Nonlinear Sigma Models and Symplectic Geometry on Loop Spaces of (Pseudo)Riemannian Manifolds

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Abstract
In this paper we consider symplectic and Hamiltonian structures of systems generated by actions of sigma-model type and show that these systems are naturally connected with specific symplectic geometry on loop spaces of Riemannian and pseudoRiemannian manifolds.

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1 Introduction
We consider two-dimensional systems generated by the sigma-model actions of the form
\[
S = \int \left( \frac{1}{2} g_{ij}(\phi) \dot{\phi}_x^i \dot{\phi}_t^j + U(\phi) \right) dx dt
\]  
(1)
where \( g_{ij}(\phi) \) is an arbitrary Riemannian or pseudoRiemannian metric in a \( N \)-dimensional target space \( M \) with local coordinates \( (\phi^1, ..., \phi^N) \), the functions \( \dot{\phi}^i(x,t) \) are bosonic fields, the field \( U(\phi) \) is arbitrary here.

It was shown in [1] (see also [2–5]) that any system generated by the action of the form (1) has always natural Hamiltonian structure given by explicit symplectic form \( \omega(\xi, \eta) = \int_\gamma < \xi, \nabla_\gamma \eta > \) on the loop space of the manifold \( M \). Correspondingly, integrable systems with the action (1), generally speaking, could be selected (see [2,3]) by description of symplectic structures (or Hamiltonian structures) compatible with the given symplectic form \( \omega(\xi, \eta) \). In fact, this is the specific problem (see [1–3]) of symplectic geometry on loop spaces. Usually, for integrable systems, second symplectic structure compatible with the first one (if it exists there) can be received from another but equivalent action for considered system.

2 Sigma Models and Symplectic Forms on Loop Spaces
Any system generated by the action (1) has the form
\[
g_{ij}(\phi) \dot{\phi}_x^i + g_{is}(\phi) \Gamma_{jk}^s(\phi) \dot{\phi}_x^j \dot{\phi}_t^k = \frac{\partial U}{\partial \phi^i}
\]  
(2)
where \( \Gamma_{jk}^s(\phi) \) are the coefficients of the Levi-Civita connection defined by the metric \( g_{ij}(\phi) \) (i.e. the only symmetric connection compatible with the metric). As it was shown in [1] the homogeneous matrix differential operator of the first order
\[
M_{ij} = g_{ij}(\phi) \frac{d}{dx} + g_{is}(\phi) \Gamma_{jk}^s(\phi) \phi_x^k
\]  
(3)
is a symplectic operator for any Riemannian or pseudoRiemannian metric \( g_{ij}(\phi) \). In other words, the inverse operator \( K^{ij} = (M^{-1})^{ij} \), such that \( K^{ij} M_{jk} = \delta_k^i \), gives always the nonlocal Poisson bracket
\[
\{ \phi^i(x), \phi^j(y) \} = K^{ij}(\phi(x)) \delta(x-y)
\]  
(4)
and hence we have always the following nonlocal Hamiltonian representation for the system (2):
\[
\phi_t^i = K^{ij} \frac{\partial U}{\partial \phi^j} \equiv \{ \phi^i(x), \int U(\phi) dx \}
\]  
(5)
The Poisson bracket (4) is nonlocal and very complicated and it is much convenient to use local symplectic representation
\[ M_{ij} \dot{\phi}_i = \frac{\partial U}{\partial \dot{\phi}_j} \]
and study the corresponding symplectic geometry. The local symplectic operator (3) gives (see [1]) a natural local symplectic form \( \omega \) on loop space of the manifold \( M \). Really, we have \( N \)-dimensional Riemannian or pseudoRiemannian manifold \( M \) with metric \( g_{ij}(\phi) \). Consider the loop space of the manifold \( M \). It means here the space \( L(M) \) of all smooth parametrized mappings \( \gamma : S^1 \to M, \gamma(x) = \{ \phi^i(x), x \in S^1 \} \). The tangent space of \( L(M) \) in its point (a loop) \( \gamma \) consists of all smooth vector fields \( \xi^i, 1 \leq i \leq N \), defined along the loop \( \gamma \). We denote it here by \( T_\gamma L(M) \) (we note that \( \xi^i(\gamma(x)) \in T_{\gamma(x)}M, \forall x \in S^1 \), where \( T_{\gamma(x)}M \) is the tangent space of the manifold \( M \) in the point \( \gamma(x) \)).

Consider natural bilinear invariant form \( \omega \) on the loop space \( L(M) \):
\[ \omega(\xi, \eta) = \int_\gamma < \xi, \nabla_\dot{\gamma} \eta > \quad (6) \]
where \( \xi, \eta \in T_\gamma L(M), < \xi, \eta > = g_{ij} \xi^i \eta^j \) is the natural scalar product on the tangent space \( TM \) of (pseudo)Riemannian manifold \( (M, g_{ij}), \dot{\gamma} = \{ \dot{\phi}_x \} \) is the velocity vector of \( \gamma(x), \nabla_\dot{\gamma} \) is the operator of covariant derivation along the loop \( \gamma \) that is generated by the Levi-Civita connection \( \Gamma^i_{jk}(\phi) \). The bilinear form \( \omega \) satisfies (see [1]) to the following necessary identities:
1. \( \omega(\xi, \eta) = -\omega(\eta, \xi) \) (skew-symmetry),
2. \( (d\omega)(\xi, \eta, \zeta) = 0 \) (i.e. the 2-form \( \omega \) is closed).

The differential \( d \) here is given by some infinite-dimensional generalization of the usual exterior differential for Lie algebra of vector fields:
\[ (d\omega)(\xi, \eta, \zeta) \equiv \sum_{(\xi, \eta, \zeta)} \left\{ \int_{S^1} \delta \omega(\eta, \zeta) dx + \omega(\xi, [\eta, \zeta]) \right\} \quad (7) \]
The sign \( \sum_{(\xi, \eta, \zeta)} \) means here that the sum is taken with respect to the all cyclic permutations of elements \( (\xi, \eta, \zeta) \).

Thus \( \omega(\xi, \eta) \) is a infinite-dimensional symplectic form on the loop space \( L(M) \). It is easy to show that \( \omega(\xi, \eta) \) is generated by the symplectic operator (3), i.e.
\[ \omega(\xi, \eta) = \int_{S^1} \xi^i M_{ij} \eta^j dx \quad (8) \]
The corresponding symplectic representation (see also [5]) of the system (2) has the form:
\[ \omega(\delta \phi, \phi_t) = \delta H, \quad (9) \]
\[ H = \int U(\phi) dx, \]

where the relation (9) is valid for arbitrary variations \( \delta \phi^i \) of the fields \( \phi^i \). \( H \) is a functional on the loop space \( L(M) \). Thus any system (2) can be given by the relation (9) on the loop space \( L(M) \).

It must be noted that the bilinear form (6) is degenerated. The null-space of (6) consists of all vector fields parallel along loop \( \gamma \). In particular, the velocity vector field \( \phi^i x^i \) belongs to the null-space of (6) if and only if \( \gamma \) is a geodesic loop on the target space \( M \). We can consider the corresponding factor space \( L(M)/G \) that means that two loops on the target space \( M \) are equivalent if they can be coincide after some parallel translation on the manifold \( M \). The form (6) is nondegenerated on the space \( L(M)/G \).

3 Sigma Models with Nontrivial Torsion and Symplectic Structures on Loop Spaces
Consider now more general systems generated by the actions of the form

\[ S = \int \left( \frac{1}{2} a_{ij}(\phi) \phi^i_x \phi^j_t + U(\phi) \right) dx dt \]

where \( a_{ij}(\phi) \) is an arbitrary tensor, i.e. \( a_{ij}(\phi) = g_{ij}(\phi) + f_{ij}(\phi) \), where \( g_{ij}(\phi) \) is a symmetric tensor (a metric on manifold \( M \)) and \( f_{ij}(\phi) \) is a skew-symmetric tensor on \( M \).

Then the corresponding Lagrangian system has also always symplectic representation defined by the symplectic form

\[ \omega(\xi, \eta) = \int_\gamma < \xi, \nabla_\gamma \eta > \]

where \( < \xi, \eta > = g_{ij} \xi^i \eta^j, \nabla_\gamma \) is a covariant derivation generated by some differential geometric connection \( \Gamma^i_{jk}(\phi) \) with nontrivial torsion. It was shown in [1] that if \( \det g_{ij}(\phi) \neq 0 \) then the form (11) is symplectic if and only if the differential geometric connection \( \Gamma^i_{jk}(\phi) \) satisfies to the following conditions:

1. the connection \( \Gamma^i_{jk}(\phi) \) is compatible with the metric \( g_{ij}(\phi) \), i.e.

\[ \nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial \phi^k} - g_{is}(\phi) \Gamma^s_{jk}(\phi) - g_{js}(\phi) \Gamma^s_{ik}(\phi) = 0; \]

2. the torsion tensor \( T_{ijk}(\phi) = g_{is}(\phi) T^s_{jk}(\phi), T^i_{jk}(\phi) = \Gamma^i_{jk}(\phi) - \Gamma^i_{kj}(\phi) \), with respect to all indices and its gradient (or exterior differential of the corresponding exterior differential form) vanishes: \( (dT)_{ijkm} = 0 \), in other words, the torsion tensor \( T_{ijk}(\phi) \) defines a closed 3-form on the manifold \( M \).
Any closed 3-form on the (pseudo)Riemannian manifold \((M, g_{ij})\) gives the only compatible with the metric \(g_{ij}(\phi)\) differential geometric connection \(\Gamma_{jk}^i(\phi)\) with nontrivial torsion tensor determined explicitly by this closed 3-form and correspondingly any closed 3-form on \(M\) generates a symplectic form \((11)\) with nontrivial torsion on \(L(M)\). The corresponding symplectic operator with nontrivial torsion have the form

\[
M_{ij} = g_{ij}(\phi) \frac{d}{dx} + g_{is}(\phi)\Gamma_{jk}^s(\phi)\phi^k_x
\]

(12)

where locally we have

\[
\Gamma_{jk}^i(\phi) = \frac{1}{2} g^{is}(\phi) \left( \frac{\partial g_{sk}}{\partial \phi^j} + \frac{\partial g_{js}}{\partial \phi^k} - \frac{\partial g_{jk}}{\partial \phi^s} + T_{sjk}(\phi) \right),
\]

\[
T_{ijk}(\phi) = \frac{1}{2} \left( \frac{\partial f_{ij}}{\partial \phi^k} + \frac{\partial f_{jk}}{\partial \phi^i} + \frac{\partial f_{ki}}{\partial \phi^j} \right),
\]

(14)

\(f_{ij}(\phi) = -f_{ji}(\phi), T_{ijk}(\phi) = \frac{1}{2} (df)_{ijk}\).

The Hamiltonian (symplectic) operators compatible with \((12)\) and \((3)\) will be considered in another publications.

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