Truncated Exponential Topp Leone Exponential Distribution: Properties and Applications

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Abstract. A new flexible compound distribution with three parameters named Truncated Exponential Topp Leone Exponential (TETLE) is introduced as a sub-model of a new generator family named Truncated Exponential Topp Leone-G. Several keys of its statistical properties, order statistics, entropies, reliability analysis, and reliability stress strength model are presented. The maximum likelihood estimation method is implemented to estimate the unknown parameters. Moreover, to assess the flexibility and usefulness, the TETLE distribution applied upon simulation and two real-life data set. The consistent and flexible performance of the maximum likelihood estimates is clearly appeared by the simulation results and the results of the real application clearly shown that the proposed distribution has superior output relative to other considered distributions for all information criteria.

1. Introduction
Lifetime data plays a significant role in an extensive variety of applications, such as public health, management, engineering, medicine, and biological sciences. Statistical distributions are used to model these data in order to examine its essential characteristics. Appropriate distribution can provide valuable information that leads to putting conclusions and decisions. When more flexible distributions are needed, many researchers are about to use the new one with more generalization. One of the supremely important distributions is the exponential distribution. It has been applied to match life data in a number of fields. However, some of these applications are restricted in part, and this restriction is certainly an opportunity for researchers to create new generalizations and to make a variety of improvements to this distribution in order to increase its flexibility. In this paper, based on the well-known exponential distribution, a new family of probability distributions is introduced through the works of Topp and Leone [1] and Eugene et al. [2].

Topp and Leone (1955) [1] developed a new distribution of empirical data with J-shaped histograms. The Topp Leone (TL) distribution had not received much attention until it discovered by Nadarajah and Kotz (2003) [3] when they studied some of its properties, moments, central moments, and characteristic function. Furthermore, among the important work of analyzing lifetime data that revealed a significant influence of the TL distribution are, Ghitany et al. (2005) [4] provided TL reliability measures, Kotz and Seier (2007) [5] offered a discussion on TL kurtosis, Vicaria et al. (2008) [6] introduced two-sided generalized TL distributions, Al-Zahrani (2012) [7] derived TL goodness of fit tests, and Sangsanit and Bodhisuwan (2016) [8] proposed the TL generated (TL-G) family of distributions.
The cumulative distribution function (cdf) and probability density function (pdf) of the TL-G family, that due to Sangsanit and Bodhisuwan [8], with one shape parameter \( \alpha \) beside the cdf and pdf of an arbitrary (parent or baseline) distribution, say \( G(x) \) and \( g(x) \), are

\[
M(x)_{TL} = (G(x))^{\alpha} (2 - G(x))^{\alpha - 1}; \quad \alpha > 0 \tag{1}
\]
\[
m(x)_{TL} = 2\alpha g(x) (G(x))^{\alpha - 1} (1 - G(x)) (2 - G(x))^{\alpha - 1} \tag{2}
\]

Eugene et al. [2] proposed a new family based on beta distribution with the cdf and pdf as below

\[
F(x)_T = \frac{1}{\beta(a,b)} \int_0^{M(x)} t^{a-1} (1 - t)^{b-1} \, dt; \quad 0 < t < 1 \text{ and } 0 < a, b < \infty \tag{3}
\]
\[
f(x)_T = \frac{1}{\beta(a,b)} (M(x))^{a-1} (1 - M(x))^{b-1} m(x) \tag{4}
\]

where \( M(x) \) and \( m(x) \) are the cdf and pdf of any distribution and \( \beta(a,b) \) represent Beta function. Then Eugene et al. derived some properties of the beta normal distribution that defined by taking \( M(x) \) to be the cdf of the normal distribution (see [2][9]).

Taking motivation from the works above, a new family named truncated exponential Topp Leone–G has been proposed (for more details about TL distribution and truncated distribution see [2] and [10]). The rest of this paper is organized as follows: the truncated exponential Topp Leone – G family is introduced in Section 2 with several general useful expressions. In Section 3, attention is given the truncated exponential Topp Leone exponential as a sub-model of the new family. In Sections 4, and 5 various properties besides order statistics, entropies, reliability measures, and reliability stress strength model are presented. The maximum likelihood estimators of the parameters are presented in Section 6. In Section 7, numerical illustrations are implemented in order to illustrate the behavior of the new distribution. Finally, in Section 8, conclusions are listed.

2. Truncated Exponential Topp Leone-G family

Consider \( H(x)_{TE} \) and \( h(x)_{TE} \) are the cdf and pdf of the truncated exponential distribution with scale parameter \( \lambda \), i.e.

\[
H(x)_{TE} = \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda}}; \quad 0 < x < 1, \lambda > 0 \tag{5}
\]
\[
h(x)_{TE} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda}} \tag{6}
\]

By composing the two continuous cdfs \((H, M)\), the cdf \( F(x) = H(M(x)) \) and corresponding pdf \( f(x) = \frac{\partial}{\partial x} F(x) \) will be

\[
F(x) = \int_0^{M(x)} \frac{\lambda e^{-\lambda v}}{1 - e^{-\lambda}} \, dv = \frac{1 - e^{-\lambda M(x)}}{1 - e^{-\lambda}} \tag{7}
\]
\[
f(x) = \frac{\lambda m(x) e^{-\lambda M(x)}}{1 - e^{-\lambda}} \tag{8}
\]

Replacing \( M(x) \) and \( m(x) \) in (7) and (8) by (1) and (2), a new generator family of probability distributions named Truncated Exponential Topp Leone-G (TETL-G) can be proposed.
The cdf and pdf of the proposed family are

\[
F(x)_{\text{TETL-G}} = \frac{1}{1 - e^{-\lambda}} \left( 1 - e^{-\lambda(G(x)) \alpha(2-G(x))} \right) \tag{9}
\]

\[
f(x)_{\text{TETL-G}} = \frac{2\lambda \alpha}{1 - e^{-\lambda}} g(x)(G(x))^{\alpha-1} \left( 1 - G(x) \right) \left( 2 - G(x) \right)^{\alpha-1} e^{-\lambda(G(x)) \alpha(2-G(x))} \tag{10}
\]

Also, the useful expansion formula of the above pdf can be got by using the expansion formulas below:

\[e^{-z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k \tag{E1}\]

\[(1-z)^a = \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\alpha-m+1)_m} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} z^m \text{ with } |z| < 1 \text{ and } a > 0 \tag{E2}\]

\[(1-z)^{-a} = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} z^m \text{ with } |z| < 1 \text{ and } a > 0 \tag{E3}\]

Now, based on the above expansion formulas, we get

\[e^{-\lambda(G(x)) \alpha} (2-G(x))^{\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (G(x))^{\alpha k} (2-G(x))^{\alpha k}, \text{ and} \]

\[(2-G(x))^{\alpha (k+1)-1} = \begin{cases} 
\sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\alpha-m+1)_m} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} \frac{G(x)^m}{(\alpha+1)_m} & ; \alpha(k+1) - 1 > 0 \\
\sum_{m=0}^{\infty} \frac{\Gamma(a+1)_m}{\Gamma(a)_m} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} \frac{\Gamma(a+1)_m}{\Gamma(a)_m} \frac{G(x)^m}{(\alpha+1)_m} & ; \alpha(k+1) - 1 < 0
\end{cases} \]

Then expansion of (10) will be

\[f^E(x)_{\text{TETL-G}} = W g(x)(G(x))^{m+\alpha(k+1)-1} \left( 1 - G(x) \right) \tag{11}\]

where

\[W = \begin{cases} 
\frac{1}{1-e^{-\lambda}} \sum_{k,m=0}^{\infty} \frac{(-1)^k m}{k!} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-m+1)} \alpha \lambda^{k+1} 2^{\alpha(k+1)-m} & ; \alpha(k+1) - 1 > 0 \\
\frac{1}{1-e^{-\lambda}} \sum_{k,m=0}^{\infty} \frac{(-1)^k m}{k!} \frac{\Gamma(\alpha+1)_m}{\Gamma(\alpha+1)_m} \frac{\Gamma(a+1)_m}{\Gamma(a)_m} \frac{\Gamma(a+1)_m}{\Gamma(a)_m} \alpha \lambda^{k+1} 2^{\alpha(k+1)-m} & ; \alpha(k+1) - 1 < 0
\end{cases} \tag{12}\]

3. Truncated Exponential Topp Leone Exponential distribution

Consider \(G(x)\) and \(g(x)\) in (9), (10), and (11) be the cdf and pdf of the exponential distribution with positive scale parameter \(\theta\), then a new distribution called Truncated Exponential Topp Leone Exponential (TETLE) distribution can be proposed as a member of TETL-G family with the cdf, pdf and expanded pdf given respectively by

\[F(x)_{\text{TETLE}} = \frac{1}{1 - e^{-\lambda}} \left( 1 - e^{-\lambda(1-e^{-2\theta x})^a} \right) \tag{13}\]

\[f(x)_{\text{TETLE}} = \frac{2\lambda a \theta}{1 - e^{-\lambda}} e^{-2\theta x} e^{-\lambda(1-e^{-2\theta x})^a} \left( 1 - e^{-2\theta x} \right)^{a-1} \tag{14}\]
\[ f_{\text{TETLE}}(x) = W \theta e^{-2x} \left(1 - e^{-\theta x}\right)^{m+\alpha(k+1)-1} \]  \hspace{1cm} (15)

with \( W \) as in (12).

The plots of the cdf and pdf of the TETLE distribution for particular parameter values are presented in Figures 1 and 2. It is noted that this new distribution is very flexible to model positive data.

4. Statistical properties and order statistics of TETLE distribution

The \( r^{th} \) moment can be found regards to (15) as follows

\[
E(X^r)_{\text{TETLE}} = \int_0^\infty x^r f(x)_{\text{TETLE}} \, dx = W \int_0^\infty x^r \theta e^{-2x} \left(1 - e^{-\theta x}\right)^{m+\alpha(k+1)-1} \, dx
\]

Now, based on (E2) and (E3), we get

\[
(1 - e^{-\theta x})^{m+\alpha(k+1)-1} = \begin{cases} 
\sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{\Gamma(m+\alpha(k+1))}{\Gamma(m+\alpha(k+1)-t)} e^{-\theta tx} & ; m + \alpha(k + 1) - 1 > 0 \\
\sum_{t=0}^{\infty} \frac{\Gamma(m+\alpha(k+1)+t-1)}{t! \Gamma(m+\alpha(k+1)-1)} e^{-\theta tx} & ; m + \alpha(k + 1) - 1 < 0
\end{cases}
\]

Now \( E(X^r)_{\text{TETLE}} \) with \( m + \alpha(k + 1) - 1 > 0 \) will be

\[
E(X^r)_{\text{TETLE}} = W \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{\Gamma(m+\alpha(k+1))}{\Gamma(m+\alpha(k+1)-t)} \int_0^\infty x^r \theta e^{-\theta(t+2)x} \, dx
\]

\[
= W \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{\Gamma(m+\alpha(k+1))}{\Gamma(m+\alpha(k+1)-t)(t+2)} \int_0^\infty x^r \theta(t+2) e^{-\theta(t+2)x} \, dx
\]

Since \( \int_0^\infty x^r \theta(t+2) e^{-\theta(t+2)x} \, dx \) represent the \( r^{th} \) moment of exponential distribution with parameters \( \theta(t+2) \), thus \( E(X^r)_{\text{TETLE}} \) becomes

**Figure 1.** The cdf plot of TETLE distribution for particular parameter values

**Figure 2.** The pdf plot of TETLE distribution for particular parameter values
\[ E(X^r)_{\text{TETLE}} = W \sum_{t=0}^{\infty} \frac{(-1)^t \Gamma(m + \alpha(k + 1))}{t!} \frac{r!}{\Gamma(m + \alpha(k + 1) - t) \Gamma(t + 2)(\theta(t + 2))^r} \]

and the \( E(X^r)_{\text{TETLE}} \) with \( m + \alpha(k + 1) - 1 < 0 \) becomes

\[ E(X^r)_{\text{TETLE}} = W \sum_{t=0}^{\infty} \frac{\Gamma(m + \alpha(k + 1) + t - 1)}{t!} \frac{r!}{\Gamma(m + \alpha(k + 1) - t) \Gamma(t + 2)(\theta(t + 2))^r} \]

Thus, the \( r^{th} \) moment of TETLE distribution is

\[ E(X^r)_{\text{TETLE}} = \begin{cases} 
W \sum_{t=0}^{\infty} \frac{(-1)^t \Gamma(m + \alpha(k + 1))}{t!} \frac{r!}{\Gamma(m + \alpha(k + 1) - t) \Gamma(t + 2)(\theta(t + 2))^r} + m + \alpha(k + 1) - 1 > 0 \\
W \sum_{t=0}^{\infty} \frac{\Gamma(m + \alpha(k + 1) + t - 1)}{t!} \frac{r!}{\Gamma(m + \alpha(k + 1) - t) \Gamma(t + 2)(\theta(t + 2))^r} + m + \alpha(k + 1) - 1 < 0 
\end{cases} \quad (16) \]

where \( W \) as in (12).

With \( (r = 1, 2, 3, 4) \), another properties such as the mean \((E(X))\), the variance \((E(X^2) - [E(X)]^2)\), the coefficient of skewness \((E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3)/[\text{var}(X)]^{3/2}\), and the coefficient of kurtosis \((E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4)/[\text{var}(X)]^2\) (see [11]) can be calculated.

The TETLE characteristic function can easily be found via \( \varphi_X(t)_{\text{TETLE}} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r)_{\text{TETLE}} \) as

\[ \varphi_X(t)_{\text{TETLE}} = \begin{cases} 
W \sum_{t,r=0}^{\infty} \frac{(-1)^t \Gamma(m + \alpha(k + 1))}{t!} \frac{r!}{\Gamma(m + \alpha(k + 1) - t) \Gamma(t + 2)(\theta(t + 2))^r} + m + \alpha(k + 1) - 1 > 0 \\
W \sum_{t,r=0}^{\infty} \frac{\Gamma(m + \alpha(k + 1) + t - 1)}{t!} \frac{r!}{\Gamma(m + \alpha(k + 1) - t) \Gamma(t + 2)(\theta(t + 2))^r} + m + \alpha(k + 1) - 1 < 0 
\end{cases} \quad (17) \]

By inverting the cdf in (13), The TETLE quantile function can be achieved as

\[ Q(q)_{\text{TETLE}} = -\frac{1}{2\theta} \ln \left( 1 - \left( \frac{-1}{\lambda} \ln \left( 1 - q \left( 1 - e^{-\lambda} \right) \right) \right)^{\frac{1}{\alpha}} \right) \quad (18) \]

By setting \( q = \frac{1}{2} \), the median of TETLE random variable can be gained as

\[ \text{median}_{\text{TETLE}} = Q\left( \frac{1}{2} \right)_{\text{TETLE}} = -\frac{1}{2\theta} \ln \left( 1 - \left( \frac{-1}{\lambda} \ln \left( \frac{1}{2} \left( 1 + e^{-\lambda} \right) \right) \right)^{\frac{1}{\alpha}} \right) \quad (19) \]
A random variable of TETLE distribution can be generated by

\[ x_{TETLE} = -\frac{1}{2\theta} \ln \left( 1 - \left( -\frac{1}{\lambda} \ln \left( 1 - U(1-e^{-\lambda}) \right) \right)^{\frac{1}{\alpha}} \right) \]  
\[(20)\]

where \( U \) follow the standard Uniform distribution.

**Order statistics:** Let \( x_{1:n}, x_{2:n}, \ldots, x_{n:n} \) the order statistics of a random sample \( x_1, x_2, \ldots, x_n \) of size \( n \) taken independently from a probability distribution. The standard formulas of the pdf and the joint pdf of order statistics are as follows (see [12])

\[ f_{k:n}(x) = \frac{n!}{(k-1)! (n-k)!} (F(x))^{k-1} (1-F(x))^{n-k} f(x) \quad 0 \leq x_{k:n} < \infty, k \leq n \]

\[ f_{j,k:n}(x,y) = \frac{n!}{(j-1)! (k-j-1)! (n-k)!} (F(x))^{j-1} (F(y) - F(x))^{k-j-1} (1-F(y))^{n-k} f(x) f(y) \quad 1 \leq j \leq k, 0 \leq x \leq y < \infty \]

Based on (13) and (14), the pdf and the joint pdf of TETLE order statistics will be

\[ f_{k:n}(x)_{TETLE} = \frac{n!}{(k-1)! (n-k)!} \left( 1 - e^{-\lambda(1-e^{-2\theta x})^\alpha} \right)^{k-1} \left( e^{-\lambda(1-e^{-2\theta x})^\alpha} - e^{-\lambda} \right)^{n-k} \]
\[ \cdot \frac{2\lambda \alpha \theta}{(1-e^{-\lambda})^n} e^{-2\theta x} e^{-\lambda(1-e^{-2\theta x})^\alpha} (1 - e^{-2\theta x})^{\alpha-1} \; ; \; 0 \leq x_{k:n} < \infty, k \leq n \]

\[ f_{j,k:n}(x,y)_{TETLE} = \frac{n!}{(j-1)! (k-j-1)! (n-k)!} \left( e^{-\lambda(1-e^{-2\theta x})^\alpha} - e^{-\lambda(1-e^{-2\theta y})^\alpha} \right)^{j-1} \left( e^{-\lambda(1-e^{-2\theta y})^\alpha} - e^{-\lambda} \right)^{n-k} e^{-2\theta (x+y)} \]
\[ \cdot e^{-\lambda((1-e^{-2\theta x})^\alpha + (1-e^{-2\theta y})^\alpha)} (1 - e^{-2\theta x})^{\alpha-1} (1 - e^{-2\theta y})^{\alpha-1} \; ; \; 1 \leq j \leq k, 0 \leq x \leq y < \infty \]

\[(21)\]

\[(22)\]

**5. The entropies, reliability measures, and reliability stress strength of TETLE distribution**

Here the entropies (Shannon entropy, and Relative entropy), reliability measures, and reliability stress strength model of TETLE distribution are obtained as follows:

**Shannon entropy:** The TETLE Shannon entropy can be achieved from (14), as follows:

\[ S_{hTETLE} = -\int_0^\infty \ln(f(x)_{TETLE}) f(x)_{TETLE} \, dx \]

with

\[ \ln(f(x)_{TETLE}) = \ln \left( \frac{2\lambda \alpha \theta}{1 - e^{-\lambda}} \right) - 2\theta x - \lambda(1-e^{-2\theta x})^\alpha + (\alpha - 1)\ln(1-e^{-2\theta x}) \]

Then

\[ S_{hTETLE} = \ln \left( \frac{1 - e^{-\lambda}}{2\lambda \alpha \theta} \right) + 2\theta E(X) + \lambda E \left( (1-e^{-2\theta X})^\alpha \right) - (\alpha - 1) E \left( \ln(1-e^{-2\theta X}) \right) \]

where \( E(X) \) as in (16) with \( r = 1 \), and

\[ E \left( (1-e^{-2\theta X})^\alpha \right) = \int_0^\infty (1-e^{-2\theta x})^\alpha f(x)_{TETLE} \, dx \]
Recall \((E2)\) and \((E1)\), we get \((1 - e^{-2\theta x})^\alpha = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i! j! \Gamma(\alpha-i+1)} (2\theta i)^j x^j\), then

\[ E\left((1 - e^{-2\theta x})^\alpha\right) = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i! j! \Gamma(\alpha-i+1)} (2\theta i)^j E(X^j) \tag{24} \]

where \(E(X^j)\) as in \((16)\) with \(r = j\).

Now for \(E\left(\ln(1 - e^{-2\theta X})\right) = \int_0^\infty \ln(1 - e^{-2\theta x}) f(x) \, dx\), using the following expansion formula

\[ \ln(1 - z) = -\sum_{i=1}^{\infty} \frac{z^i}{i}, \quad |z| < 1 \tag{E4} \]

with using \((E1)\), we get \(\ln(1 - e^{-2\theta x}) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{k! l!} (2\theta k)^l x^l\), and then

\[ E\left(\ln(1 - e^{-2\theta x})\right) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{k! l!} (2\theta k)^l E(X^l) \tag{25} \]

where \(E(X^l)\) as in \((16)\) with \(r = l\).

Relative entropy: Consider \(f(x)\) and \(f^*(x)\) are the TETLE pdf respectively, with parameters \((\lambda, \alpha, \theta)\) and \((\lambda^*, \alpha^*, \theta^*)\) then

\[
\ln \frac{f(x)}{f^*(x)} = \ln \left( \frac{\lambda \alpha \theta(1 - e^{-\lambda^*})}{\lambda^* \alpha^* \theta^*(1 - e^{-\lambda})} \right) - 2(\theta - \theta^*)X - \lambda \left(1 - e^{-2\theta x}\right)^\alpha + \lambda^* \left(1 - e^{-2\theta^* x}\right)^{\alpha^*} + (\alpha - 1) \ln(1 - e^{-2\theta x}) - (\alpha^* - 1) \ln(1 - e^{-2\theta^* x})
\]

The TETLE relative entropy is given by

\[
RE_{TETLE} = E\left(\ln \frac{f(x)}{f^*(x)}\right) = \ln \left( \frac{\lambda \alpha \theta(1 - e^{-\lambda^*})}{\lambda^* \alpha^* \theta^*(1 - e^{-\lambda})} \right) - 2(\theta - \theta^*)E(X) - \lambda E\left(\left(1 - e^{-2\theta x}\right)^\alpha\right) + \lambda^* E\left(\left(1 - e^{-2\theta^* x}\right)^{\alpha^*}\right) + (\alpha - 1) E\left(\ln(1 - e^{-2\theta x})\right) - (\alpha^* - 1) E\left(\ln(1 - e^{-2\theta^* x})\right) \tag{26}
\]

with \(E(X)\) as in \((16)\) with \(r = 1\) and other expectations with specified parameters as in \((24)\), and \((25)\).

Reliability measures: Based on \((13)\) and \((14)\), the TETLE functions of reliability, hazard, cumulative hazard, and reverse hazard can easily be found respectively as (see [13])

\[
\tau_1(x)_{TETLE} = 1 - F(x)_{TETLE} = \frac{1}{1 - e^{-\lambda}} \left( e^{-\lambda (1 - e^{-2\theta x})^\alpha} - e^{-\lambda} \right) \tag{27}
\]

\[
\tau_2(x)_{TETLE} = \frac{f(x)_{TETLE}}{R(x)_{TETLE}} = \frac{2\lambda \alpha \theta e^{-2\theta x} e^{-\lambda (1 - e^{-2\theta x})^\alpha} (1 - e^{-2\theta x})^{\alpha-1}}{e^{-\lambda (1 - e^{-2\theta x})^\alpha} - e^{-\lambda}} \tag{28}
\]

\[
\tau_3(x)_{TETLE} = -\ln(R(x)_{TETLE}) = -\ln \left(1 - \frac{1}{1 - e^{-\lambda}} \left(1 - e^{-\lambda (1 - e^{-2\theta x})^\alpha}\right)\right) \tag{29}
\]
\[ \tau_4(\alpha)_{TETLE} = \frac{f(x)_{TETLE}}{F(x)_{TETLE}} = \frac{2\lambda^2 e^{-2\theta x} e^{-\lambda (1-e^{-2\theta x})^{\alpha}} (1-e^{-2\theta x})^{\alpha-1}}{1-e^{-\lambda(1-e^{-2\theta x})^{\alpha}}} \]  
(30)

**Reliability stress strength model:** Let \( X \) (strength) and \( Y \) (stress) are two independent TETLE random variables with parameters \((\lambda', \alpha', \theta)\) and \((\lambda^*, \alpha^*, \theta^*)\) respectively, the reliability stress strength model [14] can be obtained as follows

\[ SS_{TETLE} = P(Y < X) = \int_0^{\infty} f_X(x) \int_0^{\infty} f_Y(x) \, dx \]

\[ = \frac{1}{1-e^{-\lambda^*}} \int_0^{\infty} \left(1-e^{-\lambda^*(1-e^{-2\theta^* x})^{\alpha^*}}\right) f_X(x) \, dx \]

Using \( E1, \) and \( E2 \) we get \( e^{-\lambda^*(1-e^{-2\theta^* x})^{\alpha^*}} = \sum_{i,m,j=0}^{\infty} \frac{(-1)^{i+m+j}}{i! m! j!} \Gamma(i\alpha^*+1) \Gamma(i\alpha^*-m+1) \lambda^* i (2m\theta^*)^j x^j \), then

\[ SS_{TETLE} = \frac{1}{1-e^{-\lambda^*}} \int_0^{\infty} \left[1-\sum_{i,m,j=0}^{\infty} \frac{(-1)^{i+m+j}}{i! m! j!} \Gamma(i\alpha^*+1) \Gamma(i\alpha^*-m+1) \lambda^* i (2m\theta^*)^j E(\chi^j)\right] f_X(x) \, dx \]

Thus the TETLE reliability stress strength model is

\[ SS_{TETLE} = \frac{1}{1-e^{-\lambda^*}} \left[1-\sum_{i,m,j=0}^{\infty} \frac{(-1)^{i+m+j}}{i! m! j!} \Gamma(i\alpha^*+1) \Gamma(i\alpha^*-m+1) \lambda^* i (2m\theta^*)^j E(\chi^j)\right] \]

(31)

where \( E(\chi^j) \) as in (16) with \( r = j \).

6. **Maximum likelihood estimators of TETLE parameters**

Regarding to (14), the natural logarithm likelihood function that associated with a complete random sample of size \( n \), say \((x_1, x_2, ..., x_n)\), follow TETLE distribution with the vector of parameters \( \Delta = (\lambda, \alpha, \theta)^T \) is

\[ \ell(\Delta|x) = n \ln \left(\frac{2\lambda^2 \theta^n}{1-e^{-\lambda}} - 2\theta\Sigma_{i=1}^n x_i - \lambda\Sigma_{i=1}^n (1-e^{-2\theta x_i})^\alpha + (\alpha - 1)\Sigma_{i=1}^n \ln \left((1-e^{-2\theta x_i})\right) \right) \]

(32)

The maximum likelihood (ML) estimators of three parameters can be obtained through solving the nonlinear differential equations \( \frac{\partial \ell(\Delta|x)}{\partial \lambda} = 0, \ \frac{\partial \ell(\Delta|x)}{\partial \alpha} = 0, \ \frac{\partial \ell(\Delta|x)}{\partial \theta} = 0 \). The ML estimators are not in closed forms, so numerical method is used.

7. **Numerical illustrations**

Simulation and real applications are presented here to exhibit the abilities of the proposed TETLE distribution.

7.1 **Simulation study**

The steps of the simulation process are:

1. Based on simulated formula in (20), generate i.i.d. random samples (1000 times) follow TETLE with size \( n = 10, 25, 50, 100, \) and \( 200 \) where the initial (or true) values of parameters are chosen to be as in Table 1 (also see Figure 2).

2. For parameter \( \lambda \), calculate the Bias and root mean squared error (RMSE) as

\[ Bias(\hat{\lambda}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\lambda}_i - \lambda) \]  
and \( RMSE(\hat{\lambda}) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\lambda}_i - \lambda)^2} \).
3. Repeat step 2 for parameters $\alpha$, and $\theta$.

The empirical results are listed in Table 1. It clearly appears that RMSE values decrease as the sample size increases.

| Table 1: The Bias and RMSE associated with MLE of the TETLE parameters |
|-----------------|--------|----------------|--------|----------------|--------|----------------|
| $n$             | Par.   | Init. | Bias     | RMSE   | Init. | Bias     | RMSE   |
| 10              | $\lambda$ | 1.3   | -0.988   | 2.036  | 3.1   | -0.650   | 1.815  |
|                 | $\alpha$ | 2.2   | 0.926    | 2.415  | 0.8   | 0.195    | 0.546  |
|                 | $\theta$ | 0.8   | 0.416    | 0.750  | 1.1   | 1.169    | 2.222  |
| 25              | $\lambda$ | 1.3   | -0.564   | 1.592  | 3.1   | -0.740   | 1.610  |
|                 | $\alpha$ | 2.2   | 0.238    | 0.906  | 0.8   | 0.048    | 0.223  |
|                 | $\theta$ | 0.8   | 0.193    | 0.404  | 1.1   | 0.692    | 1.256  |
| 50              | $\lambda$ | 1.3   | -0.295   | 1.357  | 3.1   | -0.595   | 1.401  |
|                 | $\alpha$ | 2.2   | 0.072    | 0.546  | 0.8   | 0.011    | 0.144  |
|                 | $\theta$ | 0.8   | 0.102    | 0.286  | 1.1   | 0.460    | 0.886  |
| 100             | $\lambda$ | 1.3   | -0.145   | 1.113  | 3.1   | -0.367   | 1.174  |
|                 | $\alpha$ | 2.2   | 0.018    | 0.352  | 0.8   | 0.002    | 0.093  |
|                 | $\theta$ | 0.8   | 0.049    | 0.205  | 1.1   | 0.267    | 0.613  |
| 200             | $\lambda$ | 1.3   | -0.068   | 0.897  | 3.1   | -0.208   | 1.003  |
|                 | $\alpha$ | 2.2   | -0.004   | 0.236  | 0.8   | -0.002   | 0.063  |
|                 | $\theta$ | 0.8   | 0.024    | 0.159  | 1.1   | 0.155    | 0.447  |
7.2 Real applications

Here we provide applications for two real data sets to illustrate the abilities and flexibility of the TETLE distribution.

Data-1: The first real data available at the early detection unit for breast cancer at Benha University Hospital in Egypt for the period from June to October 2014. The data represent the ages of 155 patients with breast tumors [15].

"46, 32, 50, 46, 44, 42, 69, 31, 25, 29, 40, 42, 24, 17, 35, 48, 49, 50, 60, 26, 36, 56, 65, 48, 66, 44, 45, 30, 28, 40, 40, 50, 41, 39, 36, 63, 40, 42, 45, 31, 48, 36, 18, 24, 35, 30, 40, 48, 50, 60, 52, 47, 50, 49, 38, 30, 52, 52, 12, 48, 50, 45, 50, 50, 53, 55, 38, 40, 42, 42, 32, 40, 50, 58, 48, 32, 45, 36, 30, 28, 38, 54, 90, 80, 60, 45, 40, 50, 50, 50, 50, 50, 60, 39, 34, 28, 18, 60, 50, 20, 40, 50, 38, 38, 42, 50, 40, 36, 38, 38, 50, 50, 31, 59, 40, 42, 38, 40, 38, 38, 50, 50, 50, 40, 65, 38, 40, 38, 38, 35, 60, 90, 48, 58, 45, 35, 38, 32, 35, 38, 34, 43, 40, 35, 54, 60, 33, 35, 36, 43, 40, 45, 56".

Data-2: The second real data represents the fatigue fracture life of Kevlar 373/epoxy, which is subject to constant pressure at a 90 per cent stress level until it has all failed [16].

"0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6656, 0.6674, 0.6751, 0.5753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.0311, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960".

The TETLE fitting is compared with six distributions: Beta Exponential (BE), Kumaraswamy Exponential (KuE), Exponentiated Generalized Exponential (EGE), Weibull Exponential (WeE), Gompertz Exponential (GoE), and Exponential (E) distributions. The R software used to calculate the MLEs of the parameters, for each distribution, along with the information criteria: (-LL) negative log-likelihood, (AIC) Akaike Information Criteria, (CAIC) Consistent Akaike Information Criteria, (BIC) Bayesian Information Criteria, (HQIC) Hanan and Quinn Information Criteria. Lowest values of these information criteria indicate the distribution with a better fit and are defined as [17]

\[
\begin{align*}
AIC &= 2p - 2\ell_{max}, \\
CAIC &= \frac{2np}{n - p - 1} - 2\ell_{max}, \\
BIC &= p \ln(n) - 2\ell_{max}, \text{ and } \text{HQIC} = 2p \ln(\ln(n)) - 2\ell_{max}.
\end{align*}
\]

where

\(p\): is the number of distribution parameters, \(\ell_{max}\): is the log-likelihood function evaluated at MLEs, and \(n\): is the sample size.

From the results that are shown in Tables 2 – 5, it is seen clearly that the lowest values of information criteria are associated with TETLE, which makes it the most fitting to represent two data sets compared to other distributions. Furthermore, the best fitting of TETLE can be seen through the plots of the histogram and empirical cdfs shown in Figures 3 - 6.
Table 2. The information criteria for fitting Data-1

| Dist. | -LL    | AIC    | CAIC   | BIC    | HQIC   |
|-------|--------|--------|--------|--------|--------|
| TETLE | 601.8285 | 1209.657 | 1209.816 | 1218.787 | 1213.365 |
| BE    | 605.1656 | 1216.331 | 1216.490 | 1225.461 | 1220.040 |
| KuE   | 602.9228 | 1211.846 | 1212.005 | 1220.976 | 1215.554 |
| EGE   | 611.2447 | 1228.489 | 1228.648 | 1237.620 | 1232.198 |
| WeE   | 610.2967 | 1226.593 | 1226.752 | 1235.724 | 1230.302 |
| GoE   | 634.5340 | 1275.068 | 1275.227 | 1284.198 | 1278.777 |
| E     | 740.3172 | 1482.634 | 1482.661 | 1485.678 | 1483.871 |

Table 3. The values of MLE to Data-1

| Dist. | \( \hat{\lambda}_{ML} \) | \( \hat{\alpha}_{ML} \) | \( \hat{\theta}_{ML} \) |
|-------|-----------------|-----------------|-----------------|
| TETLE | -4.7268         | 16.1016         | 0.0546          |
| BE    | 15.6984         | 4.0456          | 0.0386          |
| KuE   | 9.6246          | 5.8559          | 0.0372          |
| EGE   | 0.6496          | 25.6217         | 0.1333          |
| WeE   | 3.6871          | 6.0792          | 0.1264          |
| GoE   | 0.0528          | 1.0513          | 0.0585          |
| E     | ---             | ---             | 0.0229          |

Table 4. The information criteria for fitting Data-2

| Dist. | -LL    | AIC    | CAIC   | BIC    | HQIC   |
|-------|--------|--------|--------|--------|--------|
| TETLE | 121.2623 | 248.525 | 248.858 | 255.517 | 251.319 |
| BE    | 122.2276 | 250.455 | 250.788 | 257.447 | 253.249 |
| KuE   | 122.0942 | 250.188 | 250.522 | 257.181 | 252.983 |
| EGE   | 122.2436 | 250.487 | 250.820 | 257.479 | 253.282 |
| WeE   | 122.5247 | 251.049 | 251.383 | 258.042 | 253.844 |
| GoE   | 125.3744 | 256.749 | 257.082 | 263.741 | 259.543 |
| E     | 127.1143 | 256.229 | 256.283 | 258.559 | 257.160 |

Table 5. The values of MLE to Data-2

| Dist. | \( \hat{\lambda}_{ML} \) | \( \hat{\alpha}_{ML} \) | \( \hat{\theta}_{ML} \) |
|-------|-----------------|-----------------|-----------------|
| TETLE | -5.6132         | 0.4483          | 0.4180          |
| BE    | 1.6792          | 1.5236          | 0.4806          |
| KuE   | 1.5554          | 2.4608          | 0.3266          |
| EGE   | 0.4983          | 1.7095          | 1.4104          |
| WeE   | 1.3256          | 2.4310          | 1.1398          |
| GoE   | 0.5149          | 0.1521          | 0.7992          |
| E     | ---             | ---             | 0.5104          |
8 Conclusions
A newly generated family of continuous distributions with Topp Leone is introduced. Then a truncated distribution as a sub-model with three parameters called Truncated Exponential Topp Leone Exponential (TETLE) is proposed. Reliability characteristics with several main properties are presented. The TETLE distribution applied upon simulation besides two real applications with different information criteria. The results of simulation clearly shown the flexibility and consistent performance of the maximum likelihood estimates and the results of real applications clearly shown that the proposed distribution has outstanding performance than other considered distributions for all criteria. This flexibility allows using the TETLE distribution in various application areas. Furthermore, for future work, through using the same new family presented here, one can derive another new continuous probability distributions.

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