Can entanglement efficiently be weakened by symmetrization?

Keiji Matsumoto
Quantum Information Science group, National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430

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Abstract

Consider a quantum system with \( m \) subsystems with \( n \) qubits each, and suppose the state of the system is living in the symmetric subspace. It is known that, in the limit of \( m \to \infty \), entanglement between any two subsystems vanishes.

In this paper we study asymptotic behavior of the entanglement, or the minimum trace distance from the totality of separable states, as \( m \) and \( n \) grows. Our conjecture is that if \( m \) is a polynomially bounded function in \( n \), then the entanglement decreases polynomially.

The motivation of this study is a study of quantum Merlin-Arthur game. If this conjecture is true, we can prove that bipartite separable certificate does not increase the computational power of the proof system protocol.

In the paper, we provide two evidences which support the conjecture. The first one shows that the entanglement is weakened polynomially fast as the number \( m \) of subsystems grows. Second evidence suggests that \( m \) may be polynomial in the number of qubits in each subsystems.

1 Introduction

In construction of a protocol for an interactive proof system or similar proof systems \[ \mathbb{QMA}(2) \], it is often a key whether one can assume the separability between some quantum systems. For example, suppose one wants to check whether a given pair of pure states are identical with each other or not. If the given pair is separable, we can test the property by the projection onto symmetric subspace, which is realized by a simple algorithm using controlled swap. Such algorithm, however, fails if the given pair is not necessarily separable.

Kobayashi et.al. \[ \mathbb{QMA}(2) \] proposed a language class \( \mathbb{QMA}(2) \), which is accepted by a quantum Merlin-Arthur (q.m.a., hereafter) game with bipartite quantum certificate. By definition, \( \mathbb{QMA}(2) \supseteq \mathbb{QMA} \). However, it is not trivial whether
QMA(2) ⊂ QMA. Obviously, if one could check the separability of the certificate, QMA(2) = QMA. However, such test is impossible, for the totality of separable state and entangled states cannot be separated by a hyperplane in the state space. (Here, note that we are given only single copy of the quantum certificate). Therefore, the relation between these computational class is not trivial.

Watrous [7] had suggested the following q.m.a. protocol which simulate q.m.a protocol with bipartite separable certificate with the one with single certificate. In that, Merlin is asked to give \( m \) copies of the bipartite separable certificate. Arthur checks the symmetry and uses the first part of the first copy and the second part of the second copy. If this protocol is valid, QMA(2) = QMA.

For this protocol to be valid the reduced state has to be almost separable. This is intuitively true, for the following reason. One party cannot have entanglement with the rest of the system more than

\[
\log \dim \mathbb{C}^{2^n} = n,
\]

which should be equally distributed to each party. Hence, intuitively, entanglement between two parties should be roughly

\[
\frac{n}{m},
\]

which is small if \( m \) is appropriately chosen polynomial function in \( n \).

The purpose of the paper is to show two theorems which supports this conjecture. The first one shows that the entanglement, or the minimum trace distance from the totality of the separable state, is decreasing polynomially fast as the number \( m \) of subsystems grows. However, in the proof, we have to set \( m \) is an exponential function in \( n \). On the other hand, our second theorem shows that \( m \) can be made polynomial for a certain maximally entangled state.

The paper is organized as follows. First, we describe the conjectures and its use in rigorous manner. Then, we proceed to prove two evidences which supports our conjectures.

## 2 Conjectures and its application

### 2.1 Conjectures

Let us restate the problem rigorously. Let \( \mathcal{H} \otimes^m \) denote a symmetric subspace, or a subspace of \( \mathcal{H} \otimes^m \) which is spanned by vectors of the form \( |\phi\rangle \otimes^m \). Let \( \mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \cdots \simeq \mathbb{C}^{2^n} \). Given \( \rho \in \mathcal{S}(\bigotimes_{i=1}^m \mathcal{H}_i) \), we denote the reduced density matrix to the composition of the first and the second system by \( \rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} \). Also, define

\[
d_{n,m} := \dim(\mathbb{C}^{2^n}) \otimes^m = \frac{(2^n + m - 1)!}{(2^n - 1)!m!}.
\]

If one of following conjectures is true, the protocol in [7], which will be fully stated in the next subsection, is valid.
**Conjecture 1** Let $m$ be a appropriately chosen polynomial function in $n$, and $\rho \in S((\mathbb{C}^2)^{\otimes m})$. Then,

$$\min_{\sigma: \text{separable}} \| \rho|_{H_1 \otimes H_2} - \sigma \|_1 < \frac{1}{q}$$

holds for a polynomial function $q$ in $n$.

**Conjecture 2** Let $m$ be a appropriately chosen polynomial function in $n$, and suppose $\rho \in S((\mathbb{C}^2)^{\otimes m})$, such that the first $(\mathbb{C}^2)^{\otimes m}$ is a subset of $\bigotimes_{i=1}^{2m-1} H_{2i-1}$, and the second one is a subset of $\bigotimes_{i=1}^{2m} H_{2i}$. Suppose also $\rho \in S(\mathbb{C}^n \otimes \mathbb{C}^n)$, where $\mathbb{C}^n \otimes \mathbb{C}^n = H_{2m-1} \otimes H_{2m}$. Then, (1) holds for a polynomial function $q$ in $n$.

Note that Conjecture 1 is a stronger assertion than Conjecture 2.

### 2.2 How to utilize these properties

The content of this section comes from [7].

Suppose that we can fabricate the projection onto symmetric subspace with small error efficiently. (An example of quantum circuit achieving this is explained later.) Using the projection measurement, we test the given state. If the given state is in the symmetric subspace, the test accepts it with high probability. Otherwise, the test still accept it, sometimes with non-negligible property. However, upon acceptance, the output state is very close to a symmetric state. Therefore, using the projection onto symmetric subspace, we can always obtain $\rho \in S((\mathbb{C}^2)^{\otimes m})$ upon acceptance, and $\rho|_{H_1 \otimes H_2}$ is almost separable, if Conjecture 1 is true.

Suppose Conjecture 2 is true (recall Conjecture 1 is a stronger assertion than this). First, we apply projection onto $(\mathbb{C}^n \otimes \mathbb{C}^n)^{\otimes m}$. Then, the resultant state, upon acceptance, is convex combination of the pure states of the form

$$\sum_n \sqrt{a_n} |\phi_n\rangle,$$

where $|\phi_n\rangle \in W_{n,\text{odd}} \otimes W_{n,\text{even}}$ and $W_{n,\text{odd}}$ and $W_{n,\text{even}}$ are the homogeneous component of the representation of $S_m$ on $\bigotimes_{i=1}^{2m-1} H_{2i-1}$ and $\bigotimes_{i=1}^{2m} H_{2i}$, respectively, corresponding to the Young index $n$. For the proof of this fact, see the proof of Lemma 1 in [4]. Note that the assertion of this lemma uses $m$-copies of the pure state is invariant by the action of $S_m$. In our case, the given state is not $m$-copies of the pure state, but yet satisfies symmetry, which is enough to lead to the same assertion. Now, we apply the projection onto $(\mathbb{C}^2)^{\otimes m}$ to $\bigotimes_{i=1}^{2m} H_{2i-1}$. Then, upon acceptance, the state collapses to $|\phi_{n'}\rangle$, where $n'$ equals $(\frac{m}{2}, 0, \cdots, 0)$ and corresponds to symmetric subspace. Therefore, due to Conjecture 2, $\rho|_{H_1 \otimes H_2}$ is almost separable.

Such procedures can be used to study a language class called QMA(2), which is accepted by a quantum Merlin-Arthur (q.m.a., in short) game with bipartite
separable quantum certificate. By definition, QMA(2) ⊃ QMA. To show QMA(2) ⊂ QMA, we construct q.m.a. protocol with (not necessarily separable) certificate which simulates given or a q.m.a. protocol with a bipartite separable certificate. Instead of single pair of certificates, Merlin provides many pair of certificate. Arthur use the protocol above to obtain a bipartite nearly separable quantum certificate.

2.3 Testing symmetry

Here, we describe how to approximately implement a projection onto symmetric subspace. Construction is very similar to the main algorithm of [6]. Suppose there is a circuit which generates

\[ \frac{1}{\sqrt{|S_m|}} \sum_{\pi \in S_m} |\pi\rangle |\psi_\pi\rangle, \]

where \( \langle \pi | \pi' \rangle = \delta_{\pi\pi'} \). Here, we also suppose that this circuit resets working space in the end.

Given an input \( |\phi\rangle \), this gate is applied to \( |0\rangle |\phi\rangle \), to obtain

\[ \frac{1}{\sqrt{|S_m|}} \sum_{\pi \in S_m} |\pi\rangle |\psi_\pi\rangle |\phi\rangle. \]

Then, controlled \( \pi \) gate is applied, letting the control being the first register, and the target being the second register:

\[ \frac{1}{\sqrt{|S_m|}} \sum_{\pi \in S_m} |\pi\rangle |\psi_\pi\rangle \pi |\phi\rangle. \]

Finally, we apply projection onto \( \frac{1}{\sqrt{|S_m|}} \sum_{\pi \in S_m} |\pi\rangle |\psi_\pi\rangle \) to the first register. This projection is implemented by running the state generation circuit backwards, and measuring output by the computational basis.

The final state is

\[ \frac{1}{\sqrt{|S_m|}} \sum_{\pi, \pi' \in S_m} \langle \pi' | \pi \rangle \langle \psi_\pi | \psi_{\pi'} \rangle \pi |\phi\rangle \]

\[ = \frac{1}{\sqrt{|S_m|}} \sum_{\pi \in S_m} \pi |\phi\rangle, \]

which is an element of the symmetric subspace. It is easy to see that the final state equals the input \( |\phi\rangle \) if and only if \( |\phi\rangle \) is an element of a symmetric subspace. Therefore, our circuit satisfies the requirement.

Now, it still remains to show the construction of circuits which outputs \( \frac{1}{\sqrt{|S_m|}} \sum_{\pi \in S_m} |\pi\rangle |\psi_\pi\rangle \). We are done if there is a classical algorithm which generates each \( \pi \) uniformly randomly. This is easy, for each element of \( S_n \) corresponds to an ordering of \( 1, 2, \ldots, n \).
3 Evidences for the conjectures

3.1 Exponentially many sites

In this section, we prove that the minimum trace distance from the totality of separable states decreases polynomially fast as \( m \) increases.

For that, we use entanglement of formation (EoF, in short), denoted by \( E_f(\rho) \). This is an important measure of entanglement, first proposed in [1]. Given state \( \rho \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \), it is defined by

\[
E_f(\rho) := \inf_{\{p_i, |\phi_i\rangle\}} \sum_i p_i S(|\phi_i\rangle\langle\phi_i|_{\mathcal{H}_1}),
\]

where \( S(\rho) := -\text{Tr} \rho \log \rho \) and inf is taken over all the pure state ensemble \( \{p_i, |\phi_i\rangle\} \) with \( \sum_i p_i |\phi_i\rangle\langle\phi_i| = \rho \). The following lemma is of interest in its own right, due to importance of EoF.

**Lemma 3** Let \( m \) be a appropriately chosen exponential function in \( n \), and \( \rho \in \mathcal{S}((\mathbb{C}^2)^{\otimes m}) \). Then, there is an exponential function \( q \) which satisfies

\[
E_f(\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2}) \leq \frac{1}{q}
\]

**Proof.** It suffices to show the assertion when \( \rho \) is pure. Let \( \mu \) be a Haar measure in \( SU(\mathbb{C}^2) \) with normalization \( \int \mu(dU) = 1 \). Observe

\[
\rho|_{\mathbb{C}^2 \otimes \mathbb{C}^2} = S((\mathbb{C}^2)^{\otimes (m-2)}),
\]

\[
\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = S((\mathbb{C}^2)^{\otimes 2})
\]

and

\[
I_{(\mathbb{C}^2)^{\otimes m}} = d_{n,m} \int |0^n\rangle \langle 0^n| \otimes (\mathbb{C}^2)^{\otimes m} \mu(dU).
\]

Using these facts, we have

\[
\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = d_{n,m} \int (|0^n\rangle \langle 0^n| \otimes (\mathbb{C}^2)^{\otimes m}) \rho(U|0^n\rangle \otimes \rho(U|0^n\rangle)^{\otimes m-2} \mu(dU).
\]

\[
= d_{n,m} \int p_U \frac{(|0^n\rangle \langle 0^n| \otimes (\mathbb{C}^2)^{\otimes m}) \rho(U|0^n\rangle \otimes \rho(U|0^n\rangle)^{\otimes m-2}}{p_U} \mu(dU),
\]

where \( p_U = \text{Tr} \left((|0^n\rangle \langle 0^n| \otimes (\mathbb{C}^2)^{\otimes m}) \rho(U|0^n\rangle \otimes \rho(U|0^n\rangle)^{\otimes m-2}\right). \) Note the last end of equation gives a decomposition of \( \rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} \) into pure states. (\( (|0^n\rangle \langle 0^n| \otimes (\mathbb{C}^2)^{\otimes m}) \rho(U|0^n\rangle \otimes \rho(U|0^n\rangle)^{\otimes m-2}\) is of rank 1 because \( \rho \) is a pure state by assumption. ) Let \( a_{U,i} \geq a_{U,2} \geq \cdots a_{U,d_{n,2}} \) be a Schmidt coefficient of the state \( (|0^n\rangle \langle 0^n| \otimes (\mathbb{C}^2)^{\otimes m}) \rho(U|0^n\rangle \otimes \rho(U|0^n\rangle)^{\otimes m-2}\). Then
we have
\[
d_{n,m} \int p_U a_{U,1} \mu(dU) \\
= \max_{|\phi\rangle \in (C^{2^n})^{\otimes 2}} \frac{d_{n,m} \int p_U \langle \langle 0^n | U^\dagger \rangle \otimes (0^n | U^\dagger \rangle) \otimes \rho(U | 0^n \rangle) \otimes (0^n | U^\dagger \rangle) \otimes 2 \mu(dU) }{p_U} \\
\geq d_{n,m} \int p_U \langle \langle 0^n | U^\dagger \rangle \otimes (0^n | U^\dagger \rangle) \otimes \rho(U | 0^n \rangle) \otimes (0^n | U^\dagger \rangle) \otimes 2 \mu(dU) \\
= d_{n,m} \int \text{Tr} \rho \langle \langle 0^n | U^\dagger \rangle \otimes \rho(U | 0^n \rangle) \rangle \mu(dU) \\
= \frac{d_{n,m} \int \text{Tr} \rho \langle \langle 0^n | U^\dagger \rangle \otimes \rho(U | 0^n \rangle) \rangle \mu(dU) }{d_{n,m}} \\
= \frac{(2^n + m - 3)! (2^n)! (m - 1)!}{(2^n)! (m - 3)! (2^n + m - 1)!} \\
\geq \frac{(m - 1)(m - 2)}{(2^n + m - 3)(2^n + m - 2)}.
\]

Here, letting \( m = 2^n \),
\[
= \frac{(2^{2n} - 1)(2^{2n} - 2)}{(2^n + 2^{2n} - 3)(2^n + 2^{2n} - 2)} \\
\geq 1 - \frac{c}{2^n}.
\]

This implies the Schmidt coefficient sharply concentrates to the first element \( a_{U,i} \). Intuitively, this implies entanglement of \( \rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} \) is very small, for the state is decomposed into states which is close to a separable state in average. To prove our lemma, we have to upperbound \( E_f \) using this quantity. Observe that
\[
\sum_i (-a_{U,i} \log a_{U,i}) \\
\leq (-a_{U,1} \log a_{U,1}) \\
- \frac{\dim(C^{2^n})^{\otimes 2}}{\dim(C^{2^n})^{\otimes 2}} \log \frac{1 - a_{U,1}}{\dim(C^{2^n})^{\otimes 2}} \\
= h(a_{U,1}) + (1 - a_{U,1}) \log \dim(C^{2^n})^{\otimes 2},
\]
with \( h(x) = -x \log x - (1 - x) \log(1 - x) \). Therefore, if \( n \) is large enough, we have,
\[
E_f (\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2}) \\
\leq d_{n,m} \int p_U \sum_i (-a_{U,i} \log a_{U,i}) \mu(dU) \\
\leq d_{n,m} \int p_U h(a_{U,1}) \mu(dU)
\]
\[ + (\log(d_{n,2} - 1)) d_{m,n} \int p_U(1 - a_{U,1}) \mu(dU) \]
\[ \leq h \left( d_{n,m} \int p_U a_{U,1} \mu(dU) \right) \]
\[ + \log \left( \frac{(2^n + 1)!}{(2^n)!} - 1 \right) \]
\[ \leq h \left( 1 - \frac{c}{2^n} \right) + n(\log 2) \frac{c}{2^n} \]
\[ = - \left( 1 - \frac{c}{2^n} \right) \log \left( 1 - \frac{c}{2^n} \right) \]
\[ - \frac{(c \log c) n}{2^n} + n(\log 2) \frac{c}{2^n} \]
\[ = O \left( n2^{-n} \right), \]
and the proof is complete. Here, the first inequality is due to the definition of EoF. The second inequality is proved by the maximizing entropy for given \( a_{U,1} \). The maximum is achieved by setting \( a_{U,i} = \frac{1-a_{U,1}}{d_{n,2}-1} \) for \( i \geq 2 \). The third inequality is due to convexity of \( h(\cdot) \).

**Theorem 4** Let \( m \) be a appropriately chosen exponential function in \( n \), and \( \rho \in S((\mathbb{C}^{2^n})^{\otimes m}) \). Then, there is an exponential function \( q \) which satisfies (1).

**Proof.** Recall the well-known inequality

\[ E_f(\rho) \geq E_R(\rho), \]
where

\[ E_R(\rho) := \min_{\sigma: \text{separable}} D(\rho||\sigma) \]

is called relative entropy of entanglement, first defined in [4]. Also, recall well-known quantum version of Pinsker’s inequality

\[ D(\rho||\sigma) \geq \frac{1}{2} (\|\rho - \sigma\|_1)^2. \]

These inequalities lead to

\[ \min_{\sigma: \text{separable}} \|\rho - \sigma\|_1 \leq \sqrt{2E_f(\rho)}, \]

which, combined with Lemma 3 implies the theorem. ■

### 3.2 A maximally entangled state

Previous theorem may not be a good evidence for our conjecture, for \( m \) might have to be exponentially large. In this section, we supply an evidence that polynomially many copies may be enough: We prove Conjecture 2 when \( \rho \) is
a symmetric maximally entangled state. (Here entanglement is understood in terms of
\( \bigotimes_{i=1}^{m} \mathcal{H}_{2i} \otimes \bigotimes_{i=1}^{m} \mathcal{H}_{2i-1} \)-partition):

\[
\rho = |\Phi\rangle \langle \Phi|,
\]

\[
|\Phi\rangle = \sqrt{d_{n,m}} \int (U|0^n\rangle \overline{U}|0^n\rangle)^{\otimes \frac{m}{2}} \mu(dU).
\]

To see that \(|\Phi\rangle\) is maximally entangled, see partial transpose of it,

\[
|\Phi\rangle^{PT} = \sqrt{d_{n,m}} \int (U|0^n\rangle \langle 0^n| U^\dagger \langle 0^n| U^\dagger U|0^n\rangle)^{\otimes \frac{m}{2}} \mu(dU)
\]

\[
= \frac{1}{\sqrt{d_{n,m}}} I_{(2^n)^{\otimes \frac{m}{2}}},
\]

where the second identity is due to Shur’s lemma.

**Theorem 5** Let \( m \) be a appropriately chosen polynomial function in \( n \). and let \( \rho \) be as stated above. Then, \( \Box \) holds for a polynomial function \( q \) in \( n \).

**Proof.**

\[
\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = d_{n,m} \int \mu(dU) \mu(dU')
\]

\[
\times U|0^n\rangle \overline{U}|0^n\rangle \langle 0^n| U^\dagger \langle 0^n| U^\dagger U|0^n\rangle|0^n\rangle^{m-2}.
\]

Here we decompose

\[
U'|0^n\rangle = (\cos \theta) U|0^n\rangle + (\sin \theta) V|\phi_U\rangle,
\]

using a state \(|\phi_U\rangle\) and \( V \) with \( \langle \phi_U| U|0^n\rangle = 0 \) and \( \langle \phi_U| V^\dagger U|0^n\rangle = 0 \). Here, \( \theta \) and \( V \) runs all over the interval \([0, \pi]\), and \( SU(2^n-1) \). (Here, more rigorously, we are considering an embedding of \( SU(2^n-1) \) into \( SU(2^n) \) with \( \langle \phi_U| V^\dagger U|0^n\rangle = 0 \).) Also,

\[
\mu(dU') = \nu_n(d\theta) \mu'(dV),
\]

where \( \mu' \) is the Haar measure in \( SU(2^n-1) \) with \( \int \mu'(dV) = 1 \), and \( \nu_n \) is an appropriate measure. Note also

\[
\int V|\phi_U\rangle \mu'(dV) = 0.
\]

Therefore,

\[
\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = d_{n,m} \int (\cos^m \theta) \nu_n(d\theta) \overline{\theta}
\]

\[
+ d_{n,m} \int (\cos^{m-2} \theta \sin^2 \theta) \nu_n(d\theta) X.
\]
where
\[ \mathcal{P} := \int \mu(dU)U|0^n\rangle \bar{U}|0^n\rangle \langle 0^n | \bar{U}^\dagger, \]
and
\[ X := \int \mu'(dV)\mu(dU)U|0^n\rangle \bar{U}|0^n\rangle \langle \phi_U|V^\dagger \langle \phi_U|V^\dagger. \]

Note that \( \mathcal{P} \) is a separable state, and \( X \) satisfies \( \text{tr}X = 0 \). Note, letting \( \cos \theta = \langle \phi|U|0^n\rangle \),

\[ \frac{1}{d_n, \frac{m}{2}} = \frac{\langle \phi| \otimes \frac{m}{2}}{\int \langle U|0^n\rangle \langle 0^n | U^\dagger \otimes \frac{m}{2}} \mu(dU)|\phi| \otimes \frac{m}{2} \]
\[ = \int \langle \phi|U|0^n\rangle |m_n(d\theta)\mu'(dV) \]
\[ = \int \cos^m \theta \nu_n(d\theta), \]

where the second equality is due to \( \int \mu'(dV) = 1 \). Due to this,

\[ \int \sin^2 \theta \cos^{m-2} \theta \nu_n(d\theta) \]
\[ = \int (1 - \cos^2 \theta) \cos^{m-2} \theta \nu_n(d\theta) \]
\[ = \frac{1}{d_n, \frac{m}{2}} - \frac{1}{d_n, \frac{m}{2}} \]
\[ = \frac{1}{d_n, \frac{m}{2}} \frac{2(n^2 - 1)}{m}. \]

Observe that
\[ (2^n - 1) \int V|\phi_U\rangle \langle \phi_U|V^\dagger \mu'(dV) + U|0^n\rangle \langle 0^n | U^\dagger \]
is an identity operator in \( \mathbb{C}^{2^n} \). Therefore, its partial transpose with normalization factor satisfies
\[ \frac{1}{\sqrt{2^n}} \left\{ (2^n - 1) \int V|\phi_U\rangle \overline{V}|\phi_U\rangle \mu'(dV) \right\} \]
\[ = \sqrt{2^n} \int U|0^n\rangle \overline{U}|0^n\rangle \mu(dU) \]
\[ := |\Phi\rangle, \]
or
\[ \int V|\phi_U\rangle \overline{V}|\phi_U\rangle \mu'(dV) \]
\[ = \frac{1}{2^n - 2} \left( \sqrt{2^n} |\Phi\rangle - U|0^n\rangle \overline{U}|0^n\rangle \right). \]
This leads to
\[
X = \frac{1}{2^n - 1} \left( \sqrt{2^n} \int U|0^n \rangle \langle 0^n | \mu(dU) \langle \Phi | \right) \\
= \frac{1}{2^n - 1} (|\Phi \rangle \langle \Phi | - \overline{\rho}).
\]

Putting together,
\[
\rho|_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \overline{\rho} + \frac{2(2^n - 1)}{m} \frac{1}{2^n - 1} (|\Phi \rangle \langle \Phi | - \overline{\rho})
= \overline{\rho} + \frac{2}{m} (|\Phi \rangle \langle \Phi | - \overline{\rho}).
\]

For $\overline{\rho}$ is separable, $\| \frac{2}{m} (|\Phi \rangle \langle \Phi | - \overline{\rho}) \|_F \leq \frac{1}{m}$ implies our assertion. ■

4 Discussion

We had presented two evidences for our conjecture. The first one shows that the entanglement is weakened polynomially fast as the number $m$ of subsystems grows. Second evidence suggests that $m$ may be polynomial in the number of qubits in each subsystems.

Lemma 3, used to show the first evidence, is of interest in its own right, for EoF is a very important entanglement measure.

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