Higher Orders in the Colour-Octet Model of $J/\psi$ Production

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Abstract

We study the hadro- and photo-production of $c\bar{c}$ and $b\bar{b}$ mesons at low transverse momentum to high orders in the relative velocity of the pair, $v$, in non-relativistic QCD. We evaluate cross sections to order $v^7$ for $\eta_c$, sufficient for studies of photo-production in the almost-elastic region. For all other charmonium states we find the cross section to order $v^9$, sufficient for studies of the ratio of $\chi_{c1}$ and $\chi_{c2}$ production rates. We find recurrence formulæ for generating terms at even higher orders, should they be needed.

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1 Introduction

Recent progress in the understanding of cross sections for production of heavy quarkonium resonances has come through the non-relativistic QCD (NRQCD) reformulation of this problem [1]. Although the production of the pair is dominated by short distance scales of order $1/m$ (where $m$ is the heavy quark mass), longer non-perturbative scales play an important role. The projection on to a specific quarkonium state involves length scales such as $1/mv$, $1/mv^2$, etc., where $v$ is the (dimensionless) velocity of either of the heavy fermions in the rest frame of the pair.

Factorisation of the physics at the long and short scales has been proven in the NRQCD formalism for processes dominated by a large transverse momentum [2]. The resulting cross sections are a double power series in the QCD coupling $\alpha_s$ evaluated at the NRQCD factorisation scale $\mu_0$, and the velocity $v$. Often, higher orders in $v$ involve the previously neglected colour-octet states of the heavy quark pairs. For bottomonium states, $v^2 \ll \alpha_s(m^2)$ and hence colour octet contributions are often not very significant and the expansion is close to the normal perturbative expansion. For charmonium states, a numerical coincidence, $v^2 \sim \alpha_s(m^2)$, makes the double expansion more complicated.

The formalism has been successfully applied to large transverse momentum processes [3]. Interestingly, inclusive production cross sections for charmonium at low energies, dominated by low transverse momenta, also seem to have a good phenomenological description in terms of this approach [4, 5, 6]. It was argued [6] that a better understanding of such cross sections can be obtained if the higher order terms in $v$ and $\alpha_s$ are used. The argument is simple. Total inclusive $J/\psi$ cross sections arise either from direct $J/\psi$ production (which starts at order $\alpha_s v^7$) or through decays of $\chi$ states. Now $\chi_0$ and $\chi_2$ are first produced at order $\alpha_s v^5$, whereas $\chi_1$, which has the largest branching fraction into $J/\psi$, is produced only at order $\alpha_s v^9$. Hence, a full understanding of these cross sections requires the NRQCD expansion upto order $\alpha_s v^9$. We calculate these terms here. Of course, higher orders in $\alpha_s$ at lower orders in $v$ may be equally important. First attempts at computing these have been made [7].

The plan of this paper is the following. In section 2 we set out the notation with a brief review of the threshold expansion technique [8], and some extensions, which we shall use for our computation. In the next section we
perform a Taylor expansion of the transition matrix elements in the relative momentum of the heavy quark pair and give recurrence relations for the Taylor coefficients to all orders. The cross sections are listed in section 4. We conclude in section 5 with a discussion of the phenomenological applications of this computation. The appendix contains a discussion of the complete specification of states and operators required for computations beyond the leading orders of NRQCD.

2 The Threshold Expansion

The NRQCD factorisation formula for inclusive production of heavy quarkonium resonances \( H \) with 4-momentum \( P \) is

\[
d\sigma = \frac{1}{\Phi} \frac{d^3 P}{(2\pi)^3 2 E_P} \sum_{ij} C_{ij} \langle \mathcal{K}_i \Pi(H) \mathcal{K}_j^\dagger \rangle,
\]

where \( \Phi \) is a flux factor. The coefficient function \( C_{ij} \) is computable in perturbative QCD and hence has an expansion in the strong coupling \( \alpha_s \) (evaluated at the NRQCD cutoff), whereas the matrix element is non-perturbative. However, in NRQCD, it has a fixed scaling dimension in the quark velocity \( v \). Consequently, the cross section is a double power series in \( \alpha_s \) and \( v \).

The fermion bilinear operators \( \mathcal{K}_i \) are built out of heavy quark fields sandwiching colour and spin matrices and the covariant derivative \( \mathbf{D} \). The specification of the composite labels \( i \) and \( j \) is given in appendix A. They include the colour index \( \alpha \), the spin quantum number \( S \), the orbital angular momentum \( L \) (found by coupling the derivatives), the total angular momentum \( J \) and the helicity \( J_z \). At low orders in \( v \) this set is sufficient to fix the operators. Since the hadronic projection operator

\[
\Pi(H) = \sum_s |H, s\rangle \langle H, s|,
\]

(where \( s \) denotes hadronic states with energy less than the NRQCD cutoff), is diagonal in these quantum numbers, it is clear that the operators \( \mathcal{K}_i \) and \( \mathcal{K}_j \) in eq. (2.1) are restricted to have equal \( L \), \( S \), \( J \) and \( J_z \). For a more detailed discussion see the appendix.
The \( J_z \)-dependence of these matrix elements can be factored out using the Wigner-Eckart theorem—

\[
\langle \mathcal{K}_i \Pi(H) \mathcal{K}_j^\dagger \rangle = \frac{1}{2J+1} \mathcal{O}_a^H (2S+1)^N L^N_{J, J'},
\]

(2.3)

where the first factor on the right comes from a Clebsch-Gordan coefficient. This factor is conventionally included in the coefficient function. In the reduced matrix element \( \mathcal{O}_a^H \), we have introduced a new label \( N \) which is the number of derivatives in each fermion bilinear (see the appendix). In agreement with the notation of [2] we write for the off-diagonal operators

\[
\langle \mathcal{K}_i \Pi(H) \mathcal{K}_j^\dagger \rangle = \frac{1}{2J+1} \mathcal{P}_a^H (2S+1)^N L^N_{J, J'}. \tag{2.4}
\]

The power counting rule for the matrix elements in eq. (2.1) is—

\[
d = 3 + N + N' + 2(E_d + 2M_d), \tag{2.5}
\]

where \( E_d \) and \( M_d \) are the number of colour electric and magnetic transitions required to connect the hadronic state to the state \( \mathcal{K}_i |0 \rangle \). Note that at low orders in \( D \), \( N = L \), and the more familiar rules are obtained. An example is provided by the off-diagonal matrix element

\[
\mathcal{P}_1^{\eta_c} \left( ^1S_0^0, ^1S_0^2 \right) \equiv -\frac{1}{\sqrt{3}} \left\langle \psi^\dagger \chi \Pi(\eta_c) \chi^\dagger \left( -\frac{i}{2} D \right) \cdot \left( -\frac{i}{2} D \right) \chi \right\rangle + \text{h.c.} \tag{2.6}
\]

which scales as \( v^5 \). The \(-1/\sqrt{3}\) factor on the right is a trivial Clebsch-Gordan factor, explained later.

We choose to construct the coefficient functions using the “threshold expansion” technique of [3]. This consists of calculating, in perturbative QCD, the matrix element \( \mathcal{M} \) connecting the initial states to final states with a heavy quark-antiquark pair \( (\bar{Q}Q) \), and Taylor expanding the result in the relative momentum of the pair, \( q \), after performing a non-relativistic reduction of the Dirac spinors. The resulting expression is squared and matched to the NRQCD formula of eq. (2.1) by inserting a perturbative projector onto a non-relativistic \( \bar{Q}Q \) state between the two spinor bilinears. The coefficient of this matrix element is the required coefficient function.

Symbolically—

\[
\sum_{pol} |\mathcal{M}|^2 = \sum_{ij} C_{ij} \left\langle \mathcal{K}_i \Pi(\bar{Q}Q) \mathcal{K}_j^\dagger \right\rangle, \tag{2.7}
\]
where the left hand side is Taylor expanded in q. Each factor of q Fourier transforms into a factor of the covariant derivative $D$ on the right hand side. Since each matrix element on the right of eq. (2.7) corresponds to a unique matrix element in eq. (2.1), the order up to which the Taylor expansion is to be performed is determined by the scaling of the non-perturbative matrix elements with $v$. Since we require a classification of operators by the angular momentum, it turns out to be very convenient to use spherical tensor methods. These were used in an earlier paper [3] and are used more extensively here.

In this paper we evaluate the cross sections to order $\alpha_s v^9$. The Taylor expansion order, $N + N' \leq 6$ is obtained by setting $d = 9$ and $E_d = M_d = 0$ in eq. (2.5). Furthermore, since we examine the leading term in perturbation theory, the perturbative projector has only one term—

$$\Pi(\bar{Q}Q) = |\bar{Q}Q\rangle\langle Q\bar{Q}|.$$  

(2.8)

In agreement with [8] we use the relativistic normalisation

$$\langle Q(p, \xi)|\bar{Q}(q, \eta)|Q(p', \xi')\bar{Q}(q', \eta')\rangle = 4E_p E_q (2\pi)^6 \delta^3(p - p')\delta^3(q - q'),$$  

(2.9)

with the spinor normalisations $\xi^\dagger\xi = \eta^\dagger\eta = 1$. Since $E_p = E_q = \sqrt{m^2 + q^2}$, expanding this in $q^2$ allows us to write the spinor bilinears in terms of transition operators built out of the heavy quark field. For example,

$$\xi^\dagger\eta = \frac{1}{2m} \langle \bar{Q}Q(q)|\psi^\dagger\chi|0\rangle - \frac{1}{2m^3} \langle \bar{Q}Q(q)|\psi^\dagger D \cdot D \chi|0\rangle + \cdots$$  

(2.10)

Conventionally the coefficient functions and matrix elements in eq. (2.1) were written with a non-relativistic normalisation of the hadron states. In the threshold expansion technique it is more convenient to retain a relativistic normalisation similar to that in eq. (2.9). The result for the cross section is the same in either case, since a change in the definition of the matrix element is compensated by a change in the coefficient function. To leading order in $q$, the matrix elements in the notation of [3] have to be multiplied by $4m$ to obtain those in the relativistic normalisation [8]. As higher orders the relation is more complicated. In this paper, we shall work entirely with the latter.
3 The Matrix Elements

To leading order in $\alpha_s$, the kinematics is very simple. The momenta of the initial particles are $p_1$ and $p_2$. We take $p_1$ to lie in the positive $z$-direction and $p_2$ to be oppositely directed. The momentum of the meson, $P = p_1 + p_2$. As a result, $s = P^2 = M^2$, where $M$ is the meson mass.

The 4-momenta of $Q$ and $\bar{Q}$ ($p$ and $\bar{p}$ respectively) are written as

$$p = \frac{1}{2}P + L_j q^j \quad \text{and} \quad \bar{p} = \frac{1}{2}P - L_j q^j.$$  \hspace{1cm} (3.1)

Note that $p^2 = \bar{p}^2 = m^2$, where $m$ is the mass of the heavy quark. The space-like vector $q$ is always defined in the rest frame of the pair, and $L^\mu_j$ boosts it to any frame. We shall use Greek indices for Lorentz tensors and Latin indices for Euclidean 3-tensors.

The following relations are easy to prove—

$$p_1 \cdot L_j = -\frac{M}{2} \hat{z}_j, \quad \text{and} \quad p_2 \cdot L_j = \frac{M}{2} \hat{z}_j,$$  \hspace{1cm} (3.2)

where $\hat{z}$ is the unit 3-vector in the $z$-direction. They are consistent with the identity $P \cdot L_j = 0$. We shall use the two identities

$$L_j \cdot L_k = -\delta_{ij}, \quad \text{and} \quad M\epsilon^{ijk}L_{\alpha k} = \epsilon_{\mu\nu\sigma}L^\mu_i L^\nu_j P^\rho.$$  \hspace{1cm} (3.3)

Other relations can be written down, but are not important for our computations. Note our convention $\epsilon^{0123} = 1$.

This technique also depends on the usual non-relativistic reduction of Dirac spinors which gives rise to the identities

$$\bar{u}(p)\gamma^\mu v(p) = L_j^\mu \left[M\xi^\dagger\sigma^j\eta - \frac{4}{M+2m}\delta_{mn}q^j q^n \xi^\dagger \sigma^n \eta\right],$$

$$\bar{u}(p)\gamma^\mu \gamma^5 v(p) = \frac{2m}{M}P^\mu \xi^\dagger \eta - 2i L^\mu_m \epsilon_{mnj} q^n \xi^\dagger \sigma^j \eta.$$  \hspace{1cm} (3.4)

Here $\xi$ and $\eta$ are Pauli spinors and $\sigma^j$ are the usual Pauli matrices. $M$ ($M^2 = 4m^2 + 4q^2$) is the invariant mass of the $\bar{Q}Q$ system.

We work in a class of ghost-free gauges called the planar gauges. These are defined by the polarisation sum for gluons

$$\sum_{\lambda} c^\lambda_\mu(p)c_\nu^{*\lambda}(p) = d_{\mu\nu}(p) = -g_{\mu\nu} + \frac{1}{p \cdot V} (p_\mu V_\nu + p_\nu V_\mu).$$  \hspace{1cm} (3.5)
The propagator for a gluon of momentum \( p \) is then given by \( G_{\mu\nu}(p) = d_{\mu\nu}(p)/p^2 \). The vector \( V \) defines a gauge choice. We write \( V = c_1 p_1 + c_2 p_2 \), with \( c_1/c_2 \sim \mathcal{O}(1) \). We verify that all results are gauge invariant by the explicit check that they do not depend on the arbitrary coefficients \( c_1 \) and \( c_2 \).

3.1 \( \bar{q}q \rightarrow \bar{Q}Q \)

The matrix element for the subprocess \( \bar{q}q \rightarrow \bar{Q}Q \) is very simple. It is given exactly by the expression

\[
\mathcal{M} = -i g^2 f_{abc} \left[ \bar{v}(p_2) \gamma_\mu T^a u(p_1) \right] L_j^\mu \left[ M \xi^i T^c \eta - \frac{4}{M + 2m} q^j \xi^i (q \cdot \sigma) T^c \eta \right],
\]

(3.6)

where \( u \) and \( v \) are the light quark spinors. The equations of motion for the initial state quarks has been used to obtain the explicitly gauge invariant matrix element in eq. (3.6). The desired Taylor series expansion is obtained by using the relation \( M^2 = 4(m^2 + q^2) \) to expand all factors with \( M \). Converting to spherical tensors \([11]\), we find

\[
q^2 = -\sqrt{3}[qq]^0_0 \quad \text{and} \quad \sigma \cdot q = -\sqrt{3}[\sigma q]^0_0,
\]

(3.7)

where the notation \([\cdot \cdot \cdot]_M^J \) denotes a coupling to angular momentum \( J \) and helicity \( M \) of the spherical tensors inside the square brackets. The remaining Euclidean vectors are converted to spherical tensors after squaring the matrix element.

3.2 \( gg \rightarrow \bar{Q}Q \)

The s-channel gluon exchange diagram can easily be reduced to the form

\[
\mathcal{M}_s = 2g^2 f_{abc} \left( A_j + \frac{1}{2} \epsilon_1 \cdot \epsilon_2 \hat{z}_j \right) \left( \xi^i \sigma^j T^c \eta - \frac{4}{M(M + 2m)} q^j \xi^i q_i \sigma^j T^c \eta \right),
\]

(3.8)

Here \( \epsilon_i \) is the polarisation vector for the initial gluon of momentum \( p_i \), and \( T^c \) is a colour generator. For convenience we have used the notation

\[
A_i = \frac{1}{M} (\epsilon_1 \cdot L_i \epsilon_2, p_1 - \epsilon_2 \cdot L_i \epsilon_1 \cdot p_2).
\]

(3.9)

\[\text{The factor of } -\sqrt{3} \text{ in the conversion of dot products was used earlier in eq. (2.8).}\]
The $t$ and $u$ channel matrix elements require a little more work. The colour factors can be reduced using the identity
\[ T_a T_b = \frac{1}{6} \delta_{ab} + \frac{1}{2} d_{abc} T^c + \frac{i}{2} f_{abc} T^c. \] (3.10)

The matrix elements can be written in the form
\[ M_t = \frac{t}{2p \cdot p_1} \quad \text{and} \quad M_u = \frac{u}{2p \cdot p_2}. \] (3.11)

Then the factors of $1/p \cdot p_1$ and $1/p \cdot p_2$ permit a binomial expansion in powers of $q$. The resulting series in $q$ has coefficients which are simply related to $t + u$ and $t - u$. It is easy to check that
\begin{align*}
t + u &= -\left(\frac{4im}{M}\right) \epsilon_{\lambda \epsilon \mu \nu} p_1^\lambda p_2^\mu \epsilon_1^\nu \epsilon_2^\lambda (\xi^\dagger T \eta) \\
&\quad - 2M(A_j z_m - A_m z_j + B_{jm}) (q^m q^n q^p q^j \xi^\dagger T \eta) \\
&\quad + \left(\frac{8}{M + 2m}\right) \delta_{jm} B_{np} (q^m q^n q^p q^j \xi^\dagger T \eta), \quad (3.12)
\end{align*}
\begin{align*}
t - u &= -2M^2(A_j + \frac{1}{2} \epsilon_1 \cdot \epsilon_2 z_j) (\xi^\dagger T \eta) \\
&\quad + \left(\frac{8M}{M + 2m}\right) \delta_{jm} (A_n + \frac{1}{2} \epsilon_1 \cdot \epsilon_2 z_n) (q^m q^n q^j \xi^\dagger T \eta).
\end{align*}

Here $T$ stands either for the identity or a generator in the colour $SU(3)$ space, depending on which part of the colour structure of eq. (3.10) we consider. We have introduced the additional notation
\[ B_{ij} = \epsilon_1 \cdot L_i \epsilon_2 \cdot L_j + \epsilon_2 \cdot L_i \epsilon_1 \cdot L_j. \] (3.13)

Note that the binomial expansion is not the desired Taylor series expansion in $q$, since both $t + u$ and $t - u$ involve $M$, which in turn depends on $q$. However, it is an useful intermediate step, since it allows us to organise the terms neatly.

The full matrix element can be written as
\[ \mathcal{M} = \frac{1}{6} g^2 \delta_{ab} S + \frac{1}{2} g^2 d_{abc} D^c + \frac{i}{2} g^2 f_{abc} F^c. \] (3.14)

The colour amplitudes $S$ and $D$ involve only $M_t + M_u$, whereas $F$ involves $M_s$ as well as $M_t - M_u$. 

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In order to write down our results, we find it convenient to introduce the notation
\[ A = \frac{1}{M^2} \bar{\epsilon}_{\nu\sigma} p^\lambda p^\rho q^\varepsilon \epsilon_1^\mu \epsilon_2^\nu \] and
\[ S_{ij} = A_i \hat{z}_j + A_j \hat{z}_i - B_{ij} + \epsilon_1 \cdot \epsilon_2 \hat{z}_i \hat{z}_j. \] (3.15)

In order to identify all terms to order \( v^9 \) we need the colour amplitude \( S \) to order \( q^5 \)
\[ S = -\left(\frac{8im}{M^2}\right) A (\xi^\dagger \eta) + \left(\frac{4M}{M^3}\right) S_{jm} (q^m \xi^{\dagger \sigma j} \eta) - \left(\frac{32im}{M^2}\right) A m \hat{z}_m (q^m q^n \xi^{\dagger \sigma j} \eta) \]
\[ + \left(\frac{16M}{M^3}\right) [S_{jm} \hat{z}_m \hat{z}_n - \frac{M}{M + 2m} \delta_{jm} S_{np}] (q^m q^n q^p \xi^{\dagger \sigma j} \eta) \]
\[ - \left(\frac{128im}{M^6}\right) A m \hat{z}_m \hat{z}_p \hat{z}_r (q^m q^n q^p \xi^{\dagger \sigma j} \eta) \]
\[ + \left(\frac{64}{M^5}\right) [S_{jm} \hat{z}_m \hat{z}_p - \frac{M}{M + 2m} \delta_{jm} S_{np}] z_r z_s (q^m q^n q^p q^r \xi^{\dagger \sigma j} \eta) \] (3.16)

To all orders, even powers of \( q \) come with the tensor \( A \) and odd powers with \( S \). The amplitude \( D \) differs only through having colour octet matrix elements in place of the colour singlet ones shown above. For the colour amplitude \( F \) we need the expansion
\[ F^c = -\left(\frac{16im}{M^2}\right) A \hat{z}_m (q^m \xi^{\dagger T^c} \eta) + \left(\frac{8}{M^2}\right) S_{jm} \hat{z}_m (q^m q^n \xi^{\dagger T^c} \eta) \] (3.17)

In this amplitude odd powers of \( q \) come with the tensor \( A \) and even powers with \( S \). In all three colour amplitudes, the terms in \( A \) are spin singlet and those in \( S \) are spin triplet.

The decomposition into spherical tensors can be performed partially at this stage by using the identities
\[ \hat{z}_m q^m = [q]_0^1 \] and \[ \hat{z}_m \hat{z}_n q^m q^n = \sqrt{2/3} [qq]_0^2 - \sqrt{1/3} [qq]_0^0. \] (3.18)
The Euclidean indices on \( S \) are most conveniently converted after squaring the matrix element.

A recurrence relation for the \( i \)-th term, \( t_i \), in either of eqs. (3.16) or (3.17) is easy to write. We find that
\[ t_i = \frac{4}{M^2} \hat{z}_m q^m q^n t_{i-2}. \] (3.19)
This holds for all $i > 3$ in the $S$ and $D$ amplitudes and $i > 4$ for the $F$ amplitude. Also, for the $F$ amplitude this holds for $i = 3$. The $i = 4$ term in $F$ is $(2/M)\hat{z}_nq^n$ times the $i = 4$ term in $D$. The required Taylor series expansion is then obtained by expanding all factors containing $M$. This procedure is completely systematic and may be performed, for example, by a Mathematica program.

### 3.3 $\gamma g \to \bar{Q}Q$ and $\gamma\gamma \to \bar{Q}Q$

The matrix elements for the two processes $\gamma p \to \bar{Q}Q$ and $\gamma\gamma \to \bar{Q}Q$ are closely related to the $gg$ amplitudes. It is easy to check that

$$M_{\gamma g} = geD, \quad \text{and} \quad M_{\gamma\gamma} = e^2S,$$

(3.20)

where $D$ and $S$ are the colour amplitudes given in eq. (3.14), and $e$ is the charge of the heavy quark.

### 4 The Cross Sections

#### 4.1 $\bar{q}q \to \bar{Q}Q$

The squared matrix element for this process is easy to write down. After summing over initial state helicities, the amplitude square can be expressed in terms of matrix elements, over heavy-quark spinors, of products of $\sigma$ and $q$. At this stage a perturbative projector (eq. (2.8)) is introduced between the spinor bilinears in order to project on to $\bar{Q}Q$ final states. The normalisation of these states involve the energies of the quarks (see eq. (2.9)), and can be expanded in $q^2$, as shown in eq. (2.10). The computation is complete once this expansion is performed and the extra factors of $q$ arising from this appropriately absorbed into the matrix elements. The non-perturbative matrix elements needed for the cross sections of various charmonium states are listed in Table I.
Table 1: The matrix elements from the $\bar{q}q$ process contributing to the cross section for all charmonium states $H$ at order $v^d$. We use $h_c$ as shorthand for the $^1P_1$ meson.

| H     | d | Matrix Elements                      |
|-------|---|--------------------------------------|
| $\eta_c$ | 7 | $\mathcal{O}_8\left(^3S_1^0\right)$ |
| $h_c$     | 9 | $\mathcal{O}_8\left(^3S_1^0\right)$ |
| $J/\psi$ | 7 | $\mathcal{O}_8\left(^3S_1^0\right)$ |
|          | 9 | $\mathcal{P}_8\left(^3S_1^0;\,^3S_1^2\right), \mathcal{O}_8\left(^3P_1^2\right)$ |
| $\chi J$ | 5 | $\mathcal{O}_8\left(^3S_1^0\right)$ |
|          | 7 | $\mathcal{P}_8\left(^3S_1^0;\,^3S_1^2\right)$ |
|          | 9 | $\mathcal{O}_8\left(^3S_1^2\right), \mathcal{P}_8\left(^3S_1^0;\,^3S_1^4\right), \mathcal{O}_8\left(^3D_1^2\right)$ |
Finally, we list the parton level cross sections—

\[
\hat{\sigma}_{\bar{q}q}^{\eta_c} = \frac{\pi^3 \alpha_s^2}{54 m^4} \delta(\hat{s} - 4 m^2) \mathcal{O}_{8}^{\eta_c}(3 S_1^0)
\]

\[
\hat{\sigma}_{\bar{q}q}^{h_c} = \frac{\pi^3 \alpha_s^2}{54 m^4} \delta(\hat{s} - 4 m^2) \mathcal{O}_{8}^{h_c}(3 S_1^0)
\]

\[
\hat{\sigma}_{\bar{q}q}^{J/\psi} = \frac{\pi^3 \alpha_s^2}{54 m^4} \delta(\hat{s} - 4 m^2) \left[ \mathcal{O}_{8}^{J/\psi}(3 S_1^0) + \frac{1}{m^2} \left\{ \frac{2}{\sqrt{3}} \mathcal{P}_{8}^{J/\psi}(3 S_1^0, 3 S_2^0) + \frac{1}{4} \mathcal{O}_{8}^{J/\psi}(3 P_2) \right\} \right]
\]

\[
\hat{\sigma}_{\bar{q}q}^{\chi J} = \frac{\pi^3 \alpha_s^2}{54 m^4} \delta(\hat{s} - 4 m^2) \left[ \mathcal{O}_{8}^{\chi J}(3 S_1^0) + \frac{2}{\sqrt{3} m^2} \mathcal{P}_{8}^{\chi J}(3 S_1^0, 3 S_2^0) + \frac{1}{m^4} \left\{ \frac{4}{3} \mathcal{O}_{8}^{\chi J}(3 S_1^0) + \frac{5}{12} \mathcal{O}_{8}^{\chi J}(3 D_2^0) + \frac{7 \sqrt{5}}{12} \mathcal{P}_{8}^{\chi J}(3 S_1^0, 3 S_1^0) \right\} \right]
\]

(4.1)

See the appendix for details of the angular momentum coupling scheme used in this paper.

Note that in any application to hadronic collisions, the parton level centre of mass energy will be \( \hat{s} = x_1 x_2 s \), where \( s \) is the CM energy of the hadrons and \( x_1 \) and \( x_2 \) are the momentum fractions of the two partons. The contribution of this sub-process to the hadronic cross section is then obtained by convoluting the above cross sections with the appropriate parton density functions.

### 4.2 \( gg \rightarrow \bar{Q}Q \)

The squared matrix element for the \( gg \) process is more complicated, but the extraction of the cross section follows exactly the same steps as for the \( \bar{q}q \) process. Denoting the average over initial states of the product \( S^* \) by \( S \cdot S^* \), we find

\[
S \cdot S^* = \sum_{pol} S_{jm} S^*_{jm'} = \sum_{\lambda = \pm 2} \left[ \sigma q_\lambda [\sigma^\dagger q_\lambda]^0 \right]^2 + \frac{3}{2} \left[ \sigma q_0 [\sigma^\dagger q_0]^0 \right]^2.
\]

(4.2)

Although \( \sigma \) and \( q \) are self-adjoint, we have retained the more cumbersome notation in order to clarify the coupling of the different angular momenta. Also, \( A \cdot A^* = 1/8 \) and \( A \cdot S^* = 0 \). Consequently, even and odd terms in the three colour amplitudes of eq. (3.14) do not interfere with each other.
| H    | d | Colour amplitude                                      |
|------|---|-------------------------------------------------------|
|      |   | S          | D          | F          |
|      |   |            |            |            |
| $\eta_c$ | 3 | $\mathcal{O}_1 (^{1}S_0^0)$                         |            |            |
|      | 5 | $\mathcal{P}_1 (^{1}S_0^0, ^{1}S_2^0)$              |            |            |
|      | 7 | $\mathcal{O}_1 (^{1}S_0^2), \mathcal{P}_1 (^{1}S_0^0, ^{1}S_4^0)$ | $\mathcal{O}_8 (^{1}S_0^0)$ | $\mathcal{O}_8 (^{1}P_1^1)$ |
| $h_c$ | 5 | $\mathcal{O}_8 (^{1}S_0^0)$                         | $\mathcal{P}_8 (^{1}S_0^0, ^{1}S_2^0)$ |            |
|      | 7 | $\mathcal{O}_1 (^{1}S_0^0)$                         | $\mathcal{O}_8 (^{1}S_0^2), \mathcal{O}_8 (^{3}P_j^1)$, $\mathcal{O}_8 (^{1}D_2^2), \mathcal{P}_8 (^{1}S_0^0, ^{1}S_4^0)$ | $\mathcal{O}_8 (^{1}P_1^1)$ |
|      | 9 | $\mathcal{O}_1 (^{1}S_0^0)$                         | $\mathcal{O}_8 (^{1}S_0^2), \mathcal{O}_8 (^{3}P_j^1)$, $\mathcal{O}_8 (^{1}D_2^2), \mathcal{P}_8 (^{1}S_0^0, ^{1}S_4^0)$ | $\mathcal{O}_8 (^{1}P_1^1)$ |
| $J/\psi$ | 7 | $\mathcal{O}_8 (^{1}S_0^0), \mathcal{O}_8 (^{3}P_j^1)$ | $\mathcal{P}_8 (^{1}S_0^0, ^{1}S_2^0), \mathcal{P}_8 (^{3}P_j^1, ^{3}P_j^3)$ | $\mathcal{O}_8 (^{3}P_j^2)$ |
|      | 9 | $\mathcal{P}_8 (^{1}S_0^0, ^{1}S_2^0), \mathcal{P}_8 (^{3}P_j^1, ^{3}P_j^3)$ | $\mathcal{O}_8 (^{3}P_j^2)$ |            |

Table 2: The matrix elements contributing to the cross section for some charmonium states $H$ at order $v^d$. We use $h_c$ as shorthand for the $^{1}P_1$ meson. $J = 0, 2$ and $J' = 1, 2$.
| H  | d | Colour amplitude                        |
|----|----|-----------------------------------------|
|    |    | S            | D            | F            |
| $\chi_0$ | 5  | $\mathcal{O}_1(\ ^3P^1_0)$  |              |              |
|       | 7  | $\mathcal{P}_1(\ ^3P^1_0, \ ^3P^3_0)$ |              |              |
|       | 9  | $\mathcal{O}_1(\ ^3P^1_{0,2})$, $\mathcal{O}_1(\ ^3P^3_0)$, $\mathcal{P}_1(\ ^3P^1_0, \ ^3P^5_0)$ | $\mathcal{O}_8(\ ^1S^0_0)$, $\mathcal{O}_8(\ ^3P^1_{0,2})$ | $\mathcal{O}_8(\ ^1P^1_1)$, $\mathcal{O}_8(\ ^3S^2_1)$, $\mathcal{O}_8(\ ^3D^2_1)$ |
| $\chi_1$ | 9  | $\mathcal{O}_1(\ ^3P^1_{0,2})$ | $\mathcal{O}_8(\ ^1S^0_0)$, $\mathcal{O}_8(\ ^3P^1_{0,2})$ | $\mathcal{O}_8(\ ^1P^1_1)$, $\mathcal{O}_8(\ ^3S^2_1)$, $\mathcal{O}_8(\ ^3D^2_1)$ |
| $\chi_2$ | 5  | $\mathcal{O}_1(\ ^3P^1_2)$ |              |              |
|       | 7  | $\mathcal{P}_1(\ ^3P^1_2, \ ^3P^3_2)$ |              |              |
|       | 9  | $\mathcal{O}_1(\ ^3P^1_0)$, $\mathcal{O}_1(\ ^3P^3_2)$, $\mathcal{P}_1(\ ^3P^1_2, \ ^3P^5_2)$ | $\mathcal{O}_8(\ ^1S^0_0)$, $\mathcal{O}_8(\ ^3P^1_{0,2})$ | $\mathcal{O}_8(\ ^1P^1_1)$, $\mathcal{O}_8(\ ^3S^2_1)$, $\mathcal{O}_8(\ ^3D^2_{1,2,3})$ |

Table 3: The matrix elements contributing to the cross section for some charmonium states $H$ at order $v^d$. 

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In Tables 2 and 3 we list all the matrix elements which appear in charmonium cross sections to order $v^9$. The final parton level cross sections are listed in the following subsections. The coefficients in the linear combinations of matrix elements appearing there depend on the angular momentum coupling scheme. Our conventions are set out in the appendix. The contributions of these processes to hadronic cross sections are obtained by convoluting the sub-process cross sections with appropriate parton densities.

4.3 Direct $J/\psi$ cross section

The direct $J/\psi$ subprocess cross section is

$$
\hat{\sigma}_{gg}^{J/\psi}(s) = \varphi \left[ \frac{5}{48} \Theta_D^{J/\psi}(7) + \left\{ \frac{5}{48} \Theta_D^{J/\psi}(9) + \frac{3}{16} \Theta_F^{J/\psi}(9) \right\} \right]
$$

(4.3)

where

$$
\varphi = \frac{\pi^3 \alpha^2}{4 m^2} \delta(s - 4m^2),
$$

(4.4)

and $\Theta_a^{J/\psi}(n)$ denotes combinations of non-perturbative matrix elements from the colour amplitude $a$ (= $S$, $D$ or $F$) at order $v^n$. Using the notation explained in the appendix, these can be written as

$$
\Theta_D^{J/\psi}(7) = \frac{1}{2m^2} O_8^{J/\psi}(1S_0^0) + \frac{1}{2m^4} \left[ 3 O_8^{J/\psi}(3P_0^1) + \frac{4}{5} O_8^{J/\psi}(3P_1^2) \right]
$$

$$
\Theta_D^{J/\psi}(9) = \frac{1}{\sqrt{3}m^4} P_8^{J/\psi}(1S_0^0, 1S_2^0) + \frac{1}{\sqrt{15}m^6} \left[ \frac{35}{4} P_8^{J/\psi}(3P_0^1, 3P_2^0) \right.
$$

$$
\left. + 2 P_8^{J/\psi}(3P_1^3, 3P_2^3) \right]
$$

$$
\Theta_F^{J/\psi}(9) = \frac{1}{2m^6} \left[ \frac{1}{3} O_8^{J/\psi}(3P_1^2) - \frac{2}{5} O_8^{J/\psi}(3P_2^2) \right]
$$

(4.5)

Previous computations have considered the expansion only to order $v^7$. Heavy-quark spin symmetry gives the relation $O_8^{J/\psi}(3P_2^1) = 5O_8^{J/\psi}(3P_0^1)$, up to corrections of order $v^2$. Then at order $v^7$ accuracy this can be used for a further reduction of $\theta_D^{J/\psi}(7)$. Since we consider the expansion to order $v^9$, we cannot use this relation.

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4.4 $\chi_0$ cross section

The $\chi_0$ subprocess cross section is

$$\hat{\sigma}_{99}^{\chi_0}(\hat{s}) = \varphi \left[ \frac{1}{18} \Theta_S^{\chi_0}(5) + \frac{1}{18} \Theta_S^{\chi_0}(7) + \left\{ \frac{1}{18} \Theta_S^{\chi_0}(9) + \frac{5}{48} \Theta_D^{\chi_0}(9) + \frac{3}{16} \Theta_F^{\chi_0}(9) \right\} \right]. \quad (4.6)$$

The results to order $v^5$ are known from previous computations. This work identifies all the combinations of matrix elements at orders $v^7$ and $v^9$. Note that the order $v^7$ term involves an off-diagonal operator (i.e., a $P$ term), and can be found only after expanding the colour amplitude $S$ to third order in $q$.

Using the notation explained in the appendix, the combinations of non-perturbative matrix elements appearing in eq. (4.6) are

$$\Theta_S^{\chi_0}(5) = \frac{3}{2m^4} O_1^{\chi_0}(3P_1^1)$$
$$\Theta_S^{\chi_0}(7) = \frac{2}{7 \sqrt{5}} \frac{1}{4 \sqrt{3} m^6} P_1^{\chi_0}(3P_1^1, 3P_3^3)$$
$$\Theta_S^{\chi_0}(9) = \frac{2}{5m^4} O_1^{\chi_0}(3P_2^1) + \frac{1}{8m^8} \left[ \frac{245}{9} O_1^{\chi_0}(3P_0^3) + \frac{149 \sqrt{7}}{10 \sqrt{3}} P_1^{\chi_0}(3P_0^1, 3P_0^5) \right]$$
$$\Theta_D^{\chi_0}(9) = \frac{1}{2m^2} O_8^{\chi_0}(1S_0^0) + \frac{1}{2m^4} \left[ 3O_8^{\chi_0}(3P_0^1) + \frac{4}{5} O_8^{\chi_0}(3P_2^1) \right]$$
$$\Theta_F^{\chi_0}(9) = \frac{1}{6m^4} O_8^{\chi_0}(1P_1^1) + \frac{1}{18m^6} \left[ O_8^{\chi_0}(3S_1^2) + 5O_8^{\chi_0}(3D_1^2) \right]. \quad (4.7)$$

4.5 $\chi_1$ cross section

$$\hat{\sigma}_{99}^{\chi_1}(\hat{s}) = \varphi \left[ \frac{1}{18} \Theta_S^{\chi_1}(9) + \frac{5}{48} \Theta_D^{\chi_1}(9) + \frac{3}{16} \Theta_F^{\chi_1}(9) \right] \quad (4.8)$$

where the combinations of non-perturbative matrix elements are, in the no-
notation explained in the appendix,

\[
\Theta_{S}^{\chi_{1}}(9) = \frac{1}{2m^{4}} \left[ 3\mathcal{O}_{1}^{\chi_{1}}(3P_{0}^{1}) + \frac{4}{5}\mathcal{O}_{1}^{\chi_{1}}(3P_{2}^{1}) \right]
\]

\[
\Theta_{D}^{\chi_{1}}(9) = \frac{1}{2m^{2}} \mathcal{O}_{8}^{\chi_{1}}(1S_{0}^{0}) + \frac{1}{2m^{4}} \left[ 3\mathcal{O}_{8}^{\chi_{1}}(3P_{1}^{1}) + \frac{4}{5}\mathcal{O}_{8}^{\chi_{1}}(3P_{2}^{1}) \right]
\]

\[
\Theta_{F}^{\chi_{1}}(9) = \frac{1}{6m^{4}} \mathcal{O}_{8}^{\chi_{1}}(1P_{1}^{1}) + \frac{1}{3m^{6}} \left[ \frac{1}{6}\mathcal{O}_{8}^{\chi_{1}}(3S_{2}^{0}) + \frac{5}{6}\mathcal{O}_{8}^{\chi_{1}}(3D_{1}^{2}) - \frac{1}{5}\mathcal{O}_{8}^{\chi_{1}}(3D_{2}^{2}) \right].
\]

(4.9)

The \( \chi_{1} \) is produced first at order \( v^{9} \). The large branching ratio for the decay \( \chi_{1} \to J/\psi \) makes this a phenomenologically important term, and is the main motivation for this work.

### 4.6 \( \chi_{2} \) cross section

\[
\hat{\sigma}_{gg}(\hat{s}) = \varphi \left[ \frac{1}{18} \Theta_{S}^{\chi_{2}}(5) + \frac{1}{18} \Theta_{S}^{\chi_{2}}(7)
\right.
\]

\[
+ \left. \left\{ \frac{1}{18} \Theta_{S}^{\chi_{2}}(9) + \frac{5}{48} \Theta_{D}^{\chi_{2}}(9) + \frac{3}{16} \Theta_{F}^{\chi_{2}}(9) \right\} \right]
\]

(4.10)

where the combinations of non-perturbative matrix elements are

\[
\Theta_{S}^{\chi_{2}}(5) = \frac{2}{5m^{4}} \mathcal{O}_{1}^{\chi_{2}}(3P_{2}^{1})
\]

\[
\Theta_{S}^{\chi_{2}}(7) = \frac{2}{\sqrt{15m^{6}}} \mathcal{P}_{1}^{\chi_{2}}(3P_{1}^{1}, 3P_{2}^{2})
\]

\[
\Theta_{S}^{\chi_{2}}(9) = \frac{3}{2m^{4}} \mathcal{O}_{1}^{\chi_{2}}(3P_{1}^{1}) + \frac{1}{15m^{8}} \left[ \frac{262}{9} \mathcal{O}_{1}^{\chi_{2}}(3P_{2}^{1}) + \frac{141}{2\sqrt{7}} \mathcal{P}_{1}^{\chi_{2}}(3P_{2}^{1}, 3P_{2}^{2}) \right]
\]

\[
\Theta_{D}^{\chi_{2}}(9) = \frac{1}{2m^{2}} \mathcal{O}_{8}^{\chi_{2}}(1S_{0}^{0}) + \frac{1}{2m^{4}} \left[ 3\mathcal{O}_{8}^{\chi_{2}}(3P_{0}^{1}) + \frac{4}{5}\mathcal{O}_{8}^{\chi_{2}}(3P_{2}^{1}) \right]
\]

\[
\Theta_{F}^{\chi_{2}}(9) = \frac{1}{6m^{4}} \mathcal{O}_{8}^{\chi_{2}}(1P_{1}^{1}) + \frac{1}{3m^{6}} \left[ \frac{1}{6}\mathcal{O}_{8}^{\chi_{2}}(3S_{2}^{0}) + \frac{5}{6}\mathcal{O}_{8}^{\chi_{2}}(3D_{1}^{2})
\right.
\]

\[
- \frac{1}{5}\mathcal{O}_{8}^{\chi_{2}}(3D_{2}^{2}) + \frac{2}{7}\mathcal{O}_{8}^{\chi_{2}}(3D_{3}^{2}) \right].
\]

(4.11)

See the appendix for an explanation of the notation.
4.7 \( \eta_c \) cross section

The production cross section for \( \eta_c \) is—

\[
\hat{\sigma}_{gg}^{\eta_c}(\hat{s}) = \varphi \left[ \frac{1}{18} \Theta_S^{\eta_c}(3) + \frac{1}{18} \Theta_S^{\eta_c}(5) \right.
\]
\[
\left. + \left\{ \frac{1}{18} \Theta_S^{\eta_c}(7) + \frac{5}{48} \Theta_D^{\eta_c}(7) + \frac{3}{16} \Theta_F^{\eta_c}(7) \right\} \right],
\]

(4.12)

where the notation in the appendix can be used to write the combinations of non-perturbative matrix elements as

\[
\Theta_S^{\eta_c}(3) = \frac{1}{2m^2} \mathcal{O}_1^{\eta_c}(1S_0^0)
\]
\[
\Theta_S^{\eta_c}(5) = \frac{1}{\sqrt{3}m^4} \mathcal{P}_1^{\eta_c}(1S_0^0, 1S_2^0)
\]
\[
\Theta_S^{\eta_c}(7) = \frac{1}{3m^6} \left[ 2\mathcal{O}_1^{\eta_c}(1S_2^0) + \frac{4}{\sqrt{5}} \mathcal{P}_1^{\eta_c}(1S_0^0, 1S_4^0) \right]
\]
\[
\Theta_D^{\eta_c}(7) = \frac{1}{2m^2} \mathcal{O}_8^{\eta_c}(1S_0^0)
\]
\[
\Theta_F^{\eta_c}(7) = \frac{1}{6m^4} \mathcal{O}_8^{\eta_c}(1P_1^1)
\]

(4.13)

Note that the colour amplitude \( D \) first enters the cross section at order \( v^7 \). Hence, the almost-elastic cross section \( \gamma p \rightarrow \eta_c \) starts at order \( v^7 \).

4.8 \( h_c \) cross section

The production cross section for the \( 1P_1 \) charmonium state is—

\[
\hat{\sigma}_{gg}^{h_c}(\hat{s}) = \varphi \left[ \frac{5}{48} \Theta_D^{h_c}(5) + \frac{5}{48} \Theta_D^{h_c}(7) \right.
\]
\[
\left. + \left\{ \frac{1}{18} \Theta_S^{h_c}(9) + \frac{5}{48} \Theta_D^{h_c}(9) + \frac{3}{16} \Theta_F^{h_c}(9) \right\} \right],
\]

(4.14)
where the combinations of non-perturbative matrix elements can be written, using the notation in the appendix, as

\[
\Theta_D^{hc}(5) = \frac{1}{2m^2} O^{hc}(1S_0^0)
\]

\[
\Theta_D^{hc}(7) = \frac{1}{\sqrt{3}m^4} P^{hc}(1S_0^0, 1S_0^2)
\]

\[
\Theta_S^{hc}(9) = \frac{1}{2m^2} O^{hc}(1S_0^0)
\]

\[
\Theta_D^{hc}(9) = \frac{1}{3m^6} \left[ 2O^{hc}(1S_0^2) + \frac{4}{\sqrt{3}} P^{hc}(1S_0^0, 1S_0^0) \right] + \frac{1}{2m^4} \left[ 3O^{hc}(3P_0^1) + \frac{4}{5} O^{hc}(3P_2^1) \right] + \frac{1}{15m^6} O^{hc}(1D_2^2)
\]

\[
\Theta_F^{hc}(9) = \frac{1}{6m^4} O^{hc}(1P_1^1).
\]

Interestingly, only the colour amplitude $D$ enters the cross section upto order $v^7$. As a consequence, the cross sections for $pp \to h_c$ and $\gamma p \to h_c$ require the same combinations of matrix elements upto an accuracy of about 10%.

### 4.9 $\gamma g \to \bar{Q}Q$ and $\gamma \gamma \to \bar{Q}Q$

The $\gamma g$ cross sections for the production of any quarkonium state, eqs. (4.3)–(4.14), can be obtained from those for the $gg$ process by the following prescription. Replace $\alpha_s^2$ in $\varphi$ (eq. 4.4) by $\alpha^2$, delete the $\Theta_S$ and $\Theta_F$ terms, and replace the colour factor $5/48$ for the terms in $\Theta_D$ by 2. This follows from eq. (3.20).

Similarly, the $\gamma \gamma$ cross sections are obtained from the prescription—replace $\alpha_s^2$ in $\varphi$ (eq. 1.4) by $\alpha^2$, delete the $\Theta_D$ and $\Theta_F$ terms in eqs. (4.3)–(4.14), and replace the colour factor $1/18$ for the terms in $\Theta_S$ by 16.

### 5 Discussion

The number of non-perturbative matrix elements appearing in the cross sections in Section 4 is rather large. In hadron-hadron collisions the influence of the $\bar{q}q$ amplitudes is small compared to that of the $gg$ amplitudes, since the gluon luminosity is much larger. This still leaves us with the problem of determining a large number of matrix elements.
The matrix elements appearing through the colour amplitude $D$ can be isolated in almost-elastic $\gamma p$ collisions. At HERA it is now possible to separate the diffractive components of the cross section \[12\]. With this separation, it becomes possible to measure these matrix elements with greater accuracy than was possible in the past \[13\].

The matrix elements arising from the colour amplitude $S$ can be measured in $\gamma\gamma$ collisions produced in $e^+e^-$ colliders, provided the photons are point-like. Unfortunately, cross sections for $\gamma\gamma$ collisions at the LEP are likely to be dominated by resolved-photon effects \[14\]. On the other hand, double-tagged events with low hadronic energy at the LEP correspond to the scattering of two highly virtual photons, $\gamma^*\gamma^*$. The cross sections for $\gamma^*\gamma^* \rightarrow J/\psi$ (and other quarkonia) can also be computed in NRQCD. It is easy to see that these cross sections are obtained from the colour singlet part of the amplitudes in eq. (3.11), the only difference being in the denominators of the propagators. Hence, all the matrix elements from the colour amplitude $S$ which appear in Section 4 will also appear in this case, albeit in different linear combinations. Furthermore, the coefficients would depend on the virtuality of each photon, leaving us with the possibility of independent measurements of each of these matrix elements. The full computation for such $\gamma^*\gamma^*$ processes will be presented elsewhere.

The colour amplitude $F$ cannot be separated in any process, since it arises only through $gg$ initial states. The corresponding matrix elements may be obtained from $pp$, $\pi p$, $\bar{p}p$ collisions, as well as in $\gamma\gamma$ collisions where the resolved photon processes dominate.

Although quantitative estimates of low transverse momentum quarkonium cross sections involves the prior analysis of many such experiments, it is possible to make some crude but interesting estimates.

The $gg \rightarrow \eta_c$ cross section (eq. \[4.12\]) involves only the colour amplitude $S$ to next-to-leading order, $v^5$. The colour amplitude $D$ first enters at order $v^7$. As a result, the $\gamma p \rightarrow \eta_c$ cross section must be much less than the $pp \rightarrow \eta_c$ cross section. In contrast, only the colour amplitude $D$ enters the $gg \rightarrow h_c$ cross section upto the next-to-leading order, $v^7$. Thus the $pp \rightarrow h_c$ cross section can be predicted to better than 10% accuracy if the $\gamma p \rightarrow h_c$ cross section is known. If the $^1P_1$ charmonium state is identified definitively \[15\], then this fact could serve as an empirical test of NRQCD factorisation.

Further qualitative arguments depend on a more detailed analysis of the non-perturbative matrix elements. Any such matrix element $O$, with mass
dimension $D$, may be written in QCD as

$$\mathcal{O} = \Lambda^D f \left( \frac{m_f}{\Lambda} \right),$$  \hspace{1cm} (5.1)

where $\Lambda$ is the QCD scale and $m_f$ are the masses of the quarks of flavour $f$. In the chiral limit, the light quark masses may be taken to vanish, and the corresponding arguments of $f$ are then zero. For the matrix elements which appear in quarkonium cross sections, the dependence on the heavy quark mass is factored into the coefficient functions. After such a factorisation, the resulting matrix element for the production of a quarkonium $H$ can be written as

$$\mathcal{O}^{\nu} = \Lambda^D f^{\nu}(v),$$  \hspace{1cm} (5.2)

where $v$ is the dimensionless velocity which organises the NRQCD expansion. In the limit $v \to 0$, $f^{\nu} \to c^{\nu}_H v^d$, where $c^{\nu}_H$ is a constant and $d$ is the power counting dimension obtained, for example, by eq. (2.5). Power counting in NRQCD is reliable if and only if $c^{\nu}_H$ obtained from different operators (but for the same $H$) are of similar orders of magnitude.

Heavy-quark spin symmetry gives rise to further restrictions on these constants, $c^{\nu}_H$. The operator expectation values from different hadrons may be related using this approximate symmetry. Then, for the same operator, the values of the constant obtained with two different hadrons with the same radial wave-function, must be also of similar magnitude, after removing certain Clebsch-Gordan coefficients.

Such arguments may be used to study the convergence of the NRQCD series for various cross sections. In the approximation of eq. (5.2), the coefficient function times the matrix element is a function of two dimensionless variables, $v$ and $z = \Lambda^2/m^2$. With the assumption that all the constants $c^{\nu}_H$ are of similar order, the convergence properties are simple, since $z \ll v^2$. As an example, we take the cross section for $gg \to \eta_c$ (eq. 4.12). After removing all common factors, including dimensional quantities, we find

$$\hat{\sigma}^{\eta_c}_{gg} \sim \left[ \frac{1}{36} v^3 + \frac{5}{96} v^7 + \cdots \right] z^2 + \mathcal{O}(z^4).$$  \hspace{1cm} (5.3)

The first term comes from $\Theta^{5\eta_c}_S(3)$ and the second from $\Theta^{5\eta_c}_D(7)$ in eq. (4.13). The remaining terms are of higher order in $z$, and converge rapidly. The two coefficients in eq. (5.3) are of similar order, and it is quite possible that the
series in $v^2$ may be convergent for small $v$. Nevertheless, it is interesting to note that the main correction to the leading term comes, not from the order $v^5$ terms, but from one of the order $v^7$ terms.

A similar argument may be used to assert that the most important term in the cross section for $gg \rightarrow \chi_J$ is the $\Theta_D(9)$ term (eqs. 4.7, 4.9 and 4.11), since it has the operator with the lowest mass dimension. In that case, heavy-quark spin symmetry may be used to show that

$$\sigma(\chi_0) : \sigma(\chi_1) : \sigma(\chi_2) \approx 1 : 3 : 5,$$

both in $pp$ and almost-elastic $\gamma p$ collisions. The measured value of this ratio in $pp$ and $\pi p$ collisions is $0.62 \pm 0.18 \pm 0.09$ [16]. Corrections to eq. (5.4) would come from the breaking of the approximate heavy-quark symmetry at higher orders in $v$. Consequently, the same ratios should also be observed for the corresponding bottomonium states. Since the coefficients for quarkonia with unequal principal quantum numbers cannot be compared, the prediction of the direct to total $J/\psi$ cross sections is a more detailed dynamical question.

It is interesting to test such a scaling hypothesis against the matrix elements which have already been extracted. Unfortunately, a definitive test cannot yet be made, since the values quoted for the same matrix elements vary widely [17, 18]. We take the examples of the two sets

$$\langle O_H^3 (S_1) \rangle \quad \text{and} \quad \frac{1}{3} \langle O_H^1 (S_0) \rangle + \frac{1}{m^2} \langle O_H^3 (P_0) \rangle,$$

for $J/\psi$, $\psi'$, $\Upsilon(1S)$ and $\Upsilon(2S)$. The two $L = 0$ matrix elements give a common coefficient $c_H = c_1$, and the $L = 1$ matrix element gives a different coefficient $c_2$. For the same $H$, if the scaling ideas are correct, then $(c_1 - c_2)/(c_1 + c_2)$ should be approximately zero. We find that for the $\Upsilon$ states the fitted numbers [17] are within one standard deviation of this result. The values for $\psi'$ [17] are about two standard deviations away, whereas the many different fits for the $J/\psi$ [17, 18] lie between five and two standard deviations from this expectation. Although one is tempted the accept the scaling argument, it turns out that $c_2$ is systematically larger than $c_1$. This clearly calls for an inclusion of higher order effects, both in $\alpha_s$ and in $v$. The latter, specially, may give rise to unforeseen corrections in view of the pattern of coefficient functions seen here.

We would like to thank V. Ravishankar for discussions.
A The Coupling Scheme

We use spherical tensor techniques for the reduction of the matrix elements. Since all these are constructed by multiple couplings of Euclidean 3-vectors, we need the spherical tensor components for any vector $V$—

$$
V_0 = V_z, \quad V_1 = -\frac{1}{\sqrt{2}}(V_x + iV_y), \quad V_{-1} = \frac{1}{\sqrt{2}}(V_x - iV_y).
$$  \hspace{1cm} (A.1)

The labels on the spherical tensor components denote helicity. Couplings of spherical tensors require the usual Clebsch-Gordan coefficients [11]. We denote coupled tensors by the notation $[a, b]_J^M$, where $J$ is the rank of the coupled tensor and $M$ is the helicity.

Next we consider the complete specification of the state $\mathcal{K}_i\langle 0 \rangle$, where $\mathcal{K}_i$ is one of the operators in eq. (2.1). When computing the NRQCD cross section to high orders in $v$, we may have to couple a large number, $N$, of $q$’s, and, possibly, a Pauli matrix. It is possible to write down the Clebsch-Gordan series for a reduction to states of given $J$, $J_z$, $L$, and $S$, keeping explicitly the permutation symmetry of the $q$’s. However, the coupling constants are not listed in the literature, except for $N = 3$. We choose instead to first ignore the permutation symmetry, treat all the $q$’s as distinguishable operators, and use the usual methods of angular momentum recouplings. When this is completed, we explicitly symmetrise the results to get the desired expressions.

The total number of commuting operators required to specify the state in the direct product basis is $2N + 2S$. In the coupled basis we have $N$ labels from each derivative operator, 2 from the total angular momentum $J$ and the helicity $J_z$. If $S \neq 0$ then two more labels, $L$ and $S$, have to be specified. All these total to $N + 2S + 2$. Consequently another $N - 2$ labels have to be given for a complete specification of the state. It is sufficient for this purpose to fix an order of coupling the individual derivatives and the Pauli matrix and specify the tensor rank at each coupling [19].

As an illustration we take the case $S = 1$ and $N = 4$. We specify a coupling order either by the bracketed expression on the left or the binary tree on the right—

$$
\left[ \sigma \left[ [\eta q]^j_1 \left[ [\eta q]^j_2 \right]^j_3 \right]^j_s \right]_{J_z} = \sigma
$$  \hspace{1cm} (A.2)

Each node in the tree denotes a tensor whose angular momentum is to be specified. The root node (at the bottom) is the total angular momentum.
The helicity, $J_z$, is also specified at this node. All the leaf nodes (at the top) correspond to the individual Euclidean vectors being coupled. A circled node represents a Pauli matrix. A simple counting shows that all the required labels for the state are specified by this means.

In the Taylor expansion of the appropriate matrix elements (section 3) there is a natural pairing of derivatives. The factors of $\hat{z}_m \hat{z}_n$ in the recurrence relation eq. (3.19) naturally pair these operators. In addition, $\delta_{mn}$ from $q^2$ factors or the tensor $S_{np}$ in the fourth term of eq. (3.13) also pair derivatives. The tensor $S_{jm}$ pairs a Pauli matrix with a derivative. In addition, there is an unpaired derivative in the $\bar{q}q$ amplitude (eq. 3.6) and the colour amplitude $F$ (eq. 3.17). A natural set of groupings is then given by the following scheme—

1. The spin singlet terms in the colour amplitudes $S$ and $D$ have even number of derivatives. These are grouped as

$$[[qq] \cdots [qq]] = \sigma \cdot \cdots \cdot \sigma$$

(A.3)

2. The spin singlet terms in the colour amplitude $F$ have an odd number of derivatives. Exactly one is unpaired. The rest are first coupled, and this unpaired derivative is coupled at the end—

$$[q[[qq] \cdots [qq]]] = \sigma \cdot \cdots \cdot \sigma$$

(A.4)

3. The spin triplet terms in the colour amplitude $F$ or the $\bar{q}q$ amplitude have an even number of derivatives. Exactly one is unpaired, and one paired with the Pauli matrix. The $[qq]$ pairs are first coupled; the unpaired derivative is coupled to this, and finally the $[\sigma q]$ pair is recoupled to the result using a $6-J$ symbol. A further recoupling with another $6-J$ symbol then joins the two uncoupled derivatives—

$$\left[ \sigma \left[ q \left[ [qq] \cdots [qq] \right] \right] \right] \rightarrow \left[ \sigma \left[ [qq] [qq] \cdots [qq] \right] \right] = \sigma \cdot \cdots \cdot \sigma$$

(A.5)

4. The spin triplet terms in the colour amplitudes $S$ and $D$ have an odd number of derivatives. One is paired with a Pauli matrix. The rest of the derivatives are first coupled as before, and the remaining derivative is then recoupled to this. The coupling scheme is

$$\left[ \sigma \left[ q \left[ [qq] \cdots [qq] \right] \right] \right] = \sigma \cdot \cdots \cdot \sigma$$

(A.6)
In this work $N \leq 5$, and consequently, a choice of the order of couplings of more than two pairs $[qq]$ is not required. At higher orders further choices have to be made. Except at low orders ($N \leq 2$), the choice of scheme is not unique. Different coupling schemes give rise to different sets of basis states for $K_i|0\rangle$. Unitary transformations can always be found to relate these sets to each other. Although the linear combinations of matrix elements appearing in the cross sections then look different, they can be transformed into each other.

A possible phase ambiguity might remain in the definition of the states because the coupling orders $[a,b]^j$ and $[b,a]^j$ differ by the phase $(-1)^{j_a+j_b-j}$. Enumeration of all the cases arising in this problem shows that there is no such ambiguity—

1. Since $[qq]^j$ has only $j = 0$ and 2, the phase is 1, and no ordering ambiguity exists.

2. For $[[qq]^k][qq]^0]_\lambda^j$, if $\lambda = 0$, then we must have $k + j + l$ even. For integer $k$ and $j$ the phase is then 1, and no ambiguity exists. When $\lambda = \pm 2$, the phase is 1 if $j = 0$. If $j \neq 0$, then the phase is not identically 1. However, in this case we always have a sum over helicities with multiplicative factors which are independent of $\lambda$. This projects out the terms with $k + j + l$ even, and removes the ambiguity.

3. For $[[q]_0^l][qq] \cdots [qq]]_\lambda^j$, the ordering is immaterial if $\lambda = 0$, since $1+l+j$ is then even. When $\lambda = \pm 2$, the sum over $\lambda$ removes the ambiguity.

4. For $[\sigma q]^j$, since $j = 0$ or 2, the phase is 1.

5. For the final $L - S$ coupling, we always choose the Pauli matrix as the first factor.

The binary tree is thus a complete and unambiguous specification of the coupling scheme for our purposes.

After the recouplings are completed, we recognise that the tensors are symmetric under interchange of all $q$’s. A symmetric tensor with $N$ 3-d Euclidean indices reduces under the rotation group. The allowed angular momenta are $L = N$, $N - 2$, etc, down to 0 or 1, depending on whether $N$ is even or odd. Simply counting the number of components in the tensor shows that each allowed value of $L$ appears only once. As a result, it is sufficient
to label the symmetric states by $L$ and $N$ instead of all the intermediate angular momenta.

The unitary transformation between the scheme specified by all the intermediate angular momenta and the symmetric tensors can be carried out explicitly. We find that—

\[
\begin{align*}
[\sigma [qq]^{0}]_{M}^{J} & = \left(\frac{\sqrt{5}}{3}\right) 3P_{J}^{3} \\
[[qq]^{0}[qq]^{0}]_{0} & = \left(\frac{\sqrt{5}}{3}\right) 1S_{0}^{4}
\end{align*}
\]

\[
\begin{align*}
[\sigma [qq]^{1}]_{M}^{J} & = \left(\frac{2}{3}\right) 3P_{J}^{3} \\
[[qq]^{2}[qq]^{2}]_{0} & = \left(\frac{2}{3}\right) 1S_{0}^{4}
\end{align*}
\]

\[
\begin{align*}
[\sigma [qq]^{0}[qq]^{0}]_{M}^{J} & = \left(\frac{1}{3}\right) 3P_{J}^{5} \\
[\sigma [qq]^{2}[qq]^{2}]_{M}^{J} & = \left(\frac{2}{3}\right) 3P_{J}^{5}
\end{align*}
\]

\[
\begin{align*}
[\sigma [qq]^{0}[qq]^{2}]_{M}^{J} & = \left(\frac{2}{3}\right) 3P_{J}^{5} \\
[\sigma [qq]^{2}[qq]^{2}]_{M}^{J} & = \left(\frac{4}{3}\right) 3P_{J}^{5}
\end{align*}
\]

We specify the state $\mathcal{K}_{i}|0\rangle$ by a small extension of the spectroscopic notation. In most cases we will write the state, or equivalently, the operator, as $2^{S+1}L_{J}^{N}$, where $N$ gives the number of derivatives. This allows us to use the new index $N$ for power counting (see eq. 2.5).

The matrix element $\langle \mathcal{K}_{i}|\Pi(H)|\mathcal{K}_{j}\rangle$ is clearly zero unless common quantum numbers of the two states $\mathcal{K}_{i}|0\rangle$ and $\mathcal{K}_{j}|0\rangle$ agree. Thus $L$, $S$, $J$ and $J_{z}$ must be the same. Subject to these constraints, no quantum number reason prevents a non-zero matrix element for bilinears with unequal $N$. Hence we retain such off-diagonal operators in our computations.
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