Invariants and Bonnet-type theorem for surfaces in $\mathbb{R}^4$

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Abstract: In the tangent plane at any point of a surface in the four-dimensional Euclidean space we consider an invariant linear map of Weingarten-type and find a geometrically determined moving frame field. Writing derivative formulas of Frenet-type for this frame field, we obtain eight invariant functions. We prove a fundamental theorem of Bonnet-type, stating that these eight invariants under some natural conditions determine the surface up to a motion.

We show that the basic geometric classes of surfaces in the four-dimensional Euclidean space, determined by conditions on their invariants, can be interpreted in terms of the properties of two geometric figures: the tangent indicatrix, which is a conic in the tangent plane, and the normal curvature ellipse.

We construct a family of surfaces with flat normal connection.

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1. Introduction

Local invariants of surfaces in the four-dimensional Euclidean space $\mathbb{R}^4$ were studied by Eisenhart [6], Kommerell [14], Moore and Wilson [9], Schouten and Struik [18], Spivak [19], Wong [23], and Little [15]. Their study was based on a special configuration, namely a point and an ellipse lying in the normal space (the ellipse of normal curvature). This configuration leads to a theory of axial principal directions, along which the vector-valued second fundamental form points in the direction of the major and the minor axes of the curvature ellipse. In higher dimensions there is also a similar
configuration consisting of a point and a Veronese manifold. This configuration determines second order scalar invariants and generates principal axes “in general” [15]. Points where the construction of principal axes fails are regarded as singularities of the field of axes. Geometric singularities for immersions in Riemannian manifolds are considered in [1]. Special types of tangent vector fields on a surface in \( \mathbb{R}^3 \) are defined in terms of the properties of the normal curvature ellipse and families of lines determined by such tangent vector fields are studied in [10, 16].

In this paper our aim is to develop the local theory of surfaces in \( \mathbb{R}^4 \) on the base of the Weingarten map similarly to the classical case of surfaces in \( \mathbb{R}^3 \).

Let \( M^2 \) be a surface in \( \mathbb{R}^4 \) with tangent space \( T_pM^2 \) at any point \( p \in M^2 \). In [7] we introduced an invariant linear map \( \gamma \) of Weingarten-type at any \( T_pM^2 \), which generates two invariant functions \( k \) and \( \varkappa \). The sign of the function \( k \) is a geometric invariant and the sign of \( \varkappa \) is invariant under motions in \( \mathbb{R}^4 \). However, the sign of \( \varkappa \) changes under symmetries with respect to a hyperplane in \( \mathbb{R}^4 \). Analogously to \( \mathbb{R}^3 \), the invariants \( k \) and \( \varkappa \) divide the points of \( M^2 \) into four types: flat, elliptic, hyperbolic and parabolic points. The surfaces consisting of flat points, i.e. satisfying the condition \( k = \varkappa = 0 \), either lie in \( \mathbb{R}^3 \) or are developable ruled surfaces in \( \mathbb{R}^4 \). Everywhere in the present considerations, we exclude the points at which \( k = \varkappa = 0 \).

The minimal surfaces in \( \mathbb{R}^4 \) are characterized in terms of the invariants \( k \) and \( \varkappa \) by the condition \( \varkappa^2 - k = 0 \), and the surfaces with flat normal connection are characterized by \( \varkappa = 0 \).

Further, the map \( \gamma \) generates the corresponding second fundamental form \( II \) at any point \( p \in M^2 \) in the standard way. In [8] we gave a geometric interpretation of the second fundamental form \( II \) of the surface. We introduced an invariant \( \zeta_{g_1,g_2} \) of a pair of two tangents \( g_1, g_2 \) at any point \( p \) of \( M^2 \). The tangents \( g_1, g_2 \) are conjugate in terms of \( II \) if and only if \( \zeta_{g_1,g_2} = 0 \). The notions of a normal curvature and a geodesic torsion of a tangent were introduced by means of the invariant \( \zeta \). It turns out that asymptotic tangents and principal tangents in terms of \( II \) are characterized by zero normal curvature and zero geodesic torsion, respectively. The principal normal curvatures \( \nu^* \) and \( \nu'' \) are defined as the normal curvatures of the principal tangents. The invariants \( k \) and \( \varkappa \) satisfy the equalities

\[
k = \nu^* \nu'', \quad \varkappa = \frac{\nu' + \nu''}{2}.
\]

It turns out that the points at which any tangent is principal (“umbilical” points) are characterized by zero mean curvature vector, i.e. the surfaces consisting of “umbilical” points are exactly the minimal surfaces in \( \mathbb{R}^3 \).

The indicatrix of Dupin at an arbitrary (non-flat) point of a surface in \( \mathbb{R}^3 \) is introduced by means of the second fundamental form. Here, we introduce the indicatrix \( \chi \) at any point \( p \in M^2 \) in the same way:

\[
\chi : \nu'X^2 + \nu''Y^2 = \varepsilon, \quad \varepsilon = \pm 1.
\]

The conjugacy in terms of the second fundamental form coincides with the conjugacy with respect to the indicatrix \( \chi \).

In Section 3 we prove that:

The surface \( M^2 \) is minimal if and only if the indicatrix \( \chi \) is a circle.

The surface \( M^2 \) is with flat normal connection if and only if the indicatrix \( \chi \) is a rectangular hyperbola (a Lorentz circle).

In the local theory of surfaces a statement of significant importance is a theorem of Bonnet-type giving the natural conditions under which the surface is determined up to a motion. A theorem of this type was proved for surfaces with flat normal connection by B.-Y. Chen in [3]. In the class of surfaces with \( \varkappa^2 - k > 0 \) we find a geometrically determined moving frame of Frenet-type. Considering the corresponding derivative formulas, we obtain eight invariant functions. In Section 4 we prove our basic Theorem 4.1, stating that

The eight invariant functions, satisfying some natural conditions, determine the surface up to a motion in \( \mathbb{R}^4 \).

In Section 5 we construct a family of surfaces with flat normal connection, which lie on a standard rotational hypersurface in \( \mathbb{R}^3 \), and describe those of them with constant Gauss curvature, constant mean curvature, and constant invariant \( k \).
2. A geometric interpretation of the second fundamental form

Let $M^2 : z = z(u, v), (u, v) \in \mathcal{D}, \mathcal{D} \subset \mathbb{R}^2$, be a 2-dimensional surface in $\mathbb{R}^4$. The tangent space $T_p M^2$ to $M^2$ at an arbitrary point $p = z(u, v)$ of $M^2$ is span $\{z_u, z_v\}$. We choose an orthonormal frame field $\{e_1, e_2\}$ of $M^2$ so that the quadruple $\{z_u, z_v, e_1, e_2\}$ is positively oriented in $\mathbb{R}^4$. Then the following derivative formulas hold:

\[
\nabla'_u z_u = z_{uu} = \Gamma^1_{11} z_u + \Gamma^1_{12} z_v + c^1_{11} e_1 + c^1_{12} e_2, \\
\nabla'_u z_v = z_{uv} = \Gamma^1_{12} z_u + \Gamma^1_{22} z_v + c^1_{12} e_1 + c^1_{22} e_2, \\
\nabla'_v z_v = z_{vv} = \Gamma^1_{22} z_u + \Gamma^2_{22} z_v + c^2_{12} e_1 + c^2_{22} e_2,
\]

where $\Gamma^i_j$ are Christoffel's symbols and $c^i_{jk}, i, j, k = 1, 2, \text{ are functions on } M^2$.

We use the standard denotations $E, F, G$ for the coefficients of the first fundamental form and set $W = \sqrt{EG - F^2}$.

Denoting by $\sigma$ the second fundamental tensor of $M^2$, we have

\[
\sigma(z_u, z_u) = c^1_{11} e_1 + c^1_{11} e_2, \quad \sigma(z_u, z_v) = c^1_{12} e_1 + c^1_{12} e_2, \quad \sigma(z_v, z_v) = c^2_{12} e_1 + c^2_{22} e_2.
\]

The three pairs of normal vectors $\{\sigma(z_u, z_u), \sigma(z_u, z_v)\}, \{\sigma(z_u, z_v), \sigma(z_v, z_v)\}, \{\sigma(z_v, z_v), \sigma(z_v, z_u)\}$ form three parallelograms with oriented areas

\[
\Delta_1 = \begin{vmatrix} c^1_{11} & c^1_{12} \\ c^1_{11} & c^1_{12} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} c^1_{11} & c^1_{12} \\ c^1_{11} & c^1_{12} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} c^1_{12} & c^1_{22} \\ c^2_{12} & c^2_{22} \end{vmatrix},
\]

respectively. These oriented areas determine three functions $L = \frac{2\Delta_1}{W}, M = \frac{\Delta_2}{W}, N = \frac{2\Delta_3}{W}$, which change in the same way as the coefficients $E, F, G$ under any change of the parameters $(u, v)$.

Using the functions $E, F, G$ and $L, M, N$, in [7] we introduced the linear map $\gamma$ in the tangent space at any point of $M^2$

\[
\gamma : T_p M^2 \to T_p M^2,
\]

defined by the equalities

\[
\gamma(z_u) = \gamma^1_1 z_u + \gamma^1_2 z_v, \quad \gamma(z_v) = \gamma^2_1 z_u + \gamma^2_2 z_v,
\]

where

\[
\gamma^1_1 = \frac{FM - GL}{EG - F^2}, \quad \gamma^1_2 = \frac{FL - EM}{EG - F^2}, \quad \gamma^2_1 = \frac{FN - GM}{EG - F^2}, \quad \gamma^2_2 = \frac{FM - EN}{EG - F^2}.
\]

The linear map $\gamma$ of Weingarten type at the point $p \in M^2$ is invariant with respect to changes of parameters on $M^2$ as well as to motions in $\mathbb{R}^4$. This implies that the values

\[
k = \frac{LN - M^2}{EG - F^2}, \quad \varpi = \frac{EN + GL - 2FM}{2(EG - F^2)}
\]

are invariants of the surface $M^2$. The invariant $\varpi$ turns out to be the curvature of the normal connection of the surface $M^2$ in $\mathbb{R}^4$.

As in the classical case, the invariants $k$ and $\varpi$ divide the points of $M^2$ into four types: flat, elliptic, parabolic and hyperbolic. The surfaces consisting of flat points satisfy the conditions

\[
k(u, v) = 0, \quad \varpi(u, v) = 0
\]
for \((u,v) \in D\), or equivalently \(L(u,v) = 0\), \(M(u,v) = 0\), \(N(u,v) = 0\), \((u,v) \in D\). These surfaces are either planar surfaces (there exists a hyperplane \(\mathbb{R}^3 \subset \mathbb{R}^4\) containing \(M^2\)) or developable ruled surfaces in \(\mathbb{R}^4\).

Further we consider surfaces free of flat points, i.e. \((L,M,N) \neq (0,0,0)\).

Let \(X = \lambda z_u + \mu z_v\), \((\lambda,\mu) \neq (0,0)\), be a tangent vector at a point \(p \in M^2\). The Weingarten map \(\gamma\) determines a second fundamental form of the surface \(M^2\) at \(p\) as follows [7]:

\[
\|\!(\lambda,\mu)\!\| = -g(\gamma(X),X) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \quad \lambda,\mu \in \mathbb{R}.
\]

As in the classical differential geometry of surfaces in \(\mathbb{R}^3\) the second fundamental form \(\|\!\cdot\!\|\) determines conjugate tangents at a point \(p\) of \(M^2\). Two tangents \(g_1 : X = \lambda_1 z_u + \mu_1 z_v\) and \(g_2 : X = \lambda_2 z_u + \mu_2 z_v\) are said to be conjugate tangents if \(\|\!(\lambda_1,\mu_1;\lambda_2,\mu_2)\!\| = 0\), i.e.

\[
L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \lambda_2\mu_1) + N\mu_1\mu_2 = 0.
\]

A geometric interpretation of the second fundamental form and the map \(\gamma\) can be given using the geometric approach from [8] to the notion of conjugacy. Conjugate tangents are introduced in [8] in a geometric way as follows. Let \(g\) be a tangent at the point \(p \in M^2\) determined by the vector \(X = \lambda z_u + \mu z_v\). We consider the linear map \(\sigma_g : T_p M^2 \rightarrow (T_p M^2)^\perp\), defined by

\[
\sigma_g(Y) = \sigma\left(\frac{\lambda z_u + \mu z_v}{\sqrt{\|\!(\lambda,\mu)\!\|}}, Y\right), \quad Y \in T_p M^2.
\]

Let \(g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v\) and \(g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v\) be two tangents at the point \(p \in M^2\). The oriented areas of the parallelograms spanned by the pairs of normal vectors \(\sigma_{g_1}(z_v),\sigma_{g_2}(z_v)\) and \(\sigma_{g_2}(x_u),\sigma_{g_2}(x_v)\) are denoted by \(S(\sigma_{g_1}(x_u),\sigma_{g_2}(x_v))\) and \(S(\sigma_{g_2}(x_u),\sigma_{g_1}(x_v))\), respectively. Then

\[
\zeta_{g_1,g_2} = \frac{S(\sigma_{g_1}(x_u),\sigma_{g_2}(x_v))}{W} + \frac{S(\sigma_{g_2}(x_u),\sigma_{g_1}(x_v))}{W}
\]

is an invariant of the tangents \(g_1, g_2\).

**Definition 2.1 ([8]).**

Two tangents \(g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v\) and \(g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v\) are said to be conjugate tangents if \(\zeta_{g_1,g_2} = 0\).

Calculating the oriented areas in \(\zeta_{g_1,g_2}\), it can be established that

\[
\zeta_{g_1,g_2} = \frac{L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \lambda_2\mu_1) + N\mu_1\mu_2}{\sqrt{\|\!(\lambda_1,\mu_1;\lambda_2,\mu_2)\!\|}} = \frac{\|\!(\lambda_1,\mu_1;\lambda_2,\mu_2)\!\|}{\sqrt{\|\!(\lambda_1,\mu_1;\lambda_2,\mu_2)\!\|}}.
\]

Thus, \(\zeta_{g_1,g_2} = 0\) if and only if \(L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \lambda_2\mu_1) + N\mu_1\mu_2 = 0\). Hence, the tangents \(g_1\) and \(g_2\) are conjugate according to Definition 2.1 if and only if they are conjugate with respect to the second fundamental form \(\|\!\cdot\!\|\).

We define two invariants \(v_g\) and \(\alpha_g\) of any tangent \(g\) of the surface in terms of \(\zeta_{g_1,g_2}\) as follows:

\[
v_g = \zeta_{g,g}, \quad \alpha_g = \zeta_{g,g^\perp}.
\]

We call \(v_g\) the normal curvature of \(g\), and \(\alpha_g\) the geodesic torsion of \(g\). The invariant \(v_g\) is expressed by the first and the second fundamental forms of the surface in the same way as the normal curvature of a tangent in the theory of surfaces in \(\mathbb{R}^3\), i.e. \(v_g = \frac{\|\!(\lambda,\mu)\!\|}{\|\!(\lambda,\mu)\!\|}\). Further, the invariant \(\alpha_g\) can be written in the following way:

\[
\alpha_g = \frac{\lambda^2(EM - FL) + \lambda\mu(EN - GL) + \mu^2(FN - GM)}{W\|\!(\lambda,\mu)\!\|}.
\]
Hence, \( \alpha_0 \) is expressed by the coefficients of the first and the second fundamental forms in the same way as the geodesic torsion in the theory of surfaces in \( \mathbb{R}^3 \).

The notions of asymptotic tangents and principal tangents are defined in terms of the conjugacy as in \( \mathbb{R}^3 \).

A tangent \( g : X = \lambda z_0 + \mu z_2 \) is said to be asymptotic if it is self-conjugate, i.e. \( \lambda \lambda^2 + 2M\lambda\mu + N\mu^2 = 0 \). Hence, a tangent \( g \) is asymptotic if and only if \( \nu_0 = 0 \). If \( p \) is an elliptic point of \( M^2 \) (\( k > 0 \)) then there are no asymptotic tangents through \( p \); if \( p \) is a hyperbolic point (\( k < 0 \)) then there are two asymptotic tangents passing through \( p \), and if \( p \) is a parabolic point (\( k = 0 \)) then there is one asymptotic tangent through \( p \). Thus, the sign of the invariant \( k \) determines the number of asymptotic tangents at the point.

A tangent \( g : X = \lambda z_0 + \mu z_2 \) is said to be principal if it is orthogonal to its conjugate. The equation for the principal tangents at a point \( p \in M^2 \) is

\[
\begin{vmatrix}
E & F & \lambda^2 + E \\
L & M & \lambda \mu + F \\
N & M & N
\end{vmatrix} \mu^2 = 0.
\]

Hence, a tangent \( g \) is principal if and only if \( \alpha_0 = 0 \). As in the classical case we have \( \lambda^2 - k \geq 0 \) at each point of the surface. If \( \lambda^2 - k = 0 \), every tangent is principal, and if \( \lambda^2 - k > 0 \), there exist exactly two principal tangents.

A line \( c : u = u(q), v = v(q), q \in J \subset \mathbb{R} \), on \( M^2 \) is said to be an asymptotic line if its tangent at any point is asymptotic. A line \( c : u = u(q), v = v(q), q \in J \subset \mathbb{R} \), on \( M^2 \) is said to be a principal line (a line of curvature) if its tangent at any point is principal. The surface \( M^2 \) is parameterized by principal lines if and only if \( F = 0, M = 0 \).

Let \( M^2 \) be a surface with \( \lambda^2 - k > 0 \) at each point. We assume that \( M^2 \) is parameterized by principal lines and consider the unit vector fields \( x = \frac{z_0}{\sqrt{E}}, y = \frac{z_2}{\sqrt{G}} \). The equality \( M = 0 \) implies that the normal vector fields \( \sigma(x, x) \) and \( \sigma(y, y) \) are collinear. We denote by \( b \) a unit normal vector field collinear with \( \sigma(x, x) \) and \( \sigma(y, y) \), and by \( l \) the unit normal vector field such that \( \{x, y, b, l\} \) is a positive oriented orthonormal frame field of \( M^2 \) (the two vectors \( \{b, l\} \) are determined up to a sign). Thus we obtain a geometrically determined orthonormal frame field \( \{x, y, b, l\} \) at each point \( p \in M^2 \). With respect to the frame field \( \{x, y, b, l\} \) we have the following formulas:

\[
\sigma(x, x) = \nu_1 b, \quad \sigma(x, y) = \lambda b + \mu l, \quad \sigma(y, y) = \nu_2 b,
\]

where \( \nu_1, \nu_2, \lambda, \mu \) are invariant functions, whose signs depend on the orientation of \( \{b, l\} \).

Hence the invariants \( k, \lambda, \) and the Gauss curvature \( K \) of \( M^2 \) are expressed as follows:

\[
k = -4\nu_1 \nu_2 \mu^2, \quad \lambda = (\nu_1 - \nu_2)\mu, \quad K = \nu_1 \nu_2 - (\lambda^2 + \mu^2).
\]

Since \( \lambda^2 - k > 0 \), equalities (2) imply that \( \mu \neq 0 \).

The normal mean curvature vector field of \( M^2 \) is \( H = \frac{\sigma(x, x) + \sigma(y, y)}{2} = \frac{\nu_1 + \nu_2}{2} b \). Taking into account (2) we obtain that the length \( \|H\| \) of the mean curvature vector field is given by the formula

\[
\|H\| = \frac{\sqrt{\lambda^2 - k}}{2|\mu|},
\]

which shows that \( |\mu| \) is expressed by the invariants \( k, \lambda \) and the mean curvature function.

Now we shall discuss the geometric meaning of the invariant \( \lambda \). Let \( M^m \) be an \( n \)-dimensional submanifold of \((n + m)\)-dimensional Riemannian manifold \( M^{n+m} \) and \( \xi \) be a normal vector field of \( M^m \). In [3] B.-Y. Chen defined the allied vector field \( a(\xi) \) of \( \xi \) by the formula

\[
a(\xi) = \frac{\|\xi\|}{n} \sum_{i=1}^{m} \{\text{tr}(A_i A_i)\} \xi_i.
\]

where \( \{\xi_1, \xi_2, \ldots, \xi_m\} \) is a local orthonormal frame field of the normal bundle of \( M^m \), and \( A_i = A_i, \quad i = 1, \ldots, m, \)

is the shape operator with respect to \( \xi_i \). In particular, the allied vector field \( a(H) \) of the mean curvature vector field
$H$ is a well-defined normal vector field which is orthogonal to $H$. It is called the \textit{allied mean curvature vector field} of $M^n$ in $\tilde{M}^{n+m}$, B.-Y. Chen defined the $A$-submanifolds to be those submanifolds of $\tilde{M}^{n+m}$ for which $a(H)$ vanishes identically [3]. In [11, 12] the $A$-submanifolds are called \textit{Chen submanifolds}. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial $A$-submanifolds. Now let $M^n = M^2$ be a surface in $\mathbb{R}^4$. Applying the definition of the allied mean curvature vector field from equalities (1) we get

$$a(H) = \frac{\nu_1 + \nu_2}{2} \lambda \mu l = \frac{\sqrt{\kappa^2 - k^2}}{2} \lambda l.$$  

Hence, if $M^2$ is free of minimal points, then $a(H) = 0$ if and only if $\lambda = 0$. This gives the geometric meaning of the invariant $\lambda$. It is clear that $M^2$ is a non-trivial Chen surface if and only if the invariant $\lambda$ is zero.

The normal curvatures $\nu' = \frac{L}{E}$ and $\nu'' = \frac{N}{G}$ of the principal tangents are said to be \textit{principal normal curvatures} of $M^2$. The invariants $k$ and $\kappa$ of $M^2$ are expressed by the principal normal curvatures $\nu'$ and $\nu''$ as follows:

$$k = \nu' \nu'', \quad \kappa = \frac{\nu' + \nu''}{2}.$$  

Similarly to the theory of surfaces in $\mathbb{R}^3$, we consider the indicatrix $\chi$ in the tangent space $T_p M^2$ at an arbitrary point $p$ of $M^2$, defined by

$$\chi : \nu' \chi^2 + \nu'' \chi^2 = \varepsilon, \quad \varepsilon = \pm 1.$$  

If $p$ is an elliptic point ($k > 0$), then the indicatrix $\chi$ is an ellipse. The axes of $\chi$ are collinear with the principal directions at the point $p$, and the lengths of the axes are $\frac{2}{\sqrt{|\nu'|}}$ and $\frac{2}{\sqrt{|\nu''|}}$.

If $p$ is a hyperbolic point ($k < 0$), then the indicatrix $\chi$ consists of two hyperbolas. For the sake of simplicity we say that $\chi$ is a hyperbola. The axes of $\chi$ are collinear with the principal directions, and the lengths of the axes are $\frac{2}{\sqrt{|\nu'|}}$ and $\frac{2}{\sqrt{|\nu''|}}$.

If $p$ is a parabolic point ($k = 0$), then the indicatrix $\chi$ consists of two straight lines parallel to the principal direction with non-zero normal curvature.

The following statement holds:

**Proposition 2.2.**

Two tangents $g_1$ and $g_2$ are conjugate tangents of $M^2$ if and only if $g_1$ and $g_2$ are conjugate with respect to the indicatrix $\chi$.

3. Classes of surfaces characterized in terms of the tangent indicatrix and the normal curvature ellipse

The minimal surfaces in $\mathbb{R}^4$ are characterized by the following result.

**Proposition 3.1 ([7]).**

Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is minimal if and only if $\kappa^2 - k = 0$.

The surfaces with flat normal connection are characterized by the following
Proposition 3.2.
Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is a surface with flat normal connection if and only if

$$\kappa = 0.$$ 

Proof. Let $D$ be the normal connection of $M^2$. For any tangent vector fields $x, y$ and any normal vector field $n$ we have the standard decomposition

$$\nabla_n^x n = -A_n(x) + D_n n,$$

where $\langle A_n(x), y \rangle = \langle \sigma(x, y), n \rangle$.

The curvature tensor $R^x$ of the normal connection $D$ is given by

$$R^x(x, y)n = D_xD_y n - D_yD_x n - D_{[x,y]} n.$$ 

Then the normal curvature (the curvature of the normal connection) at a point $p \in M^2$ is defined by $\langle R^x(x, y) n_2, n_1 \rangle$, where $\{x, y, n_1, n_2\}$ is a right oriented orthonormal quadruple.

Without loss of generality we assume that $F = 0$ and consider the unit vector fields $x = \frac{z_u}{\sqrt{E}}, y = \frac{z_v}{\sqrt{G}}$. Then we have the formulas

$$\sigma(x, x) = \sigma(x, y) = \sigma(y, y) = \frac{c_{11}}{E} n_1 + \frac{c_{12}}{EG} n_2.$$ 

Hence,

$$A_1(x) = \frac{c_{11}}{E} x + \frac{c_{12}}{\sqrt{EG}} y, \quad A_2(x) = \frac{c_{11}}{E} x + \frac{c_{12}}{\sqrt{EG}} y.$$ 

Using (5) we calculate

$$\sigma(x, y) = \frac{c_{12}}{\sqrt{EG}} n_1 + \frac{c_{12}}{\sqrt{EG}} n_2.$$ 

Thus we get

$$\sigma(x, y) = \kappa y, \quad \sigma(y, y) = -\kappa x.$$ 

Note that $A_2 \circ A_1 - A_1 \circ A_2$ is an invariant skew-symmetric operator in the tangent space, i.e. it does not depend on the choice of the orthonormal tangent frame field $\{x, y\}$.

Since the curvature tensor $R^x$ of the connection $\nabla^x$ is zero, we have

$$\nabla_x^y n_1 - \nabla_y^x n_1 = \nabla_{[x,y]} n_1 = 0.$$ 

Therefore, the tangent component and the normal component of $R^x(x, y)n_1$ are both zero. The normal component is

$$D_xD_y n_1 - D_yD_x n_1 - \sigma(x, A_1 y) + \sigma(y, A_1 x).$$ 

Hence,

$$D_xD_y n_1 - D_yD_x n_1 - \sigma(x, A_1 y) + \sigma(y, A_1 x).$$
The left-hand side of (7) is $R^2(x, y)n_1$. Then
\[
\langle R^2(x, y)n_1, n_2 \rangle = \langle \sigma(x, A_1y), n_2 \rangle - \langle \sigma(y, A_1x), n_2 \rangle = \langle (A_2 \circ A_1 - A_1 \circ A_2)(y), x \rangle.
\]
Using (6) we obtain $\langle R^2(x, y)n_1, n_2 \rangle = -\kappa$. Since $\langle R^2(x, y)n_1, n_2 \rangle = -\langle R^2(x, y)n_2, n_1 \rangle$, we get
\[
\langle R^2(x, y)n_2, n_1 \rangle = \kappa.
\]

The last equality implies that $\kappa$ is the curvature of the normal connection. Hence, $M^2$ is a surface with flat normal connection if and only if $\kappa = 0$. \hfill \Box

We note that the condition $\kappa = 0$ implies that $k < 0$ and the surface $M^2$ has two families of orthogonal asymptotic lines. Now we shall characterize the minimal surfaces and the surfaces with flat normal connection in terms of the tangent indicatrix of the surface.

**Proposition 3.3.**
Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is minimal if and only if at each point of $M^2$ the tangent indicatrix $\chi$ is a circle.

**Proof.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. From equalities (3) it follows that
\[
\kappa^2 - k = \left(\frac{\nu - \nu^\prime}{2}\right)^2.
\]
Obviously $\kappa^2 - k = 0$ if and only if $\nu = \nu^\prime$. Applying Proposition 3.1, we get that $M^2$ is minimal if and only if $\chi$ is a circle. \hfill \Box

**Proposition 3.4.**
Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is a surface with flat normal connection if and only if at each point of $M^2$ the tangent indicatrix $\chi$ is a rectangular hyperbola (a Lorentz circle).

**Proof.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. From (3) it follows that $\kappa = 0$ if and only if $\nu^\prime = -\nu^\prime$. If $M^2$ is a surface with flat normal connection, then $k < 0$, and hence $\chi$ is a hyperbola. From $\nu^\prime = -\nu^\prime$ it follows that the semi-axes of $\chi$ are equal to $\frac{1}{\sqrt{|v^i|}}$, i.e. $\chi$ is a rectangular hyperbola. Conversely, if $\chi$ is a rectangular hyperbola, then $\nu^\prime = -\nu$, which implies that $M^2$ is a surface with flat normal connection. \hfill \Box

The minimal surfaces and the surfaces with flat normal connection can also be characterized in terms of the ellipse of normal curvature. Let us recall that the ellipse of normal curvature at a point $p$ of a surface $M^2$ in $\mathbb{R}^4$ is the ellipse $C$ in the normal space at the point $p$ given by $\{ \sigma(x, x) : x \in T_pM^2, g(x, x) = 1 \}$ [20, 21]. Let $\{ x, y \}$ be an orthonormal base of the tangent space $T_pM^2$ at $p$. Then, for any $v = \cos \psi x + \sin \psi y$, we have
\[
\sigma(v, v) = H + 2\cos \psi \frac{\sigma(x, x) - \sigma(y, y)}{2} + \sin 2\psi \sigma(x, y),
\]
where $H = \frac{\sigma(x, x) + \sigma(y, y)}{2}$ is the mean curvature vector of $M^2$ at $p$. So, when $v$ goes once around the unit tangent circle, the vector $\sigma(v, v)$ goes twice around the ellipse centered at $H$. The vectors $\frac{\sigma(x, x) - \sigma(y, y)}{2}$ and $\sigma(x, y)$ determine conjugate directions of $C$. The area of the ellipse is given by the formula $S_C = \pi |S_0|$, where $S_0$ is the oriented area of
the parallelogram spanned by the vectors $\frac{\sigma(x, x) - \sigma(y, y)}{2}$ and $\sigma(x, y)$. In view of formulas (4) the oriented area $S_0$ is expressed as $S_0 = \frac{\pi}{2} |\kappa|$. Hence, $S_\kappa = \frac{\pi}{2} |\kappa|$. A surface $M^2$ in $\mathbb{R}^4$ is called super-conformal [2] if at any point of $M^2$ the ellipse of curvature is a circle. In [5] an explicit construction is given of any simply connected super-conformal surface in $\mathbb{R}^4$ that is free of minimal and flat points.

Obviously, $M^2$ is minimal if and only if for each point $p \in M^2$ the ellipse of curvature is centered at $p$.

The minimal surfaces in $\mathbb{R}^4$ are divided into two subclasses:

- the subclass of minimal super-conformal surfaces, characterized by the condition that the ellipse of curvature is a circle;
- the subclass of minimal surfaces of general type, characterized by the condition that the ellipse of curvature is not a circle.

In [9] it is proved that on any minimal surface $M^2$ the Gauss curvature $K$ and the normal curvature $\kappa$ satisfy the following inequality

$$K^2 - \kappa^2 \geq 0.$$ 

The two subclasses of minimal surfaces are characterized in terms of the invariants $K$ and $\kappa$ as follows:

- the class of minimal super-conformal surfaces is characterized by $K^2 - \kappa^2 = 0$;
- the class of minimal surfaces of general type is characterized by $K^2 - \kappa^2 > 0$.

The class of minimal super-conformal surfaces in $\mathbb{R}^4$ is locally equivalent to the class of holomorphic curves in $\mathbb{C}^2 \equiv \mathbb{R}^4$ [6]. The inequality $K^2 - \kappa^2 \geq 0$ for minimal surfaces also follows from the Wintgen inequality $K + |\kappa| \leq \|H\|^2$, which holds for an arbitrary surface in $\mathbb{R}^4$ [22]. Following [17], a surface in $\mathbb{R}^4$ is called a Wintgen ideal surface, if it satisfies the equality $K + |\kappa| = \|H\|^2$. The Wintgen ideal surfaces are characterized by circular ellipse of normal curvature [13]. In [4] B.-Y. Chen completely classified Wintgen ideal surfaces in $\mathbb{R}^4$ with equal Gauss and normal curvatures, i.e. Wintgen ideal surfaces satisfying $|K| = |\kappa|$ identically.

The surfaces with flat normal connection are characterized in terms of the ellipse of normal curvature as follows:

**Proposition 3.5.**

Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is a surface with flat normal connection if and only if for each point $p \in M^2$ the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.

**Proof.** The formula $S_\kappa = \frac{\pi}{2} |\kappa|$ implies that $\kappa = 0$ if and only if the ellipse of curvature is a line segment.

Let $M^2$ be a surface with flat normal connection, i.e. $\kappa = 0$, $k \neq 0$. We assume that $M^2$ is parameterized by principal parameters. Then from (2) it follows that $v_1 = v_2$. Further, equalities (1) imply that for each $v = \cos \psi x + \sin \psi y$, we have $\sigma(v, v) = H + \sin 2\psi(\lambda b + \mu l)$. So, when $v$ goes once around the unit tangent circle, the vector $\sigma(v, v)$ goes twice along the line segment collinear with $\lambda b + \mu l$ and centered at $H$. The mean curvature vector field is $H = v_1 b$. Since $k \neq 0$, then $\mu \neq 0$, and the line segment is not collinear with $H$. \(\Box\)

We note that in the case $\lambda = 0$ the mean curvature vector field $H$ is orthogonal to the line segment, while in the case $\lambda \neq 0$ the mean curvature vector field $H$ is not orthogonal to the line segment. The length $d$ of the line segment is

$$d = \sqrt{\lambda^2 + \mu^2} = \sqrt{H^2 - K}.$$
So, there arises a subclass of surfaces with flat normal connection, characterized by the conditions

\[ K = 0 \quad \text{or} \quad d = \|H\|. \]

Proposition 3.4 and Proposition 3.5 give us the following:

**Corollary 3.6.**

Let \( \mathcal{M} \) be a surface in \( \mathbb{R}^4 \) free of flat points. Then the tangent indicatrix \( \chi \) is a rectangular hyperbola (a Lorentz circle) if and only if the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.

### 4. Fundamental theorem

The basic theorem in the local differential geometry of surfaces in \( \mathbb{R}^3 \) is the fundamental theorem of Bonnet. We proved a theorem of Bonnet-type for minimal surfaces in \( \mathbb{R}^4 \) in terms of their invariants [9]. We recall that minimal surfaces are characterized by the condition \( \kappa^2 - k = 0 \) at any point. In this section we consider surfaces in \( \mathbb{R}^3 \) free of minimal points, i.e. such that

\[ \kappa^2 - k > 0 \quad \text{at any point}. \]

We assume that \( \mathcal{M}^2 \) is parameterized by principal lines and consider the unit vector fields \( x = \frac{z_u}{\sqrt{E}}, \ y = \frac{z_v}{\sqrt{G}} \). If \( \{x, y, b, l\} \) is the geometrically determined moving frame field, we have the following Frenet-type formulas:

\[
\begin{align*}
\nabla_x x &= y_1 y + v_1 b, & \nabla_x b &= -v_1 x - \lambda y + \beta_1 l, \\
\nabla_y y &= -y_1 x + \lambda b + \mu l, & \nabla_y b &= -\lambda x - v_2 y + \beta_2 b, \\
\nabla_b x &= -y_2 y + \lambda b + \mu l, & \nabla_b b &= -\mu y - \beta_1 b, \\
\n\end{align*}
\]

(8)

where \( y_1 = -y_2 \ln \sqrt{E} \), \( y_2 = -x \ln \sqrt{G} \), \( v_1, v_2, \lambda, \mu, \beta_1, \beta_2 \) are geometric invariant functions. Since \( \kappa^2 - k > 0 \), then equalities (2) imply that \( \mu \neq 0 \).

Using that \( R'(x, y, x) = 0, \ R'(x, y, y) = 0, \ R'(x, y, b) = 0 \) and \( R'(x, y, l) = 0 \), from (8) we get the following integrability conditions:

\[
\begin{align*}
2\mu y_2 + v_1 \beta_2 - \lambda \beta_1 &= x(\mu); & 2\mu y_1 - \lambda \beta_2 + v_2 \beta_1 &= y(\mu); \\
v_1 v_2 - (\lambda^2 + \mu^2) &= x(y_2) + y(v_1) - (v_1 y_2 + v_2 y_1); \\
2\lambda y_2 + \mu \beta_1 - (v_1 - v_2) y_1 &= x(\lambda) - y(v_1); \\
2\lambda y_1 + \mu \beta_2 + (v_1 - v_2) y_2 &= -x(v_2) + y(\lambda); \\
v_1 \beta_1 - v_2 \beta_2 + (v_1 - v_2) \mu &= -x(\beta_2) + y(\beta_1). \\
\end{align*}
\]

(9)

Further we consider the general class of surfaces determined by the condition

\[ \mu_u \mu_v \neq 0. \]

The first two equalities of (9) imply that the condition \( \mu_u \mu_v \neq 0 \) is equivalent to

\[ (2\mu y_2 + v_1 \beta_2 - \lambda \beta_1)(2\mu y_1 - \lambda \beta_2 + v_2 \beta_1) \neq 0. \]
Then
\[
\sqrt{E} = \frac{\mu_v}{2\mu y_2 + v_1 \beta_2 - \lambda \beta_1}, \quad \sqrt{G} = \frac{\mu_v}{2\mu y_1 - \lambda \beta_2 + v_2 \beta_1},
\]
and the geometric invariants of the surface satisfy the inequalities
\[
\frac{\mu_v}{2\mu y_2 + v_1 \beta_2 - \lambda \beta_1} > 0, \quad \frac{\mu_v}{2\mu y_1 - \lambda \beta_2 + v_2 \beta_1} > 0.
\] (10)

Furthermore, taking into account (9) we get that the invariants of any surface from the general class satisfy the equalities
\[
-\gamma_1 \sqrt{E} \sqrt{G} = (\sqrt{E})_u; \quad -\gamma_2 \sqrt{E} \sqrt{G} = (\sqrt{G})_u;
\]
\[
v_1 v_2 - (\lambda^2 + \mu^2) = \frac{1}{\sqrt{E}} (y_2)_u + \frac{1}{\sqrt{G}} (y_1)_u - ((y_1)^2 + (y_2)^2);
\]
\[
2\lambda y_2 + \mu \beta_1 - (v_1 - v_2) y_1 = \frac{1}{\sqrt{E}} \lambda_u - \frac{1}{\sqrt{G}} (y_1)_u; \quad (11)
\]
\[
2\lambda y_1 + \mu \beta_2 + (v_1 - v_2) y_2 = -\frac{1}{\sqrt{E}} (y_2)_u + \frac{1}{\sqrt{G}} \lambda_u; \quad \gamma_1 \beta_1 - \gamma_2 \beta_2 + (v_1 - v_2) \mu = -\frac{1}{\sqrt{E}} (\beta_2)_u + \frac{1}{\sqrt{G}} (\beta_1)_u.
\]

We shall prove the following Bonnet-type fundamental theorem

**Theorem 4.1.**
Let \(\gamma_1, \gamma_2, v_1, v_2, \lambda, \mu, \beta_1, \beta_2\) be smooth functions defined in a domain \(D, D \subset \mathbb{R}^2\), which satisfy inequalities (10) and equalities (11). Let \(x_0, y_0, b_0, l_0\) be a positive oriented orthonormal frame at a point \(p_0 \in \mathbb{R}^4\). Then there exist a subdomain \(D_0 \subset D\) and a unique surface \(M^2 : z = z(u, v), (u, v) \in D_0\), passing through \(p_0\), such that \(\gamma_1, \gamma_2, v_1, v_2, \lambda, \mu, \beta_1, \beta_2\) are the geometric functions of \(M^2\) and \(x_0, y_0, b_0, l_0\) is the geometric frame of \(M^2\) at the point \(p_0\).

**Proof.** We consider the following system of partial differential equations for the unknown vector functions \(x = x(u, v), y = y(u, v), b = b(u, v), l = l(u, v)\) in \(\mathbb{R}^4\):
\[
\begin{align*}
x_u &= \sqrt{E} y_1 + \sqrt{E} v_1 b, & x_v &= -\sqrt{G} y_2 + \sqrt{G} \lambda b + \sqrt{G} \mu l, \\
y_u &= -\sqrt{E} y_1 x + \sqrt{E} \lambda b + \sqrt{E} \mu l, & y_v &= \sqrt{G} y_2 x + \sqrt{G} v_1 b, \\
b_u &= -\sqrt{E} v_1 x - \sqrt{E} \lambda y + \sqrt{E} \beta_1 l, & b_v &= -\sqrt{G} \lambda x - \sqrt{G} v_1 y + \sqrt{G} \beta_2 l, \\
l_u &= -\sqrt{E} \mu y - \sqrt{E} \beta_1 b, & l_v &= -\sqrt{G} \mu x - \sqrt{G} \beta_2 b.
\end{align*}
\] (12)

We denote
\[
Z = \begin{pmatrix} x \\ y \\ b \\ l \end{pmatrix}, \quad A = \sqrt{E} \begin{pmatrix} 0 & y_1 & v_1 & 0 \\ -y_1 & 0 & \lambda & \mu \\ -v_1 & -\lambda & 0 & \beta_1 \\ 0 & -\mu & -\beta_1 & 0 \end{pmatrix}, \quad B = \sqrt{G} \begin{pmatrix} 0 & -y_2 & \lambda & \mu \\ y_2 & 0 & v_2 & 0 \\ -\lambda & -v_2 & 0 & \beta_2 \\ -\mu & 0 & -\beta_2 & 0 \end{pmatrix}.
\]

Then system (12) can be rewritten in the form:
\[
Z_u = AZ, \quad Z_v = BZ.
\] (13)

The integrability conditions of (13) are
\[
Z_{uv} = Z_{wu}.
\]
Example of surfaces with flat normal connection

\[
\frac{\partial a_i^k}{\partial v} - \frac{\partial b_j^k}{\partial u} + \sum_{j=1}^{4} (a_i^j b_k^j - b_j^i a_i^k) = 0, \quad i, k = 1, \ldots, 4, \tag{14}
\]

where \(a_i^j\) and \(b_j^k\) are the elements of the matrices \(A\) and \(B\). Using (11) we obtain that the equalities (14) are fulfilled. Hence, there exists a subset \(\mathcal{D}_1 \subset \mathcal{D}\) and unique vector functions \(x = x(u, v), \quad y = y(u, v), \quad b = b(u, v), \quad l = l(u, v)\), \((u, v) \in \mathcal{D}_1\), which satisfy system (12) and the conditions:

\[
x(u_0, v_0) = x_0, \quad y(u_0, v_0) = y_0, \quad b(u_0, v_0) = b_0, \quad l(u_0, v_0) = l_0.
\]

We shall prove that \(x(u, v), \quad y(u, v), \quad b(u, v), \quad l(u, v)\) form an orthonormal frame for each \((u, v) \in \mathcal{D}_1\). Let us consider the following functions:

\[
\begin{align*}
\varphi_1 &= x^2 - 1, & \varphi_5 &= x y, & \varphi_9 &= y b, \\
\varphi_2 &= y^2 - 1, & \varphi_6 &= x b, & \varphi_{10} &= y l, \\
\varphi_3 &= b^2 - 1, & \varphi_7 &= x l, & \varphi_{11} &= b l, \\
\varphi_4 &= l^2 - 1,
\end{align*}
\]

defined for each \((u, v) \in \mathcal{D}\). Using that \(x(u, v), \quad y(u, v), \quad b(u, v), \quad l(u, v)\) satisfy (12), we obtain the system

\[
\frac{\partial \varphi_i}{\partial u} = a_i^j \varphi_j, \quad \frac{\partial \varphi_i}{\partial v} = b_i^j \varphi_j, \quad i = 1, \ldots, 10, \tag{15}
\]

where \(a_i^j, \quad b_i^j, \quad i, j = 1, \ldots, 10\), are functions of \((u, v) \in \mathcal{D}\). System (15) is a linear system of partial differential equations for the functions \(\varphi_i(u, v), \quad i = 1, \ldots, 10, \quad (u, v) \in \mathcal{D}_1\), satisfying \(\varphi_i(u_0, v_0) = 0, \quad i = 1, \ldots, 10\). Hence, \(\varphi_i(u, v) = 0, \quad i = 1, \ldots, 10\), for each \((u, v) \in \mathcal{D}_1\). Consequently, the vector functions \(x(u, v), \quad y(u, v), \quad b(u, v), \quad l(u, v)\) form an orthonormal frame field for each \((u, v) \in \mathcal{D}\).

Now, let us consider the system

\[
z_u = \sqrt{E} x, \quad z_v = \sqrt{G} y, \tag{16}
\]

of partial differential equations for the vector function \(z(u, v)\). Using (11) and (12) we get that the integrability conditions \(z_{uv} = z_{vu}\) of system (16) are fulfilled. Hence, there exists a subset \(\mathcal{D}_0 \subset \mathcal{D}_1\) and a unique vector function \(z = z(u, v)\), defined for \((u, v) \in \mathcal{D}_0\) and satisfying \(z(u_0, v_0) = p_0\).

Consequently, the surface \(M^2 : z = z(u, v), \quad (u, v) \in \mathcal{D}_0\), satisfies the assertion of the theorem. \(\square\)

5. Examples of surfaces with flat normal connection

In this section we construct a family of surfaces with flat normal connection lying on a standard rotational hypersurface in \(\mathbb{R}^4\).

Let \(\{e_1, e_2, e_3, e_4\}\) be the standard orthonormal frame in \(\mathbb{R}^3\), and \(S^2(1)\) be a 2-dimensional sphere in \(\mathbb{R}^3 = \text{span} \{e_1, e_2, e_3\}\), centered at the origin \(O\). Let \(l = l(u), \quad g = g(u)\) be smooth functions, defined in an interval \(l \subset \mathbb{R}\), such that \(l^2(u) + g^2(u) = 1, \quad u \in l\). The standard rotational hypersurface \(M^3\) in \(\mathbb{R}^4\), obtained by the rotation of the meridian curve \(m : u \mapsto (l(u), g(u))\) around the \(OE_{e_4}\)-axis, is parameterized as follows:

\[
M^3 : Z(u, w_1, w_2) = f(u) l(w_1, w_2) + g(u) e_4,
\]

where \(l(w_1, w_2)\) is the unit radius-vector of \(S^2(1)\) in \(\mathbb{R}^3\).

We consider a smooth curve \(c : l = l(v) = l(w_1^t(v), w_2^t(v)), \quad v \in J, \quad J \subset \mathbb{R}\), on \(S^2(1), \quad \text{parameterized by the arc-length, i.e.} \quad l^2(v) = 1\). We denote \(t = t^t\) and consider the moving frame field span \(\{t(v), n(v), l(v)\}\) of the curve \(c\) on \(S^2(1)\).

With respect to this orthonormal frame field the following Frenet formulas hold good:

\[
t' = t, \quad t' = \kappa n - l, \quad n' = -\kappa t, \quad \hat{\kappa} = \frac{-k}{\kappa}.
\]
where $\kappa$ is the spherical curvature of $c$.

Now we construct a surface $M^2$ in $\mathbb{R}^4$ in the following way:

$$M^2 : z(u, v) = f(u) l(v) + g(u) e_4, \quad u \in I, \quad v \in J. \quad (18)$$

The surface $M^2$ lies on the rotational hypersurface $M^2$ in $\mathbb{R}^4$. Since $M^2$ consists of meridians of $M^2$, we call $M^2$ a meridian surface.

The tangent space of $M^2$ is spanned by the vector fields:

$$z_u = l \hat{t} + \hat{g} e_4, \quad z_v = l \hat{t},$$

and hence, the coefficients of the first fundamental form of $M^2$ are $E = 1$; $F = 0$; $G = \dot{f}(u)$. Taking into account (17), we calculate the second partial derivatives of $z(u, v)$:

$$z_{uu} = \ddot{f} \hat{t} + \hat{g} e_4, \quad z_{uv} = \ddot{t}, \quad z_{vv} = f \kappa n - f l.$$

Let us denote $x = z_u, y = \frac{z_v}{f} = t$ and consider the following orthonormal normal frame field of $M^2$:

$$n_1 = n(v), \quad n_2 = -\hat{g}(u) l(v) + \ddot{f}(u) e_4.$$

Thus we obtain a positive orthonormal frame field $\{x, y, n_1, n_2\}$ of $M^2$. If we denote by $\kappa_m$ the curvature of the meridian curve $m$, i.e. $\kappa_m(u) = \dot{f}(u) \hat{g}(u) - \ddot{g}(u) - \ddot{f}(u) = \frac{-\ddot{f}(u)}{\sqrt{1 - \dot{f}(u)^2}}$, then we get the following derivative formulas of $M^2$:

$$\begin{align*}
\nabla_{x} x &= \kappa_m n_2; & \nabla_{x} y &= 0; & \nabla_{x} n_1 &= \frac{\kappa}{f} y; \\
\nabla_{y} x &= \frac{\ddot{f}}{f} y; & \nabla_{y} y &= -\kappa_m x; & \nabla_{y} n_1 &= -\frac{\hat{g}}{f} n_2; \\
\nabla_{y} n_2 &= \frac{\ddot{f}}{f} x + \kappa n_1 + \frac{\hat{g}}{f} n_2; & \nabla_{y} n_2 &= -\frac{\hat{g}}{f} y. 
\end{align*} \quad (19)$$

The coefficients of the second fundamental form of $M^2$ are $L = N = 0, M = -\kappa_m(u) \kappa(v)$. Taking into account (19), we find the invariants $k, \varkappa, K$:

$$k = -\frac{\kappa_m(u) \kappa^2(v)}{\dot{f}^2(u)}, \quad \varkappa = 0, \quad K = \frac{\kappa_m(u) \hat{g}(u)}{\dot{f}(u)}. \quad (20)$$

The equality $\varkappa = 0$ implies that $M^2$ is a surface with flat normal connection.

The mean curvature vector field $H$ is given by

$$H = \frac{\kappa}{2f} n_1 + \frac{\hat{g} + f \kappa_m}{2f} n_2. \quad (21)$$

There are three main classes of meridian surfaces:

I. $\kappa = 0$, i.e. the curve $c$ is a great circle on $S^2(1)$. In this case $n_1$ $= \text{const}$, and $M^2$ is a planar surface lying in the constant 3-dimensional space spanned by $\{x, y, n_2\}$. Particularly, if in addition $\kappa_m = 0$, i.e. the meridian curve lies on a straight line, then $M^2$ is a developable surface in the 3-dimensional space span $\{x, y, n_2\}$.

II. $\kappa_m = 0$, i.e. the meridian curve is part of a straight line. In this case $k = \varkappa = K = 0$, and $M^2$ is a developable ruled surface. If in addition $\kappa = \text{const}$, i.e. $c$ is a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in a 3-dimensional space. If $\kappa \neq \text{const}$, i.e. $c$ is not a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in $\mathbb{R}^4$.

III. $\kappa_m \kappa \neq 0$, i.e. $c$ is not a great circle on $S^2(1)$, and $m$ is not a straight line. In this general case the invariant function $k < 0$, which implies that there exist two systems of asymptotic lines on $M^2$. The parametric lines of $M^2$ given by (18) are orthogonal and asymptotic.

Let $M^2$ be a meridian surface of the general class. Now we are going to find the meridian surfaces with:
\begin{itemize}
  \item constant Gauss curvature \( K \);
  \item constant mean curvature;
  \item constant invariant function \( k \).
\end{itemize}

\textbf{Proposition 5.1.}

Let \( M^2 \) be a meridian surface in \( \mathbb{R}^4 \) from the general class. Then \( M^2 \) has constant non-zero Gauss curvature \( K \) if and only if the meridian \( m \) is given by

\[
\begin{align*}
  f(u) &= \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u, \quad K > 0; \\
  f(u) &= \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u, \quad K < 0,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are constants.

\textbf{Proof.} Using (20) and \( f^2 + g^2 = 1 \), we obtain that \( M^2 \) has constant Gauss curvature \( K \neq 0 \) if and only if the meridian \( m \) satisfies the following differential equation

\[
\ddot{f}(u) + K f(u) = 0.
\]

The general solution of the above equation is given by

\[
\begin{align*}
  f(u) &= \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u \quad \text{in case } K > 0; \\
  f(u) &= \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u \quad \text{in case } K < 0,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are constants. The function \( g(u) \) is determined by \( \dot{g}(u) = \sqrt{1 - \dot{f}^2(u)} \). \( \square \)

The equality (21) implies that the mean curvature of \( M^2 \) is given by

\[
\|H\| = \sqrt{\kappa^2(u) + (\dot{g}(u) + f(u)\kappa_m(u))^2}.
\]

(22)

The meridian surfaces with constant mean curvature (CMC meridian surfaces) are described in the next result.

\textbf{Proposition 5.2.}

Let \( M^2 \) be a meridian surface in \( \mathbb{R}^4 \) from the general class. Then \( M^2 \) has constant mean curvature \( \|H\| = a = \text{const}, a \neq 0 \), if and only if the curve \( c \) on \( S^2(1) \) is a circle with constant spherical curvature \( \kappa = \text{const} = b, b \neq 0 \), and the meridian \( m \) is determined by the following differential equation:

\[
\left( 1 - \dot{f}^2 - \ddot{f} \right)^2 = \left( 1 - \dot{f}^2 \right) \left( 4a^2 \dot{f}^2 - b^2 \right).
\]

\textbf{Proof.} From (22) it follows that \( \|H\| = a \) if and only if

\[
\kappa^2(u) = 4a^2 \dot{f}^2(u) - \left( \dot{g}(u) + f(u)\kappa_m(u) \right)^2,
\]

which implies

\[
\kappa = \text{const} = b, \quad b \neq 0; \quad 4a^2 \dot{f}^2(u) - \left( \dot{g}(u) + f(u)\kappa_m(u) \right)^2 = b^2.
\]

(23)
The first equality of (23) implies that the spherical curve \( c \) has constant spherical curvature \( \kappa = b \), i.e. \( c \) is a circle. Using that \( \dot{f}^2 + g^2 = 1 \), and \( \kappa_n = \dot{g} - g\dot{f} \) we calculate that \( g + f\kappa_n = \frac{1 - \dot{f}^2 - \ddot{f}}{\sqrt{1 - f^2}} \). Hence, the second equality of (23) gives the following differential equation for the meridian \( m \):

\[
\left( 1 - \dot{f}^2 - \ddot{f} \right)^2 = (1 - f^2) \left( 4a^2f^2 - b^2 \right).
\] (24)

Further, if we set \( \dot{f} = y(t) \) in equation (24), we obtain that the function \( y = y(t) \) is a solution of the following differential equation

\[
1 - y^2 - \frac{t}{2} (y^2)' = \sqrt{1 - y^2} \sqrt{4a^2t^2 - b^2}.
\]

The general solution of the above equation is given by

\[
y(t) = \sqrt{1 - \frac{1}{t^2}} \left( C + \frac{t}{2} \sqrt{4a^2t^2 - b^2} - \frac{b^2}{4a} \ln \left| 2at + \sqrt{4a^2t^2 - b^2} \right| \right), \quad C = \text{const}.
\] (25)

The function \( f(u) \) is determined by \( \dot{f} = y(f) \) and (25). The function \( g(u) \) is defined by \( \ddot{g}(u) = \sqrt{1 - \dot{f}^2(u)} \).

At the end of this section we find the meridian surfaces with constant invariant \( k \).

**Proposition 5.3.**

Let \( M^2 \) be a meridian surface in \( \mathbb{R}^4 \) from the general class. Then \( M^2 \) has a constant invariant \( k = \text{const} = -a^2, a \neq 0 \), if and only if the curve \( c \) on \( S^2(1) \) is a circle with spherical curvature \( \kappa = \text{const} = b, b \neq 0 \), and the meridian \( m \) is determined by the following differential equation:

\[
\ddot{f}(u) = \mp \frac{a}{b} f(u) \sqrt{1 - \dot{f}^2(u)}.
\]

**Proof.** Using (20) we obtain that \( k = \text{const} = -a^2, a \neq 0 \), if and only if \( \kappa^2(\nu)\kappa^2_n(u) = a^2f^2(u) \). Hence,

\[
\kappa(\nu) = \pm a \frac{f(u)}{\kappa_n(u)}.
\]

The last equality implies

\[
\kappa = \text{const} = b, \quad b \neq 0; \quad \pm a \frac{f(u)}{\kappa_n(u)} = b.
\] (26)

The first equality of (26) implies that the spherical curve \( c \) has constant spherical curvature \( \kappa = b \), i.e. \( c \) is a circle. The second equality of (26) gives the following differential equation for the function \( f(u) \):

\[
\frac{\ddot{f}(u)}{\sqrt{1 - \dot{f}^2(u)}} = \mp \frac{a}{b} f(u).
\] (27)

Again setting \( \dot{f} = y(f) \) in equation (27), we obtain that the function \( y = y(t) \) is a solution of the following differential equation

\[
\frac{yy'}{\sqrt{1 - y^2}} = \mp \frac{a}{b} t.
\]

The general solution of the above equation is given by

\[
y(t) = \sqrt{1 - \left( C \pm \frac{a}{b} \frac{t^2}{2} \right)^2}, \quad C = \text{const}.
\] (28)

The function \( f(u) \) is determined by \( \dot{f} = y(f) \) and (28). The function \( g(u) \) is defined by \( \ddot{g}(u) = \sqrt{1 - \dot{f}^2(u)} \).
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