Vanishing viscosity limit for Riemann solutions to a $2 \times 2$ hyperbolic system with linear damping

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Abstract

In this paper, we propose a time-dependent viscous system and by using the vanishing viscosity method we show the existence of solutions for the Riemann problem to a particular $2 \times 2$ system of conservation laws with linear damping.

Keywords: Nonstrictly hyperbolic system, linear damping, Riemann problem, time-dependent viscous system, delta shock wave solution.

1 Introduction

In this paper, we study the existence of solutions to the Riemann problem for the following hyperbolic system of conservation laws with linear damping

\[
\begin{cases}
  u_t + \frac{1}{k+1}(u^{k+1})_x = -\alpha u, \\
v_t + (vu^k)_x = 0,
\end{cases}
\]

where $\alpha > 0$ is a constant, and the sign of $v$ is assumed to be unchanging. Thus for convenience, we assume $v \geq 0$ throughout this paper. The initial data is given by

\[
(v(x, 0), u(x, 0)) = \begin{cases}
  (v_-, u_-), & \text{if } x < 0, \\
  (v_+, u_+), & \text{if } x > 0,
\end{cases}
\]

for arbitrary constant states $(v_\pm, u_\pm)$ with $v_\pm > 0$. It is well known that the system (1) is not strictly hyperbolic with eigenvalue $\lambda = u^k$ and right eigenvector $r = (1, 0)$. Moreover, $\nabla \lambda \cdot r = 0$ and therefore the system is linearly degenerate. When $k = 1$, the homogeneous

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case of the system (1) is used to model the evolution of density inhomogeneities in matter in the universe [18, B. Late nonlinear stage, 3. Sticky dust]. The system (1) belongs to the class of triangular systems. The triangular systems of conservation laws arises in a wide variety of models in physics and engineering, see for example [10, 17] and the references therein. For this reason, the triangular systems have been studied by many authors and several rigorous results have been obtained for this.

In 1993, Joseph [11] considered the Riemann problem for the homogeneous case of the system (1) with $k = 1$. He used a parabolic regularization system to obtained an explicit formulae of the Riemann solutions. So, he constructed the weak limit of the approximation solution and this is defined as a delta shock wave type solution. Recently, De la cruz [5] solved the Riemann problem to the system (1) when $k = 1$. His work include classical Riemann solution and delta shock wave solution.

In this paper, we are interested in finding solutions to the Riemann problem for the system (1) with initial data (2). Therefore, we propose the following time-dependent viscous system

$$
\begin{aligned}
&u_t + \frac{1}{k+1}(u^{k+1})_x = \varepsilon \frac{1}{\alpha k} e^{-\alpha kt}(1 - e^{-\alpha kt})u_{xx} - \alpha u, \\
v_t + (vu^k)_x = 0,
\end{aligned}
$$

(where $\alpha > 0$ is a constant) with initial data (2). Observe that when $\alpha \to 0^+$, we have that $\lim_{\alpha \to 0^+} \frac{1}{\alpha k} e^{-\alpha kt}(1 - e^{-\alpha kt}) = t$. The viscous system (3) is well motivated by scalar conservation law with time-dependent viscosity

$$
u_t + F(u)_x = G(t)u_{xx},$$

where $G(t) > 0$ for $t > 0$. When $F(u) = u^2$ the scalar equation is called the Burgers equation with time-dependent viscosity. The Burgers equation with time-dependent viscosity was studied as a mathematical model of the propagation of the finite-amplitude sound waves in variable-area ducts, where $u$ is an acoustic variable, with the linear effects of changes in the duct area taken out, and the time-dependent viscosity $G(t)$ is the duct area [2, 7, 25]. The reader can find results concerning to the existence, uniqueness and explicit solutions to the Burgers equation with time-dependent viscosity with suitable conditions for $G(t)$ in [2, 3, 7, 19, 24, 25, 28, 29] and references cited therein. The Burgers equation with time-dependent viscosity and linear damping was studied in [14] and their results include explicit solutions for differents $G(t)$.

When $G(t) = \varepsilon t$ and $\varepsilon > 0$, for systems of hyperbolic conservation laws with time-dependent viscosity we referred the works developed by Tupciev in [22] and Dafermos in [4]. The results obtained in [4] and [22] not including the delta shock waves solutions. For systems of hyperbolic conservation laws with delta shock solutions the reader may consult [6, 8, 21, 26, 27].

When $G(t)$ is nonlinear, for systems of balance laws we referred the work [5].

Note that our proposal of the time-dependent viscous system (3) is a special case of the general systems of conservation laws with time-dependent viscous system. Observe that if $(\widehat{v}, \widehat{u})$ solves

$$
\begin{aligned}
&\widehat{u}_t + \frac{1}{k+1} e^{-\alpha kt}(\widehat{u}^{k+1})_x = \varepsilon \frac{1}{\alpha k} e^{-\alpha kt}(1 - e^{-\alpha kt})\widehat{u}_{xx}, \\
&\widehat{v}_t + e^{-\alpha kt}(\widehat{v}\widehat{u}^k)_x = 0,
\end{aligned}
$$

(4)
with initial condition

\[
(\hat{v}(x,0), \hat{u}(x,0)) = \begin{cases} (v_-, u_-), & \text{if } x < 0, \\ (v_+, u_+), & \text{if } x > 0, \end{cases}
\]  

(5)

then \((v, u)\) defined by \((v, u) = (\hat{v}, \hat{u}e^{-\alpha t})\) solves the problem (3)–(2). We denote \((\hat{v}, \hat{u})\) as \((\hat{v}, \hat{u})\) when there is no confusion. In order to solve the problem (4)–(5), we introduce the similarity variable \(\xi\) and solutions to (4) should approach for large times a similarity solution \((\hat{v}, \hat{u})\) to (4) of the form \(\hat{v}(x,t) = \hat{v}(\xi), \hat{u}(x,t) = \hat{u}(\xi)\) and \(\xi = a(t)x\) for some suitable smooth function \(a(t) \geq 0\) for \(t > 0\) (more details on the similarity methods can be found in [1, 9, 13, 15, 16, 20] and references therein). Therefore, we introduce the similarity variable \(\xi = \frac{ax}{1-e^{-\alpha t}}\) and the system (4) can be written as follows

\[
\begin{align*}
-\xi \hat{u}_\xi + \frac{1}{k+1}(\hat{u}^{k+1})_\xi &= \varepsilon \hat{u}_{\xi\xi}, \\
-\xi \hat{v}_\xi + (\hat{v}\hat{u}^k)_\xi &= 0,
\end{align*}
\]

(6)

and the initial data (5) changes to the boundary condition

\[
(\hat{v}(\pm\infty), \hat{u}(\pm\infty)) = (v_\pm, u_\pm).
\]

(7)

Note that when \(\alpha \to 0+\), the similarity variable \(\xi\) converges to \(x/t\) which is well used in many methods to study the behavior and structure of solutions of nonlinear hyperbolic systems of conservation laws. Notice that when \(\varepsilon \to 0+\), the system (4) becomes

\[
\begin{align*}
\hat{u}_t + \frac{1}{k+1}e^{-\alpha k t}(\hat{u}^{k+1})_x &= 0, \\
\hat{v}_t + e^{-\alpha k t} (\hat{v}\hat{u}^k)_x &= 0.
\end{align*}
\]

(8)

Using the vanishing viscosity method, and following works by Tan, Zhang and Zheng [21] and Ercole [8] with some appropriate modifications, we show the existence of solutions for system (6) with boundary condition (7). After, we study the behavior of the solutions \((\hat{v}, \hat{u})\) as \(\varepsilon \to 0+\) to obtain classical Riemann solution and delta shock wave solution for the system (8). Finally, as \((v(x,t), u(x,t)) = (\hat{v}(x,t), \hat{u}(x,t)e^{-\alpha t})\), the solutions of (8) are used to obtain solutions of the original system (1).

The outline of the remaining of the paper is as follows. In Section 2, we show the existence of solutions to the viscous system (6) with boundary condition (7). In Section 3, we study the behavior of the solutions \((\hat{v}, \hat{u})\) as \(\varepsilon \to 0+\) and we solve the Riemann problem to the system (4) without viscosity. In Section 4, we show classical Riemann solution and delta shock solution for the nonhomogeneous system (1). Final remarks are given in Section 5.

## 2 Existence of solutions to the viscous system (6)-(7)

Considering the first equation in (6) with boundary conditions, we have

\[
\begin{align*}
-\xi \hat{u}_\xi + \frac{1}{k+1}(\hat{u}^{k+1})_\xi &= \varepsilon \hat{u}_{\xi\xi}, \\
\hat{u}(\pm\infty) &= u_\pm.
\end{align*}
\]

(9)
Now, based on the ideas of Dafermos [4], we consider the following boundary value problem
with parameters $\mu \in [0, 1]$ and $R > 1$,

$$
\begin{align*}
-\xi \hat{u}_\xi + \frac{\mu}{k+1}(\hat{u}^{k+1})_\xi &= \varepsilon \hat{u}_{\xi\xi}, \\
\hat{u}(\pm R) &= \mu u_\pm.
\end{align*}
$$

(10)

**Lemma 2.1.** Let $\hat{u}(\xi)$ be a solution of (10) on $[-R, R]$ for some $\mu > 0$. Suppose that $u_- \neq u_+$. Then, $\hat{u}$ is a strictly monotonic function on $[-R, R]$.

**Proof.** Observe that from (10) we have that

$$
\hat{u}'(\xi) = \hat{u}'(\zeta) \exp \left( \int_\zeta^\xi \frac{\mu \hat{u}^k(s) - s}{\varepsilon} ds \right)
$$

(11)

for any $\zeta \in [-R, R]$. Suppose $\xi_1 \in [-R, R]$ is a critical point of $\hat{u}(\xi)$, which implies $\hat{u}'(\xi_1) = 0$. Then, from (11) we have that $\hat{u}'(\xi) = 0$ for all $\xi \in [-R, R]$, and therefore $\hat{u}(\xi)$ is constant on $[-R, R]$. But, this contradicts the fact that $u_- \neq u_+$. Thus, $\hat{u}(\xi)$ is monotone. The monotonicity of $\hat{u}(\xi)$ depends on the value of $\hat{u}'(\xi_0)$. If $u_- > u_+$, then $\hat{u}(\xi)$ is strictly decreasing on $[-R, R]$. When $u_- < u_+$, we have that $\hat{u}(\xi)$ is strictly increasing on $[-R, R]$. 

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**Theorem 2.1.** Suppose that $u_- > u_+$. For every $\varepsilon > 0$, there exists a smooth solution (not necessarily unique) of (9).

**Proof.** From Lemma 2.1, we have $\sup_{-R < \xi < R} |u(\xi)| \leq \max\{u_-, u_+\}$ which does not depend on $\mu$ and $R$. Now, from Theorem 3.1 in [4] we conclude that there exists a solution of (9). Moreover, if $u_- > u_+$, then the solution is decreasing on $(-\infty, \infty)$. For $u_- < u_+$, the solution is increasing on $(-\infty, \infty)$. 

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**Proposition 2.1.** Let $w(\xi)$ be a smooth solution of (9). Then,

$$
|w'(\xi)| \leq |w'(0)| \exp \left( \frac{2u_-|\xi| - \xi^2}{2\varepsilon} \right), \quad -\infty < \xi < +\infty.
$$

**Proof.** Multiplying the equation of (9) by $\exp(\frac{\xi^2}{2\varepsilon})$, we have

$$
\frac{d}{d\xi} \left( w'(\xi) \exp(\frac{\xi^2}{2\varepsilon}) \right) = \frac{1}{\varepsilon} w^k(\xi) w'(\xi) \exp(\frac{\xi^2}{2\varepsilon})
$$

which yields the estimate

$$
|w'(\xi)| \leq |w'(0)| \exp \left( \frac{2u_-|\xi| - \xi^2}{2\varepsilon} \right).
$$

---

**Theorem 2.2.** Let $u_1(\xi)$ and $u_2(\xi)$ be two smooth solutions of (9). Then, $u_1 = u_2$. 
Proof. Let \( \hat{u}_1(\xi) \) and \( \hat{u}_2(\xi) \) be solutions of the problem (9) and \( \hat{U}(\xi) := \hat{u}_1(\xi) - \hat{u}_2(\xi) \). Then, from (9) we have that \( \hat{U} \) is a smooth solution of the boundary value problem

\[
\begin{cases}
-\xi \hat{U} + \mu(\hat{U}) \xi = \varepsilon \hat{U}_{\xi\xi}, \\
\hat{U}(\pm \infty) = 0,
\end{cases}
\]

(12)

where \( h(\theta) = \int_{\theta}^{1} (k + 1)(\hat{u}_1(\xi) + (\hat{u}_2(\xi) - \hat{u}_1(\xi))\theta)^k d\theta \). We note that \( h(\xi) \) is bounded. Observe that from Proposition 2.1, we have

\[ |\hat{U}'(\xi)| \leq (|u'_1(0)| + |u'_2(0)|) \exp \left( \frac{2u_k|\xi| - \xi^2}{2\varepsilon} \right) \]

and \( \hat{U}'(\xi) \) decays rapidly to zero when \( |\xi| \to \infty \) for each fixed \( \varepsilon > 0 \). Therefore, when \( \lim_{\xi \to \pm \infty} \hat{U}(\xi) = 0 \) we have \( \lim_{\xi \to \pm \infty} \varepsilon \hat{U}(\xi) = 0 \).

Let us suppose that \( \hat{U} \) is not the null function. Let \( a \) and \( b \) be consecutive zeros of \( \hat{U} \) with \(-\infty \leq a < b \leq +\infty \). So, integrating (12) by parts on \((a, b)\) we find

\[ \varepsilon (\hat{U}'(b) - \hat{U}'(a)) = \int_a^b \hat{U}(\xi) d\xi. \]

(13)

Now, if \( \hat{U} > 0 \) on \((a, b)\), then \( \hat{U}'(b) \leq 0 \leq \hat{U}'(a) \) and \( \int_a^b \hat{U}(\xi) d\xi > 0 \). But, we have a contradiction with (13) because in this case (13) implies \( \hat{U}'(b) > \hat{U}'(a) \). In similar way, if \( \hat{U} < 0 \) on \((a, b)\), then \( \hat{U}'(b) \geq 0 \geq \hat{U}'(a) \) and \( \int_a^b \hat{U}(\xi) d\xi > 0 \), which again contradicts with (13). Thus, we conclude that \( \hat{U} \equiv 0 \).

Putting \( \hat{u}(\xi) \) into the second equation of (6) with boundary conditions (7), we get

\[
\begin{cases}
-\xi \hat{v}_\xi + (\hat{v}u^k)_\xi = 0, \\
\hat{v}(\pm \infty) = v_{\pm}.
\end{cases}
\]

(14)

The singularity point of (14) is given by the unique solution of \( (\hat{v}(\xi))^k = \xi \) and it is denoted by \( \xi^k \). Observe that the solution of (14) can be obtained by pasting together the two solutions in the regions \((-\infty, \xi^k)\) and \((\xi^k, +\infty)\). Now integrating (14) from \(-\infty\) to \( \xi \) for \( \xi < \xi^k \), we obtain

\[ \hat{v}_1(\xi) = v_- \exp \left( -\int_{-\infty}^{\xi} \frac{(\hat{u}^k(s))'}{\hat{u}^k(s) - s} ds \right). \]

(15)

On the other hand, integrating (14) from \( \xi \) to \(+\infty\) for \( \xi \geq \xi^k \), we obtain

\[ \hat{v}_2(\xi) = v_+ \exp \left( \int_{\xi}^{+\infty} \frac{(\hat{u}^k(s))'}{\hat{u}^k(s) - s} ds \right). \]

(16)

Lemma 2.2. Suppose \( u_- > u_+ \). Let

\[ \hat{v}(\xi) = \begin{cases} 
\hat{v}_1(\xi), & \text{if } \xi < \xi^k, \\
\hat{v}_2(\xi), & \text{if } \xi > \xi^k,
\end{cases} \]

(17)
where \( \xi^* \) is the unique solution of the equation \( (\hat{u}(\xi))^k = \xi \) (which solution exists because \( u_- > u_+ \) and \( \hat{u} \) is decreasing), \( \hat{v}_1 \) and \( \hat{v}_2 \) are defined by (15) and (16), respectively. Then \( \hat{v} \in L^1(-\infty, +\infty) \), \( \hat{v} \) is continuous in \( (-\infty, \xi^*) \cup (\xi^*, +\infty) \) and it is a weak solution for

\[
-\xi \hat{v} + (\hat{v}^k)_\xi = 0. \tag{18}
\]

**Proof.** Note that from the formula (15), \( \hat{v}_1(\xi) \) is monotonically increasing (or decreasing) when \( v_- < 0 \) (or \( v_- > 0 \)) in the interval \( (-\infty, \xi^*) \), and from (16) that \( \hat{v}_2(\xi) \) is monotonically decreasing (or increasing) when \( v_- < 0 \) (or \( v_- > 0 \)) in the interval \( (\xi^*, +\infty) \). Also, we have

\[
\lim_{\xi \to \xi^*-} \hat{v}_1(\xi) = \pm \infty, \quad \lim_{\xi \to \xi^+} \hat{v}_2(\xi) = \pm \infty.
\]

The equation (18) can be rewritten as

\[
((\hat{u}(\xi))^k - \xi)\hat{v}' + \hat{v}(\hat{u}(\xi))^k)' = 0. \tag{19}
\]

Now, we can show that \( \hat{v} \in L^1[\xi_1, \xi_2] \) for any interval \([\xi_1, \xi_2]\) containing \( \xi^* \). In fact, integrating (19) on \([\xi, \xi]\) for \( \xi_1 < \xi < \xi^* \), we get

\[
((\hat{u}(\xi))^k - \xi)\hat{v}_1(\xi) - ((\hat{u}(\xi))^k - \xi_1)\hat{v}_1(\xi_1) + \int_{\xi_1}^{\xi} \hat{v}_1(s)ds = 0. \tag{20}
\]

Let

\[
p(\xi) = \int_{\xi_1}^{\xi} \hat{v}_1(s)ds, \quad A_1 = ((\hat{u}(\xi))^k - \xi_1)\hat{v}_1(\xi_1) \quad \text{and} \quad a(\xi) = ((\hat{u}(\xi))^k - \xi).
\]

Then (20) can be written as

\[
\begin{cases}
a(\xi)p'(\xi) + p(\xi) = A_1, \\
p(\xi_1) = 0.
\end{cases}
\]

It follows that

\[
p(\xi) = A_1 \left\{ 1 - \exp \left( - \int_{\xi_1}^{\xi} \frac{ds}{a(s)} \right) \right\}.
\]

Noting that \( a(\xi) > 0 \) and \( a(\xi) = O(|\xi - \xi^*|) \) as \( \xi \to \xi^*- \), we obtain

\[
\lim_{\xi \to \xi^*-} \int_{\xi_1}^{\xi} \hat{v}_1(s)ds = \lim_{\xi \to \xi^*-} p(\xi) = A_1. \tag{21}
\]

Hence

\[
\lim_{\xi \to \xi^*-} ((\hat{u}(\xi))^k - \xi)\hat{v}_1(\xi) = 0. \tag{22}
\]

Similarly, one can get

\[
\lim_{\xi \to \xi^+} \int_{\xi}^{\xi_2} \hat{v}_2(s)ds = A_2, \tag{23}
\]

\[
\lim_{\xi \to \xi^+} ((\hat{u}(\xi))^k - \xi)\hat{v}_2(\xi) = 0.
\]
where \( A_2 = ((\tilde{u}(\xi_2))^k - \xi_2)\tilde{v}_2(\xi_2) \). The equalities (21) and (23) imply that \( \tilde{v}(\xi) \in L^1([\xi_1, \xi_2]) \). Given an arbitrary function \( \phi \in C^{\infty}_0([\xi_1, \xi_2]) \), we can show that
\[
I \equiv -\int_{\xi_1}^{\xi_2} ((\tilde{u}(\xi))^k - \xi)\tilde{v}(\xi)\phi'(\xi) d\xi + \int_{\xi_1}^{\xi_2} \tilde{v}(\xi)\phi(\xi) d\xi = 0.
\]
Indeed, for any \( \tilde{\xi}_1, \tilde{\xi}_2 \) such that \( \xi_1 < \tilde{\xi}_1 < \xi^*_\sigma < \xi_2 < R \) we can write \( I = I_1 + I_2 + I_3 \), where
\[
I_1 = \int_{\xi_1}^{\tilde{\xi}_1} -((\tilde{u}(\xi))^k - \xi)\tilde{v}(\xi)\phi'(\xi) + \tilde{v}(\xi)\phi(\xi) d\xi,
\]
\[
I_2 = \int_{\tilde{\xi}_1}^{\tilde{\xi}_2} -((\tilde{u}(\xi))^k - \xi)\tilde{v}(\xi)\phi'(\xi) + \tilde{v}(\xi)\phi(\xi) d\xi \quad \text{and}
\]
\[
I_3 = \int_{\tilde{\xi}_2}^{\xi_2} -((\tilde{u}(\xi))^k - \xi)\tilde{v}(\xi)\phi'(\xi) + \tilde{v}(\xi)\phi(\xi) d\xi.
\]
Observe that
\[
|I_1| = \left| -((\tilde{u}(\xi))^k - \tilde{\xi}_1)\tilde{v}(\tilde{\xi}_1)\phi(\tilde{\xi}_1) + \int_{\xi_1}^{\tilde{\xi}_1}(((\tilde{u}(\xi))^k - \xi)\tilde{v}(\xi))'\phi(\xi) + \tilde{v}(\xi)\phi(\xi) ) d\xi \right| = \left| ((\tilde{u}(\xi))^k - \tilde{\xi}_1)\tilde{v}(\xi)\phi(\tilde{\xi}_1) \right|.
\]
By (22), we have that
\[
\lim_{\tilde{\xi}_1 \to \xi^*_-} |I_1| = \lim_{\tilde{\xi}_1 \to \xi^*_-} \left| ((\tilde{u}(\xi))^k - \tilde{\xi}_1)\tilde{v}(\tilde{\xi}_1)\phi(\tilde{\xi}_1) \right| = 0.
\]
In similar way, we show that
\[
\lim_{\tilde{\xi}_2 \to \xi^*_+} |I_3| = \lim_{\tilde{\xi}_2 \to \xi^*_+} \left| ((\tilde{u}(\xi))^k - \tilde{\xi}_2)\tilde{v}(\tilde{\xi}_2)\phi(\tilde{\xi}_2) \right| = 0.
\]
Since \( \tilde{v} \in L^1([\xi_1, \xi_2]) \),
\[
|I_2| \leq \int_{\xi_1}^{\tilde{\xi}_2} | -((\tilde{u}(\xi))^k - \xi)\phi'(\xi) + \phi(\xi)| \tilde{v}(\xi)| d\xi \to 0, \quad \text{as } \tilde{\xi}_1 \to \xi^*_-, \tilde{\xi}_2 \to \xi^*_+.
\]
But \( I \) is independent of \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \), so \( I = 0 \). Therefore, \( \tilde{v} \) defined in (17) is a weak solution. \( \square \)

**Lemma 2.3.** Suppose \( u_- < u_+ \). Let
\[
\tilde{v}(\xi) = \begin{cases} 
\tilde{v}_1(\xi), & \text{if } \xi < \xi^*_{\sigma_1}, \\
0, & \text{if } \xi^*_{\sigma_1} \leq \xi \leq \xi^*_{\sigma_2}, \\
\tilde{v}_2(\xi), & \text{if } \xi > \xi^*_{\sigma_2},
\end{cases}
\]
where \( \tilde{v}_1 \) and \( \tilde{v}_2 \) are defined by (15) and (16), respectively, \( \xi^*_{\sigma_1} \leq \xi^*_{\sigma_2} \) satisfying \( \xi^*_{\sigma_1} = \min\{\xi : (\tilde{u}(\xi))^k = \xi\} \), \( \xi^*_{\sigma_2} = \max\{\xi : (\tilde{u}(\xi))^k = \xi\} \) and \( \lim_{\xi \to \xi^*_{\sigma_1}^-} \tilde{v}_1(\xi) = \lim_{\xi \to \xi^*_{\sigma_2}^+} \tilde{v}_2(\xi) = 0 \). Then \( \tilde{v} \in L^1(-\infty, +\infty) \), \( \tilde{v}_1 \) is decreasing in \((-\infty, \xi^*_{\sigma_1}) \), \( \tilde{v}_2 \) is increasing in \((\xi^*_{\sigma_2}, +\infty) \), \( \tilde{v} \) is continuous on the intervals \((-\infty, \xi^*_{\sigma_1}) \) and \((\xi^*_{\sigma_2}, +\infty) \), and it is a weak solution for \(-\xi \tilde{v}_1 + (\tilde{v}_1^')_\xi = 0 \).
Proof. Observe that $u_- < u_+$ implies $\hat{u}$ is increasing. Consider now the function $s \mapsto s - \hat{u}^k(s)$ which is continuous and approaches $\pm \infty$ as $s \to \pm \infty$. Hence, there exist finite quantities $\xi_{\sigma_1}^\varepsilon = \min\{\xi : (\hat{u}(\xi))^k = \xi\}$ and $\xi_{\sigma_2}^\varepsilon = \max\{\xi : (\hat{u}(\xi))^k = \xi\}$. One has $s - (\hat{u}(s))^k < 0$ on $(-\infty, \xi_{\sigma_1}^\varepsilon)$ and $s - (\hat{u}(s))^k > 0$ on $(\xi_{\sigma_2}^\varepsilon, +\infty)$. Moreover, we can get $\xi_{\sigma_1}^\varepsilon \leq \xi_{\sigma_2}^\varepsilon$. We now claim that

$$\lim_{\xi \to \xi_{\sigma_2}^\varepsilon} \int_{\xi}^{+\infty} \frac{(\hat{u}(s))^\prime}{s - \hat{u}^k(s)} ds = +\infty. \tag{24}$$

In fact, for $R$ fixed and $\xi_{\sigma_2}^\varepsilon < \xi < R$ we have

$$\int_{\xi}^{R} \frac{(\hat{u}(s))^\prime}{s - \hat{u}^k(s)} ds = (\hat{u}^k(\xi))^\prime \int_{\xi}^{R} \frac{ds}{s - \hat{u}^k(s)} \geq (\hat{u}^k(\xi))^\prime \int_{\xi}^{R} \frac{ds}{s - \hat{u}^k(\xi)} \geq -\hat{u}(\xi)^k \ln \left( \frac{\xi - (\hat{u}^k(\xi))^k}{R - (\hat{u}^k(\xi))^k} \right) \to +\infty, \text{ as } \xi \to \xi_{\sigma_2}^\varepsilon +,$$

where $\xi \leq \zeta \leq R$. Now, from (16) and (24) we get

$$\lim_{\xi \to \xi_{\sigma_2}^\varepsilon} \hat{v}_2(\xi) = 0.$$

In a similar way, we can obtain $\lim_{\xi \to \xi_{\sigma_1}^\varepsilon} \hat{v}_1(\xi) = 0$. The monotonicity of $\hat{v}_1$ and $\hat{v}_2$ is obvious.

When $\xi_{\sigma_1}^\varepsilon \leq \xi \leq \xi_{\sigma_2}^\varepsilon$, from (19) we have

$$\int_{\xi_{\sigma_1}^\varepsilon}^{\xi_{\sigma_2}^\varepsilon} \left((\hat{u}(\xi))^k \hat{v}^\prime - \xi \hat{v}^\prime + \hat{v}((\hat{u}(\xi))^k)^\prime\right) d\xi = 0$$

or

$$((\hat{u}(\xi))^k - \xi \hat{v}(\xi))|_{\xi_{\sigma_1}^\varepsilon}^{\xi_{\sigma_2}^\varepsilon} + \int_{\xi_{\sigma_1}^\varepsilon}^{\xi_{\sigma_2}^\varepsilon} \hat{v}(\xi) d\xi = 0$$

which implies that $\hat{v}(\xi) = 0$. \hfill \Box

3 The limit solutions of (4–5) as viscosity vanishes

In this section, we are interested in analyzing the behavior of the solutions $(\hat{v}^\varepsilon, \hat{u}^\varepsilon)$ of (6–7) as $\varepsilon \to 0+$ to established the solutions of (4–5).

Case 1. $u_- > u_+$

Lemma 3.1. Let $\xi_{\sigma}^\varepsilon$ be the unique point satisfying $(\hat{u}^\varepsilon(\xi_{\sigma}^\varepsilon))^k = \xi_{\sigma}^\varepsilon$, and let $\xi_{\sigma} = \lim_{\varepsilon \to 0+} \xi_{\sigma}^\varepsilon$ (passing to a subsequence if necessary). Then for any $\eta > 0$,

$$\lim_{\varepsilon \to 0+} \hat{u}_{\xi}^\varepsilon(\xi) = 0, \text{ for } |\xi - \xi_{\sigma}| \geq \eta,$$

$$\lim_{\varepsilon \to 0+} \hat{u}^\varepsilon(\xi) = \begin{cases} u_-, & \text{if } \xi \leq \xi_{\sigma} - \eta, \\ u_+, & \text{if } \xi \geq \xi_{\sigma} + \eta, \end{cases}$$
uniformly in the above intervals. Moreover, \( \xi_\sigma = \frac{1}{k+1} \sum_{j=0}^{k} u_{-j} u^j \) and \( \xi_\sigma [u] - \frac{1}{k+1} [u^{k+1}] = 0 \).

**Proof.** To simplify the notation in this proof, we shall use \( \hat{v}, \hat{u} \) instead of \( \hat{v}^\varepsilon, \hat{u}^\varepsilon \).

Take \( \xi_3 = \xi_\sigma - \eta/2 \), and let \( \varepsilon \) be so small such that \( \xi_\sigma > \xi_3 + \eta/4 \).

Now, integrating the first equation of (6) twice on \([\xi, \xi_3]\), we get

\[
\hat{u}(\xi_3) - \hat{u}(\xi) = \hat{u}'(\xi_3) \int_\xi^{\xi_3} \exp \left( - \int_r^{\xi_3} \frac{(\hat{u}(s))^k - s}{\varepsilon} ds \right) dr \\
\leq \hat{u}'(\xi_3) \int_\xi^{\xi_3} \exp \left( - \int_r^{\xi_3} \frac{u_-^k - s}{\varepsilon} ds \right) dr \\
= \hat{u}'(\xi_3) \int_\xi^{\xi_3} \exp \left( \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) (r - \xi_3) - \frac{1}{2} (r - \xi_3)^2 \right) \right) dr \\
= \hat{u}'(\xi_3) \int_{\xi - \xi_3}^{\xi_3} \exp \left( \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) r - \frac{1}{2} r^2 \right) \right) dr.
\]

Letting \( \xi \to -\infty \), we get

\[
u_+ - \nu_- \leq \hat{u}'(\xi_3) \int_{-\infty}^{0} \exp \left( \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) r - \frac{1}{2} r^2 \right) \right) dr \\
\leq \hat{u}'(\xi_3) \int_{0}^{2\varepsilon} \exp \left( - \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) r + \frac{1}{2} r^2 \right) \right) dr \\
\leq \hat{u}'(\xi_3) \sqrt{\varepsilon} A_3
\]

for \( 0 \leq \varepsilon \leq 1 \), where \( A_3 \) is a constant independent of \( \varepsilon \). Thus

\[
|\hat{u}'(\xi_3)| \leq \frac{\nu_- - \nu_+}{\varepsilon A_3}.
\]

So

\[
|\hat{u}'(\xi)| \leq \frac{\nu_- - \nu_+}{\varepsilon A_3} \exp \left( - \int_{\xi}^{\xi_3} \frac{(\hat{u}(s))^k - s}{\varepsilon} ds \right) \tag{25}
\]

Noticing that

\[
(\hat{u}(s))^k - s = ((\hat{u}(s))^k - (\hat{u}(\xi_3))^k) - (s - \xi_3) = (k(\hat{u}(\theta))^{k-1} u'(\theta) - 1)(s - \xi_3) \geq \frac{\eta}{4}
\]

for \( s \leq \xi_3 \) and from (25) we have

\[
|\hat{u}'(\xi)| \leq \frac{\nu_- - \nu_+}{\varepsilon A_3} \exp \left( - \frac{\eta}{4\varepsilon} (\xi_3 - \xi) \right)
\]

which implies that

\[
\lim_{\varepsilon \to 0+} \hat{\eta}^\varepsilon(\xi) = 0, \text{ uniformly for } \xi \leq \xi_\sigma - \eta.
\]

Now, we choose \( \xi \) and \( \xi_4 \) such that \( \xi < \xi_4 \leq \xi_\sigma - \eta \). From

\[
\hat{u}(\xi_4) - \hat{u}(\xi) = \hat{u}'(\xi_4) \int_\xi^{\xi_4} \exp \left( - \int_r^{\xi_4} \frac{(\hat{u}(s))^k - s}{\varepsilon} ds \right) dr,
\]


we get
\[ |\hat{u}(\xi_4) - \hat{u}(\xi)| \leq |\hat{u}'(\xi_4)| \int_{\xi}^{\xi_4} \exp\left( \frac{A_4}{\epsilon} (r - \xi_4) \right) dr \leq \frac{\epsilon}{A_4} |\hat{u}'(\xi_4)| \left( 1 - \exp\left( \frac{A_4}{\epsilon} (\xi - \xi_4) \right) \right), \]
where \( A_4 = (\hat{u}(\xi_4))^k - \xi_4. \) When \( \xi \to -\infty, \) we obtain
\[ |\hat{u}(\xi_4) - u_-| \leq \frac{\epsilon}{A_4} |\hat{u}'(\xi_4)|, \]
which implies that
\[ \lim_{\epsilon \to 0^+} \hat{u}^\epsilon(\xi) = u_-, \quad \text{uniformly for } \xi \leq \xi_\sigma - \eta. \]
The results for \( \xi \geq \xi_\sigma + \eta \) can be obtained analogously.
In fact, let \( \phi \in C_0^\infty((\xi_1, \xi_2)) \) where \( \xi_1 < \xi_\sigma < \xi_2, \) From (9) we have
\[
\int_{\xi_1}^{\xi_2} \hat{u}(\xi) \left( (\xi \phi(\xi))' - \frac{1}{k+1} \hat{u}_k(\xi) \phi'(\xi) \right) d\xi = \epsilon \int_{\xi_1}^{\xi_2} \hat{u}(\xi) \phi''(\xi) d\xi. \tag{26}
\]
Passing limit \( \epsilon \to 0^+ \) in (26), we get
\[
\int_{\xi_1}^{\xi_2} u_- \left( (\xi \phi(\xi))' - \frac{1}{k+1} u_+^k \phi'(\xi) \right) d\xi + \int_{\xi_\sigma}^{\xi_2} u_+ \left( (\xi \phi(\xi))' - \frac{1}{k+1} u_+^k \phi'(\xi) \right) d\xi = 0.
\]
or
\[
u_\sigma \phi(\xi_\sigma) - \frac{1}{k+1} u_+^{k+1} \phi(\xi_\sigma) - u_+ \xi_\sigma \phi(\xi_\sigma) + \frac{1}{k+1} u_+^{k+1} \phi(\xi_\sigma) = 0
\]
which yields \( \xi_\sigma = \frac{1}{k+1} \sum_{j=0}^{k} u_-^{k-j} u_+^j \) for arbitrary \( \phi. \)

**Lemma 3.2.** For any \( \eta > 0, \)
\[
\lim_{\epsilon \to 0^+} \hat{v}^\epsilon(\xi) = \begin{cases} v_- & \text{if } \xi < \xi_\sigma - \eta, \\ v_+ & \text{if } \xi > \xi_\sigma + \eta, \end{cases}
\]
uniformly, with respect to \( \xi. \)

**Proof.** Take \( \epsilon_0 > 0 \) so small such that \( |\xi_\sigma^\epsilon - \xi_\sigma| < \frac{\eta}{2} \) whenever \( 0 < \epsilon < \epsilon_0. \) For any \( \xi \geq \xi_\sigma + \eta \) and \( \epsilon < \epsilon_0, \) we have
\[ \xi > \xi_\sigma + \frac{\eta}{2} \]
and
\[ \hat{v}^\epsilon(\xi) = v_+ \exp\left( \int_{\xi}^{\infty} \frac{((\hat{u}^\epsilon(s))^k)'^j}{(\hat{u}^\epsilon(s))^k - s} ds \right). \]
For any \( s \in [\xi, +\infty), \) we have
\[ ((\hat{u}^\epsilon(s))^k - s < ((\hat{u}^\epsilon(s))^k - \xi) = (1 - ((\hat{u}^\epsilon(s))^k)'(\xi_\sigma - \xi) \leq -\frac{\eta}{2}. \]
As \( \hat{u} \) is decreasing, we have that \((\hat{u}(\xi))^k)' = k(\hat{u}(\xi))^{k-1}\hat{u}'(\xi) < 0\), and
\[
\left(\frac{(\hat{u}(\xi))^k}'{\hat{u}(\xi)^k - s}\right) < -\frac{2}{\eta}(\hat{u}(\xi))^k, \quad \text{for any } s \in [\xi, +\infty).
\]
Now, in the last inequality, integrating on \([\xi, +\infty)\) we have
\[
0 \leq \int_{\xi}^{+\infty} \left(\frac{(\hat{u}(\xi))^k}'{\hat{u}(\xi)^k - s}\right) ds \leq -\frac{2}{\eta} \int_{\xi}^{+\infty} (\hat{u}(\xi))^k ds = -\frac{2}{\eta} (u_+^k - (\hat{u}(\xi))^k),
\]
so
\[
1 \leq \exp \left(\int_{\xi}^{+\infty} \left(\frac{(\hat{u}(\xi))^k}'{\hat{u}(\xi)^k - s}\right) ds\right) \leq \exp \left(-\frac{2}{\eta} (u_+^k - (\hat{u}(\xi))^k)\right). \tag{27}
\]
By Lemma 3.1 we have that \(\lim_{\varepsilon \to 0+} \hat{u}^{\varepsilon}(\xi) = u_+\), and from (27) we have
\[
\lim_{\varepsilon \to 0+} \exp \left(\int_{\xi}^{+\infty} \left(\frac{(\hat{u}(\xi))^k}'{\hat{u}(\xi)^k - s}\right) ds\right) = 1
\]
and
\[
\lim_{\varepsilon \to 0+} \hat{\nu}(\xi) = \lim_{\varepsilon \to 0+} v_+ \exp \left(\int_{\xi}^{+\infty} \left(\frac{(\hat{u}(\xi))^k}'{\hat{u}(\xi)^k - s}\right) ds\right) = v_+, \quad \text{uniformly for } \xi > \xi_0 + \eta.
\]
Similarly, we obtain also \(\lim_{\varepsilon \to 0} \hat{\nu}(\xi) = v_-\), uniformly for \(\xi < \xi_0 - \eta\).

\[\square\]

**Lemma 3.3.** Let \((\hat{u}^{\varepsilon}, \hat{\nu}^{\varepsilon})\) be the solution of the Riemann problem (6)-(7) Denote
\[
\sigma = \xi_0 = \lim_{\varepsilon \to 0+} \xi_0^{\varepsilon} = \lim_{\varepsilon \to 0+} (\hat{u}^{\varepsilon}(\xi_0^{\varepsilon}))^k = (\hat{u}(\sigma))^k.
\]
Then
\[
\lim_{\varepsilon \to 0+} (\hat{\nu}(\xi), \hat{\nu}(\xi)) = \begin{cases}
(v_-, u_-), & \text{if } \xi < \sigma, \\
w_0 \cdot \delta, & \text{if } \xi = \sigma, \\
(v_+, u_+), & \text{if } \xi > \sigma,
\end{cases}
\]
where \(\hat{\nu}(\xi)\) converges in the sense of the distributions to the sum of a step function and a Dirac measure \(\delta\) with weight \(w_0 = -\sigma(v_- - v_+) + (v_- u_-^k - v_+ u_+^k)\). Moreover, \(\sigma = \frac{1}{k+1} \sum_{j=0}^{k} u_{-}^{k-j} u_{+}^{j}\).

**Proof.** From Lemma 3.1 we have that \(\sigma = \xi_0 = \frac{1}{k+1} \sum_{j=0}^{k} u_{-}^{k-j} u_{+}^{j}\) and \(-\sigma(u_- - u_+) + \frac{1}{k+1}(u_{-}^{k+1} - u_{+}^{k+1}) = 0\). Moreover, observe that \(\psi_1(\theta) = \theta^k - \frac{1}{k+1} \frac{\theta^{k+1} - u_+^k}{\theta - u_+} > 0\) for all \(\theta > u_+\) and \(\psi_2(\theta) = \frac{1}{k+1} \frac{u_+^k - \theta^{k+1}}{u_+ - \theta} - \theta^k > 0\) for all \(\theta < u_-\). Then, as \(\varepsilon \to 0+\), we have \(u_+^k < \sigma < u_-^k\). Now, we need to study the limit behavior of \(\hat{\nu}^{\varepsilon}\) in the neighborhood of \(\sigma\). Let \(\xi_1\) and \(\xi_2\) be real numbers such
that \(\xi_1 < \sigma < \xi_2\) and \(\phi \in C^\infty([\xi_1, \xi_2])\) such that \(\phi(\xi) \equiv \phi(\sigma)\) for \(\xi\) in a neighborhood \(\Omega\) of \(\sigma\), 
\(\Omega \subset (\xi_1, \xi_2)\) \(^1\). Then \(\xi_2 \in \Omega\) whenever \(0 < \varepsilon < \varepsilon_0\). From (6) we have

\[
- \int_{\xi_1}^{\xi_2} \hat{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \phi' d\xi + \int_{\xi_1}^{\xi_2} \hat{\nu}^\varepsilon \phi d\xi = 0. \tag{28}
\]

For \(\alpha_1, \alpha_2 \in \Omega\), \(\alpha_1, \alpha_2\) near \(\sigma\) such that \(\alpha_1 < \sigma < \alpha_2\), we write

\[
\int_{\xi_1}^{\xi_2} \hat{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \phi' d\xi = \int_{\xi_1}^{\alpha_1} \hat{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \phi' d\xi + \int_{\alpha_2}^{\xi_2} \hat{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \phi' d\xi,
\]

and from Lemmas 3.1 and 3.2, we obtain

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} \hat{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \phi' d\xi = \int_{\xi_1}^{\alpha_1} v_- (u_-^k - \xi) \phi' d\xi + \int_{\alpha_2}^{\xi_2} v_+ (u_+^k - \xi) \phi' d\xi
\]

\[
= (v_- u_-^k - v_+ u_+^k - v_- \alpha_1 + v_+ \alpha_2) \phi(\sigma) + \int_{\xi_1}^{\xi_2} v_- \phi(\xi) d\xi + \int_{\alpha_2}^{\xi_2} v_+ \phi(\xi) d\xi
\]

Then taking \(\alpha_1 \to \sigma-\), \(\alpha_2 \to \sigma+\), we arrive at

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} \hat{\nu}^\varepsilon ((\hat{u}^\varepsilon)^k - \xi) \phi' d\xi = (-[\hat{\nu}] \sigma + [\hat{\nu}^k]) \phi(\sigma) + \int_{\xi_1}^{\xi_2} J(\xi - \sigma) \phi(\xi) d\xi \tag{29}
\]

where \([q] = q_- - q_+\) and

\[
J(x) = \begin{cases} 
  v_-, & \text{if } x < 0, \\
  v_+, & \text{if } x > 0.
\end{cases}
\]

From (28) and (29), we get

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\hat{\nu}^\varepsilon - J(\xi - \sigma)) \phi(\xi) d\xi = (-[\hat{\nu}] \sigma + [\hat{\nu}^k]) \phi(\sigma).
\]

for all sloping test functions \(\phi \in C^\infty([\xi_1, \xi_2])\).

For an arbitrary \(\psi \in C^\infty([\xi_1, \xi_2])\), we take a sloping test function \(\phi\), such that \(\phi(\sigma) = \psi(\sigma)\) and

\[
\max_{[\xi_1, \xi_2]} |\psi - \phi| < \mu,
\]

for a sufficiently small \(\mu > 0\). As \(\hat{\nu} \in L^1([\xi_1, \xi_2])\) uniformly, we obtain

\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\hat{\nu}^\varepsilon - J(\xi - \sigma)) \psi(\xi) d\xi = \lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\hat{\nu}^\varepsilon - J(\xi - \sigma)) \phi(\xi) d\xi + O(\mu)
\]

\[
= (-[\hat{\nu}] \sigma + [\hat{\nu}^k]) \phi(\sigma) + O(\mu)
\]

\[
= (-[\hat{\nu}] \sigma + [\hat{\nu}^k]) \psi(\sigma) + O(\mu).
\]

\(^1\)The function \(\phi\) is called a sloping test function [21]
Then, when $\mu \to 0^+$, we find that
\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\hat{v}^\varepsilon - J(\xi - \sigma)) \psi(\xi) d\xi = (-[\hat{v}]\sigma + [\hat{v}^k]) \psi(\sigma)
\]
holds for all test functions $\psi \in C_0^\infty([\xi_1, \xi_2])$. Thus, $\hat{v}^\varepsilon$ converges in the sense of the distributions to the sum of a step function and a Dirac delta function with strength $-[\hat{v}]\sigma + [\hat{v}^k]$. In similar way, we can show that
\[
\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} (\hat{u}^\varepsilon - \tilde{J}(\xi - \sigma)) \psi(\xi) d\xi = 0
\]
for all test functions $\psi \in C_0^\infty([\xi_1, \xi_2])$ and where $\tilde{J}(x) = \begin{cases} u_-, & \text{if } x < 0, \\ u_+, & \text{if } x > 0. \end{cases}$
Thus, $\hat{u}^\varepsilon$ converges in the sense of the distributions to a step function.

Then we get the following theorem.

**Theorem 3.1.** Suppose $u_- > u_+$. Let $(\hat{v}^\varepsilon(x,t), \hat{u}^\varepsilon(x,t))$ be the similarity solution of (4)–(5). Then the limit
\[
\lim_{\varepsilon \to 0^+} (\hat{v}^\varepsilon(x,t), \hat{u}^\varepsilon(x,t)) = (\hat{v}(x,t), \hat{u}(x,t))
\]
exists in the measure sense and $(\hat{v}, \hat{u})$ solves (8)–(5). Moreover,
\[
(\hat{v}(x,t), \hat{u}(x,t)) = \begin{cases} (v_-, u_-), & \text{if } x < \frac{\alpha k}{\sigma} (1 - e^{-\alpha kt}), \\ (\frac{w_0}{\alpha k} (1 - e^{-\alpha kt}) \delta(x - \frac{\sigma}{\alpha k} (1 - e^{-\alpha kt})), \sigma), & \text{if } x = \frac{\alpha k}{\sigma} (1 - e^{-\alpha kt}), \\ (v_+, u_+), & \text{if } x > \frac{\alpha k}{\sigma} (1 - e^{-\alpha kt}), \end{cases}
\]
where $\sigma = \frac{1}{k+1} \sum_{j=0}^{k} u_{-j}^k u_{j}^k$ and $w_0 = -\sigma (v_- - v_+) + (v_- u_{-k}^k - v_+ u_{+k}^k)$. Moreover, $\sigma$ satisfies the entropy condition $u_{+k}^k < \sigma < u_{-k}^k$.

**Case 2.** $u_- < u_+$

**Lemma 3.4.** For any $\eta > 0$,
\[
\lim_{\varepsilon \to 0^+} \hat{u}^\varepsilon_2(\xi) = 0, \text{ for } \xi \leq u_-^k - \eta \text{ or } \xi \geq u_+^k + \eta,
\]
\[
\lim_{\varepsilon \to 0^+} (\hat{v}^\varepsilon(\xi), \hat{u}^\varepsilon(\xi)) = \begin{cases} (v_-, u_-), & \text{if } \xi < \xi_{\sigma_1} - \eta, \\ (0, \xi), & \text{if } \xi_{\sigma_1} - \eta \leq \xi \leq \xi_{\sigma_2} + \eta, \\ (v_+, u_+), & \text{if } \xi > \xi_{\sigma_2} + \eta, \end{cases}
\]
uniformly in the above intervals.
Proof. Since \( \hat{u} \) is an increasing smooth function in \((-\infty, +\infty)\), then \( u_- \leq \hat{u}(\xi_{\sigma_1}) \leq \hat{u}(\xi) \leq \hat{u}(\xi_{\sigma_2}) \leq u_+ \) or \( u_-^k \leq \xi_{\sigma_1} \leq (\hat{u}(\xi))^k \leq \xi_{\sigma_2} \leq u_+^k \).

The proof of this lemma is basically similar to that of Lemma 3.1. Take \( \xi_\delta = \xi_{\sigma_1} - \eta/2 \) and let \( \varepsilon \) be so small such that \( \xi_{\sigma_1}^\varepsilon > \xi_3 + \eta/4 \). Integrating the first equation of (6) twice on \([\xi, \xi_3]\), we get

\[
\hat{u}(\xi_3) - \hat{u}(\xi) = \hat{u}'(\xi_3) \int_\xi^{\xi_3} \exp \left( - \int_r^{\xi_3} \frac{(\hat{u}(s))^k - s}{\varepsilon} \, ds \right) \, dr \\
\geq \hat{u}'(\xi_3) \int_\xi^{\xi_3} \exp \left( - \int_r^{\xi_3} \frac{u_-^k - s}{\varepsilon} \, ds \right) \, dr \\
= \hat{u}'(\xi_3) \int_\xi^{\xi_3} \exp \left( \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) (r - \xi_3) - \frac{1}{2} (r - \xi_3)^2 \right) \right) \, dr \\
= \hat{u}'(\xi_3) \int_{\xi - \xi_3}^{\xi_3} \exp \left( \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) r - \frac{1}{2} r^2 \right) \right) \, dr.
\]

Letting \( \xi \to -\infty \), we get

\[
u_+ - u_- \geq \hat{u}'(\xi_3) \int_{-\infty}^{0} \exp \left( \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) r - \frac{1}{2} r^2 \right) \right) \, dr \\
\geq \hat{u}'(\xi_3) \int_{-\infty}^{2\varepsilon} \exp \left( - \frac{1}{\varepsilon} \left( (u_-^k - \xi_3) r + \frac{1}{2} r^2 \right) \right) \, dr \\
\geq \hat{u}'(\xi_3) \sqrt{\varepsilon} A_3
\]

for \( 0 \leq \varepsilon \leq 1 \), where \( A_3 \) is a constant independent of \( \varepsilon \). Thus

\[
|\hat{u}'(\xi_3)| \leq \frac{u_+ - u_-}{\sqrt{\varepsilon} A_3}.
\]

Noticing that

\[
(\hat{u}(s))^k - s = ((\hat{u}(s))^k - (\hat{u}(\xi_{\sigma_1}))^k) - (s - \xi_{\sigma_1}^\varepsilon) = (k(\hat{u}(\theta)))^{k-1} u'(\theta) - 1)(s - \xi_{\sigma_1}^\varepsilon) \geq \frac{\eta}{4}
\]

for \( s \leq \xi_3 \) and from (25) we have

\[
|\hat{u}'(\xi)| \leq \frac{u_+ - u_-}{\sqrt{\varepsilon} A_3} \exp \left( - \frac{\eta}{4\varepsilon} (\xi_3 - \xi) \right)
\]

which implies that

\[
\lim_{\varepsilon \to 0^+} \hat{u}_\varepsilon' (\xi) = 0, \quad \text{uniformly for } \xi \leq \xi_{\sigma_1} - \eta.
\]

Now, we choose \( \xi \) and \( \xi_4 \) such that \( \xi < \xi_4 \leq \xi_{\sigma_1} - \eta \). From

\[
\hat{u}(\xi_4) - \hat{u}(\xi) = \hat{u}'(\xi_4) \int_{\xi}^{\xi_4} \exp \left( - \int_r^{\xi_4} \frac{(\hat{u}(s))^k - s}{\varepsilon} \, ds \right) \, dr,
\]
we get
\[ |\hat{u}(\xi) - \hat{\upsilon}(\xi)| \leq |\hat{u}'(\xi)| \int_{\xi}^{\xi_4} \exp \left( \frac{A_4}{\varepsilon} (r - \xi_4) \right) dr \leq \frac{\varepsilon}{A_4} |\hat{u}'(\xi_4)| \left( 1 - \exp \left( \frac{A_4}{\varepsilon} (\xi - \xi_4) \right) \right), \]
where \( A_4 = (\hat{u}(\xi_4))^k - \xi_4 \). When \( \xi \to -\infty \), we obtain \( |\hat{u}(\xi_4) - u_-| \leq \frac{\varepsilon}{A_4} |\hat{u}'(\xi_4)| \), which implies that
\[ \lim_{\varepsilon \to 0^+} \hat{\upsilon}^\varepsilon(\xi) = u_-, \quad \text{uniformly for } \xi < \xi_{\sigma_1} - \eta. \]
The results for \( \xi > \xi_{\sigma_2} + \eta \) can be obtained analogously.
Now, noticing that for \( \xi < \xi_{\sigma_1} \),
\[
\hat{v}_1(\xi) = v_- \exp \left( -\int_{-\infty}^{\xi} \left( (\hat{u}(s))^k - k s \right)' ds \right) = \lim_{R \to +\infty} v_- \exp \left( -\int_{-R}^{\xi} \left( (\hat{u}(s))^k - k s \right)' ds \right) \geq \lim_{R \to +\infty} v_- \left( \frac{u_+^k + R}{(\hat{u}(\xi))^k - \xi} \right) \exp \left( -\int_{-R}^{\xi} \frac{ds}{u_+^k - s} \right) = v_- \left( \frac{u_+^k - \xi}{(\hat{u}(\xi))^k - \xi} \right).
\]
By Lemma 2.3, \( \hat{v}_1(\xi) \) is decreasing for \( \xi < \xi_{\sigma_1} \), and from (30) we have
\[ v_- \geq \hat{v}_1(\xi) \geq v_- \left( \frac{u_+^k - \xi}{(\hat{u}(\xi))^k - \xi} \right) \to v_- \quad \text{as } \varepsilon \to 0^+. \]
Thus,
\[ \lim_{\varepsilon \to 0^+} \hat{\upsilon}^\varepsilon(\xi) = v_- \quad \text{uniformly for } \xi < \xi_{\sigma_1} - \eta. \]
Analogously, we obtain \( \lim_{\varepsilon \to 0^+} \hat{\upsilon}^\varepsilon(\xi) = v_+ \), uniformly for \( \xi > \xi_{\sigma_2} + \eta \). From Lemma 2.3, on \([\xi_{\sigma_1}, \xi_{\sigma_2}]\) we have that \( \hat{v}(\xi) = 0 \). Now, choose \( \eta_1 > 0 \) and let \( \phi \in C_c^\infty((\xi_1, \xi_2)) \) where \( \xi_1 < \xi_{\sigma_1} - \eta_1 < \xi_2 \). From (14) we have
\[ 0 = \int_{\xi_1}^{\xi_2} (\hat{v}(\xi) \phi(\xi))' - \hat{v}(\xi) (\hat{u}(\xi))^k \phi'(\xi) d\xi = \int_{\xi_1}^{\xi_{\sigma_1} - \eta_1} (v_- \phi(\xi))' - v_- u_+^k \phi'(\xi) d\xi. \]
Thus, we have \( v_- \phi(\xi_{\sigma_1} - \eta_1) - u_+^k \phi(\xi_{\sigma_1} - \eta_1) = 0 \) which yields \( \xi_{\sigma_1} = u_+^k \) for arbitrary \( \phi \) and arbitrary \( \eta_1 \). Analogously, we obtain \( \xi_{\sigma_2} = u_+^k \). For \( \xi \in [\xi_{\sigma_1} + \eta, \xi_{\sigma_2} - \eta] \), denote \( \lim_{\varepsilon \to 0^+} \hat{\upsilon}^\varepsilon(\xi) = \hat{u}(\xi) \). Thus, from the chain rule of Volpert for BV functions [23, 12], Eq. (9) and (14), we have that \( (\hat{u}(\xi))^k = \xi \) with \( \hat{u}(u_+^k) = u_- \) and \( \hat{u}(u_+^k) = u_+ \). Also, (with Lemma 2.3) we have \( \lim_{\varepsilon \to 0^+} \hat{\upsilon}^\varepsilon(\xi) = 0\).

Now, we study the limit behavior of \((\hat{v}, \hat{u})^\varepsilon\) as \( \varepsilon \to 0^+ \).

**Theorem 3.2.** Suppose \( u_+ > u_- \). Let \( (\hat{v}, \hat{u})^\varepsilon \) be the solution of the Riemann problem (6)-(7). Then, \( \lim_{\varepsilon \to 0^+} (\hat{v}^\varepsilon(x, t), \hat{u}^\varepsilon(x, t)) = (\hat{v}(x, t), \hat{u}(x, t)) \) exists in the sense of distributions and \( (\hat{v}, \hat{u}) \) solves (8)-(5). Moreover,
\[
(\hat{v}(x, t), \hat{u}(x, t)) = \begin{cases} (v_-, u_-), & \text{if } x < \frac{u_+}{\alpha k} (1 - e^{-\alpha k t}), \\ (0, (1 - e^{-\alpha k t})^{1/k}), & \text{if } \frac{u_+}{\alpha k} (1 - e^{-\alpha k t}) \leq x \leq \frac{u_+}{\alpha k} (1 - e^{-\alpha k t}), \\ (v_+, u_+), & \text{if } x > \frac{u_+}{\alpha k} (1 - e^{-\alpha k t}). \end{cases}
\]
4 Riemann problem for the system (1)

In this section, we study the Riemann problem to the original system (1). When $u_- < u_+$, the solution of (1)–(2) is directly obtained from the corresponding ones to (8)–(5) by performing the transformation of state variables $(v(x, t), u(x, t)) = (\tilde{v}(x, t), \tilde{u}(x, t)e^{-\alpha t})$, in which the positions of the contact discontinuities remain unchanged. Then, we have the following result for classical Riemann solutions.

**Theorem 4.1.** Assume that $u_- < u_+$. Then the solution for the Riemann problem is

$$(v(x, t), u(x, t)) = \begin{cases} (v_-, u_-e^{-\alpha t}), & \text{if } x < \frac{u_-}{\alpha k}(1 - e^{-\alpha kt}), \\ (0, (\frac{u_-}{1 - e^{-\alpha kt}})^{1/k}e^{-\alpha t}), & \text{if } \frac{u_-}{\alpha k}(1 - e^{-\alpha kt}) \leq x \leq \frac{u_+}{\alpha k}(1 - e^{-\alpha kt}), \\ (v_+, u_+e^{-\alpha t}), & \text{if } x > \frac{u_+}{\alpha k}(1 - e^{-\alpha kt}). \end{cases}$$

It is clear that the above theorem generalizes the Theorem 3.1 in [5]. Now, we study the case when $u_- > u_+$. We need recall the following definition:

**Definition 4.1.** A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(x(s), t(s)) : a < s < b\}$, for $w \in L^1((a, b))$, is defined as

$$\langle w(\cdot)\delta_L, \phi(\cdot, \cdot) \rangle = \int_a^b w(s)\phi(x(s), t(s)) ds, \quad \phi \in C_0^\infty(\mathbb{R} \times [0, \infty)).$$

Now, we define a delta shock wave solution for the system (1) with initial data (2).

**Definition 4.2.** A distribution pair $(v, u)$ is a delta shock wave solution of (1) and (2) in the sense of distribution if there exist a smooth curve $L$ and a function $w \in C^1(L)$ such that $v$ and $u$ are represented in the following form

$$v = \tilde{v}(x, t) + w\delta_L \quad \text{and} \quad u = \tilde{u}(x, t),$$

$\tilde{v}, \tilde{u} \in L^\infty(\mathbb{R} \times (0, \infty); \mathbb{R})$ and

$$\begin{cases} \langle u, \varphi_t \rangle + \langle u^{k+1}, \varphi_x \rangle = \int_0^\infty \int_\mathbb{R} \alpha u \varphi dxdt, \\ \langle v, \varphi_t \rangle + \langle vu^k, \varphi_x \rangle = 0, \end{cases}$$

for all the test functions $\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))$, where $u|_L = u_\delta(t)$ and

$$\langle v, \varphi \rangle = \int_0^\infty \int_\mathbb{R} \tilde{v}\varphi dxdt + \langle w\delta_L, \varphi \rangle,$$

$$\langle vG(u), \varphi \rangle = \int_0^\infty \int_\mathbb{R} \tilde{v}G(\tilde{u})\varphi dxdt + \langle wG(u_\delta)\delta_L, \varphi \rangle.$$

With the previous definitions, we are going to find a solution with discontinuity $x = x(t)$ for (1) of the form

$$(v(x, t), u(x, t)) = \begin{cases} (v_-(x, t), u_-(x, t)), & \text{if } x < x(t), \\ (w(t)\delta_L, u_\delta(t)), & \text{if } x = x(t), \\ (v_+(x, t), u_+(x, t)), & \text{if } x > x(t). \end{cases}$$
where \(v_\pm(x, t), u_\pm(x, t)\) are piecewise smooth solutions of system (1), \(\delta(\cdot)\) is the Dirac measure supported on the curve \(x(t) \in C^1\), and \(x(t), w(t)\) and \(u_\delta(t)\) are to be determined.

Since \(v(x, t) = \hat{v}(x, t)\) and \(u(x, t) = \hat{u}(x, t)e^{-at}\), from Theorem 3.1, we can establish a solution of the form (32) to the system (1) with initial data (2). Thus, we have the following result.

**Theorem 4.2.** Assume that \(u_- > u_+\). Then the Riemann problem (1)–(2) admits one and only one measure solution of the form

\[
(v(x, t), u(x, t)) = \begin{cases} 
(v_-, u_- e^{-at}), & \text{if } x < x(t), \\
(w(t)\delta(x - x(t)), \sigma e^{-at}), & \text{if } x = x(t), \\
(v_+, u_+ e^{-at}), & \text{if } x > x(t),
\end{cases}
\]

(33) where \(w(t) = \frac{m(t)}{\alpha t}(1 - e^{-akt}), x(t) = \frac{x}{\alpha t}(1 - e^{-akt}), \sigma = \sum_{j=0}^{k} u_{-j}^{k} u_+^{j}\) and \(w_0 = -\sigma(v_- - v_+) + (v_- u_0^k - v_+ u_0^k).\) Moreover, \(dx(t)/dt\) satisfies the entropy condition \(u_+ e^{-at} < dx(t)/dt < u_+ e^{-at}\) for all \(t \geq 0\).

**Proof.** We need show that (33) is a solution to the problem (1)–(2) which can be found with \((v, u) = (\hat{v}, \hat{u}e^{-at})\) and the result obtained in Theorem 3.1. Therefore, for any test function \(\varphi \in C_0^\infty(\mathbb{R} \times (0, \infty))\) we have

\[
\langle u, \varphi_t \rangle + \langle u^{k+1}, \varphi_x \rangle = \int_0^\infty \int_\mathbb{R} (u \varphi_t + u^{k+1} \varphi_x) \, dx \, dt
\]

\[
= \int_0^\infty \int_{-\infty}^{x(t)} (u_- e^{-at} \varphi_t + u_-^{k+1} e^{-\alpha(k+1)t} \varphi_x) \, dx \, dt
\]

\[
+ \int_0^\infty \int_{x(t)}^{\infty} (u_+ e^{-at} \varphi_t + u_+^{k+1} e^{-\alpha(k+1)t} \varphi_x) \, dx \, dt
\]

\[
= \int_{-\infty}^{x(t)} \left( u_-^{k+1} e^{-\alpha(k+1)t} \varphi \right) \, dt + (u_- e^{-at} \varphi) \, dx
\]

\[
+ \int_{x(t)}^{\infty} \left( u_+^{k+1} e^{-\alpha(k+1)t} \varphi \right) \, dt + (u_+ e^{-at} \varphi) \, dx + \int_0^\infty \int_\mathbb{R} \alpha u \varphi \, dx \, dt
\]

\[
= \int_0^\infty \left( (u_-^{k+1} - u_+^{k+1}) e^{-at} - \frac{dx(t)}{dt}(u_- - u_+) \right) e^{-at} \varphi \, dt + \int_0^\infty \int_\mathbb{R} \alpha u \varphi \, dx \, dt
\]

\[
= \int_0^\infty \int_\mathbb{R} \alpha u \varphi \, dx \, dt
\]

which implies the second equation of (31). A completely similar argument leads to the first
equation of (31).

\[
\langle v, \varphi_t \rangle + \langle vu^k, \varphi_x \rangle = \int_0^\infty \int_{\mathbb{R}} (v \varphi_t + vu^k \varphi_x) dx dt + \int_0^\infty w(\varphi_t + u^k \varphi_x) dt
\]

\[
= \int_0^\infty \int_{-\infty}^{x(t)} (v_- \varphi_t + v_- u^k_- e^{-\alpha k t} \varphi_x) dx dt
\]

\[
+ \int_0^\infty \int_{x(t)}^\infty (v_+ \varphi_t + v_+ u^k_+ e^{-\alpha k t} \varphi_x) dx dt + \int_0^\infty w(\varphi_t + u^k_\delta \varphi_x) dt
\]

\[
= \int_0^\infty (v_- u^k_- - v_+ u^k_+) e^{-\alpha k t} - (v_- - v_+) \frac{dx(t)}{dt} - \frac{dw(t)}{dt} \varphi dt = 0
\]

\[
\square
\]

5 Final remarks

From Theorem 4.1, we can observe that when \( \alpha \to 0^+ \), the solution converges to

\[
(v(x, t), u(x, t)) = \begin{cases} 
(v_-, u_-), & \text{if } x < u^-_t, \\
(0, (x/t)^{1/k}), & \text{if } u^- t \leq x \leq u^+_t, \\
(v_+, u_+), & \text{if } x > u^+_t,
\end{cases}
\]

which is the classical Riemann solution for the homogeneous system associated to (1). In similar way, from Theorem 4.2, we can observe that when \( \alpha \to 0^+ \), the solution converges to

\[
(v(x, t), u(x, t)) = \begin{cases} 
(v_-, u_-), & \text{if } x < \sigma t, \\
(w_0 t \delta(x - \sigma t), \sigma), & \text{if } x = \sigma t, \\
(v_+, u_+), & \text{if } x > \sigma t,
\end{cases}
\]

where \( \sigma = \frac{1}{k+1} \sum_{j=0}^{k} u^{-j}_{-} u^{j}_{+} \) and \( w_0 = -\sigma(v_- - v_+) + (v_- u^{-}_{-} - v_+ u^{+}_{-}) \). This solution is a delta shock wave solution for the homogeneous system associated to (1). The Riemann problem for the homogeneous system associated to (1) with \( k = 1 \) was solved by K.T. Joseph (see main theorem in [11]).

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