THE CALCULUS OF MULTIVECTORS
ON NONCOMMUTATIVE JET SPACES

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Abstract. We state and prove the main properties of the Schouten bracket on jet spaces in a class of noncommutative geometries. Our reasoning confirms that the traditional differential calculus does not appeal to the (graded-)commutativity assumption in the setup. We show that a chosen type of noncommutative differential calculus is essentially topological and renders an axiomatic noncommutative field theory model.

Introduction. With this report we complement the brief communication [5] in which we related a class of noncommutative geometries to associative algebras by merging the approach of [11] with that of [15]. We now prove in full detail the main message in [5]: the cyclic-invariance relation is sufficient for a construction of differential calculus of variational multivectors over (non)commutative jet spaces. In other words, we remove a much more restrictive hypothesis of the ($\mathbb{Z}_2$-graded) commutativity in the models of mathematical physics.

We formulate the definition of the Schouten bracket ([13] and [16], c.f. [2]) and establish its main properties in a way that does not appeal to the commutativity in the rings of smooth functions on the manifolds at hand. Instead, we consider the spaces of infinite jets $J^\infty(M^n_{nC} \to A)$ of mappings from (non)commutative (non)graded manifolds $M^n_{nC}$ to the factors $A$ of free associative $\mathbb{k}$-algebras (e.g., $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$) over the relation of equivalence under cyclic permutations of letters in the words. Such setup [5] is a proper noncommutative generalization of the standard jet bundle geometry for commutative manifolds [14]; this direction of research was pioneered in [15] for purely commutative domains $M^n$ and for the targets $A$ as specified above. If, at the end of the day, the target algebra $A$ is proclaimed (graded-)commutative, one restores the standard, Gel’fand–Dorfman calculus of variational multivectors [3, 4]. (We compare several definitions of the variational Schouten bracket in the commutative setup and verify their equivalence in [9].) Alternatively, under the shrinking of the source manifold $M^n$ to a point (or by postulating that the only admissible mappings $M^n \to A$ are constants), we return to the formal noncommutative geometry of [11].

We adopt the conventions of [5] and [6] and follow the notation in the papers [5, 7, 9], which contain extra references.
This note is organized as follows. We first formalize the language $A$ of cyclic words and describe the multiplicative structures in it. Next, we consider the fully noncommutative jet spaces $J^\infty(M_{n \mathbb{C}}^n \to A)$ and rephrase the standard notions of differential calculus in this setup. The noncommutative variational Schouten bracket $[,]$ is then defined as the odd Poisson bracket on the noncommutative superspace which is cotangent to $J^\infty(M_{n \mathbb{C}}^n \to A)$, c.f. [2] and [15]. Relating the odd evolutionary vector fields $Q^\xi$ to the operations $[\xi,]$, where $\xi$ is a noncommutative variational multivector [12], we affirm the shifted-graded skew-symmetry of $[,]$ and directly verify the Jacobi identity, which stems from the Leibniz rule for the derivations $Q^\xi$ acting on the bracket $[\eta,\omega]$ of multivectors; in turn, that rule holds by virtue of the conceptual equality $[Q^\xi,Q^\eta] = Q^{[\xi,\eta]}$. Having developed the calculus of noncommutative variational $k$-vectors with arbitrary $k \in \mathbb{N} \cup \{0\}$, we focus of the Poisson formalism of bi-vectors $\mathcal{P}$ satisfying the classical master-equation $[\mathcal{P},\mathcal{P}] = 0$ and rephrase the concept in terms of noncommutative Hamiltonian operators. In particular, we confirm the substitution principle that allows us to operate with identities in total derivatives (primarily, Jacobi’s identity) for variational (co)vectors which depend in an arbitrary way not only on the base coordinates but also on the jet fibre variables. Finally, we associate homological evolutionary vector fields $Q$ and bi-vectors $\mathcal{P}$ over the jet space $J^\infty(M_{n \mathbb{C}}^n \to A)$ with skew-adjoint noncommutative Noether total differential operators with involutive images and then we derive a convenient criterion which tells us whether a given differential operator of this type is Hamiltonian.

Our reasoning is canonical in a sense that it projects down to the commutative setup without modifications. Hence we conclude that – even in a purely commutative world – the determining properties of the Schouten bracket do not refer to the commutativity assumption. The crucial observation is that, due to the Leibniz rule, the derivations of $A$ do not feel any cyclic permutations of factors in the products. Because the construction of differential calculus of smooth functions and multivectors in the cyclic-invariant setup is now accomplished, it is the requirement of invariance under arbitrary shifts of the words around the circles – but not the requirement of a graded commutativity under permutations of adjacent letters – that appears as the weakest but the most natural hypothesis in the geometry of fundamental interactions at Planck scale.

1. The language of cyclic words

Let $\text{Free}(a^1, \ldots, a^m)$ be a free associative algebra over $\mathbb{R}$ with $m$ generators $a^1, \ldots, a^m$; in the meanwhile, let this algebra be not graded. By definition, the multiplication $\cdot$ in $\text{Free}(a^1, \ldots, a^m)$ obeys the identity $(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$ for the concatenation of any words $a_1$, $a_2$ and $a_3$ written in the alphabet $a^1, \ldots, a^m$. For the sake of clarity, we discard from consideration here all other relations (if any) in the algebras at hand, leaving them free, except for the following rule: We postulate that all words $a \in \text{Free}(a^1, \ldots, a^m)$ of length $\lambda(a)$ determine the equivalence classes under the cyclic permutations $t$ of the letters $[11]$:

$$a \sim t(a) \sim \ldots \sim t^{\lambda(a)-1}(a) \sim \frac{1}{\lambda(a)} \sum_{i=1}^{\lambda(a)} t^{i-1}(a).$$
Otherwise speaking, for any words $a_1, a_2 \in \text{Free} (a^1, \ldots, a^m)$ we set

$$a_1 \cdot a_2 \sim a_2 \cdot a_1$$

(no $\mathbb{Z}_2$-grading in the free associative algebra!) and denote

$$\mathcal{A} = \text{Free} (a^1, \ldots, a^m) / \sim .$$

If a word consists of just one letter, its equivalence under the cyclic permutations is trivial. If both words $a_1$ and $a_2$ are some generators of $\text{Free} (a^1, \ldots, a^m)$, then the equivalence encodes the commutativity. However, at the length three the true noncommutativity starts and it then does not retract to the commutative setup. For instance, for any $a, b, c \in \text{Free} (a^1, \ldots, a^m)$ we have that

$$aabcc \sim abcca \sim bccaa \sim ccaab \sim caabc \not\sim abacc.$$  

Remark 1.1. Let us recall that the idea of a cyclic permutation itself implicitly refers to the transcription of symbols around the circles $S^1$. By convention, the oriented circles $S^1$ which carry the letters from the alphabet of $\mathcal{A}$ also carry the marked point which is denoted by $\infty$. Thus, each cyclic-invariant word is a necklace such that the point $\infty$ is its lock; the symbols are thread on the circle and are always read from the infinity on their left to the infinity on their right. Elements of the vector space $\mathcal{A}$ are formal sums over $\mathbb{R}$ of such words. The non-graded letters $a^1, \ldots, a^m$ can freely pass through the point $\infty$ without contribution to the sign of the coefficient in front of a word.

The cyclic invariance for the classes in $\mathcal{A}$ is fundamental. It establishes a uniform geometric approach to the following multiplicative structures and notions of differential calculus over the ring $\mathcal{A}$:

- multiplication of words and word-valued functions,
- termwise action of derivations, including
- commutation of vector fields, and also
- contraction of vectors with covectors, or
- the Schouten bracket of multivectors and
- the Poisson bracket of Hamiltonian functionals.

Indeed, all these operations amount to the familiar detach-and-join construction of a pair of topological pants $S^1 \times S^1 \to S^1$. We hence conclude that noncommutative differential calculus is topology. Let us support this claim.

The natural multiplication $\times$ on the space $\mathcal{A}$ of cyclic words is as follows:

$$a_1 \times a_2 = \frac{1}{\lambda(a_1)\lambda(a_2)} \sum_{i=1}^{\lambda(a_1)} \sum_{j=1}^{\lambda(a_2)} t^{i-1}(a_1) \cdot t^{j-1}(a_2), \quad a_1, a_2 \in \mathcal{A}. \quad (1)$$

Namely, by taking the sum over all possible positions of the mark $\infty$ between the letters in each of the words, we detach the locks at $\infty$ and, preserving the orientation, join the loose ends of the first string that carries the word $a_1$ with the two ends of the other string for $a_2$. We thus endow the space $\mathcal{A}$ with the structure of an algebra over the ground field $\mathbb{R}$. The normalization by $(\lambda(a_1)\lambda(a_2))^{-1}$ matches the setup with the purely commutative world.
Example 1.1. Consider the free algebra Free$(a^1, \ldots, a^4)$ and let the two words be $a^1a^2$ and $a^3a^4$. Then we have that
\[ a^1a^2 \times a^3a^4 = \frac{1}{4}(a^1a^2a^3a^4 + a^2a^1a^3a^4 + a^3a^1a^2a^4 + a^4a^2a^1a^3) = a^3a^4 \times a^1a^2. \]

Lemma 1. The multiplication $\times$ in $A$ is commutative but in general not associative.

The proof is trivial.

2. Noncommutative Jet Spaces

Let us incorporate the space-time $M^n$ into the model (and thus, allow for a possibility to encounter the derivatives $a_\sigma = \frac{\partial^{|\sigma|}}{\partial x^\sigma}$ of the $m$ pure states $a = (a^1, \ldots, a^m)$), which motivates the introduction of the noncommutative jet bundles $J^\infty(M^n \rightarrow A)$.

Consider the set of smooth maps from an oriented $n$-dimensional real manifold $M^n$ to the algebra $A$, which is a vector space of suitable (possibly, infinite) dimension. Constructing the infinite jet space $J^\infty(M^n \rightarrow A)$ in a standard way [14], we enlarge the alphabet $a^1, \ldots, a^m$ of $A$ by the jet variables $a_\sigma$ of all finite orders $|\sigma|$, here $a_{\sigma}^j \equiv a^j$ for $1 \leq j \leq m$, and by the base variables $x$ (but we expect the theory to be invariant w.r.t. translations along $M^n$). An expansive notation for a given mapping $a = s(x)$ would be
\[ a = s^1(x) \cdot a^1 + \ldots + s^m(x) \cdot a^m, \]
where at each point $x \in M^n$ and for each $j$ we have that $s^j(x) \in \mathbb{R}$. Likewise, let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be a multi-index; the respective component of the infinite jet $j_\infty(s)$ is
\[ a_\sigma = \frac{\partial^{|\sigma|}}{\partial x^\sigma}(s^1)(x) \cdot a^1 + \ldots + \frac{\partial^{|\sigma|}}{\partial x^\sigma}(s^m)(x) \cdot a^m. \]

We denote by $C^\infty(J^\infty(M^n \rightarrow A))$ the $C^\infty(M^n)$-algebra of cyclic-invariant words written in such alphabet. For example,
\[ a^1a^2a^1 - a^2a^1a^1 \sim a^1a^1a^2 - a^1a^2a^1 \sim a^1a^1a^2a^1 \sim a^4a^2a^1a^3 \in C^\infty(J^\infty(M^n \rightarrow A)). \]

The domain of mappings to the algebra $A$ can also be a noncommutative manifold (for example, this hypothesis will simplify the proof of the substitution principle, see section 6). Anticipating a concrete application of the geometry under study in theoretical physics, we employ a proper formulation of the spectral approach $M^n \leftrightarrow C^\infty(M^n)$ to smooth manifolds, in the frames of which we interpret certain algebras as the (non)commutative analogs of the rings of smooth functions on $M^n$ (here, also on the $m$-dimensional fibres $N^m$ of the bundles $\pi: E^{m+n} \rightarrow M^n$).

Namely, we first consider the tangent bundle $TM^n \rightarrow M^n$ and denote by $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ a field of $n$-tuples of vectors which span the tangent space $T_xM^n$ at each point $x \in M^n$. We let the elements $\tilde{x}_i$ together with their inverses $\tilde{x}_i^{-1} = 1/\tilde{x}_i = (-\tilde{x}_i)$ be the $2n$ generators of (a sheaf over $M^n$ of) unital $C^\infty(M^n)$-algebras with relations including $\tilde{x}_i\tilde{x}_i^{-1} = 1 + (\text{energy})$, where $(\text{energy})$ yields a zero-length cyclic word, but possibly also including many other relations of geometric and physical origin such as, for example, $\tilde{x}_i\tilde{x}_j\tilde{x}_j^{-1}\tilde{x}_i^{-1}\tilde{x}_i = R_{ijk}(x)\tilde{x}_k$. Because we shall address the physical aspects of this model in a separate paper, let us now not consider those extra relations and the arising problem of synonyms. Nevertheless, we notice that the cyclic words
\(∞\tilde{x}_{i_1}^\pm \cdot \cdot \cdot \tilde{x}_{i_ℓ}^\pm \infty\) encode the consecutive displacements, that is, paths (in particular, knots) along the Planck-scale grid inside the fibre of the tangent bundle over each point \(x \in M^n\) of space-time.

Next, we endow each unital algebra generated by \(\tilde{x}\) and \(\tilde{x}^{-1}\) with \(n\) commuting left derivations \(\delta^j = \partial/\partial \tilde{x}_j\) so that \(\delta^j(\tilde{x}_j) = \delta^j_j\) and \(\delta^j(\tilde{x}_k) \equiv 0\). Gluing such pointwise-defined algebras over \(x \in M^n\), we obtain the \(C^∞(M^n)\)-algebra \(C^∞(M^n_{\text{NC}})\); it inherits from the ring \(C^∞(M^n)\) the \(n\) commuting left derivations \(\delta_i = \partial/\partial x^i\) such that \(\delta_i(x^i) = \delta^j_i\) and \(\delta_i(\tilde{x}_k) \equiv 0\).

Let us regard the new, noncommutative algebra as the ring of smooth functions on the noncommutative manifold which we denote by \(M^n_{\text{nc}}\) and which we take for the base of a fully noncommutative vector bundle \(\pi^{\text{nc}}\). The sections \(s \in \Gamma(\pi^{\text{nc}})\) become \(m\)-tuples of open strings,

\[a = (a^j = \sum_i s^i(x; i) \cdot \tilde{x}_{i_1}^\pm \cdot \cdot \cdot \tilde{x}_{i_ℓ}^\pm, \quad 1 \leq j \leq m),\]

where the multiplication \(\cdot\) in \(C^∞(M^n_{\text{NC}})\) is associative but not commutative if \(\ell > 0\); by default, \(\ell = 0\) means (the image under inclusion of) an element from \(C^∞(M^n)\).

Finally, we extend the alphabet of \(A \simeq C^∞(N^n_{\text{nc}})\) to the set of generators in \(C^∞(J^∞(M^n_{\text{nc}} → A))\) by incorporating the base elements \(x^1, \ldots, x^n\) and \(\tilde{x}_1, \ldots, \tilde{x}_n\) and also introducing the symbols \(\tilde{D}^\sigma(a^\sigma)\) for \(1 \leq j \leq m\) and all multi-indexes \(\sigma\) and \(\tau\), here \(\tilde{D}^\sigma(a^\sigma) \equiv a^j\).

The definition

\[
(x, \tilde{x}) \cdot (j_∞(s)^*(\tilde{D}/dx^i(f))) = \tilde{\partial}/\partial x^i((x, \tilde{x}) \cdot j_∞(s)^*(f)),
\]

\[
(x, \tilde{x}) \cdot (j_∞(s)^*(\tilde{D}^\sigma(f))) = \tilde{\partial}_i/\partial \tilde{x}_i((x, \tilde{x}) \cdot j_∞(s)^*(f))
\]

of the total derivatives \(\tilde{D}/dx^i = \tilde{\partial}_i\) and \(\tilde{D}^\sigma = \tilde{\partial}^\sigma\) for a differential function \(f\) on \(J^∞(M^n_{\text{nc}} → A)\) is reproduced literally in the fully noncommutative picture, see [2] below.

We emphasize that the chain of inclusions \(C^∞(M^n) → C^∞(M^n_{\text{NC}}) → C^∞(J^∞(M^n_{\text{nc}} → A))\) is the source of words of length zero both in \(C^∞(M^n_{\text{NC}})\), see previous paragraph, and in the fibre algebras \(A\) which now contain the necklaces with strings of \(\tilde{x}\)'s and the symbol \(∞\) at the lock.\(^1\) We note that the multiplication \(\times\) of cyclic words of zero length, i.e., a restriction of \(\times\) from \(C^∞(J^∞(M^n_{\text{nc}} → A))\) to the image under inclusion of the space \(C^∞(M^n_{\text{nc}})\) of open strings remains associative but not commutative, in contrast with Lemma [1]. Indeed, a structural difference between the variables \(\tilde{x}\) and \(\tilde{D}^\sigma(a)\) is that the former never pass through the marked point \(∞\) on \(S^1\): the intermediate order from the first \(\tilde{x}_{i_1}\) to the last \(\tilde{x}_{i_ℓ}\) is not broken. The following convention is thus important: under the multiplication \(\times\) those open strings \(\tilde{x}_{i_1}^\pm, \ldots, \tilde{x}_{i_ℓ}^\pm\) are pasted immediately after (measured counterclockwise) the lock at \(∞\) of the second factor but strictly before any other symbols of the second word (i.e., preceding a letter \(\tilde{x}_i\) or generator \(a^j\), if any).

\(^1\) The presence of zero-length cyclic words in \(C^∞(J^∞(M^n_{\text{nc}} → A))\) is manifest even if the base of \(π^{\text{nc}}\) is commutative; this is a generalization of the standard fact from a purely commutative jet bundle geometry [13]: the ring \(C^∞(M^n)\) plays the rôle of the ground field in differential calculus over \(J^∞(π)\), though known conceptual difficulties appear at exactly this point, c.f. [7].
We now upgrade necessary definitions to noncommutative geometry of the jet space $J^\infty(\pi^{nC})$; we emphasize that all the constructions and the statements which we establish in what follows for the new, wider setup obviously remain true under the stronger, more restrictive assumptions of the (graded-)commutativity.

The total derivatives w.r.t. $x^i$ and $\bar{x}_i$, $1 \leq i \leq n$, on $J^\infty(\pi^{nC})$ are expressed by the respective formulas
\begin{equation}
\begin{aligned}
\frac{\overrightarrow{d}}{dx^i} &= \frac{\overrightarrow{d}}{\partial x^i} + \sum_{|\sigma|,|\tau| \geq 0} \overrightarrow{D}^{\tau}(a_{\sigma+1}) \frac{\overrightarrow{d}}{\partial \overrightarrow{D}^\tau(a_\sigma)}, \\
\overrightarrow{D}^i &= \frac{\overrightarrow{d}}{\partial \bar{x}_i} + \sum_{|\sigma|,|\tau| \geq 0} \overrightarrow{D}^{\tau+1}(a_\sigma) \frac{\overrightarrow{d}}{\partial \overrightarrow{D}^\tau(a_\sigma)}.
\end{aligned}
\end{equation}

The evolutionary derivations
\[ \overleftarrow{\partial}_\varphi^{(a)} = \sum_{|\sigma|,|\tau| \geq 0} \left( \overrightarrow{d}^{\sigma}/\partial x^\sigma \circ \overrightarrow{D}^\tau \right)(\varphi) \cdot \overleftarrow{\partial} / \partial \overrightarrow{D}^\tau(a_\sigma) \]
act from the left by the Leibniz rule.\(^2\)

Denote by $\overrightarrow{\mathcal{H}}^n(\pi^{nC})$ the space of horizontal forms of the highest ($n$-th) degree and by $\overrightarrow{\mathcal{H}}^0(\pi^{nC})$ the respective cohomology w.r.t. the horizontal differential $\overrightarrow{d} = \sum_{i=1}^n \overrightarrow{d} \cdot \frac{\overrightarrow{d}}{dx^i}$; the Cartan differential on $J^\infty(\pi^{nC})$ is $d_\mathcal{C} = d_{\mathcal{D}} - \overrightarrow{d}$; by convention, the volume form $d\pi \in \overrightarrow{\mathcal{H}}^0(\pi^{nC})$. Denote by $\langle \cdot, \cdot \rangle$ the $\overrightarrow{\mathcal{H}}^n(\pi^{nC})$-valued coupling between the spaces of variational covectors $p$ and evolutionary vectors $\overleftarrow{\partial}_\varphi^{(a)}$. By default, we pass to the cohomology and, using the integration by parts, normalize the (non)commutative covectors as follows, $p((x, \bar{x}), [a]) = \sum_{j=1}^n \langle \text{word} \rangle \cdot d_\mathcal{C} a_j$. We note that the derivatives $\overleftarrow{\partial} / \partial a_j$ and the differentials $d_\mathcal{C} a_j$ play the rôle of the marked points $\infty$ in the cyclic words $\overleftarrow{\partial}_\varphi^{(a)}$ and $p$, respectively. The coupling $\langle p, \overleftarrow{\partial}_\varphi^{(a)} \rangle$ of the two oriented circles $\mathbb{S}^1$ carrying $p$ and $\overleftarrow{\partial}_\varphi^{(a)}$ goes along the usual lines of detaching the locks at the marked points and then joining the loose ends, preserving the orientation. (Again, this constitutes the pair of topological pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.)

Let $A: p \mapsto \overleftarrow{\partial}_A^{(p)}$ be a Noether noncommutative linear matrix operator in total derivatives, containing –apart from the total derivatives– the operators $(\cdot a)$ and $(a \cdot)$ of left- and right-multiplication by open words $a \in C^\infty(J^\infty(\pi^{nC}))$ that are always read from left to right. The adjoint operator $\overrightarrow{A}^\dagger$ is defined from the equality $\langle p_1, A(p_2) \rangle = \langle p_2, \overrightarrow{A}^\dagger(p_1) \rangle$, in which we first integrate by parts and then transport the even covector $p_2$ around the circle. Let us remark that the operators are “measured from the comma” in the coupling $\langle \cdot, \cdot \rangle$: namely, the right multiplication in the counterclockwise-acting operator $\overrightarrow{A}^\dagger$ corresponds to the left multiplication in the clockwise-acting $\overrightarrow{A}^\dagger$, and vice versa.

\(^2\)By construction, noncommutative evolutionary derivations determine flat deformations of sheafs of the algebras $A$ over the base manifolds $M^\infty_{nC}$. 

3. Noncommutative variational multivectors

The covectors \( p((x, \bar{x}), [a]) \) were even. We reverse their parity, \( \Pi: p \mapsto b((x, \bar{x}), [a]) \), preserving the topology of the bundles but postulating that all objects are at most polynomial in finitely many derivatives of \( b \). Next, we consider the noncommutative variational cotangent superspace \( \mathcal{J}^\infty(\Pi\hat{\pi}^n) = J^\infty(\Pi\hat{\pi}^n) \times_M J^\infty(\pi^n) \), see [14] and [9]. In effect, we declare that \( b, b_x, b_{xx}, \ldots, \tilde{D}^e(b) \) are the extra, odd jet variables on top of the old, even \( \tilde{D}(a_s) \)'s. The total derivatives \( \frac{\delta}{\delta x^i} \) obviously lift onto \( J^\infty(\Pi\hat{\pi}^n) \) as well as \( \delta \) that yields the cohomology \( \tilde{H}^k(\Pi\hat{\pi}^n) = \Lambda^k(\Pi\hat{\pi}^n)/({\mathrm{im}}\delta) \).

The two components of the evolutionary vector fields \( Q = \tilde{\partial}^a_{\bar{a}} + \tilde{\partial}^b_{\mu} \) now begin with \( \tilde{a} = \varphi^a((x, \bar{x}), [a], [b]) \) and \( \tilde{b} = \varphi^b((x, \bar{x}), [a], [b]) \).

The definition of noncommutative variational \( k \)-vectors, their evaluation on \( k \) covectors, the definition of the noncommutative variational Schouten bracket, and its inductive calculation are two pairs of distinct concepts.

We would like to define a noncommutative \( k \)-vector \( \xi \), \( k \geq 0 \), as a cohomology class in \( \tilde{H}^k(\Pi\hat{\pi}^n) \) whose density is \( k \)-linear in the odd \( b \)'s or their derivatives. This is inconsistent because for every \( k \) a given \( k \)-vector can be in fact cohomologically trivial. There are two ways out: either by a direct inspection of the values of multivectors on all \( k \)-tuples of covectors (the evaluation is defined in formula (3) below, but this option is inconvenient) or by using the normalization of \( \xi \).

**Definition 1.** A noncommutative \( k \)-vector \( \xi \) is the horizontal cohomology class of the element

\[
\xi = \langle b, A(b, \ldots, b) \rangle/k! ,
\]

where the noncommutative total differential operator \( A \) depends on \((k-1)\) odd entries and may have \( a \)-dependent coefficients.

Integrating by parts and pushing the letters of the word \( \xi \) along the circle, we infer that \( \langle b_1, A(b_2, \ldots, b_k) \rangle|_{b_i=b}= \langle b_2, A_1^b(b_3, \ldots, b_k, b_1) \rangle|_{b_i=b}; \) note that, each time an odd variable \( b_r \) reaches a marked point \( \infty \) on the circle, it counts the \( k-1 \) other odd variables whom it overtakes and reports the sign \((-)^{k-1} \) (in particular, \( A_1^{b_r}= A^r = -A \) if \( k=2 \)).

**Definition 2.** The value of the \( k \)-vector \( \xi \) on \( k \) arbitrary covectors \( p_i \) is

\[
\xi(p_1, \ldots, p_k) = \frac{1}{k!} \sum_{s \in S_k} (-)^{|s|} \langle p_{s(1)}, A(p_{s(2)}, \ldots, p_{s(k)}) \rangle .
\] (3)

We emphasize that we shuffle the arguments but never swap their slots, which are built into the cyclic word \( \xi \).

4. Noncommutative Schouten bracket

The commutative concatenation (1) of the densities of two multivectors provides an ill-defined product in \( \tilde{H}^k(\Pi\hat{\pi}^n) \). The genuine multiplication in the algebra \( \tilde{H}^k(\Pi\hat{\pi}^n) \) is the odd Poisson bracket (the antibracket). We fix the Dirac ordering \( \delta a \wedge \delta b \) over each \( x \) in \( J^\infty(\Pi\hat{\pi}^n) \rightarrow M^n \); note that \( \delta a \) is a covector and \( \delta b \) is an odd vector so that their coupling equals \(+1 \cdot dx\).
Definition 3. The noncommutative variational Schouten bracket of two multivectors $\xi$ and $\eta$ is

$$[\xi, \eta] = \langle \delta \xi \wedge \delta \eta \rangle.$$  \hfill (4)

In coordinates, this yields

$$[\xi, \eta] = \left[ \frac{\delta \xi}{\delta a} \cdot \frac{\delta \eta}{\delta b} - \frac{\delta \xi}{\delta b} \cdot \frac{\delta \eta}{\delta a} \right],$$

where

1. all the derivatives are thrown off the variations $\delta a$ and $\delta b$ via the integration by parts, then
2. the letters $a_\sigma$, $b_\tau$, $\delta a$, and $\delta b$, which are thread on the two circles $\delta \xi$ and $\delta \eta$, spin along these rosaries so that the variations $\delta a$ and $\delta b$ match in all possible combinations, and finally,
3. the variations $\delta a$ and $\delta b$ detach from the circles and couple, while the loose ends of the two remaining open strings join and form the new circle.

Lemma 2. The Schouten bracket is shifted-graded skew-symmetric: if $\xi$ is a $k$-vector and $\eta$ is an $\ell$-vector, then $[\xi, \eta] = -(-)^{(k-1)(\ell-1)}[\eta, \xi]$.

Proof. It is obvious that the brackets $[\xi, \eta]$ and $[\eta, \xi]$ contain the same summands which can differ only by signs; now it is our task to calculate these factors and show that the same signs appear at all the summands simultaneously. Let us compare the terms

$$\frac{\delta \xi}{\delta a} \langle \delta a, \delta b \rangle \frac{\delta \eta}{\delta b} \quad \text{and} \quad -\frac{\delta \eta}{\delta b} \langle \delta a, \delta b \rangle \frac{\delta \xi}{\delta a},$$

containing the variation of a $k$-vector $\xi$ with respect to the even variables $a$ and the variation of an $\ell$-vector $\eta$ with respect to the odd entries $b$. We first note that the left-to-right transportation of the differential $\delta b$ along the queue of $\ell - 1$ odd elements $b_\tau$ in the variation of $\eta$ produces the sign $(-)^{\ell-1}$. The variations disappear in the coupling $\langle \delta a, \delta b \rangle = +1$, so there remains to carry, via the infinity $\infty$, the object $\delta \eta/\delta b$ of parity $(-)^{\ell-1}$ around the object $\delta \xi/\delta a$, which is $k$-linear in the odd variables after the variation with respect to the even $a$. This yields the sign $(-)^{(\ell-1)-1}$. Therefore, the overall difference in the sign between the terms $\frac{\delta \xi}{\delta a} \cdot \frac{\delta \eta}{\delta b}$ and $\frac{\delta \eta}{\delta b} \cdot \frac{\delta \xi}{\delta a}$ equals

$$-(-)^{\ell-1} \cdot (-)^{(\ell-1)-1} = -(-)^{(k+1)(\ell-1)} = -(-)^{(k-1)(\ell-1)}.$$  

Due to the $(k \leftrightarrow \ell)$-symmetry of the exponent, the same sign factor matches the other pair of summands in the Schouten bracket, namely, $\frac{\delta \xi}{\delta b} \cdot \frac{\delta \eta}{\delta a}$ and $\frac{\delta \eta}{\delta a} \cdot \frac{\delta \xi}{\delta b}$. The proof is complete. \hfill \Box

Define the evolutionary vector field $Q^\xi$ on $\mathcal{T}(\Pi \tilde{\pi}^n C)$ by the rule

$$Q^\xi(\eta) = [\xi, \eta],$$  \hfill (5)

whence $Q^\xi = -\frac{\delta (a)}{\delta \xi/\delta b} + \frac{\delta (b)}{\delta \xi/\delta a}$. The normalization $\xi = \langle b, A(b, \ldots, b) \rangle / k!$ determines

$$Q^\xi = -(-)^{k-1} \frac{1}{(k-1)!} \frac{\delta (a)}{A(b, \ldots, b)} + (-)^{k-1} \frac{1}{k!} \frac{\delta (b)}{A(b, \ldots, b)} \bigg|_{b_i = b_i},$$

where for all (non)commutative objects $\varphi$ we denote by $\ell_{\varphi}^\dagger$ the adjoint to the linearization, which is $\ell_{\varphi}^\dagger(\delta a) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \varphi((x, \tilde{x}), [a + \varepsilon \delta a]).$
Example 4.1 (see [6, 7]). We have that
\[ Q^{\frac{1}{2}}_{(a), (b)} = \partial_{A}^{(a)} - \frac{1}{2} \partial_{A}^{(b)} \partial_{A}^{(a)} (b). \] (6)

Proposition 3. The equality
\[ [Q, Q] = Q^{[\xi, \eta]} \] (7)
correlates the graded commutator of graded evolutionary vector fields with the the noncommutative variational Schouten bracket of two multivectors.

Proof. To verify Eq. (7), it suffices to inspect the composition of the variations
\[ \dot{\delta} ([\xi, \eta]) = \frac{\delta ([\xi, \eta])}{\delta a} \cdot \delta a + \frac{\delta ([\xi, \eta])}{\delta b} \cdot \delta b. \]

They determine the two terms in \( Q^{[\xi, \eta]} \) and, moreover, contain the variations \( \delta a \) and \( \delta b \) standing at the right of the variational derivatives (which may themselves bear the total derivatives at their left due to the integration by parts).

Suppose that initially the \( k \)-vector \( \xi \) was written in red ink and \( \eta \) in black. Let us also agree that the colour of the ink is preserved in all the formulas which involve \( \xi \) and \( \eta \) (say, the red vector field \( Q^\xi \) inserts, by the Leibniz rule, the red sub-words in the black word \( \eta \)). It is readily seen that some variations \( \delta a \) or \( \delta b \) in \( \dot{\delta} ([\xi, \eta]) \) are red and some are black; for indeed, they stem from the argument \( [\xi, \eta] \) by the Leibniz rule, whereas some letters in the argument are red and some are black.

We now study the superposition of the two Leibniz rules: one for the variation \( \dot{\delta} \), which drags the differentials to the right, and the other rule for the evolutionary fields \( Q^\xi \) and \( Q^\eta \), which act on the arguments from the left. Consider first the application of \( Q^\xi \) to (without loss of generality) the \( a \)-component \( \dot{\delta} \eta / \delta b \) of the generating section for \( Q^\eta \) in the left-hand side of (7). The field \( Q^\xi \) is evolutionary, so it dives under all the total derivatives and acts, by the Leibniz rule, on the letters of the argument. That argument is “almost” \( \eta \) from which, by a yet another Leibniz rule for the Cartan differential, the black variation \( \delta \eta / \delta b \) of one \( b \) is taken out and transported to the right. Consequently, the graded derivation \( Q^\xi \) never overtakes that odd object \( \delta \eta / \delta b \), hence no extra sign is produced (for the even variation \( \delta a / \delta b \) in \( \dot{\delta} \eta / \delta b \), no signs would appear at all, but such permutations also never occur). The red field \( Q^\xi \) acts on the black coefficient of \( \delta \eta / \delta b \); on one hand, it differentiates — one after another — all letters in all the terms. (Note that, in particular, by the Leibniz rule it replaces the black variables \( b_{\tau} \) by their velocities, except for the variables which turned into the variations \( \delta \eta / \delta b \).)

But such elements are duly processed in the other summands of the Leibniz formula, in which other odd letters \( b_{\tau} \) yield the variations.) On the other hand, none of the red \( b \)'s from \( \xi \) in \( Q^\xi \) shows up in the form of the differential at the right. To collect the red variations \( \delta \xi / \delta b \), which contribute to the rest of the \( a \)-component of the generating section of the evolutionary field
\[ [Q^\xi, Q^\eta] = Q^\xi \circ Q^\eta - (-1)^{(k-1)(\ell-1)} Q^\eta \circ Q^\xi, \] (8)
we repeat the above reasoning for the term \( Q^\eta \circ Q^\xi \) in the graded commutator (c.f. Lemma 2).
It only remains to notice that the variations $\delta b$ of all the letters $b$ (red or black) are now properly counted. Comparing the object

$$Q^\xi(\delta n/\delta_n b) - (-)^{(k-1)(\ell-1)} Q^\eta(\delta \xi/\delta_b b)$$

with

$$\delta(Q^\xi(\eta))/\delta_n b - (-)^{(k-1)(\ell-1)} \delta(Q^\eta(\xi))/\delta_b b = \delta([\xi, \eta])/\delta b,$$

we conclude that the two expressions coincide. The same holds for the variation of $[\xi, \eta]$ with respect to $a$, which is also composed by the sum of red variations $\delta_\xi a$ and black $\delta_\eta a$. This implies equality (7).

Finally, let us act by the graded evolutionary vector field that appears in both sides of equality (7) upon a test multivector $\omega$ and then use definition (5) of the graded commutator (which captures the signs) and formula (5), which reinstates the Schouten brackets. We thus prove the other fundamental property of the bracket [ , ].

**Corollary 4.** The Leibniz rule $Q^\xi([\eta, \omega]) = [Q^\xi(\eta), \omega] + (-)^{(k-1)(\ell-1)}[\eta, Q^\xi(\omega)]$, where $\omega \in \mathcal{H}^\ell(\Pi \bar{\pi} \delta^{nc})$, is the Jacobi identity

$$[[\xi, \eta], \omega] = [[\xi, \eta], \omega] + (-)^{(k-1)(\ell-1)}[\eta, [\xi, \omega]]$$

for the noncommutative variational Schouten bracket [ , ].

## 5. Noncommutative Poisson Formalism

The differential calculus on noncommutative jet spaces which we have reported here is fully capable to serve the noncommutative Poisson formalism (but not only: other domains of its applicability are outlined in [6]). To endow the space $\mathcal{H}^\ell(\pi^{nc})$ of the functionals for the jet space $J^\infty(\pi^{nc})$ with the variational Poisson algebra structure, let us notice first that each skew-adjoint noncommutative linear total differential operator $A: p \mapsto \overrightarrow{\partial}^{(a)}_{A(p)}$ yields the bi-vector $\mathcal{P} = \frac{1}{2}(b, A(b))$. Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be zero-vectors, i.e., $\mathcal{H}_i = \int h_i((x, \bar{x}), [a]) \, dx \in \mathcal{H}^n(\pi^{nc})$. By definition, put

$$\{\mathcal{H}_i, \mathcal{H}_j\}_{\mathcal{P}} := \mathcal{P}(\overrightarrow{\partial}^{(a)}_{\mathcal{H}_i}/\delta a, \overrightarrow{\partial}^{(a)}_{\mathcal{H}_j}/\delta a),$$

which equals

$$\langle \overrightarrow{\partial}^{(a)}_{\mathcal{H}_i}/\delta a, A(\overrightarrow{\partial}^{(a)}_{\mathcal{H}_j}/\delta a) \rangle = \overrightarrow{\partial}^{(a)}_{A(\overrightarrow{\partial}^{(a)}_{\mathcal{H}_j}/\delta a)}(\mathcal{H}_i) \quad (\text{mod im } \overrightarrow{\partial}).$$

The bracket $\{ , \}_{\mathcal{P}}$ is bilinear and skew-symmetric by construction; however, the bracket $\{ , \}: \overrightarrow{\mathcal{H}}^\ell(\pi^{nc}) \times \overrightarrow{\mathcal{H}}^\ell(\pi^{nc}) \to \overrightarrow{\mathcal{H}}^\ell(\pi^{nc})$ does not in general restrict as a bi-derivation to the horizontal cohomology with respect to $\overrightarrow{\partial}$.

**Definition 4.** Bracket $[ , ]_{\mathcal{P}}$ is Poisson if it satisfies the Jacobi identity

$$\sum_{s} \{\{\mathcal{H}_1, \mathcal{H}_2\}_A, \mathcal{H}_3\}_A = 0,$$

which also is

$$\sum_{s \in S_3} (-)^{|s|} \overrightarrow{\partial}^{(a)}_{A(\overrightarrow{\partial}^{(a)}_{\mathcal{H}_s}/\delta a)}\left(\frac{1}{2} \langle \overrightarrow{\partial}^{(a)}_{\mathcal{H}_{s(1)}/\delta a}, A(\overrightarrow{\partial}^{(a)}_{\mathcal{H}_{s(2)}/\delta a}) \rangle \right) = 0;$$

the operator $A$ in $\mathcal{P} = \frac{1}{2}(b, A(b))$ is then called a Hamiltonian operator.
For each evolutionary vector field $\tilde{\partial}_{\varphi}^{(a)}$, the induced velocity $\dot{p} = L_{\tilde{\partial}_{\varphi}^{(a)}}(p)$ of a variational covector $p((x, \bar{x}), [a])$ equals $\dot{p} = \tilde{\partial}_{\varphi}^{(a)}(p) + (p)(\ell_{\varphi}^{(a)})$, where $\tilde{\partial}_{\varphi}^{(a)}$ acts on $p$ componentwise.

The proof is straightforward: it amounts to a definition of the Lie derivative $L$.

**Remark 5.1.** The tempting notation $\tilde{\partial}_{A(b)}^{(a)}(\mathcal{P})(\otimes \frac{\delta \mathcal{H}}{\delta A}) = 0$ is illegal by Lemma 5, which forbids us to set $\dot{p} \equiv 0$ at will so that the vector field $\tilde{\partial}_{\varphi}^{(a)}$ would be ill-defined on $\mathcal{T}^{\infty}(\Pi_{\mathbb{C}})$. Nevertheless, the Jacobi identity (11) is equivalent to the classical (non)-commutative master-equation

$$Q^{\mathcal{P}}(\mathcal{P}) = [\mathcal{P}, \mathcal{P}] = 0$$

upon the noncommutative variational Poisson bi-vector $\mathcal{P}$.

**Example 5.1.** Every skew-adjoint linear total differential operator $A: p \mapsto \tilde{\partial}_{A(p)}^{(a)}$ whose coefficients belong to the sub-ring $C^{\infty}(M_{nC})$—in particular, with constant coefficients—is a Hamiltonian operator.

Satisfying master-equation (12), each Poisson bi-vector $\mathcal{P}$ yields the Poisson–Lichnerowicz differential $Q^{\mathcal{P}}$ and the respective cohomology. The cohomological Lenart–Magri scheme (see § and references therein) then provides us with a concept of noncommutative infinite-dimensional completely integrable systems.

**Remark 5.2.** One has to pay due attention to the understanding of noncommutative evolution equations because of the subtle difference between the equalities of words, so that the velocities are defined up to cyclic permutations, and the equalities of corresponding evolutionary vector fields, which are uniquely defined by the derivatives that are firmly attached to the marked points $\infty$ on the circles $\mathbb{S}^1$.

**Remark 5.3.** An extension of a given commutative differential equation to the noncommutative (e.g., to cyclic-invariant) setup is a nontrivial problem if one requires that the richness of the original equation’s geometry must be preserved. Meaningful noncommutative generalizations are known for the Burgers, KdV, modified KdV, the nonlinear Schrödinger equation (NLS), and for several other commutative integrable systems (e.g., see the seminal paper [15]). Nevertheless, we stress that this particular transition between the two worlds, (graded-)commutative $\mapsto$ associative, may not be appropriately called a “quantization.”

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³Let us recall that each cyclic word describes the field $A([a(x, \bar{x})])$ which co-exists in the states $A$, $t(A)$, $\ldots$, $t^{\lambda(A)}^{-1}(A)$ at all points of the space-time $M^n$. The interaction $|A_1\rangle \otimes |A_2\rangle \mapsto |A_1 \times A_2\rangle$ is such that it does not matter (up to left $=$ right mod $O(h)$) whether $|A_1\rangle$ scatters on $|A_2\rangle$ or vice versa because the multiplication $\times$ in $A$ is commutative. (The new word $A_1 \times A_2$ could then simplify under a factorization over the extra relations, which we do not address here.) It then becomes a nontrivial task to restore the associativity of the scattering, i.e., the independence of the out-going state from the order of two pairwise collisions. The deformation quantization technique [10] which extends the multiplication $\times$ to the power series $\ast = x + h \cdot \{ , \}_{\text{Poisson}} + o(h)$ in the Planck constant $h$ is the key to solution of the triangle equation $|1\rangle \times (|2\rangle \times |3\rangle) = (|1\rangle \times |2\rangle) \times |3\rangle$. Therefore, it is the matching of the two channels in the associative scattering which has a quantum origin.
6. Noncommutative Substitution Principle

In order to make the proofs of known convenient reformulations of the Jacobi identity \([\mathcal{P}, \mathcal{P}] = 0\) (e.g., see the next section), let us establish the substitution principle [14, Problem 5.32] in the fully noncommutative setup.

**Remark 6.1.** We emphasize that for the substitution principle to hold one does not require that the setup is commutative. Moreover, the noncommutative substitution principle hints us why the base of the bundle \(\pi^n\) should itself be a noncommutative manifold: indeed, the proof then becomes particularly transparent, also specifying as the only admissible the class of horizontal bundles over \(J^\infty(\pi^n)\) whose sections \(p\) naturally bear the marked points \(\infty\). (For instance, such is the case of noncommutative variational (co)vectors: the derivatives \(\delta \partial / \partial \mathcal{D}(a)\) in a vector and the differentials \(d\mathcal{D}(a)\) in a covector play the exclusive rôle of the marked points \(\infty\) on the circles that carry the cyclic words in the formal sum \(p\). Consequently, the position of the lock on such necklaces \(S\) unambiguously determines the rules according to which the sections \(p\) are never split by any other words in the course of multiplication.) We note that the assertion is not valid for sections \(p((x, \bar{x}), [a])\) of unspecified nature.

Let us recall from section 2 that the variables \(x^a_i\), \(1 \leq i \leq n\), are the local generators of associative unital \(C^\infty(M^n)\)-algebras \(C^\infty(M^n)\). This means that we shall not attempt swapping the adjacent letters \(x^a_i\) and \(x^a_{i(a \mod n) + 1}\) in a word \(f(x) \cdot x^a_{i1} \cdot \ldots \cdot x^a_{in}\).

**Theorem 6 (The Substitution Principle).** Suppose that a tuple of identities

\[
I (\langle x, \bar{x}\rangle, [a], [p(x, \bar{x})]) \equiv 0
\]

holds on \(J^\infty(\pi^n)\) for every noncommutative variational (co)vector \(p(x, \bar{x})\) whose coefficients belong to the subring \(C^\infty(M^n) \hookrightarrow C^\infty(J^\infty(\pi^n))\). Then the identities

\[
I (\langle x, \bar{x}\rangle, [a], [p((x, \bar{x}), [a])]) \equiv 0
\]

in total derivatives with respect to \(p\) are valid on \(J^\infty(\pi^n)\) for all (local) sections \(p\) depending not only on \(x\) and \(\bar{x}\) but also endowed with arbitrary, finite differential order dependence on the unknowns \(a\) and their derivatives.

**Corollary 7.** If, under the assumptions of Theorem 6, the identities \(I (\langle x, \bar{x}\rangle, [a], [p]) \equiv 0\) in total derivatives with respect to \(p\) hold on \(J^\infty(\pi^n)\) for exact variational covectors \(p = \delta \mathcal{H} / \delta a\) which are obtained by variation of arbitrary functionals \(\mathcal{H} \in \mathcal{P}(\pi^n)\), then these identities hold for all covectors \(p \in \Gamma((\pi^n)^*_{\pi^n}(\bar{\pi^n}))\), i.e., not necessarily exact.

Indeed, it is always possible to represent locally an \((x, \bar{x})\)-dependent section \(\sum_{j=1}^m p_j(x, \bar{x}) \cdot \delta a^j\) as the right variation \(\delta \mathcal{H}\) of the functional \(\sum_{j=1}^m \int p_j(x, \bar{x}) \cdot a_j\ dx\) and then apply Theorem 6.

**Proof of Theorem 6.** For the sake of brevity, let \(p\) consist of just one word written in the alphabet of \(C^\infty(J^\infty(\pi^n))\). The crucial idea is that the position of the lock \(\infty\) is fixed on the circles which carry the word \(p\). This means that, whenever one declares an arbitrary differential dependence of \(p\) on \(a\), the words \(I\) in principle lengthen but still \(p\) is never torn in between every occurrence of the letters \(a\) under the multiplication \(\times\).
Namely, during the evaluation of \( I \) the word \( p \) is unlocked, the letters \( \bar{x} \) and \( \tilde{D}^r(a_\sigma) \) and the coefficient which depends on \( x \) are then stretched to an open string (ordered counterclockwise) and finally, this string is pasted into \( I \) without splitting, i.e., the adjacent letters of \( p \) never become separated by any other symbols. (We note that this scenario is realized irrespectively of a presence or absence of \( a \)'s in the factor \( p \), which is in contrast with formula (1).)

The total derivatives (2) then work by their definition: under a restriction of \( I \) (hence of \( p \)) onto the jet \( j_\infty(s) \) of a section \( a = s(x, \bar{x}) \), each symbol \( a' \) is replaced with the respective sum of open strings \( s^j(x, \bar{x}) \) so that left derivations (2), either in \( \tilde{D}^r(a_\sigma) \) if there is an explicit dependence of \( I \) on \( [a] \) or the derivatives falling on \( p \), then reduce to the left derivations \( \tilde{\partial}_l \) and \( \tilde{\partial}' \) in \( C^\infty(M^n_{\pi C}) \). By the initial assumption of the theorem, its assertion is valid for any strings in the basic alphabet \( (x, \bar{x}) \) replacing the entries \( p \) in \( I \). (We note that one does not even have to postulate that the sections \( a = s(x, \bar{x}) \) inserted in the explicit dependence of \( I \) on \( [a] \) coincide with the sections now standing for \( a \) in the implicit dependence \([p((x, \bar{x}), [a])]) \). Hence we conclude that the identities \( I \equiv 0 \) hold on \( J^\infty(\pi^n C) \) for the full set of arguments of the (co)vectors.

**Remark 6.2.** The proof remains literally valid in the case of evolutionary vector fields instead of variational covectors. This is important for the description of variational noncommutative symplectic structures. However, the proof reveals why this noncommutative extension of the substitution principle does not hold for arbitrary sections \( p((x, \bar{x}), [a]) \) of generic horizontal bundles over \( J^\infty(\pi^n C) \).

7. **Noncommutative \(QP\)-structures**

We now prove a criterion which states whether a given differential operator \( A \) is Hamiltonian and endows the space \( \overline{\Pi^*}(\pi^n C) \) of functionals on \( J^\infty(\pi^n C) \) with a Poisson bracket.

**Theorem 8.** Let \( A: p \in P \mapsto \hat{\delta}^{(a)}_{A(p)} \) be a skew-adjoint linear (non)commutative total differential operator with involutive image, meaning that, due to the Leibniz rule,

\[
[A(p_1), A(p_2)] = A(\hat{\delta}^{(a)}_{A(p_1)}(p_2) - \hat{\delta}^{(a)}_{A(p_2)}(p_1) + \{\{p_1, p_2\}\})
\]  

for any variational covectors \( p_1, p_2 \in \pi^n C \). Take the bi-vector \( \mathcal{P} = \frac{1}{2}\langle b, A(b) \rangle \) and let \( Q^A = \hat{\delta}^{(a)}_{A(b)} - \frac{1}{2}\hat{\delta}^{(b)}_{\{[b, b]_\pi\}} \) be an odd-parity vector field

\footnote{We recall from [7] that the odd-parity vector field \( Q^A \) is a Lie-algebroid differential.}

on the total space \( \overline{\Pi^*}(\pi^n C) \) of the (non)commutative variational cotangent bundle over \( \pi^n C \).

Then the operator \( A \) is Hamiltonian if and only if \( Q^A(\mathcal{P}) = 0 \in \overline{\Pi^*}(\pi^n C) \).

**Remark 7.1.** The equality \( A = -A^\dagger \) of the mapping \( A: P \mapsto \pi^n C \) to the space of evolutionary vector fields and, respectively, the mapping \( A^\dagger: \pi^n C \mapsto \hat{P} \) from the \( \langle , \rangle \)-dual space \( \text{Hom}_{C^\infty(J^\infty(\pi^n C))}(\pi^n C, \pi^n C) \) of variational covectors uniquely determines the domain \( P = \pi^n C \) of \( A \). For instance, such are the (non)commutative Hamiltonian operators; moreover, the “only if” statement is their well-known property (see [7] and references therein).
Proof. We note that if \( \{ , \} \rangle_{A} = 0 \), there is nothing to prove (see Example 5.1 on p. 11). From now on we suppose that \( \{ , \} \rangle_{A} \neq 0 \).

Using the Leibniz rule, we deduce that
\[
0 = Q^{A}(\mathcal{P}) = \frac{1}{2} \left( -\frac{1}{2} \langle\{\{b, b\} \rangle_{A} , A(b) \rangle - \frac{1}{2} \langle b , \delta \mathcal{A}_{A}(b) (A) \rangle + \frac{1}{2} \langle b , \frac{1}{2} A(\{\{b, b\} \rangle_{A}) \rangle \right),
\]
where we integrate by parts in the first term of the right-hand side and obtain
\[
\cong \frac{1}{2} \left( \frac{1}{2} \langle b , A(\{\{b, b\} \rangle_{A}) \rangle + \langle b , -\delta \mathcal{A}_{A}(b) (A) \rangle + \frac{1}{2} A(\{\{b, b\} \rangle_{A}) \rangle \right).
\]
The evaluation (3) of the underlined factor in the second coupling at any two covectors yields zero so that there only remains
\[
= \frac{1}{2} \cdot \langle b , A(\{\{b, b\} \rangle_{A}) \rangle.
\]
By the hypothesis of the theorem, the value of this tri-vector at any three covectors \( p_{1} \), \( p_{2} \), and \( p_{3} \in \pi(\pi) \) is equal to
\[
\frac{1}{2} \cdot \sum_{\bigcirc} \langle p_{1} , A(\{p_{2}, p_{3}\rangle_{A}) \rangle = 0 \in H^{a}(\pi^{NC}). \tag{14}
\]
We claim that the sum (14) is zero if and only if \( A \) is a Hamiltonian operator (i.e., whenever it induces the Poisson bracket \( \{ , \} \rangle_{A} \) which satisfies the Jacobi identity).

Indeed, define a bi-linear skew-symmetric bracket \( \{ , \} \rangle_{A} \) for any \( H_{i}, H_{j} \in H^{a}(\pi) \) by the rule
\[
\{ H_{i}, H_{j} \rangle_{A} := \langle \delta H_{i}/\delta a , A(\delta H_{j}/\delta a) \rangle \cong \delta A_{(H_{j} \rangle_{A})}(H_{i}).
\]
Now let \( H_{1}, H_{2}, \) and \( H_{3} \in H^{a}(\pi^{NC}) \) be three arbitrary functionals and put \( p_{i}(\{x, x\rangle_{A} , \{a\rangle_{A}) = \delta H_{i}/\delta a \) for \( i = 1, 2, 3 \). Then the left-hand side of the Jacobi identity for the bracket \( \{ , \} \rangle_{A} \) is, up to a nonzero constant factor, equal to
\[
\sum_{\bigcirc} \delta A_{(H_{i} \rangle_{A})}(p_{2}, A(p_{3}))
\]
\[
= \sum_{\bigcirc} \left( \langle \delta A_{(H_{i} \rangle_{A})}(p_{2}), A(p_{3}) \rangle + \langle p_{2}, \delta A_{(H_{i} \rangle_{A})}(A)(p_{3}) \rangle + \langle p_{2}, A(\delta A_{(H_{i} \rangle_{A})}(p_{3})) \rangle \right). \tag{15}
\]
Let us integrate by parts in the last term of the right-hand side; at the same time, using the property \( \tilde{\rho}_{\rho_{2}}^{(a)} = \tilde{\rho}_{\rho_{2}}^{(a)} \) of the exact covector \( p_{2} = \delta H_{2}/\delta a \) (we assume that the topology of the bundle \( \pi^{NC} \) matches the hypotheses of the Helmholzt theorem [14]), we conclude that the first term in the right-hand side of (15) equals
\[
\langle \tilde{\rho}_{\rho_{2}}^{(a)}(A(p_{1})) , A(p_{3}) \rangle = \langle \tilde{\rho}_{\rho_{2}}^{(a)}(A(p_{1})) , A(p_{3}) \rangle \cong \langle \tilde{\rho}_{\rho_{2}}^{(a)}(A(p_{3})) , A(p_{1}) \rangle \cong \langle \tilde{\rho}_{\rho_{2}}^{(a)}(p_{2}) , A(p_{1}) \rangle.
\]
Continuing the equality from (15), we have
\[
\cong \sum_{\bigcirc} \left( \langle p_{2}, \delta A_{(H_{i} \rangle_{A})}(A)(p_{3}) \rangle + \langle \delta A_{(H_{i} \rangle_{A})}(p_{2}), A(p_{1}) \rangle - \langle \delta A_{(H_{i} \rangle_{A})}(p_{3}), A(p_{2}) \rangle \right).
The 3 + 3 underlined terms cancel in the sum, whence we obtain the equality
\[
= \sum_{\circlearrowleft} \langle p_1, \partial_{A(p_3)}^{(a)}(A)(p_2) \rangle = \frac{1}{2} \sum_{\circlearrowleft} \langle p_1, \partial_{A(p_3)}^{(a)}(A)(p_2) - \partial_{A(p_3)}^{(a)}(A)(p_2) \rangle
\]
\[= -\frac{1}{2} \cdot \sum_{\circlearrowleft} \langle p_1, A(\{p_2, p_3\}) \rangle.\]

By (14), this expression equals zero. By (14), this expression equals zero. Therefore, the Jacobi identity for the bracket \(\{\ ,\ \}^A\) holds, whence the operator \(A\) is Hamiltonian. The proof is complete. \(\square\)

**Remark 7.2.** Theorem 8 converts the quadratic equation \([P, P] = 0 \iff (Q P)^2 = 0\) to a bi-linear condition \(Q^A(P) = 0\) upon the bi-vector \(P\) and \(Q\)-structure for the infinite jet superspace \(J^\infty(\Pi\hat{\pi})\), which is endowed with the odd Poisson bracket \([\ ,\ ]\), c.f. [1].

**Remark 7.3.** The odd-parity evolutionary vector field \(Q^A\) was introduced formally by postulating that the image of \(A\) is involutive, whence we deduced [7] that \(Q^A\) is a differential. Unfortunately, it follows from nowhere that such evolutionary vector field should at the same time be geometric, i.e., why the velocities of the variables \(b\) evaluated on true sections \(b = \Pi p((x, \bar{x}), [a])\) of the noncommutative variational cotangent bundle over \(J^\infty(\pi^{nC})\) do coincide, up to a necessity to recover the first two standard summands in the right-hand side of expansion (13), with such sections’ velocities
\[L_{\partial_{A(p)}^{(a)}(\Pi p((x, \bar{x}), [a]))}}(\Pi p((x, \bar{x}), [a]))\]
induced by the evolution \(\partial_{A(p)}^{(a)}\) on the unknowns \(a\), c.f. Lemma 5 on p. 11.

**Example 7.1.** Formal evolutionary vector fields (6) are geometric for all Hamiltonian operators (whence we deduce that Lemma 5 calculates the respective bi-differential Christoffel symbols \(\{\ ,\ \}^A\)).

On the other hand, if we accept that all homological evolutionary vector fields \(Q^A\) are geometric, then we strengthen Theorem 8 as follows: every skew-adjoint linear total differential operator with involutive image is a Hamiltonian operator.

**Conclusion.** There has been much debate about a proper construction of differential calculus for a proper choice of noncommutative geometry in models of mathematical physics. We have shown that the traditional calculus, in the form as we know it from Newton and Leibniz (or Gel’fand and Dorfman), itself is noncommutative because it does not appeal to the commutativity assumption. Moreover, we have demonstrated that within the setup of cyclic-invariant words the calculus becomes a topological theory so that all basic operations amount to the standard pair of pants. We also introduced the fully noncommutative bases of the bundles \(\pi^{nC}\) in such a way that their sections acquire the nature of paths (meaning here walks but not Feynman’s trajectories, which are grasped later by the construction of jet spaces over \(\pi^{nC}\)); consequently, the total spaces acquire the nature of knot bundles. We anticipate that our new approach and techniques are applicable to an axiomatic description of the geometry of fundamental

\[5\text{Moreover, the substitution principle (see Theorem 6) states that if such an expression vanishes for the exact covectors } p_i = \delta H_i/\delta a, \text{ then it vanishes for all sections } p_i \in \hat{\pi}(\pi).\]
interactions in (non)commutative space-time at Planck scale. The physical sense of this mathematical picture will be the subject of another paper.

Acknowledgements. The author thanks the Organizing committee of the workshop ‘Noncommutative algebraic geometry and its applications to physics’ (Lorentz Centre, The Netherlands, 2012) for partial support and a warm atmosphere during the meeting. The author is grateful to J. W. van de Leur, S. M. Natanson, and P. J. Olver for stimulating remarks and thanks Yu. I. Manin, V. N. Roubtsov, and V. V. Sokolov for helpful discussions at the Max Planck Institute for Mathematics (Bonn).

This research was supported in part by NWO VENI grant 639.031.623 (Utrecht) and JBI RUG project 103511 (Groningen). A part of this research was done while the author was visiting at the IHES (Bures-sur-Yvette); the financial support and hospitality of this institution are gratefully acknowledged.

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