Self-stabilised fractality of sea-coasts through damped erosion

B. Sapoval\textsuperscript{1,2}, A. Baldassarri\textsuperscript{1,3}, A. Gabrielli\textsuperscript{4}
\textsuperscript{1}Laboratoire de Physique de la Matière Condensée, C.N.R.S. Ecole Polytechnique, 91128 Palaiseau, France.
\textsuperscript{2}Centre de Mathématiques et de leurs Applications, Ecole Normale Supérieure, 94235 Cachan, France.
\textsuperscript{3}INFM, UdR Roma 1, Dipartimento di Fisica, Università di Roma "La Sapienza", P.le Aldo Moro 2, 00185 Rome, Italy.
\textsuperscript{4}"Enrico Fermi" Center Via Panisperna 89 A, Compendio del Viminale, Palaz. F, 00184 Rome, Italy.

(Dated: December 21, 2021)

Erosion of rocky coasts spontaneously creates irregular seashores. But the geometrical irregularity, in turn, damps the sea-waves, decreasing the average wave amplitude. There may then exist a mutual self-stabilisation of the waves amplitude together with the irregular morphology of the coast. A simple model of such stabilisation is studied. It leads, through a complex dynamics of the earth-sea interface, to the appearance of a stationary fractal seacoast with dimension close to $4/3$. Fractal geometry plays here the role of a morphological attractor directly related to percolation geometry.

Coastline morphology is of current interest in geophysical research\textsuperscript{1}, and coastline erosion has important economic consequences\textsuperscript{2}. Also, the recent concern about global warming has increased the demand for a better understanding of coastal evolution. At the same time, although the geometry of seacoasts is often used as an introductory archetype of fractal morphology in nature\textsuperscript{3,4}, there has been no explanation about which physical mechanism could justify that empirical observation. In the field litterature\textsuperscript{5}, one can read: "As a matter of some urgency, researchers concerned with coastal evolution should consider the alternative models, even if there are few supporting data. The ideas of... stochastic development, ... and criticality, all deserve investigation."

The present work propose a minimal, but robust, model of evolution of rocky coasts towards fractality.

The model describes how a stationary fractal geometry can arises spontaneously from the mutual self-stabilization of coast morphology and sea eroding power\textsuperscript{6}. If, on one hand, erosion generally increases the geometrical irregularity of the coast, on the other hand this increase creates a stronger damping of the sea and a consequent diminution of its eroding power. The increased damping argument relies on the studies of fractal acoustical cavities, which have shown that viscous damping is augmented on a longer, irregular, surface\textsuperscript{7,8}. In the following, a minimal two-dimensional model of erosion is introduced which leads to the spontaneous evolution of a smooth seashore towards fractality as shown in Fig. 1.

Rocky coasts erosion is the product of marine and atmospheric causes\textsuperscript{9}. There exist many different erosion processes: wave quarrying, abrasion, wetting and drying, frost shattering, thermal expansion, salt water corrosion, carbonation, hydrolysis. A simplified picture is used here by assuming that the different processes can be separated into two categories: “rapid” mechanical erosion (namely wave quarrying) and “slow” chemical weakening. The justification is that mechanical erosion generally occurs rapidly, mainly during storms, after rock has been altered and weakened by the slow weathering processes. We first study the supposedly rapid erosion mechanisms. Then we show that the full complex dynamics, involving fast and slow processes, build a dynamic equilibrium that preserves the fractal shape of the coast.

The sea, together with the coast, is considered to constitute a resonator. It is assumed that there exists an average excitation power of the waves $P_0$. The “force” acting on the unitary length of the coast is measured by the square of the wave amplitude $\Psi^2$. This wave amplitude is related to $P_0$ by a relation of the type $\Psi^2 \sim P_0 Q$ where $Q$ is the morphology dependent quality factor of
the system; the smaller the quality factor, the stronger the damping of the sea-waves. There are several causes to waves damping. Since the different loss mechanisms occur independently, the quality factor satisfies a relation of the type

\[ \frac{1}{Q} = \frac{1}{Q_{\text{coast}}} + \frac{1}{Q_{\text{other}}}, \]

where \( Q_{\text{coast}} \) is the quality factor due to the viscous dissipation of the fluid moving along the coast and the nearby islands and \( Q_{\text{other}} \) is related to other damping mechanisms (e.g. bulk viscous damping). Studies of fractal acoustical cavities have shown that the viscous damping increases roughly proportionally to the cavity perimeter. Therefore, one can, in first approximation, assume that \( Q_{\text{coast}} \) is inversely proportional to the coast perimeter \( L_p(t) \) whereas \( Q_{\text{other}} \) is independent of the coast morphology. In other words, the sea exerts a homogeneous erosion force \( f(t) \) on each coast element proportional to \( \Psi^2(t) \):

\[ f(t) = \frac{f_0}{1 + \frac{g L_p(t)}{L_0}}. \]

where \( L_p(t) \) is the total length of the coast at time \( t \) (then \( L_p(t = 0) = L_0 \)). The factor \( g \) measures the relative contribution to damping of a flat shore as compared with the total damping. Since \( g \) represents the importance of coast dissipation with respect to the other mechanisms, which are thought to be dominant, computations are performed under the condition \( g \ll 1 \). The quantity \( f_0 \) is the renormalized value of \( P_1 \) such that \( f(t) < 1 \) at all \( t \).

The mechano-chemical properties of the rocks constituting the coast, which are linked to structure and composition defining their “lithology”, are unknown and exhibit some dispersion. The “resisting” random earth is then modelled by a square lattice of random units of width \( L_0 \). Each site represents a small portion of the earth. The sea acts on a shoreline constituted of cells, each one characterised by a random number \( x_i \), between 0 and 1, representing its lithology. The erosion model should also take into account the fact that a site surrounded by the sea is relatively weakened as compared with a site surrounded by earth or other coast sites. Hence, the resistance to erosion \( r_i \) of a site depends on both its lithology and the number of sides exposed to the coastline, for \( f(t = 0) = 0.65 \) and different values of the scale gradients \( g \). The dynamics spontaneously stops at a value weakly dependent on \( g \). Data refer to systems with \( L_0 = 1000 \). Each curve is obtained by averaging over ten different realisations of the earth lithology distribution.

![FIG. 2: Illustration of the erosion process. The thick numbers at the square centres represent the lithology \( \{x_i\} \). The numbers in the corners are the corresponding resistances \( r_i \) which depend on the local environment as explained in the text. The sites marked with 1 in the lower corner are earth sites with no contact with the sea. On the left and on the right respectively the situations before and after a single erosion step with \( f(t) = 0.5 \). After this step resistances are updated due to the new sea environment.](image)

![FIG. 3: Time dependence of the erosion force \( f(t) \). The figure shows the evolution of the sea erosion force acting on the coastline, for \( f(t = 0) = 0.65 \) and different values of the scale gradients \( g \). The dynamics spontaneously stops at a value weakly dependent on \( g \). Data refer to systems with \( L_0 = 1000 \). Each curve is obtained by averaging over ten different realisations of the earth lithology distribution.](image)
Zero wave power is remarkable. It is a direct consequence of random percolation \[^{12}\] as discussed below. At time \(f(t_f)\) the coastline is fractal (see Figs. \[^{13}\] and \[^{14}\]) with dimension \(D_f = 1.332(3)\), very close to 4/3, up to a characteristic width \(\sigma\). This width \(\sigma\) is defined as the standard deviation of the final coastline depth \((y - Y)^{1/2}\)

\[ (\text{where } Y = \langle y \rangle \text{ is the average erosion depth}) \]

Fig. 3 shows the box-counting determination of the fractal dimension. The fact that \(D_f \approx 4/3\) confirms the relation between rapid erosion and percolation, as 4/3 is the dimension of the so-called accessible percolation cluster \[^{11}\]. Note that this value is also close to the observed fractal dimension for many seacoasts, as the United States eastern shore \[^{4}\].

A more detailed study of the coast width \(\sigma\) suggests that the model falls in the universality class of Gradient Percolation (GP) \[^{12}\]. As shown in Fig. 5 the coast width \(\sigma\) follows a scaling law with respect to \(g\)

\[ \sigma \sim g^{-\alpha_\sigma} \]  

with \(\alpha_\sigma \approx 4/7\). This law is characteristic of GP where sites are occupied at random with an occupation probability that varies in space along one fixed direction between 1 and 0 with a constant gradient \(g\), and \(\sigma\) is the width of the frontier of the infinite cluster. The fact that here \(g\) is proportional to a gradient of occupation probability is explained by the following argument. At time \(t\), the erosion power is \(f(t)\) while the sea has eroded the earth up to an average depth \(Y(t)\), an increasing function of \(t\). Inverting this function, \(f\) can be written as \(f(Y(t))\). There exists then a spatial gradient of the occupation probability by the sea. For small enough \(g\) one can write \(|df/dY| = |(g/L_0)|dL_p(Y)/dY|\). The quantity \(dL_p(Y)/dY\) is a function of \(g\) but to the lowest order it is a constant independent of \(g\) since even with \(g = 0\), there will be an erosion due to randomness and a consequent perimeter evolution \(L_p(t)\). Then to lowest order, the real gradient \(df/dY\) is linear in \(g\), which can then be called the “scale gradient”. Note that the formation of a fractal interface due to the spontaneous appearance of a gradient has already been observed in the corrosion of an aluminium film \[^{13}\], \[^{14}\].

Of course, the real dynamics of the coasts are more complex than the “rapid” process considered above. They result from the interplay with the slow weathering processes, generally attributed to carbonation or hydrolysis. These processes act on longer, geological, time scales. In order to simulate this evolution, the lithology parameter \(x_i\) of all the coast sites is decreased by a small fraction \(\epsilon\), i.e. \(x_i' = (1-\epsilon)x_i\) with \(\epsilon \ll 1\) after the erosion has stopped at \(t_f\). One or a few coast sites then become weaker than \(f(t_f)\) and the rapid erosion dynamics starts again up to a next arrest time. This procedure is then iterated. At each restart of erosion, a finite portion of the earth is eroded. The repeated effect of the “slow” weathering mechanisms gives a fluctuating behavior of \(f(t)\) and generates a coastline erosion drift with fluctuations or avalanches, but it does not alter the fractal features of the coast (see Figure 1 lower row). In that sense fractal geometry plays the role of a statistical attractor. In the language of coastal studies \[^{17}\], the system state evolves through a dynamical equilibrium where small perturbations (even without special trigger events) may stimulate large fluctuations and avalanche dynamics. This is due to the underlying criticality of percolation systems. In the authors’ mind, it is this stationary regime which corresponds to the geomorphologic observations of fractal seacoasts.

The above simple model presents limitations, but the simplifications that keep the system in the universality class of percolation are unimportant. The use of different weakening rules would eventually modify the dynamics but not the final fractal dimension. The time separation between rapid erosion and slow chemical weakening is somewhat arbitrary as both mechanisms can occur simultaneously without changing the coast fractality. Also a better model for damping should take care of a wave fre-
quency dependence as well as it should consider the existence of localization by the frontier of the waves along the irregular coast \(^1\). This would modify Eq. (2) and change the time evolution. However, since percolation possesses the universality properties of phase transitions \(^{10}\), the fractal dimension of the coast should not depend on these factors. In the context of corrosion dynamics \(^{17}\), this has been shown through arguments from dynamical field theory of absorbing states \(^{16}\). If large-scale modifications or correlations exist in the lithology properties, then the resulting geometry would be more complex. But our main result, namely the existence of irregular coasts as a result of a self-stabilisation mechanism, would remain correct even though the geometry would be more complex than that of critical percolation. We believe that it is in those terms that can be interpreted the results of several detailed study of the self-similarity properties of sea-coasts \(^{12,13}\).

A more radical change in the geometry is expected if sediments transport is taken into account. In our simplified model, the sediments are supposed to be transported offshore and disappear, while in the real erosion process these sediments are partially transported along the coast. However our approximation is justified by some examples, like the North Coast of Oregon (USA), a leading hedge coast dominated by erosion with very limited coastal deposits \(^{17}\).

One should also mention that it has been found very recently that the GP power laws apply even when the front is too narrow to be fractal \(^{20}\). This extends considerably the range of application of GP to rocky coasts which are irregular, but not to a fractal range.

This work has presented a minimal model for the formation of fractal rocky coast morphology. This model bears on the reciprocal evolution of the erosion power and the topography of the coast submitted to that erosion: The more irregularly eroded the coast is, the weaker the average sea erosion power. Note that this seems to be an empirically known effect used to build efficient breakwaters that are based on hierarchical accumulations of tetrapods piled over layers of smaller and smaller rocks, in close analogy with fractal geometry \(^{21}\). The retroaction leads to the spontaneous formation of a fractal sea-coast with a fractal dimension \(D_f = 4/3\). The fractal geometry plays here the role of a morphological attractor: whatever its initial shape, a rocky shore will end fractal when submitted to such type of erosion, forgetting its initial morphology. Note that our model suggests that, on the field, the islands which have resisted to an erosion under a force larger that the final force \(f(t_f)\), are stronger that the coast itself. This could be verified on the historical data of known seacoasts and neighbouring islands evolutions. The model reproduces at least qualitatively some of the features of real coasts using only simple ingredients: the randomness of the lithology and the decrease of the erosion power of the sea. It is worth to be noted that the use of simple geophysical ingredients leads to an evolution towards a self-organized fractality directly related to percolation theory.

A.G. wish to acknowledge the Departement of Physics of the University of Rome “La Sapienza” (Italy) for supporting this research.

[1] Eric C. F. Bird, *Coasts* (Van Nostrand Rheinhold Co., New York, 1984). Eric C. F. Bird and M. L. Schwartz (eds), *The World Coastline* (Van Nostrand Rheinhold Co., New York, 1985).
[2] E. C. Penning-Roswell, C. H. Green, P. M. Thompson, A. M. Coker, S. M. Tunstall, C. Richards, and D. J. Parker, *The Economics of Coastal Management* (Belhaven Press, London, 1992).
[3] B. B. Mandelbrot, Science **155**, 636 (1967), and *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[4] C. C. Barton, and C. Dufore, *Fractal Dimension of the Coterminous United States coastline: Scale 1:5,000,000, U.S. Geological Survey Miscellaneous Investigations Series Map I-XXX*, in press and http://coastal.er.usgs.gov/barton/pubs/fractalmap.pdf.
[5] *Coastal Evolution. Late Quaternary shoreline morphodynamics* Ed. R.W.G.Carter and C.D. Woodroffe (Cambridge University Press 1994).
[6] B. Sapoval, *Fractals* (Aditech, Paris, 1989).
[7] B. Sapoval, O. Haerblé, and S.Russ, J. Acoust. Soc. Am., 102, 2014 (1997).
[8] B. Hébert B., B. Sapoval, and S.Russ , J. Acoust. Soc. Am., 105, 1567 (1999).
[9] R. A. Davis, Jr, *Oceanography - An Introduction to the Marine Environment,* (Wm. C. Brown Publ., Dubuque, Iowa, 1986).
[10] D. Stauffer. and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1991).
[11] T. Grossman and A. Aharony, J. Phys. A, 20, L1193-1201 (1987).
[12] B. Sapoval, M. Rosso and J. F. Gouyet, J. Phys. Lett. (Paris), 46, L149 (1985).
[13] L. Balazs, Phys. Rev. E, **54**, 1183-1189 (1996).
[14] B. Sapoval, S. B. Santra, and Ph. Barboux, Europhys. Lett., 41, 297 (1998).
[15] A. Gabrielli, A. Baldassarri, and B. Sapoval, Phys. Rev. E, **62**, 3103-3115, (2000).
[16] A. Gabrielli, M. A. Munoz, and B. Sapoval, Phys. Rev. E, **64**, 016108-1, (2001).
[17] S. K. Haslett, *Coastal Systems* (Routledge, Taylor and Francis, London, 2000)
[18] M. F. Goodchild, Math. Geology, **85** (1980)
[19] R. Andre, Earth Surface Processes and Landforms, 21, 955(1996).
[20] A. Desolneux, B. Sapoval, and A. Baldassarri, in *Fractal Geometry and Applications*, Proc. Symp. Pure Math. (PSPUM) in print. See cond-mat/0302072.
[21] *Shore protection manual* (Dept. of the Army Waterways Exp. Station, Vicksburg, Mississippi, 2, 1984).