Generating Non-Linear Interpolants by Semidefinite Programming

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Abstract. Interpolation-based techniques have been widely and successfully applied in the verification of hardware and software, e.g., in bounded-model checking, CEGAR, SMT, etc., whose hardest part is how to synthesize interpolants. Various work for discovering interpolants for propositional logic, quantifier-free fragments of first-order theories and their combinations have been proposed. However, little work focuses on discovering polynomial interpolants in the literature. In this paper, we provide an approach for constructing non-linear interpolants based on semidefinite programming, and show how to apply such results to the verification of programs by examples.

Keywords: Craig interpolant, Positivstellensatz Theorem, semidefinite programming, program verification.

1 Introduction

It becomes a grand challenge how to guarantee the correctness of software, as our modern life more and more depends on computerized systems. There are lots of verification techniques based on either model-checking \cite{1}, or theorem proving \cite{2,3}, or abstract interpretation \cite{4}, or their combination, that have been invented for the verification of hardware and software, like bounded model-checking \cite{5}, CEGAR \cite{6}, satisfiability modulo theories (SMT) \cite{7}, etc. The bottleneck of these techniques is scalability, as many of real software are very complex with different features like complicated data structures, concurrency, distributed, real-time and hybrid, and so on. While interpolation-based techniques provide a powerful mechanism for local and modular reasoning, which indeed improves the scalability of these techniques, in which the notion of Craig interpolants plays a key role.

Interpolation-based local and modular reasoning was first applied in theorem proving due to Nelson and Oppen \cite{8}, called Nelson-Oppen method. The basic idea of Nelson-Oppen method is to reduce the satisfiability (validity) of a composite theory into the ones of its component theories whose satisfiability (validity) have been obtained. The hardest part of the method, which also determines the efficiency of the method, is to construct a formula using the common part of the component theories

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for a given formula of the composite theory with Craig Interpolation Theorem. In the past decade, the Nelson-Oppen method was further extended to SMT which is based on DPLL \cite{9} and Craig Interpolation Theorem \cite{10} for combining different decision procedures in order to verify a property of programs with complicated data structures. For instance, Z3 \cite{11} integrates more than 10 different decision procedures up to now, including propositional logic, equality logic with uninterpreted functions, Presburger arithmetic, array logic, difference arithmetic, bit vector logic, and so on.

In recent years, it is noted that interpolation based local and modular reasoning is quite helpful for improving the scalability of model-checking, in particular for bounded model-checking of systems with finite or infinite states \cite{5,12,13}, CEGAR \cite{14,15}, etc. McMillian first considered how to combine Craig interpolants with bounded model-checking to verify infinite state systems. The basic idea of his approach is to generate invariant using Craig interpolants so that it can be claimed that an infinite state system satisfies a property after \( k \) steps in model-checking whenever an invariant is obtained which is strong enough to guarantee the property. While in \cite{14,15,16}, how to apply the local property of Craig interpolants generated from a counter-example to refine the abstract model in order to exclude the spurious counter-example in CEGAR was investigated. Meanwhile, in \cite{17}, using interpolation technique to generate a set of atomic predicates as the base of machine-learning based verification technique was investigated by Wang et al.

Obviously, synthesizing Craig interpolants is the cornerstone of interpolation based techniques. In fact, many approaches have been proposed in the literature. In \cite{13}, McMillian presented a method for deriving Craig interpolants from proofs in the quantifier-free theory of linear inequality and uninterpreted function symbols, and based on which an interpolating theorem prover was provided. For improving the efficiency of constructing interpolant, McMillian further proposed a method based on lazy abstraction for generating interpolants. While, in \cite{15}, Henzinger et al. proposed a method to synthesizing Craig interpolants for a theory with arithmetic and pointer expressions, as well as call-by-value functions. In \cite{18}, Yorsh and Musuvathi presented a combination method for generating Craig interpolants for a class of first-order theories. While Rybalchenko and Sofronie-Stokkermans \cite{19} proposed an approach by reducing the synthesis of Craig interpolants of the combined theory of linear arithmetic and uninterpreted function symbols to constraint solving.

However, in the literature, there is little work on how to synthesize non-linear interpolants, except that in \cite{20} Kupferschmid and Becker provided a method to construct non-linear Craig Interpolant using iSAT, which is a variant of SMT solver based on interval arithmetic.

In this paper we investigate how to construct non-linear interpolants. The idea of our approach is as follows: Firstly, we reduce the problem of generating interpolants for arbitrary two polynomial formulas to that of generating interpolants for two semi-algebraic systems (SASs), which is a conjunction of a set of polynomial equations, inequations and inequalities (see the definition later). Then, according to Positivstellensatz Theorem of real algebraic geometry \cite{21}, there exists a witness to indicate the considered two SASs do not have common real solutions if their conjunction is unsatisfiable. Parrilo in \cite{22,23} gave an approach for constructing the witness by applying
semidefinite programming [24]. Our algorithm invokes Parrilo’s method as a subroutine. Our purpose is to construct Craig interpolants, so we need to obtain a special witness. In general case, we cannot guarantee the existence of the special witness, which means that our approach is only sound, but not complete. However, we discuss that if the considered two SASs meet Archimedean condition, (e.g., each variable occurring in the SASs is bounded, which is a reasonable assumption in practice), our approach is not only sound, but also complete. We demonstrate our approach by some examples, in particular, we show how to apply the results to program verification by examples.

The complexity of our approach is polynomial in $ub^{(n+b/2)} n^{n+b}$, where $u$ is the number of polynomial constraints in the considered problem, $n$ is the number of variables, and $b$ is the highest degree of polynomials and interpolants. So, the complexity of our approach is polynomial in $b$ for a given problem as in which $n$ and $u$ are fixed.

Structure of the paper: The rest of the paper is organized as follows. By a running example, we sketch our approach and show how to apply it to program verification in Section 2. Some necessary preliminaries are introduced in Section 3. A sound but incomplete algorithm for synthesizing non-linear interpolants in general case is described in Section 4. Section 5 provides a practical algorithm for systems only containing non-strict inequalities and satisfying Archimedean condition. Section 6 focuses on the correctness and complexity analysis of our approach. Our implementation and experimental results are briefly reported in Section 7. Section 8 describes more related work related to interpolant generation and its application to program verification. Our summarizes the paper and discusses future work in Section 9.

2 An Overview of Our Approach

In this section, we sketch our approach and show how to apply our results to program verification by an example.

```plaintext
1  if (x*x+y*y<1) /* initial values
2    while (x*x+y*y<3) {
3      x := x*x+y-y;  
4      y := y+x*y+y;  
5     if (x*x-2*y*y-4>0) /* unsafe area
6       error();
7     }
8   }  
```

Consider the program in Code 1.1 (left). This program tests the initial value of $x$ and $y$ at line 1, afterwards executes the while loop with $x^2 + y^2 < 3$ as the loop condition. The body of the while loop contains two assignments and an if statement in sequence. The
property we wish to check is that $\text{error()}$ procedure will never be executed. Suppose there is an execution $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 8$. We can encode such an execution by the formulas as in Code 11 (right). Note that in these formulas we use unprimed and primed versions of each variable to represent the values of the variable before and after updating respectively. Obviously, the execution is infeasible iff the conjunction of these formulas is unsatisfiable. Let $\phi \triangleq g_1 > 0 \land f_1 = 0 \land f_2 = \overline{f}$ and $\psi \triangleq g_3 > 0$. To show $\phi \land \psi$ is unsatisfiable, we need to construct an interpolant $\theta$ for $\phi$ and $\psi$, i.e., $\phi \Rightarrow \theta$ and $\theta \Rightarrow \neg \psi$. If there exist $\delta_1, \delta_2, \delta_1, h_1, h_2$ such that

$$g_1 \delta_1 + f_1 h_1 + f_2 h_2 + g_3 \delta_2 + \delta_3 = -1,$$

where $\delta_1, \delta_2, \delta_3 \in \mathbb{R}[x, y, x', y']$ are sums of squares and $h_1, h_2 \in \mathbb{R}[x, y, x', y']$, then $\theta \triangleq g_3 \delta_2 + \frac{1}{2} \leq 0$ is such an interpolant for $\phi$ and $\psi$. In this example, applying our tool A\text{ISat}, we obtain in 0.025 seconds that

$$h_1 = -290.17 - 56.86y' + 1109.95x' + 37.59y - 32.20yy' + 386.77yx' + 203.88y^2 + 107.91x^2,$$

$$h_2 = -65.71 + 0.39y' + 244.14x' + 274.80y + 69.33yy' - 193.42yx' - 88.18y^2 - 105.63x^2,$$

$$\delta_1 = 797.74 - 31.38y' + 466.12y^2 + 506.26x' + 79.87x'y' + 402.44x'^2 + 104.43y + 41.09yy' - 70.14yx' + 451.64y^2 + 578.94x^2,$$

$$\delta_2 = 436.45,$$

$$\delta_3 = 722.62 - 91.59y' + 407.17y^2 + 69.39x' + 107.41x'y' + 271.06x'^2 + 14.23y + 188.65yy' + 69.33yy'^2 - 600.47yx' - 226.01yx'y' + 142.62yx'^2 + 325.78y^2 - 156.69y^2y' + 466.12y^2y'^2 + 10.54y^2x'y' + 595.87y^2x'^2 - 11.26y^3 + 41.09y^3y' + 18.04y^3x' + 451.64y^4 + 722.52x^2 - 80.15x^2y' + 466.12x^2y'^2 - 495.78x^2x' + 79.87x^2x'y' + 402.44x^2x'^2 + 64.57x^2y + 241.99x^2y' + 73.29x^2yy' - 351.27x^2y' + 826.70x^2y'^2 + 471.03x^4.$$

Note that $\delta_1$ can be represented as $923.42(0.90 + 0.7y - 0.1y' = 0.43x')^2 + 252.84(0.42 - 0.28y + 0.21y' - 0.84x')^2 + 461.69(-0.1 - 0.83y + 0.44y' + 0.34x')^2 + 478(-0.06 + 0.48y + 0.87y' + 0.03x')^2 + 578.94(x)^2$. Similarly, $\delta_2$ and $\delta_3$ can be represented as sums of squares also.

Moreover, using the approach in [23], we can prove $\theta$ is an inductive invariant of the loop, therefore, $\text{error()}$ will never be executed.

**Remark 1.** Note that $\theta$ itself cannot be generated using quantifier elimination (QE for short) approach in [23], as it contains more than thirty monomials, which means that there are more than thirty parameters at least in any predefined template which can be used to generate $\theta$. Handling so many parameters is far beyond the capability of all the existing tool based on QE. However, the problem whether $\theta$ is an inductive variant only contains 4 variables, therefore it can be verified using QE. The detailed comparison between our approach reported in this paper and QE based technique can be seen in the related work.

### 3 Theoretical Foundations

In this section, for self-containedness, we briefly introduce some basic notions and mathematical theories, based on which our approach is developed.

\[\text{As } g_1 > 0 \Rightarrow g_2 > 0, \text{ we ignore } g_2 > 0 \text{ in } \phi.\]
Definition 1 (Interpolants). A theory $\mathcal{T}$ has interpolant if for all formulae $\phi$ and $\psi$ in the signature of $\mathcal{T}$, if $\phi \models_{\mathcal{T}} \psi$, then there exists a formula $\Theta$ that contains only symbols that $\phi$ and $\psi$ share such that $\phi \models_{\mathcal{T}} \Theta$ and $\Theta \models_{\mathcal{T}} \psi$.

An interpolant $\Theta$ of $\phi$ and $\neg \psi$ is called inverse interpolant of $\phi$ and $\psi$, i.e., $\phi \land \neg \psi \models_{\mathcal{T}} \bot$, $\phi \models_{\mathcal{T}} \Theta$ and $\Theta \land \psi \models_{\mathcal{T}} \bot$, where $\Theta$ contains only the symbols that $\phi$ and $\psi$ share.

Note that in practice, people like to abuse inverse interpolant as interpolant. Thus, as a convention, in the sequel, all interpolants are referred to inverse interpolant if not otherwise stated.

Also, in what follows, we denote by $x$ a variable vector $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$, and by $\mathbb{R}[x]$ the polynomial ring with real coefficients in variables $x$.

### 3.1 Problem Description

Here, we describe the problem we consider in this paper. Let

$$
\mathcal{T}_{1l} = \bigwedge_{j=0}^{k_l} f_{ij}(x) \triangleright 0 \quad \text{and} \quad \mathcal{T}_{2l} = \bigwedge_{j=0}^{s_l} g_{ij}(x) \triangleright' 0,
$$

be two semi-algebraic systems (SASs), where $f_{ij}$ and $g_{ij}$ are polynomials in $\mathbb{R}[x]$, and $\triangleright, \triangleright'$ belong to $\{=, \neq, \geq\}$. Clearly, any polynomial formula $\phi$ can be represented as a DNF, i.e. the disjunction of a several SASs. Let $\mathcal{T}_1 = \bigvee_{t=1}^m \mathcal{T}_{1t}$, $\mathcal{T}_2 = \bigvee_{l=1}^n \mathcal{T}_{2l}$ be two polynomial formulas and $\mathcal{T}_1 \land \mathcal{T}_2 \models \bot$, i.e., $\mathcal{T}_1$ and $\mathcal{T}_2$ do not share any real solutions. Then, the problem to be considered in this paper is how to find another polynomial formula $I$ such that $\mathcal{T}_1 \models I$ and $I \land \mathcal{T}_2 \models \bot$.

It is easy to show that if, for each $t$ and $l$, there is an interpolant $I_{tl}$ for $\mathcal{T}_{1t}$ and $\mathcal{T}_{2l}$, then $I = \bigvee_{t=1}^m \bigwedge_{l=1}^n I_{tl}$ is an interpolant of $\mathcal{T}_1$ and $\mathcal{T}_2$. Thus, we only need to consider how to construct interpolants for two SASs of the form (1) in the rest of this paper.

### 3.2 Common variables

In the above problem description, we assume $\mathcal{T}_1$ and $\mathcal{T}_2$ share a set of variables. But in practice, it is possible that they have different variables. Suppose $\mathcal{V}(\mathcal{T}_i)$ for the set of variables that indeed occur in $\mathcal{T}_i$, for $i = 1, 2$. For each $v \in \mathcal{V}(\mathcal{T}_1) \setminus \mathcal{V}(\mathcal{T}_2)$, if $v$ is a local variable introduced in the respective program, we always have an equation $v = h$ corresponding to the assignment to $v$ (possibly the composition of a sequence of assignments to $v$); otherwise, $v$ is a global variable, but only occurring in $\mathcal{T}_1$, for this case, we introduce an equation $v = v$ to $\mathcal{T}_2$; Symmetrically, each $v \in \mathcal{V}(\mathcal{T}_2) \setminus \mathcal{V}(\mathcal{T}_1)$ can be coped with similarly.

In the following, we show how to derive the equation $v = h$ from the given programs by case analysis.

- If the given program has no recursion nor loops, we can find out the dependency between the variables in $\mathcal{V}(\mathcal{T}_1) \cap \mathcal{V}(\mathcal{T}_2)$ and the variables in $\mathcal{V}(\mathcal{T}_j) \setminus \mathcal{V}(\mathcal{T}_{3-j})$ according to the order of assignments in the program segment, where $j = 1, 2$. 
Clearly, we can always represent each variable in $\mathcal{V}(T_j) - \mathcal{V}(T_{3-j})$ by an expression of $\mathcal{V}(T_1) \cap \mathcal{V}(T_2)$. Obviously, if all expressions in the program segment are polynomial, the resulted expressions are polynomial either.

- If the given program contains loops or recursion, it will become more complicated. So, we have to unwind the loop and represent each variable in $\mathcal{V}(T_j) - \mathcal{V}(T_{3-j})$ by an expression of $\mathcal{V}(T_1) \cap \mathcal{V}(T_2)$ and the number $i$ of the iterations of the loops or recursions. However, the resulted expressions may not be polynomial any more. But as proved in [26], if assignment mappings of the loops in the program segment are solvable, the resulted expressions are still polynomial.

**Definition 2 (Solvable mapping [26])**. Let $g \in \mathbb{Q}[x]^m$ be a polynomial mapping. $g$ is solvable if there exists a partition of $x$ into subvectors of variables, $x = w_1 \cup \cdots \cup w_k$, $w_i \cap w_j = \emptyset$ if $i \neq j$, such that for all $j \geq 1$, $g_{w_j}(x) = M_j w_j^T + P_j(w_1,\ldots,w_{j-1})$,

where $M_j \in \mathbb{Q}[w_j]^{|w_j|}$ is a matrix and $P_j$ is a vector of $|w_j|$ polynomials in the ring $\mathbb{Q}[w_1,\ldots,w_{j-1}]$. For $j = 1$, $P_1$ must be a constant vector, implying that $g_{w_1}$ is an affine mapping.

For example, in the Code 1.2, $a, b$ are common variables, $x, y, z, w$ are local variables. Let $T_1$ be related to the left and $T_2$ to the right of Code 1.2. $T_1$ uses an order $y \succ x \succ b \succ a$ on variables, and $T_2$ uses an order $z \succ w \succ b \succ a$ on variables. Obviously, in every iteration of the loops, variable with higher precedence can only be assigned with a polynomial of variables with lower precedence. In order to prove the assert, we unwind the first loop $i$ times, and obtain the values of $a, b, x, y$ are $a, b, a^{i+1}, \sum_{j=0}^{i+1} a^{i+1-j}b^j$, respectively. Similarly, unwind the second loop $j$ times, and obtain the values of $a, b, z, w$ are $a + j \cdot b, b \cdot z, ja + \frac{(j+1)j}{2}b, (ja + \frac{(j+1)j}{2})b^2$, respectively. Thus, in the first loop, the local variables $x, y$ are represented by expressions of $a, b$, so are $z, w$ in the second loop. Using such replacements, we can obtain an interpolant $I_{ij}$ only concerning the common variables $a, b$ w.r.t. the $i$-th unwinding of the first loop and the $j$-th unwinding of the second loop. Whenever we can prove that $I_{ij}$ is an invariant of Code 1.2, then the assert is guaranteed. This is a procedure of BMC.

In what follows, we use subvariable to denote the above procedure to transform two SASs that may not share same variables to two SASs that share same variables.
3.3 Real Algebraic Geometry

In this subsection, we introduce some basic notions and results on real algebraic geometry, that will be used later.

**Definition 3 (ideal).** Let \( \mathcal{I} \) be an ideal in \( \mathbb{R}[x] \), that is, \( \mathcal{I} \) is an additive subgroup of \( \mathbb{R}[x] \) satisfying \( fg \in \mathcal{I} \) whenever \( f \in \mathcal{I} \) and \( g \in \mathbb{R}[x] \). Given \( h_1, \ldots, h_m \in \mathbb{R}[x] \), 
\[
\langle h_1, \ldots, h_m \rangle = \{ \sum_{j=1}^m u_j h_j \mid u_1, \ldots, u_m \in \mathbb{R}[x] \}
\]
denotes the ideal generated by \( h_1, \ldots, h_m \).

**Definition 4 (multiplicative monoid).** Given a polynomial set \( P \), let \( \text{Mult}(P) \) be the multiplicative monoid generated by \( P \), i.e., the set of finite products of the elements of \( P \) (including the empty product which is defined to be 1).

**Definition 5 (Cone).** A cone \( C \) of \( \mathbb{R}[x] \) is a subset of \( \mathbb{R}[x] \) satisfying the following conditions: (i) \( p_1, p_2 \in C \Rightarrow p_1 + p_2 \in C \); (ii) \( p_1, p_2 \in C \Rightarrow p_1 p_2 \in C \); (iii) \( p \in \mathbb{R}[x] \Rightarrow p^2 \in C \).

Given a set \( P \subseteq \mathbb{R}[x] \), let \( C(P) \) be the smallest cone of \( \mathbb{R}[x] \) that contains \( P \). It is easy to see that \( C(\emptyset) \) corresponds to the polynomials that can be represented as a sum of squares, and is the smallest cone in \( \mathbb{R}[x] \), i.e., \( \{ \sum_{i=1}^r p_i^2 \mid p_i \in \mathbb{R}[x] \} \), denoted by \( \text{SOS} \). For a finite set \( P \subseteq \mathbb{R}[x] \), \( C(P) \) can be represented as:
\[
C(P) = \{ q + \sum_{i=1}^r q_i p_i \mid q, q_1, \ldots, q_r \in C(\emptyset), p_1, \ldots, p_r \in \text{Mult}(P) \}.
\]

**Positivstellensatz Theorem,** due to Stengle [21], is an important theorem in real algebraic geometry. It states that, for a given SAS, either the system has a solution in \( \mathbb{R}^n \), or there exists a certain polynomial identity which bears witness to indicate that the system has no solutions.

**Theorem 1 (Positivstellensatz Theorem, [21]).** Let \( (f_j)_{j=1}^s, (g_k)_{k=1}^t, (h_l)_{l=1}^n \) be finite families of polynomials in \( \mathbb{R}[x] \). Denote by \( C \) the cone generated by \( (f_j)_{j=1}^s \), \( \text{Mult} \) the multiplicative monoid generated by \( (g_k)_{k=1}^t \), and \( \mathcal{I} \) the ideal generated by \( (h_l)_{l=1}^n \). Then the following two statements are equivalent:
\[
\begin{align*}
1. & \text{ the SAS } \begin{cases} f_1(x) \geq 0, & \cdots, \ f_s(x) \geq 0, \\
 \text{g_1(x) \neq 0,} & \cdots, \ g_t(x) \neq 0, \text{ has no real solutions;}
\end{cases} \\
2. & \text{there exist } f \in C, g \in \text{Mult}, h \in \mathcal{I} \text{ such that } f + g^2 + h \equiv 0.
\end{align*}
\]

3.4 Semidefinite Programming

In [21], Stengle did not provide a constructive proof to Theorem 1. However, Parrilo in [22,23] provided a constructive way to obtain the witness, which is based on semidefinite programming. Parrilo’s result will be the starting point of our method, so we briefly review semidefinite programming below. We use \( \text{Sym}_n \) to denote the set of \( n \times n \) real symmetric matrices, and \( \text{deg}(f) \) the highest total degree of \( f \) for a given polynomial \( f \) in the sequel.
Definition 6 (Positive semidefinite matrices). A matrix \( M \in \text{Sym}_n \) is called \( \text{positive semidefinite} \), denoted by \( M \succeq 0 \), if \( x^T M x \geq 0 \) for all \( x \in \mathbb{R}^n \).

Definition 7 (Inner product). The inner product of two matrices \( A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n} \), denoted by \( \langle A, B \rangle \), is defined by \( \text{Tr}(A^T B) = \sum_{i,j=1}^{n} a_{ij}b_{ij} \).

Definition 8 (Semidefinite programming (SDP)). The standard (primal) and dual forms of a SDP are respectively given in the following:

\[
p^\ast = \inf_{X \in \text{Sym}_n} \langle C, X \rangle \quad \text{s.t.} \quad X \succeq 0, \quad \langle A_j, X \rangle = b_j \quad (j = 1, \ldots, m) \tag{2}
\]

\[
d^\ast = \sup_{y \in \mathbb{R}^m} b^T y \quad \text{s.t.} \quad \sum_{j=1}^{m} y_j A_j + S = C, \quad S \succeq 0, \tag{3}
\]

where \( C, A_1, \ldots, A_m, S \in \text{Sym}_n \) and \( b \in \mathbb{R}^m \).

There are many efficient algorithms to solve SDP such as interior-point method. We present a basic path-following algorithm to solve (2) in the following.

Definition 9 (Interior point for SDP).

\[
\text{intF}_p = \{ X : \langle A_i, X \rangle = b_i \quad (i = 1, \ldots, m), \quad X \succ 0 \},
\]

\[
\text{intF}_d = \{ (y, S) : S = C - \sum_{i=1}^{m} A_i y_i \succ 0 \},
\]

\[
\text{intF} = \text{intF}_p \times \text{intF}_d.
\]

Obviously, \( \langle C, X \rangle - b^T y = \langle X, S \rangle > 0 \) for all \( (X, y, S) \in \text{intF} \). Especially, we have \( d^\ast \leq p^\ast \). So the soul of interior-point method to compute \( p^\ast \) is to reduce \( \langle X, S \rangle \) incessantly and meanwhile guarantee \( (X, y, S) \in \text{intF} \).

Algorithm 1: Interior Point Method

```
input : C, A_j, b_j (j = 1, \ldots, m) as in (2) and a threshold c
output: p^\ast

1 Given a \( (X, y, S) \in \text{intF} \) and \( XS = \mu I \);
   /* \( \mu \) is a positive constant and \( I \) is the identity matrix. */
2 while \( \mu > c \) do
3     \( \mu = \gamma \mu \);
   /* \( \gamma \) is a fixed positive constant less than one */
4     use Newton iteration to solve \( (X, y, S) \in \text{intF} \) with \( XS = \mu I \);
5 end
```
3.5 Constructive Proof of Theorem 1 Using SDP

Given a polynomial \( f(x) \) of degree no more than \( 2d \), \( f \) can be rewritten as \( f = Z^T Q Z \) where \( Z \) is a vector consists of all monomials of degrees no more than \( d \), e.g.,

\[
Z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, x_2 x_3, \ldots, x_n]^T, \quad \text{and} \quad Q = \begin{pmatrix}
\frac{a_1}{2} & \frac{a_2}{2} & \ldots & \frac{a_n}{2} \\
\frac{a_1 x_1}{2} & \frac{a_2 x_2}{2} & \ldots & \frac{a_n x_n}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_1 x_1^d}{2} & \frac{a_2 x_2^d}{2} & \ldots & \frac{a_n x_n^d}{2}
\end{pmatrix}
\]

is a symmetric matrix. Note that here \( Q \) is not unique in general. Moreover, \( f \in \mathcal{C}(\emptyset) \) iff there is a positive semidefinite constant matrix \( Q \) such that \( f(x) = Z^T Q Z \). The following lemma is an obvious fact on how to use the above notations to express the polynomial multiplication.

Lemma 1. For given polynomials \( f_1, \ldots, f_n, g_1, \ldots, g_n \), assume \( \sum_{i=1}^n f_i g_i = \sum_{i=1}^s c_i m_i \), where \( c_i \in \mathbb{R} \) and \( m_i \)s are monomials. Suppose \( g_i = Z_i^T Q_i Z \) and \( Q_2 = \text{diag}(Q_{21}, \ldots, Q_{2n}) \). Then there exist symmetric matrices \( Q_{11}, \ldots, Q_{1s} \) such that \( c_i = \langle Q_{1i}, Q_2 \rangle \), i.e., \( \sum_{i=1}^s f_i g_i = \sum_{i=1}^s \langle Q_{1i}, Q_2 \rangle m_i \), in which \( Q_{1i} \) can be constructed from the coefficients of \( f_1, \ldots, f_n \).
The following two specific SASs

4 Synthesizing Non-linear Interpolants in General Case

As discussed before, we only need to consider how to synthesize interpolants for the following two specific SASs

$$T_1 = \begin{cases} f_1(x) \geq 0, \ldots, f_s(x) \geq 0, \\ g_1(x) \neq 0, \ldots, g_t(x) \neq 0, \\ h_1(x) = 0, \ldots, h_{u_1}(x) = 0 \end{cases}$$

$$T_2 = \begin{cases} f_{s+1}(x) \geq 0, \ldots, f_s(x) \geq 0, \\ g_{s+1}(x) \neq 0, \ldots, g_t(x) \neq 0, \\ h_{u_1+1}(x) = 0, \ldots, h_u(x) = 0 \end{cases}$$

where $T_1$ and $T_2$ do not share any real solutions.

5 For example, let $q_{u_1} = (\frac{1}{2}p_i + 1)^2, q_{u_2} = (\frac{1}{2}p_i - 1)^2$. 

Example 1. Let $f = a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2$ and $g = b_0 + b_{10}x_1 + b_{01}x_2$. Then, $f = \langle Q_{11}, Q_2 \rangle x_1^2 + \langle Q_{12}, Q_2 \rangle x_1x_2 + \langle Q_{13}, Q_2 \rangle x_2^2 + \langle Q_{14}, Q_2 \rangle x_1^2x_2 + \langle Q_{15}, Q_2 \rangle x_1x_2^2 + \langle Q_{16}, Q_2 \rangle x_2^3 + \langle Q_{17}, Q_2 \rangle x_1^3$, where

$$Q_2 = \begin{pmatrix} b_{00} & b_{02} & b_{04} \\ b_{02} & 0 & 0 \\ b_{04} & 0 & 0 \end{pmatrix},$$

$$Q_{11} = \begin{pmatrix} a_{20} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_{12} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_{13} = \begin{pmatrix} a_{02} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Back to Theorem 1. We show how to find $f \in \mathcal{C}, g \in \text{Mult}, h \in \mathcal{I}$ such that $f + g^2 + h \equiv 0$ via SDP solving. First, since $f \in \mathcal{C}, f$ can be written as a sum of the products of some known polynomials and some unknown SOSs. Second, $h \in \mathcal{I} \{h_1, \ldots, h_u\}$ is equivalent to $h = h_1p_1 + \cdots + h_up_u$, which is further equivalent to $h = h_{1}(q_{11} - q_{12}) + \cdots + h_{u}(q_{u1} - q_{u2})$, where $p_i, q_{ij} \in \mathbb{R}[x]$ and $q_{ij} \in \text{SOS}$. Third, fix an integer $d > 0$, let $g = (H^{i=1}_i g_i)^d$, and then $f + g^2 + h \equiv 0$ can be written as $\sum_{i=1}^l f_i^t \delta_i$, where $l$ is a constant integer, $f_i^t \in \mathbb{R}[x]$ are known polynomials and $\delta_i \in \text{SOS}$ are undermined SOS polynomials. Therefore, Theorem 1 is reduced to fixing a sufficiently large integer $d$ and finding undermined SOS polynomials $\delta_i$ occurring in $f, h$ with degrees less than or equal to $\deg(g^2)$, which satisfies $f + g^2 + h \equiv 0$. Based on Lemma 1 this is a SDP problem of form (2). The constraints of the SDP are of the form $\langle A_j, X \rangle = 0$, where $A_j$ and $X$ correspond to $Q_{1j}$ and $Q_2$ in Lemma 1 respectively. And $Q_2$ is a block diag matrix whose blocks correspond to the undermined SOS polynomials in the above discussion. That is,

**Theorem 2 (22).** Consider a system of polynomial equalities and inequalities of the form in Theorem 1. Then the search for bounded degree Positivstellensatz refutations can be done using semidefinite programming. If the degree bound is chosen to be large enough, then the SDPs will be feasible, and the certificates can be obtained from its solution.

Algorithm 2 is an implementation of Theorem 2 and we will invoke Algorithm 2 as a subroutine later. Note that Algorithm 2 is a little different from the original one in [23], as here we require that $f$ has 1 as a summand for our specific purpose.
By Theorems [1&2], there exist $f \in C(\{f_1, \ldots, f_s\})$, $g \in \text{Multi}(\{g_1, \ldots, g_t\})$ and $h \in I(\{h_1, \ldots, h_u\})$ such that $f + g^2 + h \equiv 0$, where

$$g = \prod_{i=1}^{t} g_i^{2m},$$

$$h = q_1 h_1 + \cdots + q_u h_u,$$

$$f = p_0 + p_1 f_1 + \cdots + p_s f_s + p_{12} f_1 f_2 + \cdots + p_{1s} f_1 \cdots f_s,$$

in which $q_i$ and $p_i$ are in SOS.

If $f$ can be represented by three parts: the first part is an SOS polynomial that is greater than 0, the second part is from $C(\{f_1, \ldots, f_s\})$, and the last part is from $C(\{f_{s+1}, \ldots, f_s\})$, i.e., $f = p_0 + \sum_{v \subseteq \{1, \ldots, s\}} p_v (\Pi_{i \in v} f_i) + \sum_{v \subseteq \{s+1, \ldots, s\}} p_v (\Pi_{i \in v} f_i)$, where $\forall x \in \mathbb{R}^n, p_0(x) > 0$ and $p_v \in$ SOS. Then let

$$f_{T_1} = \sum_{v \subseteq 1, \ldots, s_1} p_v \Pi_{i \in v} f_i, \quad h_{T_1} = q_1 h_1 + \cdots + q_u h_u,$$

$$f_{T_2} = \sum_{v \subseteq s_1 + 1, \ldots, s} p_v \Pi_{i \in v} f_i, \quad h_{T_2} = h - h_{T_1},$$

$$q = f_{T_1} + g^2 + h_{T_1} + \frac{g_0}{2} = -(f_{T_2} + h_{T_2}) - \frac{g_0}{2}.$$  

Obviously, we have $\forall x \in T_1, q(x) > 0$ and $\forall x \in T_2, q(x) < 0$. Thus, let $I = q(x) > 0$. We have $T_1 \models I$ and $I \wedge T_2 \models \bot$.

Notice that because the requirement on $f$ cannot be guaranteed in general, the above approach is not complete generally. We will discuss under which condition the requirement can be guaranteed in the next section. We implement the above method for synthesizing non-linear interpolants in general case by Algorithm 3.

**Example 2.** Consider

$$\mathcal{T}_1 = \begin{cases} x_1^2 + x_2^2 + x_3^2 - 2 \geq 0, \\ x_1 + x_2 + x_3 \neq 0, \\ 1.2x_1^2 + x_2^2 + x_1x_3 = 0 \end{cases}$$

and

$$\mathcal{T}_2 = \begin{cases} -3x_1^2 - 4x_2^3 - 10x_3^2 + 20 \geq 0, \\ 2x_1 + 3x_2 - 4x_3 \neq 0, \\ x_1^2 + x_2^2 - x_3 - 1 = 0 \end{cases}$$
In this section, we show, under Archimedean condition, the requirement can be indeed true, the other two polynomials are respectively from $C(\{f_1, \ldots, f_s\})$ and $C(\{f_s, \ldots, f_s\})$. In this section, we show, under Archimedean condition, the requirement can be indeed guaranteed. Thus, our approach will become complete. In particular, we shall argue Archimedean condition is a necessary and reasonable restriction in practice.

Clearly, $T_1$ and $T_2$ do not share any real solutions, see Fig. 1 (left). By setting $b = 2$, after calling Certificate Generation, we obtain an interpolant $I$ with 30 monomials $-14629.26 + 2983.44x_3 + 10972.97x_3^2 + 297.62x_2 + 297.64x_2x_3 + 0.02x_2x_3^2 + 9625.61x_2^2 - 1161.80x_2x_3^3 + 0.01x_2^2x_3^3 + 811.93x_2^3 + 2745.14x_2^4 - 10648.11x_1 + 3101.42x_1x_3 + 8646.17x_1x_3^2 + 511.84x_1x_3^3 - 1034.31x_1x_2x_3 - 0.02x_1x_2x_3^2 + 9233.66x_2^2 + 1342.55x_1x_2x_3^2 - 138.70x_1x_2^2 + 11476.61x_2^3 - 3737.70x_2^4x_3 + 4071.65x_2^4x_3^2 - 2153.00x_1x_2^4 + 373.14x_2^4x_2x_3 + 7616.18x_2^4x_2^2 + 8950.77x_1^3 + 1937.92x_1^4x_3 - 64.07x_1^3x_2 + 4827.25x_1^4$, whose figure is depicted in Fig. 1 (right).

5 A Complete Algorithm Under Archimedean Condition

Our approach to synthesizing non-linear interpolants presented in Section 4 is incomplete generally as it requires that the polynomial $f$ in $C(\{f_1, \ldots, f_s\})$ produced by Algorithm 2 can be represented by the sum of three polynomials, one of which is positive, the other two polynomials are respectively from $C(\{f_1, \ldots, f_s\})$ and $C(\{f_s, \ldots, f_s\})$. In this section, we show, under Archimedean condition, the requirement can be indeed guaranteed. Thus, our approach will become complete. In particular, we shall argue Archimedean condition is a necessary and reasonable restriction in practice.

6 For simplicity, we do not draw $x_1 + x_2 + x_3 \neq 0$, nor $2x_1 + 3x_2 - 4x_3 \neq 0$ in the figure.
5.1 Archimedean Condition

To the end, we need more knowledge of real algebraic geometry.

**Definition 10 (quadratic module).** For \( g_1, \ldots, g_m \in \mathbb{R}[x] \), the set

\[
\mathcal{M}(g_1, \ldots, g_m) = \{ \delta_0 + \sum_{j=1}^{m} \delta_j g_j | \delta_0, \delta_j \in C(0) \} \tag{5}
\]

is called the quadratic module generated by \( g_1, \ldots, g_m \). A quadratic module \( \mathcal{M} \) is called proper if \(-1 \notin \mathcal{M} \) (i.e. \( \mathcal{M} \neq \mathbb{R}[x] \)). A quadratic module \( \mathcal{M} \) is maximal if for any \( p \in \mathbb{R}[x] \cap \mathcal{M} \), \( \mathcal{M} \cup \{ p \} \) is not a quadratic module.

In what follows, we will use \(-\mathcal{M}\) to denote \( \{-p | p \in \mathcal{M}\} \) for any given quadratic module \( \mathcal{M} \).

The following results are adapted from [27] and will be used later, whose proofs can be found in [27].

**Lemma 2 ([21][27]).**

1) If \( \mathcal{M} \subseteq \mathbb{R}[x] \) is a quadratic module, then \( I = \mathcal{M} \cap -\mathcal{M} \) is an ideal. 

2) If \( \mathcal{M} \subseteq \mathbb{R}[x] \) is a maximal proper quadratic module, then \( \mathcal{M} \cup -\mathcal{M} = \mathbb{R}[x] \).

3) \( \{ x \in \mathbb{R}^n | f(x) \geq 0 \} \) is a compact set \( \mathcal{T} \) for some \( f \in \mathcal{M}(\{ f_1, \ldots, f_s \}) \) iff

\[
\forall p \in \mathbb{R}[x], \exists n \in \mathbb{N}. n \pm p \in \mathcal{M}(f_1, \ldots, f_s). \tag{6}
\]

**Definition 11 (Archimedean).** For \( g_1, \ldots, g_m \in \mathbb{R}[x] \), the quadratic module \( \mathcal{M}(g_1, \ldots, g_m) \) is said to be Archimedean if the condition \( (6) \) holds.

Let

\[
\mathcal{T}_1 = f_1(x) \geq 0, \ldots, f_{s_1}(x) \geq 0 \quad \text{and} \quad \mathcal{T}_2 = f_{s_1+1}(x) \geq 0, \ldots, f_s(x) \geq 0 \tag{7}
\]

be two SASs, where \( \{ f_i(x) | i = 1, \ldots, s \} \) contains constraints on the upper and lower bounds of every variable \( x_i \), and \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) do not share real solutions.

**Remark 2.** Regarding \( \{ f_1, \ldots, f_s \} \) in \( (7) \), as every variable is bounded, assume \( N - \sum_{i=1}^{b} x_i^2 \in \{ f_1, \ldots, f_s \} \) for a const \( N \), then \( \mathcal{M}(f_1, \ldots, f_s) \) is Archimedean.

**Lemma 3.** [21][27] Let \( \mathcal{M} \subseteq \mathbb{R}[x] \) be a maximal proper quadratic module which is Archimedean, \( I = \mathcal{M} \cap -\mathcal{M} \), and \( f \in \mathbb{R}[x] \), then there exists \( a \in \mathbb{R} \) such that \( f - a \in I \).

**Lemma 4.** If \( I \) is an ideal and there exists \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) such that \( x_i - a_i \in I \) for \( i = 1, \ldots, n \), then for any \( f \in \mathbb{R}[x] \), \( f - f(a) \in I \).

**Proof.** Because \( x_i - a_i \in I \) for \( i = 1, \ldots, n \), \( \langle x_1 - a_1, \ldots, x_n - a_n \rangle \subseteq I \). For any \( f \in \mathbb{R}[x] \), \( \langle x_1 - a_1, \ldots, x_n - a_n \rangle \) is a radical ideal\(^7\) and \( (f - f(a))(a) = 0 \), so \( f - f(a) \in \langle x_1 - a_1, \ldots, x_n - a_n \rangle \subseteq I \).

\(^7\) \( S \) is a compact set in \( \mathbb{R}^n \) iff \( S \) is a bounded closed set.

\(^8\) Ideal \( I \) is a radical ideal if \( I = \sqrt{I} = \{ f | f^k \in I \} \) for some integer \( k \geq 0 \).
**Theorem 3.** Suppose \( \{ f_1(x), \ldots, f_s(x) \} \) is given in \( \mathbb{R}^n \). If \( \bigwedge_{i=1}^s (f_i \geq 0) \) is unsatisfiable, then \(-1 \in \mathcal{M}(f_1, \ldots, f_s)\).

**Proof.** By Remark 2, \( \mathcal{M}(f_1, \ldots, f_s) \) is Archimedean. Thus, we only need to prove that the quadratic module \( \mathcal{M}(f_1, \ldots, f_s) \) is proper.

Assume \( \mathcal{M}(f_1, \ldots, f_s) \) is proper. By Zorn’s lemma, we can extend \( \mathcal{M}(f_1, \ldots, f_s) \) to a maximal proper quadratic module \( \mathcal{M} \supseteq \mathcal{M}(f_1, \ldots, f_s) \). As \( \mathcal{M}(f_1, \ldots, f_s) \) is Archimedean, \( \mathcal{M} \) is also Archimedean. By Lemma 3 there exists \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) such that \( x_i - a_i \in I = \mathcal{M} - \mathcal{M} \) for all \( i \in \{1, \ldots, n\} \). From Lemma 4, \( f - f(a) \in I \) for any \( f \in \mathbb{R}[x] \). In particular, for \( f = f_j \), we have \( f_j(a) = f_j - (f_j - f_j(a)) \in \mathcal{M} \) since \( f_j \in \mathcal{M}(f_1, \ldots, f_s) \subseteq \mathcal{M} \) and \(- (f_j - f_j(a)) \in \mathcal{M} \), which implies \( f_j(a) \geq 0 \), for \( j = 1, \ldots, s \). This contradicts to the unsatisfiability of \( \bigwedge_{i=1}^s (f_i \geq 0) \).

By Theorem 3 we have \(-1 \in \mathcal{M}(f_1, \ldots, f_s) \). So, there exist \( \sigma_0, \ldots, \sigma_s \in C(\emptyset) \) such that \(-1 = \sigma_0 + \sigma_1 f_1 + \cdots + \sigma_s f_s + \sigma_{s+1} f_{s+1} + \cdots + f_s \sigma_s \). It follows

\[
- \left( \frac{1}{2} + \sigma_{s+1} f_{s+1} + \cdots + \sigma_s f_s \right) = \frac{1}{2} + \sigma_0 + \sigma_1 f_1 + \cdots + \sigma_s f_s. \tag{8}
\]

Let \( q(x) = \frac{1}{2} + \sigma_0 + \sigma_1 f_1 + \cdots + \sigma_s f_s \), we have \( \forall x \in T_1, q(x) > 0 \) and \( \forall x \in T_2, q(x) < 0 \). Thus, let \( I = q(x) > 0 \). According to Definition 1, \( I \) is an interpolant of \( T_1 \) and \( T_2 \). So, under Archimedean condition, we can revise Algorithm 3 as Algorithm 4.

---

**Algorithm 4: RSN Interpolants**

```plaintext
input: \( T_1 \) and \( T_2 \) as in \( \{ h_{u+1}, \ldots, h_u \} \)
/* \( h_{u+1}, \ldots, h_u \) are the equality occur in \( T_2 \) */
output: \( I \)
1 \( b=2; \)
2 \( V_1 = \forall \{ f_1, \ldots, f_s \}; \)
/* Get all variables of \( T_1 \) */
3 \( V_2 = \forall \{ f_{s+1}, \ldots, f_u \}; \)
4 \( V = V_1 \cap V_2; \)
5 \( \{ f_{s+1}, \ldots, f_u \} = \text{subvariable}(\{ f_{s+1}, \ldots, f_u \}, V, \{ h_{u+1}, \ldots, h_u \}); \)
/* Replacing every uncommon variable \( v \) by polynomial \( h \)
where \( v \geq h \) and \( v \leq h \) as described in Section 3.2 */
6 while true do
7    \( \text{sdp=Certificate Generation}(\{ f_1, \ldots, f_s \}, \emptyset, \{ b \}); \)
8    if \( \text{sdp} \neq \text{NULL} \) then
9       \( I = \{ \frac{1}{2} + \sum_{i=1}^s p_i f_i > 0 \}; \)
10      \( I' = \text{subvariable}(I, V, \{ h_{u+1}, \ldots, h_u \}); \)
11     return \( I' \);
12    else
13       \( b=b+2; \)
14   end
15 end
```
Example 3. Let $\Psi = \bigwedge_{i=1}^{3} x_i \geq -2 \land -x_i \geq -2$, $f_1 = -x_1^2 - 4x_2^3 - x_3^2 + 2$, $f_2 = x_1^2 - x_2^2 - x_3 - 1$, $f_3 = -x_1^2 - 4x_2^3 - x_3^2 + 3x_1x_2 + 0.2$, $f_4 = -x_1^2 + x_2^2 + x_3 + 1$. Consider $T_1 = \Psi \land f_2 \geq 0 \land f_3 \geq 0$ and $T_2 = \Psi \land f_3 \geq 0 \land f_4 \geq 0$. Obviously, $T_1 \land T_2$ is unsatisfiable, see Fig. 2 (left).

By applying RSN\_Interpolants, we can get an interpolant as $-33.7255 x_1^3 + 61.1309 x_1^2 + 6.4818 x_1 x_3 - 57.927 x_1^2 x_3 + 13.4887 x_1^2 x_2 x_3 - 48.9983 x_1^2 x_3^2 - 8.1444 x_3^2 - 48.1049 x_1 x_3^2 - 6.7143 x_1^2 x_3^2 + 29.8965 x_1^2 x_2 x_3 + 61.5932 x_1 x_2 + 0.051659 x_1 x_3 - 0.88593 x_1 x_3 - 34.7211 x_3^2 - 7.8128 x_2 x_3 - 71.9085 x_1^2 x_3^2 - 60.5361 x_2 x_3^2 - 1.6845 x_2 x_3 - 0.5856 x_2 x_3 - 15.2929 x_3^2 - 9.7563 x_2^2 + 6.7326$, which is depicted in Fig. 2 (right). In this example, the final value of $b$ is 2.

5.2 Discussions

1. **Reasonability of Archimedean condition**: Considering only bounded numbers can be represented in computer, so it is reasonable to constraint each variable with upper and lower bounds in practice. Not allowing strict inequalities indeed reduce the expressiveness from a theoretical point of view. However, as only numbers with finite precision can be represented in computer, we always can relax a strict inequality to an equivalent non-strict inequality in practice. In a word, we believe Archimedean condition is reasonable in practice.

2. **Necessity of Archimedean condition**: In Theorem 3, Archimedean condition is necessary. For example, let $T_1 = \{ x_1 \geq 0, x_2 \geq 0 \}$ and $T_2 = \{ -x_1 x_2 - 1 \geq 0 \}$. Obviously, $T_1 \land T_2 = \emptyset$ is not Archimedean and unsatisfiable, but $-1 \notin M(x_1, x_2, -x_1 x_2 - 1)$.

**Theorem 4.** $-1 \notin M(x_1, x_2, -x_1 x_2 - 1)$

**Proof.** Suppose these exist $\delta_0, \delta_1, \delta_2, \delta_3 \in C(\emptyset)$ such that $h = \delta_0 + x_1 \delta_1 + x_2 \delta_2 - (x_1 x_2 + 1) \delta_3 = -1$. Let $c_0 x_1^{2a_0} x_2^{2b_0}, c_1 x_1^{2a_1 + 1} x_2^{2b_1}, c_2 x_1^{2a_2} x_2^{2b_2 + 1}$, and $c_3 x_1^{2a_3 + 1} x_2^{2b_3 + 1}$ be the leading terms of $\delta_0, x_1 \delta_1, x_2 \delta_2$ and $(x_1 x_2 + 1) \delta_3$, respectively, according to the total degree order of monomials, where $c_i \geq 0$ and $a_i, b_i \in \mathbb{N}$. Obviously, the four terms are pairwise different. So, the leading term of $h$ must be one of the four terms if they are not zero. This, together with $h = -1$, imply that $c_1 = c_2 = c_3 = 0$ and thus $\delta_1 = \delta_2 = \delta_3 = 0$. Therefore, $\delta_0 = -1$, a contradiction. $\square$
6 Correctness and Complexity Analysis

The correctness of the algorithm $S_N$-Interpolants is obvious according to Theorem 2 and the discussion of Section 4. Its complexity just corresponds to one iteration of the algorithm $R_SN$-Interpolants. The correctness of the algorithm $R_SN$-Interpolants is guaranteed by Theorem 3 and Theorem 4. The cost of each iteration of $R_SN$-Interpolants depends on the number of the variables $n$, the number of polynomial constraints $u$, and the current value of $b_f$. The size of $X$ in (2) is $u^{(n+b_f/2)}$ and the $m$ in (2) is $(n+b_f)$. So, the complexity of applying interior method to solve the SDP is polynomial in $u^{(n+b_f/2)}(n+b_f)$. Hence, the cost of each iteration of $R_SN$-Interpolants is $u^{(n+b_f/2)}(n+b_f)$. Therefore, the total cost of $R_SN$-Interpolants is $b_fu^{(n+b_f/2)}(n+b_f)$. For a given problem, $n, u$ are fixed, so the complexity of the algorithm becomes polynomial in $b_f$. The complexity of Algorithm $S_N$-Interpolants is the same as above discussions, except that the number of polynomial constraints is about $2^{s1} + 2^{s1}$.

As indicated in [23], there are upper bounds on $b_f$, which are at least triply exponential. So our approach can enumerate all possible instances, but can not be done in polynomial time.

7 Implementation and Experimental Results

We have implemented a prototypical tool of the algorithms described in this paper, called $AiSat$, which contains 6000 lines of C++ codes. $AiSat$ calls Singular [28] to deal with polynomial input and CSDP to solve SDPs. In $AiSat$, we design a specific algorithm to transform polynomial constraints to matrices constraints, which indeed improves the efficiency of our tool very much, indicated by the comparison with SOSTOOLS [29] (see the table below). As a future work, we plan to implement a new SDP solver with more stability and convergence efficiency on solving SDPs.

In the following, we report some experimental results by applying $AiSat$ to some benchmarks.

The first example is from [30], see the source code in Code 1.3. We show its correctness by applying $AiSat$ to the following two possible executions.

- Subproblem 1: Suppose there is an execution starting from a state satisfying the assertion at line 13 (obviously, the initial state satisfies the assertion), after $→ 6 → 7 → 8 → 9 → 11 → 12 → 13$, ending at a state that does not satisfy the assertion. Then the interpolant synthesized by our approach is $716.77 + 1326.74(ya) + 1.33(ya)^2 + 433.90(ya)^3 + 668.16(xa) - 155.86(xa)(ya) + 317.29(xa)(ya)^2 + 222.00(xa)^2 + 592.39(xa)^2(ya) + 271.11(xa)^3$, which guarantees that this execution is infeasible.
- Subproblem 2: Assume there is an execution starting from a state satisfying the assertion at line 13, after $→ 6 → 7 → 8 → 10 → 11 → 12 → 13$, ending at a state that does not satisfy the assertion. The interpolant generated by our approach is $716.95 + 1330.91(ya) + 67.78(ya)^2 + 551.51(ya)^3 + 660.66(xa) - 255.52(xa)(ya) + 199.84(xa)(ya)^2 + 155.63(xa)^2 + 386.87(xa)^2(ya) + 212.41(xa)^3$, which guarantees this execution is infeasible either.
can obtain an interpolant. The correctness is obtained. 

The second example accelerate (see Code 1.4) is from [20]. Taking the air resistance into account, the relation between the car’s velocity and the physical drag contains quadratic functions. Due to air resistance the velocity of the car cannot be beyond $49.61 \text{ m/s}$, which is a safety property. Assume that there is an execution ($vc < 49.61$) → $8 \rightarrow 10 \rightarrow 11 \rightarrow 12 \rightarrow 13 (vc \geq 49.61)$. By applying Applying AiSat, we can obtain an interpolant $-1.3983vc + 69.358 > 0$, which guarantees $vc < 49.61$. So, accelerate is correct. we can synthesize an interpolant, which guarantees the safety property.

The last example logistic is also from [20]. Mathematically, the logistic loop is written as $x_{n+1} = rx_n(1 - x_n)$, where $0 \leq x_n \leq 1$. When $r = 3.2$, the logistic loop oscillates between two values. The verification obligation is to guarantee that it is within the safe region $(0.79 \leq x \land x \leq 0.81) \lor (0.49 \leq x \land x \leq 0.51)$. By applying AiSat to the following four possible executions, the correctness is obtained.

- Subproblem 1: $\{x \geq 0.79 \land x \leq 0.81\}$ logistic $\{x > 0.51\}$ is invalidated by the synthesized interpolant $108.92 - 214.56x > 0$.
- Subproblem 2: $\{x \geq 0.79 \land x \leq 0.81\}$ logistic $\{x < 0.49\}$ is outlawed by the synthesized interpolant $-349.86 + 712.97x > 0$.
- Subproblem 3: $\{x \geq 0.49 \land x \leq 0.51\}$ logistic $\{x > 0.81\}$ is excluded by the generated interpolant $177.21 - 219.40x > 0$.
- Subproblem 4: $\{x \geq 0.49 \land x \leq 0.51\}$ logistic $\{x < 0.79\}$ is denied by the generated interpolant $244.85 + 309.31x > 0$.

Some experimental results of applying AiSat to the above three examples on a desktop (64-bit Intel(R) Core(TM) i5 CPU 650 @ 3.20GHz, 4GB RAM memory and Ubuntu 12.04 GNU/Linux) are listed in the table below. Meanwhile, as a comparison, we apply the SOSTOOLS to the three examples with the same computer.

```
1 int main () {
2 int x,y;
3 int xa := 0;
4 int ya := 0;
5 while (nondet()) {
6 x := xa + 2*ya;
7 y := -2*xa + ya;
8 x++;
9 if (nondet()) y= y+x;
10 else y := y-x;
11 xa := x - 2*y;
12 ya := 2*x + y;
13 assert (xa + 2*ya >= 0);
14 return 0;
15 }
```

Code 1.3: ex1

```
1 vc:=0;
2 /* the initial velocity */
3 fr:=1000;
4 /* the initial force */
5 ac:=0.0005*fr;
6 /* the initial acceleration */
7 while (1) {
8 x++;
9 if (nondet()) y= y+x;
10 else y := y-x;
11 xa := x - 2*y;
12 ya := 2*x + y;
13 assert (xa + 2*ya >= 0);
14 return 0;
15 }
```

Code 1.4: An accelerating car
Benchmark | #Subproblems | AiSat (milliseconds) | SOSTOOLS (milliseconds) |
--- | --- | --- | --- |
ex1 | 2 | 60 | 3229 |
accelerate | 1 | 940 | 879 |
logistic | 4 | 20 | 761 |

8 Related work

In Introduction, we have introduced many work related to interpolant generation and its application to program verification. In this section, we will mention some existing work on program verification, so that we give a comparison between our approach and them.

Work on program verification can date back to the late sixties (or early seventies) of the 20th century when the so-called Floyd-Hoare-Naur’s inductive assertion method [31,32,33] was invented, which was thought as the dominant approach on automatic program verification. The method is based on Hoare Logic [32], by using pre- and post-conditions, loop invariants and termination analysis through ranking functions, etc. Therefore, the discovery of loop invariants and ranking functions plays a central role in proving the correctness of programs and is also thought of as the most challenging part of the approach.

Since then, there have been lots of attempts to handle invariant generation of programs, e.g. [34,35,36,37], but only with a limited success. Recently, due to the advance of computer algebra, several methods based on symbolic computation have been applied successfully to invariant generation, for example the techniques based on abstract interpretation [38,39,40,41], quantifier elimination [42,43,25] and polynomial algebra [44,45,26,46].

The basic idea behind the approaches based on abstract interpretation is to perform an approximate symbolic execution of a program until an assertion is reached that remain unchanged by further executions of the program. However, in order to guarantee termination, the method introduces imprecision by use of an extrapolation operator called widening/narrowing. This operator often causes the technique to produce weak invariants. Moreover, proposing widening/narrowing operators with certain concerns of completeness is not easy and becomes a key challenge for abstract interpretation based techniques [38,39].

In contrast, approaches by exploiting the theory of polynomial algebra to discover invariants of polynomial programs were proposed in [44,45,26,46]. In [44], Mueller-Olm and Seidl applied the technique of linear algebra to generate polynomial equations of bounded degree as invariants of programs with affine assignments. In [45,26], Rodriguez-Carbonell and Kapur first proved that the set of polynomials serving as loop invariants has the algebraic structure of ideal, then proposed an invariant generation algorithm by using fixpoint computation, and finally implemented the algorithm by the Gröbner bases and the elimination theory. The approach is theoretically sound and complete in the sense that if there is an invariant of the loop that can be expressed as a conjunction of polynomial equations, applying the approach can indeed generate it. While in [46], the authors presented a similar approach to finding invariants represented by a polynomial equation whose form is priori determined (called templates) by using an
extended Gröbner basis algorithm. The complexity of the above approaches are double exponential as Gröbner base technique is adopted.

Compared with polynomial algebra based approaches that can only generate invariants represented as polynomial equations, Colón et al in [42] proposed an approach to generate linear inequalities as invariants for linear programs, based on Farkas’ Lemma and nonlinear constraint solving. The complexity depends on the complexity of linear programming, which is in general is polynomial in the number of variables.

In addition, Kapur in [43] proposed a very general approach for automatic generation of more expressive invariants by exploiting the technique of quantifier elimination, and applied the approach to Presburger Arithmetic and quantifier-free theory of conjunctively closed polynomial equations. Theoretically speaking, the approach can also be applied to the theory of real closed fields, but Kapur also pointed out in [43] that this is impractical in reality because of the high complexity of quantifier elimination, which is doubly exponential [47] in the number of quantifiers. While in [25], we improved Kapur’s approach by using the theory of real root classification of SASs [48], with the complexity singly exponential in the number of variables and doubly exponential in the number of parameters.

Comparing with the approaches based on polynomial algebra, or Farkas’ Lemma or Gröbner, our approach is more powerful, also more efficient except for Farkas’ Lemma based approach. Comparing with quantifier elimination based approach [43,25], our approach is much more efficient, even according to the complexity analysis of quantifier elimination given in [49], which is doubly exponential in the number of the quantifier alternation, and becomes singly exponential in the number of variables and constraints in our setting.

9 Conclusion

The main contributions of the paper include:

- We give a sound but not incomplete algorithm SN_Interpolants for the generation of interpolants for non-linear arithmetic in general.
- If the two systems satisfy Archimedean condition, we provide a more practical algorithm RSN_Interpolants, which is not only sound but also complete, for generating Craig interpolants.
- We implement the above algorithms as a prototypical tool AiSat, and demonstrate our approach by applying the tool to some benchmarks.

In the future, we will focus on how to combine non-linear arithmetic with other well-established decidable first order theories. In particular, we believe that we can use the method of [51,20] to extend our algorithm to uninterpreted functions. To investigate errors caused by numerical computation in SDP is quite interesting. In addition, to investigate the possibility to apply our results to the verification of hybrid systems is very significant.

9 Hoon Hong in [50] pointed out the existing algorithms for the existential real theory which are singly exponential in the number of variables is far from realization, even worse than the general algorithm for quantifier elimination.
References

1. Clarke, E.M., Emerson, E.A.: Design and synthesis of synchronization skeletons using branching-time temporal logic. In: Logic of Programs. (1981) 52–71
2. Nipkow, T., Wenzel, M., Paulson, L.C.: Isabelle/HOL: a proof assistant for higher-order logic. Springer-Verlag, Berlin, Heidelberg (2002)
3. Owre, S., Rushby, J., Shankar, N.: Pvs: A prototype verification system. In: CADE. Volume 607 of LNCS. (1992) 748–752
4. Cousot, P., Cousot, R.: Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In: Proceedings of POPL’77. (1977) 238–252
5. Biere, A., Cimatti, A., Clarke, E., Zhu, Y.: Symbolic model checking without bdds. In: TACAS’99. Volume 1579 of LNCS. (1999) 193–207
6. Clarke, E., Grumberg, O., Jha, S., Lu, Y., Veith, H.: Counterexample-guided abstraction refinement. In: CAV’00. Volume 1855 of LNCS. (2000) 154–169
7. Nieuwenhuis, R., Oliveras, A., Tinelli, C.: Solving sat and sat modulo theories: From an abstract davis–putnam–logemann–loveland procedure to dpll(). J. ACM 53(6) (2006) 937–977
8. Nelson, G., Oppen, D.C.: Simplification by cooperating decision procedures. ACM Trans. Program. Lang. Syst. 1(2) (October 1979) 245–257
9. Davis, M., Logemann, G., Loveland, D.: A machine program for theorem-proving. Commun. ACM 5(7) (1962) 394–397
10. Craig, W.: Linear reasoning: A new form of the herbrand-gentzen theorem. J. Symb. Log. 22(3) (1957) 250–268
11. de Moura, L., Bjørner, N.: Z3: An efficient smt solver. In: TACAS’08. Volume 4963 of LNCS. (2008) 337–340
12. McMillan, K.L.: Interpolation and sat-based model checking. In: CAV’03. Volume 3920 of LNCS. (2003) 1–13
13. McMillan, K.L.: An interpolating theorem prover. Theor. Comput. Sci. 345(1) (2005) 101–121
14. Graf, S., Saidi, H.: Construction of abstract state graphs with pvs. In: CAV’97. Volume 1254 of LNCS. (1997) 72–83
15. Henzinger, T.A., Jhala, R., Majumdar, R., McMillan, K.L.: Abstractions from proofs. In: POPL’04. (2004) 232–244
16. McMillan, K.L.: Lazy abstraction with interpolants. In: CAV’06. Volume 4144 of LNCS. (2006) 123–136
17. Jung, Y., Lee, W., Wang, B.Y., Yi, K.: Predicate generation for learning-based quantifier-free loop invariant inference. In: TACAS’11. Volume 6605 of LNCS. (2011) 205–219
18. Yorsh, G., Musuvathi, M.: A combination method for generating interpolants. In: CADE’05. Volume 3632 of LNCS. (2005) 353–368
19. Rybalchenko, A., Sofronie-Stokkermans, V.: Constraint solving for interpolation. J. Symb. Comput. 45(11) (2010) 1212–1233
20. Kupferschmid, S., Becker, B.: Craig interpolation in the presence of non-linear constraints. In: FORMATS’11. Volume 6919 of LNCS. (2011)
21. Bochnak, J., Coste, M., Roy, M.F.: Real Algebraic Geometry. Springer (1998)
22. Parrilo, P.A.: Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. PhD thesis, California Inst. of Tech. (2000)
23. Parrilo, P.A.: Semidefinite programming relaxations for semialgebraic problems. Mathematical Programming 96 (2003) 293–320
24. Vandenberghe, L., Boyd, S.: Semidefinite programming. SIAM Review 38(1) (1996) 49–95
25. Chen, Y., Xia, B., Yang, L., Zhan, N.: Generating polynomial invariants with discoverer and qepcad. In: Formal Methods and Hybrid Real-Time Systems. Volume 4700 of LNCS. (2007) 67–82
26. Rodriguez-Carbonell, E., Kapur, D.: Generating all polynomial invariants in simple loops. Journal of Symbolic Computation 42 (2007) 443–476
27. Laurent, M.: Sums of squares, moment matrices and optimization over polynomials. In: Emerging Applications of Algebraic Geometry, Volume 149 of The IMA Volumes in Mathematics and its Applications. (2009) 157–270
28. Greuel, G.M., Pfister, G., Schönemann, H.: Singular: a computer algebra system for polynomial computations. ACM Commun. Comput. Algebra 42(3) (2009) 180–181
29. Prajna, S., Papachristodoulou, A., Seiler, P., Parrilo, P.A.: SOSTOOLS: Sum of squares optimization toolbox for MATLAB. (2004)
30. Gulavani, B., Chakraborty, S., Nori, A., Rajamani, S.: Automatically refining abstract interpretations. In: TACAS’08. Volume 4963 of LNCS. (2008) 443–458
31. Floyd, R.W.: Assigning meanings to programs. In: Proc. Symposia in Applied Mathematics 19. (1967) 19–37
32. Hoare, C.: An axiomatic basis for computer programming. Comm. ACM 12(10) (1969) 576–580
33. Naur, P.: Proofs of algorithms by general snapshops. BIT 6 (1966) 310–316
34. Wegbreit, B.: The synthesis of loop predicates. Communications of the ACM 17(2) (1974) 102–112
35. German, S., Wegbreit, B.: A synthesizer of inductive assertions. IEEE Transactions on Software Engineering 1(1) (1975) 68–75
36. Katz, S., Manna, Z.: Logical analysis of programs. Communications of the ACM 19(4) (1976) 188–206
37. Karr, M.: Affine relationships among variables of a program. Acta Informatica 6 (1976) 133–151
38. Cousot, P., Halbwachs, N.: Automatic discovery of linear restraints among the variables of a program. In: ACM POPL’78. (1978) 84–97
39. F. Besson, T.J., Talpin, J.P.: Polyhedral analysis of synchronous languages. In: SAS’99. Volume 1694 of LNCS. (1999) 51–69
40. Rodriguez-Carbonell, E., Kapur, D.: An abstract interpretation approach for automatic generation of polynomial invariants. In: SAS’04. Volume 3148 of LNCS. (2004) 280–295
41. Cousot, P.: Proving program invariance and termination by parametric abstraction, lAGRangian relaxation and semidefinite programming. In: VMCAI’05. Volume 3385 of LNCS. (2005) 1–24
42. M. Colón, S.S., Sipma, H.: Linear invariant generation using non-linear constraint solving. In: CAV’03. Volume 2725 of LNCS. (2003) 420–432
43. Kapur, D.: Automatically generating loop invariants using quantifier elimination. In: Intl. Conf. on Applications of Computer Algebra (ACA’04). (2004)
44. Müller-Olm, M., Seidl, H.: Precise interprocedural analysis through linear algebra. In: ACM POPL’04. (2004) 330–341
45. Rodriguez-Carbonell, E., Kapur, D.: Automatic generation of polynomial loop invariants: algebraic foundations. In: ISSAC’04. (2004)
46. S. Sankaranarayanan, H.S., Manna, Z.: Non-linear loop invariant generation using gröbner bases. In: ACM POPL’04. (2004) 318–329
47. J. H., D., Heintz, J.: Real elimination is doubly exponential. J. of Symbolic Computation 5 (1988) 29–37
48. Xia, B., Yang, L.: An algorithm for isolating the real solutions of semi-algebraic systems. J. Symbolic Computation 34 (2002) 461–477
49. Brown, C.W.: The Complexity of Quantifier Elimination and Cylindrical Algebraic Decomposition. United States Naval Academy
50. Hong, H.: Comparison of several decision algorithms for the existential theory of the reals (1991)
51. Sofronie-Stokkermans, V.: Interpolation in local theory extensions. In: Automated Reasoning. (2006) 235–250