History entanglement entropy

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Abstract

A formalism is proposed to describe entangled quantum histories, and their entanglement entropy. We define a history vector, living in a tensor space with basis elements corresponding to the allowed histories, i.e. histories with nonvanishing amplitudes. The amplitudes are the components of the history vector, and contain the dynamical information. Probabilities of measurement sequences, and resulting collapse, are given by generalized Born rules: they are all expressed by means of projections and scalar products involving the history vector. Entangled history states are introduced, and a history density matrix is defined in terms of ensembles of history vectors. The corresponding history entropies (and history entanglement entropies for composite systems) are explicitly computed in two examples taken from quantum computation circuits.
1 Introduction

Formulations of quantum mechanics based on histories, rather than on states at a given time, have their logical roots in the work of Feynman [1, 2] (see also the inspirational Chapter 32 of Dirac’s book [3]), and could be seen as generalizations of the path-integral approach. A list with the references more relevant for the present work is given in [4]-[20].

We have seen in [20] how to define a history operator on the Hilbert space $\mathcal{H}$ of a physical system, in terms of which to compute probabilities of successive measurements at times $t_1, \ldots, t_n$. In the present note we introduce a history vector, living in a tensor space $\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$, where every $\mathcal{H}$ corresponds to a particular $t_i$. This vector contains the same information as the history operator, but is more suited to define entanglement of histories, and compute their density matrices and corresponding von Neumann entropies.

This approach is similar in spirit to the one advocated in refs. [15]-[19], but with substantial differences. In [15]-[19] the scalar product between history states depends on chain operators containing information on evolution and measurements. In our framework the algebraic structure does not depend on the dynamics, and all possible histories (not only “consistent” sets) correspond to orthonormal vectors in $\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$. The dynamical information is instead encoded in the coefficients (amplitudes) multiplying the basis vectors.

The Born rules for probabilities and collapse are extended to history vectors in a straightforward way. Every history vector has a pictorial representation in terms of allowed histories, and its collapse after a measurement sequence entails the disappearance of some histories. In this sense measurement “alters the past”,

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but never in a way to endanger causality. As an illustration, the formalism is applied to the entangler-disentangler and the teleportation quantum circuits.

The content of the paper is as follows. Chain operators and probabilities for multiple measurements at different times are recalled in Section 2. Section 3 introduces history amplitudes, an essential ingredient in the definition of the history vector, given in Section 4. The generalized Born rules for probabilities of outcome sequences and collapse are derived, using appropriate projectors on the history vector. In Section 5 we propose a definition for history entanglement, based on a tensor product between history states. Section 6 deals with density matrices, constructed using ensembles of history vectors. This allows the computation of history entropy, and history entanglement entropy for composite systems. Two examples based on quantum computation circuits are provided in Section 7, and we calculate their history entanglement entropy. Section 8 contains some conclusions.

2 Chain operators and probabilities

As recalled in [20], probabilities of obtaining sequences $\alpha = \alpha_1, \alpha_2, \ldots \alpha_n$ of measurement results, starting from an initial state $|\psi\rangle$, can all be expressed in terms of a chain operator $C_{\psi,\alpha}$. This operator encodes measurements at times $t_1, \ldots, t_n$ corresponding to projectors $P_{\alpha_1}, \ldots P_{\alpha_n}$, and unitary time evolution between measurements:

$$C_{\psi,\alpha} = P_{\alpha_n} U(t_n, t_{n-1}) P_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P_{\alpha_1} U(t_1, t_0) P_{\psi}$$

(2.1)

with $t_0 < t_1 < \cdots < t_{n-1} < t_n$ and $P_{\psi} = |\psi\rangle \langle \psi|$. The $P_{\alpha_i}$ are projectors on eigensubspaces of observables, satisfying orthogonality and completeness relations:

$$P_{\alpha_i} P_{\beta_i} = \delta_{\alpha_i,\beta_i} P_{\alpha_i}, \quad I = \sum_{\alpha_i} P_{\alpha_i}$$

(2.2)

and $U(t_{i+1}, t_i)$ is the evolution operator between times $t_i$ and $t_{i+1}$.

The probability of obtaining the sequence $\alpha$ is given by

$$p(\psi, \alpha_1, \ldots \alpha_n) = Tr(C_{\psi,\alpha} C_{\psi,\alpha}^d)$$

(2.3)

and could be considered the “probability of the history” $\psi, \alpha_1, \ldots \alpha_n$. We can easily prove that the sum of all these probabilities gives 1:

$$\sum_{\alpha} Tr(C_{\psi,\alpha} C_{\psi,\alpha}^d) = 1$$

(2.4)

by using the completeness relations in (2.2) and unitarity of the $U(t_{i+1}, t_i)$ operators. We also find

$$\sum_{\alpha_n} p(\psi, \alpha_1, \alpha_2, \ldots, \alpha_n) = p(\psi, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$

(2.5)
However other standard sum rules for probabilities are not satisfied in general. For example relations of the type
\[ \sum_{\alpha_2} p(\psi, \alpha_1, \alpha_2, \alpha_3) = p(\psi, \alpha_1, \alpha_3) \] (2.6)
hold only if the so-called \textit{decoherence condition} is satisfied:
\[ Tr(C_{\psi,\alpha} C_{\psi,\beta}^\dagger) + \text{c.c.} = 0 \quad \text{when} \ \alpha \neq \beta \] (2.7)
as can be checked on the example (2.6) written in terms of chain operators, and easily generalized. Note that for chain operators the following is trivially true:
\[ \sum_{\alpha_i} C_{\psi,\alpha_1,...\alpha_n} = C_{\psi,\alpha_1,...\alpha_i,...\alpha_n} \] (2.8)
due to \[ \sum_{\alpha_i} P_{\alpha_i} = I. \]

If all the histories we consider are such that the decoherence condition holds, they are said to form a \textit{consistent} set, and can be assigned probabilities satisfying all the standard sum rules.

In general, histories do not form a consistent set: interference effects between them can be important, as in the case of the double slit experiment. For this reason we will not limit ourselves to consistent sets. Formula (2.3) for the probability of successive measurement outcomes holds in any case.

3 Amplitudes

If \( P_{\alpha_n} = |\alpha_n\rangle\langle\alpha_n| \), i.e. the eigenvalue \( \alpha_n \) is nondegenerate, the chain operator can be written as
\[ C_{\psi,\alpha} = |\alpha_n\rangle A(\psi, \alpha) \langle \psi | \] (3.1)
where
\[ A(\psi, \alpha) = \langle \alpha_n| U(t_n, t_{n-1}) P_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P_{\alpha_1} U(t_1, t_0) |\psi\rangle \] (3.2)
is the \textit{amplitude} of the history \( \psi, \alpha \), and
\[ |A(\psi, \alpha)|^2 = Tr(C_{\psi,\alpha} C_{\psi,\alpha}^\dagger) = p(\psi, \alpha) \] (3.3)
This easily generalizes to the case of a \( g_n \)-degenerate eigenvalue \( \alpha_n \), with corresponding (orthonormal) eigenvectors \( |\alpha_n, i\rangle \) \( (i = 1, ... g_n) \):
\[ C_{\psi,\alpha} = \sum_i |\alpha_n, i\rangle A_i(\psi, \alpha) \langle \psi |, \quad \sum_i |A_i(\psi, \alpha)|^2 = Tr(C_{\psi,\alpha} C_{\psi,\alpha}^\dagger) = p(\psi, \alpha) \] (3.4)
The amplitudes \( A_i(\psi, \alpha) \) are given by the formula (3.2) where \( \langle \alpha_n \rangle \) is substituted by \( \langle \alpha_n, i\rangle \).
A scalar product between chain operators can be defined as
\[
(C_\psi, \alpha, C_\psi, \beta) \equiv \text{Tr}(C_\psi, \alpha C^\dagger_\psi, \beta) \tag{3.5}
\]
All the properties of a (complex) scalar product hold, in particular
\[
(C_\psi, \alpha, C_\psi, \alpha) = p(\psi, \alpha) = 0 \Rightarrow C_\psi, \alpha = 0 \tag{3.6}
\]
\textbf{Note:} if we divide the set \(\alpha_1, \ldots, \alpha_{n-1}\) into two complementary sets \(\alpha_1, \ldots, \alpha_m\) and \(\alpha_{j1}, \ldots, \alpha_{jp}\) with \(m + p = n - 1\), then
\[
\sum_{\alpha_{j1}, \ldots, \alpha_{jp}} A(\psi, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = A(\psi, \alpha_1, \ldots, \alpha_m, \alpha_n) \tag{3.7}
\]
because of the completeness relations in (2.2). This just rephrases property (2.8) for chain operators, with the difference that \(\alpha_n\) is never summed on since it enters the amplitude (3.2) as a bra rather than as a projector.

## 4 History vector, probabilities and collapse

Consider a physical system in the state \(|\psi\rangle\) at time \(t_0\) and devices that can be activated at times \(t_1, \ldots, t_n\) to measure given observables, with projectors on eigensubspaces as in (2.2). Before any measurement, the system can be described by a \textit{history vector}, living in \(n\)-tensor space
\[
|\Psi\rangle = \sum_{\alpha} A(\psi, \alpha)|\alpha_1\rangle \otimes \ldots \otimes |\alpha_n\rangle \tag{4.1}
\]
where the coefficients \(A(\psi, \alpha)\) are given by the amplitudes of the histories \(\alpha = \alpha_1, \ldots, \alpha_n\), computed as in the previous Section, and \(|\alpha_k\rangle\) are a basis of orthonormal vectors at each time \(t_k\). If no degeneracy was present, these vectors would be just the eigenvectors of the observable(s) measured at time \(t_k\). If the \(\alpha_k\) \((k < n)\) eigenvalues are degenerate, the information on degeneracy is lost in the symbol \(|\alpha_k\rangle\), but is contained in the amplitude \(A(\psi, \alpha)\), where the projectors \(P_{\alpha_k}\) on the eigensubspaces are present. In case \(\alpha_n\) is degenerate, the sum on \(\alpha\) in (4.1) must include the degeneracy index \(i\), and (4.1) will be short for
\[
|\Psi\rangle = \sum_{\alpha, i} A_i(\psi, \alpha)|\alpha_1\rangle \otimes \ldots \otimes |\alpha_{n-1}\rangle \otimes |\alpha_n, i\rangle \tag{4.2}
\]
\textbf{Note}: In the following we will assume for simplicity that \(\alpha_n\) is nondegenerate: all the results generalize easily to the degenerate case, usually by summing on the index \(i\).

The “time product” \(\otimes\) has all the properties of a tensor product. The symbol \(\otimes\) (or just a blank) will be reserved for tensor products between states of subsystems at the same time \(t_k\). The vector is normalized since
\[
\langle \Psi | \Psi \rangle = \sum_{\alpha} |A(\psi, \alpha)|^2 = 1 \tag{4.3}
\]
The *history content* of the system is defined to be the set of histories \( \alpha = \alpha_1, ... \alpha_n \) contained in \( | \Psi \rangle \), i.e. all histories having nonvanishing amplitudes.

Probabilities of measuring sequences \( \alpha = \alpha_1, ... \alpha_n \) are given by the familiar formula
\[
p(\psi, \alpha) = \langle \Psi | P_\alpha | \Psi \rangle = |A(\psi, \alpha)|^2.
\]
(4.4)

with
\[
P_\alpha = |\alpha_1 \rangle \langle \alpha_1| \otimes ... \otimes |\alpha_n \rangle \langle \alpha_n|.
\]
(4.5)

Formula (4.4) holds for sequences of measurements occurring at all times \( t_1, ... t_n \).

Such a multiple measurement with results \( \alpha_1, ... \alpha_n \) projects the state into the basis vector \( |\alpha_1 \rangle \otimes ... \otimes |\alpha_n \rangle \):
\[
|\Psi \rangle \rightarrow \frac{P_\alpha |\Psi \rangle}{\sqrt{\langle \Psi | P_\alpha | \Psi \rangle}} = |\alpha_1 \rangle \otimes ... \otimes |\alpha_n \rangle
\]
(4.6)

up to a phase.

A partial measurement at times \( t_{i_1}, ... t_{i_m} \) \((m < n)\) yielding the sequence \( \alpha_{i_1}, ... \alpha_{i_m} \) likewise projects the state vector \( |\Psi \rangle \) into
\[
|\Psi_\alpha \rangle = \frac{P_\alpha |\Psi \rangle}{\sqrt{\langle \Psi | P_\alpha | \Psi \rangle}}
\]
(4.7)

where \( P_\alpha \) is the projector on the sequence \( \alpha_{i_1}, ... \alpha_{i_m} \), i.e. a tensor product of identity operators and projectors at times \( t_{i_1}, ... t_{i_m} \):
\[
P_\alpha = I \otimes ... \otimes |\alpha_{i_1} \rangle \langle \alpha_{i_1}| \otimes I \otimes ... \otimes |\alpha_{i_m} \rangle \langle \alpha_{i_m}| \otimes I \otimes ...
\]
(4.8)

Then \( |\Psi_\alpha \rangle \) is given by the expression (4.1) where the sum on \( \alpha \) involves only the times \( t_j \) different from \( t_{i_1}, ... t_{i_m} \), the rest of the \( \alpha \)'s being fixed to the values \( \alpha_{i_1}, ... \alpha_{i_m} \).

The projected history vector \( |\Psi_\alpha \rangle \) can be used to compute conditional probabilities. The probability of obtaining the results \( \beta_{j_1}, ... \beta_{j_p} \) at times \( t_{j_1}, ... t_{j_p} \), having already obtained \( \alpha_{i_1}, ... \alpha_{i_m} \) at times \( t_{i_1}, ... t_{i_m} \) (with \( j_1, ... j_p \) and \( i_1, ... i_m \) having no intersection, and union coinciding with \( 1, ... n \)), is given by
\[
p(\beta | \alpha) = \langle \Psi_\alpha | P_\beta | \Psi_\alpha \rangle
\]
(4.9)

Finally, to compute probabilities for partial measurements at times \( t_{i_1}, ... t_{i_m}, t_n \) \((t_n \text{ always included})\), we need a “shorter” history vector \( |\Psi' \rangle \) with \( \otimes \) products corresponding to the subset \( t_{i_1}, ... t_{i_m}, t_n \). This vector can be obtained from \( |\Psi \rangle \) by using a different type of projection \( P \), defined on the basis vectors as:
\[
P_{i_1, ... i_m, n} |\alpha_1 \rangle \otimes ... \otimes |\alpha_n \rangle \equiv |\alpha_{i_1} \rangle \otimes ... \otimes |\alpha_{i_m} \rangle \otimes |\alpha_n \rangle
\]
(4.10)
and extended by linearity on any $|\Psi\rangle$. This projection reduces the number of factors in the $\odot$ tensor product. For example

$$P_{1,3,5}|\alpha_1\rangle \odot |\alpha_2\rangle \odot |\alpha_3\rangle \odot |\alpha_4\rangle \odot |\alpha_5\rangle = |\alpha_1\rangle \odot |\alpha_3\rangle \odot |\alpha_5\rangle \quad (4.11)$$

Then $|\Psi\rangle$ is given by

$$|\Psi\rangle = P_{i_1,\ldots,i_m,n}|\Psi\rangle = P_{i_1,\ldots,i_m,n} \sum_\alpha A(\psi, \alpha)|\alpha_1\rangle \odot \cdots \odot |\alpha_{n-1}\rangle \odot |\alpha_n\rangle =$$

$$= \sum_{\alpha_1\ldots,\alpha_m,\alpha_n} \left( \sum_{\alpha_{j_1},\ldots,\alpha_{j_p}} A(\psi, \alpha) |\alpha_{i_1}\rangle \odot \cdots \odot |\alpha_{i_m}\rangle \odot |\alpha_n\rangle \right) =$$

$$= \sum_{\alpha_1\ldots,\alpha_m,\alpha_n} A(\psi, \alpha_{i_1}, \ldots, \alpha_{i_m}, \alpha_n) |\alpha_{i_1}\rangle \odot \cdots \odot |\alpha_{i_m}\rangle \odot |\alpha_n\rangle \quad (4.12)$$

where we have used eq. (3.7) in the third line. The subsets $j_1,\ldots,j_p$ and $i_1,\ldots,i_m$ in (4.12) have no intersection, and union coinciding with $1,\ldots,n-1$. The probability of obtaining the sequence $\alpha_{i_1},\ldots,\alpha_{i_m},\alpha_n$ is

$$p(\psi, \alpha_{i_1}, \ldots, \alpha_{i_m}, \alpha_n) = \langle \Psi'|P_{\alpha_{i_1},\ldots,\alpha_{i_m},\alpha_n}|\Psi\rangle = |A(\psi, \alpha_{i_1}, \ldots, \alpha_{i_m}, \alpha_n)|^2. \quad (4.13)$$

with

$$P_{\alpha_{i_1},\ldots,\alpha_{i_m},\alpha_n} = |\alpha_{i_1}\rangle \langle \alpha_{i_1} \odot \cdots \odot |\alpha_{i_m}\rangle \langle \alpha_{i_m} | \odot |\alpha_n\rangle \langle \alpha_n| \quad (4.14)$$

We can also compute the probabilities $p(\psi, \alpha_{i_1}, \ldots, \alpha_{i_m})$ of sequences that do not include $\alpha_n$, simply by summing (4.13) on $\alpha_n$, due to eq. (2.5). Therefore:

$$p(\psi, \alpha_{i_1}, \ldots, \alpha_{i_m}) = \langle \Psi'|P_{\alpha_{i_1},\ldots,\alpha_{i_m}}|\Psi\rangle = \sum_{\alpha_n} |A(\psi, \alpha_{i_1}, \ldots, \alpha_{i_m}, \alpha_n)|^2 \quad (4.15)$$

with

$$P_{\alpha_{i_1},\ldots,\alpha_{i_m}} = |\alpha_{i_1}\rangle \langle \alpha_{i_1} \odot \cdots \odot |\alpha_{i_m}\rangle \langle \alpha_{i_m} | \odot I \quad (4.16)$$

**Note:** for $m = 0$, $P$ projects on $t_n$, and the procedure yields the probability $p(\psi, \alpha_n)$ of obtaining $\alpha_n$ at time $t_n$ without other measurements. The projected vector $|\Psi\rangle = \sum_{\alpha_n} A(\psi, \alpha_n)|\alpha_n\rangle$ is just the (usual) state vector $|\psi(t_n)\rangle$ of the system at time $t_n$, since

$$|\psi(t_n)\rangle = U(t_n, t_0)|\psi\rangle = \sum_{\alpha_n} |\alpha_n\rangle \langle \alpha_n|U(t_{i_1}, t_0)|\psi\rangle = \sum_{\alpha_n} A(\psi, \alpha_n)|\alpha_n\rangle \quad (4.17)$$

In conclusion, probabilities for (sequences of) measurements at any times can be computed via scalar products involving appropriate projections of the history vector $|\Psi\rangle$. 

6
5 History entanglement

It is useful to define a tensor product between history vectors of subsystems. On the basis history vectors the product acts as

\[(|\alpha_1\rangle \odot \ldots \odot |\alpha_n\rangle)(|\beta_1\rangle \odot \ldots \odot |\beta_n\rangle) \equiv |\alpha_1\rangle |\beta_1\rangle \odot \ldots \odot |\alpha_n\rangle |\beta_n\rangle \quad (5.1)\]

and is extended by bilinearity on all combinations of these vectors. No symbol is used for this tensor product, to distinguish it from the tensor product \(\odot\) encoding time information.

This allows a definition of product history states, which are defined to be expressible in the form:

\[
(\sum_\alpha A(\psi,\alpha)|\alpha_1\rangle \odot \ldots \odot |\alpha_n\rangle)(\sum_\beta A(\psi,\beta)|\beta_1\rangle \odot \ldots \odot |\beta_n\rangle) \quad (5.2)
\]

or, using bilinearity:

\[
\sum_{\alpha,\beta} A(\psi,\alpha)A(\psi,\beta)|\alpha_1\beta_1\rangle \odot \ldots \odot |\alpha_n\beta_n\rangle \quad (5.3)
\]

with \(|\alpha_i\beta_i\rangle \equiv |\alpha_i\rangle |\beta_i\rangle\) for short. A product history state is thus characterized by factorized amplitudes \(A(\psi,\alpha,\beta) = A(\psi,\alpha)A(\psi,\beta)\).

If the history state cannot be expressed as a product, it is said to be history entangled\(^1\). In this case, results of measurements on system A are correlated with those on system B and viceversa. Indeed if the amplitudes \(A(\psi,\alpha,\beta)\) in the history state are not factorized, the probability for Alice to obtain the sequence \(\alpha\) if Bob obtains the sequence \(\beta\) depends on \(\beta\), and viceversa, this probability being proportional to \(|A(\psi,\alpha,\beta)|^2\). On the other hand, if the history state is a product (5.2), the probability for Alice is \(|A(\psi,\alpha)|^2\) and does not depend on \(\beta\) (and likewise for Bob).

6 Density matrix and history entropy

A system in the history state \(|\Psi\rangle\) has the density matrix:

\[\rho = |\Psi\rangle \langle \Psi| \quad (6.1)\]

a positive operator satisfying \(Tr(\rho) = 1\) (due to \(\langle \Psi|\Psi\rangle = 1\)). Probabilities of measuring sequences \(\alpha = \alpha_1, \ldots \alpha_n\) are given by the standard formula:

\[p(\alpha_1, \ldots \alpha_n) = Tr(\rho \mathbb{P}_\alpha) \quad (6.2)\]

\(^1\)This entanglement is quite different from the one considered in refs. [15]-[18], where it involves superpositions of history states (without need of a composite system), and should be considered as a temporal entanglement.
cf. equation (4.4). A measurement as the one considered in eq. (4.7) projects the density matrix in the usual way:

$$\rho \rightarrow \rho_\alpha = |\Psi_\alpha\rangle\langle\Psi_\alpha| = \frac{P_\alpha \rho P_\alpha}{Tr(\rho P_\alpha)}$$ (6.3)

If a measurement is performed, but the result remains unknown, the density matrix becomes

$$\rho \rightarrow \rho' = \sum_\alpha |A(\psi, \alpha)|^2 |\Psi_\alpha\rangle\langle\Psi_\alpha|$$ (6.4)

Consider now a system AB composed by two subsystems A and B. The history state of AB has the general form

$$|\Psi_{AB}\rangle = \sum_{\alpha, \beta} A(\psi, \alpha, \beta) |\alpha_1\beta_1\rangle \otimes \cdots \otimes |\alpha_n\beta_n\rangle$$ (6.5)

with corresponding density matrix

$$\rho^{AB} = |\Psi_{AB}\rangle\langle\Psi^{AB}|$$ (6.6)

As usual, we can define reduced density matrices by partially tracing on the subsystems:

$$\rho^A \equiv Tr_B(\rho^{AB}), \quad \rho^B \equiv Tr_A(\rho^{AB})$$ (6.7)

In general $\rho^A$ and $\rho^B$ will not describe pure history states anymore.

Finally, we can define the system history (von Neumann) entropy as

$$S(\rho^{AB}) = -\rho^{AB} \log \rho^{AB}$$ (6.8)

and the history entanglement entropies for subsystems A and B:

$$S(\rho^A) = -\rho^A \log \rho^A, \quad S(\rho^B) = -\rho^B \log \rho^B$$ (6.9)

### 7 Examples

In this Section we examine two examples of quantum systems evolving from a given initial state, and subjected to successive measurements. They are taken from simple quantum computation circuits\(^2\) where unitary gates determine the evolution between measurements. Only two gates are used: the Hadamard one-qubit gate $H$ defined by:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$ (7.1)

and the two-qubit $CNOT$ gate:

$$CNOT|00\rangle = |00\rangle, \quad CNOT|01\rangle = |01\rangle, \quad CNOT|10\rangle = |11\rangle, \quad CNOT|11\rangle = |10\rangle$$ (7.2)

Quantum computing circuits in the consistent history formalism have been discussed for example in in ref.s [5, 22].

\(^2\)A review on quantum computation can be found for ex. in [21].
7.1 Entangler-disentangler

Fig. 1 The entangler - disentangler circuit, and some history diagrams: a) no measurements, or Bob measures 0 at $t_1$; b) Alice measures 0 at $t_1$; c) Alice measures 1 at $t_2$. Black triangles indicate measurements.

If the initial state (at $t_0$) is $|00\rangle$, the history state of the system before any measurements (at times $t_1, \ldots, t_4$) is given by

$$|\Psi\rangle = \frac{1}{2}(|00\rangle \otimes |00\rangle \otimes |00\rangle \otimes |00\rangle + |00\rangle \otimes |00\rangle \otimes |00\rangle \otimes |10\rangle +$$

$$+ |10\rangle \otimes |11\rangle \otimes |10\rangle \otimes |00\rangle - |10\rangle \otimes |11\rangle \otimes |10\rangle \otimes |10\rangle)$$

(7.3)

the amplitudes being given by formula (3.2), i.e.

$$A(00, 00, 00, 00, 00) = \langle 00|(H \otimes I)|00\rangle\langle 00|CNOT|00\rangle\langle 00|CNOT|00\rangle\langle 00|(H \otimes I)|00\rangle = +\frac{1}{2}$$

$$A(00, 00, 00, 00, 10) = \langle 10|(H \otimes I)|00\rangle\langle 00|CNOT|00\rangle\langle 00|CNOT|00\rangle\langle 00|(H \otimes I)|00\rangle = +\frac{1}{2}$$

$$A(00, 10, 11, 10, 00) = \langle 00|(H \otimes I)|10\rangle\langle 10|CNOT|11\rangle\langle 11|CNOT|10\rangle\langle 10|(H \otimes I)|00\rangle = +\frac{1}{2}$$

$$A(00, 10, 11, 10, 10) = \langle 10|(H \otimes I)|10\rangle\langle 10|CNOT|11\rangle\langle 11|CNOT|10\rangle\langle 10|(H \otimes I)|00\rangle = -\frac{1}{2}$$

(7.4)

These amplitudes (or equivalently the history vector $|\Psi\rangle$) encode all the necessary information to compute probabilities, according to the rules of Section 4. For example the probability for measuring any of those four sequences is $1/4$, whereas the
probability of measuring 10 at $t_4$ without measurements at $t_1, t_2, t_3$ is zero (the two histories with 10 at $t_4$ have opposite amplitudes and therefore interfere).

The history content of the system before measurements is displayed in diagram a) of Fig. 1. Measurements by Alice project the state $|\Psi\rangle$ and reduce its history content as shown in diagrams b) and c).

The unmeasured state $|\Psi\rangle$ is history entangled, whereas the projected $|\Psi_\alpha\rangle$ after Alice measurements in diagrams b) and c) is a product history state.

The reduced density operator for Alice before measurements is

$$\rho^A \equiv Tr_B(\rho^{AB}) = Tr_B|\Psi\rangle\langle\Psi| =$$

$$\frac{1}{4} |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \langle 0| \otimes |0\rangle \otimes |0\rangle + \frac{1}{4} |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \langle 0| \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle$$

$$\frac{1}{4} |1\rangle \otimes |1\rangle \otimes |0\rangle \langle 1| \otimes |1\rangle \otimes |1\rangle \otimes |0\rangle + \frac{1}{4} |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \langle 1| \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle$$

(7.5)

describing a mixed history state, with equal probabilities for the four histories available to Alice. This reduced density matrix can be used to compute statistics for Alice measurements. The system entropy is zero, since it is in a pure state, but the entropy corresponding to $\rho^A$ (the entropy “seen” by Alice) is

$$S(\rho^A) = -Tr(\rho^A \log \rho^A) = -4(\frac{1}{4} \log \frac{1}{4}) = 2$$

(7.6)

since $\rho^A$ has four eigenvalues equal to $\frac{1}{4}$.

Note that without measurements the circuit is simply the identity circuit for two qubits, so the initial state $00$ can only propagate to $00$ at time $t_4$. The situation is different when intermediate measurements are performed, as depicted in diagrams b) and c). In these cases also the state 10 at time $t_4$ becomes available.

### 7.2 Teleportation

The teleportation circuit [23] is the three-qubit circuit given in Fig. 3, where the upper two qubits belong to Alice, and the lower one to Bob.
The initial state is a three-qubit state, tensor product of the single qubit $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$ to be teleported and the 2-qubit entangled Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Before any measurement, the history vector contains 8 histories:

$$|\Psi\rangle = \frac{1}{2}(\alpha|000\rangle \circ |000\rangle \circ |000\rangle - \alpha|000\rangle \circ |000\rangle \circ |100\rangle +$$
$$+ \beta|100\rangle \circ |110\rangle \circ |010\rangle - \beta|100\rangle \circ |110\rangle \circ |110\rangle)$$
$$+ \alpha|011\rangle \circ |011\rangle \circ |011\rangle - \alpha|011\rangle \circ |011\rangle \circ |111\rangle$$
$$+ \beta|111\rangle \circ |101\rangle \circ |001\rangle - \beta|111\rangle \circ |101\rangle \circ |101\rangle)$$

(7.7)

The amplitudes being given by

$$A(\chi \otimes \beta_{00}, \alpha_1, \alpha_2, \alpha_3) = \langle \alpha_3|H_1P_{\alpha_2}CNOT_{1,2}P_{\alpha_1}|\chi \otimes \beta_{00}\rangle$$

(7.8)

For example

$$A(\chi \otimes \beta_{00}, 000, 000, 000) = \langle 000|H_1|000\rangle\langle 000|CNOT_{1,2}|000\rangle\langle 000|\chi \otimes \beta_{00}\rangle = \alpha/2$$

(7.9)
where \( H_1 \equiv H \otimes I \otimes I \) and \( \text{CNOT}_{1,2} \equiv \text{CNOT} \otimes I \). For the moment we do not take into account the \( X \) and \( Z \) gates, activated by the results of Alice measurements at \( t_3 \). The history vector has the representation given in Fig. 2a.

Suppose now that Alice measures her two qubits at time \( t_3 \), without any prior measurement. To compute probabilities we need the projection of \( |\Psi\rangle \) on \( t_3 \), i.e.

\[
|\Psi\rangle = \mathcal{P}_3 |\Psi\rangle = \frac{\alpha}{4} (|000\rangle - |100\rangle + |011\rangle - |111\rangle) + \frac{\beta}{4} (|010\rangle - |110\rangle + |001\rangle - |101\rangle)
\]

(7.10) cf. (4.10). There are four possible outcomes, each with probability \( 1/4 \). For example

\[
p(00) = \langle \Psi | P_{00} \otimes I | \Psi \rangle = \frac{1}{4}(|\alpha|^2 + |\beta|^2) = \frac{1}{4}
\]

(7.11)

Once Alice has obtained 00 at \( t_3 \), corresponding to the projector \( P_{\alpha_3} = P_{00} \otimes I \), the history vector collapses into

\[
|\Psi\rangle = \frac{I \otimes I \otimes (P_{00} \otimes I)|\Psi\rangle}{\sqrt{\langle \Psi | I \otimes I \otimes (P_{00} \otimes I)|\Psi\rangle}} = \alpha |000\rangle \otimes |000\rangle \otimes |000\rangle + \beta |111\rangle \otimes |101\rangle \otimes |001\rangle
\]

(7.12)

and corresponds to the diagram b) in Fig. 2. With this vector we can compute the conditional probability that Bob measures 0 or 1 at \( t_3 \), given that Alice has measured 00:

\[
p(0_B|00_A) = \langle \Psi | I \otimes I \otimes P_0 | \Psi_{\alpha} \rangle = |\alpha|^2
\]
\[
p(1_B|00_A) = \langle \Psi | I \otimes I \otimes P_1 | \Psi_{\alpha} \rangle = |\beta|^2
\]

(7.13)

To find the (usual) state vector of the system at time \( t_3 \) we project \( |\Psi_{\alpha}\rangle \) on \( t_3 \) with the use of the \( \mathcal{P}_3 \) projector:

\[
|\Psi\rangle = \mathcal{P}_3 |\Psi_{\alpha}\rangle = \alpha |000\rangle + \beta |001\rangle = |00\rangle (\alpha |0\rangle + \beta |1\rangle)
\]

(7.14)

and we see that Bob’s qubit is in the correctly teleported state \( |\chi\rangle = \alpha |0\rangle + \beta |1\rangle \).

Similar arguments hold if Alice obtains 01 or 10 or 11. In these cases Bob’s qubit at time \( t_3 \) is found to be in states that can be transformed into \( |\chi\rangle \) using \( X \) and \( Z \) gates, represented by the Pauli matrices \( \sigma_x \) and \( \sigma_z \) on the \( (|0\rangle, |1\rangle) \) basis.

Finally, if at time \( t_3 \) Alice measures 00 and Bob measures 1, the history vector \( |\Psi\rangle \) collapses to

\[
|\Psi\rangle = \frac{I \otimes I \otimes (P_{00} \otimes P_1)|\Psi\rangle}{\sqrt{\langle \Psi | I \otimes I \otimes (P_{00} \otimes P_1)|\Psi\rangle}} = |111\rangle \otimes |101\rangle \otimes |001\rangle.
\]

(7.15)

and corresponds to the diagram c) in Fig. 2.

The unmeasured history vector \( |\Psi\rangle \) in (7.14) is entangled. The history vector \( |\Psi_{\alpha}\rangle \) in (7.12) after Alice measures 00 is likewise entangled, even if the (usual) state
of the system at \( t_3 \) is a product state. Only the history state \((7.15)\) is a product history state \(|11⟩ ⊙ |10⟩ ⊙ |00⟩⟩ ⊙ (|1⟩ ⊙ |1⟩ ⊙ |1⟩⟩

### Density matrix and entropy

The von Neumann entropy for the system before measurements is zero, since the system is in a pure history state. The reduced history density matrix for Bob, before any measurement, is given in terms of the history vector \(|Ψ⟩ \) in (7.7):

\[
ρ^B = Tr_A(|Ψ⟩⟨Ψ|) = \frac{1}{2} (|0⟩ ⊙ |0⟩ ⊙ |0⟩⟩ ⊙ (|0⟩ ⊙ |0⟩ ⊙ |0⟩⟩ + |1⟩ ⊙ |1⟩ ⊙ |1⟩⟩ ⊙ (|1⟩ ⊙ |1⟩ ⊙ |1⟩⟩)
\]

(7.16)

and does not depend on \( α \) and \( β \). The corresponding von Neumann entropy is \( S(ρ^B) = \log 2 = 1 \).

If Alice measures her two qubits, without communicating her result, the density matrix of the system becomes

\[
ρ^{AB} = \sum_α |A(ψ, α)|^2 |α⟩⟨α|
\]

(7.17)

(the sum on \( α \) is over the 8 histories contained in the history vector \(|Ψ⟩\)) yielding a matrix with 4 eigenvalues equal to \(|α|^2/4\) and 4 eigenvalues equal to \(|β|^2/4\). Then the von Neumann entropy is

\[
S(ρ^{AB}) = -|α|^2 \log \frac{|α|^2}{4} - |β|^2 \log \frac{|β|^2}{4} = -|α|^2 \log |α|^2 - |β|^2 \log |β|^2 + 2 \quad (7.18)
\]

Setting \( p = |α|^2 \), the entropy \( S(p) = 2 - p \log p - (1 - p) \log(1 - p) \) is maximum and equal to \( \log 2 + 2 = 3 \) when \( p = 1/2 \), and is minimum and equal to 2 when \( p = 0, 1 \).

The reduced density matrix for Bob coincides with the one before measurements by Alice given in (7.16), as expected, since Alice’s act of measuring cannot be detected by Bob. The corresponding von Neumann entropy is therefore the same: \( S(ρ^B) = -\log(1/2) = 1 \).

### 8 Conclusions

History amplitudes, or equivalently chain operators, encode all the information necessary to compute probabilities of outcome sequences when measuring a given physical system. In the paper [20] we proposed a pictorial way to represent the history content (i.e. the set of all histories with nonvanishing amplitudes) encoded in a history operator, acting on the Hilbert space \( \mathcal{H} \) of physical states. In the present paper amplitudes are used to construct a history vector, living in a tensor product of multiple \( \mathcal{H} \) copies, in terms of which all probabilities can be expressed via projections and scalar products.

The formalism proposed here has two advantages with respect to the usual state vector description of a physical system:

1. **Avoiding the State Vector**
   - Historically, the state vector description has been the standard approach in quantum mechanics, but it has limitations. For instance, it cannot easily handle situations where parts of the system remain fixed after measurements, which is common in many physical scenarios.
   - With the history vector description, we can maintain a more direct connection to the underlying physical processes, allowing for a clearer interpretation of the system's evolution.

2. **Simplifying Probabilistic Calculations**
   - The history vector description simplifies the computation of probabilities, especially when dealing with complex systems involving multiple measurements or interactions.
   - By encoding the system's history in a vector form, it becomes easier to perform calculations and derive the probabilities of various outcomes without having to explicitly work with the full state vector.

These advantages make the history vector formalism a powerful tool for understanding and analyzing quantum systems, particularly in scenarios where traditional state vector methods might be cumbersome or less intuitive.
1) it provides a convenient way to keep track of all possible histories of the system, and their reduction due to measurements. This can be translated into graphs that facilitate intuition on how the system behaves under unitary time evolution and measurements at different times.

2) it allows the definition of history entanglement, history entropy, and history entanglement entropy for composite systems.

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