Finite polynomial cohomology with coefficients

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Abstract. We introduce a theory of finite polynomial cohomology with coefficients in this paper. We prove several basic properties and introduce an Abel–Jacobi map with coefficients. As applications, we use such a cohomology theory to study arithmetics of compact Shimura curves over $\mathbb{Q}$, and simplify proofs of the works of Darmon–Rotger and Bertolini–Darmon–Prasanna.

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1 Introduction

In this paper, we provide and study a theory of finite polynomial cohomology with coefficients. Such a cohomology theory with trivial coefficient was first introduced by A. Besser in [Bes00] for a proper smooth scheme. In what follows, in order to motivate our study, we begin with a brief review of Besser’s work. An
overview of the present paper will be provided thereafter. In the end of this introduction, we discuss several further research directions and possible applications of our theory.

1.1 Besser’s finite polynomial cohomology and the $p$-adic Abel–Jacobi map

Fix a rational prime $p$ and let $K$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ be the ring of integer of $K$. Let $X$ be a proper smooth scheme over $\mathcal{O}_K$ of relative dimension $d$. Given $m, n \in \mathbb{Z}$, Besser defined a cohomology theory $R\Gamma_{\text{fp},m}(X, n)$, whose $i$-th cohomology group is denoted by $H^{i}_{\text{fp},m}(X, n)$. The novelties of this cohomology theory are the following:

- One can view the theory of finite polynomial cohomology as a generalisation of Coleman’s integration.
- One can use finite polynomial cohomology to understand the $p$-adic Abel–Jacobi map explicitly.

Let us explain these in more details.

From the definition of finite polynomial cohomology, one can easily deduce a fundamental exact sequence

$$0 \to H^{i-1}_{\text{dR}}(X_K) \xrightarrow{\iota_{\text{fp}}} H^{i}_{\text{fp}}(X, n) \xrightarrow{pr_{\text{fp}}} F^nH^{i}_{\text{dR}}(X_K) \to 0, \quad (1)$$

where $H^{i}_{\text{fp}}(X, n)$ stands for $H^{i}_{\text{fp},i}(X, n)$, and $\langle F^n \rangle$ stands for the $n$-th filtration of the de Rham cohomology groups. When $i = n = 1$, Besser explained in [Bes00, Theorem 1.1] that, given $\omega \in F^1H^{1}_{\text{dR}}(X_K)$, any of its lift $\bar{\omega} \in H^{1}_{\text{fp}}(X, 1)$ can be viewed as a Coleman integration of $\omega$. In particular, for any $x \in X(\mathcal{O}_K)$, the map $x \mapsto x^*\bar{\omega} \in H^{1}_{\text{fp}}(\text{Spec} \mathcal{O}_K, 1) \cong K$ is the evaluation at $x$ of a Coleman integral $F_{\omega}$ of $\omega$. Hence the finite polynomial cohomology can viewed as a generalisation of Coleman’s integration theory.

Let us turn our attention to the $p$-adic Abel–Jacobi map. We denote by $Z^i(X)$ the set of smooth irreducible closed subschemes of $X$ of codimension $i$ and let

$$A^i(X) := \text{the free abelian group generated by } Z^i(X).$$

Besser constructed a cycle class map

$$\eta_{\text{fp}} : A^i(X) \to H^{2i}_{\text{dR}}(X, i),$$

which, after composing with the projection $pr_{\text{fp}} : H^{2i}_{\text{fp}}(X, i) \to F^iH^{2i}_{\text{dR}}(X_K)$, agrees with the usual de Rham cycle class map. The $p$-adic Abel–Jacobi map is then defined to be

$$\text{AJ} : A^i(X)_0 := \ker pr_{\text{fp}} \circ \eta_{\text{fp}} \xrightarrow{\eta_{\text{fp}}} \frac{H^{2i-1}_{\text{dR}}(X_K)}{F^iH^{2i-1}_{\text{dR}}(X_K)} \cong (F^{d-i+1}H^{2d-2i+1}_{\text{dR}}(X_K))^\vee,$$

where the last isomorphism is given by the Poincaré duality. Besser then proved the following theorem.

**Theorem 1.1.1** ([Bes00, Theorem 1.2]). For any $Z = \sum_{j} n_jZ_j \in A^i(X)_0$, $\text{AJ}(Z)$ is the functional on $F^{d-i+1}H^{2d-2i+1}_{\text{dR}}(X_K)$ such that, for any $\omega \in F^{d-i+1}H^{2d-2i+1}_{\text{dR}}(X_K)$,

$$\text{AJ}(Z)(\omega) = \int_Z \omega := \sum_{j} n_j \text{tr}_{Z_j} \iota_{Z_j}^*\bar{\omega},$$

where

- $\bar{\omega} \in H^{2d-2i+1}_{\text{dR}}(X, d - i + 1)$ is a lift of $\omega$ via the projection $pr_{\text{fp}}$;
- $\iota_{Z_j} : Z_j \hookrightarrow X$ is the natural closed immersion;
- $\text{tr}_{Z_j} : H^{2d-2i+1}_{\text{dR}}(Z_j, d - i + 1) \cong H^{2d-2i}_{\text{dR}}(Z_j, K) \cong K$ is the canonical isomorphism induced from $\Omega$. 


1.2 An overview of the present paper

Immediately from the original construction, there are two natural questions:

- Is there a theory of finite polynomial cohomology for general varieties?
- Is there a theory of finite polynomial cohomology with non-trivial coefficients?

The former is studied by Besser, D. Loeffler and S. Zerbes in [BLZ16] for varieties that is not required to have good reduction, by using methods developed by J. Nekovář and W. Nizioł in [NN16].

However, to the authors’s knowledge, there seems to be no literature on finite polynomial cohomology with non-trivial coefficients. It is then reasonable to expect the existence of a theory of finite polynomial cohomology with non-trivial coefficients. Instead, they worked with finite polynomial cohomology (with trivial coefficient) of the Kuga–Sato variety over the modular curve. Nonetheless, the methods in loc. cit. already use implicitly the idea of finite polynomial cohomology with non-trivial coefficients. It is then

The purpose of this paper is to initiate the study of finite polynomial cohomology with non-trivial coefficients. The first question one encounters is what the eligible coefficients are. Fortunately, the work of K. Yamada ([Yam20]) provides a suitable candidate. Indeed, given a proper weak formal scheme \( X \) over \( \mathcal{O}_K \) such that its special fibre \( X_0 \) is of strictly semistable and its dagger generic fibre is smooth over \( K \), Yamada defined a category \( \text{Syn}(X_0, X, \mathcal{X}) \) of syntomic coefficients. Objects of this category are certain overconvergent \( F \)-isocrystals that admits a filtration that satisfies Griffiths’s transversality. Then, given an object \( (\mathscr{E}, \Phi, \text{Fil}^i) \in \text{Syn}(X_0, X, \mathcal{X}) \), \( n \in \mathbb{Z} \), and a (suitable) polynomial \( P \), we are able provide a definition of syntomic \( P \)-cohomology with coefficients in \( \mathscr{E} \), denoted by \( \text{R} \Gamma^\text{syn,P}(X, \mathscr{E}, n) \).

Now, suppose that there is a proper smooth scheme \( X \) over \( \mathcal{O}_K \) such that its \( \varpi \)-adic weak completion agrees with \( \mathcal{X} \). In this case, we are able provide a definition of finite polynomial cohomology groups \( H^i_n(X, \mathscr{E}, n) \) for any \( (\mathscr{E}, \Phi, \text{Fil}^i) \in \text{Syn}(X_0, X, \mathcal{X}) \) and any \( n \in \mathbb{Z} \). We summarise some of its basic properties in the following theorem.

**Theorem 1.2.1** (Corollary [1.1.8], Corollary [1.2.3] and Proposition [4.2.5]). Let \( X \) be a proper smooth scheme over \( \mathcal{O}_K \) of relative dimension \( d \) with \( \varpi \)-adic weak completion \( \mathcal{X} \). Let \( (\mathscr{E}, \Phi, \text{Fil}^i) \in \text{Syn}(X_0, X, \mathcal{X}) \).

(i) For any \( i \in \mathbb{Z}_{\geq 0} \) and any \( n \in \mathbb{Z} \), we have a fundamental short exact sequence as in (1)

\[
0 \to \frac{H^{i-1}_\text{dR}(\mathcal{X}, \mathscr{E})}{H^n\text{Fil}^{i-1}_\text{dR}(\mathcal{X}, \mathscr{E})} \xrightarrow{\text{pr}_e} H^i_{\text{tp}}(\mathcal{X}, \mathscr{E}, n) \xrightarrow{\text{pr}_e} F^nH^i_{\text{dR}}(\mathcal{X}, \mathscr{E}) \to 0.
\]

(ii) There is a perfect pairing

\[
H^i_{\text{tp}}(\mathcal{X}, \mathscr{E}, n) \times H^{2d-i+1}_{\text{tp}}(\mathcal{X}, \mathscr{E}^\vee, d - n + 1) \to K.
\]

(iii) For any irreducible closed subscheme \( t : Z \to X \) which is smooth over \( \mathcal{O}_K \) and of codimension \( i \), we have a pushforward map \( \nu_* : H^i_{\text{tp}}(Z, t^* \mathscr{E}, n) \to H^{i+2i}_{\text{tp}}(\mathcal{X}, \mathscr{E}, n + i) \), where \( Z \) is the dagger space associated to the \( \varpi \)-adic weak completion of \( Z \).

The next question we asked is whether there exists an Abel–Jacobi map for finite polynomial cohomology with coefficients. We first describe the relation between finite polynomial cohomology with coefficients and Coleman’s integration of modules with connections. After this is accomplished, we try to interpret the Abel–Jacobi map as certain kind of integration similar to the complex geometry case. An immediate problem
one encounters is which group should take place of $A^i(X)$ when non-trivial coefficients are involved. More precisely, to the author’s knowledge, there does not exist a notion of ‘cycle class group with coefficients’. In this paper, we proposed a candidate for this purpose:

$$A^i(X, \mathcal{E}) := \bigoplus_{Z \in Z^i(X)} H^0_{\text{dR}}(Z, \mathcal{E}).$$

We then consider certain subgroup $A^i(X, \mathcal{E})_0 \subset A^i(X, \mathcal{E})$, which is an analogue to $A^i(X)_0$.

We have the following result.

**Theorem 1.2.2** (Theorem 5.2.3). There exists a finite polynomial Abel–Jacobi map for $(\mathcal{E}, \Phi, \text{Fil}^*)$,

$$\text{AJ}_{\text{fp}} = \text{AJ}_{\text{fp}, \mathcal{E}} : A^i(X, \mathcal{E})_0 \to \left( F^{d-i+1} H^2_{\text{dR}}(X, \mathcal{E}^\vee) \right)^{\vee}$$

such that for any $(\theta_Z)_Z \in A^i(X, \mathcal{E})_0$ and any $\omega \in F^{d-i+1} H^2_{\text{dR}}(X, \mathcal{E}^\vee)$,

$$\text{AJ}_{\text{fp}}((\theta_Z)_Z)(\omega) = \sum_Z \left( \int_Z \omega \right)(\theta_Z).$$

The theory of finite polynomial cohomology with coefficients has potential applications to study the arithmetic of automorphic forms. In the final section of this paper, we illustrate this by reproducing the formula in [DR14, Theorem 3.8] and the formula in [BDP13, Proposition 3.18 & Proposition 3.21] in the case of compact Shimura curves over $\mathbb{Q}$. More precisely, let $X$ be the compact Shimura curve over $\mathbb{Q}$, parametrising false elliptic curves of level away from $p$ and let $\pi : A_{\text{univ}} \to X$ be the universal false elliptic curve over $X$. After fixing an idempotent $\varepsilon \in M_2(\mathbb{Z}_p) \setminus \{1, 0\}$, we consider $\omega := \varepsilon \pi_\ast \Omega^1_{A_{\text{univ}}/X}$, $\mathcal{H} := \varepsilon R^1 \pi_\ast \Omega^*_{A_{\text{univ}}/X}$ and

$$\omega^k := \omega \otimes k, \quad \mathcal{H}^k := \text{Sym}^k \mathcal{H}.$$

The following theorem summarises the application in the direction of diagonal cycles:

**Theorem 1.2.3** (Theorem 6.2.5 and Corollary 6.2.7). Let $(k, \ell, m) \in \mathbb{Z}^3$ such that

- $k + \ell + m \in 2 \mathbb{Z}$,
- $2 < k \leq \ell \leq m$ and $m < k + \ell$,
- $\ell + m - k = 2t + 2$ for some $t \in \mathbb{Z}_{\geq 0}$.

and write

$$r_1 := k - 2, \quad r_2 := \ell - 2, \quad r_3 := m - 2.$$

Then, there exists a cycle $\Delta_{2,2,2}^{k,\ell,m} \in A^2(X, (\mathcal{H}^{r_1} \otimes \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3})^{\vee})_0$, which is called the diagonal cycle with coefficients in $(\mathcal{H}^{r_1} \otimes \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3})^{\vee}$, such that for any

$$\eta \in H^1_{\text{dR}}(X, \mathcal{H}^{r_1}) \quad \omega_2 \in F^1 H^1_{\text{dR}}(X, \mathcal{H}^{r_2}) = H^0(X, \omega^\ell), \quad \text{and} \quad \omega_3 \in F^1 H^1_{\text{dR}}(X, \mathcal{H}^{r_3}) = H^0(X, \omega^m),$$

we have the formula

$$\text{AJ}_{\text{fp}}(\Delta_{2,2,2}^{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3) = (\eta, \xi(\omega_2, \omega_3))_{\text{dR}},$$

where $\xi(\omega_2, \omega_3)$ is an element in $H^1_{\text{dR}}(X, \mathcal{H}^{r_1}(-t))$ whose definition only depends on $\omega_2$ and $\omega_3$.

The following theorem summarises the direction of application to cycles attached to isogenies:
Theorem 1.2.4 (Theorem 6.3.1). Let \( x = (A, i, \alpha) \) and \( y = (A', i', \alpha') \) be two \( K \)-rational points on \( X \) and suppose there exists an isogeny \( \varphi : (A, i, \alpha) \to (A', i', \alpha') \). For any \( r \in \mathbb{Z}_{>0} \), define the sheaf \( \mathcal{H}^r := \mathcal{H}^r \otimes \text{Sym}^r \varepsilon H^1_{\text{dR}}(A) \) on \( X \). Then, there exists a unique cycle \( \Delta_x \in A^1(X, \mathcal{H}^r(r))_0 \) such that for any \( \omega \in H^0(X, \omega^{r+2}) \) and any \( \alpha \in \text{Sym}^r \varepsilon H^1_{\text{dR}}(A) \), we have
\[
AJ_{fp}(\Delta_x)(\omega \otimes \alpha) = \langle \varphi^*(F_\omega(y)), \alpha \rangle,
\]
where \( F_\omega \) is the Coleman integral of the form \( \omega \) and the pairing is the Poincaré pairing on \( \text{Sym}^r \varepsilon H^1_{\text{dR}}(A) \).

We remark that the formulae we obtained in Theorem 1.2.3 and Theorem 1.2.4 can be linked to special \( L \)-values as in [DR14] and [BDP13] respectively. As our purpose is to demonstrate how finite polynomial cohomology with coefficients can be used in practice, we do not pursue such a relation in this paper.

1.3 Some further remarks

Our theory of finite polynomial cohomology with coefficients leads to several further questions that we would like to study in future projects. Let us briefly discuss them in the following remarks.

A comparison conjecture. When the polynomial \( P = 1 - q^{-n}T \), where \( q \) is the cardinality of the residue field of \( \mathcal{O}_K \), we follow the traditional terminology and call \( R\Gamma_{\text{syn}}(X, \mathcal{E}, n) := R\Gamma_{\text{syn}, 1-q^{-n}T}(X, \mathcal{E}, n) \) the syntomic cohomology of \( X \) with coefficients in \( \mathcal{E} \) twisted by \( n \). When the coefficient \( \mathcal{E} \) is the trivial coefficient, it follows from the result of J.-M. Fontaine and W. Messing ([FM89]) that there is a canonical quasi-isomorphism
\[
\tau_{\leq n} R\Gamma_{\text{syn}}(X, n) \simeq \tau_{\leq n} R\Gamma_{\text{et}}(X_{\mathbb{C}_p}, \mathbb{Q}_p(n)).
\]
Inspired by this classical result, we make the following conjecture.

Conjecture 1 (Syntomic–étale comparison with coefficients). There exists a suitable subcategory of étale local systems on \( X \), such that for any such étale local system \( \mathcal{E} \),
- there exists an associated overconvergent \( F \)-isocrystal \((\mathcal{E}, \Phi)\);
- for any \( n \in \mathbb{Z} \), there exists a well-defined syntomic cohomology \( R\Gamma_{\text{syn}}(X, \mathcal{E}, n) \); and
- there is a canonical quasi-isomorphism
  \[
  \tau_{\leq n} R\Gamma_{\text{syn}}(X, \mathcal{E}, n) \simeq \tau_{\leq n} R\Gamma_{\text{et}}(X_{\mathbb{C}_p}, \mathbb{E}(n)).
  \]

We remark that, although we employed the category of syntomic coefficients introduced in [Yam20], we believe this category is too restrictive. More precisely, Yamada’s syntomic coefficients required the sheaves to be unipotent. However, for applications to the arithmetic of automorphic forms, we encounter sheaves that are not unipotent but still with some nice properties. Therefore, we do not directly conjecture a ‘syntomic étale local system’ that corresponds to Yamada’s syntomic coefficients. Instead, we formulate the above conjecture in a rather vague manner. But we hope that it will be more useful for arithmetic applications.

A motivic expectation. Readers might feel that our candidate \( A^i(X, \mathcal{E}) \) of cycle class group with coefficients is too ad hoc. However, when \( \mathcal{E} = \mathcal{O}_{X_0/\mathcal{O}_{K_0}} \) is the trivial coefficient (i.e., the structure sheaf of the overconvergent site of the special fibre \( X_0 \)), one can check that the classical \( p \)-adic Abel–Jacobi map factors as
\[
AJ : A^i(X)_0 \to A^i(X, \mathcal{O}_{X_0/\mathcal{O}_{K_0}})_0 \to (F^{d-i+1} H^d_{\text{dR}}(X))^\vee.
\]
This justifies the use of \( A^i(X, \mathcal{E}) \).

However, the classical cycle class group \( A^i(X) \) is strongly linked to the theory of motivic cohomology and algebraic \( K \)-theory, while it is not obvious that our cycle class group \( A^i(X, \mathcal{E}) \) has such a link at the first glance. Nevertheless, we still believe such a link may exist as stated in the next conjecture.
Conjecture 2. There exists a suitable category of coefficients for the motivic cohomology of $X$ such that for any coefficient $E_{\text{mot}}$ in this category

- there exists an associated overconvergent $F$-isocrystal $(E, \Phi)$;
- there exists a canonical map from the motivic cohomology of $X$ with coefficients in $E_{\text{mod}}$ to $A^i(X, E)$, i.e., $H^i_{\text{mot}}(X, E_{\text{mot}}) \to A^i(X, E_{\text{mod}})$; and
- there exists an Abel–Jacobi map $\text{AJ}: H^i_{\text{mot}}(X, E_{\text{mot}}) \to (F^{d-i+1}H^{2d-2i+1}_{d\text{R}}(X, E^\vee))^\vee$ such that it factors as

$$
\begin{array}{ccc}
H^i_{\text{mot}}(X, E_{\text{mot}}) & \xrightarrow{\text{AJ}} & (F^{d-i+1}H^{2d-2i+1}_{d\text{R}}(X, E^\vee))^\vee \\
& & \xrightarrow{\text{AJ}_{fp}} A^i(X, E)_{0}
\end{array}
$$

We remark that the relation between syntomic cohomology and motivic cohomology without coefficients is directly established by Ertl–Nizioł in [EN19]. It is shown in loc. cit. that such a link between syntomic cohomology and motivic cohomology is tightened with the relation between syntomic cohomology and étale cohomology. It could be possible that the method of Ertl–Nizioł is generalisable to the setting with non-trivial coefficients and provide satisfactory answers to Conjecture 1 and Conjecture 2 simultaneously. We wish to come back to this in our future study.

Arithmetic applications. To the authors’s knowledge, there are at least two possible directions for arithmetic applications:

(I) As mentioned above, we applied our theory to study the arithmetic of compact Shimura curves over $\mathbb{Q}$. However, the relation between the Abel–Jacobi map and arithmetic has been widely investigated for more general Shimura varieties. Note that, without the theory of finite polynomial cohomology with coefficients, the process of obtaining arithmetic information is quite indirect: often one starts with a non-PEL-type Shimura variety, then one needs to use the Jacquet–Langlands correspondence to move to a PEL-type Shimura variety and use ‘Liebermann’s trick’. We believe that, with the theory of finite polynomial cohomology with coefficients and with some mild modification if necessary, one should be able to gain arithmetic information directly without bypassing the aforementioned process.

(II) In [BdS16], Besser and E. de Shalit used techniques of finite polynomial cohomology to construct $L$-invariants for $p$-adically uniformised varieties. When such $p$-adically uniformised variety is a Shimura variety, it is a natural question to ask whether the method of Besser–de Shalit can be generalised to obtain $L$-invariants of automorphic forms on this Shimura variety of higher weights. The first step to answer this question is, of course, to introduce coefficients to the cohomology groups they study. We hope that our work will shed some light in this direction and we wish to come back to this question in our future study.

Structure of the paper

In [2] and [3], we recall some preliminaries of the rigid Hyodo–Kato theory developed by V. Ertl and Yamada in [EY21; Yam20]. More precisely, we recall the definition of weak formal schemes in [2.1] and its relation with dagger spaces in [2.2]. For the convenience of the readers, we briefly discuss the log structures that
one can put on weak formal schemes. Then, we follow the strategy of [Yam20] to introduce the category of overconvergent $F$-isocrystals in §3.1 and the Hyodo–Kato theory with coefficients in §3.2.

In §4, we provide a definition of syntomic $P$-cohomology with coefficients. Our definition works not only for proper smooth weak formal schemes over $\mathcal{O}_K$, but also for proper weak formal schemes over $\mathcal{O}_K$ with strictly semistable reduction. Such a definition is inspired by [BLZ16]. In §4, we prove some basic properties of this cohomology theory. By applying the results of Ertl–Yamada in [EY20], we discuss the cup products and pushforward maps for syntomic $P$-cohomology with coefficients in §4.2.

We start with §5 with some general definition. In §5.1, we motivate the definition of the Abel–Jacobi map by explaining the relation between the finite polynomial cohomology and Coleman’s integration, in the case that $X$ is a proper smooth curve over $\mathcal{O}_K$. Our Abel–Jacobi map with coefficients is constructed in §5.2. For this construction, we assume that our weak formal scheme is the weak formal completion of a proper smooth scheme $X$ over $\mathcal{O}_K$. In §5.3, we briefly discuss a possible direction on how the Abel–Jacobi maps can be generalised to the case when $X$ has semistable reduction. We remark here that such a theory is far from satisfactory.

Finally, we apply our theory to study arithmetic of compact Shimura curves over $\mathbb{Q}$ in §6. We will recall the definition and basic properties of compact Shimura curves over $\mathbb{Q}$ in §6.1. The construction of the diagonal cycle $\Delta^k_{2,2,2}$ and the formula in Theorem 1.2.3 are provided in §6.2 while the construction of the cycle $\Delta_x$ and the formulae in Theorem 1.2.4 are shown in §6.3. Note that, as mentioned above, the coefficients introduced in [Yam20] are too restrictive for arithmetic applications. However, our theory works for more general coefficients that satisfies certain conditions. These conditions are stated in the beginning of §6. In fact, the syntomic coefficients introduced in [Yam20] and $\mathcal{H}^k$ satisfy these conditions.

Acknowledgement

The project initially grew out from the authors’s curiosity on whether a theory of finite polynomial cohomology with coefficients exist during a working seminar with Giovanni Rosso and Martí Roset. We would like to thank them for intimate discussions. We thank Adrian Iovita for suggesting the computation in §6.3. Many thanks also go to Antonio Cauchi for explaining the motivic viewpoint of our cycle class group, which leads to the formulation of Conjecture 2. We would also like to thank Kazuki Yamada for answering our questions about his work.

Conventions and notations

Throughout this paper, we fix the following:

- Let $p$ be a positive rational prime number and let $K$ be a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field $k$. We let $q := \# k$ and so $q = p^e$ for some $e \in \mathbb{Z}_{>0}$. We also fix a uniformiser $\wp \in \mathcal{O}_K$.

- We denote by $K_0 := W(k)[1/p]$ the maximal unramified field extension of $\mathbb{Q}_p$ inside $K$ and write $\mathcal{O}_{K_0} = W(k)$ for its ring of integers.

- Choose once and forever an algebraic closure $\overline{K}$ of $K$ and denote by $\mathbb{C}_p$ the $p$-adic completion of $\overline{K}$. We normalise the $p$-adic norm $| \cdot |$ on $\mathbb{C}_p$ so that $|p| = 1/p$.

- For any commutative square

$$
\begin{array}{ccc}
A^* & \xrightarrow{\alpha} & B^* \\
\downarrow & & \downarrow \\
C^* & \xrightarrow{\gamma} & D^*
\end{array}
$$

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- For any commutative square

$$
\begin{array}{ccc}
A^* & \xrightarrow{\alpha} & B^* \\
\downarrow & & \downarrow \\
C^* & \xrightarrow{\gamma} & D^*
\end{array}
$$
of complexes (of abelian groups), we write

\[
\begin{bmatrix}
A^\bullet & \xrightarrow{\alpha} & B^\bullet \\
\downarrow & & \downarrow \\
C^\bullet & \xrightarrow{\gamma} & D^\bullet
\end{bmatrix}
\]

:= Cone (\text{Cone}(\alpha)[−1] \to \text{Cone}(\gamma)[−1]) [−1],

where the map \text{Cone}(\alpha)[−1] \to \text{Cone}(\gamma)[−1] is induced from the vertical maps from the square.

• In principle, symbols in Gothic font (e.g., \mathcal{X}, \mathcal{Y}, \mathcal{Z}) stand for formal schemes or weak formal schemes; symbols in calligraphic font (e.g., \mathcal{X}, \mathcal{Y}, \mathcal{Z}) stand for rigid analytic spaces or dagger spaces; and symbols in script font (e.g., \mathcal{O}, \mathcal{F}, \mathcal{E}) stand for sheaves (over various geometric objects).

2 Preliminaries I: Geometry

The purpose of this section is to recall some terminologies in algebraic geometry that will play essential roles in this paper. More precisely, we recall the theory of weak formal schemes in §2.1 by following [EY21, §1] and recall the notion of dagger spaces in the language of adic spaces in §2.2 by following [Vez18, §2]. Finally, we briefly discuss log structures in §2.3. We claim no originality in this section.

2.1 Weak formal schemes

In this subsection, we fix a noetherian ring \( R \) with an ideal \( I \).

**Definition 2.1.1.** Let \( A \) be an \( R \)-algebra and let \( \hat{A} \) be the \( I \)-adic completion of \( A \).

(i) The \( I \)-adic weak completion \( A^\dagger \) of \( A \) is the \( R \)-subalgebra of \( \hat{A} \), consisting of elements for the form

\[
f = \sum_{i=0}^{\infty} P_i(a_1, ..., a_r),
\]

where \( a_1, ..., a_r \in A \) and \( P_i(T_1, ..., T_r) \in I^i R[T_1, ..., T_r] \), such that there exists a constant \( C > 0 \) satisfying

\[
C(i + 1) \geq \deg P_i
\]

for all \( i \geq 0 \).

(ii) We say \( A \) is weakly complete (resp., weakly complete finitely generated (wcfg)) if \( A^\dagger = A \) (resp., there exists a surjective \( R \)-algebra morphism \( R[X_1, ..., X_n]^\dagger \to A \)).

Given a weakly complete \( R \)-algebra \( A \), it is regarded naturally as a topological \( R \)-algebra with respect to the \( I \)-adic topology. More generally, if \( A \) is wcfg, we also regard \( A \) as a topological \( R \)-algebra with respect to the \( I \)-adic topology. Hence, the surjection

\[
R[X_1, ..., X_n]^\dagger \to A
\]

is a continuous \( R \)-algebra morphism.

**Definition 2.1.2 ([EY21, Definition 1.3]).** A topological \( R \)-algebra \( A \) is said to be pseudo-weakly complete finitely generated (pseudo-wcfg) if there exists an ideal of definition \( J \subset A \) and a finite generating system \( a_1, ..., a_m \) of \( J \) such that the morphism

\[
R[T_1, ..., T_m] \to A, \quad T_i \mapsto a_i
\]

makes \( A \) an \((I, T_1, ..., T_m)\)-adically wcfg \( R[T_1, ..., T_m]\)-algebra. Pseud-wcfg \( R \)-algebras form a category whose morphisms are given by continuous \( R \)-algebra homomorphisms.
Unwinding this definition, given a pseudo-wcfg $R$-algebra $A$, there exists $n \in \mathbb{Z}_{\geq 0}$ and a surjective continuous $R[T_1, \ldots, T_m]$-algebra morphism

$$\xi : R[T_1, \ldots, T_m][X_1, \ldots, X_n] \rightarrow A,$$

where the weak completion is taken with respect to the $(I, T_1, \ldots, T_m)$-adic topology. We call such a $\xi$ a representation of $A$. Note that, by [EY21, Corollary 1.5], the condition of being pseudo-wcfg is independent to the choice of the ideals of definition and the choice of generating systems.

Our next goal is to globalise the aforementioned terminology. To this end, we have to study localisations of pseudo-wcfg $R$-algebras, resembling the theory of schemes. Let $A$ be a pseudo-wcfg $R$-algebra with a representation $R[T_1, \ldots, T_m][X_1, \ldots, X_n] \rightarrow A$. For any $f \in A$, define

$$A_f^\dagger := \text{the } (I, T_1, \ldots, T_m)\text{-adic weak completion of } A_f.$$

Since the topology of $A$ is independent to the choice of ideals of definitions and independent to the choice of generating systems, $A_f^\dagger$ is independent to the choice of representations $\xi$. Consequently, we can define the following ringed space

$$\text{Spwf } A := (\text{Spec } A/J, \mathcal{O}_{\text{Spwf } A}),$$

where the structure sheaf $\mathcal{O}_{\text{Spwf } A}$ is defined by

$$\mathcal{O}_{\text{Spwf } A} : \text{Spec}(A/J)_f \mapsto A_f^\dagger$$

for any $f \in A$. Note that the underlying topological space Spec $A/J$ is independent to the choice of ideals of definition and so $\mathcal{O}_{\text{Spwf } A}$ is well-defined.

**Definition 2.1.3** ([EY21, Definition 1.9]). A weak formal scheme $\mathfrak{X}$ over $R$ is a ringed space which admits an open covering $\{\mathfrak{U}_\lambda\}_{\lambda \in \Lambda}$ such that each $\mathfrak{U}_\lambda$ is isomorphic to Spwf $A$ for some pseudo-wcfg $R$-algebra $A$. The category of weak formal schemes over $R$ is denoted by $\text{FSch}_R^\dagger$.

**Remark 2.1.4.** From the construction, one sees that we have the following natural functors:

(i) Let $\text{Sch}_{R/I}$ be the category of schemes over $R/I$. Then, we have a functor

$$\text{FSch}_R^\dagger \rightarrow \text{Sch}_{R/I}, \quad \mathfrak{X} \mapsto X_0$$

locally given by

$$\text{Spwf } A = (\text{Spec } A/J, \mathcal{O}_{\text{Spwf } A}) \mapsto \text{Spec } A/J.$$

(ii) Let $\text{FSch}_R$ be the category of formal schemes over $R$. Then, we have a functor

$$\text{FSch}_R^\dagger \rightarrow \text{FSch}_R, \quad \mathfrak{X} \mapsto \hat{\mathfrak{X}}$$

locally given by

$$\text{Spwf } A = (\text{Spec } A/J, \mathcal{O}_{\text{Spwf } A}) \mapsto \text{Spf } A = (\text{Spec } A/J, \mathcal{O}_{\text{Spf } A}),$$

where the structure sheaf $\mathcal{O}_{\text{Spf } A}$ assigns each distinguished open subspace Spec$(A/J)_f$ to the completion $\hat{A}_f$ of $A_f$.

Consequently, for any $\mathfrak{X} \in \text{FSch}_R^\dagger$, we shall refer $X_0$ as the associated scheme over $R/I$ and $\hat{\mathfrak{X}}$ as the associated formal scheme over $R$. ■
2.2 Dagger spaces

In this subsection, let $F$ be a nonarchimedean field of mixed characteristic $(0, p)$ with valuation ring $\mathcal{O}_F$. Let $m_F$ be the maximal ideal of $\mathcal{O}_F$ and let $\varpi_F \in m_F$ be a pseudouniformiser of $F$ that divides $p$. For any $n \in \mathbb{Z}_{\geq 0}$, define

$$F[X_1, ..., X_n]^\dagger := \lim_{h \to 0} F(\varpi_F^{1/h} X_1, ..., \varpi_F^{1/h} X_n)$$

$$= \left\{ \sum_{(i_1, ..., i_n) \in \mathbb{Z}_{\geq 0}^n} a_{i_1, ..., i_n} X_1^{i_1} \cdots X_n^{i_n} \in F[X_1, ..., X_n] : \lim_{\delta \to 0} |a_{i_1, ..., i_n}| \delta^{\sum i_j} = 0 \text{ for some } \delta \in \mathbb{R}_{> 0} \right\}.$$

The algebra $F[X_1, ..., X_n]^\dagger$ is equipped with the $p$-adic topology and the $p$-adic completion of $F[X_1, ..., X_n]^\dagger$ is the Tate algebra $F(X_1, ..., X_n)$.

More generally, a **dagger algebra** $A$ is a topological $F$-algebra which is isomorphic to $F[X_1, ..., X_n]^\dagger/(f_1, ..., f_r)$ for some $n \in \mathbb{Z}_{\geq 0}$ and some $f_i \in F(\varpi_F^{1/N} X_1, ..., \varpi_F^{1/N} X_n)$ for some sufficiently large $N$. Note that, if $\hat{A}$ denotes the completion of $A$, then $\hat{A} \simeq F(X_1, ..., X_n)/(f_1, ..., f_r)$.

For any dagger algebra $A$ over $F$, we define the topological space

$$\langle \text{Spa}^\dagger A := |\text{Spa}(\hat{A}, \hat{A}^\circ)|,$$

where $\hat{A}^\circ$ is the ring of power bounded elements of $\hat{A}$. For any $f_1, ..., f_n, g \in A \subset \hat{A}$, we define the rational subspace

$$|\text{Spa}^\dagger A[f_1, ..., f_n/g]| := \{|f_i| \leq |g| \text{ for all } i = 1, ..., n\} \subset |\text{Spa}^\dagger A|.$$ 

By [GK00, Proposition 2.8], we see that rational subspaces in $|\text{Spa}^\dagger A|$ form a basis of the topology. Consequently, we can consider the following ringed space

$$\text{Spa}^\dagger A := (|\text{Spa}^\dagger A|, \mathcal{O}_{\text{Spa}^\dagger A}),$$

where the structure sheaf $\mathcal{O}_{\text{Spa}^\dagger A}$ is defined by

$$\mathcal{O}_{\text{Spa}^\dagger A} : |\text{Spa}^\dagger A[f_1, ..., f_n/g]| \mapsto A[f_1, ..., f_n/g] : A[X_1, ..., X_n] / (gX_i - f_i).$$

Here,

$$A[X_1, ..., X_n] : A \otimes^\dagger F[X_1, ..., X_n]^\dagger$$

with the tensor product $\otimes^\dagger$ in the category of dagger algebras over $F$ (see [op. cit., Paragraph 1.16]).

**Definition 2.2.1.** A **dagger space** $\mathcal{X}$ over $F$ is a ringed space which admits an open covering $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ such that each $\mathcal{U}_\lambda$ is isomorphic to $\text{Spa}^\dagger A$ for some dagger algebra $A$ over $F$. We denote by $\text{Rig}_F^\dagger$ the category of dagger spaces over $F$.

**Remark 2.2.2.** Let $\text{Rig}_F$ be the category of rigid analytic spaces over $F$. Then, from the construction, one sees that there is a natural functor

$$\text{Rig}_F^\dagger \to \text{Rig}_F, \quad \mathcal{X} \mapsto \hat{\mathcal{X}}$$

locally given by

$$\text{Spa}^\dagger A \mapsto \text{Spa}(\hat{A}, \hat{A}^\circ).$$

Hence, for any dagger space $\mathcal{X}$ over $F$, we call $\hat{\mathcal{X}}$ the **associated rigid analytic space**. 

---

\[\text{In this article, we will always view rigid analytic spaces as adic spaces.}\]
As an analogue of the relationship between formal schemes and rigid analytic spaces, one can also associate a dagger space over $F$ to a weak formal scheme over $\mathcal{O}_F$. This phenomenon was first established in [LM13]. We recall this construction by following [EY21] for the notion of weak formal schemes introduced in the previous subsection.

**Proposition 2.2.3 ([EY21, Proposition 1.24]).** Let $A$ be a pseudo-wcfg $\mathcal{O}_F$-algebra with ideal of definition $J$ and a generating system $a_1, \ldots, a_n \in J$. For any $m \in \mathbb{Z}_{\geq 0}$, define

$$A_m := A \left[ \frac{a_1^{m_1} \cdots a_n^{m_n}}{p} : m_i \in \mathbb{Z}_{\geq 0}, \sum m_i = m \right]^\dagger$$

$$= A \left[ X_{m_1, \ldots, m_n} : m_i \in \mathbb{Z}_{\geq 0}, \sum m_i = m \right]^\dagger / (pX_{m_1, \ldots, m_n} - a_1^{m_1} \cdots a_n^{m_n}).$$

(i) Each $A_m$ is wcfg over $\mathcal{O}_F$ and is independent to the choice of $a_1, \ldots, a_n$.

(ii) The wcfg $\mathcal{O}_F$-algebras $A_m$ form an inductive system. Define

$$A_\eta := \left( \lim_{\longrightarrow} A_m \right) [1/p].$$

Then $A_\eta$ is a dagger algebra over $F$ and is independent to the choice of $J$.

Thanks to the independence in the proposition above, we have a natural functor

$$\text{FSch}^\dagger_{\mathcal{O}_F} \rightarrow \text{Rig}^\dagger_F, \quad \mathfrak{X} \mapsto \mathfrak{X} = \mathfrak{X}_\eta,$$

locally given by

$$\text{Spwf} A \mapsto \text{Spa}^\dagger A_\eta.$$ 

For any $\mathfrak{X} \in \text{FSch}^\dagger_{\mathcal{O}_F}$, we then call $\mathfrak{X} = \mathfrak{X}_\eta$ the **associated dagger space** or the **dagger generic fibre** of $\mathfrak{X}$.

### 2.3 Log structure

The purpose of this subsection is to briefly discuss how one can equip a weak formal scheme a log structure. In particular, we shall make precise the definition of strictly semistable weak formal schemes. We encourage readers who are unfamiliar with the language of log geometry to consult [Ill02] for more details.

Let $R$ be a noetherian ring with an ideal $I$. Let $\mathfrak{X}$ be a weak formal scheme over $R$ with respect to the $I$-adic topology. Recall that a **pre-log structure** on $\mathfrak{X}$ is a sheaf of (commutative) monoids $\mathcal{M}$ on the étale site $\mathfrak{X}_{\text{et}}$ of $\mathfrak{X}$ together with a morphism of sheaves of monoids $\alpha : \mathcal{M} \rightarrow \mathcal{O}_{\mathfrak{X}_{\text{et}}}$.

It is furthermore called a **log structure** if the induced morphism

$$\alpha : \alpha^{-1} \mathcal{O}_X^\times \rightarrow \mathcal{O}_{\mathfrak{X}_{\text{et}}}^\times$$

is an isomorphism. Recall also that given a pre-log structure $\mathcal{M}$ on $\mathfrak{X}$, there is an **associated log structure** $\mathcal{M}^\circ$ on $\mathfrak{X}$, constructed by the pushout diagram

$$\begin{array}{ccc}
\alpha^{-1} \mathcal{O}_{\mathfrak{X}_{\text{et}}}^\times & \xrightarrow{\alpha} & \mathcal{O}_{\mathfrak{X}_{\text{et}}} \\
\downarrow & & \downarrow \\
\mathcal{M} & \longrightarrow & \mathcal{M}^\circ
\end{array}$$
Finally, recall that morphisms of weak formal schemes with log structures are those that are compatible with the log structures. The morphisms that induced isomorphisms on the log structures are called strict morphisms or exact morphisms.\footnote{For other adjectives for morphisms of log weak formal schemes, e.g., smooth and étale morphisms, we refer the readers to [Ill02 §1.5].}

**Definition 2.3.1.** Let $\mathfrak{X}$ be a weak formal scheme with a log structure $\alpha : \mathcal{M} \to \mathcal{O}_{X, \text{et}}$. Let $P$ be a monoid and we abuse the notation to denote the associated constant sheaf of monoids on $\mathfrak{X}$ again by $P$. A chart of $\mathfrak{X}$ modeled on $P$ is a morphism of sheaves of monoids $\theta : P \to \mathcal{M}$ such that the log structure associated with $\alpha \circ \theta$ is isomorphic to $\mathcal{M}$.

The following example play an essential role in this paper.

**Example 2.3.2.** Let $R = \mathcal{O}_K$ with $I = (\varpi)$. Let $n \in \mathbb{Z}_{\geq 0}$ and $r \leq n$, consider

$$\mathfrak{X} = \text{Spwf} \mathcal{O}_K[X_1, \ldots, X_n]^{\dagger}/(X_1 \cdots X_r - \varpi).$$

Then, $\mathfrak{X}$ can be equipped with a log structure given by the chart

$$\mathbb{Z}_{\geq 0} \to \mathcal{O}_K[X_1, \ldots, X_n]^{\dagger}/(X_1 \cdots X_r - \varpi), \quad (a_1, \ldots, a_n) \mapsto X_1^{a_1} \cdots X_n^{a_n}.$$

In this case, the normal crossing divisor in $\mathfrak{X}$ defined by $X_{r+1} \cdots X_n$ is called the horizontal divisor of $\mathfrak{X}$.

Finally, by letting $\mathcal{O}_K^{\log = \varpi}$ be the log weak formal scheme $\text{Spwf} \mathcal{O}_K$ with the log structure given by the chart

$$\mathbb{Z}_{\geq 0} \to \mathcal{O}_K, \quad 1 \mapsto \varpi,$$

one sees easily that $\mathfrak{X}$ is a log formal scheme over $\mathcal{O}_K^{\log = \varpi}$.\hfill \blacksquare

**Definition 2.3.3.** Let $\mathfrak{X}$ be a log formal scheme over $\mathcal{O}_K^{\log = \varpi}$. We say $\mathfrak{X}$ is strictly semistable if, Zariski locally, there is a strict smooth morphism

$$\mathfrak{X} \to \text{Spwf} \mathcal{O}_K[X_1, \ldots, X_n]^{\dagger}/(X_1 \cdots X_r - \varpi),$$

of log weak formal schemes over $\mathcal{O}_K^{\log = \varpi}$. Moreover, the normal crossing divisor locally generated by $X_{r+1} \cdots X_n$ is called the horizontal divisor of $\mathfrak{X}$.

### 3 Preliminaries II: Hyodo–Kato theory

In this section, we recall the theory of Hyodo–Kato cohomology with coefficients by following [Yam20]. To this end, we fix the following notations though out this section:

- We denote by $k^{\log = 0}$ be the log scheme $\text{Spec} k$ with the log structure given by

$$\mathbb{Z}_{\geq 0} \to k, \quad 1 \mapsto 0.$$

- Recall $\mathcal{O}_{K_0} := W(k)$. We similarly denote by $\mathcal{O}_{K_0}^{\log = 0}$ the log weak formal scheme $\text{Spwf} \mathcal{O}_{K_0}$ with log structure given by

$$\mathbb{Z}_{\geq 0} \to \mathcal{O}_{K_0}, \quad 1 \mapsto 0.$$

Moreover, we write $\mathcal{O}_{K_0}^{\log = 0}$ for the weak formal scheme $\text{Spwf} \mathcal{O}_{K_0}$ with the trivial log structure. Moreover, the Frobenius $\phi$ on $W(k)$ induces natural Frobenii on $\mathcal{O}_{K_0}^{\log = 0}$ and $\mathcal{O}_{K_0}^{\log = 0}$, which are still denoted by $\phi$.\hfill \[12]
• As before, we write $O_{K, log=\varnothing}$ for the log weak formal scheme $Spw\ O_K$ with the log structure given by
\[ Z_{\geq 0} \to O_K, \quad 1 \mapsto \varnothing. \]

• Let $S$ be the log weak formal scheme $Spw\ O_{K, 0}$ with the log structure given by
\[ Z_{\geq 0} \to O_{K, 0}, \quad 1 \mapsto T. \]

Consequently, we have natural morphisms of fine weak formal schemes
\[ O_{K, log=0} \xrightarrow{\sigma_0} S \xleftarrow{h_T} O_{K, log=\varnothing} \]
given by
\[ 0 \leftarrow T \mapsto \varnothing. \]
Moreover, we extend the Frobenius $\phi$ to $S$ by setting $\phi : T \mapsto T^p$.

• We fix a scheme $X_0$ over $k$, which is assumed to be strictly semistable over $k_{log=0}$, i.e., Zariski locally, we have a strict smooth morphism
\[ X_0 \to \text{Spec } k[X_1, \ldots, X_n]/(X_1 \cdots X_r) \]
of log schemes over $k_{log=0}$.

### 3.1 Overconvergent $F$-isocrystals and rigid cohomology with coefficients

Our goal in this subsection is to define and study overconvergent $F$-isocrystals. We follow the strategy in [Yam20], introducing first the so-called log overconvergent site of $X_0$. The idea of such a site goes back to B. Le Stum’s work [LS11] without log structure.

**Definition 3.1.1** ([Yam20, Definition 2.18]). Let $\Xi$ be either $S$, $O_{K, log=0}$, $O_{K, log=\varnothing}$ or $O_{K, log=\varnothing}$ and so we have a natural homeomorphic exact closed immersion $i : k_{log=0} \to \Xi$. The **log overconvergent site** $OC(X_0/\Xi)$ of $X_0$ relative to $\Xi$ is defined as follows:

- An object in $OC(X_0/\Xi)$ is a commutative diagram
  \[
  \begin{array}{ccc}
  Z & \xrightarrow{i} & \Xi \\
  \downarrow{h_Z} & & \downarrow{h_{\Xi}} \\
  X_0 & \xrightarrow{\theta} & k_{log=0} \\
  \end{array}
  \]

  where
  - $i : Z \to \Xi$ is a homeomorphic exact closed immersion from a fine log scheme $Z$ over $k_{log=0}$ into a log weak formal scheme $\Xi$ that is flat over $Z_p$;
  - $h_Z : Z \to k_{log=0}$ (resp., $h_{\Xi} : \Xi \to \Xi$) is a morphism of schemes (resp., weak formal schemes);
  - $\theta : Z \to X_0$ is a morphism of log schemes over $k_{log=0}$.

  We shall usually abbreviate such an object as a quintuple $(Z, \Xi, i, h, \theta)$.

- Morphisms in $OC(X_0/\Xi)$ are the obvious ones.

- A cover in $OC(X_0/\Xi)$ is a collection of morphisms $\{f_\lambda : (Z_\lambda, \Xi_\lambda, i_\lambda, h_\lambda, \theta_\lambda) \to (Z, \Xi, i, h, \theta)\}_{\lambda \in \Lambda}$ such that
the induced morphism $Z_\lambda \to Z$ is strict;

- the induced family of morphisms on the generic dagger fibres $\{Z_\lambda \to Z\}_{\lambda \in \Lambda}$ is an open cover for $Z$;

- the induced morphism $Z_\lambda \to Z \times \mathbb{A}$ is an isomorphism for every $\lambda$.

**Remark 3.1.2.** One can also consider the absolute log overconvergent site $OC(k_{\log=0}/T)$, whose objects are just commutative diagrams

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & Z \\
\downarrow^{h_k} & & \downarrow^{h_T} \\
k_{\log=0} & \xrightarrow{\iota} & k_{\log=0}
\end{array}
\]

as in Definition 3.1.1 and whose morphisms and covers are defined similarly as above. Hence, we can view $X_0$ as a presheaf on $OC(k_{\log=0}/T)$, defined as

\[
\begin{aligned}
X_0(Z, 3, i, h) &= \left\{ \text{commutative diagrams} \right. \\
&\left. \begin{array}{ccc}
Z & \xrightarrow{i} & Z \\
\downarrow^{h_k} & & \downarrow^{h_T} \\
k_{\log=0} & \xrightarrow{\iota} & k_{\log=0}
\end{array} \right\}.
\end{aligned}
\]

Consequently, we then can view $OC(X_0/T)$ as the sliced site of $OC(k_{\log=0}/T)$ over the presheaf $X_0$. 

**Lemma 3.1.3.** Let $T$ be either $S$, $O_{K_0}^{log=0}$, $O_{K_0}^{log=\emptyset}$ or $O_K^{log=\emptyset}$.

(i) The category of sheaves on $OC(X_0/T)$ is equivalent to the following category:

- Objects are collections of sheaves $F_Z = F(Z, 3, i, h, \theta)$ on $Z$ for each $(Z, 3, i, h, \theta) \in OC(X_0/T)$ and morphisms

  \[
  \tau_f : f^{-1}F_Z \to F_Z
  \]

  for each morphism $f : (Z, 3, i, h, \theta) \to (Z', 3', i', h', \theta')$ satisfying the usual cocycle condition and $f$ is an isomorphism if $Z$ is an open subset of $Z'$. Here, $f_{\eta}$ is the morphism on the dagger generic fibres induced by $f$.

- Morphisms are compatible morphisms between collections of sheaves.

For each sheaf $F$ on $OC(X_0/T)$, we call $F_Z$ the **realisation** of $F$ on $Z$.

(ii) The presheaf $\mathcal{O}_{X_0/T}$ on $OC(X_0/T)$ defined by

\[
\mathcal{O}_{X_0/T} : (Z, 3, i, h, \theta) \mapsto \mathcal{O}_Z(Z)
\]

is a sheaf. We then call $\mathcal{O}_{X_0/T}$ the **structure sheaf** on $OC(X_0/T)$.

**Proof.** The first assertion follows from [LS11, Proposition 2.1.9] while the second assertion follows from [op. cit., Corollary 2.3.3].

**Definition 3.1.4.** (Yam20, Definition 2.22). Let $T$ be either $S$, $O_{K_0}^{log=0}$, $O_{K_0}^{log=\emptyset}$ or $O_K^{log=\emptyset}$.

(i) A **log overconvergent isocrystal** on $X_0$ over $T$ is an $\mathcal{O}_{X_0/T}$-module $\mathcal{E}$ such that,

- for any $(Z, 3, i, h, \theta) \in OC(X_0/T)$, the realisation $\mathcal{E}_Z$ is a coherent locally free $\mathcal{O}_Z$-module; and
• for any morphism \( f : (Z, Z', i, h, \theta) \to (Z', Z', i', h', \theta') \), the induced morphism

\[ \tau_f : f^* : E \to \epsilon \]

is an isomorphism.

Log overconvergent isocrystals naturally form a category, which is denoted by \( \text{Isoc}(X_0/\Xi) \).

(ii) When \( \Xi \neq O^\log=0_K \), a log overconvergent F-isocrystal on \( X_0 \) over \( \Xi \) is a pair \((\epsilon, \Phi)\), where \( \epsilon \in \text{Isoc}(X_0/\Xi) \) and \( \Phi : \phi^* \epsilon \to \epsilon \) is an isomorphism\(^3\).

Lemma 3.1.5. Let \( \Xi \) be either \( \mathcal{E} \), \( O^\log=0_K \), \( O^\log=0_K \) or \( O^\log=\varpi \). Then, the category \( \text{Isoc}(X_0/\Xi) \) (resp., \( \text{FIsoc}(X_0/\Xi) \)) admits internal Hom’s and tensor products, denoted by \( \text{Hom} \) and \( \otimes \) respectively. In particular, given \( \epsilon, \lambda \in \text{Isoc}(X_0/\Xi) \) (resp., \( (\epsilon, \Phi) \in \text{FIsoc}(X_0/\Xi) \)), the dual \( \epsilon^\vee \) (resp., \( (\epsilon, \Phi)^\vee \)) is well-defined.

Proof. The constructions are given in [Yam20, Definition 2.23]. \( \square \)

Let \( \Xi \) be either \( \mathcal{E} \), \( O^\log=0_K \), \( O^\log=0_K \) or \( O^\log=\varpi \) and let \( \epsilon \in \text{Isoc}(X_0/\Xi) \) (resp., \((\epsilon, \Phi) \in \text{FIsoc}(X_0/\Xi)\)) when \( \Xi \neq O^\log=\varpi \). Our next goal is to introduce the notion of log rigid cohomology of \( X_0 \) relative to \( \Xi \) with coefficients in \( \epsilon \) (resp., \((\epsilon, \Phi)\)). To this end, we fix a collection \( \{ (Z_\lambda, 3_\lambda, i_\lambda, h_\lambda, \theta_\lambda) \}_{\lambda \in \Lambda} \), where \( (Z_\lambda, 3_\lambda, i_\lambda, h_\lambda, \theta_\lambda) \in \text{OC}(X_0/\Xi) \) such that

- each \( Z_\lambda \) is of finite type over \( k \),
- \( \{ \theta_\lambda : Z_\lambda \to X_0 \}_{\lambda \in \Lambda} \) is a Zariski open cover by exact open immersions, and
- every \( h_\lambda, \Xi : 3_\lambda \to \Xi \) is smooth.

When \( \Xi \neq O^\log=\varpi \), we further fix a Frobenius lift \( \varphi_\lambda \) on \( 3_\lambda \) which is compatible with \( \phi \). The existence of such a collection is guaranteed by [Yam20, Proposition 2.29].

For any finite subset \( \Xi \subset \Lambda \), choose an.exactification

\[ Z_\Xi := \cap_{\lambda \in \Xi} Z_\lambda \xrightarrow{i_\Xi} 3_\Xi \to \prod_{\lambda \in \Xi} 3_\lambda \]

of the diagonal embedding \( Z_\Xi \to \prod_{\lambda \in \Xi} 3_{\lambda} \). By letting \( h_{\Xi, 0} : Z_\Xi \to k^\log=0 \) and \( h_{\Xi, \Xi} : 3_\Xi \to \Xi \) be the structure morphisms and \( \theta_\Xi := \cap_{\lambda \in \Xi} \theta_\lambda \), we see that \( (Z_\Xi, 3_\Xi, i_{\Xi}, h_\Xi, \theta_\Xi) \in \text{OC}(X_0/\Xi) \).

Furthermore, if \( \Xi \neq O^\log=\varpi \), the Frobenii \( \varphi_\lambda \)'s induces a Frobenius \( \varphi_\Xi \) on \( 3_\Xi \) by \( \varphi_\Xi := (\prod_{\lambda \in \Xi} \varphi_\lambda)|_{3_\Xi} \).

Let \( \Omega^\log_{3_\Xi/\Xi} \) be the complex of log differential forms of \( 3_\Xi \) over \( \Xi \). By tensoring with \( K_0([T]) \) (resp., \( K_0; \) resp., \( K \)) if \( \Xi = \mathcal{E} \) (resp., \( \Xi = O^\log=0_K \) or \( O^\log=0_K \); resp., \( \Xi = O^\log=\varpi \)), we obtain a complex \( \Omega^\log_{3_\Xi/\Xi, 0} \) of sheaves on the dagger space \( Z_\Xi \). Finally, we write

\[ \Omega^\log_{3_\Xi/\Xi, \eta} := \Omega^\log_{3_\Xi/\Xi, 0}|_{Z_\Xi} \]

By [GK01, Lemma 1.2], we know that \( \Omega^\log_{3_\Xi/\Xi, \eta} \) is independent of the choice of exactification \( 3_\Xi \).

\(^3\)Here, we abuse the notation, write \( \phi \) to be the endomorphism on \( \text{Isoc}(X_0/\Xi) \) induced from the Frobenius on \( \Xi \) and the absolute Frobenius on \( X_0 \) (see [Yam20, (2.21)]).
For $\mathcal{E} \in \text{lsoc}^\dagger(X_0/\mathfrak{T})$, we have the realisation $\mathcal{E}_\Xi$ on $Z_\Xi$. By \cite[Corollary 2.27]{Yam20}, we know that $\mathcal{E}_\Xi$ is equipped with an integrable connection

$$\nabla : \mathcal{E}_\Xi \to \mathcal{E}_\Xi \otimes \Omega_{3\Xi}^{\log,1}/\mathcal{T}_\eta.$$  

Moreover, if $(\mathcal{E}, \Phi) \in \text{Flsoc}^\dagger(X_0/\mathfrak{T})$, then $\Phi$ induces a commutative diagram

$$
\begin{array}{ccc}
\varphi_{\Xi}^* \mathcal{E}_\Xi & \xrightarrow{\nabla} & \varphi_{\Xi}^* \mathcal{E}_\Xi \otimes \varphi_{\Xi}^* \Omega_{3\Xi}^{\log,1}/\mathcal{T}_\eta \\
\downarrow \Phi & & \downarrow \Phi \otimes \varphi_{\Xi}^* \\
\mathcal{E}_\Xi & \xrightarrow{\nabla} & \mathcal{E}_\Xi \otimes \Omega_{3\Xi}^{\log,1}/\mathcal{T}_\eta
\end{array}
$$

In particular, by restricting to $|Z_\Xi|_{Z_\Xi}$, we can consider the complex $R\Gamma(|Z_\Xi|_{Z_\Xi}, \mathcal{E}_\Xi \otimes \Omega_{Z_\Xi}^{\log,\bullet})$.

For any finite subsets $\Xi_1 \subset \Xi_2 \subset \Lambda$, one has a natural map $\delta_{\Xi_2, \Xi_1} : |Z_{\Xi_2}|_{Z_{\Xi_2}} \to |Z_{\Xi_1}|_{Z_{\Xi_1}}$, which induces $\delta_{\Xi_2, \Xi_1}^{-1}(\mathcal{E}_{\Xi_1} \otimes \Omega_{Z_{\Xi_1}}^{\bullet}) \to \mathcal{E}_{\Xi_2} \otimes \Omega_{Z_{\Xi_2}}^{\bullet}$. Consequently, after fixing an order on $\Lambda$, one obtains a simplicial dagger space $|Z_{\Xi}|_{Z_\Xi}$ and a complex of sheaves $\mathcal{E}_{\Xi} \otimes \Omega_{Z_{\Xi}}^{\bullet}$ on $|Z_{\Xi}|_{Z_\Xi}$. Consequently, $\Phi$ induces a Frobenius action on the complex $R\Gamma(|Z_{\Xi}|_{Z_\Xi}, \mathcal{E}_{\Xi} \otimes \Omega_{Z_{\Xi}}^{\log,\bullet})$. We still denote by $\Phi$ the induced operator.

**Definition 3.1.6.** Let $\mathfrak{T}$ be either $\mathfrak{S}$, $\mathcal{O}_K^{\log=0}$, $\mathcal{O}_K^{\log=\emptyset}$ or $\mathcal{O}_K^{\log=\infty}$ and let $\mathcal{E} \in \text{lsoc}^\dagger((X_0/\mathfrak{T})$. The **log rigid cohomology of $X_0$ relative to $\mathfrak{T}$ with coefficients in $\mathcal{E}$** is defined to be the complex in the derived category $D(K_0((T)))$ (resp., $D(K_0)$; resp., $D(K)$) if $\mathfrak{T} = \mathfrak{S}$ (resp., $\mathfrak{T} = \mathcal{O}_K^{\log=0}$ or $\mathcal{O}_K^{\log=\emptyset}$; resp., $\mathfrak{T} = \mathcal{O}_K^{\log=\infty}$)

$$R\Gamma_{\text{rig}}(X_0/\mathfrak{T}, \mathcal{E}) := \text{hocolim} R\Gamma(|Z_{\Xi}|_{Z_\Xi}, \mathcal{E}_{\Xi} \otimes \Omega_{Z_{\Xi}}^{\log,\bullet}),$$

where the homotopy colimit is taken over the category of hypercovers of $X_0$ built from the simplicial dagger spaces described above (see \cite[Tag 01FH]{Sta24}).

**Remark 3.1.7.** Similar notion apply to $(\mathcal{E}, \Phi) \in \text{Flsoc}^\dagger(X_0/\mathfrak{T})$ when $\mathfrak{T} \neq \mathcal{O}_K^{\log=\infty}$. We leave the detailed definition to the readers.

**Remark 3.1.8.** Readers who are more familiar with the traditional definition of rigid cohomology (introduced by P. Berthelot in \cite{Ber80}) may wonder how to compare Berthelot’s definition with the aforementioned definition. For this, we refer the readers to \cite[Theorem 5.1]{GK00}, \cite[Remark 4.2.5]{Ked06}, and \cite[Corollary 3.6.8]{LS11}.

We conclude this subsection by giving the definition of log rigid cohomology with compact supports by following \cite{EY20}. To this end, recall that $X_0$ is strictly semistable, i.e., Zariski locally, we have a strictly smooth morphism

$$X_0 \to \text{Spec} k[X_1, \ldots, X_n]/(X_1 \cdots X_r)$$

of log schemes over $k_{\log=0}$. Denote by $D_0 \subset X_0$ the horizontal divisor, locally defined by $X_{r-1} \cdots X_n$.

Let $\mathfrak{T}$ be either $\mathfrak{S}$, $\mathcal{O}_K^{\log=0}$, $\mathcal{O}_K^{\log=\emptyset}$ or $\mathcal{O}_K^{\log=\infty}$. Given $(Z, \mathfrak{Z}, i, h, \theta) \in \text{OC}(X_0/\mathfrak{T})$, we define the sheaf $\mathcal{O}_{\mathfrak{Z}}(D_0)$ on $\mathfrak{Z}$ to be the locally free $\mathcal{O}_{\mathfrak{Z}}$-module, locally generated by $i_\ast \theta^e X_{r+1} \cdots X_n$. This sheaf then induces a locally free $\mathcal{O}_{\mathfrak{Z}}$-module $\mathcal{O}_{\mathfrak{Z}}(D_0)$ on the dagger generic fibre $\mathfrak{Z}$ of $\mathfrak{Z}$. Consequently, by Lemma 3.1.3 there is a sheaf $\mathcal{O}_{X_0/\mathfrak{T}}(D_0)$ on $\text{OC}(X_0/\mathfrak{T})$ whose realisation on each $(Z, \mathfrak{Z}, i, h, \theta)$ is $\mathcal{O}_{\mathfrak{Z}}(D_0)$.

\footnote{Recall from Remark 3.1.2 that we can view $X_0$ as a presheaf of the absolute log overconvergent site $\text{OC}(k_{\log=0}/\mathfrak{T})$. Then, we can consider hypercovers of $X_0$ in $\text{OC}(k_{\log=0}/\mathfrak{T})$ (see, for example, \cite[Definition 2.1]{LMM20}).}
Definition 3.2.1. Let $\mathcal{X}$ be either $\mathcal{S}$, $\mathcal{O}_{K_0}^{\log=0}$, $\mathcal{O}_{K_0}^{\log=\emptyset}$ or $\mathcal{O}_{K_0}^{\log=\infty}$ and let $\mathcal{E}$ be in $\text{Isoc}^+(X_0 / \mathcal{X})$. Denote by $\mathcal{E}(D_0)$ the tensor product $\mathcal{E} \otimes_{\mathcal{O}_{X_0/\mathcal{X}}} \mathcal{O}_{X_0/\mathcal{X}}(D_0)$. Then, the log rigid cohomology with compact support of $X_0$ relative to $\mathcal{X}$ with coefficients in $\mathcal{E}$ is defined to be the complex in the derived category $D(K_0((T)))$ (resp., $D(K_0)$; resp., $D(K)$) if $\mathcal{X} = \mathcal{S}$ (resp., $\mathcal{X} = \mathcal{O}_{K_0}^{\log=0}$ or $\mathcal{O}_{K_0}^{\log=\emptyset}$; resp., $\mathcal{X} = \mathcal{O}_{K_0}^{\log=\infty}$)

$$
R_{\text{rig},c}(X_0 / \mathcal{X}, \mathcal{E}) := R_{\text{rig}}(X_0 / \mathcal{X}, \mathcal{E}(D_0)).
$$

Remark 3.1.10. When $\mathcal{X} \neq \mathcal{O}_{K_0}^{\log=\infty}$ and $(\mathcal{E}, \Phi) \in \text{Flsoc}^+(X_0 / \mathcal{X})$, then the log rigid cohomology with compact support of $X_0$ relative to $\mathcal{X}$ with coefficients in $(\mathcal{E}, \Phi)$ is defined in a similar manner. We, again, leave the detailed definition to the readers. $
$
### 3.2 Hyodo–Kato theory with coefficients

Following [Yam20], to discuss the Hyodo–Kato theory with coefficients, we have to refine the coefficients. Such a refinement relies on the notion of residue maps, which we now recall from [Ked07] Definition 2.3.9:

Let $\mathcal{E} \in \text{Isoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})$. For any $(Z, 3, i, h, \theta) \in \text{OC}(X_0 / \mathcal{S})$, we can regard $(Z, 3, i, h', \theta) \in \text{OC}(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})$, where $h' : 3 \overset{\eta}{\to} \mathcal{S} \to \mathcal{O}_{K_0}^{\log=\emptyset}$. Zariski locally on $3$, we have a strictly smooth morphism

$$
3 \to \text{Spf} \mathcal{O}_{K_0}[T][x_1, \ldots, x_n]/(T - x_1 \cdots x_r).
$$

Let $W$ be one of the $X_i$'s and let $\mathcal{D}_W \subset \mathcal{Z}$ be the closed dagger subspace defined by $W$. Denote by $\Omega^1_{Z/K_0}$ the sheaf of differential one-forms on $Z$ over $K_0$, then we have

$$
\text{coker} \left( \mathcal{E}_Z \otimes (\Omega^1_{Z/K_0} \oplus \mathcal{O}_Z \log=\emptyset) \to \mathcal{E}_Z \otimes \Omega^1_{\mathcal{O}_{K_0}^{\log=\emptyset} / \mathcal{O}_{K_0}^{\log=\emptyset} \eta} \right) = \mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W} \log W.
$$

Hence, we have a map

$$
\mathcal{E}_Z \overset{\nabla}{\to} \mathcal{E}_Z \otimes \Omega^1_{\mathcal{O}_{K_0}^{\log=\emptyset} / \mathcal{O}_{K_0}^{\log=\emptyset} \eta} \to \mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W} \log W,
$$

which induces a map $\mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W} \to \mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W} \log W$. After identifying $\mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W} \log W$ with $\mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W}$, one obtains the residue map

$$
\text{Res}_W : \mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W} \to \mathcal{E}_Z \otimes \mathcal{O}_{\mathcal{D}_W}
$$

along $\mathcal{D}_W$.

**Definition 3.2.1.** Let $\mathcal{E} \in \text{Isoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})$.

(i) We say $\mathcal{E}$ has nilpotent residues, if for any $(Z, 3, i, h, \theta) \in \text{OC}(X_0 / \mathcal{S})$, the residue maps $\text{Res}_W$ on the realisation $\mathcal{E}_Z$ are nilpotent. We denote by $\text{Isoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})_{nr}$ the full subcategory of overconvergent isocrystals having nilpotent residues.

(ii) Suppose $\mathcal{E} \in \text{Isoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})_{nr}$. We say $(\mathcal{E}, \Phi)$ is unipotent if $\mathcal{E}$ is an iterated extension of $\mathcal{O}_{X_0/\mathcal{S}}^{\log=\emptyset}$. We denote by $\text{Flsoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})_{\text{unip}}$ the full subcategory of unipotent overconvergent isocrystals.

(iii) If $(\mathcal{E}, \Phi) \in \text{Flsoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})$, we say it has nilpotent residue (resp., is unipotent) if $\mathcal{E}$ has nilpotent residue (resp., is unipotent). We let $\text{Flsoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})_{\text{unip}}$ (resp., $\text{Flsoc}^+(X_0 / \mathcal{O}_{K_0}^{\log=\emptyset})_{\text{unip}}$) be the full subcategory of overconvergent $F$-isocrystals having nilpotent residues (resp., being unipotent.)

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Note that there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_K^{\log=0} & \xrightarrow{T} & \mathcal{O}_K^{\log=\varpi} \\
\downarrow & & \downarrow \\
\mathcal{O}^{\log=0}_K & \xleftarrow{T^{-1}} & \mathcal{O}_S^{\log=0}.
\end{array}$$

Therefore, via base change, we have functors of overconvergent isocrystals

$$\text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=0}) \xleftrightarrow{\text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=\varpi})} \text{Isoc}^\dagger(X_0/\mathcal{O}_S^{\log=0}).$$

In particular, for any $E \in \text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=\varpi})$, we can view it as an overconvergent isocrystal in $\text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=0})$, $\text{Isoc}^\dagger(X_0/\mathcal{O}_S^{\log=0})$, or $\text{Isoc}^\dagger(X_0/\mathcal{O}_S^{\log=\varpi})$. We shall abuse the notation and still denote its images by $E$. Similar for overconvergent $F$-isocrystals.

To define the Hyodo–Kato cohomology with coefficients, we fix a collection $\{(Z_\lambda, Z_\lambda^i, h_\lambda, \theta_\lambda)\}_{\lambda \in \Lambda}$ where $(Z_\lambda, Z_\lambda^i, h_\lambda, \theta_\lambda) \in \text{OC}(X_0/\mathfrak{S})$ as in §3.1. Resuming the notation in loc. cit., for any finite subset $\Xi \subset \Lambda$, we can consider the Kim–Hain complex

$$\Omega^{\log, \bullet}_{Z_{\Xi, \eta}}[u]_{\text{naive}} := \Omega^{\log, \bullet}_{Z_{\Xi, \eta}}[u]_{Z_{\Xi, \eta}[u]},$$

generated by $\Omega^{\log, \bullet}_{Z_{\Xi, \eta}}[u]_{\text{naive}}$, and degree-0 elements $u^i$ (for $i \in \mathbb{Z}_{\geq 0}$) such that

- $u^0 = 1,$
- $u^i \wedge u^j = \frac{(i+j)!}{i!j!} u^{i+j},$
- $du^{i+1} = -(d \log T) u^i.$

We let $\Omega^{\log, \bullet}_{Z_{\Xi, \eta}}[u]_k$ be the subcomplex of $\Omega^{\log, \bullet}_{Z_{\Xi, \eta}}[u]_{\text{naive}}$, consisting of sections of $\Omega^{\log, \bullet}_{Z_{\Xi, \eta}}$ and $u^0, ..., u^k$.

**Definition 3.2.2.** Let $E \in \text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=\varpi})_{\text{unip}}$.

(i) The **Hyodo–Kato cohomology of $X_0$ with coefficients in $E$** is defined to be the complex

$$R\Gamma_{\text{HK}}(X_0, E) := \text{hocolim}_k \text{hocolim}_k R\Gamma(\mathcal{Z}_{\eta}[Z_{\bullet}], E \otimes \Omega^{\log, \bullet}_{Z_{\Xi, \eta}}[u]_k),$$

(ii) The **Hyodo–Kato cohomology of $X_0$ with compact supports and coefficients in $(E, \Phi)$** is defined to be the complex

$$R\Gamma_{\text{HK}, c}(X_0, E) := R\Gamma_{\text{HK}}(X_0/\mathcal{O}_K^{\log=0}, (E, \Phi)).$$

**Remark 3.2.3.** We again leave the similar definition for $(E, \Phi) \in \text{Flsoc}^\dagger(X_0/\mathcal{O}_K^{\log=\varpi})_{\text{unip}}$ to the readers.
Let $E \in \text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=0})$. On both $R\Gamma_{HK}(X_0, E)$ and $R\Gamma_{HK,c}(X_0, E)$, there is a monodromy operator $N$ defined by

$$N(u[i]) = u[i-1].$$

Moreover, if $(E', \Phi) \in \text{Flsoc}^\dagger(X_0/\mathcal{O}_K^{\log=0})$, the complex $R\Gamma_{HK}(X_0, (E', \Phi))$ and $R\Gamma_{HK,c}(X_0, (E', \Phi))$ admit Frobenius actions induced by $\Phi$ and

$$\varphi(u[i]) = p^i u[i].$$

We again denote the Frobenius action on the complexes by $\Phi$. Therefore, one sees immediately that

$$\Phi N = p N \Phi. \quad (4)$$

Our next goal is to briefly discuss the so-called Hyodo–Kato morphisms. To this end, we assume from now on that there exists a flat weak formal scheme $X$ over $\mathcal{O}_K$ which is strictly semistable over $\mathcal{O}_K^{\log=\varpi}$ such that its dagger generic fibre $\mathcal{X}$ is proper smooth over $K$ and the associated scheme of $X$ over $k$ is $X_0$.

Let $\iota : X_0 \hookrightarrow \mathcal{X}$ be the canonical inclusion and let $h_k : X_0 \to k^{\log=0}$ and $h_{\mathcal{O}_K^{\log=\varpi}} : \mathcal{X} \to \mathcal{O}_K^{\log=\varpi}$ be the structure morphisms, then $(X_0, \mathcal{X}, \iota, h, \text{id}) \in \mathcal{OC}(X_0/\mathcal{O}_K^{\log=0})$. For any $E \in \text{Isoc}^\dagger(X_0/\mathcal{O}_K^{\log=0})$, we see immediately from construction that

$$R\Gamma_{\text{rig}}(X_0/\mathcal{O}_K^{\log=\varpi}, E) = R\Gamma_{\text{dR}}(\mathcal{X}, E).$$

Moreover, since $\mathcal{X}$ is assumed to be proper smooth, we see that

$$R\Gamma_{\text{rig},c}(X_0/\mathcal{O}_K^{\log=\varpi}, E) = R\Gamma_{\text{dR}}(\mathcal{X}, E).$$

**Theorem 3.2.4.** For any $(E', \Phi) \in \text{Flsoc}^\dagger(X_0/\mathcal{O}_K^{\log=0})^{\text{unip}}$, we have the following quasi-isomorphisms:

(i) The comparison quasi-isomorphisms

$$R\Gamma_{HK}(X_0, (E', \Phi)) \to R\Gamma_{\text{rig}}(X_0, (E', \Phi)), \quad u[i] \mapsto \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else} \end{cases}.$$

It is moreover compatible with the Frobenius structures on both sides.

(ii) The Hyodo–Kato quasi-isomorphism

$$\Psi_{HK} : R\Gamma_{HK}(X_0, (E', \Phi)) \otimes_{K_0} K \to R\Gamma_{\text{dR}}(\mathcal{X}, E').$$

(iii) The Hyodo–Kato quasi-isomorphism

$$\Psi_{HK,c} : R\Gamma_{HK,c}(X_0, (E', \Phi)) \otimes_{K_0} K \to R\Gamma_{\text{rig},c}(X_0/\mathcal{O}_K^{\log=\varpi}, E') = R\Gamma_{\text{dR}}(\mathcal{X}, E').$$

**Proof.** The first assertion is [Yam20, Theorem 4.8]. The second and the third quasi-isomorphism is given by a choice of $p$-adic logarithm and

$$\Psi_{HK}, \Psi_{HK,c} : u[i] \mapsto \frac{(- \log \varpi)^i}{i!}.$$

For the proof, see [Yam20, Proposition 8.8] and [EY20, Corollary 3.8].
Recall that the de Rham complex \( R^\Gamma_{dR}(X) \) admits a filtration induced by 
\[
\tau_{\geq i} : (0 \to \mathcal{O}_X \to \Omega^1_{X/K} \to \cdots),
\]
where \( \tau_{\geq i} \) is the truncation functor. Suppose \( \mathcal{G} \) is a vector bundle on \( X \) with integrable connection. Assume that \( \mathcal{G} \) admits a finite descending filtration \( \text{Fil}^i \mathcal{G} \). Recall that such a filtration satisfies Griffiths’s transversality if and only if 
\[
\nabla(\text{Fil}^i \mathcal{G}) \subset \text{Fil}^{i-1} \mathcal{G} \otimes \Omega^1_{X/K}.
\]
In this case, we define a filtration \( F^n R^\Gamma_{dR}(X, \mathcal{G}) \) on \( R^\Gamma_{dR}(X, \mathcal{G}) \) induced by 
\[
\text{Fil}^n \left( 0 \to \mathcal{G} \subset \mathcal{G} \otimes \Omega^1_{X/K} \to \cdots \right) := \oplus_{i+j=n} \tau_{\geq i} \left( 0 \to \text{Fil}^i \mathcal{G} \subset \text{Fil}^{i-1} \mathcal{G} \otimes \Omega^1_{X/K} \to \cdots \right).
\]
Together with the Hyodo–Kato isomorphism, they inspire the following definition.

**Definition 3.2.5** ([Yam20, Definition 9.6]). The category of syntomic coefficients \( \text{Syn}(X_0, \mathfrak{X}, X) \) is defined to be the category of triples \((\mathcal{E}, \Phi, \text{Fil}^\bullet)\), where

- \((\mathcal{E}, \Phi) \in \text{FIsoc}^1(X_0/\mathcal{O}_{K_0}^\log=\emptyset)_{\text{unip}}\),
- \(\text{Fil}^\bullet\) is a filtration of \(\mathcal{E}\) on \(X\) (i.e., after base change to \(\text{OC}(X_0/\mathcal{O}_{K_0}^\log=\emptyset)\)) that satisfies Griffiths’s transversality.

Morphisms are morphisms of overconvergent \(F\)-isocrystals that preserve the filtrations.

**Remark 3.2.6.** Given \((\mathcal{E}, \Phi, \text{Fil}^\bullet) \in \text{Syn}(X_0, \mathfrak{X}, X)\), we define its dual \((\mathcal{E}^\vee, \Phi^\vee, \text{Fil}^{\vee\bullet}) \in \text{Syn}(X_0, \mathfrak{X}, X)\) as follows. By [Yam20, Proposition 3.6], we know that we have a dual \((\mathcal{E}^\vee, \Phi^\vee) \in \text{FIsoc}^1(X_0/\mathcal{O}_{K_0}^\log=\emptyset)_{\text{unip}}\) with \(\mathcal{E}^\vee = \text{Hom}(\mathcal{E}, \mathcal{O}_{X_0/\mathcal{O}_{K_0}^\log=\emptyset})\). Hence, on \(X\), we have surjective morphisms

\[
\mathcal{E}^\vee \to (\text{Fil}^i \mathcal{E})^\vee
\]

for any \(i\). Hence, we define

\[
\text{Fil}^{\vee, i+1} \mathcal{E}^\vee := \ker(\mathcal{E}^\vee \to (\text{Fil}^i \mathcal{E})^\vee).
\]

One can easily check that this filtration satisfies Griffiths’s transversality. Consequently, \((\mathcal{E}^\vee, \Phi^\vee, \text{Fil}^{\vee\bullet}) \in \text{Syn}(X_0, \mathfrak{X}, X)\).

## 4 Finite polynomial cohomology with coefficients

The theory of finite polynomial cohomology was first introduced by Besser in [Bes00] and generalised to general varieties in [BLZ16]. Such a theory was first introduced to solve the problem that the theory of syntomic cohomology lack of cup products. The aim of this section is to generalise the aforementioned works to a cohomology theory with coefficients.

Throughout this section, we resume the notations used in \[3\] and \[32\]. In particular, we have a fixed proper weak formal scheme \(\mathfrak{X}\) over \(\mathcal{O}_{K_0}^\log=\emptyset\), which is strictly semistable, with proper smooth dagger generic fibre \(X\) over \(K\) and strictly semistable special fibre \(X_0\) over \(k^\log=0\). We also recall the horizontal divisor \(D_0 \subset X_0\).
4.1 Definitions and some basic properties

Following [Bes00], consider

\[ \text{Poly} := \text{the multiplicative monoid of all polynomials } P(X) = \prod_{i=1}^{n}(1-\alpha_i T) \in \mathbb{Q}[T] \text{ with constant term 1} \]

The following definition is inspired by [BLZ16].

**Definition 4.1.1.** Let \((\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathcal{X}, \mathcal{X})\).

(i) Given \(P \in \text{Poly} \) and \(n \in \mathbb{Z} \), the **syntomic \(P\)-cohomology of \(\mathcal{X}\) with coefficients in \(\mathcal{E}\) twisted by \(n\) is defined to be

\[
\Gamma_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n) := \begin{bmatrix}
\Gamma_{\text{HK}}(\mathcal{X}_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \\
\mathcal{N} \otimes \text{id}
\end{bmatrix}
\begin{bmatrix}
(P(\Phi^i) \otimes \text{id}) \oplus (\Phi_{\text{HK}} \otimes \text{id}) \\
N \otimes \text{id}
\end{bmatrix}
\begin{bmatrix}
\Gamma_{\text{HK}}(\mathcal{X}_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \\
\mathcal{N} \otimes \text{id} \oplus 0
\end{bmatrix}
\begin{bmatrix}
\Gamma_{\text{HK}}(\mathcal{X}_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K
\end{bmatrix},
\]

where

\[
\Delta(\mathcal{X}, \mathcal{E}, n) := \Gamma_{\text{dR}}(\mathcal{X}, \mathcal{E})/F^n \Gamma_{\text{dR}}(\mathcal{X}, \mathcal{E})
\]

and recall that \(q = p^c = \#k\). The \(i\)-th cohomology group of \(\Gamma_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n)\) is then denoted by \(H^i_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n)\).

(ii) When \(P = 1-q^{-n}T\), we write

\[
\Gamma_{\text{syn}}(\mathcal{X}, \mathcal{E}, n) := \Gamma_{\text{syn}, 1-q^{-n}T}(\mathcal{X}, \mathcal{E}, n)
\]

and call it the **syntomic cohomology of \(\mathcal{X}\) with coefficients in \(\mathcal{E}\) twisted by \(n\). The \(i\)-th cohomology group of \(\Gamma_{\text{syn}}(\mathcal{X}, \mathcal{E}, n)\) is then denoted by \(H^i_{\text{syn}}(\mathcal{X}, \mathcal{E}, n)\).

For any \((\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathcal{X}, \mathcal{X})\), the \(\mathcal{O}_{\mathcal{X}}\)-module \(\mathcal{E}(D_0)\) on \(\mathcal{X}\) also admits a filtration defined as

\[
\text{Fil}^i(\mathcal{E}(D_0)) := (\text{Fil}^i \mathcal{E}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}(X_0).
\]

By definition, this filtration on \(\mathcal{E}(D_0)\) also satisfies Griffiths’s transversality. It then consequently defines a filtration \(F^n \Gamma_{\text{dR}, \mathcal{E}}(\mathcal{X}, \mathcal{E})\) on \(\Gamma_{\text{dR}, \mathcal{E}}(\mathcal{X}, \mathcal{E})\). This observation then leads to the following definition.

**Definition 4.1.2.** Let \((\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathcal{X}, \mathcal{X})\).

(i) Given \(P \in \text{Poly} \) and \(n \in \mathbb{Z} \), the **compactly supported syntomic \(P\)-cohomology of \(\mathcal{X}\) with coefficients in \(\mathcal{E}\) twisted by \(n\) is defined to be

\[
\Gamma_{\text{syn}, P, c}(\mathcal{X}, \mathcal{E}, n) := \begin{bmatrix}
\Gamma_{\text{HK}, c}(\mathcal{X}_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \\
N \otimes \text{id}
\end{bmatrix}
\begin{bmatrix}
(P(\Phi^i) \otimes \text{id}) \oplus (\Phi_{\text{HK}} \otimes \text{id}) \\
N \otimes \text{id}
\end{bmatrix}
\begin{bmatrix}
\Gamma_{\text{HK}, c}(\mathcal{X}_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \\
\mathcal{N} \otimes \text{id} \oplus 0
\end{bmatrix}
\begin{bmatrix}
\Gamma_{\text{HK}, c}(\mathcal{X}_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K
\end{bmatrix},
\]

The \(i\)-th cohomology group of \(\Gamma_{\text{syn}, P, c}(\mathcal{X}, \mathcal{E}, n)\) is denoted by \(H^i_{\text{syn}, P, c}(\mathcal{X}, \mathcal{E}, n)\).

(ii) When \(P = 1-q^{-n}T\), we write

\[
\Gamma_{\text{syn}, c}(\mathcal{X}, \mathcal{E}, n) := \Gamma_{\text{syn}, 1-q^{-n}T, c}(\mathcal{X}, \mathcal{E}, n)
\]

and call it the **syntomic cohomology of \(\mathcal{X}\) with coefficients in \(\mathcal{E}\) twisted by \(n\). The \(i\)-th cohomology group of \(\Gamma_{\text{syn}, c}(\mathcal{X}, \mathcal{E}, n)\) is then denoted by \(H^i_{\text{syn}}(\mathcal{X}, \mathcal{E}, n)\).
**Notation 4.1.3.** For complexes $R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi))$, $R\Gamma_{dR}(\mathcal{X}, \mathcal{E})$, $R\Gamma_{syn, P}(\mathcal{X}, \mathcal{E}, n)$, ... etc, we drop the ‘$\mathcal{E}$’ in the notation if $\mathcal{E}$ is nothing but the structure sheaf. Similar notations apply to cohomology groups. Moreover, we will often abuse the notations and write $P(\Phi^e)$ and $N$ for $P(\Phi^e) \otimes \text{id}$ and $N \otimes \text{id}$ on vector spaces over $K$.

**Proposition 4.1.4.** Given $(\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathfrak{x}, \mathcal{X})$ and two polynomials $P, Q \in \text{Poly}$, we have a natural morphism

$$R\Gamma_{syn, P}(\mathcal{X}, \mathcal{E}, n) \to R\Gamma_{syn, PQ}(\mathcal{X}, \mathcal{E}, n).$$

In particular, if $P$ is divided by $1 - q^{-n}T$, then there is a natural morphism

$$R\Gamma_{syn}(\mathcal{X}, \mathcal{E}, n) \to R\Gamma_{syn, P}(\mathcal{X}, \mathcal{E}, n).$$

Similar statements hold for the compactly supported version.

**Proof.** This is immediate from definition. \qed

To understand the syntomic $P$-cohomology better, we begin with some simplification of notations. We let

$$C^* := \text{Cone} \left( R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \to R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \oplus DR(\mathcal{X}, \mathcal{E}, n) \right) [-1]$$

$$D^* := \text{Cone} \left( R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi)) \to R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \right) [-1]$$

be the mapping fibres of the horizontal rows in the definition of $R\Gamma_{syn, P}(\mathcal{X}, \mathcal{E}, n)$. Then, by definition, we have

$$R\Gamma_{syn, P}(\mathcal{X}, \mathcal{E}, n) = \text{Cone} \left( C^* \xrightarrow{\alpha} D^* \right) [-1],$$

where $\alpha$ is the map induced by the vertical maps in the definitions of $R\Gamma_{syn, P}(\mathcal{X}, \mathcal{E}, n)$ and $R\Gamma_{syn, P, c}(\mathcal{X}, \mathcal{E}, n)$ respectively. Therefore, we have a long exact sequence

$$\cdots \to H^{i-1}(C^*) \xrightarrow{\gamma} H^{i-1}(D^*) \xrightarrow{\delta} H^i_{syn, P}(\mathcal{X}, \mathcal{E}, n) \xrightarrow{\tau} H^i(C^*) \xrightarrow{\alpha} H^i(D^*) \to \cdots$$

**Proposition 4.1.5.** Let $(\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathfrak{x}, \mathcal{X})$ and $P \in \text{Poly}$.

(i) A class in $H^i_{syn, P}(\mathcal{X}, \mathcal{E}, n)$ is represented by a quintuple $(x, y, z, u, v)$ with

$$x \in R\Gamma_{HK}^i(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K, \quad u \in R\Gamma_{HK}^{i-1}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K,$$

$$y \in R\Gamma_{HK}^{i-2}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K, \quad v \in R\Gamma_{HK}^{i-2}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K,$$

$$z \in DR^{i-1}(\mathcal{X}, \mathcal{E}, n),$$

such that

$$d_{HK} \otimes \text{id}(x) = 0, \quad N \otimes \text{id}(x) - d_{HK} \otimes \text{id}(u) = 0,$$

$$P(\Phi^e) \otimes \text{id}(x) + d_{HK} \otimes \text{id}(y) = 0, \quad N \otimes \text{id}(y) + P(q^{-1}\Phi^e) \otimes \text{id}(u) + d_{HK} \otimes \text{id}(v) = 0,$$

$$\Psi_{HK} \otimes \text{id}(x) + d_{dr}(z) \in F^n R\Gamma_{dr}^i(\mathcal{X}, \mathcal{E}).$$

Here, $d_{HK}$ and $d_{dr}$ are differentials of $R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi))$ and $R\Gamma_{dr}(\mathcal{X}, \mathcal{E})$ respectively.
(ii) We have a diagram

\[
\begin{array}{ccc}
H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi) \otimes_{K_0} \mathbb{K}) & \xrightarrow{\alpha} & H^i_{\text{syn}, p}(X, \mathcal{E}, \mathcal{N}) \xrightarrow{\beta} H^i_{\text{syn}, p}(X, \mathcal{E}, n) \\
\downarrow & & \downarrow \\
\ker \alpha & \xrightarrow{\gamma} & \text{ker} \alpha \\
\downarrow & & \downarrow \\
\text{image } N + \text{image } P(q^{-1} \Phi) & \xrightarrow{\delta} & \text{image } N + \text{image } P(q^{-1} \Phi)
\end{array}
\]

where the middle row and two vertical columns are all exact. Here,

\[B^i := \left\{ (x, y) \in H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \oplus H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K : N x \in \text{image } P(q^{-1} \Phi) \right\} \]

(iii) A similar statements hold for the compactly supported version.

Proof. Unwinding the construction of mapping fibres, one sees that \(R \Gamma^i_{\text{syn}, p}(X, \mathcal{E}, n)\) is equal to the direct sum

\[R \Gamma^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \oplus R \Gamma^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \oplus \text{DR}^i_{\text{HK}}(X_0, (\mathcal{E}, \mathcal{N})) \oplus R \Gamma^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \oplus R \Gamma^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K\]

with differentials given by

\[d_{\text{syn}} = \begin{pmatrix}
d_{\text{HK}} \\
-P(\Phi^e) \\
-\Psi_{\text{HK}} \\
N \\
-N \\
-N \otimes 0 \\
N \otimes 0
\end{pmatrix}.
\]

When \(\mathcal{E}\) is the trivial overconvergent \(F\)-isocrystal, this description is exact the one discussed in [BLZ16, §2.4].

For (ii), by definition, we have a commutative diagram

\[
\begin{array}{cccccccc}
\cdots & \xrightarrow{N} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{N \otimes 0} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{\alpha} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{N} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{N \otimes 0} & \cdots \\
\cdots & \xrightarrow{H^i (\mathcal{E}^e)} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{H^i (\mathcal{E}^e)} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{H^i (\mathcal{E}^e)} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{H^i (\mathcal{E}^e)} & H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \xrightarrow{H^i (\mathcal{E}^e)} & \cdots
\end{array}
\]
where the horizontal rows are exact sequences. One then deduces a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i-1}_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \longrightarrow & H^{i-1}_{\text{HK}}(\mathcal{E}, \Phi) & \longrightarrow & H^i(C^\ast) & \longrightarrow & H^i(X_0, (\mathcal{E}, \Phi)) & \longrightarrow & 0 \\
0 & \downarrow & \alpha & & & & & & \downarrow N & & \\
0 & \longrightarrow & H^{i-1}_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K & \longrightarrow & H^i(D^\ast) & \longrightarrow & H^i(X_0, (\mathcal{E}, \Phi)) & \longrightarrow & 0 \\
\end{array}
\]

The desired diagram can then be concluded directly by snake lemma.

\[\square\]

**Corollary 4.1.6.** Let \((\mathcal{E}, \Phi, \text{Fil}^\ast) \in \text{Syn}(X_0, \mathcal{X}, \mathcal{X})\) and \(P \in \text{Poly}\). For any \(n \in \mathbb{Z}\) and any \(i \in \mathbb{Z}_{\geq 0}\), there is a 3-step descending filtration \(F^\ast = F^\ast_P\) on \(H^i_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n)\) such that

\[
\begin{align*}
F^0/F^1 &= \ker \left( H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi))^{P(\Phi^e)=0, N=0} \cap F^n H^i_{\text{dR}}(\mathcal{X}, \mathcal{E}) \to H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \right) \\
F^1/F^2 &= \left\{ (x, y) \in H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \oplus H^{i-1}_{\text{HK}}(\mathcal{E}, \Phi) : N x \in \text{image} P(q^{-1} \Phi^e) \right\} \\
F^2/F^3 &= \text{coker} \left( H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi))^{P(\Phi^e)=0, N=0} \cap F^n H^{i-1}_{\text{dR}}(\mathcal{X}, \mathcal{E}) \to \text{image} N + \text{image} P(q^{-1} \Phi^e) \right).
\end{align*}
\]

Similar for the compactly supported version, whose 3-step filtrations are denoted by \(F^\ast_c = F^\ast_P\).

**Proof.** The filtrations are defined as follows:

\[
\begin{align*}
F^1 &= \ker \left( H^i_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n) \to H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi))^{P(\Phi^e)=0, N=0} \otimes_{K_0} K \right) \\
F^2 &= \gamma^{-1}(B^i) \\
F^3 &= \text{image} \left( H^i_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes_{K_0} K \right) \to \text{coker} \alpha \to H^i_{\text{syn}, P}(\mathcal{X}, P, n).
\end{align*}
\]

Remark that this gives an explicit construction of the 3-step filtration in [BLZ16, Definition 2.3.3] when \(\mathcal{E}\) is the trivial overconvergent \(F\)-isocrystal.

**Corollary 4.1.7.** (i) For any \((\mathcal{E}, \Phi, \text{Fil}^\ast) \in \text{Syn}(X_0, \mathcal{X}, \mathcal{X}), P \in \text{Poly}\) and any small enough \(n\), we have

\[
\begin{align*}
H^0_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n) &\cong H^0_{\text{HK}}(X_0, (\mathcal{E}, \Phi))^{P(\Phi^e)=0, N=0} \otimes_{K_0} K \\
H^0_{\text{syn}, P,c}(\mathcal{X}, \mathcal{E}, n) &\cong H^0_{\text{HK}, c}(X_0, (\mathcal{E}, \Phi))^{P(\Phi^e)=0, N=0} \otimes_{K_0} K.
\end{align*}
\]

(ii) Let \(d = \dim \mathcal{X}\). Suppose \(P \in \text{Poly}\) is a polynomial such that \(P(\Phi^e) = 0\) on \(K_0\). Then, for any \(n \in \mathbb{Z}\),

\[
H^{2d+2}_{\text{syn}, P,c}(\mathcal{X}, n) \cong K.
\]

**Proof.** Apply the diagram in Proposition 4.1.5 to \(i = 0\), one easily obtains the first assertion. For the second statement, apply loc. cit. again to \(i = 2d + 2\) and \(P\) yields

\[
H^{2d+2}_{\text{HK}, c}(X_0) \otimes_{K_0} K \cong H^{2d+2}_{\text{syn}, P,c}(\mathcal{X}, n)
\]

because of degree reasons. Note that \(H^{2d}_{\text{HK}, c}(X_0) \cong K_0\) with \(N\) acting by 0. Due to the choice of \(P\), the result then follows. \(\square\)
Corollary 4.1.8. Suppose now that $\mathcal{X}$ is smooth over $\mathcal{O}_K$ and so $X_0$ is smooth over $k$. Then, we have a short exact sequence

$$0 \to \frac{H_{dR}^{1}(\mathcal{X},\mathcal{E})}{P(\Phi^r)}(F^n H_{dR}^{1}(\mathcal{X},\mathcal{E})) \xrightarrow{i_n} H_{syn,P}(\mathcal{X},\mathcal{E},n) \xrightarrow{pr_{\beta_0}} H_{HK}^{0}(X_0,(\mathcal{E},\Phi))^{P(\Phi^r)=0} \otimes K_0 \cap F^n H_{dR}^{1}(\mathcal{X},\mathcal{E}) \to 0.$$  

In particular, if $\text{Fil}^* = (\mathcal{E} = \text{Fil}^r \mathcal{E} \supset \cdots \supset \text{Fil}^\ell \mathcal{E} \supset \text{Fil}^{\ell+1} \mathcal{E} = 0)$ for some $r, \ell \in \mathbb{Z}$ with $\ell \geq r$ and $d = \dim \mathcal{X}$, then

$$H_{syn,P}(\mathcal{X},\mathcal{E},0) \cong H_{HK}^{0}(X_0,(\mathcal{E},\Phi))^{P(\Phi^r)=0} \otimes K_0 \quad \text{and} \quad H_{dR}^{2d}(\mathcal{X},\mathcal{E}) \cong H_{syn,P}(\mathcal{X},\mathcal{E},\ell + d + 1).$$

Proof. Since $\mathcal{X}$ is now smooth over $\mathcal{O}_K$, the monodromy operator $N = 0$ and the compactly supported Hyodo–Kato cohomology agrees with the Hyodo–Kato cohomology, which both agree with the usual rigid cohomology (see Yam20, Proposition 8.9). Thus, we see that $H_{syn,P}(\mathcal{X},\mathcal{E},n) \cong H^i(C^*)$. The desired results then follows from the diagram in Proposition 4.1.5 and the isomorphism

$$\frac{H_{HK}^{i-1}(X_0,(\mathcal{E},\Phi)) \otimes K_0}{(P(\Phi^r)x,\Psi_{HK}(x)) : x \in H_{HK}^{i-1}(X_0,(\mathcal{E},\Phi))} \xrightarrow{\text{pr}_{\beta_0}} H_{dR}^{i-1}(\mathcal{X},\mathcal{E})^\vee,$$

where $(x,y) \mapsto x - P(\Phi^r)y$.

Note that when $\mathcal{E}$ is the trivial overconvergent $F$-isocrystal, the short exact sequence is exactly the one in Bes00, Proposition 2.5).

4.2 Cup products and pushforward maps

Proposition 4.2.1. Let $(\mathcal{E}, \Phi_\mathcal{E}, \text{Fil}_{\mathcal{E}}^\ast), (\mathcal{G}, \Phi_\mathcal{G}, \text{Fil}_{\mathcal{G}}^\ast) \in \text{Syn}(X_0, \mathcal{X}, \mathcal{X}), P, Q \in \text{Poly}$ and $n, m \in \mathbb{Z}$, we have a natural map

$$R\Gamma_{\text{syn},P}(\mathcal{X},\mathcal{E},n) \otimes R\Gamma_{\text{syn},Q,c}(\mathcal{X},\mathcal{G},m) \to R\Gamma_{\text{syn},P \ast Q,c}(\mathcal{X},\mathcal{E} \otimes \mathcal{G},n + m).$$

Here, for $P(X) = \prod_i(1 - \alpha_i X), Q(X) = \prod_j(1 - \beta_j X)$, we set $P \ast Q(X) = \prod_{i,j}(1 - \alpha_i \beta_j X)$.

Proof. Note that we have natural maps induced by tensor products

$$R\Gamma_{dR}(\mathcal{X},\mathcal{E}) \otimes R\Gamma_{dR}(\mathcal{X},\mathcal{G}) \to R\Gamma_{dR}(\mathcal{X},\mathcal{E} \otimes \mathcal{G})$$

$$R\Gamma_{HK}(X_0,(\mathcal{E},\Phi_\mathcal{E})) \otimes R\Gamma_{HK,c}(X_0,(\mathcal{G},\Phi_\mathcal{G})) \to R\Gamma_{HK,c}(X_0,(\mathcal{E} \otimes \mathcal{G},\Phi_\mathcal{E} \otimes \Phi_\mathcal{G}))$$

Note also that

$$(\text{Fil}_{\mathcal{E}}^i \mathcal{E} \otimes \Omega^s_{\mathcal{X}/K}) \times (\text{Fil}_{\mathcal{G}}^j \mathcal{G} \otimes \Omega^t_{\mathcal{X}/K}) \to (\text{Fil}_{\mathcal{E}}^i \mathcal{E} \otimes \text{Fil}_{\mathcal{G}}^j \mathcal{G}) \otimes \Omega^{s+t}_{\mathcal{X}/K},$$

induces a map

$$F^n R\Gamma_{dR}(\mathcal{X},\mathcal{E}) \otimes F^m R\Gamma_{dR}(\mathcal{X},\mathcal{G}) \to F^{n+m} R\Gamma_{dR,c}(\mathcal{X},\mathcal{E} \otimes \mathcal{G})$$

and so a map

$$\text{DR}(\mathcal{X},\mathcal{E},n) \otimes \text{DR}(\mathcal{X},\mathcal{G},m) \to \text{DR}(\mathcal{X},\mathcal{E} \otimes \mathcal{G},n + m).$$

On the other hand, for any $P, Q \in \text{Poly}$, we have a commutative diagram

$$\begin{array}{ccc}
R\Gamma_{HK}(X_0,(\mathcal{E},\Phi_\mathcal{E})) \otimes R\Gamma_{HK,c}(X_0,(\mathcal{G},\Phi_\mathcal{G})) & \longrightarrow & R\Gamma_{HK,c}(X_0) \\
\downarrow R\Gamma \quad & & \downarrow R\Gamma \\
R\Gamma_{HK}(X_0,(\mathcal{E},\Phi_\mathcal{E})) \otimes R\Gamma_{HK,c}(X_0,(\mathcal{G},\Phi_\mathcal{G})) & \longrightarrow & R\Gamma_{HK,c}(X_0,(\mathcal{E} \otimes \mathcal{G},\Phi_\mathcal{E} \otimes \Phi_\mathcal{G}))
\end{array}$$

This diagram is compatible with the monodromy operator $N$. These two observations then induce the desired pairing.

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For any \((\mathcal{E}, \Phi, \text{Fil}^\bullet) \in \text{Syn}(X_0, \mathfrak{X}, \mathcal{X})\), the maps in (6) induce pairings
\[
\begin{align*}
R\Gamma_{\text{dr}}(\mathcal{X}, \mathcal{E}) \otimes R\Gamma_{\text{dr}}(\mathcal{X}, \mathcal{E}^\vee) & \to R\Gamma_{\text{dr}}(\mathcal{X}) \\
R\Gamma_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes R\Gamma_{\text{HK},c}(X_0, (\mathcal{E}^\vee, \Phi^\vee)) & \to R\Gamma_{\text{HK},c}(X_0)
\end{align*}
\]
which yield quasi-isomorphisms (see [EY20, §6])
\[
\begin{align*}
R\Gamma_{\text{dr}}(\mathcal{X}, \mathcal{E}) & \simeq R\text{Hom}(R\Gamma_{\text{dr}}(\mathcal{X}, \mathcal{E}), K)[-2 \dim \mathcal{X}] & \text{in } D(\text{Mod}_K) \\
R\Gamma_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) & \simeq R\text{Hom}(R\Gamma_{\text{HK},c}(X_0, (\mathcal{E}^\vee, \Phi^\vee)), K_0)[-2 \dim \mathcal{X}] & \text{in } D(\text{Mod}_{K_0}(\varphi, N))
\end{align*}
\]
where \(D(\text{Mod}_K)\) is the derived category of \(K\)-vector spaces and \(D(\text{Mod}_{K_0}(\varphi, N))\) is the derived category of \((\varphi, N)\)-modules over \(K_0\).

**Corollary 4.2.2.** For any \(P, Q \in \text{Poly}\), there is a natural pairing
\[
R\Gamma_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n) \otimes R\Gamma_{\text{syn}, Q,c}(\mathcal{X}, \mathcal{E}^\vee, m) \to R\Gamma_{\text{syn}, P \ast Q,c}(\mathcal{X}, n + m),
\]
which is compatible with the pairings in (5) via the natural morphisms
\[
\begin{align*}
R\Gamma_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n) \otimes R\Gamma_{\text{syn}, Q,c}(\mathcal{X}, \mathcal{E}^\vee, m) & \to R\Gamma_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \otimes R\Gamma_{\text{HK},c}(X_0, (\mathcal{E}^\vee, \Phi^\vee)) \\
R\Gamma_{\text{dr}}(\mathcal{X}, \mathcal{E})[-1] \otimes R\Gamma_{\text{dr}}(\mathcal{X}, \mathcal{E}^\vee)[-1] & \to R\Gamma_{\text{syn}, P}(\mathcal{X}, \mathcal{E}, n) \otimes R\Gamma_{\text{syn}, Q,c}(\mathcal{X}, \mathcal{E}^\vee, m)
\end{align*}
\]

**Proof.** This is an immediate consequence of Proposition 4.2.1.

**Corollary 4.2.3.** Suppose \(\mathfrak{X}\) is a smooth weak formal scheme over \(\mathcal{O}_K\). Let \((\mathcal{E}, \Phi, \text{Fil}^\bullet) \in \text{Syn}(X_0, \mathfrak{X}, \mathcal{X})\). Suppose \(P, Q \in \text{Poly}\) are chosen such that
\begin{itemize}
\item \(P(\Phi^e)\) annihilates \(H_{\text{HK}}^i(X_0, (\mathcal{E}, \Phi))\) but acts invertibly on \(H_{\text{HK}}^{i-1}(X_0, (\mathcal{E}, \Phi))\);
\item \(Q(\Phi^{e \vee})\) annihilates \(H_{\text{HK}}^{2d-i+1}(X_0, (\mathcal{E}^\vee, \Phi^\vee))\) but acts invertibly on \(H_{\text{HK}}^{2d-i}(X_0, (\mathcal{E}^\vee, \Phi^\vee))\);
\item \(P \ast Q(\Phi^e)\) annihilates \(K_0\).
\end{itemize}
Then, we have a perfect pairing
\[
H_{\text{syn}, P}^i(\mathcal{X}, \mathcal{E}, n) \times H_{\text{syn}, Q}^{2d-i+1}(\mathcal{X}, \mathcal{E}^\vee, d - n + 1) \to K.
\]

**Proof.** Immediately from Corollary 4.2.2 we have
\[
H_{\text{syn}, P}^i(\mathcal{X}, \mathcal{E}, n) \times H_{\text{syn}, Q}^{2d-i+1}(\mathcal{X}, \mathcal{E}^\vee, d - n + 1) \to H_{\text{syn}, P \ast Q}^{2d+1}(\mathcal{X}, d + 1).
\]
One deduces easily from Corollary 4.1.8 that
\[
H_{\text{syn}, P \ast Q}^{2d+1}(\mathcal{X}, d + 1) \cong K
\]
and so we have a morphism
\[
H_{\text{syn}, P}^i(\mathcal{X}, \mathcal{E}, n) \to H_{\text{syn}, Q}^{2d-i+1}(\mathcal{X}, \mathcal{E}^\vee, d - n + 1) \to H_{\text{syn}, P \ast Q}^{2d+1}(\mathcal{X}, d + 1) \cong K.
\]

\[\text{By a } (\varphi, N)\text{-modules over } K_0 \text{ we mean a } K_0\text{-vector space equipped with a } \varphi\text{-semilinear action } \phi \text{ and a } K_0\text{-linear endo-}\]

morphism }\]
Applying loc. cit. again, we obtain a commutative diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & H^{i-1}_{\text{dR}}(X, \mathcal{E}) & \rightarrow & H^i_{\text{syn}, p}(X, \mathcal{E}, n) & \rightarrow & F^n H^i_{\text{dR}}(X, \mathcal{E}) & \rightarrow & 0 \\
\quad \quad \downarrow \cong & & \downarrow & & \downarrow \cong & & & & \\
0 & \rightarrow & (F^{d-n+1}H^{2d-i+1}_{\text{dR}}(X, \mathcal{E}^\vee)) & \rightarrow & H^{2d-i+1}_{\text{syn}, Q}(X, \mathcal{E}^\vee, d-n+1) & \rightarrow & \left(\frac{H^{2d-i}_{\text{dR}}(X, \mathcal{E}^\vee)}{F^{d-n+1}H^{2d-i+1}_{\text{dR}}(X, \mathcal{E}^\vee)}\right)^\vee & \rightarrow & 0
\end{array}
$$

The assertion then follows.

\[\square\]

**Remark 4.2.4.** When \( \mathfrak{X} \) is a general strictly semistable weak formal scheme over \( \mathcal{O}_K \), \((\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathfrak{X}, \mathcal{X}) \) and \( P, Q \in \text{Poly} \) are chosen so that \( P \ast Q(\mathcal{E}^\vee) \) annihilates \( K_0 \), then we have a pairing

\[ H^i_{\text{syn}, p}(X, \mathcal{E}, n) \times H^{2d-i+2}_{\text{syn}, Q,c}(X, \mathcal{E}^\vee, m) \rightarrow H^{2d+2}_{\text{syn}, P \ast Q,c}(X, m+n) \cong K. \]

However, we do not know whether this pairing is perfect. In fact, it seems to the authors that it is too optimistic to expect this pairing to be perfect.

Our next task is to establish a proper pushforward map for syntomic \( P \)-cohomology groups. In \[\text{Bes00},\] such a map is constructed by using the perfect pairing on finite polynomial cohomology groups. In our situation, we are not able to prove the pairing introduced above is a perfect pairing. However, one may still establish a proper pushforward map in our case by using the perfect pairings on the Hyodo–Kato cohomology groups and the perfect pairing on the de Rham cohomology groups. More precisely, we have the following result.

**Proposition 4.2.5.** Suppose \( \mathfrak{X} \) is a flat strictly semistable weak formal scheme over \( \mathcal{O}_{\text{log}=0}^{\text{log}=\infty} \) together with a exact closed immersion \( \iota : \mathfrak{X} \rightarrow \mathcal{X} \) of codimension \( i \). We write \( Z_0 \) for its special fibre and \( \mathcal{Z} \) for its dagger generic fibre. Assume also that \( \mathcal{Z} \) is also smooth over \( K \). Let \((\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathfrak{X}, \mathcal{X}) \) such that \( \text{Fil}^* = (\mathcal{E} = \text{Fil}^r \mathcal{E} \supset \cdots \supset \text{Fil}^\ell \mathcal{E} \supset \text{Fil}^\ell+1 \mathcal{E} = 0) \) for some \( r, \ell \in \mathbb{Z} \) with \( \ell \geq r \). Then, for any \( P \in \text{Poly} \) and any \( j \in \mathbb{Z}_{\geq 0} \), we have a pushforward map

\[ \iota_* : H^j_{\text{syn}, p}(Z_0, \iota^* \mathcal{E}, n) \rightarrow H^{j+i}_{\text{syn}, p}(X, \mathcal{E}, n+i). \]

**Proof.** The proof follows from the following three steps.

**Step 1.** We claim that there is a pushforward map

\[ \iota_* : H^j_{\text{HK}}(Z_0, \iota^* \mathcal{E}, \Phi) \rightarrow H^{j+i}_{\text{HK}}(X_0, (\mathcal{E}, \Phi)). \]

Note that \( H^j_{\text{HK}}(Z_0, \iota^* \mathcal{E}, \Phi) \) is the dual of \( H^{2d-2i-j}_{\text{HK,c}}(Z_0, (\iota^* \mathcal{E}^\vee, \Phi^\vee)) \) under the Poincaré pairing and we have a pullback map

\[ \iota^* : H^{2d-2i-j}_{\text{HK,c}}(X_0, (\mathcal{E}^\vee, \Phi^\vee)) \rightarrow H^{2d-2i-j}_{\text{HK,c}}(Z_0, (\iota^* \mathcal{E}^\vee, \Phi^\vee)). \]

Therefore, for any \( x \in H^{j}_{\text{HK}}(Z_0, \iota^* \mathcal{E}, \Phi) \), we define \( \iota_* x \) to be the class in \( H^{j+i}_{\text{HK}}(X_0, (\mathcal{E}, \Phi)) \) such that for any \( y \in H^{2d-2i-j}_{\text{HK,c}}(X_0, (\mathcal{E}^\vee, \Phi^\vee)) \),

\[ \langle \iota_* x, y \rangle = \langle x, \iota^* y \rangle. \]

Note that this formula uniquely defines \( \iota_* x \) due to the Poincaré duality on Hyodo–Kato cohomology groups.
Step 2. We claim that there is a pushforward map
\[ \iota_* : H^j_{dR}(Z, \iota^* E) \to H^{j+2i}_{dR}(X, E) \]
such that
\[ \iota_* (F^n H^j_{dR}(Z, \iota^* E)) \subset F^{n+i} H^{j+2i}_{dR}(X, E). \]
Indeed, the procedure is as above. We only have to check the condition on the filtration holds. However, $F^n H^j_{dR}(Z, \iota^* E)$ is the dual of $H^{2d-2i-j}_{dR}(Z, \iota^* E)$ and the pullback map
\[ \iota^* : \frac{H^{2d-2i-j}_{dR}(X, E^\vee)}{F^{d-i-n+1} H^{2d-2i-j}_{dR}(X, E^\vee)} \to \frac{H^{d-i-j}_{dR}(Z, \iota^* E^\vee)}{F^{d-2i-n+1} H^{2d-2i-j}_{dR}(Z, \iota^* E^\vee)} \]
allows us to conclude the result as $\frac{H^{2d-2i-j}_{dR}(X, E^\vee)}{F^{d-i-n+1} H^{2d-2i-j}_{dR}(X, E^\vee)}$ is the dual of $F^{n+i} H^{j+2i}_{dR}(X, E^\vee)$.

Step 3. Finally, note that the Poincaré pairings are compatible with the Frobenius action and the monodromy operator. Therefore, everything can be put together and one obtains the desired pushforward map for syntomic $P$-cohomology groups.

5 The Abel–Jacobi map and $p$-adic integration

The main goal of this section is to provide a construction of the Abel–Jacobi map. Following Besser’s strategy in [Bes00], we first show that finite polynomial cohomology with coefficients can be viewed as a generalisation of Coleman’s integration theory for modules with connections in [Col94, Section 10]. Then we construct the Abel–Jacobi map in §5.2. As mentioned in the introduction, the idea is that the Abel–Jacobi map should admit a interpretation as certain kind of integration. Since we wish to utilise the Coleman integration theory, we will restrict ourselves to varieties with good reduction in the aforementioned discussions. In the final subsection §5.3 we shall briefly discuss the case for varieties with semistable reduction.

We begin with some general construction.

Let $X$ be a proper smooth scheme over $\mathcal{O}_K$, which is of relative dimension $d$. We write $X_0$ (resp., $\mathfrak{X}$; resp., $X$) for its special fibre (resp., $\varpi$-adic weak completion; resp., dagger generic fibre). For each $i \in \mathbb{Z}_{\geq 0}$, we define
\[ Z^i(X) := \{ Z : \text{smooth irreducible closed subscheme of } X \text{ of codimension } i \}. \]
For each $Z \in Z^i(X)$, we again use the similar notation and denote by $Z_0$ (resp., $\mathfrak{Z}$; resp., $Z$) its special fibre (resp., $\varpi$-adic weak completion; resp., dagger generic fibre). For each such $Z$, we write $\iota_Z : \mathfrak{Z} \hookrightarrow \mathfrak{X}$ for the induced closed immersion from $\mathfrak{Z}$ to $\mathfrak{X}$.

Lemma 5.0.1. Let $(\mathfrak{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, X, X)$. Let $i, j \in \mathbb{Z}_{\geq 0}$ and let $Z \in Z^i(X)$. Then, the Frobenius action on $H^j_{HK}(Z_0/K_0, (\mathfrak{E}, \Phi)) := H^j_{HK}(Z_0/K_0, (\iota^*_Z \mathfrak{E}, \iota^*_Z \Phi))$ has eigenvalues of Weil weight $j$.

Proof. Note that, by definition, $\mathfrak{E}$ is a unipotent $F$-isocrystal. Thus, it suffices to show the statement for the trivial coefficient. However, this is a result in [CLS98].

Let
\[ \text{Poly}_j := \text{the submonoid of Poly, consisting of polynomials of weight } j, \]
i.e., those having roots $\alpha$ with $|\alpha|_C = q^{j/2}$. 

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For any $Z \in Z^i(X)$, any $n \in \mathbb{Z}$ and any $j \in \mathbb{Z}$, we write
\[ H^j_{\text{fp}}(Z, \mathcal{E}, n) := \lim_{P \in \text{Poly}} H^j_{\text{syn}, P}(Z, \iota^* \mathcal{E}, n). \] (9)

Then, we have the following immediate corollary.

**Corollary 5.0.2.** Let $(\mathcal{E}, \Phi, \text{Fil}) \in \text{Syn}(X_0, \mathfrak{X}, \mathcal{X})$. For any $Z \in Z^i(X)$, any $n \in \mathbb{Z}$ and any $j \in \mathbb{Z}$, we have a short exact sequence
\[ 0 \to \frac{H^j_{\text{dR}}(Z, \mathcal{E})}{F^n H^j_{\text{dR}}(Z, \mathcal{E})} \xrightarrow{i_{\text{dR}}} H^j_{\text{fp}}(Z, \mathcal{E}, n) \xrightarrow{pr_{\text{dR}}} F^n H^j_{\text{dR}}(Z, \mathcal{E}) \to 0. \]
Here, $H^j_{\text{dR}}(Z, \mathcal{E}) := H^j_{\text{dR}}(Z, \iota^* \mathcal{E})$.

**Proof.** The assertion follows immediately from Corollary 4.1.8 and Lemma 5.0.1. \[ \square \]

### 5.1 A connection with Coleman’s $p$-adic integration

Suppose in this subsection that $X$ is a proper smooth curve over $\mathcal{O}_K$. We also assume that $K = K_0$ for the convenience of the exposition. Suppose moreover that $(\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, \mathfrak{X}, \mathcal{X})$ is of trivial filtration, i.e.,
\[ \text{Fil}^i \mathcal{E} = \begin{cases} \mathcal{E}, & i \leq 0 \\ 0, & i > 0 \end{cases} \]

For any affine open $Y \subset X$, we again similarly write $Y_0$ (resp., $\mathfrak{Y}$; resp., $\mathcal{Y}$) for its special fibre (resp., $\mathfrak{O}$-adic weak completion; resp., dagger generic fibre). For any polynomial $P \in \text{Poly}$, we similarly consider
\[ R\Gamma_{\text{syn}, P}(\mathcal{Y}, \mathcal{E}, n) := \text{Cone} \left( R\Gamma_{\text{HK}}(Y_0/K, (\mathcal{E}, \Phi)) \xrightarrow{P(\Phi^e) \otimes \Psi_{\text{HK}}} R\Gamma_{\text{HK}}(Y_0/K, (\mathcal{E}, \Phi)) \oplus R\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{E})/F^n R\Gamma_{\text{dR}}(\mathcal{Y}, \mathcal{E}) \right) [-1]. \]
Note that, since $X$ is smooth over $\mathcal{O}_K$, we do not have the monodromy operator; also, because of the assumption $K = K_0$, the Hyodo–Kato map $\Phi_{\text{HK}}$ is, in fact, the identity map. Immediately from the construction, we see that there is a restriction map
\[ R\Gamma_{\text{syn}, P}(X, \mathcal{E}, n) \to R\Gamma_{\text{syn}, P}(\mathcal{Y}, \mathcal{E}, n), \quad x \mapsto x|_{\mathcal{Y}}. \]

Our main results of this subsection reads as follows.

**Lemma 5.1.1.** Let $(\mathcal{E}, \Phi, \text{Fil}^*)$ be as above. Then, for any affine open $Y \subset X$, we have an isomorphism
\[ H^1_{\text{syn}, P}(\mathcal{Y}, \mathcal{E}, 1) \cong \left\{ (f, \omega) \in \mathcal{E}(\mathcal{Y}) \oplus \mathcal{E}(\mathcal{Y}) \otimes \mathcal{E}(\mathcal{Y}) \Omega^1_{\mathcal{Y}/K} : \nabla f = P(\Phi^e)\omega \right\}. \]

**Proof.** By the assumption on the filtration of $\mathcal{E}$ and the construction, the complex defining $R\Gamma_{\text{syn}, P}(\mathcal{Y}, \mathcal{E}, 1)$ is given by
\[ \mathcal{E}(\mathcal{Y}) \xrightarrow{\nabla x = (-\nabla x, P(\Phi^e)x, x)} \left( \mathcal{E}(\mathcal{Y}) \otimes \mathcal{E}(\mathcal{Y}) \Omega^1_{\mathcal{Y}/K} \right) \oplus \mathcal{E}(\mathcal{Y}) \oplus \mathcal{E}(\mathcal{Y}) \xrightarrow{(x, y, z) \mapsto P(\Phi^e)x + \nabla y} \mathcal{E}(\mathcal{Y}) \otimes \mathcal{E}(\mathcal{Y}) \Omega^1_{\mathcal{Y}/K}. \]
Therefore,
\[ H^1_{\text{syn}, P}(\mathcal{Y}, \mathcal{E}, 1) = \left\{ (x, y, z) \in \left( \mathcal{E}(\mathcal{Y}) \otimes \mathcal{E}(\mathcal{Y}) \Omega^1_{\mathcal{Y}/K} \right) \oplus \mathcal{E}(\mathcal{Y}) \oplus \mathcal{E}(\mathcal{Y}) : P(\Phi^e)x + \nabla y = 0 \right\} \bigcup \left\{ (-\nabla x, P(\Phi^e)x, x) : x \in \mathcal{E}(\mathcal{Y}) \right\}. \]
However, one observes that there is an isomorphism
\[ H^1_{\text{syn}, P}(\mathcal{Y}, \mathcal{E}, 1) \to \left\{ (f, \omega) \in \mathcal{E}(\mathcal{Y}) \oplus \mathcal{E}(\mathcal{Y}) \otimes \mathcal{E}(\mathcal{Y}) \Omega^1_{\mathcal{Y}/K} : \nabla f = \omega \right\}, \quad (x, y, z) \mapsto (-x - \nabla z, y - P(\Phi^e)z), \]
the assertion then follows. \[ \square \]
To link our theory with Coleman integration, we have to introduce more terminologies. Let \( \hat{X} \) be the \( \varpi \)-adic completion of \( X \), which is viewed as a rigid analytic space. Recall that there is a specialisation map

\[
sp : |\hat{X}| \to |X|_0,
\]

locally given by

\[
|\Sp(A, A^\varpi)| \to |\Spec A^\varpi/\varpi|, \quad \cdot|_x \mapsto p_x := \{a \in A^\varpi/\varpi : |\tilde{a}|_x < 1 \text{ for some } \tilde{a} \in A^\varpi \text{ s.t. } \tilde{a} \equiv a \mod \varpi\}.
\]

Then, for any affine open \( U_0 \subset X_0 \), we define the tube

\[
|U_0|_{\hat{X}} := \text{the interior of } sp^{-1}(U_0).
\]

Note that any affine open \( U_0 \subset X_0 \) is of the form \( U_0 = X_0 \setminus \{x_1, \ldots, x_t\} \) where \( x_i \) are points in \( X_0 \). Thus,

\[
|U_0|_{\hat{X}} = \hat{X} \setminus (\bigcup_{j=1}^r D(\tilde{x}_j, 1))
\]

where \( \tilde{x}_j \in \hat{X} \) is a lift of \( x_j \) and \( D(\tilde{x}_j, 1) \) is the disc of radius 1 centred at \( \tilde{x}_j \), i.e., \( D(\tilde{x}_j, 1) = |x_j|_{\hat{X}} \). An open subset \( \mathcal{W} \subset \hat{X} \) of the form

\[
\hat{X} \setminus (\bigcup_{j=1}^r D(\tilde{x}_j, r_j))
\]

with \( 0 < r_j < 1 \) is called a strict neighbourhood of \( |U_0|_{\hat{X}} \) (in \( \hat{X} \)).

Moreover, if we start with \( \mathscr{E} \in \text{Isoc}^1(X_0/\mathcal{O}_K) \), then its \( \varpi \)-adic completion \( \widehat{\mathscr{E}} \) defines a coherent sheaf on \( \hat{X} \) with integrable connection

\[
\widehat{\mathcal{E}} \xrightarrow{\nabla} \widehat{\mathcal{E}} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/K}.
\]

One sees immediately that there is a natural map

\[
H^1_{\text{dR}}(X, \mathscr{E}) \to H^1_{\text{dR}}(\hat{X}, \widehat{\mathcal{E}})
\]

given by the \( \varpi \)-adic completion. However, since both spaces are finite-dimensional \( K \)-vector spaces, this morphism is, in fact, an isomorphism.

**Corollary 5.1.2.** Let \( (\mathcal{E}, \Phi, \Fil^*) \in \text{Syn}(X_0, \mathfrak{X}, X) \) be as above and let \( P \in \text{Poly} \) be a polynomial such that \( P(\Phi^\varrho) \) annihilates \( H^1_{\text{dR}}(X, (\mathscr{E}, \Phi)) \) and is an isomorphism on \( H^0_{\text{HK}}(X, (\mathscr{E}, \Phi)) \). So we have a projection

\[
pr_{\text{Tr}} : H^1_{\text{syn}, P}(X, \mathcal{E}, 1) \to F^1 H^1_{\text{dR}}(X, \mathcal{E}) = \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K}).
\]

Then, for any \( \omega \in \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K}) \), a lift \( \tilde{\omega} \in H^1_{\text{syn}, P}(X, \mathcal{E}, 1) \) of \( \omega \) can be viewed as an Coleman integral of \( \omega \) in the following sense:

\[
\text{Let } Y_0 \subset X_0 \text{ be an affine open subscheme. Then the pullback of } \tilde{\omega} \text{ in } H^1_{\text{syn}, P}(\overline{Y_0|_{\hat{X}}}, \mathcal{E}, 1) \text{ corresponds uniquely to a locally analytic section } F_\omega \text{ (c.f. } \text{[Col94] §10}) \text{ of } \mathcal{E} \text{ over some strict neighbourhood } \mathcal{W} \text{ of } |\overline{Y_0|_{\hat{X}}}| \text{ such that } \nabla F_\omega = \omega|_W. \text{ Moreover, if } \tilde{\omega}' \text{ is another lift of } \omega \text{ that corresponds to the integration } F_\omega' \text{ on } W, \text{ then } F_\omega - F_\omega' \in \mathcal{E}(W)^{\nabla=0}.
\]

**Proof.** By Lemma 5.1.1 we see that

\[
\tilde{\omega}|_Y = (f, \omega) \text{ with } \nabla f = P(\Phi^\varrho)\omega.
\]

Then, by applying [Col94, Theorem 10.1], we have the desired \( F_\omega \). □

**Remark 5.1.3.** Careful readers will find that there is a condition of \( \mathcal{E} \) having regular singular annuli in [Col94, Theorem 10.1]. We remark that, since we start with \( (\mathcal{E}, \Phi, \Fil^*) \in \text{Syn}(X_0, \mathfrak{X}, X) \), \( \mathcal{E} \) is, in particular, unipotent. This then implies that Coleman’s condition of \( \mathcal{E} \) being regular singular is fulfilled. □
5.2 The Abel–Jacobi map

Let $X$ be a proper smooth scheme over $O_X$, which is of relative dimension $d$. We fix $(\mathcal{E}, \Phi, \text{Fil}^*) \in \text{Syn}(X_0, X, \mathcal{E})$ such that $\text{Fil}^* = (\mathcal{E} = \text{Fil}^0 \mathcal{E} \supset \cdots \supset \text{Fil}^i \mathcal{E} \supset \text{Fil}^i+1 \mathcal{E} = 0)$. Consequently, for any $i \in \mathbb{Z}_{\geq 0}$, $Z \in Z^i(X)$, we have

$$H^0_{\text{fp}}(Z, \mathcal{E}, 0) \cong H^0_{\text{dR}}(Z, \mathcal{E}).$$

We shall then always identify $H^0_{\text{fp}}(Z, \mathcal{E}, 0)$ with $H^0_{\text{dR}}(Z, \mathcal{E})$ via this isomorphism.

Inspired by the classical definition of cycle class groups, we have the following definition.

**Definition 5.2.1.** Let $(\mathcal{E}, \Phi, \text{Fil})$ be as above. Let $i \in \mathbb{Z}_{\geq 0}$. Then, the (de Rham) cycle class group of codimension $i$ with respect to $\mathcal{E}$ is defined to be

$$A^i(X, \mathcal{E}) := \oplus_{Z \in Z^i(X)} H^0_{\text{dR}}(Z, \mathcal{E}).$$

An element in $A^i(X, \mathcal{E})$ will be written in the form $(\theta_Z)_Z$ or $\sum_Z \theta_Z \cdot Z$.

Note that, by applying Proposition 4.2.5, we have a pushforward map

$$\iota_{3,*} : H^0_{\text{fp}}(Z, \mathcal{E}) \rightarrow H^{2i}_{\text{fp}}(X, \mathcal{E}, i).$$

It then induces a morphism

$$\eta_{\text{fp}} : A^i(X, \mathcal{E}) \rightarrow H^{2i}_{\text{fp}}(X, \mathcal{E}, i), \quad (\theta_Z)_Z \mapsto \sum_{Z \in Z^i(X)} \iota_{3,*} \theta_Z.$$
Proof. Observe that
\[
\eta_{\text{fp}}(A^i(X, \mathcal{E})) \subset \ker pr_{\text{fp}} = \frac{H^{2i-1}_{\text{dR}}(X, \mathcal{E})}{F^i H^{2i-1}_{\text{dR}}(X, \mathcal{E})}.
\]
However, the Poincaré duality for de Rham cohomology implies that

\[
\frac{H^{2i-1}_{\text{dR}}(X, \mathcal{E})}{F^i H^{2i-1}_{\text{dR}}(X, \mathcal{E})} \cong (F^{d-i+1} H^{2d-2i+1}_{\text{dR}}(X, \mathcal{E}^\vee))^{\vee}.
\]

Therefore, we arrive at the desired map

\[
\text{AJ}_{\text{fp}} : A^i(X, \mathcal{E})_0 \to \frac{H^{2i-1}_{\text{dR}}(X, \mathcal{E})}{F^i H^{2i-1}_{\text{dR}}(X, \mathcal{E})} \cong (F^{d-i+1} H^{2d-2i+1}_{\text{dR}}(X, \mathcal{E}^\vee))^{\vee}.
\]

To show the formula, it is enough to look at one \( Z \in Z^i(X) \) and any \( \theta_Z \in H^0_{\text{dR}}(Z, \mathcal{E}) \). For any \( \omega \in F^{d-i+1} H^{2d-2i+1}_{\text{dR}}(X, \mathcal{E}^\vee) \), we choose a lift \( \widetilde{\omega} \in H^{2d-2i+1}_{\text{fp}}(X, \mathcal{E}^\vee, d - i + 1) \) via the short exact sequence

\[
0 \to \frac{H^{2d-2i}_{\text{dR}}(X, \mathcal{E}^\vee)}{F^{d-i+1} H^{2d-2i}_{\text{dR}}(X, \mathcal{E}^\vee)} \to H^{2d-2i}_{\text{fp}}(X, \mathcal{E}^\vee, d - i + 1) \to F^n H^{2d-2i+1}_{\text{dR}}(X, \mathcal{E}^\vee) \to 0.
\]

Then, we have

\[
\text{AJ}_{\text{fp}}(\theta_Z \cdot Z)(\omega) = (i^*_Z \widetilde{\omega}, \theta_Z)_{\text{fp}, Z} = (\widetilde{\omega}, \iota_3, \theta_Z)_{\text{fp}, X},
\]

where the first equation follows from the compatibility in Corollary 5.2.2, the second equation follows from the definition of \( \text{AJ}_{\text{fp}} \) and the final equation follows from the construction of the pushforward map.

\[\square\]

Remark 5.2.4. Comparing the Abel–Jacobi map with coefficients with the classical one (recalled in §1.1), one might find \( A^i(X, \mathcal{E}) \) a bit ad hoc. However, one can check that the classical Abel–Jacobi map, in fact, factors as

\[
\text{AJ} : A^i(X) \to A^i(X, \mathcal{O}_{X_0/\mathcal{O}_K})_0 \to (F^{d-i+1} H^{2d-2i+1}_{\text{dR}}(X))^{\vee}.
\]

One then sees that our construction actually agrees with the classical one when the coefficient is trivial. •

Remark 5.2.5. It is clear from Corollary 5.1.2 that the lift \( \tilde{\omega} \) should be viewed as a (generalised) ‘Coleman integral’. This justifies the notation \( \int_Z \omega \) in Theorem 5.2.3. •

5.3 Towards a semistable theory

In this subsection, we would like to discuss a direction how one could generalise the finite polynomial Abel–Jacobi map to varieties with semistable reduction. Therefore, we fix a proper scheme \( X \) over \( \mathcal{O}_K \), which is of relative dimension \( d \). We assume its special fibre \( X_0 \) is strictly semistable and its \( \varpi \)-adic weak formal completion \( \hat{X} \) has a proper smooth dagger generic fibre \( X' \).

For each \( i \in \mathbb{Z}_{\geq 0} \), we define

\[
Z^i(X) := \left\{ Z \to X \text{ exact closed immersion of codimension } i : \text{Z is strictly semistable over } \mathcal{O}_K^{\log=\varpi} \text{ the dagger generic fibre } Z \text{ of } X \text{ is smooth over } K \right\}.
\]

For each \( Z \in Z^i(X) \), we similarly denote its special fibre (resp., \( \varpi \)-adic weak formal completion) by \( Z_0 \) (resp., \( \mathfrak{z} \)). Also, we write \( \iota_3 : \mathfrak{z} \to \hat{X} \) for the induced exact closed immersion.

Let \((\mathcal{E}, \Phi, \text{Fil}^s) \in \text{Syn}(X_0, \hat{X}, X')\). We again assume \( \text{Fil}^s = (\mathcal{E} = \text{Fil}^0 \mathcal{E} \supset \cdots \supset \text{Fil}^t \mathcal{E} \supset \text{Fil}^{t+1} \mathcal{E} = 0) \) to simplify the notation in computation. Hence, by Corollary 4.1.7 we have

\[
H^0_{\text{syn}, \mathcal{P}}(Z, \mathcal{E}, 0) \equiv H^0_{\text{HK}}(Z_0, (i^*_3 \mathcal{E}, \iota_3^* \Phi))^{P(\mathcal{E}^s) = 0, N = 0} \otimes_{\mathcal{O}_K} K
\]

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for any \( Z \in Z^i(X) \) and any \( P \in \text{Poly} \).

From now on, we fix a \( P \in \text{Poly} \). We similarly consider the (Hyodo–Kato) cycle class group of codimension \( i \) with respect to \( P \) and \( \mathcal{E} \), defined to be

\[
A^i(X, P, \mathcal{E}) := \oplus_{Z \in Z^i(X)} H^0_{\text{HK}}(Z_0, (i^*_Z \mathcal{E}, i^*_Z \Phi))^{P(\Phi^\vee) = 0, N = 0} \otimes K.
\]

Then, via the pushforward map in Proposition 4.2.5, one obtains a map

\[
\eta_P : A^i(X, P, \mathcal{E}) \to H^{2i}_{\text{syn}, P}(X, \mathcal{E}, i).
\]

Recall from Proposition 4.1.5 that we have a natural map

\[
\eta_P : A^i(X, P, \mathcal{E}) \to H^{2i}_{\text{HK}}(X_0, (\mathcal{E}, \Phi))^{P(\Phi^\vee) = 0, N = 0} \otimes K \cap F^i H^d_{\text{dR}}(X, \mathcal{E}).
\]

Consequently, by applying the 3-step filtration in Corollary 4.1.6, we can define the null-homologous cycles to be elements in

\[
A^i(X, P, \mathcal{E})_0 := \ker \eta_P \circ \eta_P.
\]

Composing \( \eta_P \) with \( \eta_P \), we define the syntomic \( P \)-Abel–Jacobi map

\[
A_{\text{syn}, P}(X, P, \mathcal{E})_0 \to F^1_P.
\]

To understand \( A_{\text{syn}, P} \), choose \( Q \in \text{Poly} \) such that \( P \circ Q(\mathcal{E}) \) annihilates \( K_0 \). Then, the pairing in Remark 4.2.4 yields a map

\[
H^{2i}_{\text{syn}, P}(X, \mathcal{E}, i) \to \left( H^{2d-2i+2}_{\text{syn}, Q, c}(X, \mathcal{E}^\vee, m) \right) ^* \quad \text{for any} \ m \in \mathbb{Z}.
\]

Therefore, for any \( \omega \in H^{2d-2i+2}_{\text{syn}, Q, c}(X, \mathcal{E}^\vee, m) \) and for any \( (\theta_Z)_Z \in A^i(X, P, \mathcal{E})_0 \), we have

\[
A_{\text{syn}, P}((\theta_Z)_Z)(\omega) = \sum_{Z \in Z^i(X)} \langle \iota^*_Z \theta_Z, \omega \rangle = \sum_{Z \in Z^i(X)} \langle \theta_Z, \iota^*_Z \omega \rangle, \quad (10)
\]

where the pairing \( \langle \cdot, \cdot \rangle \) is the pairing in Remark 4.2.4.

We finally remark that the formalism above is far from satisfactory for the following reasons:

(i) Since we do not have a perfect pairing in this situation, equation (10) does not uniquely determine the class \( A_{\text{syn}, P}((\theta_Z)_Z) \).

(ii) In practice, the cohomology classes that we are interested in usually come from the de Rham cohomology. However, we can only work out a formula for the syntomic \( P \)-cohomology groups. Of course, if one could choose \( Q \) such that \( Q(\mathcal{E}^\vee) \) further acts isomorphically on \( H^{2d-2i+1}_{\text{HK}, \mathcal{E}}(X, \mathcal{E}^\vee, m) \), then we have a surjection

\[
H^{2d-2i+2}_{\text{syn}, Q, c}(X, \mathcal{E}^\vee, m) \to H^{2d-2i+2}_{\text{HK}, \mathcal{E}}(X_0, (\mathcal{E}^\vee, \Phi^\vee))^{Q(\Phi^\vee) = 0, N = 0} \otimes K \cap F^m H^{2d-2i+2}_{\text{dR}}(X, \mathcal{E}^\vee).
\]

Hence, for any \( \omega \in H^{2d-2i+2}_{\text{HK}, \mathcal{E}}(X_0, (\mathcal{E}^\vee, \Phi^\vee))^{Q(\Phi^\vee) = 0, N = 0} \otimes K \cap F^m H^{2d-2i+2}_{\text{dR}}(X, \mathcal{E}^\vee) \), one can choose a lift \( \tilde{\omega} \in H^{2d-2i+2}_{\text{syn}, Q, c}(X, \mathcal{E}^\vee, m) \). However, (10) still suggests that the resulting calculation will still depend on the choice \( \tilde{\omega} \).

(iii) Finally, it would be more pleasant to have a theory that does not depend on the choice of \( P \) and \( Q \). However, unlike in the good reduction case, we do not have a good control on the eigenvalues of the Frobenius actions on the cohomology groups. To the authors’ knowledge, these eigenvalues could be pathological for general coefficients. Even for certain nice enough coefficients, we still know little on how one can control the Frobenius actions (see, for example, [Yam20, Proposition 6.6]).
6 Applications

In this section, we aim to carry out an arithmetic application of the previous theory. More precisely, we would like to establish a relation between the finite polynomial Abel–Jacobi map and the arithmetic of compact Shimura curves over \( \mathbb{Q} \). Our result is inspired by the works of Darmon–Rotger in [DR14] and Bertolini–Darmon–Prasanna in [BDP13].

To establish such an arithmetic application, one immediately encounters the problem that, in the previous sections, we consider our coefficients in \( \text{Syn}(X_0, \mathcal{X}, \mathcal{X}') \), which are required to be unipotent, while the sheaves that one often encounters in the theory of automorphic forms are not unipotent. However, taking a closer look at our theory, one finds that the condition of unipotence can be loosened once the following conditions are satisfied:

**Conditions.** Suppose \( X_0, \mathcal{X} \) and \( \mathcal{X}' \) are as in (4). Let \( \text{Syn}^\#(X_0, \mathcal{X}, \mathcal{X}') \) be the category consisting of triples \((\mathcal{E}, \Phi, \text{Fil}^*)\), where \((\mathcal{E}, \Phi) \in \text{Fls}_c(X_0/O_{K_0}^{\log=\emptyset})\) and \( \text{Fil}^* \) be a filtration of \( \mathcal{E} \) on \( \mathcal{X} \) that satisfies Griffiths’s transversality.

- (HK-dR). There exists an morphism

\[
\Psi_{HK} : R\Gamma_{HK}(X_0, (\mathcal{E}, \Phi)) \to R\Gamma_{dR}(\mathcal{X}, \mathcal{E})
\]

such that, after base change to \( K \), \( \Psi_{HK} \otimes \text{id} \) is a quasi-isomorphism.

- (Purity). Suppose moreover that there is a smooth scheme \( X \) over \( \mathcal{O}_K \) whose \( \mathbb{Q} \)-adic weak completion is \( \mathcal{X} \). Then, for every closed irreducible smooth subscheme \( Z \subset X \), the characteristic polynomial of \( \Phi \) on \( H^1_{\text{HK}}(Z_0, (\mathcal{E}, \Phi)) \) is of pure Weil weight \( s(i) \in \mathbb{Z} \), which is independent of \( Z \), and \( s(i) < s(i+1) \) for all \( i \geq 0 \).

Remark that the hypothesis (HK-dR) allows us to consider the diagram in Definition 4.1.1 while the hypothesis (Purity) allows us to consider

\[
H^1_{\text{fp}}(Z, \mathcal{E}, n) := \lim_{\substack{\text{poly}(i) \to \mathcal{X}}} H^i_{\text{syn}, p}(Z, \mathcal{E}, n)
\]

as in (9). In this situation, the formalism in §5.2 immediately goes through.

6.1 Compact Shimura curves over \( \mathbb{Q} \)

In this subsection, we briefly recall the theory of compact Shimura curves over \( \mathbb{Q} \) as a preparation of the next subsection. Our main references are [DT94, Buz97, Kas99] and readers are encouraged to consult op. cit. for more detail discussions.

Let \( D \) be an indefinite non-split quaternion algebra over \( \mathbb{Q} \). We assume that \( p \) does not divide the discriminant \( \text{disc}(D) \) of \( D \). In particular, \( D \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p) \). We fix a maximal order \( \mathcal{O}_D \subset D \) and an isomorphism \( \mathcal{O}_D \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p) \). We further denote by \( \tilde{G} \) the algebraic group over \( \mathbb{Z} \), whose \( R \)-points are given by

\[
\tilde{G}(R) := (\mathcal{O}_D \otimes_{\mathbb{Z}} R)^\times
\]

for any ring \( R \). We fix once and for all a neat compact open subgroup \( \Gamma^p \subset G(\tilde{\mathbb{Z}}^p) \) such that \( \Gamma^p = \prod_{\ell \neq p} \Gamma_\ell \) with \( \Gamma_\ell \subset G(\mathbb{Z}_\ell) \) being compact open and \( \det \Gamma^p = \tilde{\mathbb{Z}}^{r \times} \). We set \( \Gamma := \Gamma^p G(\mathbb{Z}_p) = \Gamma^p \text{GL}_2(\mathbb{Z}_p) \).

By a **false elliptic curve** over a \( \mathbb{Z}[1/\text{disc}(D)] \)-scheme \( S \), we mean a pair \((A, i)\), where \( A \) is an abelian surface over \( S \) and \( i : \mathcal{O}_D \to \text{End}_S(A) \). Note that on \((A, i)\), there is a unique principal polarisation ([Buz97, §1]). By the discussion of §2 of op. cit., we know that the functor

\[
\text{Sch}_{\mathbb{Z}_p} \to \text{Sets}, \quad S \mapsto \left\{ (A, i, \alpha) : \begin{array}{l}
(A, i) \text{ is a false elliptic curve over } S \\
\alpha \text{ is a } \Gamma \text{-level structure}
\end{array} \right\} / \simeq
\]

is
is representable by a smooth projective scheme over $\mathbb{Z}_p$ of relative dimension 1. We denote this curve by $X$. As usual, we write $X_0$ for the special fibre of $X$ over $\mathbb{F}_p$, $\mathcal{X}$ for the $p$-adic weak completion of $X$, and $\bar{\mathcal{X}}$ the dagger space associated with $\mathcal{X}$.

Let $\pi : A^\text{univ} \to X$ be the universal false elliptic curve over $X$. We consider three sheaves

$$\tilde{\omega} := \pi^* \Omega^1_{A^\text{univ}/X}, \quad \tilde{\mathcal{H}} := R^1 \pi_* \Omega^1_{A^\text{univ}/X}, \quad \text{and} \quad \tilde{\omega}^{-1} := R^1 \pi_* \mathcal{O}_{A^\text{univ}}$$

with a canonical exact sequence

$$0 \to \tilde{\omega} \to \tilde{\mathcal{H}} \to \tilde{\omega}^{-1} \to 0 \quad (11)$$

given by the Hodge–de Rham spectral sequence. We choose once and for all an idempotent $\varepsilon \in M_2(\mathbb{Z}_p) = \mathcal{O}_D$ as in [DT94, §4] (e.g., $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$) and define

$$\omega := \varepsilon \tilde{\omega}, \quad \mathcal{H} := \varepsilon \tilde{\mathcal{H}}, \quad \text{and} \quad \omega^{-1} := \varepsilon \tilde{\omega}^{-1}.$$

**Lemma 6.1.1.** The sheaves $\omega$, $\mathcal{H}$ and $\omega^{-1}$ enjoy the following properties.

(i) We have $\omega^{-1} \cong \omega^\vee$ and $\omega \cong \omega^{-1} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{Z}_p}$.

(ii) There exists a exact sequence

$$0 \to \omega \to \mathcal{H} \to \omega^{-1} \to 0.$$

In particular, $\mathcal{H}$ is of locally free of rank 2 over $X$ with a filtration

$$\text{Fil}^i \mathcal{H} = \begin{cases} \mathcal{H}, & i \leq 0 \\ \omega, & i = 1 \\ 0, & i > 1 \end{cases}.$$

(iii) The sheaf $\mathcal{H}$ is self-dual.

(iv) There is an integrable connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{Z}_p}$ such that the filtration in (ii) satisfies Griffiths’s transversality.

**Proof.** The first assertion is nothing but [DT94, Lemma 7]. For the second assertion, one applies $\varepsilon$ to the exact sequence (11). For (iii), note that $\tilde{\mathcal{H}}$ is self-dual, whose duality comes from the principal polarisation on $A^\text{univ}$. Together with the fact that $\tilde{\mathcal{H}}$ is dual to $(1 - \varepsilon)\tilde{\mathcal{H}}$ and these two subsheaves are isomorphic to each other (see, for example, [Bra12, §1.4] or [Bra14, §1]), we conclude (iii). Finally, (iv) follows from that $\tilde{\mathcal{H}}$ admits an integrable connection, i.e., the Gauss–Manin connection, that satisfies Griffiths’s transversality.

For any $k \in \mathbb{Z}_{\geq 0}$, we define

$$\omega^k := \omega^k \quad \text{and} \quad \mathcal{H}^k := \text{Sym}^k \mathcal{H}.$$

By Lemma 6.1.1 we see that

- $\mathcal{H}^k$ admits a filtration of length $k$, i.e.,

$$\mathcal{H}^k = \text{Fil}^0 \mathcal{H}^k \supset \text{Fil}^1 \mathcal{H}^k \supset \cdots \supset \text{Fil}^k \mathcal{H}^k \supset \text{Fil}^{k+1} \mathcal{H}^k = 0$$

with

$$\text{Fil}^i \mathcal{H}^k / \text{Fil}^{i+1} \mathcal{H}^k \cong \omega^i \otimes_{\mathcal{O}_X} \omega^{-1} \otimes^{k-i}.$$
• \( \mathcal{H}^k \) admits an integrable connection which satisfies Griffith’s transversality (with respect to the filtration above).

It turns out that the Hodge filtration on \( H^1_{dR}(X, k, \mathcal{H}^k) \) is given by

\[
F^n H^1_{dR}(X, \mathcal{H}^k) = \begin{cases} 
H^1_{dR}(X, \mathcal{H}^k), & n \leq 0 \\
H^0(X, \mathcal{H}^k), & n = 1, \ldots, k + 1 \\
0, & n > k + 1 
\end{cases}
\]

where the equation \( H^0(X, \mathcal{H}^k) = H^0(X, A) \) follows from the Kodaira–Spencer isomorphism (see [Kas99, Corollary 3.2]).

**Lemma 6.1.2.** Let \( k \in \mathbb{Z}_{>0} \). The sheaf \( \mathcal{H}^k \) induces an object in \( \text{Syn}^*(X, \chi) \), denoted by \( (\mathcal{H}^k, \Phi, \text{Fil}^*) \), such that (HK-dR) is satisfied.

**Proof.** Note first that \( \mathcal{H}^k \) induces a locally free \( \mathcal{O}_X \)-module on \( X \), which we still denote by \( \mathcal{H}^k \). On the overconvergent site \( \text{OC}(X_0/\mathbb{Z}_p) \), we consider the sheaf

\[(Z, 3, i, h, \theta) \mapsto h^*(\mathcal{H}^k(Z)).\]

One checks easily that this sheaf is an overconvergent isocrystal. We abuse the notation and denote it again by \( \mathcal{H}^k \). By the discussion above, it is equipped with a natural filtration that satisfies Griffith’s transversality. Moreover, the reason that \( \mathcal{H}^k \) is an overconvergent \( F \)-isocrystal lies on the fact that \( \mathcal{H}^k \) is isomorphic to the relative crystalline cohomology of \( A_{univ} \) over \( X \) (see [Ogu84, Theorem 3.10] and [CI10, Lemma 5.14]) allows one to conclude the result. Finally, note that \( (X_0, \chi) \in \text{OC}(X_0/\mathbb{Z}_p) \) and so, as discussed in §3.2 the condition (HK-dR) is satisfied.

**Remark 6.1.3.** In fact, one sees that the cohomology groups

\[ H^i_{HK}(X_0, (\mathcal{H}^k, \Phi)), \quad H^i_{dR}(X, \mathcal{H}^k), \quad H^i_{dR}(X_{Q_p}, \mathcal{H}^k) \]

are all isomorphic to each other. Therefore, in what follows, we shall abuse the notation and denote all of them simply by \( H^i_{dR}(X, \mathcal{H}^k) \). We also make no distinction between \( X, X_{Q_p} \), and \( \chi \) by simply writing \( X \) when it causes no confusion.

**Lemma 6.1.4.** The eigenvalue of the Frobenius \( \Phi \) acting on \( H^i_{dR}(X, \mathcal{H}^k) \) is of Weil weight \( i + k \). In particular, (Purity) is satisfied.

**Proof.** When \( k = 0 \), this is the case without coefficient and one just apply the result of [CLS98]. So, let’s assume \( k > 0 \). Note that, since \( k > 0 \), \( \mathcal{H}^k \) has no global horizontal sections ([Bro12, Lemma VII.4]). Thus, we only need to show the assertion for \( H^1_{dR}(X, \mathcal{H}^k) \). Note also that it is enough to show the claim for \( \text{Sym}^k \tilde{\mathcal{H}} \) since \( \mathcal{H}^k \) is a direct summand of \( \text{Sym}^k \tilde{\mathcal{H}} \).

Observe first that there is a surjection

\[ H^1_{dR}(X, \tilde{\mathcal{H}}^\otimes k) \to H^1_{dR}(X, \text{Sym}^k \tilde{\mathcal{H}}) \]

and so it reduces to show the statement for \( H^1_{dR}(X, \tilde{\mathcal{H}}^\otimes k) \). Let

\[ (A_{univ})^k := A_{univ} \times_X \cdots \times_X A_{univ} \]

Using the relative Künneth formula ([Sta22, Tag 0FMC]), one can deduce that \( H^1_{dR}(X, \tilde{\mathcal{H}}^\otimes k) \) is a subspace of \( H^{k+1}_{dR}((A_{univ})^k) \). However, we know that the eigenvalues of the Frobenius acting on \( H^{k+1}_{dR}((A_{univ})^k) \) is of Weil weight \( k + 1 \) (again, by [CLS98]), we then conclude the result.

\[ \square \]
6.2 Diagonal cycles and a formula of Darmon–Rotger

Our goal in this subsection is to achieve the formula in [DR14, Theorem 3.8] in the case of compact Shimura curve over $\mathbb{Q}$ without the use of the Kuga–Sato variety.

Let $(k, \ell, m) \in \mathbb{Z}^3$ be such that

- $k + \ell + m \in 2\mathbb{Z}$,
- $2 < k \leq \ell \leq m$ and $m < k + \ell$,
- $\ell + m - k = 2t + 2$ for some $t \in \mathbb{Z}_{\geq 0}$.

We write $r_1 := k - 2$, $r_2 := \ell - 2$, $r_3 := m - 2$, and $r := \frac{r_1 + r_2 + r_3}{2}$.

Note that we have $r_2 + r_3 \geq r_1 \geq r_3 - r_2$ and $r - r_1 = t$.

We fix three elements

\[ \eta \in H^1_{dR}(X, \mathcal{H}^{r_1}) \]
\[ \omega_2 \in H^0(X, \omega^\ell) = F^1H^1_{dR}(X, \mathcal{H}^{r_2}) \]
\[ \omega_3 \in H^0(X, \omega^m) = F^1H^1_{dR}(X, \mathcal{H}^{r_3}) \]

and view $\eta \otimes \omega_1 \otimes \omega_2 \in F^2H^3_{dR}(X^3, \mathcal{H}^{r_1} \otimes \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3})$ by applying Künneth formula.

Now, consider the triple product $X^3$ and let $\pi_i : X^3 \to X$ be the projection of the $i$-th component. Fix a base point $o \in X$ and consider the following embeddings of $X$ into $X^3$

\[ \iota_{123} : X \xrightarrow{\Delta} X \times X \times X \]
\[ \iota_{12} : X \xrightarrow{\Delta} X \times X \to X \times X \times \{o\} \hookrightarrow X \times X \times X \]
\[ \iota_{13} : X \xrightarrow{\Delta} X \times X \to X \times \{o\} \times X \hookrightarrow X \times X \times X \]
\[ \iota_{23} : X \xrightarrow{\Delta} X \times X \to \{o\} \times X \times X \hookrightarrow X \times X \times X \]
\[ \iota_1 : X \to X \times \{o\} \times X \hookrightarrow X \times X \times X \]
\[ \iota_2 : X \to \{o\} \times X \times \{o\} \hookrightarrow X \times X \times X \]
\[ \iota_3 : X \to \{o\} \times \{o\} \times X \hookrightarrow X \times X \times X \]

We also consider the embeddings

\[ \iota'_{23} : X \xrightarrow{\Delta} X \times X \]
\[ \iota'_2 : X \xrightarrow{\Delta} X \times \{o\} \hookrightarrow X \times X \]
\[ \iota'_3 : X \xrightarrow{\Delta} \{o\} \times X \hookrightarrow X \times X \]

which can be viewed as cutting off the first component of the maps $\iota_{123}$, $\iota_{12}$ and $\iota_{13}$.

**Lemma 6.2.1.** As sheaves over $X$, we have a Frobenius-equivariant decomposition

\[ \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \cong \bigoplus_{j \geq 0} \mathcal{H}^{r_3+r_2-2j}(-j). \]

In particular, $\mathcal{H}^{r_1}(-t)$ is a direct summand of $\mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3}$ and we have a projection

\[ pr_{r_1} : \mathcal{H}^{r_2} \otimes \mathcal{H}^{r_3} \to \mathcal{H}^{r_1}(-t). \]
Thus, by applying Künneth decomposition, we see that
\[
\Sym^2 V \otimes \Sym^3 V \cong \bigoplus_{r=0}^2 \Sym^{3+r^2-2} V.
\]

See, for example, [FH04, Exercise 11.11]. We note that we have the twists in the statement so that the decomposition is Frobenius-equivariant.

**Corollary 6.2.2.** Over $X$, we have a Frobenius-equivariant decomposition

\[
\mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3 \cong \mathcal{H}^r_1 \otimes \left( \bigoplus_{j=0}^{r^2-2} \mathcal{H}^{r^3+r^2-2j} \right).
\]

Now we let $X_{123} := \iota_{123}(X) \in Z^2(X^3)$. For any $\tau \in H^3_{fp}(X_{123}, \mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3, 2)$, we write $\tau = \tau_1 \otimes \tau_2 \otimes \tau_3$, where $\tau_i$ corresponds to the $\mathcal{H}^r_i$-component. Applying the perfect pairing

\[
\langle \cdot, \cdot \rangle_{fp} : H^0_{fp}(X_{123}, \mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3, 2) \times H^0_{fp}(X_{123}, (\mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3)^\vee, 0) \to \mathbb{Q}_p,
\]

we know that there exists a unique element

\[
1_{X_{123}} \in H^0_{fp}(X_{123}, (\mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3)^\vee, 0) = H^0_{dr}(X_{123}, (\mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3)^\vee)
\]

such that

\[
\langle \tau, 1_{X_{123}} \rangle_{fp} = \langle \tau_1, pr_{r_1}(\tau_2 \otimes \tau_3) \rangle_{fp}.
\]

**Definition 6.2.3.** The diagonal cycle with coefficients in $(\mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3)^\vee$ is defined to be

\[
\Delta^{k,f,m}_{2,2,2} := 1_{X_{123}} \cdot X_{123} \in A^2(X^3, (\mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3)^\vee).
\]

**Lemma 6.2.4.** The diagonal cycle $\Delta^{k,f,m}_{2,2,2}$ is null-homologous.

**Proof.** Note that the sheaf $\mathcal{H}^r_i$ has no global horizontal sections unless $r_i = 0$ ([Bro13, Lemma VII.4]). Thus, by applying Künneth decomposition, we see that $H^2_{dr}(X^3, \mathcal{H}^r_1 \otimes \mathcal{H}^r_2 \otimes \mathcal{H}^r_3) = 0$. This then implies the desired result.

**Theorem 6.2.5.** For $\eta, \omega_2, \omega_3$ as above, we have

\[
\AJ_{fp}(\Delta^{k,f,m}_{2,2,2})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \tilde{\eta}, \psi_{23}(\omega_2 \otimes \omega_3) \rangle_{fp},
\]

where

- $\tilde{\eta}$ is a lift of $\eta$ via the projection $H^1_{fp}(X, \mathcal{H}^r_1, 0) \to H^1_{dr}(X, \mathcal{H}^r_1)$,
- $(\omega_2 \otimes \omega_3)^{\sim}$ is a lift of $\omega_2 \otimes \omega_3$ via the projection $H^2_{fp}(X^2, \mathcal{H}^r_2 \otimes \mathcal{H}^r_3, 2) \to F^2 H^2_{dr}(X^2, \mathcal{H}^r_2 \otimes \mathcal{H}^r_3)$ (alternatively, it can be written as $\pi^2_{23} \omega_2 \otimes \pi^2_{3} \omega_3$ where $\omega_j \in H^1_{fp}(X, \mathcal{H}^r_j, 1)$ is a lift of $\omega_j$),
- $\psi_{23} : H^0_{fp}(X^2, \mathcal{H}^r_2 \otimes \mathcal{H}^r_3, 2) \to H^0_{fp}(X, \mathcal{H}^r_1(-t), 2)$ is the map given by $pr_{r_1} \circ \iota_{23}^*$.

Here, we use that the self-duality of $\mathcal{H}^r_1$ gives a pairing $\mathcal{H}^r_1 \otimes \mathcal{H}^r_1(-t) \to O_X(-r)$.

**Proof.** By the definition of $\Delta^{k,f,m}_{2,2,2}$, we have

\[
\AJ_{fp}(\Delta^{k,f,m}_{2,2,2})(\eta \otimes \omega_2 \otimes \omega_3) = \langle \iota_{123}^*(\eta \otimes \omega_2 \otimes \omega_3), 1_{X_{123}} \rangle_{fp}
\]

\[
= \langle \iota_{123}^*(\pi^1_{123} \tilde{\eta} \cup \pi^2_{23} \omega_2 \cup \pi^3_{3} \omega_3), 1_{X_{123}} \rangle_{fp}
\]

\[
= \langle \tilde{\eta}, pr_{r_1} \circ \iota_{23}^*(\omega_2 \otimes \omega_3)^{\sim} \rangle_{fp}
\]

\[
= \langle \tilde{\eta}, \psi_{23}(\omega_2 \otimes \omega_3)^{\sim} \rangle_{fp}.
\]
**Remark 6.2.6.** Readers might notice that our choice of the diagonal cycle $\Delta_{2,2,2}^{k,\ell,m}$ only involves $X_{123}$, which is simpler than the diagonal cycle considered in [BDP13] for the case $k = \ell = m = 2$. It could be thought as that the terms other than $X_{123}$ are needed only to make the cycle null-homologous. As shown in [Bes16], those terms do not contribute to the Abel–Jacobi map in the end. So, it makes sense to consider only $X_{123}$, the image of the diagonal embedding.

Note that, because of Lemma 6.1.4, we can choose a polynomial $P(T) \in \mathbb{Q}[T]$ such that

- $P(\Phi)(\omega_2 \otimes \omega_3) = 0 \in H^2_{\text{dR}}(X^2, \mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3})$ and
- $P(\Phi)$ acts isomorphically on $H^1_{\text{dR}}(X^2, \mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3})$.

As a consequence, there exists a section $\rho(P, \omega_2, \omega_3) \in H^0(X^2, (\mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3}) \otimes \Omega^1_{X^2/\mathbb{Q}_p})$ such that

$$\nabla \rho(P, \omega_2, \omega_3) = P(\Phi)(\omega_2 \otimes \omega_3)$$

and it is well-defined up to a horizontal section. Moreover, as $P(\Phi)$ acts isomorphically on $H^1_{\text{dR}}(X, \mathbb{H}^{r_1}(-t))$, we may set

$$\xi(\omega_2, \omega_3) := P(\Phi)^{-1}\psi_{23}^*\rho(P, \omega_2, \omega_3) \in H^1_{\text{dR}}(X, \mathbb{H}^{r_1}(-t)).$$

Similar as in [DR14, Proposition 3.7], this element is independent to the choice of the polynomial $P$.

**Corollary 6.2.7.** Notation as above. We have

$$\text{AJ}_{\text{ip}}(\Delta_{2,2,2}^{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3) = (\eta, \xi(\omega_2, \omega_3))_{\text{dR}}.$$  

In particular,

$$\text{AJ}_{\text{ip}}(\Delta_{2,2,2}^{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3) = \text{AJ}_{\text{p}}(\Delta_{k,\ell,m})(\eta \otimes \omega_2 \otimes \omega_3)$$

where $\text{AJ}_{\text{p}}$, the $(p\text{-adic})$ Abel–Jacobi map, and $\Delta_{k,\ell,m}$, the generalised Gross–Kudla–Schoen cycle, are both defined similarly as in [DR14] via Kuga–Sato varieties.

**Proof.** The proof is similar to the one of [DR14, Theorem 3.8] (more precisely, the passage after Lemma 3.11 of op. cit.). We just translate it into the language of finite polynomial cohomology with coefficients.

First, observe that the element $(\omega_2 \otimes \omega_3)^\sim$ can be represented by

$$(\rho(P, \omega_2, \omega_3), \omega_2 \otimes \omega_3) \in H^2_{\text{syn}, \mathbb{p}}(X^2, \mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3}, 2)$$

Now consider the following commutative diagram coming from the exact sequences in Corollary 4.1.8

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{\text{dR}}(X^2, \mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3}) & \longrightarrow & H^2_{\text{dR}}(X^2, \mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3}, 2) & \longrightarrow & F^2H^3_{\text{dR}}(X^2, \mathbb{H}^{r_2} \otimes \mathbb{H}^{r_3}) & \longrightarrow & 0 \\
& & \uparrow{\psi_{23}^*} & & \uparrow{\psi_{23}^*} & & \downarrow{\psi_{23}^*} & & \\
0 & \longrightarrow & H^1_{\text{dR}}(X, \mathbb{H}^{r_1}(-t)) & \longrightarrow & H^2_{\text{syn}, \mathbb{p}}(X, \mathbb{H}^{r_1}(-t), 2) & \longrightarrow & F^2H^3_{\text{dR}}(X, \mathbb{H}^{r_1}(-t)) & \longrightarrow & 0
\end{array}
$$

As $F^2H^3_{\text{dR}}(X, \mathbb{H}^{r_1}(-t)) = 0$, the element $\psi_{23}^*(\omega_2 \otimes \omega_3)^\sim = i_{\text{ip}}(\xi(\omega_2, \omega_3))$. Hence we see that $\psi_{23}^*(\omega_2 \otimes \omega_3)^\sim = i_{\text{ip}}(\xi(\omega_2, \omega_3))$. 

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Now the result follows from the compatibility of the Poincaré pairings and the maps \( i_{fp} \) and \( pr_{fp} \) (Corollary 4.2.2), which reads

\[
\langle \tilde{\eta}, \psi_2 \ast (\omega_2 \otimes \omega_3) \rangle_{fp} = \langle pr_{fp}^{-1}(\eta), i_{fp}(\xi(\omega_2, \omega_3)) \rangle_{fp}
\]

\[
= \langle pr_{fp}^{-1}(\eta), i_{fp}(\xi(\omega_2, \omega_3)) \rangle_{fp}
\]

\[
= \langle \eta, \xi(\omega_2, \omega_3) \rangle_{dR}.
\]

Remark 6.2.8. Although the computations above are in the setting of compact Shimura curves over \( \mathbb{Q} \), our strategy should directly apply to the setting of elliptic modular curves (as in [DR14]) after taking care of the log structure on the compactified modular curves defined by the cusps.

Remark 6.2.9. Of course, by choosing \( \eta, \omega_2 \) and \( \omega_3 \) carefully and applying the strategy in [DR14], one can deduce that the formula in Theorem 6.2.5 computes special values of triple product \( L \)-functions associated to certain eigenforms. As the purpose of this section is to indicate how the theory of finite polynomial cohomology with coefficients can be applied in practice, we do not intend to link our formula with special \( L \)-values in this paper.

6.3 Cycles attached to isogenies and formulae of Bertolini–Darmon–Prasanna

In this subsection, we establish a second application of finite polynomial cohomology with coefficients in the spirit of [BDP13]. To this end, we fix a finite extension \( K \) of \( \mathbb{Q}_p \) and base change \( X \) to \( K \). We abuse the notation and still denote it by \( X \). We also fix two \( K \)-rational points \( x = (A, i, \alpha) \) and \( y = (A', i', \alpha') \) in \( X(K) \) such that there is an isogeny \( \varphi : (A, i, \alpha) \to (A', i', \alpha') \). Finally, we fix an integer \( r \in \mathbb{Z}_{>0} \) and consider the sheaf

\[
\mathcal{H}^{r, r} := \mathcal{H}^r \otimes \text{Sym}^r \epsilon H^1_{dR}(A).
\]

Our result in this subsection is the following theorem:

**Theorem 6.3.1.** There is a unique cycle \( \Delta_\varphi = \theta \cdot y \in A^1(X, \mathcal{H}^{r, r}(r))_0 \) such that for any \( \omega \in H^0(X, \omega^{r+2}) \) and any \( \alpha \in \text{Sym}^r \epsilon H^1_{dR}(A) \), we have

\[
AJ_{fp}(\Delta_\varphi)(\omega \otimes \alpha) = \langle F_\omega(y) \otimes \alpha, \theta \rangle = \langle \varphi^*(F_\omega(y)), \alpha \rangle.
\]

Here

(i) \( \theta \) is an element in \( \mathcal{H}^{r, r}(r)(y) \) that will be specified later;

(ii) \( F_\omega \) is the Coleman integral of the form \( \omega \);

(iii) the middle pairing in on \( \mathcal{H}^{r, r}(y) = \text{Sym}^r \epsilon H^1_{dR}(A') \otimes \text{Sym}^r \epsilon H^1_{dR}(A) \); and

(iv) the last pairing is on \( \text{Sym}^r \epsilon H^1_{dR}(A) \).

In particular, we obtain the same formulae as in [BDP13, Proposition 3.18 & Proposition 3.21].

**Remark 6.3.2.** (i) Similarly as before, we have \( H^2_{dR}(X, \mathcal{H}^{r, r}) = 0 \), which implies that \( A^1(X, \mathcal{H}^{r, r}(r)) = A^1(X, \mathcal{H}^{r, r}(r))_0 \).

(ii) To be more precise, the pairing in Theorem 6.3.1 (iii) is between \( \mathcal{H}^{r, r}(y) \) and its twist \( \mathcal{H}^{r, r}(y)(r) \). We omit the twist \( (r) \) for notational simplicity.
Let $\omega$ and $\alpha$ be as in the theorem. Recall that we have a short exact sequence
\[
0 \to \frac{H^0_{\text{dR}}(X, \mathcal{H}_r^r)}{F^1H^0_{\text{dR}}(X, \mathcal{H}_r^r)} \xrightarrow{i_{\text{ip}}} H^1_{\text{ip}}(X, \mathcal{H}_r^r, 1) \xrightarrow{pr_{\text{ip}}} F^1H^1_{\text{dR}}(X, \mathcal{H}_r^r) \to 0.
\]
Choose a lift $(\omega \otimes \alpha)^\sim \in H^1_{\text{ip}}(X, \mathcal{H}_r^r, 1)$ of $\omega \otimes \alpha \in F^1H^1_{\text{dR}}(X, \mathcal{H}_r^r)$. Then, for any $\theta \in H^0_{\text{ip}}(y, \mathcal{H}_r^r(r), 0) = H^0_{\text{dR}}(y, \mathcal{H}_r^r(r))$, we have
\[
\text{AJ}_{\text{ip}}(\theta)(\omega \otimes \alpha) = \langle (\omega \otimes \alpha)^\sim, \iota_y, \theta \rangle_{\text{ip},X} = \langle i_y^* (\omega \otimes \alpha)^\sim, \theta \rangle_{\text{ip},y},
\]
where $\iota_y : y \hookrightarrow X$ is the natural closed embedding. Note that $i_y^* (\omega \otimes \alpha)^\sim = i_y^* \omega \otimes \alpha$, where $\omega \in H^1_{\text{ip}}(X, \mathcal{H}_r^r, 1)$ is a lift of $\omega$.

On the other hand, the short exact sequence
\[
0 \to \frac{H^0_{\text{dR}}(y, \mathcal{H}_r^r)}{F^1H^0_{\text{dR}}(y, \mathcal{H}_r^r)} \xrightarrow{i_{\text{ip}}} H^1_{\text{ip}}(y, \mathcal{H}_r^r, 1) \xrightarrow{pr_{\text{ip}}} F^1H^1_{\text{dR}}(y, \mathcal{H}_r^r) \to 0
\]
and the fact that $H^1_{\text{dR}}(y, \mathcal{H}_r^r) = 0$ imply there exists $s_\omega \in H^0_{\text{dR}}(y, \mathcal{H}_r^r)$ such that $i_{\text{ip}}(s_\omega) = i_y^* \omega$. Hence, we have
\[
\text{AJ}_{\text{ip}}(\theta)(\omega \otimes \alpha) = \langle i_y^* (\omega \otimes \alpha)^\sim, \theta \rangle_{\text{ip},y} = \langle i_y^* \omega \otimes \alpha, \theta \rangle_{\text{ip},y} = \langle i_y^* (s_\omega) \otimes \alpha, \theta \rangle_{\text{ip},y} = \langle s_\omega \otimes \alpha, \theta \rangle_{\text{dR},y}.
\]
(12)

**Lemma 6.3.3.** In $\mathcal{H}_r^r(y)$, we have
\[
s_\omega = F_\omega(y),
\]
where $F_\omega$ is the Coleman integration corresponds to $\omega$. Note that $F_\omega$ is unique since $\mathcal{H}_r^r$ has no global horizontal section on $X$.

**Proof.** We first choose a polynomial $P \in \text{Poly}$ such that $H^1_{\text{syn},P}(X, \mathcal{H}_r^r, 1) = H^1_{\text{ip}}(X, \mathcal{H}_r^r, 1)$. Then, we have the commutative diagram
\[
\begin{array}{ccccccccc}
0 \to & H^0_{\text{dR}}(X, \mathcal{H}_r^r) & \xrightarrow{i_{\text{ip}}} & H^1_{\text{ip}}(X, \mathcal{H}_r^r, 1) & \xrightarrow{pr_{\text{ip}}} & F^1H^1_{\text{dR}}(X, \mathcal{H}_r^r) & \to & 0 \\
0 \to & H^0_{\text{dR}}(y, \mathcal{H}_r^r) & \xrightarrow{i_{\text{ip}}} & H^1_{\text{ip}}(y, \mathcal{H}_r^r, 1) & \xrightarrow{pr_{\text{ip}}} & F^1H^1_{\text{dR}}(y, \mathcal{H}_r^r) & \to & 0 \\
\end{array}
\]
Note that $\frac{H^0_{\text{dR}}(X, \mathcal{H}_r^r)}{F^1H^0_{\text{dR}}(X, \mathcal{H}_r^r)}$ and $F^1H^1_{\text{dR}}(y, \mathcal{H}_r^r)$ both vanish.

Let $\mathcal{V}$ be a strict neighbourhood of the rigid analytic space associated to an affinoid inside $X$ such that $\mathcal{V}$ contains the residue disc of $y$ and all its Frobenius translations. By Corollary $5.1.2$, the restriction of $\omega$ to $\mathcal{V}$ is (uniquely) represented by an element of the form $(G', \omega)$ where $G' \in H^0(\mathcal{V}, \mathcal{H}_r^r)$ is such that $P(\Phi)\omega = \nabla G'$. Hence we have $P(\Phi)F_\omega = G'$.

The equation $s_\omega = F_\omega(y)$ is clear due to the construction of Coleman integration ([Col82, Col94]) and that
\[
i_{\text{ip}} : H^0_{\text{dR}}(y, \mathcal{H}_r^r) = \mathcal{H}_r^r(y) \to H^1_{\text{syn},P}(y, \mathcal{H}_r^r, 1)
\]
is the map $s \mapsto (P(\Phi)(s), 0)$. \hfill $\square$
Remark 6.3.4. We remark that the term $(\Phi F_\omega)(y)$ is not the naive $F_\omega(\phi(y))$, where $\phi : X \to X$ is the Frobenius. Rather, it is understood as the following explanation:

Since $\mathcal{H}^r$ is an overconvergent $\mathcal{F}$-isocrystal, there is a Frobenius neighbourhood $W' \subset W$ (see [Co94, §10]) and a horizontal map $\Phi : \phi^* \mathcal{H}^r \to \mathcal{H}^r|_{W'}$. This induces a map $\Phi$ from $(\mathcal{H}^r)_{\phi(y)}$ to $(\mathcal{H}^r)_y$, which may be thought as the 'parallel transport along the Frobenius' from $\phi(y)$ to $y$. Hence $P(\Phi)(y)$ takes value in $\mathcal{H}^r(y)$.

Now we construct the desired cycle $\Delta_\varphi$ by specifying the coefficient $\theta$. Observe that, since $\epsilon H^1_{dR}(A)$ is self-dual, we have a natural perfect pairing (see, for example, [BDP13, (1.1.14)])

$$\text{Sym}^r \epsilon H^1_{dR}(A) \otimes \text{Sym}^r \epsilon H^1_{dR}(A) \to K, \quad \beta \otimes \gamma \mapsto \langle \beta, \gamma \rangle.$$ 

Moreover, we also have a perfect pairing between $\text{Sym}^r \epsilon H^1_{dR}(A) \otimes \text{Sym}^r \epsilon H^1_{dR}(A)$ and itself. Hence, there is an element $1_A \in \text{Sym}^r \epsilon H^1_{dR}(A) \otimes \text{Sym}^r \epsilon H^1_{dR}(A)$ such that

$$\langle \beta \otimes \gamma, 1_A \rangle = \langle \beta, \gamma \rangle.$$

Definition 6.3.5. The cycle attached to the isogeny $\varphi$ is defined to be

$$\Delta_\varphi := \theta \cdot y \in A^1(X, \mathcal{H}^r(r)) = A^1(X, \mathcal{H}^r(r))_0,$$

where $\theta = \varphi_* 1_A$ and $\varphi_*$ is the pushforward map

$$\varphi_* : \mathcal{H}^r(r)(x) = \text{Sym}^r \epsilon H^1_{dR}(A) \otimes \text{Sym}^r \epsilon H^1_{dR}(A) \to \text{Sym}^r \epsilon H^1_{dR}(A') \otimes \text{Sym}^r \epsilon H^1_{dR}(A) = \mathcal{H}^r(r)$$

induced by the isogeny $\varphi : (A, i, \alpha) \to (A', i', \alpha')$.

Proof of Theorem 6.3.1. The proof now follows easily by combining [BDP13, §10], Lemma 6.3.3 and the definition of $\Delta_\varphi$. \hfill \Box

Remark 6.3.6. As before, our strategy can be directly applied to the setting of elliptic modular curves as in [BDP13] after taking care of the log structure on the compactified modular curves. In this case, readers should be able to compare this to the proof of Proposition 3.21 in op. cit.. The pushforward of the fundamental class $[A'] \in H^1_{dR}(A')$ under the map $(\varphi^*, \text{id}') : A \to X_r$ (notations as in op. cit.) corresponds to our element $1_A$.

Remark 6.3.7. Similarly as before, if we choose $x$ and $y$ to be Heegner points on the Shimura curve (or modular curve), it is expected that the formula in Theorem 6.3.1 computes special values of the anti-cyclotomic $p$-adic $L$-functions as in [BDP13]. Since we only wanted to demonstrate the use of finite polynomial cohomology with coefficients in practice, we do not intend to link our formula with special $L$-values. \hfill \Box

Remark 6.3.8. As it could be seen in the two applications, one advantage of using the finite polynomial cohomology with coefficients is that our cycles are much simpler. In particular, we can get rid of the idempotents used to define the cycles in [DR14] and [BDP13], which makes the computations simpler. \hfill \Box

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