EXISTENCE OF PARABOLIC BOUNDARY POINTS OF CERTAIN DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. In this paper, the existence of parabolic boundary points of certain convex domains in $\mathbb{C}^2$ is given. On the other hand, the nonexistence of parabolic boundary points of infinite type of certain domains in $\mathbb{C}^2$ is also shown.

1. INTRODUCTION

Let $\Omega$ be a domain in $\mathbb{C}^n$. Denote by $Aut(\Omega)$ the group of holomorphic automorphisms of $\Omega$. The group $Aut(\Omega)$ is a topological group with the natural topology of uniform convergence on compact sets of $\Omega$ (i.e., the compact-open topology).

It is a standard and classical result of H. Cartan that if $\Omega$ is a bounded domain in $\mathbb{C}^n$ and the automorphism group of $\Omega$ is noncompact then there exist a point $x \in \Omega$, a point $p \in \partial \Omega$, and automorphisms $\varphi_j \in Aut(\Omega)$ such that $\varphi_j(x) \to p$. In this circumstance we call $p$ a boundary orbit accumulation point. The classification of domains with noncompact automorphism group relies deeply on the study the geometry of the boundary at an orbit accumulation point $p$. For instance, Wong and Rosay [15], [16] showed that if $p$ is a strongly pseudoconvex point, then the domain is biholomorphic to the ball. In [1], [2], [3] and [5], E. Bedford, S. Pinchuk and F. Berteloot showed that if $p$ is of finite type, then the domain is biholomorphic to the domain of the following form

$$M_P = \{(w, z) \in \mathbb{C}^2 : \text{Re}w + P(z, \bar{z}) < 0\}$$

where $P$ is an homogeneous polynomial in $z$ and $\bar{z}$. Each domain $M_P$ is called a model of $\Omega$ at $p$. To prove this, they first applied the Scaling method to point out that $Aut(\Omega)$ contains a parabolic subgroup, i.e.,
there is a point \( p_\infty \in \partial \Omega \) and a one-parameter subgroup \( \{ h^t \}_{t \in \mathbb{R}} \subset \text{Aut}(\Omega) \) such that for all \( z \in \Omega \)
\[
\lim_{t \to \pm \infty} h^t(z) = p_\infty.
\] (1.1)

Each boundary point satisfying (1.1) is called a parabolic boundary point of \( \Omega \). After that, the local analysis of a holomorphic vector field \( H \) which generates the above subgroup \( h^t \) was carried out to show that \( \Omega \) is biholomorphic to the desired homogeneous model.

We now consider a bounded domain \( \Omega \subset \mathbb{C}^2 \). Suppose that \( \Omega \) is biholomorphic to the domain \( D \) defined by \( D = \{ (w, z) \in \mathbb{C}^2 : \text{Re}w + \sigma(z) < 0 \} \) with some smooth real-valued function on the complex plane. The one-parameter of translations \( \{ L^t \}_{t \in \mathbb{R}} \) given by \( L^t(w, z) = (w + it, z) \) acts on the domain \( D \). The transformation \( \psi : D \to \Omega \) allows us to define the one-parameter group of biholomorphic mappings \( \{ h^t := \psi^{-1} \circ L^t \circ \psi \}_{t \in \mathbb{R}} \) acting on \( \Omega \). The first aim of this paper is to show that this one-parameter group is parabolic. Namely, we prove the following theorem.

**Theorem 1.1.** Let \( \Omega \) be a \( \mathcal{C}^1 \)-smooth, bounded, strictly geometrically convex domain in \( \mathbb{C}^2 \). Let \( \psi : \Omega \to D \) be a biholomorphism, where \( D := \{ (w, z) \in \mathbb{C}^2 : \text{Re}w + \sigma(z) < 0 \} \), \( \sigma \) is a \( \mathcal{C}^1 \)-smooth nonnegative function on the complex plane such that \( \sigma(0) = 0 \). Then, there exists some point \( a_\infty \in \partial \Omega \) such that \( \lim_{t \to +\infty} \psi^{-1}(w \pm it, z) = a_\infty \) for any \( (w, z) \in D \).

On the other hand, R. Greene and S. G. Krantz \[8\] suggested the following conjecture.

**Greene-Krantz Conjecture.** If the automorphism group \( \text{Aut}(\Omega) \) of a smoothly bounded pseudoconvex domain \( \Omega \subset \mathbb{C}^n \) is noncompact, then any orbit accumulation point is of finite type.

The main results around this conjecture are due to R. Greene and S. G. Krantz \[8\], K. T. Kim \[11\], K. T. Kim and S. G. Krantz \[12, 13\], H. B. Kang \[10\], M. Landucci \[14\], J. Byun and H. Gaussier \[6\].

Let \( P_\infty(\partial \Omega) \) be the set of all points in \( \partial \Omega \) of infinite type. In \[14\], M. Landucci proved that the automorphism group of a domain is compact
if $P_{\infty}(\partial \Omega)$ is a closed interval on the real "normal" line in a complex space with dimension 2. In [6], J. Byun and H. Gaussier also proved that there is no parabolic boundary point if $P_{\infty}(\partial \Omega)$ is a closed interval transerval to the complex tangent space at one boundary point. For the case which $P_{\infty}(\partial \Omega)$ is a closed curve on the boundary, is not there exist any parabolic boundary point? In [10], H. B. Kang showed that the automorphism group of the bounded domain $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|^2 + P(w) < 1\}$ is compact, where the function $P(w)$ is smooth and vanishes to infinite order at $w = 0$. In [13], K. T. Kim and S. G. Kantz considered the pseudoconvex domain $\Omega \subset \mathbb{C}^2$ where the local defining function of $\Omega$ in a neighborhood of the point of infinite type $(0, 0)$ takes the form $\rho(z) = \text{Re}z_1 + \psi(z_2, \text{Im}z_1)$. They pointed out that the origin is not a parabolic boundary point (see [13, Theorem 4.1]). Their proof based on the "fact" that the function $\psi$ vanishes to infinite order at $(0, 0)$. But, in general, it is not true, e.g., $\psi(z_2, \text{Im}z_1) = e^{-1/|z_2|^2} + |z_2|^4.|\text{Im}z_1|^2$.

The second aim of this paper is to prove the following theorem which shows that there is no parabolic boundary point of infinite type if $P_{\infty}(\partial \Omega)$ is a closed curve.

Theorem 1.2. Let $\Omega \subset \mathbb{C}^2$ be a bounded, pseudoconvex domain in $\mathbb{C}^2$ and $0 \in \partial \Omega$. Assume that

1. $\partial \Omega$ is $C^{\infty}$- smooth satisfying Bell’s condition $R$,
2. There exists a neighborhood $U$ of $0 \in \partial \Omega$ such that

$$\Omega \cap U = \{(z_1, z_2) \in \mathbb{C}^2 : \rho = \text{Re}z_1 + P(z_2) + Q(z_2, \text{Im}z_1) < 0\},$$

where $P$ and $Q$ satisfy the following conditions

(i) $P$ is smooth, subharmonic and strictly positive at all points different from the origin, where it vanishes at any order, i.e., $\lim_{z_2 \to 0} \frac{P(z_2)}{|z_2|^N} = 0$, $\forall N \geq 0$,

(ii) $Q(z_2, \text{Im}z_1)$ is smooth and can be written as $Q(z_2, \text{Im}z_1) = |z_2|^4|\text{Im}|^2 R(z_2, \text{Im}z_1)$ with some smooth function $R(z_2, \text{Im}z_1)$.

Then, $(0, 0)$ is not a parabolic boundary point.
Remark 1. By a simple computation, we see that \((0, 0)\) is of infinite type, \((it, 0)\) with \(t\) small enough, are of type greater than or equal to 4 and the other boundary points in a neighborhood of the origin, are strictly pseudoconvex.

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2. Existence of the parabolic boundary point

In this section, we prove Theorem 1.1. To do this, first of all, we recall some notations and some definitions.

For two domains \(D, \Omega\) in \(\mathbb{C}^n\), denote by \(\text{Hol}(D, \Omega)\) the set of all holomorphic maps from \(D\) into \(\Omega\). Moreover, we denote by \(d(z, \Omega)\) the distance from the point \(z \in \Omega\) to \(\partial \Omega\) and by \(\Delta\) the open unit disk in the complex plane.

Definition 2.1. Let \(p, q\) be two points in a domain \(\Omega\) in \(\mathbb{C}^n\) and \(X\) a vector in \(\mathbb{C}^n\).

(a) The kobayashi infinitesimal pseudometric \(F_\Omega(p, X)\) is defined by

\[
F_\Omega(p, X) = \inf \{\alpha > 0 \mid \exists g \in \text{Hol}(\Delta, \Omega), \ g(0) = 0, \ g'(0) = X/\alpha\}
\]

(b) The kobayashi pseudodistance \(k_\Omega(p, q)\) is defined by

\[
k_\Omega(p, q) = \inf \int_a^b F_\Omega(\gamma(t), \gamma'(t))dt,
\]

where the infimum is taken over all differentiable curves \(\gamma : [a, b] \to \Omega\) such that \(\gamma(a) = p\) and \(\gamma(b) = q\).

Before proceeding to prove the Theorem 1.1 we need some following lemmas.

Lemma 2.1. Let \(\Omega\) be a \(C^1\)-smooth, bounded, strictly geometrically convex domain in \(\mathbb{C}^2\). Then, there exists \(\epsilon_0 > 0\) such that for any \(\eta \in \partial \Omega\) and for any \(\epsilon \in (0, \epsilon_0]\), there exists a constant \(K(\epsilon) > 0\) such
that the following holds:

\[ k_\Omega(z, w) \geq -\frac{1}{2} \ln d(z, \partial \Omega) - K(\epsilon) \]

for any \( z, w \in \Omega \) with \( |z - \eta| < \epsilon, \ |w - \eta| > 3\epsilon \).

**Proof.** Since \( \partial \Omega \) is strictly geometrically convex, there exists a family of holomorphic peak functions

\[ F : \Omega \times \partial \Omega \rightarrow \mathbb{C} \]

\[(z, \eta) \mapsto F(z, \eta)\]

such that

(i) \( F \) is continuous and \( F(\cdot, \eta) \) is holomorphic;

(ii) \( |F| < 1 \);

(iii) There exist a positive constant \( A \) and a positive constant \( \epsilon_0 \) such that

\[ |1 - F(\eta + t\vec{n}_\eta, \eta)| \leq At \text{ for } t \in [0, \epsilon_0], \text{ where } \vec{n}_\eta \text{ is the normal to } \partial \Omega \text{ at } \eta. \]

Taking \( \epsilon_0 > 0 \) small enough, we may assume that \( \partial B(\eta, 3\epsilon) \cap \partial \Omega \neq \emptyset \) for \( \epsilon \leq \epsilon_0 \) and for any \( \eta \in \partial \Omega \).

Let \( \gamma \) be some smooth part in \( \Omega \) such that \( \gamma(0) = z, \ \gamma(1) = w \) and

\[ \int_0^1 F_\Omega[\gamma(t), \gamma'(t)]dt \leq k_\Omega(z, w) + 1. \]

Let \( z_0 \in \gamma \) such that \( |z_0 - \eta| = 3\epsilon \).

We have

\[ k_\Omega(z, w) \geq \int_0^1 F_\Omega[\gamma(t), \gamma'(t)]dt - 1 \geq k_\Omega(z, z_0) - 1. \]  \( (2.2) \)

Let \( \tilde{\eta} \in \partial \Omega \) be such that \( z = \tilde{\eta} + t\tilde{n}_\eta, \ t > 0 \). We set \( u_0 := F(z_0, \tilde{\eta}) \) and \( u := F(z, \tilde{\eta}) \), \( u \) and \( u_0 \) are in the unit disk \( \Delta \). Then we have

\[ k_\Omega(z, z_0) \geq k_\Delta(u, u_0) = \frac{1}{2} \ln \frac{1 + |\tau_{u_0}(u)|}{1 - |\tau_{u_0}(u)|} \geq -\frac{1}{2} \ln(1 - |\tau_{u_0}(u)|), \]  \( (2.3) \)

where \( \tau_{u_0}(u) = \frac{u - u_0}{1 - \bar{u}_0u} \). One easily checks that

\[ 1 - |\tau_{u_0}(u)| \leq \frac{2|1 - u|}{1 - |u_0|}. \]  \( (2.4) \)
Using the properties of $F$ we have

$$|1 - u| = |1 - F(z, \bar{\eta})| \leq At = Ad(z, b\Omega). \quad (2.5)$$

Since $|\eta - \bar{\eta}| \leq |\eta - z| + |z - \bar{\eta}| < 2\epsilon$ and $|z_0 - \eta| = 3\epsilon$, we have $|z_0 - \bar{\eta}| \leq \epsilon$.

Let $M(\epsilon) = \sup_{\eta \in \partial \Omega, z \in \Omega} |F(z, \eta)|$, $M(\epsilon) < 1$ and therefore

$$1 - |u_0| = 1 - |F(z_0, \bar{\eta})| \geq 1 - M(\epsilon) > 0. \quad (2.6)$$

From (2.4), (2.5) and (2.6) we get

$$1 - \tau_{u_0}(u) \leq \frac{2A}{1 - M(\epsilon)} d(z, \partial \Omega). \quad (2.7)$$

Then form (2.2), (2.3) and (2.7) we obtain

$$k_{\Omega}(z, w) \geq \frac{1}{2} \ln d(z, \partial \Omega) \geq \frac{1}{2} \frac{2A}{1 - M(\epsilon)} - 1 \quad (2.8)$$

and this completes the proof. □

**Lemma 2.2.** Let $\Omega$ be a $C^1$-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^2$ and let $\eta, \eta' \in \partial \Omega$ with $\eta \neq \eta'$. Then there exist $\epsilon > 0$ and a constant $K$ such that

$$k_{\Omega}(z, w) \geq -\frac{1}{2} \ln d(z, \partial \Omega) - \frac{1}{2} \ln d(w, \partial \Omega) - K,$$

for any $z \in B(\eta, \epsilon)$ and any $w \in B(\eta', \epsilon)$.

**Proof.** Let $\eta$ and $\eta'$ be two distinct points on $\partial \Omega$. Suppose that $|z - \eta| < \epsilon$ and $|w - \eta'| < \epsilon$ and let $\gamma$ be a $C^1$ part in $\Omega$ connecting $z$ and $w$ such that $k_{\Omega}(z, w) \geq \int_{0}^{1} F_{\Omega}[\gamma(t), \gamma'(t)]dt - 1$. If $\epsilon$ is small enough we may find $z_0 \in \gamma$ such that $|z_0 - \eta| > 3\epsilon$ and $|z_0 - \eta'| > 3\epsilon$. Let $z_0 = \gamma(t_0)$, then

$$k_{\Omega}(z, w) \geq \int_{0}^{t_0} F_{\Omega}(\gamma(t), \gamma'(t))dt + \int_{t_0}^{1} F_{\Omega}(\gamma(t), \gamma'(t))dt - 1$$

$$\geq k_{\Omega}(z, z_0) + k_{\Omega}(z_0, w) - 1$$

$$\geq -\frac{1}{2} \ln d(z, \partial \Omega) - \frac{1}{2} \ln d(w, \partial \Omega) - 2K(\epsilon) - 1,$$
where the last inequality is obtained by applying two times Lemma 2.1.

We now recall the definition of horosphere

**Definition 2.2.** Let \( a \in \Omega, \eta \in \partial \Omega, \ R > 0 \). The big horosphere with pole \( a \), center \( \eta \) and radius \( R \) in \( \Omega \) is defined as follows

\[
F^\Omega_a(\eta, R) = \{z \in \Omega : \liminf_{w \to \eta} [k_\Omega(z, w) - k_\Omega(a, w)] < \frac{1}{2} \ln R\}.
\]

**Lemma 2.3.** If \( \Omega \) is a \( C^1 \)-smooth, bounded, strictly geometrically convex domain in \( \mathbb{C}^2 \), then \( F^\Omega_a(\eta, R) \cap \partial \Omega \subset \{\eta\} \) for any \( a \in \Omega, \ \eta \in \partial \Omega, \ R > 0 \).

**Proof.** If there exists \( \eta' \in \partial \Omega \cap F^\Omega_a(\eta, R) \) then we can find a sequence \( \{z_n\} \subset \Omega \) with \( z_n \to \eta' \) and a sequence \( \{w_n\} \subset \Omega \) with \( z_n \to \eta \) such that

\[
k_\Omega(z_n, w_n) - k_\Omega(a, w_n) < \frac{1}{2} \ln R \tag{2.9}
\]

By Lemma 2.2, the following estimate occurs if \( \eta \neq \eta' \) and \( n \) great enough.

\[
k_\Omega(z_n, w_n) \geq -\frac{1}{2} \ln d(z_n, \partial \Omega) - \frac{1}{2} \ln d(w_n, \partial \Omega) - K, \quad (2.10)
\]

where \( K \) is a constant.

On the other hand, we have

\[
k_\Omega(a, w_n) \leq -\frac{1}{2} \ln d(w_n, \partial \Omega) + K(a), \quad (2.11)
\]

since \( \partial \Omega \) is smooth.

From (2.9), (2.10) and (2.11) we get

\[
-\frac{1}{2} \ln d(z_n, \partial \Omega) \lesssim 1, \quad (2.12)
\]

which is absurd.

**Proof of Theorem 1.1.** Set \( a_n := \psi^{-1}(-t_n, 0) \) where \( \lim t_n = +\infty \), after taking a subsequence we may assume that \( \lim a_n = a_\infty \in \partial \Omega \). We may also assume that \( a_\infty \) is the origin in \( \mathbb{C}^2 \).
Set \( b_t := \psi^{-1}(-1 + it, 0) \), according to Lemma 2.3 it suffices to show that there exists \( R_0 > 0 \) such that

\[ \{ b_t : t \in \mathbb{R} \} \subset F^\omega_{a_0}(a_\infty, R_0). \] (2.13)

Since \( a_n \to a_\infty \), we have

\[ \liminf_{w \to a_\infty} [k_\Omega(b_t, w) - k_\Omega(a_0, w)] \leq \liminf_{n \to +\infty} [k_\Omega(b_t, a_n) - k_\Omega(a_0, a_n)]. \] (2.14)

Then by the invariance of the Kobayashi metric and the convexity of \( D \) we have

\[ k_\Omega(b_t, a_n) - k_\Omega(a_0, a_n) = k_D([-1 + it, 0], (-t_n, 0)] - k_D([-t_0, 0], (-t_n, 0]) = k_H(-1 + it, -t_n) - k_H(-t_0, -t_n), \] (2.15)

where \( H \) is the left half plane \( \{ \text{Re} w < 0 \} \).

Let \( \sigma : H \to \Delta \) be a biholomorphism between \( H \) and the disk \( \Delta \) given by \( \sigma(w) = \frac{w + 1}{w - 1} \). Set \( z_t := \sigma(-1 + it) = \frac{it}{-2 + it} \) and \( x_n := \sigma(-t_n) = \frac{-t_n + 1}{-t_n - 1} \). Then we have

\[
k_H(-1 + it, -t_n) - k_H(-t_0, -t_n) = k_\Delta(z_t, x_n) - k_\Delta(x_0, x_n) \\
= \ln \left( \frac{|1 - x_n z_t| + |x_n - z_t|}{|1 - x_n z_t| - |x_n - z_t|} \right) \left( \frac{|1 - x_n x_0| + |x_n - x_0|}{|1 - x_n x_0| - |x_n - x_0|} \right) \\
= \ln \left( \frac{|1 - x_n z_t|^2 - |x_n - z_t|^2}{|1 - x_n z_t|^2 + |x_n - z_t|^2} \right) \left( \frac{|1 - x_n x_0|^2 - |x_n - x_0|^2}{|1 - x_n x_0|^2 + |x_n - x_0|^2} \right) \\
= \ln \left( \frac{1 - x_0^2}{|1 - z_t|^2} \right) \left( \frac{|1 - x_n x_0|^2 - |x_n - x_0|^2}{|1 - x_n x_0|^2 + |x_n - x_0|^2} \right).
\] (2.16)

From (2.15) and (2.16) we have

\[
\lim_{n \to -\infty} [k_\Omega(b_t, a_n) - k_\Omega(a_0, a_n)] = \ln \left( \frac{1 - x_0^2}{|1 - z_t|^2} \right) \left( \frac{|1 - x_n x_0|^2}{|1 - x_n x_0|^2 - |x_n - x_0|^2} \right). \] (2.17)

Finally, (2.13) follows directly from (2.14) and (2.17) when \( \ln \frac{1 - x_0^2}{|1 - x_0|^2} < \frac{1}{2} \ln R_0 \).

\( \square \)
3. NON-EXISTENCE OF THE PARABOLIC BOUNDARY POINT OF INFINITE TYPE

Let \( \Omega \) be a domain satisfying conditions given in Theorem 1.2. In this section, the non-existence of the parabolic boundary point of infinite type of \( \Omega \) is proved. First of all, we need some following lemmas.

**Lemma 3.1.** There do not exist \( a, b \in \mathbb{C} \) with \( \text{Re} a \neq 0 \) and \( b \neq 0 \) such that

\[
\text{Re}[aP(z) + bz^kP'(z)] = \gamma(z)P(z),
\]

for some \( k \in \mathbb{N}, \ k > 1 \) and for every \( |z| < \epsilon_0 \) with \( \epsilon_0 > 0 \) small enough, where \( \gamma(z) \) is smooth and \( \gamma(z) \to 0 \) as \( z \to 0 \).

**Proof.** Suppose that there exist \( a, b \in \mathbb{C} \) with \( \text{Re} a \neq 0 \) and \( b \neq 0 \) such that

\[
\text{Re}[aP(z) + bz^kP'(z)] = \gamma(z)P(z),
\]

for some \( k \in \mathbb{N}, \ k > 1 \) and for every \( |z| < \epsilon_0 \) with \( \epsilon_0 > 0 \) small enough. This equation is equivalent to

\[
1 + \text{Re}\left[\frac{b}{\text{Re}a}z^kP'(z)\right] = \gamma_1(z), \quad \forall \ 0 < |z| < \epsilon_0,
\]

where \( \gamma_1(z) = \gamma(z)/\text{Re}a \). Let \( F(z) = \ln P(z) \) and write \( z = re^{i\varphi} \), \( \frac{b}{2\text{Re}a} = \frac{1}{R}e^{i\psi} \). Then, by (3.20), we get

\[
\frac{\partial F}{\partial x}(z)\cos(k\varphi + \psi) + \frac{\partial F}{\partial y}(z)\sin(k\varphi + \psi) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(z).
\]

If we set \( \varphi_0 = \frac{2\pi - \psi}{k - 1} \), then

\[
\frac{\partial F}{\partial x}(re^{i\varphi_0})\cos(\varphi_0) + \frac{\partial F}{\partial y}(re^{i\varphi_0})\sin(\varphi_0) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).
\]

Let \( g(r) := F(re^{i\varphi_0}) \). It is easy to see that

\[
g'(r) = -\frac{R}{r^k} + \frac{R}{r^k}\gamma_1(re^{i\varphi_0}).
\]
Let $h(r) := g(r) + \frac{R}{1 - kr^k - 1}$. Then

$$h'(r) = \frac{R}{r^k} \gamma_1(re^{i\phi_0}).$$

We may assume that there exists $r_0$ small enough such that $|h'(r)| \leq \frac{R}{2r^k}$, for every $0 < r \leq r_0$. Thus, we have the following estimate

$$|h(r)| \leq |h(r_0)| + \left| \int_{r_0}^r |h'(r)| dr \right|$$

$$\leq |h(r_0)| + \frac{R}{2} \left| \int_{r_0}^r r^{-k} dr \right|$$

$$\leq |h(r_0)| - \frac{R}{2(k - 1)} r_0^{1-k} + \frac{R}{2(k - 1)} r^{1-k}.$$

Hence,

$$g(r) \geq \frac{R}{k - 1} r^{1-k} - |h(r_0)| + \frac{R}{2(k - 1)} r_0^{1-k} - \frac{R}{2(k - 1)} r^{1-k}.$$

It implies that $\lim_{r \to 0^+} g(r) = +\infty$. This means that $P(re^{i\phi_0}) \not\to 0$ as $r \to 0^+$. It is impossible. □

**Lemma 3.2.** There do not exist $a, b \in \mathbb{C}$ with $Re a \neq 0$ and $b \neq 0$ such that

$$Re[aP^{n+1}(z) + bz^k P'(z)] = \gamma(z) P^{n+1}(z), \quad (3.21)$$

for some $k \in \mathbb{N}$, $k > 1$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough, where $\gamma(z) \to 0$ as $z \to 0$.

**Proof.** Suppose that there exist $a, b \in \mathbb{C}$ with $Re a \neq 0$ and $b \neq 0$ such that

$$Re[aP^{n+1}(z) + bz^k P'(z)] = \gamma(z) P^{n+1}(z), \quad (3.22)$$

for some $k \in \mathbb{N}$, $k > 1$ and for every $|z| < \epsilon_0$ with $\epsilon_0 > 0$ small enough. This equation is equivalent to

$$1 + Re\left[ \frac{b}{Rea} z^k \frac{P'(z)}{P^{n+1}(z)} \right] = \gamma_1(z), \quad \forall 0 < |z| < \epsilon_0, \quad (3.23)$$
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where \( \gamma_1(z) = \gamma(z)/\text{Re}a \). Let \( F(z) = \frac{1}{P_n(z)} \) and write \( z = re^{i\varphi} \), \( \frac{-b}{2n\text{Re}a} = \frac{1}{R}e^{i\psi} \). By (3.23), we get

\[
\frac{\partial F}{\partial x}(z) \cos(k\varphi + \psi) + \frac{\partial F}{\partial y}(z) \sin(k\varphi + \psi) = -\frac{R}{r^k} + \frac{R}{r^k} \gamma_1(z).
\]

If we set \( \varphi_0 = \frac{2\pi}{k} - \psi k - 1 \), then

\[
\frac{\partial F}{\partial x}(re^{i\varphi_0}) \cos(\varphi_0) + \frac{\partial F}{\partial y}(re^{i\varphi_0}) \sin(\varphi_0) = -\frac{R}{r^k} + \frac{R}{r^k} \gamma_1(re^{i\varphi_0}).
\]

Let \( g(r) := F(re^{i\varphi_0}) \). Then we see that

\[
g'(r) = -\frac{R}{r^k} + \frac{R}{r^k} \gamma_1(re^{i\varphi_0}).
\]

Let \( h(r) := g(r) + \frac{R}{1 - k} \frac{1}{r^k} \). Then we may assume that there is \( r_0 \) small enough such that

\[
|h'(r)| \leq \frac{3R}{2r^k},
\]

for every \( 0 < r \leq r_0 \). Thus, we have the following estimate

\[
|g(r)| \leq |g(r_0)| + \left| \int_{r_0}^{r} |g'(r)|dr \right|
\]

\[
\leq |g(r_0)| + \frac{3R}{2} \left| \int_{r_0}^{r} r^{-k}dr \right|
\]

\[
\leq |g(r_0)| - \frac{3R}{2(k-1)} r_0^{1-k} + \frac{3R}{2(k-1)} r^{1-k}.
\]

Therefore, we obtain

\[
\frac{1}{P_n(re^{i\varphi_0})} \lesssim \frac{1}{r^{1-k}},
\]

\[
P(re^{i\varphi_0}) \gtrsim r^{\frac{1}{n-1}}.
\]

This means that \( P(re^{i\varphi_0}) \) does not vanish to infinite order at \( r = 0 \). It is a contradiction. \( \square \)

**Lemma 3.3.** There do not exist \( a, b \in \mathbb{C} \) with \( \text{Re}a \neq 0 \) and \( b \neq 0 \) such that

\[
\text{Re}[aP^{n+1}(z) + bzP'(z)] = \gamma(z)P^{n+1}(z),
\]

(3.24)
for some \( n \geq 0 \) and for every \( |z| < \epsilon_0 \) with \( \epsilon_0 > 0 \) small enough, where \( \gamma(z) \to 0 \) as \( z \to 0 \).

**Proof.** Suppose that there exist \( a, b \in \mathbb{C} \) with \( \text{Re}a \neq 0 \) and \( b \neq 0 \) such that (3.24) holds. We first consider the case \( n = 0 \). Then the equation (3.24) is equivalent to

\[
\text{Re}\left[ \frac{b}{\text{Re}a}z \frac{\partial}{\partial z} \ln P(z) \right] = -1 + \gamma_1(z),
\]

where \( \gamma_1(z) := \gamma(z)/\text{Re}a \). Let \( u(z) := \ln P(z) \) and write \( \frac{b}{2\text{Re}a} = \alpha + i\beta \), \( z = x + iy \). Then, by (3.25), we have the following first order partial differential equation

\[
(\alpha x - \beta y) \frac{\partial}{\partial x} u(x, y) + (\beta x + \alpha y) \frac{\partial}{\partial y} u(x, y) = -1 + \gamma_1(x, y).
\]

In order to solve this partial differential equation, we need to solve the following system of differential equation.

\[
\begin{cases}
x'(t) = \alpha x - \beta y \\
y'(t) = \beta x + \alpha y, \ t \in \mathbb{R}.
\end{cases}
\]

By a simple computation, we obtain

\[
\begin{cases}
x(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \\
y(t) = -c_2 e^{\alpha t} \cos(\beta t) + c_1 e^{\alpha t} \sin(\beta t), \ t \in \mathbb{R},
\end{cases}
\]

where \( c_1, c_2 \) are two constant real numbers. Let \( g(t) := u(x(t), y(t)) \). Then \( g'(t) = -1 + \gamma_1(x(t), y(t)) \). Thus, \( g(t) = -t + \int_{t_0}^{t} \gamma_1(x(s), y(s)) \, ds + t_0 + g(t_0) \). From (3.27), we get

\[
x^2 + y^2 = (c_1^2 + c_2^2) e^{2\alpha t}, \ t \in \mathbb{R}.
\]

Consider three following cases

**Case 1.** \( \alpha = 0 \). In this case, take \( c_1 = r > 0, c_2 = 0 \), where \( r \) small enough. Then, on each small circle \( \{x(t) = r \cos(t), y(t) = r \sin(t), \ t \in [0, 2\pi]\} \), \( g(t) = -t + \int_{0}^{t} \gamma_1(x(s), y(s)) \, ds + u(r, 0) \). Taking \( r \) small enough, we may assume that \( |\gamma_1(x(s), y(s))| \leq 1/2 \) for all \( s \in [0, 2\pi] \). It is easy
to see that $|g(2\pi) - g(0)| \geq \pi$. This is absurd since $g(2\pi) = g(0) = u(r, 0)$.

**Case 2.** $\alpha > 0$. By (3.28), $(x(t), y(t)) \to 0$ as $t \to -\infty$. Then, $u(x(t), y(t)) \to +\infty$ as $t \to -\infty$. This is a contradiction.

**Case 3.** $\alpha < 0$. By (3.28), we have $(x(t), y(t)) \to 0$ as $t \to +\infty$ and $t = \frac{1}{2\alpha} \ln \frac{x^2 + y^2}{c_1^2 + c_2^2}$. Taking $t_0 > 0$ big enough, we may assume that $|\gamma_1(x(s), y(s))| \leq 1$ for all $s \geq t_0$. Then for all $t \geq t_0$, we have

$$g(t) \geq -(t - t_0) - \int_{t_0}^{t} |\gamma_1(x(s), y(s))| ds - |g(t_0)|$$

$$\geq -(t - t_0) - \int_{t_0}^{t} |\gamma_1(x(s), y(s))| ds - |g(t_0)|$$

$$\geq -(t - t_0) - |t - t_0| - |g(t_0)|$$

$$\geq -2(t - t_0) - |g(t_0)|.$$  

Hence, for all $t \geq t_0$, we obtain

$$P(z(t)) \geq e^{-2t}$$

$$\geq |z(t)|^{-1/\alpha},$$

where $z(t) := x(t) + iy(t)$. It is impossible since $P$ vanishes to infinite order at 0.

We now consider the case $n > 0$. Then the equation (3.24) is equivalent to

$$\Re\left[\frac{b}{-n\Re a} z \frac{1}{\partial z P^n(z)} \right] = -1 + \gamma_1(z), \quad (3.29)$$

where $\gamma_1(z) := \gamma(z)/\Re a$. Let $u(z) := \frac{1}{P^n(z)}$ and write $\frac{b}{-2n\Re a} = \alpha + i\beta$, $z = x + iy$. Then, by (3.29), we have the following first order partial differential equation

$$(\alpha x - \beta y) \frac{\partial}{\partial x} u(x, y) + (\beta x + \alpha y) \frac{\partial}{\partial y} u(x, y) = -1 + \gamma_1(x, y). \quad (3.30)$$
In order to solve this partial differential equation, we need to solve the following system of differential equations.

\[
\begin{align*}
  x'(t) &= \alpha x - \beta y \\
  y'(t) &= \beta x + \alpha y, \ t \in \mathbb{R}.
\end{align*}
\]

By a simple computation, we obtain

\[
\begin{align*}
  x(t) &= c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \\
  y(t) &= -c_2 e^{\alpha t} \cos(\beta t) + c_1 e^{\alpha t} \sin(\beta t), \ t \in \mathbb{R},
\end{align*}
\]

where \(c_1, c_2\) are two constant real numbers. Let \(g(t) := u(x(t), y(t))\). Then \(g'(t) = -1 + \gamma_1(x(t), y(t))\). Thus, \(g(t) = -t + \int_{t_0}^t \gamma_1(x(s), y(s))ds + t_0 + g(t_0)\). From (3.31), we get

\[
x^2 + y^2 = (c_1^2 + c_2^2)e^{2\alpha t}, \ t \in \mathbb{R}.
\]

Consider three following cases

**Case 1.** \(\alpha = 0\). In this case, take \(c_1 = r > 0, c_2 = 0\), where \(r\) small enough. Then, on each small circle \(\{x(t) = r \cos(t), y(t) = r \sin(t), \ t \in [0, 2\pi]\}\), \(g(t) = -t + \int_0^t \gamma_1(x(s), y(s))ds + u(r, 0)\). Taking \(r\) small enough, we may assume that \(|\gamma_1(x(s), y(s))| \leq 1/2\) for all \(s \in [0, 2\pi]\). It is easy to see that \(|g(2\pi) - g(0)| \geq \pi\). This is not possible since \(g(2\pi) = g(0) = u(r, 0)\).

**Case 2.** \(\alpha < 0\). By (3.32), \((x(t), y(t)) \to 0\) as \(t \to +\infty\). Then, \(u(x(t), y(t)) \to -\infty\) as \(t \to -\infty\). It is a contradiction.

**Case 3.** \(\alpha > 0\). By (3.32), we have \((x(t), y(t)) \to 0\) as \(t \to -\infty\) and \(t = \frac{1}{2\alpha} \ln \frac{x^2 + y^2}{c_1^2 + c_2^2}\). Taking \(t_0 < 0\) such that \(|t_0|\) big enough, we may assume that \(|\gamma_1(x(s), y(s))| \leq 1\) for all \(s \leq t_0\). Then for all \(t \leq t_0\), we...
have the following estimate
\[ g(t) \leq -(t - t_0) + \left| \int_{t_0}^{t} \gamma_1(x(s), y(s)) \, ds \right| + |g(t_0)| \]
\[ \leq -(t - t_0) + \left| \int_{t_0}^{t} \gamma_1(x(s), y(s)) \, ds \right| + |g(t_0)| \]
\[ \leq -(t - t_0) + |t - t_0| + |g(t_0)| \]
\[ \leq -2(t - t_0) + |g(t_0)|. \]
Hence, for all \( t \leq t_0 \), we obtain
\[ P^n(z(t)) \gtrsim \frac{1}{-2t} \]
\[ \gtrsim \frac{-1}{\ln |z(t)|}, \]
where \( z(t) := x(t) + iy(t) \). This implies that
\[ \lim_{t \to -\infty} \frac{P(z(t))}{|z(t)|} = +\infty. \]
This is impossible since \( P \) vanishes to infinite order at 0. \( \Box \)

Let \( F = (f, g) \in Aut(\Omega) \) be such that \( F(0, 0) = (0, 0) \). Because of Bell’s condition R of \( \partial \Omega \), \( F \) extends smoothly to the boundary of \( \Omega \).
Let \( U \) be a a neighborhood of \((0, 0)\). Then, there exists a neighborhood \( V \) of \((0, 0)\) such that
\[ F(\overline{\Omega} \cap V) \subset \overline{\Omega} \cap U. \] (3.33)
The following lemma is similar to Lemma 2.5 of [14].

**Lemma 3.4.** Let \( F = (f, g) \in Aut(\Omega) \). Let \( U, V \) be two neighborhoods of \((0, 0)\) such that [3.33] holds. Then, for any \((z_1, z_2) \in V\),

(i) \( g(z_1, 0) = 0 \).

(ii) \( f(z_1, z_2) = f(z_2) \).

**Proof.** a) Let \( U, V \) be two neighborhoods of \((0, 0)\) such that [3.33] holds. Let \( \gamma \) be the set of all points \((it, 0) \in \partial \Omega \cap U \). By Bell’s condition R, the restriction to \( \partial \Omega \) of the extension of \( F \) to \( \overline{\Omega} \) defines a C-R automorphism of \( \partial \Omega \). Since the D’Angelo type is a C-R invariant, we
have $F(\gamma \cap V) \subset \gamma$. Hence, $g(it, 0) = 0$ and $Re(f(it, 0)) = 0$. Since $h(z_1) := g(z_1, 0) \in Hol(H) \cap C^\infty(\overline{H})$, $g(z_1, 0) \equiv 0$. Here, we denote $H$ by $H = \{z_1 \in \mathbb{C} : Re z_1 < 0\}$.

b) A classical argument based on the Hopf’s lemma shows that $(\rho \circ F)(z_1, z_2)$ is also a defining function on $V$. In particular, there exists a smooth function $k(z_1, z_2)$ which is strictly positive and such that, for any $(z_1, z_2) \in V$,

$$Re z_1 + P(z_2) + Q(z_2, Im z_1)$$

$$= k(z_1, z_2)[Re(f(z_1, z_2) + P(g(z_1, z_2)) + Q(g(z_1, z_2), Im f(z_1, z_2))].$$

(3.34)

We claim that for any $N \geq 1$ and any $(it, 0) \in \gamma \cap V$

$$\frac{\partial^N}{\partial z_2^N}(Re(z_1, z_2) + P(g(z_1, z_2)) + Q(g(z_1, z_2), Im f(z_1, z_2)))\bigg|_{(it,0)} = 0.$$ (3.35)

In fact, for any $(it, 0) \in \gamma \cap V$ we have that

$$Re f(it, 0) + P(g(it, 0)) + Q(g(it, 0), Im f(it, 0)) = 0.$$ (3.35)

From (3.34), it follows that

$$\frac{\partial}{\partial z_2}(Re(z_1, z_2) + P(g(z_1, z_2)) + Q(g(z_1, z_2), Im f(z_1, z_2)))\bigg|_{(it,0)} = 0.$$ (3.35)

which implies (3.35) for $N = 1$. Taking the $N$-th derivative with respect to $z_2$ of (3.34) and using an inductive argument, it follows that (3.35) holds also for any $N > 1$. From a), (3.35) and the property (2.i) we get that for any $N \geq 1$ and any $(it, 0) \in \partial \Omega \cap V$

$$\frac{\partial^N}{\partial z_2^N} f(it, 0) = 0.$$ (3.36)

Using the same arguments as for (a), we see that (3.36) implies (b). □

Proof of theorem 1.2. Suppose that $(0, 0) \in \partial \Omega$ be a parabolic boundary point associated with a one-parameter group $\{F_\theta\}_{\theta \in \mathbb{R}} \subset Aut(\Omega)$. Let $H$ be the vector field generating the group $\{F_\theta\}_{\theta \in \mathbb{R}}$, i.e.,

$$H(z) = \frac{d}{d\theta} F_\theta(z)\bigg|_{\theta=0}.$$
Since $\Omega$ satisfies Bell’s condition $R$, each automorphism of $\Omega$ extends to be of class $C^\infty$ on $\overline{\Omega}$. Therefore, $H \in Hol(\Omega) \cap C^\infty(\overline{\Omega})$. Furthermore, since $F_\theta(\partial \Omega) \subset \partial \Omega$, it follows that $H(z) \in T_z(\partial \Omega)$ for all $z \in \partial \Omega$, i.e.,

\[(\text{Re}H)\rho(\zeta) = 0, \ \forall \zeta \in \partial \Omega. \tag{3.37}\]

A vector field $H \in Hol(\Omega) \cap C^\infty(\overline{\Omega})$ satisfying (3.37) is called to be a holomorphic tangent vector field for domain $\Omega$. Since $F_\theta(0, 0) = (0, 0)$, it follows from Lemma 3.4 that $F_\theta(z_1, z_2) = (f_\theta(z_1), z_2 g_\theta(z_1, z_2))$, where $f_\theta$ and $g_\theta$ are holomorphic on $U \cap \Omega$, where $U$ is a neighborhood of $(0, 0)$. Hence, the vector field $H$ has the form

$$H(z_1, z_2) = h_1(z_1) \frac{\partial}{\partial z_1} + z_2 h_2(z_1, z_2) \frac{\partial}{\partial z_2},$$

where $h_1$ and $h_2$ is holomorphic on $\Omega$ and, is of class $C^\infty$ up to the boundary $\partial \Omega$. Moreover, $h_1$ vanishes at the origin. By a simple computation, we get

$$\frac{\partial}{\partial z_1} \rho(z_1, z_2) = \frac{1}{2} + \frac{\partial}{\partial z_1} Q(z_2, \text{Im} z_1),$$

$$\frac{\partial}{\partial z_2} \rho(z_1, z_2) = P'(z_2) + \frac{\partial}{\partial z_2} Q(z_2, \text{Im} z_1).$$

Since $H(z)$ is a tangent vector field to $\partial \Omega$, we have

\[
\text{Re} \left[ \left( \frac{1}{2} + \frac{\partial}{\partial z_1} Q(z_2, \text{Im} z_1) \right) h_1(z_1) + (P'(z_2) + \frac{\partial}{\partial z_2} Q(z_2, \text{Im} z_1)) z_2 h_2(z_1, z_2) \right] = 0, \tag{3.38}
\]

for all $(z_1, z_2) \in \partial \Omega$. For any $(it, 0) \in \partial \Omega \cap U$, we have

$$\text{Re} h_1(it) = 0. \tag{3.39}$$

Since $h_1 \in Hol(H) \cap C^\infty(\overline{H})$, where $H$ is the left half-plane, by the Schwarz reflection principle, $h_1$ can be extended to be a holomorphic on a neighborhood of $z_1 = 0$. From (3.38), it follows that, for any $(-P(z_2), z_2) \in \partial \Omega \cap U$,

$$\text{Re} \left[ \frac{1}{2} h_1(-P(z_2)) + z_2 P'(z_2) h_2(-P(z_2), z_2) \right] = 0. \tag{3.40}$$
Expanding $h_1$ and $h_2$ into Taylor series about the origin, we get $h_1(z_1) = \sum_{n=0}^{\infty} a_n z_1^n$ and $h_2(z_1, z_2) = \sum_{k=0}^{\infty} b_k(z_1) z_2^k$, where $a_n \in \mathbb{C}$, $b_k \in Hol(H) \cap \mathcal{C}^\infty(\mathbb{P})$, for any $n, k \in \mathbb{N}$. Note that $a_0 = 0$ since $h_1(0) = 0$. If there exists an integer number $n \geq 1$ such that $\text{Re} a_n \neq 0$, then the biggest term in $\text{Re}[\frac{1}{2} h_1(-P(z_2))]$ has the form $\text{Re} a_n P^n(z_2)$. Therefore, there exists at least $k \in \mathbb{N}$ such that either $b_k(0) \neq 0$ or $b_k(z_1)$ vanishes to finite order at $z_1 = 0$. Then the biggest term in $\text{Re}[z_2 P'(z_2) h_2(-P(z_2), z_2)]$ has the form $\text{Re}[b z_2^k P'(z_2) P^l(z_2)]$, where $b \in \mathbb{C}^*$, $l \in \mathbb{N}$. By (3.40), there exists $\epsilon_0 > 0$ such that

$$\text{Re}[a_n P^{n-l}(z_2) + b z_2^k P'(z_2)] = o(P^{n-l}(z_2)), \quad (3.41)$$

for all $|z_2| < \epsilon_0$. It is easy to see that $n > l$. Thus, by Lemma 3.1, Lemma 3.2, and Lemma 3.3, we get $\text{Re} a_n = b = 0$. This is a contradiction. Therefore, $\text{Re} a_n = 0$ for every $n \geq 1$ and thus, we can write $h_1(z_1) = i \sum_{n=1}^{\infty} \alpha_n z_1^n$, where $\alpha_n \in \mathbb{R}$, $n = 1, 2, \ldots$. Let $u(z_1) := \text{Re} h_1(z_1)$. Then the function $u$ is harmonic on the left half-plane $H$ and is smooth up to the boundary $\partial H$. By (3.39), we have, for any real number $t$ small enough, $u(it) = 0$. Moreover, $u(-t) = 0$ for any $t$ small enough since $h_1(z_1) = i \sum_{n=1}^{\infty} \alpha_n z_1^n$. Hence, by the maximum principle, we conclude that $u(z_1) \equiv 0$. Consequentially, $h_1(z_1) \equiv 0$ and hence, $H$ becomes a planar vector field. This is impossible since $\partial \Omega$ is not flat near the origin. So the proof is completed. □

References

[1] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^2$ with noncompact groups of automorphisms*, Math. USSR Sbornik 63(1989), 141-151.
[2] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group*, J. Geom. Anal. 1 (1991), 165-191.
[3] E. Bedford and S. Pinchuk, *Domains in $\mathbb{C}^2$ with noncompact automorphism groups*, Indiana Univ. Math. Journal 47(1998), 199-222.
[4] S. Bell, *Compactness of families of holomorphic mappings up to the boundary*, Lecture Notes in Math. Vol. 1268, Springer-Verlag, Berlin/New Jork, 1987, 29-42.
[5] F. Berteloot, *Characterization of models in $\mathbb{C}^2$ by their automorphism groups*, Internat. J. Math. 5(1994), 619-634.

[6] J. Byun and H. Gaussier, *On the compactness of the automorphism group of a domain*, C. R. Acad. Sci. Paris, Ser. 1341 (2005), 545-548.

[7] J. P. D’Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. Math. 115 (1982), 615-637.

[8] R. Greene and S. G. Krantz, *Techniques for studying automorphisms of weakly pseudoconvex domains*, Math. Notes, Vol 38, Princeton Univ. Press, Princeton, NJ, 1993, 389-410.

[9] A. Isaev and S. G. Krantz, *Domains with non-compact automorphism group: A survey*, Adv. Math. 146 (1999), 1-38.

[10] H. B. Kang, *Holomorphic automorphisms of certain class of domains of infinite type*, Tohoku Math. J. 46 (1994), 345-422.

[11] K. T. Kim, *On a boundary point repelling automorphism orbits*, J. Math. Anal. Appl. 179 (1993), 463-482.

[12] K. T. Kim and S. G. Krantz, *Convex scaling and domains with non-compact automorphism group*, Illinois J. Math. 45 (2001), 1273-1299.

[13] K. T. Kim and S. G. Krantz, *Some new results on domains in complex space with non-compact automorphism group*, J. Math. Anal. Appl. 281 (2003), 417-424.

[14] M. Landucci, *The automorphism group of domains with boundary points of infinite type*, Illinois J. Math. 48 (2004), 33-40.

[15] B. Wong, *Characterization of the ball in $\mathbb{C}^n$ by its automorphism group*, Invent. Math. 41 (1977), 253-257.

[16] J. P. Rosay, *Sur une caractérisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d’automorphismes*, Ann. Inst. Fourier 29 (4) (1979), 91-97.

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