Nonparametric Expected Shortfall Forecasting
Incorporating Weighted Quantiles

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Abstract

A new semi-parametric Expected Shortfall (ES) estimation and forecasting framework is proposed. The proposed approach is based on a two-step estimation procedure. The first step involves the estimation of Value-at-Risk (VaR) at different levels through a set of quantile time series regressions. Then, the ES is computed as a weighted average of the estimated quantiles. The quantiles weighting structure is parsimoniously parameterized by means of a Beta function whose coefficients are optimized by minimizing a joint VaR and ES loss function of the Fissler-Ziegel class. The properties of the proposed approach are first evaluated with an extensive simulation study using various data generating processes. Two forecasting studies with different out-of-sample sizes are conducted, one of which focuses on the 2008 Global Financial Crisis (GFC) period. The proposed models are applied to 7 stock market indices and their forecasting performances are compared to those of a range of parametric, non-parametric and semi-parametric models, including GARCH, Conditional AutoRegressive Expectile (CARE, \(^\text{Taylor 2008}\)), joint VaR and ES quantile regression models (\(^\text{Taylor 2019}\)) and simple average of quantiles. The results of the forecasting experiments provide clear evidence in support of the proposed models.

Keywords: Value-at-Risk, Expected Shortfall, quantile regression, Beta function, joint loss.
INTRODUCTION

Value-at-Risk (VaR) is employed by many financial institutions as an important risk management tool. Representing the market risk as one number, VaR has been used as a standard risk measurement metric for the past two decades. However, as recently recognized by the Basel Committee for Banking Supervision, VaR suffers from a number of weaknesses affecting its reliability as a reference metric for determining regulatory capital requirements (Basel Committee on Banking Supervision, 2013). First, VaR cannot measure the expected loss for extreme (violating) returns. In addition, it can be shown that VaR is not always a coherent risk measure, due to failure to match the subadditivity property. For these reasons, the Committee proposed in May 2012 to replace VaR with the Expected Shortfall (ES, Artzner 1997; Artzner et al. 1999). Thus, in recent years ES has been increasingly employed for tail risk measurement. However, still there is much less existing research on modeling ES compared with VaR.

ES calculates the expected value of return being below the quantile (VaR) of its distribution. Differently from VaR, it is a coherent measure and it "measures the riskiness of a position by considering both the size and the likelihood of losses above a certain confidence level" (Basel Committee on Banking Supervision, 2013).

The Basel III Accord, which was implemented in 2019, places new emphasis on ES. Its recommendations for market risk management are illustrated in the 2019 document “Minimum capital requirements for market risk” that, on page 89, mentions: “ES must be computed on a daily basis for the bank-wide internal models to determine market risk capital requirements. ES must also be computed on a daily basis for each trading desk that uses the internal models approach (IMA).”; “In calculating ES, a bank must use a 97.5th percentile, one-tailed confidence level” (Basel Committee on Banking Supervision 2019). Therefore, in the empirical application of our paper, we mainly focus on one-step-ahead tail risk forecasting at 2.5% quantile level.

The literature on ES modelling and forecasting is closely related to previous research on VaR. The quantile regression type model, e.g. the Conditional Autoregressive Value-
at-Risk (CAViaR) model of Engle and Manganelli (2004), is a popular semi-parametric approach to forecast VaR. Gerlach et al. (2011) generalize the CAViaR models to a fully nonlinear family.

However, the CAViaR type models cannot directly estimate and forecast ES. A semi-parametric model that directly estimates quantiles and expectiles, and implicitly ES, called the Conditional Autoregressive Expectile (CARE) model, is proposed by Taylor (2008). To select the appropriate expectile level, a grid search process is required for the CARE type models which is relatively computationally expensive (dependent on the model complexity and the size of the grid).

Taylor (2019) proposes a joint ES and quantile regression framework (ES-CAViaR) which employs the Asymmetric Laplace (AL) density to build a likelihood function whose Maximum Likelihood Estimates (MLEs) coincide with those obtained by minimisation of a strictly consistent joint loss function for VaR and ES. The frameworks in Taylor (2019) assume that the difference or ratio between VaR and ES follow specific dynamics, also in order to guarantee that VaR and ES do not cross with each other. Essentially, this implies additional assumptions on ES dynamics.

Fissler and Ziegel (2016) develop a family of joint loss functions (or “scoring rules”) that are strictly consistent for the true VaR and ES, i.e. they are uniquely minimized by the true VaR and ES series. Under specific choices of the functions involved in the joint loss function of Fissler and Ziegel (2016), it can be shown that the negative of AL log-likelihood function, presented in Taylor (2019), can be derived as a special case of the Fissler and Ziegel (2016) class of loss functions. Patton et al. (2019) propose new dynamic models for VaR and ES, through adopting the generalized autoregressive score (GAS) framework (Creal et al. 2013 and Harvey 2013) and utilizing the loss functions in Fissler and Ziegel (2016).

In our paper, a new ES estimation and forecasting framework is proposed where the ES is modelled as an affine function of tail quantiles. Hence, we refer to our approach as the Weighted Quantile estimator. The quantiles are produced from the CAViaR model of Engle and Manganelli (2004), by grid search of a range of equally spaced quantile
levels below the target VaR level, i.e. 2.5%. We will discuss the selection details of these quantile levels later. The weighting pattern of the selected quantiles is based on a two parameter Beta function, also called the ”Beta lag”, borrowed from the literature on Mixed Data Sampling (MIDAS, Ghysels et al. 2007) regression models. The Beta lag function is a parsimonious but yet flexible choice and is able to reproduce a variety of different behaviours such as declining, increasing or hump shaped patterns.

We estimate the parameters of the Beta Lag, determining the Beta weights assigned to the selected quantiles, by minimizing strictly consistent VaR and ES joint loss functions of the class defined in Fissler and Ziegel (2016). In particular we focus on the negative AL loss.

Our framework has some important advantages. First, the proposed estimator does not require any additional assumption on the ES process but it only relies on the natural definition of ES as the tail conditional expectation of the conditional distribution of returns\footnote{Under the assumption that this distribution is continuous, as later explained.}. Furthermore, the dynamics of the ES, that can be explicitly derived by standard algebraic manipulations, are those naturally implied by its definition in terms of the expectation of tail-quantiles. This implies that the ES can be predicted without having to specify an additional dynamic equation, so reducing model uncertainty and risk of potential mis-specification on the ES side.

Our method has some interesting connections with the existing literature. First, there are some evident affinities between our method and CARE models. Namely, both our framework and CARE models involve a two-step estimation procedure and a grid search process. Later, we will show that our framework can produce more accurate ES forecasting results than CARE, by using a significantly lower number of grid search quantile levels. In the empirical section, we have shown that our framework using grid size of 3 (quantile levels) can have clearly improved performance compared to CARE with grid size of 50 (expectile levels). Further, it is closely related to literature on forecasts combination. Taylor (2020) has recently proposed to use a forecast combination of different VaR&ES models of the same order. However, our strategy is substantially different since we are
combining forecasts from a list of VaR models (CAViaR) of different quantile orders, instead of a list of different joint VaR&ES models.

The paper is organized as follows. Section 2 defines the motivating framework for our proposal. A review of the ES-CAViaR and CARE models is then presented in Section 3. Section 4 formalizes the proposed weighted quantile framework while Section 5 investigates the dynamical properties of the proposed conditional ES estimator. Finally, we present empirical evidence on both simulated and real stock market data. First, Section 7 presents the results of a Monte Carlo simulation aimed at providing an appraisal of the finite-sample statistical properties of the proposed estimators. Section 8 then, assesses the effectiveness of the proposed estimation approach in standard risk management applications by presenting the results of two out-of-sample tail risk forecasting exercises in which the performances of the weighted quantile estimator are compared with those of some state-of-the-art competitors. Section 9 concludes the paper and discusses future work.

2 FRAMEWORK AND MOTIVATION

To start, let $\mathcal{I}_t$ be the information available at time $t$ and

$$F_t(r) = Pr(r_t \leq r | \mathcal{I}_{t-1})$$

be the Cumulative Distribution Function (CDF) of $r_t$ conditional on $\mathcal{I}_{t-1}$. We assume that $F_t(.)$ is strictly increasing and continuous on the real line $\mathbb{R}$. Under this assumption, the one-step-ahead $\alpha$ level Value at Risk at time $t$ can be defined as

$$Q_{t,\alpha} = F_t^{-1}(\alpha) \quad 0 < \alpha < 1.$$  

Within the same framework, the one-step-ahead $\alpha$ level Expected Shortfall can be shown (see Acerbi and Tasche, 2002, among others) to be equal to the tail conditional expectation of $r_t$

$$ES_{t,\alpha} = E(r_t | \mathcal{I}_{t-1}, r_t \leq Q_{t,\alpha}).$$
This is equivalent to state that $ES_{t,\alpha}$ is related to $F_t(.)$ by the following integral

$$ES_{t,\alpha} = \frac{1}{F_t(Q_{t,\alpha})} \int_{-\infty}^{Q_{t,\alpha}} rdF_t(r) = \frac{1}{\alpha} \int_{-\infty}^{Q_{t,\alpha}} rdF_t(r),$$  \hspace{1cm} (1)$$

that, after a simple change of variable, can be rewritten as

$$ES_{t,\alpha} = \frac{1}{\alpha} \int_0^\alpha Q_{t,p} dp. \hspace{1cm} (2)$$

The integral in (2) can be approximated over a discrete grid by means of standard numerical integration techniques. Namely, given the target quantile level $\alpha$, assume that an equally spaced grid of quantile levels of size $M$ is selected,

$$\alpha_M = [\alpha_1, \alpha_2, \ldots, \alpha_M = \alpha],$$

where, setting $\alpha_0 = 0$, 

$$\alpha_m = \alpha_{m-1} + \eta,$$

with $\eta = (\alpha_M - \alpha_1)/(M - 1) = \alpha/M$, for $m = 1, \ldots, M$. A simple rectangular rule would then lead to the following approximation

$$ES_{t,\alpha} \approx \frac{1}{\alpha} \eta \sum_{i=1}^M Q_{t,\alpha_i} = \frac{1}{M} \sum_{i=1}^M Q_{t,\alpha_i} = \sum_{i=1}^M w_i Q_{t,\alpha_i} \hspace{1cm} (3)$$

with $w_i = 1/M$, for $i = 1, \ldots, M$. It is easy to show that, in theory, as $M \to \infty$ the above approximation asymptotically tends to the “true” $ES_{t,\alpha}$ value. In general, it can be shown (see [Davis and Rabinowitz 1984]) that many higher order integration rules, such as the trapezoidal and Simpson’s rule, can be represented as weighted averages of the form in (3) where, modulating the choice of the weights $w_i$, one can obtain different integration rules as special cases. For example, the set of weights ($w_1 = 1/2M, w_2 = 1/M, \ldots, w_{M-1} = 1/M, w_M = 1/2M$) would lead to a trapezoidal rule (for more details see [Davis and Rabinowitz 1984, page 57, Section 2.1.5]). It is however worth noting that, in real data applications, data scarcity prevents accurate estimation of VaR for extreme quantile orders, posing constraints on the choice of the minimum grid value $\alpha_1$.

It follows that a correction for this left-tail truncation bias should be considered when designing an estimator for $ES_{t,\alpha}$ based on the representation in Equation (2). Furthermore, referring to an appropriately defined strictly consistent scoring rule, the weights
could be estimated rather than fixed a priori. This approach would bring some important advantages. First, it would be possible to modulate, in a data-driven fashion, the weights assigned to each tail-quantile in (3) in order to optimally match the tail properties of returns and, eventually, down-weight less accurately estimated extreme quantiles. Second, it would allow to control the left-tail truncation bias. Last, working with estimated weights would reduce the impact, to some extent, of the subjective choice of the lower bound $\alpha_1$.

In Section 4, starting from a set of consistent estimators of $Q_{t,p}$ ($0 < p \leq \alpha$), these ideas will be elaborated to define a semi-parametric two-step estimation strategy for $ES_{t,\alpha}$. Before moving to the illustration of our proposal, in the next section, we present a review of the main approaches for joint semi-parametric estimation of conditional VaR and ES. In order to simplify notation, in the remainder, unless differently specified, the following notational conventions will be adopted: $ES_{t,\alpha} \equiv ES_t$ and $Q_{t,\alpha} \equiv Q_t$.

3 JOINT MODELLING OF VaR AND ES: ES-CAViaR AND CARE MODELS

3.1 Models based on strictly consistent scoring rules

Koenker and Machado (1999) show that the quantile regression estimator is equivalent to a maximum likelihood estimator when assuming that the data are conditionally distributed as an Asymmetric Laplace (AL) with a mode at the quantile of interest. If $r_t$ is the return on day $t$ and $Pr(r_t < Q_t|I_{t-1}) = \alpha$, then the parameters in the model for $Q_t$ can be estimated maximizing a quasi-likelihood based on:

$$p(r_t|I_{t-1}) = \frac{\alpha(1 - \alpha)}{\sigma} \exp\left(-\frac{(r_t - Q_t) \cdot (\alpha - I(r_t < Q_t))}{\sigma}\right),$$

for $t = 1, \ldots, n$ and where $\sigma$ is a nuisance parameter.

Taylor (2019) extends this result to incorporate the associated ES quantity into the likelihood expression, noting a link between $ES_t$ and a dynamic $\sigma_t$, resulting in the con-
ditional density function:

\[
p(r_t | I_{t-1}) = \frac{\alpha(1 - \alpha)}{ES_t} \exp \left( - \frac{(r_t - Q_t)(\alpha - I(r_t < Q_t))}{\alpha ES_t} \right),
\]

allowing a likelihood function to be built and maximised, given model expressions for \((Q_t, ES_t)\). Taylor (2019) notes that the negative logarithm of the resulting likelihood function is strictly consistent for \((Q_t, ES_t)\) considered jointly, e.g. it fits into the class of jointly consistent scoring functions for VaR & ES developed by Fissler and Ziegel (2016).

Taylor (2019) incorporates two different ES components that describe the dynamics of VaR and ES and also avoid ES estimates crossing the corresponding VaR estimates, as presented below in Model (5) (ES-CAViaR-Add: ES-CAViaR with an additive VaR to ES component) and Model (6) (ES-CAViaR-Mult: ES-CAViaR with a multiplicative VaR to ES component):

**ES-CAViaR-Add:**

\[
Q_t = \beta_1 + \beta_2 |r_{t-1}| + \beta_3 Q_{t-1},
\]

\[
ES_t = Q_t - w_t,
\]

\[
w_t = \begin{cases} 
\gamma_0 + \gamma_1 (Q_{t-1} - r_{t-1}) + \gamma_2 w_{t-1} & \text{if } r_{t-1} \leq Q_{t-1}, \\
w_{t-1} & \text{otherwise,}
\end{cases}
\]

where, to ensure that the VaR and ES series do not cross, Taylor (2019) imposes the following constraints: \(\gamma_0 \geq 0, \gamma_1 \geq 0, \gamma_2 \geq 0\).

**ES-CAViaR-Mult:**

\[
Q_t = \beta_1 + \beta_2 |r_{t-1}| + \beta_3 Q_{t-1},
\]

\[
ES_t = (1 + \exp(\gamma_0)) Q_t,
\]

where \(\gamma_0\) is unconstrained.
3.2 Models based on expectiles

The concept of expectile is closely related to the concept of quantile. The \( \tau \) level expectile \( \mu_{\tau} \), as defined by Aigner et al. (1976), can be estimated through minimizing the following Asymmetric Least Squares (ALS) criterion (Taylor 2008):

\[
\sum_{t=1}^{N} |\tau - I(r_t < \mu_{\tau})|(r_t - \mu_{\tau})^2,
\]  

(7)

no distributional assumption is required to estimate \( \mu_{\tau} \) here.

As discussed in Section 1, conditional ES is defined as \( \text{ES}_{t,\alpha} = E(r_t|r_t < Q_{t,\alpha}, \mathcal{I}_{t-1}) \). Newey and Powell (1987) and Taylor (2008) show that this is related to the conditional \( \tau \) level expectile \( \mu_{t,\tau} \) by the relationship:

\[
\text{ES}_{t,\alpha}\tau = \left(1 + \frac{\tau}{(1-2\tau)\alpha_{\tau}}\right)\mu_{t,\tau},
\]  

(8)

where \( \mu_{t,\tau} = Q_{t,\alpha_{\tau}} \), i.e. \( \tau \) level expectile \( \mu_{t,\tau} \) occurs at the quantile level \( \alpha_{\tau} \) of \( r_t \). \( \alpha_{\tau} \) is used here to emphasize this relationship. Thus, \( \mu_{t,\tau} \) can be used to estimate the \( \alpha_{\tau} \) level conditional quantile \( Q_{t,\alpha_{\tau}} \), and then scaled to estimate the associated \( \text{ES}_{t,\alpha_{\tau}} \).

Exploiting this relationship, Taylor (2008) proposes the CARE type models which have a similar form to the CAViaR models of Engle and Manganelli (2004), where lagged returns drive the expectiles and model parameters are estimated minimizing an ALS criterion. The general Symmetric Absolute Value (SAV) form of this model is:

**CARE-SAV:**

\[
\mu_{t,\tau} = \beta_1 + \beta_2|r_{t-1}| + \beta_3\mu_{t-1,\tau}
\]

where \( \mu_{t,\tau} \) is the \( \tau \) level expectile on day \( t \). The CARE-type model produces one-step-ahead forecasts of expectiles \( (\mu_{t,\tau}) \), that can be employed as VaR estimates \( (Q_{t,\alpha_{\tau}}) \), by an appropriate choice of \( \tau \). The VaR estimates can be further scaled, using Equation (8), to produce forecasts of ES which cannot be directly calculated under the CAViaR framework.

However, the selection of the appropriate expectile level \( \tau \) requires a grid search, based on the optimization of the violation rate (VRate, the percentage of returns exceeding
VaR estimates) or of the aggregated quantile loss function (Gerlach and Wang, 2020).

Specifically, in the first case, for each grid value of $\tau$, the ALS estimator of the CARE equation parameters $\beta_j$ ($j = 1, 2, 3$) is found, yielding an associated VRate($\tau$). $\hat{\tau}$ is then set to the grid value of $\tau$ s.t. VRate is closest to the desired $\alpha_\tau$. Differently, when the aggregated quantile loss is chosen as an objective function, the selected $\alpha_\tau$ is chosen to minimize over the selected grid the value of the quantile loss wrt to the quantile level, see Gerlach and Wang (2020). In real applications, this grid search approach can be computationally expensive (dependent on the size of the grid), and the performance can be affected by the size and gap of the grid which is normally decided by means of an ad-hoc approach.

Fissler and Ziegel (2016) develop a family of joint loss functions whose value depends on the associated VaR and ES. Members of this family are strictly consistent for $(Q_t, ES_t)$, i.e. their expectations are uniquely minimized by the true VaR and ES series. The general form of this functional family is:

$$S_t(r_t, Q_t, ES_t) = (I_t - \alpha)G_1(Q_t) - I_tG_1(r_t) + G_2(ES_t) \left( ES_t - Q_t + \frac{I_t}{\alpha}(Q_t - r_t) \right) - H(ES_t) + a(r_t),$$

where $I_t = 1$ if $r_t < Q_t$ and 0 otherwise, for $t = 1, \ldots, N$, $G_1(.)$ is increasing, $G_2(.)$ is strictly increasing and strictly convex, $G_2 = H'$ and $\lim_{x \to -\infty} G_2(x) = 0$ and $a(\cdot)$ is a real-valued integrable function.

As discussed in Taylor (2019), assuming $r_t$ to have zero mean, making the choices: $G_1(x) = 0$, $G_2(x) = -1/x$, $H(x) = -\log(-x)$ and $a = 1 - \log(1 - \alpha)$, which satisfy the required criteria, returns the scoring function:

$$S_t(r_t, Q_t, ES_t) = -\log \left( \frac{\alpha - 1}{ES_t} \right) - \frac{(r_t - Q_t)(\alpha - I(r_t \leq Q_t))}{\alpha ES_t},$$

where the aggregated loss is indicates as $S = \sum_{t=1}^{N} S_t$. Taylor (2019) refers to expression (9) as the AL log score. The negative of Equation (9) then can be treated as the AL log-likelihood, and is a strictly consistent scoring rule that is jointly minimized by the true VaR and ES series.
4 THE WEIGHTED QUANTILE ESTIMATOR

In this section, we illustrate the proposed two-step approach for semi-parametric estimation of Expected Shortfall. At step 1, we obtain semi-parametric estimates of VaR over a pre-defined set of risk levels \( \leq \alpha \). Then, at step-2, conditional on the estimates obtained at step 1 and relying on the representation of ES in Equation (2), an estimate of the conditional ES is obtained as an affine function of the 1st-stage VaR estimates given by a weighted average plus a constant. For these reasons we refer to our approach as the Weighted Quantile estimator.

Compared to a simple average, working with an affine transformation offers some important advantages. First, this specification allows to easily control for left-tail truncation bias. Second, since the weights are fitted via the optimization of a strictly consistent scoring rule for \((Q_t, ES_t)\), it is potentially possible to obtain relevant gains in terms of accuracy in the estimation of VaR and ES.

Next, we provide a detailed description of the two steps of the proposed estimation procedure. Although, for ease of explanation, we focus on the standard risk level \( \alpha = 2.5\% \), the method can be immediately extended to other values of \( \alpha \).

Step-1:

Under step 1, given the target quantile level \( \alpha = 2.5\% \), an equally spaced grid of quantile levels of size \( M \) is selected,

\[
\alpha_M = [\alpha_1, \alpha_2, \ldots, \alpha_M],
\]

where \( \alpha_m = \alpha_{m-1} + \eta \), with \( \eta = (\alpha - \alpha_1)/(M - 1) \) and \( \alpha_M = \alpha \), for \( m = 2, \ldots, M \). The value of the lower bound \( \alpha_1 \) can be selected on a case-by-case basis, mainly taking into account the length of the available in-sample returns series. As an example, with \( M = 10 \) and \( \alpha = 2.5\% \), fixing \( \alpha_1 = 0.005 \) we have \( \eta = 0.0022 \) and the following grid of quantile levels

\[
\alpha_M = [0.005, 0.0072, 0.0094, 0.0117, 0.0139, 0.0161, 0.0183, 0.0206, 0.0228, 0.025].
\]
Then, $M$ CAViaR models are separately estimated for each of the quantile orders in $\alpha_M$. For illustrative purposes, without implying any loss of generality, we here refer to the CAViaR symmetric absolute value (CAViaR-SAV) framework

$$Q_t = \beta_0 + \beta_1|r_{t-1}| + \beta_2 Q_{t-1}. \quad (10)$$

The proposed procedure can however be immediately extended to consider different conditional quantile models of different nature and complexity, such as CAViaR with asymmetric specification (CAViaR-AS) or nonlinear threshold specification.

For each trial quantile level $\alpha_m \in \alpha_M$, the above CAViaR-SAV model is then used to produce the time series of conditional in-sample quantiles $Q_{1:N,\alpha_m}$ and quantile forecasts $\hat{Q}_{N+1,\alpha_m}$, for $m = 1, \ldots, M$. The set of in-sample quantiles at all trial quantile levels, from $\alpha_1$ to $\alpha_M$, is collected in the $N \times M$ matrix $Q_{1:N}$. Here, the last ($M$-th) column of $Q_{1:N}$ corresponds to the time series of 2.5% in-sample conditional quantiles: $Q_{1:N,0.025}$. Similarly, we use the notation $\hat{Q}_{N+1}$ to indicate the $1 \times M$ vector of 1-step-ahead VaR forecasts at all trial quantile levels. The last element in $\hat{Q}_{N+1}$ represents the VaR forecast at the target 2.5% level: $\hat{Q}_{N+1,0.025}$.

Lastly, we would like to emphasize that the proposed framework can actually incorporate quantile estimates obtained from any model (not necessarily CAViaR), while we leave this for future research.

**Step-2:**

In the second stage of our approach, we predict the conditional ES as an affine function of the elements of $\hat{Q}_{N+1}$.

More precisely, the in-sample conditional ES at time $t$ is modelled as

$$ES_t^{(wp)} = w_0 + \sum_{i=1}^{M} w_i Q_{t,\alpha_i}, \quad (11)$$

where the weights $w_i$, $i = 1, \ldots, M$, are generated by some flexible and parsimonious function. A suitable choice is given by the Beta lag function borrowed from the literature on mixed data sampling and distributed lag models (see Ghysels et al. 2007, among
others). Namely, for $i = 1, \ldots, M$, $w_i = w \left( \frac{i}{M}; a, b \right)$ with

$$w(x; a, b) = \frac{x^{a-1}(1-x)^{b-1}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}.$$ \hspace{1cm} (12)

As previously discussed, the estimation of ES based on numerical integration of the tail quantiles is inherently affected by truncation bias since the summation on the RHS of (11) does not involve conditional quantiles of order below $\alpha_1$. In Equation (11) we control for truncation bias in two different ways. First, we include the intercept term $w_0$ in order to control for fixed bias. Second, the sum of weights appearing on the RHS of (11), $\theta = \sum_{i=1}^{M} w_i$, has been deliberately left unconstrained in order to allow the size of bias to depend on the average tail VaR level.

**Remark 1.** It is worth noting that, letting $\tilde{w}_i = w_i/\theta$, Equation (11) can be alternatively written as

$$ES_{t}^{(wq)} = w_0 + \theta \sum_{i=1}^{M} \tilde{w}_i Q_{t,\alpha_i},$$ \hspace{1cm} (13)

where $\sum_{i=1}^{M} \tilde{w}_i = 1$ by construction. The reparameterization in (13) makes evident the role of $\theta$ for bias correction.

The main reasons for adopting the Beta Lag specification to model the weights behaviour in Equation (11) are its parsimony, since it only depends on two parameters, and flexibility. Figure 1 displays various patterns that can be generated from the weight structure defined in Equation (12) for different values of the coefficients $a$ and $b$. To facilitate comparison among different patterns, the weights in the plots have been normalized so that they sum up to unity ($\theta = 1$). Constraining $a$ or $b$ to be equal to 1, a zero-modal behaviour is observed. Namely, for $a = 1$ and $b > 1$, the Beta Lag function returns declining weights while, choosing $a > 1$ and $b = 1$, increasing weights are obtained. The value of the unconstrained coefficient determines the speed of decay of the curve. Removing the unity constraint on either $a$ or $b$ makes the curve more flexible and allows to reproduce unimodal, hump-shaped behaviors such as those observed in the lower panel of Figure 1. Mixtures of Beta Lag polynomials could also be used to further increase the flexibility of
Figure 1: The figure displays various weighting patterns generated by the Beta Lag function for different values of the parameters $a$ and $b$ (from left to right and from top to bottom): $(a = 1, b = 4)$, $(a = 1, b = 20)$, $(a = 4, b = 1)$, $(a = 20, b = 1)$, $(a = 3, b = 8)$, $(a = 6, b = 8)$, $(a = 8, b = 3)$, $(a = 8, b = 6)$. 
the curve, as explored in Ghysels et al. (2007).

The only unknown parameters in Equation (11) are \((w_0, a, b)\). Conditional on the fitted VaR series, these can be estimated minimizing a strictly consistent scoring rule that is

\[
(\hat{w}_0, \hat{a}, \hat{b}) = \arg \min_{(w_0, a, b)} \sum_{t=1}^{N} S_t(r_t, ES_t; w_0, a, b|Q_{1:N})
\]

where

\[
S_t(r_t, ES_t; w_0, a, b|Q_t) = (I_t - \alpha)G_1(Q_t) - I_tG_1(r_t) + G_2(ES_t) \left( ES_t - Q_t + \frac{I_t}{\alpha}(Q_t - r_t) \right)
- H(ES_t) + a(r_t),
\]

and \(ES_t\) is defined as in (11).

One-step-ahead forecasts of \(ES_t\) can then be easily computed by replacing estimated in-sample quantiles, on the RHS of (11), by their out-of-sample forecasts obtained from the associated CAViaR models \((\hat{Q}_{N+1})\). Formally, the ES predictor at time \(N+1\), conditional on in-sample information available at time \(N\), is obtained as

\[
\hat{ES}_{N+1,\alpha} = w_{0,N} + \sum_{i=1}^{M} w_{i,N}\hat{Q}_{N+1,\alpha},
\]

where the subscript \(N\) in \(w_{i,N}\) indicates that the weight function is estimated using information up to time \(N\).

The number of grid points \(M\) is a hyper-parameter that we need to choose. Comprehensive simulation and empirical studies are conducted on testing the effects of incorporating various \(M\). Overall, \(M\) affects the trade off between accuracy and computational cost. However, our weighted quantile framework is turned out to be capable of accurately predicting the ES using a very small value of \(M\), i.e. \(M = 3\). In the simulation study, to demonstrate the effect of \(M\), we have tested \(M = 3, M = 5, M = 10\) and \(M = 50\) respectively.

In our empirical investigations, as a robustness check, we compare the performance of the weighted quantile framework in Equation (11) with a simpler approach replacing
the weighted average with an equally weighted average of quantiles:

\[ ES_t^{(avg)} = w_0 + \frac{\sum_{i=1}^{M} Q_{t,\alpha_i}}{M}. \]  

(15)

In the empirical section, we found that this framework is however consistently outperformed by the more complex weighted quantile estimator.

**Remark 2.** As presented in Figure 1, the last element of the sequence of weights generated from the Beta Lag function (i.e. the one corresponding to the risk level \( \alpha_M = \alpha \)) is by construction equal to 0, except when parameter \( b \) equals to 1. To address this issue, in the implementation, we set the number of grid points equal to \( M + 1 \), so that the weight of the \( \alpha_M \)-quantile is not 0 by construction. Referring to the previous example, to estimate ES at level \( \alpha = 2.5\% \) as a weighted average of \( M = 10 \) tail quantiles of order \( \alpha_i \leq \alpha \), setting \( \alpha_1 = 0.005 \), we simply use an equally spaced grid of \( M + 1 = 11 \) points, with \( \alpha_M = 0.025 \).

\[ \alpha_{M+1} = [0.005, 0.0072, 0.0094, 0.0117, 0.0139, 0.0161, 0.0183, 0.0206, 0.0228, 0.025, 0.0272]. \]

At no additional cost in terms of estimated parameters, this simple solution guarantees that the estimated weight for quantile at level \( \alpha_{M+1} = 0.0272 \) will always be 0 while assigning a positive weight to the \( \alpha_M = 0.025 \) quantile. In this way, consistently with its theoretical definition, the ES will be estimated as a weighted average of \( M \) quantiles of order \( \alpha_i \leq \alpha \) \((i = 1, \ldots, M)\) without systematically excluding the estimated VaR at the target level \( \alpha \).

## 5 Implied ES Dynamics

In this section we investigate the dynamic properties of the estimated conditional ES series obtained through the Weighted Quantile estimator. Assuming, for ease of presentation,
that the first-stage VaR series are generated by a CAViaR-SAV model, it is easy to show that Equation (11) can be rewritten as

$$ES_t = w_0 + \bar{\beta}_0 + \bar{\beta}_1 |r_{t-1}| + \sum_{i=1}^{M} w_i \beta_{2,i} Q_{t-1,\alpha_i}$$

(16)

where $\bar{\beta}_k = \sum_{i=1}^{M} w_i \beta_{k,i}$, for $k = 0, 1$, and $(\beta_{0,i}, \beta_{1,i}, \beta_{2,i})$ are the parameters of the CAViaR-SAV model for the conditional $\alpha_i$-quantile of $r_t$, for $i = 1, \ldots, M$. If the CAViaR-SAV model is well specified, i.e. if log-returns are generated by the following GARCH-type process

$$r_t = h_t z_t, \quad z_t \sim \text{i.i.d.} (0, 1),$$

$$h_t = \omega + \gamma |r_{t-1}| + \delta h_{t-1}, \quad \omega > 0, \gamma > 0, \delta > 0.$$ 

$\beta_{2,i}$ will be constant across different quantile orders i.e. $\beta_{2,i} = \bar{\beta}_2 = \delta$, for $i = 1, \ldots, M$, as also largely confirmed by our empirical results on real financial data.

Equation (16) will then simplify to the following

$$ES_t = w_0 + \bar{\beta}_0 + \bar{\beta}_1 |r_{t-1}| + \bar{\beta}_2 \sum_{i=1}^{M} w_i Q_{i,t-1}$$

(17)

$$= w_0 + \bar{\beta}_0 + \bar{\beta}_1 |r_{t-1}| + \bar{\beta}_2 (ES_{t-1} - w_0)$$

(18)

$$= \beta^*_0 + \bar{\beta}_1 |r_{t-1}| + \bar{\beta}_2 ES_{t-1}$$

(19)

where $\beta^*_0 = w_0 (1 - \bar{\beta}_2) + \bar{\beta}_0$. These derivations show that, in our approach, the ES is allowed to have dynamics that are separate from those of VaR. At the same time, these are automatically implied by the dynamics of conditional quantiles in the tail below VaR, without requiring any additional ad-hoc assumptions.

Comparing our approach to other existing proposals, it should be remarked that our weighted quantile estimator is more flexible than that proposed in Model (6) by Taylor (2019), based on the assumption that the conditional ES is a multiplicative rescaling of the fitted VaR model. Also, it differs from the “additive” approach proposed in Model (5) of the same paper under two main respects. First, we directly model the dynamics of the ES rather than the difference between ES and VaR. Second, the ES estimates are continuously updated, and not only when VaR is violated, as in Taylor (2019).
As discussed in Section 4, the proposed framework involves two estimation steps: the first for VaR and the second for ES. These are described in detail below.

Step-1: tail VaRs

This step aims to estimate the CAViaR models at the proposed quantile levels $\alpha_M$ using a Quasi Maximum Likelihood (QML) approach, following Engle and Manganelli (2004). Although, for ease of presentation, we focus on the CAViaR-SAV model, the same procedure can be immediately extended to other variants of the CAViaR framework.

In the first step, the quantile regression equation parameters $(\beta_{0,\alpha_m}, \beta_{1,\alpha_m}, \beta_{2,\alpha_m})$ are separately estimated, for each $\alpha_i \in \alpha_M$, by minimizing the quantile loss function:

$$\frac{1}{N} \sum_{t=1}^{N} (\alpha_m - I(r_t < Q_{t,\alpha_m})) (r_t - Q_{t,\alpha_m})$$

$$m = 1, \ldots, M, \quad (20)$$

whose negative, as shown in Giacomini and Komunjer (2005) among others, can be interpreted as a quasi-likelihood function.

As documented by Engle and Manganelli (2004), solutions to the optimization of the quantile loss objective function can be heavily dependent on the chosen initial values. To account for this issue, we adopt a multi-start optimization procedure inspired by that suggested in the paper by Engle and Manganelli (2004). First, multiple (10,000 in our paper) candidate parameter starting vectors are generated from adequately chosen uniform random variables, leading to multiple and different locally optimal QML estimates. Then the top 2 (out of 10,000) sets of the parameters that produced the highest likelihood function values are used as starting values for another optimization round. Lastly, the final parameter estimates are selected as the ones producing higher objective function values from the 2 sets of starting values.

Step-2: ES

In the second step of the optimization, when the weighted quantile estimator $ES_t^{(wq)}$
in (11) is considered, the parameters to be estimated are the intercept term \( w_0 \) and the coefficients of the Beta lag function \((a, b)\). Conditional on first stage VaR estimates, these are estimated minimizing the AL log score function defined in (9). Differently, for the bias corrected simple average estimator \( ES_t^{(avg)} \) in (15), the only parameter to be estimated is intercept term \( w_0 \) which is also estimated by unconstrained optimization.

7 SIMULATION

In this section, simulation studies are conducted to assess the statistical properties and performances of the proposed models, with respect to the one-step-ahead VaR and ES estimation accuracy.

Namely, to compare the bias and efficiency of the proposed weighted and simple average quantile methods, both the mean and Root Mean Squared Error (RMSE) values are calculated over the replicated data sets.

The simulation design is structured as follows: 1000 replicated return series are generated from a Absolute Value (AV) GARCH-t model, considering various degrees of freedom (DoFs) in order to reproduce different tail behaviours. The simulated Data Generating Process (DGP) is specified in the vignette below as Simulation Model (21).

**Simulation Model: (AV GARCH-t)**

\[
\begin{align*}
  r_t &= \sigma_t \varepsilon_t, \quad (21) \\
  \sigma_t &= 0.02 + 0.10|r_{t-1}| + 0.85 \sigma_t,
\end{align*}
\]

where \( \varepsilon_t \overset{i.i.d.}{\sim} t_\nu(0, 1) \) with \( \nu \) indicating the DoFs parameter, equal to 5, 10 and 50 respectively.

To facilitate the comparisons with the findings of our real data application, the simulation has been performed considering as sample size \( n = 1900 \), that has been chosen to approximately match the length of the available in-sample period in our empirical application in Section 8.
The true one-step-ahead level \( \alpha \) VaR forecasts from the above simulation model are calculated as:

\[
\text{VaR}_{t+1} = \sigma_{t+1} t_{\nu}^{-1}(\alpha) \sqrt{\frac{\nu - 2}{\nu}},
\]

where \( t_{\nu}^{-1} \) is the inverse of Student-t’s CDF with the \( \nu \) degrees of freedom. Similarly, ES forecasts from the same model are calculated as:

\[
\text{ES}_{t+1} = -\sigma_{t+1} \left( \frac{g_{\nu}(t_{\nu}^{-1}(\alpha))}{\alpha} \right) \left( \frac{\nu + (t_{\nu}^{-1}(\alpha))^2}{\nu - 1} \right) \sqrt{\frac{\nu - 2}{\nu}},
\]

where \( g_{\nu} \) is the Student-t PDF.

These true VaR and ES forecasts are calculated for each data set and used to compute RMSE for the VaR and ES forecasts obtained from a CAViaR-SAV, for the VaR, and both the \( ES^{(weq)} \) and the \( ES^{(avg)} \) estimators, for the ES. The averages of the true and estimated VaR & ES, over the 1000 data sets, are given in Table 1 (“True” column).

Namely, the proposed weighted quantile estimators in (11), named as WQ-M with \( M \in \{3, 5, 10, 50\} \), are then computed for each of the simulated data sets\(^3\). For comparison, the simple average with bias correction in (15), denoted as SA-BC-M, and the simple average without bias correction in (15) with the constraint \( w_0 = 0 \), denoted as SA-No-BC-M, are also included in the simulation study.

The VaR and ES forecasting simulation results are summarized in Table 1. Since both the weighted and simple average quantile approaches used the exactly same “Step-1” quantile estimation process, the VaR\(_{n+1}\) results for both approaches are identical. We can clearly see that, as expected, the quantile forecasts have mean values that are quite close to true values with relatively small RMSE. This is evidently due to the fact that the CAViaR-SAV model is correctly specified under an AV GARCH-t DGP.

Focusing on the ES forecasting, the bias results clearly favor the weighted quantile estimator, compared to the simple average approaches, for all the values of \( \nu \) considered. Due to the extra uncertainty introduced when estimating the Beta lag function parameters, the simple average (including the bias correction term \( w_0 \)) approach produces smaller

\(^3\)Reminding the considerations in Remark 2, here \( M \) indicates the number of quantiles involved in the weighted average with non-zero weights.
RMSE. The simple average of quantiles, without any bias correction, is characterized by a substantially worse performance than the other approaches.

Regarding the impact of the number of averaged quantiles, it is interesting to note that its choice is not critical and, in general, the $ES^{(wq)}$ and $ES^{(avg)}$ estimators based on only $M = 3$ grid points (WQ-3 and SA-BC-3) are already characterized by good performances. In addition, using $M = 5, 10$ and $50$, we can still observe accurate and very close ES forecasting results. However, the $M = 50$ setting requires much higher computational cost compared to other options. Therefore, we have selected $M = 3, 5$ and $10$ for the empirical study.

The estimated Beta Lag weights for the 1000 simulated data sets of each data generation process are presented in Figure 2. For all the values of $\nu$, the fitted weights distribution is characterized by two modes respectively occurring at the lower truncation point of the selected grid of $\alpha$ values and immediately before the upper truncation point, that is the target ES order. The lower mode is evidently accounting for the truncation bias arising from the omission of extreme left quantiles and, as it could have been reasonably argued, its effect is more substantial for $\nu = 5$. Furthermore, it is worth noting that, in the absence of left truncation bias, the pattern of the Beta Lag weights would be expected to match the profile of the returns tail distribution.

Table 2 summarizes the simulated distribution of the estimated coefficients for the different settings of the $ES^{(wq)}$ and $ES^{(avg)}$ estimators that have been here considered. For SA-BC estimators, the average estimated $w_0$ intercept is, as expected, negative, correcting for the left tail truncation bias. Confirming our intuition, this is more substantial for heavy tailed processes that are for low values of $\nu$.

WQ estimators are more flexible since the bias correction takes place through both $w_0$ and $\theta = \sum_{i=1}^{M} w_i$. The average value of the estimated intercept is positive being compensated by the fact that the average estimated $\theta$ is greater than 1, as expected. Again, in line with our findings for the SA-BC estimator, the difference $(\theta - 1)$ is higher for lower values of the degrees of freedom parameter $\nu$.

Overall, the simulation results illustrate the validity of the proposed models and the
corresponding estimation process. The performance of the weighted and simple average quantile approaches will be further compared in the empirical section.

Table 1: Simulation results with $M = 3, 5, 10,$ and 50 equally spaced averaged quantiles Summary statistics for proposed models, with data simulated from Model (21). Note that the estimation grid actually includes $M + 1$ values, that is $M + 1 = 4, 6, 11$ and 51, with the last Beta weight being equal to 0 by construction. See Remark 2 for details.

| $\nu$ = 5 | True | WQ-3 | SA-BC-3 | SA-No-BC-3 |
|-----------|------|------|---------|------------|
| VaR$_{t+1}$ | -1.3032 | -1.3096 | 0.1572 | -1.3096 | 0.1572 |
| ES$_{t+1}$ | -1.7853 | -1.8093 | 0.2504 | -1.7759 | 0.2290 | -1.6381 | 0.2653 |

| $\nu$ = 10 | True | WQ-3 | SA-BC-3 | SA-No-BC-3 |
|-----------|------|------|---------|------------|
| VaR$_{t+1}$ | -1.3775 | -1.3798 | 0.1271 | -1.3798 | 0.1271 | -1.3798 | 0.1271 |
| ES$_{t+1}$ | -1.7428 | -1.7591 | 0.1738 | -1.7287 | 0.1677 | -1.6348 | 0.1954 |

| $\nu$ = 50 | True | WQ-3 | SA-BC-3 | SA-No-BC-3 |
|-----------|------|------|---------|------------|
| VaR$_{t+1}$ | -1.3821 | -1.3785 | 0.1162 | -1.3785 | 0.1162 | -1.3785 | 0.1162 |
| ES$_{t+1}$ | -1.6657 | -1.6760 | 0.1408 | -1.6525 | 0.1373 | -1.5830 | 0.1595 |

Note: A box indicates the favored estimators, based on mean and RMSE.
Figure 2: Estimated Beta weights for the 1000 simulated data sets of each data generating process, $M = 10$. 
Table 2: Average of the estimated parameters (for WQ and SA-BC approaches) and Beta weights sum (for WQ approach), across the 1000 simulated data sets from Model (21).

| $n = 1900$ | WQ-3 | SA-BC-3 |
|------------|------|---------|
| $\nu = 5$ | $0.1379$ | $1.2358$ | $3.6619$ | $1.1597$ | $-0.1378$ |
| $\nu = 10$ | $0.0805$ | $1.3804$ | $3.0601$ | $1.1052$ | $-0.0919$ |
| $\nu = 50$ | $0.0469$ | $1.5393$ | $2.5897$ | $1.0812$ | $-0.0695$ |

| $\nu = 5$ | $0.2037$ | $1.7807$ | $3.8162$ | $1.2029$ | $-0.1656$ |
| $\nu = 10$ | $0.0834$ | $1.9007$ | $3.0300$ | $1.1091$ | $-0.1104$ |
| $\nu = 50$ | $0.0473$ | $1.9821$ | $2.7865$ | $1.0775$ | $-0.0822$ |

| $\nu = 5$ | $0.1821$ | $2.5134$ | $3.9378$ | $1.2109$ | $-0.1810$ |
| $\nu = 10$ | $0.0800$ | $2.5743$ | $3.3084$ | $1.1206$ | $-0.1216$ |
| $\nu = 50$ | $0.0348$ | $2.5173$ | $3.2810$ | $1.0835$ | $-0.0899$ |

| $\nu = 5$ | $0.1548$ | $3.5111$ | $4.6744$ | $1.2128$ | $-0.1913$ |
| $\nu = 10$ | $0.0641$ | $3.4378$ | $4.3128$ | $1.1191$ | $-0.1285$ |
| $\nu = 50$ | $0.0306$ | $3.4039$ | $4.1765$ | $1.0785$ | $-0.0945$ |
8 DATA and EMPIRICAL STUDY

8.1 Data and empirical study design

The daily data including open, high, low and closing prices, are downloaded from Thomson Reuters Tick History and cover the period from the beginning of 2000 to the end of 2015. Data are collected for 7 market indices: S&P500, NASDAQ (both US), Hang Seng (Hong Kong), FTSE 100 (UK), DAX (Germany), SMI (Swiss) and ASX200 (Australia).

A rolling window with fixed in-sample size is employed for estimation to produce each one-step-ahead forecast in the forecasting period. Table 3 reports the in-sample size for each series, which differs due to different non-trading days occurring in each market.

Two forecasting studies with different out-of-sample sizes are conducted. The first study aims to assess the performance of the models specifically for the 2008 Global Financial Crisis (GFC) period, thus the initial date of the out-of-sample forecasting period is chosen as January 2008. Then for each index the out-of-sample size $m$ is chosen as 400, meaning that the end of the forecasting period is approximately falling around August 2009.

The second forecasting study incorporates a 8 year out-of-sample period, with the start date of the out-of-sample still chosen as Jan 2008 and out-of-sample size $m$ as 2000. Therefore, the end of the forecasting period is around end of 2015.

Both daily one-step-ahead Value-at-Risk (VaR) and Expected Shortfall (ES) forecasts are considered for the returns on the 7 indices, using $\alpha = 2.5\%$, as recommended by the Basel Committee on Banking Supervision (2019).

For ES prediction, we have implemented the weighted quantile approach WQ-M, with $M \in \{3, 5, 10\}$. Similar to the simulation section, we have also considered the simple average with bias correction, SA-BC-M, and the simple average without bias correction, SA-No-BC-M. For the prediction of first stage quantile forecasts, two different regression specifications, CAViaR-SAV and CAViaR-AS, have been implemented. The estimation of 1st-stage CAViaR models has been performed following the procedure described in Section

24
Furthermore, the forecasting performances of the methods proposed in this paper have been compared with those yielded by other previously proposed approaches. Namely, the ES-CAViaR models of Taylor (2019) are also included in the study, again employing the CAViaR-SAV and CAViaR-AS models for the specification of the quantile regression component. These models are estimated following the suggestions from Taylor (2019).

To make the models comparable, the CAViaR components of our proposed weighted quantile and ES-CAViaR models have used exactly the same set up. Then, to assist the optimization in the estimation of ES-CAViaR models, the initial values of the parameters of the ES component are also selected by means of an additional random sampling procedure. When the ES-CAViaR-Add model of expression (5) is used for the ES, $10^4$ candidate parameter vectors are incorporated. For the simpler ES-CAViaR-Mult Model (6), $10^3$ candidate parameter vectors are used.

In addition, the following models have also been included in the forecasting comparison: the conventional GARCH (Bollerslev (1986)), EGARCH (Nelson (1991)) and GJR-GARCH (Glosten et al. (1993)), all with Student-t errors; the GARCH employing Hansen’s skewed-t distribution (Hansen (1994)); the CARE (using the size of the expectile level grid as 50) with Symmetric Absolute Value (CARE-SAV) and asymmetric specifications (CARE-AS). The GARCH-t, EGARCH-t and GJR-GARCH-t models are estimated using the Econometrics toolbox included in the Matlab 2019b release. The GARCH-Skew-t and CARE models are estimated by maximum likelihood using the Matlab code developed by the authors.

### 8.2 Evaluation of forecasting performance: Quantile Loss

One-step-ahead forecasts of VaR and ES are generated for each day in the forecast period for each data series.

The standard quantile loss function is also employed to compare the models for VaR forecast accuracy. Since the standard quantile loss function is strictly consistent, i.e. the
expected loss is a minimum at the true quantile series. Thus, the most accurate VaR forecasting model should produce the minimized quantile loss function, given as:

\[
\sum_{t=n+1}^{n+m} (\alpha - I(r_t < Q_t))(r_t - Q_t),
\]

where \( n \) is the in-sample size, and \( m \) is the out-of-sample size with \( m \in \{400, 2000\} \). \( Q_{n+1}, \ldots, Q_{n+m} \) is a series of quantile forecasts at level \( \alpha = 2.5\% \) for the observations \( r_{n+1}, \ldots, r_{n+m} \).

The quantile loss results are presented in Table 3 for each model for each series. The average loss is included in the “Avg Loss” column. The average rank based on ranks of quantile loss across 7 markets is calculated and shown in the “Avg Rank” column. Box indicates the favoured model and dashed box indicates the 2nd ranked model based on the average loss and rank.

It is worth reminding that, in the first stage of the weighted quantile estimation, VaR predictions are obtained via the estimation of either CAViaR-SAV or CAViaR-AS models, named as WQ-SAV or WQ-AS in Table 3. Depending on their ability to account for leverage effects in VaR dynamics, the tested models can be grouped into two categories: symmetric and asymmetric. For example, the GARCH-t, CARE-SAV, ES-CAViaR-Add-SAV, ES-CAViaR-Mult-SAV and WQ-SAV models have symmetric volatility or quantile (expectile) component, while the EGARCH-t, GJR-GARCH-t, CARE-AS, ES-CAViaR-Add-AS, ES-CAViaR-Mult-AS and WQ-AS have asymmetric ones.

Based on the quantile loss results, we can see that the proposed weighted quantile and ES-CAViaR type models are characterized by very close performances.

Also, in general and as expected, asymmetric models tend to perform slightly better than symmetric ones. For the SAV type models, the average quantile loss is around 58, while, for the AS type models, this average stays around 56, regarding the forecasting study on GFC period. For the study with longer forecasting horizon, the SAV and AS type models have average quantile loss around 172 and 167 respectively. This is not surprising since, as presented in Section 8.1, we have exactly the same CAViaR component for the weighted quantile and ES-CAViaR models, to make the ES comparison a fair one.
Therefore, the empirical results lend evidence on this. The ES-CAViaR framework has the CAViaR parameters re-estimated when estimating ES, thus we observe minor quantile loss differences between the WQ and ES-CAViaR frameworks.

For both forecasting studies, EGARCH-t, GJR-GARCH-t and CARE-AS have slightly higher average quantile loss values, compared with ES-CAViaR-Add-AS, ES-CAViaR-Add-AS and WQ-AS type models. The symmetric GARCH-t and CARE-SAV have relatively less preferred performance compared with the ES-CAViaR-Add-SAV, ES-CAViaR-Mult-SAV and WQ-SAV models.

Table 3: 2.5% quantile loss function values across the markets.

| Model              | S&P500 | NASDAQ | HangSeng | FTSE  | DAX   | SMI   | ASX200 | Avg Loss | Avg Rank |
|--------------------|--------|--------|----------|-------|-------|-------|--------|----------|----------|
| GARCH-t            | 57.6   | 62.4   | 72.8     | 56.1  | 55.6  | 55.1  | 48.1   | 58.2     | 8.29     |
| EGARCH-t           | 59.8   | 62.8   | 67.1     | 54.7  | 53.4  | 51.4  | 47.8   | 56.7     | 4.57     |
| GJR-GARCH-t        | 55.4   | 62.0   | 67.3     | 54.9  | 54.0  | 52.3  | 47.2   | 56.2     | 4.13     |
| GARCH-Skew-t       | 56.5   | 61.8   | 72.6     | 55.3  | 54.4  | 54.9  | 47.0   | 57.5     | 5.71     |
| CARE-SAV           | 60.2   | 65.3   | 66.3     | 57.0  | 58.6  | 53.9  | 51.8   | 59.0     | 10.14    |
| CARE-AS            | 61.1   | 66.8   | 61.7     | 55.0  | 55.9  | 54.9  | 48.9   | 57.7     | 8.71     |
| ES-CAViaR-Add-AS   | 60.6   | 63.0   | 63.2     | 52.9  | 53.7  | 51.4  | 47.0   | 56.0     | 4.00     |
| ES-CAViaR-Mult-AS  | 59.1   | 63.7   | 65.0     | 52.8  | 54.4  | 51.6  | 46.0   | 56.1     | 4.14     |
| ES-CAViaR-Add-SAV  | 59.6   | 63.8   | 68.5     | 55.7  | 56.8  | 53.3  | 48.1   | 56.0     | 8.71     |
| ES-CAViaR-Mult-SAV | 58.7   | 63.4   | 79.3     | 55.3  | 57.1  | 53.3  | 47.8   | 58.0     | 7.43     |
| WQ-AS              | 59.9   | 63.3   | 63.8     | 53.1  | 53.6  | 51.8  | 46.6   | 56.0     | 4.29     |
| WQ-SAV             | 59.4   | 63.1   | 69.6     | 55.7  | 56.8  | 53.3  | 47.9   | 58.0     | 7.86     |

Out of sample: m=400, n=400
In sample: n=2000, m=2000

| Model              | 2000   | 2000   | 2000    | 2000   | 2000   | 2000   | 2000   |
|--------------------|--------|--------|---------|--------|--------|--------|--------|
| GARCH-t            | 168.3  | 187.2  | 205.3   | 160.4  | 188.8  | 164.4  | 145.2  | 174.2    | 11.29    |
| EGARCH-t           | 166.9  | 184.0  | 194.9   | 155.2  | 181.6  | 159.3  | 140.3  | 168.9    | 4.71     |
| GJR-GARCH-t        | 162.9  | 183.1  | 196.4   | 156.4  | 182.9  | 159.4  | 141.7  | 169.0    | 4.71     |
| GARCH-Skew-t       | 165.4  | 183.7  | 204.6   | 158.7  | 185.4  | 163.3  | 142.7  | 172.0    | 7.14     |
| CARE               | 168.8  | 185.9  | 200.0   | 159.4  | 188.1  | 165.7  | 146.7  | 173.5    | 10.71    |
| CARE-AS            | 168.2  | 184.2  | 189.8   | 154.6  | 181.0  | 163.5  | 142.8  | 169.2    | 6.00     |
| ES-CAViaR-Add-AS   | 166.5  | 181.5  | 190.4   | 153.2  | 179.2  | 158.6  | 140.6  | 167.1    | 2.41     |
| ES-CAViaR-Mult-AS  | 165.2  | 182.7  | 192.4   | 152.7  | 179.8  | 158.4  | 139.2  | 167.2    | 2.18     |
| ES-CAViaR-Add-SAV  | 169.2  | 185.4  | 202.0   | 158.8  | 187.7  | 162.5  | 143.3  | 172.7    | 9.14     |
| ES-CAViaR-Mult-SAV | 168.1  | 184.9  | 203.1   | 158.5  | 187.7  | 162.9  | 143.5  | 172.7    | 8.57     |
| WQ-AS              | 167.0  | 182.4  | 191.0   | 153.2  | 179.3  | 159.0  | 139.7  | 167.4    | 2.86     |
| WQ-SAV             | 168.7  | 184.3  | 204.1   | 158.3  | 187.7  | 161.9  | 142.8  | 172.5    | 8.29     |

Note: Box indicates the favoured model and dashed box indicates the 2nd ranked model based on the average loss and rank.
8.3 Evaluation of forecasting performance: VaR and ES Joint Loss

In this section we assess the ability of the different models under comparison to forecast VaR and ES jointly. To this purpose, Table 4 reports, for each model and data series, the value of the loss function in Equation (9) aggregated over the out-of-sample period: 

\[ S = \sum_{t=n+1}^{n+m} S_t, \quad \text{with} \quad m \in \{400, 2000\}. \]

We use this to jointly compare the VaR and ES forecasts from all models, because the AL log-score in Equation (9) is a strictly consistent scoring rule that is jointly minimized by the true VaR and ES series.

As mentioned in Section 8.1, for ES prediction, incorporating \( M \in \{3, 5, 10\} \) we have implemented the weighted quantile approach WQ-M, the simple average with bias correction SA-BC-M, and the simple average without bias correction SA-No-BC-M. By further incorporating the CAViaR-SAV and CAViaR-AS, we have 18 frameworks to be tested. Including the other 10 competing models, we have 28 models in total in Table 4.

With respect to the forecasting study on GFC period, based on the average VaR & ES joint loss values the proposed WQ-3-AS produces the smallest loss, followed by ES-CAViaR-Mult-AS. The WQ-3-AS model also on average ranks as the best, followed by the WQ-5-AS. As discussed in Section 8.2, we have employed the same CAViaR component for weighted quantile and ES-CAViaR models. Therefore, the good performance of the proposed weighted quantile framework lends evidence on its validity for predicting ES.

In addition, the WQ-AS models on average rank better and produce lower loss compared to EGARCH-t and GJR-GARCH-t models. Comparing the ES-CAViaR-SAV type models with the WQ-SAV frameworks, we can still see that the WQ-SAV models, based on different numbers of grid points \( M \), have lower average loss and rank better than ES-CAViaR-Add-SAV and similar performance compared with ES-CAViaR-Mult-SAV. On the other end, the CARE-SAV model on average produces the highest average joint loss. The SA-No-BC frameworks produce slightly smaller loss values and rank similar, compared to CARE-SAV.

Lastly, the weighted quantile framework has consistently improved performance than
the SA-BC, which demonstrates the usefulness of weighted average scheme. In addition, the performance of SA-BC is clearly better compared with SA-No-BC, demonstrating the effectiveness of the bias correction.

Regarding the forecasting study with out-of-sample size 2000, the top 2 performed models are ES-CAViaR-Mult-AS and ES-CAViaR-Add-AS models. The SA-BC-3-AS and WQ-3-AS frameworks rank as the 3rd and 4th, with clear better performance compared to EGARCH-t, GJR-GARCH-t and CARE-AS.

Comparing the ES-CAViaR-SAV type models with the WQ-SAV type models, we still observe the proposed WQ-SAV frameworks have improved performance compared to ES-CAViaR-Add-SAV and close performance compared to ES-CAViaR-Mult-SAV.

Finally, we would like to emphasize that the WQ framework using $M = 3$ can already generate very competitive performance, for both forecasting studies. Such results lend evidence on the fact the proposed WQ framework can work effectively without having significantly increased computation cost, compared to other models. Compared with ES-CAViaR models, the WQ type models estimate and forecast ES nonparametrically, without assuming the relationships between the ES and VaR dynamics.

### 8.4 Model Confidence Set

In this section, the Model Confidence Set (MCS) ([Hansen et al., 2011](https://www.kevinsheppard.com/code/matlab/mfe-toolbox/)) is used to assess the statistical significance of differences in the values of the AL log-score observed for the various model under comparison, avoiding multiple testing biases.

A MCS is a set of models that is constructed such that it will contain the best model with a given level of confidence (75% is used in our paper). All computations were performed using the Matlab code for MCS testing included in Kevin Sheppard’s MFE toolbox. The R and SQ methods which use absolute and squared values sum respectively during the calculation of test statistic are employed in our paper, details as

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4The code can be downloaded at the url “https://www.kevinsheppard.com/code/matlab/mfe-toolbox/”
Table 5 presents the 75% MCS using both the R and SQ methods, for two forecasting studies with out-of-sample size \( m \) as 400 and 2000 respectively. Columns “R-Total-GFC” and “SQ-Total-GFC”, “R-Total” and “SQ-Total” count the total number of times that a model is included in the 75% MCS across the 7 return series.

Overall, we observe our weighted quantile models are more or equally likely to be included in MCS, in comparison with other models. For both R and SQ methods across two forecasting studies, ES-CAViaR-Mult-AS, WQ-3-AS and WQ-10-AS are the only 3 models that are included in MCS for all 7 series.

More specifically for the forecasting study on GFC period, via the R method, ES-CAViaR-Mult-AS, WQ-3-AS and WQ-10-AS are included in the MCS for all 7 markets, followed by EGARCH-t, GJR-GARCH-t, ES-CAViaR-Add-AS, SA-BC-3-AS, WQ-5-AS and SA-BC-5-AS models. Via the SQ method, all the WQ-AS type models are included in the MCS for all the 7 markets, as well as the EGARCH-t, GJR-GARCH-t and ES-CAViaR-AS type models.

With respect to the forecasting study with \( m = 2000 \), for both R and SQ method, all the WQ-AS and SA-BC-AS models are included in MCS for 7 times, together with ES-CAViaR-AS models. We can see that EGARCH-t and GJR-GARCH-t are less likely to be included in MCS, compared to our proposed frameworks. The GARCH-t is least likely to be included in MCS for both R and SQ methods.
9 CONCLUSION

In this paper, we propose an innovative semi-parametric weighted quantile framework for ES estimation and forecasting. The proposed approach relies on a two step estimation procedure. The quantiles weighting scheme is parsimoniously parameterized by incorporating a Beta function whose coefficients are optimized by minimizing a consistent joint VaR and ES loss function. Through simulation study, we have demonstrated the effectiveness of the proposed framework. In an empirical study, focusing on the high volatile GFC period, improvements in the out-of-sample forecasting of ES are observed, compared to traditional GARCH and CARE models, as well as the ES-CAViaR models. Empirical evidence on a longer forecasting period confirms the superiority of the WQ framework over GARCH-type and CARE models and its competitiveness with state-of-the-art approaches such as the ES-CAViaR models.

The proposed framework can be extended in a number of ways. First, at the moment the first stage of the framework only uses quantile estimates from CAViaR. However, the proposed framework is quite flexible, so it can actually incorporate quantile estimates obtained from any models. Second, during the second stage of the estimation (when estimating the parameters of Beta function), we can also re-estimate the CAViaR parameters to potentially further improve the VaR and ES estimation accuracy, similar to the ES-CAViaR estimation. Third, the framework can be also extended by the idea of forecasting combination, see Taylor (2020) as example.

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## Table 4: 2.5% VaR and 2.5% ES joint loss function values across the markets.

| Model                        | SP500 | NASDAQ | HangSen | FTSE   | DAX    | SMI    | ASX200 | Avg Loss | Avg Rank |
|------------------------------|-------|--------|---------|--------|--------|--------|--------|----------|----------|
| GARCH-t                      | 3424.8| 4568.2 | 4708.7  | 4244.3 | 4658.4 | 4301.1 | 4071.6 | 4409.7   | 27.0     |
| EGARCH-t                     | 4290.6| 4571.6 | 4586.9  | 4192.1 | 4565.2 | 4245.2 | 3990.4 | 4344.0   | 16.6     |
| GJR-GARCH-t                  | 4292.9| 4490.2 | 4601.7  | 4183.8 | 4601.4 | 4253.6 | 4009.1 | 4339.9   | 13.1     |
| GARCH-Skew-t                 | 4254.7| 4491.0 | 4681.8  | 4194.0 | 4595.4 | 4245.3 | 4009.5 | 4353.1   | 15.3     |
| CARE                         | 4304.8| 4528.0 | 4664.2  | 4204.9 | 4606.2 | 4294.2 | 4080.8 | 4383.3   | 23.3     |
| CARE-AS                      | 4274.9| 4470.5 | 4550.7  | 4158.5 | 4518.5 | 4250.7 | 4031.3 | 4322.2   | 12.6     |
| ES-CAYiar-Add-AS             | 4242.3| 4439.2 | 4551.9  | 4313.8 | 4506.8 | 4192.1 | 3992.9 | 4259.5   | 17.7     |
| ES-CAYiar-Mult-AS            | 4242.3| 4451.2 | 4546.2  | 4175.8 | 4509.1 | 4188.3 | 3977.4 | 4259.2   | 14.4     |
| ES-CAYiar-Add-SAV            | 4304.4| 4505.2 | 4678.6  | 4199.8 | 4612.0 | 4246.3 | 4042.7 | 4399.6   | 22.9     |
| ES-CAYiar-Mult-SAV           | 4289.8| 4498.8 | 4673.7  | 4190.4 | 4609.0 | 4239.3 | 4042.4 | 4363.8   | 18.7     |
| Wq-3-AS                      | 4252.6| 4448.2 | 4553.1  | 4323.0 | 4507.8 | 4194.1 | 3982.0 | 4295.4   | 4.9      |
| SA-BC-3-AS                   | 4253.7| 4450.8 | 4551.9  | 4313.1 | 4507.3 | 4196.6 | 3979.5 | 4296.1   | 4.7      |
| SA-BC-3-AS-AS                | 4265.5| 4468.2 | 4555.2  | 4145.6 | 4520.3 | 4241.5 | 3985.7 | 4307.8   | 9.7      |
| Wq-5-AS                      | 4255.0| 4451.5 | 4552.3  | 4146.1 | 4509.6 | 4196.9 | 3977.7 | 4297.4   | 5.9      |
| SA-BC-5-AS                   | 4261.2| 4452.5 | 4549.5  | 4140.0 | 4505.4 | 4240.0 | 3976.3 | 4298.7   | 5.3      |
| SA-BC-5-AS-AS                | 4275.3| 4471.8 | 4555.0  | 4176.8 | 4520.9 | 4222.3 | 3984.5 | 4312.1   | 10.5     |
| Wq-16-AS                     | 4254.7| 4450.7 | 4552.7  | 4125.8 | 4509.3 | 4139.9 | 3985.9 | 4296.1   | 5.6      |
| SA-BC-10-AS                  | 4259.3| 4452.4 | 4549.8  | 4125.0 | 4508.5 | 4198.5 | 3983.4 | 4296.7   | 5.6      |
| SA-BC-10-AS-AS               | 4274.7| 4473.0 | 4554.2  | 4139.6 | 4525.3 | 4223.2 | 3993.2 | 4311.9   | 10.9     |
| Wq-3-SAV                     | 4292.5| 4492.5 | 4699.4  | 4191.9 | 4606.2 | 4237.4 | 4030.9 | 4364.4   | 17.3     |
| Wq-3-SAV-AS                  | 4293.5| 4495.5 | 4704.4  | 4190.3 | 4608.4 | 4240.2 | 4028.7 | 4365.9   | 18.9     |
| SAV                          | 4306.2| 4514.2 | 4716.8  | 4199.7 | 4619.5 | 4256.9 | 4030.0 | 4377.6   | 23.6     |
| Wq-5-SAV                     | 4291.1| 4492.4 | 4697.0  | 4189.6 | 4608.4 | 4238.5 | 4037.5 | 4364.9   | 17.7     |
| SA-BC-5-SAV                  | 4296.9| 4494.2 | 4701.6  | 4188.2 | 4610.0 | 4243.3 | 4033.6 | 4366.8   | 19.9     |
| SA-BC-5-SAV-AS               | 4311.1| 4513.9 | 4715.3  | 4199.4 | 4623.4 | 4262.7 | 4037.1 | 4380.4   | 24.6     |
| Wq-16-SAV                    | 4291.5| 4492.5 | 4697.6  | 4198.3 | 4606.6 | 4238.5 | 4037.6 | 4366.1   | 18.7     |
| SA-BC-10-SAV                 | 4296.5| 4494.1 | 4703.3  | 4198.0 | 4607.9 | 4242.4 | 4031.3 | 4367.5   | 19.7     |
| Wq-3-SAV-AS-AS               | 42931.2| 4515.3 | 4717.6  | 4211.5 | 4622.7 | 4240.3 | 4036.0 | 4382.5   | 25.7     |

**Note:** Box indicates the favoured model and dashed box indicates the 2nd ranked model based on the average loss and rank.
Table 5: 75% model confidence set results summary with R and SQ methods.

| Model              | R-Total-GFC | SQ-Total-GFC | R-Total | SQ-Total |
|--------------------|-------------|--------------|---------|----------|
| GARCH-t            | 4           | 3            | 1       | 2        |
| EGARCH-t           | 4           | 7            | 5       | 5        |
| GJR-GARCH-t        | 4           | 7            | 5       | 5        |
| GARCH-Skew-t       | 5           | 5            | 4       | 5        |
| CARE               | 4           | 6            | 2       | 3        |
| CARE-AS            | 4           | 5            | 5       | 7        |
| ES-CAViaR-Add-AS   | 4           | 7            | 7       | 7        |
| ES-CAViaR-Mult-AS  | 4           | 7            | 7       | 7        |
| ES-CAViaR-Add-SAV  | 5           | 5            | 2       | 4        |
| ES-CAViaR-Mult-SAV | 3           | 5            | 3       | 4        |
| WQ-3-AS            | 2           | 7            | 7       | 7        |
| SA-BC-3-AS         | 2           | 7            | 7       | 7        |
| SA-No-BC-3-AS      | 2           | 7            | 7       | 7        |
| WQ-5-AS            | 4           | 6            | 5       | 6        |
| SA-BC-5-AS         | 4           | 7            | 7       | 7        |
| SA-No-BC-5-AS      | 4           | 6            | 5       | 6        |
| WQ-10-AS           | 7           | 7            | 7       | 7        |
| SA-BC-10-AS        | 4           | 7            | 7       | 7        |
| SA-No-BC-10-AS     | 4           | 6            | 5       | 5        |
| WQ-3-SAV           | 3           | 5            | 5       | 4        |
| SA-BC-3-SAV        | 3           | 5            | 4       | 4        |
| SA-No-BC-3-SAV     | 2           | 3            | 3       | 4        |
| WQ-5-SAV           | 3           | 5            | 3       | 4        |
| SA-BC-5-SAV        | 3           | 5            | 4       | 4        |
| WQ-10-SAV          | 2           | 2            | 3       | 3        |
| SA-BC-10-SAV       | 2           | 5            | 3       | 4        |
| SA-No-BC-10-SAV    | 2           | 3            | 2       | 3        |

Out-of-sample m  | 400         | 400          | 2000    | 2000     |

Note: Boxes indicate the favoured model and dashed box indicates the 2nd ranked model, based on the number of times that a model is included in the MCS across the 7 markets (higher is better).