Perfect Retroreflectors and Billiard Dynamics

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Abstract. We construct semi-infinite billiard domains which reverse the direction of most incoming particles. We prove that almost all particles will leave the open billiard domain after a finite number of reflections. Moreover, with high probability the exit velocity is exactly opposite to the entrance velocity, and the particle’s exit point is arbitrarily close to its initial position. The method is based on asymptotic analysis of statistics of entrance times to a small interval for irrational circle rotations. The rescaled entrance times have a limiting distribution in a limit when the number of iterates tends to infinity and the length of the interval vanishes. The proof of the main results follows from the study of related limiting distributions and their regularity properties.

1. Introduction

The present paper is motivated by the problem of constructing open billiard domains with exact velocity reversal (EVR), which means that the velocity of every incoming particle is reversed when the particle eventually leaves the domain. This problem arises in the construction of perfect retroreflectors—optical devices that exactly reverse the direction of an incident beam of light and preserve the original image. A well-known example of a perfect retroreflector is the Eaton lens [3, 12] which is a spherically symmetric lens that, unlike our model, also reverses the original image. A second application lies in the search for domains that maximize the pressure of a flow of particles [10]: for a particle of mass $m > 0$, which moves towards a wall with velocity $\vec{v}$, the impulse transmitted to the wall at the moment of reflection is equal to $2m|\vec{v}_n|$, where $\vec{v}_n$ is the normal component of $\vec{v}$. It is maximized when $\vec{v} = \vec{v}_n$, i.e. when the direction of the particle is reversed.

We construct a family of domains $D_\varepsilon$, for which EVR holds up to a set of initial condition whose measure tends to zero in the limit $\varepsilon \to 0$. 

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The domain \( D_\epsilon \) is the semi-infinite tube \([0, \infty) \times [0, 1]\) with vertical barriers of height \( \epsilon/2 \) at the points \((n, 0)\) and \((n, 1)\), \(n \in \mathbb{N}\) as illustrated in Fig. 1. Inside the domain the particle moves with the constant speed and elastic reflections from the boundary. Since the kinetic energy of the particle is preserved, we can assume that the speed of the particle is equal to one.

The motion of the particle is determined by the point \( y_{\text{in}} \in [0, 1] \), where it enters the tube and the initial velocity \( v_{\text{in}} = (\cos(\pi \varphi), \sin(\pi \varphi)) \) at this point. The measure \( \mathbb{P} \) on the initial conditions \((y_{\text{in}}, \varphi)\) considered below is a Borel probability measure absolutely continuous with respect to the Lebesgue measure on \( \Omega = [0, 1] \times [-1/2, 1/2] \).

![Figure 1. The Model](image)

**Theorem 1.** For every \( \epsilon \in (0, 1) \) there exists a set \( \Omega(\epsilon) \subset \Omega \) of full Lebesgue measure, such that for every \((y_{\text{in}}, \varphi) \in \Omega(\epsilon)\), the particle eventually leaves the tube.

The position and the velocity with which it leaves the tube are denoted by \((y_{\text{out}}, v_{\text{out}})\). By Theorem 1 for every \( \epsilon \in (0, 1) \) the functions \( y_{\text{out}} = y_{\text{out}}(y_{\text{in}}, v_{\text{in}}) \) and \( v_{\text{out}} = v_{\text{out}}(y_{\text{in}}, v_{\text{in}}) \) are defined \( \mathbb{P} \)-almost everywhere.

**Theorem 2.** For any \( \delta > 0 \),

\[
\mathbb{P}\{(y_{\text{in}}, v_{\text{in}}) : v_{\text{out}} = -v_{\text{in}}, |y_{\text{out}} - y_{\text{in}}| < \delta\} \to 1 \text{ as } \epsilon \to 0
\]

Theorem 2 follows from the results on the existence of certain limiting distributions for the exit statistics of the billiard particle as \( \epsilon \to 0 \). Below we formulate these results as Theorem 3 and Theorem 4. In the last section of the paper we show how they imply Theorem 2.

Let \( Q_\epsilon = Q_\epsilon(y_{\text{in}}, v_{\text{in}}) \) be the number of reflections from the vertical walls before the particle leaves the tube. Let \( T_\epsilon = T_\epsilon(y_{\text{in}}, v_{\text{in}}) \) be the time that particle spends inside the tube. By Theorem 1 both \( Q_\epsilon \) and \( T_\epsilon \) are finite \( \mathbb{P} \)-a.e.

Consider also a bi-infinite tubular domain similar to the one described above. It consists of two horizontal lines at the unit distance from each other and a one-periodic configuration of vertical walls of height \( \epsilon/2 \).
Let $x$ be the horizontal coordinate and assume that the particle starts at $x = 0$. Let $\xi^0_\varepsilon = 0$ and $\xi^k_\varepsilon \in \mathbb{Z}$ be $x$-coordinate of the particle at the moment of $k$’th reflection from a vertical wall. Since the tube is now bi-infinite, $\{\xi^k_\varepsilon\}$ is a discrete time process on $\mathbb{Z}$, defined for any $k \in \mathbb{N}$. We also define a continuous version of this process: $\{\xi_\varepsilon(t)\}$ is the projection of the trajectory of a billiard particle in the bi-infinite tube to the $x$-axis normalized to have constant speed $1/\varepsilon$.

**Theorem 3.**

1. The process $\{\varepsilon \xi^k_\varepsilon\}$ converges in distribution (w.r.t. $\mathbb{P}$) to a stochastic process $\{\xi^k\}$ as $\varepsilon \to 0$.

2. There exists a limiting probability distribution function $G : \mathbb{N} \to [0, 1]$ such that for every $k \in \mathbb{N}$, $\mathbb{P}\{Q_\varepsilon(y_{\text{in}}, v_{\text{in}}) = k\} \to G(k)$ as $\varepsilon \to 0$.

The second part of Theorem 3 says that for the limiting stochastic process $\{\xi^k\}$, with probability one there exists $k \in \mathbb{N}$, such that $\xi^k < 0$. Similar results are true for the continuous process $\{\xi_\varepsilon(s)\}$ as well:

**Theorem 4.**

1. The process $\{\varepsilon \xi_\varepsilon(s)\}$ converges in distribution w.r.t. $\mathbb{P}$ to a stochastic process $\xi(s)$ as $\varepsilon \to 0$.

2. There exists a limiting probability distribution function $H : \mathbb{R}_\geq 0 \to [0, 1]$, such that for every $t \geq 0$, $\mathbb{P}\{\varepsilon T_\varepsilon(y_{\text{in}}, v_{\text{in}}) < t\} \to H(t)$ as $\varepsilon \to 0$.

### 2. Reduction to Circle Rotations and Point-Wise Exits

We first reformulate the problem in terms of circle rotations. Let us identify $[0, 1)$ with $S^1 = \mathbb{R}/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, let $R_\alpha : S^1 \to S^1$ be the circle rotation by angle $\alpha$:

$$R_\alpha x = x + \alpha \mod 1.$$ 

Always assume that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Let $I_\varepsilon = [-\varepsilon/2, \varepsilon/2] \subset S^1$.

We define several sequences measuring the return times to the interval $I_\varepsilon$, which will be used throughout the proofs. The hitting times $m^k_\varepsilon = m^k_\varepsilon(x, \alpha)$, $k = 0, 1, 2, \ldots$ are defined for $x \in S^1$ by:

$$m^0_\varepsilon = 0, \quad m^k_\varepsilon(x) = \min\{l > m^{k-1}_\varepsilon : R^l_\alpha x \in I_\varepsilon\}$$

The sequence $n^k_\varepsilon = n^k_\varepsilon(x, \alpha)$, $k = 1, 2, \ldots$ of relative return times to the interval $I_\varepsilon$ is defined for $x \in S^1$ by:

$$n^k_\varepsilon = m^k_\varepsilon(x) - m^{k-1}_\varepsilon(x)$$

We shall also use the sequence $\{\xi^k\}$ defined in the introduction as the sequence of the horizontal coordinates of points of the reflection from the vertical walls.
Note that if \( x = y \in S^1 \), and \( \alpha = \tan(\pi \varphi) \), then \( n_i^j(x) \) is the distance between horizontal coordinate of the place of the \((i - 1)\)'st and the \(i\)'th reflections of the particle from vertical walls. Therefore,

\[
\xi^k_\varepsilon = n_1^\varepsilon - n_2^\varepsilon + \ldots + (-1)^{k+1}n_k^\varepsilon,
\]

\[
Q_\varepsilon = Q_\varepsilon(x, \alpha) = \min\{ j \in \mathbb{N} : n_1^\varepsilon(x) - n_2^\varepsilon(x) + \ldots + (-1)^{j+1}n_j^\varepsilon(x) \leq 0 \} - 1.
\]

Let \( \bar{n}_k^\varepsilon = (n_1^\varepsilon, \ldots, n_k^\varepsilon)^T \), \( \bar{m}_k^\varepsilon = (m_1^\varepsilon, \ldots, m_k^\varepsilon)^T \) and \( \bar{\xi}_k^\varepsilon = (\xi^1_\varepsilon, \ldots, \xi^k_\varepsilon)^T \), then

\[
(2.2) \quad \bar{\xi}_k^\varepsilon = A \bar{n}_k^\varepsilon, \quad \text{and} \quad \bar{m}_k^\varepsilon = B \bar{n}_k^\varepsilon,
\]

where \( A \) and \( B \) are two \( k \times k \) matrices with

\[
A_{i,j} = \begin{cases} 
0, & \text{if } i < j, \\
(-1)^{j+1}, & \text{if } i \geq j,
\end{cases}
\]

and

\[
B_{i,j} = \begin{cases} 
0, & \text{if } i < j, \\
1, & \text{if } i \geq j.
\end{cases}
\]

The probability measure \( P \) on the initial conditions \((y_\varepsilon, \varphi_\varepsilon) \in [0, 1] \times [-\pi/2, \pi/2]\) for the billiard particle induces a probability measure on the initial conditions \((x, \alpha) \in [0, 1] \times [0, 1) \simeq T^2\) for the circle rotation \( R_\alpha \), which is absolutely continuous w.r.t. to the Lebesgue measure on \( T^2 \) and which will be also denoted by \( P \).

We now prove Theorem 1.

Let \( \hat{T}_{\alpha,\varepsilon} : I_\varepsilon \to I_\varepsilon \) be the map induced on \( I_\varepsilon \) by the circle rotation \( R_\alpha : T^2 \to T^2 \).

**Proposition 5.** For every \( \varepsilon \in (0, 1) \) there exists a set of full Lebesgue measure \( \Lambda(\varepsilon) \subset S^1 \), such that for every \( \alpha \in \Lambda \), the map \( \hat{T}_{\alpha,\varepsilon} \) is weakly mixing.

**Proof.** The proof of Proposition will follow from a combination of results of [1] and [2].

For every \( \varepsilon > 0 \) there exists a full Lebesgue measure set \( \Lambda'(\varepsilon) \subset S^1 \), such that for every \( \alpha \in \Lambda'(\varepsilon) \), the corresponding map \( \hat{T}_{\alpha,\varepsilon} \) is an interval exchange transformation of three intervals of combinatorial type \((3, 2, 1)\).

Recall Property P introduced by Boshernitzan in [1]:

**Definition 1.** A set \( \mathcal{A} \subset \mathbb{N} \) is called essential, if for any integer \( l \geq 2 \) there exists \( \lambda > 1 \), such that the system

\[
\begin{cases} 
n_{i+1} > 2n_i, & \text{for } 1 \leq i \leq l - 1, \\
n_1 < \lambda n_1, \\
n_i \in \mathcal{A}, & \text{for } 1 \leq i \leq l
\end{cases}
\]

has an infinite number of solutions \((n_1, n_2, \ldots, n_l)\).

Let \( m_n(\hat{T}_{\alpha,\varepsilon}) \) be the length of the smallest interval of continuity of \( \hat{T}_{\alpha,\varepsilon}^n \).
Definition 2. An interval exchange map $\hat{T}_{\alpha, \varepsilon}$ has property $P$, if for some $\delta > 0$ the set
\[
\mathcal{A}(\alpha, \varepsilon, \delta) = \{ n \in \mathbb{N} | m_n(\hat{T}_{\alpha, \varepsilon}) > \frac{\delta}{n} \}
\]
is essential.

Proposition 6. ([1], Theorem 9.4 (a)) For every $\varepsilon \in (0, 1)$ there exist a full Lebesgue measure set $\Lambda(\varepsilon) \subset S^1$, such that for every $\alpha \in \Lambda$, the map $\hat{T}_{\alpha, \varepsilon}$ has property $P$.

By Theorem 5.3 of [2], property $P$ implies weak-mixing for an interval exchange of three intervals with combinatorics $(3, 2, 1)$ (and more generally, for any combinatorics of a so-called $W$-type, see [2]).

This implies Proposition 5. □

The next two statements are well-known. We include their proofs to keep the exposition self-contained.

Lemma 7. For every $\varepsilon \in (0, 1)$ there exists a set of full Lebesgue measure $\Lambda(\varepsilon) \subset S^1$, such that for every $\alpha \in \Lambda$, the map $\hat{T}_{\alpha, \varepsilon}^2$ is ergodic.

Proof. Assume now that $\hat{T}_{\alpha, \varepsilon}^2$ is not ergodic. Then there exists a bounded $f \neq \text{const}$, such that $\hat{T}_{\alpha, \varepsilon}^2 f = f$, and therefore $\hat{T}_{\alpha, \varepsilon}(f + \hat{T}_{\alpha, \varepsilon} f) = \hat{T}_{\alpha, \varepsilon} f + f$.

Since $\hat{T}_{\alpha, \varepsilon}$ is ergodic, this implies that $\hat{T}_{\alpha, \varepsilon} f + f = C$, or $f - C/2 = -(\hat{T}_{\alpha, \varepsilon} f - C/2)$. If $g = f - C/2$, then $g$ is not identically zero, and $\hat{T}_{\alpha, \varepsilon} g = -g$. Therefore $\lambda = -1$ is an eigenvalue of $\hat{T}_{\alpha, \varepsilon}$ and so $\hat{T}_{\alpha, \varepsilon}$ is not weakly mixing. □

Proposition 8. Let $T$ be an ergodic transformation on $(X, \mu)$, $\mu(X) = 1$, and let $f \in L^1(X, \mu)$, $\int f \, d\mu = 0$ and $S_n(f, x) = f(x) + f(Tx) + \ldots + f(T^{n-1}x)$ be its Birkhoff sums. Then either $S_n(f, x)$ is unbounded from below for almost every $x \in X$, or $f$ is a co-boundary, i.e. there exists a measurable $g(x)$, such that $f(x) = g(x) - g(Tx)$.

Proof. Since $T$ is ergodic, the set of points $x$ for which $S_n(f, x)$ is bounded from below has measure either equal to zero or one. In the first case, Proposition is proved, so assume that it has measure one. Then the function $g(x) = \inf_{n \geq 1} S_n(f, x)$ is finite almost everywhere.

We have $g(Tx) + f(x) = \inf_{n \geq 2} S_n(f, x)$, so $h(x) := g(Tx) - g(x) + f(x) \geq 0$.

It is enough to show that $\int h \, d\mu = 0$. If $g(x) \in L^1(X, \mu)$, then $\int h \, d\mu = 0$ by the definition of $h(x)$ above. If not, then by Birkhoff ergodic theorem, for $\mu$-a.e. $x \in X$,
\[ \lim_{n \to \infty} \frac{S_n(h, x)}{n} = \int_X hd\mu \]

where the integral can be equal to infinity.

We write
\[ \frac{S_n(h, x)}{n} = \frac{S_n(f, x)}{n} + \frac{g(T^n x) - g(x)}{n} \]

Since \( g(x) \) is finite almost everywhere, we can choose a set \( Y \subset X \), such that \( \mu(Y) > 0 \) and for every \( y \in Y \), \( |g(y)| < M \) for some constant \( M \). Then by ergodicity of \( T \), there exists a subsequence \( n_k \), such that \( T^{n_k} x \in Y \), and therefore, by Birkhoff ergodic theorem for \( \mu \)-almost all \( x \in X \) we have
\[ \lim_{k \to \infty} \frac{S_{n_k}(h, x)}{n_k} = \lim_{k \to \infty} \frac{S_{n_k}(f, x)}{n_k} + \frac{g(T^{n_k} x) - g(x)}{n_k} = 0 \]

which implies \( \int_X hd\mu = 0 \). \( \square \)

Proof of Theorem 1. For any \( \varepsilon > 0 \) choose an \( \alpha \in \Lambda(\varepsilon) \), so that the map \( \hat{T}^{2\varepsilon}_{\alpha, \varepsilon} \) is ergodic. Let \( x \in I_\varepsilon \) and \( f(x) = n^1_\varepsilon(x) - n^1_\varepsilon(\hat{T}^{2\varepsilon}_{\alpha, \varepsilon}x) \).

Then the Birkhoff sums for \( \hat{T}^{2\varepsilon}_{\alpha, \varepsilon} \) and \( f(x) \) are
\[ S_m(f, x) = f(x) + f(\hat{T}^{2\varepsilon}_{\alpha, \varepsilon}x) + f(\hat{T}^{4\varepsilon}_{\alpha, \varepsilon}x) + \ldots + f(\hat{T}^{2m\varepsilon}_{\alpha, \varepsilon}x) = n^1_\varepsilon(x) - n^2_\varepsilon(x) + \ldots - n^{2m}_\varepsilon(x) \]

By Proposition 8 for Lebesgue almost every \( x \in I_\varepsilon \), either there exists \( m_0 \in \mathbb{N} \), such that \( S_{m_0}(f, x) \leq 0 \) (and therefore \( Q_\varepsilon(x, \alpha) < \infty \)) or \( f(x) \) is a co-boundary. But in the second case, \( S_m(f, x) = g(x) - g(\hat{T}^{2m\varepsilon}_{\alpha, \varepsilon}x) \) for a measurable \( g(x) \). Either \( g(x) < \text{esssup} \ g(x) \), or \( g(x) = \text{esssup} \ g(x) \) on a positive Lebesgue measure set. In either case, the ergodicity of \( \hat{T}^{2\varepsilon}_{\alpha, \varepsilon} \), implies that for Lebesgue a.e. \( x \), there exists \( m_0 \in \mathbb{N} \), such that \( S_{m_0}(f, x) \leq 0 \) and so \( Q_\varepsilon(x, \alpha) < \infty \).

Now let \( x \in S^1 \setminus I_\varepsilon \). Since \( \alpha \notin \mathbb{Q} \), there exists \( n_0 > 0 \), such that \( R^{-n_0}_\alpha x \in I_\varepsilon \).

Then for Lebesgue a.e. \( x \in S^1 \)
\[ Q_\varepsilon(x, \alpha) \leq Q_\varepsilon(\hat{T}^{-n_0}_\alpha x, \alpha) < \infty \]

\( \square \)

3. Limiting distributions

We now prove theorems 3 and 4.

\[ \frac{S_n(h, x)}{n} = \frac{S_n(f, x)}{n} + \frac{g(T^n x) - g(x)}{n} \]

where the integral can be equal to infinity.
3.1. Notations and the formulation of the main limiting distribution result. Let \( F_{\varepsilon}^{(n)}(t_1, \ldots, t_n) = \mathbb{P}\{\varepsilon m_1 > t_1, \varepsilon m_2 > t_2, \ldots, \varepsilon m_n > t_n\} \) be the joint distribution function of the vector \( \varepsilon \vec{m}_\varepsilon^n = (\varepsilon m_1, \varepsilon m_2, \ldots, \varepsilon m_n)^T \).

It is also convenient to introduce

\[
\mathcal{N}_\varepsilon(x, \alpha, T) = \#\{k \in \mathbb{Z} \cap (0, \varepsilon^{-1}T] : k\alpha + x \in I_\varepsilon + \mathbb{Z}\},
\]

the number of times the particle hits vertical walls during the time \( \varepsilon^{-1}T \).

Note that

\[(3.3) \quad \mathbb{P}\{\varepsilon m_k^\varepsilon(x, \alpha) > t_k, k = 1, \ldots, n\} = \mathbb{P}\{\mathcal{N}_\varepsilon(x, \alpha, t_k) \leq k - 1, k = 1, \ldots, n\}.\]

Let \( \chi_I \) denote the characteristic function of the interval \( I \subset \mathbb{R} \) and \( \psi_T(x, y) = \chi_{[0,1]}(x)\chi_{[-T/2,T/2]}(y) \) be the characteristic function of a corresponding rectangle.

Then

\[
\mathcal{N}_\varepsilon(x, \alpha, T) = \sum_{m=1}^{\lfloor \varepsilon^{-1}T \rfloor} \sum_{n \in \mathbb{Z}} \chi_I \varepsilon \alpha + n + x = \\
= \sum_{(m,n) \in \mathbb{Z}^2} \chi_{[0,1]} \left( \frac{m}{\varepsilon^{-1}T} \right) \chi_{[-T/2,T/2]}(\varepsilon^{-1}T(\alpha m + n + x)) = \\
= \sum_{(m,n) \in \mathbb{Z}^2} \psi_T \left( (m, n + x) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^{-1}T & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \right)
\]

Therefore,

\[(3.4) \quad \mathcal{N}_\varepsilon(x, \alpha, T) = \# \left\{ (m, n) \in \mathbb{Z}^2 : (m, n + x) \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \in \mathcal{R}(T) \right\},\]

where \( \mathcal{R}(T) = (0, T] \times [-1/2, 1/2] \).

Let \( ASL(2, \mathbb{R}) = SL(2, \mathbb{R}) \ltimes \mathbb{R}^2 \) be the semidirect product group with multiplication law

\[(M, \mathbf{v})(M', \mathbf{v}') = (MM', \mathbf{v}M' + \mathbf{v}').\]

The action of an element \( (M, \mathbf{v}) \) of this group on \( \mathbb{R}^2 \) is defined by

\[(3.5) \quad \mathbf{w} \mapsto \mathbf{w}M + \mathbf{v}\]

Each affine lattice of covolume one in \( \mathbb{R}^2 \) can then be represented as \( \mathbb{Z}^2 g \) for some \( g \in ASL(2, \mathbb{R}) \), and the space of affine lattices is represented by \( X = ASL(2, \mathbb{Z}) \backslash ASL(2, \mathbb{R}) \), where \( ASL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \). Denote by \( \nu \) the Haar probability measure on \( X \).
Theorem 9. As $\varepsilon \to 0$, the limit of (3.3) exists and is equal to
(3.6) $F^{(n)}(t_1, \ldots, t_n) = \nu(\{g \in X : \#(\mathbb{Z}^2 g \cap R(t_k)) \leq k - 1 (k = 1, \ldots, n)\})$, which is a $C^1$ function $\mathbb{R}^n_{\geq 0} \to [0, 1]$.

We define the associated limiting probability density $\phi^{(n)}(t_1, \ldots, t_n)$ by

$$F^{(n)}(t_1, \ldots, t_n) = \int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} \phi^{(n)}(t_1, \ldots, t_n) dt_1, \ldots, dt_n$$

3.2. The reduction of Theorem 3 to Theorem 9. Because of the relation (2.2), Theorem 9 implies the convergence in distribution for the sequences $\{\varepsilon n_k\}$ and $\{\varepsilon \bar{m}_k\}$ (part (1) of Theorem 3).

Indeed, let $k \in \mathbb{N}$ and $I_1, \ldots, I_k$ be a collection of $k$ intervals on the real line. Let $I = I_1 \times \ldots \times I_k \subset \mathbb{R}^k$. Then

$$\lim_{\varepsilon \to 0} P\{\varepsilon n_1^k \in I_1, \ldots, \varepsilon n_k^k \in I_k\} = \lim_{\varepsilon \to 0} P\{\varepsilon \bar{m}_k^k \in B I\} = \int_{B I} \phi^{(k)}(t_1, \ldots, t_k) dt_1 \ldots dt_k$$

and

$$\lim_{\varepsilon \to 0} P\{\varepsilon \xi_1^k \in I_1, \ldots, \varepsilon \xi_k^k \in I_k\} = \lim_{\varepsilon \to 0} P\{\varepsilon \bar{m}_k^k \in B A^{-1} I\} = \int_{B A^{-1} I} \phi^{(k)}(t_1, \ldots, t_k) dt_1 \ldots dt_k$$

The convergence for the random variable $Q_{\varepsilon}(x, \alpha)$ also follows from Theorem 9.

Indeed, for any $k \geq 1$ let $\chi_{A_k}$ be the characteristic function of the set

$$\Delta_k = \{(y_1, \ldots, y_k) \in \mathbb{R}^k : y_1 > 0, \ldots, y_{k-1} > 0, y_k < 0\}.$$

Then for every $\varepsilon > 0$ we have

$$Q_{\varepsilon}(x, \alpha) = \min\{j \in \mathbb{Z}_+ : \xi_{\varepsilon}^j \leq 0\} - 1$$

Therefore,

$$P\{Q_{\varepsilon}(x, \alpha) = k\} = P\{\varepsilon \xi_1^k > 0, \ldots, \varepsilon \xi_{k-1}^k > 0, \varepsilon \xi_k^k \leq 0\} =$$

$$= P\{\varepsilon \bar{m}_k^k \in B A^{-1} \Delta_k\} = \int_{B A^{-1} \Delta_k} dF^{(k)}_{\varepsilon \bar{m}_k^k},$$

and by Theorem 9 and the Helly-Bray Theorem ([5], p.183), there exists the limit
(3.7) \[ G(k) = \lim_{\varepsilon \to 0} \mathbb{P}\{Q_{\varepsilon}(x, \alpha) = k\} = \int_{BA^{-1}\Delta_k} \phi^{(k)}(t_1, \ldots, t_k) dt_1 \ldots dt_k \]

Notice that the representation (3.7) implies that \( \sum_{k=1}^{\infty} G(k) \leq 1 \)

**Proposition 10.**

(3.8) \[ \sum_{k=1}^{\infty} G(k) = 1 \]

*Proof.* Let \( \{\eta^k\} \) be the limiting process for the sequence \( \{\varepsilon n^k_{\varepsilon}\} \). From the explicit description (3.6) of the limiting distribution in Theorem 9, we have the following description of the process \( \{\eta^k\} \).

![Figure 2](attachment:image.png)

**Figure 2.** The horizontal ray through \((0, 0)\) generates the sequence \(-y_k\) as an orbit of an interval exchange map.

Let \( g \in X \) be an affine lattice which has no points either with the same horizontal coordinates, or on the boundary of the semi-infinite tube \( \mathcal{R}_\infty = [0, +\infty) \times [-1/2, 1/2] \). The set of such lattices has full Haar measure in \( X \). Let us enumerate points of \( g \) which lie in \( \mathcal{R}_\infty \) according to their horizontal coordinates: if the coordinates of the \( k \)th lattice point of \( g \) in \( \mathcal{R}_\infty \) are \((x_k, y_k) = (x_k(g), y_k(g))\) \((k = 1, 2, \ldots)\), then \( x_k < x_{k+1} \) for any \( k = 1, 2, \ldots \). Notice, that \( \nu \)-almost every lattice \( g \) has infinitely many points in \( \mathcal{R}_\infty \).

The sequence of random variables \( \varepsilon n^k_{\varepsilon} = \varepsilon n^k_{\varepsilon}(x, \alpha) \) w.r.t. the probability measure \( \mathbb{P} \) on \( \mathbb{T}^2 \) converges in distribution to the sequence \( \eta^1 = \eta^1(g) = x^1(g) \), and \( \eta^k = \eta^k(g) = x_k(g) - x_{k-1}(g) \) for \( k \geq 2 \) w.r.t Haar measure \( \nu \) on \( X \).
Therefore, in order to prove (3.8), it is enough to show that for \( \nu \)-almost every affine lattice \( g \in X \), there exists an even \( k > 0 \), such that

\[
(3.9) \quad \eta^1 - \eta^2 + \eta^3 - \ldots - \eta^k = x_1 - (x_2 - x_1) + (x_3 - x_2) - \ldots - (x_k - x_{k-1}) \leq 0
\]

We will now show that the sequence \( y_k(g) \) is an orbit of a certain map of an interval into itself, reduce (3.9) to a Birkhoff sum over this map and treat it in the way as in Section 2.

First, we describe the map. Consider set \( \mathcal{I} \subset \mathbb{R}^2 \) of vertical segments of unit length centered at every lattice point of \( g \). We identify each segment in \( \mathcal{I} \) with the base \( I = [-1/2, 1/2] \) of the tube \( \mathcal{R}_\infty \) by parallel translation. Let \( \pi : \mathcal{I} \to I \) be the projection, which sends a point on some interval through lattice point to the corresponding point in \( I \).

Consider a unit speed flow in the positive horizontal direction on \( \mathbb{R}^2 \). Its first return map to \( \mathcal{I} \) is a well-defined map \( \hat{T} = \hat{T}(g) \) of \( \mathcal{I} \) into itself. We define the corresponding invertible map \( T : I \to I \), so that \( \pi \circ \hat{T} = T \circ \pi \). It is easy to see, that the map \( T \) is an exchange of three intervals. For \( \nu \)-almost every lattice \( g \) it has combinatorial type \((3, 2, 1)\).

For every \( y \in I \), we let \( \psi(y) \) to be the Euclidean distance between \( \hat{y} \in \pi^{-1}(y) \) and its image under \( \hat{T} \). Clearly, this does not depend on the choice of \( \hat{y} \in \pi^{-1}(y) \).

Notice that the sequence \( \{y_k\} \) of the vertical coordinates of the lattice points of \( g \) in \( \mathcal{R}_\infty \) is related to the map \( T \) described above: for \( k \in \mathbb{N} \), \( y_k = -T^{k-1}(-y_1) \) (see Figure 2). Also for \( k \in \mathbb{N} \), we have \( \psi(-y_k) = x_{k+1} - x_k \). Let \( -y_0 = T^{-1}(-y_1) \). Then the sum in (3.9) has the form (recall, \( k \) is even)

\[
(3.10) \quad x_1 - \psi(-y_1) + \psi(-y_2) - \ldots - \psi(-y_{k-1}) \leq \psi(-y_0) - \psi(-T(-y_0)) + \psi(T(-y_0)) - \ldots - \psi(T^{k-1}(-y_0))
\]

Therefore similarly to Section 2, the alternating sum (3.9) is reduced to a Birkhoff sum for the function \( f(y) = \psi(-y) - \psi(-T(-y)) \) and the map \( T^2 \).

Let the lengths of the interval exchange map \( T \) be equal to \( (\lambda_1, \lambda_2, 1 - \lambda_1 - \lambda_2) \). Denote the simplex of possible \( \lambda_i \)'s by

\[
\Lambda = \{(\lambda_1, \lambda_2) | \lambda_1 > 0, \lambda_2 > 0, \lambda_1 + \lambda_2 < 1\} \subset \mathbb{R}^2
\]

and the corresponding interval exchange map of combinatorial type \((3, 2, 1)\) by \( T_{\lambda_1, \lambda_2} \). The following theorem was first proved by Katok and Stepin in [4].

**Theorem 11.** For Lebesgue almost every pair \((\lambda_1, \lambda_2) \in \Lambda\), the map \( T_{\lambda_1, \lambda_2} \) of the interval \( I \) onto itself is weakly-mixing.

Similarly to the proof of Theorem 11, Theorem 11 and Proposition 8 imply that there exists a full Lebesgue measure subset \( \Lambda_1 \subset \Lambda \), such that for every.
(\lambda_1, \lambda_2) \in \Lambda_1$, there exists a full Lebesgue measure subset $I' = I'(\lambda_1, \lambda_2) \subset I$, such that for every $y \in I'$ there exists $k > 0$, such that
\[
\psi(-y_0) - \psi(-T_{\lambda_1, \lambda_2}(-y_0)) + \psi(T_{\lambda_1, \lambda_2}^2(-y_0)) - \ldots - \psi(T_{\lambda_1, \lambda_2}^{k-1}(-y_0)) \leq 0.
\]

Let $\tilde{X} \subset X$ be the set of lattices, for which the construction above gives an interval exchange transformation of combinatorial type $(3 2 1)$. Then $\tilde{X}$ is open and $\nu(\tilde{X}) = 1$.

Notice that for any $g \in \tilde{X}$, the map $X : g \mapsto (\lambda_1, \lambda_2, y_0)$ is differentiable and its differential is surjective. Therefore, the preimage of any Lebesgue measure zero set under $X$ has Haar measure zero in $X$.

Therefore, the set of lattices $g \in X$, such that $X(g) \in \{(\lambda_1, \lambda_2, y_0) \mid (\lambda_1, \lambda_2) \in \Lambda_1, y_0 \in I'(\lambda_1, \lambda_2)\}$ has full Haar measure in $X$ and so (3.8) is proved. □

Remark 12. The condition (3.8) is equivalent to the tightness of the family of distributions $\{Q_\varepsilon\}$ as $\varepsilon \to 0$. Namely, for any $\delta > 0$ there exists $N = N(\delta)$ and $\varepsilon_1 = \varepsilon_1(\delta)$, such that for $\varepsilon < \varepsilon_1$
\[
1 - \delta \leq \sum_{k=1}^N \mathbb{P}\{Q_\varepsilon(x, \alpha) = k\} \leq 1
\]

3.3. Continuous case.

Proposition 13. For any $s > 0$ and $\delta > 0$, there exists $\varepsilon_0 > 0$ and $k \in \mathbb{N}$, such that
\[
\mathbb{P}\{(\varepsilon m_\varepsilon^k \leq s)\} < \delta
\]
for all $\varepsilon < \varepsilon_0$.

Proof. We have
\[
\mathbb{P}\{\varepsilon m_\varepsilon^k \leq s\} = \mathbb{P}\{N_\varepsilon(x, \alpha, s) \geq k\}
\]
The limit, as $\varepsilon \to 0$, exists and, in view of [7] (p.1131, first equation), is bounded by
\[
\leq C_s k^{-3}
\]
for some constant $C_s$. □

We now prove part (1) of Theorem 4.

For any $N \in \mathbb{N}$ and intervals $I_1, \ldots, I_N \subset \mathbb{R}$,
\[
\mathbb{P}\{\varepsilon \xi_\varepsilon(s_1) \in I_1, \ldots, \varepsilon \xi_\varepsilon(s_N) \in I_N\} = \\
= \sum_{k \in \mathbb{Z}_{\geq 0}^N} \mathbb{P}\{\varepsilon \xi_\varepsilon(s_j) \in I_j, \varepsilon m_\varepsilon^k \leq s_j < \varepsilon m_\varepsilon^{k+1} (j = 1, \ldots, N)\}
\]
Notice that
\[
\varepsilon \xi_\varepsilon(s) = \begin{cases} 
  s & \text{if } 0 \leq s < \varepsilon m_\varepsilon^1, \\
  \varepsilon m_\varepsilon^k + (-1)^k(s - \varepsilon m_\varepsilon^k) & \text{if } \varepsilon m_\varepsilon^k \leq s < \varepsilon m_\varepsilon^{k+1},
\end{cases}
\]
and

\[ \xi_k^\varepsilon = \sum_{i=1}^{k} (-1)^{i-1}(k-i+1)m_i^\varepsilon. \]

Therefore, by Theorem 9, for every fixed \( k \in \mathbb{Z}_N^+ \) we have

\[
\lim_{\varepsilon \to 0} \mathbb{P}\{\varepsilon \xi^\varepsilon(s_j) \in I_j, \varepsilon m^k_j \leq s_j < \varepsilon m^{k+1}_j \ (j = 1, \ldots, N)\} =
\int_{B_k} \phi^{(k+1)}(t_1, \ldots, t_{k+1}) dt_1 \ldots dt_{k+1},
\]

with \( k = \max(k) \), and the range of integration restricted to the set

\[
(3.13) \quad B_k = \left\{ (t_1, \ldots, t_{k+1}) : \sum_{i=1}^{k} (-1)^{i-1}(k-i+1)t_i + (-1)^k(s_j - t_{k+1}) \in A_j \right\}
\]

Furthermore,

\[
\sum_{k \in \mathbb{Z}_N^+} \sum_{\max(k) \geq R} \mathbb{P}\{\varepsilon \xi^\varepsilon(s_j) \in I_j, \varepsilon m^k_j \leq s_j < \varepsilon m^{k+1}_j \ (j = 1, \ldots, N)\} \leq \mathbb{P}\{\varepsilon m^R \leq s_1\}.
\]

Part (1) of Theorem 4 now follows from Proposition 13.

For the part (2) of Theorem 4 we have,

\[
(3.14) \quad \mathbb{P}\{\varepsilon T^\varepsilon \leq s\} = \sum_{k \in \mathbb{N}} \mathbb{P}\{\varepsilon T^\varepsilon \leq s, Q^\varepsilon(x, \alpha) = k\}
\]

Notice that if \( Q^\varepsilon(x, \alpha) = k \), then the time which the particle spends in the tube is equal to

\[ T^\varepsilon = T^\varepsilon(x, \alpha) = 2\sqrt{1 + \alpha^2(n_1^\varepsilon + n_2^\varepsilon + \ldots + n_k^\varepsilon)}, \]

and so,

\[
(3.15) \quad \mathbb{P}\{\varepsilon T^\varepsilon \leq s, Q^\varepsilon(x, \alpha) = k\} = \mathbb{P}\{2\varepsilon \sqrt{1 + \alpha^2(n_1^\varepsilon + n_2^\varepsilon + \ldots + n_k^\varepsilon)} < s, Q^\varepsilon(x, \alpha) = k\}
\]
By Theorem 8, for any $s > 0$ there exists joint limiting distribution of
\[ P\{\alpha < s, \varepsilon m^k_\varepsilon(x, \alpha) > t_k \ (k = 1, \ldots, n)\}, \]
as $\varepsilon \to 0$, and therefore, of (3.15) as well.

On the other hand,
\[ P\{\varepsilon T_\varepsilon \leq s, \mathcal{Q}_\varepsilon(x, \alpha) \geq k\} \leq P\{\varepsilon m^k_\varepsilon \leq s\}, \]
and so, Proposition 13 and the convergence of (3.15) imply the existence
of the limit
\[ H(s) = \lim_{\varepsilon \to 0} P\{\varepsilon T_\varepsilon \leq s\} \]
Also, since
\[ P\{\varepsilon T_\varepsilon \leq s\} \geq \sum_{k=1}^{N} P\{\varepsilon T_\varepsilon \leq s, \mathcal{Q}_\varepsilon(x, \alpha) = k\}, \]
the tightness (3.11) implies that $H(s) \to 1$ as $s \to \infty$.
This finishes the proof of part (2) of Theorem 4.

3.4. The proof of Theorem 9. By (3.3) it is enough to show that for any
$n \in \mathbb{N}$ and any $n$-tuples $(t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ there exists the limit
\[ G^{(n)}(t_1, \ldots, t_n) = \lim_{\varepsilon \to 0} P\{\mathcal{N}_\varepsilon(x, \alpha, t_j) = k_j, \ (j = 1, \ldots, n)\} = 
\nu(\{g \in X : \#(\mathbb{Z}^2 g \cap \mathcal{R}(t_j)) = k_j \ (j = 1, \ldots, n)\}) \]
and that $G^{(n)}(t_1, \ldots, t_n)$ is a $C^1$-function of $(t_1, \ldots, t_n)$. For $n = 1$ the convergence in (3.16) was first proved by Mazel and Sinai ([9]). It was later reproved and generalized by the third author ([6], [7]) using different methods. The proof of the convergence in (3.16) follows the
one in [6]. The proof of the regularity of the limiting function is similar to
the one in [8].

We reduce the convergence in (3.16) to an equidistribution result for the
gedesic flow on $X$.

Recall, that the action of the geodesic flow on $X$ is given by right action
of a one-parameter subgroup of $X$:
\[ \Phi^t = \left( \begin{array}{cc} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{array} \right), (0,0) \right). \]

The unstable horocycle of the flow $\Phi^t$ on $X$ is then parametrized by the
subgroup $H = \{n_-(x, \alpha)\}_{(x, \alpha) \in \mathbb{T}^2}$ :
\[ n_-(x, \alpha) = \left( \begin{array}{cc} 1 & \alpha \\ 0 & 1 \end{array} \right), (0, x) \right). \]

For $g \in X$ let $F_T(g)$ be equal to the number of lattice points of $\mathbb{Z}^2 g$ in the
rectangle $\mathcal{R}(T)$. 
Then by (3.4)
\[ N_\varepsilon(x, \alpha, T) = F_T(n_-(x, \alpha)\Phi^t) \]
with \( t = -2\ln(\varepsilon) \).

**Theorem 14.** [6] For any bounded \( f : ASL(2, \mathbb{Z}) \setminus ASL(2, \mathbb{R}) \to \mathbb{R} \), such that the discontinuities of \( f \) are contained in a set of \( \nu \)-measure zero and any Borel probability measure \( \mathbb{P} \), absolutely continuous with respect to Lebesgue measure on \([0, 1) \times [0, 1)\)

\[ \lim_{t \to \infty} \int_0^1 \int_0^1 f(n_-(x, \alpha)\Phi^t) \, d\mathbb{P}(x, \alpha) = \int_{ASL(2, \mathbb{Z}) \setminus ASL(2, \mathbb{R})} f \, d\nu \]

Let 
\[ D(g) = \begin{cases} 1 & \text{if } F_{t_j}(g) = k_j, \ (j = 1, \ldots, n), \\ 0 & \text{otherwise} \end{cases} \]

Then \( D(g) \) satisfies the conditions of Theorem 14. The convergence in (3.16) now follows from theorem 14 applied to the function \( D(g) \).

We now prove \( C^1 \) regularity of the limiting function \( G^{(n)}(t_1, \ldots, t_n) \). It is enough to consider the case when all \( t_j \) are different. We also assume that all \( k_j > 0 \). The case when some \( k_j = 0 \) is similar.

Let \( X_1 = SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R}) \) be the homogeneous space of lattices of covolume one and let \( \nu_1 \) be the probability Haar measure on \( X_1 \). For a given \( y \in \mathbb{R}^2 \) let
\[ X(y) = \{ g \in X : y \in \mathbb{Z}^2 g \}, \]
where the action of \( X \) on \( \mathbb{R}^2 \) is given by the formula (3.5).

There is a natural identification of the sets \( X(y) \) and \( X_1 \) through
\[ X(y) = \{ (M, y) : M \in X_1 \} \]
Under this identification the probability Haar measure \( \nu_1 \) on \( X_1 \) induces a probability Borel measure \( \nu_y \) on \( X(y) \).

We will need the following two results.

**Proposition 15.** (Siegel’s formula, [11]) Let \( f \in L^1(\mathbb{R}^2) \), then
\[ (3.17) \quad \int_{X_1} \sum_{k \in \mathbb{Z}^2 \setminus 0} f(kM) \, d\nu_1(M) = \int_{\mathbb{R}^2} f(x) \, dx \]

**Proposition 16.** (8) Let \( \mathcal{E} \subset X \) be any Borel set; then \( y \to \nu_y(\mathcal{E} \cap X(y)) \) is a measurable function from \( \mathbb{R}^2 \) to \( \mathbb{R} \). If \( U \subset \mathbb{R}^2 \) is any Borel set such that \( \mathcal{E} \subset \bigcup_{y \in U} X(y) \), then
\[ (3.18) \quad \nu(\mathcal{E}) \leq \int_U \nu_y(\mathcal{E} \cap X(y)) \, dy \]
Furthermore, if \( \forall y_1 \neq y_2 \in U : X(y_1) \cap X(y_2) \cap \mathcal{E} = \emptyset \), then equality holds in (3.18).

Notice that Propositions 15 and 16 imply that if there are two different indices \( 1 \leq i, j \leq n \), such that

\[
\Delta_{ij}(h_i, h_j) = \{ g \in X : |\mathbb{Z}^2 g \cap (\mathcal{R}(t_i) \triangle \mathcal{R}(t_i + h_i))| > 0 \} \cap \\
\cap \{ g \in X : |\mathbb{Z}^2 g \cap (\mathcal{R}(t_j) \triangle \mathcal{R}(t_j + h_j))| > 0 \} \neq \emptyset,
\]

then

\[
\nu\{\Delta_{ij}(h_i, h_j)\} = \tilde{o}(\|h\|) \text{ as } \|h\| \to 0
\]

Therefore,

\[
(3.19) \quad G^{(n)}(t_1 + h_1, \ldots, t_n + h_n) - G^{(n)}(t_1, \ldots, t_n) = \\
= \sum_{j=1}^{n} (G^{(n)}(t_1, t_2, \ldots, t_{j-1}, t_j + h_j, t_{j+1}, \ldots, t_n) - G^{(n)}(t_1, \ldots, t_n)) + \tilde{o}(\|h\|) = \\
= \sum_{j=1}^{n} (\nu\{g \in X : |\mathbb{Z}^2 g \cap \mathcal{R}(t_j)\} = k_j - 1, |\mathbb{Z}^2 g \cap \mathcal{R}(t_j + h_j)| = k_j, \\
|\mathbb{Z}^2 g \cap \mathcal{R}(t_i)\} = k_i, i \neq j\} - \\
- \nu\{g \in X : |\mathbb{Z}^2 g \cap \mathcal{R}(t_j)\} = k_j, |\mathbb{Z}^2 g \cap \mathcal{R}(t_j + h_j)| = k_j + 1, \\
|\mathbb{Z}^2 g \cap \mathcal{R}(t_i)\} = k_i, i \neq j\} + \tilde{o}(\|h\|)
\]

Consider a single term in the expression above. Let

\[
\mathcal{E}_j = \mathcal{E}_j(h_j) = \{ g \in X : |\mathbb{Z}^2 g \cap \mathcal{R}(t_j)\} = k_j, \\
|\mathbb{Z}^2 g \cap \mathcal{R}(t_j + h_j)| = k_j + 1, |\mathbb{Z}^2 g \cap \mathcal{R}(t_i)\} = k_i, i \neq j\},
\]

and let \( U = \mathcal{R}(t_j + h_j) \setminus \mathcal{R}(t_j) \). Then by the proposition 16

\[
\nu(\mathcal{E}_j) = \int_U \nu_Y(\mathcal{E}_j \cap X(y)) dy = \int_{t_j}^{t_j + h_j} \int_{-1/2}^{1/2} \nu_{(x,y)}(\mathcal{E}_j \cap X(x,y)) dx dy
\]

Therefore, by proposition 15

\[
\lim_{h_j \to 0} \frac{1}{h_j} \nu(\mathcal{E}_j(h_j)) = \\
= \int_{-1/2}^{1/2} \nu_1(\{ g \in X_1 : |\mathbb{Z}^2 g \cap (\mathcal{R}(t_i) - (t_j, y))\} = k_i, (i = 1, \ldots, n\}) dy
\]

For every fixed \( y \in [-1/2, 1/2] \) continuity of the expression under the integral sign with respect to \( (t_1, \ldots, t_n) \) again follows from Proposition 15.

It is clearly uniform in \( y \) and therefore the integral is continuous with respect...
Each term in (3.19) can be treated in a similar way and this proves $C^1$ regularity of the function $G^{(n)}(t_1, \ldots, t_n)$ and finishes the proof of Theorem 9.

4. **Proof of Theorem 2**

We now deduce (1.1) from part (2) of Theorem 3.

Consider the unfolding of the tube to $\mathbb{R}^2$ obtained by the reflections from the horizontal boundary of the tube. Let $p_k = (x^k, \zeta^k)$ be the position of the particle at the moment of the $k$’th reflection from the wall in this unfolding.

![Figure 3](image)

**Figure 3.** An unfolded trajectory. In this example, $Q_\varepsilon = 3$, $[\zeta] = 2$ and $n^1_\varepsilon = 2$, $n^2_\varepsilon = 1$, $n^3_\varepsilon = 3$, $n^4_\varepsilon > 4$

Then

$$\xi^k = n^1_\varepsilon - n^2_\varepsilon + \ldots + (-1)^{k+1}n^k_\varepsilon$$

and

$$\zeta^k = y_{in} + \alpha(n^1_\varepsilon + n^2_\varepsilon + \ldots + n^k_\varepsilon)$$

At the moment of the exit from the tube, the vertical coordinate of the particle is

(4.20) $\zeta = 2(y_{in} + \alpha n^1_\varepsilon + \alpha n^3_\varepsilon + \ldots + \alpha n^{Q_\varepsilon}_\varepsilon) - y_{in}$

Let

$$z = y_{in} + \alpha n^1_\varepsilon + \alpha n^3_\varepsilon + \ldots + \alpha n^{Q_\varepsilon}_\varepsilon$$

and let $|| \cdot ||$ denote the distance to the nearest integer.

Then

$$||y_{in} + \alpha n^1_\varepsilon|| \leq \varepsilon/2, ||\alpha n^i_\varepsilon|| \leq \varepsilon \text{ for } i > 1$$

Therefore,

(4.21) $||z|| \leq \varepsilon Q_\varepsilon/2$
Notice that \( v_{\text{out}} = -v_{\text{in}} \), if both the number of reflections from vertical walls and from horizontal walls is odd. The former is obviously odd at the moment of exit. The number of reflections from the horizontal walls is equal to the integer part \( \lfloor \bar{\zeta} \rfloor \).

If \( z - \lfloor z \rfloor \leq 1/2 \), then by (4.20), \( \lfloor \bar{\zeta} \rfloor \) is odd provided that \( 2||z|| < y_{\text{in}} \), and if \( z - \lfloor z \rfloor > 1/2 \), then \( \lfloor \bar{\zeta} \rfloor \) is odd provided that \( 1 - 2||z|| > y_{\text{in}} \).

By (4.21) this is the case, when
\[
\varepsilon Q_\varepsilon < \min\{y_{\text{in}}, 1 - y_{\text{in}}\}
\]

By the assumption, the probability measure \( \mathbb{P} \) on the initial conditions \((y_{\text{in}}, \alpha)\) is absolutely continuous with respect to the Lebesgue measure, therefore for any \( k \in \mathbb{N} \),
\[
\mathbb{P}\{Q_\varepsilon = k, \varepsilon k < \min\{y_{\text{in}}, 1 - y_{\text{in}}\}\} = \mathbb{P}\{Q_\varepsilon = k\} - \mathbb{P}\{Q_\varepsilon = k, \min\{y_{\text{in}}, 1 - y_{\text{in}}\} \leq \varepsilon k\} \to G(k) \text{ as } \varepsilon \to 0
\]
Together with the tightness condition (3.11) this implies
\[
\mathbb{P}\{\varepsilon Q_\varepsilon < \min\{y_{\text{in}}, 1 - y_{\text{in}}\}\} \to 1 \text{ as } \varepsilon \to 0
\]
and so,
\[
\mathbb{P}\{v_{\text{out}} = -v_{\text{in}}\} \to 1 \text{ as } \varepsilon \to 0.
\]
Note that the existence of the limiting probability distribution for \( \{Q_\varepsilon\} \) as \( \varepsilon \to 0 \) also implies that for any \( \delta > 0 \),
\[
\mathbb{P}\{|y_{\text{out}} - y_{\text{in}}| > \delta\} \to 0 \text{ as } \varepsilon \to 0.
\]
Indeed, after each reflection from a vertical wall, the particle backtracks itself with an error at most \( \varepsilon \), so at the moment of exit it backtracks the incoming trajectory with total error of at most \( \varepsilon Q_\varepsilon \).

This finishes the proof of Theorem 2.

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