The Stueckelberg Field

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In 1938, Stueckelberg introduced a scalar field which makes an Abelian gauge theory massive but preserves gauge invariance. The Stueckelberg mechanism is the introduction of new fields to reveal a symmetry of a gauge–fixed theory. We first review the Stueckelberg mechanism in the massive Abelian gauge theory. We then extend this idea to the standard model, stueckelberging the hypercharge $U(1)$ and thus giving a mass to the physical photon. This introduces an infrared regulator for the photon in the standard electroweak theory, along with a modification of the weak mixing angle accompanied by a plethora of new effects. Notably, neutrinos couple to the photon and charged leptons have also a pseudo-vector coupling. Finally, we review the historical influence of Stueckelberg’s 1938 idea, which led to applications in many areas not anticipated by the author, such as strings. We describe the numerous proposals to generalize the Stueckelberg trick to the non-Abelian case with the aim to find alternatives to the standard model. Nevertheless, the Higgs mechanism in spontaneous symmetry breaking remains the only presently known way to give masses to non-Abelian vector fields in a renormalizable and unitary theory.

Keywords: Stueckelberg, Proca, photon, electroweak, massive Yang–Mills

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I. INTRODUCTION

Research on the theory of massive vector fields started with (Procă, 1936), and reached a major milestone with the standard electroweak theory which is unitary and renormalizable, and successful.

In this paper, we would like to review a contribution of Ernst C.G. Stueckelberg in 1938, namely his introduction of the scalar “$B$–field” of positive–definite metric, accompanying a massive Abelian vector field (Stueckelberg, 1938a, b, c). We shall see that this idea had many different applications which went far beyond the original motivations of its author. Stueckelberg also invented in that year the general formulation of baryon number conservation (Stueckelberg, 1938c, p. 317).

We will not review other important contributions by Stueckelberg. It is, nevertheless, noteworthy that among his discoveries one can include the picture of antiparticles as particles moving backwards in time implying pair creation and annihilation (Stueckelberg, 1941, 1942), the causal propagator (Stueckelberg, 1946), (Rivier, 1949; Rivier and Stueckelberg, 1948; Stueckelberg and Rivier, 1950) and the renormalization group (Stueckelberg and Petermann, 1951, 1953).

We shall mainly discuss three important topics, (1) hidden symmetry, (2) renormalizability, and (3) infrared divergences.

It was believed by many that only massless vector theories were gauge–invariant. Then (Pauli, 1941) showed that Stueckelberg’s formalism for a massive vector field satisfied a restricted $U(1)$ gauge invariance, similar to the one encountered in quantum electrodynamics, but with the gauge function $\Lambda(x)$ restricted by the massive Klein–Gordon equation.

Much later, Delbourgo, Twisk and Thompson found that Stueckelberg’s lagrangian for real vector fields, complemented with ghost terms, is actually BRST invariant (Delbourgo et al., 1988). The BRST symmetry (Becchi et al., 1974, 1975; Tvhst, 1975) allows for a systematic and convenient exploration of gauge symmetries, and in fact the $S$–matrix elements of a BRST–invariant theory are independent of the gauge–fixing terms (Led, 1976). The BRST symmetry facilitates considerably the effort to prove the perturbative renormalizability of a theory, as well as its unitarity (Alvarez-Gaume and Witten, 1982, Becchi et al., 1981), see also (Zinn-Justin, 1975), (Kraus, 1998) and the textbook (Collins, 1984).

Charged vector theories (for example non-Abelian gauge theories) are trickier. The electroweak theory, with spontaneously broken $SU(2)_L \times U(1)_Y$ symmetry, is renormalizable (t Hoold, 1971a). As is well known, this theory comprises two charged and two neutral vector bosons. A technical problem remains: the infrared divergences of massless (vector) field theories.

One important aim of the present paper, therefore, is to construct a BRST invariant $SU(2)_L \times U(1)_Y$ theory with a massive photon, which calls for a Stueckelberg field, along with the appropriate ghost terms. This development was suggested in (Stora, 2000). A complication is that the BRST–invariant infrared regulator not only gives a mass to the photon, but also changes its couplings to the fermions, the weak mixing angle, and more. But the new terms in the lagrangian (or their modifications) are proportional at least to the photon’s mass squared, and thus very small. Indeed, from a Cavendish experiment and the known value of the galactic magnetic field one finds the stringent upper limit $m_\gamma < 10^{-16}$ eV (Hagiwara et al., 2002, p. 249).
This “stueckelberged” standard model has the important advantage of providing an infrared cut-off for photon interactions, while preserving BRST invariance. Thus the infrared catastrophe should be avoided without spoiling ultraviolet renormalizability and unitarity. An explicit proof of this fact has not been fully worked out yet. The brute force alternative, to give an explicit mass to the photon which becomes zero at the end of the calculation and not modifying any of the other couplings in the electroweak theory, works well for the computation of $S$–matrix elements [Itzykson and Zuber [1980], Passarino and Veltman [1979]]. Dimensional regularization can be used also to provide an infrared regulator [Gastmans et al., 1971], but then the spacetime dimension must be extended to a higher value, instead of a lower one as required to regulate ultraviolet divergences, whereby the computation is tediously long but straightforward [Collins, 1984].

This paper is organized as follows.

In section II, we review the original formalism and motivations of Proca and Stueckelberg.

In section III, we review the BRST invariance properties of the massive neutral vector field together with the Stueckelberg $B$–field.

In section IV we consider in some detail the case of a $U(1)$ gauge field coupled to matter, and study the Stueckelberg mass mechanism in the absence and in the presence of spontaneous symmetry breaking and the usual Higgs mechanism. This section is a warm–up for the core of the paper, in section V, where we write out the $SU(2)_L \times U(1)_Y$ lagrangian with a Stueckelberg field and a mass term for the hypercharge vector boson, as well as with the usual spontaneous symmetry breaking. We write out the full ghost sector and check the BRST invariance of the full lagrangian. We then turn to some of the phenomenological consequences of the model. In particular, we study mass matrices, mixing angles and currents, and we scratch the surface of some conundrums related to anomalies and the electric charge. Curiously, one should now distinguish the quantum field $A_\mu$ responsible for photon scattering from the external $A_\mu^{\text{e.m.}}$ which enters in the calculation of, e.g. the $(g - 2)$ value of the electron: for a massless photon the two fields coincide.

Section VI is devoted to a historical review of the influence of Stueckelberg’s three 1938 papers. In the forties and fifties, the renormalizability of the massive Abelian Stueckelberg theory was painfully established. There was a long debate in the sixties and seventies about the non-Abelian case, which waned when massive Yang–Mills with spontaneous symmetry breakdown and the Higgs mechanism was shown to be unitary and renormalizable [t Hooft, 1971a]. The problem is nowadays essentially solved: no renormalizable and unitary non-Abelian Stueckelberg model has been found, and [Hurth, 1997] has claimed that it is impossible to do so in perturbation theory. Renormalizable models of massive Yang–Mills without Higgs mechanism nor Stueckelberg field have been exhibited, but they are not unitary.

On the other hand, Stueckelberg fields were introduced very early in string theory by Pierre Ramond and collaborators, both for the formulation of the antisymmetric partner to the graviton [Kalb and Ramond, 1974] and in covariant string field theory [Marshall and Ramond, 1975, Ramond, 1980]. Stueckelberg fields turned out to be crucial in the covariant quantization of the spacetime supersymmetric string [Bergshoeff and Kallosh, 1990a] and in the destruction of unwanted $U(1)$s in string phenomenology [Aldazabal et al., 2000a]. They have also proven useful in the study of dualities in field and string theory, see section VI.D.

II. QUANTIZATION OF THE MASSIVE VECTOR FIELD

The electromagnetic potential is described by a neutral vector field $A_\mu$ obeying Maxwell’s equations. Its quantization gives rise to a massless particle, the photon, which has only two physical degrees of freedom, its two transverse polarizations or, equivalently, its two helicities (+1 and -1). The vector field $A_\mu$ has, however, four components. This is an example of how physicists introduce apparently unphysical entities in order to simplify the theory. Indeed, with the four–vector $A_\mu$ one can construct a manifestly Lorentz–invariant theory of the potential $A_\mu$ interacting with a current $j_\mu$.

How can one reduce the four components of $A_\mu$ to the two physical degrees of freedom of the photon? First, four becomes three in a Lorentz–invariant way by imposing the Lorentz subsidiary condition $\partial^\nu A_\nu = 0$ (for quantum subtleties, see below). With gauge invariance and the mass–shell condition, only two components survive.

Contrariwise, if one adds to the wave equation of $A_\mu$, a mass term, the gauge invariance is lost, because the field $A_\mu$ transforms inhomogeneously and thus the mass term in the lagrangian is not invariant. The three components of $A_\mu$ left by the Lorentz condition are then interpreted as belonging to a massive vector field, that is a massive particle of spin one. This spin–one object has now a longitudinal polarization, in addition to the two transverse ones.

Stueckelberg’s wonderful trick consists in introducing an extra physical scalar field $B$, in addition to the four components $A_\mu$, for a total of five fields, to describe covariantly the three polarizations of a massive vector field. With the Stueckelberg mechanism, which we shall exhibit in more detail below, not only is Lorentz covariance manifest, but also, and most interestingly, gauge invariance is also manifest. The Stueckelberg field restores the gauge symmetry which had been broken by the mass term.
A. Proca

The original aim of (Proca, 1936) was to describe the four states of electrons and positrons by a Lorentz four-vector. The motivation was to imitate the procedure of (Pauli and Weisskopf, 1934), who had quantized the scalar field obeying the Klein–Gordon equation and had interpreted the conserved current as carrying electric charge rather than probability. This, they thought, eliminated negative probabilities. Of course, Proca’s choice is inadequate for describing spin–1/2 particles, and it didn’t make much sense either a decade after Dirac’s equation. Nevertheless, Proca’s mathematical formalism describes well a massive real or complex vector field. The original papers are devoted to the complex case, and we shall present it now, although later on we shall consider the real case exclusively.

Proca’s equation of motion for a free complex vector field \( V_\mu \) reads as follows (Proca, 1936):

\[
\partial_\mu F^{\mu\nu} + m^2 V_\nu = 0 \tag{1}
\]

where the field strength is

\[
F^{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{2}
\]

with \( \partial_\mu = \partial/\partial x^\mu \) and the sign in (1) depends on the metric, which is (+−−−) throughout this paper. Differentiating the equations of motion (1) with respect to \( x^\nu \) yields immediately the Lorentz condition

\[
\partial_\mu V_\mu = 0 \tag{3}
\]

Notice that a non-zero mass is crucial for (3) to follow from the equations of motion.

Hence, \( V_\mu(x) \) describes a spin–one particle of non-zero mass \( m \). Following the lucid presentation in (Wentzel, 1943, 1948), equation (1) can be derived from a lagrangian density for the complex \( V_\mu \):

\[
\mathcal{L} = -\frac{1}{2} F^{\mu\nu} F^{\nu\mu} + m^2 V_\mu V^\dagger_\mu \tag{4}
\]

where \( \dagger \) means hermitian conjugation. The hamiltonian density following from the above lagrangian density has three positive terms involving the spatial components \( V_i \) (\( i = 1, 2, 3 \)), and one negative term depending on \( V_0 \). This last term can be eliminated using the Lorentz condition (3) and the resulting hamiltonian is thus positive definite.

It is then possible to use the canonical formalism to find, first, the commutation relations for the spatial components of the vector field, and then, using again the Lorentz condition (3), the commutations relations for its temporal component. The result is (Wentzel, 1943)

\[
[V_\mu(x), V_\nu(y)] = [V_\mu^\dagger(x), V_\nu^\dagger(y)] = 0 \tag{5}
\]

\[
[V_\mu(x), V_\nu^\dagger(y)] = -i \left( \partial_\mu g_{\mu\nu} + \frac{1}{m^2} \partial_\mu \partial_\nu \right) \Delta_m(x-y) \tag{6}
\]

Here, the massive generalization \( \Delta_m(x) \) of the Jordan–Pauli function obeys

\[
(\partial^2 + m^2) \Delta_m(x) = 0 \tag{7}
\]

where \( \partial^2 = \partial_\mu \partial^\mu \).

The commutator (6) differs from the corresponding expression in QED by the second term in the right–hand side, proportional to \( 1/m^2 \), which is either absent (Stueckelberg–Feynman gauge) or with coefficient \( -1/\partial^2 \) instead of \( 1/m^2 \) (Landau gauge). After 1945 it became clear that the term \( m^{-2}\partial_\mu \partial_\nu \) gives rise to (quadratic) divergences at high energies which cannot be eliminated by the renormalization procedure

B. Stueckelberg

Stueckelberg’s formalism for the vector field differs from Proca’s. His motivations made sense at the time, so let us sketch briefly the historical framework. Recall that Yukawa, in order to explain the nuclear forces, postulated the existence of a massive particle which would mediate them, just as the photon mediates the Coulomb force between charged particles\(^1\). The first attempt (Yukawa, 1935) called for the exchanged particle to be a component of a Lorentz

\(^1\) Late in his life, in 1979, Stueckelberg wrote a letter to V. Telegdi stating that “I had the same idea, probably before Yukawa.” But it was never published.
four-vector (all computations were carried in the static approximation). Then, (Yukawa and Sakata, 1937) proposed a scalar particle instead. (Stueckelberg, 1938a) showed that choosing a scalar would lead to a repulsive instead of attractive nuclear interaction and then turned to reconsider the exchange of a massive charged vector particle. The guiding principle of this research was to develop a formalism as close as possible to QED.

Instead of Proca’s equation of motion (11), (Stueckelberg, 1938b) wrote simply

\[(\partial^2 + m^2) A_\mu(x) = 0\]  

which follows from the covariant lagrangian density

\[\mathcal{L} = -\partial_\mu A_\nu^\dagger \partial^\nu A_\mu + m^2 A_\mu^\dagger A_\mu\]  

The difference between this lagrangian and Proca’s (other than the change in notation between \(V_\mu\) and \(A_\mu\)) is a term \(\partial_\mu A_\nu^\dagger \partial^\nu A_\mu\) which, up to total derivatives, is \((\partial^\mu A_\nu^\dagger)(\partial^\nu A_\mu)\). This term, present in Proca’s lagrangian but not in Stueckelberg’s, is responsible for being able to derive the Lorentz condition (13) from Proca’s lagrangian.

So following the QED track leads, not surprisingly, to the fact that the gauge condition must be imposed as a supplementary ingredient besides the covariant lagrangian: a disadvantage of Stueckelberg’s procedure is thus that the Lorentz subsidiary condition does not follow from the equation of motion, as was the case with Proca. This feature has terrible consequences on the positivity of the hamiltonian, which is now

\[\mathcal{H} = -\sum_{\lambda=0,1,2,3} (\partial_\lambda A_\mu^\dagger)(\partial_\lambda A_\mu) - m^2 A_\mu^\dagger A_\mu\]  

The explicit sum over \(\lambda\) gives no trouble, but the implicit sum over \(\mu\) does, since \(A_\mu = g_{\mu\nu} A^\nu\) and thus the contribution from \(A_\mu\) to the energy density is negative, whereby it is not possible to conclude that (10) is positive definite. In Proca’s formalism, the negative term with \(A_\mu\) can be eliminated with the subsidiary condition which follows from the field equations, but now we do not have it automatically. Where does the subsidiary condition come from, then?

In QED, the same problem arises: \(\partial^\mu A_\mu = 0\) does not follow from the equations of motion. (Fermi, 1932) proposed to impose instead \(\partial^\mu A_\mu |\text{phys}\rangle = 0\), with |\text{phys}\rangle an admissible physical state of the system; see also the discussion in (Pais, 1986, pp. 354–355). Even this condition is too strong, however, because it restricts the space of physical states to nothing. But its spirit is correct. In fact, we only need (Gupta, 1950) the weaker, and sufficient, condition

\[\partial^\mu A_\mu^{-\dagger} |\text{phys}\rangle = 0\]  

where \(A_\mu^{-\dagger}\) involves only free-field annihilation operators (\(A_\mu = A_\mu^{-\dagger} + A_\mu^{\dagger}\), with \(A_\mu^{\dagger}\) involving only creation operators); the only requirement on the choice of polarization is that the hamiltonian be bounded below. The Hilbert space still has indefinite metric, but the space of physical states is of positive definite norm.

Could the same trick be used in the massive vector field theory? Surprisingly, because of the non-zero mass, one cannot impose the operator condition (14), since it comes into conflict with the canonical commutation relations. The commutation relations of Stueckelberg’s vector field are the same as those of the QED’s photon in what later was to be called the Feynman gauge:

\[[A_\mu(x), A_\nu(y)] = 0\]
\[[A_\mu(x), A_\nu^\dagger(y)] = -ig_{\mu\nu}\Delta_m(x-y)\]  

except that \(\Delta_m\) obeys the Klein–Gordon equation with \(m \neq 0\). It is easy to derive from (15) the commutator

\[[\partial^\mu A_\mu(x), \partial^\nu A_\nu^\dagger(y)] = i\partial^2 \Delta_m(x-y)\]  

In QED, \(\partial^2 \Delta_0 = 0\), so (16) is consistent indeed. But in the massive vector case, since \(\partial^2 \Delta_m = -m^2 \Delta_m\), the subsidiary condition (14) is inconsistent with the commutation relations.

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2 To find phenomenological agreement, one needs to consider an isospin triplet of pseudoscalar intermediate particles (Kemmer, 1938). (Stueckelberg, 1937) and (Bhabha, 1938) first noticed that the Yukawa particle could decay into electrons (this is wrong, of course, since a muon is not a pion; it was a major discovery later on (Lattes et al., 1947) that the “\(\mu\)-meson” was a different particle from the “\(\pi\)-meson”. See also the criticisms of Yukawa’s scalar proposal in (Kemmer, 1938a,b) and (Serber, 1933) and the comments in (Pais, 1986, p. 434).
Stueckelberg brilliantly solved this puzzle by introducing a new additional scalar field \( B(x) \) which is now known as the Stueckelberg field \((\text{Stueckelberg}, 1938\text{a}, \text{p.} 243), \text{(Stueckelberg}, 1938\text{b}, \text{p.} 302)\). Note that the Hilbert space norm of the Stueckelberg field is positive, a simple fact which has caused much confusion when overlooked.

In the original formulation, Stueckelberg’s \( B \) field obeys the same equation \((8)\) as the vector field \( A_\mu \); both fields are complex and with the same mass \( m \):

\[
(\partial^2 + m^2) B(x) = 0 \quad (14)
\]

In close analogy with \( (22) \),

\[
[B(x), B(y)] = 0 \\
[B(x), B^\dagger(y)] = i\Delta_m(x - y) \quad (15)
\]

The subsidiary condition \( (11) \) is replaced (in the Gupta–Bleuler version), however, by

\[
S(x) |\text{phys}\rangle \equiv (\partial^\mu A_\mu(x) + m B(x)) |\text{phys}\rangle = 0 \quad (16)
\]

and one verifies easily that \( S(x) \) commutes both with \( S(y) \) and with \( S^\dagger(y) \). A short explicit calculation shows that, after imposing the subsidiary condition \( (14) \), the Hamiltonian is positive definite.

Instead of \( (9) \), the Stueckelberg Lagrangian density is now \((\text{Stueckelberg}, 1938\text{c}, \text{p.} 313)\)

\[
\mathcal{L}_{\text{Stueck}} = -\partial_\mu A_\mu^\dagger \partial^\nu A^\nu + m^2 A_\mu^\dagger A^\mu + \partial_\mu B^\dagger \partial^\mu B - m^2 B^\dagger B \quad (17)
\]

which describes consistently and covariantly a free massive charged vector field, accompanied by the Stueckelberg scalar. An enormous advantage of Stueckelberg’s formalism is the absence of derivatives in the commutation relations. These derivatives would make the theory more singular at higher energies.

On the other hand, the number of degrees of freedom has now been increased to five, instead of the required three for a massive vector field. The situation is somewhat similar to that encountered in QED. The subsidiary condition \( (10) \) can be used to decrease the number of components to four. In addition, Stueckelberg’s theory satisfies a new gauge invariance \((\text{Pauli}, 1941)\), a feature that explains the lasting success of this formalism in the literature up to our days. The fact that a supplementary condition has to be imposed on physical states, like in QED, just means that the gauge invariance \((\text{Pauli}, 1941)\), a feature that explains the lasting success of this formalism in the literature up to our days. The important difference between the above gauge transformation and the usual Abelian gauge transformation in QED is that here the gauge parameter \( \Lambda(x) \) is restricted by \( (20) \). This dynamical restriction does not change the number of degrees of freedom, however. Note, in passing, that the vector field in QED is real, whereas here we are following the original papers which dealt with complex fields. The discussion on the number of fields and physical degrees of freedom does not change if these fields and degrees of freedom are complex or real.
The second term in equation (21), which gives rise to the mass term for the vector and to the kinetic term for the Stueckelberg scalar, clearly displays another way of thinking of the Stueckelberg mechanism, in terms of representations of the Lorentz group. Spin one representations can be built from a vector or, with the help of the momentum operator, from a scalar. The Stueckelberg trick is to couple the spin one $A_\mu$ with the spin one $\partial_\mu B$ to have enough degrees of freedom for a gauge-invariant massive vector field. Alternatively, it compensates the scalar piece $\partial_\mu A^\mu$ of the vector field with the Stueckelberg scalar $B$.

To conclude the comparison between Proca’s and Stueckelberg’s formalism, let us note that the latter can be brought quite close to the former. Indeed, if one defines (Pauli, 1941)

$$V_\mu \equiv A_\mu - \frac{1}{m} \partial_\mu B$$

one sees that Stueckelberg’s lagrangian (21) is the sum of Proca’s (4) plus an extra term:

$$\mathcal{L}_{\text{Stueck}} = \mathcal{L}_{\text{Proca}} - (\partial^\mu A^\mu_\mu + m B)(\partial^\nu A^\nu_\nu + m B)$$

Note that $V_\mu(x)$ is gauge–invariant under (18) ($V_\mu'(x) = V_\mu(x)$) and so is $F_{\mu\nu} = F^V_{\mu\nu}$. The supplementary condition (16) is the same as Proca’s (5) on the Stueckelberg field’s mass shell.

On physical states, for which

$$\langle \text{phys} | \partial_\nu A^\nu_\nu + m B | \text{phys} \rangle' = 0$$

Stueckelberg’s lagrangian (24) coincides with Proca’s. As a general rule, keep in mind that for renormalization purposes it will turn out to be advantageous to keep $A_\mu$ and $B$ independent as long as possible: it is not a good policy to eliminate $B$ and recover the cumbersome Proca lagrangian!

We shall discuss the physical relevance of Stueckelberg’s $B$ field in the next section, in the context of real vector (and Stueckelberg) fields. We have presented the complex formulation here because that is what was originally studied by Proca and Stueckelberg.

III. BRST INVARIANCE

We confine ourselves to real vector fields from now on. Stueckelberg’s theory of a massive vector field $A_\mu$ accompanied by the real Stueckelberg scalar field $B$ (Stueckelberg, 1938a,b) satisfies a gauge invariance despite the presence of the mass term for $A_\mu$ (Pauli, 1941):

$$\delta A_\mu(x) = \partial_\mu \Lambda(x)$$
$$\delta B(x) = m \Lambda(x)$$

where the real gauge parameter $\Lambda(x)$ is restricted by (20).

To see this, we start from the lagrangian (17) specialized to real fields

$$\mathcal{L}_{\text{Stueck}} = -\frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} m^2 A^\mu_\mu A_\mu + \frac{1}{2} \partial^\mu A^\mu \partial_\mu B - \frac{1}{2} m^2 B^2$$

and rewrite it (up to total derivatives) as

$$\mathcal{L}_{\text{Stueck}} = -\frac{1}{4} F^2_{\mu\nu} + \frac{1}{2} m^2 \left( A^\mu - \frac{1}{m} \partial^\mu B \right)^2 - \frac{1}{2} \left( \partial_\mu A^\mu + m B \right)^2$$

Let us consider now the supplementary condition (16) on physical states, which involve both the Stueckelberg and the vector fields. The Stueckelberg field actually participates in the definition of asymptotic states, so there one certainly cannot ignore it (Picariello and Quadri, 2002)! (Glauber, 1953) pointed out that the condition (16) can be viewed as a practical definition of the Stueckelberg $B$ field. The massive photon’s three physical components fix the value of $B$ through the physical state condition, consistently. (Note that in the original literature, these massive photons were called mesons.) See section VI.B below for a more detailed discussion of these points.

Consider now the expression

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha} (\partial_\mu A^\mu + \alpha m B)^2$$

Note that $V_\mu(x)$ is gauge–invariant under (18) ($V_\mu'(x) = V_\mu(x)$) and so is $F_{\mu\nu} = F^V_{\mu\nu}$. The supplementary condition (16) is the same as Proca’s (5) on the Stueckelberg field’s mass shell.

On physical states, for which

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which is just like the ’t Hooft gauge–fixing term in the Abelian Higgs model, where $B$ is replaced by the Goldstone boson$^3$.

We see that the Stueckelberg lagrangian \[ \mathcal{L}_{\text{Stueck}} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 \left( A^\mu - \frac{1}{m} \partial^\mu B \right)^2 - \frac{1}{2\alpha} (\partial^\mu A^\mu + \alpha m B)^2 \] (31)

The restriction (20) on $\Lambda$ is now $(\partial^2 + \alpha m^2)\Lambda = 0$, satisfied because of the equation of motion for $B$, which is simply the same $(\partial^2 + \alpha m^2)B = 0$.

The gauge fixing term (30) could lead to non–acceptable physics, negative-norm modes remaining coupled to physical modes, but [Delbourgo et al., 1988] showed that it does not. Indeed, if one adds the appropriate terms with the well–known ghosts (Faddeev and Popov, 1967) to Stueckelberg’s lagrangian (28), one obtains a theory invariant under the BRST symmetry (Becchi et al., 1974, 1975; Tyutin, 1975), as we shall show shortly.

For completeness, and to come closer to QED, consider also the lagrangian for a fermion minimally coupled to the massive vector field

\[ \mathcal{L}_f = \bar{\psi} \left[ \gamma^\mu (i\partial_\mu + g A_\mu) - M \right] \psi \] (32)

It is invariant under the gauge transformation

\[ A_\mu'(x) = A_\mu(x) + \frac{1}{m} \partial_\mu B(x) \] (33)
\[ \psi'(x) = e^{iB(x)/m} \psi(x) \] (34)
\[ \bar{\psi}'(x) = e^{-iB(x)/m} \bar{\psi}(x) \] (35)

On the other hand, consider the Proca vector interaction

\[ \mathcal{L}'_f = \bar{\psi} \left[ \gamma^\mu (i\partial_\mu + g V_\mu) - M \right] \psi \] (36)

The substitution $V_\mu = A_\mu + \frac{1}{m} \partial_\mu B$ yields

\[ \mathcal{L}'_f = \bar{\psi} \left[ \gamma^\mu \left( i\partial_\mu + g A_\mu + \frac{9}{m} \partial_\mu B \right) - M \right] \psi \] (37)

This shows that the bad high–energy behavior of the Proca lagrangian comes from the term $\partial_\mu B$. However, the transformations (33) and (34) eliminate this term and leave us with (32) (Stora, 2000).

This is an instance of a general $U(1)$ gauge transformation (27) where the function $\Lambda(x)$ is chosen proportional to $B(x)$. The elimination of $\partial_\mu B$ from the interaction is an important step in the proof of renormalizability. Let us stress also that the field $B(x)$ must be renormalized in a gauge–invariant way (Glauber, 1953). The Stueckelberg massive Abelian model was rigorously proved to be renormalizable and unitary by (Lowenstein and Schroer, 1972), see also (van Hees, 2003).

We now turn to the BRST symmetry for the Stueckelberg theory coupled to a fermion, $\mathcal{L} = \mathcal{L}_{\text{Stueck}} + \mathcal{L}_f$, given by eqs. (31) and (32) above.

Let $\omega(x)$ and $\omega^*(x)$ be independent scalar anticommuting fields. First, read off from the infinitesimal gauge transformations the following BRST transformation s:

\[ s A_\mu = \partial_\mu \omega \] (38)
\[ s B = m \omega \] (39)
\[ s \psi = i g \omega \psi \] (40)
\[ s \bar{\psi} = -i g \omega \bar{\psi} \] (41)
\[ s \omega = 0 \] (42)

---

$^3$ More generally, one could use two different parameters, $(2\alpha_1)^{-1}(\partial_\mu A^\mu + \alpha_2 m)^2$. This generalization is useful in checking gauge independence of observables in some calculations, but it has the disadvantage that a mixing quadratic term $A^\mu \partial_\mu B$ survives in the lagrangian.
Note that $\omega(x)$ is an anticommuting scalar, $\omega(x)^2 = 0$, and thus the BRST transformation $s$ is nilpotent, $s^2 = 0$, even off-shell.

The crucial property of the BRST transformation $s$ is that it is nilpotent, $s^2 = 0$, even off-shell, i.e. without using the field equations. The important point is that the gauge parameter $\Lambda$ was constrained by the Klein–Gordon equation, whereas $\omega$ is free.

The fermionic lagrangian $\mathcal{L}_f$ and the first two terms of (31), which we can denote simply as $\mathcal{L}_g$, are invariant under $s$. Letting

$$L_{\text{Stueck}} = L_g + L_{gf} \quad (43)$$

we can consider instead of the particular gauge–fixing term (30), which is Stueckelberg’s original one for $\alpha = 1$, a more general one (Delbourgo et al., 1988) with an arbitrary local functional

$$L_{gf} = s \left[ \omega^* \left( G(A_\mu, B, \psi, \bar{\psi}) + \frac{\alpha}{2} b \right) \right] \quad (44)$$

which is invariant under the nilpotent $s$ with

$$s \omega^* = b$$

$$s b = 0 \quad (46)$$

The auxiliary field $b$ is just a Lagrange multiplier. The local functional $G(A_\mu, B, \psi, \bar{\psi})$ will be chosen in a specific form only for calculational convenience. The global parameter $\alpha$ does not transform under $s$ and labels a family of different but equivalent gauge slices.

In the above definition of the BRST operator $s$, we have implicitly set $s \alpha = 0$. One could introduce instead an additional anticommuting scalar $\beta$ and define (Piguet and Sorella, 1995) $s \alpha = \beta$, $s \beta = 0$. This extended BRST symmetry allows one to find an extended Slavnov identity which clarifies the gauge-independence of observables.

To check the invariance, observe that $s$ anticommutes with both $\omega$ and $\omega^*$. Using (46), the gauge–fixing lagrangian (44) can be rewritten as

$$L_{gf} = -\omega^* (s G) + b G + \frac{\alpha}{2} b^2 \quad (47)$$

Note that, crucially,

$$s L_{gf} = \omega^* (s^2 G) - b (s G) + b (s G) = 0 \quad (48)$$

Explicitly,

$$L_{gf} = \frac{1}{2} \left( \sqrt{\alpha} b + \frac{1}{\sqrt{\alpha}} G \right)^2 - \frac{1}{2\alpha} G^2 - \omega^* (s G) \quad (49)$$

The auxiliary scalar $b$ field (Lautrup, 1967; Nakanishi, 1966) has indefinite metric, in sharp contrast to Stueckelberg’s $B$ field, which propagates and whose metric is positive. It can be eliminated using its own algebraic equations of motion, which is equivalent to the gaussian redefinition in the functional formalism:

$$\frac{\delta L_{gf}}{\delta b} = 0 \Rightarrow b = -\frac{1}{\alpha} G \quad (50)$$

so that

$$L_{gf} = -\omega^* (s G) - \frac{1}{2\alpha} G^2 \quad (51)$$

There are infinitely many gauge choices for $G$, all providing the same $S$–matrix. Popular are the covariant gauge $G = \partial^\mu A_\mu$ and the ’t Hooft–like gauge $G = \{ \partial^\mu A_\mu + \alpha m B \}$, which gives (40). In these cases, the high energy behavior of the vector field propagator goes like $g_{\mu \nu}/k^2$, so these Stueckelberg theories are power–counting renormalizable (being also unitary). For $G = 0$, one recovers the Proca theory, which is not power–counting renormalizable. The resolution of this apparent contradiction lies in the fact that the fermion field, although local, is not renormalizable in the Jaffee sense (Zimmermann, 1969): in the Proca gauge, the fermion field is something like $e^{-\partial A_\psi}$, and this field is not locally renormalizable because the exponential kills all tempered test functions. The Green’s functions are OK, since the physics is indeed gauge-invariant anyway, and the bothersome exponential disappear from all fermion bilinears, but the theory is not renormalizable.
We choose
\[ G = \partial^\mu A_\mu + \alpha m B \] (52)
so that
\[ sG = (\partial^\mu \partial_\mu \omega + \alpha m^2 \omega) \] (53)
and thus
\[ \mathcal{L}_{gf} = -\omega^* \left( \partial^2 + \alpha m^2 \right) \omega \frac{1}{2\alpha} (\partial^\mu A_\mu + \alpha m B)^2 \] (54)

The ghost term decouples as in QED. In the Stueckelberg or Feynman gauge \( \alpha = 1 \), we recover the lagrangian \( \mathcal{L}_{\text{Stueck}} \) in eq. (29), whose BRST invariance is thus established. It is now a canonical exercise to show the renormalizability and unitarity of the Stueckelberg model (van Hees, 2003; Lowenstein and Schroer, 1972; Picariello and Quadri, 2002).

IV. MASSIVE U(1) GAUGE FIELD

Let us analyze the full \( U(1) \) massive gauge field theory, including spontaneous symmetry breakdown, before turning to the standard model in the next section. This will allow us to highlight the differences and similarities between the Higgs and Stueckelberg mechanisms, which can be implemented simultaneously. The starting lagrangian has three components
\[ \mathcal{L}_o = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}_f \] (55)
where the gauge, scalar and fermion pieces are as follows:
\[ \mathcal{L}_g = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2} (\partial_\mu B - m A_\mu)^2 \] (56)
\[ \mathcal{L}_s = |\partial_\mu \Phi - ie A_\mu \Phi|^2 - \lambda \left( \Phi^\dagger \Phi - \frac{f^2}{2} \right)^2 \] (57)
\[ \mathcal{L}_f = \bar{\psi} (i\gamma^\mu + g A^\mu - M) \psi \] (58)

In this lagrangian, \( e, g, \lambda, m, f \) and \( M \) are parameters (the first three massless, the last three of mass dimension one). They are all customary except for the photon mass \( m \), accompanied by the Stueckelberg field \( B(x) \), which is a scalar commuting field with positive metric. It is useful to introduce the covariant derivatives
\[ D_\mu \Phi = \partial_\mu \Phi - ie A_\mu \Phi \] (59)
and
\[ D_\mu \psi = \partial_\mu \psi - ig A_\mu \psi \] (60)

The vacuum expectation value of the complex scalar field, \( \langle \Phi \rangle = f/\sqrt{2} \) is taken to be a real modulus. We will distinguish between the cases with \( f \) zero or non-zero, since in the latter case the \( U(1) \) gauge symmetry is spontaneously broken (Englert and Brout, 1964; Guralnik et al., 1964; Higgs, 1964). Since \( m \neq 0 \), in both cases the photon is massive.

Each of the three pieces of the above lagrangian is invariant under the BRST transformation \( s \)
\[ s \mathcal{L}_g = s \mathcal{L}_s = s \mathcal{L}_f = 0 \] (61)
defined by (38–42) and
\[ s \Phi = ie \omega \Phi \] (62)
\[ s \Phi^\dagger = -iei \omega \Phi^\dagger \] (63)

Note also that \( s F_{\mu \nu} = 0, s D_\mu \Phi = ie \omega D_\mu \Phi, s D_\mu \psi = ig \omega D_\mu \psi, \) and \( f, \) just like all other parameters, is inert, \( s f = 0. \)
In order to quantize $\mathcal{L}_o$, we must fix the gauge, as discussed in section III. To do so, we add to the lagrangian a gauge–fixing piece $\mathcal{L}_{gf}$ given by (44). After eliminating the auxiliary $b$, the result is a gauge–fixed lagrangian
\[ \mathcal{L} = \mathcal{L}_o - \frac{1}{2\alpha} G^2 - \omega^*(s \mathcal{G}) \] (64)
and we can choose $\mathcal{G}$ as we wish. It is convenient to include a covariant gauge condition in $\mathcal{G}$, as well as a term that cancels, up to total derivatives, the quadratic mixing terms in $\mathcal{L}_o$ involving one derivative.

Define the local current $j^\mu$ as
\[ j^\mu(x) = \frac{\delta \mathcal{L}_o}{\delta A^\mu(x)} \] (65)
By explicit computation, this current is
\[ j^\mu = m (m A^\mu - \partial_\mu B) + ie (\Phi^\dagger D^\mu \Phi - \Phi D^\mu \Phi^\dagger) + g \bar{\psi} \gamma^\mu \psi \] (66)
This current is BRST invariant, $sj^\mu = 0$, and conserved, $\partial^\mu j_\mu = 0$ (from the field equations for $A_\mu$). Since physical states 1) contain no ghosts nor antighosts and 2) are annihilated by the gauge condition $\mathcal{G}$, the field equations for $A_\mu$ from the full gauge-fixed lagrangian (64) imply that the expectation of the divergence of the current vanishes between physical states:
\[ \langle \text{phys} | \partial^\mu j_\mu | \text{phys} \rangle = 0 \] (67)
So the current is indeed conserved in the quantum theory.

Before ending this section, let us note that the Stueckelberg model can be viewed as a free Abelian Higgs model,
\[ \mathcal{L} = -\frac{1}{4} F^2_{\mu\nu} + |(\partial_\mu - ieA_\mu)\Phi|^2 \] (68)
where the module of the complex scalar field is fixed, and its phase is the Stueckelberg field,
\[ \Phi = \frac{m}{\sqrt{2}} e^{ieB(x)/m} \] (69)
This cute formulation is due to [Kibble, 1965]. We shall not exploit it in what follows.

A. Massive electrodynamics

Let us first work out the simple case with $f = 0$, so that the $U(1)$ symmetry is unbroken in perturbation theory – and still the photon is massive. We choose
\[ \mathcal{G} = \partial_\mu A^\mu + \alpha m B \] (70)
to cancel the cross-term between $A_\mu$ and $B$ in $\mathcal{L}_o$, whereby
\[ \mathcal{L} = -\frac{1}{4} F^2_{\mu\nu} + \frac{m^2}{2} A^2_\mu - \frac{1}{2\alpha} (\partial \cdot A)^2 \]
\[ + \frac{1}{2} (\partial_\mu B)^2 - \frac{\alpha m^2}{2} B^2 \]
\[ - \omega^* (\partial^2 + \alpha m^2) \omega \]
\[ + \mathcal{L}_s + \mathcal{L}_f - m \partial_\mu (B A^\mu) \] (71)
with $\mathcal{L}_s$ given by (57) with $f = 0$, and $\mathcal{L}_f$ given by (58). Since there is no spontaneous symmetry breakdown, the gauge–fixing is conveniently chosen independent of the matter fields, the complex scalar and the Dirac fermion. The gauge sector contains first one massive vector field, which decomposes into three physical components of mass $m$ (one longitudinal and two transverse) and one spin–zero piece $\partial A$ of mass $\sqrt{\alpha} m$. It also contains a commuting scalar Stueckelberg $B$–field with mass $\sqrt{\alpha} m$, and a pair of anticommuting ghost–antighost scalars, also with mass $\sqrt{\alpha} m$. All these fields must be kept to prove renormalizability to all orders in perturbation theory. For the computation of $S$–matrix elements, however, we can integrate out the two conjugate Faddeev–Popov ghosts $\omega$ and $\omega^*$, since they do not couple to other fields and they never appear in external asymptotic states. We cannot, however, integrate out the Stueckelberg $B$–field: it is a free field but, as discussed above, it plays a role in the definition of physical states and it undergoes a non-trivial renormalization. It is still possible to gauge–fix it to $B = 0$, recovering a Proca massive Abelian gauge field minimally coupled to a charged scalar and a charged fermion.
B. Spontaneously broken $U(1)$

What happens if $f \neq 0$, that is if the photon acquires a mass both through the Stueckelberg trick and through the Higgs mechanism? It is now convenient to choose a different gauge–fixing functional $G$, similar to 't Hooft’s, to cancel not only the mixing between the photon and the Stueckelberg field, but also that between the photon and the Goldstone. The price we pay is a quadratic mixing term between the Stueckelberg and Goldstone fields, which then have to be redefined through a global rotation. We can parameterize $\Phi$ in cartesian or polar forms.

1. Cartesian parametrization

Although awkward in the Abelian case, this parametrization is the one we will use in the standard model. Recalling that the vacuum expectation value $f/\sqrt{2}$ of $\Phi$ is real, we write

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i \phi_2 + f)$$  \hspace{1cm} (72)

It is worth noting explicitly that

$$s \phi_1 = -e\omega \phi_2$$ \hspace{1cm} (73)
$$s \phi_2 = e\omega(\phi_1 + f)$$ \hspace{1cm} (74)

We choose the gauge–fixing function

$$G = \partial^\mu A_\mu + \alpha(mB + ef\phi_2)$$ \hspace{1cm} (75)

Up to a total derivative (proportional to $\partial_\mu [A^\mu S]$), this gauge–fixing function eliminates the mixing terms between the vector field and the gradients $\partial_\mu B$ and $\partial_\mu \phi_2$ of the scalars.

We find the tremendous lagrangian

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_{gh} + \mathcal{L}_f$$ \hspace{1cm} (76)

where the quadratic piece

$$\mathcal{L}_2 = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2\alpha}(\partial \cdot A)^2 + \frac{1}{2}m^2_\gamma A^2_\mu + \frac{1}{2}(\partial_\mu G)^2$$
$$+ \frac{1}{2}(\partial_\mu S)^2 - \frac{\alpha}{2}m^2_\gamma S^2 + \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}m^2_H \phi_1^2$$ \hspace{1cm} (77)

has been diagonalized into the Stueckelberg ($S$) and Goldstone ($G$) mass eigenstate scalar fields through

$$\begin{pmatrix} S \\ G \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} B \\ \phi_2 \end{pmatrix}$$ \hspace{1cm} (78)

with the angle $\beta$ given by

$$\tan \beta = \frac{ef}{m}$$ \hspace{1cm} (79)

and the short-hands for the Higgs and photon masses

$$m^2_H = 2\lambda f^2$$ \hspace{1cm} (80)
$$m_\gamma = \sqrt{m^2 + e^2f^2}$$ \hspace{1cm} (81)

Note that $\sin \beta = \frac{ef}{m_\gamma}$ and $\cos \beta = \frac{m}{m_\gamma}$. The gauge–fixing function is just

$$G = \partial^\mu A_\mu + \alpha m_\gamma S$$ \hspace{1cm} (82)

whereas

$$s S = \left(m_\gamma + \frac{e^2f}{m_\gamma} \phi_1 \right) \omega$$ \hspace{1cm} (83)
$$s G = \frac{em}{m_\gamma} \phi_1 \omega$$ \hspace{1cm} (84)
Note that the two contributions in quadrature to the photon mass are always positive, so we cannot envisage a cancellation. The phase conventions are such that $e, m$ and $f$ are always positive.

The cubic, quartic and ghost lagrangians are

$$L_3 = -e A'' \phi_1 \partial_\mu \phi_1 - \phi_2 \partial_\mu \phi_1 + e^2 f \phi_1 A^2_\mu - \lambda f \phi_1 \left( \phi_1^2 + \phi_2^2 \right)$$

$$= -e A'' \left[ \cos \beta (\phi_1 \partial_\mu G - G \partial_\mu \phi_1) + \sin \beta (\phi_1 \partial_\mu S - S \partial_\mu \phi_1) \right]$$

$$+ e^2 f \phi_1 A^2_\mu - \lambda f \phi_1 \left( \phi_1^2 + (\cos \beta G + \sin \beta S)^2 \right)$$

$$L_4 = \frac{e^2}{2} A^2_\mu \left( \phi_1^2 + \phi_2^2 \right) - \frac{\lambda}{4} \left( \phi_1^2 + \phi_2^2 \right)^2$$

$$= \frac{e^2}{2} A^2_\mu \left( \phi_1^2 + (\cos \beta G + \sin \beta S)^2 \right) - \frac{\lambda}{4} \left( \phi_1^2 + (\cos \beta G + \sin \beta S)^2 \right)^2$$

$$L_{gh} = -\omega^* \left[ \partial^2 + \alpha m^2 + \alpha e^2 f \phi_1 \right] \omega$$

The couplings of the two scalars $G$ and $S$ (identical except for a $\sin \beta$ or $\cos \beta$ weight) are both derivative and non-derivative. Furthermore, the Faddeev–Popov ghosts couple to the Higgs field $\phi_1$.

In the limit $m \to 0, \beta \to \pi/2$ and the massless $G$ decouples, whereas the surviving $S$ coincides with the original $\phi_2$. In this limit, of course, the photon mass is due only to the Higgs mechanism. Contrariwise, when $f \to 0, \beta \to 0$, the field $S \to B$ decouples, and only $G$ remains coupled. Curiously, in this limit, the surviving field is again $\phi_2$. So in both extreme limits, $f \to 0$ and $m \to 0$, the original Stueckelberg field decouples. Also, the lagrangian in both limits is identical, except, of course, that the photon mass $\frac{81}{8}$ is either $ef$ or $m$. In general, for $f \neq 0$ and $m \neq 0$, there are altogether three propagating scalar fields with different masses, the Higgs $\phi_1$, the Goldstone $G$, and the Stueckelberg $S$.

2. Polar parametrization

Letting

$$\Phi(x) = \frac{1}{\sqrt{2}} e^{i \theta(x)/f} (H(x) + f)$$

the scalar part of the lagrangian is

$$L_s = \frac{1}{2} (\partial_\mu H)^2 + \frac{1}{2} (\partial_\mu \theta - e f A_\mu)^2 \left( 1 + \frac{H}{f} \right)^2 - \lambda H^2 \left( f + \frac{H}{2} \right)^2$$

Note that the Goldstone field $\theta$ is massless and couples only through its derivatives. Due to spontaneous symmetry breaking, the photon $A_\mu$ has acquired a mass $ef$, which adds in quadrature to the Stueckelberg mass $m$.

It would be tempting to carry out the gauge transformation

$$\Phi \to e^{-i \theta/f} \Phi = \frac{1}{\sqrt{2}} (f + H)$$

$$A_\mu \to A_\mu - \frac{1}{ef} \partial_\mu \theta$$

$$B \to B - \frac{m}{ef} \theta$$

$$\psi \to e^{-i \varphi / (ef)} \psi$$

whereby the Goldstone $\theta$ would disappear completely from the classical lagrangian:

$$L = -\frac{1}{4} F_{\mu \nu}^2 + \frac{1}{2} (m A_\mu - \partial_\mu B)^2 + \frac{1}{2} (\partial_\mu H)^2$$

$$+ \frac{e^2}{2} A_\mu^2 (H + f)^2 - 2 \frac{\lambda}{4} (H^2 + 2 f H)^2 + L_f$$

(94)
Note, however, that the choice \( \Lambda = -\theta/(e f) \) in \[93\] means that we must require \((\partial^2 + m^2)\theta = 0\), which is not consistent with the field equations for \( \theta \) from the original lagrangian \[55\]. Thus, this “unitary” gauge is not allowed due to the presence of the Stueckelberg field.Scalars are, indeed, trickier than fermions.

Therefore, we choose

\[
\mathcal{G} = \partial_\mu A^\mu + \alpha m B + \alpha e f \theta
\]

(95)

whereby the quadratic piece of the gauge-fixed lagrangian is

\[
\mathcal{L}_2 = -\frac{1}{4} F_{\mu \nu}^2 + \frac{m^2}{2} A_\mu^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 = \bar{\psi}(i\partial - M)\psi \\
+ \frac{1}{2} (\partial_\mu B)^2 + \frac{1}{2} (\partial_\mu \theta)^2 - \frac{\alpha}{2} (m B + e f \theta)^2 \\
+ \frac{1}{2} (\partial_\mu H)^2 - \frac{m_0^2}{2} H^2 \\
+ \omega^2 \left( \partial^2 + \alpha m_0^2 \right) \omega - \partial_\mu [A_\mu(e f \theta + m B)]
\]

(96)

where the photon and Higgs masses are given by \[51\] and \[50\]. To derive this expression, it is useful to keep in mind that in the polar parametrization, \( s H = 0 \) and \( s \theta = e f \omega \) (and, as usual, \( s f = 0 \)). Note that the last line of \[95\] can be ignored in the computation of \( S \)-matrix elements.

Observe that the photon mass \[51\] squared is the sum of two contributions, one from the Stueckelberg mechanism and the other from the Higgs mechanism. Note also that the \( B \) and \( \theta \) fields mix, so we must rotate them into mass eigenstates. One of these is massless, and the other has its mass squared equal to the gauge parameter times the photon mass squared. We will call them \( G' \) and \( S' \) (after Goldstone and Stueckelberg, distinguishing them from \( G \) and \( S \) in the cartesian parametrization above):

\[
\begin{pmatrix}
S' \\
G'
\end{pmatrix} = \begin{pmatrix}
\cos \beta & \sin \beta \\
-\sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
B \\
\theta
\end{pmatrix}
\]

(97)

with the angle \( \beta \) defined by eq. \[73\].

Dropping the total derivative \(-m_\gamma \partial_\mu (A^\mu S')\) and the non-interacting ghost-antighost system, we end up with the following gauge-fixed quantum lagrangian, appropriate for perturbative evaluations of \( S \)-matrix elements:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu}^2 + \frac{m^2}{2} A_\mu^2 - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \mathcal{L}_f \\
+ \frac{1}{2} (\partial_\mu S')^2 - \frac{\alpha}{2} m_\gamma^2 S'^2 + \frac{1}{2} (\partial_\mu G')^2 \\
+ \frac{1}{2} (\partial_\mu H)^2 - \frac{m_0^2}{2} H^2 - \frac{\lambda}{4} H^4 - \lambda f H^3 \\
+ \left( \frac{H}{f} + \frac{H^2}{2f^2} \right) \left[ \frac{1}{m_\gamma} (e f \partial_\mu S' + m \partial_\mu G') - e f A_\mu \right]^2
\]

(98)

The most salient feature of this lagrangian is that there are two independent fields with derivative couplings to the Higgs. The two scalars \( G' \) and \( S' \) have the same masses as \( G \) and \( S \) in the cartesian parametrization, but now their couplings (identical except for a \( \sin \beta \) or \( \cos \beta \) weight) are only derivative. Furthermore, as customary in the Abelian case, the Faddeev–Popov ghosts decouple.

Note that in the limit \( m \to 0 \), that is when the photon mass is due solely to the spontaneous symmetry breakdown, the massless field \( G' \) decouples and the massive field \( S' \) coincides with the original Goldstone \( \theta \). This reproduces, of course, the usual Higgs mechanism. The limit \( f \to 0 \) is rather singular in this polar parametrization \[83\], but in this limit the massive \( S' \) coincides with \( B \) and decouples whereas the massless \( G' \) is just \( \theta \) and remains coupled.

V. ELECTROWEAK THEORY WITH A MASSIVE PHOTON

We now come to one of the main points of this paper which is to allow a consistent regularization of the infrared divergences due to the photon by giving it a finite mass. To achieve this, we use the Stueckelberg mechanism described earlier of introducing an auxiliary scalar field \( B \) of positive metric while preserving the BRST invariance of the standard electroweak theory \[Glashow 1961\] \[Salam 1968\] \[Weinberg 1967\]. This allows a separate treatment of infrared and
ultraviolet divergences in the perturbative expansion. It does not suffice to add an explicit mass term for the photon, however, even with the Stueckelberg trick, because it would spoil the $SU(2) \times U(1)$ symmetry. Instead, following Stora (2000) and Grassi and Hurth (2001), we give a Stueckelberg mass to the vector field $V_\mu$ corresponding to the hypercharge Abelian factor $U(1)_Y$ of the gauge group. After the symmetry breakdown $SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}$, the photon field $A_\mu$ inherits a mass proportional to the original Stueckelberg mass for the hypercharge vector boson. Empirically, this mass is strictly bound. As we shall show, it is necessary to modify many of the parameters in the electroweak theory, albeit by very small amounts. The spontaneously broken electroweak theory is still BRST invariant, and in addition it is free of infrared divergences. Of course, infrared divergences in QCD remain.

It is perhaps worth stressing at this early point that the Stueckelberg mechanism is implemented in the Abelian factor of the standard model gauge group to give mass to the Abelian gauge boson without any symmetry breaking. We do not know of a mechanism for generating a Stueckelberg mass if none is present to start with. Thus, if there is a grand unification of the standard model into a gauge group without Abelian factors, then there is no reason to expect a Stueckelberg mechanism at low energies. On the other hand, if at high energies the gauge group contains a $U(1)$ factor, its gauge boson can acquire a Stueckelberg mass without symmetry breakdown and this mass could then tumble down. In the framework of string theory, such an Abelian mass generation is not forbidden. String phenomenologists use the Stueckelberg trick assiduously to get rid of spurious divergences. Of course, infrared divergences in QCD remain.

Our approach is rather that since gauge invariance allows for an Abelian vector mass term, it should be considered. Accordingly, the Stueckelberg mass $m$ for the Abelian $U(1)_Y$ factor is a free parameter of the standard model, just like $\theta_{QCD}$. We are aware, however, that the introduction of this new mass scale entails unavoidably a new hierarchy problem, notably with respect to the electroweak breaking scale $f$ ($m \ll f$). Of course, dimensional transmutation yields an intrinsic mass scale for strong interactions, $\Lambda_{QCD}$, so it is somewhat symmetric, perhaps, that the Stueckelberg mechanism produces a mass scale for the Abelian factor of the standard model gauge group. These three unrelated masses, associated with each of the three gauge group factors of the standard model, accompany generically the three distinct phases of a gauge theory (confinement, spontaneous symmetry breaking, or Abelian). This could bring up an additional problem, since this third phase is usually called Coulomb precisely because the static potential is of infinite range. But a massive photon implies that the electrostatic potential deviates from Coulomb, so it is not of infinite range. In fact, we are not quite clear about what properties the external “classical” electromagnetic fields inherit from the modified massive quantum photon. The issue does not arise in Stueckelberg QED, but it does in the Stueckelberg standard model: the Noether current leaving invariant the vacuum does not couple to the asymptotic (and massive) photon. We shall return to these tricky issues below.

Other authors have investigated in different directions. Cveti´c and Kogerler, 1991 exploited the Stueckelberg formalism, but they constructed a standard model with the gauge symmetry realized non-linearly, which is equivalent to taking the Higgs mass to infinity. In a similar vein, Grosse-Knetter and Kogerler, 1993 integrated out the Higgs field from the standard model and obtained an effective lagrangian close to Proca’s (Dittmaier and Grosse-Knetter, 1995, 1996, 1992, Grosse-Knetter, 1993, 1994, Grosse-Knetter et al., 1993). Our approach is orthogonal to these viewpoints, since we keep the physical Higgs field and our photon is not massless. Also, we ensure exact quantum unitarity and renormalizability.

The hypercharge is normalized such that $Q = T_3 + Y/2$. We follow the careful notation of Taylor, 1976, and leave to the Appendix some details of our conventions, as well as the long formulas of interest only to the conscientious scholar.

For starters, we concentrate in section V.A on the gauge sector, which consists of the vectors, scalars and ghosts. We turn to the fermion matter fields, with their lagrangian and BRST transformations, in section V.B. For simplicity, we present here the Stueckelberg modification of the minimal standard theory, without any neutrino masses: we do not include right–handed gauge singlets. Including them, with their phenomenologically required large Majorana mass, does not change the analysis in any substantial way: it would be tantalizing to speculate that neutrino masses are related to the Stueckelberg mechanism.

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4 Raymond Stora informed us that his original idea to use the Stueckelberg field in the standard model arose in the course of a conversation with Tobias Hurth.
A. The gauge sector

The starting gauge-invariant lagrangian is a sum of pieces associated with the gauge, the scalar, and the fermion fields:

\[ \mathcal{L}_o = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}_f \]  

(99)

The gauge lagrangian contains the usual kinetic terms for the vector fields \( \vec{W}^{\mu} \) and \( V^{\mu} \), as well as the Stueckelberg mass for \( V^{\mu} \), along with the kinetic term for the Stueckelberg field \( B \) (of zero hypercharge and weak isospin):

\[ \mathcal{L}_g = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (m V^\mu - \partial^\mu B)^2 \]  

(100)

where the gauge potential field strengths are

\[ F_{\mu\nu} = \partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu} \]  

(101)

\[ \vec{F}_{\mu\nu} = \partial_{\mu} \vec{W}_{\nu} - \partial_{\nu} \vec{W}_{\mu} + g \vec{W}_{\mu} \times \vec{W}_{\nu} \]  

(102)

The scalar lagrangian, including Higgs and Goldstones, is

\[ \mathcal{L}_s = |D_{\mu} \Phi|^2 - \lambda \left( \Phi^\dagger \Phi - \frac{f^2}{2} \right)^2 \]  

(103)

where the scalar weak isodoublet is

\[ \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} H + f + i \vec{\tau} \cdot \vec{\phi} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{2}} \phi_- \\\ \sqrt{2} (H - i \phi_3 + f) \end{pmatrix} \]  

(104)

and its covariant derivative is

\[ D_{\mu} \Phi = \left( \partial_{\mu} - i \frac{g}{2} \vec{\tau} \cdot \vec{\omega} \right) \vec{W}_{\mu} + \frac{i g'}{2} V_{\mu} \]  

(105)

The minima of the potential are located at \( |\Phi^\dagger \Phi| = f^2/2 \), and we choose the vacuum to be given by \( < \Phi > = f/\sqrt{2} \), with \( f \) real. The scalar lagrangian is spelled out in section A.2 of the Appendix.

The gauge lagrangian (100) is invariant under the BRST transformation \( s \) defined by

\[ s V_{\mu} = \partial_{\mu} \omega \]  

(106)

\[ s B = m \omega \]  

(107)

\[ s \vec{W}_{\mu} = \partial_{\mu} \vec{\omega} - g \vec{\omega} \times \vec{W}_{\mu} \]  

(108)

where \( \omega \) and \( \vec{\omega} \) are anticommuting scalars. The BRST operator \( s \) is nilpotent, \( s^2 = 0 \), with

\[ s \omega = 0 \]  

(109)

\[ s \vec{\omega} = -\frac{g}{2} \vec{\omega} \times \vec{\omega} \]  

(110)

Note that \( s F_{\mu\nu} = 0 \) and \( s \vec{F}_{\mu\nu} = -g \vec{\omega} \times \vec{F}_{\mu\nu} \).

The scalar lagrangian (103) is also BRST invariant with the additional definition

\[ s \Phi = \frac{i}{2} \left( g \vec{\tau} \cdot \vec{\omega} + g' \omega \right) \Phi \]  

(111)

which implies that \( s (D_{\mu} \Phi) = (i/2) (g \vec{\tau} \cdot \vec{\omega} + g' \omega) D_{\mu} \Phi \). The BRST transforms of the components of the scalar doublet defined in equation (104) are

\[ s \phi_\pm = \frac{g}{2} (i \omega_\pm \phi_\mp \mp i \omega_\mp \phi_\pm + \omega_\pm H) \mp i \frac{g'}{2} \omega_\pm \phi_\pm + \frac{f^2}{2} \omega_\pm \]  

(112)

\[ s \phi_3 = \frac{g}{2} (i \omega_- \phi_+ + i \omega_+ \phi_- + \omega_3 H) - \frac{g'}{2} \omega H + \frac{f}{2} (g \omega_3 - g' w) \]  

(113)
and

\[ sH = -\frac{g}{2} \vec{\omega} \cdot \vec{\phi} + \frac{g'}{2} \omega \phi_3 \]  

(114)

The fermion lagrangian, discussed in the next section, is also BRST–invariant. The nilpotency of the BRST operator \( s \) (without having to use the equations of motion) and the invariance of the full lagrangian under \( s \) ensure the renormalizability of the theory to all orders in perturbation theory, hopefully. To check this explicitly, the full Slavnov–Taylor identities would have to be proved. This is a significant piece of work which has not been done.

We add to \( \mathcal{L}_0 \) a gauge–fixing lagrangian

\[ \mathcal{L}_{gf} = s \left[ \vec{\omega}^* \cdot \left( \vec{G} + \frac{\alpha}{2} \vec{b} \right) + \omega^* \left( G + \frac{\alpha'}{2} b \right) \right] \]  

(115)

with more Faddeev–Popov ghosts \( \omega^*, \vec{\omega}^* \) (independent from \( \omega \) and \( \vec{\omega} \)) and Nakanishi–Lautrup ghosts \( b, \vec{b} \), subject to

\[ s \vec{\omega}^* = \vec{b} \]  

(116)

\[ s \vec{b} = 0 \]  

(117)

\[ s \omega^* = b \]  

(118)

\[ s b = 0 \]  

(119)

Notice that we keep two different gauge parameters \( \alpha \) and \( \alpha' \) for each of the factors in the electroweak gauge group, namely \( SU(2)_L \) and \( U(1)_Y \), respectively. Below, we will equate them to simplify some tree–level expressions, but in general they must be kept distinct because under renormalization they behave differently, since there is no symmetry which favors their equality.

Eliminating the auxiliary Nakanishi–Lautrup ghosts \( b \) and \( \vec{b} \) is not a very good idea, since once they are gone the BRST operator is nilpotent only on-shell. Of course, at tree level it’s safe and instructive to get rid of them algebraically, finding the lagrangian

\[ \mathcal{L}_{ph} = \mathcal{L}_0 + \mathcal{L}_{gf}' + \mathcal{L}_{gh} \]  

(120)

with

\[ \mathcal{L}_{gf}' = -\frac{1}{2\alpha} G^2 - \frac{1}{2\alpha'} \vec{G}^2 \]  

(121)

and

\[ \mathcal{L}_{gh} = -\omega^* s G - \vec{\omega}^* \cdot s \vec{G} \]  

(122)

Following ’t Hooft, we choose the following gauge functions \( \text{Grassi and Hurth} \) \{2001\}

\[ G = \partial_\mu V^\mu + \alpha' m B - \alpha' \frac{g'}{2} f \phi_3 \]  

(123)

\[ \vec{G} = \partial_\mu \vec{W}^\mu + \alpha \frac{g}{2} f \vec{\phi} \]  

(124)

Notice that the \( SU(2) \) gauge function is just ’t Hooft’s, whereas the \( U(1) \) function \( G \) contains also a term involving the Stueckelberg field. These gauge functions have been chosen to give total derivatives when combined with the terms in the lagrangian with one gauge boson, one scalar and one derivative. All total derivatives in the lagrangian we just drop.

1. Mass eigenstates

Let us collect terms in the lagrangian \( \mathcal{L}_{ph} \) in eq. (120) into three pieces: the fermionic lagrangian \( \mathcal{L}_f \) that we have not even written out yet, a quadratic lagrangian \( \mathcal{L}_2 \) with at most two fields, and the rest, which we call the interaction lagrangian \( \mathcal{L}_{int} \). They are all spelled out in the Appendix. We concentrate now on \( \mathcal{L}_2 \) and diagonalize it.

Just like in the usual standard theory, the charged vector fields \( W^\pm_\mu \) have mass

\[ M_W = \frac{f g}{2} \]  

(125)
whereas the charged scalars, $\phi_{\pm}$, and the charged ghost-antighost pairs, $\omega_{\pm}^{(v)}$, have mass $\sqrt{\alpha} M_W$. Neutral boson fields mix in pairs. Explicitly, in the bases $(W_3^\mu, V^\mu)$, $(\phi_3, B)$ and $(\omega_3, \omega)$, the respective square mass matrices for vectors, scalars and ghosts are

$$M_v^2 = \frac{f^2}{4} \begin{pmatrix} g^2 & -g' \mu \\ -g' \mu & g'^2 + \mu^2 \end{pmatrix}$$  
$$M_s^2 = \frac{f^2}{4} \begin{pmatrix} \alpha g^2 + \alpha^2 g'^2 & -\alpha' g' \mu \\ -\alpha' g' \mu & \alpha' \mu^2 \end{pmatrix}$$  
$$M_g^2 = \frac{f^2}{4} \begin{pmatrix} \alpha g^2 & -\alpha' g g' \\ -\alpha' g g' & \alpha' (g'^2 + \mu^2) \end{pmatrix}$$

where the last matrix is understood to be sandwiched between $(\omega_3^*, \omega^*)$ and $(\omega_3, \omega)$, and we have introduced the rescaled Stueckelberg mass of the hypercharge vector boson

$$\mu = \frac{2 m}{f}$$

which behaves like a coupling constant. These mass matrices reduce to the usual ones of the standard model when $\mu = 0$.

The mass eigenstates are obtained by rotations

$$\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} W_3^\mu \\ V^\mu \end{pmatrix}$$  
$$\begin{pmatrix} G \\ S \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \phi_3 \\ B \end{pmatrix}$$  
$$\begin{pmatrix} \chi_Z \\ \chi_A \end{pmatrix} = \begin{pmatrix} \cos \tilde{\theta}_w & -\sin \tilde{\theta}_w \\ \sin \tilde{\theta}_w & \cos \tilde{\theta}_w \end{pmatrix} \begin{pmatrix} \omega_3 \\ \omega \end{pmatrix}$$  
$$\begin{pmatrix} \chi_Z^* \\ \chi_A^* \end{pmatrix} = \begin{pmatrix} \cos \tilde{\theta}_w & -\sin \tilde{\theta}_w \\ \sin \tilde{\theta}_w & \cos \tilde{\theta}_w \end{pmatrix} \begin{pmatrix} \omega_3^* \\ \omega^* \end{pmatrix}$$

where the mixing angles are defined by

$$\tan 2\theta_w = \frac{2 g g'}{g^2 - g'^2 - \mu^2}$$  
$$\tan 2\beta = \frac{2 \mu g'}{\alpha g g' + \mu^2 - \mu^2}$$  
$$\tan 2\tilde{\theta}_w = \frac{2 g g'}{\alpha g g' - \alpha \mu^2 + \mu^2}$$

The last two expressions simplify if we set $\alpha' = \alpha$, which we are allowed to do at tree level. In this simpler case, $\tilde{\theta}_w = \theta_w$.

Let us point out the main differences with the usual electroweak theory. Since the hypercharge gauge field $V_\mu$ has a Stueckelberg mass, the weak mixing angle $\theta_w$ is modified. Just as in the usual theory, the mixing angle between the associated ghost fields is also the weak mixing angle if the two gauge parameters $\alpha$ and $\alpha'$ are equal, but not otherwise. The new mixing angle is $\beta$, between the longitudinal degrees of freedom of the $SU(2)_L$ and of the $U(1)$ neutral vector bosons. This is reasonable, since the latter is the Stueckelberg field, which does not exist in the minimal electroweak theory. The angle $\beta$ is tiny, proportional to the ratio of the Stueckelberg mass to the electroweak vacuum expectation value. Again, let us stress that the charged sector does not change.

To expand in powers of the Stueckelberg mass $m$ or, better, in terms of the rescaled Stueckelberg mass $\mu = 2 m/f$, it is useful to introduce the convenient parameter

$$\epsilon = \frac{\mu^2}{g^2 + g'^2} = \frac{m^2}{f^2(g^2 + g'^2)}$$

Note that $\mu$ has mass dimension zero but behaves as a coupling constant, whereas $\epsilon$ is truly dimensionless. The following trigonometric functions of the modified weak mixing angle are handy:

$$\tan \theta_w = \frac{g'}{g} (1 + \epsilon) + O(\epsilon^2)$$
\[ s_w = \sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} \left( 1 + \epsilon \frac{g'^2}{g^2 + g'^2} \right) + \mathcal{O}(\epsilon^2) \]  
(139)

\[ c_w = \cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}} \left( 1 - \epsilon \frac{g'^2}{g^2 + g'^2} \right) + \mathcal{O}(\epsilon^2) \]  
(140)

Finally, the small mixing angle \( \beta \) between the Goldstone \( \phi_3 \) and the Stueckelberg \( B \) is approximately

\[ \beta = \frac{g'}{\sqrt{g^2 + g'^2}} \sqrt{\epsilon} + \mathcal{O}(\epsilon^{3/2}) \simeq \frac{g'}{\sqrt{g^2 + g'^2}} \sqrt{\epsilon} \simeq \cos \theta_w \sqrt{\epsilon} \]  
(141)

where the last expressions hold at tree level if we choose \( \alpha = \alpha' \).

As was to be expected, the photon has a non-vanishing mass \( M_A \) proportional, to first order, to the original Stueckelberg mass \( m = m f / 2 \) of the hypercharge vector boson:

\[ M_A = m \cos \theta_w + \mathcal{O}(m^3) = M_W \sqrt{\epsilon} + \mathcal{O}(\epsilon^{3/2}) \]  
(142)

The \( Z^\mu \) vector boson mass \( M_Z \) differs slightly from the usual one because the weak mixing angle is slightly different. Now it becomes

\[ M_Z = \frac{f}{2} \sqrt{g^2 + g'^2} \left( 1 + \frac{\epsilon}{2} \frac{g'^2}{g^2 + g'^2} \right) + \mathcal{O}(\epsilon^2) \]

\[ = \frac{M_W}{\cos \theta_w} \left( 1 - \frac{\epsilon}{2} \sin^2 \theta_w \right) \]  
(143)

The exact mass eigenvalues are given in the Appendix, eq. (A11).

Very nicely, after rotating by \( \beta \) the Goldstone \( \phi_3 \) and Stueckelberg \( B \) fields, the mass eigenstates \( G \) and \( S \) have exactly the same masses as the anticommuting ghosts \( \chi_Z \) and \( \chi_A \), obtained by rotating through \( \theta_w \) the ghosts \( \omega_3 \) and \( \omega \).

If we set \( \alpha' = \alpha \) (valid at tree level), then we find the simple formulas

\[ M_S = M_{\chi_A} = \sqrt{\alpha} M_A \]  
(144)

\[ M_G = M_{\chi_Z} = \sqrt{\alpha} M_Z \]  
(145)

The full expressions are given in section A.4 of the Appendix.

**B. Matter**

The fermion lagrangian is the sum of a lepton and a quark lagrangians. Added to the gauge and scalar lagrangians discussed above, it yields the full classical lagrangian.

The lepton lagrangian is

\[ L_\ell = \bar{R} \left( i \partial - g' \gamma \right) R \]

\[ + \bar{L} \left( i \partial - \frac{g'}{2} \gamma + \frac{g}{2} \tilde{\gamma} \cdot \tilde{W} \right) L \]

\[ - (y_e R (\Phi^\dagger L) + \text{h.c.}) \]  
(146)

where \( y_e \) is a Yukawa matrix, the hypercharges of \( R \) and \( L \) are \(-2\) and \(-1\), we have suppressed family indices, and

\[ R = e_R = \frac{1 - \gamma_5}{2} \]  
(147)

\[ L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} 1 + \gamma_5 \\ 2 \end{pmatrix} \begin{pmatrix} \nu \\ e \end{pmatrix} \]  
(148)
The BRST transformations of the lepton fields are

\[ s_R = -ig' \omega R \]  \hspace{1cm} (149)

\[ s_L = \frac{i}{2} (g \vec{r} \cdot \vec{\omega} - g' \omega) L \]  \hspace{1cm} (150)

The quark lagrangian is

\[ L_q = i \bar{Q} \left( \partial - ig' \frac{\omega}{6} V - ig' \frac{2}{3} \vec{r} \cdot \vec{W} \right) Q \\
+ i \bar{U} \left( \partial_u - ig' \frac{6}{3} \right) U \\
+ i \bar{D} \left( \partial + ig' \frac{3}{3} \right) D \\
- \left( y_d \bar{D}(\Phi^1 Q) + y_u(\bar{Q}i\tau^2 \Phi)U + \text{h.c.} \right) \]  \hspace{1cm} (151)

where

\[ Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \]  \hspace{1cm} (152)

\[ U = u_R \]  \hspace{1cm} (153)

\[ D = d_R \]  \hspace{1cm} (154)

with hypercharges 1/3, 4/3 and -2/3, and we have suppressed both family and color indices.

The BRST transformations of the quark fields are

\[ s_U = i \frac{2g'}{3} \omega U \]  \hspace{1cm} (155)

\[ s_D = -i \frac{g'}{3} \omega D \]  \hspace{1cm} (156)

\[ s_Q = i \frac{1}{2} \left( g \vec{r} \cdot \vec{\omega} + g' \right) \]  \hspace{1cm} (157)

The Yukawa interactions in (146) and (151) give the usual mass matrices to the fermions,

\[ M_e = \frac{y_e f}{\sqrt{2}} , \quad M_u = \frac{y_u f}{\sqrt{2}} , \quad M_d = \frac{y_d f}{\sqrt{2}} \]  \hspace{1cm} (158)

so that the free fermion lagrangian is

\[ L_{ff} = \bar{\nu}_L \Phi + \sum_{\psi=e,d,u} \left\{ i \bar{\psi}_L \partial \psi_L + i \bar{\psi}_R \partial \psi_R - M_{\psi} \bar{\psi}_R \psi_L - M_{\psi}^* \bar{\psi}_L \psi_R \right\} \]  \hspace{1cm} (159)

They also give rise to the interactions between the scalars and the fermions:

\[ L_y = -\frac{1}{\sqrt{2}} (H + i \cos \beta G + i \sin \beta S) (y_e \bar{e}_R e_L + y_d \bar{d}_R d_L + y_u \bar{u}_R u_L) \\
+ i \phi^+ \left( y_e \bar{e}_R \nu_L + y_d \bar{d}_R u_L - y_u \bar{u}_R d_L \right) \\
+ \text{h.c.} \]  \hspace{1cm} (160)

where we have already eliminated \( \phi_3 \) and \( B \) in favor of the mass eigenstates \( G \) and \( S \).

The interaction between the fermions and the gauge bosons is cute. From the covariant kinetic terms for the fermions we find the usual charged current lagrangian

\[ L_{cc} = \frac{g}{\sqrt{2}} (\bar{\nu}_L \partial - e_L \nu_L + \bar{u}_L \partial - d_L u_L) + \text{h.c.} \]  \hspace{1cm} (161)

The neutral currents are rather funny due to the massiveness of the photon and the modified weak mixing angle. Indeed, the neutral current lagrangian can be written as

\[ L_{nc} = \sum_{\psi} \bar{\psi} \left( n_\psi A + n_\psi^Z Z \right) \psi \]  \hspace{1cm} (162)
where the sum runs over all the two-component fermionic fields with non-zero isospin, $\psi \in \{\nu_L, e_L, e_R, d_L, d_R, u_L, u_R\}$. All the exact couplings and their expansion to first order in $\epsilon$ are given in section A.7 of the Appendix.

To illustrate the novel features, it is instructive to display the approximate lepton couplings to the photon, with the help of (137) and the traditional

$$ e = \frac{gg'}{\sqrt{g^2 + g'^2}} $$

which has no particular physical meaning when the Stueckelberg mass does not vanish:

$$ n^A_\nu \simeq \frac{\epsilon}{2} e $$

$$ n^A_{eL} \simeq -e \left( 1 + \frac{\epsilon}{2} \frac{g^2 - g'^2}{g^2 + g'^2} \right) $$

$$ n^A_{eR} \simeq -e \left( 1 - \frac{\epsilon}{2} \frac{g^2 - g'^2}{g^2 + g'^2} \right) $$

These neutral currents can be rewritten in Dirac spinor notation as follows:

$$ \mathcal{L}_{nc} = \sum_{\psi} \left\{ \bar{\psi}_A (v^A_\psi + a^A_\psi \gamma_5) \psi + \bar{\psi}_A (v^A_\psi + a^A_\psi \gamma_5) \psi \right\} $$

where the sum runs over $\psi \in \{\nu, e, u, d\}$. The leptonic couplings to the photon are

$$ v^A_\nu = a^A_\nu = -a^A_e \simeq \frac{\epsilon}{4} $$

$$ v^A_e \simeq -e \left( 1 + \frac{\epsilon}{4} \frac{g^2 - 3g'^2}{g^2 + g'^2} \right) $$

Notice the universality of the fermionic axial coupling to the photon, only of order $\epsilon$. There is also a universality in the fermionic axial couplings to the $Z$, which remain unchanged at first order from the standard value. These universalities extend to the quarks, with the couplings shown in section A.7.

A BRST-consistent mass for the photon in the standard electroweak theory implies then that it has not only the massive photon, $W^\pm$, $Z$, and massless gluon. The triangle graph with three external photons, for instance, is proportional to the sum of the cubes of the photonic couplings of the left-handed fermion fields minus the sum of the cubes of the photonic couplings of the right-handed fields. Using the two-component form of the fermion couplings to the neutral vector bosons, the charged current (161), and the fact that quarks come in triplets of $SU(3)$ whereas leptons are singlets thereof, the following exact relationships between the fermion couplings are verified, where we note on the left the three gauge bosons at the vertices of the fermion loop, including the gluons $G$ and the gravitons $h$:

$$ AAA \quad (n^A_\nu)^3 + (n^A_{eL})^3 - (n^A_{eR})^3 + 3(n^A_{uL})^3 + 3(n^A_{dL})^3 - 3(n^A_{uR})^3 - 3(n^A_{dR})^3 = 0 $$

$$ AAZ \quad (n^A_\nu)^2(n^A_\nu) + (n^A_{eL})^2(n^A_{eL}) - (n^A_{eR})^2(n^A_{eR}) + 3(n^A_{uL})^2(n^A_{uL}) $$

C. Anomalies

The theory we have constructed has exact BRST symmetry, even at the quantum level, so it is free of anomalies. In fact, since the only change in the BRST structure with respect to that of the standard theory is the addition of the doublet $sB = \omega$, $s\omega = 0$ with trivial cohomology, the anomalies in the Stueckelberg modified standard model are the same as those in the usual standard model. But we can check the vanishing of all anomalies in the modified model directly.

It is quite remarkable that anomalies cancel directly in the basis of physical vector bosons (the propagating degrees of freedom, namely the massive photon, $W^\pm$, $Z$, and massless gluon). The triangle graph with three external photons, for instance, is proportional to the sum of the cubes of the photonic couplings of the left-handed fermion fields minus the sum of the cubes of the photonic couplings of the right-handed fields. Using the two-component form of the fermion couplings to the neutral vector bosons, the charged current (161), and the fact that quarks come in triplets of $SU(3)$ whereas leptons are singlets thereof, the following exact relationships between the fermion couplings are verified, where we note on the left the three gauge bosons at the vertices of the fermion loop, including the gluons $G$ and the gravitons $h$:
and all four currents are conserved (171).

\[ AZZ \quad (n_u^A)^2(n_{dL}^Z) + (n_{dL}^A)(n_{dR}^Z) - (n_u^A)(n_{dR}^Z) + 3(n_{uR}^A)(n_{dL}^Z) = 0 \]

\[ ZZZ \quad (n_u^Z)^3 + (n_{dL}^Z)^3 - (n_{dR}^Z)^3 = 0 \]

\[ AWW \quad (n_u^A) + 3(n_{dL}^A) + 3(n_{dR}^A) = 0 \]

\[ ZWW \quad (n_u^Z) + 3(n_{dL}^Z) + 3(n_{dR}^Z) = 0 \]

\[ AGG \quad (n_u^A) + (n_{dL}^A) - (n_{dR}^A) = 0 \]

\[ ZGG \quad (n_u^Z) + (n_{dL}^Z) - (n_{dR}^Z) = 0 \]

\[ Ahh \quad (n_{dR}^A) + 3(n_{uR}^A) = 0 \]

\[ Zhh \quad (n_u^Z) + 3(n_{dR}^Z) = 0 \]

Triangle graphs other than those listed here vanish trivially. Note that the actual condition from the cancellation of the \( Ahh \) and \( Zhh \) anomalies (Witten, 1985) is really the displayed relation minus the relation from \( AWW \) (or \( ZWW \)).

This exact cancellation takes place independently of the value of the Stueckelberg mass, and independently of the values of \( g \) and \( g' \) or, more accurately, independently of the value of the “massive” weak mixing angle \( \theta_w \).

All these anomalies cancel family by family. The cancellation of anomalies requires, of course, the color factor 3 for quarks.

D. Currents

Recapitulating, the classical lagrangian for the Stueckelberg–modified standard electroweak theory is

\[ L_0 = L_g + L_s + L_\ell + L_q \]

where the gauge, scalar, lepton and quark lagrangians are given by eqs. (100), (103), (146) and (151), respectively.

1. Classical currents

Let us define the classical currents as

\[ j_\mu = \frac{\delta L_0}{\delta \nabla_\mu} \]

\[ \bar{j}_\mu = \frac{\delta L_0}{\delta \bar{W}_\mu} \]

The equations of motion for \( V_\mu \) and \( \bar{W}_\mu \) then read as follows:

\[ \partial_\mu F^{\mu\nu} = -j^\nu \]

\[ \partial_\mu \bar{F}^{\mu\nu} = -\bar{j}^\nu \]

and all four currents are conserved \((\partial_\mu j^\mu = \partial_\mu \bar{j}^\mu = 0)\).

Explicitly,

\[ j_\mu = m(mV_\mu - \partial_\mu B) + \frac{ig'}{2} (\Phi^D_\mu \phi - (D_\mu \Phi)^\dagger \Phi) + \sum g' Y_f \bar{\psi}_\mu \psi \]

\[ \bar{j}_\mu = g F^{\mu\nu} \times \bar{W}_\nu \]
It turns out that $\partial j = \partial \vec{J} = 0$ and $s_j = 0$, whereas $s\vec{J} \neq 0$.

Of course, it is not surprising that the conserved $SU(2)$ current $\vec{J}$ is not BRST–invariant (neither is the covariantly conserved current): it forms an $SU(2)$ triplet! Under the restricted BRST transformations with $\omega^3 = \omega^2 = 0$, the third component $J_3$ of the $SU(2)$ current is invariant.

After spontaneous symmetry breaking, the current coupled to the physical massive photon is

$$J^\mu_A = c_w j^\mu + s_w J_3^\mu \tag{187}$$

We are interested in linear combinations of $j$ and $J_3$. Note that any linear combination of $j_\mu$ and $J_3^\mu$ is conserved, in particular the Noether current associated with global $SU(2) \times U(1)$ transformations leaving the vacuum invariant

$$J^\mu_{em} = \frac{1}{\sqrt{g^2 + g'^2}} (g_j^\mu + g' J_3^\mu) \tag{188}$$

Somewhat more explicitly, the fermionic parts of these currents are

$$J_{em}^{(f)\mu} = e \sum_{U,D,R,Q,L} \bar{\psi} Q_\psi \gamma^\mu \psi \tag{189}$$
$$J^{(f)\mu}_A = \sum_{U,D,R,Q,L} \bar{\psi} (g' c_w Y_\psi / 2 + g s_w T_3^3) \gamma^\mu \psi \tag{190}$$

with $e$ given by eq. 103, $Q_\psi = Y_\psi / 2 + T_3^3$, and $T_3^3 = T_3^L = \tau^3 / 2$, $T_3^R = T_3^D = T_3^3 = 0$. Since $\tan \theta_w = (1 + c) g' / g + O(\epsilon^2)$, the two currents differ for a non-zero $e = 4 m_j (f^2 (g^2 + g'^2))$.

The gauge and scalar pieces of these two currents are related just like the above. Indeed, from the expressions

$$J^{(g)\mu}_A = m c_w (m V^\mu - \partial^\mu B) + ig s_w (F^+_{\mu \nu} W^{\nu -} - F^-_{\mu \nu} W^{\nu +}) \tag{191}$$
$$J^{(s)\mu}_A = c_w g' + s_w g \frac{1}{2} (g' V^\mu + g W_{\mu}^3) \phi_+ \phi_+ - \frac{c_w g g' \phi_3}{2} (W^\mu_{\mu} \phi_+ + W_{\mu}^\mu \phi_-)$$
$$+ \frac{c_w g' - s_w g}{2} \left[ \frac{1}{2} (\phi_3^2 + (H + f)^2) (g' V^\mu - g W_{\mu}^3) + H \partial^\mu \phi_3 - \phi_3 \partial^\mu H + f \partial^\mu \phi_3 \right] \tag{192}$$

it suffices to replace the weak mixing angle by its traditional or Stueckelberg–free value, $c_w \rightarrow g' / \sqrt{g^2 + g'^2}$ and $s_w \rightarrow g' / \sqrt{g^2 + g'^2}$ to find the expressions for $J^{(g)\mu}_A$ and $J^{(s)\mu}_A$. Note the enormous simplification in the scalar current, where the second line drops off, including in particular the term linear in $\partial^\mu \phi_3$.

The Noether current which is associated with the global transformation leaving invariant the vacuum expectation value of the scalar field is just $J^\mu_{em}$, and thus the $v.e.v.$ is invariant under the action of the electric charge $Q = \int d^3 x J^\mu_{em} = Y / 2 + T_3$, but not under that of the charge associated with $J_A$.

2. Quantum currents

Adding the gauge fixing terms to the lagrangian 181, so that

$$\mathcal{L} = \mathcal{L}_w + \mathcal{L}_{gh} + \mathcal{L}_{gf} \tag{193}$$

where the last two terms are given by eqs. 121 and 122, the field equations for $V_\mu$ and $W_{3\mu}$ are now

$$\partial_\mu F^{\mu \nu} + \frac{1}{\alpha} \partial^\nu \mathcal{G} = j^\nu \tag{194}$$
$$\partial_\mu F_{3\mu}^{\nu} + \frac{1}{\alpha} \partial^\nu \mathcal{G}_{3\mu} - ig \partial^\nu (\omega^+ \omega^- - \omega^- \omega^+) = J_3^\nu + ig (\omega^+ \partial^\nu \omega^- - \omega^- \partial^\nu \omega^+) \tag{195}$$
with the usual
\[ G = \left( \partial V + \alpha' mB - \frac{\alpha' g f}{2} \phi_3 \right) \] (196)
\[ \bar{G} = \left( \partial W_3 + \frac{\alpha g f}{2} \phi_3 \right) \] (197)

Since physical states satisfy the supplementary conditions
\[ \langle \text{phys} | G | \text{phys} \rangle = 0 \] (198)
\[ \langle \text{phys} | \bar{G} | \text{phys} \rangle = 0 \] (199)

and they do not contain any ghosts, it is clear that the divergence of the currents \( j^\mu \) and \( \bar{J}^\mu \) are zero sandwiched between physical states:
\[ \langle \text{phys} | \partial j | \text{phys} \rangle = 0 \] (200)
\[ \langle \text{phys} | \partial \bar{J} | \text{phys} \rangle = 0 \] (201)

Of course, this result is fully expected. Since the gauge–fixing lagrangian is \( s \) of something, the difference between the quantum and classical currents is again the \( s \) of something, which cannot make any difference for physical states.

3. Conclusions

From the above analysis, it would seem that the current associated with external classical fields (the background fields?) is \( J^{\mu}_{em} \), whereas the current coupled to the quantum asymptotic photon field is \( J^{\mu}_{A} \). The latter can be measured in scattering processes like Bhabha, Compton or bremsstrahlung. The former’s physical relevance stems from the fact that it is the Noether current of the global \( U(1)_{em} \) symmetry leaving invariant the vacuum.

E. Some phenomenology

Before proceeding to an overview of some of the phenomenological issues associated with our modification of the standard model, it is perhaps wise to recall the stringent experimental limits on the mass of the photon. The analysis of experimental data is based on the Maxwell–Proca equations
\[ \partial_{\mu} F^{\mu\nu}_V = j^\nu + m_\gamma^2 V^\nu \] (202)
\[ \partial_{\mu} \tilde{F}^{\mu\nu}_V = 0 \] (203)

where in addition to the Lorentz gauge following from the field equations one has imposed the Proca gauge \( B = 0 \). These equations are not gauge-invariant, but they are the gauge-fixed version of gauge-invariant field equations, as we have discussed in section II. Since gauge-fixing does not change the physics, they are perfectly valid starting points for experimentalists. Interestingly, in an experiment of size \( L \), photon mass effects scale like \( (m_\gamma L)^2 \), without any resonance effects in the photon frequency (Goldhaber and Nieto, 1971). Measurements of the energy density with a Cavendish torsion experiment lead to the strong limit (Luo et al., 2003)
\[ m_\gamma < 1.2 \times 10^{-17} \text{ eV} \] (204)

Direct measurements of the speed of light are five orders of magnitude worse (Schaefer, 1999). Other limits and methods, as well to references to the early literature, notably (Schrödinger, 1949), can be found in (Hagiwara et al., 2002; Lakes, 1998).

It is important to keep in mind that the direct limit on the photon’s mass is very strong, so that the modifications to the standard model stemming from the consistent application of the Stueckelberg mechanism to the hypercharge Abelian factor, in particular the modified weak mixing angle and fermion couplings to the photon and the \( Z \), are not expected to be competitive.
It is useful to view the introduction of the BRST–consistent mass for the photon in the standard model as a tiny modification of the latter. Charged currents do not change, whereas to lowest order in the Stueckelberg mass parameter, the weak neutral currents remain essentially undisturbed: the photon acquires a mass and changes its couplings without affecting much the rest of the theory. This is, of course, fortunate, since the experimental success of the standard model constitutes the culmination of the quantum understanding of nature.

There are no flavor–changing neutral currents in the theory. A major problem arises in the computation of charges for bound states. Consider for example the neutron, or rather the baryon with valence quarks $uud$. Depending on the chirality of the three quarks, we find different charges or, more precisely, couplings to the photon. The results can be summarized in terms of the neutrino’s coupling to the photon,

$$Q'_\nu = n'_{\nu} = \frac{1}{2}(g_{sw} - g'c_w) \simeq \frac{e}{2}\epsilon$$

with $\epsilon$ defined in eq. (103). In the following table, we denote by $Q(B)$ the coupling of the bound state $B$ (labelled by its valence quarks) to the physical asymptotic massive photon.

| $Q(u_Ld_Ld_L)$ | $Q(u_Ld_Ld_R)$ | $Q(u_Rd_Ld_L)$ | $Q(u_Rd_Ld_R)$ |
|----------------|----------------|----------------|----------------|
| $-Q'_\nu$      | $0$            | $2Q'_\nu$      | $0$            |
| $Q(ulld_Ld_L)$ | $Q(uld_Ld_R)$  | $Q(urd_Ld_R)$  | $Q(urd_Rd_R)$  |
| $-Q'_\nu$      | $2Q'_\nu$      | $Q'_\nu$       | $0$            |

That the charge of $\Delta^0$ is not exactly zero is of no particular experimental relevance, but one should have serious problems accepting the fact that a neutron’s charge depends on its spin. It is comforting that $(207)$, one of the true neutron states, is neutral, but disquieting that the coupling of $(210)$ to the photon does not vanish.

The $uud$ bound state (proton) has similarly three different couplings to the photon depending on the handedness of the valence quarks, with $u_Lu_Ld_L$ and $u_Lu_Rd_R$ degenerate. The fact that left- and right-handed electrons have different couplings to the photon has the same origin as the difference in couplings to the photon for the various bound states of three valence quarks. This situation is very problematic not only conceptually, but also for the stability of matter. Indeed, it is very hard to escape catastrophic and observable consequences (for examples, electric fields near grounded metallic conductors) if matter is not neutral. The total charge of the hydrogen atoms can be read off from the following table, showing the total coupling to the photon of the bound states of a $uud$ baryon and an electron.

$$p_L = [u_L(1)u_R(2) - u_R(1)u_L(2)]d_L(3)$$

its photon charge $g'c_w$ is equal and opposite to the charge of the right-handed electron $e_R$. Similarly, if

$$p_R = [u_L(1)u_R(2) - u_R(1)u_L(2)]d_R(3)$$

its charge $(g'c_w + gsw)/2$ is compensated by the $e_L$ charge. For the neutron, only the photonic charge of $n_L$ vanishes. The neutrality of normal matter is thus assured. This calculation is too naive, however, in the absence of a realistic three–quark model.
VI. THE INFLUENCE OF STUECKELBERG’S 1938 PAPERS

Stueckelberg’s 1938 papers, rather difficult to read then and now, have been continuously cited from 1941 to the present. The following domains of influence will be reviewed: renormalization of massive vector field interactions, hidden symmetry, and electroweak theory without spontaneous symmetry breaking.

Other topics worthy of attention which have developed from Stueckelberg’s 1938 papers include baryon number, as emphasized in [Wigner, 1967, footnote on p.25], broken chiral symmetry, and electromagnetic properties of vector mesons. We shall not review them here.

We must distinguish two aspects of Stueckelberg’s formalism for massive gauge vector fields (vector mesons in those days):

1) The decomposition of the massive Proca vector field $V_\mu$, as in equation (23) above, namely

$$ V_\mu = A_\mu - \frac{1}{m} \partial_\mu B $$

(220)

The $\partial_\mu B$ term is responsible for the singular character of Proca’s theory: the commutation relations of the massive vector $A_\mu$ and of the Stueckelberg field $B$ are local, but the presence of the derivative of $B$ makes the commutation relations of $V_\mu$ non-local.

2) The replacement of Proca’s free lagrangian by Stueckelberg’s:

$$ L_{Stueck}(A_\mu, B) = L_{Proca}(A_\mu, B) + L_{gf} = -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}m^2 A_\mu^2 + \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}m^2 B^2 $$

(221)

(We consider the neutral case (28), which follows from (31) with $\alpha = 1$; for charged vector fields the Proca and Stueckelberg lagrangians are given by the historical (9) and (17), respectively). Previously, it was believed that a massive vector theory could not be gauge invariant, but [Pauli, 1941] showed that $L_{Stueck}$ was a counter-example to such belief, still surprisingly common nowadays. We have seen in section III that the theory of real massive vector fields is even BRST invariant, which is relevant for its renormalizability ([Delbourgo et al., 1988]).

We now review the historical development of these ideas in three different but complementary directions: (A) renormalizability, (B) hidden symmetries, and (C) massive theories without Higgs. We also mention (D) some related applications of the Stueckelberg trick.

A. The question of renormalizability

1. Power–counting renormalizability

It was found in the 1930s that the quantum field theory of electrons and photons (quantum electrodynamics, QED) was plagued by infinities, already at low orders in perturbation theory.

In 1949, Dyson showed that, in QED, renormalization of mass and charge of the electron and renormalization of the wave-functions (or better, the rescaling of the field operators) removed all the divergences from the $S$–matrix to all orders in perturbation theory ([Dyson, 1949]). This is now known as the power–counting procedure, because it is based on counting the powers of four–momenta over which one integrates. Dyson’s proof was later made more rigorous by ([Weinberg, 1960], [Bogoliubov and Shirkov, 1976] and others.

After Dyson, it was natural to ask whether massive vector field interactions were also renormalizable. Vector mesons were first considered, following Yukawa, as mediators of strong nuclear interactions, with little phenomenological success.

[Miyamoto, 1948] was the first to use Stueckelberg’s lagrangian (221) extensively in the theory of charged massive vector mesons interacting with nucleons, mimicking the treatment of QED in the super–many–time formalism ([Tomonaga, 1946]), which has the advantage of being manifestly Lorentz invariant. Tomonaga and collaborators applied this formalism to the interaction of electrons with photons ([Koba et al., 1947]) and of mesons with photons ([Kanesawa and Tomonaga, 1948]). In the latter case, an additional interaction term was necessary to satisfy relativistic invariance. Miyamoto showed that, for the interaction of mesons and nucleons, the additional term is provided automatically in the Stueckelberg formalism by the scalar $B$–field.

Miyamoto then derived the generalized Schrödinger equation, the integrability conditions, Stueckelberg’s auxiliary condition, and the passage to the Heisenberg picture. Proca’s theory is not well adapted to the many-time formalism, and it has problems with the integrability conditions. Miyamoto’s careful and extensive work paved the way for further research.
In a different vein, using the many–time formalism of (Dirac et al., 1932), (Podolsky and Schwed, 1948) considered a non-renormalizable modification of QED with higher derivatives of the massless $A_\mu$ field. They claimed to get a finite self-energy for a point source. In the process of trying to quantize the theory, they introduced the Stueckelberg field $B$ in order to get a consistent subsidiary condition, similar to Stueckelberg’s (20).

2. 1949–1954 : lessons from QED

The first definite answer to the question of renormalizability of vector interactions with the nucleons was provided by (Matthews, 1949a,b). This problem depends crucially on the high–energy behavior of the $S$–matrix, which depends in turn on the power of the energy–momentum factors. Four–momenta are the Fourier transform of derivatives, and they may appear in the commutation relations of the quantized fields, as well as in the interaction lagrangian.

This point can be illustrated by comparing Proca’s commutation relations for real massive vector fields,

\[ [V_\mu(x), V_\nu(y)] = -i \left( g_{\mu\nu} + \frac{1}{m^2} \partial_\mu \partial_\nu \right) \Delta_m(x - y) \]  

(222)

with Stueckelberg’s:

\[ [A_\mu(x), A_\nu(y)] = -i g_{\mu\nu} \Delta_m(x - y) \]  

(223)

Similarly, (6) can be compared with (12) for charged (i.e. non-hermitian) vector fields. Proca’s theory is clearly more divergent at high energies than Stueckelberg’s. On the other hand, the interaction of Proca’s massive vector field with a charged fermion field $\psi$ is the harmless

\[ \mathcal{L}^I_{\text{Proca}} = e \bar{\psi} \gamma^\mu V_\mu \psi \]  

(224)

whereas Stueckelberg’s is

\[ \mathcal{L}^I_{\text{Stueck}} = e \bar{\psi} \gamma^\mu \psi \left( A_\mu - \frac{1}{m} \partial_\mu B \right) \]  

(225)

The last vertex diverges like $p_\mu$, so it seems that Stueckelberg’s interacting theory is also singular.

According to Matthews, however, the bad terms with $\partial_\mu B$ can be eliminated from the interaction by a unitary transformation, as follows. Working in the Dirac or interaction picture, Matthews (quoting Miyamoto) writes, instead of (225), the interaction term

\[ \mathcal{L}^I_{\text{Stueck}} = j^\mu \left( A_\mu - \frac{1}{m} \partial_\mu B \right) + \frac{1}{2m^2} (j^n n_\mu)^2 \]  

(226)

where $j^\mu = e \bar{\psi} \gamma^\mu \psi$ and $n_\mu$ is a normal unit vector to a general space-like surface. The point is that the last term, quadratic in $j^n$, is absolutely necessary for what they called “integrability” in those days; note that it does not look renormalizable. The physical states are defined using Stueckelberg’s subsidiary condition (16),

\[ (\partial^\mu A_\mu + m B)^{-1})\vert_{\text{phys}} > = 0 \]  

(227)

Now Matthews performs the unitary transformation (Case, 1949; Dyson, 1948)

\[ \vert_{\text{phys}} \rightarrow \vert_{\text{phys}}' = e^{-igG} \vert_{\text{phys}} \]  

(228)

\[ G = \frac{1}{m} \int d\sigma^\mu j_\mu(x) B(x) \]  

(229)

where the integral is over the space–like reference surface. Note the similarity to the gauge transformation (33–35). This redefinition eliminates the last two terms in (226) and thus we end up with an interaction lagrangian exactly like that of the massless photon interacting with the electron current in QED, which is renormalizable (Dyson, 1948). For charged (non hermitian) vector fields, an additional term spoils the renormalizability (Case, 1949). See also (Belinfante, 1949a,b) and (Gupta, 1951).

(Phillips, 1954) worked in the same framework as Matthews (Stueckelberg lagrangian, with the unitary transformation (228)), but criticized a technicality concerning the integrability conditions. Introducing a “quasi interaction...
representation”, in which the massive field $A_\mu$ obeys free field equations but the “free” equations for the $B$ and $\psi$ fields include the term $j_\mu(x)\partial^\mu B(x)$, these can be eliminated just as proposed by Matthews.

The review [Matthews and Salam, 1951] established to which meson interactions Dyson’s proof of finiteness of QED could be applied. The result is that the only interaction of vectors or pseudovectors with fermions satisfying Dyson’s criteria is the vector interaction of a neutral vector [Matthews, 1949]. On the other hand, the scalar interactions of scalars and the pseudoscalar interactions of pseudoscalars require only a finite number of renormalizations, as in the case of QED. In addition to the counterterms analogous to those occurring in QED, one needs a quartic (pseudo)scalar term and, in the case of scalar mesons, a further cubic term. See also the textbook presentation of (Umezawa, 1956) and the discussions of [Fujii, 1954].

3. 1960–1962: equivalence theorems

In the late 1950s and early 1960s, the interest in intermediate vector theories was revived by work on an isospin $SU(2)$ gauge–invariant theory of massless vector fields (Yang and Mills, 1954). For a history of gauge fields, see (O’Raifeartaigh and Straumann, 2000). Furthermore, (Feynman and Gell-Mann, 1958), (Sakurai, 1958) and (Sudarshan and Marshak, 1958) proposed the universal $V − A$ theory of weak nuclear interactions, which “can most beautifully be formulated by assuming an intermediate vector particle,” as stated by (Kamefuchi, 1960). Indeed, (Feynman and Gell-Mann, 1958), (Sudarshan and Marshak, 1958) suggested that the $V − A$ interaction could be mediated by a charged spin–one particle. (Bludman, 1958) proposed to add a neutral field in the framework of an $SU(2)$ Yang–Mills invariance, and (Glashow, 1961) added yet another neutral particle to achieve $SU(2) \times U(1)$ invariance. On the other hand, (Fujii, 1959) proposed a massive vector meson to mediate strong interactions, and (Sakurai, 1960) identified it with the $\rho$. (Lee et al., 1949) had already proposed the idea that the exchange of a boson of non-specified spin could explain the approximate equality of the $\beta$–decay and muon interaction couplings (the universality of weak interactions). (Schwinger, 1957) had “freely invented” an intermediate vector boson in strong and weak interactions (with some hints from experiment). It was tempting to identify the latter with the Yang–Mills field. But for empirical reasons, related to the Fermi theory of weak interactions, this particle ought to be massive. And yet, the theory of massive charged vector fields seemed not to be renormalizable. The main reason for this singularity seemed to be the lack of gauge invariance of such theories.

In this context, (Glashow, 1959) conjectured that a “partially conserved” vector current could lead to a renormalizable theory. This proposal was refuted independently by (Salam, 1960) and (Kamefuchi, 1960). Both used the decomposition $V_\mu = A_\mu - m^{-1}\partial_\mu B$, with $m$ the mass of the vector meson, and found a general equivalence theorem for vector meson interactions, from which they deduced “a precise criterion for renormalizability in the conventional sense” (Kamefuchi et al., 1961). They were inspired by (Dyson, 1948), who had already shown that the pseudovector interaction

$$g\bar{\psi}\gamma^\mu\gamma_5\partial_\mu B\psi$$

of a pseudoscalar field $\tilde{B}$ with a nucleon field $\psi$ was equivalent to an exponential pseudoscalar interaction of $\tilde{B}$ with $\psi$, using the unitary transformation

$$\psi \rightarrow \psi' = e^{ig\gamma_\mu\tilde{B}}\psi$$

which eliminated the pseudovector interaction and took the mass term $M\bar{\psi}\psi$ of the nucleon into

$$M\bar{\psi}'(1 - e^{-2ig\gamma_\mu\tilde{B}})\psi'$$

(Salam, 1960) applied the same procedure to a real pseudovector Proca field $\tilde{V}_\mu = A_\mu - \frac{1}{m}\partial_\mu \tilde{B}$. The pseudovector interaction of $\tilde{V}_\mu$ with fermions contains the pseudovector interaction of $\tilde{B}$. The elimination of the latter yields, again, a non-renormalizable exponential interaction. The current

$$j_\mu = g\bar{\psi}\gamma_\mu\gamma_5\psi$$

is “partially conserved:”

$$\partial^\mu j_\mu = 2igM\bar{\psi}\gamma_5\psi$$

5 In 1951 parity conservation was still unquestioned.
(Kamefuchi [1960] treated the general case of the interaction of the neutral vector field $V_\mu$ with an arbitrary complex field of spin $0$, $1/2$, or $1$, and assumed that the interaction Hamiltonian was of the form $H_1 + H_2$, with $H_1$ gauge invariant and $H_2$ not gauge invariant. Writing $V_\mu = A_\mu - \frac{1}{m} \partial_\mu B$ and applying the analogue of (231), the $B$ field is successfully eliminated from $H_1$, but reappears in an exponential in $H_2$.

Both Salam’s and Kamefuchi’s examples contradict Glashow. They had only considered, nevertheless, the case of Abelian $U(1)$ gauge invariance.

Shortly thereafter, (Umezawa and Kamefuchi [1961] generalized the equivalence theorem of (Salam, 1960) and (Kamefuchi, 1960) to the non-Abelian isospin $SU(2)$ Yang–Mills gauge theory, with massive vector mesons. They found that the mass terms spoil renormalizability.

To prove the equivalence theorems, they used Stueckelberg’s lagrangian, which they introduced in a new and elegant way. Then they extended it to the Yang–Mills case.

Their starting point is the Proca lagrangian for a real vector field $V_\mu$, with interactions. Introduce in addition to the real scalar Stueckelberg field $B(x)$ an extra real scalar field $C(x)$, with the wrong energy and the wrong metric in Hilbert space:

$$L_{UK} = -\frac{1}{4} F_{\mu\nu}(V)^2 + \frac{1}{2} m^2 V_\mu^2 + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 B^2 - \frac{1}{2} (\partial_\mu C)^2 + \frac{1}{2} m^2 C^2 + L_{\text{int}}(U_\mu, \phi)$$

(235)

The non-vanishing commutation relations are

$$[V_\mu(x), V_\nu(y)] = -i \left( g_{\mu\nu} + \frac{1}{m^2} \partial_\mu \partial_\nu \right) \Delta_m(x - y)$$
$$[B(x), B(y)] = i \Delta_m(x - y)$$
$$[C(x), C(y)] = -i \Delta_m(x - y)$$

(236)

and the interaction lagrangian depends on some other “matter” fields $\phi(x)$ and on the vector field

$$U_\mu(x) = V_\mu(x) + \frac{1}{m} \partial_\mu (B(x) - C(x))$$

(237)

It follows from (236) and (237) that $E = B - C$ is a free massive field:

$$(\partial^2 + m^2) E(x) = 0$$

(238)

To ensure positivity of the physical Hilbert space, one can impose the subsidiary condition

$$E^{(-)}|_{\text{phys}} >= 0$$

(239)

for physical states in the Heisenberg representation, since it is consistent with the field equations. Define now $A_\mu$ by

$$V_\mu(x) = A_\mu(x) - \frac{1}{m} \partial_\mu C(x)$$

(240)

Then (237) implies that the interacting vector field is

$$U_\mu(x) = A_\mu(x) + \frac{1}{m} \partial_\mu B(x)$$

(241)

Substituting (240) into (235), we find, after integrating by parts and dropping total derivatives,

$$L'_{UK} = -\frac{1}{2} (\partial_\mu A_\nu)^2 + \frac{1}{2} m^2 A_\mu^2 + \frac{1}{2} (\partial_\mu B)^2 - \frac{1}{2} m^2 B^2 - \frac{1}{2} D^2 + L_{\text{int}}(U_\mu, \phi)$$

(242)

where the auxiliary field

$$D(x) = \partial_\mu A_\mu(x) + mC(x)$$

(243)

satisfies the algebraic equation of motion $D = 0$. Using this fact, the subsidiary condition (239) reads now

$$(B + \frac{1}{m} \partial^\mu A_\mu)^{(-)}|_{\text{phys}} >= 0$$

(244)
so that both Stueckelberg’s lagrangian and Stueckelberg’s subsidiary condition are recovered.

Umezawa and Kamefuchi proceeded to extend the Stueckelberg lagrangian to isovector fields. They decomposed the lagrangian into a free piece:

\[
L_0 = -\frac{1}{2}(\partial_\mu \vec{A}_\nu)^2 + \frac{1}{2}m^2 \vec{A}_\mu^2 + \frac{1}{2}(\partial_\mu \vec{B})^2 - \frac{1}{2}m^2 \vec{B}^2
\]

(245)

and an interaction:

\[
L_I = \frac{1}{2}(m^2 - m'^2)(V^3_\mu)^2 - \frac{g}{2}\vec{A}_{\mu\nu} \cdot \vec{V}_\mu \times \vec{V}_\nu - \frac{g^2}{4}(\vec{V}_\mu \times \vec{V}_\nu)^2
\]

(246)

where

\[
\vec{V}_\mu = \vec{A}_\mu - \frac{1}{m}\partial_\mu \vec{B}
\]

(247)

and the short-hand

\[
\vec{A}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu
\]

(248)

is not the non-Abelian field strength of \(\vec{A}_\mu\).

Notice the astute first term in the interaction, with \(m'\) a free parameter. For \(m = m'\), the full isospin symmetry is restored, and the isovector current

\[
\vec{j}_\nu = \partial_\mu \vec{A}_{\mu\nu} - m^2 \vec{V}_\nu
\]

(249)

is conserved, \(\partial_\nu \vec{j}_\mu = 0\).

The derivation of the equivalence theorems for the isovector lagrangians, for \(m = m' \neq 0\) or for \(m \neq m'\), is rather lengthy. It turns out that, if \(m \neq 0\), the theory is not renormalizable, even when the current (249) is conserved.

In the classic paper [Salam, 1962] entitled *Renormalizability of Gauge Theories*, Salam gave a simpler and more general discussion of the renormalizability condition, which we now summarize. Salam’s paper is also remarkably modern in its notation and outlook. Consider a set of spinor fields \(\psi\) on which acts a Lie group with generators \(T_i\):

\[
\psi(x) \rightarrow \psi'(x) = e^{igT_i b(x)}\psi(x) \equiv U(x)\psi(x)
\]

(250)

with structure constants defined by

\[
[T_i, T_j] = if_{ijk} T_k
\]

(251)

and a set of vector fields

\[
V_\mu(x) = T_i V^i_\mu(x)
\]

(252)

transforming inhomogeneously:

\[
V_\mu(x) \rightarrow V'_\mu(x) = U^{-1}\partial_\mu U(x) + \frac{ig}{4} U^{-1}(x) \partial_\mu U(x)
\]

(253)

The following lagrangian is invariant under the gauge transformation \(U(x)\):

\[
L_S(\psi, V_\mu) = i\bar{\psi}(\slashed{\partial} - ig\slashed{V}) \psi + M\bar{\psi}\psi - \frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu}
\]

(254)

with

\[
F_{\mu\nu} = (\partial_\mu - igV_\mu)V_\nu - (\partial_\nu - igV_\nu)V_\mu
\]

(255)

the covariant field strength, which transforms homogeneously:

\[
F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = U^{-1}(x)F_{\mu\nu}(x)U(x)
\]

(256)

Add now a vector mass term, which is not invariant under the gauge transformation (253),

\[
L_{\text{mass}} = -\frac{1}{2}m^2 \text{tr} V^2_\mu
\]

(257)
To study the renormalizability of this theory, Salam proposed two steps. First, introduce the Stueckelberg fields $B^i$ through $V'_\mu = A^\mu - \frac{1}{m} \partial_\mu B$, with $A^\mu = T_i A^\mu_i$ and $B = T_1 B^1$. Secondly, change $\psi$ to $\psi'$ and $V'_\mu$ to $V'^{in}_\mu$, using the gauge transformations $\chi_i$ and $\phi_i$, with the gauge parameters chosen as $b^i = B^i$. Under this transformation, $L_S(\psi, V'_\mu)$ is invariant, but $L_{mass}$ is not.

On the other hand, it follows from (258) that $V'^{in}_\mu = A^{in}_\mu$, and hence, in the weak coupling limit $g \rightarrow 0$ where asymptotic states are defined, one obtains

$$V'^{in}_\mu = A^{in}_\mu$$

The $S$–matrix in the new variables has contributions from two pieces: those from $L_S(A^{in}_\mu, \psi^{in})$ yield only renormalizable infinities (the derivative couplings of $B$ with $\psi$ have been eliminated), whereas those from $L_{mass}(A^{in}_\mu, B^{in})$ produce exponential infinities unless either

$$m = 0$$

or the following two conditions hold:

$$\text{tr} \left[ \frac{m^2}{g^2} (\partial_\mu U) (\partial^\mu U)^{-1} - (\partial_\mu B^{in})^2 \right] = 0 \quad (260)$$

and

$$\text{tr} \left[ A^{in}_\mu \left( U^{-1} \partial^\mu U - i \frac{g}{m} \partial^\mu B^{in} \right) \right] = 0 \quad (261)$$

This is a powerful theorem. For a massive neutral vector field interacting with fermions, for example with the nucleons, there is only one $B$ field, and $U$ is Abelian. Then both (260) and (261) are satisfied, and the theory is renormalizable even with a massive vector.\footnote{This result can be understood easily in terms of the BRST invariance of section III.}

In general, (260) and (261) can be satisfied provided $\text{tr} T_i T_j = 0$. However, for simple Lie groups $\text{tr} T_i T_j = \lambda \delta_{ij}$, with the normalization $\lambda \neq 0$, and thus the last two conditions are not satisfied.

The only term in the lagrangian considered which is not gauge invariant is the vector mass term. Clearly, any other non-invariant term in the lagrangian, of the generic form $L(V)$, will transform into $L(S \psi')$, with $S$ containing non-renormalizable exponentials of Stueckelberg’s $B$ field. This checks, for instance, with the fermion mass term $M \psi \psi'$ under the transformation (231) above, which yielded the horrible (232).

Salam concluded that renormalizability of a gauge theory requires vanishing masses. Salam then developed the ideas of (Nambu and Jona-Lasinio, 1961) to get masses in a self–consistent way, and later co-birthed the concept of broken symmetry (Goldstone et al., 1962), which eventually gave mass to the vectors of a broken gauge invariance (Englert and Brout, 1964; Higgs, 1964). The first discussion of a non-Abelian spontaneous symmetry breakdown in (Kibble, 1967), whereas (t Hooft, 1971a,b) provided the proof of renormalizability.

Earlier, (Komar and Salam, 1969) had calculated explicitly to lowest order the self–energy and vertex correction of the isospin $SU(2)$ Yang–Mills theory, in agreement with the above result (Salam, 1962). However, to prove non-renormalizability one must consider not the Green’s functions in general, but the Green’s functions on-shell, i.e. the $S$-matrix elements (Veltman, 1968; Zimmermann, 1968) used Stueckelberg’s lagrangian and its invariance under the Pauli gauge transformations to study the renormalization of masses of real vector fields. The renormalization of massive chiral $U(1)$ theory was carried out by (Lee, 1966). For a generalization of the Stueckelberg formalism see (Fujii and Kamefuchi, 1964). A later discussion of these subjects can be found in (Ito, 1976).

**B. Hidden symmetries**

As mentioned above, it was widely assumed that giving a mass to the photon would spoil gauge invariance. But the Stueckelberg lagrangian with a physical scalar field $B$ in addition to the massive photon (Stueckelberg, 1938) enjoys indeed gauge invariance (Pauli, 1941) and even BRST invariance (Delbourgo et al., 1988). The Proca lagrangian does not have this symmetry, so it is absolutely necessary to include the Stueckelberg field $B$ which does not play, however, a dynamical role. One may call this state of affairs a “hidden symmetry.” As we shall see below, the same trick has been used by several authors in more general contexts.
Glauber, who visited Pauli at Zürich in 1950, was the first to emphasize the close relationship between Stueckelberg’s gauge–invariant scheme and QED (Glauber, 1953). We also owe him the remark that the (Dyson, 1948) transformation is just a gauge transformation, which eliminates the $B$–field from the interaction. For vanishing photon mass $m$, the Stueckelberg field $B$ disappears as well from the supplementary condition, which is then the same as in QED, and eliminates the longitudinal polarization of the vector field $A_\mu$. For non-vanishing mass $m \neq 0$, this condition can be considered as a definition of the $B$–field, while the massive photon is no longer restricted to be transverse.

Glauber then proceeds to calculate radiative corrections to the photon mass. These can be separated in two parts, according to whether they are gauge–invariant or not. Both pieces are formally divergent. As in electrodynamics, the non-gauge–invariant integrals must be presumed to vanish (to second order, the contribution is identical to the one of QED). To renormalize the theory, one must remember that the $B$–field is still present in the free–field hamiltonian. To preserve gauge invariance of the corrections, one has to reintroduce $B$ through the supplementary condition in the last step. The photon mass correction is logarithmically divergent, and vanishes when $m$ goes to zero.

Aware of Glauber’s preprint, Umezawa (1952) generalized these results to tensor representations, studied the transition when the photon’s mass vanishes (see also Umezawa, 1956, pp. 113 and 204), and gave a classification of the renormalizable and non-renormalizable interactions of neutral and charged particles of spin 0, 1/2 and 1. Bonometto (1963) considers the gauge invariance of massive vector theories in a five–dimensional formalism and shows the connection between Stueckelberg’s formalism and that of Ogievetskii and Polubarinov (1961). The latter start from the lagrangian for the real field $A_\mu$ interacting with a conserved (Dirac) current

$$\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu - m^2 A_\mu^2 + j_\mu A^\mu$$

This is invariant under the transformation $\delta A_\mu = \partial_\mu \Lambda(x)$ subject to $(\partial^2 - m^2)\Lambda = 0$. The $A_\mu$ field can be split into an invariant spin–1 part, and a non-invariant spin–zero part. They show that the scalar has no interactions, so one can forget it. Furthermore, the total energy operator is positive definite up to an irrelevant constant. Hence, the supplementary condition imposed by Stueckelberg to ensure positivity is no longer required.

Many other papers deal with the relation of massive to massless QED, focusing on a variety of questions, independently of Stueckelberg’s $B$ field. For example, Coester (1951) and Stueckelberg (1957) find that after a suitable canonical transformation, the contributions of the scalar and longitudinal components of the vector field to the $S$–matrix compensate each other in the limit $m \to 0$. Schwinger (1962a,b) exhibits a gauge invariant massive field theory which has no continuous limit to QED when $m \to 0$. Bouwhare and Gilbert (1962) invent a soluble field theory in which they then carry out the limit as the bare mass vanishes, but in which the vector particle remains massive. This toy model is gauge invariant, since a Stueckelberg massless scalar field is also introduced. Feldman and Matthews (1963) also show that “gauge invariance does not require the bare photon mass to be zero”. Kanefuchi and Umezawa (1964) use the original formalism of Stueckelberg (1938a,b) to show that the representation of gauge transformations for massive vector fields is inequivalent to that for zero mass. As discussed below, the limit $m \to 0$ was also studied by van Dam and Veltman (1970) and Slavnov and Faddeev (1970).

Ramond (1986) applied Stueckelberg’s scheme to a completely new domain, in order “to obtain the fully covariant and gauge invariant field theory for free open bosonic strings in [the critical] 26 dimensions. [This] approach […] is based on very simple analogies with local field theory. […] Stueckelberg fields arise naturally and are shown to be unrestricted for the most general gauge transformations.”

Ramond remarks that “in any theory which is known in a specific gauge, one can always reconstruct the original gauge invariant theory provided one knows the form of the gauge transformations and the gauge conditions.” This was precisely the situation for the first quantized string, where the gauge symmetry is given by (half of) the Virasoro algebra.

In massless QED, the gauge transformation is of course $\delta A_\mu(x) = \partial_\mu \Lambda(x)$, and the gauge condition is $\partial^\mu A_\mu = 0$. From this, and the equation of motion $\partial^2 A_\mu = 0$, one can deduce the Lorentz invariant and gauge invariant equation

$$\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial^\mu F_{\mu\nu} = 0.$$

In the Proca theory, one would start with

$$-(\partial^2 + m^2)A_\mu(x) = 0$$

and

$$\partial^\mu A_\mu = 0$$

One could try the gauge transformation

$$\delta A_\mu = \partial_\mu \Lambda(x)$$
The variation of (264) gives
\[ \partial^{\mu} A_{\mu} + \partial^{\mu} \partial_{\mu} \Lambda = 0 \] (266)
However, the variation of (263) implies
\[ (\partial^2 + m^2) A_{\mu} + \partial_{\mu} (\partial^2 + m^2) \Lambda = 0 \] (267)
which is not compatible with (266). Now, Ramond rewrites (266) as
\[ \partial^{\mu} A_{\mu} + (\partial^2 + m^2) \Lambda - m^2 \Lambda = 0 \] (268)
and interprets the last term as the variation of the Stueckelberg scalar field \(-mB\). Equation (268) then becomes
\[ \partial^{\mu} A_{\mu} + mB + (\partial^2 + m^2) \Lambda = 0 \] (269)
On the mass-shell, \((\partial^2 + m^2) \Lambda = 0\), hence the supplementary condition is now
\[ \langle \text{phys} | \partial^{\mu} A_{\mu} + mB | \text{phys} \rangle = 0 \] (270)
which is gauge invariant provided one completes (265) with
\[ \delta B = m \Lambda \] (271)
Substituting (269) into (267), one gets the covariant equation of motion
\[ \partial^{\mu} F_{\mu \nu} + m^2 A_{\nu} - m \partial_{\nu} B = 0 \] (272)
"which is the Stueckelberg equation for a massive vector field."
Ramond proceeds "to apply these tricks" to the string equation of motion
\[ (L_0 - 1) \Phi = 0 \] (273)
with gauge conditions
\[ L_n \Phi = 0 \quad (n \geq 1) \] (274)
and gauge transformation
\[ \delta \Phi = \sum_{n \geq 1} L_{-n} \Lambda^{(n)} \] (275)
The Virasoro operators \(L_n\) satisfy the algebra
\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{D}{12} n(n^2 - 1) \delta_{n,-m} \] (276)
where \(D = 26\) is the number of spacetime dimensions. The Stueckelberg fields \(\Phi_{n}^{(p)}\) are then introduced, with variations
\[ \delta \Phi_{n}^{(p)} = -L_n \Lambda^{(p)} + (2n + p) \Lambda^{(n+p)} \] (277)
and equations of motion
\[ L_0 \Phi_{n}^{(p)} = -L_0 \left( L_n \Lambda^{(p)} + (2n + p) \Lambda^{(n+p)} \right) \] (278)
We leave to the reader the pleasure of exploring the rest of the paper, which concludes as follows: "It should be clear that the subsidiary (Stueckelberg) fields lead to much simpler looking expressions." For technical details, see [Pfeffer et al. 1986] and its superpartner [Kleppe et al. 1989].
The ten–dimensional "superstring" [Green and Schwarz 1984] is equivalent to the "fermionic string" [Neveu and Schwarz 1971; Ramond 1971]. Its covariant quantization turns out to be tricky, so people started by quantizing the ten–dimensional superparticle [Brink and Schwarz 1981; Casalbuoni 1976], which describes the dynamics of the zero–modes of the ten–dimensional superstring. As shown by [Bergshoeff and Kallosh 1990a,b], Stueckelberg symmetries appear also in this context.
The lagrangian of the classical superparticle in first order formalism is given by

$$L_{cl} = P_\mu \dot{X}^\mu - \Theta \Gamma^\mu P_\mu \dot{\Theta} - \frac{1}{2} g P^2$$  \hspace{1cm} (279)

Here, \((X^\mu, \Theta)\) are the classical coordinates of the superparticle, with \(X^\mu\) a vector of the ten-dimensional Lorentz group and \(\Theta\) a 16–component Majorana–Weyl spinor of positive chirality, \(P_\mu\) is the canonical momentum conjugate to \(X^\mu\), \(g\) is the einbein, and the dot denotes a time derivative. In order to quantize \((279)\), \(\text{Bergshoeff and Kallosh, 1990a}\) introduce an infinity of ghosts, antighosts, and Lagrange multipliers. They propose a new lagrangian which is BRST invariant. Besides, it is also invariant under Stueckelberg transformations.

To illustrate these symmetries, we write down the transformation of the coordinate \(X^\mu\):

$$\delta_{St} X^\mu = \sum_{p \geq 0} \theta_{p+1,0} \Gamma^\mu \epsilon^{p,0}$$  \hspace{1cm} (280)

where \(\theta_{p,0}\) are some of the ghosts, and \(\epsilon^{p,0}\) are local parameters with have the same commutation properties and reality and chirality conditions as the antighosts \(\bar{\theta}^{0,0}\).

The Stueckelberg symmetries show that, effectively, the antighosts \(\bar{\theta}^{0,0}\) do not occur in the new lagrangian. Hence, they can be eliminated by field redefinitions.

Commenting on a previous paper on the quantization of the superparticle, \(\text{Bergshoeff and Kallosh, 1990b}\) showed that “the mysterious gauge symmetry found by \(\text{Fisch and Henneaux, 1989}\) is a Stueckelberg symmetry \((280)\).”

The final result of \(\text{Bergshoeff and Kallosh, 1990a}\) is a free quadratic lagrangian, BRST invariant without any constraints. The Noether BRST charge \(Q\) is nilpotent \((Q^2 = 0)\) off–shell.

### C. Massive gauge theories without Higgs

#### 1. Successes and problems of the standard theory

The standard theory of electroweak interactions \(\text{Glashow, 1961; Salam, 1968; Weinberg, 1967}\) has many virtues. It is a gauge theory with BRST invariance \(\text{Becchi et al., 1974, 1975; Tyutin, 1975}\) and its gauge group, \(SU(2) \times U(1)\), is spontaneously broken through the Higgs mechanism \(\text{Englert and Brout, 1964; Guralnik et al., 1964; Higgs, 1964; Kibble, 1967}\) to the \(U(1)\) invariance of quantum electrodynamics. The theory is unitary and renormalizable \(\text{Becchi et al., 1974; 1981; 't Hooft, 1974}\). The massive gauge vector bosons \(W^\pm\) and \(Z\) corresponding to the broken symmetries have been discovered \(\text{Arnison et al., 1983; Bagnaia et al., 1983; Banner et al., 1983}\) and have been abundantly produced at LEP and SLAC, and of course the \(U(1)\) gauge boson is the massless photon. The standard theory is well suited for perturbative computations, allowing detailed calculations of cross–sections and decay rates, in remarkable agreement with experiment; a good review is \(\text{Altarelli, 2000}\).

In spite of its extraordinary and complete success, the standard theory has some weaknesses, though what they are is somewhat a matter of taste. The remarkable agreement of all known data with the standard theory has prompted theoreticians to look for alternatives of it which preserve such valuable virtue and overcome its shortcomings.

To begin with, one should point out that the spin–zero Higgs particle has not yet been discovered, although indications exist of \(M_{Higgs} = 115\) GeV \(\text{Abbiendi et al., 2001; Abreu et al., 2001; Achatz et al., 2001; Heister et al., 2002}\). Even if this result is falsified, this is not worrisome, and the experimental results available to date can be reinterpreted as bounding \(M_{Higgs} > 114\) GeV. The LEP measurements are of such precision that they verify the radiative corrections of the standard theory, and thus bound the Higgs mass (through its logarithm) to around \(M_{Higgs} < 215\) GeV at 95% C.L. \(\text{Hagiwara et al., 2002}\), so not finding the Higgs at the Tevatron would not be catastrophic, in sharp contrast to what would happen if it were not found at the LHC. At any rate, the theory does not predict the Higgs mass, which is quite an independent parameter (subject to more or less educated bounds, less stringent than the experimental ones). Let us note, however, that if the Higgs were heavier than around 300 GeV, then it would be strongly coupled, so we could not calculate in perturbation theory \(\text{Casalbuoni et al., 1988; 1996; 1997}\) and the above bounds would have to be reinterpreted. It is important to stress that there is a logical difference between the Higgs mechanism and the existence of the Higgs particle: the latter provides an elegant and simple implementation of the former. In terms of parameters, the non-zero vacuum expectation value is independent of the (fundamental or effective) scalar field’s mass. Nevertheless, let us emphasize right away that all efforts to implement the Higgs mechanism of the standard model without a physical Higgs boson have failed so far.

There are other theoretical misgivings about the standard theory. Paramount is the hierarchy problem. What stabilizes the energy scale of electroweak symmetry breaking, \(m \sim O(10^2)\) GeV, with that of gravity, \(M \sim O(10^{18})\) GeV? Equivalently, the likely unification of the electroweak and strong couplings, should take place at a comparably remote energy \(\text{Georgi et al., 1974}\). Why should it be so different from the electroweak scale? Typically, the radiative
corrections to the $Z$ mass would be of the order of $\log M/m$, so in order to reach phenomenological agreement and keep $m << M$, one needs a careful and unnatural fine tuning of the mass parameters in the theory, order by order in perturbation theory. (The more dramatic quadratic divergence problem is an artifact of regularizations breaking gauge invariance.) The hierarchy problem is, thus, the wide difference between the Higgs vacuum expectation value and the superstring (or quantum gravity, or unification) scale. Supersymmetry stabilizes the hierarchy problem but does not solve it. It could well be that the hierarchy problem is related to the cosmological constant problem, the solution to which has not yet been found, even in the framework of string theory.

Another problem, called the infrared catastrophe, arises in quantum chromodynamics, the non-Abelian gauge theory of strong interactions. Its gauge group, $SU(3)$, is unbroken, and its massless vector bosons, the gluons, interact with colored particles, namely quarks and themselves. When the energy of the scattering process becomes small, the number of gluons emitted by a colored particle diverges. The same problem occurs in quantum electrodynamics, where the number of photons of low energy emitted by a charged particle diverges. In QED, a remedy to this situation has been found, since the photon is neutral. But in QCD, the gluons carry color and therefore interact with themselves. This makes the infrared problem untractable, and the divergences are laboriously removed only at the very end of the computation of suitable observables. (Weinberg, 1965), see however Slavnov (1981). On the other hand, a theory of massive vector bosons would not be plagued by these infrared divergences; the Higgs mechanism does not help since it would break asymptotic freedom. (t’Hooft, 1978; van Nieuwenhuizen, 1995; Ojima, 1982).

2. Alternative models

For the above reasons, among others, models have been proposed where the vector boson mass is put in by hand, in a variety of more or less astute ways. One could call them “genuine” massive non-Abelian gauge theories. All such models proposed so far are either not unitary or not renormalizable. This could well be the end of the story. Nevertheless, the history of physics is full of no–go theorems which turned out to be wrong. Hence the continued theoretical interest in these theories. On the other hand, massive vector boson models which are renormalizable but not unitary could still provide valuable hints towards a solution of the infrared problem.

On the more theoretical side, one might ask whether a “non-renormalizable” theory is really useless. For instance, it is conceivable that the divergences in the Green’s functions would cancel in the on–shell $S$–matrix elements (van Nieuwenhuizen, 1993): this happy situation does not seem to be realized in any of the models constructed so far. A quite unconventional proposal is to work in the Euclidean region of space–like external momenta, to consider a massive scalar theory with an exponential self–interaction which seems non-renormalizable, and then to suggest a method for constructing an $S$–matrix finite to all orders in perturbation theory (Efimov, 1965). See also Delbourgo et al. 1969, Fradkin, 1963, Ghose and Das, 1972, Ginibre and Velo, 1975. This method was developed in the Yang–Mills case by Lehmann and Pohlmeyer, 1971; Salam, 1971; Taylor, 1971b, and later by Fukuda et al. 1981, 1982, 1983.

In a different vein, Georgi (1993) discussed composite Higgs fields, chiral symmetry, and technicolor, whereas Niemi (1996) discussed how massive Yang–Mills could arise from massless Yang–Mills coupled to a topological field theory.

As emphasized, among others, by Reiff and Veltman, 1969; Veltman, 1968, 1970 and Slavnov, 1972a), the problem of unitarity and renormalizability is very delicate without having at our disposal a consistent and parameter–free method of regularizing the perturbative expansion. Also, as epitomized by the neutral vector boson theory, hidden symmetries may be responsible for the cancellation of divergences in individual graphs. Let us just emphasize in passing that the standard theory is indeed unitary and renormalizable in the dimensionally regularized perturbative expansion (Bollini and Giambiagi, 1972; ’t Hooft and Veltman, 1972), see also Akvampong and Delbourgo, 1973b, 1974a, Breitenlohner and Maison, 1977.

Our aim now is not to provide an exhaustive history of “genuine” massive vector models, but only to focus on the influence of Stueckelberg’s seminal papers (Stueckelberg, 1938a,b,c) in this general line of research. His crucial contribution was the introduction of a bona fide auxiliary scalar field with positive metric and positive energy, modifying Proca’s original model.

We shall limit ourselves to sketching the context in which Stueckelberg’s idea has been generalized or applied to various domains. In the following, we rely mostly on the lucid reviews by Delbourgo et al. 1988 and by van Nieuwenhuizen, 1997. Let us recall that Delbourgo et al. 1988 proved that the original Stueckelberg theory for neutral massive vector fields (with the addition of Faddeev–Popov ghosts and the Nakanishi–Lautrup Lagrange multiplier) was invariant under nilpotent BRST transformations. This ensures unitarity and renormalizability. The upshot of both reviews for existing massive Yang–Mills theories without a Higgs mechanism is the following: those which use a suitable generalization of the Stueckelberg mechanism are unitary order by order, but not perturbatively renormalizable, due to the non-polynomial interaction, whereas those which do not use the Stueckelberg mechanism are renormalizable but not unitary, because of physical ghosts (de Boer et al., 1996; Curci and Ferrari, 1976a).
What is the generalization of the Stueckelberg trick to non-Abelian massive gauge theories? Remember that (Salam, 1962) and (Umezawa and Kamefuchi, 1961) kept the substitution $A_\mu \rightarrow A_\mu - m^{-1}i\partial_\mu B$ of the Abelian case, letting $A_\mu = A^0_\mu T_i$, $B = B^i T_i$ with $T_i$ the generators of a Lie algebra. The new lagrangian was then gauge–invariant. The divergences due to the Proca vector propagator were then explicit in the $\partial_\mu B^i$ terms. The latter was transformed away by a unitary redefinition of the fields which yielded a non-polynomial (actually, exponential) term in the Stueckelberg field $B$.

(Kunitama and Goto, 1967) proceeded in a slightly different way, starting from

$$\mathcal{L} = -\frac{1}{4} \left( \partial_\mu A^\nu_\mu - \partial_\nu A^\mu_\mu + gf_{ijk} A^j_\mu A^k_\nu \right)^2 + \frac{m^2}{2} \text{Tr} \left[ A^i_\mu T^i - i g U^{-1} \partial_\mu U \right]^2$$

(281)

with

$$U = \exp \left( i \frac{g}{m} B^i T_i \right)$$

(282)

The lagrangian (281) is invariant under the gauge transformation

$$\delta A^i_\mu = (D_\mu \Lambda)^i = \partial_\mu A^i_\mu + gf^{ijk} A^{i}_\nu A^k_\nu$$

$$\delta B^i = m \Lambda^i$$

(283)

The end result is very similar to that of (Salari, 1962) and (Umezawa and Kamefuchi, 1961). A common weakness of these schemes is the absence of ghosts in the lagrangian (281), as emphasized by (Slavnov, 1972a): in the language of path integrals, the ghosts compensate the propagation of unphysical states of the gauge fields (Faddeev and Popov, 1967). It turns out that the same $S$–matrix is obtained with or without the Stueckelberg fields. Indeed, using Pauli–Villars regularization and the preprint (t’Hooft, 1971a), Slavnov showed that the $S$–matrix is independent of the longitudinal part of the Green’s function. Crucially, the arguments for the renormalizability of the massive neutral vector theory do not apply to the massive Yang–Mills case. The symmetry of the theory, however, ensures a partial cancellation of divergences. Indeed, as shown by (Slavnov and Faddeev, 1970; Veltman, 1968), the one–loop diagrams of the massive Yang–Mills theory do not generate any divergences other than the usual ones associated with mass, charge, and wave–function renormalization. It is interesting to note that (Veltman, 1992) introduced a triplet of scalar fields to give mass to the $W$ vector bosons which look like non-Abelian Stueckelberg fields.

What about higher orders? Since the massless Yang–Mills field theory is renormalizable, one could expect that the massive theory is also renormalizable, if the $m \rightarrow 0$ limit exists. Alas, this limit is sick (Boulware, 1970; van Dam and Veltman, 1970; Slavnov and Faddeev, 1970): the matrix elements of massive Yang–Mills theories are discontinuous in the limit $m \rightarrow 0$. The reason for this singularity is easy to understand from counting physical fields: a massive vector particle has three physical degrees of freedom whereas a massless one has only two. If the vector field happens to be neutral, as in QED, all the matrix elements which are not diagonal in the number of longitudinal photons vanish as $m \rightarrow 0$, and thus the massless limit is well–defined. In the non-Abelian case, however, where the gauge fields interact with themselves, this is not so, and the limit contains a charged massless scalar field in addition to the transverse vector modes.

One would be tempted to conclude that the massive Yang–Mills theory is not renormalizable in the usual sense, and indeed (Reiff and Veltman, 1969) found new divergences at two loops. To settle this issue satisfactorily, an invariant regularization is required. Attempts have been made (Delbourgo et al., 1969) to apply the method of (Efimov, 1963; Fradkin, 1963), but they have been criticized by Slavnov: ambiguity in the summation procedure, unreliable transition to the pseudo–Euclidean region, and the fact that the solutions obtained do not, in general, have the symmetry built into the original lagrangian.

The extremely clear paper (Salari and Stratheie, 1970) precedes (Slavnov, 1972a) and covers roughly the same ground as it. After recalling the Stueckelberg formalism for neutral vector fields, they recast it in the language of path integrals. The advantage is that field redefinitions can be tracked more carefully, including non-trivial Jacobians in the measure (Faddeev and Popov, 1967). The Stueckelberg substitution

$$V_\mu = A_\mu = V_\mu + \frac{1}{m} \partial_\mu B$$

(284)

of the neutral Proca field $V_\mu$ by a vector field $A_\mu$ and a scalar $B$ yields the generating functional

$$Z[I^\mu, \eta, \bar{\eta}] = \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp i \int \left\{ \frac{1}{2} A^\mu \left( \partial^2 + m^2 \right) A_\mu - \frac{1}{2} B \left( \partial^2 + m^2 \right) B \right\}$$
\[ +\mathcal{L}_f + I^\mu \left( A_\mu - \frac{1}{m} \partial_\mu B \right) + \bar{\psi} \eta + \bar{\eta} \psi \]  

(285) where \( \mathcal{L}_f \) contains all the terms involving fermions. One can now introduce a Lagrange multiplier \( C \) to get an equivalent expression, using the functional identity

\[ \int D C \delta \left( \partial_\mu V^\mu + \frac{1}{m} \partial^2 B \right) = 1 \]  

(286) Dropping, for the sake of notational convenience, the fermionic fields and their sources, the above generating functional is equivalent to

\[ Z[I^\mu] = \int D A^\mu DBDC \exp i \left\{ \mathcal{L}[A^\mu - \frac{1}{m} \partial^\mu B] + C \partial_\mu A^\mu + I^\mu \left( A_\mu - \frac{1}{m} \partial_\mu B \right) \right\} \]  

(287) The propagators are now

\[ \langle TA_\mu(x)A_\nu(y)\rangle = - \left( g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \Delta_m(x-y) \]

\[ \langle TA_\mu(x)B(y)\rangle = 0 \]

\[ \langle TB(x)B(y)\rangle = \Delta_0(x-y) \]  

(288) and thus

\[ \left\langle T(A_\mu(x) - \frac{1}{m} \partial_\mu B(x))(A_\nu(y) - \frac{1}{m} \partial_\nu B(y)) \right\rangle = - \left( g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \Delta_m(x-y) \]  

(289) This language was then generalized by \cite{Salam and Strathdee, 1970} to charged vector fields. The first example is provided by an isotriplet \( V^\mu_i \) of vector fields. Let \( A_\mu \) and \( \Omega \) denote the transverse and longitudinal parts of \( V^\mu_i \), defined by

\[ V^\mu_i \tau^i = A^\mu_i \Omega \tau^i \Omega^{-1} + \frac{i}{g} \Omega \partial_\mu \Omega^{-1} \]  

(290) with \( \tau^i \) the Pauli matrices and

\[ \partial^\mu A^i_\mu = 0 \]  

(291) Equation (290) can be viewed as a non-Abelian gauge transformation, where \( \Omega \) is a \( 2 \times 2 \) unitary matrix, conveniently expanded as

\[ \Omega(x) = \frac{g}{m} \left[ \sigma(x) - i \tau^i B^i(x) \right] \]  

(292) in terms of the constrained field variables subject to

\[ \sigma(x)^2 + \vec{B}(x)^2 = m^2/g^2 \]  

(293) The change of field variables is done according to

\[ D V = \int D A D \Omega J(A^\Omega) \exp \left\{ - \frac{i}{2} \int (\partial A + mB)^2 \right\} \]  

(294) where the Jacobian is given by \cite{Faddeev and Popov, 1967}

\[ (J(A^\Omega))^{-1} = \int D \Omega \exp \left\{ - \frac{i}{2} \int (\partial A^\Omega + mB)^2 \right\} \]  

(295) and \( A^\Omega \) is defined by the right–hand side of (290). The generating functional is then

\[ Z[I] = \int D V \exp i \int \left( \mathcal{L}[V] + I V \right) \]

\[ = \int D A D \Omega J(A^\Omega) \exp i \int \left( \mathcal{L}[A^\Omega] - \frac{1}{2} (\partial A + mB)^2 + I A^\Omega \right) \]  

(296)
and the chronological pairings read as follows:

\[
\langle TV^{\mu}_i(x)V^j(y)\rangle = -\left(g_{\mu\nu} + \frac{\partial_{\mu}\partial_{\nu}}{\alpha^2}\right)\Delta_m(x-y)\delta^{ij}
\]
\[
\langle TA^{\mu}_i(x)A^j(y)\rangle = -g_{\mu\nu}\Delta_m(x-y)\delta^{ij}
\]
\[
\langle TA^{\mu}_i(x)B^j(y)\rangle = 0
\]
\[
\langle TB^{\mu}(x)B^j(y)\rangle = \Delta_0(x-y)\delta^{ij}
\]

(297)

From this, (Salam and Strathdee, 1974) developed a perturbation theory.

Note that the above procedure can be applied to a subset of the charged vector fields. For example, if \(V^3_\mu\) is not present, then merely replace \(DV \rightarrow DV\delta(V^3)\).

(Fukuda et al., 1981) describe a quantum theory of massive Yang–Mills fields. They start from the lagrangian of Kunimasa and Goto (1967), to which they add Faddeev–Popov ghosts and the Nakanishi–Lautrup Lagrange multiplier. The latter is quantized in a Hilbert space of indefinite metric, following (Kugo and Ojima, 1978, 1979). Their scalar field \(\xi\) is in fact the generalized Stueckelberg field \(B\) defined by (Kunimasa and Goto, 1967) and used by (Slavnov, 1972a). Fukuda et al. (1981) introduce, however, a different subsidiary condition, again following (Kugo and Ojima, 1978). The resulting theory is claimed to be invariant under a nilpotent BRST transformation and unitary; this statement is questioned by (Delbourgo et al., 1988, p. 442). Its lagrangian contains an exponential in the scalar field, just like Kunimasa and Goto (1967) and hence, it is not renormalizable in the conventional sense, although Fukuda et al. (1981) claim, quoting Salam (1974), that it is renormalizable in the sense of Efimov (1965).

An inconclusive extension to massive two forms in this direction has been considered in (Lahiri, 1992).

Based on Kunimasa and Goto (1967) and Fukuda et al. (1981, 1982, 1983), (Sonoda and Tsai, 1984) introduce a Stueckelberg scalar for the \(U(1)_V\) vector boson in the usual way, plus an isovector Stueckelberg field for the \(SU(2)_L\) sector. They find, interestingly, that the ratio of the masses of these scalars had to be proportional to the weak mixing angle, that is to \(g'/g\). The theory is, however, not renormalizable.

(Burnel, 1986a) formulates the Abelian theory of massive vector bosons in a gauge invariant way, without Higgs fields, but using the theory of constrained systems (Dirac, 1964). The Proca and Stueckelberg formulations appear as particular gauges, unitarity being obvious in the former, and renormalizability in the latter (Matthews, 1949a,b). BRST invariance is also discussed, after introducing Faddeev–Popov ghosts.

(Burnel, 1986b) extends this method to the non-Abelian case, constructing a gauge invariant lagrangian, following Fukuda et al. (1981, 1982, 1983), Slavnov and Faddeev (1970). There exists a gauge with only physical particles, which can be called the unitary gauge. However, it is not power–counting renormalizable. In a different gauge, there are Faddeev–Popov ghosts and a Stueckelberg scalar field. Due to the nilpotent BRST and anti–BRST invariance, the fundamental Ward–Takahashi–Slavnov–Taylor identities are satisfied (Slavnov, 1972b, 1973; Takahashi, 1957; Taylor, 1971a; Ward, 1954), and thus unitarity is ensured, but not renormalizability. In yet another gauge, the theory is renormalizable but the BRST transformation is not nilpotent, and unitarity is not satisfied (Curci and d’Emilio, 1973; Curci and Ferrari, 1976). This emphasizes the importance of the nilpotency of the BRST invariance. (Curci and d’Emilio, 1973) apply this model (renormalizable but not unitary) to the infrared problem.

(Burnel, 1986b) then abandons the lagrangian formalism and extends the Stueckelberg Abelian field equations

\[
\partial^{\mu}F_{\mu\nu} + m^2A_\nu + \partial_\nu B = 0
\]
(298)

\[
\partial^{\mu}A_\mu = \alpha B
\]
(299)
to the non-Abelian case as follows:

\[
D^{ij}_\mu F^j_{\mu\nu} + m^2A^j_\nu + D^j_\nu B_j = 0
\]
(300)

\[
\partial^{\mu}A^j_\mu = \alpha B^j
\]
(301)

With canonical commutation relations, the gauge field propagator is now

\[
\Delta_{ij}^{\mu\nu} = -i\delta^{ij} \left\{ \frac{g_{\mu\nu} - k_\mu k_\nu/m^2}{k^2 - m^2 + i\epsilon} + \frac{k_\mu k_\nu/m^2}{k^2 - \alpha m^2 + i\epsilon} \right\}
\]
(302)

Although it is possible to maintain gauge invariance, unitarity cannot be satisfied. The conclusion, therefore, is rather pessimistic. Burnel (1986b), however, ends the paper with the following remark: “We have also emphasized that the nonrenormalizable couplings always involve unphysical fields. Since, in general, ghosts of the type used here do not contribute at all to physical amplitudes, it is quite plausible that the physical sector of massive Yang–Mills theory be
4. 1988: unitarity versus renormalizability reassessed

We now present the important review \cite{Delbourgo1988}. We have already mentioned several times their treatment of the Stueckelberg theories of the Abelian massive vector field. In their discussion of a variety of massive gauge–invariant non-Abelian models without Higgs mechanism, the themes are unitarity, renormalizability and BRST invariance. They also provide a rich bibliography up to 1988.

Their first example is the non-Abelian generalization of the Stueckelberg formalism by \cite{Kunimasa1967}, supplemented by \cite{Slavnov1972, Slavnov1970}. Following the latter, they establish unitarity for the gauge propagator to one loop, working in the Landau gauge. They show explicitly that one half of the contribution of the Faddeev–Popov ghosts to the imaginary part of this propagator is compensated by the spin–zero part of the gauge vector field, whereas the other half is compensated by the Stueckelberg scalar field. This means that in the zero–mass limit one does not recover the massless Yang–Mills theory, as already emphasized by \cite{Slavnov1970}. Repeating their analysis of unitarity for fermion–antifermion scattering to order $g^4$, \cite{Delbourgo1988} find again that the Stueckelberg scalar contributes the essential factor of $1/2$ with the correct sign. They then turn to the high–energy behavior of longitudinally polarized vector bosons, computing their elastic scattering. In a theory with Higgs bosons, the amplitude is bound, in agreement with unitarity. In the Stueckelberg case, even if $S^1S = 1$ is satisfied order by order in $g^2$, it turns out that the amplitude in increasing orders of $g^2$ scales with an increasing power of $E^2/m^2$ (they show this explicitly up to $E^4/m^4$). They conclude that renormalizability is not satisfied perturbatively in the generalized Stueckelberg scheme. Nevertheless, \cite{Delbourgo1988} point out that \cite{Shizuya1975} established in a non-conventional manner the renormalizability of two–dimensional massive Yang–Mills, elaborated upon by \cite{Bardeen1978}.

A more complete discussion of unitarity bounds can be found in \cite{Cornwall1973, Cornwall1974, Llewellyn-Smith1973}. They introduce the concept of “tree unitarity”, holding when the $N$–particle $S$–matrix elements in the tree approximation diverge no more rapidly than $E^{4–N}$ in the high–energy limit, and discuss its relation to gauge invariance and renormalizability. They remark that “a big advantage of the Stueckelberg formalism is that all bad behavior is now isolated in the vertices.” Curiously, they quote \cite{Stueckelberg1938} instead of \cite{Stueckelberg1938a}.

They conclude that tree unitarity is only satisfied in models with spontaneously broken symmetry.

The second model discussed by \cite{Delbourgo1988} was proposed by \cite{Fradkin1969} and \cite{Curci1976b} as a possible candidate for a theory of massive Yang–Mills fields; see also \cite{Ojima1980} and the particularly clear \cite{Ojima1982}. The lagrangian involving no Stueckelberg fields but including Faddeev–Popov ghosts is

\begin{equation}
\mathcal{L} = -\frac{1}{4} F^2 + \frac{m^2}{2} A^2 - \frac{1}{2\alpha} (\partial \cdot A)^2 + \omega^* \partial \cdot D\omega + \alpha m^2 \omega^* \omega + \frac{\alpha}{8} (\bar{\omega} \times \omega)^2
\end{equation}

where all fields carry $SU(2)$ indices. In \cite{Fradkin1969}, the Landau gauge $\alpha = 0$ is chosen. This lagrangian is invariant under the extended “BRST transformation”

\begin{equation}
\delta A_\mu = D_\mu \omega
\end{equation}

\begin{equation}
\delta \omega = \frac{1}{2} \omega \times \omega
\end{equation}

\begin{equation}
\delta \omega^* = -\frac{1}{\alpha} \partial \cdot A + \omega^* \times \omega
\end{equation}

which is not nilpotent:

\begin{equation}
\delta^2 \neq 0
\end{equation}

The theory is gauge invariant and has a good high–energy behavior, albeit only in the Landau gauge. Indeed, thanks to the gauge-fixing term, the vector propagator has a $k_\mu k_\nu/k^2$ term, instead of $k_\mu k_\nu/m^2$. Hence, the model is power–counting renormalizable. It is not unitary, for at least three reasons:

\footnote{A recurrent problem in the historical record is that many citations to Stueckelberg’s work are incorrect, a sad reflection of the fact that his papers have not been read.}
1) The proof of unitarity for massive gauge theories in \cite{Kugo_Ojima_1979} rests on the nilpotency of the BRST charge $Q$, to whose cohomology physical states belong. Nilpotency could be enforced using the Nakanishi–Lautrup Lagrange multiplier $b$, such that $\delta \omega^* = b + \omega^* \times \omega$ and $\delta b = b \times \omega$. This modification would spoil the invariance, however.

2) The ghost and gauge–fixing terms, that is the last four terms of (303) are not the $\delta$ of something, and the physical lagrangian is not by itself gauge invariant. Hence, the ghosts cannot be eliminated from the physical $S$–matrix.

3) In the Landau gauge, $\mathcal{L}$ is just the effective action in \cite{Kunimasa_Gotd_1967, Slavnov_Faddeev_1970}, without the Stueckelberg terms, which were shown to be crucially necessary for unitarity at one loop.

A pretty variant of the Stueckelberg model was presented by two of the authors in \cite{Delbourgo_Thompson_1986}. Using the field equations, the scalar Stueckelberg field $B$ was eliminated in favor of a gauge–fixing functional of the vector field in such a way that the gauge invariance of the mass terms was preserved; the inherent non-polynomiality could then be disregarded in some particular gauge. Whereas the renormalizability of the scheme was not cast in doubt, it turns out that unitarity is violated \cite{Kosinski_Szymanski_1987, Kubo_1987}. For example, \cite{Kubo_1987} shows that the one–loop correction to the imaginary part of the gauge boson self–energy is the same as in the spontaneously broken theory in the Landau gauge, except that the would–be Goldstone boson is missing. But its contribution is necessary to compensate the Faddeev–Popov ghosts.

\cite{Delbourgo_etal_1988} reexamine this model in an equivalent formulation. They find that it is invariant under a nilpotent BRST transformation. It is also power–counting renormalizable and gauge covariant (that is to say, the $S$–matrix does not depend on the gauge). Nevertheless, unitarity is violated. According to the authors, “we learn that gauge invariance and gauge covariance of a theory are not strong enough conditions to ensure unitarity.” This result is paradoxical since it comes into conflict with the proof of unitarity by \cite{Kugo_Ojima_1979}. \cite{Delbourgo_etal_1988} resolve this contradiction by pointing out that, contrary to the Stueckelberg model \cite{Kunimasa_Gotd_1967}, the model in \cite{Delbourgo_Thompson_1986} does not tend to the massive Yang–Mills theory when the Stueckelberg field is taken to zero. They conclude that “the original Stueckelberg model is just right to ensure (one loop) unitarity, and any tampering leads us astray.”

Finally, \cite{Delbourgo_etal_1988} describe the completely different model of \cite{Batalin_1974}, which avoids several of the problems of the previous ones. The lagrangian is

\[
\mathcal{L} = -\frac{1}{4} F^2 + \frac{m^2}{2} A^2 + \frac{\xi}{2} (\partial \cdot A)^2 + \mathcal{G}(\xi, A)
\]

with $\mathcal{G}(0, A) = 0$. Under a gauge transformation one has in particular

\[
\xi \rightarrow \xi + \delta\xi
\]

\[
\mathcal{L} \rightarrow \mathcal{L} + \frac{\delta\xi}{2} (\partial \cdot A)^2 + \mathcal{G}(\xi + \delta\xi, A) - \mathcal{G}(\xi, A)
\]  

(307)

Batalin gives a procedure to calculate $\mathcal{G}$, but this model is not perturbatively renormalizable and, as far as we know, its unitarity has not been established either.

In conclusion, \cite{Delbourgo_etal_1988} notice that renormalizability and unitarity seem to be competing qualities of massive non Abelian theories. “The original Stueckelberg formulation, with its inherent non-polynomiality, is unitary but not renormalizable. This is in itself quite interesting, implying that the naive massive Yang–Mills action is of the correct form to ensure unitarity, and as we have seen any tampering with this leads us astray.”

\cite{Burnel_1986b} has argued that in the Stueckelberg model one can find gauges in which ultraviolet divergences are confined to vertices that always involve unphysical fields, so that these divergences may cancel in the physical sector. “Finally, it must be admitted that the Higgs mechanism remains the most complete method for giving mass to the vector bosons.”

5. 1995–2003: new viewpoints

Van Nieuwenhuizen and collaborators have reexamined several aspects of the review of \cite{Delbourgo_etal_1988} in a series of papers.

\cite{van Nieuwenhuizen_1995} emphasizes the importance of Veltman’s and Slavnov’s work, comprising explicit computations of one-loop \cite{Veltman_1968} and two-loop \cite{Reiff_Veltman_1969} divergences, the relation between massive and massless Yang-Mills theories \cite{van Dam_Veltman_1974, Slavnov_Faddeev_1970}, and the generalized Ward identity \cite{Veltman_1970}. The \cite{Curci_Ferrari_1976} model is then rederived in detail by requiring a (not nilpotent) BRST invariance. Renormalizability holds, but not unitarity, because one cannot enforce nilpotent BRST invariance.
invariant way, then it is renormalizable. Renormalizability, and it relies only on the scaling properties of the theory. If a theory can be normalized in a gauge by physical conditions stays the same at any order of perturbation theory. This is generally called power-counting.

Due to nilpotency, the Ward identity of non-Abelian gauge theories acquires an additional term which calls for a more careful enquiry into renormalizability. This is carried out using the full beauty of the BRST formalism, with or without Lagrange multipliers. They find again that the model is indeed renormalizable, with five multiplicative renormalization factors. This checks with an explicit one-loop computation. The authors then “determine the physical states, extending the work of (Ojima 1982). Many of these states have, for arbitrary values of the parameters of the theory, a negative norm, and from this we conclude that the model is not unitary”. This statement contradicts the curious claim of (Periwal, 1995) that the theory is unitary and renormalizable, with three Z factors.

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Already [Curci and Ferrari 1976I] and [Ojima 1982] noticed that if one takes over the formula for $Q$ from the massless case in the general $\alpha$–gauge

$$Q_{CF} = \int \frac{\partial_\nu A^\nu}{\alpha} \partial_\mu \omega \, d^3 x$$  \hspace{1cm} (315)$$

one gets an operator which is not nilpotent in the massive case. But, as noted above, nilpotency is a necessary condition for unitarity. So a different $Q$ is needed.

The generalization in [Kunimasa and Goto 1967], [Fukuda et al. 1981, 1982, 1983] of the Stueckelberg formalism is to add scalar fields $B^i(x)$ satisfying

$$[B^i(x), B^j(y)] = i \delta^{ij} \Delta_M(x - y)$$  \hspace{1cm} (316)$$

$$\left( \nabla^2 + M^2 \right) B^i(x) = 0$$  \hspace{1cm} (317)$$

The BRST generator is then

$$Q_S = \int \eta(x) \partial_\mu \omega(x) \, d^3 x$$  \hspace{1cm} (318)$$

where the local quantities

$$\eta^i(x) = \alpha^{-1} \partial^\mu A_\mu^i(x) + m B^i(x)$$  \hspace{1cm} (319)$$

are the algebraically solved Nakanishi–Lautrup ghosts or, equivalently, the supplementary condition. The corresponding BRST transformation looks familiar:

$$[Q_S, A_\mu^i] = i \partial_\mu \omega^i$$  \hspace{1cm} (320)$$

$$[Q_S, B^i] = i m \omega^i$$  \hspace{1cm} (321)$$

$$[Q_S, \omega^i] = 0$$  \hspace{1cm} (322)$$

$$[Q_S, \partial_\mu A_\mu^i] = -i M^2 \omega^i$$  \hspace{1cm} (323)$$

$$[Q_S, \omega^{*i}] = -i \eta^i$$  \hspace{1cm} (324)$$

$$[Q_S, \eta^i] = 0$$  \hspace{1cm} (325)$$

[Hurtl 1997] proceeds to construct the most general gauge–invariant coupling $T_1$ such that $[Q_S, T_1]$ is a total derivative as in [305]. Lorentz– and $SU(N)$–invariant, with ghost number zero, and with maximal mass dimension equal to four. There are a certain number of trilinear couplings in the fields $A_\mu^i, B^i, \omega^i, \omega^{*i}$ and their derivatives. This has thus defined a manifestly normalizable theory which is gauge invariant to first order in perturbation theory, and respects further symmetry conditions. Can one prove the condition of gauge invariance, namely that $[Q_S, T_n]$ be a total derivative, inductively to all orders in perturbation theory? By explicit calculation, [Hurtl 1997] shows that this condition fails already at second order. The constraint of normalizability is essential for this conclusion. Hence, the Stueckelberg generalization for non-Abelian massive gauge theories is not perturbatively renormalizable. More precisely: no (non-linear) deformation of the (linear) asymptotic BRST invariance (implemented on the asymptotic Fock space) can lead to a renormalizable and unitary theory.

Non-Abelian Stueckelberg lagrangians are plagued by nonpolynomial terms. Perhaps a correct understanding of these terms could lead to a unitary and renormalizable non-Abelian Stueckelberg model. The work of [Dragon et al. 1997] rules out this possibility by showing that the non-polynomial structure can be reduced algebraically to a polynomial version of the Stueckelberg model. But then the results of [Hurtl 1997] are applicable, and thus this model is not unitary and renormalizable.

[Dragon et al. 1997] start with the [Kunimasa and Goto 1967] generalization of the Stueckelberg model to a massive Yang–Mills theory: the vector fields $A_\mu^i$ and scalar fields $B^i$ belong to the adjoint representation of a Lie group $G$. The kinetic and mass terms of the lagrangian are separately gauge–invariant. To compensate the unphysical degrees of freedom of $A_\mu^i$ and $B$, they introduce Faddeev–Popov ghosts $\omega^i$ and $\omega^{*i}$, also in the adjoint. The lagrangian, including a gauge–fixing term, is invariant under a BRST operator which is nilpotent if one adds further Nakanishi–Lautrup Lagrange multipliers $b^i$ to the set of fields. The gauge–fixing term is such that the propagators of $A_\mu^i$ and $B$ fall off like $k^{-2}$ for large momenta $k^\mu$. Unfortunately, the exponential of $B$ appears in the lagrangian. They then redefine the vector field $A_\mu^i$ into a new $\tilde{A}_\mu^i$ such that the $S$–matrix is unchanged [Coleman et al. 1968]. The BRST
transformation, $s$ is then given by

\begin{align}
  s\hat{A}_\mu &= 0 \\
  sB &= m\omega \\
  sw &= 0 \\
  sw^* &= b \\
  sb &= 0
\end{align}

and the BRST–invariant lagrangian is of the form

$$
\mathcal{L} = \mathcal{L}_{\text{phys}}(\hat{A}_\mu, \partial^\nu \hat{A}_\nu) + s(\omega X)
$$

where one can choose

$$
X = \frac{1}{2} \left[ b - \partial^\mu \hat{A}_\mu - (\partial^2 + m^2)\frac{B}{m} \right]
$$

so that the Faddeev–Popov ghosts $\omega$ and $\omega^*$ are free and the Nakanishi–Lautrup auxiliary field $b$ can be solved from $X = 0$. Redefining again $\hat{A}_\mu = A_\mu + m^{-1}\partial_\mu B$, the physical lagrangian containing the gauge and the Stueckelberg fields becomes

$$
\mathcal{L}_{\text{phys}} = -\frac{1}{4} \left( \text{tr} \left[ F_{\mu\nu}(\hat{A} - m^{-1}\partial B) \right] \right)^2
$$

with the usual notation $F_{\mu\nu}^i(Y) = \partial_\mu Y_\nu^i - \partial_\nu Y_\mu^i + gf^{ij}_k Y^j_\mu Y^k_\nu$.

The result of all these redefinitions is to replace the original exponential in $B$ by a polynomial in $\partial_\mu B^i$, of mass dimension eight. The propagators of $\hat{A}_\mu$ and $B$ are well–behaved at high energies, but the derivative interaction of $B$ is still not power–counting renormalizable. Of course, (333) reminds us very much of expressions in [Salam 1962, Umezawa and Kamefuchi 1961].

### D. Related applications

We end this chapter mentioning a variety of applications of the Stueckelberg formalism which fall outside our main line of exposition, but might be of interest to the reader.

Early phenomenological applications of charged vector mesons in the Proca and Stueckelberg theories are (Young and Bliedman 1963), (Bailin 1964a, 1965), and also (Abe 1969, Santhanam 1968), (Watanabe et al. 1967) applied the Stueckelberg formalism to the Rarita–Schwinger (spin 3/2) field, whereas (Arkani-Hamed et al. 2003, Delbourgo and Salam 1975, Luty et al. 2003) did it for the graviton (spin 2) field. These last two references illustrate nicely the rich possibilities of D-brane backgrounds.

The Stueckelberg theory was supersymmetrized ($N = 1$) very early (Delbourgo 1975). It was found that the condition for the super-Stueckelberg mechanism to work was that $\bar{D}D\bar{J} = 0$, with $J$ the external supercurrent coupled to the superphoton. In the following, we use Wess-Bagger notation (i.e. two-component fermions) but with our usual metric $+ - - -$. The standard kinetic term (in terms of $W^a = \bar{D}^2 D^a V$, with $V = V^\dagger$ a vector superfield)

$$
\mathcal{L}_0 = \frac{1}{4} \int d^2\theta W^2 + \frac{1}{4} \int d^2\bar{\theta} \bar{W}^2
$$

which in the Wess-Zumino gauge reduces to

$$
\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^2 - i\lambda\sigma^\mu\partial_\mu\lambda + \frac{1}{2}D^2
$$

is supplemented with

$$
\mathcal{L}_m = -m^2 \int d^2\theta d^2\bar{\theta}[V + \frac{i}{m}(\Phi - \Phi^\dagger)]^2
$$

mimicking the (bosonic) Stueckelberg starting point. Note that we must introduce a chiral and an antichiral superfield. The mass term above gives a mass to the vector, yields the kinetic terms for the complex Stueckelberg scalar fields $a$ and $a^*$ and their spin 1/2 superpartners $\psi$ and $\bar{\psi}$, induces a mixed mass term between the photino $\lambda$ and the
Stueckelberg fermion $\psi$, and provides the cross term $mA^a \partial_a (a + a^*)$. When the auxiliary field $D$ is eliminated, a mass term for $(a - a^*)$ comes out as well. We still must add a gauge-fixing term. In the massless case, it is of the form $\int d^4 \theta (D^2 V) (D^2 V)$, whereas now it is better to take

$$\mathcal{L}_{gf} = \xi \int d^2 \theta d^2 \theta (D^2 V + \frac{1}{m^2} \Phi)(D^2 + \frac{1}{m^2} \Phi^*)$$

The Stueckelberg mechanism has been used extensively in the pedagogical presentations of (Gates et al., 1983; Siegel, 1999) where the Stueckelberg or "compensator" fields are introduced as a simple example of Goldstone fields, and numerous applications to supersymmetry are discussed. See also (Guerdane and Lagrange, 1991).

The non-renormalizability of the non-Abelian Stueckelberg models are crucial to establish the non-renormalizability of the $N = 2$ massive Yang–Mills theory (Khelashvili and Ogievetsky, 1991), and useful for its analysis in harmonic superspace (Volkov and Maslikov, 1994).

Stueckelberg’s trick has been used to reformulate chiral (Banerjee et al., 1994a; Kulshreshtha et al., 1994; Kulshreshtha, 1998; Kulshreshtha and Kulshreshtha, 2001a; Kulshreshtha et al., 2002), and other two-dimensional models (Kulshreshtha, 2001a,b, 2002; Kulshreshtha and Kulshreshtha, 2002b), as well as the massive three-dimensional gauge theory (Dilkes and McKeon, 1993; Schonfeld, 1981). (Banerjee et al., 1996) established a three-dimensional duality similar to the well-known duality between sine–Gordon and Thirring (Coleman, 1975) exploiting the Stueckelberg formulation. Attempts to extend the three-dimensional topological Yang–Mills mass (Deser et al., 1982) to four dimensions used the Stueckelberg trick (Allen et al., 1991; Hwang and Lee, 1997; Lahiri, 1997), as did a powerful no-go theorem that pretty much ended such attempts (Henneaux et al., 1997), except perhaps for the provocative (Lahiri, 2001) which claims, using the duality with massive two-forms, that the non-Abelian massive vector boson renormalizability although its unitarity is not discussed.

The two–dimensional Stueckelberg theory has been studied in a Robertson–Walker background, in a black hole metric, and in a Rindler wedge (Chimento and Cossarini, 1992, 1993; Chimento et al., 1994). Janssen and Dullin (1985) formulated the Stueckelberg theory in anti–de Sitter space.

Deguchi and Nakajima (1994, 1995, 1996) derived Stueckelberg’s construction starting from loop space, and extended it to a classical non-Abelian version for rank–two tensor fields. Deguchi (1997) found, in particular, that only null strings interact with massive vector fields, and no strings interact with massive three–rank tensor fields at the classical level. A similar application of the Stueckelberg trick to string theory is (Aleksandrova and Bozhilov, 2003), whereas a different one to membrane theory is (Pavsic, 1998).

The idea of inventing new fields in order to uncover or make manifest hidden symmetries has been applied in many contexts. The extension of the Stueckelberg formalism for a massive antisymmetric field (Kalb and Ramond, 1974) has been the subject of intensive research (Banerjee and Banerjee, 1996; Barcelos-Neto et al., 1994; Deguchi and Kokubo, 2002; Deguchi et al., 1999; Kuzmin and McKeon, 2002; Sawavanaghi, 1993; Smalilagio and Spallucci, 2001), approaching a clear understanding (Diamantini, 2001). The generalization to $p$–forms ($p = 2$ for the Kalb–Ramond field) has met with success (Bizdadea, 1996; Bizdadea et al., 1999; Bizdadea and Salitri, 1998). (Deguchi et al., 1997) show that a $U(1)$ gauge theory defined in the configuration space for closed $p$–branes yields the gauge theory of a massless $(p + 1)$ antisymmetric tensor field and a Stueckelberg massive vector field. The $p$–form extension of the Stueckelberg formalism has been used to establish dualities between field theories (Freedman and Townsend, 1984; Sawavanaghi, 1996, 1997; Smalilagio and Spallucci, 2000) and to study symmetry breaking in $D$–brane theories (Ansaldi et al., 2000).

Dualities between massive three–dimensional non-commutative field theories appear elegantly if care is taken to apply the Stueckelberg mechanism first to make explicit the gauge symmetries (Ghosi et al., 2001, 2003).

The Stueckelberg theory has been analyzed in the Batalin–Fradkin–Vilkovisky formalism (Banerjee et al., 1996; Barcelos-Neto et al., 1994a,b; Davi, 1988, 1993; Sawavanaghi, 1995, 1997), in axiomatic field theory (Morchio and Strocchi, 1988, 1987) and in a new quantization procedure in algebraic field theory (Wiedemann and Landsman, 1996). The relationship between spin zero and spin one has been emphasized in (Kruglov, 2001, 2003).

(Banerjee and Barcelos-Neto, 1997) consider the (classical) hamiltonian formulation of a Higgs–free massive Yang–Mills theory with the Stueckelberg trick, and (Barcelos-Neto and Rabelde, 1997) apply it to the standard model. (Deguchi, 1999) interpolates between various classical lagrangians, with or without Higgs and Stueckelberg fields, and (Deguchi and Kokubo, 2003) applies it to the Abelian projection of the simplest $SU(2)$.

A particularly nice application of the Stueckelberg mechanism has appeared in the construction of string-derived particle field theories (Aldazabal et al., 2000). In general, after compactifying a ten-dimensional string, or choosing a suitable brane background, there are too many $U(1)$s left. Non-Abelian gauge groups can be broken by a variety of mechanisms, but the rank of the gauge group does not change in general. Polchinski’s observation that all continuous symmetries in string theories must be gauge symmetries [for a discussion, see (Polchinski, 1998)] precludes the possibility of arranging moduli in such a way that these spurious $U(1)$s become mere global symmetries. The Stueckelberg mechanism is then applied to them, with a large Stueckelberg mass, whereby the additional Abelian gauge
vectors become very massive (without breaking any gauge symmetry) and disappear from the low-energy spectrum, not leaving even a global symmetry behind.

VII. CONCLUSIONS AND OUTLOOK

Stueckelberg’s original idea was to introduce a physical scalar field \( B(x) \) into the Abelian massive vector field lagrangian to make the theory as similar as possible to QED. It was shown by several authors that this proposal facilitated the discussion of the renormalizability of massive vector field theories and made manifest some hidden symmetries. The neutral massive Abelian vector field theory is gauge invariant and BRST invariant, because the transformation of the Stueckelberg \( B \) field compensates the transformation of the mass term. This explains the renormalizability of this theory. The field \( B \) plays a role similar to that of the Goldstone boson in spontaneously broken theories. Charged or non-Abelian theories without the Higgs mechanism have been shown to be not renormalizable, because the derivative couplings of the Stueckelberg field can only be eliminated at the expense either of exponential couplings or of unitarity. Work on these issues has not been completely abandoned, however.

Stueckelberg’s mechanism, simple as it seems today and always elegant, inspired numerous imitations, ranking from non-Abelian massive vector theories without Higgs fields to supersymmetric, topological and string theories. In many cases, new symmetries were discovered.

Additionally, in this paper we have constructed a standard electroweak theory with a massive photon, preserving the \( SU(2)_L \times U(1)_Y \) BRST symmetry. The neutral scalar Stueckelberg field \( B \) appears together with a massive hypercharge vector field, and the photon inherits a mass after the spontaneous symmetry breaking. This can be interpreted in two ways. The photon mass can be considered as an infrared cut-off, a mere calculational trick, allowing one to deal cleanly and separately with the infrared divergences. This would require, of course, the zero mass limit to be smooth. A less conservative point of view calls for taking the photon mass seriously, albeit limited by empirical data to a very small value. In this case, new phenomena appear, proportional to the photon’s mass (squared): neutrino photon couplings, parity violation of the electron photon couplings, slightly different Z mass, etc. None of them seem comparable, in precision, to the direct limits on the photon mass, but more research in this direction seems warranted, in particular for neutrino cosmology. Finally, the physical definition of the electric current appears now in a new light.

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APPENDIX A

We have collected here some long formulae on the Stueckelberg modification of the standard model, section V.

1. Notation

We use the metric \((+,-,-,-)\). Let us point out in passing that Stueckelberg used a real metric but with the opposite sign (standard in modern, string–oriented, notation), while Pauli and others used Euclidean metric with an imaginary fourth component.

We use throughout the notations

\[
A_\pm = \frac{1}{\sqrt{2}}(A_1 \pm iA_2)
\]  
\[ (A1) \]

\[
\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i = A_+B_- + A_-B_+ + A_3B_3
\]  
\[ (A2) \]
and

$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$ (A3)

with $\varepsilon_{123} = 1$ and cyclic; in the $\{+, -, 3\}$ basis the non-zero elements of the $\varepsilon_{ijk}$ are

$$\varepsilon_{3+} = -\varepsilon_{3-} = \varepsilon_{+3} = -\varepsilon_{-3} = -\varepsilon_{-3} = i$$ (A4)

Sometimes we use the short-hand $\bar{A}^2 = \vec{A} \cdot \vec{A}$.

2. Scalar lagrangian

Explicitly,

$$|D_\mu \Phi|^2 = \frac{1}{2} (\partial_\mu H)^2 + \frac{1}{2} (\partial_\mu \phi_3)^2 + \partial_\mu \phi_+ \partial_\mu \phi_-$$

$$+ \frac{1}{2} \partial_\mu H (g W_3^\mu - g' V^\mu) \phi_3 - \frac{1}{2} \partial_\mu \phi_3 (g W_3^\mu - g' V^\mu) (H + f)$$

$$+ \frac{g}{2} (\partial_\mu H - i \partial_\mu \phi_3) W_\mu^\nu \phi_+ + \frac{g}{2} (\partial_\mu H + i \partial_\mu \phi_3) W_\mu^\nu \phi_-$$

$$- \frac{g}{2} \partial_\mu \phi_+ W_\mu^\nu (H + f - i \phi_3) - \frac{g}{2} \partial_\mu \phi_- W_\mu^\nu (H + f + i \phi_3)$$

$$+ \frac{1}{4} \left( (H + f)^2 + \phi_3^2 \right) \left[ g^2 W_\mu^\nu W_\nu^\mu + \frac{1}{2} (g W_3^\mu - g' V^\mu)^2 \right]$$

$$- i \frac{gg'}{2} (H + f - i \phi_3) \phi_+ V_\mu W_3^\nu + i \frac{gg'}{2} (H + f + i \phi_3) \phi_- V_\mu W_3^\nu$$

$$+ \frac{1}{2} \phi_+ \phi_- \left[ g^2 W_\mu^\nu W_\nu^\mu + \frac{1}{2} (g W_3^\mu + g' V^\mu)^2 \right]$$ (A5)

3. Quadratic lagrangian

Using the choices of gauge–fixing functions (124), and dropping three total derivatives, the quadratic part of the lagrangian is

$$\mathcal{L}_2 = -\frac{1}{2} (\partial_\mu \bar{W}_\nu)^2 + \frac{f^2 g^2}{8} \bar{W}_\mu^2 + \frac{f^2 g^2}{4} \left( \frac{1}{2} - \frac{1}{\alpha'} \right) (\partial_\mu \bar{W}_\mu)^2 - \frac{gg'}{4} W_3^\mu V_\mu$$

$$- \frac{1}{2} (\partial_\mu V_\nu)^2 + \frac{1}{2} \left( m^2 + f^2 g^2 \right) \bar{V}_\mu^2 + \frac{1}{2} \left( 1 - \frac{1}{\alpha'} \right) (\partial_\mu V_\mu)^2$$

$$+ \frac{1}{2} \left( \partial_\mu \phi \right)^2 - \frac{\alpha' f^2 g^2}{8} \phi^2 - \frac{\alpha' f^2 g^2}{8} \phi_3^2$$

$$+ \frac{1}{2} \left( \partial_\mu B \right)^2 - \frac{\alpha' m^2}{2} B^2 + \frac{\alpha' g' mf}{2} \phi_3 B$$

$$+ \frac{1}{2} \left( \partial_\mu H \right)^2 - \lambda f^2 H^2$$

$$- \omega^* \left( \partial_\mu \phi \right)^2 + \frac{\alpha' f^2 g^2}{4} \omega^* \omega + \frac{f^2 gg'}{4} (\alpha' \omega^* \omega_3 + \alpha \omega^* \omega)$$

$$- \bar{\omega}^* \left( \partial_\mu \phi \right)^2 + \frac{\alpha' f^2 g^2}{4} \bar{\omega} \bar{\omega}$$ (A6)

It might be necessary to expand $\epsilon$ in (164) to second order for some computations, but here we restrict ourselves to first order. The various rotations of $g$ and $g'$ to order $\epsilon$ are

$$g c_w + g' s_w \simeq \sqrt{g^2 + g'^2} + \mathcal{O}(\epsilon^2)$$ (A7)

$$g c_w - g' s_w \simeq \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} \left( 1 - 2 \frac{g^2 g'^2}{g^2 - g'^2} \epsilon \right) + \mathcal{O}(\epsilon^2)$$ (A8)
\[
g s_w - g' c_w \simeq \epsilon - \frac{g g'}{\sqrt{g^2 + g'^2}} + O(\epsilon^2) \quad (A9)
\]
\[
g s_w + g' c_w \simeq \frac{2 g g'}{\sqrt{g^2 + g'^2}} \left(1 + \frac{1}{2} \frac{g^2 - g'^2}{g^2 + g'^2} \epsilon\right) + O(\epsilon^2) \quad (A10)
\]

4. Mass formulas

The exact mass eigenvalues of the neutral vectors, scalars and ghosts are
\[
M_{\chi^2} = \frac{f^2}{8} \left(g^2 + g'^2\right) \left(1 + \epsilon \pm \sqrt{1 - 2 \epsilon \frac{g^2 - g'^2}{g^2 + g'^2} + \epsilon^2}\right) \quad (A11)
\]
\[
M_{\chi^2} = M_{\chi^2} = \frac{\alpha' f^2}{8} \left(\frac{\alpha}{\alpha'} g^2 + g'^2 + \epsilon (g^2 + g'^2)\right) \pm \sqrt{\left(\frac{\alpha}{\alpha'} g^2 + g'^2\right)^2 - 2 \epsilon \left(\frac{\alpha}{\alpha'} g^2 - g'^2\right) \left(g^2 + g'^2\right) + \epsilon^2 \left(g^2 + g'^2\right)^2} \quad (A12)
\]
This formula is the same as (A11), with the substitutions \(g^2 \rightarrow \alpha'^2, g'^2 \rightarrow \alpha g'^2\), and \(\epsilon \rightarrow \alpha' \epsilon\). In general, to lowest order in \(\epsilon\), the masses of the neutral longitudinal scalars and the neutral ghosts are
\[
M_G = M_{\chi^2} \simeq \frac{f^2}{4} \left(\alpha g^2 + \alpha' g'^2\right) \left(1 + \epsilon \frac{g^2 (g^2 + g'^2)}{\alpha g^2 + g'^2}\right) \quad (A13)
\]
\[
M_s = M_{\chi^2} \simeq \epsilon \frac{\alpha f^2 g^2}{4} \frac{g^2 + g'^2}{\alpha g^2 + g'^2} \quad (A14)
\]
These expressions simplify to those in the main text by setting \(\alpha' = \alpha\).

5. Interaction lagrangian

We turn now to the interaction lagrangian \(\mathcal{L}_{\text{int}}\), which added to the quadratic lagrangian \(\mathcal{A}6\) makes up the full physical gauge-fixed (but still matter-free) lagrangian useful for quantum computations. We quote it in terms of the original variables: these fields do not have a well-defined propagator!

\[
\mathcal{L}_{\text{int}} = -ig \left[\partial_{\mu} W^+_{\nu} (W^{\mu \nu} - W^{\mu \lambda} W^{\nu \lambda}) + \text{cyclic}\right]
+ \frac{1}{2} \left(g W^3_{\nu} - g' V_\nu\right) (\phi_3 \partial_{\mu} H - H \partial_{\mu} \phi_3)
+ \frac{g}{2} W^{-} \left[(\phi_+ \partial_{\mu} H - H \partial_{\mu} \phi_+) + i (\phi_3 \partial_{\mu} \phi_+ - \phi_+ \partial_{\mu} \phi_3)\right]
+ \frac{g}{2} W^{-} \left[(\phi_- \partial_{\mu} H - H \partial_{\mu} \phi_-) - i (\phi_3 \partial_{\mu} \phi_- - \phi_- \partial_{\mu} \phi_3)\right]
+ f H \left[\frac{g^2}{2} (W^+ W^-) + \frac{1}{4} (g W^3 - g' V)^2 - \lambda (2 \phi_+ \phi_- + H^2 + \phi_3^2)\right]
+ i g g' f \left(\phi_- - \phi_+\right) (V W^3)
+ g^2 \left[(W^3 W^+) (W^3 W^-) - (W^3)^2 (W^+ W^-) - \frac{1}{2} (W^+ W^-)^2 + \frac{1}{2} (W^+) (W^-)^2\right]
+ \frac{g^2}{2} \left(\phi_+ \phi_- + \frac{1}{2} H^2 + \frac{1}{2} \phi_3^2\right) (W^+ W^-)
\]
The general expressions are quite unwieldy, even in the simplified case. The neutral current lagrangian is

\[ L_{\text{nc}} = \sum_{\psi} \bar{\psi} (n^A_{\psi} A + n^Z_{\psi} Z) \psi \]  

where the sum runs over all the fermionic fields with non-zero isospin, \( \psi \in \{\nu_L, e_L, e_R, d_L, d_R, u_L, u_R\} \). The couplings are

\[ n^A_{\psi} = -\frac{1}{2} g' c_w + \frac{1}{2} g s_w \approx \frac{\epsilon}{2} \frac{g g'}{\sqrt{g'^2 + g^2}} \]  

\[ n^Z_{\psi} = \frac{1}{2} g' s_w + \frac{1}{2} g c_w \approx \frac{1}{2} \sqrt{g'^2 + g^2} \]  

\[ n^A_{\psi} = -\frac{1}{2} g' c_w - \frac{1}{2} g s_w \approx -\frac{g g'}{\sqrt{g'^2 + g^2}} \left( 1 + \frac{\epsilon g^2 - g'^2}{2 g^2 + g'^2} \right) \]
\[ n^A_{\epsilon R} = -g' c_{\epsilon w} \simeq -\frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 - \epsilon \frac{g^2}{g^2 + g'^2} \right) \]  
\[ n^Z_{e L} = \frac{1}{2} g' s_w - \frac{1}{2} g c_{\epsilon w} \simeq -\frac{1}{2} \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} \left( 1 - 2\epsilon \frac{g^2 g'^2}{g^4 - g'^4} \right) \]  
\[ n^Z_{\epsilon R} = g' s_w \simeq \frac{g^2}{\sqrt{g^2 + g'^2}} \left( 1 + \epsilon \frac{g^2}{g^2 + g'^2} \right) \]  
\[ n^A_{\epsilon L} = \frac{1}{6} g' c_{\epsilon w} + \frac{1}{2} g s_w \simeq \frac{2}{3} \frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 + \epsilon \frac{3g^2 - g'^2}{4 \frac{g^2}{g^2 + g'^2}} \right) \]  
\[ n^A_{\epsilon R} = \frac{2}{3} g' c_{\epsilon w} \simeq \frac{2}{3} \frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 - \epsilon \frac{g^2}{g^2 + g'^2} \right) \]  
\[ n^Z_{\epsilon L} = -\frac{1}{6} g' s_w + \frac{1}{2} g c_{\epsilon w} \simeq \frac{3}{6} \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} \left( 1 - 4\epsilon \frac{g^2 g'^2}{(g^2 + g'^2)(3g^2 - g'^2)} \right) \]  
\[ n^Z_{\epsilon R} = -\frac{2}{3} g' s_w \simeq \frac{2}{3} \frac{g^2}{\sqrt{g^2 + g'^2}} \left( 1 + \epsilon \frac{g^2}{g^2 + g'^2} \right) \]  
\[ n^A_{\epsilon L} = \frac{1}{6} g' c_{\epsilon w} - \frac{1}{2} g s_w \simeq -\frac{1}{3} \frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 + \epsilon \frac{3g^2 - g'^2}{4 \frac{g^2}{g^2 + g'^2}} \right) \]  
\[ n^A_{\epsilon R} = -\frac{1}{3} g' c_{\epsilon w} \simeq -\frac{1}{3} \frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 - \epsilon \frac{g^2}{g^2 + g'^2} \right) \]  
\[ n^Z_{\epsilon L} = -\frac{1}{6} g' s_w - \frac{1}{2} g c_{\epsilon w} \simeq -\frac{1}{6} \frac{3g^2 + g'^2}{\sqrt{g^2 + g'^2}} \left( 1 - 2\epsilon \frac{g^2 g'^2}{(g^2 + g'^2)(3g^2 + g'^2)} \right) \]  
\[ n^Z_{\epsilon R} = \frac{1}{3} g' s_w \simeq \frac{1}{3} \frac{g^2}{\sqrt{g^2 + g'^2}} \left( 1 + \epsilon \frac{g^2}{g^2 + g'^2} \right) \]  

These neutral currents can be rewritten in Dirac spinor notation as follows:

\[ \mathcal{L}_{nc} = \sum_{\psi} \left\{ \bar{\psi} A(v^A_{\psi} + a^A_{\psi} \gamma_5) \psi + \bar{\psi} Z(v^Z_{\psi} + a^Z_{\psi} \gamma_5) \psi \right\} \]  

where the sum runs over \( \psi \in \{\nu, e, u, d\} \), and we recall that the left–handed projector is \((1 + \gamma_5)/2\). The various couplings are the following:

\[ v^A_{\psi} = a^A_{\psi} = a^A_{\psi} = -a^A_{\psi} = -a^A_{\psi} = \frac{1}{4} (g' s_w - g' c_{\epsilon w}) \simeq \frac{\epsilon}{4} \frac{gg'}{\sqrt{g^2 + g'^2}} \]  
\[ v^Z_{\psi} = a^Z_{\psi} = a^Z_{\psi} = -a^Z_{\psi} = -a^Z_{\psi} = \frac{1}{2} (g c_{\epsilon w} + g' s_w) \simeq \frac{1}{2} \frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} \]  
\[ v^A_{u} = -\frac{3}{4} g s_w - \frac{3}{4} g' c_{\epsilon w} \simeq -\frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 + \frac{\epsilon g^2 - 3g'^2}{4 \frac{g^2}{g^2 + g'^2}} \right) \]  
\[ v^A_{d} = -\frac{3}{4} g s_w - \frac{3}{4} g' c_{\epsilon w} \simeq -\frac{gg'}{\sqrt{g^2 + g'^2}} \left( 1 + \frac{3g^2 - g'^2}{8 \frac{g^2}{g^2 + g'^2}} \right) \]  
\[ v^A_{\epsilon} = -\frac{1}{4} g c_{\epsilon w} - \frac{3}{4} g' s_w \simeq \frac{gg'}{4 \sqrt{g^2 + g'^2}} \left( -g^2 + 3g'^2 + 4\epsilon \frac{g'^2}{g^2 + g'^2} \right) \]  
\[ v^Z_{\epsilon} = -\frac{1}{4} g c_{\epsilon w} - \frac{3}{4} g' s_w \simeq \frac{gg'}{4 \sqrt{g^2 + g'^2}} \left( -g^2 + 3g'^2 + 4\epsilon \frac{g'^2}{g^2 + g'^2} \right) \]  
\[ v^A_{\nu} = \frac{1}{4} (g' s_w - g' c_{\epsilon w}) \simeq \frac{1}{12} \frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} \left( 3g^2 - 5g'^2 - 8\epsilon \frac{g^2 g'^2}{g^2 + g'^2} \right) \]  
\[ v^Z_{\nu} = -\frac{1}{4} g s_w + \frac{1}{12} g' \sin \theta_w \simeq \frac{1}{12} \frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} \left( -3g^2 + g'^2 + 4\epsilon \frac{g^2 g'^2}{g^2 + g'^2} \right) \]  
\[ v^A_{d} = -\frac{1}{4} (g' s_w - g' c_{\epsilon w}) \simeq -\frac{1}{12} \frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} \left( 3g^2 - 5g'^2 - 8\epsilon \frac{g^2 g'^2}{g^2 + g'^2} \right) \]  
\[ v^Z_{d} = -\frac{1}{4} g s_w + \frac{1}{12} g' \sin \theta_w \simeq -\frac{1}{12} \frac{g^2 + g'^2}{\sqrt{g^2 + g'^2}} \left( -3g^2 + g'^2 + 4\epsilon \frac{g^2 g'^2}{g^2 + g'^2} \right) \]
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