SUBHARMONIC SOLUTIONS OF FIRST ORDER HAMILTONIAN SYSTEMS WITH SUBQUADRATIC CONDITION

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Abstract. Using a homologically link theorem in variational theory and iteration inequalities of Maslov-type index, we prove the existences of a sequence of subharmonic solutions for one type of sub-quadratic non-autonomous Hamiltonian systems. Moreover, we also study the minimal period problem of some autonomous Hamiltonian systems with sub-quadratic condition.

Keywords: Maslov-type index, Morse index, homologically link, subharmonic solutions, Hamiltonian systems

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1. Introduction and main results

In this paper, we first consider subharmonic solutions of the following non-autonomous Hamiltonian system

$$\left\{ \begin{array}{l} \dot{z} = J H'_z(t, z), \forall z \in \mathbb{R}^{2n}, \\
\quad \quad z(kT) = z(0), \quad k \in \mathbb{N}, \end{array} \right. \quad (1.1)$$

where $T > 0$, $H'_z$ denotes the gradient of $H$ with respect to the variable $z \in \mathbb{R}^{2n}$, and $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with $I_n$ being the identity matrix on $\mathbb{R}^n$. Without confusion, we shall omit the subindex of the identity matrix.

Let $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$ and $z = (p, q)$. Set $H'_z(t, z) = (H'_{p}(t, z), H'_{q}(t, z))$. We denote $|p|$ and $p \cdot q$ the norm and inner product in $\mathbb{R}^n$ respectively. Denote any principal diagonal matrix $\text{diag}(a, \ldots, a, b, \ldots, b) \in \mathbb{R}^{2n}$ by $V(a, b)$ with $a, b \in \mathbb{R}$, then $V(a, b)(z) = (a p_1, \ldots, a p_n, b q_1, \ldots, b q_n)$.

For a subquadratic Hamiltonain, now we assume the Hamiltonian satisfying the following hypotheses as in [15] with a bit difference

(H1) $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, [0, +\infty))$ and $H(t + T, z) = H(t, z)$, $\forall t \in \mathbb{R}, \forall z \in \mathbb{R}^{2n}$;

(H2) There exist constants $\sigma, \omega > 0$ such that

$$\lim_{|z| \to +\infty} \frac{H(t, z)}{|p|^{1+\frac{\sigma}{\omega}} + |q|^{1+\frac{\sigma}{\omega}}} = 0;$$

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Moreover, if 

\[ \frac{1}{\mu} H'_p(t, z) \cdot p + \frac{1}{v} H'_q(t, z) \cdot q \geq -c_1, \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}; \]

(H4) There exist constants \( c_2, c_3 > 0 \) and \( \beta \in (1, 2) \) such that

\[ H(t, z) - \frac{1}{\mu} H'_p(t, z) \cdot p - \frac{1}{v} H'_q(t, z) \cdot q \geq c_2|z|^\beta - c_3, \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}; \]

(H5) There are constants \( \lambda \in (1, \frac{\beta^2}{2+\beta}) \) and \( b_0 > 0 \) such that

\[ |H''(t, z)| \leq b_0(|z|^{\lambda-1} + 1), \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}; \]

(H6) \( H(t, 0) = 0 \) and \( |H'_z(t, z)| > 0, \forall z \neq 0. \)

We now state the main results of this paper.

**Theorem 1.1.** Suppose \( H \) satisfies (H1)-(H6). Set \( \alpha = T/2\pi \). Then there exists \( \alpha_0 > 0 \) such that for any \( T \geq 2\pi \alpha_0 \) and each integer \( k \geq 1 \), the system \((\ref{eq:1.1})\) possesses a nontrivial \( kT \)-periodic solution \( z_k \) with its Maslov-type index satisfying

\[ i_{kT}(z_k) \leq n \leq i_{kT}(z_k) + \nu_{kT}(z_k). \]

Moreover, if \( i_{kT}(z_k) + \nu_{kT}(z_k) > n \), then \( z_k \) and \( z_{lk} \) are geometrically distinct provided \( l > \frac{2n}{i_{kT}(z_k) + \nu_{kT}(z_k) - n} \).

For the Hamiltonian \( H \) contains a quadratic term, i.e., \( H(t, z) = \frac{1}{2}(\tilde{B}(t)z, z) + \tilde{H}(t, z) \), we state the result as follows.

We set \( w = \max_{t \in \mathbb{R}} |\tilde{B}(t)| \).

**Theorem 1.2.** Suppose \( \tilde{H} \) satisfies (H1)-(H6) and

(H7) \( \tilde{B}(t) \) is a \( T \)-periodic, symmetric and continuous matrix function and satisfies

\[ (\tilde{B}(t)z, z) = 2(\tilde{B}(t)z, V(\frac{1}{\mu}, \frac{1}{v})(z)), \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}. \]

We also require there exists an unbounded sequence \( \{\rho_m\} \subset (0, +\infty) \) with \( \inf \rho_m = 0 \) such that

\[ (\tilde{B}(t) B_{\rho}z, B_{\rho}z) = \rho^{\sigma-2} (\tilde{B}(t)z, z), \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2n}. \]

Hold for \( \rho \in \{\rho_m\} \), where \( B_{\rho}z = (\rho^{-1}p, \rho^{-1}q) \) for \( \rho > 0 \) with \( \rho, \tilde{\omega}, \tilde{\sigma} \) defined as in Section 3. Set \( \alpha = T/2\pi \). Then there exists \( \alpha_0 > 0 \) such that for any \( T \geq 2\pi \alpha_0 \) and each integer \( 1 \leq k \leq \frac{2\pi}{\sigma T} \), the system \((\frac{1}{\tilde{H}})\) possesses a nontrivial \( kT \)-periodic solution \( z_k \) with its Maslov-type index satisfying

\[ i_{kT}(z_k) \leq n \leq i_{kT}(z_k) + \nu_{kT}(z_k). \]

Moreover, if \( i_{kT}(z_k) + \nu_{kT}(z_k) > n \), then \( z_k \) and \( z_{lk} \) are geometrically distinct provided \( l > \frac{2n}{i_{kT}(z_k) + \nu_{kT}(z_k) - n} \) and \( lk \leq \frac{2\pi}{\alpha T} \).
We also consider the minimal periodic solutions of the following autonomous Hamiltonian systems

\[
\begin{aligned}
\dot{z} &= JH'(z), \quad z \in \mathbb{R}^{2n}, \\
z(T) &= z(0).
\end{aligned}
\]  

(1.2)

**Theorem 1.3.** Suppose the autonomous Hamiltonian \( H(z) \) satisfies (H1)-(H6) and (H8) \( H''_{zz}(z) \) is strictly positive for every \( z \in \mathbb{R}^{2n} \setminus \{0\} \). Set \( \alpha = T/2\pi \). Then there exists \( \alpha_0 > 0 \) such that for any \( T \geq 2\pi\alpha_0 \), the system (1.2) possesses a nontrivial solution \( z \) with minimal period \( T \).

The first result for existence of subharmonic periodic solutions of the system (1.1) was obtained by Rabinowitz in [29]. Since then, many mathematicians made their contributions in this topic. See for example [5, 8, 10, 17, 19, 25, 27, 31]. For the brake subharmonic solutions of Hamiltonian systems we refer to [16, 33]. For the \( P \)-symmetric subharmonic solutions we refer to [24].

In [28], Rabinowitz proposed a conjecture on whether a superquadratic Hamiltonian system possesses a non-constant periodic solution having any prescribed minimal period. After paper [28], much work has been done in this field. We refer to [6, 9, 7, 11, 12] for the minimal periodic solutions. For the minimal periodic problem of brake solutions of Hamiltonian systems, we refer to [20, 33]. For the minimal \( P \)-symmetric periodic solutions of Hamiltonian systems, we refer to [21, 23, 32].

Linking theorems provides a simple but extremely powerful method to prove the existence of critical points. We follow the ideas in [25] which use the linking Theorem 2.11 to look for the critical points and estimate the corresponding Morse index, based on those, we study the subharmonic solutions and minimal periodic solutions under subquadratic conditions by the method in [19] respectively. The main difficult is to construct two sets and prove that they are homological linking which is the content of Lemma 3.2. Our idea comes from [15]. In [15], F. Guo and Q. Xing constructed a similar linking structure to study the existence of periodic solutions for subquadratic Hamiltonian systems.

This paper is divided into 3 sections. In Section 2, we briefly sketch some notions about the Maslov-type index and the iteration inequalities developed by C. Liu and Y. Long in [22]. We also recall the homologically link theorem in [11] from which we can find a critical point of the corresponding functional together with Morse index information. We prove that there is a homologically link structure for the functional under the conditions of Theorem 1.1. In Section 3, we give a proof of Theorem 1.1-1.3.

2. Preliminaries

We first review the Maslov-type index and some iteration properties. Here we use the notions and results in [22] and [26].

For \( \tau > 0 \), we recall that symplectic group is defined as \( Sp(2n) = \{ M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^TJM = J \} \), where \( \mathcal{L}(\mathbb{R}^{2n}) \) is the space of \( 2n \times 2n \) real matrices, the set of symplectic paths is defined by \( \mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], Sp(2n)) \mid \gamma(0) = I \} \).

Let \( S_\tau = \mathbb{R}/\tau\mathbb{Z} \) and \( \mathcal{L}_s(\mathbb{R}^{2n}) \) denotes all symmetric real \( 2n \times 2n \) matrices. For \( B(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n})) \), suppose \( \gamma \) is the fundamental solution of the linear Hamiltonian systems

\[
\dot{y}(t) = JB(t)y, \quad y \in \mathbb{R}^{2n}.
\]  

(2.1)
Then the Maslov-type index pair of $\gamma$ is defined as a pair of integers

$$(i_\tau, \nu_\tau) \equiv (i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},$$

where $i_\tau$ is the index part and

$$\nu_\tau = \dim \ker(\gamma(\tau) - I)$$

is the nullity. We also call $(i_\tau, \nu_\tau)$ the Maslov-type index of $B(t)$, just as in \cite{22,23}. If $(z, \tau)$ is a $\tau$-periodic solution of (1.1), then the Maslov-type index of the solution $z$ is defined to be the Maslov-type index of $B(t) = H''_{zz}(t, z(t))$ and denoted by $(i_\tau(z), \nu_\tau(z))$. We call the solution $z$ is non-degenerate if $\nu_\tau(z) = 0$.

For $\gamma \in \mathcal{P}_\tau(2n)$, we define the $m$-th iteration path $\gamma_m : [0, m\tau] \to Sp(2n)$ of $\gamma$ by

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \forall j\tau \leq t \leq (j + 1)\tau, \quad 0 \leq j \leq m - 1.$$ 

We denote the corresponding Maslov-type index of $\gamma^m$ on $[0, m\tau]$ by $(i_{m\tau}, \nu_{m\tau}) \equiv (i_{m\tau}(\gamma^m), \nu_{m\tau}(\gamma^m))$.

**Proposition 2.1** \cite{19}. If $z$ is a $k\tau$-periodic solution of the system (1.1), then $i_{k\tau}(z) = i_{k\tau}(z)$ and $\nu_{k\tau}(z) = \nu_{k\tau}(z)$ for all integers $0 \leq j \leq k$.

**Proposition 2.2** \cite{19}. For $m \in \mathbb{N}$, there holds

$$m(i_\tau + \nu_\tau - n) - n \leq i_{m\tau} \leq m(i_\tau + n) + n - \nu_{m\tau}.$$ 

**Proposition 2.3** \cite{22}. For $m \in \mathbb{N}$, there holds

$$m(i_\tau + \nu_\tau - n) - n - \nu_\tau \leq i_{m\tau} \leq m(i_\tau + n) - (\nu_{m\tau} - \nu_\tau).$$

**Proposition 2.4** \cite{11}. Let $B(t) \in C(\mathbb{R}, \mathcal{L}_\sigma(\mathbb{R}^{2n}))$ be $\tau$-periodic and positive definite for all $t \in [0, \tau]$. Suppose that $B(t_0)$ is strictly positive for some $t_0 \in [0, \tau]$. Then $i_\tau(B)$ is strictly positive for some $t_0 \in [0, \tau]$.

**Proposition 2.5** \cite{7}. Let $B(t) \in C(\mathbb{R}, \mathcal{L}_\sigma(\mathbb{R}^{2n}))$ be $\tau$-periodic. Suppose there exists some $m \in \mathbb{N}$ such that $i_{m\tau}(B) \leq n + 1$, $i_\tau(B) \geq n$ and $\nu_\tau(B) \geq 1$. Then $m = 1$.

As is in \cite{13}, by making change of variables $\varsigma = \frac{t}{\alpha}$ with $\alpha = \frac{\tau}{2\pi}$, seeking for $T$-periodic solutions of the system (1.1) diverts to searching for $2\pi$-periodic solutions of the system

$$\begin{cases}
\dot{p} = -\alpha H'(\alpha, z), \\
\dot{q} = \alpha H''(\alpha, z).
\end{cases} \quad (2.2)$$

Hence we can focus our attention on $2\pi$-periodic solutions of the system (1.1). In the following we always assume $\tau = 2\pi$.

Now we introduce some concepts and conclusions which are used later. For $S_\tau = \mathbb{R}/\tau\mathbb{Z}$, let $E = W^{1,2}(S_\tau, \mathbb{R}^{2n})$. Recall that $E$ consists of all the elements $z \in L^2(S_\tau, \mathbb{R}^{2n})$ satisfying

$$z(t) = \sum_{j \in \mathbb{Z}} \exp\left(\frac{2j\pi t}{\tau}\right) a_j, a_j \in \mathbb{R}^{2n},$$

$$\|z\|^2 = \tau|a_0|^2 + \tau \sum_{j \in \mathbb{Z}} |j||a_j|^2 < +\infty.$$ 

The inner product in $E$ is given by

$$\langle z_1, z_2 \rangle = \tau a_0^1 \cdot a_0^2 + \tau \sum_{j \in \mathbb{Z}} |j|a_j^1 \cdot a_j^2 \quad \text{for} \quad z_k = \sum_{j \in \mathbb{Z}} \exp\left(\frac{2j\pi t}{\tau}\right) a_j^k, k = 1, 2.$$
Lemma 2.6 ([30]). For each $s \in [1, +\infty)$, $E$ is compactly embedded in $L^s(S_r, \mathbb{R}^{2n})$. In particular there is an $C_s > 0$ such that $\|z\|_{L^s} \leq C_s \|z\|$ for all $z \in E$.

Let $\mathcal{L}_s(E)$ and $\mathcal{L}_s(E)$ denote the spaces of the bounded self-adjoint linear operator and compact linear operator on $E$ respectively. For $B(t) \in C(S_r, \mathcal{L}_s(\mathbb{R}^{2n}))$, we define two operators $A$, $B \in \mathcal{L}_s(E)$ by extending the bilinear forms:

$$
\langle Ax, y \rangle = \int_0^\tau -J\dot{x}(t) \cdot y(t) dt, \quad \langle Bx, y \rangle = \int_0^\tau B(t)x(t) \cdot y(t) dt.
$$

(2.3)
on $E$. Then $\ker A = \mathbb{R}^{2n}$, the Fredholm index $\text{ind} A = 0$ and $B \in \mathcal{L}_s(E)$. Using the Floquet theory, we have

$$
\nu_\tau = \dim \ker(A - B).
$$

For $m \in \mathbb{N}$, set $E^0 = \mathbb{R}^{2n}$,

$$
E_m = \sum_{j=-m}^{m} \exp\left(\frac{2j\pi t}{\tau}\right) \mathbb{R}^{2n},
$$

$$
E^\pm = \sum_{z_j \in \mathbb{Z}} \exp\left(\frac{2j\pi t}{\tau}\right) \mathbb{R}^{2n},
$$

and $E^+ = E_m \cap E^+$, $E^- = E_m \cap E^-$. Obviously, $E = E^+ \oplus E^0 \oplus E^-$ and $E_m = E^+_m \oplus E^0 \oplus E^-_m$.

It is easy to check that $E^+$, $E^0$, $E^-$ are respectively the subspaces of $E$ on which $A$ is positive definite, null, and negative definite, and these spaces are orthogonal with respect to $A$. For $z = z^+ + z^0 + z^-$ and $z^0 \in E^0$, we have $\langle Az, z \rangle = \langle Az^+, z^+ \rangle + \langle Az^-, z^- \rangle$ and $\|z\|^2 = |z^0|^2 + \frac{1}{2}(\langle Az^+, z^+ \rangle - \langle Az^-, z^- \rangle)$.

Let $P_0$ be the orthogonal projection from $E$ to $E^0$ and $P_m$ be the orthogonal projection from $E$ to $E_m$ for $m \in \mathbb{N}$. Then $\{P_m\}_{m=0}^\infty$ is a Galerkin approximation sequence respect to $A$.

For $S \in \mathcal{L}_s(E)$ and $d > 0$, we denote by $M^+_d(S)$, $M^-_d(S)$ and $M^0_d(S)$ the eigenspace corresponding to the eigenvalue belonging to $[d, +\infty)$, $(-\infty, -d]$ and $(-d, d)$, respectively, and denote by $M^+_S(S)$, $M^-_S(S)$, and $M^0_S(S)$, respectively, the positive, negative definite, and null subspace of $S$. Set $S^d = (S|_{Em})^{-1}$, and $P_mSP_m = (P_mSP_m)|_{Em}: Em \rightarrow Em$.

In [11], Fei and Qiu studied the relation between Maslov-type index and Morse index by Galerkin approximation method and got the following theorem.

Theorem 2.7 ([11]). For $B(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n}))$ with the Maslov-type index $(i_\tau, \nu_\tau)$ and any constant $0 < d \leq \frac{1}{4}(A - B)^2\|^{-1}$, there exists an $m_0 > 0$ such that for $m \geq m_0$, there holds

$$
\dim M^+_d(P_m(A - B)P_m) = \frac{1}{2} \dim E_m - i_\tau - \nu_\tau,
$$

$$
\dim M^-_d(P_m(A - B)P_m) = \frac{1}{2} \dim E_m + i_\tau,
$$

$$
\dim M^0_d(P_m(A - B)P_m) = \nu_\tau,
$$

(2.4)

where $B$ is the operator defined by (2.3) corresponding to $B(t)$.

Definition 2.8 ([14]). Let $M$ be a Hilbert manifold. Suppose that $Q$ is a closed $q$-dimensional ball topologically embedded into $M$ and $S$ is a closed subset such that $\partial Q \cap S = \emptyset$. We say that $\partial Q$ and $S$ homotopically link if $\varphi(Q) \cap S \neq \emptyset$ for each $\varphi \in C(Q, M)$ such that $\varphi|_{\partial Q} = id|_{\partial Q}$.
Definition 2.9 ([1][4]). Let $M$ be a Hilbert manifold. Suppose that $Q$ is a closed $q$-dimensional ball topologically embedded into $M$ and $S$ is a closed subset such that $\partial Q \cap S = \emptyset$. We say that $\partial Q$ and $S$ homologically link if $\partial Q$ is the support of a non-vanishing homology class in $H_{q-1}(M \setminus S)$.

A new linking structure is given in Lemma 2.10 which is different from the structures in [25] and [31]. We will prove it by the method in [25] with subtle changes.

Lemma 2.10. Let $M = M_1 \oplus M_2$ be a Hilbert space with $\dim M_2 = q$ and $P_2 : M \rightarrow M_2$ be the orthogonal projection. For $\vartheta > 0$, let $B_\vartheta$ be a bounded linear invertible operator on $M$ such that $\vartheta > \|B_\vartheta^{-1}P_2^{-1}\|$ and $P_2B_\vartheta : M_2 \rightarrow M_2$ is invertible. Suppose $S = M_1 + u_0$ with $u_0 \in M_2$ and $\|u_0\| = 1$, $\tilde{Q} = \{z \mid \|z\| \leq \vartheta, z \in M_2\}$. Then $B_\vartheta(\partial Q)$ and $S$ homologically link.

Proof. We claim that $B_\vartheta(\partial Q)$ and $S$ homotopically link. First we prove that $B_\vartheta(Q) \cap S \neq \emptyset$, it is equivalent to prove that $\psi_0(t,v) = (0,0)$ has a solution in $[0,\vartheta] \times M_2$, where

$$\psi_0(t,v) = (\|v\| - t, P_2B_\vartheta v - u_0), \quad (t,v) \in [0,\vartheta] \times M_2.$$  

Note that $t = \|B_\vartheta^{-1}u_0\|(< \vartheta)$ and $v = B_\vartheta^{-1}u_0(\in M_2)$ is the unique solution of $\psi_0$ in $[0,\vartheta] \times M_2$, where $B_\vartheta = P_2B_\vartheta$ and $B_\vartheta^{-1}$ denotes the inverse of $B_\vartheta|_{M_2}$. Therefore $(0,0) \notin \vartheta_0(\partial([0,\vartheta] \times M_2))$.

For $\varphi \in C(B_\vartheta(Q),M)$ with $\varphi|_{B_\vartheta(\partial Q)} = id|_{B_\vartheta(\partial Q)}$, define $\psi : [0,\vartheta] \times M_2 \rightarrow \mathbb{R} \times M_2$ as

$$\psi(t,v) = (\|v\| - t, P_2\varphi B_\vartheta v - u_0).$$

It remains to show that $\varphi(B_\vartheta(Q)) \cap S \neq \emptyset$, it is equivalent to prove $\psi(t,v) = (0,0)$ has a solution in $[0,\vartheta] \times M_2$. Since $\psi = \psi_0$ on $\partial([0,\vartheta] \times M_2)$, the Brouwer degree theory shows that $\deg(\psi_0, (0,\vartheta) \times M_2,(0,0)) = \pm 1$. Thus $\psi(t,v) = (0,0)$ has a solution in $[0,\vartheta] \times M_2$. By the above, $B_\vartheta(\partial Q)$ and $S$ homotopically link.

Finally, we have $B_\vartheta(\partial Q)$ and $S$ homologically link by Theorem II.1.2 in [4].

Let $f$ be a $C^2$ functional on a Hilbert manifold $M$. Denote by $D^2f$ the Hessian of $f$. Recall that the Morse index $m(x)$ of $f$ at a critical point $x$ is the dimension of a maximal subspace on which $D^2f(x)$ is strictly negative and the large Morse index $m^*(x)$ of $x$ is $m(x) + \dim \ker D^2f(x)$.

In order to find the critical points and get the corresponding Morse index estimates, we need the following homologically link theorem which was proved in [1].

Theorem 2.11 ([1]). Let $M$ be a Hilbert manifold. Let $Q \subset M$ be a topologically embedded closed $q$-dimensional ball and let $S \subset M$ be a closed subset such that $\partial Q \cap S = \emptyset$. Assume that $\partial Q$ and $S$ homologically link. Let $f \in C^2(M)$ be a function with Fredholm gradient such that

(i) $\sup_{\partial Q} f < \inf_S f$;

(ii) $f$ satisfies $(PS)$ condition on some open interval containing $[\inf_S f, \sup_Q f]$.

Then, if $\Gamma$ denotes the set of all $q$-chains in $M$ whose boundary has support $\partial Q$, the number

$$c := \inf_{\xi \in \Gamma} \sup_{|\xi|} f$$

belongs to $[\inf_S f, \sup_Q f]$ and is a critical value of $f$. Moreover, $f$ has a critical point $\bar{x}$ such that $f(\bar{x}) = c$ and the following estimate on Morse index of $\bar{x}$ holds

$$m(\bar{x}) \leq q \leq m^*(\bar{x}).$$
3. Subquadratic Hamiltonian systems

In this section, we suppose that \( H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, [0, +\infty)) \) satisfies conditions (H1)-(H6) defined in Section 1. We now consider following non-autonomous Hamiltonian system

\[
\begin{aligned}
\dot{z} &= JH_z'(t, z), \quad z \in \mathbb{R}^{2n}, \\
\quad z(T) &= z(0), \quad T > 0.
\end{aligned}
\]  

(3.1)

Define \( f(z) = \frac{1}{2}\langle Az, z \rangle - \int_0^T H(t, z)dt \) for \( z \in E \). Form (H5), we have \( f \in C^2(E, \mathbb{R}) \). Let \( G(z) = -f(z) \). Looking for solutions of (3.1) is equivalent to looking for critical points of \( G \) on \( E \).

Let \( G_m = G|_{E_m} \). Set \( X_m = E^- \oplus E^0 \) and \( Y_m = E_m^+ \). Next we will show that \( G_m \) satisfies the hypotheses of Theorem 1.1 when \( H \) satisfies (H1)-(H5). The proofs are similar to those in [15].

**Lemma 3.1.** Suppose \( H \) satisfies (H3)-(H5). Then \( G_m \) satisfies the (P.S) condition on \( E_m \) for any \( m \in \mathbb{N} \), i.e., any sequence \( \{z_j\} \subset E_m \) possesses a convergent subsequence in \( E_m \), provided \( G_m(z_j) \) is bounded and \( G_m(z_j) \to 0 \) as \( j \to \infty \).

**Proof.** Let \( \{z_j\} \) be such a sequence. Thus suppose \( |G_m(z_j)| \leq K_1 \) and \( G'_m(z_j) \to 0 \) as \( j \to \infty \). We claim that \( \{z_j\} \) is bounded. Otherwise, there exists a subsequence \( \{z_{j_k}\} \) such that \( \|z_{j_k}\| \to +\infty \) as \( k \to +\infty \). For simplicity, we still denote \( \{z_{j_k}\} \) by \( \{z_j\} \). For large \( j \) and \( z_j = (p_j, q_j) \), there exists constant \( b \) such that

\[
K_1 + b\|z_j\| \geq G_m(z_j) - \frac{1}{\mu}G'_m(z_j) \cdot (p_j, 0) - \frac{1}{\nu}G'_m(z_j) \cdot (0, q_j)
\]

\[
= \int_0^\tau \left( H(t, z_j) - \frac{1}{\mu}H'_p(t, z_j) \cdot p_j - \frac{1}{\nu}H'_q(t, z_j) \cdot q_j \right) dt
\]

\[
\geq \int_0^\tau (c_2|z_j|^\beta - c_3)dt = c_2\|z_j\|^{\beta}_L - \tau c_3
\]

via (H4) and the form of \( G_m \). From (3.2), there is constant \( K_2 > 0 \) such that

\[
\|z_j\|^{\beta}_L \leq K_2(1 + \|z_j\|^\frac{\beta}{2}), \quad \text{for } m \in \mathbb{N} \text{ large enough.}
\]

(3.3)

Set \( z_j = z_j^+ + z_j^0 + z_j^- \), simple computation shows

\[
\|z_j^+\| \geq |G'_m(z_j) \cdot z_j^+| = \left| \int_0^\tau [H'_z(t, z_j) \cdot z_j^+ - (-Jz_j \cdot z_j^+)dt] \right|
\]

which implies

\[
\int_0^\tau -Jz_j \cdot z_j^+ dt \leq \left| \int_0^\tau [H'_z(t, z_j) \cdot z_j^+ dt] + \|z_j^+\| \right|, \quad \text{for } m \in \mathbb{N} \text{ large enough.}
\]

(3.4)
By (H5), Hölder inequality and Lemma 2.6, we obtain
\[ \|z_j^+\|^2 = \frac{1}{2} \langle Az_j^+, z_j^+ \rangle = \frac{1}{2} \int_0^\tau -J \dot{z}_j^+ \cdot z_j^+ dt \]
\[ \leq \int_0^\tau \left[ b_0(|z_j|^\lambda + 1)|z_j^+| dt + \|z_j^+\| \right] \]
\[ \leq \int_0^\tau b_0(|z_j|^\lambda + 1)|z_j^+| dt + \|z_j^+\| \]
\[ \leq \left( \int_0^\tau |z_j|^\beta dt \right)^{\frac{\lambda}{\beta}} \left( \int_0^\tau |z_j|^{\frac{\alpha}{\beta}} dt \right)^{\frac{\beta}{\alpha}} + c_4 \|z_j^+\|_{L^1} + \|z_j^+\| \]
\[ \leq C_\alpha \|z_j\|_{L^\alpha} \|z_j^+\| + c_4 C_\lambda \|z_j^+\| + \|z_j^+\| \leq K_3(1 + \|z_j\|_{L^\alpha}^\lambda)\|z_j^+\|, \]
where \( C_1 \) and \( C_{\frac{\alpha}{\beta}} \) are the embedding constants in Lemma 2.6 and \( K_3 > 0 \). Combining (3.3) with (3.5), we have
\[ \|z_j^+\| \leq K_4(1 + \|z_j\|_{L^\alpha}^\lambda), \text{ for } m \in \mathbb{N} \text{ large enough,} \]
where \( K_4 > 0 \). Similarly, we have
\[ \|z_j^-\| \leq K_4(1 + \|z_j\|_{L^\alpha}^\lambda), \text{ for } m \in \mathbb{N} \text{ large enough.} \]

Next we estimate the boundedness of \( \{z_j^0\} \). Set \( \tilde{z}_j = z_j - z_j^0 = z_j^+ + z_j^- \). By (H5), (3.6)-(3.7) and Lemma 2.6, we obtain
\[ \int_0^\tau [H(t, z_j) - H(t, z_j^0)] dt = \int_0^\tau \int_0^\tau H_s(t, z_j^0 + s\tilde{z}_j) \cdot \tilde{z}_j ds dt \]
\[ \leq \int_0^\tau 2^\lambda b_0 \left( |z_j^0|^\lambda + |\tilde{z}_j|^\lambda + 1 \right) |\tilde{z}_j| dt \]
\[ \leq 2^\lambda b_0 C_1 |z_j^0|^\lambda \|\tilde{z}_j\| + 2^\lambda b_0 C_\lambda \|\tilde{z}_j\|^\lambda + 1 + 2^\lambda b_0 C_1 \|\tilde{z}_j\| \]
\[ \leq K_5 \left( \|z_j\|^\lambda + \|z_j\|^\lambda + \frac{\lambda^2}{\beta} \right) + K_6 \left( 1 + \|z_j\|_{L^\alpha}^\lambda \right) + K_7 \left( 1 + \|z_j\|_{L^\alpha}^\lambda \right), \]
where \( K_5, K_6 \) and \( K_7 \) are positive constants, \( C_1 \) and \( C_\lambda \) are the embedding constants in Lemma 2.6. Since \( \lambda < \beta \), for \( m \in \mathbb{N} \) large enough, we have
\[ \int_0^\tau [H(t, z_j) - H(t, z_j^0)] dt \leq K_8 \left( 1 + \|z_j\|_{L^\alpha}^\lambda \right), \]
where \( K_8 > 0 \). Simple computation shows
\[ K_1 \geq G_m(z_j) = \int_0^\tau [H(t, z_j) - H(t, z_j^0)] dt - \|z_j^+\|^2 + \|z_j^-\|^2 + \int_0^\tau H(t, z_j^0) dt. \]
Notice that \( \lambda + \frac{\lambda^2}{\beta} \geq \frac{2\lambda}{\beta} \). According to (3.6)-(3.7) and (3.9)-(3.10), we obtain
\[ \int_0^\tau H(t, z_j^0) dt \leq K_9 \left( 1 + \|z_j\|_{L^\alpha}^\lambda \right), \text{ for } m \in \mathbb{N} \text{ large enough,} \]
\[ \int_0^\tau H(t, z_j^0) dt \leq K_9 \left( 1 + \|z_j\|_{L^\alpha}^\lambda \right), \text{ for } m \in \mathbb{N} \text{ large enough,} \]
where \( K_9 > 0 \). From (H3) and (H4), it follows that
\[
\int_0^\tau H(t, z_j^0) dt = \int_0^\tau \left( \frac{1}{\mu} H_p(t, z_j^0) \cdot y_j^0 + \frac{1}{v} H_q(t, z_j^0) \cdot q_j^0 \right) dt + \int_0^\tau \left( c_2 |z_j^0|^{\beta} - c_3 \right) dt
\]
\[
\geq \int_0^\tau \left( c_2 |z_j^0|^{\beta} - c_7 \right) dt = c_2 \tau |z_j^0|^{\beta} - c_7 \tau,
\]
where \( c_7 = c_1 + c_3 \). From (3.11)-(3.12), we see that
\[
|z_j^0| \leq K_{10} \left( 1 + \|z_j\| \frac{\beta \lambda + \lambda \beta}{\beta^2} \right),
\]
where \( K_{10} \) is a proper and positive constant. We conclude from (3.13)-(3.14) that
\[
\|z_j\|^2 = \|z_j^+\|^2 + \|z_j^0\|^2 + \|z_j^-\|^2 \leq \hat{K}_4 \left( 1 + \|z_j\|^{\frac{2\lambda}{\beta}} \right) + \hat{K}_10 \left( 1 + \|z_j\|^{\frac{2(\lambda + \lambda \beta)}{\beta^2}} \right),
\]
where \( \hat{K}_4 \) and \( \hat{K}_{10} \) are proper and positive constants. Since \( \frac{\lambda}{\beta} < \frac{\lambda + \lambda \beta}{\beta^2} \) and \( \lambda \in [1, \frac{2}{\beta + 1}) \), \( \frac{\lambda + \lambda \beta}{\beta^2} < 1 \). From this and (3.14), we have \( \|z_j\|^2 \leq K_{11} \left( 1 + \|z_j\|^{\frac{2(\lambda + \lambda \beta)}{\beta^2}} \right) \) with a proper and positive constant \( K_{11} \). Hence
\[
1 = \frac{\|z_j\|^2}{\|z_j\|^2} \leq K_{11} \frac{1 + \|z_j\|^{\frac{2(\lambda + \lambda \beta)}{\beta^2}}}{\|z_j\|^2} \to 0, \text{ as } j \to +\infty,
\]
a contradiction.

Thus \( \{z_j\} \) is bounded in \( E_m \). Since \( E_m \) is finite dimensional, \( \{z_j\} \) is precompact and possesses a convergent subsequence in \( E_m \). □

Applying the same process of the proof Lemma 3.1 and the standard argument in the appendix of [2], we have the following Lemma 3.2.

**Lemma 3.2.** Suppose \( H \) satisfies (H3)-(H5). Then \( G \) satisfies (P.S)* condition with respect to \( \{E_m\}_{m \in \mathbb{N}} \), i.e., for any sequence \( \{z_m\} \subset E \) satisfying \( z_m \in E_m \), \( G_m(z_m) \) is bounded and \( G_m'(z_m) \to 0 \) possesses a convergent subsequence in \( E \).

There is a constant \( q \) such that \( \tilde{\sigma} = \frac{\omega}{\sigma + \omega} \geq 1 \) and \( \tilde{\omega} = \frac{\omega}{\sigma + \omega} \geq 1 \). For \( \rho > 0 \) and \( z = (p, q) \in E \), we define an operator \( B_\rho : E \to E \) by
\[
B_\rho z = (\rho \tilde{\sigma}^{-1} p, \rho \tilde{\omega}^{-1} q).
\]
It is easy to see that \( B_\rho \) is a linear bounded and invertible operator and \( \|B_\rho\| \leq 1 \) if \( \rho \leq 1 \).

For \( z = z^+ + z^0 + z^- \in E \), we have
\[
\langle AB_\rho z, B_\rho z \rangle = \rho^{\sigma - 2} \langle Az, z \rangle = \rho^{\sigma - 2} (\|z^+\|^2 - \|z^-\|^2).
\]

**Lemma 3.3.** Suppose \( H \) satisfies (H2), (H3) and (H4). Let \( u_0 \in Y_1 \) and \( S_m = E^- \oplus E^0 \oplus u_0 \). Then there exist \( \tilde{\sigma} > 1 \), \( \overline{\sigma} \) and \( \kappa \) with \( \kappa > \overline{\sigma} \) independent of \( m \) such that \( G_m|_{B_\rho(\partial Q_m)} \leq \overline{\sigma} \) and \( G_m|_{S_m} \geq \kappa \), where \( Q = \{z \mid \|z\| \leq \tilde{\sigma}, z \in E^+\} \), \( Q_m = Q \cap E_m \), and \( \partial Q_m \) refers to the boundary of \( Q_m \) relative to \( \{z \mid z \in E^+\} \cap E_m \).

**Proof.** It is sufficient to show that \( G|_{B_\rho(\partial Q)} \leq \overline{\sigma} \) and \( G|_{S_m} \geq \kappa \) for any \( m \in \mathbb{N} \).

By (H2), for any \( \varepsilon > 0 \), there exists \( M_\varepsilon \) such that
\[
H(t, z) \leq \varepsilon \left( |p|^{1+\frac{\sigma}{\sigma-1}} + |q|^{1+\frac{\sigma}{\sigma-1}} \right) + M_\varepsilon, \forall (t, z) \in \mathbb{R} \times \mathbb{R}^n.
\]
For $z \in \partial Q = \{z = (p, q) \mid z \in E^+, \|z\| = \vartheta\}$. From (3.16)-(3.17), we have
\[
G(B_{\vartheta}z) = \int_0^T H(t, B_{\vartheta}z) - \frac{1}{2}\langle AB_{\vartheta}z, B_{\vartheta}z \rangle \leq \varepsilon C(\vartheta) \left(\vartheta^{\alpha-1}(1+\vartheta)\|p\|^{1+\vartheta} + \vartheta^{\beta-1}(1+\vartheta)\|q\|^{1+\vartheta}\right) + M_{\varepsilon}\tau - \vartheta^e
\]
\[
\leq 2\varepsilon C(\vartheta) \vartheta^e + M_{\varepsilon}\tau - \vartheta^e = (2\varepsilon C(\vartheta) - 1)\vartheta^e + M_{\varepsilon}\tau,
\]
where $C(\vartheta)$ is the embedding constant. Choose $\varepsilon > 0$ such that $0 < 2\varepsilon C(\vartheta) - 1 < 1$. Set $\varpi = (2\varepsilon C(\vartheta) - 1)\vartheta^e + M_{\varepsilon}\tau$, hence we have $G|_{B_{\vartheta}(\partial Q)} \leq \varpi$.

For any $m \in \mathbb{N}$ and $z \in S_m$, (H1) implies
\[
G(z) = \int_0^T H(t, z)dt + \|z\| \geq -\|u_0\|^2.
\]

Hence we can choose $\kappa = -\|u_0\|^2 < 0$ and $\vartheta > 1$ large enough such that $G|_{S_m} \geq \kappa > \varpi$.

**Lemma 3.4.** Under the conditions of Lemma 3.3 for $\vartheta > 1$, we have $B_{\vartheta}(\partial Q_m)$ and $S_m$ homologically link.

**Proof.** Since $\vartheta > 1$, $\vartheta > \|B_{\vartheta}^{-1}\| = \|B_{\vartheta}\|$. Let $P : E \to E^+$ denote the orthogonal projection. Then $PB_{\vartheta} : E^+ \to E^+$ is linear bounded and invertible. Note that $B_{\vartheta}(E_m) \subset E_m$ and $B_{\vartheta}|E_m : E_m \to E_m$ is linear bounded and invertible. Then $P_mB_{\vartheta}|E_m : Y_m \to Y_m$ is also linear bounded and invertible, where $P_m : E_m \to Y_m$ is the orthogonal projection. We complete the proof by Lemma 2.10. \hfill \square

**Theorem 3.5.** Suppose $H$ satisfies (H1)-(H6). Set $\alpha = T/2\pi$. Then there exists $\alpha_0 > 0$ such that for any $T \geq 2\pi\alpha_0$ and each integer $k \geq 1$, the system (1.1) possesses a nontrivial $kT$-periodic solution $z_k$ with its Maslov-type index satisfying
\[
i_{kT}(z_k) \leq n \leq i_{kT}(z_k) + \nu_{kT}(z_k).
\]

**Proof.** We follow the ideas of [15] and [19]. The proof falls naturally into three parts.

**Step 1.** We claim that the system (3.1) possesses a classic $\tau$-periodic solution $z$ under the conditions (H1)-(H5).

From Lemma 3.1, 3.3, 3.4 we see that $G_m \in C^2(E, \mathbb{R})$ satisfies all the hypotheses of Theorem 2.11 if $H$ satisfies (H1)-(H5) and then $G_m$ has a critical point $z_m$ satisfying
\[
G_m(z_m) \geq \kappa \quad \text{and} \quad m(z_m) \leq \dim Y_m \leq m^*(z_m).
\]

By Lemma 3.2 we assume $z_m \to z \in E$ with $G(z) \geq \kappa$ and $\nabla G(z) = 0$. It is obvious that $z$ is a critical point of $f$. Hence $z$ is a weak solution of (3.1) and $G(z) \geq \kappa$. Finally $z$ is a classical $\tau$-periodic solution of (3.1) by similar argument in [30].

**Step 2.** We claim that for $T$ large enough, the equation $\dot{z} = JH'_z(t, z)$ possesses a non-constant classical $T$-periodic solution $z$ under the conditions (H1)-(H6).

For $\alpha = \frac{T}{\pi} > 0$, we set $G_\alpha(z) = \alpha \int_0^T H(at, z)dt - \frac{1}{2}\langle Az, z \rangle$ and $G_{\alpha, m} = G_\alpha|_{E_m}$.

By the same argument in Lemma 3.3 there exist $\vartheta > 1$, $\varpi$ with $\varpi < 1$ independent of $m$ such that
\[
G_\alpha|_{B_{\vartheta}(\partial Q)} \leq \varpi, \quad \text{where} \quad Q = \{z \mid \|z\| \leq \vartheta, z \in E^+\}.
\]
Following [3], we divide into three cases to show that for $\alpha$ large enough, $G_\alpha(z) \geq 1$ with $z = z^\perp + z^0 + u_0 \in S_m = X_m + u_0$ and $u_0 \in Y_1$.

Case 1 If $\|z^\perp\|^2 > \|u_0\|^2 + 1$, then $G_\alpha(z) = \alpha \int_0^\tau H(\alpha, z)dt + \|z^\perp\|^2 - \|u_0\|^2 \geq 1$.

Case 2 If $\|z^\perp\|^2 \leq \|u_0\|^2 + 1$ and $|z_0| < \tilde{k}$, where $\tilde{k}$ is a proper and positive constant. Then by (H3)-(H5) and Lemma 2.6 we obtain

$$
\begin{align*}
\int_0^\tau H(\alpha, z^\perp + z^0 + u_0)dt = & \int_0^\tau H(\alpha, u_0)dt + (\int_0^\tau H(\alpha, z^\perp + z^0 + u_0)dt - \int_0^\tau H(\alpha, z^0)dt) \\
= & \int_0^\tau H(\alpha, u_0)dt + \int_0^1 \int_0^\tau H'_z(\alpha, z^0 + s(z^\perp + u_0)) \cdot (z^\perp + u_0)dsdt \\
\geq & \int_0^\tau (c_2|z^0|^\beta - c_1 - c_3)dt - \int_0^\tau (b_0|z^0|^\lambda + b_0)|z^\perp + u_0|dt \\
\geq & \tau c_2|z^0|^\beta - \int_0^\tau \bar{c}_4|z^\perp + u_0|^\lambda + 1 dt - \int_0^\tau \bar{c}_4|z^0|^\lambda |z^\perp + u_0|dt \\
& - c_4 \int_0^\tau |z^\perp + u_0|dt - \tau(c_1 + c_3) \\
\geq & \tau c_2|z^0|^\beta - \bar{c}_4|z^\perp + u_0|^\lambda + 1 - \bar{c}_4(|z^0|^\lambda + 1)|z^\perp + u_0| - \tau(c_1 + c_3).
\end{align*}
$$

Since $\|z^\perp + u_0\| \leq \sqrt{2}\|u_0\|^2 + 1$ and $\beta > \lambda$, $\int_0^\tau H(\alpha, z^\perp + z^0 + u_0)dt \geq 1$ for $\tilde{k}$ large enough. Hence we have $G_\alpha(z) \geq \alpha - \|u_0\|^2 \geq 1$ if $\alpha \geq \|u_0\|^2 + 1$.

Case 3 If $\|z^\perp\|^2 \leq \|u_0\|^2 + 1$ and $|z_0| > \tilde{k}$. Now it is the same as the case (iii) in the proof of Theorem 4.7 in [3]. So we have $G_\alpha(z) \geq 1$ for $\alpha$ large enough.

Combining the three cases, there exists an $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$,

$$
G_\alpha(z) \geq 1, \quad \forall z \in S_m.
$$

(3.22)

For $\alpha \geq \alpha_0$ fixed, by (3.21)-(3.22), using the same argument as in step 1, we have that $G_{\alpha, m}$ has a critical point $z_{\alpha, m}$ satisfying

$$
G_{\alpha, m}(z_{\alpha, m}) \geq \inf_{z \in S_m} G_{\alpha, m}(z) \geq 1 \quad \text{and} \quad m(z_{\alpha, m}) \leq \dim Y_m \leq m^*(z_{\alpha, m}).
$$

(3.23)

By Lemma 3.2 we assume $z_m \to z \in E$ with $G_\alpha(z_m) \geq 1$ and $\nabla G_\alpha(z_m) = 0$.

According to (H6), it is easy to see that $z = 0$ is the unique trivial solution of (3.1). By the above, for $\alpha \geq \alpha_0$, we have the critical value $G_\alpha(z_m) \geq 1 > 0 = G_\alpha(0)$ and the corresponding critical point $z_{\alpha}$ is nontrivial. Using the same argument in [3], $z_{\alpha}$ is a a nontrivial classic $T$-periodic solution of (3.1).

**Step 3.** We claim that the Maslov-type index of $z_{\alpha}$ satisfies $i_T(z_{\alpha}) \leq n \leq i_T(z_{\alpha}) + v_T(z_{\alpha})$.

Let $B$ be the operator for $B(t) = H''_{zz}(t, z_{\alpha})$ defined by (2.3). By direct computation, we get

$$
\langle G''_\alpha(x)w, w \rangle - \langle (B - A)w, w \rangle = \int_0^\tau [(H''_{zz}(t, x(t))w, w) - (H''_{zz}(t, z_{\alpha}(t))w, w)]dt, \quad \forall w \in W.
$$

Then by the continuous of $H''_{zz}$,

$$
\|G''_\alpha(x) - (B - A)\| \to 0 \quad \text{as} \quad \|x - z_{\alpha}\| \to 0.
$$

(3.24)

Let $d = \frac{1}{2}\|(A - B)^2\|^{-1}$. By (3.21), there exists $r_0 > 0$ such that

$$
\|G''_\alpha(x) - (B - A)\| < \frac{1}{2}d, \quad \forall x \in V_{r_0} = \{x \in E : \|x - z_{\alpha}\| \leq r_0\}.
$$
Hence for \( m \) large enough, there holds
\[
\|G''_\alpha(x) - P_m(B - A)P_m\| < \frac{1}{2}d, \quad \forall x \in V_{r_0} \cap E_m.
\]
(3.25)
For \( x \in V_{r_0} \cap W_m, \forall w \in M^+_d(P_m(B - A)P_m) \setminus \{0\}, \) (3.25) implies that
\[
\langle G''_\alpha(x)w, w \rangle \leq \langle P_m(B - A)P_mw, w \rangle + \|G''_\alpha(x) - P_m(B - A)P_m\| \cdot \|w\|^2
\leq -d\|w\|^2 + \frac{1}{2}d\|w\|^2 = -\frac{1}{2}d\|w\|^2 < 0.
\]
Then
\[
\dim M^-(G''_\alpha(x)) \geq \dim M^+_d(P_m(B - A)P_m), \quad \forall x \in V_{r_0} \cap E_m.
\]
(3.26)
Similarly, we have
\[
\dim M^+(G''_\alpha(x)) \geq \dim M^+_d(P_m(B - A)P_m), \quad \forall x \in V_{r_0} \cap E_m.
\]
(3.27)
Note that
\[
\dim M^+_d(P_m(B - A)P_m) = \dim M^+_d(P_m(A - B)P_m),
\]
(3.28)
\[
\dim M^0_d(P_m(B - A)P_m) = \dim M^0_d(P_m(A - B)P_m)
\]
By (3.29), (3.26) and Theorem 2.11 for large \( m \) we have
\[
\frac{1}{2} \dim E_m - n = \dim Y_m \geq m(z_{\alpha,m})
\geq \dim M^+_d(P_m(B - A)P_m) = \dim M^+_d(P_m(A - B)P_m)
= \frac{1}{2} \dim E_m - iT(z_{\alpha}) - \nu T(z_{\alpha})
\]
(3.29)
and
\[
\frac{1}{2} \dim E_m - n = \dim Y_m \leq m^*(z_{\alpha,m})
\leq \dim(M^+_d(P_m(A - B)P_m) \oplus M^0_d(P_m(A - B)P_m))
= \dim(M^+_d(P_m(B - A)P_m) \oplus M^0_d(P_m(B - A)P_m))
= \frac{1}{2} \dim E_m - iT(z_{\alpha}).
\]
(3.30)
Thus we obtain (1.2) by (3.29) - (3.30). \( \square \)

**Proof of Theorem 1.1**

Since \( H \) is \( kT \)-periodic, by Theorem 3.5 the system (1.1) possesses a nontrivial \( k\tau \)-periodic solution \( z_k \) satisfying
\[
i_{kT}(z_k) \leq n \leq i_{kT}(z_k) + \nu_{kT}(z_k).
\]
(3.31)
If \( z_k \) and \( z_{pk} \) are not geometrically distinct, then there exist integers \( l \) and \( m \) such \( l \ast z_k = m \ast z_{pk} \) by definition. By Proposition 2.1 we have
\[
i_{kT}(l \ast z_k) = i_{kT}(z_k), \quad \nu_{kT}(l \ast z_k) = \nu_{kT}(z_k),
i_{pkT}(m \ast z_{pk}) = i_{pkT}(z_{pk}), \quad \nu_{pkT}(m \ast z_{pk}) = \nu_{pkT}(z_{pk}).
\]
(3.31) implies that \( i_{pkT}(z_{pk}) \leq n \) and \( i_{kT}(z_k) + \nu_{kT}(z_k) \geq n \). Since \( i_{kT}(z_k) + \nu_{kT}(z_k) > n \), Proposition 2.2 shows that \( l \leq \frac{2n}{i_{kT}(z_k) + \nu_{kT}(z_k) - n} \) which contradicts with the assumption \( l > \frac{2n}{i_{kT}(z_k) + \nu_{kT}(z_k) - n} \). We complete the proof of Theorem 1.1.
Proof of Theorem 1.2
At the moment, $H(t, z)$ is defined by $H(t, z) = \frac{1}{2}(\tilde{B}(t)z, z) + \tilde{H}(t, z)$. Since (H7) holds, we have
\[ G(z) - G'(z)V\left(\frac{1}{\mu}, \frac{1}{v}\right)(z) = \int_{0}^{\tau} (\tilde{H}(t, z) - \tilde{H}'(t, z)V\left(\frac{1}{\mu}, \frac{1}{v}\right)(z))dt \]
and
\[ (\tilde{B}(t)B_{\rho}z, B_{\rho}z) = \rho^{\varepsilon-2}(\tilde{B}(t)z, z), \forall \ z \in E, \]
where $G, g$ and $B_{\rho}$ ($\rho > 0$) are defined in Section 3. Note that $H$ satisfies (H1)-(H6) if $\tilde{H}$ does. We can define $B_{\delta}$ for small $\delta \in \{\rho_m\}$ as in Section 3 and then complete the proof by applying the same arguments as above.

Proof of Theorem 1.3
For $\alpha = T/2\pi$. By Theorem 3.5, there exists $\alpha_0 > 0$ such that for any $T \geq 2\pi\alpha_0$, the system (1.2) possesses a nontrivial $T$-periodic solution $z$ with
\[ i_T(z) \leq n. \] (3.32)

The rest proof is almost the same as that in [22]. For readers’ convenience, we estimate the iteration number of the solution $(z, T)$.

Suppose $(z, T)$ has minimal period $\frac{T}{k}$, i.e., its iteration number is $k \in \mathbb{N}$. Since the Hamiltonian system in (1.2) is autonomous and the condition (H8) holds, we have
\[ \nu_{\frac{T}{k}}(z) \geq 1 \text{ and } i_{\frac{T}{k}}(z) \geq n. \] (3.33)
by Proposition 2.4. Thus by (3.32) and (3.33) and Proposition 2.5 we obtain $k = 1$ and complete the proof.

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