Computing the $p$-adic Canonical Quadratic Form in Polynomial Time

Chandan Dubey

Thomas Holenstein

chandan.dubey@inf.ethz.ch	thomas.holenstein@inf.ethz.ch

Institut für Theoretische Informatik, ETH Zürich

Abstract

An $n$-ary integral quadratic form is a formal expression $Q(x_1, \ldots, x_n) = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j$ in $n$-variables $x_1, \ldots, x_n$, where $a_{ij} = a_{ji} \in \mathbb{Z}$. We present a randomized polynomial time algorithm that given a quadratic form $Q(x_1, \ldots, x_n)$, a prime $p$, and a positive integer $k$ outputs a $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U$ transforms $Q$ to its $p$-adic canonical form.

1 Introduction

Let $R$ be a commutative ring with unity and $R^\times$ be the set of units (i.e., invertible elements) of $R$. A quadratic form over the ring $R$ in $n$-formal variables $x_1, \ldots, x_n$ in an expression $\sum_{1 \leq i, j \leq n} a_{ij} x_i x_j$, where $a_{ij} = a_{ji} \in R$. A quadratic form can equivalently be represented by a symmetric matrix $Q^n = (a_{ij})$ such that $Q(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^T Q (x_1, \ldots, x_n)$. The quadratic form is called integral if $R = \mathbb{Z}$ and the determinant of the quadratic form $Q$ is defined as $\det(Q)$. In this paper, we concern ourselves with integral quadratic forms, henceforth referred only as quadratic forms.

Let $p$ be a prime and $k$ be a positive integer. Then, two quadratic forms $Q_1, Q_2$ are said to be $p^k$-equivalent if there is a $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^T Q_1 U \equiv Q_2 \mod p^k$ (denoted $Q_1 \cong_{p^k} Q_2$). Two quadratic forms are $p^*$-equivalent if they are $p^k$-equivalent for every $k > 0$ (denoted $Q_1 \cong_{p^*} Q_2$). Intuitively, $p^k$-equivalence means that there exists an invertible linear change of variables over $\mathbb{Z}/p^k\mathbb{Z}$ that transforms one form to the other.

For a quadratic form $Q$ and prime $p$, the set $\{S^n \mid S \cong_{p^*} Q\}$ is the set of $p^*$-equivalence forms of $Q$ (also called $p^*$-equivalence class of $Q$). In this paper, we are interested in defining and computing a “canonical” quadratic form for the $p^*$-equivalence class of a given quadratic form $Q$. In particular, we are interested in showing the existence of a function $\text{can}_p$ such that for all integral quadratic forms $Q$, $\text{can}_p(Q) \in \{S \mid S \cong_{p^*} Q\}$; with the property that if $Q_1 \cong_{p^*} Q_2$ then $\text{can}_p(Q_1) = \text{can}_p(Q_2)$. We also consider a related problem of coming up with a canonicalization procedure. In particular, we want a polynomial time algorithm that given $Q, p$ and a positive integer $k$, finds $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^T Q U \equiv \text{can}_p(Q) \mod p^k$.

It is not difficult to show the existence of a canonical form. For example, we can go over the $p^*$-equivalence class of $Q$ and output the form which is lexicographically the smallest one. But, this form gives us no meaningful information about $Q$ or the $p^*$-equivalence class of $Q$.

Gauss [Gau86] gives a complete classification of binary quadratic forms (i.e., $n = 2$). Mathematicians have been more interested in coming up with a list of necessary and sufficient conditions for $p^*$-equivalence (also called equivalence over the $p$-adic integers $\mathbb{Z}_p$). There are several competing but equivalent candidates for the set of conditions [Cas78, O’M73, CS99, Kit99]. We choose to use the same set of conditions as Conway-Sloane [CS99], called the $p$-symbol of a quadratic form.

For odd prime $p$, the $p$-canonical form is implicit in Conway-Sloane [CS99] and is also described explicitly by Hartung [Har08, Cas78]. The canonicalization algorithm in this case is not complicated and can be claimed to be implicit in Cassels [Cas78].
The definition of canonical form for the case of prime 2 is quite involved and needs careful analysis\footnote{Cassels (page 117, Section 4, [Cas78]), referring to the canonical forms for $p = 2$ observes that “only a masochist is invited to read the rest”}. Jones [Jon44] presents the most complete description of the 2-canonical form. His method is to come up with a small 2-canonical forms and then showing that every quadratic form is 2-equivalent to one of these. Unfortunately, a few of his transformations are existential i.e., he shows that a transformations with certain properties exists without explicitly finding them.

Conway-Sloane [CS99], instead, compute a description of a quadratic form (called canonical 2-symbol) with the property that two quadratic forms are 2-equivalent iff they have the same canonical 2-symbol. They do not provide a 2-canonical form i.e., can$_2(q) \in \{s \mid s \cong^2 q\}$. The p-canonical forms are very useful in the study of quadratic forms and their equivalence, see [Sie32, Jon44, Jon50, Cas78, CS99, Kit99, Har08].

**Our Contribution.** We give polynomial time $p$-canonicalization algorithm. In particular, we present an algorithm that given an integral quadratic form $Q$ runs in time poly($n, \log(\det(Q))$) and outputs can$_p(q)$.

Given an integral quadratic form $Q$, a positive integer $k$ and a prime $p$, we also provide a randomized poly($n, \log(\det(Q)), \log(p), k$) algorithm that outputs a matrix $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^TQU \equiv \text{can}_p(Q) \pmod{p^k}$. This algorithm is especially useful if we want to find a transformation that maps $Q_1$ to $Q_2$ over $\mathbb{Z}/p^k\mathbb{Z}$, where $Q_1 \cong^2 Q_2$. In this case, the required transformation is $U_1U_2^{-1} \pmod{p^k}$, where $U_1^TQ_1U_1 \equiv \text{can}_p(Q_1) \equiv U_2^TQ_2U_2 \pmod{p^k}$.

## 2 Preliminaries

Integers and ring elements are denoted by lowercase letters, vectors by bold lowercase letters and matrices by typewriter uppercase letters. The $i$'th component of a vector $v$ is denoted by $v_i$. We use the notation $(v_1, \cdots, v_n)$ for a column vector and the transpose of matrix $A$ is denoted by $A'$. The matrix $A^n$ will denote a $n \times n$ square matrix. The scalar product of two vectors will be denoted $v \cdot w$ and equals $\sum_i v_iw_i$. The standard Euclidean norm of the vector $v$ is denoted by $||v||$ and equals $\sqrt{\mathbf{v}^T\mathbf{v}}$.

If $Q^1_1, Q^2_2$ are matrices, then the direct product of $Q_1$ and $Q_2$ is denoted by $Q_1 \oplus Q_2$ and is defined as diag$(Q_1, Q_2) = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. Given two matrices $Q_1$ and $Q_2$ with the same number of rows, $[Q_1, Q_2]$ is the matrix which is obtained by concatenating the two matrices columnwise. A matrix is called unimodular if it is an integer $n \times n$ matrix with determinant $\pm 1$. If $Q^n$ is a $n \times n$ integer matrix and $q$ is a positive integer then $Q \pmod{q}$ is defined as the matrix with all entries of $Q$ reduced modulo $q$.

Let $R$ be a commutative ring with unity and $R^\times$ be the set of units (i.e., invertible elements) of $R$. If $Q \in R^{n \times n}$ is a square matrix, the adjugate of $Q$ is defined as the transpose of the cofactor matrix and is denoted by adj$(Q)$. The matrix $Q$ is invertible if and only if $\det(Q)$ is a unit of $R$. In this case, $\text{adj}(Q) = \det(Q)Q^{-1}$.

The set of invertible $n \times n$ matrices over $R$ is denoted by $\text{GL}_n(R)$. The subset of matrices with determinant 1 will be denoted by $\text{SL}_n(R)$.

**Fact 1** A matrix $U$ is in $\text{GL}_n(R)$ iff $\det(U) \in R^\times$.

The set of odd primes is denoted by $\mathbb{P}$. We define $\mathbb{Q}/(-1)\mathbb{Q} = \mathbb{Z}/(-1)\mathbb{Z} := \mathbb{R}$. For every prime $p$ and positive integer $k$, we define the ring $\mathbb{Z}/p^k\mathbb{Z} = \{0, \cdots, p^k - 1\}$, where product and addition is defined modulo $p^k$.

Let $p$ be a prime, and $a, b$ be integers. Then, ord$_p(a)$ is the largest integer exponent of $p$ such that $p^{\text{ord}_p(a)} \mid a$. We let ord$_p(0) = \infty$. The $p$-coprime part of $a$ is then cpr$_p(a) = a/p^{\text{ord}_p(a)}$. Note that cpr$_p(a)$ is, by definition, a unit of $\mathbb{Z}/p\mathbb{Z}$. For $\frac{a}{b}$, a rational number, we define ord$_p(\frac{a}{b}) = \text{ord}_p(a) - \text{ord}_p(b)$. The $p$-coprime part of $\frac{a}{b}$ is denoted as cpr$_p(\frac{a}{b})$ and equals $a/p^{\text{ord}_p(a)}$. For a positive integer $q$, one writes $a \equiv b \pmod{q}$, if $q$
divides $a - b$. By $x := a \mod q$, we mean that $x$ is assigned the unique value $b \in \{0, \ldots, q - 1\}$ such that $b \equiv a \mod q$. An integer $t$ is called a quadratic residue modulo $q$ if $\gcd(t, q) = 1$ and $x^2 \equiv t \mod q$ has a solution.

**Definition 1** Let $p$ be an odd prime, and $t$ be a positive integer with $\gcd(t, p) = 1$. Then, the Legendre-symbol of $t$ with respect to $p$ is defined as follows.

$$\left( \frac{t}{p} \right) = t^{(p-1)/2} \mod p = \begin{cases} 1 & \text{if } t \text{ is a quadratic residue modulo } p \\ -1 & \text{otherwise.} \end{cases}$$

For the prime 2, there is an extension of Legendre symbol called the Kronecker symbol. It is defined for odd integers $t$ and $\left( \frac{t}{2} \right)$ equals 1 if $t \equiv \pm 1 \mod 8$, and $-1$ if $t \equiv \pm 3 \mod 8$.

The $p$-sign of $t$, denoted $\operatorname{sgn}_p(t)$, is defined as $\left( \frac{\operatorname{cpr}_2(t)}{p} \right)$ for odd primes $p$ and $\operatorname{cpr}_2(t) \mod 8$ otherwise. We also define $\operatorname{sgn}_p(0) = 0$, for all primes $p$. Thus,

$$\operatorname{sgn}_p(0) = 0 \quad \operatorname{sgn}_p(t > 0) \in \begin{cases} \{+1, -1\} & \text{if } p \text{ is odd} \\ \{1, 3, 5, 7\} & \text{otherwise} \end{cases}$$

The following lemma is well known.

**Lemma 1** Let $p$ be an odd prime. Then, there are $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues modulo $p$. Also, every quadratic residue in $\mathbb{Z}/p\mathbb{Z}$ can be written as a sum of two quadratic non-residues and every quadratic non-residue can be written as a sum of two quadratic residues.

An integer $t$ is a square modulo $q$ if there exists an integer $x$ such that $x^2 \equiv t \pmod q$. The integer $x$ is called the square root of $t$ modulo $q$. If no such $x$ exists, then $t$ is a non-square modulo $q$.

The following lemma is folklore and gives the necessary and sufficient conditions for an integer $t$ to be a square modulo $p^k$. For completeness, a proof is provided in Appendix [3]

**Lemma 2** Let $p$ be a prime, $k$ be a positive integer and $t \in \mathbb{Z}/p^k\mathbb{Z}$ be a non-zero integer. Then, $t$ is a square modulo $p^k$ if and only if $\text{ord}_p(t)$ is even and $\operatorname{sgn}_p(t) = 1$.

**Definition 2** Let $p$ be an odd prime. Then, $\sigma_p$ is the smallest quadratic non-residue modulo $p$.

Assuming GRH, $\sigma_p$ is a number less than $3(\ln p)^2/2$ [Ank52, Wed01] and hence can be found deterministically in $O(\log^3 p)$ ring operations over $\mathbb{Z}/p\mathbb{Z}$.

**Definition 3** Let $p$ be a prime and $\frac{a}{b}$ be a rational number. Then, $\frac{a}{b}$ can be uniquely written as $\frac{a}{b} = p^\alpha \frac{a'}{b'}$, where $a, b$ are units of $\mathbb{Z}/p\mathbb{Z}$. We say that $\frac{a}{b}$ is a $p$-antisquare if $\alpha$ is odd and $\operatorname{sgn}_p(a) \neq \operatorname{sgn}_p(b)$.

For convenience we define integers $k_p$, and a completion of an integer $q$ (denoted $\bar{q}$), as follows.

$$k_p = \begin{cases} 3 & \text{if } p = 2, \text{ and} \\ 1 & p \text{ odd prime.} \end{cases} \quad \bar{q} = q \prod_{p|2q} p^{k_p} \quad (1)$$

We also introduce the following notations.

$$\operatorname{sgn}^X = \{1, 3, 5, 7\}$$

$$\mathbb{Q}_1 \rightarrow \mathbb{Q}_2 \text{ denotes } \mathbb{Q}_2 \equiv U \mathbb{Q}_1 U \mod p^k$$

$$T^+ = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad T^- = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad T \in \{T^-, T^+\}$$
**Definition 4** Let $p^k$ be a prime power. A vector $v \in (\mathbb{Z}/p^k\mathbb{Z})^n$ is called primitive if there exists a component $v_i$, $i \in [n]$, of $v$ such that $\gcd(v_i, p) = 1$. Otherwise, the vector $v$ is non-primitive.

Our definition of primitiveness of a vector is different but equivalent to the usual one in the literature. A vector $v \in (\mathbb{Z}/q\mathbb{Z})^n$ is called primitive over $\mathbb{Z}/q\mathbb{Z}$ for a composite integer $q$ if it is primitive modulo $p^\text{ord}_p(q)$ for all primes that divide $q$.

**Lemma 3** Let $p$ be a prime, $k$ be a positive integer and $x \in (\mathbb{Z}/p^k\mathbb{Z})^n$ be a primitive vector. Then, an $A$ can be found in $O(n^2)$ ring operation such that $[x, A] \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z})$.

**Proof:** The column vector $x = (x_1, \ldots, x_n)$ is primitive, hence there exists a $x_i$, $i \in [n]$ such that $x_i$ is invertible over $\mathbb{Z}/p^k\mathbb{Z}$. It is easier to write the matrix $U$, which equals $[x, A]$ where the row $i$ and 1 or $[x, A]$ are swapped.

$$U = \begin{pmatrix} x_i & 0 \\ x^{-1} \pmod{p} & I_{n-2} \end{pmatrix} \quad x_i = (x_1, \ldots, x_i-1, x_{i+1}, \ldots, x_n)$$

The matrix $U$ has determinant 1 modulo $p^k$ and hence is invertible over $\mathbb{Z}/p^k\mathbb{Z}$. The lemma now follows from the fact that the swapped matrix is invertible if and only if the original matrix is invertible. 

**Quadratic Form.** An $n$-ary quadratic form over a ring $R$ is a symmetric matrix $Q \in R^{n \times n}$, interpreted as the following polynomial in $n$ formal variables $x_1, \ldots, x_n$ of uniform degree 2.

$$\sum_{1 \leq i, j \leq n} Q_{ij}x_ix_j = Q_{11}x_1^2 + Q_{12}x_1x_2 + \cdots = x^TQx$$

The quadratic form is called integral if it is defined over the ring $\mathbb{Z}$. It is called positive definite if for all non-zero column vectors $x$, $x^TQx > 0$. This work deals with integral quadratic forms, henceforth called simply quadratic forms. The determinant of the quadratic form is defined as $\text{det}(Q)$. A quadratic form is called diagonal if $Q$ is a diagonal matrix.

Given a set of formal variables $x = (x_1, \ldots, x_n)^T$ one can make a linear change of variables to $y = (y_1, \ldots, y_n)^T$ using a matrix $U \in R^{n \times n}$ by setting $y = Ux$. If additionally, $U$ is invertible over $R$ i.e., $U \in \text{GL}_n(R)$, then this change of variables is reversible over the ring. We now define the equivalence of quadratic forms over the ring $R$ (compare with Lattice Isomorphism).

**Definition 5** Let $Q_1^n, Q_2^n$ be quadratic forms over a ring $R$. They are called $R$-equivalent if there exists a $U \in \text{GL}_n(R)$ such that $Q_2 = U^TQ_1U$.

If $R = \mathbb{Z}/q\mathbb{Z}$, for some positive integer $q$, then two integral quadratic forms $Q_1^n$ and $Q_2^n$ will be called $q$-equivalent (denoted, $Q_1 \sim_q Q_2$) if there exists a matrix $U \in \text{GL}_n(\mathbb{Z}/q\mathbb{Z})$ such that $Q_2 = U^TQ_1U \pmod{q}$. For a prime $p$, they are $p^n$-equivalent (denoted, $Q_1 \sim_p^q Q_2$) if they are $p^k$-equivalent for every positive integer $k$. Additionally, $(−1)^r$-equivalence as well as $(−1)$-equivalence mean equivalence over the reals $\mathbb{R}$.

Let $Q^n$ be an $n$-ary integral quadratic form, and $q, t$ be positive integers. If the equation $x^TQx \equiv t \pmod{q}$ has a solution then we say that $t$ has a $q$-representation in $Q$ (or $t$ has a representation in $Q$ over $\mathbb{Z}/q\mathbb{Z}$). Solutions $x \in (\mathbb{Z}/q\mathbb{Z})^n$ to the equation are called $q$-representations of $t$ in $Q$. We classify the representations into two categories: primitive and non-primitive (see Definition 4).

For the following result, see Theorem 2, [Jon55].

**Theorem 4** An integral quadratic form $Q^n$ is equivalent to a quadratic form $q_1 \oplus \cdots \oplus q_a \oplus q_{a+1} \oplus \cdots \oplus q_n$ over the field of rationals $\mathbb{Q}$, where $a \in [n]$, $q_1, \ldots, q_a$ are positive rational numbers and $q_{a+1}, \ldots, q_n$ are negative rational numbers.
Theorem 5
For completeness, we provide a proof in Appendix A.

Definition 6
A matrix over integers is in a block diagonal form if it is a direct sum of type I and type II forms; where type I form is an integer while type II is a matrix of the form \( \begin{pmatrix} 2^k a & b^2 \\ 2^k b & 2^{k+1} c \end{pmatrix} \) with \( b \) odd.

Theorem 5
Let \( Q^n \) be an integral quadratic form, \( p \) be a prime, and \( k \) be a positive integer. Then, there is an algorithm that performs \( O(n^{1+\omega} \log k) \) ring operations and produces a matrix \( U \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \) such that \( U^T Q U \equiv Q \pmod{p^k} \), is a diagonal matrix for odd primes \( p \) and a block diagonal matrix (in the sense of Definition 6) for \( p = 2 \).
Definition 7 Let $D^n = \oplus D^m$ be a block diagonal quadratic form (Definition 2). A local transformation is a matrix $U \in GL_n(\mathbb{Z}/p^k\mathbb{Z})$ which applies a sub-transformation $V \in GL_n(\mathbb{Z}/p^k\mathbb{Z})$ on a contiguous sequence of blocks $B^a = D_j \oplus D_{j+1} \oplus \cdots$ turning it into $V^T B V \mod p^k$, leaving rest of the blocks in $D$ unchanged. A local transformation transforms a block diagonal form to a $p^k$-equivalent quadratic form. Given a matrix $V \in GL_n(\mathbb{Z}/p^k\mathbb{Z})$ and a contiguous sequence $B^a = D_j \oplus D_{j+1} \oplus \cdots$ of blocks, to apply $V$ on; the local transformation $U \in GL_n(\mathbb{Z}/p^k\mathbb{Z})$ is given by $U = \Gamma^{a_1+\cdots+a_{j-1}} \oplus V \oplus \Gamma^{a_j+\cdots+a_n}$.

2.1 Canonical Blocks

In this section, we describe the canonical form for a single Type I and a single Type II block. For convenience, we introduce the following Type II matrices.

Definition 8 Let $B = 2^\ell \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ be a Type II block with $b \, \text{odd}$. Then, $\text{ord}_2(B) = \ell$.

Next, we define something called the $p^k$-symbol of an integer.

Definition 9 The $p^k$-symbol of an integer $t$ is $\text{SYM}_{p^k}(t) = (\text{ord}_p(t \mod p^k), \text{sgn}_p(t \mod p^k))$.

The next lemma shows the importance of the $p^k$-symbol.

Lemma 6 For integers $a,b$ and prime $p$: $b \equiv a \mod p^k$ iff $\text{SYM}_{p^k}(a) = \text{SYM}_{p^k}(b)$.

Proof: The lemma is true if $\text{ord}_p(a) = \text{ord}_p(b)$ is at least $k$. Hence, we assume that $\text{ord}_p(a), \text{ord}_p(b) < k$.

We first show that $b \equiv a \mod p^k$ implies $\text{SYM}_{p^k}(a) = \text{SYM}_{p^k}(b)$. If $b \equiv a \mod p^k$ then there exists a $u \in (\mathbb{Z}/p^k\mathbb{Z})^\times$ such that $b = u^2a \mod p^k$. But, multiplying by a square of a unit does not change the sign i.e., $\text{sgn}_p(a \mod p^k) = \text{sgn}_p(u^2 a \mod p^k) = \text{sgn}_p(b \mod p^k)$. Also, $\text{ord}_p(u) = 0$ implies that $\text{ord}_p(a) = \text{ord}_p(b)$. This shows that $\text{SYM}_{p^k}(a) = \text{SYM}_{p^k}(b)$.

We now show the converse. Suppose $a$ and $b$ be such that $\text{SYM}_{p^k}(a) = \text{SYM}_{p^k}(b)$. Let $\text{ord}_p(a) = \text{ord}_p(b) = \alpha$. By definition of $p^k$-symbol, $\text{sgn}_p(a \mod p^k) = \text{sgn}_p(b \mod p^k)$. But then,

$$\text{sgn}_p(\text{cpr}_p(a) \mod p^{k-\alpha}) = \text{sgn}_p(\text{cpr}_p(b) \mod p^{k-\alpha})$$

$$\iff \text{sgn}_p(\text{cpr}_p(a) \text{cpr}_p(b)^{-1} \mod p^{k-\alpha}) = 1$$

By Lemma 2, $\text{cpr}_p(a) \text{cpr}_p(b)^{-1} \mod p^{k-\alpha}$ is a quadratic residue modulo $p^{k-\alpha}$. But then, there exists a unit $u$ such that

$$u^2 \equiv \text{cpr}_p(a) \text{cpr}_p(b)^{-1} \mod p^{k-\alpha}.$$ Multiplying this equation by $\text{cpr}_p(b)^p$ yields $u^2 b \equiv a \mod p^k$ or $b \equiv a \mod p^k$.

Let $p$ be a prime and $B$ be a single block, according to the Definition 6. If $B$ is of Type I then $B$ is an integer and the canonical function $\text{can}_p(B)$ is defined as follows.

$$\text{can}_p(B) = \begin{cases} 
 p^{\text{ord}_p(B)} & \text{if } p \text{ odd, } \frac{\text{cpr}_p(B)}{p} = 1 \\
 p^{\text{ord}_p(B)} \sigma_p & \text{if } p \text{ odd, } \frac{\text{cpr}_p(B)}{p} = -1 \\
 2^{\text{ord}_2(B)(\text{cpr}_2(B) \mod 8)} & \text{if } p = 2
\end{cases}$$

The uniqueness of the canonical form follows from Lemma 6. Otherwise, $B$ is a Type II block and $p = 2$. Let $B = 2^\ell \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$, $b \text{ odd}$. The square of an odd integer is always equal to 1 modulo 8. But then, the quantity $4ac - b^2 \mod 8 \in \{3, 7\}$. The 2-canonical form for a Type II block $B$ is defined as follows.

$$\text{can}_2(2^\ell \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}) = \begin{cases} 
 2^\ell & 4ac - b^2 \equiv 3 \mod 8 \\
 2^\ell & 4ac - b^2 \equiv 7 \mod 8
\end{cases}$$

The uniqueness follows from Lemma 6, [Jon42].
2.2 Primitive Representations Modulo $p^k$

The following theorem gives an algorithmic handle on the question of deciding if an integer $t$ has a primitive $p^r$-representation in $\mathbb{Q}^n$. The theorem is implicit in Siegel [Sie33].

**Theorem 7** Let $\mathbb{Q}^n$ be an integral quadratic form, $t$ be an integer, $p$ be a prime and $k = \max\{\text{ord}_p(q), \text{ord}_p(t)\} + k_p$. Then, if $t$ has a primitive $p^k$-representation in $\mathbb{Q}$ then $t$ has a primitive $p^r$-representation in $\bar{\mathbb{Q}}$ for all $\bar{\mathbb{Q}} \sim \mathbb{Q}$.

**Proof:** We do the proof in two steps: (i) if $t$ has a primitive $p^k$-representation in $\mathbb{Q}$ then $t$ has a primitive $p^r$-representation in $\mathbb{Q}$, and (ii) if $t$ has a primitive $p^s$-representation in $\mathbb{Q}$ then $t$ has a primitive $p^r$-representation in $\bar{\mathbb{Q}}$ for all $\bar{\mathbb{Q}}$ such that $\bar{\mathbb{Q}} \sim \mathbb{Q}$.

The proof of (i) follows. By assumption, there exists a primitive $x \in (\mathbb{Z}/p^k\mathbb{Z})^n$ such that $x^TQx \equiv t \pmod{p^k}$. Let $a = x^TQx$ be an integer, then by definition of symbols $a$ and $t$ have the same $p$-symbol. This implies that for all $i \geq k$ there exists a unit $u_i \in \mathbb{Z}/p^i\mathbb{Z}$ such that $u_i^2a \equiv t \pmod{p^i}$. It follows that $u_i x$ is a primitive representation of $t$ in $\mathbb{Z}/p^i\mathbb{Z}$. But, if $x$ is a primitive representation of $t$ by $\mathbb{Q}$ over $\mathbb{Z}/p^r\mathbb{Z}$ then $x$ is also a primitive representation of $t$ by $\mathbb{Q}$ over $\mathbb{Z}/p^r\mathbb{Z}$, for all positive integers $j \leq i$. This completes the proof of (i).

The proof of (ii) follows. Let $K$ be an arbitrary positive integer and $x \in (\mathbb{Z}/p^K\mathbb{Z})^n$ be a primitive vector such that $x^TQx \equiv t \pmod{p^K}$. As $\bar{\mathbb{Q}} \sim \mathbb{Q}$, there exists $U \in GL_n(\mathbb{Z}/p^K\mathbb{Z})$ such that $Q \equiv U \bar{Q}U \pmod{p^K}$. Thus, $(Ux)^T\bar{Q}(Ux) \equiv t \pmod{p^K}$ and $Ux$ is a $p^K$-representation of $t$ in $\bar{\mathbb{Q}}$. If $x$ is primitive then so is $Ux$. As $K$ is arbitrary, the proof of (ii) and hence the theorem is complete.

Next, we give several results from [DH14]. This paper deals with the following problem. Given a quadratic form $Q$ in $n$-variables, a prime $p$, and integers $k, t$ find a solution of $x^TQx \equiv t \pmod{p^k}$, if it exists. Note that it is easy (i.e., polynomial time tester exists) to test if $t$ has a $p^k$-representation in $\mathbb{Q}$.

**Theorem 8** Let $\mathbb{Q}^n$ be an integral quadratic form, $p$ be a prime, $k$ be a positive integer, $t$ be an element of $\mathbb{Z}/p^k\mathbb{Z}$. Then, there is a polynomial time Las Vegas algorithm that performs $O(n^{1+\omega}\log k + nk^3 + n\log p)$ ring operations over $\mathbb{Z}/p^k\mathbb{Z}$ and fails with constant probability (say, at most $\frac{1}{5}$). Otherwise, the algorithm outputs a primitive $p^k$-representation of $t$ by $\mathbb{Q}$, if such a representation exists. The time complexity can be improved for the following special cases.

| Type | $p$ | Complexity |
|------|-----|-------------|
| $I$, $p$ odd | $O(\log k + \log p)$ |
| $I$, $p = 2$ | $O(k)$ |
| $II$ | $O(k \log k)$ |

Next, we give necessary and sufficient conditions for a Type II block to represent an integer $t$. A proof of this result can also be found in [DH14].

**Lemma 9** Let $Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$, $b$ odd be a type II block, and $t, k$ be positive integers. Then, $Q$ represents $t$ primitively over $\mathbb{Z}/2^k\mathbb{Z}$ if $\text{ord}_2(t) = 1$.

3 Symbol of a Quadratic Form

There are several equivalent ways of giving a description of the $p^r$-equivalence [CS99, Kit99, O'M73, Cas78]. In this work, we go with a modified version of the Conway-Sloane description, called the $p$-symbol of a quadratic form. Our modification gets rid of the need to use the $p$-adic numbers. Note that $p$-adic numbers are a staple in this area and we are not aware of any work which does not use them [Kit99, O'M73, Sie33].

By definition, two quadratic forms are $p^r$-equivalent if they are $p^k$-equivalent for all positive integers $k$. In an algorithmic sense, this is problematic because there are infinitely many possibilities for $k$. Recall the
definition of $k_p$. It equals 1 if $p$ is an odd prime and 3, otherwise. The following theorem shows that it is enough to test equivalence for just one value of $k$.

**Theorem 10** Let $\mathbb{Q}^n$ be an integral quadratic form, $p$ be a prime and $k = \text{ord}_p(\det(\mathbb{Q}^n)) + k_p$. If $D^n$ is a block diagonal form which is equivalent to $\mathbb{Q}$ over $\mathbb{Z}/p^k\mathbb{Z}$, then $D \sim \mathbb{Q}$.

**Proof:** Let $D^n$ be the block diagonal form equivalent to $\mathbb{Q}$ over $\mathbb{Z}/p^k\mathbb{Z}$; $k = \text{ord}_p(\det(\mathbb{Q}^n)) + k_p$. Then, we show that $D$ is $p^\ell$-equivalent to $\mathbb{Q}$ for all $\ell > k$.

Let $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ be such that $D \equiv U\mathbb{Q}U^{-1} \mod p^k$. Then, every entry of $U\mathbb{Q}U^{-1}$ must be divisible by $p^k$. Let $p^k\mathbb{Q} \equiv U\mathbb{Q}U^{-1} \mod p^\ell$. Consider the quadratic form $D + p^k\mathbb{Q}$. As $k = \text{ord}_p(\det(\mathbb{Q}^n)) + k_p = \text{ord}_p(\det(D)) + k_p$, it follows that all off-diagonal entries have higher $p$-order than the diagonal entries. It is hence possible to diagonalize $D + p^k\mathbb{Q}$ over $\mathbb{Z}/p^\ell\mathbb{Z}$ to a quadratic form $D + p^k\mathbb{Q}_1$, where $D$ is also a block diagonal form with matching Type i.e., if the first block of $D$ is Type II then so is the first block of $\mathbb{Q}$ (see proof of Theorem 5).

Let $B, \tilde{B}$ be single Type I or Type II blocks. If $k = \text{ord}_p(d) + k_p$, then $B + p^k\tilde{B} \sim B$ (see Lemma 6 for Type I blocks and Lemma 6, Jon42 for Type II blocks). Thus, we conclude that $D + p^k\tilde{B} \sim \mathbb{Q}$.

The rest of this section follows from Conway-Sloane [CS99] and Theorem 10.

Let $\mathbb{Q}^n$ be an integral quadratic form. The $p$-symbol of $\mathbb{Q}$ is defined as follows.

(-1)-symbol. For $p = -1$ the (-1)-symbol is the same as the (-1)-signature of $\mathbb{Q}$.

### 3.1 $p$-symbol, $p$ odd prime

Let $k = \text{ord}_p(\det(\mathbb{Q}^n)) + 1$ and $D$ be the diagonal quadratic form which is $p^k$-equivalent to $\mathbb{Q}$ (see Theorem 5). Then, $D$ can be written as follows.

$$D = D_0^{n_0} + pD_1^{n_1} + \cdots + p^iD_i^{n_i} + \cdots \quad i \leq \text{ord}_p(\det(\mathbb{Q}^n)), \quad (3)$$

where $D_0, \cdots, D_{k-1}$ are diagonal quadratic forms, $\sum_i n_i = n$ and $p$ does not divide $\det(D_0) \cdots \det(D_{k-1})$.

Let $\llbracket p \rrbracket(\mathbb{Q})$ is the set of $p$-orders $i$ with non-zero $n_i$. Then, the $p$-symbol of $\mathbb{Q}$ is defined as the set of scales $i$ occurring in Equation (3) with non-zero $n_i$, dimensions $n_i = \text{dim}(D_i)$ and signs $\epsilon_i = \left(\frac{\det(D_i)}{p}\right)$.

$$\text{sym}_p(\mathbb{Q}) = \left\{ (p, i, \left(\frac{\det(D_i)}{p}\right), n_i) \mid i \in \llbracket p \rrbracket(\mathbb{Q}) \right\} \quad (4)$$

The following fundamental result follows from Theorem 9, page 379 [CS99] and Theorem 10.

**Theorem 11** For $p \in \{-1\} \cup \mathbb{P}$, two quadratic forms are $p^\ast$-equivalent iff they have the same $p$-symbol.

### 3.2 2-symbol

Let $k = \text{ord}_2(\det(\mathbb{Q}^n)) + 3$ and $D$ be the block diagonal form which is $2^k$-equivalent to $\mathbb{Q}$ (Theorem 5). Then, $D$ can be written as follows.

$$D = D_0^{n_0} + 2D_1^{n_1} + \cdots + 2^iD_i^{n_i} + \cdots \quad i \leq \text{ord}_2(\det(\mathbb{Q}^n)), \quad (5)$$
where \( \det(D_0), \cdots, \det(D_i), \cdots \) are odd, \( \sum_i n_i = n \) and each \( D_i \) is in block diagonal form according to Definition 6. The 2-symbol of \( 2^i D_i \) are the following quantities.

\[
\begin{pmatrix}
  i & \text{scale of } D_i \\
n_i = \dim(D_i) & \text{dimension of } D_i \\
\epsilon_i = \frac{\det(D_i)}{2} & \text{sign of } D_i \\
\text{type}_i = \text{I or II} & \text{type of } D_i \\
\odt_i \in \{0, \cdots, 7\} & \text{oddity of } D_i
\end{pmatrix}
\]

(6)

Let the set of scales \( i \), with non-zero \( n_i \), be denoted \( \mathbb{I}_2(\mathbb{Q}) \). Then, the 2-symbol of \( \mathbb{Q} \) is written as follows.

\[
\text{sym}_2(\mathbb{Q}) = \{(2, i, \epsilon_i, n_i, \text{type}_i, \odt_i) \mid i \in \mathbb{I}_2(\mathbb{Q})\}
\]

(7)

In contrast to the \( p \in \{-1\} \cup \mathbb{P} \) case, two \( 2^* \)-equivalent quadratic forms may produce two different 2-symbols. These symbols are then said to be \( 2^* \)-equivalent. There is a transformation which maps all equivalent 2-symbols to a unique description (see Conway-Sloane [CS99], page 381). We repeat this transformation for the sake of completeness.

**Compartments and Trains.** Let \( T_2 \) be a 2-symbol. Let us define an interval as a consecutive sequence of forms \( 2^i D_i \), even including those with dimension 0. The form with dimension 0 is treated as a form with Type II and Legendre symbol +1. A compartment is then a maximal interval in which all forms are of Type I. A train is a maximal interval with the property that for each pair of adjacent forms, at least one is of Type I. There are two ways in which the symbol maybe altered without changing the equivalence class.

(i) **Oddity fusion.** The oddities inside a compartment can be changed in such a way that the total sum over any compartment remains the same i.e., the sum of oddities in a compartment is an invariant. For example, \( 3 \oplus 5 \) and \( 1 \oplus 7 \) are \( 2^* \)-equivalent.

(ii) **Sign walking.** A 2-symbol remains in the same equivalence class if the signs of any two terms in the same train are simultaneously changed, provided certain oddities are changed by 4. Let us suppose that we want to flip the signs of terms at 2-scale \( i \) and 2-scale \( j \), \( i < j \). We imagine walking the train from \( i \) to \( j \), taking steps between adjacent forms of scales \( r \) and \( r + 1 \). Because we are in the same train during the entire walk, at least one of \( D_r \) and \( D_{r+1} \) is of Type I. The rule is that the total oddity of the compartment must be changed by 4 modulo 8, each time in the walk when either \( D_r \) or \( D_{r+1} \) is in that compartment.

An example from Conway-Sloane is as follows. Here, instead of considering the 2-symbol as a tuple \( (2, 2^i, \epsilon_i, n_i, \odt_i) \), it is easier to consider it as a list with the corresponding term \( (2^i)^{\epsilon_i n_i} \). Let us suppose that we have the following symbol.

\[
1^0 + 2^2 [2^{-2} 4^{-3}] 3 8^{+0} [16^{+1}] 3 2_0^{+2}
\]

The compartments have been denoted by square brackets [ ] and the symbol at scale 3 has dimension 0. Suppose we want to flip the signs at scale 1 and 4. We have to take the steps 1 \( \rightarrow 2 \rightarrow 3 \rightarrow 4 \). The steps 1 \( \rightarrow 2 \) and 2 \( \rightarrow 3 \) uses the first compartment while the step 3 \( \rightarrow 4 \) uses the second. The oddity of the first compartment remains unchanged (used twice) and the oddity of the second changes by 4 modulo 8. The final equivalent form is as follows.

\[
1^0 + 2^2 [2^{-2} 4^{-3}] 3 8^{+0} [16^{-1}] 3 2_0^{+2}
\]

Using sign walking, one can show that \( 1 \oplus 2^2 \) and \( 5 \oplus 2^2 \cdot 5 \) are \( 2^* \)-equivalent.
2-canonical symbol. Using these rules, a 2-canonical symbol can be computed. This is done as follows. Compute the 2-symbol and use oddity fusion and sign walking to make sure that there is at most one minus sign per train and this is on the earliest nonzero dimensional form in the train. Using this convention and only mentioning the total oddities of the compartments, the resulting description is unique and can be taken as a canonical symbol for the form (see page 382, [CS99]). Thus, the 2-canonical symbol for $1_6^{-2}[2^3 4^3]7^{16}1; 32_4^{-2}$ is $1^{-2}[2^3 4^3]7^{16}; 32^2$.

4 Canonicalization: $p$ odd prime

In this section, we describe the function $\text{can}_p$, which maps an integral quadratic form $Q$ to its unique canonical form $\text{can}_p(Q)$. Then, we prove the following theorem.

**Theorem 12** Let $Q^n$ be an integral quadratic form, $p$ be an odd prime and $k > \text{ord}_p(\det(Q))$. Then, there is an algorithm (Las Vegas with constant probability of success) that given $(Q^n, p, k)$ performs $O(n^{1+\omega}\log k + n\log p + \log^3 p)$ ring operations over $\mathbb{Z}/p^k\mathbb{Z}$ and outputs $U \in \text{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that $U^tQU \equiv \text{can}_p(Q) \mod p^k$.

The $O(\log^3 p)$ ring operations are needed to compute $\sigma_p$, assuming GRH. Note that one can fix any quadratic non-residue and define the canonical form with that instead of $\sigma_p$. This will obviate the need to use GRH and the number of ring operations will be $O(n^{1+\omega}\log k + n\log p)$.

The canonical form for an odd prime $p$ is defined as follows.

**Definition 10** A quadratic form is $p$-canonical for an odd prime $p$, if it is of the form $\oplus_i p^i D_i^n$, where $D_i^n$ is a diagonal quadratic form equal to $1^n_i$ or $1^{n-1}_i \oplus \sigma_p$, and the $p$-scales $i$ of the diagonal entries of the quadratic form are non-decreasing.

The uniqueness of the $p$-canonical form follows directly from the definition of $p$-symbol and Theorem [11]. We now describe the canonicalization algorithm. Let $Q^n$ be an integral quadratic form, $p$ be a prime and $k$ be a positive integer. For $n = 1$, the canonicalization algorithm follows from Theorem [8].

Suppose $n = 2$ and the input $Q^2$ is of the form $\tau_1 \oplus \tau_2$, where $\tau_1, \tau_2$ are units of $\mathbb{Z}/p\mathbb{Z}$. The $p$-canonical form of $Q$ is $1 \oplus \{1, \sigma_p\}$. The following lemma shows how to canonicalize in this case.

**Lemma 13** Let $\tau_1, \tau_2 \in (\mathbb{Z}/p\mathbb{Z})^\times$, $p$ odd prime and $k$ be a positive integer. Then, there is a $U \in \text{GL}_2(\mathbb{Z}/p^k\mathbb{Z})$ which transforms $\tau_1 \oplus \tau_2$ to $1 \oplus \{1, \sigma_p\}$ modulo $p^k$. The transformation $U$ can be found by a Las Vegas algorithm which performs $O(\log k + \log p)$ ring operations over $\mathbb{Z}/p^k\mathbb{Z}$ and fails with constant probability.

**Proof:** Consider the situation when $\tau_1$ is a quadratic residue modulo $p$. Then, we use Theorem [8] to find a primitive $x$ such that $x^2 \tau_1 \equiv 1 \mod p^k$. The number $\tau_2$ is either a non-residue or a residue modulo $p$. In either case, we find a $y$ using Theorem [8] again such that $y^2 \tau_2 \equiv 1 \mod \{1, \sigma_p\}$.

If $\tau_2$ is a quadratic residue then we make the following transformation and reduce to the previous case.

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\tau_1 & 0 \\
0 & \tau_2
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\equiv
\begin{pmatrix}
\tau_2 & 0 \\
0 & \tau_1
\end{pmatrix}
\mod p^k
$$

Otherwise, both $\tau_1$ and $\tau_2$ are quadratic non-residues. From Lemma [11] 1 can be written as a sum of two non-residues. Let $(\tau_1^-, \tau_2^-)$ be one such pair. Then, we can write 1 as $(\tau_1^-) + (p^k + 1 - \tau_1^-)$ over $\mathbb{Z}/p^k\mathbb{Z}$, where both $\tau_1^-$ and $p^k + 1 - \tau_1^-$ are quadratic non-residues as $\left(\frac{p^k + 1 - \tau_1^-}{p}\right) = \left(\frac{1 - \tau_1^-}{p}\right) = \left(\frac{\tau_2^-}{p}\right)$. Thus, 1 has a primitive $p^k$-representation in $\tau_1 \oplus \tau_2$ over $\mathbb{Z}/p^k\mathbb{Z}$.
We now use \( \tau_1 \) to represent \( \tau_1^- \) primitively and \( \tau_2 \) to represent \( p^k + 1 - \tau_1 \) primitively over \( Z/p^kZ \) (use Theorem 8). Let \((x, y) \in (Z/p^kZ)^2\) be the primitive representation. Then, we extend it to a matrix 
\[
\begin{pmatrix}
x & a \\
y & b
\end{pmatrix} \in \text{GL}_2(Z/p^kZ),
\]
using Lemma 8. Applying this transformation on \( \tau_1 \oplus \tau_2 \) yields the following matrix.
\[
\begin{pmatrix}
1 & a\tau_1 x + b\tau_2 y \\
a\tau_1 x + b\tau_2 y & a^2\tau_1 + b^2\tau_2
\end{pmatrix} \mod p^k
\]
This matrix can be diagonalized using Theorem 5 keeping the 1 unchanged; to a matrix of the following form.
\[
\begin{pmatrix}
1 & 0 \\
0 & a
\end{pmatrix} \mod p^k
\]
But all these transformation are from \( \text{GL}_2(Z/p^kZ) \) and hence do not change the symbol of the matrix \( \tau_1 \oplus \tau_2 \). Thus, \( a \) must be a unit of \( Z/p^kZ \). One can now use Theorem 5 to find a \( z \) such that \( z^2a \mod p^k \in \{1, \sigma_p\} \). The \( U \) in this case, is the product of all transformations in \( \text{GL}_2(Z/p^kZ) \) we have used so far.

We are now ready to prove Theorem 12.

**Proof:** (Theorem 12) The algorithm makes a sequence of transformations, each from \( \text{GL}_n(Z/p^kZ) \).

(i.) Use Theorem 5 to find \( U_0 \in \text{GL}_n(Z/p^kZ) \) such that \( U_0 QU_0 \mod p^k \) is a diagonal matrix. Use transformations \( V_1, \ldots, V_n \in \text{SL}_n(Z/p^kZ) \) to transform the diagonal matrix to the form \( d_1 \oplus \cdots \oplus d_n \), such that \( \text{ord}_{p}(d_1) \leq \cdots \leq \text{ord}_{p}(d_n) \). Note that each \( V_i \) exchanges two diagonal entries. The total number of ring operations for this step is \( O(n^{1+\omega} \log k) \).

(ii.) The matrix is now of the form \( p^{m_1}D_1^{n_1} \oplus \cdots \oplus p^{m_m}D_m^{n_m} \), where \( D_1, \ldots, D_m \) are diagonal matrices with unit determinants. We next use transformations to transform \( D_i \) to \( \text{can}_p(D_i) \). This is done as follows. If \( D_i \) is of dimension 1 then we use Theorem 5 to canonicalize it. Otherwise, \( D_i^{n_i, \sigma_i} \). The matrix is of the form \( (\tau_1, \ldots, \tau_n) \), where \( \tau_1, \ldots, \tau_n \) are units of \( Z/p^kZ \). We apply a transformation on \( \tau_1 \oplus \tau_2 \) to turn it into \( 1 \oplus \tau \) over \( Z/p^kZ \), where \( \tau \in \{1, \sigma_p\} \) (Lemma 13). Continuing in a similar way, we end up with the transformation \( \tau_i \in \text{GL}_n(Z/p^kZ) \), for each \( i \) such that \( \tau_i D_i \tau_i^{-1} \equiv I^{n_i - 1} \oplus \{1, \sigma_p\} \mod p^k \). Let \( U_i \in \text{GL}_n(Z/p^kZ) \) be the corresponding local transformation (see Equation 7). This step takes \( O(n\log k + \log p^k) \) ring operations over \( Z/p^kZ \).

Then, the transformation \( U = U_0V_1 \cdots V_nU_1 \cdots \) is a product of matrices, each from \( \text{GL}_n(Z/p^kZ) \) and turns \( Q \) to its canonical form. The algorithm performs \( O(n^{1+\omega} \log k + \log p^k) \) ring operations over \( Z/p^kZ \). It additionally needs \( O(\log^3 p) \) ring operations to compute \( \sigma_p \), assuming GRH.

### 5 Canonicalization: \( p = 2 \)

The 2-canonical form is non-trivial and requires care. We follow the same procedure (sign walking and oddity fusion) as Conway-Sloane [CS99] given in Section 3.2. Then, we prove the following theorem.

**Theorem 14** Let \( Q^n \) be an integral quadratic form, and \( k \geq \text{ord}_{2}(\det(Q)) + 3 \). Then, there is an algorithm that given \( (Q^n, k) \) performs \( O(n^{1+\omega} \log k + n \log p) \) ring operations over \( Z/2^kZ \) and outputs \( U \in \text{GL}_n(Z/2^kZ) \) such that \( U^T Q U \equiv \text{can}_2(Q) \mod 2^k \).

#### 5.1 Type II Block

Our next step is to give a transformation which maps any Type II matrix to its canonical form. Recall that the canonical form of a Type II matrix \( Q \) is either \( 2^{\text{ord}_{2}(Q)T^-} \) or \( 2^{\text{ord}_{2}(Q)T^+} \).
Lemma 15 Let $Q$ be a Type II matrix of 2-order $0$. Then, there is an algorithm that given $(Q, k \geq 3)$ as input; performs $O(k \log k)$ ring operations and outputs a $U \in \text{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ such that $U'QU \mod 2^k \in \{T^+, T^-\}$.

Proof: By definition, $Q$ is of the form $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$, where $b$ is odd. The integer $b$ is odd and so $b^2 \mod 8 = 1$. Thus, $\det(Q) \mod 4 = 4ac - b^2 \mod 4 = 3$ and $\det(Q) \mod 8 \in \{3, 7\}$. For convenience, suppose that

$$\lambda = \det(Q) \mod 8, \quad \lambda \in \{3, 7\}.$$  

Let $s = k + 1$. We now give a transformation which maps $Q$ to its canonical form.

(i.) From Lemma 9, 2 has a primitive representation in $Q$ over $\mathbb{Z}/2^s\mathbb{Z}$. Use Theorem 8 to find one such primitive representation $(x_1, x_2) \in (\mathbb{Z}/2^s\mathbb{Z})^2$. Without loss of generality assume $x_1$ is odd and define $U \in \text{GL}_2(\mathbb{Z}/2^s\mathbb{Z})$ as follows.

$$U = \begin{pmatrix} x_1 & 0 \\ x_2 & x_1^{-1} \mod 2^s \end{pmatrix}, \quad U'QU \equiv \begin{pmatrix} 2 & b + 2cx_2x_1^{-1} \\ b + 2cx_2x_1^{-1} & 2cx_1^{-2} \end{pmatrix} \mod 2^s$$

(ii.) The matrix $U$ is in $\text{GL}_2(\mathbb{Z}/2^s\mathbb{Z})$. Thus, $\det(U)$ is a unit of $\mathbb{Z}/2^s\mathbb{Z}$ and $\det(U'QU) \mod 8 = \lambda = \det(Q) \mod 8$. Thus, the following equation has a solution.

$$x^2 \det(U'QU) \equiv \lambda \pmod{2^s} \quad (8)$$

(iii.) A primitive solution of Equation 8 can be found using Theorem 8. Let us denote the solution by $x$. Then, $x$ is primitive and the matrix $V$ defined by $1 \oplus x$ is in $\text{GL}_2(\mathbb{Z}/2^s\mathbb{Z})$.

(iv.) Let $S := V'QUV \mod 2^s$. Then, by construction, $S_{11} = 2$ and $\det(S) = x^2 \det(U'QU) \mod 2^s$ which equals $\lambda$ (Equation 8).

(v.) By assumption, $b$ is odd. But then, $S_{12}$ is odd and $(1 - S_{12})/2$ is an element of $\mathbb{Z}/2^s\mathbb{Z}$. It follows that the matrix $W := \begin{pmatrix} 1 & 1-6ix \\ 0 & 1 \end{pmatrix}$ is in $\text{GL}_2(\mathbb{Z}/2^s\mathbb{Z})$. But then, for some integer $y$,

$$W'SW \equiv \begin{pmatrix} 2 & 1 \\ 1 & 2y \end{pmatrix} \mod 2^s \quad (9)$$

(vi.) By construction, $\det(W) \equiv 1 \mod 2^s$ and from item (iv),

$$\det(W'SW) \equiv \det(S) \equiv \lambda \pmod{2^s} \quad (10)$$

(vii.) By Equation 8 and Equation 10, $4y - 1 \equiv \lambda \pmod{2^s}$. Recall $\lambda \in \{3, 7\}$. This implies that $\lambda + 1$ is divisible by 4 and

$$2y \mod 2^s \in \{\{2, 2 + 2^{s-1}\} \mid \lambda = 3 \quad, \quad \{4, 4 + 2^{s-1}\} \mid \lambda = 7\} \quad \Rightarrow \quad 2y \mod 2^k = \begin{cases} 2 & \lambda = 3 \\ 4 & \lambda = 7 \end{cases}$$

(viii.) Thus, the transformation $U'W$ is in $\text{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ and transforms $Q$ to its canonical form over $\mathbb{Z}/2^k\mathbb{Z}$, by construction.

By Lemma 9 and Theorem 8, the transformation can be constructed in $O(k \log k)$ ring operations. □
5.2 Dimension = 3, with one Type II block

Let us suppose that the input matrix $Q^3$ is of the form $\tau \oplus T$, where $\tau$ is a unit of $\mathbb{Z}/2^k\mathbb{Z}$. In this case, we show that the matrix can be transformed into a diagonal matrix $\tau_1 \oplus \tau_2 \oplus \tau_3$ over $\mathbb{Z}/2^k\mathbb{Z}$ such that $\tau_1, \tau_2, \tau_3$ are all units of $\mathbb{Z}/2^k\mathbb{Z}$.

**Lemma 16** Let $k \geq 3$ be an integer and $\tau$ be a unit of $\mathbb{Z}/2^k\mathbb{Z}$. Then, there is an algorithm that performs $O(k \log k)$ ring operations and transforms $\tau \oplus T$ to $\tau_1 \oplus \tau_2 \oplus \tau_3$, where $\tau_1, \tau_2, \tau_3$ are units of $\mathbb{Z}/2^k\mathbb{Z}$.

**Proof:** As usual, we provide a sequence of transformations, each from $\text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$.

(i.) Let $V \in \text{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ be the matrix that transforms $\begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}$ to $T^+$ over $\mathbb{Z}/2^k\mathbb{Z}$. This matrix can be found using Lemma 15. Suppose $U$ is defined as follows.

$$U = \begin{cases} 1 \oplus U^{-1}_1 \mod 2^k & \text{if } T = T^+ \\ I^3 & \text{otherwise} \end{cases}$$

By construction, $U$ transforms $\tau \oplus T$ to the following form.

$$\tau \oplus T \xrightarrow{U,2^k} \begin{pmatrix} \tau & 0 & 0 \\ 0 & x & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \text{where } x \in \{2,8\}$$

(ii.) Consider the matrix $V$ defined as follows.

$$V = \begin{pmatrix} 1 & 1 & 1 \\ r & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Whatever integer value $r$ takes, the matrix $V$ has determinant 1 and hence $V \in \text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$. The matrix $V$ makes the following transformation.

$$\begin{pmatrix} \tau & 0 & 0 \\ 0 & x & 1 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{V,2^k} \begin{pmatrix} r^2x + \tau & rx + \tau + r & \tau + r \\ rx + r + \tau & \tau + 4 + x & \tau + 3 \\ \tau + r & \tau + 3 & \tau + 2 \end{pmatrix}$$

(iii.) The integer $x \in \{2,8\}$ and so $(x + 1)$ is a unit of $\mathbb{Z}/2^k\mathbb{Z}$. If we set $r := \frac{-x}{x+1} \mod 2^k$, then $rx + r + \tau \equiv 0 \mod 2^k$. Thus, the matrix has been transformed into the following form, where $\tau_1 = r^2x + \tau$ and $\tau_2 = \tau + 4 + x$.

$$\begin{pmatrix} \tau_1 & 0 & \tau + r \\ 0 & \tau_2 & \tau + 3 \\ \tau + r & \tau + 3 & \tau + 2 \end{pmatrix}$$

(iv.) The numbers $\tau_1 = r^2x + \tau$ as well as $\tau_2 = \tau + 4 + x$ are odd because $x \in \{2,8\}$ and $\tau$ is odd. Thus, by Theorem 5 we can find a matrix $W \in \text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$ such that $W$ transforms the matrix to the following final form, where $\tau_3$ is also a unit of $\mathbb{Z}/2^k\mathbb{Z}$.

$$\begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{pmatrix}$$

□

Lemma 16 implies that one does not need to have Type I and Type II matrices on the same 2-scale. Thus, for every 2-symbol there is an equivalent 2-symbol in which all 2-scales are either exclusively Type I or Type II.
5.3 Sign Walking

We now make the transformations required to perform sign walking. Note that the sign walking is done on symbols but we make the corresponding transformations between $2^k$-equivalent quadratic forms. Also, sign walking involves a single train and hence, we never walk between two Type II forms.

By definition, $\left( \frac{x}{y} \right) = 1$ iff $x \mod 8 \in \{1, 7\}$. Thus, $\tau + 4$ always has the opposite sign of $\tau$. In the lemma below, notice that in every transformation the negative sign propagates to the front.

**Lemma 17** There exists invertible transformations over $\mathbb{Z}/2^k\mathbb{Z}$, computable in $O(k \log k)$ ring operations, such that

- (i.) $\tau_1 + 4\tau_2 \rightarrow (\tau_1 + 4) \oplus 4(\tau_2 + 4) \quad k \geq 5$
- (ii.) $\tau \oplus 2T^- \rightarrow \tau \oplus 2T^+ \quad k \geq 4$
- (iii.) $T_1 \oplus 2\tau_1^- \rightarrow T_2 \oplus 2\tau_2^- \quad k \geq 4$
- (iv.) $T_1 \oplus T \rightarrow T_2 \oplus T^+ \quad k \geq 3$

where $\tau_1, \tau_2 \in \{1, 3, 5, 7\}$, $\tau_1^- \in \{3, 5\}$, $\tau_2^+ \in \{1, 7\}$ and $T_1, T_2 \in \{T^-, T^+\}$.

**Proof:** We itemize the transformation and show how to find them under the corresponding item.

(i.) By assumption, $\tau_1$ and $\tau_2$ are odd and so $\tau_2$ is invertible over $\mathbb{Z}/2^k\mathbb{Z}$. Let $U$ be defined as follows.

$$U := \begin{pmatrix} 1 & 4 \\ 1 & \frac{\tau_1}{\tau_2} \mod 2^{k-2} \end{pmatrix} \mod 2^k$$  \hspace{1cm} (11)

Then, $\det(U) = -4 - \frac{\tau_1}{\tau_2} \mod 2^{k-2}$; $\det(U)$ is odd and $U \in \text{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$. If $x = \frac{\tau_1}{\tau_2} \mod 2^{k-2}$, then $4(\tau_2 x + \tau_1) \equiv 0 \mod 2^k$. The transformation $U$ has the following effect on our input matrix $\tau_1 + 4\tau_2$.

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & 4\tau_2 \end{pmatrix} \rightarrow_{2^k} \begin{pmatrix} \tau_1 + 4\tau_2 & 0 \\ 0 & 4(\tau_2 x^2 + 4\tau_1) \end{pmatrix}$$

The integers $\tau_1, \tau_2$ are odd by hypothesis and $x$ is odd by construction. But then,

$$\tau_1 + 4\tau_2 \equiv \tau_1 + 4 \mod 8$$
$$\tau_2 x^2 + 4\tau_1 \equiv \tau_2 + 4 \mod 8$$

Thus, we can find $2^k$-primitive transformations that map $\tau_1 + 4\tau_2$ to $\tau_1 + 4$ and $4(\tau_2 x^2 + 4\tau_1)$ to $4(\tau_2 + 4)$ using Lemma 8. These transformations take the matrix to the final form.

(ii.) We first transform our input matrix $\tau \oplus T^-$ by the matrix $V \in \text{GL}_2(\mathbb{Z}/p^k\mathbb{Z})$ defined below.

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad V' (\tau \oplus T^-) V = \begin{pmatrix} \tau + 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

This transformation brings the first entry to the correct sign. We block diagonalizing this matrix using our algorithm in the proof of Theorem 7 and find a matrix $U \in \text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$ such that

$$U' \begin{pmatrix} \tau + 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} U \equiv (\tau + 4) \oplus X \mod 2^k ,$$

where $X$ is a Type II block. The transformations made so far are from $\text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$, and so, $\det(\tau_1 \oplus 2T^-) \not\sim (\tau + 4) \det(X)$. In particular, this implies that ord$_{2}(\det(X)) = 2$ and $\frac{\text{cpr.}(\det(X))}{2} = +$. Because, $X$ is a Type II block, it is $2^k$-equivalent to $2T^+$ and such a transformation can be found using Lemma 8.
(iii.) Given the input matrix $T_1 \oplus 2\tau_1^-$ we apply the transformation $V \in \text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$ given below.

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & b \in \{2, 4\} & 0 \\ 0 & 0 & 2\tau \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 1 & b + 2\tau & 2\tau \\ 0 & 2\tau & 2\tau \end{pmatrix}$$

This transformation maps $b$ to $b + 2\tau$, where $\tau$ is odd. But then,

$$\det(T_1) = 2b - 1 \quad \text{det} \left( \begin{array}{ccc} 2 & 1 & 0 \\ 1 & b + 2\tau & 0 \\ 0 & 0 & x \end{array} \right) = 2 - 4\tau$$

Thus, the sign of the Type II matrix has been switched. Next we use the block diagonalization algorithm from the proof of Theorem 5. If $\mathbf{U}$ is the output then, for some integer $x$,

$$\mathbf{U'} \begin{pmatrix} 2 & 1 & 0 \\ 1 & b + 2\tau & 0 \\ 0 & 2\tau & 2\tau \end{pmatrix} U \equiv \begin{pmatrix} 2 & 1 & 0 \\ 1 & b + 2\tau & 0 \\ 0 & 0 & x \end{pmatrix} \pmod{2^k}$$

As before, the $2^k$-symbol of the determinant of the quadratic form does not change when transformed by $U \in \text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$; implying, $\text{ord}_2(x) = 1 \quad \text{sgn}_2(\text{cpr}_2(x)) = +$. Using Lemma 15 and Theorem 8 the following transformation over $\mathbb{Z}/2^k\mathbb{Z}$ can be found.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & b + 2\tau & 0 \\ 0 & 0 & x \end{pmatrix} \rightarrow T_2 \oplus 2\tau_2^+$$

where $T_2$ has the opposite sign as $T_1$ and $\tau_2^+ \in \{1, 7\}$.

(iv.) Let $b \in \{2, 4\}$, then the input matrix $T_1 \oplus T^-\oplus T^-$ is in the following form.

$$\begin{pmatrix} 2 & 1 \\ 1 & b \end{pmatrix} \oplus \begin{pmatrix} 2 & 1 \\ 1 & b \end{pmatrix}$$

If $b = 4$ then we swap the two matrices. Otherwise, we apply the following transformation.

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad T^- \oplus T^- \rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix} \oplus \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

Diagonalization (see Theorem 5) of this matrix from top down, will yield a quadratic form of the form $T^+ \oplus Y$, where $Y$ is also equivalent to $T^+$. We then make a local transformation to convert $Y$ to $T^+$.

\[\square\]

5.4 Oddity Fusion

The transformations under the oddity fusion step deal with a single compartment. A compartment is a consecutive sequence of Type I forms. Two adjacent quadratic forms in the same compartment differ by at most 1 in terms of their 2-scale. In this case, we want to find the minimum lexicographically possible set of integers, that can be represented.

Lemma 18 Let $\tau, \tau_1, \tau_2, \tau_3 \in \text{SGN}^\times, i, i_1, i_2, i_3 \in \text{positive integers and } T \in \{T^-, T^+\}$ be a Type II matrix. If $2^{i_1} \tau_1 \oplus 2^{i_2} \tau_2 \oplus 2^{i_3} \tau_3 \rightarrow 2^{i_1} \tau \oplus 2^i T$, then $i_1 = i_2 = i_3 = i = 1$. 

15
Proof: Suppose that $i_1 = i_2 = i$ is not true. Then, the 2-symbols (see Section 3.2) of the first and the second quadratic form are not equivalent because one cannot be transformed into the other by a combination of oddity fusion and sign walking steps. □

Lemma 19 Let $\tau_1, \tau_2, \tau_3$ be odd integers and $k$ be a positive integer. Then, there is an algorithm that transforms $D = \tau_1 \oplus \tau_2 \oplus \tau_3$ to one of the forms in Table 1 in $O(k^2 \log k)$ ring operations, where $\epsilon = \left( \frac{\tau_1 \tau_2 \tau_3}{2} \right)$ and $odt := \tau_1 + \tau_2 + \tau_3 \mod 8$.

| $\epsilon$  | odt | Form   | $\epsilon$  | odt | Form   |
|------------|-----|--------|------------|-----|--------|
| + 1        | 1 1 7 | 3 3 3 3  | - 1        | 3 3 7 |
| + 3        | 1 1 1 | 3 3 7  | - 3        | 1 3 7 |
| + 5        | 3 3 7 | - 5    | 1 1 3  |      |
| + 7        | 1 3 3 | - 7    | 1 1 5  |      |

Proof: The forms listed in Table 1 are exhaustive. The transformation from $D$ to one of these forms can be done using Theorem 8 as follows.

(i.) Read the canonical form from Table 1 using the oddity and the value of $\epsilon$ of the quadratic form $D$. Let it be $\tilde{\tau}_1 \oplus \tilde{\tau}_2 \oplus \tilde{\tau}_3$.

(ii.) Use Theorem 8 to represent $\tilde{\tau}_1$ using $D$ over $\mathbb{Z}/2^k\mathbb{Z}$. Let $x \in (\mathbb{Z}/2^k\mathbb{Z})^3$ be the representation and $U \in GL_3(\mathbb{Z}/2^k\mathbb{Z})$ be the corresponding primitive extension as in Lemma 3.

(iii.) The integer $\tilde{\tau}_1$ is odd and hence using Theorem 5 the matrix $U'DU$ can be block diagonalized over $\mathbb{Z}/2^k\mathbb{Z}$ by matrix $V \in GL_3(\mathbb{Z}/2^k\mathbb{Z})$ such that;

$$V'U'DU \equiv \tilde{\tau}_1 \oplus B,$$
where $B \in (\mathbb{Z}/2^k\mathbb{Z})^{2 \times 2}$.

(iv.) The oddity and Legendre symbol for the matrix $B$ can be computed exhaustively using $D$ and $\tilde{\tau}_1$ as follows.

$$odt(B) = odt(D) - \tilde{\tau}_1 \mod 8$$

$$\left( \frac{\det(B)}{2} \right) = \epsilon \left( \frac{\tilde{\tau}_1}{2} \right)$$

(v.) If $odt(B) = 0$ then $B$ might be of Type II. The exhaustive list of such matrices $\tilde{\tau}_1 \oplus B$, where $B$ is Type II is given in Table 2 below.

(vi.) The bad cases are problematic because it is impossible to transform $T^-$ or $T^+$ to a form $\tilde{\tau}_2 \oplus \tilde{\tau}_3$ using transformations from $GL_2(\mathbb{Z}/2^k\mathbb{Z})$. Fortunately, the strategy to represent the smallest possible $\tilde{\tau}_1$ fails i.e., results in one of the bad cases; only when $D \not\sim 1 \oplus 1 \oplus 7$. For all other forms in Table 1 it can be checked that $\tilde{\tau}_2 + \tilde{\tau}_3 \mod 8 \neq 0$.

For the special case of $1 \oplus 1 \oplus 7$, we represent $7$ instead of $1$ under item (ii.). Then, $odt(B) = 1 + 1 = 2$ and $B$ is not of Type II.

(vii.) If $B$ is not of Type II then we transform $B$ to $\tilde{\tau}_2 \oplus \tilde{\tau}_3$ using Theorem 8 and Theorem 5. Note that the transformation exists because the 2-symbol of $B$ matches the 2-symbol of $\tilde{\tau}_2 \oplus \tilde{\tau}_3$. In case of $D = 1 \oplus 1 \oplus 7$ we end up with $7 \oplus 1 \oplus 1$ instead. We then swap $7$ and $1$ using a transformation from $GL_3(\mathbb{Z}/2^k\mathbb{Z})$. □
5.5 Canonicalizing a Single Compartment

By definition, all forms in a compartment are of scaled Type I i.e., the compartment is of the form $2^{i_1} \tau_1 \oplus 2^{i_2} \tau_2 \oplus \cdots$, where $\tau_1, \tau_2, \cdots \in \operatorname{SGN}^x$ and $i_1, i_2, \cdots$ are positive integers.

**Definition 11** Let $D = 2^{i_1} \tau_1 \oplus 2^{i_2} \tau_2 \oplus \cdots \oplus 2^{i_n} \tau_n$ be a single compartment, where $\tau_1, \cdots, \tau_n \in \operatorname{SGN}^x$ and $i_1 \leq \cdots \leq i_n$ are positive integers such that any two consecutive ones differ by at most 1. Then, the canonical form of $D$ is $2^{i_1} \tau_1 \oplus \cdots \oplus 2^{i_n} \tau_n$, where $(\tau_1, \cdots, \tau_n)$ is lexicographically minimum possible option in the $2^*$-equivalence class of $D$.

**Lemma 20** Let $k \geq 3$ be an integer, $\tau \in \operatorname{SGN}^x$ and $D^n = \tau_1 \oplus 2^{i_2} \tau_2 \oplus \cdots \oplus 2^{i_n} \tau_n$ be a diagonal form with $\tau_1, \cdots, \tau_n \in \operatorname{SGN}^x$, and $i_2 \leq \cdots \leq i_n$. Then, $\tau$ is primitively representable in $D$ over $\mathbb{Z}/2^k\mathbb{Z}$, if it is primitively representable in $\tau_1 \oplus \cdots \oplus 2^{i_4} \tau_4$ over $\mathbb{Z}/2^k\mathbb{Z}$.

**Proof:** A primitive representation of $\tau$ exists iff a primitive representation exists modulo 8 (Theorem 7). Let $A = \tau_1 \oplus \cdots \oplus 2^{i_4} \tau_4$. We can make two simplifications to the form $D$: (i) we can only use Type I blocks for which 2-order is $\leq 2$, and (ii) there is no need to have more than 1 element of order 2 as we can add at most 4 modulo 8 to the result using any number of such elements. Item (ii) implies that if $2 \not\in \{i_2, i_3, i_4\}$ then we do not need to use $2^{i_5}, \cdots, 2^{i_n}$ i.e., the lemma is true in this case.

Thus, (i)+(ii) imply that the only possible values for the vector $(i_2, i_3, i_4)$ are $(0, 0, 0), (0, 0, 1), (0, 1, 1),$ and $(1, 1, 1).$ In all these cases, and for all possible values of $\tau_1, \cdots, \tau_4 \in \operatorname{SGN}^x$, we verify by brute-force that 1 can be represented modulo 8. Let $x \in (\mathbb{Z}/2^k\mathbb{Z})^4$ be a primitive $2^k$-representation of 1 and $U \in \operatorname{GL}_4(\mathbb{Z}/2^k\mathbb{Z})$ be an extension of $x$ given by Lemma 3. By construction, $(U^T A U)_{11} \equiv 1 \mod 2^k$. Using Theorem 5 we can find $V \in \operatorname{GL}_4(\mathbb{Z}/2^k\mathbb{Z})$ such that

$$V^T U A U V \equiv 1 \oplus X \mod 2^k,$$ where $X$ is in block diagonal form.

If $X$ does not have a Type II block then we are done. Depending on the value of $(i_2, i_3, i_4)$ we proceed as follows.

(0, 0, 0). If a Type II form appears within $X$ then $X = T \oplus \tau$, where $T$ is a Type II block of 2-order 0. In this case, we can locally get rid of the Type II block using Lemma 16 as follows.

$$1 \oplus X \to 1 \oplus a \oplus b \oplus c \quad (12)$$

(1, 1, 1). If a Type II form appears within $X$ then $X = 2T \oplus 2T$, where $T$ is a Type II block of 2-order 0. We locally get rid of the Type II block using Lemma 16 as in Equation (12).

(0, 1, 1). It is impossible for $X$ to contain a Type II form because then the 2-symbols of $A$ and $1 \oplus X$ will not be equivalent i.e., they cannot be transformed using sign walking and oddity fusion.
We can now define the function can. The matrix $A$ is of the form $\tau_1 \oplus \tau_2 \oplus \tau_3 \oplus 2\tau_4$, in this case. If $\tau_1 + \tau_2 + \tau_3 \mod 8 \not\in \{3, 7\}$ then we apply the following transformation.

$$W = I^{2 \times 2} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$(\tau_1 \oplus \tau_2 \oplus \tau_3 \oplus 2\tau_4) \rightarrow (\tau_3 + 2\tau_4) \oplus (2\tau_4)$$

But then, $\tau_1 + \tau_2 + \tau_3 + 2\tau_4 \equiv \tau_1 + \tau_2 + \tau_3 + 2 \mod 4$. Thus, we may assume that the sum of the first three Type I entries of the input quadratic form $A$ is in the set $\{3, 7\}$ modulo 8 i.e., if $A = \tau_1 \oplus \tau_2 \oplus \tau_3 \oplus 2\tau_4$, then $\tau_1 + \tau_2 + \tau_3 \mod 8 \in \{3, 7\}$. In this case, we exhaustively check that 1 can be represented primitively using only $\tau_1 \oplus \tau_2 \oplus \tau_3$. It also follows that the oddity of the leftover $2 \times 2$ matrix must be 2 or 6. But then, this matrix cannot be Type II.

Lemma 21 Let $D = 2^{i_1}\tau_1 \oplus \cdots \oplus 2^{i_n}\tau_n$ be a single compartment, where $\tau_1, \ldots, \tau_n \in \text{sgn}^\times$ and $i_1 \leq \cdots \leq i_n, k$ are positive integers. Then, there is an algorithm that performs $O(nk^3)$ ring operations and finds $U \in \text{GL}_n(\mathbb{Z}/2^k\mathbb{Z})$ that transforms $D$ into can$_2(D)$ over $\mathbb{Z}/2^k\mathbb{Z}$.

**Proof:** We divide the proof in several cases, depending on the value of the dimension $n$.

$n = 2$. We exhaustively try to primitively represent the smallest integer of the form $2^{i_1}\tau$, where $\tau \in \text{sgn}^\times$ using Theorem 8. Let $x \in (\mathbb{Z}/2^k\mathbb{Z})^2$ be a representation, $U \in \text{GL}_2(\mathbb{Z}/2^k\mathbb{Z})$ be the corresponding extension and $V$ be the transformation given by the block diagonalization Theorem 5. Then

$$V'U'AV \equiv 2^{i_1}\tau \oplus 2^{i_2}\tau \mod 2^k,$$ where $\tau$ is odd.

We now use Theorem 8 to transform $\tau$ locally to something from the set $\text{sgn}^\times$ over $\mathbb{Z}/2^k\mathbb{Z}$. By construction, the resulting matrix is in its 2-canonical form.

$n = 3$. The sequence of transformations is as follows: (i) find the smallest primitively $2^k$-representable integer of the form $2^{i_1}\tilde{\tau}_1$ with $\tilde{\tau}_1 \in \text{sgn}^\times$ by doing an exhaustive search for primitive representation of $\tilde{\tau}_1$ by $\tau_1 \oplus 2^{i_2-i_1}\tau_2 \oplus 2^{i_3-i_1}\tau_3$ over $\mathbb{Z}/8\mathbb{Z}$ (Theorem 7), (ii) Find a primitive representation $x \in (\mathbb{Z}/2^k\mathbb{Z})^3$ using Theorem 8 and extend it to $U \in \text{GL}_3(\mathbb{Z}/2^k\mathbb{Z})$ using Lemma 19 (iii) Block diagonalize $U'DU$ using $V$ given by Theorem 5. Then

$$V'U'DUV \equiv 2^{i_1}\tilde{\tau}_1 \oplus X, \text{ where } X \in (\mathbb{Z}/2^k\mathbb{Z})^{2 \times 2}.$$

The type of $X$ is II only when $i_1 = i_2 = i_3$ (Lemma 18). If this is the case, then we can apply Lemma 19 to canonicalize instead. Otherwise, $X$ is of Type I and we have reduced to the case of $n = 2$.

$n \geq 4$. By Lemma 20 we can represent the smallest possible integer of the form $2^{i_1}\tau$, with $\tau \in \text{sgn}^\times$. This way we reduce to one smaller dimension. Finally, we reduce to the case of dimension 3.

The number of ring operations follows from using Theorem 8 at most $O(n)$ times on diagonal matrices of dimensions at most 4. 

5.6 Canonical Form, any dimension

We can now define the function can$_2(Q^n)$. The uniqueness follows from Conway-Sloane [CS99], see Section 3.2.

**Proof:** (Theorem 14) We perform the following sequence of transformations over $\mathbb{Z}/2^k\mathbb{Z}$.
1. Block diagonalize the quadratic form.
2. For each type II block, apply the transform it to $T^+$ or $T^-$ using Lemma [15].
3. For each 2-scale, apply the transformation in Lemma [16] to transform the matrix to a block diagonal form where all scales have either only type I matrices, or only type II matrices.
4. For each train, do a sign walk to move all minus signs to the front of the train (see Lemma [17]). Also, from Lemma [17], the canonical form for each type II part has at most one $T^-$ i.e., it is either $2^i(T^-, T^+, \cdots, T^+)$ or $2^i(T^+, T^+, \cdots, T^+)$. 
5. Transform each compartment to its corresponding canonical form (Definition [11]) using Lemma [21].

The final transformation is the multiplication of all the local transformations which have been constructed above. The number of local transformations is bounded by $O(n)$. Thus, the algorithm performs at most $O(n^{1+\omega} \log k + nk^3)$ ring operations. □

References

[Ank52] NC Ankeny. The least quadratic non residue. *Annals of mathematics*, pages 65–72, 1952.

[Cas78] John WS Cassels. Rational quadratic forms. *London and New York*, 1978.

[CS99] John Conway and Neil JA Sloane. *Sphere packings, lattices and groups*, volume 290. Springer, 1999.

[DH14] Chandan Dubey and Thomas Holenstein. Sampling a uniform random solution of a quadratic equation modulo $p^k$. *arXiv preprint arXiv:1404.0281*, 2014.

[Gau86] Carl Friedrich Gauß. Disquisitiones arithmeticae, 1801. English translation by arthur a. clarke, 1986.

[Har08] Rupert Hartung. *Computational problems of quadratic forms: complexity and cryptographic perspectives*. PhD thesis, Ph. D. thesis, Goethe-Universität Frankfurt a. M., 2008, http://publikationen.ub.uni-frankfurt.de/volltexte/2008/5444/pdf/HartungRupert.pdf, 2008.

[Jon42] Burton W Jones. Related genera of quadratic forms. *Duke Mathematical Journal*, 9(4):723–756, 1942.

[Jon44] Burton W Jones. A canonical quadratic form for the ring of 2-adic integers. *Duke Math. J*, 11(715):e727, 1944.

[Jon50] Burton Wadsworth Jones. *The arithmetic theory of quadratic forms*, volume 10. Mathematical Association of America, distributed by Wiley [New York, 1950.

[Kit99] Yoshiyuki Kitaoka. *Arithmetic of quadratic forms*, volume 106. Cambridge University Press, 1999.

[O’M73] Onorato Timothy O’Meara. *Introduction to quadratic forms*, volume 117. Springer, 1973.

[Sie35] Carl Ludwig Siegel. Über die analytische theorie der quadratischen formen. *The Annals of Mathematics*, 36(3):527–606, 1935.

[Wat60] George Leo Watson. *Integral quadratic forms*. Cambridge, 1960.

[Wed01] Sebastian Wedeniwski. *Primality Tests on Commutator Curves*. PhD thesis, Eberhard-Karls-Universität Tübingen, 2001.
A Diagonalizing a Matrix

In this section, we provide a proof of Theorem 5.

Module. There are quadratic forms which have no associated lattice e.g., negative definite quadratic forms. To work with these, we define the concept of free modules (henceforth, called module) which behave as vector space but have no associated realization over the Euclidean space $\mathbb{R}^n$.

If $M$ is finitely generated $R$-module with generating set $x_1, \cdots, x_n$ then the elements $x \in M$ can be represented as $\sum_{i=1}^n r_ix_i$, such that $r_i \in R$ for every $i \in [n]$. By construction, for all $a, b \in R$, and $x, y \in M$;

$$a(x + y) = ax + ay \quad (a + b)x = ax + bx \quad a(bx) = (ab)x \quad 1x = x$$

Note that, if we replace $R$ by a field in the definition then we get a vector space (instead of a module). Any inner product $\beta : M \times M \to R$ gives rise to a quadratic form $Q \in R^{n \times n}$ as follows;

$$Q_{ij} = \beta(x_i, x_j).$$

Conversely, if $R = \mathbb{Z}$ then by definition, every symmetric matrix $Q \in \mathbb{Z}^{n \times n}$ gives rise to an inner product $\beta$ over every $\mathbb{Z}$-module $M$; as follows. Given $n$-ary integral quadratic form $Q$ and a $\mathbb{Z}$-module $M$ generated by the basis $\{x_1, \cdots, x_n\}$ we define the corresponding inner product $\beta : M \times M \to \mathbb{Z}$ as;

$$\beta(x, y) = \sum_{i,j} c_id_jQ_{ij} \text{ where, } x = \sum_i c_ix_i \quad y = \sum_j d_jx_j.$$.

In particular, any integral quadratic form $Q^n$ can be interpreted as describing an inner product over a free module of dimension $n$.

For studying quadratic forms over $\mathbb{Z}/p^k\mathbb{Z}$, where $p$ is a prime and $k$ is a positive integer; the first step is to find equivalent quadratic forms which have as few mixed terms as possible (mixed terms are terms like $x_1x_2$).

Proof(Theorem 5) The transformation of the matrix $Q$ to a block diagonal form involves three different kinds of transformation. We first describe these transformations on $Q$ with small dimensions (2 and 3).

1) Let $Q$ be a $2 \times 2$ integral quadratic form. Let us also assume that the entry with smallest $p$-order in $Q$ is a diagonal entry, say $Q_{11}$. Then, $Q$ is of the following form; where $\alpha_1, \alpha_2$ and $\alpha_3$ are units of $\mathbb{Z}/p\mathbb{Z}$.

$$Q = \begin{pmatrix} p^{i\alpha_1} & p^{j\alpha_2} \\ p^{j\alpha_2} & p^{k\alpha_3} \end{pmatrix} \quad i \leq j, s$$

The corresponding $U \in \text{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$, that diagonalizes $Q$ is given below. The number $\alpha_1$ is a unit of $\mathbb{Z}/p\mathbb{Z}$ and so $\alpha_1$ has an inverse in $\mathbb{Z}/p^k\mathbb{Z}$.

$$U = \begin{pmatrix} 1 & -p^{i-1}\alpha_2 \mod p^k \\ 0 & \alpha_1 \end{pmatrix} \quad U^tQU \equiv \begin{pmatrix} p^{i\alpha_1} & 0 \\ 0 & p^{k\alpha_3} - p^{2j-i}\alpha_1^2 \end{pmatrix} \text{ (mod } p^k)$$

2) If $Q^2$ does not satisfy the condition of item (1) i.e., the off diagonal entry is the one with smallest $p$-order, then we start by the following transformation $V \in \text{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$.

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad V^tQV = \begin{pmatrix} Q_{11} + 2Q_{12} + Q_{22} & Q_{12} + Q_{22} \\ Q_{12} + Q_{22} & Q_{22} \end{pmatrix}$$

If $p$ is an odd prime then $\text{ord}_p(Q_{11} + 2Q_{12} + Q_{22}) = \text{ord}_p(Q_{12})$, because $\text{ord}_p(Q_{11})$, $\text{ord}_p(Q_{22}) > \text{ord}_p(Q_{12})$. By definition, $S = V^tQV$ is equivalent to $Q$ over the ring $\mathbb{Z}/p^k\mathbb{Z}$. But now, $S$ has the property that $\text{ord}_p(S_{11}) = \text{ord}_p(S_{12})$, and it can be diagonalized using the transformation in (1). The final transformation in this case is the product of $V$ and the subsequent transformation from item (1). The product of two matrices from $\text{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$ is also in $\text{SL}_2(\mathbb{Z}/p^k\mathbb{Z})$, completing the diagonalization in this case.
(3) If \( p = 2 \), then the transformation in item (2) fails. In this case, it is possible to subtract a linear combination of these two rows/columns to make everything else on the same row/column equal to zero over \( \mathbb{Z}/2^k\mathbb{Z} \). The simplest such transformation is in dimension 3. The situation is as follows. Let \( Q^3 \) be a quadratic form whose off diagonal entry has the lowest possible power of 2, say \( 2^3 \) and all diagonal entries are divisible by at least \( 2^{\ell+1} \). In this case, the matrix \( Q \) is of the following form.

\[
Q = \begin{pmatrix} 2^{\ell+1}a & 2^b & 2^d \\ 2^b & 2^{\ell+1}c & 2^e \\ 2^d & 2^e & 2^{\ell+1}f \\
\end{pmatrix} \quad \text{b odd, } \ell \leq i, j
\]

In such a situation, we consider the matrix \( U \in \text{SL}_3(\mathbb{Z}/2^k\mathbb{Z}) \) of the form below such that if \( S = U'QU \pmod{2^k} \) then \( S_{13} = S_{23} = 0 \).

\[
U = \begin{pmatrix} 1 & 0 & -r \\ 0 & 1 & -s \\ 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
(U'QU)_{13} \equiv 0 \pmod{2^k} \implies 2^r + s \equiv 2^{\ell}d \pmod{2^{k-\ell}}
\]

\[
(U'QU)_{23} \equiv 0 \pmod{2^k} \implies 2^r + s \equiv 2^{\ell}e \pmod{2^{k-\ell}}
\]

For \( i, j \geq \ell \) and \( b \) odd, the solution \( r \) and \( s \) can be found by the Cramer’s rule, as below. The solutions exist because the matrix \( \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \) has determinant \( 4ac - b^2 \), which is odd and hence invertible over the ring \( \mathbb{Z}/2^{k-\ell}\mathbb{Z} \).

\[
r = \frac{\det(2^{\ell-\ell}d \quad s)}{\det(2a \quad b \quad 2c)} \pmod{2^{k-\ell}} \quad \text{and} \quad s = \frac{\det(2a \quad 2^{\ell-\ell}d)}{\det(2a \quad b \quad 2c)} \pmod{2^{k-\ell}}
\]

This completes the description of all the transformations we are going to use, albeit for \( n \)-dimensional \( Q \) they will be a bit technical. The full proof for the case of odd prime follows.

Our proof will be a reduction of the problem of diagonalization from \( n \) dimensions to \( (n-1) \)-dimensions, for the odd primes \( p \). We now describe the reduction.

Given the matrix \( Q^r \), let \( M \) be the corresponding \( (\mathbb{Z}/p^k\mathbb{Z}) \)-module with basis \( B = \{b_1, \ldots, b_n\} \) i.e., \( Q = B'B \). We first find a matrix entry with the smallest \( p \)-order, say \( Q_{1,1} \). The reduction has two cases: (i) there is a diagonal entry in \( Q \) with the smallest \( p \)-order, and (ii) the smallest \( p \)-order occurs on an off-diagonal entry.

We handle case (i) first. Suppose it is possible to pick \( Q_{i,j} \) as the entry with the smallest \( p \)-order. Our first transformation \( U_1 \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \) is the one which makes the following transformation i.e., swaps \( b_1 \) and \( b_i \),

\[
\{b_1, \ldots, b_n\} \xrightarrow{U_1 \cdot p^k} \{b_i, b_2, \ldots, b_{i-1}, b_1, b_{i+1}, \ldots, b_n\}
\]

Let us call the new set of elements \( B_1 = \{v_1, \ldots, v_n\} \) and the new quadratic form \( Q_1 = Q^iB_1 \pmod{p^k} \). Then, \( v_i^\prime v_1 \) has the smallest \( p \)-order in \( Q_1 \) and \( U_1 QU_1 \equiv Q_1 \pmod{p^k} \). The next transformation \( U_2 \in \text{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \) is as follows.

\[
U_2 = \begin{cases} v_1 \\ v_i - \frac{v_i^\prime v_1}{p^{\text{ord}_p((Q_1)_{11})}} \cdot \left( \frac{1}{\text{cpr}_p((Q_1)_{11})} \right) \pmod{p^k} \cdot v_1 & \text{if } i = 1 \\ v_i & \text{otherwise} \end{cases}
\]

By assumption, \((Q_1)_{11}\) is the matrix entry with the smallest \( p \)-order and so \( p^{\text{ord}_p((Q_1)_{11})} \) divides \( v_i^\prime v_1 \). Furthermore, \( \text{cpr}_p((Q_1)_{11}) \) is invertible modulo \( p^k \). Thus, the transformation in Equation (14) is well defined. Also note that it is a basis transformation, which maps one basis of \( B_1 = \{v_1, \ldots, v_n\} \) to another basis
$B_2 = [w_1, \cdots, w_n]$. Thus, the corresponding basis transformation $U_2$ is a unimodular matrix over integers, and so $U_2 \in SL_n(\mathbb{Z}/p^k\mathbb{Z})$. Let $Q_2 = U_2^t Q_1 U_2 \mod p^k$. Then, we show that the non-diagonal entries in the entire first row and first column of $Q_2$ are 0.

\[
(Q_2)_{i(\neq 1)} = (Q_2)_{11} = w_i^t \varepsilon_1 \mod p^k \\
\equiv \varepsilon_1 v_i - \frac{v_i^t \varepsilon_1}{p^{\text{ord}_p((Q_1)_{11})}} \cdot \left(\frac{1}{\text{cpr}_p((Q_1)_{11})} \mod p^k\right) \cdot v_i^t \varepsilon_1 \\
\equiv 0 \mod p^k
\]

Thus, we have reduced the problem to $(n - 1)$-dimensions. We now recursively call this algorithm with the quadratic form $S = [w_2, \cdots, w_n]^t [w_2, \cdots, w_n] \mod p^k$ and let $V \in SL_{n-1}(\mathbb{Z}/p^k\mathbb{Z})$ be the output of the recursion. Then, $V^t S V \mod p^k$ is a diagonal matrix. Also, by construction $Q_2 = \text{diag}((Q_2)_{11}, S)$. Let $U_3 = 1 \oplus V$, and $U = U_1 U_2 U_3$, then, by construction, $U^t Q U \mod p^k$ is a diagonal matrix; as follows.

\[
U^t Q U \equiv U_3^t U_2^t U_1^t Q U_1 U_2 U_3 \equiv U_3^t Q U_3 \equiv (1 \oplus V)^t \text{diag}((Q_2)_{11})(1 \oplus V) \\
\equiv \text{diag}((Q_2)_{11}, V^t S V) \mod p^k
\]

Otherwise, we are in case (ii) i.e., the entry with smallest $p$-order in $Q$ is an off diagonal entry, say $Q_{i^* j^*}, i^* \neq j^*$. Then, we make the following basis transformation from $[b_1, \cdots, b_n]$ to $[v_1, \cdots, v_n]$ as follows.

\[
v_i = \begin{cases} b_{i^*} + b_{j^*} & \text{if } i = i^* \\ b_i & \text{otherwise}. \end{cases}
\]

The transformation matrix $U_0$ is from $SL_n(\mathbb{Z}/p^k\mathbb{Z})$. Recall, $\text{ord}_p(Q_{i^* j^*}) < \text{ord}_p(Q_{i^* i^*}), \text{ord}_p(Q_{j^* j^*})$, and so $\text{ord}_p(v_{i^*}, v_{j^*}) = \text{ord}_p(b_{i^*}, b_{j^*})$. Furthermore, $\text{ord}_p(v_{i^*} v_{j^*}) \geq \text{ord}_p(b_{i^*} b_{j^*})$, and so the minimum $p$-order does not change after the transformation in Equation (15). This transformation reduces the problem to the case when the matrix entry with minimum $p$-order appears on the diagonal. This completes the proof of the theorem for odd primes $p$.

For $p = 2$, exactly the same set of transformations works, unless the situation in item (3) arises. In such a case, we use the type II block to eliminate all other entries on the same rows/columns as the type II block. Thus, in this case, the problem reduces to one in dimension $(n - 2)$.

The algorithm uses $n$ iterations, reducing the dimension by 1 in each iteration. In each iteration, we have to find the minimum $p$-order, costing $O(n^2 \log k)$ ring operations and then 3 matrix multiplications costing $O(n^3)$ operations over $\mathbb{Z}/p^k\mathbb{Z}$. Thus, the overall complexity is $O(n^4 + n^3 \log k)$ or $O(n^4 \log k)$ ring operations.

\[\square\]

B Missing Proofs

**Proof:** (proof of Lemma [2]) We split the proof in two parts: for odd primes $p$ and for the prime 2.

**Odd Prime.** If $0 \neq t \in \mathbb{Z}/p^k\mathbb{Z}$ then $\text{ord}_p(t) < k$. If $t$ is a square modulo $p^k$ then there exists a $x$ such that $x^2 \equiv t \mod p^k$. Thus, there exists $a \in \mathbb{Z}$ such that $x^2 = t + ap^k$. But then, $2 \text{ord}_p(x) = \text{ord}_p(t + ap^k) = \text{ord}_p(t)$. This implies that $\text{ord}_p(t)$ is even and $\text{ord}_p(x) = \text{ord}_p(t)/2$. Substituting this into $x^2 = t + ap^k$ and dividing the entire equation by $p^{\text{ord}_p(t)}$ yields that $\text{cpr}_p(t)$ is a quadratic residue modulo $p$; as follows.

\[
\text{cpr}_p(x)^2 = \text{cpr}_p(t) + ap^{k-\text{ord}_p(t)} \equiv \text{cpr}_p(t) \mod p
\]
Conversely, if $\text{cpr}_p(t)$ is a quadratic residue modulo $p$ then there exists a $u \in \mathbb{Z}/p^k\mathbb{Z}$ such that $u^2 \equiv \text{cpr}_p(t) \pmod{p^k}$, by Lemma 2. If $\text{ord}_p(t)$ is even then $x = p^{\text{ord}_p(t)/2} u$ is a solution to the equation $x^2 \equiv t \pmod{p^k}$.

**Prime 2.** If $0 \neq t \in \mathbb{Z}/2^k\mathbb{Z}$ then $\text{ord}_2(t) < k$. If $t$ is a square modulo $2^k$ then there exists an integer $x$ such that $x^2 \equiv t \pmod{2^k}$. Thus, there exists an integer $a$ such that $x^2 = t + a 2^k$. But then, $2 \text{ord}_2(x) = \text{ord}_2(t + a 2^k) = \text{ord}_2(t)$. This implies that $\text{ord}_2(t)$ is even and $\text{ord}_2(x) = \text{ord}_2(t)/2$.

Substituting this into the equation $x^2 = t + a 2^k$ and dividing the entire equation by $2^{\text{ord}_2(t)}$ yields,

\[
\text{cpr}_2(x)^2 = \text{cpr}_2(t) + a 2^{k - \text{ord}_2(t)} \quad \text{cpr}_2(t) < 2^{k - \text{ord}_2(t)}.
\]

But $\text{cpr}_2(x)$ is odd and hence $\text{cpr}_2(x)^2 \equiv 1 \pmod{8}$. If $k - \text{ord}_2(t) > 2$, then $\text{cpr}_2(t) \equiv 1 \pmod{8}$. Otherwise, if $k - \text{ord}_2(t) \leq 2$ then $\text{cpr}_2(t) < 2^{k - \text{ord}_2(t)}$ implies that $\text{cpr}_2(t) = 1$.

Conversely, if $\text{cpr}_2(t) \equiv 1 \pmod{8}$ then there exists a $u \in \mathbb{Z}/2^k\mathbb{Z}$ such that $u^2 \equiv \text{cpr}_2(t) \pmod{2^k}$, by Lemma 2. If $\text{ord}_2(t)$ is even then $x = 2^{\text{ord}_2(t)/2} u$ is a solution to the equation $x^2 \equiv t \pmod{2^k}$.

$\square$