RIGHT (OR LEFT) INVERTIBILITY OF BOUNDED AND UNBOUNDED OPERATORS AND APPLICATIONS TO THE SPECTRUM OF PRODUCTS

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Abstract. This paper is mainly concerned with proving $\sigma(AB) = \sigma(BA)$ for two linear and non necessarily bounded operators $A$ and $B$. The main tool is left and right invertibility of bounded and unbounded operators.

1. Introduction

All operators considered here are linear and defined on a complex separable Hilbert space $H$. In order to avoid trivialities in the bounded case, we further assume that $\dim H = \infty$. Also, we assume that the reader is well aware of the basic notions of bounded and unbounded operators as well as the algebraic notions of right and left invertibility.

It is known that if no condition is imposed on either of the operators $A$ or $B$, then we are only sure that:

$$\sigma(AB) - \{0\} = \sigma(BA) - \{0\}.$$

We would like to know when

$$\sigma(AB) = \sigma(BA) \cdot \cdot \cdot (E)$$

holds for two linear bounded operators.

We already know that if one of the operators is invertible, then it may be shown that $AB$ and $BA$ are similar, hence they have the same spectrum. $(E)$ is also satisfied when one of the operators is compact.

Hladnik-Omladič [5] proved the following:

Theorem 1.1. Let $A, B \in B(H)$ be such that $B$ is positive. Let $P$ be the (unique) square root of $B$. Then

$$\sigma(AB) = \sigma(BA) = \sigma(PAP).$$

Another case for which the equality $\sigma(AB) = \sigma(BA)$ holds is when one of the operators is normal:

Theorem 1.2. (Barraa-Boumaazghour, [1]) Let $A, B \in B(H)$ be such that one of them is normal. Then

$$\sigma(AB) = \sigma(BA).$$

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The proof of the preceding theorem relied on the following:

**Theorem 1.3.** (Spain, [11]) A normal unilaterally invertible element of a complex unital Banach algebra is invertible.

But, as quoted in [11] the result "appears to have escaped notice up to now". However, Spain [11] did miss that in Conway’s [2] (Fredholm Theory Chapter).

The first observation in our work is that all of the three previous results become just a mere consequence (at least in $B(H)$!) of the next proposition whose simple proof is left to the reader.

**Proposition 1.4.** Let $A, B \in B(H)$ be such that $A$ is self-adjoint. If $AB = I$ (or $BA = I$), then $A$ is invertible and $B$ is self-adjoint.

**Corollary 1.5.** Let $A, B \in B(H)$ be such that $B$ is positive. Let $P$ be the (unique) square root of $B$. Then

$$
\sigma(AB) = \sigma(BA) = \sigma(PAP).
$$

**Proof.** To establish $\sigma(AB) = \sigma(BA)$, we have to show that $AB$ is invertible iff $BA$ is invertible. This is done as follows:

1. **$BA$ is invertible** $\Rightarrow$ $B$ is right invertible
2. $B$ is invertible (by Proposition 1.4)
3. $B^{-1}$ is invertible
4. $B^{-1}BA = A$ is invertible
5. $AB$ is invertible
6. $B$ is left invertible
7. $B$ is invertible (by Proposition 1.4)
8. $B^{-1}$ is invertible
9. $A = ABB^{-1}$ is invertible
10. $BA$ is invertible.

This settles the first equality. To prove the second equality, just apply the first equality to obtain

$$
\sigma(BA) = \sigma(PPA) = \sigma(PAP).
$$

□

Using the polar decomposition of a normal operator, Proposition 1.4 yields

**Corollary 1.6.** Let $A \in B(H)$ be a right (or left) invertible normal operator. Then $A$ is invertible.

**Proof.** Left to the reader. □

This (combined with Corollary 1.5) gives

**Corollary 1.7.** Let $A, B \in B(H)$ be such that one of them is normal. Then

$$
\sigma(AB) = \sigma(BA).
$$
Unfortunately, we cannot go up to the class of hyponormal operators. Indeed, consider the usual (unilateral) shift $S$ on $\ell^2$. Then $S^*S = I$, $SS^* \neq I$ and $S$ is hyponormal. Hence

1. $S$ is left invertible without being invertible;
2. Also, $\sigma(S^*S) = \{1\} \neq \sigma(SS^*) = \{0, 1\}$.

We can, however, generalize the previous results to non necessarily bounded operators. Moreover, normality is not indispensable. Only a condition of the type $\ker(A) = \ker(A^*)$ (or even an inclusion in some cases) will suffice. See Theorem 2.3.

It is worth noticing, that the works on the spectra of unbounded products are only numbered. For instance, see [3], [4], [8] and [10].

We conclude this introduction with an application of Corollary 1.5 (cf. [9]).

**Proposition 1.8.** (cf. Theorem 3.6) Let $A, B \in B(H)$ be such that $A$ is positive and $B$ is self-adjoint. If $AB$ (or $BA$) is hyponormal, then $AB$ (or $BA$) is self-adjoint.

**Proof.** By Corollary 1.5, $\sigma(AB)$ (or $\sigma(BA)$) is real. The result then follows by remembering that a hyponormal operator with a real spectrum is self-adjoint (see e.g. [12]).

**Corollary 1.9.** Let $A, B \in B(H)$ be such that $A$ is positive and $B$ is self-adjoint. If $AB$ (or $BA$) is hyponormal, then $A + iB$ is normal.

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2. LEFT OR RIGHT INVERTIBLE UNBOUNDED OPERATORS

First, recall (cf. [2]):

**Definition.** An unbounded linear operator $A$ with domain $D(A) \subset H$, is said to be invertible if there exists an everywhere defined $B \in B(H)$ such that $AB = I$ and $BA \subset I$.

**Remark.** It is known that if $A$ and $B$ are two unbounded and invertible operators, then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

**Remark.** The invertibility of $A$ is equivalent to requiring $A$ to be injective and $A^{-1} \in B(H)$. Hence, if $A$ is invertible, then $A$ is closed and if $A$ is closed and densely defined, then $A$ is invertible iff $A^*$ is so.

Based on the definition just above and the bounded case, we introduce:

**Definition.** Let $H$ be a Hilbert space and let $A$ be an unbounded operator with domain $D(A) \subset H$. We say that $A$ is **right invertible** if there exists an everywhere defined $B \in B(H)$ such that $AB = I$; and we say that $A$ is **left invertible** if there is an everywhere defined $C \in B(H)$ such that $CA \subset I$.

**Remark.** It is easily seen that if $A$ is closed and densely defined, then $A$ is left (resp. right) invertible iff $A^*$ is right (resp. left) invertible.

To show the importance of the notion of left or right invertibility, we give the following result:

**Proposition 2.1.** Let $A$ be a non necessarily bounded operator with domain $D(A) \subset H$. If $A$ is left and right invertible simultaneously, then $A$ is invertible.
Proof. By assumption \( AB = I \) and \( CA \subseteq I \) for some bounded \( B, C \in B(H) \). Then \( C = C(AB) = (CA)B \subseteq B \), and hence \( B = C \) as they are both everywhere defined.

\[ \square \]

**Corollary 2.2.** A right (or left) invertible non necessarily bounded self-adjoint operator is invertible.

We now turn to non necessarily bounded normal operators. Fortunately, Corollary [1.6] also holds for unbounded operators. In fact, the result is true for a more general class of operators.

**Theorem 2.3.** A right (resp. left) invertible closed and densely defined operator \( A \) such that \( \ker(A) \subseteq \ker(A^*) \) (resp. \( \ker(A^*) \subseteq \ker(A) \)) is invertible. In particular, if \( A \) is closed and densely defined and \( \ker(A) = \ker(A^*) \), then \( A \) is left invertible iff \( A \) is right invertible iff \( A \) is invertible.

Proof. By the two remarks just above, it suffices to consider the case of right invertibility. So assume that \( A \) is right invertible, i.e. \( AB = I \) for some \( B \in B(H) \). Hence

\[ \text{ran}(A) = H \implies \ker(A^*) = \{0\} \implies \ker(A) = \{0\}. \]

Since \( A^{-1} \) is closed and \( D(A^{-1}) = \text{ran}(A) = H \), the Closed Graph Theorem yields \( A^{-1} \in B(H) \), that is, \( A \) is invertible.

\[ \square \]

**Remark.** Plainly, self-adjoint and normal operators \( A \) are closed densely defined and they obey \( \ker(A) = \ker(A^*) \).

On the other hand, if \( A \) is closed and hyponormal, then \( \ker(A) \subseteq \ker(A^*) \). So a right invertible closed hyponormal operator is invertible. Similarly, a left invertible closed cohyponormal operator is invertible.

3. Spectra of Products of Unbounded Operators

In this paper, we use the following definition (cf. [2]) of the spectrum:

**Definition.** Let \( A \) be a non necessarily bounded operator with domain \( D(A) \subseteq H \). We say that \( \lambda \) is not in \( \sigma(A) \) if \( A - \lambda \) is injective and \( (A - \lambda)^{-1} \) is in \( B(H) \).

**Remark.** Using the previous definition, we easily see that if \( \sigma(A) \neq \mathbb{C} \), then \( A \) is closed.

In [3], it is shown that if \( A \) and \( B \) are two non necessarily bounded operators such that \( \sigma(AB) \neq \mathbb{C} \) and \( \sigma(BA) \neq \mathbb{C} \) (hence both \( AB \) and \( BA \) are closed), then

\[ \sigma(AB) - \{0\} = \sigma(BA) - \{0\}. \]

It is clear that if we want to obtain the equality \( \sigma(AB) = \sigma(BA) \), we must show that \( AB \) is invertible iff \( BA \) is invertible. We reserve a substantial part to this equivalence.

**Theorem 3.1.** Assume \( A \) is a closed and densely defined operator in \( H \) and \( B \in B(H) \) is such that \( BA \) is invertible. If either \( \ker(A^*) \subseteq \ker(A) \) or \( \ker(B) \subseteq \ker(B^*) \), then the operators \( A, B \) and consequently \( AB \) are invertible.

To prove this theorem we need a lemma.

**Lemma 3.2.** If \( A \) is closed and \( B \) is an operator such that \( BA \) is right invertible, then \( B \) too is right invertible.
Proof. Since \( BAC = I \) for some \( C \in B(H) \), it follows that \( D(AC) = H \). Hence by the Closed Graph Theorem, \( AC \in B(H) \), and we are done. □

Now we give a proof of Theorem 3.1.

Proof.

- \( \ker(A^*) \subseteq \ker(A) \): We may write
  \[
  BA \text{ invertible } \implies DBA \subseteq I \text{ for some } D \in B(H) \\
  \implies A \text{ left invertible} \\
  \implies A^{-1} \in B(H) \text{ (Theorem 2.3)} \\
  \implies B = (BA)A^{-1} \text{ injective} \\
  \implies A^{-1}B^{-1} = (BA)^{-1} \in B(H) \\
  \implies D(B^{-1}) = H.
  \]
  In fine, the Closed Graph Theorem gives \( B^{-1} \in B(H) \), as required.

- \( \ker(B) \subseteq \ker(B^*) \): We can write:
  \[
  BA \text{ invertible } \implies B \text{ right invertible (Lemma 3.2)} \\
  \implies B^{-1} \text{ invertible} \\
  \implies A = B^{-1}(BA) \text{ invertible.}
  \]

Accordingly, \( A, B \) and \( AB \) are all invertible.

Interchanging the roles of \( BA \) and \( AB \) in the assumptions of the foregoing theorem does not lead to the invertibility of \( BA \). An extra condition has to be added. We have:

**Theorem 3.3.** Assume \( A \) is a closed densely defined operator and \( B \in B(H) \) is such that \( B^*A^* \) is closed and \( AB \) is invertible. If either \( \ker(A) \subseteq \ker(A^*) \) or \( \ker(B^*) \subseteq \ker(B) \), then the operators \( A, B \) and consequently \( BA \) are invertible.

Proof. This is a consequence of Theorem 3.1. Indeed, since \( AB = (B^*A^*)^* \) is invertible and \( B^*A^* \) is closed and densely defined, we infer from the second remark in the beginning of Section 2 that \( B^*A^* \) is invertible. Applying Theorem 3.1, we see that \( A^* \) and \( B^* \) are invertible, and by the same remark again so are \( A \) and \( B \). □

**Corollary 3.4.** Let \( A \) be a closed densely defined operator in \( H \) and let \( B \in B(H) \) be such that \( \sigma(B^*A^*) \neq \mathbb{C} \) and \( \sigma(BA) \neq \mathbb{C} \). If either \( \ker(A^*) = \ker(A) \) or \( \ker(B) = \ker(B^*) \), then
  \[
  \sigma(BA) = \sigma(AB).
  \]

Proof. Since \( \sigma(AB) = \sigma((B^*A^*)^*) = [\sigma(B^*A^*)]^* \neq \mathbb{C} \), we see that the required result follows from Theorem 3.1, Theorem 3.3 and [3]. □

Remark. A condition of the type \( \sigma(BA) \neq \mathbb{C} \) is not unnatural. Remember that if \( A, B \in B(H) \), then we always have \( \sigma(BA) \neq \mathbb{C} \)!
**Corollary 3.5.** Let $A$ and $B$ be two self-adjoint operators such that $B$ is bounded. If $\sigma(BA) \neq \mathbb{C}$, then 
$$\sigma(AB) = \sigma(BA).$$

We finish this paper with the following result (cf. [9]).

**Theorem 3.6.** Let $A$ and $B$ be two self-adjoint operators such that $B$ is bounded and positive. If $BA$ is hyponormal, then both $BA$ and $AB$ are self-adjoint (and $AB = BA$!) whenever $\sigma(BA) \neq \mathbb{C}$.

**Remark.** The foregoing theorem was first shown with the extra assumption "$B$ being injective". Then, we discussed with Professor Jan Stochel whether the closedness of $P^2A$ would imply that of $PA$, whenever $P \in B(H)$ is self-adjoint? The answer turned out to be positive and here is the result:

**Proposition 3.7.** Let $P \in B(H)$ be self-adjoint and let $A$ be an arbitrary operator such that $P^2A$ is closed. Then $PA$ is closed.

**Proof.** Let $(x_n)$ be in $D(PA) = D(A)$ such that 
$$PAx_n \rightarrow y \text{ and } x_n \rightarrow x.$$ 
Then it is clear that $y \in \text{ran}(P)$. Since $P$ is continuous, we obtain 
$$P^2Ax_n \rightarrow Py \text{ and } x_n \rightarrow x.$$ 
As $P^2A$ is closed, we then obtain 
$$P^2Ax = Py \text{ and } x \in D(P^2A) = D(A).$$

Hence $P(y - PAx) = 0$, that is, $y - PAx \in \ker(P)$. Since also $y - PAx \in \text{ran}(P)$ and $P$ is self-adjoint, we get $y - PAx \in \mathbb{R}$. Thus $y - PAx = 0$ or $PAx = y$. Since we already know that $x \in D(A) = D(PA)$, the proof of the closedness of $PA$ is complete. □

Now, we prove Theorem 3.6

**Proof.** Let $P$ be the unique square root of $B$. Since $\sigma(BA) \neq \mathbb{C}$, $BA$ or $P^2A$ is closed so that $PA$ is closed by Proposition 3.7. The rest of the proof is divided into two parts.

1. First, $PAP$ is self-adjoint: Since $P$ is bounded and $PA$ is closed, we have 
$$\sigma(PAP) = \sigma(PA) = \sigma(PAP),$$ 
i.e. $PAP$ is surely self-adjoint so that $\sigma(PAP) \neq \mathbb{C}$.

2. Second, we show that $BA$ and $AB$ are self-adjoint: Since $\sigma(P^2A) \neq \mathbb{C}$ and $\sigma(PAP) \neq \mathbb{C}$, by Corollary 3.4

$$\sigma(BA) = \sigma(PAP) \subset \mathbb{R}.$$ 

Now, if $W(BA)$ denotes the numerical range of $BA$, then from [6] we know that 
$$W(BA) \subset \text{conv} \sigma(BA) \subset \mathbb{R}$$ 
for $BA$ is hyponormal. Thus $BA$ is closed, symmetric and with real spectrum, it is self-adjoint! Accordingly, 
$$AB = (BA)^* = BA.$$ □
Remark. If we assume that $BA$ is subnormal (which is stronger than hyponormal), then we can obtain the self-adjointness of $BA$ and $AB$ without using the machinery of the preceding proof, we just apply Theorem 4.2 of [13], and other known properties.

4. Conclusion

It was the referee’s idea to improve the results in the case of unbounded operators by using conditions on kernels. Indeed, in the first version of the paper we only dealt with normal and self-adjoint operators. Needless to say that some of the results in the bounded case are particular cases of some of those of their unbounded counterparts.

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