SPATIAL DEGENERACY VS FUNCTIONAL RESPONSE

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Abstract. In this paper, we are concerned with a predator-prey model with Beddington-DeAngelis functional response in heterogeneous environment. By the bifurcation theory and some estimates, the global bifurcation of positive stationary solution is shown. Our result shows that new stationary patterns are produced by the spatial degeneracy and the Beddington-DeAngelis functional response. Essentially different from the known results, the two factors generate two critical values for the prey growth rate $\lambda$. As $\lambda$ crosses each critical value, the positive stationary solution set undergoes a drastic change. In particular, when $\lambda$ is suitably large, the interaction between the two factors yields nonexistence of positive stationary solutions for any $\mu$, which is in strong contrast to the existence for suitable ranges of $\mu$ corresponding to the Lotka-Volterra or Holling-II functional response. Moreover, which one of the two factors plays a dominating role in the stationary patterns is shown. In addition, we give the asymptotic behavior of the positive stationary solutions as $\mu \to \infty$. Finally, both uniqueness and multiplicity of the positive stationary solutions are shown as well as their stability.

1. Introduction. Due to its existence in biology and rich dynamics in mathematics, the predator-prey model has been extensively studied in the past few decades[2, 17, 43]. It is well known that the predator-prey model is usually sensitive to the change of the functional response, which refers to the change in the density of the prey attached per unit time per predator as the prey density changes. The first functional response is the classical Lotka-Volterra functional response, which was adopted by Lotka when he studied a hypothetical chemical reaction in 1925, and by Volterra when he modeled a predator-prey interaction in 1926. The Lotka-Volterra functional response function is a straight line through the origin and is unbounded, which implies that the consumption rate of the prey by each predator will become arbitrarily large if the prey density is sufficiently high. However, the rate at which a predator can consume the prey is limited[4]. Thus, more reasonable functional
response has been proposed, such as Holling-II functional response and so on. Observing that the predator and prey species interact with each other, the mutual interference between predator also contributes to the functional response. Thus, in order to include the effects of feeding saturation and intraspecific consumer interference, Beddington[1] and DeAngelis et al.[8] introduced the following functional response

\[ \frac{u}{1 + mu + kv} \]

which is called Beddington-DeAngelis functional response. Here, the parameter \( k \) measures the mutual interference between predators, one can refer to [9] and references therein for more detailed biological background.

Taking the dispersal of the prey and predator species into account, a prototypical predator-prey model with Beddington-DeAngelis functional response can be modeled by the following reaction-diffusion system

\[
\begin{align*}
  u_t - d_1 \Delta u &= u \left( \lambda - au - \frac{bv}{1 + mu + kv} \right), \\
  v_t - d_2 \Delta v &= v \left( \mu - dv + \frac{cu}{1 + mu + kv} \right).
\end{align*}
\]

In (1), all the parameters are traditionally assumed to be spatially homogeneous. When \( k = 0 \), (1) reduces to the classical predator-prey model with Holling-II functional response, which has been studied extensively, one can refer to [2, 17, 16] and references therein. When \( k > 0 \), as far as we know, there are not many works, one can refer to [3, 5, 27, 26] and references therein.

It is well known that the environment in which the species lives is usually heterogeneous, and what is more important is that the spatial heterogeneity may have profound effects on the ecosystems. Thus, it is more realistic to assume that \( \lambda, \mu, a, b, c, d, m \) and \( k \) are functions of the spatial variable \( x \). Therefore, we consider the following system

\[
\begin{align*}
  u_t - d_1 \Delta u &= u \left( \lambda(x) - a(x)u - \frac{b(x)v}{1 + m(x)u + k(x)v} \right), \\
  v_t - d_2 \Delta v &= v \left( \mu(x) - dv + \frac{c(x)u}{1 + m(x)u + k(x)v} \right), \\
  \partial_\nu u = \partial_\nu v &= 0, \\
  u(x, 0) &= u_0(x) \geq 0, \\
  v(x, 0) &= v_0(x) \geq 0, \\
  x &\in \Omega, t > 0,
\end{align*}
\]

In system (2), \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \) with smooth boundary \( \partial \Omega \), \( \nu(x) \) is the outward unit normal on \( \partial \Omega \) and \( \partial_\nu u = \nabla u(x) \cdot \nu(x) \). Thus, one sees that system (2) is self-contained, and there is no flux on the boundary \( \partial \Omega \). All the coefficient functions, except \( \mu(x) \), are nonnegative continuous functions on \( \Omega \), while \( \mu(x) \) is a continuous function on \( \Omega \) and may take negative values.

As pointed out in [13] and [22], it is generally not easy to capture the influence of the spatial heterogeneity on the ecosystems. During the process of the coefficient varying from constant to nonconstant, two problems occur: the mathematical techniques are rather sensitive to the change, such as the Lyapunov function technique, and become unapplicable in the heterogeneous case; the mathematical techniques are insensitive to the change, such as the bifurcation theory and the topological degree approach, and can also be applied, while the mathematical result obtained is similar to that found in the homogeneous case, and it is difficult to observe the
effects of the spatial heterogeneity. Then how can we overcome the two problems? One way is to develop new mathematical technique to study the models, which is very challenging. The other way is to improve the model appropriately by the sensitiveness of the model to its coefficients.

For the latter way, Du et al. \cite{7, 10, 11, 13, 15, 21, 20, 19, 23} has done a series of work. On the one hand, observing that in most predator-prey interactions, the prey population would extinguish if the growth rate of the predator is too large or the predation rate is too high. Then human interference is often needed to save the endangered prey species and a natural idea is to set protection zones for the prey, where the prey species can enter and leave freely while the predator is blocked out. By introducing protection zones to the competition model and the predator-prey model, Du et al.\cite{21, 19, 15} have shown some interesting results about the effects of the spatial heterogeneity on the positive stationary solution and dynamical behavior. Precisely, a critical patch size for the protection zone is found, which is described by the principal eigenvalue $\lambda^D_{1}(\Omega_0)$ of $-\Delta$ over the protection zone $\Omega_0$ with homogeneous Dirichlet boundary condition. When the protection zone is below the critical patch size, then the observed dynamics of the predator-prey is qualitatively similar to the case without protection zones; whereas, once the protection zone is over the critical patch size, then fundamentally different dynamics can be found. Additionally, we point out that when the protection zone is below the critical patch size, a strikingly different behavior to the no-protection zone for the competition model can still be shown, the reason may be that more detailed result can by obtained for the competition model due to its monotonicity. On the other hand, it has been observed that the behavior of ecosystems is usually very sensitive to certain coefficient functions becoming small in part of the underlying spatial region. Then Du et al. has successfully applied this observation for the prey-predator model \cite{7, 20} and the competition model \cite{10, 11} to reveal the effects of spatial heterogeneity. It is shown that if the chosen parameter is below a critical value, then the degeneracy of the coefficient functions does not seem to have a significant effect on the model; while a drastic change occurs once it crosses the critical value. In addition, positive stationary solutions with certain prescribed spatial patterns were obtained by Du and Hsu\cite{13} for a diffusive Leslie predator-prey model. One can also see \cite{12, 36, 39, 40} and references therein for more details about spatial heterogeneity. Finally, We point out that either of the two observations also requires the authors to develop new techniques.

Motivated by their work, we assume that the coefficient functions of (2) have certain degeneracies and try to show some results about its positive stationary solution set. To focus our attention on the effect of the degeneracy of $a(x)$ on the positive stationary solution set, we assume that all the other parameters are spatially homogeneous. Moreover, for the convenience of notation, we assume that $d_1 = d_2 = d = 1$. Then the corresponding stationary problem of (2) becomes the following problem

$$
\begin{cases}
- \Delta u = u \left( \lambda - a(x)u - \frac{b v}{1 + mu + kv} \right), & x \in \Omega, \\
- \Delta v = v \left( \mu - v + \frac{c u}{1 + mu + kv} \right), & x \in \Omega, \\
\partial_{\nu} u = \partial_{\nu} v = 0, & x \in \partial \Omega.
\end{cases}
$$

\tag{3}

Here, $\lambda, b, c, m$ and $k$ are positive constants, $\mu$ is a real constant and may be negative.
The spatially heterogeneous function $a(x)$ is a nonnegative continuous function on $\Omega$ and satisfies

$$a(x) \equiv 0, \quad x \in \bar{\Omega}_0, \quad a(x) > 0, \quad x \in \Omega \setminus \bar{\Omega}_0,$$

where $\Omega_0$ satisfying $\bar{\Omega}_0 \subset \Omega$ is a connected subdomain of $\Omega$ with smooth boundary $\partial \Omega_0$. It is clear that $\Omega_0$ is a favorable subregion for the prey, in which the growth of the prey is governed by the Malthusian law and is unbounded. Then a natural question is to investigate the effects of the spatial degeneracy on the stationary patterns. More importantly, can the spatial degeneracy together with the Beddington-DeAngelis functional response generate new spatial patterns? Between the Beddington-DeAngelis functional response and the spatial heterogeneity, which one of them plays a dominating role in the stationary problem?

To achieve the above goals, we mainly take advantage of the global bifurcation solution set of (3) bifurcating from its semitrivial solution set $\Gamma_u$ and $\Gamma_v$, which are given in Section 2. Precisely, the result shows that if $0 < \lambda < \min \{ b/k, \lambda^D_1(\Omega_0) \}$, then a bounded global continuum of positive solutions of (3) connects $\Gamma_u$ with $\Gamma_u$ at certain points; if $\min \{ b/k, \lambda^D_1(\Omega_0) \} < \lambda < \max \{ b/k, \lambda^D_1(\Omega_0) \}$, then an unbounded global continuum of positive solutions bifurcates from $\Gamma_u$ when $b/k < \lambda^D_1(\Omega_0)$ and from $\Gamma_v$ when $\lambda^D_1(\Omega_0) < b/k$; if $\lambda \geq \max \{ b/k, \lambda^D_1(\Omega_0) \}$, then no positive solutions bifurcate from either $\Gamma_u$ or $\Gamma_v$. In particular, if $\lambda \geq \lambda^D_1(\Omega_0) + b/k$, it is sure that (3) has no positive solutions. Thus, we see that two critical values for the prey growth rate $\lambda$ exist, that is, $\lambda^D_1(\Omega_0)$ and $b/k$, which are induced by the spatial degeneracy and the Beddington-DeAngelis functional response, respectively. As $\lambda$ crosses each critical value, the positive stationary solution set undergoes a drastic change.

First, we show the effects of the spatial degeneracy on the positive stationary solution set. Comparing to the result revealed in the homogeneous environment, one can see that if the size of $\Omega_0$ is small enough such that $\lambda^D_1(\Omega_0) > \max \{ b/k, \lambda \}$, then the spatial degeneracy seems to have little effect on the positive stationary solution set, the result obtained is similar to that in the homogeneous case. If the size of $\Omega_0$ is appropriate, then either similar or strikingly different result can be revealed. Precisely, when $\lambda^D_1(\Omega_0) \in (b/k, \lambda)$, then the spatial degeneracy causes the unbounded bifurcation continuum existing in the homogeneous case to vanish; while if $\lambda^D_1(\Omega_0) \in (\lambda, b/k)$, then the spatial degeneracy has little effect, where bounded bifurcation continuum can be founded in both cases. If the size of $\Omega_0$ is large enough such that $\lambda^D_1(\Omega_0) < \min \{ b/k, \lambda \}$, then essentially different structure of positive stationary solution set can be seen. Precisely, if $\lambda^D_1(\Omega_0) < b/k < \lambda$, then the spatial degeneracy causes the original unbounded bifurcation continuum to vanish; and if $\lambda^D_1(\Omega_0) < \lambda < b/k$, then the spatial degeneracy forces the bounded bifurcation continuum existing in the homogeneous case to become unbounded. Thus, one sees that there are two critical patch sizes for the degenerative region. Therefore, more complicated result about the spatial degeneracy is shown, and new phenomena occurs.

Second, comparing to either the Lotka-Volterra or Holling-II functional response [7, 20], the Beddington-DeAngelis functional response produces one more critical value $b/k$, which implies that the Beddington-DeAngelis functional response can behave rather different behavior. Here, we point out that homogeneous Dirichlet boundary condition is considered in [7] rather than Neumann boundary condition, but the result can be carried over to the Neumann boundary condition with some minor modifications. When $\lambda > \max \{ \lambda^D_1(\Omega_0), b/k \}$, then no positive stationary
solution bifurcates from either $\Gamma_0$ or $\Gamma_L$ for the Beddington-DeAngelis functional response; whereas, an unbounded bifurcation continuum exists for both the Lotka-Volterra and Holling-II functional response. In particular, if $\lambda \geq \lambda_D^D(\Omega_0) + b/k$, then no positive stationary solution exists for the Beddington-DeAngelis functional response, while at least one positive stationary solution exists for the Lotka-Volterra functional response if $\mu < \lambda/b$, and at least one positive stationary solution exists for the Holling-II functional response if $\mu > \lambda/b$. Thus, one sees that the Beddington-DeAngelis functional response makes the predation ability a little weaker than that of the Lotka-Volterra and Holling-II functional response. Then if there is a favorable subregion for the prey species at the same time, the prey species and predator species will not have coexistence states as the growth rate of the prey becomes large. Thus, the Beddington-DeAngelis functional response can also generate completely new stationary patterns.

Third, concerned about the role of the Beddington-DeAngelis functional response and the spatial degeneracy, one can know that if $k$ and $\Omega_0$ are small enough such that $0 < \lambda < \min \{b/k, \lambda_1^D(\Omega_0)\}$, then both of them seem to have little effect on the positive stationary solution. As $k$ becomes suitable large such that $b/k < \lambda_1^D(\Omega_0)$ and $\lambda \in (b/k, \lambda_1^D(\Omega_0))$, then the positive stationary solution set is essentially different from that of the Lotka-Volterra or Holling-II functional response, and the Beddington-DeAngelis functional response plays a dominating role; while if $\Omega_0$ is suitable large such that $\lambda_1^D(\Omega_0) < b/k$ and $\lambda \in (\lambda_1^D(\Omega_0), b/k)$, then the positive stationary solution set is rather different from that in the homogeneous environment, and the spatial degeneracy has a dominating role. If both $b/k$ and $\Omega_0$ are large such that $\lambda > \max \{b/k, \lambda_1^D(\Omega_0)\}$, then the Beddington-DeAngelis functional response and the spatial degeneracy play a dominating role at the same time. In this case, as a result of which, they cause no coexistence state of the prey and predator species. In this case, the positive stationary solutions may be determined by certain boundary blow-up systems, and needs a further and careful study.

In addition, it should be pointed out that one optimal condition for the coexistence region of (3) is given in certain circumstance. In fact, if $b/k < \lambda_1^D(\Omega_0)$ and $\lambda \in (b/k, \lambda_1^D(\Omega_0))$, then (3) has a positive solution if and only if $\mu > \mu_2$, which is defined by (11). Moreover, as $\mu$ becomes sufficiently large, the positive solution is unique and linearly stable. Besides the uniqueness of the positive solution, multiple existence of positive solutions of (3) can be shown under suitable conditions. Actually, for $\lambda_1^D(\Omega_0) < b/k$ and $\lambda \in (\lambda_1^D(\Omega_0), b/k)$, if the bifurcation direction at $(\mu_1, 0, \mu_1) \in \Gamma_v$ is subcritical, where $\mu_1$ is defined by (8), then there exists a small positive number $\varepsilon$ such that (3) has at least two positive solutions for $\mu \in (\mu_1 - \varepsilon, \mu_1)$. Moreover, one of the multiple existing positive solutions is exactly the one bifurcating from $(\mu_1, 0, \mu_1)$, which is linearly stable.

At the end of the introduction, we remark that besides the two ways introduced by Du et al. to examine the effects of spatial heterogeneity, Hutson et al. and Lou et al. have also done a series of works to investigate the effects of spatial heterogeneity for some specific diffusive competition models, where the spatial heterogeneity is created by the birth rates of the two competing species. One can refer to their work [24, 28, 33] and references therein. Additionally, for the cross-diffusive system in heterogeneous environment, one can see [30, 34, 42, 41] and references therein.
Finally, we give the notations used in this paper. Let $O$ be a bounded domain with smooth boundary. The usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\Omega)$ are defined by

$$
\|u\|_p = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{1/p} \quad \text{and} \quad \|u\|_{\infty, \Omega} = \max_{\Omega} |u(x)|.
$$

If $O$ is omitted in $\|u\|_{\infty, \Omega}$, then we understand that $O = \Omega$.

For a continuous function $\phi$, let $\lambda_1^D(\phi, O)$ and $\lambda_1^N(\phi, O)$ be the principal eigenvalues of $-\Delta + \phi$ subject to Dirichlet and Neumann boundary conditions over a domain $O$, respectively. If $O$ is omitted, then we understand that $O = \Omega$. If the potential function $\phi$ is omitted, then we understand that $\phi = 0$. It is well known that the following properties hold:

(i): $\lambda_1^D(\phi, O) > \lambda_1^N(\phi, O)$;
(ii): $\lambda_1^D(\phi_1, O) > \lambda_1^D(\phi_2, O)$ if $\phi_1 \geq \phi_2$ and $\phi_1 \neq \phi_2$, where $B = D$ or $B = N$;
(iii): $\lambda_1^D(\phi, O_1) \geq \lambda_1^D(\phi, O_2)$ if $O_1 \subset O_2$.

This paper is organized as follows. In Section 2, we show the global bifurcation of positive solutions of (3) bifurcating from the semitrivial solution set. In Section 3, we give the asymptotic behavior of the positive solutions as $\mu \to \infty$ when the bifurcation curve is unbounded with respect to $\mu$. Moreover, the positive solution set of the limiting equation is also shown. In Section 4, we give some stability and multiplicity results about the positive solution of (3). In particular, the existence of the positive solutions can be either unique or multiple.

2. Bifurcation of positive stationary solutions. In this section, we regard $\mu$ as the bifurcation parameter and investigate the positive solution set of (3) bifurcating from its semitrivial solution set. It is hoped that the result can reveal some interesting phenomena.

First, it is well known that (see [14, 35]), if $0 < \lambda < \lambda_1^D(\Omega_0)$, then the following problem

$$
-\Delta u = u(\lambda - a(x)u), \quad x \in \Omega, \quad \partial_{\nu}u|_{\partial\Omega} = 0 \quad (4)
$$

has a unique positive solution $u_\lambda(x)$; while if $\lambda \geq \lambda_1^D(\Omega_0)$, then (4) has no positive solutions. Thus, it is clear that as $\lambda \in (0, \lambda_1^D(\Omega_0))$, (3) has a unique semitrivial solution $(u_\lambda, 0)$ of the form $(u, 0)$ with $u > 0$; as $\lambda \geq \lambda_1^D(\Omega_0)$, (3) has no such semitrivial solutions.

On the other hand, (3) has a unique semitrivial solution $(0, \mu)$ of the form $(0, v)$ with $v > 0$ for any $\mu > 0$, and no such semitrivial solutions for any $\mu \leq 0$.

To show the positive solution set of (3) bifurcating from the semitrivial solution set, we need the following a priori estimates.

**Theorem 2.1.** Assume that $\lambda \neq \lambda_1^D(\Omega_0)$ is a fixed positive constant, $M$ is any given positive constant. If $|\mu| \leq M$, then there exists a positive constant $C$ independent of $\mu$ such that any positive solution $(u, v)$ of (3) satisfies

$$
\|u\|_{\infty} + \|v\|_{\infty} \leq C.
$$

**Proof.** Suppose that the assertion is false. Then we can find a positive constant $M$ and a sequence of $\{\mu_n\}$ with $|\mu_n| \leq M$ such that (3) with $\mu = \mu_n$ has a positive solution $(u_n, v_n)$, which satisfies

$$
\|u_n\|_{\infty} + \|v_n\|_{\infty} \to \infty \quad \text{as} \quad n \to \infty.
$$
Then it follows that
\[ -\Delta v_n = v_n \left( \mu_n - v_n + \frac{c u_n}{1 + m u_n + k v_n} \right) \leq \left( M + \frac{c}{m} - v_n \right) v_n, \quad x \in \Omega, \]
we obtain that
\[ \|v_n\|_{\infty} \leq M + \frac{c}{m}. \]
Then it must hold that \( \|u_n\|_{\infty} \to \infty \) as \( n \to \infty \).

Let \( \tilde{u}_n = \frac{u_n}{\|u_n\|}. \) Then \( \tilde{u}_n \) satisfies
\[ -\Delta \tilde{u}_n = \tilde{u}_n \left( \lambda - a(x)u_n - \frac{b v_n}{1 + m u_n + k v_n} \right) \leq \lambda \tilde{u}_n. \]
Then it follows that
\[ \int_{\Omega} |\nabla \tilde{u}_n|^2 + \int_{\Omega} \tilde{u}_n^2 \leq (\lambda + 1) \int_{\Omega} \tilde{u}_n^2 \leq (\lambda + 1) |\Omega|. \]
Thus, \( \{\tilde{u}_n\} \) is a bounded sequence in \( H^1(\Omega) \). Then subject to a subsequence, \( \tilde{u}_n \) converges to some function \( \tilde{u} \) weakly in \( H^1(\Omega) \) and strongly in \( L^p(\Omega) \) for any \( p > 1 \). Moreover, \( 0 \leq \tilde{u} \leq 1 \) and \( \tilde{u} \neq 0 \) in \( \Omega \).

Since \( -\Delta v_n \leq \left( M + \frac{c}{m} \right) v_n \) and \( \|v_n\|_{\infty} \leq M + \frac{c}{m} \), we can also know that subject to a subsequence, \( v_n \) converges to some \( v \) weakly in \( H^1(\Omega) \) and strongly in \( L^p(\Omega) \) for \( p > 1 \).

Note that
\[ -\Delta \tilde{u}_n = \tilde{u}_n \left( \lambda - a(x)\|u_n\|_{\infty} \tilde{u}_n - \frac{b v_n}{1 + m u_n + k v_n} \right). \tag{5} \]
Multiplying both sides of (5) by \( \tilde{u}_n \) and integrating over \( \Omega \), we obtain that
\[ \frac{1}{\|u_n\|_{\infty}} \int_{\Omega} |\nabla \tilde{u}_n|^2 = \frac{\lambda}{\|u_n\|_{\infty}} \int_{\Omega} \tilde{u}_n^2 - \int_{\Omega} a(x)\tilde{u}_n^3 - b \|u_n\|_{\infty} \int_{\Omega} \frac{\tilde{u}_n^2 v_n}{1 + m u_n + k v_n}. \]

Then as \( n \to \infty \), it holds that
\[ \int_{\Omega} a(x)\tilde{u}_n^3 = \int_{\Omega \setminus \Omega_0} a(x)\tilde{u}_n^3 = 0. \]
Since \( a(x) > 0 \) on \( \Omega \setminus \Omega_0 \), we obtain that \( \tilde{u} = 0 \) almost everywhere in \( \Omega \setminus \Omega_0 \). Due to the smoothness of \( \partial \Omega \), one knows that \( \tilde{u} \in H^1_0(\Omega_0) \).

Multiplying both sides of (5) by \( \varphi \in C_0^\infty(\Omega_0) \) and integrating over \( \Omega_0 \), it follows that
\[ \int_{\Omega_0} \nabla \tilde{u}_n \cdot \nabla \varphi = \lambda \int_{\Omega_0} \tilde{u}_n \varphi - b \int_{\Omega_0} \frac{\tilde{u}_n v_n}{1 + m u_n + k v_n} \varphi. \tag{6} \]
Because
\[ \left\| \frac{\tilde{u}_n v_n}{1 + m u_n + k v_n} \right\|_{\infty} = \left\| \frac{\tilde{u}_n v_n}{1 + m \|u_n\|_{\infty} u_n + k v_n} \right\|_{\infty} \leq \frac{\|v_n\|_{\infty}}{m \|u_n\|_{\infty}} \to 0, \]
we see that as \( n \to \infty \),
\[ \int_{\Omega_0} \nabla \tilde{u} \cdot \nabla \varphi = \lambda \int_{\Omega_0} \tilde{u} \varphi. \]
Therefore, \( \tilde{u} \geq 0 \) is a weak solution of
\[ -\Delta \tilde{u} = \lambda \tilde{u}, \quad x \in \Omega_0. \]
Then either \( \tilde{u} > 0 \) in \( \Omega_0 \) or \( \tilde{u} \equiv 0 \) in \( \Omega_0 \). We have known that \( \tilde{u} = 0 \) almost everywhere in \( \Omega \setminus \Omega_0 \) and \( \tilde{u} |_{\Omega_0} \in H^1_0(\Omega_0) \). Then the fact that \( \tilde{u} \neq 0 \) yields that \( \tilde{u} > 0 \).
in \( \Omega \). Then it follows that \( \lambda = \lambda^D_1(\Omega_0) \), which is a contradiction. Thus, the proof of the theorem is complete. 

To show the positive solution set of (3) bifurcating from the semitrivial solution set, it also requires to know some estimates of the bifurcation parameter \( \mu \). Precisely, we have the following two lemmas.

**Lemma 2.2.** Assume that \( 0 < \lambda < \min \{b/k, \lambda^D_1(\Omega_0)\} \). If (3) has a positive solution \((u, v)\), then there exists a positive number \( \hat{\mu} \) such that \( \mu \leq \hat{\mu} \).

**Proof.** Assume that the conclusion is false. Then without loss of generality, we may assume that there exists a sequence of \( \{\mu_n\} \) with \( \mu_n > 0 \) and \( \mu_n \to \infty \) as \( n \to \infty \) such that (3) at \( \mu = \mu_n \) has a positive solution \((u_n, v_n)\).

Since \( 0 < \lambda < \lambda^D_1(\Omega_0) \), it follows that \( 0 < u_n < u_\lambda(x), \ x \in \bar{\Omega} \).

Moreover, it is clear that \( v_n > \mu_n > 0, \ x \in \bar{\Omega} \).

From the equation of \( u_n \), one sees that

\[
\lambda = \lambda^N_1 \left( a(x)u_n + \frac{bv_n}{1 + mu_n + kv_n} \right) > \lambda^N_1 \left( \frac{bv_n}{1 + m\|u\|_{\infty} + kv_n} \right) > \lambda^N_1 \left( \frac{b\mu_n}{1 + m\|u\|_{\infty} + k\mu_n} \right) = \frac{b\mu_n}{1 + m\|u\|_{\infty} + k\mu_n}.
\]

However, as \( n \to \infty \), the right hand side of (7) converges to \( b/k \), which is a contradiction to the assumption \( \lambda < b/k \). Thus, the proof of the lemma is complete.

**Lemma 2.3.** A necessary condition for (3) to possess a positive solution \((u, v)\) is that \( \mu > -c/m \). In particular, if \( 0 < \lambda < \lambda^D_1(\Omega_0) \), then a more precise necessary condition holds, that is,

\[
\mu > \lambda^N_1 \left( -\frac{cu_\lambda}{1 + mu_\lambda} \right).
\]

The proof of Lemma 2.3 is quite standard, and we omit it.

Now we apply the bifurcation result of Crandall-Rabinowitz\[6\] to show the positive solution set of (3) bifurcating from the semitrivial solution set \( \Gamma_v = \{ (\mu, u, v) = (\mu, 0, \mu) : \mu > 0 \} \). Since the argument is quite standard, we only give the main steps.

Let \( X = \{ u \in W^{2,p}(\Omega) : \partial_\nu u = 0, \ x \in \partial \Omega \} \), \( Y = L^p(\Omega) \), \( p > 1 \). Define \( v = w + \mu \). Then for \( \mu > 0 \), we further define a mapping \( F : \mathbb{R} \times X \times X \to Y \times Y \) by

\[
F(\mu, u, w) = \begin{pmatrix}
\Delta u + u \left( \lambda - a(x)u - \frac{b(w+\mu)}{1 + mu + kw + (w+\mu)} \right) \\
\Delta w - \mu w - w^2 + \frac{b\mu}{1 + mu + kw + (w+\mu)}
\end{pmatrix}.
\]

Then simple computations deduce that

\[
F_{(u, w)}(\mu, 0, 0) = \begin{pmatrix}
\Delta \phi + \left( \lambda - \frac{b\mu}{1 + k\mu} \right) \phi \\
\Delta \psi - \mu \psi + \frac{c\mu}{1 + k\mu} \phi
\end{pmatrix}.
\]
It should be pointed out that if $0 < \lambda < b/k$, then there exists a unique positive number $\mu_1$ such that
\[
\lambda = \frac{b\mu_1}{1 + k\mu_1},
\] (8)
so $\mu_1 = \frac{\lambda}{b - \lambda k}$; while if $\lambda \geq b/k$, we cannot find such a positive number of $\mu$ satisfying (8). Then when $0 < \lambda < b/k$, it can be verified that
\[
\ker \left( F_{(u, w)}(\mu_1, 0, 0) \right) = \text{span} \{ (\phi_1, \psi_1) \} = \text{span} \left\{ \left( 1, \frac{c}{1 + k\mu_1} \right) \right\},
\] (9)
and
\[
\text{range} \left( F_{(u, w)}(\mu_1, 0, 0) \right) = \left\{ (f, g) \in Y \times Y : \int_{\Omega} f = 0 \right\}.
\]
Moreover, one can further verify that
\[
F_{(u, w)}(\mu_1, 0, 0)(\phi_1, \psi_1) = \left( -\frac{b}{(1 + k\mu_1)^2} \phi_1 - \psi_1 + \frac{c}{1 + k\mu_1} \phi_1 \right) \notin \text{range} \left( F_{(u, w)}(\mu_1, 0, 0) \right).
\]
Then the bifurcation result of Crandall-Rabinowitz [6] concludes that the positive solution set of (3) near $(\mu_1, 0, 0)$ is a smooth curve
\[
\{ (\mu(s), u(s), v(s)) = (\mu_1 + s\tilde{\mu}_1 + o(s), s\phi_1 + o(s), \mu(s) + s\psi_1 + o(s)) : 0 < s < \delta \},
\]
where $\delta > 0$ is a small number. By virtue of the direction formula of the bifurcation by Shi [38, Theorem 2.1 and (4.5)], $\tilde{\mu}_1$ is expressed by
\[
\tilde{\mu}_1 = -\int_{\Omega} a(x) \phi_1^2 + \frac{b m\mu_1}{(1 + k\mu_1)^2} \int_{\Omega} \phi_1^2 - \frac{b}{(1 + k\mu_1)^2} \int_{\Omega} \phi_1 \psi_1
\]
\[
= -\frac{(1 + k\mu_1)^2}{b|\Omega|} \int_{\Omega} a(x) + m\mu_1 - \frac{c}{1 + k\mu_1}
\]
\[
= -\frac{b}{(b - \lambda k)^2|\Omega|} \int_{\Omega} a(x) + \frac{m\lambda}{b - \lambda k} - \frac{c b - \lambda k}{b}.
\] (10)
Thus, we obtain the following lemma.

**Lemma 2.4.** If $0 < \lambda < b/k$, then positive solutions of (3) bifurcate from $\Gamma_v$ if and only if $\mu = \mu_1$, which is defined by (8). If $\lambda \geq b/k$, then no positive solutions of (3) bifurcate from $\Gamma_v$.

Similarly, we can show the following lemma.

**Lemma 2.5.** If $0 < \lambda < \lambda_1^D(\Omega_0)$, then positive solutions of (3) bifurcate from $\Gamma_u = \{ (\mu, u, v) = (\mu, u_\lambda, 0) : \mu \in \mathbb{R} \}$ if and only if
\[
\mu = \mu_2 = \lambda_1^N \left( -\frac{c u_\lambda}{1 + m u_\lambda} \right).
\] (11)
Furthermore, all positive solutions of (3) near $(\mu_2, u_\lambda, 0)$ is a smooth curve
\[
\{ (\mu(s), u(s), v(s)) = (\mu_2 + s\tilde{\mu}_2 + o(s), u_\lambda - s\phi_2 + o(s), s\psi_2 + o(s)) : 0 < s < \delta \},
\]
where $\delta > 0$ is a small number, $\psi_2$ is the corresponding positive eigenfunction of $\mu_2$, $\phi_2 = (-\Delta + 2a(x)u_\lambda - \lambda)^{-1} \frac{6u_\lambda}{1 + m u_\lambda} \psi_2 > 0$, and
\[
\tilde{\mu}_2 = \frac{\int_{\Omega} \psi_2^3 + kc \int_{\Omega} \frac{u_\lambda}{(1 + m u_\lambda)^3} \psi_2^3 + \int_{\Omega} \frac{c}{(1 + m u_\lambda)^2} \phi_2 \psi_2^2}{\int_{\Omega} \psi_2^3} > 0.
\] (12)
Remark 1. By virtue of Lemma 2.4, one sees that corresponding to either the Lotka-Volterra or Holling-II functional response, there always exists a smooth curve of positive stationary solutions bifurcating from $\Gamma_v$ at $\mu = \mu_1 = \lambda/b$ for all $\lambda > 0$, which is in strong contrast to the case of the Beddington-DeAngelis functional response. Moreover, detailed computations yield that the bifurcation direction keeps negative for the Lotka-Volterra functional response. Then once $\mu$ exists and becomes large, the direction varies from negative to positive, where the relation between $\mu$ and the bifurcation direction is monotone. However, the relation between $k$ and the bifurcation direction is rather complicated. In particular, if $\mu$ is sufficiently large, then it is certain that the bifurcation direction varies from positive to negative as $k$ crosses a critical value $k_0$. Since the bifurcation direction may affect the positive stationary solution set, it is quite meaningful to investigate the Beddington-DeAngelis functional response.

Due to the estimates given by Theorem 2.1, Lemmas 2.2, 2.3 and the local bifurcation results, we can further obtain the global continuum of positive solutions of (3) bifurcating from $\Gamma_u$ or $\Gamma_v$ through a standard global bifurcation analysis [32, 37].

Theorem 2.6. Assume that $b/k < \lambda_P^2(\Omega_0)$. If $0 < \lambda < b/k$, then a bounded global continuum $\Gamma_1$ of positive solutions of (3) bifurcates from $\Gamma_v$ at $\mu = \mu_1$ and meets $\Gamma_u$ at $\mu = \mu_2$; if $b/k \leq \lambda < \lambda_P^1(\Omega_0)$, then an unbounded global continuum $\Gamma_2$ of positive solutions of (3) bifurcates from $\Gamma_u$ at $\mu = \mu_2$ and $\text{Proj}_{\mu_2} \Gamma_2 = (\mu_2, \infty)$; if $\lambda \geq \lambda_P^1(\Omega_0)$, then no positive solutions of (3) bifurcate from either $\Gamma_u$ or $\Gamma_v$.

The bifurcation diagrams of Theorem 2.6 are shown in Figures 1 and 2.

From Lemmas 2.2, 2.3 and Theorem 2.6, we see that if $b/k < \lambda_P^1(\Omega_0)$ and $\lambda \in (0, b/k)$, then (3) has a positive solution as $\mu_2 < \mu < \mu_1$ and no positive solutions as $\mu \leq \mu_2$ or $\mu > \mu_1$; However, if $\lambda \in (b/k, \lambda_P^1(\Omega_0))$, then the optimal condition for the coexistence region of (3) can be shown. That is, if $\lambda \in (b/k, \lambda_P^1(\Omega_0))$, then (3) has a positive solution if and only if $\mu > \mu_2$.

Theorem 2.7. Assume that $\lambda_P^2(\Omega_0) < b/k$. If $0 < \lambda < \lambda_P^2(\Omega_0)$, then a bounded global continuum $\Gamma_3$ of positive solutions of (3) bifurcates from $\Gamma_v$ at $\mu = \mu_1$ and meets $\Gamma_u$ at $\mu = \mu_2$; if $\lambda_P^2(\Omega_0) < \lambda < b/k$, then an unbounded global continuum $\Gamma_4$ of positive solutions of (3) bifurcates from $\Gamma_v$ at $\mu = \mu_1$ and $\text{Proj}_{\mu_1} \Gamma_4 = (\mu_1, \infty)$; if $\lambda \geq b/k$, then no positive solutions of (3) bifurcate from either $\Gamma_u$ or $\Gamma_v$.

As $\lambda_P^2(\Omega_0) < b/k$ and $\lambda \in (0, \lambda_P^2(\Omega_0))$, the bifurcation diagram is the same to that shown in Figure 1. The bifurcation diagram as $\lambda \in (\lambda_P^1(\Omega_0), b/k)$ is shown in Figure 3.

Finally, we point out that at the presence of the Beddington-DeAngelis functional response and the spatial degeneracy, if the prey growth rate $\lambda$ becomes suitably large, then not only can we know that no positive stationary solutions bifurcate from the semitrivial solution set, but also know that no positive stationary solutions exist.

Theorem 2.8. If $\lambda \geq \lambda_P^1(\Omega_0) + b/k$, then (3) has no positive solutions.

Proof. If the conclusion fails. Then there exists a certain number $\lambda_0 \geq \lambda_P^1(\Omega_0) + b/k$ such that (3) at $\lambda = \lambda_0$ has a positive solution $(u, v)$. Then from the first equation
of (3), it follows that

$$
\lambda_0 = \lambda_1^N \left( a(x)u + \frac{bv}{1 + mu + kv} \right).
$$

The properties of the principal eigenvalue deduces that

$$
\lambda_0 < \lambda_1^D \left( a(x)u + \frac{bv}{1 + mu + kv} \right) < \lambda_1^D \left( a(x)u + \frac{bv}{1 + mu + kv}, \Omega_0 \right).
$$
\[ \lambda^D_1 \left( \frac{bv}{1 + mu + kv}, \Omega_0 \right) < \lambda^D_1(\Omega_0) + b/k, \]

which is a contradiction. Thus, the proof of the theorem is complete. \( \square \)

**Remark 2.** To conclude this section, we examine the effects of either the Beddington-DeAngelis functional response or the spatial heterogeneity on the positive stationary solution set.

(i) First, we show that surrounding by the same heterogeneous environment, whether the positive stationary solution set of the predator-prey model with Beddington-DeAngelis functional response is different from that of the predator-prey model with Lotka-Volterra or Holling-II functional response. It should be pointed out that applying similar technique to that used by Dancer and Du\([7]\), one can deduce that under Neumann boundary condition, if \(0 < \lambda < \lambda^D_1(\Omega_0)\), then the predator-prey model with Lotka-Volterra functional response possesses a bounded global continuum of positive stationary solutions connecting \(\Gamma_u\) and \(\Gamma_v\); if \(\lambda > \lambda^D_1(\Omega_0)\), then it has an unbounded global continuum \(\Gamma\) of positive stationary solutions bifurcating from \(\Gamma_u\) with \(\text{Proj} \Gamma = (-\infty, \lambda/b)\); due to \([20]\), one can know that corresponding to the Holling-II functional response, if \(0 < \lambda < \lambda^2_1(\Omega_0)\), then a bounded global continuum of positive stationary solution connecting \(\Gamma_u\) and \(\Gamma_v\) also exists; if \(\lambda > \lambda^2_1(\Omega_0)\), then an unbounded global continuum \(\Gamma\) of positive stationary solutions bifurcates from \(\Gamma_v\), but \(\text{Proj} \Gamma = (\lambda/b, \infty)\). Thus, we see that the Beddington-DeAngelis functional response yields another critical value \(\lambda = b/k\) except \(\lambda = \lambda^D_1(\Omega_0)\), and generates new stationary patterns. If the mutual interference between predators \(k\) is large such that \(b/k < \lambda^D_1(\Omega_0)\) and \(\lambda \in (b/k, \lambda^D_1(\Omega_0))\), then the bounded bifurcation continuum connecting \(\Gamma_u\) and \(\Gamma_v\) becomes unbounded as the functional response changes to the Beddington-DeAngelis functional response; if \(\lambda \geq \lambda^D_1(\Omega_0)\), then the unbounded bifurcation continuum bifurcating from \(\Gamma_v\) disappears. In particular, when \(\lambda \geq \lambda^D_1(\Omega_0) + b/k\), then the predator-prey model with Lotka-Volterra functional response has at least one positive stationary solution for any \(\mu < \lambda/b\), and the predator-prey model with Holling-II functional response has at least one positive stationary solution for any \(\mu > \lambda/b\). This is in strong contrast to the nonexistence of positive stationary solutions for any \(\mu\) corresponding to the Beddington-DeAngelis functional response.

(ii) Next, we show the effects of the spatial heterogeneity on the positive stationary solution set. In the homogeneous environment, that is, \(\Omega_0 = \emptyset\), one can know that if \(0 < \lambda < b/k\), then there exists a bounded global continuum of positive stationary solutions connecting \(\Gamma_u\) and \(\Gamma_v\); if \(\lambda \geq b/k\), then the continuum of positive stationary solutions bifurcating from \(\Gamma_u\) no longer joins with \(\Gamma_v\) and becomes unbounded.

Comparing to Theorems 2.6 and 2.7, if the size of \(\Omega_0\) is small such that \(\lambda^D_1(\Omega_0) > \max\{b/k, \lambda\}\), then the spatial heterogeneity has little effect on the positive stationary solution set. Precisely, if \(0 < \lambda < b/k < \lambda^D_1(\Omega_0)\), then the bifurcation continuum is bounded in either homogeneous or heterogeneous environment; if \(b/k < \lambda < \lambda^D_1(\Omega_0)\), then the bifurcation continuum is unbounded in both two cases. If the size of \(\Omega_0\) is suitably large such that \(\lambda^D_1(\Omega_0) \in (b/k, \lambda)\) or \(\lambda^D_1(\Omega_0) \in (\lambda, b/k)\), then the result is subtle. Precisely, the spatial heterogeneity causes the unbounded bifurcation continuum to vanish when \(\lambda^D_1(\Omega_0) \in (b/k, \lambda)\), whereas it seems to have little effect on the positive stationary solutions when \(\lambda^D_1(\Omega_0) \in (\lambda, b/k)\). If the size
of $\Omega_0$ is large such that $\lambda^D(\Omega_0) < \min\{b/k, \lambda\}$, then it can be seen that the spatial heterogeneity has profound effect on the positive stationary solution. To be precise, if $\lambda^D(\Omega_0) < b/k < \lambda$, then the spatial heterogeneity forces the unbounded bifurcation continuum to vanish; if $\lambda^D(\Omega_0) < \lambda < b/k$, then it forces the bounded bifurcation continuum to be unbounded. Moreover, if $\lambda \geq \lambda^D(\Omega_0) + b/k$, then the degeneracy of $a(x)$ further causes the nonexistence of any positive stationary solution.

Thus, we see that the spatial heterogeneity caused by the degeneracy of $a(x)$ can yield quite complicated stationary patterns.

3. Asymptotic behavior of positive stationary solutions. Due to Theorems 2.6 and 2.7, we know that when $b/k < \lambda^D(\Omega_0)$ and $b/k < \lambda < \lambda^D(\Omega_0)$, the positive solution set of (3) bifurcating from $\Gamma_u$ is unbounded with respect to $\mu$; when $\lambda^D(\Omega_0) < b/k$ and $\lambda^D(\Omega_0) < \lambda < b/k$, then the positive solution set of (3) bifurcating from $\Gamma_v$ is also unbounded with respect to $\mu$. Thus, it is natural and interesting to study the asymptotic behavior of positive solutions of (3) as $\mu \to \infty$.

**Theorem 3.1.** Assume that $b/k < \lambda^D(\Omega_0)$ and $\lambda \in (b/k, \lambda^D(\Omega_0))$. Let $\{\mu_n\}$ be an increasing sequence of positive numbers such that $\mu_n \to \infty$ and $(u_n, v_n)$ be an arbitrary positive solution of (3) with $\mu = \mu_n$. Then subject to a subsequence, it holds that

$$\lim_{n \to \infty} u_n = u_{\lambda - \epsilon}(x), \quad \lim_{n \to \infty} v_n = 1.$$  

Moreover, the convergence is uniform in $\bar{\Omega}$.

**Proof.** Since $\lambda \in (b/k, \lambda^D(\Omega_0))$, we know that

$$0 < u_n < u_{\lambda}(x), \quad x \in \bar{\Omega}.$$  

Moreover, since

$$\frac{bv_n}{1 + mu_n + kv_n} \leq \frac{b}{k},$$  

applying the standard elliptic regularity result [25], it follows that subject to a subsequence if necessary, $u_n \to \hat{u} \geq 0$ in $C(\bar{\Omega})$.

We claim that $\hat{u} \neq 0$ for $x \in \bar{\Omega}$. Otherwise, setting $\bar{u}_n = \frac{u_n}{\|u_n\|_{\infty}}$. Then it is clear that $\bar{u}_n$ satisfies

$$-\Delta \bar{u}_n = \bar{u}_n \left( \lambda - a(x)u_n - \frac{bv_n}{1 + mu_n + kv_n} \right), \quad x \in \Omega, \quad \partial_\nu \bar{u}_n|_{\partial \Omega} = 0.$$  

Applying the standard elliptic regularity result, we also know that subject to a subsequence if necessary, $\bar{u}_n \to \bar{u}$ in $C(\bar{\Omega})$ with $\bar{u} \neq 0$. Since

$$\mu_n < v_n < \mu_n + \frac{c}{m}, \quad x \in \bar{\Omega},$$  

we have that $v_n \to \infty$ uniformly in $\Omega$ as $n \to \infty$. Then

$$\frac{bv_n}{1 + mu_n + kv_n} = \frac{b}{v_n + \frac{m}{m_n} + k} \to \frac{b}{k} \text{ uniformly in } \Omega.$$  

Then it follows that $\bar{u}$ satisfies

$$-\Delta \bar{u} = \bar{a}(\lambda - b/k), \quad x \in \Omega, \quad \partial_\nu \bar{u} = 0, \quad x \in \partial \Omega.$$  

Since $\bar{u} \neq 0$, we have that $\bar{u} > 0$, $x \in \bar{\Omega}$. Then $\lambda = b/k$, which is a contradiction to the assumption. Thus, $\hat{u} \neq 0$ for $x \in \bar{\Omega}$. 


It can be verified that $\hat{u}$ satisfies
\begin{equation}
-\Delta \hat{u} = \hat{u}(\lambda - b/k - a(x)\hat{u}), \quad x \in \Omega, \quad \partial_n \hat{u} = 0, \quad x \in \partial \Omega.
\end{equation}
Since $\hat{u} \neq 0$, it holds that $\hat{u} > 0$ for $x \in \bar{\Omega}$, which implies that $\hat{u}$ is a positive solution of (13). While the assumption asserts that
\begin{equation*}
0 < \lambda - b/k < \lambda^D(\Omega_0),
\end{equation*}
which deduces that (13) has a unique positive solution $u_{\lambda - \frac{b}{k}}(x)$. Thus, it follows that $\hat{u} = u_{\lambda - \frac{b}{k}}(x)$.

Set $w_n = \frac{\hat{u}}{\mu_n}$. Then $w_n$ satisfies
\begin{equation*}
-\Delta w_n = \mu_n w_n \left(1 - w_n + \frac{c u_n}{\mu_n + m u_n + k v_n}\right).
\end{equation*}
Then it is clear that
\begin{equation*}
-\Delta w_n \geq \mu_n w_n (1 - w_n), \quad x \in \Omega.
\end{equation*}
On the other hand, since $\frac{c u_n}{\mu_n + m u_n + k v_n}$ is uniformly bounded, for any small $\varepsilon > 0$, when $n$ is sufficiently large, it holds that
\[\frac{1}{\mu_n + m u_n + k v_n} < \varepsilon.\]
Thus, for sufficiently large $n$, it holds that
\[\Delta w_n \leq \mu_n w_n (1 + \varepsilon - w_n), \quad x \in \Omega.
\]
Due to the proof of Lemma 2.2 in [18], we can know that
\begin{equation}
-\Delta z = \mu_n z(1 - z), \quad x \in \Omega, \quad \partial_n z = 0, \quad x \in \partial \Omega,
\end{equation}
has a unique positive solution $w_n$, and
\begin{equation}
-\Delta z = \mu_n z(1 + \varepsilon - z), \quad x \in \Omega, \quad \partial_n z = 0, \quad x \in \partial \Omega,
\end{equation}
has a unique positive solution $\overline{w_n}$. Then for sufficiently large $n$, $w_n$ is a supersolution of (14) and a subsolution of (15). Then a super-sub solution argument and the uniqueness of the positive solution of (14) and (15) deduce that for sufficiently large $n$,
\[w_n \leq \overline{w_n}.
\]
Moreover, as $n \to \infty$, it holds that $w_n \to 1$ and $\overline{w_n} \to 1 + \varepsilon$ uniformly in $\bar{\Omega}$. While $\varepsilon > 0$ can be arbitrarily small, we obtain that as $n \to \infty$, $w_n \to 1$ uniformly in $\bar{\Omega}$. Thus, the proof of the theorem is complete.

**Theorem 3.2.** Assume that $\lambda^D(\Omega_0) < b/k$ and $\lambda \in (\lambda^D(\Omega_0), b/k)$. Let $\{\mu_n\}$ be an increasing sequence of positive numbers such that $\mu_n \to \infty$ and $(u_n, v_n)$ be an arbitrary positive solution of (3) with $\mu = \mu_n$. Then subject to a subsequence, it holds that
\[\lim_{n \to \infty} \frac{\|u_n\|_{\mu_n}}{\mu_n} = \sigma \in \left(\frac{b - k \lambda}{m \lambda}, \infty\right), \quad \lim_{n \to \infty} \frac{v_n}{\mu_n} = 1.
\]
Moreover, $\frac{u_n}{\sigma \mu_n} \to \hat{u}$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for $p > 1$. Here, $\hat{u}$ with $\|\hat{u}\|_{\infty} = 1$ is a nonnegative function satisfying $\hat{u} = 0$ for $x \in \Omega \setminus \Omega_0$. Moreover, $\hat{u}|_{\Omega_0} \in H^1_0(\Omega_0)$ is a positive weak solution of
\begin{equation}
-\Delta u = u \left(\lambda - \frac{b}{k + m \sigma u}\right), \quad x \in \Omega_0, \quad u = 0, \quad x \in \partial \Omega_0.
\end{equation}
Proof. The argument for $v_n$ in Theorem 3.1 still holds true, so it holds that $\frac{v_n}{\mu_n} \to 1$ uniformly in $\Omega$ as $n \to \infty$.

Next, we claim that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Otherwise, we can find a positive constant $M$ independent of $n$ such that $\|u_n\|_\infty \leq M$. Then similar to the proof of Theorem 3.1, we can obtain that subject to a subsequence if necessary, $u_n \to \hat{u}$ in $C(\bar{\Omega})$ with $\hat{u} \neq 0$. Moreover, $\hat{u}$ satisfies (13). Since $\hat{u} \neq 0$, the Harnack inequality[31] shows that $\hat{u} > 0$ for $x \in \bar{\Omega}$. Then it holds that

$$0 < \lambda - b/k < \lambda^P(\Omega_0),$$

which is a contradiction to the assumption. Thus, it must hold that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

Setting $\tilde{u}_n = \frac{u_n}{\|u_n\|_\infty}$. Since $-\Delta \tilde{u}_n \leq \lambda \tilde{u}_n$, similar argument to that in Theorem 2.1 yields that $\tilde{u}_n$ converges to some $\tilde{u} \neq 0$ weakly in $H^1(\Omega)$ and strongly in $L^p(\Omega)$ for $p > 1$. Moreover, $\|\tilde{u}\|_\infty = 1$, $\tilde{u} = 0$ almost everywhere in $\Omega \setminus \Omega_0$. Due to the smoothness of $\partial \Omega_0$, $\tilde{u}|_{\partial \Omega_0} \in H^1_0(\Omega)$. 

In the following, we first show that by taking a subsequence,

$$\lim_{n \to \infty} \frac{\|u_n\|_\infty}{\mu_n} = \sigma \in \left(\frac{b - k\lambda}{m\lambda}, \infty\right).$$

If $\lim_{n \to \infty} \frac{\|u_n\|_\infty}{\mu_n} = \infty$, then there exists a subsequence, which is still denoted by $\left\{ \frac{\|u_n\|_\infty}{\mu_n} \right\}$, such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Multiplying the equation of $\tilde{u}_n$ by $\varphi \in C^0_{\bar{\Omega}}(\Omega_0)$ and integrating over $\Omega$, we obtain that (6) holds true. Note that

$$\frac{\tilde{u}_n v_n}{1 + m u_n + k v_n} = \frac{\frac{\tilde{u}_n}{\mu_n} \frac{v_n}{\mu_n}}{1 + m \frac{\|u_n\|_\infty}{\mu_n} \tilde{u}_n + k \frac{v_n}{\mu_n}} \leq \frac{v_n/\mu_n}{m \|u_n\|_\infty/\mu_n} \to 0$$

uniformly in $\bar{\Omega}$, we obtain that $\tilde{u}$ is a weak solution of

$$-\Delta \tilde{u} = \lambda \tilde{u}, \quad x \in \Omega_0.$$

Then we can know that $\tilde{u} > 0$ for $x \in \Omega_0$. Thus, we have that $\lambda = \lambda^P_1(\Omega_0)$, which is a contradiction to the assumption. Thus, $\lim_{n \to \infty} \frac{\|u_n\|_\infty}{\mu_n} < \infty$.

On the other hand, from the equation of $u_n$, we can obtain that

$$\lambda = \lambda^N \left( a(x) u_n + \frac{b v_n}{1 + m u_n + k v_n} \right) > \lambda^N \left( \frac{b \mu_n}{1 + m \|u_n\|_\infty + k \mu_n} \right) = \frac{b \mu_n}{1 + m \|u_n\|_\infty + k \mu_n}.$$

Since $0 < \lambda < b/k$, it follows that

$$\frac{\|u_n\|_\infty}{\mu_n} > \frac{b - k\lambda}{m\lambda} - \frac{1}{m\mu_n}.$$

Thus,

$$\liminf_{n \to \infty} \frac{\|u_n\|_\infty}{\mu_n} \geq \frac{b - k\lambda}{m\lambda} = \sigma_0 > 0.$$

Then it suffices to show that $\liminf_{n \to \infty} \frac{\|u_n\|_\infty}{\mu_n} > \sigma_0$. Otherwise, there exits a subsequence, which is still denoted by $\left\{ \frac{\|u_n\|_\infty}{\mu_n} \right\}$, such that $\frac{\|u_n\|_\infty}{\mu_n} \to \sigma_0$ as $n \to \infty$. 


Multiplying the equation of $\tilde{u}_n$ by $\tilde{u}_n$ and integrating over $\Omega$, we obtain that
\[
\int_\Omega |\nabla \tilde{u}_n|^2 = \lambda \int_\Omega \tilde{u}_n^2 - \int_\Omega a(x)u_n\tilde{u}_n^2 - b \int_\Omega \frac{\tilde{u}_n^2 v_n}{1 + mu_n + kv_n}.
\]
Then
\[
\int_{\Omega_0} |\nabla \tilde{u}_n|^2 \leq \lambda \int_{\Omega_0} \tilde{u}_n^2 - b \int_{\Omega_0} \frac{\tilde{u}_n^2 v_n}{1 + mu_n + kv_n}.
\]
Since
\[
\frac{v_n}{1 + mu_n + kv_n} = \frac{v_n/\mu_n}{\frac{1}{\mu_n} + m\|u_n\|_\infty \tilde{u}_n + k\frac{v_n}{\mu_n}} \to \frac{1}{k + m\sigma_0 \tilde{u}_n}
\]
in $L^p(\Omega)$, it follows that
\[
\int_{\Omega_0} |\nabla \tilde{u}|^2 \leq \int_{\Omega_0} \tilde{u}^2 \left( \lambda - \frac{b}{1 + mu_n + kv_n} \right) \leq \int_{\Omega_0} \tilde{u}^2 \left( \lambda - \frac{b}{k + m\sigma_0 \tilde{u}_n} \right) = 0.
\]
Thus, we obtain that $\tilde{u}_n \equiv 0$ in $\Omega$, which is impossible. Thus, subject to a subsequence, $\|u_n\|_\infty \to \sigma \in (\sigma_0, \infty)$.

Since $u_n$ satisfies
\[
-\Delta \tilde{u}_n = \tilde{u}_n \left( \lambda - \frac{bv_n}{1 + mu_n + kv_n} \right), \quad x \in \Omega_0,
\]
moreover,
\[
\lambda - \frac{bv_n}{1 + mu_n + kv_n} \to \lambda - \frac{b}{m\sigma_0 + k} \quad \text{in} \quad L^p(\Omega),
\]
we see that $\tilde{u}_n$ is a weak solution of (16). Furthermore, we can see that $\tilde{u}_n \in C(D)$ for any compact subset of $\Omega_0$. Since $\tilde{u}_n \neq 0$, we also see that $\tilde{u}_n > 0$ for $x \in \Omega_0$. The proof of the theorem is complete.

At the end of this section, we show some results about the positive solution set of (16). First, it is clear that a necessary condition for (16) to possess a positive solution is that
\[
\lambda_1^p(\Omega_0) < \lambda < \lambda_1^p(\Omega_0) + b/k.
\]
(17)

To show the sufficient condition, we need the following two lemmas.

**Lemma 3.3.** Assume that $M$ is any fixed positive constant and $\varepsilon > 0$ is a small number. If $\lambda_1^p(\Omega_0) + \varepsilon < \lambda \leq M$, then there exists a positive constant $C$ independent of $\lambda$ such that any positive solution $u$ of (16) satisfies $\|u\|_\infty, \Omega_0 \leq C$.

**Proof.** If the conclusion fails, then there exist a positive number $M$, a small positive number $\varepsilon$ and a sequence of $\{\lambda_n\}$ such that $\lambda_n \to \lambda_0 \in [\lambda_1^p(\Omega_0) + \varepsilon, M]$ such that (16) at $\lambda = \lambda_n$ has a positive solution $u = u_n$ satisfying $\|u_n\|_\infty, \Omega_0 \to \infty$.

Let $\tilde{u}_n = \frac{u_n}{\|u_n\|_\infty, \Omega_0}$. Then $\tilde{u}_n$ satisfies
\[
-\Delta \tilde{u}_n = \tilde{u}_n \left( \lambda_n - \frac{b}{k + m\sigma u_n} \right), \quad x \in \Omega_0, \quad \tilde{u}_n|_{\partial \Omega_0} = 0.
\]
Since
\[
\left| \lambda_n - \frac{b}{k + m\sigma u_n} \right| \leq M + b/k,
\]
and
\[
\frac{\tilde{u}_n}{k + ma u_n} = \frac{\tilde{u}_n}{k + ma \|u_n\|_{\infty, \Omega_0}} \tilde{u}_n \leq \frac{1}{ma \|u_n\|_{\infty, \Omega_0}} \to 0,
\]
we obtain that subject to a subsequence if necessary, \(\tilde{u}_n \to \tilde{u}\) uniformly in any compact subset of \(\Omega_0\). Moreover, \(\tilde{u}\) satisfies
\[
- \Delta \tilde{u} = \lambda_0 \tilde{u}, \quad x \in \Omega_0, \quad \tilde{u}|_{\partial \Omega_0} = 0.
\]
Since \(\tilde{u} \neq 0\), it follows that \(\tilde{u} > 0\) for \(x \in \Omega_0\). Thus, \(\lambda_0 = \lambda_1^D(\Omega_0)\), a contradiction. Thus, the proof of the lemma is complete.

**Lemma 3.4.** Let \(\{\lambda_n\}\) be a sequence of positive numbers with \(\lambda_n \to \lambda_1^D(\Omega_0)\). Then any positive solution \(u = u_n\) of (16) at \(\lambda = \lambda_n\) satisfies
\[
\|u_n\|_{\infty, \Omega_0} \to \infty \quad \text{and} \quad \frac{u_n}{\|u_n\|_{\infty, \Omega_0}} \to \Phi,
\]
where \(\Phi\) normalized by \(\|\Phi\|_{\infty, \Omega_0} = 1\) is the positive eigenfunction of
\[
- \Delta \Phi = \lambda_1^D(\Omega_0)\Phi, \quad x \in \Omega_0, \quad \Phi|_{\partial \Omega_0} = 0.
\]
Moreover, the convergence is uniform in any compact subset of \(\Omega_0\).

**Proof.** First, we prove that \(\|u_n\|_{\infty, \Omega_0} \to \infty\) as \(n \to \infty\). If not, we can find a positive constant \(M\) independent of \(n\) such that \(\|u_n\|_{\infty, \Omega_0} \leq M\). Then subject to a subsequence, \(u_n \to \tilde{u}\) uniformly in any compact subset of \(\Omega_0\). Moreover, \(\tilde{u} \neq 0\). Otherwise, by setting \(\tilde{u}_n = \frac{u_n}{\|u_n\|_{\infty, \Omega_0}}\), we can obtain that \(\tilde{u}_n \to \tilde{u}\) uniformly in any compact subset of \(\Omega_0\), where \(\tilde{u} \neq 0\) satisfies
\[
- \Delta \tilde{u} = (\lambda_1^D(\Omega_0) - b/k)\tilde{u}, \quad x \in \Omega_0, \quad \tilde{u}|_{\partial \Omega_0} = 0.
\]
Then the Harnack inequality shows that \(\tilde{u} > 0\). Then it follows that
\[
\lambda_1^D(\Omega_0) - b/k = \lambda_1^D(\Omega_0),
\]
which is impossible. Thus, we obtain that \(\tilde{u} \neq 0\), which satisfies
\[
- \Delta \tilde{u} = \tilde{u} \left(\lambda_1^D(\Omega_0) - \frac{b}{k + ma \tilde{u}}\right), \quad x \in \Omega_0, \quad \tilde{u}|_{\partial \Omega_0} = 0. \tag{19}
\]
Then the Harnack inequality also deduces that \(\tilde{u} > 0\). However, by the necessary condition, we know that (19) has no positive solution. Thus, we obtain a contradiction. Thus, it follows that (19) has no positive solution. Thus, we obtain a contradiction. Then we obtain that \(\tilde{u}_n \to \tilde{u}\) uniformly in any compact subset of \(\Omega_0\). Moreover, \(\tilde{u}\) satisfies (18) with \(\lambda_0 = \lambda_1^D(\Omega_0)\). Thus, the proof of the lemma is complete.

Applying Lemmas 3.3 and 3.4, a standard bifurcation analysis asserts that (16) has a positive solution for \(\lambda_1^D(\Omega_0) < \lambda < \lambda_1^D(\Omega_0) + b/k\). Thus, we obtain the following theorem.

**Theorem 3.5.** (16) has a positive solution if and only if (17) holds.

By the property of the eigenvalue, we can also know that any positive solution of (16) is unstable.
4. Stability and multiplicity of positive stationary solutions. In this section, we show some results about the stability and multiplicity of the positive solutions of (3).

The first result deals with the stability and uniqueness of the positive solution as \( \mu \) is sufficiently large.

**Theorem 4.1.** Assume that \( b/k < \lambda_1^D(\Omega_0) \) and \( \lambda \in (b/k, \lambda_1^D(\Omega_0)) \). Then there exists a positive constant \( \mu^* \) such that (3) has a unique positive solution for \( \mu > \mu^* \). Moreover, the unique positive solution is linearly stable.

The proof of Theorem 4.1 is given by the following Propositions 1 and 2.

**Proposition 1.** Assume that \( b/k < \lambda_1^D(\Omega_0) \) and \( \lambda \in (b/k, \lambda_1^D(\Omega_0)) \). Then as \( \mu \) is sufficiently large, any positive solution \((u_n, v_n)\) of (3) is linearly stable.

**Proof.** If the conclusion is false, then we can find a sequence of \( \{\mu_n\} \) with \( \mu_n \to \infty \) and a corresponding positive solution \((u_n, v_n)\) of (3) at \( \mu = \mu_n \), such that the following linearized problem of (3) at \((u, v) = (u_n, v_n)\)

\[
\begin{aligned}
-\Delta \phi_n &= \left( \lambda - 2a(x)u_n - \frac{bv_n(1 + kv_n)}{(1 + mu_n + kv_n)^2} \right) \phi_n - \frac{bu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} \psi_n \\
&\quad + \eta_n \phi_n, \ x \in \Omega, \\
-\Delta \psi_n &= \left( \mu_n - 2v_n + \frac{cu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} \right) \psi_n + \frac{cv_n(1 + kv_n)}{(1 + mu_n + kv_n)^2} \phi_n \\
&\quad + \eta_n \psi_n, \ x \in \Omega, \\
\partial_\nu \phi_n &= \partial_\nu \psi_n = 0, \ x \in \partial \Omega,
\end{aligned}
\]  

has an eigenvalue solution pair \((\phi_n, \psi_n, \eta_n)\) with \(\|\phi_n\|_2 + \|\psi_n\|_2 = 1\) and \(\Re \eta_n \leq 0\) for any \(n \geq 1\). Note that \(\phi_n\) and \(\psi_n\) may be complex-valued.

**Step 1.** First, we claim that \(\Re \eta_n\) is uniformly bounded. Otherwise, since \(\Re \eta_n \leq 0\), we may assume that \(\Re \eta_n \to -\infty\) as \(n \to \infty\).

By virtue of Kato’s inequality, we obtain that

\[
-\Delta |\phi_n| \leq -\Re \left( \frac{\phi_n}{|\phi_n|} \Delta |\phi_n| \right) \\
\leq \left( \lambda - 2a(x)u_n - \frac{bv_n(1 + kv_n)}{(1 + mu_n + kv_n)^2} \right) |\phi_n| \tag{21}
\]

and

\[
-\Delta |\psi_n| \leq -\Re \left( \frac{\psi_n}{|\psi_n|} \Delta |\psi_n| \right) \\
\leq \left( \mu_n - 2v_n + \frac{cu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} \right) |\psi_n| + \Re \eta_n |\phi_n|, \tag{22}
\]
Applying Theorem 3.1, it holds that

\[ 0 \leq \int_\Omega |\nabla \phi_n|^2 \leq \int_\Omega \left( \lambda - 2a(x)u_n - \frac{b v_n (1 + kv_n)}{(1 + mu_n + kv_n)^2} \right) |\phi_n|^2 + \int_\Omega \frac{b u_n (1 + mu_n)}{(1 + mu_n + kv_n)^2} |\phi_n| |\psi_n| + Re \eta_n \int_\Omega |\phi_n|^2 \]

\[ \leq \lambda \int_\Omega |\phi_n|^2 + \frac{b}{m} \left( \frac{1}{\int_\Omega |\phi_n|^2} \right)^{1/2} \left( \int_\Omega |\psi_n|^2 \right)^{1/2} + Re \eta_n \int_\Omega |\phi_n|^2. \]

Then it follows that

\[ (-\lambda - Re \eta_n) \int_\Omega |\phi_n|^2 \leq \frac{b}{m} \left( \frac{1}{\int_\Omega |\phi_n|^2} \right)^{1/2} \left( \int_\Omega |\psi_n|^2 \right)^{1/2}. \]

From (20), one sees that \( \phi_n \not\equiv 0 \), otherwise \( \psi_n \equiv 0 \), which is a contradiction to \( |\phi_n|_2 + |\psi_n|_2 = 1 \). Moreover, since \( Re \eta_n \to -\infty \), we know that for sufficiently large \( n \), it holds that \( -\lambda - Re \eta_n > 0 \). Thus, for sufficiently large \( n \), it holds that

\[ |\phi_n|_2 \leq \frac{b}{m(-\lambda - Re \eta_n)} |\psi_n|_2. \] (23)

Let \( n \to \infty \) in (23), it follows that

\[ |\phi_n|_2 \to 0 \quad \text{and} \quad |\psi_n|_2 \to 1. \]

Multiplying both sides of (22) by \( |\psi_n| \) and integrating over \( \Omega \), we obtain that

\[ 0 \leq \int_\Omega |\nabla \psi_n|^2 \leq \mu_n \int_\Omega \left( 1 - 2 \frac{v_n}{\mu_n} + \frac{1}{\mu_n} \frac{c u_n (1 + mu_n)}{(1 + mu_n + kv_n)^2} \right) |\psi_n|^2 + c \int_\Omega \frac{v_n (1 + kv_n)}{(1 + mu_n + kv_n)^2} |\phi_n| |\psi_n| + Re \eta_n \int_\Omega |\psi_n|^2. \]

Noting that

\[ \frac{v_n (1 + kv_n)}{(1 + mu_n + kv_n)^2} \leq \frac{1}{k}. \]

Applying Theorem 3.1, it holds that

\[ 1 - 2 \frac{v_n}{\mu_n} + \frac{1}{\mu_n} \frac{c u_n (1 + mu_n)}{(1 + mu_n + kv_n)^2} \to -1 \]

uniformly in \( \bar{\Omega} \). Thus, for sufficiently large \( n \),

\[ 0 \leq \frac{1}{2} \mu_n \int_\Omega |\psi_n|^2 + \frac{c}{k} \left( \frac{\int_\Omega |\phi_n|^2}{\int_\Omega |\psi_n|^2} \right)^{1/2} \left( \int_\Omega |\psi_n|^2 \right)^{1/2} + Re \eta_n \int_\Omega |\psi_n|^2. \] (25)

However, the right hand side of (25) tends to \( -\infty \) as \( n \to \infty \), which is impossible. Thus, \( Re \eta_n \) is uniformly bounded.

**Step 2.** Next, we prove that \( |Im \eta_n| \) is uniformly bounded. If not, then we may assume that \( |Im \eta_n| \to \infty \) as \( n \to \infty \).

Multiplying the first equation of (20) by \( \bar{\phi}_n \) and integrating over \( \Omega \), one sees that

\[ \int_\Omega |\nabla \phi_n|^2 = \int_\Omega \left( \lambda - 2a(x)u_n - \frac{b v_n (1 + kv_n)}{(1 + mu_n + kv_n)^2} \right) |\phi_n|^2 - \int_\Omega \frac{b u_n (1 + mu_n)}{(1 + mu_n + kv_n)^2} \bar{\phi}_n |\psi_n| + \eta_n \int_\Omega |\phi_n|^2. \]
Then
\[
\text{Im}\eta_n \int_{\Omega} |\phi_n|^2 = \text{Im} \int_{\Omega} \frac{bu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} \phi_n \psi_n.
\]

Thus,
\[
|\text{Im}\eta_n| \int_{\Omega} |\phi_n|^2 \leq \frac{b}{m} \int_{\Omega} |\phi_n| |\psi_n|.
\]

Then
\[
|\text{Im}\eta_n| \int_{\Omega} |\phi_n|^2 \leq \frac{b}{m} \|\phi_n\|_2 \|\psi_n\|_2.
\]

Thus,
\[
|\text{Im}\eta_n| \|\phi_n\|_2 \leq \frac{b}{m} \|\psi_n\|_2.
\]

Since \(|\text{Im}\eta_n| \to \infty\), it follows that
\[
\|\phi_n\|_2 \to 0 \quad \text{and} \quad \|\psi_n\|_2 \to 1.
\]

Then similar to step 1, we can also obtain (25). In this case, \(\text{Re} \eta_n\) is uniformly bounded. But the right hand side of (25) also tends to \(\pm \infty\) as \(n \to \infty\), which is also impossible. Thus, \(|\text{Im}\eta_n|\) is uniformly bounded.

**Step 3.** Since \(\eta_n\) is uniformly bounded, we can assume that \(\eta_n \to \eta\) as \(n \to \infty\) with \(\text{Re}\eta \leq 0\). In the following, we show that \(\text{Re}\eta > 0\), which yields a contradiction.

Note that (24) holds. Then for sufficiently large \(n\), it holds that
\[
\mu_n - 2v_n + \frac{cu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} < 0.
\]

Multiplying the equation of \(\psi_n\) in (20) by \(\psi_n\) and integrating over \(\Omega\), we obtain that
\[
\int_{\Omega} |\nabla \psi_n|^2 = \int_{\Omega} \left( \mu_n - 2v_n + \frac{cu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} \right) |\psi_n|^2
+ c \int_{\Omega} \frac{v_n(1 + kv_n)}{(1 + mu_n + kv_n)^2} \phi_n \psi_n + \eta_n \int_{\Omega} |\psi_n|^2.
\]

Then due to the uniform boundedness of \(\eta_n\), \(\|\phi_n\|_2\) and \(\|\psi_n\|_2\), we obtain that for sufficiently large \(n\),
\[
\int_{\Omega} |\nabla \psi_n|^2 \leq \frac{c}{k} \|\phi_n\|_2 \|\psi_n\|_2 + |\eta_n| \int_{\Omega} |\psi_n|^2 \leq \frac{c}{k} + |\eta| + 1.
\]

Then \(\psi_n\) converges to some \(\psi\) weakly in \(H^1(\Omega)\) and strongly in \(L^2(\Omega)\). From the equation of \(\phi_n\) in (20), applying the standard \(L^p\) theory for elliptic equations, we can know that \(\|\phi_n\|_{W^{2,2}(\Omega)}\) is uniformly bounded. Then one can also see that \(\phi_n\) converges to \(\phi\) weakly in \(H^1(\Omega)\) and strongly in \(L^2(\Omega)\).

From (26), it holds that
\[
\int_{\Omega} |\nabla \psi_n|^2 = \int_{\Omega} \left( 1 - \frac{2v_n}{\mu_n} + \frac{\mu_n cu_n(1 + mu_n)}{(1 + mu_n + kv_n)^2} \right) |\psi_n|^2
+ \frac{c}{\mu_n} \int_{\Omega} \frac{v_n(1 + kv_n)}{(1 + mu_n + kv_n)^2} \phi_n \psi_n + \frac{\eta_n}{\mu_n} \int_{\Omega} |\psi_n|^2.
\]

Then as \(n \to \infty\), it follows that
\[
\int_{\Omega} |\psi|^2 = 0.
\]
Thus, \( \psi \equiv 0 \). Then it follows that \( \phi \not\equiv 0 \). Thus, applying Theorem 3.1, we obtain that \( \phi \) is a nontrivial solution of the following problem
\[
-\Delta \phi = \left( \lambda - \frac{b}{k} - 2a(x)u_{\lambda - \frac{b}{k}}(x) \right) \phi + \eta \phi, \quad x \in \Omega, \quad \partial_\nu \phi |_{\partial \Omega} = 0.
\]
Then it follows that
\[
\text{Re}\eta \geq \lambda_1^N \left( -\lambda + \frac{b}{k} + 2a(x)u_{\lambda - \frac{b}{k}}(x) \right).
\]
However, from the equation of \( u_{\lambda - \frac{b}{k}}(x) \), we can see that
\[
\lambda_1^N \left( -\lambda + \frac{b}{k} + a(x)u_{\lambda - \frac{b}{k}}(x) \right) = 0.
\]
Then it holds that
\[
\text{Re}\eta \geq \lambda_1^N \left( -\lambda + \frac{b}{k} + 2a(x)u_{\lambda - \frac{b}{k}}(x) \right)
> \lambda_1^N \left( -\lambda + \frac{b}{k} + a(x)u_{\lambda - \frac{b}{k}}(x) \right) = 0,
\]
which is a contradiction. Thus, the proof of the proposition is complete. \( \Box \)

**Proposition 2.** Assume that \( b/k < \lambda_1^D(\Omega_0) \) and \( \lambda \in (b/k, \lambda_1^D(\Omega_0)) \). Then as \( \mu \) is sufficiently large, (3) has a unique positive solution \( (u_\mu, v_\mu) \).

**Proof.** Due to Theorem 2.6, the existence is obvious. In the following, we prove the uniqueness by the topological degree argument.

Let \( \mu \) be any fixed sufficiently large positive constant. For \( t \in [0, 1] \), define an operator
\[
A_\mu(t, u, v) = (-\Delta + I)^{-1} \left( u \left( \lambda - a(x)u - \frac{tbv}{1+mu+kv} \right) + u \right),
\]
where \((-\Delta + I)^{-1}\) is the inverse operator of \(-\Delta + I\) subject to Neumann boundary condition over \( \Omega \). Then if \((u, v)\) is a fixed point of \(A_\mu(t, u, v)\) if and only if it satisfies
\[
\begin{cases}
-\Delta u = u \left( \lambda - a(x)u - \frac{tbv}{1+mu+kv} \right), & x \in \Omega, \\
-\Delta v = v \left( \mu - v + \frac{tcu}{1+mu+kv} \right), & x \in \Omega, \\
\partial_\nu u = \partial_\nu v = 0, & x \in \partial \Omega.
\end{cases}
\]
Since \( \lambda \in (b/k, \lambda_1^D(\Omega_0)) \), we have that for any \( t \in [0, 1] \),
\[
0 < u < u_\lambda(x) \leq \|u_\lambda\|_\infty, \quad 0 < \mu < v < \mu + \frac{c\|u\|_\infty}{1 + m\|u\|_\infty} \leq \mu + \frac{c}{m}.
\]
Then applying the Harnack inequality, we obtain that there exists a positive constant \( C \) independent of \( t \) such that for any \( t \in [0, 1] \),
\[
\max_{\Omega} u \leq C \min_{\Omega} u.
\]
In the following, we show that there exists a positive constant \( \kappa \) independent of \( t \), such that \( \min_{\Omega_t} u \geq \kappa \) for any \( t \in [0, 1] \). If not, we can find a sequence of \( \{t_n\} \) with \( t_n \to t \in [0, 1] \) such that (28) with \( t = t_n \) has a positive solution \( (u_n, v_n) \) satisfying \( \min_{\Omega} u_n \to 0 \). Then it holds that \( u_n \to 0 \) uniformly in \( \Omega \). So (29) deduces
that \( v_n \to \mu \) uniformly in \( \Omega \). Setting \( \bar{u}_n = \frac{\bar{u}_n}{\|\bar{u}_n\|_\infty} \). Then subject to a subsequence, \( \bar{u}_n \to \bar{u} \) in \( C(\Omega) \). Moreover, \( \bar{u} \) is a nontrivial nonnegative solution of the following problem
\[
- \Delta \bar{u} = \bar{u} \left( \lambda - \frac{tb\mu}{1 + k\mu} \right), \quad x \in \Omega, \quad \partial_\nu \bar{u}|_{\partial\Omega} = 0. \tag{30}
\]
Since \( \bar{u} \) is nontrivial, it follows that \( \bar{u} > 0 \) for \( x \in \bar{\Omega} \). Thus, we have that
\[
\lambda = \frac{tb\mu}{1 + k\mu} < \frac{b}{k},
\]
which is a contradiction to the assumption. Thus, for any \( t \in [0,1] \), \( \min_{\bar{\Omega}} u \) has a positive lower bound \( \kappa \).

Now, we define
\[
O_\mu = \left\{ (u,v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \frac{\kappa}{2} < u < \|u_\lambda\|_\infty + 1, \quad \frac{\mu}{2} < v < \mu + \frac{c}{m} + 1 \right\}.
\]
Then it is clear that \( A_\mu \) is completely continuous from \( [0,1] \times O_\mu \) to \( C(\bar{\Omega}) \times C(\bar{\Omega}) \). Moreover, for any \( t \in [0,1] \),
\[
A_\mu(t,u,v) \neq (u,v), \quad \forall (u,v) \in \partial O_\mu.
\]
Thus, \( \deg(I - A_\mu(t,\cdot),O_\mu,(0,0)) \) is well defined. Moreover, by the homotopy invariance, it follows that
\[
\deg(I - A_\mu(0,\cdot),O_\mu,(0,0)) = \deg(I - A_\mu(1,\cdot),O_\mu,(0,0)).
\]
When \( t = 0 \), (28) has a unique positive solution \( (u_\lambda, \mu) \). Then
\[
\deg(I - A_\mu(0,\cdot),O_\mu,(0,0)) = \text{index}(I - A_\mu(0,\cdot),(u_\lambda, \mu)).
\]
Since \( (u_\lambda, \mu) \) is nondegenerate and linearly stable, it can be verified that
\[
\text{index}(I - A_\mu(0,\cdot),(u_\lambda, \mu)) = 1.
\]
Thus,
\[
\deg(I - A_\mu(1,\cdot),O_\mu,(0,0)) = 1.
\]
When \( t = 1 \), Theorem 1 yields that each positive fixed point of \( A_\mu \) is isolated and has index 1. Moreover, by the compactness of \( A_\mu \), there exist only finitely many isolated positive fixed points, which are denoted by \( (u_1, v_1), \ldots, (u_j, v_j) \). Then it follows that
\[
1 = \deg(I - A_\mu(1,\cdot),O_\mu,(0,0)) = \sum_{i=1}^j \text{index}(I - A_{\mu}(1,\cdot),(u_i, v_i)) = j,
\]
which asserts that \( j = 1 \). Thus, the uniqueness is proved for any fixed sufficiently large \( \mu \). The proof of the proposition is complete.

Next, we show the stability of the positive solution bifurcating from \((\mu_1,0,\mu_1)\) given by Lemma 2.4.

**Theorem 4.2.** Assume that \( 0 < \lambda < b/k, \mu_1 \) is defined by (10). If \( \tilde{\mu}_1 \neq 0 \), then there exists a small positive number \( \sigma \) such that the positive solution \((\mu(s),u(s),v(s))\) of (3) bifurcating from \((\mu_1,0,\mu_1)\) is nondegenerate for \( s \in (0,\sigma) \). Moreover, if \( \tilde{\mu}_1 > 0 \), then \((u(s),v(s))\) is unstable; if \( \tilde{\mu}_1 < 0 \), then \((u(s),v(s))\) is linearly stable.

**Proof.** For the positive number \( \delta \) defined in the proof of Lemma 2.4, we know that for \( s \in (0,\delta) \), there exists a branch of positive solutions \((\mu(s),u(s),v(s))\) of (3) bifurcating from \((\mu_1,0,\mu_1)\).
Thus, the proof of the theorem is complete.

Therefore, we can find a small number \( \delta > 0 \)
and Lemma 2.4 asserts that as \( s \to 0^+ \),

\[
L(s) \to L_0 = \begin{pmatrix}
-\Delta - \frac{b_{\mu_1}}{1 + k_{\mu_1}} & 0 \\
-\frac{c_{\mu_1}}{1 + k_{\mu_1}} & -\Delta + \mu_1
\end{pmatrix}.
\]

Then it is clear that 0 is the least eigenvalue of \( L_0 \), and the corresponding positive
eigenfunction is \( (\phi_1, \psi_1) \) defined by (9). Moreover, the real parts of all the other
eigenvalues of \( L_0 \) are positive and bounded away from 0. Thus, the perturbation
theory of linear operators\(^{[29]} \) asserts that there exists a small positive number \( \sigma_1 < \delta \)
such that for \( s \in (0, \sigma_1) \), \( L(s) \) has a unique eigenvalue \( \eta(s) \) satisfying \( \eta(s) \to 0 \) as
\( s \to 0^+ \). Moreover, all the other eigenvalues of \( L(s) \) have positive real parts and
apart from 0. Without loss of generality, we may assume that \( (\phi(s), \psi(s)) \to (\phi_1, \psi_1) \)
as \( s \to 0^+ \).

Note that \( \phi(s) \) satisfies

\[
-\Delta \phi(s) = \left( \lambda - 2u(x)u(s) - \frac{bv(s)(1 + kv(s))}{(1 + mu(s) + kv(s))^2} \right) \phi(s)
- \frac{bu(s)(1 + mu(s))}{(1 + mu(s) + kv(s))^2} \psi(s) + \eta(s)\phi(s).
\]  

(31)

Multiplying both sides of (31) by \( u(s) \) and integrating over \( \Omega \), it follows that

\[
\int_{\Omega} \phi(s) (-\Delta u(s)) = \int_{\Omega} \left( \lambda - 2u(x)u(s) - \frac{bv(s)(1 + kv(s))}{(1 + mu(s) + kv(s))^2} \right) \phi(s)u(s)
- \int_{\Omega} \frac{bu(s)(1 + mu(s))}{(1 + mu(s) + kv(s))^2} \psi(s) + \eta(s) \int_{\Omega} \phi(s)u(s).
\]

By the equation of \( u(s) \) and some computations, we further obtain that

\[
\eta(s) \int_{\Omega} \phi(s)u(s) = \int_{\Omega} \left( a(x) - \frac{bmv(s)}{(1 + mu(s) + kv(s))^2} \right) \phi(s)u^2(s)
+ b \int_{\Omega} \frac{u^2(s)(1 + mu(s))}{(1 + mu(s) + kv(s))^2} \psi(s).
\]

Applying Lemma 2.4 again, we know that \( u(s)/s \to \phi_1 \). Then as \( s \to 0^+ \), we have that

\[
\lim_{s \to 0^+} \frac{\eta(s)}{s} = \frac{\int_{\Omega} \left( a(x) - \frac{bmv(s)}{(1 + mu(s) + kv(s))^2} \right) \phi_1^2 + b \int_{\Omega} \frac{\phi_1^2 \psi_1}{(1 + k\mu_1)^2}}{\int_{\Omega} \phi_1^2}.
\]

Thus,

\[
\lim_{s \to 0^+} \frac{\eta(s)}{s} = -\frac{b}{(1 + k\mu_1)^2} \mu_1 \neq 0.
\]

Therefore, we can find a small number \( \sigma < \min \{ \delta, \sigma_1 \} \) such that for \( s \in (0, \sigma) \),
the positive solution \( (u(s), v(s)) \) of (3) at \( \mu = \mu(s) \) is nondegenerate. Moreover, if \( \tilde{\mu}_1 > 0 \), then \( (u(s), v(s)) \) is unstable; if \( \tilde{\mu}_1 < 0 \), then \( (u(s), v(s)) \) is linearly stable.

The proof of the theorem is complete. \( \square \)
Due to Remark 1, one sees that if the functional response changes from the Holling-II one to the Beddington-DeAngelis one, then the positive solution \((\mu(s), u(s), v(s))\) bifurcating from \(\Gamma_v\) may vary from unstable to linearly stable.

Similar to the proof of Theorem 4.2, we can also obtain the stability of the positive solutions of (3) bifurcating from \(\Gamma_u\).

**Theorem 4.3.** Assume that \(0 < \lambda < \lambda^1_1(\Omega_0)\). Then there exists a positive constant \(\sigma\) such that for \(s \in (0, \sigma)\), the positive solution \((\mu(s), u(s), v(s))\) of (3) bifurcating from \((\mu_1, u_0, v_0)\) is always nondegenerate and linearly stable.

By virtue of Theorem 2.7, we know that if \(\lambda^0(\Omega_0) < b/k\) and \(\lambda \in (\lambda^0(\Omega_0), b/k)\), then (3) has at least one positive solution for \(\mu > \mu_1\). This is only one sufficient condition but not necessary condition for the existence of positive solutions. Then it is natural to ask what will happen when \(\mu < \mu_1\). Actually, we can show that (3) may possess at least two positive solutions if \(\mu\) is slightly smaller than \(\mu_1\). Precisely, we have the following theorem.

**Theorem 4.4.** Assume that \(\lambda^0(\Omega_0) < b/k\) and \(\lambda \in (\lambda^0(\Omega_0), b/k)\). If the number \(\mu_1\) defined by (10) is negative, then there exists a small positive constant \(\varepsilon\) such that (3) has at least two positive solutions for \(\mu \in (\mu_1 - \varepsilon, \mu_1)\).

**Proof.** By virtue of Lemma 2.4, there exists a smooth curve \(\{(\mu(s), u(s), v(s)) : 0 < s < \delta\}\) of positive solutions of (3) bifurcating from \((\mu_1, 0, \mu_1)\), where \(\delta\) is the positive constant given in Lemma 2.4.

Since \(\mu_1 < 0\), one sees that \(\mu'(0) = \tilde{\mu}_1 < 0\). Thus, there exists a small positive constant \(\delta_1 < \delta\) such that \(\mu'(s) < 0\) for \(s \in (0, \delta_1)\). Thus, \(\mu(\delta_1) < \mu(s) < \mu_1\) for \(s \in (0, \delta_1)\). Moreover, we choose \(\delta_1\) sufficiently small such that \(\mu(\delta_1) > 0\). Then for any fixed \(s \in (0, \delta_1)\), that is, for any fixed \(\mu = \mu(s) \in (\mu(\delta_1), \mu_1)\), (3) at \(\mu = \mu(s)\) has a positive solution \((u, v) = (u(s), v(s))\).

In the following, we claim that for any fixed \(\mu \in (\mu(\delta_1), \mu_1)\), (3) has at least two positive solutions. Otherwise, there exists some \(\mu_0 \in (\mu(\delta_1), \mu_1)\) such that (3) with \(\mu = \mu_0\) has a unique positive solution. We assume that \(\mu_0 = \mu(s_0)\), \(s_0 \in (0, \delta_1)\). Then one sees that the unique positive solution of (3) is given by \((u(s_0), v(s_0))\).

For any \(t \in [0, 1]\), we define an operator \(A(t, u, v)\) by (27) with \(\mu = \mu_0\). Then similar to the proof of Theorem 2.1, we can show that for any \(t \in [0, 1]\), any positive fixed point of \(A(t, u, v)\) satisfies

\[
0 < u \leq M, \quad 0 < \mu_0 < v < \mu_0 + \frac{tc\|u\|_{\infty}}{1 + m\|u\|_{\infty}} \leq \mu_0 + \frac{c}{m},
\]

where \(M\) is a positive constant independent of \(t\). In the following, we prove that there exists a positive constant \(\kappa\) independent of \(t\) such that for any \(t \in [0, 1]\), any positive fixed point of \(A(t, u, v)\) satisfies \(\min_u u \geq \kappa\). If not, there exists a sequence of \(\{t_n\}\) with \(t_n \to t \in [0, 1]\) such that \(A(t_n, u, v)\) has a positive fixed point \((u_n, v_n)\) satisfying \(\min_u u_n \to 0\) as \(n \to \infty\). Then the Harnack inequality deduces that \(u_n \to 0\) uniformly in \(\Omega\) as \(n \to \infty\). Then by setting \(\tilde{u}_n = \frac{u_n}{\|u_n\|_{\infty}}\), we know that subject to a subsequence if necessary, \(\tilde{u}_n \to \tilde{u}\) in \(C(\bar{\Omega})\). Moreover, \(\tilde{u} \geq 0\) is a nontrivial solution of (30) with \(\mu = \mu_0\). Thus, we further know that \(\tilde{u} > 0\) for \(x \in \Omega\). Then

\[
\lambda = \frac{tb\mu_0}{1 + k\mu_0} \leq \frac{b\mu_0}{1 + k\mu_0} < \frac{b\mu_1}{1 + k\mu_1}.
\]
However, Lemma 2.4 asserts that \( \lambda = \frac{b\mu_1}{M + \mu_1} \), which is a contradiction. Thus, there exists a positive constant \( \kappa \) independent of \( t \) such that for any \( t \in [0, 1] \), any positive fixed point of \( A(t, u, v) \) satisfies \( \min_{\Omega} u \geq \kappa \).

Define

\[
O = \left\{ (u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) : \frac{K}{2} < u < M + 1, \quad \frac{\mu_0}{2} < v < \mu_0 + \frac{c}{m} + 1 \right\}.
\]

Then \( A(t, u, v) \) is completely continuous from \([0, 1] \times O \) to \( C(\bar{\Omega}) \times C(\bar{\Omega}) \). Moreover, for any \( t \in [0, 1] \),

\[
A(t, u, v) \neq (u, v), \quad \forall (u, v) \in \partial O.
\]

Thus, \( \deg \left( I - A(\cdot, \cdot), O, (0, 0) \right) \) is well defined and independent of \( t \in [0, 1] \). Thus,

\[
\deg \left( I - A(0, \cdot), O, (0, 0) \right) = \deg \left( I - A(1, \cdot), O, (0, 0) \right).
\]

As \( t = 0 \), since \( \lambda > \lambda^2_1(\Omega_0) \), we see that \( A(0, \cdot) \) has no positive fixed point in \( O \). Thus,

\[
\deg \left( I - A(0, \cdot), O, (0, 0) \right) = 0.
\]

As \( t = 1 \), \( A(1, \cdot) \) has a unique positive fixed point \((u(s_0), v(s_0))\). Moreover, Theorem 4.2 implies that \((u(s_0), v(s_0))\) is nondegenerate and linearly stable. Thus, we have that

\[
\deg \left( I - A(1, \cdot), O, (0, 0) \right) = \text{index} \left( A(1, \cdot), (u(s_0), v(s_0)) \right) = 1.
\]

Thus, we obtain a contradiction. Therefore, for any fixed \( \mu \in (\mu(\delta_1), \mu_1) \), (3) has at least two positive solutions. Then by choosing \( \varepsilon = \mu_1 - \mu(\delta_1) \), the proof of the theorem is complete.

It should be pointed out that when the protection zone \( \Omega_0 \) is large such that \( \lambda^2_1(\Omega_0) < b/k \) and \( \lambda \in (\lambda^2_1(\Omega_0), b/k) \), the assumptions in Theorem 4.4 may hold true due to Remark 1.

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