SHIFTED SCHUR FUNCTIONS

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Abstract

The classical algebra $\Lambda$ of symmetric functions has a remarkable deformation $\Lambda^*$, which we call the algebra of shifted symmetric functions. In the latter algebra, there is a distinguished basis formed by shifted Schur functions $s^*_\mu$, where $\mu$ ranges over the set of all partitions. The main significance of the shifted Schur functions is that they determine a natural basis in $Z(gl(n))$, the center of the universal enveloping algebra $U(gl(n))$, $n = 1, 2, \ldots$.

The functions $s^*_\mu$ are closely related to the factorial Schur functions introduced by Biedenharn and Louck and further studied by Macdonald and other authors.

A part of our results about the functions $s^*_\mu$ has natural classical analogues (combinatorial presentation, generating series, Jacobi–Trudi identity, Pieri formula). Other results are of different nature (connection with the binomial formula for characters of $GL(n)$, an explicit expression for the dimension of skew shapes $\lambda/\mu$, Capelli–type identities, a characterization of the functions $s^*_\mu$ by their vanishing properties, ‘coherence property’, special symmetrization map $S(gl(n)) \to U(gl(n))$).

The main application that we have in mind is the asymptotic character theory for the unitary groups $U(n)$ and symmetric groups $S(n)$ as $n \to \infty$.

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References
Introduction

1. Shifted Schur polynomials and factorial Schur polynomials. Recall that the Schur function (or Schur polynomial) in \( n \) variables can be defined as ratio of two \( n \times n \) determinants

\[
s_\mu(x_1, \ldots, x_n) = \frac{\det[x_{i,j}^{\mu_j+n-j}]}{\det[x_{i,j}^{n-j}]},
\]

(0.1)

where \( \mu \), the parameter of the polynomial, is an arbitrary partition \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0 \) of length \( \leq n \).

Denote by the symbol \((x \downarrow k)\) the \( k \)-th falling factorial power of a variable \( x \),

\[
(x \downarrow k) = \begin{cases} x(x-1)\ldots(x-k+1), & \text{if } k = 1, 2, \ldots, \\ 1, & \text{if } k = 0. \end{cases}
\]

(0.2)

In the present paper we study the following Schur–type polynomials:

\[
s_\mu^*(x_1, \ldots, x_n) = \frac{\det[(x_i + n - i \downarrow \mu_j + n - j)]}{\det[(x_i + n - i \downarrow n - j)]}.
\]

(0.3)

We call them the shifted Schur polynomials.

These new polynomials differ by a shift of arguments only from the factorial Schur polynomials

\[
t_\mu(x_1, \ldots, x_n) = \frac{\det[(x_i \downarrow \mu_j + n - j)]}{\det[(x_i \downarrow n - j)]}.
\]

(0.4)

The polynomials \( t_\mu \) were introduced by Biedenharn and Louck [BL1], [BL2]. Then they were studied in Chen–Louck [CL], Goulden–Greene [GG], Goulden–Hamel [GH], and Macdonald [M2] (see also Macdonald [M1], 2nd edition, Ch. 1, section 3, Examples 20–21). In particular, Macdonald developed a theory of more general ‘factorial’ polynomials including as special cases the polynomials \( s_\mu \) and \( t_\mu \).

In these works it was shown that several important facts about the ordinary Schur polynomials (e.g., the Jacobi–Trudy identity and the combinatorial presentation) can be transferred to the factorial polynomials.

Our results about the shifted Schur polynomials \( s_\mu^* \) can be restated as certain new results about the factorial polynomials \( t_\mu \). However, as it will be shown, the use of the shifted polynomials has many advantages and provides a new insight.

2. Stability and the algebra \( \Lambda^* \) of shifted Schur functions. Recall that the ordinary Schur polynomials are stable in the following sense:

\[
s_\mu(x_1, \ldots, x_n, 0) = s_\mu(x_1, \ldots, x_n).
\]

(0.5)

This stability property holds also for the shifted polynomials,

\[
s_\mu^*(x_1, \ldots, x_n, 0) = s_\mu^*(x_1, \ldots, x_n).
\]

(0.6)
but fails for the factorial polynomials \( t_\mu \).

The stability property (0.6) allows us to introduce the functions \( s^*_\mu \) in infinitely many variables — just in the same way as for the classical Schur functions. These functions \( s^*_\mu \) form a distinguished basis in a certain new algebra which we denote by \( \Lambda^* \) and call the algebra of shifted symmetric functions.

As in the classical context of symmetric functions, elements of the algebra \( \Lambda^* \) may be viewed as functions \( f(x_1, x_2, \ldots) \) on infinite sequences of arguments such that \( x_i = 0 \) for \( i \) large enough. But the ordinary symmetry is replaced by the ‘shifted symmetry’:

\[
f(x_1, \ldots, x_i, x_{i+1}, \ldots) = f(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots), \quad i = 1, 2, \ldots \tag{0.7}
\]

As examples of shifted symmetric functions one can take the complete shifted functions \( h^*_r = s^*_r \) and the elementary shifted functions \( e^*_r = s^*_1 \), \( r = 1, 2, \ldots \):

\[
h^*_r(x_1, x_2, \ldots) = \sum_{1 \leq i_1 \leq \ldots \leq i_r < \infty} (x_{i_1} - r + 1)(x_{i_2} - r + 2) \ldots x_{i_r}, \tag{0.8}
\]

\[
e^*_r(x_1, x_2, \ldots) = \sum_{1 \leq i_1 < \ldots < i_r < \infty} (x_{i_1} + r - 1)(x_{i_2} + r - 2) \ldots x_{i_r}. \tag{0.9}
\]

The algebra \( \Lambda^* \) (but yet without the functions \( s^*_\mu \)) appeared in Olshanski’s papers [O1], [O2] in connection with a construction of Laplace operators on the infinite–dimensional classical groups. Then this algebra was studied in the note [KO] by Kerov and Olshanski. We will often call the functions \( s^*_\mu \in \Lambda^* \) the \( s^*-\text{functions} \).

3. The \( s^*\)-functions and the center of the universal enveloping algebra \( \mathcal{U}(\mathfrak{gl}(n)) \). Let \( \Lambda^*(n) \) denote for the algebra of shifted symmetric polynomials in \( n \) variables, and note that the polynomials \( s^*_\mu \) form a basis in \( \Lambda^*(n) \). Let \( \mathfrak{Z}(\mathfrak{gl}(n)) \) denote the center of \( \mathcal{U}(\mathfrak{gl}(n)) \), the universal enveloping algebra of \( \mathfrak{gl}(n) = \mathfrak{gl}(n, \mathbb{C}) \).

There exists a canonical algebra isomorphism

\[
\mathfrak{Z}(\mathfrak{gl}(n)) \rightarrow \Lambda^*(n), \quad A \mapsto f_A, \tag{0.10}
\]

which plays the key role in our paper. This isomorphism is simply the well–known Harish–Chandra isomorphism; it takes a central element \( A \in \mathfrak{Z}(\mathfrak{gl}(n)) \) to its eigenvalue \( f_A(\lambda_1, \ldots, \lambda_n) \) in a highest weight module \( V_\lambda \), where \( \lambda \) varies over \( \mathbb{C}^n \).

By taking the preimage of the polynomials \( s^*_\mu \in \Lambda^*(n) \) under the isomorphism (0.10) we obtain a certain distinguished basis \( \{ S_\mu \} \) in the center \( \mathfrak{Z}(\mathfrak{gl}(n)) \). It will be shown that the central elements \( S_\mu \) possess many remarkable properties.

In Okounkov’s paper [Ok1] an explicit formula expressing the elements \( S_\mu \) in terms of the standard generators \( E_{ij} \in \mathfrak{gl}(n) \) was found; then it was improved in Nazarov [N2] and in [Ok2]. In the particular case of \( \mu = (1^k) \), \( k = 1, \ldots, n \), this formula turns into a classical formula occurring in the well–known Capelli identity (see Howe [H], Howe–Umeda [HU]). We would like to note that the paper [HU] was one of the starting points of our work.

Note that a kind of isomorphism (0.10) also exists for the algebra \( \Lambda^* \) although the naive \( \infty \)-dimensional analogue of the algebras \( \mathcal{U}(\mathfrak{gl}(n)) \), the inductive limit algebra \( \lim \mathcal{U}(\mathfrak{gl}(n)) \), has trivial center (see [O1], [O2]).

\[\text{The results of the present paper also provide us with an expression for } S_\mu \text{ but it is less satisfactory.}\]
4. The $s^*$-functions and Vershik–Kerov’s asymptotic theory of characters. Suppose we have an infinite increasing chain

$$G(1) \subset G(2) \subset \ldots$$

(0.11)

of finite or compact groups. In the asymptotic theory of characters (see Vershik–Kerov [VK1], [VK2], [VK3]) a central place is taken by the problem of the limit behavior of the expression

$$\chi_\pi(g) = \frac{\text{tr} \pi(g)}{\dim \pi},$$

(0.12)

where $g$ is an arbitrary but fixed element of a group $G(k)$ while $\pi = \pi_n$ is an irreducible representation of $G(n)$ (where $n \geq k$) which varies as $n \to \infty$.

When the groups $G(n)$ are the symmetric groups $S(n)$ or the unitary groups $U(n)$, Vershik and Kerov found necessary and sufficient conditions on sequences $\{\pi_n\}$ under which the limit of the expression (0.12) exists for any $g \in \lim_{\to} G(k)$.

It turns out that in the symmetric group case, for the normalized character (0.12) there exists an explicit formula in terms of $s^*$-functions; this is a corollary of a new formula for dimension of skew Young diagrams. In the case of $G(n) = U(n)$ an explicit formula in terms of $s^*$-functions also exists provided group elements $g \in U(k)$ are replaced by elements of the algebra $U(gl(k))$.

The present paper is much obliged to Vershik–Kerov’s work: in fact, our initial aim was to analyze the sketch of proof of the main theorem in [VK2] about the characters of $U(\infty)$.

We plan to present a detailed exposition of this fundamental theorem (as well as its generalization to other classical groups), based on the machinery of $s^*$-functions, in next papers.

5. Main results.

I. Binomial formula. Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n) \in \mathbb{Z}^n$ be a dominant weight for the group $GL(n, \mathbb{C})$ and let $gl(n)_\lambda(z_1, \ldots, z_n)$ stand for the corresponding irreducible character, viewed as a function on the subgroup of diagonal matrices. Then we have the following expansion (Theorem 5.1)

$$\frac{gl(n)_\lambda(1 + x_1, \ldots, 1 + x_n)}{gl(n)_\lambda(1, \ldots, 1)} = \sum_\mu \frac{s^*_\mu(\lambda_1, \ldots, \lambda_n)s_\mu(x_1, \ldots, x_n)}{c(n, \mu)},$$

(0.13)

where $\mu$ ranges over the set of partitions of length $\leq n$ and $c(n, \mu)$ are certain number factors not depending on $\lambda$. We call (0.13) the binomial formula for the (normalized) characters of the group $GL(n)$.

In a very different form, the expansion (0.13) can be found in Macdonald [M1], Ch.I, Section 3, Example 10 (this example is due to Lascoux). But the new point here is the observation that the binomial formula turns out to be related to the $s^*$-functions.

The binomial formula plays an essential role in applications to the asymptotic character theory. In the next paper we shall present similar formulas for other classical groups.

Note that the idea to replace in (0.12) the compact groups by the universal enveloping algebras is present, in an implicit form, in Vershik–Kerov’s note [VK2].
II. Dimension of skew Young diagrams. Let \( \mu \vdash k \) and \( \lambda \vdash n \) be two partitions, also viewed as Young diagrams. Let us assume \( k \leq n \) and \( \mu \subset \lambda \), and denote by \( \dim \lambda/\mu \) the number of standard tableaux of shape \( \lambda/\mu \); in particular, \( \dim \lambda = \dim \lambda/\varnothing \). Recall that for \( \dim \lambda \) nice explicit formulas are known. Now we have

\[
\frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s^*_\mu(\lambda)}{n(n-1)\ldots(n-k+1)},
\]

where \( s^*_\mu(\lambda) = s^*_\mu(\lambda_1, \lambda_2, \ldots, \lambda_k) \).

To our knowledge, this is a new formula, which is a quite surprising fact.

Formula (0.14) can be applied to the asymptotic character theory of the symmetric groups.

III. Higher Capelli identities. We shall consider differential operators with polynomial coefficients on the space \( M(n, m) \) of \( n \times m \) matrices. Let \( x_{ij} \) denote the natural coordinates in \( M(n, m) \) and let \( \partial_{ij} \) be the corresponding partial derivatives. Let \( \mu \vdash k \) be an arbitrary partition of length \( \leq \min(n, m) \). We define the higher Capelli operator, indexed by \( \mu \), as the following differential operator on \( M(n, m) \):

\[
\Delta_{\mu}^{(n, m)} = (k!)^{-1} \sum_{i_1, \ldots, i_k=1}^n \sum_{j_1, \ldots, j_k=1}^m \sum_{s \in S(k)} \chi^\mu(s) \cdot x_{i_1 j_1} \cdots x_{i_k j_k} \partial_{s(1)j_1} \cdots \partial_{s(k)j_k},
\]

where \( \chi^\mu(s) \) is the value of the irreducible character of \( S(k) \), indexed by \( \mu \), at the permutation \( s \in S(k) \).

These operators are invariant with respect to the action of the groups \( GL(n) \) and \( GL(m) \) by left and right multiplications on \( M(n, m) \), respectively. In the particular case \( n = m, \mu = (1^n) \), the operator (0.15) reduces to the well-known Capelli operator

\[
\det \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nm} \end{bmatrix} \det \begin{bmatrix} \partial_{11} & \cdots & \partial_{1n} \\ \vdots & \ddots & \vdots \\ \partial_{n1} & \cdots & \partial_{nn} \end{bmatrix}
\]

(0.16)

Other Capelli operators are obtained when \( n \) and \( m \) are arbitrary and \( \mu = (1^k) \). (About Capelli operators and Capelli identities see Howe [H], Howe–Umeda [HU].) When \( \mu = (k) \), we obtain Capelli–type operators found by Nazarov [N1].

The action of the groups \( GL(n) \) and \( GL(m) \) on the space \( M(n, m) \) induces two homomorphisms, \( L \) and \( R \), of the universal enveloping algebras \( \mathcal{U}(\mathfrak{gl}(n)) \) and \( \mathcal{U}(\mathfrak{gl}(m)) \), respectively, in the algebra of differential operators on \( M(n, m) \) with polynomial coefficients. Let us use a more detailed notation \( \mathbb{S}_\mu[n] \) for the central elements \( \mathbb{S}_\mu \) defined above. Then we have the following result (Corollary 6.8)

\[
L(\mathbb{S}_\mu[n]) = R(\mathbb{S}_\mu[m]) = \Delta_{\mu}^{(n, m)}
\]

(0.17)

for all partitions \( \mu \) of length \( \leq \min(n, m) \).

Together with the explicit formula for the quantum immanants \( \mathbb{S}_\mu[n] \), obtained in [Ok1] (see also [N2] and [Ok2]) the relations (0.17) provide higher analogues of the classical Capelli identity.

IV. Characterization Theorem for the \( s^* \)-functions. An important idea (already used in [KO]) is to interpret elements of the algebra \( \Lambda^* \) as functions \( f(\lambda) = f(\lambda_1, \lambda_2, \ldots) \) on the set of partitions.
Characterization Theorem [Ok1]. (See also Theorems 3.3 and 3.4 below.) Fix a partition $\mu$. Then $s^*_\mu$ is the unique, within a scalar factor, element of the algebra $\Lambda^*$ such that $s^*_\mu(\lambda) = 0$ for all $\lambda \neq \mu$ with $|\lambda| \leq |\mu|$.

This characterization of $s^*$-functions turns out to be an efficient tool for proving various results about the $s^*$-functions. Its role is especially important in the proof of the identities (0.17).

V. Combinatorial presentation of $s^*$-functions. Recall that the ordinary Schur function $s_\mu$ admits a nice combinatorial description in terms of tableau $x$. There exists a similar description for the shifted Schur functions (Theorem 11.1):

$$ s^*_\mu(x_1, x_2, \ldots) = \sum_T \sum_{\alpha \in \mu} (x_{T(\alpha)} - c(\alpha)), \quad (0.18) $$

summed over all reverse tableaux $T$ of shape $\mu$ and over all boxes $\alpha$ of $\mu$, where $c(\alpha)$ is the content of the box $\alpha$ and in a reverse tableau, in contrast to the conventional one, the entries decrease left to right along each row (weakly) and down each column (strictly).

Note that (0.8) and (0.9) are particular cases of (0.18).

Formula (0.18) can be easily derived from the combinatorial presentation of the polynomials $t_\mu$ (see Chen–Louck [CL] and Macdonald [M2]), but we also give an independent proof, based on the Characterization Theorem.

VI. The coherence property of shifted Schur polynomials. There is one more stability property of the shifted Schur polynomials, which is best stated in terms of the central elements $S_{\mu|n} \in \mathfrak{Z}(\mathfrak{gl}(n))$.

Note that for each $m = 1, 2, \ldots$ there exists a canonical projection

$$ U(\mathfrak{gl}(m)) \rightarrow \mathfrak{Z}(\mathfrak{gl}(m)), \quad (0.17) $$

commuting with the adjoint representation. It turns out that if we apply to $S_{\mu|n}$ the $m$-th projection (0.17), where $m > n$, then the result will be proportional to $S_{\mu|m}$ (Theorem 10.1). We call this the coherence property. When expressed in terms of the polynomials $s^*_\mu$ the coherence property leads to an interesting identity, see (10.30) below.

We think the coherence property is an important argument in favor of the thesis that the elements $S_{\mu|n}$ constitute a distinguished basis of the center.

VII. Generating series for $h^*$- and $e^*$-functions. There are nice generating series for the complete symmetric functions $h_1, h_2, \ldots$ and elementary symmetric functions $e_1, e_2, \ldots$. In Theorem 12.1 we present their analogues for $h^*_1, h^*_2, \ldots$ and $e^*_1, e^*_2, \ldots$. An interesting feature of these new series is that they involve inverse factorial powers of the formal parameter.

VIII. Jacobi–Trudi formula. For factorial Schur polynomials $t_\mu$ a Jacobi–Trudi–type formula was given in Chen–Louck [CL], Goulden–Hamel [GH], and Macdonald [M2], [M1], Ch. I, section 3, Examples 20–21. However, this formula, being rewritten in terms of the polynomials $s^*_{\mu|n}$, turns out to be not stable as $n \rightarrow \infty$ and so makes no sense in the algebra $\Lambda^*$. In Theorem 13.1 we obtain a different formula, which is stable and so expresses the $s^*$-functions in terms of $h^*_1, h^*_2, \ldots$. There is also a dual formula expressing $s^*_\mu$ through $e^*_1, e^*_2, \ldots$. Then a general result due to Macdonald implies that there is a determinantal expression of the $s^*$-functions in terms of the ‘hook’ $s^*$-functions which is just the same as in the classical Giambelli formula.
6. Notes. First observations about the $s^*$-functions were made by Olshanski [O3]. The results of the present work were the subject of a talk at the 7-th Conference on formal power series and algebraic combinatorics (Université Marne–la–Vallée, May 29–June 2, 1995).

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1. The algebra of shifted symmetric functions

Throughout the paper we assume that the ground field is $\mathbb{C}$ (although most results hold over any field of characteristic zero).

Recall the definition of the algebra $\Lambda$ of symmetric functions, see Macdonald [M1]. Let $\Lambda(n)$ denote the algebra of symmetric polynomials in $x_1, \ldots, x_n$. This algebra is graded by degree of polynomials. The specialization $x_{n+1} = 0$ is a morphism of graded algebras

$$\Lambda(n + 1) \to \Lambda(n).$$

(1.1)

By definition $\Lambda$ is the projective limit

$$\Lambda = \varprojlim \Lambda(n), \quad n \to \infty,$$

in the category of graded algebras, taken with respect to morphisms (1.1). An element $f \in \Lambda$ is by definition a sequence $(f_n)_{n \geq 1}$ such that:

1. $f_n \in \Lambda(n)$, $n = 1, 2, \ldots$,
2. $f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n)$ (the stability condition),
3. $\sup_n \deg f_n < \infty$.

Now let us denote by $\Lambda^*(n)$ the algebra of polynomials in $x_1, \ldots, x_n$ that become symmetric in new variables

$$x'_i = x_i - i + \text{const}, \quad i = 1, \ldots, n.$$  

(1.2)

Here ‘const’ is an arbitrary fixed number; note that the definition does not depend on its choice. We call such polynomials shifted symmetric. The algebra $\Lambda^*(n)$ is filtered by degree of polynomials, and the specialization $x_{n+1} = 0$ is a morphism of filtered algebras

$$\Lambda^*(n + 1) \to \Lambda^*(n).$$

(1.3)

Definition 1.1. Let

$$\Lambda^* = \varprojlim \Lambda^*(n), \quad n \to \infty,$$

(1.4)

be the projective limit in the category of filtered algebras, taken with respect to morphisms (1.3). We call $\Lambda^*$ the algebra of shifted symmetric functions.

Throughout this paper we use the notation

$$(x \downarrow k) = \begin{cases} x(x-1) \ldots (-k+1), & \text{if } k = 1, 2, \ldots, \\ 1, & \text{if } k = 0, \end{cases}$$

(1.5)

for the $k$-th falling factorial power of a variable $x$. By $\ell(\mu)$ we denote the length of a partition $\mu$. 

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Definition 1.2. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition, $\ell(\mu) \leq n$. We define the shifted Schur polynomial in $n$ variables, indexed by $\mu$ as ratio of two $n \times n$ determinants,

$$s_{\mu}(x_1, \ldots, x_n) = \frac{\det[(x_i + n - i \mid \mu_j + n - j)]}{\det[(x_i + n - i \mid n - j)]}, \quad (1.6)$$

where $1 \leq i, j \leq n$.

Note that the denominator in (1.6) equals the Vandermonde determinant in the variables (1.2). Since the numerator is skew–symmetric in these variables, the ratio is indeed a polynomial. Sometimes we will denote the polynomial $s_{\mu}(x_1, \ldots, x_n)$ by $s_{\mu|n}$. Let us agree that $s_{\mu|n} = 0$ if $\mu$ is a partition with $\ell(\mu) > n$.

Let us show that the sequence $\{s_{\mu|n}\}$ defines an element of the algebra $\Lambda^\ast$. It is clear that $s_{\mu|n}$ is shifted symmetric and $\deg s_{\mu|n} = |\mu|$ for $n \geq \ell(\mu)$, so the degree is bounded as $n \rightarrow \infty$. Let us verify the stability condition:

Proposition 1.3. For each partition $\mu$ and each $n$

$$s_{\mu}(x_1, \ldots, x_n, 0) = s_{\mu}(x_1, \ldots, x_n). \quad (1.7)$$

Proof. By definition,

$$s_{\mu}(x_1, \ldots, x_{n+1}) = \frac{\det [(x_i + n + 1 - i \mid \mu_j + n + 1 - j)]}{\det [(x_i + n + 1 - i \mid n + 1 - j)]}, \quad (1.8)$$

where $i, j = 1, 2, \ldots, n + 1$.

First suppose $\ell(\mu) \leq n$; that is $\mu_{n+1} = 0$. Then substituting $x_{n+1} = 0$ in the numerator of (1.8) we obtain

$$\det \begin{bmatrix} (x_1 + n \mid \mu_1 + n) & \ldots & (x_1 + n \mid \mu_n + 1) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ (x_n + 1 \mid \mu_1 + n) & \ldots & (x_n + 1 \mid \mu_n + 1) & 1 \\ 0 & \ldots & 0 & 1 \end{bmatrix}.$$ 

Clearly this determinant equals

$$(x_1 + n) \ldots (x_n + 1) \det \begin{bmatrix} (x_1 + n - 1 \mid \mu_1 + n - 1) & \ldots & (x_1 + n - 1 \mid \mu_n) \\ \vdots & \ddots & \vdots \\ (x_n \mid \mu_1 + n - 1) & \ldots & (x_n \mid \mu_n) \end{bmatrix}.$$

Likewise, the denominator of (1.8) becomes

$$(x_1 + n) \ldots (x_n + 1) \det [(x_i + n - i \mid n - j)], \quad i, j = 1 \ldots n.$$

This yields (1.7) provided $\ell(\mu) \leq n$. If $\ell(\mu) = n + 1$ then substituting $x_{n+1} = 0$ in the numerator of (1.8) we obtain

$$\det \begin{bmatrix} (x_1 + n \mid \mu_1 + n) & \ldots & (x_1 + n \mid \mu_n + 1) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ (x_n + 1 \mid \mu_1 + n) & \ldots & (x_n + 1 \mid \mu_n + 1) & 0 \\ 0 & \ldots & 0 & 0 \end{bmatrix}.$$

This determinant clearly vanishes and hence the left–hand side of (1.7) vanishes. On the other hand the right–hand side equals zero by definition. If $\ell(\mu) > n + 1$ then both sides equal zero by definition. This completes the proof. $\qed$
Definition 1.4. By Proposition 1.3, for each partition \( \mu \) the sequence (1.6), where \( n \to \infty \), defines an element of the algebra \( \Lambda^* \). We denote it by \( s^*_\mu \) and call it the \textit{shifted Schur function}, indexed by \( \mu \). These functions will also be called \textit{s*-functions} for short.

Note that the shift of variables (1.2) establishes an isomorphism between \( \Lambda^*(n) \) and \( \Lambda(n) \) but no shift of variables maps \( \Lambda^* \) to \( \Lambda \). However \( \Lambda^* \) and \( \Lambda \) are related in the following way:

Proposition 1.5. \textit{The graded algebra} \( \text{gr} \Lambda^* \) \textit{corresponding to the filtered algebra} \( \Lambda^* \) \textit{is canonically isomorphic to the algebra} \( \Lambda \).

Proof. Note that if \( f \in \Lambda^*(n) \) is of degree \( \leq d \) then its \( d \)-th homogeneous component is a symmetric polynomial. It follows that the algebras \( \text{gr} \Lambda^*(n) \) and \( \Lambda(n) \) are canonically isomorphic for each \( n \). Since the isomorphisms \( \text{gr} \Lambda^*(n) \to \Lambda(n) \) are compatible with the specialization maps \( x_n = 0 \), we obtain a canonical algebra isomorphism \( \text{gr} \Lambda^* \to \Lambda \). □

If \( f \) is an element of \( \Lambda^* \) (resp., of \( \Lambda^*(n) \)) of degree \( d \) then its image in the \( d \)-th homogeneous component of the graded algebra \( \Lambda \) (resp., \( \Lambda(n) \)) will be called the \textit{highest term} of \( f \).

Note that the highest term of \( s^*_\mu \in \Lambda^* \) is the ordinary Schur function \( s_\mu \in \Lambda \) and the highest term of \( s^*_{\mu|n} \in \Lambda^*(n) \) is \( s_{\mu|n} \in \Lambda(n) \), the Schur polynomial in \( n \) variables. This follows at once from the comparison of (1.6) with the well–known expression for \( s_{\mu|n} \),

\[
s_\mu(x_1, \ldots, x_n) = \frac{\det[x_i^{\mu_j+n-j}]}{\det[x_i^{n-j}]}.
\] (1.9)

By definition, put

\[
h^*_k = s^*_{(k)}, \quad k = 1, 2, \ldots
\] (1.10)

\[
e^*_k = s^*_{(1k)}, \quad k = 1, 2, \ldots
\] (1.11)

These are shifted analogues of the complete homogeneous symmetric functions and the elementary symmetric functions.

Put also

\[
p^*_k = \sum_i ((x_i - i)^k - (-i)^k).
\] (1.12)

These are certain analogues of Newton power sums, which appeared in Olshanski’s papers [O1] and [O2].

Corollary 1.6.

1. The shifted Schur functions \( \{s^*_\mu\} \) form a linear basis in \( \Lambda^* \).
2. The algebra \( \Lambda^* \) is the algebra of polynomials in \( h^*_1, h^*_2, \ldots \) or in \( e^*_1, e^*_2, \ldots \).
3. The algebra \( \Lambda^* \) is the algebra of polynomials in \( p^*_1, p^*_2, \ldots \).

Proof. Immediately follows from Proposition 1.5 and the similar well–known claims for the algebra \( \Lambda \). □

Note also that for any fixed \( n \) the shifted Schur polynomials \( s^*_{\mu|n} \) form a linear basis in \( \Lambda^*(n) \).
Remark 1.7. The algebra $\Lambda^*$ also may be regarded as a deformation of the algebra $\Lambda$. Indeed, let $\theta$ be a number parameter. For each $n = 1, 2, \ldots$ let $\Lambda_\theta^*(n)$ be the algebra of polynomials in $x_1, \ldots, x_n$ which become symmetric in variables
\begin{equation}
  x'_i = x_i + \text{const} - i\theta, \quad i = 1, \ldots, n,
\end{equation}
and define $\Lambda_\theta^* = \lim \leftarrow \Lambda_\theta^*(n)$. Then $\Lambda_1^* = \Lambda^*$ and $\Lambda_0^* = \Lambda$. Note that the scaling
\begin{equation}
  x_i \mapsto x_i / \theta
\end{equation}
establishes an isomorphism $\Lambda_\theta^* \cong \Lambda^*$ for all nonzero $\theta$.

2. Quantum immanants

Throughout the paper we will use the following notation:

- $\mathfrak{gl}(n)$ is the general linear Lie algebra $\mathfrak{gl}(n, \mathbb{C})$,
- $E_{ij}$ are the standard generators of $\mathfrak{gl}(n)$ — the matrix units,
- $\mathcal{U}(\mathfrak{gl}(n))$ is the universal enveloping algebra of $\mathfrak{gl}(n)$,
- $\mathfrak{z}(\mathfrak{gl}(n))$ is the center of $\mathcal{U}(\mathfrak{gl}(n))$,
- $\mathcal{S}(\mathfrak{gl}(n))$ is the symmetric algebra of $\mathfrak{gl}(n)$,
- $GL(n)$ is the general linear group $GL(n, \mathbb{C})$.

Recall the construction of the Harish-Chandra isomorphism for the case of $\mathfrak{gl}(n)$, see, e.g., Bourbaki [Bou], Ch. VIII, 8.5, or Dixmier [D], 7.3.

Suppose $X \in \mathfrak{z}(\mathfrak{gl}(n))$. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, we consider an arbitrary (cyclic) highest weight $\mathfrak{gl}(n)$-module with highest weight $\lambda$ (relative to the upper triangular Borel subalgebra). Then $X$ acts in this module as a scalar operator, say $f_X(\lambda) \text{id}$. (Note that $f_X(\lambda)$ does not depend on the choice of the module.) The assignment
\[
X \mapsto f_X(\cdot)
\]
is an isomorphism of the algebra $\mathfrak{z}(\mathfrak{gl}(n))$ onto the algebra of polynomials in $\lambda \in \mathbb{C}^n$ that are symmetric in the coordinates of $\lambda' = \lambda + \rho$, where $\rho$ stands for the half–sum of positive roots. This isomorphism is called the Harish–Chandra isomorphism.

Equivalently, the Harish–Chandra isomorphism can be defined by making use of the projection
\[
\mathcal{U}(\mathfrak{gl}(n)) = (n_- \mathcal{U}(\mathfrak{gl}(n)) + \mathcal{U}(\mathfrak{gl}(n))n_+) \oplus \mathcal{U}(\mathfrak{h}) \mapsto
\quad \mapsto \mathcal{U}(\mathfrak{h}) = \mathcal{S}(\mathfrak{h}) = \mathbb{C}[\lambda_1, \ldots, \lambda_n],
\]
where $n_+$ and $n_-$ are the upper and lower triangular nilpotent subalgebras of $\mathfrak{gl}(n)$ and $\mathfrak{h}$ is the diagonal subalgebra.

Note also that
\[
\deg X = \deg f_X
\]
where $\deg X$ is the degree of $X$ with respect to the natural filtration in $\mathcal{U}(\mathfrak{gl}(n))$.

Proposition 2.1. The Harish–Chandra isomorphism is an algebra isomorphism
\[
\mathfrak{z}(\mathfrak{gl}(n)) \to \Lambda^*(n).
\]

Proof. This follows at once from the definitions, because the shift $\lambda \mapsto \lambda' = \lambda + \rho$ is of the form (1.2). (We recall that
\[
\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-3}{2}, -\frac{n-1}{2}\right).
\]
\[\square\]
Example 2.2. Take as $X$ the Casimir element

\[ C = \sum_{i,j} E_{ij} E_{ji} \in \mathfrak{z}(\mathfrak{gl}(n)). \]

Since

\[ C = \sum_i E_{ii}^2 + 2 \sum_{i>j} E_{ij} E_{ji} + \sum_{i<j} (E_{ii} - E_{jj}), \]

its image under the projection (2.1) is

\[ \sum_i E_{ii}^2 + \sum_{i<j} (E_{ii} - E_{jj}), \]

whence

\[ f_C(\lambda) = \sum \lambda_i^2 + \sum_{i<j} (\lambda_i - \lambda_j) = \sum (\lambda_i^2 + (n+1-2i)\lambda_i) \]

\[ = \sum ((\lambda_i + \frac{n+1}{2} - i)^2 - (\frac{n+1}{2} - i)^2), \]

which is a shifted symmetric polynomial in $\lambda$. 

Definition 2.3. By virtue of Proposition 2.1, for each partition $\mu$ with $\ell(\mu) \leq n$ there exists a central element

\[ S_\mu = S_{\mu|n} \in \mathfrak{z}(\mathfrak{gl}(n)) \tag{2.3} \]

corresponding to the shifted Schur polynomial $s^*_{\mu|n} \in \Lambda^*(n)$, i.e.,

$S_{\mu|n}$ is defined by

\[ f_{S_\mu}(\lambda_1, \ldots, \lambda_n) = s^*_{\mu}(\lambda_1, \ldots, l_n), \quad (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n. \tag{2.4} \]

We will call $S_\mu$ the quantum $\mu$-immanant (an explanation of this term will be given below in Remark 2.6).

Let us calculate the highest term of $S_\mu$ with respect to the natural filtration in $\mathfrak{u}(\mathfrak{gl}(n))$. Recall that there is a canonical isomorphism

\[ \text{gr } \mathfrak{u}(\mathfrak{gl}(n)) \cong S(\mathfrak{gl}(n)), \tag{2.5} \]

and denote by $I(\mathfrak{gl}(n))$ the subalgebra of invariants in $S(\mathfrak{gl}(n))$ under the adjoint action of the group $GL(n)$. We have

\[ \text{gr } \mathfrak{z}(\mathfrak{gl}(n)) \cong I(\mathfrak{gl}(n)). \tag{2.6} \]

By means of the basis $\{ E_{ij} \}$ of $\mathfrak{gl}(n)$ we can identify $\mathfrak{gl}(n)$ with its dual space (under this identification each $E_{ij}$ becomes the coordinate function $x_{ij}$). Then $S(\mathfrak{gl}(n))$ can be identified with the algebra $\mathbb{C}[\mathfrak{gl}(n)]$ of polynomial functions on $\mathfrak{gl}(n)$ and $I(\mathfrak{gl}(n))$ turns into the algebra of invariant polynomial functions.

Next, remark that the algebra $I(\mathfrak{gl}(n))$ is isomorphic to the algebra $\Lambda(n)$: the isomorphism

\[ I(\mathfrak{gl}(n)) \cong \Lambda(n) \tag{2.7} \]
is simply the restriction of invariant polynomials to the diagonal subalgebra of $\mathfrak{gl}(n)$ (the Chevalley restriction map, see [Bou], Ch. VIII, 8.3, or [D], 7.3). It is well-known that the Harish-Chandra homomorphism and the Chevalley restriction map are compatible within lower terms, i.e., the following diagram is commutative

$$\begin{array}{ccc}
gr \mathfrak{z}(\mathfrak{gl}(n)) & \longrightarrow & I(\mathfrak{gl}(n)) \\
\downarrow & & \downarrow \\
gr \Lambda^*(n) & \longrightarrow & \Lambda(n),
\end{array}$$

(2.8)

where the bottom arrow in (2.8) is the canonical isomorphism mentioned in Proposition 1.5.

Finally, let us introduce the element

$$S_\mu = S_{\mu|n} \in I(\mathfrak{gl}(n)) \subset S(\mathfrak{gl}(n)),$$

(2.9)

which corresponds to the Schur polynomial $s_{\mu|n}$ under the isomorphism (2.7). If we consider $S_\mu$ as a polynomial function $S_\mu(X)$ on $\mathfrak{gl}(n)$ then $S_\mu(X)$ is simply the Schur polynomial $s_\mu$ in the eigenvalues of the matrix $X$.

Now look at the commutative diagram (2.8). Since the highest term of $s^*_\mu$ equals $s_\mu$ we see that $S_\mu$ is the highest term of $S_\mu$.

In the next proposition we present an explicit formula for $S_\mu$. Note that this is essentially the well-known definition of the Schur function via the characteristic map, see Macdonald [M1], Ch. I, section 7.

**Proposition 2.4.** Let $\mu \vdash k$ be a partition, $\ell(\mu) \leq n$, and let $\chi^\mu$ denote the corresponding irreducible character of the symmetric group $S(k)$. Then we have

$$S_\mu = (k!)^{-1} \sum_{i_1,\ldots,i_k=1}^n \sum_{s \in S(k)} \chi^\mu(s) E_{i_1,i_{s(1)}} \cdots E_{i_k,i_{s(k)}},$$

(2.10)

or, as a polynomial invariant,

$$S_\mu(X) = (k!)^{-1} \sum_{i_1,\ldots,i_k=1}^n \sum_{s \in S(k)} \chi^\mu(s) x_{i_1,i_{s(1)}} \cdots x_{i_k,i_{s(k)}},$$

(2.11)

where $X$ is a $n \times n$ matrix and $x_{ij}$ are its entries.

**Corollary 2.5.** We have

$$S_\mu = (k!)^{-1} \sum_{i_1,\ldots,i_k=1}^n \sum_{s \in S(k)} \chi^\mu(s) E_{i_1,i_{s(1)}} \cdots E_{i_k,i_{s(k)}} + \text{lower terms}$$

(2.12)

**Proof of Proposition 2.4.** Let $V$ denote the space $\mathbb{C}^n$ considered as a natural $\mathfrak{gl}(n)$-module and let $V_\mu$ be the irreducible polynomial $\mathfrak{gl}(n)$-module, indexed by $\mu$. Both $V$ and $V_\mu$ may also be regarded as modules over the semigroup $M(n)$ of all $n \times n$ matrices. Since the Schur polynomials coincide with the characters of the irreducible polynomial modules, we have

$$S_\mu(X) = \text{tr}_{V_\mu}(X), \quad X \in M(n).$$

(2.13)
On the other hand, by the Schur–Weyl duality, $V_\mu$ occurs in the decomposition of the module $V^\otimes k$ with multiplicity $\dim \mu$, where $\dim \mu = \chi^\mu(e)$ is the dimension of the character $\chi^\mu$. It follows

$$S_\mu(X) = \frac{1}{\dim \mu} \tr_{V^\otimes k}(X^\otimes k \cdot P_\mu),$$

where $P_\mu$ stands for the projection in the tensor space $V^\otimes k$ onto the isotypic component of $V_\mu$. This projection is given by the central idempotent

$$(1/k!) \dim \mu \sum_{s \in S(k)} \chi^\mu(s) \cdot s \in \mathbb{C}[S(k)],$$

whence

$$s_\mu(X) = \tr(X^\otimes k \cdot \chi^\mu/k!)$$

$$= (1/k!) \sum_{i_1,\ldots,i_k=1}^n \sum_{s \in S(k)} \chi^\mu(s) \cdot x_{i_1,i_{s(1)}}, \ldots, x_{i_k,i_{s(k)}},$$

which proves (2.11). □

**Remark 2.6.** Given a matrix $A = [a_{ij}]$, $i, j = 1, \ldots, k$, the number

$$\sum_{s \in S(k)} \chi^\mu(s) \cdot a_{1,s(1)} \cdots a_{k,s(k)}$$

is called the $\mu$-immanant of the matrix $A$, see Littlewood [L], 6.1. If $\mu = (1^k), (k)$ then the $\mu$-immanant turns into determinant and permanent respectively. Note that (2.11) expresses the invariant polynomial $S_\mu(X)$ as sum of $\mu$-immanants of principal $k$-submatrices (possibly with repeated rows and columns) of the matrix $X$. This was one of the reasons why the central element $S_\mu$ (which may be viewed as a ‘quantum analogue’ of $S_\mu$) was called the ‘quantum immanant’. Another reason is that in the special case $\mu = (1^k), k = 1, 2, \ldots, n$, the corresponding central elements appear when expanding the so-called quantum determinant for the Yangian $Y(gl(n))$ (about this, see Nazarov [N1] and Molev–Nazarov–Olshanski [MNO]).

To find an explicit formula for $S_\mu$ is a much less trivial problem. Such a formula was obtained in [Ok1], [N2], and [Ok2]; see also Remark 14.6 and section 15.

By virtue of the Harish-Chandra isomorphism, properties of $s^*$-functions can be interpreted as properties of quantum immanants and vice versa. For example, Proposition 1.3 asserts a kind of stability for quantum immanants. This stability will be discussed below in Remark 6.9.

### 3. Characterization of $s^*$-functions

For any element $f \in \Lambda^*$ its value $f(x_1, x_2, \ldots)$ on an arbitrary infinite sequence $(x_1, x_2, \ldots)$ is well-defined provided $x_i = 0$ as $i$ is large enough. In particular, the value $f(\lambda)$ exists for any partition $\lambda$; moreover, $f$ is uniquely determined by its values on all partitions. Thus, the algebra $\Lambda^*$ can be realized as a certain function
algebra on the set of partitions. This new point of view (usually partitions appear rather as parameters than arguments) turns out to be extremely fruitful.

In this section we review some results of Okounkov’s paper [Ok1] about the $s^*$-functions viewed as functions on partitions. For completeness we give proofs.

Let $\mu$ be a partition, also viewed as a Young diagram. Denote by $H(\mu)$ the product of the hook lengths of all boxes of $\mu$,

$$H(\mu) = \prod_{\alpha \in \mu} h(\alpha). \quad (3.1)$$

Let $\lambda$ be another partition. Write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all $i = 1, 2, \ldots$.

**Theorem 3.1 (Vanishing Theorem [Ok1]).** We have

$$s^*_\mu(\lambda) = 0 \quad \text{unless} \quad \mu \subset \lambda, \quad (3.2)$$

$$s^*_\mu(\mu) = H(\mu). \quad (3.3)$$

**Proof.** Observe that

$$(a \downarrow b) = 0 \quad \text{if} \quad a, b \in \mathbb{Z}_+, b > a.$$  

Suppose $\lambda_l < \mu_l$ for some $l$ and choose an arbitrary $n \geq \max(\ell(\mu), \ell(\lambda))$. Then in the $n \times n$ matrix

$$[(\lambda_i + n - i \downarrow \mu_j + n - j)] \quad (3.4)$$

all entries with $i \geq l$ and $j \leq l$ vanish. Let us expand the determinant of the matrix (3.4) into an alternating sum of monomials in the entries of (3.4); then each monomial will contain at least one entry with $i \geq l$ and $j \leq l$. Hence the determinant of (3.4) vanishes. Since the denominator in (1.6) does not vanish when $(x_1, \ldots, x_n)$ is a partition $(\lambda_1, \ldots, \lambda_n)$, (3.2) follows.

Now take $\lambda = \mu$, and let $n \geq \ell(\mu)$. Then in the matrix

$$[(\mu_i + n - i \downarrow \mu_j + n - j)] \quad (3.5)$$

all entries with $i > j$ vanish. Hence the determinant of (3.5) equals

$$\prod_i (\mu_i + n - i)!. \quad (3.6)$$

Therefore

$$s^*_\mu(\mu) = \prod_i (\mu_i + n - i)! / \prod_{i < j} (\mu_i - \mu_j - i + j). \quad (3.7)$$

Recall that there are two formulas for $\dim \mu$ (the dimension of the irreducible representation of the symmetric group indexed by $\mu$),

$$\dim \mu = |\mu|! / H(\mu) \quad \text{(the hook formula)}$$

$$= |\mu|! \prod_{i < j} (\mu_i - \mu_j - i + j) / \prod_i (\mu_i + n - i)!. \quad (3.8)$$

Thus (3.6) equals $H(\mu)$.  \[\square\]

---

5 This realization has been considered by Kerov and Olshanski [KO].
Theorem 3.2 (Characterization Theorem I [Ok1]). The function $s^*_\mu$ is the unique element of $\Lambda^*$ such that $\deg s^*_\mu \leq |\mu|$ and
\[ s^*_\mu(\lambda) = \delta_{\mu\lambda} H(\mu) \]
for all $\lambda$ such that $|\lambda| \leq |\mu|$.

This follows from a more strong claim:

Theorem 3.3 (Characterization Theorem I' [Ok1]). Suppose $\ell(\mu) \leq n$. Then the polynomial $s^*_{\mu|n}$ is the unique element of $\Lambda^*(n)$ such that $\deg s^*_{\mu|n} \leq |\mu|$ and
\[ s^*_{\mu|n}(\lambda) = \delta_{\mu\lambda} H(\mu) \]
for all $\lambda$ such that $|\lambda| \leq |\mu|$ and $\ell(\lambda) \leq n$.

Proof. We have to prove that if $f \in \Lambda^*(n)$, $\deg f \leq |\mu|$, and $f(\lambda) = 0$ for all $\lambda$ such that $|\lambda| \leq |\mu|$, $\ell(\lambda) \leq n$, then $f = 0$. Put $k = |\mu|$. The polynomials $\{s^*_\lambda\}$, where $|\lambda| \leq k$, $\ell(\lambda) \leq n$, constitute a linear basis in the space of shifted symmetric polynomials in $n$ variables of degree $\leq k$. Hence
\[ f = \sum c_\lambda s^*_\lambda, \quad |\lambda| \leq k, \quad \ell(\lambda) \leq n, \tag{3.9} \]
for some coefficients $c_\lambda$. Let us show that $c_\lambda = 0$ for all $\lambda$. Suppose there are partitions $\nu$ such that $c_\nu \neq 0$. Choose such a partition $\nu$ so that $c_\nu \neq 0$ and $c_\eta = 0$ for all $\eta$, $|\eta| < |\nu|$, and evaluate (3.9) at $\nu$. By the Vanishing Theorem we obtain
\[ 0 = c_\nu H(\nu). \]
Thus $c_\nu = 0$, which leads to contradiction. \(\square\)

There exists a slightly different version of the Characterization Theorem:

Theorem 3.4 (Characterization theorem II [Ok1]). The function $s^*_\mu$ is the unique element of $\Lambda^*$ such that the highest term of $s^*_\mu$ is the ordinary Schur function $s_\mu$ and
\[ s^*_\mu(\lambda) = 0 \]
for all $\lambda$ such that $|\lambda| < |\mu|$.

Proof. Suppose there are two such elements $f_1$ and $f_2$ of $\Lambda^*$. Then $\deg(f_1 - f_2) < |\mu|$ and $(f_1 - f_2)(\lambda) = 0$ for all $\lambda$ such that $|\lambda| < |\mu|$. By Characterization Theorem I we have $f_1 - f_2 = 0$. \(\square\)

Note that there is an obvious analogue of Theorem 3.4 for the polynomials $s^*_{\mu|n}$.

Example 3.5. As a first application of the Characterization Theorem let us give a new proof of Proposition 1.3.

Suppose $\ell(\mu) \leq n + 1$ and consider the polynomial
\[ f = s^*_\mu(x_1, \ldots, x_n, 0). \tag{3.10} \]
Clearly it is shifted symmetric. By the Vanishing Theorem
\[ f(\lambda) = 0 \quad \text{for all } \lambda \text{ such that } \mu \not\subseteq \lambda, \ell(\lambda) \leq n, \]
\[ f(\mu) = H(\mu) \quad \text{if } \ell(\mu) \leq n. \]
Hence by Characterization Theorem I'
\[ f = \begin{cases} s^*_\mu(x_1, \ldots, x_n), & \ell(\mu) \leq n, \\ 0, & \ell(\mu) = n + 1. \end{cases} \]
Remark 3.6. The Vanishing and Characterization Theorems are a way to control lower terms of the inhomogeneous polynomials $s^*_{\mu_1|n}$. In particular cases similar arguments were used by several people (see, e.g. Howe–Umeda [HU]). In full generality these arguments were developed by Sahi [Sa1] (we learned about this important paper after work on the present paper was completed). For an arbitrary fixed decreasing sequence of real numbers $\rho = (\rho_1, \ldots, \rho_n)$, Sahi established existence and uniqueness of symmetric polynomials $p_{\mu}(y_1, \ldots, y_n)$ such that $\deg p_{\mu} \leq |\mu|$, $p_{\mu}(\lambda + \rho) = 0$ when $|\lambda| \leq |\mu|$, $\lambda \neq \mu$, and $p_{\mu}(\mu + \rho) \neq 0$. Sahi also described an inductive procedure to construct the polynomials $p_{\mu}$.

Theorem 3.3 shows that in the particular case $\rho_i - \rho_{i+1} = 1$ Sahi’s polynomials $p_{\mu}$ reduce to the factorial Schur polynomials $t_{\mu}(y_1, \ldots, y_n)$ and so admit a nice closed expression. (It seems difficult to see this directly from Sahi’s general construction.)

4. Duality

Given a partition $\mu$, we denote by $\mu'$ the dual partition (i.e., the transposed Young diagram). Recall that in the algebra $\Lambda$ of symmetric functions there exists an involutive automorphism $\omega$ such that

$$\omega(s_{\mu}) = s_{\mu'} \quad \text{for all } \mu, \quad (4.1)$$

and

$$\omega(p_k) = (-1)^{k-1}p_k \quad \text{for } k = 1, 2, \ldots, \quad (4.2)$$

see [M1], Ch. I, (2.13) and (3.8).

In this section we aim to construct a similar automorphism for the algebra $\Lambda^*$. As we will see it admits a nice interpretation when the algebra $\Lambda^*$ is realized as an algebra of functions on the set of partitions.

Consider the elements $p^*_{k}$ introduced in (1.12) and combine them into the following generating series

$$P^*(u) = \sum_{k>0} p^*_{k} u^{-(k+1)}. \quad (4.3)$$

We have

$$P^*(u) = \sum_i \sum_{k>0} ((x_i - i)^k - (-i)^k) u^{-(k+1)}$$

$$= \sum_i \left( \frac{1}{u - x_i + i} - \frac{1}{u + i} \right)$$

$$= \frac{d}{du} \log \prod_i \frac{u - x_i + i}{u + i}. \quad (4.4)$$

Define an endomorphism $\omega$ of the algebra $\Lambda^*$ on the generators $p^*_{k}$ of $\Lambda^*$ by

$$\omega P^*(u) = P^*(-u - 1), \quad (4.5)$$

where

$$\omega P^*(u) = \sum_{k>0} \omega(p^*_{k}) u^{-(k+1)}. \quad (4.6)$$
Remark that (4.5) makes sense because a formal power series in \((u + 1)^{-1}\) can be reexpanded as a formal power series in \(u^{-1}\). The mapping

\[
    u \mapsto -u - 1
\]

is involutive. Hence \(\omega\) is an involutive automorphism of \(\Lambda^*\). We have

\[
    \omega P^*(u) = \sum_i \left( \frac{1}{-u - (x_i + 1 - i)} - \frac{1}{-u - (1 - i)} \right)
\]

\[
    = \sum_i \sum_{k>0} (-1)^{k-1} ((x_i + 1 - i)^k - (1 - i)^k) u^{-(k+1)},
\]

whence

\[
    \omega(p^*_k) = (-1)^{k-1} \sum_i ((x_i + 1 - i)^k - (1 - i)^k). \tag{4.7}
\]

This is an analogue of (4.2).

**THEOREM 4.1.** Suppose \(\lambda\) is a partition. Then

\[
    [\omega(f)](\lambda) = f(\lambda') \tag{4.8}
\]

for all \(f \in \Lambda^*\).

**Proof.** Introduce two factorial analogues of the functions \(p^*_k\)

\[
    \hat{p}_k(x) = \sum_i \left( (x - i \uparrow k) - (-i \uparrow k) \right), \tag{4.9}
\]

where \((x \uparrow k) = x(x + 1)\ldots(x + k - 1)\) stands for the \(k\)-th raising factorial power of \(x\), and

\[
    \check{p}_k(x) = \sum_i \left( (x_i + 1 - i \downarrow k) - (1 - i \downarrow k) \right). \tag{4.10}
\]

Let us expand \((x \downarrow k)\) in ordinary powers of \(x\),

\[
    (x \downarrow k) = \sum_{l \leq k} c_{kl} x^l, \quad c_{kl} \in \mathbb{Z}. \tag{4.11}
\]

It is clear that

\[
    (x \downarrow k) = \sum_{l \leq k} (-1)^{k-l} c_{kl} x^l. \tag{4.12}
\]

The numbers \((-1)^{k-l} c_{kl}\) are known as Stirling numbers of the first kind. We have

\[
    \omega(\hat{p}_k) = \sum_l c_{kl} \omega(p^*_l)
\]

\[
    = \sum_l c_{kl} (-1)^{l-1} \sum_i \left( (x_i + 1 - i)^l - (1 - i)^l \right)
\]

\[
    = (-1)^{k-1} \sum_l (-1)^{k-l} c_{kl} \sum_i \left( (x_i + 1 - i)^l - (1 - i)^l \right)
\]

\[
    = (-1)^{k-1} \check{p}_k. \tag{4.13}
\]
Since the functions \( \hat{p}_k \) are generators of \( \Lambda^* \) it suffices to check that

\[
\hat{p}_k(\lambda) = (-1)^{k-1}\hat{p}_k(\lambda')
\]  

(4.14)

for all \( \lambda \) and all \( k > 0 \). In order to do this let us calculate the sum

\[
k \sum_{(i,j) \in \lambda} (j - i \mid k - 1)
\]  

(4.15)

in two ways, by making use of the following elementary fact

\[
k \sum_{l=a}^b (l \mid k - 1) = (b + 1 \mid k) - (a \mid k).
\]  

(4.16)

First sum (4.15) along the rows. Then we obtain

\[
\sum_i ((\lambda_i + 1 - i \mid k) - (1 - i \mid k)) = \hat{p}_k(\lambda).
\]

Next sum (4.15) along the columns. Then we obtain

\[
\sum_j ((j \mid k) - (j - \lambda'_j \mid k)) = (-1)^{k-1}\sum_j \left(- (j \mid k) + (\lambda'_j - j \mid k)\right) = (-1)^{k-1}\hat{p}_k(\lambda').
\]

This proves (4.14) and the theorem. □

Note that the trick with two way summation of (4.15) was used in [KO] in the proof of another identity involving \( \hat{p}(\lambda) \).

**Theorem 4.2.** For each partition \( \mu \)

\[
\omega(s^*_\mu) = s^*_{\mu'}.
\]  

(4.17)

**Proof.** This immediately follows from Theorem 4.1, Theorem 3.2, and the following obvious equality

\[
H(\mu) = H(\mu').
\]  

□

This is an exact analogue of (4.1). As for ordinary symmetric functions, we have

\[
\omega(h^*_k) = e^*_k, \quad \omega(e^*_k) = h^*_k \quad \text{for } k = 1, 2, \ldots.
\]  

(4.18)
5. Binomial Theorem

The aim of this section is to write a Taylor–type expansion of the irreducible characters of $GL(n)$ at $1 \in GL(n)$ in terms of $s^*$-functions.

Introduce some notation. Let $\mu$ be a Young diagram and $\alpha = (i, j)$ be a box of $\mu$. The number
$$c(\alpha) = j - i$$
(5.1)
is called the content of $\alpha$. Put
$$(n \uparrow \mu) = \prod_{\alpha \in \mu} (n + c(\alpha)).$$
(5.2)

This is a generalization both of raising and falling factorial powers. Indeed,
$$(n \uparrow (k)) = n(n+1)\ldots(n-k+1) \quad \text{and} \quad (n \uparrow (1^k)) = (n \downarrow k).$$

Clearly
$$(n \uparrow \mu) = \prod_i (\mu_i + n - i)! / (n - i)!.$$ 
(5.3)

By a signature we mean an arbitrary highest weight for the group $GL(n)$, i.e., an $n$-tuple
$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n,$$
such that $\lambda_1 \geq \ldots \geq \lambda_n$.

To each signature corresponds an irreducible finite-dimensional $GL(n)$-module; let $gl(n)_\lambda(X)$ denote its character and $\dim_{GL(n)} \lambda$ its dimension. We also consider $gl(n)_\lambda$ as a (rational) function $gl(n)_\lambda(z_1, \ldots, z_n)$ on the torus of diagonal matrices in $GL(n)$. When $\lambda_n \geq 0$, the character $gl(n)_\lambda$ coincides with the polynomial function $S_\lambda$ on $n \times n$ matrices, and $gl(n)_\lambda(z_1, \ldots, z_n)$ becomes the Schur polynomial $s_{\lambda|n}$ (in the general case $gl(n)_\lambda(z_1, \ldots, z_n)$ is often called the rational Schur function). The character $gl(n)_\lambda(z_1, \ldots, z_n)$ is given by exactly the same formula as $s_{\lambda|n}$:
$$gl(n)_\lambda(z_1, \ldots, z_n) = \frac{\det[z_i^{\lambda_j+n-j}]}{\det[z_i^{n-j}]}, \quad 1 \leq i, j \leq n;$$
(5.4)

this is Weyl’s character formula for $GL(n)$. For the dimension $\dim_{GL(n)} \lambda$ there are two useful expressions:
$$\dim_{GL(n)} \lambda = (n \uparrow \lambda) / H(\lambda)$$
(5.5)

$$= \prod_{i<j} (\lambda_i - \lambda_j + j - i) / \prod_i (n - i)!$$
(5.6)

(recall that $H(\lambda)$ is the product of hook lengths in $\lambda$, see (3.1)). The first expression is called the hook formula for $GL(n)$, and the second one is Weyl’s dimension formula.

**Theorem 5.1 (Binomial Theorem for $GL(n)$).** Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a signature for $GL(n)$. Then
$$\frac{gl(1 + x_1, \ldots, 1 + x_n)}{\dim_{GL(n)} \lambda} = \sum_\mu \frac{1}{(n \uparrow \mu)} s^*_\mu(\lambda_1, \ldots, \lambda_n) s_\mu(x_1, \ldots, x_n)$$
(5.7)

Recall that the symbol $(n \uparrow \mu)$ is defined by (5.2) and (5.1).
Comments to (5.7).

(1) If \( n = 1 \) then (5.7) turns into

\[
(1 + x)^k = \sum_{m \geq 0} \frac{1}{m!} (k \downarrow m) x^m.
\]  

(5.8)

This is why (5.7) is called the binomial formula.

(2) Let us regard the character \( gl(n)\lambda \) as a function on the whole group \( GL(n) \).

Then (5.7) may be written as

\[
gl(1 + X) = \dim_{GL(n)} \lambda \sum_{\mu} \frac{1}{(n \mid \mu)} s^*_\mu(\lambda) S_\mu(X),
\]  

(5.9)

where \( X \) is a \( n \times n \) matrix. This may be viewed as a Taylor–type expansion of the character at \( 1 \in GL(n) \).

(3) The binomial formula (5.7) is not new. In a different form it can be found in [M1], Ch. I, section 3, Example 10. What is new is the observation that the coefficients of the expansion are essentially the \( s^* \)-functions in \( \lambda \).

(4) Note a remarkable symmetry between \( (\lambda_1, \ldots, \lambda_n) \) and \( (x_1, \ldots, x_n) \) in the right–hand side of (5.7).

Now we will give two different proofs of Theorem 5.1: the first proof follows the arguments sketched in Macdonald [M1], Ch. I, section 3, Example 10, while the second proof is based on the Characterization Theorem.

First Proof. Put

\[
l_i = \lambda_i + n - i, \quad i = 1, \ldots, n.
\]  

(5.10)

Then by (5.8)

\[
\det \left[(1 + x_i)^{\lambda_j + n - j}\right] = \det \left[ \sum_{m \geq 0} \frac{(l_j \downarrow m_i)}{m_i!} x_i^{m_i} \right], \quad 1 \leq i, j \leq n,
\]  

(5.11)

\[
= \sum_{m_1, \ldots, m_n \geq 0} \frac{\det[(l_j \downarrow m_i)]}{m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n}.
\]  

(5.12)

Remark that the determinant in (5.12) vanishes unless the numbers \( m_1, \ldots, m_n \) are pairwise distinct; moreover it is antisymmetric with respect to permutations of these numbers. Hence

\[
\det[(1 + x_i)^{\lambda_j + n - j}] = \sum_{m_1 > \ldots > m_n \geq 0} \frac{\det[(l_j \downarrow m_i)]}{m_1! \ldots m_n!} \det[x_i^{m_i}].
\]  

(5.13)

Put \( \mu_i = m_i - n + i, \quad i = 1, \ldots, n \). We obtain from (5.4), (5.13), and (1.9)

\[
gl(n)\lambda(1 + x_1, \ldots, 1 + x_n) = \sum_{\mu} \frac{\det[(l_j \downarrow m_i)]}{m_1! \ldots m_n!} s_\mu(x_1, \ldots, x_n)
\]  

(5.14)
In particular, if \( x_1 = \ldots = x_n = 0 \) then all summands vanish except those corresponding to \( \mu = 0 \), and we obtain Weyl’s dimension formula (5.6)

\[
\dim_{GL(n)} \lambda = \frac{\det[(\lambda_j + n - j \mid n - i)]}{\prod_i(n-i)!} = \frac{\prod_{i<j}(\lambda_i - \lambda_j + j - i)}{\prod_i(n-i)!}.
\]  

(5.15)

It follows from (5.14) and (5.15) that

\[
gl(n)(1 + x_1, \ldots, 1 + x_n) = \dim_{GL(n)} \lambda \sum_{\mu} \prod_{i=1}^{n} \frac{(n-i)!}{(\mu_i + n - i)!} s_{\mu}^*(\lambda) s_{\mu}(x_1, \ldots, x_n).
\]  

(5.16)

By virtue of (5.3) this coincides with the desired formula. □

For the second proof we need some notation which will also be used later. Let \( C[GL(n)] \) be the algebra of regular functions on the affine complex algebraic group \( GL(n) \). There is a canonical pairing

\[
\langle \cdot, \cdot \rangle : \mathcal{U}(gl(n)) \otimes C[GL(n)] \to \mathbb{C},
\]  

(5.17)

which arises when one looks at elements of \( \mathcal{U}(gl(n)) \) as distributions supported at the unity.\(^6\)

Note that \( C[GL(n)] \) is formed by matrix coefficients of finite–dimensional \( GL(n) \)-modules. If \( X \) is an element of \( GL(n) \), \( V \) is a finite–dimensional \( GL(n) \)-module (also viewed as a \( gl(n) \)-module), \( \xi \in V \), \( \eta \in V^* \), and \( f_{\xi \eta} = \langle (\cdot)\xi, \eta \rangle \) is the corresponding matrix coefficient, then

\[
\langle X, f_{\xi \eta} \rangle = (X\xi, \eta),
\]  

(5.18)

where \( X \) in the right–hand side is viewed as an operator in \( V \).

By \( I(GL(n)) \) we denote the subspace of central functions in \( C[GL(n)] \); it is spanned by the irreducible characters \( gl(n)_\lambda \), where \( \lambda \) is an arbitrary signature. Note that there exists a canonical projection

\[
C[GL(n)] \to I(GL(n)),
\]  

(5.19)

the unique projection commuting with the action of \( GL(n) \) by conjugations.

Recall that by \( C[gl(n)] \) we denote the algebra of polynomial functions on \( gl(n) \), and by \( I(gl(n)) \) the subspace of invariant functions. This subspace is spanned by the polynomials \( S_{\mu \mid n} \). We have again a canonical projection (introduced in section 2)

\[
C[gl(n)] \to I(gl(n)).
\]  

(5.20)

Note that the both projections, (5.19) and (5.20), can be defined as averaging with respect to the action of the compact group \( U(n) \) by conjugations.

\(^6\)Since \( GL(n) \) is a complex Lie group and distributions are usually considered on real manifolds, one could interpret \( \mathcal{U}(gl(n)) \) as the algebra of distributions on the real form \( U(n) \subset GL(n) \), supported at \( 1 \in U(n) \). Similarly, the elements of \( C[GL(n)] \) can be viewed as functions on \( U(n) \). Finally, as the space of test functions one could take, instead of \( C[GL(n)] \), the larger space of smooth functions on the group \( U(n) \).
Second Proof of Theorem 5.1. Suppose $\mu$ ranges over the set of partitions with $\ell(\mu) \leq n$. There exist central distributions $\psi_\mu \in \mathfrak{z}(\mathfrak{gl}(n))$ such that for any $F \in I(GL(n))$
\[ F(1 + X) = \sum_\mu \langle \psi_\mu, F \rangle S_\mu(X), \quad X \in \mathfrak{gl}(n). \quad (5.21) \]
Indeed, to derive (5.21) we write the Taylor expansion of $F(1 + X)$ and then ‘average’ it using the projection (5.20).

Let $\tilde{\psi}_\mu \in \Lambda^*(n)$ correspond to $\psi_\mu$ under the Harish–Chandra isomorphism $\mathfrak{z}(\mathfrak{gl}(n)) \cong \Lambda^*(n)$, see section 2. It follows from (5.21) that $\deg \psi_\mu \leq |\mu|$, hence
\[ \deg \tilde{\psi}_\mu \leq |\mu|. \quad (5.22) \]

Let $\lambda$ be a signature, let $V_\lambda$ be the corresponding irreducible $GL(n)$-module (also viewed as a $\mathfrak{gl}(n)$-module), and let $gl(n)_\lambda(X)$ be the character of $V_\lambda$. Take $F = gl(n)_\lambda$ and remark that
\[ \langle \psi_\mu, gl(n)_\lambda \rangle = \text{tr}_{V_\lambda}(\psi_\mu) = \dim_{GL(n)} \lambda \cdot \tilde{\psi}_\mu(\lambda). \quad (5.23) \]
It follows
\[ gl(n)_\lambda(1 + X) = \dim_{GL(n)} \lambda \cdot \sum_\mu \tilde{\psi}_\mu(\lambda) S_\mu(X). \quad (5.24) \]

Now it suffices to prove that
\[ \tilde{\psi}_\mu = \frac{1}{(n \uparrow \mu) s^*_\mu |n}. \quad (5.25) \]
To do this suppose $\lambda$ is a partition ($\ell(\lambda) \leq n$), write $S_\lambda$ instead of $gl(n)_\lambda$ and rewrite (5.24) as
\[ S_\lambda(1 + X) = \dim_{GL(n)} \lambda \cdot \sum_\mu \tilde{\psi}_\mu(\lambda) S_\mu(X). \quad (5.26) \]
On the other hand, observe that
\[ S_\lambda(1 + X) = S_\lambda(X) + \text{terms of degree strictly less than } |\lambda|. \quad (5.27) \]
By comparing (5.26) and (5.27) we see that
\[ \tilde{\psi}_\mu(\lambda) = 0 \quad \text{if } |\lambda| \leq |\mu|, \lambda \neq \mu, \]
\[ \dim_{GL(n)} \mu \cdot \tilde{\psi}_\mu(\mu) = 1. \]
By using (5.22) and applying Characterization Theorem I' (Theorem 3.3) we conclude that
\[ \tilde{\psi}_\mu = \frac{H(\mu)}{\dim_{GL(n)} \mu} s^*_\mu |n = \frac{1}{(n \uparrow \mu) s^*_\mu |n} \]
(here we have used (5.5)). This proves (5.25) and completes the proof. \(\square\)
6. Eigenvalues of higher Capelli operators

Let $M(n, m)$ denote the space of $n \times m$ matrices over $\mathbb{C}$ and let $\mathbb{C}[M(n, m)]$ denote the algebra of polynomial functions on $M(n, m)$. The groups $GL(n)$ and $GL(m)$ act on $M(n, m)$ by left and right multiplications, respectively. Thus, $\mathbb{C}[M(n, m)]$ becomes a bi–module over $(GL(n), GL(m))$ and also over $(\mathcal{U}(\mathfrak{gl}(n)), \mathcal{U}(\mathfrak{gl}(m)))$ (we may arrange the actions so that this module would decompose on polynomial irreducible submodules over $\mathcal{U}(\mathfrak{gl}(n))$ and $\mathcal{U}(\mathfrak{gl}(m)))$.

Let $D(M(n, m))$ stand for the algebra of (analytic) differential operators on $M(n, m)$ with polynomial coefficients. The space $\mathbb{C}[M(n, m)]$ has a natural structure of a $D(M(n, m))$-module, and the actions of $\mathcal{U}(\mathfrak{gl}(n))$ and $\mathcal{U}(\mathfrak{gl}(m))$ in the same space can be presented as algebra morphisms

$$L : \mathcal{U}(\mathfrak{gl}(n)) \rightarrow D(M(n, m)),$$  \hspace{1cm} (6.1)

$$R : \mathcal{U}(\mathfrak{gl}(m)) \rightarrow D(M(n, m));$$  \hspace{1cm} (6.2)

$L$ and $R$ are defined on the generators of both algebras (the matrix units) as follows:

$$L(E_{pq}) = \sum_{j=1}^{m} x_{pj} \partial_{qj}, \quad 1 \leq p, q \leq n,$$  \hspace{1cm} (6.3)

$$R(E_{rs}) = \sum_{i=1}^{n} x_{ir} \partial_{is}, \quad 1 \leq r, s \leq m.$$  \hspace{1cm} (6.4)

(Here and below we denote by $x_{ij}$ the natural coordinate functions on $M(n, m)$ and by $\partial_{ij}$ — the corresponding partial derivatives.)

The aim of this section is to produce a certain differential operator $\Delta^{(n,m)}_{\mu}$, indexed by an arbitrary partition $\mu$ with $\ell(\mu) \leq \min(n, m)$, and to prove the following identity:

$$L(S_{\mu|n}) = R(S_{\mu|m}) = \Delta^{(n,m)}_{\mu}.$$  \hspace{1cm} (6.5)

Introduce two formal $n \times m$ matrices $X = [x_{ij}]$ and $D = [\partial_{ij}]$ and denote by $D'$ the transposed $m \times n$ matrix. To each partition $\mu \vdash k$, $\ell(\mu) \leq \min(n, m)$, we associate the following differential operator

$$\Delta^{(n,m)}_{\mu} = \text{tr}(X^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\mu / k!)
= (1/k!) \sum_{i_1, \ldots, i_k=1}^{n} \sum_{j_1, \ldots, j_k=1}^{m} \sum_{s \in S(k)} \chi^\mu(s) \cdot x_{i_1j_1} \ldots x_{i_kj_k} \partial_{i_{s(1)}j_1} \ldots \partial_{i_{s(k)}j_k},$$  \hspace{1cm} (6.6)

where $\chi^\mu$ is the irreducible character of the symmetric group $S(k)$, indexed by $\mu$. We call $\Delta^{(n,m)}_{\mu}$ the higher Capelli operator.

A few comments to this formula: we can write

$$X = \sum x_{ij} \otimes E_{ij} \in D(M(n, m)) \otimes \text{Hom}(\mathbb{C}^m, \mathbb{C}^n),$$  \hspace{1cm} (6.7)

$$D = \sum \partial_{ij} \otimes E_{ij} \in D(M(n, m)) \otimes \text{Hom}(\mathbb{C}^m, \mathbb{C}^n),$$  \hspace{1cm} (6.8)

$$D' = \sum \partial_{ij} \otimes E_{ji} \in D(M(n, m)) \otimes \text{Hom}(\mathbb{C}^n, \mathbb{C}^m),$$  \hspace{1cm} (6.9)
and
\[ X^{\otimes k} \in D(M(n, m)) \otimes \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k}), \]  
\[ (D')^{\otimes k} \in D(M(n, m)) \otimes \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k}), \]  
so that
\[ X^{\otimes k} \cdot (D')^{\otimes k} \in D(M(n, m)) \otimes \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k}). \]

Further, the character
\[ \chi^\mu = \sum_{s \in S(k)} \chi^\mu(s) \cdot s \in \mathbb{C}[S(k)] \]
is identified with its image in \( \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k}) \) under the action of \( S(k) \) in the tensor space \((\mathbb{C}^n)^{\otimes k}\). Thus, the product
\[ X^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\mu \]
is well-defined as an element of
\[ D(M(n, m)) \otimes \text{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes k}). \]

Finally, after taking the trace with respect to the space \((\mathbb{C}^n)^{\otimes k}\) we obtain an element of \( D(M(n, m)) \).

**Proposition 6.1.** The operator \( \Delta^{(n,m)}_{\mu} \) is \( GL(n) \times GL(m) \)-invariant.

**Proof.** Under the action of an element \((g, h) \in GL(n) \times GL(m)\) the matrices \( X \) and \( D' \) are transformed as follows
\[ X \xrightarrow{(g,h)} gXh', \quad D \xrightarrow{(g,h)} (g')^{-1}Dh^{-1}. \]

Therefore the action of \((g, h)\) maps \( \Delta^{(n,m)}_{\mu} \) to
\[
\text{tr} \left( g^{\otimes n} \cdot X^{\otimes n} \cdot (h')^{\otimes n} \cdot \left( ((h')^{-1})^{\otimes n} \cdot (D')^{\otimes n} \cdot (g^{-1})^{\otimes n} \right) \cdot \chi^\mu / k! \right) = \text{tr} \left( X^{\otimes n} \cdot (D')^{\otimes n} \cdot \chi^\mu / k! \right). \quad \square
\]

**Example 6.2.** Suppose \( \mu = (1^k) \), where \( k = 1, \ldots, \min(n,m) \), and let the indices \( i_1, \ldots, i_k \) range over \( \{1, \ldots, n\} \) while the the indices \( j_1, \ldots, j_k \) range over \( \{1, \ldots, m\} \). Then \( \Delta^{(n,m)}_{\mu} \) turns into the \( k \)-th classical Capelli operator (see Howe–Umeda [HU], Nazarov [N1])
\[
\Delta^{(n,m)}_{(1^k)} = (1/k!) \sum_{i_1, \ldots, i_k} \sum_{j_1, \ldots, j_k} \sum_{s \in S(k)} \text{sgn}(s) \cdot x_{i_1j_1} \cdots x_{i_kj_k} \partial_{i_{s(1)}j_1} \cdots \partial_{i_{s(k)}j_k}
\]
\[
= (1/k!)^2 \sum_{i_1, \ldots, i_k} \sum_{j_1, \ldots, j_k} \sum_{s,t \in S(k)} \text{sgn}(st) \cdot x_{i_1j_1} \cdots x_{i_kj_k} \partial_{i_{s(t)}j_1} \cdots \partial_{i_{s(k)}j_k}
\]
\[
= (1/k!)^2 \sum_{i_1, \ldots, i_k} \sum_{j_1, \ldots, j_k} \sum_{s,t \in S(k)} \text{sgn}(s) \text{sgn}(t) \cdot x_{i_{s(t)}j_1} \cdots x_{i_{s(k)}j_k} \partial_{i_{s(t)}j_1} \cdots \partial_{i_{s(k)}j_k}
\]
\[
= (1/k!)^2 \sum_{i_1, \ldots, i_k} \sum_{j_1, \ldots, j_k} \det \begin{bmatrix} x_{i_1j_1} & \cdots & x_{i_kj_k} \\ \vdots & \ddots & \vdots \\ x_{i_1j_1} & \cdots & x_{i_kj_k} \end{bmatrix} \det \begin{bmatrix} \partial_{i_{1}j_1} & \cdots & \partial_{i_{1}j_k} \\ \vdots & \ddots & \vdots \\ \partial_{i_{k}j_1} & \cdots & \partial_{i_{k}j_k} \end{bmatrix}
\]
\[
= \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_k} \det \begin{bmatrix} x_{i_1j_1} & \cdots & x_{i_kj_k} \\ \vdots & \ddots & \vdots \\ x_{i_1j_1} & \cdots & x_{i_kj_k} \end{bmatrix} \det \begin{bmatrix} \partial_{i_{1}j_1} & \cdots & \partial_{i_{1}j_k} \\ \vdots & \ddots & \vdots \\ \partial_{i_{k}j_1} & \cdots & \partial_{i_{k}j_k} \end{bmatrix}
\]

This is why the operators \( \Delta^{(n,m)}_{\mu} \) are called the higher Capelli operators.
Example 6.3. Suppose $\mu = (k)$, where $k = 1, 2, \ldots$. The operator $\Delta^{(n,m)}_{(k)}$ can be written as follows

$$\Delta^{(n,m)}_{(k)} = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \sum_{1 \leq j_1 \leq \ldots \leq j_k \leq m} \frac{1}{p_1!p_2!\ldots q_1!q_2!\ldots}$$

$$\cdot \text{per} \left[ \begin{array}{ccc} x_{i_1j_1} & \cdots & x_{i_kj_k} \\ \vdots & \ddots & \vdots \\ x_{i_kj_1} & \cdots & x_{i_kj_k} \end{array} \right] \text{per} \left[ \begin{array}{ccc} \partial_{i_1j_1} & \cdots & \partial_{i_1j_k} \\ \vdots & \ddots & \vdots \\ \partial_{i_kj_1} & \cdots & \partial_{i_kj_k} \end{array} \right],$$

(6.13)

where $p_1, p_2, \ldots$ and $q_1, \ldots, q_2, \ldots$ stand for the multiplicities in the multisets \{i_1, \ldots, i_k\} and \{j_1, \ldots, j_k\}, respectively, and ‘per’ means ‘permanent’. The differential operators (6.13) first appeared in Nazarov’s work [N1].

From the very definition of the operators $\Delta^{(n,m)}_{\mu}$ it readily follows that their dependence on $n$ and $m$ is not essential. Suppose $N \geq n$ and $M \geq m$. We have the natural projection

$$M(N, M) \to M(n, m)$$

(excision of the upper left corner of size $n \times m$ from a $N \times M$ matrix) and the corresponding inclusion (lifting)

$$\mathbb{C}[M(n, m)] \to \mathbb{C}[M(N, M)].$$

(6.15)

Proposition 6.4. The subspace $\mathbb{C}[M(n, m)]$ is an invariant subspace of the operator $\Delta^{(N,M)}_{\mu}$ and

$$\Delta^{(N,M)}_{\mu}|_{\mathbb{C}[M(n, m)]} = \Delta^{(n,m)}_{\mu}.$$  

(6.16)

Proof. This follows at once from the fact that for each summand occurring in the definition of the operator $\Delta^{(n,m)}_{\mu}$ (see (6.6)) the coefficient and the derivatives depend on the same set of indices. \qed

In other words, the expression (6.6) is stable as $n, m \to \infty$.

It is well known that the space $\mathbb{C}[M(n, m)]$ as $GL(n) \times GL(m)$ bi–module decomposes into a multiplicity free sum,

$$\mathbb{C}[M(n, m)] = \bigoplus_{\lambda} V_{\lambda|n} \otimes V_{\lambda|m}, \quad \ell(\lambda) \leq \min(n, m),$$

(6.17)

and the highest vectors are

$$v_\lambda = \prod_i \det \left[ \begin{array}{ccc} x_{i1} & \cdots & x_{i\ell} \\ \vdots & \ddots & \vdots \\ x_{i1} & \cdots & x_{i\ell} \end{array} \right]^{\lambda_i - \lambda_{i+1}}.$$  

(6.18)

By virtue of Proposition 6.1 each higher Capelli operator $\Delta^{(n,m)}_{\mu}$ acts in the subspace $V_{\lambda|n} \otimes V_{\lambda|m}$ as a scalar operator, say $f_\mu(\lambda)$. We can obtain the scalar $f_\mu(\lambda)$ by applying $\Delta^{(n,m)}_{\mu}$ to the highest vector $v_\lambda$. Now remark that the polynomial function $v_\lambda$ does not depend on $n$ and $m$. It follows that the scalar $f_\mu(\lambda)$ does not depend on $n, m$, too.
Proposition 6.5. For each $\mu$ the eigenvalue $f_{\mu}(\lambda)$ of the higher Capelli operator $\Delta_{\mu}^{(n,m)}$ in $V_{\lambda|n} \otimes V_{\lambda|m}$ is a shifted symmetric polynomial in $\lambda$ of degree $\leq |\mu|$.

Proof. By a well-known result (see, e.g., Howe [H]), each of the two subalgebras $L(\mathfrak{gl}(n))) \subset D(M(n,m))$, $R(\mathfrak{gl}(m)) \subset D(M(n,m))$

coincides with the centralizer of the other subalgebra in $D(M(n,m))$. Hence $\Delta_{\mu}^{(n,m)}$ belongs both to the center of $L(\mathfrak{gl}(n)))$ and $R(\mathfrak{gl}(m)))$. Take the minimal of the numbers $n, m$; assume this is $n$. Then the morphism $L$ is an embedding, so that $\Delta_{\mu}^{(n,m)}$ is the image of a central element of $\mathcal{U}(\mathfrak{gl}(n))$.\footnote{In fact the latter claim holds even if $n > m$, see Remark 6.10 below.}

But we know that the eigenvalue of each central element in $V_{\lambda|n}$ is a shifted symmetric polynomial in $\lambda$ (see section 2).

Further, we have $f_{\mu}(\lambda) = (\Delta_{\mu}^{(n,m)} v_{\lambda}(1)$. Since $\Delta_{\mu}^{(n,m)}$ is of order $|\mu|$, it follows from the form of $v_{\lambda}$ that the polynomial $f_{\mu}(\lambda)$ has degree $\leq |\mu|$. This completes the proof. \hfill \Box

Propositions 6.4 and 6.5 imply the following

Corollary 6.6. The eigenvalue $f_{\mu}(\lambda)$ is a shifted symmetric function in $\lambda$. \hfill \Box

Theorem 6.7. For each partition $\mu$ the eigenvalue $f_{\mu}(\lambda)$ of the higher Capelli operator $\Delta_{\mu}^{(n,m)}$ in the irreducible component $V_{\lambda|n} \otimes V_{\lambda|m}$ of the decomposition (6.17) coincides with $s_{\mu}^*(\lambda)$.

(It is tacitly assumed that $n, m \geq \ell(\mu)$.)

Proof. We will apply Characterization Theorem I (Theorem 3.2). Observe that the component $V_{\lambda|n} \otimes V_{\lambda|m}$ is contained in the subspace of polynomials of degree $|\lambda|$. From (6.6) it is clear that $\Delta_{\mu}^{(n,m)}$ annihilates all polynomials of degree $< |\mu|$. Therefore

$$f_{\mu}(\lambda) = 0, \quad |\lambda| < |\mu|.$$ 

Suppose $|\lambda| = |\mu| = k$. By virtue of Proposition 6.4 we may assume $m = k$. Let $\{e_i\}$, $i = 1, \ldots, n$, be the standard basis of $\mathbb{C}^n$. Embed $(\mathbb{C}^n)^{\otimes k}$ in $\mathbb{C}[M(n,k)]$ as follows:

$$e_{i_1} \otimes \cdots \otimes e_{i_k} \rightarrow x_{i_1} \cdots x_{i_k}.$$ 

This embedding is $GL(n)$-equivariant. Hence it takes the isotypic component of $V_{\lambda|n}$ in the tensor space $(\mathbb{C}^n)^{\otimes k}$ to $V_{\lambda|n} \otimes V_{\lambda|k}$.

On the other hand, it follows from (6.6) (and the fact that $\chi^\mu(s^{-1}) = \chi^\mu(s)$) that the restriction of the operator $\Delta_{\mu}^{(n,k)}$ to the image of the tensor space is reduced to the following action:

$$e_{i_1} \otimes \cdots \otimes e_{i_k} \rightarrow \sum_{s \in S(k)} \chi^\mu(s) e_{i_{s(1)}} \otimes \cdots \otimes e_{i_{s(k)}}.$$ 

I.e., the action of $\Delta_{\mu}^{(n,k)}$ turns into the action of the central element $\chi^\mu \in \mathbb{C}[S(k)]$ in $(\mathbb{C}^n)^{\otimes k}$. By the Schur–Weyl duality, the eigenvalue of $\chi^\mu$ in the isotypic component of $V_{\lambda|n}$ is the same as in the irreducible $S(k)$-module, indexed by $\lambda$. Thus, this eigenvalue (which coincides with $f_{\mu}(\lambda)$) is equal to

$$\delta_{\lambda, \mu} k! / \dim \mu = \delta_{\lambda, \mu} H(\mu).$$

By Characterization Theorem I we conclude that $f_{\mu}(\lambda) = s_{\mu}^*(\lambda)$. \hfill \Box
Corollary 6.8. The identity (6.5) holds □.

Note that in the proof of Theorem 6.7, in place of Characterization Theorem I one could use Characterization Theorem II (Theorem 3.4). Then one has to check that the highest term of \( f_\mu \) equals \( s_\mu \). This is equivalent to the fact that \( \Delta_{\mu}^{(n,m)} \) and \( L(S_\mu|_n) \) coincide up to lower terms, which can be deduced from (2.10) and (6.6), by making use of (6.3).

The identity (6.5) plus an explicit expression for the quantum immanants (which is obtained in [Ok1], [N2], and [Ok2]) can be called the higher analogue of the classical Capelli identity. (The well–known classical Capelli identity corresponds to the case \( \mu = (1^k) \).)

Remark 6.9. Let

\[
M(\infty, \infty) = \lim_{\leftarrow} M(n, m)
\]

be the space of all \( \infty \times \infty \) matrices and let

\[
\mathbb{C}[M(\infty, \infty)] = \lim_{\leftarrow} \mathbb{C}[M(n, m)]
\]

be the space of (cylindrical) polynomial functions on \( M(\infty, \infty) \). If \( \mu \) is fixed and \( n, m \to \infty \) then the collection \( \Delta_\mu = \{ \Delta_\mu^{(n,m)} \} \) can be viewed as an ‘infinite–dimensional differential operator’, acting in \( \mathbb{C}[M(\infty, \infty)] \). It was shown in Olshanski’s paper [O2] that the algebra \( \Lambda^* \) can be realized as the algebra of all differential operators on \( M(\infty, \infty) \) commuting with the action of the group

\[
GL(\infty) \times GL(\infty) = \lim_{\rightarrow} GL(n) \times GL(m).
\]

In this realization, the operators \( \Delta_\mu \) just correspond to the shifted Schur functions \( s_\mu^* \in \Lambda^* \).

Remark 6.10. Let us return to the proof of Proposition 6.5 and note that the center of the algebra \( L(U(gl(n))) \) always coincides with \( L(\mathfrak{z}(gl(n))) \), even if \( L \) is not an embedding.

To see this we use the canonical projection

\[
U(gl(n)) \to \mathfrak{z}(gl(n)), \quad (6.20)
\]

the unique projection commuting with the adjoint action of \( gl(n) \); it exists because \( U(gl(n)) \) is a semisimple \( gl(n) \)-module (cf. [Bou], Ch. VIII, 8.5, Proposition 7). Next, consider \( D(M(n, m)) \) as a \( gl(n) \)-module with respect to commutation with elements of \( L(gl(n)) \); this module also is semisimple, whence there exists a unique equivariant projection of \( D(M(n, m)) \) onto the subspace of \( gl(n) \)-invariants (i.e., onto differential operators commuting with \( L(U(gl(n))) \)). The map \( L : U(gl(n)) \to D(M(n, m)) \) is clearly compatible with these two projections.

Now take an arbitrary element \( A \) in the center of \( L(U(gl(n))) \) and choose \( B \in U(gl(n)) \) such that \( L(B) = A \). Applying to \( B \) the first projection we obtain a central element \( B' \), and \( L(B') \) coincides with the image \( A' \) of \( A \) under the second projection. But we have \( A' = A \), whence \( L(B') = A \), so that \( A \) lies in the image of the center.

Finally, note that the both projections can also be defined as averaging over the compact form \( U(n) \subset GL(n) \).
7. Capelli–type identity for Schur–Weyl duality

Recall that the Schur–Weyl duality between the general linear and symmetric groups is based on the following decomposition of the tensor space \((\mathbb{C}^n)^{\otimes l}\) as a bi–module over \((GL(n), S(l))\) \((n, l = 1, 2, \ldots)\)

\[
(\mathbb{C}^n)^{\otimes l} = \sum_{\lambda} (V_{\lambda|n} \otimes W_{\lambda}), \quad \lambda \vdash l, \ell(\lambda) \leq n. \tag{7.1}
\]

Here \(W_{\lambda}\) denotes the irreducible \(S(l)\)-module, indexed by \(\lambda\), and \(V_{\lambda|n}\), as above, is the irreducible polynomial \(GL(n)\)-module (or \(U(gl(n))\)-module), also indexed by \(\lambda\).

Let \(\tau_{GL(n)} : U(gl(n)) \to \text{End}((\mathbb{C}^n)^{\otimes l})\) \((7.2)\)

and \(\tau_{S(l)} : \mathbb{C}[S(l)] \to \text{End}((\mathbb{C}^n)^{\otimes l})\) \((7.3)\)

denote the algebra morphisms defined by the respective actions. The decomposition \((7.1)\) implies that each of the subalgebras \(\tau_{GL(n)}(U(gl(n))) \subset \text{End}((\mathbb{C}^n)^{\otimes l})\)

and \(\tau_{S(l)}(\mathbb{C}[S(l)]) \subset \text{End}((\mathbb{C}^n)^{\otimes l})\) \((7.4)\)
is the centralizer of the other. It follows that the centers of the both subalgebras \((7.4)\) are the same. But these two centers coincide with the images of the centers of the algebras \(U(gl(n))\) and \(\mathbb{C}[S(l)]\), respectively (this can be shown as in Remark 6.10 above), whence we obtain that

\[
\tau_{GL(n)}(3(gl(n))) = \tau_{S(l)}(Z(S(l))), \tag{7.5}
\]

where \(Z(S(l))\) stands for the center of the group algebra \(\mathbb{C}[S(l)]\).

A natural problem arising from \((7.5)\) is to produce ‘sufficiently many’ couples \((A, a)\) of central elements \(A \in 3(gl(n)), a \in Z(S(l))\) satisfying the identity

\[
\tau_{GL(n)}(A) = \tau_{S(l)}(a). \tag{7.6}
\]

By analogy with the duality between \(GL(n)\) and \(GL(m)\) acting in \(\mathbb{C}[M(n,m)]\) we call such identities the Capelli-type identities for the Schur–Weyl duality.

A family of identities of type \((7.6)\) was found in [KO] (see also section 15 below). Now we present another family.

**Theorem 7.1.** Let \(\mu \vdash k\) be a partition such that \(k \leq l\) and \(\ell(\mu) \leq n\), and let

\[
\text{Ind} \chi^\mu = \sum_{t \in S(l)/S(k)} t \cdot \chi^\mu \cdot t^{-1} \in Z(S(l)) \tag{7.7}
\]

be the induced character. Then

\[
\tau_{GL(n)}(S_{\mu|n}) = \tau_{S(l)}(\text{Ind} \chi^\mu / (l-k)!). \tag{7.8}
\]

**Proof.** The idea is to reduce the problem to the duality between two general linear groups, studied in section 6.
As in the proof of Theorem 6.7, let us embed the tensor space \((C^n)^\otimes l\) into the space of polynomials \(C[M(n, l)]\):

\[
\Phi : e_{r_1} \otimes \cdots \otimes e_{r_l} \to x_{r_1} \cdots x_{r_l}.
\] (7.9)

The map \(\Phi\) commutes with the action of \(U(gl(n))\). By Corollary 6.8, \(\Phi\) intertwines the operators \(\tau_{GL(n)}(S_{\mu|n})\) and \(\Delta_{\mu}^{(n,l)}\). Hence it suffices to prove that \(\Phi\) also intertwines \(\tau_{S(l)}\) and \(\Delta_{\mu}^{(n,l)}\), and this can be done by a direct computation.

Indeed, write

\[
\Delta_{\mu}^{(n,l)} = \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^n \sum_{j_1, \ldots, j_k=1}^l \sum_{s \in S(k)} \chi_{\mu}(s) \cdot x_{i_{s(1)}j_1} \cdots x_{i_{s(k)}j_k} \partial_{i_1j_1} \cdots \partial_{i_kj_k}
\]

and apply this operator to the monomial

\[
x_{r_1} \cdots x_{r_l}.
\] (7.10)

Then the result can be described as follows. We choose an arbitrary subset \(J = \{j(1) < \ldots < j(k)\} \subset \{1, \ldots, l\}\), next replace in (7.10) each of the letters \(x_{r_{j(1)}j(1)}, \ldots, x_{r_{j(k)}j(k)}\) by \(x_{r_{j(s(1))j(1)}}, \ldots, x_{r_{j(s(k))j(k)}}\), respectively, where \(s\) is an arbitrary permutation of \(1, \ldots, k\), then multiply by \(\chi_{\mu}(s)\) and sum over all \(s\) and all \(J\).

On the other hand, the same result is obtained if we first apply to the tensor \(e_{r_1} \otimes \cdots \otimes e_{r_l}\) the operator \(\tau_{S(l)}(\text{Ind} \chi_{\mu}/(l - k)!))\) and then apply the map \(\Phi\). This completes the proof.

\[\square\]

8. Dimension of skew Young diagrams

Let \(\mu \vdash k\) and \(\lambda \vdash l\) be two Young diagrams such that \(k \leq l\) and \(\mu \subset \lambda\), and let \(\lambda/\mu\) be the corresponding skew diagram. The number \(\dim \lambda/\mu\), the dimension of \(\lambda/\mu\), equals the number of standard tableaux of shape \(\lambda/\mu\), or, that is the same, the number of paths from \(\mu\) to \(\lambda\) in the Young graph (see Vershik–Kerov [VK1], [VK3]). Recall that the Young graph is the oriented graph whose vertices are Young diagrams and two diagrams \(\mu\) and \(\nu\) are connected by an edge if \(\nu\) is obtained from \(\mu\) by adding a single box.

Equivalently, in terms of the irreducible characters \(\chi_{\lambda}\) and \(\chi_{\mu}\), indexed by \(\lambda\) and \(\mu\),

\[
\dim \lambda/\mu = \langle \text{Res} \chi_{\lambda}, \chi_{\mu} \rangle_{S(k)},
\] (8.1)

where \(\text{Res} \chi_{\lambda}\) is the restriction of \(\chi_{\lambda}\) to \(S(k)\) and \(\langle \cdot, \cdot \rangle_{S(k)}\) is the standard scalar product of functions on \(S(k)\)

\[
\langle \phi, \psi \rangle_{S(k)} = \frac{1}{k!} \sum_{s \in S(k)} \phi(s) \overline{\psi(s)}.
\] (8.2)

**Theorem 8.1.** We have

\[
\frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s^*_{\mu}(\lambda)}{(l \mid k)}.
\] (8.3)

Note that this is an explicit formula for \(\dim \lambda/\mu\), because for \(\dim \lambda\) there are nice formulas (see (3.7), (3.8)). We will prove (8.3) in three different ways. The first
proof is an immediate application of the Capelli–type identity (7.8). The second proof is a direct calculation which uses only formula (8.1) and the definition (1.6) of shifted Schur polynomials. The third proof is given in section 9 below as a corollary of a Pieri–type formula for the $s^\ast$-functions.

**First Proof.** Choose an arbitrary natural $n \geq \ell(\lambda)$ and let us calculate the eigenvalues of the both sides of the identity (7.8) in the irreducible component $V_{\lambda|n} \otimes W_{\lambda}$ of the decomposition (7.1). The eigenvalue of $\tau_{GL(n)}(S_{\mu|n})$ equals the eigenvalue of $S_{\mu|n}$ in $V_{\lambda|n}$, which equals $s^\ast_{\mu}(\lambda)$ by the very definition of $S_{\mu|n}$.

The eigenvalue of $\tau_{S(l)}(\text{Ind } \chi^\mu)$ is equal to

$$\frac{1}{\dim \lambda} \text{tr}_{W_{\lambda}}(\text{Ind } \chi^\mu) = \frac{l!}{\dim \lambda} \frac{\langle \chi^\lambda, \text{Ind } \chi^\mu \rangle_{S(l)}}{\dim \lambda} = \frac{l!}{\dim \lambda} \langle \text{Res } \chi^\lambda, \chi^\mu \rangle_{S(k)} = \frac{l! \dim \lambda/\mu}{\dim \lambda},$$

(8.4)

where the two last equalities are given by Frobenius reciprocity and formula (8.1), respectively.

By virtue of the identity (7.8), we have

$$s^\ast_{\mu}(\lambda) = \frac{\dim \lambda/\mu}{(l-k)!} l!,$$

which is equivalent to (8.3). \[\square\]

**Second Proof.** By using formula (8.1), Frobenius reciprocity and the characteristic map (see Macdonald’s book [M1], Ch. I, section 7) we express $\dim \lambda/\mu$ in terms of Schur functions,

$$\dim \lambda/\mu = \langle \text{Res } \chi^\lambda, \chi^\mu \rangle_{S(k)} = \langle \chi^\lambda, \text{Ind } \chi^\mu \rangle_{S(l)} = (s_\lambda, s_\mu p^l_{1-k}),$$

(8.5)

where $(\cdot, \cdot)$ denotes the canonical inner product in the algebra $\Lambda$.

Further, we fix $n \geq \ell(\lambda)$ and remark that by a standard argument (see e.g., [M1], Ch. I, section 7, Example 7), (8.5) equals the coefficient of

$$x_{1}^{\lambda_{1}+n-1} \ldots x_{n}^{\lambda_{n}}$$

in the expansion of

$$\det[x_{i}^{\mu_{j}+n-j}]_{1 \leq i, j \leq n} \cdot (x_{1} + \ldots + x_{n})^{l-k}.$$  

(8.6)

By expanding the both factors of (8.6) we find that (8.6) can be written as

$$\sum_{s \in S(n)} \text{sgn}(s) \sum_{r_{1}+\ldots+r_{n}=l-k} \frac{(l-k)!}{r_{1}! \ldots r_{n}!} \prod_{i=1}^{n} x_{i}^{\mu_{s(i)}+n-s(i)+r_{i}},$$

In fact this was the first proof we found, and it was inspired by Macdonald’s letter [M3], see a discussion in section 15.
so that the desired coefficient is equal to

\[
\sum_{s \in S(n)} \text{sgn}(s) \frac{(l-k)!}{\prod_{i=1}^{n}(\lambda_i - \mu s(i) - i + s(i))!}
\]

\[
= (l-k)! \det \left[ \frac{1}{(\lambda_i - \mu j - i + j)!} \right]
\]

\[
= (l-k)! \det \left[ \frac{1}{((\lambda_i + n - i) - (\mu j + n - j))!} \right]
\]

\[
= \frac{(l-k)!}{l!} \prod_{p < q} (\lambda_i - q - p) \cdot \det \left[ \frac{(\lambda_i + n - i)!}{(\lambda_i + n - i)!} \right] \prod_{p < q} (\lambda_i - q - p)
\]

\[
= \frac{\dim \lambda}{(l \downarrow k)} s^*_\mu(\lambda),
\]

where we have used the definition (1.6) of the shifted Schur polynomials and formula (3.8) for \(\dim \lambda\). This completes the proof. \(\square\)

9. Pieri–type formula for \(s^*\)-functions

In this section we present one more proof of Theorem 8.1. It is based on a Pieri–type formula (9.2) which is of independent interest.

For two Young diagrams \(\mu\) and \(\nu\), let the symbol \(\mu \nearrow \nu\) mean that \(|\nu| = |\mu| + 1\) and \(\mu \subseteq \nu\) (then \(\mu\) and \(\nu\) are connected by an edge in the Young graph).

Recall that

\[
s^*_\mu(1)(x) = s_{(1)}(x) = \sum_i x_i.
\]

**Theorem 9.1.** For any Young diagram \(\mu\)

\[
s^*_\mu(p_1 - |\mu|) = \sum_{\nu, \mu \nearrow \nu} s^*_\nu.
\]

**Proof.** We shall apply Characterization Theorem II (Theorem 3.4). Let \(k = |\mu|\). The highest terms of both sides of (9.2) (which are of degree \(k + 1\)) agree by the classical Pieri formula. So it suffices to verify that the left–hand side of (9.2) vanishes at each diagram \(\eta\) with \(|\eta| \leq k\). We have

\[
s^*_\mu(\eta)(p_1(\eta) - |\mu|) = 0, \quad \text{if} \quad \eta \neq \mu,
\]

because \(s^*_\mu(\eta) = 0\) then (Theorem 3.1). Further, the same is also true for \(\eta = \mu\), because then

\[
s^*_\mu(\eta) - |\mu| = s^*_\mu(\mu) - |\mu| = 0,
\]

by (9.1). This completes the proof. \(\square\)

Now we shall deduce Theorem 8.1 from Theorem 9.1.
Third Proof of Theorem 8.1. Recall the equality to be proved: if $\mu \vdash k$, $\lambda \vdash l$, and $\mu \subset \lambda$ then
\[
\frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s^*_\mu(\lambda)}{(l \downarrow k)}.
\] (9.3)

Using the identity (9.2) $l - k$ times we get
\[
s^*_\mu(p_1 - k) \cdots (p_1 - l + 1) = \sum_{\nu \triangleright \mu, |\nu| = l} \dim \nu/\mu \cdot s^*_\nu.
\] (9.4)

Let us evaluate the both sides of (9.4) at $\lambda$. We obtain
\[
s^*_\mu(\lambda)(l - k)! = \dim \lambda/\mu \cdot s^*_\lambda(\lambda).
\] (9.5)

In particular for $\mu = \emptyset$ we obtain
\[
l! = \dim \lambda \cdot s^*_\lambda(\lambda).
\]
(This is a different proof of (3.3).) By substituting this into (9.5) we arrive to (9.3). □

10. Coherence property of quantum immanants and shifted Schur polynomials

Recall (Proposition 1.3) that the shifted Schur polynomials (just as the ordinary ones) possess the stability property
\[
s^*_{\mu|n+1}(x_1, \ldots, x_n, 0) = s^*_{\mu|n}(x_1, \ldots, x_n),
\] (10.1)

which can also be formulated in terms of bi–invariant differential operators on matrix spaces (Remark 6.9). The aim of this section is to establish another stability property of the polynomials $s^*_{\mu|n}$, which we call the coherence property. This new property is best stated in terms of the quantum immanants $S_{\mu|n}$.

Let $n = 1, 2, \ldots$. Recall (Remark 6.10) that since the adjoint representation of the Lie algebra $\mathfrak{gl}(n)$ in its enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$ is completely reducible, there exists a unique $\mathfrak{gl}(n)$-equivariant projection
\[
\mathcal{U}(\mathfrak{gl}(n)) \rightarrow \mathfrak{z}(\mathfrak{gl}(n))
\] (10.2)
on the center, which can also be defined as averaging over the compact form $U(n) \subset GL(n)$ with respect to its adjoint action,
\[
X \mapsto \int_{U(n)} \operatorname{Ad}(u) \cdot X \, du, \quad X \in \mathcal{U}(\mathfrak{gl}(n)),
\] (10.3)

where $du$ is the normalized Haar measure.

For each couple $n < N$ define the averaging operator $\operatorname{Avr}_{n,N}$,
\[
\operatorname{Avr}_{n,N} : \mathfrak{z}(\mathfrak{gl}(n)) \rightarrow \mathfrak{z}(\mathfrak{gl}(N)),
\] (10.4)
as the composition of three maps,

\[ \mathfrak{z}(\mathfrak{gl}(n)) \to \mathcal{U}(\mathfrak{gl}(n)) \to \mathcal{U}(\mathfrak{gl}(N)) \to \mathfrak{z}(\mathfrak{gl}(N)), \tag{10.5} \]

where the first and second arrows are natural embeddings and the third arrow is the projection (10.2) for \( \mathfrak{gl}(N) \).

For each \( n \) we consider the canonical pairing (5.17) between the enveloping algebra \( \mathcal{U}(\mathfrak{gl}(n)) \) and the space \( \mathbb{C}[GL(n)] \) of regular functions on the group \( GL(n) \). For \( N > n \) the embedding \( GL(n) \to GL(N) \) induces the restriction map

\[ \text{Res}_{Nn} : \mathbb{C}[GL(N)] \to \mathbb{C}[GL(n)], \tag{10.6} \]

which is dual to the embedding

\[ \mathcal{U}(\mathfrak{gl}(n)) \to \mathcal{U}(\mathfrak{gl}(N)). \tag{10.7} \]

Similarly, for each \( n \) the embedding

\[ I(GL(n)) \to \mathbb{C}[GL(n)] \tag{10.8} \]

is dual to the projection (10.2) (recall that by \( I(GL(n)) \) we denoted the subspace of central functions in \( \mathbb{C}[GL(n)] \)).

Therefore, given \( Z \in \mathfrak{z}(\mathfrak{gl}(n)) \), its image \( \text{Avr}_{nN} Z \) under (10.4) can be characterized as the unique element of \( \mathfrak{z}(\mathfrak{gl}(N)) \) satisfying

\[ \langle \text{Avr}_{nN} Z, F \rangle = \langle Z, \text{Res}_{Nn} F \rangle, \quad F \in I(GL(N)). \tag{10.9} \]

**Theorem 10.1 (Coherence Theorem).** Let \( \mu \) be a partition and let \( N > n \geq \ell(\mu) \). Then

\[ \text{Avr}_{nN} S_{\mu|n} = \frac{(n \uparrow \mu)}{(N \uparrow \mu)} S_{\mu|N}. \tag{10.10} \]

(Recall that by \( S_{\mu|n} \) we denote the quantum immanant, see Definition 2.3; the symbol \( (n \uparrow \mu) \) is defined in (5.2).)

We call the relation (10.10) the coherence property of the quantum immanants (or equivalently, of the shifted Schur polynomials). Three different proofs of this property will be given. The first and second ones are obtained by making use of the Binomial Theorem and the Characterization Theorem, respectively. The third proof is a direct computation which uses only the initial definition (1.6) of the shifted Schur polynomials; it leads to an interesting identity involving these polynomials.

**First Proof.** For arbitrary \( n \) and \( \mu \) \((\ell(\mu) \leq n)\) set

\[ \tilde{S}_{\mu|n} = \frac{1}{(n \uparrow \mu)} S_{\mu|n}. \tag{10.11} \]

We have to prove that

\[ \text{Avr}_{nN} \tilde{S}_{\mu|n} = \tilde{S}_{\mu|N}, \quad N > n. \tag{10.12} \]
By duality, this can be restated as the relation
\begin{equation}
\langle \tilde{S}_\mu|_N, F \rangle = \langle \tilde{S}_\mu|_n, \text{Res}_N F \rangle \quad \text{for each } F \in I(GL(N)). \tag{10.13}
\end{equation}

By virtue of the Binomial Theorem (Theorem 5.1) and the argument used in its second proof (see, in particular, formula (5.21)), we have the expansion
\begin{equation}
F(1 + X) = \sum_{\mu, \ell(\mu) \leq n} \langle \tilde{S}_\mu|_n, F \rangle S_{\mu|_n}(X), \quad X \in \mathfrak{gl}(n), \tag{10.14}
\end{equation}
where the invariant polynomials \( S_{\mu|_n}(X) \) were introduced in section 2.

Let us show that the desired relation (10.10) is a formal consequence of the relation (10.14) and the stability property:
\begin{equation}
X \in \mathfrak{gl}(n) \subset \mathfrak{gl}(N) \Rightarrow S_{\mu|_N}(X) = \begin{cases} S_{\mu|_n}(X), & \ell(\mu) \leq n, \\ 0, & \ell(\mu) > n. \end{cases} \tag{10.15}
\end{equation}

Indeed, let us write the \( N \)-th relation of the form (10.14),
\begin{equation}
F(1 + X) = \sum_{\mu, \ell(\mu) \leq N} \langle \tilde{S}_\mu|_N, F \rangle S_{\mu|_N}(X), \quad X \in \mathfrak{gl}(n), \quad F \in I(GL(N)), \tag{10.16}
\end{equation}
and then take \( X \in \mathfrak{gl}(n) \). By virtue of (10.15) we obtain then
\begin{equation}
\text{Res}_N F(1 + X) = \sum_{\mu, \ell(\mu) \leq n} \langle \tilde{S}_\mu|_n, F \rangle S_{\mu|_n}(X), \quad X \in \mathfrak{gl}(n). \tag{10.17}
\end{equation}

On the other hand, the \( n \)-th relation (10.14), applied to the function \( \text{Res}_N F \), gives
\begin{equation}
\text{Res}_N F(1 + X) = \sum_{\mu, \ell(\mu) \leq n} \langle \tilde{S}_\mu|_n, \text{Res}_N F \rangle S_{\mu|_n}(X), \quad X \in \mathfrak{gl}(n). \tag{10.18}
\end{equation}

Since the polynomials \( S_{\mu|_n}, \ell(\mu) \leq n \), are linearly independent, the comparison of (10.17) and (10.18) implies (10.13). This completes the proof. \( \square \)

Below we shall need the well-known branching rule for the general linear groups. Let \( \lambda \) be a signature for \( GL(n+1) \) and let \( V_{\lambda|n+1} \) be the corresponding irreducible \( GL(n+1) \)-module; then its decomposition under the action of the subgroup \( GL(n) \subset GL(n+1) \) looks as follows
\begin{equation}
V_{\lambda|n+1}|_{GL(n)} \sim \bigoplus_{\nu < \lambda} V_{\nu|n}, \tag{10.19}
\end{equation}
where \( \nu < \lambda \) means
\begin{equation}
\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \ldots \lambda_n \geq \nu_n \geq \lambda_{n+1} \tag{10.20}
\end{equation}
(see, e.g., Želobenko [Z]). In terms of characters the branching rule looks as follows
\begin{equation}
gl(n+1)_\lambda|_{GL(n)} = \sum_{\nu < \lambda} gl(n)_\nu. \tag{10.21}
\end{equation}
Second Proof. We apply Characterization Theorem I (Theorem 3.2). Let \( k = |\mu| \). The central element \( \tilde{S}_{\mu | N} \) has degree \( k \). Since the averaging operator does not raise degree, the central element \( \text{Avr}_{n N} \tilde{S}_{\mu | n} \) has degree \( \leq k \). Hence, by Characterization Theorem, it suffices to prove that the both central elements have the same eigenvalues in any irreducible polynomial module \( V_{\lambda | N} \) such that \( |\lambda| \leq k \).

The eigenvalue of \( \tilde{S}_{\mu | N} \) is equal to \( s^*_\mu(\lambda)/(N \uparrow \lambda) \). For \( \lambda \neq \mu \) this is zero, and for \( \lambda = \mu \) this number equals
\[
\frac{s^*_\mu(\mu)}{(N \uparrow \mu)} = \frac{H(\mu)}{(N \uparrow \mu)} = \frac{1}{\dim_{GL(N)} \mu}.
\] (10.22)

The eigenvalue of \( \text{Avr}_{n N} \tilde{S}_{\mu | n} \) can be written as
\[
\frac{\langle \text{Avr}_{n N} \tilde{S}_{\mu | n}, \text{gl}(N) \rangle_{\lambda}}{\dim_{GL(N)} \lambda} = \frac{\langle \tilde{S}_{\mu | n}, \text{Res}_{N n} \text{gl}(N) \rangle_{\lambda}}{(n \uparrow \mu) \dim_{GL(N)} \lambda}.
\] (10.23)

By the branching rule (10.19)
\[
\text{Res}_{N n} \text{gl}(N)_{\lambda} = \begin{cases} \text{gl}(n)_{\lambda} + \text{lower terms}, & \text{if } \ell(\lambda) \leq n, \\ \text{lower terms}, & \text{if } \ell(\lambda) > n, \end{cases}
\] (10.24)
where ‘lower terms’ means ‘a linear combination of characters \( \text{gl}(n)_{\nu} \) with \( |\nu| < k \).'

Recall that
\[
\langle \tilde{S}_{\mu | n}, \text{gl}(n)_{\nu} \rangle = \dim_{GL(n)} \nu \cdot s^*_\mu(\nu).
\]
Hence (10.24) implies that the expression (10.23) vanishes when \( \lambda \neq \mu \). If \( \lambda = \mu \) then (10.23) equals
\[
\frac{s^*_\mu(\mu) \dim_{GL(n)} \mu}{(n \uparrow \mu) \dim_{GL(N)} \mu} = \frac{H(\mu) \dim_{GL(n)} \mu}{(n \uparrow \mu) \dim_{GL(N)} \mu} = \frac{1}{\dim_{GL(N)} \mu}.
\]
Thus, we obtain the same result as for \( \tilde{S}_{\mu | N} \), which completes the proof. \( \square \)

For the third proof of Theorem 10.1 we need the following claim, which is of independent interest.

**Proposition 10.2.** Fix \( n = 1, 2, \ldots \) and introduce a linear map \( \Lambda^*(n) \rightarrow \Lambda^*(n+1) \) making the following diagram commutative:
\[
\begin{array}{ccc}
\mathfrak{g}l(n) & \xrightarrow{\text{Avr}_{n,n+1}} & \mathfrak{g}l(n+1) \\
\downarrow & & \downarrow \\
\Lambda^*(n) & \xrightarrow{=} & \Lambda^*(n+1)
\end{array}
\] (10.25)

(here the vertical arrows are the canonical algebra isomorphisms discussed in section 2). Given \( f \in \Lambda^*(n) \), let \( f' \in \Lambda^*(n+1) \) denote its image under that map. Then for each signature \( \lambda \) for \( GL(n+1) \) the following relation holds
\[
f'(\lambda_1, \ldots, \lambda_{n+1}) = \sum_{\nu < \lambda} \frac{\dim_{GL(n)} \nu}{\dim_{GL(n+1)} \lambda} f(\nu_1, \ldots, \nu_n).
\] (10.26)
Moreover, \( f' \in \Lambda^*(n+1) \) is uniquely determined by these relations.

**Proof.** Let \( Z \in \mathfrak{z}(\mathfrak{gl}(n)) \) and \( Z' \in \mathfrak{z}(\mathfrak{gl}(n+1)) \) be the central elements corresponding to \( f \) and \( f' \). Then, by virtue of (10.9), \( Z' \) is characterized by the relations

\[
\langle Z', F \rangle = \langle Z, \text{Res}_{n+1,n} F \rangle, \quad F \in I(\text{GL}(n+1)).
\]  

(10.27)

We may assume \( F \) is an irreducible character of \( \text{GL}(n+1) \), i.e., \( F = \text{gl}(n+1)_\lambda \), where \( \lambda \) is an arbitrary signature. By the branching rule (10.19),

\[
\text{Res}_{n+1,n} \text{gl}(n+1)_\lambda = \sum_{\nu \prec \lambda} \text{gl}(n)_\nu.
\]  

(10.28)

Hence (10.27) can be rewritten as

\[
\dim_{\text{GL}(n+1)} \lambda \cdot f'(\lambda_1, \ldots, \lambda_{n+1}) = \sum_{\nu \prec \lambda} \dim_{\text{GL}(n)} \nu \cdot f(\nu_1, \ldots, \nu_n),
\]  

(10.29)

which is equivalent to (10.26). \( \square \)

Note that in Theorem 10.1 we may assume without loss of generality that \( N = n+1 \). It follows that the claim of Theorem 10.1 is equivalent to the following family of relations:

\[
\frac{s^*_{\mu|n+1}(\lambda_1, \ldots, \lambda_{n+1})}{(n+1 \upharpoonright \mu)} = \sum_{\nu \prec \lambda} \frac{\dim_{\text{GL}(n)} \nu}{\dim_{\text{GL}(n+1)} \lambda} \cdot \frac{s^*_{\mu|n}(\nu_1, \ldots, \nu_n)}{(n \upharpoonright \mu)},
\]  

(10.30)

where \( n \geq \ell(\mu) \) and \( \lambda \) is an arbitrary signature for \( \text{GL}(n+1) \).

We call these the *coherence relations* for the shifted Schur polynomials.

**Third Proof of Theorem 10.1.** We aim to give a direct proof of the coherence relations (10.30). We have

\[
s^*_{\mu|n+1}(\lambda_1, \ldots, \lambda_{n+1}) = \frac{\det[(\lambda_i + n + 1 - i \upharpoonright \mu_j + n + 1 - j)]}{\prod_{i < j} (\lambda_i - \lambda_j + j - i)}, \quad 1 \leq i, j \leq n + 1,
\]

\[
s^*_{\mu|n}(\nu_1, \ldots, \nu_n) = \frac{\det[(\nu_i + n - i \upharpoonright \mu_j + n - j)]}{\prod_{i < j} (\nu_i - \nu_j + j - i)}, \quad 1 \leq i, j \leq n,
\]

\[
\dim_{\text{GL}(n+1)} \lambda = \frac{\prod_{i < j} (\lambda_i - \lambda_j + j - i)}{n!(n-1)\ldots0!},
\]

\[
\dim_{\text{GL}(n)} \nu = \frac{\prod_{i < j} (\nu_i - \nu_j + j - i)}{(n-1)\ldots0!},
\]

\[
\frac{(n+1 \upharpoonright \mu)}{(n \upharpoonright \mu)} = \prod_{i=1}^{n} \frac{\mu_i + n + 1 - i}{n - i + 1} = \frac{1}{n!} \prod_{i=1}^{n} (\mu_i + n + 1 - i)
\]

(in the last formula we have used the assumption \( \ell(\mu) \leq n \)).

It follows that the coherence relations (10.30) can be rewritten as

\[
\det[(\lambda_i + n + 1 - i \upharpoonright \mu_j + n + 1 - j)]_{1 \leq i, j \leq n+1} = \left( \prod_{j=1}^{n} (\mu_j + n + 1 - j) \right) 
\]

\[
\cdot \sum_{\nu \prec \lambda} \det[(\nu_i + n - i \upharpoonright \mu_j + n - j)]_{1 \leq i, j \leq n}.
\]  

(10.31)
Now consider the first determinant in (10.31) and remark that its last column is equal to \((1, \ldots, 1)\), because \(\mu_{n+1} = 0\). For \(i = 1, \ldots, n\) let us subtract from the \(i\)-th row the \(i+1\)-th one. Then we get the determinant of a \(n \times n\) matrix \(A = [A_{ij}]\), where

\[
A_{ij} = (\lambda_i + n + 1 - i | \mu_j + n + 1 - j) - (\lambda_{i+1} + n - i | \mu_j + n + 1 - j). \tag{10.32}
\]

Note the following identity (which we already used above, see (4.16))

\[
(b + 1 | k) - (a | k) = k \sum_{c=a}^{b} (c | k - 1), \quad b \geq a, \quad a, b \in \mathbb{Z}. \tag{10.33}
\]

By applying this identity with 
\[
a = \lambda_{i+1} + n - i, \quad b = \lambda_i + n - i, \quad c = \nu_i + n - i, \quad k = \mu_j + n + 1 - j,
\]

we get

\[
A_{ij} = (\mu_j + n + 1 - j) \sum_{\lambda_i \geq \nu_i \geq \lambda_{i+1}} (\nu_i + n - i | \mu_j + n - j). \tag{10.34}
\]

This gives an expansion of \(\det A\) which coincides with the right-hand side of (10.31). □

**Corollary 10.3.** Let \(\mu\) be a partition, \(\ell(\mu) \leq n\), and let \(\lambda\) be a signature for \(GL(N)\), where \(N > n\). Let \(V_{\lambda|N}\) be the irreducible \(GL(N)\)-module, indexed by \(\lambda\). Then

\[
\frac{\text{tr} V_{\lambda|N} S_{\mu|n}}{\dim_{GL(N)} \lambda} = \frac{(n | \mu)}{(N | \mu)} s^*_{\mu}(\lambda). \tag{10.35}
\]

□

## 11. Combinatorial formula for \(s^*\)-functions

Let \(\mu/\nu\) be a skew diagram. By definition, put

\[
(x | \mu/\nu) = \prod_{\alpha \in \mu/\nu} (x - c(\alpha)). \tag{11.1}
\]

This is a generalization of the falling factorial powers. It is clear that

\[
(x | \mu/\nu) = (x | \mu)(x | \nu)^{-1}.
\]

We have \((x | \mu) = (x | k)\) if \(\mu = (k)\). According to (10.20) we write \(\mu \succ \nu\) if \(\mu_i \geq \nu_i \geq \mu_{i+1}, \quad i = 1, 2, \ldots\). Fix some \(n\) and denote by \(\text{RTab}(\mu, n)\) the set of sequences

\[
\mu = \mu^{(1)} \succ \mu^{(2)} \succ \cdots \succ \mu^{(i)} \succ \cdots \succ \mu^{(n+1)} = \emptyset. \tag{11.2}
\]

Equivalently, elements of \(\text{RTab}(\mu, n)\) can be considered as tableaux \(T\) of shape \(\mu\) whose entries \(T(\alpha)\) (where \(\alpha \in \mu\)) belong to \(\{1, \ldots, n\}\) and weakly decrease along each row and strictly decrease down each column. We call such tableaux *reverse tableaux* of shape \(\mu\).

There is a natural inclusion

\[
\text{RTab}(\mu, n) \subset \text{RTab}(\mu, n + 1), \tag{11.3}
\]

where \(\text{RTab}(\mu, n)\) is identified with the set of sequences (11.2) such that \(\mu^{(n+2)} = \emptyset\). Denote by \(\text{RTab}(\mu)\) the union of all the sets \(\text{RTab}(\mu, n), \quad n = 1, 2, \ldots\). This set consists of all infinite sequences

\[
\mu = \mu^{(1)} \succ \mu^{(2)} \succ \cdots \succ \mu^{(i)} \succ \cdots, \tag{11.4}
\]

such that \(\mu^{(i)} = \emptyset\) for all sufficiently large \(i\).
Theorem 11.1. We have
\[ s^*_{\mu}(x_1, x_2, \ldots) = \sum_{R \in \text{Tab}(\mu)} \prod_{i \in \mu} (x_i \downarrow \mu^{(i+1)}) \]
that is,
\[ s^*_{\mu|n}(x_1, \ldots, x_n) = \sum_{T \in \text{Tab}(\mu, n)} \prod_{\alpha \in \mu} (x_T(\alpha) - c(\alpha)) \]
for all \( n \geq \ell(\mu) \).

Formula (11.5) is equivalent to the combinatorial presentation of factorial Schur polynomials (0.4),

\[ t_{\mu}(x_1, \ldots, x_n) = \sum_{T \in \text{Tab}(\mu, n)} \prod_{\alpha \in \mu} (x_T(\alpha) - T(\alpha) - c(\alpha) + 1), \]

where Tab(\( \mu \), \( n \)) denotes the set of ordinary tableaux of shape \( \mu \), see (6.16) in Macdonald [M2] and Theorem 3.2 in Chen–Louck [CL]. To establish the equivalence of (11.6) and (11.7) it suffices to use the obvious relation between both kinds of polynomials and reverse in (11.7) the order of variables (which is possible due to symmetry of the factorial Schur polynomials).

Below we present two proofs of formula (11.6). The first one is essentially the proof of Chen and Louck [CL], rewritten in terms of the shifted Schur polynomials; we give it for sake of completeness. The second proof is based on the Characterization Theorem and closely follows the argument of [Ok1], Proposition 3.8; the only difference is that we directly verify that the right–hand side of (11.6) is shifted symmetric.

Let us consider first the simplest case when \( \mu = (k) \) and we have only two variables \( x \) and \( y \). By virtue of the definition (1.6) of the shifted Schur polynomials formula (11.6) reduces in this case to the following elementary claim:

**Lemma 11.2.**

\[ \frac{(x + 1 \downarrow k + 1) - (y \downarrow k + 1)}{x - y + 1} = \sum_{l=0}^{k} (y \downarrow l)(x - l \downarrow k - l). \]

This is Lemma 2.1 in Chen–Louck [CL]; we will use it in the both proofs of the theorem.

**Proof of Lemma.** We have
\[ (x - y + 1) \sum_{l=0}^{k} (y \downarrow l)(x - l \downarrow k - l) \]
\[ = \sum_{l=0}^{k} ((y \downarrow l)(x - l \downarrow k - l)(x + 1 - l) - (y \downarrow l)(x - l \downarrow k - l)(y - l)) \]
\[ = \sum_{l=0}^{k} (y \downarrow l)(x + 1 \downarrow k - l + 1) - \sum_{l=0}^{k} (y \downarrow l + 1)(x - l \downarrow k - l). \]
It is easy to see that all summands cancel each other except

\[(x + 1 \mid k + 1) - (y \mid k + 1).\]  

**First Proof of Theorem 11.1.** It is clear that the theorem is equivalent to the following branching rule for \(s^*\)-functions:

\[
s^*_{\mu}(x_1, x_2, \ldots) = \sum_{\nu < \mu} (x_1 \mid \mu/\nu) s^*_\nu(x_2, x_3, \ldots), \tag{11.9}
\]

or equivalently,

\[
s^*_{\mu}(x_1, \ldots, x_n) = \sum_{\nu < \mu} (x_1 \mid \mu/\nu) s^*_\nu(x_2, \ldots, x_n), \quad n \geq \ell(\mu). \tag{11.10}
\]

Let us check (11.10) using the definition (1.6) of the polynomials \(s^*_\mu|n\). Consider the numerator of (1.6),

\[
\det \left[
\begin{array}{cccc}
(x_1 + n - 1 \mid \mu_1 + n - 1) & \ldots & (x_1 + n - 1 \mid \mu_n) \\
\vdots & & \vdots \\
(x_n \mid \mu_1 + n - 1) & \ldots & (x_n \mid \mu_n)
\end{array}
\right]. \tag{11.11}
\]

For all \(j = 1, \ldots, n - 1\) subtract from the \(j\)-th column of (11.11) the \((j + 1)\)-th column, multiplied by

\[
(x_1 - \mu_{j+1} + j \mid \mu_j - \mu_{j+1} + 1).
\]

Then for all \(j < n\) the \((i, j)\)-th entry of (11.11) becomes

\[
(x_i + n - i \mid \mu_{j+1} + n - j - 1)[(x_i - \mu_{j+1} + j + 1 - i \mid \mu_j - \mu_{j+1} + 1) \\
- (x_1 - \mu_{j+1} + j \mid \mu_j - \mu_{j+1} + 1)]. \tag{11.12}
\]

In particular the first row of (11.11) becomes

\[(0, \ldots, 0, (x_1 + n - 1 \mid \mu_n)).\]

Let us apply the lemma, where we substitute

\[
x = x_1 - \mu_{j+1} + j - 1, \quad k = \mu_j - \mu_{j+1},
\]

\[
y = x_i - \mu_{j+1} + j + 1 - i, \quad l = \nu_j - \mu_{j+1}.
\]

Then we obtain

\[
x - y + 1 = x_1 - x_i + i - 1
\]

\[
(x - l \mid k - l) = (x_1 - \nu_j + j - 1 \mid \mu_j - \nu_j)
\]

\[
(y \mid l) = (x_i - \mu_{j+1} + j + 1 - i \mid \nu_j - \mu_{j+1})
\]

\[
(x_i + n - i \mid \mu_{j+1} + n - j - 1)(y \mid l) = (x_i + n - i \mid \nu_j + n - j - 1),
\]

\[\square\]
whence the entry (11.12) equals
\[-(x_1 - x_i + i - 1) \sum_{\nu_j = \mu_{j+1}}^{\mu_j} (x_1 - \nu_j + j - 1 \downarrow \mu_j - \nu_j) (x_i + n - i \downarrow \nu_j + n - j - 1).\]

Therefore the determinant (11.11) equals
\[
\prod_{1 < i} (x_1 - x_i + i - 1) \sum_{\nu < \mu} (x_1 \downarrow \mu/\nu) \det [(x_{i+1} + n - i - 1 \downarrow \nu_j + n - j - 1)]_{i,j=1}^{n-1},
\]
which implies (11.10). □

Second Proof of Theorem 11.1. Denote the right-hand side of (11.6) by \(\Sigma_{\mu|n}\) or by \(\Sigma_{\mu}(x_1, \ldots, x_n)\) (below we will also need an evident generalization of this notation to skew diagrams).

To apply Theorem 3.4 we have to check that, first, \(\Sigma_{\mu|n}\) is a shifted symmetric polynomial and, second, that it vanishes at certain partitions and differs from \(s_{\mu|n}^*\) in lower terms only.

For the first claim we shall suitably modify the well-known argument of Bender and Knuth [BK], which proves symmetry of the combinatorial formula for ordinary Schur polynomials.

Let us show that \(\Sigma_{\mu|n}\) is a shifted symmetric polynomial, i.e.,
\[
\Sigma_{\mu}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \Sigma_{\mu}(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots, x_n)
\]
for \(i = 1, \ldots, n - 1\).

Let us fix some \(i\) and put \(y_1 = x_i, y_2 = x_{i+1}\). Consider the skew diagram \(\nu = \mu^{(i)}/\mu^{(i+2)}\). It suffices to prove that the expression
\[
\Sigma_{\nu|2} = \Sigma_{\nu}(y_1, y_2) = \sum_{T \in \text{RTab}(\nu, 2)} \prod_{\alpha \in \nu} (y_{T(\alpha)} - c(\alpha))
\]
satisfies the shifted symmetry condition
\[
\Sigma_{\nu}(y_1, y_2) = \Sigma_{\nu}(y_2 - 1, y_1 + 1).
\]

Note that each column in \(\nu\) may contain either 2 or 1 boxes. For each column with 2 boxes, the entries of the boxes do not depend on \(T\): by the definition of reverse tableaux we always have to put 2 over 1. Hence the contribution of any such column in the product in (11.14) has the form \((y_2 - c)(y_1 - c + 1)\), which is clearly a shifted symmetric expression. Thus, we may strike out from \(\nu\) all the columns with 2 boxes. After this operation the shape \(\nu\) becomes a horizontal strip, and the property (11.15) can be verified separately for each row of \(\nu\). This essentially means that we have reduced the problem to the case \(\nu = (k)\) when (11.15) follows from Lemma 11.2.

Let us turn to the second claim. We have to show that if \(\lambda\) is a partition with \(\ell(\lambda) \leq n\) then \(\Sigma_{\mu|n}(\lambda) = 0\) unless \(\mu \subset \lambda\). In fact we will check a stronger fact: for each \(T \in \text{RTab}(\mu, n)\)
\[
\prod_{\alpha \in \mu} (\lambda_{T(\alpha)} - c(\alpha)) = 0
\]
(11.16)
unless $\mu \subset \lambda$.

Let $\Pi_T$ denote the product in (11.16). Put $\lambda_{(i,j)} = \lambda_{T(i,j)}$. Since $T \in \text{RTab}(\mu)$ we have

$$\lambda(1,1) \leq \lambda(1,2) \leq \cdots \leq \lambda(1,\mu_1). \quad (11.17)$$

If $\Pi_T(\lambda) \neq 0$ then

$$\lambda(1,1) \neq 0, \quad \lambda(1,2) \neq 1, \quad \lambda(1,3) \neq 2, \ldots \quad (11.18)$$

By (11.17) and (11.18) we have

$$\lambda(1,1) \geq 1, \quad \lambda(1,2) \geq 2, \quad \lambda(1,3) \geq 3, \ldots \quad (11.19)$$

Again since $T \in \text{RTab}(\mu)$ we have for each $i$

$$T(1,i) < T(2,i) < \cdots < T(\mu_i',i), \quad (11.20)$$

whence by (11.19)

$$i \leq \lambda(1,i) \leq \lambda(2,i) \leq \cdots \leq \lambda(\mu_i',i) \quad (11.21)$$

By virtue of (11.20) and (11.21), for each $i$, in the diagram $\lambda$, there are at least $\mu_i'$ rows of length $\geq i$, so that $\lambda_i' \geq \mu_i'$. Thus, $\Pi_T(\lambda) \neq 0$ implies $\mu \subset \lambda$. By Theorem 3.4, $\Sigma_{\mu|n}$ equals $s_\mu^*$ up to a constant factor. In order to see that this factor equals 1 we can either compare the highest terms or calculate explicitly the unique non-vanishing summand in $\Sigma_{\mu|n}(\mu)$. □

**Corollary 11.3.** The complete shifted functions $h_r^* = s_{(k)}^*$ and the elementary shifted functions $e_r^* = s_{(1^r)}^*$ ($r = 1, 2, \ldots$) can be written as

$$h_r^*(x_1, x_2, \ldots) = \sum_{1 \leq i_1 \leq \cdots \leq i_r < \infty} (x_{i_1} - r + 1)(x_{i_2} - r + 2) \cdots x_{i_r}, \quad (11.22)$$

$$e_r^*(x_1, x_2, \ldots) = \sum_{1 \leq i_1 < \cdots < i_r < \infty} (x_{i_1} + r - 1)(x_{i_2} + r - 2) \cdots x_{i_r}. \quad (11.23)$$

□

Our next aim is to interpret formula (11.6) in terms of Gessel-Viennot nonintersecting lattice paths (see Gessel–Viennot [GV], Sagan [S]). It will be convenient for us to consider south-western paths instead of more customary north-eastern ones. We define a path in $\mathbb{Z}^2$ as a sequence

$$p = (p(l)) \in \mathbb{Z}^2, \quad l = 1, 2, \ldots,$$

such that

$$p(l + 1) - p(l) \in \{(-1, 0), (0, -1)\}.$$

According to these two possibilities we speak of a horizontal step or a vertical step, respectively. We shall say that the path $p$ ends at $(a, -\infty)$ if the $x$-coordinate of $p(l)$ equals $a$ for all sufficiently large $l$.

For a partition $\mu$, we denote by $P(\mu)$ the set of sequences $P = (p_i)$, $i = 1, 2, \ldots$, such that $p_i$ starts at $(\mu_i - i, -1)$, ends at $(-i, -\infty)$, and $p_i \cap p_j = \emptyset$ if $i \neq j$. Remark that only finitely many paths $p_i$ are not purely vertical.
To each element of $\text{RTab}(\mu)$

$$\mu = \mu^{(1)} \succ \mu^{(2)} \succ \cdots \succ \mu^{(i)} \succ \cdots$$

assign an element $P = (p_i)$ of $P(\mu)$ such that the vertical steps of the path $p_i$ are

$$\{(\mu^{(j)}_i - i, 1 - j), (\mu^{(j)}_i - i, -j)\}, \quad j = 2, 3, \ldots.$$  

By the standard arguments of the Gessel-Viennot theory (see [GV] or [S]) this is a bijection between the set of sequences (11.24) (or, equivalently, the set $\text{RTab}(\mu)$) and the set $P(\mu)$.

Next, the product over all boxes of $T \in \text{RTab}$ in (11.5) is equal to the product over all horizontal steps $\{((a, -b), (a - 1, -b))\}$ of $P$ of the following factors

$$x_b - a. \quad (11.25)$$

Denote this product by $\Pi_P(x)$. Then Theorem 11.1 can be restated as follows:

**Proposition 11.4.** In the above notation

$$s_{\mu}^*(x) = \sum_{P \in P(\mu)} \Pi_P(x). \quad (11.26)$$

□

**Example.** Suppose $\mu = (3, 2)$ and $T$ is the following element of $\text{RTab}(\mu)$

$$T = \begin{array}{ccc}
3 & 2 & 2 \\
1 & 1 &
\end{array}$$

Then the corresponding nonintersecting paths are drawn thick in the following picture

On the horizontal steps are written the corresponding factors (11.25).

Denote by $P(\mu, n)$ the set of $n$-tuples $P = (p_i)$ of non-intersecting lattice paths $p_i$ such that $p_i$ starts at $(\mu_i - i, -1)$, ends at $(-i, -n)$, and $p_i \cap p_j = \emptyset$ if $i \neq j$. Then it follows from (11.6) that

$$s_{\mu}^*(x_1, \ldots, x_n) = \sum_{P \in P(\mu, n)} \Pi_P(x). \quad (11.27)$$
Proposition 11.5. We have the following Jacobi–Trudy–type formula

\[ s_\mu(x_1, \ldots, x_n) = \det \left[ h_{\mu_i - i + j}^*(x_1 + j - 1, \ldots, x_n + j - 1) \right]. \quad (11.28) \]

Proof. The standard arguments of the Gessel–Viennot theory, see [GV] and [S], imply

\[ \sum_{P \in P(\mu, n)} \Pi_P(x) = \det \left[ \Sigma(i, j) \right], \quad (11.29) \]

where

\[ \Sigma(i, j) = \sum_p \Pi_p(x), \]

summed over all paths \( p \) from \((\mu_i - i, -1)\) to \((-j, -n)\). It follows from (11.27) that

\[ \Sigma(i, j) = h_{\mu_i - i + j}^*(x_1 + j - 1, \ldots, x_n + j - 1), \]

provided \( \mu_i - i + j \geq 0 \) and \( \Sigma(i, j) = 0 \) otherwise, which implies (11.29). \( \square \)

Another possible way to prove (11.28) is to derive it directly from the determinantal formula (1.6) for shifted Schur polynomials: this can be done using the transformations of determinants, described (in case of ordinary Schur functions) in Littlewood [L], 6.3. Note that formula (11.28) can easily be rewritten in terms of the factorial Schur polynomials \( t_\mu(x_1, \ldots, x_n) \). For these polynomials, more general results are contained in Chen–Louck [CL], Theorem 4.1; Goulden–Hamel [GH], Theorem 4.2; Macdonald [M2], (6.7), and [M1], 2nd edition, Ch. I, section 3, Example 20 (c).

However, formula (11.28) does not express the \( s^* \)-functions in terms of the generators \( h_k^* \) of the algebra \( \Lambda^* \), because the shift of arguments is not defined in \( \Lambda^* \) (in other words, formula (11.28) is not stable as \( n \to \infty \)). In section 13 below we shall derive from (11.28) a true Jacobi-Trudi formula for \( s^* \)-functions.

12. Generating series for \( h^* \) and \( e^* \)-functions

Let \( u \) be a formal variable. We shall deal with series like

\[ c_0 + \frac{c_1}{u} + \frac{c_2}{u(u - 1)} + \frac{c_3}{u(u - 1)(u - 2)} + \cdots, \quad (12.1) \]

which make sense because they can be rewritten as formal power series in \( u^{-1} \).

Introduce the following generating series for the \( h^* \) and \( e^* \)-functions:

\[ H^*(u) = \sum_{r \geq 0} \frac{h_r^*(x_1, x_2, \ldots)}{(u \downarrow r)}, \quad E^*(u) = \sum_{r \geq 0} \frac{e_r^*(x_1, x_2, \ldots)}{(u \downarrow r)}. \quad (12.2) \]

Theorem 12.1.

\[ H^*(u) = \prod_{i=1}^{\infty} \frac{u + i}{u + i - x_i}, \quad (12.3) \]

\[ E^*(u) = \prod_{i=1}^{\infty} \frac{u - i + 1 + x_i}{u - i + 1}. \quad (12.4) \]
We present two proofs: one is a direct computation and another one is based on the Characterization Theorem.

First Proof. Let us prove (12.3). If \( n = 1 \) then (12.3) takes the form (we write \( x \) instead of \( x_1 \))

\[
\sum_{r \geq 0} \frac{(x \mid r)}{(u \mid r)} = \frac{u + 1}{u + 1 - x}. \tag{12.5}
\]

This is a particular case of Gauss’ formula for the value of the hypergeometric function \( F(a, b; c; z) \) at \( z = 1 \) (see, e.g., Whittaker–Watson [WW], Part II, Section 14.11); in our case \( a = -x, \ b = 1, \ c = -u \). One also can check (12.5) directly. Indeed, let us show by induction on \( m = 1, 2, \ldots \) that

\[
\frac{u + 1}{u + 1 - x} = \sum_{r=0}^{m} \frac{(x \mid r)}{(u \mid r)} + O\left(\frac{1}{u^{m+1}}\right). \tag{12.6}
\]

For \( m = 1 \) this is trivial. Next for \( m > 1 \)

\[
\sum_{r=0}^{m} \frac{(x \mid u)}{(u \mid r)} = 1 + \frac{x}{u} \sum_{r=0}^{m-1} \frac{(x - 1 \mid r - 1)}{(u - 1 \mid r - 1)} = 1 + \frac{x}{u} \left(\frac{u}{u + 1 - x} + O\left(\frac{1}{u^m}\right)\right) \text{ by assumption}
\]

\[
= 1 + \frac{x}{u + 1 - x} + O\left(\frac{1}{u^{m+1}}\right).
\]

Thus we have checked (12.3) for \( n = 1 \).

Now we use induction on \( n \). It follows from (11.22) that

\[
h^*_r(x_1, \ldots, x_n) = \sum_{\substack{p, q \geq 0 \atop p+q=r}} h^*_p(x_1 - q, \ldots, x_{n-1} - q)(x_n \mid q). \tag{12.7}
\]

Therefore,

\[
\sum_{r \geq 0} \frac{h^*_r(x_1, \ldots, x_n)}{(u \mid r)} = \sum_{\substack{p, q \geq 0 \atop p+q=r}} \frac{h^*_p(x_1 - q, \ldots, x_{n-1} - q)(x_n \mid q)}{(u \mid p + q)}
\]

\[
= \sum_{q \geq 0} \frac{(x_n \mid q)}{(u \mid q)} \sum_{p \geq 0} \frac{h^*_p(x_1 - q, \ldots, x_{n-1} - q)}{(u - q \mid p)}
\]

\[
= \sum_{q \geq 0} \frac{(x_n \mid q)}{(u \mid q)} \prod_{i=1}^{n-1} \frac{u - q + i}{u + i - x_i}, \tag{12.8}
\]

where the last equality holds by the inductive assumption. It is easy to see that

\[
\frac{(u - q + n - 1) \ldots (u - q + 1)}{u(u - 1) \ldots (u - q + 1)} = \frac{(u + n - 1) \ldots (u + 1)}{(u + n - 1) \ldots (u - q + n)}. \]
Therefore, (12.8) equals
\[
\left( \prod_{i=1}^{n-1} \frac{u+i}{u+i-x_i} \right) \sum_{q \geq 0} \frac{(x_n \mid q)}{(u+n-1 \mid q)}
= \left( \prod_{i=1}^{n-1} \frac{u+i}{u+i-x_i} \right) \frac{u+n}{u+n-x_n} = \prod_{i=1}^{n} \frac{u+i}{u+i-x_i},
\]
as desired. This proves (12.3).

The proof of (12.4) is similar and even simpler. It suffices to check the identity
\[
\sum_{r=0}^{n} \frac{e_r^*(x_1, \ldots, x_n)}{(u \mid r)} = \prod_{i=1}^{n} \frac{u-i+1+x_i}{u-i+1}.
\]

We again use induction on \(n\). For \(n = 1\) the identity is trivial. By virtue of (11.23),
\[
e_r^*(x_1, \ldots, x_n) = e_r^*(x_1, \ldots, x_{n-1}) + e_{r-1}^*(x_1, \ldots, x_{n-1}+1)x_n.
\]

It follows
\[
\sum_{r=0}^{n} \frac{e_r^*(x_1, \ldots, x_n)}{(u \mid r)} = \sum_{r=0}^{n-1} \frac{e_r^*(x_1, \ldots, x_{n-1})}{(u \mid r)} + x_n \sum_{r=1}^{n} \frac{e_{r-1}^*(x_1, \ldots, x_{n-1}+1)}{(u \mid r)}.
\]

Using the relation \((u \mid r) = u(u-1 \mid r-1), r \geq 1\), we can rewrite this as
\[
\sum_{r=0}^{n-1} \frac{e_r^*(x_1, \ldots, x_{n-1})}{(u \mid r)} + \frac{x_n}{u} \sum_{r=0}^{n-1} \frac{e_{r}^*(x_1+1, \ldots, x_{n-1}+1)}{(u-1 \mid r)}.
\]

By the inductive assumption, (12.12) is equal to
\[
\prod_{i=1}^{n-1} \frac{u-i+1+x_i}{u-i+1} + \frac{x_n}{u} \prod_{i=1}^{n-1} \frac{u-i+1+x_i}{u-i} = (1 + \frac{x_n}{u-n+1}) \prod_{i=1}^{n-1} \frac{u-i+1+x_i}{u-i+1} = \prod_{i=1}^{n} \frac{u-i+1+x_i}{u-i+1}.
\]

\[\square\]

Note that (12.4) also follows from a relation proved by Macdonald, see [M2], (6.5), or [M1], 2nd edition, Chapter I, Section 3, Example 20 (a).

For the second proof of Theorem 12.1 we need the following evident lemma.

**Lemma 12.2.** Let \(f(u)\) be a polynomial of degree \(\leq k+1\) and write
\[
F(u) = \frac{f(u)}{u(u-1) \ldots (u-k)}.
\]

Then
\[
F(u) = c_0 + \frac{c_1}{u} + \frac{c_2}{u(u-1)} + \ldots + \frac{c_{k+1}}{u(u-1) \ldots (u-k)},
\]

where \(c_0, c_1, \ldots, c_{k+1}\) are constants.
where \(c_i\) are some coefficients.

**Proof.** It suffices to write

\[
f(u) = c_{k+1} + c_k(u-k) + c_{k-1}(u-k)(u-k+1) + \ldots + c_0(u-k)(u-k+1) \ldots (u-1)u.
\]

\[\square\]

**Second Proof of Theorem 12.1.** Let us prove (12.3). It is clear that

\[
\prod_{i=1}^{\infty} \frac{u + i}{u + i - x_i} = c_0(x) + \frac{c_1(x)}{u} + \frac{c_2(x)}{u(u-1)} + \frac{c_3(x)}{u(u-1)(u-2)} + \ldots ,
\]

(12.15)

where \(c_i(x)\) are certain shifted symmetric functions in \(x\). By Theorem 3.4, it suffices to check two claims: first, the highest term of \(c_k\) is equal to \(h_k\), \(k = 0, 1, 2, \ldots\), and, second,

\[c_k(\lambda) = 0 \quad \text{for a partition } \lambda \text{ such that } |\lambda| < k.\]

(12.16)

Put \(x_i = ut\xi_i\) and let \(u \to \infty\). Then the left–hand side of (12.15) turns into

\[
\prod_i (1 - t\xi_i)^{-1},
\]

which implies the first claim.

Next, substitute \(x = \lambda\) into (12.15). Fix an arbitrary \(n \geq \ell(\lambda)\). Then the left–hand side of (12.15) can be written as

\[
\frac{(u + 1)(u + \ldots + u + n)}{(u - (\lambda_n - n)) \ldots (u - (\lambda_1 - 1))}.
\]

(12.17)

Note that the factors in the denominator are of the form \(u - r\), where the \(r\)'s are pairwise distinct integers from \(-n, -n+1, \ldots, \lambda_1 - 1\). If \(r < 0\) then the corresponding factor cancels with a certain factor in the numerator. After all cancellations only factors with \(r \in \{0, \ldots, \lambda_1 - 1\}\) will remain (in the denominator). Hence the product (12.17) can be rewritten in the form (12.13) with \(k \leq \lambda_1 - 1\), so that, by Lemma 12.2,

\[c_{\lambda_1+1}(\lambda) = c_{\lambda_1+2}(\lambda) = \ldots = 0.\]

Thus, \(c_k(\lambda) = 0\) when \(\lambda_1 < k\), which implies our second claim.

For (12.4) the argument is similar and even a bit simpler. The key observation is that if \(\lambda\) is a partition and \(n = \ell(\lambda)\) then

\[
\prod_{i=1}^{\infty} \frac{u - i + 1 + \lambda_i}{u - i + 1} = \frac{(u + \lambda_1) \ldots (u - n + 1 + \lambda_n)}{u(u-1) \ldots (u - n + 1)},
\]

so that, by Lemma 12.2, the corresponding coefficients \(c_k\) vanish at \(\lambda\) if \(k > \ell(\lambda)\) and hence if \(|\lambda| < k.\) \[\square\]
Corollary 12.3. The generating series for the $h^*$ - and $e^*$ -functions satisfy the relation

$$H^*(u)E^*(-u - 1) = 1. \quad (12.18)$$

□

Recall that in section 1 we have introduced certain generators $p_k^*$ of the algebra $\Lambda^*$ (see (1.12)). We have also introduced their generating series $P^*(u)$, see (4.3) and (4.4). By comparing (4.4) and (12.3) we see that

$$P^*(u) = \frac{d}{du} \log H^*(u). \quad (12.19)$$

Similarly, (4.4) and (12.4) imply

$$P^*(u) = -\frac{d}{du} \log E^*(-u - 1). \quad (12.20)$$

Now we would like to remark that this agrees with the following facts, proved in section 4:

$$\omega P^*(u) = P^*(-u - 1), \quad \omega H^*(u) = E^*(u). \quad (12.21)$$

Thus, each of identities (12.3), (12.4), combined with the relation $\omega H^*(u) = E^*(u)$, implies the other.

Besides $\{p_k^*\}$, there is one more family of elements in $\Lambda^*$ which also may be viewed as an analogue of the Newton power sums. We denote these new elements by $p_k^0$, $k = 1, 2, \ldots$, and define them in terms of their generating series

$$P^0(u) = \sum_{k \geq 1} p_k^0 u^{-k} \quad (12.22)$$

as follows

$$1 + u^{-1} P^0(u) = \frac{H^*(u)}{H^*(u - 1)} = \frac{E^*(-u)}{E^*(-u - 1)}. \quad (12.23)$$

Clearly, the highest term of $p_k^0$ (as that of $p_k^*$) is the Newton power sum $p_k$. An advantage of the elements $p_k^0$ is that we have from (12.23) and the relation $\omega H^*(u) = E^*(u)$

$$\omega P^0(u) = -P^0(-u), \quad (12.24)$$

so that

$$\omega p_k^0 = (-1)^{k-1} p_k^0, \quad k = 1, 2, \ldots, \quad (12.25)$$

just as for the customary power sums.

Remark 12.4. The generating series $E^*(u)$ is closely related to the classical Capelli identity and the so called quantum determinant for the Yangian of $\mathfrak{gl}(n)$, see Howe–Umeda [HU] and Nazarov [N1]. As was shown by Nazarov [N1], the series $H^*(u)$ also appears in a Capelli identity; it arises when inverting the quantum determinant. It should be mentioned that the form of the generating series $H^*(u)$ and $E^*(u)$ was suggested us by Nazarov’s work [N1]. Finally, note that the series $P^0(u)$ appears in the computation of the square of the antipode in the Yangian, see section 6 in Molev–Nazarov–Olshanski [MNO].
13. Jacobi–Trudi formula for \( s^* \)-functions

Define an automorphism \( \phi \) of the algebra \( \Lambda^* \) by

\[
\phi H^*(u) = H^*(u - 1),
\]

where \( H^*(u) \) is the generating series (12.2) and

\[
\phi H^*(u) = \sum_{k \geq 0} \phi(h^*_k) (u \downarrow k)^{-1}.
\]

From the identity

\[
\frac{1}{(u - 1 \downarrow k)} = \frac{1}{(u \downarrow k)} + \frac{k}{(u \downarrow k + 1)}
\]

it follows that

\[
\phi(h^*_k) = h^*_k + (k - 1)h^*_k-1, \quad k = 1, 2, \ldots.
\]

Iterating (13.4) we obtain

\[
\phi^r(h^*_k) = \sum_{i=0}^{r} \binom{r}{i} (k - 1 \downarrow i) h^*_{k-i}, \quad r = 1, 2, \ldots
\]

From (12.18) it follows that

\[
\phi E^*(u) = E^*(u + 1).
\]

Therefore, by the very definition of \( E^*(u) \), see (12.2),

\[
\phi^{-1}(e^*_k) = e^*_k + (k - 1)e^*_{k-1}, \quad r = 1, 2, \ldots
\]

and

\[
\phi^{-r}(e^*_k) = \sum_{i=0}^{r} \binom{r}{i} (k - 1 \downarrow i) e^*_{k-i}.
\]

We have the following analogues of the classical Jacobi–Trudi and Nägelsbach–Kostka formulas.

**Theorem 13.1.**

\[
s^*_\mu = \det \left[ \phi^{j-i}h^*_{\mu_i-i+j} \right]_{1 \leq i,j \leq l}
\]

\[
s^*_\mu = \det \left[ \phi^{1-j}e^*_{\mu_i' - i+j} \right]_{1 \leq i,j \leq m}
\]

for arbitrary \( l, m \) such that \( l \geq \ell(\mu) \) and \( m \geq \mu_1 \).

**Proof.** By virtue of (13.4) and (13.7), the action of \( \phi \) on the \( h^*_k \)'s is the same as that of \( \phi^{-1} \) on the \( e^*_k \)'s. Hence, by duality (4.17), it suffices to check one of these two formulas. We shall deduce formula (13.9) from (11.28).
We can assume that we have a finite number of variables $x_1, \ldots, x_n$. Define in the space $\Lambda^*(n)$ the ‘shift operator’ $\tau$,

$$[\tau f](x_1, \ldots, x_n) = f(x_1 + 1, \ldots, x_n + 1). \quad (13.11)$$

Then

$$\tau^r H^*(u) = \prod_{i=1}^n \frac{u + i}{u + i - x_i - r} = \prod_{i=1}^n \frac{u + i}{u + i - r} \prod_{i=1}^n \frac{u - r + i}{u - r + i - x_i} = \frac{(u + n \downarrow r)}{(u \downarrow r)} H^*(u - r). \quad (13.12)$$

Next consider the following ‘lowering’ operator, acting in the linear span of the polynomials $h^*_k \in \Lambda^*(n)$,

$$\sigma : h^*_k \mapsto h^*_{k-1}. \quad (13.13)$$

It is readily seen that

$$\sigma^r H^*(u) = \frac{1}{(u \downarrow r)} H^*(u - r). \quad (13.14)$$

Let us view $\tau$ as an operator in the linear span of the $h^*_k$. Clearly $\tau$ and $\sigma$ commute.

Now we claim that for any $r = 1, 2, \ldots$

$$H^*(u - r) = \tau^r H^*(u) + \sum_{i=1}^r c_i \tau^{r-i} \sigma^i H^*(u), \quad (13.15)$$

where $c_i$ are certain number coefficients which depend on $r$ and $n$. Indeed, by (13.12) and (13.14)

$$\tau^{r-i} \sigma^i H^*(u) = \frac{(u + n \downarrow r - i)}{(u \downarrow r)} H^*(u - r)$$

and hence (13.15) reduces to the evident identity

$$(u \downarrow r) = (u + n \downarrow r) + \sum_{i=1}^r c_i (u + n \downarrow r - i).$$

Now we are in a position to derive (13.9) from (11.28). Abbreviate $h^*_r = h^*_r(x_1, \ldots, x_n)$. We have to prove that

$$\det[\phi^{j-1} h^*_{\mu_i - i+j}]_{1 \leq i, j \leq l} = \det[\tau^{j-1} h^*_{\mu_i - i+j}]_{1 \leq i, j \leq l}. \quad (13.16)$$

To do this we shall show that for any $j = 2, 3, \ldots, l$, the $j$-th column of the matrix in the left–hand side of (13.16) is equal to the $j$-th column of the matrix in the right–hand side of (13.16) plus a linear combination of the preceding columns.

Fix some $j$ and denote by $H_j$ the $j$-th column of the matrix $[h^*_{\mu_i - i+j}]$. Then the first $j$ columns of the matrix $[\tau^{j-1} h^*_{\mu_i - i+j}]$ equal

$$\sigma^{j-1} H_j, \tau \sigma^{j-2} H_j, \ldots, \tau^{j-1} H_j.$$ 

By (13.15)

$$\phi^{j-1} H_j = \tau^{j-1} H_j + \sum_{i=1}^{j-1} c_i \tau^{j-i} \sigma^i H_j.$$ 

This concludes the proof. □
Remark 13.2. The equivalence of (13.9) and (13.10) can also be deduced from a general result due to Macdonald ([M2], Section 9, or [M1], 2nd edition, Chapter I, Section 3, Example 21). Macdonald’s argument also implies that for the $s^*$-functions there exists a precise analogue of the classical Giambelli formula: write $\mu$ in Frobenius notation,

$$\mu = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r);$$

then

$$s^*_\mu = \det[s^*_{\alpha_i \mid \beta_j}]. \quad (13.17)$$

14. Special symmetrization

In this section we discuss a relation between the results of section 6 and a certain linear isomorphism

$$\sigma : S(\mathfrak{gl}(n)) \rightarrow \mathcal{U}(\mathfrak{gl}(n)). \quad (14.1)$$

This isomorphism was introduced by Olshanski [O2] in connection with infinite-dimensional Lie algebras and Yangians. Then $\sigma$ was used in Kerov–Olshanski [KO]; there it was called the special symmetrization. We also present some new results about the map $\sigma$.

The construction of the special symmetrization is based on the fact that the Lie algebra structure of $\mathfrak{gl}(n)$ is obtained from the associative algebra structure on the space of $n \times n$ matrices. To define $\sigma$ let us view the enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$ as the algebra of the (complex) distributions on the real Lie group $GL(n, \mathbb{R})$, supported at the unity, and, similarly, identify the symmetric algebra $S(\mathfrak{gl}(n))$ with the algebra of (complex) distribution on the additive Lie group $\mathfrak{gl}(n)$, supported at zero. Further, introduce a local chart on $GL(n, \mathbb{R})$, centered at the unity, by using the map $X \rightarrow 1 + X$ from a neighborhood of $0 \in \mathfrak{gl}(n)$ to $GL(n, \mathbb{R})$. This chart allows us to identify the both algebras of distributions as vector spaces, and we take this identification as the map $\sigma$. (This definition is equivalent to that given in [O2], 2.2.11.)

Note that $\sigma$ differs from the customary symmetrization map $S(\mathfrak{gl}(n)) \rightarrow \mathcal{U}(\mathfrak{gl}(n))$ (see, e.g., Dixmier [D], 2.4.6) but shares several its properties. In particular, $\sigma$ preserves highest terms and commutes with the adjoint action of the group $GL(n)$.

We maintain the notation of sections 2 and 6. Let us use the map $R$ to identify $\mathcal{U}(\mathfrak{gl}(n))$ with the algebra of left-invariant differential operators on the space of $n \times n$ matrices (see (6.4), where we shall assume $m = n$). Then we have a surprisingly simple formula

$$R(\sigma(E_{i_1j_1} \ldots E_{i_kj_k})) = \sum_{\alpha_1, \ldots, \alpha_k = 1}^n x_{\alpha_1i_1} \ldots x_{\alpha_ki_k} \partial_{\alpha_1j_1} \ldots \partial_{\alpha_kj_k} \quad (14.2)$$

(see [O2], Lemma 2.2.12).

Since $\sigma$ is $GL(n)$-equivariant, it establishes a linear isomorphism

$$\sigma : I(\mathfrak{gl}(n)) \rightarrow \mathfrak{z}(\mathfrak{gl}(n)). \quad (14.3)$$

We recall that $I(\mathfrak{gl}(n))$ stands for the subalgebra of $GL(n)$-invariants in $S(\mathfrak{gl}(n))$. Recall also that we have defined natural bases $\{S_{\mu|n}\} \subset I(\mathfrak{gl}(n))$ and $\{S_{\mu|n}\} \subset \mathfrak{z}(\mathfrak{gl}(n))$, see (2.9) and Definition 2.3; here $\mu$ ranges over the set of partitions with $\ell(\mu) \leq n$. 

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**Theorem 14.1.** The special symmetrization \( \sigma \) takes each \( S_{\mu|\nu} \in I(\mathfrak{gl}(n)) \) to \( S_{\mu|\nu} \in \mathfrak{S}(\mathfrak{gl}(n)) \).

**Proof.** It suffices to prove that \( R(\sigma(S_{\mu|\nu})) \) coincides with \( R(S_{\mu|\nu}) \).

Let \( k = |\mu| \). By Proposition 2.4 and formula (14.2), \( R(\sigma(S_{\mu|\nu})) \) is equal to to

\[
(k!)^{-1} \sum_{\alpha_1, \ldots, \alpha_k=1}^{n} \sum_{i_1, \ldots, i_k=1}^{n} \sum_{s \in \mathcal{S}(k)} \chi^\mu(s) \cdot x_{\alpha_1 i_1} \ldots x_{\alpha_k i_k} \partial_{\alpha_1 i_{s(1)}} \ldots \partial_{\alpha_k i_{s(k)}}. \tag{14.4}
\]

Next, by the identity (6.5), proved in Corollary 6.8, \( R(S_{\mu|\nu}) \) is the differential operator \( \Delta_{\mu}^{(n,n)} \), which is given by (6.6), where we have to set \( m = n \):

\[
\Delta_{\mu}^{(n,n)} = (k!)^{-1} \sum_{i_1, \ldots, i_k=1}^{n} \sum_{j_1, \ldots, j_k=1}^{n} \sum_{s \in \mathcal{S}(k)} \chi^\mu(s) \cdot x_{i_1 j_1} \ldots x_{i_k j_k} \partial_{i_{s(1)} j_1} \ldots \partial_{i_{s(k)} j_k}. \tag{14.5}
\]

Now it is easy to see that the both expressions, (14.4) and (14.5), are the same (it suffices to reorder the derivatives in (14.5) and then change the notation of the subscripts). This completes the proof. \( \square \)

There is a formula for the inverse map \( \sigma^{-1} : U(\mathfrak{gl}(n)) \to S(\mathfrak{gl}(n)) \), obtained in [O2], 2.2.13. Let us fix \( k = 1, 2, \ldots \) and denote by \( \Xi(k) \) the set of all partitions \( \xi \) of the set \( \{1, \ldots, k\} \) into disjoint subsets \( \{i_1 < i_2 < \ldots\}, \{j_1 < j_2 < \ldots\}, \ldots \), called the clusters of \( \xi \). Suppose we have a monomial \( A_1 \circ \ldots \circ A_k \in U(\mathfrak{gl}(n)) \) (we write “\( \circ \)” for the multiplication in \( U(\mathfrak{gl}(n)) \) to distinguish it from the multiplication in \( S(\mathfrak{gl}(n)) \)). Put

\[
\Pi_\xi(A_1, \ldots, A_k) = \langle A_{i_1} A_{i_2} \ldots \rangle \cdot \langle A_{j_1} A_{j_2} \ldots \rangle \ldots \in S(\mathfrak{gl}(n)), \tag{14.6}
\]

where \( \langle A_{i_1} A_{i_2} \ldots \rangle \in \mathfrak{gl}(n) \) is the ordinary matrix product of \( A_{i_1}, A_{i_2}, \ldots \) (that is, the product in the associative algebra of \( n \times n \) matrices). Then

\[
\sigma^{-1}(A_1 \circ \ldots \circ A_k) = \sum_{\xi \in \Xi(k)} \Pi_\xi(A_1, \ldots, A_k). \tag{14.7}
\]

For instance,

\[
\sigma^{-1}(A_1) = A_1,
\]
\[
\sigma^{-1}(A_1 \circ A_2) = A_1 A_2 + \langle A_1 A_2 \rangle,
\]
\[
\sigma^{-1}(A_1 \circ A_2 \circ A_3) = A_1 A_2 A_3 + \langle A_1 A_2 \rangle A_3 + \langle A_1 A_3 \rangle A_2 + \langle A_2 A_3 \rangle A_1 + \langle A_1 A_2 A_3 \rangle.
\]

We call (14.7) the **cluster formula.** Note that this formula is used by Okounkov [Ok2] in computation of the quantum immanants. There it is also remarked that the cluster formula can be derived from the classical Wick formula.

Our next aim is to inverse the cluster formula and so find an explicit expression for the special symmetrization.

For \( A \in \mathfrak{gl}(n) \) we write \( A^k \) for the \( k \)-th power of \( A \) in \( S(\mathfrak{gl}(n)) \) and \( \langle A \rangle^k \) for its \( k \)-th power in the matrix algebra (so that \( \langle A \rangle^k \in \mathfrak{gl}(n) \)). Recall a standard notation: if \( \lambda \) is a partition, \( \lambda = (1^{r_1} 2^{r_2} \ldots) \), then

\[
z_{\lambda} = 1^{r_1} r_1! 2^{r_2} r_2! \ldots. \tag{14.8}
\]
Theorem 14.2. For any $A \in \mathfrak{gl}(n)$

$$
\sigma(A^k) = \sum_{\lambda \vdash k} (-1)^{k-\ell(\lambda)} \frac{k!}{z_\lambda} \langle A \rangle^{\lambda_1} \circ \langle A \rangle^{\lambda_2} \circ \ldots .
$$

(14.9)

Using polarization we easily obtain from (14.9) the following inversion formula (with respect to (14.7)).

Theorem 14.3. For any $A_1, \ldots, A_k \in \mathfrak{gl}(n)$

$$
\sigma(A_1 \ldots A_k)
= \sum_{s \in S(k)} \sum_{\lambda \vdash k} (-1)^{k-\ell(\lambda)} z_\lambda^{-1} \langle A_{s(1)} \ldots A_{s(\lambda_1)} \rangle \circ \langle A_{s(\lambda_1+1)} \ldots A_{s(\lambda_1+\lambda_2)} \rangle \circ \ldots .
$$

(14.10)

\[\square\]

Note that the first version of the inversion formula was obtained by Postnikov [P]. He did not use the polarization argument, and his formula involved an extra symmetrization.

Proof of Theorem 14.2. We remark first that our inversion problem makes sense for any associative algebra. Indeed, let $M$ be such an algebra, also viewed as a Lie algebra with bracket $[A, B] = AB - BA$. Then formula (14.7) defines a linear isomorphism

$$
\sigma^{-1} : U(M) \to S(M).
$$

(14.11)

Indeed, it is readily verified that for $i = 1, \ldots, k - 1$

$$
\sigma^{-1}(A_1 \circ \ldots \circ A_i \circ A_{i+1} \circ \ldots \circ A_k) - \sigma^{-1}(A_1 \circ \ldots \circ A_{i+1} \circ A_i \circ \ldots \circ A_k)
= \sigma^{-1}(A_1 \circ \ldots \circ [A_i, A_{i+1}] \circ \ldots \circ A_k),
$$

(14.12)

so that the map $\sigma^{-1}$ is correctly defined.

The next observation is that without loss of generality we may assume $M$ to be commutative. Indeed, given $A \in M$ we consider the associative subalgebra $M_A$ in $M$, spanned by the powers of $A$, and remark that $\sigma^{-1}$ maps $U(M_A)$ onto $S(M_A)$. So we may replace $M$ by $M_A$ and reduce the problem to the case when $M$ is a commutative associative algebra and the corresponding Lie structure is trivial. Then $U(M)$ is canonically identified with $S(M)$, so we may view $\sigma$ as a linear isomorphism of the vector space $S(M) = U(M)$.

The set $\Xi(k)$ is partially ordered: $\xi_1 \leq \xi_2$ if each cluster of $\xi_2$ is a subset of a cluster of $\xi_1$. Note that this order is inverse to the order usually used in combinatorics. The partition

$$
\widehat{1} = \{\{1\}, \{2\}, \ldots, \{k\}\}
$$

is the maximal element of $\Xi(k)$.

It follows from (14.7) that

$$
\sigma^{-1}(\Pi_\xi(A, \ldots, A)) = \sum_{\zeta \subseteq \xi} \Pi_\zeta(A, \ldots, A).
$$

(14.13)
Hence we can use the Möbius inversion in the poset $\Xi(k)$ to obtain a formula for $\sigma$. The Möbius function $\mu$ of the poset $\Xi(k)$ is well-known (see Stanley [St], section 3.10.4). In particular,

$$\mu(\xi, \hat{1}) = (-1)^{k-\ell(\lambda)} \prod (\lambda_i - 1)!,$$

(14.14)

where $\lambda_i$ are the cardinalities of the clusters of $\xi$. Therefore

$$\sigma(A \cdot \ldots \cdot A) = \sigma(\Pi(\lambda)(A, \ldots, A))
= \sum_{\xi \in \Xi(k)} (-1)^{k-\ell(\lambda)} \prod (\lambda_i - 1)! \Pi(\lambda)(A, \ldots, A).$$

(14.15)

It is easy to see that there are

$$\frac{k!}{z_{\lambda} \prod (\lambda_i - 1)!}$$

elements of $\Xi(k)$ with cardinalities of clusters equal to $\lambda$, which implies (14.9).

**Example 14.4.** Suppose $A \in \mathfrak{gl}(n)$ is a matrix projection, $\langle A \rangle^2 = A$. Then

$$\sigma(A^k) = A \circ (A - 1) \circ \ldots \circ (A - k + 1).$$

(14.16)

**Remark 14.5.** In the right-hand side of (14.9), for any $\lambda$, the factors

$$\langle A \rangle^{\lambda_1}, \langle A \rangle^{\lambda_2}, \ldots$$

can be rewritten in an arbitrary order. After polarization we can obtain in this way a number of various formulas which differ from (14.10) but are equivalent to it.

**Remark 14.6.** By applying formula (14.10) to the expression (2.10) for the elements $S_{\mu|n}$ we obtain, by virtue of Theorem 14.1, a certain explicit expression for the quantum immanants $S_{\mu|n}$. However, this expression seems to be more complicated than the expression obtained in the papers [Ok1], [N2], [Ok2].

We conclude with one more interpretation of the special symmetrization. Note that, by virtue of (14.2), the linear isomorphism $\sigma : S(\mathfrak{gl}(n)) \to U(\mathfrak{gl}(n))$ can be obtained via realizing both $S(\mathfrak{gl}(n))$ and $U(\mathfrak{gl}(n))$ by differential operators on $n \times n$ matrices. Now we aim to describe a different realization which leads to the same result.

As in the proof of Theorem 14.2 we shall work with an arbitrary associative algebra $M$, also viewed as a Lie algebra.

First, we form the algebra $\widetilde{M} = C1 \oplus M$, where 1 is a formally adjoint unity. For each $k = 1, 2, \ldots$ we consider the algebra $\widetilde{M} \otimes^k$. Its elements are linear combinations of the tensors $A_1 \otimes \ldots \otimes A_k$, where each $A_i$ belongs either to $C1$ or to $M$. We equip $\widetilde{M}$ with a filtration by setting

$$\deg(A_1 \otimes \ldots \otimes A_k) = \text{card}\{i | A_i \in M\}.$$  

(14.17)

Note that this filtration is compatible with the algebra structure.
Next, for each $k$ we define a projection $\tilde{M}^\otimes (k+1) \to \tilde{M}^\otimes k$ by setting
\[
A_1 \otimes \ldots \otimes A_{k+1} \mapsto \begin{cases} 
\alpha A_1 \otimes \ldots \otimes A_k, & \text{if } A_{k+1} = \alpha 1, \\
0, & \text{if } A_{k+1} \in M.
\end{cases}
\tag{14.18}
\]
Let $\Omega(M)$ be the projective limit of the algebras $\tilde{M}^\otimes k$, taken with respect to the projections (14.18) in the category of filtered algebras.

Finally, we define the embeddings
\[
\mathcal{U}(M) \to \Omega(M), \quad S(M) \to \Omega(M)
\tag{14.19}
\]
as follows.

Given $A \in M$, put
\[
A^{(i)} = 1^\otimes (i-1) \otimes A \otimes 1^\otimes \infty, \quad i = 1, 2, \ldots.
\tag{14.20}
\]
Then the first arrow in (14.19) is an algebra morphism, defined on elements $A \in M$ by
\[
A \mapsto \sum_{i=1}^{\infty} A^{(i)}
\tag{14.21}
\]
(note that the right–hand side of (14.11) is a well–defined element of $\Omega(M)$).

As for the second arrow in (14.19), we define it on monomials in the symmetric algebra as follows:
\[
A_1 \ldots A_k \mapsto \sum_{i_1, \ldots, i_k} A_1^{(i_1)} \ldots A_k^{(i_k)},
\tag{14.22}
\]
summed over all $k$-tuples of pairwise distinct indices. We again note that the right–hand side is well–defined in $\Omega(M)$.

**Proposition 14.7.** The mappings (14.19) defined above are embeddings with the same image, so that they define a linear isomorphism $S(M) \to \mathcal{U}(M)$. This isomorphism coincides with the map $\sigma$, defined by the cluster formula (14.7).

**Proof.** This follows at once from (14.7). □

15. **Concluding remarks**

1. **The map $\varphi : \Lambda \to \Lambda^*$.** By definition, this is a linear isomorphism such that
\[
\varphi(s_\mu) = s_\mu^* \quad \text{for each partition } \mu.
\tag{15.1}
\]
In case of finitely many variables $\varphi$ turns into the linear isomorphisms
\[
\varphi : \Lambda(n) \to \Lambda^*(n), \quad n = 1, 2, \ldots,
\tag{15.2}
\]
such that
\[
\varphi(s_{\mu|n}) = s_{\mu|n}^*, \quad \ell(\mu) \leq n.
\tag{15.3}
\]
Equivalently, by Theorem 14.1, the map $\varphi$ can be defined via the commutative diagram
\[
\begin{array}{ccc}
I(\mathfrak{gl}(n)) & \xrightarrow{\sigma} & 3(\mathfrak{gl}(n)) \\
\downarrow & & \downarrow \\
\Lambda(n) & \xrightarrow{\varphi} & \Lambda^*(n)
\end{array}
\tag{15.4}
\]
where $\sigma$ is the special symmetrization and the vertical arrows are the canonical isomorphisms discussed in section 2. Thus, the special symmetrization is a natural extension of the map $\varphi$. 55
2. The basis \( \{ p_\mu^\# \} \subset \Lambda^* \). Let \( p_\mu \in \Lambda \) denote the ordinary power sum functions, where \( \mu \) is an arbitrary partition. We define the elements \( p_\mu^\# \in \Lambda^* \) by

\[
p_\mu^\# = \varphi(p_\mu).
\]

(15.5)

In particular,

\[
p_k^\# = \varphi(p_k), \quad k = 1, 2, \ldots.
\]

(15.6)

Note that

\[
\omega(p_k^\#) = (-1)^{k-1} p_k^#,
\]

(15.7)

where \( \omega : \Lambda^* \to \Lambda^* \) is the involution of section 4.

Thus, in the algebra \( \Lambda^* \), we have three different families of generators, \( \{ p_k^* \} \), \( \{ p_k^\# \} \), and \( \{ p_k^\circ \} \); each of them can be regarded as an analogue of Newton power sums \( p_k \).

Recall the following fundamental identity in the theory of symmetric functions,

\[
p_\mu = \sum_{\lambda \vdash k} \chi_\mu^\lambda s_\lambda,
\]

(15.8)

where \( k = |\mu| \), \( \chi_\lambda^\mu \) is the irreducible character of the symmetric group \( S(k) \), indexed by \( \lambda \), and \( \chi_\mu^\lambda \) is its value on any permutation with cycle–type \( \mu \). It follows

\[
p_\mu^\# = \sum_{\lambda \vdash k} \chi_\mu^\lambda s_\lambda^*.
\]

(15.9)

3. The functions \( p_\mu^\# \) and a Capelli–type identity for the Schur–Weyl duality. Let \( \rho = (\rho_1, \ldots, \rho_m) \) be a partition of \( k \) and let \( l \geq k \) and \( n \) be natural numbers. As in section 7 we consider the tensor space \( (\mathbb{C}^n)^\otimes l \) as a bimodule over \( GL(n) \) and \( S(l) \). In the paper [KO] there were introduced central elements

\[
a_{\rho,l} \in Z(S(l)), \quad A_{\rho,n} \in Z(\mathfrak{gl}(n)),
\]

(15.10)

such that, in notation (7.2) and (7.3),

\[
\tau_{GL(n)}(A_{\rho,n}) = \tau_{S(l)}(a_{\rho,l})
\]

(15.11)

([KO], Theorem 2).

The element \( a_{\rho,l} \) is defined as a sum of products of cycles,

\[
a_{\rho,l} = \sum_{i_1, \ldots, i_k} (i_1, \ldots, i_{\rho_1})(i_{\rho_1+1}, \ldots, i_{\rho_1+\rho_2}) \ldots (i_{\rho_1+\ldots+\rho_{m-1}+1}, \ldots, i_k),
\]

(15.12)

where summation is taken over all \( k \)-tuples of pairwise distinct indices from the set \( \{1, \ldots, n\} \). That is, \( a_{\rho,l} \) is a scalar multiple of the conjugacy class in \( S(l) \) corresponding to the partition \( \rho \cup 1^{l-k} \).

The element \( A_{\rho,n} \) is defined via the special symmetrization map as follows:

\[
A_{\rho,n} = \sigma(P_{\rho|n}),
\]

(15.13)
where the element $P_{\rho|n} \in I(\mathfrak{gl}(n))$, viewed as an invariant polynomial function on $\mathfrak{gl}(n)$, is defined by

$$P_{\rho|n}(X) = \text{tr} X^{\rho_1} \cdot \text{tr} X^{\rho_2} \cdot \ldots \cdot \text{tr} X^{\rho_m}, \quad X \in \mathfrak{gl}(n).$$  \hspace{1cm} (15.14)

In other words, under the identification $I(\mathfrak{gl}(n)) = \Lambda(n)$, $P_{\rho|n}$ becomes the power sum function $p_{\rho}$ in $n$ variables.

For a partition $\lambda \vdash l$, let $f_{\rho}(\lambda)$ be the eigenvalue of the central element $a_{\rho,l}$ in $W_{\lambda}$ (the irreducible $S(l)$-module, indexed by $\lambda$). By virtue of (15.11) and the Schur–Weyl duality (7.1), $f_{\rho}(\lambda)$ also is the eigenvalue of $A_{\rho,n}$ in $V_{\lambda|n}$, the irreducible polynomial $\mathfrak{gl}(n)$-module corresponding to $\lambda$ (we assume $\ell(\lambda) \leq n$). It follows ([KO], Proposition 3) that $f_{\rho}(\cdot)$ is a shifted symmetric function with highest term $p_{\rho}$.

Now we remark that in notation of section 7

$$a_{\rho,l} = \sum_{\mu \vdash k} \chi_{\rho}^\mu \text{Ind} \chi_{\mu}^\mu / (l - k)!$$  \hspace{1cm} (15.15)

and

$$A_{\rho,n} = \sum_{\mu \vdash k} \chi_{\mu}^\rho S_{\mu|n}.$$  \hspace{1cm} (15.16)

Therefore, the relation (15.11) is equivalent to the relation (7.8) of Theorem 7.1.

An important consequence of this is that

$$f_{\rho} = \varphi(p_{\rho}),$$  \hspace{1cm} (15.17)

or, in notation (15.5),

$$f_{\rho} = p_{\rho}^\#.$$  \hspace{1cm} (15.18)

Formula (15.17) first appeared in Macdonald’s letter [M3] addressed to Olshanski (this letter was written as a comment to the papers [KO] and [O3]). There Macdonald introduced the map $\varphi$ and gave a simple direct proof of (15.17). At that moment we were already aware of the Characterization Theorem, of formula (6.5) for the differential operators corresponding to the quantum immanants, and of Theorem 14.1, which is essentially equivalent to the relation (15.17). However, the elegant argument of Macdonald suggested us formula (8.3) for the dimension of skew Young diagrams (Theorem 8.1).

Theorem 8.1 is one more result equivalent to (15.17). As we already noted in section 8, the second proof of Theorem 8.1 is nothing but a slight modification of Macdonald’s proof of the relation (15.17).

Note also that the eigenvalue $f_{\rho}(\lambda)$ can be written in the ‘Gibbsian form’

$$f_{\rho}(\lambda) = \frac{(s_{\lambda}, p_{\rho} e^{p_1})}{(s_{\lambda}, e^{p_1})}.$$  \hspace{1cm} (15.19)

This fact was communicated by Kerov to Olshanski when working on the paper [KO]; it is close to expressions used by Macdonald [M3].
4. Character values on small cycle–types. By the very definition of $f_\rho(\lambda)$, we have

$$f_\rho(\lambda) = \frac{(l \downarrow k)}{\dim \lambda} \chi^\lambda_{\rho \cup 1^{l-k}}. \quad (15.20)$$

Therefore, in view of (15.18) and (15.9) we obtain

$$\chi^\lambda_{\rho \cup 1^{l-k}} = \frac{\dim \lambda}{(l \downarrow k)} \sum_{\mu \vdash k} \chi^\mu_{\rho} s^*_\mu(\lambda), \quad \rho \vdash k, \lambda \vdash l, l \geq k. \quad (15.21)$$

This implies that the values of an irreducible character of $S(l)$ on cycle–types of the form $\rho \cup 1^{l-k}$, $k < l$, can be expressed in terms of the character table of the smaller group $S(k)$ and the $s^*$-functions. Hence, for small $k$ and arbitrary $l$ and $\lambda \vdash l$ we can obtain from (15.21) explicit formulas for the character values. Such formulas were derived by several authors: Frobenius, Murnaghan, and Ingram (see [I] and references therein), Kerov (unpublished), Wassermann [W].

4. Explicit formula for quantum immanants. The following formula for the quantum immanant $S_\mu$ was found by Okounkov [Ok1].

Denote by $E = (E_{ij})$ the $n \times n$ matrix formed by the standard generators $E_{ij}$ of $U(\text{gl}(n))$. (In section 6 we already used matrices with non-commutative entries.)

Let $T$ be a Young tableau of shape $\mu$. Put $k = |\mu|$. Let

$$P_T \in \mathbb{C}[S(k)]$$

be the orthogonal projection onto the corresponding Young basis vector in the irreducible $S(k)$-module indexed by $\mu$; this projection is proportional to the corresponding diagonal matrix element. Denote by $c_T(i)$ the content of the $i$-th box in the tableau $T$. We have the following

**Theorem[Ok1].**

$$S_\mu = \text{tr} \left( (E - c_T(1)) \otimes \cdots \otimes (E - c_T(k)) \cdot P_T \right).$$

*In particular, the right–hand side does not depend on the particular choice of the tableau $T$.*

Further discussions of this topic can be found in [Ok1], [N2], and [Ok2].

5. A connection between the binomial formula (5.7) and the dimension formula (8.3) for skew shapes. Here we aim to explain why the coefficients in the binomial formula (5.7) are equal, within simple factors, to the numbers $\dim \lambda/\mu$. Our argument will follow an idea due to Bingham [Bi], Theorem I (see also Lassalle [La], Macdonald [M4]).

**Lemma.** Consider the algebra $\Lambda(n)$ of symmetric polynomials in $x_1, \ldots, x_n$ and endow it with the inner product $\langle , \rangle$ such that

$$\langle s_\mu|n, s_\nu|n \rangle = \delta_{\mu\nu}(n \uparrow \mu).$$

Further, introduce two operators in $\Lambda(n)$, $D$ and $M$, where

$$D = \partial/\partial x_1 + \cdots + \partial/\partial x_n, \quad M = \text{multiplication by } x_1 + \cdots + x_n.$$
Then $D$ and $M$ are mutually adjoint with respect to this inner product.

**Proof.** The simplest way to check this is to use the basic formula (0.1) for the Schur polynomials. □

An explanation of this fact is as follows. Consider the reproducing kernel of the inner product,

$$F(x, y) = \sum_{\ell(\mu) \leq n} \frac{s_{\mu|n}(x)s_{\mu|n}(y)}{\langle s_{\mu|n}, s_{\mu|n} \rangle},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and let

$$X = \text{diag}(x_1, \ldots, x_n), \quad Y = \text{diag}(y_1, \ldots, y_n)$$

be the diagonal matrices corresponding to $x$ and $y$. Then it turns out that

$$F(x, y) = \int_{u \in U(n)} e^{\text{tr} X u Y u^{-1}} d\mu,$$

where $d\mu$ is the normalized Haar measure on the unitary group $U(n)$. From this formula one easily obtains the relation

$$(\partial/\partial x_1 + \ldots + \partial/\partial x_n)F(x, y) = (y_1 + \ldots + y_n)F(x, y),$$

which is equivalent to the claim of the lemma. Finally, note that the formula

$$\int_{u \in U(n)} e^{\text{tr} X u Y u^{-1}} d\mu = \sum_{\ell(\mu) \leq n} \frac{s_{\mu|n}(x)s_{\mu|n}(y)}{\langle s_{\mu|n}, s_{\mu|n} \rangle}$$

can be obtained from the binomial formula by a limit transition.

Now we apply the lemma to establish a close connection between (5.7) and (8.3). Let $\lambda \vdash l$ and $\mu \vdash k$ be two partitions such that $l > k$, $\ell(\lambda) \leq n$, $\ell(\mu) \leq n$. We remark that

$$s_{\lambda|n}(1 + x_1, \ldots, 1 + x_n) = (e^D s_{\lambda|n})(x_1, \ldots, x_n),$$

hence the coefficient of $s_{\mu|n}$ in the decomposition of the left-hand side is equal to

$$\frac{\langle e^D s_{\lambda|n}, s_{\mu|n} \rangle}{\langle s_{\mu|n}, s_{\mu|n} \rangle} = \frac{\langle s_{\lambda|n}, e^M s_{\mu|n} \rangle}{\langle s_{\mu|n}, s_{\mu|n} \rangle} = \frac{(l - k)! \langle s_{\lambda|n}, s_{\lambda|n} \rangle}{(l - k)! \langle s_{\mu|n}, s_{\mu|n} \rangle} \dim \lambda/\mu.$$

Thus,

$$s_{\lambda|n}(1 + x_1, \ldots, 1 + x_n) = \sum_{\mu} \frac{(n \uparrow \lambda)}{(\lambda \downarrow [\mu])! (n \uparrow \mu)} \dim \lambda/\mu \cdot s_{\mu|n}(x_1, \ldots, x_n).$$
6. Factorial Schur polynomials from Schubert polynomials. The following remark is due to Alain Lascoux.

Consider double Schubert polynomials: these are polynomials in two sets of variables, $X$ and $Y$, which are indexed by permutations, see Lascoux [Lasc], Macdonald [M5]. Some special permutations (Grassmannian permutations) give Schubert polynomials which are symmetric in $x_1, \ldots, x_n$. In this case, when one specializes $Y$ to $\{0, 1, 2, \ldots\}$, these Schubert polynomials become the factorial Schur polynomials in $x_1, \ldots, x_n$.

One can also recover factorial Schur polynomials (or rather generalized binomial coefficients $\binom{\lambda}{\mu}$, which are closely connected to them) by specializing $X = \{1, 1, \ldots\}$, $Y = \{0, 0, \ldots\}$ and considering the specialized Schur polynomial as a function of its indexing permutation. The permutations that one has to take are defined, in a certain way, by a pair $\lambda, \mu$ of partitions. Then one obtains the coefficients $\binom{\lambda}{\mu}$.

7. Further developments. Recently Ivanov and Okounkov [Iv], [IO] studied factorial analogues of Schur $Q$-functions. Factorial analogues of super Schur functions are studied in Molev’s paper [Mo]. Certain factorial functions play an important role in Molev–Nazarov work [MN] on Capelli operators for the orthogonal and symplectic groups. The study of general shifted Macdonald polynomials was started in [KnS,Kn,Sa2] and [Ok3].

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