Bifurcation and dynamical behaviour of a fractional order Lorenz-Chen-Lu like chaotic system with discretization

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Abstract. In this article, the dynamical behaviour of a discrete fractional order Lorenz-Chen-Lu like chaotic system is investigated. A discretization process is used to obtain a discrete version. The system has three constants, three fixed points and generates typical chaotic attractors. Numerical verifications are presented for the analysis of dynamical system. The chaotic attractors, bifurcation diagrams and phase portraits are provided for distinctive values of parameters. The sensitivity to initial conditions of the parameter values for the fixed points is also discussed.

1. Introduction
A fractional order system is a dynamical system which can be formulated by fractional differential equations containing derivatives of non-integer order. Its applications are found in several fields of engineering and science such as biology, diffusion processes, viscoelastic materials, magnetic fields, electro chemistry and finance systems. In modern years fractional order systems have attracted many researchers due of their applications [6, 8, 10].

Chaos theory as a new division of mathematics and physics has given birth to a new technique of exploring the universe and is a key tool to be aware of the behavior of the processes in the world. In current years, chaotic behaviors have been useful in special areas of engineering and science such as physics, economics, medicine, biology, electronics, mechanics, ecology and etc., [6, 10].

In 1950s, the meteorologist E.N.Lorenz acquired the LGP-30 computer having internal memory16KB, using computer, in 1963, he discovered the chaotic motion on a chaotic attractor [1, 2]. The model studied by E.N.Lorenz arising in weather prediction was described by autonomous system of differential equations containing nonlinear terms. It is famous for having chaotic solutions for few parameters and initial values represented by

\[ x' = a(y - x), \quad y' = cx - y - xz, \quad z' = xy - bz. \]

These equations contain three state variables \( x, y, z \). For typical values of \( a = 10, b = 8/3 \) and \( c = 28 \), the famous butterfly attractor evolves. On the other hand, the Chen’s system [3, 4, 9] is described by the following set of equations

\[ x' = a(y - x), \quad y' = (c - a)x + cy - xz, \quad z' = xy - bz \]

which has a set of chaotic parameters \( a = 35, b = 3, c = 28 \). The Chen’s system is a dual of the Lorenz system but evolves more sophisticated attractors. The Lu system [5, 14] is described by the following set of equations

\[ x' = a(y - x), \quad y' = cy - xz, \quad z' = xy - bz \]

which has a set of chaotic parameters \( a = 36, b = 3, c = 20 \).
The article is organized as follows. In section 2, we present the basic definitions of the fractional derivatives and integrals. In section 3, we describe the chaotic system and explain the discretization process of fractional order system. In section 4, we analyze the equilibrium points with stability. Further we present illustrative examples and numerical simulations. In section 5, we present the bifurcation diagrams and chaotic attractors in $x, y, z$-space. Section 6, deals with the sensitivity of the system to initial values.

2. Fractional Derivatives and Integrals
We present some important definitions of the fractional derivatives and integrals [6].

**Definition 1.** The Riemann-Liouville fractional derivative is expressed as follows:

$$D^\alpha f(\tau) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{d\tau} \right)^n \int_0^\tau \frac{f(\xi)}{(\tau-\xi)^{\alpha-n+1}} d\xi.$$  

**Definition 2.** The Caputo fractional derivative is defined by

$$D^\alpha f(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau \frac{f^n(\xi)}{(\tau-\xi)^{\alpha-n+1}} d\xi,$$

where $\Gamma$ is the gamma function. The Riemann-Liouville and Caputo fractional derivatives are widely used in the application of real life problems [6, 8, 10].

3. Fractional Order Lorenz-Chen-Lu Like Chaotic System and Discretization
In some known literatures, authors have analyzed the Yang system similar to Lorenz like system [12].

$$\begin{align*}
x'(t) &= a(y(t) - x(t)) \\
y'(t) &= cx(t) - x(t)z(t) \\
z'(t) &= x(t)y(t) - bz(t)
\end{align*}$$

These equations contain three state variables $x, y, z$ and three state parameters $a, b, c$. They think that found the system to have two interesting chaotic attractors (called as Yang–Chen attractor) when $(a, b, c) = \left(\frac{10}{3}, \frac{8}{3}, 16\right)$ and $(a, b, c) = (35, 335, 35)$ respectively.

Here we are concerned with the fractional order chaotic system given by

$$\begin{align*}
D^\alpha x(t) &= a(y(t) - x(t)) \\
D^\alpha y(t) &= cx(t) - x(t)z(t) \\
D^\alpha z(t) &= x(t)y(t) - bz(t)
\end{align*}$$

where $\alpha$ is the fractional order. Now we are interested in discretizing fractional order chaotic system given in the form.

$$\begin{align*}
D^\alpha x(t) &= a \left( y \left( r \left[ \frac{t}{r} \right] \right) - x \left( r \left[ \frac{t}{r} \right] \right) \right) \\
D^\alpha y(t) &= cx \left( r \left[ \frac{t}{r} \right] \right) - x \left( r \left[ \frac{t}{r} \right] \right)z \left( r \left[ \frac{t}{r} \right] \right) \\
D^\alpha z(t) &= x \left( r \left[ \frac{t}{r} \right] \right)y \left( r \left[ \frac{t}{r} \right] \right) - bz \left( r \left[ \frac{t}{r} \right] \right)
\end{align*}$$

with initial condition $x(0) = x_0, y(0) = y_0, z(0) = z_0$.

Applying the discretization process [7, 11, 13] and letting $t \rightarrow (n + 1)r$, we end up with the discretized system as given below

$$\begin{align*}
x_{n+1} &= x_n + s(a(y_n - x_n)), \\
y_{n+1} &= y_n + s(cx_n - x_nz_n), \quad (4) \\
z_{n+1} &= z_n + s(x_ny_n - bz_n),
\end{align*}$$

where $s = \frac{r^n}{\Gamma(1+\alpha)}$. 
4. Equilibrium Points and Numerical Simulations

Now we analyze the stability of the system (4) which has the following three equilibrium points.

(i) \( F_0 = (0,0,0) \),
(ii) \( F_1 = (\sqrt{bc}, \sqrt{bc}, c) \),
(iii) \( F_2 = (-\sqrt{bc}, -\sqrt{bc}, c) \)

Linearization of the system (4) about \( F_0 \) yield the characteristic equation is

\[
P(\lambda) = \lambda^3 + \lambda^2[s(a+b) - 3] + \lambda \left[3 - 2s(a+b) + s^2(a-b-c)\right] + [s(a+b) - s^2a(b-c) - s^3abc - 1]
\]

Let

\[
a_1 = s(a+b) - 3, \quad a_2 = 3 - 2s(a+b) + s^2a(b-c),
\]

\[
a_3 = s(a+b) - s^2a(b-c) - s^3abc - 1.
\]

From the Jury test [15], if \( P(1) > 0, P(-1) < 0 \) and \( P(0) = a_3 < 0, |b_3| > b_1, c_2 |c_3| \), where \( b_3 = 1 - a_0^2, b_2 = a_1 - a_0 a_2, b_1 = a_2 - a_3 a_1, c_3 = b_3^2 - b_1^2 \) and \( c_2 = b_3 b_2 - b_1 b_2 \) then the roots of \( P(\lambda) \) satisfy \( |\lambda| < 1 \) and thus \( F_0 \) is asymptotically stable. Suppose \( P(1) < 0 \) then \( F_0 \) is unstable.

Linearization of the system (4) about \( F_1 \) or \( F_2 \) yield the characteristic equation is

\[
P(\lambda) = \lambda^3 + \lambda^2[s(a+b) - 3] + \lambda \left[3 - 2s(a+b) + s^2b(a+c)\right] + [2s^3abc - s^2b(a+c) + s(a+b) - 1]
\]

Let

\[
a_{11} = s(a+b) - 3, \quad a_{22} = 3 - 2s(a+b) + s^2b(a+c),
\]

\[
a_{33} = 2s^3abc - s^2b(a+c) + s(a+b) - 1.
\]

From the Jury test [15], if \( P(1) > 0, P(-1) < 0 \) and \( P(0) = a_{33} < 0, |b_{33}| > b_{11}, c_{22} > |c_{33}| \), where \( b_{33} = 1 - a_{22}^2, b_{22} = a_{11} - a_{33} a_{22}, b_{11} = a_{22} - a_{33} a_{11}, c_{33} = b_{33}^2 - b_{11}^2 \) and \( c_{22} = b_{33} b_{22} - b_{11} b_{22} \) then the roots of \( P(\lambda) \) satisfy \( |\lambda| < 1 \) and thus \( F_1 \) or \( F_2 \) is asymptotically stable. Suppose \( P(1) < 0 \) then \( F_1 \) or \( F_2 \) is unstable.

We discuss the time plots, phase portraits of 3D chaotic attractor in \( xyz \) – space and 2D chaotic attractor \( xy, xz, yz \) – planes of the system (4) with initial values \( (x_0, y_0, z_0) = (0.1, 0.1, 0.1) \).

Example 1. Taking \( a = 0.3, b = 0.1, c = 0.8, r = 0.1, \alpha = 0.9 \) and applying Jury conditions, we get \( P(1) = 0.00010765 > 0, \ P(-1) = -7.7942 < 0 \) and \( a_{33} = -0.9494 < 0 \). Then the equilibrium point \( F_1 \) is asymptotically stable. Figure 1 (a – e) are the time plots, phase portraits of 2D and 3D chaotic attractors.

![Figure 1](image-url)
Example 2. Taking $\alpha = 0.3$, $b = 0.1$, $c = 0.9$, $r = 0.1$, $\alpha = 0.9$ and applying Jury conditions, we get $P(1) = 0.00012111 > 0$, $P(-1) = -7.7946 < 0$ and $a_{33} = -0.9496 < 0$. Then the equilibrium point $F_1$ is asymptotically stable. Figure 2(a – e) are the time plots, phase portraits of 2D and 3D chaotic attractors.

Example 3. Taking $\alpha = 0.3$, $b = 0.2$, $c = 1.4$, $r = 0.1$, $\alpha = 0.9$ and applying Jury conditions, we get $P(1) = 0.0003769 > 0$, $P(-1) = -7.7495 < 0$ and $a_{33} = -0.9400 < 0$. Then the equilibrium point $F_1$ is asymptotically stable. Figure 3(a – e) are the time plots, phase portraits of 2D and 3D chaotic attractors.
Example 4. Taking $a = 0.3$, $b = 0.07$, $c = 4.0$, $r = 0.1$, $\alpha = 0.9$ and applying Jury conditions, we get $P(1) = 0.00037679 > 0$, $P(-1) = -7.8162 < 0$ and $a_{33} = -0.9563 < 0$. Then the equilibrium point $F_1$ is asymptotically stable. Figure 4(a–e) are the time plots, phase portraits of 2D and 3D chaotic attractors.

![Figure 4](image)

**Figure 4.** Time plots and phase portraits of the Lu like chaotic system

Example 5. Taking $a = 0.9$, $b = 0.07$, $c = 3.5$, $r = 0.1$, $\alpha = 0.9$ and applying Jury conditions, we get $P(1) = 0.00098908 > 0$, $P(-1) = -7.5017 < 0$ and $a_{33} = -0.8773 < 0$. Then the equilibrium point $F_1$ is asymptotically stable. Figure 5(a–e) are the time plots, phase portraits of 2D and 3D chaotic attractors.

![Figure 5](image)

**Figure 5.** Time plots and phase portraits of the Chen like chaotic system
Example 6. Taking $a = 0.28, b = 0.07, c = 3.5, r = 0.1, \alpha = 0.9$ and applying Jury conditions, we get $P(1) = 0.00030771 > 0, P(-1) = -7.8255 < 0$ and $a_{33} = -0.9584 < 0$. Then the equilibrium point $F_3$ is asymptotically stable. Figure 6(a – e) are the time plots, phase portraits of 2D and 3D chaotic attractors.

Figure 6. Time plots and phase portraits of the Lu like chaotic system

5. Bifurcation and Chaotic Attractor
In dynamical systems, it is observed that a slight smooth change in the bifurcation parameter values of a system results in a sudden qualitative or topological change in its behaviour and thus bifurcation occurs. Generally, at bifurcation point, it is noticed that the local stability properties of equilibria, other invariant sets or periodic orbits are set to alter [13, 15]. Now, we analyze the bifurcation for the variables $x, y, z$ and the parameter values $b = 0.5, c = 3, r = 0.1, \alpha = 0.9$ fixing and varying $a \in [0, 3]$ (see figure.7). We present phase portraits for $x, y, z$ - space corresponding to Figure. 7 with varying $a \in [0, 3]$, (see Figure.8). It is quite interesting to discuss the phase portraits of various $a$ corresponding to the bifurcation diagram in figure.7 presented in figure.8. For $a \in (0.25, 0.31)$ the fixed point is stable as in Figure.8(a,b,c), but it moves from stable to unstable through the occurrence of limit cycle for $a = 0.32$ in Figure.8(d). For $a = 1.2$ is the Lu like chaotic attractor as in Figure.8(e). For $a \in (1.5, 3.0)$ is the Chen like chaotic attractor as in Figure.8(f,g,h).

Figure 7. Bifurcation diagrams with the initial values $(x_0 = 0.1, y_0 = 0.1, z_0 = 0.1)$
The bifurcation diagrams for the parameter values $a = 0.5, c = 3.0, r = 0.1, \alpha = 0.9$ fixing and varying $b \in [0, 1.5]$, (see Figure 9). We present phase diagrams for $x, y, z$ - space corresponding to Figure 9. For $b \in (0.1, 0.2)$ is the Lu like chaotic attractor as in Figure 10(a-e). For $b = 0.78$ the fixed point is unstable through the occurrence of limit cycle as in Figure 10(f), but it moves from unstable to stable for $a \in (0.8, 0.9)$ as in Figure 10(g,h).

Figure 8. Phase portraits for $x, y, z$-space corresponding to Figure 7.

Figure 9. Bifurcation diagrams with the initial values $(x_0 = 0.1, y_0 = 0.1, z_0 = 0.1)$

Figure 10. Phase portraits for $x, y, z$-space corresponding to Figure 9.
The bifurcation diagrams for the parameter values \( a = 0.3, b = 0.08, r = 0.1, \alpha = 0.9 \) are fixing and varying \( c \in [0, 3] \) (see figure.11). It is quite interesting to discuss the phase portraits of various \( c \) corresponding to the bifurcation diagram in figure.11 presented in figure.12. For \( c = 0.3 \) is the fixed point is stable as in Figure.12(a). For \( c = 0.5 \) is the Lorenz like chaotic attractor as in Figure.12(b). For \( c \in (1.0, 1.5) \) is the Chen like chaotic attractor as in Figure.12(c,d). For \( c \in (1.7, 3.0) \) is the Lu like chaotic attractor as in Figure.12(e-h).

![Figure 11. Bifurcation diagrams with the initial values \((x_0 = 0.1, y_0 = 0.1, z_0 = 0.1)\)](image1)

![Figure 12. Phase portraits for \(x, y, z\)-space corresponding to Figure 11.](image2)

6. Sensitivity to Initial Values
In this section, we discuss the sensitivity of Lorenz-Chen-Lu like chaotic system (4) to initial values. We consider time plots for 3-sets of initial conditions \((x_0, y_0, z_0), (x_0 + 0.0001, y_0, z_0); (x_0, y_0, z_0), (x_0, y_0 + 0.0001, z_0)\) and \((x_0, y_0, z_0), (x_0, y_0, z_0 + 0.0001)\) respectively. Sensitivity to initial values implies that a small change in the initial values leads to fluctuations which may be negligible initially but builds up rapidly. Thus, small change in initial values of the current path may lead to significantly different future behavior [15]. Case (i). We consider the parameter values \( a = 0.3, b = 0.1, c = 0.8, r = 0.1, \alpha = 0.9 \) and analyze the sensitivity to initial values on 3D Lorenz like chaotic system with \((x_0 = 0.1, y_0 = 0.1, z_0 = 0.1), (x_0 = 0.1, y_0 = 0.1001, z_0 = 0.1)\) and \((x_0 = 0.1, y_0 = 0.1, z_0 = 0.1001)\), see figure-13. Case (ii). We take the values \( a = 0.9, b = 0.07, c = 3.5, r = 0.1, \alpha = 0.9 \) and analyze the sensitivity to initial values on 3D Chen like chaotic system with
\( (x_0 = 0.1, y_0 = 0.1, z_0 = 0.1), (x_0 = 0.1, y_0 = 0.1001, z_0 = 0.1) \) and \( (x_0 = 0.1, y_0 = 0.1, z_0 = 0.1001) \), see figure-14.

**Figure 13.** Sensitivity to initial values on 3D Lorenz like chaotic system

**Figure 14.** Sensitivity to initial values on 3D Chen like chaotic system

**Figure 15.** Sensitivity to initial values on 3D Lu like chaotic system
Case (iii), we consider the parameter values $\alpha = 0.01, b = 0.5, h = 0.1, \alpha = 0.9$ and analyze the sensitivity to initial values on 3D Lu like chaotic system with $(x_0 = 0.1, y_0 = 0.1, z_0 = 0.1), (x_0 = 0.1, y_0 = 0.1001, z_0 = 0.1)$ and $(x_0 = 0.1, y_0 = 0.1, z_0 = 0.1001)$, see figure-15.

7. Conclusion

This article investigated the dynamics of a discrete fractional order Lorenz-Chen-Lu like chaotic system with order in $(0,1)$ which is developed from the system presented in (4) by using a discretization method. We established the existence of three equilibrium points and Jury condition is applied to analyze the stability of the system (4). Numerically, we have illustrated the dynamical behavior of the system (4) by means of chaotic attractors, bifurcation diagrams and phase diagrams for different set of parameter values and we also discussed the sensitivity analysis of the system.

References
[1] Edward Lorenz N 1963 Deterministic Non-periodic Flow Journal of the Atmospheric Sciences 20 pp 130 – 141
[2] Etienne Ghys 2013 The Lorenz Attractor, a Paradigm for Chaos Chaos 154 Springer Basel
[3] Chen G and Ueta T 1999 Yet another chaotic attractor International Journal of Bifurcation and Chaos 9 7 pp 1465 – 1466
[4] Ueta T and Chen G 2000 Bifurcation analysis of Chen’s equation International Journal of Bifurcation and Chaos 10 8 pp 19 – 31
[5] Jinhu Lu and Guanrong Chen 2002 A New Chaotic Attractor Coined International Journal of Bifurcation and Chaos 12 3 pp 659 – 661
[6] Keith B. Oldham and Jerome Spanier 1974 The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order Dover Publications INC
[7] Ravi P Agarwal, Ahmed MA El-Sayed and Sanaa M Salman 2013 Fractional-order Chaos system: discretization bifurcation and chaos Advances in Difference Equations, 2013:320.
[8] Miller K.S and Ross B 1993 An Introduction to the Fractional Calculus and Fractional Differential Equations John Wiley & Sons New York NY USA
[9] Leonov G A and Kuznetsov N V 2015 On differences and similarities in the analysis of Lorenz, Chen and Lu systems Applied Mathematics and Computation 256 pp 334 – 343
[10] Ivo Petras 2010 Fractional order Nonlinear Systems - Modeling Analysis and Simulation Higher Education Press Springer International Edition
[11] George Maria Selvam A, Paul Loganathan M, Janagaraj R and Abraham Vianny D 2018 Chaotic behavior in discrete Chen system with fractional order International Journal of Technical Innovation in Modern Engineering & Science 4 11 pp 310 – 315
[12] Haijum Wang and Xianyi Li 2015 True and False Chaotic Attractors in a 3-D Lorenz-type System Annual Review of Chaos Theory, Bifurcations and Dynamical Systems 5 pp 33–41
[13] George Maria Selvam A and Abraham Vianny D 2019 Dynamic analysis of a discrete fractional order 3D chaotic system American International Journal of Research in Science Technology Engineering & Mathematics pp 243 – 251
[14] Jinhu Lu, Tianshou Zhou and Suochun Zhang 2002 Chaos synchronization between linearly coupled chaotic systems Chaos, Solutions & Fractals 14 pp 529 – 541
[15] George Maria Selvam A and Abraham Vianny D 2019 Bifurcation and Chaotic Behavior of a Discrete Fractional Order Lorenz System AIP Conference Proceedings 2112 pp 020052-1–020052-13