LOCALIZATION THEORY FOR DERIVATORS

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ABSTRACT. We outline the theory of reflections for prederivators, derivators and stable derivators. In order to parallel the classical theory valid for categories, we outline how reflections can be equivalently described as categories of fractions, reflective factorization systems, and categories of algebras for idempotent monads. This is a further development of the theory of monads and factorization systems for derivators.

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1. INTRODUCTION

The notion of co/reflection (or co/reflective localization) $i : \mathcal{B} \rightleftarrows \mathcal{C}$ combines together two of the most natural and pervasive notions in category theory: it is an adjunction between $\mathcal{C}$ and one of its full subcategories $\mathcal{B}$.

Since (echoing [ML98]) ‘adjoints are everywhere’, it is easy to believe that reflections pop up quite often in Mathematics: indeed, many theorems inside and outside category theory admit nifty translations in terms of the existence of a certain co/reflective localization. Even more, certain theorems are all about describing certain classes of well-behaved categories in terms of reflective localizations of ‘prototypical’ such categories: for example, a fundamental characterization of locally finitely presentable categories [AR94] is that they all arise as localizations of presheaf categories $[\mathcal{A}^{op}, \text{Set}]$ on small categories $\mathcal{A}$, whereas Giraud theorem refines
this statement asserting that \textit{Grothendieck toposes} all arise as \textit{left-exact} localizations of such presheaf categories (this means that the left adjoint preserves finite limits). Moreover, we currently have a great deal of ways to characterize reflective subcategories of a given \mathcal{C} in terms of other data or structures on it: more precisely,

1.1) co/reflective subcategories of \mathcal{C} arise as special \textit{categories of fractions}, i.e. categories where a given class of arrows in \mathcal{C} has been formally inverted into a category \mathcal{C}[S^{-1}], initial with this property. More in detail, a co/reflective subcategory arises inverting all the arrows that the left adjoint sends to isomorphisms. Somehow understandably, homotopy theorists feel comfortable with the nomenclature “reflective localization” because these are particular cases of homotopy categories;

1.2) co/reflective subcategories of \mathcal{C} are in bijection with (categories of co/algebras for) idempotent co/monads on \mathcal{C} \cite[§4.2]{Bor94b} (recall that a monad \textit{T} is \textit{idempotent} if its multiplication \mu : T^2 \Rightarrow T is invertible);

1.3) co/reflective subcategories of \mathcal{C} are in bijection with co/reflective \textit{prefactorization systems} \cite{CHK85} on \mathcal{C} (recall that a prefactorization system is \textit{reflective} if its right class satisfies the ‘two out of three’ property).

Somehow, this series of equivalences draws the state of the art on the subject: our aim in the present paper is to provide a similar description of localization theory in a the framework of pre/derivators.

The current explosive development of higher category theory forces (not only, but especially) the community of category theorists, geometers and topologists to reenact many classical statements into the language of (\infty, 1)-category theory, aiming to provide a robust framework in which to find newer and newer applications for higher category theory.

Now, approaching this subject everybody notices quite quickly that there are many models we can choose to work in: among many, we cite simplicial categories \cite{Ber07}, model categories \cite{Qui67}, quasicategories \cite{Joy08, Lur09}; of course, choosing the ‘right’ framework to address a particular problem is a matter of taste, mathematical experience, and –to a certain extent– fashion. What is even more clear though is that category theorists are quite interested in comparing the same construction in different models, and (even better) having ways to prove their mutual equivalence. Even though there are formal ways to address this issue, there is no general recipe to build these dictionaries between models. As a result, it can be easy, difficult or extremely difficult to define the same notion, according to the framework we chose to work within, let alone to compare it to others.
The theory of co/reflective localizations makes no exception in this respect: in the setting of simplicial categories it is quite easy to rephrase what is a reflection functor, with the only care that not all unenriched functors $F : |\mathcal{A}| \to |\mathcal{B}|$ between the underlying categories of two simplicial categories lift to simplicial functors $F : \mathcal{A} \to \mathcal{B}$; in the setting of model categories the ‘homotopy invariant’ notion of co/reflective localization is that of a left/right Bousfield localization [Hir03], and this can be seen as solving the universal problem of enlarging the class of weak equivalences $W$ of a model category $\mathcal{M}$ into $W \subset W'$, while maintaining the co/fibrations fixed. This results in the homotopy category $\text{Ho}(\mathcal{M}, W)$ being a localization (in the 1-categorical sense) of $\text{Ho}(\mathcal{M}, W')$. Finally, in the realm of quasicategories the process of localization is captured by the following construction: if $W \subset \mathcal{C}$ determines a class of edges of the quasicategory $\mathcal{C}$, then the localization $\mathcal{C}[W^{-1}]$ of $\mathcal{C}$ at $W$ is defined by the homotopy pushout

$$
\begin{array}{ccc}
W & \xrightarrow{\rho} & \mathcal{C} \\
\downarrow & & \downarrow \\
W & \longrightarrow & \mathcal{C}[W^{-1}]
\end{array}
$$

in the category of simplicial sets, where $\tilde{W}$ is the fibrant replacement of the simplicial subset $W \subset \mathcal{C}$ in the Kan-Quillen model structure (a model for which is $\text{Ex}_\infty(W)$).

Now, the theory of derivators, initiated in order to grasp a more intrinsic description of the construction exhibiting the derived category of an abelian category $\mathcal{A}$, is able to get rid of simplicial machineries in $(\infty,1)$-category theory (or at least, it reduces them to the bare minimum with which a category theorist or an algebraist is comfortable). Even though there is no clear evidence that the 2-category of prederivators, as defined in [Gro], but especially in [Gro13] can be made (weakly) equivalent to one of the other models for $(\infty,1)$-categories (but see [Car16] for a partial result in this direction), there is a certain effort to establish to which extent this comparison of models is possible, were it only because derivators are ‘friendlier’ in that they only appeal 2-dimensional category theory. The present paper is part of this effort: here, we study the theory of co/reflections for prederivators, and we provide equivalent characterizations for these co/reflections in terms of objects of fractions, algebras for idempotent monads, and reflective factorization systems echoing characterizations L1—L3 above.

**Organization of the paper.** In section 2 we briefly review the fundamental definitions of 0-, 1- and 2-cells in the 2-category of prederivator, and the subsequent refinement to the full sub-2-category of derivators, as well as the notions of co/sieve,
homotopy exact and $\mathbb{D}$-exact square, etc.; this is meant to be a quick reference for the reader, but it may appear terse if approached without a previous knowledge of these definitions. In section 3 we provide the basic definition of a reflection in $\mathcal{P}\mathcal{D}\mathcal{e}\mathcal{r}$; it is precisely what we expect it to be in a generic 2-category: an adjunction $L \xrightarrow{\varepsilon} R$ with invertible counit $\varepsilon$ (co-reflections are defined dually, of course). Here, in Proposition 3.27 we establish the equivalence between (left exact) reflections of $\mathbb{D}$ and derivators of fractions with respect to sub-prederivators $\mathcal{S} \subset \mathbb{D}$ (admitting a left calculus of fractions), and their description (Proposition 3.37) in terms of coherently orthogonal classes as defined in Definition 3.34 (but more extensively used in [LV17a]). Animated by the desire to get rid of the stability assumption in our previous work, we had to refine our notion of factorization system: this yield the notion of choric factorization Definition 3.38, and its equivalence with choric reflections (Definition 3.14). In section 4 we investigate the connection between co/localizations and co/algebras for idempotent co/monads: the equivalence result is easily reached (Definition-Proposition 4.13), but leaves us with the feeling that outlining such a pervasive and useful theory as that of monads is an urgent matter: we take advantage of the theory initiated in [LN] expanding it and polishing some of its corners, and we prove in 4.15 the equivalence between left exact localization and algebras for left exact idempotent monads.

Addendum. A more general version of some results presented here has been obtained independently by I. Coley [Col18] (see §3.3.3, §4.4.10, the account of regularity property, and 7.14, that works as a motivation for that paper). We remain available to the author and the readers of both papers, welcoming the notification of any other potential overlap to properly acknowledge it.

Notation and terminology.

Foundations. Among different foundational conventions that one may adopt, in this paper we assume that every set lies in a suitable Grothendieck universe [GV72]. This choice can nevertheless be safely replaced by the more popular (albeit less powerful) foundation using sets and classes.

More in detail we implicitly fix a universe $\mathcal{U}$, whose elements are termed sets; small categories have a set of morphisms; locally small categories are always considered to be small with respect to some universe: treating with derivators it is a common choice to employ the so-called two-universe convention, where we postulate the existence of a universe $\mathcal{U}^+ \succ \mathcal{U}$ in which all the classes of objects of non-$\mathcal{U}$-small, locally small categories live.
Categories and functors. Possibly large categories will be usually denoted as calligraphic letters like \( A, B, C \) and suchlike; classes of morphisms in a category, often confused with the subcategory they generate, are denoted as calligraphic letters \( E, \mathcal{X}, \mathcal{Y}, \ldots \) as well; when they are considered as objects of the category of categories, small categories are usually denoted as capital Latin letters like \( I, J, K \) and suchlike: we denote in the same way an object of a possibly large category \( \mathcal{C} \); this slight abuse of notation causes no harm whatsoever. The 2-categories of diagrams, small categories, categories, prederivators and derivators, and more generally all 2-categories, are denoted in a sans-serif typeface like \( \text{Dia}, \text{Cat}, \text{CAT}, \text{PDer}, \text{Der} \). The correspondence that inverts 1-cells of a 2-category is denoted \( \text{op} \), whereas the correspondence that inverts 2-cells is denoted \( \text{co} \). We denote \( J / j \) the slice category of \( J \) at the object \( j \), and having objects the arrows with codomain \( j \); dually, \( J j / \) denotes the category of morphisms having domain \( j \).

Functors between small categories are usually denoted as lowercase Latin letters like \( u, v, w, \ldots \) and suchlike (there must be of course numerous deviations to this rule); an hom-object in a category \( K \) or higher category \( K \) is often denoted \( K(A, B) \) or \( \text{Cat}(A, B) \); the symbols \( \_ \_ \_ \_ \_ \) are used as placeholders for the “generic argument” of any kind of mapping; natural transformations between functors; or more generally 2-cells in a 2-category, are often written in Greek, or Latin lowercase alphabet, and collected in the set \( \text{Nat}(F, G) \).

Whenever there is an adjunction \( F \dashv G \) between functors, that we denote \( F : A \rightleftarrows B : G \) in its domain are called mates or adjuncts; so, the notation “the mate/adjunct of \( f : Fa \rightarrow b \)” means “the unique arrow \( g : a \rightarrow Gb \) determined by \( f : Fa \rightarrow b \)”. When there is an adjunction between two functors \( F, G \) we adopt \( F \dashv \text{co} \) \( G \) as a compact notation to denote all at once that \( F \) is left adjoint to \( G \), with unit \( \eta : 1 \rightarrow GF \) and counit \( \epsilon : FG \rightarrow 1 \). A customary choice of notation for the whiskering between a 1-cell \( F \) and a 2-cell \( \alpha \) is \( F \ast \alpha \) or \( \alpha \ast F \). In order to avoid confusion with the many occurrences of an ‘upper-star’ besides a morphism, we choose to denote the pre- and post-composition morphisms induced by \( F : \mathbb{D} \rightarrow \mathbb{E} \) as \( F_{\downarrow} : \text{PDer}(\mathbb{X}, \mathbb{D}) \rightarrow \text{PDer}(\mathbb{X}, \mathbb{E}) \) and \( F_{\uparrow} : \text{PDer}(\mathbb{E}, \mathbb{X}) \rightarrow \text{PDer}(\mathbb{D}, \mathbb{X}) \) respectively.

Special categories. Derivator theory forces to work with a huge variety of category shapes, and forces to choose clever notation to denote these categories \( I, J, \ldots \) as well as the functors \( I \rightleftarrows J \). In our work, many choices are classical: for example, the simplex category \( \Delta \) is the topologist’s delta, having objects nonempty finite ordinals \( [n] = \{0 < 1 < \cdots < n\} \) (this is opposed to the algebraist’s delta \( \Delta_+ \) which
has an additional initial object \([-1]\)]; we denote \(\Delta^n\) the representable presheaf on \([n] \in \Delta\), i.e. the image of \([n]\) under the Yoneda embedding of \(\Delta\) in the category of simplicial sets \(\hat{\Delta}\). More often though, the objects of \(\Delta\) are considered as categories via the obvious embedding: as a consequence, certain objects have many names (for example, the terminal object \([0]\) of \(\Delta\) and \(\text{Cat}\) is called \(e\) in \(\text{Dia}\)).

The notation for other common categories deserves to be explained; the "generic span" \(\{2 \leftarrow 0 \rightarrow 1\}\) will be denoted as \(\Gamma\), where the opposite category "generic cospan" \(\{0 \rightarrow 2 \leftarrow 1\}\) will be denoted as \(\gamma\). The nerves of these two categories are the simplicial sets \(\Lambda^2_0\) and \(\Lambda^2_2\) (as it is customary to blur the distinction between a category and its nerve, we don’t insist in keeping these notation separated). The completions of these two categories to "generic commutative squares" are obtained introducing a terminal (resp., initial) object into \(\Gamma\) (resp., \(\gamma\)), in such a way that these two categories have objects labeled \(\begin{array}{c} 0 \\ \downarrow \\ 1 \\ \downarrow \\ \infty \end{array}\) (resp., \(\begin{array}{c} -\infty \\ \downarrow \\ 0 \\ \downarrow \\ +\infty \end{array}\)); this choice permits agile notation as \(X\ldots\), \(\ldots\), \(\ldots\), etc. to refer to the various sides of \(X \in \mathbb{D}(\square)\). Of course, these two categories are isomorphic, hence indistinguishable: it is only context that gives to their objects different labels. Another useful convention, employed from time to time to refer to the sides of \(X \in \mathbb{D}(\square)\) is the following: we write the sides \(X_{00} \rightarrow X_{10} \downarrow \downarrow X_{01} \rightarrow X_{11}\) as \(X_N\) ("X north"), \(X_S\) ("X south"), etc., with evident meaning. The category \(\Rightarrow\) denotes the "generic pair of parallel arrows" \(\{0 \Rightarrow 1\}\); its completion to a category with initial object is \(\text{eq} = \{-\infty \rightarrow 0 \Rightarrow 1\}\); whereas its completion to a category with terminal object is \(\text{coeq} = \{0 \Rightarrow 1 \rightarrow +\infty\}\). The category \([1] \times [1] \times [1]\), that looks like a cube, appears in certain arguments involving an adjunction “lifted” from \(L \dashv R\) to \(L^{[1]} \dashv R^{[1]}\) and restricted to a certain subcategory of the domain of \(R^{[1]}\); in these cases, we employ the build a roof notation used in Rubik’s cube theory that assigns to the faces of a cube letters \(\text{BLDRUF}\) as in

![Diagram](image.png)

according to the position of the face with respect to the observer. We choose to orient the cube in such a way there is a morphism of squares \(F \rightarrow B\); when needed, we refer to the various edges of each face according to the aforementioned NSWE notation for \(X \in \mathbb{D}(\square)\). The subsets of the cube \(\square\) are then depicted \(\mathcal{C}, \mathcal{D}, \mathcal{R}, \mathcal{G}, \mathcal{B}, \ldots\), etc.
2. Review of derivator theory

**Definition 2.1** (Category of diagrams). Let us start recalling that a *category of diagrams* is a full 2-subcategory $\text{Dia}$ of $\text{Cat}$, such that

- Dia1) all finite posets, considered as categories, belong to Dia;
- Dia2) given $I \in \text{Dia}$ and $i \in I$, the slice constructions $I_i$ and $I_i^\text{op}$ belong to Dia;
- Dia3) if $I \in \text{Dia}$, then $I^\text{op} \in \text{Dia}$;
- Dia4) for every Grothendieck fibration $u : I \to J$, if all fibers $I_j$, for $j \in J$, and the base $J$ belong to Dia, then so does $I$.

The minimal example for a category of diagrams is the locally posetal 2-category $\text{fPos}$ of finite posets, while the maximal is $\text{Cat}$, containing all small categories. There are other possible choices, like the 2-category of finite categories.

**Definition 2.2** (the 2-category of prederivators). If Dia is a category of diagrams, we call a *prederivator of type Dia* a 2-functor $D : \text{Dia}^\text{op} \to \text{CAT}$; the 2-category of prederivators has objects the prederivators, 1-cells the pseudonatural transformations, and 2-cells the modifications between pseudonatural transformations.

The notion of derivator is a refinement of the notion of prederivator, motivated by the desire to provide a satisfactory axiomatization for triangulated categories that only appeals 2-categorical language. A *derivator* is then a prederivator that satisfies the following additional conditions (we mimic the labeling convention of [Gro13]):

- Der1) The functor $\mathbb{D}(I \sqcup J) \to \mathbb{D}(I) \times \mathbb{D}(J)$ obtained from the canonical inclusions $i_I : I \to I \sqcup J \leftarrow J : i_J$ is an equivalence.
- Der2) Each object $j : e \to J$ induces a family of functors $\mathbb{D}(J) \xrightarrow{j^*} \mathbb{D}(e)$; this family is jointly reflective, i.e. a morphism $f \in \mathbb{D}(J)$ is invertible if and only if each $j^*f$ is invertible in $\mathbb{D}(e)$.
- Der3) Each functor $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ induced by $u : I \to J$ admits both a left adjoint $u_!$ and a right adjoint $u_*$. These functors are called, respectively, the homotopy left Kan extension and homotopy right Kan extension along $u$.

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1Taking [Gro13] as a standard reference for all the unexplained notation, we stick to the choice to call “functors” between derivators the 1-cells of $\text{PDer}$, and “natural transformations” between morphisms of derivators its 2-cells.
Given a functor \( u : J \to K \), there exist two squares in CAT, induced by the colax pullbacks defining the slice and coslice categories, i.e. by 2-cells in Dia

\[
\begin{array}{ccc}
J/k & \xrightarrow{t} & e \\
p \downarrow \mathcal{G} & k & \sim & p^* \downarrow \mathcal{K} \\
J & \xrightarrow{u} & K \\
\end{array}
\]

\[
\begin{array}{ccc}
D(J/k) & \xrightarrow{t_*} & D(e) \\
p^* \downarrow \mathcal{K} & k^* & \sim & p^* \downarrow \mathcal{K} \\
D(J) & \xrightarrow{u_*} & D(K) \\
\end{array}
\]

\[
\begin{array}{ccc}
J_{k/} & \xrightarrow{t} & e \\
p \downarrow \mathcal{G}' & k & \sim & p^* \downarrow \mathcal{K}' \\
J & \xrightarrow{u} & K \\
\end{array}
\]

\[
\begin{array}{ccc}
D(J_{k/}) & \xrightarrow{t_!} & D(e) \\
p^* \downarrow \mathcal{K}' & k^* & \sim & p^* \downarrow \mathcal{K}' \\
D(J) & \xrightarrow{u_!} & D(K) \\
\end{array}
\]

These squares are filled by invertible 2-cells \( \varpi' : t_! p^* \Rightarrow k^* u_* \), and \( \varpi : k^* u_* \Rightarrow t_* p^* \) for every derivator \( D \).

**Remark 2.3.** Taken all together, the axioms of derivator are meant to ensure that we can build a category theory which is expressive enough for concrete applications, and in particular, they are meant to express the fact that we can compute left and right Kan extensions for every functor \( u : I \to J \) (Der3), and that these extensions are pointwise (Der4).

**Remark 2.4.** It is helpful to keep in mind the two paradigmatic examples of constructions giving rise to derivators:

- Let \( \mathcal{M} \) be a model category; then the association \( J \mapsto \text{Ho}(\mathcal{M}^J) \) defines a derivator.
- Let \( \mathcal{C} \) be a quasicategory; then the association \( J \mapsto \text{Ho}(\mathcal{C}^N J) \) defines a prederivator, which is a derivator if \( \mathcal{C} \) is complete and cocomplete.

**Remark 2.5.** Even though we call them functors, the 1-cells \( F : \mathbb{E} \to \mathbb{D} \) of PDer are pseudonatural transformations between 2-functors; every such transformation comes equipped with coherence data as part of its definition, and in particular with invertible 2-cells \( \gamma_{F,u} : u^* \circ F_J \Rightarrow F_I \circ u^* \), one for each \( u : I \to J \), suitably compatible with composition.
Remark 2.6. Every such transformation induces additional 2-cells (the ‘adjuncts’ of \( \gamma_{F,u} \), or the 2-cells obtained by ‘base change’ from \( \gamma_{F,u} \): see [Gro13, §1.2])

\[
\begin{array}{ccc}
\mathbb{E}(I) & \xrightarrow{u_*} & \mathbb{E}(J) \\
\downarrow & \gamma_F \downarrow & \downarrow \\
\mathbb{D}(I) & \xrightarrow{u_*} & \mathbb{D}(J)
\end{array}
\quad
\begin{array}{ccc}
\mathbb{E}(I) & \xrightarrow{u_!} & \mathbb{E}(J) \\
\downarrow & \gamma_F^{-1} \downarrow & \downarrow \\
\mathbb{D}(I) & \xrightarrow{u_!} & \mathbb{D}(J)
\end{array}
\]

We denote these 2-cells obtained pasting the units and counits of the adjunctions \( u_! \dashv u^* \dashv u_* \) with \( \gamma_{F,u} \), respectively \( (\gamma_{F,u})_* \) and \( (\gamma_{F,u})! \). Somewhat sloppily, these morphisms measure how far is \( F \) from commuting with the Kan extensions \( u_!, u_* \) (see [Gro13, §2.2])

Remark 2.7. The 2-cells \( \varpi_* \), \( \varpi_! \) of axiom (Der4) can be regarded as the 2-cells obtained by base change from the 2-cell \( \varpi \).

We recall a few classical definitions from [Gro13] that will be useful later:

Definition 2.8. Let \( u: J \to K \) be a fully faithful functor which is injective on objects.

i) The functor \( u \) is called a cosieve if whenever we have a morphism \( u(j) \to k \) in \( K \) then \( k \) lies in the image of \( u \).

ii) The functor \( u \) is called a sieve if whenever we have a morphism \( k \to u(j) \) in \( K \) then \( k \) lies in the image of \( u \).

Definition 2.9. Let \( \mathbb{D} \) be a derivator and let us consider a natural transformation \( \alpha \) as indicated in the following square in \( \mathbf{Cat} \):

\[
\begin{array}{ccc}
J_1 & \xrightarrow{u} & J_2 \\
\downarrow \gamma & \downarrow \gamma \\
K_1 & \xrightarrow{w} & K_2
\end{array}
\]

The square is \( \mathbb{D} \)-exact if the base change \( \alpha_! : u_1! \circ v^* \to w^* \circ u_2! \) (or, by [Gro13, Lemma 1.20], equivalently \( \alpha_* : u_2^* \circ w_* \to v_* \circ u_1^* \)) is a natural isomorphism. The square is called homotopy exact if it is \( \mathbb{D} \)-exact for all derivators \( \mathbb{D} \).
Remark 2.10. Note that axiom (Der\textsubscript{4}) can be rephrased as ‘the squares

\[
\begin{array}{cccc}
J/k & \xrightarrow{t} & e & J/k' \\
p & \downarrow & k & p \\
J & \xrightarrow{u} & K & J \\
\end{array}
\]

are homotopy exact.’

Lemma 2.11. Let \( F : \mathcal{D} \to \mathcal{E} \) be a morphism of prederivators; then \( F \) is an equivalence (i.e. it admits an adjoint inverse \( G \) such that \( GF \cong 1_\mathcal{D} \) and \( FG \cong 1_\mathcal{E} \)) if and only if each component \( F_J : \mathcal{D}(J) \to \mathcal{E}(J) \) is an equivalence of categories in \( \text{CAT} \).

Proof. It is clear that the condition is necessary. To show that it is also sufficient, let \( G_J \) be the adjoint inverse of \( F_J \), in such a way that each \( F_J \eta^J \) is an equivalence of categories; in particular, each unit \( \eta^J \) and each counit \( \varepsilon_J \) are invertible natural transformations. Given \( u : I \to J \) now it’s easily seen that the coherence morphisms \( \gamma^u_{F,u} : u^*F_J \Rightarrow F_Iu^* \), when pasted with \( \eta, \varepsilon \), give isomorphisms

\[
\begin{array}{ccc}
\mathcal{E}(J) & \xrightarrow{G_J} & \mathcal{D}(I) \\
\downarrow \gamma^u_{F,u} & \nearrow \gamma^u_{F,u} & \downarrow \eta \\
\mathcal{E}(I) & \xrightarrow{u^*} & \mathcal{D}(I) \\
\end{array}
\]

testifying that the components \( G_J \) glue to a pseudonatural transformation \( \mathcal{E} \to \mathcal{D} \).

Definition 2.12 ([Gro13], 2.2). Let \( F : \mathcal{D} \to \mathcal{E} \) be a morphism of derivators and let \( u : J \to K \) be a functor. The morphism \( F \) preserves homotopy left respectively homotopy right Kan extensions along \( u \) if the natural transformation

\[
(\gamma^F_u)_! : u_!F \Rightarrow Fu_! \quad \text{respectively} \quad \gamma^F_{u *} : Fu_* \Rightarrow u_*F
\]
is an isomorphism.

In order to discuss the interaction of limits and reflections more systematically, we include the following definition. These will turn useful in the proof of general facts about co/limits preservation properties of reflections.

Definition 2.13. Let \( F : \mathcal{D} \to \mathcal{E} \) be a morphism of derivators and let \( u : A \to B \) be in \( \text{Cat} \).
(1) The morphism $F$ reflects left (resp. right) Kan extensions along $u$ if for every diagram $X \in \mathcal{D}(B)$ such that $FX \in \mathcal{E}(B)$ lies in the essential image of $u_! : \mathcal{E}(A) \to \mathcal{E}(B)$ (resp., of $u_* : \mathcal{E}(A) \to \mathcal{E}(B)$) then already $X$ lies in the essential image of $u_! : \mathcal{D}(A) \to \mathcal{D}(B)$ (resp., of $u_* : \mathcal{D}(A) \to \mathcal{D}(B)$).

(2) The morphism $F$ reflects colimits of shape $A$ if it reflects left Kan extensions along $A \to A^\triangleright$ (resp., $A \to A^\triangleleft$).

**Remark 2.14.** (1) A morphism $F$ of derivators reflects left Kan extensions along $u$ if and only if the opposite morphism $F^{op}$ reflects right Kan extensions along $u^{op}$.

(2) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between complete and cocomplete categories. Then $F$ reflects colimits of shape $A$ if and only if $y_F$ preserves colimits of shape $A$. In fact, it suffices to note that $X : A^\triangleright \to \mathcal{C}$ is a colimiting cocone if and only if it lies in the essential image of the left Kan extension along $i_A : A \to A^\triangleright$.

**Remark 2.15.** By [Gro16b, Prop. 3.9] the following are equivalent for a morphism $F : \mathcal{D} \to \mathcal{E}$ of derivators and $A \in \text{Cat}$.

(1) The morphism $F$ preserves and reflects colimits of shape $A$.

(2) A cocone $X \in \mathcal{D}(A^\triangleright)$ is colimiting if and only if the cocone $FX \in \mathcal{E}(A^\triangleright)$ is colimiting.

**Definition 2.16.** A morphism of prederivators is conservative if every component is conservative.

Thus, a morphism $F : \mathcal{D} \to \mathcal{E}$ is conservative if a morphism $f : X \to Y$ in $\mathcal{D}(A)$ is an isomorphism as soon as $Ff : FX \to FY$ is an isomorphism in $\mathcal{E}(A)$.

**Examples 2.17.** Fully faithful morphism of prederivators are conservative. In particular, reflections, co-reflections, and equivalences are conservative.

**Definition 2.18.** Let $F : \mathcal{D} \to \mathcal{E}$ be a morphism of prederivators; we say that $F$ creates homotopy left (resp., right) Kan extensions along $u : A \to B$ if for every $X \in \mathcal{D}(B)$,

- the object $X$ lies in the essential image of $u_!^\triangleright$ (resp, $u_*^\triangleright$) if and only if $F_B X$ lies in the essential image of $u_!^\triangleleft$ (resp, $u_*^\triangleleft$);
- $F$ preserves homotopy left (resp., right) Kan extensions along $u$.

**Definition 2.19.** We say that a prederivator has initial (resp. terminal) objects if $t^* : \mathcal{D}(e) \to \mathcal{D}(\varnothing)$ (where $t : \varnothing \to e$ is the unique arrow) has a left (resp., right) adjoint. Of course the notion is meaningful when $\mathcal{D}$ is a derivator, as (see [Gro13, 1.12.i]) $\mathcal{D}(\varnothing)$ is the terminal category.
Here, we record a rather useful characterization of morphisms of derivators commuting with finite homotopy limits.

**Lemma 2.20.** The following conditions are equivalent for an adjunction \( L : D \rightleftarrows E : R \) of derivators (the assumption that these are derivators can’t be removed):

1. The left adjoint \( L \) commutes with finite limits;
2. The left adjoint \( L \) commutes with terminal objects and homotopy pullbacks;
3. The left adjoint \( L \) commutes with products and homotopy equalizers;

**Proof.** The proof that \( \text{CL}_2 \Rightarrow \text{CL}_1 \) is the dual of [PS14, 7.1], where a non-elementary argument proved that one can actually construct homotopy colimits over \( J \in \text{Dia} \) out of coproducts and homotopy pushouts if \( J \) is homotopy finite.

We show that \( \text{CL}_3 \) implies \( \text{CL}_2 \): given \( X \in D(\_\_\_) \) there is a diagram \( \bar{X} \in D(\Rightarrow) \) such that \( \text{holim} X \cong \text{holim}_{\Rightarrow} \bar{X} \).

To this end, define the following functors between finite categories:

- \( i_{\_\_\_} : \_\_\_ \rightarrow \square \) is the inclusion of the left and bottom edge of \( \_\_\_ \);
- the functor \( i_\perp \) adjoins an initial object to the category \( \Rightarrow \);
- the coequalizer \( q_{\_\_\_} \) of the pair \( d_0, d_1 : [0] \Rightarrow \_\_\_ \), i.e. the functor

\[
\begin{bmatrix}
0 \\
1 \rightarrow 2
\end{bmatrix} \xrightarrow{(0 \equiv 1) \Rightarrow 2}
\]

- the coequalizer \( q_{\square} \) of the pair \( (\_)_N, (\_)_W : [1] \Rightarrow \square \), i.e. the functor

\[
\begin{bmatrix}
-\infty \rightarrow 0 \\
1 \rightarrow 2
\end{bmatrix} \xrightarrow{\square\square} \begin{bmatrix}
-\infty \rightarrow (0 \equiv 1) \Rightarrow 2
\end{bmatrix}
\]

Notice that these functors arrange in a commutative square so that \( q_{\square} i_{\_\_\_} = i_{\_\_\_} q_\perp \), hence \( (q_{\square})_*(i_{\_\_\_})_* = (q_\perp i_{\_\_\_})_* = (i_{\_\_\_}q_\perp)_* \). Thus, we are reduced to verify that \( (-\infty)^*(q_{\square})_* \cong (-\infty)^* \). Let \( Y \in D(\square) \), to compute \( (-\infty)^*(q_{\square})_* Y \) one can use the axiom (Der4) (Kan extensions are pointwise in a derivator), so that

\[
(-\infty)^*(q_{\square})_* Y \cong \text{holim} \text{pr}^* Y
\]

where \( \text{pr} : (-\infty/q_{\square}) \rightarrow \square \) is the canonical functor from the comma category. It is easy to notice that the category \( (-\infty/q_{\square}) \) admits an initial object \( \emptyset \). Hence,

\[
\text{holim} \text{pr}^* Y \cong \emptyset^* \text{pr}^* Y \cong (-\infty)^* Y,
\]

since taking the homotopy limit over a category with an initial object \( \emptyset \) is the same as evaluating at \( \emptyset \). This concludes the proof, as this argument gives a canonical
morphism
\[ L \Rightarrow q_{L, *} X \rightarrow q_{L,*} L_{*} X \]
that can be easily seen to be invertible using (Der2).  \[ \Diamond \]

3. Reflective sub-prederivators

3.1. Generalities about reflections. In this section we study some basics concerning co/reflections of derivators. Related to this topic, see also [Hel88, CT11, Tab08]. A rather precise study of reflection theory for derivators has also been given in [Col18] (see in particular §3.3.3 –that we proved independently as Proposition 3.12–, §4.4.10, the account of regularity property).

We begin by recalling the following notion from category theory.

Definition 3.1. An adjunction \( L: C \rightleftarrows D: R \) in \( \text{CAT} \) is a reflection if the right adjoint is fully faithful and it is a co-reflection if the left adjoint is fully faithful.

Duality allows us to focus on reflections only. We will rarely mention co-reflections, as this dualization process is straightforward.

Remark 3.2. Let \( L: C \rightleftarrows D: R \) be a reflection in \( \text{CAT} \).

1. If \( \Sigma_L \) denotes the class of morphisms in \( C \) which are inverted by \( L \), then \( C \to D \) is a model for the “category of fractions” \( C \to C[\Sigma_L^{-1}] \) of \( C \) at the class \( S \) (see [Bor94a, Prop. 5.3.1]), and this localization is reflective.

2. If \( C \) is complete or cocomplete, then so is \( D \) ([Bor94a, Prop. 3.5.3, Prop. 3.5.4]).

In the notation of the above remark, one can give a description of the essential image of the fully faithful right adjoint in terms of \( S = \Sigma_L \), and we quickly recall the relevant notions.

Definition 3.3. Let \( L: C \rightleftarrows D: R \) be a reflection in \( \text{CAT} \) and let \( S \) be the class of morphisms inverted by \( L \).

\[ [\text{Lur09, 4.4.3.2}] \] gives a proof that homotopy pullbacks can be constructed using equalizers in the setting of quasicategories. The present proof can be seen as the analogue statement adapted to derivators (whenever \( D = D_C \) for a quasicategory \( C \), it is really equivalent): starting with \( X \in D(\mathbb{J}) \) we build a diagram whose incoherent image in \( D(\mathbb{J})^{-1} \) is
\[ X_0 \times X_1 \rightrightarrows X_2. \]
This is exactly \( \text{dia}_{\Rightarrow(q_{L,*})}X \) constructed above.
(1) An object \(X \in \mathcal{C}\) is \(S\)-local if for every \(f : Y \to Z\) in \(S\) the precomposition
\[
\mathcal{C}(Z, X) \xrightarrow{- \circ f} \mathcal{C}(Y, X)
\]
is a bijection.

(2) A morphism \(f : Y \to Z\) in \(\mathcal{C}\) is an \(S\)-local equivalence if
\[
\mathcal{C}(Z, X) \xrightarrow{- \circ f} \mathcal{C}(Y, X)
\]
is a bijection for every \(S\)-local object \(X \in \mathcal{C}\).

We denote the class of \(S\)-local objects in a category \(\mathcal{C}\) as \(\mathcal{S}_L\); sending a class of morphisms to its class of local objects, and then to the class of \(S\)-local equivalences sets up a closure operator \((\_)^{lw}\) on \(\text{hom}(\mathcal{C})\); among the classes of \((\_)^{lw}\)-closed there are all \(\Sigma\)'s. More precisely,

**Lemma 3.4.** Let \(L : \mathcal{C} \rightleftarrows \mathcal{D} : R\) be a reflection in \(\text{CAT}\) and let \(\mathcal{S}_L\) be the class of morphisms inverted by \(L\) (we shortly refer to these morphisms as the \(L\)-local equivalences).

1. The counit \(\varepsilon : LR \xrightarrow{\sim} \text{id}\) is an isomorphism as is the image \(L\eta : L \xrightarrow{\sim} LRL\) of the unit \(\eta\) under \(L\) (which is to say that \(\eta_X \in \mathcal{S}_L\) for \(X \in \mathcal{D}\)).
2. An object \(X \in \mathcal{C}\) lies in the essential image of \(R\) if and only if \(\eta_X\) is invertible if and only if \(X\) is \(\mathcal{S}_L\)-local.
3. A morphism is an \(\mathcal{S}_L\)-local equivalence if and only if it lies in \(S\).
4. An object \(X \in \mathcal{C}\) is \(\mathcal{S}_L\)-local if and only if
\[
\mathcal{C}(RLX, X) \xrightarrow{- \circ \eta_X} \mathcal{C}(X, X)
\]
is a bijection.

**Proof.** A right adjoint is fully faithful if and only if the counit is invertible. The invertibility of \(L\eta\) follows from the triangular identities. Statements (2) and (3) are left as exercises. Finally, since \(\eta_X\) lies by (1) in \(S\), the map \(\_ \circ \eta_X\) is bijective for every \(S\)-local \(X\). Conversely, suppose that \(\_ \circ \eta_X\) is bijective and let \(\alpha : RLX \to X\) such that \(\alpha \circ \eta_X = \text{id}_X\). In order to check that \(\eta_X \circ \alpha = \text{id}_{RLX}\) it suffices to consider the bijection
\[
\mathcal{C}(RLX, RLX) \xrightarrow{- \circ \eta_X} \mathcal{C}(X, RLX)
\]
for the \(S\)-local object \(RLX\) and to note that \(\eta_X \circ \alpha\) and \(\text{id}_{RLX}\) are both sent to \(\eta_X\).

We are now ready to define the main notion we will handle throughout the paper:
**Definition 3.5.** Let $L : D \leftrightarrows E : R$ be an adjunction in $\text{PDer}$ or $\text{Der}$.

1. The adjunction $L \dashv R$ is a reflection if $R$ is fully faithful, i.e. the counit $\varepsilon : LR \Rightarrow \text{id}$ is invertible.
2. The adjunction $L \dashv R$ is a co-reflection if $L$ is fully faithful, i.e. the unit $\eta : \text{id} \Rightarrow RL$ is invertible.

The notions of $S$-local objects and $S$-local equivalences can be extended in a straightforward way to this setting, to establish an analogue of Lemma 3.4. In particular, the essential image of $R$ coincides with the class of $L$-local objects; this suggests the existence of an identification $E \simeq D[\mathcal{S}^{-1}]$ for the domain of the right adjoint in a reflection, with respect to a suitable notion of “object of fractions” in $\text{PDer}$. We establish such an equivalence in Proposition 3.8 below.

If $L$ is a morphism of prederivators, we let $\mathcal{S}_L$ denote the sub-prederivator $\mathcal{S}_L(J)$ of all $X \in D[1](J)$ such that $\text{dia}[1]LX \in D(J)[1]$ is an isomorphism. It is called the prederivator of $S$-locals (but our crossed reference remains Definition 3.3, where the notion is defined for categories, as we will use quite often the prederivator analogue of Lemma 3.4).

**Definition 3.6.** Let $S \subseteq D[1]$ be a sub-prederivator. We say that a morphism of prederivators $F : D \to E$ inverts $S$ if for each $J \in \text{Dia}$ and $X \in S(J)$, $\text{dia}[1]F^{[1]}_J(X)$ is an isomorphism in $E(J)$.

**Definition 3.7** (prederivators of fractions). Let $D$ be a prederivator, and $S \subseteq D[1]$ a sub-prederivator. Define $\text{PDer}_S(D, E)$ to be the subcategory of $\text{PDer}(D, E)$ made by all $F : D \to E$ that invert $S$.

We call the prederivator of fractions of $D$ with respect to $S$ the prederivator (when it exists, unique up to isomorphism) $D[S^{-1}]$ with a canonical morphism $\gamma : D \to D[S^{-1}]$ having the property that $L^! : \text{PDer}(D[S^{-1}], E) \to \text{PDer}(D, E)$ induces an equivalence of the domain with the subcategory $\text{PDer}_S(D, E)$, so that

$$\text{PDer}_S(D, E) \cong \text{PDer}(D[S^{-1}], E).$$

It is worth to notice that the usual care is needed about the presence of a derivator of fractions $D[S^{-1}]$ in our universe. The standard choice amounts to choose a big enough universe of sets $\text{SET}$ for which $D[S^{-1}] : \text{Dia} \to \text{CAT}$ is a 2-functor.

---

3Recall from [Gro13] that there is a natural functor $J \to [D(J), D(e)]$ regarding every morphism $\varphi : i \to j$ as a natural transformation of the corresponding classifying functors; in the cartesian closed structure of $\text{CAT}$ this corresponds to a functor $D(J) \to [J, D(e)]$, and we call this the underlying diagram functor.
Proposition 3.8. The following conditions are equivalent for an adjunction $L : D \rightleftarrows E : R$ of prederivators:

- $L \dashv R$ is a reflection of $D$ onto $E$;
- The pair $(E, L)$ exhibits the prederivator of fractions $D[\Sigma L^{-1}]$.

Proof. The proof is a slick adaptation of a 2-categorical argument: let $L \xrightarrow{\eta} R$ be the adjunction in subject. Then $PDer(R, X) \xrightarrow{\eta^!} PDer(L, X)$ is an adjunction

$$PDer(R, X) : PDer(D, X) \rightleftarrows PDer(E, X) : PDer(L, X),$$

which is moreover still a reflection (because $\eta^!$ remains an isomorphism); the essential image of $L^!$ can be then easily characterized as the subcategory $PDer_{SL}(D, X)$.

The notion of reflection can be specialized in several different ways:

Definition 3.9. An adjunction of prederivators $L : D \rightleftarrows E : R$ is said to be

- **essential** if $L$ has a left adjoint $Z$;
- **Frobenius** if it is essential and $Z \cong R$;
- **(left) exact** if $L$ commutes with finite right Kan extensions.

Definition 3.10 (regular prederivator). Let $D$ be a prederivator satisfying axiom (Der3). We say that $D$ is **regular** when given a finite category $K$ and when $\omega$ is regarded as a category in the obvious way, we have that the diagram

$$\begin{array}{ccc}
\text{holim}_K \times \omega & \xrightarrow{\text{holim}_K} & \text{holim}_K \\
D(K \times \omega) & \xleftarrow{K \times \text{pt}_{\omega^!}} & D(K) \\
\text{pt}_{\omega^!} & \xrightarrow{\rho_D} & D(\omega)
\end{array}$$

commutes up to an invertible 2-cell $\rho_D$.

This definition comes from [Hel88, IV.5]; see also [CT11, CT12]. Note that [Gro16a] every stable derivator is regular.

Proposition 3.11. Let $D$ be a prederivator, and $L \dashv R : D \to E$ a left exact reflection of $D$. If $D$ is regular, then so is $E$. 

Proof. The proof is simple, and follows analyzing the pasting diagram

\[
\begin{array}{ccc}
\mathbb{E}(K \times \omega) & \xrightarrow{K \times pt_{\omega,t}} & K \times \mathbb{E}(K) \\
\downarrow & & \downarrow \\
\mathbb{D}(K \times \omega) & \xrightarrow{\gamma} & \mathbb{D}(K) \\
\downarrow & & \downarrow \\
\mathbb{D}(\omega) & \xrightarrow{\imath_{\omega}} & \mathbb{D}(e) \\
\downarrow & & \downarrow \\
\mathbb{E}(\omega) & \xrightarrow{\gamma''} & \mathbb{E}(e) \\
\end{array}
\]

(the coincidence of this pasting square with the canonical filling \(\rho_{\mathbb{E}}\) is an easy check).

The most important closure property enjoyed by a reflection of a prederivator in \(\text{PDer}\) is, however, under the properties that define a derivator: this appears as [Col18, 3.3], where the general theory of localizations is laid down in order to give a more modern account of the main theorem in [Hel97].

Proposition 3.12. Let \(L : \mathbb{D} \rightleftharpoons \mathbb{E} : R\) be an adjunction of prederivators.

1. If \(\mathbb{D}\) is a derivator and \(R\) is fully faithful, then also \(\mathbb{E}\) is a derivator.
2. If \(\mathbb{E}\) is a derivator and \(L\) is fully faithful, then also \(\mathbb{D}\) is a derivator.

Proof. By duality it suffices to take care of the first statement, and we establish each of the axioms \((\text{Der}_1)-(\text{Der}_4)\) individually.

\((\text{Der}_1)\) Let \(J_1, J_2 \in \text{Dia}\), and consider the obvious inclusions \(J_1 \xrightarrow{m} J_1 \sqcup J_2 \xleftarrow{n} J_2\).

Consider now the following commuting diagram:

\[
\begin{array}{ccc}
\mathbb{D}(J_1 \sqcup J_2) & \xrightarrow{[u_1^* \circ u_2]^t} & \mathbb{D}(J_1) \times \mathbb{D}(J_2) \\
\downarrow_{L_{J_1 \sqcup J_2}} & & \downarrow_{R_{J_1 \sqcup J_2}} \\
\mathbb{E}(J_1 \sqcup J_2) & \xrightarrow{[u_1^* \circ u_2]^t} & \mathbb{E}(J_1) \times \mathbb{E}(J_2) \\
\end{array}
\]

By \((\text{Der}_1)\) for \(\mathbb{D}\), the arrow \(\mathbb{D}(J_1 \sqcup J_2) \rightarrow \mathbb{D}(J_1) \times \mathbb{D}(J_2)\) is an equivalence. By commutativity, and using Lemma 3.4, it is easy to check that this implies that the arrow \(\mathbb{E}(J_1 \sqcup J_2) \rightarrow \mathbb{E}(J_1) \times \mathbb{E}(J_2)\) is also an isomorphism. Furthermore,
by (Der1) for \( \mathcal{D} \), \( \mathcal{D}(\emptyset) \) is not the empty category and so, by the existence of the functor \( R_\emptyset : \mathcal{D}(\emptyset) \to \mathcal{E}(\emptyset) \), we deduce that \( \mathcal{E}(\emptyset) \) is not the empty category.

(Der2) Let \( f \) be a morphism in \( \mathcal{E}(J) \) for some \( J \in \text{Dia} \). Then, \( f \) is an isomorphism if and only if \( R_J(f) \) is an isomorphism which happens, by (Der2), if and only if \( R_J(f)_j \) is an isomorphism for any \( j \in J \). This last condition is clearly equivalent to the statement that \( R_\emptyset(f)_j \) is an isomorphism for any \( j \in J \), which means exactly that \( f_j \) is an isomorphism for any \( j \in J \).

(Der3) Consider a functor \( u : A \to B \) in \( \text{Dia} \) and let \( u_* : \mathcal{D}(A) \to \mathcal{D}(B) \) be the right Kan extension in \( \mathcal{D} \). For \( X \in \mathcal{E}(A) \) we show that \( u_* R_A X \) lies in the essential image of \( R_B \). By Lemma 3.4 it suffices to show that the top horizontal morphism in

\[
\begin{array}{ccc}
\mathcal{D}(B)(R_B L_B u_* R_A X, u_* R_A X) & \xrightarrow{\eta u_* R_A} & \mathcal{D}(B)(u_* R_A X, u_* R_A X) \\
\downarrow & & \downarrow \\
\mathcal{D}(A)(u^* R_B L_B u_* R_A X, R_A X) & \xrightarrow{\iota} & \mathcal{D}(A)(u^* u_* R_A X, R_A X) \\
\downarrow & & \downarrow \\
\mathcal{E}(A)(L_A u^* R_B L_B u_* R_A X, X) & \xrightarrow{\iota} & \mathcal{E}(A)(L_B u^* u_* R_A X, X) \\
\downarrow & & \downarrow \\
\mathcal{E}(A)(u^* L_B R_B L_B u_* R_A X, X) & \xrightarrow{\cong} & \mathcal{E}(A)(u^* L_B u_* R_A X, X)
\end{array}
\]

is an isomorphism. By naturality this diagram commutes, and the bottom horizontal is invertible by Lemma 3.4. For every \( X \in \mathcal{E}(B) \) and \( Y \in \mathcal{E}(A) \) there is a chain of natural isomorphisms

\[
\mathcal{E}(B)(X, L_B u_* R_A Y) \cong \mathcal{D}(B)(R_B X, R_B L_B u_* R_A Y) \\
\cong \mathcal{D}(B)(R_B X, u_* R_A Y) \\
\cong \mathcal{D}(A)(u^* R_B X, R_A Y) \\
\cong \mathcal{D}(A)(R_A u^* X, R_A Y) \\
\cong \mathcal{E}(A)(u^* X, Y),
\]

showing that \( L_B u_* R_A \) is a model for the right Kan extension along \( u \) in \( \mathcal{E} \).

For the existence of left Kan extensions in \( \mathcal{E} \), let \( u_! : \mathcal{D}(A) \to \mathcal{D}(B) \) be a left Kan extension functor in \( \mathcal{D} \), \( X \in \mathcal{E}(A) \) and \( Y \in \mathcal{E}(B) \). The natural
bijections
\[ \mathbb{E}(B)(L_B u R_A X, Y) \cong \mathbb{D}(B)(u_R A X, R_B Y) \]
\[ \cong \mathbb{D}(A)(R_A X, u^* R_B Y) \]
\[ \cong \mathbb{D}(A)(R_A X, R_A u^* Y) \]
\[ \cong \mathbb{E}(A)(X, u^* Y) \]

show that \( L_B u R_A \) is a model for the left Kan extension along \( u \) in \( \mathbb{E} \).

(Der) Let \( u : A \to B \) be a functor in \( \text{Dia} \) and let \( b \in B \). Using the explicit form of homotopy Kan extensions in \( \mathbb{E} \) described above, it is not difficult to use the fact that homotopy Kan extensions are computed point-wise in \( \mathbb{D} \) to show that the same happens in \( \mathbb{E} \).

Remark 3.13. Let \( L : \mathbb{D} \rightleftarrows \mathbb{E} : R \) be a reflection of derivators and let \( u : A \to B \). Left and right Kan extension functors in \( \mathbb{E} \) along \( u \) are respectively given by

\[ L_B u R_A : \mathbb{E}(A) \to \mathbb{E}(B) \]
\[ L_B u^* R_A : \mathbb{E}(A) \to \mathbb{E}(B). \]

We record here the definition of a choric reflection: sharpening Definition 3.5 in this way ensures that there is some control on the coherence with which the unit map \( \eta : 1 \Rightarrow RL \) is given.

Definition 3.14 (Choric reflection). Let \( L \overset{\eta}{\to} R : \mathbb{D} \rightleftarrows \mathbb{E} \) be a reflection; we define \( \text{Chr}_L \) to be the sub-prederivator

\[ \text{Chr}_L = \{ X \in \mathbb{D}^{[1]} | X_1 \in \mathbb{E}(e) \} \subseteq \mathbb{D}^{[1]} \]

Now, we define \( R^t : \text{Chr}_L \to \mathbb{D} \) to be the restriction of evaluation at 0, i.e. \( R^t := 0^*|_{\text{Chr}_L} \).

We say that the reflection \((L, R)\) is choric if \( R^t \) admits a left adjoint \( L^s \).

Remark 3.15. It is worth to spell out explicitly how some incoherent diagrams associated to the adjunction \( L^t \eta^t \Rightarrow R^s \) look like. Every \( X \in \mathbb{D}(\Box) \) induces \( L^t X \in \text{Chr}_L(\Box) \subseteq \mathbb{D}^{[1]}(\Box) = \mathbb{D}(\Box) \), and

\[ \text{dia}(X) = \]

\[ \text{RLX}_{oo} \]
\[ \Rightarrow \]
\[ \text{RLX}_{o1} \]
\[ \downarrow \]
\[ \text{X}_{o0} \]
\[ \text{RLX}_{10} \]
\[ \Rightarrow \]
\[ \text{RLX}_{11} \]
\[ \downarrow \]
\[ \text{X}_{10} \]
\[ \text{RLX}_{11} \]
\[ \Rightarrow \]
\[ \text{X}_{11} \]

in such a way the L and R faces of the cube correspond to naturality squares of the unit of \( L \dashv R \), and the B face correspond to the rule “apply \( L \) to the square \( X \)".
Remark 3.16. As noticed, if \( L \dashv R \) is a choric reflection, so that there is an induced adjunction \( L^\triangleright \dashv R^\triangleright \), the latter adjunction gives a coherent choice of liftings of unit components for the former adjunction. This motivates the notation, as \( (\_)^\triangleright \) “lifts the adjunction (by a semitone)”.

It may appear as if this notion is given with the only purpose to obtain an ad-hoc control of coherence for the unit \( \eta \); it is nevertheless possible to prove that many reflections are choric (see Proposition 3.40, where we prove that choric reflections arise as those associated to choric factorization systems (Definition 3.38), and every factorization system which is defined by an algebra structure for the squaring monad ([LV17a, Thm III]) on \( \text{PDer} \) is in fact choric).

We end this introductory subsection gathering a few examples of reflections of prederivators and derivators:

Example 3.17. Let \( i : \text{Dia} \to \text{CAT} \) be the inclusion functor. For every category \( \mathcal{C} \) there is the represented prederivator \( \mathcal{y}_C \) obtained via the map \( \mathcal{C} \mapsto \mathcal{D}_C := \text{CAT}(i(\_), \mathcal{C}) \) (the “restricted Yoneda embedding”), so that an adjunction \( L : \mathcal{C} \leftrightarrow \mathcal{D} : R \) in \( \text{CAT} \) is a reflection if and only if the induced adjunction \( \mathcal{y}_F : \mathcal{y}_C \leftrightarrow \mathcal{y}_D : \mathcal{y}_G \) is.

This has interesting specific sub-examples:

Example 3.18. As a special case of particular relevance later there are the following adjunctions

\[
\text{Sp}_{\geq 0} \leftrightarrows \text{Sp} \quad \text{and} \quad \text{Sp} \leftrightarrows \text{Sp}_{\leq 0}
\]

exhibiting the derivator \( \text{Sp}_{\geq 0} \) of connective spectra as a co-reflection of \( \text{Sp} \) and the derivator \( \text{Sp}_{\leq 0} \) of coconnective spectra as a reflection of \( \text{Sp} \). More generally, let \( k \in \mathbb{Z} \). The full sub-derivator \( \text{Sp}_{\leq k} \subseteq \text{Sp} \) of \( k \)-coconnective spectra is closed under arbitrary limits and the full sub-derivator \( \text{Sp}_{\geq k} \subseteq \text{Sp} \) of \( k \)-connective spectra is closed under arbitrary colimits. This translates into a reflection between the associated derivators.

Example 3.19. If \( p \) is a prime number, the class of \( p \)-acyclic chain complexes of abelian groups \( \{X \_ \mid X_\_ \otimes \mathbb{Z}/p\mathbb{Z} \cong 0\_\} \) is a co-reflection of \( \text{Ch}(\mathbb{Z}) \). This translates into a reflection between the associated derivators.

Example 3.20. Let \( F : M \leftrightarrows N : G \) be a Quillen adjunction between Quillen model categories and let

\[
LF : \text{Ho}_M \leftrightarrows \text{Ho}_N : RG
\]

be the induced derived adjunction. If \( (F,G) \) is a left Bousfield localization, then the derived adjunction is a reflection. Similarly, right Bousfield localizations induce co-reflections.
Example 3.21. The Kan extension adjunctions associated to fully faithful functors $u : A \to B$ yield co-reflections and reflections,

$$u_! : D^A \rightleftarrows D^B : u^* \quad \text{and} \quad u^* : D^B \rightleftarrows D^A : u_*.$$  

Example 3.22 (Recollements of stable derivators). Let $j : I \to J$ be a sieve (Definition 2.8), and let $i : K \to J$ be the subcategory spanned by the complement of the image of $j$. Let $D$ be a stable derivator. Then we can obtain recollements [BBD82] of triangulated categories

$$D(I) \xrightarrow{j^*} D(J) \xrightarrow{i_!} D(K) \quad \text{and} \quad D(K) \xrightarrow{i^*} D(J) \xrightarrow{j_*} D(I)$$

where the functors $j^*, i^*$ are respectively called co-exceptional and exceptional inverse images of $j$ and $i$.

This motivates the following definition: a **recollement of derivators** is an arrangement of morphisms of derivators

$$D' \xrightarrow{i_L} D \xleftarrow{i_R} D'' \xrightarrow{q_L} \xleftarrow{q_R} D'$$

satisfying the following conditions

R1) There are adjunctions $i_L \dashv i \dashv i_R$ and $q_L \dashv q \dashv q_R$;

R2) The functors $i, q_L, q_R$ are all fully faithful;

R3) The image of $i$ equals the **essential kernel** of $q$, i.e. the full subcategory of $D$ such that $qX \cong 0$ in $D^1$;

R4) The natural homotopy commutative diagrams

$$
\begin{array}{ccc}
qLq & \xrightarrow{\epsilon(qL^{-q})} & \text{id}_D \\
\downarrow & & \downarrow \eta(iL^{-i}) \\
0 & \xrightarrow{\eta(qiL^{-q})} & qRq
\end{array} \quad \quad \begin{array}{ccc}
iqR & \xrightarrow{\epsilon(iR^{-i})} & \text{id}_D \\
\downarrow & & \downarrow \eta(q^{-qR}) \\
0 & \xrightarrow{\eta(q^{-qR})} & qRq
\end{array}
$$

induced by the previous axioms are both cartesian and cocartesian.
It is immediate to see that every sieve induces two recollements of derivators between the shifted derivators

\[
\begin{array}{ccc}
\mathcal{D}^J & \xrightarrow{j^*} & \mathcal{D}^K \\
\downarrow{j} & & \downarrow{i} \\
\mathcal{D}^I & \xrightarrow{i^*} & \mathcal{D}^J
\end{array}
\]

such that the recollements in (⋆) correspond to the evaluation of this diagram on the base. Recollements situation are quite natural ways to build reflections (and in fact more: a pair of reflective and bireflective sub-prederivators) of a given stable derivator \(\mathcal{D}\).

More conceptual examples of reflections are the following:

**Example 3.23.** Regarded as a 2-category, \(\text{PDer}\) supports a calculus of Kan extensions;\(^4\) in particular, we can define a 1-cell \(G : \mathcal{D} \to \mathcal{E}\) in \(\text{PDer}\) to be dense if the left extension \(\text{lan}_G : \mathcal{E} \to \mathcal{E}\) exists, is pointwise and isomorphic to the identity \(\text{id}_\mathcal{E}\).

In presence of a Yoneda structure on a 2-category \(\mathcal{K}\), we can characterize a dense \(G : \mathcal{D} \to \mathcal{E}\) as a 1-cell such that \(\mathcal{E}(G, 1) = \text{lan}_G \mathcal{Y}_\mathcal{D}\) is fully faithful:

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{G} & \mathcal{E} \\
\downarrow{\mathcal{Y}_\mathcal{D}} & & \downarrow{\text{lan}_G \mathcal{Y}_\mathcal{D}} \\
\mathcal{D} & \xleftarrow{\text{lan}_G \mathcal{Y}_\mathcal{D}} & \mathcal{E}
\end{array}
\]

when the 1-cell \(\mathcal{E}(G, 1)\) admits a left adjoint, this determines a reflection of \(\mathcal{D}\) (the *Yoneda object* [SW78] associated to \(\mathcal{D}\)). It is tempting to extend this characterization to the 2-category of prederivators (or a suitable sub-2-category thereof): a thorough discussion of all these issues will be the subject of a separate work.

\(^4\)Every now and then we will implicitly rely on the following abstract result: \(\text{PDer}\) is at the same time the category of algebras for a 2-monad, and the 2-category of coalgebras for a 2-comonad, on the 2-category \([\text{Dia}_0, \text{CAT}]\) (\(\text{Dia}_0\) is the class of objects of \(\text{Dia}\)). More explicitly, there is a triple of adjoints

\[
\begin{array}{c}
[\text{Dia}^\text{op}, \text{Cat}] \\
\downarrow{\text{Lan}_H} \\
[\text{Dia}^\text{op}, \text{Cat}]_{\text{Ran}_H}
\end{array}
\]

induced by the inclusion \(H : \text{Dia}_0 \to \text{Dia}\). The monad of the adjunction \(\text{Lan}_H \dashv (\_ \circ H)\) and the comonad of the adjunction \((\_ \circ H) \dashv \text{Ran}_H\) do the job.
3.2. **Calculus of fractions.** Left exact reflections can be characterized via a derivator-theoretic analogue of [Bor94a, 5.6.1]: we first introduce the necessary terminology.

**Definition 3.24.** Let \( S \subseteq \mathbb{D}^{[1]} \) be a sub-prederivator; we say that \( S \) is **wide** if it is closed under composition \(^5\) and contains all isomorphisms.

**Definition 3.25.** A sub-prederivator \( S \subseteq \mathbb{D}^{[1]} \) is said to **satisfy the right Ore condition** if for every \( J \in \text{Dia} \) the following two conditions are satisfied:

1. for every \( X \in \mathbb{D}^{J(\square)} \) such that \( X(0,1) \in S(J) \), there exists an \( X' \in \mathbb{D}^{J(\square)} \) such that \( X' \cong X \) and \( X(-\infty,0) \in S(J) \).

2. for every \( X \in \mathbb{D}^{J(\text{coeq})} \) such that \( X(1,\infty) \) is in \( S(J) \), there exists an \( X' \in \mathbb{D}^{J(\text{coeq})} \) such that \( X(\infty,0) \in S(J) \).

**Definition 3.26.** A sub-prederivator \( S \hookrightarrow \mathbb{D}^{[1]} \) is said to **admit a right calculus of fractions** if it is wide and satisfies the right Ore condition.

It turns out that choric reflection allow to find an analogue of [Bor94a, 5.6.1] for left exact reflections of derivators:

**Proposition 3.27.** The following conditions are equivalent for a choric reflection \( L : \mathbb{D} \rightleftarrows \mathbb{E} : R \) of derivators:

1. **cf1)** The left adjoint \( L \) commutes with finite limits, so the reflection \( L \dashv R \) is left exact (Definition 3.9);

2. **cf2)** The sub-prederivator \( S_L \) is homotopy pullback stable: if \( X \in \mathbb{D}(\square) \) is cartesian and such that \( X_E \in S_L(e) \), then \( X_W \in S_L(e) \).

3. **cf3)** The sub-prederivator \( S_L \) is closed under finite right Kan extensions in \( \mathbb{D} \).

**Proof.** The proof proceeds in various steps that we state as different items: first, notice that Lemma 2.20 above shows that \( \text{cf1} \) is equivalent to the request that \( L \) commutes with terminal objects and homotopy pullbacks, and to the request that \( L \) commutes with products and homotopy equalizers. We will freely use this equivalence throughout the present proof.

(1) To show that \( \text{cf1} \) implies \( \text{cf3} \) assume that \( L \dashv R \) is a left exact reflection. Recall that \( \text{cf3} \) says that whenever we have a diagram \( Y \in \mathbb{D}(\mathcal{B}) \) whose

\(^5\)This means that whenever \( X \in \mathbb{D}^{[2]}(J) \) is such that \( X(0,1), X(1,2) \in S(J) \) then also \( X(0,2) \in S(J) \); as a side note, we remark that in a similar fashion we can define cancellation properties for a sub-prederivator \( S \): see [LV17b, 3.17] for more on this.
underlying diagram is

\[
\begin{array}{c}
Y_{001} \quad Y_{011} \\
Y_{000} \quad Y_{010} \quad Y_{100} \quad Y_{110} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y_{000} \quad Y_{100} \quad Y_{011} \quad Y_{111} \\
\end{array}
\]

where each vertical arrow \(s_1, s_2, s_3\) is in \(S_L\), and the \(U, D\) faces are cartesian, then also \(s_4 \in S_L\). Assuming \(\text{CF}1\) and applying \(L\) to it, we get that the \(U, D\) faces remain cartesian, and \(Ls_4\) is the unique morphism connecting two pullbacks of isomorphic diagrams; this entails that \(Ls_4\) is invertible.

(2) We show that \(\text{CF}2\) implies \(\text{CF}1\). The functor \(L\) commutes with finite right Kan extensions if and only if it commutes with finite limits [Gro13, 2.4], and by our Lemma 2.20, this latter condition is true if and only if \(L\) commutes with homotopy pullbacks (as \(L\) already preserves terminal objects).

We show that given a cartesian square \(i_{\Delta^n} Y = \begin{array}{c} Y_{-\infty} \rightarrow Y_0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y_1 \quad Y_2 \end{array}\) where \(Y_E\) lies in \(S_L^{\pm}\), then \(L\) preserves it, and then we show that this is in fact sufficient. Given such a \(Y\), note that also \(Y_E\) lies in \(S_L^{\pm}\), as this class is closed under pullback. Now, we can embed \(Y\) as the \(F\) face of a cube \(L^\#(Y)\) (as \(L \dashv R\) is a choric reflection)

\[
\begin{array}{c}
RLY_{-\infty} \quad RLY_0 \\
Y_{-\infty} \quad Y_0 \\
\downarrow \quad \downarrow \\
Y_1 \quad Y_2 \\
\end{array}
\]

Now \(L\) commutes with pullbacks if and only if the \(B\) face of this cube is cartesian; unwinding the definitions, and employing (Der2), this is equivalent to say that in the diagram \(L^\#Y \rightarrow i_{\Box^n} L^\#i_{\Box^n} Y\), whose underlying diagram is
the arrow $\zeta : LY_{-\infty} \to P$ is an isomorphism; this is equivalent to ask that $\zeta$ lies in $S_L \cap S_L^+$ (since $(S_L, S_L^+)$ is a prefactorization). Now since $P \in E(e)$, the arrow $\zeta$ lies in $E(e)[1] = (\perp E(e)[1])\perp = \mathcal{M}$, so we only have to show that $\zeta$ gets inverted by $L$. To see this, consider the juxtaposition of squares

$$
\begin{array}{ccc}
Y_{-\infty} & \xrightarrow{\eta_{\infty}} & RLY_{-\infty} \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\eta_1} & RLY_1
\end{array}
$$

the whole diagram is homotopy cartesian, so that (since we assume that $S_L$ is closed under pullback) $\zeta \circ \eta_{\infty}$ lies in $S_L$; but then, so do $\eta_1$ and $P \to RLY_1$, so (since $S_L$ is a 3-for-2 class), $\zeta \in S_L$.

It remains to show that now $L$ preserves pullbacks: given any $i_{\gamma} Y = Y_{-\infty} \xrightarrow{\gamma_{\infty}} Y_0 \xrightarrow{\gamma} Y_1 \xrightarrow{\gamma} Y_2$ we can always $\mathcal{F}$-factor its columns (because $(S_L, S_L^+)$ is a DFS and the functor $\mathcal{D}\mathcal{F}([1]) \to \mathcal{D}[1]([1])$ is essentially surjective), to obtain

$$
\begin{array}{ccc}
Y_{-\infty} & \xrightarrow{s_{\infty}} & Y_0 \\
\downarrow & & \downarrow \\
F & \xrightarrow{s''} & F'
\end{array}
$$

now the upper square is cartesian (and the left column exhibits the factorization of $Y_W$) since $S_L$ is closed under pullback, so that $L$ preserves it; the lower square is cartesian, and the above argument shows that $L$ preserves it as well.

(3) Now we prove that $\text{CF}_3$ implies $\text{CF}_2$: let $i_{\delta} Y = Y_{-\infty} \xrightarrow{\delta_{\infty}} Y_0 \xrightarrow{\delta} Y_1 \xrightarrow{\delta} Y_2$ be a cartesian square such that $Y_E \in S_L$, and consider the cube
it is easy to see that its \( U = B, F \) and \( D \) faces are cartesian and that the west arrow of its \( U \) face and the west arrow of its \( L \) face are isomorphic. The latter arrow lies in \( S_L \) since the class is 3-for-2, and we can conclude. \( \diamondsuit \)

**Remark 3.28.** It would be really tempting to prove that the three conditions above are in turn equivalent to the following statement, as in categories:

\( \text{cf}_4 \) The sub-prederivator \( S_L \subseteq D \) of \( L \)-locals (Definition 3.3 adapted to prederivators) admits a right calculus of fractions.

It is in fact quite easy to show that any of the three conditions implies \( \text{cf}_4 \). The equivalence is of course true in case \( D \) is a discrete prederivator; it seems to be a difficult task to prove that condition \( \text{cf}_4 \) is sufficient to entail the left exactness of \( L \) (and to build a concrete presentation of the prederivator \( D[J^\infty] \)).

**Lemma 3.29.** Let \( F: D \to E \) be a conservative morphism of derivators and let \( u: A \to B \) be fully faithful. If \( F \) preserves left Kan extensions along \( u \), then \( F \) reflects left Kan extensions along \( u \).

**Proof.** Since \( u: A \to B \) is fully faithful, the same is true for the left Kan extension functors \( u_! \) in \( D \) and \( E \), and we can hence characterize the respective essential images by the invertibility of the counits \( \varepsilon: u_!u^* \to \text{id} \). Given a diagram \( X \in D(B) \) by [Gro16b, 3.10] there is a commutative diagram

\[
\begin{array}{ccc}
  u_!F\gamma^* & \xrightarrow{\cong} & Fu_!u^* \\
  \gamma^{-1} \downarrow \cong & & \downarrow F\varepsilon \\
  u_!u^*F & \xrightarrow{\varepsilon F} & F
\end{array}
\]

in which the unlabeled morphism is the canonical mate expressing the compatibility of \( F \) with \( u_! \). Since \( F \) is assumed to preserve left Kan extensions along \( u \), this canonical mate is an isomorphism. Thus, \( FX \) lies in the essential image of \( u_! : E(A) \to E(B) \) if and only if \( \varepsilon F \) is an isomorphism if and only if \( F\varepsilon \) is an isomorphism. Since \( F \) is conservative this is the case if and only if \( \varepsilon: u_!u^*X \to X \) is an isomorphism which is to say that \( X \) lies in the essential image of \( u_! : D(A) \to D(B) \). \( \diamondsuit \)

**Corollary 3.30.** Let \( F: D \to E \) be a conservative morphism of derivators and let \( A \in \text{Cat} \). If \( F \) preserves colimits of shape \( A \), then \( F \) reflects colimits of shape \( A \).

**Proof.** Since \( F \) preserves colimits of shape \( A \) if and only if it preserves left Kan extensions along the fully faithful functor \( i_A: A \to A^\circ \) ([Gro16b, Prop. 3.9]), this statement is immediate from Lemma 3.29. \( \diamondsuit \)
Corollary 3.31.  
(1) Co-reflections of derivators preserve and reflect colimits.  
(2) Reflections of derivators preserve and reflect limits.  
(3) Equivalences of derivators preserve and reflect limits and colimits.

Proof. This is immediate from Corollary 3.30.

Remark 3.32. Let \( L : \mathcal{D} \rightleftharpoons \mathcal{E} : R \) be a reflection of derivators.

(1) The left adjoint \( L \) preserves left Kan extensions.

(2) The left adjoint \( L \) preserves right Kan extensions of diagrams in the essential image of \( R \).

(3) The right adjoint \( R \) preserves and reflects right Kan extensions.

It only remains to verify the second claim, and for that purpose let \( u : A \to B \) be in \( \text{Cat} \). In order to show that the canonical mate \( Lu_*R \to u_*LR \) is invertible, it suffices to consider the following diagram

\[
\begin{array}{ccc}
L(Ru_*) \sim & \to & Lu_*R \to u_*LR \\
\varepsilon \sim & \overset{\text{id}}{\downarrow} & \sim \varepsilon \\
\overset{u_*}{\downarrow} & & \overset{u_*}{\downarrow}
\end{array}
\]

which commutes by functoriality of canonical mates and [Gro16b, 3.11].

Definition 3.33. Let \( F : \mathcal{D} \to \mathcal{E} \) be a morphism of prederivators; we say that \( F \) creates homotopy left (resp., right) Kan extensions along \( u : A \to B \) if for every \( X \in \mathcal{D}(B) \),

- the object \( X \) lies in the essential image of \( u_!^D \) (resp., \( u_*^D \)) if and only if \( F_BX \) lies in the essential image of \( u_!^F \) (resp., \( u_*^D \));
- \( F \) preserves homotopy left (resp., right) Kan extensions along \( u \).

3.3. Orthogonality and co/reflections. The classical theory motivates a deeper glance at the interaction between reflection of (pre)derivators and the orthogonality relation that can be defined for the objects of \( \text{PDer} \); such theory has been used in [LV17a] to study \( t \)-structures on stable derivators, finding a derivator analogue of the main theorem in [Lor16], that “\( t \)-structures are normal torsion theories”.

We first recall the definition of coherent orthogonality from [LV17b]: let \( \mathcal{D} \) be a prederivator satisfying axiom (Der3). Let us consider the string of adjoints

\[
\begin{array}{c}
\mathcal{D}([1]) \overset{pr_*}{\longrightarrow} \mathcal{D}(e) \\
\overset{pt_*}{\longrightarrow}
\end{array}
\]
where \( pt_1 \) (resp. \( pt_\ast \)) sends a coherent diagram \( X = \left[ \begin{array}{c} X_0 \\ X_1 \end{array} \right] \in D([1]) \) into its target object \( X_1 \) (resp. source object \( X_0 \)) in the base of the derivator. Given this, we define coherent orthogonality as follows.

**Definition 3.34 (coherent orthogonality).** Two coherent diagrams \( X, Y \in D([1]) \) are called (coherently) orthogonal if the unit morphism \( \eta_X : X \to pt^\ast pt_1 X \) becomes invertible once the functor \( D([1])(_,Y) \) is applied, i.e. if the arrow

\[
D(X_1,Y_0) = D([1])(pt_1 X, pt_\ast Y) \cong D([1])(pt^\ast pt_1 X, Y) \xrightarrow{D([1])(\eta_X,Y)} D([1])(X,Y)
\]

is an isomorphism.

The orthogonality relation is denoted \( X \perp Y \) and defines the condition that “every commutative square having \( X \) on the left and \( Y \) on the right admits a unique filler, and coherently so” in the context of derivators. This paves the way to the following definition, based on the fact that, dealing with classical orthogonality, we can blur the distinction between objects and their initial or terminal arrows.

**Definition 3.35 (co/locality and orthogonality).** Let \( X, Y \in D([1]), X_1, Y_0 \in D(e) \).

1. We say that \( Y_0 \) is \( X \)-local, or \( X \dashv Y_0 \), if \( X \vdash \left[ \begin{array}{c} Y_0 \\ X_1 \end{array} \right] \);
2. We say that \( X_1 \) is \( Y \)-colocal, or \( X_1 \leq Y \), if \( \left[ \begin{array}{c} \emptyset \\ X_1 \end{array} \right] \leq Y \);
3. We say that \( X_1 \) and \( Y_0 \) are mutually orthogonal, and write \( X_1 \leq Y_0 \), if \( X_1 \) is \( \left[ \begin{array}{c} Y_0 \\ X_1 \end{array} \right] \)-colocal, or equivalently \( Y_0 \) is \( \left[ \begin{array}{c} \emptyset \\ X_1 \end{array} \right] \)-local.

This last condition means in particular that \( D(e)(X_1,Y_0) \) is reduced to a singleton.

Notice that, in the above notation, \( X_1 \leq Y_0 \) if and only if \( D(e)(X_1,Y_0) = 0 \) and so, depending on the context, we sometimes also use the more common notation \( X_1 \perp Y_0 \) to mean the same as \( X_1 \leq Y_0 \). Certain slight abuses of notation are now straightforward to understand: we can define orthogonality, as well as co/locality, with respect to a chosen class of \( \{X_\alpha\}_{\alpha \in A} \in D([1]) \) and this gives the usual Galois connection

\[
\dashv (-) \dashv (-)^\perp,
\]

which also allows to speak about the pairs \((S,S^\perp)\) and \((\dashv S,S)\) generated by a sub-prederivator \( S \). The notion of coherent orthogonality is used in [LV17b] to lay the foundation of the theory of factorization systems on derivators, and then a theory of coherent \( t \)-structures as a consequence (see [FL16] for the characterization of \( t \)-structures as “normal torsion theories”). A general survey of the main features of
factorization systems will be the subject of a subsequent work; for the moment we only record that [GLV17] already contains a rather general result, since it characterizes certain factorization systems as algebra structures for the “squaring” 2-monad of [KT93].

**Definition 3.36** (pre/factorizations and crumblings). A derivator prefactorization system \( \text{dpfs} \) for short) on a derivator \( D \) is defined to be a pair \( E, M \) such that \( E = p = M \) and \( M = E^p \). A derivator factorization system \( \text{dfs} \) for short) on a derivator \( D \) is defined to be a pair \( (F, \Psi F) \) where \( F \) is a dpfs and \( \Psi F \) is a functorial factorization morphism, namely ([LV17b, Def. 3.16]) an equivalence of derivators \( \Psi : D F \to D[1] \) having domain those \( X \in D(J) \) such that \( X(0,1) \in E(J) \) and \( X(1,2) \in M(J) \). More generally, if \( S \) is a sub-prederivator of \( D \), we call \( S \)-crumbling a prefactorization system \( (E, M) \) with a functorial factorization morphism which is an equivalence restricted to \( S \); of particular importance for us is the case when \( S \) is the essential image of \( pt_* \); in that case, somewhat sloppily, we say that “there are factorizations of all terminal arrows” and we speak about \( \tau \)-crumbling factorization.

With **Definition 3.34** in hand, we can prove the derivator analogue of [Bor94a, 5.4.4]:

**Proposition 3.37.** Let \( L \dashv R : D \rightleftarrows E \) be a reflection of the derivator \( D \). Consider the prederivator \( S_L \subseteq D[1] \), where \( f \in S_L(J) \) if and only if \( L(f) \) is an isomorphism. Then the following conditions are equivalent for a given \( X \in D(J) \):

1. \( X \) lies in the essential image of \( R_J^I : E^J([1]) \to D^J([1]) \);
2. \( f \perp X \) for every \( f \in S_L(J) \);
3. given \( Y \in D(J) \), consider \( \xi_J,Y \in D^J([1]) \) such that \( \text{dia}_{[1]}(\xi_J,Y) : Y \to R_J L_J Y \) is the unit of our reflection, then \( \xi_J,Y \perp X \).

Moreover, we can intrinsically characterize the sub-prederivator of \( L \)-locals (**Definition 3.3** adapted to prederivators); the following conditions are equivalent for a diagram \( X \in D^J([1]) \):

1. \( X \in S_L(J) \);
2. \( X \perp Y_0 \) for every \( Y_0 \in E^J(e) \);
3. \( X \perp Y \) for every \( Y \in E^J([1]) \).

**Proof.** A coherent morphism \( X \in D([1]) \) is an isomorphism (i.e., by definition, \( \text{dia}_{[1]}(X) \) is an isomorphism in \( D(e) \)) if and only if \( \eta_X : X \to pt^* pt_* X \) is an isomorphism. In fact, \( \eta_X \) is an isomorphism in \( D([1]) \) if and only if both \( (\eta_X)_0 \) and \( (\eta_X)_1 \) are isomorphisms in \( D(e) \); now, \( (\eta_X)_1 \) is conjugated to the identity of \( X_1 \), so
it is always an isomorphism, while $(\eta_X)_0$ is conjugated to $\text{dia}_{[1]}X$, so it is an isomorphism exactly when $X$ is so. With this characterization, let us verify the equivalence of statements (1–3):

(1)⇒(2). Let $Y \in \mathcal{E}([1])$ be such that $X \cong R_{[1]}^f Y$. Given $f \in \mathcal{S}_[1][e]$, we should verify that the canonical map

$$\mathcal{D}^f([1])(\text{pt}\ast \text{pt}! f, R_{[1]}^f Y) \cong \mathcal{D}^f([1])(\text{pt}\ast \text{pt}! f, X) \to \mathcal{D}^f([1])(f, R_{[1]}^f Y).$$

is an isomorphism. By adjointness, we can equivalently verify that the following map is an isomorphism:

$$\mathcal{E}^f([1])(\text{pt}\ast \text{pt}! L_{[1]}^f f, Y) \cong \mathcal{E}^f([1])(L_{[1]}^f \text{pt}\ast \text{pt}! f, Y) \to \mathcal{E}^f([1])(L_{[1]}^f f, Y).$$

where the first isomorphism holds since $L_{[1]}^f$ is a morphism of derivators (so it commutes with pt$\ast$) and it is a left adjoint (so it commutes with pt$!$). Now, the above map is induced by the canonical map $\eta_{L_{[1]}^f f} : L_{[1]}^f f \to \text{pt}\ast \text{pt}! L_{[1]}^f f$, which is an isomorphism if and only if $L_{[1]}^f f$ is an isomorphism, but this is true by hypothesis.

(2)⇒(3). It is clear that $L_J$ sends the unit $Y \to R_J L_J Y$ to an isomorphism, that is, $L_J(\text{dia}_{[1]}(\tilde{\xi}_{J,Y}))$ is an isomorphism. Since in a derivator isomorphisms can be checked pointwise, this means that $L_J(\tilde{\xi}_{J,Y})$ is an isomorphism, that is, $\tilde{\xi}_{J,Y} \in \mathcal{S}_L^f(e)$, so the thesis follows.

(3)⇒(1). It is enough, assuming (3), to show that the unit $\xi^f_{[1],X} : X \to R_{[1]}^f L_{[1]}^f X$ is an isomorphism or, equivalently, that $(\xi^f_{[1],X})_0 = \xi_{J,X_0} : X_0 \to R_J L_J X_0$ and $(\xi^f_{[1],X})_1 = \xi_{J,X_1} : X_1 \to R_J L_J X_1$ are both isomorphisms. Let $i = 0, 1$, and let $\tilde{\xi}_{J,X_i}$ be an object lifting $\xi_{J,X_i}$, so by (3), $\tilde{\xi}_{J,X_i} \perp X_i$ is clearly an isomorphism.

We now draw a definition parallel to Definition 3.14, and in Proposition 3.40 we prove the equivalence between the two notions. This is the best approximation to the classical result [CHK85] connecting reflective subcategories $\mathcal{A} \subseteq \mathcal{C}$ with reflective prefactorization systems on $\mathcal{C}$.

**Definition 3.38** (Choric factorization). Let $D$ be a prederivator, $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ a pair of sub pre-derivators of $D^{[1]}$ and let $\mathfrak{D}_\mathfrak{F} : \text{Dia}^{\text{op}} \to \text{Cat}$ be a pre-derivator such that $\mathfrak{D}_\mathfrak{F}(I) \subseteq D(I \times [3])$ is the full subcategory spanned by those $X \in D(I \times [3])$ such that $X_{(0,1)} \in \mathcal{E}(I)$ and $X_{(1,2)} \in \mathcal{M}(I)$. Denote by

$$\Psi_{\mathfrak{F}} : \mathfrak{D}_\mathfrak{F} \longrightarrow D^{[1]}$$

the restriction of the morphism of derivators $D^{[3]} \rightarrow D^{[1]}$ induced by $(0, 2)$. 

In this notation, $\mathfrak{F}$ is said to be choric if $\Psi_\mathfrak{F}$ is fully faithful. More generally, if $S$ is a sub-prederivator of $D$ we say that $\mathfrak{F}$ is $S$-choric if $\Psi_\mathfrak{F}$ is fully faithful when restricted to $S$; particularly important for us is the case when $S$ is the essential image of $p_{t^*}$; in this case we speak about $\tau$-choric dpfs.

As we observe in [LV17a], the dpfs induced by a pair $(E,F)$ is a choric dfs, and any choric dfs on $D$ arises this way from an Eilenberg-Moore factorization. So, we are currently unable to exhibit an example example of non-choric dpfs.

Remark 3.39. Remind the notation of Definition 3.14. In the representable case the functor

$$D \leftarrow \text{Chr}_L = \{X \in D[1] \mid X_1 \in M/\ast\} \subseteq D[1]$$

always admits a left adjoint, given by a choice, for a coherent morphism $X \in C[1]$, of a dotted arrow in

\[
\begin{array}{ccc}
X_0 & \xrightarrow{X} & X_1 \\
\downarrow & & \downarrow \\
RX_0 & \longrightarrow & RX_1.
\end{array}
\]

Strict orthogonality, and the uniqueness of such a dotted arrow, entail that this choice is in fact unique. But in general, even when liftable to a coherent diagram in $D^1(\Box)$, the incoherent diagram in $D([1])^\Box$ can’t be lifted uniquely, and additional conditions on a family of functors $D(I) \to \text{Chr}_L(I)$ that exist separately must be imposed in order to ensure that these are the components of a pseudonatural transformation $D \to \text{Chr}_L$.

Proposition 3.40. Given a derivator $D$, there exists an equivalence between

- the posets of choric reflections of $D$ (Definition 3.14) and
- the poset of reflective, $\tau$-choric dpfs on $D$.

Proof. Let $L \dashv R$ be a choric reflection, where $R : E \to D$; we define $S_L$ to be the sub-prederivator of $L$-locals, as we did elsewhere. Our aim is to prove that the dpfs right-generated by $S_L$ is a $\tau$-choric, reflective dpfs on $D$. It is obvious that $(S_L, S_L^\perp)$ is reflective, as $S_L$ is a 3-for-2 class. It is also obvious that it is a dpfs, so we are only left to prove that it is $\tau$-choric.

Now, every terminal arrow $\left[ X \xrightarrow{X} \ast \right]$ can be $(S_L, S_L^\perp)$-factored: if the reflection is choric, there is a well-defined way to attach to $X \in D$ its reflection $\left[X \xrightarrow{LX} \right]$, and the
terminal arrow $\left[ \frac{LX}{*} \right]$ lies in $\mathbb{S}_L^\perp$; now, this latter arrow lies in $\mathbb{E}(e) \subseteq (\perp \mathbb{E}(e)[1])^\perp$, so we have the result, and the unit arrow is evidently inverted by $L$.

Conversely, let $(\mathbb{E}, \mathbb{M})$ be a $\tau$-choric dpfs on $\mathbb{D}$. We define $\mathbb{M}/*$ to be the class of objects $X \in \mathbb{D}(e)$ such that $t_*X = \left[ \frac{X}{*} \right]$ lies in $\mathbb{M}$. We have to prove that sending $X \in \mathbb{D}(J)$ into $(\Psi_\mathbb{S}(0_*X))_1$ determines a reflection of $\mathbb{D}$, or more precisely that the composition

$$L_\mathbb{S}: \mathbb{D} \xrightarrow{0_*} \mathbb{D}[1] \xrightarrow{\Psi_\mathbb{S}^{-1}} \mathbb{D}_\mathbb{S} \xrightarrow{1_*} \mathbb{D}$$

when corestricted to its essential image, works as left adjoint to the inclusion $\mathbb{M}/* \hookrightarrow \mathbb{D}$ (and so in particular this essential image coincides with $\mathbb{M}/*$). But this is straightforward, as $L_\mathbb{S}$ acts on incoherent diagrams as follows:

$$X \mapsto \left[ \frac{X}{*} \right] \mapsto \left[ \frac{X}{L_\mathbb{S}X} \right] \mapsto L_\mathbb{S}X$$

so that the object $L_\mathbb{S}X$ falls onto $\mathbb{M}/*$.

\begin{prop}
Let $L \dashv R$ be a choric reflection of derivators; its associated dpfs $(\mathbb{S}_L, \mathbb{S}_L^\perp)$ is then $\tau$-choric, and if $\mathbb{S}_L$ is pullback stable then it is also a DFS.
\end{prop}

\begin{proof}
We prove that $(\mathbb{E}, \mathbb{M}) = (\mathbb{S}_L, \mathbb{S}_L^\perp)$ is a factorization. Let $i : \varnothing \to \mathcal{E}$ and $j : \mathcal{E} \to \mathcal{B}$ be the canonical inclusions of the $F$ face and of the $DRF$ faces. Given $X \in \mathbb{D}(1)$ we consider the coherent morphism $L^1X \to j_*i_!L^2X$ whose incoherent underlying diagram looks like

\[
\begin{array}{c}
\begin{tikzcd}
X_0 \rar{P} \drar{RLX_0} & RLX_0 \\
X_1 \urar{RLX_1} \rar{RLX_1} & RLX_1
\end{tikzcd}
\end{array}
\]

Now $RLX \in \mathbb{D}(1)$ lies in $\mathbb{E}(e)[1] = (\perp \mathbb{E}(e)[1])^\perp = \mathbb{M}$, so (since this class is always closed under pullback) also $P \to X_1$ lies in $\mathbb{M}$. In order to show that the $R$ face of the cube is the incoherent diagram associated to the $(\mathbb{E}, \mathbb{M})$-factorization of $X$, we must show that $X_0 \to P$ lies in $\mathbb{S}_L$, i.e. that $L$ inverts this morphism: but this follows from the fact that $X_i \to RLX_i$ lies in $\mathbb{S}_L$ for $i = 0, 1$ and from the assumption that $\mathbb{S}_L$ is closed under pullback (recall that $\mathbb{S}_L$ is a 3-for-2 class).
\end{proof}
4. Monads and their algebras

In this section we introduce some useful terminology and notation about monads on a (pre)derivator. The excellent [LN] investigates the theory of monads on derivators, with applications to stable derivators, but also lays the foundation of the general theory in an elegant and readable way. We advise the reader to consult this reference, that we follow quite nearly (in particular Definition 4.3 and the previous discussion on the derivators of algebras for a monad comes from there).

**Definition 4.1 (Monad on a prederivator).** Let \( D \) be a prederivator. We define a monad on \( D \) to be a morphism \( T: D \to D \) equipped with two natural transformations

\[
\begin{align*}
(1) & \quad \mu: T \circ T \Rightarrow T \quad \text{(the multiplication of the monad)}; \\
(2) & \quad \eta: \text{id} \Rightarrow T \quad \text{(the unit of the monad)},
\end{align*}
\]

satisfying the usual associativity and unitality conditions expressed by the commutativity of the following diagrams of 2-cells: the compatibility of \( \mu, \eta \) with the structure of \( T \) as a pseudonatural transformation, as well as the associativity and unitality constraints, will usually remain hidden; the relevant diagram can be easily drawn and translated in equational form.

**Remark 4.2.** It’s easy to see that a monad on \( D \) induces a monad on each category \( D(J) \), whose multiplication and unit are the components of the modifications \( \mu, \eta \) respectively. The fact that the assignment \( J \mapsto D(J) \) lifts to a suitable category of categories with monad \( \text{Mon}_l(\text{Cat}) \) (this terminology is better than that in [LN], where these are called monadic categories) is a coherence request that can be packed in the following lifting criterion (see [LN] again):

\[
\begin{array}{ccc}
\text{Mon}_l(\text{Cat}) & \xrightarrow{U} & \text{Cat} \\
\text{Dia}^{\text{op}} & \xrightarrow{D} & \\
\end{array}
\]

(it is possible to obtain \( \text{Mon}_l(\text{Cat}) \) as a suitable Grothendieck 2-construction, see [Gro11]).

Of course, a similar strategy yields the definition of an algebra for the monad \( T \): Remark 4.2 above entails that we can consider, for every object \( J \in \text{Cat} \), the category of algebras for the monad \( T_J \) on \( D(J) \). The next result states that all these categories glue together to form a derivator which is the derivator of \( T \)-algebras on \( D \).
Definition 4.3 (EM-algebra for a monad). Let $T$ be a monad on the prederivator $\mathbb{D}$; the assignment $J \mapsto \mathbb{D}(J)^T$ defines a prederivator $\mathbb{D}^T$ which is the **prederivator of $T$-algebras** for $\mathbb{D}$, or the Eilenberg-Moore prederivators. Each of the free-forgetful adjunctions $F_T \dashv U_T$ glue together as components of an adjunction of prederivators $F^T \dashv U^T$.

We now define a 2-category whose 0-cells are the monads over $\mathbb{D} \in \text{PDer}$ (this is of course a particular instance of the 2-category of monads in a 2-category $\mathcal{K}$, whose 0-cells are the monads over $K \in \mathcal{K}$).

**Definition 4.4** (The 2-category of monads in $\text{PDer}$). Given pairs $(\mathbb{E}, T), (\mathbb{D}, S)$ where $\mathbb{E}, \mathbb{D}$ are prederivators endowed with monads $T, S$, a **morphism of monads** $(F, \sigma) : (\mathbb{E}, T) \to (\mathbb{D}, S)$ is a pair where $F : \mathbb{E} \to \mathbb{D}$ is a morphism of prederivators, and $\sigma : SF \Rightarrow FT$ is a 2-cell that fills the square

$$
\begin{array}{ccc}
\mathbb{E} & \xrightarrow{F} & \mathbb{D} \\
\downarrow T & \sigma & \downarrow S \\
\mathbb{E} & \xrightarrow{F} & \mathbb{D}.
\end{array}
$$

This pair is such that the following diagrams of 2-cells commute in $\text{PDer}$:

$$
\begin{array}{ccc}
SSF & \xrightarrow{\mu_{SF}} & SF \\
\downarrow \sigma & \downarrow \sigma & \downarrow \sigma \\
FTT & \xrightarrow{F \eta_T} & FT
\end{array}
$$

$$
\begin{array}{ccc}
F & \xrightarrow{\eta_T} & FT \\
\downarrow \sigma & \downarrow \sigma & \downarrow \sigma \\
SF & \xrightarrow{F \eta_T} & SF.
\end{array}
$$

A 2-cell $\theta : (F, \sigma) \Rightarrow (G, \tau)$ consists of a 2-cell $\theta : F \Rightarrow G$ in $\text{PDer}$ such that $\tau \circ (S \ast \theta) = (\theta \ast T) \circ \sigma$.

This defines the **2-category of monads in $\text{PDer}$** that we denote $\text{Mnd}(\text{PDer})$.

**Remark 4.5.** As observed in [LN], it is easy to see that if $T$ is a monad on $\mathbb{D}$ then each $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ induced from $u : I \to J$ becomes a strong monad morphism (i.e. a monad morphism where $\sigma$ is invertible; the $\sigma$ here is of course $\gamma_{T,u}$ of Remark 2.5).

**Remark 4.6.** Note that there exists a 2-functor $\text{Mnd}(\text{PDer}) \to \text{PDer}$ that sends the monad $T$ to its domain and projects $(F, \sigma)$ to $F$. We denote, with a slight imprecise notation, the fiber over $\mathbb{D}$ as $\text{Mnd}(\mathbb{D})$, the sub-2-category of monads $T : \mathbb{D} \to \mathbb{D}$ on a fixed domain, monad morphisms and monad 2-cells.

The following remark is the content of 2.10 and 2.11 on [LN].
Remark 4.7. A monad on \( \mathbb{D} \) can equivalently be defined as a lax functor \( T : * \to \text{PDer} \) from the terminal 2-category. In this picture, as we recall below, the Eilenberg-Moore category of \( T \) coincides with the lax limit of \( T \).

Remark 4.8. Let \( T = (T, \eta, \mu) \) be a monad on a derivator \( \mathbb{E} \); let \( \text{Split}(T) \) the category of those adjoint pairs \( F \dashv G \) such that \( GF = T, G\varepsilon F = \mu \). The terminal object in this category is called the “free-forgetful adjunction”; \( \text{Split}(T) \) can be regarded as the category of cones for \( T \), when it is regarded as a lax functor, and the Eilenberg-Moore category of \( T \) as its lax limit.

Proposition 4.9. The 2-category \( \text{PDer} \) of prederivators admits the construction of algebras in the sense of [Str72].

Proof. Recall that a 2-category “admits the construction of algebras” if the canonical functor \( \text{PDer} \to \text{Mnd}(\text{PDer}) \) has a right adjoint; unraveling the definition of a monad morphism, we see that a morphism \( (F, \sigma) : (\mathbb{E}, \text{id}_\mathbb{E}) \to (\mathbb{D}, T) \) in \( \text{Mnd}(\text{PDer}) \) is precisely a morphism \( \mathbb{E} \to \mathbb{D}^T \).

In this discussion we are guided by the principle that as cumbersome as it may seem, the internal category theory of prederivators cannot stray much from the category theory of \( \text{CAT} \), given the tight relation between the two objects.\(^6\) this suggests that there may exist different way to re-enact monad theory inside \( \text{PDer} \), more reminiscent of 1-category theory. Fortunately, it turns out that this is true, and that we can fruitfully borrow many ideas from the treatment of monads in enriched category theory.

Proposition 4.10. Let \( T \) be a monad on the prederivator \( \mathbb{A} \), and \( S : \mathbb{C} \to \mathbb{A} \) a morphism of prederivators; then \( S \) admits a lifting to a morphism of prederivators \( \bar{S} : \mathbb{C} \to \mathbb{A}^T \) along the forgetful functor \( U^T : \mathbb{A}^T \to \mathbb{A} \) if and only if \( S \) is a \( T \)-module, i.e. if and only if there exists a 2-cell \( \zeta : TS \Rightarrow S \) such that the diagrams

\[
\begin{array}{ccc}
TST & \xrightarrow{T\zeta} & TS \\
\downarrow{\mu S} & & \downarrow{\zeta} \\
TS & \xrightarrow{\zeta} & S
\end{array}
\quad\quad\quad
\begin{array}{ccc}
S & \xrightarrow{\eta S} & TS \\
\downarrow{\zeta} & & \downarrow{\zeta} \\
S & \xrightarrow{\zeta} & S
\end{array}
\]

commute. We denote the subcategory of \( T \)-modules, inside the whole category \( \text{hom}(TS, S) \) of 2-cells \( TS \Rightarrow S \), as \( \text{Act}(TS, S) \)

\(^6\)A prederivator can be intuitively represented as a “variable category” in much the same way in which a presheaf can be thought as a variable sets; this intuition is taken further in [Str81], in the case of pseudofunctors \( \mathcal{C} \to \text{CAT} \) where \( \mathcal{C} \) is a small bicategory.
Let $F : \mathcal{X} \rightleftarrows \mathcal{Y} : G$ be an adjunction in $\mathbf{PDer}$; then $G$ is a $GF$-module with $\sigma = G\varepsilon$. As a consequence, if we denote $T = GF$ the associated monad, we have

**Corollary 4.11.** There exists a unique morphism of prederivators $K = K_{F,G} : \mathcal{Y} \rightarrow \mathcal{X}^T$, that moreover has the property $K\, F = F^T$. This morphism is called the comparison morphism between $T$-algebras and $\mathcal{Y}$.

**Remark 4.12.** The assignment $\mathcal{G} : \text{Mnd}(\mathcal{A}) \rightarrow \mathbf{PDer}/\mathcal{A} : T \mapsto \left[ U_T : \frac{T}{\mathcal{A}} \right]$ defines a contravariant functor that realizes a bijection (natural in $T$, with respect to monad morphisms)

$$
\mathbf{PDer}(\mathcal{X}, \mathcal{A})(S, \mathcal{G}(T)) \cong \text{Act}(TS, S)
$$

where in the right hand side we denoted the subcategory of $T$-modules $TS \Rightarrow S$ defined in Proposition 4.10. Moreover, if we track the image of $\text{Act}(TS, S)$ under the chain of isomorphisms

$$
\mathbf{PDer}(\mathcal{X}, \mathcal{A})(S, \mathcal{G}(T)) \cong \text{Act}(TS, S) \subset \text{hom}(TS, S) \cong \text{hom}(T, \text{Ran}_S S)
$$

we get that this corresponds exactly to the subcategory of monad morphisms $T \rightarrow \text{Ran}_S S$ between $T$ and the codensity monad $\langle S, S \rangle = (\text{Ran}_S S, u_S, \varepsilon_S)$. Thus we established the equivalence

$$
\mathbf{PDer}(\mathcal{X}, \mathcal{A})(S, \mathcal{G}(T)) \cong \text{Mnd}(\mathcal{A})(T, \langle S, S \rangle),
$$

that in high-sounding terms can be called [Dub70, Lac10] the “semantic $\dashv$ structure” adjunction in $\mathbf{PDer}$ between the 2-category $\mathbf{PDer}/\mathcal{A}$ of those $S : \mathcal{X} \rightarrow \mathcal{A}$ admitting a codensity monad.$^7$

It is interesting to single out the monads whose multiplication is an isomorphism; because of their property these are called idempotent, and it turns out that they correspond to reflective subcategories of their domain, under the correspondence $T \mapsto \mathcal{U}^T$.

**Definition-Proposition 4.13.** (this is the derivator analogue of [Bor94a, 4.2.4]) The following conditions are equivalent

1. the counit of the adjunction $F^T \dashv U^T$ is an isomorphism;
2. the multiplication $\mu : T \circ T \Rightarrow T$ is an invertible modification;
3. For every $T$-algebra $(A, a)$, the structure map $TA \rightarrow A$ is an isomorphism.

If any of these conditions is satisfied, $T$ is called an idempotent monad.

---

$^7$Read “heavy $\mathbf{PDer}/\mathcal{A}$” for the subcategory of functors $\mathcal{X} \rightarrow \mathcal{A}$ admitting a codensity monad; $\mathfrak{b}$ is the alchemical token symbolizing lead (or planet Saturn) [Hol68, p. 153].
Remark 4.14. According to Definition 3.5, condition IM1 entails that there is a reflection $\mathbb{D}^T \rightleftarrows \mathbb{D}$; we remark that [Bor94a, 4.2.4] actually contains another equivalent condition (1′): a monad $T$ is idempotent if the forgetful $U^T : \mathbb{D}^T \to \mathbb{D}$ is fully faithful. In the category of prederivators, we take condition IM1 as a definition of $U^T$ being fully faithful, so the equivalence (1′) ⇔ (1) is true by definition.

Proof. We show the chain of implications (2) ⇒ (3) ⇒ (1) ⇒ (2) and then conclude.

By definition, an invertible modification is such if and only if it has invertible components: since a $T_j$-algebra is defined to be the result of gluing together all the $T_j$-algebras on $\mathbb{D}(J)$ we can think of a $T_j$-algebra as a $\text{Cat}$-indexed family of morphisms $a_j \in \mathbb{D}(J)(T_j A_J, A_J)^8$

A $T_j$-algebra now satisfies the first two commutativity conditions of this list, and the unit of the monad relates to the multiplication as depicted in the third diagram:

Assuming (2), from these relations we get that $\eta_J \ast T_J = T_J \ast \eta_J = \mu_J^{-1}$ (since this is true on each component, we can safely write $\eta \ast T = T \ast \eta = \mu^{-1}$). But then $T_J a_J$ is invertible and equal to $(T_J \ast \eta_J)^{-1} = (\mu_J^{-1})^{-1} = \mu_J$. Finally, naturality for $\eta_J$ gives that

Since $\eta_{A_J} \circ a_J = T_J a_J \circ \eta_J \ast T_J = \text{id}_{T_J A_J}$, this gives that $a_J$ is forced to be the inverse (hence unique) for $\eta_{A_J}$.

To show that (3) ⇒ (1), and in fact that (1) ⇔ (2), it suffices to write the components of the counit $F^T U^T \Rightarrow 1$: it is evidently diagram II above. $\diamond$

This result proves part of the following, equivalent characterizations of localizations of a prederivator:

---

8Even though at the end of this proof we will see that the $a_J$ can be chosen naturally in $J$, we must note that this nice behaviour is a consequence of the idempotency of the monad $T$: in a few words, a classical argument shows how the uniqueness of the $T$-algebra structure on an object forces it to be natural.
Theorem 4.15. [Bor94a, 5.5.6] There is a bijection between
1) τ-choric localizations of \( D \);
2) (categories of algebras for) idempotent monads on \( D \).
3) reflective, τ-choric dpps on \( D \).

This bijection restricts to a bijection between
1) left exact τ-choric reflections of \( D \);
2) (categories of algebras for) left exact idempotent monads on \( D \).
3) reflective, τ-choric dpps on \( D \), closed under finite Kan extensions.

Proof. The equivalence 1 ⇔ 2 follows from Definition-Proposition 4.13, and the equivalence 2 ⇔ 3 is the content of Proposition 3.40.

We now show how this bijection restricts when we consider left exact reflections.\(^9\)

Let \( u : I \to J \) be a functor between homotopy finite categories, \( T \) and idempotent monad on \( D \in \text{PDer} \), and \( F^T : D^T \rightleftarrows D : U^T \) the free-forgetful adjunction of Definition 4.3. Since \( F^T \) and \( U^T \) and \( T = U^T F^T \) are morphisms of derivators, they come equipped with 2-cells \( (\gamma_{F^T,u})_* , (\gamma_{U^T,u})_* , (\gamma_T)_* \) as in Remark 2.6 such that

\[
\begin{array}{ccc}
\mathbb{D}(I) & \xrightarrow{u_*} & \mathbb{D}(J) \\
F^T \downarrow^{(\gamma_{F^T,u})_*} & & \downarrow^{(\gamma_T)_*} \\
D^T(I) & \xrightarrow{u_*} & D^T(J) \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{D}(I) & \xrightarrow{u_*} & \mathbb{D}(J) \\
U^T \downarrow^{(\gamma_{U^T,u})_*} & & \downarrow^{(\gamma_{U^T,u})_*} \\
D(I) & \xrightarrow{u_*} & D(J) \\
\end{array}
\]

but now the 2-cell \( (\gamma_T)_* \) is invertible (i.e. \( T \) commutes with homotopy finite Kan extensions) if and only if \( U^T ((\gamma_{F^T,u})_*) \) is invertible; since \( U^T \) has each component fully faithful, it is conservative, and this latter condition is true if and only if \( (\gamma_{F^T,u})_* \) is invertible (i.e. if \( F^T \) commutes with homotopy finite Kan extensions). \(\Box\)

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\(^9\)Recall from Definition 3.9 that a left exact localization is a localization whose left adjoint commutes with finite right Kan extensions; and that a finite right Kan extension is a functor of the form \( u_* \) for \( u : I \to J \) a functor between homotopy finite categories – categories whose nerve has a finite number of nondegenerate simplices as a simplicial set.
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