A VARIATIONAL MODEL WITH FRACTIONAL-ORDER
REGULARIZATION TERM ARISING IN REGISTRATION OF
DIFFUSION TENSOR IMAGE

HUAN HAN*

School of Mathematics and Physics, China University of Geosciences,
Wuhan 430076, China

(Communicated by Yunmei Chen)

ABSTRACT. In this paper, a new variational model with fractional-order reg-
ularization term arising in registration of diffusion tensor image (DTI) is pre-
sented. Moreover, the existence of its solution is proved to ensure that there is
a regular solution for this model. Furthermore, three numerical tests are also
performed to show the effectiveness of this model.

1. Introduction. Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded domain, i.e., \( \Omega = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \). Suppose \( T \) and \( D \) are two functions defined from \( \Omega \) to the set of
3 \times 3 real Symmetric Positive Definite matrixes (\( SPD(3) \) in short). That is,
\[
T : \Omega \rightarrow SPD(3), \quad D : \Omega \rightarrow SPD(3).
\]

In DTI registration, \( T \) and \( D \) are viewed as two images defined on \( \Omega \), where \( T \) is called floating image and \( D \) is called target image. The goal of registration is
to find a 1-to-1 spatial transformation \( h : \Omega \rightarrow \Omega \) such that
\( T \circ h(\cdot) \) is close to \( D(\cdot) \) in some sense. On the other hand, in order to keep \( T \circ h(\cdot) \) align with spatial
transformation, reorientation of \( T \circ h(\cdot) \) must be additionally considered. For this
purpose, Alexander[1] put forward two reorientation strategies: finite strain (FS)
strategy and preservation principle direction (PPD) strategy. Based on FS strategy,
Li[12] introduced a new operator “\( \Diamond \)” defined by
\[
T \Diamond h(x) = R[T \circ h(x)]R^T \quad \text{with} \quad R = J^T(JJJ^T)^{-\frac{1}{2}} \quad \text{and} \quad J = \nabla h^{-1}(x).
\]

With the help of this operator, the Large Deformation Diffeomorphic (LDD) DTI
registration model (cf. [2, 3, 8, 12]) can be formulated as
\[
\tilde{v} = \arg \min_v \tilde{H}(v),
\]
where \( \tilde{H}(v) = \int_0^T \|Lv(\cdot,t)\|^2_{L^2(\Omega)} dt + \|T \circ h(\cdot) - D(\cdot)\|^2_{L^2(\Omega)} \), \( L : [H_0^3(\Omega)]^3 \rightarrow [L^2(\Omega)]^3 \)
is a linear differential operator satisfying
\[
\|Lv(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{i=1}^3 \int_{\Omega} |(Lv)_i(x,t)|^2 dx \geq c\|v(\cdot,t)\|^2_{[H_0^3(\Omega)]^3},
\]

2010 Mathematics Subject Classification. Primary: 68U10, 62H35, 74G65, 94A08, 97M10, 58E05; Secondary: 49J45, 49J35.

Key words and phrases. Variation, DTI registration, fractional-order derivatives.

The first author is supported by NSFC under grant No.11471331 and partially supported by National Center for Mathematics and Interdisciplinary Sciences.

* Corresponding author: Huan Han.
employed integer-order derivatives in linear differential operator some suitable space. Note that almost all the DTI registration model\cite{12, 18} have

\[ (5) \quad \frac{d\eta(t; t, x)}{ds} = v(\eta(t; t, x), s), \quad \eta(t; t, x) = x \quad \text{and} \quad h(x) = \eta(0; \tau, x). \]

**Remark 1.** Note that in (2), \( R, J \) and \( T \circ h(x) \) are all \( 3 \times 3 \) matrixes with \( h^{-1}(x) = \eta(\tau; 0, x) \) and \( \mathbf{J} = \nabla_x \eta(\tau; 0, x) \) (cf. \cite{8}). Here the existence of function \( h^{-1} \) is given by (ii) in Lemma 3.6 and the definition of \((JJ^T)^{-\frac{1}{2}}\) can refer to Appendix in \cite{8}. \( \square \)

Here (5) is used as a constraint to guarantee that deformation \( h(x) \) is the flow of a diffeomorphism\cite{2, 3, 8, 12} (i.e., \( h \) is an invertible function that maps \( \Omega \) onto itself and both \( h \) and \( h^{-1} \) are differentiable\cite{12}). This ensures that disjoint set remains disjoint. That is, no fusion of point occurs under LDD.

The objective of LDD is not just to find an \( h(x) \) such that \( T \circ h(\cdot) \) approximates \( D(\cdot) \). Rather, the objective of LDD is to find an optimal path \( \eta(0; s, x) \) continuously parameterized by time \( s \) that smoothly deform \( T(\cdot) \) through \( T_s = T \circ \eta(0; s, x) \) to \( T_\tau = T \circ \eta(0; \tau, x) \).

As to LDD registration model, there are many pioneering works\cite{2, 3, 5, 8, 12, 18}. In \cite{8}, authors prove that there exists a solution to variational problem (3)-(5) on some suitable space. Note that almost all the DTI registration model\cite{12, 18} have employed integer-order derivatives in linear differential operator \( L \). During the last decades, it has been showed that many problems involving science and engineering can be modeled more accurately by employing fractional-order derivatives\cite{15, 17, 20} than integer-order derivatives. Especially in image processing, texture of many images is of fractional order smoothness. Fractional order model is expected to obtain a better result. Motivated by this fact, the aim of this paper is to employ fractional-order derivatives in DTI registration model.

Before giving our results, we introduce some notations and definitions.

Throughout this paper, we define \( \Omega \triangleq (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3) \subset \mathbb{R}^3 \). Moreover, for \( x \in \Omega \), the inner product and modulus of matrix \( A(x) = (a_{ij}(x))_{n \times m}, B = (b_{ij}(x))_{n \times m} \) are defined as

\[
A(x) : B(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}(x) b_{ij}(x), \quad ||A(x)|| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2(x)},
\]

respectively.

Furthermore, we say \( A(x) \) is continuous on \( \Omega \) if \( a_{ij}(x)(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m) \) are continuous on \( \Omega \). What’s more, we define \( \partial_\beta A(x) = \left( \frac{\partial a_{ij}(x)}{\partial x_\beta} \right)_{n \times m} \) for \( \beta = 1, 2, 3 \).

Moreover, for matrix sequence \( A_k(x) = (a_{ij}^k(x))_{n \times m} \), we say \( A_k(x) \xrightarrow{k} A(x) \) if \( a_{ij}^k(x) \xrightarrow{k} a_{ij}(x)(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m) \).

Based on definition of Riemann-Liouville derivative in \cite{6}, for \( x = (x_1, x_2, x_3) \in \Omega \) and function \( f : \Omega \rightarrow \mathbb{R} \), define

\[
\frac{\partial^\alpha f(x)}{\partial x_i^\alpha} \triangleq D_{[a_i, x_i]}^\alpha f(x) = \frac{1}{\Gamma([\alpha] + 1 - \alpha)} \left( \frac{d}{dx_i} \right)^{[\alpha]+1} \int_{a_i}^{x_i} f(i)(x, t) \frac{t^{\alpha-\alpha}}{(x_i-t)^{\alpha-\alpha-\alpha}} \, dt,
\]

\[
\frac{\partial^{*\alpha} f(x)}{\partial x_i^{*\alpha}} \triangleq D_{[x_i, b_i]}^{*\alpha} f(x) = \frac{1}{\Gamma([\alpha] + 1 - \alpha)} \left( -\frac{d}{dx_i} \right)^{[\alpha]+1} \int_{x_i}^{b_i} f(i)(x, t) \frac{t^{\alpha-\alpha}}{(t-x)^{n-\alpha-\alpha}} \, dt,
\]
where $\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx$, \( [\cdot] \) is round down function, here and in what follows, \( f^{(i)}(x, t) = f(t, x, t_3), \ f^{(2)}(x, t) = f(x_1, t, x_3), \ f^{(3)}(x, t) = f(x_1, x_2, t) \) and \( i = 1, 2, 3 \).

**Definition 1.1.** For \( \alpha > 0 \) and function \( g : \Omega \to \mathbb{R} \), define semi-norms

\[
|g|_{F^2_\alpha(\Omega)} = \left( \int_\Omega \|\nabla^\alpha g(x)\|^2 dx \right)^{\frac{1}{2}},
\]

and norms

\[
\|g\|_{F^2_\alpha(\Omega)} = \left( \int_\Omega \|\nabla^\alpha g(x)\|^2 dx \right)^{\frac{1}{2}},
\]

where \( \nabla^\alpha g(x) = \left( \frac{\partial^\alpha g(x)}{\partial x_i^\alpha} \right)_{1 \times 3} \) and \( \nabla^\alpha g(x) = \left( \frac{\partial^\alpha g(x)}{\partial x_i^\alpha} \right)_{1 \times 3} \).

Based on Definition 1.1, define space \( F^\infty_{L,0}(\Omega) \) and \( F^\infty_{R,0}(\Omega) \) as the closure of \( C^\infty_0(\Omega) \) under the norm \( \| \cdot \|_{F^\infty_{L,0}(\Omega)} \) and \( \| \cdot \|_{F^\infty_{R,0}(\Omega)} \), respectively (see Remark 3 for details).

**Definition 1.2.** For \( \alpha > 0 \) and \( u \in L^1(\mathbb{R}^3) \), define the semi-norm and norm

\[
|u|_{H^\alpha(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \|\xi\|^2 |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad \|u\|_{H^\alpha(\mathbb{R}^3)} = \left( \|u\|^2_{L^2(\mathbb{R}^3)} + |u|^2_{H^\alpha(\mathbb{R}^3)} \right)^{\frac{1}{2}},
\]

where here and in what follows, \( \hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} u(x)e^{-ix \cdot \xi} dx \).

Define Sobolev space \( H^\alpha(\mathbb{R}^3) \) as the closure of \( C^\infty_0(\mathbb{R}^3) \) under the norm \( \| \cdot \|_{H^\alpha(\mathbb{R}^3)} \) (cf. [4, 7]).

In Definition 1.2, if we restrict \( \mathbb{R}^3 \) to \( \Omega \), then \( H^\alpha_m(\Omega) \) is the closure of \( C^\infty_0(\Omega) \) under the norm \( \| \cdot \|_{H^\alpha_m(\Omega)} \) (cf. [4, 7]).

For \( \tau > 0, \alpha > 2.5, \alpha \neq m + 0.5, m \in \mathbb{N} \) and \( u : \Omega \times [0, \tau] \to \mathbb{R}^3 \), define a separable Hilbert space

\[
\mathcal{F} \triangleq \{ u(x, t) = (u_i(x, t))_{1 \times 3} : u_i(x, t) \in F^\infty_{L,0}(\Omega) \text{ for any } t \in [0, \tau] \text{ and } i = 1, 2, 3 \},
\]

endowing with the following inner product and norm

\[
(u, v)_{\mathcal{F}} = \int_0^\tau \int_\Omega \nabla^\alpha u(x, t) \cdot \nabla^\alpha v(x, t) dx dt, \quad \|u\|^2_{\mathcal{F}} = \int_0^\tau \|\nabla^\alpha u(\cdot, t)\|^2_{L^2(\Omega)} dt,
\]

where \( \nabla^\alpha u(x, t) = \left( \frac{\partial^\alpha u(x, t)}{\partial x_i^\alpha} \right)_{3 \times 3} \).

Based on the above notations and definitions, the variational model with fractional-order regularization term arising in registration of diffusion tensor image (DTI) can be formulated as

\[
(6) \quad \bar{v} = \arg \min_{v \in \mathcal{F}} H(v),
\]

where \( H(v) = H_1(v) + H_2(v) \), \( H_1(v) = \int_0^\tau \|\nabla^\alpha v(\cdot, t)\|^2_{L^2(\Omega)} dt, \ H_2(v) = \|T \circ h(\cdot) - D(\cdot)\|^2_{L^2(\Omega)} \) and \( h(x) \) is defined by (5).

Another purpose of this paper is to give a rigorous proof on the existence of solution to (6). As to this problem, we have the following result:

**Theorem 1.3.** Let \( T \) and \( D \) be two functions defined by (1), and let the set \( \triangle T \triangleq \{ x : T(\cdot) \text{ is discontinuous at } x \} \) be a set of measure zero. If \( \max_{x \in \Omega} \|T(x)\| < +\infty \), \( G \triangleq \max_{x \in \Omega} \|T(x) - D(x)\| < +\infty \), then the variational problem (6) admits a solution
Moreover, if $H(v)$ in (6) is formulated as
\[ H(v) = \int_0^\tau \| \nabla^\alpha u(\cdot, t) \|_{L^2(\Omega)}^2 dt + \| T \circ h(\cdot) - D(\cdot) \|_{L^2(\Omega)}^2, \]
then there also exists a global minimizer to $H(v)$ on space
\[ \mathcal{F}_1 \ni \{ u(x, t) = (u_i(x, t))_{1 \times 3} : u_i(x, t) \in F^{\alpha}_{R,0}(\Omega) \text{ for any } t \in [0, \tau] \text{ and } i = 1, 2, 3 \}, \]
endowing with the following inner product and norm
\[ (u, v)_{\mathcal{F}_1} = \int_0^\tau \int_\Omega \nabla^\alpha u(x, t) \cdot \nabla^\alpha v(x, t) dx dt, \quad \| u \|_{\mathcal{F}_1}^2 = \int_0^\tau \| \nabla^\alpha u(\cdot, t) \|_{L^2(\Omega)}^2 dt, \]
where $\alpha > 2.5$, $\alpha \neq m + 0.5$, $m \in \mathbb{N}$ and $\nabla^\alpha u(x, t) = \left( \frac{\partial^{\alpha} u_i(x, t)}{\partial x_j^{\alpha}} \right)_{3 \times 3}$.

2. Equivalence of $F^\alpha_{L,0}(\Omega)$, $F^\alpha_{R,0}(\Omega)$ and $H^\alpha_0(\Omega)$. In [8], authors impose the condition (4) on $L$ such that $v(\cdot, t) \in [H^3_0(\Omega)]^3 \hookrightarrow [C^1(\Omega)]^3$ which ensures the existence and uniqueness of solution to (5). As the basic space of this paper, $F^\alpha_{L,0}(\Omega)$ and $F^\alpha_{R,0}(\Omega)$ are also needed to embedded into $C^1(\Omega)$. Otherwise, the uniqueness of solution to (5) can not be guaranteed[16].

For this purpose, we will prove the equivalence of $F^\alpha_{L,0}(\Omega)$, $F^\alpha_{R,0}(\Omega)$ and $H^\alpha_0(\Omega)$, since $H^\alpha_0(\Omega) \hookrightarrow C^1(\Omega)(\alpha > 2.5)$(cf. [4, Theorem 4.57]).

First, we introduce some definitions.

**Definition 2.1.** For $\alpha > 0$ and function $g : \mathbb{R}^3 \to \mathbb{R}$, define the semi-norms
\[ |g|_{F^\alpha_L(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \| D^\alpha g(x) \|_2^2 dx \right)^{\frac{1}{2}}, \quad |g|_{F^\alpha_R(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \| D^\alpha g(x) \|_2^2 dx \right)^{\frac{1}{2}}, \]
and norms
\[ \| g \|_{F^\alpha_L(\mathbb{R}^3)} = \left( |g|_{F^\alpha_L(\mathbb{R}^3)}^2 + |g|_{F^\alpha_R(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}}, \quad \| g \|_{F^\alpha_R(\mathbb{R}^3)} = \left( |g|_{F^\alpha_L(\mathbb{R}^3)}^2 + |g|_{F^\alpha_R(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}}, \]
where $D^\alpha g(x) = (D_j^\alpha g(x))_{1 \times 3}$, $D^\alpha g(x) = (D_j^\alpha g(x))_{1 \times 3}$ and
\[ D_j^\alpha g(x) = \frac{1}{\Gamma((\alpha + 1) - \alpha)} \left( \frac{d}{dx_j} \right)^{[\alpha]+1} \int_{-\infty}^{x_j} \frac{g^{(j)}(x, t)}{(x_j - t)^{\alpha - [\alpha]}} dt, \]
\[ D_j^\alpha g(x) = \frac{1}{\Gamma((\alpha + 1) - \alpha)} \left( - \frac{d}{dx_j} \right)^{[\alpha]+1} \int_{x_j}^{+\infty} \frac{g^{(j)}(x, t)}{(t - x_j)^{\alpha - [\alpha]}} dt. \]

Define $F^\alpha_L(\mathbb{R}^3)$, $F^\alpha_R(\mathbb{R}^3)$ as the closure of $C^\infty_0(\mathbb{R}^3)$ under the norm $\| \cdot \|_{F^\alpha_L(\mathbb{R}^3)}$ and $\| \cdot \|_{F^\alpha_R(\mathbb{R}^3)}$, respectively.

**Definition 2.2.** For $\alpha > 0$ and function $g : \mathbb{R}^3 \to \mathbb{R}$, define semi-norm
\[ |g|_{F^\alpha_S(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} D^\alpha g(x) \cdot D^\alpha g(x) dx \right)^{\frac{1}{2}}, \]
and norm
\[ \| g \|_{F^\alpha_S(\mathbb{R}^3)} = \left( |g|_{F^\alpha_L(\mathbb{R}^3)}^2 + |g|_{F^\alpha_R(\mathbb{R}^3)}^2 \right)^{\frac{1}{2}}. \]
Define $F^\alpha_S(\mathbb{R})$ as the closure of $C_0^\infty(\mathbb{R})$ under the norm $\| \cdot \|_{F^\alpha_S(\mathbb{R})}$.

If we restrict $\mathbb{R}$ to $\Omega$ and replace $D^\alpha g(x)$, $D^{\alpha \ast} g(x)$ with $\nabla^\alpha g(x)$, $\nabla^{\alpha \ast} g(x)$ in (9) respectively, then $F^\alpha_{S,0}(\Omega)$ is defined similarly.

**Lemma 2.3.** For $\alpha > 0$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, there holds
\[
\frac{1}{3} \| \xi^\alpha \|^2 \leq \| \xi \|^{2\alpha} \leq 3 \| \xi^\alpha \|^2,
\]
where $\xi^\alpha = (\xi_1^\alpha, \xi_2^\alpha, \xi_3^\alpha)$.

**Proof.** Since $\| \xi \|^{2\alpha} = (\xi_1^2 + \xi_2^2 + \xi_3^2)^\alpha \geq \xi_j^{2\alpha} (j = 1, 2, 3)$, then there holds
\[
\frac{1}{3} \| \xi^\alpha \|^2 \leq \frac{1}{3} (\xi_1^{2\alpha} + \xi_2^{2\alpha} + \xi_3^{2\alpha}) \leq \| \xi \|^{2\alpha}.
\]
On the other hand, $\| \xi \|^{2\alpha} \leq 3 \max_{j=1,2,3} \xi_j^{2\alpha}$ implies that
\[
\| \xi \|^{2\alpha} \leq 3(\xi_1^{2\alpha} + \xi_2^{2\alpha} + \xi_3^{2\alpha}) = 3 \| \xi^\alpha \|^2.
\]

**Lemma 2.4.** Assume $\alpha > 0$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, then
\[
(D_j^\alpha f)(\xi) = (i \xi_j)^\alpha \hat{f}(\xi), \quad (D_j^{\alpha \ast} f)(\xi) = (-i \xi_j)^\alpha \hat{f}(\xi) \quad (j = 1, 2, 3).
\]

**Proof.** For function $g : \mathbb{R} \rightarrow \mathbb{R}$ and $\chi \in \mathbb{R}$, by Appendix in [6], we know
\[
(D_j \hat{g})(\chi) = (i \chi)^\alpha \hat{g}(\chi), \quad (D_j^{\alpha \ast} \hat{g})(\chi) = (-i \chi)^\alpha \hat{g}(\chi),
\]
where $\hat{g}(\chi) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} g(y) e^{-i y \chi} dy$.

Based on this conclusion, we have
\[
(D_j^\alpha \hat{f})(\xi) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D_j^\alpha f(x_1, x_2, x_3)e^{-i \sum_{j=1}^{3} x_j \xi_j} \, dx_1 dx_2 dx_3
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} D_j^\alpha f(x_1, x_2, x_3)e^{-i x_1 \xi_1} \, dx_1 \right] e^{-i \sum_{j=2}^{3} x_j \xi_j} \, dx_2 dx_3
\]
\[
= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (i \xi_1)^\alpha \int_{-\infty}^{+\infty} f(x_1, x_2, x_3)e^{-i x_1 \xi_1} \, dx_1 \right] e^{-i \sum_{j=2}^{3} x_j \xi_j} \, dx_2 dx_3
\]
\[
= (i \xi_1)^\alpha \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x_1, x_2, x_3)e^{-i \sum_{j=1}^{3} x_j \xi_j} \, dx_1 dx_2 dx_3
\]
(16)
\[
= (i \xi_1)^\alpha \hat{f}(\xi).
\]
Similarly, we can prove that $(D_j^{\alpha \ast} f)(\xi) = (i \xi_j)^\alpha \hat{f}(\xi)(j = 2, 3)$.

With the help of the second equality of (15), we can prove that
\[
(D_j^\alpha \hat{f})(\xi) = (-i \xi_j)^\alpha \hat{f}(\xi) \quad (j = 1, 2, 3).
\]

**Lemma 2.5.** Assume $\alpha > 0$, then $F^\alpha_L(\mathbb{R})$, $F^\alpha_R(\mathbb{R})$ and $H^\alpha(\mathbb{R})$ are equivalent.
Proof. By Lemma 2.4 and Plancherel Theorem[7, Theorem 1 in Section 4.3],
\[
|f|^2_{F^\alpha_2(\mathbb{R}^3)} = \sum_{j=1}^3 \|D^\alpha_j f\|^2_{L^2(\mathbb{R}^3)} = \sum_{j=1}^3 \|\hat{(D^\alpha_j f)}\|^2_{L^2(\mathbb{R}^3)} = \sum_{j=1}^3 \|\xi_j^{\alpha} \hat{f}(\xi)\|^2_{L^2(\mathbb{R}^3)}
\]
(18)
\[
= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \sum_{j=1}^3 |\xi_j|^{2\alpha} d\xi = \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \|\xi\|^2 d\xi.
\]
By Lemma 2.3, we know that $F^\alpha_2(\mathbb{R}^3)$ and $H^\alpha(\mathbb{R}^3)$ are equivalent.
In a similar way, we can prove that $F^\alpha_3(\mathbb{R}^3)$ and $H^\alpha(\mathbb{R}^3)$ are equivalent. □

Lemma 2.6. Assume $\alpha > 0$ and let $f(x)$ be a function defined from $\mathbb{R}^3$ to $\mathbb{R}$, then
\[
\int_{\mathbb{R}^3} D^\alpha f(x) \cdot D^{\alpha*} f(x) dx = \cos(\pi \alpha) \|D^\alpha f\|^2_{L^2(\mathbb{R}^3)}.
\]
(19)
Proof. By Parseval equality[7, Theorem 2 in Section 4.3]
\[
\int_{\mathbb{R}^3} u\bar{v} dx = \int_{\mathbb{R}} \tilde{u}(\xi) \tilde{\bar{v}}(\xi)d\xi,
\]
we have,
\[
\int_{\mathbb{R}^3} D^\alpha_1 f(x) \cdot D^{\alpha*}_1 f(x) dx = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D^\alpha_1 f(x) \overline{D^{\alpha*}_1 f(x)} dx_1 dx_2 dx_3
\]
\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} D^\alpha_1 f(x) \overline{D^{\alpha*}_1 f(x)} dx_1 \right] dx_2 dx_3
\]
\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} (\hat{D^\alpha_1 f}(\xi_1)(\hat{D^{\alpha*}_1 f}(\xi_1)) d\xi_1 \right] dx_2 dx_3,
\]
(21)
where, $(\hat{D^\alpha_1 f})(\xi_1) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} D^\alpha_1 f(x) e^{-i\xi_1 x_1} dx_1$, $(\hat{D^{\alpha*}_1 f})(\xi_1) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} D^{\alpha*}_1 f(x) e^{-i\xi_1 x_1} dx_1$.
On the other hand, we know
\[
(iw)^\alpha = \begin{cases} 
  e^{-i\pi\alpha}(-iw)^\alpha & \text{if } w \geq 0, \\
  e^{i\pi\alpha}(-iw)^\alpha & \text{if } w < 0.
\end{cases}
\]
(22)
It follows from (22) that
\[
\int_{-\infty}^{+\infty} (\hat{D^\alpha_1 f})(\xi_1)(\hat{D^{\alpha*}_1 f})(\xi_1) d\xi_1
\]
\[
= \int_{-\infty}^{0} (i\xi_1)^\alpha f^{(1)}(\xi_1)(-i\xi_1)^\alpha \overline{f^{(1)}(\xi_1)} d\xi_1
\]
\[
+ \int_{0}^{+\infty} (i\xi_1)^\alpha f^{(1)}(\xi_1)(-i\xi_1)^\alpha \overline{f^{(1)}(\xi_1)} d\xi_1
\]
\[
= \cos(\pi \alpha) \int_{-\infty}^{+\infty} (i\xi_1)^\alpha f^{(1)}(\xi_1)(i\xi_1)^\alpha \overline{f^{(1)}(\xi_1)} d\xi_1
\]
\[
+ i \sin(\pi \alpha) \left( \int_{0}^{+\infty} (i\xi_1)^\alpha f^{(1)}(\xi_1)(i\xi_1)^\alpha \overline{f^{(1)}(\xi_1)} d\xi_1
\]
\[
- \int_{-\infty}^{0} (i\xi_1)^\alpha \overline{f^{(1)}(\xi_1)(i\xi_1)^\alpha \overline{f^{(1)}(\xi_1)}} d\xi_1 \right),
\]
where \( \hat{f}^{(1)}(\xi_1) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} f^{(1)}(x,t)e^{-i\xi_1t}dt. \)

What’s more, by \( f(-\xi_1) = \hat{f}(\xi_1) \), we obtain that
\[
\int_{-\infty}^{+\infty} (i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)(i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)d\xi_1 = \int_{-\infty}^{0} (i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)(i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)d\xi_1.
\]

Therefore, we have
\[
\int_{-\infty}^{+\infty} (D^\xi_1 f)(\xi_1)(D^\xi_1 f)(\xi_1)d\xi_1 = \cos(\pi \alpha) \int_{-\infty}^{+\infty} (i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)(i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)d\xi_1.
\]
Substitute (23) into (21) yields
\[
\int_{\mathbb{R}^3} D_1^\alpha f(x) D_2^\alpha f(x)dx = \cos(\pi \alpha) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} (i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)(i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1)d\xi_1 \right] dx_2 dx_3
\]
\[
= \cos(\pi \alpha) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} \left| (i\xi_1)^{\alpha} \hat{f}^{(1)}(\xi_1) \right|^2 d\xi_1 \right] dx_2 dx_3
\]
\[
= \cos(\pi \alpha) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| D_1^\alpha f(x) \right|^2 dx_1 \right] dx_2 dx_3
\]
\[
= \cos(\pi \alpha) \| D_1^\alpha f \|^2_{L^2(\mathbb{R}^3)}.
\]

In a similar way, we can prove \( \int_{\mathbb{R}^3} D_j^\alpha f(x) D_j^\alpha f(x)dx = \cos(\pi \alpha) \| D_j^\alpha f \|^2_{L^2(\mathbb{R}^3)} \) for \( j = 2, 3 \), which concludes (19).

**Lemma 2.7.** Assume \( \alpha > 0 \) and \( \alpha \neq m + \frac{1}{2}, m \in \mathbb{N}, \) then \( F^\alpha_L(\mathbb{R}^3), F^\alpha_R(\mathbb{R}^3), F^\alpha_S(\mathbb{R}^3) \) and \( H^\alpha(\mathbb{R}^3) \) are equivalent.

**Proof.** By Lemma 2.6, \( F^\alpha_S(\mathbb{R}^3) \) and \( F^\alpha_L(\mathbb{R}^3) \) are equivalent. On the other hand, by Lemma 2.5, we prove this Lemma.

**Lemma 2.8.** Assume \( \alpha > 0 \) and \( \alpha \neq m + \frac{1}{2}, m \in \mathbb{N}, \) then \( F^\alpha_{S,0}(\Omega) \) and \( H^\alpha_0(\Omega) \) are equivalent.

**Proof.** Let \( \hat{f} \) be extension of \( f \in C^\infty_0(\Omega) \) by zero outside \( \Omega, \) then \( \text{supp}(\hat{f}) \subset \Omega, \) where \( \text{supp}(\hat{f}) = \{ x : \hat{f}(x) \neq 0 \}. \) What’s more, \( \text{supp}(D^\alpha_1 \hat{f}) \subset (a_1, +\infty) \times (a_2, b_2) \times (a_3, b_3), \) \( \text{supp}(D^\alpha_2 \hat{f}) \subset (a_1, b_1) \times (a_2, +\infty) \times (a_3, b_3), \) \( \text{supp}(D^\alpha_3 \hat{f}) \subset (a_1, b_1) \times (a_2, b_2) \times (a_3, +\infty), \) \( \text{supp}(D^\alpha_1 \hat{f}) \subset (-\infty, b_1) \times (a_2, b_2) \times (a_3, b_3), \) \( \text{supp}(D^\alpha_2 \hat{f}) \subset (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3), \) \( \text{supp}(D^\alpha_3 \hat{f}) \subset (a_1, b_1) \times (a_2, b_2) \times (-\infty, b_3). \)

Therefore, \( \text{supp}(D^\alpha_1 \hat{f} \cdot D^\alpha_2 \hat{f}) \subset \Omega. \) This implies,
\[
\| f \|_{F^\alpha_{S,0}(\Omega)} = \| \hat{f} \|_{F^\alpha_{S,0}(\mathbb{R}^3)}, \| f \|_{H^\alpha_0(\Omega)} = \| \hat{f} \|_{H^\alpha(\mathbb{R}^3)}.
\]
By Lemma 2.7, we obtain that \( F^\alpha_{S,0}(\Omega) \) and \( H^\alpha_0(\Omega) \) are equivalent.

Based on above Lemmas, we give the main result of this section.

**Theorem 2.9.** Assume \( \alpha > 0 \) and \( \alpha \neq m + \frac{1}{2}, m \in \mathbb{N}, \) then \( F^\alpha_{L,0}(\Omega), F^\alpha_R(\Omega) \) and \( H^\alpha_0(\Omega) \) are equivalent.
Proof. Let $\tilde{f}$ be extension of $f \in C^\infty_0(\Omega)$ by zero outside $\Omega$, then
\[ \|f\|^2_{L^2_0(\Omega)} + \|\nabla^\alpha f\|^2_{L^2(\Omega)} \leq \|\tilde{f}\|^2_{L^2(\mathbb{R}^3)} + \|D^\alpha \tilde{f}\|^2_{L^2(\mathbb{R}^3)} = 2 \|\tilde{f}\|^2_{L^2(\mathbb{R}^3)} \leq C \|f\|^2_{H^\alpha(\mathbb{R}^3)} = C \|f\|^2_{H^\alpha(\mathbb{R}^3)}. \]
That is, $H^\alpha_0(\Omega) \subseteq F^\alpha_{L_0}(\Omega)$.

On the other hand, by Lemma 2.7, Lemma 2.8 and Cauchy inequality [7] $ab \leq \frac{a^2}{2} + \varepsilon b^2$ ($\varepsilon > 0$),
\[ |f|^2_{H^\alpha(\Omega)} \leq C |f|^2_{F^\alpha_{L_0}(\Omega)} = C \left| \int_\Omega \nabla^\alpha f(x) \cdot \nabla^\alpha f(x) dx \right| \leq \frac{C}{4\varepsilon} \|\nabla^\alpha f\|^2_{L^2(\Omega)} + C \varepsilon \|\nabla^\alpha f\|^2_{L^2(\Omega)} \]
\[ = \frac{C}{4\varepsilon} |f|^2_{F^\alpha_{L_0}(\Omega)} + C \varepsilon |f|^2_{F^\alpha_{L_0}(\Omega)} \leq \frac{C}{4\varepsilon} |f|^2_{F^\alpha_{L_0}(\Omega)} + C \varepsilon |f|^2_{H^\alpha(\mathbb{R}^3)} \]
\[ \leq \frac{C}{4\varepsilon} |f|^2_{F^\alpha_{L_0}(\Omega)} + C \varepsilon |f|^2_{H^\alpha(\mathbb{R}^3)} = C \frac{1}{4\varepsilon} |f|^2_{F^\alpha_{L_0}(\Omega)} + C \varepsilon |f|^2_{H^\alpha(\mathbb{R}^3)}. \]
Let $\varepsilon = \frac{1}{4C}$ in (26), then
\[ |f|^2_{H^\alpha(\Omega)} \leq C C_1 |f|^2_{F^\alpha_{L_0}(\Omega)}. \]
That is, $F^\alpha_{L_0}(\Omega) \subseteq H^\alpha_0(\Omega)$.

Now, we conclude $F^\alpha_{L_0}(\Omega) = H^\alpha_0(\Omega)$.

Similarly, we can prove that $F^\alpha_{R_0}(\Omega)$ and $H^\alpha_0(\Omega)$ are equivalent. $\square$

Remark 3. By classical results on Sobolev space [4, 7], $H^\alpha_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ under the norm $\|\cdot\|_{H^\alpha_0(\Omega)}$. On the other hand, it follows from Lemma 2.8 and Theorem 2.9 that $F^\alpha_{L_0}(\Omega)$, $F^\alpha_{R_0}(\Omega)$, $F^\alpha_{S_0}(\Omega)$ and $H^\alpha_0(\Omega)$ are equivalent. Therefore, $F^\alpha_{L_0}(\Omega)$, $F^\alpha_{R_0}(\Omega)$ and $F^\alpha_{S_0}(\Omega)$ are the closure of $C^\infty_0(\Omega)$ under the norm $\|\cdot\|_{F^\alpha_{L_0}(\Omega)}$, $\|\cdot\|_{F^\alpha_{R_0}(\Omega)}$ and $\|\cdot\|_{F^\alpha_{S_0}(\Omega)}$, respectively.

3. Existence of solution to (6).

Definition 3.1. (cf. [6]) For $\alpha > 0$, $x \in [a, b]$ and function $f : [a, b] \rightarrow \mathbb{R}$, define
\[ D_{[x, t]}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - x)^{-\alpha} f(t) dt, \]
\[ D_{[x, b]}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{-\alpha} f(t) dt. \]
By Appendix in [6], we know that
\[ D_{[x, x]}^{-\alpha} D_{[x, x]}^\alpha f(x) = f(x), \quad D_{[x, b]}^{-\alpha} D_{[x, b]}^\alpha f(x) = f(x). \]

Based on Definition 3.1, for $x = (x_1, x_2, x_3) \in \Omega$, $i = 1, 2, 3$ and function $f : \Omega \rightarrow \mathbb{R}$, define
\[ D_{[x_i, x_i]}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_i} (x_i - t)^{-\alpha} f(t) dt, \]
\[ D_{[x_i, b_i]}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_i}^{b_i} (t - x_i)^{-\alpha} f(t) dt. \]
By (28), we know that
\[ D_{[x_i, x_i]}^{-\alpha} D_{[x_i, x_i]}^\alpha f(x) = f(x), \quad D_{[x_i, b_i]}^{-\alpha} D_{[x_i, b_i]}^\alpha f(x) = f(x)(i = 1, 2, 3). \]

Based on above definitions and notations, we obtain the following property of operators $D_{[x_i, x_i]}^{-\alpha}$ and $D_{[x_i, b_i]}^{-\alpha}$ ($i = 1, 2, 3$).
Lemma 3.2. For \( \alpha > 0 \), \( x = (x_1, x_2, x_3) \in \Omega \) and function \( f \in L^2(\Omega) \), there exists a constant \( C = C(\alpha, \Omega) \) such that

\[
\| D_{[a_i, x_i]}^\alpha f \|_{L^2(\Omega)} \leq C\| f \|_{L^2(\Omega)}, \quad \| D_{[x_i, b_i]}^\alpha f \|_{L^2(\Omega)} \leq C\| f \|_{L^2(\Omega)} \quad (i = 1, 2, 3).
\]

Proof. Since \( D_{[a_i, x_i]}^\alpha f(x) = \frac{x_i^{\alpha - 1}}{\Gamma(\alpha)} \ast f(x_1, x_2, x_3) \), by Young’s inequality[7]

\[
\| v \ast w \|_{L^2([a_i, b_i])}^2 \leq \| v \|_{L^1([a_i, b_i])}^2 \| w \|_{L^2([a_i, b_i])}^2
\]

(30) it yields,

\[
\| D_{[a_i, x_i]}^\alpha f \|_{L^2(\Omega)}^2
\]

\[
= \int_{a_i}^{b_i} \int_{a_j}^{b_j} \int_{a_k}^{b_k} | D_{[a_i, x_i]}^\alpha f(x_1, x_2, x_3) |^2 \, dx_3 \, dx_2 \, dx_1
\]

\[
= \int_{a_j}^{b_j} \int_{a_k}^{b_k} \left( \int_{a_i}^{b_i} | D_{[a_i, x_i]}^\alpha f(x_1, x_2, x_3) |^2 \, dx_1 \right) \, dx_2 \, dx_3
\]

\[
\leq \int_{a_j}^{b_j} \int_{a_k}^{b_k} \left( \frac{1}{\Gamma(\alpha)^2} \int_{a_i}^{b_i} | f(x_1, x_2, x_3) |^2 \, dx_1 \right) \, dx_2 \, dx_3
\]

\[
\leq \int_{a_j}^{b_j} \int_{a_k}^{b_k} \left( \frac{(|a_i| + |b_i|)^2}{\Gamma(\alpha + 1)^2} \int_{a_i}^{b_i} | f(x_1, x_2, x_3) |^2 \, dx_1 \right) \, dx_2 \, dx_3
\]

\[
= \frac{(|a_i| + |b_i|)^2}{\Gamma(\alpha + 1)^2} \int_{a_i}^{b_i} \int_{a_j}^{b_j} \int_{a_k}^{b_k} | f(x_1, x_2, x_3) |^2 \, dx_3 \, dx_2 \, dx_1
\]

(31) Similarly, \( \| D_{[a_i, x_i]}^\alpha f \|_{L^2(\Omega)}^2 \leq \frac{(|a_i| + |b_i|)^2}{\Gamma(\alpha + 1)^2} \| f \|_{L^2(\Omega)}^2 \quad (i = 2, 3). \)

Let \( C = \max_{i=1,2,3} \left\{ \frac{|a_i| + |b_i|}{\Gamma(\alpha + 1)} \right\} \), then

\[
\| D_{[a_i, x_i]}^\alpha f \|_{L^2(\Omega)} \leq C\| f \|_{L^2(\Omega)} \quad (i = 1, 2, 3).
\]

In a similar way, we can prove that

\[
\| D_{[x_i, b_i]}^\alpha f \|_{L^2(\Omega)} \leq C\| f \|_{L^2(\Omega)} \quad (i = 1, 2, 3).
\]

\[
\square
\]

Lemma 3.3. Assume \( \alpha > 0 \), \( u \in F_{L,0}^\alpha(\Omega) \) and \( v \in F_{R,0}^\alpha(\Omega) \), then there exists a constant \( C = C(\alpha, \Omega) \) such that

\[
\| u \|_{L^2(\Omega)} \leq C\| u \|_{F_{L,0}^\alpha(\Omega)}, \quad \| v \|_{L^2(\Omega)} \leq C\| v \|_{F_{R,0}^\alpha(\Omega)}.
\]

Proof. It follows from Lemma 3.2 and (29) that

\[
\| u \|_{L^2(\Omega)} = \| D_{[a_j, x_j]}^\alpha D_{[a_i, x_i]}^\alpha u \|_{L^2(\Omega)} \leq \tilde{C}\left\| \frac{\partial^\alpha u}{\partial x_j^\alpha} \right\|_{L^2(\Omega)} \quad (j = 1, 2, 3),
\]

where \( \tilde{C} = \tilde{C}(\alpha, \Omega) \).
It follows from (35) that

$$
(36) \quad \|u\|^2_{L^2(\Omega)} \leq \frac{C^2}{3} \sum_{j=1}^{3} \left\| \frac{\partial^3 u}{\partial x_j^3}\right\|^2_{L^2(\Omega)} = \frac{C^2}{3} |u|_{H^6_{\text{loc}}(\Omega)}^2.
$$

Let $C = \frac{C}{\sqrt{3}}$, this concludes the first equation of (34).

Similarly, we can prove the second equation of (34).

**Lemma 3.4.** Assume $\alpha > 2.5$, $\alpha \neq m + 0.5$, $m \in \mathbb{N}$ and $u \in F^\alpha_{L,0}(\Omega)$, then there exists a constant $K = K(\alpha, \Omega)$ such that

(i). $\|u(x) - u(y)\| \leq K\|\nabla^\alpha u\|_{L^2(\Omega)} \|x - y\|.$

(ii). $\|\nabla u(x) - \nabla u(y)\| \leq K\|\nabla^\alpha u\|_{L^2(\Omega)} \|x - y\|^\lambda$, where $0 < \lambda \leq \alpha - 1.5 - [\alpha - 1.5]$ as $2.5 < \alpha < 3.5$ and $\lambda = 1$ as $\alpha \geq 3.5$.

**Proof.** (i). By Theorem 2.9, $u \in F^\alpha_{L,0}(\Omega) = H^\alpha_0(\Omega) \hookrightarrow C^1(\Omega)$(cf. [4, Theorem 4.57]). This implies

$$
(37) \quad \|u(x) - u(y)\| = \|\nabla u(\xi) \cdot (x - y)\| \leq \|\nabla u\|_{C^1(\Omega)} \|x - y\| \leq C_1\|u\|_{H^\alpha(\Omega)} \|x - y\|.
$$

On the other hand, by Lemma 3.3, we know there exists a constant $C_3 = C_3(\alpha, \Omega)$ such that

$$
(38) \quad \|u\|_{F^\alpha_{L,0}(\Omega)} \leq C_3\|\nabla^\alpha u\|_{L^2(\Omega)}.
$$

By (37) and (38), we obtain that

$$
(39) \quad \|u(x) - u(y)\| \leq K\|\nabla^\alpha u\|_{L^2(\Omega)} \|x - y\|.
$$

This concludes (i).

(ii). Here we divide our discussion into two different cases:

**Case 1** ($2.5 < \alpha < 3.5$). Since $F^\alpha_{L,0}(\Omega) = H^\alpha_0(\Omega) \hookrightarrow C^{1,\lambda}(\Omega)$ ($0 < \lambda < \alpha - 1.5 - [\alpha - 1.5]$)[4, Theorem 4.57], then

$$
(40) \quad \frac{\|\nabla u(x) - \nabla u(y)\|}{\|x - y\|^\lambda} \leq \|u\|_{C^{1,\lambda}(\Omega)} \leq C\|u\|_{F^\alpha_{L,0}(\Omega)} \leq K\|\nabla^\alpha u\|_{L^2(\Omega)}.
$$

**Case 2** ($\alpha \geq 3.5$). Since $F^\alpha_{L,0}(\Omega) = H^\alpha_0(\Omega) \hookrightarrow C^2(\Omega)$[4, Theorem 4.57], then

$$
(41) \quad \|\nabla u(x) - \nabla u(y)\| \leq \|u\|_{C^2(\Omega)} \leq C\|u\|_{F^\alpha_{L,0}(\Omega)} \leq K\|\nabla^\alpha u\|_{L^2(\Omega)}.
$$

This concludes (ii).
As to the existence of $h^{-1} : \Omega \to \Omega$, we have the following result.

**Lemma 3.6.** Assume $v(x,s) \in \mathcal{F}$ and $v(\cdot,s)|_{\mathbb{R}^3 \setminus \Omega} = 0$ for each $s \in [0,\tau]$, then for $s,t \in [0,\tau]$ and $x \in \overline{\Omega}$, $\eta(s,t,x)$ defined by (5) is differential with respect to $x$.

Define $\Theta(s,t,x) \triangleq \nabla_x \eta(s,t,x)$, then

(i). $\Theta(s,t,x)$ is the solution of

$$
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d\Theta(s,t,x)}{ds} = \nabla_\eta v(\eta(s,t,x),s)\Theta(s,t,x), \\
\Theta(t,t,x) = I,
\end{array} \right.
\end{aligned}
$$

and

$$
\det(\Theta(s,t,x)) = e^{\int_{s}^{t} \sum_{i=1}^{3} v_{i,n}(\eta(r,t,x),r)dr},
$$

where $v_{i,n}(r,t,x),r = \frac{\partial v_{i,n}(\eta(r,t,x),r)}{\partial \eta_{n}} (i = 1,2,3)$.

(ii). Define $h(x) \triangleq \eta(0;\tau,x)$ as (5), then $h$ is a 1-to-1 and onto mapping which ensures the existence of $h^{-1}(x)$ in (2).

**Proof.** (i). Based on Theorem 2.9, this conclusion can be obtained in a similar way with Lemma 2.3 in [8].

(ii). For any $x \in \Omega$, $\det (\nabla_x h(x)) = e^{-\int_{0}^{\tau} \sum_{i=1}^{3} v_{i,n}(\eta(s,t,x),s)ds} \neq 0$, by Inverse Function Theorem[7, Theorem 7 in Appendix C], we know $h$ is a 1-to-1 and onto mapping. This ensures the existence of $h^{-1}(x)$ in (2).

**Lemma 3.7.** Assume $\{v_n(x,s)\}$ is a bounded sequence on $\mathcal{F}$ with

$$
\|v_n\|_{\mathcal{F}}^2 = \int_{0}^{\tau} \|\nabla^\alpha v_n(\cdot,s)\|_{L^2(\Omega)}^2 ds \leq M < +\infty,
$$

and $v_n(\cdot,s)|_{\mathbb{R}^3 \setminus \Omega} = 0$ for each $s \in [0,\tau]$, then

(i). $\{v_n(x,s)\}$ is a weakly compact set on $\mathcal{F}$.

(ii). If we denote $n_k$ as the sequence number of a weakly convergent subsequence $\{v_{n_k}(x,s)\}$ with weak limit $v(x,s)$, then

$$
M \geq \lim_{n_k \to \infty} \inf \int_{0}^{\tau} \|\nabla^\alpha v_{n_k}(\cdot,s)\|_{L^2(\Omega)}^2 ds \geq \int_{0}^{\tau} \|\nabla^\alpha v(\cdot,s)\|_{L^2(\Omega)}^2 ds.
$$

(iii). Let $v_{n_k}(x,s)$, $v(x,s)$ be functions defined in (ii). Consider the equations

$$
\frac{d\eta_{n_k}(s,t,x)}{ds} = v_{n_k}(\eta_{n_k}(s,t,x),s) \text{ with } \eta_{n_k}(t,t,x) = x,
$$

and

$$
\frac{d\eta(s,t,x)}{ds} = v(\eta(s,t,x),s) \text{ with } \eta(t,t,x) = x,
$$

then for each $(x,t) \in \Omega \times [0,\tau]$, these two equations have a unique solution $\eta_{n_k}(s,t,x)$, $\eta(s,t,x) \in C([0,\tau],\Omega)$, respectively. Furthermore, for each $s \in [0,\tau]$, $\eta_{n_k}(s,t,x) \in C^{[\alpha-1.5,\lambda]}(\Omega)$ with $\eta_{n_k}(s,t,x) \xrightarrow{k} \eta(s,t,x)$ uniformly on $[0,\tau]$, where $0 < \lambda \leq \alpha - 1.5 - [\alpha - 1.5]$.

**Proof.** (i). By Lemma 3.3, we know $\| \cdot \|_{\mathcal{F}}$ is a norm and $\mathcal{F}$ is a separable Hilbert space. This implies (i) for the fact that any closed ball in a separable Hilbert space is a weakly compact set.
Furthermore, in a similar way with proof of Lemma 2.7 in [8], we can prove that then there exists a weakly convergent subsequence \( \eta \) uniformly on \([0, \tau] \times \Omega \). By Lemma 3.7, we know that \( \eta_n(s; t, x) \xrightarrow{k} \eta(s; t, x) \) uniformly on \([0, \tau] \times \Omega \). □

Before we give a proof of Theorem 1.3, let’s recall the following Lemma.

**Lemma 3.8.** (cf. [14, Theorem 1.4.4]) Let \( E \) be a weakly compact set on Banach space \( X \). If \( H : E \to \mathbb{R} \) is a lower weak semi-continuous (l.w.c) functional, then there exists \( v_0 \in E \) such that \( H(v_0) = \inf_{v \in E} H(v) \).

**Proof of Theorem 1.3.** Define function \( \tilde{v}(x, s) \equiv 0 \) on \( \Omega \times [0, \tau] \). Since \( \tilde{v}(x, s) \in F \) and \( H(\tilde{v}) = \|T(\cdot) - D(\cdot)\|^2_{L^2(\Omega)} \leq G(\Omega) \triangleq M \), we only need to show the existence of global minimizer of \( H(v) \) on the ball

\[
B_M \triangleq \{v(x, s) : \|v\|^2_F \leq M\}.
\]

By (i) in Lemma 3.7, we know \( B_M \) is a weakly compact set. Choose \( \{v_n\} \in B_M \), then there exists a weakly convergent subsequence \( \{v_{n_k}\} \) such that

\[
v_{n_k} \xrightarrow{k} v \in B_M.
\]

By (ii) in Lemma 3.7, we know that

\[
M \geq \lim_{n_k \to \infty} \inf \int_0^\tau \|\nabla^\alpha v_{n_k}(\cdot, t)\|_{L^2(\Omega)}^2 dt \geq \int_0^\tau \|\nabla^\alpha v(\cdot, t)\|_{L^2(\Omega)}^2 dt.
\]

By (iii) in Lemma 3.7, \( \eta_{n_k}(s; t, x) \xrightarrow{k} \eta(s; t, x) \) for all \((x, s, t) \in \Omega \times [0, \tau] \times [0, \tau]\), where \( \eta_{n_k}(s; t, x), \eta(s; t, x) \) are the solution of (46) and (47), respectively. Furthermore, we have

\[
h_{n_k}(x) = \eta_{n_k}(0; \tau, x) \xrightarrow{k} \eta(0; \tau, x) = h(x) \quad \text{for all} \quad x \in \Omega.
\]

Define \( \Theta_{n_k}(s; 0, x) \triangleq \nabla_x \eta_{n_k}(s; 0, x), \quad \Theta(s; 0, x) \triangleq \nabla_x \eta(s; 0, x) \), then by Lemma 3.6, \( \Theta_{n_k}(s; 0, x) \) and \( \Theta(s; 0, x) \) are the solutions of

\[
\begin{cases}
\frac{d\Theta_{n_k}(s; 0, x)}{ds} = \nabla_{\eta_{n_k}} v_{n_k}(\eta_{n_k}(s; 0, x), s) \Theta_{n_k}(s; 0, x), \\
\Theta_{n_k}(0; 0, x) = I,
\end{cases}
\]

and

\[
\begin{cases}
\frac{d\Theta(s; 0, x)}{ds} = \nabla_\eta v(\eta(s; 0, x), s) \Theta(s; 0, x), \\
\Theta(0; 0, x) = I,
\end{cases}
\]

respectively.

Based on these notations, here we claim that the functional

\[
H : B_M \to \mathbb{R} \quad \text{is l.w.c.}
\]

The proof of claim (54) can be divided into following five steps.

**Step 1.** We claim that there exists a constant \( 0 < \bar{M} < +\infty \) such that

\[
\|\Theta_{n_k}(s; 0, x)\| \leq \bar{M}, \quad \|\Theta(s; 0, x)\| \leq \bar{M}.
\]
By (52), we have
\begin{equation}
\Theta_{n_k}(s; 0, x) = I + \int_0^s \nabla_{\eta_{n_k}} v_{n_k}(\eta_{n_k}(r; 0, x), r) \Theta_{n_k}(r; 0, x) dr,
\end{equation}
this yields
\begin{equation}
\|\Theta_{n_k}(s; 0, x)\| = \|I\| + \int_0^s \|\nabla_{\eta_{n_k}} v_{n_k}(\eta_{n_k}(r; 0, x), r)\|\|\Theta_{n_k}(r; 0, x)\| dr.
\end{equation}
By Grownwall inequality and Lemma 3.3, we have
\begin{align*}
\|\Theta_{n_k}(s; 0, x)\| &\leq \|I\| e^{\int_0^s \|\nabla_{\eta_{n_k}} v_{n_k}(\eta_{n_k}(r; 0, x), r)\| dr} \\
&\leq \|I\| e^{\int_0^s \|\nabla_{\eta_{n_k}} v_{n_k}(\cdot, r)\|_{C^1(\Omega)} dr} \\
&\leq \|I\| e^{\int_0^s C\|\nabla_{\eta_{n_k}} v_{n_k}(\cdot, r)\|_{L^2(\Omega)} dr} \\
&\leq \|I\| e^{C \int_0^s \|\nabla_{\eta_{n_k}} v_{n_k}(\cdot, r)\|_{L^2(\Omega)}^2 dr} \leq \|I\| e^{C(t-M)^{\frac{1}{2}}} \approx M.
\end{align*}
Similarly, we can prove that \(\|\Theta(s; 0, x)\| \leq M\).

**Step 2.** We claim that \(\nabla_x \left[ \int_0^t [v_{n_k}(x, r) - v(x, r)] dr \right] \rightarrow O_{3 \times 3}\) uniformly on \(\Omega \times [0, \tau]\), where \(O_{3 \times 3}\) is a 3 \(\times\) 3 matrix whose elements are all zero.

Let \(w_{n_k}(x, s) \triangleq v_{n_k}(x, s) - v(x, s), z_{n_k}(x, s) \triangleq \int_0^s w_{n_k}(x, r) dr\), then \(w_{n_k} \xrightarrow{k} 0\) with \(\|w_{n_k}\|_{L^2} \leq 2M\).

If \(x, y \in \Omega\) and \(s, t \in [0, \tau]\), then by (i) in Lemma 3.4,
\begin{align*}
\|z_{n_k}(x, s) - z_{n_k}(y, t)\| &= \left\| \int_0^s [w_{n_k}(x, r) - w_{n_k}(y, r)] dr \right\| + \left\| \int_s^t w_{n_k}(y, r) dr \right\| \\
&\leq K \int_0^s \|\nabla_z v_{n_k}(\cdot, r)\|_{L^2(\Omega)} \|x - y\| \\
&\quad + K \int_0^s \|\nabla_z v(\cdot, r)\|_{L^2(\Omega)} \|x - y\| + \left\| \int_s^t w_{n_k}(y, r) dr \right\| \\
&\leq K \tau^{\frac{1}{2}} \left( \int_0^s \|\nabla_z v_{n_k}(\cdot, r)\|_{L^2(\Omega)}^2 dr \right)^{\frac{1}{2}} \|x - y\| \\
&\quad + K \tau^{\frac{1}{2}} \left( \int_0^s \|\nabla_z v(\cdot, r)\|_{L^2(\Omega)}^2 dr \right)^{\frac{1}{2}} \|x - y\| + \left\| \int_s^t w_{n_k}(y, r) dr \right\| \\
&\leq 2K M^{\frac{1}{2}} \tau^{\frac{1}{2}} \|x - y\| + \left\| \int_s^t w_{n_k}(y, r) dr \right\| \\
&\leq 2K M^{\frac{1}{2}} \tau^{\frac{1}{2}} \|x - y\| + C \tau - s^{\frac{1}{2}} \|w_{n_k}\|_{L^2} \\
&\leq 2K M^{\frac{1}{2}} \tau^{\frac{1}{2}} \|x - y\| + 2CM \|t - s\|^{\frac{1}{2}}.
\end{align*}
By Arzela-Ascoli Theorem[13], \(\{z_{n_k}\}\) is relative compact on \([C(\Omega \times [0, \tau])]^3\).
By [5, 8], \(z_{n_k} \xrightarrow{k} 0\) uniformly on \(\Omega \times [0, \tau]\). On the other hand, by (ii) in Lemma 3.4, we have
\begin{align*}
\|\nabla_x z_{n_k}(x, s) - \nabla_y z_{n_k}(y, t)\| &= \left\| \int_0^s [\nabla_x w_{n_k}(x, r) - \nabla_y w_{n_k}(y, r)] dr \right\| + \left\| \int_s^t \nabla_y w_{n_k}(y, r) dr \right\|
\end{align*}
Therefore, \( \{ \nabla_x z_{n_k}(x, s) \} \) is relative compact on \([C(\Omega \times [0, \tau])]^3\).

Now, choose any convergent subsequence of \( \{ \nabla_x z_{n_k}(x, s) \} \) with limit \( P_{3 \times 3} \). That is,

\[
(57) \quad \nabla_x z_{n_k}(x, s) \xrightarrow{k} P_{3 \times 3} \text{ uniformly on } \Omega \times [0, \tau].
\]

Since \( z_{n_k} \xrightarrow{k} 0 \) uniformly on \( \Omega \times [0, \tau] \), then \( P_{3 \times 3} = \nabla_x 0 = O_{3 \times 3} \).

This implies, \( \nabla_x [\int_0^s [v_n(x, r) - v(x, r)]dr] \xrightarrow{k} O_{3 \times 3} \) uniformly on \( \Omega \times [0, \tau] \).

**Step 3.** We claim that \( \| \Theta_{n_k} \Theta^T_{n_k} - \Theta \Theta^T \| \xrightarrow{k} 0 \). Here for the sake of simplicity, we denote \( \Theta_{n_k}(s; 0, x) \) and \( \Theta(s; 0, x) \) by \( \Theta_{n_k} \) and \( \Theta \), respectively.

\[
\| \Theta_{n_k}(s; 0, x) - \Theta(s; 0, x) \|
\leq \int_0^s \| [\nabla_{\eta_{n_k}} v_{n_k}(\eta_{n_k}(r; 0, x), r) - \nabla_{\eta} v_{n_k}(\eta(r; 0, x), r)] \Theta_{n_k}(r; 0, x) \| dr
\]

\[
+ \int_0^s \| \nabla_{\eta} v_{n_k}(\eta(r; 0, x), r) [\Theta_{n_k}(r; 0, x) - \Theta(r; 0, x)] \| dr
\]

\[
+ \| \int_0^s [\nabla_{\eta} v_{n_k}(\eta(r; 0, x), r) - \nabla_{\eta} v(\eta(r; 0, x), r)] \Theta(r; 0, x) dr \|
\]

\[
\leq \int_0^s K \| \nabla^\alpha v_{n_k}(., r) \|_{L^2(\Omega)} \| \eta_{n_k}(r; 0, x) - \eta(r; 0, x) \|^\lambda \| \Theta_{n_k}(r; 0, x) \| dr
\]

\[
+ \int_0^s \| \nabla_{\eta} v_{n_k}(\eta(r; 0, x), r) \| \| \Theta_{n_k}(r; 0, x) - \Theta(r; 0, x) \| dr
\]

\[
+ \| \int_0^s [\nabla_{\eta} v_{n_k}(\eta(r; 0, x), r) - \nabla_{\eta} v(\eta(r; 0, x), r)] \Theta(r; 0, x) dr \|
\]

\[
\leq K \tilde{M} \tau^{\frac{1}{2}} \left( \int_0^s \| \nabla^\alpha v_{n_k}(., r) \|_{L^2(\Omega)}^2 dr \right)^{\frac{1}{2}} \| \eta_{n_k}(r; 0, x) - \eta(r; 0, x) \|^\lambda \| C([0, \tau]; \mathbb{R})^3
\]

\[
+ \int_0^s \| \nabla_{\eta} v_{n_k}(\eta(r; 0, x), r) \| \| \Theta_{n_k}(r; 0, x) - \Theta(r; 0, x) \| dr
\]

\[
+ \| \int_0^s [\nabla_{\eta} v_{n_k}(\eta(r; 0, x), r) - \nabla_{\eta} v(\eta(r; 0, x), r)] \Theta(r; 0, x) dr \|
\]
\[ \leq K \tilde{M}^\frac{1}{2} M^\frac{1}{2} \| \eta_{n_k}(r; 0, x) - \eta(r; 0, x) \|^\lambda_{C([0, \tau]; \mathbb{R})} \]
\[+ \left\| \int_0^\tau [\nabla \eta v_{n_k}(\eta(r; 0, x), r) - \nabla \eta v(\eta(r; 0, x), r)] \Theta(r; 0, x) dr \right\| \]
\[+ \int_0^\tau \| \nabla \eta v_{n_k}(\eta(r; 0, x), r) \| \| \Theta_{n_k}(r; 0, x) - \Theta(r; 0, x) \| dr \]
\[\triangleq I_1 + I_2 + I_3. \tag{58} \]

On the other hand, by (iii) in Lemma 3.7, we have \( \| \eta_{n_k}(s; 0, x) - \eta(s; 0, x) \|_{C([0, \tau]; \mathbb{R})} \overset{k}{\to} 0. \)

This leads to \( I_1 \overset{k}{\to} 0 \). By **Step 2**, we know that \( \nabla_x \left[ \int_0^s v_{n_k}(x, r) - v(x, r) dr \right] \overset{k}{\to} 0 \) \( O_{3,2,3} \) uniformly on \( \Omega \times [0, \tau] \). This implies that \( I_2 \overset{k}{\to} 0. \)

Hence, \( I_1 + I_2 \overset{k}{\to} 0. \) Then there exist \( N = N(\varepsilon) \), such that \( I_1 + I_2 < \varepsilon \) as \( n_k > N. \)

Therefore, it follows from (58) that
\[
\| \Theta_{n_k}(s; 0, x) - \Theta(s; 0, x) \| \leq \varepsilon + \int_0^\tau \| \nabla \eta v_{n_k}(\eta(r; 0, x), r) \| \| \Theta_{n_k}(r; 0, x) - \Theta(r; 0, x) \| dr,
\]
and Gronwall inequality implies that
\[
\| \Theta_{n_k}(s; 0, x) - \Theta(s; 0, x) \| \leq \varepsilon e^{\int_0^\tau \| \nabla \eta v_{n_k}(\eta(r; 0, x), r) \| dr} \leq \varepsilon e^{K \tau^\frac{1}{2} (f_0^\tau \| \nabla \eta v_{n_k}(\cdot, r) \|_{L^2(\Omega)} dr)} \frac{1}{2} \leq \varepsilon e^{K \tau^\frac{1}{2} M^\frac{1}{2}}.
\]

Let \( \varepsilon \to 0 \), then \( \Theta_{n_k}(s; 0, x) \overset{k}{\to} \Theta(s; 0, x) \). Then,
\[
\| \Theta_{n_k}^T \Theta_{n_k}^T - \Theta \Theta^T \| \leq \| \Theta_{n_k} \Theta_{n_k}^T - \Theta \Theta^T \| + \| \Theta \Theta_{n_k}^T - \Theta \Theta^T \|
\]
\[
\leq \| \Theta_{n_k} - \Theta \| \| \Theta_{n_k}^T \| + \| \Theta \| \| \Theta_{n_k}^T - \Theta^T \|
\]
\[
\leq \tilde{M} \| \Theta_{n_k} - \Theta \| + \tilde{M} \| \Theta_{n_k}^T - \Theta^T \| \overset{k}{\to} 0,
\tag{59}
\]

since \( \Theta_{n_k} \overset{k}{\to} \Theta \) and \( \Theta_{n_k}^T \overset{k}{\to} \Theta^T \).

**Step 4.** We claim that \( R_{n_k} \overset{k}{\to} R. \)

Define
\[
A_{n_k} = \Theta_{n_k}(\tau; 0, x) \Theta_{n_k}^T(\tau; 0, x) = \left( a_{n_k}^{ij}(\tau; 0, x) \right)_{3 \times 3},
\]
\[
A = \Theta(\tau; 0, x) \Theta^T(\tau; 0, x) = \left( a_{ij}(\tau; 0, x) \right)_{3 \times 3}.
\]

By (59), we know \( a_{n_k}^{ij}(\tau; 0, x) \overset{k}{\to} a_{ij}(\tau; 0, x) \) for \( i, j = 1, 2, 3. \) Now, we simply denote \( a_{n_k}^{ij}(\tau; 0, x) \) and \( a_{ij}(\tau; 0, x) \) by \( a_{n_k}^{ij} \) and \( a_{ij} \), respectively.

By Lemma 3.6, we obtain that
\[
\det(\Theta_{n_k}(\tau; 0, x)) = e^{-\int_0^\tau \frac{1}{3} \sum_{i=1}^3 (v_{n_k}(\eta_{n_k}(s, 0, x), s))_{i, (n_{n_k})}, ds} \neq 0.
\tag{60}
\]

By (60), we know that \( \Theta_{n_k} \Theta_{n_k}^T, \Theta \Theta^T \in SPD(3). \) Let \( \lambda^{(1)}_{n_k} \geq \lambda^{(2)}_{n_k} \geq \lambda^{(3)}_{n_k} > 0, \lambda^{(1)} \geq \lambda^{(2)} \geq \lambda^{(3)} > 0 \) be the eigenvalues of \( \Theta_{n_k} \Theta_{n_k}^T, \Theta \Theta^T \), respectively.
What’s more, by (60), we obtain that
\[
\det(A_{nk}) = e^{-\int_0^T \left( \sum_{i=1}^3 (v_{nk}(t, x, s)) \right) ds} \leq e^{\int_0^T \| \nabla v(t, x, s) \| ds}
\]
\[
\leq e^{\int_0^T \| v_{nk}(t, x, s) \| \lambda_{C(1)}^2 \| ds} \leq e^{2C \int_0^T \| v_{nk}(t, x, s) \| \lambda_{F_{\infty}}^{\Omega_1} \| ds}
\]
\[
\leq e^{2K \int_0^T \| \nabla v_{nk}(t, x, s) \| _{L^2(\Omega)}^2 \| ds} \leq e^{2K^2 \| \nabla v_{nk}(t, x, s) \| _{L^2(\Omega)}^2 \| ds}^{\frac{1}{2}}
\]
(61)
\[
\leq e^{2K^2 M^2} \triangleright M_1 < +\infty.
\]

In a similar way, we can obtain that
(62)
\[
\det(A_{nk}) \geq \frac{1}{M_1}.
\]

By singularity decomposition theorem[11], we can find two $3 \times 3$ orthogonal matrix $U_{nk}, V_{nk}$ such that $\Theta_{nk} = U_{nk} S_{nk} V_{nk}^T$, where $S_{nk} = \text{diag} \left( \sqrt{\lambda_{nk}^{(1)}}, \sqrt{\lambda_{nk}^{(2)}}, \sqrt{\lambda_{nk}^{(3)}} \right)$, the columns of $U_{nk}, V_{nk}$ are orthogonal eigenvectors of $\Theta_{nk} \Theta_{nk}^T$ and $\Theta_{nk}^T \Theta_{nk}$, respectively.

Similarly, $\Theta = USV^T$, where $S = \text{diag} \left( \sqrt{\lambda^{(1)}}, \sqrt{\lambda^{(2)}}, \sqrt{\lambda^{(3)}} \right)$, the columns of $U, V$ are orthogonal eigenvectors of $\Theta \Theta^T$ and $\Theta^T \Theta$, respectively.

Then, $A_{nk} = \Theta_{nk} \Theta_{nk}^T = U_{nk} S_{nk}^2 U_{nk}^T$, $A_{nk}^{-1} = U_{nk} S_{nk}^{-2} U_{nk}^T$ and $A = \Theta \Theta^T = US^2 U^T, A^{-1} = US^{-2} U^T$. Hence,
\[
\| A_{nk}^{-1} \| \leq \| U_{nk} \| \| S_{nk}^{-2} \| \| U_{nk}^T \| \leq \| U_{nk} \|^2 \left[ \frac{1}{\lambda_{nk}^{(1)}} + \frac{1}{\lambda_{nk}^{(2)}} + \frac{1}{\lambda_{nk}^{(3)}} \right]
\]
\[
\leq \| U_{nk} \|^2 \frac{\lambda^{(1)}_{nk} \lambda^{(2)}_{nk} + \lambda^{(1)}_{nk} \lambda^{(3)}_{nk} + \lambda^{(2)}_{nk} \lambda^{(3)}_{nk}}{\lambda^{(1)}_{nk} \lambda^{(2)}_{nk} \lambda^{(3)}_{nk}} = \| U_{nk} \|^2 \frac{\lambda^{(1)}_{nk} \lambda^{(2)}_{nk} \lambda^{(3)}_{nk}}{\det(A_{nk})}
\]
\[
\leq \| U_{nk} \|^2 \frac{\lambda^{(1)}_{nk} + \lambda^{(2)}_{nk} + \lambda^{(3)}_{nk}^2}{\det(A_{nk})} = \| U_{nk} \|^2 \frac{\text{tr}(A_{nk})^2}{\det(A_{nk})} \leq \| U_{nk} \|^2 \| A_{nk} \|^2 \| \det(A_{nk}) \|
\]
(63)
\[
\leq 27 M^2 M_1 \triangleright M_2 < +\infty
\]
by (61), (62) and Step 1, where $\text{tr}(A)$ denote the trace of matrix $A$. Note that here we use the equalities
(64)
\[
\lambda^{(1)}_{nk} + \lambda^{(2)}_{nk} + \lambda^{(3)}_{nk} = \text{tr}(A_{nk}), \quad \lambda^{(1)}_{nk} \lambda^{(2)}_{nk} \lambda^{(3)}_{nk} = \det(A_{nk}).
\]

Similarly, we know that $\| A^{-1} \| \leq M_2$.

By (63), we obtain that
(65)
\[
\| A_{nk}^{-1} - A^{-1} \| = \| A_{nk}^{-1}(A - A_{nk}) A^{-1} \| \leq \| A_{nk}^{-1} \| \| A - A_{nk} \| \| A^{-1} \| \xrightarrow{k} 0,
\]
since $A - A_{nk} = \Theta_{nk} \Theta_{nk}^T - \Theta \Theta^T \xrightarrow{k} 0$ by (59).

Hence, $A_{nk}^{-1} \xrightarrow{k} A^{-1}$.

Since $A_{nk}^{-1} = U_{nk} S_{nk}^{-2} U_{nk}^T = [U_{nk} S_{nk}^{-1} U_{nk}^T] [U_{nk} S_{nk}^{-1} U_{nk}^T] \triangleq B_{nk} B_{nk}^T$, $A^{-1} = US^{-2} U^T = [US^{-1} U^T] [US^{-1} U^T] \triangleq BB$, then $B_{nk}, B \in SPD(3)$. Then, the Minkowskii inequality[18, Equation (1.1)] implies
(66)
\[
\| \det(B_{nk} + B) \|^2 \geq \| \det(B_{nk}) \|^2 + \| \det(B) \|^2 \geq \| \det(B) \|^2.
\]
This is, \( \det(B_{nk} + B) \geq \det(B) = \frac{1}{\lambda_{(1)}^{\lambda_{(2)}} \lambda_{(3)}} > 0 \).

Further, we have

\[
\|B_{nk} + B\| \leq \|B_{nk}\| + \|B\| \leq \|U_{nk}\|^2 \left[ \frac{1}{\sqrt{\lambda_{nk}^{(1)}}} + \frac{1}{\sqrt{\lambda_{nk}^{(2)}}} + \frac{1}{\sqrt{\lambda_{nk}^{(3)}}} \right]
\]

\[
\quad + \|U\|^2 \left[ \frac{1}{\sqrt{\lambda_{nk}^{(1)}}} + \frac{1}{\sqrt{\lambda_{nk}^{(2)}}} + \frac{1}{\sqrt{\lambda_{nk}^{(3)}}} \right]
\]

\[
\leq \|U_{nk}\|^2 \frac{\lambda_{nk}^{(1)} + \lambda_{nk}^{(2)} + \lambda_{nk}^{(3)}}{\sqrt{\lambda_{nk}^{(1)}} \lambda_{nk}^{(2)}} + \|U\|^2 \frac{\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}}{\sqrt{\lambda^{(1)}} \lambda^{(2)} \lambda^{(3)}}
\]

\[
= \|U_{nk}\|^2 \frac{\text{tr}(A_{nk})}{\sqrt{\det(A_{nk})}} + \|U\|^2 \frac{\text{tr}(A)}{\sqrt{\det(A)}}
\]

\[
(67) \quad \leq 9M_2 \sqrt{M_1} + 9M_2 \sqrt{M_1} \leq 18M_2 \sqrt{M_1} \triangleq M_3 < +\infty.
\]

By (65), we obtain that

\[
(68) \quad A_{nk}^{-1} - A^{-1} = B_{nk}^2 - B^2 = (B_{nk} + B)(B_{nk} - B) \xrightarrow{k} 0.
\]

By (68) and Lemma 2.5 in [8], we obtain that \((\Theta_{nk} \Theta_{nk}^T)^{-\frac{1}{2}} = B_{nk} \xrightarrow{k} B = (\Theta \Theta^T)^{-\frac{1}{2}}.\)

This yields

\[
\|\Theta_{nk}(\Theta_{nk} \Theta_{nk}^T)^{-\frac{1}{2}} - \Theta(\Theta \Theta^T)^{-\frac{1}{2}}\|
\]

\[
\leq \|\Theta_{nk}\|(\|(\Theta_{nk} \Theta_{nk}^T)^{-\frac{1}{2}} - (\Theta \Theta^T)^{-\frac{1}{2}}\| + \|\Theta_{nk} - \Theta\|((\Theta \Theta^T)^{-\frac{1}{2}})) \xrightarrow{k} 0.
\]

So, \( R_{nk} \xrightarrow{k} R.\)

**Step 5.** We claim that \(\|T \circ h_{nk}(\cdot) - D(\cdot)\|_{L^2(\Omega)}^2 \xrightarrow{k} \|T \circ h(\cdot) - D(\cdot)\|_{L^2(\Omega)}^2.\)

Let \(x \in \Omega \setminus h^{-1}(\Delta T).\) by (51) and **Step 4**, it yields

\[
\|T \circ h_{nk}(x) - T \circ h(x)\| = \|R_{nk}[T \circ h_{nk}(x)]R_{nk} - R[T \circ h(x)]R^T\|
\]

\[
\leq \|R_{nk}[T \circ h_{nk}(x)]R_{nk} - R[T \circ h_{nk}(x)]R_{nk}^T\|
\]

\[
+ \|R[T \circ h_{nk}(x)]R_{nk} - R[T \circ h_{nk}(x)]R_{nk}^T\|
\]

\[
+ \|R[T \circ h(x)]R_{nk}^T - R[T \circ h(x)]R_{nk}^T\|
\]

\[
\leq \|R_{nk} - R\|\|T \circ h_{nk}(x)\| \|R_{nk}^T\|
\]

\[
+ \|R\|\|T \circ h_{nk}(x) - T \circ h(x)\| \|R_{nk}^T\|
\]

\[
+ \|R\|\|T \circ h(x)\| \|R_{nk}^T - R_{nk}^T\| \xrightarrow{k} 0.
\]

This implies, \(\|T \circ h_{nk}(x) - D(x)\|_{L^2(\Omega)}^2 \xrightarrow{k} \|T \circ h(x) - D(x)\|_{L^2(\Omega)}^2\) for all \(x \in \Omega \setminus h^{-1}(\Delta T).\)

On the other hand, \(\|T \circ h_{nk}(\cdot) - D(\cdot)\|^2 \leq \max_{x \in G} \|T(x) - D(x)\|^2 = G \in L^1(\Omega).\)

Besides, by (ii) in Lemma 3.6, \(h^{-1}(\Delta T)\) is a set of measure 0. Now by Dominance Theorem[7, Theorem 5 in Appendix E], it yields

\[
(71) \quad ||T \circ h_{nk}(\cdot) - D(\cdot)||_{L^2(\Omega)}^2 \xrightarrow{k} ||T \circ h(\cdot) - D(\cdot)||_{L^2(\Omega)}^2.
\]
It follows from (50) and (71) that, \( H(v) \) is a l.w.c functional. That is, 

\[
\lim_{n_k \to \infty} \inf H(v_{n_k}) \geq H(v).
\]

This concludes the claim (54).

By Lemma 3.8, there exists a global minimizer \( \bar{v}(x, s) \in B_M \) such that 
\[
H(\bar{v}) = \inf_{v \in B_M} H(v) = \inf_{v \in F} H(v) = \min_{v \in F} H(v).
\]
That is, \( \bar{v}(x, s) \) is a solution of (6).

For the above minimizer \( \bar{v}(x, s) \in F \), by Lemma 3.5, we know that there exists a unique \( \bar{\eta}(s; t, x) \in C([0, \tau], \Omega) \) such that 

\[
\frac{d\bar{\eta}(s; t, x)}{ds} = \bar{v}(\bar{\eta}(s; t, x), s), \quad \bar{\eta}(t; t, x) = x.
\]

Furthermore, by Lemma 3.5 and (5), we know the mapping \( \bar{h}(x) = \bar{\eta}(0; \tau, x) \in [C^{[\alpha-1.5, \lambda]}(\Omega)]^3 \) with \( \nabla_x \bar{h}(x) \) given by Lemma 3.6, where \( 0 < \lambda < \alpha - [\alpha] \). Moreover, by (ii) in Lemma 3.6, we know \( \bar{h} \) is a 1-to-1 and onto mapping. \( \square \)

4. Application. In this section, we mainly focus on deriving the Euler-Lagrange equation of (6) and studying effective numerical method to validate my fractional order model.

**Theorem 4.1.** For \( v \in F \), the Euler-Lagrange equation of (6) can be written as 

\[
\begin{cases}
\text{div}^\alpha \left( \nabla^\alpha v(x, s) \right) = f(\eta), \\
\frac{d\eta(s; t, x)}{ds} = v(\eta(s; t, x), s), \quad \eta(t; t, x) = x, \quad \bar{h}(x) = \eta(0; \tau, x),
\end{cases}
\]

where \( f(\eta) = (f_1(\eta), f_2(\eta), f_3(\eta)) \) \( = \text{det}(D\eta(\tau; s, x))\bar{W}, \bar{W} = (\bar{W}_\alpha)_{3 \times 1}, \bar{W}_\alpha = \bar{A} : \{ \bar{R}\partial_\alpha[T \circ \eta(0; s, x)]\bar{R}_\tau \}, \bar{A} = T \circ \eta(0; s, x) - D(\eta(\tau; s, x)), T \circ \eta(0; s, x) = \bar{R}(T \circ \eta(0; s, x))\bar{R}_\tau, \quad \bar{J}_1 = D\Psi(x, s), \quad \Psi(x, s) = \eta(\tau; 0, \eta(\tau; s, x))
\]

and for any vector \( g = (g_1, g_2, g_3) \), \( \text{div}^\alpha(g) = \sum_{i=1}^3 \frac{\partial^{\alpha} g_i}{\partial x_i^\alpha} \).

*Proof.* Assume \( v \in F \) is perturbed along \( w \in F \), then 

\[
\partial_v H_1(v) = 2 \int_0^\tau \int_\Omega \text{div}^\alpha : \nabla^\alpha w dxds = 2 \int_0^\tau \int_\Omega \text{div}^\alpha \left( \nabla^\alpha v(x, s) \right) \cdot w dxds.
\]

On the other hand, it follows from the results of Lemma 2.1 we have obtained in [9](see Appendix A for details) that 

\[
\partial_v H_2(v) = \int_0^\tau \int_\Omega -2 \text{det}(D\eta(\tau; s, x))\bar{W} \cdot w dxds.
\]

Then \( \partial_v H(v) = \partial_v H_1(v) + \partial_v H_2(v) = 0 \) implies, 

\[
\begin{cases}
\text{div}^\alpha \left( \nabla^\alpha v(x, s) \right) = f(\eta), \\
\frac{d\eta(s; t, x)}{ds} = v(\eta(s; t, x), s), \quad \eta(t; t, x) = x, \quad \bar{h}(x) = \eta(0; \tau, x),
\end{cases}
\]

where \( f(\eta) = (f_1(\eta), f_2(\eta), f_3(\eta)) \) \( = \text{det}(D\eta(\tau; s, x))\bar{W} \). \( \square \)

**Remark 4.** \( g \in F_{L,0}^\alpha(\Omega) = H_0^\alpha(\Omega) \) implies, \( \frac{\partial^{\alpha} g(x)}{\partial x_i^{\alpha}} |_{x \in \partial \Omega} = 0, \) \( (k = 0, 1, 2, \cdots, [\alpha]; i = 1, 2, 3) \)(cf. Sobolev space in [4, 7]). That is, any function on \( F_{L,0}^\alpha(\Omega) \) satisfies the homogeneous boundary conditions. This implies two important properties of fractional derivatives we used to derive (75):

(i). In this sense, Riemann-Liouville derivatives, Grunwald-Letnikov derivatives and Caputo derivatives are equivalent[20].

Inverse Problems and Imaging Volume 12, No. 6 (2018), 1263–1291
(ii). Integration by parts formula

\[
\int_{a_i}^{b_i} \xi(x) \cdot \frac{\partial^{\alpha} f(x)}{\partial x_i^{\alpha}} \, dx_i = \int_{a_i}^{b_i} \frac{\partial^{\alpha+\gamma} \xi(x)}{\partial x_i^{\alpha+\gamma}} \cdot f(x) \, dx_i \quad i = 1, 2, 3.
\]

Based on Theorem 4.1, we begin to give a numerical scheme of (74).

We define a spatial partition \( p_{l,m,n} = (x_l, y_m, z_n) \) (for all \( l = 0, 1, \cdots, N_1 + 1; m = 0, 1, \cdots, N_2 + 1; n = 0, 1, \cdots, N_3 + 1 \)) of image \( \Omega \) and a time partition \( s_k \) (for all \( k = 0, 1, 2, \cdots, N \)) of interval \([0, \tau]\). To overcome the large computational cost of 3D model and keep the stability of scheme, we use the alternative direction implicit (ADI) scheme[21] to solve (74). From time \( s_k \) to \( s_{k+1} \), the ADI scheme is divided into following four steps:

**Step 1.** On \( x_1 \) direction \( (s_k \rightarrow s_{k+\frac{1}{3}})(\beta = 1, 2, 3) \).

\[
\frac{\partial^{\alpha+\beta}}{\partial x_1^{\alpha+\beta}} \left( \frac{\partial^\alpha v_\beta(p_{l,m,n}, s_{k+\frac{1}{3}})}{\partial x_1^\alpha} \right) = - \sum_{j=1, j \neq 1}^{3} \frac{\partial^{\alpha+\beta}}{\partial x_j^{\alpha+\beta}} \left( \frac{\partial^\alpha v_\beta(p_{l,m,n}, s_k)}{\partial x_j^\alpha} \right) + f_\beta(\eta(s_k; t, p_{l,m,n})).
\]

**Step 2.** On \( x_2 \) direction \( (s_{k+\frac{1}{3}} \rightarrow s_{k+\frac{2}{3}})(\beta = 1, 2, 3) \).

\[
\frac{\partial^{\alpha+\beta}}{\partial x_2^{\alpha+\beta}} \left( \frac{\partial^\alpha v_\beta(p_{l,m,n}, s_{k+\frac{2}{3}})}{\partial x_2^\alpha} \right) = - \sum_{j=1, j \neq 2}^{3} \frac{\partial^{\alpha+\beta}}{\partial x_j^{\alpha+\beta}} \left( \frac{\partial^\alpha v_\beta(p_{l,m,n}, s_{k+\frac{1}{3}})}{\partial x_j^\alpha} \right) + f_\beta(\eta(s_k; t, p_{l,m,n})).
\]

**Step 3.** On \( x_3 \) direction \( (s_{k+\frac{2}{3}} \rightarrow s_{k+1})(\beta = 1, 2, 3) \).

\[
\frac{\partial^{\alpha+\beta}}{\partial x_3^{\alpha+\beta}} \left( \frac{\partial^\alpha v_\beta(p_{l,m,n}, s_{k+1})}{\partial x_3^\alpha} \right) = - \sum_{j=1, j \neq 3}^{3} \frac{\partial^{\alpha+\beta}}{\partial x_j^{\alpha+\beta}} \left( \frac{\partial^\alpha v_\beta(p_{l,m,n}, s_{k+\frac{2}{3}})}{\partial x_j^\alpha} \right) + f_\beta(\eta(s_k; t, p_{l,m,n})).
\]

**Step 4.** Calculation of \( \eta(s_{k+1}; t, p_{l,m,n}) \) and \( \Psi(p_{l,m,n}, s_{k+1}) \).

\[
\begin{cases}
\eta(\tau; s_{k+1}, p_{l,m,n}) = \eta(\tau; s_k, p_{l,m,n} + \beta_{l,m,n}), \\
\eta(0; s_{k+1}, p_{l,m,n}) = \eta(0; s_k, p_{l,m,n} - \beta_{l,m,n}),
\end{cases}
\]

\[
\Psi(p_{l,m,n}, s_{k+1}) = \eta(\tau; \tau - s_k, \eta(\tau; s_k, p_{l,m,n}) - \beta_{l,m,n}),
\]

where \( \Delta s_k = s_k - s_{k-1} \) and \( \beta_{l,m,n}, \beta_{l,m,n} \) are iteratively calculated using the following formula(cf. [2]):

\[
\beta_{l,m,n} = \frac{\Delta s_k}{2} v(p_{l,m,n} - \beta_{l,m,n}, s_{k+1}), \quad \beta_{l,m,n} = \frac{\Delta s_k}{2} v(\eta(\tau; s_k, p_{l,m,n}) - \beta_{l,m,n}, s_{k+1}).
\]

Now we consider the discretization of \( \alpha \)-order derivatives at point \( p_{l,m,n} \) along \( x_1 \) direction(similar discretization for \( x_2 \) and \( x_3 \) direction)using Grunwald approximation as[17]:

\[
\frac{\partial^\alpha f(p_{l,m,n})}{\partial x_1^\alpha} = \frac{\delta^\alpha f(p_{l,m,n})}{h^\alpha} + O(h) = h^{-\alpha} \sum_{j=0}^{l+1} \rho_j^{(\alpha)} f_{l-j+1,m,n} + O(h),
\]
and

\[ \frac{\partial^{\alpha} f(p_{l,m,n})}{\partial x_1^{\alpha}} \approx \frac{\partial R f(p_{l,m,n})}{\partial x_1} + O(h) = h^{-\alpha} \sum_{j=0}^{N_1-l+2} \rho_j^{(\alpha)} f_{l+j-1,m,n} + O(h), \]

where \( f_{l,m,n} = f(x_1,y_m,z_n) \) and \( \rho_0^{(\alpha)} = 1, \rho_j^{(\alpha)} = \left(1 - \frac{j+\alpha}{j}\right) \rho_{j-1}^{(\alpha)} \) for \( j = 1, 2, \ldots, \max\{N_1, N_2, N_3\} \).

Let \( U_{m,n} = (f_{1,m,n}, f_{2,m,n}, \ldots, f_{N_1,m,n})^T \), \( \frac{\partial^\alpha U_{m,n}}{\partial x_1^\alpha} = \left(\frac{\partial^{\alpha} f_{1,m,n}}{\partial x_1^\alpha}, \frac{\partial^{\alpha} f_{2,m,n}}{\partial x_1^\alpha}, \ldots, \frac{\partial^{\alpha} f_{N_1,m,n}}{\partial x_1^\alpha}\right)^T \), then it follows from (84) and (85) that

\[ \frac{\partial^\alpha U_{m,n}}{\partial x_1^\alpha} \approx B_{N_1} U_{m,n}, \quad \frac{\partial^\alpha U_{m,n}^T}{\partial x_1^\alpha} \approx B_{N_1}^T U_{m,n}, \]

where \( B_{N_1} = \frac{1}{h^\alpha} \)

\[ \begin{pmatrix} \rho_1^{(\alpha)} & \rho_0^{(\alpha)} & 0 & \cdots & 0 \\ \rho_2^{(\alpha)} & \rho_1^{(\alpha)} & \rho_0^{(\alpha)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \rho_{N_1-1}^{(\alpha)} & \rho_{N_1-2}^{(\alpha)} & \cdots & \rho_1^{(\alpha)} & \rho_0^{(\alpha)} \\ \rho_{N_1}^{(\alpha)} & \rho_{N_1-1}^{(\alpha)} & \cdots & \rho_2^{(\alpha)} & \rho_1^{(\alpha)} \end{pmatrix} \]

Therefore,

\[ \frac{\partial^{\alpha} U_{m,n}^\beta}{\partial x_1^{\alpha}} = B_{N_1}^T U_{m,n}^\beta \overset{\triangle}{=} A_{N_1} U_{m,n}^\beta, \quad \beta = 1, 2, 3, \]

where here and in what follows \( U_{m,n}^\beta = (v_{\beta,1,m,n}^{k+\frac{1}{2}}, v_{\beta,2,m,n}^{k+\frac{1}{2}}, \ldots, v_{\beta,N_1,m,n}^{k+\frac{1}{2}})^T \).

Based on these notations, the Step 1-Step 3 of ADI scheme can be rewritten as:

**Step 1.** On \( x_1 \) direction (\( s_k \to s_{k+\frac{1}{2}} \)).

\[ A_{N_1} U_{m,n}^\beta = E_{m,n}^\beta = (e_{\beta,1,m,n}^k, e_{\beta,2,m,n}^k, \ldots, e_{\beta,N_1,m,n}^k)^T, \]

\[ m = 1, 2, \ldots, N_2; n = 1, 2, \ldots, N_3. \]

where \( e_{\beta,l,m,n}^k = -\sum_{j=1}^3 \sum_{j \neq 1} \frac{\partial^\alpha \left( \frac{\partial^{\alpha} e_{\beta,l,m,n}}{\partial x_j^\alpha} \right)}{\partial x_j^\alpha} + f_{\beta}(\eta(s_k; t, p_{l,m,n})). \)

Solving linear systems (88), we will obtain \( v_{\beta,l,m,n}^{k+\frac{1}{2}} \) for \( l = 1, 2, \ldots, N_1, m = 1, 2, \ldots, N_2, n = 1, 2, \ldots, N_3 \) and \( \beta = 1, 2, 3 \).

**Step 2.** On \( x_2 \) direction (\( s_k+\frac{1}{2} \to s_k+\frac{1}{4} \)).

\[ A_{N_2} V_{l,n}^\beta = F_{l,n}^\beta = (f_{\beta,1,l,n}^{k+\frac{1}{2}}, f_{\beta,2,l,n}^{k+\frac{1}{2}}, \ldots, f_{\beta,N_2,l,n}^{k+\frac{1}{2}})^T, \quad l = 1, 2, \ldots, N_1; n = 1, 2, \ldots, N_3, \]

where \( V_{l,n}^\beta = (v_{\beta,1,l,n}^{k+\frac{1}{2}}, v_{\beta,2,l,n}^{k+\frac{1}{2}}, \ldots, v_{\beta,N_2,l,n}^{k+\frac{1}{2}})^T \). \( f_{\beta,l,m,n}^{k+\frac{1}{2}} = -\sum_{j=1}^3 \frac{\partial^\alpha \left( \frac{\partial^{\alpha} v_{\beta,l,m,n}}{\partial x_j^\alpha} \right)}{\partial x_j^\alpha} \)

+ \( f_{\beta}(\eta(s_k; t, p_{l,m,n})). \)

Solving linear systems (89), we will obtain \( v_{\beta,l,m,n}^{k+\frac{1}{4}} \) for \( l = 1, 2, \ldots, N_1, m = 1, 2, \ldots, N_2, n = 1, 2, \ldots, N_3 \) and \( \beta = 1, 2, 3 \).

**Step 3.** On \( x_3 \) direction (\( s_k+\frac{1}{4} \to s_{k+1} \)).

\[ A_{N_3} W_{l,m}^\beta = G_{l,m}^\beta = (g_{\beta,1,l,m}^{k+\frac{1}{4}}, g_{\beta,2,l,m}^{k+\frac{1}{4}}, \ldots, g_{\beta,N_3,l,m}^{k+\frac{1}{4}})^T, \quad l = 1, 2, \ldots, N_1; m = 1, 2, \ldots, N_2, \]
where $W_{i,m}^s = (v_{j,l,m,1}^{k+1}, \ldots, v_{j,l,m,N_3}^{k+1})^T$, $g_{\beta,l,m,n}^{k+2} = -\sum_{j=1, j \neq 3}^N \frac{\partial^2 v_{j,l,m,n}}{\partial x_j}$.

Solving linear systems (90), we will obtain $v_{\beta,l,m,n}^{k+1}$ for $l = 1, 2, \cdots, N_1$, $m = 1, 2, \cdots, N_2$, $n = 1, 2, \cdots, N_3$ and $\beta = 1, 2, 3$.

A pseudo-code of the ADI scheme is summarized as follows:

**Algorithm 4.1.** (ADI scheme for fractional order DTI registration model)

1. Given initial values $\eta(\tau; s_0, p_{l,m,n}) = p_{l,m,n}$, $\eta(s_0; s_0, p_{l,m,n}) = p_{l,m,n}$, and $v_\beta(p_{l,m,n}, s_0) = 0$ for $l = 1, 2, \cdots, N_1$, $m = 1, 2, \cdots, N_2$, $n = 1, 2, \cdots, N_3$ and $\beta = 1, 2, 3$.

2. Solving linear systems (88) to obtain $v_{\beta,l,m,n}^{k+\frac{1}{2}}$ for $l = 1, 2, \cdots, N_1$, $m = 1, 2, \cdots, N_2$, $n = 1, 2, \cdots, N_3$ and $\beta = 1, 2, 3$.

3. Solving linear systems (89) to obtain $v_{\beta,l,m,n}^{k+2}$ for $l = 1, 2, \cdots, N_1$, $m = 1, 2, \cdots, N_2$, $n = 1, 2, \cdots, N_3$ and $\beta = 1, 2, 3$.

4. Solving linear systems (90) to obtain $v_{\beta,l,m,n}^{k+1}$ for $l = 1, 2, \cdots, N_1$, $m = 1, 2, \cdots, N_2$, $n = 1, 2, \cdots, N_3$ and $\beta = 1, 2, 3$.

5. Using (82) to calculate $\eta(0; s_{k+1}, p_{l,m,n})$ and $\eta(\tau; s_{k+1}, p_{l,m,n})$ for $l = 1, 2, \cdots, N_1$, $m = 1, 2, \cdots, N_2$, $n = 1, 2, \cdots, N_3$.

6. Calculate deformed images $T \circ \eta(0; s_{k+1}, \cdot)$ and $D \circ \eta(\tau; s_{k+1}, \cdot)$.

7. Check the stop condition. If satisfied, stop and update the registered image $T \circ \eta(0; s_{k+1}, \cdot)$; else set $k = k + 1$ and return to (2).

Next, we will perform three numerical tests to show the effectiveness of Algorithm 4.1. In these three tests, two different DTI data sets coming from brain of two different individuals are collected with dimension $= 128 \times 128 \times 45$, respectively. Fractional Anisotropy (FA) [10] of one slice of $T(\cdot)$ and $D(\cdot)$ are shown as Figure 1.

(a) FA of the 22th slice of $T(\cdot)$

(b) FA of the 22th slice of $D(\cdot)$

**Figure 1.** One slice of $T(\cdot)$ and $D(\cdot)$

In this paper, we use the following three criterions [10, 20] to evaluate the effectiveness of Algorithm 4.1.

**Criterion 1.** Let $\vec{e}_i$ and $\vec{e}_i'$ be the first eigenvector of the target image $D(\cdot)$ and the floating image $T \circ \eta(0; \tau, \cdot)$ at voxel $i$, respectively. The angle $a_i$ between $\vec{e}_i$ and $\vec{e}_i'$ can be calculated as follows:

$$a_i = \arccos \left( \frac{\vec{e}_i \cdot \vec{e}_i'}{||\vec{e}_i|| \cdot ||\vec{e}_i'||} \right).$$
Based on this notation, we define

\[ a = \frac{1}{\sqrt{V}} \sum_{i=1}^{V} a_i, \]  

(92)

where \( V \) is the total number of white matter voxels whose FA is greater than 0.45.

**Criterion 2.** Define \( \text{SSD}(T, D) = \frac{1}{2} \sum_{i=1}^{128} \sum_{j=1}^{128} \sum_{k=1}^{45} ||T_{i,j,k} - D_{i,j,k}||^2 \) and

\[ \text{Re} - \text{SSD}(T, D, T \circ h(\cdot)) = \frac{\text{SSD}(T \circ h(\cdot), D)}{\text{SSD}(T, D)}. \]  

(93)

Note that here \( T_{i,j,k}, D_{i,j,k} \in \text{SPD}(3) \).

**Criterion 3.** Define

\[ J_h = \min_{x \in \Omega} \det(\nabla h(x)). \]  

(94)

In practice, the smaller \( a \) and \( \text{Re} - \text{SSD}(T, D, T \circ h(\cdot)) \), the better registration effectiveness. Moreover, \( J_h \) is also expected to be positive which ensures no fusion of points.

**Remark 5.** In DTI registration, orientation of nerve fiber in white matter region is a very important evaluation index. **Criterion 1** aims to evaluate orientation of nerve fiber. Therefore, **Criterion 1** is a commonly used criterion in DTI registration. On the other hand, **Criterion 2** is also commonly used in scalar image registration. What’s more, \( J_h \) is an important index to reflex whether the fusion of points occurs. Here we regard \( J_h \) as a criterion to show the advantage of diffeomorphic registration. Therefore, we list three different evaluation criterions to offer readers with reference.

Based on these two DTI data sets and evaluation criterions (92),(93),(94), the following three numerical tests are performed to show the effectiveness of our model (6).

![Figure 2](image)

(a) \( a \) changes with differential order \( \alpha \)
(b) \( \text{Re} - \text{SSD} \) changes with differential order \( \alpha \)

**Figure 2.** \( a \) and \( \text{Re} - \text{SSD} \) change with differential order \( \alpha \)

**Test 1.** (Sensitivity of Algorithm 4.1 with respect to \( \alpha \)) First, we give a numerical test to show how sensitive our fractional-order DTI registration model is with respect to the value of differential order \( \alpha \) in (6). Here \( \alpha \) varies from 2.55 to 2.95. For each \( \alpha \), we calculate a value of \( a \) and \( \text{Re} - \text{SSD} \) based on the final registration results \( T \circ h(\cdot) \). The result(see FIGURE 2) shows, the smaller \( \alpha \) will lead to much
more staircase effect of deformation $h(\cdot)$ and larger $\alpha$ will make $h(\cdot)$ much smoother.

Test 2. (Comparison with original model (3) with $L = -\Delta + \gamma$). Here we take integer-order operator $L = -\Delta + \gamma$ (cf. [2]) in original model (3) and compare the registration results with our fractional-order DTI registration model (6) ($\alpha = 2.95$). During each iteration process, we calculate a value of $a$ and Re–SSD based on image $T \circ \eta(0; s, x)$ and $D(\cdot)$. The value of $a$ and Re–SSD changes with time $s$ in iteration process are shown as FIGURE 3. This shows that our fractional-order DTI registration model (6) can improve the registration result compared with commonly used DTI registration model (3) with linear integer-order differential operator $L = -\Delta + \gamma$ in practice. Note that here $\tau = 1$ and the final registration result $T \circ h(\cdot)$ are shown as FIGURE 4(a)-(b).

Test 3. (Comparison between $\alpha = 1.9$ and $\alpha = 2.95$ in model (6)) In this test, we compare the registration results between $\alpha = 1.9$ and $\alpha = 2.95$ in model (6). During each iteration process, we calculate a value of $a$ and Re–SSD based on image $T \circ \eta(0; s, x)$ and $D(\cdot)$. The value of $a$ and Re–SSD changes with time $s$ in iteration process are shown as FIGURE 3. This shows that the registration result of $\alpha = 2.95$ is much better than $\alpha = 1.9$ in model (6). Note that here $\tau = 1$ and the final registration result $T \circ h(\cdot)$ are shown as FIGURE 4(c). Moreover, it follows from FIGURE 3(a)-(b) that both $\alpha = 2.95$ and $\alpha = 1.9$ in (6) can improve model (3) with $L = -\Delta + \gamma$.

Remark 6. It follows from Theorem 1.3 that $\alpha$ needs to be greater than 2.5 to ensure the existence, uniqueness of solution to (5) and well-defined of $T \circ h(\cdot)$ in (2). However, in numerical implementation, $\alpha \in (1, 2.5]$ may also work because of the fact that any two discretized data points can be connected by a function smooth enough. For this reason, we take $\alpha = 1.9$ in Test 3 to keep align with some published models ($\alpha \in (1, 2)$).

It follows from these three numerical tests that our fractional-order DTI registration model (6) improves the traditional integer-order DTI registration model (3) with $L = -\Delta + \gamma$.

Acknowledgments. This paper is partially based on some results of my P.H.D dissertation supervised by Professor Huan-Song Zhou in Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences. The authors of this paper would
like to thank Professor Fuchun Lin in Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences for providing the DTI data for Test 1, Test 2 and Test 3. Thanks also to the Referee for her/his very helpful remarks. This work was supported by NSFC(No.11471331) and partially supported by National Center for Mathematics and Interdisciplinary Sciences.

Appendix A. Calculation of $\partial_{w} H_{2}(v)$. By calculation,

$$
\partial_{w} H_{2}(v) = 2 \int_{\Omega} A : \partial_{w}[T \circ \eta(0; \tau, x)]dx = 2 \int_{\Omega} A : \{ \partial_{w} R [T \circ \eta(0; \tau, x)] R^{T} \\
+ R \partial_{w} [T \circ \eta(0; \tau, x)] R^{T} + R [T \circ \eta(0; \tau, x)] \partial_{w} R^{T} \} dx,
$$

where $A = (a_{ij}(x))_{3 \times 3} = T \circ \eta(0; \tau, x) - D(x)$.

What’s more, by [12], we know

$$
\partial_{w} R = \frac{1}{2}(\partial_{w} J^{T} - \partial_{w} J).
$$

Substitute (96) into (95), it yields

$$
\partial_{w} H_{2}(v) = 2Y_{0} + (Y_{2} - Y_{1} + Y_{4} - Y_{3}),
$$

where

$$
Y_{0} = \int_{\Omega} [T \circ \eta(0; \tau, x) - D(x)] : \{ R[\partial_{w}(T \circ \eta(0; \tau, x))] R^{T} \} dx,
$$
\[
Y_1 = \int_\Omega A : \partial_w J [T \circ \eta(0; \tau, x)] R^T dx, \quad Y_2 = \int_\Omega A : \partial_w J^T [T \circ \eta(0; \tau, x)] R^T dx, \\
Y_3 = \int_\Omega A : R [T \circ \eta(0; \tau, x)] \partial_w J^T dx, \quad Y_4 = \int_\Omega A : R [T \circ \eta(0; \tau, x)] \partial_w J dx.
\]

The calculation of (97) is divided into following five steps:

**Step 1.** \( Y_0 = \int_\Omega \{ R \partial_w (T \circ \eta(0; \tau, x)) \} R^T \} dx. \)

Let \( R = (r_{ij}(x))_{3 \times 3} \), \( \Theta = (\theta_{ij}(x))_{3 \times 3} = \partial_w [T \circ \eta(0; \tau, x)] \). By the fact

\[
(r_{ij}(x))_{3 \times 3} \cdot (\theta_{ij}(x))_{3 \times 3} \cdot (r_{ji}(x))_{3 \times 3} = \left( \sum_{i,j,m=1}^3 r_{ij} r_{jm} \theta_{tm} \right)_{3 \times 3},
\]
we obtain that

\[
Y_0 = \int_\Omega \sum_{i,j=1}^3 a_{ij} \left( \sum_{l,m=1}^3 r_{il} r_{jm} \theta_{tm} \right) dx = \int_\Omega \sum_{i,j,l,m=1}^3 a_{ij} r_{il} r_{jm} \theta_{tm} dx
\]

\[
= \int_\Omega \sum_{i,j,l,m=1}^3 a_{ij} r_{il} r_{jm} D_{l,m} \eta \circ \eta(0; \tau, x) \cdot \partial_w \eta(0; \tau, x) dx
\]

\[
= - \int_0^\tau \int_\Omega \sum_{i,j,l,m=1}^3 a_{ij} r_{il} r_{jm} D_{l,m} \eta \circ \eta(0; \tau, x) \cdot D_{l,m} \eta(0; \tau, x) [D \eta(0; \tau, x)]^-1 dx ds
\]

where \( W = (W_\alpha)_{1 \times 3} = \{ R \partial_w [T \circ \eta(0; \tau, x)] R^T \} \). Note that here and in what follows, we use the formula (see Lemma 2.1 in [2]):

\[
\partial_w \eta(t; s, x) = D \eta(t; s, x) \int_s^t [D \eta(r; s, x)]^{-1} w \circ \eta(r; s, x) dr.
\]

Let \( y = \eta(s; \tau, x) \), then \( x = \eta(\tau; s, y) \). It follows from the fact \( \eta(0; \tau, \eta(\tau; s, x)) = \eta(0; s, x) \) that

\[
Y_0 = \int_0^\tau \int_\Omega | \det(D \eta(\tau; s, x))| \tilde{W} \cdot w(x, s) dx ds
\]

\[
= \int_0^\tau \int_\Omega \det(D \eta(\tau; s, x)) \tilde{W} \cdot w(x, s) dx ds,
\]
where \( \tilde{W} = (\tilde{W}_\alpha)_{3 \times 3}, \tilde{W}_\alpha = \tilde{A} \cdot (R \partial_w [T \circ \eta(0; s, x)] R^T), \tilde{A} = T \circ \eta(0; s, x) - D(\eta(\tau; s, x)), T \circ \eta(0; s, x) = \tilde{R} [T \circ \eta(0; s, x)] \tilde{R}^T, \tilde{R} = J_1 (J_1 J_1^T)^{-\frac{1}{2}}, J_1 = D \Psi(x, s) \) and \( \Psi(x, s) = \eta(\tau; 0, \eta(\tau; s, x)) \).

Note that here and in what follows, we use the fact \( \det(D \eta(t; s, x)) > 0 \) for each \( s, t \in [0, \tau] \) (see Lemma 2.3 in [8]).

**Step 2.** \( Y_1 = \int_\Omega A : \partial_w J [T \circ \eta(0; \tau, x)] R^T dx. \)

Let \( B = (b_{ij})_{3 \times 3} = \partial_w J, C = (c_{ij})_{3 \times 3} = [T \circ \eta(0; \tau, x)] R^T, \) by the fact

\[
A : [BC] = \sum_{i,j,l=1}^3 a_{ij} b_{il} c_{lj},
\]
we obtain that
\[ Y_1 = \int_{\Omega} \sum_{i,j,l=1}^{3} a_{ij} c_{ij} b_{ij} \, dx = \int_{\Omega} \sum_{i,j,l=1}^{3} a_{ij} c_{ij} \partial_w \left( \frac{\partial \eta_0(\tau;0,x)}{\partial x_l} \right) \, dx \]
\[ = - \int_{\Omega} \sum_{i,j,l=1}^{3} \frac{\partial(a_{ij} c_{ij})}{\partial x_l} \partial_w \eta_0(\tau;0,x) \, dx = - \int_{\Omega} P \cdot \partial_w \eta_0(\tau;0,x) \, dx \]
\[ = - \int_{0}^{T} \int_{\Omega} P \cdot D\eta(\tau;0,x) \left[ D\eta(s;0,x) \right]^{-1} \circ \eta(s;0,x) \, dx \, ds, \]
where \( P = \text{div}(AC^T) \), here and in what follows \( \text{div}(E) = \left( \sum_{j=1}^{3} \frac{\partial e_{ij}}{\partial x_j} \right) \) for any matrix \( E = (e_{ij})_{3 \times 3}. \)

Note that here and in what follows, we use the equation \( \eta_0(\tau;0,x) = \frac{\partial}{\partial x_1} \left( \partial_w \eta_0(\tau;0,x) \right) \) and \( \partial_w \eta_0(\tau;0,x) \big|_{x \in \partial \Omega} = 0 \) for the fact \( \eta(s,t;x) \big|_{x \in \partial \Omega} = x \) for any \( s,t \in [0,T]. \)

Let \( y = \eta(s;0,x) \), then \( x = \eta(0,s,y). \) It follows from the fact \( \eta(\tau;0,\eta(0;s,x)) = \eta(\tau;s,x) \) that
\[ Y_1 = - \int_{0}^{T} \int_{\Omega} \det(D\eta(0;s,x)) \hat{P} \cdot D\eta(\tau;s,x) w(x,s) \, dx \, ds, \]
where \( \hat{P} = \text{div}(A\hat{C}^T), \ \hat{A} = T \circ \Phi(x,s) - D(\eta(0;s,x)), \ \hat{C} = [T \circ \Phi(x,s)] \hat{R}^T, \ T \circ \Phi(x,s) = \hat{R} [T \circ \Phi(x,s)] \hat{R}^T, \ \hat{R} = J_2^T (J_2 J_2^T)^{-\frac{3}{2}}, \ J_2 = D\eta(\tau;s,x), \ \Phi(x,s) = \eta(0;\tau,\eta(0;s,x)). \)

**Step 3.** \( Y_2 = \int_{\Omega} A : \partial_w J^T [T \circ \eta(0;\tau,x)] \hat{R}^T \, dx. \)

By the fact
\[ A : [B^T C] = \sum_{i,j,l=1}^{3} a_{ij} b_{ij} c_{ij}, \]
we obtain that
\[ Y_2 = \int_{\Omega} \sum_{i,j,l=1}^{3} a_{ij} b_{ij} c_{ij} \, dx = \int_{\Omega} \sum_{i,j,l=1}^{3} a_{ij} c_{ij} \partial_w \left( \frac{\partial \eta_0(\tau;0,x)}{\partial x_l} \right) \, dx \]
\[ = - \int_{\Omega} \sum_{i,j,l=1}^{3} \frac{\partial(a_{ij} c_{ij})}{\partial x_l} \partial_w \eta_0(\tau;0,x) \, dx = - \int_{\Omega} P_1 \cdot \partial_w \eta_0(\tau;0,x) \, dx \]
\[ = - \int_{0}^{T} \int_{\Omega} P_1 \cdot D\eta(\tau;0,x) \left[ D\eta(s;0,x) \right]^{-1} \circ \eta(s;0,x) \, dx \, ds, \]
where \( P_1 = \text{div}(CA^T). \)

Let \( y = \eta(s;0,x) \), then \( x = \eta(0;s,y). \) It follows from the fact \( \eta(\tau;0,\eta(0;s,x)) = \eta(\tau;s,x) \) that
\[ Y_2 = - \int_{0}^{T} \int_{\Omega} \det(D\eta(0;r,x)) \hat{P}_1 \cdot D\eta(\tau;s,x) w(x,s) \, dx \, ds, \]
where \( \tilde{P}_1 = \text{div}(\tilde{C} \tilde{A}^T) \).

**Step 4.** \( Y_3 = \int_\Omega A : R[T \circ \eta(0; \tau, x)] \partial_w J^T dx. \)

By the fact

\[
(107) \quad A : [C^T B^T] = \sum_{i,j,l=1}^{3} a_{ij} b_{jl} c_{li},
\]

we obtain that

\[
Y_3 = \int_\Omega \sum_{i,j,l=1}^{3} a_{ij} b_{jl} c_{li} dx = \int_\Omega \sum_{i,j,l=1}^{3} a_{ij} c_{li} \partial_w \left( \frac{\partial \eta_j(\tau; 0, x)}{\partial x_i} \right) dx
= -\int_\Omega \sum_{i,j,l=1}^{3} \frac{\partial(a_{ij} c_{li})}{\partial x_l} \partial_w \eta_j(\tau; 0, x) dx = -\int_\Omega Q \cdot \partial_w \eta_j(\tau; 0, x) dx
= -\int^{\tau}_0 \int_\Omega Q \cdot D\eta(\tau; 0, x) [D\eta(s; 0, x)]^{-1} w \circ \eta(s; 0, x) dx ds,
\]

where \( Q = \text{div}(A^T C^T) \).

Let \( y = \eta(s; 0, x) \), then \( x = \eta(0; s, y) \). It follows from the fact \( \eta(\tau; 0, \eta(0; s, x)) = \eta(\tau; s, x) \) that

\[
(108) \quad Y_3 = -\int^{\tau}_0 \int_\Omega \det(D\eta(0; s, x)) \tilde{Q} \cdot D\eta(\tau; s, x) w(x, s) dx ds,
\]

where \( \tilde{Q} = \text{div}(\tilde{A}^T \tilde{C}^T) \).

**Step 5.** \( Y_4 = \int_\Omega A : R[T \circ \eta(0; \tau, x)] \partial_w J dx. \)

By the fact

\[
(109) \quad A : [C^T B] = \sum_{i,j,l=1}^{3} a_{ij} b_{jl} c_{li},
\]

we obtain that

\[
Y_4 = \int_\Omega \sum_{i,j,l=1}^{3} a_{ij} b_{jl} c_{li} dx = \int_\Omega \sum_{i,j,l=1}^{3} a_{ij} c_{li} \partial_w \left( \frac{\partial \eta_j(\tau; 0, x)}{\partial x_j} \right) dx
= -\int_\Omega \sum_{i,j,l=1}^{3} \frac{\partial(a_{ij} c_{li})}{\partial x_j} \partial_w \eta_j(\tau; 0, x) dx = -\int_\Omega Q_1 \cdot \partial_w \eta_j(\tau; 0, x) dx
= -\int^{\tau}_0 \int_\Omega Q_1 \cdot D\eta(\tau; 0, x) [D\eta(s; 0, x)]^{-1} w \circ \eta(s; 0, x) dx ds,
\]

where \( Q_1 = \text{div}(CA) \).

Let \( y = \eta(s; 0, x) \), then \( x = \eta(0; s, x) \). It follows from the fact \( \eta(\tau; 0, \eta(0; s, x)) = \eta(\tau; s, x) \) that

\[
(110) \quad Y_4 = -\int^{\tau}_0 \int_\Omega \det(D\eta(0; s, x)) \tilde{Q}_1 \cdot D\eta(\tau; s, x) w(x, s) dx ds,
\]

where \( \tilde{Q}_1 = \text{div}(\tilde{C} \tilde{A}) \).

By (102), (104), (106), (108), (110) and (97), we obtain

\[
(111) \quad \partial_w H_2(v) = \int^{\tau}_0 \int_\Omega \delta H_2(v) \cdot w(x, s) dx ds,
\]
where
\[ \delta H_2(v) = -\det(D\eta(0; s, x))[\bar{P}_1 - \bar{P} + \bar{Q} - \bar{Q}_1]D\eta(\tau; s, x) - 2 \det(D\eta(\tau; s, x))\bar{W}. \]

Since \( T \circ \Phi(x, s), D(\eta(0; s, x)) \in SPD(3) \), this implies \( \hat{A} = \hat{A}^T \), then we obtain
\[ (112) \quad \bar{P}_1 = \bar{Q}_1, \bar{P} = \bar{Q}. \]

This implies,
\[ (113) \quad \delta H_2(v) = -2 \det(D\eta(\tau; s, x))\bar{W}. \]

That is,
\[ (114) \quad \partial_w H_2(v) = \int_0^\tau \int_\Omega -2 \det(D\eta(\tau; s, x))\bar{W} \cdotwdxds. \]

REFERENCES

[1] D. C. Alexander, C. Pierpaoli, P. J. Basser and J. C. Gee, Spatial transformations of diffusion tensor magnetic resonance images, IEEE Transaction on Medical Imaging, 20 (2001), 1131–1139.

[2] M. F. Beg, M. I. Miller, A. Trouve and L. Younes, Computing large deformation metric mappings via geodesic flows of diffeomorphisms, International Journal of Computer Vision, 61 (2005), 139–157.

[3] M. Bruveris, F. Gay-Balmaz, D. D. Holm and T. S. Ratiu, The momentum map representation of images, Journal of Nonlinear Science, 21 (2011), 115–150.

[4] F. Demengel and G. Demengel, Functional spaces for the theory of elliptic partial differential equations, Springer, (2011), 219–224.

[5] P. Dupuis, U. Grenander and M. I. Miller, Variational problems on flows of diffeomorphisms for image matching, Quarterly of Applied Mathematics, 56 (1998), 587–600.

[6] V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numerical Method for Partial Differential Equations, 22 (2006), 558–576.

[7] L. C. Evans, Partial differential equations, American Mathematical Society, (1997), 251–308.

[8] H. Han and H. Zhou, A variational problem arising in registration of diffusion tensor image, Acta Mathematica Scientia, 37 (2017), 539–554.

[9] H. Han and H. Zhou, Spectral representation of solution of a variational model in diffusion tensor images registration, preprint.

[10] W. V. Hecke and A. Leemans, Nonrigid coregistration of diffusion tensor images using a viscous fluid model and mutual information, IEEE Transaction on Medical Imaging, 26 (2007), 1598–1612.

[11] C. R. Johnson, K. Okubo and R. Reams, Uniqueness of matrix square roots and application, Linear Algebra and it Applications, 323 (2001), 51–60.

[12] J. Li, Y. Shi, G. Tran, I. Dinov, D. Wang and A. Toga, Fast local trust region for diffusion tensor registration using exact reorientation and regularization, IEEE Transaction on Medical Imaging, 33 (2014), 1–43.

[13] R. Li, S. Zhong and C. Swartz, An improvement of the Arzela-Ascoli theorem, Topology and Its Applications, 150 (2012), 2058–2061.

[14] F. O’Sullivan, The Analysis of Some Penalized Likelihood Schemes, Statistics Department Technical Report No.726, University of Wisconsin, 1983.

[15] I. Podlubny, Fractional Differential Equations: An introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and some of Their Applications, Math. Sci. Eng. Elsevier Science, (1999), 50–90.

[16] G. Teschl, Ordinary differential equations and Dynamical systems, American Mathematical Society, (2012), 50–230.

[17] H. Wang and N. Du, Fast solution methods for space-fractional diffusion equations, Journal of Computational and Applied Mathematics, 255 (2014), 376–383.

[18] T. Yeo, T. Vercauteren, P. Ficlard, J. Peyrat, X. Pennec, P. Golland, N Ayache and O. Clatz, DTREFinD: Diffusion tensor registration with exact finite-strain differential, IEEE Transaction on Medical Imaging, 28 (2009), 1914–1928.
[19] S. Zhan, On the determinantal inequalities, *Journal of Inequalities in Pure and Applied Mathematics*, 6 (2005), Article 105, 7 pp.

[20] J. Zhang and K. Chen, Variational image registration by a total fractional-order variation model, *Journal of Computational Physics*, 293 (2015), 442–461.

[21] Y. Zhang and Z. Sun, Error analysis of a compact ADI scheme for the 2D fractional subdiffusion equation, *Journal of Scientific Computing*, 59 (2014), 104–128.

Received December 2016; revised August 2018.

_E-mail address:_ hanhuan11@mails.ucas.ac.cn