RATIONAL REPRESENTATIONS AND PERMUTATION REPRESENTATIONS OF FINITE GROUPS

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Abstract. We investigate the question which $\mathbb{Q}$-valued characters and characters of $\mathbb{Q}$-representations of finite groups are $\mathbb{Z}$-linear combinations of permutation characters. This question is known to reduce to that for quasi-elementary groups, and we give a solution in that case. As one of the applications, we exhibit a family of simple groups with rational representations whose smallest multiple that is a permutation representation can be arbitrarily large.

1. Introduction

Many rational invariants of a finite group $G$ are encoded in the ring $\operatorname{Char}_\mathbb{Q}(G)$ of rationally-valued characters, the ring $R_{\mathbb{Q}}(G)$ of rational representations, and the ring $\operatorname{Perm}(G)$ of virtual permutation representations. All three have the same $\mathbb{Z}$-rank, and there are natural inclusions with finite cokernels

$$\operatorname{Perm}(G) \rightarrow R_{\mathbb{Q}}(G) \rightarrow \operatorname{Char}_\mathbb{Q}(G).$$

The quotient $\operatorname{Char}_\mathbb{Q}(G)/R_{\mathbb{Q}}(G)$ is studied by the theory of Schur indices, and the purpose of this paper is to investigate the other two,

$$C(G) = \frac{R_{\mathbb{Q}}(G)}{\operatorname{Perm}(G)} \quad \text{and} \quad \hat{C}(G) = \frac{\operatorname{Char}_\mathbb{Q}(G)}{\operatorname{Perm}(G)}.$$

They have exponent dividing $|G|$ by Artin’s induction theorem, and Serre remarked that $C(G)$ need not be trivial [14, Exc. 13.4]. It is trivial for $p$-groups [6, 12, 13], and it is known for nilpotent groups [11] (see also [2], for Weyl groups of Lie groups [15, 9] and in other special cases [2, 7]. It follows from the general results of Dress, Kletzing, and Hambleton-Taylor-Williams [4, 5, 9, 8], that the study of $C(G)$ for a group $G$ reduces, in principle, to that of its quasi-elementary subgroups, or of its ‘basic’ quasi-elementary subquotients. Specifically, for subgroups the statement is that of the two maps

$$\prod_{Q \leq G \text{ quasi-elem.}} C(Q) \xrightarrow{\text{Ind}} C(G) \xrightarrow{\text{Res}} \prod_{Q \leq G \text{ quasi-elem.}} C(Q),$$

the first one is surjective and the second one injective, and similarly for $\hat{C}$. This is also an immediate consequence of Solomon’s induction theorem, see [3].

Our first observation is that the composite map allows us to describe $C(G)$ and $\hat{C}(G)$ explicitly, in a way that bypasses the representation theory of $G$ — purely in terms of quasi-elementary subgroups and the ‘Res Ind’ maps between them; in fact, it is enough to consider maximal quasi-elementary subgroups, i.e. $p$-normalizers of cyclic subgroups of $G$. In [3] we give a simple formula for the Res Ind maps, and in [4] we prove one of the main results of the paper, which describes $C(Q)$ and $\hat{C}(Q)$ for a $p$-quasi-elementary group $Q = C \rtimes P$.

Its simplest formulation is:
Theorem 1.1 (=Theorem 4.6). Let \( \rho \) be a rational irreducible representation of a \( p \)-quasi-elementary group \( Q = C \rtimes P \). (So \( C \) is cyclic, \( P \) a \( p \)-group, and \( p \mid |C| \).

The order of \( \rho \) in \( C(G) \) is
\[
\frac{\dim \hat{\rho} \dim \hat{\pi}}{\dim \rho},
\]
where \( \hat{\rho} \) is the (unique) rational irreducible constituent of \( \rho|_C \) and \( \hat{\pi} \) the rational irreducible constituent of \( \rho|_P \) of minimal dimension.

Together with the aforementioned ‘Res Ind’ formula, it gives a way to compute \( C(G) \) and \( \hat{C}(G) \) efficiently in a given finite group \( G \). Incidentally, it also gives an algorithm to find \( \text{Perm}(G) \subset R_q(G) \) without computing the subgroup lattice. In \( \S 3 \) and \( \S 4 \) we illustrate applications of this approach to proving both triviality and non-triviality of \( C(G) \), as we shall now describe.

In general, \( C(G) \) remains somewhat mysterious, especially in non-soluble groups. Already Frobenius knew that \( C(A_n) \) is trivial for all \( n \). It was announced by Solomon in \( [15] \) that \( C(\text{PSL}_2(\mathbb{F}_q)) \) is trivial for all prime powers \( q \). In \( \S 5 \) we explain how this, and the same statement for \( \text{GL}_2(\mathbb{F}_q) \) and \( \text{PGL}_2(\mathbb{F}_q) \), follow from the results of \( \S 3 \) and \( \S 4 \).

There is, to our knowledge, no example in the literature of a simple group with non-trivial \( C(G) \). In \( \S 6 \) we show:

Theorem 1.2 (=Theorem 6.1 and Corollary 6.6). The exponent of the 2-part of \( C(G) \) is unbounded in the families \( G = \text{PSL}_k(\mathbb{F}_p) \) and \( G = \text{SL}_k(\mathbb{F}_p) \). Moreover, \( \hat{C}(\text{PSL}_k(\mathbb{F}_p)) \neq \{1\} \) for all even \( k \geq 4 \) and all odd primes \( p \).

Notation. Throughout the paper, \( G \) denotes a finite group. We write

\[
\begin{align*}
\text{Char}(G) &= \text{the character ring } G, \\
\text{Char}_{\mathbb{Q}}(G) &= \text{the ring of } \mathbb{Q}\text{-valued characters}, \\
R_q(G) &= \text{the ring of characters of virtual } \mathbb{Q}G\text{-representations}, \\
\text{Perm}(G) &= \text{the ring of characters of virtual permutation representations}, \\
C(G) &= R_q(G)/\text{Perm}(G), \\
\hat{C}(G) &= \text{Char}_{\mathbb{Q}}(G)/\text{Perm}(G), \\
\mathbb{Q}(\chi) &= \text{the field of of character values of } \chi, \\
m(\chi) &= \text{the Schur index of a complex character } \chi \text{ over } \mathbb{Q}(\chi).
\end{align*}
\]

For a complex character \( \chi \) of \( G \), define its trace and, when \( \chi \) is irreducible, its rational hull as

\[
\begin{align*}
\text{Tr} \chi &= \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \chi^\sigma \in \text{Char}_{\mathbb{Q}}(G), \\
\hat{\chi} &= m(\chi) \text{Tr} \chi \in R_q(G).
\end{align*}
\]

If \( \chi \) is irreducible, then \( \text{Tr} \chi \) is a \( \mathbb{Q} \)-irreducible character and \( \hat{\chi} \) a rational irreducible representation. We write

\[
\begin{align*}
\text{Irr}(G) &= \text{the set of (complex) irreducible characters of } G, \\
\text{Irr}_{\mathbb{Q}}(G) &= \text{the set of } \mathbb{Q}\text{-irreducible characters of } G, \\
\mu(\alpha, \beta) &= \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \text{multiplicity of } \alpha \text{ in } \beta, \\
&\quad \text{used for characters } \alpha \in \text{Irr}_{\mathbb{Q}}(G), \beta \in \text{Char}_{\mathbb{Q}}(G), \text{ and} \\
&\quad \text{also for rational representations } \alpha, \beta \text{ with } \alpha \text{ irreducible.}
\end{align*}
\]

We write \( x \sim y \) for conjugate elements. A \( p \)-quasi-elementary group is one of the form \( G = C \rtimes P \) with \( C \) cyclic, and \( P \) a \( p \)-group; throughout the paper we adopt the convention that \( p \nmid |C| \).
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2. Basic facts

Lemma 2.1. An inclusion $N \triangleleft G$ induces injections $C(G/N) \to C(G)$, $\hat{C}(G/N) \to \hat{C}(G)$.

Proof. Suppose $\bar{\rho}$ is a representation of $G/N$, which lifts to $\rho \in \text{Perm} G$. Write

$$\rho = \bigoplus C[G/H_{i}]^{ \oplus n_{i}}, \quad n_{i} \in \mathbb{Z}.$$ 

For a subgroup $H < G$ recall that $C[G/H]^{N} \cong C[G/NH]$, as $G$-representations (see e.g. [3], proof of Thm. 2.8(5)). Therefore,

$$\bar{\rho} = \rho^{N} = \bigoplus C[G/NH_{i}]^{ \oplus n_{i}} \in \text{Perm}(G/N),$$

as required. □

Lemma 2.2. Let $\rho$ be an irreducible rational representation and $\tau \in \text{Irr} G$ its constituent, so $\text{Tr} \tau \in \text{Irr} \mathbb{Q}(G)$ and $\rho = m(\tau) \text{Tr} \tau$. The order of $\text{Tr} \tau$ in $\hat{C}(G)$ is $m(\tau)$ times the order of $\rho$ in $C(G)$.

Proof. Clear from the definitions of $C(G)$ and $\hat{C}(G)$. □

This allows us to immediately deduce results about $\hat{C}(G)$ from those about $C(G)$, and conversely.

Nilpotent groups. Some statements seem to have a cleaner formulation for $C(G)$ than for $\hat{C}(G)$, and for some it is the other way around. Let us briefly illustrate this with an example of nilpotent groups:

Theorem 2.3 (Rasmussen [11] Thm 5.2). Let $G = G_{2} \times G_{2}'$ be a nilpotent group, where $G_{2}$ is its Sylow 2-subgroup. Then $C(G)$ is trivial, unless $G_{2}' \neq \{1\}$ and there exists an irreducible character $\chi$ of $G_{2}$ with $m(\chi) = 2$ and such that one of the following holds:

1. $\mathbb{Q}(\chi) \neq \mathbb{Q}$, or
2. $\mathbb{Q}(\chi) = \mathbb{Q}$ and there exists a prime divisor $q$ of $|G|$ such that the order of $2$ (mod $q$) is even.

The conditions turn out to be much simpler if one transforms this into a result about $\hat{C}(G)$. The following follows easily from [11] Thm. 4.2 and standard facts about Schur indices:

Theorem 2.4. Let $\chi = \chi_{2}\chi_{2}'$ be an irreducible character of a nilpotent group $G = G_{2} \times G_{2}'$ as above. Then the order of $\text{Tr} \chi$ in $\hat{C}(G)$ is $m(\chi_{2})$ (which is 1 or 2).
Metabelian and supersolvable groups. The following theorem will be of central importance in what follows. It implies that knowing the order of every $\mathbb{Q}$-irreducible representation in $\hat{C}(G)$ determines the structure of $\hat{C}(G)$ completely when $G$ is metabelian or supersoluble (e.g. nilpotent or quasi-elementary). It does not hold in arbitrary groups, as first noted by Berz [2]; the smallest counterexample is $G = C_3 \times SL_2(F_3)$.

**Theorem 2.5** (Berz [2]). If $G$ is metabelian or supersoluble, then $\text{Perm}(G) \subseteq R_{\mathbb{Q}}(G)$ is freely generated by $n_p \rho$, as $\rho$ ranges over irreducible rational representations of $G$, and

$$n_p = \gcd_{H \leq G} \mu(\rho, \mathbb{Q}[G/H]).$$

**Lemma 2.6.** If $G = A \times V$ with $A$ abelian and $V$ an elementary abelian $p$-group, then $\hat{C}(G) = \{1\}$.

**Proof.** By Theorem 2.5, it is enough to show that every complex irreducible character $\tau$ of $G$ occurs exactly once in $\mathbb{C}[G/H]$ for a suitable $H < G$. This is clear when $\dim \tau = 1$. Otherwise $\tau = \text{Ind}_{AU}^G \chi$, for some subgroup $U$ of $V$ and a $1$-dimensional character $\chi$ of $AU$ (see [13, Part II, §8.2]). Let $H$ be a subgroup of $V$ that is complementary to $U$, i.e. $HU = V$ and $H \cap U = \{1\}$. By Mackey’s formula, we have

$$\langle \tau, \mathbb{C}[G/H] \rangle = \langle \chi, \text{Res}_{AU} \text{Ind}_{H}^{G} 1 \rangle = \langle \chi, \text{Ind}_{AU \cap H}^{AU} 1 \rangle = \langle \chi, \mathbb{C}[AU] \rangle = 1.$$  

Recall that a $p$-quasi-elementary group $G = C \times P$ is basic if the kernel $K$ of $P \rightarrow \text{Aut}(C)$ is trivial or isomorphic to $D_8$ or has normal $p$-rank one.

**Proposition 2.7** ([7], Proposition 5.2). Let $G = C \times P$ be basic $p$-quasi-elementary. Let $A_p$ be a maximal cyclic subgroup of $K = \ker(P \rightarrow \text{Aut}(C))$ that is normal in $P$ (it is all of $K$ if $K$ is cyclic, and has index 2 in $K$ otherwise), let $A = CA_p$, and let $\chi$ be a faithful one-dimensional character of $A$. Then $\rho = \text{Tr} \text{Ind}_{A}^{G} \chi$ is a $\mathbb{Q}$-irreducible character, and

$$\text{order of } \rho \text{ in } \hat{C}(G) = \frac{|P|}{|A_p|} \cdot \max_{H \leq G, A_p \cap H = 1} |H|.$$  

3. $\hat{C}(G)$ as a Mackey functor

Let $\mathcal{R}$ be a Mackey subfunctor of the character ring Mackey functor $\text{Char}(G)$. This simply means that for any finite group $G$, $\mathcal{R}(G)$ is a subgroup of $\text{Char}(G)$ such that if $H \leq G$ are finite groups, then

- for all $\rho \in \mathcal{R}(H)$, $\text{Ind}_{H}^{G} \rho \in \mathcal{R}(G)$,
- for all $\tau \in \mathcal{R}(G)$, $\text{Res}_{H} \tau \in \mathcal{R}(H)$,
- for all $\rho \in \mathcal{R}(H)$ and $g \in G$, $\rho^g \in \mathcal{R}(H^g)$.

Here are some examples:

- $R_K(G)$, the representation ring of $G$ over a fixed subfield $K$ of $\mathbb{C}$,
- $\text{Char}_K(G)$, the ring generated by $K$-valued characters, with fixed $K \subset \mathbb{C}$,
- $\text{Perm}(G)$, the ring generated by permutation characters,
- the subgroup of $\text{Char}(G)$ generated by characters of degree divisible by a fixed integer $n$.

If $p$ is a prime number, write $\mathcal{R}(G)_p$ for $\mathcal{R}(G) \otimes \mathbb{Z}_p$. 

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**thm:Berz**

**A:Cp**

**prop:mainresfaithful**

**sMackey**
Proposition 3.1. Let $G$ be a finite group, fix a prime number $p$, and let $\mathcal{F}_p$ be a family of subgroups of $G$ such that every $p$-quasi-elementary subgroup of $G$ is conjugate to a subgroup of some $Q \in \mathcal{F}_p$. Then
\[
\prod_{Q \in \mathcal{F}_p} \text{Res}_Q : \mathcal{R}(G)_p \to \prod_{Q \in \mathcal{F}_p} \mathcal{R}(Q)_p
\]
is injective. Dually,
\[
\sum_Q \text{Ind}_Q^G : \prod_{Q \in \mathcal{F}_p} \mathcal{R}(Q)_p \to \mathcal{R}(G)_p
\]
is surjective.

Proof. By Solomon’s induction theorem, a prime-to-$p$ multiple $d$ of the trivial representation can be written as
\[
d_1 = \sum_i n_i \text{Ind}_{H_i}^G 1_{H_i}
\]
for some $p$-quasi-elementary subgroups $H_i$ and integers $n_i$. Because $\text{Ind}_{H_i}^G 1_{H_i} \cong \text{Ind}_{H_i}^G 1_{H_i}$, we may assume that each $H_i$ is contained in some $Q_i \in \mathcal{F}_p$. Taking tensor products with any $\rho \in \mathcal{R}(G)$ yields
\[
d\rho = \sum_i n_i \text{Ind}_{H_i}^G \text{Res}_{H_i} \rho.
\]
If all $\text{Res}_{H_i} \rho$ were 0, then so would be $d\rho$, and therefore also $\rho$. This proves injectivity. Also, the equation shows that $d\rho \in \text{Im} \left( \sum_Q \text{Ind}_Q^G \mathcal{R}(Q) \right)$, which proves surjectivity, since $d$ is invertible in $\mathbb{Z}_p$.

Corollary 3.2. For $S, T \in \mathcal{F}_p$ write $\alpha_{S,T} = \text{Res}_T \text{Ind}_S^G : \hat{C}(S) \to \hat{C}(T)$. Then
\[
\hat{C}(G)_p \cong \text{Image} \left( \prod_{S \in \mathcal{F}_p} \alpha_{S,T} : \prod_{S \in \mathcal{F}_p} \hat{C}(S) \to \prod_{T \in \mathcal{F}_p} \hat{C}(T) \right).
\]
In particular, $\hat{C}(G)_p = 1$ if and only if for all pairs $S, T \in \mathcal{F}_p$ and all $\rho \in \mathcal{R}_Q(S)$ (equivalently, for those $\rho$ that have a non-trivial class in $C(S)$), we have $\text{Res}_T \text{Ind}_S^G \rho \in \text{Perm}(T)$. The same also holds for $C(G)$.

Proof. Apply Proposition 3.1 to $\mathcal{R}$ being $\text{Perm}, \mathcal{R}_Q,$ and $\text{Char}_Q$.

Corollary 3.3. Let $\mathcal{F}$ be a family of subgroups of $G$ such that every quasi-elementary subgroup is conjugate to a subgroup of some $Q \in \mathcal{F}$. Then
\[
\hat{C}(G) \hookrightarrow \prod_{Q \in \mathcal{F}} \hat{C}(Q)
\]
via the (product of) restriction maps. Consequently, the kernel of the composition
\[
\mathcal{R}_Q(G) \xrightarrow{\prod \text{Res}} \prod_{Q \in \mathcal{F}} \mathcal{R}_Q(Q) \to \prod_{Q \in \mathcal{F}} \hat{C}(Q)
\]
is $\text{Perm}(G)$. Dually, the composition
\[
\prod_{Q \in \mathcal{F}} \mathcal{R}_Q(Q) \xrightarrow{\text{Ind}} \mathcal{R}_Q(G) \to \hat{C}(G)
\]
is onto. The same also holds for $\text{Char}_Q(G)$ and $C(G)$.
Lemma 3.7. \(\text{lem:innerprod} \)

\[ \rho F = \text{Notation 3.6. } \]

Proof. First, note that by definition of inner products and of induced class functions, computing \(\hat{C}(G)\) as a subring of \(R_Q(G) \leq \text{Char}_Q(G)\), without computing the full lattice of subgroups of \(G\).

Remark 3.4. The theorem and the two corollaries give a very efficient way of computing \(\hat{C}(G)\), \(\rho(G)\), \(\rho_x(G)\), and \(\rho_y(G)\) and of finding \(\text{Perm}(G)\) as a subring of \(R_Q(G) \leq \text{Char}_Q(G)\), without computing the full lattice of subgroups of \(G\).

Remark 3.5. One possible family \(\mathcal{F}_p\) is the set of maximal \(p\)-quasi-elementary subgroups of \(G\). These are of the form

\[ Q = C \rtimes Syl_p(N_G(C)), \]

where \(C\) is cyclic of order prime to \(p\). Possible families \(\mathcal{F}\) in Corollary 3.3 are \(\mathcal{F} = \bigcup_p \mathcal{F}_p\), as \(p\) ranges over prime divisors of \(|G|\), or alternatively \(\mathcal{F} = \{N_G(C)\}\) as \(C\) ranges over (representatives of conjugacy classes of) cyclic subgroups of \(G\).

Notation 3.6. For the remainder of this section we use the following notation:

\[
\begin{align*}
CC(G) &= \text{ the set of conjugacy classes of } G, \\
CC_{cyc}(G) &= \text{ the set of conjugacy classes of cyclic subgroups of } G, \\
[x] &= \text{ the conjugacy class of } x, \text{ when } x \text{ is either an element of } G \\
&\quad \quad \text{ or a cyclic subgroup}, \\
\text{Tr}^* \chi &= \text{ the normalised trace } \text{Tr}^* \chi = \frac{1}{|\mathbb{Q}^\times|} \text{Tr} \chi \text{ of a character } \chi, \\
\tau(D) &= \tau(y), \text{ where } D \leq G \text{ is a cyclic subgroup, } y \text{ is any generator of } D, \text{ and } \tau \in \text{Char}_Q(G) \otimes \mathbb{Q}. \text{ The rationality of } \tau \text{ ensures that } \tau(y) \text{ only depends on } D \text{ and not on the generator } y. \\
\end{align*}
\]

Note in particular, that for any character \(\chi\) of \(G\) and any cyclic subgroup \(D\) of \(G\), \(\text{Tr}^* \chi(D)\) is the average value of \(\chi\) on the generators of \(D\).

Lemma 3.7. Let \(H_1, H_2\) be two subgroups of \(G\). Let \(\tau_i\) be a character of \(H_i\), \(i = 1, 2\), and assume that \(\tau_1\) is \(\mathbb{Q}\)-valued. Then

\[
\langle \text{Ind}_{H_1}^G, \tau_1, \text{Ind}_{H_2}^G, \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{cyc}(G)} |N_G(C)|\phi(|C|) \cdot \sum_{\delta_1 \leq H_1} \tau_1(D_1) \cdot \sum_{\delta_2 \leq H_2} \text{Tr}^* \tau_2(D_2).
\]

Proof. First, note that by definition of inner products and of induced class functions,

\[
\langle \text{Ind}_{H_1}^G, \tau_1, \text{Ind}_{H_2}^G, \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \left( \sum_{y \in [x] \cap H_1} \tau_1(y) \right) \left( \sum_{y \in [x] \cap H_2} \tau_2(y) \right).
\]

The main idea of the proof is to partition the set of conjugacy classes of elements of \(G\) according to conjugacy classes of cyclic subgroups they generate, and to use the fact that for a rational character \(\tau\), \(\tau(x) = \tau(x')\) whenever \(x\) and \(x'\) generate conjugate cyclic subgroups. We get

\[
\langle \text{Ind}_{H_1}^G, \tau_1, \text{Ind}_{H_2}^G, \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \left( \sum_{y \in [x] \cap H_1} \tau_1(y) \right) \left( \sum_{y \in [x] \cap H_2} \tau_2(y) \right) = \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{cyc}(G)} f(C),
\]

where \(f(C)\) is the rationality of \(\tau\) on the generator of \(C\).

where

\[ f(C) = |Z(G)(C)| \cdot \sum_{\chi \in CG(C)} \left( \sum_{y \in [x] \cap H_1} \tau_1(y) \right) \left( \sum_{y \in [x] \cap H_2} \tau_2(y) \right) \]

\[ = |Z(G)(C)| \cdot \sum_{d_1 \leq H_1} \tau_1(D_1) \cdot \sum_{d_2 \leq H_2} \tau_2(D_2) \]

\[ = |N_G(C)| \cdot \sum_{d_1 \leq H_1} \tau_1(D_1) \cdot \sum_{d_2 \leq H_2} \phi(|C|) \cdot \text{Tr}_{\mathbb{Q}/\mathbb{Q}} \tau_2(D_2), \]

as claimed.

\[ \square \]

**Corollary 3.8.** Suppose \( H_1 < Q_1 < G, \ H_2 < Q_2 < G, \) and let \( \chi_i \) be irreducible characters of \( H_i. \) Set \( \tau_i = \text{Ind}_{H_i}^{Q_i} \chi_i, \) and \( \rho_i = \text{Tr} \tau_i. \) Assume that \( \tau_2 \) is irreducible. Then

\[
\mu(\rho_2, \text{Res}_{Q_2} \text{Ind}_{Q_1}^{G} \rho_1) = \frac{|\{\tau_1, \chi \}|}{|H_1| |H_2|} \sum_{[\chi] \in CG} |N_G(C)| \phi(|C|) \cdot \sum_{d_1 \leq H_1} \tau_1(D_1) \cdot \sum_{d_2 \leq H_2} \text{Tr}^* \chi_2(D_2).
\]

**Proof.**

\[
\mu(\rho_2, \text{Res}_{Q_2} \text{Ind}_{Q_1}^{G} \rho_1) = \frac{1}{|\mathbb{Q}(\chi) : \mathbb{Q}|} \sum_{[\chi] \in CG} |N_G(C)| \phi(|C|) \cdot \sum_{d_1 \leq H_1} \tau_1(D_1) \cdot \sum_{d_2 \leq H_2} \text{Tr}^* \chi_2(D_2).
\]

\[ \square \]

**Lemma 3.9.** If \( C \) is a cyclic group, \( \chi \) is an 1-dimensional character of \( C, \) then \( (\text{Tr}^* \chi)(C) = \mu(\text{ord}(\chi))/\phi(\text{ord}(\chi)), \) where \( \mu \) is the Möbius mu function, and \( \text{ord}(\chi) \) is the smallest natural number \( n \) such that \( \chi^n = 1. \)

**Proof.** It is enough to prove the lemma for faithful characters \( \chi, \) since we may, without loss of generality, replace \( C \) by \( C/\ker \chi. \) Let \( g \) be a generator of \( C. \) Then

\[
(\text{Tr}^* \chi)(C) = \frac{1}{|\mathbb{Q}(\chi) : \mathbb{Q}|} \text{Tr} \chi(g) = \frac{1}{\phi(\text{ord}(\chi))} \text{Tr} \chi(g).
\]

If \( |C| = n, \) then \( \chi(g) \) is a primitive \( n \)-th root of unity, and the fact that its trace is \( \mu(n) \) is classical.

\[ \square \]

**Corollary 3.10.** Let \( G \) be a group and \( p^r \) a prime power. Then \( \hat{C}(G) \) has an element of order \( p^r \) if and only if there exist two \( p \)-quasi-elementary subgroups \( Q_1, Q_2 \) of \( G, \) irreducible monomial characters \( \tau_i = \text{Ind}_{H_i}^{Q_i} \chi_i \) of \( Q_i, \) and an integer \( k, \) such that

- the rational character \( \text{Tr} \tau_2 \) has order divisible by \( p^{k+r} \) in \( \hat{C}(Q_2), \) and
Remark 3.11.

- Note that it is enough to take the last two sums in the above formula only over those $D_i$ for which $D_i \cap \ker \chi_i$ has square-free index in $D_i$, since for the others $\mu(\ord(\Res_{D_i} \chi_i)) = 0$. For example if $\chi_i$ are faithful, then the outer sum may be taken over $U$ of square free order.
- If, say, $H_1$ is cyclic, the sum $\sum_{D_1 \leq U}^{\neq} \mu([D_1 : D_1 \cap \ker \chi_1])$ has at most one term for every $U$.
- If $Q_1$ and $Q_2$ are basic, then in addition to $H_1$ being cyclic, Proposition 2.7 gives a simple expression for the order of $\Tr \tau_2$ in $\hat{C}(G)$.

Proof of Corollary 3.10. By Corollary 3.7, $\hat{C}(G)_p$ has an element of order $p^r$ if and only if there exist $p$-quasi-elementary subgroups $Q_1, Q_2$, and characters $\rho_i \in \Irr_{Q_i}(Q_1)$, such that $\rho_2$ has order $p^{k+r}$ in $\hat{C}(Q_2)$ for some $k$, and $\mu(\rho_2, \Res_{Q_2} \Ind_{Q_1}^G \rho_1)$ has $p$-adic valuation at most $k$. Quasi-elementary groups are M-groups, so if $\tau_1$ is a complex irreducible constituent of $\rho_i$, then there exist subgroups $H_i \leq Q_i$ such that $\tau_i = \Ind_{H_i}^{Q_i} \chi_i$ for 1-dimensional characters $\chi_i \in \Irr(H_i)$. The result therefore follows from Corollary 3.8 in combination with Lemma 3.9. 

4. Quasi-elementary groups

The aim of this section is to provide several formulae of theoretical and algorithmic interest for the orders of characters in $\hat{C}(G)$ and $C(G)$ when $G$ is quasi-elementary. Let $G = C \times P$ with $P$ a $p$-group and $C$ cyclic of order coprime to $p$; we identify $P$ with a Sylow subgroup of $G$.

Lemma 4.1. Let $N$ be a normal subgroup of a finite group $G$, let $\eta$ be an irreducible character of $N$, and let $\theta$ be a complex irreducible constituent of $\Ind_N^G \eta$. Write $G_\eta = \Gal(Q(\eta)/Q)$, and similarly for $G_\theta$. Then

$$\frac{[Q(\eta) : Q]}{[Q(\theta) : Q]} = \frac{\# \{ \gamma \in G_\eta \mid (\eta^\gamma, \Res_N \theta) \neq 0 \}}{\# \{ \gamma \in G_\theta \mid (\Ind_N^G \eta, \theta^\gamma) \neq 0 \}}.$$ 

In particular, if $\Ind_N^G \eta$ is irreducible, then

$$\frac{[Q(\eta) : Q]}{[Q(\theta) : Q]} = \# \{ \gamma \in G_\eta : (\eta^\gamma, \Res_N \theta) \neq 0 \}.$$ 

Proof. The $G$-action on the characters of $N$ commutes with the Galois action. Every Galois conjugate of $\theta$ is a constituent of $\Ind_N^G \eta^\gamma$ for some $\gamma \in G_\eta$, and moreover the number of distinct Galois conjugates of $\theta$ in $\eta^\gamma$ is independent of $\gamma$. Also, the number of Galois conjugates of $\eta$ in $\Res_N \theta^\gamma$ is independent of $\gamma \in G_\theta$. So an inclusion–exclusion count gives

$$\# G_\theta = \# G_\eta \cdot \frac{\# \{ \gamma \in G_\theta \mid (\Ind_N^G \eta, \theta^\gamma) \neq 0 \}}{\# \{ \gamma \in G_\eta : (\eta^\gamma, \Res_N \theta) \neq 0 \}}.$$
Lemma 4.2. Let $\eta$ be an irreducible complex representation of $G$, with rational hull $\hat{\eta}$. Then
\[
\dim \hat{\eta} = \dim \eta \cdot m(\eta) \cdot [\mathbb{Q}(\eta) : \mathbb{Q}].
\]
Proof. The rational hull of $\eta$ is given by
\[
\hat{\eta} = m(\eta) \sum_{\gamma \in \text{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})} \eta^\gamma,
\]
whence the claim follows.

Theorem 4.3. Let $G = C \rtimes X$ with $C$ cyclic of order coprime to $|X|$. Let $\tau$ be a complex irreducible character of $G$ with rational hull $\rho = \hat{\tau}$, let $\pi$ be a complex irreducible constituent of $\text{Res}_X \tau$ with rational hull $\hat{\pi}$, let $\hat{\psi}$ be the unique rational irreducible constituent of $\text{Res}_C \rho$, let $K_\psi$ the stabiliser of $\hat{\psi}$ under the $X$-action on $\text{Irr}_\mathbb{Q}(C)$, and let $\xi$ be a complex irreducible constituent of $\text{Res}_K \hat{\psi} \pi$. Then
\[
\mu(\rho, \text{Ind}_G^X \hat{\pi}) = \frac{m(\pi)}{m(\tau)} \langle \tau, \text{Res}_K \pi \rangle \cdot \# \{\text{Galois conjugates } \pi' \text{ of } \pi \text{ s.t. } \langle \text{Res}_K \pi', \xi \rangle \neq 0\}
\]
\[
= \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}.
\]
Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker \rho \cap C)$. So $K = K_\psi$ is assumed to be the kernel of the $X$-action on $C$. Let $\psi$ be a complex constituent of $\tau|_C$. In particular, $\tau = \text{Ind}_{C \cap K}^G \psi \xi$, as explained in [14, Part II, §8.2]. We have
\[
\rho = m(\tau) \sum_{\gamma \in \text{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})} \tau^\gamma; \quad \dim \rho = m(\tau)[\mathbb{Q}(\tau) : \mathbb{Q}] \dim \tau,
\]
\[
\hat{\pi} = m(\pi) \sum_{\gamma \in \text{Gal}(\mathbb{Q}(\pi)/\mathbb{Q})} \pi^\gamma; \quad \dim \hat{\pi} = m(\pi)[\mathbb{Q}(\pi) : \mathbb{Q}] \dim \pi.
\]
Thus
\[
\mu(\rho, \text{Ind}_G^X \hat{\pi}) = \frac{1}{m(\tau)} \langle \tau, \text{Ind}_G^X \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \text{Ind}_{C \cap K}^G \psi \xi, \text{Ind}_G^X \hat{\pi} \rangle
\]
\[
= \frac{1}{m(\tau)} \langle \text{Res}_X \text{Ind}_{C \cap K}^G \psi \xi, \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \text{Ind}_K^X \xi, \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \xi, \text{Res}_K \hat{\pi} \rangle,
\]
where the last line follows from Mackey’s formula, noting that $CK\backslash G/X$ consists of one double coset, and that $CK \cap X = K$.

Next, $X$ acts on the representations of $K$ by conjugation, and there is a Clifford theory decomposition
\[
\text{Res}_K \pi = e \sum_{g \in X/\text{Stab}_X \xi} \xi^g.
\]
Recall that the constituents of $\hat{\pi}$ are Galois conjugates of $\pi$, and we select those whose restriction to $K$ contains $\xi$:
\[
\Omega = \{ \gamma \in \text{Gal}(\mathbb{Q}(\pi)/\mathbb{Q}) \mid \langle \text{Res}_K \pi^\gamma, \xi \rangle \neq 0 \}.
\]
The inner product $\langle \text{Res}_K \pi^\gamma, \xi \rangle = \langle \text{Res}_K \pi, \xi^{\gamma^{-1}} \rangle$ is the same (and equals $e$) for every $\gamma \in \Omega$, since $\xi^{\gamma^{-1}}$ is irreducible and so must be one of $\xi^\mu$ in (4.4). So we have

$$\frac{1}{m(\tau)} \langle \xi, \text{Res}_K \hat{\pi} \rangle = \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \text{Res}_K \pi \rangle,$$

which proves the first equality.

It remains to show that

$$\frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \text{Res}_K \pi \rangle = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}. \tag{4.5}$$

By comparing dimensions in (4.4), and since $\tau = \text{Ind}_{C/K}^G \psi \xi$, we see that

$$\langle \xi, \text{Res}_K \pi \rangle = e = \frac{\dim \pi}{|X : \text{Stab}_X \xi| \dim \xi} = \frac{|X : K| \dim \pi}{|X : \text{Stab}_X \xi| \dim \tau} = \frac{[\text{Stab}_X \xi : K] \dim \pi}{\dim \tau},$$

so

$$\mu(\rho, \text{Ind}_C^G \hat{\pi}) = \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \text{Res}_K \pi \rangle = |\Omega| \cdot [\text{Stab}_X \xi : K] \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau}.$$

Consider the two groups

$$H_1 = \{ \gamma \in \text{Gal}(Q(\psi \xi)/Q) \mid \langle (\psi \xi)^{\gamma}, \text{Res}_C \pi \rangle \neq 0 \},$$

$$H_2 = \{ \gamma \in \text{Gal}(Q(\xi)/Q) \mid \langle \xi^{\gamma}, \text{Res}_K \pi \rangle \neq 0 \}.$$

There is a natural projection $H_1 \to H_2$ given by the restriction of Galois action to $Q(\xi)$, whose kernel consists of precisely those elements of $\text{Gal}(Q(\psi \xi)/Q)$ that act trivially on $\xi$, and through the action of some $g \in X$ on $\psi$ (this last condition is equivalent to the Galois element being in $H_1$). Thus, the kernel is isomorphic to the subgroup of $G/CK$ that acts trivially on $\xi$, i.e. to $\text{Stab}_X \xi/K$. We deduce that

$$\mu(\rho, \text{Ind}_C^G \hat{\pi}) = |\Omega| \frac{|H_1| m(\pi) \dim \pi}{|H_2| m(\tau) \dim \tau}.$$

Now, by applying Lemma 4.1 first to $CK \triangleleft G$ with $\theta = \tau$, $\eta = \psi \xi$, and then to $K \triangleleft X$ with $\theta = \pi$, $\eta = \xi$, we find that

$$|H_1| = \frac{|Q(\xi) : Q| |Q(\psi) : Q|}{|Q(\tau) : Q|} \quad \text{and} \quad |H_2| = |\Omega| \frac{|Q(\xi) : Q|}{|Q(\pi) : Q|},$$

so that

$$\mu(\rho, \text{Ind}_C^G \hat{\pi}) = |\Omega| \cdot \frac{|H_1| m(\pi) \dim \pi}{|H_2| m(\tau) \dim \tau} \cdot \frac{|Q(\xi) : Q|}{|Q(\psi) : Q|} \cdot \frac{|Q(\psi) : Q|}{|Q(\pi) : Q|} \cdot \frac{|Q(\pi) : Q| m(\pi) \dim \pi}{|Q(\tau) : Q| m(\tau) \dim \tau} \cdot \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho},$$

where the last equality follows from Lemma 4.2. 

**Theorem 4.6.** Let $G = C \times P$ be $p$-quasi-elementary, let $\rho$ be a rational irreducible representation of $G$. Write $\hat{\psi}$ for the unique rational irreducible constituent of $\rho|_C$, and let $\hat{\pi}$ be the rational irreducible constituent of $\rho|_P$ of minimal dimension. Denote by $\pi$ a complex constituent of $\hat{\pi}$, by $\xi$ a complex irreducible constituent of
Theorem 5.1. \( \pi_{|K_\psi} \), where \( K_\psi \leq P \) is the stabiliser in \( P \) of \( \hat{\psi} \), and by \( \tau \) a complex constituent of \( \rho \) such that \( \tau \mid_P \) contains \( \pi \). Then

\[
\text{order of } \rho \text{ in } C(G) = \mu(\rho, \text{Ind}_P^G \hat{\pi}) = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho} = \frac{m(\pi)}{m(\tau)}(\xi, \text{Res}_{K_\psi} \pi) \cdot \#\{\text{Galois conjugates } \pi' \text{ of } \pi \text{ s.t. } \xi \subset \text{Res}_{K_\psi} \pi'\}.
\]

Proof. We may assume that \( \rho|_C \) is faithful, otherwise we prove the result in the quotient \( G/(\ker \rho \cap C) \) (see Lemma 2.1). Thus, \( K = K_\psi \) is assumed to be the kernel of the \( P \)-action on \( C \). Under this assumption, if \( H \leq G \) intersects \( C \) non-trivially, then

\[
(\rho, C[G/H])_G = (\text{Res}_H \rho, 1)_H = 0.
\]

Write \( o \) for the order of \( \rho \) in \( C(G) \). By Theorem 2.5, we have

\[
o \cdot (\rho, \rho) = \text{gcd}(\rho, C[G/H])_G = \text{gcd}(\rho, C[G/H])_G \leq P = \text{gcd}(\rho|_P, C[H/P])_P.
\]

Because \( C(P) = 1 \) by the Ritter-Segal theorem [12, 13], we can replace the permutation representations \( C[P/H] \) by all rational representations of \( P \) in the last term. This is clearly the same as just taking the rational irreducible constituents \( \hat{\pi}_1, ..., \hat{\pi}_k \) of \( \rho|_P \), so

\[
o = \frac{1}{(\rho, \rho)} \text{gcd}(\rho|_P, \hat{\pi}_j) = \text{gcd}(\rho|_P, \text{Ind}_P^G \hat{\pi}_j).
\]

The theorem will therefore follow from Theorem 4.3 once we show that the gcd may be replaced by the term corresponding to \( \hat{\pi} \) of minimal dimension. Now, by Theorem 2.2 and by Lemma 3.2,

\[
\mu(\rho, \text{Ind}_P^G \hat{\pi}_j) = \frac{\dim \hat{\psi} \dim \hat{\pi}_j}{\dim \rho} = \frac{\dim \hat{\psi} m(\pi_j)}{\dim \rho} \frac{\dim \pi_j |Q(\pi_j) : Q|}{\dim \rho},
\]

where \( \pi_j \) is a complex irreducible constituent of \( \hat{\pi}_j \). We argue as in [17, \S 2]: if \( p = 2 \), then all the terms \( m(\pi_j), \dim \pi_j, |Q(\pi_j) : Q| \) are powers of 2, so gcd and minimum are the same. If \( p \) is odd, then \( m(\pi_j) = 1 \), and moreover, either some \( \pi_j = 1 \), in which case the claim is clear, or else all \( \dim \pi_j \) are powers of \( p \), while all \( |Q(\pi_j) : Q| \) are \((p - 1)\) times powers of \( p \) (17, Lemma 2.1), so again \( \text{gcd and minimum are the same.} \)

\[
\tag{4.7}
o = \frac{1}{(\rho, \rho)} \text{gcd}(\rho|_P, \hat{\pi}_j) = \text{gcd}(\rho|_P, \text{Ind}_P^G \hat{\pi}_j).
\]

5. Examples: \( \text{GL}_2(\mathbb{F}_q) \), \( \text{PGL}_2(\mathbb{F}_q) \), \( \text{SL}_2(\mathbb{F}_q) \) and \( \text{PSL}_2(\mathbb{F}_q) \)

Theorem 5.1. For every prime power \( q = p^a \), the group \( G = \text{GL}_2(\mathbb{F}_q) \) has \( \hat{C}(G) = \{1\} \).

Proof. By Corollary 3.2 it suffices to show that every maximal quasi-elementary subgroup \( Q = C \times P \) of \( G = \text{GL}_2(\mathbb{F}_q) \) is contained in some \( Q < G \) with \( \hat{C}(Q) = 1 \).

Pick \( C = \langle g \rangle \) cyclic, and let \( P = \text{Syl}_l(N_G(C)) \) for some prime number \( l \). Write \( f(t) \) for the characteristic polynomial of \( g \).

Case 1 (split Cartan). Suppose \( f(t) \) has distinct roots \( a, b \in \mathbb{F}_q^\times \). Then \( g \) is conjugate to \( (0, a) \), and its centraliser is the split Cartan subgroup:

\[
Z_G(C) \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times, \quad N_G(C) = (\mathbb{F}_q^\times \times \mathbb{F}_q^\times) \rtimes C_2,
\]
Theorem 6.1. Let \( k \geq 4 \) be an integer, and \( p \) a prime. The groups \( \text{PSL}_k(\mathbb{F}_p) \), and therefore also \( \text{SL}_k(\mathbb{F}_p) \), have \( \hat{C}(G) \) of order divisible by \( 2^{\min(\text{ord}_2(k),\text{ord}_2(p-1))} \).
Lemma 6.2 (Zsigmondy’s Theorem [LS]). If \( k > 4 \) or if \( p + 1 \) is not a power of 2, then there exists a prime number \( l \) that divides \( p^{k-2} - 1 \) but does not divide \( p^s - 1 \) for any \( s < k - 2 \).

Write \( Q_{2N} \) for the generalised quaternion group of order \( 2^N \).

**Lemma 6.3.** The group \( SL_2(\mathbb{F}_q), q = p^k \) has a 2-Sylow subgroup of the form

- \( S = \{ (1, 0)^t \} \cong C_p^k \) if \( p = 2 \);
- \( S = \langle c, h \rangle \cong Q_{2N}, c = (\alpha 0, 0 \alpha^{-1}), h = (0 1, 1 0) \) with \( \alpha \in \mathbb{F}_q^\times \) of exact order \( 2^{N-1} \) if \( q \equiv 1 \mod 4 \);
- \( S = \langle c, h \rangle \cong Q_{2N}, c = (\alpha \beta, 1 \alpha^{-1}), h = (\gamma \delta, -\gamma) \) with \( \alpha + \beta \sqrt{-1} \in \mathbb{F}_q^\times \) of exact order \( 2^{N-1} \) if \( q \equiv 3 \mod 4 \).

Conjugation by the matrix \( \iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) is an automorphism of \( S \), acting as \(-1\) in the first case, as \( c \mapsto c, h \mapsto h^{-1} \) in the second case, and as \( c \mapsto c^{-1}, h \mapsto hc^{2m+1} \) for some \( m \) in the last case.

**Proof.** Direct computation. \( \square \)

From now on, \( G \) will denote \( PSL_k(\mathbb{F}_p) \). The theorem only has content when \( k \) is even and \( p \) is odd, so we will assume this. Write

\[
 n = \text{ord}_2(p - 1) \geq 1, \quad N = \text{ord}_2(p^{k-2} - 1) \geq 3, \quad m = \text{ord}_2(k - 2) \geq 1.
\]

**Case A:** Either \( k > 4 \) or \( p = 1 \mod 4 \). Let \( A \) be a generator of a non-split Cartan subgroup \( \mathbb{F}_{p^{k-2}}^\times = GL_1(\mathbb{F}_{p^{k-2}}) < GL_{k-2}(\mathbb{F}_p) \), and \( l \) a prime divisor of \( p^{k-2} - 1 \) as in Zsigmondy’s Theorem. The conditions on \( l \) imply that the normaliser of \( \langle A^{\frac{k-2}{l}} \rangle \cong C_l \) in \( GL_{k-2}(\mathbb{F}_p) \) is generated by \( A \) and by the Frobenius automorphism \( F \in \text{Gal}(\mathbb{F}_{p^{k-2}}/\mathbb{F}_p) \) of order \( k - 2 \). Note that \( F \) has determinant \(-1\), since it is an odd permutation on a normal basis of \( \mathbb{F}_{p^{k-2}}/\mathbb{F}_p \). Define

\[
 c_p = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad c_l = \begin{pmatrix} 1 & 1 \\ 1 & A^{\frac{k-2}{2}} \end{pmatrix},
\]

\[
 x = \begin{pmatrix} d^{-1} & 1 \\ 1 & U \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} F^{(k-2)/2m},
\]

where \( U = A^{\frac{k-2}{2}} \) and \( d = \det U \). We view these matrices as representing elements of \( G = PSL_k(\mathbb{F}_p) \). Write

\[
 C = \langle c_p c_l \rangle \cong C_{pl}, \quad P = \langle x, f \rangle \cong C_{2^N} \times C_{2m}, \quad Q = CP \cong (C_p \times C_{l}) \times (C_{2N} \times C_{2m}).
\]

Note that \( C_{2N} \) acts trivially on \( C_l \), and through a \( C_{2n} \) quotient on \( C_p \), while \( C_{2m} \) acts through a \( C_2 \) quotient on \( C_p \) and faithfully on \( C_l \).

**Case B:** \( p \equiv 3 \mod 4 \) and \( k = 4 \). We take the same \( c_p \) as in Case A, and \( C = \langle c_p \rangle \). A 2-Sylow of the centralizer of \( C \) in \( G \) is isomorphic to \( \{1\} \times \text{Syl}_2(SL_2(\mathbb{F}_p)) \),
which is isomorphic to $Q_{2N}$ by the last case of Lemma 6.3. A 2-Sylow of the normalizer is

$$P = \text{Syl}_2 N_G(C) = \text{Syl}_2 Z_G(C) \rtimes \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cong Q_{2N} \rtimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is in fact isomorphic to the semi-dihedral group $SD_{2N+1}$. Again, we let $Q = CP$.

In both cases, write $K$ for the centralizer of $C$ in $P$. Thus, $K \cong C_{2N-m}$ in case A, and $K \cong Q_{2N}$ in case B, where the isomorphism is that of Lemma 6.3. Let $A_p$ be $K$ in case A, and a cyclic subgroup of index 2 in $K$ that is normal in $Q$ in case B. Let $\chi$ be faithful irreducible characters of $CA_p$, $\tau = \text{Ind}_{CA_p}^Q \chi$ and $\rho = \text{Tr} \tau \in \text{Irr}(Q)$.

Lemma 6.4. The character $\rho$ has order $2^n$ in $\hat{C}(Q)$.

Proof. We will use Proposition 2.7. The biggest subgroup of $P$ that intersects $CA_p$ trivially is of order 2 in case B, and of order $2^m$ in case A. So the order of $\rho$ in $\hat{C}(G)$ is $2^{N+1-(N-1)-1} = 2$ in case B, and $2^{N+m-(N-n)-m} = 2^n$ in case A. \end{proof}

Finally, we show that $\text{Ind}_{CA_p}^Q \rho$ has order divisible by $2^{\min\{\text{ord}_2(k), \text{ord}_2(p-1)\}}$ in $\hat{C}(G)$. We will use Corollary 3.10 with $Q_1 = Q_2 = Q$ and $\chi_1 = \chi_2 = \chi$. In view of Lemma 6.4, we therefore need to show that

$$\sum_{[U] \in CC_{cyc}(G)} S(U)$$

has 2-adic valuation at most $n - \min\{\text{ord}_2(k), \text{ord}_2(p-1)\}$, where for $U \leq CA_p$,

$$S(U) = \frac{[Q(\tau) : Q]}{|CA_p|^2} |N_G(U)| \phi(|U|) \cdot \left( \sum_{D \leq CA_p, D \neq U} \frac{\mu([D : D \cap \ker \chi])}{\phi([D : D \cap \ker \chi])} \right)^2.$$

Note that since $CA_p$ is cyclic and $\chi$ is faithful, the formula for $S(U)$ simplifies to

$$S(U) = \frac{[Q(\tau) : Q]}{|CA_p|^2 \phi(|U|)} |N_G(U)| \mu(|U|)^2,$$

see Remark 3.11. In particular, $S(U) = 0$ if $U$ has non-square-free order.

Case A.

The subgroups of $CK$ of square-free order are $C_{2lp}$, $C_{lp}$, $C_{2l} C_l$, $C_{2p}$, $C_p$, $C_2$, and $C_1$. We will show that $S(C_{lp}) + S(C_{2lp})$ has a strictly lower 2-adic valuation than the rest of the sum, and that this valuation is $n - \min\{\text{ord}_2(k), n\}$. A summary of the below calculations is:
\[
\begin{align*}
\text{ord}_2[Q(\tau) : Q] &= \text{ord}_2(l - 1) + N - n - 1 - m, \\
\text{ord}_2|CK|^2 &= 2(N - n), \\
\phi(|C_{1p}|) &= (l - 1)(p - 1), \\
|N_C(C_{1p})| &= |N_C(C_{21p})| = \frac{(k - 2)p(p^{k-2} - 1)(p - 1)}{\gcd\{k, p - 1\}}, \\
\text{ord}_2(S(C_{1p}) + S(C_{21p})) &= \text{ord}_2(2S(C_{1p})) \\
&= 1 + \text{ord}_2(l - 1) + N - n - 1 - m - 2(N - 2) + N + n + m - \min\{\text{ord}_2(k), n\} - \text{ord}_2(l - 1) + n \\
&= n - \min\{\text{ord}_2(k), n\}.
\end{align*}
\]

The assertions concerning \(|CK|\) and \(\phi(|C_{1p}|)\) are clear.

Since the conjugation action of \(P\) on \(\text{Irr}(CK)\) is through Galois automorphisms, and \(\ker(P \to \text{Aut}(CK))\) has index \(2^{n+m}\) in \(P\), we have

\[
[Q(\tau) : Q] = 2^{-n-m}[Q(\chi) : Q] = \frac{p - 1}{2^n} \frac{(l - 1)2^{N-n-1}}{2^m},
\]

with \(2\)-adic valuation \(\text{ord}_2(l - 1) + N - n - 1 - m\).

The normaliser \(N_{GL_k(p)}\) of the preimage of \(C_{1p}\) under \(SL \to PSL\) consists of block diagonal matrices, with the normaliser of non-split Cartan in the lower right corner (order \((k - 2)(p^{k-2} - 1)\)), and a Borel subgroup in the top left (order \(p(p - 1)^2\)).

The determinant is surjective on \(N_{GL_k(p)}\), and \(N_{GL_k(p)}\) contains \(Z(GL_k(p))\), so the normaliser of \(C_{1p}\) in \(PSL\) has order \(\frac{(k - 2)p^{k-2} - 1)p(p - 1)}{\gcd\{k, p - 1\}}\), with \(2\)-adic valuation \(N + n + m - \min\{\text{ord}_2(k), n\}\). This is also the normaliser of \(C_{21p}\).

It remains to show that the rest of the sum in equation \((\mathfrak{m})\) has strictly greater \(2\)-adic valuation than \(\text{ord}_2(S(C_{1p}) + S(C_{21p}))\). If \(U \leq C\), then \(|N_C(U)|\) and \(|N_C(UC_2)|\) agree up to a power of \(p\), \(\phi(|U|) = \phi(|UC_2|)\), while \(\mu(|U|) = -\mu(|UC_2|)\). It follows that the \(2\)-adic valuation of \(S(U) + S(UC_2)\) is at least 1 greater than that of \(S(U)\).

Moreover, for any \(U \leq C_{1p}\), the normaliser of \(U\) in \(G\) contains that of \(C_{1p}\), while \(1/\phi(|U|)\) has strictly greater \(2\)-adic valuation than \(1/\phi(|C_{1p}|)\) whenever \(U \neq C_{1p}\).

This establishes the claim.

**Case B.**

The subgroups of \(CA_p\) of square-free order are \(C_1\), \(C_2\), \(C_p\), and \(C_{2p}\). We will show that \(\text{ord}_2(\sum S(U)) = \text{ord}_2(S(C_p) + S(C_{2p})) = 0\). Again, we summarise the
below calculations as follows:

\[
\begin{align*}
\text{ord}_2 [Q(\tau) : \mathbb{Q}] &= N - 3, \\
\text{ord}_2 |CA_p|^2 &= 2N - 2, \\
\phi(|C_p|) &= \phi(|C_{2p}|) = p - 1, \\
|N_G(C_p)| &= p^4|N_G(C_{2p})| = p^4 \cdot \frac{(p - 1)p^2(p + 1)}{2}, \\
\text{ord}_2 (S(C_p) + S(C_{2p})) &= \text{ord}_2 ((1 + p^4)S(C_{2p})) = 1 + N - 3 - 2N + 2 - 1 + N + 1 = 0.
\end{align*}
\]

The assertions concerning $|CA_p|$ and $\phi$ are clear.

It follows from the description of the $P$-action on Irr$(CK)$ that $[Q(\tau) : \mathbb{Q}] = \frac{N}{p}[Q(\chi) : \mathbb{Q}]$, and has $2$-adic valuation $2^{N-3}$.

The normaliser of $C_{2p}$ in GL$_4$ is block diagonal, with all invertible matrices in the bottom right corner, and Borel in the top left. So its order in PSL is $(p^2 - 1)p^2(p + 1)^2$ with $2$-adic valuation $N + 1$. Finally, $|N(C_p)| = p^4|N(C_{2p})|$, e.g. see Murray [10] §4.

It remains to show that the $2$-adic valuation of $S(C_1) + S(C_2)$ is positive. The normaliser of $C_2$ in GL$_4$ is GL$_2 \times$ GL$_2$, so the order of the normaliser in PSL is $\frac{(p^2 - 1)p^2(p + 1)^2}{2}$, with $2$-adic valuation $2N + 1$, and the normaliser of $C_1$ is even bigger. So the $2$-adic valuations of $S(C_1)$ and of $S(C_2)$ are positive.

**Corollary 6.6.** As $G$ ranges over the simple groups PSL$_k(\mathbb{F}_p)$, and therefore also over SL$_k(\mathbb{F}_p)$, the exponent of $C(G)_2$ is unbounded.

**Proof.** If $\text{ord}_2 (k) > \text{ord}_2 (p - 1)$, then by [10] Lemma 5.6(1)] all Schur indices in PSL$_k(\mathbb{F}_p)$ are trivial. So the assertion follows from Theorem 6.1. □

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