Constraining the deformed dispersion relation with the hydrogen atom 1S-2S transition

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Abstract

In this paper, we use the latest results of the ultra-high accuracy 1S-2S transition experiments in hydrogen atom to constrain the forms of the deformed dispersion relation in the nonrelativistic limit. For the leading correction in the nonrelativistic limit, the experiments set a limit within a single order of magnitude of the desired Planck-scale level, thereby providing another example of the Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments. And for the next-to-leading term, bound has two orders of magnitude away from the Planck scale, but it still amounts to the best limit, in contrast to previously obtained bound in the nonrelativistic limit from the cold-atom-recoil experiments.

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I. INTRODUCTION

Establishing a complete and self-consistent quantum theory of gravity is one of the main challenges in modern physics. Till now, a full understanding of quantum gravity is lacking, but some phenomenological attempts to explore quantum gravity effects have attracted many people’s attentions [1–56]. Especially, gaining experimental insights on quantum gravity realm is very challenging, because most quantum gravity effects are expected to occur on the ultra-high Planck energy scale \( E_p = \sqrt{\hbar c^5/G} \approx 1.2 \times 10^{19}\text{GeV} \), leaving only minuscule traces on processes we can access experimentally. But thanks to a large and determined efforts made over the last decade, we do now have at least a few research lines in quantum gravity phenomenology where it is established that quantum properties of gravity could be investigated with the desired Planck-scale sensitivity. For example, due to the ultra-high levels of accuracy of atom interferometry, the cold-atom-recoil experiments have been used to establish meaningful bounds on parameters characterizing quantum gravity effects, and the exceptional sensitivity of the experiments set a limit within a single order of magnitude of the desired Planck-scale level, thereby providing the first example of the Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments [45, 57]. In this paper, we attempt to find another example to close to or reach the desired Planck-scale sensitivity by using the latest results of the hydrogen atom 1S-2S transition experiments to constrain the forms of the deformed dispersion relation in the nonrelativistic limit. In [58, 59], quantum gravity corrections to Lamb Shift have computed in the framework of Generalized Uncertainty Principle (GUP) where the accuracy of precision measurement of Lamb Shift of about \( 1 \times 10^{-12} \) leads to the upper bounds on parameters of quantum gravity effects \( \beta_0 < 10^{36} \). On the other hand, the progress of frequency conversion technology, such as frequency doubling and frequency division in laser research, makes precision of Lamb Shift experiments in hydrogen atom and deuterium atom ultra high. In Ref. [60, 61], the accuracy of precision measurement of the hydrogen 1S-2S frequency (Lamb Shift experiments) reaches \( 4.5 \times 10^{-15} \). In our case, we use the latest results of the hydrogen atom 1S-2S transition experiments to observe the Planck-sale sensitivity of quantum gravity.

The remainder of this paper is organized as follows. In Sec. II we briefly introduce the deformed dispersion relation in nonrelativistic limit. Then, by comparing the results of a detailed calculation of the deformed dispersion relation effects on the 1S-2S transition in
hydrogen atom with its accuracy of precision measurement, upper bounds on the parameters of the deformed dispersion relation are obtained in Sec. III. Sec. IV ends up with some conclusions.

II. THE DEFORMED DISPERSION RELATION IN THE NONRELATIVISTIC LIMIT

In 2002, Amelino-Camelia has constructed the famous Doubly Special Relativity (DSR), which has two observer-independent constants, i.e. speed of light $c$ and Planck length $L_p$, of relativity [1]. In the DSR, the deformed dispersion relation naturally leads to the Planck scale departure from Lorentz symmetry, which is referred to as the Lorentz invariance violation of dispersion relations. The related studies were advocating that a general implication of spacetime quantization is a modification of the classical-spacetime dispersion relation between energy $E$ and (modulus of) momentum $p$ of a microscopic particle with mass $m$, usually of the form

$$E^2 = p^2 + m^2 + p^2(\xi_n \frac{E}{M_p})^n.$$ (1)

where the speed of light $c$ is set to 1. The past decade of vigorous investigations of these modifications of the dispersion relation reached its most noteworthy results in analyses of observational astrophysics data, which of course concern the ultrarelativistic ($p \gg m$) regime of particle kinematics [33, 44, 62–66].

In the nonrelativistic limit ($p \ll m$), the deformed dispersion relation (1) should be taken the form [45, 48]

$$E \simeq m + \frac{p^2}{2m} + \frac{1}{2M_p} \left( \xi_1 mp + \xi_2 p^2 + \xi_3 \frac{p^3}{m} \right).$$ (2)

The dispersion relation includes correction terms that are linear in $1/M_p$. The model-dependent dimensionless parameters $\xi_1$, $\xi_2$, $\xi_3$ should (when different from zero) have values roughly of order one, so that indeed the new effects are introduced in some neighborhood of the Planck scale. Evidence that at least some of these parameters should be non-zero has been found most notably in Loop Quantum Gravity [67–69], and in particular the framework introduced in Refs. 67, 70, which was inspired by Loop Quantum Gravity, produces a term linear in $p$ in the nonrelativistic limit (the effect here parameterized by $\xi_1$). In our case,
it is reasonable to use the deformed dispersion relation in the nonrelativistic limit because $p \ll m$ (the energy of the electron for $n = 1$ state of hydrogen is about $13.6\,eV$, but its mass $m \cong 0.5 \times 10^6\,eV$).

Unfortunately, as usual in quantum gravity research, the theoretically-favoured range of values of the parameters of the dispersion relation translates into a range of possible magnitudes of the effects that is extremely challenging. If the Planck scale is the characteristic scale of quantum gravity effects, the values of these parameters such as $\xi_1$, $\xi_2$, $\xi_3$ in the deformed dispersion relation (2) should indeed be close to 1. It means that the effects of the deformed terms characterized quantum gravity effects is extremely small as a result of the overall factor $1/M_p$. Some researches have suggested that, the quantum gravity scale might be somewhat smaller than the Planck scale, and might even coincide with the grand unification scale in particle physics, which is 3 orders of magnitude smaller (see Refs. [48, 71, 72] and references therein). This means that these parameters such as $\xi_1$, $\xi_2$, $\xi_3$ gain 3 orders of magnitude, but the prospect of detectable quantum gravity effects remains very small.

Recently, The Planck-scale sensitivity in the deformed dispersion relation (2) has been studied by using cold atom recoil experiments in Ref. [45], and the meaningful bounds on the parameters $\xi_1$ and $\xi_2$ have been obtained. The results shown that $\xi_1 = -1.8 \pm 2.1$ and $|\xi_2| < 10^9$, by using the experiments data of Caesium-atom recoil measurements in Ref. [73] and electron-anomaly measurements in Ref. [74]. As discussed above, the range of values of $\xi_1$ indicates that the cold-atom recoil experiments can be considered as the first example of controlled laboratory experiments probing the form of the dispersion relation with sensitivity that is meaningful from a Planck scale perspective. But the bound on parameter $\xi_2$ in the dispersion relation was still a few orders of magnitude away from the Planck scale.

Therefore, our main objective here is to show that the experiment of the ultra-high accuracy 1S–2S transition in hydrogen atom can be used to establish improved bounds on the parameters $\xi_1$ and $\xi_2$ that characterized the nonrelativistic limit of the deformed dispersion relation (2).
III. BOUNDS ON THE PARAMETERS OF THE DEFORMED DISPERSION RELATION

The hydrogen atom has played a central position in the development of quantum mechanics. As it is the simplest of atoms, it has played an important role for the development and testing of fundamental theories through ever-refined comparisons between experimental data and theoretical predictions, and hydrogen spectroscopy is closely related to the successive advances in the understanding of atomic structure. In recent years, with the advance of experimental technology, the absolute frequency of the 1S-2S transition in atomic hydrogen via two photon spectroscopy has been measured with particularly high precision, so that it can be used to achieve various accurate measurement. For example, the Rydberg constant $R_\infty$ and the proton charge radius have been further improved through the advance of measurement precision of the 1S-2S two photo transition \[^{15}\]. A value of $R_\infty = 10973731.56854(10) m^{-1}$ was obtained. The 1S-2S hydrogen spectroscopy can also be used to search new limits on the drift of fundamental constants \[^{76, 77}\]. Another important application of the 1S-2S two photo transition is used to test electron boost invariance \[^{77}\]. Inspiring by these achievements with the absolute 1S-2S transition frequency in atomic hydrogen, we think about the possibility of studying quantum gravity effects on the hydrogen atomic spectroscopy.

Neglecting hyperfine structure, the hydrogen energy levels are given by the sum of the following contributions \[^{78}\]

$$E(n, J, L) = E_{DC}(n, J) + E_{RM}(n, J) + E_{LS}(n, J, L), \quad (3)$$

where $E_{DC}$ denotes the Dirac-Coulomb energy as the main energy contribution, $E_{RM}$ the leading recoil corrections due to the finite mass of the nucleus, and $E_{LS}$ the Lamb shift. The first two energy contributions are the main one, which are well-known functions of the Rydberg constant $R_\infty$, the fine structure constant $\alpha$ and the ratio of the electron and nuclear mass $m_e/m_N$. The Lamb shift contains QED corrections and corrections for the finite size and polarizability of the nucleus. Reviews of the contributions in hydrogenic atoms have been given by in \[^{78-80}\]. We follow here the expression derived by Bethe for the energy level shift. It has been pointed out by Bethe \[^{81}\] that the displacement of the 2S level of hydrogen observed by Lamb and Retherford \[^{82}\], can be simply explained as a shift in the energy of the atom arising from its interaction with the radiation field. Subsequently, by
calculating the mean square amplitude of oscillation of an electron coupled to the zero-point fluctuations of the electromagnetic field, the shift of \( nS \) energy levels has been given by \[83\]

\[
\Delta E_n = \frac{4\alpha^2}{3m^2} (\ln \frac{1}{\alpha}) |\psi_n(0)|^2
= \frac{8\alpha^3}{3\pi n^3} (\ln \frac{1}{\alpha}) \left( \frac{1}{2} \alpha^2 m \right) \delta_{00}.
\]  

Since the scale of quantum electrodynamical effect is related to the principle quantum number \( n \) as \( 1/n^3 \), so the 1S Lamb shift is the largest in atomic hydrogen.

Our main objects here is to expose sensitivity to a meaningful range of values of the parameters \( \xi_1 \) and \( \xi_2 \), let us focus on the Planck scale corrections with coefficient \( \xi_1 \) and \( \xi_2 \). Thus, the Planck scale correction terms are regarded as the perturbation terms of the levels energy of hydrogen atom with a well defined quantum Hamiltonian. In the deformed dispersion relation \[2\], the leading correction and the next-to-leading correction are respectively denoted by Hamiltonian \( \hat{H}' \) and \( \hat{H}'' \), where

\[
\hat{H}' = \xi_1 \frac{m}{2M_p} \hat{p},
\]
\[
\hat{H}'' = \xi_2 \frac{\hat{p}^2}{2M_p}.
\]

Now, we compute the bounds on parameters \( \xi_1 \) and \( \xi_2 \) by studying the Planck scale correction of the hydrogen energy levels.

### A. Bound on the parameter \( \xi_1 \)

Since the hydrogen atom is spherically symmetric, the Coulomb potential of the hydrogen atom is given by

\[
V(r) = -\frac{k}{r},
\]

where \( k = e^2/4\pi\varepsilon_0 = \alpha \hbar \), \( e \) is electronic charge. To first order, the perturbing Hamiltonian \( \hat{H}' \) shift the energy to

\[
E_n = E_n^{(0)} + \xi_1 \frac{m}{2M_p} \langle nlm|\hat{p}|nlm \rangle,
\]

where \( E_n^{(0)} = -k/2an^2 \), \( a \) is the Bohr radius. As discussed above, the 1S Lamb shift is the largest in atomic hydrogen, so we are concerned only with the effects of the Planck scale
TABLE I: Quantities used in our calculation

| Quantity | Value |
|----------|-------|
| $\alpha$ | $137.035999139(31)$ |
| $R_\infty$ | $10973731.568508(65) m^{-1}$ |
| $a$ | $0.52917721067(12) \times 10^{-10} m$ |
| $m$ | $0.5109989461(31) / c^2 MeV$ |
| $M_p$ | $1.220910(29) \times 10^{19} / c^2 GeV$ |

correction on the shift of 1S energy levels. We have $l = m = 0$, and utilize the following to calculate the energy shift: $R_{10}(r) = 2a^{-3/2}e^{-r/a}$, $Y_{00} = 1/\sqrt{4\pi}$. We derive

$$<100|\hat{p}|100> = -i\hbar\langle 100|\frac{\partial}{\partial r}|100\rangle = \frac{i\hbar}{a}. \quad (8)$$

Thus, the shift of energy levels due to the leading correction in the DSR framework is expressed as

$$\Delta E = \left| \xi_1 \frac{m}{2M_p} \langle 100|\hat{p}|100\rangle \right| = \frac{m\hbar}{2M_p a}. \quad (9)$$

The additional contribution due to the correction of the parameter $\xi_1$ term in proportion to the original value 1S Lamb shift is given by

$$\frac{\Delta E}{\Delta E_1} = \xi_1 \frac{3m}{32M_p aR_\infty a^3 \ln \frac{1}{\alpha}} \approx 3.5 \times 10^{-15} \xi_1, \quad (10)$$

where some values in Table 1 have been used. As discussed above, if the Planck scale is the characteristic scale of quantum gravity effects, parameter $\xi_1$ should indeed be close to 1, and then the additional contribution in proportion to the original value (10) is approximately equal to $3.5 \times 10^{-15}$. The current accuracy of precision measurement of the hydrogen 1S-2S transition reach the $4.5 \times 10^{-15}$ regime [50]. It interestingly means that the hydrogen 1S-2S transition experiment we here considered can indeed probe the Planck-scale sensitivity on basis of the deformed dispersion relation [2].
B. Bound on the parameter $\xi_2$

Following the same steps that we performed above for the correction term with coefficient $\xi_1$, it is easy to verify that the correction term with coefficient $\xi_2$ would produce the following modification of the hydrogen 1S energy levels

$$\Delta E' = |\langle 100|\hat{H}'|100\rangle| = \xi_2 \frac{1}{2M_p}|\langle 100|p^2|100\rangle|. \tag{11}$$

Using the expression

$$\hat{p}^2 = 2m[\hat{H}_0 + \frac{k^2}{r}], \tag{12}$$

where $\langle 100|\hat{H}_0|100\rangle = E_1^{(0)}$, we have

$$\langle 100|\hat{p}^2|100\rangle = \frac{mk}{a} = \frac{m\hbar\alpha}{a}. \tag{13}$$

The shift of energy levels due to the next-to-leading correction in the DSR framework is expressed as

$$\Delta E' = \xi_2 \frac{m\alpha\hbar}{2M_p a}. \tag{14}$$

Thus, the additional contribution due to the correction of the parameter $\xi_2$ term in proportion to the original value 1S Lamb shift is given by

$$\frac{\Delta E'}{\Delta E_1} = \xi_2 \frac{3m}{32M_p aR_\infty\alpha^2 \ln \frac{1}{\alpha}} \approx 2.6 \times 10^{-17}\xi_2. \tag{15}$$

According to the current accuracy of precision measurement of the hydrogen 1S-2S transition, the result allow us to establish that $|\xi_2| < 10^2$, which means that we indeed can probe the spacetime structure down to length scales of order $10^{-33}m \sim \xi_2/M_p$. This bound is the best limit on the scenario for the deformation of Lorentz symmetry in the nonrelativistic limit, since previous attempts to constrain the parameter $\xi_2$ is at level $|\xi_2| < 10^9$ by using the cold atom recoil experiments $[45]$. By comparing (9) with (14), it is easy to find that the magnitude of the energy shifts of the hydrogen atom caused by the leading correction term and the next-to-leading correction term differs by the fine structure constant $\alpha \sim 10^2$. However, in the study of constraining bounds on quantum gravity effects in the deformed dispersion relation by using the cold atom recoil experiment, the leading correction term and the next-to-leading correction term cause the energy correction to differ by a factor $m/(\hbar\nu_{\ast} + p) \sim 10^9$(see details in [45]). Thus, the hydrogen 1S-2S transition experiments can be considered to be able to investigate the desired Planck scale sensitivity.
IV. CONCLUSIONS

We use the latest results of the ultra-high accuracy 1S-2S transition experiments in hydrogen atom to establish upper bounds on parameters $\xi_1$ and $\xi_2$ characterizing the nonrelativistic limits of the deformed dispersion relation. The results show that the exceptional sensitivity of the experiments sets a limit on parameter $\xi_1$ within a single order of magnitude of the desired Planck-scale level, thereby providing another example of the Planck-scale sensitivity in the study of the dispersion relation in controlled laboratory experiments. At the same time, bound of parameter $\xi_2$ has two orders of magnitude away from the Planck scale, but it still amounts to the best limit, in contrast to previously obtained bounds in the nonrelativistic limit from the cold-atom-recoil experiments [45, 57]. We can expect that, as the hydrogen atom 1S-2S transition experiments continue to improve, more stringent bounds on parameters $\xi_1$ and $\xi_2$ could be found in the near future.

V. ACKNOWLEDGEMENTS

This work is supported by the Program for NCET-12-1060, by the Sichuan Youth Science and Technology Foundation with Grant No. 2011JQ0019, and by FANEDD with Grant No. 201319, and by the Innovative Research Team in College of Sichuan Province with Grant No. 13TD0003, and by Sichuan Natural Science Foundation with Grant No. 16ZB0178, and by the starting funds of China West Normal University with Grant No.17YC513 and No.17C050.

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