A look to differential

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Abstract. This paper reveals results of a research related to the presentation of the concept of differential for functions of a variable, from some calculus textbooks. Given that the concept of differential is closely linked to the concepts of derivative, integral and differential equation, the objective is to analyze the different ways of presenting the concept from those instances, in order to determine whether the presentation facilitates its understanding. The research follows a qualitative methodology based on a bibliographic review and conceptual didactic analysis, proceeding from content analysis corresponding to the notion of differential. It was found that the concept of differential in textbooks is presented in a geometric context, as an increment of a variable and as a function, dispossessed of its arithmetic meaning, being impossible to suppress the concept in the formal construction of the calculus.

1. Introduction

The teaching of calculus and differential equations in university courses is characterized by the execution of algebraic processes, consisting of applying formulas for derivatives, integration techniques and searching for a family of functions that correspond to the general solution of a differential equation. Any of the issues mentioned involves differentials, a concept that appears with different forms of representation according to the situation addressed, whose significance in some cases goes unnoticed by students and in others it is presented as an essential component in the expression of the derivative or the integral, without deepen their importance very much, fact reflected in their exposition in textbooks.

When studying calculus, the concept of differential is taught after introducing the concept of derivative, but the calculation texts usually define the concept after having developed the applications of the derivative, or reserve it for the applications of the integral.

Historically, the concept of differential preceded the derivative, it was glimpsed in the exhaustion method developed by Eudoxus, in Pascal's work related to areas and centroids, and others until the invention of the calculation by Leibniz [1], being used since then in the applications of calculation and the analytical solution of differential equations. The differential of a function really arose from the concept of indivisible, a concept that from a modern point of view was never clearly defined, although it was widely used for several centuries; later, the differential of a function was cemented in the infinitesimals, understood as something of an extremely small constant magnitude that, however, was not zero.

2. Background

Several works on the teaching of the calculus [2] point out that, the difficulties encountered in the presentation of the differential and integral calculus originate in the algorithmic form as the concept of differential through the texts is faced, in addition to a lack of explanation about what is represented by symbols to express derivatives and differentials, however, the important thing for the student is to know
when and why it is necessary to use them, to know the strategies to apply differentials in solving problems, be aware of the hypothetical nature of the differentials and effectively assess the role played by the differentials in physics.

The expressions $\int f(x) \, dx$, $f(x, y) \, dx + g(x, y) \, dy=0$ or $dy=\frac{f'(x)}{x} \, dx$, involve differentials that the student sees as a form of mathematical syntax, expressions symbolic, fictitious entities or representation to make sense of said expressions [3]; therefore, when studying calculus, in the initial courses does not develop a true understanding of this concept [4], for the student the differential has meaning to indicate the variable of integration, or when interpreted geometrically under the infinitesimal perspective, such is the case of the area differential, where texts intended for teaching make use of this fact in modeling, but the syntax is inherited from the contributions of the Bernoulli to the calculation, $\int f(x) \, dx=u$ means $du=f(x)$ and $\frac{dy}{dx}=k \frac{dx}{x}$ transcribe $y=k \log \frac{x}{a}$.

Orton [5] examined the conceptual errors of a group of calculus students, and found that students, when solving problems of calculus applications, often confuse the differential $dx$ with the rate of change $\Delta x$ of the variable $x$, there is difficulty in understanding the meaning of $dx$ in $\int f(x) \, dx$ and $\frac{dy}{dx}$.

Lady [3] points out some objections regarding the definition of differential given in the calculation books; Valdivé [6], presented a proposal for the teaching of calculus, based on the importance of understanding the meaning of the differential, attributes the difficulty in learning the calculus to the complexity and nature of the concepts, difficulty linked in some cases to knowledge of the professors in the epistemological aspects, among them the infinitesimal. The teaching of calculus must focus on showing the difficulties that the construction of the mathematical notion of differential has had in its evolution, from the increment and the infinitesimals as the infinitely small.

Hernández [7] shows cognitive obstacles present in texts of differential equations, one of them occurs when defining the first-order differential equation uses the notation $\frac{dy}{dx}=f(x, y)$ with the derivative being an operator, and then When obtaining the solution with the separable variable method, the same operator is considered as a ratio of differentials, and there is an inconsistency between the two situations.

For Thurston [8], the derivative is understood by students in seven ways, one of which is the relationship between the infinitesimal change in the value of a function and the infinitesimal change in the function; Hu and Rebello [4] attribute meaning to the differential learning by students, when it represents a physical attribute, a small length $dx$, an infinitesimal charge $dq$ or electric field $dE$ or a small portion of a whole; Bajracharya [9], identifies the difficulties related to the fundamental theorem of calculation when it should be applied in argumentation and solving physical problems, teaching based on infinitesimals is realistic and helps to involve students by allowing their intuition to guide them correctly [10].

3. Theoretical framework

The differential notion is linked to the infinitesimal being a fundamental part of the structure of the calculation [11]. Informally an infinitesimal is a very small quantity close to zero, or an infinitely small quantity [12]; a number is said to be infinitely small or infinitesimal, if $-a < \varepsilon < a$ for each positive real number a being $\varepsilon$ different from zero [13]. Bell [14] defines the infinitesimal as the smallest part in which a continuum can be fractioned, for example, a line segment, "infinitesimals other than zero can exist and will exist only in a potential sense, however, as we will see, this potential existence will be sufficient for the development of infinitesimal analysis in smooth worlds".

For Leibniz, the infinitesimal has a metaphysical connotation, they are infinitely small quantities, smaller than any other fixed quantity whose value is smaller than any other number without ever being zero [15], from that infinitesimal, the differential of a magnitude $x$, denoted by $dx$, is an infinitesimal variation of that magnitude, which can be as small as you want without being never zero. The variation of a variable is called increment, so things, the increase of the variable $x$ is $\Delta x$, which is a difference between two values of $x$.

According to this epistemological foundation, the differential is subordinated to the infinitesimal approximation of the increment of a variable, if $y$ is a quantity (variable) dependent on the quantity $x$,
then the differential \( \text{d} \) must also be small, therefore the increase \( \Delta y \) it is measured on the function, while the differential \( \text{d}y \) is measured on the tangent by virtue of the definition of the derivative of Leibniz \( \text{dy}/\text{dx} \) as a quotient of infinitesimals. This same strategy was followed by Newton, with the peculiarity that the character of its difference with respect to \( \Delta y \), can only be recognized 'in motion', so to speak, if we consider an increase \( \Delta x \) that approaches zero as infinitesimal, then the difference between \( \text{d}y \) and \( \Delta y \) will be as small as desired even when compared with \( \Delta x \), this interpretation corresponds to the so-called evanescent quantities.

Leibniz justified the existence of infinitesimals using infinitely large integers, larger than all the usual integers, \( \text{d}x = \frac{1}{N} \) where \( N > n \) for all natural number \( n \), these infinitesimals were replaced in modern Cauchy analysis with the notation \( \delta/\varepsilon \) used in the definition of limit and continuity, \( f: [a, b] \rightarrow \mathbb{R} \) is continuous at point \( x_0 \) if: \( (\forall \varepsilon > 0) (\exists \delta > 0) (\forall x \epsilon [a, b], |f(x) - f(x_0)| < \varepsilon \rightarrow |x - x_0| < \delta) \). The infinitesimal of Leibniz was closer to the metaphysical than to the mathematical, it was enough to take an infinitely small amount as necessary without damaging the essence of the infinitesimal, made sustainable by the Archimedean property, consequently for each \( \varepsilon > 0 \) there is a number natural \( n \) such that \( \varepsilon \), or also, if \( 0 < \varepsilon < b \), there exists a natural number \( n \) such that \( ne > b \), and although Leibniz interprets the derivative \( \text{dy}/\text{dx} \) as the quotient of infinitesimals, this expression is only a notation, because of being a quotient in the definition of derivative with step to the limit, the limit of a quotient is the quotient of the limits if they exist, and the derivative could not be reached, this is fundamental in the conceptualization of the differential [16].

In the times of Newton and Leibniz there was no concept of function, therefore both referred to the derivative of a variable, thus it is established with the formula of the derivative of the product of two Leibniz variables, \( d(xy) = y\text{d}x + x\text{d}y \), and in the expressions for the derivatives or fluxions of Newton \( x, y, z \), the increase \( \Delta \) by extrapolation becomes the differential [17], and from there \( \Delta y/\Delta x \) transforms \( \text{dy}/\text{dx} \), this mutation being justified by Cauchy when defining the derivative with the step to the limit; all this, looking for an explanation to the fiction of the infinitesimals, because "the infinitesimal seems to constitute an intermediate mathematical entity between zero and a finite quantity, without being in reality neither of the two [18]".

For Cauchy the infinitely small quantity is a variable whose value decreases indefinitely converging to zero, for this reason infinitesimals are variables or functions characterized by converging towards the zero limit when \( x \) tends to zero; Cauchy later defines the differential based on the derivative \( \text{d}y = f'(x) \text{d}x \), "\( \text{d}x \) being an arbitrary increase (large or small) of the variable and thus became a simple formal instrument, necessary to justify and abbreviate certain demonstrations " [2].

From the geometric point of view, the differential can be seen as a linear approximation (Dray and Manogue, 2010). Since the slope of the tangent line to the graph of \( y = f(x) \), \( \tan \theta = f'(x_0) = \Delta R/h, \Delta R = f'(x)h \), then \( \Delta y \approx f'(x)h \); Now, taking the step to the limit in the leibnizian sense when \( h \rightarrow 0 \), the increment becomes an infinitesimal, \( \Delta R \) is an approximation of \( \Delta y \), and being small, the same \( h \), are interpreted as differentials, then it would be \( \text{d}y = f'(x)\text{d}x \).

According to Courant and Robbins [19], "no difficulty arises, and no mystery exists if we understand well that the symbol \( \text{d} \) of the differential only indicates that the step must be made to the limit: \( \Delta x \rightarrow 0 \) and consequently, \( \Delta y \rightarrow 0 \). Leibniz's notation is, at least, excellent shorthand, which replaces the more complicated explicit notion of the step to the limit. In fact, it is indispensable in certain more advanced aspects of the theory.

4. Methodology
The research presented here follows the qualitative approach of exploratory, descriptive and explanatory type, is based on the conceptual didactic analysis of regressive or scrutinizing type [20], applied to the definition of differential according to four texts of calculation edited at different times, during the last forty years, widely used in university education. It is exploratory because it seeks a first approach to the subject investigated from its relationship with the infinitesimals, through the documentary search of texts and articles related to the subject, carefully chosen to form the state of art and the theoretical
framework; it is descriptive, because it shows the different conceptions about the differential present in the selected texts; and it is explanatory, because it gives pedagogical or didactic reasons about the type of definition exposed by each text based on the antecedent and consequent theory; In addition, the definition of differential exposed in each one of them is analyzed from the epistemological, the geometric, searching for each one the guiding principle by carefully examining its meaning, establishing similarities and differences in the exhibition, and placing each definition in a conceptual category of according to its origin.

The research presented here follows the qualitative approach of exploratory, descriptive and explanatory type, is based on the conceptual didactic analysis of regressive or scrutinizing type [20], and applied to the definition of differential according to four texts of calculation edited at different times, during the last fifty years, widely used in university education. It is exploratory because it seeks a first approach to the subject investigated from its relationship with the infinitesimals, through the documentary search of texts and articles related to the subject, carefully chosen to form the state of art and the theoretical framework; it is descriptive, because it shows the different conceptions about the differential present in the selected texts; and it is explanatory, because it gives pedagogical or didactic reasons about the type of definition exposed by each text based on the antecedent and consequent theory; In addition, the definition of differential exposed in each one of them is analyzed from the epistemological, the geometric, searching for each one the guiding principle by carefully examining its meaning, establishing similarities and differences in the exhibition, and placing each definition in a conceptual category of according to its origin.

The criteria for the choice of the texts were: number of editions, frequency of use in the university courses and recommendations of university professors who have given courses of calculation and differential equations several times; in this way it was determined that the texts with enough editions are Thomas [21], thirteen editions; Stewart [22], eight editions and Leithold [23], seven editions; and Apostol [24], a traditional and widely used text recommended by mathematics teachers. Four reference categories were considered; the first corresponds to the epistemological character when the definition is associated with the infinitesimals, the second considers the differential from a geometric perspective as an approximation of the increment, the third characterizes the differential as a function, and the fourth describes the differential as a notation starting of the increase.

5. Results
The conceptual didactic analysis applied to the nine selected textbooks allowed extracting the information shown below. Thomas defines the derivative using limits and exposes the different notations to be used; for this purpose, he introduces two new variables, dx and dy, "with the property that, if his reason exists, it will be equal to the derivative". Then enter the differential with the following definition: Let \( y = f(x) \) be a differentiable function. The differential \( dx \) is an independent variable. The differential \( dy \) is \( dy = f'(x) dx \). The variable \( dy \) depends on \( x \) and \( dx \).

For Thomas a differential \( dy \) is a dependent variable defined by \( dy = f'(x) dx \), where \( dx \) is an independent variable, a concept that is reached through the function of approximation or linearization of a function \( f \) defined as \( L(x) = f(a) + f'(a)(x - a) \); approximating \( L(x) \) to \( f(x) \) we have the standard linear approximation of \( f \) at \( x = a \); Obviously, \( L(x) = f(a) + f'(a)(x - a) \) is translated in the vicinity of \( x = a \) as the differential, taking \( x - a = h \) arrive at the definition of differential as it is announced: If \( f \) is a derivable function, the differential of a function corresponding to the increment \( h \) of the independent variable is the product \( f'(x)\cdot h \).

Geometrically, for Thomas, the differential \( d \) is the change given by the tangent in the linearization of \( f \) when \( x = a \) changes by a quantity \( dx = \Delta x = h \); additionally, the differential \( dy = f'(x) dx \) expresses how sensitive is the result of before a change in the data for different values of \( x \), \( d \) is the change in the linearization or amount of tangent line that rises or falls according to the case; In other words, the differential gives an approximation to the error in \( f \) when there is an error \( dx = \Delta x \) in the independent variable \( x \).
Regarding the antiderivative presentation process, Thomas ignores the differential when he explains the meaning of $\int f(x)\,dx$; but in spite of this situation, in the substitution formula he is forced to exhibit it when he takes $u = g(x)$ and then $du = g'(x)\,dx$.

One of the texts most used in the eighties was Leithold, the geometric interpretation of the differential is presented in two ways, $dy$ is the linear approximation of the tangent to the secant, the closer is $x$ to $x_0$, or similarly, while $\Delta x$ is closer from zero, closer is $\Delta y$ a $dy$. Defined the derivative in terms of a differential ratio, there is no problem for the calculation of antiderivatives when substitutions are used, although the author first states a theorem called chain rule for antiderivation in order to justify the substitution which expresses $\int f(g(x))(g'(x)\,dx) = F(x) + C$.

The exhibition of the concept of differential in Leithold is presented from three points of view, first considers the differential as an approximation of the differential, then associates the definition with the differential ratio as visualized by Leibniz, and finally, defines the differential by means of a function of two independent variables $x$ and $\Delta x$, covering almost all the proposed categories; this way, this text tries to locate the differential from different alternatives allowing mobility in the learning of said concept.

Apostol is a different book from the other texts of calculation, since it studies the integral before the derivative. For this author, the differential is only a historical aspect devised by Leibniz to justify the derivative defined by a quotient of infinitesimals called differential; however, it attributes advantages of didactic type to the quotient of differentials, as expressed in the following words: The symbol $dy/dx$ has the obvious advantage of summarizing the complete process of calculating a difference quotient and subsequent step-to-limit. Later it will be observed that the use of the ratio of differentials allows to operate more easily and the formulas that are obtained are remembered without difficulty.

According to Apostol, the concept of differential is not necessary to define the derivative or represent it, it makes clear that with the Lagrange notations and the Cauchy limit step, it expresses the derivative with $f$ and defines it as a new function obtained from $f$; also proposes as an alternative to express the derivative with the Arbogast notation, the function $Df$ is obtained from $f$ through the derivation process, the symbol $D$ is the differential or derivative operator, thus things, $DF = f' = \lim_{h \to 0} ((f(x + h) - f(x))/h)$; although inevitably, sooner rather than later, it must resort to the differential in the integration process, for issues that the author states are easy to handle the integral and its applications, this corroborates the impossibility of ignoring the concept of differential, for therefore in this text the differential is a notation derived from the increment and the infinitesimals. The results obtained on the presentation of the differential concept in the different texts according to the established categories are summarized in the following table.

| Categories books | Differential as |
|------------------|----------------|
|                  |    infinitesimal | increase | function | notation |
| Thomas [21]      | X               | X        | X        |
| Stewart [22]     | X               | X        | X        |
| Leithold [23]    | X               | X        | X        |
| Apostol [24]     | X               | X        | X        |

The presentation of the results shows some things in common for all the texts. The derivative is assumed from the quotient of increments and then the step to the limit, without forgetting the geometric or physical origin that originated said concept; use Leibniz's notation to represent the derivative, as well as Lagrange's notation; each text defines in a separate section the differential according to the established categories, but it is observed that the differential is stripped of its arithmetical origin. Apostol's text does not recognize the differential from the increments, and insists on solving the problem of substitution in the indefinite integral by using the rule of the chain, but its theoretical scaffolding collapses before the need to use the differential as infinitesimal in the technique of integration through substitutions.

The most versatile text is Leithold, it proposes a variety of representations of the differential, in order to avoid obstacles in its learning and subsequent use; The most formal texts in terms of presentation are
Spivak and Apostol, the most austere text is Kitchen, however they allow the student to use differentials in terms of infinitesimals, evoking their usefulness according to what was stated in previous citations.

Analyzing the content of the definitions of differential around the didactic transposition, for students, especially those of engineering and economic sciences, it is more profitable to present the differential in terms of infinitesimals or increments as do the texts whose representation is located in the two first categories, defining the infinitesimal as a function has little relevance, limiting the differential to a notation makes its usefulness invisible, obscures its meaning and denies its geometric interpretation with which the difference between increment and differential can be observed.

6. Conclusions
Although the definition of the Cauchy limit discards the concept of infinitesimal, separating the geometry from the analysis, the infinitely small is still latent in the understanding of the concept of derivative. The approach that apparently diverts the differential calculation as a fundamental concept inspired by its inventors Leibniz, Bernoulli and L'Hôpital, still lives in a significant number of current calculation texts, the differential has withstood all attempts to be completely eliminated from the analysis, it still appears in mathematics, either introduced in the infinitesimal presentation of the calculation in an informal way allowing easy learning, or now introduced rigorously as infinitesimal in the non-standard analysis; even the differential allows to establish partitions in the calculation curriculum, the curricula propose differential calculus, integral calculus and differential equations.

Then a discussion arises between the rigor raised in the text of Apostol and the didactic raised by the other texts. Obviously a mathematician can not deprive the theoretical construction of rigor, nor can it be said that what is teachable is easily understood by students, even when transposition distorts concepts; It is necessary to show the two sides of the coin, to examine the content of the study texts is a priority task for the teacher, for this reason should be careful to use a single text where not all approaches are present.

Regarding the above, it is pertinent to show the concepts from their epistemology and the didactic; hence the point of view of differential in the presentation of the calculation has a didactic advantage, allows to understand with greater ease the concept of derivative, is an element of great use in the calculation of antiderivatives, allows algebraic management in the solution of equations Differentials and is used successfully in the solution of geometric and physical problems.

Finally it is stated that it is impossible to eliminate the concept of differential in the calculation, because sooner than later this must be used in error theory and the calculation of indefinite integrals applying substitutions; based on this conclusion, Leithold's text, without being totally formal, does not spare any resources in the presentation of the differentials, being according to the analysis of results exposed, the text that facilitates the greater understanding of said concept.

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