Estimates for the Kolmogorov widths of weighted
Sobolev classes with conditions on the 0-th and the
greatest derivatives∗

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Abstract

In this article, we obtain the order estimates for the Kolmogorov widths
of sets with conditions on the norm in the weighted Sobolev space $W^r_{p_1}$ and
in the weighted space $L_{p_0}$.

1 Introduction

In this paper, we continue the investigation from [30] about estimating the Kolmogorov widths of the weighted Sobolev classes with conditions on the derivatives of order 0 and the greatest order. Such classes were studied by Oinarov [17], Stepanov and Ushakova [23] (in these papers, sharp two-sided estimates for the norm of the embedding operator of the weighted Sobolev class on an interval and half-axis with conditions on 0-th and the first derivatives were obtained), Triebel [26], Lizorkin and Otelbaev [13–15], Mynbaev and Otelbaev [16], Aitenova and Kusainova [1, 2] (in these papers, the problem on estimating the Kolmogorov and the linear widths of weighted Sobolev classes on a domain with conditions on the greatest and 0-th derivatives in the weighted $L_q$-space was studied; the conditions on the both derivatives were given in weighted $L_p$-spaces). In addition, Boykov [3] studied the problem on estimating the Kolmogorov and linear widths of weighted Sobolev classes on a cube with conditions on the derivatives of order from 0 to $r$; the conditions on lower derivatives are given in weighted $L_\infty$-spaces, and the conditions on higher derivatives are given in weighted $L_p$-spaces. Boykov and Ryazantsev [4] studied the problem on estimating the Kolmogorov widths of the infinite intersection of weighted Sobolev classes on a cube (the conditions on the derivatives are given in weighted $L_\infty$-spaces).

For details, see [30].

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First we recall the definition of the Kolmogorov widths. Let $X$ be a normed space, $C \subset X$, and let $n \in \mathbb{Z}_+$. The Kolmogorov widths of the set $C$ in $X$ are defined by

$$d_n(C, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in C} \inf_{y \in L} \|x - y\|,$$

where $\mathcal{L}_n(X)$ is the family of linear subspaces in $X$ of dimension at most $n$. For details, see [18] and [25].

In [30], the problem on estimating the widths $d_n(M, L_{q,v}(\Omega))$ was studied, where

$$M = \left\{ f : \Omega \to \mathbb{R}, \left\| \frac{\nabla^r f}{g} \right\|_{L_{p_1}(\Omega)} \leq 1, \left\| w f \right\|_{L_{p_0}(\Omega)} \leq 1 \right\},$$

$$L_{q,v}(\Omega) = \left\{ f : \Omega \to \mathbb{R}, \left\| f \right\|_{L_{q,v}(\Omega)} := \left\| v f \right\|_{L_q(\Omega)} < \infty \right\}.$$

In particular, there was considered the following example: $\Omega$ is a John domain, the weights have the form

$$g(x) = \text{dist}^{-\beta}(x, \Gamma), \ w(x) = \text{dist}^{-\sigma}(x, \Gamma), \ v(x) = \text{dist}^{-\lambda}(x, \Gamma),$$

where $\Gamma \subset \partial \Omega$ is an $h$-set,

$$h(t) = t^\theta, \quad 0 \leq \theta < d.$$ (3)

The definition of a John domain and of an $h$-set will be given in §3. Here we notice that John domains have no zero angles, and that the Sobolev embedding condition for such domains is the same as for a cube [20][21]. The examples of John domains are domains with Lipschitz boundary and Koch’s snowflake. The examples of $h$-sets are $k$-dimensional Lipschitz manifolds (with $h(t) = t^k$), Koch’s curve, some Cantor-type sets.

In [30] order estimates for $d_n(M, L_{q,v}(\Omega))$ were obtained, where the weight functions are given by (2), (3), and the additional conditions on the parameters hold:

$$r + \frac{d - d}{p_0} > 0, \quad r + \frac{d - d}{p_1} > 0, \quad \beta + \sigma - r - \frac{d - \theta}{p_0} + \frac{d - \theta}{p_1} > 0.$$

(4)

The case when some of these inequalities does not hold was not studied. Here we consider this case, but instead of $M$ we take the set

$$\widehat{M} = \left\{ f : \Omega \to \mathbb{R}, \left\| \frac{\nabla^r f}{g} \right\|_{L_{p_1}(\Omega)} + \left\| \frac{f}{g_0} \right\|_{L_{p_0}(\Omega)} \leq 1, \left\| w f \right\|_{L_{p_0}(\Omega)} \leq 1 \right\},$$

(5)

where $g, w, v$ are given by (2), (3),

$$g_0(x) = \text{dist}^{-\beta}(x, \Gamma)$$

(6)

(for $M$, there are difficulties with embedding theorems).
In this paper, we obtain the order estimates for $d_n(\widehat{M}, L_{q,v}(\Omega))$. The main idea of the proof is the same as in [30]. The estimating from above is reduced to estimating the sum of the Kolmogorov widths of the intersections of two finite-dimensional balls; they can be estimated by the widths of balls in $p_0$, $p_1$, 2- or $q$-metrics. In order to estimate the widths from below, it is sufficient to estimate the widths of some intersections of two balls. In [30], we reduced this problem to considering the multi-dimensional cube or octahedron; the estimates for the widths of such sets are well-known [9, 19, 24]. If (4) does not hold, it is not sufficient to consider the cube or the octahedron; we apply the results of [29].

The paper is organized as follows. In §2 we obtain the estimates for the widths of the intersection of two balls in some general function spaces. In §3 we apply this result for the intersection of two weighted Sobolev classes. Besides the set $\hat{M}$ given by (2), (3), (5), (6), we consider the analogues of other examples from [30]. In addition, we will study some cases when the widths of $M$ and $\hat{M}$ have the same orders.

2 Estimates for the widths of the intersection of two balls in function spaces

Let $(\Omega, \Sigma, \mathrm{mes})$ be a measure space. We say that $A, B \subset \Omega$ do not overlap if $\mathrm{mes}(A \cap B) = 0$. Let $E, E_1, \ldots, E_m \subset \Omega$ be measurable sets, $m \in \mathbb{N} \cup \{\infty\}$. We say that $\{E_i\}_{i=1}^m$ is a partition of $E$ if the sets $E_i$ do not overlap and $\mathrm{mes}\left(\bigcup_{i=1}^m E_i \setminus E\right) = 0$.

We denote by $\chi_E(\cdot)$ the indicator function of a set $E$.

Let $1 < p_0, p_1 \leq \infty$, $1 \leq q < \infty$. In [30] the spaces $X_{p_i}(\Omega)$ ($i = 0, 1$) and $Y_q(\Omega)$ were defined. We recall their properties. For each measurable set $E \subset \Omega$, we define

- the spaces $X_{p_i}(E)$ with seminorms $\| \cdot \|_{X_{p_i}(E)}$, $i = 0, 1$,
- the Banach space $Y_q(E)$ with norm $\| \cdot \|_{Y_q(E)}$.

which satisfy the following conditions:

1. $X_{p_i}(E) = \{f|_E : f \in X_{p_i}(\Omega)\}$, $i = 0, 1$, $Y_q(E) = \{f|_E : f \in Y_q(\Omega)\}$;
2. if $\mathrm{mes} E = 0$, then $\dim Y_q(E) = \dim X_{p_i}(E) = 0$, $i = 0, 1$;
3. if $E \subset \Omega$, $E_j \subset \Omega$ ($j \in \mathbb{N}$) are measurable sets, $E = \bigcup_{j \in \mathbb{N}} E_j$, then
   \[
   \|f\|_{X_{p_i}(E)} = \left\|\{\|f|_{E_j}\|_{X_{p_i}(E_j)}\}_{j \in \mathbb{N}}\right\|_{l_{p_i}}, \quad f \in X_{p_i}(E), \ i = 0, 1,
   \]
   \[
   \|f\|_{Y_q(E)} = \left\|\{\|f|_{E_j}\|_{Y_q(E_j)}\}_{j \in \mathbb{N}}\right\|_{l_q}, \quad f \in Y_q(E);
   \]
4. if $E \in \Sigma$, $f \in Y_q(\Omega)$, then $f \cdot \chi_E \in Y_q(\Omega)$. 

3
Let $\mathcal{P}(\Omega)$ be a subspace of dimension $r_0 \in \mathbb{N}$ in the space of equivalence classes of measurable functions on $(\Omega, \Sigma, \mu)$. For each set $E \in \Sigma$ we denote

$$\mathcal{P}(E) = \{ P|_E : P \in \mathcal{P}(\Omega) \}.$$ 

Let $G \subset \Omega$ be a measurable set, and let $T$ be a partition of $G$. We write

$$\mathcal{S}_T(\Omega) = \{ f : \Omega \to \mathbb{R} : f|_E \in \mathcal{P}(E), \ E \in T, \ f|_{\Omega \setminus G} = 0 \}.$$ 

If $T$ is finite and for each $E \in T$ the inclusion $\mathcal{P}(E) \subset Y_q(E)$ holds, then $\mathcal{S}_T(\Omega) \subset Y_q(\Omega)$ (see property 4).

For each finite partition $T = \{E_j\}_{j=1}^n$ of a set $E$ and for each function $f \in Y_q(\Omega)$ we write

$$\|f\|_{p,q,T} = \left( \sum_{j=1}^n \|f|_{E_j}\|^p_{Y_q(E_j)} \right)^{\frac{1}{p}}, \quad i = 0, 1. \quad (7)$$

We suppose that there are a partition $\{\Omega_{t,j}\}_{t \geq t_0, j \in \hat{J}_t}$ of $\Omega$ into measurable subsets (here $t_0 \in \mathbb{Z}_+$) and numbers $c \geq 1, s_* > 0, k_*, k_* \in \mathbb{N}, \gamma_* \geq 0, \alpha_* \in \mathbb{R}, \mu_* \in \mathbb{R}$, such that the following assumptions hold.

**Assumption A.** For each $t \geq t_0, j \in \hat{J}_t$, the inclusion $\mathcal{P}(\Omega_{t,j}) \subset X_{p_1}(\Omega_{t,j}) \cap X_{p_0}(\Omega_{t,j})$ holds. If $s_* + \frac{1}{q} - \frac{1}{p_1} > 0$ or $p_0 \geq q$, then $X_{p_1}(\Omega_{t,j}) \cap X_{p_0}(\Omega_{t,j}) \subset Y_q(\Omega_{t,j})$.

**Assumption B.** The following estimate holds:

$$\text{card } \hat{J}_t \leq c \cdot 2^{\gamma_* k_* t}, \quad t \geq t_0. \quad (8)$$

**Assumption C.** For each $t \geq t_0, j \in \hat{J}_t$, there is a sequence of partitions $\{T_{t,j,m}\}_{m \in \mathbb{Z}_+}$ of the set $\Omega_{t,j}$ such that

$$T_{t,j,0} = \{\Omega_{t,j}\}, \quad \text{card } T_{t,j,m} \leq c \cdot 2^m, \quad (9)$$

and for all $E \in T_{t,j,m}$

$$\text{card } \{E' \in T_{t,j,m+1} : \text{mes } (E \cap E') > 0\} \leq c. \quad (10)$$

**Assumption D.** If $p_0 \geq q$, then for each $t \geq t_0, j \in \hat{J}_t, m \in \mathbb{Z}_+, E \in T_{t,j,m}$, we have

$$\|f\|_{Y_q(E)} \leq c \cdot 2^{-\alpha_* k_* t} \cdot 2^m \left( \frac{1}{p_0} - \frac{1}{q} \right) \|f\|_{X_{p_0}(E)}. \quad (11)$$

**Assumption E.** For each $t \geq t_0, j \in \hat{J}_t, m \in \mathbb{Z}_+, E \in T_{t,j,m}$, there is a linear continuous projection $P_E : Y_q(\Omega) \to \mathcal{S}_{\{E\}}(\Omega)$ with the following properties:
1. For each function \( f \in X_{P_1}(\Omega) \cap X_{P\Omega}(\Omega) \) we have
\[
\|P_E(f \cdot \chi_E)\|_{Y_q(E)} \leq c \cdot 2^{-\alpha \cdot k_1 t} \cdot 2^{m(\frac{1}{p_0} - \frac{1}{q})}\|f\|_{X_{P\Omega}(E)};
\]
if \( m = 0 \), then, in addition,
\[
\|P_E(f \cdot \chi_E)\|_{Y_q(E)} \leq c \cdot 2^{\mu \cdot k_1 t}\|f\|_{X_{P_1}(E)}.
\] (13)

2. For each \( t \geq t_0, \ j \in \hat{J}_t, f \in Y_q(\Omega) \)
\[
\sum_{E \in t_{j,m}} P_E(f \cdot \chi_E) \to f \cdot \chi_{\Omega_{t,j}} \text{ in the space } Y_q(\Omega).
\] (14)

3. Let \( E \in T_{t,j,m}, E' \in T_{t,j,m+1}, \mes(E \cap E') > 0, f \in X_{P_1}(\Omega) \cap X_{P\Omega}(\Omega), \hat{P}_E f, \hat{P}_E f \in \mathcal{P}(\Omega), (\hat{P}_E f)|E = P_E(f \cdot \chi_E)|E, (\hat{P}_E f)|E' = P_E(f \cdot \chi_{E'})|E'. \) Then
\[
\|\hat{P}_E f - \hat{P}_E f\|_{Y_q(E \cup E')} \leq c \cdot 2^{\alpha \cdot k_1 t} \cdot 2^{-m(s + \frac{1}{q} - \frac{1}{p})}\|f\|_{X_{P_1}(E \cup E')}.
\] (15)

In addition, we suppose that the following assumption holds.

**Assumption F.** For each \( t \geq t_0, m \in \mathbb{Z}_+ \), there are functions \( \varphi_j^{t,m} \in X_{P\Omega}(\Omega) \cap X_{P_1}(\Omega) \) \((1 \leq j \leq \nu_j^{t,m})\) with pairwise disjoint supports such that
\[
\nu_j^{t,m} = \left[c^{-1}2^{\gamma \cdot k_1 t} \cdot 2^m\right],
\]
\[
\|\varphi_j^{t,m}\|_{Y_q(\Omega)} = 1, \quad \|\varphi_j^{t,m}\|_{X_{P\Omega}(\Omega)} \leq c \cdot 2^{\alpha \cdot k_1 t} \cdot 2^{m(1/q - 1/p_0)},
\]
\[
\|\varphi_j^{t,m}\|_{X_{P_1}(\Omega)} \leq c \cdot 2^{-\mu \cdot k_1 t} \cdot 2^{m(s + 1/q - 1/p_1)}.
\]

We write
\[
BX_{P_1}(\Omega) = \{ f \in X_{P_1}(\Omega) : \|f\|_{X_{P_1}(\Omega)} \leq 1 \}, \quad i = 0, 1,
\]
\[
M = BX_{P\Omega}(\Omega) \cap BX_{P_1}(\Omega),
\]
\[
\mathfrak{J}_0 = (p_0, p_1, q, r_0, c, k_*, k_\infty, s_*, \gamma_*, \mu_*, \alpha_*).
\]

In [30] the order estimates for \( d_q(M, Y_q(\Omega)) \) were obtained; it was supposed that Assumptions A - E without (13) and (14) hold, and (15) was replaced by
\[
\|f - P_E(f \cdot \chi_E)\|_{Y_q(E)} \leq c \cdot 2^{\alpha \cdot k_1 t} \cdot 2^{-m(s + \frac{1}{q} - \frac{1}{p_1})}\|f\|_{X_{P_1}(E)};
\]
in addition, it was supposed that
\[
s_* + \frac{1}{q} - \frac{1}{p_1} > 0, \quad s_* + \frac{1}{p_0} - \frac{1}{p_1} > 0,
\]
\[
\mu_* + \alpha_* > 0, \quad \mu_* + \alpha_* + \gamma_*/p_0 + \gamma_*/p_1 > 0.
\] (16)
Here we obtain the estimates for $d_n(M, Y_q(\Omega))$ when at least one of the values in (16) is negative. In addition, we notice the cases when it is possible to obtain the estimates without the condition (13).

**Auxiliary assertions.** First we write the discretization lemmas.

Let $1 \leq s \leq \infty$. We denote by $l^N_s$ the space $\mathbb{R}^N$ with norm

$$
\|(x_1, \ldots, x_N)\|_{l^N_s} = \begin{cases}
\left(\sum_{j=1}^N |x_j|^s\right)^{1/s}, & s < \infty, \\
\max_{1 \leq j \leq N} |x_j|, & s = \infty,
\end{cases}
$$

and by $B^N_s$, the unit ball in $l^N_s$.

Given $t, m \in \mathbb{Z}_+$, we denote

$$
\nu_{t,m} = \lceil 2 \cdot 2^{\gamma^* k t} \cdot 2^m \rceil,
$$

(17)

$$
W_{t,m} = 2^{\nu_{t,m}} \cdot 2^{-m(s + 1/q - 1/p_1)} B_{p_1}^{\nu_{t,m}} \cap 2^{-\alpha k t} \cdot 2^{-m(1/q - 1/p_0)} B_{p_0}^{\nu_{t,m}}.
$$

(18)

The following assertion is obtained in [30, p. 30].

**Lemma 1.** Let Assumption F hold. Then for all $t \geq t_0, m \geq 0$

$$
d_n(M, Y_q(\Omega)) \gtrsim d_n(W_{t,m}, l^{\nu_{t,m}}).
$$

(19)

As in [30, p. 10], we write

$$
\Omega_t = \bigcup_{j \in J_t} \Omega_{t,j},
$$

define the partitions $T_{t,m}$ and $\hat{T}_{t,m}$ of the set $\Omega_t$ by

$$
T_{t,m} = \{ E \in T_{t,j,m} : j \in J_t \}, \quad \hat{T}_{t,m} = \{ E \cap E' : E \in T_{t,m}, E' \in T_{t,m+1} \},
$$

(20)

and obtain

$$
\text{card } T_{t,m} \lesssim 2^{\gamma^* k t} \cdot 2^m, \quad \text{card } \hat{T}_{t,m} \lesssim 2^{\gamma^* k t} \cdot 2^m.
$$

(21)

We also define the operator $P_{t,m} : Y_q(\Omega) \to Y_q(\Omega)$ by

$$
P_{t,m}f = \sum_{j \in J_t} \sum_{E \in T_{t,j,m}} P_E(f \cdot \chi_E)
$$

and obtain the following estimates [30, p. 10]

$$
\text{rk } P_{t,m} \lesssim 2^{\gamma^* k t} \cdot 2^m, \quad \text{rk } (P_{t,m+1} - P_{t,m}) \lesssim 2^{\gamma^* k t} \cdot 2^m
$$

(22)
Indeed, by (7) and (20) we get
\[ \|P_{t,m}f\|_{p_0,q,T_{t,m}} \leq 2^{-\alpha_k t} \cdot 2^{m(1/p_0 - 1/q)}, \quad f \in M; \] (23)

\[ \|P_{t,m+1}f - P_{t,m}f\|_{p_0,q,T_{t,m}} \leq 2^{-\alpha_k t} \cdot 2^{m(1/p_0 - 1/q)}, \quad f \in M. \] (24)

Similarly we obtain the estimate
\[ \|P_{t,0}f\|_{p_1,q,T_{t,0}} \leq 2^\mu_k t, \quad f \in M. \] (25)

We prove that
\[ \|P_{t,m+1}f - P_{t,m}f\|_{p_1,q,T_{t,m}} \leq 2^\mu_k t \cdot 2^{-m(s+1/q - 1/p_1)} \cdot 2^{m(s+1/q - 1/p_1)}, \quad f \in M. \] (26)

Indeed, by (17) and (20) we get
\[
\|P_{t,m+1}f - P_{t,m}f\|_{p_1,q,T_{t,m}} = \left( \sum_{E \in T_{t,m}, E' \in T_{t,m+1}} \|P_E(f \cdot \chi_E) - P_{E'}(f \cdot \chi_{E'})\|_{Y_q(E \cap E')}^{p_1} \right)^{1/p_1} \\
\leq \left( \sum_{E \in T_{t,m}} \sum_{E' \in T_{t,m+1}, \text{mes}(E \cap E') > 0} \| \hat{P}_E f - \hat{P}_{E'} f \|_{Y_q(E')}^{p_1} \right)^{1/p_1} \\
\leq 2^{\mu_k t} \cdot 2^{-m(s+1/q - 1/p_1)} \left( \sum_{E \in T_{t,m}} \sum_{E' \in T_{t,m+1}, \text{mes}(E \cap E') > 0} \|f\|_{X_{p_1}(E \cup E')}^{p_1} \right)^{1/p_1} \\
\leq 2^{\mu_k t} \cdot 2^{-m(s+1/q - 1/p_1)} \|f\|_{X_{p_1}(\Omega_t)} \leq 2^{\mu_k t} \cdot 2^{-m(s+1/q - 1/p_1)}. \]

Let \( \hat{\Omega}_t = \cup_{i \geq t} \cup_{i \in J_i} \hat{\Omega}_{t,i} \).

It follows from Assumption A and (14) that if \( s + \frac{1}{q} - \frac{1}{p_1} > 0 \) or \( p_0 > q \), then for \( f \in X_{p_0}(\Omega) \cap X_{p_1}(\Omega) \)
\[ \|f - P_{t,m}f\|_{Y_q(\Omega_i)} \to 0 \quad \text{as} \quad m \to \infty. \]

Hence, for each \( j \geq t_0 \)
\[ f = \sum_{t=t_0}^j P_{t,0}f + \sum_{t=t_0}^j \sum_{m=0}^\infty (P_{t,m+1}f - P_{t,m}f) + f \cdot \chi_{\hat{\Omega}_{t+1}}. \]
In \cite{30} Proposition 3] it was proved that for \( l \in \mathbb{Z}_+ \)
\[
d_t((P_{t,m+1} - P_{t,m})M, Y_q(\Omega)) \lesssim d_t(W_{t+\tau_0,m}, l_q^{\nu_1+\tau_0,m}),
\]
\[
d_t(P_{t,0}M, Y_q(\Omega)) \lesssim 2^{-\alpha_1 k_1}d_t(B_{p_0}^{\nu_1+\tau_0,m}, l_q^{\nu_1+\tau_0,m}),
\]
where \( \tau_0 \in \mathbb{Z}_+ \) depends only on \( 3_0 \). Here we applied \cite{17, 22, 23, 24, 26}. Employing \cite{25} together with \cite{23}, we similarly obtain that
\[
d_t(P_{t,0}M, Y_q(\Omega)) \lesssim d_t(W_{t+\tau_0,0}, l_q^{\nu_1+\tau_0,0}).
\]
This yields the following assertion.

**Lemma 2.** Let Assumptions [A, E] hold, let \( n \in \mathbb{Z}_+ \), \( \hat{i}(n) \geq t_0 \), \( k_{t,m} \in \mathbb{Z}_+ \), \( C \in \mathbb{N} \),
\[
\sum_{t=0}^{\hat{i}(n)} \sum_{m=0}^{\infty} k_{t,m} \lesssim Cn.
\]
Then there is \( C_1 = C_1(3_0) \in \mathbb{N} \) such that
\[
d_{C_1 C_n}(M, Y_q(\Omega)) \lesssim \sum_{t=0}^{\hat{i}(n)} \sum_{m=0}^{\infty} d_{k_{t,m}}(W_{t,m}, l_q^{p_r,m}) + \sup_{f \in M} \| f \|_{Y_q(\Omega, 1)}.
\]
In addition,
\[
\sup_{f \in M} \| f \|_{Y_q(\Omega)} \lesssim \sum_{t=0}^{\infty} \sum_{m=0}^{\infty} d_0(W_{t,m}, l_q^{p_r,m}).
\]

**Proposition 1.** Let Assumptions [A, E] hold without \cite{13}, let \( n \in \mathbb{Z}_+ \), \( \hat{i}(n) \geq t_0 \),
\[
k_{t,m} \in \mathbb{Z}_+, s_t \in \mathbb{Z}_+, C \in \mathbb{N}, \sum_{t=t_0}^{\hat{i}(n)} \sum_{m=0}^{\infty} k_{t,m} + \sum_{t=t_0}^{\hat{i}(n)} s_t \lesssim Cn.
\]
Then there is \( C_1 = C_1(3_0) \) such that
\[
d_{C_1 C_n}(M, Y_q(\Omega)) \lesssim \sum_{t=0}^{\hat{i}(n)} \sum_{m=0}^{\infty} d_{k_{t,m}}(W_{t,m}, l_q^{p_r,m}) +
\]
\[
\sum_{t=0}^{\hat{i}(n)} d_{s_t}(2^{\alpha_2 k_1} B_{p_0}^{\nu_1+\tau_0,m}, l_q^{p_1,0}) + \sup_{f \in M} \| f \|_{Y_q(\Omega, 1)}.
\]
In fact, the analogue of Proposition 1 was proved in \cite{30}.
In \cite{29}, estimates for the Kolmogorov widths of \( B_{p_0}^{N} \cap \nu B_{p_1}^{N} \) for \( p_0 > p_1 \) were obtained. Rewrite this result for the widths of \( \nu_0 B_{p_0}^{N} \cap \nu_1 B_{p_1}^{N} \).

In what follows, we define the numbers \( \lambda \) and \( \hat{\lambda} \) by
\[
\frac{1}{q} = \frac{1 - \lambda}{p_1} + \frac{\lambda}{p_0}, \quad \frac{1}{2} = \frac{1 - \hat{\lambda}}{p_1} + \frac{\hat{\lambda}}{p_0}.
\]

\[
(30)
\]
Notice that

\[(1 - \lambda) \left( s^* + \frac{1}{q} - \frac{1}{p_1} \right) + \lambda \left( \frac{1}{q} - \frac{1}{p_0} \right) = (1 - \lambda)s^*. \]  

(31)

If \( p_0 \leq q \leq p_1 \) or \( p_1 \leq q \leq p_0 \), then

\[\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N \subset \nu_0 \nu_1^{1 - \lambda} B_q^N, \]  

(32)

if \( p_0 \leq 2 \leq p_1 \) or \( p_1 \leq 2 \leq p_0 \), then

\[\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N \subset \nu_0 \nu_1^{1 - \lambda} B_q^N.\]

It follows from Hölder’s inequality or from Galeev’s result [6, Theorem 2], [7, Theorem 1].

We formulate the corollary of the main result of [29] (see Theorem 1 and remarks before and after it).

**Theorem A.** Let \( 1 \leq p_1 < p_0 \leq \infty \), \( 1 \leq q < \infty \), \( N \in \mathbb{N} \), \( n \in \mathbb{Z}_+ \), \( n \leq N/2 \). We define the numbers \( \lambda \) and \( \tilde{\lambda} \) by [29].

If \( \nu_1/\nu_0 \leq 1 \), then

\[d_n(\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N, l_q^N) = d_n(\nu_1 B_{p_1}^N, l_q^N).\]  

(33)

If \( \nu_1/\nu_0 \geq N^{1/p_1 - 1/p_0} \), then

\[d_n(\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N, l_q^N) = d_n(\nu_0 B_{p_0}^N, l_q^N).\]  

(34)

Let \( 1 < \nu_1/\nu_0 < N^{1/p_1 - 1/p_0} \). We set \( \kappa = (\nu_1/\nu_0)^{\frac{q}{p_1 - p_0}} \). Then the following estimates hold.

1. Let \( 1 \leq p_1 < p_0 \leq q \leq 2 \) or \( 1 \leq p_1 < p_0 \leq 2 < q \). Then

\[d_n(\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N, l_q^N) \asymp d_n(\nu_0 B_{p_0}^N, l_q^N).\]

2. Let \( 1 \leq p_1 < 2 < p_0 < q < \infty \). Then

\[d_n(\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N, l_q^N) \asymp \begin{cases} 
  d_n(\nu_0 B_{p_0}^N, l_q^N) & \text{if } n^{\frac{1}{2}} \cdot N^{-\frac{1}{q}} \leq \kappa, \\
  d_n(\nu_1^{1 - \lambda} \nu_0^{\lambda} B_2^N, l_q^N) & \text{if } n^{\frac{1}{2}} \cdot N^{-\frac{1}{q}} \geq \kappa.
\end{cases}\]

3. Let \( 2 \leq p_1 < p_0 < q < \infty \). Then

\[d_n(\nu_0 B_{p_0}^N \cap \nu_1 B_{p_1}^N, l_q^N) \asymp \begin{cases} 
  d_n(\nu_0 B_{p_0}^N, l_q^N) & \text{if } n^{\frac{1}{2}} \cdot N^{-\frac{1}{q}} \leq \kappa, \\
  d_n(\nu_1 B_{p_1}^N, l_q^N) & \text{if } n^{\frac{1}{2}} \cdot N^{-\frac{1}{q}} \geq \kappa.
\end{cases}\]
4. Let $2 \leq p_1 \leq q \leq p_0$. Then
\[
d_n(\nu_0 B_{p_0}^{N} \cap \nu_1 B_{p_1}^{N}, t_q^{N}) \asymp \begin{cases} 
  d_n(\nu_1^{-1/\lambda} \nu_0^\lambda B_q^{N}, t_q^{N}) & \text{if } n^{\frac{1}{p}} \cdot N^{-\frac{1}{q}} \leq \kappa, \\
  d_n(\nu_1 B_{p_1}^{N}, t_q^{N}) & \text{if } n^{\frac{1}{p}} \cdot N^{-\frac{1}{q}} \geq \kappa.
\end{cases}
\]

5. Let $1 \leq p_1 < 2 < q \leq p_0$. Then
\[
d_n(\nu_0 B_{p_0}^{N} \cap \nu_1 B_{p_1}^{N}, t_q^{N}) \asymp \begin{cases} 
  d_n(\nu_1^{-1/\lambda} \nu_0^\lambda B_q^{N}, t_q^{N}) & \text{if } n^{\frac{1}{p}} \cdot N^{-\frac{1}{q}} \leq \kappa, \\
  d_n(\nu_1^{-1/\lambda} \nu_0^\lambda B_2^{N}, t_q^{N}) & \text{if } n^{\frac{1}{p}} \cdot N^{-\frac{1}{q}} \geq \kappa.
\end{cases}
\]

6. Let $q \leq 2$, $1 \leq p_1 < q < p_0$. Then
\[
d_n(\nu_0 B_{p_0}^{N} \cap \nu_1 B_{p_1}^{N}, t_q^{N}) \asymp d_n(\nu_1^{-1/\lambda} \nu_0^\lambda B_q^{N}, t_q^{N}).
\]

7. Let $1 \leq q < p_1 < p_0 \leq \infty$. Then
\[
d_n(\nu_0 B_{p_0}^{N} \cap \nu_1 B_{p_1}^{N}, t_q^{N}) \asymp d_n(\nu_1 B_{p_1}^{N}, t_q^{N}).
\]

For $p_0 < p_1$, the estimates can be rewritten by rearranging the indices 0 and 1.

The estimates of the widths of $B_p^{N}$ in $l_q^{N}$ were obtained in [8, 12, 19, 22, 24] (for details, see [18, 25]). We formulate these results for the cases we need below.

**Theorem B.** [8] Let $1 \leq p \leq q < \infty$, $0 \leq n \leq N/2$.

1. Let $1 \leq q \leq 2$. Then $d_n(B_p^{N}, t_q^{N}) \asymp 1$.

2. Let $2 < q < \infty$, $\lambda_{pq} = \min \left\{ 1, \frac{1/p - 1/q}{1/2 - 1/q} \right\}$. Then
\[
d_n(B_p^{N}, t_q^{N}) \asymp \min \{ 1, n^{-1/2} N^{1/q} \} \lambda_{pq}.
\]

**Theorem C.** [18, 24] Let $1 \leq q \leq p \leq \infty$, $0 \leq n \leq N$. Then
\[
d_n(B_p^{N}, t_q^{N}) = (N - n)^{1/q - 1/p}.
\]

The embedding theorems for $X_{p_1}(\Omega) \cap X_{p_0}(\Omega)$ into $Y_q(\Omega)$.

Notice that if $p_0 \leq q$, then the condition $s_\ast + \frac{1}{q} - \frac{1}{p_1} > 0$ is necessary for the compact embedding. Indeed, if $s_\ast + \frac{1}{q} - \frac{1}{p_1} \leq 0$, then for each $m \in \mathbb{Z}_+$ we have $2^{-m(s_\ast + 1/q - 1/p_1)} \geq 1, 2^{-m(1/q - 1/p_0)} \geq 1$; hence
\[
d_n(W_{l_0, m}, t_{q^{\nu_0, m}}) \geq \begin{cases} 
  d_n(B_1^{\nu_0, m}, t_{q^{\nu_0, m}}) & \text{if } q \geq 1
\end{cases}
\]

for sufficiently large $m$ (see Theorem [13] and [19]). Hence by Lemma [1] we have $d_n(M, Y_q(\Omega)) \geq 1$. In what follows we assume that $s_\ast + \frac{1}{q} - \frac{1}{p_1} > 0$ or $p_0 > q$.

We denote $x_+ = \max\{ x, 0 \}$ for $x \in \mathbb{R}$.

Let the number $\nu_\ast$ be defined as follows.
1. Let one of the following conditions hold: a) \( \mu_* + \alpha_* \leq 0, \mu_* + \alpha_* + \gamma_/p_0 - \gamma_/p_1 \leq 0, s_* + 1/q - 1/p_1 > 0; \) b) \( p_0 < p_1 \leq q, \mu_* + \alpha_* \leq 0, s_* + 1/q - 1/p_1 > 0; \) c) \( p_0 > p_1 \geq q, \mu_* + \alpha_* + \gamma_/p_0 - \gamma_/p_1 \leq 0. \) Then

\[
\nu_* = -\mu_* - \left( \frac{\gamma_*}{q} - \frac{\gamma_0}{p_1} \right).
\] (35)

2. Let one of the following conditions hold: a) \( p_1 < q < p_0, \mu_* + \alpha_* + \gamma_/p_0 - \gamma_/p_1 \leq 0 \leq \mu_* + \alpha_*, \) b) \( p_0 < q < p_1, \mu_* + \alpha_* \leq 0 \leq \mu_* + \alpha_*/p_0 - \gamma_/p_1. \) Then

\[
\nu_* = \frac{\alpha_*(1/p_1 - 1/q) + \mu_*(1/p_0 - 1/q)}{1/p_1 - 1/p_0}.
\] (36)

3. Let \( p_0 \geq q, \mu_* + \alpha_* + \gamma_/p_0 - \gamma_/p_1 \geq 0. \) Then

\[
\nu_* = \alpha_* + \frac{\gamma_*}{p_0} - \frac{\gamma_*}{q}.
\] (37)

4. Let one of the following conditions hold: a) \( \mu_* + \alpha_* + \gamma_/p_0 - \gamma_/p_1 < 0 \leq \mu_* + \alpha_* + p_1 \leq p_0 \leq q, s_* + \frac{1}{q} - \frac{1}{p_1} > 0, \) b) \( \mu_* + \alpha_* \leq 0, p_1 < q < p_0, s_* + \frac{1}{q} - \frac{1}{p_1} < 0. \) Then

\[
\nu_* = \frac{\alpha_*(s_* + 1/q - 1/p_1) + \mu_*(1/q - 1/p_0)}{s_* + 1/p_0 - 1/p_1}.
\] (38)

**Proposition 2.** Let \( \nu_* > 0 \) be defined by (35)-(38). Then for each function \( f \in M \) and for each \( t \geq t_0 \)

\[
\|f\|_{Y_q(H_t)} \lesssim 2^{-\nu_* k_* t}.
\] (39)

**Proof.** Notice that \( d_0(B^N_{p_0}, t^N_q) = N^{(1/q - 1/p)_+}. \)

In case 1, we have \( s_* + \frac{1}{q} - \frac{1}{p_1} > 0; \) hence

\[
\|f\|_{Y_q(H_t)} \lesssim 5_0 \sum_{l \geq t} \sum_{m \geq 0} d_0(W_{l,m}, t^{q,m}_q) \leq \sum_{l \geq t} \sum_{m \geq 0} 2^{\mu_* k_* l} \cdot 2^{-(s_* + 1/q - 1/p_1)} \lesssim 3_0,
\] (28)

\[
\lesssim 3_0 \sum_{l \geq t} \sum_{m \geq 0} 2^{\mu_* k_* l} \cdot 2^{-(s_* + 1/q - 1/p_1)} \cdot 2^{(\gamma_* k_* l + m)(1/q - 1/p_1)_+} \leq 2^{(\mu_* + (\gamma_* /q - \gamma_/p_1)_+) k_* t}.
\] (17)

Consider case 2. We define \( \lambda \in (0, 1) \) by (30). Then

\[
\|f\|_{Y_q(H_t)} \lesssim 5_0 \sum_{l \geq t} \sum_{m \geq 0} d_0(W_{l,m}, t^{q,m}_q) \leq 31_0.
\] (31, 32)
\[ \leq \sum_{l \geq t} \sum_{m \geq 0} 2^{(\mu_*(1-\lambda)-\alpha_* \lambda)k_* l} \cdot 2^{-m_*(1-\lambda)} \lesssim 2^{(\mu_*(1-\lambda)-\alpha_* \lambda)k_* l}. \]

In case 3, the assertion follows from (11) and Hölder’s inequality (see [30, Proposition 4]).

Consider case 4. We define the number \( m_l \) by

\[ 2^{-\alpha_* k_* l} \cdot 2^{-m_l(1/q-1/p_0)} = 2^{\mu_* k_* l} \cdot 2^{-m_l(s_* + 1/q - 1/p_1)}. \]  \( \tag{40} \)

Let a) hold. If \( p_0 = q \), then as in case 3 we get \( \|f\|_{Y_q(\hat{\Omega}_t)} \lesssim 2^{-\alpha_* k_* l} \). It is equivalent to (39), where \( \nu_* \) is given by (38). Let \( p_0 < q \). Notice that \( s_* + \frac{1}{p_0} - \frac{1}{p_1} > s_* + \frac{1}{q} - \frac{1}{p_1} > 0 \). Since \( \mu_* + \alpha_* \geq 0 \), we have \( m_l \geq 0 \) for \( l \geq 0 \) (see (40)). Hence,

\[ \|f\|_{Y_q(\hat{\Omega}_t)} \lesssim \sum_{l \geq t} \sum_{m \geq 0} d_0(W_{l,m}, t_q^\nu) \leq \frac{28}{30} \]  \( \tag{38} \)

\[ \lesssim \sum_{l \geq t} \sum_{0 \leq m \leq m_l} 2^{\alpha_* k_* l} \cdot 2^{-m(1+1/p_0)} + \sum_{l \geq t} \sum_{m > m_l} 2^{\mu_* k_* l} \cdot 2^{-m(s_* + 1/q - 1/p_1)} \leq \frac{30}{30} \]  \( \tag{40} \)

\[ \lesssim \sum_{l \geq t} 2^{\mu_* k_* l} \cdot 2^{-m_l(s_* + 1/q - 1/p_1)} \leq 2^{-\nu_* k_* l}. \]

Let b) hold. Since \( \mu_* + \alpha_* \leq 0 \), \( s_* + \frac{1}{p_0} - \frac{1}{p_1} < s_* + \frac{1}{q} - \frac{1}{p_1} < 0 \), we have \( m_l \geq 0 \) for \( l \geq 0 \) (see (40)). We define \( \lambda \in (0, 1) \) by (30) and get

\[ \|f\|_{Y_q(\hat{\Omega}_t)} \leq \sum_{l \geq t} \sum_{m \geq 0} d_0(W_{l,m}, t_q^\nu) \leq \frac{28}{30} \]  \( \tag{18} \)

\[ \lesssim \sum_{l \geq t} \sum_{0 \leq m \leq m_l} 2^{\mu_* k_* l} \cdot 2^{-m(s_* + 1/q - 1/p_1)} + \sum_{l \geq t} \sum_{m > m_l} 2^{(\mu_*(1-\lambda)-\alpha_* \lambda) k_* l} \cdot 2^{-s_* (1-\lambda) m} \leq \frac{30}{30} \]  \( \tag{40} \)

\[ \lesssim \sum_{l \geq t} 2^{\mu_* k_* l} \cdot 2^{-m_l(s_* + 1/q - 1/p_1)} \leq 2^{-\nu_* k_* l}. \]

This completes the proof.

\[ \square \]

**Estimates for the widths.**

We denote

\[ \tilde{\theta} = s_* \frac{\alpha_* + \gamma_* / p_0 - \gamma_* / q}{\mu_* + \alpha_* + \gamma_*(s_* + 1/p_0 - 1/p_1)}, \]  \( \tag{41} \)

\[ \hat{\theta} = \frac{\mu_*(1/q - 1/p_0) + \alpha_*(s_* + 1/q - 1/p_1)}{\mu_* + \alpha_* + \gamma_*(s_* + 1/p_0 - 1/p_1)}, \]  \( \tag{42} \)
\[
\hat{\nu} = \frac{1}{2} \cdot \frac{\alpha_\ast (1/p_1 - 1/q) + \mu_\ast (1/p_0 - 1/q)}{(\mu_\ast + \alpha_\ast) (1/2 - 1/q) + \gamma_\ast (1/p_1 - 1/p_0)/q},
\]
\[
\hat{\nu} = \frac{\mu_\ast (1/p_0 - 1/2) + \alpha_\ast (1/p_1 - 1/2)}{\gamma_\ast (1/p_1 - 1/p_0)} + \frac{1}{2} - \frac{1}{q}.
\]

We define the numbers \( j_0 \) and \( \theta_j \) \((1 \leq j \leq j_0)\) as follows.

First we consider the case \( s_\ast + \frac{1}{\max(p_0, q)} - \frac{1}{p_i} > 0 \) or \( \min\{p_0, p_1\} \geq q \).

**Notation 1.** Let \( s_\ast + \frac{1}{\max(p_0, q)} - \frac{1}{p_1} > 0 \), \( \mu_\ast + \alpha_\ast \leq 0 \), \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 \leq 0 \); if \( \gamma_\ast = 0 \), we suppose that \( \mu_\ast < 0 \) and set \( -\mu_\ast/\gamma_\ast = +\infty \).

- If \( p_1 \geq q \) or \( q \leq 2 \), we set \( j_0 = 2 \), \( \theta_1 = s_\ast - \left(\frac{1}{p_1} - \frac{1}{q}\right) \), \( \theta_2 = -\frac{\mu_\ast}{\gamma_\ast} - \left(\frac{1}{q} - \frac{1}{p_1}\right) \).
- If \( p_1 < q, q > 2 \), we set \( j_0 = 4 \), \( \theta_1 = s_\ast + \min \left\{ 0, \frac{1}{2} - \frac{1}{p_1} \right\} \), \( \theta_2 = \frac{\theta}{2} \left(s_\ast + \frac{1}{q} - \frac{1}{p_1}\right) \), \( \theta_3 = -\frac{\mu_\ast}{\gamma_\ast} + \min \left\{ \frac{1}{2} - \frac{1}{q}, \frac{1}{p_1} - \frac{1}{q}\right\} \), \( \theta_4 = -\frac{\mu_\ast}{2\gamma_\ast} \).

**Notation 2.** Let \( p_0 \geq q, p_1 \geq q \); if \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 \leq 0 \), \( \gamma_\ast = 0 \), we suppose that \( \mu_\ast < 0 \) and set \( -\mu_\ast/\gamma_\ast = +\infty \). Then \( j_0 = 2 \), \( \theta_1 = s_\ast \).

\[
\theta_2 = \begin{cases} \tilde{\theta} & \text{if } \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 > 0, \\ -\frac{\mu_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} & \text{if } \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 \leq 0. \end{cases}
\]

**Notation 3.** Let \( s_\ast + \frac{1}{\max(p_0, q)} - \frac{1}{p_1} > 0 \), \( \mu_\ast + \alpha_\ast < 0 \), \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 > 0 \).

1. Let \( p_0 < p_1 < q \).

- If \( q \leq 2 \), then \( j_0 = 2 \), \( \theta_1 = s_\ast + \frac{1}{q} - \frac{1}{p_1} \), \( \theta_2 = -\frac{\mu_\ast}{\gamma_\ast} \).

- If \( q > 2, p_1 \leq 2 \), then \( j_0 = 4 \), \( \theta_1 = s_\ast + \frac{1}{2} - \frac{1}{p_1} \), \( \theta_2 = \frac{\theta}{2} \left(s_\ast + \frac{1}{q} - \frac{1}{p_1}\right) \), \( \theta_3 = -\frac{\mu_\ast}{\gamma_\ast} + \frac{1}{2} - \frac{1}{q} \), \( \theta_4 = -\frac{\mu_\ast}{2\gamma_\ast} \), \( \theta_5 = \tilde{\nu} \).

- If \( q > 0, p_0 \geq 2 \), then \( j_0 = 5 \), \( \theta_1 = s_\ast, \theta_2 = \frac{\theta}{2} \left(s_\ast + \frac{1}{q} - \frac{1}{p_1}\right) \), \( \theta_3 = \tilde{\theta} \), \( \theta_4 = -\frac{\mu_\ast}{2\gamma_\ast} \), \( \theta_5 = \tilde{\nu} \), \( \theta_6 = \tilde{\nu} \).

2. Let \( p_0 < q < p_1 \).

- If \( q \leq 2 \), then \( j_0 = 3 \), \( \theta_1 = s_\ast, \theta_2 = \tilde{\theta}, \theta_3 = \frac{\alpha_\ast (1/q - 1/p_1) + \mu_\ast (1/q - 1/p_0)}{\gamma_\ast (1/p_0 - 1/p_1)} \).

- If \( q > 2, p_0 \geq 2 \), then \( j_0 = 3 \), \( \theta_1 = s_\ast, \theta_2 = \tilde{\theta}, \theta_3 = \tilde{\nu} \).

- If \( q > 2, p_0 < 2 \), then \( j_0 = 4 \), \( \theta_1 = s_\ast, \theta_2 = \tilde{\theta}, \theta_3 = \tilde{\nu}, \theta_4 = \tilde{\nu} \).
Notation 4. Let \( s_\ast + \frac{1}{\max(p_0, q)} - \frac{1}{p_1} > 0 \), \( \mu_\ast + \alpha_\ast > 0 \), \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 < 0 \).

1. Let \( p_1 < p_0 < q \).
   - If \( q \leq 2 \), then \( j_0 = 2 \), \( \theta_1 = s_\ast + \frac{1}{q} - \frac{1}{p_1} \), \( \theta_2 = \tilde{\theta} \).
   - If \( q > 2 \), \( p_0 \leq 2 \), then \( j_0 = 4 \), \( \theta_1 = s_\ast + \frac{1}{2} - \frac{1}{p_1} \), \( \theta_2 = \frac{2}{q} \left( s_\ast + \frac{1}{q} - \frac{1}{p_1} \right) \), \( \theta_3 = \tilde{\theta} + \frac{1}{2} - \frac{1}{q} \), \( \theta_4 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \).
   - If \( q > 2 \), \( p_1 \geq 2 \), then \( j_0 = 5 \), \( \theta_1 = s_\ast \), \( \theta_2 = \frac{2}{q} \left( s_\ast + \frac{1}{q} - \frac{1}{p_1} \right) \), \( \theta_3 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \), \( \theta_4 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \), \( \theta_5 = \tilde{\nu} \).
   - If \( q > 2 \), \( p_1 < 2 < p_0 \), then \( j_0 = 6 \), \( \theta_1 = s_\ast + \frac{1}{2} - \frac{1}{p_1} \), \( \theta_2 = \frac{2}{q} \left( s_\ast + \frac{1}{q} - \frac{1}{p_1} \right) \), \( \theta_3 = \tilde{\theta} + \frac{1}{2} - \frac{1}{q} \), \( \theta_4 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \), \( \theta_5 = \tilde{\nu} \), \( \theta_6 = \tilde{\nu} \).

2. Let \( p_1 < q < p_0 \).
   - If \( q \leq 2 \), then \( j_0 = 3 \), \( \theta_1 = s_\ast + \frac{1}{q} - \frac{1}{p_1} \), \( \theta_2 = \tilde{\theta} \), \( \theta_3 = \frac{\alpha_\ast(1/p_1-1/q)+\mu_\ast(1/p_0-1/q)}{\gamma_\ast(1/p_1-1/p_0)} \).
   - If \( q > 2 \), \( p_1 \geq 2 \), then \( j_0 = 5 \), \( \theta_1 = s_\ast \), \( \theta_2 = \frac{2}{q} \left( s_\ast + \frac{1}{q} - \frac{1}{p_1} \right) \), \( \theta_3 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \), \( \theta_4 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \), \( \theta_5 = \tilde{\nu} \).
   - If \( q > 2 \), \( p_1 < 2 \), then \( j_0 = 6 \), \( \theta_1 = s_\ast + \frac{1}{2} - \frac{1}{p_1} \), \( \theta_2 = \frac{2}{q} \left( s_\ast + \frac{1}{q} - \frac{1}{p_1} \right) \), \( \theta_3 = \tilde{\theta} + \frac{1}{2} - \frac{1}{q} \), \( \theta_4 = \frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \), \( \theta_5 = \tilde{\nu} \), \( \theta_6 = \tilde{\nu} \).

Now we consider the case \( p_0 > q > p_1 \), \( s_\ast + \frac{1}{p_0} - \frac{1}{p_1} < 0 \). We set

\[
\hat{\sigma} = s_\ast \cdot \frac{1}{q} - \frac{1}{p_0} - \frac{1}{p_1} + \frac{2s_\ast}{q}.
\]  

Notation 5. Let \( p_0 > q > p_1 \), \( s_\ast + \frac{1}{p_0} - \frac{1}{p_1} < 0 \).

1. Let \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 \geq 0 \).
   - If \( q \leq 2 \), then \( j_0 = 2 \), \( \theta_1 = s_\ast \frac{1/q-1/p_0}{1/p_1-1/p_0} \), \( \theta_2 = \tilde{\theta} \).
   - If \( q > 2 \), \( p_1 \leq 2 \), then \( j_0 = 2 \), \( \theta_1 = \hat{\sigma} \), \( \theta_2 = \tilde{\theta} \).
   - If \( q > 2 \), \( p_1 > 2 \), then \( j_0 = 3 \), \( \theta_1 = s_\ast \), \( \theta_2 = \hat{\sigma} \), \( \theta_3 = \tilde{\theta} \).

2. Let \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 < 0 \), \( \mu_\ast + \alpha_\ast > 0 \).
   - If \( q \leq 2 \), then \( j_0 = 2 \), \( \theta_1 = s_\ast \frac{1/q-1/p_0}{1/p_1-1/p_0} \), \( \theta_2 = \frac{\alpha_\ast(1/p_2-1/q)+\mu_\ast(1/p_0-1/q)}{\gamma_\ast(1/p_1-1/p_0)} \).
   - If \( q > 2 \), \( p_1 \leq 2 \), then \( j_0 = 2 \), \( \theta_1 = \hat{\sigma} \), \( \theta_2 = \tilde{\nu} \).
   - If \( q > 2 \), \( p_1 > 2 \), then \( j_0 = 4 \), \( \theta_1 = s_\ast \), \( \theta_2 = \hat{\sigma} \), \( \theta_3 = \tilde{\nu} \), \( \theta_4 = -\frac{\mu_\ast}{\gamma_\ast} - \frac{1}{q} + \frac{1}{p_1} \).
3. Let \( \mu_* + \alpha_* < 0, s_* + 1/q - 1/p_1 < 0 \).
   - If \( q \leq 2 \), then \( j_0 = 2, \theta_1 = s_* \frac{1/q - 1/p_0}{1/p_1 - 1/p_0}, \theta_2 = \hat{\theta} \).
   - If \( q > 2 \), then \( j_0 = 2, \theta_1 = \hat{\sigma}, \theta_2 = \frac{\phi}{2} \).

4. Let \( \mu_* + \alpha_* < 0, s_* + 1/q - 1/p_1 > 0 \); if \( \gamma_* = 0 \), we suppose that \( \mu_* < 0 \) and set \( -\mu_*/\gamma_* = +\infty \).
   - If \( q \leq 2 \), then \( j_0 = 3, \theta_1 = s_* \frac{1/q - 1/p_0}{1/p_1 - 1/p_0}, \theta_2 = \hat{\theta}, \theta_3 = -\frac{\mu_*}{\gamma_*} \).
   - If \( q > 2 \), \( p_1 \leq 2 \), then \( j_0 = 3, \theta_1 = \hat{\sigma}, \theta_2 = -\frac{\mu_*}{2\gamma_*}, \theta_3 = \frac{\phi}{2} \).
   - If \( q > 2 \), \( p_1 > 2 \), then \( j_0 = 5, \theta_1 = s_* \), \( \theta_2 = \hat{\sigma}, \theta_3 = -\frac{\mu_*}{2\gamma_*}, \theta_4 = \frac{\phi}{2} \),
     \( \theta_5 = -\frac{\mu_*}{\gamma_*} - \frac{1}{q} + \frac{1}{p_1} \).

Theorem 1. Let \( j_0, \theta_j (1 \leq j \leq j_0) \) be defined by Notations\[\text{[2]} \text{[3]}. \] Suppose that there exists \( j_* \in \{1, \ldots, j_0\} \) such that \( \theta_{j_*} < \min_{j \neq j_*} \theta_j \), and \( \theta_{j_*} > 0 \). Then
\[
d_n(M, Y_q(\Omega)) \asymp n^{-\theta_{j_*}}.
\]

Proof. We define the numbers \( \hat{m}_t, \overline{m}_t, \hat{m}_t, m_t, m'_t \in \mathbb{R} \) by
\[
2^{\gamma_* k_t} \cdot \hat{m}_t = n, \quad \overline{m}_t = n^{q/2}, \quad \hat{m}_t s_* = 2^{(\mu_* + \alpha_* + \gamma_*/p_0 - \gamma_*/p_1)k_t}, \quad m_t (s_* + 1/p_0 - 1/p_1) = 2^{(\mu_* + \alpha_*)k_t},
\]
\[
2^{(\mu_* + \alpha_*)k_t} \cdot 2^{-s_* (1/p_0 - 1/p_1)} m'_t \left( n^{-1/2} \cdot 2^{\gamma_* k_t/q} \cdot 2^{m'_t/q} \right)^{1/p_0 - 1/p_1} = 1
\]
(the numbers \( \overline{m}_t \) and \( m'_t \) will be used only for \( q > 2 \)).

The numbers \( \lambda \) and \( \tilde{\lambda} \) are given by \([30]\).

The following equations hold:
\[
2^{\mu_* k_t} \cdot 2^{-m'_t (s_* + 1/q - 1/p_1)} \left( n^{-1/2} 2^{\gamma_* k_t/q} \cdot 2^{m'_t/q} \right)^{1/p_0 - 1/q} = 2^{-\alpha_* k_t} \cdot 2^{-m'_t (1/q - 1/p_0)} \left( n^{-1/2} 2^{\gamma_* k_t/q} \cdot 2^{m'_t/q} \right)^{1/p_0 - 1/q} = 2^{(\mu_* (1 - \tilde{\lambda}) - \alpha_* \lambda) k_t} \cdot 2^{-m'_t ((s_* + 1/q - 1/p_1) (1 - \tilde{\lambda}) + (1/q - 1/p_0) \lambda)} n^{-1/2} 2^{\gamma_* k_t/q} \cdot 2^{m'_t/q};
\]
if \( m'_t = \overline{m}_t \), then \( m'_t = m_t \); if \( m'_t = \hat{m}_t \), then \( m'_t = \hat{m}_t \); (51)
if \( m_t = \hat{m}_t \), then
\[
2^{\mu_* k_t} \cdot 2^{-m_t (s_* + 1/q - 1/p_1)} = 2^{-\alpha_* k_t} \cdot 2^{-m_t (1/q - 1/p_0)} \asymp n^{-\theta};
\]
if \( m_t = \overline{m}_t \), then
\[
2^{\mu_+ k_+ t} . 2^{-m_t(s_*+1/q-1/p_1)} = 2^{-\alpha_+ k_+ t} . 2^{-m_t(1/q-1/p_0)} \text{ if } m'_t = 0, \text{ then}
\]
\[
2^{\mu_+ k_+ t} . 2^{-m_t(s_*+1/q-1/p_1) n^{1/q-1/p_1}} = 2^{-\alpha_+ k_+ t} . 2^{-m_t(1/q-1/p_0) n^{1/q-1/p_0}} \text{ if } \hat{m}_t = \hat{m}_t, \text{ then}
\]

if \( m'_t = 0 \), then
\[
2(\mu_+ (1-\lambda) - \alpha_+) k_+ t \equiv n^{-\beta};
\]
if \( \hat{m}_t = 0 \), then
\[
2(\mu_+ (1-\lambda) - \alpha_+) k_+ t . n^{-1/2} \equiv n^{-\beta}.
\]

From (31) it follows that for \( p_0 > q, s_* + \frac{1}{p_0} - \frac{1}{p_1} < 0 \) we have
\[
2^{-m'_t((1-\lambda)(s_*+1/q-1/p_1)+\lambda(1/q-1/p_0))} = 2^{-m'_t(1-\lambda)s_*} \equiv n^{-\delta}.
\]

**Proposition 3.** Let \( q > 2, \hat{m}_{t_1} = 0, m'_{t_2} = 0, \overline{m}_{t_3} = 0 \). Suppose that one of the following conditions hold:

1. \( p_0 < p_1, \mu_+ + \alpha_+ < 0 < \mu_+ + \alpha_+ + \frac{2_0}{p_0} - \frac{2_1}{p_1} \);

2. \( p_0 > p_1, \mu_+ + \alpha_+ + \frac{2_0}{p_0} - \frac{2_1}{p_1} < 0 < \mu_+ + \alpha_+ \).

Then \( t_1 < t_2 < t_3 \).

**Proof.** We have
\[
2^{\gamma_+ k_+ t_1} = n, \quad 2^{(2(\mu_+ + \alpha_+) \frac{1/2-1/q}{1/p_1-1/p_0} + \frac{2_0}{q}) k_+ t_2} = n, \quad 2^{\frac{2_0 k_+ t_3}{q}} = n.
\]

In both cases \( \frac{\mu_+ + \alpha_+}{1/p_1-1/p_0} > 0 \); therefore, \( (\mu_+ + \alpha_+) \frac{1/2-1/q}{1/p_1-1/p_0} + \frac{2_0}{q} > \frac{2_0}{q} \) and \( t_2 < t_3 \).

The inequality \( t_1 < t_2 \) is equivalent to \( (\mu_+ + \alpha_+) \frac{1/2-1/q}{1/p_1-1/p_0} + \frac{2_0}{q} < \frac{2_0}{q} \), or
\[
\frac{\mu_+ + \alpha_+}{1/p_1-1/p_0} < \gamma_+.
\]

If \( p_0 < p_1 \), it is equivalent to \( \mu_+ + \alpha_+ + \frac{2_0}{p_0} - \frac{2_1}{p_1} > 0 \); if \( p_0 > p_1 \), it is equivalent to \( \mu_+ + \alpha_+ + \frac{2_0}{p_0} - \frac{2_1}{p_1} < 0 \).

The upper estimates for the widths are obtained as in [30]. Here we write the sketch of the proof.
According to \(3_{0}\), we choose the numbers \(\hat{t}(n)\) and \(t_{*}(n) \in [0, \hat{t}(n)]\) (here \(2^{k_{t} \cdot \hat{t}(n)}\) is a positive degree of \(n\), \(2^{\gamma_{k} \cdot \hat{t}(n)} \leq n^{\max\{1, q/2\}}\); also we choose a sufficiently small \(\varepsilon > 0\) (we will write later how to choose these numbers). We set

\[
m_{t}^{*} = \max\{\hat{m}_{t} - \varepsilon |t - t_{*}(n)|, 0\}.
\]  

(58)

If \(q \leq 2\) or \(\min\{p_{0}, p_{1}\} \geq q\), we set \(k_{t,m} = \nu_{t,m}\) for \(0 \leq m < m_{t}^{*}\), \(t \leq \hat{t}(n)\), and \(k_{t,m} = 0\) for \(m \geq m_{t}^{*}\), \(t \leq \hat{t}(n)\). Then

\[
\sum_{0 \leq t \leq \hat{t}(n)} \sum_{m \geq 0} k_{t,m} = \sum_{0 \leq t \leq \hat{t}(n)} \sum_{0 \leq m < m_{t}^{*}} \nu_{t,m} \leq 3_{0} \sum_{0 \leq t \leq \hat{t}(n)} \sum_{0 \leq m < m_{t}^{*}} 2^{\gamma_{k} \cdot k_{t} \cdot 2^{m_{t}^{*}} \cdot 2^{\varepsilon |t - t_{*}(n)|}} \leq n.
\]  

(56)

(59)

Let \(q > 2\) and \(\min\{p_{0}, p_{1}\} < q\). We choose \(m_{*}(n) \in [(\hat{m}_{t_{*}(n)})_{+}, \bar{m}_{t_{*}(n)}]\) according to \(3_{0}\) and set \(k_{t,m} = \nu_{t,m}\) for \(0 \leq m < m_{t}^{*}\), \(k_{t,m} = [\nu_{t,m} \cdot 2^{\varepsilon |t - t_{*}(n)|} + |m_{t}^{*} - m_{*}(n)|]\) for \(0 \leq t \leq \hat{t}(n), m_{t}^{*} \leq m \leq \bar{m}_{t}, k_{t,m} = 0\) for \(m > \bar{m}_{t}.\) As above, we get

\[
\sum_{0 \leq t \leq \hat{t}(n)} \sum_{m \geq 0} k_{t,m} \leq n.
\]  

(60)

Then we apply the following method. Consider the domain

\[
A = A(n) = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq (\hat{m}_{t})_{+}\}.
\]

According to Theorems \(A_{1}\), \(A_{2}\) we write the order estimates of \(d_{n}(W_{t,m}, l_{q}^{\nu_{m}})\) for \((t, m) \in A \cap \mathbb{Z}_{+}^{2}\) (from \(1_{7}\) and \(1_{6}\) it follows that \(\nu_{t,m} \geq 2n\)). We obtain the partition of \(A\) into polygonal subdomains \(A_{i} = A_{i}(n), 1 \leq i \leq i_{0}\); for \((t, m) \in A_{i}\), one of the following estimates holds:

\[
d_{n}(W_{t,m}, l_{q}^{\nu_{m}}) \asymp 2^{\alpha_{*} k_{t} \cdot 2^{m(1/q - 1/p_{0})}} d_{n}(B_{p_{0}}^{\nu_{m}}, l_{q}^{\nu_{m}}),
\]

(61)

\[
d_{n}(W_{t,m}, l_{q}^{\nu_{m}}) \asymp 2^{\alpha_{*} k_{t} \cdot 2^{m(1/q - 1/p_{1})}} d_{n}(B_{p_{1}}^{\nu_{m}}, l_{q}^{\nu_{m}}),
\]

(62)

\[
d_{n}(W_{t,m}, l_{q}^{\nu_{m}}) \asymp 2^{((1 - \lambda) \mu_{*} - \lambda \alpha_{*}) k_{t} \cdot 2^{m((1 - \lambda)(s_{*} + 1/q - 1/p_{1}) + \lambda(1/q - 1/p_{0}))}} d_{n}(B_{2}^{\nu_{m}}, l_{q}^{\nu_{m}}),
\]

(63)

\[
d_{n}(W_{t,m}, l_{q}^{\nu_{m}}) \asymp 2^{(1 - \lambda) \mu_{*} - \lambda \alpha_{*}) k_{t} \cdot 2^{m(1 - \lambda)s_{*}},
\]

(64)
where \( \lambda, \tilde{\lambda} \) are defined by (30). Taking into account the estimates of \( d_n(B_{t,m}^{\nu,m}, l_q^{\nu,m}) \), we get that for \((t, m) \in A_i\) the following estimate holds:

\[
d_n(W_{t,m}, p_q^{\nu,m}) \leq \varphi_i(t, m, n) := 2^{\kappa_1,t + \kappa_2,m} n^{\sigma_i},
\]

(65)

(here \( \kappa_{1,i}, \kappa_{2,i}, \sigma_i \) are real numbers); in addition, if \((t, m) \in A_i \cap A_j\), we have \( \varphi_i(t, m, n) = \varphi_j(t, m, n) \). Let

\[
\varphi(t, m, n) = \varphi_i(t, m, n) \quad \text{for} \quad (t, m) \in A_i.
\]

(66)

We estimate from above the sum

\[
S := \sum_{(t, m) \in A_i \cap \mathbb{Z}_+^2} \varphi(t, m, n) = \sum_{i=1}^{i_0} \sum_{(t, m) \in A_i \cap \mathbb{Z}_+^2} \varphi_i(t, m, n).
\]

Suppose that on each unbounded \( A_i \) we have the progression which strictly decreases with \( m \). We calculate \( \varphi_i(t, m, n) \) in the vertices of \( A_i \), \( 1 \leq i \leq i_0 \). Suppose that they have the form \( n^{-\beta_j}, 1 \leq j \leq k \), where \( \beta_j = \beta_j(30) \). If there exists \( j_* \in \{1, \ldots, k\} \) such that \( \beta_{j_*} < \min_{j \neq j_*} \beta_j \), then

\[
S \approx n^{-\beta_{j_*}} = \varphi(t_n, m_n, n);
\]

(67)

here \((t_n, m_n)\) is a vertex of \( A_i = A_i(n) \) for some \( i \).

Now we set \( t_*(n) = t_n, m_*(n) = m_n \). Consider the domain

\[
A^e_i = A^e_i(n) = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m_*(n)\}
\]

(68)

and its partition into subdomains \( A^e_i = A^e_i(n) \supset A_i, 1 \leq i \leq i_0 \): if

\[
(\partial A_i) \cap \{(t, (\hat{m}_t)_+) : t \geq 0\} = \{(t, (\hat{m}_t)_+) : t_{i,1} \leq t \leq t_{i,2}\} \neq \emptyset,
\]

then

\[
A^e_i = A_i \cup \{(t, m) : t_{i,1} \leq t \leq t_{i,2}, m_{i}^* \leq m \leq (\hat{m}_t)_+\};
\]

otherwise, \( A^e_i = A_i \).

For \((t, m) \in A^e_i\), we estimate \( d_{kt,m}(W_{t,m}, l_q^{\nu,m}) \) from above as follows: if for \((t, m) \in A_i\) we get (61), (62), (63) or (64), then we write, respectively,

\[
\begin{align*}
\frac{d_{kt,m}(W_{t,m}, l_q^{\nu,m})}{30} & \leq 2^{\mu_* k, t} : 2^{-m(s_*+1/q-1/p_1)} d_{kt,m}(B_{p_1}^{\nu,m}, l_q^{\nu,m}), \\
\frac{d_{kt,m}(W_{t,m}, l_q^{\nu,m})}{30} & \leq 2^{\alpha_* k, t} : 2^{-m(1/q-1/p_0)} d_{kt,m}(B_{p_0}^{\nu,m}, l_q^{\nu,m}), \\
\frac{d_{kt,m}(W_{t,m}, l_q^{\nu,m})}{30} & \leq 2^{2((1-\lambda_0)\mu_* - \tilde{\lambda}_0) k, t} : 2^{-m(1-\lambda_0)(s_*+1/q-1/p_1)+\tilde{\lambda}_0(1/q-1/p_0)} d_{kt,m}(B_{p_1}^{\nu,m}, l_q^{\nu,m}), \\
\frac{d_{kt,m}(W_{t,m}, l_q^{\nu,m})}{30} & \leq 2^{2(1-\tilde{\lambda}_0)\mu_* \tilde{\lambda}_0 k, t} : 2^{-m(1-\lambda_0)(s_*+1/q-1/p_1)+\tilde{\lambda}_0(1/q-1/p_0)} d_{kt,m}(B_{p_1}^{\nu,m}, l_q^{\nu,m}),
\end{align*}
\]

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$$d_{k_{t,m}}(W_{t,m}, l_q^{n,m}) \lesssim 2^{((1-\lambda)\mu_\ast - \lambda \alpha_\ast)k_{t,t} \cdot 2^{-m(1-\lambda)s_\ast}}.$$  

Notice that $2k_{t,m} \leq \nu_{t,m}$ for $(t, m) \in A^c$; indeed, we have $k_{t,m} = 0$ or, by (17), (16), (58), (68), we have

$$2n \cdot 2^{-\varepsilon(|t-t_\ast(n)|+|m-m_\ast(n)|)} \leq 2 \cdot 2^{\gamma_\ast k_{t,t} \cdot 2^{-m_\ast \varepsilon|t-t_\ast(n)|}}.$$  

By the definition of $k_{t,m}$, the inequality $2k_{t,m} \leq \nu_{t,m}$ and Theorems B, C, we get that for $(t, m) \in A_{i_0}^c$

$$d_{k_{t,m}}(W_{t,m}, l_q^{n,m}) \lesssim \varphi_i(t, m, n) \cdot 2^{c_1 \varepsilon |t-t_\ast(n)|+|m-m_\ast(n)|},$$  

where $c_1 = c_1(3_0)$. If $\varepsilon > 0$ is sufficiently small (we choose it according to $3_0$), then

$$\sum_{(t, m) \in A^c \cap \mathbb{Z}_2^2} d_{k_{t,m}}(W_{t,m}, l_q^{n,m}) \lesssim n^{-\beta_\ast}.$$  

Now we estimate $\sup_{f \in M} \|f\|_{Y_q(\hat{t}(n))}$ according to Proposition 2. Let

$$\sup_{f \in M} \|f\|_{Y_q(\hat{t}(n))} \lesssim n^{-\beta_\ast}$$  

for some $j \in \{1, \ldots, k\}$. We apply (27), (59), (60), (69), take into account that $d_{k_{t,m}}(W_{t,m}, l_q^{n,m}) = 0$ for $m < m_\ast^i$ and get: there is $C = C(3_0) \in \mathbb{N}$ such that

$$d_{Cn}(M, Y_q(\Omega)) \lesssim n^{-\beta_\ast} + n^{-\beta_\ast} \leq n^{-\beta_\ast}.$$  

It implies that

$$d_n(M, Y_q(\Omega)) \lesssim n^{-\beta_\ast}.$$  

In order to get the lower estimate, we apply (19). Taking into account that $\nu_{t,m} \geq 2n$ for $(t, m) \in A$, we have

$$d_n(M, Y_q(\Omega)) \gtrsim d_n(W_{[t_n],[m_n]}, l_q^{n,m}, n) \gtrsim \varphi(t_n, m_n, n) \gtrsim n^{-\beta_\ast}.$$  

Now we consider the cases. We write:

1. $\hat{t}(n)$ and the estimate for $\sup_{f \in M} \|f\|_{Y_q(\hat{t}(n))}$;
2. the polygonal subdomains $A_i$, $1 \leq i \leq i_0$;
3. the estimates (11)–(14) for each $A_i$;
4. notice if the progression $\varphi_i(t, m, n)$ strictly increases or decreases with $m$ or $t$ (if we can see it immediately);
5. the vertices of $A_i$ in which it is sufficient to calculate $\varphi(t, m, n)$;
6. the values of $\varphi(t, m, n)$ in these vertices.

First we suppose that $s_* + \frac{1}{\max(p_0, p)} - \frac{1}{p_1} > 0$ or $\min\{p_0, p_1\} \geq q$. Then from (33), (34), (48), (49) it follows that for $m \geq \max\{m_t, \tilde{m}_t\}$ we have (61), and for $m \leq \min\{m_t, \tilde{m}_t\}$ we have (62).

**Case 1.** Let $\mu_* + \alpha_* \leq 0$, $\mu_* + \alpha_* + \frac{m}{p_0} - \frac{m}{p_1} \leq 0$. By (48), (49), we have $m_t \leq 0$, $\tilde{m}_t \leq 0$ for all $t \geq 0$. Therefore, for all $m \in \mathbb{Z_+}$ we get (61).

First we consider $\gamma_* > 0$.

If $p_1 > q$ or $q = 2$, then we define $\hat{t}(n)$ by $\hat{m}_{i(n)} = 0$. Then

$$\sup_{f \in M} \|f\|_{Y_q(\tilde{\Omega}_{i(n)\hat{t}})} \lesssim 36, 39, 46 n^{\mu_* + (1/q - 1/p_1)_+}. \quad (70)$$

For all $(t, m) \in A$ we have (61):

$$d_n(W_{t, m}, p_{q, m}) \lesssim \frac{1}{36} 2^{\mu_* k_* t} \cdot 2^{-m(s_* + 1/q - 1/p_1)} \cdot 2^{\gamma_* k_* (1/q - 1/p_1)_+} \cdot 2^{m(1/q - 1/p_1)_+}; \quad (71)$$

the right-hand side strictly decreases with $m$. The polygonal domain $A$ has two vertices: $(0, \tilde{m}_0)$ and $(\hat{t}(n), 0)$. We substitute these points into (71) and get $n^{-s_* + (1/p_1 - 1/q)_+}$, $n^{\mu_* + (1/q - 1/p_1)_+}$. This together with (70) yields the order estimates for the widths (see Notation [1]).

Let $p_1 < q$, $q > 2$. We define the number $\hat{t}(n)$ by $\overline{m}_{i(n)} = 0$. Then

$$\sup_{f \in M} \|f\|_{Y_q(\tilde{\Omega}_{\hat{t}(n)\hat{t}})} \lesssim 35, 39, 47 n^{\mu_* / 2\gamma_*}. \quad (72)$$

The domain $A$ is divided into two subdomains:

$$A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \overline{m}_t\},$$

$$A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq \overline{m}_t\}$$

(see Fig. 1).

In both subdomains (61) holds; in $A_2$ the progression strictly decreases with $m$. The vertices of $A_i$ are as follows: $(0, \tilde{m}_0)$, $(0, \overline{m}_0)$, $(t_1(n), 0)$, $(\hat{t}(n), 0)$, where $t_1(n)$ is defined by $\hat{m}_{t_1(n)} = 0$. We substitute these points into

$$2^{\mu_* k_* t} \cdot 2^{-m(s_* + 1/q - 1/p_1)} \cdot \left(n^{-\frac{t}{2} \frac{\gamma_*}{2 - 1/q}} \cdot 2^{\frac{\gamma_*}{2 - 1/q}}\right)^{\lambda_{p_1 q}} \quad (73)$$

(see Theorem [3]) and get

$$n^{-s_* - \min(0, 1/2 - 1/p_1)}, \ n^{-\frac{t}{2} (s_* + 1/q - 1/p_1)}, \ n^{\mu_* / 2 \gamma_* - \min(1/p_1 - 1/q, 1/2 - 1/q)}, \ n^{\mu_* / 2 \gamma_*}.$$
By (72), we get the desired estimate for the widths (see Notation 1).

If \( \gamma_\ast = 0 \), we choose the number \( \hat{t}(n) \) such that \( 2^{k_\ast \hat{t}(n)} \) is a sufficiently large degree of \( n \), and argue as above. Notice that by \( \mu_\ast < 0 \) the right-hand side of (71) and (73) strictly decreases with \( t \); hence it is sufficient to substitute \((0, \hat{m}_0)\) into (71) for \( q \leq 2 \), and \((0, \hat{m}_0), (0, \overline{m}_0)\) into (73) for \( q > 2 \).

**Case 2.** Let \( p_0 \geq q, p_1 \geq q \). Then for \( m \geq \hat{m}_t \) we have (61), and for \( m \leq \hat{m}_t \) we have (62). Indeed, let \( p_1 \leq p_0 \). Then \( m_\ast \geq \hat{m}_t \). If \( m \geq m_\ast \), then (61) follows from (33); if \( m \leq m_\ast \), then (62) follows from (34). If \( \hat{m}_t \leq m \leq m_\ast \), by assertion 7 of Theorem A we get (61). Similarly we can consider the case \( p_1 \geq p_0 \).

If \( \mu_\ast + \alpha_\ast + \frac{s_\ast}{p_0} - \frac{s_\ast}{p_1} \leq 0 \), then \( \hat{m}_t \leq 0 \). Therefore, (61) holds for all \( m \in \mathbb{Z}_+ \); in addition, we have (39), where \( \nu_\ast \) is defined by (35). Hence the widths can be estimated as in Case 1.

Let \( \mu_\ast + \alpha_\ast + \frac{s_\ast}{p_0} - \frac{s_\ast}{p_1} > 0 \). Then \( \hat{m}_t \geq 0 \) for \( t \geq 0 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = \hat{m}_{t(n)} \). Then

\[
A = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq \hat{m}_t\}.
\]

By (37), (39), (41), (46), (48), we get

\[
\sup_{f \in M} \|f\|_{Y_q^{\hat{m}_{\hat{t}(n)}}} \lesssim n^{-\hat{\beta}}.
\]

For all \((t, m) \in A\) we have (61); the right-hand side is as follows:

\[
2\mu_\ast k_\ast t_2^{-m(s_\ast+1/q-1/p_1)}2^{(\gamma_\ast k_\ast t+m)/(1/q-1/p_1)};
\]

it strictly decreases with \( m \). Hence it is sufficient to calculate (75) in \((0, \hat{m}_0)\) and \((\hat{t}(n), \hat{m}_{\hat{t}(n)})\). Taking into account (46) and (54), we get \( n^{-s_\ast} \) and \( n^{-\hat{\beta}} \). This together with (74) yields the desired estimate for \( d_n(M, Y_q(\hat{\Omega})) \) (see Notation 2).

**Case 3.** Let \( \mu_\ast + \alpha_\ast < 0, \mu_\ast + \alpha_\ast + \frac{s_\ast}{p_0} - \frac{s_\ast}{p_1} > 0, p_0 < p_1 < q \). Then \( \gamma_\ast > 0, m_\ast \leq 0, \overline{m}_t > 0 \) for \( t \geq 0 \). In addition, (39) holds with \( \nu_\ast \) defined by (35).
We apply Theorem A (since $p_0 < p_1$, we rearrange 0 and 1).

If $q \leq 2$ or $p_1 \leq 2$, we get (61) for $m \geq m_t$ (see assertion 1 of Theorem A). Hence this estimate holds for all $m \geq 0$. Further we argue as in Case 1.

Let $p_1 > 2, q > 2$. The number $\hat{t}(n)$ is defined by $\hat{m}_{\hat{t}(n)} = 0$. Then

$$\sup_{f \in M} \|f\|_{Y_q(\hat{m}_{\hat{t}(n)})} \lesssim n^{\mu_*/2\gamma_*}. \quad (76)$$

We define the points $t_1(n), t_2(n), t_3(n)$ by $\hat{m}_{t_1(n)} = m_{t_2(n)} = \hat{m}_{t_3(n)} = 0$. By Proposition 3, we have $t_3(n) < t_2(n) < \hat{t}(n)$.

Taking into account (51), we get that $A$ is divided into

$$A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \hat{m}_t, m \geq m'_t\},$$

$$A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq \hat{m}_t\},$$

$$A_3 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m'_t\}$$

(see Fig. 2).

![Figure 2: The partition of A.](image)

For $(t, m) \in A_1 \cup A_2$ we get (61) by (34) and assertions 2, 3 of Theorem A (recall that we rearrange 0 and 1); for $(t, m) \in A_3$ we have (62) if $p_0 \geq 2$, and (63) if $p_0 < 2$. In $A_2$ the progression strictly decreases with $m$. If $p_0 \geq 2$, then in $A_3$ the progression strictly increases with $m$.

Hence, for $p_0 \geq 2$ it is sufficient to calculate the right-hand side of (61) in $(0, \hat{m}_0), (0, \hat{m}_0), (\hat{t}(n), 0), (t_1(n), \hat{m}_{t_1}(n)), (t_2(n), 0)$, and for $p_0 < 2$, the same values and the right-hand side of (63) in $(t_3(n), 0)$. Applying (46), (47), (50), (54), (55), (56), we get $n^{-\theta_j}, 1 \leq j \leq j_0$ (see Notation 3, case 1, subcase $q > 2, 2 < p_1 < q$). This together with (76) yields the desired estimate for the widths.

**Case 4.** Let $\mu_* + \alpha_* < 0, \mu_* + \alpha_* + \frac{2\gamma}{p_0} > \frac{2\gamma}{p_1}$, $p_0 < q < p_1$. Again we get $\gamma_* > 0, \hat{m}_t \geq 0, m_t \leq 0$ for $t \geq 0$.

Let $q \leq 2$. We define the number $\hat{t}(n)$ by $\hat{m}_{\hat{t}(n)} = 0$. Then

$$\sup_{f \in M} \|f\|_{Y_q(\hat{m}_{\hat{t}(n)})} \lesssim n^{\mu_*/2\gamma_*} \lesssim 2^{(1-\lambda)\mu_* - \lambda^*} k_{\hat{t}(n)} \quad (\text{assertion 2 of Theorem A})$$

$$n^\gamma \lesssim n^{-\theta_3}. \quad (77)$$
We define the number \( t_1(n) \) by \( \hat{m}_{t_1(n)} = \tilde{m}_{t_1(n)} \).

The domain \( A \) is divided into
\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq (\hat{m}_t)_+, m \geq \hat{m}_t\},
\]
\[
A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \hat{m}_t\}.
\]

By (34) and assertion 6 of Theorem A (with rearranged 0 and 1), we get (61) in \( A_1 \), and (64) in \( A_2 \). Everywhere the progression strictly decreases with \( m \). Hence it is sufficient to calculate
\[
2^{\gamma_1 k_s \ell(n)} \leq n^{\theta_j}, 1 \leq j \leq 3 \quad \text{(see Notation 3, case 2, subcase \( q \leq 2 \)).}
\]

This together with (77) yields the desired estimate for the widths.

Let \( q > 2 \). We define \( \hat{t}(n) \) by \( m'_{\hat{t}(n)} = 0 \). Then
\[
2^{\gamma_1 k_s \ell(n)} \leq n^{\theta_j/2} \quad \text{(see Proposition 3),}
\]
\[
\sup_{f \in M} \| f \|_{Y_q(\tilde{\Omega}_{\hat{t}(n)})} \leq n^{-\hat{\nu}}. \tag{78}
\]

We define the numbers \( t_1(n) \) and \( t_2(n) \) by \( \tilde{m}_{t_1(n)} = \tilde{m}_{t_1(n)} \) and \( \tilde{m}_{t_2(n)} = 0 \). By Proposition 3, \( t_2(n) < \hat{t}(n) \).

Taking into account (51), we get that \( A \) is divided into
\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq \tilde{m}_t, m \geq (\hat{m}_t)_+\},
\]
\[
A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m'_l \leq m \leq \hat{m}_t\},
\]
\[
A_3 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\tilde{m}_t)_+ \leq m \leq \tilde{m}_t, m \leq m'_t\}
\]
(see Fig. 4).

We apply Theorem A (assertions 4–5 and (34)); since \( p_0 < p_1 \), we rearrange 0 and 1. We get (61) in \( A_1 \) and (64) in \( A_2 \). The progressions strictly decrease with
m. If \( p_0 \geq 2 \), we get (62) in \( A_3 \); the progression strictly increases with \( m \). If \( p_0 < 2 \), we get (63) in \( A_3 \).

For \( p_0 \geq 2 \) it is sufficient to calculate

\[
2^{\mu_* k_* t} \cdot 2^{-m(s_* + 1/q - 1/p_1)} \cdot (2^{\gamma_* k_* t} \cdot 2^m)^{1/q - 1/p_1}
\]

in \((0, \hat{\mu}_0), (t_1(n), \hat{m}_{t_1(n)})\) and \(2^{((1 - \lambda)\mu_* - \lambda \alpha_\ast)k_* t} \cdot 2^{-m(1 - \lambda)s_*}\) in \((\hat{t}(n), 0)\). For \( p_0 < 2 \), in addition, we calculate the right-hand side of (63) in \((t_2(n), 0)\). Applying (46), (50), (54), (55), (56), we get

\[
\frac{1 - \lambda}{p_0} - \frac{2}{p_1} < 0, p_1 < p_0 < q.
\]

Then \( \gamma_* > 0 \), \( \hat{m}_t \leq 0 \), \( m_t \geq 0 \) for \( t \geq 0 \).

If \( q \leq 2 \), then we define the number \( \hat{t}(n) \) by \( \hat{m}_t(n) = m_t(n) \). From (38), (39), (42), (47), (49) we get

\[
\sup_{f \in \mathcal{M}} \| f \|_{X_q(\hat{t}(n))} \lesssim n^{-\theta/2}.
\]

(79)

From (33) it follows that (61) holds for all \((t, m) \in A\). The right-hand side is equal to \(2^{\mu_* k_* t} \cdot 2^{-m(s_* + 1/q - 1/p_1)}\) and strictly decreases with \( m \). Therefore, it is sufficient to calculate this value in \((0, \hat{\mu}_0)\) and \((\hat{t}(n), \hat{m}_{t_1(n)})\). Taking into account (16) and (52), we get \( n^{-s_* - 1/ q + 1/p_1} \) and \( n^{-\theta} \). This together with (79) yields the desired estimate for the widths (see Notation 4 case 1, subcase \( q \leq 2 \)).

Let \( q > 2 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_t(n) = m_t(n) \). From (38), (39), (42), (47), (49) we get

\[
\sup_{f \in \mathcal{M}} \| f \|_{Y_q(\hat{t}(n))} \lesssim n^{-\theta/2}.
\]

(80)

Let \( p_0 \leq 2 \). We define the number \( t_1(n) \) by \( \hat{m}_{t_1(n)} = m_{t_1(n)} \).

The domain \( A \) is divided into

\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \hat{m}_t, m \geq m_t\}.
\]
\[ A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m_t\}, \]
\[ A_3 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq (\hat{m}_t)_+, m \leq m_t\} \]

(see Fig. 5).

Figure 5: The partition of \( A \).

From (33) it follows that (61) holds for \((t, m) \in A_1 \cup A_2\); in \( A_2 \) the progression strictly decreases with \( m \); from assertion 1 of Theorem A it follows that (62) holds in \( A_3 \) (the progression strictly increases with \( m \)). Hence it is sufficient to calculate
\[
2^{n-k \cdot t} \cdot 2^{-m(s+1/q-1/p_1)} \cdot n^{-1/2} 2^{t (k+1)} 2^{m/q} \quad \text{in} \quad (0, \hat{m}_0), (0, \overline{m}_0), (t_1(n), \hat{m}_{t_1(n)}) \text{and} \quad (\hat{t}(n), \overline{m}_{\hat{t}(n)}). 
\]
Taking into account (46), (47), (52) and (53), we get \( n^{-\theta_j}, 1 \leq j \leq 4 \) (see Notation 4, case 1, subcase \( q > 2, p_0 \leq 2 \)).

Let \( q > 2, p_0 > 2 \).

If \( p_1 \geq 2 \), we define the numbers \( t_1(n) \) and \( t_2(n) \) by \( \hat{m}_{t_1(n)} = 0 \) and \( m'_{t_2(n)} = 0 \).

By Proposition 3, we have \( t_1(n) < t_2(n) \).

The domain \( A \) is divided into
\[ A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \overline{m}_t; m \geq m'_t \text{ for } t_2(n) < t \leq \hat{t}(n)\}, \]
\[ A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq \overline{m}_t\}, \]
\[ A_3 = \{\{(t, m) : 0 \leq t \leq \hat{t}(n), 0 \leq m \leq m'_t\} \quad \text{if} \quad m'_t \uparrow, \}
\[ \varnothing \quad \text{otherwise} \]

(see Fig. 6).

By Theorem A (see (33) and assertion 3), (61) holds in \( A_1 \cup A_2 \) (in \( A_2 \) the progression strictly decreases with \( m \)), (62) holds in \( A_3 \) (the progression strictly increases with \( m \)). Hence it is sufficient to calculate the right-hand side of (61) in \((0, \hat{m}_0), (0, \overline{m}_0), (t_1(n), 0), (t_2(n), 0)\) and \((\hat{t}(n), \overline{m}_{\hat{t}(n)})\). Taking into account (46), (47), (50), (53) and (55), we get \( n^{-\theta_j}, 1 \leq j \leq 5 \) (see Notation 4 case 1, subcase \( q > 2, p_1 \geq 2 \)).

Let \( p_1 < 2 \). We define the numbers \( t_1(n), t_2(n) \) and \( t_3(n) \) by \( m_{t_1(n)} = \hat{m}_{t_1(n)}, \hat{m}_{t_2(n)} = 0 \) and \( m'_{t_3(n)} = 0 \). By Proposition 3 we have \( t_2(n) < t_3(n) \).
The domain $A$ is divided into subsets

\[ A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \overline{m}_t, \ m \geq m_t \}, \]
\[ A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), \ m \geq m_t \}, \]
\[ A_3 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m_t, \ m \geq m_t' \quad \text{for} \quad t_2(n) < t \leq \hat{t}(n) \}, \]
\[ A_4 = \begin{cases} \{ (t, m) : 0 \leq t \leq \hat{t}(n), \ 0 \leq m \leq m'_t \} & \text{if} \ m'_t \uparrow \uparrow, \\ \emptyset & \text{otherwise} \end{cases} \]

(see Fig. 7).

From (33) and assertion 2 of Theorem A it follows that (61) holds in $A_1 \cup A_2$ (in $A_2$ the progression strictly decreases with $m$), (63) holds in $A_3$, (62) holds in $A_4$ (the progression strictly increases with $m$). Hence it is sufficient to calculate the right-hand side of (61) in $(0, \hat{m}_0)$, $(0, \overline{m}_0)$, $(t_1(n), \hat{m}_{t_1(n)})$ and $(\hat{t}(n), \overline{m}_{t(n)})$, and the right-hand side of (63) in $(t_2(n), 0)$ and $(t_3(n), 0)$. Taking into account (46), (47),
\[ \gamma (50), (53), (55), 56 \), we get \( n^{-\theta_j}, 1 \leq j \leq 6 \) (see Notation 4 case 1, subcase \( q > 2, p_1 < 2 < p_0 \)).

This together with (56) yields the desired estimate for the widths.

**Case 6.** Let \( \mu_\ast + \alpha_\ast > 0, \mu_\ast + \alpha_\ast + \frac{2}{p_0} - \frac{2}{p_1} < 0, p_1 < q < p_0 \). Then \( \gamma_\ast > 0 \), \( \hat{m}_t \leq 0, m_t \geq 0 \) for \( t \geq 0 \).

Let \( q < 2 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = 0 \). By (36) and (39), we have (77); the number \( \theta_3 \) is the same. In addition, we define \( t_1(n) \) by \( m_{t_1(n)} = \hat{m}_{t_1(n)} \).

The domain \( A \) is divided into

\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq (\hat{m}_t)_+, m \geq m_t\},
\]
\[
A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq (\hat{m}_t)_+, m \leq m_t\}.
\]

By (33) and assertion 6 of Theorem A (61) holds in \( A_1 \), (61) holds in \( A_2 \) (in both domains the progression strictly decreases with \( m \)). Therefore it is sufficient to calculate \( 2^{\mu_\ast k_t}, 2^{-n(\mu_\ast + 1/\theta - 1/p_1)} \) in \( (0, \hat{m}_0) \) and \( (t_1(n), \hat{m}_{t_1(n)}) \), and \( 2^{((1-\lambda)\mu_\ast - \lambda \alpha_\ast) k_t} \cdot 2^{-m(1-\lambda)\alpha_\ast} \) in \( (\hat{t}(n), 0) \). Taking into account (46) and (52), we get \( n^{-\theta_j}, 1 \leq j \leq 3 \) (see Notation 4 case 2, subcase \( q < 2 \)). This together with (77) yields the desired estimate for the widths.

Let \( q > 2 \). We define the number \( \hat{t}(n) \) by \( m'_{\hat{t}(n)} = 0 \). Then (78) holds. By Proposition 3, we have \( 2^{2\gamma_\ast k_t(\hat{t}(n))} \leq n^{q/2} \).

Let \( p_1 \geq 2 \).

We define the numbers \( t_1(n) \) and \( t_2(n) \) by \( m_{t_1(n)} = m_{t_2(n)} = \hat{m}_{t_1(n)} = 0 \).

By Proposition 3, we have \( t_2(n) < \hat{t}(n) \).

The domain \( A \) is divided into

\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m_t; m \leq m'_t \text{ for } t_1(n) < t \leq \hat{t}(n)\},
\]
\[
A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m_t, m \geq m'_t\},
\]
\[
A_3 = \begin{cases} 
\{(t, m) : 0 \leq t \leq \hat{t}(n), m'_t \leq m \leq m_t, \quad \text{if } m'_t \downarrow\downarrow, \\
\emptyset, \quad \text{otherwise}
\end{cases}
\]

(see Fig. 8).

By (33) and assertion 4 of Theorem A (61) holds in \( A_1 \cup A_2 \) (in \( A_2 \) the progression strictly decreases with \( m \)). Therefore it is sufficient to calculate the right-hand side of (61) in \( (0, \hat{m}_0), (0, m_0), (t_1(n), m_{t_1(n)}), (t_2(n), 0) \) and \( (\hat{t}(n), 0) \). Taking into account (46), (47), (50), (53), (55), we get \( n^{-\theta_j}, 1 \leq j \leq 5 \) (see Notation 4 case 2, subcase \( q > 2, p_1 \geq 2 \)). This together with (78) yields the desired estimate for the widths.

Let \( p_1 < 2 \). We define the numbers \( t_1(n), t_2(n) \) and \( t_3(n) \) by \( m_{t_1(n)} = \hat{m}_{t_1(n)}, m_{t_2(n)} = \hat{m}_{t_2(n)}, m_{t_3(n)} = 0 \).

By Proposition 3, we have \( t_3(n) < \hat{t}(n) \).

The domain \( A \) is divided into

\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m_t, m \geq m_t\},
\]
\[
A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m_t, m \geq m_t\},
\]

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Figure 8: The partition of $A$.

$$A_3 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq \langle \bar{m}_t \rangle_+, m \leq m'_t \text{ for } t_1(n) < t \leq \hat{t}(n)\},$$

$$A_4 = \begin{cases} \{(t, m) : 0 \leq t \leq \hat{t}(n), m'_t \leq m \leq m_t \} & \text{if } m'_t \downarrow \downarrow, \\ \emptyset & \text{otherwise} \end{cases}$$

(see Fig. 9).

Figure 9: The partition of $A$.

By (33) and assertion 5 of Theorem A (61) holds in $A_1 \cup A_2$ (in $A_2$ the progression strictly decreases with $m$), (63) holds in $A_3$, (64) holds in $A_4$ (the progression strictly decreases with $m$). Therefore, it is sufficient to calculate the right-hand side of (61) in $(0, \hat{m}_0)$, $(0, \bar{m}_0)$, $(t_1(n), \bar{m}_{t_1(n)})$, $(t_2(n), \hat{m}_{t_2(n)})$ and the right-hand side of (63) in $(t_3(n), 0)$ and $(\hat{t}(n), 0)$. Taking into account (46), (47), (50), (52), (53), (55), (56), we get $n^{-\theta_j}, 1 \leq j \leq 6$ (see Notation 4, case 2, subcase $q > 2, p_1 < 2$).

This together with (78) yields the desired estimate of the widths.

Now we consider the case $p_0 > q > p_1$, $s_+ + \frac{1}{p_0} - \frac{1}{p_1} < 0$. Notice that if $q > 2$, then by $s_+ > 0$ and $s_+ + \frac{1}{p_0} - \frac{1}{p_1} < 0$ we get

$$2^{\hat{m}_0} = n < 2m'_0 \leq \frac{n^{q/2}}{2} = 2\bar{m}_0. \quad (81)$$
Notice that

\[ 2^{(\mu + \alpha)t} \cdot 2^{-m(s_1 + 1/p_0 - 1/p_1)} \geq 1 \iff m \geq \hat{m}_t, \quad (82) \]

\[ 2^{(\mu + \alpha + \gamma/p_0 - \gamma/p_1)k_1t} \cdot 2^{-ms} \leq 1 \iff m \geq \tilde{m}_t, \quad (83) \]

\[ n^{1/2}(2^{\gamma k_1t} \cdot 2^m)^{-1/q} \geq (2^{\mu + \alpha + \gamma/p_0 - \gamma/p_1})k_1t \cdot 2^{-m(s_1 + 1/p_0 - 1/p_1)} (n^{-1/2})^{1/2-1/q} \iff m \leq m_t^\prime. \quad (84) \]

**Case 1.** Let \( \mu + \alpha + \gamma/p_0 - \gamma/p_1 \geq 0 \). Then \( \hat{m}_t \geq 0, m_t \leq 0 \) for \( t \geq 0 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = \tilde{m}_{\tilde{t}(n)} \). By \( (57), (39), (11), (46), (48) \), we get \( (74) \).

Let \( q \leq 2 \). If \( (t, m) \in A \), we have \( m \geq \hat{m}_t \). By \( (82), (83) \) and assertion 6 of Theorem \( A \) we have \( (64) \); the progression strictly decreases with \( m \). Hence it is sufficient to calculate \( 2^{((1-\lambda)\mu_1 - \lambda \alpha_1)k_1t} \cdot 2^{-m((1-\lambda)s_1)} \) in \( (0, \tilde{m}_0) \) and \( (\hat{t}(n), \tilde{m}_{\hat{t}(n)}) \).

Applying \( (54) \), we get \( n^{-s_1} \cdot \frac{1}{q} \cdot \frac{1}{p_0 - 1/p_0} \) and \( n^{-\delta} \).

Let \( q > 2 \). Taking into account \( (51) \) and \( (81) \), we get that \( A \) is divided into

\[ A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m_t^\prime \}, \]

\[ A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m_t^\prime \} \]

(see Fig. 10).

![Figure 10: The partition of A.](image)

We apply \( (82), (83), (84) \) and assertions 4, 5 of Theorem \( A \). We get that \( (64) \) holds in \( A_1 \) (the progression strictly decreases with \( m \)), and in \( A_2 \), \( (61) \) holds for \( p_1 \geq 2 \), \( (63) \) holds for \( p_1 < 2 \) (since \( s_1 + 1/p_0 - 1/p_1 \geq 0 \), the progression strictly increases with \( m \) for \( p_1 \leq 2 \)). For \( p_1 \leq 2 \), it is sufficient to calculate \( 2^{((1-\lambda)\mu_1 - \lambda \alpha_1)k_1t} \cdot 2^{-m((1-\lambda)s_1)} \) in \( (0, m_0^\prime) \) and \( (\hat{t}(n), \tilde{m}_{\hat{t}(n)}) \); for \( p_1 > 2 \), in addition, we calculate \( 2^{\mu_1 k_1t} \cdot 2^{-m(s_1 + 1/q - 1/p_1)} d_n(B_{\mu_1 t, m_{\hat{t}(n)}}, l_{\mu_1 t, m_{\hat{t}(n)}}) \) in \( (0, \tilde{m}_0) \). Taking into account \( (50), (52) \) and \( (57) \), we get \( n^{-\delta} \) and \( n^{-\varphi} \), and for, \( p_1 > 2 \), in addition, we get \( n^{-s_1} \).

This together with \( (74) \) yields the desired estimates for the widths (see Notation 5 case 1).

**Case 2.** Let \( \mu + \alpha + \gamma/p_0 - \gamma/p_1 < 0, \mu + \alpha > 0 \). Then \( \hat{m}_t \leq 0, m_t \leq 0 \) for \( t \geq 0 \).
Let \( q \leq 2 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = 0 \). Then

\[
\sup_{f \in M} \|f\|_{Y_p(\tilde{\Omega}[\hat{t}(n)])} \leq \frac{2^{((1-\lambda)\mu_\ast-\lambda\alpha_\ast)k,\hat{t}(n)}}{\gamma_\ast} n^{((1-\lambda)\mu_\ast-\lambda\alpha_\ast)/\gamma_\ast} = n^{-\theta_2} \tag{85}
\]

(see Notation 5 case 2, subcase \( q \leq 2 \)).

By (82), (83) and assertion 6 of Theorem A we get (64) for \((t, m) \in A\); the progression strictly decreases with \( m \). Hence it is sufficient to calculate \( 2^{((1-\lambda)\mu_\ast-\lambda\alpha_\ast)k,\hat{t}(n)} \cdot 2^{-m(1-\lambda)s_\ast} \) in \((0, \hat{m}_0)\) and \((\hat{t}(n), 0)\). We get \( n^{-\theta_1} \) and \( n^{-\theta_2} \) (see Notation 5 case 2, subcase \( q \leq 2 \)). This together with (85) yields the estimate for the widths.

Let \( q > 2 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = 0 \). From Proposition 3 it follows that \( t_1(n) \leq \hat{t}(n), \hat{m}_{t_1(n)} > 0 \).

Taking into account (81), we get that \( A \) is divided into

\[
A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m'_1 \},
\]

\[
A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), \hat{m}_{t_1} \leq m \leq m'_t \}
\]

(see Fig. 11).

![Figure 11: The partition of A.](image)

By (82)–(84) and assertions 4, 5 of Theorem A (64) holds in \( A_1 \) (the progression strictly decreases with \( m \)), and in \( A_2 \), (61) holds for \( p_1 \geq 2 \), (63) holds for \( p_1 < 2 \) (if \( p_1 \leq 2 \), the progression strictly increases with \( m \)). Hence, for \( p_1 \leq 2 \), it is sufficient to calculate \( 2^{((1-\lambda)\mu_\ast-\lambda\alpha_\ast)k,\hat{t}(n)} \cdot 2^{-m(1-\lambda)s_\ast} \) in \((0, m'_0)\) and \((\hat{t}(n), 0)\), and for \( p_1 > 2 \), in addition, \( 2^{2s_\ast} \cdot 2^{-m(s_\ast+1/q-1/p_1)}d_n(B_{p_1}^{m, k, t}, l_q^{m, m}) \) in \((0, \hat{m}_0)\) and \((t_1(n), 0)\). Taking into account (50), (55) and (57), we get \( n^{-\theta_j}, j = 1, \ldots, j_0 \) (see Notation 5 case 2, subcase \( q > 2 \)).

This together with (78) yields the desired estimate for the widths.

**Case 3.** Let \( \mu_\ast + \alpha_\ast < 0, s_\ast + \frac{1}{q} - \frac{1}{p_1} < 0 \). Then \( \mu_\ast + \alpha_\ast + \gamma_\ast/p_0 - \gamma_\ast/p_1 < 0 \), \( \hat{m}_t \leq 0 \), \( m_t \geq 0 \) for \( t \geq 0 \).
Let \( q \leq 2 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = m_{\hat{t}(n)} \). By (38), (39), (46), (49), we have (79).

From (82), (83) and assertion 6 of Theorem A it follows that (64) holds in \( A \); the progression strictly decreases with \( m \). Hence it is sufficient to calculate \( 2^{((1-\lambda)\mu_{\ast}-\lambda \alpha_{\ast})k_t} \cdot 2^{-m(1-\lambda)s_{\ast}} \) in \((0, \hat{m}_0)\) and \((\hat{t}(n), \hat{m}_{\hat{t}(n)})\). Taking into account (49) and (52), we get \( n^{-\theta_{\ast}} \) and \( n^{-\theta_{\ast}} \) (see Notation 5, case 3, subcase \( q \leq 2 \)). This together with (79) yields the estimate for the widths.

Let \( q > 2 \). We define the number \( \hat{t}(n) \) by \( \hat{m}_{\hat{t}(n)} = m_{\hat{t}(n)} \). By (38), (39), (47), (49), we get (80).

Notice that \( m'_{\hat{t}} > \hat{m}_t \) for \( 0 \leq t \leq \hat{t}(n) \). Indeed, otherwise there exists \( t^* \in [0, \hat{t}(n)] \) such that \( \hat{m}_{t^*} = m'_{t^*} \) (it follows from (81)). By (51), we have \( m'_{\hat{t}(n)} = m_{\hat{t}(n)} > 0 \) (this together with (81) implies that \( m'_{t^*} > 0 \), \( m'_{t^*} = \tilde{m}_{t^*} \leq 0 \). We arrive to a contradiction.

Taking into account (51), we get that \( A \) is divided into

\begin{align*}
A_1 &= \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m'_{t}\}, \\
A_2 &= \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m < m'_{t}, m \geq m_t\}, \\
A_3 &= \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m_t\}
\end{align*}

(see Fig. 12).

![Figure 12: The partition of \( A \).](image)

We apply (82)–(84), (34) and assertions 4–5 of Theorem A. In \( A_1 \), (63) holds (the progression strictly decreases with \( m \)), in \( A_2 \), we have (51) for \( p_1 \geq 2 \) and (63) for \( p_1 < 2 \) (since \( s_{\ast} + \frac{1}{q} - \frac{1}{p_1} < 0 \), in both cases, in the right-hand sides of (61) and (63) the progression strictly increases with \( m \)), and in \( A_3 \), we have (61) (the progression strictly increases with \( m \)). Hence it is sufficient to calculate \( 2^{((1-\lambda)\mu_{\ast}-\lambda \alpha_{\ast})k_t} \cdot 2^{-m(1-\lambda)s_{\ast}} \) in \((0, m'_0)\) and \((\hat{t}(n), \hat{m}_{\hat{t}(n)})\). Taking into account (50), (53) and (57), we get \( n^{-\theta_{\ast}}, j = 1, 2 \) (see Notation 5 case 3, subcase \( q > 2 \)). This together with (80) yields the desired estimates for the widths.

**Case 4.** Let \( \mu_{\ast} + \alpha_{\ast} < 0, s_{\ast} + \frac{1}{q} - \frac{1}{p_1} > 0 \). Then \( \mu_{\ast} + \alpha_{\ast} + \gamma_{\ast}/p_0 - \gamma_{\ast}/p_1 < 0 \), \( \hat{m}_t \leq 0, m_t \geq 0 \) for \( t \geq 0 \).

First we consider the case \( \gamma_{\ast} > 0 \).
Let $q \leq 2$. We define the number $\hat{t}(n)$ by $\hat{m}_{\hat{t}(n)} = 0$. By (35), (39), we have (70).

The domain $A$ is divided into

$$A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq (\hat{m}_t)_+, m \geq m_t\},$$

$$A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m_t\}$$

(see Fig. 13).

![Figure 13: The partition of $A$.](image)

By (82)–(84), (33) and assertions 4, 5 of Theorem A holds in $A_1$, (61) holds in $A_2$; the progressions strictly decrease with $m$. We define the number $t_1(n)$ by $\hat{m}_{t_1(n)} = m_{t_1(n)}$. It is sufficient to calculate $2^{((l-1)\lambda_t-\lambda^{(m)k})t} \cdot 2^{m(1-\lambda^{(m)k})}$ in $(0, \hat{m}_0)$, and $2^{m_{t_1(n)+1/q-1/p}}$ in $(t_1(n), \hat{m}_{t_1(n)})$ and $(\hat{t}(n), 0)$. Taking into account (46) and (52), we get $n^{-\delta_j}$, $1 \leq j \leq 3$ (see Notation 5, case 4, subcase $q \leq 2$). This together with (70) yields the estimate for the widths.

Let $q > 2$. We define $\hat{t}(n)$ by $\hat{m}_{\hat{t}(n)} = 0$. We apply (35), (39) and obtain (72). As in the previous case, we get that if $t \geq 0$, $m'_t \geq m_t$, then $m'_t > \hat{m}_t$.

Taking into account (51), we get that $A$ is divided into

$$A_1 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \geq m'_t, m \geq m_t\},$$

$$A_2 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq m'_t, m \geq m_t\},$$

$$A_3 = \{(t, m) : 0 \leq t \leq \hat{t}(n), (\hat{m}_t)_+ \leq m \leq \hat{m}_t, m \leq m_t\},$$

$$A_4 = \{(t, m) : 0 \leq t \leq \hat{t}(n), m \leq m'_t, m \geq m_t\}$$

(see Fig. 14).

By (82)–(84), (33) and assertions 4, 5 of Theorem A holds in $A_1$ (the progression strictly decreases with $m$), in $A_2$, (61) holds for $p_1 \geq 2$, and (63) holds for $p_1 < 2$ (if $p_1 \leq 2$, the progression strictly increases with $m$), in $A_3 \cup A_4$, we get (61) (in $A_3$ the progression strictly increases for $p_1 \leq 2$, in $A_4$, the progression strictly decreases). Notice that, for $p_1 \geq 2$, in $A_2 \cup A_3$ the right-hand side of (61) has the order

$$2^{m_{t_1(n)+1/q-1/p}}k \cdot 2^{m_{t_1(n)+1/q-1/p}}(n^{-\frac{\lambda^{(m)k}t-1}{2}} \cdot 2^{m_{t_1(n)+1/q-1/p}})^{1/2-1/q}.$$

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We define the numbers $t_1(n)$ and $t_2(n)$ by $m_{t_1(n)} = m_{t_2(n)}$. It is sufficient to calculate $2(1-\lambda)\mu_s - \lambda \alpha_s)^{k_s} t \cdot 2-m(1-\lambda)s$ in $(0, m')$ and $2\mu_s^{k_s} t \cdot 2-m(s+1/q-1/p_1)$ in $(\hat{t}(n), 0)$ and $(t_1(n), m_{t_1(n)})$, and for $p_1 > 2$, in addition, we calculate the value $2\mu_s^{k_s} t \cdot 2-m(s+1/q-1/p_1)d_{s}(B_p^{\nu}, l_q^{\nu})$ in $(0, m_0)$ and $(t_2(n), 0)$. Taking into account (10), (53) and (57), we get $n^{-\theta_j}, 1 \leq j \leq j_0$ (see Notation 6, case 4, subcase $q > 2$).

This together with (72) yields the estimate for the widths.

For $\gamma = 0$, the proof is similar; the number $\hat{t}(n)$ is such that $2^{\hat{t}(n)}$ is a sufficiently large degree of $n$.

**Remark 1.** Let $\mu_s + \alpha_s + \frac{\gamma}{p_0} - \frac{\gamma}{p_1} > 0$ and one of the following conditions holds:

1. $p_0 > q$, $p_1 > q$;
2. $p_0 > q > p_1$, $s_s + \frac{1}{p_0} - \frac{1}{p_1} < 0$.

Then the assertion of Theorem 1 holds without the condition (13).

The proof is similar; here we use (29). Since $p_0 > q$ and $\mu_s + \alpha_s + \frac{\gamma}{p_0} - \frac{\gamma}{p_1} > 0$, we have (30), where $\nu_s$ is defined by (57); recall that it follows from (11) and Hölder’s inequality. We get that $\hat{m}_t > 0$ holds for $t < \hat{t}(n)$; we set $s_t = \nu_t,0$ and obtain $d_{s_t}(B_p^{\nu}, l_q^{\nu}) = 0$.

### 3 The estimates for the widths of the intersection of weighted Sobolev classes

Recall the definitions of a John domain and of an $h$-set.

We denote by $B_a(x)$ the Euclidean ball of radius $a$ centered at the point $x$.

**Definition 1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $a > 0$. We write $\Omega \in FC(a)$ if there is a point $x_s = x_s(\Omega) \in \Omega$ such that for each $x \in \Omega$ there are a number $T(x) > 0$ and a curve $\gamma_x : [0, T(x)] \rightarrow \Omega$ with the following properties:

1. $\gamma_x$ has the natural parametrization with respect to the Euclidean norm on $\mathbb{R}^d$;
2. $\gamma_x(0) = x, \gamma_x(T(x)) = x_s$,
3. $B_{at}(\gamma_x(t)) \subset \Omega$ for all $t \in [0, T(x)]$.

We say that $\Omega$ is a John domain if $\Omega \in \text{FC}(a)$ for some $a > 0$.

**Definition 2.** (see [3]). Let $\Gamma \subset \mathbb{R}^d$ be a non-empty compact set, let $h : (0, 1] \to (0, \infty)$ be a non-decreasing function. We say that $\Gamma$ is an $h$-set if there are $c_* \geq 1$ and a finite $\sigma$-additive measure $\mu$ on $\mathbb{R}^d$ such that $\text{supp} \mu = \Gamma$ and

$$c_*^{-1}h(t) \leq \mu(B_t(x)) \leq c_*h(t)$$

for all $x \in \Gamma$ and $t \in (0, 1]$.

Below $\text{mes}$ is the Lebesgue measure on $\mathbb{R}^d$, $Y_q(\Omega) = L_{q,v}(\Omega)$, $\mathcal{P}(\Omega) = \mathcal{P}_{r-1}(\Omega)$ is the space of polynomials of degree at most $r-1$, $X_{p_0}(\Omega) = L_{p_0,w}(\Omega)$. When estimating the widths of $\widehat{M}$ defined by (5), we set

$$X_{p_1}(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \left\| \frac{\nabla f}{g} \right\|_{L_{p_1}(\Omega)}^{p_1} + \left\| \frac{\nabla f}{f_0} \right\|_{L_{p_1}^{1/p_1}(\Omega)}^{p_1} < \infty \right\},$$

$$\|f\|_{X_{p_1}(\Omega)} = \left( \left\| \frac{\nabla f}{g} \right\|_{L_{p_1}(\Omega)}^{p_1} + \left\| \frac{\nabla f}{f_0} \right\|_{L_{p_1}^{1/p_1}(\Omega)}^{p_1} \right)^{1/p_1}.$$ 

(86)

When estimating the widths of $M$ defined by (1), we set

$$X_{p_1}(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \left\| \frac{\nabla f}{g} \right\|_{L_{p_1}(\Omega)} < \infty \right\},$$

$$\|f\|_{X_{p_1}(\Omega)} = \left\| \frac{\nabla f}{g} \right\|_{L_{p_1}(\Omega)}.$$ 

(87)

First we consider $\overline{M}$ in $L_{q,v}(\Omega)$ defined by (5), where $g, g_0, w, v$ are defined by [2], [3], $\Omega \subset (-\frac{1}{2}, \frac{1}{2})^d$ is a John domain, $\Gamma \subset \partial \Omega$ is an $h$-set, $h$ is defined by (3).

We show that for $\overline{M}$ Assumptions $[\text{A1} \text{ E}]$ hold with

$$\gamma_* = \theta, \quad s_* = \frac{r}{d}, \quad \mu_* = \beta + \lambda - r - \frac{d}{q} + \frac{d}{p_1}, \quad \alpha_* = \sigma - \lambda + \frac{d}{q} - \frac{d}{p_0}. \quad (88)$$

We define the partitions $\{\Omega_{t,j}\}_{t \geq t_{0,j} \in J_j}$ and $T_{t,j,m}$ as in [30] (§4, proof of Theorem 1). Notice that there are numbers $b_* = b_*(a, d) > 0$, $\overline{\sigma} = \overline{\sigma}(a, d) \in \mathbb{N}$ such that $\Omega_{t,j} \in \text{FC}(b_*)$, $\Omega_{t,j} \ni 0, \quad g(x) \asymp 2^{\beta \overline{\sigma} m}$, $g_0(x) \asymp 2^{(\beta - r)\overline{\sigma}m}$, $\sigma - \lambda + \frac{d}{q} - \frac{d}{p_0}. \quad (89)$

If $E \in T_{t,j,m}$, then there is $b_{**}(a, d) > 0$ such that $E \in \text{FC}(b_{**}(a, d))$,

$$\text{mes} E \leq b_* \left( \frac{d}{p_0} \right)^{-1}, \quad \text{diam} E \leq b_* \left( \frac{d}{p_0} \right)^{-1}. \quad (90)$$

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Assumption A follows from (89), the embedding theorem \[20, 21\] and Hölder’s inequality.

Assumptions B–D, F can be proved as in \[30\] (see §4–5, proofs of the upper and the lower estimates for Theorem 1).

Let us prove that Assumption E holds.

We show that for \(E \in T_{t,j,m}\)
\[
\operatorname{mes} E \gtrsim a, d^{-m}, \quad \operatorname{diam} E \gtrsim a, d^{-m}. \tag{91}
\]

To this end, we recall how to construct the partitions \(T_{t,j,m}\) according to \[27\].

Let \(t \geq t_0, j \in \hat{J}\) be fixed; we set \(G = \Omega_{t,j}\). Let \(\Theta(G)\) be the Whitney’s covering of \(G\). Each element of \(\Theta(G)\) is a dyadic cube; if the intersection of \(\Delta, \Delta' \in \Theta(G)\) has dimension \(d - 1\), then
\[
\frac{\operatorname{mes}(\Delta)}{\operatorname{mes}(\Delta')} \asymp 1. \tag{92}
\]

In \[27\] Lemma 3, a tree \(T\) with the root \(\omega_*\) was constructed, as well as the bijection \(F: V(T) \to \Theta(G)\) (here \(V(T)\) is the set of all vertices of \(T\)) with the following properties:

1. If the vertices \(\omega\) and \(\omega'\) are adjacent, then \(F(\omega)\) and \(F(\omega')\) have the \(d - 1\)-dimensional intersection.

2. There are numbers \(c_1 = c_1(a, d) \in \mathbb{N}\) and \(c_2 = c_2(a, d) \in \mathbb{N}\) such that for all vertices \(\omega > \omega'\)
\[
\rho(\omega, \omega') \leq c_1(m_\omega - m_{\omega'}) + c_2, \tag{93}
\]
where \(\rho(\omega, \omega')\) is the distance between \(\omega\) and \(\omega'\), \(2^{-m_\omega}\) and \(2^{-m_{\omega'}}\) is the length of the edge of \(F(\omega)\) and \(F(\omega')\), respectively.

Let \(T'\) be a subtree of \(T\) with the minimal vertex \(\omega_*\), let \(V(T')\) be its vertex set, and let \(\tilde{G}_{T'} = \cup_{\omega \in V(T')} F(\omega)\). In \[27\] (see formula (3.1) and Corollary 1) the domain
\[
G_{T'} \in \mathbb{FC}(b_{**}(a, d)), \quad x_*(G_{T'}) \in F(\omega'), \tag{94}
\]
was constructed; \(G_{T'} \cap \tilde{G}_{T'}\) has the zero measure, the point \(x_*(G_{T'})\) is from Definition 1.

Let \(\omega\) be a vertex of \(T\). We denote by \(T_\omega\) the tree with vertex set \(\{\omega' : \omega' \geq \omega\}\), and by \(V_1(\omega)\), the set of vertices that follow the vertex \(\omega\):
\[
V_1(\omega) = \{\omega' : \omega' \geq \omega\ \text{is adjacent with} \ \omega\}.
\]

From \(92\) and property 1 of \(F\) we get that there is a number \(k(d) \in \mathbb{N}\) such that \(\operatorname{card} V_1(\omega) \leq k(d)\) for each vertex \(\omega\).

The partition \(T_{t,j,m}\) was constructed in two steps.
1. First we construct the partition $T_{t,j,m}'$ (see Lemma 5 from [27]). For each vertex $\omega$ we construct the partition $P_\omega$ of $G_{T_\omega}$ (see Lemma 4 from [27]). If
\[
\mes(G_{T_\omega}) \leq (k(d) + 1)\mes(G) \cdot 2^{-m},
\]
we set $P_\omega = \{G_{T_\omega}\}$. If
\[
\mes(G_{T_\omega}) > (k(d) + 1)\mes(G) \cdot 2^{-m},
\]
we find a vertex $\hat{\omega} \geq \omega$ such that
\[
\mes(G_{T_{\hat{\omega}}}) > \mes(G_{T_\omega}) - 2^{-m}\mes(G) \overset{(95)}{\geq} k(d) \cdot 2^{-m}\mes(G),
\]
and for each $\omega' \in V_1(\hat{\omega})$,
\[
\mes(G_{T_{\omega'}}) \leq \mes(G_{T_\omega}) - 2^{-m}\mes(G).
\]
By (94), (95) and (96), $\mes(F(\omega)) \gtrsim a,d 2^{-m}\mes(G)$,
\[
\mes(F(\hat{\omega})) \gtrsim a,d 2^{-m}\mes(G);
\]
from (92) and property 1 of $F$ it follows that if $\omega' \in V_1(\hat{\omega})$, then $\mes(F(\omega')) \gtrsim 2^{-m}\mes(G)$. Hence
\[
\mes(G_{T_{\omega'}}) \gtrsim a,d 2^{-m}\mes(G), \omega' \in V_1(\hat{\omega}),
\]
and if $\hat{\omega} > \omega$, then
\[
\mes(G_{T_{\omega'} \setminus T_\omega}) \gtrsim a,d 2^{-m}\mes(G).
\]
The partition $P_\omega$ consists of $G_{T_{\omega'} \setminus T_\omega}$ (if $\hat{\omega} > \omega$), $F(\hat{\omega})$ and $G_{T_{\omega'}}$, $\omega' \in V_1(\hat{\omega})$.

Now the partition $T_{t,j,m}'$ is defined as follows. First we construct $P_{\omega'}$. If $P_{\omega'} \neq \{G_{T_{\omega'}}\}$, then for each $\omega' \in V_1(\hat{\omega})$ we construct $P_{\omega'}$, and so on. When we stop dividing the subtrees, we get the desired partition $T_{t,j,m}'$. From (97)–(99) it follows that
\[
\mes(E) \gtrsim a,d 2^{-m}\mes(G), \ E \in T_{t,j,m}'.
\]

2. Now we construct $T_{t,j,m}$; it is a subdivision of $T_{t,j,m}'$. If $F(\omega) \in T_{t,j,m}'$, $\mes(F(\omega)) > 2^{-m}\mes(G)$, then we take the uniform division of $F(\omega)$ into dyadic cubes $\Delta$,
\[
2^{-m}\mes(G) \lesssim \mes(\Delta) \leq 2^{-m}\mes(G).
\]
From (100)–(101) we get (91). This together with (90) implies that
\[ \text{mes } E \gtrless 2^{-\delta d - m}, \quad \text{diam } E \gtrless 2^{-\delta - \frac{m}{d}}, \quad E \in T_{t,j,m}. \] (102)

Now we prove that Assumption \( \mathcal{E} \) holds.

The inequality (12) is checked as in [30] (see §4, proof of the upper estimate for Theorem 1). The estimate (13) can be proved similarly as (12). Here we use the following fact: if \( E \) is a ball of radius \( r \), \( f \in W^r_1(\bar{B}) \), then by the Hölder’s inequality
\[
\rho^{-d + \frac{d}{q}} \|f\|_{L_1(B)} \lesssim \rho^{r + \frac{d}{q} - \frac{d}{r}} (\|\nabla f\|_{L^q(B)} + \rho^{-r} \|f\|_{L^q(B)}).
\]

Let us prove (14). Recall how the operator \( P_E \) is defined. Let \( x_*(E) \in E \) be the point from Definition 1, let \( B_E \subset E \) be the ball centered at \( x_*(E) \), \( \text{diam } B_E \gtrsim \text{diam } E \). We take the orthogonal projection from \( L_2(B_E) \) onto \( P_{r-1}(B_E) \), then extend it onto \( L_1(B_E) \) as a continuous operator. After that we extend each polynomial onto \( \mathbb{R}^d \) and multiply it by \( \chi_E \). Since \( L_{q,v}(B_E) \subset L_1(B_E) \), the operator \( P_E \) is well-defined on \( L_{q,v}(\Omega) \).

If \( f \in L_{q,v}(\Omega) \), then
\[
\|P_E(f \cdot \chi_E)\|_{L_q(E)} \lesssim_{a,d,q} \|P_E(f \cdot \chi_E)\|_{L_q(B_E)} \lesssim_{d,q} \lesssim (\text{mes } B_E)^{1/q-1} \|P_E(f \cdot \chi_E)\|_{L_1(B_E)} \lesssim_{d,q} \lesssim (\text{mes } B_E)^{1/q-1} \|f\|_{L_1(B_E)} \lesssim \|f\|_{L_q(E)}.
\]

This together with (89) implies that
\[
\left\| \sum_{E \in T_{t,j,m}} P_E(f \cdot \chi_E) \right\|_{L_{q,v}(\Omega_{t,j})} \lesssim_{a,d,q} \|f\|_{L_{q,v}(\Omega_{t,j})}. \] (103)

If \( f \in C_0^\infty(\Omega_{t,j}) \), then (14) follows from (89), (90) and the estimate
\[
\|f - P_E(f \cdot \chi_E)\|_{L_q(E)} \lesssim_{q,d,a} (2^{-m} \text{mes } \Omega_{t,j})^{1/d} \|\nabla f\|_{L_q(E)}, \quad E \in T_{t,j,m}
\]
(see [28] Lemma 8). The space \( C_0^\infty(\Omega_{t,j}) \) is dense in \( L_{q,v}(\Omega_{t,j}) \). This together with (103) yields that (14) holds for each function \( f \in L_{q,v}(\Omega_{t,j}) \).

Let us prove (15). Let \( E \in T_{t,j,m}, E' \in T_{t,j,m+1}, \text{mes}(E \cap E') > 0 \). First we define the domain \( G_{E,E'} \subset E \cup E' \). According to the construction of \( T_{t,j,m} \), we have the following cases.

1. Let \( E = G_A, E' = G_{A'} \), where \( A \) and \( A' \) are subtrees of \( \mathcal{T} \) with the minimal vertices \( \omega \) and \( \omega' \). Then \( \omega \sim \omega' \) comparable; hence they can be joint by a chain \( S \). Let \( G_{E,E'} = G_S \). Notice that \( \text{mes } F(\omega) \gtrsim \text{mes } (E) \gtrsim \text{mes } F(\omega') \). By (93), \( S \) has at most \( C(a,d) \) vertices. Therefore \( G_{E,E'} \in \mathbf{FC}(b_1(a,d)) \), where \( b_1(a,d) > 0 \); we can take as \( x_*(G_{E,E'}) \) from Definition 1 the center of \( F(\omega) \), as well as the center of \( F(\omega') \).
2. Let \( E' \subset F(\omega') \) be a dyadic cube, \( E = G_\mathcal{A} \), where \( \mathcal{A} \) is a subtree of \( \mathcal{T} \) with the minimal vertex \( \omega \) (the case when \( E \subset F(\omega) \), \( E' = G_{\mathcal{A}'} \) is considered similarly). Then \( \omega' \) is a vertex of \( \mathcal{A} \). As in the previous case we join \( \omega \) and \( \omega' \) by a chain \( \mathcal{S} \) and set \( G_{E,E'} = G_{\mathcal{S}} \). Again \( G_{E,E'} \in \mathcal{FC}(b_1(a, d)) \), \( \operatorname{mes} F(\omega) \stackrel{\text{[102]}}{\asymp} \operatorname{mes} E' \); hence we can take as the point \( x_*(G_{E,E'}) \) from Definition \([1]\) the center of \( F(\omega) \), as well as the center of \( E' \).

3. Let \( E \) and \( E' \) be dyadic cubes, \( E \subset E' \) or \( E' \subset E \). By \([102]\), \( \operatorname{mes} E \asymp \operatorname{mes} E' \).

We set \( G_{E,E'} = E \cup E' \). Then \( G_{E,E'} \in \mathcal{FC}(b_2(a, d)) \); we can take as the point \( x_*(G_{E,E'}) \) from Definition \([1]\) the center of \( E \), as well as the center of \( E' \).

In all cases
\[
\operatorname{diam} G_{E,E'} \asymp \operatorname{diam} (E \cup E').
\] (104)

Let \( B_{E,E'} \supset E \cup E' \) be a ball,
\[
\operatorname{diam} B_{E,E'} \asymp \operatorname{diam}(E \cup E');
\] (105)

we extend the polynomials \( \tilde{P}_Ef \) and \( \tilde{P}_{E'}f \) onto \( \mathbb{R}^d \) (the notation will be the same).

Applying the embedding theorem, we get
\[
\| \tilde{P}_Ef - \tilde{P}_{E'}f \|_{L_q(E \cup E')} \lesssim \| \tilde{P}_Ef - \tilde{P}_{E'}f \|_{L_q(B_{E,E'})} \stackrel{\text{[104], [105]}}{\lesssim} (\operatorname{mes} B_{E,E'})^{1/q-1} \| \tilde{P}_Ef - \tilde{P}_{E'}f \|_{L_1(B_{E,E'})} \stackrel{\text{[104], [105]}}{\lesssim} (\operatorname{mes} B_{E,E'})^{1/q-1} \| \tilde{P}_Ef - \tilde{P}_{E'}f \|_{L_1(G_{E,E'})} \lesssim (\operatorname{mes} B_{E,E'})^{1/q-1} \| f - \tilde{P}_{E'}f \|_{L_1(G_{E,E'})} + \| f - \tilde{P}_Ef \|_{L_1(G_{E,E'})} \lesssim \| f - \tilde{P}_{E'}f \|_{L_1(G_{E,E'})} \lesssim \| f - \tilde{P}_Ef \|_{L_1(G_{E,E'})} \lesssim 2^{-\left(\frac{\pi d+m}{2}+\frac{1}{q'}+\frac{1}{d}+\frac{1}{q'}\right)} \| \nabla' f \|_{L_{p_1}(E \cup E')} \lesssim 2^{-\left(\frac{\pi d+m}{2}+\frac{1}{q'}+\frac{1}{d}+\frac{1}{q'}\right)} \| \nabla' f \|_{L_{p_1}(E \cup E')}.
\]

This together with \([89]\) yields \([15]\).

This completes the proof of Assumption \([13]\). Notice that in \([15]\) the space \( X_{p_1}(\Omega) \) can be defined by \([86]\), as well as by \([87]\). We obtain

**Theorem 2.** Let \( \hat{M} \) be defined by \([5]\), let \( g, g_0, w, v \) be defined by \([2], [6]\), let \( \Omega \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d \) be a John domain, let \( \Gamma \subset \partial \Omega \) be an h-set, where h is defined by \([3]\). Let \( s_*, \gamma_*, \mu_*, \alpha_* \) be defined according to \([88]\). Then for \( M := \hat{M} \) Theorem \([7]\) holds.

If one of the conditions from Remark \([7]\) holds, the same estimates are true for the set \( M \) defined by \([7]\).
As in the previous cases (see also [30], §ϕg), the Kolmogorov widths of 
\[ \{ - \log t, 1 \} \]; here we assume that
\[
\beta + \lambda = r + \frac{d}{q} - \frac{d}{p_1}, \quad \sigma - \lambda = \frac{d}{p_0} - \frac{d}{q},
\]
where \( s + \frac{1}{\max\{p_0, q\}} - \frac{1}{p_1} > 0 \), and let \( \lambda < \frac{d-\mu}{q} \) (in the second example, \( \theta := 0 \)). Then for the set \( M \) the estimate from Theorem I holds (here \( j_0 \) and \( \theta_j \) are as in Notation I).

It follows from the inclusions \( \hat{M} \subset M \subset W_{p_1,\frac{d}{q}}(\Omega) \), where
\[
W_{p_1,\frac{d}{q}}(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \left\| \nabla^r f \right\|_{L_{p_1}(\Omega)} \leq 1 \right\}.
\]

Indeed, we have
\[
d_{n}(\hat{M}, L_{q,v}(\Omega)) \leq d_n(M, L_{q,v}(\Omega)) \leq d_n(W_{p_1,\frac{d}{q}}(\Omega), L_{q,v}(\Omega)).
\]
The left-hand side can be estimated from below according to Theorem I. The right-hand side can be estimated from above by [28, Theorem 1, cases 1, 3, 4].

Now we consider Example 3 from [30]. The set \( \hat{M} \) is defined by (5), where
\[
g(x) = (1 + |x|^\beta), \quad g_0(x) = (1 + |x|)^{\beta+r},
\]
\[
w(x) = (1 + |x|^\sigma), \quad v(x) = (1 + |x|)^{\lambda}.
\]
As in the previous cases (see also [30], §4, 5, proof of Theorem 3), we check Assumptions A[F] with \( s = \frac{d}{q}, \gamma = 0, \mu = \beta + \lambda + r + \frac{\sigma - \frac{d}{q}}{p_1}, \alpha = \sigma - \lambda - \frac{d}{q} + \frac{d}{p_0} \). Hence the Kolmogorov widths of \( \hat{M} \) can be estimated according to Theorem I if one of the conditions of Remark I holds, the same estimates are true for the set \( M \) defined by (1).

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