A no-go theorem for scalar fields with couplings from Ginzburg–Landau models

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Abstract Recently Hod proved a no-go theorem that static scalar fields cannot form spherically symmetric boson stars in the asymptotically flat background. On the other side, scalar fields can be coupled to the gradient according to next-to-leading order Ginzburg–Landau models. In the present work, we extend Hod’s discussions by considering couplings between static scalar fields and the field gradient. For a non-negative coupling parameter, we show that there is no asymptotically flat spherically symmetric boson stars made of coupled static scalar fields.

1 Introduction

It is widely believed that fundamental scalar fields exist in nature. The systems composed of scalar fields have attracted a lot of attention from mathematicians and physicists. One of the most famous properties of such systems is the no hair theorem, which states that static scalar fields cannot exist outside asymptotically flat black hole horizons, see Refs. [1–19] and reviews [20,21]. Lately, it was found that no scalar field property also appears in curved horizonless spacetimes [22–38].

A well known horizonless configuration is the boson star made of scalar fields. Theoretically, the scalar fields can be stationary or static. It was found that boson stars can be constructed with stationary scalar fields [39,40]. Then it is interesting to examine whether static scalar fields can form boson stars. In the flat spacetime, boson stars cannot be made of static scalar fields due to Derrick’s theorem [41]. In the asymptotically flat background, Hod proved that boson stars cannot be composed of static scalar fields [42,43]. In the asymptotically flat background, this no boson star property also appears [44]. On the other side, in next-to-leading order Ginzburg–Landau family of models for a BCS superconductor, there is a new term with scalar fields coupled to the gradient [45–47] and the couplings play an important role in scalar condensations [48]. Along this line, it is interesting to study the no boson star property with coupled static scalar fields.

In the following, we plan to extend the discussion in [42] by considering static scalar fields coupled to the gradient in the asymptotically flat background. We show that boson stars cannot be made of static scalar fields with a non-negative coupling parameter. We summarize main results in the last section.

2 No static boson star for coupled scalar fields

In the extended Ginzburg–Landau models, a new term with scalar fields coupled to the gradient appears [45–47]. In this work, we study the system with coupled static scalar fields in the curved spacetime. The Lagrangian density describing scalar fields coupled to the gradient is [49–51]

\[ \mathcal{L} = R - |\nabla \psi|^2 - \xi \psi^2 |\nabla \psi|^2 - V(\psi^2), \]

where \( R \) corresponds to the Ricci curvature and \( \xi \) is the coupling parameter. In order to obtain results in this work, we assume that the coupling parameter is non-negative as \( \xi \geq 0 \). We study radial direction depending scalar fields expressed as \( \psi = \psi(r) \).

The potential \( V(\psi^2) \) is positive semidefinite and increases as a function of \( \psi^2 \) satisfying

\[ V(0) = 0 \quad \text{and} \quad \dot{V} = \frac{dV(\psi^2)}{d(\psi^2)} > 0. \]

The free scalar fields with \( V(\psi^2) = \mu^2 \psi^2 \) apparently satisfy the relation (2) as \( V(0) = 0 \) and \( \dot{V} = \mu^2 > 0 \). Here \( \mu \) is the nonzero scalar field mass.

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The metric of spherically symmetric boson star can be expressed as [52–57]

\[ ds^2 = -e^{-\chi} dt^2 + \frac{dr^2}{g} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]  

(3)

\( \chi \) and \( g \) are functions of the coordinate \( r \). A and \( \phi \) are angular coordinates.

The Lagrangian density (1) yields the scalar field equation

\[ (1 + \xi \psi^2) \psi'' + [(1 + \xi \psi^2) \left( \frac{2}{r} - \frac{\chi'}{2} + \frac{g'}{g} \right) \]
\[ + 2\xi \psi \psi'] \psi' - \left( \xi \psi^2 + \frac{V}{g} \right) \psi = 0. \]  

(4)

At spatial infinity, metric functions asymptotically behave as [42]

\[ \chi \to 0, \quad g \to 1 \quad \text{for} \quad r \to \infty. \]  

(5)

And near the origin, metric functions are characterized by [42]

\[ \chi' \to 0, \quad g \to 1 + O(r^2) \quad \text{for} \quad r \to 0. \]  

(6)

The corresponding energy density is

\[ \rho = -T_t^t = g\psi'^2 + \xi g\psi^2\psi' + V(\psi^2). \]  

(7)

The finite gravitational mass condition \( M = \int_0^\infty 4\pi r^2 \rho dr < \infty \) implies that \( r^2 \rho \) decreases faster than \( \frac{1}{r} \) around the infinity. Then we arrive at the asymptotical behavior

\[ r^3 \rho \to 0 \quad \text{for} \quad r \to \infty. \]  

(8)

According to relations (2), (7) and (8), we obtain the infinity vanishing condition

\[ \psi(\infty) = 0. \]  

(9)

The scalar field equation near the origin is

\[ (1 + \xi \psi^2) \psi'' + \frac{2}{r} (1 + \xi \psi^2) \psi' - (\xi \psi^2 + \dot{V}) \psi = 0. \]  

(10)

Around the origin, we set the scalar field in the general form \( \psi(r) = r^a (a_0 + a_1 r + a_2 r^2 + \cdots) \) [58]. Putting this expression into Eq. (10) and considering the leading order, we find the relation \( s = 0 \) and the solution satisfies the near origin asymptotical behavior

\[ \psi(r) = a \left[ 1 + \frac{\dot{V}(a^2)}{6(1 + \xi a^2)} \cdot r^2 \right] + O(r^3), \]  

(11)

where \( a \) is the scalar field value \( \psi(0) \) at the origin.

In the case of \( a = \psi(0) = 0 \), also considering the relation (9), we deduce that the eigenfunction \( \psi(r) \) must possess at least one extremum point \( r_{peak} \) [22]. The scalar fields are characterized by

\[ \{ \psi^2 > 0, \quad \psi' = 0 \text{ and } \psi'' \leq 0 \} \quad \text{for} \quad r = r_{peak}. \]  

(12)

At \( r = r_{peak} \), the relation (12) yields an inequality

\[ (1 + \xi \psi^2) \psi'' + \left( 1 + \xi \psi^2 \right) \left( \frac{2}{r} - \frac{\chi'}{2} + \frac{g'}{g} \right) \]
\[ + 2\xi \psi \psi' \left[ \psi' - \left( \xi \psi^2 + \frac{\dot{V}}{g} \right) \psi \right] < 0. \]  

(13)

In particular, Eq. (4) is invariant under the transformation \( \psi \to -\psi \). Without loss of generality, the scalar field at the origin can be assumed to have a non-negative value. The case of \( a = \psi(0) = 0 \) has been studied in the front. In another case of \( a > 0 \), (11) yields \( \psi'(0) = 0 \) and \( \psi''(0) = \frac{a\dot{V}(a^2)}{3(1 + \xi a^2)} > 0 \), which implies \( \psi' > 0 \) for \( r > 0 \) in the near origin region. The scalar field eigenfunction \( \psi(r) \) increases to be more positive around the origin and finally approaches zero at the infinity. So one deduces that the scalar field \( \psi(r) \) reaches a positive local maximum value at one extremum point \( \tilde{r}_{peak} \). At this point, the eigenfunction is characterized by

\[ \{ \psi > 0, \quad \psi' = 0 \text{ and } \psi'' \leq 0 \} \quad \text{for} \quad r = \tilde{r}_{peak}. \]  

(14)

At \( r = \tilde{r}_{peak} \), the relation (14) yields the inequality

\[ (1 + \xi \psi^2) \psi'' + \left( 1 + \xi \psi^2 \right) \left( \frac{2}{r} - \frac{\chi'}{2} + \frac{g'}{g} \right) \]
\[ + 2\xi \psi \psi' \left[ \psi' - \left( \xi \psi^2 + \frac{\dot{V}}{g} \right) \psi \right] < 0. \]  

(15)

Relations (13) and (15) are in contradiction with the scalar field equation (4). So the scalar field equation cannot be respected at the corresponding extremum points. We therefore conclude that spherically symmetric asymptotically flat boson stars cannot be constructed with static coupled scalar fields with non-negative coupling parameters.

### 3 Conclusions

We studied the nonexistence of boson stars composed of static scalar fields in the spherically symmetric asymptotically flat background. We considered static scalar fields coupled to the field gradient, where the coupling term also appears in the next-to-leading order Ginzburg–Landau models. For non-negative coupling parameters, we obtained the characteristic inequalities (13) and (15) at corresponding extremum points. However, these relations are in contradiction with the static scalar field equation (4). As a summary, we proved the nonexistence of spherically symmetric asymptotically flat static boson stars made of scalar fields with a non-negative coupling parameter. In a future work, we would try to prove the non-existence of static boson stars that are made of scalar fields with negative values of the coupling parameter \( \xi \). Alternatively, if such self-gravitating field con-
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