A Hausdorff Operator with Commuting Family of Perturbation Matrices Is a Non-Riesz Operator

A. R. Mirotin

Department of Mathematics and Programming Technologies,
Francisk Skorina Gomel State University,
Sovietskaya, 104, 246019 Gomel, Belarus
E-mail: amirotin@yandex.ru

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Abstract. We consider a generalization of the notion of Hausdorff operator on Lebesgue spaces and, under natural conditions, prove that such an operator is not a Riesz operator provided it is non zero. In particular, it cannot be represented as a sum of a quasinilpotent and a compact operators.

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1. INTRODUCTION AND PRELIMINARIES

Hausdorff operators are deeply rooted in Fourier analysis. The one-dimensional Hausdorff operator

\( (\mathcal{H}_1 f)(x) = \int_{\mathbb{R}} f(xt) d\chi(t) \)

(\( \chi \) is a measure supported by \([0, 1]\)) was introduced by Hardy [?], Section 11.18] as a continuous variable analog of a regular Hausdorff transformation (or Hausdorff means) for series. Its \( n \)-dimensional generalization is as follows:

\( (\mathcal{H} f)(x) = \int_{\mathbb{R}^n} K(u)f(A(u)x) du, \)

(1)

where \( K : \mathbb{R}^m \to \mathbb{C} \) is a locally integrable function, \( f : \mathbb{R}^n \to \mathbb{C} \), \( A(u) \) stands for a family of nonsingular \( n \times n \) matrices with real entries defined on \( \mathbb{R}^m \), and \( x \in \mathbb{R}^n \) is a column vector [?].

Note that the mapping \( x \mapsto Ax \) \( (A \in GL(n, \mathbb{R})) \), which enters formula (1), is a general form of an automorphism of the additive group \( \mathbb{R}^n \). This observation leads to the definition of a (generalized) Hausdorff operator over a general group \( G \) via the automorphisms of \( G \) that was introduced and studied by the author in [?] and [?]. For the additive group \( \mathbb{R}^n \), this definition is as follows.

Definition 1. Let \((\Omega, \mu)\) be a \( \sigma \)-compact topological space endowed with a positive regular Borel measure \( \mu \), let \( K \) be a locally \( \mu \)-integrable function on \( \Omega \), and let \( (A(u))_{u \in \Omega} \) be a \( \mu \)-measurable family of nonsingular \( n \times n \) matrices that are nonsingular for \( \mu \)-almost every \( u \), with \( K(u) \neq 0 \). We define the Hausdorff operator with the kernel \( K \) by (recall that \( x \in \mathbb{R}^n \) is a column vector)

\( (\mathcal{H}_{K,A} f)(x) = \int_{\Omega} K(u)f(A(u)x) d\mu(u). \)

The general form of a Hausdorff operator given by Definition 1 (with an arbitrary measure space \((\Omega, \mu)\) instead of \( \mathbb{R}^m \)) gives us, in particular, the opportunity to consider (in the case \( \Omega = \mathbb{Z}^m \)) discrete Hausdorff operators (see [?], [?], and Example 3 below).

Many important operators of harmonic analysis are special cases of Hausdorff operators. For example, the Hardy and the adjoint Hardy operators, the Cesàro operator, the Hardy–Littlewood–Pólya operator, the Calderón operator, the Riemann–Liouville fractional derivatives can be derived from (or reduced to) the Hausdorff operator (see, e.g., [?]).

The modern theory of Hausdorff operators was initiated by Liflyand and Moricz [?]. This new class of integral operators is an active area of research now. See the survey articles [?] and [?] for historical remarks and the state-of-the-art up to 2014. Particularly, the boundedness of Hausdorff operators on Lebesgue spaces and Hardy spaces has received extensive study in recent years. At the same time, structural and spectral...
properties of Hausdorff operators remain unexplored or very little studied (see [? , ?]). Among them we can list the properties of compactness, Fredholm property, and others. Thus, the study of such properties of Hausdorff operators remains an urgent problem.

The problem of compactness of Hausdorff operators was posed by Liflyand [?] (see also [?]). There is a conjecture that a nonzero Hausdorff operator on \( L^p(\mathbb{R}^n) \) is not compact. In the case \( p = 2 \) and for commuting \( A(u) \), this conjecture was confirmed in [?] (and, for the diagonal \( A(u) \), in [?]). Moreover, we conjectured in [?] that a nontrivial Hausdorff operator on \( L^p(\mathbb{R}^n) \) is not a Riesz operator.

The notion of Riesz operator was introduced by Ruston [?]. Recall that a bounded operator \( T \) on a Banach space is a Riesz operator if it possesses spectral properties like those of a compact operator; i.e., \( T \) is a noninvertible operator whose nonzero spectrum consists of eigenvalues of finite multiplicity with no limit points other than 0. This is equivalent to the fact that \( T - \lambda I \) is Fredholm for all scalars \( \lambda \neq 0 \) [?, [?, Section 9.6]. For example, a sum of a quasinilpotent and a compact operators is a Riesz operator [?, Theorem 3.29]. See [?], [?, Section 9.6], [?] and the bibliography therein for other interesting characterizations of Riesz operators.

In this note, we prove the aforementioned conjecture for the case in which \( A(u) \) is a commuting family of self-adjoint matrices. The result was announced in [?]. The case of positive or negative definite perturbation matrices was considered in [?].

2. MAIN RESULT

We need several lemmas to prove our main result.

**Lemma 1.** [cf. [? , (11.18.4), [?]]]. Let \( \left| \det A(u)^{-1/p}K(u) \right| \in L^1(\Omega, \mu) \). Then the operator \( \mathcal{H}_{K,A} \) is bounded in \( L^p(\mathbb{R}^n) \) (\( 1 \leq p \leq \infty \)) and

\[
\|\mathcal{H}_{K,A}\| \leq \int_{\Omega} |K(u)| \left| \det A(u) \right|^{-1/p} d\mu(u).
\]

This estimate is sharp (see Theorem 1 in [?]).

**Lemma 2.** [cf.[?]]. Under the assumptions of Lemma 1, the adjoint for a Hausdorff operator \( \mathcal{H}_{K,A} \) on \( L^p(\mathbb{R}^n) \) is of the form

\[
(\mathcal{H}_{K,A}^*f)(x) = \int_{\Omega} K(v)|\det A(v)|^{-1} f(A(v)^{-1}x) dv(v).
\]

Thus, the adjoint of a Hausdorff operator is also a Hausdorff operator.

**Lemma 3.** Let \( S \) be a ball in \( \mathbb{R}^n \), \( q \in [1, \infty) \), and let \( R_{q,S} \) denote the restriction operator \( L^q(\mathbb{R}^n) \to L^q(S) \), \( f \mapsto f|S \). If we identify the dual \( (L^q)^* \) of \( L^q \) with \( L^p \) \( (1/p + 1/q = 1) \), then the adjoint \( R_{q,S}^* \) is the operator of natural embedding \( L^p(S) \hookrightarrow L^p(\mathbb{R}^n) \).

**Proof.** For \( g \in L^p(S) \), let

\[
g^*(x) = \begin{cases} g(x), & \text{for } x \in S, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus S. \end{cases}
\]

Then the mapping \( g \mapsto g^* \) is the natural embedding \( L^p(S) \hookrightarrow L^p(\mathbb{R}^n) \).

By definition, the adjoint \( R_{q,S}^* : L^q(S)^* \to L^q(\mathbb{R}^n)^* \) acts according to the rule

\[
(R_{q,S}^* \Lambda)(f) = \Lambda(R_{q,S} f) \quad (\Lambda \in L^q(S)^*, f \in L^q(\mathbb{R}^n)).
\]

If we identify (by the Riesz theorem) the dual of \( L^q(S) \) with \( L^p(S) \) via the formula \( \Lambda \leftrightarrow g \), where

\[
\Lambda(h) = \int_S g(x)h(x) \, dx \quad (g \in L^p(S), h \in L^q(S)),
\]

and proceed similarly for the dual of \( L^q(\mathbb{R}^n) \), then the definition of \( R_{q,S}^* \) becomes

\[
\int_{\mathbb{R}^n} (R_{q,S}^* g)(x) f(x) \, dx = \int_S g(x)(f|S)(x) \, dx.
\]
However,
\[ \int_S g(x)(f|S)(x)dx = \int_{\mathbb{R}^n} g^*(x)f(x)dx \quad (f \in L^q(\mathbb{R}^n)). \]

The right-hand side of the last formula is a linear functional on \( L^q(\mathbb{R}^n)^* \). If we identify (again by the Riesz theorem) this functional with the function \( g^* \), then the result follows.

Consider the modified \( n \)-dimensional Mellin transform for the \( n \)-hyperoctant \( U \) of \( \mathbb{R}^n \) in the form
\[ (\mathcal{M} f)(s) := \frac{1}{(2\pi)^{n/2}} \int_U |x|^{-\frac{n}{2}+is} f(x)dx, \quad s \in \mathbb{R}^n. \]

In what follows, for \( 1 < q \leq \infty \), we assume that
\[ |x|^{-\frac{n}{2}+is} := \prod_{j=1}^n |x_j|^{-\frac{1}{q}+is_j}, \quad \text{where} \quad |x_j|^{-\frac{1}{q}+is_j} := \exp\left(\left(-\frac{1}{q} + is_j\right) \log |x_j|\right). \]

**Lemma 4.** (1) The mapping \( \mathcal{M} \) is a bounded operator between \( L^p(U) \) and \( L^q(\mathbb{R}^n) \) for
\[ 1 \leq p \leq 2 \quad (1/p + 1/q = 1). \]

(2) If we identify the dual \( (L^p)^* \) of \( L^p \) with \( L^q \) \((1/p + 1/q = 1)\), then the adjoint for the operator \( \mathcal{M} \) on the space \( L^p(U) \) \((1 \leq p \leq 2)\) is as follows:
\[ (\mathcal{M}^* g)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |x|^{-\frac{n}{2}+is} g(s)ds, \quad x \in U. \]

**Proof.** (1) This can readily be seen from the Hausdorff–Young inequality for the \( n \)-dimensional Fourier transform by using the exponential change of variables (see [3]).

(2) To compute \( \mathcal{M}^* \), for \( g \in L^p(\mathbb{R}^n) \), consider the operator
\[ (\mathcal{M}' g)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |x|^{-\frac{n}{2}+is} g(s)ds, \quad x \in U. \]

This is a bounded operator taking \( L^p(\mathbb{R}^n) \) into \( L^q(U) \). Indeed, since
\[ |x|^{-\frac{n}{2}+is} = \prod_{j=1}^n \left|x_j^{-\frac{1}{q}+is_j}\exp(is_j \log |x_j|)\right|, \]
we have
\[ (\mathcal{M}' g)(x) = |x|^{-\frac{1}{q}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(is \cdot (\log |x_j|))g(s)ds, \quad x \in U, \]
where \( |x| := |x_1| \cdots |x_n| \), \( (\log |x_j|) := (\log |x_1|, \ldots, \log |x_n|) \), and the dot stands for the inner product in \( \mathbb{R}^n \). Thus, we can express the function \( \mathcal{M}' g \) via the Fourier transform \( \hat{g} \) of \( g \) as
\[ (\mathcal{M}' g)(x) = |x|^{-1/q} \hat{g}(-\log |x_j|), \quad x \in U, \]
and therefore,
\[ \|\mathcal{M}' g\|_{L^q(U)} = \left( \int_U |x|^{-1} |\hat{g}(-\log |x_j|)|^q dx \right)^{1/q}. \]

Putting \( y_j := -\log |x_j| (j = 1, \ldots, n) \) in the last integral and taking into account the fact that the modulus of the Jacobian of this transformation is
\[ \left|\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}\right| = \det \text{diag}(e^{-y_1}, \ldots, e^{-y_n}) = \exp \left(-\sum_{j=1}^n y_j\right), \]
we see by the Hausdorff–Young inequality that
\[ \|\mathcal{M}' g\|_{L^q(U)} = \|\hat{g}\|_{L^q(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)}. \]
If $f \in L^p(U)$, and $f(x)|x|^{-1/q} \in L^1(U)$, $g \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then the Fubini–Tonelli theorem implies that
\[
\int_{\mathbb{R}^n} (\mathcal{M} f)(s)g(s)ds = \int_U f(x)(\mathcal{M}' g)(x)dx.
\]
Since the bilinear dual pairing $(\varphi, \psi) \mapsto \int \varphi\psi d\nu$ is continuous on $L^p(\nu) \times L^q(\nu)$, it follows that the last equation holds for all $f \in L^p(U)$, $g \in L^p(\mathbb{R}^n)$ by continuity. Thus, $\mathcal{M}' = \mathcal{M}^*$.

Now we can prove our main result.

**Theorem 1.** Let $A(u)$ be a commuting family of real self-adjoint $n \times n$-matrices ($u$ satisfies the condition $K(u) \neq 0$), and let $(\text{det } A(u))^{-1/p}K(u) \in L^1(\Omega)$. Then the Hausdorff operator $\mathcal{H}_{K,A}$ in $L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) is a Riesz operator and it is also nontrivial. For the proof of the Riesz property, it is sufficient to
\[
\mathcal{H}_{K,A} f = \int_{\mathbb{R}^n} K(u)f(A(u)x)d\mu(u).
\]

Proof. Assume the contrary. Since $A(u)$ form a commuting family, it follows that there are an orthogonal $n \times n$ matrix $B$ and a family of diagonal nonsingular real matrices $A'_i(u) = \text{diag}(a_{i1}(u), \ldots, a_{in}(u))$ such that $A'_i(u) = B^{-1}A(u)B$ for $u \in \Omega$. Consider the bounded and invertible operator $(Bf)(x) := f(Bx)$ on $L^p(\mathbb{R}^n)$. By the equality
\[
\mathcal{H}_{K,A}B^{-1} = \mathcal{H}_{K,A'},
\]
which proves the Riesz condition for $\mathcal{H}_{K,A'}$.

As in [?], let us consider some fixed enumeration $U_j$ ($j = 1; \ldots; 2^n$) of the family of all open hyperoctants in $\mathbb{R}^n$. For every pair $(i, j)$ of indices, there is a unique sequence $\varepsilon(i,j) \in \{-1, 1\}^n$ such that
\[
\varepsilon(i,j)U_i := \{\varepsilon(i,j)1; \ldots; \varepsilon(i,j)n\} : x = (x_k)_{k=1}^n \in U_i \equiv U_j.
\]

Then $\varepsilon(i,j)U_j = U_i$ (and vice versa) and $\varepsilon(i,j)U_i \cap U_i = \emptyset$ for $l \neq j$.

We set
\[
\Omega_{ij} := \{u \in \Omega : (\text{sgn}(a_{i1}(u)); \ldots; \text{sgn}(a_{in}(u))) = \varepsilon(i,j)\};
\]
let
\[
(H_{ij}f)(x) := \int_{\Omega_{ij}} H_{ij} f(A'(u)x)d\mu(u).
\]

Since $A'(u)U_i = U_j$ if and only if $u \in \Omega_{ij}$ and $A'(u)U_j \cap U_i = \emptyset$ for $l \neq j$, we have $f(A'(u)x) = 0$ for $f \in L^p(U_j)$ and $x \notin U_i$. Thus, every $H_{ij}$ takes $L^p(U_j)$ to $L^p(U_i)$. Moreover, if $f \in L^p(\mathbb{R}^n)$ and $f := \chi_{U_j}$ ($\chi_E$ denotes the indicator of a subset $E \subset \mathbb{R}^n$), then (as in the proof of formula (1) in [?]), for a.e. (with respect to the Lebesgue measure) $x \in \mathbb{R}^n$,
\[
(\mathcal{H}_{K,A'}f)(x) = \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} (H_{ij}f_j)(x).
\]

Indeed, it is clear that
\[
(\mathcal{H}_{K,A'}f)(x) = \sum_{j=1}^{2^n} \int_{\Omega} K(u)f_j(A'(u)x)d\mu(u).
\]

For a.e. $x \in \mathbb{R}^n$, there is a unique $i$ such that $x \in U_i$. Taking into account the fact that $f_j(A'(u)x) = 0$ for all $u \notin \Omega_{ij}$, we have
\[
\int_{\Omega} K(u)f_j(A'(u)x)d\mu(u) = \int_{\Omega_{ij}} K(u)f_j(A'(u)x)d\mu(u).
\]

Since
\[
\int_{\Omega_{ij}} K(u)f_j(A'(u)x)d\mu(u) = 0 \quad \text{for} \quad l \neq i,
\]

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Consider the mapping \( Jx := (\varepsilon(i,j)k_{x_k})_{k=1}^n \) \((x = (x_k)_{k=1}^n \in \mathbb{R}^n)\). Then \( J : U_j \to U_i \), and the operator \((\tilde{J}f)(x) := f(Jx)\) takes \( L^p(U_i) \) onto \( L^p(U_j) \) isometrically. It follows that the operator

\[
\mathcal{K} := \tilde{J} \mathcal{H}
\]

acts on \( L^p(U_j) \) and is bounded. From now on, we write \( U := U_j \) for simplicity. Then \( \mathcal{K} \) is a nontrivial Riesz operator on \( L^p(U) \). Indeed, \( \tilde{J} \mathcal{H}_{K,A} = \mathcal{K}_{K,A} \tilde{J} \) is a Riesz operator on \( L^p(\mathbb{R}^n) \) \([?], \text{Lemma 5}\). Since the closed subspace \( L^p(U) \) of \( L^p(\mathbb{R}^n) \) is invariant with respect to this operator, it follows that the restriction \( \tilde{J} \mathcal{H}_{K,A}|L^p(U) \) (which is equal to \( \mathcal{K} \)) is a Riesz operator on \( L^p(U) \) by \([?], \text{p. 80, Theorem 3.21}\), as well.

Let \( 1 \leq p < \infty \). To arrive at a contradiction, we use the modified \( n \)-dimensional Mellin transform \( \mathcal{M} \) for the \( n \)-hyperoctant \( U \). By Lemma 4, the mapping \( \mathcal{M} \) is a bounded operator between \( L^p(U) \) and \( L^q(\mathbb{R}^n) \) for \( 1 \leq p \leq 2 \) \((1/p + 1/q = 1)\).

Let \( f \in L^p(U) \). Note that

\[
(\mathcal{K}f)(x) = \int_{\Omega_i} K(u)f(A'(u)(Jx))d\mu(u) = \int_{\Omega_i} K(u)f(A''(u)x)d\mu(u),
\]

where

\[
A''(u) = \text{diag}(\varepsilon(i,j)A_1(u), \ldots, \varepsilon(i,j)A_n(u)).
\]

Assume first that \( |y|^{-1/p}f(y) \in L^1(U) \). Then, as in the proof of Theorem 1 of \([?] \) (or \([?] \)), using the Fubini–Tonelli theorem and integrating via the substitution \( x = (A(u'))^{-1}y \), we obtain

\[
(\mathcal{M} \mathcal{K} f)(s) = \varphi(s)(\mathcal{M} f)(s) \quad (s \in \mathbb{R}^n),
\]

where the function

\[
\varphi(s) := \int_{\Omega_i} K(u)|a(u)|^{-1/p-1}d\mu(u)
\]

(the \((i,j)\) entry of the matrix symbol of a Hausdorff operator \([?, \text{Definition 2}], [?] \)) is bounded and continuous on \( \mathbb{R}^n \).

Therefore,

\[
\mathcal{M} \mathcal{K} f = \varphi \mathcal{M} f.
\]

By continuity, the last equation holds for all \( f \in L^p(U) \).

Let \( 1 \leq p \leq 2 \). There exists a constant \( c > 0 \) such that the set \( \{ s \in \mathbb{R}^n : |\varphi(s)| > c \} \) contains an open ball \( S \). Formula (3) implies that

\[
M_\psi R_{q,S} \mathcal{M} \mathcal{K} = R_{q,S} \mathcal{M},
\]

where \( \psi = (1/\varphi)|S, M_\psi \) stands for the operator of multiplication by \( \psi \), and \( R_{q,S} : L^q(\mathbb{R}^n) \to L^q(S)' \), \( f \mapsto f|S \), is the restriction operator. Let \( T = R_{q,S} \mathcal{M} \). Passing to conjugates gives

\[
\mathcal{M}^* T^* M_\psi^* = T^*.
\]

By \([?, \text{Theorem 1}] \), this implies that the operator \( T^* = \mathcal{M}^* R_{q,S}^* \) is of finite rank. However, by Lemma 3, \( R_{q,S}^* \) is the natural embedding operator \( L^p(S) \to L^p(\mathbb{R}^n) \). Thus, the restriction of the operator \( \mathcal{M}^* \) to \( L^p(S) \) is of finite rank. Since, by Lemma 4, \( \mathcal{M}^* \) can easily be reduced to a Fourier transform, this contradicts the Paley–Wiener theorem on the Fourier image of the space \( L^2(S) \), see, e.g., \([?, \text{Theorem III.4.9}] \) (in our case, \( L^2(S) \subset L^p(S) \)).

Finally, if \( 2 < p \leq \infty \), then one can use duality arguments. Indeed, by Lemma 2, the adjoint operator \( \mathcal{H}_{K,A}^* \) (as an operator on \( L^q(\mathbb{R}^n) \)) is also of Hausdorff type. More precisely, it is equal to \( \mathcal{K}_{q,B} \), where

\[
B(u) = A(u)^{-1} = \text{diag}(1/a_1(u), \ldots, 1/a_n(u))
\]

and

\[
\Psi(u) = K(u)|\det A(u)|^{-1} = K(u)/\prod_j a_j(u).
\]

It is easy to see that \( \mathcal{H}_{q,B} \) satisfies all the conditions of Theorem 1 (with \( q, \Psi, \) and \( B \) instead of \( p, K, \) and \( A \), respectively). Since \( 1 \leq q < 2 \), it follows that the operator \( \mathcal{K}_{q,B} \) is not a Riesz operator on \( L^q(\mathbb{R}^n) \). The same holds for \( \mathcal{H}_{K,A} \), because \( T \) is a Riesz operator if and only if its conjugate \( T^* \) is a Riesz operator \([?, \text{p. 81, Theorem 3.22}] \). This completes the proof.
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3. COROLLARIES AND EXAMPLES

**Corollary 1.** Under the same assumptions as in Theorem 1, \( \mathcal{H}_{K,A} - \lambda \) is not a Fredholm operator for some scalar \( \lambda \neq 0 \).

**Corollary 2.** Let the conditions of Theorem 1 hold. Then either there is a nonzero point of \( \sigma(\mathcal{H}_{K,A}) \) which is not a pole of the resolvent of \( \mathcal{H}_{K,A} \) or there is a nonzero point \( \lambda \) of \( \sigma(\mathcal{H}_{K,A}) \) such that the spectral projection \( E(\lambda) \) has an infinite-dimensional range.

**Proof.** This follows from Theorem 1 and the characterization of Riesz operator given in [7, Theorem 3.17]. □

**Corollary 3.** [7]. Let \( A(v) \) be a commuting family of real positive or negative definite \( n \times n \) matrices (\( v \) ranges over the support of \( K \)), and let \( \det A(v)^{-\frac{1}{p}} K(v) \in L^1(\Omega, \mu) \). Then the Hausdorff operator \( \mathcal{H}_{K,A} \) in \( L^p(\mathbb{R}^n) \) \((1 \leq p \leq \infty)\) is not a Riesz operator (and, in particular, it is not a sum of a quasinilpotent and a compact operators) if it is nonzero.

**Corollary 4.** Let \( \phi: \Omega \to \mathbb{C} \), and let \( a: \Omega \to \mathbb{R} \) be \( \mu \)-measurable functions for which we have \( |a(u)|^{-1/p} \phi(u) \in L^1(\Omega, \mu) \). Then the one-dimensional Hausdorff operator

\[
(\mathcal{H}_{\phi,a} f)(x) = \int_{\Omega} \phi(u) f(a(u)x) d\mu(u) \quad (x \in \mathbb{R})
\]

on \( L^p(\mathbb{R}) \) \((1 \leq p \leq \infty)\) is not a Riesz operator (and, in particular, it is not a sum of a quasinilpotent and a compact operators) if it is nonzero.

**Example 1.** Let \( t^{-1/q} \psi_1(t) \in L^1(0, \infty) \). Then, by Corollary 4, the operator

\[
(\mathcal{H}_{\psi_1} f)(x) = \int_{0}^{\infty} \psi_1(t) f\left(\frac{x}{t}\right) dt
\]

is not a Riesz operator in \( L^p(\mathbb{R}) \) \((1 \leq p \leq \infty)\) if it is nonzero.

**Example 2.** Let \( (u_1 \cdots u_n)^{-1/p} \psi(u_1, \ldots, u_n) \in L^1(\mathbb{R}^n_+) \). Then, by Corollary 3, the operator

\[
(\mathcal{H}_{\psi} f)(x) = \frac{1}{x_1 \cdots x_n} \int_{\mathbb{R}^n_+} \psi \left( \frac{t_1}{x_1} \cdots \frac{t_n}{x_n} \right) f(t) dt
\]

is not a Riesz operator in \( L^p(\mathbb{R}^n_+) \) \((1 \leq p \leq \infty)\) if it is nonzero.

**Example 3.** (Discrete Hausdorff operators, cf. [7, Example 3]). Let \( \Omega = \mathbb{Z}^n_+ \), and let \( \mu \) be the counting measure. Then Definition 1 turns into

\[
(\mathcal{H}_{K,A} f)(x) = \sum_{u \in \mathbb{Z}^n_+} K(u)f(A(u)x)
\]

(Here \( A(u) \) form a family of real nonsingular \( n \times n \) matrices; the right-hand side is regarded as the Lebesgue integral with respect to the counting measure on \( \mathbb{Z}^n_+ \)). Assume that \( 1 \leq p \leq \infty \) and \( \sum_{u \in \mathbb{Z}^n_+} |K(u)||\det A(u)|^{-1/p} < \infty \). Then the operator \( \mathcal{H}_{K,A} \) is well defined and bounded on \( L^p(\mathbb{R}^n) \) by Lemma 1, and it is not a Riesz operator by Theorem 1 if it is nonzero and the matrices \( A(u) \) are permutable and self-adjoint.

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