Abstract. In this paper, we prove the local well-posedness of the Ericksen-Leslie system, and the global well-posedness for small initial data under the physical constrain condition on the Leslie coefficients, which ensures that the energy of the system is dissipated. Instead of the Ginzburg-Landau approximation, we construct an approximate system with the dissipated energy based on a new formulation of the system.

1. Introduction

The hydrodynamic theory of liquid crystals was established by Ericksen [4, 5] and Leslie [9] in the 1960’s. This theory treats the liquid crystal material as a continuum and completely ignores molecular details. Moreover, this theory considers perturbations to a presumed oriented sample. The configuration of the liquid crystals is described by a director field \( n(t, x) \in S^2, x \in \mathbb{R}^3 \).

The general Ericksen-Leslie system takes the form

\[
\begin{cases}
  v_t + v \cdot \nabla v = -\nabla p + \frac{\gamma}{Re} \Delta v + \frac{1 - \gamma}{Re} \nabla \cdot \sigma, \\
  \nabla \cdot v = 0, \\
  n \times (n - \gamma_1 N - \gamma_2 D \cdot n) = 0,
\end{cases}
\]

where \( v \) is the velocity of the fluid, \( p \) is the pressure, \( Re \) is the Reynolds number and \( \gamma \in (0, 1) \). The stress \( \sigma \) is modeled by the phenomenological constitutive relation

\[ \sigma = \sigma^L + \sigma^E, \]

where \( \sigma^L \) is the viscous (Leslie) stress

\[ \sigma^L = \alpha_1 (nn : D)nn + \alpha_2 nN + \alpha_3 NN + \alpha_4 D + \alpha_5 nn \cdot D + \alpha_6 D \cdot nn \]

with \( D = \frac{1}{2} (\kappa^T + \kappa), \kappa = (\nabla v)^T \), and

\[ N = n_t + v \cdot \nabla n + \Omega \cdot n, \quad \Omega = \frac{1}{2} (\kappa^T - \kappa). \]

The six constants \( \alpha_1, \cdots, \alpha_6 \) are called the Leslie coefficients. While, \( \sigma^E \) is the elastic (Ericksen) stress

\[ \sigma^E = -\frac{\partial E_F}{\partial (\nabla n)} \cdot (\nabla n)^T, \]

where \( E_F = E_F(n, \nabla n) \) is the Oseen-Frank energy with the form

\[ E_F = \frac{k_1}{2} (\nabla \cdot n)^2 + \frac{k_2}{2} |n \times (\nabla \times n)|^2 + \frac{k_3}{2} |n \cdot (\nabla \times n)|^2. \]
Here $k_1, k_2, k_3$ are the elastic constant. For the simplicity, we will consider the case $k_1 = k_2 = k_3 = 1$. In such case, $E_F = \frac{1}{2} |\nabla n|^2$, and the molecular field $h$ is given by

$$h = -\frac{\delta E_F}{\delta n} = \nabla \cdot \frac{\partial E_F}{\partial (\nabla n)} - \frac{\partial E_F}{\partial n} = -\Delta n,$$

$$(\sigma^E)_{ij} = -(\nabla n \odot \nabla n)_{ij} = -\partial_i n_k \partial_j n_k.$$ 

Finally, the Leslie coefficients and $\gamma_1, \gamma_2$ satisfy the following relations

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \tag{1.4}$$
$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \tag{1.5}$$

where (1.4) is called Parodi’s relation derived from the Onsager reciprocal relation [15]. These two relations ensure that the system has a basic energy law.

As the general Ericksen-Leslie system is very complicated, most of earlier works treated the simplified(or approximated) system of (1.1). Motivated by the work on the harmonic heat flow, Lin and Liu [12] add the penalty term $\frac{1}{4\varepsilon^2}(|n|^2 - 1)^2$ in $W$ in order to remove some higher-order nonlinearities due to the constraint $|n| = 1$. In such case, the system becomes

$$\begin{cases}
    v_t + v \cdot \nabla v = -\nabla p + \frac{\gamma}{Re} \Delta v + \frac{1}{Re} \nabla \cdot \sigma, \\
    n_t + v \cdot \nabla n + \Omega \cdot n - \mu_1 \Delta n - \mu_2 D \cdot n - \frac{1}{\varepsilon^2}(|n|^2 - 1) n = 0.
\end{cases} \tag{1.6}$$

This is so called the Ginzburg-Landau approximation. They proved the global existence of weak solution and the local existence and uniqueness of strong solution of the system (1.6) under certain strong constrains on the Leslie coefficients. We refer to [18] for a recent result about the role of Parodi’s relation in the well-posedness and stability. However, whether the solution of (1.6) converges to that of (1.1) as $\varepsilon$ tends to zero is still a challenging question. When neglecting the Leslie stress $\sigma_L$ in (1.1), a simplest system preserving the basic energy law is the following

$$\begin{cases}
    v_t + v \cdot \nabla v - \Delta v + \nabla p = -\nabla \cdot (\nabla n \odot \nabla n), \\
    n_t + v \cdot \nabla n - \Delta n = |\nabla n|^2 n. \tag{1.7}
\end{cases}$$

For this system, the local existence and uniqueness of strong solution can be proved by using the standard energy method; see [16] for the well-posedness result with rough data. Huang and Wang [7] give the following BKM type blow-up criterion: Let $T^*$ be the maximal existence time of the strong solution. If $T^* < \infty$, then it is necessary

$$\int_0^{T^*} \|\nabla \times v(t)\|_{L^\infty} + \|\nabla n(t)\|^2_{L^\infty} dt = +\infty.$$ 

In two dimensional case, the global existence of weak solution has been independently proved by Lin, Lin and Wang [13] and Hong [6], where they construct a class of weak solution with at most a finite number of singular times. The uniqueness of weak solution is proved by Lin-Wang [14] and Xu-Zhang [19]. The global existence of weak solution of (1.7) is a challenging open problem in three dimensional case. On the other hand, in the case when $|\nabla n|^2 n$ in (1.7) is replaced by $\frac{1}{\varepsilon^2}(|n|^2 - 1) n$, the global existence and partial regularity of weak solution were studied in [10, 11].

The purpose of this paper is to study the well-posedness of the general Ericksen-Leslie system. The first step is to understand the complicated energy-dissipation law of the system arising from the Leslie stress. Moreover, whether the energy defined in (2.1) is dissipated remains unknown in physics, since the Leslie coefficients are difficult to determine by using experimental results. We present a sufficient and necessary condition on the Leslie coefficients to ensure that the energy of the system is dissipated. The next step is to construct an approximate system with the dissipated energy under the physical condition on the Leslie coefficients. However, the Ginzburg-Landau approximation does not satisfy our requirement. We introduce a new equivalent formulation of the
system (1.1). Based on this formulation, we can construct an approximate system such that the energy is still dissipated, although the key property $|n| = 1$ is destroyed.

Our main results are stated as follows.

**Theorem 1.1.** Let $s \geq 2$ be an integer. Assume that the Leslie coefficients satisfy (2.6), and the initial data $\nabla n_0 \in H^{2s}(\mathbb{R}^3), v_0 \in H^{2s}(\mathbb{R}^3)$. There exist $T > 0$ and a unique solution $(v, n)$ of the Ericksen-Leslie system (1.1) such that 

$$v \in C([0, T]; H^{2s}(\mathbb{R}^3)) \cap L^2(0, T; H^{2s+1}(\mathbb{R}^3)), \quad \nabla n \in C([0, T]; H^{2s}(\mathbb{R}^3)).$$

Let $T^*$ be the maximal existence time of the solution. If $T^* < +\infty$, then it is necessary

$$\int_0^{T^*} \|
abla \times v(t)\|_{L^\infty} + \|
abla n(t)\|_{L^\infty}^2 dt = +\infty.$$

For small initial data, we prove the following global well-posedness.

**Theorem 1.2.** With the same assumptions as in Theorem 1.1, there exists an $\varepsilon_0 > 0$ such that if

$$\|\nabla n_0\|_{H^{2s}} + \|v_0\|_{H^{2s}} \leq \varepsilon_0,$$

then the solution obtained in Theorem 1.1 is global in time.

The other sections of this paper are organized as follows. In section 2, we derive the basic energy law of the system and give the physical constrain condition on the Leslie coefficients. In section 3, we introduce a new equivalent formulation. Section 4 is devoted to the proof of local well-posedness. In section 5, we prove the global well-posedness of the system for small initial data.

2. Basic energy-dissipation law

We first derive the basic energy law of the system (1.1).

**Proposition 2.1.** If $(v, n)$ is a smooth solution of (1.1), then it holds that

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{Re}{2(1 - \gamma)}|v|^2 + E_F dx = -\int_{\mathbb{R}^3} \left( \frac{\gamma}{1 - \gamma} |\nabla v|^2 + (\alpha_1 + \frac{\gamma^2}{\gamma_1}) |D : nn|^2 + \alpha_4 D : D \right. + (\alpha_5 + \alpha_6 - \frac{\gamma^2}{\gamma_1}) |D : n|^2 + \frac{1}{\gamma_1} |n \times h|^2 dx. \tag{2.1}$$

**Proof.** Using the first equation of (1.1) and $\nabla \cdot v = 0$, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{Re}{2(1 - \gamma)}|v|^2 + E_F dx$$

$$= \int_{\mathbb{R}^3} \frac{Re}{1 - \gamma} v \cdot v_t dx + \int_{\mathbb{R}^3} \frac{\delta E_F}{\delta n} \cdot n_t dx$$

$$= -\int_{\mathbb{R}^3} \frac{\gamma}{1 - \gamma} |\nabla v|^2 + (\sigma^E + \sigma^E) : \nabla v dx + \int_{\mathbb{R}^3} \frac{\delta E_F}{\delta n} \cdot (\dot{n} - v \cdot \nabla n) dx, \tag{2.2}$$

where $\dot{n} = n_t + v \cdot \nabla n$. Using $\nabla \cdot v = 0$ again, we have

$$\int_{\mathbb{R}^3} \sigma^E : \nabla v + \frac{\delta E_F}{\delta n} \cdot (v \cdot \nabla n) dx$$

$$= \int_{\mathbb{R}^3} (\frac{\partial E_F}{\partial (\nabla n)} \cdot (\nabla n)^T) : \nabla v - \left( \nabla \cdot \frac{\partial E_F}{\partial (\nabla n)} - \frac{\partial E_F}{\partial n} \right) \cdot (v \cdot \nabla n) dx$$

$$= \int_{\mathbb{R}^3} (\frac{\partial E_F}{\partial (\nabla n)} : (v \cdot \nabla^2 n) + \frac{\partial E_F}{\partial n} \cdot (v \cdot \nabla n) dx$$

$$= \int_{\mathbb{R}^3} v \cdot \nabla E_F(n, \nabla n) dx = 0. \tag{2.3}$$
Due to (1.2), (1.4) and (1.5), we find
\[
\int_{\mathbb{R}^3} \sigma^L : \nabla v \, dx = \int_{\mathbb{R}^3} \left( (\alpha_1(nn \cdot D))nn + \alpha_2nn + \alpha_3nn + \alpha_4D + \alpha_5nn \cdot D + \alpha_6D \cdot nn \right) : (D + \Omega) \, dx
\]
\[
= \int_{\mathbb{R}^3} \left( (\alpha_1(nn : D))^2 + \alpha_4D : D + (\alpha_5 + \alpha_6)|D \cdot n|^2 + (\alpha_2 + \alpha_3)n \cdot (D \cdot n) \right.
\]
\[
+ (\alpha_2 - \alpha_3)n \cdot (\Omega \cdot N) - (\alpha_5 - \alpha_6)(D \cdot n) \cdot (\Omega \cdot n) \bigg) \, dx
\]
\[
= \int_{\mathbb{R}^3} \left( (\alpha_1(nn : D))^2 + \alpha_4D : D + (\alpha_5 + \alpha_6)|D \cdot n|^2 + \gamma_2n \cdot (D \cdot N) \right.
\]
\[
- \gamma_1n \cdot (\Omega \cdot N) + \gamma_2(D \cdot n) \cdot (\Omega \cdot n) \bigg) \, dx
\]
\[
= \int_{\mathbb{R}^3} \left( (\alpha_1(nn : D))^2 + \alpha_4D : D + (\alpha_5 + \alpha_6)|D \cdot n|^2 + \gamma_2N \cdot (D \cdot n) \right.
\]
\[
+ \gamma_1N \cdot (\Omega \cdot n) + \gamma_2(D \cdot n) \cdot (\Omega \cdot n) \bigg) \, dx,
\]
and
\[
- \int_{\mathbb{R}^3} \frac{\delta E_F}{\delta n} \cdot \dot{n} \, dx = \int_{\mathbb{R}^3} h \cdot \dot{n} \, dx = \int_{\mathbb{R}^3} h \cdot (N - \Omega \cdot n) \, dx.
\]
The third equation of (1.1) implies that
\[
\int_{\mathbb{R}^3} (\Omega \cdot n) \cdot (\gamma_1N + \gamma_2(D \cdot n) - h) \, dx = 0,
\]
and direct calculations show that
\[
\int_{\mathbb{R}^3} (\gamma_2N \cdot (D \cdot n) + h \cdot N) \, dx = \int_{\mathbb{R}^3} (n \times N) \cdot (n \times h + \gamma_2n \times D \cdot n) \, dx
\]
\[
= \int_{\mathbb{R}^3} \frac{1}{\gamma_1} \left( n \times h \right) \cdot \left( n \times h + \gamma_2n \times D \cdot n \right) \, dx
\]
\[
= \int_{\mathbb{R}^3} \frac{1}{\gamma_1} |n \times h|^2 - \frac{\gamma_2}{\gamma_1} |D \cdot n|^2 + \frac{\gamma_2}{\gamma_1} |n \cdot D \cdot n|^2 \, dx.
\]
Thus, we have
\[
\int_{\mathbb{R}^3} \sigma^L : \nabla v - \frac{\delta E_F}{\delta n} \cdot \dot{n} \, dx
\]
\[
= \int_{\mathbb{R}^3} \left( (\alpha_1 + \frac{\gamma_2}{\gamma_1})|D : nn|^2 + \alpha_4D : D + (\alpha_5 + \frac{\gamma_2}{\gamma_1} - \frac{\gamma_2}{\gamma_1})|D \cdot n|^2 + \frac{1}{\gamma_1} |n \times h|^2 \right) \, dx.
\]
Then the energy law (2.1) follows from (2.2)-(2.4).

\[\square\]

The following proposition presents a sufficient and necessary condition on the Leslie coefficients to ensure that the energy is dissipated; see also [12] for the related discussions on the choice of the Leslie coefficients. We denote
\[
\beta_1 = \alpha_1 + \frac{\gamma_2}{\gamma_1}, \quad \beta_2 = \alpha_4, \quad \beta_3 = \alpha_5 + \alpha_6 - \frac{\gamma_2}{\gamma_1}.
\]

**Proposition 2.2.** The following dissipation relation holds
\[
\beta_1(nn : D)^2 + \beta_2D : D + \beta_3|D \cdot n|^2 \geq 0
\]
for any symmetric trace free matrix $D$ and unit vector $n$, if and only if
\[
\beta_2 \geq 0, \quad 2\beta_2 + \beta_3 \geq 0, \quad \frac{3}{2}\beta_2 + \beta_3 + \beta_1 \geq 0.
\]

**Proof.** By the rotation invariance, we may assume $n = (0, 0, 1)^T$ and $D = (D_{ij})_{3 \times 3}$ with $D_{11} + D_{22} + D_{33} = 0$. It is easy to get
\[
\beta_1(n : D)^2 + \beta_2 D : D + \beta_3 |D \cdot n|^2
\]
\[
= \beta_1 D_{33}^2 + \beta_2 (D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2 + 2D_{31}^2) + \beta_3 (D_{11}^2 + D_{22}^2 + D_{33}^2)
\]
\[
= 2\beta_2 D_{12}^2 + (2\beta_2 + \beta_3)(D_{31}^2 + D_{32}^2) + \beta_2 (D_{11}^2 + D_{22}^2) + (\beta_1 + \beta_2 + \beta_3)D_{33}^2
\]
\[
= 2\beta_2 D_{12}^2 + (2\beta_2 + \beta_3)(D_{31}^2 + D_{32}^2) + \beta_2 (D_{11}^2 + D_{22}^2) + (\beta_1 + \beta_2 + \beta_3)(D_{11} + D_{22})^2.
\]

The inequality holds
\[
2\beta_2 D_{12}^2 + (2\beta_2 + \beta_3)(D_{31}^2 + D_{32}^2) \geq 0
\]
for all $D_{12}, D_{31}$, and $D_{32}$, if and only if $\beta_2 \geq 0$, and $2\beta_2 + \beta_3 \geq 0$.

As $D_{11} + D_{22} \geq \frac{1}{2}(D_{11} + D_{22})^2$, the inequality holds
\[
\beta_2 (D_{11}^2 + D_{22}^2) + (\beta_1 + \beta_2 + \beta_3)(D_{11} + D_{22})^2 \geq 0
\]
for all $D_{11}$ and $D_{22}$, if and only if $\frac{3}{2}\beta_2 + \beta_3 + \beta_1 \geq 0$. \hfill $\Box$

In [17], we show that if the Ericksen-Leslie system is derived from the Doi-Onsager equation, then the energy (2.1) is indeed dissipated. Let us make it precise. The nondimensional Doi-Onsager equation takes as follows
\[
\begin{align*}
\frac{\partial f^\varepsilon}{\partial t} + v^\varepsilon \cdot \nabla f^\varepsilon &= \frac{1}{\varepsilon} \mathcal{R} \cdot (\mathcal{R} f^\varepsilon + f^\varepsilon \mathcal{R} \mu^\varepsilon f^\varepsilon) - \mathcal{R} \cdot (m \times \kappa^\varepsilon \cdot m f^\varepsilon), \\
\frac{\partial v^\varepsilon}{\partial t} + v^\varepsilon \cdot \nabla v^\varepsilon &= -\nabla p^\varepsilon + \frac{\gamma}{Re} \Delta v^\varepsilon + \frac{1}{2Re} \nabla \cdot (D^\varepsilon : \langle mmmm \rangle f^\varepsilon) \\
&\quad + \frac{1}{\varepsilon Re} (\nabla \cdot \tau^\varepsilon + \mathbf{F}^\varepsilon),
\end{align*}
\]

where $\varepsilon$ is the Deborah number, $\kappa^\varepsilon = (\nabla v^\varepsilon)^T, D^\varepsilon = \frac{1}{2} (\kappa^\varepsilon + (\kappa^\varepsilon)^T)$, and $\tau^\varepsilon = -\langle m m \times \mathcal{R} \mu^\varepsilon f^\varepsilon \rangle, \quad \mathbf{F}^\varepsilon = -\langle \nabla \mu^\varepsilon \rangle f^\varepsilon, \quad \mu^\varepsilon = \ln f^\varepsilon + \mathcal{U} \varepsilon f$.

\[
\mathcal{U} \varepsilon f = \alpha \int_{\mathbb{R}^3} \int_{S^2} |m \times m'|^2 \frac{1}{\sqrt{\varepsilon}} g\left(\frac{x - x'}{\sqrt{\varepsilon}}\right) f(x', m', t) \, dm' \, dx'.
\]

When $\varepsilon$ is small, the solution $(f^\varepsilon, v^\varepsilon)$ of the system (2.7) has the expansion
\[
f^\varepsilon = f_0(m \cdot n) + \varepsilon f_1 + \cdots,
\]
\[
v^\varepsilon = v_0 + \varepsilon v_1 + \cdots,
\]

where $(v_0, n)$ is determined by (1.1) with the Leslie coefficients given by
\[
\begin{align*}
\alpha_1 &= -\frac{S_4}{2}, \quad \alpha_2 = -\frac{1}{2} (1 + \frac{1}{\lambda}) S_2, \quad \alpha_3 = -\frac{1}{2} (1 - \frac{1}{\lambda}) S_2, \\
\alpha_4 &= \frac{4}{15} - \frac{5}{21} S_2 - \frac{1}{35} S_4, \quad \alpha_5 = \frac{7}{7} S_4 + \frac{6}{7} S_2, \quad \alpha_6 = \frac{7}{7} S_4 - \frac{1}{7} S_2.
\end{align*}
\]

Here $S_2 = \langle P_2(m \cdot n) \rangle_{h_{\eta_1 n}}, S_4 = \langle P_4(m \cdot n) \rangle_{h_{\eta_1 n}}$ with $P_k(x)$ the $k$-th Legendre polynomial and
\[
h_{\eta_1 n}(m) = \frac{e^{\eta_1 (m \cdot n)^2}}{\int_{S^2} e^{\eta_1 (m \cdot n)^2} \, dm}.
\]

Here $\eta_1$ and $\lambda$ are constants depending only on $\alpha$. When the Leslie coefficients are given by (2.8), we show that the dissipation relation (2.5) holds; see [17][5][8] for the details.
3. A New Formulation of the Ericksen-Leslie System

Set $\mu_1 = \frac{1}{\mu_1}, \mu_2 = -\frac{2\mu_1}{\mu_2}$. The third equation of (1.1) is equivalent to

$$n_t + v \cdot \nabla n + \Omega \cdot n - (I - nn) \cdot (\mu_1 h + \mu_2 D \cdot n) = 0,$$

which can be written as

$$n_t + v \cdot \nabla n + n \times ((\Omega \cdot n - \mu_1 h - \mu_2 D \cdot n) \times n) = 0. \tag{3.1}$$

Substituting them into (1.2), we get

$$\sigma^L = \beta_1 (nn : D) nn - \frac{1}{2} (1 + \mu_2)n(I - nn) \cdot h + \frac{1}{2} (1 - \mu_2)(I - nn) \cdot h n$$

$$+ \beta_2 D + \frac{\beta_3}{2} (nD \cdot n + D \cdot nn)$$

$$\begin{align*}
&= \beta_1 (nn : D) nn - \frac{1}{2} (1 + \mu_2) nn \times (h \times n) + \frac{1}{2} (1 - \mu_2) n \times (h \times n) n \\
&\quad + \beta_2 D + \frac{\beta_3}{2} (nD \cdot n + D \cdot nn). \tag{3.2}
\end{align*}$$

With the new formulation (3.1) and (3.2), we can derive the same energy law (2.1) without using the constrain $|n| = 1$. To see it, we need the following important cancelation relations.

**Lemma 3.1.** It holds that

$$\begin{align*}
&\left( -\frac{1}{2} nn \times (h \times n) + \frac{1}{2} n \times (h \times n)n \right) : (D + \Omega) - ((\Omega \cdot n) \times n) \cdot (h \times n) = 0, \\
&\left( -\frac{1}{2} nn \times (h \times n) + \frac{1}{2} n \times (h \times n)n \right) : (D + \Omega) + (h \times n) \cdot ((D \cdot n) \times n) = 0.
\end{align*}$$

**Proof.** Direct calculations show that

$$\begin{align*}
&\left( -\frac{1}{2} nn \times (h \times n) + \frac{1}{2} n \times (h \times n)n \right) : (D + \Omega) - ((\Omega \cdot n) \times n) \cdot (h \times n) \\
&= (n \times (h \times n)n) : \Omega - ((\Omega \cdot n) \times n) \cdot (h \times n) \\
&= (n \times (h \times n)) \cdot (\Omega \cdot n) - ((\Omega \cdot n) \times n) \cdot (h \times n) = 0,
\end{align*}$$

and

$$\begin{align*}
&\left( -\frac{1}{2} nn \times (h \times n) + \frac{1}{2} n \times (h \times n)n \right) : (D + \Omega) + (h \times n) \cdot ((D \cdot n) \times n) \\
&= - (n \times (h \times n)n) : D + ((D \cdot n) \times n) \cdot (h \times n) \\
&= -(n \times (h \times n)) \cdot (D \cdot n) + ((D \cdot n) \times n) \cdot (h \times n) = 0.
\end{align*}$$

The proof is finished. \qed

Now we derive the energy law (2.1) by using (3.1) and (3.2), since the derivation will be helpful to understand the energy estimates in the next section. Thanks to (1.1) and (3.1), we have

$$\begin{align*}
&-\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{Re}{1 - \gamma} |v|^2 + |\nabla n|^2 dx \\
&= \int_{\Omega} \frac{\gamma}{1 - \gamma} |\nabla v|^2 + (\sigma^L + \sigma^E) : \nabla v - (v \cdot \nabla n) \cdot h \\
&\quad + (n \times ((\mu_1 h + \mu_2 D \cdot n - \Omega \cdot n) \times n)) \cdot h dx \\
&= \int_{\Omega} \frac{\gamma}{1 - \gamma} |\nabla v|^2 + |\mu_1 h \times n|^2 + \sigma^E : \nabla v - (v \cdot \nabla n) \cdot h \\
&\quad + \sigma^L : \nabla v - ((\Omega \cdot n) \times n) \cdot (h \times n) + \mu_2 (h \times n) \cdot ((D \cdot n) \times n) dx.
\end{align*}$$
For the Ericksen stress term, we have
\[
\int_{\mathbb{R}^3} \sigma^E : \nabla v - (v \cdot \nabla n) \cdot h dx = \int_{\mathbb{R}^3} -\partial_i n_k \partial_j n_k \partial_i v_j - v_j \partial_j n_k \partial_i n_k dx
\]
\[
= \int_{\mathbb{R}^3} v_j \partial_j \partial_i n_k \partial_i n_k - \partial_i (v_j \partial_j n_k \partial_i n_k) dx = 0,
\]
while for the Leslie stress term, we get by (3.2) and Lemma 3.1 that
\[
\int_{\mathbb{R}^3} \sigma^L : \nabla v - ((\Omega \cdot n) \times n) \cdot (h \times n) + \mu_2 (h \times n) \cdot ((D \cdot n) \times n) dx
\]
\[
= \int_{\mathbb{R}^3} \left( \beta_1 (nn : D) nn + \frac{1}{2} (-1 - \mu_2) nn \times (h \times n) + \frac{1}{2} (1 - \mu_2) n \times (h \times n) n + \beta_2 D
\]
\[
+ \frac{\beta_3}{2} (nD \cdot n + D \cdot nn) : (D + \Omega) - ((\Omega \cdot n) \times n) \cdot (h \times n) + \mu_2 (h \times n) \cdot ((D \cdot n) \times n) dx
\]
\[
= \int_{\mathbb{R}^3} \beta_1 (nn : D)^2 + \beta_2 D : D + \beta_3 |D \cdot n|^2 - ((\Omega \cdot n) \times n) \cdot (h \times n) + \mu_2 (h \times n) \cdot ((D \cdot n) \times n)
\]
\[
+ \left( \frac{1}{2} (-1 - \mu_2) nn \times (h \times n) + \frac{1}{2} (1 - \mu_2) n \times (h \times n) n \right) : (D + \Omega) dx
\]
\[
= \int_{\mathbb{R}^3} \beta_1 (nn : D)^2 + \beta_2 D : D + \beta_3 |D \cdot n|^2 dx.
\]
Then the energy law (2.1) follows from the above identities.

Although the energy law can be derived without using the property \(|n| = 1\), this property is vital for the dissipation relation (2.5) under the condition (2.6). Hence, it is important to construct an approximate system preserving the energy law and \(|n| = 1\) in order to prove the local well-posedness of (1.1). It is usually difficult. For this, we introduce a modified stress tensor so that the energy is still dissipated for the modified system under the condition (2.6). The modified Leslie stress tensor takes the form
\[
\tilde{\sigma}^L = \beta_1 (nn : D) nn + \frac{1}{2} (-1 - \mu_2) nn \times (h \times n) + \frac{1}{2} (1 - \mu_2) n \times (h \times n) n
\]
\[
+ \beta_2 |n|^4 D + \frac{\beta_3}{2} |n|^2 (nD \cdot n + D \cdot nn).
\]
It is obvious that \(\tilde{\sigma}^L = \sigma^L\) if \(|n| = 1\). An important fact is that for any traceless symmetric \(D\) and vector \(n\) (not necessary unit), it still holds
\[
\langle \beta_1 (nn : D) nn + \beta_2 |n|^4 D + \frac{\beta_3}{2} |n|^2 (nD \cdot n + D \cdot nn), D \rangle \geq 0
\]
under the condition (2.6). We denote
\[
\sigma_1 (v, n) = \beta_1 (nn : D) nn + \beta_2 |n|^4 D + \frac{\beta_3}{2} |n|^2 (nD \cdot n + D \cdot nn),
\]
\[
\sigma_2 (n) = \frac{1}{2} (-1 - \mu_2) nn \times (h \times n) + \frac{1}{2} (1 - \mu_2) n \times (h \times n) n.
\]
The reformulated new system takes
\[
\begin{align*}
\begin{cases}
    v_t + v \cdot \nabla v = -\nabla p + \nu \Delta v + \nabla \cdot (\sigma_1 (v, n) + \sigma_2 (n) + \sigma^E), \\
    n_t + v \cdot \nabla n + n \times ((\Omega \cdot n - \mu_1 h - \mu_2 D \cdot n) \times n) = 0.
\end{cases}
\end{align*}
\]
Here we set $\nu = \frac{3}{10}$ and take $\frac{1}{2\nu} = 1$. Similar to Proposition 2.1, we can show that the system (3.3) obeys the following energy-dissipation law:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |v|^2 + |\nabla n|^2 \, dx = - \int_{\mathbb{R}^3} \left( \nu |\nabla v|^2 + \beta_1 |D : n| + \beta_2 |n|^4 : D : D 
+ \beta_3 |n|^2 D \cdot n| + \mu_1 |n \times h|^2 \right) \, dx,$$

which is dissipated under the condition (2.6) by (3.3).

4. LOCAL WELL-POSEDNESS AND BLOW-UP CRITERION

This section is devoted to proving the local well-posedness of the system (1.1). The following lemma will frequently used.

**Lemma 4.1.** For any $\alpha, \beta \in \mathbb{N}^3$, it holds that

$$\|D^\alpha (fg)\|_{L^2} \leq C \sum_{|\gamma| = |\alpha|} \left( \|f\|_{L^\infty} \|D^\gamma g\|_{L^2} + \|g\|_{L^\infty} \|D^\gamma f\|_{L^2} \right),$$

$$\|D^\alpha f D^\beta g\|_{L^2} \leq C \left( \sum_{|\gamma| = |\alpha| + |\beta|} \|D^\gamma f\|_{L^2} \|g\|_{L^\infty} + \sum_{|\gamma| = |\alpha| + |\beta| - 1} \|D^\gamma f\|_{L^\infty} \|D^\gamma g\|_{L^2} \right).$$

This lemma can be easily proved by using Bony’s decomposition; see [1] for example. The proof of Theorem 1.1 is split into several steps.

**Step 1.** Construction of the approximate solutions

The construction is based on the classical Friedrich’s method. Define the smoothing operator

$$(4.6) \quad J_\varepsilon f = \mathcal{F}^{-1}(1_{|\xi| \leq \frac{1}{\varepsilon}} \mathcal{F} f),$$

where $\mathcal{F}$ is the usual Fourier transform. Let $\mathcal{P}$ be the operator which projects a vector field to its solenoidal part. We introduce the following approximate system of (3.3):

$$\begin{cases}
\frac{\partial v_\varepsilon}{\partial t} + J_\varepsilon \mathcal{P}(J_\varepsilon v_\varepsilon \cdot \nabla J_\varepsilon v_\varepsilon) = \nu \Delta J_\varepsilon v_\varepsilon + \nabla \cdot J_\varepsilon \mathcal{P}(\sigma_1 (J_\varepsilon v_\varepsilon, J_\varepsilon n_\varepsilon) + \sigma_2 (J_\varepsilon v_\varepsilon) + \sigma^E (J_\varepsilon n_\varepsilon)), \\
\frac{\partial n_\varepsilon}{\partial t} + J_\varepsilon \left( J_\varepsilon v_\varepsilon \cdot \nabla J_\varepsilon n_\varepsilon + J_\varepsilon n_\varepsilon \times [(J_\varepsilon \Omega_\varepsilon \cdot J_\varepsilon n_\varepsilon - \mu_1 J_\varepsilon h_\varepsilon - \mu_2 J_\varepsilon D_\varepsilon \cdot J_\varepsilon n_\varepsilon) \times J_\varepsilon n_\varepsilon] \right) = 0, \\
(v_\varepsilon, n_\varepsilon)|_{t=0} = (J_\varepsilon v_0, J_\varepsilon n_0).
\end{cases}$$

The above system can be viewed as an ODE system on $L^2(\mathbb{R}^3)$. Then we know by the Cauchy-Lipschitz theorem that there exist a strictly maximal time $T_\varepsilon$ and a unique solution $(v_\varepsilon, n_\varepsilon)$ which is continuous in time with value in $H^k(\mathbb{R}^3)$ for any $k \geq 0$. As $J_\varepsilon^2 = J_\varepsilon$, we know that $(J_\varepsilon v_\varepsilon, J_\varepsilon n_\varepsilon)$ is also a solution. Therefore, $(v_\varepsilon, n_\varepsilon) = (J_\varepsilon v_0, J_\varepsilon n_0)$. Thus, $(v_\varepsilon, n_\varepsilon)$ satisfies the following system

$$\begin{cases}
\frac{\partial v_\varepsilon}{\partial t} + J_\varepsilon \mathcal{P}(v_\varepsilon \cdot \nabla v_\varepsilon) = \nu \Delta v_\varepsilon + \nabla \cdot J_\varepsilon \mathcal{P}(\sigma_1 (v_\varepsilon, n_\varepsilon) + \sigma_2 (n_\varepsilon) + \sigma^E (n_\varepsilon)), \\
\frac{\partial n_\varepsilon}{\partial t} + J_\varepsilon \left( v_\varepsilon \cdot \nabla n_\varepsilon + n_\varepsilon \times [(\Omega_\varepsilon \cdot n_\varepsilon - \mu_1 h_\varepsilon - \mu_2 D_\varepsilon \cdot n_\varepsilon) \times n_\varepsilon] \right) = 0, \\
(v_\varepsilon, n_\varepsilon)|_{t=0} = (J_\varepsilon v_0, J_\varepsilon n_0).
\end{cases}$$

**Step 2.** Uniform energy estimates

We define

$$E_h(v, n) \overset{\text{def}}{=} ||n - n_0||^2_{L^2} + ||\nabla n||^2_{L^2} + ||\nabla \Delta^s n||^2_{L^2} + ||v||^2_{L^2} + ||\Delta^s v||^2_{L^2}.$$
First of all, we get by the second equation of (4.7) that
\[
\frac{d}{dt}|\mathbf{n}_\varepsilon - \mathbf{n}_0|^2_{L^2} = 2\langle \partial_t \mathbf{n}_\varepsilon, \mathbf{n}_\varepsilon - \mathbf{n}_0 \rangle \\
= \langle \mathbf{v}_\varepsilon \cdot \nabla \mathbf{n}_\varepsilon + \mathbf{n}_\varepsilon \times [(\Omega_\varepsilon \cdot \mathbf{n}_\varepsilon - \mu_1 \mathbf{h}_\varepsilon - \mu_2 \mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon], \mathcal{J}_\varepsilon(\mathbf{n}_\varepsilon - \mathbf{n}_0) \rangle \\
= \langle \mathbf{v}_\varepsilon \cdot \nabla \mathbf{n}_\varepsilon + \mathbf{n}_\varepsilon \times [(\Omega_\varepsilon \cdot \mathbf{n}_\varepsilon - \mu_1 \mathbf{h}_\varepsilon - \mu_2 \mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon], \mathcal{J}_\varepsilon(\mathbf{n}_\varepsilon - \mathbf{n}_0) \rangle \\
\leq C\left(\|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}\|\mathbf{v}_\varepsilon\|_{L^2} + \|\mathbf{n}_\varepsilon\|^2_{L^\infty}\left(\|\nabla \mathbf{v}_\varepsilon\|_{L^2} + \|\Delta \mathbf{n}_\varepsilon\|_{L^2}\right)\right)\|\mathbf{n}_\varepsilon - \mathbf{n}_0\|_{L^2} \\
\leq C\left(\|\nabla \mathbf{n}_\varepsilon\|_{L^\infty} + \|\mathbf{n}_\varepsilon\|^3_{L^\infty}\right)E_\varepsilon(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon).
\]

(4.8)

The following energy law still holds for the approximate system (4.7):
\[
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{v}_\varepsilon|^2 + |\nabla \mathbf{n}_\varepsilon|^2 dx = -\int_{\mathbb{R}^3} \left(\nu |\nabla \mathbf{v}_\varepsilon|^2 + \beta_1 |\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon|^2 + \beta_2 |\mathbf{n}_\varepsilon|^4 |\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon|^2 + \beta_3 |\mathbf{n}_\varepsilon|^2 |\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon|^2 + \mu_1 |\mathbf{n}_\varepsilon \times \mathbf{h}_\varepsilon|^2\right) dx.
\]

(4.9)

Now we turn to the estimate of the higher order derivative for \(\mathbf{n}_\varepsilon\).
\[
\frac{1}{2}\frac{d}{dt}\langle \nabla \Delta^s \mathbf{n}_\varepsilon, \nabla \Delta^s \mathbf{n}_\varepsilon \rangle = \langle \Delta^s(\mathbf{v}_\varepsilon \cdot \nabla \mathbf{n}_\varepsilon), \Delta^s \mathbf{n}_\varepsilon \rangle + \langle \Delta^s \left[\mathbf{n}_\varepsilon \times \left((\Omega_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right)\right], \Delta^{s+1} \mathbf{n}_\varepsilon \rangle \\
- \mu_2 \langle \Delta^s \left[\mathbf{n}_\varepsilon \times \left((\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right)\right], \Delta^{s+1} \mathbf{n}_\varepsilon \rangle - \mu_1 \langle \Delta^s \left[\mathbf{n}_\varepsilon \times \left(\Delta \mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\right)\right], \Delta^{s+1} \mathbf{n}_\varepsilon \rangle
\]

(4.10)

As \(\nabla \cdot \mathbf{v}_\varepsilon = 0\), we get by Lemma 4.1 that
\[
I_1 = -\langle \nabla \Delta^s(\mathbf{v}_\varepsilon \cdot \nabla \mathbf{n}_\varepsilon), \nabla \Delta^s \mathbf{n}_\varepsilon \rangle + \langle \mathbf{v}_\varepsilon \cdot \nabla (\nabla \Delta^s \mathbf{n}_\varepsilon), \nabla \Delta^s \mathbf{n}_\varepsilon \rangle \\
= -\langle ||\nabla \Delta^s, \mathbf{v}_\varepsilon|| \cdot \nabla \mathbf{n}_\varepsilon, \nabla \Delta^s \mathbf{n}_\varepsilon \rangle \\
\leq ||\nabla \Delta^s, \mathbf{v}_\varepsilon|| \cdot ||\nabla \Delta^s \mathbf{n}_\varepsilon||_{L^2} \\
\leq C\left(||\nabla \mathbf{n}_\varepsilon||_{H^{2s}} + ||\nabla \mathbf{v}_\varepsilon||_{H^{2s}}\right)\|\nabla \Delta^s \mathbf{n}_\varepsilon\|_{L^2} \\
\leq C\left(||\nabla \mathbf{v}_\varepsilon||_{L^\infty} + ||\nabla \mathbf{n}_\varepsilon||^2_{L^\infty}\right)||\nabla \mathbf{n}_\varepsilon||_{H^{2s}} + \delta ||\nabla \mathbf{v}_\varepsilon||_{H^{2s}}.
\]

(4.11)

Here and in what follows, \(\delta\) denotes a positive constant to be determined later. We rewrite \(I_2\) as
\[
I_2 = \langle \mathbf{n}_\varepsilon \times \left((\Delta^s \Omega_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right), \Delta^{s+1} \mathbf{n}_\varepsilon \rangle \\
- \langle \nabla \Delta^s \left[\mathbf{n}_\varepsilon \times \left((\Omega_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right)\right], \nabla \Delta^s \mathbf{n}_\varepsilon \rangle + \langle \mathbf{n}_\varepsilon \times \left((\nabla \Delta^s \Omega_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right), \nabla \Delta^s \mathbf{n}_\varepsilon \rangle \\
+ \langle \mathbf{n}_\varepsilon \times \left((\Delta^s \Omega_\varepsilon \cdot \nabla \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right), \nabla \Delta^s \mathbf{n}_\varepsilon \rangle \\
+ \langle \mathbf{n}_\varepsilon \times \left((\nabla \Omega_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \nabla \mathbf{n}_\varepsilon\right), \nabla \Delta^s \mathbf{n}_\varepsilon \rangle,
\]

from which and Lemma 4.1 it follows that
\[
I_2 \leq \langle \mathbf{n}_\varepsilon \times \left((\Delta^s \Omega_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right), \Delta^{s+1} \mathbf{n}_\varepsilon \rangle \\
+ C_3\left(||\mathbf{n}_\varepsilon||^2_{L^\infty} ||\nabla \mathbf{v}_\varepsilon||_{L^\infty} + ||\mathbf{n}_\varepsilon||^4_{L^\infty} ||\nabla \mathbf{n}_\varepsilon||^2_{L^\infty}\right)||\nabla \mathbf{n}_\varepsilon||^2_{H^{2s}} + \delta ||\nabla \mathbf{v}_\varepsilon||^2_{H^{2s}}.
\]

(4.12)

Similarly, we have
\[
I_3 \leq -\mu_2 \langle \mathbf{n}_\varepsilon \times \left((\Delta^s \mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\right), \Delta^{s+1} \mathbf{n}_\varepsilon \rangle \\
+ C_4\left(||\mathbf{n}_\varepsilon||^2_{L^\infty} ||\nabla \mathbf{v}_\varepsilon||_{L^\infty} + ||\mathbf{n}_\varepsilon||^4_{L^\infty} ||\nabla \mathbf{n}_\varepsilon||^2_{L^\infty}\right)||\nabla \mathbf{n}_\varepsilon||^2_{H^{2s}} + \delta ||\nabla \mathbf{v}_\varepsilon||^2_{H^{2s}}.
\]

(4.13)
For $I_4$, we have
\[
I_4 = -\mu_1 \langle \mathbf{n}_e \times (\nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e) , \nabla^{s+1} \mathbf{n}_e \rangle \\
- \left[ \mu_1 \langle \nabla^{s} \mathbf{n}_e \times (\nabla \mathbf{n}_e \times \mathbf{n}_e) , \nabla^{s+1} \mathbf{n}_e \rangle - \mu_1 \langle \mathbf{n}_e \times (\nabla^{s} \mathbf{n}_e \times \mathbf{n}_e) , \nabla^{s+1} \mathbf{n}_e \rangle \right] \\
- \left[ \mu_1 \langle \mathbf{n}_e \times \nabla^{s}(\nabla \mathbf{n}_e \times \mathbf{n}_e) , \nabla^{s+1} \mathbf{n}_e \rangle - \mu_1 \langle \mathbf{n}_e \times (\nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e) , \nabla^{s+1} \mathbf{n}_e \rangle \right] \\
= \mu_1 \langle \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e , \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \rangle + I_{41} + I_{42}.
\]
We get by Lemma [1.1] that
\[
I_{42} \leq C \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}} \| \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \|_{L^2} \\
\leq C_\delta \| \nabla \mathbf{n}_e \|_{L^\infty}^2 \| \nabla \mathbf{n}_e \|_{H^{2s}}^2 + \delta \| \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \|_{L^2}^2,
\]
and
\[
I_{41} = \mu_1 \left[ \langle \nabla^{s} \mathbf{n}_e \times (\nabla \mathbf{n}_e \times \mathbf{n}_e) , \nabla^{s} \mathbf{n}_e \rangle - \langle \nabla \mathbf{n}_e \times (\nabla^{s} \mathbf{n}_e \times \mathbf{n}_e) , \nabla \mathbf{n}_e \rangle \right] \\
= C \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla^{s} \mathbf{n}_e \|_{L^2} + \| \nabla \mathbf{n}_e \|_{H^{2s}} \| \nabla^{s} \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}} \\
\leq C \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla^{s} \mathbf{n}_e \|_{L^2} + \| \nabla \mathbf{n}_e \|_{H^{2s}} \| \nabla^{s} \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}} \\
\leq C \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla^{s} \mathbf{n}_e \|_{L^2} + \| \nabla \mathbf{n}_e \|_{H^{2s}} \| \nabla^{s} \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}}. \\
\]
which imply that
\[
I_4 \leq - \mu_1 \langle \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e , \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \rangle \\
+ C_\delta \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}}^2 + \delta \| \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \|_{L^2}^2. \\
\tag{4.14}
\]
Substituting (4.11) into (4.10), we infer that
\[
\frac{1}{2} \frac{d}{dt} \langle \nabla^{s} \mathbf{v}_\mathbf{e} , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle + \mu_1 \langle \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e , \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \rangle \\
\leq \langle \mathbf{n}_e \times ((\nabla^{s} \mathbf{\Omega}_\mathbf{e} \cdot \mathbf{n}_e) \times \mathbf{n}_e) , \nabla^{s+1} \mathbf{n}_e \rangle - \mu_2 \langle \mathbf{n}_e \times (\nabla^{s} \mathbf{D}_\mathbf{e} \cdot \mathbf{n}_e) \times \mathbf{n}_e , \nabla^{s+1} \mathbf{n}_e \rangle \\
+ C_\delta \| \nabla \mathbf{v}_\mathbf{e} \|_{L^\infty} + \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}}^2 \\
+ \delta \| \nabla^{s+1} \mathbf{n}_e \times \mathbf{n}_e \|_{L^2}^2. \\
\tag{4.15}
\]
Next we consider the estimate of the higher order derivative for $\mathbf{v}_\mathbf{e}$.
\[
\frac{1}{2} \frac{d}{dt} \langle \nabla^{s} \mathbf{v}_\mathbf{e} , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle + \nu \langle \nabla^{s} \mathbf{v}_\mathbf{e} , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle \\
= - \langle \nabla^{s} (\mathbf{v}_\mathbf{e} \cdot \nabla \mathbf{v}_\mathbf{e}) , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle + \langle \nabla^{s} (\nabla \mathbf{v}_\mathbf{e} \cdot \nabla \mathbf{v}_\mathbf{e}) , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle \\
- \langle \nabla^{s} \left( \beta_1 (\mathbf{n}_e \times \mathbf{D}_\mathbf{e}) \mathbf{n}_e \mathbf{n}_e + \beta_2 |\mathbf{n}_e|^2 \mathbf{D}_\mathbf{e} + \frac{\beta_3}{2} |\mathbf{n}_e|^2 \mathbf{D}_\mathbf{e} \cdot \mathbf{n}_e + \mathbf{D}_\mathbf{e} \cdot \mathbf{n}_e \mathbf{n}_e \right) \\
- \frac{1}{2} (1 + \mu_2) \mathbf{n}_e \times (\mathbf{h}_\mathbf{e} \times \mathbf{n}_e) + \frac{1}{2} (1 - \mu_2) \mathbf{n}_e \times (\mathbf{h}_\mathbf{e} \times \mathbf{n}_e) \nabla^{s} \mathbf{v}_\mathbf{e} \rangle \\
= - \langle \nabla^{s} (\mathbf{v}_\mathbf{e} \cdot \nabla \mathbf{v}_\mathbf{e}) , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle + \langle \nabla^{s} (\nabla \mathbf{v}_\mathbf{e} \cdot \nabla \mathbf{v}_\mathbf{e}) , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle \\
- \langle \nabla^{s} (\beta_1 (\mathbf{n}_e \times \mathbf{D}_\mathbf{e}) \mathbf{n}_e \mathbf{n}_e + \beta_2 |\mathbf{n}_e|^2 \mathbf{D}_\mathbf{e} + \frac{\beta_3}{2} |\mathbf{n}_e|^2 (\mathbf{n}_e \mathbf{D}_\mathbf{e} \cdot \mathbf{n}_e + \mathbf{D}_\mathbf{e} \cdot \mathbf{n}_e \mathbf{n}_e) \rangle , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle \\
+ \mu_2 \langle \nabla^{s} (\mathbf{n}_e \times (\mathbf{h}_\mathbf{e} \times \mathbf{n}_e)) , \nabla^{s} \mathbf{D}_\mathbf{e} \rangle - \langle \nabla^{s} (\mathbf{n}_e \times (\mathbf{h}_\mathbf{e} \times \mathbf{n}_e)) , \nabla^{s} \mathbf{D}_\mathbf{e} \rangle \\
= II_1 + II_2 + II_3 + II_4 + II_5.
\]
It follows from Lemma [4.1] that
\[
II_1 = \langle [\nabla^{s} , \mathbf{v}_\mathbf{e} ] \cdot \nabla \mathbf{v}_\mathbf{e} , \nabla^{s} \mathbf{v}_\mathbf{e} \rangle \leq C \| \nabla \mathbf{v}_\mathbf{e} \|_{L^\infty} \| \mathbf{v}_\mathbf{e} \|_{H^{2s}}^2, \\
II_2 \leq C_\delta \| \nabla \mathbf{n}_e \|_{L^\infty} \| \nabla \mathbf{n}_e \|_{H^{2s}}^2 + \delta \| \nabla \mathbf{v}_\mathbf{e} \|_{L^\infty}^2,
\]
and
\[ II_4 \leq \mu_2 \langle \nabla \cdot (\Delta D \cdot \nabla \varepsilon), \Delta \varepsilon \rangle + C_\delta \| \nabla \varepsilon \|_{L^\infty}^2 \| \nabla \varepsilon \|_{L^2}^2 + \delta \| \nabla \varepsilon \|_{H^2}^2, \]
\[ II_5 \leq - \langle \nabla \cdot (\Delta \varepsilon), \Delta \varepsilon \rangle + C_\delta \| \nabla \varepsilon \|_{L^\infty}^2 \| \nabla \varepsilon \|_{H^2}^2 + \delta \| \nabla \varepsilon \|_{H^2}^2, \]
and by (3.3),
\[ II_3 \leq - \langle (\beta_1 (\nabla \cdot D) \varepsilon) \varepsilon \varepsilon, (\beta_2 |\nabla \varepsilon|^2 \varepsilon + \beta_3 \| \nabla \varepsilon \|^2 \varepsilon), \Delta \varepsilon \rangle + C_\delta \| \nabla \varepsilon \|_{L^\infty}^2 \| \nabla \varepsilon \|_{L^2}^2 + \delta \| \nabla \varepsilon \|_{H^2}^2, \]
\[ \langle (\varepsilon \cdot \nabla \varepsilon) \varepsilon, (\varepsilon \cdot \nabla \varepsilon), \Delta \varepsilon \rangle + C_\delta \| \varepsilon \|_{L^\infty}^2 \| \nabla \varepsilon \|_{L^2}^2 + \delta \| \nabla \varepsilon \|_{H^2}^2, \]
\[ \leq C_\delta \| \nabla \varepsilon \|_{L^\infty}^2 (\| \nabla \varepsilon \|_{L^\infty}^2 + \| \varepsilon \|_{L^\infty}^2) (\| \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2 + \delta \| \nabla \varepsilon \|_{H^2}^2). \]

Summing up, we conclude that
\[ \frac{1}{2} \frac{d}{dt} \| \Delta \varepsilon \|_{L^\infty}^2 + \mu \| \nabla \Delta \varepsilon \|_{L^\infty}^2 \leq \mu \| \nabla \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2, \]
\[ \frac{1}{2} \frac{d}{dt} \| \varepsilon \|_{L^\infty}^2 + \nu \| \nabla \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2 \leq \mu \| \nabla \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2, \]
\[ \frac{1}{2} \frac{d}{dt} \| \nabla \varepsilon \|_{L^\infty}^2 + \mu \| \nabla \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2, \]
\[ \leq C (\| \nabla \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2) (\| \nabla \varepsilon \|_{L^\infty}^2 + \| \nabla \varepsilon \|_{H^2}^2) \]

**Step 3.** Existence of the solution
As $s \geq 2$, we deduced from Sobolev embedding and (4.17) that
\[ \frac{d}{dt} E_s (\varepsilon, \nabla \varepsilon) + \nu \| \nabla \varepsilon \|_{L^\infty}^2 + \nu \| \nabla \varepsilon \|_{H^2}^2 \leq \mathcal{F} (E_s (\varepsilon, \nabla \varepsilon)), \]
where $\mathcal{F}$ is an increasing function with $\mathcal{F}(0) = 0$. This implies that there exists $T > 0$ depending only on $E_s (\varepsilon, \nabla \varepsilon) = 0$ such that for any $t \in [0, \min(T, T_{\delta})]$, $E_s (\varepsilon, \nabla \varepsilon) \leq 2 E_s (\varepsilon, \nabla \varepsilon)$, which in turn ensures that $T_{\delta} \geq T$ by a continuous argument. Thus, we obtain an uniform estimate for the approximate solution on $[0, T]$. Then the existence of the solution can be deduced by a standard compactness argument.

**Step 4.** Uniqueness of the solution
Let $(\varepsilon_1, n_1)$ and $(\varepsilon_2, n_2)$ be two solutions of the system (1.1) with the same initial data. We denote
\[ \delta_\varepsilon = \varepsilon_1 - \varepsilon_2, \quad \delta_n = n_1 - n_2, \quad \delta_h = h_1 - h_2, \quad \delta_D = D_1 - D_2, \quad \delta_\Omega = \Omega_1 - \Omega_2. \]
Then $(\delta_\varepsilon, \delta_n)$ satisfies
\[ \frac{\partial \delta_\varepsilon}{\partial t} + \varepsilon_1 \cdot \nabla \delta_\varepsilon + \delta_\varepsilon \cdot \nabla \varepsilon_2 = -\nu \Delta \delta_\varepsilon + \Delta \cdot (\sigma_1 (\varepsilon_1, n_1) - \sigma_1 (\varepsilon_2, n_2)), \]
\[ \frac{\partial \delta_n}{\partial t} + \varepsilon_1 \cdot \nabla \delta_n + \delta_\varepsilon \cdot \nabla n_2 = -n_1 \times (\Omega_1 \cdot n_1 - \delta n) \times n_1 + n_2 \times (\Omega_2 \cdot n_2 - \delta n) \times n_2). \]
We make $L^2$ energy estimate for $\delta_\nu$ to get
\[
\frac{1}{2} \frac{d}{dt} \| \delta_\nu \|^2_{L^2} + \nu \| \nabla \delta_\nu \|^2_{L^2} = -\langle \delta_\nu \cdot \nabla \nu_2, \delta_\nu \rangle + \langle \nabla \cdot (\sigma_1(\nu_1, n_1) - \sigma_1(\nu_2, n_2)), \delta_\nu \rangle \\
+ \langle \nabla \cdot (\sigma_2(n_1) - \sigma_2(n_2)), \delta_\nu \rangle + \langle \nabla \cdot (\sigma^E(n_1) - \sigma^E(n_2)), \delta_\nu \rangle \\
= R_1 + R_2 + R_3 + R_4,
\]
and make $H^1$ energy estimate for $\delta_n$ to get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \delta_n \|^2_{L^2} = \langle \nu_1 \cdot \nabla \delta_n + \delta_\nu \cdot \nabla n_2, \Delta \delta_n \rangle \\
+ \langle n_1 \times ((\Omega_1 \cdot n_1 - \mu_1 h_1 - \mu_2 D_1 \cdot n_1) \times n_1) \\
- n_2 \times ((\Omega_2 \cdot n_2 - \mu_1 h_2 - \mu_2 D_2 \cdot n_2) \times n_2), \Delta \delta_n \rangle \\
= S_1 + S_2.
\]
Now we estimate $R_1, \cdots, R_4$. It is easy to see that
\[
R_1 \leq \| \nabla \nu_2 \|_{L^\infty} \| \delta_\nu \|^2_{L^2}, \\
R_4 \leq C (\| \nabla n_1 \|_{L^\infty} + \| \nabla n_2 \|_{L^\infty}) \| \nabla \delta_n \|_{L^2} \| \nabla \delta_\nu \|_{L^2}.
\]
By (3.3), we have
\[
R_2 = -\langle \sigma_1(\delta_n, n_1), \nabla \delta_\nu \rangle - \langle \sigma_1(\nu_2, n_1) - \sigma_1(\nu_2, n_2), \nabla \delta_\nu \rangle \\
\leq C \| \nabla \nu_2 \|_{L^3} \| \nabla \delta_n \|_{L^2} \| \nabla \delta_\nu \|_{L^2}.
\]
For $R_3$, we have
\[
R_3 = \mu_2 \langle n_1 \times (h_1 \times n_1) n_1 - n_2 \times (h_2 \times n_2) n_2, \delta_D \rangle \\
+ \langle n_1 \times (h_1 \times n_1) n_1 - n_2 \times (h_2 \times n_2) n_2, \delta_\Omega \rangle \\
= \mu_2 \langle n_1 \times (\delta_h \times n_1) n_1, \delta_D \rangle + \langle n_1 \times (\delta_h \times n_1) n_1, \delta_\Omega \rangle \\
+ \mu_2 \langle n_2 \times (h_2 \times n_2) n_1 - n_2 \times (h_2 \times n_2) n_2, \delta_D \rangle \\
+ \langle n_1 \times (h_2 \times n_1) n_1 - n_2 \times (h_2 \times n_2) n_2, \delta_\Omega \rangle \\
\leq \mu_2 \langle n_1 \times (\delta_h \times n_1) n_1, \delta_D \rangle + \langle n_1 \times (\delta_h \times n_1) n_1, \delta_\Omega \rangle \\
+ C \| \Delta n_2 \|_{L^3} \| \nabla \delta_n \|_{L^2} \| \nabla \delta_\nu \|_{L^2}.
\]
Let us turn to estimate $S_1$ and $S_2$. It is easy to see that
\[
S_1 \leq \| \nabla \nu_1 \|_{L^\infty} \| \nabla \delta_n \|^2_{L^2} + C (\| \nabla n_2 \|_{L^\infty} + \| \Delta n_2 \|_{L^3}) \| \nabla \delta_\nu \|_{L^2} \| \nabla \delta_n \|_{L^2},
\]
and for $S_2$, we have
\[
S_2 = \langle n_1 \times ((\sigma_1 \cdot n_1 - \mu_1 \delta_h - \mu_2 \sigma_D \cdot n_1) \times n_1), \Delta \delta_n \rangle \\
+ \langle n_1 \times ((\Omega_1 \cdot n_1 - \mu_1 h_2 - \mu_2 D_2 \cdot n_1) \times n_1) \\
- n_2 \times ((\Omega_2 \cdot n_2 - \mu_1 h_2 - \mu_2 D_2 \cdot n_2) \times n_2), \Delta \delta_n \rangle \\
\leq \langle n_1 \times ((\delta_\Omega \cdot n_1 - \mu_\delta_h - \mu_2 \sigma_D \cdot n_1) \times n_1), \Delta \delta_n \rangle \\
+ C (\| \nabla \nu_2 \|_{L^\infty} + \| \Delta \nu_2 \|_{L^3} + \| \Delta n_2 \|_{L^2} + \| \nabla \Delta n_2 \|_{L^3}) \| \nabla \delta_n \|^2_{L^2}.
\]
Summing up all the above estimates, we obtain
\[
\frac{d}{dt} \| \delta_\nu \|^2_{L^2} + \| \nabla \delta_n \|^2_{L^2} \leq C (\| \delta_\nu \|^2_{L^2} + \| \nabla \delta_n \|^2_{L^2}),
\]
which implies that $\delta_\nu(t) = 0$ and $\delta_n(t) = 0$ on $[0, T]$.

**Step 5.** Blow-up criterion
First of all, the solution of (1.1) satisfies $|n| = 1$ if $|n_0| = 1$. Thus, it holds that
\[(4.18)\quad n \times (\Delta n \times n) = \Delta n + |\nabla n|^2 n.\]

Hence, $I_4$ in (4.10) can be written as
\[I_4 = -\mu_1 \langle \Delta^{s+1} n, \Delta^s (|\nabla n|^2 n), \Delta^{s+1} n \rangle\]
\[= -\mu_1 \langle \Delta^{s+1} n, \Delta^s + 1 n \rangle + \mu_1 \langle \Delta^s (|\nabla n|^2) n, \Delta^s (|\nabla n|^2 n), \nabla \Delta^s n \rangle,\]
which along with Lemma 4.1 gives
\[I_4 \leq -\mu_1 \langle \Delta^{s+1} n, \Delta^{s+1} n \rangle + C \|\nabla n\|_{H^L} \|\nabla n\|_{H^{L+2s}} \|\Delta^{s+1} n\|_{L^2}\]
\[+ C \|\nabla n\|^2_{L^{L+1}} \|\nabla n\|^2_{H^{2s}}.\]

On the other hand, we can bound $I_{3}$ as
\[I_{3} \leq C (\|\nabla n\|^2_{H^{L+1}} + \|\nabla v\|_{H^{L+1}}) (\|\nabla n\|^2_{H^{L+1}} + \|v\|^2_{H^{L+2s}} + \delta \|\nabla \Delta^s v\|_{L^2}^2\]
by using the commutator estimate like
\[\|\nabla [\Delta^s, f] g\|_{L^2} \leq C (\|\Delta^s \nabla f\|_{L^2} \|\nabla g\|_{L^\infty} + \|\nabla f\|_{L^\infty} \|\Delta^s \nabla g\|_{L^2}).\]

From the proof in Step 2, we can deduce that
\[\frac{d}{dt} E_s(v, n) \leq C (1 + \|\nabla n\|^2_{L^{L+1}} + \|\nabla v\|_{L^{L+1}}) E_s(v, n).\]

Recall the following Logarithmic Sobolev inequality from [2]:
\[\|\nabla v\|_{L^\infty} \leq C (1 + \|\nabla v\|_{L^2} + \|\nabla \times v\|_{L^\infty}) \log (2 + \|v\|_{H^k})\]
for any $k \geq 3$. Thus, we have
\[\frac{d}{dt} E_s(v, n) \leq C (1 + \|\nabla v\|_{L^2} + \|\nabla n\|^2_{L^{L+1}} + \|\nabla \times v\|_{L^\infty}) \log (2 + E_s(v, n)) E_s(v, n).\]

Applying Gronwall’s inequality twice, we infer that
\[E_s(v, n) \leq E_s(v_0, n_0) \exp \exp \left( C \int_0^t (1 + \|\nabla v\|_{L^2} + \|\nabla n\|^2_{L^{L+1}} + \|\nabla \times v\|_{L^\infty}) dt \right)\]
for any $t \in [0, T^*)$. Especially, if $T^* < +\infty$ and
\[\int_0^{T^*} (\|\nabla n\|^2_{L^{L+1}} + \|\nabla \times v\|_{L^\infty}) dt < +\infty,\]
then $E_s(v, n)(t) \leq C$ for any $t \in [0, T^*)$. Thus, the solution can be extended after $t = T^*$, which contradicts the definition of $T^*$. The blow-up criterion follows.

5. Global well-posedness for small initial data

This section is devoted to the proof of Theorem 1.2. Assume that $(v, n)$ is the solution of the system (1.1) on $[0, T]$ obtained in Theorem 1.1. We define
\[E_s(v, n) \equiv \|\nabla n\|^2_{L^2} + \|\nabla \Delta^s n\|^2_{L^2} + \|v\|^2_{L^2} + \|\Delta^s v\|^2_{L^2},\]
\[D_s(v, n) \equiv \mu_1 \|\Delta n\|^2_{L^2} + \mu_1 \|\Delta^{s+1} n\|^2_{L^2} + \nu \|\nabla v\|^2_{L^2} + \nu \|\Delta^s \nabla v\|^2_{L^2}.\]

By the interpolation, there exist $c_0 > 0$ and $C_0 > 0$ such that
\[c_0 (\|\nabla n\|^2_{H^{L+2s}} + \|v\|^2_{H^{L+2s}}) \leq E_s(v, n) \leq C_0 (\|\nabla n\|^2_{H^{L+2s}} + \|v\|^2_{H^{L+2s}}),\]
\[c_0 (\|\Delta n\|^2_{H^{L+2s}} + \|\nabla v\|^2_{H^{L+2s}}) \leq D_s(v, n) \leq C_0 (\|\Delta n\|^2_{H^{L+2s}} + \|\nabla v\|^2_{H^{L+2s}}).\]
The basic energy-dissipation law tells us that
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\mathbf{v}|^2 + |\nabla \mathbf{n}|^2 d\mathbf{x} + \int_{\mathbb{R}^3} (\nu |\nabla \mathbf{v}|^2 + \mu_1 |\mathbf{n} \times \mathbf{v}|^2) d\mathbf{x} \leq 0, \]
which along with (4.18) implies that
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\mathbf{v}|^2 + |\nabla \mathbf{n}|^2 d\mathbf{x} + \int_{\mathbb{R}^3} (\nu |\nabla \mathbf{v}|^2 + \mu_1 |\Delta \mathbf{n}|^2) d\mathbf{x} \leq \mu_1 \int_{\mathbb{R}^3} |\nabla \mathbf{n}|^4 d\mathbf{x} \leq C \|\nabla \mathbf{n}\|_{L^2} \|\Delta \mathbf{n}\|^2_{L^2} \leq C E_s(\mathbf{v}, \mathbf{n}) D_s(\mathbf{v}, \mathbf{n}). \]

Similar to (4.111), we have
\[ \frac{1}{2} \frac{d}{dt} \langle \nabla \Delta^s \mathbf{n}, \nabla \Delta^s \mathbf{n} \rangle = \langle \Delta^s (\mathbf{v} \cdot \nabla \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle + \langle \Delta^s \nabla \mathbf{n}, \Delta^s \nabla \mathbf{n} \rangle - \mu_2 \langle \Delta^s [\mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n})], \Delta^s \nabla \mathbf{n} \rangle - \mu_1 \langle \Delta^s [\mathbf{n} \times (\mathbf{n} \times \mathbf{n})], \Delta^s \nabla \mathbf{n} \rangle \]
\[ = I_1 + I_2 + I_3 + I_4. \]

We get by Lemma 4.1 and Sobolev embedding that
\[ I_1 \leq \|\Delta^s (\mathbf{v} \cdot \nabla \mathbf{v})\|_{L^2} \|\Delta^s \nabla \mathbf{n}\|_{L^2} \leq C \|\nabla \mathbf{n}\|_{L^\infty} \|\Delta^s \nabla \mathbf{n}\|_{L^2} \|\Delta^s \nabla \mathbf{n}\|_{L^2} \leq C E_s(\mathbf{v}, \mathbf{n}) \frac{1}{2} D_s(\mathbf{v}, \mathbf{n}); \]
and
\[ I_2 = \langle \mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle \]
\[ + \langle \Delta^s [\mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n})], \Delta^s \nabla \mathbf{n} \rangle - \langle \mathbf{n} \times ((\Delta^s \nabla \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle \]
\[ \leq \langle \mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle \]
\[ + C \|\nabla \mathbf{n}\|_{L^\infty} \|\nabla \mathbf{v}\|_{H^{2s-1}} \|\Delta^s \nabla \mathbf{n}\|_{L^2} \|\Delta^s \nabla \mathbf{n}\|_{L^2} \]
\[ \leq \langle \mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle + C E_s(\mathbf{v}, \mathbf{n}) \frac{1}{2} D_s(\mathbf{v}, \mathbf{n}); \]
\[ I_3 \leq - \mu_2 \langle \mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle + C E_s(\mathbf{v}, \mathbf{n}) \frac{1}{2} D_s(\mathbf{v}, \mathbf{n}); \]

Similar to Step 4 in Section 4, we have
\[ I_4 \leq - \mu_1 \langle \Delta^s \nabla \mathbf{n}, \Delta^s \nabla \mathbf{n} \rangle + C \|\nabla \mathbf{n}\|_{L^\infty} \|\nabla \mathbf{v}\|_{L^2} \|\Delta^s \nabla \mathbf{n}\|_{H^1} \|\Delta^s \nabla \mathbf{n}\|_{L^2} \]
\[ \leq - \mu_1 \langle \Delta^s \nabla \mathbf{n}, \Delta^s \nabla \mathbf{n} \rangle + C (E_s(\mathbf{v}, \mathbf{n}) \frac{1}{2} + E_s(\mathbf{v}, \mathbf{n})) D_s(\mathbf{v}, \mathbf{n}). \]

Summing up (5.20)–(5.24), we obtain
\[ \frac{1}{2} \frac{d}{dt} \langle \nabla \Delta^s \mathbf{n}, \nabla \Delta^s \mathbf{n} \rangle + \mu_1 \langle \Delta^s \nabla \mathbf{n}, \Delta^s \nabla \mathbf{n} \rangle \]
\[ \leq \langle \mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle - \mu_2 \langle \mathbf{n} \times ((\nabla \cdot \mathbf{n}) \times \mathbf{n}), \Delta^s \nabla \mathbf{n} \rangle \]
\[ + C (E_s(\mathbf{v}, \mathbf{n}) \frac{1}{2} + E_s(\mathbf{v}, \mathbf{n})) D_s(\mathbf{v}, \mathbf{n}). \]

Now we consider the estimate for the velocity. By Step 2 in Section 4, we have
\[ \frac{1}{2} \frac{d}{dt} \langle \Delta^s \mathbf{v}, \Delta^s \mathbf{v} \rangle + \nu \langle \nabla \Delta^s \mathbf{v}, \nabla \Delta^s \mathbf{v} \rangle \]
\[ = - \langle \Delta^s (\mathbf{v} \cdot \nabla \mathbf{v}), \Delta^s \mathbf{v} \rangle + \langle \Delta^s (\nabla \mathbf{n} \cdot \nabla \mathbf{n}), \Delta^s \nabla \mathbf{v} \rangle \]
\[ - \langle \Delta^s (\mathbf{n} \cdot \mathbf{n}) \nabla \mathbf{n} + \mathbf{n} \cdot \mathbf{n} \nabla \mathbf{n}, \Delta^s \mathbf{D} \rangle \]
\[ + \frac{\mu_2}{2} \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n}) \nabla \mathbf{n} + \mathbf{n} \times (\mathbf{h} \times \mathbf{n}) \nabla \mathbf{n}), \Delta^s \mathbf{D} \rangle \]
\[ - \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n}) \nabla \mathbf{n} + \mathbf{n} \times (\mathbf{h} \times \mathbf{n}) \nabla \mathbf{n}), \Delta^s \mathbf{D} \rangle \]
\[ = II_1 + II_2 + II_3 + II_4 + II_5. \]
We get by Lemma 4.1 and Sobolev embedding that
\begin{equation}
I_1 \leq C \|v\|_{L^\infty} \|\Delta^s v\|_{L^2} \|\Delta^s \nabla v\|_{L^2} \leq CE_s(v,n)^{\frac{1}{2}} D_s(v,n);
\end{equation}
\begin{equation}
I_2 \leq C \\|\nabla v\|_{L^\infty} \|\Delta^s \nabla v\|_{L^2} \|\Delta^s \nabla v\|_{L^2} \leq CE_s(v,n)^{\frac{1}{2}} D_s(v,n);
\end{equation}
and by Proposition 2.2,
\begin{equation}
I_3 \leq -\left( (\beta_1(nn : \Delta^s D)n + \beta_2 \Delta^s D + \frac{\beta_3}{2} (n \Delta^s D \cdot n + \Delta^s D \cdot nn)), \Delta^s D \right) \\
+ C(\|\nabla v\|_{L^\infty} \|\Delta^s n\|_{L^2} + \|\nabla n\|_{L^\infty} \|\nabla v\|_{H^{2s-1}}) \|\Delta^s \nabla v\|_{L^2}
\end{equation}
\begin{equation}
\leq CE_s(v,n)^{\frac{1}{2}} D_s(v,n);
\end{equation}
Similarly, we have
\begin{equation}
I_4 + I_5 \leq \mu_2 \langle n \times (\Delta^{s+1} n \times n), \Delta^s D \cdot n \rangle - \langle n \times (\Delta^{s+1} n), \Delta^s \Omega \cdot n \rangle \\
+ CE_s(v,n)^{\frac{1}{2}} D_s(v,n).
\end{equation}
Summing up (5.26)-(5.30), we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \langle \Delta^s v, \Delta^s v \rangle + \nu \langle \nabla \Delta^s v, \nabla \Delta^s v \rangle \\
\leq \mu_2 \langle n \times (\Delta^{s+1} n \times n), \Delta^s D \cdot n \rangle - \langle n \times (\Delta^{s+1} n), \Delta^s \Omega \cdot n \rangle \\
+ CE_s(v,n)^{\frac{1}{2}} D_s(v,n).
\end{equation}
It follows from (5.19), (5.25) and (5.31) that
\begin{equation}
\frac{1}{2} \frac{d}{dt} E_s(v,n) + D_s(v,n) \leq C (E_s(v,n)^{\frac{1}{2}} + E_s(v,n)) D_s(v,n).
\end{equation}
This implies that there exists an \( \varepsilon_0 > 0 \) such that if \( E_s(v_0,n_0) \leq \varepsilon_0 \), then
\begin{equation}
E_s(v,n)(t) \leq E_s(v_0,n_0) \quad \text{for any } t \in [0,T].
\end{equation}
Thus, the solution is global in time by blow-up criterion in Theorem 1.1.

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