ENDOMORPHISM ALGEBRAS OF GEOMETRICALLY SPLIT
ABELIAN SURFACES OVER \( \mathbb{Q} \)

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ABSTRACT. We determine the set of geometric endomorphism algebras of geometrically split abelian surfaces defined over \( \mathbb{Q} \). In particular we find that this set has cardinality 92. The essential part of the classification consists in determining the set of quadratic imaginary fields \( M \) with class group \( C_2 \times C_2 \) for which there exists an abelian surface \( A \) defined over \( \mathbb{Q} \) which is geometrically isogenous to the square of an elliptic curve with CM by \( M \). We first study the interplay between the field of definition of the geometric endomorphisms of \( A \) and the field \( M \). This reduces the problem to the situation in which \( E \) is a \( \mathbb{Q} \)-curve in the sense of Gross. We can then conclude our analysis by employing Nakamura’s method to compute the endomorphism algebra of the restriction of scalars of a Gross \( \mathbb{Q} \)-curve.

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1. Introduction

Let \( A \) be an abelian variety of dimension \( g \geq 1 \) defined over a number field \( k \) of degree \( d \). Let us denote by \( A_{\overline{\mathbb{Q}}} \) its base change to \( \overline{\mathbb{Q}} \). We refer to \( \text{End}(A_{\overline{\mathbb{Q}}}) \), the \( \mathbb{Q} \)-algebra spanned by the endomorphisms of \( A \) defined over \( \overline{\mathbb{Q}} \), as the \( \overline{\mathbb{Q}} \)-endomorphism algebra of \( A \). For a fixed choice of \( g \) and \( d \), it is conjectured that the set of possibilities for \( \text{End}(A_{\overline{\mathbb{Q}}}) \) is finite. A slightly stronger form of this conjecture, applying
to endomorphism rings of abelian varieties over number fields, has been attributed to Coleman in [BFGR06].

Hereafter, let $A$ denote an abelian surface defined over $\mathbb{Q}$. In the case that $A$ is geometrically simple (that is, $A_{\overline{\mathbb{Q}}}$ is simple), the previous conjecture stands widely open. If $A$ is principally polarized and has CM it has been shown (see [MU01], [BS17], and [KS23]) that $\text{End}(A_{\overline{\mathbb{Q}}})$ is one of 13 possible quartic CM fields. However, narrowing down to a finite set the possible quadratic real fields and quaternion division algebras over $\mathbb{Q}$ which occur as $\text{End}(A_{\overline{\mathbb{Q}}})$ for some $A$ has escaped all attempts of proof. See also [OS18] for recent more general results which prove Coleman’s conjecture for CM abelian varieties.

In the present paper, we focus on the case that $A$ is geometrically split, that is, the case in which $A_{\overline{\mathbb{Q}}}$ is isogenous to a product of elliptic curves, which we will assume from now on. Let $A$ be the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$, where $A$ is a geometrically split abelian surface over $\mathbb{Q}$.

Let us briefly recall how scattered results in the literature ensure the finiteness of $A$ (we will detail the arguments in Section 4). Indeed, if $A_{\overline{\mathbb{Q}}}$ is isogenous to the product of two non-isogenous elliptic curves, then the finiteness (and in fact the precise description) of the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$ follows from [FKRS12, Proposition 4.5]. If, on the contrary, $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve, then the finiteness of the set of possibilities for $\text{End}(A_{\overline{\mathbb{Q}}})$ was established by Shafarevich in [Sha96] (see also [Cre92] and [Gon11] for the determination of the precise subset corresponding to modular abelian surfaces). In the present work, we aim at an effective version of Shafarevich’s result. Our starting point is [FG18, Theorem 1.4], which we recall in our particular setting.

**Theorem 1.1** ([FG18]). If $A$ is an abelian surface defined over $\mathbb{Q}$ such that $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve $E/\mathbb{Q}$ with CM by a quadratic imaginary field $M$, then the class group of $M$ is $1$, $C_2$, or $C_2 \times C_2$.

It should be noted that several other works can be used to see that, in the situation of the theorem, the exponent of the class group of $M$ divides 2 (see [Sch07] or [Kan11], for example).

While it is an easy observation that an abelian surface $A$ as in the theorem can be found for each quadratic imaginary field $M$ with class group $1$ or $C_2$, or $C_2 \times C_2$ (see [FG18, Remark 2.20] and also Section 4), the question whether such an $A$ exists for each of the fields $M$ with class group $C_2 \times C_2$ is far from trivial. The aforementioned results are thus not sufficient for the determination of the set $A$. The main contribution of this article is the following theorem.

**Theorem 1.2.** Let $M$ be a quadratic imaginary field with class group $C_2 \times C_2$. There exists an abelian surface defined over $\mathbb{Q}$ such that $A_{\overline{\mathbb{Q}}}$ is isogenous to the square of an elliptic curve $E/\mathbb{Q}$ with CM by $M$ if and only if the discriminant of $M$ belongs to the set

$$\{ -84, -120, -132, -168, -228, -280, -372, -408, -435, -483, -520, -532, -595, -627, -708, -795, -1012, -1435 \}. \quad (1.1)$$

The only imaginary quadratic fields with class group $C_2 \times C_2$ whose discriminant does not belong to $\{1, 11\}$ are

$$\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760}). \quad (1.2)$$
With Theorem 1.2 at hand, the determination of the set $A$ follows as a mere corollary (see §4 for the proof).

**Corollary 1.3.** The set $A$ of $\mathbb{Q}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:

i) $\mathbb{Q} \times \mathbb{Q}$, $\mathbb{Q} \times M$, $M_1 \times M_2$, where $M$, $M_1$ and $M_2$ are quadratic imaginary fields of class number 1;

ii) $M_2(\mathbb{Q})$, $M_2(M)$, where $M$ is a quadratic imaginary field with class group $1 \times C_2$, or $C_2 \times C_2$ and distinct from those listed in (1.2).

In particular, the set $A$ has cardinality 92.

The paper is organized in the following manner. In Section 2 we attach a $c$-representation $\varrho_V$ of degree 2 to an abelian surface $A$ defined over $\mathbb{Q}$ such that $A_{\mathbb{Q}}$ is isogenous to the square of an elliptic curve $E/\mathbb{Q}$ with CM by $M$. It is well known that $E$ is a $\mathbb{Q}$-curve and that one can associate a 2-cocycle $c_E$ to $E$. A $c$-representation is essentially a representation up to scalar and it is thus a notion closely related to that of projective representation. In the case of the $c$-representation $\varrho_V$ attached to $A$, the scalar that measures the failure of $\varrho_V$ to be a proper representation is precisely the 2-cocycle $c_E$. Choosing the language of $c$-representations instead of that of projective representations has an unexpected payoff: the tensor product of a $c$-representation $\varrho_V$ and its contragradient $\varrho_V^*$ is again a proper representation. We show that $\varrho_V \otimes \varrho_V^*$ coincides with the representation of $G_{\mathbb{Q}}$ on the 4 dimensional $M$-vector space $\text{End}(A_{\mathbb{Q}}^+)$. This representation has been studied in detail in [FS14] and the tensor decomposition of $\text{End}(A_{\mathbb{Q}})$ is exploited in Theorems 2.20 and 2.27 to obtain obstructions on the existence of $A$. These obstructions extend to the general case those obtained in [FG18, §3.1, §3.2] under very restrictive hypotheses. The $c$-representation point of view also allows us to understand in a unified manner what we called group theoretic and cohomological obstructions in [FG18]. It should be noted that one can define analogues of $\varrho_V$ in other more general situations. For example, a parallel construction in the context of geometrically isotypic abelian varieties potentially of $GL_2$-type has been exploited in [FG19] to determine a tensor factorization of their Tate modules. This can be used to deduce the validity of the Sato-Tate conjecture for them in certain cases.

In Section 3, we describe a method of Nakamura to compute the endomorphism algebra of the restriction of scalars of certain Gross $\mathbb{Q}$-curves (see Definition 2.9 below for the precise definition of these curves). Then we apply this method to all Gross $\mathbb{Q}$-curves with CM by a field $M$ of class group $C_2 \times C_2$. This computation plays a key role in the proof of Theorem 1.2, both in proving the existence of the abelian surfaces for the fields $M$ different from those listed in (1.2), and in proving the non-existence for the fields of (1.2).

In Section 4 we culminate the proofs of Theorem 1.2 and Corollary 1.3 by assembling together the obstructions and existence results from Sections 2 and 3. We essentially show that we can use the results of Section 2 to reduce to the case of Gross $\mathbb{Q}$-curves, and then we deal with this case using the results of Section 3.

**Notations and terminology.** For $k$ a number field, we will work in the category of abelian varieties up to isogeny over $k$. Note that isogenies become invertible in this category. Given an abelian variety $A$ defined over $k$, the set of endomorphisms $\text{End}(A)$ of $A$ defined over $k$ is endowed with a $\mathbb{Q}$-algebra structure. More generally,
if $B$ is an abelian variety defined over $k$, we will denote by $\text{Hom}(A,B)$ the $\mathbb{Q}$-vector space of homomorphisms from $A$ to $B$ that are defined over $k$. We note that for us $\text{End}(A)$ and $\text{Hom}(A,B)$ denote what some other authors call $\text{End}^0(A)$ and $\text{Hom}^0(A,B)$. We will write $A \sim B$ to mean that $A$ and $B$ are isogenous over $k$. If $L/k$ is a field extension, then $A_L$ will denote the base change of $A$ from $k$ to $L$. In particular, we will write $A_L \sim B_L$ if $A$ and $B$ become isogenous over $L$, and we will write $\text{Hom}(A_L,B_L)$ to refer to what some authors write as $\text{Hom}_L(A,B)$.

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2. $c$-representations and $k$-curves

The goal of this section is to obtain obstructions to the existence of abelian surfaces defined over $\mathbb{Q}$ such that $\text{End}(A_{\mathbb{Q}}) \cong M_2(M)$, where $M$ is a quadratic imaginary field. To this purpose, we analyze the interplay between the $k$-curves and $c$-representations that arise from them.

2.1. $c$-representations: general definitions. Let $V$ be a vector space of finite dimension over a field $k$ and let $G$ be a finite group. We say that a map $\varrho_V : G \to \text{GL}(V)$ is a $c$-representation (of the group $G$) if $\varrho_V(1) = 1$ and there exists a map $c_V : G \times G \to k^\times$ such that for every $\sigma, \tau \in G$ one has

$$\varrho_V(\sigma)\varrho_V(\tau) = \varrho_V(\sigma\tau)c_V(\sigma, \tau).$$

(2.1)

Remark 2.1. The following properties follow easily from the definition:

i) Note that we have $\varrho_V(\sigma^{-1}) = \varrho_V(\sigma)^{-1}c_V(\sigma^{-1}, \sigma)$ and $\varrho_V(\sigma^{-1}) = \varrho_V(\sigma)^{-1}c_V(\sigma, \sigma^{-1})$.

In particular, $c_V(\sigma, \sigma^{-1}) = c_V(\sigma^{-1}, \sigma)$.

ii) Note that if $c_V(\cdot, \cdot) = 1$, the notion of $c$-representation corresponds to the usual notion of representation.
Let $V$ and $W$ be $c$-representations of the group $G$. Let $T = \text{Hom}(V, W)$ denote the space of $k$-linear maps from $V$ to $W$. A homomorphism of $c$-representations from $V$ to $W$ is a $k$-linear map $f \in T$ such that
\[ f(v) = \varrho_W(\sigma)(f(\varrho_V(\sigma)^{-1}v)) \]
for every $v \in V$ and $\sigma \in G$.

Consider now the map
\[ \varrho_T : G \to \text{GL}(\text{Hom}(V, W)), \]
declared by
\[ (\varrho_T(\sigma)f)(v) = \varrho_W(\sigma)(f(\varrho_V(\sigma)^{-1}v)). \]

**Proposition 2.2.** The map $\varrho_T$ together with the map $c_T : G \times G \to k^\times$ defined by $c_T = \varrho_V^{-1} \cdot \varrho_W$ equip $T$ with the structure of a $c$-representation.

Before proving the proposition we show a particular case. In the case that $W$ is $k$ equipped with the trivial action of $G$, let us write $V^* = T$ and $\varrho^* = \varrho_T$. In this case, $\varrho^*(\sigma)$ is the inverse transpose of $\varrho_V(\sigma)$. The assertion of the proposition is then immediate from (2.1).

The following two lemmas, whose proof is straightforward, imply the proposition.

**Lemma 2.3.** The maps $\varrho_\otimes : G \to \text{GL}(V \otimes W)$, defined by $\varrho_\otimes(\sigma)(v \otimes w) = \varrho_V(\sigma)v \otimes \varrho_W(\sigma)w$ and $c_\otimes = \varrho_V \cdot \varrho_W$ endow $V \otimes W$ with a structure of $c$-representation.

**Lemma 2.4.** The map
\[ \phi : W \otimes V^* \to T \]
declared by $\phi(w \otimes f)(v) = f(v)w$ is an isomorphism of $c$-representations.

**Corollary 2.5.** When $V = W$, the $c$-representation $T$ is in fact a representation.

### 2.2. $k$-curves: general definitions

We briefly recall some definitions and results regarding $\mathbb{Q}$-curves and, more generally, $k$-curves with complex multiplication. More details can be found in [FG18, §2.1] and the references therein (especially [Que00], [Rib92], and [Nak04]).

Let $E/\mathbb{Q}$ be an elliptic curve and let $k$ be a number field, whose absolute Galois group we denote by $G_k$.

**Definition 2.6.** We say that $E$ is a $k$-curve if for every $\sigma \in G_k$ there exists an isogeny $\mu_\sigma : \sigma E \to E$.

**Definition 2.7.** We say that $E$ is a Ribet $k$-curve if $E$ is a $k$-curve and the isogenies $\mu_\sigma$ can be taken to be compatible with the endomorphisms of $E$, in the sense that the following diagram
\[
\begin{array}{ccc}
\sigma E & \xrightarrow{\mu_\sigma} & E \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
\sigma E & \xrightarrow{\mu_\sigma} & E \\
\end{array}
\]
commutes for all $\sigma \in G_k$ and all $\varphi \in \text{End}(E)$.
Remark 2.8.  i) Observe that if \( E \) does not have CM, then \( E \) is a \( k \)-curve if and only if it is a Ribet \( k \)-curve. If \( E \) has CM (say by a quadratic imaginary field \( M \)), it is well known that \( E \) is isogenous to all of its Galois conjugates and hence it is always a \( k \)-curve; it is a Ribet \( k \)-curve if and only if \( M \subseteq k \) (cf. [Shi94, Theorem 2.2]).

ii) We warn the reader that in the present paper we are using a slightly different terminology from that of [FG18]: as in [FG18] the only relevant notion was that of a Ribet \( k \)-curve, we called Ribet \( k \)-curves simply \( k \)-curves.

Let \( K \) be a number field containing \( k \). We say that an elliptic curve \( E/K \) is a \( k \)-curve defined over \( K \) (resp. a Ribet \( k \)-curve defined over \( K \)) if \( E \subseteq K \) is a \( k \)-curve (resp. a Ribet \( k \)-curve). We will say that \( E \) is completely defined over \( K \) if, in addition, all the isogenies \( \mu: \sigma E \to E \) can be taken to be defined over \( K \).

Definition 2.9. Let \( H \) denote the Hilbert class field of \( M \) and let \( E/H \) be an elliptic curve with CM by \( M \). We say that \( E \) is a Gross \( Q \)-curve if \( E \) is completely defined over \( H \).

The next proposition characterizes the existence of Gross \( Q \)-curves and Ribet \( M \)-curves with CM by \( M \) defined over the Hilbert class field \( H \).

Proposition 2.10. Let \( M \) be a quadratic imaginary field and let \( D \) denote its discriminant. Then:

i) There exists a Ribet \( M \)-curve \( E^* \) with CM by \( M \) and completely defined over \( H \).

ii) There exists a Gross \( Q \)-curve \( E^* \) with CM by \( M \) (and completely defined over \( H \)) if and only if \( D \) is not of the form

\[
D = -4p_1 \ldots p_{t-1},
\]

where \( t \geq 2 \) and \( p_1, \ldots, p_{t-1} \) are primes congruent to 1 modulo 4.

The first part of the previous proposition is a weaker form of [Shi71, Proposition 5, p. 521] (see also [Nak01, Remark 1]) and [Nak04, Proposition 5]. Discriminants of the form \( D \) are called exceptional.

Suppose from now on that \( E \) is a \( k \)-curve defined over \( K \) with CM by an imaginary quadratic field \( M \). Fix a system of isogenies \( \{ \mu_\sigma: \sigma E \to E \}_{\sigma \in G_k} \). By enlarging \( K \) if necessary, we can always assume that \( K/k \) is Galois and that \( E \) is completely defined over \( K \). We will equip \( \text{End}(E) \) with the following action. For \( \sigma \in \text{Gal}(K/k) \) and \( \varphi \in \text{End}(E) \) define

\[
\sigma \ast \varphi = \mu_\sigma \circ \varphi \circ \mu_\sigma^{-1}.
\]

Note that if \( E \) is a Ribet \( k \)-curve, then this action is trivial. If we regard \( M \) as a \( \text{Gal}(K/k) \)-module by means of the natural Galois action (which is actually the trivial action when \( k \) contains \( M \)) and \( \text{End}(E) \) endowed with the action defined above, then the identification of \( \text{End}(E) \) with \( M \) becomes a \( \text{Gal}(K/k) \)-equivariant isomorphism. The map

\[
c_K^E: \quad \text{Gal}(K/k) \times \text{Gal}(K/k) \to M^\times, \\
(\sigma, \tau) \mapsto \mu_\sigma \circ \sigma \circ \mu_\tau^{-1} \circ \mu_\tau^{-1}
\]

satisfies the condition

\[
(\varphi \ast c_K^E(\sigma, \tau)) \cdot c_K^E(\varphi, \sigma, \tau)^{-1} \cdot c_K^E(\varphi, \sigma) \cdot c_K^E(\varphi, \sigma)^{-1} = 1,
\]
for \( \varrho, \sigma, \tau \in \text{Gal}(K/k) \), and is then a 2-cocycle\(^1\). Denote by \( \gamma_E^K \) the cohomology class in \( H^2(\text{Gal}(K/k), M^\times) \) corresponding to \( c_E^K \). The class \( \gamma_E^K \) only depends on the \( K \)-isogeny class of \( E \).

The next result is a consequence of Weil’s descent criterion, extended to varieties up to isogeny by Ribet (Ribet 92 §8).

**Theorem 2.11** (Ribet–Weil). Suppose that \( E \) is a Ribet \( k \)-curve completely defined over \( K \) (and hence \( M \subseteq k \)). Let \( L \) be a number field with \( k \subseteq L \subseteq K \), and consider the restriction map

\[
\text{res}: H^2(\text{Gal}(K/k), M^\times) \to H^2(\text{Gal}(K/L), M^\times).
\]

If \( \text{res}(\gamma_E^K) = 1 \), there exists an elliptic curve \( C/L \) such that \( E \sim C_K \).

### 2.3. \( M \)-curves from squares of CM elliptic curves

Let \( M \) be a quadratic imaginary field. Let \( A \) be an abelian surface defined over \( \mathbb{Q} \) such that \( A \mathbb{Q} \) is isogenous to \( E^2 \), where \( E \) is an elliptic curve defined over \( \mathbb{Q} \) with CM by \( M \). Let \( K/\mathbb{Q} \) denote the minimal extension over which

\[
\text{End}(A \mathbb{Q}) \simeq \text{End}(A_K).
\]

By the theory of complex multiplication, \( K \) contains the Hilbert class field \( H \) of \( M \). Note also that \( K/\mathbb{Q} \) is Galois and the possibilities for \( \text{Gal}(K/\mathbb{Q}) \) can be read from [FKRS12, Table 8]. For our purposes, it is enough to recall that

\[
\text{Gal}(K/M) \simeq \begin{cases} 
C_r & \text{for } r \in \{1, 2, 3, 4, 6\}, \\
D_r & \text{for } r \in \{2, 3, 4, 6\}, \\
A_4, S_4 & \text{for } r \in \{4\}.
\end{cases}
\]

Here, \( C_r \) denotes the cyclic group of \( r \) elements, \( D_r \) denotes the dihedral group of \( 2r \) elements, and \( A_4 \) (resp. \( S_4 \)) stands for the alternating (resp. symmetric) group on 4 letters.

We can (and do) assume that \( E \) is in fact defined over \( K \), and then we have that \( A_K \sim E^2 \). For \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) we have that \( (\sigma E)^2 \sim \sigma A_K = A_K \sim E^2 \). Therefore, Poincaré’s decomposition theorem implies that \( E \) is a \( \mathbb{Q} \)-curve completely defined over \( K \).

For the purposes of this article, we need to consider the following (slightly more general) situation: Let \( N/M \) be a Galois subextension of \( K/M, \) and let \( E^* \) be a Ribet \( M \)-curve which is completely defined over \( N \) and such that \( E_\mathbb{Q}^* \sim E^{*}_\mathbb{Q} \). Observe that there always exist \( N \) and \( E^* \) satisfying these conditions, for example by taking \( N = K \) and \( E^* = E \); but in [2.4] and [2.5] below we will exploit certain situations where \( N \subseteq K \) and \( E^* \neq E \).

Then we can consider two cohomology classes: the class \( \gamma_E^N \) attached to \( E \), and the class \( \gamma_N^E \) attached to \( E^* \). We recall the following key result about \( \gamma_E^N \), which is a particular case of [FG18, Corollary 2.4].

**Theorem 2.12.** The cohomology class \( \gamma_E^K \) is 2-torsion.

Denote by \( U \) the set of roots of unity of \( M \) and put \( P = M^\times/U \). The same argument of [FG18, Proof of Theorem 2.14] proves the following decomposition of

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\(^1\)Actually, this is the inverse of the cocycle considered in [FG18], but this does not affect any of the results that we will use.
the 2-torsion of $H^2(\Gal(K/M), M^\times)$:

$$(2.6) \quad H^2(\Gal(K/M), M^\times)[2] \simeq H^2(\Gal(K/M), U)[2] \times \Hom(\Gal(K/M), P/P^2).$$

If $M \neq \Q(i), \Q(\sqrt{-3})$ this particularizes to

$$(2.7) \quad H^2(\Gal(K/M), M^\times)[2] \simeq H^2(\Gal(K/M), \{\pm 1\}) \times \Hom(\Gal(K/M), P/P^2).$$

For $\gamma \in H^2(\Gal(K/M), M^\times)[2]$ we will denote by $(\gamma_{\pm}, \bar{\gamma})$ its components under the isomorphism (2.7); we will refer to $\gamma_{\pm}$ as the sign component and to $\bar{\gamma}$ as the degree component.

In order to study the relation between $\gamma^K_E$ and $\gamma^N_{E^*}$, define $L/K$ to be the smallest extension such that $E^*_L$ and $E_L$ are isogenous. Since all the endomorphisms of $E$ are defined over $K$, this is also the smallest extension $L/K$ such that $\Hom(E^*_L, E_L) = \Hom(E^*_Q, E_Q)$. The extension $L/Q$ is Galois. Indeed, for $\sigma \in G_Q$ put $L' = \sigma L$ and let $\beta_\sigma : \sigma E^* \to E^*$ and $\mu_\sigma : \sigma E \to E$ be isogenies defined over $N$ and over $K$ respectively; then, if $\phi : E^*_L \to E_L$ is an isogeny defined over $L$ we find that $\mu_\sigma \circ \sigma \circ \beta_\sigma^{-1}$ is an isogeny defined over $L'$ between $E^*_L$ and $E_{L'}$, so that $L \subseteq L'$ and therefore $L = L'$.

One can also characterize $L/K$ as the minimal extension such that

$$\Hom(E^*_Q, A_Q) \simeq \Hom(E^*_L, A_L).$$

Denote by

$$\inf^K_N : H^2(\Gal(N/M), M^\times) \longrightarrow H^2(\Gal(K/M), M^\times)$$

the inflation map in Galois cohomology.

**Lemma 2.13.** Suppose that $M \neq \Q(i), \Q(\sqrt{-3})$. Then

$$\inf^K_N(\gamma^N_{E^*}) = w \cdot \gamma^K_E,$$

for some $w \in H^2(\Gal(K/M), \{\pm 1\})$.

**Proof.** Since $E_L \simeq (E_*)_L$ we have that

$$(2.8) \quad \inf^L_N(\gamma^N_{E^*}) = \inf^L_K(\gamma^K_E).$$

Now consider the following piece of the inflation–restriction exact sequence

$$(2.9) \quad H^1(\Gal(L/K), M^\times) \overset{\iota}{\longrightarrow} H^2(\Gal(K/M), M^\times) \overset{\inf^L_K}{\longrightarrow} H^2(\Gal(L/M), M^\times).$$

Equality (2.8) implies that $\inf^L_K(\gamma^N_{E^*})$ and $\gamma^K_E$ have the same image under the inflation map $\inf^K_K$, and therefore

$$\inf^K_N(\gamma^N_{E^*}) = t(v) \cdot \gamma^K_E,$$

for some $v \in H^1(\Gal(L/K), M^\times)$. If $M \neq \Q(i), \Q(\sqrt{-3})$ we have that

$$H^1(\Gal(L/K), M^\times) \simeq \Hom(\Gal(L/K), \{\pm 1\})$$

and therefore $t(v)$ belongs to $H^2(\Gal(K/M), \{\pm 1\})$. ☐

Observe that from Theorem 2.12 one cannot deduce that the class $\gamma^N_{E^*}$ is 2-torsion, since $A_N$ is not isogenous to $(E^*)^2$ in general. By Lemma 2.13, what we do
Lemma 2.16. We sometimes will write as \( t \) \( \rho \) weis isogenies to define a tautological:

\[
\text{Proof.}
\]

The following technical lemma will be used in (2.5) below.

Lemma 2.14. Suppose that \( N/M \) is abelian and that \( M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}) \). Let \( c^N_{\mu^*} \) be a cocycle representing the class \( \gamma^N_{E^*} \). Then \( c^N_{\mu^*}(\sigma, \tau) = c^N_{\mu^*}(\tau, \sigma) \) for all \( \sigma, \tau \in \text{Gal}(N/M) \).

**Proof.** Since \( M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}) \) we have that

\[
\text{H}^1(\text{Gal}(K/N), M^\times) = \text{Hom}(\text{Gal}(K/N), \{\pm 1\}).
\]

By (2.11) and (2.12) we can suppose that there exists a map \( d : \text{Gal}(N/M) \to M^\times \) such that

\[
c^N_{\mu^*}(\sigma, \tau) = d(\sigma)d(\tau)d(\sigma\tau)^{-1} \cdot t(\mu)(\sigma, \tau),
\]

where \( t(\mu)(\sigma, \tau) \in \{\pm 1\} \). Therefore

\[
c^N_{\mu^*}(\sigma, \tau) = \pm d(\sigma)d(\tau)d(\sigma\tau)^{-1} = \pm d(\sigma)d(\tau)d(\tau\sigma)^{-1} = \pm c^N_{\mu^*}(\tau, \sigma)^2.
\]

We see that \( c^N_{\mu^*}(\sigma, \tau) \) is a root of unity in \( M \), hence \( \pm 1 \). \( \Box \)

2.4. \( \rho \)-representations from squares of CM elliptic curves. Keep the notations from Section 2.3. We will denote by \( V \) the \( M \)-module \( \text{End}(E_L^*) \). Fix a system of isogenies \( \{\mu_{\sigma} : \sigma \to E^* \}_{\sigma \in \text{Gal}(L/M)} \). We do not have a natural action of \( \text{Gal}(L/M) \) on \( V \), but the next lemma says that we can use the chosen system of isogenies to define a \( \rho \)-action on \( V \).

**Lemma 2.15.** The map \( \varrho_V : \text{Gal}(L/M) \to \text{GL}(V) \)

defined by

\[
\varrho_V(f) = \sigma f \circ \mu_{\sigma}^{-1}
\]

for \( \sigma \in \text{Gal}(L/M) \), \( f \in V \)

and the 2-cocycle \( c^L_{\mu^*} \) endow the module \( V \) with a structure of a \( \rho \)-representation.

**Proof.** This is tautological:

\[
\varrho_V(\sigma)\varrho_V(\tau)(f) = \sigma f \circ \mu_{\sigma}^{-1} \circ \mu_{\tau}^{-1} = \sigma f \circ \mu_{\sigma\tau}^{-1} \cdot c^L_{\mu^*}(\sigma, \tau) = \varrho_V(\sigma\tau)(f)c^L_{\mu^*}(\sigma, \tau).
\]

Let now \( R \) denote the \( M \)-module \( \text{End}(A_K) \). It is equipped with the natural Galois conjugation action of \( \text{Gal}(L/M) \), which factors through \( \text{Gal}(K/M) \) and which we sometimes will write as \( \varrho_\varphi(\sigma)(\psi) = \sigma\psi \). Let \( T \) denote \( \text{Hom}(V, V) \), equipped with the \( \rho \)-representation structure given by Lemma 2.15 and Proposition 2.2. Note that by Corollary 2.5, we know that \( T \) is actually a \( M[\text{Gal}(L/M)] \)-module.

**Lemma 2.16.** The map \( \Phi : R \to T \simeq V \otimes V^* \)

\[
\Phi(\psi)(f) = \psi \circ f, \text{for } f \in V, \psi \in \text{End}(A_K)
\]

is an isomorphism of \( \rho \)-representations (and thus of \( M[\text{Gal}(L/M)] \))-modules.
Proof. The fact that $\Phi$ is a morphism of $c$-representations is straightforward:
\[
\rho_T(\sigma)(\Phi(c^{-1}\psi))(f) = \rho_V(\sigma)(\Phi(c^{-1}\psi)(\rho_V(\sigma)^{-1}(f))) = \rho_V(\sigma)(c^{-1}\psi \circ \rho_V(\sigma)^{-1})(f) c_E(\sigma^{-1}) = \psi \circ f \circ c^{-1}(\sigma^{-1}) = \Phi(\psi)(f),
\]
where we have used Remark 2.1 in the second and last equalities. The lemma follows by noting that $\Phi$ is clearly injective and that both $R$ and $T$ have dimension 4 over $M$. \hfill $\Box$

We now describe the $M[\Gal(K/M)]$-module structure of $R$. It follows from (2.5) that the order $r$ of an element $\sigma \in \Gal(K/M)$ is 1, 2, 3, 4, or 6.

Lemma 2.17. $\Tr \rho_R(\sigma) = 2 + \zeta_r + \overline{\zeta}_r$, where $\zeta_r$ is a primitive $r$-th root of unity.

Remark 2.18. Note that this lemma is proven in [FS14, Proposition 3.4] under the strong running hypothesis of that paper: in our setting that hypothesis would say that there exists $E^*$ defined over $M$ such that $A_{\overline{\mathbb{Q}}_L} \simeq E^*_2$ (i.e., that $N$ can be taken to be $M$, in the notation of the previous section).

Proof. We claim that $\Tr(\rho_R) \in M$ is in fact rational. Let us postpone the proof of this claim until the end of the proof of the lemma. Assuming it, we have that
\[
(2.13) \quad \Tr_{M/\overline{\mathbb{Q}}}(\Tr(\rho_R(\sigma))) = 2 \Tr(\rho_R)(\sigma).
\]
But if $\rho_{R_0}$ is the representation afforded by $R$ regarded as an 8 dimensional module over $\mathbb{Q}$, we have
\[
(2.14) \quad \Tr_{M/\overline{\mathbb{Q}}}(\Tr(\rho_R(\sigma))) = \Tr(\rho_{R_0})(\sigma) = 2(2 + \zeta_r + \overline{\zeta}_r),
\]
where the last equality is [FKRS12, Proposition 4.9]. The comparison of (2.13) and (2.14) concludes the proof of the lemma.

We turn now to prove the rationality of $\Tr \rho_R$. We first recall the aforementioned proof (that of [FS14, Proposition 3.4]) which uses the fact that we can choose $E^*$ to be defined over $M$. In this case, we have that $V$ is an $M[\Gal(L/M)]$-module, that $\Tr(\rho_{V^*})$ is a sum of roots of unity so that $\Tr(\rho_{V^*}) = \overline{\Tr(\rho_V)}$, and hence that $\Tr(\rho_R) = \Tr(\rho_V) \cdot \overline{\Tr(\rho_V)} \in \mathbb{Q}$. For the general case, assume that $\Tr \rho_R$ does not belong to $\mathbb{Q}$. Since it is a sum of roots of unity of orders 4 or 6, then $M$ would be $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, but then we could take a model of $E^*$ defined over $M$, and by the above paragraph, the trace $\Tr \rho_R$ would be rational, which is a contradiction. \hfill $\Box$

2.5. Obstructions. Keep the notations from Section 2.4 and Section 2.6. Let $S$ denote the normal subgroup of $\Gal(K/M)$ generated by the square elements. In this section, we make the following hypotheses.

Hypothesis 2.19. \begin{itemize} \item[i)] There exists a Ribet $M$-curve $E^*$ with CM by $M$ completely defined over $N$, where $N/M$ is the subextension of $K/M$ fixed by $S$. \item[ii)] $M \neq \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$. \end{itemize}

Let $\sigma \in \Gal(K/M)$ be an element of order $r \in \{4, 6\}$. Let
\[
(2.15) \quad \tau : \Gal(K/M) \to \Gal(N/M) \simeq \Gal(K/M)/S
\]
denote the natural projection map. Note that \( \text{Gal}(N/M) \) is a group of exponent dividing 2.

**Theorem 2.20.** Under Hypothesis 2.19 we have:

i) If \( r = 4 \), then \( 2c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) \) belongs to \( \pm(M^\times)^2 \).

ii) If \( r = 6 \), then \( 3c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) \) belongs to \( \pm(M^\times)^2 \).

**Proof.** First of all, note that \( E^* \) is completely defined over \( N \). Thus we can, and do, assume that \( c_{E_r}^N \) is the inflation of \( c_{E_r}^M \). Let \( s \in \text{Gal}(L/M) \) be a lift of \( \sigma \). By part ii) of Hypothesis 2.19 we have that \( [L : K] \leq 2 \). Therefore, the order of \( s \) divides \( 2r \). We then have

\[
(2.16) \quad \varphi_V(s)^{2r} = \varphi_V(s^r)^r c_{E_r}^N(\bar{\sigma}, \bar{\sigma})^r = \varphi_V(s^{2r}) c_{E_r}^N(\bar{\sigma}, \bar{\sigma})^r = c_{E_r}^N(\bar{\sigma}, \bar{\sigma})^r,
\]

where we have used that \( c_{E_r}^N(\bar{\sigma}^{2e}, \bar{\sigma}^{2e'}) = 1 \) for any pair of integers \( e, e' \). Let \( \alpha \) and \( \beta \) be the eigenvalues of \( \varphi_V(s) \). By (2.16), we have that \( \alpha^{2r} = c_{E_r}^N(\bar{\sigma}, \bar{\sigma})^r \), from which we deduce that \( \omega_r \alpha^2 = c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) \in M^\times \), where \( \omega_r \) is a (not necessarily primitive) \( r \)-th root of unity.

Since the eigenvalues of \( \varphi_V(s) \) are \( 1/\alpha \) and \( 1/\beta \), by Lemmas 2.17 and 2.18 we have that

\[
(2.17) \quad 2 + \zeta_r + \bar{\zeta}_r = (\alpha + \beta) \left( \frac{1}{\alpha} + \frac{1}{\beta} \right); \text{ equivalently, } \alpha^2 + \beta^2 = (\zeta_r + \bar{\zeta}_r)\alpha\beta.
\]

This means that \( \alpha/\beta \) satisfies the \( r \)-th cyclotomic polynomial and thus, by reordering \( \alpha \) and \( \beta \) if necessary, we have that \( \alpha = \beta\zeta_r \).

Combining this with (2.17), we get

\[
(2 + \zeta_r + \bar{\zeta}_r) c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) = (2 + \zeta_r + \bar{\zeta}_r)\omega_r \alpha^2 = (2 + \zeta_r + \bar{\zeta}_r)\alpha\beta\omega_r \zeta_r = (\alpha + \beta)^2 \omega_r \zeta_r.
\]

Since the left-hand side is in \( M^\times \), the fact that \( \alpha + \beta \in M^\times \) tells us that \( \omega_r \zeta_r \in M^\times \).

If \( \omega_r \zeta_r \) is not rational, then \( M = \mathbb{Q}(\zeta_r) \), which contradicts part ii) of Hypothesis 2.19.

If \( \omega_r \zeta_r \in \mathbb{Q} \), since it is a root of unity, it must be \( \pm1 \) and thus we get

\[
\pm(2 + \zeta_r + \bar{\zeta}_r) c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) = (\alpha + \beta)^2.
\]

Therefore, \( (2 + \zeta_r + \bar{\zeta}_r) c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) \) belongs to \( \pm(M^\times)^2 \). \( \square \)

**Remark 2.21.** Note that it follows from the above proof that if \( r = 4 \), then any lift \( s \in \text{Gal}(L/M) \) of \( \sigma \) has order \( 2r = 8 \). Indeed, if the order of \( s \) was \( r \), then arguing as in (2.10), we would obtain \( \varphi_V(s)^r = c_{E_r}^N(\bar{\sigma}, \bar{\sigma})^{r/2} \), from which we would infer \( \omega_r \alpha^2 = c_{E_r}^N(\bar{\sigma}, \bar{\sigma}) \), for some (not necessarily primitive) \( r/2 \)-th root of unity. We could then run the same argument as above, but since \( \omega_r \alpha^2 \) is never rational, we would deduce now that \( M = \mathbb{Q}(i) \). Note that if \( r = 6 \) it can certainly happen that \( \omega_r \zeta_r \in \mathbb{Q} \).

Until the end of this section, we make the following additional assumption on \( M \).

**Hypothesis 2.22.**

i) \( \text{Gal}(K/M) \simeq D_4 \) or \( D_6 \).

ii) \( M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}) \).

Hypothesis i) implies that \( N/M \) is a biquadratic extension. By part i) of Proposition 2.10 there exists a Ribet \( M \)-curve \( E^* \) with CM by \( M \) completely defined over the Hilbert class field \( H \) of \( M \). Using [FGT8, Theorem 2.14], it is immediate to see that \( H \subset N \), so that Hypothesis 2.22 implies Hypothesis 2.19.

The next two propositions describe the structure of the group \( \text{Gal}(L/M) \).
We will first see that and consider the restriction map \( \gamma \) and the notation \( \text{Recall the cohomology class} \).

**Proof.** If \( \text{Gal}(K/M) \simeq D_4 \), then by Remark we have that any element of \( \text{Gal}(L/M) \) projecting onto an element of \( \text{Gal}(K/M) \) of order 4 must have order 8. The groups of order 16 with a quotient isomorphic to \( D_4 \) satisfying the previous property are those in the statement of the proposition.

**Proposition 2.24.** If \( \text{Gal}(K/M) \simeq D_6 \), there exists a Ribet M-curve \( E^* \) completely defined over \( N \) with CM by \( M \) such that \( E \sim E^*_K \) and hence \( L = K \) and \( \text{Gal}(L/M) \simeq D_6 \).

**Proof.** Recall the cohomology class \( \gamma^K_E \in H^2(\text{Gal}(K/M),M^\times)[2] \) attached to \( E \) and consider the restriction map

\[
\text{res} : H^2(\text{Gal}(K/M),M^\times) \to H^2(\text{Gal}(K/N),M^\times).
\]

We will first see that \( \gamma = \text{res}\gamma^K_E \) is trivial. Recall the decomposition of the 2-torsion cohomology classes into degree and sign components

\[
H^2(\text{Gal}(K/N),M^\times)[2] \simeq H^2(\text{Gal}(K/N),\{\pm 1\}) \times \text{Hom}(\text{Gal}(K/N),P/P^2),
\]

and the notation \( \gamma_\pm \) (resp. \( \bar{\gamma} \)) for the sign component (resp. degree component) of \( \gamma \). Since \( \text{Gal}(K/N) \simeq C_3 \) is the subgroup of \( \text{Gal}(K/M) \) generated by the squares, we have that \( \bar{\gamma} \) is trivial. Since

\[
H^2(\text{Gal}(K/N),\{\pm 1\}) \simeq H^2(C_3,\{\pm 1\}) = 0,
\]

we see that \( \gamma_\pm \) is also trivial. By Theorem there exists an elliptic curve \( E^* \) defined over \( N \) such that \( E^*_K \sim E \). To see that \( E^* \) is completely defined over \( N \), on the one hand, note that since \( M \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}) \), then \( E^* \) and any Galois conjugate \( \sigma E^* \) of it are isogenous over a quadratic extension of \( N \). On the other hand, since \( E^*_K \sim E \) and \( E \) is completely defined over \( K \), we have that the smallest field of definition of \( \text{Hom}(E^*_E,\sigma E^*_E) \) is contained in \( K \). Since \( K/N \) is a cubic extension, we deduce that \( E^* \) and \( \sigma E^* \) are in fact isogenous over \( N \).

**Corollary 2.25.** If \( \text{Gal}(K/M) \simeq D_r \) for \( r = 4 \) or \( 6 \), there exists a Ribet M-curve \( E^* \) with CM by \( M \) completely defined over \( N \) for which \( \text{Gal}(L/M) \) contains

i) an element \( s \) of order 8 if \( r = 4 \) and of order 6 if \( r = 6 \);

ii) an element \( t \) such that \( ts^{a}t^{-1} = t^{a} \) for \( 1 \leq a \leq 2r \) such that \( a \equiv -1 \mod r \).

**Proof.** This is obvious when \( \text{Gal}(L/M) \) is dihedral. For the other options allowed by Proposition recall that

\[
\text{QD}_8 \simeq \langle s,t \mid s^8, t^2, tsts^5 \rangle, \quad \text{Q}_{16} \simeq \langle s,t \mid s^8, t^2s^4, tsts^{-1}s \rangle.
\]

**Remark 2.26.** It is clear from the proof of Proposition that, in the case that \( N = H \) and \( H \) is not exceptional, we can choose \( E^* \) in the above corollary to be a Gross \( \mathbb{Q} \)-curve.

\footnote{The gap identification numbers of \( \text{QD}_8 \) and \( \text{Q}_{16} \) are \((16,8)\) and \((16,9)\), respectively.}
Until the end of this section, we will assume that $E^*$ is as in the previous corollary. Let $s$ and $t$ be also as in the corollary, and let $\sigma$ and $\tau$ be the images of $s$ and $t$ under the projection map 

$$\text{Gal}(L/M) \to \text{Gal}(K/M).$$

Recall also the projection map $\tilde{\tau} : \text{Gal}(K/M) \to \text{Gal}(N/M)$ and note that $\tilde{\sigma}$ and $\tilde{\tau}$ are non-trivial elements of $\text{Gal}(N/M)$.

**Theorem 2.27.** Under Hypothesis 2.22 we have $c^N_{E^*}(\tilde{\tau}, \tilde{\tau}) = \pm 1$.

**Proof.** By Lemma 2.14 we have that $c^N_{E^*}(g, g') = \pm c^N_{E^*}(g', g)$ for every $g, g' \in \text{Gal}(N/M)$. Moreover, the 2-cocycle condition (2.4) asserts that 

$$c^N_{E^*}(\tilde{\tau}, \tilde{\tau}) = c^N_{E^*}(\tilde{\tau}, \tilde{\tau})c^N_{E^*}(\tilde{\sigma}, 1) = c^N_{E^*}(\tilde{\sigma}, \tilde{\tau})c^N_{E^*}(\tilde{\sigma}, \tilde{\tau}).$$

Then, we have 

$$\varrho_V(t)\varrho_V(s)\varrho_V(t^{-1}) = \varrho_V(t)\varrho_V(s)\varrho_V(t^{-1})c^N_{E^*}(\tilde{\tau}, \tilde{\tau}) =$$

$$= \varrho_V(ts)\varrho_V(t^{-1})c^N_{E^*}(\tilde{\tau}, \tilde{\tau})c^N_{E^*}(\tilde{\tau}, \tilde{\tau}) =$$

$$= \varrho_V(ts^-1)c^N_{E^*}(\tilde{\tau}, \tilde{\tau})c^N_{E^*}(\tilde{\tau}, \tilde{\tau}) =$$

$$\pm \varrho_V(s^a)c^N_{E^*}(\tilde{\tau}, \tilde{\tau}).$$

It is easy to observe that 

$$\varrho_V(s)^a = \varrho_V(s^a)c^N_{E^*}(\tilde{\sigma}, \tilde{\sigma})^{a^{-1}/2}.$$  

Letting $\alpha$ and $\beta$ be the eigenvalues of $\varrho_V(s)$, taking traces of (2.18), and applying (2.19), we obtain 

$$(\alpha + \beta) = \pm (\alpha^a + \beta^a)c^N_{E^*}(\tilde{\sigma}, \tilde{\sigma})^{-(a-1)/2}c^N_{E^*}(\tilde{\tau}, \tilde{\tau})^2$$

But as in the proof of Theorem 2.20 we have $\beta = \zeta_r\alpha$ and $c^N_{E^*}(\tilde{\sigma}, \tilde{\sigma}) = \omega_r\alpha^2$, where $\zeta_r$ and $\omega_r$ are $r$-th roots of unity and $\zeta_r$ is primitive. This, together with the fact that $a \equiv -1 \pmod{r}$, permits to write the above equation as 

$$\pm \frac{1 + \zeta_r}{\omega_r(a-1)/2(1 + \zeta_r)} = c^N_{E^*}(\tilde{\tau}, \tilde{\tau})^2 \in (M^\times)^2.$$  

One easily verifies that $(1 + \zeta_r)/(1 + \zeta_r^r)$ is an $r$-th root of unity. Therefore, the left-hand side of the above equation is a root of unity in $M^\times$, and hence it must be $\pm 1$. \qed

### 3. Restriction of Scalars of Gross $\mathbb{Q}$-curves

For the convenience of the reader, in this section we review some results of Nakamura [Nak04] on Gross $\mathbb{Q}$-curves, to which we refer for more details and proofs. Let $M$ be an imaginary quadratic field. Throughout this section, we make the following hypothesis.

**Hypothesis 3.1.**

i) $M$ is non-exceptional.

ii) $M$ has class group isomorphic to $C_2 \times C_2$.

**Remark 3.2.** If $M$ has class group isomorphic to $C_2 \times C_2$, then the discriminant $D$ of $M$ belongs to the set 

$$\{-84, -120, -132, -168, -195, -228, -280, -312, -340, -372, -408, -435, -483, -520, -532, -555, -595, -627, -708, -715, -760, -795, -1012, -1435\}.$$
This list can be easily obtained from [Wat04], for example. Among them, only $-340$ is exceptional.

Then, by Proposition 2.10 there exists a Gross $\mathbb{Q}$-curve $E$ with CM by $M$, which is thus completely defined over the Hilbert class field $H$ of $M$. The aim of the present section is to describe Nakamura’s method for computing the endomorphism algebra of the restriction of scalars of a Gross $\mathbb{Q}$-curve, which we will then apply to all Gross $\mathbb{Q}$-curves attached to $M$ satisfying Hypothesis 3.1. Our account of Nakamura’s method will be only in the particular case where $M$ has class group $C_2 \times C_2$, which is the case of interest to us.

As seen in Section 2.2 one can associate to $E$ a cohomology class $\gamma_E := \gamma_E^H$ in the group $H^2(\text{Gal}(H/\mathbb{Q}), M^\times)$. The set of cohomology classes arising from Gross $\mathbb{Q}$-curves over $H$ has cardinality 8 (cf. [Nak04, Proposition 4]), and we regard the set of Gross $\mathbb{Q}$-curves over $H$ as partitioned into 8 equivalence classes according to their cohomology class.

Let $\text{Res}_{H/M}(E)$ denote Weil’s restriction of scalars of $E$. This variety is a priori defined over $M$, but it can be defined over $\mathbb{Q}$, in the sense that $\text{Res}_{H/M}(E) \simeq (B_E)_M$ for some variety $B_E/\mathbb{Q}$. It turns out that the endomorphism algebra $D_E = \text{End}(B_E)$ only depends on the cohomology class $\gamma_E$ [Nak04, Proposition 6]. Nakamura devised a method for computing $D_E$ in terms of the Hecke character attached to $E$, which he applied to compute all the endomorphism algebras arising in this way from Gross $\mathbb{Q}$-curves in the cases where $D = -84$ and $D = -195$.

We extend his computation to the remaining 21 non-exceptional discriminants of Remark 3.2.

3.1. Hecke characters of Gross $\mathbb{Q}$-curves. The first step is to compute a set of Hecke characters whose associated elliptic curves represent all the equivalence classes of Gross $\mathbb{Q}$-curves.

Local characters. We begin by defining certain local characters that will be used to describe the Hecke characters. Let $\mathbb{I}_M$ be the group of ideles of $M$. If $p$ is a prime of $M$, we denote by $U_p = \mathcal{O}_{M,p}^\times$ the group of local units. Also, for a rational prime $p$ put $U_p = \prod_{p \mid p} U_p$.

Suppose that $p$ is odd and inert in $M$. Then define $\eta_p$ as the unique character $\eta_p : U_p \rightarrow \{\pm 1\}$ such that $\eta_p(-1) = (-1)^{\frac{p-1}{2}}$.

Suppose now that 2 is ramified in $M$ and write $D = 4m$. If $m$ is odd, then

$$U_2/U_2^2 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \simeq \langle \sqrt{m}, 3 - 2\sqrt{m}, 5 \rangle.$$

Define $\eta_{-8}: U_2 \rightarrow \{\pm 1\}$ to be the character with kernel $\langle 3 - 2\sqrt{m}, 5 \rangle$. If $m$ is even then

$$U_2/U_2^2 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \simeq \langle 1 + \sqrt{m}, -1, 5 \rangle.$$

Define $\eta_8$ to be the character with kernel $\langle 1 + \sqrt{m}, -1 \rangle$ and $\eta_{-8}$ the character with kernel $\langle 1 + \sqrt{m}, -5 \rangle$.

Hecke characters. Let $U_M = \prod_p U_p$ be the maximal compact subgroup of $\mathbb{I}_M$. Let $S$ be a finite set of primes of $M$ and put $U_S = \prod_{p \in S} U_p$. Suppose that $\eta : U_S \rightarrow \{\pm 1\}$ is a continuous homomorphism such that $\eta(-1) = -1$. Next, we explain how to construct from $\eta$ a Hecke character $\phi : \mathbb{I}_M \rightarrow \mathbb{C}^\times$ (not uniquely determined) that gives rise, in certain cases, to a Gross $\mathbb{Q}$-curve.
Lemma 3.3. Suppose that \( \eta \) extended to a character \( \tilde{\eta} \) following formula for extensions, cf. [Shi71, p. 523]). For future reference, it will be useful to have the following formula for \( \phi \) evaluated at certain principal ideals.

Lemma 3.3. Suppose that \((\alpha)\) is a principal ideal of \( M \) such that \( v_p(\alpha) = 0 \) for all \( p \in S \), and denote by \( \alpha_S \in U_S \) the natural image of \( \alpha \) in \( U_S \). Then

\[
\phi((\alpha)) = \eta(\alpha_S)\alpha_\infty,
\]

where \( \alpha_\infty \) denotes the image of \( \alpha \) in \( M_\infty = \mathbb{C} \).

Proof. If we write \( \alpha = \prod_{q \in T} q^{e_q(\alpha)} \), where \( T \) denotes the support of \( \alpha \), then

\[
\phi((\alpha)) = \prod_{q \in T} \phi_q(\alpha_q),
\]

where \( \phi_q \) denotes the restriction of \( \phi \) to \( M_q \) and \( \alpha_q \) the image of \( \alpha \) in \( M_q \). Observe that by hypothesis \( S \cap T = \emptyset \), and that if \( q \not\in S \cup T \), then \( \phi_q(\alpha_q) = 1 \), since \( \alpha_q \) belongs to \( U_q \) and \( \phi_q|_{U_q} = \tilde{\eta}|_{U_q} = 1 \). Therefore, we can write

\[
\phi((\alpha)) = \prod_{q \in T} \phi_q(\alpha_q) \prod_{q \not\in T} \phi_q(\alpha_q) \prod_{q \in S} \phi_q^{-1}(\alpha_q) = \left( \prod_{q} \phi_q(\alpha_q) \right) \eta(\alpha_S),
\]

where we have used that \( \eta \) has order 2. Then, by (3.1) we have that

\[
\phi((\alpha)) = \left( \phi_\infty(\alpha_\infty) \prod_{q} \phi_q(\alpha_q) \right) \phi_\infty^{-1}(\alpha_\infty)\eta(\alpha_S) = \phi(\alpha)\alpha_\infty \eta(\alpha_S) = \alpha_\infty \eta(\alpha_S).
\]

Define now a Hecke character of \( H \) by means of \( \psi = \phi \circ N_{H/M} \), where

\[
N_{H/M} : I_H \to I_M
\]
denotes the norm on ideles. By a result of Shimura [Shi71, Proposition 9], the Hecke character \( \psi \) is attached to a Gross \( \mathbb{Q} \)-curve if and only if \( \tilde{\phi} = \phi \), where the bar denotes the action of complex conjugation.

For example, if \( D \) has some prime factor \( q \equiv 3 \pmod{4} \), put \( \eta_0 = \eta_q \). If all the odd primes dividing \( D \) are congruent to 1 modulo 4, then \( D = 8m \) for some odd \( m \) and we define \( \eta_0 \) to be \( \eta_{-8} \). If we denote by \( \phi_0 : I_M \to \mathbb{C}^\times \) a Hecke character attached to \( \eta_0 \) by the above construction, then the Hecke character \( \psi_0 = \phi_0 \circ N_{H/M} \) is the Hecke character attached to a Gross \( \mathbb{Q} \)-curve over \( H \).

Let \( W \) be the set of characters \( \theta : U_M \to \{ \pm 1 \} \) such that \( \theta(-1) = 1 \) and \( \bar{\theta} = \theta \). Denote also by \( W_0 \) the set of \( \theta \in W \) such that \( \theta = \kappa \circ N_{M/\mathbb{Q}} \) for some Dirichlet character \( \kappa \). By [Nak04, Proposition 3], the group \( W/W_0 \) is generated by two characters that can be described explicitly in terms of the characters \( \eta_p, \eta_{-4}, \eta_{-8}, \) and \( \eta_q \). More precisely:

1. If \( D = -pqr \) with \( p, q, \) and \( r \) primes congruent to 3 modulo 4, then \( W/W_0 = \langle \eta_p \eta_q, \eta_p \eta_r \rangle \).
(2) If \( D = -pqr \) with \( p \) and \( q \) primes congruent to 1 modulo 4, and \( r \equiv 3 \) (mod 4), then \( W/W_0 = \langle \eta_p, \eta_q \rangle \).

(3) If \( D = -4pq \) with \( p \) and \( q \) congruent to 3 modulo 4, then \( W/W_0 = \langle \eta_{-4}, \eta_p \eta_q \rangle \).

(4) If \( D = -8pq \) with \( p \) and \( q \) congruent to 3 modulo 4 then \( W/W_0 = \langle \eta_{-8}, \eta_p \eta_q \rangle \).

(5) If \( D = -8pq \) with \( p \equiv 1 \) (mod 4) and \( q \equiv 3 \) (mod 4) then \( W/W_0 = \langle \eta_8, \eta_p \rangle \).

(6) If \( D = -8pq \) with \( p \) and \( q \) congruent to 1 modulo 4, then \( W/W_0 = \langle \eta_p, \eta_q \rangle \).

Denote by \( \tilde{\omega}_1, \tilde{\omega}_2 \) the generators of \( W/W_0 \), and define \( \omega_i = \tilde{\omega}_i \circ N_{H/M} \).

Now let \( k/H \) be a quadratic extension such that \( k/\mathbb{Q} \) is Galois and \( k/M \) is non-abelian. Such quadratic extensions exist by [Nak04, Theorem 1]. Denote by \( \chi: \mathbb{F} \rightarrow \{0,1\} \) the Hecke character attached to \( k/H \).

By [Nak04, Theorem 2], the eight equivalence classes of \( \mathbb{Q} \)-curves over \( H \) are represented by the Hecke characters \( \psi_0 \cdot \omega \) with \( \omega \in \langle \omega_1, \omega_2, \chi \rangle \). Observe that, in particular, this set of Hecke characters does not depend on the choice of \( k \) (any \( k \) which is Galois over \( \mathbb{Q} \) and non-abelian over \( M \) will produce the same set of Hecke characters).

### 3.2. Method for computing the endomorphism algebra

Let \( p_1 \) and \( p_2 \) be prime ideals of \( M \) that generate the class group and that are coprime to the conductors of \( \psi_0, \omega_1, \omega_2, \) and \( \chi \). Let \( L_i \) be the decomposition field of \( p_i \) in \( H \), and \( F_i \) the maximal totally real subfield of \( L_i \).

Suppose that \( E \) is a Gross \( \mathbb{Q} \)-curve over \( H \) with Hecke character of the form \( \psi = \psi_0 \omega_1^a \omega_2^b \) for some \( a, b \in \{0,1\} \). We can write \( \psi = \phi \circ N_{H/M} \), where \( \phi = \phi_0 \tilde{\omega}_1^a \tilde{\omega}_2^b \). Then \( \phi(p_i) + \phi(p_i) \) generates a quadratic number field \( \mathbb{Q}(\sqrt{m_i}) \), and the endomorphism algebra \( D_E = \text{End}(B_E) \) is isomorphic to the biquadratic field \( \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}) \) (cf. [Nak04, Proposition 7, Theorem 3]).

**Remark 3.4.** Observe that \( \phi(p_i) + \phi(p_i) \) can be computed if one knows the two quantities \( \phi(p_i^2) \) and \( \phi(p_i p_i) \). Since \( p_i^2 \) and \( p_i p_i \) are principal, one can compute \( \phi(p_i^2) \) and \( \phi(p_i p_i) \) by means of (3.2).

Suppose now that the Hecke character of \( E \) is of the form \( \psi = \psi_0 \chi_1^a \chi_2^b \). Then \( D_E \) is a quaternion algebra over \( \mathbb{Q} \), say \( D_E \simeq \left( \frac{t_1 t_2}{\mathbb{Q}} \right) \). The \( t_i \) can be computed as follows (see [Nak04, Proposition 7]). First of all, let \( n_1 \) and \( n_2 \) be the rational numbers defined as in the previous paragraph for the character \( \psi/\chi = \psi_0 \chi_1^a \chi_2^b \).

1. Suppose that \( \text{Gal}(k/L_i) \simeq C_2 \times C_2 \). Then:
   (a) If \( k/F_i \) is abelian then \( t_i = n_i \).
   (b) If \( k/F_i \) is non-abelian, then \( t_i = D/n_i \).

2. Suppose that \( \text{Gal}(k/L_i) \simeq C_4 \). Then:
   (a) If \( k/F_i \) is abelian, then \( t_i = -n_i \).
   (b) If \( k/F_i \) is non-abelian, then \( t_i = -D/n_i \).

### 3.3. Computations and tables

For each of the 23 non-exceptional imaginary quadratic fields of class group \( C_2 \times C_2 \), we have computed the 8 endomorphism algebras arising from restriction of scalars of Gross \( \mathbb{Q} \)-curves. The results are displayed in Table 1. The notation is as follows: for the biquadratic fields, the notation \( (a,b) \) indicates the field \( \mathbb{Q}(\sqrt{a}, \sqrt{b}) \); for the quaternion algebras, we write the discriminant of the algebra.
For a Gross $\mathbb{Q}$-curve $E$, recall that we denote by $B_E$ the abelian variety over $\mathbb{Q}$ such that $\text{Res}_{H/M} E \sim (B_E)_M$. Since $B_E$ is isogenous to its quadratic twist over $M$, this implies that

$$\text{Res}_{H/Q} E \sim (B_E)^2.$$ 

We observe in Table 1 that for all discriminants except $-195, -312, -555, -715,$ and $-760$, at least one of the quaternion algebras is the split algebra $\mathbb{M}_2(\mathbb{Q})$ of discriminant 1. This implies that for the corresponding Gross $\mathbb{Q}$-curve $E$ the variety $B_E$ decomposes as

$$B_E \sim A^2,$$

with $A/\mathbb{Q}$ an abelian surface. Therefore, $\text{Res}_{H/Q} E$ decomposes as the fourth power of an abelian surface.

On the other hand, for the discriminants $-195, -312, -555, -715,$ and $-760$ we see that $B_E$ is always simple: its endomorphism algebra is either a biquadratic field or a quaternion division algebra over $\mathbb{Q}$. Therefore, $\text{Res}_{H/Q} E \sim W^2$ for some simple variety $W$ of dimension 4. We record these findings in the following statement.

**Theorem 3.5.** Let $M$ be an imaginary quadratic field of discriminant $D$ and Hilbert class field $H$. Suppose that $D$ is non-exceptional and that $\text{Gal}(H/M) \cong C_2 \times C_2$. If $D \neq -195, -312, -555, -715, -760$, there exists a Gross $\mathbb{Q}$-curve $E/H$ such that

$$\text{Res}_{H/Q} E \sim A^4,$$

for some simple abelian surface $A/\mathbb{Q}$.

If $D = -195, -312, -555, -715, -760$, then for every Gross $\mathbb{Q}$-curve $E/H$ we have that

$$\text{Res}_{H/Q} E \sim W^2,$$

for some simple abelian variety $W/\mathbb{Q}$ of dimension 4.

**Remark 3.6.** As mentioned above, the cases of $D = -84$ and $D = -195$ were already computed by Nakamura ([Nak04, §6]). We note what appears to be a typo in Nakamura’s table on page 647: the last biquadratic field should be $\mathbb{Q}(\sqrt{-14}, \sqrt{42})$, instead of $\mathbb{Q}(\sqrt{-14}, \sqrt{-42})$.

We have used the software Sage [S+14] and Magma [BCP97] to perform the computations of Table 1. The interested reader can find the code we used in [https://github.com/xguitart/restriction_of_scalars_of_Q_curves](https://github.com/xguitart/restriction_of_scalars_of_Q_curves).
### Biquadratic fields

| $D$   | Biquadratic fields | Quaternion Algebras |
|-------|--------------------|---------------------|
| $-84$ | $(-14, -2), (-6, 2), (-6, -42), (-14, 42)$ | $2, 1, 2, 1$       |
| $-120$| $(-5, 10), (5, -10), (-5, -10), (5, 10)$   | $1, 6, 3, 1$       |
| $-132$| $(22, -2), (-6, -2), (6, -66), (-22, -66)$ | $1, 2, 1, 2$       |
| $-168$| $(-14, -2), (3, -21), (14, 21), (-3, 2)$  | $2, 1, 1, 1$       |
| $-195$| $(13, -5), (-13, -5), (-13, 5), (13, 5)$  | $13, 39, 26, 39$   |
| $-228$| $(-38, -2), (6, -2), (-6, -114), (38, -114)$ | $2, 1, 2, 1$       |
| $-280$| $(-10, -5), (-10, 5), (10, -5), (10, 5)$  | $2, 1, 1, 14, 14$  |
| $-312$| $(13, -26), (-13, 26), (-13, -26), (13, 26)$ | $13, 39, 26, 39$   |
| $-372$| $(-62, 31), (-6, -3), (-6, 31), (-62, -3)$ | $2, 1, 2, 1$       |
| $-408$| $(-17, 34), (-17, -34), (17, -34), (17, 34)$ | $2, 1, 1, 1$       |
| $-435$| $(-29, -5), (-29, 5), (29, -5), (29, 5)$  | $2, 1, 1, 1$       |
| $-483$| $(-23, 7), (23, -69), (-21, 7), (21, 69)$ | $2, 1, 1, 1$       |
| $-520$| $(-13, -5), (13, -5), (-13, 5), (13, 5)$  | $1, 1, 1, 2$       |
| $-532$| $(-38, -19), (-14, 7), (-14, -19), (-38, 7)$ | $1, 2, 1, 2$       |
| $-555$| $(37, -5), (-37, -5), (-37, 5), (37, 5)$  | $37, 111, 74, 111$ |
| $-595$| $(-17, 85), (17, -85), (-17, -85), (17, 85)$ | $7, 1, 1, 14$      |
| $-627$| $(19, -11), (-19, -57), (-33, 11), (33, 57)$ | $1, 2, 1, 1$       |
| $-708$| $(118, -59), (-6, -3), (6, -59), (-118, 3)$ | $1, 2, 1, 2$       |
| $-715$| $(-13, -65), (13, -65), (-13, 65), (13, 65)$ | $5, 10, 55, 55$    |
| $-760$| $(-10, 5), (10, -5), (-10, -5), (10, 5)$  | $5, 95, 10, 95$    |
| $-795$| $(-53, -5), (53, -5), (-53, 5), (53, 5)$  | $6, 1, 1, 3$       |
| $-1012$| $(-46, 23), (-22, -11), (-22, 23), (46, -11)$ | $2, 1, 2, 1$       |
| $-1435$| $(-41, 205), (-41, -205), (41, -205), (41, 205)$ | $2, 1, 1, 1$       |

Table 1. Endomorphism algebras of the restriction of scalars of Gross $\mathbb{Q}$-curves. For the biquadratic fields, the notation $(a, b)$ indicates the field $\mathbb{Q}(\sqrt{a}, \sqrt{b})$; for the quaternion algebras, we write the discriminant of the algebra.
4. Proof of the main theorems

We begin with a Lemma that will be used in the proof of Theorem 1.2.

**Lemma 4.1.** Let $E$ be a Gross $\mathbb{Q}$-curve with CM by a field $M$ of discriminant $D$, and suppose that $\text{Gal}(H/M)$ is isomorphic to $C_2 \times C_2$. Denote by $\gamma^H_E$ the class in $H^2(\text{Gal}(H/M), M^\times)$ attached to $E$, and by $c_E$ a cocycle representing $\gamma^H_E$. If $\sigma \in \text{Gal}(H/M)$ is non-trivial, then $\pm d \cdot c_E(\sigma, \sigma) \in (M^\times)^2$ for some divisor $d$ of $D$ such that $d$ is not a square in $M^\times$.

*Proof.* Let $\mathcal{O}_M$ denote the ring of integers of $M$. Denote by $p_1, p_2, p_3$ the primes dividing $D$. Observe that $p_i \mathcal{O}_M = p_i^2$, with $p_i$ a non-principal prime ideal of $\mathcal{O}_M$. It is clear that we can always find $p_i, p_j$ such that $\pm p_i p_j$ is not a square in $M^\times$, and therefore $p_i p_j$ is not principal. Thus $p_i, p_j$ generate the class group. Therefore, we can assume that any non-trivial element of $\text{Gal}(H/K)$ is of the form $\sigma_q$ for some unramified prime $q$ which is equivalent to either $p_i, p_j$ or $p_i \cdot p_j$. Here $\sigma_q$ stands for the Frobenius automorphism of $H/K$ at $q$.

Now we argue (and use the same notation) as in [Nak04, Proof of Theorem 3]. Namely, denote by $u(q)$ the $q$-multiplication isogenies
\[ u(q) : \gamma^q E \to E, \]
and denote by $c$ the 2-cocycle associated to $E$ using the system of isogenies $u(q)$ (together with the identity isogeny for $1 \in \text{Gal}(H/M)$). Note that $c_E$ is any cocycle representing $\gamma^H_E$, and it may be different from $c$. But in any case they are cohomologous, which in particular implies that
\[ c(\sigma_q, \sigma_q) = b_q^2 \cdot c_E(\sigma_q, \sigma_q) \text{ for some } b_q \in M^\times. \]

From display (6) and the display after that of loc. cit., since the order $n$ of $\sigma_q$ is 2 in our case, we see that
\[ c(\sigma_q, \sigma_q) \mathcal{O}_M = q^2. \]

The proof is finished by observing that $q^2 = \alpha \mathcal{O}_M$, where $\alpha \in M^\times$ is, up to an element of $(M^\times)^2$, equal to $\pm p_i, \pm p_j$, or $\pm p_i \cdot p_j$.

*Proof of Theorem 1.2.* For all the quadratic imaginary fields not listed in (1.2), we have constructed in the first part of Theorem 1.5 abelian surfaces defined over $\mathbb{Q}$ satisfying the hypothesis of the theorem. To rule out the remaining 6 fields, we proceed in the following way.

Let $M$ be one of the fields in the list (1.2) and suppose that an abelian surface $A$ satisfying the hypothesis of the theorem exists for $M$. Resume the notations from Section 2.4. As $\text{Gal}(H/M) \simeq C_2 \times C_2$ and $H \subseteq K$ (by [FG18, Theorem 2.14]), the only possibilities for $\text{Gal}(K/M)$ are $C_2 \times C_2, D_4$, and $D_6$.

Suppose that $\text{Gal}(K/M)$ is $C_2 \times C_2$. Then $K = H$ and thus $E$ is a Gross $\mathbb{Q}$-curve. By Proposition 2.10 we have that $M$ is not exceptional and thus we cannot have $M = \mathbb{Q}(\sqrt{-340})$. For the other possibilities for $M$, we have seen in the second part of Theorem 1.5 that $\text{Res}_{H/Q} E$ does not have any simple factor of dimension 2, but this is a contradiction with the fact that $A$ should be a factor of $\text{Res}_{H/Q} E$ (indeed, the universal property of Weil’s restriction of scalars implies that $\text{Hom}(A, \text{Res}_{H/Q} E) = \text{Hom}(A_H, E) \simeq M^2$, and thus $\text{Hom}(A, \text{Res}_{H/Q} E) \neq 0$).

Suppose that $\text{Gal}(K/M)$ is $D_4$ or $D_6$. Resume the notations of Section 2.5. Let $E^*$ be a Ribet $M$-curve completely defined over $H$ with CM by $M$ which we
chose as in Corollary 2.25 (and which exists because of Proposition 2.10). Note that Hypothesis 2.22 is satisfied. Then, by Theorem 2.24 there is a non-trivial element \( \gamma \in \text{Gal}(N/M) = \text{Gal}(H/N) \) such that
\[
(4.2) \quad c_E^H(\gamma, \gamma) = \pm 1.
\]
If \( M \) is non-exceptional, as noted in Remark 2.26 we can suppose that \( E^* \) is in fact a Gross \( \mathbb{Q} \)-curve. Then (4.2) is a contradiction with Lemma 4.1.

It remains to show that (4.2) also brings a contradiction if \( M = \mathbb{Q}(\sqrt{-340}) \) is the exceptional field. Put \( T = H^{(\gamma)} \), the fixed field by \( \gamma \). Observe that \( M \nsubseteq T \nsubseteq H \). If \( c_E^H(\tau, \tau) = 1 \) then by Theorem 2.11 the curve \( E^* \) is isogenous to a curve defined over \( T \), and this is a contradiction with the fact that \( M(j_{E^*}) = H \).

Suppose now that \( c_E^H(\tau, \tau) = -1 \). We will see that we can apply the above argument to an appropriate quadratic twist of \( E^* \).

**Claim 4.2.** There exists a quadratic extension \( S/H \) such that \( S/M \) is Galois with \( \text{Gal}(S/M) \simeq D_4 \) and such that \( \tau \) lifts to an element of order 4 of \( \text{Gal}(S/M) \).

We now show how this claim allows us to produce the appropriate twisted curve (and we will prove the claim later on). Define \( C \) to be the \( S/H \) quadratic twist of \( E^* \). By [EG13, Lemma 3.13], the curve \( C \) is an \( M \)-curve completely defined over \( H \) and the cohomology classes of \( E^* \) and \( C \) are related by
\[
\gamma_C^H = \gamma_E^H \cdot \gamma_S,
\]
where \( \gamma_S \in H^2(\text{Gal}(H/M), \{\pm 1\}) \) is the cohomology class attached to the exact sequence
\[
(4.3) \quad 1 \rightarrow \text{Gal}(S/H) \simeq \{\pm 1\} \rightarrow \text{Gal}(S/M) \simeq D_4 \rightarrow \text{Gal}(H/M) \rightarrow 1.
\]
If we identify \( \text{Gal}(S/M) \simeq \langle s, \bar{t}|s^4, t^2, stst \rangle \), then \( \text{Gal}(S/H) \) can be identified with the subgroup generated by \( s^2 \) and we can assume that \( \tilde{\tau} \) lifts to \( s \). Let \( c_S \) be a cocycle representing \( \gamma_S \). The usual construction that associates a cohomology class to \( \langle \tilde{\tau}, \bar{\tau} \rangle \) gives that \( c_S(\tilde{\tau}, \bar{\tau}) = s \cdot s \). Since \( s^2 \) is the non-trivial element of \( \text{Gal}(S/H) \), it corresponds to \( -1 \) under the isomorphism \( \text{Gal}(S/H) \simeq \{\pm 1\} \), so that \( c_S(\tilde{\tau}, \bar{\tau}) = -1 \).

We conclude that \( c_E^H(\tilde{\tau}, \bar{\tau}) = c_E^H(\tilde{\tau}, \bar{\tau})c_S(\tilde{\tau}, \bar{\tau}) = 1 \), and as before this implies that \( C \) can be defined over \( T \), which is a contradiction.

**Proof of Claim 4.2.** The Hilbert class field of \( M \) is \( H = \mathbb{Q}(i, \sqrt{5}, \sqrt{17}) \). If we write \( H = M(\sqrt{a}, \sqrt{b}) \) and suppose that \( \tilde{\tau}(\sqrt{b}) = \sqrt{b} \), it is well known (see, e.g. [Led01, §0.4]) that the obstruction to the existence of \( S \) is given by the quaternion algebra \( \left( \frac{a, b}{M} \right) \) being nonsplit. There are 3 possibilities for \( T \), namely \( T = M(\sqrt{5}) \), \( T = M(\sqrt{17}) \), or \( T = M(\sqrt{5\cdot 17}) \), each one giving a different obstruction. The resulting quaternion algebras giving the obstruction are
\[
\left( \frac{17 \cdot 5, 5}{M} \right), \left( \frac{17 \cdot 5, 17}{M} \right), \left( \frac{17, 5}{M} \right).
\]
Since they are all the split, the field \( S \) does exist in all three cases.

**Remark 4.3.** As a byproduct of the above proof, we see that there do not exist abelian surfaces over \( \mathbb{Q} \) such that \( \text{End}(A_{\mathbb{Q}}) \simeq M_2(M) \) with \( M \) a quadratic imaginary field with class group \( C_2 \times C_2 \) and \( \text{Gal}((K/M) \simeq D_4 \) or \( D_6 \). As shown by the table of [Car01, p. 112], there do exist abelian surfaces over \( \mathbb{Q} \) such that \( \text{End}(A_{\mathbb{Q}}) \simeq M_2(M) \).
with $M$ a quadratic imaginary field with class group $C_2$ and $\text{Gal}(K/M) \simeq D_4$ (resp. $D_6$). If $M$ is not exceptional, Theorem 2.20 and Lemma 4.1 imply that 2 (resp. 3) divide the discriminant of $M$ is a necessary condition for the existence of such an $A$. The examples of the table of [Car01, p. 112] show that this is actually a necessary and sufficient condition.

Proof of Corollary 1.3. Suppose that $A$ is an abelian surface defined over $\mathbb{Q}$ such that $A_{\mathbb{Q}} \sim E \times E'$, where $E$ and $E'$ are elliptic curves defined over $\mathbb{Q}$. If $E$ and $E'$ are not isogenous, then $\text{End}(A_{\mathbb{Q}})$ is

$$\mathbb{Q} \times \mathbb{Q}, \quad M \times \mathbb{Q} \quad \text{or} \quad M_1 \times M_2,$$

where $M, M_1 \not\simeq M_2$ are quadratic imaginary fields, depending on whether none of $E$ and $E'$ has CM, only one of $E$ and $E'$ has CM, or both of $E$ and $E'$ have CM. In any case, note that by [FKRS12 Proposition 4.5], both $E$ and $E'$ can be defined over $\mathbb{Q}$, whereby the class number of $M, M_1$, and $M_2$ must be 1. Recalling that there are 9 quadratic imaginary fields of class number 1, this accounts for 46 distinct $\mathbb{Q}$-endomorphism algebras.

If $E$ and $E'$ are isogenous, we have that $\text{End}(A_{\mathbb{Q}})$ is $M_2(M)$ or $M_2(\mathbb{Q})$, where $M$ is a quadratic imaginary field, depending on whether $E$ has CM or not. Assume that we are in the former case. By Theorem 1.1, we have that $M$ has class group $1, C_2$, or $C_2 \times C_2$. As explained in [FG18 Remark 2.20], for all fields $M$ with class group $1$ (resp. $C_2$), abelian surfaces $A$ over $\mathbb{Q}$ with $\text{End}(A_{\mathbb{Q}}) \simeq M_2(\mathbb{Q})$ can be easily found. Indeed, let $E$ be an elliptic curve over $\mathbb{Q}$ with CM by the maximal order of $M$ and defined over $\mathbb{Q}$ (resp. $\mathbb{Q}(j_E)$). Then consider the square (resp. the restriction of scalars from $\mathbb{Q}(j_E)$ down to $\mathbb{Q}$) of $E$. If $M$ has class group $C_2 \times C_2$, invoke Theorem 1.2 to obtain 18 possibilities for $M$. Taking into account that there are 18 quadratic imaginary fields of class group $1$ (see [Wat04] for example), we obtain 46 possibilities for the endomorphism algebra of a geometrically split abelian surface over $\mathbb{Q}$ with $\mathbb{Q}$-isogenous factors.

An open problem. We wish to conclude the article with an open question.

**Question 4.4.** Which is the subset of $A$ made of the $\mathbb{Q}$-endomorphism algebras $\text{End}(\text{Jac}(C)_{\mathbb{Q}})$ of geometrically split Jacobians of genus 2 curves $C$ defined over $\mathbb{Q}$?

Again the most intriguing case is to determine how many of the 45 possibilities for $M_2(M)$, with $M$ a quadratic imaginary field, allowed by Theorem 1.2 for geometrically split abelian surfaces defined over $\mathbb{Q}$ still occur among geometrically split Jacobians of genus 2 curves $C$ defined over $\mathbb{Q}$. Looking at the more restrictive setting that requires $\text{Jac}(C)$ to be isomorphic to the square of an elliptic curve with CM by the maximal order of $M$, Gélin, Howe, and Ritzenthaler [GHR19] have shown that there are 13 possibilities for such an $M$ (all with class number $\leq 2$).

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