ON LAPLACE TRANSFORM AND (IN) STABILITY OF EXTERNALLY DAMPED AXIALLY MOVING STRING

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Abstract

This paper examines an (in) stability of an axially moving string system under the effect of external (viscous) damping. The string is taken to be fixed at both ends and general initial conditions are taken into consideration. The belt (string) speed is assumed to be non-constant harmonically varying about a relatively large means speed. The external damping is also considered to be small. Mathematically, the transverse vibrations of damped axially moving string system are modeled as second order linear homogeneous partial differential equation with variable coefficients. The approximate-analytic solution of the given initial-boundary value problem has been obtained by the application of two timescales perturbation method in conjunction of with Laplace transform method. It is found out that there are infinitely many values of resonant frequency parameter that gives rise to internal resonance in the system. However, in this study only non-resonant and the fundamental resonant cases has been studied. It turned out that the mode-response and the energy of system exhibits stability under certain values of damping parameter and mode-truncation for those parametric values is not problematic.

Keywords: axially moving string, viscous damping, mode response, internal resonance, Laplace transform method

I. Introduction

Axially moving devices, such as conveyor belt, elevator cables, aerial cables, power transmission belts, plastic films, oil pipeline, magnetic tapes, band-saw blades, crane and mine hoists, play very important role in our lives. These systems have been studied for many years and still of great interest for researchers due to their theoretical and industrial importance. Despite having many advantages of these mechanical systems, the vibrations, particularly transverse vibrations associated with these devices have limited their applications. Irregular speed of driven motor, eccentricity of a pulley, non-uniformity of material properties, and environmental
disturbance such as wind, storm and earthquake can lead to severe vibrations which may not only cause human discomfort but also damage to the structures. The understanding of transversal vibrations of axially moving continua is important for the design and manufacture of these devices. In order to mitigate the vibrations in these mechanical structures both material [VIII, XI, XIII, XIV, XVIII, XXVII, XXVIII] and boundary dampers [III, X, XIX, XXV, XXVI] are effectively used for many years. Furthermore the study of dynamics of an axially moving string with time-dependent velocity has been gaining much attention for last two decades see for instance see [V, VI, XV, XX]. In [IV] authors, studied weakly non-linear string-like equation by keeping one end of the string fixed, while spring-mass-dashpot system attached at other boundary. A two timescales perturbation method has been used to construct the formal asymptotic approximation of solution and it was shown that solution tends to zero by increasing value of damping parameter. In [XXI] authors studied beam-like equation by considering one end of the beam to be simply supported whereas spring-dashpot system attached on other end. Two timescales perturbation method is used to construct formal asymptotic solutions. It has been shown that how different oscillations are damped. In [XXII] authors studied beam-like equation under the influence of material damping by considering beam as simply supported at both ends. The solution of initial-boundary value problem was obtained by two timescales perturbation method. It has been shown, that oscillations modes are damped out and damping rates are in fact depending on mode number. In [XXX] authors studied nonlinear parametric vibrations of an axially moving string under the influence of fractional viscoelastic damping. A two timescales perturbation method has been applied to investigate the steady-state responses of the fractional damping and numerical results are presented to exhibit the influence of different parameters. In [I, XXIII] authors studied string-like equations under the influence of viscous and internal damping. A two timescales perturbation method is used to obtain formal asymptotic approximation of solutions. It has been shown that amplitude response has been clearly damped out. In [XVI] authors studied transverse vibrations of an axially moving string under the boundary damping. The asymptotic solution of initial-boundary value problem was obtained by using two timescales perturbation method together with method of characteristic coordinates. It has been shown that the motion of the travelling string in terms of vertical displacement is damped out by increasing the damping in the system. In [XII] authors studied free transverse vibrations of an axially moving string with a fixed end and a spring-mass-dashpot system attached at other boundary. A two timescales perturbation method and Laplace transform method has been used to investigate the system. The numerical results show that the increases of the damping factor considerably reduce the amplitude of the transverse vibrations. In [XVII] authors studied the string-like equation with fixed boundary conditions with time-varying velocity. A two timescales perturbation method together with Laplace transform method has been employed for constructing the formal asymptotic solutions. The energy of system was obtained from infinite dimensional system of ODE’s. It has been shown that Galerkin’s truncation method cannot be applied to
obtain approximation valid on long timescales. However, this study was restricted to un-damped system.

This paper aims to examine the (in) stabilities of axially moving system under the effect of external (viscous) damping. The two timescales perturbation method together with Laplace transform method is used to construct the solution of the initial-boundary value problem.

II. Equations of Motion

An axially translating string under the influence of small viscous damping $\delta$, moving with non-constant velocity $V$ is represented in Fig. 1. Both ends of string are kept fixed i.e., $u = 0$ at $x = 0, x = L$ and pulleys are apart by distance $L$. The equations of motion of an axially moving string under small viscous damping are formulated by extended Hamilton’s principle [XXIV] under the assumptions:

- The density $\rho$ of belt (string) is considered to be constant.
- The tension $P$ of string is considered to be constant and to be large compared to weight of string.
- Bending stiffness, external excitations and effect of gravity are neglected.
- Viscous damping is considered as small.
- Belt velocity is considered as harmonically time-varying velocity function $V(t) = V_0 + \alpha \sin(\omega t)$.
- Belt always moves in one direction: $|V_0| > \alpha$.
- Only transversal motion of string between fixed ends is taken.
- The small displacement is considered that is $|u_0| \ll 1$.

Based on the above assumptions the following governing equations of motion are developed:

$$\rho (u_{tt} + 2Vu_{xt} + Vu_x + V^2 u_{xx}) - Pu_{xx} + \delta (u_t + Vu_x) = 0 \quad t \geq 0,$$

$$u(0, t) = 0, \quad \text{and} \quad u(L, t) = 0, \quad t \geq 0, \quad (2)$$

Fig. 1: The schematic model of a viscously damped axially moving belt
$u(x, 0) = f(x), \quad \text{and} \quad u_t(x, 0) = g(x), \quad 0 < x < L,$ \hspace{1cm} (3)

Where $\rho$ is mass density per unit length of string, $u(x, t)$ represents transversal displacement, $P$ is constant tension in string, and the coefficient of small viscous damping is denoted by $\delta$. The initial displacement and velocity are represented by $f(x)$ and $g(x)$ respectively. With the following dimensionless quantities, we can rewrite initial-boundary value problem Eq. (1) to (3) in non-dimensional form:

$$
\begin{align*}
x^* &= \frac{x}{L}, \quad V^* = \frac{V}{c}, \quad t^* = \frac{ct}{L}, \quad u^*(x^*, t^*) = \frac{u(x, t)}{L}, \quad \omega^* = \frac{L \omega}{c}, \quad f^*(x) = \frac{f(x)}{L},

g^*(x^*) = \frac{g(x)}{c}, \quad \delta^* = \frac{\delta L}{\rho c},
\end{align*}
$$
\hspace{1cm} (4)

Where, $\sqrt{\frac{P}{\rho}}$, by substitution Eq. (4) and related derivatives into the initial-boundary value problem (1)-(3) yields the dimensionless form (where asterisks have been neglected) as under:

$$
u_{tt} + 2V v_{xt} + V_t v_x + (V^2 - 1)u_{xx} + \delta (u_t + V u_x) = 0 \quad t \geq 0,$$ \hspace{1cm} (5)

With the fixed boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(1, t) = 0, \quad t \geq 0,$$ \hspace{1cm} (6)

And the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < 1.$$ \hspace{1cm} (7)

**Formal Approximations of the Solution**

In this section, we will solve the initial-boundary value problem (5)-(7). The velocity of belt varies with time due to many reasons, such as eccentricity of pulleys, irregular speed of driven motor, non-uniformity of material properties. Thus we have considered harmonically high mean non-constant velocity and is expressed as $V(t) = V_0 + \alpha \varepsilon \sin(\omega t)$ where $\varepsilon$ is small dimensionless parameter with $0 < \varepsilon \ll 1$, and $V_0$ and $\alpha$ are positive constants. The velocity fluctuation parameter of belt is denoted by $\omega$. We have also considered small viscous damping, that is, $\delta = O(\varepsilon)$. With these assumptions Eq. (5) can be rewritten as under:
\[
\frac{\partial^2 u}{\partial t^2} + 2V_0 \frac{\partial^2 u}{\partial x \partial t} + (V_0^2 - 1) \frac{\partial^2 u}{\partial x^2} = \varepsilon \left[ -2\alpha \sin(\omega t) u_{xt} - 2V_0 \alpha \sin(\omega t) u_{xx} - \alpha \Omega \cos(\omega t) u_x - \delta (u_t + V_0 u_x) \right] + O(\varepsilon^2)
\]

(8)

A straightforward expansion method in \( \varepsilon \) may have secular terms which cause the non-uniformity in the solution, therefore such unbounded terms must be avoided. Thus it is reasonable to use two timescales perturbation method for further investigation of Eq. (8) and readers are referred to see [II, VII].

\[
u(x,t;\varepsilon) = v_k(x,t_0,t_1;\varepsilon) = v_0(x,t_0,t_1) + \varepsilon v_1(x,t_0,t_1) + O(\varepsilon^2)
\]

(9)

In which \( t_0 = t \) and \( t_1 = \varepsilon t \) are the usual fast and slow timescales respectively. Based upon these new timescales and by using chain rule we have following transformations:

\[
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t_0} + \varepsilon \frac{\partial v}{\partial t_1}
\]

(10)

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t_0^2} + 2\varepsilon \frac{\partial^2 v}{\partial t_0 \partial t_1} + \varepsilon^2 \frac{\partial^2 v}{\partial t_1^2}
\]

(11)

Substitution of Eqs. (9) - (11) into Eq. (8) and by collecting coefficients of \( \varepsilon^0 \) and \( \varepsilon^1 \) it yields:

\[
O(1) - \text{Problem: } \frac{\partial^2 v_0}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_0}{\partial t_0 \partial x} + (V_0^2 - 1) \frac{\partial^2 v_0}{\partial x^2} = 0
\]

(12)

\[
O(\varepsilon) - \text{Problem: } \frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - 1) \frac{\partial^2 v_1}{\partial x^2} = -2\alpha \sin(\omega t) \frac{\partial^2 v_0}{\partial t_0 \partial x} - 2V_0 \alpha \sin(\omega t) \frac{\partial^2 v_0}{\partial x^2} - \alpha \Omega \cos(\omega t) \frac{\partial v_0}{\partial x} - \delta (\frac{\partial v_0}{\partial t_0} + V_0 \frac{\partial v_0}{\partial x})
\]

(13)
The solution of Eq. (12) is obtained by application of Laplace transform method [IX] and for the complete solution $O(1)$-problem readers are referred to [XXIX]. Thus we have following solution as under:

\[
v_0(x, t_0, t_1) = \sum_{n=1}^{\infty} \left[ F_{[1]n}(x)(A_n(t_1) \cos(\Omega_n t_0) - B_n(t_1) \sin(\Omega_n t_0)) \\
+ F_{[2]n}(x)(A_n(t_1) \sin(\Omega_n t_0) + B_n(t_1) \cos(\Omega_n t_0)) \right]
\]  

(14)

Where the functions $F_{[1]n}(x)$ and $F_{[2]n}(x)$ are given as under:

\[
F_{[1]n}(x) = \cos(n\pi(V_0 + 1)x) - \cos(n\pi(V_0 - 1)x)
\]

and

\[
F_{[2]n}(x) = \sin(n\pi(V_0 + 1)x) - \sin(n\pi(V_0 - 1)x)
\]  

(15)

And $\Omega_n = n\pi(1 - V_0^2)$, $n \in N$ are the known as the natural frequencies of the belt system. In Eq. (14) $A_n(t_1)$ and $B_n(t_1)$ are still arbitrary functions of slow timescale and these functions will be used to remove secular terms in the solution of the $O(\varepsilon)$ problem. Substitution Eq. (14) and Eq. (15) into Eq. (13), the solution of $O(\varepsilon)$ problem becomes:

\[
\frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - 1) \frac{\partial^2 v_1}{\partial x^2}
\]

\[
= \sum_{n=1}^{\infty} \left[ \sin(\Omega_n t_0) \phi_n + \cos(\Omega_n t_0) \bar{\phi}_n \right]
\]

\[
+ \sum_{n=1}^{\infty} \left[ \sin(\Omega_n t_0) \sin(\Omega_n t_0) \psi_n \right.
\]

\[
+ \sin(\Omega_n t_0) \cos(\Omega_n t_0) \bar{\psi}_n + \cos(\Omega_n t_0) \sin(\Omega_n t_0) \varphi_n
\]

\[
+ \cos(\Omega_n t_0) \cos(\Omega_n t_0) \bar{\varphi}_n \]

(16)

Where

\[
\phi_n(x, t_1) = (2 \frac{\partial A_{n0}}{\partial t_1} + \delta A_{n0}) \left( \Omega_n F_{[1]n} - V_0 \frac{\partial F_{[2]n}}{\partial x} \right)
\]

\[
+ \left( 2 \frac{\partial B_{n0}}{\partial t_1} + \delta B_{n0} \right) \left( \Omega_n F_{[2]n} + V_0 \frac{\partial F_{[1]n}}{\partial x} \right)
\]
\[
\bar{\varphi}_n(x, t_1) = \left( 2 \frac{\partial A_{n0}}{\partial t_1} + \delta A_{n0} \right) \left( -\Omega_n F_{[2]n} - V_0 \frac{\partial F_{[1]n}}{\partial x} \right) \\
+ \left( 2 \frac{\partial B_{n0}}{\partial t_1} + \delta B_{n0} \right) \left( \Omega_n F_{[1]n} - V_0 \frac{\partial F_{[2]n}}{\partial x} \right) \\
\]
\[
\Psi_n(x, t_1) = 2\alpha \left\{ A_{n0} \left( \Omega_n \frac{\partial F_{[1]n}}{\partial x} - V_0 \frac{\partial^2 F_{[2]n}}{\partial x^2} \right) \\
+ B_{n0} \left( \Omega_n \frac{\partial F_{[2]n}}{\partial x} + V_0 \frac{\partial^2 F_{[1]n}}{\partial x^2} \right) \right\} \\
\]
\[
\tilde{\Psi}_n(x, t_1) = 2\alpha \left\{ A_{n0} \left( -\Omega_n \frac{\partial F_{[2]n}}{\partial x} - V_0 \frac{\partial^2 F_{[1]n}}{\partial x^2} \right) \\
+ B_{n0} \left( \Omega_n \frac{\partial F_{[1]n}}{\partial x} - V_0 \frac{\partial^2 F_{[2]n}}{\partial x^2} \right) \right\} \\
\]
\[
\varphi_n(x, t_1) = \alpha \omega \left( -A_{n0} \frac{\partial F_{[2]n}}{\partial x} + B_{n0} \frac{\partial F_{[1]n}}{\partial x} \right) \\
\tilde{\varphi}_n(x, t_1) = \alpha \omega \left( -A_{n0} \frac{\partial F_{[1]n}}{\partial x} - B_{n0} \frac{\partial F_{[2]n}}{\partial x} \right) \\
\]

III. Non Resonant Case

In this section, we will study the Eq. (16) for the non-resonant case that is, when frequency of the belt velocity is not in neighbored of any natural frequency \((\omega \neq \Omega_n)\). Thus only the first summation on R.H.S of Eq. (16) will lead to unbounded behavior of \(O(\varepsilon)\). problem,

\[
\frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - 1) \frac{\partial^2 v_1}{\partial x^2} = \sum_{n=1}^{\infty} \left[ \sin(\Omega_n t_0) \varphi_n + \cos(\Omega_n t_0) \bar{\varphi}_n \right] \\
\]

After the very long but an elementary calculation the solution of Eq. (18) is obtained by means of Laplace transform method and is given as under:

\[
v_1(x, t_0, t_1) = -\frac{1}{4} \sum_{n=1}^{\infty} \left[ \left( f_{[1]n} + \tilde{f}_{[2]n} \right) t_0 \sin(\Omega_n t_0) \\
+ \left( f_{[1]n} - f_{[2]n} \right) t_0 \cos(\Omega_n t_0) \right] + \text{NST} \\
\]

Where

\[
f_{[1]n}(x, t_1) = p_n(x, t_1) F_{[2]n}(x) + w_n(x, t_1) F_{[1]n}(x) \\
f_{[2]n}(x, t_1) = -p_n(x, t_1) F_{[1]n}(x) + w_n(x, t_1) F_{[2]n}(x) \\
\]

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\begin{align*}
\ddot{f}_{1n}(x, t_1) &= \ddot{p}_n(x, t_1)F_{[2]n}(x) + \ddot{w}_n(x, t_1)F_{[1]n}(x) \\
\ddot{f}_{2n}(x, t_1) &= -\ddot{p}_n(x, t_1)F_{[1]n}(x) + \ddot{w}_n(x, t_1)F_{[2]n}(x)
\end{align*}

(20)

And

\begin{align*}
p_n(x, t_1) &= -\frac{1}{2} \int_0^\frac{1}{n\pi} \frac{\phi_n(x, t_1)}{-F_{[1]n}(x)} dx \\
w_n(x, t_1) &= \frac{1}{2} \int_0^\frac{1}{n\pi} \frac{\phi_n(x, t_1)}{-F_{[2]n}(x)} dx \\
\ddot{p}_n(x, t_1) &= -\frac{1}{2} \int_0^\frac{1}{n\pi} \frac{\ddot{\phi}_n(x, t_1)}{-F_{[1]n}(x)} dx \\
\ddot{w}_n(x, t_1) &= \frac{1}{2} \int_0^\frac{1}{n\pi} \frac{\ddot{\phi}_n(x, t_1)}{-F_{[2]n}(x)} dx
\end{align*}

(21)

And where NST stands for non-secular terms. The coefficients of \( t_0\sin(n\Omega x_0) \) and \( t_0\cos(n\Omega x_0) \) in Eq. (19) must be equate to zero to avoid from secular terms, thus we get:

\begin{align*}
(f_{[1]n}(x, t_1) + \ddot{f}_{[2]n}(x, t_1)) &= 0 \\
(f_{[1]n}(x, t_1) - \ddot{f}_{[2]n}(x, t_1)) &= 0
\end{align*}

(22)

By using Eq. (20) into Eq. (22) we get

\begin{align*}
p_n(x, t_1)F_{[2]n}(x) + w_n(x, t_1)F_{[1]n}(x) + \ddot{w}_n(x, t_1)F_{[2]n}(x) \\
- \ddot{p}_n(x, t_1)F_{[1]n}(x) &= 0 \\
p_n(x, t_1)F_{[1]n}(x) - w_n(x, t_1)F_{[2]n}(x) + \ddot{w}_n(x, t_1)F_{[1]n}(x) \\
+ \ddot{p}_n(x, t_1)F_{[2]n}(x) &= 0
\end{align*}

(23)

By using Eq. (15), Eq. (17) and Eq. (21) into Eq. (23), after long but elementary calculations finally we get following coupled system of ordinary differential equations.

\[ \left( \frac{dA_{n0}}{dt_1} + \frac{\delta}{2} A_{n0} \right) \cos(n\pi V_0 x) - \left( \frac{dB_{n0}}{dt_1} + \frac{\delta}{2} B_{n0} \right) \sin(n\pi V_0 x) = 0 \]

\[ \left( \frac{dA_{n0}}{dt_1} + \frac{\delta}{2} A_{n0} \right) \sin(n\pi V_0 x) + \left( \frac{dB_{n0}}{dt_1} + \frac{\delta}{2} B_{n0} \right) \cos(n\pi V_0 x) = 0 \]

(24)
It should be observed that for \( V_0 = 0 \) in above system we get same system as obtained for non-resonant case for velocity \( V(t) = \epsilon(V_0 + \alpha \sin \omega t) \) given in [XXIV]. Further for \( \delta = 0 \) system (24) reduces to same as obtained in [XX]. The solution can easily be obtained for unknown quantities \( A_{n0} \) and \( B_{n0} \) and that is given as under:

\[
A_{n0}(t_1) = C_1 e^{-\frac{\delta t_1}{2}} \quad \text{and} \quad B_{n0}(t_1) = C_2 e^{-\frac{\delta t_1}{2}}
\]  

(25)

Where \( C_1 \) and \( C_2 \) are arbitrary constants and can be obtained by using given conditions of the problem. Eq. (25) clearly shows that the amplitude-response of string-like equation under the influence of viscous damping for non-resonant case is clearly damped out.

IV. Fundamental Resonant Case

In this section, we will remove the unbounded terms in the solution of Eq. (16) for fundamental resonant case \( (\omega = \Omega) \) that is, \( \omega = \pi(1 - V_0^2) \). By putting \( \omega = \pi(1 - V_0^2) \) into Eq. (16) and by re-arranging terms we get:

\[
\frac{\partial^2 v_1}{\partial t_0^2} + 2V_0 \frac{\partial^2 v_1}{\partial t_0 \partial x} + (V_0^2 - 1) \frac{\partial^2 v_1}{\partial x^2} = \sum_{n=1}^{\infty} \left[ \sin(\Omega_n t_0) \phi_n + \cos(\Omega_n t_0) \varphi_n \right]
\]

\[
+ \frac{1}{2} \sum_{n=1}^{\infty} \left[ \cos(\Omega_{n-1} t_0) (\Psi_n + \varphi_n) \right] + \cos(\Omega_{n+1} t_0) (\Psi_n + \varphi_n)
\]

(26)

Where \( \phi_n, \varphi_n, \Psi_n, \Psi_n, \Psi_n, \varphi_n \) are given in Eq. (17). By applying the Laplace transform method the solution of Eq. (26) is obtained as under:

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\[ v_1(x, t_0, t_1) = -\left[ \frac{1}{2} f_{[1]} \right] + \frac{1}{4} f_{[2]} \right] t_0 \sin(\Omega t_0) \]
\[-\left[ \frac{1}{2} f_{[1]} - f_{[2]} \right] t_0 \cos(\Omega t_0) \]
\[- \sum_{n=2}^{\infty} \left[ \frac{1}{2} f_{1n} + f_{2n} \right] t_0 \sin(\Omega_n t_0) \]
\[+ \left[ \frac{1}{4} f_{[1]} - f_{[2]} \right] t_0 \cos(\Omega_n t_0) \right] + NST \]

Where \( f_{[1]} \) and \( f_{[2]} \) are given in Eq. (20), and the functions \( f_{[1]} \) and \( f_{[2]} \) are represented as under:

\[ f_{[1]} = w_{[1]}(x, t_1) F_{[1]}(x) + p_{[1]}(x, t_1) F_{[2]}(x) \]
\[ f_{[2]} = w_{[2]}(x, t_1) F_{[2]}(x) + p_{[2]}(x, t_1) F_{[1]}(x) \]

Where \( k = 1, 2, 3, 4 \), respectively, the functions \( F_{[1]}(x) \) and \( F_{[2]}(x) \) are given in Eq. (15) and \( w_{[k]}(x, t_1) \) and \( p_{[k]}(x, t_1) \) are defined as under:

\[ w_{[1]} = \frac{1}{2} \int_0^1 \psi_{n+1} + \phi_{n+1} \left[ -F_{[2]}(x) \right] dx \]
\[ p_{[1]} = \frac{1}{2} \int_0^1 \psi_{n+1} + \phi_{n+1} \left[ -F_{[1]}(x) \right] dx \]
\[ w_{[2]} = \frac{1}{2} \int_0^1 -\psi_{n-1} + \phi_{n-1} \left[ -F_{[2]}(x) \right] dx \]
\[ p_{[2]} = \frac{1}{2} \int_0^1 -\psi_{n-1} + \phi_{n-1} \left[ -F_{[1]}(x) \right] dx \]
\[ w_{[3]} = \frac{1}{2} \int_0^1 -\psi_{n+1} + \phi_{n+1} \left[ -F_{[2]}(x) \right] dx \]
\[ p_{[3]} = \frac{1}{2} \int_0^1 -\psi_{n+1} + \phi_{n+1} \left[ -F_{[1]}(x) \right] dx \]
\[ w_{[4]} = \frac{1}{2} \int_0^1 \psi_{n-1} + \phi_{n-1} \left[ -F_{[2]}(x) \right] dx \]

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\[ p_n^{[4]} = -\frac{1}{2} \int_0^1 \frac{\Psi_{n-1} \Phi_{n-1}}{n\pi} [-F_{1[n]}(x)] dx \]

Where \( \Psi_n(x, t_1), \Phi_n(x, t_1) \) and \( \Phi_n(x, t_1) \) are given in Eq. (17). The solvability condition to remove the secular terms in Eq. (27) is given as under:

\[
\left( f_{[1]n} + \tilde{f}_{[2]n} \right) + \frac{1}{2} \left( f_{[1][1]} + f_{[3][1]} \right) = 0 \\
\left( \tilde{f}_{[1]n} - f_{[2]n} \right) + \frac{1}{2} \left( f_{[1][1]} - f_{[3][1]} \right) = 0
\]  

(30)

and for \( n \geq 2 \), we have following coupled equations:

\[
\left( f_{[1]n} + \tilde{f}_{[2]n} \right) + \frac{1}{2} \left( f_{[1][n]} + f_{[3][n]} + f_{[2][n]} - f_{[1][n]} \right) = 0 \\
\left( \tilde{f}_{[1]n} - f_{[2]n} \right) + \frac{1}{2} \left( f_{[1][n]} - f_{[3][n]} + f_{[2][n]} - f_{[1][n]} \right) = 0
\]  

(31)

Defining \( A_{00}(t_1) \equiv 0 \) and \( B_{00}(t_1) \equiv 0 \) and for \( n = 1, 2, 3, \ldots \), we have following coupled equations as under:

\[
F_{[1]n}(x)\sigma_{[1]n}(t_1) + F_{[2]n}(x)\sigma_{[2]n}(t_1) = 0, \\
F_{[2]n}(x)\sigma_{[2]n}(t_1) - F_{[1]n}(x)\sigma_{[1]n}(t_1) = 0
\]  

(32)

Where

\[
\sigma_{[1]n}(t_1) = w_n(t_1) - \tilde{p}_n(t_1) + \frac{1}{2} \left( w_n^{[3]}(t_1) - p_n^{[1]}(t_1) + w_n^{[4]}(t_1) - p_n^{[2]}(t_1) \right)
\]

\[
\sigma_{[2]n}(t_1) = \tilde{w}_n(t_1) + p_n(t_1) + \frac{1}{2} \left( w_n^{[1]}(t_1) + p_n^{[3]}(t_1) + w_n^{[2]}(t_1) + p_n^{[4]}(t_1) \right)
\]  

(33)

Where \( \sigma_{[1]n}(t_1) \) and \( \sigma_{[2]n}(t_1) \) are unknowns in above coupled system. The determinant of this system is non-zero so it yields that \( \sigma_{[1]n}(t_1) = 0 \) and \( \sigma_{[2]n}(t_1) = 0 \), so we get

\[
w_n(t_1) - \tilde{p}_n(t_1) + \frac{1}{2} \left( w_n^{[3]}(t_1) - p_n^{[1]}(t_1) + w_n^{[4]}(t_1) - p_n^{[2]}(t_1) \right) = 0
\]

\[
\tilde{w}_n(t_1) + p_n(t_1) + \frac{1}{2} \left( w_n^{[1]}(t_1) + p_n^{[3]}(t_1) + w_n^{[2]}(t_1) + p_n^{[4]}(t_1) \right) = 0
\]  

(34)

We solve the coupled system (34) for \( \frac{dA_{n0}}{dt}, \frac{dB_{n0}}{dt}, A_{n0}, B_{n0} \) and finally get following coupled system of ODE’s as under:

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The Eq. (35) represents the infinite dimensional system of coupled ordinary differential equations. It can clearly be seen that for $\delta = 0$ above system coincides with the system as studied in [XX] and for $V_0 = 0$ it coincides with the system studied in [XXIV] and finally for $\delta = 0, V_0 = 0$ it reduces to system as given in [V].

System (35) can be rewritten as under:

$$\frac{dA_{n0}}{dt_1} = -\delta A_{n0} + \gamma(n + 1)A_{(n+1)0} - \sigma(n + 1)B_{(n+1)0} - \gamma(n - 1)A_{(n-1)0} - \sigma(n - 1)B_{(n-1)0}$$

$$\frac{dB_{n0}}{dt_1} = -\delta B_{n0} + \alpha(n + 1)A_{(n+1)0} + \gamma(n + 1)B_{(n+1)0} + \sigma(n - 1)A_{(n-1)0} + \gamma(n - 1)B_{(n-1)0}$$

where $\bar{\ell}_1 = \frac{\alpha}{2} t_1, \gamma = \sin \pi V_0, \sigma = 1 + \cos \pi V_0$ and $\bar{\delta} = \frac{\delta}{2}$. For sake of convenience we drop the bar from $\bar{\ell}_1$ and $\bar{\delta}$.

V. Mathematical Analysis of System (36)

Consider that $X_{n0}(t_1) = nA_{k0}(t_1)$ and $Y_{n0}(t_1) = nB_{k0}(t_1)$, system (36) yields

$$\begin{align*}
\frac{dX_{n0}}{dt_1} &= -\delta X_{n0} + n\left[\gamma X_{(n+1)0} - \sigma Y_{(n+1)0} - \gamma X_{(n-1)0} - \sigma Y_{(n-1)0}\right] \\
\frac{dY_{n0}}{dt_1} &= -\delta Y_{n0} + n\left[\sigma X_{(n+1)0} + \gamma Y_{(n+1)0} + \sigma X_{(n-1)0} - \gamma Y_{(n-1)0}\right]
\end{align*}$$

(37)

For $n = 1, 2, 3, \ldots$ and the functions $X_{n0}(t_1)$ and $Y_{n0}(t_1)$ are zero for non-positive indices $n$. By multiplying first equation by $X_{n0}(t_1)$ and second by $Y_{n0}(t_1)$ on both sides of Eq. (37) we get,

$$\begin{align*}
X_{n0} \frac{dX_{n0}}{dt_1} &= -\delta X_{n0}^2 + n[X_{n0}(\gamma X_{(n+1)0} - \sigma Y_{(n+1)0} - \gamma X_{(n-1)0} - \sigma Y_{(n-1)0})] \\
Y_{n0} \frac{dY_{n0}}{dt_1} &= -\delta Y_{n0}^2 + n[Y_{n0}(\sigma X_{(n+1)0} + \gamma Y_{(n+1)0} + \sigma X_{(n-1)0} - \gamma Y_{(n-1)0})]
\end{align*}$$

(38)
By adding both sides of Eq. (38) and take sum from \( n = 1 \) to \( \infty \), thus one obtains

\[
\frac{1}{2} \sum_{n=1}^{\infty} (X_{n0}^2 + Y_{n0}^2) + \delta \sum_{n=1}^{\infty} (X_{n0}^2 + Y_{n0}^2) = \sum_{n=1}^{\infty} \left[ n\gamma (X_{n0}X_{(n+1)0} + Y_{n0}Y_{(n+1)0}) - n\sigma (X_{n0}Y_{(n+1)0} - Y_{n0}X_{(n+1)0}) \right]
\]

Differentiate Eq. (39) both sides with respect to \( t_1 \), one get

\[
\frac{d^2W(t_1)}{dt_1^2} + 4\delta \frac{dW(t_1)}{dt_1} + 4(\delta^2 - \gamma^2 - \sigma^2)W(t_1) = 0 \tag{40}
\]

where \( \sum_{n=1}^{\infty} (X_{n0}^2 + Y_{n0}^2) = W(t_1) \). For damping parameter \( \delta = 0 \) coincides the equation obtained in [XX]. The roots of Eq. (40) are \(-2\delta \pm 2\sqrt{\sigma^2 + \gamma^2}\) thus the solution of Eq. (40) is given as under:

\[
W(t_1) = e^{-2\delta t_1} \left[ C_1 e^{2\sqrt{\sigma^2 + \gamma^2} t_1} + C_2 e^{-2\sqrt{\sigma^2 + \gamma^2} t_1} \right] \tag{41}
\]

Where \( C_1 \) and \( C_2 \) are arbitrary constants. Three cases for energy hold as under

**Case I:** when \( \delta = \sqrt{\sigma^2 + \gamma^2} \) the energy of system first decreases in smaller time domain and then become constant, such behaviour of the energy shows that system is stable in this case.

**Case II:** when \( \delta > \sqrt{\sigma^2 + \gamma^2} \) the energy of the system tends to zero as time grow without bound and in this case system remains stable in nature.

**Case III:** when \( \delta < \sqrt{\sigma^2 + \gamma^2} \) the energy of system grow exponentially as time progresses, thus for this case the system becomes unstable.

**Case IV:** when \( \delta = 0 \) yields similar behaviour as obtained in [XX].

**VI. Galerkin’s Truncation Method**

In this section, we study the infinite dimensional coupled system (36) by truncating it to finite number of modes. The eigenvalues of system has been obtained up to 10 vibration modes by using Maple16 and are listed in Table.1. It can be observed that the eigenvalues of the truncated system are multiplicity of 2 and are always either real or complex with negative real part. The obtained modes are stable in nature due to negative real part and will clearly be damped out. For \( \delta = 0 \) we get similar eigenvalues see [XX].
Table 1: Eigenvalues of truncated coupled system (36)

| No. of Modes | Eigenvalues of matrix | Order of matrix |
|--------------|-----------------------|-----------------|
| 1            | $-\delta$             | 2               |
| 2            | $-\delta \pm \sqrt{2(\sigma^2 + \gamma^2)}i$ | 4               |
| 3            | $-\delta, -\delta \pm 2\sqrt{2(\sigma^2 + \gamma^2)}i$ | 6               |
| 4            | $-\delta \pm 1.13\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 4.33\sqrt{\sigma^2 + \gamma^2}i$ | 8               |
| 5            | $-\delta, -\delta \pm 2.302\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 5.89\sqrt{\sigma^2 + \gamma^2}i$ | 10              |
| 6            | $-\delta \pm 3.56\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 7.50\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 1.00\sqrt{\sigma^2 + \gamma^2}i$ | 12              |
| 7            | $-\delta, -\delta \pm 9.15\sqrt{\sigma^2 + \gamma^2}i - \delta \pm 2.05\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 4.90\sqrt{\sigma^2 + \gamma^2}i$ | 14              |
| 8            | $-\delta \pm 10.83\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 0.93\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 6.30\sqrt{\sigma^2 + \gamma^2}i$ | 16              |
| 9            | $-\delta, -\delta \pm 12.54\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 1.89\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 7.74\sqrt{\sigma^2 + \gamma^2}i$ | 18              |
| 10           | $-\delta \pm 14.26\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 0.87\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 5.65\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 9.23\sqrt{\sigma^2 + \gamma^2}i, -\delta \pm 2.93\sqrt{\sigma^2 + \gamma^2}i$ | 20              |

VII. Concluding Remarks

In this paper, (in) stability of axially moving system under the effect of external damping has been studied. It has been assumed that the string moves in one direction with non-constant velocity $V(t) = V_0 + \alpha \varepsilon \sin(\omega t)$, where $0 < \varepsilon \ll 1$ and $V_0, \alpha, \omega$ are positive constants. A two timescales perturbation method with conjugation of Laplace transform method has been employed in search of infinite mode approximate solutions. It has been found that there are infinitely many values of $\omega$ giving rise to resonance in the axially moving string. This paper is restricted to non-resonant and fundamental resonance case that is, $\omega = \pi(1 - V_0^2)$. It has been obtained that for non-resonant case system is stable in nature, whereas, for fundamental resonant case the energy of system is obtained from infinite dimensional system of coupled ordinary differential equation, it has been observed that for $\delta = \sqrt{\sigma^2 + \gamma^2}$ and $\delta > \sqrt{\sigma^2 + \gamma^2}$, the energy of system decreases and the system remains stable. However for $\delta < \sqrt{\sigma^2 + \gamma^2}$, the energy of system grows without bound. For damping parameter $\delta = 0$ it yields similar solution as obtained in [XX].

Infinite coupled system of ODE’s has also been analysed by using Galerkin’s truncation method by truncating it to finite number of modes. Eigenvalues are obtained up to 10 modes of vibration, which are stable in nature due to negative real part in each eigenvalue. Finally, it can concluded that the truncation of modes is not problematic for two cases of damping, that is $\delta = \sqrt{\sigma^2 + \gamma^2}$ and $\delta > \sqrt{\sigma^2 + \gamma^2}$.
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