LINEARISATION OF WEAK VECTOR-VALUED FUNCTIONS

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Abstract. We study the problem of linearisation of vector-valued functions which are defined in a weak way, e.g. weakly holomorphic or harmonic vector-valued functions. The two main approaches to such a problem are the Dixmier–Ng theorem and the $\varepsilon$-product of Schwartz. We generalise the Dixmier–Ng theorem and show that in the setting of the classical Dixmier–Ng theorem the two mentioned approaches to linearisation coincide except for the topologies involved. We apply our results to generalise known statements on weakly compact composition operators and extension problems of weak vector-valued functions.

1. Introduction

In the present paper we consider the problem of linearising weak vector-valued functions. Suppose that $F(\Omega)$ is a locally convex Hausdorff space of real- or complex-valued functions on some non-empty set $\Omega$ and for a locally convex Hausdorff space $E$ define its weak vector-valued counterpart by

$$F(\Omega, E) \sigma := \{f: \Omega \to E \mid \forall e' \in E': e' \circ f \in F(\Omega)\}$$

where $E'$ is the topological linear dual space of $E$. Linearising the functions in $F(\Omega, E) \sigma$ means that we want to find a locally convex Hausdorff space $G$ such that the point evaluations $\delta_x$ belong to $G$ for all $x \in \Omega$ and the map

$$\chi: L(G, E) \to F(\Omega, E) \sigma, \ T \mapsto [x \mapsto T(\delta_x)]$$

becomes a well-defined linear isomorphism, and with a suitable choice of topologies on $L(G, E)$ and $F(\Omega, E) \sigma$ a topological isomorphism as well.

Such linearisations become useful in transferring results from the scalar-valued to the vector-valued case like extension results [7, 22, 27, 29], surjectivity of linear partial differential operators [23], and series representations [26]. Apart from the approach in [8] there are basically two main routes that one may take to linearisation. On the one hand, linearisation based on the $\varepsilon$-product of Schwartz where $G = F(\Omega)'$ is the used, which is the topological linear dual of $F(\Omega)$ equipped with the topology of uniform convergence on absolutely convex compact subsets of $F(\Omega)$, and $L(G, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of $G$, see e.g. [2, 5, 21, 25, 30, 42, 43]. On the other hand, linearisation based on the Dixmier–Ng theorem where $F(\Omega)$, equipped with a norm $\| \cdot \|$, is a Banach space such that the $\| \cdot \|$-closed unit ball is compact w.r.t. to some auxiliary locally convex Hausdorff topology $\tau$ on $F(\Omega)$, one uses a predual of $F(\Omega)$ constructed from this assumption as $G$ and equips $L(G, E)$ with the topology of

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uniform convergence on bounded subsets of $G$, see e.g. [22, 33, 35, 54]. The two main points of our paper are that one can generalise the Dixmier–Ng theorem to obtain a linearisation in a more general setting, see Theorem 4.3, and that in the setting of the old Dixmier–Ng theorem the two linearisation approaches mentioned above coincide except for the topologies on $L(G, E)$ and $\mathcal{F}(\Omega, E)_\sigma$ if $E$ is complete, see Theorem 4.3.

Let us give an outline of our paper. In Section 2 we briefly recall the notion of a (pre-)Saks space being a triple $(X, \|\cdot\|, \tau)$ of a normed space $(X, \|\cdot\|)$ with an additional coarser locally convex Hausdorff topology $\tau$, which is in the case of a Saks space also norming, see [10]. Further, we recap the mixed topology $\gamma = \gamma(\|\cdot\|, \tau)$ associated to a pre-Saks space, which was introduced in [45] and is the finest locally convex, even linear, Hausdorff topology between the $\|\cdot\|$-topology and $\tau$. Then we generalise the Dixmier–Ng theorem in Proposition 2.6 to the setting of semi-reflexive $(X, \|\cdot\|, \tau)$ pre-Saks spaces (semi-reflexive w.r.t. $\gamma$) and note that in the original Dixmier–Ng theorem the pre-Saks space is a semi-Montel space w.r.t. $\gamma$.

In Section 3 we give examples of semi-Montel Saks spaces $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ of functions to which we may apply our linearisation results and show that in these specific cases $\mathcal{F}(\Omega, E)$ coincides with a strong version of this space (under some restrictions on $E$). Our examples include weighted sequence spaces, kernels of hypoelliptic linear partial differential operators in weighted spaces of continuous functions, in particular weighted spaces of holomorphic or harmonic functions, weighted Bloch spaces, spaces of Lipschitz continuous functions and spaces of $k$-times continuously partially differentiable functions on some open bounded set $\Omega \subset \mathbb{R}^d$ whose partial derivatives extend continuously to the boundary of $\Omega$ and whose partial derivatives of order $k$ are $\alpha$-Hölder continuous for some $0 < \alpha \leq 1$.

In Section 4 we prove our already mentioned main results Theorem 4.3 for semi-reflexive pre-Saks function spaces $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ and Theorem 4.5 for semi-Montel pre-Saks function spaces. Further, we characterise the approximation property of $(\mathcal{F}(\Omega), \gamma)$ in Corollary 4.6.

In Section 5 we present applications of our linearisations to derive results on weakly compact composition operators in Theorem 5.1 and extension results for weak vector-valued functions in Theorem 5.3.

2. Notions and preliminaries

In this short section we recall some basic notions from the theory of locally convex spaces, (pre-)Saks spaces and mixed topologies. For a vector space $X$ over the field $\mathbb{R}$ or $\mathbb{C}$ with a Hausdorff locally convex topology $\tau$ we denote by $(X, \tau')$ the topological linear dual space and just write $X' = (X, \tau')$ if $(X, \tau)$ is a normed space. By $\Gamma_\tau$ we always denote a directed system of continuous seminorms that generates the Hausdorff locally convex topology $\tau$ on $X$. For two locally convex Hausdorff spaces $(X, \tau)$ and $(E, \tau_E)$ we denote by $L((X, \tau), (E, \tau_E))$ the space of continuous linear maps from $(X, \tau)$ to $(E, \tau_E)$ and write $L_t((X, \tau), (E, \tau_E))$ for this space equipped with the locally convex topology of uniform convergence on the bounded subsets of $X$ if $t = b$, and on the absolutely convex compact subsets of $X$ if $t = \kappa$. We usually omit the topology $\tau_E$ and just write $L_t((X, \tau), E)$ instead of $L_t((X, \tau), (E, \tau_E))$. We do the same with $\tau$ if $(X, \tau)$ is a normed space.

By [44, Chap. I, §1, Définition, p. 18] the $\varepsilon$-product of Schwartz of two locally convex Hausdorff spaces $(X, \tau)$ and $(E, \tau_E)$ is defined by $X \varepsilon E := L_\varepsilon(X', E)$ where $X' := (X, \tau)'$ and $L(X', E)$ is equipped with the topology of uniform convergence on the equicontinuous subsets of $(X, \tau)'$. We identify the tensor product $X \otimes E$ with the linear finite rank operators in $X \varepsilon E$ and recall that $X$ has the approximation
property of Schwartz if and only if \( X \otimes E \) is dense in \( X \varepsilon E \) for all Banach spaces \( E \) (see e.g. [23, Satz 10.17, p. 250]).

Further, we recall the definition of local completeness of a locally convex Hausdorff space \( E \), which we need in some examples of vector-valued functions. For a disk \( D \subset E \), i.e. a bounded, absolutely convex set, the vector space \( E_D = \bigcup_{n \in \mathbb{N}} nD \) becomes a normed space if it is equipped with the gauge functional of \( D \) as a norm (see [20, p. 151]). The space \( E \) is called \textit{locally complete} if \( E_D \) is a Banach space for every closed disk \( D \subset E \) (see [24, 10.2.1 Proposition, p. 197]).

Let us recall the definition of the mixed topology, [48, Section 2.1], and the notion of a Saks space, [10, I.3.2 Definition, p. 27–28], which will be important for the rest of the paper.

2.1. \textbf{Definition} ([30, Definition 2.2, p. 3]). Let \((X, \| \cdot \|)\) be a normed space and \(\tau\) a Hausdorff locally convex topology on \(X\) that is coarser than the \(\| \cdot \|\)-topology \(\tau_{\| \cdot \|}\). Then

(a) the \textit{mixed topology} \(\gamma := \gamma(\| \cdot \|, \tau)\) is the finest linear topology on \(X\) that coincides with \(\tau\) on \(\| \cdot \|\)-bounded sets and such that \(\tau \subset \gamma \subset \tau_{\| \cdot \|}\);

(b) the triple \((X, \| \cdot \|, \tau)\) is called a \textit{pre-Saks space}. It is called a \textit{Saks space} if there exists a directed system of continuous seminorms \(\Gamma_{\tau}\) that generates the topology \(\tau\) such that

\[
\|x\| = \sup_{q \in \Gamma_{\tau}} q(x), \quad x \in X. \tag{1}
\]

In comparison to [30, Definition 2.2, p. 3] we dropped the assumption that the space \((X, \| \cdot \|)\) should be complete and added the notion of a pre-Saks space in Definition 2.1. The mixed topology is actually Hausdorff locally convex and the definition given above is equivalent to the one introduced by Wiweger [48, Section 2.1] due to [48, Lemmas 2.2.1, 2.2.2, p. 51].

It is often useful to have a characterisation of the mixed topology by generating systems of continuous seminorms, e.g. the definition of dissipativity in Lumer–Phillips generation theorems for bi-continuous semigroups depends on the choice of the generating system of seminorms of the mixed topology (see [31]). For that purpose we introduce the following auxiliary topology.

2.2. \textbf{Definition} ([30, Definition 3.9, p. 9]). Let \((X, \| \cdot \|, \tau)\) be a Saks space and \(\Gamma_{\tau}\) a directed system of continuous seminorms that generates the topology \(\tau\) and fulfils [11]. We set

\[
\mathcal{N} := \{ (q_n, a_n)_{n \in \mathbb{N}} \mid (q_n)_{n \in \mathbb{N}} \subset \Gamma_{\tau}, (a_n)_{n \in \mathbb{N}} \in c_0^* \}
\]

where \(c_0^*\) is the family of all real non-negative null-sequences. For \((q_n, a_n)_{n \in \mathbb{N}} \in \mathcal{N}\) we define the seminorm

\[
\|x\|_{(q_n, a_n)_{n \in \mathbb{N}}} := \sup_{n \in \mathbb{N}} q_n(x) a_n, \quad x \in X.
\]

We denote by \(\gamma_s := \gamma_s(\| \cdot \|, \tau)\) the locally convex Hausdorff topology that is generated by the system of seminorms \((\|x\|_{(q_n, a_n)_{n \in \mathbb{N}}})_{(q_n, a_n)_{n \in \mathbb{N}} \in \mathcal{N}}\) and call it the \textit{submixed topology}.

By [10, 1.1.10 Proposition, p. 9], [10, 1.4.5 Proposition, p. 41–42] and [17, Lemma A.1.2, p. 72] we have the following relation between the mixed and the submixed topology.

2.3. \textbf{Remark} ([30, Remark 3.10, p. 9]). Let \((X, \| \cdot \|, \tau)\) be a Saks space, \(\Gamma_{\tau}\) a directed system of continuous seminorms that generates the topology \(\tau\) and fulfils [11]. \(\gamma := \gamma(\| \cdot \|, \tau)\) the mixed and \(\gamma_s := \gamma_s(\| \cdot \|, \tau)\) the submixed topology.

(a) We have \(\tau \subset \gamma_s \subset \gamma\) and \(\gamma_s\) has the same convergent sequences as \(\gamma\).
I is an isometry, by [10, I.3.1 Lemma, p. 27] and [32, Lemma 5.5 (a), p. 2680–2681].

Proof. Part (a) follows from [10, I.1.18 Proposition, p. 15]. The first part of (b) follows from (a), the open mapping theorem and the estimate

\[ \|x\| \leq q(x) + \varepsilon, \]

for all \( x \in X \). If \((X, \|\cdot\|, \tau)\) is in addition a Saks space, then \( \|\cdot\| \) is an equality, thus \( \mathcal{I} \) is an isometry, by [10, I.3.1 Lemma, p. 27] and [32, Lemma 5.5 (a), p. 2680–2681]. Part (c) follows from Remark 2.5.

\[ \square \]

The submixed topology \( \gamma' \) was originally introduced in [48, Theorem 3.1.1, p. 62] where a proof of Remark 2.3 (b) can be found, too. The following notions will also be needed.

2.4. Definition. Let \((X, \|\cdot\|, \tau)\) be a pre-Saks space.

(a) We call \((X, \|\cdot\|, \tau)\) complete if \((X, \gamma)\) is complete.

(b) We call \((X, \|\cdot\|, \tau)\) semi-reflexive if \((X, \gamma)\) is semi-reflexive.

(c) We call \((X, \|\cdot\|, \tau)\) semi-Montel if \((X, \gamma)\) is a semi-Montel space.

For Saks spaces the notions of completeness and semi-reflexivity were already introduced in [10, I.3.2 Definition, p. 27–28] and [31, Definition 2.2, p. 2]. The preceding notions may be characterised by topological properties of \( B_{1\|}\) w.r.t. \( \tau \).

2.5. Remark. Let \((X, \|\cdot\|, \tau)\) be a pre-Saks space.

(a) \((X, \|\cdot\|, \tau)\) is a Saks space if and only if \( B_{1\|} = \{ x \in X \mid |x| \leq 1 \} \) is \( \tau \)-closed by [10, I.3.1 Lemma, p. 27].

(b) \((X, \|\cdot\|, \tau)\) is complete if and only if \( B_{1\|} \) is \( \tau \)-complete by [10, I.1.14 Proposition, p. 11].

(c) \((X, \|\cdot\|, \tau)\) is semi-reflexive if and only if \( B_{1\|} \) is \( \sigma(X, (X, \tau)') \)-complete by [10, I.1.21 Corollary, p. 16], where \( \sigma(X, (X, \tau)') \) denotes the weak topology of the dual pair \((X, (X, \tau)')\).

(d) \((X, \|\cdot\|, \tau)\) is semi-Montel if and only if \( B_{1\|} \) is \( \tau \)-compact by [10, I.1.13 Proposition, p. 11].

We note the following generalisation of the Dixmier–Ng theorem [38, Theorem 1, p. 279], which links the existence of a Banach predual to the mixed topology.

2.6. Proposition. Let \((X, \|\cdot\|, \tau)\) be a pre-Saks space, \( X'_\gamma := (X, \gamma)' \) and \( \|\cdot\|_{X'_\gamma} \) denote the restriction of the operator norm \( \|\cdot\|_X \) to \( X'_\gamma \).

(a) Then \( X'_\gamma \) is a \( \|\cdot\|_{X_\gamma} \)-closed subspace of \( X' \) and \( (X, \gamma)'_{X'_\gamma} = (X'_\gamma, \|\cdot\|_{X'_\gamma}) \).

(b) If \((X, \|\cdot\|, \tau)\) is semi-reflexive, then \((X, \|\cdot\|)\) is a Banach dual space with predual \( X'_\gamma \), more precisely, the evaluation map

\[ \mathcal{I}: (X, \|\cdot\|) \to (X'_\gamma, \|\cdot\|_{X'_\gamma})_{X'_\gamma}, x \mapsto \{ x' \mapsto x'(x) \}, \]

is a topological isomorphism, in particular, \((X, \|\cdot\|)\) is complete. If \((X, \|\cdot\|, \tau)\) is in addition a Saks space, then the evaluation map is an isometry.

(c) If \( B_{1\|} \) is \( \tau \)-compact, then \((X, \|\cdot\|, \tau)\) is a complete semi-Montel Saks space, in particular, semi-reflexive.

Proof. Part (a) follows from [10, I.1.18 Proposition, p. 15]. The first part of (b) follows from (a), the open mapping theorem and the estimate

\[ \|\mathcal{I}(x)\|_{(X'_\gamma, \|\cdot\|_{X'_\gamma})'} = \sup_{x' \in X'_\gamma, \|x'\|_{X'_\gamma} \leq 1} |x'(x)| \leq \|x\| \]

(2)

for all \( x \in X \). If \((X, \|\cdot\|, \tau)\) is in addition a Saks space, then (2) is an equality, thus \( \mathcal{I} \) an isometry, by [10, I.3.1 Lemma, p. 27] and [32, Lemma 5.5 (a), p. 2680–2681]. Part (c) follows from Remark 2.5.
In the Dixmier–Ng theorem [38, Theorem 1, p. 279] $B_{\beta\gamma}$ is assumed to be $\tau$-compact and the predual constructed there coincides with our predual $X'_\gamma$ since

$$X'_\gamma = \{ x' : X \to \mathbb{K} \mid x' \text{ linear on } X \text{ and } \tau \text{-continuous on } B_{\beta\gamma} \}$$

by [10, I.1.7 Corollary, p. 8] where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the scalar field of $X$.

We close this section with the following observation concerning the approximation property of $(X, \gamma)$ in the semi-Montel case, whose proof is an adaptation of [21, Theorem 4.6 (i)$\iff$(ii), p. 651–652] in combination with [35, Theorem 4.1, p. 43] where $X = \text{Lip}_0(\Omega)$ (see Theorem 3.7).

### 2.7. Proposition

Let $(X, \| \cdot \|, \tau)$ be a semi-Montel pre-Saks space. Then the following assertions are equivalent.

1. $(X, \gamma)$ has the approximation property.
2. $(X'_\gamma, \| \cdot \|_{X'_\gamma})$ has the approximation property.

**Proof.** (a)$\iff$(b): Due to Remark 2.5 (d) and [10, I.4.1 Proposition, p. 38] or [10, I.4.2 Corollary (d), p. 38] in combination with [35, Theorem 4.1, p. 43] we have $(X, \gamma) = (X'_\gamma, \| \cdot \|_{X'_\gamma})$. Hence $E := (X'_\gamma, \| \cdot \|_{X'_\gamma})$ has the approximation property by Corollary 1.3, p. 144 because $(X, \gamma) = E_\kappa'$ has the approximation property.

(b)$\iff$(a): We note that $(X'_\gamma, \| \cdot \|_{X'_\gamma}) = (X, \gamma)'_\kappa = (X, \gamma)'_{\kappa}$ by Remark 2.5 (d), and Proposition 2.6 (a) and (c). Thus $E := (X, \gamma)$ has the approximation property by Corollary 1.3, p. 144 because $(X'_\gamma, \| \cdot \|_{X'_\gamma}) = E_\kappa'$ has the approximation property.

In particular, the proof above shows that $(X, \gamma) = (X'_\gamma, \| \cdot \|_{X'_\gamma}) = ((X, \gamma)'_{\kappa})$, i.e. $(X, \gamma)$ is a DFC-space by 33, Theorem 4.1, p. 43, if $(X, \| \cdot \|, \tau)$ is a semi-Montel pre-Saks space. Sufficient conditions that guarantee that $(X, \gamma)$ has the approximation property may be found in [10, I.4.20 Proposition, p. 53] and [10, I.4.21, I.4.22 Corollaries, p. 54].

### 3. Examples of Saks function spaces and their weak vector-valued versions

In this section we give some examples of spaces of functions which are complete semi-Montel Saks spaces. Since point evaluation functionals will be important for our linearisation results, we introduce the following notion.

#### 3.1. Definition

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\Omega$ a non-empty set and $(\mathcal{F}(\Omega), \| \cdot \|, \tau)$ a pre-Saks space of functions $f: \Omega \to \mathbb{K}$. We call $(\mathcal{F}(\Omega), \| \cdot \|, \tau)$ a pre-Saks function space if $\delta_{x'} \in \mathcal{F}(\Omega)'$ for all $x' \in \Omega$ where $\delta_{x'}(f) := f(x)$ for $f \in \mathcal{F}(\Omega)$ (see [23, p. 235]).

We will need a weak vector-valued version of such spaces, too. Let $(\mathcal{F}(\Omega), \| \cdot \|, \tau)$ be a pre-Saks function space. For a locally convex Hausdorff space $E$ over the field $\mathbb{K}$ with fundamental system of seminorms $\Gamma_E$ we define the space of weak $E$-valued $\mathcal{F}$-functions by

$$\mathcal{F}(\Omega, E)_\sigma := \{ f: \Omega \to E \mid \forall e' \in E' : e' \circ f \in \mathcal{F}(\Omega) \}.$$  

For $p \in \Gamma_E$ we set $U_p = \{ x \in E \mid e(x) < 1 \}$ and denote by $U_p^e$ the polar of $U_p$. If $(\mathcal{F}(\Omega), \| \cdot \|)$ is a Banach space, thus webbed, the supremum $\|f\|_{\sigma,p} := \sup_{e \in U_p^e} |e' \circ f|$ is finite for every $f \in \mathcal{F}(\Omega, E)_\sigma$ and $p \in \Gamma_E$ by [23, Satz 10.6 a), p. 237] (cf. [23, 3.12 Proposition, p. 10]). Hence the space $\mathcal{F}(\Omega, E)_\sigma$ equipped with the system of seminorms $(\| \cdot \|_{\sigma,p})_{p \in \Gamma_E}$ is a locally convex Hausdorff space if $(\mathcal{F}(\Omega), \| \cdot \|)$ is a Banach space.

Our first examples are certain (sub)spaces of the weighted space of continuous functions on a completely regular Hausdorff space. Let us recall the following observations. For a completely regular Hausdorff space $\Omega$ (see [19, Definition 11.1, p. 180]) we denote by $\mathcal{W}_{0,0}(\Omega)$ the family of all bounded functions $w: \Omega \to [0, \infty)$
that vanish at infinity, i.e. for every \( \varepsilon > 0 \) the set \( \{ x \in \Omega \mid w(x) \geq \varepsilon \} \) is compact. Further, we denote by \( W_{\text{usc,0}}^+(\Omega) \) resp. \( C_0^0(\Omega) \) the family of all upper semicontinuous resp. continuous functions \( w: \Omega \to [0, \infty) \) that vanish at infinity. We note that \( C_0^0(\Omega) \subset W_{\text{usc,0}}^+(\Omega) \subset W_{\text{b,b,0}}^+(\Omega) \) because upper semicontinuous functions are bounded on compact sets. By the proofs of \textit{[10]} I.1.11 Proposition, p. 82–83 and \textit{[9]} Proposition 3, p. 590 we have the following proposition.

3.2. Proposition. Let \( \Omega \) be a completely regular Hausdorff space, \((K_n)_{n \in \mathbb{N}}\) a strictly increasing sequence of compact subsets of \( \Omega \) and \((a_n)_{n \in \mathbb{N}}\) a strictly decreasing positive null-sequence. Then there is \( w \in W_{\text{usc,0}}^+(\Omega) \) such that \( \supp w \subseteq \bigcup_{n \in \mathbb{N}} K_n \) and \( w(x) = a_n \) for \( x \in K_1 \) and \( a_{n+1} \leq w(x) \leq a_n \) for \( x \in K_{n+1} \setminus K_n \). If \( \Omega \) is locally compact and \( K_n \subset K_{n+1} \) for every \( n \in \mathbb{N} \), then we may choose \( w \in C_0^0(\Omega) \).

Here \( \supp w \) denotes the support of \( w \) and \( K_{n+1} \) the set of inner points of \( K_{n+1} \).

3.3. Remark. For a completely regular Hausdorff space \( \Omega \) and a continuous function \( v: \Omega \to (0, \infty) \) we set

\[
C_v(\Omega) = \{ f \in C(\Omega) \mid \| f \| := \sup_{x \in \Omega} |f(x)|v(x) < \infty \}
\]

where \( C(\Omega) \) is the space of scalar-valued continuous functions on \( \Omega \). Due to \textit{[18]} Example D), p. 65–66 and \textit{[18]} Lemma A.5, p. 44 we have that \( (C_v(\Omega), \| \cdot \|, \tau_{\text{co}}) \) is a Saks space and \( \gamma(\| \cdot \|, \tau_{\text{co}}) = \gamma(\| \cdot \|, \tau_{\text{co}}) \) (cf. \textit{[18]} Lemma A.1, p. 44) where \( \tau_{\text{co}} \) denotes the \textit{compact-open topology}, i.e. the topology of uniform convergence on compact subsets of \( \Omega \). Hence the the mixed topology \( \gamma(\| \cdot \|, \tau_{\text{co}}) \) is generated by the system of seminorms

\[
\| f \|_{(K_n,a_n)_{n \in \mathbb{N}}} = \sup_{n \in \mathbb{N}} \sup_{x \in K_n} |f(x)|v(x)a_n, \quad f \in C_v(\Omega),
\]

where \((K_n)_{n \in \mathbb{N}}\) is a sequence of compact subsets of \( \Omega \), and \((a_n)_{n \in \mathbb{N}} \in c_0^0(\Omega) \) by Definition \textit{[22]} 2.2. Furthermore, let \( W_0 = W_{\text{b,b,0}}^+(\Omega) \) or \( W_{\text{usc,0}}^+(\Omega) \). By \textit{[44]} Theorem 2.4, p. 316 and \textit{[18]} Lemma A.5, p. 44 another system of seminorms that generates the mixed topology \( \gamma(\| \cdot \|, \tau_{\text{co}}) \) is given by

\[
\| f \|_w = \sup_{x \in \Omega} |f(x)|v(x)w(x), \quad f \in C_v(\Omega),
\]

for \( w \in W_0 \) (cf. \textit{[18]} Proposition A.3, p. 44 for \( W_0 = W_{\text{b,b,0}}^+(\Omega) \)). If \( \Omega \) is locally compact, we may replace \( W_0 \) by \( C_0^0(\Omega) \) due to \textit{[44]} Theorem 2.3 (b), p. 316 and \textit{[18]} Lemma A.5, p. 44. Thus \( \gamma(\| \cdot \|, \tau_{\text{co}}) \) coincides with the general strict topology defined in \textit{[45]} p. 145 by \textit{[43]} Theorem 3.1, p. 146 (see also \textit{[16]} Theorem B, p. 277) if \( \Omega \) is locally compact.

Now, let \( \mathcal{F}(\Omega) \) be a linear subspace of \( C_v(\Omega) \) and denote by \( \tau_{\text{co},\mathcal{F}(\Omega)} \) the relative compact-open topology on \( \mathcal{F}(\Omega) \) induced by \( (C_v(\Omega), \tau_{\text{co}}) \). If the \( \| \cdot \|_{\mathcal{F}(\Omega)} \)-closed unit ball \( B_{\mathcal{F}(\Omega)} = \{ f \in \mathcal{F}(\Omega) \mid \| f \| \leq 1 \} \) is \( \tau_{\text{co},\mathcal{F}(\Omega)} \)-compact, then \( \mathcal{F}(\Omega), \| \cdot \|_{\mathcal{F}(\Omega)} \) is a Banach space by \textit{[10]} I.1.2 Lemma, p. 4, \( \mathcal{F}(\Omega), \| \cdot \|_{\mathcal{F}(\Omega)}, \tau_{\text{co},\mathcal{F}(\Omega)} \) a complete semi-Montel Saks space by \textit{[10]} I.1.13, I.1.14 Propositions, p. 11 and \textit{[10]} I.3.1 Lemma, p. 27 and

\[
\gamma(\| \cdot \|_{\mathcal{F}(\Omega)}, \tau_{\text{co},\mathcal{F}(\Omega)}) = \gamma(\| \cdot \|, \tau_{\text{co}})_{\mathcal{F}(\Omega)}
\]

by \textit{[10]} I.4.6 Lemma, p. 44. In the following we use the conventions

\[
(\mathcal{F}(\Omega), \| \cdot \|, \tau_{\text{co}}) = (\mathcal{F}(\Omega), \| \cdot \|_{\mathcal{F}(\Omega)}, \tau_{\text{co},\mathcal{F}(\Omega)})
\]

and \( \gamma(\| \cdot \|, \tau_{\text{co}}) = \gamma(\| \cdot \|_{\mathcal{F}(\Omega)}, \tau_{\text{co},\mathcal{F}(\Omega)}) \) for such subspaces \( \mathcal{F}(\Omega) \) of \( C_v(\Omega) \) (see Corollary 3.5).
Examples of completely regular Hausdorff spaces are metrisable spaces by Proposition 11.5, p. 181 and locally convex Hausdorff spaces by Proposition 3.27, p. 95. Moreover, a topological space Ω is called $k$-space if it fulfills the following condition: $A \subset Ω$ is closed if and only if $A \cap K$ is closed in $K$ for every compact $K \subset Ω$. In particular, metrisable spaces and DFM-spaces, i.e., strong duals of Fréchet–Montel spaces, are completely regular Hausdorff $k$-spaces by Proposition 3.3.20, p. 152 and 4.11 Theorem (5), p. 39, respectively. Further, every subspace of a completely regular Hausdorff space is completely regular and Hausdorff as well.

We see in Proposition 2.6 (c) and Remark 3.3 that the compactness of norm-closed unit balls in weaker (relative) topologies plays a prominent role. Therefore we note the following observation which we will use repeatedly.

3.4. Proposition. Let $(Y, τ)$ be a Hausdorff space, $X \subset Y$ and denote by $τ_X := τ|_X$ the relative topology on $X$ induced by $(Y, τ)$ on $X$. Suppose that $B \subset X$ is relatively compact in $(Y, τ)$, i.e., the closure $\overline{B}$ of $B$ w.r.t. $τ$ is compact in $(Y, τ)$. Then $B$ is compact in $(X, τ_X)$ if $B$ is closed in $(Y, τ)$.

Proof. Due to our assumptions $B = \overline{B}$ is compact in $(Y, τ)$. Let $I$ be a non-empty set, $O_i \subset X$ $τ_X$-open for all $i \in I$ and $B \subset \bigcup_{i \in I} O_i$. Since $τ_X$ is the relative topology induced by $(Y, τ)$ on $X$, there are $τ$-open sets $U_i \subset Y$ such that $O_i = U_i \cap X$ for all $i \in I$. Thus we have

$$B \subset \bigcup_{i \in I}(U_i \cap X) \subset \bigcup_{i \in I}U_i,$$

implying that there is $n \in \mathbb{N}$ such that $B \subset \bigcup_{k=1}^n U_{i_k}$ as $B$ is compact in $(Y, τ)$.

Hence $B = (B \cap X) \subset \bigcup_{k=1}^n O_{i_k}$, which means that $B$ is compact in $(X, τ_X)$. □

Let us introduce the (sub)spaces of the weighted space of continuous functions and their vector-valued versions that we want to consider. Let $E$ be a locally convex Hausdorff space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with fundamental system of seminorms $Γ_E$.

For a discrete space $Ω$ and a function $v : Ω \to (0, ∞)$ we set

$$ℓv(Ω, E) = \{f : Ω \to E \mid \forall p \in Γ_E : \|f\|_p := \sup_{x \in Ω}|f(x)|v(x) < ∞\}.$$

First, we note that $v \in C(Ω)$ and $ℓv(Ω) = Cv(Ω)$ since $Ω$ is discrete (in particular, metrisable by Proposition 4.1.4, Example, p. 251).

For an open set $Ω \subset \mathbb{R}^d$ we define the kernel

$$C^∞(Ω, E) = \{f \in C^∞(Ω, E) \mid f \in ker P(\partial)^E\}$$

of a linear partial differential operator $P(\partial)^E : C^∞(Ω, E) \to C^∞(Ω, E)$ which is hypoelliptic if $E = \mathbb{K}$. For a continuous weight $w : Ω \to (0, ∞)$ we define the weighted kernel

$$C_P(Ω, E) := \{f \in C^∞(Ω, E) \mid \forall p \in Γ_E : \|f\|_p := \sup_{x \in Ω}|f(x)|v(x) < ∞\}.$$

For an open subset $Ω$ of a complex locally convex Hausdorff space let $H(Ω, E)$ be the space of holomorphic functions $f : Ω \to E$, i.e., the space of Gâteaux-holomorphic and continuous functions $f : Ω \to E$ (see Definition 3.6, p. 152), and for a continuous weight $w : Ω \to E$ we set

$$Hw(Ω, E) := \{f \in H(Ω, E) \mid \forall p \in Γ_E : \|f\|_p := \sup_{x \in Ω}|f(x)|v(x) < ∞\}.$$

Further, we set $F(Ω) := F(Ω, \mathbb{K})$ if $F = ℓv, C_P(Ω), C_P(Ω)v$ or $Hv$, and omit the index $p$ of $\|f\|_p$ if $E = \mathbb{K}$.  

3.5. Corollary. Let $E$ be a locally convex Hausdorff space with fundamental system of seminorms $Γ_E$ and $Ω$ a non-empty set. If
validity of the statements (a), (b) and (c) that

\[ \text{Proof.} \]

First, we note that for all

\[ p. 8–9 \] and Proposition 3.4 with the Montel space

\[ H \]

\[ K. \]

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Example 3.8 (g), p. 160 and

\[ \text{and for part (d) below assume in addition that } X \text{ is metrisable or a DFM-space, and } E \text{ complete,} \]

and \( v: \Omega \to (0, \infty) \) continuous, then the following assertions hold.

(a) \( \mathcal{F}(\Omega, E) = \mathcal{F}(\Omega, E)_\sigma \) is a complete semi-Montel Saks function space.

(b) The mixed topology \( \gamma(\cdot \| \cdot, \tau_{co}) \) is generated by the system of seminorms

\[ \| f \|_{(K_n, a_n)_{n \in \mathbb{N}}} := \sup_{n \in \mathbb{N}} \| f(x) a_n \|, \quad f \in \mathcal{F}(\Omega), \]

where \( (K_n)_{n \in \mathbb{N}} \) is a sequence of compact subsets of \( \Omega \) and \( (a_n)_{n \in \mathbb{N}} \in c_0^\mathbb{N} \).

(c) Let \( \mathcal{W}_0 = W^0_0(\Omega) \) or \( W^\infty_0(\Omega) \). The mixed topology \( \gamma(\cdot \| \cdot, \tau_{co}) \) is also generated by the system of seminorms

\[ |f|_w := \sup_{z \in \Omega} |f(x)| v(x) w(x), \quad f \in \mathcal{F}(\Omega), \]

for \( p \in \Gamma_E \) and \( w \in \mathcal{W}_0 \). If \( \Omega \) is locally compact, we may replace \( \mathcal{W}_0 \) by \( C^0_0(\Omega) \).

(d) \( \mathcal{F}(\Omega, E) = \mathcal{F}(\Omega, E)_\sigma \) and \( \| \cdot \|_p = \| \cdot \|_{\sigma, p} \) for all \( p \in \Gamma_E \).

\[ \text{Proof.} \]

First, we note that \( \delta_x \in \mathcal{F}(\Omega)_\delta \) for every \( x \in \Omega \) since \( \delta_x \in (\mathcal{F}(\Omega), \tau_{co})' \) and \( \tau_{co} \subset \gamma(\cdot \| \cdot, \tau_{co}) \). Due to Proposition 2.24 (c) and Remark 3.3 we only need for the validity of the statements (a), (b) and (c) that \( B_{\| \cdot \|} \) is \( \tau_{co} \)-compact in each case.

(i) The \( \| \cdot \| \)-closed unit ball \( B_{\| \cdot \|} \) is \( \tau_{co} \)-compact by [11, II.1.24 Remark 4], p. 88–89 and [18, Lemma A.5], p. 44. Further, we have \( \ell v(\Omega, E) = \ell v(\Omega, E)_\sigma \) and \( \| \cdot \|_p = \| \cdot \|_{\sigma, p} \) for all \( p \in \Gamma_E \) by [34, Proposition 24.10], p. 282.

(ii) The \( \| \cdot \| \)-closed unit ball \( B_{\| \cdot \|} \) is \( \tau_{co} \)-compact by the proof of [28, 3.10 Corollary, p. 8–9] and Proposition 2.24 with the Montel space \( (Y, \tau) = (C^\infty_0(\Omega), \tau_{co}) \).

If \( E \) is locally complete, then we have \( \mathcal{C}(\Omega, E) = \mathcal{C}(\Omega, E)_\sigma \) by the weak-strong principle [4, Theorem 9], p. 232] and \( \| \cdot \|_p = \| \cdot \|_{\sigma, p} \) for all \( p \in \Gamma_E \) by [34, Proposition 24.10], p. 282.

(iii) It is easily seen that \( \| \cdot \| \)-closed unit ball \( B_{\| \cdot \|} \) is closed in \( (H(\Omega), \tau_{co}) \). Further, \( (H(\Omega), \tau_{co}) \) is a semi-Montel space by [11, Proposition 3.37], p. 130] since \( \Omega \) is an open subset of a locally convex Hausdorff k-space. This implies that \( B_{\| \cdot \|} \) is \( \tau_{co} \)-compact by Proposition 2.24 with \( (Y, \tau) = (H(\Omega), \tau_{co}) \).

Let us turn to (d). Since \( E \) is complete and \( \Omega \) an open subset of a metrisable locally convex space or of a DFM-space, we have \( H(\Omega, E) = H(\Omega, E)_\sigma \) by [12, Example 3.8 (g), p. 160] and \( \| \cdot \|_p = \| \cdot \|_{\sigma, p} \) for all \( p \in \Gamma_E \) by [34, Proposition 24.10], p. 282].

Corollary 3.3 (a) and (c) for \( \mathcal{W}_0 = C^0_0(\Omega) \) in case (iii) are already contained in [4, Proposition 3.1, 3.2], p. 77–78 if \( \Omega \) is an open subset of \( C^d \). In case (iii) for an open subset \( \Omega \) of a Banach space another system of seminorms that generates the mixed topology \( \gamma(\cdot \| \cdot, \tau_{co}) \) in the spirit of Corollary 3.3 (b) on \( H^\infty(\Omega) = H(\Omega) \) with \( v(z) := 1 \) for all \( z \in \Omega \) is given in [36, 4.5 Theorem], p. 875].

For a complex locally convex Hausdorff space \( E \) with fundamental system of seminorms \( \Gamma_E \) and a continuous function \( v: \mathbb{D} \to (0, \infty) \) with \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) we define the Bloch type space

\[ Bv(\mathbb{D}, E) := \{ f \in H(\mathbb{D}, E) \mid \forall p \in \Gamma_E: \| f \|_p < \infty \} \]

with

\[ \| f \|_p := p(f(0)) + \sup_{z \in \mathbb{D}} p(f'(z)) v(z). \]
Further, we set \( B_v(\mathbb{D}) := B_v(\mathbb{D}, \mathbb{C}) \) and omit the index \( p \) of \( \| \cdot \|_p \) if \( E = \mathbb{C} \).

3.6. **Theorem.** Let \( E \) be a locally complete complex locally convex Hausdorff space with fundamental system of seminorms \( \Gamma_E \) and \( v: \mathbb{D} \to (0, \infty) \) continuous. Then the following assertions hold.

(a) \((B_v(\mathbb{D}), \| \cdot \|, \tau_{co})\) is a complete semi-Montel Saks function space.
(b) The mixed topology \( \gamma(\| \cdot \|, \tau_{co}) \) is generated by the system of seminorms

\[
\| f \|_{(K_n, \omega_n)_{n \in \mathbb{N}}} := \sup_{n \in \mathbb{N}} \sup_{z \in K_n} (|f(0)| + |f'(z)|v(z))\omega_n, \quad f \in B_v(\mathbb{D}),
\]

where \((K_n)_{n \in \mathbb{N}}\) is a sequence of compact subsets of \( \mathbb{D} \) and \((\omega_n)_{n \in \mathbb{N}} \in c_0^\prime\).
(c) Let \( \mathcal{W}_0 := \mathcal{W}_{c_0}^0(\mathbb{D}) \) or \( \mathcal{W}_{c_0}^0(\mathbb{D}) \) or \( C^\prime(\mathbb{D}) \). The mixed topology \( \gamma(\| \cdot \|, \tau_{co}) \) is also generated by the system of seminorms

\[
|f|_{w} := \sup_{z \in \mathbb{D}} (|f(0)| + |f'(z)|v(z))w(z), \quad f \in B_v(\mathbb{D}),
\]

for \( p \in \Gamma_E \) and \( w \in \mathcal{W}_0 \).

(d) \((B_v(\mathbb{D}), E) = B_v(\mathbb{D}, E)_{\sigma} \) and \( \| \cdot _{\sigma, p} \leq \| \|_p \leq 2\| \cdot _{\sigma, p} \) for all \( p \in \Gamma_E \).

**Proof.** (a) First, we note that \( \tau_{co} \) is for every \( z \in \mathbb{D} \) since \( \tau_{co} \in (\| \cdot \|, \tau_{co}) \). Due to Proposition 2.6 (c) we only need for the validity of statement (a) that \( B_{\| \cdot \|} \) is \( \tau_{co} \)-compact. By the Weierstraß theorem the \( \| \cdot \| \)-closed unit ball \( B_{\| \cdot \|} \) is closed in \((H(\mathbb{D}), \tau_{co})\). The set \( B_{\| \cdot \|} \) is \( \tau_{co} \)-compact by \([14, p. 4]\) for radial, non-increasing \( v \), and by the proof of \([29, Corollary 3.8, p. 9–10]\) in combination with Proposition 2.3 (b) for general \( v \).

(b) For compact \( K \subset \mathbb{D} \) we set

\[
qu_K(f) := \sup_{z \in K} |f(0)| + |f'(z)|v(z), \quad f \in B_v(\mathbb{D}).
\]

We claim that the seminorms \((q_K)\) where \( K \) runs through the compact subsets of \( \mathbb{D} \) generates the topology \( \tau_{co} \). For every \( 0 < r < 1 \) we have

\[
\max_{|z| \leq r} |f(z)| \leq \left( 1 + \frac{r}{\min_{|\zeta| \leq r} v(\zeta)} \right) \left( |f(0)| + \sup_{|\zeta| \leq r} |f'(\zeta)|v(\zeta) \right)
\]

for all \( f \in B_v(\mathbb{D}) \) by the proof of \([29, Corollary 3.8, p. 9–10]\), and for every \( 0 < s < r < 1 \)

\[
|f(0)| + \max_{|z| \leq s} |f'(z)|v(z) \leq |f(0)| + \frac{1}{r} \max_{|\zeta| \leq s} v(\zeta) \max_{|\zeta| \leq r} |f(\zeta)| \leq \left( 1 + \frac{1}{r} \max_{|\zeta| \leq s} v(\zeta) \right) \max_{|\zeta| \leq r} |f(\zeta)|
\]

for all \( f \in B_v(\mathbb{D}) \) by Cauchy’s inequality, which proves our claim.

Since

\[
\| f \| = \sup_{K \subset \mathbb{D}} \sup_{z \in K} |f(0)| + |f'(z)|v(z), \quad f \in B_v(\mathbb{D}),
\]

we have by condition (ii) of Remark 2.3 (b) that the mixed topology \( \gamma(\| \cdot \|, \tau_{co}) \) is generated by the seminorms

\[
\| f \|_{(K_n, \omega_n)_{n \in \mathbb{N}}} := \sup_{n \in \mathbb{N}} \sup_{z \in K_n} (|f(0)| + |f'(z)|v(z))\omega_n, \quad f \in B_v(\mathbb{D}),
\]

where \((K_n)_{n \in \mathbb{N}}\) is a sequence of compact subsets of \( \mathbb{D} \) and \((\omega_n)_{n \in \mathbb{N}} \in c_0^\prime\).

(c) We denote by \( \omega_0 \) and \( \omega_{uc} \) the locally convex Hausdorff topologies generated by \( \| \cdot \|_{ucW_{c_0}^0(\mathbb{D})} \) and \( \| \cdot \|_{ucW_{c_0}^0(\mathbb{D})} \), respectively. First, we prove that the identity map \( \text{id}: (B_v(\mathbb{D}), \gamma(\| \cdot \|, \tau_{co})) \to (B_v(\mathbb{D}), \omega_0) \) is continuous. Due to \([14, I.1.7 Corollary, p. 8]\) and \([14, I.1.8 Lemma, p. 8]\) we only need to prove that its restriction to \( B_{\| \cdot \|} \) is continuous at zero. Let \( \varepsilon > 0, w \in \mathcal{W}_{c_0}^0(\mathbb{D}) \) and set \( V := \{ f \in B_v(\mathbb{D}) \mid |f|_w \leq \varepsilon \} \). Then there is a compact set \( K \subset \mathbb{D} \) such that \( w(z) < \frac{\varepsilon}{2} \) for \( z \in \mathbb{D} \setminus K \). We define
\[ U := \{ f \in Bv(\mathbb{D}) \mid q_{K}(f) \leq \frac{\varepsilon}{2(1 + |w|_{\infty})} \} \]  
where \( |w|_{\infty} = \sup_{z \in \mathbb{D}} w(z) \) and note for all \( f \in U \cap B_{\| \cdot \|} \) it holds that
\[
|f|_{w} \leq \sup_{z \in \mathbb{D} \setminus K} ((f(0)) + |f'(z)||v(z)|)w(z) + \sup_{z \in K} (|f(0)| + |f'(z)||v(z)|)w(z) \\
\leq \frac{\varepsilon}{2} \|f\| + \|w\|_{\infty} q_{K}(f) \leq \frac{\varepsilon}{2} + \|w\|_{\infty} \frac{\varepsilon}{2(1 + \|w\|_{\infty})} \leq \varepsilon,
\]
yielding \((U \cap B_{\| \cdot \|}) \subset V\) and so the continuity of \(id\).

Second, we prove that \(id : (Bv(\mathbb{D}), \omega_{usc}) \rightarrow (Bv(\mathbb{D}), \gamma(\| \cdot \|, \tau_{co}))\) is continuous. Let \((K_n)_{n \in \mathbb{N}}\) be a sequence of compact subsets of \(\mathbb{D}\) and \((a_n)_{n \in \mathbb{N}} \in c_{0}^{+}\). W.l.o.g. \(K_n \subset K_{n+1}\) and \(0 < a_{n+1} < a_{n}\) for all \(n \in \mathbb{N}\). Then there is \(w \in W_{usc,0}(\mathbb{D})\) with \(\text{supp} w \subset \bigcup_{n \in \mathbb{N}} K_n\) such that \(w(z) = a_1\) for \(z \in K_1\) and \(a_{n+1} \leq w(z) \leq a_n\) for \(z \in K_{n+1} \setminus K_n\) by Proposition 3.2. It follows that
\[
\|f|_{(K_n,a_n)_{n \in \mathbb{N}}} \leq \sup_{z \in \Omega} |f(0)| + |f'(z)||v(z)|w(z) = |f|_w
\]
for all \(f \in Bv(\mathbb{D})\), which yields the continuity of \(id\) by part (b). Due to \(C^{+}_0(\mathbb{D}) \subset W_{usc,0}(\mathbb{D}) \subset W_{usc,0}^{+}(\mathbb{D})\), Proposition 5.2 and the local compactness of \(\mathbb{D}\) this proves statement (c).

(d) Since \(E\) is locally complete, we have \(Bv(\mathbb{D}, E) = Bv(\mathbb{D}, E)_0\) by the weak-strong principle [3, Theorem 9, p. 232] and \(\| \cdot \|_{\sigma,p} \leq \| \cdot \| \leq 2\| \cdot \|_{\sigma,p}\) for all \(p \in \Gamma_E\) by [34, Proposition 24.10, p. 282].

The proof of Theorem 5.1(c) is just an adaptation of the proofs of [3, Proposition 3, p. 590] and [10, II.1.11 Proposition, p. 82].

For a locally convex Hausdorff space \(E\) over the field \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\) with fundamental system of seminorms \(\Gamma_{E}\) and a metric space \((\Omega, d)\) with a base point denoted by \(0\), i.e. a pointed metric space in the sense of [47, p. 1], we define the space of \(E\)-valued Lipschitz continuous on \((\Omega, d)\) that vanish at \(0\) by
\[
\text{Lip}_{0}(\Omega, E) := \text{Lip}_{0}(\Omega, d, E) : = \{ f : \Omega \rightarrow E \mid f(0) = 0 \text{ and } \|f\|_{p} < \infty \forall p \in \Gamma_{E} \}
\]
where
\[
\|f\|_{p} := \sup_{x, y \in \Omega} \frac{p(f(x) - f(y))}{d(x, y)}.
\]
Further, we set \(\text{Lip}_{0}(\Omega) := \text{Lip}_{0}(\Omega, \mathbb{K})\) and omit the index \(p\) of \(\| \cdot \|_{p}\) if \(E = \mathbb{K}\).

**3.7. Theorem.** Let \(E\) be a locally convex Hausdorff space with fundamental system of seminorms \(\Gamma_{E}\), \((\Omega, d)\) a pointed metric space and \(\Omega_{wd} := \{(x, y) \in \Omega^{2} \mid x \neq y \}\). Then the following assertions hold.

(a) \(\text{Lip}_{0}(\Omega, \| \cdot \|, \tau_{co})\) is a complete semi-Montel Saks function space.

(b) The mixed topology \(\gamma(\| \cdot \|, \tau_{co})\) is generated by the system of seminorms
\[
\|f\|_{(K_n,a_n)_{n \in \mathbb{N}}} := \sup_{(x, y) \in K_n} \frac{|f(x) - f(y)|}{d(x, y)} a_n, \quad f \in \text{Lip}_{0}(\Omega),
\]
where \((K_n)_{n \in \mathbb{N}}\) is a sequence of compact subsets of \(\Omega_{wd}\) and \((a_n)_{n \in \mathbb{N}} \in c_{0}^{+}\).

(c) Let \(W_0 := W_{usc,0}^{+}(\Omega_{wd})\) or \(W_{usc,0}(\Omega_{wd})\). The mixed topology \(\gamma(\| \cdot \|, \tau_{co})\) is also generated by the system of seminorms
\[
|f|_{w} := \sup_{x, y \in \Omega} \frac{|f(x) - f(y)|}{d(x, y)} w(x, y), \quad f \in \text{Lip}_{0}(\Omega),
\]
for \(p \in \Gamma_{E}\) and \(w \in W_0\). If \(\Omega\) is locally compact, we may replace \(W_0\) by \(C_{0}^{+}(\Omega_{wd})\).

(d) \(\text{Lip}_{0}(\Omega, E) = \text{Lip}_{0}(\Omega, E)_{\sigma}\) and \(\| \cdot \|_{p} = \| \cdot \|_{\sigma,p}\) for all \(p \in \Gamma_{E}\).
Proof. (a) First, we note that \( \delta_x \in \text{Lip}_0(\Omega)' \), for every \( x \in \Omega \) since \( \delta_x \in (\text{Lip}_0(\Omega), \tau_{co})' \) and \( \tau_{co} \subset \gamma(\| \cdot \|, \tau_{co}) \). Due to Proposition 2.6 (c) we only need for the validity of statement (a) that \( B_{|f|} \) is \( \tau_{co} \)-compact. For all \( f \in B_{|f|} \) we have

\[
|f(x) - f(y)| \leq d(x, y), \quad x, y \in \Omega,
\]

which implies

\[
|f(x)| = |f(x) - f(0)| \leq d(x, 0), \quad x \in \Omega.
\]

It follows that \( B_{|f|} \) is (uniformly) equicontinuous and \( \{ f(x) \mid f \in B_{|f|} \} \) is bounded in \( \mathbb{K} \) for all \( x \in \Omega \). Ascoli’s theorem (see e.g. [37] Theorem 47.1, p. 290) implies that \( B_{|f|} \) is compact in \((C(\Omega), \tau_{co})\) and so \( \tau_{co} \)-compact in \( \text{Lip}_0(\Omega) \) as well by Proposition 3.4 with \( (Y, \tau) \equiv (C(\Omega), \tau_{co}) \).

(b) Let \( \tau_k \) be the locally convex Hausdorff topology generated by the directed system of seminorms given by

\[
q_K(f) := \sup_{(x,y) \in K} \frac{|f(x) - f(y)|}{d(x,y)}, \quad f \in \text{Lip}_0(\Omega),
\]

for compact \( K \subset \Omega_{wd} \). Then we have

\[
\|f\| = \sup_{K \subset \Omega_{wd}} q_K(f), \quad f \in \text{Lip}_0(\Omega).
\]

Furthermore, for compact \( K \subset \Omega_{wd} \) the projections \( \pi_1(K) \) and \( \pi_2(K) \) on the first and second component, respectively, are compact in \( \Omega \) and

\[
q_K(f) \leq 2 \max_{(x,y) \in K} \frac{1}{d(x, y)} \sup_{\pi_1(K) \subset \pi_2(K)} |f(x)|
\]

for all \( f \in \text{Lip}_0(\Omega) \), which implies \( \tau_k \subset \tau_{co} \). On the other hand, for every \( \varepsilon > 0 \) and compact \( K \subset \Omega \) we have with \( B_{|f|} := \{ x \in \Omega \mid d(x, 0) < \varepsilon \} \) that

\[
\sup_{x \in K} |f(x)| = \sup_{x \in K, x \neq 0} \frac{|f(x) - f(0)|}{d(x, 0)} \leq \max_{x \in K} d(x, 0) \sup_{x \in K, x \neq 0} \frac{|f(x) - f(0)|}{d(x, 0)} + \varepsilon \sup_{x \in K, x \neq 0} \frac{|f(x) - f(0)|}{d(x, 0)}
\]

for all \( f \in \text{Lip}_0(\Omega) \). Thus the topologies \( \tau_k \) and \( \tau_{co} \) coincide on \( \| \cdot \| \)-bounded sets.

It follows that \( \gamma(\| \cdot \|, \tau_k) = \gamma(\| \cdot \|, \tau_{co}) \) by [10] 1.1.6 Corollary (ii), p. 6. Due to condition (ii) of Remark 2.3 (b) this yields that the mixed topology \( \gamma(\| \cdot \|, \tau_{co}) \) is generated by the seminorms

\[
\|f\|_{(K_n, a_n)_{n \in \mathbb{N}}} = \sup_{n \in \mathbb{N}} \sup_{(x,y) \in K_n} \frac{|f(x) - f(y)|}{d(x,y)} a_n, \quad f \in \text{Lip}_0(\Omega),
\]

where \((K_n)_{n \in \mathbb{N}}\) is a sequence of compact subsets of \( \Omega_{wd} \) and \((a_n)_{n \in \mathbb{N}} \in c_0^\prime \).

(c) The proof is similar to the one of Theorem 3.6 (c). We denote by \( \omega_{b} \) and \( \omega_{usc} \) the locally convex Hausdorff topologies generated by \((|\cdot|_w)_{w \in \mathcal{W}^{\prime}_{b, \omega}(\Omega_{wd})}\) and \((|\cdot|_w)_{w \in \mathcal{W}^{\prime}_{usc}(\Omega_{wd})}\), respectively. First, we prove that the continuity of the identity map \( \text{id} : (\text{Lip}_0(\Omega), \gamma(\| \cdot \|, \tau_{co})) \to (\text{Lip}_0(\Omega), \omega_{b}) \). Due to [10] 1.1.7 Corollary, p. 8 and [10] 1.1.8 Lemma, p. 8 we only need to prove that its restriction to \( B_{|f|} \) is continuous at zero. Let \( \varepsilon > 0 \), \( w \in \mathcal{W}^{\prime}_{b, \omega}(\Omega_{wd}) \) and set \( V := \{ f \in \text{Lip}_0(\Omega) \mid |f|_w \leq \varepsilon \} \). Then there is a compact set \( K \subset \Omega_{wd} \) such that \( w(x, y) < \frac{\varepsilon}{2} \) for \( (x, y) \in \Omega_{wd} \setminus K \). We define \( U := \{ f \in \text{Lip}_0(\Omega) \mid |f|_w \leq \frac{\varepsilon}{2} \} \) and note that for all \( f \in U \cap B_{|f|} \) it holds that

\[
|f|_w \leq \sup_{(x,y) \in \Omega_{wd} \setminus K} \frac{|f(x) - f(y)|}{d(x,y)} w(x,y) + \sup_{(x,y) \in K} \frac{|f(x) - f(y)|}{d(x,y)} w(x,y)
\]
of pointwise convergence 3.4, p. 647] where it is used that (W.l.o.g.)
dervatives up to order τ with partially differentiable E for Proposition 24.10, p. 282].

\[ \operatorname{supp} \subset \Gamma \text{ system of seminorms} \]

\[ \sup \{ f \mid (x,y) \in \Omega, E \} \]

Theorem 3.7 (a) is already contained in [21, Theorem 2.1 (7), p. 642] but the proof
of \( \tau_\text{co} \)-compactness of \( B_{\| \cdot \|} \) is only sketched (see [21, p. 641]), which is why we
included it here. Another system of seminorms that generates the mixed topology
\( \gamma(\| \cdot \|, \tau_\text{co}) \) on \( \operatorname{Lip}_0(\Omega) \) in the spirit of Theorem 3.7 (b) is given in [21, Theorem
3.4, p. 647] where it is used that \( \gamma(\| \cdot \|, \tau_\omega) = \gamma(\| \cdot \|, \tau_\text{co}) \) holds for the topology of
pointwise convergence \( \tau_\omega \) (see [21, p. 642]). Theorem 3.7 (c) for \( \mathcal{W}_0 = C^1_0(\Omega_w) \)
generals [21, Theorem 3.3, p. 645] from compact to locally compact \( \Omega \).

For a locally convex Hausdorff space \( E \) over the field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) with fundamental
system of seminorms \( \Gamma_0 \) and \( k \in \mathbb{N}_0 \) we define the space of \( k \)-times continuously
partially differentiable \( E \)-valued functions on an open set \( \Omega \subset \mathbb{R}^d \) whose partial
derivatives up to order \( k \) are continuously extendable to the boundary of \( \Omega \) by

\[ \mathcal{C}^k(\overline{\Omega}, E) := \{ f \in C^k(\Omega, E) \mid (\partial^\beta)^E f \text{ cont. extendable on } \overline{\Omega} \text{ for all } \beta \in \mathbb{N}^d_0, |\beta| \leq k \} \]

which we equip with the system of seminorms given by

\[ |f|_{C^k(\overline{\Omega}), p} := \sup_{x \in \Omega} \sup_{\beta \in \mathbb{N}^d_0, |\beta| \leq k} p(\partial^\beta f(x)), \quad f \in C^k(\overline{\Omega}, E), \]

for \( p \in \Gamma_0 \). The space of functions in \( C^k(\overline{\Omega}, E) \) such that all its \( k \)-th partial
derivatives are \( \alpha \)-Hölder continuous with \( 0 < \alpha \leq 1 \) is given by

\[ \mathcal{C}^{k,\alpha}(\overline{\Omega}, E) := \{ f \in C^k(\overline{\Omega}, E) \mid \forall p \in \Gamma_0 : |f|_p < \infty \} \]

where

\[ |f|_p := |f|_{C^k(\overline{\Omega}), p} + \sup_{\beta \in \mathbb{N}^d_0, |\beta| \leq k} \| \partial^\beta f \|_{C^{\alpha,0}(\Omega), p} \]

with

\[ |f|_{C^{\alpha,0}(\Omega), p} := \sup_{x, y \in \Omega} \frac{p(f(x) - f(y))}{|x - y|^\alpha}. \]

Further, we set \( \mathcal{C}^k(\overline{\Omega}) := \mathcal{C}^k(\overline{\Omega}, \mathbb{K}) \), \( \mathcal{C}^{k,\alpha}(\overline{\Omega}) := \mathcal{C}^{k,\alpha}(\overline{\Omega}, \mathbb{K}) \) and omit the index \( p \) of
\( |\cdot|_{C^k(\overline{\Omega})}, |\cdot|_{C^{\alpha,0}(\Omega)} \) if \( E = \mathbb{K} \).
3.8. Theorem. Let $E$ be a locally complete locally convex Hausdorff space with fundamental system of seminorms $\Gamma_E$, $\Omega \subset \mathbb{R}^d$ a non-empty open bounded set, $\overline{\Omega}_{wd} := \{ (x, y) \in \mathbb{R}^d \mid x \neq y \}$, $k \in \mathbb{N}_0$ and $0 < \alpha \leq 1$. If $k \geq 1$, assume additionally that $\Omega$ has Lipschitz boundary. Then the following assertions hold.

(a) $(C^{k, \alpha}(\overline{\Omega}), \| \cdot \|, \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ is a complete semi-Montel Saks function space.

(b) The mixed topology $\gamma(\| \cdot \|, \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ is generated by the system of seminorms

$$
\| f \|_{(K_n, \alpha_n), \text{rel.} n} := \sup_{x \neq y} \sup_{K_n \beta \in N_0^d, |\beta|=k} \left( |f|_{C^{k, \alpha}(\overline{\Omega})} + \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} \right) a_n, \quad f \in C^{k, \alpha}(\overline{\Omega}),
$$

where $(K_n)_{n \in \mathbb{N}}$ is a sequence of compact subsets of $\overline{\Omega}_{wd}$ and $(\alpha_n)_{n \in \mathbb{N}} \in \epsilon_0^\circ$. Furthermore, for compact $\Omega$ is relatively compact in $\overline{\Omega}_{wd}$.

(c) Let $W_0 = W_{b, 0}(\overline{\Omega}_{wd})$ or $W_{usc, 0}(\overline{\Omega}_{wd})$ or $C^0(\overline{\Omega}_{wd})$. The mixed topology $\gamma(\| \cdot \|, \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ is also generated by the system of seminorms

$$
|f|_w := \sup_{x \neq y} \sup_{\beta \in N_0^d, |\beta|=k} \left( |f|_{C^{k, \alpha}(\overline{\Omega})} + \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} \right) w(x, y), \quad f \in C^{k, \alpha}(\overline{\Omega}),
$$

for $p \in \Gamma_E$ and $w \in W_0$.

(d) $C^{k, \alpha}(\overline{\Omega}), E) = C^{k, \alpha}(\overline{\Omega}), \Gamma_E)$ and $\| \cdot \|_{C^{k, \alpha}(\overline{\Omega}), E) \leq \| \cdot \|_p \leq 2 \| \cdot \|_{C^{k, \alpha}(\overline{\Omega}), E}$ for all $p \in \Gamma_E$.

Proof. (a) First, we note that $\delta_x \in C^{k, \alpha}(\overline{\Omega})$ for every $x \in \Omega$ since it holds that $\delta_x \in (C^{k, \alpha}(\overline{\Omega}), \| \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ and $\tau_{\infty} \subset \gamma(\| \cdot \|, \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ where $\tau_{\infty}$ denotes the topology induced by $\| \cdot \|_{C^{k, \alpha}(\overline{\Omega})}$. Due to Proposition 2.6 (c) we only need for the validity of statement (a) that $B_{l, k}$ is $\| \cdot \|_{C^{k, \alpha}(\overline{\Omega})}$-compact. The $\| \cdot \|_p$-closed unit ball $B_{l, k}$ is relatively compact in $(C^{k, \alpha}(\overline{\Omega}), \| \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ by [1], 8.6 Einbettungssatz in Hölder-Räumen, p. 338, and it is easily seen that is also closed in $(C^{k, \alpha}(\overline{\Omega}), \| \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$ by a pointwise argument. It follows that $B_{l, k}$ is $\| \cdot \|_{C^{k, \alpha}(\overline{\Omega})}$-compact in $C^{k, \alpha}(\overline{\Omega})$ by Proposition 3.4 with $(Y, \tau) = (C^{k, \alpha}(\overline{\Omega}), \| \cdot \|_{C^{k, \alpha}(\overline{\Omega})})$.

(b) First, we note that for every $\beta \in \mathbb{N}_0^d$ with $|\beta|=k$ and $f \in C^{k, \alpha}(\overline{\Omega})$ the partial derivative $\partial^\beta f$ has an $\alpha$-Hölder continuous extension to $\overline{\Omega}$, which we denote by the same symbol, and the extension has the same Hölder constant by [17], Proposition 1.6, p. 5 [17] and Proposition 2.50, p. 66, i.e.

$$
|\partial^\beta f|_{C^{k, \alpha}(\overline{\Omega})} = \sup_{x, y \in \Omega} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha}.
$$

Let $\tau_\alpha$ be the locally convex Hausdorff topology generated by the directed system of seminorms given by

$$
q_K(f) := \sup_{(x, y) \in K} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha}, \quad f \in C^{k, \alpha}(\overline{\Omega}),
$$

for compact $K \subset \overline{\Omega}_{wd}$. Then we have

$$
\| f \| = \sup_{K \subset \overline{\Omega}_{wd}} \sup_{K \text{ compact}} q_K(f), \quad f \in C^{k, \alpha}(\overline{\Omega}).
$$

Furthermore, for compact $K \subset \overline{\Omega}_{wd}$ the projections $\pi_1(K)$ and $\pi_2(K)$ on the first and second component, respectively, are compact in $\overline{\Omega}$ and

$$
q_K(f) \leq |f|_{C^{k, \alpha}(\overline{\Omega})} + 2 \max_{(x, y) \in K} \frac{1}{|x-y|^\alpha} \sup_{\beta \in N_0^d, |\beta|=k} |\partial^\beta f(x)|
$$
for all \( f \in C^{k,\alpha}(\Omega) \), which implies \( \tau_k \subset \tau_{C^k} \). On the other hand, fix some \( y_0 \in \Omega \). For every \( \varepsilon > 0 \) we have with \( B_{\varepsilon} := \{ x \in \Omega \mid |x - y_0|^\alpha < \varepsilon \} \) that

\[
|f|_{C^k(\Omega)} \leq \sup_{x \in \partial \Omega, x \not\in y_0} \frac{|\partial^\beta f(y_0)|}{|x - y_0|^\alpha} + \sup_{x \in \Omega, x \not\in y_0} \frac{|\partial^\beta f(x) - \partial^\beta f(y_0)|}{|x - y_0|^\alpha}
\]

\[
\leq |f|_{C^k(\Omega)} + \sup_{x \in \Omega, x \not\in y_0} \frac{|\partial^\beta f(x) - \partial^\beta f(y_0)|}{|x - y_0|^\alpha} + \varepsilon \frac{\sup_{x \in B_{\varepsilon}, x \not\in y_0} \frac{|\partial^\beta f(x) - \partial^\beta f(y_0)|}{|x - y_0|^\alpha}}{\varepsilon}
\]

\[
\leq 1 + \max_{x \in \Omega} |x - y_0|^\alpha \frac{\sup_{x \in B_{\varepsilon}, x \not\in y_0} \frac{|\partial^\beta f(x) - \partial^\beta f(y_0)|}{|x - y_0|^\alpha}}{\varepsilon}
\]

for all \( f \in C^{k,\alpha}(\Omega) \). Thus the topologies \( \tau_k \) and \( \tau_{C^k} \) coincide on \( \| \cdot \| \)-bounded sets. It follows that \( \gamma(\| \cdot \|, \tau_k) = \gamma(\| \cdot \|, \| \cdot \|_{C^k(\Omega)}) \) by [11, I.1.6 Corollary (ii), p. 6]. Due to condition (ii) of Remark 2.3 (b) this yields that the mixed topology \( \gamma(\| \cdot \|, \| \cdot \|_{C^k(\Omega)}) \) is generated by the seminorms

\[
\|f\|_{(K_n, \gamma_n, \rho_n)} = \sup_{n \in \mathbb{N}} \sup_{(x,y) \in K_n} \left( |f|_{C^k(\Omega)} + \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x - y|^\alpha} \right) \gamma_n, \quad f \in C^{k,\alpha}(\Omega),
\]

where \( (K_n)_{n \in \mathbb{N}} \) is a sequence of compact subsets of \( \overline{\Omega}_{med} \) and \( (\gamma_n)_{n \in \mathbb{N}} \in c^0_\gamma \).

(c) Observing that \( \overline{\Omega}_{med} \) is locally compact, the proof of statement (c) is analogous to the one of Theorem 3.8 (c).

(d) Since \( E \) is locally complete, we have \( C^{k,\alpha}(\Omega, E) = C^{k,\alpha}(\Omega, E) \) by [29, Corollary 5.3, p. 16]. Further, we have \( \| \cdot \|_{\sigma, p} \leq \| \cdot \|_p \leq 2 \| \cdot \|_{\sigma, p} \) for all \( p \in \Gamma_E \) by [31, Proposition 24.10, p. 282] and because of \( \partial^\beta (e^\prime \circ f) = e^\prime \circ (\partial^\beta f) \) for all \( e^\prime \in E^\prime \), \( f \in C^{k,\alpha}(\Omega, E) \) and \( \beta \in \mathbb{N}^d_0 \) with \( |\beta| \leq k \).

If \( k = 0 \), then Theorem 3.8 still holds if \( \Omega \) is replaced by any compact subset of \( \mathbb{R}^d \) by [1], 8.6 Einbettungssatz in H"older-R"aumen, p. 338].

4. Linearisation

This section is dedicated to our main results on linearisation of weak vector-valued functions.

4.1. Proposition. Let \( (\mathcal{F}(\Omega), \| \cdot, \|) \) be a semi-reflexive pre-Saks function space and \( E \) a locally convex Hausdorff space with fundamental system of seminorms \( \Gamma_E \).

(a) Let \( \delta \cdot \Omega \rightarrow \mathcal{F}(\Omega)' \), \( \delta(x) := \delta_x \). Then \( \delta \in \mathcal{F}(\Omega, \mathcal{F}(\Omega)')_\sigma \). 

(b) The map

\[
\chi : L_b(\mathcal{F}(\Omega)' \cdot, E) \rightarrow \mathcal{F}(\Omega, E)_\sigma, \quad \chi(T) := T \circ \delta,
\]
is a topological isomorphism into, i.e. a topological isomorphism to its range, and for all \( p \in \Gamma_E \)
\[
\| \chi(T) \|_{\sigma,p} = \sup_{f' \in \mathcal{F} \Omega \gamma} \sup_{\| f' \|_{\mathcal{F} \Omega \gamma} \leq 1} p(T(f')) = T \in L(\mathcal{F} \Omega \gamma, E).
\]

**Proof.** (a) Since \((\mathcal{F} \Omega \gamma, \| \cdot \|, \tau)\) is semi-reflexive, we have \((\mathcal{F} \Omega \gamma)' = \mathcal{F} \Omega\) (as linear spaces). Hence it follows for all \( f \in (\mathcal{F} \Omega \gamma)' = \mathcal{F} \Omega\) that \((f \circ \delta)(x) = f(x)\) for all \( x \in \Omega\) and thus \( f \circ \delta = f \in \mathcal{F} \Omega\), implying \( \delta \in \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\).

(b) Due to the semi-reflexivity of \((\mathcal{F} \Omega \gamma, \| \cdot \|, \tau)\) we have \( \gamma \circ T \in (\mathcal{F} \Omega \gamma)' = \mathcal{F} \Omega\) for all \( \gamma' \in E'\) and \( T \in L(\mathcal{F} \Omega \gamma, E)\). Therefore \( \gamma' \circ \chi(T) = (\gamma' \circ T) \circ \delta = (\gamma' \circ T) \in (\mathcal{F} \Omega \gamma)' = \mathcal{F} \Omega\) for all \( \gamma' \in E'\) and \( T \in L(\mathcal{F} \Omega \gamma, E)\), which yields that \( \chi(T) \in \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\). Thus \( \chi \) is well-defined, and obviously linear as well. Let \( p \in \Gamma_E \). Then we have
\[
\| \chi(T) \|_{\sigma,p} = \sup_{e \in \mathcal{U}_p} \| e' \circ T \circ \delta \| = \sup_{e \in \mathcal{U}_p} \sup_{\gamma' \in \mathcal{F} \Omega \gamma} \| (f', \gamma' \circ T)(f') \| = \sup_{e \in \mathcal{U}_p} \sup_{\gamma' \in \mathcal{F} \Omega \gamma} p(T(f'))
\]
for all \( T \in L(\mathcal{F} \Omega \gamma, E)\) where we used that \( \mathcal{F} \Omega \gamma \) is norming by [32, Lemma 5.5 (a), p. 2680–2681] for the second equation, and [34, Proposition 22.14, p. 256] for the last equation. From this we deduce that \( \chi \) is a topological isomorphism into. \( \square \)

**4.2. Remark.** If \( E \) is metrisable, i.e. we may choose \( \Gamma_E \) countable, then \( \chi \) is an isometry of the induced metrics on \( L_0(\mathcal{F} \Omega \gamma, E) \) and \( \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\).

Next, we prove the surjectivity of the map \( \chi \) if \( E \) is complete. The proof is quite similar to the one given in [23, Theorem 14 (i), p. 1524]. We identify \( E \) with a linear subspace of the algebraic dual \( E'^* \) of \( E' \) by the canonical injection \( x \mapsto [e' \mapsto e'(x)] =: (x, e') \).

**4.3. Theorem.** Let \((\mathcal{F} \Omega, \| \cdot \|, \tau)\) be a semi-reflexive pre-Saks function space and \( E \) a complete locally convex Hausdorff space. Then the map
\[
\chi: L_0(\mathcal{F} \Omega \gamma, E) \to \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\gamma, \chi(T) := T \circ \delta,
\]
is a topological isomorphism and its inverse fulfills
\[
(\chi^{-1}(f))(f'), \gamma' \circ f' = f(\gamma' \circ f), \quad f \in \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\gamma, f' \in \mathcal{F} \Omega \gamma, e' \in E'.
\]
In particular, \( \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\gamma \) is complete.

**Proof.** Due to Proposition [4.3] we only need to show that \( \chi \) is surjective. Fix \( f \in \mathcal{F} \Omega, (\mathcal{F} \Omega \gamma)'\gamma \). The map \( \Psi(f)(f'): E' \to K, (\Psi(f)(f'), e') := f'(\gamma' \circ f) \), is clearly linear, hence \( \Psi(f)(f') \in E'^* \), where \( K = \mathbb{R} \) or \( \mathbb{C} \) is the scalar field of \( E \). We set for \( p \in \Gamma_E \)
\[
\| y \|_{\mathcal{U}_p} = \sup_{e \in \mathcal{U}_p} \| y(e') \| \leq \infty, \quad y \in E'^*,
\]
and note that \( p(x) = \| x \|_{\mathcal{U}_p} \) for every \( x \in E \). We observe that
\[
\| \Psi(f)(f') \|_{\mathcal{U}_p} = \sup_{e \in \mathcal{U}_p} \| f'(\gamma' \circ f) \| = \| f' \|_{\mathcal{F} \Omega \gamma} \sup_{e \in \mathcal{U}_p} \| e' \circ f \| = \| f' \|_{\mathcal{F} \Omega \gamma} \| f \|_{\sigma, p}.
\] (3)

Next, we show that \( \Psi(f)(f') \in E \). First, we remark that
\[
(\Psi(f)(\delta_x), \gamma') = f'(\gamma(x)) = f(x), \quad x \in \Omega,
\]
for every $c' \in E'$, yielding $\Psi(f)(\delta_x) \in E$ for every $x \in \Omega$. By the Hahn–Banach theorem the span of $\{\delta_x \mid x \in \Omega\}$ is $\|\cdot\|_{\mathcal{F}(\Omega)}$-dense in $\mathcal{F}(\Omega)'$. Thus there is a net $(f'_i)_{i \in I}$ in this span such that it $\|\cdot\|_{\mathcal{F}(\Omega)}$-converges to $f'$ and $\Psi(f)(f'_i) \in E$ for each $i \in I$ as well as

$$|\Psi(f)(f'_i) - \Psi(f)(f')|_{\sigma,p} \leq |f'_i - f'|_{\mathcal{F}(\Omega)} \|f\|_{\sigma,p} \to 0$$

for all $p \in \Gamma_E$. Hence $(\Psi(f)(f'_i))_{i \in I}$ is a Cauchy net in the complete space $E$ with a limit $g \in E$. From

$$|g - \Psi(f)(f')|_{\sigma,p} \leq |g - \Psi(f)(f'_i)|_{\sigma,p} + |\Psi(f)(f'_i) - \Psi(f)(f')|_{\sigma,p}$$

for all $p \in \Gamma_E$ we deduce that $\Psi(f)(f') = g \in E$. In combination with (3) we derive that $\Psi(f) \in L(\mathcal{F}(\Omega)'_\gamma, E)$. Finally, we note that

$$\chi(\Psi(f))(x) = \Psi(f)(\delta_x) = f(x), \quad x \in \Omega,$$

implying the surjectivity of $\chi$. 

Special cases of Theorem 4.3 for Banach spaces $E$ were already obtained before. In [36] Theorem 2.1, p. 869 for $\mathcal{F}(\Omega) = H^\infty(\Omega)$ where $\Omega$ is an open subset of a Banach space, and in [5, 3.7 Proposition, p. 292] combined with [5, 1.5 Theorem (e), p. 277–278] for $\mathcal{F}(\Omega) = H_{\nu}(\Omega)$ such that the polynomials are dense in $H_{\nu}(\Omega) := \{f \in H_{\nu}(\Omega) \mid f v \in C^d(\Omega)\}$ where $\Omega$ is a balanced open subset of $C^d$ and $v : \Omega \to (0, \infty)$ a radial continuous weight (more general, families of weights $V$ and quasi-complete locally convex Hausdorff spaces $E$ are allowed in [3, 3.7 Proposition, p. 292] if $H_{\nu}(\Omega)$ is bornological). Furthermore, in [4] Lemma 10, p. 243] for $\mathcal{F}(\mathbb{D}) \subset H(\mathbb{D})$ that its closed unit ball is compact in the compact-open topology $\tau_{co}$, in [33] Lemma 5.2, p. 14] for $\mathcal{F}(\mathbb{D}) \subset h(\mathbb{D}) := C^\infty(\mathbb{D})$ that its closed unit ball is compact in the compact-open topology $\tau_{co}$, and in [22] Proposition 6, p. 3] for $\mathcal{F}(\Omega) = A_{\nu}(\Omega) \subset H_{\nu}(\Omega)$ such that its closed unit ball is compact in the compact-open topology $\tau_{co}$ where $\Omega$ is an open connected subset of a Banach space and $v : \Omega \to (0, \infty)$ a continuous weight.

Let us consider another linearisation method for $\mathcal{F}(\Omega, E)_\sigma$, namely, $\varepsilon$-products. Let $(\mathcal{F}(\Omega), \|\cdot\|, \tau)$ a Saks function space such that $\gamma = \gamma_\sigma$. Then, by Definition 2.2 a system of seminorms that generates $\gamma$ is given by

$$\|f\|_{(q_n, a_n)_{n \in \mathbb{N}}} := \sup_{n \in \mathbb{N}} q_n(f) a_n, \quad f \in \mathcal{F}(\Omega),$$

for $(q_n)_{n \in \mathbb{N}} \subset \Gamma_\tau$, where $\Gamma_\tau$ is a directed system of continuous seminorms that generates the topology $\tau$ and fulfills (1), and $(a_n)_{n \in \mathbb{N}} \in c_0^\ast$. For a locally convex Hausdorff space $E$ with fundamental system of seminorms $\Gamma_E$ we set

$$\|f\|_{\sigma, (q_n, a_n)_{n \in \mathbb{N}}, p} := \sup_{c' \in U_p^\ast} \sup_{n \in \mathbb{N}} q_n(c' \cdot f) a_n, \quad f \in \mathcal{F}(\Omega, E)_\sigma,$$

for $p \in \Gamma_E$, $(q_n)_{n \in \mathbb{N}} \subset \Gamma_\tau$, and $(a_n)_{n \in \mathbb{N}} \in c_0^\ast$. If $\mathcal{F}(\Omega, \|\cdot\|)$ is additionally a Banach space, then for $p \in \Gamma_E$, $(q_n)_{n \in \mathbb{N}} \subset \Gamma_\tau$ and $(a_n)_{n \in \mathbb{N}} \in c_0^\ast$ it holds

$$\|f\|_{\sigma, (q_n, a_n)_{n \in \mathbb{N}}, p} \leq \|f\|_{\sigma, p} < \infty$$

for all $f \in \mathcal{F}(\Omega, E)_\sigma$. So the system of seminorms $(\|f\|_{\sigma, (q_n, a_n)_{n \in \mathbb{N}}, p})_{(q_n, a_n)_{n \in \mathbb{N}}, p \in \Gamma_E}$ induces a locally convex Hausdorff topology on $\mathcal{F}(\Omega, E)_\sigma$ which we denote by $\gamma_{\mathcal{F}}$.

4.4 Remark. Let $E$ be a locally convex Hausdorff space with fundamental system of seminorms $\Gamma_E$. For the spaces $\mathcal{F}(\Omega)$ from Corollary 3.5 Theorem 3.6 Theorem
and Theorem 3.5 another system of seminorms that induces $\gamma_E$ on $F(\Omega, E)_\sigma$ is given by
\[
|f|_{w,p} := \sup_{x \in \Omega} p(f(x))v(x)w(x), \quad f \in F(\Omega, E)_\sigma,
\]
in the case $F = \ell v$, $C_p(\Omega)v$ or $Hv$, and
\[
|f|_{w,p} := \sup_{x \in \Omega} (p(f(0)) + p(f'(z))v(z))w(z), \quad f \in Bv(D, E)_\sigma,
\]
if $E$ is locally complete, and
\[
|f|_{w,p} := \sup_{x,y \in \Omega} \frac{p(f(x) - f(y))}{d(x,y)}w(x,y), \quad f \in Lip_0(\Omega, E)_\sigma,
\]
as well as
\[
|f|_{w,p} := \sup_{x \in \Omega} \sup_{y \in \Omega} \left( |f|_{C^1(\Omega),p} + p(\frac{\partial^2 f(x) - \partial^2 f(y)}{|x-y|^\gamma}) \right)w(x,y), \quad f \in C^{k,\gamma}(\Omega, E)_\sigma,
\]
if $E$ is locally complete, for $p \in \Gamma_E$ and $w \in \mathcal{W}_0$, respectively. This follows in combination with [34, Proposition 24.10, p. 282] from part (c) of Corollary 3.5, Theorem 3.8, and part (d) of Theorem 3.6 and Theorem 3.7.

4.5. Theorem. Let $(F(\Omega), \| \cdot \|, \tau)$ be a semi-Montel pre-Saks function space and $E$ a complete locally convex Hausdorff space. Then the map
\[
\chi : F(\Omega)_\gamma \times \mathcal{E} \rightarrow (F(\Omega, E)_\sigma, \gamma_E), \quad \chi(T) := T \circ \delta,
\]
is a topological isomorphism. In particular, $(F(\Omega, E)_\sigma, \gamma_E)$ is complete.

Proof. Since $(F(\Omega), \| \cdot \|, \tau)$ is semi-Montel, it is a complete semi-reflexive Saks space with $\gamma = \gamma_\sigma$ and $(F(\Omega), \| \cdot \|)$ a Banach space by Remark 2.3, Remark 2.5, and Proposition 2.6 (b). Furthermore, we have $(F(\Omega))_\gamma = (F(\Omega))_\sigma = (F(\Omega))_\gamma$ because $F(\Omega, \gamma_\sigma)$ is a semi-Montel space and by Proposition 2.6 (a). This yields
\[
F(\Omega)_\gamma \times \mathcal{E} = L_c((F(\Omega))_\gamma), E) = L_{\nu}(F(\Omega, E)_\sigma, E),
\]
implying that $\chi$ is a linear isomorphism by Theorem 4.3.

Next, we show that $\chi$ and its inverse are continuous. Let $T$ be a directed system of continuous seminorms that generates the topology $\tau$ and fulfils (11). We note that for $(q_n)_{n \in \mathbb{N}} \subset \Gamma_T$ and $(a_n)_{n \in \mathbb{N}} \subset e_0^\ast$ the sets
\[
V_{(q_n, a_n)} := \bigcap_{n \in \mathbb{N}} \left\{ a_n f \mid f \in U_{q_n} \right\}
\]
form a base of $\gamma$-neighbourhoods of zero by Definition 2.2 since $\gamma = \gamma_\sigma$, where $U_{q_n} := \{ f \in F(\Omega) \mid q_n(f) < 1 \}$. By the bipolar theorem we have
\[
V_{(q_n, a_n)}^\circ = \overline{\text{acx}}(\bigcup_{n \in \mathbb{N}} \left\{ a_n f' \mid f' \in U_{q_n}^\circ \right\}) = \overline{\text{acx}}(W_{(q_n, a_n)}),
\]
where $\overline{\text{acx}}(W_{(q_n, a_n)})$ denotes the $\kappa(F(\Omega)_\gamma, F(\Omega)_\sigma)$-closure of the absolutely convex hull $\text{acx}(W_{(q_n, a_n)})$ of $W_{(q_n, a_n)}$ (see [20, 8.4.4, p. 152, 8.5, p. 156–157] the topology of $F(\Omega)_\gamma \times \mathcal{E}$ is generated by the seminorms
\[
|T|_{(q_n, a_n), \kappa, p} := \sup_{\overline{\text{acx}}(W_{(q_n, a_n)})} p(T(y)) \sup_{\overline{\text{acx}}(W_{(q_n, a_n)})} p(T(y)), \quad T \in F(\Omega)_\gamma \times \mathcal{E},
\]
for $(q_n)_{n \in \mathbb{N}} \subset \Gamma_T$, $(a_n)_{n \in \mathbb{N}} \subset e_0^\ast$ and $p \in \Gamma_E$. By the continuity of $T \in F(\Omega)_\gamma \times \mathcal{E}$ we have
\[
|T|_{(q_n, a_n), \kappa, p} = \sup_{\overline{\text{acx}}(W_{(q_n, a_n)})} p(T(y)) \sup_{\overline{\text{acx}}(W_{(q_n, a_n)})} p(T(y)).
\]
On the other hand, for \( y \in \text{acx}(W(q_n,a_n)_{n \in \mathbb{N}}) \) there are \( m \in \mathbb{N}, \lambda_k \in \mathbb{K}, f'_k \in U_{q_n}^* \), \( 1 \leq k \leq m \) with \( \sum_{k=1}^{m} |\lambda_k| = 1 \) such that \( y = \sum_{k=1}^{m} \lambda_k a_k f'_k \). It follows that for all \( T \in \mathcal{F}(\Omega), \varepsilon \in E \)
\[
p(T(y)) \leq \sum_{k=1}^{m} |\lambda_k| p(T(a_k f'_k)) \leq \sup_{1 \leq k \leq m} p(T(a_k f'_k)) \leq \sup_{z \in W(q_n,a_n)_{n \in \mathbb{N}}} p(T(z))
\]
and we deduce
\[
|T|_{(q_n,a_n)_{n \in \mathbb{N}},p} = \sup_{y \in W(q_n,a_n)_{n \in \mathbb{N}}} p(T(y)) = \sup_{a_n} p(T(f') a_n).
\]
Using that \( (e' \circ T) \circ \delta = (e' \circ T) \in (\mathcal{F}(\Omega)')' = \mathcal{F}(\Omega) \) (see the proof of Proposition 4.4), we conclude that
\[
\|\chi(T)\|_{\sigma,(q_n,a_n)_{n \in \mathbb{N}},p} = \sup_{q_n}(e' \circ T) a_n = \sup_{q_n} |(f', e' \circ T)| a_n
\]
\[
= \sup_{q_n} |(e' \circ T, f')| a_n = \sup_{q_n} p(T(f') a_n)
\]
\[
= |T|_{(q_n,a_n)_{n \in \mathbb{N}},p}
\]
for all \( T \in \mathcal{F}(\Omega), \varepsilon \in E \). Hence \( \chi \) and its inverse are continuous.

Since \( \mathcal{F}(\Omega), E \) is also complete by \( 23 \) Satz 10.3, p. 234\], implying the completeness of \( (\mathcal{F}(\Omega), E)_{\sigma, \gamma_E} \).

The proof of the continuity of \( \chi \) and its inverse in Theorem 4.5 is similar to the proof of \( 23 \) Lemma 7, p. 151\]. Moreover, Theorem 4.5 in combination with Theorem 3.7 and Remark 4.4 generalises \( 21 \) Theorem 4.4, p. 648\] where \( \mathcal{F}(\Omega, E) = \text{Lip}_0(\Omega, E) \) and \( E \) a Banach space. Due to Remark 4.4 for the spaces in Corollary 3.7 the result of Theorem 4.5 is already contained in [23, 3.1 Bemerkung, p. 141\] (cf. [27, 3.14 Proposition, p. 12\] and [36, 4.8 Theorem, p. 878\]) even for quasi-complete locally convex Hausdorff spaces \( E \). However, the proof is different. Theorem 4.5 allows us to characterise \( (\mathcal{F}(\Omega), \gamma) \) having the approximation property by approximation in \( (\mathcal{F}(\Omega), E)_{\sigma, \gamma_E} \).

4.6. Corollary. Let \( (\mathcal{F}(\Omega), \|\cdot\|, \tau) \) be a semi-Montel pre-Saks function space. Then the following assertions are equivalent:

(a) \( (\mathcal{F}(\Omega), \gamma) \) has the approximation property.

(b) \( \mathcal{F}(\Omega)'_{\sigma, \gamma_E} \) has the approximation property.

(c) \( \mathcal{F}(\Omega) \otimes E \) is dense in \( (\mathcal{F}(\Omega), E)_{\sigma, \gamma_E} \) for every Banach space \( E \).

(d) \( \mathcal{F}(\Omega) \otimes E \) is dense in \( (\mathcal{F}(\Omega), E)_{\sigma, \gamma_E} \) for every complete locally convex Hausdorff space \( E \).

Proof. The equivalence (a) \( \iff \) (b) follows from Proposition 2.7. The remaining equivalences are a consequence of Theorem 4.5 and \( 23 \) Satz 10.17, p. 250\].

[23] Theorem 4.6, p. 651–652\] is a special case of Corollary 4.6 for \( \mathcal{F}(\Omega) = \text{Lip}_0(\Omega) \). For \( \mathcal{F}(\Omega) = \mathcal{H}^\infty(\Omega) \) Corollary 4.6 is contained in [36, 5.4 Theorem, p. 883\] where \( \Omega \) is a balanced bounded open subset of a Banach space. Further, it is known that \( (\mathcal{H}^\infty(\Omega), \gamma) \) and \( (\mathcal{H}^\infty(\Omega)'_{\gamma, \|\cdot\|_{\mathcal{H}^\infty(\Omega)'}}) \) with \( \gamma = \gamma_{\|\cdot\|, \tau_{\text{co}}} \) have the approximation property by [3, Satz 3.9, p. 145\] for simply connected open \( \Omega \subset \mathbb{C} \) (cf. [11, V.2.4 Proposition, p. 233\] for \( \Omega = \mathbb{D} \)). The same is true for \( (\ell^1(\Omega), \gamma) \) and \( (\ell^1(\Omega)'_{\gamma, \|\cdot\|_{\ell^1(\Omega)'}}) \) for \( v: \Omega \to (0, \infty) \) and discrete \( \Omega \) by [2], 5.5 Theorem (3), (4), p. 205\].
5. Applications

Our first application of the linearisation results from the preceding section concerns (weakly) compact operators. Let \((X, \cdot \| \cdot \|)\) and \((Y, \cdot \| \cdot \|)\) be Banach spaces. We recall that a linear map \(T: X \to Y\) is called \emph{compact} if \(T(B_{\| \cdot \|})\) is relatively \(\| \cdot \|\)-compact. A linear map \(T: X \to Y\) is called \emph{weakly compact} if \(T(B_{\| \cdot \|})\) is relatively \(\sigma(Y, Y')\)-compact (see e.g. [8, p. 235]).

Let \((\mathcal{F}(\Omega), \| \cdot \|, \tau)\) be a pre-Saks function space such that \((\mathcal{F}(\Omega), \| \cdot \|)\) is complete and \(\varphi: \Omega \to \Omega\) such that \(f \circ \varphi \in \mathcal{F}(\Omega)\) for all \(f \in \mathcal{F}(\Omega)\). Then the weighted composition operators \(C_{\varphi}: \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)\) and \(C_{\varphi}: \mathcal{F}(\Omega, E)_{\sigma} \to \mathcal{F}(\Omega, E)_{\sigma}\) given by \(C_{\varphi}(f) := f \circ \varphi\) are well-defined linear maps since \(e' \circ f \in \mathcal{F}(\Omega)\) for every \(e' \in E'\) and \(f \in \mathcal{F}(\Omega, E)_{\sigma}\). The following theorem generalises [6, Proposition 11, p. 244] and [33, Theorem 5.3, p. 14]. Its proof is based on the proof of [6, Proposition 11, p. 244].

5.1. Theorem. Let \((\mathcal{F}(\Omega), \| \cdot \|, \tau)\) be a semi-reflexive pre-Saks function space, \(\varphi: \Omega \to \Omega\) such that \(f \circ \varphi \in \mathcal{F}(\Omega)\) for all \(f \in \mathcal{F}(\Omega)\) and \((E, \| \cdot \|)\) a Banach space.

(a) If \(C_{\varphi} \in L(\mathcal{F}(\Omega))\), then \(C_{\varphi} \in L(\mathcal{F}(\Omega, E)_{\sigma})\).

(b) If \(C_{\varphi}: \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)\) is compact and \(E\) reflexive, then \(C_{\varphi}: \mathcal{F}(\Omega, E)_{\sigma} \to \mathcal{F}(\Omega, E)_{\sigma}\) is weakly compact.

Proof. To distinguish the two composition operators we denote the one on \(\mathcal{F}(\Omega, E)_{\sigma}\) by \(C_{\varphi}^{E}\). Further, we set \(U_{E} = \{x \in E \mid \|x\|_{E} < 1\}\).

(a) We note that

\[
|C_{\varphi}^{E}(f)_{\sigma, \| \cdot \|} - C_{\varphi}^{E}(f)_{\sigma, \| \cdot \|}| = \left|\sup_{e' \in U_{E}} \langle e' \circ (f \circ \varphi) \rangle \right| \leq \|f \circ \varphi\| \leq \|C_{\varphi}(f)\| \leq \|C_{\varphi}^{E}(f)_{\sigma, \| \cdot \|}\|
\]

for all \(f \in \mathcal{F}(\Omega)\), which implies the continuity of \(C_{\varphi}^{E}\).

(b) First, we observe that the dual map \(C_{\varphi}': \mathcal{F}(\Omega)' \to \mathcal{F}(\Omega)'\) leaves the \(\| \cdot \|_{\mathcal{F}(\Omega)'}\)-closed linear subspace \(\mathcal{F}(\Omega)'_{\sigma}\) invariant (see Proposition 2.7 (a)). Indeed, we have \(C_{\varphi}''(\delta_{x}) = \delta_{x}(x)\) for all \(x \in \Omega\). Since the span of \(\{\delta_{x} \mid x \in \Omega\}\) is \(\| \cdot \|_{\mathcal{F}(\Omega)'}\)-dense in \(\mathcal{F}(\Omega)'\), by the proof of Theorem 4.3, we obtain that \(C_{\varphi}''\) leaves \(\mathcal{F}(\Omega)'_{\sigma}\) invariant.

Second, we claim that

\[
C_{\varphi}^{E} = \chi \circ W_{\varphi} \circ \chi^{-1}
\]

where \(\chi\) is the isomorphism from Theorem 4.3 and

\[
W_{\varphi}: L_{0}(\mathcal{F}(\Omega)'_{\sigma}, E) \to L_{0}(\mathcal{F}(\Omega)'_{\sigma}, E), \ T \mapsto \text{id}_{E} \circ T \circ (C_{\varphi})_{\mathcal{F}(\Omega)_{\sigma}},
\]

with the identity map \(\text{id}_{E}\) on \(E\). Indeed, we have by Theorem 4.3 that

\[
(\chi \circ W_{\varphi} \circ \chi^{-1})(f)(x) = \chi(\chi^{-1}(f) \circ (C_{\varphi})_{\mathcal{F}(\Omega)'_{\sigma}})(x) = (\chi^{-1}(f) \circ (C_{\varphi})_{\mathcal{F}(\Omega)'_{\sigma}})(\delta_{x}) = \chi^{-1}(f)(\delta_{x}(x)) = f(\varphi(x)) = C_{\varphi}^{E}(f)(x)
\]

for all \(f \in \mathcal{F}(\Omega, E)_{\sigma}\) and \(x \in \Omega\). Since \(\text{id}_{E}\) is weakly compact by [31, Proposition 23.25, p. 272] due to the reflexivity of \(E\) and \((C_{\varphi})_{\mathcal{F}(\Omega)_{\sigma}}\) is compact as it is the restriction to an invariant closed subspace of \(\mathcal{F}(\Omega)'\) of the compact operator \(C_{\varphi}\) by [31, Schauder’s theorem 15.3, p. 141], we get that \(W_{\varphi}\) is weakly compact by [11, Theorem 2.9, p. 100]. We conclude that \(C_{\varphi}^{E} = \chi \circ W_{\varphi} \circ \chi^{-1}\) is weakly compact.

Let \(\varphi: D \to D\) be holomorphic and \(B_{a} = B_{v_{a}}(D)\) with \(v_{a}(z) = (1 - |z|^{2})^{a}\) for all \(z \in D\) and some \(a > 0\). Necessary and sufficient conditions such that \(C_{\varphi} \in L(B_{a})\) resp. \(C_{\varphi}\) is compact are given in [39, Theorem 2.1, p. 193] resp. [39, Theorem 3.1, p. 198–199].

Next, we generalise the extension result [22, Theorem 10, p. 5]. For this purpose we need to recall some definitions. Let \(E\) be a locally convex Hausdorff space. A
linear subspace $G \subset E'$ is said to determine boundedness of $E$ if every $\sigma(E,G)$-bounded set $B \subset E$ is already bounded in $E$ (see [11, p. 231]). In particular, such a $G$ is $\sigma(E',E)$-dense in $E'$. Further, by Mackey’s theorem $G = E'$ determines boundedness. Another example is the following one.

5.2. Remark. Let $(E, \| \cdot \|_E)$ be a pre-Saks space. Then $E' \subset E'$ determines boundedness of $(E, \| \cdot \|_E)$. Indeed, let $B \subset E$ be $\sigma(E,E')$-bounded. Then $B$ is $\gamma$-bounded by the Mackey’s theorem. It follows that $B$ is $\| \cdot \|_E$-bounded by [10, 1.1.11 Proposition, p. 10].

Moreover, let $(\mathcal{F}(E), \| \cdot \|, \tau)$ be a pre-Saks function space. A set $U \subset \Omega$ is called a set of uniqueness for $\mathcal{F}(E)$ if for each $f \in \mathcal{F}(E)$ the validity of $f(x) = 0$ for all $x \in U$ implies $f = 0$ on $\Omega$ (see [22, p. 3]). We note that $U \subset \Omega$ is a set of uniqueness for $\mathcal{F}(E)$ if and only if the span of $\{\delta_x \mid x \in U\}$ is $\sigma(\mathcal{F}(E)'_\gamma, \mathcal{F}(E))$-dense. For instance, set $H = H_v(\mathbb{D})$ with $v(z) = 1$ for all $z \in \mathbb{D}$. Then a sequence $U = \{z_n\}_{n \in \mathbb{N}} \subset \mathbb{D}$ of distinct elements is a set of uniqueness for $H$ if and only if it satisfies the Blaschke condition $\sum_{n \in \mathbb{N}} (1-|z_n|)^2 = \infty$ (see e.g. [40, 15.23 Theorem, p. 303]). Further examples of sets of uniqueness for the spaces $C^p(\Omega) v(\Omega)$ are given in [28, p. 12–13]. Now, we only need to adapt the proof of [24, Theorem 10, p. 5] a bit to get the following theorem.

5.3. Theorem. Let $(\mathcal{F}(E), \| \cdot \|, \tau)$ be a semi-reflexive pre-Saks function space, $U \subset \Omega$ a set of uniqueness for $\mathcal{F}(E)$, $E$ a complete locally convex Hausdorff space and $E' \subset E'$ determine boundedness. If $f: U \to E$ is a function such that $f' \in \mathcal{F}(E,E'_\gamma)$ for each $e' \in E$, then there exists a unique extension $F \in \mathcal{F}(E,E'_\gamma)$ of $f$. 

Proof. Let $X_U$ denote the span of $\{\delta_x \mid x \in U\}$. Since $U$ is a set of uniqueness for $\mathcal{F}(E)$, $X_U$ is $\sigma(\mathcal{F}(E)'_\gamma, \mathcal{F}(E))$-dense and thus also $\| \cdot \|_{\mathcal{F}(E)'_\gamma}$-dense by [24, 8.2.5 Proposition, p. 149] as $(\mathcal{F}(E)'_\gamma, \sigma(\mathcal{F}(E)'_\gamma, \mathcal{F}(E)))' = \mathcal{F}(E) = (\mathcal{F}(E)'_\gamma, \| \cdot \|_{\mathcal{F}(E)'_\gamma})'$. We note that the linear map $T: X_U \to E$ determined by $T(\delta_x) = f(x)$ for $x \in U$ is well-defined because $G$ is $\sigma(E',E)$-dense. Let $x' \in B_{\| \cdot \|_{\mathcal{F}(E)'_\gamma}}$ and $e' \in G$. Then there are $k \in \mathbb{N}$, $\alpha_i \in \mathbb{K}$ and $x_i \in U$, $1 \leq i \leq k$, such that $x' = \sum_{i=1}^k \alpha_i \delta_{x_i}$, and

$$|e'(T(x'))| = \sum_{i=1}^k |\alpha_i e'(f(x_i))| = \sum_{i=1}^k |\alpha_i f'(x_i)| = \left| \sum_{i=1}^k \alpha_i f'(x_i) \right| \leq \|f'\|.$$ 

Since this holds for any $x' \in B_{\| \cdot \|_{\mathcal{F}(E)'_\gamma}}$ and $e' \in G$, we deduce that $T(B_{\| \cdot \|_{\mathcal{F}(E)'_\gamma}})$ is $\sigma(E,G)$-bounded and hence bounded in $E$ because $G$ determines boundedness. We conclude that $T: (X_U, \| \cdot \|_{\mathcal{F}(E)'_\gamma}) \to E$ is continuous by [34, Proposition 24.10, p. 282] as the normed space $(X_U, \| \cdot \|_{\mathcal{F}(E)'_\gamma})$ is bornological by [34, Proposition 24.13, p. 283]. Since $X_U$ is $\| \cdot \|_{\mathcal{F}(E)'_\gamma}$-dense in $\mathcal{F}(E)'_\gamma$, there is a unique continuous linear extension $\bar{T}: \mathcal{F}(E)'_\gamma \to E$ of $T$ by [24, 3.4.2 Theorem, p. 61–62]. Setting $F := \chi \circ \bar{T} \in \mathcal{F}(E,E'_\gamma)$, we observe that $F(x) = (\chi \circ \bar{T})(x) = \bar{T}(\delta_x) = T(\delta_x) = f(x)$ for all $x \in U$ by Theorem [11, 1.1.3] which proves the existence of the extension of $f$. The uniqueness of the extension follows from $U$ being a set of uniqueness for $\mathcal{F}(E)$ and $G$ being $\sigma(E',E)$-dense.

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