INVARIANT METRIC ON THE EXTENDED SIEGEL-JACOBI
UPPER HALF SPACE

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Abstract. The real Jacobi group $G^J_n(\mathbb{R})$, defined as the semidirect product of the Heisenberg group $H_n(\mathbb{R})$ with the symplectic group $Sp(n, \mathbb{R})$, admits a matrix embedding in $Sp(n+1, \mathbb{R})$. The modified pre-Iwasawa decomposition of $Sp(n, \mathbb{R})$ allows us to introduce a convenient coordinatization $S_n$ of $G^J_n(\mathbb{R})$, which for $G^J_1(\mathbb{R})$ coincides with the S-coordinates. Invariant one-forms on $G^J_n(\mathbb{R})$ are determined. The formula of the 4-parameter invariant metric on $G^J_1(\mathbb{R})$ obtained as sum of squares of 6 invariant one-forms is extended to $G^J_n(\mathbb{R})$, $n \in \mathbb{N}$. We obtain a three parameter invariant metric on the extended Siegel-Jacobi upper half space $\tilde{\mathcal{X}}^J_n \approx \mathcal{X}^J_n \times \mathbb{R}$ by adding the square of an invariant one-form to the two-parameter balanced metric on the Siegel-Jacobi upper half space $\mathcal{X}^J_n = \frac{G^J_n(\mathbb{R})}{U(n) \times \mathbb{R}}$.

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1. Introduction

The real Jacobi group \( G_n^J(\mathbb{R}) := H_n(\mathbb{R}) \rtimes \text{Sp}(n, \mathbb{R}) \) of degree \( n \) is defined as \( G_n^J(\mathbb{R}) := H_n(\mathbb{R}) \rtimes \text{Sp}(n, \mathbb{R}) \), where \( H_n(\mathbb{R}) \) denotes the real Heisenberg group. The Siegel-Jacobi upper half space is the \( G_n^J(\mathbb{R}) \)-homogeneous manifold \( \mathcal{X}_n^J := \frac{G_n^J(\mathbb{R})}{U(n) \times \mathbb{R}} \approx \mathcal{X}_n \times \mathbb{R}^n \) [14, 15], \([87]-[89]\),

where \( \mathcal{X}_n \) denotes the Siegel upper half space realized as \( \text{Sp}(n, \mathbb{R}) \) [14, p 398].

The Jacobi group \( G_n^J := H_n \rtimes \text{Sp}(n, \mathbb{R}) \), \( H_n(\mathbb{R}) \), \( \mathcal{X}_n^J \), and \( \mathcal{X}_n \) are isomorphic with \( G_n^J \), \( \text{Sp}(n, \mathbb{R}) \), \( H_n \), \( \mathcal{D}_n^J \), respectively \( \mathcal{D}_n \), see [8]–[12], [14, 23, 35, 87, 88].

The dimensions of the enumerated manifolds are: \( \dim \text{Sp}(n, \mathbb{R}) = 2n^2 + n \), \( \dim H_n(\mathbb{R}) = 2n + 1 \), \( \dim G_n^J(\mathbb{R}) = (2n + 1)(n + 1) \), \( \dim U(n) = n \), \( \dim \mathcal{X}_n^J = n(n + 3) + 1 \), \( \dim \mathcal{X}_n = n(n + 1) \).

The Jacobi group, as a unimodular, non-reductive, algebraic group of Harish-Chandra type [16, 55], [72]–[75], also a coherent state (CS) type group [4, 56, 57, 63, 64, 66] is an interesting object in Mathematics [11]–[14], [67], or Weyl-symplectic group [85]. The Jacobi group is responsible for the squeezed states [9, 10], we are expecting that the manifold \( \mathcal{X}_n^J \) to have applications in quantum optics. We recall that the squeezed states are a particular class of “minimum uncertainty states” (MUS) [62] and that “Gaussian pure states” (“Gaussons”) [78] are more general MUSs.

The invariant metrics on homogeneous manifolds associated to the real Jacobi group \( G_n^J(\mathbb{R}) \) were obtained in [3, 15], applying Cartan’s moving frame method [28, 29]. We have determined a 3-parameter invariant metric on the extended Siegel-Jacobi upper half space of order \( n \), \( \tilde{\mathcal{X}}_n^J := \frac{G_n^J(\mathbb{R})}{U(n) \times \mathbb{R}} \approx \mathcal{X}_n \times \mathbb{R}^n \) [14, [87]-[89]].
half-plane \([3,15]\). To get the invariant metric on \(\tilde{X}_1^J\), we have determined the invariant one-forms \(\lambda_1, \ldots, \lambda_6\) on \(G_1^J(\mathbb{R})\). Then we have calculated the invariant vector fields \(L^j\) verifying the relations \(<\lambda_i, L^j> = \delta_{ij}, \ i, j = 1, \ldots, 6\), such that \(L^j\) are orthonormal with respect to the 4-parameter invariant metric \(d s^2_{G_1^J(\mathbb{R})}\) expressed in the \(S\)-coordinates \((x, y, \theta, p, q, \kappa)\) \([23, \text{p} \ 10]\), where \(\theta \in [0, 2\pi)\) and the other \(S\)-coordinates are in \(\mathbb{R}\).

In the present paper we apply to \(G_n^J(\mathbb{R})\), \(n \in \mathbb{N}\), the method applied in [14] to \(G_1^J(\mathbb{R})\). Firstly we determine the invariant one-forms on \(G_n^J(\mathbb{R})\). If a point \(g \in G_n^J(\mathbb{R})\) is parametrized by the coordinates \((M, X, \kappa)\), where \(M \in \text{Sp}(n, \mathbb{R})\), \(X := (\lambda, \mu) \in \text{M}(1, 2n, \mathbb{R})\), \(\kappa \in \mathbb{R}\), and \((p, q) = XM^{-1}\), then we have the following representation of the real Jacobi group embedded in \(\text{Sp}(n+1, \mathbb{R})\) \([86, 92]\).

\[
(1.1) \quad g = \begin{pmatrix} a & 0_{n1} & b & q^t \\ \lambda & 1 & \mu & \kappa \\ c & 0_{n1} & d & -p^t \\ 0_{1n} & 0 & 0_{1n} & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(n, \mathbb{R}).
\]

In this paper we parametrize the group \(G_n^J(\mathbb{R})\) with a system of coordinates \((x, y, X, Y, p, q, \kappa)\), where \(x + i y \in \mathcal{X}_n\), \(X + i Y \in \mathcal{U}(n)\), while \((p, q, \kappa)\) characterize the Heisenberg group \(\text{H}_n(\mathbb{R})\). This system of coordinates, denoted \(S_n, n \in \mathbb{N}\), coincides for \(n = 1\) with the \(S\)-coordinates of \(G_1^J(\mathbb{R})\) \([23, \text{p} \ 10]\). The main ingredient of the \(S_n\)-parametrization of \(G_n^J(\mathbb{R})\) is the modified pre-Iwasawa decomposition of the symplectic group \(\text{Sp}(n, \mathbb{R})\), inspired by [2] [42]. We obtain a 4-parameter invariant metric on \(G_n^J(\mathbb{R})\), which in the case \(n = 1\) coincides with the 4-parameter invariant metric determined in [14]. However, the explicit expressions for the metrics in Proposition 2 obtained from the invariant one forms on \(G_n^J(\mathbb{R})\) are quite complicated, so in order to obtain the invariant metric on the odd dimensional extended Siegel-Jacobi space \(\tilde{X}_n^J\) we just add the square of an invariant one-form attached to \(\kappa\) to the 2-parameter balanced metric of the Siegel-Jacobi upper half space obtained via the CS method in [11, 14].

The paper is organized as follows. Section 2 summarizes the embedding of the Heisenberg group \(\text{H}_n(\mathbb{R})\) in \(\text{Sp}(n+1, \mathbb{R})\). Section 3 describes the symplectic group. The pre-Iwasawa decomposition is introduced in Lemma 4, while Lemma 5 shows that the modified pre-Iwasawa decomposition is compatible with the linear fractional action of \(\text{Sp}(n, \mathbb{R})\) on \(\mathcal{X}_n\). Section 4 considers the real Jacobi group \(G_n^J(\mathbb{R})\). The embedding of \(G_n^J(\mathbb{R})\) in \(\text{Sp}(n+1, \mathbb{R})\) is described in Remark 6. After choosing a base of the Lie algebra \(\mathfrak{g}_n^J(\mathbb{R})\) which in particular for \(n = 1\) coincides with that in [15], Lemma 8 describes the action of the Jacobi group on the homogeneous manifolds \(\mathcal{X}_n^J\) and \(\tilde{X}_n^J\). In Section 4.3 are calculated the fundamental vector fields (FVF) associated to the generators of the Jacobi group on \(\mathcal{X}_n^J\) and \(\tilde{X}_n^J\). In Section 4.5 are obtained the invariant one-forms on \(G_n^J(\mathbb{R})\) in the \(S_n\)-coordinates, see Lemma 10 and (4.32). The difficulties to calculate the invariant vector fields once the invariant one-forms are known are exemplified in Section 4.6. Proposition 2 expresses the 4-parameter invariant metric on \(G_n^J(\mathbb{R})\). Proposition 3, an extension to \(n \in \mathbb{N}\) of [3] Proposition 1, expresses the Kähler two-form on \(\mathcal{X}_n^J\) in several types of variables. Remark 11 gives a CS-meaning to the \(S_n\)-parameters \(p, q\) describing \(G_n^J(\mathbb{R})\). The invariant metric on the odd dimensional manifold \(\tilde{X}_n^J\) is given in Theorem 1. Finally, other parametrizations of the Jacobi algebra \(\mathfrak{g}_n^J(\mathbb{R})\) are recalled.
in § III while Section IV summarizes the method of calculating the differential of square root of a symmetric matrix.

To conclude, the new results of this paper are contained in Lemma 4, Lemma 5, parts of Lemma 8, the base \((4.6)\) of \(g^t_n(\mathbb{R})\), Lemma 10, Propositions 1–3 and Remark 11.

The main result of the present investigation is stated in Theorem 1.

**Notation** We denote by \(\mathbb{R}, \mathbb{C}, \mathbb{Z},\) and \(\mathbb{N}\) the field of real numbers, the field of complex numbers, the ring of integers, and the set of positive integers, respectively. We denote by \(i\) the imaginary unit \(\sqrt{-1}\), and the complex conjugate of \(z\) by \(\bar{z}\). We denote the set of \(m \times n\) matrices with entries in the field \(\mathbb{F}\) as \(M(m,n;\mathbb{F})\) and if \(n = m\) we write \(M(n,\mathbb{F})\). \(M(n,\mathbb{F})\) for \(\mathbb{F}\) equal with \(\mathbb{R}\) or \(\mathbb{C}\) is denoted by \(M(n)\). We denote the transpose (the Hermitian conjugate) of the matrix \(A\) by \(A^t\), (respectively \(A^\dagger\)). \(1_n\) denotes the identity matrix of \(M(n,\mathbb{F})\), while \(0_{nm} \in M(n, m, \mathbb{F})\) denotes the matrix with all elements zero and \(0_n\) means \(0_{nn}\). \(E_p \in M(1, n, \mathbb{R})\) denotes the matrix with 1 on the position \(p\), \((E_p)_i = \delta_{pi}\) and similarly for \(E_q, p, q = 1, \ldots, n\). \(E_{ij}\) denotes the square matrix with entry 1 at the intersection of the \(i\)th row with the \(j\)th column, \((E_{ij})_{kl} = \delta_{ik}\delta_{jl}\), and \(E_{ij}E_{kl} = \delta_{jk}E_{il}\). When the dimension of a submatrix of a block matrix is not evident, the subindices \(pq\) specify that the respective submatrix is in \(M(p,q,\mathbb{R})\). We denote by \(d\) the differential. We use Einstein convention that repeated indices are implicitly summed over. We denote by \(dg(a_1, \ldots, a_n)\) the diagonal matrix which has on diagonal \(a_1, \ldots, a_n\). We denote by \(<\lambda|L>\) the pairing of the one-form \(\lambda\) with the vector field \(L\). We consider a complex separable Hilbert space \(\mathcal{H}\) endowed with a scalar product \((\cdot, \cdot)\) which is antilinear in the first argument, \((\lambda x, y) = \bar{\lambda}(x, y)\) \(x, y \in \mathcal{H}\), \(\lambda \in \mathbb{C} \setminus 0\). If \(\pi\) is a representation of a Lie group \(G\) on the Hilbert space \(\mathcal{H}\), we denote \(X := d\pi(X)\) for \(X \in g\).

2. The Heisenberg group \(H_\mathbb{R}(\mathbb{R})\) as subgroup of \(Sp(n+1,\mathbb{R})\)

The real Heisenberg group \(H_\mathbb{R}(\mathbb{R})\), parametrized by \((\lambda, \mu, \kappa), \lambda, \mu \in M(1, n, \mathbb{R}), \kappa \in \mathbb{R}\), has the composition law \[16, 55, 74, 75, 86, 92\]

\[
(\lambda, \mu, \kappa) \times (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda').
\]

\(H_\mathbb{R}(\mathbb{R})\) is a particular case of the Heisenberg group \(H_\mathbb{R}^{(n,m)}\) for \(m = 1\), see \[87\] and \[90\]. If \(g \in H_\mathbb{R}(\mathbb{R})\), we represent it \[86, 92\] and its inverse embedded in \(Sp(n+1,\mathbb{R})\) as

\[
g = \begin{pmatrix}
1 & 0 & 0 & \mu^t \\
\lambda & 1 & \mu & \kappa \\
0 & 0 & 1 & -\lambda^t \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad g^{-1} = \begin{pmatrix}
1 & 0 & 0 & -\mu^t \\
-\lambda & 1 & -\mu & -\kappa \\
0 & 0 & 1 & \lambda^t \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

see also notation in \((1.1)\) and Lemma 6.
If the generators $P_p, Q_q, p, q = 1, \ldots, n, R$, of the Heisenberg group are defined in (2.3), see also the last three equations in (4.5) and Lemma 6.

\[
(2.3a) \quad P_p = \begin{pmatrix}
0 & 0 & 0 & 0 \\
E_p & 0 & 0 & 0 \\
0 & 0 & -E_p^t & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad p = 1, \ldots, n
\]

\[
(2.3b) \quad Q_q = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & E_q \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad q = 1, \ldots, n
\]

\[
(2.3c) \quad R = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

then

\[
(2.4) \quad g^{-1} \, dg = P_p \lambda^p + Q_q \lambda^q + R \lambda^r.
\]

With (2.2) and (2.4), the left invariant one-forms on $H_n(\mathbb{R})$ are

\[
(2.5) \quad \lambda^p = d \lambda_p, \quad \lambda^q = d \mu_q, \quad \lambda^r = d \kappa - \lambda d \mu^t + \mu d \lambda^t.
\]

The left action of the Heisenberg group on itself is obtained from (2.1)

\[
\exp(\lambda P + \mu^t Q + \kappa R)(\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda \mu_0^t - \mu \lambda_0^t).
\]

The left invariant metric on the Heisenberg group is

\[
g^L(\lambda, \mu, \kappa) = d \lambda^2 + d \mu^2 + (d \kappa - \lambda d \mu^t + \mu d \lambda^t)^2.
\]

The fundamental vector fields, see \[44\] p. 121, Ch II § 3], [53] p. 42], or [15] § 6.1, v1], on the Heisenberg group $H_n(\mathbb{R})$ are

\[
P^* = \frac{\partial}{\partial \lambda} + \mu^t \frac{\partial}{\partial \kappa}, \quad Q^* = \frac{\partial}{\partial \mu} - \lambda \frac{\partial}{\partial \kappa}, \quad R^* = \frac{\partial}{\partial \kappa}.
\]

See also \[1.25\].

3. The symplectic group $Sp(n, \mathbb{R})$

3.1. Basics. The group $Sp(n, \mathbb{K})$ admits a matrix realization in $M \in M(2n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, verifying the relation

\[
(3.1) \quad M^t J_n M = J_n, \quad J_n := \begin{pmatrix}
0_n & 1_n \\
-1_n & 0_n
\end{pmatrix}.
\]

If there is no possibility of confusion, we denote $J_n$ just with $J$.

Let us consider a matrix

\[
(3.2) \quad M = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad a, b, c, d \in M(n, \mathbb{R}).
\]

It is easy to prove \[38\]-\[40\], \[77\] that
Remark 1. a) If $M \in \text{Sp}(n, \mathbb{R})$, then $M$ is similar with $M^t$ and $M^{-1}$ and also $J \in \text{Sp}(n, \mathbb{R})$.

b) If $M \in \text{Sp}(n, \mathbb{R})$ is as in (3.2), then the matrices $a, b, c, d$ in (3.2) verify the sets of equivalent conditions

\[(3.3a) \quad ab^t - ba^t = 0_n, \quad ad^t - bc^t = 1_n, \quad cd^t - dc^t = 0_n;\]
\[(3.3b) \quad a^t c - c^t a = 0_n, \quad a^t d - c^t b = 1_n, \quad b^t d - d^t b = 0_n.\]

c) If $M \in \text{Sp}(n, \mathbb{R})$ has the form (3.2), then

\[(3.4) \quad M^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}.\]

d) The matrices in $\text{Sp}(n, \mathbb{R})$ have the determinant 1.
e) The following subsets of $\text{GL}(2n, \mathbb{R})$ are subgroups of $\text{Sp}(n, \mathbb{R})$

\[N = \left\{ \begin{pmatrix} 1_n & A \\ 0_n & 1_n \end{pmatrix} : A = A^t \right\}, \quad \tilde{N} = \left\{ \begin{pmatrix} 1_n & 0_n \\ B & 1_n \end{pmatrix} : B = B^t \right\},
D = \left\{ \begin{pmatrix} C & 0_n \\ 0_n & (C^t)^{-1} \end{pmatrix} : C \in \text{GL}(n, \mathbb{R}) \right\}.
\]

$\text{Sp}(n, \mathbb{R})$ is generated by $D \cup \tilde{N} \cup \{J\}$ and $D \cup N \cup \{J\}$.

Using (3.4) it can be shown that the matrix $M \in \text{Sp}(n, \mathbb{R}) \cap O_{2n}$ has the expression

\[(3.5) \quad M = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}, \quad X^t X + Y^t Y = XX^t + YY^t = 1_n, \quad X^t Y = Y^t X, \quad Y X^t = XY^t.
\]

If $M \in M(2n, \mathbb{R})$ has the properties (3.5), let

\[(3.6) \quad M' := X + i Y \in M(n, \mathbb{C}),\]

and

Remark 2. The correspondence $M \rightarrow M'$ of (3.5) with (3.6) is a group isomorphism $\text{Sp}(n, \mathbb{R}) \cap O_{2n} \approx \text{U}(n)$.

3.2. The real symplectic algebra $\mathfrak{sp}(n, \mathbb{R})$. The real symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ is a real form of the simple Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ of type $\mathfrak{c}_n$ and $X \in \mathfrak{sp}(n, \mathbb{R})$ if $X^t J + JX = 0$, or equivalently

\[(3.7) \quad X = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \quad b = b^t, \quad c = c^t,
\]

where $a, b, c \in M(n, \mathbb{R})$, and similarly for $\mathfrak{sp}(n, \mathbb{C})$.

We write an element $X$ (3.7) as

\[(3.8) \quad H_{ij} := \begin{pmatrix} E_{ij} & 0_n \\ 0_n & -E_{ji} \end{pmatrix}, \quad 2F_{ij} := \begin{pmatrix} 0_n & E_{ij} + E_{ji} \\ 0_n & 0_n \end{pmatrix}, \quad 2G_{ij} := \begin{pmatrix} 0_n & E_{ij} + E_{ji} \\ E_{ij} + E_{ji} & 0_n \end{pmatrix}.
\]
In the matrix realization (3.7), the real algebra \( \mathfrak{sp}(n, \mathbb{R}) \) has the \( 2n^2 + n \) generators
\[
H_{ij}, \quad F_{ij}, \quad G_{ij}, \quad 1 \leq i \leq j \leq n.
\]

3.3. \( X_n \) as Hermitian symmetric space. We briefly recall some well known facts about Hermitian symmetric spaces \cite{8,44,84}. We use the notation

- \( X_n \): Hermitian symmetric space of noncompact type, \( X_n = G_0/K \);
- \( X_c \): compact dual of \( X_n \), \( X_c = G_c/K \);
- \( G_0 \): real Hermitian group;
- \( G = G^c_0 \): the complexification of \( G_0 \);
- \( P \): a parabolic subgroup of \( G \);
- \( K \): maximal compact subgroup of \( G_0 \);
- \( G_c \): compact real form of \( G \).

The compact manifold \( X_c \) of \( \frac{n(n+1)}{2} \) complex dimension has a complex structure inherited from the identification of \( X_c \) with \( G/P \). The group \( G_c \) acts transitively on \( X_c \) with isotropy group \( K = G_0 \cap P = G_c \cap P \).

\[ X_n = G_0/K = G_n(x_0) \]

is open in \( X_c \), where \( x_0 \) is a base point of \( G \) corresponding to \( K \). If \( \{e_1, \ldots, e_{2n}\} \) is a base of \( \mathbb{C}^{2n} \), in our case we take \( x_0 = e_{n+1} \wedge \cdots \wedge e_{2n} \in X_n \) as base point, and \( G_0 = \text{Sp}(n, \mathbb{R}) \approx \text{Sp}(n, \mathbb{R})_c \).

\( X_c \) includes \( X_n \) under Borel embedding \( X_n \subset X_c : gK \mapsto gP, \ g \in G_0 \).

The hermitian form on \( \mathbb{C}^{2n} \)
\[
< u, v > = - \sum_{j=1}^n u^j \bar{v}^j + \sum_{k=1}^n u^{n+k} \bar{v}^{n+k}
\]
specifies the indefinite unitary group \( U(n, n) \), hence the transformation group \( \text{Sp}(n, \mathbb{R})_c \) acting on \( X_n \)
\[
G := \text{Sp}(n, \mathbb{C}), \ G_c := \text{Sp}(n) = \text{Sp}(n, \mathbb{C}) \cap U(2n) \subset SU(2n), \ K := U(n).
\]

We have also
\[
P := \{ g \in G : g(x_0) = x_0 \} = \left\{ \begin{pmatrix} a & 0_n \\ c & d \end{pmatrix} : a'c = c'a, \ a'd = \mathbb{1}_n \right\}.
\]

Let us consider also
\[
m^+ := \left\{ \begin{pmatrix} 0_n & b \\ 0_n & 0_n \end{pmatrix} : b' = b \right\}, \ b \in M(n, \mathbb{C}).
\]

Then
\[
(3.10) \quad W \mapsto \hat{W} = \begin{pmatrix} 0_n & W \\ 0_n & 0_n \end{pmatrix}, \xi(W) = (\exp \hat{W})x_0 = v_1 \wedge \cdots \wedge v_n, (v_1, \cdots, v_n) = \begin{pmatrix} W \\ \mathbb{1}_n \end{pmatrix},
\]

and \( \xi \) maps the symmetric \( n \times n \) matrices \( W \) of \( m^+ \) such that \( \mathbb{1}_n - W \hat{W} > 0 \) onto a dense open subset of \( X_c \) that contains \( X_n \).

\( X_n \) is a Hermitian symmetric space of type CI (cf. Table V, p. 518, in \cite{44}), identified with the symmetric bounded domain of type II, \( \mathcal{R}_{II} \) in Hua’s notation \cite{51}.

Let us denote by \( X_n \) the set
\[
(3.11) \quad X_n := \{ v \in M(n, \mathbb{C}) | v = s + ir, s, r \in M(n, \mathbb{R}), r > 0, s^t = s; r^t = r \}.
\]
Remark 3. The action \((3.12)\) of \(\text{Sp}(n, \mathbb{R})\) on the Siegel upper half space \(X_n\)
\[(3.12) \quad v_1 = M(v) = (av + b)(cv + d)^{-1} = (vc^t + d^t)^{-1}(va^t + b^t),\]
is a transitive one. The correspondence
\[\zeta : X_n \rightarrow X_n = \text{Sp}(n, \mathbb{R})/K, \quad K = \text{Sp}(n, \mathbb{R}) \cap O_{2n}; v \mapsto M_{X+1Y}K,\]
where \(M_{X+1Y}\) is defined in \((3.13)\), is a 1-1 map which realizes the Siegel upper half space \((3.11)\) as the homogenous manifold \(X_n\).

**Proof.** Firstly it is proved that the matrix \(cv + d\) in \((3.12)\) is invertible, see e.g. [39, pp 1-11]. Then it is proved that \(M(v) \in X_n\) [39, 77].

It is find a symplectic map that sends \(i1_n\) to \(X + iY \in X_n, Y > 0\) as the composition of the symplectic maps \(V \rightarrow \sqrt{Y}V\sqrt{Y}\) and \(V \rightarrow V + X\) associated with the symplectic matrices
\[\begin{pmatrix} \sqrt{Y} & 0_n \\ 0_n & \sqrt{Y}^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_n & X \\ 0_n & 1_n \end{pmatrix}.
\]

We introduce the notation
\[(3.13) \quad M_{X+1Y} := \begin{pmatrix} 1_n & X \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0_n \\ 0_n & \sqrt{Y}^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{Y} & X \sqrt{Y}^{-1} \\ 0_n & \sqrt{Y}^{-1} \end{pmatrix}.
\]

The subgroup of \(\text{Sp}(n, \mathbb{R})\) which stabilizes \(i1_n \in X_n\) is the subgroup of orthogonal symplectic matrices of the form \((3.5)\). \(\square\)

Note that an argument similar with that used in Remark 3 was given in [1], following [78].

3.4. Pre-Iwasawa and modified pre-Iwasawa decompositions. We recall that the Iwasawa decomposition [44, Ch VI, §3] of \(\text{SL}(2, \mathbb{R})\) is used for the so called \(S\)-parametrization of the Jacobi group \(G_J^1(\mathbb{R}), \text{see [21, p 4], [22, p 15], [23, p 7]}.\)

In the present paper we find a similar decomposition for \(G_J^n(\mathbb{R})\).

We recall the Iwasawa decomposition [44, Ch. VI, §3] of \(\text{Sp}(n, \mathbb{R}) \ni G = KAN\ [83, p. 285] \) corresponds to

\[K := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in U(n) \right\},\]
\[A := \{\text{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}); a_1, \ldots, a_n > 0\},\]
\[N := \left\{ \begin{pmatrix} A & B \\ 0_n & (A^{-1})^t \end{pmatrix} : \text{A real unit upper triangular}, AB^t = BA^t \right\}.
\]

For Cholesky factorisation see [83, p. 287] and [82]; for QR decomposition see [76, p. 143]. We also mention the Iwasawa decomposition for \(G_J^n(\mathbb{R})\) was considered in [87, § 9.1.2].

Following the method of [2] and [42, §2.2.2], we find similarly

**Lemma 4. Pre-Iwasawa decomposition** Let as consider the pre-Iwasawa decomposition of \(M \in \text{Sp}(n, \mathbb{R})\)
\[(3.14) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_n & x \\ 0_n & 1_n \end{pmatrix} \begin{pmatrix} y & 0_n \\ 0_n & y^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.
\]
where the last matrix in (3.14) in $U(n)$ verify (3.5).

We find

\begin{equation}
(3.15) \quad y = (dd^t + cc^t)^{-1/2}, \quad X - iY = y(d + ic).
\end{equation}

Let us also define

\[ t := y^2(db^t + ca^t)y^{-1} = (bd^t + ac^t)y. \]

The matrices $y$ and $x := ty$ are symmetric, $y$ is positive definite, and all the factors in (3.14) are unique. We have also

\begin{equation}
(3.16) \quad x = (dd^t + cc^t)^{-1}(db^t + ca^t) = (bd^t + ac^t)(dd^t + cc^t)^{-1}.
\end{equation}

The inverse of the transform $(a,b,c,d) \rightarrow (x,y,X,Y)$ in equations (3.15), (3.16) is

\begin{equation}
(3.17) \quad a = yX - xy^{-1}Y, \quad b = yY + xy^{-1}X, \quad c = -y^{-1}Y, \quad d = y^{-1}X.
\end{equation}

In the case of $SL(2, \mathbb{R})$, the expression (3.17) corresponds to (46.b) in [13] if we replace $y \rightarrow y^{1/2}$ and take $X = \cos \theta$, $Y = \sin \theta$.

The first factor in (3.14) corresponds to the “free propagation subgroup” [2].

In the next Lemma we modify the pre-Iwasawa decomposition of $Sp(n, \mathbb{R})$ so that it coincides with Iwasawa decomposition of the group $Sp(1, \mathbb{R}) \approx SL(2, \mathbb{R})$ in [23] p 9.

We get

**Lemma 5. Modified pre-Iwasawa decomposition** The action of $M \in Sp(n, \mathbb{R})$ (3.2) on $X_n$, expressed in the parameters of the pre-Iwasawa decomposition in Lemma 4

\begin{equation}
(3.18) \quad (a,b,c,d) \times (x',y',X',Y') \rightarrow (x_1,y_1,X_1,Y_1),
\end{equation}

where $x', y' \in M(n, \mathbb{R})$, $x' = (x')^t, y' = (y')^t, y' > 0$, and

\begin{equation}
(3.19a) \quad a = y^{1/2}X - xy^{-1/2}Y, \quad b = y^{1/2}Y + xy^{-1/2}X, \quad c = -y^{-1}Y, \quad d = y^{-1}X,
\end{equation}

\begin{equation}
(3.19b) \quad x = y(db^t + ca^t), \quad y = (dd^t + cc^t)^{-1}, \quad X - iY = y^{1/2}(d + ic),
\end{equation}

is given by the formulas

\begin{equation}
(3.20) \quad x_1 + i y_1 = [c(y' + x'y'^{-1}x')c^t + d(y')^{-1}d^t + cx'(y')^{-1}d^t + d(y')^{-1}x'^t]^{-1} \times [c(y' + x(y')^{-1}x')a^t + cx'(y')^{-1}b^t + d(y')^{-1}x'a^t + d(y')^{-1}b^t + i],
\end{equation}

\begin{equation}
X_1 - iY_1 = (y_1)^{1/2}[(c(x' + d)(y')^{-1}2X' + c(y')^{1/2}Y')] + i[(c(y')^{1/2}X' - (cx' + d)(y')^{-1/2}Y')],
\end{equation}

while the action given by (3.12) $M \times (x',y') \rightarrow (x_1,y_1), v_1 := x_1 + iy_1$ expressing the linear fractional (3.12) transformation is

\begin{equation}
(3.22) \quad x_1 + iy_1 = (\tilde{v}'c^t + d^t)^{-1}(B + iy')(cv' + d)^{-1},
\end{equation}

\[ B = 2\tilde{v}'ca'cv' + \tilde{v}'(c'b + a'd) + (b'c + d'a)v' + 2bd. \]

The modified pre-Iwasawa decomposition (3.19) is compatible with the Möbius transform (3.22), i.e.

\begin{equation}
(3.23) \quad x_1 + iy_1 \equiv x_1 + iy_1.
\end{equation}
The transformation of the matrices associated as in Remark 2 to the pair \((X,Y)\) defined in (3.21) under the action (3.18) reads

\[
\begin{pmatrix}
X_1 & Y_1 \\
-Y_1 & X_1
\end{pmatrix} = y_1 \left[ (cx' + d)(y')^{-\frac{1}{2}} \mathbb{1}_{2n} - c(y')^{\frac{1}{2}}J_{2n} \right] \begin{pmatrix}
X' & Y' \\
-Y' & X'
\end{pmatrix}.
\]

**Proof.** This is an easy but long calculation and we indicate only the main steps.

We write (3.20) as

\[
x_1 + iy_1 = A^{-1}M,
\]

where

\[
A := c(y' + x'y^{-1}x')c' + d(y')^{-1}d' + cx'(y')^{-1}d' + d(y')^{-1}x'c',
\]

\[
M := c(y' + x'(y')^{-1}x')a' + cx'(y')^{-1}b' + d(y')^{-1}x'a + d(y')^{-1}b'.
\]

Firstly it is proved that \(y_1\) defined in (3.20) is equal with \(y_1\) in (3.22) and then it is obtained

\[
A = (cv' + d)y'^{-1}(\bar{v}c' + d'),
\]

or

(3.24) \[\]

\[
A^{-1} = (\bar{v}c' + d')^{-1}y'(cv' + d)^{-1}.
\]

In order to prove that \(x_1\) in (3.20) is equal with \(x_1\) in (3.22), with (3.24), we have to verify that

\[
y'(cv' + d)^{-1}M = \frac{B}{2}(cv' + d)^{-1},
\]

i.e.

(3.25) \[\]

\[
M(cv' + d) = (cv' + d)(y')^{-1}\frac{B}{2}.
\]

Using equations (3.3), it is verified the identity of the imaginary parts of both sides of (3.25) and then the identity of the real parts. \(\square\)

4. The real Jacobi group \(G_n^J(\mathbb{R})\)

The real Jacobi group \(G_n^J(\mathbb{R})\) has the composition law

(4.1) \[\]

\[
(M, (\lambda, \mu, \kappa)) \times (M', (\lambda', \mu', \kappa')) = (MM', (\bar{\lambda} + \lambda', \bar{\mu} + \mu', \kappa + \kappa' + \bar{\lambda}\mu' - \bar{\mu}\lambda'')),
\]

where \(M, M' \in \text{Sp}(n, \mathbb{R})\) have the form (3.2) and verifies the conditions of (3.3) \((\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_n(\mathbb{R}), \) and \((\bar{\lambda}, \bar{\mu}) = (\lambda, \mu)M' \) [16, 55, 74, 75, 87, 92].

4.1. The Jacobi group \(G_n^J(\mathbb{R})\) as subgroup of \(\text{Sp}(n+1, \mathbb{R})\). Let us consider a matrix \(M \in \text{Sp}(n, \mathbb{R})\) as in (3.2) verifying (3.3). Let us introduce the block matrix

(4.2) \[\]

\[
g := \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in M(2n + 2, \mathbb{R}),
\]

where the submatrices \(A, B, C, D \in M(n + 1, \mathbb{R})\) are defined as in (1.1), i.e.

(4.3) \[\]

\[
A := \begin{pmatrix}
an_n & \varnothing_{n1} \\
\lambda_{1n} & 1_{11}
\end{pmatrix},
B := \begin{pmatrix}
b_{mn} & g_{n1}^t \\
\mu_{1n} & \kappa_{11}
\end{pmatrix},
C := \begin{pmatrix}
c_{mn} & \varnothing_{n1} \\
\varnothing_{1n} & 0
\end{pmatrix},
D := \begin{pmatrix}
d_{nn} & -p_{1n}^t \\
\varnothing_{1n} & 1_{11}
\end{pmatrix}.
\]

We verify that indeed
Lemma 6. The matrix $g$ defined in (4.2), (4.3) is in $\text{Sp}(n+1, \mathbb{R})$.

Proof. We calculate the submatrices of the matrix $L$

$$L := gJ_{n+1}g^t = \begin{pmatrix} U & V \\ Z & T \end{pmatrix}.$$ 

We find

$$U = 0_{n+1}, \ V = 1_{n+1}, \ Z = -1_{n+1}, \ T = 0_{n+1},$$

i.e. $L = J_{n+1}$ and the conditions (3.1) are verified. \hfill $\blacksquare$

If $g = (M, X, \kappa) \in G^J_n(\mathbb{R})$, then $g^{-1} = (M^{-1}, -Y, -\kappa)$, i.e., with the conventions in (4.2), (4.3), we have the following representation in $\text{Sp}(n+1, \mathbb{R})$, see also (1.1)

$$(4.4) \quad g = \begin{pmatrix} a & 0 & b & q^t \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -p^t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} d^t & 0 & -b^t & -\mu^t \\ -p & 1 & -q & -\kappa \\ -c^t & 0 & a^t & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(n, \mathbb{R}).$$

4.2. The Lie algebra $\mathfrak{g}^J_n(\mathbb{R})$. Now we introduce a set of matrices that form a base for the Lie algebra $\mathfrak{g}^J_n(\mathbb{R})$ embedded in $\mathfrak{sp}(n+1, \mathbb{R})$ as in Lemma 6 which in the case $n = 1$ corresponds to the base $F, G, H, P, Q, R$ in [15]

$$(4.5a) \quad 2(F_{IJ})_{ij} := \delta_{I,i}\delta_{J,n+1+j} + \delta_{I,j}\delta_{J,n+1+i}, \ I, J = 1, \ldots, 2n+2; i, j = 1, \ldots, n;$$

$$(4.5b) \quad 2(G_{IJ})_{ij} := \delta_{I,n+1+i}\delta_{J,j} + \delta_{I,n+1+j}\delta_{J,i},$$

$$(4.5c) \quad (H_{IJ})_{ij} := \delta_{I,i}\delta_{J,j} - \delta_{I,j}\delta_{J,i},$$

$$(4.5d) \quad (P_{IJ})_{ij} := \delta_{I,n+1}\delta_{J,j} - \delta_{I,j}\delta_{J,n+1+i},$$

$$(4.5e) \quad (Q_{IJ})_{ij} := \delta_{I,i}\delta_{J,2n+2} + \delta_{I,n+1}\delta_{J,n+1+i},$$

$$(4.5f) \quad R_{IJ} := \delta_{I,n+1}\delta_{J,2n+2}.$$ 

With the conventions (4.2), (4.3), the first three matrices in (4.5) can be written down as

$$(4.6a) \quad 2F_{ij} := \begin{pmatrix} 0 & 0 & E_{ij} + E_{ji} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ i, j = 1, \ldots, n;$$

$$(4.6b) \quad 2G_{ij} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_{ij} + E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(4.6c) \quad H_{ij} := \begin{pmatrix} E_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -E_{ji} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

while the matrices $P_p, Q_q, p, q = 1, \ldots, n, R$ have been already defined in (2.3).
An element $X \in \mathfrak{g}^J_n(\mathbb{R})$ can be written as matrix of $\text{Sp}(n+1, \mathbb{R})$ in the base (4.6) as

$$X = \sum_{i,j=1}^{n} a_{ij} H_{ij} + 2 \sum_{1 \leq i < j \leq n} (b_{ij} F_{ij} + c_{ij} G_{ij})$$

$$+ \sum_{1 \leq i = j \leq n} (b_{ij} F_{ij} + c_{ij} G_{ij}) + \sum_{i=1}^{n} (p_i P_i + q_i Q_i) + rR, \quad b = b^t, \ c = c^t.$$ 

It can be verified that

**Lemma 7.** The commutation relations of the generators (4.6) of the Jacobi algebra $\mathfrak{g}^J_n(\mathbb{R})$ are

(4.7a) $[H_{kl}, F_{ij}] = \delta_{ij} F_{ik} + \delta_{kl} F_{kj}$,

(4.7b) $[G_{ij}, H_{kl}] = \delta_{ij} G_{lk} + \delta_{kl} G_{ij}$,

(4.7c) $4[F_{ij}, G_{kl}] = \delta_{kl} H_{kj} + \delta_{ij} H_{ik} + \delta_{jk} H_{il} + \delta_{ik} H_{jl}$,

(4.7d) $[P_p, Q_q] = 2\delta_{pq} R$,

(4.7e) $2[P_p, F_{ij}] = \delta_{pi} Q_j + \delta_{pj} Q_i$,

(4.7f) $2[Q_q, G_{ij}] = \delta_{iq} P_j + \delta_{jq} P_i$,

(4.7g) $[P_p, H_{ij}] = \delta_{pi} P_j$,

(4.7h) $[H_{ij}, Q_q] = \delta_{jq} Q_i$.

The commutation relations (4.7) of the generators of $G^J_n(\mathbb{R})$ represent the generalization of the corresponding commutation relations (3.4), (5.1) and (8.20) of the generators of $G^J_n(\mathbb{R})$ in (1.5).

4.3. **The action.** Following [11] §5, let us consider the restricted real group $G^J_n(\mathbb{R})_0$ consisting of elements of the form defined in (4.1), but $g = (M, X)$, where $X = (\lambda, \mu)$.

We consider the Siegel-Jacobi upper half space $\mathcal{X}^J_n$ realised as in (3.11).

We introduce for $\mathcal{X}^J_n$ the analog of parametrization used in [21 p 7], [23 p. 11], [40 § 38] for $\mathcal{X}^J_1$

(4.8) $u := pv + q, \ v := x + iy, \ v = v^t, \ y > 0, \ p, q \in M(1, n, \mathbb{R})$.

It should be noted that there is an isomorphism $G^J_n(\mathbb{R})_0 \cong (M, X, K) \to (M, X) \in G^J_n(\mathbb{R})_0$ through which the action of $G^J_n(\mathbb{R})_0$ on $\mathcal{X}^J_n$ can be defined as in [11] Proposition 2.

It is easy to prove that

**Lemma 8. a)** If $\mathcal{X}^J_n \ni v = x + iy$, then the action of $G^J_n(\mathbb{R})_0$ on $\mathcal{X}^J_n$: $(M, X) \times (v', u') \to (v_1, u_1)$, where $M \in \text{Sp}(n, \mathbb{R})$ has the expression (3.22), is given by the formulas

(4.9a) $v_1 = (av' + b)(cv' + d)^{-1} = (v'c^t + d^t)^{-1}(v'a^t + b^t)$,

(4.9b) $u_1 = (u' + \lambda v' + \mu)(cv' + d)^{-1}$.

If the modified pre-Iwasawa decomposition (3.19) is used, $v_1$ in (4.9a) has the equivalent expressions (3.20), (3.22) via the identification (3.23).
b) For \( \lambda, \mu \in M(1, n, \mathbb{R}) \), let us consider \((p, q)\) such that

\[
(4.10a) \quad (p, q) = (\lambda, \mu) M^{-1} = (\lambda d^t - \mu c^t, -\lambda b^t + \mu a^t),
\]

\[
(4.10b) \quad (\lambda, \mu) = (p, q) M = (pa + qc, pb + qd), \quad p, q, \lambda, \mu \in M(1, n, \mathbb{R}).
\]

Then the action of \( G^J_n(\mathbb{R}) \) on \( \mathcal{X}^J_n \): \((M, X) \times (x', y', p', q') \rightarrow (x_1, y_1, p_1, q_1)\) is given by \((4.9a)\), while

\[
(4.11) \quad (p_1, q_1) = (p, q) + (p', q') \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = (p + p'd^t - q'c^t, q - p'b^t + q'a^t).
\]

c) The action of \( \tilde{G}^J_n(\mathbb{R}) \) on \( \tilde{\mathcal{X}}^J_n \approx \mathcal{X}^J_n \times \mathbb{R} \):

\[
(4.12a) \quad (M, (\lambda, \mu), \kappa) \times (v', u', \kappa') \rightarrow (v_1, u_1, \kappa_1),
\]

\[
(4.12b) \quad (M, (\lambda, \mu), \kappa) \times (x', y', p', q', \kappa') \rightarrow (x_1, y_1, p_1, q_1, \kappa_1)
\]

is given by \((4.9), (4.11)\) and

\[ \kappa_1 = \kappa + \kappa' + \lambda q'^t - \mu p'^t. \]

d) The 1-form

\[
(4.13) \quad \lambda^R = d \kappa - p \, d q^t + q \, d p^t
\]

is invariant to the action \((4.12)\) of \( \tilde{G}^J_n(\mathbb{R}) \) on \( \tilde{\mathcal{X}}^J_n \).

e) The action of \( G^J_n(\mathbb{R}) \) on \( G^J_n(\mathbb{R}) \)

\[
(M, (\lambda, \mu), \kappa) \times (S_n)^I \rightarrow (S_n)^I,
\]

is given in \((3.21)\) for \( X', Y' \), while the other actions are given in \( a)-d)\) of the present Lemma.

### 4.4. Fundamental vector fields on \( \mathcal{X}^J_n \) and \( \tilde{\mathcal{X}}^J_n \)

We calculate FVF associated to the generators of the Jacobi group on homogenous manifolds attached to \( G^J_n(\mathbb{R}) \).

For a symmetric matrix \( x \in M(n) \) we introduce the notation

\[
(4.14) \quad \partial_x := \left( (2 - \delta_{ij}) \frac{\partial}{\partial x_{ij}} \right)_{i,j=1,\ldots,n}.
\]

If \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \) are two \( n \)-vectors, we introduce also the notation

\[
(4.15) \quad (a \odot b)_{ij} := a_i b_j + a_j b_i - a_i b_j \delta_{ij}, \quad i, j = 1, \ldots, n.
\]

Note the isomorphism of the representations \((3.8)\) and \((4.6)\) of \( \mathfrak{sp}(n, \mathbb{R}) \). To a matrix \( A \) as in \((4.6)\), let us denote by \( \hat{A} \) the corresponding matrix in the representation \((3.8)\).

We make the following

**Remark 9.** Let \( z \in M(n) \). Then we have the relation \((4.16)\)

\[
(4.16) \quad \frac{\partial}{\partial z} \, dz = 1_n, \text{ i.e. } \frac{\partial z_{ij}}{\partial z_{pq}} = \delta_{ip} \delta_{jq}.
\]

If the matrix \( z \) is symmetric, instead of \((4.16)\) we have \((4.17)\)

\[
(4.17) \quad D_z \, dz = 1_n, \quad z = z^t, \text{ i.e. } (D_z)_{\mu \nu} \, dz_{\nu \chi} = \delta_{\mu \chi}, \quad z_{\mu \nu} = z_{\nu \mu}.
\]
where

\[(D_z)_{\mu\nu} := e_{\mu\nu} \frac{\partial}{\partial z_{\mu\nu}}, \quad e_{\mu\nu} := \frac{1 + \delta_{\mu\nu}}{2}, \]  
no summation!

**Proof.** (4.16) is evident.

Using equation [11, (4.5)] which says that for a symmetric matrix \( w \) we have

\[(4.19) \quad \frac{\partial w_{ij}}{\partial w_{pq}} = \delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp} - \delta_{ij} \delta_{pq}, \quad w_{ij} = w_{ji}, \]

(4.17) it is verified, where the symbol \( D \) in (4.18) was introduced in [11, (3.39)]. □

We obtain the following representations of the FVF associated to the base (4.6), (2.3) of the Lie algebra \( \mathfrak{g}_J^n(\mathbb{R}) \)

**Proposition 1.** a) The fundamental vector fields in the coordinates \((v, u)\) of \( X_J^n \) on which \( G_J^n(\mathbb{R}) \) acts as in Lemma 8 a) are given by the holomorphic FVF

\[(4.20a) \quad F_{ij}^* = \hat{F}_{ij} \frac{\partial}{\partial v}, \quad i, j = 1, \ldots, n; \]

\[(4.20b) \quad G_{ij}^* = -v \hat{G}_{ij} v \frac{\partial}{\partial v} - (\frac{\partial}{\partial u})^t u \hat{G}_{ij} v; \]

\[(4.20c) \quad H_{ij}^* = (\hat{E}_{ij} v + v \hat{E}_{ji}) \frac{\partial}{\partial v} + (\frac{\partial}{\partial u})^t u \hat{E}_{ji}; \]

\[(4.20d) \quad P_p^* = \hat{E}_p x (\frac{\partial}{\partial u})^t; \quad Q_q^* = \hat{E}_q (\frac{\partial}{\partial u})^t; \quad R^* = 0, \quad p, q = 1, \ldots, n. \]

b) The real holomorphic FVF associated to (4.20b) in the variables \((x, y, \xi, \rho)\) on \( X_J^n \), where \( v := x + iy, \ y > 0, \ u := \xi + i\rho \) as in (4.8), are

\[(4.21a) \quad F_{ij}^* = (F_1^*)_{ij}, \]

\[(4.21b) \quad G_{ij}^* = (G_1^*)_{ij} + (\frac{\partial}{\partial \xi})^t (\rho \hat{G}_{ij} y - \xi \hat{G}_{ij} x) - (\frac{\partial}{\partial \rho})^t (\xi \hat{G}_{ij} y + (\rho \hat{G}_{ij} x); \]

\[(4.21c) \quad H_{ij}^* = (H_1^*)_{ij} + (\frac{\partial}{\partial \xi})^t \xi \hat{E}_{ji} + (\frac{\partial}{\partial \rho})^t \rho \hat{E}_{ij}; \]

\[(4.21d) \quad P_p^* = \hat{E}_p x (\frac{\partial}{\partial \xi})^t + \hat{E}_p y (\frac{\partial}{\partial \rho})^t; \quad Q_q^* = \hat{E}_q (\frac{\partial}{\partial \xi})^t, \quad R^* = 0, \]

where

\[(4.22a) \quad (F_1^*)_{ij} = \hat{F}_{ij} \frac{\partial}{\partial x}; \]

\[(4.22b) \quad (G_1^*)_{ij} = \alpha \frac{\partial}{\partial x} - \beta \frac{\partial}{\partial y}; \]

\[(4.22c) \quad (H_1^*)_{ij} = (\hat{E}_{ij} x + x \hat{E}_{ji}) \frac{\partial}{\partial x} + (\hat{E}_{ij} y + y \hat{E}_{ji}) \frac{\partial}{\partial y}; \]

\[(4.22d) \quad \alpha := (y \hat{G}_{ij} y - x \hat{G}_{ij} x), \quad \beta := (x \hat{G}_{ij} y + y \hat{G}_{ij} x), \]

are FVF associated with the generators of \( \mathfrak{sp}(n, \mathbb{R}) \) corresponding to the action (4.9a) of \( \text{Sp}(n, \mathbb{R}) \) on \( X_n \).
c) The FVF (4.21) in the variables \((x, y, p, q)\) on \(\mathcal{X}'_n\), where

\begin{align}
\tag{4.23a}
v &= x + iy, \quad u = pv + q = \xi + i\rho, \\
\tag{4.23b}
p &= \rho y^{-1}, \quad q = \xi - \rho y^{-1}x,
\end{align}

are

\begin{align}
\tag{4.24a}
(F^*)_{ij} &= \hat{F}_{ij}(\frac{\partial}{\partial x} - \frac{\partial}{\partial q} \circ p), \\
(G^*)_{ij} &= (G^*_1)_{ij} - (\frac{\partial}{\partial p}^t) p y^{-1} \beta + \beta (\frac{\partial}{\partial p} y^{-1}) \circ p - \alpha \frac{\partial}{\partial p} \circ p + ((\frac{\partial}{\partial q}^t) p) \alpha \\
\tag{4.24b}
&= -\beta (\frac{\partial}{\partial q} xy^{-1}) \circ p + ((\frac{\partial}{\partial q}^t) p)(y^{-1} x \beta) \\
&\quad - ((\frac{\partial}{\partial p})^t q) y^{-1} \hat{G}_{ij} y + ((\frac{\partial}{\partial q})^t q) y^{-1} \alpha,
\end{align}

\begin{align}
(H^*)_{ij} &= (H^*_1)_{ij} + (\hat{E}_{ij} y + y \hat{E}_{ji}) \big[ - (\frac{\partial}{\partial p} y^{-1}) \circ p + (\frac{\partial}{\partial q} xy^{-1}) \circ p \big] \\
\tag{4.24c}
&= - (\hat{E}_{ij} x + x \hat{E}_{ji}) \frac{\partial}{\partial q} \circ p + ((\frac{\partial}{\partial q})^t q)(x \hat{E}_{ji} - y^{-1} x y \hat{E}_{ji}) \\
&\quad + ((\frac{\partial}{\partial q})^t q) \hat{E}_{ji} + ((\frac{\partial}{\partial q})^t q) \hat{E}_{ij},
\end{align}

\begin{align}
\tag{4.24d}
P^*_p &= E_p(\frac{\partial}{\partial p})^t, \quad Q^*_q = E_q(\frac{\partial}{\partial q})^t, \quad R^* = 0.
\end{align}

d) Now we consider the action of \(G^*_n(R)\) on \((u', v', \kappa')\) in \(\tilde{\mathcal{X}}'_n\) as in Lemma 8c). We find for FVF \(F^*_i, G^*_i, H^*_i\) the expressions (4.20a), (4.20b), respectively (4.20c), while instead of (4.20d), we find

\begin{align}
P^*_p &= \hat{E}_p v(\frac{\partial}{\partial u})^t + q \hat{\partial}_\kappa, \quad Q^*_q = \hat{E}_q(\frac{\partial}{\partial u})^t - p \hat{\partial}_\kappa, \quad R^* = \partial_\kappa.
\end{align}

e) The FVF on \(\tilde{\mathcal{X}}'_n\) in the variables \((x, y, \xi, \rho, \kappa)\) are given by (4.21a), (4.21b), respectively (4.21c) for \(F^*_i, G^*_i, H^*_i\), while (4.21d) became

\begin{align}
P^*_p &= E_p(x \frac{\partial}{\partial \xi} + y \frac{\partial}{\partial \rho}) + q \hat{\partial}_\kappa, \quad Q^*_q = E_q \frac{\partial}{\partial \xi} - p \hat{\partial}_\kappa, \quad R^* = \partial_\kappa, \quad p = \rho y^{-1}, \quad q = \xi - \rho y^{-1} x.
\end{align}

f) We express the FVF \(F^*_i, G^*_i, H^*_i\) on \(\tilde{\mathcal{X}}'_n\) in the variables \((x, y, p, q, \kappa)\) as in (4.24a), (4.24b), respectively (4.24c), and

\begin{align}
\tag{4.25}
P^*_p &= E_p(x \frac{\partial}{\partial q} - y \frac{\partial}{\partial q} xy^{-1} + y \frac{\partial}{\partial p} y^{-1}) + q \hat{\partial}_\kappa, \quad Q^*_q = E_q \frac{\partial}{\partial q} - p \hat{\partial}_\kappa, \quad R^* = \partial_\kappa.
\end{align}

Proof. a) We apply the definition of fundamental vector fields. For \(P_p, Q_q, R\) on components, we find

\begin{align}
(P^*_p)_{ij} &= (\hat{E}_pv)_{ij}, \quad (Q^*_q)_{ij} = (\hat{E}_q)_{ij}, \quad R^* = 0,
\end{align}

which we write as in (4.20d).
b) In order to determine the real holomorphic FVF associated to the holomorphic FVF (4.20), let \( Z \) be a holomorphic vector field on a complex \( n \)-dimensional manifold
\[
Z := \sum_{i=1}^{n} Z_j \frac{\partial}{\partial z_j}, \quad Z_j := A_j + i B_j, \quad A_j, B_j \in C^\infty(M).
\]
Then the real holomorphic field \( X = Z + \bar{Z} \) in coordinates \((x_j, y_j)\), \( z_j = x_j + i y_j \) is, see [15, Proposition 22 in v1] or [54, Proposition 2.11],
\[
X = \sum_{i=1}^{n} A_j \frac{\partial}{\partial x_j} + B_j \frac{\partial}{\partial y_j}.
\]

\[\square\]

c) In order to make the change of variables \((x, y, \xi, \rho) \to (x, y, p, q)\) as in (4.23), firstly it is observed that the Jacobian of the transformation is non-zero:
\[
\frac{\partial(x, y, \xi, \rho)}{\partial(x, y, p, q)} = -y < 0.
\]
With formula (4.19), we get the following formulas
\[
\begin{align*}
\frac{\partial}{\partial x_{ij}} & \to (2 - \delta_{ij}) \frac{\partial}{\partial x_{ij}} - (p \odot \frac{\partial}{\partial q})_{ij}; \\
\frac{\partial}{\partial y_{ij}} & \to (2 - \delta_{ij}) \frac{\partial}{\partial y_{ij}} - (\frac{\partial}{\partial p} y^{-1} \odot p)_{ij} + (\frac{\partial}{\partial q} x y^{-1} \odot p)_{ij}; \\
\frac{\partial}{\partial \xi_i} & \to \frac{\partial}{\partial q_i}; \\
\frac{\partial}{\partial \rho_l} & \to (\frac{\partial}{\partial p} y^{-1})_l - (\frac{\partial}{\partial q} x y^{-1})_l;
\end{align*}
\]
which can be written down in the conventions (4.14), (4.15) as
\[
\begin{align*}
\frac{\partial}{\partial x} & = \partial_x - \frac{\partial}{\partial q} \odot p; \\
\frac{\partial}{\partial y} & = \partial_y + [(- \frac{\partial}{\partial p} + \frac{\partial}{\partial q} x y^{-1}) \odot p]; \\
\frac{\partial}{\partial \xi} & = \frac{\partial}{\partial q}; \\
\frac{\partial}{\partial \rho} & = \frac{\partial}{\partial p} y^{-1} - \frac{\partial}{\partial q} x y^{-1}.
\end{align*}
\]

4.5. **Invariant one-forms on the Jacobi group.** From (4.4), we obtain
\[
(4.28) \quad g^{-1} \, \text{d} \, g = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{pmatrix},
\]
Lemma 10. With (4.28) and (4.30), we get

\begin{align}
A_{21} &= d\lambda - p d a - q d c; A_{22} = 0; A_{23} = d\mu - p d b - q d d; A_{24} = d\kappa - p d q^t + q d p^t; \\
A_{31} &= -c^t d a + a^t d c; A_{32} = 0; A_{33} = -c^t d b + a^t d d; A_{34} = -c^t d q^t - a^t d p^t; \\
A_{41} &= A_{42} = A_{43} = A_{44} = 0.
\end{align}

With (4.10) and (3.3), we get from (4.29) the relations

\begin{align}
A_{24} &= d\kappa - p d q^t + q d p^t; \quad A_{34} = -A_{21}; \quad A_{23} = A_{14}; \quad A_{11} = -A_{33}.
\end{align}

With (4.28) and (4.30), we get

Lemma 10. For $g \in g_n^I(\mathbb{R})$ as in (4.1), we have in the basis (4.6), (2.3) the expression

\[
g^{-1} d g = \sum_{i,j=1}^{n} (\lambda^H)_{ij} H_{ij} + \sum_{1 \leq i \leq j \leq n} [(\lambda^F)_{ij} F_{ij} + (\lambda^G)_{ij} G_{ij}] + \sum_{i=1}^{n} [(\lambda^P)_i P_i + (\lambda^Q)_i Q_i] + \lambda^R R,
\]

where the invariant one-forms corresponding to the generators (4.6) are

\begin{align}
(4.31a) & \quad \lambda^F = d^t d b - b^t d d = (\lambda^F)^t, \\
(4.31b) & \quad \lambda^G = -c^t d a + a^t d c = (\lambda^G)^t, \\
(4.31c) & \quad \lambda^H = d^t d a - b^t d c = d b^t c - d d^t a = (\lambda^H)^t, \\
(4.31d) & \quad \lambda^P = d\lambda - p d a - q d c = d p a + d q c = \lambda^P - \lambda H - \mu^G, \\
(4.31e) & \quad \lambda^Q = d q d + d p b = d\mu - p d b - q d d = \lambda^Q - \lambda F + \mu^H, \\
(4.31f) & \quad \lambda^R = d\kappa - p d q^t + q d p^t = \lambda^R + \lambda L^F \lambda^t - \mu \lambda G \mu^t - 2\lambda H \mu^t,
\end{align}

and $\lambda^P, \lambda^Q, \lambda^R$ are given by (2.3).

Let us introduce the notation

\[
L := y^{-1} d y, \quad R := d y y^{-1}, \quad C := y^{-1} d x y^{-1}.
\]

With Lemma 4, we rewrite the invariant one-forms (4.31) for $G_n^I(\mathbb{R})$ as

\begin{align}
(4.32a) & \quad \lambda^F = X^t d Y - Y^t d X + X^t CY + Y^t RX, \\
(4.32b) & \quad \lambda^G = -X^t d Y + Y^t d X + Y^t LX - Y^t CY + X^t RY, \\
(4.32c) & \quad \lambda^H = X^t d X + Y^t d Y + X^t LX - X^t CY - Y^t RY, \\
(4.32d) & \quad \lambda^P = d p(y X - xy^{-1} Y) - d q y^{-1} Y, \\
(4.32e) & \quad \lambda^Q = d q y^{-1} X + d p(y Y + xy^{-1} X), \\
(4.32f) & \quad \lambda^R = d\kappa - d q p^t + d p q^t.
\end{align}

We have also

\begin{align}
(4.33) & \quad \lambda^F + \lambda^G = X^t (L + R) Y + Y^t (L + R) X + X^t CX - Y^t CY, \\
& \quad \lambda^F - \lambda^G = 2(X^t d Y - Y^t d X) + 2X^t (L - R) Y + X^t CX + Y^t CY.
\end{align}
Equations (4.31) generalize to $G^J_n(\mathbb{R})$, $n \in \mathbb{N}$, the corresponding equations (4.4) and (5.19) in [15] for $G^J(\mathbb{R})$. The last expression of $\lambda^R$ was obtained previously in (4.13) just in analogy to [15, (5.5f)] for the Jacobi group $G^J_n(\mathbb{R})$ and the invariance of the 1-form was verified.

We see in (4.33) that for any $n \in \{\mathbb{N}\} \setminus \{1\}$, $\lambda^F + \lambda^G$ does not depend on $dX, dY$, but $\lambda^H$ does, while in the case $n = 1$ both $\lambda^F + \lambda^G$ and $\lambda^H$ they do not depend on $d\theta$.

Indeed, in the case of $G^J_1(\mathbb{R})$, $X = \cos \theta$, $Y = \sin \theta$, $y \to y^2$, we get equations (4.11) in [15]

\begin{align*}
(4.34a) \quad \lambda^F &= \frac{dx}{y} \cos^2 \theta + \frac{dy}{2y} \sin 2\theta + d\theta, \\
(4.34b) \quad \lambda^G &= -\frac{dx}{y} \sin^2 \theta + \frac{dy}{2y} \sin 2\theta - d\theta, \\
(4.34c) \quad \lambda^H &= -\frac{dx}{2y} \sin 2\theta + \frac{dy}{2y} \cos 2\theta, \\
(4.34d) \quad \lambda^F + \lambda^G &= \frac{dx}{y} \cos 2\theta + \frac{dy}{y} \sin 2\theta, \\
(4.34e) \quad \lambda^F - \lambda^G &= \frac{dx}{y} + 2d\theta, \\
(4.34f) \quad 2\lambda^H &= -\frac{dx}{y} \sin 2\theta + \frac{dy}{y} \cos 2\theta.
\end{align*}

With the first equation (4.33), we get

\begin{equation}
(\lambda^F + \lambda^G)^2 = \text{tr}[2(X'LYX'R + X'LYX'R + X'LXYY'X - X'LXYY'X)]
+ X'LYY'X + X'LYX'CX
+ 2(X'RYY'X + X'RYY'X - X'RYY'X)
+ 2(Y'LXX'CXX) + Y'LXX'CXX - Y'LXX'CXX + Y'LXX'CXX
+ 2(Y'RXX'CXX - Y'RXX'CXX) + Y'RXX'CXX + Y'RXX'CXX
+ X'CXY'RX + X'CXY'RX.
\end{equation}

With (4.32c), we get

\begin{align*}
(\lambda^H)^2 &= (\lambda^H_1)^2 + (\lambda^H_2)^2, \\
(\lambda^H_1)^2 &= \text{tr}[X'(LX - CY)X'dX + X'X'Y(dX + LX)] + Y'(dY - RY)Y'dY
- Y'dY(Y'R + X'C)Y + X'dXY'(dY - RY) - X'dXY'C
+ X'(LX - CY)Y'dY + Y'dYX'(dX + LX) - Y'RXY'tdX; \\
(\lambda^H_2)^2 &= \text{tr}[X'(LX - CY)X'LX + Y'RXY'(Y'R + X'C)Y
+ X'((CY - LX)Y'R - RXY'tL + (CY - LX)X'C)Y].
\end{align*}
With the second equation $(4.38)$, we get
\[
(\lambda^F - \lambda^G)^2 = \text{tr} \{4[(X^t d Y - Y^t d X)^2 + (X^t d Y - Y^t d X)(X^t C X + Y^t C Y)]
\]
\[
+ 2X^t C X^t Y^t C Y + (X^t C X)^2 + (Y^t C Y)^2 + [X^t (L - R) Y]^2
\]
\[
2(X^t C X + Y^t C Y)[X^t (L - R) Y + Y^t (R - L) X]
\]
\[
+ 4X^t (L - R)(X^t d Y - Y^t d X)].
\]
(4.37)

In the case of the Jacobi group $G_1^1(\mathbb{R})$, when $X = \cos \theta, Y = \sin \theta$ and $y \to y^{1/2}$, $(4.35)$, $(4.37)$, respectively $(4.36)$ become what is obtained from $(4.34d)$, i.e.
\[
(\lambda^F + \lambda^G)^2 = \frac{(\cos 2\theta d x)^2 + (\sin 2\theta d y)^2 + \sin 4\theta d x d y}{y^2};
\]
\[
(\lambda^H_1)^2 = 0;
\]
\[
(\lambda^H_2)^2 = \frac{(\sin 2\theta d x)^2 + (\cos 2\theta d y)^2 - \sin 4\theta d x d y}{4y^2};
\]
\[
(\lambda^F - \lambda^G)^2 = 4 d \theta^2 + \frac{d x^2}{y} + \frac{d x^2}{y^2}.
\]

4.6. Invariant vector fields on the Jacobi group. Once we have determined the invariant one-forms $(3.31)$, we have to determine the invariant vector fields orthogonal to them solving the equations
\[
(L^\alpha)^t = \delta_{\alpha\beta}, \alpha, \beta = F, G, H; \quad <(L^\alpha)^t|\lambda^\beta> = \delta_{\alpha\beta}, \alpha, \beta = P, Q, R.
\]

We find
\[
(L^F)^t = (\frac{\partial}{\partial b}) a + (\frac{\partial}{\partial c}) b,
\]
(4.39a)
\[
(L^G)^t = (\frac{\partial}{\partial a}) b + (\frac{\partial}{\partial c}) d,
\]
(4.39b)
\[
(L^H)^t = (\frac{\partial}{\partial a}) a + (\frac{\partial}{\partial b}) b + (\frac{\partial}{\partial c}) c + (\frac{\partial}{\partial d}) d,
\]
(4.39c)
\[
L^P = (\frac{\partial}{\partial p}) d - (\frac{\partial}{\partial q}) b - (\frac{\partial}{\partial k}) (pb + qd),
\]
(4.39d)
\[
L^Q = - (\frac{\partial}{\partial p}) c + (\frac{\partial}{\partial q}) a + (pa + qc)\frac{\partial}{\partial k},
\]
(4.39e)
\[
L^R = \frac{\partial}{\partial k}.
\]
(4.39f)

In order to determine the invariant vector fields orthogonal to the invariant one-forms $(4.32)$ as in $(4.38)$, we have to calculate the derivative of $(a, b, c, d)$ expressed as in pre-Iwasawa decomposition $(3.15)$, $(3.16)$ or modified pre-Iwasawa decomposition $(3.19)$, but this is not an easy task.

For example, let’s take the simpler case $d = y^{-1}X$ in $(3.15)$, then $d d = -y^{-1} d y y^{-1} + y^{-1} d X$. More generally, let us consider the one-forms
\[
F_{pq} := A_{pq} d y_{ij} B_{jq} + C_{pq} d X_{ij} D_{jq}, \quad A, B, C, D \in M(n, \mathbb{R}).
\]
We have to determine the invariant vector field

\[ f_{qr} := M_{qm}D_{mn}(y)N_{nr} + P_{qm}D_{mn}(X)Q_{nr} \]

such that

\[ <F_{pq}, f_{qr}> = \delta_{pr}. \]

The matrices \( M, N, P, Q \) such that satisfy the following matrix equation

\[ \text{tr}[(MB)(AN) + (CP^tQ)] = 1_n, \]

must be determined, which is generally a difficult problem. If we consider the expression of \( d \) in (3.19a) the situation is even more complicated because of the difficulties to calculate the differential of the square root of a matrix, see Appendix 7, and we abandon the task of explicitly determining the invariant vector fields orthogonal to the left invariant one-forms (4.31).

5. Invariant metrics on homogeneous manifolds associated to \( G^J_n(\mathbb{R}) \)

We follow the notation in [15, (4.15), (5.21)] for the invariant one-forms on \( G^J_1(\mathbb{R}) \).

**Proposition 2.** Let us introduce the invariant one-forms on \( G^J_n(\mathbb{R}) \)

\[
\begin{align*}
\lambda_1 &:= \sqrt{\alpha}(\lambda^F + \lambda^G), \\
\lambda_2 &:= \sqrt{\alpha}\lambda^H, \\
\lambda_3 &:= \sqrt{\beta}(\lambda^F - \lambda^G), \\
\lambda_4 &:= \sqrt{\gamma}\lambda^P, \\
\lambda_5 &:= \sqrt{\gamma}\lambda^Q, \\
\lambda_6 &:= \sqrt{\delta}\lambda^R,
\end{align*}
\]

(5.1)

where we use the expressions (4.32) for \( \lambda^F, \ldots, \lambda^R \). The composition law (4.1) in the variables \( (x, y, X, Y) \) is given in Lemma 4 or in Lemma 5 and for \( p, q, \kappa \) in Lemma 8. Let us consider the 4-parameter left invariant metric on \( G^J_n(\mathbb{R}) \), which coincides with metric (5.32) on \( G^J_1(\mathbb{R}) \) in [15]

\[ \text{d}s^2_{G^J_n(\mathbb{R})} = \sum_{i=1}^{6} \lambda_i^2, \]

(5.2)

where the square of the invariant one-forms \( \lambda_1, \lambda_2, \lambda_3 \) in (5.2) are given in (4.35), (4.36), respectively (4.37), and the squares of \( \lambda_4, \lambda_5, \lambda_6 \) are given taking the square of (4.32d) . . . (4.32e).

Depending of the values of the parameters \( \alpha, \beta, \gamma, \delta \), (5.2) gives the invariant metric on the following manifolds:

1. if \( \beta, \gamma, \delta = 0 \) - the Siegel upper half-plane \( \mathcal{X}_n \);
2. if \( \gamma, \delta = 0, \alpha \beta \neq 0 \) - the group \( \text{Sp}(n, \mathbb{R}) \);
3. if \( \beta, \delta = 0 \) - the Siegel-Jacobi half space \( \mathcal{X}^J_n \);
4. if \( \beta = 0 \) - the extended Siegel-Jacobi extended half space \( \tilde{\mathcal{X}}^J_n \);
5. if \( \alpha \beta \gamma \delta \neq 0 \) - the Jacobi group \( G^J_n(\mathbb{R}) \).

The invariant vector fields (4.39), orthonormal with respect the invariant one-forms (5.1) in the sense of (4.38), are orthonormal with respect to the metric (5.2).

Proposition 2 is an extension to \( G^J_n(\mathbb{R}) \), \( n \in \mathbb{N} \), of [15, Theorem 1] for \( G^J_1(\mathbb{R}) \). However, the expressions (4.35), (4.36), (4.37) are complicated and also the invariant vector fields (4.39) are in fact not explicitly calculated due to the difficulties signaled in Section 4.6.
Even the metric on the Siegel upper-half space given at (1) in Proposition 2 is difficult to recognize.

We give a simple expression of the invariant metric on \( \mathcal{X}_n^J \) without the invariant one-forms, using the metric determined on the Siegel-Jacobi upper half space \( \mathcal{X}_n^J \).

5.1. Invariant metrics on \( \mathcal{X}_n^J \) and \( \tilde{\mathcal{X}}_n^J \). Below \( k, 2k \in \mathbb{N} \) indexes the holomorphic discrete series of Sp\((n, \mathbb{R})\) and \( \nu > 0 \) indexes the representations of the Heisenberg group. We reformulate for \( G_n^J(\mathbb{R}), n \in \mathbb{N}, [\text{13}]\) Proposition 1 for \( G_1^J(\mathbb{R}) \). The starting point is [\text{14}] Proposition 3, see also [\text{14}] Theorem 3.2.

**Proposition 3.** a) The Kähler two-form

\[
-W \omega_{\mathcal{D}_\mathbb{H}}(W, \bar{z}) = \frac{k}{2} \operatorname{tr}(B \wedge B) + \nu \operatorname{tr}(A^t \bar{M} \wedge A),
\]

where

\[
B(W) := M \, d \, W, \quad M := (1_n - W \bar{W})^{-1}, \quad z \in M(1, n, \mathbb{C}), \quad \eta \in M(n, 1, \mathbb{C}),
\]

is \((G_n^J)_0\) invariant to the action \( \text{Sp}(n, \mathbb{R})_C \times \mathbb{C}^n : (W, z^t) \to (W_1, z_1^t) \).

\[
(5.4) \quad \begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ \bar{\mathcal{Q}} & \mathcal{P} \end{pmatrix} (\alpha) \times (W, z^t) = ((W \mathcal{Q}^t + \mathcal{P}^t)^{-1}(\mathcal{Q}^t + W \mathcal{P}^t), (W \mathcal{Q}^t + \mathcal{P}^t)^{-1}(z^t + \alpha^t - W \alpha^t)),
\]

where \((5.3a)\) are verified, i.e.

\[
(5.5) \quad \mathcal{P} \mathcal{P}^t - \mathcal{Q} \mathcal{Q}^t = 1_n, \quad \mathcal{P} \mathcal{Q} = \mathcal{Q} \mathcal{P}, \quad \mathcal{P}^t \mathcal{P} = \mathcal{Q}^t \mathcal{Q} = 1_n, \quad \mathcal{P}^t \bar{\mathcal{Q}} = \mathcal{Q}^t \mathcal{P}.
\]

We have the change of variables \((W, z) \to (W, \eta)\)

\[
(5.6) \quad FC : \quad z^t = \eta - W \bar{\eta}; \quad FC^{-1} : \quad \eta = M(z^t + Wz^t),
\]

and

\[
A(W, z) \to A(W, \eta) = d \eta - W \, d \bar{\eta}.
\]

The complex two-form

\[
\omega_{\mathcal{D}_\mathbb{H}}(W, \eta) := FC^*(\omega_{\mathcal{D}_\mathbb{H}}(W, z))
\]

is not a Kähler two-form.

The symplectic two-form \(\omega_{\mathcal{D}_\mathbb{H}}(W, \eta)\) is invariant to the action \((g, \alpha) \times (W, \eta) \to (W_1, \eta_1)\) of \((G_n^J)_0\) on \(\mathcal{D}_n \times \mathbb{C}^n\)

\[
\eta_1^t = \mathcal{P}(\eta + \alpha)^t + \mathcal{Q}(\eta + \alpha)^t,
\]

where \(W_1\) is defined in \((5.4)\) and \((\mathcal{P}, \mathcal{Q})\) verify \((5.5)\).

b) Using the partial Cayley transform

\[
(5.7a) \quad \Phi^{-1} : v = i(1_n - W)^{-1}(1_n + W); \quad u^t = (1_n - W)^{-1}z^t, \quad W \in \mathcal{D}_n, \quad v \in \mathcal{X}_n;
\]

\[
(5.7b) \quad \Phi : W = (v - i1_n)^{-1}(v + i1_n), \quad z^t = 2i(v + i1_n)^{-1}u^t, \quad z, u \in M(1, n, \mathbb{C}),
\]

we obtain

\[
(5.8) \quad A(W, z) = 2i(v + i1_n)^{-1}G(v, u), \quad G(v, u) = d \, u^t - d \, v(v - \bar{v})^{-1}(u - \bar{u})^t.
\]

The Kähler two-form on \(\mathcal{X}_n^J\) depending on two parameters, invariant to the action \((4.9)\) of \(G_n^J(\mathbb{R})_0\), is

\[
- i \omega_{\mathcal{X}_n^J}(v, u) = \frac{k}{2} \operatorname{tr}(H \wedge \bar{H}) + \frac{2\nu}{i} \operatorname{tr}(G^t D \wedge \bar{G}), \quad D := (\bar{v} - v)^{-1}, \quad H := D \, d \, v.
\]
We have the change of variables $\mathcal{C}_1: (v, \eta) \rightarrow (v, u)$, where
\begin{align}
\eta &= (\bar{v} - i 1_n)D(v - i 1_n)[(v - i 1_n)^{-1}u^t - (\bar{v} - i 1_n)^{-1}u^t], \\
u^t &= \frac{1}{2i}[(v + i 1_n)\eta - (v - i 1_n)\bar{\eta}].
\end{align}

\(5.9a\)

\(5.9b\)

c) If we make the change of variables (4.8), then (5.8) becomes
\[ G^t(v, u) = d u - p d v, \]
and
\[ G^t(v, u) = G^t(x, y, p, q) = d p v + d q = d p(x + i y) + d q. \]

d) With (5.9), (4.8) and
\(5.10\)
\[ M(1, n, C) \ni \eta := \chi + i \psi, \chi, \psi \in M(1, n, R), \]
we have the change of coordinates
\[ (x, y, p, q) \rightarrow (x, y, \chi, \psi), \quad p^t = \psi, q^t = \chi, \]
and
\[ G^t(v, \eta) = G^t(x, y, \chi, \psi) = d \psi^t x + d \chi^t + i d \psi^t y. \]

We obtain
\[ \eta = (q + i p)^t; \quad q^t = \frac{1}{2}(\eta + \bar{\eta}), \quad p^t = \frac{1}{2i}(\eta - \bar{\eta}). \]

Given the change of variables (4.8) and
\[ u := \xi + i \rho, \]
we have the change of variables
\[ (x, y, \xi, \rho) \rightarrow (x, y, p, q), \quad \xi = px + q, \quad \rho = py, \]
and (5.11) becomes
\[ G^t(v, u) = G^t(x, y, \xi, \rho) = d \xi - \rho y^{-1} d x + i(d \rho - \rho y^{-1} d y). \]

With (4.8), (5.10) and 5.9, we have the change of coordinates
\[ (x, y, \xi, \rho) \rightarrow (x, y, \chi, \rho), \quad \xi = \psi^t x + \chi^t, \rho = \psi^t y. \]

We recall that in Perelomov’s approach to CS it is considered the triplet $(G, \pi, \mathfrak{g})$, where $\pi$ is a unitary, irreducible representation of the Lie group $G$ on the separable complex Hilbert space $\mathcal{H}$. We can introduce the normalized (un-normalized) CS-vector $e_x$ (respectively, $e_z$) defined in $z \in M = G/H$
\[ e_x = \exp(\sum_{\phi \in \Delta^+} x_\phi X^+_{\phi} - \bar{x}_\phi X^-_{\phi})e_0, \quad e_z = \exp(\sum_{\phi \in \Delta^+} z_\phi X^+_{\phi})e_0, \]
where $e_0$ is the extremal weight vector of the representation $\pi$, $\Delta^+$ denotes the set of positive roots of the Lie algebra $\mathfrak{g}$ of $G$, and $X_\phi, \phi \in \Delta$ are the generators. $X^+_{\phi}$ ($X^-_{\phi}$) corresponds to the positive (respectively, negative) generators. See details in (4, 13, 69).

Let us denote by $\mathcal{C}$ the change of variables $x \rightarrow z$ in formula (5.12) such that
\[ e_x = (e_z, e_z)^{-\frac{1}{2}}e_z, \quad z = FC(x). \]
Lemma 2] verifies the assertion above for CS defined on $\mathcal{D}_n^J$, see also [5] Lemma 3, [7] Lemma 6.11 and Remark 6.12. But the same assertions are true for CS defined on $\mathcal{D}_n^J$, see [6] Lemma 7 and Comment 8] and [8] Lemma 3.6 and Remark 3.7.

Next remark generalizes [3] Remark 1] established on $G^J_n(\mathbb{R})$ to $G^J_n(\mathbb{R})$, $n \in \mathbb{N}$.

**Remark 11.** The FC-transform (5.6) relates the un-normalized CS-vector $e_{Wz}$ to the normalized one $\mathbb{w}_\eta$

$$\mathbb{w}_\eta = (e_{Wz}, e_{Wz})^{-\frac{1}{2}} e_{Wz}, \quad W \in \mathcal{D}_n, \ z, \ \eta^J \in M(1, n, \mathbb{C}),$$

and the $S_n$-variables $p, q$ are related to parameter $\eta$ defined in (5.6) by the relation

$$\eta = (q + ip)^J.$$

If we denote $\alpha := \frac{\xi}{\tau}$, $\gamma =: \nu$ and take into consideration assertion d) in Lemma 8 it is obtained.

**Theorem 1.** The metric on $\mathcal{X}_n^J$, $G^J_n(\mathbb{R})_0$-invariant to the action in Lemma 8 has the expressions

$$d s_{\mathcal{X}_n^J}(x, y, p, q) = \alpha \operatorname{tr}[(y^{-1} d x)^2 + (y^{-1} d y)^2]$$

(5.13a)

$$+ \gamma[d p(xy^{-1}x + yy^{-1}y) dp^t + d qy^{-1}dq^t + 2d pxy^{-1}dq^t];$$

$$d s_{\mathcal{X}_n^J}(x, y, \chi, \psi) = \alpha \operatorname{tr}[(y^{-1} d x)^2 + (y^{-1} d y)^2]$$

(5.13b)

$$+ \gamma[d \psi^t(xy^{-1}x + yy^{-1}y) d \psi + d \chi t y^{-1}d \chi + 2d \psi txy^{-1}d \chi];$$

$$d s_{\mathcal{X}_n^J}(x, y, \xi, \rho) = \alpha \operatorname{tr}[(y^{-1} d x)^2 + (y^{-1} d y)^2]$$

(5.13c)

$$+ \gamma[d \xi y^{-1}d \xi^t + d \rho y^{-1}d \rho^t + d \rho y^{-1}d xy^{-1}(\rho y^{-1}d x)^t + \rho y^{-1}d yy^{-1}(\rho y^{-1}d x)^t - 2\rho y^{-1}d xy^{-1}d \xi^t - 2\rho y^{-1}d y^{-1}d \rho^t]$$

The three parameter metric on $\tilde{\mathcal{X}}_n^J$, $G^J_n(\mathbb{R})$-invariant to the action c) in Lemma 8 is

$$d s_{\tilde{\mathcal{X}}_n^J}(x, y, p, q, \kappa) = d s_{\mathcal{X}_n^J}(x, y, p, q) + \lambda^2$$

(5.14)

$$= \alpha \operatorname{tr}[(y^{-1} d x)^2 + (y^{-1} d y)^2]$$

$$+ \gamma[d p(xy^{-1}x + yy^{-1}y) dp^t + d qy^{-1}dq^t + 2d pxy^{-1}dq^t]$$

$$+ \delta(d \kappa - p d q^t + q d p^t)^2.$$

Formula (5.13) (5.14) is a generalisation to $\mathcal{X}_n^J$ ($\tilde{\mathcal{X}}_n^J$), $n \in \mathbb{N}$, of equation (5.25b) (respectively, (5.30)) in [15] corresponding to $n = 1$.

6. Appendix: Other representations of the Jacobi algebra

We remind that the Jacobi algebra $\mathfrak{g}_n^J$, also denoted $\mathfrak{st}(n, \mathbb{R})$ by Kirillov in [48 §18.4] or $\mathfrak{osp}(2n + 2, \mathbb{R})$ in [19], is isomorphic with the subalgebra of Weyl algebra $A_n$ (see also [31]) of polynomials of degree maximum 2 in the variables $p_1, \ldots, p_n, q_1, \ldots, q_n$.

In [8] we have considered complex and biboson realization of Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ as $\mathfrak{sp}(n, \mathbb{R})_C$

$$\mathfrak{sp}(n, \mathbb{R})_C = \left\langle \sum_{i,j=1}^{n} (2a_{ij}K_{ij}^0 + b_{ij}K_{ij}^+ - \bar{b}_{ij}K_{ij}^-) \right\rangle,$$
where matrices \( a = (a)_{ij}, b = (b)_{ij}, i, j = 1, \ldots, n \) verify conditions \( a^\dagger = -a, b^\dagger = b \).

The realization of the generators of the symplectic group in biboson operators was observed firstly in \cite{41, 61}.

The correspondence between the generators \((4.6)\) and the generators in \cite[p. 248]{87} of \( G_n^J(\mathbb{R}) \) is

\[
2H \rightarrow A + S, \quad 4F \rightarrow B + T, \quad 4G \rightarrow B - T, \quad D^0 \rightarrow R, \quad D_{1q} \rightarrow P_q, \quad \hat{D}_{1q} \rightarrow Q_q.
\]

The algebra \( \text{wsp}(2N, \mathbb{R}) \) in \cite{70}, the semidirect product of \( \text{sp}(n, \mathbb{R}) \) and Heisenberg, is essentially the algebra in \cite{8}, except a factor 2.

The algebra of the inhomogeneous symplectic group \( \text{ISp}(2, \mathbb{R}) \) in \cite{52} is the same as our Jacobi algebra \( g_n^J \) in \cite{7}.

The Jacobi algebra \( g_n^J \) of the Jacobi group \( G_n^J \) is realized as two-foton algebra \cite{93} and \( G_n^J \) is embedded in \( \text{Sp}(n + 1, \mathbb{R})_C \) in the context of mean-field theory in Nuclear Physics \cite{68}.

7. Appendix: Differential of square root of a symmetric matrix

Let us consider a matrix \( A \in M(n, \mathbb{R}) \) with the eigenvalues \( \lambda_1, \ldots, \lambda_n \), and let \( \mathbb{R} \in \alpha > 0 \). Then there exists a unitary matrix \( U \) such that

\[
A^\alpha = U \text{diag}(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) U^\dagger.
\]

For \( \alpha = 1/2 \), i.e. \( A^{1/2}A^{1/2} = A \), we have

\[
(7.1) \quad dA^{1/2}A^{1/2} + A^{1/2}dA^{1/2} = dA.
\]

\((7.1)\) is a particular case of the matrix Sylvester equation

\[
(7.2) \quad AX + XB = C,
\]

where \( A \in M(n), B \in M(m) \) and \( X, C \in M(m, n) \). Then the solution \( X \) of the matrix equation \((7.2)\) can be written as \cite{30}

\[
(7.3) \quad (1_m \otimes A + B^t \otimes 1_n)\text{vec}(X) = \text{vec}(C),
\]

and the solution of the differential equation \((7.1)\) becomes

\[
(7.4) \quad \text{vec}(dA^{1/2}) = ((A^{1/2})^\dagger \oplus A^{1/2})^{-1}\text{vec}(dA).
\]

\( \otimes \) denotes in \((7.3)\), the Kronecker product, \( \oplus \) in \((7.4)\) denotes the Kronecker sum, while \( \text{vec}(X) \) denotes the vectorization of the matrix \( X \) \cite{30, 59}.

If the matrix \( A \) is symmetric and positive definite, we introduce the notation \cite{30}

\[
\alpha := \text{vech}(A) = L_n A, \quad a := D_n \alpha
\]

\[
\sigma := \text{vech}(A^{1/2}) = L_n A^{1/2}, \quad A^{1/2} = D_n \sigma,
\]

where \( \text{vech}(X) \) denotes the half-vectorization of the matrix \( X \), while \( D_n \) and \( L_n \) denotes the duplication, respectively elimination matrix, see \cite{59} for definitions.

It is obtained

\[
dl \sigma = L_n((A^{t})^{1/2} \oplus A^{1/2})^{-1} D_n \dl \alpha, \quad \frac{\dl \sigma}{\dl \alpha} = L_n((A^{t})^{1/2} \oplus A^{1/2})^{-1} D_n.
\]
In our case of (4.32), due to (3.19), we have to replace for the symmetric positive definite matrix $y \rightarrow y^{1/2}$, and formula (7.4) reads

$$\text{vec}(d y^{1/2}) = (y^{1/2} \otimes y^{1/2})^{-1} \text{vec}(d y) = (y^{1/2} \otimes \mathbb{1}_n + \mathbb{1}_n \otimes y^{1/2})^{-1} \text{vec}(d y).$$

$$\text{vech}(d y^{1/2}) = L_n(y^{1/2} \otimes y^{1/2})^{-1} \text{vech}(d y) = (y^{1/2} \otimes \mathbb{1}_n + \mathbb{1}_n \otimes y^{1/2})^{-1} \text{vech}(d y).$$

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