MAXIMALITY OF LINEAR OPERATORS

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Abstract. We present maximality results in the setting of non necessarily bounded operators. In particular, we discuss and establish results showing when the "inclusion" between operators becomes a full equality.

1. Introduction

In the theory of non necessarily bounded linear operators on a complex Hilbert space $H$, we say that an operator $T$ with domain $D(T) \subset H$ is an extension of $S$ with domain $D(S) \subset H$ when: $D(S) \subset D(T)$ and $Sx = Tx$ for all $x \in D(S)$. We then write $S \subset T$. We say that $S$ is closed if it possess a closed graph in $H \oplus H$.

The product of $S$ and $T$ is defined $(ST)x = S(Tx)$ for each $x$ on the natural domain $D(ST) = \{x \in D(T) : Tx \in D(S)\}$.

We say that $T$ is invertible if there exists an $S \in B(H)$ (we then write $T^{-1} = S$) such that

$$ST \subset I \text{ and } TS = I$$

where $I$ is the identity operator on $H$. It is known that the product $ST$ is closed if for instance $S$ is closed and $T \in B(H)$, or if $S^{-1} \in B(H)$ and $T$ is closed.

We also recall that an operator $S$ is said to be densely defined if its domain $D(S)$ is dense in $H$. It is known that in such case its adjoint $S^*$ exists and is unique. Notice that if $S$, $T$ and $ST$ are all densely defined, then we are only sure of

$$T^*S^* \subset (ST)^*,$$

and a full equality occurring if e.g. $T^{-1} \in B(H)$ or $S \in B(H)$.

A densely defined operator $S$ is said to be symmetric if $S \subset S^*$. We say that $S$ is normal if $S$ is densely defined, closed and $SS^* = S^*S$. Recall that the previous is equivalent to $\|Sx\| = \|S^*x\|$ for all $x \in D(S) = D(S^*)$. We say that $S$ is formally normal if $\|Sx\| = \|S^*x\|$ for all $x \in D(S) \subset D(S^*)$. Other classes of operators are defined in the usual fashion.

Let us also agree that any operator is linear and non necessarily bounded unless we specify that it belongs to $B(H)$. We also assume the basic theory of operators (see e.g. [2] or [17]). We do recall the celebrated Fuglede-Putnam Theorem though:

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Theorem 1.1. (for a proof, see e.g. [2]) Let $T \in \mathcal{B}(H)$ and let $M, N$ be two normal non necessarily bounded operators. Then

$$TN \subset MT \implies TN^* \subset M^*T.$$  

One of the main aims of this work is to seek conditions which transform $S \subset T$ into $S = T$ (which we call a maximality condition) for some classes of operators, and also in the case of a product of two operators. This type of results is a powerful tool when proving results on unbounded operators. For instance, Statement (3) of the next theorem is used in the proof of the "unbounded" version of the spectral theorem of normal operators (see e.g. [12]). For other uses, see e.g. [6] or [9].

Let us now list some known (see e.g. [12] or [13]) maximality results:

Theorem 1.2. Let $S, T$ be two operators with (dense when necessary) domains $D(S)$ and $D(T)$ respectively such that $S \subset T$. Then $S = T$ when one of the following occurs:

1. $S$ is surjective and $T$ is injective.
2. $T$ is symmetric and $S$ is self-adjoint (resp. normal). We then say that self-adjoint (resp. normal) operators are maximally symmetric.
3. $T$ and $S$ are normal (we say that normal operators are maximally normal).
   
   Hence, self-adjoint (resp. normal) operators are maximally normal (resp. self-adjoint).
4. $S$ is normal and $T$ is formally normal.

In fact, Statements (2) to (4) of the preceding result are all simple consequences of the following readily verified result:

Proposition 1.3. Let $S$ and $T$ be two operators of domains $D(S)$ and $D(T)$ respectively. If $S$ is densely defined and $D(S^*) \subset D(S)$, then

$$S \subset T \implies S = T$$

whenever $D(T) \subset D(T^*)$.

Let us now say a few words about "double maximality". A known property (Theorem 5.31, [17]) states that if $S$ is a symmetric operator such that $S \subset R$ and $S \subset T$ where $R, T$ are self-adjoint and $D(R) \subset D(T)$, then $T = R$. Observe that the assumption $S$ symmetric is tacitly assumed in $S \subset R$ so there was no need to assume it. What is more, is that the assumption $S$ being symmetric is not used in the proof of the previous result. So, we restate this result as (cf. Proposition 5.1):

Proposition 1.4. Let $S$ be a densely defined operator such that $S \subset R$ and $S \subset T$ where $R, T$ are both self-adjoint. If $D(R) \subset D(T)$, then $T = R$.

Closely related to what has just been said, we have:

Proposition 1.5. (see [9], cf. [16]) Let $R, S, T$ be three densely defined operators on a Hilbert space $H$ with respective domains $D(R)$, $D(S)$ and $D(T)$. Assume that

$$\begin{align*}
T &\subset R, \\
T &\subset S.
\end{align*}$$

Assume further that $R$ and $S$ are self-adjoint. Let $D \subset D(T)$ ($\subset D(R) \cap D(S)$) be dense. Let $D$ be a core, for instance, for $S$. Then $R = S$.

Finally, we recall results on the case when we have a product on one side of the "inclusion":
Theorem 1.6. Let $R, S, T, A, B, C$ be operators such that $T \subset RS$ and $AB \subset C$. Then:

1. $T = RS$ if all $R, S, T$ are self-adjoint (see [3]).
2. $T = RS$ if $R, S, T$ are self-adjoint and $T_0 \subset RS$ instead of $T \subset RS$ where $T_0$ is the restriction of $T$ to some domain $D_0(T)$ (see [10]).
3. $AB = C$ when $A$ and $B$ are self-adjoint, $B$ is positive and $B^{-1} \in B(H)$ and $C$ is normal ([8]).
4. $C = BA$ whenever $A, B$ are self-adjoint and $B^{-1} \in B(H)$ and $C$ is closed and symmetric ([10]).

Remark. As observed in [4], the first statement in the previous theorem does not extend to normal operators. Indeed, just in the naive case of unitary operators, we have that a product of any two unitary operators is always unitary even when the two factors of the product do not commute. This observation motivates the investigation in the case where one operator is normal.

Remark. Another natural question may pop up. In [3], the authors before showing that $T = RS$, they first showed that $R$ and $S$ commute strongly (i.e. the corresponding spectral measures commute). So what if we have $T \subset ABC$, do we still expect $T = ABC$ when all of $T, A, B, C$ are self-adjoint? The answer is negative (at least as far as the idea of their proof is concerned). Indeed, we can have a self-adjoint product of three self-adjoint operators which do not commute pairwise. In $\mathbb{R}^2$, consider the following self-adjoint matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Then

$$ABC = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

is self-adjoint. Nevertheless, none of the products $AB, AC$ and $BC$ is self-adjoint, that is,

$$AB \neq BA, \quad AC \neq CA \quad \text{and} \quad BC \neq CB.$$ 

2. Some Results on Normality

The normality of unbounded products of normal operators has been studied recently. See e.g. [5] and the references therein. We recall

Lemma 2.1. (cf. [4]) Let $A, B$ be normal operators with $B^{-1} \in B(H)$. If $AB \subset BA$, then $AB$ and $BA$ are both normal.

The chosen idea of proof of the following result is via the Fuglede-Putnam Theorem (for a different proof, we may proceed as in [1]).

Theorem 2.2. Let $A, B$ be normal operators with $B \in B(H)$. If $BA \subset AB$, then $AB$ and $BA$ are both normal (and so $AB = BA$).

Proof. Since $BA \subset AB$, Fuglede Theorem yields $BA^* \subset A^*B$. Hence (since also $AB$ is densely defined),

$$B^*BAA^* \subset B^*ABA^* \subset B^*AA^*B = B^*A^*AB \subset (AB)^*AB.$$ 


Since \( AB \) is closed, it follows that \((AB)^* AB\) is self-adjoint, and by the boundedness of \( B^* B \), we get
\[
(AB)^* AB \subset AA^* B^* B \text{ or merely } (AB)^* AB = AA^* B^* B = AA^* BB^*
\]
by Theorem 1.6. Similarly, we obtain
\[
AB(AB)^* = AA^* BB^*,
\]
and this marks the end of the proof of the normality of \( AB \).

To show that \( BA \) is normal, we first observe that
\[
BA = (BA)^{*} = (A^* B^*)^*.
\]
Now, since \( BA \subset AB \), clearly \( B^* A^* \subset A^* B^* \). The first part of the proof leads to the normality of \( A^* B^* \) because both \( A^* \) and \( B^* \) are normal. Accordingly, \((A^* B^*)^*\) too is normal, that is, \( BA \) is normal. □

The following result is known to most readers (a proof based on the spectral theorem may be found in [1]). We can equally regard it as a consequence of the preceding theorem:

**Corollary 2.3.** Let \( A, B \) be self-adjoint operators with \( B \in B(H) \). If \( BA \subset AB \), then \( AB \) and \( BA \) are both self-adjoint.

**Proof.** Since \( BA \subset AB \), and \( A \) and \( B \) are self-adjoint, the previous theorem yields the normality of \( BA \). But
\[
BA \subset AB \implies BA \subset AB = (BA)^*.
\]
i.e. \( BA \) is symmetric as well. Therefore, \( BA \) is self-adjoint. Accordingly,
\[
AB = (BA)^* = BA,
\]
and so \( AB \) is also self-adjoint, as required. □

3. **Main Results on Maximality**

The same idea of proof of (Theorem 5.31, [17], discussed above) may lead to the following result which seems to have escaped notice up to now.

**Proposition 3.1.** Let \( S \) be a densely defined operator such that \( S \subset T \) and \( S \subset T^* \). If \( D(T) = D(T^*) \), then \( T \) is self-adjoint.

**Proof.** For all \( x \in D(T) = D(T^*) \) and for all \( y \in D(S) \subset D(T) = D(T^*) \) we may write
\[
<Tx, y> = <x, T^*y> = <x, Sy> = <x, Ty> = <T^*x, y>.
\]
Since \( D(S) \) is dense, it follows that \( Tx = T^*x \) for all \( x \in D(T) = D(T^*) \), that is, \( T \) is self-adjoint. □

**Corollary 3.2.** Let \( S \) be a densely defined operator such that \( S \subset T \) and \( S \subset T^* \). If \( T \) is normal, then it is self-adjoint.

The next result is perhaps known:
Proposition 3.3. Let \( A, B \) be two linear operators on a Hilbert space \( H \). Assume also that \( B \in B(H) \). Assume further that \( A \) has a domain \( D(A) \) and that \( A \subset B \).

1. We do not necessarily have \( A = B \) if \( A \) is densely defined but not closed.
2. We do not necessarily have \( A = B \) if \( A \) is closed but not densely defined.
3. Assume now that \( A \) is closed. Then

\[
A = B \iff \overline{D(A)} = H.
\]

Particularly, if \( C \) is invertible, then

\[
AC \subset B \implies AC = B.
\]

Proof. First, remember that \( A \subset B \) means that \( Ax = Bx \) for all \( x \in D(A) \), i.e. \( A \) is bounded on \( D(A) \).

1. We only have \( B = \overline{A} \). Since \( A \) is densely defined, from \( A \subset B \), we get that \( B^* \subset A^* \). But \( D(B^*) = H \) and so \( B^* = A^* \). Hence

\[
B = A^{**} = \overline{A}.
\]

For a counterexample, just consider \( A = B|D \) (\( B \) restricted to \( D \)) where \( D \) is dense in \( H \) but not closed. Since \( D \) is not closed, \( A \), which is bounded on \( D \), cannot be closed. Observe in the end that \( A \neq B \) because \( D \neq H \).

2. Just consider \( A = 0 \) (the zero operator) on the trivial domain \( D(A) = \{0\} \). Take \( B \) to be any non-zero bounded operator. Since \( A(0) = 0 = B(0) \), we see plainly that \( A \subset B \). Finally, it is clear that \( A \) is closed on \( D(A) \), that \( D(A) \) is not dense in \( H \) and that \( A \neq B \).

3. The implication "\( \Rightarrow \)" is evident. One way of proving the reverse implication is as follows: As mentioned above, \( A \) is bounded on \( D(A) \). Since \( A \) is closed, \( D(A) \) is closed. By hypothesis, \( \overline{D(A)} = H \) and so \( D(A) = H \). This leads to \( A = B \).

Finally, observe that as \( AC \subset B \) and \( C \) is invertible, we then get that \( A \subset BC^{-1} \). By the first part of this answer and since \( BC^{-1} \in B(H) \), we obtain \( A = BC^{-1} \). Thus,

\[
AC = BC^{-1}C = B,
\]

as required.

\( \square \)

Closely related to the foregoing theorem, we have:

Lemma 3.4. Assume that \( S \) is closed and densely defined in \( H \), \( B \in B(H) \) is self-adjoint and \( SB \subset I \). Then \( B \) is injective, \( M = D(SB) \) is closed and \( SB = I_M \).

Proof. Since \( S \) is closed and \( B \in B(H) \), the general theory says that \( SB \) is closed. This combined with \( SB \subset I \) completes the proof.

\( \square \)

Proposition 3.5. Assume that \( B \in B(H) \) is injective and self-adjoint, and \( B^{-1} \) is not bounded. Then there exists a closed, densely defined and injective operator \( S \) in \( H \) such that \( SB \subset I \) and \( SB \) is not densely defined.

Proof. Since, by assumption, \( A := B^{-1} \) is self-adjoint and unbounded, we see that \( D(A^2) \subset D(A) \) (by applying Lemma A.1 in [14] to \( R = |B| \)). Then, take a (necessarily nonzero) vector \( e \in D(A) \setminus D(A^2) \). It follows from Lemma 3.2 of [14], that \( M := D(A) \ominus_A < e > \) is a vector subspace of \( D(A) \) which is dense in \( H \), where
\( \oplus \) designates the orthogonal difference with respect to the graph inner product of \( A \) (cf. [14]) and \( < e > = \mathbb{C} \cdot e \). Set \( S = A|_M \). Since \( M \) is a closed vector subspace of \( D(A) \) with respect to the graph norm of \( A \), we see that the operator \( S \) is closed, densely defined and injective. Then clearly \( SB = (B^{-1}|_M) B \subset B^{-1}B = I \) and, because \( A \) is injective and \( D(A) \ominus_A < e > \neq D(A) \), we have \( D(SB) = B^{-1}(D(S)) = A(D(A) \ominus_A < e >) \subseteq A(D(A)) \subset H \).

Since, by Lemma 3.4, \( D(SB) \) is closed, we are done. \( \Box \)

The following gives more information about Theorem 1.6 is:

**Theorem 3.6.** Let \( A, B, T \) be non necessarily bounded operators such that \( A \) is self-adjoint, \( B \) is symmetric with \( B^{-1} \in B(H) \) (hence \( B \) is self-adjoint) and \( T \) is symmetric. Assume further that \( AB \subset T \). Then:

1. \( AB \subset BA \).
2. \( BA \) is normal.
3. \( T = (BA)^* \).
4. \( T \) is essentially self-adjoint.

If \( T \) is also closed, then \( BA \) is self-adjoint and \( T = BA \) and \( T = \overline{AB} \).

**Proof.**

- Since \( T \) is densely defined, so is \( AB \) and so \( T^* \subset (AB)^* = B^*A^* = BA \) since also \( B^{-1} \in B(H) \) and \( A \) and \( B \) are self-adjoint. Since \( T \) is symmetric, we obtain \( AB \subset T \subset T^* \subset BA \).

Lemma 2.1 (or else) then yields the normality of \( BA \).

Now, since \( T^* \subset BA \), we get \( (BA)^* \subset T^{**} = \overline{T} \). Because \( BA \) is normal, so is \( (BA)^* \). But, normal operators are maximally symmetric. Therefore, we finally infer that \( (BA)^* = \overline{T} \), i.e. \( T \) is essentially self-adjoint (for \( \overline{T} \) is normal and symmetric).

- Suppose now that \( T \) is also closed. From above, it is self-adjoint and \( (BA)^* = T \). Hence \( T = (BA)^* = (BA)^{**} = \overline{BA} = BA \) since \( BA \) is closed.

In fine, \( \overline{AB} = (AB)^{**} = (BA)^* = T \).

**Corollary 3.7.** Let \( A, B, T \) be non necessarily bounded operators such that \( A \) is self-adjoint, \( B \) is symmetric with \( B^{-1} \in B(H) \) (hence \( B \) is self-adjoint) and \( T \) is symmetric. Assume further that \( AB \subset T \). Then \( A = BAB^{-1} \).
Proof. From Theorem 3.6, we have $AB \subset BA$. Left and right multiplying by $B^{-1}$ give
\[ B^{-1}A \subset AB^{-1}. \]
Since $B^{-1} \in B(H)$, Corollary 2.3 yields the self-adjointness of $AB^{-1}$. We may also write
\[ AB \subset BA \implies A \subset B(AB^{-1}). \]
Finally, Theorem 1.6 yields
\[ A = BAB^{-1}, \]
finishing the proof.

Remark. In general,
\[ BA \subset T \not\Rightarrow BA = T \]
even when $A$, $B$ and $T$ are all self-adjoint. Indeed, just consider an invertible self-adjoint $A$ with a domain $D(A) \subseteq H$ such that $A^{-1} = B \in B(H)$ and $T = I_H$ (the identity operator on the whole space $H$). Then
\[ BA = A^{-1}A = I_{D(A)} \subset I_H = T \]
where $I_{D(A)}$ is the identity operator on $D(A)$.

We also have:

**Theorem 3.8.** Let $A$, $B$ and $T$ be operators where $B \in B(H)$. If $T^*$ is symmetric, $B$ is self-adjoint and $A$ is normal, then
\[ T \subset AB \implies \overline{T} = AB. \]
In particular, if we further assume that $T$ is closed, then we obtain $T = AB$.

**Proof.** Clearly,
\[ T \subset AB \implies \overline{T} \subset AB. \]
Hence
\[ T \subset AB \implies BA^* \subset (AB)^* \subset T^* \subset T^{**} = \overline{T} \subset AB. \]
The Fugelde-Putnam Theorem then gives
\[ BA \subset A^*B. \]
Reasoning as in the proof of Theorem 2.2, we may prove
\[ (AB)^*AB = AB(AB)^* = AA^*B^2, \]
i.e. $AB$ is normal. Hence $(AB)^*$ too is normal. Since normal operators are maximally symmetric, we get
\[ (AB)^* \subset T^* \implies (AB)^* = T^* \implies AB = \overline{AB} = (AB)^{**} = T^{**} = \overline{T}. \]

**Corollary 3.9.** Let $A$, $B$ and $T$ be operators where $B \in B(H)$. If $T$ is symmetric, $B$ is self-adjoint and $A$ is normal, then
\[ BA \subset T \implies \overline{T} = BA. \]
Proof. As above, we get
\[ BA(BA)^* = (BA)^*BA = A^*AB. \]
Since normal operators are maximally symmetric, we obtain
\[ BA \subset T \implies BA \subset T = BA, \]
as needed. \(\square\)

From the proof of Theorem 3.8 we have:

**Corollary 3.10.** Let \( A, B \) and \( T \) be operators where \( B \in B(H) \). If \( T \) is symmetric, \( B \) is self-adjoint and \( A \) is normal, then
\[ T \subset AB \implies BA = A^*B. \]

**Proof.** We have already obtained:
\[ BA \subset A^*B \text{ and } BA^* \subset AB. \]
These two inequalities allow us to establish the normality of both \( BA \) and \( A^*B \) (cf. Theorem 2.2). Therefore,
\[ BA = A^*B \]
\(\square\)

**Corollary 3.11.** Let \( A, B, T \) be non necessarily bounded operators such that \( A \) is self-adjoint, \( B \) is symmetric with \( B^{-1} \in B(H) \) and \( T \) is normal. Then:
\[ AB \subset T \implies A = TB^{-1}. \]

**Proof.** Obviously,
\[ AB \subset T \implies A \subset TB^{-1} \implies B^{-1}T^* \subset A \subset TB^{-1} \implies B^{-1}T \subset T^*B^{-1} \]
where we used the Fuglede-Putnam Theorem in the last implication. As in the preceding corollary, we may show the normality of \( TB^{-1} \). This, combined with the self-adjointness of \( A \) and \( A \subset TB^{-1} \) lead finally to \( A = TB^{-1} \), as needed. \(\square\)

**Remark.** We already observed in the remark just above Theorem 3.8 that if \( A, B \) and \( T \) are as in the previous corollary, then we must not have \( T = AB \). The same counterexample may be reused here.

The following is also worth stating.

**Corollary 3.12.** Let \( A, B, T \) be operators such that \( A \) is normal, \( B \) is bounded and self-adjoint and \( T \) is self-adjoint. Then
\[ T \subset AB \implies T = AB. \]

**Proof.** As in the proofs above, we can easily show that \( AB \) is normal. Then Theorem 1.2 does the remaining job. \(\square\)

**Theorem 3.13.** Let \( A, B \) and \( T \) be non-necessarily bounded operators. Assume that \( B \) is normal, that \( A \) is symmetric and invertible (hence \( A \) is self-adjoint) and that \( T \) is self-adjoint. Then
\[ T \subset AB \implies T = AB. \]
Proof. We claim that $AB$ is normal. First we have:

\[ T \subset AB \implies B^*A \subset T \subset AB \]
\[ \implies A^{-1}B^*AA^{-1} \subset A^{-1}ABA^{-1} \]
\[ \implies A^{-1}B^* \subset BA^{-1} \]
\[ \implies A^{-1}B \subset B^*A^{-1} \text{ (by Fuglede-Putnam Theorem)} \]
\[ \implies BA \subset AB^*. \]

Hence

\[ (AB)^*AB \supset B^*BA^2 \text{ or } (AB)^*AB \subset A^2B^*B \]

as $(AB)^*AB$ is self-adjoint since $AB$ is closed because also $A^{-1} \in B(H)$. Therefore,

\[ (AB)^*AB = A^2B^*B \]

by Theorem 1.6. Similarly, we may prove that

\[ AB(AB)^* = A^2B^*B. \]

Accordingly, $AB$ is normal. In the end, since self-adjoint operators are maximally normal, we obtain

\[ T \subset AB \implies T = AB, \]

as required. \qed

4. A Conjecture

Unfortunately, if we switch the roles of $A$ and $B$ in Corollary 3.12, then we have not been able so far to find a complete answer. Indeed, we need a version of Fuglede-Putnam Theorem which is not available in the literature yet. Even with help from Bent Fuglede himself, we have only got as far as the following (we have chosen not to include the proof in this paper):

**Theorem 4.1.** Let $B$ be a bounded normal operator with a (finite) pure point spectrum and let $A$ be a closed (possibly unbounded) operator on a separable complex Hilbert space $H$. Let $f, g : \mathbb{C} \to \mathbb{C}$ be two continuous functions. Then

\[ BA \subset Af(B) \implies g(B)A \subset A(g \circ f)(B). \]

**Corollary 4.2.** With $A$ and $B$ as above, we have

\[ BA \subset AB^* \implies B^*A \subset AB. \]

**Proof.** Just apply Theorem 4.1 to the functions $f, g : z \mapsto z$ (so that $g \circ f$ becomes the identity map on $\mathbb{C}$). \qed

**Corollary 4.3.** With $A$ and $B$ as above, we have

\[ T \subset AB \implies T = AB \]

if we also suppose that $A$ and $T$ are self-adjoint.

**Proof.** Apply Corollary 4.2. \qed

Related to what has just been discussed, we propose the following conjecture:

**Conjecture 4.4.** Let $A$ be an operator (densely defined and closed if necessary) and let $B \in B(H)$ be normal. Then

\[ BA \subset AB^* \implies B^*A \subset AB. \]
Remark. What makes the previous conjecture interesting is that it is known to hold if $A \in B(H)$ (Fuglede-Putnam Theorem), and as it is posed, it is covered by none of the known (unbounded) generalizations of Fuglede-Putnam Theorem (see e.g. [2], [11] and [15]).

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