SU(3) and Isospin Breaking Effects on $B \rightarrow PPP$ Amplitudes

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Abstract

Several modes of $B$ decays into three pseudoscalar octet mesons PPP have been measured. These decays have provided useful information for $B$ decays in the standard model (SM). Some of powerful tools in analyzing $B$ decays are flavor $SU(3)$ and isospin symmetries. Such analyses are usually hampered by $SU(3)$ breaking effects due to a relatively large strange quark mass which breaks $SU(3)$ symmetry down to isospin symmetry. The isospin symmetry also breaks down when up and down quark mass difference is non-zero. It is therefore interesting to find relations which are not sensitive to $SU(3)$ and isospin breaking effects. We find that the relations among several fully-symmetric $B \rightarrow PPP$ decay amplitudes are immuned from $SU(3)$ breaking effects due a non-zero strange quark mass, and also some of them are not affected by isospin breaking effects. Measurements for these relations can provide important information about $B$ decays in the SM.

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I. INTRODUCTION

Several decay modes of $B$ decays into three pseudoscalar octet mesons $PPP$ have been measured\cite{1,2}. $B \rightarrow PPP$ has been a subject of theoretical studies\cite{3}. The new data have raised new interests in related theoretical studies\cite{4,5,6,7,8}. With more data from LHCb, one can expect that the study of $B \rightarrow PPP$ will provide more important information for $B$ decays in the standard model (SM).

A powerful tool to analyze $B$ decays is flavor $SU(3)$ symmetry\cite{10}. Some of the interesting features of using flavor $SU(3)$ are the predictions of relations among different decay modes which can be experimentally tested. The flavor $SU(3)$ symmetry is, however, expected to be only an approximate symmetry because $u$, $d$ and $s$ quarks have different masses. Since the strange quark has a relative larger mass compared with those of up and down quarks, it is the larger source of symmetry breaking. If up and down quark masses are neglected, a non-zero strange quark mass breaks flavor $SU(3)$ symmetry down to the isospin symmetry. When up and down quark mass difference is kept, isospin symmetry is also broken. The $SU(3)$ breaking effect is at the level of 20 percent for the $\pi$ and $K$ decay constants $f_\pi$ and $f_K$. For 2-body pseudoscalar octet meson $B$ decays, although there are some $SU(3)$ breakings\cite{11}, it works reasonably well, such as rate differences between some of the $\Delta S = 0$ and $\Delta S = 1$ two-body pseudoscalar meson $B$ decays\cite{12,13}. Analysis has also been carried out for $B \rightarrow PPP$ decays using flavor $SU(3)$ recently. It has been shown that the decay and CP asymmetry patterns for the charged $B^+$ decays into $K^+K^-K^+$, $K^+K^-\pi^+$, $K^+\pi^-\pi^+$ and $\pi^+\pi^-\pi^+$ do not follow $SU(3)$ predictions. To explain data, large $SU(3)$ breaking effects are needed\cite{6,7}. Usually isospin breaking effects are much smaller because up and down quark masses are much smaller than the strange quark mass and the QCD scale.

Because of possible large flavor $SU(3)$ breaking effects for $B \rightarrow PPP$, the predicted relations among different decay modes can only provide limited information. One wonders whether there exist relations which are immune from $SU(3)$ or even isospin breaking effects due to $u$, $d$ and $s$ quark mass differences. To this end we carried out an analysis for $B \rightarrow PPP$ decays using flavor $SU(3)$ symmetry to identify possible relations, and then include $SU(3)$ breaking effects due to a strange quark mass, and also up and down quark masses to see whether some relations still remain to hold. We find that the relations between several fully-symmetric $B \rightarrow PPP$ decay amplitudes studied in Ref.\cite{9} are not affected by the flavor $SU(3)$ breaking effects due a non-zero strange quark mass, and some of them are not even affected by isospin breaking effects. These relations when measured experimentally can provide useful information about $B$ decays in the SM. In the following we provide some details.

II. $SU(3)$ CONSERVING AMPLITUDES

We start with the description of $B$ decays into three pseudoscalar octet mesons from flavor $SU(3)$ symmetry. The leading quark level effective Hamiltonian up to one loop level
in electroweak interaction for hadronic charmless $B$ decays in the SM can be written as

$$H_{eff}^q = \frac{4G_F}{\sqrt{2}} [V_{ub}V_{ud}^* (c_1O_1 + c_2O_2) - \sum_{i=3}^{12} (V_{ub}V_{uq}^{*}e_i^{uc} + V_{tb}V_{tq}^{*}e_i^{tc})O_i], \quad (2.1)$$

where $q$ can be $d$ or $s$, the coefficients $c_{1,2}$ and $c_{j,k}^{ij} = c_j^i - c_k^i$, with $j$ and $k$ indicate the internal quark, are the Wilson Coefficients (WC). The tree WC's are of order one with, $c_1 = -0.31$, and $c_2 = 1.15$. The penguin WC's are much smaller with the largest one $c_6$ to be $-0.05$. These WC's have been evaluated by several groups [14]. $V_{ij}$ are the KM matrix elements. In the above the factor $V_{eb}V_{eq}^*$ has been eliminated using the unitarity property of the KM matrix.

The operators $O_i$ are given by

$$O_1 = (\bar{q}_i u_j)_{V-A} (\bar{u}_k b_j)_{V-A} , \quad O_2 = (\bar{q}u)_{V-A} (\bar{u}b)_{V-A} ,$$

$$O_{3,5} = (\bar{q}b)_{V-A} \sum_q (\bar{q}^c q')_{V-A} , \quad O_{4,6} = (\bar{q}b j)_{V-A} \sum_q (\bar{q}^c q')_{V-A} ,$$

$$O_{7,9} = \frac{3}{2} (\bar{q}b)_{V-A} \sum_q e_q (\bar{q}^c q')_{V-A} , \quad O_{8,10} = \frac{3}{2} (\bar{q}b j)_{V-A} \sum_q e_q (\bar{q}^c q')_{V-A} ,$$

$$O_{11} = \frac{q}{16\pi^2} \bar{q}\sigma_{\mu\nu} G^{\mu\nu} (1 + \gamma_5)b , \quad O_{12} = \frac{q}{16\pi^2} \bar{q}\sigma_{\mu\nu} F^{\mu\nu} (1 + \gamma_5)b . \quad (2.2)$$

where $(\bar{a}b)_{V-A} = \bar{a}\gamma_\mu (1 - \gamma_5)b$. $G^{\mu\nu}$ and $F^{\mu\nu}$ are the field strengths of the gluon and photon, respectively.

At the hadron level, the decay amplitude can be generically written as

$$A = \langle \text{final state} | H^q_{eff} | \bar{B} \rangle = V_{ub}V_{ud}^* T(q) + V_{tb}V_{td}^* P(q) , \quad (2.3)$$

where $T(q)$ contains contributions from the tree as well as penguin due to charm and up quark loop corrections to the matrix elements, while $P(q)$ contains contributions purely from one loop penguin contributions. $B$ indicates one of the $B^+, B^0$ and $B^0_s$. $B_i = (B^+, B^0, B^0_s)$ forms an $SU(3)$ triplet.

The flavor $SU(3)$ symmetry transformation properties for operators $O_{1,2}, O_{3-6,11,12}$, and $O_{7-10}$ are: $3_a + 3_b + 6 + \bar{15}$, $3$, and $3_a + 3_b + 6 + \bar{15}$, respectively. We indicate these representations by matrices in $SU(3)$ flavor space by $H(3)$, $H(6)$ and $H(\bar{15})$. For $q = d$, the non-zero entries of the matrices $H(i)$ are given by [12]

$$H(3)^2 = 1 , \quad H(6)^{12} = H(6)^{32} = 1 , \quad H(6)^{21} = H(6)^{31} = -1 ,$$

$$H(\bar{15})^2 = H(\bar{15})^3 = 3 , \quad H(\bar{15})^2 = -2 , \quad H(\bar{15})^3 = H(\bar{15})^{23} = -1 . \quad (2.4)$$

And for $q = s$, the non-zero entries are

$$H(3)^3 = 1 , \quad H(6)^{13} = H(6)^{33} = 1 , \quad H(6)^{31} = H(6)^{32} = -1 ,$$

$$H(\bar{15})^3 = H(\bar{15})^{31} = 3 , \quad H(\bar{15})^3 = -2 , \quad H(\bar{15})^2 = H(\bar{15})^{23} = -1 . \quad (2.5)$$

These properties enable one to write the decay amplitudes for $B \to PPP$ decays in only a few $SU(3)$ invariant amplitudes [10]. Here $P$ is one of the mesons in the pseudoscalar
octet meson $M = (M_{ij})$ which is given by,

$$\begin{pmatrix}
\frac{\pi^0}{\sqrt{2}} + \frac{\eta_s}{\sqrt{6}} & \pi^+ & K^+

\pi^- & \frac{\pi^0}{\sqrt{2}} + \frac{\eta_s}{\sqrt{6}} & K^0

K^- & K^0 & -2\frac{\eta_s}{\sqrt{6}}
\end{pmatrix}.$$  \tag{2.6}

Construction of $B \rightarrow PPP$ decay amplitude can be done order by order using three $M'$s, $B$, and the Hamiltonian $H$, and also derivatives on the mesons to form $SU(3)$. The $SU(3)$ conserving momentum independent amplitudes can be constructed by the following.

For the $T(q)$ amplitude, we have

$$T(q) = a^T(3)B_iH^i(3)M_{ij}^1M_j^1M_k^1 + b^T(3)H^i(3)M_{ij}^1M_j^1M_k^1 + c^T(3)H^i(3)M_{ij}^1M_j^1M_k^1B_k$$

$$+ a^T(6)B_iH^i(6)M_{ij}^1M_n^1M_l^1 + b^T(6)B_iH^i(6)M_j^1M_n^1M_l^1$$

$$+ c^T(6)B_iH^i(6)M_j^1M_n^1M_l^1 + d^T(6)B_iH^i(6)M_j^1M_n^1M_l^1$$

$$+ a^T(15)B_iH^i(15)M_j^1M_n^1M_l^1 + b^T(15)B_iH^i(15)M_j^1M_n^1M_l^1$$

$$+ c^T(15)B_iH^i(15)M_j^1M_n^1M_l^1.$$ \tag{2.7}

One can write similar amplitude $P(q)$ for the penguin contributions.

The coefficients $a(i)$, $b(i)$, $c(i)$ and $d(i)$ are constants which contain the WCs and information about QCD dynamics. Expanding the above $T(q)$ amplitude, one can extract the decay amplitudes for specific decays in terms of these coefficients.

In the above we have described how to obtain flavor $SU(3)$ amplitudes which are momentum independent. However, due to the three body decay nature, in general, there are momentum dependence in the decay amplitudes. The momentum dependence can be determined by analysing Dalitz plots for the decays. The lowest order terms with derivatives lead to two powers of momentum dependence. One can obtain relevant terms by taking two times of derivatives on each of the terms in Eq. (2.7) and then collecting them together. It has been shown that there are six independent ways of taking derivatives for each of the terms listed in eq. (2.7). For example after taking derivatives for $B_iH^i(3)M_j^1M_k^1M_l^1$, we have the following independent terms

$$\begin{align*}
(\partial_\mu B_i)H^i(3)(\partial^\mu M_j^1)M_k^1M_l^1, & \quad (\partial_\mu B_i)H^i(3)M_j^1(\partial^\mu M_k^1)M_l^1, & \quad (\partial_\mu B_i)H^i(3)M_j^1M_k^1(\partial^\mu M_l^1), \\
B_iH^i(3)(\partial_\mu M_j^1)(\partial^\mu M_k^1)M_l^1, & \quad B_iH^i(3)(\partial_\mu M_j^1)M_k^1(\partial^\mu M_l^1), & \quad B_iH^i(3)M_j^1M_k^1(\partial_\mu M_l^1).
\end{align*} \tag{2.8}$$

The full list of the possible terms have been obtained in Ref. [6] in the Appendix B. We will not repeat them here.

Using the above $SU(3)$ decay amplitudes, one can find some interesting relations among different decays [6]. It has been recently pointed out that there are additional relations among the fully-symmetric final states B decay amplitudes $A_{FS}$ [9]. Study of these relations can provide further information about flavor $SU(3)$ symmetry in $B$ decays.

The fully-symmetric $B \rightarrow PPP$ amplitudes $A_{FS}$ is related to the usual decay amplitudes $A(P_1(p_1)P_2(p_2)P_3(p_3))$ for the final mesons $P_{1,2,3}$ carrying momenta $p_{1,2,3}$, for all three final
mesons are distinctive, by

\[
A_{FS}(P_1 \ P_2 \ P_3) = 
\frac{1}{\sqrt{3}} (A(P_1(p_1)P_2(p_2)P_3(p_3)) + A(P_1(p_2)P_2(p_3)P_3(p_1)) + A(P_1(p_3)P_2(p_1)P_3(p_2))) ,
\]

For the cases that two of them or all three of them are identical particles, the identical particle factorial factors should be taken cared.

To understand that why there are new relations between the fully-symmetric amplitudes for different decay modes, let us consider \( B^+ \rightarrow K^0\pi^+\pi^0 \) and \( B_d^0 \rightarrow K^+\pi^0\pi^- \) decays as examples.

Expanding eq. (2.7), one obtains

\[
T(B^+ \rightarrow K^0\pi^+\pi^0) = \sqrt{2} \left( c(6) + d(6) + 2c(15) + 2d(15) \right) ,
\]

and

\[
T(B^0 \rightarrow K^+\pi^0\pi^-) = T(B^+ \rightarrow K^0\pi^+\pi^0) .
\]

From which we get \( T_{FS}(B^+ \rightarrow K^0\pi^+\pi^0) = T_{FS}(B^0 \rightarrow K^+\pi^0\pi^-) \).

As the decay ampli\(tudes \) may have momentum dependence, we should also check if the equality of the above two amplitudes are equal when taking into account of momentum dependence in the amplitudes. Expanding terms in Appendix B of Ref.[6], we find

\[
T^p(B^+ \rightarrow K^0\pi^+\pi^0) = 
\alpha_1 p_B \cdot p_1 + \alpha_2 p_B \cdot p_2 + \alpha_3 p_B \cdot p_3 + \alpha_4 p_1 \cdot p_2 + \alpha_5 p_1 \cdot p_3 + \alpha_6 p_2 \cdot p_3 ,
\]

\[
T^p(B_d^0 \rightarrow K^+\pi^0\pi^-) = 
\beta_1 p_B \cdot p_1 + \beta_2 p_B \cdot p_2 + \beta_3 p_B \cdot p_3 + \beta_4 p_1 \cdot p_2 + \beta_5 p_1 \cdot p_3 + \beta_6 p_2 \cdot p_3 .
\]

The coefficients \( \alpha_i \) and \( \beta_i \) are given by,

\[
\alpha_1 = \sqrt{2} \left( c'(6) + 2c'(15) + 2d'(6) + 2d'(15) \right) ,
\]

\[
\begin{align*}
\alpha_2 &= \frac{1}{\sqrt{2}} \left( -b'(6) + b'(6) + 3b'(15) + 3b'(15) + c'(3)^2 + c'(3)^2 + c'(6)_1 \\
&+ c'(6)_3 + c'(15) + 3c'(15) + d'(6) + d'(6) + 5d'(15) - d'(15)_3 \right) , \\
\alpha_3 &= \frac{1}{\sqrt{2}} \left( b'(6) - b'(6) + 3b'(15) - 3b'(15) + c'(3)^2 + c'(3)_3 + c'(6)_1 + c'(6)_3 \\
&+ 3c'(15)_1 + c'(15)_3 + d'(6) + 2d'(6) - d'(6) - d'(15) + 4d'(15)_2 + d'(15)_3 \right) , \\
\alpha_4 &= \frac{1}{\sqrt{2}} \left( b''(6) - b''(6) + 3b''(15) + 3b''(15) + c''(3)^2 + c''(3)^2 + c''(6)_1 + c''(6)_3 \\
&+ c''(15)^2 + 3c''(15) - d''(6) + 2d''(6) + d''(6) + d''(15) - d''(15)^2 + 4d''(15)_3 \right) , \\
\alpha_5 &= \frac{1}{\sqrt{2}} \left( b''(6) - b''(6) + 3b''(15) - 3b''(15) + c''(3)^2 + c''(3)_3 + c''(6)_1 \\
&+ c''(6)_3 + 3c''(15)_1 + c''(15)_2 + d''(6) + d''(6) + d''(15) + d''(15)_1 + d''(15)_2 \right) , \\
\alpha_6 &= \sqrt{2} \left( c''(6) + 2c''(15) + d''(6) + 2d''(15) \right) .
\end{align*}
\]
and

\[
\beta_1 = \sqrt{2} (c'(6)_2 + 2c'(\bar{15})_2 + d'(6)_3 + 2d'(\bar{15})_3),
\]
\[
\beta_2 = \frac{1}{\sqrt{2}} \left( b'(6)_1 - b'(6)_2 + b'(\bar{15})_1 - b'(\bar{15})_2 + c'(3)_2 - c'(\bar{3})_2 + c'(6)_1 + c'(6)_3 \right.
\]
\[
+ c'(\bar{15})_1 + 3c'(\bar{15})_3 + d'(6)_1 + 2d'(6)_2 - d'(\bar{6})_3 - 3d'(\bar{15})_1 + 4d'(\bar{15})_2 + 3d'(\bar{15})_3),
\]
\[
\beta_3 = \frac{1}{\sqrt{2}} \left( -b'(6)_1 + b'(6)_2 - b'(\bar{15})_1 + b'(\bar{15})_2 - c'(3)_2 + c'(\bar{3})_2 + c'(6)_1 \right)
\]
\[
+ c'(\bar{15})_1 + 3c'(\bar{15})_3 + d'(6)_1 + d'(6)_3 + 7d'(\bar{15})_1 - 3d'(\bar{15})_3),
\]
\[
\beta_4 = \frac{1}{\sqrt{2}} \left( b''(6)_2 - b''(6)_3 + b''(\bar{15})_2 - b''(\bar{15})_3 + c''(3)_1 - c''(\bar{3})_2 + c''(6)_1 \right)
\]
\[
+ c''(\bar{15})_1 + 3c''(\bar{15})_2 + d''(6)_1 + d''(6)_3 - 3d''(\bar{15})_1 + 7d''(\bar{15})_2),
\]
\[
\beta_5 = \frac{1}{\sqrt{2}} \left( -b''(6)_2 + b''(6)_3 - b''(\bar{15})_2 + b''(\bar{15})_3 - c''(3)_1 + c''(\bar{3})_2 + c''(6)_1 + c''(6)_3 \right)
\]
\[
+ 3c''(\bar{15})_1 + c''(\bar{15})_2 - d''(6)_1 + 2d''(6)_2 + d''(6)_3 + 3d''(\bar{15})_1 - 3d''(\bar{15})_2 + 4d''(\bar{15})_3),
\]
\[
\beta_6 = \sqrt{2} \left( c''(6)_2 + 2c''(\bar{15})_3 + d''(6)_1 + 2d''(\bar{15})_1 \right).
\]

One can see from the above that \( T^p(B^+ \to K^0\pi^+\pi^0) \) is no longer equal to \( T^p(B^0_{d} \to K^+\pi^0\pi^-) \). However, one can readily see from the above equations, that

\[
\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3, \quad \alpha_4 + \alpha_5 + \alpha_6 = \beta_4 + \beta_5 + \beta_6.
\]

This fact makes the fully-symmetric amplitudes to satisfy

\[
T^p(B^+ \to K^0\pi^+\pi^0)_{FS} = T^p(B^0_{d} \to K^+\pi^0\pi^-)_{FS}.
\]

Similarly, the penguin amplitudes \( P \) and \( P^p \) have the same properties discussed above for the tree amplitudes, \( T \) and \( T^p \).

The total fully-symmetric amplitudes \( A_{FS} = V_{ub}V^*_{uq}(T_{FS} + T^p_{FS}) + V_{tb}V^*_{tq}(P_{FS} + P^p_{FS}) \) then have the relation

\[
A(B^+ \to K^0\pi^+\pi^0)_{FS} = A(B^0_{d} \to K^+\pi^0\pi^-)_{FS}.
\]

Enlarging the amplitudes to fully-symmetric ones, indeed produce more relations.

Expanding eq. (2.17) and equations in Appendix B of Ref. [9], we obtain the following relations confirming those obtained in Ref. [9]. For \( b \to s \) induced \( B \to PPP \) amplitudes, we have

1. \( B \to K\pi\pi \)

\[
S1.1 = A(B^+ \to K^0\pi^+\pi^0)_{FS} - A(B^0 \to K^+\pi^0\pi^-)_{FS} = 0,
\]
\[
S1.2 = \sqrt{2} A(B^+ \to K^0\pi^+\pi^0)_{FS} - A(B^0 \to K^0\pi^+\pi^-)_{FS} + 2 A(B^0 \to K^0\pi^0\pi^0)_{FS} = 0,
\]
\[
S1.3 = \sqrt{2} A(B^0 \to K^+\pi^0\pi^-)_{FS} + A(B^+ \to K^+\pi^+\pi^-)_{FS} - 2 A(B^+ \to K^+\pi^0\pi^-)_{FS} = 0.
\]

2. \( B \to KK\bar{K} \)
S2.1 = -A(B^+ \to K^+K^+K^-)_{FS} + A(B^+ \to K^+K^0\bar{K}^0)_{FS} \\
+ A(B^0 \to K^0K^+K^-)_{FS} - A(B^0 \to K^0K^0\bar{K}^0)_{FS} = 0.

3. \( B^0_s \to \pi K\bar{K} \)

S3.1 = \sqrt{2}A(B^0_s \to \pi^0K^+K^-)_{FS} - \sqrt{2}A(B^0_s \to \pi^0K^0\bar{K}^0)_{FS} \\
- A(B^0_s \to \pi^-K^+\bar{K}^0)_{FS} - A(B^0_s \to \pi^+K^-\bar{K}^0)_{FS} = 0.

4. \( B^0_s \to \pi\pi\pi \)

S4.1 = 2A(B^0_s \to \pi^0\pi^0\pi^0)_{FS} - A(B^0_s \to \pi^0\pi^+\pi^-)_{FS} = 0.

For \( \bar{b} \to \bar{d} \) induced \( B \to PPP \) amplitudes, we have

1. \( B \to \pi K\bar{K} \)

D1.1 = -\sqrt{2}A(B^0 \to \pi^0K^+K^-)_{FS} + A(B^0 \to \pi^+K^0\bar{K}^0)_{FS} - A(B^+ \to \pi^+K^+K^-)_{FS} \\
+ \sqrt{2}A(B^0 \to \pi^0K^0\bar{K}^0)_{FS} + A(B^0 \to \pi^-K^+\bar{K}^0)_{FS} \\
+ A(B^+ \to \pi^+K^0\bar{K}^0)_{FS} - \sqrt{2}A(B^+ \to \pi^0K^+\bar{K}^0)_{FS} = 0.

2. \( B \to \pi\pi\pi \)

D2.1 = 2A(B^0 \to \pi^0\pi^0\pi^0)_{FS} - A(B^0 \to \pi^+\pi^0\pi^-)_{FS} = 0, \\
D2.2 = 2A(B^+ \to \pi^+\pi^0\pi^0)_{FS} - A(B^+ \to \pi^-\pi^+\pi^+)_{FS} = 0.

3. \( B^0_s \to K\pi\pi \)

D3.1 = -2A(B^0_s \to \bar{K}^0\pi^0\pi^0)_{FS} + A(B^0_s \to K^0\pi^+\pi^-)_{FS} - \sqrt{2}A(B^0_s \to K^-\pi^+\pi^0)_{FS} = 0.

Note that we have different normalizations than those used in Ref.[9] for some of the final meson states and also identical particle combinatorial factors. One can easily obtain relations in the form in Ref. [8] by multiplying a "-1" to the amplitudes when \( \pi^- \), \( K^- \), \( \pi^0 \) appear each time as one of the final states, and a factor \( 1/\sqrt{2} \) and \( 1/\sqrt{6} \) in our formulation for the corresponding amplitudes, respectively, when the decays involve two and three identical particles.

III. \( SU(3) \) AND ISOSPIN BREAKING DUE TO QUARK MASS DIFFERENCES

The main source for flavor \( SU(3) \) symmetry breaking effects comes from difference in masses of \( u, d \) and \( s \) quarks. Under \( SU(3) \), the mass matrix can be viewed as combinations of representations from \( 3 \times 3 \), to matching the \( (u, d, s) \) transformation property as a fundamental representation, which contains an 1 and an 8 irreducible representations. The diagonalized mass matrix can be expressed as a linear combination of the identity matrix \( I \),
and the Gell-Mann matrices $\lambda_3$ and $\lambda_8$. We have

$$
\begin{pmatrix}
m_u & 0 & 0 \\
0 & m_d & 0 \\
0 & 0 & m_s
\end{pmatrix}
= \frac{1}{3}(m_u + m_d + m_s) I + \frac{1}{2}(m_u - m_d) X + \frac{1}{6}(m_u + m_d - 2m_s) W ,
$$

(3.1)

with $X$ and $W$ given by

$$
X = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
W = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 & -2
\end{pmatrix} .
$$

(3.2)

Compared with $s$-quark mass $m_s$, the $u$ and $d$ quark masses $m_{u,d}$ are much smaller, $SU(3)$ breaking effects due to a non-zero $m_s$ dominates the $SU(3)$ breaking effects. When up and down quark mass difference is neglected, the residual symmetry of $SU(3)$ becomes the isospin symmetry. In that case when studying $SU(3)$ breaking effects, the term proportional to $X$ can be dropped. The identity $I$ part contributes to the $B$ decay amplitudes in a similar way as that given in eq. (2.7) which can be absorbed into the coefficients $a(i)$ to $d(i)$. Only $W$ piece will contribute to the $SU(3)$ breaking effects. We will first discuss this case, and then also study the isospin breaking effects by including the term proportional to $X$.

### A. $SU(3)$ breaking due a non-zero $m_s$

To construct relevant decay amplitudes for $B \to PPP$ decays, one first breaks the contraction of indices at any joint in eq. (2.7), and inserts a $W$ in between, and then contracts all indices appropriately. For example corresponding to the first term in eq. (2.7), there are two ways to insert $W$,

$$
B_i H^a(\bar{3}) W^i_a M^i_k M^l_j , \quad B_i H^i(\bar{3}) M^i_k M^l_j W^k_a .
$$

(3.3)

The full list of possible independent terms are given in Appendix A of Ref.[6].

Extracting the $SU(3)$ breaking terms for $B^+\to K^0\pi^+\pi^0$ and $B^0\to K^+\pi^0\pi^-$ decays, we have the corrections for the decay amplitudes, $\Delta T$, as

$$
\Delta T(B^+\to K^0\pi^+\pi^0) = \sqrt{2} \left( c^T_1(6) + \sqrt{2} c^T_2(6) - 2 c^T_3(6) + \sqrt{2} c^T_4(6) + c^T_5(6) + c^T_1(15) 
+ c^T_2(15) - 2 c^T_3(15) + c^T_4(15) + d^T_1(6) + d^T_2(6) - 2 d^T_3(6) + d^T_4(6)
+ d^T_5(6) + d^T_1(15) + d^T_2(15) - 2 d^T_3(15) + d^T_4(15) + d^T_5(15) \right) ,
$$

(3.4)

and

$$
\Delta T(B^0\to K^+\pi^0\pi^-) = \Delta T(B^+\to K^0\pi^+\pi^0) ,
$$

(3.5)
which leads to the equality of the fully-symmetric amplitudes for these two decays. Therefore,

\[ S1.1 = A(B^0 \rightarrow K^+\pi^0\pi^-)_{FS} - A(B^+ \rightarrow K^0\pi^+\pi^-)_{FS} = 0, \]

still holds.

Note that even \( SU(3) \) breaking effects affect each of the decay amplitudes, the relation of the fully-symmetric amplitudes of these two decays are not affected. Expanding terms in Appendix B of Ref.[6], one can study which relations discussed above. We find that all the relations among the fully-symmetric amplitudes still hold, that is

\[ S1.1 = 0, \quad S2.1 = 0, \quad S1.3 = 0, \quad S2.1 = 0, \quad S3.1 = 0, \quad S4.1 = 0, \quad D1.1 = 0, \quad D2.1 = 0, \quad D2.2 = 0, \quad D3.1 = 0, \]

are still true even if one include \( SU(3) \) breaking effects due a non-zero strange quark mass. This actually is not a surprise because the relations discussed can be obtained by isospin symmetry considerations.

Experimental verification of these relations may provide important tests for the validity of flavor \( SU(3) \) for \( B \) decays.

B. Isospin breaking due to up and down quark mass difference

It would be interesting to investigate what happens when mass difference between up and down quark, which breaks isospin symmetry, is also included. We now discuss these isospin breaking effects for the relations discussed before.

One can obtain the corrections by replacing \( W \) by \( X \) in Appendix A of Ref.[4]. We indicate the coefficients in a similar way as that \( SU(3) \) breaking effects due to a non-zero \( m_s \), but with a superscript \( I \) to indicate the effects of isospin breaking, for example for tree operator corrections by \( a_{ij}^T, b_{ij}^T, c_{ij}^T, \) and \( d_{ij}^T \). The correction to the decay amplitude will also be indicated by a superscript \( I, \Delta T^I \).

Expanding all terms, we obtain the corrections due to isospin breaking effects for all the decay amplitudes discussed previously. We find that except that the relation S4.1 still holds, all other relations for the \( B \rightarrow PPP \) decay amplitudes induced by \( \bar{b} \rightarrow \bar{s} \) and \( \bar{b} \rightarrow \bar{d} \) interactions discussed earlier are broken.

In fact each of the decay modes relevant in S4.1 is affected by isospin breaking effects, but they are affected in such a way that the equality of the amplitudes is not affected. That is, we still have:

\[ S4.1 = 2A(B_s^0 \rightarrow \pi^0\pi^0\pi^0)_{FS} - A(B_s^0 \rightarrow \pi^0\pi^+\pi^-)_{FS} = 0. \]

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This makes this relation special so that test of this prediction is independent of SU(3)
breaking due to a non-zero strange quark and isospin breaking due to up and down quark
mass difference.

We also found some other interesting relations even isospin violating effects are included,
namely the corrections for some of the relations discussed above are related to others. For
$b \rightarrow s$ interaction induced decay modes, we have an additional relation which relate $S1.2$
and $S1.3$ because the isospin breaking effects satisfy

$$[\sqrt{2}\Delta T^I(B^+ \rightarrow K^{0}\pi^{+}\pi^{0}) - \Delta T^I(B^0 \rightarrow K^{0}\pi^{+}\pi^{-}) + 2\Delta T^I(B^0 \rightarrow K^{0}\pi^{+}\pi^{0})]$$

$$- [\sqrt{2}\Delta T^I(B^0 \rightarrow K^{+}\pi^{-}\pi^{0}) + \Delta T^I(B^+ \rightarrow K^{+}\pi^{+}\pi^{-}) - 2\Delta T^I(B^+ \rightarrow K^{+}\pi^{0}\pi^{0})].$$

(3.10)

Although the right hand sides of $S1.2$ and $S1.3$ are not zero anymore, the above relation
leads to,

$$S1.2 = -S1.3 \neq 0.$$  \hspace{1cm} (3.11)

For $b \rightarrow d$ interactions induced decay modes, we have

$$[2\Delta T^I(B^+ \rightarrow \pi^{0}\pi^{0}\pi^{+}) - \Delta T^I(B^+ \rightarrow \pi^{-}\pi^{+}\pi^{+})]$$

$$- [-2\Delta T^I(B^0_s \rightarrow K^{0}\pi^{0}\pi^{0}) + \Delta T^I(B^0_s \rightarrow K^{0}\pi^{+}\pi^{-}) - \sqrt{2}\Delta T^I(B^0_s \rightarrow K^{-}\pi^{+}\pi^{0})],$$

$$\sqrt{2}[2\Delta T^I(B^0 \rightarrow \pi^{0}\pi^{0}\pi^{0}) - \Delta T^I(B^0 \rightarrow \pi^{+}\pi^{-}\pi^{0})]$$

$$- [-2\Delta T^I(B^+ \rightarrow \pi^{0}\pi^{0}\pi^{+}) - \Delta T^I(B^+ \rightarrow \pi^{-}\pi^{+}\pi^{+})].$$

(3.12)

Due to isospin breaking effects, the right hand sides of $D2.1$, $D2.2$ and $D3.1$ are non-zero.
However, the above relations imply

$$\sqrt{2}D2.1 = -D2.2 \neq 0, \hspace{1cm} D2.2 = -D3.1 \neq 0.$$  \hspace{1cm} (3.13)

We would like to emphasize that since the above relations hold even when isospin effects
have been taken into account, they can provide useful information about $B$ decays in the
SM in a way independent of flavor $SU(3)$ and isospin breaking effects.

IV. CONCLUSIONS AND DISCUSSIONS

Flavor $SU(3)$ and isospin symmetries have been considered to be powerful tools in an-
alyzing B decays. Such analyses are usually hampered by a relatively large strange quark
mass which breaks $SU(3)$ symmetry down to isospin symmetry. The isospin symmetry also
breaks down when up and down quark mass difference is kept. It is therefore interesting to
find relations which are not sensitive to $SU(3)$ and isospin breaking effects. We have carried
out detailed analyses including $SU(3)$ and isospin breaking effects due to u, d and s quark
mass differences for $B \rightarrow PPP$ decays. We find that a class of relations in fully-symmetric
amplitudes are not broken by $SU(3)$ breaking effects due to a non-zero strange quark mass,
and the relations

$$S4.1 = 0, \hspace{1cm} S1.2 + S1.3 = 0, \hspace{1cm} \sqrt{2}D2.1 + D2.2 = 0, \hspace{1cm} D2.2 + D3.1 = 0.$$  \hspace{1cm} (4.1)
hold even isospin breaking effects due to up and down quark mass difference is included. Measurements for these relations will provide important information about B decays in the SM.

We would like to end the paper by commenting $SU(3)$ breaking effects on the $U$-spin symmetry relations in the following

$$T_{\Delta s=-1}(B^+ \to K^+ K^K^-) = T_{\Delta s=0}(B^+ \to \pi^+\pi^\pi^-),$$
$$T_{\Delta s=-1}(B^+ \to K^+\pi^+\pi^-) = T_{\Delta s=0}(B^+ \to \pi^+K^K^-).$$  \hspace{1cm} (4.2)

The momentum dependent terms also respect the above relations. The above equalities also hold for the fully-symmetric amplitudes for corresponding pairs of decay modes in the $SU(3)$ limit. These relations imply in the SM that the CP violating rate asymmetries defined by $A_{asy} = \Gamma(B \to PPP) - \Gamma(\bar{B} \to \bar{P}\bar{P}\bar{P})$ are equal but opposite in sign for each pair of decay modes above.

For the fully-symmetric amplitudes of these decays modes, we also have

$$T_{\Delta s=-1}(B^+ \to K^+ K^+K^-)^{FS} = T_{\Delta s=-1}(B^+ \to K^+\pi^+\pi^-)^{FS},$$
$$T_{\Delta s=0}(B^+ \to \pi^+\pi^\pi^-)^{FS} = T_{\Delta s=0}(B^+ \to \pi^+K^K^-)^{FS}.$$ \hspace{1cm} (4.3)

Unlike the other fully-symmetric amplitudes studied in previous sections, the relations in eq. (4.2) and eq. (4.3) are broken when $SU(3)$ breaking effects due to a non-zero strange quark mass is included. Therefore there may be sizeable deviation for these relations. Relations in eq. (4.2) have been discussed recently. It was found that indeed there are large $SU(3)$ breaking effects \cite{5,6}. The relations in eq. (4.2) and eq. (4.3) will not provide as much insight as those from the fully-symmetric amplitudes which still hold when isospin breaking effects are included discussed earlier.

However, we find that the $SU(3)$ breaking effects due to a non-zero strange quark mass and the isospin breaking effects due to the difference of up and down quark masses are equal for some of the above relations with

$$\Delta T(B^+ \to K^+\pi^+\pi^-) - \Delta T(B^+ \to K^+K^K^-)$$
$$= -[\Delta T(B^+ \to \pi^+K^K^-) - \Delta T(B^+ \to \pi^+\pi^\pi^-)]$$
$$= -[a_4^T(6) + 3a_4^T((15)) + b_3^T(3) + b_4^T(6) + 3b_4^T((15)) + c_2^T(3) - c_2^T(6) + c_3^T(6)$$
$$- c_5^T(6) - c_3^T((15)) - c_3^T((15)) - 2c_4^T((15)) + 3c_5^T((15)) - d_3^T(6) + 3d_4^T((15))].$$ \hspace{1cm} (4.4)

and the isospin breaking effects satisfy

$$\Delta T_I(B^+ \to K^+\pi^+\pi^-) - \Delta T_I(B^+ \to K^+K^K^-)$$
$$= -[\Delta T_I(B^+ \to \pi^+K^K^-) - \Delta T_I(B^+ \to \pi^+\pi^\pi^-)]$$
$$= -[a_4^T(6) + 3a_4^T((15)) + b_3^T(3) + b_4^T(6) + 3b_4^T((15)) + c_2^T(3) - c_2^T(6) + c_3^T(6)$$
$$- c_5^T(6) - c_3^T((15)) - c_3^T((15)) - 2c_4^T((15)) + 3c_5^T((15)) - d_3^T(6) + 3d_4^T((15)].$$ \hspace{1cm} (4.5)

The above leads to the following relation which is independent of $SU(3)$ breaking effects and isospin breaking effects due to strange, up and down quark mass differences,

$$T(B^+ \to K^+\pi^+\pi^-)^{FS} - T(B^+ \to K^+K^K^-)^{FS}$$
$$= T(B^+ \to \pi^+\pi^\pi^-)^{FS} - T(B^+ \to \pi^+K^K^-)^{FS} \neq 0.$$ \hspace{1cm} (4.6)
and similarly for penguin amplitudes $P_{FS}$.

When the relevant decay amplitudes are measured precisely, one can also obtain useful information for $B$ decays in the SM.

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[1] J.P. Lees et al. (BaBar Collaboration), arXiv:1305.4218[hep-ex].
[2] R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 111, 101801 (2013) [arXiv:1306.1246 [hep-ex]]; R. Aaij et al. [LHCb Collaboration], JHEP 1310, 143 (2013) [arXiv:1307.7648 [hep-ex]]; R. Aaij et al. [LHCb Collaboration], Phys. Rev. Lett. 112, no. 1, 011801 (2014) [arXiv:1310.4740 [hep-ex]]; R. Aaij et al. [LHCb Collaboration], arXiv:1408.5373 [hep-ex].
[3] N. G. Deshpande, G. Eilam, X. -G. He and J. Trampetic, Phys. Rev. D 52, 5354 (1995) [hep-ph/9503273]; S. Fajfer, T. -N. Pham and A. Prapotnik, Phys. Rev. D 70, 034033 (2004) [hep-ph/0405065]; H. -Y. Cheng and K. -C. Yang, Phys. Rev. D 66, 054015 (2002) [hep-ph/0205133]; H. -Y. Cheng, C. -K. Chua and A. Soni, Phys. Rev. D 76, 094006 (2007) [arXiv:0704.1049 [hep-ph]]; N. R. -L. Lorier, M. Imbeault and D. London, Phys. Rev. D 84, 034040 (2011) [arXiv:1011.4972 [hep-ph]].
[4] Z. -H. Zhang, X. -H. Guo and Y. -D. Yang, Phys. Rev. D 87, 076007 (2013) [arXiv:1303.3676 [hep-ph]]; Z. H. Zhang, X. H. Guo and Y. D. Yang, arXiv:1308.5242 [hep-ph].
[5] M. Gronau, Phys. Lett. B 727, 136 (2013) [arXiv:1308.3448 [hep-ph]].
[6] D. Xu, G. N. Li and X. G. He, Int. J. Mod. Phys. A 29, 1450011 (2014) [arXiv:1307.7186 [hep-ph]].
[7] D. Xu, G. -N. Li and X. -G. He, Phys. Lett. B 728, 579 (2014) [arXiv:1311.3714 [hep-ph]].
[8] H. Y. Cheng and C. K. Chua, Phys. Rev. D 88, 114014 (2013) [arXiv:1308.5139 [hep-ph]]; H. Y. Cheng, Nucl. Phys. Proc. Suppl. 246-247, 109 (2014); W. F. Wang, H. C. Hu, H. n. Li and C. D. L, Phys. Rev. D 89, 074031 (2014) [arXiv:1402.5280 [hep-ph]]; Y. Li, arXiv:1401.5948 [hep-ph]; Y. Li, Phys. Rev. D 89, 094007 (2014) [arXiv:1402.6052 [hep-ph]].
[9] B. Bhattacharya, M. Gronau, M. Imbeault, D. London and J. L. Rosner, Phys. Rev. D 89, 074043 (2014) [arXiv:1402.2909 [hep-ph]].

[10] D. Zeppenfeld, Z. Phys. C 8, 77 (1981); M. J. Savage and M. B. Wise, Phys. Rev. D 39 (1989) 3346 [Erratum-ibid. D 40 (1989) 3127]; L. -L. Chau, H. -Y. Cheng, W. K. Sze, H. Yao and B. Tseng, Phys. Rev. D 43, 2176 (1991) [Erratum-ibid. D 58, 019902 (1998)]; M. Gronau, O. F. Hernandez, D. London and J. L. Rosner, Phys. Rev. D 50, 4529 (1994).

[11] A. S. Dighe, M. Gronau and J. L. Rosner, Phys. Rev. Lett. 79, 4333 (1997); X. G. He, Y. K. Hsiao, J. Q. Shi, Y. L. Wu and Y. F. Zhou, Phys. Rev. D 64, 034002 (2001); H. -K. Fu, X. -G. He, Y. -K. Hsiao and J. -Q. Shi, Chin. J. Phys. 41, 601 (2003); H. -K. Fu, X. -G. He and Y. -K. Hsiao, Phys. Rev. D 69, 074002 (2004). X. -G. He and B. Mckellar, arXiv: hep-ph/0410098; C -W Chiang, M. Gronau, J. Rosner and D. Suprun, Phys. Rev. D70, 034020(2004); C. -W. Chiang, M. Gronau, Z. Luo, J. Rosner and D. Suprun, Phys. Rev. D69, 034001(2004).

[12] N. G. Deshpande and X. -G. He, Phys. Rev. Lett. 75, 1703 (1995); X. -G. He, Eur. Phys. J. C 9, 443 (1999); X. G. He, S. F. Li and H. H. Lin, JHEP 1308, 065 (2013) [arXiv:1306.2658 [hep-ph]].

[13] R. Fleischer, Eur. Phys. J. C 51, 849 (2007); M. Gronau and J. L. Rosner, Phys. Lett. B 482, 71 (2000); M. A. Dariescu, N. G. Deshpande, X. -G. He and G. Valencia, Phys. Lett. B 557, 60 (2003); M. Beneke, eConf C 0304052, FO001 (2003) [hep-ph/0308040]; N. G. Deshpande, X. -G. He and J. -Q. Shi, Phys. Rev. D 62, 034018 (2000); A. Ali, G. Kramer, Y. Li, C. -D. Lu, Y. -L. Shen, W. Wang and Y. -M. Wang, Phys. Rev. D 76, 074018 (2007).

[14] G. Buchalla, A. J. Buras and M. E. Lautenbacher, Rev. Mod. Phys. 68, 1125 (1996); M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Nucl. Phys. B 415, 403 (1994); N. G. Deshpande and X. -G. He, Phys. Lett. B 336, 471 (1994).