BOUNDS ON LEAVES OF FOLIATIONS OF THE PLANE

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Abstract. This paper contributes to the solution of the Poincaré problem, which is to bound the degree of a (generalized algebraic) leaf of a (singular algebraic) foliation of the complex projective plane. The first theorem gives a new sort of bound, which involves the Castelnuovo–Mumford regularity of the singular locus of the leaf. The second theorem gives a bound in terms of two singularity numbers of the leaf: the total Tjurina number, and the number of non-quasi-homogeneous singularities. If such singularities are present, then this bound improves one due to du Plessis and Wall, at least when the curve is irreducible.

1. Introduction

When does a singular foliation of $\mathbb{P}_C^2$ defined by polynomials have a leaf defined by a polynomial? This question is fundamental, but difficult, and it has stimulated a lot of research for well over a century.

Notably, in 1891, Poincaré [P], p. 161, observed that, once we possess a bound on the degree $d$ of a polynomial $F$ defining a leaf, we can try to find $F$ by making purely algebraic computations. The problem of finding such bounds is now known as the Poincaré problem. The substantial current interest in it was stimulated by Cerveau and Lins Neto [CL] in 1991.

The available bounds on the degree $d$ of $F$ depend on the degree $m$ of the foliation and on the number and type of the singularities of the Zariski closure $C$ of the leaf, the curve defined by $F$. For example, Campillo and Carnicer [CC] found bounds depending on the topological type as encoded in the Enriques resolution diagram; their results are improved in [EK1], Theorem 5.3. Du Plessis and Wall [dPW] found bounds depending on the analytic type as encoded in the Tjurina number; their results are generalized to curves in any projective space in [EK1], Section 6, via a second approach, and they are improved below via a third approach. Additional work is cited in [EK1].

The present note introduces a new sort of bound, one that also takes into account the distribution of the singularities as measured by the Castelnuovo–Mumford regularity $\sigma$ of the singular locus of $C$. In a similar vein, [ES] introduced the study of the regularity of a

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variety invariant under a singular vector field of projective space. And [EK2] introduced the study of the regularity of the intersection of a solution of a Pfaff system with the singular locus of the system.

In the present note, the first main result is Theorem 2.5, which asserts this: if \( \sigma \leq d - 2 \), then \( d \leq m + 1 \); otherwise, \( d \leq m + 1 + \rho \) where \( \rho := \sigma - d - 2 \); furthermore, \( d = m + 1 + \rho \) if \( d \geq 2m + 2 \) and if the foliation has finitely many singularities. The proof requires Corollary 4.5 of [EK2], but only its assertion that \( h^1(\mathcal{O}_C(m-1)) = 0 \) if \( h^1(\Omega_C) = 1 \).

The regularity \( \sigma \) satisfies the following upper bound, given in Lemma 3.1:

\[
\sigma \leq d - 2 + (\tau - u)/(d - 1)
\]

where \( \tau \) is the total Tjurina number, the sum of the local Tjurina numbers of \( C \), and where \( u \) is the number of non-quasi-homogeneous singularities of \( C \).

Lemma 3.1 and Theorem 2.5 yield the second main result, Theorem 3.2, which asserts that

\[
(d - 1)(d - m - 1) + u \leq \tau.
\]

By incorporating \( u \), this bound improves that of du Plessis and Wall [dPW], Theorem 3.2, at least when \( C \) is irreducible.

In singularity theory, upper bounds on \( \tau \) are also important. Proposition 6.3 of [EK1] asserts a bound for curves in any projective space, if the foliation has finitely many singularities. In the present case, the bound is this:

\[
\tau \leq (d - 1)(d - m - 1) + m^2.
\]

If \( d \leq 2m \), then the bound can be improved, provided \( C \) is irreducible and is not the closure of a leaf of a foliation of smaller degree than \( m \); namely, then

\[
\tau \leq (d - 1)(d - m - 1) + m^2 - \binom{2m+2-d}{2}.
\]

At least when \( C \) is irreducible, these two bounds agree with those proved by du Plessis and Wall [dPW], Theorem 3.2. Both bounds are asserted in Proposition 3.3 below, and proved via a rather different and more conceptual approach, similar to that of [EK1] and [EK2].

Du Plessis and Wall proved, with consummate skill, their lower and upper bounds on \( \tau \) under different, possibly weaker hypotheses, which are discussed in Remark 3.4 below. However, seen from the viewpoint of foliation theory, our hypotheses, stronger or not, do not appear to be unreasonable.

All the proofs below are purely algebraic, and work over any algebraically closed field of characteristic 0, not just \( \mathbb{C} \). In fact, Section 2 and Proposition 3.3 are set over an algebraically closed field of arbitrary characteristic \( p \), but the proof of the Theorem 2.5 requires \( p \nmid d \).

2. Bounds on the degree

2.1. Foliations of the plane. By definition, a singular foliation of \( \mathbb{P}^2 \) is a nonzero map \( \eta: \Omega_{\mathbb{P}^2} \to \mathcal{L} \) with \( \mathcal{L} \) invertible. Its singular locus is the subscheme \( S \subset \mathbb{P}^2 \) with ideal

\[
\mathcal{I}_S := \text{Im}(\eta \otimes \mathcal{L}^{-1}).
\]

The singular locus \( S \) is never empty. Indeed, its degree is given by Formula (3.3.2) below, and plainly never 0. (This formula was, according to Poincaré [P], p. 165, known before 1870 to Darboux.)
The singular foliation \( \eta \) defines an actual foliation of \( \mathbb{P}^2 - S \). For convenience, let’s call a singular foliation simply a \textit{foliation}.

Note \( \dim S \leq 1 \) since \( \eta \neq 0 \). If \( \dim S = 1 \), let \( D \subset \mathbb{P}^2 \) be the largest curve (effective divisor) contained in \( S \). Then \( \eta \) factors through a foliation \( \eta' : \Omega^1_{\mathbb{P}^2} \to \mathcal{L}(-D) \), and its singular locus is finite. Thus questions about foliations of \( \mathbb{P}^2 \) can often be reduced to the case of foliations with finite singular locus.

Let \( P \in \mathbb{P}^2 \). If \( P \notin S \), then \( \eta(P) : \Omega^1_{\mathbb{P}^2}(P) \to \mathcal{L}(P) \) is a surjection of vector spaces; hence, \( \eta(P) \) defines a 1-dimensional subspace of the tangent space \( T_P \mathbb{P}^2 \), so a line \( L_P \subset \mathbb{P}^2 \) through \( P \). Enumerate the \( P \) on a general line \( M \subset \mathbb{P}^2 \) with \( P \in S \) or \( L_P = M \); the total is called the \textit{degree} of \( \eta \) and denoted by \( \deg \eta \). More precisely, \( \deg \eta \) is the degree of the degeneracy locus of \( \Omega^1 \) on \( \mathbb{P}^2 \), set \( \deg \eta = \deg \mathcal{L} + 1 \), or
\[
\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(m - 1) \quad \text{where} \quad m := \deg \eta. \tag{2.1} \]

Let \( C \subset \mathbb{P}^2 \) be a reduced curve. Call \( C \) a \textit{generalized leaf} of \( \eta \) if no component of \( C \) lies in \( S \) and if \( \eta|C \) factors through the natural surjection \( \beta_C : \Omega^1_{\mathbb{P}^2}|C \to \Omega^1_C \).

Assume \( C \cap S \) is finite. Then \( C \) is a generalized leaf if and only if, for every simple point \( P \) of \( C \) off \( S \), the line \( L_P \) is the tangent to \( C \) at \( P \). Indeed, if \( C \) is a generalized leaf, then the induced map \( \Omega^1_C \to \mathcal{L}|C \) is, at \( P \), a surjection from one invertible sheaf to another, so an isomorphism; hence, \( L_P \) is the tangent line. Conversely, suppose \( \eta|C \) does not factor through \( \beta_C \). Set \( \mathcal{K} := \Ker \beta_C \). Then \( \eta|\mathcal{K} \) is nonzero. Hence, since \( C \) is reduced and \( \mathcal{L} \) is invertible, \( \eta|\mathcal{K} \) is surjective at infinitely many points of a component of \( C \). So \( \eta|\mathcal{K} \) is surjective at some simple point \( P \) of \( C \). Then \( P \notin S \), and \( L_P \) is not the tangent to \( C \) at \( P \).

A generalized leaf is thus a union of actual leaves of the foliation defined by \( \eta \) on \( \mathbb{P}^2 - S \). For added convenience, from now on, let’s call a generalized leaf simply a \textit{leaf}.

2.2. \textit{Singularities of plane curves.} Let \( C \subset \mathbb{P}^2 \) be a reduced curve. By definition, the \textit{singular locus} of \( C \) is the subscheme \( \Sigma \subset C \) whose ideal \( \mathcal{I}_{\Sigma,C} \) is the first Fitting ideal of \( \Omega^1_C \).

Set \( d := \deg C \). Then the ideal \( \mathcal{I}_{\Sigma,C} \) can be computed using the standard presentation
\[
\mathcal{O}_C(-d) \to \Omega^1_{\mathbb{P}^2}|C \to \Omega^1_C \to 0. \tag{2.2.1} \]
It yields a map \( \mathcal{O}_C(-d) \otimes \Omega^1_C \to \wedge^2 \Omega^1_{\mathbb{P}^2}|C \), whence a map \( \Omega^1_C \to \mathcal{O}_C(d - 3) \). The latter is an isomorphism off \( \Sigma \), and its image is \( \mathcal{I}_{\Sigma,C}(d - 3) \). Hence
\[
\Omega^1_C/\text{Torsion} \hookrightarrow \mathcal{I}_{\Sigma,C}(d - 3). \tag{2.2.2} \]

Let \( P \in \Sigma \). Take local coordinates \( x, y \) for \( \mathbb{P}^2 \) at \( P \), and let \( f(x, y) = 0 \) be a local equation for \( C \). By definition, the \textit{Tjurina number} \( \tau_P \) of \( C \) at \( P \) is the colength in \( \mathcal{O}_{\mathbb{P}^2,P} \) of the ideal generated by \( f \) and its partials \( f_x, f_y \). And the \textit{Milnor number} \( \mu_P \) of \( C \) at \( P \) is the colength in \( \mathcal{O}_{\mathbb{P}^2,P} \) of the ideal generated only by \( f_x, f_y \). Plainly, both \( \tau_P \) and \( \mu_P \) are analytic (formal) invariants.

By definition, \( P \) is a \textit{quasi-homogeneous} singularity of \( C \) if, after a suitable analytic change of variables, \( f(x, y) \) becomes a weighted-homogeneous polynomial. Saito [Sa], second “Satz” on p. 123, proved that, in characteristic zero, \( \bar{P} \) is quasi-homogeneous if and only if \( \tau_P = \mu_P \); the result is also discussed in [D], see Theorem 7.42, p. 120.
From (2.2.1) and (2.2.2), it is clear that the ideal \( I_{\Sigma, C} \) is generated at \( P \) by \( f_x, f_y \). Hence the Tjurina number \( \tau_P \) is the length of \( \mathcal{O}_\Sigma \) at \( P \). Since \( C \) is reduced, \( \Sigma \) is finite. Set 
\[
\tau := \sum_{P \in \Sigma} \tau_P,
\]
and call \( \tau \) the total Tjurina number of \( C \). Note \( \tau = \deg \Sigma \).

By definition, the polar system of \( C \) is the linear system on \( \mathbb{P}^2 \) generated by the three partial derivatives of the homogeneous polynomial \( F \) defining \( C \) (see \[De\], Section 3). If the characteristic is 0, or if it is positive and does not divide \( \deg C \), then Euler’s formula shows that \( F \) belongs to the ideal generated by its partial derivatives. Therefore, the base locus of the polar system is, scheme-theoretically, just the singular locus \( \Sigma \) of \( C \).

**Proposition 2.3.** Let \( \eta \) be a foliation of \( \mathbb{P}^2 \). Let \( S \) be its singular locus, and \( m \) its degree. If \( S \) is finite and if \( m > 0 \), then \( \text{reg} \, S = 2m \).

**Proof.** Owing to (2.1.1) and (2.1.2), the Koszul complex on \( \eta \otimes \mathcal{L}^{-1} \) yields a short sequence
\[
0 \to \Omega^2_{\mathbb{P}^2}(2 - 2m) \to \Omega^1_{\mathbb{P}^2}(1 - m) \to \mathcal{I}_{S, \mathbb{P}^2} \to 0.
\]
(2.3.1)

It is exact since \( S \) is finite and \( \mathbb{P}^2 \) is Cohen-Macaulay.

Twisting (2.3.1) by \( i - 1 \), and taking cohomology, we obtain the following exact sequence:
\[
H^1(\Omega^1_{\mathbb{P}^2}(i - m)) \to H^1(\mathcal{I}_{S, \mathbb{P}^2}(i - 1)) \to H^2(\Omega^2_{\mathbb{P}^2}(i + 1 - 2m)) \to H^2(\Omega^1_{\mathbb{P}^2}(i - m)).
\]
(2.3.2)

Now, \( H^1(\Omega^1_{\mathbb{P}^2}(i - m)) \neq 0 \) if and only if \( i = m \), by \[De\], Théorème 1.1, p. 40. By duality, \( H^2(\Omega^2_{\mathbb{P}^2}(i + 1 - 2m)) = 0 \) if and only if \( i \geq 2m \). By hypothesis, \( m > 0 \), or \( 2m > m \). Take \( i = 2m \) in (2.3.2). Its exactness now yields \( H^1(\mathcal{I}_{S, \mathbb{P}^2}(2m - 1)) = 0 \). Thus \( \text{reg} \, S \leq 2m \).

On the other hand, by \[De\], Théorème 1.1, p. 40, again, \( H^2(\Omega^1_{\mathbb{P}^2}(m - 1)) = 0 \) as \( m \geq 0 \). Now, \( H^2(\Omega^2_{\mathbb{P}^2}) \neq 0 \). Take \( i = 2m - 1 \) in (2.3.2). Its exactness yields \( H^1(\mathcal{I}_{S, \mathbb{P}^2}(2m - 2)) \neq 0 \). Thus \( \text{reg} \, S \geq 2m \). \( \Box \)

**Lemma 2.4.** Let \( C \subset \mathbb{P}^2 \) be a curve of degree \( d \), and \( T \subset C \) a finite subscheme. Then
\[
h^2(\mathcal{I}_{T, \mathbb{P}^2}(i)) = 0 \text{ for } i \geq -2,
\]
(2.4.1)
\[
h^1(\mathcal{I}_{T, \mathbb{P}^2}(i)) = h^1(\mathcal{I}_{T, \mathbb{P}^2}(i)) + \epsilon \text{ where } \epsilon := \begin{cases} 1, & \text{if } i = d - 3; \\ 0, & \text{if } i > d - 3. \end{cases}
\]
(2.4.2)

**Proof.** For any \( i \), consider the two twisted standard exact sequences:
\[
0 \to \mathcal{I}_{T, \mathbb{P}^2}(i) \to \mathcal{O}_{\mathbb{P}^2}(i) \to \mathcal{O}_T(i) \to 0,
\]
\[
0 \to \mathcal{O}_{\mathbb{P}^2}(i - d) \to \mathcal{I}_{T, \mathbb{P}^2}(i) \to \mathcal{I}_{T, \mathbb{P}^2}(i) \to 0.
\]
Since \( T \) is finite, \( h^1(\mathcal{O}_T(i)) = 0 \). So Serre’s computation of \( h^q(\mathcal{O}_{\mathbb{P}^2}(i)) \) yields the formulas. \( \Box \)

**Theorem 2.5.** Let \( C \subset \mathbb{P}^2 \) be a reduced curve of degree \( d \). Assume either the characteristic is 0 or it is positive and does not divide \( d \). Assume \( C \) is a leaf of a foliation of \( \mathbb{P}^2 \) of degree \( m \). Let \( \Sigma \) be the singular locus of \( C \), and set \( \sigma := \text{reg} \, (\Sigma) \) and \( \rho := \sigma - d + 2 \). Then
\[
d \leq \begin{cases} m + 1, & \text{if } \rho \leq 0; \\ m + 1 + \rho, & \text{if } \rho > 0. \end{cases}
\]
Furthermore, if \( S \) is finite and \( d \geq 2m + 2 \), then \( d = m + 1 + \rho \).
Proof. Suppose \( \rho \leq 0 \). Then \( \mathcal{I}_{\Sigma, \mathbb{P}^2} \) is \((d - 2)\)-regular; in particular, \( \operatorname{h}^1(\mathcal{I}_{\Sigma, \mathbb{P}^2}(d - 3)) = 0 \). So \( \operatorname{h}^1(\mathcal{I}_{\Sigma, \mathbb{C}}(d - 3)) = 1 \) by (2.4.2). It now follows from (2.2.2) that \( \operatorname{h}^1(\Omega^1_{\mathbb{C}}) = 1 \). Hence Corollary 4.5, yields

\[
\operatorname{h}^1(\mathcal{O}_{C}(m - 1)) = 0.
\]  

(2.5.1)

Suppose \( \rho > 0 \). Then \( \sigma > d - 2 \). So (2.4.2) applies with \( i := \sigma - 1 \) and \( \epsilon := 0 \). Now, \( \operatorname{h}^1(\mathcal{I}_{\Sigma, \mathbb{P}^2}(\sigma - 1)) = 0 \) by definition of \( \sigma \). Hence

\[
\operatorname{h}^1(\mathcal{I}_{\Sigma, \mathbb{C}}(\sigma - 1)) = 0.
\]  

(2.5.2)

By hypothesis, \( \mathbb{C} \) is a leaf of a foliation \( \eta: \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(m - 1) \). So \( \eta \) induces a map \( \varphi: \Omega^1_{\mathbb{C}} \to \mathcal{O}_{C}(m - 1) \). In fact, \( \eta|C = \varphi \beta_C \) where \( \beta_C: \Omega^1_{\mathbb{C}}|C \to \Omega^1_{\mathbb{C}} \) is the standard surjection. Let \( S \) be the singular locus of \( \eta \). By (2.1.1) and (2.1.2), the image of \( \eta \) is \( \mathcal{I}_{S, \mathbb{P}^2}(m - 1) \). Hence the image of \( \varphi \) is \( \mathcal{I}_{S \cap C, \mathbb{C}}(m - 1) \).

Since \( \mathbb{C} \) is reduced, \( \mathcal{O}_{C}(m - 1) \) is torsion free. Hence, \( \varphi \) factors through \( \Omega^1_{\mathbb{C}}/\text{Torsion} \), which is equal to \( \mathcal{I}_{\Sigma, \mathbb{C}}(d - 3) \) by (2.2.2). Thus there is a map \( \mathcal{I}_{\Sigma, \mathbb{C}}(d - 3) \to \mathcal{O}_{C}(m - 1) \). It is injective since no component of \( \mathbb{C} \) lies in \( S \). Its image is \( \mathcal{I}_{S \cap C, \mathbb{C}}(m - 1) \) by the preceding paragraph. In other words, there is an exact sequence,

\[
0 \to \mathcal{I}_{\Sigma, \mathbb{C}}(d - 3) \to \mathcal{O}_{C}(m - 1) \to \mathcal{O}_{S \cap C}(m - 1) \to 0.
\]  

(2.5.3)

Twist by \( \rho \), and take cohomology. Then use (2.5.2) to obtain

\[
\operatorname{h}^1(\mathcal{O}_{C}(m + \rho - 1)) = 0.
\]  

(2.5.4)

By duality, \( \operatorname{h}^1(\mathcal{O}_{C}(i)) = \operatorname{h}^0(\mathcal{O}_{C}(d - 3 - i)) \) for any \( i \). Hence \( \operatorname{h}^1(\mathcal{O}_{C}(i)) = 0 \) if and only if \( i \geq d - 2 \). Therefore, (2.5.1) and (2.5.4) yield the first assertion.

Assume now that \( d \geq 2m + 2 \). Then \( d \geq m + 2 \). So the first assertion yields \( d \leq m + 1 + \rho \). It remains to prove \( d \geq m + 1 + \rho \).

The hypothesis on the characteristic implies that \( \Sigma \) is the base locus of the polar system; see the end of §2.2. Hence \( \Sigma \) is contained in the finite intersection \( T \) of two members of the system. Now, given any two curves \( E, F \subset \mathbb{P}^2 \) of degrees \( e, f \), their intersection \( Z \), if finite, has regularity \( e + f - 1 \), because the ideal \( \mathcal{I}_{Z, \mathbb{P}^2} \) has the Koszul resolution

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-E - F) \to \mathcal{O}_{\mathbb{P}^2}(-E) \oplus \mathcal{O}_{\mathbb{P}^2}(-F) \to \mathcal{I}_{Z, \mathbb{P}^2} \to 0.
\]

Therefore, \( \operatorname{reg}(T) = 2d - 3 \). However, \( T \supset \Sigma \) and \( T \) is finite; whence, \( \sigma \leq \operatorname{reg}(T) \). Hence \( \rho := \sigma - d + 2 \leq d - 1 \). Thus if \( m = 0 \), then \( d \geq m + 1 + \rho \).

Assume \( m > 0 \). Then \( d \geq 2m + 2 \geq m + 3 \). But \( d \leq m + 1 + \rho \) as observed just above. So \( d \leq m + \sigma - d + 3 \leq \sigma \). Therefore, we may take \( i := \sigma - 2 \) in (2.4.2). Now, \( \operatorname{h}^1(\mathcal{I}_{\Sigma, \mathbb{P}^2}(\sigma - 2)) \neq 0 \) by definition of \( \sigma \). Hence \( \operatorname{h}^1(\mathcal{I}_{\Sigma, \mathbb{C}}(\sigma - 2)) \neq 0 \). The exactness of (2.5.3) means \( \mathcal{I}_{\Sigma, \mathbb{C}}(d - 3) = \mathcal{I}_{S \cap C, \mathbb{C}}(m - 1) \). Twisting by \( \rho - 1 \) and taking cohomology, we conclude

\[
\operatorname{h}^1(\mathcal{I}_{S \cap C, \mathbb{C}}(m + \rho - 2)) \neq 0.
\]  

(2.5.5)

Proposition 2.3 implies \( S \) is \( i \)-regular for all \( i \geq 2m \). So \( \operatorname{h}^1(\mathcal{I}_{S, \mathbb{P}^2}(i - 1)) = 0 \) for all \( i \geq 2m \). Assume \( S \) is finite. Then \( \mathcal{I}_{S, \mathbb{P}^2} \) has finite colength in \( \mathcal{I}_{S \cap C, \mathbb{P}^2} \). Hence

\[
\operatorname{h}^1(\mathcal{I}_{S \cap C, \mathbb{P}^2}(i - 1)) = 0 \text{ for all } i \geq 2m.
\]  

(2.5.6)
Set $j := m + \rho - 2$. Then $j + 1 \geq d - 2 \geq 2m$. Form the twisted exact sequence of ideals

$$0 \to \mathcal{I}_{C, \mathbb{P}^2}(j) \to \mathcal{I}_{S \cap C, \mathbb{P}^2}(j) \to \mathcal{I}_{S \cap C, C}(j) \to 0.$$  

The first term is equal to $\mathcal{O}_{\mathbb{P}^2}(j - d)$. Take cohomology. Then (2.5.6) and (2.5.5) yield $h^2(\mathcal{O}_{\mathbb{P}^2}(j - d)) \neq 0$. Hence $j - d \leq -3$, or $d \geq m + 1 + \rho$.

\[\square\]

3. Bounds on the total Tjurina number

**Lemma 3.1.** Let $C \subset \mathbb{P}^2$ be a reduced curve of degree $d \geq 2$. Let $\tau$ be its total Tjurina number, and $u$ its number of non-quasi-homogeneous singularities. Let $\Sigma$ be its singular locus, and set $\sigma := \text{reg} \Sigma$. Assume the characteristic is 0. Then

$$\sigma \leq d - 2 + (\tau - u)/(d - 1).$$

**Proof.** Let $M$ be a general polar of $C$. Then $M$ is smooth off the base locus of the polar system by Bertini’s First Theorem, Theorem (3.2) in [K2], because the characteristic is 0. This base locus is $\Sigma$, again because the characteristic is 0; see the end of §2.2. But $\Sigma$ is finite, since $C$ is reduced. Hence, $M$ is generically smooth. So, since $M$ is Cohen–Macaulay, $M$ is reduced.

Either $M$ is irreducible or $C$ is the union of concurrent lines by [GL], Lemma 3.8, p. 333. An alternative proof of this fact is given at the end of the present proof.

First, assume $C$ is the union of concurrent lines, say meeting at $P$. Then $P$ is a quasi-homogeneous singularity of $C$, and its only singularity; so $u = 0$. Since $C$ is a cone, its polar system has dimension 1. Hence the base locus $\Sigma$ is the complete intersection of two polars. So $\tau = (d - 1)^2$ by Bezout’s theorem. In addition, $\sigma = 2d - 3$; see the middle of the proof of Theorem 2.5. Thus, in the present case, equality holds in the asserted inequality.

From now on, assume $M$ is irreducible. The lemma holds if $\sigma \leq d - 2$ since $\tau - u \geq 0$; so we may assume $\sigma \geq d - 1$.

Since $\sigma \geq 1$, we have $h^2(\mathcal{I}_{\Sigma, \mathbb{P}^2}(\sigma - 3)) = 0$ by (2.4.1). Hence $h^1(\mathcal{I}_{\Sigma, \mathbb{P}^2}(\sigma - 2)) \neq 0$ by definition of $\sigma$. So (2.4.2) yields $h^1(\mathcal{I}_{\Sigma, M}(\sigma - 2)) \neq 0$. Hence duality yields

$$\text{Hom}(\mathcal{I}_{\Sigma, M}(\sigma - 2), \mathcal{O}_M(d - 4)) \neq 0. \quad (3.1.1)$$

Given $P \in \Sigma$, denote by $\varepsilon_P$ the multiplicity of the ideal of $\Sigma$ in the local ring $\mathcal{O}_{\mathbb{P}^2, P}$. Then two general polars $M$ and $N$ meet at $P$ with multiplicity $\varepsilon_P$. On the one hand, the Tjurina number $\tau_P$ of $C$ at $P$ is the length at $P$ of the scheme cut out by all the polars. On the other hand, the Milnor number $\mu_P$ of $C$ at $P$ is equal to the intersection number at $P$ of two special polars. Hence

$$\tau_P \leq \varepsilon_P \leq \mu_P. \quad (3.1.2)$$

Saito’s theorem (see §2.2) asserts $\tau_P = \mu_P$ if and only if $P$ is quasi-homogeneous.

Suppose $\mathcal{I}_{\Sigma, M}$ is invertible at $P \in \Sigma$. Let’s prove $P$ is quasi-homogeneous on $C$. (We don’t need the converse, but it holds as $M$ is general. Indeed, if $\tau_P = \mu_P$, then $\tau_P = \varepsilon_P$ by (3.1.2), and the equation $\tau_P = \varepsilon_P$ just means $\Sigma$ is cut out of $\mathbb{P}^2$ at $P$ by two general polars.)

Take local coordinates $x, y$ for $\mathbb{P}^2$ at $P$, and let $f(x, y) = 0$ be a local equation for $C$. Then $\mathcal{I}_{\Sigma, \mathbb{P}^2}$ is generated at $P$ by $f$ and its partials $f_x, f_y$. If the latter alone generate, then $\tau_P = \mu_P$, and so $P$ is quasi-homogeneous on $C$ by Saito’s theorem, recalled above. Suppose $f_x, f_y$ don’t generate, and let’s achieve a contradiction.
At any rate, \( \mathcal{I}_{\Sigma, P^2} \) is 2-generated at \( P \) as \( \mathcal{I}_{\Sigma, M} \) is invertible there. Hence \( \mathcal{I}_{\Sigma, P^2} \) is generated at \( P \) either by \( f, f_x \) or by \( f, f_y \). Either way, \( \mathcal{I}_{\Sigma, C} \) is invertible at \( P \). But then, (2.2.2) implies \( \text{Hom}(\Omega_C, \mathcal{O}_C) \) is invertible at \( P \). So \( P \notin \Sigma \) by [L], Theorem 1, p. 879, a contradiction.

Alternatively, since \( \mathcal{I}_{\Sigma, C} \) is invertible at \( P \), it follows that \( \tau_P = e_P \) where \( e_P \) denotes the multiplicity of \( \mathcal{I}_{\Sigma, C} \) at \( P \). However, Teissier proved \( e_P = m_P + \delta_P - 1 \) where \( m_P \) denotes the multiplicity of \( C \) at \( P \); see Corollaire 1.5 on p. 320 and Remarque 1.6 1) on p. 300 in [Te]; alternatively, see pp. 358–359 of [K1], where Teissier’s formula is derived from the Milnor–Jung formula and another formula for \( m_P \), due to Piene. Since \( \tau_P \leq m_P \) by (3.1.2), it follows that \( m_P = 1 \). So again \( P \notin \Sigma \), a contradiction.

Owing to (3.1.1), there is a nonzero map \( w: \mathcal{I}_{\Sigma, M} \to \mathcal{O}_M(d - \sigma - 2) \). Since \( M \) is reduced and irreducible, \( w \) is injective and its cokernel \( \text{Cok} \) is supported by a finite subset \( W \subset M \). If \( P \in C \) is one of the \( u \) singularities that are not quasi-homogeneous, then \( \mathcal{I}_{\Sigma, M} \) is not invertible at \( P \), as we just proved. Hence \( w \) is not surjective at \( P \); in other words, \( P \in W \). It follows that

\[
\begin{align*}
\mu_P &\leq \deg W \leq \deg \text{Cok} w = \chi(\mathcal{O}_M(d - \sigma - 2)) - \chi(\mathcal{I}_{\Sigma, M}) \\
&= (\chi(\mathcal{O}_M) - \chi(\mathcal{I}_{\Sigma, M})) - (\chi(\mathcal{O}_M) - \chi(\mathcal{O}_M(d - \sigma - 2))) \\
&= \tau - (d - 1)(\sigma - d + 2).
\end{align*}
\]

The asserted bound on \( \sigma \) follows directly, and the proof is complete.

Here’s an alternative proof that either \( M \) is irreducible or \( C \) is the union of concurrent lines. The Milnor–Jung formula says \( \mu_P = 2\delta_P - (r_P - 1) \) where \( \delta_P \) is the genus diminution at \( P \) and \( r_P \) is the number of branches of \( C \) at \( P \). Set \( \delta := \sum \delta_P \), and let \( r \) be the number of irreducible components of \( C \). Since \( C \) is connected, \( \sum (r_P - 1) \geq r - 1 \). Hence,

\[
\sum \mu_P \leq 2\delta - (r - 1). \tag{3.1.3}
\]

Let \( p_a \) be the arithmetic genus of \( C \), and \( g \) its geometric genus. Then \( \delta = p_a - g + r - 1 \).

In addition, \( p_a = (d - 1)(d - 2)/2 \) because \( C \subseteq P^2 \) is of degree \( d \). Since \( g \geq 0 \), we get

\[
2\delta - (r - 1) \leq (d - 1)(d - 2) + (r - 1).
\]

But \( r \leq d \). So the above inequality and (3.1.3) yield

\[
\sum \mu_P \leq (d - 1)^2. \tag{3.1.4}
\]

By Bezout’s theorem, two general polars \( M \) and \( N \) meet in \((d - 1)^2 \) points \( P \) counted with multiplicity. If \( P \in \Sigma \), then this multiplicity is \( \varepsilon_P \). Hence

\[
\sum_{P \in \Sigma} \varepsilon_P \leq (d - 1)^2. \tag{3.1.5}
\]

In addition, either equality holds in (3.1.5) or \( M \) and \( N \) meet outside \( \Sigma \).

First, assume equality holds in (3.1.5). Then equality holds in (3.1.4) because of (3.1.2). Hence also \( r = d \) and \( \sum (r_P - 1) = r - 1 \). The first equation says \( C \) is a union of lines. Then the second says these lines are concurrent.

Finally, assume \( M \) and \( N \) meet outside \( \Sigma \). Then the polar system is not composite with a pencil. Hence, by Bertini’s second theorem, Theorem (5.1) in [K2], a general polar is irreducible. \( \square \)
Theorem 3.2 (du Plessis–Wall). Let \( C \subset \mathbb{P}^2 \) be a reduced curve of degree \( d \geq 2 \). Let \( \tau \) be its total Tjurina number, and \( u \) its number of non-quasi-homogeneous singularities. Let \( \Sigma \) be its singular locus, and set \( \sigma := \text{reg} \Sigma \). Assume the characteristic is 0. Assume \( C \) is a leaf of a foliation of \( \mathbb{P}^2 \) of degree \( m \). Then

\[
(d-1)(d-m-1) + u \leq \tau.
\]

If equality holds, then either \( d = m + 1 \) and \( C \) is smooth, or \( d > m + 1 \) and \( \sigma = 2d - m - 3 \).

Proof. First, assume \( d \leq m + 1 \). Then the asserted inequality holds because \( \tau \geq u \). Furthermore, equality holds if and only if \( d = m + 1 \) and \( \tau = u \). Now, \( \tau = u \) if and only if every singularity of \( C \) is non-quasi-homogeneous and has Tjurina number 1. However, only an ordinary node has Tjurina number 1, and it is quasi-homogeneous. Thus \( \tau = u \) if and only if \( \tau = 0 \).

Finally, assume \( d > m + 1 \). Then \( d - m - 1 \leq \sigma + 2 - d \) by Theorem 2.5. In addition, \((d-1)(\sigma + 2 - d) \leq \tau - u \) by Lemma 3.1. The asserted inequality follows directly. Furthermore, if equality holds, then \( d - m - 1 = \sigma + 2 - d \).

\( \square \)

Proposition 3.3 (du Plessis–Wall). Let \( C \subset \mathbb{P}^2 \) be a reduced curve of degree \( d \) and total Tjurina number \( \tau \). Let \( m \) be the least degree of a foliation of \( \mathbb{P}^2 \) with \( C \) as leaf. Then

\[
m \leq d - 1 \quad \text{and} \quad \tau \leq (d-1)(d-m-1) + m^2.
\]

Moreover, if \( d \leq 2m \) and if \( C \) is irreducible, then

\[
\tau \leq (d-1)(d-m-1) + m^2 - \binom{2m+2-d}{2}.
\]

Proof. To see \( m \leq d - 1 \), take homogeneous coordinates \( x, y, z \) for \( \mathbb{P}^2 \), and let \( F(x, y, z) = 0 \) define \( C \). Given a component of \( C \), there is a partial of \( F \) that doesn’t vanish along it since \( C \) is reduced. So some linear combination of the partials doesn’t vanish along any component. Changing coordinates, we may assume neither \( F_x \) nor \( z \) vanishes along any component.

Form the Hamilton foliation \( \theta : \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2}(d-2) \) where the second map is \((F_y, -F_x, 0)\). Then \( \theta|C \) factors through \( \beta_C : \Omega^1_{\mathbb{P}^2}|C \to \Omega^1_C \) because \( F_y F_x = F_x F_y \). Suppose \( \theta \) vanishes along some component \( C' \) of \( C \). Then \((F_y, -F_x, 0)\) is a polynomial multiple of \((x, y, z)\) on \( C' \) since the restriction of the Euler sequence \( 0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2} \to 0 \) is exact where the third map is \((x, y, z)\). Hence, either \( z \) or \( F_x \) must vanish along \( C' \), but neither do. Thus \( C \) is a leaf of \( \theta \). Now, \( \theta \) is of degree \( d - 1 \). Therefore \( m \leq d - 1 \).

The first bound on \( \tau \) is proved for curves in higher space in [EK1], Proposition 6.3. In the present case, the proof becomes shorter. We give it now mainly to recall its ingredients.

Let \( \eta : \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(m - 1) \) be a foliation with \( C \) as leaf, and \( S \) its singular locus. Since \( m \) is minimal, \( S \) is finite. Let \( \Sigma \) be the singular locus of \( C \). The exactness of (2.5.3) means \( \mathcal{I}_{\Sigma,C}(d - 3) = \mathcal{I}_{S \cap C,C}(m - 1) \). So \( \mathcal{I}_{\Sigma,C}(d - m - 2) = \mathcal{I}_{S \cap C,C} \).

Hence

\[
\tau = \chi(\mathcal{O}_C) - \chi(\mathcal{I}_{\Sigma,C}) = \chi(\mathcal{O}_C(d - m - 2)) - \chi(\mathcal{I}_{\Sigma,C,d - m - 2})
\]

\[
= \left(\chi(\mathcal{O}_C(d - m - 2)) - \chi(\mathcal{O}_C)\right) + \left(\chi(\mathcal{O}_C) - \chi(\mathcal{I}_{S \cap C,C})\right)
\]

\[
d(d - m - 2) + \deg(S \cap C) \quad (3.3.1)
\]

Since \( S \) is finite, \( S \) represents the top Chern class of \((\Omega^1_{\mathbb{P}^2})^*(m - 1)\); so

\[
\deg S = m^2 + m + 1. \quad (3.3.2)
\]
Alternatively, $\deg S$ can be computed from Sequence (2.3.1). The first assertion now follows from (3.3.1), from (3.3.2) and from the trivial bound $\deg(S \cap C) \leq \deg S$.

Better lower bounds for $\deg S - \deg(S \cap C)$ yield better upper bounds for $\tau$. For example, if $d \leq 2m$ and if $C$ is irreducible, then the following bound obtains:

$$\deg S - \deg(S \cap C) \geq \left(\frac{2m+2-d}{2}\right). \tag{3.3.3}$$

And it, (3.3.1), and (3.3.2) yield the second assertion. This bound is proved next.

Assume $d \leq 2m$. Then $\mathcal{I}_{C,\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-d)$ yields $h^0(\mathcal{I}_{C,\mathbb{P}^2}(2m)) = \left(\frac{2m+2-d}{2}\right)$. Now, $\mathcal{I}_{C,\mathbb{P}^2}$ generates $\mathcal{I}_{S \cap C, S}$ in $\mathcal{O}_S$. Hence (3.3.3) obtains if the following composition is injective:

$$u: H^0(\mathcal{I}_{C,\mathbb{P}^2}(2m)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2m)) \to H^0(\mathcal{O}_S(2m)).$$

Extend the exact sequence (2.3.1) to an augmented resolution:

$$0 \to \Omega^2_{\mathbb{P}^2}(-2m+2) \to \Omega^1_{\mathbb{P}^2}(-m+1) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_S \to 0.$$ Twist it by $2m$ and take global sections to obtain this complex:

$$H^0(\Omega^1_{\mathbb{P}^2}(m+1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2m)) \to H^0(\mathcal{O}_S(2m)). \tag{3.4.4}$$

It is exact because $H^1(\Omega^2_{\mathbb{P}^2}(2)) = 0$.

Given $s \in H^0(\mathcal{I}_{C,\mathbb{P}^2}(2m))$ with $u(s) = 0$, we must show $s = 0$.

Since (3.4.4) is exact, $s$ is the image of some $s' \in H^0(\Omega^1_{\mathbb{P}^2}(m+1))$. Let $\pi: \Omega^1_{\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2} \to \Omega^2_{\mathbb{P}^2}$ be the natural pairing, and consider the composition,

$$\eta': \Omega^1_{\mathbb{P}^2} \xrightarrow{1 \otimes s'} \Omega^1_{\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(m+1) \xrightarrow{\pi(m+1)} \Omega^2_{\mathbb{P}^2}(m+1).$$

It turns out that $\eta'$ is a foliation with $C$ as leaf if $s \neq 0$ and if $C$ is irreducible. But then $\eta'$ has degree $m-1$, contradicting the minimality of $m$. Thus necessarily $s = 0$, as desired.

So assume $s \neq 0$. Then $s' \neq 0$. Also $\eta' \neq 0$ because $\pi$ is a perfect pairing. Thus $\eta'$ is a foliation. And it remains to show $C$ is a leaf of $\eta'$.

Let $\mathcal{K}$ be the kernel of $\beta_C: \Omega^1_{\mathbb{P}^2}|C \to \Omega^1_{\mathbb{P}^2}$. Since $C$ is reduced, $C$ is generically smooth and $\mathcal{O}_C(-d)$ is torsion free. So the first map in (2.2.1) is injective. Thus $\mathcal{K} = \mathcal{O}_C(-d)$.

Note $s'|C \in H^0(\mathcal{K}(\eta(m+1)|C))$ because $s \in H^0(\mathcal{I}_{C,\mathbb{P}^2}(2m))$. Since $C$ is a leaf of $\eta$, there is, by definition, a map $\varphi: \Omega^1_C \to \mathcal{O}_C(m-1)$ such that $\eta|C = \varphi \beta_C$. Since $S$ is finite and $C$ is generically smooth, $\varphi$ is generically an isomorphism. Thus $\mathcal{K}(\eta(m+1)|C)$ and $\mathcal{K}(m+1)$ agree generically, and hence $\text{Im}(s'|C)$ generically lies in $\mathcal{K}(m+1)$.

It follows that $\eta'|C$ factors through $\beta_C$, or $\eta'|C) \mathcal{K} = 0$. Indeed, since $\text{Im}(s'|C)$ generically lies in $\mathcal{K}(m+1)$, also $\eta'|C$ generically lies in $(\pi|C)(m+1)(\mathcal{K} \otimes \mathcal{K}(m+1))$. But $\mathcal{K}$ is invertible; whence, $(\pi|C)(\mathcal{K} \otimes \mathcal{K}) = 0$. So $\eta'|C \mathcal{K}$ is generically trivial. But $\eta'|C \mathcal{K}$ lies in $\Omega^2_{\mathbb{P}^2}|C(m+1)$, and $\Omega^2_{\mathbb{P}^2}|C(m+1)$ is torsion free. Therefore, $(\eta'|C) \mathcal{K} = 0$.

Finally, $C$ does not lie in the singular locus of $\eta'$; otherwise, $\eta'$ would factor through a foliation of degree $m - 1 - d$, but $m - 1 - d < 0$. So, since $C$ is irreducible, no component of $C$ lies in the singular locus. Thus $C$ is indeed a leaf of $\eta'$.

$$\square$$

**Remark 3.4.** Over $\mathbb{C}$, let $C \subset \mathbb{P}^2$ be a reduced curve of degree $d$ and with total Tjurina number $\tau$. Let $m$ be the least degree of a foliation $\eta$ of $\mathbb{P}^2$ such that $\eta|C$ factors through
the natural surjection $\beta_C \colon \Omega^1_{\mathbb{P}^2} | C \to \Omega^1_C$. In essentially this setting, Du Plessis and Wall, in [dPW], Theorem 3.2, p. 263, proved $d \geq m + 1$ and
\[(d - 1)(d - m - 1) \leq \tau \leq (d - 1)(d - m - 1) + m^2; \tag{3.4.1}\]
furthermore, if $d \leq 2m$, then
\[\tau \leq (d - 1)(d - m - 1) + m^2 - \binom{2m + 2 - d}{2}. \tag{3.4.2}\]

Let $m'$ be the least degree of a foliation of $\mathbb{P}^2$ with $C$ as leaf. Clearly, $m \leq m'$. If $m = m'$, then (3.4.1) follows from Theorem 3.2 and Proposition 3.3. In this case, Theorem 3.2 is stronger, since $u \geq 0$. If also $C$ is irreducible, then (3.4.2) follows from Proposition 3.3.

In fact, if $C$ is irreducible, then $m = m'$. Indeed, let $\eta$ be a foliation of degree $m$ of $\mathbb{P}^2$ such that $\eta | C$ factors through $\beta_C$. If $C$ were not a leaf of $\eta$, then $C$ would be contained in the singular locus of $\eta$, and hence $m \geq d$. But $m \leq d - 1$ by Proposition 3.3.

However, if $C$ is reducible then possibly $m < m'$. For instance, suppose $C$ is the union of $d$ lines, $d - 1$ concurrent. Let $K$ be the union of the latter, $M$ the additional line. Since $C$ is not a cone, $m > 0$. But $K$ is a leaf of a foliation of degree 0. So there is a foliation $\eta$ of degree 1, with $M$ in its singular locus, and such that $\eta | C$ factors through $\beta_C$. Thus $m = 1$. However, given a foliation $\eta'$ of $\mathbb{P}^2$ of degree $m'$ with $C$ as leaf, it follows from [dPW], Corollary 3.1.1, p. 263, that either $m + m' \geq d - 1$ or both $\eta$ and $\eta'$ factor through a foliation $\eta''$ such that $\eta'' | C$ factors through $\beta_C$. If the latter were true, then $\eta$ would be a scalar multiple of $\eta''$, because $m$ is minimal. Then $\eta'$ would factor through $\eta$, and hence the singular locus of $\eta$ would be contained in that of $\eta'$. But $M$ is in the singular locus of $\eta$ and not in that of $\eta'$. So $m + m' \geq d - 1$, and hence $m' \geq d - 2$.

It is possible to recover Du Plessis and Wall’s lower bound on $\tau$ with a bit more work. Given a foliation $\eta$ such that $\eta | C$ factors through $\beta_C$, let $B \subseteq C$ be the union of all the 1-dimensional components of the singular locus of $\eta$ contained in $C$. Then $\eta$ factors through a foliation $\eta'$ whose degree is $d - \deg B$ and for which $A := C - B$ is a leaf. Then Theorem 3.2 applies to $A$ and $\eta'$ in place of $C$ and $\eta$. The resulting lower bound on the Tjurina number of $A$ can now be used to obtain the desired lower bound on $\tau$. This argument is described in detail in the proof of [EK1], Corollary 6.4. However, the argument does not seem to yield the upper bounds.

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