COUNTING IDEALS IN POLYNOMIAL RINGS

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Abstract. We investigate properties of zeta functions of polynomial rings and their quotients, generalizing and extending some classical results about Dedekind zeta functions of number fields. By an application of Delange’s version of the Ikehara Tauberian Theorem, we are then able to determine the asymptotic order of the ideal counting function in such rings. As a result, we produce counting estimates on ideal lattices of bounded determinant coming from fixed number fields, as well as density estimates for any ideal lattices among all sublattices of \( \mathbb{Z}^d \). We conclude with some more general speculations and open questions.

1. Introduction

A classical arithmetic problem in the theory of finitely generated groups and rings is the study of the asymptotic order of growth of the number of subgroups of bounded index. A common approach to this problem involves studying the analytic properties of a corresponding zeta function (a Dirichlet series generating function) and then applying a Tauberian theorem to deduce information about the number in question, represented by the coefficients of this zeta function. This research direction received a great deal of attention over the years as can be seen from \([11], [8], [6], [7], [3], [17]\) and the references within. In the recent years, a similar approach has also been applied to the more geometric setting of counting sublattices in lattices, e.g. \([15], [10], [9], [1], [14]\).

In this note, we consider some special cases of the following general setting. Let \( R \) be a commutative ring with identity such that for every natural number \( n \) the set of ideals in \( R \) of index \( n \) is a finite number, call this number \( a_n(R) \). One is often interested in the asymptotic behaviour of the sequence \( a_n(R) \) or the summatory sequence \( A_N(R) := \sum_{n \leq N} a_n(R) \). To that end one can introduce the zeta function

\[
\zeta(R, s) := \sum_n \frac{a_n(R)}{n^s}
\]

and calculate its abscissa of convergence, so that one might apply a Tauberian theorem.

We will make use of the following well-known result, which is a consequence of Delange’s extended version of Ikehara’s Tauberian Theorem (\([5]\)):

Theorem 1.1. Let \( (a_n)_{n \geq 1} \) be any sequence of non-negative real numbers and

\[
Z(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.
\]

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Assume that $Z(s)$ has abscissa of convergence $\sigma > 0$ and admits a meromorphic extension to some neighborhood of the line $\Re(s) = \sigma$ with a pole of order $w$ at $s = \sigma$ and no other singularity. Then

$$\sum_{n=1}^{N} a_n \sim c \cdot N^\sigma \cdot (\log N)^{w-1}$$

holds for $c = \frac{\text{Res}(Z, \sigma)}{\sigma \cdot \Gamma(w)}$, where $\text{Res}(Z, \sigma)$ stands for the residue of $Z(s)$ at $s = \sigma$ and $\Gamma(w)$ is the value of gamma function at $w$.

The classical situation is that of rings of integers in number fields, where the zeta function $\zeta(R, s)$ is Dedekind’s zeta function which converges for $\Re(s) > 1$ and is meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$. In this situation $A_N(R) \sim c \cdot N$, the constant being the residue of $\zeta(R, s)$ at $s = 1$.

One of our general results is the calculation of the zeta-function for polynomial rings:

**Theorem 1.2.** Let $K$ be a number field with ring of integers $\mathcal{O}_K$. Then we have

$$\zeta(\mathcal{O}_K[X], s) = \prod_{d=1}^{\infty} \zeta_K(d(s - 1)),$$

where $\zeta_K := \zeta(\mathcal{O}_K, \cdot)$ is the Dedekind zeta-function of the number field $K$.

In particular, this function has abscissa of convergence $\sigma = 2$, meromorphic extension to $\Re(s) > 1$ and simple poles at $s = \frac{d+1}{d}$, $d \in \mathbb{N}$. As the poles accumulate towards 1, $\zeta(\mathcal{O}_K[X], s)$ cannot be extended meromorphically beyond $\Re(s) > 1$. The largest pole is at $s = 2$ and therefore the number of ideals in $\mathcal{O}_K[X]$ of index less than $N$ grows like a multiple of $N^2$.

We will, however, be more generally interested in ideals in quotient rings of $\mathcal{O}_K[X]$, which are related to lattices in $\mathcal{O}_K^d$ for some $d$. To that end, we will need some more machinery. Let $R$ again be any commutative ring. If $I \triangleleft R$ is an ideal of index $mn$ with coprime $m$ and $n$, then by the Chinese Remainder Theorem $R/I$ is a direct product of two rings of orders $m$ and $n$, respectively, and therefore $I$ is the intersection of two ideals of indices $m$ and $n$, respectively. These are uniquely determined by $I$, and therefore

$$a_{mn}(R) = a_m(R) \cdot a_n(R).$$

This means that the sequence $(a_n(R))_n$ is multiplicative, and so $\zeta(R, s)$ formally has a decomposition as an Euler product:

$$\zeta(R, s) = \prod_{p \in \mathcal{P}} E_p(R, s), \quad E_p(R, s) := \sum_{k=0}^{\infty} \frac{a_{p^k}(R)}{p^{ks}}.$$

If $\hat{R} \subseteq R$ is a subring of finite index, then for almost all prime numbers the Euler-factors $E_p(R, s)$ and $E_p(\hat{R}, s)$ coincide, and therefore the zeta functions $\zeta(R, s)$ and $\zeta(\hat{R}, s)$ have the same convergence behaviour, if these finitely many Euler factors do not behave very abnormally. We will have to control this behaviour, when it comes to applications of this observation.

This paper is organized as follows. In Section 2, we describe the construction of lattices from ideals in rings of integers of number fields and quotient polynomial rings via the coefficient embedding. We then use the Dedekind zeta function to
obtain counting and density estimates on the numbers of ideal lattices coming from a fixed ring of algebraic integers, improving on previous results [2] in some cases. Our main result in this section (Theorem 2.3) is an asymptotic estimate on the number of ideals in the quotient polynomial ring \( \mathbb{Z}[X]/(f) \), where \( f(X) \in \mathbb{Z}[X] \) is a monic separable polynomial. We also briefly comment on the situation when \( f(X) \) is not separable. In Section 3 we discuss the question of which sublattices of \( \mathbb{Z}^d \) arise as ideal lattices from quotient polynomial rings \( \mathbb{Z}[X]/(f) \) for some monic polynomial \( f \) of degree \( d \). Specifically, we investigate the analytic properties of the corresponding zeta function (Theorem 3.2) and, as a consequence of this theorem, show that the number of such ideal sublattices of index \( \leq N \) grows asymptotically like \( O(N^2) \) (Corollary 3.3). Finally in Section 4, we prove Theorem 1.2 and use it to deduce that the proportion of ideal lattices of index \( \leq N \) among all sublattices of \( \mathbb{Z}^d \) for \( d \geq 3 \) tends to 0 as \( N \to \infty \) (Corollary 4.2).

Remark 1.1. After finishing our calculations we became aware of the so-called Kähler zeta-functions, in particular the results of Lustig in [12]. With some additional work, Lustig’s results can be used to prove our Theorem 1.2. His approach however is purely local, while we employ more global methods. Furthermore, our result is more specific and neither our Theorem 3.2 nor Remark 4.1 (b) seem to have a parallel there.

2. Ideal lattices from fixed rings

In this section, we motivate our subsequent questions by looking at rings of integers in number fields.

Let \( K \) be a number field of degree \( d > 1 \) with \( r_1 \) real and \( r_2 \) pairs of complex conjugate embeddings (so \( d = r_1 + 2r_2 \)), and write \( \mathcal{O}_K \) for its ring of integers. There exists \( \theta \in \mathcal{O}_K \) such that \( K = \mathbb{Q}(\theta) \). Fix this \( \theta \), then for each \( \alpha \in K \) there exist \( a_0, \ldots, a_{d-1} \in \mathbb{Q} \) such that

\[
\alpha = \sum_{n=0}^{d-1} a_n \theta^n.
\]

Notice that \( \mathbb{Z}[\theta] \) is a finite-index subring of \( \mathcal{O}_K \), possibly proper. Define an embedding \( \rho_K : K \to \mathbb{Q}^d \) by

\[
\rho_K(\alpha) = (a_0, \ldots, a_{d-1}),
\]

then for every nonzero ideal \( I \subseteq \mathcal{O}_K \) the image \( \rho_K(I) \) is a full-rank lattice in \( \mathbb{R}^d \), and for every ideal \( I \subseteq \mathbb{Z}[\theta] \) the image \( \rho_K(I) \) is a finite index sublattice of \( \mathbb{Z}^d \). Furthermore, if some finite index sublattice \( \Lambda \subseteq \mathbb{Z}^d \) is equal to \( \rho_K(I) \) for some ideal \( I \subseteq \mathcal{O}_K \), we will say that \( \Lambda \) is an \( \mathcal{O}_K \)-ideal lattice, or just an ideal lattice when the choice of \( K \) and \( \theta \) is fixed. The first observation (see equation (1) of [2]) is that for each ideal \( I \subseteq \mathcal{O}_K \), its norm is given by

\[
\nu(I) = D_K^{-1} \det(\rho_K(I)),
\]

where \( D_K := \det(\rho_K(\mathcal{O}_K)) \).

For \( T \in \mathbb{R}_{>0} \), define

\[
\mathcal{N}_K(T) = \left| \{ \Lambda \subseteq \mathbb{Z}^d : \Lambda = \rho_K(I) \text{ for some ideal } I \subseteq \mathcal{O}_K, \ det \Lambda \leq T \} \right| \leq \left| \{ I \subseteq \mathcal{O}_K : \nu(I) \leq TD_K^{-1} \} \right|.
\]

(1)
An upper bound on $N_K(T)$ has been obtained in Theorem 2 of \cite{2}, which was then used in Corollary 1 of \cite{2} to establish a density estimate on ideal lattices from $O_K$ among all sublattices of $\mathbb{Z}^d$. In the special case when $O_K = \mathbb{Z}[\theta]$, one can easily use the standard analytic method to prove an asymptotic formula for $N_K(T)$ as $T \to \infty$ and also deduce a more precise density estimate.

**Lemma 2.1.** Suppose that $O_K = \mathbb{Z}[\theta]$. As $T \to \infty$,

\begin{equation}
N_K(T) \sim \frac{2^{r_1 + r_2} \pi^{r_2} h_K R_K}{\omega_K \sqrt{|\Delta_K|}} T,
\end{equation}

where $h_K$ is the class number, $R_K$ the regulator, $\Delta_K$ the discriminant, and $\omega_K$ the number of roots of unity of the number field $K$.

**Proof.** Note that $D_K = \mathbb{Z}[\theta]$. Furthermore, every ideal lattice from $O_K$ is a sublattice of $\mathbb{Z}^d$, and so there is equality in the inequality of (1). Let $s \in \mathbb{C}$ and define the Dedekind zeta-function of $K$ by

$$\zeta_K(s) = \sum_{I \subseteq O_K} \nu(I)^{-s} = \sum_{n=1}^{\infty} a_n n^{-s},$$

where $a_n$ is the number of ideals of norm $n$ in $O_K$, and so, by \cite{1},

$$N_K(T) = \sum_{n=1}^{T} a_n.$$

It is well-known that $\zeta_K(s)$ is analytic in the half-plane $\Re(s) > 1$ and has only a simple pole at $s = 1$ with residue given by the analytic class number formula:

$$\lim_{s \to 1} (s-1)^{-1} \zeta_K(s) = \frac{2^{r_1 + r_2} \pi^{r_2} h_K R_K}{\omega_K \sqrt{|\Delta_K|}}.$$

Combining this formula with Theorem 1.1 yields the result. \hfill \square

**Corollary 2.2.** Define

$$M(T) = \left| \{ \Lambda \subseteq \mathbb{Z}^d : \det \Lambda \leq T \} \right|.$$

Suppose that $O_K = \mathbb{Z}[\theta]$. As $T \to \infty$,

$$\frac{N_K(T)}{M(T)} \sim \frac{2^{r_1 + r_2} \pi^{r_2} h_K R_K d}{\omega_K \sqrt{|\Delta_K|} \prod_{n=2}^{d} \zeta(n)} T^{1-d}.$$

**Proof.** The zeta-function of all finite index sublattices of $\mathbb{Z}^d$ is

$$\zeta_{\mathbb{Z}^d}(s) = \sum_{\Lambda \subseteq \mathbb{Z}^d} (\det \Lambda)^{-s}.$$

It is a well-known fact (see, for instance p. 793 of \cite{1}) that

$$\zeta_{\mathbb{Z}^d}(s) = \zeta(s) \zeta(s-1) \cdots \zeta(s-d+1),$$

where $\zeta(s)$ is the Riemann zeta-function. Hence $\zeta_{\mathbb{Z}^d}(s)$ is analytic in the half-plane $\Re(s) > d$ and has a simple pole at $s = d$ with the residue

$$\zeta(d) \zeta(d-1) \cdots \zeta(2).$$

Combining this formula with Theorem 1.1 implies that

$$M(T) \sim \prod_{n=2}^{d} \zeta(n) T^d.$$. 
This estimate together with Lemma 2.1 yields the result. □

In the special case when \( \mathcal{O}_K = \mathbb{Z}[\theta] \) there is also another way to look at ideal lattices from \( \mathcal{O}_K \). Let \( f(X) \) be the minimal polynomial of \( \theta \), which is a monic irreducible polynomial in \( \mathbb{Z}[X] \) of degree \( d \). Then \( \mathcal{O}_K \cong \mathbb{Z}_f := \mathbb{Z}[X]/(f(X)) \) and ideal lattices from \( \mathcal{O}_K \) correspond to ideal lattices in \( \mathbb{Z}_f \), which is a special case of the ideal lattice construction of [13]. Define a \( \mathbb{Z} \)-module isomorphism \( \rho_f : \mathbb{Z}_f \to \mathbb{Z}^d \), given by

\[
\rho_f \left( \sum_{n=0}^{d-1} a_n x^n \right) = (a_0, \ldots, a_{d-1}).
\]

Let \( I \subseteq \mathbb{Z}_f \) be an ideal, then it is known that \( \rho(I) \) is a finite index sublattice of \( \rho(\mathbb{Z}_f) = \mathbb{Z}^d \) (Lemma 3 of [13]). A counting estimate on such ideal sublattices is then given by our Lemma 2.1 above.

This leads us to the more general question: if \( f(X) \in \mathbb{Z}[X] \) is a monic polynomial of degree \( d \), what is the asymptotic behaviour of the number of ideals of index at most \( N \) in \( \mathbb{Z}_f := \mathbb{Z}[X]/(f) \) as \( N \) goes to infinity? We again define

\[
\zeta(\mathbb{Z}_f, s) = \sum_{I \subseteq \mathbb{Z}_f} \nu(I)^{-s},
\]

where the sum goes over finite index ideals and \( \nu(I) := |I/\mathbb{Z}_f| \).

**Theorem 2.3.** Let \( f \in \mathbb{Z}[X] \) be monic and assume that \( f = g_1 \cdots g_k \) with \( g_1, \ldots, g_k \) irreducible, monic and pairwise distinct. Then the zeta-function \( \zeta_f(s) \) converges for \( \Re(s) > 1 \), has a meromorphic extension to the halfplane

\[
\{ s \in \mathbb{C} \mid \Re(s) > 0 \},
\]

and has a pole of order \( k \) at \( s = 1 \). In particular, \( A_N(\mathbb{Z}_f) \sim cN(\log N)^{k-1} \) for some constant \( c \).

**Proof.** For \( 1 \leq i \leq k \), let \( \theta_i \in \mathbb{C} \) be a root of \( g_i \) and \( \mathcal{O}_i \subseteq \mathbb{C} \) be the integral closure of \( \mathbb{Z}[\theta_i] \). We first observe that by a variant of the Chinese Remainder Theorem, the ring \( \mathbb{Z}_f \) is isomorphic to a subring of finite index in \( S := \mathcal{O}_1 \times \cdots \times \mathcal{O}_k \). If \( d \) is the degree of \( f \), then both rings have an additive group isomorphic to \( \mathbb{Z}^d \). Let \( M \) be the index of \( \mathbb{Z}_f \) in \( S \). We denote by \( a_n(\mathbb{Z}_f) \) the number of ideals of index \( n \) in \( \mathbb{Z}_f \) and by \( a_n(S) \) the number of ideals of index \( n \) in \( S \) respectively.

Whenever \( I \subseteq S \) is an ideal of index \( n \) in \( S \), then \( MI \subseteq \mathbb{Z}_f \) is an ideal of index \( M^{d-1} \cdot n \) in \( \mathbb{Z}_f \). As multiplication by \( M \) is injective on \( S \), this multiplication map from ideals in \( S \) to ideals in \( \mathbb{Z}_f \) is injective, and for every natural number \( N \)

\[
\sum_{n=1}^{NM^{d-1}-1} a_n(\mathbb{Z}_f) \geq \sum_{n=1}^{N} a_n(S).
\]

Therefore the abscissa of convergence of \( \zeta(\mathbb{Z}_f, s) \) is at least as large as that of \( \zeta(S, s) \). This only depends on the fact, that the rings have an additive group isomorphic to \( \mathbb{Z}^d \) and therefore holds for all subrings of finite index in \( S \). Furthermore, as remarked in Section 1, the Euler-factors of both zeta-functions do coincide for all prime numbers not dividing the index.

We now consider Euler-factors of the ring \( \mathbb{Z} + MS \), and show that they have the same convergence behaviour as Euler-factors for \( S \). Since

\[
\mathbb{Z} + MS \subseteq \mathbb{Z}_f \subseteq S,
\]
it would follow that Euler-factors of $Z_f$ must also have the same convergence behaviour. Writing $M = p_1 \cdot p_2 \cdots p_l$ as a product of prime numbers, we see that

$$S \supseteq \mathbb{Z} + p_1 S \supseteq \mathbb{Z} + p_2 (\mathbb{Z} + p_1 S) = \mathbb{Z} + p_1 p_2 S \supseteq \cdots ,$$

and this shows by the same argument as above that it suffices to treat the case $M = p$ and the Euler-factor at the prime $p$.

Therefore let $S$ be a ring additively isomorphic to $\mathbb{Z}^d$, $p$ a prime number and $R = \mathbb{Z} + pS$. For every ideal $I$ of index $p^k$ in $R$, $SI$ is an ideal of $p$-power index in $S$. As $pS \subseteq R$, we see that $pSI \subseteq I$ and that the index of $SI$ in $S$ is at least $p^{-d} p^k$. For two ideals $I_1, I_2$ in $R$ the equality $SI_1 = SI_2$ implies $pI_2 \subseteq pSI_2 = pSI_1 \subseteq I_1$ and by symmetry $pI_1 \subseteq I_2$. Hence $p^2 I_1 \subseteq pI_2 \subseteq I_1$. The cardinality of the fibre of the map $I \mapsto SI$ therefore is at most the number of subgroups in $I_1/p^2 I_1 \cong \mathbb{Z}^d/p^2 \mathbb{Z}^d$, which is finite and independent of $I_1$. Call this number $\gamma$.

Then

$$\sum_{k=0}^{K} a_{p^k} (S) \geq \frac{1}{\gamma} \sum_{k=0}^{K+d} a_{p^k} (\mathbb{Z} + pS),$$

thereby showing that the Euler-factor in $\zeta(\mathbb{Z} + pS, s)$ converges whenever that of $\zeta(S, s)$ does. Coming back to our statement, the Euler-factors in the zeta-function for $\mathcal{O}_1 \times \cdots \times \mathcal{O}_k$ are just the products of the Euler-factors of $\zeta(\mathcal{O}_1, s), \ldots, \zeta(\mathcal{O}_k, s)$, which all converge for $\Re(s) > 0$. This completes the proof.

**Remark 2.1.** What if $f(X)$ is not separable? Here are some speculations. Going through the proof of Theorem 2.3, it is sufficient to understand the zeta-functions for rings of the type $\mathbb{Z}[X]/(f^n)$, where $f$ is monic and irreducible. We calculated this for $f(X) = X, e = 2$ and obtained

$$\zeta(\mathbb{Z}[X]/(X^2), s) = \zeta(s) \cdot \zeta(2s - 1),$$

which again has a double pole at $s = 1$. We expect

$$\zeta(\mathbb{Z}[X]/(X^n), s) = \zeta(s) \cdot \zeta(2s - 1) \cdot \zeta(3s - 2) \cdots \zeta(es - (e - 1)),$$

which has a pole of order $e$ at $s = 1$. More generally, we would expect that the order of the pole of $\zeta(\mathbb{Z}[X]/(f), s)$ at $s = 1$ is the sum of the multiplicities of the irreducible monic factors of $f$.

### 3. Ideal Lattices in Dimension $d$

In this section, we aim to understand which subgroups of $\mathbb{Z}^d$ arise as images of finite index ideals in $Z_f := \mathbb{Z}[X]/(f)$ under $\rho_f$ for some monic polynomial $f(X)$ of degree $d$. Again, we fix the group isomorphism $\rho_f : Z_f \to \mathbb{Z}^d$, given by

$$\rho_f \left( \sum_{n=0}^{d-1} a_n X^n \right) = (a_0, \ldots, a_{d-1}).$$

Like every finite index subgroup of $\mathbb{Z}^d$, such a subgroup is generated by the columns of an integral full-rank upper triangular matrix

$$A := \begin{pmatrix}
    a_{0,0} & a_{0,1} & \cdots & a_{0,d-1} \\
    0 & a_{1,1} & \cdots & a_{1,d-1} \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & a_{d-1,d-1}
\end{pmatrix},$$
the \( j \)-th column corresponding to a generator of degree \( j \) in the ideal.

Multiplication by \( X \) is an endomorphism of the ideal. This shows that for each \( 0 \leq j \leq d - 2 \) the \( j \)-th column of \( A \) shifted down by one belongs to the subgroup generated by the 0-th to \((j + 1)\)-th column. For \( j = d - 1 \) it shows, that there exists some monic polynomial \( f \) of degree \( d \) such that the \((d - 1)\)-th column shifted by one (in \( \mathbb{Z}^{d+1} \)) belongs to the subgroup generated by the columns of \( A \) and the column containing the coefficients of \( f \). This last demand is always satisfied, and we therefore only have to deal with the first set of conditions!

We call such a matrix an \textit{idealizing matrix}, and two idealizing matrices are called \textit{indifferent}, whenever they correspond to the same ideal. One can check by induction on \( d \) that an idealizing upper triangular matrix satisfies:

\[
\forall 1 \leq j \leq d - 1 : a_{j,j} \text{ divides } a_{i,k} \text{ whenever } 0 \leq i, k \leq j.
\]

Details of this calculation can be found in [10]. The idealizing matrices for which in every \( i \)-th row the entries are between 0 and \( a_{i,i} - 1 \) give a set of representatives for the indifferency-classes. We call these matrices \textit{reduced} idealizing matrices, and now count them by induction in the following way.

**Lemma 3.1.** Let \( A \in \mathbb{Z}^{d \times d} \) be a reduced idealizing matrix. Then the number of \( \beta \in \mathbb{Z}^d \) such that

\[
\tilde{A} := \begin{pmatrix} A & \beta \\ 0 & 1 \end{pmatrix} \in \mathbb{Z}^{d+1 \times d+1}
\]

is a reduced idealizing matrix is \( a_d^{d-1,d-1} \).

**Proof.** As the first \( d - 1 \) columns of \( \tilde{A} \) already satisfy the conditions imposed on idealizing matrices, we only have to care about the condition that the shifted \( d \)-th column is contained in the \( \mathbb{Z} \)-span of the columns of \( (\delta_\alpha^i) \) with \( \delta \in \mathbb{Z}^{d-1}, a = a_{d-1,d-1} \in \mathbb{Z} \).

Then our condition is

\[
\begin{pmatrix} 0 \\ \delta \end{pmatrix} - a\beta \in AZ^d.
\]

Note that by \([10] a \) divides every entry of \( A \) and that therefore we get

\[
\beta \in \frac{1}{a} \left( \begin{pmatrix} 0 \\ \delta \end{pmatrix} + AZ^d \right).
\]

Two different values of \( \beta \) define the same indifferency class if and only if their difference is in \( AZ^d \). As the index of \( AZ^d \) in \( \frac{1}{a}AZ^d \) is \( a^d \), we get exactly \( a^d \) possible indifferency classes of matrices \( \tilde{A} \). \( \square \)

**Theorem 3.2.** Let \( d \in \mathbb{N} \) be some natural number, and denote by \( c^{(d)}_n \) the number of subgroups in \( \mathbb{Z}^d \) of index \( n \) which are in the image of \( \rho_f \) for some monic polynomial \( f \in \mathbb{Z}[X] \) of degree \( d \). Then

\[
\zeta^{(d)}(s) := \sum_{n=0}^{\infty} \frac{c^{(d)}_n}{n^s} = \prod_{i=1}^{d-1} \zeta(i(s - 1)) \cdot \zeta(ds),
\]

where \( \zeta = \zeta^{(1)} \) is the Riemann zeta-function.

**Proof.** The proof is by induction on \( d \). We have to count indifferency classes of matrices with given determinant \( n \). The diagonal entries of such a matrix are numbers...
$a_0, \ldots, a_{d-1}$, where each $a_j$ divides the $a_j$ with $j < i$. It is more convenient to write these numbers as products

$$b_1 b_2 \cdots b_d, b_2 \cdots b_d, \ldots, b_{d-1} b_d, b_d,$$

where $b_1, \ldots, b_d$ are any natural numbers.

For $d = 1$, we just have one indifferency class of matrices of determinant $n$ for every $n$, hence

$$\zeta^{(1)}(s) = \zeta(s).$$

For $d = 2$ and every value of $b_1$, by Lemma 3.1, we obtain exactly $b_1$ classes of matrices

$$\begin{pmatrix} b_1 & \beta \\ 0 & 1 \end{pmatrix},$$

and every matrix with arbitrary $b_1, b_2$ is $b_2$ times one of these, therefore we have $b_1$ matrices with fixed $b_1$. This gives

$$\zeta^{(2)}(s) = \sum_{b_1} \frac{b_1}{b_1^2} = \sum_{b_1} \frac{1}{b_1^{s-1}} = \zeta(s-1) \times \zeta(2s).$$

Going on like this, using Lemma 3.1 repeatedly, we find that for fixed $b_1, \ldots, b_d$ we have $b_1 \cdot b_2^2 \cdot b_3^3 \cdot \cdots \cdot b_d^{d-1}$ indifferency classes of idealizing matrices with diagonal

$$(b_1 \cdots b_d, b_2 \cdots b_d, \ldots, b_{d-1} b_d, b_d),$$

and this leads to

$$\zeta^{(d)}(s) = \sum_{b_1, \ldots, b_d=1} \frac{b_1 \cdot b_2^2 \cdot b_3^3 \cdot \cdots \cdot b_d^{d-1}}{(b_1 \cdot b_2^2 \cdot b_3^3 \cdot \cdots \cdot b_d^{d-1})^s} = \prod_{i=1}^{d-1} \zeta(i(s-1)) \times \zeta(ds).$$

\[\square\]

**Corollary 3.3.** For $d \geq 2$, the number $A_N$ of subgroups in $\mathbb{Z}^d$ of index less than or equal to $N$, which are in the image of $\rho_f$ for some monic polynomial $f \in \mathbb{Z}[X]$ of degree $d$, grows asymptotically as $cN^{2d}$, where

$$c = \frac{\text{Res}(\zeta^{(d)}, 2)}{2} = \left[ \prod_{i=2}^{d-1} \zeta(i) \right] \cdot \zeta(2d).$$

**Proof.** Apply the Tauberian Theorem 1.1 to $\zeta^{(d)}$. The abscissa of convergence is 2 and $\Gamma(2) = 1$; use the fact that the Riemann zeta-function $\zeta(s)$ has a simple pole at $s = 1$ and no other poles in $\mathbb{C}$. \[\square\]

### 4. Zeta-functions of Polynomial Rings

In this section we prove Theorem 1.2. We start with its version for $K = \mathbb{Q}$, and then explain how to generalize it to any number field.

**Proposition 4.1.** We have

$$\zeta(\mathbb{Z}[X], s) = \prod_{i=1}^{\infty} \zeta(i(s-1)).$$
Proof. Every ideal of finite index in \( \mathbb{Z}[X] \) contains some monic polynomial, and therefore \( a_n(R) = \lim_{d \to \infty} a^d_n \). The discussion in the previous section shows that for a fixed \( n \) the sequence \( \langle a^d_n \rangle \) becomes constant for \( d > n \) (very rough estimate). This leads to

\[
\zeta(\mathbb{Z}[X], s) = \lim_{d \to \infty} \zeta^{(d)}(s),
\]

which gives the desired result. The infinite product on the right hand side in fact converges absolutely for \( \Re(s) > 2 \).

Corollary 4.2. For \( d \geq 3 \), the proportion of the ideal lattices of index \( \leq N \) in \( \mathbb{Z}^d \) among all subgroups of index \( \leq N \) tends to zero as \( N \) goes to infinity.

Proof. The zeta-function \( \zeta(\mathbb{Z}[X], s) \) converges for \( \Re(s) > 2 \) and has a simple pole at \( s = 2 \). Theorem 1.1 says that the number of ideal lattices has quadratic growth, while that for all subgroups has growth of order \( N^d \).

Proof of Theorem 1.2. We now explain how the calculations in the proof of the last Proposition can be extended to rings of integers in arbitrary number fields instead of \( \mathbb{Z} \).

(a) Let \( R \) be an infinite principal ideal domain such that every nonzero ideal in \( R \) has finite index. Here, the method using idealizing matrices can be extended directly. Repeating \textit{mutatis mutandis} the arguments from the last section shows the formal identity

\[
\zeta(R[X], s) = \prod_{i=1}^{\infty} \zeta(R, i(s - 1)).
\]

(b) If \( \mathcal{O}_K \) is the ring of integers in a fixed number field \( K \) of finite degree over \( \mathbb{Q} \), we use the Euler-product

\[
\zeta(\mathcal{O}_K[X], s) = \prod_{p \in \mathcal{P}} \zeta(\mathbb{Z}(p) \otimes \mathcal{O}_K, s)\zeta(\mathcal{O}_K[X], s),
\]

as the ideals of \( p \)-power index in \( \mathcal{O}_K[X] \) correspond bijectively and index preservingly to ideals of finite index in \( \mathbb{Z}(p) \otimes \mathcal{O}_K \). But this last ring is a Dedekind domain with finitely many prime ideals only (i.e. semilocal) and therefore is a principal ideal domain which satisfies the condition from part (a) of this proof. Hence

\[
\zeta(\mathcal{O}_K[X], s) = \prod_{p \in \mathcal{P}} \prod_{i=1}^{\infty} \zeta(\mathbb{Z}(p) \otimes \mathcal{O}_K, i(s - 1)).
\]

For every prime number \( p \), the factor \( \zeta(\mathbb{Z}(p) \otimes \mathcal{O}_K, i(s - 1)) \) is the product of the “true” Euler factors of \( \zeta(\mathcal{O}_K, i(s - 1)) \) for prime ideals in \( \mathcal{O}_K \) containing \( p \), and therefore, using the Euler-decomposition of the Dedekind zeta-function \( \zeta(\mathcal{O}_K, \cdot) \), we arrive at the assertion of Theorem 1.2.

Remark 4.1. We conclude this paper with some remarks.

(a) The first factor in \( \zeta(\mathcal{O}_K[X], s) \) is \( \zeta(\mathcal{O}_K, s - 1) \), which is the Hasse-Weil zeta-function of the affine line \( \mathbb{A}^1_{\mathcal{O}_K} \).
This corresponds to those ideals in \( \mathcal{O}_K[X] \) where the quotient ring has squarefree characteristic. It would be interesting to understand the zeta-function we have calculated from this point of view and perhaps describe it as that of some deformation of the affine line.

There is another way to discover \( \zeta(\mathcal{O}_K, s-1) \) as a factor in \( \zeta(\mathcal{O}_K[X], s) \), namely by counting only ideals which contain a linear monic polynomial.

(b) If \( a_n(\mathfrak{O}) \) denotes the number of isomorphism classes of abelian groups of order \( n \), then an argument based on the structure theorem for finitely generated abelian groups shows, that

\[
\sum_{n=1}^{\infty} \frac{a_n(\mathfrak{O})}{n^s} = \prod_{d=1}^{\infty} \zeta(ds) = \zeta(\mathbb{Z}[X], s+1).
\]

Namely, for every choice \( b_1, \ldots, b_d \in \mathbb{N} \) there is exactly one abelian group with elementary divisors \( b_d, b_db_{d-1}, b_db_{d-2}, \ldots, b_d, \ldots, b_1 \). This group has order \( b_1 \cdot b_2^2 \cdot \ldots \cdot b_d^d \), and therefore we get

\[
\sum_{n=1}^{\infty} \frac{a_n(\mathfrak{O})}{n^s} = \lim_{d \to \infty} \sum_{b_1, \ldots, b_d} \frac{1}{(b_1 \cdot b_2^2 \cdot \ldots \cdot b_d^d)^s} = \lim_{d \to \infty} \zeta(s) \cdot \zeta(2s) \cdots \zeta(ds).
\]

This is closely related to the Cohen-Lenstra heuristics, cf. [4], although there the product \( \prod_{i \geq 1} \zeta(s+i) \) plays a more important role. However, the residue of our \( \zeta(\mathcal{O}_K[X], s) \) at \( s = 2 \) is the number \( C_\infty \) on page 35 of loc. cit. We thank Ernst-Ulrich Gekeler for kindly guiding us towards [4].

Maybe it is possible by such means to obtain a nice interpretation of \( \zeta(\mathbb{Z}[X], s) \) as the zeta-function counting finite abelian groups endowed with some endomorphism satisfying certain properties. The number \( a_n(\mathfrak{O}) \) is the number of orbits of subgroups of index \( n \) in \( \mathbb{Z}^\infty = \bigoplus_{i \in \mathbb{N}} \mathbb{Z} \) under the action of the automorphism group of \( \mathbb{Z}^\infty \). It would also be interesting to count the number of orbits of \( \text{Aut}(\mathbb{Z}[X]) \) acting on ideals of index \( n \). We did not yet carry out the corresponding calculations.

(c) We should mention the short note [18] of Witt. His results imply that the sum formally defining the \( \zeta \)-function for the polynomial ring \( \mathbb{Z}[X,Y] \) does not converge anywhere in the complex plane.

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