Fractal scale Hilbert spaces and scale Hessian operators

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Abstract
Scale spaces were defined by H. Hofer, K. Wysocki, and E. Zehnder. In this note we introduce a subclass of scale spaces and explain why we believe that this subclass is the right class for a general setup of Floer theory.

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1 Introduction
The definition of a scale Hilbert structure is due to H. Hofer, K. Wysocki, and E. Zehnder [3, 4, 5].

Definition 1.1 A scale Hilbert space is a tuple
\[ \mathcal{H} = \{(H_k, \langle \cdot, \cdot \rangle_k)\}_{k \in \mathbb{N}_0} \]
where for each \( k \in \mathbb{N}_0 \) the pair \((H_k, \langle \cdot, \cdot \rangle_k)\) is a real Hilbert space and the vector spaces \( H_k \) build a nested sequence \( H = H_0 \supset H_1 \supset H_2 \supset \ldots \) such that the following two axioms hold.

(i) For each \( k \in \mathbb{N} \) the inclusion \((H_k, \langle \cdot, \cdot \rangle_k) \hookrightarrow (H_{k-1}, \langle \cdot, \cdot \rangle_{k-1})\) is compact.

(ii) For each \( k \in \mathbb{N}_0 \) the subspace \( H_\infty = \bigcap_{n=0}^\infty H_n \) is dense in \( H_k \) with respect to the topology induced from \( \langle \cdot, \cdot \rangle_k \).

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Remark 1.2 Scale structures can as well be defined on Banach spaces but in this paper we restrict ourselves to the case of Hilbert spaces.

We further recall the notion of scale continuous map from $[3, 4, 5]$. Suppose that $\mathcal{H} = \{(H_k, \langle \cdot, \cdot \rangle_k)\}$ and $\mathcal{H}' = \{(H'_k, \langle \cdot, \cdot \rangle'_k)\}$ are two scale Hilbert spaces.

**Definition 1.3** A map $f : H_0 \to H'_0$ is called scale continuous if for each $k$ the map $f$ restricts to a continuous map $f_k : H_k \to H'_k$.

**Remark 1.4** Using the same notion one can also define scale continuity for maps which are only defined on an open subset of $H_0$.

There are two equivalence relations for scale Hilbert spaces which we explain next.

**Definition 1.5** A linear, scale continuous map $\Phi$ between scale Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ is called a scale isomorphism if for each $k \in \mathbb{N}_0$ its restriction $\Phi_k : H_k \to H'_k$ is a bijection.

**Remark 1.6** It follows from the open mapping theorem that the inverse of a scale isomorphism is a scale isomorphism as well.

**Definition 1.7** A scale isometry $\Phi$ from $\mathcal{H}$ to $\mathcal{H}'$ is a bijective linear map which restricts for each $k$ to an isometry $\Phi_k$ between the Hilbert spaces $H_k$ and $H'_k$.

**Definition 1.8** Two scale Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are called scale isomorphic, if there exists a scale isomorphism from $\mathcal{H}$ to $\mathcal{H}'$. They are called scale isometric, if there exists a scale isometry between them.

**Remark 1.9** Since each scale isometry is a special case of a scale isomorphism, scale isometric scale Hilbert spaces are automatically scale isomorphic.

We also recall from $[2]$ the notion of a scale Hilbert pair.

**Definition 1.10** A scale Hilbert pair is a pair of Hilbert spaces $\mathcal{H} = (H_0, H_1)$ such that $H_1 \subset H_0$ is a dense subset and the inclusion $H_1 \hookrightarrow H_0$ is compact.

The notions of scale isomorphism and scale isometry for scale Hilbert pairs is the same as the one for scale Hilbert spaces.

An example of a scale Hilbert pair is the following. Let $f : \mathbb{N} \to (0, \infty)$ be a monotone unbounded function. By $\ell^2$ we refer as usual to the Hilbert space of square summable sequences. We say that a sequence $x = (x_1, x_2, \ldots)$ is in $\ell^2_f$ if

$$\sum_{\nu=1}^{\infty} f(\nu)x_{\nu}^2 < \infty.$$ 

We give $\ell^2_f$ the structure of a Hilbert space by introducing for $x, y \in \ell^2_f$ the inner product

$$\langle x, y \rangle_f = \sum_{\nu=1}^{\infty} f(\nu)x_{\nu}y_{\nu}.$$
Then the pair $\left(\ell^2, \ell^2_f\right)$ is a scale Hilbert pair. It can be shown, see [2], that each infinite dimensional scale Hilbert pair is scale isomorphic to a pair $\left(\ell^2, \ell^2_f\right)$. In particular, each Hilbert space $H_k$ arising in an infinite dimensional scale Hilbert space $\mathcal{H}$ is separable and hence isometric to $\ell^2$. This reduces the geography problem for scale Hilbert spaces to the problem of how infinitely many $\ell^2$-spaces can be nested into each other to produce a scale Hilbert space. For $f$ as above we introduce the scale Hilbert space $\ell^2_{f^k}$ given by

$$\ell^2_{f^k} = \ell^2_f, \quad k \in \mathbb{N}_0.$$  

**Remark 1.11** The standard orthogonal basis of $\ell^2$ is a common orthonormal basis of $\ell^2_{f^k}$ for $k \in \mathbb{N}_0$. This is a nontrivial issue already in finite dimensions. While two scalar products always admit a common orthogonal basis this in general fails for three scalar products.

We are now in position to define the notion of a fractal scale Hilbert space.

**Definition 1.12** A scale Hilbert space $\mathcal{H}$ is called fractal if there exists an unbounded monotone function $f: \mathbb{N} \to (0, \infty)$ such that $\mathcal{H}$ is scale isomorphic to $\ell^2_f$.

**Example 1.13** As an example of a fractal scale Hilbert space let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle and

$$\mathcal{H} = \left\{W^{k,2}(S^1, \mathbb{R})\right\}_{k \in \mathbb{N}_0}$$

the scale Hilbert space of Sobolev maps from the circle to the reals. Here we understand that $W^{0,2}(S^1, \mathbb{R}) = L^2(S^1, \mathbb{R})$ the space of square integrable maps. An orthogonal basis for $L^2(S^1, \mathbb{R})$ is the Fourier basis $\{e_\nu\}_{\nu \in \mathbb{N}}$ defined by

$$e_1(t) = 1, \quad e_{2m}(t) = \sqrt{2} \sin(2\pi mt), \quad e_{2m+1}(t) = \sqrt{2} \cos(2\pi mt), \quad m \in \mathbb{N}, \quad t \in S^1.$$  

The Fourier basis is a common orthogonal basis for all $W^{k,2}(S^1, \mathbb{R})$. Indeed, one computes for $\nu, \nu' \in \mathbb{N}$

$$\langle e_\nu, e_{\nu'} \rangle_k = \delta_{\nu,\nu'} \left( \sum_{j=0}^{k} \left(2\pi \frac{\nu}{2^j}\right)^2\right).$$

We conclude that $\mathcal{H}$ is scale isomorphic to $\ell^{2,\sigma}$ where $\sigma: \mathbb{N} \to (0, \infty)$ is the function

$$\sigma(\nu) = \nu^2 + 1, \quad \nu \in \mathbb{N}.$$  

**Remark 1.14** If $\mathcal{H} = \{H_k\}_{k \in \mathbb{N}_0}$ is a scale Hilbert space then following H. Hofer, K. Wysocki, and E. Zehnder we denote for $m \in \mathbb{N}_0$ by $\mathcal{H}^m$ the shifted scale Hilbert space

$$\mathcal{H}^m = \{H_{k+m}\}_{k \in \mathbb{N}_0}. \quad (1)$$
If $\mathcal{H}$ is fractal then $\mathcal{H}^m$ is scale isomorphic to $\mathcal{H} = \mathcal{H}^0$ for every $m \in \mathbb{N}_0$. On the other hand this property is not sufficient to characterize fractal scale Hilbert spaces as can be shown with the methods from [2].

We next justify why we believe that the scale Hilbert spaces which arise in semiinfinite dimensional Morse theories as introduced by Floer [1] are expected to be fractal. This reasoning is inspired by the paper [6] of J. Robbin and D. Salamon.

Recall that if $H_1$ is a vector space and $H_0$ is a Hilbert space, then a linear operator $A: H_1 \to H_0$ is called Fredholm if the following three conditions hold

(i) $\ker(A)$ is finite dimensional.
(ii) $\text{im}(A)$ is a closed subspace of the Hilbert space $H_1$.
(iii) The orthogonal complement $\text{im}(A) \perp \subset H_0$ is finite dimensional.

**Remark 1.15** One usually requires that $H_1$ is itself a Hilbert space or at least a Banach space and $A$ is continuous. However, note that the Fredholm property is independent of the norm on $H_1$ and only needs the scalar product on $H_0$.

If $A: H_1 \to H_0$ is a Fredholm operator, then the integer

$$\text{ind}(A) = \dim(\ker(A)) - \dim(\text{im}(A) \perp)$$

is called the *Fredholm index* of $A$. For the following definition recall also the notation $\mathcal{H}^m$ for the shifted scale Hilbert spaces from (1).

**Definition 1.16** Let $\mathcal{H}$ be a scale Hilbert space and $A: \mathcal{H}^1 \to \mathcal{H}^0$ be a linear scale continuous operator. Then $A$ is called a scale Hessian operator if the following three axioms hold.

**Symmetry:** The operator $A: H_1 \to H_0$ is symmetric, i.e. $\langle \xi, A\eta \rangle_0 = \langle A\xi, \eta \rangle_0$ for all $\xi, \eta \in H_1$.

**Fredholm:** The operator $A: H_1 \to H_0$ is a Fredholm operator of index 0.

**Regularity:** If $\xi \in H_1$ and $A\xi \in H_n$ for $n \in \mathbb{N}_0$, then actually $\xi \in H_{n+1}$.

**Remark 1.17** The motivation for introducing the notion of scale Hessian operator is that in Floer theory the Hessian at a critical point is supposed to be a scale Hessian operator. The Hessian should obviously be symmetric. The requirement that the Hessian is a Fredholm operator of index 0 can be interpreted that the critical point equation represents a well-posed problem which is neither overdetermined nor underdetermined. The regularity axiom can be thought of as to guarantee smoothness of the solutions of the critical point equation.

**Theorem A:** Assume that for the scale Hilbert space $\mathcal{H}$ a scale Hessian operator exists. Then $\mathcal{H}$ is fractal.

**Proof:** Theorem A is an immediate consequence of Corollary 2.7 proved in the following section. □
2 Proof of the main result

Let $A$ be a scale Hessian operator on a scale Hilbert space $H$. We can also interpret the linear operator $A: H_1 \to H_0$ as an unbounded operator $A: H_0 \to H_0$ with dense domain $\text{dom}(A) = H_1 \subset H_0$. The following Lemma tells us that $A$ interpreted in this way is a self-adjoint operator.

Lemma 2.1 Assume that $H_0$ is a Hilbert space and $A$ is a symmetric unbounded operator on $H_0$ with dense domain $\text{dom}(A) = H_1 \subset H_0$. Suppose further that $A: H_1 \to H_0$ is a Fredholm operator of index 0. Then $A$ is selfadjoint.

Proof: We first note that the symmetry of $A$ implies that

$$\ker(A) \subset \text{im}(A)^\perp.$$ 

Since the index of $A$ is 0 by assumption we conclude that

$$\dim(\ker(A)) = \dim(\text{im}(A)^\perp)$$

and hence

$$\ker(A) = \text{im}(A)^\perp. \quad (2)$$

Now choose

$$v \in \text{dom} A^*.$$ 

This means that there exists $y \in H_0$ such that the following holds

$$\langle y, w \rangle = \langle v, Aw \rangle, \quad \forall \ w \in H_1.$$ 

Choose $\eta \in H_1$ and $\xi \in \text{im}(A)^\perp$ satisfying

$$y = A\eta + \xi.$$ 

For $q \in H_1 \cap \text{im}(A)$ we compute

$$\langle v, Aq \rangle = \langle y, q \rangle = \langle A\eta, q \rangle + \langle \xi, q \rangle = \langle \eta, Aq \rangle.$$ 

We conclude that

$$\langle v - \eta, Aq \rangle = 0, \quad \forall \ q \in H_1 \cap \text{im}(A) = H_1 \cap \ker(A)^\perp.$$ 

But this is equivalent to the assertion

$$\langle v - \eta, w \rangle = 0, \quad \forall \ w \in \text{im}(A).$$ 

We deduce using (2)

$$v - \eta \in \text{im}(A)^\perp = \ker(A).$$ 

Hence

$$v \in \eta + \ker(A) \subset \text{dom}(A).$$
In particular, $A$ is selfadjoint. This finishes the proof of the Lemma. □

If $A$ is a scale Hessian operator on a scale Hilbert space $H$, we can endow $H_1 = \text{dom}(A)$ with another inner product defined by

$$\langle \xi, \eta \rangle_A = \langle \xi, \eta \rangle_0 + \langle A\xi, A\eta \rangle_0, \quad \xi, \eta \in H_1.$$ 

The norm $\| \cdot \|_A$ induced from $\langle \cdot, \cdot \rangle_A$ is the graph norm on $\text{dom}(A)$, and since $A$ is self-adjoint by Lemma 2.1 we conclude that the pair $(\text{dom}(A), \langle \cdot, \cdot \rangle_A)$ is complete, i.e. itself a Hilbert space.

**Lemma 2.2** The two scalar products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_A$ on $H_1 = \text{dom}(A)$ give rise to equivalent norms.

**Proof:** We prove the Lemma in two Steps.

**Step 1:** There exists a constant $c > 0$ such that $\| \xi \|_1 \leq c \| \xi \|_A$ for all $\xi \in H_1$.

We complexify the real Hilbert spaces $H_1$ and $H_0$ to complex Hilbert spaces $H_1^C$ and $H_0^C$. Since $A$ interpreted as unbounded operator on $H_0^C$ is selfadjoint by Lemma 2.1 we conclude that the spectrum of $A$ is contained on the real axis. In particular there exists $\lambda \in \mathbb{C}$ such that $(A - \lambda \text{id}): H_1^C \to H_0^C$ is invertible. Since $A: H^1 \to H^0$ is scale continuous we conclude that $A - \lambda \text{id}$ is a continuous bijective map from $H_1^C$ to $H_0^C$. Hence by the open mapping theorem its inverse is also continuous. In particular, there exists $c_0 > 0$ such that

$$\| (A - \lambda \text{id})^{-1} \eta \|_1 \leq c_0 \| \eta \|_0, \quad \forall \eta \in H_0^C.$$ 

Hence we estimate for $\xi \in H_1$

$$\| \xi \|_1 = \| (A - \lambda \text{id})^{-1}(A - \lambda \text{id})\xi \|_1$$
$$\leq c_0 \| (A - \lambda \text{id})\xi \|_0$$
$$\leq c_0 (\| A\xi \|_0 + \| \lambda \xi \|_0)$$
$$\leq \max\{c_0, |\lambda|c_0\} (\| \xi \|_0 + \| A\xi \|_0).$$

Hence Step 1 follows with $c = \max\{c_0, |\lambda|c_0\}$.

**Step 2:** We prove the Lemma.

By Step 1 the identity map is a continuous map from $(H_1, \langle \cdot, \cdot \rangle_A)$ to $(H_1, \langle \cdot, \cdot \rangle_1)$. Hence by the open mapping theorem the two scalar products are equivalent. □

**Definition 2.3** An unbounded selfadjoint operator $A$ on a Hilbert space $H_0$ is called cocompact if the inclusion $I: (\text{dom}(A), \langle \cdot, \cdot \rangle_A) \to (H_0, \langle \cdot, \cdot \rangle_0)$ is a compact operator.
Lemma 2.4 Assume that $A$ is a cocompact selfadjoint operator on a Hilbert space $H_0$. Then the spectrum of $A$ is discrete and consists of eigenvalues of finite multiplicity.

**Proof:** We set $H_1 = \text{dom}(A)$ and endow it with the scalar product $\langle \cdot, \cdot \rangle_A$. As in the proof of Lemma 2.2 we again complexify the Hilbert spaces $H_0$ and $H_1$ to complex Hilbert spaces $H_0^C$ and $H_1^C$ and note that there exists $\lambda \in \mathbb{C}$ such that $A_\lambda = A - \lambda \text{id}: H_1^C \to H_0^C$ is invertible. We consider the operator

$$B_\lambda : H_0^C \to H_0^C$$

defined by

$$B_\lambda = I \circ A_\lambda^{-1}.$$

By the open mapping theorem $A_\lambda^{-1}$ is continuous and $I$ is compact by assumption. We conclude that $B_\lambda$ is a compact operator. We next show that $B_\lambda$ is normal. In order to do that we first determine its adjoint. For $\xi_1, \xi_2 \in H_0$ we compute with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $H_0$ using the fact that $A$ is selfadjoint

$$\langle B_\lambda^* \xi_1, \xi_2 \rangle = \langle \xi_1, B_\lambda \xi_2 \rangle = \langle \xi_1, I A_\lambda^{-1} \xi_2 \rangle = \langle A_\lambda^{-1} \xi_1, A_\lambda^{-1} \xi_2 \rangle = \langle A_\lambda^{-1} \xi_1, A_\lambda^{-1} \xi_2 \rangle = \langle IA_\lambda^{-1} \xi_1, \xi_2 \rangle.$$

We deduce

$$B_\lambda^* = I \circ A_\lambda^{-1} = B_\lambda.$$

Using this formula we obtain

$$B_\lambda^* B_\lambda = I A_\lambda^{-1} I A_\lambda^{-1} = I A_\lambda^{-1} A_\lambda^{-1} = I (A_\lambda A_\lambda)^{-1} = I (A^2 - 2 \text{Re}(\lambda) A + |\lambda|^2 \text{id})^{-1} = B_\lambda B_\lambda^*.$$

In particular, $B_\lambda$ is normal. Summing up, we have shown that $B_\lambda$ is a compact, invertible, normal operator. We apply next the spectral theorem to $B_\lambda$ and conclude that there exists an orthonormal basis $\{e_\nu\}_{\nu \in \mathbb{N}}$ of $H_0^C$ consisting of eigenvectors of $B_\lambda$ and a zero sequence $\{\mu_\nu\}_{\nu \in \mathbb{N}}$ of corresponding eigenvalues of $B_\lambda$ such that for $\xi \in H_0$ the formula

$$B_\lambda \xi = \sum_{\nu = 1}^{\infty} \mu_\nu \langle \xi, e_\nu \rangle e_\nu.$$
holds. If \( \nu \in \mathbb{N} \) we get by applying \( A_\lambda \) to the eigenvalue equation

\[
B_\lambda e_\nu = \mu_\nu e_\nu
\]

the equation

\[
A_\lambda e_\nu = \frac{1}{\mu_\nu} e_\nu.
\]

By the definition of \( A_\lambda \) we conclude

\[
A e_\nu = \left( \frac{1}{\mu_\nu} + \lambda \right) e_\nu =: \gamma_\nu e_\nu.
\]  

(3)

In particular, if \( \xi \in H_1^C \) we have the formula

\[
A \xi = \sum_{\nu=1}^{\infty} \gamma_\nu \langle \xi, e_\nu \rangle e_\nu.
\]  

(4)

Since \( A \) is selfadjoint it follows that \( \gamma_\nu \) is real for each \( \nu \in \mathbb{N} \), so that we can replace the basis \( \{ e_\nu \}_{\nu \in \mathbb{N}} \) by a real orthonormal basis of \( H_0 \) such that (4) continues to hold. This finishes the proof of the Lemma.

We continue the notation of the proof of the previous Lemma. After reordering the basis vectors \( \{ e_\nu \}_{\nu \in \mathbb{N}} \) we can assume without loss of generality that the function \( \nu \mapsto |\gamma_\nu| \) is monotone. We define \( f_A \in \tilde{F} \) by the formula

\[
f_A(\nu) = 1 + \gamma_\nu^2, \quad \nu \in \mathbb{N}.
\]

Corollary 2.5 Assume that \( A \) is a cocompact selfadjoint operator on a Hilbert space \( H_0 \). Then the scale Hilbert pair \( (H_0, \text{dom}(A)) \) is scale isometric to \( (\ell^2, \ell^2 f_A) \).

Proof: Identify \( H_0 \) with \( \ell^2 \) using an orthonormal basis for which (4) holds with monotone increasing \( |\gamma_\nu| \). We compute for \( \nu, \mu \in \mathbb{N} \)

\[
\langle e_\nu, e_\mu \rangle = \langle e_\nu, e_\mu \rangle + \langle Ae_\nu, Ae_\mu \rangle = \delta_{\nu\mu} + \gamma_\nu \gamma_\mu \delta_{\nu\mu}.
\]

This proves the Corollary.

Corollary 2.6 Assume that \( \mathcal{H} \) is a scale Hilbert space and \( A \) is a scale Hessian operator on \( \mathcal{H} \). Then the restriction of \( A \) to \( \mathcal{H}^2 \) is a scale Hessian operator on \( \mathcal{H}^1 \).

Proof: The symmetry axiom and the regularity axiom for \( A|_{\mathcal{H}^2} \) are trivial. The only nontrivial axiom to check is the Fredholm axiom. By the regularity axiom we obtain

\[
\text{dom}(A|_{\mathcal{H}^1}) = H_2.
\]

Let \( \{ e_\nu \}_{\nu \in \mathbb{N}} \) be an orthonormal basis of \( H_0 \) as constructed in the proof of Lemma 2.4 such that (4) holds true. Since the Fredholm property is further
unchanged if we replace the scalar product by an equivalent one we can thanks to Lemma 2.2 assume without loss of generality that \( H_1 = \text{dom}(A) \) is endowed with the scalar product \( \langle \cdot, \cdot \rangle_A \). For \( \nu \in \mathbb{N} \) we set
\[
e^A_\nu = \frac{1}{\sqrt{f_A(\nu)}} e_\nu.
\]

Then \( \{e^A_\nu\}_{\nu \in \mathbb{N}} \) is an orthonormal basis of \( H_1 \). By (4) we get for \( \xi \in H_2 \) the formula
\[
A\xi = \sum_{\nu=1}^{\infty} \gamma_\nu \langle \xi, e^A_\nu \rangle_A e^A_\nu.
\]

In particular, if one identifies \( H_1 \) with \( H_0 \) under the isomorphism which maps the orthonormal basis \( \{e^A_\nu\}_{\nu \in \mathbb{N}} \) of \( H_1 \) to the orthonormal basis \( \{e_\nu\}_{\nu \in \mathbb{N}} \) of \( H_0 \) the operators \( A|_{H_1} \) and \( A \) are identified. Since \( A : \text{dom}(A) \to H_0 \) is a Fredholm operator of index zero, the same has to be true for \( A|_{H_1} \). This finishes the proof of the Corollary. □

**Corollary 2.7** Assume that \( A \) is a scale Hessian operator on a scale Hilbert space \( \mathcal{H} \). Then \( \mathcal{H} \) is scale isomorphic to \( \ell^2, f^A \).

**Proof:** By regularity we obtain for \( k \in \mathbb{N}_0 \) unbounded operators
\[
A_k = A|_{H_k} : \text{dom}(A_k) = H_{k+1} \to H_k.
\]

In particular, we have
\[
A_{k+1} = A_k|_{\text{dom}(A_k)}.
\]

By induction on Lemma 2.2 we can assume that after replacing the scale structure of \( H_1 \) with \( H_0 \) under the isomorphism which maps the orthonormal basis \( \{e^A_\nu\}_{\nu \in \mathbb{N}} \) of \( H_1 \) to the orthonormal basis \( \{e_\nu\}_{\nu \in \mathbb{N}} \) of \( H_0 \) the operators \( A|_{H_1} \) and \( A \) are identified. Since \( A : \text{dom}(A) \to H_0 \) is a Fredholm operator of index zero, the same has to be true for \( A|_{H_1} \). This finishes the proof of the Corollary.
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