

I. INTRODUCTION

Primordial black holes (PBHs) can be formed from the gravitational collapse of overdense regions with their density contrasts exceeding the threshold value at the horizon reentry during radiation domination [1, 2]. PBHs with stellar masses may be the black holes in the gravitational waves (GWs) events detected by the Laser Interferometer Gravitational Wave Observatory (LIGO) Scientific Collaboration and the Virgo Collaboration [3–16]. PBHs with planetary masses can explain the ultrashort-timescale microlensing events in the OGLE data [17], and can act as the Planet 9 which is a hypothetical astrophysical object in the outer solar system used to explain the anomalous orbits of trans-Neptunian objects [18]. PBHs are also proposed to account for dark matter (DM) [19–28], and those with masses around $10^{-17} - 10^{-15} M_\odot$ and $10^{-13} - 10^{-12} M_\odot$ can make up almost all of DM for there are no observational constraints on the abundances of PBHs at these mass windows.

The overdense regions, collapsing to PBHs by gravitational force, originate from the primordial curvature perturbations generated during Inflation [29–39]. From the threshold value of the density contrasts for PBHs formation, the amplitude of the power spectrum of the primordial curvature perturbations is constrained to $A_s \sim O(0.01)$, which is seven orders of magnitude larger [40–42] than the large scale constraints $A_s = 2.1 \times 10^{-9}$ [43] from the observation of cosmic microwave background (CMB) anisotropy measurements. Therefore, the allowed way to produce enough PBHs DM is by enhancing the power spectrum by about seven orders of magnitude at small scales.

The traditional slow-roll inflation model is hard to enhance the power spectrum at small scales while keeping the model consistent with the large scale constraints. To solve this difficult, we need consider the ultra-slow-roll inflation model that transiently satisfies the condition $\ddot{\phi} + 3H\dot{\phi} \approx 0$ [44–46]. For the canonical inflation models with a single field, a simple way to realize the ultra-slow-roll inflation is by introducing an inflection point in the potential [40, 47–52]. However, it is not easy to achieve the big enhancement on the power spectrum while keeping the total number of e-folds around 50 – 60 [53, 54]. Noncanonical kinetic terms inflation or other kinds of noncanonical inflation models were then considered [55–72]. For example, with the coupling function $f(\phi)$ and potential satisfying $(V_\phi + V^2 f_\phi / 6)|_{\phi = \phi_0} \approx 0$, the Gauss-Bonnet inflation model has a transient ultra-slow-roll process at the critical point $\phi_c$ [69, 70] and succeeds in enhancing the power spectrum and produce PBHs. The noncanonical kinetic term inflation model with coupling function $G(\phi) = h/[1 + (|\phi - \phi_c|/w)^q]$ can realize a large enhancement on the power spectrum and produce PBHs if the parameter $h$ is large enough [62, 67]. In this paper, we focus on the scalar-tensor inflation and find that with the Ricci scalar coupling function being a quadratic form $\Omega(\phi) = 1 - 2\phi/\phi_c + (1 + \delta^2)(\phi/\phi_c)^2$, the ultra-slow-roll condition can be satisfied transiently at the critical point $\phi_c$, and the power spectrum can be enhanced enough to produce PBHs. These masses and abundances of the PBHs can be adjusted by the parameters $\phi_c$ and $\delta$, respectively.

With the formation of PBHs, the large scalar perturbations at small scales induce secondary gravitational waves after the horizon reentry during the radiation dominated epoch [73–103]. These scalar-induced gravitational waves (SIGWs) have wide frequency distribution and can be detected by pulsar timing arrays (PTA) [104–108] and the space-based GW detectors such as Laser Interferometer Space Antenna (LISA) [109, 110], Taiji [111], and TianQin [112] in the future. For example, the stochastic process with a common amplitude and a common spectral slope across pulsars detected by the North American Nanohertz Observatory for Gravitational Wave (NANOGrav) Collaboration recently [113] may be the SIGWs with nHz frequencies [68, 114–117].

The paper is organized as follows. In Sec. II, we show the enhancement mechanism on the power spectrum of the scalar-tensor inflation in detail. We discuss the pro-
duction of PBH DM and the generation of SIGWs from this mechanism in Sec. III. We conclude the paper in Sec. V.

II. THE MODEL

The action for the scalar-tensor theory in Jordan frame is

\[ S = \int \sqrt{-g} dx^4 \left[ \frac{\Omega(\phi)}{2} R - \frac{\omega(\phi)}{2} (\partial \phi)^2 - V(\phi) \right], \]  

where \( \Omega(\phi) \) and \( \omega(\phi) \) are the coupling functions, and \( V(\phi) \) is the potential, \((\partial \phi)^2 = g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \) and \( 8\pi G = 1 \). For the homogeneous and isotropic background, the Friedmann equation and the equation of motion for the scalar field are

\[ H^2 = \frac{1}{3\Omega} \left( \frac{1}{2} \dot{\phi}^2 + V \right) - \frac{\dot{H}}{\Omega} \, , \]  

\[ \left[ \omega + \frac{3}{2\Omega} \left( \frac{d\Omega}{d\phi} \right)^2 \right] (\dot{\phi} + 3H\phi) + \frac{dV}{d\phi} - 2V \frac{d\Omega}{d\phi} \frac{1}{\Omega} \frac{d\Omega}{d\phi} + \left( \frac{3}{2\Omega} \frac{d\Omega}{d\phi} \frac{d\phi}{d\phi} + \frac{1}{2} \frac{d\omega}{d\phi} + \frac{1}{\Omega} \frac{d\Omega}{d\phi} \right) \phi^2 = 0, \]

where a “dot” denotes the derivative with respect to cosmic time. Under the slow-roll conditions \([118]\)

\[ \frac{1}{2} \dot{\phi}^2 \ll V(\phi), \quad |\dot{g}| \ll Hg, \]  

where the function \( g \) denotes an arbitrary function, such as \( \Omega(\phi) \) and \( \omega(\phi) \), the background equations (2) and (3) become

\[ H^2 \approx \frac{V(\phi)}{3\Omega(\phi)}, \]

\[ (\dot{\phi} + 3H\phi) + \Gamma(\phi) \frac{dU(\phi)}{d\phi} \approx 0, \]

with the effective potential \( U(\phi) = V(\phi)/\Omega(\phi)^2 \) and

\[ \Gamma(\phi) = \frac{\Omega(\phi)^2}{\omega(\phi) + \frac{3}{2\Pi} \left( \frac{d\Omega}{d\phi} \right)^2}. \]

The condition of forming PBHs requires the amplitude of the power spectrum of the primordial curvature to reach around \( A_\zeta \sim O(0.01) \), while the constraints on power spectrum at large scales from the observation of CMB anisotropy measurements is \( A_\zeta = 2.1 \times 10^{-9} \) \([119] \). Therefore, to produce PBHs, the power spectrum should be enhanced by about seven orders of magnitude at small scales, and this is hard to realize in the slow-roll inflation model. For the ultra-slow-roll inflation with condition \([44]\)

\[ \ddot{\phi} + 3H \dot{\phi} \approx 0, \]

the power spectrum can be enhanced enough to produce PBHs. From equation (6), to obtain the ultra-slow-roll condition (8), we need

\[ \Gamma(\phi) \frac{dU(\phi)}{d\phi} \ll 1. \]  

If the effective potential \( U(\phi) \) has a near inflection point, \( dU(\phi)/d\phi|_{\phi_c} \approx 0 \), the condition (9) can be satisfied easily \([40, 47] \). In addition to the method of near inflection point, the other way to obtain condition (9) is by making the coupling functions satisfy

\[ \Gamma(\phi) = \frac{\Omega(\phi)^2}{\omega(\phi) + \frac{3}{2\Pi} \left( \frac{d\Omega}{d\phi} \right)^2} \ll 1, \]

which requires \( \omega(\phi) \gg 1 \) or \( \Omega(\phi) \ll 1 \) at the ultra-slow-roll point. The situation \( \omega(\phi) \gg 1 \) has been researched in papers \([62, 67] \) with the form

\[ \omega(\phi) = 1 + \frac{h}{1 + (|\phi - \phi_c|/w)^q}, \]

and \( h \gg 1 \). At the point \( \phi_c \), the coupling function satisfies \( \omega(\phi_c) = 1 + h \gg 1 \), and the ultra-slow-roll condition (10) is satisfied.

In this paper, we consider the other case, \( \Omega(\phi) \ll 1 \). To obtain this case, we extend the usual coupling function \( \Omega(\phi) = 1 + \xi \phi^2 \) in the Higgs inflation \([120] \) to

\[ \Omega(\phi) = 1 - \frac{2\phi}{\phi_c} + (1 + \delta^2) \left( \frac{\phi}{\phi_c} \right)^2, \]

where the term \( \delta^2 \ll 1 \) is used to keep the coupling function \( \Omega(\phi) \) from exact zero. At the critical point, the coupling function becomes \( \Omega(\phi_c) = \delta^2 \ll 1 \), the condition (10) is satisfied, the inflaton evolves into a transitory ultra-slow-roll phase where the power spectrum of the curvature perturbations is enhanced. The critical point \( \phi_c \) controls the position of the peak in the power spectrum and \( \delta \) determines the amplitude of the peak. For the coupling function of the kinetic term, we choose

\[ \omega(\phi) + \frac{3}{2\Omega(\phi)} \left( \frac{d\Omega(\phi)}{d\phi} \right)^2 = a\Omega(\phi)^b + c\Omega(\phi), \]

with \( b < 1 \). The first term \( a\Omega(\phi)^b \) is used to adjust the shape of the peak, and the second term \( c\Omega(\phi) \) is used to keep \( w(\phi) = 1 \) at the lower energy scales \( \phi \ll 1 \), which requires

\[ c = 1 - a + \frac{6}{\phi_c^2}. \]

The potential is \([63] \)

\[ V(\phi) = \frac{\lambda\phi^4}{4} \left( \frac{\Omega(\phi)}{1 + \xi \phi^2} \right)^2, \]
with $\xi = 10$. At the lower energy scales $\phi \ll 1$, the potential reduces to the Higgs potential with the form $\lambda \phi^4/4$.

In the other hand, taking the conformal transformation,

$$\bar{g}_{\mu\nu} = \Omega(\phi)g_{\mu\nu},$$

(16)

and changing the Jordan frame to the Einstein frame, action (1) becomes

$$S = \int d^4x \sqrt{-\bar{g}} \left[ \frac{\bar{R}}{2} - \frac{1}{2}k(\phi)(\bar{\partial}\phi)^2 - \frac{V(\phi)}{\Omega(\phi)^2} \right],$$

(17)

with $(\bar{\partial}\phi)^2 = g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi$ and

$$k(\phi) = \frac{3}{2} \left( \frac{d\Omega/d\phi}{\Omega} \right)^2 + \frac{\omega(\phi)}{\Omega(\phi)},$$

(18)

Combining equation (12) and (13), the coupling function of the kinetic term becomes

$$k(\phi) = 1 + \left( -a + \frac{6}{\phi_c^2} \right) + \frac{a}{\Omega^{1-\delta}} = 1 + G(\phi),$$

(19)

with

$$G(\phi) = \left( \frac{6}{\phi_c^2} - a \right) + \frac{a(\phi_c/\delta)^{2-2b}}{\left( \frac{2-\phi_c}{\delta} + \phi_c^2 \right)^{1-\delta}},$$

(20)

which is similar with equation (11). As pointed out in Refs. [62, 67], this kind of kinetic term can succeed in enhancing the power spectrum and producing enough PBHs.

### III. THE RESULTS

#### A. power spectra

The quadratic action for the curvature perturbation $\zeta$ of the scalar-tensor inflation (1) is

$$S^{(2)} = \frac{1}{2} \int d^3x d^3y \frac{a^2[\omega\phi^2 + 3\Omega^2/2\Omega]}{(H + \Omega/2\Omega)^2} \left[ (\zeta')^2 - (\nabla\zeta)^2 \right],$$

(21)

where $\eta = \int dt/a$ is the conformal time and $\zeta' = d\zeta/d\eta$. The equation for the curvature perturbation in k-space is

$$\frac{d^2u_k}{d\eta^2} + \left( k^2 - \frac{1}{z} \frac{d^2z}{d\eta^2} \right) u_k = 0,$$

(22)

with $u_k = z\zeta_k$ and [121]

$$z^2 = \frac{a^2[\omega\phi^2 + 3\Omega^2/2\Omega]}{(H + \Omega/2\Omega)^2}.$$

(23)

The power spectrum of the curvature perturbation is

$$P_\zeta = \frac{k^3}{2\pi^2} |\zeta_k|^2,$$

(24)

which can be obtain by solving the background equations (2) and (3), and the perturbation equation (22).

By choosing the values of parameters $a$, $\delta$, $\phi_c$, $\lambda$, and the scalar field $\phi_s$ at the pivot scale, we can numerically obtain the power spectrum. For the values of the parameter sets listed in table I, the numerical results of the power spectra of the curvature perturbation are shown in figure 1. The $e$-folding numbers $N$ of these models in table I are about $N \in (55, 65)$. The scalar tilt and tensor-to-scalar ratio of these models are listed in table I which are around

$$n_s \approx 0.965, \quad r \approx 0.004,$$

(25)

which are consistent with the observational constraints [43, 122],

$$n_s = 0.9649 \pm 0.0042 \quad (68\% \text{ CL}),$$

(26)

$$r_{0.05} < 0.06 \quad (95\% \text{ CL}).$$

(27)

The position of the peak of the power spectra in figure 1 is controlled by the parameter $\phi_s$ in equation (12). The power spectra with peak scale around $k_{\text{peak}} \approx 10^6 \text{ Mpc}^{-1}$, $k_{\text{peak}} \approx 10^9 \text{ Mpc}^{-1}$, and $k_{\text{peak}} \approx 10^{12} \text{ Mpc}^{-1}$ are given in figure 1, and denoted as red lines, green lines, and black lines, respectively. In table I, they are labeled as “1”, “2”, and “3”, respectively. The shape of the peak is determined by the index $b$ in equation (13). The narrow peak denoted by the dashed line in figure 1 is from the model with $b = 1/2$ and labeled as “Mn” in table I, the broad peak denoted by the solid line in figure 1 is from the model with $b = 2/5$ and labeled as “Mb” in table I.

#### B. primordial black holes

If the amplitude of the power spectrum is enhanced to $A_c \sim \mathcal{O}(0.01)$ at small scales, it may form PBHs from gravitational collapse during the radiation domination. The mass fraction of the Universe that collapses to form PBHs at formation is denoted by

$$\beta = \frac{\rho_{\text{PBH}}}{\rho_b},$$

(28)

where $\rho_b$ is the energy density of the background and $\rho_{\text{PBH}}$ is the energy density of the PBHs at formation. From the peak theory, the energy density of the PBHs is [126–131],

$$\rho_{\text{PBH}} = \int_{v_c}^\infty M_{\text{PBH}}(v)N_{pk}(v)dv,$$

(29)

where the number density of the PBHs is [126]

$$N_{pk}(v) = \frac{1}{a^3 \sqrt{2\pi}} \left( \frac{\sigma_1}{3\sigma_0} \right)^3 \nu^4 \exp \left( -\frac{\nu^2}{2} \right).$$

(30)
there are usual three choices, the real-space top-hat win-

domination. 

where $P$ is the power spectrum of the density contrast $\delta$ and $\sigma_0$ is the threshold for the formation of PBHs, and $\sigma_0$ is the variance of the smoothed density contrast. The moment of the smoothed density power spectrum $\sigma_1$ is defined by 

$$
\sigma_n^2 = \int_0^\infty \frac{dk}{k} k^{2n} T^2(k, R_H) W^2(k, R_H) P_\delta(k),
$$

(31)

where $P_\delta$ is the power spectrum of the density contrast which is related to the power spectrum of primordial curvature perturbations $P_\zeta$ by 

$$
P_\delta(k) = \frac{4(1 + w)^2}{(5 + 3w)^2} \left(\frac{k}{a H}\right)^4 P_\zeta(k),
$$

(32)

with the state equation $w = 1/3$ during the radiation domination.

For the window function $W(k, R_H)$ in equation (31), there are usual three choices, the real-space top-hat window function, the Gauss window function, and the $k$-space top-hat window function [132]. In this paper, we choose the real-space top-hat window function, in the $k$-

space it is 

$$
W(k, R_H) = 3 \left[ \frac{\sin (kR_H) - (kR_H) \cos (kR_H)}{(kR_H)^3} \right],
$$

(33)

with the smoothed scale $R_H \sim 1/\alpha H$. The threshold $\delta_c$ of the PBHs formation is dependent on the window function and the shape of density perturbations [129, 130, 133]. For the real space top-hat window function, in this paper, we choose $\delta_c = 0.51$ [133, 134]. During radiation domination with constant degrees of freedom, the transfer function in equation (31) is

$$
T(k, R_H) = 3 \left[ \frac{\sin \left( \frac{kR_H}{\sqrt{3}} \right) - \left( \frac{kR_H}{\sqrt{3}} \right) \cos \left( \frac{kR_H}{\sqrt{3}} \right)}{(kR_H/\sqrt{3})^3} \right].
$$

(34)

The masses of primordial black holes in equation (29) obey the critical scaling law with the formula [135–137]

$$
M_{PBH} = \kappa M_H (\delta - \delta_c)^\gamma,
$$

(35)

where $\kappa = 3.3$ for the real space top-hat window function and $\gamma = 0.36$ in the radiation domination [135, 136]. The horizon mass related to the horizon scale is 

$$
M_H \approx 13 \left( \frac{g_*}{106.75} \right)^{-1/6} \left( \frac{k}{10^6 \text{Mpc}^{-1}} \right)^{-2} M_\odot,
$$

(36)

where $g_*$ is the number of relativistic degrees of freedom at the formation. With the help of the background equations of the energy density during radiation domination, $\rho_b \propto a^{-4}$ and $\rho_{PBH} \propto a^{-3}$, we obtain the relation of the density parameter of the PBHs at present and fraction of PBHs in the Universe at formation [138],

$$
\Omega_{PBH} = \int_{M_{min}}^{M_{max}} d \ln M_H \left( \frac{M_{eq}}{M_H} \right)^{1/2} \beta(M_H),
$$

(37)

where $M_{eq} = 2.8 \times 10^{17} M_\odot$ is the horizon mass at the matter-radiation equality. In our model, $\beta(M_H) \to 0$ at the condition $M_H \to 0$ or $M_H \to \infty$, so we take the lower limit of integral as $M_{min} = 0$ and the upper limit of that as $M_{max} = \infty$, for the sake of simplicity. The fraction of primordial black holes in the dark matter at present is 

$$
f_{PBH} = \frac{\Omega_{PBH}}{\Omega_{DM}} = \int f(M_{PBH}) d \ln M_{PBH},
$$

(38)
where the PBHs mass function is defined as

$$f(M_{\text{PBH}}) = \frac{1}{\Omega_{\text{DM}} d \ln M_{\text{PBH}}}$$  \hspace{1cm} (39)$$

Combining relation (37) and definition (39), using equation (35) and \(d\delta/d\ln M_{\text{PBH}} = \mu^{1/\gamma}/\gamma\), the mass function (39) becomes \[138\]

$$f(M_{\text{PBH}}) = \frac{1}{\Omega_{\text{DM}}} \int_{M_{\text{min}}}^{M_{\text{max}}} dM_H \frac{M_{\text{PBH}}}{M_H} \frac{\sqrt{M_\Omega}}{M_H} \times \frac{1}{3\pi} \left( \frac{\sigma_1}{\sqrt{3\sigma_0aH}} \right)^3 \frac{1}{\sigma_0^2} \left( \mu^{1/\gamma} + \delta_c \right)^3 \times \mu^{1/\gamma} \exp \left[ - \left( \frac{\mu^{1/\gamma} + \delta_c}{2\sigma_0^2} \right)^2 \right],$$  \hspace{1cm} (40)$$

with \(\mu = M_{\text{PBH}}/(\kappa M_H)\).

Using the numerical results of the power spectra of the models listed in table I, combining equations (31) and (40), we obtain the mass function of the PBHs and the results are displayed in figure 2, the corresponding PBHs abundance \(f_{\text{PBH}}\) and PBHs masses \(M_{\text{peak}}\) at the peak are listed in table II. The PBHs with stellar masses, planetary masses, and \(10^{-12} M_\odot\) are produced and denoted by red lines, green lines, and black lines in figure 2, respectively. The PBHs with stellar masses, labeled as “1” in table II, can explain the black holes in LIGO/Virgo events [3–5]. The PBHs with planetary masses, labeled as “2” in table II, can explain the ultra short-timescale microlensing events in OGLE data [17] and the anomalous orbits of trans-Neptunian objects[18]. The PBHs with masses around \(10^{-12} M_\odot\), labeled as “3” in table II, can account for almost all of the dark matter, and their abundances are \(f_{\text{PBH}} = 0.95\) for model “Mn3” and \(f_{\text{PBH}} = 0.96\) for model “Mb3”.

C. scalar-induced gravitational waves

In addition to supplying the condition of the formation of PBHs, during radiation domination and after reentering the horizon, the large scalar perturbations can induce the gravitational waves with frequencies ranging from nHz to mHz. The SIGWs with nHz can be detected by PTA and account for the NANOGrav 12.5yrs signal, and those with mHz can be detected by the space-based GW detectors like LISA, Taiji and TianQin in the future. In the cosmological background and neglecting the anisotropic stress, the perturbed metric in the Newtonian gauge is

$$ds^2 = -a^2(\eta)(1 + 2\Phi)d\eta^2 + a^2(\eta) \left( 1 - 2\Phi \delta_{ij} + \frac{1}{2} h_{ij} \right) dx^i dx^j,$$  \hspace{1cm} (41)$$

where \(\eta\) is the conformal time, \(\Phi\) is the Bardeen potential. The tensor perturbations \(h_{ij}\) expressed in the Fourier space are

$$h_{ij}(x, \eta) = \int d^3k e^{ik \cdot x} \frac{1}{(2\pi)^{3/2}} [\tilde{h}_k(\eta)e_{ij}(k) + \tilde{\epsilon}_k(\eta)e_{ij}(k)],$$  \hspace{1cm} (42)$$

where \(e_{ij}(k)\) and \(\tilde{\epsilon}_{ij}(k)\) are the plus and cross polarization tensors which can be expressed as

$$e_{ij}(k) = \frac{1}{\sqrt{2}} [\epsilon_i(k)e_j(k) - \epsilon_j(k)e_i(k)];$$  \hspace{1cm} (43)$$

$$\tilde{\epsilon}_{ij}(k) = \frac{1}{\sqrt{2}} [\epsilon_i(k)e_j(k) + \epsilon_j(k)e_i(k)];$$  \hspace{1cm} (44)$$

with \(e \cdot \tilde{\epsilon} = e \cdot k = \tilde{e} \cdot k\).

For either polarization, the tensor perturbations induced from linear scalar perturbations in the Fourier space satisfy [75, 76]

$$h_{kk}' + 2\mathcal{H}h_k'' + k^2h_k = 4S_k,$$  \hspace{1cm} (45)$$

where a prime denotes the derivative with respect to the conformal time, \(h_k' = dh_k/d\eta\), and \(\mathcal{H} = a'/a\) is the conformal Hubble parameter. \(S_k\) is the second order source from the linear scalar perturbations,

$$S_k = \int \frac{d^3k}{(2\pi)^{3/2}} e_{ij}(k)\tilde{k}^i\tilde{k}^j \left[ 2\Phi_k \Phi_{k-\tilde{k}} + \frac{1}{\mathcal{H}^2} \left( \Phi_k' + \mathcal{H}\Phi_k \right) \left( \Phi_{k-\tilde{k}}' + \mathcal{H}\Phi_{k-\tilde{k}} \right) \right].$$  \hspace{1cm} (46)$$

FIG. 2. The corresponding PBHs mass function of the models listed in table I. The shaded regions show the observational constraints on the PBH abundance: the cyan region from accretion constraints by CMB [139, 140], the red region from LIGO-Virgo Collaboration measurements [141–146], the gray region from the EROS/MACHO [147], the green region from microlensing events with Subaru HSC [148], the blue region from the Kepler satellite [149], the orange region from white dwarf explosion (WD) [154], the yellow region from galactic center 511 keV gamma-ray line (INTEGRAL) [151–153], the brown region from extragalactic gamma-rays by PBH evaporation (EG\(\gamma\)) [150]. The purple region are the allowed PBH abundance from the ultrashort-timescale microlensing events in the OGLE data [17].
The relation between Bardeen potential $\Phi_k$ and the primordial curvature perturbation $\zeta_k$ in Fourier space is
\begin{equation}
\Phi_k = \frac{3 + 3w}{5 + 3w} T(k, \eta) \zeta_k, \tag{47}
\end{equation}
where $T(k, \eta)$ is the transfer function (34). The definition of the power spectrum $P_h(k, \eta)$ for the SIGWs is
\begin{equation}
\langle h_k(\eta)h_k(\eta') \rangle = \frac{2\pi^2}{k^3} \delta^{(3)}(k - \tilde{k}) P_h(k, \eta). \tag{48}
\end{equation}

The tensor perturbation (45) can be solved by the Green function method and the solution is
\begin{equation}
h_k(\eta) = \frac{4}{a(\eta)} \int_{\eta_k}^\eta d\eta' g_k(\eta, \eta') a(\eta') S_k(\eta'), \tag{49}
\end{equation}
where the corresponding Green function is
\begin{equation}
g_k(\eta, \eta') = \frac{\sin[k(\eta - \eta')]}{k}. \tag{50}
\end{equation}

Substituting the result (49) into definition (48), we obtain [41, 75, 76, 90, 91]
\begin{equation}
P_h(k, \eta) = 4 \int_0^\infty dv \int_{1-v}^{1+v} du \left[ \frac{4u^2 - (1 - u^2 + v^2)^2}{4uv} \right]^2 \times I_{RD}^2(u, v, x) P_c(kv) P_c(ku), \tag{51}
\end{equation}
where $u = |k - \tilde{k}|/k$, $v = \tilde{k}/k$, $x = k\eta$ and the integral kernel $I_{RD}$ is
\begin{equation}
I_{RD}(u, v, x) = \int_1^x dy y \sin(x - y) \{3T(uy)T(vy) + y[T(vy)uT'(uy) + vT'(vy)T(uy)] + y^2uvT''(uy)T'(vy) \}. \tag{52}
\end{equation}

Substituting equation (51) into the definition of energy density of SIGWs,
\begin{equation}
\Omega_{GW}(k, \eta) = \frac{1}{24} \left( \frac{k}{aH} \right)^2 P_h(k, \eta), \tag{53}
\end{equation}
we get [41, 91]
\begin{equation}
\Omega_{GW}(k, \eta) = \frac{1}{6} \left( \frac{k}{aH} \right)^2 \int_0^\infty dv \int_{1-v}^{1+v} du \left[ \frac{4u^2 - (1 - u^2 + v^2)^2}{4uv} \right]^2 \times \frac{T_{RD}^2(u, v, x) P_c(kv) P_c(ku)}{4uv}, \tag{54}
\end{equation}
where $T_{RD}$ is the oscillation time average of the integral kernel. After formation during the radiation domination, the SIGWs behave like radiation, so the energy density of the SIGWs is in direct proportion to the energy density of the radiation. Using this property, we can obtain the energy density of the SIGWs at present easily and it is
\begin{equation}
\Omega_{GW}(k, \eta_0) = c_g \Omega_{r,0} \Omega_{GW}(k, \eta), \tag{55}
\end{equation}
where $\Omega_{r,0}$ is the energy density of radiation at present, and [114, 116]
\begin{equation}
c_g = 0.387 \left( \frac{g_*^3}{106.75} \right)^{-1/3}. \tag{56}
\end{equation}

Substituting the numerical results of the power spectra of the models listed in table I into equation (54), we obtain the energy density of SIGWs and the numerical results are displayed in figure 3, the corresponding peak frequencies $f_\ast$ are listed in table II. The SIGWs of models labeled as “1” in table II are denoted by the red lines in figure 3 with frequencies around $10^{-9}$ Hz. They are consistent with $2\sigma$ region of the NANOGrav 12.5 yrs signal, which indicates that the NANOGrav 12.5 yrs signal may be SIGWs. The SIGWs of models labeled as “2” in table II are denoted by the green lines in figure 3 with frequencies around $10^{-6}$ Hz. The broad peak can be detected by the space-based detectors LISA and Taiji. The SIGWs of models labeled as “3” in table II are denoted by the black lines in figure 3 with frequencies around $10^{-3}$ Hz, and can be detected by the LISA, Taiji, and TianQin in the future.

### IV. CONCLUSION

PBHs and SIGWs can be produced from the inflation models with a transient ultra-slow-roll process, where the
equation of motion for the scalar field is $\ddot{\phi} + 3H \dot{\phi} \approx 0$. For the scalar-tensor inflation, the ultra-slow-roll condition can be realized by taking $Ω^2/|\omega + 3\Omega^2/(2Ω)| \ll 1$ with $Ω(\phi)$ coupling to the Ricci scalar and $\omega(\phi)$ to the kinetic term. For the coupling function $Ω(\phi)$ with quadratic form $Ω(\phi) = 1 - 2\phi/ϕ_ε + (1 + δ^2) (ϕ/ϕ_ε)^2$, under the condition $δ^2 \ll 1$, the ultra-slow condition is satisfied at the point $ϕ_ε$ and the power spectra can be enhanced enough to produce PBHs and generate SIGWs. The parameter $ϕ_ε$ controls the position of the peak in power spectra and also governs the masses of PBHs and the frequencies of SIGWs. The kinetic coupling function is $ω(ϕ) + 3Ω^2/(2Ω) = aΩ^b + cΩ$, determining the shape of the peak in power spectra.

In this paper, we produce three kinds of PBHs: the PBHs with stellar masses, those with planetary masses, and those with masses around $10^{-12}M_⊙$. The cases with a narrow peak and a broad peak are both given for each kind. The first kind PBHs have the peak masses $M_{peak} = 38M_⊙$ (narrow peak) and $M_{peak} = 25M_⊙$ (broad peak), which are consistent with the Planck 2018 observational data. The second kind PBHs have the peak masses $M_{peak} = 1.95 \times 10^{-5}M_⊙$ (narrow peak) and $M_{peak} = 6.93 \times 10^{-9}M_⊙$ (broad peak), which can account for almost all of the dark matter. The corresponding SIGWs have the peak frequencies around $10^{-3}$ Hz and can be detected by the space-based detectors LISA and Taiji. The third kind PBHs have peak masses $M_{peak} = 1.23 \times 10^{-12}M_⊙$ (narrow peak) with the PBHs abundance $f_{PBH} = 0.95$, and $M_{peak} = 7.87 \times 10^{-13}M_⊙$ (broad peak) with the PBHs abundance $f_{PBH} = 0.96$, they can account for almost all of the dark matter. The corresponding SIGWs have the peak frequencies around $10^{-6}$ Hz, and the broad peak case can be detected by the space-based detectors LISA and Taiji.

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