Abstract

We describe a simple way of incorporating fluctuations of the Hubble scale during the horizon exit of scalar perturbations into the $\delta N$ formalism. The dominant effect comes from the dependence of the Hubble scale on low-frequency modes of the inflaton. This modifies the coefficient of the log-enhanced term appearing in the curvature spectrum at second order in field fluctuations. With this modification, the relevant coefficient turns out to be proportional to the second derivative of the tree-level spectrum with respect to the inflaton $\varphi$ at horizon exit. A logarithm with precisely the same coefficient appears in a calculation of the log-enhancement of the curvature spectrum based purely on the geometry of the reheating surface. We take this agreement as strong support for the proposed implementation of the $\delta N$ formalism. Moreover, our analysis makes it apparent that the log-enhancement of the inflationary power-spectrum is indeed physical if this quantity is defined using a global coordinate system on the reheating surface (or any other post-inflationary surface of constant energy density). However, it can be avoided by defining the spectrum using invariant distances on this surface.
1 Introduction

It is well known that curvature perturbations created during cosmological inflation \([1,2]\) are affected by infrared (IR) divergences \([3,5]\) (see \([6]\) for some recent discussions). These divergences are closely related to the familiar divergence of the scalar-field correlator in de Sitter space \([7]\). While, at leading order, the divergence can be absorbed into the definition of the background and is hence unobservable, higher orders in the curvature perturbation lead to corresponding log-enhanced corrections to the power spectrum. Since the IR cutoff appearing in these logarithms is provided by the size of the observed universe (rather than by, for example, the size of the universe created by inflation), these logarithms are, however, not particularly large in practice \([5]\). Nevertheless, it is conceivable that the power spectrum is measured by a very late observer, who has access to the entire region of the universe created in ‘our’ inflationary patch. For such a ‘late’ observer, the infrared logarithms can be extremely large and it is an interesting question of principle how to achieve consistency between his and our (i.e. the ‘early’ observer’s) measurement of the power spectrum.

We approach these issues starting from the \(\delta N\) formalism \([8,9]\). It turns out that the problem can be resolved rather easily if a simple modification of the \(\delta N\) formalism, which takes fluctuations of the Hubble scale during slow-roll inflation into account, is implemented. In essence, this modification consists of treating the Hubble scale, which defines the normalization of the scalar-field correlator, as a function of the perturbed background value of \(\varphi\) relevant at the time of horizon exit of a given mode. Since this background perturbation depends only on modes with a smaller wave number, our proposal can be implemented in an unambiguous and straightforward way.

The Hubble-scale fluctuations discussed above lead to slow-roll suppressed but log-enhanced contributions to the power-spectrum, similar to the familiar ‘c-loop’ effects of the \(\delta N\) formalism. Combining both these contributions, the coefficient of the first log-enhanced correction to the power spectrum takes a very simple form. It is essentially given by the second derivative in \(\varphi\) of the leading order power spectrum, \(N_\varphi(\varphi)^2H(\varphi)^2/(2\pi)^2\). Here the argument \(\varphi\) is the value of the inflaton corresponding to the horizon exit of the wave number \(k\) under consideration.

The physical significance of this proposal can be understood as follows: Consider the reheating surface (or any other surface of constant energy density after the end of inflation). Disregarding vector and tensor modes, the geometry of this surface, parameterized by coordinates \(\vec{y} = (y^1, y^2, y^3)\), can be characterized by a single scalar function \(\zeta(\vec{y})\), known as the curvature perturbation. The power spectrum may be defined as the logarithmic derivative of the correlation function \(\langle \zeta(\vec{x})\zeta(\vec{x}+\vec{y})\rangle\) with respect to \(y = |\vec{y}|\). Alternatively, a closely related spectrum based on a fixed invariant distance \(s\) between pairs of points in the correlator can be defined. This latter power spectrum is an entirely local quantity and its expectation value does not depend on the size of the region over which the correlator is measured (i.e. on the ‘age’ of the observer). The two spectra, which we denote by \(P_\zeta(y)\) and \(P_\zeta(s)\), are related by \(P_\zeta(y) = \langle P_\zeta(ye^{\bar{\zeta}})\rangle\), where \(\bar{\zeta}\) is a coarse-grained value of \(\zeta\) relevant for the distance-measurement between a given pair of points. In the relation between the two spectra, the averaging is over the potentially large observed region, with large expectation values of \(\bar{\zeta}\) and \(\bar{\zeta}^2\) in the case of a late observer. This leads to a log-enhancement of \(P_\zeta(y)\) which is easily seen to be

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1 Of course, this possibility is in practice limited by the presently observed dark energy or cosmological constant.
2 The physical importance of higher-order terms in the \(\delta N\) formalism was first appreciated in \([9]\).
precisely the log-enhancement found earlier in (our implementation of) the \( \delta N \) formalism.

On the one hand, this agreement provides support for the implementation of the \( \delta N \) formalism that we advocate. On the other hand, it makes the ‘physical reality’ of large logs from IR divergences particularly clear: The log-enhancement arises due to the use of global coordinates in a very large region, where these deviate significantly from the invariant distance. It can be avoided if one measures the power spectrum \( \tilde{P}_\zeta(s) \), which is defined using a two-point correlation function based on the invariant distance between each pair of points appearing in the spatial average.

While our work was being finalized, Ref. [10] appeared, which overlaps with part of our analysis. We will comment on this in more detail at the end of Sec. 3.

2 Hubble scale fluctuations in the \( \delta N \) formalism

During inflation, the amplitude of scalar field fluctuations is controlled by the Hubble parameter \( H \). The latter is usually evaluated at the value \( \varphi(t_k) \) of the classical homogeneous solution \( \varphi \), where \( t_k \) is the time of horizon exit of the mode \( k: H = H(\varphi(t_k)) \). However, this approach does not account for the fact that perturbations with wavelength larger than \( k^{-1} \) have already left the horizon. These perturbations modify the value of the scalar field relevant for the mode \( k \) and need to be taken into account.

We do so by writing the scalar field perturbation in a quasi-de-Sitter background as

\[
\delta \varphi(\vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{2k^3}} H(\varphi(t_k) + \delta \tilde{\varphi}(\vec{x})) \, \tilde{a}_{\vec{k}},
\]

where

\[
\delta \tilde{\varphi}(\vec{x}) = \int_{l \ll k} \frac{d^3 l}{(2\pi)^3} \frac{e^{-i\vec{l}\cdot\vec{x}}}{\sqrt{2l^3}} H(\varphi(t_l)) \, \tilde{a}_{\vec{l}},
\]

where \( \tilde{a}_{\vec{k}} \) is a normalized Gaussian random variable \(^3\). In writing eq. (1) and in the rest of the paper, we only include the leading order contributions to the fluctuations and neglect slow-roll corrections that are not enhanced by potentially large logarithms. The \( \delta \tilde{\varphi}(\vec{x}) \) contribution in the argument of the Hubble parameter accounts for the backreaction of long wavelength modes on the scalar fluctuations \( \delta \varphi(\vec{x}) \) generated inside the horizon.

In order to analyze their effect, we expand the Hubble parameter up to second order in \( \delta \tilde{\varphi} \),

\[
\delta \varphi(x) = \int \frac{d^3 k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \chi_{\vec{k}} \left[ 1 + A_1 \int_{l \ll k} \frac{d^3 l}{(2\pi)^3} e^{-i\vec{l}\cdot\vec{x}} \chi_{\vec{l}} + A_2 \int_{l,m \ll k} \frac{d^3 l}{(2\pi)^3} \frac{d^3 m}{(2\pi)^3} e^{-i(\vec{l}+\vec{m})\cdot\vec{x}} \chi_{\vec{l}} \chi_{\vec{m}} \right]
\]

where we have introduced the Gaussian field

\[
\chi_{\vec{k}} = \frac{H(\varphi(t_k))}{\sqrt{2k^3}} \tilde{a}_{\vec{k}}.
\]

\(^3\)It satisfies the relation \( \langle \tilde{a}_{\vec{k}} \tilde{a}_{\vec{p}} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{p}) \).
We use the abbreviations $A_1 = H_φ/H$ and $A_2 = H_{φφ}/(2H)$ for coefficients involving the first and second derivative of $H$ with respect to $φ$. Note that as a result of the corrections proportional to $A_1$ and $A_2$, the scalar fluctuation in Fourier space, $δφ_κ$, are not Gaussian at Hubble exit. A similar idea was considered to second order in the scalar field perturbations, in the context of the tree-level bispectrum in [11]. Note that we neglect the quantum mechanically generated non-Gaussianity of the fields which is present at horizon exit [12].

We now turn to the curvature fluctuation $ζ$ evaluated on a surface of uniform energy density (for our purposes, the reheating surface). The $δN$ formalism calculates $ζ$ starting from scalar fluctuations on a flat surface,

$$ζ(⃗x) = N(φ + δφ(⃗x)) - N(φ) = N_φ δφ(⃗x) + \frac{1}{2} N_{φφ} δφ(⃗x)^2 + \frac{1}{6} N_{φφφ} δφ(⃗x)^3 + \cdots.$$  (5)

Here $N_φ$, $N_{φφ}$ are derivatives of the number of e-folds $N$ as a function of $φ$, evaluated on the classical background trajectory. The resulting two point function for curvature perturbations in Fourier space is given by

$$⟨ζ_κζ_ρ⟩ = N_φ^2 δ_{κρ} + N_{φφ} ⟨δφ_κ(δφ^2)⟩_κ + \frac{N_φ^2}{4} ⟨(δφ^2)_κ(δφ^2)⟩_κ + \frac{N_φ N_{φφφ}}{3} ⟨δφ_κ(δφ^3)⟩_κ.$$  (6)

Note that correlators of odd powers of $δφ$ are in general non-zero since $δφ_κ$ is not Gaussian. However, one may check that nevertheless $⟨δφ_κ⟩ = 0$. We have not set $<ζ> = 0$, though making this choice would not change in an essential way our results (we will discuss this in more detail later).

We now define the spectrum $P_χ$ through $⟨χ_κχ_ρ⟩ = (2π)^3 δ^3(⃗k + ⃗p) 2π^2 P_χ(k)/k^3$. This implies that $P_χ(k) = (H/2π)^2$. The spectrum of curvature perturbations $P_ζ$ is defined in an analogous way. It follows from a straightforward evaluation of eq. (6) (see Appendix A for details) that

$$P_ζ(k) = P_χ \left\{ N_φ^2 + P_χ \ln(kL) \left[ (A_1^2 + 2A_2) N_φ^2 + 4A_1 N_φ N_{φφ} + N_φ^2 N_{φφφ} \right] \right\}$$

$$= P_χ N_φ^2 + P_χ \ln(kL) \frac{d^2}{dφ^2} \left( N_φ^2 P_χ \right).$$  (7)

Here it is crucial that the IR cutoff $L$ is set by the size of the region in which the correlator is measured. As mentioned before, we take it for granted that perturbations on even larger scales can be absorbed in the background and are hence unobservable. In other words, the classical evolution starts with the horizon exit of modes of scale $k \sim 1/L$. Our main interest is the dependence of the spectrum on the size $L$ of the observed region.

We can express the second derivative along the scalar field, appearing in equation (7), in terms of slow-roll parameters. We define as usual $ε = V_φ^2/(2V)$, $η = V_{φφ}/V$, $ξ^2 = V_φ V_{φφφ}/V^2$, and we set $M_ϕ^2 = 8π$. Using the well known relations $n_κ - 1 = 2η - 6ε$ for the spectral index, and $α_κ = 16εη - 24ε^2 - 2ξ^2$ for its running, i.e. $α_κ \equiv dn_κ/d\ln k$ (all quantities evaluated at Hubble exit) we find

$$\frac{d^2}{dφ^2} \left( N_φ^2 P_χ \right) = \frac{P_χ N_φ^4}{4} \left[ (η - 2ε)(n_κ - 1) + (n_κ - 1)^2 + α_κ \right]$$  (8)

This quantity vanishes in the case of scale invariance, i.e. when $n_κ = 1$ and $α_κ = 0$. In this case logarithmic corrections to the power spectrum drop out, at leading order in the slow-roll expansion.

We will discuss an interpretation of this result in the next section.
At our level of accuracy, we can rewrite eq. (7) as

\[ P_\zeta(k) = P_\chi N_\varphi^2 + \frac{1}{2} \frac{d^2}{d\varphi^2} (N_\varphi^2 P_\chi), \] (9)

where

\[ \langle \delta\varphi^2 \rangle_{1/k} = \int_{L^{-1}}^k \frac{d^3 k'}{(2\pi)^3} \frac{H^2}{2k'^3} = P_\chi \ln(kL) \] (10)

is the expectation value of \( \delta\varphi^2 \) measured on a length scale \( 1/k \). We see that the log-enhanced correction takes the suggestive form of the second term in a Taylor expansion. The physical meaning of this structure will be clarified below.

We note that the integration in eq. (10) was performed assuming that \( \varphi \) has an exactly scale invariant spectrum. More generally, if the power spectrum of \( \varphi \) has a constant spectral index \( n_\varphi \), the \( \ln(kL) \) term above should be replaced by

\[ \ln(kL) \rightarrow \frac{1}{n_\varphi - 1} \left( 1 - (kL)^{-(n_\varphi - 1)} \right). \] (11)

However, provided that we do not consider an exponentially large range of scales and the field fluctuations are reasonably close to scale invariant, then \(|n_\varphi - 1| \ln(kL) \ll 1\), and the more general result above is well approximated by \( \ln(kL) \). Since the running of the spectral index is expected to be of order \((n_\varphi - 1)^2\), further corrections associated with this running are suppressed by a similar argument.

3 Infrared divergences from the geometry of the reheating surface

We now provide a physical interpretation for the log-enhanced correction to the power spectrum given in eq. (7). An observer, in order to make a measurement, specifies a coordinate system on the reheating surface (or on any other surface of constant energy density after the end of inflation). This can be done as follows. Choose a coordinate system where slices of constant time \( t \) have uniform energy density. Neglecting vector and tensor modes, we write the metric as

\[ ds^2_4 = -dt^2 + a^2(t) e^{2\zeta(x)} \delta_{ij} dx^i dx^j. \] (12)

We use conventions where reheating occurs at time \( t_f \), at which the homogeneous scale factor \( a(t_f) = 1 \). This leads to the following metric on the three-dimensional reheating surface:

\[ ds^2_3 = e^{2\zeta(x)} \delta_{ij} dx^i dx^j. \] (13)

Important consequences for the power spectrum of \( \zeta \) can be derived just from the geometry of the reheating surface specified above. To see this, let us define the power spectrum as the logarithmic
derivative of the two point function,
\[
\mathcal{P}_\zeta(y) \equiv \frac{d}{d \ln y} \frac{1}{2} \langle (\zeta(\vec{x}) - \zeta(\vec{x} + \vec{y}))^2 \rangle = -\frac{d}{d \ln y} \langle \zeta(\vec{x}) \zeta(\vec{x} + \vec{y}) \rangle, \tag{14}
\]
where \( y \) is the length of a coordinate vector on the reheating surface, \( y^2 = (y_1)^2 + (y_2)^2 + (y_3)^2 \). It is not difficult to see that eq. (14) gives the correct value for the power spectrum associated with the fluctuations of a slowly-rolling scalar in quasi-de Sitter background (see Appendix B).

Alternatively, we can define the power spectrum averaging over pairs of points separated by a fixed invariant distance \( s \),
\[
\hat{\mathcal{P}}_\zeta(s) \equiv \frac{d}{d \ln s} \frac{1}{2} \langle (\zeta(\vec{x}) - \zeta(\vec{x} + \vec{y}(s)))^2 \rangle. \tag{15}
\]
Here \( \vec{y}(s) \) denotes a coordinate vector with invariant length \( s \). We have introduced a tilde to distinguish this spectrum from eq. (14). It is clear that long-wavelength background fluctuations of \( \zeta \), which affect only the parameterization but not the physics of any local patch of the reheating surface, are irrelevant for this new spectrum. In other words, \( \hat{\mathcal{P}}_\zeta(s) \) is a purely local quantity which, by its very definition, cannot depend on the size \( L \) of the observed region.

Defining \( \bar{\zeta} \) as the average value of \( \zeta \) characteristic of a small region containing a particular pair of points \( \vec{x} \) and \( \vec{x} + \vec{y}(s) \), we can write
\[
\hat{\mathcal{P}}_\zeta(s) \equiv \frac{d}{d \ln s} \frac{1}{2} \left\langle (\zeta(\vec{x}) - \zeta(\vec{x} + e e^{-\bar{\zeta}}))^2 \right\rangle. \tag{16}
\]
Here \( e \) is a coordinate unit vector. We see that the two spectra differ only by the selection of pairs of points in the averaging procedure. In one case, pairs with fixed coordinate distance \( y \), in the other case pairs with fixed invariant distance
\[
s = e \bar{\zeta} y \tag{17}
\]
are chosen. Hence we can express \( \mathcal{P}_\zeta \) in terms of the locally defined spectrum \( \hat{\mathcal{P}}_\zeta \) through
\[
\mathcal{P}_\zeta(y) = -\frac{d}{d \ln y} \langle \zeta(\vec{x}) \zeta(\vec{x} + \vec{y}) \rangle = -\frac{d}{d \ln s} \langle \zeta(\vec{x}) \zeta(\vec{x} + e(\vec{y}e^{-\bar{\zeta}})) \rangle = \langle \hat{\mathcal{P}}_\zeta(\vec{y}e^{-\bar{\zeta}}) \rangle. \tag{18}
\]
Here the second equality holds because logarithmic derivatives in \( y \) and \( s \) agree. The third equality relies on the fact that \( \hat{\mathcal{P}}_\zeta \) is a local quantity, as explained before. The averaging in the last term remains non-trivial because of the variation of \( \bar{\zeta} \) over the large patch of size \( L \) on which \( \mathcal{P}_\zeta \) was originally defined.

Expanding to second order in \( \bar{\zeta} \), we find
\[
\mathcal{P}_\zeta(y) = \hat{\mathcal{P}}_\zeta(y) + \langle \bar{\zeta} \rangle \frac{d}{d \ln y} \hat{\mathcal{P}}_\zeta(y) + \frac{1}{2} \langle \bar{\zeta}^2 \rangle \frac{d^2}{(d \ln y)^2} \hat{\mathcal{P}}_\zeta(y). \tag{19}
\]
This quantifies the deviation of the observer-patch-dependent spectrum \( \mathcal{P}_\zeta \) from the locally defined spectrum \( \hat{\mathcal{P}}_\zeta \) for large patches. It also clarifies the origin of the large logarithms of \( L \) found in the \( \delta N \)
formalism. They arise because the expectation values of $\bar{\zeta}$ and $\bar{\zeta}^2$ grow logarithmically with $L$. We note that, if the power spectrum $\tilde{P}_\zeta$ is scale independent, the two last terms in eq. (19) vanish. This is compatible with eqs. (7), (8) and has a simple intuitive reason: The log-enhancement occurs because in regions with a large background value of $\bar{\zeta}$ the scale of a given mode is effectively misidentified. However, in the scale-invariant case, such a misidentification has no consequences.

We now compare eqs. (19) and (7) quantitatively. To achieve this, we can rewrite derivatives in $\ln y$ in terms of derivatives in $\varphi$ using $d\ln(y)/d\varphi = N_\varphi$ (alternatively we can directly use the results of Appendix B) and express expectation values of $\bar{\zeta}$ and $\bar{\zeta}^2$ in terms of $\varphi$-correlators using eq. (5). Equivalently, we can observe that eq. (19) is simply a second order Taylor expansion around the classical trajectory. Since there is an unambiguous functional relation between fluctuations in $\varphi$ and $\ln y$ (the latter being equivalent to $\ln \frac{L}{y}$), we immediately conclude that

$$P_\zeta(y) = \tilde{P}_\zeta(y) + \frac{1}{2} \frac{d^2 \tilde{P}_\zeta}{d\varphi^2} \langle \delta \varphi^2 \rangle = \tilde{P}_\zeta(y) + \frac{1}{2} P_\chi \ln(L/y) \frac{d^2 \tilde{P}_\zeta}{d\varphi^2}. \quad (20)$$

Here $\langle \delta \varphi^2 \rangle$ is the zero-momentum $\delta \varphi$ correlator with UV cutoff $y$ and IR cutoff $L$, in analogy to $\langle \bar{\zeta}^2 \rangle$ above. Furthermore, we used the fact that the expectation value of $\delta \varphi$ vanishes. Thus, we have achieved complete agreement with the IR divergences or, more precisely, with the log-enhanced corrections which arise in the $\delta N$ formalism.

To summarize, we have obtained a simple physical interpretation for the logarithmic contributions to the power spectrum that were determined in the previous section: The log-enhancement arises due to the use of global coordinates in a very large region, where these deviate significantly from the invariant distance. It can be completely avoided if the power spectrum $\tilde{P}_\zeta$ is considered, which is based on the two-point correlation function defined in terms of the invariant distance between each pair of points appearing in the spatial average.

As mentioned in the Introduction, a discussion closely related to the present section appeared in [10] while this paper was being finalized. In particular, using an argument slightly different from ours, a log-enhancement arising from the large-scale geometry of the reheating surface was derived. Our findings can be brought into agreement with [10] if we drop the term $\sim \langle \bar{\zeta} \rangle$ in eq. (19) and set $\alpha_\zeta = \left. \frac{dn_\zeta}{d\ln k} \right|_{k=1}$. This corresponds to keeping just the $(n_\zeta - 1)^2$ term in our eq. (8). Indeed, the term proportional to $\langle \bar{\zeta} \rangle$ disappears if we rescale our coordinates on the reheating surface according to $y \rightarrow ye^{-\bar{\zeta}}$. This can be directly checked from eq. (19) and corresponds to the fact that, in these new coordinates, the average of $\bar{\zeta}$ vanishes. While this is certainly a rather natural coordinate choice, it does not serve our purpose of comparing with the $\delta N$ formalism: Indeed, in this new coordinate system the value $\bar{\zeta} = 0$ does not any more correspond to the endpoint of the classical trajectory. Nevertheless, setting questions related to the $\delta N$ formalism aside, the term $\sim \langle \bar{\zeta} \rangle$ appears just to be matter of different conventions between [10] and our discussion. By contrast, we believe that our term $\sim \alpha_\zeta$ is real and, in general, no more slow-roll suppressed than the $(n_\zeta - 1)^2$ term.

The agreement of the findings of this section (and of [10], subject to the caveats mentioned above) with the previously discussed calculation in the $\delta N$ formalism is a non-trivial result. In particular, [10] argue that any attempt to understand the IR divergences using the $\delta N$ approach is incomplete due to certain divergences of the $\delta \varphi$ correlation function which are already present at horizon crossing. One way of interpreting our modification of the $\delta N$ formalism is by saying that, via Eq. (3), we
include such divergences in the formalism.\textsuperscript{4} It thus appears that our purely classical modification to the $\delta N$ formalism is sufficient to fully capture the leading logarithmic behaviour.

\section{Discussion}

We have studied IR divergences during inflation using both the $\delta N$ formalism and a simple, phenomenological approach based just on the geometry of the reheating surface. By implementing a simple modification of the $\delta N$ formalism, we took into account the effect of modes that left the horizon long before the scales we are observing on the Hubble scale. Including this effect provides new log-enhanced contributions to the power spectrum, at the same order in $H$ and slow-roll as the standard classical loop corrections. We found that the combination of all contributions can be assembled in an elegant formula, in which the log-enhanced contributions are weighted by the second derivative of the tree level power spectrum, with respect to the inflaton field.

This result can be understood intuitively by considering two power spectra: One is defined locally on the surface of reheating, using invariant distances to define the correlator. The other is based on the coordinate distance on this surface and depends on global features of this surface, in particular on long-wavelength modes. When expressed in terms of the local spectrum, this latter, global spectrum exhibits an IR divergence associated to the size of the region on which it is measured. It is, in fact, this latter spectrum that is calculated in the $\delta N$ formalism and the log-divergence found in both approaches is precisely the same. This provides strong support for the modification of the $\delta N$ formalism we propose. In the case of an exactly scale invariant spectrum, the IR logarithms are absent. For an observer dealing with a scale-dependent spectrum and having a very large region available for his measurement, the use of the local spectrum, which is not affected by our IR effects, appears to be clearly favored.

In single-field, slow-roll inflation the coefficient multiplying the logarithm is heavily suppressed. While it is usually also suppressed in multi-field models, this statement does not hold any more in complete generality. In fact, in some special cases the coefficient could be large enough to compensate for the power spectrum suppression found in single-field models \cite{13, 14}. It would therefore clearly be of interest to extend our results to multi-field inflation. In these models, which contain isocurvature perturbations during inflation, the Hubble scale will depend on several fields, even if the primordial curvature perturbation depends on only one field. In cases where the loop correction is not small, it has been argued they could give an observable contribution to $\zeta$ through non-Gaussianity \cite{9, 15}, in particular through a special type of scale dependence of the bispectrum non-linearity parameter \cite{13, 14}. For a discussion of the scale dependence of the tree level bispectrum see \cite{16}. It would thus be interesting to study the effect of our loop corrections on the bispectrum (and higher order $n$-point correlators).

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Appendix A: Correlator of the curvature perturbation

In this appendix, we present the calculation relating the correlator of curvature perturbations $\zeta$ to correlators of the Gaussian random field $\chi$. The terms which need to be calculated are

$$
\langle \zeta_{\vec{k}} \zeta_{\vec{p}} \rangle = N_{\varphi}^2 \langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \rangle + N_{\varphi} N_{\varphi \varphi} \langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \rangle + \frac{N_{\varphi}^2}{4} \langle \delta \varphi_{\vec{k}}^2 \delta \varphi_{\vec{p}}^2 \rangle + \frac{N_{\varphi} N_{\varphi \varphi}}{3} \langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \delta \varphi_{\vec{q}} \rangle.
$$

(21)

In the previous equations, expressions like $(\delta \varphi_{\vec{k}}^2 \delta \varphi_{\vec{p}}^2)$ indicate convolutions. The terms in the last two lines contain correlators between convolutions, that provide (see the first reference in [15] for more details)

$$
\langle (\delta \varphi_{\vec{k}}^2) \delta \varphi_{\vec{p}}^2 \rangle = 4 N_{\chi} \ln(kL), \langle \chi_{\vec{k}} \chi_{\vec{p}} \rangle
$$

and

$$
\langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \delta \varphi_{\vec{q}} \rangle = 3 N_{\chi} \ln(kL) \langle \chi_{\vec{k}} \chi_{\vec{p}} \rangle.
$$

(22)

Using the expansion of eq. (3), written in momentum space, the 2-point correlator of the scalar field $\delta \varphi$ is given by

$$
\langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \rangle = \langle \chi_{\vec{k}} \chi_{\vec{p}} \rangle \left[ 1 + (A_1^2 + 2A_2) \int \frac{d^3 l}{(2\pi)^3} \frac{d^3 m}{(2\pi)^3} \langle \chi_{\vec{l}} \chi_{\vec{m}} \rangle \right].
$$

(23)

Note that, after applying Wick’s theorem, only one term survives the conditions $l, m \ll k$. The remaining correlator $\langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \rangle$, appearing in the second term of eq. (21), is of order three in $\delta \varphi$. Therefore, only terms at next-to-leading order, in the expansion of eq. (3), contribute to it. One finds

$$
\langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \rangle = A_1 \int \frac{d^3 q}{(2\pi)^3} \left[ \frac{d^3 l}{(2\pi)^3} \langle \chi_{\vec{k} - \vec{l}} \chi_{\vec{p} - \vec{q}} \chi_{\vec{q}} \rangle \right]
$$

$$
+ 2A_1 \int \frac{d^3 q}{(2\pi)^3} \left[ \frac{d^3 l}{(2\pi)^3} \langle \chi_{\vec{k}} \chi_{\vec{l}} \chi_{\vec{q} - \vec{q}} \chi_{\vec{p} - \vec{q}} \rangle \right].
$$

(24)

Decomposing the four point functions appearing in the integrands by means of Wick’s theorem, and taking care of the conditions on the size of $l$, one finds

$$
\langle \delta \varphi_{\vec{k}} \delta \varphi_{\vec{p}} \rangle = 4A_1 \mathcal{P}_\chi \ln(kL) \langle \chi_{\vec{k}} \chi_{\vec{p}} \rangle.
$$

(25)
Putting all terms together, the final result for the spectrum of curvature perturbations reads

\[
P_\zeta(k) = \mathcal{P}_\chi \left\{ N_\varphi^2 + \mathcal{P}_\chi \ln(kL) \left[ (A_1^2 + 2A_2) N_\varphi^2 + 4A_1 N_\varphi N_{\varphi\varphi} + N_{\varphi\varphi}^2 + N_\varphi N_{\varphi\varphi\varphi} \right] \right\}
= \mathcal{P}_\chi N_\varphi^2 + \mathcal{P}_\chi \ln(kL) \frac{1}{2} \frac{d^2}{d\varphi^2} (N_\varphi^2 \mathcal{P}_\chi). \tag{26}
\]

**Appendix B: Power spectrum in coordinate space**

In this appendix, we discuss properties of the power spectrum in coordinate space, defined in eq. (14) as

\[
P_\zeta(y) \equiv -\frac{d}{d \ln y} \langle \zeta(\bar{x}) \zeta(\bar{x} + \bar{y}) \rangle. \tag{27}
\]

The power spectrum in coordinate space is no more difficult to handle than the usual power spectrum in momentum space. Moreover, as discussed in the main text, it allows for a simpler and more direct analysis of log-enhanced contributions associated with this quantity.

However, for the purpose of this appendix, we consider only observed regions which are not much larger than \( y \) (i.e. IR cutoffs not much smaller than the relevant momentum scale \( k \)). Under this assumption, \( \mathcal{P}_\zeta \) and \( \tilde{\mathcal{P}}_\zeta \) coincide and all that follows applies to both spectra.

We can compare the two definitions of the power spectra as follows. Fourier expanding the two-point function of curvature perturbations, we can write

\[
P_\zeta(y) = -\frac{d}{d \ln y} \int \frac{d^3p}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} e^{-i[\bar{p}\bar{x} + \bar{k}(\bar{x} + \bar{y})]} \langle \zeta(\bar{p})\zeta(\bar{k}) \rangle
\]

\[
= -\frac{d}{d \ln y} \int \frac{k^2 dk}{2} \int_{-1}^{1} d(cos \theta) e^{i ky \cos \theta} \mathcal{P}_F^F(k)
= -\frac{d}{d \ln y} \int_{k_0}^{k} \frac{\sin(ky)}{ky} \mathcal{P}_F^F(k) \, d \ln k. \tag{28}
\]

To pass from the first to the second line, we used the definition of the power spectrum in momentum space, given before eq. (7). It is labeled by an \( F \), to distinguish it from the analogous quantity in real space. In the third line, we made the IR cutoff explicit introducing a scale \( k_0 \). We can expand \( \mathcal{P}_F^F \) as

\[
\mathcal{P}_F^F(k) = \mathcal{P}_F^F(k_p) + \ln \left( \frac{k}{k_p} \right) \frac{d\mathcal{P}_F^F}{d \ln k} (k_p) + \cdots, \tag{29}
\]

where \( k_p \) is some pivot scale, the inverse of which is of the order of \( y \): we can set \( y \approx k_p^{-1} \). The second and higher terms are slow-roll suppressed with respect to the first term. Substituting this expansion in eq. (28), we find

\[
P_\zeta(y) = \mathcal{P}_F^F(k_p) \frac{\sin(k_0 y)}{k_0 y} + \text{slow-roll suppressed terms} \tag{30}
\]
where the slow-roll suppressed terms are weighted by logarithms of order \( \ln(k_0y) \). We choose \( k_0 \) such that \( k_0y \simeq k_0/k_p \ll 1 \), but not as small as to lead to large logarithms in the slow-roll suppressed terms. This implies that the power spectra in coordinate and momentum space coincide, up to negligible slow-roll corrections:

\[
P_\zeta(y) \simeq P_\zeta^F(1/y).
\]  

(31)

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