrows \Gamma_0 \to \{G \equiv *\} | P \text{ является правым } G \text{ простым бибунделем} \}.$$ 

любые бибундель $P : \Gamma_1 \rightrightarrows \Gamma_0 \to \{G \equiv *\}$ является простым $G$ пучком над группоидом $\Gamma$ (см. [14]).

- Функция категория $[[\Gamma_0/\Gamma_1], BG]$ также эквивалентна категории кокилей $BG(\Gamma_1 \rightrightarrows \Gamma_0)$ (см. Определение 6.5 и Правило 6.6 ниже). Объекты категории кокилей $BG(\Gamma_1 \rightrightarrows \Gamma_0)$ снова являются простыми $G$-пучками над группоидом $\Gamma$. В частности, если группоид $\Gamma$ является покрытием группоида, то покрытие, полученный из покрытия $\{U_\alpha\}$ над пространством $M$ тогда объекты категории кокилей являются Čech кокилами с значениями в линейной группе $G$. Поэтому они “являются” простыми $G$-пучками над пространством $M$.

По аналогии с определением простого $G$ пучка над стакком $\mathcal{X}$ мы определяем простые пучки с соединениями и параллельные транспортные функции над $\mathcal{X}$ следующим образом.

Определение 6.2. Пусть $\mathcal{X}$ — стакк над $\text{Man}$ и $G$ — линейная группа. Мы определяем категорию простых $G$ пучков с соединениями над стакком $\mathcal{X}$ быть функция категория $[\mathcal{X}, B^\nabla G]$.

В частности простым пучком с соединением над стакком $\mathcal{X}$ является 1-морфизм стакков $F : \mathcal{X} \to B^\nabla G$. 

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Definition 6.3. Let $\mathcal{X}$ be a stack over $\text{Man}$ and $G$ a Lie group. We define the category of parallel transport functors over the stack $\mathcal{X}$ to be the functor category $[\mathcal{X}, \text{Trans}_G]$.

In particular a parallel transport functor on a stack $\mathcal{X}$ is a 1-morphism of stacks $T : \mathcal{X} \to \text{Trans}_G$.

We have the following extension of Theorem 4.1 from manifolds to stacks.

Theorem 6.4. For any stack $\mathcal{X}$ over the category of manifolds the functor $\text{hol} : B\nabla G \to \text{Trans}_G$ induces an equivalence of categories

$$\text{hol}^* : [\mathcal{X}, B\nabla G] \to [\mathcal{X}, \text{Trans}_G],$$

where the functor $\text{hol}^*$ is defined by

$$\text{hol}^*(\alpha : F \Rightarrow H) := (\text{hol} \circ \alpha : \text{hol} \circ F \Rightarrow \text{hol} \circ H)$$

for a morphism $(\alpha : F \Rightarrow H) \in [\mathcal{X}, B\nabla G]$.

Proof. Since $\text{hol}$ is an equivalence of categories so is $\text{hol}^*$.

We now interpret the results of the above theorem more concretely in terms of the cocycle category. Recall that a Lie groupoid $\Gamma = \{\Gamma_1 \rightrightarrows \Gamma_0\}$ gives rise to a simplicial manifold. In particular we have three face maps $d_i : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$, $i = 0, 1, 2$ which are defined by

$$d_0(\gamma_1, \gamma_2) = \gamma_2, \quad d_1(\gamma_1, \gamma_2) = \gamma_1\gamma_2 \quad \text{and} \quad d_2(\gamma_1, \gamma_2) = \gamma_1.$$

Definition 6.5 (The category $\mathcal{X}(\Gamma)$ of $\Gamma$ cocycles). Let $\Gamma = \{\Gamma_1 \rightrightarrows \Gamma_0\}$ be a Lie groupoid and $\mathcal{X} \to \text{Man}$ a category fibered in groupoids. We define the category $\mathcal{X}(\Gamma)$ of $\Gamma$ cocycles with values in $\mathcal{X}$ as follows. The objects of $\mathcal{X}(\Gamma)$ are pairs $(p, \varphi)$ where $p$ is an object of $\mathcal{X}(\Gamma_0)$ and $\varphi : s^*p \to t^*p$ is an arrow in $\mathcal{X}(\Gamma_1)$. The morphism $\varphi$ is subject to the cocycle condition

$$d_2^*\varphi \circ d_0^*\varphi = d_1^*\varphi,$$

where $d_i : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$ are the face maps defined above. A morphism from $(p, \varphi)$ to $(p', \varphi')$ is a morphism $\alpha : p \to p'$ in $\mathcal{X}(\Gamma_0)$ such that the diagram

$$\begin{array}{ccc}
  t^*p & \xrightarrow{\varphi} & s^*p \\
  t^*\alpha \downarrow & & \downarrow s^*\alpha \\
  t^*p' & \xrightarrow{\varphi'} & s^*p'
\end{array}$$

commutes.

The following fact is well-known to experts.

Proposition 6.6. Let $\Gamma$ be a Lie groupoid and $\mathcal{X} \to \text{Man}$ a category fibered in groupoids. There is a canonical functor

$$\Sigma : [[\Gamma_0/\Gamma_1], \mathcal{X}] \to \mathcal{X}(\Gamma)$$

from the functor category $[[\Gamma_0/\Gamma_1], \mathcal{X}]$ to the cocycle category $\mathcal{X}(\Gamma)$.

If moreover $\mathcal{X}$ is a stack then $\Sigma$ is an equivalence of categories (i.e., an isomorphism of stacks).

Proof. We follow the custom of identifying a manifold $M$ with the stack $\text{Hom}(\cdot, M)$ without further comment. Recall that the stack quotient $[\Gamma_0/\Gamma_1]$ is a geometric stack. The canonical atlas $p : \Gamma_0 \to [\Gamma_0/\Gamma_1]$ is characterize by the fact that $p(id_{\Gamma_0})$ is the principal $\Gamma$ bundle $t : \Gamma_1 \to \Gamma_0$ (see...
Recall further that the manifold $\Gamma_1$ is the 2-categorical fiber product $\Gamma_0 \times_{p,[\Gamma_0/\Gamma_1],p} \Gamma_0$ and the diagram

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{t} & \Gamma_0 \\
\downarrow{s} & \downarrow{\beta} & \downarrow{p} \\
\Gamma_0 & \xrightarrow{p} & [\Gamma_0/\Gamma_1]
\end{array}
\]


2-commutes. Also

\[
(\beta \circ d_2) * (\beta \circ d_0) = \beta \circ d_1
\]

as natural isomorphisms from $p \circ s \circ d_0 = p \circ s \circ d_1$ to $p \circ t \circ d_2 = p \circ t \circ d_1$ (here $*$ denotes the vertical composition of natural transformations). Consequently for any CFG $X \to \text{Man}$ and any 1-morphism $f : [\Gamma_0/\Gamma_1] \to X$ of CFGs we have a natural isomorphism

\[
f \circ \beta : f \circ p \circ s \Rightarrow f \circ p \circ t
\]

satisfying

\[
(f \circ \beta \circ d_2) * (f \circ \beta \circ d_0) = f \circ \beta \circ d_1.
\]

Consider now the object $P := (f \circ p) (id_{\Gamma_0}) \in \mathcal{X}(\Gamma_0)$. For any map $h : \Gamma_1 \to \Gamma_0$

\[
(f \circ p \circ h) (id_{\Gamma_1}) = h^*((f \circ p) (id_{\Gamma_0}))
\]

Consequently

\[
\varphi := (f \circ \beta) (id_{\Gamma_1})
\]

is an isomorphism in $\mathcal{X}(\Gamma_0)$ from $s^*P$ to $t^*P$. Equation (6.2) translates then into the cocycle condition

\[
(d_2^* \varphi) \circ (d_0^* \varphi) = d_1^* \varphi.
\]

Therefore the pair $(P = (f \circ p) (id_{\Gamma_0}), \varphi = (f \circ \beta) (id_{\Gamma_1}))$ is an object of the cocycle category $\mathcal{X}(\Gamma)$. This defines the functor $\Sigma$ on objects. Similarly given a morphism $\gamma : f \Rightarrow h$ in the functor category $[[\Gamma_0/\Gamma_1],\mathcal{X}]$ we get a morphism

\[
\alpha = (\gamma \circ p) (id_{\Gamma_0}) : (f \circ p) (id_{\Gamma_0}) \to (h \circ p) (id_{\Gamma_0})
\]

in $\mathcal{X}(\Gamma_0)$. It is not hard to check $\alpha$ is a morphism in $\mathcal{X}(\Gamma)$ from $\Sigma(f)$ to $\Sigma(h)$. This defines the functor $\Sigma$ on morphisms.

A proof that $\Sigma$ is an equivalence of categories if $\mathcal{X}$ is a stack is a bit more involved. We refer an interested reader to [19][Proposition 3.19].

We are now in position to reformulate Theorem 6.4 for geometric stacks in terms of cocycles.

**Theorem 6.7.** For any Lie groupoid $\Gamma$ the functor $\text{hol}$ induces an equivalence of categories

\[
\text{hol}_\Gamma : B^\nabla G(\Gamma) \to \text{Trans}_G(\Gamma).
\]

**Proof.** By Theorem 6.4 the functor categories $[[\Gamma_0/\Gamma_1],B^\nabla G]$ and $[[\Gamma_0/\Gamma_1],\text{Trans}_G]$ are equivalent. By Proposition 6.6 the first functor category is equivalent to the cocycle category $B^\nabla G(\Gamma)$ and the second functor category is equivalent to the cocycle category $\text{Trans}_G(\Gamma)$. 

Alternatively one can view Theorem 6.7 as an instant of the following general fact:

**Proposition 6.8.** Suppose $\Gamma$ is a Lie groupoid, $\mathcal{X}, \mathcal{Y} \to \text{Man}$ are two categories fibered in groupoids and $F : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of fibered categories. Then the functor $F$ induces a functor

\[
F_\Gamma : \mathcal{X}(\Gamma) \to \mathcal{Y}(\Gamma)
\]

between cocycle categories. Moreover if $F$ is an equivalence of categories then so is $F_\Gamma$.

We discuss a proof of Proposition 6.8 in Appendix B.
Part 3. Appendices

APPENDIX A. DIFFEOREGICAL SPACES AND DIFFEOREGICAL GROUPOIDS

The goal of this section is to recall two definitions and some properties of the category Diff\text{eol} of diffeological spaces and to prove the folklore result that the thin fundamental groupoid is a groupoid internal to the category of diffeological spaces. We start by recalling the “traditional” definition of a diffeological space, which is due to Souriau [21]. A similar notion was independently introduced by K.-T. Chen [8, 9]. Our primary references are [11] and [1].

Definition A.1 (Diffeology). A diffeology on a set \( X \) is a collection of functions \( D_X \subset \{ f : U \to X \mid U \subset \mathbb{R}^n \text{ open}, \ n \in \mathbb{N} \} \) satisfying the following three conditions:

1. every constant map is in \( D_X \);
2. if \( V \subset \mathbb{R}^m \) is open, \( f : U \to X \) is in \( D_X \), and \( g : V \to U \) is a smooth map, then the composite \( f \circ g : V \to X \) is also in \( D_X \);
3. if \( \{ U_i \} \) is an open cover of \( U \subset \mathbb{R}^n \) and \( f : U \to X \) is a map of sets such that \( f|_{U_i} \to X \) is in \( D_X \), then \( f : U \to X \) is in \( D_X \).

The pair \( (X, D_X) \) is called a diffeological space, and the elements of \( D_X \) are called plots.

Remark A.2. Just as in the case of topological spaces, it is common to refer to a diffeological space \( (X, D_X) \) simply as \( X \) with the choice of a diffeology \( D_X \) suppressed from the notation.

Example A.3. Any smooth manifold \( M \) is a diffeological space. The set in question is the underlying set of the manifold \( M \), and the collection of plots \( D_M \) consists of all smooth maps from all open subsets \( U \) of \( \bigsqcup_{n=0}^{\infty} \mathbb{R}^n \) to the manifold \( M \).

Definition A.4. A map of diffeological spaces or a smooth map from a diffeological space \( (X, D_X) \) to a space \( (Y, D_Y) \) is a map of sets \( f : X \to Y \) such that for any plot \( p : U \to X \) in \( D_X \), the composite \( f \circ p \) is in \( D_Y \).

Remark A.5 (The category Diff\text{eol} of diffeological spaces). The composite of two smooth maps between diffeological spaces is smooth. Thus, diffeological spaces and smooth maps form a category which we denote by Diff\text{eol}. It is well-known [11] that the category of manifolds embeds into the category of diffeological spaces. That is, a map \( f : M \to N \) between two manifolds is smooth in the diffeological sense if and only if it is a smooth map of manifolds.

The category Diff\text{eol} has many nice properties [16, 1]. For instance, it has all small limits and colimits. Also the space of maps between two diffeological spaces is again naturally a diffeological space. It will be useful for us to write down explicitly several corresponding constructions. We start with a definition.

Definition A.6 (Subspace diffeology). Let \( (X, D_X) \) be a diffeological space and \( Y \subset X \) a subset. The \textit{subspace diffeology} \( D_Y \) is the set:

\[
D_Y := \{ (p : U \to X) \in D_X \mid p(U) \subset Y \}
\]

To introduce the quotient diffeology it is convenient to switch our point of view and think of diffeological spaces as certain kinds of sheaves of sets.

Definition A.7 (The category Open). The objects of the category Open are by definition all open subsets of all coordinate vector spaces \( \mathbb{R}^n, n \geq 0 \), or, equivalently, open subsets of \( \bigsqcup_{n=0}^{\infty} \mathbb{R}^n \). A morphism in Open from an open set \( U \) to an open set \( V \) is a smooth map \( f : U \to V \).
Definition A.8. A (set-valued) presheaf $\mathcal{R}$ on a category $\text{Open}$ is a contravariant functor from $\text{Open}$ to the category $\text{Set}$ of sets:

$$\mathcal{R} : \text{Open}^{\text{op}} \to \text{Set}.$$ 

Notation A.9. Given a presheaf $\mathcal{R} : \text{Open}^{\text{op}} \to \text{Set}$ and an arrow $f : U \to V$ in $\text{Open}$, we get a map of sets $\mathcal{R}(f) : \mathcal{R}(V) \to \mathcal{R}(U)$, which we think of as a pullback along $f$. Thus, given an element $s \in \mathcal{R}(V)$, we denote $\mathcal{R}(f)s \in \mathcal{R}(U)$ by $f^*s$. If $U \subset V$ and $f$ is the inclusion, we may also write $s|_{U}$ for $f^*s$.

Definition A.10. A presheaf $\mathcal{S} : \text{Open}^{\text{op}} \to \text{Set}$ is a sheaf if for any open set $U \in \text{Open}$, any open cover $\{U_i\}_{i \in I}$ of $U$, and any collection of elements $s_i \in \mathcal{S}(U_i)$ with

$$s|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

there exists a unique element $s \in \mathcal{S}(U)$ with

$$s|_{U_i} = s_i.$$

Example A.11. Any set $X$ defines a sheaf $X$ on $\text{Open}$:

$$X(U \xrightarrow{f} V) := \text{Map}(U, X) \xleftarrow{f^*} \text{Map}(V, X),$$

where $\text{Map}(U, X)$ denotes the set of maps from $U$ to $X$ and $f^*$ denotes the pullback by $f$. Note that for the one element set $\ast := \mathbb{R}^0 \in \text{Open}$ the evaluation map

$$X(\ast) = \text{Map}(\ast, X) \xrightarrow{\text{Eval}} X$$

is a bijection of sets.

Definition A.12. A presheaf $\mathcal{R} : \text{Open}^{\text{op}} \to \text{Set}$ is a subpresheaf of a presheaf $\mathcal{S} : \text{Open}^{\text{op}} \to \text{Set}$ if for every open set $U \in \text{Open}$

$$\mathcal{R}(U) \subset \mathcal{S}(U).$$

We are now in position to define diffeologies in terms of sheaves.

Definition A.13 (Diffeology, as a sheaf). Let $\ast$ be the one element set $\mathbb{R}^0$. A diffeology $D = D_X$ on a set $X$ is a subsheaf of the sheaf $X$, defined in Example A.11, such that $D(\ast) = X(\ast)$.

Remark A.14. If $D$ is a diffeology on a set $X$ in the sense of Definition A.13, then $D(U) \subset X(U) \equiv \text{Map}(U, X)$ for any $U \in \text{Open}$. It is for this reason that elements of $D(U)$ may be thought of as the plots of Definition A.1.

Since the category of diffeological spaces has all small colimits (see [1]), it has quotients. Explicitly they can be constructed as follows.

Construction A.15 (Quotient diffeology). Let $(X, D_X)$ be a diffeological space, $R \subset X \times X$ an equivalence relation, $Y = X/R$ the set of equivalence classes and $q : X \to Y$ the quotient map. The quotient diffeology $D_Y$ on $Y$ is constructed as the sheafification of the presheaf $D_{pre}(Y)$ defined by

$$D_{pre}(Y) := \{p : U \to Y \mid \text{there is } (\tilde{p} : U \to X) \in D_X \text{ with } q \circ \tilde{p} = p\}.$$ 

Explicitly, for any open $U \subset \mathbb{R}^n$, a map $p : U \to Y$ is a plot in the quotient diffeology $D_Y$ if and only if for every $u \in U$ there is an open neighborhood $V$ of $u$ in $U$ and $\tilde{p} : V \to X$ with

$$q \circ \tilde{p} = p|_{V}.$$ 

The quotient map $q : (X, D_X) \to (Y, D_Y)$ has the following universal property. Give the product $X \times X$ the product diffeology $D_X \times D_X$ (see Remark A.18 below) and $R \subset X \times X$ the subspace diffeology. Then $X \xrightarrow{\Delta} Y$ is the coequalizer of the diagram $R \rightrightarrows X$ in the category $\text{Diffeol}$ of diffeological spaces.
Lemma A.16. Let $q: X \to Y$ be a quotient map between two diffeological spaces (that is, $q$ is the coequalizer of a diagram $R \rightrightarrows X$ in Diff$\text{eol}$ for some equivalence relation $R$ on $X$), and let $Z$ be another diffeological space. Then a map $f: Y \to Z$ is smooth if and only if for any plot $p: U \to X$ the composite $f \circ q \circ p: U \to Z$ is a plot on $Z$.

Proof. If $f$ is smooth, then $q \circ f$ is smooth. Therefore for any plot $p: U \to X$ the composite $f \circ q \circ p: U \to Z$ is a plot on $Z$.

Conversely suppose that for any plot $p: U \to X$ the composite $f \circ q \circ p: U \to Z$ is a plot on $Z$ and suppose $r: U \to Y$ is a plot. By definition of quotient diffeology there is an open cover $\{U_\alpha\}$ of $U$ and a collection of plots $r_\alpha: U_\alpha \to X$ so that

$$q \circ r_\alpha = r|_{U_\alpha}.$$  

By assumption, 

$$(f \circ r)|_{U_\alpha} = f \circ (r|_{U_\alpha}) = f \circ q \circ r_\alpha: U_\alpha \to Z$$

are plots. Since a diffeology is a sheaf, $f \circ r$ is a plot. Therefore $f$ is smooth. □

Since the category of diffeological spaces has all small limits, it has fiber products. Explicitly, they can be constructed as follows.

Construction A.17 (Fiber product diffeology). Let $f: (X, D_X) \to (Z, D_Z)$ and $g: (Y, D_Y) \to (Z, D_Z)$ be two maps of diffeological spaces. We construct their fiber product as follows: the underlying set is the fiber product

$$X \times_{f,Z,g} Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

and the set of plots is

$$D(X \times_{f,Z,g} Y) := \{(p_X, p_Y) \in D_X \times D_Y \mid f \circ p_X = g \circ p_Y\}.$$  

It is not hard to check that the diffeological space $(X \times_{f,Z,g} Y, D(X \times_{f,Z,g} Y))$ together with the obvious maps to $(X, D_X)$ and $(Y, D_Y)$ is a fiber product in the category of diffeological spaces. □

Remark A.18. If $Z$ is a single point then the fiber product $(X \times_{f,Z,g} Y, D(X \times_{f,Z,g} Y))$ is the product of $(X, D_X)$ and $(Y, D_Y)$. Thus the construction of the fiber product diffeology includes the construction of the (binary) product diffeology as a special case.

Next we construct a diffeology on the space $\mathcal{P}(M)$ of paths with sitting instances in a manifold $M$ (q.v. Notation 2.3). By the quotient construction this then defines a diffeology on the set $\mathcal{P}(M)/\sim$ of paths modulo thin homotopy.

Definition A.19 (Path space diffeology). As before, denote the set of paths with sitting instances in a manifold $M$ by $\mathcal{P}(M)$ (all paths in $\mathcal{P}(M)$ are parameterized by $[0, 1]$). Let $U$ be an open set in some $\mathbb{R}^n$. Define a map of sets $p: U \to \mathcal{P}(M)$ to be a plot if the associated map

$$\hat{p}: U \times [0, 1] \to M, \quad \hat{p}(u, x) := p(u)(x)$$  

is smooth. □

Remark A.20. Strictly speaking, one should check that the collection of plots on the set of paths $\mathcal{P}(M)$ given in Definition A.19 forms a sheaf on $\text{Open}$. This is straightforward, and we leave it to an interested reader.

Remark A.21. If $X$ and $Y$ are two diffeological spaces, then the space $\text{Hom}(X, Y)$ of smooth maps from $X$ to $Y$ is also a diffeological space: $p: U \to \text{Hom}(X, Y)$ is a plot if and only if $\hat{p}: U \times X \to Y$ is smooth. One can show that the mapping space diffeology on $\mathcal{P}(M)$ agrees with the diffeology defined in Definition A.19. The key issue is that a map $\hat{p}: U \times [0, 1] \to M$ is smooth as a map of manifolds with boundary if and only if it is smooth as a map of diffeological spaces, see [11].
Lemma A.22. Let $\mathcal{P}(M)$ be the set of paths with sitting instances in a manifold $M$ with path space diffeology (q.v. A.19).

1. The evaluation maps

$$ev_0 : \mathcal{P}(M) \to M, \quad \gamma \mapsto \gamma(0) \quad \text{and} \quad ev_1 : \mathcal{P}(M) \to M, \quad \gamma \mapsto \gamma(1)$$

are smooth.

2. The concatenation map

$$\tilde{m} : \mathcal{P}(M) \times_{ev_0, M, ev_1} \mathcal{P}(M) \to \mathcal{P}(M)$$

defined by:

$$\tilde{m}(\gamma, \tau)(t) := \begin{cases} \tau(2t) & \text{if} \quad t \in [0, 1/2] \\ \gamma(2t - 1) & \text{if} \quad t \in [1/2, 1] \end{cases} \quad (A.2)$$

is smooth.

Proof. By Definition A.19 of the path space diffeology, for any plot $p : U \to \mathcal{P}(M)$, we have $ev_0 \circ p = \tilde{p}|_{\{0\}}$, where $\tilde{p} : U \times [0, 1] \to M$ is the associated map. Since $\tilde{p}$ is smooth the map $ev_0$ is smooth as well. A similar argument shows that the map $ev_1$ is smooth.

Recall that a plot $p$ of the fiber product $\mathcal{P}(M) \times_M \mathcal{P}(M)$ is given by a pair of plots

$$p = (p_1 : U \to \mathcal{P}(M), \quad p_2 : U \to \mathcal{P}(M))$$

with $p_1(x)(1) = p_2(x)(0)$ for all $x \in U$ (Construction A.17). We have

$$\tilde{m} \circ p(u, t) = \begin{cases} p_1(u, 2t) & \text{if} \quad t \in [0, 1/2] \\ p_2(u, 2t - 1) & \text{if} \quad t \in [1/2, 1] \end{cases}$$

Since the map $\tilde{m} \circ p$ is smooth for all plots $p$, the map $\tilde{m} \circ p$ is smooth for all plots $p$. Hence the map $\tilde{m}$ is smooth. \qed

Definition A.23 (Diffeology on the space of thin homotopy classes of paths). As before, denote the set of thin homotopy classes of paths with sitting instances in a manifold $M$ by $\mathcal{P}(M)/\sim$. We define the diffeology on $\mathcal{P}(M)/\sim$ to be the quotient diffeology induced by the map $\mathcal{P}(M) \to \mathcal{P}(M)/\sim$.

Definition A.24 (q.v. [11, 8.3]). A diffeological groupoid $\Gamma$ is a groupoid object in the category $\mathbf{Diff}$ of diffeological spaces. That is, the sets of objects and arrows, $\Gamma_0$ and $\Gamma_1$, are diffeological spaces, and the structure maps $s, t, m, i, u$ (q.v. Notation 2.10) are maps of diffeological spaces.

As observed in the Definition/Proposition 2.9, for a manifold $M$, the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ is a groupoid with the set of objects the manifold $M$ and the set of morphisms the set $\mathcal{P}(M)/\sim$ of paths modulo thin homotopy. We are now in position to state and prove the following folklore result.

Proposition A.25. The thin fundamental groupoid $\Pi^{\text{thin}}(M)$ of a manifold $M$ is a diffeological groupoid.

Proof. We need to show that the five structure maps $s, t, m, u, and i$ of the groupoid $\Pi^{\text{thin}}(M) = \{\mathcal{P}(M)/\sim\} \rightrightarrows M$ are maps of diffeological spaces, i.e. smooth (q.v. Notation 2.10).

We start with the source map $s$. Recall that it is defined by

$$s[\gamma] = \gamma(0)$$

for any class $[\gamma] \in \mathcal{P}(M)/\sim$. Let $q : \mathcal{P}(M) \to \mathcal{P}(M)/\sim$ be the quotient map. By Lemma A.16, it is enough to show that for any plot $p : U \to \mathcal{P}(M)$ the composite map $f(u) := (s \circ q \circ p)(u)$ is smooth. However, $s \circ q$ is just the map $ev_0 : \mathcal{P}(M) \to M$ which is smooth by Lemma A.22. So $s$ is smooth. A similar argument shows that $t$ is smooth.
To show that the multiplication map $m$ is smooth, we need to show that for any plot $p : U \to \mathcal{P}(M)/\sim \times_M \mathcal{P}(M)/\sim$ the composite map $m \circ p : U \to \mathcal{P}(M)/\sim$ is a plot for the quotient diffeology on $(\mathcal{P}(M)/\sim)$. Recall that $m : (\mathcal{P}(M)/\sim) \times_M (\mathcal{P}(M)/\sim) \to \mathcal{P}(M)/\sim$ is defined by

$$m([\gamma], [\tau]) := [\tilde{m}(\gamma, \tau)]$$

where $\tilde{m} : \mathcal{P}(M) \times_M \mathcal{P}(M) \to \mathcal{P}(M)$ is the concatenation:

$$\tilde{m}(\gamma, \tau)(t) := \begin{cases} \tau(2t) & \text{if } t \in [0, 1/2] \\ \gamma(2t - 1) & \text{if } t \in [1/2, 1] \end{cases}$$

(q.v. (2.1)). In other words, $m$ is defined to make the diagram

$$\begin{array}{ccc} \mathcal{P}(M) \times_M \mathcal{P}(M) & \xrightarrow{\tilde{m}} & \mathcal{P}(M) \\ (q,q) \downarrow & & \downarrow q \\ (\mathcal{P}(M)/\sim) \times_M (\mathcal{P}(M)/\sim) & \xrightarrow{m} & \mathcal{P}(M)/\sim \end{array}$$

commute. Recall once more that a plot $r : U \to \mathcal{P}(M)/\sim \times_M \mathcal{P}(M)/\sim$ is of the form $r = (r_1, r_2)$ where $r_1 : U \to \mathcal{P}(M)/\sim$ are plots with $s \circ r_1 = t \circ r_2$. Since both $r_1$ and $r_2$ locally lift with respect to the quotient map $q : \mathcal{P}(M) \to \mathcal{P}(M)/\sim$ to plots of $\mathcal{P}(M)$, it follows that for each point $x \in U$, we can find an open neighborhood $V$ of $x$ in $U$ and plots $s_i : V \to \mathcal{P}(M)$, such that $q \circ s_i = r_i | V$. Furthermore, since $s \circ q = ev_0$ and $t \circ q = ev_1$, the map $s = (s_1, s_2)$ is a plot for the fiber product diffeology on $\mathcal{P}(M) \times_M \mathcal{P}(M)$.

By Lemma A.22 the concatenation $\tilde{m}$ is smooth. Hence $\tilde{m} \circ s$ is a plot for $\mathcal{P}(M)$ which descends to the plot $q \circ \tilde{m} \circ s$ for $\mathcal{P}(M)/\sim$. Since $\text{A}(3)$ commutes, it follows that $m \circ r$ is locally a plot for $\mathcal{P}(M)$. Since diffeologies are sheaves, $r$ is globally a plot. Therefore the map $m$ is smooth. The proof that the structure maps $i$ and $u$ are smooth is similar. \qed

We end the appendix with two technical results.

**Lemma A.26.** For any manifold $M$ the target map $t : \mathcal{P}(M)/\sim \to M$ of the thin fundamental groupoid $\Pi^{\text{thin}}(M)$ has local sections. More precisely, for any $[\gamma] \in \mathcal{P}(M)/\sim$, there is a neighborhood $U$ of $x = \gamma(1) = t([\gamma])$ and a smooth section $\sigma : U \to \mathcal{P}(M)/\sim$ of $t$ with $\sigma(x) = [\gamma]$.

**Proof.** Choose a coordinate chart $\varphi : U \to \mathbb{R}^n$ ($n = \dim M$) with $x \in U$, $\varphi(x) = 0$ and $\varphi(U) = \mathbb{R}^n$. The map

$$p : U \times [0, 1] \to M, \quad p(y, t) := \varphi^{-1}(\beta(t) \cdot \varphi(y))$$

is smooth. Here $\beta \in C^\infty([0, 1])$ is the function in Remark 2.2. Then

$$\tilde{p} : U \to \mathcal{P}(M), \quad \tilde{p}(y)(t) := \varphi^{-1}(\beta(t) \cdot \varphi(y))$$

is smooth. By construction, $\tilde{p}(x)$ is the constant curve $1_x$. Also $\tilde{p}(y)(0) = x$ and $\tilde{p}(y)(1) = y$ for all $y \in U$. Hence

$$[\tilde{p}] := q \circ \tilde{p} : U \to \mathcal{P}(M)/\sim$$

is a section of $t$ with $[\tilde{p}](x) = [1_x]$ and $s([\tilde{p}(y)]) = x$ for all $y \in U$. Since the multiplication $m : \mathcal{P}(M)/\sim \times_M \mathcal{P}(M)/\sim \to \mathcal{P}(M)/\sim$ is smooth, the map

$$\sigma : U \to \mathcal{P}(M)/\sim, \quad \sigma(y) := [\tilde{p}(y)][\gamma]$$

is smooth. By construction, $\sigma$ is a desired section. \qed

**Proposition A.27.** The assignment $M \mapsto \Pi^{\text{thin}}(M)$ extends to a functor

$$\Pi^{\text{thin}} : \text{Man} \to \text{DiffGpd}$$

from the category $\text{Man}$ of manifolds to the category $\text{DiffGpd}$ of diffeological groupoids.
**Proof.** Let \( f : M \to N \) be a smooth map between two manifolds. We need to define a smooth functor \( \Pi^\text{thin}(f) : \Pi^\text{thin}(M) \to \Pi^\text{thin}(N) \) so that on objects it is the map \( f : M \to N \).

Define \( f_* : \mathcal{P}(M) \to \mathcal{P}(N) \) by

\[
f_*(\gamma) = f \circ \gamma
\]

for all paths \( \gamma \in \mathcal{P}(M) \). If \( p : U \to \mathcal{P}(M) \) is a plot, then

\[
((f_* \circ p)(u))(t) = f(p(u)(t)) = f(\hat{p}(u,t)),
\]

where, as before, \( \hat{p}(u,t) = p(u)(t) \). By definition of the diffeology on the space of paths, the map \( \hat{p} \) is smooth. Hence \( (u,t) \mapsto ((f_* \circ p)(u))(t) \) is smooth. It follows

\[
f_* \circ p : U \to \mathcal{P}(N)
\]

is a plot. Therefore \( f_* \) is smooth. Follow \( f_* \) by the quotient map \( q_N : \mathcal{P}(N) \to \mathcal{P}(N)/\sim \). If \( H : \gamma_0 \Rightarrow \gamma_1 \) is a thin homotopy between two paths in \( M \) then \( f \circ H : f_*\gamma_0 \Rightarrow f_*\gamma_1 \) is a thin homotopy between their images in \( \mathcal{P}(N) \). Hence \( q_N \circ f_* \) induces a smooth map \( [f_*] : \mathcal{P}(M)/\sim \to \mathcal{P}(N)/\sim \) making the diagram

\[
\begin{array}{ccc}
\mathcal{P}(M) & \xrightarrow{f_*} & \mathcal{P}(N) \\
q_M \downarrow & & \downarrow q_N \\
\mathcal{P}(M)/\sim & \xrightarrow{[f_*]} & \mathcal{P}(N)/\sim
\end{array}
\] (A.4)

commute. Note that

\[
s([f_*]([\gamma])) = s([f \circ \gamma]) = f(\gamma(0)) = f(s([\gamma])).
\]

Similarly \( s([f_*]([\gamma])) = f(t([\gamma])) \). Hence

\[
\begin{array}{ccc}
\mathcal{P}(M)/\sim & \xrightarrow{[f_*]} & \mathcal{P}(N)/\sim \\
(s,t) \downarrow & & \downarrow (s,t) \\
M \times M & \xrightarrow{f} & N \times N
\end{array}
\] (A.5)

commutes. If \( \gamma, \tau : [0,1] \to M \) are two paths with \( \gamma(0) = \tau(1) \) then \( f \circ \gamma(0) = f \circ \tau(1) \). Moreover the concatenation of \( f \circ \gamma \) and \( f \circ \tau \) is the concatenation of \( \gamma \) and \( \tau \) followed by \( f \):

\[
(f \circ \gamma)(f \circ \tau) = f \circ (\gamma \tau).
\]

It follows from the definition of multiplication in the thin fundamental groupoid and the map \([f_*]\) that

\[
[f_*]([\gamma][\tau]) = [f \circ \gamma][f \circ \tau].
\]

Thus \([f_*]\) preserves multiplication. Since \( f_* \) applied to a constant path \( 1_x \) is the constant path \( 1_{f(x)} \), \([f_*]\) preserves identities. Define \( \Pi^\text{thin}(f) \) on objects to be \( f \) and on arrows \([f_*]\). Then \( \Pi^\text{thin}(f) \) is a functor. It is smooth by construction. \( \square \)

**Appendix B. Functors between the cocycle categories**

In this appendix we prove

**Proposition 6.8** Suppose \( \Gamma \) is a Lie groupoid, \( \mathcal{X}, \mathcal{Y} \to \text{Man} \) are two categories fibered in groupoids and \( F : \mathcal{X} \to \mathcal{Y} \) is a 1-morphism of fibered categories. Then the functor \( F \) induces a functor

\[
F_{\Gamma} : \mathcal{X}(\Gamma) \to \mathcal{Y}(\Gamma)
\]
between cocycle categories. Moreover if $F$ is an equivalence of categories then so is $F_{\Gamma}$.

**Proof.** Recall that objects and arrows of a CFG $\varpi : \mathcal{U} \to \text{Man}$ pull back along maps in $\text{Man}$. More precisely suppose $f : M \to N$ is a morphism in $\text{Man}$ and $x$ is an object of $\mathcal{U}(N)$. Then there is an object $f^* x$ of $\mathcal{U}(M)$ (“a pullback of $x$ by $f$”) and an arrow $\bar{f} : f^* x \to x$ with $\varpi(\bar{f}) = f$.

The pullback $f^* x$ is not unique, but any two pullbacks are isomorphic. If $x_1 \xrightarrow{\xi} x_2$ is a morphism in $\mathcal{U}(M)$ then given the choices of pullbacks $\bar{f}_i : f^* x_i \to x_i$, $i = 1, 2$ there is an unique arrow $f^* \xi : f^* x_1 \to f^* x_2$ making the diagram

$$
\begin{array}{ccc}
  f^* x_1 & \xrightarrow{\bar{f}_1} & x_1 \\
  f^* \xi & \downarrow \quad & \downarrow \xi \\
  f^* x_2 & \xrightarrow{\bar{f}_2} & x_2
\end{array}
$$

commute.

At this point it is convenient to completely switch to simplicial notion and set $d_0 : \Gamma_1 \to \Gamma_0$ to be the source map and $d_1 : \Gamma_1 \to \Gamma_0$ to be the target map.

Suppose $(x, \varphi : d_0^* x \to d_1^* x)$ is an object of $\mathcal{X}(\Gamma)$. Then $F(x)$ is an object of $\mathcal{Y}(\Gamma_0)$. Now choose $d_i^* F(x) := F(d_i^* x)$.

Then $F(\varphi)$ is a morphism in $\mathcal{Y}(\Gamma_0)$ from $d_0^* F(x)$ to $d_1^* F(x)$. We need to check that the pair $(F(x), F(\varphi))$ is an object of $\mathcal{Y}(\Gamma)$, i.e., that $F(\varphi)$ satisfies the cocycle condition. We set

$$
d_i^* d_j^* F(x) := F(d_i^* d_j^* x) \quad i = 0, 1, 2, j = 0, 1.
$$

By definition of an object in $\mathcal{X}(\Gamma)$ the pullbacks

$$
d_i^* \varphi : d_i^* d_0^* x \to d_i^* d_1^* x, \quad i = 0, 1, 2
$$

satisfy the cocycle equation

$$
d_2^* \varphi d_0^* \varphi = d_1^* \varphi.
$$

Since the diagrams

$$
\begin{array}{ccc}
  F(d_i^* d_0^* x) & \xrightarrow{F(d_i^* \varphi)} & F(d_i^* x) \\
  \downarrow & & \downarrow F(\varphi) \\
  F(d_i^* d_1^* x) & \xrightarrow{F(d_i^* d_1^* \varphi)} & F(d_i^* x)
\end{array}
$$

commutes in $\mathcal{Y}$ and since the horizontal arrows project to $d_i : \Gamma_1 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$ $(i = 0, 1, 2)$ we have

$$
F(d_i^* \varphi) = d_i^* F(\varphi).
$$

Hence

$$
d_2^* F(\varphi) d_0^* F(\varphi) = F(d_2^* \varphi) F(d_0^* \varphi) = F(d_2^* \varphi d_0^* \varphi) = F(d_1^* \varphi) = d_1^* F(\varphi).
$$

We conclude that the pair $(F(x), F(\varphi))$ is an object of $\mathcal{Y}(\Gamma)$. Similarly if $\alpha : (x, \varphi) \to (x', \varphi')$ is a morphism in $\mathcal{X}(\Gamma)$ then $F(\alpha) : (F(x), F(\varphi)) \to (F(x'), F(\varphi'))$ is a morphism in $\mathcal{Y}$. It is routine to check that these maps on objects and morphisms do assemble into a functor $F_{\Gamma} : \mathcal{X}(\Gamma) \to \mathcal{Y}(\Gamma)$.

If $F : \mathcal{X} \to \mathcal{Y}$ is an equivalence of categories then it has a weak inverse $H : \mathcal{Y} \to \mathcal{X}$ so that $\gamma : H \circ F \Rightarrow id_{\mathcal{X}}$ and $\delta : F \circ H \Rightarrow id_{\mathcal{Y}}$ for some natural isomorphism $\gamma$ and $\delta$. These natural isomorphisms then give rise to natural isomorphisms

$$
\begin{align*}
\gamma_{\Gamma} & : H_{\Gamma} \circ F_{\Gamma} \Rightarrow id_{\mathcal{X}(\Gamma)}, \\
\delta_{\Gamma} & : F_{\Gamma} \circ H_{\Gamma} \Rightarrow id_{\mathcal{Y}(\Gamma)}
\end{align*}
$$
and the proposition follows.

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