VIRASORO SYMMETRIES OF MULTICOMPONENT GELFAND–DICKEY SYSTEMS

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We study the additional symmetries and \(\tau\)-functions of multicomponent Gelfand–Dickey hierarchies, which include classical integrable systems such as the multicomponent Korteweg–de Vries and Boussinesq hierarchies. Using various reductions, we derive B- and C-type multicomponent Gelfand–Dickey hierarchies. We show that not all flows of their additional symmetries survive. We find that the generators of the additional symmetries of the B- and C-type multicomponent Gelfand–Dickey hierarchies differ but the forms of their additional flows are the same.

Keywords: multicomponent Gelfand–Dickey hierarchy, additional symmetry, string equation, \(\tau\)-function, Virasoro constraint

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1. Introduction

The Gelfand–Dickey (GD) hierarchy was introduced in [1], attracted wide attention, and then became one of the hottest topics in classical integrable systems. The Hamiltonian theory of the GD hierarchy gradually developed in terms of the Lax pairs; it became a powerful basis for studying integrable systems. This approach was presented in detail in [2]. Broad investigations of soliton solutions of the GD hierarchy, its additional symmetries, \(\tau\)-function, Bäcklund transformations, and other related properties were conducted [3]. In addition, much research was devoted to the supersymmetric GD hierarchy, the \(q\)-deformed GD hierarchy, etc. [4]–[10]. But there has been not so much research on the multicomponent GD (mcGD) hierarchy, generally speaking. Here, based on previous research, we study the additional symmetries and \(\tau\)-functions of the mcGD hierarchy and also B- and C-type mcGD hierarchies.

When the theory of additional symmetries first appeared, it was on the periphery of the theory of integrable systems. As a result of long studies of string equations [11]–[13], Virasoro constraints [14], [15], and other aspects of the theory, the additional symmetries began to play an extremely important role and gradually attracted the attention of researchers [16]–[18]. The additional symmetries of the mcGD hierarchy are directly used to derive the string equation, which appears in studying string theory. In the general case, this equation is written as \([P, Q] = 1\), where \(P\) and \(Q\) are differential operators.

One of the important characteristics of the string equation is its close relation to hierarchies of some integrable equations. This relation is that the string equation is invariant under flows generated by equations

\[\frac{d}{dt} \text{string equation} = \{P, \text{string equation}\}\]

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in the hierarchy [2]. Some of the additional symmetries are reducible to Virasoro symmetries with a wide range of applications, among which their role in the $\tau$-function is particularly important [19], [20] because the $\tau$-function appears as a partition function or as a generating function in many modern problems in mathematics and physics. Using the action of Virasoro symmetries on the $\tau$-function, we can also obtain explicit solutions of nonlinear equations with Virasoro constraint conditions in the form of matrix integrals. For example, solutions satisfying the Virasoro conditions were obtained for the Toda chain and the KdV equation in [21], [22]. The additional symmetries can also be used to find eigenfunctions of the linearized problem and to solve the stability problem [23].

Here, we present the definition and properties of the mcGD hierarchy and the B- and C-type mcGD hierarchies. The structure of this paper is as follows. We first write the Lax equations for the studied hierarchies, introduce the Lax operators $L$ and $R_\alpha$ and the corresponding constraint equations, and also introduce the wave operator $\phi$ and discuss some properties of the wave functions that naturally lead to the Sato equations. Introducing the Orlov–Shulman operator $M$, we then define additional symmetries, from which we derive several properties that are important from the practical standpoint. We find that only a small part of the additional flows survive, and we present expressions for them. Moreover, we analyze and calculate a special additional flow and obtain an important application of additional symmetries in string theory. Finally, based on the existence theorem for $\tau$-functions, we discuss additional symmetries of $\tau$-functions and obtain an equivalent form of the string equation, the Virasoro condition. We note that the C-type mcGD hierarchy can be divided into even and odd forms.

2. Multicomponent Gelfand–Dickey hierarchies

The GD hierarchy is one of the most important topics discussed in the theory of classical integrable systems. The definition of the mcGD hierarchy is based on differential operators $L$ and $R_\alpha$ of the $N$th order. The operator $L$ has the form

$$L = A \partial^N + u_2 \partial^{N-2} + u_3 \partial^{N-3} + \cdots + u_0,$$

with $A = \text{diag}(a_1, a_2, \ldots, a_n)$, where $a_i$ are nonzero constants. The diagonal elements of $u_0$ are all equal to zero except in the case $n = 1$, and the $u_i$ are arbitrary $n \times n$ matrices. The operator $R_\alpha$ has the form

$$R_\alpha = \sum_{i=1}^{\infty} R_{i\alpha} \partial^{-i}, \quad \alpha = 1, 2, \ldots, n,$$

with $R_{0\alpha} = E_{\alpha}$, where $E_{\alpha}$ is the matrix with the single element with the index $(\alpha, \alpha)$ equal to unity and all other elements equal to zero. The operator $R_\alpha$ must satisfy the condition $[L, R_\alpha] = 0$. It can be shown that such operators exist. Their elements are differential polynomials constructed from the $u_i$, and these operators satisfy the relations

$$R_\alpha R_\beta = \delta_{\alpha\beta} R_\alpha, \quad \sum_{\alpha=1}^{n} R_\alpha = I.$$

The operators $L$ and $R_\alpha$ are called the Lax operators of the mcGD hierarchy.

The mcGD hierarchy can be defined in terms of the Lax equations

$$\partial_{n,\alpha} L = [B_{n,\alpha}, L], \quad \partial_{n,\alpha} R_\beta = [B_{n,\alpha}, R_\beta], \quad n \neq 0 \text{ (mod } N),$$

$$\sum_{\alpha=1}^{n} \partial_{jN,\alpha} L = 0, \quad \sum_{\alpha=1}^{n} \partial_{jN,\alpha} R_\beta = 0, \quad j = 1, 2, \ldots,$$

(2.4)
where \( \partial_{n,\alpha} = \partial / \partial t_{n,\alpha} \) and the operators \( B_{n,\alpha} \) are related to the differential part of the operator \( L^{n/N} R_\alpha \),

\[
B_{n,\alpha} = (L^{n/N} R_\alpha)_+.
\]

The mcGD hierarchy can also be defined in terms of the zero-curvature equation

\[
\partial_{n,\alpha} B_{m,\alpha} - \partial_{m,\alpha} B_{n,\alpha} + [B_{m,\alpha}, B_{n,\alpha}] = 0, \quad n, m \neq 0 \text{ (mod } N),
\]

which is derived from the Lax equations. We note that the variables \( x \) and \( t_{n,\alpha} \) are not independent but they are also not equivalent. The relation between them is given by the equality

\[
\partial = \sum_{\alpha=1}^{n} a_\alpha^{-1/N} \partial_{1,\alpha}.
\]

We obtain the multicomponent KdV hierarchy at \( N = 2 \) and the multicomponent Boussinesq hierarchy at \( N = 3 \).

The Lax operators of the mcGD hierarchy can also be expressed in the dressing form

\[
L = \phi A \phi^{-N}, \quad R_\alpha = \phi E_\alpha \phi^{-1},
\]

where the quasidifferential operator

\[
\phi = \phi(A\partial^N) = \sum_{i=0}^{\infty} \alpha_i (A\partial^N)^{-i}, \quad \alpha_0 = I,
\]

is called the dressing or wave operator.

**Proposition 2.1.** Under the dressing transformation, the vector field transforms as

\[
\partial_{n,\alpha} \phi = -(L^{n/N} R_\alpha)_- \phi, \quad n \neq 0 \text{ (mod } N),
\]

\[
\sum_{\alpha=1}^{n} \partial_{jN,\alpha} \phi = 0, \quad j = 1, 2, \ldots.
\]

The obtained equations are called the Sato equations of the mcGD hierarchy.

In what follows, we discuss the wave function of the mcGD hierarchy. We introduce the series

\[
\xi(t, z) = \sum_{i=1}^{\infty} \sum_{\alpha=1}^{n} t_{i,\alpha} E_\alpha z^i.
\]

It has the properties

\[
\partial^m \xi(t, z) = z^m A^{-m/N} \xi(t, z), \quad \partial_{n,\alpha} \xi(t, z) = z^n E_\alpha \xi(t, z), \quad n \neq 0 \text{ (mod } N),
\]

\[
\sum_{\alpha=1}^{n} \partial_{jN,\alpha} \xi(t, z) = z^{jN} \xi(t, z), \quad j = 1, 2, \ldots.
\]

The wave function of the mcGD hierarchy is then defined as

\[
W(t, z) = \phi e^{\xi(t, z)} = \omega(t, z) e^{\xi(t, z)}.
\]

The function \( \omega \) is the symbol of the dressing operator \( \phi \).
Corollary 2.1. The wave function of the mcGD hierarchy satisfies the equations

\[ L^{m/N}W = z^m W, \quad \partial_{n,\alpha} W = (L^{n/N} R_\alpha)_+ W, \quad n \neq 0 \text{ (mod } N), \]

\[ \sum_{\alpha=1}^n \partial_{jN,\alpha} W = L^j W, \quad j = 1, 2, \ldots. \] (2.10)

**Proof.** From properties (2.9), we obtain

\[ A^{m/N} \partial^m e^{\xi(t,z)} = z^m e^{\xi(t,z)}, \]

hence

\[ A^{m/N} \partial^m = z^m, \]

and consequently

\[ L^{m/N}W = \phi A^{m/N} \partial^m e^{\xi(t,z)} = z^m W, \]

and

\[ \partial_{n,\alpha} W = \partial_{n,\alpha} (\phi e^{\xi(t,z)}) = (\partial_{n,\alpha} \phi) e^{\xi(t,z)} + \phi (\partial_{n,\alpha} e^{\xi(t,z)}) = \]

\[ = -(L^{n/N} R_\alpha) - \phi e^{\xi(t,z)} + \phi z^n E_\alpha e^{\xi(t,z)} = \]

\[ = -(L^{n/N} R_\alpha) - W + \phi A^{n/N} \partial^n E_\alpha \phi^{-1} \phi e^{\xi(t,z)} = \]

\[ = -(L^{n/N} R_\alpha) - W + L^{n/N} R_\alpha W = (L^{n/N} R_\alpha)_+ W, \]

\[ \sum_{\alpha=1}^n \partial_{jN,\alpha} W = \sum_{\alpha=1}^n \partial_{jN,\alpha} (\phi e^{\xi(t,z)}) = \phi \left( \sum_{\alpha=1}^n \partial_{jN,\alpha} e^{\xi(t,z)} \right) = \phi (z^j N e^{\xi(t,z)}) = L^j W. \]

We now consider the adjoint wave function \( W^*(t,z) \), where the asterisk denotes formal conjugation, \( \partial^* = -\partial, (\partial^{-1})^* = -\partial^{-1} \), and \( (AB)^* = B^* A^* \). The adjoint wave function \( W^*(t,z) \) of the mcGD hierarchy is defined as

\[ W^*(t,z) = (\phi^*)^{-1} e^{-\xi(t,z)}. \] (2.11)

We previously presented the Lax operators of the mcGD hierarchy, and it is easy to prove that the operators \( \partial_{n,\alpha} = (L^{n/N} R_\alpha)_+ \) and \( \sum_{\alpha=1}^n \partial_{jN,\alpha} - L^j \) can be written in the dressed form

\[ \partial_{n,\alpha} = (L^{n/N} R_\alpha)_+ = \phi (\partial_{n,\alpha} - A^{n/N} \partial^n E_\alpha) \phi^{-1}, \quad n \neq 0 \text{ (mod } N), \]

\[ \sum_{\alpha=1}^n \partial_{jN,\alpha} - L^j = \phi \left( \sum_{\alpha=1}^n \partial_{jN,\alpha} - A^j \partial^{jN} \right) \phi^{-1}, \quad j = 1, 2, \ldots. \] (2.12)

Applying the dressing transformation to both sides of the equations

\[ \left[ \partial_{n,\alpha} - A^{n/N} \partial^n E_\alpha, A \partial^N \right] = 0, \quad \left[ \sum_{\alpha=1}^n \partial_{jN,\alpha} - A^j \partial^{jN}, A \partial^N \right] = 0, \]

we obtain the Lax equations for the mcGD hierarchy:

\[ \left[ \partial_{n,\alpha} - (L^{n/N} R_\alpha)_+, L \right] = 0, \quad n \neq 0 \text{ (mod } N), \]

\[ \left[ \sum_{\alpha=1}^n \partial_{jN,\alpha} - L^j, L \right] = 0, \quad j = 1, 2, \ldots. \] (2.13)
2.1. Additional symmetries of the mcGD hierarchies. We introduce the operator

\[ \Gamma = \sum_{k=1}^{\infty} \sum_{\alpha=1}^{n} k t_{k,\alpha} A^{(k-1)/N} \partial^{k-1} E_\alpha, \quad k \neq 0 \pmod{N}. \]  

(2.14)

It has the properties

\[ \partial_{n,\alpha} \Gamma = nA^{(n-1)/N} \partial^{n-1} E_\alpha, \quad n \neq 0 \pmod{N}, \quad \partial^k \Gamma = \Gamma \partial^k + kA^{-1/N} \partial^{k-1}. \]

It is easy to verify that the operators \( \Gamma \) and \( \partial_{n,\alpha} - A^{n/N} \partial^n E_\alpha \) commute,

\[ [\partial_{n,\alpha} - A^{n/N} \partial^n E_\alpha, \Gamma] = 0, \quad n \neq 0 \pmod{N}. \]  

(2.15)

Applying the dressing transformation to the equation \( [\partial_{n,\alpha} - A^{n/N} \partial^n E_\alpha, \Gamma] = 0 \), we obtain

\[ \partial_{n,\alpha} M = [(L^{n/N} R_\alpha)_+, M], \quad n \neq 0 \pmod{N}, \]

(2.16)

where we call \( M = \phi \Gamma \phi^{-1} \) an Orlov–Shulman operator.

We next define the additional symmetries of the mcGD hierarchies. We first introduce additional independent variables \( t_{l,m,\alpha}^* \) and define the action of the additional flows on the wave operator \( \phi \) as

\[ \partial_{l,m,\alpha}^* \phi = -(M^m L^{1/N} R_\alpha)_- \phi, \]

(2.17)

where \( \partial_{l,m,\alpha}^* = \partial / \partial t_{l,m,\alpha}^* \) here and hereafter.

We now consider a case related to the Virasoro conditions. For the special differential operator \( M^m L^{1/N} R_\alpha \), assuming that its negative part vanishes and acting with the operator \( (M^m L^{1/N} R_\alpha)_+ \) on \( W \), we obtain a differential equation related to \( z \),

\[ (M^m L^{1/N} R_\alpha)_+ W = z^l E_\alpha \partial_z^m W. \]  

(2.18)

We note that this system can be rewritten as a linear equation for an isomonodromy problem.

**Corollary 2.2.** Taking the definition of an additional symmetry of the mcGD hierarchy into account, we obtain the equations

\[ \partial_{l,m,\alpha}^* L = -[(M^m L^{1/N} R_\alpha)_-, L], \quad \partial_{l,m,\alpha}^* R_\beta = -[(M^m L^{1/N} R_\alpha)_-, R_\beta], \]

(2.19)

which imply that

\[ \partial_{l,m,\alpha}^* (L^{n/N} R_\beta)_- = -[(M^m L^{1/N} R_\alpha)_-, (L^{n/N} R_\beta)_- + \partial_{n,\beta}]_-, \quad n \neq 0 \pmod{N}. \]  

(2.20)

**Proof.** By direct calculations, we obtain

\[ \partial_{l,m,\alpha}^* L = \partial_{l,m,\alpha}^* (\phi A \partial^N \phi^{-1}) = (\partial_{l,m,\alpha}^* \phi) A \partial^N \phi^{-1} + \phi A \partial^N (\partial_{l,m,\alpha}^* \phi^{-1}) =
\]

\[ = -(M^m L^{1/N} R_\alpha)_- L + L(M^m L^{1/N} R_\alpha)_- = -[(M^m L^{1/N} R_\alpha)_-, L]. \]

We similarly obtain the second equation in (2.19). It hence follows that

\[ \partial_{l,m,\alpha}^* (L^{n/N} R_\beta)_- = -[(M^m L^{1/N} R_\alpha)_-, (L^{n/N} R_\beta)_-]_- - [(M^m L^{1/N} R_\alpha)_-, \partial_{n,\beta}]_- =
\]

\[ = -[(M^m L^{1/N} R_\alpha)_-, (L^{n/N} R_\beta)_- + \partial_{n,\beta}]_-.
\]
Only some of the additional flows of the mcGD hierarchy survive.

**Theorem 2.1.** Only the additional flows of the mcGD hierarchy that satisfy the condition

\[ (M^{m-1} L^{(N+l-1)/N})_- = 0 \]

and have the form \( \sum_{\alpha=1}^{n} \partial^*_l m, \alpha \) survive.

**Proof.** In the Lax operator \( L \) of the mcGD hierarchy, the negative part vanishes. We know that \( \partial^*_l m, \alpha L = -[\{M^m L^{l/N} R_\alpha\}_-, L] \). We therefore consider

\[ \partial^*_l m, \alpha L_- = -[\{M^m L^{l/N} R_\alpha\}_-, L_-] = -(\phi[\Gamma^m A^{l/N} \partial^l E_\alpha, A \partial^N]\phi^{-1})_- \]

After some calculations, this equation reduces to

\[ \partial^*_l m, \alpha L_- = \frac{N}{N+1} [L^{(N+l)/N} R_\alpha, M^m]_- = n N (M^{m-1} L^{(N+l-1)/N} R_\alpha)_- \]

If and only if \( (M^{m-1} L^{(N+l-1)/N})_- = 0 \), we obtain

\[ \sum_{\alpha=1}^{n} \partial^*_l m, \alpha L_- = n N (M^{m-1} L^{(N+l-1)/N})_- = 0 \]

Therefore, we should consider only additional flows \( \sum_{\alpha=1}^{n} \partial^*_l m, \alpha \) satisfying the condition

\[ (M^{m-1} L^{(N+l-1)/N})_- = 0 \]

According to Theorem 2.1, we can conclude as follows. If we assume that a solution of the mcGD hierarchy is defined by the Virasoro condition \( \sum_{\alpha=1}^{n} (M^{m-1} L^{(N+l-1)/N} R_\alpha)_- = 0 \) under the condition that the solution satisfies the constraint \( (L^{(N+l)/N})_- = 0 \), then it must also satisfy the \( W_{1+\infty} \) symmetry \( (M^m)_- = 0 \). In other words, the \( W_{1+\infty} \) symmetry is compatible with both the Virasoro condition \( (M^{m-1} L^{(N+l-1)/N})_- = 0 \) and the constraint \( (L^{(N+l)/N})_- = 0 \).

**Proposition 2.2.** The additional flows \( \sum_{\alpha=1}^{n} \partial^*_l m, \alpha \) satisfying \( (M^{m-1} L^{(N+l-1)/N})_- = 0 \) commute with the flows \( \partial_{n, \beta} \), \( n \neq 0 \) (mod \( N \)), of the mcGD hierarchy.

**Proof.** We have

\[ \left[ \sum_{\alpha=1}^{n} \partial^*_l m, \alpha, \partial_{n, \beta} \right] \phi = \left( \sum_{\alpha=1}^{n} [\partial^*_l m, \alpha, \partial_{n, \beta}] \right) \phi = \sum_{\alpha=1}^{n} ([\partial^*_l m, \alpha, \partial_{n, \beta}] \phi) \]

and

\[ [\partial^*_l m, \alpha, \partial_{n, \beta}] \phi = \partial^*_l m, \alpha (\partial_{n, \beta} \phi) - \partial_{n, \beta} (\partial^*_l m, \alpha \phi) = \]

\[ = \partial^*_l m, \alpha (L^{n/N} R_\beta)_- \phi + \partial_{n, \beta} (M^m L^{l/N} R_\alpha)_- \phi = \]

\[ = (\partial^*_l m, \alpha (L^{n/N} R_\beta)_-) \phi - (L^{n/N} R_\beta)_- (\partial^*_l m, \alpha \phi) + \]

\[ + (\partial_{n, \beta} (M^m L^{l/N} R_\alpha)_-) \phi + (M^m L^{l/N} R_\alpha)_- (\partial_{n, \beta} \phi) = \]

\[ = [(M^m L^{l/N} R_\alpha)_-, (L^{n/N} R_\beta)_- + \partial_{n, \beta}]_\phi + (L^{n/N} R_\beta)_- (M^m L^{l/N} R_\alpha)_- \phi + \]

\[ + [(\partial_{n, \beta}, (M^m L^{l/N} R_\alpha)_-) \phi - (M^m L^{l/N} R_\alpha)_- (L^{n/N} R_\beta)_- \phi = \]

\[ = [(L^{n/N} R_\beta)_-, (M^m L^{l/N} R_\alpha)_-] \phi - [(L^{n/N} R_\beta)_- (M^m L^{l/N} R_\alpha)_-] \phi = 0, \]
whence we obtain \[ \sum_{\alpha=1}^{n} \partial_{\alpha,m,\alpha} \theta_{n,\alpha} \phi = 0. \] These calculations show that the flows commute on the operator \( \phi \) and hence commute on the whole differential algebra generated by coefficients of \( \phi \). The proposition is proved.

A direct application of the additional symmetry of the mcGD hierarchy yields the string equation appearing in string theory. We first note the relation between the Lax operators and the Orlov–Shulman operator of the mcGD hierarchy

\[
[L^{1/N}, M] = \phi[A^{1/N} \partial, \Gamma] \phi^{-1} = \phi(A^{1/N} A^{-1/N}) \phi^{-1} = I,
\]

\[
[R_{\alpha}, M] = \phi[E, \Gamma] \phi^{-1} = 0.
\]

We hence obtain

\[
[L^{n/N}, M] = \phi[A^{n/N} \partial, \Gamma] \phi^{-1} = \phi(A^{n/N} nA^{-1/N} \partial^{n-1}) \phi^{-1} = nL^{(n-1)/N}.
\]  

(2.21)

Moreover, we have \([L^{n/N}, ML^{-(n-1)/N} R_{\alpha}] = nR_{\alpha} \).

Further, we consider the special additional flow with \( l = -(n-1), l - 1 = 0 \) (mod \( N \)),

\[
\sum_{\alpha=1}^{n} \partial^{*}_{-(n-1),1,\alpha} L^{n/N} = - \sum_{\alpha=1}^{n} [(ML^{-(n-1)/N} R_{\alpha})_{-}, L^{n/N}], \quad n = 0 \) (mod \( N \)).

Combining this expression with (2.21), we rewrite it as

\[
\sum_{\alpha=1}^{n} \partial^{*}_{-(n-1),1,\alpha} L^{n/N} = \sum_{\alpha=1}^{n} [(ML^{-(n-1)/N} R_{\alpha})_{+}, L^{n/N}] + nI.
\]

Moreover, \( L^{n/N} = (L^{n/N})_{+} \) for \( n = 0 \) (mod \( N \)), and therefore

\[
\sum_{\alpha=1}^{n} \left[ L^{n/N} - \frac{1}{n} (ML^{-(n-1)/N} R_{\alpha})_{+} \right] = I,
\]

whence we obtain the so-called string equation of the mcGD hierarchy:

\[
\left[ L^{n/N} - \frac{1}{n} \left( \sum_{\alpha=1}^{n} (ML^{-(n-1)/N} R_{\alpha})_{+} \right) \right] = I.
\]  

(2.22)

Based on the arguments presented above, we can find that the string equation is related to the condition that the operator is independent of the additional variables.

2.2. The \( \tau \)-function and Virasoro constraint of the mcGD hierarchies. We formulate the existence theorem [2] for the \( \tau \)-function.

**Theorem 2.2.** We assume that the \( \tau \)-function is a matrix \( T = (\tau_{\alpha,\beta}) \), and we write \( \tau \) instead of \( \tau_{\alpha,\alpha} \) for brevity. Then there exists a function \( \tau(\ldots, t_{s,\gamma}, \ldots) \) such that

\[
\omega_{\alpha,\alpha}(t, z) = \frac{\tau(\ldots, t_{s,\gamma} - \delta_{\gamma,\beta}(1/s^*)\ldots)}{\tau(\ldots, t_{s,\gamma}, \ldots)},
\]

\[
\omega_{\alpha,\beta}(t, z) = \frac{\tau_{\alpha,\beta}(\ldots, t_{s,\gamma} - \delta_{\gamma,\beta}(1/s^*)\ldots)}{z \cdot \tau(\ldots, t_{s,\gamma}, \ldots)}, \quad \alpha \neq \beta, \quad s \neq 0 \) (mod \( N \))

(2.23)

(the time variable \( t_{s,\gamma} \) shifts only if \( \gamma = \beta \)). The equation

\[
\sum_{\alpha=1}^{n} \partial_{kN,\alpha} \tau = 0, \quad k = 1, 2, \ldots,
\]

(2.24)

is satisfied in all cases except \( s \neq 0 \) (mod \( N \)) noted above.
In what follows, starting from the additional symmetries of the wave operator \( \phi \) and taking the existence theorem for the \( \tau \)-function into account, we study the additional symmetries of the \( \tau \)-function.

To facilitate subsequent calculations, we first appropriately split the sum \((l - 1 = 0 \mod N)\):

\[
\sum_{\alpha=1}^{n} \partial_{l,1,\alpha} \phi = - \sum_{\alpha=1}^{n} (ML^{l/N} R_{\alpha}) \phi = - \sum_{\alpha=1}^{n} \left( \phi \sum_{k=1, k \not\equiv 0 \mod N}^{\infty} kt_{k,\alpha} A^{(k+l-1)/N} \partial E_{\alpha} \phi^{-1} \right) \phi.
\]

Depending on the value of \( k \), we can represent the expression in the right-hand side of the last equality with \( l < 0 \) as the sum of three terms:

\[
O = [\phi, t_{1,\alpha}] A^{l/N} \partial E_{\alpha} + \sum_{k=1}^{-l} kt_{k,\alpha} \phi A^{(k+l-1)/N} \partial E_{\alpha}, \]

\[
P = (\phi (-l + 1) t_{-l+1,\alpha} E_{\alpha} \phi^{-1}) \phi = (-l + 1) [\phi, t_{-l+1,\alpha} E_{\alpha}], \]

\[
Q = - \sum_{k=-l+2, k \not\equiv 0 \mod N}^{\infty} kt_{k,\alpha} \partial \tau_{k+l-1,\alpha} \phi.
\]

After this partition, we obtain

\[
\sum_{\alpha=1}^{n} \partial_{l,1,\alpha} \omega = - z^l \partial \omega I - \sum_{\alpha=1}^{n} \left( \sum_{k=1, k \not\equiv 0 \mod N}^{\infty} kt_{k,\alpha} z^{k+l-1} \omega E_{\alpha} + (-l + 1) [\omega, t_{-l+1,\alpha} E_{\alpha}] \right) - \sum_{k=-l+2, k \not\equiv 0 \mod N}^{\infty} kt_{k,\alpha} \partial \tau_{k+l-1,\alpha} \omega.
\]

Consequently,

\[
\sum_{\alpha=1}^{n} \partial_{l,1,\alpha} \omega_{\gamma,\beta} = \sum_{\alpha=1}^{n} \left( - z^l \partial \omega_{\gamma,\beta} \delta_{\alpha,\beta} - \sum_{k=1}^{-l} kt_{k,\alpha} z^{k+l-1} \omega_{\gamma,\beta} \delta_{\alpha,\beta} - (-l + 1) [\omega, t_{-l+1,\alpha} E_{\alpha}] \right) + \sum_{k=-l+2, k \not\equiv 0 \mod N}^{\infty} kt_{k,\alpha} \partial \tau_{k+l-1,\alpha} \omega_{\gamma,\beta}.
\]

Based on the obtained results, we easily obtain the proposition in [2] on the additional symmetries of the \( \tau \)-function.

**Proposition 2.3.** The action of the additional flows \( \sum_{\alpha=1}^{n} \partial_{l,1,\alpha} (l < 0, l - 1 = 0 \mod N) \) on \( \tau_{\gamma,\beta} \) is given by

\[
\sum_{\alpha=1}^{n} \partial_{l,1,\alpha} \tau_{\gamma,\beta} = \sum_{\alpha=1}^{n} \left( \sum_{k=-l+1}^{\infty} kt_{k,\alpha} \partial \tau_{k+l-1,\alpha} + \frac{1}{2} \sum_{k+s=-l+1} kst_{k,\alpha} t_{s,\alpha} \right) \tau_{\gamma,\beta} + \left( \sum_{\alpha=1}^{n} c_{\alpha,\beta} \right) \tau_{\gamma,\beta},
\]

where \( t_{k,\alpha} \) and \( t_{s,\alpha} \) are the arguments of the element \( \tau_{\gamma,\beta} \).
The method for proving this proposition is similar to that in [2], and we do not repeat it here. Because the string equation is the condition that the operator is independent of the additional variables, we can also find that the operators are independent of the additional variables by the same method as in the proof of the proposition. Below, we prove this statement by showing that \( L_l \tau = 0, l - 1 = 0 \pmod{N}, l < 0, \) where

\[
L_l = \sum_{\alpha=1}^{n} \left( \sum_{k=-l+1}^{\infty} k t_{l,\alpha} \partial_{k+l-1,\alpha} + \frac{1}{2} \sum_{k+s=-l+1}^{\infty} k s t_{l,\alpha} t_{s,\alpha} + c_{l,\alpha} \right),
\]

which is obviously equivalent to the string equation. We call this relation the Virasoro constraint of the mcGD hierarchy. The operator \( L_l \) satisfies the Virasoro commutation relation

\[
[L_{-m}, L_{-n}] = (-m + n) L_{-(m+n)}, \quad m, n = 1, 2, \ldots.
\]

There are ordinary differential equations related to the solutions of the Virasoro constraints, similar to the Painlevé equations appearing in the theory of the nonlinear Schrödinger and the KdV equations. We briefly describe these equations based on the classical KdV equation.

**Example** [24]. The classical KdV equation

\[
u_t + 6uu_x + u_{xxx} = 0
\]

is obtained from the Lax pair \( L = A \partial^2 + u_0 \) and \( A = -4 \partial^3 - 3u \partial - 3u_x \), where \( A = \text{diag}(1, \ldots, 1) \) and \( u_0 = \text{diag}(u(x,t), \ldots, u(x,t)) \). We introduce the Virasoro operator

\[
\hat{L} = L_{-1} = \frac{t_{1,1}^2}{2} + \sum_{k=2}^{\infty} k t_{k,\alpha} \partial_{k-2,\alpha}.
\]

Let \( \tau(x) \) be a solution of the mcGD hierarchy for which \( \hat{L} \tau(x) = 0 \). We add the condition \( \partial \tau / \partial t_{2,1} = 0 \) \((i = 1, 2, \ldots)\) on the function \( \tau(x) \) to make it also a solution of the multicomponent KdV hierarchy. For example, \( \tau(x) \) satisfies the bilinear differential equation

\[
(D_{t_{1,1}}^4 + D_{t_{1,1}} D_{t_{3,1}}) \tau \cdot \tau = 0,
\]

corresponding to (2.26). Further, setting \( \delta(x) = \tau(x) \) under the conditions \( t_{1,1} = x, \quad t_{5,1} = -1/5, \) and \( t_{j,1} = 0 \) for \( j \neq 1 \) and \( j \neq 5 \), we find that \( \delta(x) \) satisfies the bilinear differential equation

\[
(D_x^4 - 2x) \delta \cdot \delta = 0.
\]

Let \( p = -d^2 \log \delta / dx^2 \). Then Eq. (2.28) can be rewritten as the first Painlevé equation \( P_1 \)

\[
\frac{d^2 p}{dx^2} = 6p^2 + x.
\]

**3. B-type multicomponent Gelfand–Dickey hierarchies**

This section is devoted to the mcGD hierarchy where the operators \( L \) and \( R_{\alpha} \) given by (2.1) and (2.2) also satisfy the conditions \( L^* = -\partial L \partial^{-1} \) and \( R_{\alpha}^* = \partial R_{\alpha} \partial^{-1} \). In this case, \( L \) and \( R_{\alpha} \) are called the Lax operators of the B-type mcGD hierarchy. Based on these operators, we define this hierarchy as

\[
\partial_{n,\alpha} L = [B_{n,\alpha}, L], \quad \partial_{n,\alpha} R_{\beta} = [B_{n,\alpha}, R_{\beta}], \quad n \neq 0 \pmod{N},
\]

\[
\sum_{\alpha=1}^{n} \partial_{jN,\alpha} L = 0, \quad \sum_{\alpha=1}^{n} \partial_{jN,\alpha} R_{\beta} = 0, \quad j = 1, 2, \ldots.
\]
Proposition 3.1. The B-type mcGD hierarchy has only odd flows.

Proof. Based on the foregoing, we see that the Lax equations of the B-type mcGD hierarchy are

\[ \partial_{n,\alpha} L = [B_{n,\alpha}, L], \quad \partial_{n,\alpha} R_\beta = [B_{n,\alpha}, R_\beta], \quad n \neq 0 \pmod{N}. \]

Taking the conditions on these Lax equations into account, we obtain

\[ (L_n^{n/N})^* = (L^*)^n/N = (-1)^n \partial L^{n/N} \partial^{-1}, \]
\[ (L_n^{n/N} R_\alpha)^* = R_\alpha^*(L^*)^n/N = (-1)^n \partial R_\alpha L^{n/N} \partial^{-1}. \]

We then simplify \( \partial_{n,\alpha} L^* \) and \( \partial_{n,\alpha} R_\alpha^* \) in two ways:

\[ \partial_{n,\alpha} L^* = (\partial_{n,\alpha} L)^* = [B_{n,\alpha}, L]^* = -\partial [B_{n,\alpha}, L] \partial^{-1} = [\partial B_{n,\alpha} \partial^{-1}, L^*], \]
\[ \partial_{n,\alpha} L^* = [B_{n,\alpha}, L]^* = (B_{n,\alpha} L - L B_{n,\alpha})^* = L^* B_{n,\alpha}^* - B_{n,\alpha}^* L^* = [-B_{n,\alpha}^*, L^*] \tag{3.2} \]

and

\[ \partial_{n,\alpha} R_\beta^* = (\partial_{n,\alpha} R_\beta)^* = [B_{n,\alpha}, R_\beta]^* = \partial [B_{n,\alpha}, R_\beta] \partial^{-1} = [\partial B_{n,\alpha} \partial^{-1}, R_\beta^*], \]
\[ \partial_{n,\alpha} R_\beta^* = [B_{n,\alpha}, R_\beta]^* = (B_{n,\alpha} R_\beta - R_\beta B_{n,\alpha})^* = R_\beta^* B_{n,\alpha}^* - B_{n,\alpha}^* R_\beta^* = [-B_{n,\alpha}^*, R_\beta^*] \tag{3.3} \]

Comparing these equations, we obtain

\[ B_{n,\alpha}^* = -\partial B_{n,\alpha} \partial^{-1}, \quad n \neq 0 \pmod{N}. \tag{3.4} \]

Further, we take the definition of \( B_{n,\alpha} (n \neq 0 \pmod{N}) \) into account and find its adjoint operator

\[ B_{n,\alpha}^* = ((L_n^{n/N} R_\alpha)_+)^* = (-1)^n \partial (R_\alpha L^{n/N}_+) \partial^{-1} = (-1)^n \partial B_{n,\alpha} \partial^{-1}. \tag{3.5} \]

Analyzing relations (3.4) and (3.5), we find that \( n \) can take only odd values. The proposition is proved.

Therefore, we can write the Lax equations of the B-type mcGD hierarchy more concisely,

\[ \partial_{2n+1,\alpha} L = [B_{2n+1,\alpha}, L], \quad \partial_{2n+1,\alpha} R_\beta = [B_{2n+1,\alpha}, R_\beta], \quad 2n + 1 \neq 0 \pmod{N}, \]
\[ \sum_{\alpha=1}^{n} \partial_{(2j'+1)N,\alpha} L = 0, \quad \sum_{\alpha=1}^{n} \partial_{(2j'+1)N,\alpha} R_\beta = 0, \quad j' = 0, 1, 2, \ldots, \]

where \( \partial_{2n+1,\alpha} = \partial/\partial t_{2n+1,\alpha} \) and \( B_{2n+1,\alpha} = (L_{(2n+1)/N}^R R_\alpha)_+ \).

The Lax operators of the B-type mcGD hierarchy can also be expressed in the dressed form

\[ L = \phi A \delta^N \phi^{-1}, \quad R_\alpha = \phi E_\alpha \phi^{-1}, \tag{3.6} \]

with the dressing operator

\[ \phi = \phi(A \delta^N) = \sum_{i=0}^{\infty} \alpha_i (A \delta^N)^{-i}, \quad \alpha_0 = I, \]

and \( \phi^* = \partial \phi^{-1} \partial^{-1} \), which distinguishes the considered case from the mcGD hierarchy.
Further, we obtain the Sato equations of the B-type mcGD hierarchy,

$$\partial_{2n+1,\alpha} \phi = -(B_{2n+1,\alpha} - \phi), \quad 2n + 1 \neq 0 \text{ (mod } N),$$

$$\sum_{\alpha=1}^{n} \partial_{(2j+1)N,\alpha} \phi = 0, \quad j' = 0, 1, 2, \ldots.$$ 

The wave function $W(t, z)$ and the adjoint wave function $W^*(t, z)$ in this case are given by

$$W(t, z) = \phi e^{\xi(t, z)} = \omega(t, z) e^{\xi(t, z)},$$

$$W^*(t, z) = (\phi^*)^{-1} e^{-\xi(t, z)} = (\partial \phi \partial^{-1}) e^{-\xi(t, z)},$$

where

$$\xi(t, z) = \sum_{i=1}^{\infty} \sum_{\alpha=1}^{n} t_{2i-1,\alpha} E_{\alpha} z^{2i-1}.$$ 

We note that the Lax equations of the B-type mcGD hierarchy can also be derived from the equations

$$L^{(2n+1)/N} W = z^{2n+1} W, \quad \partial_{2n+1,\alpha} W = B_{2n+1,\alpha} W, \quad 2n + 1 \neq 0 \text{ (mod } N),$$

$$\sum_{\alpha=1}^{n} \partial_{(2j+1)N,\alpha} W = L^{2j+1} W, \quad j' = 0, 1, 2, \ldots.$$ 

### 3.1. Additional symmetry of the B-type mcGD hierarchies

We first give the Orlov–Shulman operator $M = \phi \Gamma \phi^{-1}$ for the B-type mcGD hierarchy, where

$$\Gamma = \sum_{i=1}^{\infty} \sum_{\alpha=1}^{n} (2i-1)t_{2i-1,\alpha} A^{(2i-2)/N} \partial^{2i-2} E_{\alpha}, \quad 2i - 1 \neq 0 \text{ (mod } N).$$

We easily find that $[M, L^{(2n+1)/N}] = -(2n + 1) L^{2n/N}.$

We next define the additional symmetries of the B-type mcGD hierarchies. We first introduce additional independent variables $t_{l,m,\alpha}^*$ and define the action of the additional flows on the wave operator $\phi$ as

$$\partial_{l,m,\alpha}^* \phi = -(D_{l,m,\alpha})_+ \phi,$$

where $D_{l,m,\alpha} = M^m L^{l/N} R_{\alpha} - (-1)^l R_{\alpha} L^{(l-1)/N} M^m L^{1/N}.$

Similarly to Sec. 2, assuming that the differential operator $D_{l,m,\alpha}$ has no negative part and acting with the operator $(D_{l,m,\alpha})_+$ on $W$, we obtain an equation related to $z$,

$$(D_{l,m,\alpha})_+ W = (z^l - (-1)^l z^{2l-1}) E_{\alpha} \partial_z^m W - (-1)^l m(l - 1) z^{2(l-1)} E_{\alpha} \partial_z^{m-1} W.$$ (3.9)

We note that this system can also be rewritten as a linear equation for an isomonodromy problem.

Using simple calculations, based on the definition of an additional symmetry of the B-type mcGD hierarchy, we obtain the formulas

$$\partial_{l,m,\alpha}^* L = -[(D_{l,m,\alpha})_-, L], \quad \partial_{l,m,\alpha}^* R_{\beta} = -[(D_{l,m,\alpha})_-, R_{\beta}].$$ (3.10)

Only some of the additional flows of the B-type mcGD hierarchy survive.
Theorem 3.1. Only those additional flows of the B-type mcGD hierarchy survive that satisfy either the condition \((M^{m-1}L^{(N+l-1)/N})_-=0\) or the condition \(l=2i\ (i \in \mathbb{Z})\) and, moreover, in both cases have the form \(\sum_{\alpha=1}^{n} \partial^*_{l,m,\alpha}\).

Proof. The negative part of the Lax operator \(L\) of the B-type mcGD hierarchy is equal to zero, and, as is known, \(\partial^*_{l,m,\alpha} L = -(D_{l,m,\alpha})_-.\) Therefore, we consider

\[
\partial^*_{l,m,\alpha} L_- = -[(D_{l,m,\alpha})_-, L] = \end{align*}

\[
= -(\phi[\Gamma^m A^{l/N} \partial^l E_\alpha, A\partial^N]|\phi^{-1})_+ + (-1)\frac{l}{N}(\phi[E_\alpha A^{l-1/N} \partial^l \Gamma^m A^{1/N} \partial, A\partial^N]|\phi^{-1})_-. \]

After some calculations, this formula reduces to

\[
\partial^*_{l,m,\alpha} L_- = (1 - (-1)^l)mN(M^{m-1}L^{(N+l-1)/N} R_{\alpha})_. \]

We obtain

\[
\sum_{\alpha=1}^{n} \partial^*_{l,m,\alpha} L_- = (1 - (-1)^l)mN(M^{m-1}L^{(N+l-1)/N})_-= 0
\]

only if \((M^{m-1}L^{(N+l-1)/N})_- = 0\) or \(l=2i\ (i \in \mathbb{Z})\).

Proposition 3.2. Additional symmetric flows \(\sum_{\alpha=1}^{n} \partial^*_{l,m,\alpha}\) satisfying \((M^{m-1}L^{(N+l-1)/N})_- = 0\) or \(l=2i\ (i \in \mathbb{Z})\) commute with the flows \(\partial_{2k+1, \beta}\) \((2k+1 \neq 0 \bmod N)\) of the B-type mcGD hierarchy.

Proof. We have

\[
\left[ \sum_{\alpha=1}^{n} \partial^*_{l,m,\alpha}, \partial_{2k+1, \beta} \right] \phi = \left( \sum_{\alpha=1}^{n} [\partial^*_{l,m,\alpha}, \partial_{2k+1, \beta}] \right) \phi = \sum_{\alpha=1}^{n} [\partial^*_{l,m,\alpha}, \partial_{2k+1, \beta}] \phi,
\]

and

\[
[\partial^*_{l,m,\alpha}, \partial_{2k+1, \beta}] \phi = -\partial^*_{l,m,\alpha} ((B_{2k+1, \beta})_- \phi) + \partial_{2k+1, \beta} ((D_{l,m,\alpha})_- \phi) =
\]

\[
= [(D_{l,m,\alpha})_-, (B_{2k+1, \beta})_- + \partial_{2k+1, \beta}]_- \phi + (B_{2k+1, \beta})_- (D_{l,m,\alpha})_- \phi +
\]

\[
+ [\partial_{2k+1, \beta}, (D_{l,m,\alpha})]_- \phi - (D_{l,m,\alpha})_- (B_{2k+1, \beta})_- \phi =
\]

\[
= [(D_{l,m,\alpha})_-, (B_{2k+1, \beta})_-]_- \phi - [(D_{l,m,\alpha})_-, (B_{2k+1, \beta})]_- \phi = 0,
\]

whence \(\sum_{\alpha=1}^{n} \partial^*_{l,m,\alpha}, \partial_{2k+1, \beta} \phi = 0\), which proves the proposition.

We now consider the additional flow with \(l = -(2k-1), \ l - 1 = 0 \ (\bmod N)\),

\[
\sum_{\alpha=1}^{n} \partial^*_{-(2k-1), 1, \alpha} L^{2k/N} = -\sum_{\alpha=1}^{n} [(D_{-(2k-1), 1, \alpha})_-, L^{2k/N}], \ 2k = 0 \ (\bmod N).
\]

After some calculations, we obtain

\[
\sum_{\alpha=1}^{n} \partial^*_{-(2k-1), 1, \alpha} L^{2k/N} = \sum_{\alpha=1}^{n} [(D_{-(2k-1), 1, \alpha})_+ , L^{2k/N}] + 4kI.
\]

If \(2k = 0 \ (\bmod N)\), then \(L^{2k/N}\) is a differential operator, and

\[
\sum_{\alpha=1}^{n} \left[ L^{2k/N}, \frac{1}{4k} (D_{-(2k-1), 1, \alpha})_+ \right] = I,
\]

whence we obtain the string equation for the B-type mcGD hierarchy

\[
\left[ L^{2k/N}, \frac{1}{4k} \sum_{\alpha=1}^{n} (D_{-(2k-1), 1, \alpha})_+ \right] = I.
\]
3.2. The $\tau$-function and Virasoro constraint of the B-type mcGD hierarchies. We formulate the existence theorem for the $\tau$-function [2].

**Theorem 3.2.** We assume that the $\tau$-function is a matrix $T = (\tau_{\alpha,\beta})$, and we write $\tau$ instead of $\tau_{\alpha,\alpha}$ for brevity. There exists a function $\tau(\ldots,t_{2s-1},\ldots)$ such that

$$
\omega_{\alpha,\alpha}(t,z) = \frac{\tau(\ldots,t_{2s-1},\gamma - 1/(2s-1)z^{2s-1}),\ldots)}{\tau(\ldots,t_{2s-1},\gamma,\ldots)},
$$

and

$$
\omega_{\alpha,\beta}(t,z) = \frac{\tau_{\alpha,\beta}(\ldots,t_{2s-1},\gamma - 1/(2s-1)z^{2s-1}),\ldots)}{z \cdot \tau(\ldots,t_{2s-1},\gamma,\ldots)},
$$

where $\alpha \neq \beta$ and $2s - 1 \neq 0 \pmod{N}$ (the time $t_{2s-1}$ shifts only if $\gamma = \beta$). The equation

$$
\sum_{\alpha=1}^{n} \partial_{k,\alpha} \tau = 0, \quad k = 0 \pmod{N} \text{ or } k = 0 \pmod{2},
$$

is satisfied in all cases except $2s - 1 \neq 0 \pmod{N}$ noted above.

Further, starting from the additional symmetries of the wave operator $\phi$ and taking the existence theorem for the $\tau$-function into account, we study the additional symmetries of the $\tau$-function.

**Proposition 3.3.** We can bring the operator $D_{-(2k-1),1,\alpha}$ ($2k = 0 \pmod{N}$, $k > 0$) to the form

$$
(D_{-(2k-1),1,\alpha})_- = 2\phi \left( \sum_{i=1}^{k} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} E_{\alpha} \right) \phi^{-1} + 2 \sum_{i=k+1}^{\infty} (2i-1)t_{2i-1,\alpha} (L^{(2i-2k-1)/N} R_{\alpha})_- - 2kL^{-2k/N} R_{\alpha}. \tag{3.12}
$$

**Proof.** According to the definition of an additional symmetry of the B-type mcGD hierarchy, we have

$$
(D_{-(2k-1),1,\alpha})_- = (ML^{-(2k-1)/N} R_{\alpha} - (1)^{-(2k-1)} R_{\alpha} L^{(l-1)/N} M^{m} L^{1/N})_- =
$$

$$
= (ML^{-(2k-1)/N} R_{\alpha})_- + (R_{\alpha} L^{(l-1)/N} M^{m} L^{1/N})_- =
$$

$$
= 2(\phi \Gamma A^{-(2k-1)/N} \partial^{-(2k-1)} E_{\alpha} \phi^{-1})_- - 2k(\phi E_{\alpha} A^{-2k/N} \partial^{-2k} \phi^{-1})_- =
$$

$$
= 2 \left( \phi \sum_{i=1}^{\infty} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} E_{\alpha} \phi^{-1} \right)_- - 2kL^{-2k/N} R_{\alpha} =
$$

$$
= 2\phi \left( \sum_{i=1}^{k} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} E_{\alpha} \right) \phi^{-1} + 2 \sum_{i=k+1}^{\infty} (2i-1)t_{2i-1,\alpha} (L^{-(2i-2k-1)/N} R_{\alpha})_- - 2kL^{-2k/N} R_{\alpha}.
$$

The proposition is proved.
Based on the proposition proved above and the additional symmetries of the B-type mcGD hierarchy, we can calculate

\[
\sum_{\alpha=1}^{n} \partial^*_{\alpha(2k-1),1,\alpha} \phi = - \sum_{\alpha=1}^{n} (D_{-(2k-1),1,\alpha})_{-} \phi = \\
\sum_{\alpha=1}^{n} \left( -2\phi \sum_{i=1}^{k} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} E_{\alpha} + \\
+ 2 \sum_{i=k+1}^{\infty} (2i-1)t_{2i-1,\alpha} (\partial_{2(i-k)-1,\alpha}) \phi \right) + 2k\phi A^{-2k/N} \partial^{-2k} I, \tag{3.13}
\]

where \(2k = 0 \pmod{N}\). Obviously, if we apply both sides of Eq. (3.13) to the function \(\exp(A^{-1/N}xz)\), the equation still holds. Further calculation yields

\[
[\phi, t_{1,\alpha}] \exp(A^{-1/N}xz) = (\partial \omega) \exp(A^{-1/N}xz),
\]

\[
\phi \partial^{-1} \exp(A^{-1/N}xz) = \phi z^{-1} \exp(A^{-1/N}xz) = z^{-1} \omega(\exp(A^{-1/N}xz)),
\]

and we obtain

\[
\left( \sum_{\alpha=1}^{n} \partial^*_{\alpha(2k-1),1,\alpha} \right) \exp(A^{-1/N}xz) = \left( - \sum_{\alpha=1}^{n} (D_{-(2k-1),1,\alpha})_{-} \phi \right) \exp(A^{-1/N}xz) = \\
- 2z^{-2k+1}(\partial \omega) \exp(A^{-1/N}xz) + 2kz^{-2k}\omega(\exp(A^{-1/N}xz)) - \\
- 2 \left( \sum_{i=1}^{k} \sum_{\alpha=1}^{n} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} E_{\alpha} \omega \right) \exp(A^{-1/N}xz) + \\
+ 2 \sum_{i=k+1}^{\infty} \sum_{\alpha=1}^{n} (2i-1)t_{2i-1,\alpha} (\partial_{2(i-k)-1,\alpha}) \exp(A^{-1/N}xz)
\]

and consequently

\[
\sum_{\alpha=1}^{n} \partial^*_{\alpha(2k-1),1,\alpha} \omega = - 2z^{-2k+1}(\partial \omega) I + 2kz^{-2k}\omega I - \\
- 2 \sum_{i=1}^{k} \sum_{\alpha=1}^{n} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} E_{\alpha} \omega + \\
+ 2 \sum_{i=k+1}^{\infty} \sum_{\alpha=1}^{n} (2i-1)t_{2i-1,\alpha} (\partial_{2(i-k)-1,\alpha}) \omega.
\]

Correspondingly, we have

\[
\sum_{\alpha=1}^{n} \partial^*_{\alpha(2k-1),1,\alpha}\omega_{\gamma,\beta} = \sum_{\alpha=1}^{n} \left( -2z^{-2k+1} \delta_{\alpha,\beta} (\partial \omega_{\gamma,\beta}) + 2kz^{-2k} \delta_{\alpha,\beta} \omega_{\gamma,\beta} - \\
- 2 \sum_{i=1}^{k} (2i-1)t_{2i-1,\alpha} A^{(2i-2k-1)/N} \partial^{2i-2k-1} \delta_{\alpha,\beta} \omega_{\gamma,\beta} + \\
+ 2 \sum_{i=k+1}^{\infty} (2i-1)t_{2i-1,\alpha} (\partial_{2(i-k)-1,\alpha}) \omega_{\gamma,\beta} \right).
\]
We can show that $\sum_{\alpha=1}^{n} \partial_{-(2k-1),1,\alpha}^{*} \phi = 0$ and $\sum_{\alpha=1}^{n} \partial_{-(2k-1),1,\alpha}^{*} \omega = 0$ are equivalent. As a result, we obtain constraints on $\tau$-function (3.11).

It is difficult to find a unified simple form for $\sum_{\alpha=1}^{n} \partial_{-(2k-1),1,\alpha}^{*}$ ($2k = 0 \ (\text{mod} \ N)$, $k > 0$), and we therefore restrict ourselves here to the case where $k$ is a positive integer.

**Proposition 3.4.** The additional symmetries $\sum_{\alpha=1}^{n} \partial_{-(2k-1),1,\alpha}^{*}$ (where $2k = 0 \ (\text{mod} \ N)$, $k > 0$) act on $(\tau_{\gamma,\beta})$ as

$$\sum_{\alpha=1}^{n} \partial_{-(2k-1),1,\alpha}^{*} \tau_{\gamma,\beta} = \sum_{\alpha=1}^{n} \left( \frac{1}{2} \sum_{m=2k}^{\infty} (2m-1)t_{2m-1,\alpha} \partial_{2(m-k)-1,\alpha} + \frac{1}{8} \sum_{m+s=2k} (2m-1)(2s-1) t_{2m-1,\alpha} t_{2s-1,\alpha} \right) \tau_{\gamma,\beta} + \left( \sum_{\alpha=1}^{n} c_{\alpha,\beta} \right) \tau_{\gamma,\beta}, \quad (3.14)$$

where $t_{2m-1,\alpha}$ and $t_{2s-1,\alpha}$ are arguments of the element $\tau_{\gamma,\beta}$.

Setting

$$L_{-(2k-1)} = \sum_{\alpha=1}^{n} \left( \frac{1}{2} \sum_{m=2k}^{\infty} (2m-1)t_{2m-1,\alpha} \partial_{2(m-k)-1,\alpha} + \frac{1}{8} \sum_{m+s=2k} (2m-1)(2s-1) t_{2m-1,\alpha} t_{2s-1,\alpha} + c_{\alpha,\beta} \right),$$

where $2k = 0 \ (\text{mod} \ N)$ and $k > 0$, we can verify that $L_{-(2k-1)}$ satisfies Virasoro commutation relation (2.25), and the Virasoro constraint on the $\tau$-function of the B-type mcGD hierarchy is hence

$$L_{-(2k-1)} \tau = 0, \quad 2k = 0 \ (\text{mod} \ N), \quad k > 0. \quad (3.15)$$

### 4. C-type multicomponent Gelfand–Dickey hierarchies

The Lax operators $L$ and $R_{\alpha}$ of the C-type mcGD hierarchy are obtained from Lax operators (2.1) and (2.2) of the mcGD hierarchy if we impose the conditions $L^{*} = (-1)^{N} L$ and $R_{\alpha}^{*} = R_{\alpha}$. Clearly, we can distinguish two classes of the C-type mcGD hierarchy depending on the parity of $N$. In the case of odd $N$, the differential operator $L$ given by (2.1) satisfies the condition $L^{*} = -L$. In the case of even $N$, we have $L^{*} = L$. Although these constraints differ, the symmetry generators, the surviving flows, the string equation, etc., for the two classes of C-type mcGD hierarchies coincide.

Similarly to case of the B-type mcGD hierarchy, the Lax equations of the C-type mcGD hierarchy do not have even flows, i.e., $u_{i} = u_{i}(x; t_{1}, t_{3}, \ldots)$.

**Proposition 4.1.** The C-type mcGD hierarchy has only odd flows.

**Proof.** From the analysis presented above, we already know the Lax equations of the C-type mcGD hierarchy. Combined with the constraints on its Lax equations, we obtain

$$(L^{n/N})^{*} = (-1)^{n} L^{n/N}, \quad (L^{n/N} R_{\alpha})^{*} = (-1)^{n} R_{\alpha} L^{n/N}.$$  

Further, we write $\partial_{n,\alpha} L^{*}$ and $\partial_{n,\alpha} R_{\alpha}^{*}$ in a simpler form in two ways:

$$\partial_{n,\alpha} L^{*} = \partial_{n,\alpha} L^{*} = [B_{n,\alpha}, L^{*}] = -[B_{n,\alpha}, L] = [B_{n,\alpha}, L^{*}], \quad (4.1)$$

$$\partial_{n,\alpha} L^{*} = [B_{n,\alpha}, L]^{*} = (B_{n,\alpha} L - LB_{n,\alpha})^{*} = L^{*} B_{n,\alpha}^{*} - B_{n,\alpha}^{*} L^{*} = [B_{n,\alpha}^{*}, L^{*}].$$

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\[ \partial_{n,\alpha} R_{\beta}^* = \partial_{n,\alpha} (R_{\beta})^* = [B_{n,\alpha}, R_{\beta}] = [B_{n,\alpha}, R_{\beta}^*], \quad 2n + 1 \neq 0 \pmod{N}, \]

\[ \partial_{n,\alpha} R_{\beta}^* = [B_{n,\alpha}, R_{\beta}]^* = (B_{n,\alpha} R_{\beta} - R_{\beta} B_{n,\alpha})^* = R_{\beta}^* B_{n,\alpha}^* - B_{n,\alpha}^* R_{\beta}^* = [-B_{n,\alpha}, R_{\beta}^*]. \]

Comparing these representations, we obtain

\[ B_{n,\alpha}^* = -B_{n,\alpha}, \quad n \neq 0 \pmod{N}. \]

We then take the definition of \( B_{n,\alpha} \) \((n \neq 0 \pmod{N})\) into account and obtain the expression for its adjoint,

\[ B_{n,\alpha}^* = \left((-1)^n (R_{\alpha} L n/N) + \right) = (-1)^n B_{n,\alpha}, \quad n \neq 0 \pmod{N}. \]

Analyzing relations (4.3) and (4.4), we find that \( n \) can take only odd values.

Therefore, we can write the Lax equations as

\[ \partial_{2n+1,\alpha} L = [B_{2n+1,\alpha}, L], \quad \partial_{2n+1,\alpha} R_{\beta} = [B_{2n+1,\alpha}, R_{\beta}], \quad 2n + 1 \neq 0 \pmod{N}, \]

\[ \sum_{\alpha=1}^{n} \partial_{(2j'+1)N,\alpha} L = 0, \quad \sum_{\alpha=1}^{n} \partial_{(2j'+1)N,\alpha} R_{\beta} = 0, \quad j' = 0, 1, 2, \ldots. \]

Taking a dressing operator of the form

\[ \phi = \phi(A \partial^N) = \sum_{i=0}^{\infty} \alpha_i (A \partial^N)^{-i}, \quad \alpha_0 = 1, \quad \phi^* = \phi^{-1}, \]

we obtain the wave function \( W(t, z) = \phi e^{\xi(t,z)} = \omega(t, z) e^{\xi(t,z)} \) of the C-type mcGD hierarchy, where

\[ \xi(t,z) = \sum_{i=1}^{\infty} \sum_{\alpha=1}^{n} t_{2i-1,\alpha} F_{\alpha} z^{2i-1}. \]

In addition, we have the adjoint wave function

\[ W^*(t, z) = (\phi^*)^{-1} e^{-\xi(t,z)} = \phi e^{-\xi(t,z)}. \]

Further, we obtain the Sato equations of the C-type mcGD hierarchy:

\[ \partial_{2n+1,\alpha} \phi = -(B_{2n+1,\alpha}) \phi, \quad 2n + 1 \neq 0 \pmod{N}, \]

\[ \sum_{\alpha=1}^{n} \partial_{(2j'+1)N,\alpha} \phi = 0, \quad j' = 0, 1, 2, \ldots. \]

The Lax equations of the C-type mcGD hierarchy can also be obtained from the compatibility conditions for the linear partial differential equations

\[ L^{(2n+1)/N} W = z^{2n+1} W, \quad \partial_{2n+1,\alpha} W = B_{2n+1,\alpha} W, \quad 2n + 1 \neq 0 \pmod{N}, \]

\[ \sum_{\alpha=1}^{n} \partial_{(2j'+1)N,\alpha} W = L^{(2j'+1)} W, \quad j' = 0, 1, 2, \ldots, \]

where \( W(t, z) \) is the wave function of the C-type mcGD hierarchy.
Additional symmetry of the C-type mcGD hierarchies. Based on the results obtained above related to the C-type mcGD hierarchy, we can observe that many concepts and properties of this hierarchy are similar to those of the B-type mcGD hierarchy and the biggest difference is related to the definitions and operations of the adjoint hierarchy. Therefore, we mainly discuss this topic in detail in what follows. Similarly to what was done above, we first define the Orlov–Shulman operator $M = \phi \Gamma \phi^{-1}$ of the C-type mcGD hierarchy, where

$$
\Gamma = \sum_{i=1}^{\infty} \sum_{\alpha=1}^{n} (2i-1)t_{2i-1,\alpha} A^{(2i-2)/N} \partial^{2i-2} E_{\alpha}, \quad 2i - 1 \not\equiv 0 \,(\text{mod} \, N),
$$

whence we obtain

$$
M^* = (\phi \Gamma \phi^{-1})^* = \phi \Gamma \phi^{-1} = M.
$$

Keeping some necessary information in mind, we define the additional symmetries of the C-type mcGD hierarchy. The generators $C_{l,m,\alpha}$ obviously differ from the generators $D_{l,m,\alpha}$ of the additional symmetries of the B-type mcGD hierarchy, and the basic reason for this is the different constraints imposed on the Lax operators.

**Proposition 4.2.** In the C-type mcGD hierarchy, the generators $C_{l,m,\alpha}$ of the additional symmetries must satisfy the condition $(C_{l,m,\alpha})^* = -C_{l,m,\alpha}$, and these generators can then be taken in the form

$$
C_{l,m,\alpha} = M^m L^{1/N} R_{\alpha} - (-1)^{l} R_{\alpha} L^{1/N} M^m. \quad (4.7)
$$

**Proof.** In accordance with the properties of the additional symmetries of the C-type mcGD hierarchy, we know that

$$
\partial*_{l,m,\alpha} \phi^* = (\partial*_{l,m,\alpha} \phi)^* = (-1)^{-l} (C_{l,m,\alpha})^* = -\phi^*(C_{l,m,\alpha})^* - \phi^* (C_{l,m,\alpha})^* - 
$$

and because $\phi^* = \phi^{-1}$, we hence derive the relation

$$
\partial*_{l,m,\alpha} \phi^* = \partial*_{l,m,\alpha} \phi^{-1} = -\phi^{-1} (\partial*_{l,m,\alpha} \phi)^{-1} = \phi^{-1} (C_{l,m,\alpha})^* = \phi^* (C_{l,m,\alpha})^*.
$$

Comparing these two results, we find that $(C_{l,m,\alpha})^* = - (C_{l,m,\alpha})^*$ and consequently $C_{l,m,\alpha} = -(C_{l,m,\alpha})^*$. For the generators $M^m L^{1/N} R_{\alpha}$ of the additional symmetries of the mcGD hierarchy, we have

$$
(M^m L^{1/N} R_{\alpha})^* = R_{\alpha}^* (L^{1/N})^* (M^m)^* = (-1)^{l} R_{\alpha} L^{1/N} M^m.
$$

It is hence easy to obtain

$$
C_{l,m,\alpha} = M^m L^{1/N} R_{\alpha} - (-1)^{l} R_{\alpha} L^{1/N} M^m.
$$

The proposition is proved.

We next define the additional symmetries of the c-type mcGD hierarchies. We first introduce additional independent variables $t*_{l,m,\alpha}$ and define the action of the additional flows on the wave operator $\phi$ as

$$
\partial*_{l,m,\alpha} \phi = -(C_{l,m,\alpha})^* \phi, \quad \text{where } C_{l,m,\alpha} = M^m L^{1/N} R_{\alpha} - (-1)^{l} R_{\alpha} L^{1/N} M^m. \quad (4.8)
$$
Similarly to what was done above, assuming that the differential operator \( C_{l,m,\alpha} \) has no negative part and acting with the operator \((C_{l,m,\alpha})_+\) on \( W \), we obtain the equation related to \( z \)

\[
(C_{l,m,\alpha})_+ W = (1 - (-1)^i) z^l E_\alpha \partial_z^m W - (-1)^i m l z^{-1} E_\alpha \partial_z^{m-1} W.
\] (4.9)

We note that this system can also be rewritten as a linear equation for an isomonodromy problem.

Further, we briefly describe the method for constructing \( C_{l,m,\alpha} \). In accordance with the properties of the additional symmetries, we have

\[
\partial^a_{l,m,\alpha} L = -[(C_{l,m,\alpha})_-, L], \quad \partial^a_{l,m,\alpha} R_\beta = -[(C_{l,m,\alpha})_-, R_\beta].
\]

Only some of the additional flows of the C-type mcGD hierarchy survive.

**Theorem 4.1.** Only those additional flows of the C-type mcGD hierarchy survive that satisfy either \((M^{m-1} L^{(N+l-1)/N})_- = 0\) or \( l = 2i \ (i \in \mathbb{Z})\) and, moreover, have the form \( \sum_{\alpha=1}^n \partial^a_{l,m,\alpha} \) in both cases.

**Proof.** The Lax operator \( L \) of the C-type mcGD hierarchy has a negative part equal to zero. Because \( \partial^a_{l,m,\alpha} L = -[(C_{l,m,\alpha})_-, L] \), we consider

\[
\partial^a_{l,m,\alpha} L_+ = -[(C_{l,m,\alpha})_-, L]_+ = -\left(\phi[\Gamma^m A^{l/N} \partial^l E_\alpha, A \partial^N]\phi^{-1}\right)_+ + (-1)^i \left(\phi[E_\alpha A^{l/N} \partial^l \Gamma^m, A \partial^N]\phi^{-1}\right)_+.
\]

After some calculations, this formula reduces to

\[
\partial^a_{l,m,\alpha} L_- = \partial^a_{l,m,\alpha} L_+ = (1 - (-1)^i) m N \left( M^{m-1} L^{(N+l-1)/N} R_\alpha \right)_-.
\]

We obtain

\[
\sum_{\alpha=1}^n \partial^a_{l,m,\alpha} L_- = (1 - (-1)^i) m N \left( M^{m-1} L^{(N+l-1)/N} \right)_- = 0
\]

only if \( (M^{m-1} L^{(N+l-1)/N})_- = 0 \) or \( l = 2i \ (i \in \mathbb{Z}) \).

The surviving additional flows of the C-type mcGD hierarchy satisfy the following proposition.

**Proposition 4.3.** Additional symmetric flows \( \sum_{\alpha=1}^n \partial^a_{l,m,\alpha} \) satisfying \((M^{m-1} L^{(N+l-1)/N})_- = 0\) or \( l = 2i \ (i \in \mathbb{Z}) \) commute with the flows \( \partial_{2k+1,\beta} \) \((2k+1 \neq 0 \ (\text{mod} \ N))\) of the C-type mcGD hierarchy.

**Proof.** We have

\[
\left[ \sum_{\alpha=1}^n \partial^a_{l,m,\alpha}, \partial_{2k+1,\beta} \right] \phi = \left( \sum_{\alpha=1}^n \left[ \partial^a_{l,m,\alpha}, \partial_{2k+1,\beta} \right] \right) \phi = \sum_{\alpha=1}^n \left[ \partial^a_{l,m,\alpha}, \partial_{2k+1,\beta} \right] \phi
\]

and

\[
[\partial^a_{l,m,\alpha}, \partial_{2k+1,\beta}] \phi = -\partial^a_{l,m,\alpha} ((B_{2k+1,\beta})_- \phi) + \partial_{2k+1,\beta} ((C_{l,m,\alpha})_- \phi) =
\]

\[
= [(C_{l,m,\alpha})_-, (B_{2k+1,\beta})_- + \partial_{2k+1,\beta}]_- \phi + (B_{2k+1,\beta})_- (C_{l,m,\alpha})_- \phi +
\]

\[
+ [\partial_{2k+1,\beta}, (C_{l,m,\alpha})_-]_- \phi - (C_{l,m,\alpha})_- (B_{2k+1,\beta})_- \phi =
\]

\[
= [(C_{l,m,\alpha})_-, (B_{2k+1,\beta})_-] \phi - [(C_{l,m,\alpha})_-, (B_{2k+1,\beta})_-]_- \phi = 0.
\]

Then

\[
\left[ \sum_{\alpha=1}^n \partial^a_{l,m,\alpha}, \partial_{2k+1,\beta} \right] \phi = 0,
\]

which proves the proposition.
After some calculations, we find that independently of whether the order of the C-type mcGD hierarchy is even or odd, the string equation is the same. We present the procedure for deriving this equation below.

We first consider the additional flow

$$\sum_{\alpha=1}^{n} \partial^*_{-(2k-1),1,\alpha} L^{2k/N} = -\sum_{\alpha=1}^{n} \left( C_{-(2k-1),1,\alpha}, L^{2k/N} \right), \quad 2k = 0 \pmod{N}.$$  

After some calculations, we obtain the relation

$$\sum_{\alpha=1}^{n} \partial^*_{-(2k-1),1,\alpha} L^{2k/N} = \sum_{\alpha=1}^{n} \left( C_{-(2k-1),1,\alpha} + L^{2k/N} \right) + 4kI.$$  

If $2k = 0 \pmod{N}$, then $L^{2k/N}$ is a differential operator, and

$$\sum_{\alpha=1}^{n} \left[ L^{2k/N}, \frac{1}{4k} (C_{-(2k-1),1,\alpha})^+ \right] = I,$$

and

$$\left[ L^{2k/N}, \frac{1}{4k} \left( \sum_{\alpha=1}^{n} (C_{-(2k-1),1,\alpha})^+ \right) \right] = I$$

is therefore the string equation of the C-type mcGD hierarchy.

In the literature, the string equation is sometimes written as $[P, Q] = 1$, where $P$ and $Q$ are differential operators, which is equivalent to the equation obtained here, and relates to the case where the operator is independent of the variables.

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