Hausdorff–Lebesgue dimension of attractors

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1. INTRODUCTION. HAUSDORFF MEASURE AND DIMENSION AND HAUSDORFF–LEBESGUE MEASURE AND DIMENSION

In the present paper the classical ideas of Hausdorff and Lebesgue are combined and the Hausdorff–Lebesgue measure is introduced. This makes it possible to obtain new results in chaotic dynamics.

Consider a compact $K \subset \mathbb{R}^n$ and the numbers $d \geq 0, \varepsilon > 0$.

Define the Hausdorff measure and Hausdorff dimension of a compact $K$ [1,2].

Consider all coverings of $K$ by the balls $B_i$ of radii $r_i \leq \varepsilon$.

Suppose, $\mu_H(K, d, \varepsilon) = \inf \sum r_i^d$, where infimum is taken over all $\varepsilon$-coverings of compact $K$.

Obviously, $\mu_H(K, d, \varepsilon)$ increases with increasing $\varepsilon$. Therefore there exists a limit $\mu_H(K, d) = \lim_{\varepsilon \to 0} \mu_H(K, d, \varepsilon)$.

**Definition 1.** The value $\mu_H(K, d)$ is called a Hausdorff measure of compact $K$.

We introduce $\dim_H K = \inf\{ d | \mu_H(K, d) = 0 \}$.

**Definition 2.** The value $\dim_H K$ is called a Hausdorff measure of $K$.

Note that a set of balls $\{B_i\}$ can be chosen as a set of cubes with sides $2r_i \leq 2\varepsilon$. In this case the dimensions $\dim_H K$ coincide.

If the covering $\{B_i\}$ involves the balls of equal radii $r_i = \delta \leq \varepsilon$, we say about fractal measure $\mu_F(K, d)$ and fractal dimension $\dim_F K$.

The Hausdorff measure and fractal measure are outer measures. But in my view this measure is also outer. Therefore in this paper we combine the ideas of Hausdorff and Lebesgue.

Consider all coverings of $K$ by disjoint cubes $C_i$ with sides $2\delta_i \leq 2\varepsilon$.

Also, as in the theory of Lebesgue measure, in the case of intersection of boundaries $\partial C_i \cap \partial C_j$ such a set of intersections is included only in $C_i$ or in $C_j$.

Suppose that $\mu_{HL}(K, d, \varepsilon) = \inf \sum \delta_i^d$, where the infimum is taken over all $2\varepsilon$-coverings of compact $K$. It is obvious that $\mu_{HL}(K, d, \varepsilon)$ increases with decreasing $\varepsilon$. Consequently there exists the limit $\mu_{HL}(K, d) = \lim_{\varepsilon \to 0} \mu_{HL}(K, d, \varepsilon)$. 

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**Definition 3.** The value $\mu_{HL}(K, d)$ is called a Hausdorff–Lebesgue measure of compact $K$.

We introduce

$$\dim_{HL} K = \inf \{ d \mid \mu_{HL}(K, d) = 0 \}.$$ 

**Definition 4.** The value $\dim_{HL} K$ is called a Hausdorff–Lebesgue dimension of compact $K$.

Consider now all coverings of $K$ by disjoint cubes $C_i$ with sides $2\varepsilon$

**Definition 5.** The value

$$\mu_{FHL}(K, d) = \limsup_{\varepsilon \to 0} \sum_i \varepsilon^d$$

is called a Hausdorff–Lebesgue fractal measure of compact $K$.

**Definition 6.** The value

$$\dim_{FHL} K = \inf \{ d \mid \mu_{FHL}(K, d) = 0 \}$$

is called a Hausdorff–Lebesgue fractal dimension.

It is obvious that

$$\mu_H(K, d, \sqrt{n}\varepsilon) \leq \mu_{HL}(K, d, \varepsilon)(n)^{d/2},$$

$$\mu_H(K, d) \leq \mu_{HL}(K, d)(n)^{d/2} \leq \mu_{FHL}(K, d)(n)^{d/2},$$

$$\dim_H K \leq \dim_{HL} K \leq \dim_{FHL} K,$$

and for $k$-dimensional manifold $K$

$$\dim_H K = \dim_{HL} K = \dim_{FHL} K = k.$$ 

The following relations are also obvious.

For compacts $K_i$ such that $K \subset \bigcup_i K_i$ the inequality

$$\mu_{HL}(K \bigcap (\bigcup_i K_i), d, \varepsilon) \leq \sum_i \mu_{HL}(K \bigcap K_i, d, \varepsilon)$$ \hspace{1cm} (1)

is satisfied. For disjoint compacts $K_i$ such that $K \supset \bigcup_i K_i$, the inequality

$$\mu_{HL}(K \bigcap (\bigcup_i K_i)d, \varepsilon) \geq \sum_i \mu_{HL}(K \bigcap K_i, d, \varepsilon)$$ \hspace{1cm} (2)

is satisfied. Similar relations are satisfied for $\mu_{FHL}(K, d)$. 
2. Upper Estimates of Hausdorff–Lebesgue Dimension

Recall [3] that a linear operator $A$ can be represented in the form of a product $A = SQ$, where $S$ is symmetric nonnegative and $Q$ are orthogonal operators. Recall also [3] that $S$ always has in $\mathbb{R}^n$ orthonormal system of eigenvectors $e_i$ ($j = 1, \ldots, n$) with real characteristic numbers that coincide with singular values $\alpha_j$ ($\alpha_1 \geq \cdots \geq \alpha_n \geq 0$) of operator $A$.

**Definition 7.** A cube $C$ is called oriented if the sides of cube $QC$ are parallel vectors $e_1, \ldots, e_n$.

Consider now a continuously differentiable mapping $F(x) : \mathbb{R}^n \to \mathbb{R}^n$

$$F(x + h) - F(x) = (T_x F)h + o(h).$$

(3)

Suppose that $\alpha_1(T_x F) \geq \cdots \geq \alpha_n(T_x F)$ are singular values of matrices $T_x F$ at the point $x$,

$$\omega_d(T_x F) = \alpha_1(T_x F) \cdots \alpha_k(T_x F) \alpha_{k+1}(T_x F)^s, \quad d = k + s$$

**Theorem 1.** Suppose that $FK = K$ and

$$\sup_K \omega_d(T_x F) < 1.$$ 

(4)

Then

$$\dim_{HL} K \leq d.$$ 

(5)

**Proof.** From condition (4) it follows the existence of a number $\nu < 1$ such that

$$\sup_K \omega_d(T_x F) \leq \nu.$$ 

(6)

It is well known [2] that for a natural number $p$ it is valid the inequality

$$\sup_K \omega_d(T_x F^p) \leq \nu^p.$$ 

(7)

We introduce the denotation

$$\beta = \sup_K \alpha_1(T_x F),$$

$$\gamma = \sup_K \alpha_{k+1}(T_x F).$$

It is obvious that $\gamma^d \leq \nu$,

$$\sup_K \alpha_1(T_x F^p) \leq \beta^p, \quad \sup_K \alpha_{k+1}(T_x F^p) \leq \nu^{p/d}.$$ 

Choose $p$ in such a way that

$$\sqrt{n} \nu^{p/d} \leq 1, \quad 2^k n^{d/2} \nu^p < 2^{-2}$$

and $\varepsilon$ such that in the $\beta^p \varepsilon$-neighborhoods of all points of compact $K$ there exists linearization procedure (3).

Consider a covering $K$ by the cubes $C_i$ with sides $2\delta_i$ and centers at the points $x_i$. Consider also the oriented with respect to $T_{x_i} F^p$ cubes $\bar{C}_i \supset C_i$ with centers $x_i$ and sides $2\sqrt{n} \delta_i$. 
Obviously, $T_{x_i}F^p \tilde{C}_i$ is a parallelipiped with sides $2\sqrt{n} \delta_i \alpha_j(T_{x_i}F^p)$, $j = 1, \ldots, n$, and

$$T_{x_i}F^p C_i \subset T_{x_i}F^p \tilde{C}_i.$$ We cover this parallelipiped by cubes with sides $2\sqrt{n} \delta_i \alpha_{k+1}(T_{x_i}F^p)$. The number of such cubes is less than or equal to

$$\left( \frac{\alpha_1(T_{x_i}F^p)}{\alpha_{k+1}(T_{x_i}F^p)} + 1 \right) \cdots \left( \frac{\alpha_k(T_{x_i}F^p)}{\alpha_{k+1}(T_{x_i}F^p)} + 1 \right).$$

Consequently

$$\mu_{HL}(T_{x_i}F^p \tilde{C}_i, d, \varepsilon) \leq 2^k n^{d/2} \omega_d(T_{x_i}F^p) \delta_i^d \leq \frac{\delta_i}{4}.$$ Then by (1) we have

$$\mu_{HL}(K, d, \varepsilon) = \mu_{HL}(F^p K, d, \varepsilon) < \frac{1}{2} \mu_{HL}(K, d, \varepsilon).$$ However in this case $\mu_{HL}(K, d, \varepsilon) = 0$ and $\mu_{HL}(K, d) = 0$. This implies the assertion of theorem.

**Theorem 2.** Suppose that for the compacts $	ilde{K} \supset K$ it is valid the following conditions $F^m(K) \subset \tilde{K}$, $\forall m \geq 1$,

$$\sup_{\tilde{K}} \omega_d(T_x F) < 1,$$

$$\mu_{HL}(K, d) < \infty.$$ Then

$$\lim_{m \to \infty} \mu_{HL}(F^m(K), d) = 0.$$ This theorem is an analog of the theorems on Hausdorff measure, proved in [4]. The proof of Theorem 2 is similar to the scheme, used in [4] with applying the estimates, obtained in proving Theorem 1.

The upper estimate of measure and dimension of Hausdorff–Lebesgue is Lyapunov dimension. Recall the definition of Lyapunov dimension [2,5].

**Definition 8.** The local Lyapunov dimension of the map $F$ at the point $x$ is the number

$$\dim_L(F, x) = j + s,$$ where $j$ is the largest integer from interval $[1, n]$ such that

$$\alpha_1(T_x F) \ldots \alpha_j(T_x F) \geq 1$$

and $s$ is such that $s \in [0, 1]$ and

$$\alpha_1(T_x F) \ldots \alpha_j(T_x F) \alpha_{j+1}(T_x F)^s = 1.$$ By definition, in the case $\alpha_1(T_x F) < 1$ we have $\dim_L(F, x) = 0$ and in the case $\alpha_1(T_x F) \ldots \alpha_n(T_x F) \geq 1$ we have $\dim_L(F, x) = n$. 
Definition 9. The Lyapunov dimension of the map $F$ on the set $K$ is the number
$$\dim_L(F, K) = \sup_K \dim_L(F, x).$$

Definition 10. A local Lyapunov dimension of the sequence of maps $F^m$ at the point $x$ is a number
$$\dim_L x = \limsup_{m \to +\infty} \dim_L(F^m, x).$$

Definition 11. The Lyapunov dimension of maps $F^m$ on the set $K$ is a number
$$\dim_L K = \sup_K \dim_L x.$$  

Theorem 1 implies the following result.

Theorem 3. Suppose that $F(K) = K$. Then $\dim_{HL} K \leq \dim_L K$.

Hypothesis. If $F(K) = K$, then $\dim_{FHL} K \leq \dim_L K$.

The theory of Lyapunov dimension of attractors is well developed [2,5–8]. For many classical attractors the estimates and formulas of Lyapunov dimension are obtained. Consider such attractors.

Consider the dynamical systems generated by the differential equations
$$\frac{dX}{dt} = f(X), \quad X \in \mathbb{R}^n, \quad t \in \mathbb{R}^1 \quad (8)$$
or by the difference equations
$$X(t + 1) = f(X(t)), \quad X \in \mathbb{R}^n, \quad t \in \mathbb{Z}. \quad (9)$$
Here $\mathbb{Z}$ is a set of integers, $f(X)$ is a vector-function: $\mathbb{R}^n \to \mathbb{R}^n$. We assume that the trajectory $X(t, X_0)$ of equation (8) is uniquely determined for $t \in \mathbb{R}^1$. Here $X(0, X_0) = X_0$.

Definition 12. We say that $K$ is invariant if $X(t, K) = K, \forall t \in \mathbb{R}^1$. Here
$$X(t, K) = \{X(t, X_0) \mid X_0 \in K\}.$$

Definition 13. We say that the invariant set $K$ is locally attractive if for a certain $\varepsilon$-neighborhood $K(\varepsilon)$ of $K$ the relation
$$\lim_{t \to +\infty} \rho(K, X(t, x_0)) = 0, \quad \forall x_0 \in K(\varepsilon)$$
is satisfied.
Here $\rho(K, x)$ is a distance from the point $x$ to the set $K$, defined as
$$\rho(K, X) = \inf_{Y \in K} |Y - X|,$$
$| \cdot |$ is Euclidian norm in $\mathbb{R}^n$,
$$K(\varepsilon) = \{Y \mid \rho(K, Y) \leq \varepsilon\}.$$  

Definition 14. We say that the invariant set $K$ is globally attractive if
$$\lim_{t \to +\infty} \rho(K, X(t, x_0)) = 0, \quad \forall x_0 \in \mathbb{R}^n.$$
**Definition 15.** We say that $K$ is
1) an attractor if it is an invariant closed and locally attractive set
2) a global attractor if it is an invariant closed and globally attractive set. Consider a Lorenz system [9]
\[
\begin{align*}
\dot{x} &= -\sigma(x - y), \\
\dot{y} &= r x - y - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]
where $\sigma > 0$, $r > 1$, $b \in [0, 4]$.

**Theorem 4.** [8] If
\[
\frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4\sigma r}} > 1,
\]
then any solution of system (8) tends to equilibrium as $t \to +\infty$. If
\[
\frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4\sigma r}} \leq 1,
\]
then
\[
\dim_{L} K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4\sigma r}}.
\]
Here $K$ is a global attractor of system (10).

From Theorems 3 and 4 it follows that for a global attractor $K$ of system (10) with $\sigma = 10$, $b = 8/3$, $r = 28$ we have
\[
\dim_{HL} K \leq 2.4014.
\]

Consider now a local attractor $K$ of system (10), which does not involve equilibria. It is well known that for system (10) we have
\[
\omega_3(T_x F^t) = e^{-(\sigma+b+1)t}, \quad \forall x \in R^3.
\]
Here $F^t$ is a shift operator along trajectories of system (10).

It is also well known that if
\[
\sup_{K} \alpha_1(T_x F^t) \leq e^{at},
\]
where $a$ is positive number, then
\[
\sup_{K} [\alpha_1(T_x F^t) \alpha_2(T_x F^t)] \leq e^{at}.
\]

It follows from the fact that in this case either the first, either the second Lyapunov exponent is equal to zero.

Theorem 1 and [10] implies that in this case we have
\[
\dim_{HL} K \leq 2 + a/(\sigma + b + 1 + a).
\]

The numerical results give for $\sigma = 10$, $b = 8/3$, $r = 28$, $a \leq 0.829$.

Consequently in this case we have
\[
\dim_{H} K \leq \dim_{HL} K \leq 2.058.
\]
3. LOWER ESTIMATES OF HAUSDORFF-LEBESGUE MEASURE

Consider one-parameter group of diffeomorphisms $F^t$, $k$-dimensional smooth manifold $K$, $k$-dimensional segment of surface $K_1 \subset K$.

**Theorem 5.** Suppose that $F^t K_1 \subset K$, $\forall t > 0$ $\mu_{HL}(K_1, k) > 0$, and for a certain $t_i \to +\infty$ the following conditions

\[
\lim_{t_j \to +\infty} \inf_{K} (\alpha_k(T_xF^{t_j}) - 2\alpha_{k+1}(T_xF^{t_j})) > 0, \tag{15}
\]

\[
\lim_{t_j \to +\infty} (\inf_{K} \omega_k(T_xF^{t_j})) = +\infty \tag{16}
\]

are satisfied. Then

\[
\mu_{HL}(K, k) = +\infty.
\]

**Proof.** It is obvious that here $\mu_{FHL}(K, k) = \mu_{HL}(K, k)$.

Relation (16) implies that for any $R > 0$ there exists $\tau > 0$ such that

\[
\inf_{K} \omega_k(T_xF^\tau) \geq R.
\]

We choose a number $\tilde{\varepsilon}$ such that in any $[\sup K \alpha_1(T_xF^\tau)]\tilde{\varepsilon}$-neighborhood of the point $x \in K$ there exists a linearization procedure

\[
F^\tau(x + h) - F^\tau(x) = (T_xF^\tau)h + o(h).
\]

Suppose that $\mu_{FHL}(K_1, k) = \mu_{HL}(K_1, k) = \nu > 0$. Consider a covering $K_1$ by disjoint cubes $C_i$ with sides $2\varepsilon < 2\tilde{\varepsilon}$ and centers at the points $x_i$. Definition $\mu_{FHL}$ implies that the number of these cubes is as follows

\[
N \approx \frac{\nu}{\varepsilon_k}.
\]

These cubes involve oriented cubes $\tilde{C}_i$ with sides $2\varepsilon/\sqrt{n}$. Obviously, $T_xF^\tau \tilde{C}_i$ is a parallelepiped with sides $2\varepsilon\alpha_j(T_xF^\tau)/\sqrt{n}$ and

\[
T_xF^\tau C_i \supset T_xF^\tau \tilde{C}_i.
\]

This parallelepiped contains

\[
\left(\frac{\alpha_1(T_xF^\tau)}{\alpha_{k+1}(T_xF^\tau)} - 1\right) \cdots \left(\frac{\alpha_k(T_xF^\tau)}{\alpha_{k+1}(T_xF^\tau)} - 1\right)
\]

disjoint cubes with sides $2\alpha_{k+1}(T_xF^\tau)\varepsilon$. Any such cube contains points from $F^\tau K_1$.

Then from (15) it follows that

\[
\mu_{HL}(F^\tau K_1, k) \geq \left(\frac{\nu}{\varepsilon_k}\right) \frac{1}{2^k n^{k/2}} \inf_{K} \omega_k(T_xF^\tau) \varepsilon^k \geq \frac{R\nu}{2^k n^{k/2}} \tag{17}
\]

From (17) and inclusion $F^\tau K_1 \subset K$ it follows that $\mu_{HL}(K, k) = +\infty$.

Theorem 5 implies that if (15) and (16) are satisfied, then the compact $K$ cannot be bounded closed manifold. We show numerically that for the Lorenz system with $\sigma = 10$, 

\( b = 2/3, \ r = 28 \) the estimation \( \inf_K \alpha_1(T_x F^\tau) \geq \exp(0.788t) \) is valid. Consequently a smooth manifold cannot be an attractor of Lorenz system.

Let us give a geometric interpretation of the proof of theorem.

![Diagram 1](image1.png)

**Рис. 1.**

In Fig.1 the curve \( K_1 \) is covered by disjoint cubes \( C_i \) with centers \( x_i \) on a curve and the lengths of sides \( 2\varepsilon \). Inside these cubes there are oriented cubes \( \tilde{C}_i \) with sides \( \sqrt{2}\varepsilon \). The number of these cubes is \( N \approx \frac{\nu}{\varepsilon} \).

![Diagram 2](image2.png)

**Рис. 2.**

In Fig.2 the curve \( F^\tau K_1 \) is partially covered by parallelograms \( T_{x_i} F^\tau \tilde{C}_i \). They involve the cubes with sides \( \sqrt{2}\alpha_2(T_{x_i} F^\tau)\varepsilon \). The number of all such cubes is as follows

\[
\tilde{N} \approx \frac{\nu}{\varepsilon} \frac{\alpha_1(T_{x_i} F^\tau)}{\sqrt{2}\alpha_2(T_{x_i} F^\tau)}.
\]

It is clear that the length of curve \( F^\tau K_1 \) is greater than or equal to

\[
\tilde{N}\alpha_2(T_{x_i} F^\tau) = \nu \inf_K \omega_1(T_x F^\tau) \geq \nu R/\sqrt{2}.
\]

Similar consideration is valid for \( k = 2, \ldots, n - 1 \).

**Problem.** To extend Theorem 5 to any \( d \in (1, n) \) and more wide class of sets \( K \).
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