Abstract: We study \( N = 2 \) supersymmetric theory on a large family of squashed 4-spheres preserving \( SU(2) \times U(1) \) isometry and determine the conditions under which this background is supersymmetric. We then compute the partition function of this theory using localization technique. The results indicate that for \( N = 2 \) SUSY including both vector-multiplets and hypermultiplets, the partition function is independent of the arbitrary squashing functions as well as of the other supergravity background fields.
1 Introduction

Supersymmetric theories equipped with Localization techniques furnish a rich ground for exact computation of various quantities in Quantum Field Theory. This program started with the work of [1, 2] and was generalized to curved spaces by [3] and [4, 5]. A systematic way to put Rigid SUSY on curved spaces was worked out by [6, 7] and has been applied extensively to calculate partition functions and expectation values of Wilson loops on squashed spheres and more general manifolds in various dimensions [8–12]. Partition function on the squashed spheres depend in general on the squashing parameters. However for some squashing preserving a particular isometry of the manifold the partition function comes out to be independent of squashing parameters. For example [8] considered squashed $S^3$ with $SU(2) \times U(1)$ isometry and showed that the $N = 2$ supersymmetric partition function was independent of the squashing. Similarly results has been obtained by [13] for $S^4$ squashed in a specific way.

For the case of squashed $S^3$, the squashing independence of the partition function was due to the fact [8] that the eigenmodes of the kinetic operators for scalars and fermions form multiplets of same eigenvalues under $SU(2)$ isometry group. Komargodski et al. [14] analyzed the properties of $N = 2$ supersymmetric partition functions on on general 3-dimensional compact manifolds and showed that the partition function depends only on a geometrical structure called transversely holomorphic foliation. The $SU(2) \times U(1)$
squashing of $S^3$ does not change this geometrical structure and hence the partition function remains independent of the squashing. For the four dimensional case such analysis has been done only for $N = 1$ SUSY.

The squashing independence of the SUSY partition function has been used in an interesting way in [13, 15], who, using this independence, showed that the partition function on squashed $S^4$ is equal to that on a branched $S^4$. The SUSY partition function on the branched $S^4$ calculates the SUSY Renyi entropy of a circular region in a 4-dimensional space [16, 17]. Similar results for squashed 3-spheres have been obtained in [18, 19].

In this paper we calculate $N = 2$ supersymmetric partition function on a family of squashed $S^4$s with $SU(2) \times U(1)$ isometry including the instanton contribution and show that it is independent of the squashing parameters and the background. In the case of $N = 2$ on ellipsoid considered in [8] the isometry is $U(1) \times U(1)$. In the limit $l = r$ this symmetry is enhanced to $SO(3) \times SO(2)$ which is to be seen as the subgroup of $SO(5)$. However the $SU(2) \times U(1)$ isometry in our case is a subgroup of $SO(4) \equiv SU(2)_L \times SU(2)_R$. The paper is organized as follows. In section 2, the Killing spinor equations for $N = 2$ Rigid SUSY on squashed $S^4$ are given, in section 3 the squashed $S^4$ metric and spin connection components are given, section 4 we solve the Killing spinor equations, calculate various background fields and then we will give the conditions for the regularity of background fields. Section 5 contains the calculation of $Q^2$ parameters for the fields of vector-multiplet and hyper-multiplet. In sections 6 and 7, we find the saddle point configurations and one-loop determinants respectively for vector-multiplet and hyper-multiplet, closely following [20] and [3]. In section 8 we comment on the contribution of point instantons and anti-instantons to the supersymmetric partition function. The brief summary of the main results is given in section 9.

2 Rigid Supersymmetric Theories on Curved Spaces

By now the standard way [6] to put Rigid SUSY on a curved space is to couple the flat space version of the SUSY to a background supergravity multiplet. The supergravity multiplet contains graviton, gravitino and some auxiliary scalar and tensor field. To obtain a Rigid SUSY theory on the curved space, the quantum fluctuations of the gravitational background have to be decoupled by taking the Planck mass limit $M_P \to \infty$, and setting gravitinos to zero. Following this procedure [20] obtained a set of Killing spinor equations which have to be satisfied in order to obtain Rigid $N = 2$ SUSY on ellipsoid. For our case of squashed-$S^4$ the formalism remains effectively the same.

The Killing spinor equation consists of two sets of equations. The first set of equations is obtained by setting the SUSY variation of the gravitino to zero [20]:

\begin{align}
D_m \xi_A &+ T^{kl} \sigma_{kl} \sigma_m \xi_A = -i \sigma_m \xi_A', \\
D_{m} \xi_A + \bar{T}^{kl} \bar{\sigma}_{kl} \bar{\sigma}_m \xi_A = -i \bar{\sigma}_m \xi_A' \quad \text{for some} \quad \xi_A', \xi_A
\end{align}

and second set is obtained from SUSY variation of the dilatino [20]:

\begin{align}
\sigma^m \bar{\sigma}^n D_m D_n \xi_A + 4 D_l T_{mn} \sigma^{mn} \sigma^l \xi_A = M \xi_A,
\bar{\sigma}^m \sigma^n D_m D_n \bar{\xi}_A + 4 D_l T_{mn} \bar{\sigma}^{mn} \bar{\sigma}^l \bar{\xi}_A = M \bar{\xi}_A
\end{align}

where $M$ is a real scalar background field.

Where $\xi_A$ and $\bar{\xi}_A$ are chiral and anti-chiral Killing spinors satisfying reality conditions and are the parameters of $N = 2$ SUSY. The index $A$ belongs to the $SU(2)_R$ $R$-symmetry of the background. Here
$T^{kl}, T^{kl}$ are a self-dual and anti-self-dual real tensor background fields, and the covariant derivatives contain a background $SU(2)_{R}$ gauge field $(V_{m})^{b}_{A}$, in addition to spin connection $\Omega^{ab}_{m}$ (we use the conventions of[20]). In four component notation the main equation is written compactly as

$$D_{m}\xi + T.\Gamma_{m}\xi = -i\Gamma_{m}\xi'$$

(2.3)

where $T.\Gamma \equiv T_{k}\Gamma^{kl}$. Now multiplying from left by $\Gamma^{m}$ and using the Fierz identity $\Gamma_{m}\Gamma_{kl}\Gamma^{m} = 0$ we get

$$\xi_{p} \equiv \Gamma^{m}D_{m}\xi = -4\xi'$$

(2.4)

Here a new spinor $\xi_{p}$ is defined which will be useful later on when we will calculate the square of supersymmetry transformation $Q^{2}$ acting on different fields of $N = 2$ theory.

3 Squashed $S^{4}$

The family of squashed 4-spheres which we will consider are defined by the following metric or vielbein one-forms:

$$ds^{2} = dr^{2} + f(r)^{2}(d\theta^{2} + d\phi^{2} \sin^{2}(\theta)) + h(r)^{2}(d\psi + d\phi \cos(\theta))^{2}$$

$$e^{1} = dr, \quad e^{2} = f(r)\left(-\frac{1}{2}d\theta \cos(\psi) - \frac{1}{2}d\phi \sin(\theta) \sin(\psi)\right),$$

$$e^{3} = f(r)\left(-\frac{1}{2}d\theta \cos(\psi) - \frac{1}{2}d\phi \sin(\theta) \sin(\psi)\right), \quad e^{4} = h(r)\left(\frac{d\psi}{2} + \frac{1}{2}d\phi \cos(\theta)\right)$$

(3.1)

where $f(r)$ and $h(r)$ are smooth arbitrary functions of $r$ and are well defined near north and south poles. The above metric has $SU(2) \times U(1)$ isometry. The spin connection $\Omega^{ab}_{m} = dx^{m}\Omega^{ab}_{m}$ has the following components,

$$\Omega^{21}_{1} = 1 - \frac{h(r)^{2}}{2f(r)^{2}}, \quad \Omega^{41}_{1} = \frac{h'(r)}{2}, \quad \Omega^{31}_{2} = \frac{h(r)\sin(\psi)}{2f(r)}, \quad \Omega^{32}_{2} = \frac{h(r)\cos(\psi)}{2f(r)}$$

$$\Omega^{21}_{3} = \frac{1}{2}\cos(\psi)f'(r), \quad \Omega^{22}_{3} = -\frac{1}{2}\sin(\psi)f'(r), \quad \Omega^{31}_{3} = \cos(\theta) - \frac{h(r)^{2}\cos(\theta)}{2f(r)^{2}}, \quad \Omega^{32}_{3} = h(r)\sin(\theta)\sin(\psi)\sin(\psi)\sin^{2}(\theta)$$

$$\Omega^{33}_{3} = \frac{1}{2}\sin(\theta)\cos(\psi)\sin(\psi)\sin(\psi)\sin(\psi)\sin(\theta)$$

(3.2)

where $a,b = 1,...,4$ are flat indices and $m = 1,...,4$ is curved space index.

4 Solution of Killing Spinor Equation on the Squashed $S^{4}$

When a sphere is squashed, as in our case, the Killing spinor equation remains the same except that the covariant derivative contains additional background gauge fields[20]. The purpose of this section is to show that if these background fields $V_{mAB}, T_{mn}, \tilde{T}_{mn}, M$ are chosen appropriately, the squashed $S^{4}$
admits a Killing spinor which is solution of the two sets of Killing spinor equations 2.1 and 2.2. The complexified general form of background field $T$ and $V$ is the following

$$
V_m = \begin{pmatrix}
iv_{3m} & i(v_{1m} + iv_{2m}) \\
(i(v_{1m} - iv_{2m}) & -iv_{3m}
\end{pmatrix}
$$

$$
T = \begin{pmatrix}
ix_3 & i(t_1 - it_2) & 0 & 0 \\
ix(t_1 + it_2) & -ix_3 & 0 & 0 \\
0 & 0 & ix_3 & i(t_1 - it_2) \\
0 & 0 & i(t_1 + it_2) & -ix_3
\end{pmatrix}
$$

(4.1)

We will consider the following ansatz for the Killing spinor and will calculate the background fields $T$, $V$ and $M$ such that this ansatz satisfies the set of Killing spinor equations.

$$
\xi = \begin{pmatrix}
s_1(r) & 0 \\
0 & t_2(r) \\
s_3(r) & 0 \\
0 & t_4(r)
\end{pmatrix}
$$

(4.2)

The Killing spinor satisfies the reality condition[20]:

$$
(\xi_\alpha A) = \epsilon^{AB} \epsilon^{\alpha\beta} \xi_{\beta B}, \quad (\bar{\xi}_\alpha A) = \epsilon^{AB} \bar{\epsilon}^{\alpha\beta} \bar{\xi}_{\beta B}
$$

(4.3)

The parameters in the Killing spinor are smooth functions of $r$, but otherwise arbitrary. However after solving the Killing spinor equations, it turns out that some of these parameters are constrained. The most general simultaneous solution to the main and auxiliary equations takes the following form, where only the non-zero part of the background fields and Killing spinor components are given:

$$
s_1(r) = s(r)e^{-is_1p(r)}, \quad s_3(r) = \frac{ich(r)e^{is_1p(r)}}{s(r)}, \quad t_2(r) = s(r)e^{-is_1p(r)},
$$

$$
t_4(r) = -\frac{ich(r)e^{-is_1p(r)}}{s(r)}
$$

$$
t_3 = \frac{s(r)(f(r)(2f(r)s'(r) - s(r)f'(r)) + h(r)s(r))}{4ef(r)^2h(r)}
$$

$$
\bar{t}_3 = \frac{c(f(r)h(r)(s(r)f'(r) + 2f(r)s'(r)) - 2f(r)^2s(r)h'(r) + h(r)^2s(r))}{4f(r)^2s(r)^2}
$$

$$
V_{33} = \frac{1}{2}\left(\frac{h(r)}{f(r)} - 2 + \frac{h'(r) - 2}{h(r)} - \frac{2s'(r)}{s(r)}\right)
$$

$$
M = \frac{2f''(r)}{f(r)} + \frac{f'(r)^2 - 2h'(r) + \frac{4h(r)s'(r)}{s(r)}}{f(r)^2} + \frac{h(r)^2}{f(r)^4} + \frac{4s'(r)(s(r)h'(r) - h(r)s'(r))}{h(r)s(r)^2}
$$

$$
s_{1p}(r) = 0
$$

Here $c$ is a real constant with units of inverse length and the background fields $T$ and $V_m$ are indexed by flat tangent space indices. For these background fields to be well defined on the squashed $S^4$, it is necessary that $s(r)$ be non-zero between the two Poles.

We thus determined the form of all the background fields in order for $N = 2$ SUSY theory on the squashed 4-sphere to admit a rigid supersymmetry. We have set $v_{12} = 0$ and this choice of background preserves $SU(2) \times U(1) \times U(1)_R$ symmetry. With $v_{12} \neq 0$ it can be shown that the symmetry is reduced to $SU(2) \times U(1)'$ where $U(1)' \equiv (U(1) \times U(1)_R)_{\text{diagonal}}$. 

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$$
4.1 Regularity of the Background Fields

The requirement that our background reduces to $\Omega \rightarrow \text{background}$ at the North and South Poles implies that $f(r) = 0 \quad h(r) = 0$ at $r = 0$ and $r = \pi$. Moreover for our metric to be non-singular in the interval $\pi > r > 0$, the functions $f(r)$ and $h(r)$ are strictly non-zero and does not change sign inside the interval.

North Pole ($r = 0$): Near the North Pole the regularity of invariant quantities $R, R_{\mu \nu}R^{\mu \nu}$ and of the background fields both in flat tangent space indices and curved space indices, fixes $f(r)$, $h(r)$ and $s(r)$ in the following form

$$h(r) = r + h_n r^3$$
$$f(r) = r + f_n r^3$$
$$s(r) = s_{n0} + s_{n2} r^2 + s_{n3} r^3$$

(4.5)

There are higher order terms, but those are irrelevant to the present analysis.

South Pole ($r = \pi$): Similarly near the South Pole the regularity requirements fix $f(r)$, $h(r)$ and $s(r)$ in the following way

$$h(r) = \pi - r + h_s (\pi - r)^3$$
$$f(r) = \pi - r + f_s (\pi - r)^3$$
$$s(r) = (\pi - r)s_{s1} + (\pi - r)^2 s_{s3}$$

(4.6)

Where $h_n, f_n, s_{n0}, s_{n2}, s_{n3}, h_s, f_s, s_{s1}, s_{s3}$ are arbitrary real constants.

A quantity of interest which we want to calculate now is $(s(r)^2 - \frac{c^2 h(r)^2}{s(r)^2})$. At the north pole it evaluates to $s_{s0}^2$, whereas at the South Pole it evaluates to $-\frac{s_{s0}^2}{s_{s1}}$. So it has the interesting property that it changes sign between North and South poles and hence passes through zero. This result will have important consequences later on in section 7 when we will calculate the one-loop determinant.

Before passing on, we want to comment that there is an ambiguity in the choice of the functions $f(r), h(r)$ and $s(r)$ at the North and South Poles, that is, if we take following choice for these functions at the North Pole

$$h(r) = -r + h_n r^3$$
$$f(r) = r + f_n r^3$$
$$s(r) = s_{n1} + s_{n3} r^3$$

(4.7)

and the following choice at the South pole

$$h(r) = \pi - r + h_s (\pi - r)^3$$
$$f(r) = \pi - r + f_s (\pi - r)^3$$
$$s(r) = s_{s0} + s_{s2} (\pi - r)^2 + s_{s3} (\pi - r)^3$$

(4.8)

all the background fields are still regular there. The only difference is that the quantity $(s(r)^2 - \frac{c^2 h(r)^2}{s(r)^2})$ evaluates to $-\frac{s_{s0}^2}{s_{s1}}$ at the North pole and to $s_{s0}^2$ at the South pole. Every other result remains the same.

5 Multiplets

Vector multiplet

In N=2 SUSY, vector multiplets are made of a gauge field $A_m$, two independent gauginos(in the Euclidean signature) $\lambda_{\alpha A}, \bar{\lambda}_{\dot{\alpha} A}$, two scalar fields $\phi, \bar{\phi}$ and an auxiliary field $D_{AB} = D_{BA}$ all Lie algebra valued. These
fields satisfy reality conditions compatible with supersymmetry transformations \cite{20}. The supersymmetric Yang-Mills Lagrangian is given by \cite{20}:

\[ L_{YM} = \text{Tr} \left( \frac{1}{2} F_{mn}^a F^{mn} + 16 F_{mn}^a (\phi T^{mn} + \phi^T T^{mn}) + 64 \phi^2 T_{mn} T^{mn} + 64 \phi^2 T^{mn} T_{mn} \right) \]

\[ - 4D_m \phi D^m \phi + 2M \phi \bar{\phi} - 2\lambda^A \sigma^m D_m \lambda_A - 2\lambda^A [\phi, \lambda_A] + 2\lambda^A [\phi, \bar{\lambda}] + 4[\phi, \bar{\phi}]^2 - \frac{1}{2} D^{AB} D_{AB} \]

and is supersymmetric with respect to the SUSY transformation rules given in \cite{20}. Including the topological term the action is \cite{20}:

\[ S_{YM} = \frac{1}{g^2_{YM}} \int d^4 x \sqrt{g_{YM}} + \frac{\ell}{8\pi^2} \int \text{Tr}(F \wedge F) \]

\[ (5.2) \]

Hypermultiplet

The hypermultiplet consists of scalars \( q_{AI} \) and fermions \( \psi_{\alpha A}, \bar{\psi}_{\dot{\alpha} I} \) satisfying the reality conditions \cite{20}. The index \( I \) runs from 1 to \( 2r \). There is also an auxiliary scalar \( F_{IA} \) transforming as a doublet under a local \( SU(2)_R \) symmetry. This symmetry is introduced in the theory by the requirement that the SUSY algebra of matter multiplet is closed off shell \cite{20}. The gauge covariant kinetic Lagrangian for the hypermultiplet is \cite{20}:

\[ L_{mat} = \frac{1}{2} D_m q^A D^m q_A - q^A \{ \phi, \psi \} q_A + \frac{\ell}{2} q^A D_{AB} q_B + \frac{1}{8} (R + M) q^A q_A - \frac{\ell}{2} \bar{\psi} \sigma^m D_m \psi - \]

\[ \frac{1}{2} \psi \bar{\psi} + \frac{1}{2} \bar{\psi} \bar{\psi} + \frac{\ell}{2} \psi^{kl} T_{kl} \psi = \frac{\ell}{2} \bar{\psi} \sigma^{kl} T_{kl} \bar{\psi} - q^A \lambda_A \psi + \bar{\psi} \lambda_A q_A - \frac{1}{2} F^A F_A \]

\[ (5.3) \]

Closure of SUSY Algebra

For localization computation we need a continuous fermionic symmetry. So we take supercharge \( Q \) to be fermionic, so that the corresponding Killing spinor is bosonic. The supersymmetry transformation \( Q \) acting on the fields of \( N = 2 \) SUSY theory squares into a sum of bosonic symmetries:

\[ Q^2 \equiv L_v + \text{Gauge}(\hat{\Phi}) + \text{Lorentz}(L_{ab}) + \text{Scale}(\omega) + R_{U(1)}(\Theta) + R_{SU(2)}(\hat{\Theta}_{AB}) + \hat{R}_{SU(2)}(\hat{\Theta}) \]

\[ (5.4) \]

with various parameters defined as in \cite{20}. For vector multiplets the SUSY algebra is closed off shell, the only requirement being that the Killing spinor equations be satisfied. For the hypermultiplet the closure of full \( N = 2 \) SUSY algebra requires the existence of infinite number of auxiliary spinors and some auxiliary fields (see \cite{20}). But for localization computation we need only one supercharge corresponding to a particular Killing spinor and in this case only finite number of auxiliary spinors are required.

Next we compute these transformation parameters for our squashed 4-sphere background. First of all we observe that \( \xi^A \zeta_{pA} = \bar{\xi}^A \zeta = 0 \). This condition implies that \( \omega = \Theta = 0 \). In other words the square of the supersymmetry transformation does not give rise to dilation or \( U(1)_R \) transformation. This condition is necessary because the non-zero values of the background fields \( T_{ab} \) and \( \hat{T}_{ab} \) break the \( U(1)_R \) symmetry anyway.
The explicit expression for other transformation parameters are given below

\[
L_{ab} = \begin{pmatrix} 0 & -8c & 0 & 0 \\ 8c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
\Theta_{AB} = \begin{pmatrix} 2c \left( \frac{h(r)^2}{f(r)^2} - \frac{2s'(r)h(r)}{s(r)} + h'(r) \right) & 2c \left( \frac{h(r)^2}{f(r)^2} - \frac{2s'(r)h(r)}{s(r)} + h'(r) \right) \\ 0 & 0 \end{pmatrix}
\]

\[
\hat{\Theta}^{A}_B = \begin{pmatrix} 4c & 0 \\ 0 & -4c \end{pmatrix}
\]

\[
\text{Lie}_v \xi = \begin{pmatrix} -2c s(r) f'(r)^2 + h(r)^2 & 0 \\ 0 & 2c s(r) f'(r)^2 + h(r)^2 \\ 2c^2 h(r) f'(r)^2 h'(r) - h(r)^2 & 0 \\ 0 & 2c^2 h(r) f'(r)^2 h'(r) - h(r)^2 \end{pmatrix}
\]

where the Lie derivative \( \text{Lie}_v \) is defined as \( L_v \xi \equiv \nu^m D_m \xi + \frac{1}{4} D_m \nu \Gamma^{ab} \xi \). The non-zero \( L_{ab} \) implies the fact that the \( U(1) \) group which is used to find the fixed points of the manifold, belongs to the cartan of \( SU(2) \) part of the isometry group \( SU(2) \times U(1) \). Therefore it follows that our Killing spinor is invariant under \( Q^2 \). In 4-component notation

\[
Q^2 \xi = i \text{Lie}_v \xi - \xi \hat{\Theta} = 0
\]

The general solution to the constraint equations of auxiliary spinors is

\[
\hat{\xi} = \begin{pmatrix} \frac{ch(r)}{s(r)} & 0 \\ 0 & \frac{ch(r)}{s(r)} \\ -is(r) & 0 \\ 0 & is(r) \end{pmatrix}
\]

To fix the background \( SU(2)_R \), we have to fix the corresponding gauge field \( \hat{V}_m \)

\[
\hat{V}_m = \begin{pmatrix} i \hat{v}_{3m} \\ i(\hat{v}_{1m} + i \hat{v}_{2m}) \\ i(\hat{v}_{1m} - i \hat{v}_{2m}) \\ -i \hat{v}_{3m} \end{pmatrix}
\]

The requirement that all the background fields be invariant under the action of \( Q^2 \) fixes all the components of \( \hat{V}_m \) to zero except \( \hat{v}_{33}, \hat{v}_{34} \), which remain arbitrary.

After the fixing of gauge, \( \hat{\Theta}^{A}_B \) becomes

\[
\hat{\Theta}^{A}_B = \begin{pmatrix} -4(h(r)\hat{v}_{33}(r)c + c) & 0 \\ 0 & 4(h(r)\hat{v}_{33}(r)c + c) \end{pmatrix}
\]

Also auxiliary spinor \( \hat{\xi} \) is invariant under \( Q^2 \)

\[
Q^2 \hat{\xi} = 0
\]
6 Path Integration

6.1 $SYM$ Saddle Points

The path integral computation of the expectation value of an observable of a supersymmetric theory which is invariant under a supercharge $Q$ localizes to a subset $S_Q$ of the entire field space. The zero locus of the supercharge $Q$ coincides with the set of bosonic configurations for which the supersymmetry variations of the fermions vanish.

$$Q\Psi = 0 \quad \text{for all fermions } \Psi. \quad (6.1)$$

To take into account the gauge fixing, the supercharge $Q$ is generalized to $\hat{Q} \equiv Q + Q_B$, where $Q_B$ is the BRST-supercharge. However as pointed out in [3], this does not affect the zero locus. An indirect way to calculate the zero locus of the supercharge, is to add to the Lagrangian a $Q$-exact quantity $QV$, whose critical point set includes $S_Q$ as a subset and whose bosonic part is semi-positive definite. Now either solving the localization equation

$$\hat{Q}\lambda = 0 \quad (6.2)$$

directly or analyzing the $\hat{Q}$-transform of the following quantity,

$$V = \text{Tr}[(\hat{Q}\lambda_A)\lambda_A + (\hat{Q}\lambda_A^\dagger)\lambda_A^\dagger] \quad (6.3)$$

which has semi-positive definite bosonic part [20], we get following partial differential equations for $\phi_2(\psi, \theta, \varphi, r)$

$$\partial_\psi \phi_2(\psi, \theta, \varphi, r) = 0. \quad (6.4)$$

and

$$(-c^2 f(r)h(r)^3s(r)f''(r) - c^2f(r)h(r)^2s(r)f'(r)h'(r) + 2c^2f(r)h(r)^3f'(r)s'(r) - c^2h(r)^3s(r)f'(r)^2 + 2c^2h(r)^3s(r)h'(r) - 2c^2h(r)^3s'(r) - f(r)h(r)s(r)^3f''(r) + f(r)s(r)^3f'(r)h'(r) - 2f(r)h(r)s(r)^4f'(r)s'(r) - h(r)s(r)^5f'(r)^2 - 2h(r)^2s(r)^4s'(r)) \frac{\sin(\theta)}{2ch(r)^2s(r)^3} \phi_2(\theta, \varphi, r) + (-3c^2f(r)h(r)^3s(r)f'(r) - c^2f(r)^2h(r)^3s(r)h'(r) + 2c^2f(r)^2h(r)^3s'(r) + c^2h(r)^4s(r)^2 - 3f(r)h(r)s(r)^5f'(r) + f(r)^2s(r)^5h'(r) - 2f(r)^2h(r)s(r)^4s'(r) - h(r)^2s(r)^3f'(r)^2) \frac{\sin(\theta)}{2ch(r)^2s(r)^3} \partial_\varphi \phi_2(\theta, \varphi, r) + \frac{c^2h(r)^2 + s(r)^4}{ch(r)s(r)^2} \partial_r \phi_2(\theta, \varphi, r) \quad (6.5)$$

where in the second equation we used the fact that $\phi_2(\psi, \theta, \varphi, r)$ is independent of $\psi$-coordinate. For the round sphere

$$f(r) = \text{Sin}(r), \quad b(r) = \text{Sin}(r), \quad s(r) = \frac{1}{\sqrt{2}} \text{Cos}\left[\frac{r}{2}\right], \quad c = \frac{1}{4} \quad (6.6)$$

the field $\phi_2 = 0$ at the localization locus, which will also ensure that $A_m = 0$ at the locus[3, 20]. This result is true in an open neighborhood of the round $S^4$. 
The saddle points are thus labeled by a Lie Algebra valued constant \( a_0 \), and are given by the equations \([3, 20]\):

\[
A_m = 0, \quad \phi = \bar{\phi} = a_0, \quad D_{AB} = -\imath a_0 \omega_{AB}.
\]

The value of the Super-Yang-Mills action on this saddle point is \([3, 20]\):

\[
\frac{1}{g_{YM}^2} \int d^4x \sqrt{|g_{YM}|_{\text{saddlepoint}}} = \frac{2\pi^3 \text{Tr} |a_0^3|}{c^2 g_{YM}^2}.
\]

### 6.2 Saddle points for Matter multiplet

To find the saddle points of the matter multiplet we will use the following fermionic functional

\[
V_{\text{mat}} = \text{Tr}[(\hat{Q} \psi_{\alpha I})^\dagger \bar{\psi}^{\dot{\alpha}}_{\dot{I}} + (\hat{Q} \bar{\psi}^{\dot{\alpha}}_{\dot{I}})^\dagger \psi_{\alpha I}]
\]

The bosonic part of \( \hat{Q} V_{\text{mat}} \) is

\[
\hat{Q} V_{\text{mat}}|_{\text{bos}} = \text{Tr}[(\hat{Q} \psi_{\alpha I})^\dagger \hat{Q} \psi_{\alpha I} + (\hat{Q} \bar{\psi}^{\dot{\alpha}}_{\dot{I}})^\dagger \hat{Q} \bar{\psi}^{\dot{\alpha}}_{\dot{I}}]
\]

Following \([3]\) it is easy to check that

\[
\hat{Q} V_{\text{mat}}|_{\text{bos}} = 4 \|\xi\|^2 \left( \frac{1}{2} (D m q^{A I} - P m q^{A I})^2 + M q(r) q^{A I} q_{I A} - \frac{1}{2} F^{A I} F_{I A} \right)
\]

where

\[
P^{B}_{mA} = \frac{1}{\|\xi\|^2} \left( 2(\epsilon \xi_{\gamma m} \xi_{p} + \epsilon \xi T \gamma_{m} \xi) B^{A} + D^m \text{Log}(\|\xi\|)^2 (\epsilon \xi_{\gamma m} \xi) B^{A} \right)
\]

and

\[
M q = -\frac{1}{4} R + \frac{1}{\|\xi\|^2} \left( 8 \epsilon^{A}_{\gamma p A} \xi_{p} + \xi^{A} \gamma^{m} T \gamma_{m} \xi_{A} - D^m \text{Log}(\|\xi\|)^2) \xi^{A} (3 \gamma_{m} \xi_{p A} + T \gamma_{m} \xi_{A}) + \frac{1}{2} (P^{m A}_{B} P^{B}_{m A}) - \frac{1}{2\|\xi\|^2} P^{m A}_{A} P^{B}_{m B} \right)
\]

where \( \xi_{A} = (\xi_{\alpha A}, \bar{\xi}_{\alpha A}) \), \( \epsilon^{AB} \) is the \( SU(2)_{R} \) tensor and \( R \) is the Ricci scalar. As a result of the condition \( F_{I A}^{T} = -F^{A I} \) which is imposed along the contour of path integration, all the bosonic terms are manifestly positive definite except the term containing \( M q(r) \). For the round \( S^4 \)

\[
M q(r) = \frac{7}{8} + \frac{\cos(2r)}{8}
\]

and it is bounded from below by \( \frac{7}{8} \). Therefore there is a large open neighborhood of the round sphere for which \( M q(r) \) is positive definite. So we get the result for the saddle points of the hypermultiplet as

\[
q_{I A} = 0, \quad F_{I A} = 0
\]

Hence there will be no classical contribution from the hypermultiplet sector.
7 One-loop determinant

To calculate the one-loop determinant we have to first fix the local gauge redundancy of the path integral. To perform the gauge fixing we follow [3] and [20] and choose the following gauge:

\[ G = i \partial_m A^m + i L_v ((\xi^A \xi_A - \bar{\xi}_A \bar{\xi}_A) \phi_2 - \nu^m A_m) \]  

(7.1)

The saddle point conditions do not change under the new supercharge \( \hat{Q}^2 \equiv (Q_B + Q)^2 \) and the zero mode of \( \phi_1 = a_0 \) at the saddle point.

7.1 Vector multiplet contribution

The value of the path integral is invariant under the \( \hat{Q} \)-exact deformation \( L \to L + s \hat{Q}(V + V_{GF}) \). By choosing the bosonic part of \( L \to L + s \hat{Q}(V + V_{GF}) \) positive definite and sending \( s \to \infty \), Gaussian approximation becomes exact for the path integral over the fluctuations around the locus. The Gaussian integral gives the square root of the ratio between the determinant of fermionic kinetic operator \( K_{\text{fermion}} \) and that of the bosonic kinetic operator \( K_{\text{boson}} \). These kinetic operators follow from the quadratic part of the \( \hat{Q} \)-exact regulator.

Following the notation of [20], the quadratic part of \( V + V_{GF} \) can be written as:

\[ (V + V_{GF})_{\text{quadratic}} = (\hat{Q}X, \Xi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ Q \Xi \end{pmatrix} \]  

(7.2)

where \( D_{ij} \) are differential operators and \( X, \Xi \) are bosonic and fermionic fields respectively, specified by [20]:

\[ \Xi \equiv (\Xi_{AB}, \bar{C}, C), \quad X = (\phi_2, A_m; \bar{a}_0, B_0) \]  

(7.3)

where

\[ \Xi_{AB} \equiv 2\xi(A \lambda_B) - 2\xi(A \bar{\lambda} B) \]  

(7.4)

Where \( \bar{C}, C, \bar{a}_0, B_0 \) belong to the ghost multiplets as introduced in [8],[3] for fixing the local gauge redundancy of the path integral. The fields \( X \) and \( \Xi \) can be regarded as sections of bundles \( E_0, E_1 \) on the manifold and hence \( D_{10} \) acts on the complex as \( D_{10} : \Gamma(E_0) \to \Gamma(E_1) \). The invariance of the deformation term \( \hat{Q}(V + V_{GF}) \) under the action of \( \hat{Q} \) and the pairing of the fields under \( \hat{Q}^2 = H \) leads to the cancellations between bosonic and fermionic fluctuations, which gives the following result [3, 20]:

\[ \frac{\det K_{\text{fermion}}}{\det K_{\text{boson}}} = \frac{\det \Xi H}{\det X H} = \frac{\det \text{Coker} D_{10} H}{\det \text{Ker} D_{10} H} \]  

(7.5)

The fact that \( \hat{Q}^2 \) commutes with the differential operators \( D_{ij} \) is used in the derivation of the last expression and is a result of the invariance of \( (V + V_{GF}) \) under \( \hat{Q}^2 \). This can readily be seen by considering \( \hat{Q}^2(V + V_{GF})_{\text{quad}} \).

\[ \hat{Q}(V + V_{GF})_{\text{quad}} = \begin{pmatrix} X & \hat{Q} \Xi \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \hat{Q} \Xi \end{pmatrix} - \begin{pmatrix} \hat{Q} X & \Xi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q} X \\ \Xi \end{pmatrix} \]  

(7.6)
\[
\mathcal{D} \equiv \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}
\]

Then
\[
\dot{Q}^2(V + V_{GF})_{quad} = \left( \dot{Q} X \dot{Q}^2 \Sigma \right) \begin{pmatrix} -\dot{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{D} \begin{pmatrix} X & \dot{Q} \Sigma \end{pmatrix} + \left( X \dot{Q} \Sigma \right) \begin{pmatrix} -\dot{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{D} \begin{pmatrix} \dot{Q} X \\ \dot{Q}^2 \Sigma \end{pmatrix} - \\
\left( \dot{Q}^2 X \dot{Q} \Sigma \right) \mathcal{D} \begin{pmatrix} 1 & 0 \\ 0 & -\dot{Q}^2 \end{pmatrix} \begin{pmatrix} \dot{Q} X \\ \dot{Q} \Sigma \end{pmatrix} + \left( \dot{Q} X \Sigma \right) \mathcal{D} \begin{pmatrix} 1 & 0 \\ 0 & -\dot{Q}^2 \end{pmatrix} \begin{pmatrix} \dot{Q} X \\ \dot{Q} \Sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\dot{Q}^2 \end{pmatrix} \mathcal{D} \begin{pmatrix} 1 & 0 \\ 0 & -\dot{Q}^2 \end{pmatrix} \begin{pmatrix} \dot{Q} X \\ \dot{Q} \Sigma \end{pmatrix} + \\
\mathcal{D} \begin{pmatrix} 1 & 0 \\ 0 & -\dot{Q}^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\dot{Q}^2 \end{pmatrix} \begin{pmatrix} \dot{Q} X \\ \dot{Q} \Sigma \end{pmatrix}
\]
(7.7)

Now with the requirement that \([\dot{Q}^2, D_{ij}] = 0\), different terms cancel among each other and we get
\[
\dot{Q}^2(V + V_{GF})_{quad} = 0
\]  (7.8)

### 7.2 Index of \(D_{10}\)

In computing this index, the constant fields \(B_0, \bar{a}_0\) are regarded as sitting in the kernel of \(D_{10}\) and making a contribution 2 to the index. This is because the weights of \(B_0, \bar{a}_0\) are 0 under the action of \(U(1)\) at the Poles. To obtain the remaining contribution, we read off the differential operator \(D_{10}\) from [20]:
\[
\Xi D_{10} X + \Xi D_{11} \dot{Q} \Xi = \text{Tr}[cG - D_m c(\dot{Q} \Psi^m)^\dagger + \frac{1}{2} \Xi_{AB} (\dot{Q} \Xi_{AB})^\dagger]
\]  (7.9)

Up to non-linear terms [20]
\[
(\dot{Q} \Psi_m)^\dagger = -i L_v A_m + D_m (\Phi - 2i (\xi^A \xi_A - \bar{\xi}^A \bar{\xi}_A) \phi_{21} u^m A_m),
\]
\[
(\dot{Q} \Xi_{AB})^\dagger = -\xi^A \sigma^k \xi^B (F_{kl} - 8 \delta T_{kl} + 8 \delta S_{kl}) + \xi^A \sigma^k \xi^B (F_{kl} - 8 \delta T_{kl} + 8 \delta S_{kl}) - 4 \xi^A \sigma^m \xi^B D_n \phi_{2} - D^{AB}
\]
(7.10)

As in [3, 20], this operator \(D_{10}\) will be shown to be transversally elliptic with respect to the isometry \(L_v\) of the squashed \(S^4\) by calculating its symbol. To compute the symbol \(\sigma(D_{10})\) we follow closely [20]. The Fourier transform of the operator \(D_{10}\) is taken and then only the highest order derivative terms are retained. To write the symbol explicitly, as in [20] it is convenient to introduce four unit vector fields \(\mu^m_\alpha (\alpha = 1, 2, 3, 4)\) by the formula:
\[
-2i (r^A)_B \xi^B \sigma^m \xi_A = 4ch(r) \mu^m_\alpha, \quad 2 \xi^A \sigma^m \xi_A = 4ch(r) \mu^m_\alpha \quad (\alpha = 1, 2, 3),
\]  (7.11)

and parametrize the momenta in the local orthonormal frame defined by the vielbein \(\mu^m_\alpha\). Here \(c\) is the constant appearing in the definition of the Killing spinor.
The determinant of the above symbol is

\[
\Xi \sigma(D_{10}) X = (\Xi_1, \Xi_2, \Xi_3, -\epsilon, w)
\]

where \( W(r) \equiv 2s_1a(r)^2 - \frac{2e_2(r)^2}{s_1a(r)^2} \). The 5 × 5 matrix in the middle is block diagonalized by a change of variables in \( X \) and \( \Xi \), as in [20].

\[
\sigma(D_{10}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & p_4
\end{pmatrix}, \quad \begin{pmatrix}
p_4W(r) & p_3 & -p_2 & -p_1 & 0 \\
-p_3 & p_4W(r) & p_1 & -p_2 & 0 \\
p_2 & -p_1 & p_4W(r) & -p_3 & 0 \\
p_1 & p_2 & p_3 & p_4W(r) & 0 \\
0 & 0 & 0 & 0 & 4cp_2^2h(r) - 2(p_1^2 + p_2^2 + p_3^2)
\end{pmatrix}
\]

(7.13)

We get similar results as in [20], also in our case the lower-right 1 × 1 block gives a trivial contribution to the index, since the corresponding differential operator should have just one-dimensional kernel and cokernel of constant functions. The non-trivial contribution to the index comes just from the upper-left 4 × 4 block of the matrix in the middle of (7.13),

\[
\sigma(D'_{10}) = \begin{pmatrix}
p_4W(r) & p_3 & -p_2 & -p_1 \\
-p_3 & p_4W(r) & p_1 & -p_2 \\
p_2 & -p_1 & p_4W(r) & -p_3 \\
p_1 & p_2 & p_3 & p_4W(r)
\end{pmatrix}
\]

(7.14)

Following [3, 20] a differential operator is called elliptic if its symbol is invertible for non-zero \( p_1, p_2, p_3, p_4 \). The determinant of the above symbol is

\[
\text{Det}(\sigma(D'_{10})) = \left( \frac{4cp_2^2h(r)^4}{s(r)^2} - 8e_2p_1^2h(r)^2 + p_2^2 + p_3^2 + 4p_2^2s(r)^2 \right)^2
\]

(7.15)

For \( p_1 = p_2 = p_3 = 0 \) and \( p_4 \neq 0 \), this value of determinant changes sign between North and South Poles and hence has at least one zero. Therefore the symbol is not invertible at the location of that zero. But if we restrict the momentum to be orthogonal to the vector \( v \), i.e. \( p_4 = 0 \), then \( \sigma \) is invertible always that \( (p_1, p_2, p_3) \) are not zero simultaneously. The corresponding differential operator is dubbed transversally elliptic respect to the symmetry generated by \( v \). The kernel and cokernel of transversally elliptic operators are generically infinite dimensional, but since \([\hat{Q}^2, D_{ij}] = 0\), they can both be decomposed into finite dimensional eigenspaces of \( H \) [3].
The index theorem localizes the contributions to the fixed points of the action of $H$, that is to the North and South poles of the squashed $S^4$. According to the Atiyah-Bott [21] formula, the index is given by

$$
\text{ind}(D'_{10}) = \sum_{\text{fixed points}} \frac{\text{Tr} E_0(\gamma) - \text{Tr} E_1(\gamma)}{\det(1 - \frac{\partial}{\partial x})}
$$

(7.16)

where $\gamma$ denotes the eigenvalue the action of the operator $e^{iHt}$ on the vector and $SU(2)_R$ indices of the fields. Near the North pole, the operator $e^{iHt}$ acts on the local coordinates $z_1 = x_1 + \iota x_2, z_2 = x_3 + \iota x_4$ where in polar coordinates, near the North Pole

$$
x_1 + \iota x_2 = r \cos \left( \frac{\theta}{2} \right) e^{\iota \frac{\pi \Omega}{4}},
$$

$$
x_3 + \iota x_4 = r \sin \left( \frac{\theta}{2} \right) e^{\iota \frac{\pi \Omega}{4}}
$$

(7.17)

Following the notation of [20]:

$$
\tilde{z}_1 = e^{\iota \mu c t} z_1 \equiv q_1 z_1, \quad \tilde{z}_2 = e^{\iota \mu c t} z_2 \equiv q_2 z_2.
$$

(7.18)

Here $0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi, 0 \leq \psi \leq 4\pi$ and in our case $q_1 = q_2$. Therefore

$$
\det(1 - \frac{\partial}{\partial z}) = (1 - q_1)(1 - \bar{q}_1)(1 - q_2)(1 - \bar{q}_2).
$$

(7.19)

The eigenvalues of $Q^2$ for the fields of vector multiplet at the North pole read exactly as in [20],

$$
\gamma(A_{z_1}) = q_1, \quad \gamma(A_{z_2}) = q_2, \quad \gamma(\Xi_{11}) = \bar{q}_1, \quad \gamma(\Xi_{12}) = \bar{q}_2, \quad \gamma(c) = 1.
$$

(7.20)

The one loop determinant can be easily computed by extracting the spectrum of eigenvalues of $H$ from the index. For a non-abelian group $G$, with $a_0$ in its Cartan sub algebra, the one loop contribution can be written as [20]:

$$
Z^\text{re}\text{c}_{1-\text{loop}} = \frac{\text{det} K_{\text{fermion}}}{\text{det} K_{\text{boson}}} = \prod_{a \in \Delta^+} \left( \frac{1}{(a_0, a)^2} \right) \prod_{m,n \geq 0} \frac{((m + n) + \iota a_0, a)((m + n + 2) + \iota a_0, a)}{((m + n) - \iota a_0, a)((m + n + 2) - \iota a_0, a)}
$$

(7.21)

$$
\times \prod_{a \in \Delta^+} \frac{\Upsilon(\iota a_0, a)\Upsilon(-\iota a_0, a)}{(a_0, a)^2}
$$

where $a_0 \equiv \frac{2m}{b}$. The function $\Upsilon(x)$ has zeros at $x = -(m + n), (m + n + 2)$ and is used to express the appropriately regularized infinite products. It is defined by the following infinite product

$$
\Upsilon_b(x) = \prod_{n_1, n_2 \geq 0} \frac{(bn_1 + n_2/b + x)(bn_1 + n_2/b + b + 1/b - x)}{(bn_1 + n_2/b + 1/b - x)}
$$

(7.22)

where $b$ is an arbitrary constant and for our case $b = 1$. 



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7.3 Hypermultiplet one-loop contribution

After certain redefinition of the fields [20], the computation of the one-loop determinant reduces to that of the index of an operator $D_{10}^{\text{mat}}$. This operator corresponds to the terms bilinear in the fields $\Xi$ and $q_{I A}$ in the functional $V_{\text{mat}}$. Its symbol $\sigma(D_{10}^{\text{mat}})$ is given by

$$
\sigma(D_{10}^{\text{mat}}) = \left( \begin{array}{c}
\frac{2((p_3 - ip_4)\alpha(r)^4 + c^2 h(r)^2(p_3 + ip_4))}{s(r)^4 + c^2 h(r)^2} & 2(p_1 + ip_2) \\
2((p_2 + ip_4)\alpha(r)^4 + c^2 h(r)^2(p_2 - ip_4)) & \frac{2(p_1 - ip_2)}{s(r)^4 + c^2 h(r)^2}
\end{array} \right)
$$

(7.23)

The determinant of this symbol is

$$
\text{Det}[\sigma(D_{10}^{\text{mat}})] = -\frac{4(\alpha(r)^4 - c^2 h(r)^2)^2}{(c^2 h(r)^2 + s(r)^4)^2} p_1^2 - 4p_1^2 - 4p_2^2 - 4p_3^2
$$

(7.24)

So for $p_1 = p_2 = p_3 = 0, p_4 \neq 0$, the determinant changes sign somewhere between North and South Poles and hence at least one zero. Therefore the operator $D_{10}^{\text{mat}}$ is again transversally elliptic with respect to the isometry generated by $L_v$ in the $p_4$ direction.

The index for the action of $H$ on different fields at the poles will be calculated by using Atiyah-Bott formula. The eigenvalues for the action of $Q^2$ on the matter multiplet case are given below and read the same as in [20]:

$$
\gamma(q_{A=1}^I) = q_1^I q_2^\dagger \; , \quad \gamma(q_{A=2}^I) = q_1^I q_2^\dagger \; , \quad \gamma(\psi_\alpha^I) = q_1^I q_2^\dagger \; , \quad \gamma(\psi_\alpha^I) = q_1^I q_2^\dagger
$$

(7.25)

These weights can easily be read from the $Q^2$ SUSY transformation parameters At the two poles of the squashed $S^4$, the operator $D_{10}^{\text{mat}}$ reduces to the Dirac operator $D_{\text{Dirac}}$ which maps the space of positive chirality spinors $S^+$ to negative chirality spinor $S^-$

$$
D_{\text{Dirac}} : S^+ \rightarrow S^-
$$

(7.26)

Its index is

$$
\text{ind}(D_{\text{Dirac}}, q_1, q_2) = \frac{(q_1^I q_2^\dagger + q_1^I q_2^\dagger) - (q_1^I q_2^\dagger + q_1^I q_2^\dagger)}{(1 - q_1)(1 - q_2)} = \frac{q_1^I q_2^\dagger}{(1 - q_1)(1 - q_2)}
$$

(7.27)

For the hypermultiplets coupled to gauge symmetry, in the representation $R \oplus \bar{R}$, the total index with the contributions from North and South poles, is given by

$$
\text{ind}(D_{10}^{\text{mat}}, q_1, q_2) = -\sum_{\rho \in R} (e^{t_{\rho 0} + e^{-t_{\rho 0}}} (q_1^I q_2^\dagger (1 - q_1)(1 - q_2) + q_1^I q_2^\dagger (1 - q_1)(1 - q_2)))
$$

(7.28)

where $\rho$ runs over all the weights in a given representation and the minus sign corresponds to the fact that for the hypermultiplet, the chirality of the complex is opposite to the chirality of the $U(1)$ rotation near each of the fixed points [3]. In this expression for the index, the first term is the contribution of North Poles and so is expanded in positive power series, whereas the second term, the South Pole contribution,
is expanded in the negative power series. The index can be easily translated into the one-loop determinant for the hypermultiplet

\[ Z^{hyp}_{1-\text{loop}} = \prod_{\rho \in R} \prod_{m,n \geq 0} \frac{((m + n + 1) - i\hat{a}_0.\alpha)^{-1}((m + n + 1) + i\hat{a}_0.\alpha)^{-1}}{\prod_{\rho \in R} \Upsilon(i\hat{a}_0.\rho + 1)^{-1}} \]  

(7.29)

8 Instanton contribution

Near the North pole the Killing spinor evaluates to

\[ \xi = \begin{pmatrix} s_{n0} & 0 \\ 0 & s_{n0} \\ \frac{s_{n0}}{g} & s_{n0} \\ 0 & -\frac{s_{n0}}{g} \end{pmatrix} \]  

(8.1)

so that \( \xi^A \xi_A = 2s_{n0}^2 \) and \( \bar{\xi}^A \bar{\xi}_A = \frac{2s_{n0}^2}{r^2} \). Since \( \bar{\xi}^A \bar{\xi}_A \to 0 \) at the North Pole, the localization equation has to be evaluated away from the North pole to have smooth gauge field configuration. Similarly near the South pole

\[ \xi = \begin{pmatrix} (\pi - r)s_{s1} & 0 \\ 0 & (\pi - r)s_{s1} \\ \frac{ic}{s_{s1}} & \frac{ic}{s_{s1}} \\ 0 & -\frac{ic}{s_{s1}} \end{pmatrix} \]  

(8.2)

and \( \xi^A \xi_A = 2(\pi - r)^2s_{s1}^2 \) and \( \bar{\xi}^A \bar{\xi}_A = \frac{2s_{s1}^2}{r^2} \). In this case \( \xi^A \xi_A \to 0 \). Therefore South pole has also to be excluded if smooth gauge field configuration is assumed.

To include the contribution from the poles, we first notice that because \( \bar{\xi}^A \bar{\xi}_A \to 0 \) at the North Pole, in general \( F^{+}_{mn} \neq 0, F^{-}_{mn} = 0 \) there and still solve the localization equation. This configuration is the point anti-instanton contribution. Also at the North pole the following condition is satisfied for our background

\[ \frac{1}{4} \Omega^{ab} \sigma_{ab} \xi_A + i\xi_B V^B_{mA} = 0 \]  

(8.3)

On the other hand near the South pole \( \xi^A \xi_A \to 0 \), and we get the point instanton contribution \( F^{+}_{mn} = 0, F^{-}_{mn} \neq 0 \) and the following twisting condition is satisfied

\[ \frac{1}{4} \Omega^{ab} \bar{\sigma}_{ab} \bar{\xi}_A + i\bar{\xi}_B \bar{V}^B_{mA} = 0 \]  

(8.4)

The Killing vector near the North Pole can be written as

\[ \nu^m \frac{\partial}{\partial x_m} = 4c(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}) + 4c(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}) \]  

(8.5)

So near the South pole our \( N = 2 \) theory on squashed \( S^4 \) approaches topologically twisted theory with Omega deformation parameters \( \epsilon_1 = 4c, \epsilon_2 = 4c \) \([1, 2]\), and the contribution of these point-instanton is given by \( Z_{inat}(\eta_0, \epsilon_1, \epsilon_2, \tau) \), where the parameter \( \tau \) is defined by \( \tau \equiv \frac{\bar{\theta}}{2\pi} + \frac{\alpha_1}{g_{YM}} \).
Whereas near the North Pole, the contribution of point anti-instantons is given by $Z_{\text{inst}}(a_0, \epsilon_1, \epsilon_2, \bar{\tau})$. The final form of the squished $S^4$ partition function is:

$$Z = \int \frac{d\hat{a}_0 e^{-2\pi i \hat{a}_0 \phi}}{e^{\hat{a}_0}} |Z_{\text{inst}}|^2 \prod_{\alpha \in \Delta_+} \Upsilon(i\hat{a}_0, \alpha) \Upsilon(-i\hat{a}_0, \alpha) \prod_{\rho \in R} \Upsilon(i\hat{a}_0, \rho + 1)$$  \hspace{1cm} (8.6)

### 9 Conclusions

We have computed the partition function of $N = 2$ SUSY on squashed $S^4$ which admits $SU(2) \times U(1)$ isometry, using SUSY Localization technique. We find that the full partition function is independent of the squashing as well as the background fields. It will be interesting to explain this independence along the same lines as given in [14]. That is to say, if we deform the vector multiplet and hypermultiplet actions around the round $S^4$ with respect to e.g. $f(r)$, it may be possible to write these deformed actions as $Q$-exact terms separately. This $Q$-exactness of the deformed action will explain the independence of partition function of the parameter $f(r)$ in the sense of [14]. This analogy is only suggestive because the authors in [14] discussed only the case of $N = 1$ SUSY on compact four manifolds which are complex, whereas $S^4$ is not a complex manifold. Moreover we have to consider perturbations around the round $S^4$ unlike [14], where they perturb around $R^4$. We will report on this issue in future. It will also be interesting to use this more general squashing independence to study the properties of supersymmetric Renyi entropy on these four manifolds as suggested in [13, 15]

**Acknowledgements**  We would like to thank Giulio Bonelli for discussions.

### References

[1] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv.Theor.Math.Phys. 7 (2004) 831–864, [hep-th/0206161].

[2] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

[3] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun.Math.Phys. 313 (2012) 71–129, [arXiv:0712.2824].

[4] A. Kapustin, B. Willett, and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, JHEP 1003 (2010) 089, [arXiv:0909.4559].

[5] I. Yaakov, *Localization of gauge theories on the three-sphere*, .

[6] G. Festuccia and N. Seiberg, *Rigid Supersymmetric Theories in Curved Superspace*, JHEP 1106 (2011) 114, [arXiv:1105.0689].

[7] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, *Exploring Curved Superspace*, JHEP 1208 (2012) 141, [arXiv:1205.1115].

[8] N. Hama, K. Hosomichi, and S. Lee, *Notes on SUSY Gauge Theories on Three-Sphere*, JHEP 1103 (2011) 127, [arXiv:1012.3512].

[9] T. Nosaka and S. Terashima, *Supersymmetric Gauge Theories on a Squashed Four-Sphere*, JHEP 1312 (2013) 001, [arXiv:1310.5939].

[10] J. Gomis and S. Lee, *Exact Kahler Potential from Gauge Theory and Mirror Symmetry*, JHEP 1304 (2013) 019, [arXiv:1210.6022].
[11] Y. Imamura, *Perturbative partition function for squashed $S^5$, arXiv:1210.6308*.

[12] C. Klare and A. Zaffaroni, *Extended Supersymmetry on Curved Spaces, JHEP 1310* (2013) 218, [arXiv:1308.1102].

[13] X. Huang and Y. Zhou, *N = 4 Super-Yang-Mills on Conic Space as Hologram of STU Topological Black Hole, arXiv:1408.3393*.

[14] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, *The Geometry of Supersymmetric Partition Functions, JHEP 1401* (2014) 124, [arXiv:1309.5876].

[15] M. Crossley, E. Dyer, and J. Sonner, *Super-Rényi Entropy & Wilson Loops for N=4 SYM and their Gravity Duals, arXiv:1409.0542*.

[16] H. Casini, M. Huerta, and R. C. Myers, *Towards a derivation of holographic entanglement entropy, JHEP 1105* (2011) 036, [arXiv:1102.0440].

[17] I. R. Klebanov, S. S. Pufu, S. Sachdev, and B. R. Safdi, *Rényi Entropies for Free Field Theories, JHEP 1204* (2012) 074, [arXiv:1111.6290].

[18] T. Nishioka and I. Yaakov, *Supersymmetric Rényi Entropy, JHEP 1310* (2013) 155, [arXiv:1306.2958].

[19] X. Huang, S.-J. Rey, and Y. Zhou, *Three-dimensional SCFT on conic space as hologram of charged topological black hole, JHEP 1403* (2014) 127, [arXiv:1401.5421].

[20] N. Hama and K. Hosomichi, *Seiberg-Witten Theories on Ellipsoids, JHEP 1209* (2012) 033, [arXiv:1206.6359].

[21] M. F. Atiyah and R. Bott, *A lefschetz fixed point formula for elliptic complexes: I, Annals of Mathematics 86* (1967), no. 2 pp. 374–407.