A MATCHED ALTERNATING DIRECTION IMPLICIT (ADI) METHOD FOR SOLVING THE HEAT EQUATION WITH INTERFACES

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Abstract. A novel Douglas alternating direction implicit (ADI) method is proposed in this work to solve a two-dimensional (2D) heat equation with interfaces. The ADI scheme is a powerful finite difference method for solving parabolic equations, due to its unconditional stability and high efficiency. However, it suffers from a serious accuracy reduction in space for interface problems with different materials and nonsmooth solutions. If the jumps in a function and its derivatives are known across the interface, rigorous ADI schemes have been successfully constructed in the literature based on the immersed interface method (IIM) so that the spatial order can be restored. Nevertheless, the development of accurate and stable ADI methods for general parabolic interface problems with physical interface conditions that describe jumps of a function and its flux, is essentially an open problem. To overcome this difficulty, a novel tensor product decomposition is proposed in this paper to decouple 2D jump conditions into essentially one-dimensional (1D) ones. These 1D conditions can then be incorporated into the ADI central difference discretization, using the matched interface and boundary (MIB) technique. The proposed matched ADI scheme is unconditionally stable and first order accurate in time. Fast algebraic solvers for perturbed tridiagonal systems are developed to maintain the computational efficiency. The second order of accuracy in space is achieved in all tested parabolic interface problems with complex geometries and spatial-temporal dependent jump conditions.

Key words. Heat equation; Parabolic interface problem; Jump conditions; Alternating direction implicit (ADI); Matched interface and boundary (MIB).

AMS subject classifications. 65M06, 65M12, 35K05.

1. Introduction. In this paper, we propose a new alternating direction implicit (ADI) method for solving two-dimensional (2D) parabolic interface problems

\[
\frac{\partial u}{\partial t} = \nabla \cdot (\alpha \nabla u) + f, \quad \text{in } \Omega = \Omega^+ \cup \Omega^-
\]

where \(u(x,y,t)\) is a function of interests, e.g. the temperature, \(\alpha\) is the diffusion coefficient and \(f(x,y,t)\) is a source. For simplicity, the domain \(\Omega\) is assumed to be a rectangle one, with proper boundary conditions prescribed for \(u\) on \(\partial \Omega\). Across a material interface \(\Gamma\) separating two media \(\Omega^-\) and \(\Omega^+\), the diffusion coefficient \(\alpha\) is discontinuous, while the source term \(f(x,y,t)\) may be even singular. Physically, the solution \(u\) on both sides of \(\Gamma\) is related analytically via jump conditions

\[
[u] = u^+ - u^- = \phi(s,t), \quad [\alpha u_n] = \alpha^+ \frac{\partial u^+}{\partial n} - \alpha^- \frac{\partial u^-}{\partial n} = \psi(s,t),
\]

where \(s\) is the arc-length parametrization of the interface \(\Gamma\), and \(n\) the unit normal direction. The superscript, \(-\) or \(+\), denotes the limiting value of a function from one side or the other of the interface. We note that (1.2) takes a quite general form, while for many applications, we have simply \(\phi = \psi = 0\).

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The parabolic interface problem governed by (1.1) and (1.2) appears in many physical and engineering applications, such as the continuous casting in the metallurgical industry, the freezing process of perishable foodstuffs in the food engineering, and the magnetic fluid hyperthermia treatment of cancer. The analysis of conductive heat transfer process over composite media is indispensable in these applications.

Since the physical solution is non-smooth or even discontinuous across the interface, the standard numerical methods often perform poorly for the parabolic interface problem. To restore the accuracy near the interface, the jump conditions (1.2) have to be incorporated into the numerical discretization in certain manner. For finite element methods [6, 19, 20, 2] and finite volume methods [23], many rigorous interface treatments have been proposed to deliver high accuracy in solving parabolic interface problems.

As one of the most successful finite difference methods for solving material interface problems, the immersed interface method (IIM) was originally introduced by LeVeque and Li [12] for solving elliptic equations with discontinuous coefficients and singular sources. By rigorously imposing jump conditions via local Taylor expansions, the IIM achieves the second order of accuracy for complex elliptic interfaces. The development of the IIM for solving parabolic equations has been considered in [10, 11, 4, 5, 1]. In [10, 11], the authors constructed the IIM together with the implicit Euler time integration for parabolic equations with singular sources. The IIM has been applied in [4, 5] for solving the Poisson equation over moving irregular domains or over fixed domains with the Neumann boundary condition. The maximum principle preserving IIM has been proposed in [1] by Adams and Li for solving the convection-diffusion equation with general jump conditions. In this scheme, the advection term is discretized explicitly while the diffusion term is treated implicitly. A fast multigrid method is implemented to efficiently solve the linear system of equations for the implicit time stepping [1].

As one of the most successful finite difference methods for solving parabolic equations, the classical ADI method [8, 7, 17] can be written as some perturbations of multidimensional implicit methods, such as the Crank-Nicolson and backward Euler. In general, the ADI method is unconditionally stable for parabolic problems without interfaces so that a large time increment is admissible, which in turn can lead to a faster simulation. The major attraction of the ADI method, as compared with other implicit methods, is that it reduces a multidimensional problem to sets of independent one-dimensional (1D) problems of tridiagonal structures, and such matrices can be efficiently solved using the Thomas algorithm [21]. Neglecting jump conditions, the classical ADI method has been applied to solve interface problems with sharp or smeared interfaces, see for example [3, 9, 21, 22]. However, for sharp interface problems, the central finite difference approximation degrades to the first order of accuracy in space and the precious unconditional stability may be lost [9, 24]. Therefore, it is highly desired to construct novel ADI methods for solving parabolic interface problems without compromising stability and spatial accuracy.

The first rigorous interface treatment in the ADI framework was due to Li and Mayo [13], in which a homogeneous 2D heat equation with a constant diffusion coefficient and a singular source is solved. For such an interface setting, the jump conditions (1.2) take a simpler form with $\alpha = 1$. Thus, higher order jump conditions can be simply derived [13]. This allows the construction of a second order accurate IIM-ADI method by adding some correction terms into the classical ADI scheme for irregular points near the interface. Considering the same jump conditions, Liu and Zheng
have extended the IIM-ADI method to solve a 2D homogeneous convection-diffusion equation [15] and a three-dimensional (3D) homogeneous heat equation [16].

However, grand difficulties are encountered when the IIM-ADI method [13, 15, 16] is generalized in [14] to solve a 2D heat equation with nonhomogeneous media, i.e., \( \alpha \) being a piecewise constant. By still assuming simple jump conditions with prescribed function and derivative jumps, i.e., given \([u]\) and \([u_n]\) values, the second order jump condition for \([u_{nn}]\) cannot be simply derived from the governing equation now. A rather complicated second order jump condition is considered in [14] so that the SOR iterative method has to be used to solve a 2D linear system in each step of the Crank-Nicolson time integration. We note that, on the other hand, if the general jump conditions like (1.2) are used, the second order jump conditions could be naturally derived. Nevertheless, the construction of the IIM-ADI scheme through introducing correction terms remains a challenge for general jump conditions. Therefore, the development of accurate and stable ADI methods for the parabolic interface problem (1.1) and (1.2) is essentially an open problem.

The objective of this paper is to propose a novel matched ADI method to overcome the aforementioned difficulties for solving general parabolic interface problems. The proposed matched ADI method is formulated based on our previous interface scheme, the matched interface and boundary (MIB) method, originally developed for solving elliptic and hyperbolic interface problems [25, 27]. One distinction between the MIB and IIM is that the MIB interface modeling just needs zeroth and first order jump conditions, i.e., (1.2) so that the difficulty associated with the second order jump conditions of the IIM is simply bypassed in the MIB approach. However, the MIB scheme has never been applied to a parabolic interface problem before. Moreover, the existing MIB scheme cannot be directly utilized in the ADI formulation, because the 2D MIB interface treatment will couple \( x \) and \( y \) directions simultaneously. The most significant contribution of this work is the introduction of a novel tensor-product decomposition of jump conditions (1.2), which decouples 2D jump conditions into 1D ones, in the same spirit of the ADI method. Then, 1D MIB interface treatments will be developed in space to secure a second order of accuracy without affecting the stability. Fast algebraic solvers based on the Thomas algorithm will be developed to solve 1D linear systems efficiently.

The rest of this paper is organized as follows. Section 2 is devoted to the theory and algorithm of the proposed matched ADI method. Numerical tests are carried out to validate the proposed method by considering various particular forms for the jump conditions (1.2). Finally, a conclusion ends this paper.

2. Theory and Algorithm. Consider an interface problem, in which \( \Omega^- \) is interior to \( \Omega^+ \). Define a uniform mesh partition of the computational domain \( \Omega \). Without the loss of generality, we assume that the grid spacing \( h \) in both \( x \) and \( y \) directions is the same and one grid line cuts the interface \( \Gamma \) at most twice. Denote the time increment to be \( \Delta t \) and take \( N_x \) and \( N_y \) as the number of grid points in each direction. To facilitate the following discussions, we adopt a notation at node \((x_i, y_j, t_k)\): \( u_{i,j}^{k} = u(x_i, y_j, t_k) \).

We first rewrite the heat equation (2.1) by dividing \( \alpha \) throughout

\[
\frac{1}{\alpha} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{f}{\alpha}, \quad \text{in } \Omega^- \text{ or } \Omega^+.
\]

If the jump conditions (1.2) are rigorously enforced in the numerical discretization, the numerical solution to (2.1) will be identical to that of (1.4), whereas (2.1) allows
an easier formulation for the ADI method. The semi-discretization of (2.1) using the implicit Euler time integration at a general spatial node \((x_i, y_j)\) reads

\[
\frac{u^{k+1} - u^k}{\alpha \Delta t} = \left[ \frac{\partial^2 u}{\partial x^2} \right]^{k+1} + \left[ \frac{\partial^2 u}{\partial y^2} \right]^{k+1} + \frac{f^{k+1}}{\alpha},
\]

which is first order accurate in time. We propose a first order Douglas ADI method for solving (2.1),

\[
\begin{align*}
\left( \frac{1}{\alpha} - \Delta t \delta_{xx} \right) u^{*}_{i,j} &= \left( \frac{1}{\alpha} + \Delta t \delta_{yy} \right) u^{k}_{i,j} + \frac{\Delta t}{\alpha} f^{k+1}_{i,j}, \\
\left( \frac{1}{\alpha} - \Delta t \delta_{yy} \right) u^{k+1}_{i,j} &= \frac{1}{\alpha} u^{*}_{i,j} - \Delta t \delta_{yy} u^{k}_{i,j},
\end{align*}
\]

which can be shown to be a higher order perturbation of the implicit Euler scheme (2.2). Here \(\delta_{xx}\) and \(\delta_{yy}\) are discrete matrix operators for finite difference approximations in \(x\) and \(y\) directions. For nodes away from the interface, a central difference approximation is used, e.g.,

\[
\delta_{yy} u^{k+1}_{i,j} := \frac{1}{h^2} (u^{k+1}_{i,j} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j+1}).
\]

In the present study, a proper Dirichlet or Neumann boundary condition for \(u\) is assumed to be given on the boundary \(\partial \Omega\). Such a boundary condition is implemented as in the classical ADI schemes [8, 7, 17]. If a nontrivial boundary condition is encountered, the advanced MIB boundary closure method [26] can be utilized to enclose such a condition into the finite difference discretization.

For nodes near the interface \(\Gamma\), novel interface treatments will be developed to correct discrete finite difference operators \(\delta_{xx}\) and \(\delta_{yy}\) via rigorously imposing the jump conditions (1.2). To this end, we consider some tensor product decompositions of jump conditions (1.2) in the ADI framework. At an interface point, we denote the outer normal and tangential directions as \(n\) and \(\tau\), respectively. Denote the angle between \(n\) and the \(x\)-axis as \(\theta\). Coordinate transformations can be employed to convert between the derivatives

\[
\frac{\partial}{\partial n} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \tau} = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}.
\]

Based on (1.2), one more jump condition can be derived by differentiating along the interface \(\Gamma\): \([u_\tau] = \frac{\partial \phi}{\partial x} = \phi_\tau\). We thus have three zeroth and first order jump conditions

\[
\begin{align*}
[u] &= \phi, \quad [u_\tau] = \phi_\tau, \quad [\alpha u_n] = \psi.
\end{align*}
\]

However, jump conditions given in (2.6) cannot be applied in a 1D manner in the ADI algorithm. To illustrate this, we transform the flux jump condition into Cartesian directions

\[
\cos \theta [\alpha u_x] + \sin \theta [\alpha u_y] = \psi.
\]

A strong coupling in \(x\) and \(y\) directions is clearly seen.

In this paper, we propose to decompose the 2D jump conditions (2.6) into two sets of essentially 1D jump conditions. We illustrate the idea by considering the \(x\)
direction formulation. The $y$ direction can be similarly treated. Consider an interface point $(x_i, y_j)$ which is the intersection point between one $x$ grid line and $\Gamma$. If the normal direction $n$ at this point happens to be along the $x$ direction, we have simply 1D jump conditions $[u] = \phi$ and $[\alpha u_x] = \pm \psi$. For more general scenario, in which $n$ is not along the $x$ direction, we will analytically derive a hybrid jump condition using $x$ and $\tau$. In particular, we have $u_y = \sec \theta u_\tau + \tan \theta u_x$. By substituting this into (2.7), we arrive at

$$[\alpha u_x] + \sin \theta [\alpha u_\tau] = \cos \theta \psi.$$  \hfill (2.8)

Even though Eq. (2.8) looks similar to Eq. (2.7), Eq. (2.8) allows 1D numerical approximation, whereas Eq. (2.7) does not. This is because the jump value of $[u]$ is known, while that of $[u_y]$ is not. In fact, Eq. (2.8) can be further rewritten as

$$[\alpha u_x] = \cos \theta \psi - \sin \theta (\alpha^+ - \alpha^-)u^+_\tau - \sin \theta \alpha^+ \phi_\tau = \psi_x,$$

where $u^+_\tau$ shall be evaluated based on finite difference approximations using grid nodes exclusively from the positive side of $\Gamma$, i.e., $\Omega^+$. Once $u^+_\tau$ is accurately estimated, $\psi_x$ is known. We thus have essentially 1D jump conditions $[u] = \phi$ and $[\alpha u_x] = \psi_x$. Similarly, we derive the following essentially 1D jump conditions in the $y$ direction,

$$[u] = \phi, \quad [\alpha u_y] = \sin \theta \psi + \cos \theta (\alpha^+ - \alpha^-)u^+_\tau + \cos \theta \alpha^+ \phi_\tau = \psi_y.$$ \hfill (2.9)

We note that $\psi_x$ and $\psi_y$ can also be evaluated through calculating $u^+_\tau$ from the negative side of the interface.

We propose a new MIB scheme to impose the decomposed 1D jump conditions in the vicinity of the interface $\Gamma$. Comparing with other established interface methods, the MIB method \cite{25, 27, 26} is ideally-suited to the present problem because in the MIB, jump condition enforcement is disassociated with the derivative discretization and can be conducted in a 1D manner along each Cartesian direction. Here, we consider the MIB modification to $\delta_{yy} u_{i,j}^{k+1}$ in the ADI scheme (2.3) as an example, where $(x_i, y_j)$ is an irregular node near the interface $\Gamma$. A typical situation is shown in Fig. 2.1(a). In the MIB scheme, to approximate function or its derivatives on one side of interface, one never directly refers to function values from the other side. Instead, fictitious values from the other side of the interface will be supplied. Referring to Fig. 2.1(a), $\delta_{yy} u_{i,j}^{k+1}$ will be corrected as

$$\delta_{yy} u_{i,j}^{k+1} = \frac{1}{h^2}(u_{i,j-1}^{k+1} - 2u_{i,j}^{k+1} + \tilde{u}_{i,j+1}^{k+1}),$$ \hfill (2.10)

where $\tilde{u}_{i,j+1}^{k+1}$ is a fictitious value at the node $(x_i, y_{j+1})$. Similarly, $\delta_{yy} u_{i,j+1}^{k+1}$ will be modified. This calls for two fictitious values $\tilde{u}_{i,j}^{k+1}$ and $\tilde{u}_{i,j+1}^{k+1}$, which will be resolved based on the jump conditions (2.10).

In order to impose the jump conditions (2.10), we first need to approximate $u^+_\tau$. Consider the situation shown in Fig. 2.1(a). We calculate $u^+_\tau$ in two steps. First, we calculate the intersection points between the tangential line $\tau$ and two grid lines $x = x_{i-1}$ and $x = x_{i+1}$. There two auxiliary nodes are shown as open squares in Fig. 2.1(a). A central difference is conducted to approximate the $\tau$ derivative at the interface point $(x_i, y_j)$ using two auxiliary values of $u$. Second, each of these two auxiliary values will be further interpolated by using three on-grid function values, all selecting from $\Omega^+$. These six nodes are shown as filled squares in Fig. 2.1(a). In this
manner, $u_{i,j}^+$ is actually approximated by six grid values of $u$, with the spatial accuracy being second order. In the present study, we will make use of the known values of $u$ at the current time instant $t_k$ to estimate $u_{i,j}^+$, which avoids the introduction of a coupling among different $y$ grid lines at the future time instant $t_{k+1}$. Otherwise, the 1D linear systems of the ADI algorithm are not independent, which cannot be solved efficiently. We note that the present approximation is of first order accurate in time when it is applied to correct $\delta_{yy}u_{i,j}^{k+1}$ and $\delta_{xx}u_{i,j}^*$ in (2.3). This is acceptable, since the temporal order of the ADI scheme (2.3) is also one.

With the calculated $u_{i,j}^+, \psi_y$ is then known at the interface point $(x_i, y_k)$. We will next determine $\delta_{ij}^{k+1}$ and $\delta_{ij}^{k+1}$ by using four function values at the future time $t_{k+1}$, i.e., $u_{i,j-1}^{k+1}$, $u_{i,j}^{k+1}$, $u_{i,j+1}^{k+1}$, and $u_{i,j+2}^{k+1}$, see Fig. 2.1 (a). For this purpose, (2.10) will be discretized in the same manner of (2.11), i.e., never referring to function values across the interface $\Gamma$

$$
(2.12) \quad w_{1,0}^+ u_{i,j}^{k+1} + w_{0,1}^+ u_{i,j+1}^{k+1} + w_{0,3}^+ u_{i,j+2}^{k+1} = w_{1,0}^- u_{i,j}^{k+1} + w_{0,2}^- u_{i,j+1}^{k+1} + w_{0,3}^- u_{i,j+2}^{k+1} + \phi,
$$

$$
= \alpha^+ \left( w_{1,1}^+ u_{i,j}^{k+1} + w_{1,2}^+ u_{i,j+1}^{k+1} + w_{1,3}^+ u_{i,j+2}^{k+1} \right) - \alpha^- \left( w_{1,1}^- u_{i,j}^{k+1} + w_{1,2}^- u_{i,j+1}^{k+1} + w_{1,3}^- u_{i,j+2}^{k+1} \right) + \psi_y,
$$

where $w_{I,J}^+$ and $w_{I,J}^-$ for $I = 0, 1$ and $J = 1, 2, 3$ are one-sided finite difference weights, respectively, for left and right subdomains. Here the subscript $I$ represents interpolation ($I = 0$) and the first derivative approximation ($I = 1$), and $J$ is for grid index. After the discretization, (2.12) actually represents two algebraic equations. By solving (2.12), one can determine $\delta_{ij}^{k+1}$ and $\tilde{\delta}_{ij}^{k+1}$ as linear combinations of $u_{i,j-1}^{k+1}$, $u_{i,j+1}^{k+1}$, $u_{i,j+2}^{k+1}$, $\phi$, and $\psi_y$. By substituting such combinations into (2.11) to eliminate the fictitious value, $\delta_{yy}u_{i,j}^{k+1}$ is now a second order accurate finite difference approximation to the double $y$ derivative, involving four $u$ values along the $y$ direction and two nonhomogeneous values $\phi$ and $\psi_y$.  

Fig. 2.1. Illustration of the MIB grid partitions. (a) For a regular interface; (b) For a corner case. In both charts, the jump conditions will be discretized by using fictitious values (open circles) and function values (filled circles). In (a), the approximation of $u_{i,j}^0$ is also shown, i.e., it is approximated by two auxiliary values (open squares), then interpolated by six function values (filled squares).
When one grid line intersects the interface $\Gamma$ near a rounded or sharp corner, the interface could be cut twice within a short distance. If in between these two intersection points there is no grid node, a grid refinement is necessary. If there are at least two grid nodes, the aforementioned matched ADI algorithm can be conducted. However, additional corner treatments are called for the case where only one node locates in between two intersection points, see Fig. 2.1 (b). A MIB corner scheme is proposed to solve this problem. Denote two intersection points as $(x_i, y_{T1})$ and $(x_i, y_{T2})$ with $y_{T1} < y_j < y_{T2}$. At $(x_i, y_{T1})$ and $(x_i, y_{T2})$, $u_x^+$ can be calculated individually as outlined above, so that the nonhomogeneous values $(\phi_1, \psi_{y1})$ and $(\phi_2, \psi_{y2})$ are known, respectively. The jump conditions (2.10) can then be imposed at $(x_i, y_{T1})$ and $(x_i, y_{T2})$ to form four algebraic equations. Nevertheless, only three fictitious values are needed to correct $\delta_{yy}u$, i.e., $\tilde{u}_{i,j-2}^+, \tilde{u}_{i,j}^+$, and $\tilde{u}_{i,j+1}^+$. This difficulty can be trivially avoided by introducing one more fictitious value. Referring to Fig. 2.1 (b), we denote $D_1 = |y_{T1} - y_{j-1}|$ and $D_2 = |y_{j+1} - y_{T2}|$. If $D_1 < D_2$, the fourth fictitious value is chosen as $\tilde{u}_{i,j-2}^+$. Otherwise, it is selected as $\tilde{u}_{i,j+2}^-$. With such a grid partition, each term of the jump conditions (2.10) is approximated through a third order finite difference approximation involving four points. This ensures the overall accuracy of the matched ADI scheme, since the solution usually undergoes a rapid change near the corner. In particular, for the case shown in Fig. 2.1 (b), the positive and negative terms at $(x_i, y_{T1})$ are approximated based on $(u_{i,j-2}^{k+1}, u_{i,j-1}^{k+1}, \tilde{u}_{i,j}^{k+1}, u_{i,j+1}^{k+1})$ and $(u_{i,j-1}^{k+1}, u_{i,j}^{k+1}, \tilde{u}_{i,j+1}^{k+1}, \tilde{u}_{i,j+2}^{k+1})$, respectively, while those at $(x_i, y_{T2})$ are based on $(u_{i,j-1}^{k+1}, \tilde{u}_{i,j}^{k+1}, u_{i,j+1}^{k+1}, u_{i,j+2}^{k+1})$ and $(\tilde{u}_{i,j-2}^{k+1}, \tilde{u}_{i,j-1}^{k+1}, u_{i,j}^{k+1}, \tilde{u}_{i,j+1}^{k+1})$, respectively. The details of the discretization are omitted here. By substituting the solved linear combinations into (2.11), $\delta_{yy}u_{i,j}^{k+1}$ is approximated by five function values $(u_{i,j-2}^{k+1}, u_{i,j-1}^{k+1}, u_{i,j}^{k+1}, u_{i,j+1}^{k+1}, u_{i,j+2}^{k+1})$ and four nonhomogeneous values $(\phi_1, \psi_{y1}, \phi_2, \psi_{y2})$.

We note that the proposed MIB treatments for the regular interface case and the corner case need to be conducted only once at the beginning of the simulation, because the geometric domain, grid, and spatial discretization of the jump conditions are all time invariant. In fact, the entries of the discrete operators $\delta_{xx}u$ and $\delta_{yy}u$ are all time independent and can be pre-determined. At each time step, one just needs to update nonhomogeneous values $\phi$, $\psi_x$ and $\psi_y$.

However, the discrete matrices $\delta_{xx}u$ and $\delta_{yy}u$ of the matched ADI algorithm are no longer tridiagonal, so that new algebraic solvers have to be developed to maintain the efficiency. For a regular interface case, one grid line cuts the interface $\Gamma$ twice at a well
separated distance. There are totally four irregular nodes sandwiched the interface. For each of them, the actual band width becomes four, while two nonhomogeneous values $\phi$ and $\psi_y$ shall be added into the right hand side of the ADI scheme (2.3). A typical matrix structure is shown in Fig. 2.2 (a), where $N$ is the dimension of matrix $A$, i.e., $N = N_x$ or $N = N_y$. For the corner interface case, a similar analysis indicates that the band width of three consecutive irregular nodes are changed from three to five, see Fig. 2.2 (b).

Since the change of the band-structure is not too dramatic, the corresponding linear system $AU = b$ could be solved by the Woodbury formula [18]. We consider the regular interface case as an example. Denote the indices of four irregular nodes to be $I, I + 1, J, \text{and} J + 1$. See Fig. 2.2 (a). The extra coefficients of $A$ can be accounted for by defining two $N \times 2$ matrices $P$ and $Q$ with four nonzero elements each:

\[
P_{I,1} = 1, \quad P_{I+1,1} = 1, \quad P_{J,2} = 1, \quad P_{J+1,2} = 1,
Q_{I-1,1} = A_{I+1,I-1}, \quad Q_{I+2,1} = A_{I,I+2}, \quad Q_{I-1,2} = A_{J+1,J-1}, \quad Q_{J+2,2} = A_{J,J+2}.
\]

We then have $A = T + PQ^T$, where $T$ is a tridiagonal matrix. Thus, we have analytically

\[(2.13) \quad A^{-1} = (T + PQ^T)^{-1} = T^{-1} - [T^{-1}P(1 + Q^T T^{-1} P)^{-1} Q^T T^{-1}],\]

by the Woodbury formula [18]. In other words, the inversion of $A$ can be carried out through applying the Thomas algorithm three times by solving some auxiliary systems about $T$.

However, the Woodbury formula is not used in our computation, because a more efficient algebraic procedure is available. For example, for the regular interface case, four elementary row operations are simply conducted. In particular, by denoting the $I^{th}$ row of $A$ as $R_I$, the elementary row operation $R_{I+1} = A_{I+1,I-1} / A_{I,I-1} R_I$ will zero the entry at the position $(I + 1, I - 1)$. Similar row operations are conducted to vanish other three positions, so that a tridiagonal system is formed. In a similar manner, the matrix $A$ of the corner interface can be treated by six row operations to reduce to a tridiagonal one. After the Gauss elimination, the Thomas algorithm [21] is applied only once to solve the adjusted system.

The proposed matched ADI algorithm is very efficient. To solve the 1D linear system within each inner ADI step, the flop counts are essentially due to the Thomas algorithm, i.e., on the order of $O(N)$, because the overhead for the Gauss elimination is very small and does not grow with $N$. Thus, like the standard 2D ADI algorithm, the complexity of advancing one time step in the proposed matched ADI schemes is about $O(N^2)$. Moreover, the proposed matched ADI scheme is unconditionally stable for solving the heat equation. In practice, one can simply fix $\Delta t$ to be on the order of $h$. Consequently, the complexity of entire time integration will be on the order of $O(N^3)$ for solving parabolic interface problems with $N^2$ unknowns.

3. Numerical experiments. In this section, we investigate the stability and accuracy of the proposed matched ADI algorithm for solving 2D parabolic interface problems with different jump conditions and interface geometries. Piecewisely defined analytical solutions will be constructed in each example. The initial solution is chosen according the analytical solution at $t = 0$. The ADI time stepping will be carried out until a stopping time $t = T$. Without the loss of generality, a square domain $[-D, D] \times [-D, D]$ with the Dirichlet boundary condition is considered in all examples. Here,
the boundary data is simply given by the analytical solutions. Similarly, the jump conditions defining function and flux jumps across the interfaces are also calculated according to the given analytical solutions. For simplicity, the mesh sizes in both $x$ and $y$ directions are chosen to be the same, i.e., $N = N_x = N_y$ with $h = \Delta x = \Delta y$.

The domain size $D$ is usually chosen as a non-integer so that the corner interface case could be encountered in a coarse grid. This enables us to fully validate the proposed matched ADI algorithm. Numerical errors in $L_\infty$ and $L_2$ norms are reported in all examples.

**Example 1.** We first study a circular interface problem with a continuous solution. Consider a square domain $[-D,D] \times [-D,D]$ with a circular interface $r^2 = x^2 + y^2 = 1$. The piecewise coefficient is defined to be $\alpha^- = 1$ and $\alpha^+ = 10$, respectively, for $r < 1$ and $r \geq 1$. The analytical solution to the heat equation is designed to be

$$u(x,y,t) = \begin{cases} \left(\frac{r^6 - 1}{\alpha^-} - \frac{3}{\alpha^+}\right) \cos(t), & \text{if } r < 1 \\ -\frac{3}{\alpha^+ r^2} \cos(t), & \text{if } r \geq 1 \end{cases}$$

so the jump conditions are simply $[u] = 0$ and $[\alpha u_n] = 0$. The source term is then given as

$$f(x,y,t) = \begin{cases} -\left(\frac{r^6 - 1}{\alpha} - \frac{3}{\alpha}\right) \sin(t) - 36r^4 \cos(t), & \text{if } r < 1 \\ \frac{3}{\alpha^+ r^2} \sin(t) + 12r^{-4} \cos(t), & \text{if } r \geq 1 \end{cases}$$

In this example, the domain size is set to be $D = 1.99$ and the stopping time is fixed as $T = 2$.

We first examine the temporal convergence. The proposed matched ADI algorithm is found to be unconditionally stable for all tested $h$ and $\Delta t$. By taking $N_x = N_y = 321$, the numerical errors generated by using different $\Delta t$ are shown in Fig. 3.1 (a). A similar pattern can be observed for both $L_\infty$ and $L_2$ errors. i.e., the temporal errors become smaller and smaller until they are limited by the accuracies of the spatial discretization. The temporal convergence order of the matched ADI algorithm can be analyzed via considering errors before reaching the limiting precision. For these errors, a linear least-squares fitting [26] is conducted in the log-log scale. The fitted convergence lines are shown as solid lines in Fig. 3.1 (a). Moreover, the
Table 3.1
Spatial convergence tests for first two examples.

| N   | Example 1 | | Example 2 | |
|-----|-----------|---|-----------|---|
|     | $L_\infty$ | $L^2$ | $L_\infty$ | $L^2$ |
| 21  | 1.92E-2  | 6.75E-3 | 1.78E-4 | 9.15E-5 |
| 41  | 4.49E-3  | 2.09  | 4.38E-5 | 2.02  |
| 81  | 1.34E-3  | 1.74  | 1.77E-3 | 2.16  |
| 161 | 3.78E-4  | 1.83  | 3.07E-6 | 2.53  |
| 321 | 9.47E-5  | 2.00  | 5.38E-7 | 2.51  |

The fitted slope essentially represents the numerical convergence rate $r$ of the scheme. The temporal order in $L_\infty$ and $L^2$ norms is found to be $r = 1.58$ and $r = 1.48$, respectively. In other word, the numerically detected temporal order of the matched ADI algorithm is about half order higher than its theoretical design. This is perhaps because the present solution is continuous. The superconvergence of the Douglas ADI scheme for smooth solutions has also been observed in other literature studies \[9, 24\].

We next quantitatively examine the spatial accuracy. A small enough $\Delta t = 10^{-4}$ is employed so that the temporal error can be neglected in the present study. The numerical errors of the matched ADI method for different mesh sizes $N = N_x = N_y$ are listed in Table 3.1. Based on successive mesh refinements, the numerically calculated convergence rates are also reported for both error measurements. It can be seen that the matched ADI algorithm achieves the second order of accuracy in both $L_\infty$ and $L^2$ norms for the present parabolic interface problem.

**Example 2.** We next consider a circular interface problem with constant jump values. The interface $\Gamma$ is defined as $r = 0.5$ and the domain size is set as $D = 0.99$. The diffusion coefficient is chosen as $\alpha^- = 2$ and $\alpha^+ = 10$, respectively, for $r < 0.5$ and $r \geq 0.5$. The source term is defined to be

$$f(x, y, t) = \begin{cases} -\sin(t) - 4\alpha^- & \text{if } r < 0.5 \\ -\sin(t) - 8r^2 - 4 & \text{if } r \geq 0.5 \end{cases}$$

The analytical solution can then be given as

$$u(x, y, t) = \begin{cases} \cos(t) + r^2 - 1 & \text{if } r < 0.5 \\ \cos(t) + \frac{1}{4}(1 - \frac{9}{8\alpha^+}) + \frac{1}{\alpha^+}(\frac{1}{4} + r^2) & \text{if } r \geq 0.5 \end{cases}$$

It can be verified that the jump values are constants along the interface $\Gamma$ and are time invariant. In particular, we have $[u] = 1$ and $[\alpha u_n] = -0.75$.

For all tested $h$ and $\Delta t$, the matched ADI method is again found to be unconditionally stable for this example. By choosing $N_x = N_y = 321$ and $T = 1$, the temporal accuracies are analyzed in Fig. 3.1(b). It can be seen that both $L_\infty$ and $L^2$ errors decrease uniformly until the limiting precisions of the spatial discretization are reached. After that, by using a smaller $\Delta t$, the error become slightly larger. The least-squares error analysis is also conducted. With a rate of 1.11 for both error norms, the matched ADI method clearly attains the first order of accuracy in time for this example with a discontinuous solution. By using a sufficiently small $\Delta t = 10^{-4}$, the spatial accuracies of the matched ADI method are investigated in Table 3.1. The
spatial convergence is not very uniform for this example. However, the overall order of the matched ADI method is still around two in both error measurements.

**Example 3.** To further explore the potential of the proposed ADI method, we consider a circular interface problem with general jump values. The interface is also defined as $r = 0.5$ with domain size $D = 0.99$. By taking $\alpha^- = 1$ and $\alpha^+ = 10$, the analytical solution is chosen as

$$u(x, y, t) = \begin{cases} \cos(t) + \exp(x^2 + y^2), & \text{if } r < 0.5 \\ \cos(t) + \sin(kx) \cos(ky), & \text{if } r \geq 0.5 \end{cases},$$

where the wavenumber is chosen as $k = 2$. The source term is given as

$$f(x, y, t) = \begin{cases} -\sin(t) - 4\alpha^- \exp(x^2 + y^2)(x^2 + y^2 + 1), & \text{if } r < 0.5 \\ -\sin(t) + 2\alpha^+ k^2 \sin(kx) \cos(ky), & \text{if } r \geq 0.5 \end{cases}.$$

The jump conditions at a interface point $(x, y) = (\frac{1}{2} \cos \theta, \frac{1}{2} \sin \theta)$ can be derived from the analytical solution

$$[u] = \sin(k \frac{1}{2} \cos \theta) \cos(k \frac{1}{2} \sin \theta) - \exp(\frac{1}{4}),$$

$$[\alpha u_n] = \alpha^+ k \cos \theta \cos(k \frac{1}{2} \cos \theta) \cos(k \frac{1}{2} \sin \theta) - \alpha^- \exp(\frac{1}{4}),$$

$$[u_L] = -k \sin \theta \cos(k \frac{1}{2} \cos \theta) \cos(k \frac{1}{2} \sin \theta) - k \cos \theta \sin(k \frac{1}{2} \cos \theta) \sin(k \frac{1}{2} \sin \theta),$$

where the third jump condition $[u_L] = \phi_L$ is derived from the function jump $[u] = \phi$. We note that the present jump conditions are quite general in the sense that all jump values are functions of space, even though they are time independent.

Again, the matched ADI algorithm is unconditionally stable for all tested $h$ and $\Delta t$ in this example. By using $N_x = N_y = 321$ and $T = 1$, both $L_\infty$ and $L_2$ errors immediately begin to decay as $\Delta t$ becomes smaller, and the limiting precisions are approached when $\Delta t$ is small enough. It can be seen from Fig. 3.2 (a) that the slope
**Table 3.2**

Spatial convergence tests for Example 3 and Example 4.

| $N$ | $L_\infty$ Error | $L_2$ Error | $L_\infty$ Error | $L_2$ Error |
|-----|------------------|-------------|------------------|-------------|
| 21  | 9.12E-3          | 1.61E-3     | 4.77E-3          | 8.54E-4     |
| 41  | 2.51E-3          | 1.86E-4     | 3.76E-4          | 1.96E-4     |
| 81  | 4.93E-4          | 2.35E-4     | 7.24E-4          | 3.57E-4     |
| 161 | 7.47E-5          | 2.72E-4     | 1.39E-5          | 6.66E-4     |
| 321 | 1.36E-5          | 2.45E-4     | 3.04E-5          | 1.57E-4     |

**Fig. 3.3.** Numerical solution with $N_x = N_y = 81$ at $T = 1$ (a). Example 3; (b). Example 4.

for the $L_\infty$ and $L_2$ error curve is, respectively, 1.07 and 1.06. Thus, the matched ADI delivers the first order of accuracy in time for parabolic interface problems with general jump values. By taking $\Delta t = 10^{-4}$, the spatial errors are reported in Table 3.2. The matched ADI method clearly achieves second order in space for this example.

The matched ADI solution based on a mesh $N_x = N_y = 81$ at the time $T = 1$ is shown in Fig. 3.3 (a). The jump values clearly change with respect to the angle $\theta$.

**Example 4.** We next extend the Example 3 to a more general situation, by considering time dependent jumps. The parameters for the interface and domain are fixed to be $r = 0.5$, $D = 0.99$, $\alpha^- = 1$, and $\alpha^+ = 10$. The analytical solution is constructed as

$$u(x, y, t) = \begin{cases} 
\sin(kx) \cos(ky) \cos(t), & \text{if } r < 0.5 \\
\cos(kx) \sin(ky) \cos(t), & \text{if } r \geq 0.5,
\end{cases}$$

where the wavenumber is chosen as $k = 2$. The source term is given as

$$f(x, y, t) = \begin{cases} 
(2k^2 \alpha^- \cos(t) - \sin(t)) \sin(kx) \cos(ky), & \text{if } r < 0.5 \\
(2k^2 \alpha^+ \cos(t) - \sin(t)) \cos(kx) \sin(ky), & \text{if } r \geq 0.5.
\end{cases}$$

Now, the jump conditions at a interface point $(x, y) = (\frac{1}{2} \cos \theta, \frac{1}{2} \sin \theta)$ depend on
both space and time

\[ [u] = \cos(\frac{k}{2} \cos \theta) \sin(\frac{k}{2} \sin \theta) \cos(t) - \sin(\frac{k}{2} \cos \theta) \cos(\frac{k}{2} \sin \theta) \cos(t), \]

\[ [\alpha u_n] = k \cos(t) (\alpha^- \sin \theta - \alpha^+ \cos \theta) \sin(\frac{k}{2} \cos \theta) \sin(\frac{k}{2} \sin \theta) \]
\[ + k \cos(t) (\alpha^+ \sin \theta - \alpha^- \cos \theta) \cos(\frac{k}{2} \cos \theta) \cos(\frac{k}{2} \sin \theta), \]

\[ [u_\tau] = k \cos(t) (\cos \theta + \sin \theta) (\cos(\frac{k}{2} \cos \theta) \cos(\frac{k}{2} \sin \theta) + \sin(\frac{k}{2} \cos \theta) \sin(\frac{k}{2} \sin \theta)). \]

Such jump conditions are the most general ones for parabolic interface problems. The temporal convergence pattern now turns out to be significantly different from those of the previous examples. By using \( N_x = N_y = 321 \) and \( T = 1 \), the \( L_\infty \) and \( L_2 \) errors are depicted in Fig. 3.2(b). It can be observed that the error curve does not immediately decay for large \( \Delta t \) values. In other word, the temporal convergence of the matched ADI method is somehow polluted by the time dependent jump conditions. Nevertheless, for those non-convergent \( \Delta t \) values, the error remains to be bounded. This suggests that the matched ADI algorithm is still unconditionally stable in this case. When a rather small \( \Delta t = 10^{-3} \) is employed, the matched ADI method begins to converge. Moreover, once the convergence starts, the rate is pretty high. The least-squares fitting shows that the rate for the descending parts is \( r = 1.88 \) and \( r = 1.91 \), respectively, for the \( L_\infty \) and \( L_2 \) errors. Thus, excluding the polluted region, the matched ADI method yields a second order of accuracy in time. On the other hand, if the polluted region is included in the least-squares analysis, the overall temporal order will become about one.

The spatial orders are not affected by the time dependent jumps. By using a sufficiently small \( \Delta t = 10^{-6} \), the numerical errors based on different meshes are listed in Table 3.2. The numerical orders are all around two and are comparable to those of the Example 3. This demonstrates the robustness of the proposed MIB interface treatment in solving time dependent jumps. The matched ADI solution with \( N_x = N_y = 81 \) and \( T = 1 \) is plotted in Fig 3.3(b). The jump values shown in the figure will oscillate with respect to the time \( t \).

**Example 5.** At last, we explore the performance of the proposed matched ADI algorithm for interfaces of general shape. To this end, the following interface which is parameterized with the polar angle \( s \) will be studied

\[ \Gamma : \quad r = \frac{1}{2} + b \sin(ms), \quad s \in [0, 2\pi]. \]

Here the parameter \( m \) determines the number of “leaves” of the core region \( \Omega^- \) and \( b \) controls the magnitude of the curvature. Two independent cases with parameters \((m, b) = (2, 1/4)\) and \((4, 1/10)\) are considered. A square domain with \( D = 0.99 \) is also employed. The resulting configurations of the two leaves and four leaves cases can be seen from the contour plots of numerical solutions given in Fig. 3.4. It is clear that concave segments or negative curvatures are involved in the present interfaces. In both cases, we set \( \alpha^- = 1 \) and \( \alpha^+ = 10 \). The analytical solution is constructed as in the Example 4

\[ u(x,y,t) = \begin{cases} 
\sin(kx) \cos(ky) \cos(t), & \text{in } \Omega^- \\
\cos(kx) \sin(ky) \cos(t), & \text{in } \Omega^+, 
\end{cases} \]
with $k = 2$. The source term is also given as
\[
    f(x, y, t) = \begin{cases} 
    (2k^2 \alpha^- \cos(t) - \sin(t)) \sin(kx) \cos(ky), & \text{in } \Omega^- \\
    (2k^2 \alpha^+ \cos(t) - \sin(t)) \cos(kx) \sin(ky), & \text{in } \Omega^+. 
    \end{cases}
\]

The jump conditions can be similarly calculated according to the analytical solution. The details are omitted here.

By using $N_x = N_y = 321$ and $T = 1$, the $L_\infty$ and $L_2$ errors for both two leaves and four leaves cases are shown in Fig. 3.5. The temporal convergences of these two cases are very similar to that of Example 4, because the present jump conditions are also time dependent. In particular, the convergence begins only when $\Delta t$ is small enough and is of second order once occurs, while the stability is maintained for all $\Delta t$ values. The overall temporal order is also around one, if the polluted region is also included in the least-squares analysis. By taking $\Delta t = 2.5 \times 10^{-6}$ and $\Delta t = 10^{-5}$, respectively, for
Table 3.3
Spatial convergence tests for Example 5.

| N  | Two leaves case | Four leaves case |
|----|----------------|-----------------|
|    | $l_\infty$ error | $l_\infty$ order | $l_2$ error | $l_2$ order |
| 21 | 3.06E-3       | 7.49E-4     | 5.97E-3    | 1.48E-3    |
| 41 | 5.37E-4       | 2.51        | 2.48E-3    | 1.27       |
| 81 | 1.80E-4       | 1.47E-4     | 9.95E-4    | 1.32       |
| 161| 3.92E-5       | 1.03E-5     | 1.32E-4    | 2.91       |
| 321| 1.08E-5       | 2.55E-6     | 4.91E-5    | 1.72       |

The two leaves case and four leaves case, the spatial errors are reported in Table 3.3. It can be observed that when the geometrical structure becomes more complicated or the number of leaves $m$ is larger, the convergence pattern of the matched ADI method becomes more oscillatory and the overall numerical order becomes slightly smaller. Nevertheless, the proposed matched ADI method still can secure a second order of accuracy for these challenging parabolic interface problems of complicated geometry. The contour plots of the matched ADI solutions based on a mesh $N_x = N_y = 81$ at time $T = 1$ are illustrated in Fig. 3.4. The solutions clearly undergo sharp changes across the interface $\Gamma$, and such changes are time variant.

4. Conclusion. This paper presents a novel matched ADI method for solving parabolic interface problems with general jump conditions and complex geometries. The development of accurate and stable ADI schemes for such interface problems is essentially an open problem, because the existing IIM-ADI schemes require the second order jump conditions and have difficulties to correct finite difference approximations based on general flux jumps. The second order jump conditions are not needed in our interface treatment. Moreover, a novel tensor product decomposition is proposed to decouple 2D jump conditions into essentially 1D ones. By enforcing these 1D conditions, rigorous 1D MIB schemes are developed to treat regular interfaces and corner interfaces. The resulting matched ADI scheme achieves second order of accuracy in space for interfaces of different shapes. The efficiency of the ADI scheme is well maintained, because the MIB interface treatment needs to be conducted only once at the beginning of the computation and fast algebraic solvers are developed for perturbed tridiagonal systems. The proposed matched ADI scheme is unconditionally stable and first order accurate in time. Nevertheless, the temporal convergence is somewhat polluted when dealing with time-dependent jump conditions. The development of robust matched ADI schemes for unsteady jumps and more general parabolic equations is currently under investigation.

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