AN INTEGRAL EQUATION INVOLVING BESSEL POTENTIALS ON HALF SPACE

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Abstract. In this article we consider the following integral equation involving Bessel potentials on a half space \( \mathbb{R}^n_+ \):

\[
 u(x) = \int_{\mathbb{R}^n_+} \left\{ g_\alpha(x - y) - g_\alpha(\bar{x} - y) \right\} u^\beta(y)dy, \quad x \in \mathbb{R}^n_+, 
\]

where \( \alpha > 0, \beta > 1 \), \( \bar{x} \) is the reflection of \( x \) about \( x_n = 0 \), and \( g_\alpha(x) \) denotes the Bessel kernel. We first enhance the regularity of positive solutions for the integral equation by regularity-lifting-method, which has been extensively used by many authors. Then, employing the method of moving planes in integral forms, we demonstrate that there is no positive solution for the integral equation.

1. Introduction. As we know, a Liouville type theorem has been widely applied to derive a priori estimate for the solutions of boundary value problems either on bounded domains in Euclidean space or on Riemannian manifolds with boundaries. Similar Liouville type theorems can also be established on the half space. For instance, Chen, Fang and Li ([6]) recently investigated the following integral equation on the upper half space \( \mathbb{R}^n_+ \):

\[
 u(x) = c_n \int_{\mathbb{R}^n_+} \left( \frac{1}{|x - y|^{n-2m}} - \frac{1}{|\bar{x} - y|^{n-2m}} \right) u^p(y)dy, \quad (1)
\]

where \( p > 1 \) and \( m \) is a positive integer with \( 1 < m < \frac{n}{2} \). They showed that, under almost no assumptions on \( u \), the integral equation (1) is equivalent to the poly-harmonic semi-linear equation with Navier boundary conditions on the half-space \( \mathbb{R}^n_+ \):

\[
 \begin{cases} 
 (-\Delta)^m u = u^p, \quad u > 0 & \text{in } \mathbb{R}^n_+, \n u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial \mathbb{R}^n_+. 
\end{cases} \quad (2)
\]

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Especially, by using the equivalence result they obtained the following Liouville type theorem.

**Proposition 1 (Chen-Fang-Li).** Let \( \frac{n}{n-2m} < p < \infty \). If \( u \in L^{\frac{n(p-1)}{n}}(\mathbb{R}^n_{+}) \) is a nonnegative classical solution of PDE (2), then \( u \) is identically equal to zero.

Under local integrability condition on the solution, the authors ([4]) applied the method of moving planes in integral forms to demonstrate the non-existence of equation with Navier boundary conditions in \( \mathbb{R}^n \)_book [35]. Closely related to the integral equation (3) is the higher-order differential equation involving Bessel potentials on the half space \( \mathbb{R}^n_+ \) (Cao-Chen). Proposition 1

**Proposition 2 (Cao-Chen).** Let \( \frac{n}{n-\alpha} < p \leq \frac{n+\alpha}{n-\alpha} \). If \( u \in L^{\frac{n(p-1)}{n}}(\mathbb{R}^n_{+}) \) is a non-negative solution of integral equation (1), then \( u \equiv 0 \). Here \( 0 < \alpha < n \) if \( n > 3 \), \( 1 < \alpha < n \) if \( n = 3 \).

Motivated by the works mentioned above, in the present paper we study the integral equation involving Bessel potentials on the half space \( \mathbb{R}^n_+ \),

\[
\begin{align*}
   u(x) = \int_{\mathbb{R}^n_+} \{g_\alpha(x-y) - g_\alpha(\bar{x}-y)\}u_\beta(y)dy, \quad x \in \mathbb{R}^n_+, \tag{3}
\end{align*}
\]

where \( \alpha > 0, \beta > 1 \), \( \bar{x} = (x',-x_n) \) is the reflection of \( x = (x',x_n) \) about the hyperplane \( x_n = 0 \), and \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \), and Bessel kernel

\[
g_\alpha(x) = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_0^\infty \exp\left\{ -\frac{|x|^2}{t} - \frac{t}{4\pi} \right\} \frac{dt}{t^{(n-\alpha)/2+1}}.
\]

The reader interested in properties of Bessel potentials is referred to Stein’s excellent book [35]. Closely related to the integral equation (3) is the higher-order differential equation with Navier boundary conditions in \( \mathbb{R}^n_+ \),

\[
\begin{align*}
   &\left\{ (I - \Delta)^\frac{\alpha}{2} u = u^\beta \quad \text{in} \quad \mathbb{R}^n_+, \\
   &u = (-\Delta)u = \cdots = (-\Delta)^{\frac{\alpha}{2}-1}u = 0 \quad \text{on} \quad \partial \mathbb{R}^n_+. \tag{4}
\end{align*}
\]

The corresponding works that are similar to ours, but in the whole Euclidean space \( \mathbb{R}^n \), can be found in [28, 22]. Ma and Chen ([28]) dealt with the integral equation

\[
u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y)u_\beta(y)dy, \quad x \in \mathbb{R}^n. \tag{5}
\]

They established that all the positive solutions of (5) are radially symmetric and monotone decreasing about some point. The regularity of positive solutions to (5) were investigated by Han and Lu ([22]). It should be emphasized that under some appropriate decay conditions of the solutions at infinity, (5) is equivalent to the following semi-linear partial differential equation

\[
(I - \Delta)^\frac{\alpha}{2} u = u^\beta, \quad x \in \mathbb{R}^n,
\]

which has been studied in several earlier papers. In particular, when \( \alpha = 2, \beta = 3 \), and \( n = 3 \), the uniqueness of the positive solution was a well-known result of Coffman [3], which was extended by Mcleod and Serrin [33] to general \( n \) and all values of \( \beta \) below a certain bound depending on \( n \). Later, for \( \alpha = 2 \) and \( \beta \) up to the critical exponent \( \frac{2(n+2)}{n-2} \), Kwong [23] also established the uniqueness of the positive solution in a bounded or unbounded annular region.

In this paper we investigate the qualitative properties of positive solutions to integral equation (3). Our first objective is the following:
Theorem 1.1. Assume that \( u(x) \) is a positive smooth solution of integral equation (3). Then \( u(x) \) satisfies PDE (4).

Then we apply the regularity lifting by contracting operators to boost the positive solutions for integral equation (3) from \( L^q \) to \( L^\infty \).

Theorem 1.2. For any given \( q > \max\{\beta, \frac{n(\beta-1)}{\alpha}\} \), if \( u(x) \in L^q(\mathbb{R}^n_+) \) is a positive solution of integral equation (3), then \( u(x) \) belongs to \( L^\infty(\mathbb{R}^n_+) \).

We further use the regularity lifting by combination of contracting and shrinking operators to arrive at Lipschitz continuity.

Theorem 1.3. Let \( q, \alpha, \beta, \) and \( u \) be as in Theorem 1.2, and assume that \( \alpha > 1 \). Then the solution \( u(x) \) of integral equation (3) belongs to \( C^{0,1}(\mathbb{R}^n_+) \).

Next, by using the method of moving planes in integral forms we show that the positive solution of integral equation (3) is monotone increasing with respect to the variable \( x_n \).

Theorem 1.4. Assume that \( q > \max\{\beta, \frac{n(\beta-1)}{\alpha}\} \), and let \( u(x) \in L^q(\mathbb{R}^n_+) \) be a positive solution of (3). Then \( u(x) \) is strictly monotone increasing with respect to the variable \( x_n \).

Based upon Theorem 1.4 we present nonexistence of positive solutions to integral equation (3), and hence derive a new lieouville-type theorem on \( \mathbb{R}^n_+ \).

Theorem 1.5. Assume that \( q > \max\{\beta, \frac{n(\beta-1)}{\alpha}\} \). If the nonnegative solution \( u(x) \) of (3) satisfies \( u(x) \in L^q(\mathbb{R}^n_+) \), then \( u(x) \equiv 0 \).

It should be noticed that Liouville type results strongly rely on the type of boundary conditions considered. There are many references about the Liouville type results for different boundary conditions. For example, the Dirichlet Boundary condition [19, 20, 34]; the nonlinear Neumann boundary condition [2, 27]; the mixed boundary condition [17]; the Navier boundary conditions [5, 37].

Finally, we demonstrate the non-existence of positive solution for the integral equation in even more general form

\[
  u(x) = \int_{\mathbb{R}^n_+} \{g_\alpha(x-y) - g_\alpha(\bar{x}-y)\} f(y, u(y)) dy, \quad x \in \mathbb{R}^n_+, \tag{6}
\]

where \( f(x, u) \) is a nonnegative function on the half space \( \mathbb{R}^n_+ \).

Theorem 1.6. Let \( q > \max\{\beta, \frac{n(\beta-1)}{\alpha}\} \). Suppose that \( u(x) \in L^q(\mathbb{R}^n_+) \) is the nonnegative solution of integral equation (6). If \( f(x, u) \) satisfies the following conditions:

(i) \( f(x, u) \) is non-decreasing in the variable \( x_n \) and non-decreasing with respect to \( u \),

(ii) \( \frac{\partial f}{\partial u} \in L^{\frac{n}{\beta-1}}(\mathbb{R}^n_+) \) is non-decreasing with respect to \( u \),

then \( u \) is identically equal to zero.

In order to show Theorems 1.4 and 1.6, we apply the method of moving planes in integral forms, which was initially observed by Chen, Li and Ou ([14]). It is significantly different from the traditional methods of moving planes used for partial differential equations. One outstanding feature of this method is that it works for all real values of \( \alpha \) indiscriminately.
For more related results regarding the method of moving planes and integral
equations, please refer the reader to [5, 7, 8, 10, 11, 12, 13, 15, 16, 24, 25, 26, 29,
30, 31, 32, 36] and the references therein.

The organization of this article is as follows. Section 2 is devoted to the regu-
larity of solutions and prove Theorems 1.2 and 1.3. In section 3, we first establish
two lemmas which are very important in carrying the method of moving plane in
integral forms, then deal with Theorems 1.4, 1.5 and 1.6 consecutively. The proof
of Theorem 1.1 is exhibited in Section 4.

2. Regularity of solutions. The main goal of this section is to show our regularity
results, which need the following several propositions. To begin with we list some
important properties about the Bessel kernel $g_\alpha$; see, for example, [21](Chap. III,
§56) and [38](Chap. II, §6).

Proposition 3. (i) For $1 \leq p \leq \infty$, $\|g_\alpha \ast f\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$.
(ii) For any real number $\alpha > 2$, $(I - \Delta)g_\alpha = g_\alpha - 2$.
(iii) $(I - \Delta)g_\alpha(x - y) = \delta(x - y)$ for any $x, y \in \mathbb{R}^n$, where $\delta(x)$ is the famous Dirac’s
delta function.

The following proposition displays some fundamental facts of Bessel kernel that
it possesses the essential behavior of Riesz kernel near zero but exponential decay
at infinity.

Proposition 4. Given $\alpha > 0$, the Bessel kernel $g_\alpha$ is a smooth function on $\mathbb{R}^n \setminus \{0\}$
that satisfies $g_\alpha(x) > 0$ for all $x \in \mathbb{R}^n$. Moreover, there exist positive finite constants $C_\alpha, c_\alpha$
such that

\[
g_\alpha(x) \sim C_\alpha |x|^{(\alpha - n - 1)/2} e^{-|x|/2} \quad \text{as} \quad |x| \to \infty,
\]

and such that

\[
g_\alpha(x) \sim c_\alpha h_\alpha(x) \quad \text{as} \quad |x| \to 0,
\]

where $h_\alpha$ is equal to

\[
h_\alpha(x) = \begin{cases} |x|^{\alpha - n}, & 0 < \alpha < n, \\ \log \frac{|x|}{\alpha}, & \alpha = n, \\ 1, & \alpha > n. \end{cases}
\]

Proof. The estimate (7) can be found in [18], and the other results follow directly
from [21, Proposition 6.1.5].

To demonstrate Theorems 1.2 and 1.3 we require the next results of Regularity
Lifting from Chen and Li ([9]), which play a crucial role in our proof.

Assume $V$ is a Hausdorff topological vector space with two extended norms,

\[
\| \cdot \|_X, \| \cdot \|_Y : V \to [0, \infty].
\]

Let $X := \{ v \in V : \| v \|_X < \infty \}$ and $Y := \{ v \in V : \| v \|_Y < \infty \}$. We also suppose
that the topology in $V$ is weaker than the topology of $X$ and the weak topology $Y$.
A map $F$ from $X$ into itself is called a contracting map if there exists a positive
number $\theta < 1$ such that

\[
\|Fg - Fh\|_X \leq \theta \|g - h\|_X \quad \text{for all} \quad g, h \in X.
\]

And $F$ from $X$ into itself is called a shrinking map if there exists a number $\eta < 1$
such that

\[
\|Fg\|_X \leq \eta \|g\|_X \quad \text{for all} \quad g \in X.
\]
It is easy to see that, if $\mathcal{F}$ is a linear operator, then these two conditions above are equivalent.

**Proposition 5** (Regularity Lifting I). Assume $X$ and $Y$ are Banach spaces, and that $\mathcal{F}$ is a contracting map from $X$ into itself and from $Y$ into itself. If $g \in X$ and there exists $h \in X \cap Y$ such that $g = Fg + h$ in $X$, then $g \in X \cap Y$.

In this paper we choose $V$ to be the space of distributions and $X, Y$ to $L^p(\mathbb{R}^n_+)$ for $1 \leq p \leq \infty$.

**Definition 2.1** ("XY-pair"). Assume $X, Y$ are two normed spaces described above. If whenever the sequence $\{g_n\} \subset X$ with $g_n \to g$ in $X$ and $\|g_n\|_Y \leq C$ implies $g \in Y$, we say that $X$ and $Y$ are an "XY-pair".

**Proposition 6** (Regularity Lifting II). Assume Banach spaces $X, Y$ is an "XY-pair", and that $X_0$ and $Y_0$ are closed subsets of $X$ and $Y$, respectively. Let $\mathcal{F}$ be a contracting operator from $X_0$ to $X$ and a shrinking operator from $Y_0$ to $Y$. Define $Sg = Fg + U$ for some $U \in X_0 \cap Y_0$, and suppose that $S : X_0 \cap Y_0 \to X_0 \cap Y_0$. Then the equation $g = Fg + U$ admits a unique solution $u$ in $X_0$, and more importantly $u \in Y$.

Let us point out that "XY-pairs" are quite familiar. In this note we take $X = L^\infty(\mathbb{R}^n_+)$, and $Y = C^{0,1}(\mathbb{R}^n_+)$. Here $C^{0,1}(\mathbb{R}^n_+)$ denotes the space of Lipschitz continuous functions equipped with the norm

$$
\|f\|_{C^{0,1}} = \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.
$$

For later applications, we provide the following Hardy-Littlewood-Sobolev (HLS) type inequality for Bessel potential, which was established recently in [28, Theorem 9].

**Proposition 7**. Assume $q > \max\{\beta, \frac{n(\beta - 1)}{\alpha}\}$ with $\alpha > 0$ and $\beta > 1$. If $f \in L^\frac{n}{\beta}(\mathbb{R}^n)$, then $g_{\alpha} * f \in L^q(\mathbb{R}^n)$. Moreover, we have $\|g_{\alpha} * f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^\frac{n}{\beta}(\mathbb{R}^n)}$, where $C = C(\alpha, \beta, n, q)$ and $*$ denotes the convolution of functions.

Now we are ready to show Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** The proof consists of two steps.

**Step 1.** We first manifest that $u(x) \in L^p(\mathbb{R}^n_+)$ for any $1 < p < \infty$. For simplify the notation, denote

$$
K_{\alpha}(x, y) = g_{\alpha}(x - y) - g_{\alpha}(\bar{x} - y).
$$

Inspired by [22], we introduce the linear operator

$$
\mathcal{F}_u w(x) = \int_{\mathbb{R}^n_+} K_{\alpha}(x, y)|v(y)|^{\beta - 1} w(y)dy.
$$

For any given $a > 0$, we define

$$
u_a(x) = \begin{cases} 
u(x), & \text{if } |\nu(x)| > a \text{ or } |x| > a, \\ 0, & \text{elsewhere}, \end{cases}
$$

and set $u_a(x) = u(x) - \nu_a(x)$. It is obvious that, for any $\gamma > 0$, $u^\gamma = (u_a + u_b)^\gamma = u_a^\gamma + u_b^\gamma$. Thus, integral equation (3) can be written as

$$
u = u_a + u_b = \mathcal{F}_{u_a} u_a + \mathcal{F}_{u_b} u_b,
$$
and hence \( u_a = F_{u_a} u_a + F_{u_b} u_b - u_b \). Denote \( U(x) = F_{u_a} u_b(x) - u_b(x) \), then

\[
U(x) = \int_{\mathbb{R}^n_+} K_\alpha(x, y)|u_b(y)|^{\beta-1} u_b(y)dy - u_b(x)
\]

\[
= \int_{\mathbb{R}^n_+} K_\alpha(x, y)u_b^\beta(y)dy - u_b(x)
\]

Notice that \( |u_b(x)| \leq a \) and \( u_b(x) \) vanishes outside the ball \( B_a(0) \). It is not difficult to verify that \( U(x) \in L^\infty(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+) \), and then applying interpolation inequality gives \( U(x) \in L^p(\mathbb{R}^n_+) \) for any \( 1 < p < \infty \).

Next, it will be showed that, for any \( \beta < p < \infty \), \( F_{u_a} : w \to F_{u_a} w \) is a contraction mapping on \( L^p(\mathbb{R}^n_+) \). Take

\[
\gamma = \frac{(\beta - 1)p + q}{q},
\]
i.e.,

\[
\gamma = \frac{\beta - 1}{p} + \frac{1}{p},
\]
and \( \gamma = \frac{(\beta-1)p + 1}{q} > 1 \). Because \( q > \max\{\beta, \frac{n(\beta-1)}{\alpha}\} \), it can be deduced that \( p > \max\{\gamma, \frac{n(\gamma-1)}{\alpha}\} \).

Obviously,

\[
|F_{u_a} w(x)| \leq \int_{\mathbb{R}^n_+} K_\alpha(x, y)||u_a(y)||^{\beta-1} w(y)dy
\]

\[
\leq \int_{\mathbb{R}^n_+} g_\alpha(x-y)||u_a(y)||^{\beta-1} |w(y)|dy.
\]

Using Hardy-Littlewood-Sobolev inequality (Proposition 7) and Hölder inequality, we obtain

\[
\|F_{u_a} w\|_{L^p(\mathbb{R}^n_+)} \leq C\|u_a\|^{\beta-1} w\|_{L^{p/\gamma}(\mathbb{R}^n_+)}
\]

\[
\leq C\|u_a\|^{\beta-1}\|w\|_{L^{p/(\gamma-1)}(\mathbb{R}^n_+)}\|w\|_{L^p(\mathbb{R}^n_+)}
\]

\[
= C\|u_a\|^{\beta-1}\|w\|_{L^{p/(\gamma-1)}(\mathbb{R}^n_+)}\|w\|_{L^p(\mathbb{R}^n_+)}
\]

\[
= C\|u_a\|^{\beta-1}\|w\|_{L^p(\mathbb{R}^n_+)}.
\]

Since \( u \in L^q(\mathbb{R}^n_+) \), by the definition of \( u_a(x) \), one can choose sufficiently large \( a > 0 \) such that

\[
C\|u_a\|^{\beta-1} \leq 1/2,
\]
and hence achieve

\[
\|F_{u_a} w\|_{L^q(\mathbb{R}^n_+)} \leq \frac{1}{2}\|w\|_{L^p(\mathbb{R}^n_+)}.
\]
That is, \( F_{u_a} : L^p(\mathbb{R}^n_+) \to L^p(\mathbb{R}^n_+) \) is a contracting operator for all \( \beta < p < \infty \) (including \( p = q \)).

By the Regularity Lifting I (Proposition 5), we can conclude that the solution of \( w = F_{u_a} w + U \) belongs to \( L^q(\mathbb{R}^n_+) \cap L^p(\mathbb{R}^n_+) \). Since \( u_a \) solves \( w = F_{u_a} w + U \), by uniqueness of solutions, it can be deduced that \( u_a \in L^q(\mathbb{R}^n_+) \cap L^p(\mathbb{R}^n_+) \). And it then follows that \( u \in L^p(\mathbb{R}^n_+) \) for any \( \beta < p < \infty \).
We further derive, for any $\beta < p < \infty$, $u^\beta$ belongs to $L^p_\beta(\mathbb{R}^n_+)$, i.e., $u^\beta \in L^p_\beta(\mathbb{R}^n_+)$ for $1 < p < \infty$. Note that $g_\alpha(x) > 0$ for all $x \in \mathbb{R}^n_+$, integral equation (3) leads to
\[
u(x) \leq \int_{\mathbb{R}^n_+} g_\alpha(x - y)u^\beta(y)dy.
\]

Then employing (i) of Proposition 3, we get
\[
\|u\|_{L^p_\alpha(\mathbb{R}^n_+)} \leq \|g_\alpha * u^\beta\|_{L^p_\alpha(\mathbb{R}^n_+)} \leq \|u^\beta\|_{L^p_\alpha(\mathbb{R}^n_+)}. 
\]

Consequently it follows that $u(x)$ belongs to $L^p(\mathbb{R}^n_+)$ for all $1 < p < \infty$.

**Step 2.** In this step, we are going to demonstrate that $u(x) \in L^\infty(\mathbb{R}^n_+)$. Taking advantage of inequality (8) and Hölder inequality, we easily see
\[
|u(x)| \leq \left( \int_{\mathbb{R}^n_+} |g_\alpha(x - y)|^\beta dy \right)^{1/p'} \left( \int_{\mathbb{R}^n_+} |u^\beta(y)|^\beta dy \right)^{1/p} 
\]
\[
= \|g_\alpha(x - y)\|_{L^p(\mathbb{R}^n_+)} \|u\|_{L^{\beta p}(\mathbb{R}^n_+)}^{\beta} 
\]
\[
\leq \|g_\alpha\|_{L^p(\mathbb{R}^n_+)} \|u\|_{L^{\beta p}(\mathbb{R}^n_+)}^{\beta},
\]

where the numbers $p$ and $p'$ are Hölder conjugates of each other. We know from Step 1 that $\|g_\alpha\|_{L^{\beta p}(\mathbb{R}^n_+)} < \infty$ for arbitrary $1 < p < \infty$. If $p$ can be suitably chosen such that $\|g_\alpha\|_{L^p(\mathbb{R}^n_+)}$ is finite, it then follows that $u(x)$ is uniformly bounded in $\mathbb{R}^n_+$.

To finish the proof of Step 2, it now suffices to find the appropriate $p$ such that $\|g_\alpha\|_{L^p(\mathbb{R}^n_+)} < \infty$. Since $g_\alpha$ vanishes exponentially at infinity, we only require to consider $g_\alpha$ at the origin, i.e., $\int_{B_\varepsilon(0)} |g_\alpha(x)|^{p'} dx$ for $\varepsilon$ small enough. In view of Proposition 4, we have three cases.

i) When $0 < \alpha < n$, we take $\frac{n}{\alpha} < p < \infty$, then $1 < p' < \frac{n}{n-\alpha}$, and $p'(\alpha - n) > -n$. Therefore, we have, for sufficiently small $\varepsilon$,
\[
\int_{B_\varepsilon(0)} |g_\alpha(x)|^{p'} dx \sim c_\alpha \int_{B_\varepsilon(0)} |x|^{p'(\alpha - n)} dx < \infty.
\]

ii) When $\alpha = n$, choose $p > 1$, then $1 < p' < \infty$. Evidently, for sufficiently small $\varepsilon$,
\[
\int_{B_\varepsilon(0)} |g_\alpha(x)|^{p'} dx \sim c_\alpha \int_{B_\varepsilon(0)} \left| \log \frac{2}{|x|} \right|^{p'} dx < \infty.
\]

iii) When $\alpha > n$, it is easy to see that $\int_{B_\varepsilon(0)} |g_\alpha(x)|^{p'} dx$ is finite for small positive number $\varepsilon$ and any $1 < p' < \infty$, since $g_\alpha$ is bounded near the origin.

This completes the proof of Theorem 1.2. 

\textbf{Corollary 1.} Under the same conditions of Theorem 1.2, $u(x)$ is continuous.
Proof. One easily computes that
\[
\begin{align*}
u(x) - u(z) &= \int_{\mathbb{R}^n_+} [K_\alpha(x,y) - K_\alpha(z,y)] u^\beta(y) dy \\
&= \int_{B^+_M(x)} [K_\alpha(x,y) - K_\alpha(z,y)] u^\beta(y) dy \\
&\quad + \int_{\mathbb{R}^n_+ \setminus B^+_M(x)} [K_\alpha(x,y) - K_\alpha(z,y)] u^\beta(y) dy \\
&\equiv I(x,z) + II(x,z),
\end{align*}
\]
where \(B^+_M(x)\) denotes the intersection between the ball \(B_M(x)\) and the half space \(\mathbb{R}^n_+\).

It follows from Theorem 1.2 that
\[
\int_{\mathbb{R}^n_+} K_\alpha(x,y) u^\beta(y) dy < \infty.
\]
Thus we can choose \(M\) sufficiently large such that \(II(x,z)\) is small enough.

To estimate \(I(x,z)\), we first consider
\[
F(x, z) := \int_{B^+_M(x)} [g_\alpha(x,y) - g_\alpha(z,y)] u^\beta(y) dy.
\]
In view of Proposition 4, it is easy to obtain that, when \(|x - z| \to 0\),
\[
g_\alpha(x,y) - g_\alpha(z,y) \to 0 \quad \text{for} \quad y \in B^+_M(x).
\]
Hence \(\lim_{x \to z} F(x, z) = 0\). On the other hand, since \(|\bar{x} - \bar{z}| \to 0\) as \(|x - z| \to 0\), we have
\[
\int_{B^+_M(x)} [g_\alpha(\bar{x}, y) - g_\alpha(\bar{z}, y)] u^\beta(y) dy \to 0 \quad \text{as} \quad |x - z| \to 0.
\]
Consequently, it can be deduced that
\[
\lim_{x \to z} \int_{B^+_M(x)} [K_\alpha(x,y) - K_\alpha(z,y)] u^\beta(y) dy = 0.
\]
Therefore we can conclude that the solution \(u(x)\) to integral equation (3) is continuous.

The proof is finished. \(\square\)

Remark 1. Corollary 1 illustrates that the positive solution \(u(x)\) of (3) is continuous for all \(\alpha > 0\) under the assumption \(u \in L^q(\mathbb{R}^n_+)\) with \(q > \max\{\beta, \frac{n(\beta - 1)}{\alpha}\}\). However, the Lipschitz continuity is obtained only for \(\alpha > 1\) in Theorem 1.3.

Proof of Theorem 1.3. It is readily seen that
\[
u(x) = \int_{\mathbb{R}^n_+} K_\alpha(x,y) u^\beta(y) dy
\]
\[
= \int_{B^+_3(x)} K_\alpha(x,y) u^\beta(y) dy + \int_{\mathbb{R}^n_+ \setminus B^+_3(x)} K_\alpha(x,y) u^\beta(y) dy,
\]
where \(B^+_3(x) = B_3(x) \cap \mathbb{R}^n_+\).

For a fixed \(\delta > 0\), define
\[
\mathcal{F} g(x) = \int_{B^+_\delta(x)} K_\alpha(x,y) g^\beta(y) dy
\]
and
\[ U(x) = \int_{\mathbb{R}^n_+ \setminus B^n_+(x)} K_\alpha(x, y)u^\beta(y)dy. \]

Then one can easily check that \( u(x) \) satisfies the equation \( g(x) = \mathcal{F}g(x) + U(x) \). Denote \( \mathcal{S}g = \mathcal{F}g + U \).

In order to apply the Regularity Lifting II, and then promote the regularity of positive solutions to integral equation (3) from \( L^\infty \) to \( C^{0,1}\), we first define
\[ X_0 = \{ g \in L^\infty(\mathbb{R}^n_+) \mid g \geq 0, g(x', 0) = 0, \|g\|_\infty \leq 2\|u\|_\infty \}, \]
\[ Y_0 = \{ g \in C^{0,1}(\mathbb{R}^n_+) \mid g \geq 0, g(x', 0) = 0, \|g\|_\infty \leq 2\|u\|_\infty \}. \]

It is not difficult to verify that \( X_0 \) and \( Y_0 \) are closed subsets of \( L^\infty(\mathbb{R}^n_+) \) and \( C^{0,1}(\mathbb{R}^n_+) \), respectively. In what follows we will prove that, for \( \delta \) sufficiently small,
(i) \( \mathcal{F} \) is a contracting operator from \( X_0 \) to \( L^\infty(\mathbb{R}^n_+) \),
(ii) \( \mathcal{F} \) is a shrinking operator from \( Y_0 \) to \( C^{0,1}(\mathbb{R}^n_+) \),
(iii) \( U \in X_0 \cap Y_0 \),
(iv) \( \mathcal{S} : X_0 \cap Y_0 \to X_0 \cap Y_0 \).

(i) For any \( h_1, h_2 \in X_0 \) and any \( x \in \mathbb{R}^n_+ \), it can be deduced that
\[
|\mathcal{F}h_1(x) - \mathcal{F}h_2(x)| \leq \int_{B^n_+(x)} K_\alpha(x, y)|h_1^\beta(y) - h_2^\beta(y)|dy
= \int_{B^n_+(x)} K_\alpha(x, y)|\varphi^{\beta-1}(y)|h_1(y) - h_2(y)|dy
\leq C\|u\|^{\beta-1}_\infty|h_1 - h_2|_\infty \int_{B^n_+(x)} K_\alpha(x, y)dy
\leq C\|u\|^{\beta-1}_\infty|h_1 - h_2|_\infty \int_{B^n_+(x)} K_\alpha(x, y)dy
\leq C\|u\|^{\beta-1}_\infty|h_1 - h_2|_\infty \int_{B^n_+(x)} g_\alpha(x - y)dy
= C\|u\|^{\beta-1}_\infty|h_1 - h_2|_\infty \int_{B^n_+(x)} g_\alpha(y)dy,
\]

where \( \varphi(y) \) is valued between \( h_1(y) \) and \( h_2(y) \) according to the mean value theorem.

To estimate the integral \( \int_{B^n_+(0)} g_\alpha(x)dy \) for sufficiently small \( \delta \), we require to consider three cases in light of Proposition 4.

(i.i) Since \( g_\alpha(y) \sim c_\alpha|y|^{\alpha-n} \) for \( 0 < \alpha < n \), we obtain \( \int_{B^n_+(0)} g_\alpha(x)dy \sim c_\alpha \delta^n \).

Thus we can take \( \delta \) to be sufficiently small such that
\[
C\|u\|^{\beta-1}_\infty \int_{B^n_+(0)} g_\alpha(y)dy \leq 1/4;
\]

(i.ii) Since \( g_\alpha(y) \sim c_\alpha \log \frac{2}{|y|} \) for \( \alpha = n \), we obtain
\[
\int_{B^n_+(0)} g_\alpha(y)dy \sim c_\alpha \delta^n \log \frac{2}{\delta}.
\]

Therefore the desired result is derived:
(i.iii) Since \( g_\alpha(y) \sim c_\alpha \) for \( \alpha > n \), we can take \( \delta \) to be sufficiently small such that
\[
\int_{B^n_+(0)} g_\alpha(y)dy \leq 1/(4C\|u\|^{\beta-1}_\infty).\]
In general, we can choose \( \delta \) sufficiently small such that, for any \( \alpha > 0 \),

\[
C\|u\|_{L^\infty}^{\beta-1} \int_{B_\delta(0)} g_\alpha(y)dy \leq \frac{1}{4},
\]
and then

\[
\|\mathcal{F}h_1 - \mathcal{F}h_2\|_{L^\infty} \leq \frac{1}{4}\|h_1 - h_2\|_{L^\infty}.
\]

Consequently, for such a small \( \delta \), \( \mathcal{F} \) is a contracting operator from \( X_0 \) to \( L^\infty(\mathbb{R}_+^n) \).

(ii) Notice that \( h \in Y_0 \) can be extended by 0 to the whole space. For any \( h \in Y_0 \) and any \( x, z \in \mathbb{R}_+^n \), we have

\[
|\mathcal{F}h(x) - \mathcal{F}h(z)| = \left| \int_{B_\delta(z)} K_\alpha(x,y)h^\beta(y)dy - \int_{B_\delta(z)} K_\alpha(z,y)h^\beta(y)dy \right|
= \left| \int_{B_\delta(z)} g_\alpha(x-y)h^\beta(y)dy - \int_{B_\delta(z)} g_\alpha(z-y)h^\beta(y)dy \right|
\leq \left| \int_{B_\delta(z)} g_\alpha(x-z)h^\beta(y)dy - \int_{B_\delta(z)} g_\alpha(z-z)h^\beta(y)dy \right|
+ \int_{B_\delta(z)} g_\alpha(x-z)h^\beta(y)dy - \int_{B_\delta(z)} g_\alpha(z-z)h^\beta(y)dy
\equiv I(x,z) + II(x,z).
\]

For the first part, the variables transformation allows us to deduce

\[
\int_{B_\delta(z)} g_\alpha(z-y)h^\beta(y)dy = \int_{B_\delta(z)} g_\alpha(x-y)h(y+z-x)dy,
\]
so it follows that

\[
I(x,z) = \left| \int_{B_\delta(z)} g_\alpha(x-y)[h^\beta(y) - h^\beta(y+z-x)]dy \right|
\leq \int_{B_\delta(z)} \beta g_\alpha(x-y)\varphi^{\beta-1}(y)h(y) - h(y+z-x)dy
\leq C\|u\|^\beta_{L^\infty} \|h\|_{C^{0,1}} |x-z| \int_{B_\delta(z)} g_\alpha(y)dy
= C\|u\|^\beta_{L^\infty} \|h\|_{C^{0,1}} |x-z| \int_{B_\delta(0)} g_\alpha(y)dy,
\]

where \( \varphi(y) \) is valued between \( h(y) \) and \( h(y+z-x) \). In the same way as the argument of (9), we can choose sufficiently small \( \delta \) such that, for any \( h \in Y_0 \) and \( x, z \in \mathbb{R}_+^n \),

\[
I(x,z) \leq \frac{1}{8}\|h\|_{C^{0,1}} |x-z|.
\]
It is easy to see that

\[ II(x, z) = \left| \int_{B_{\delta}(x)} g_{\alpha}(\bar{x} - y)h^\beta(y)dy - \int_{B_{\delta}(x)} g_{\alpha}(\bar{z} - y - z + x)h^\beta(y + z - x)dy \right| \]

\[ \leq \left| \int_{B_{\delta}(x)} [g_{\alpha}(\bar{x} - y) - g_{\alpha}(\bar{z} - y - z + x)]h^\beta(y)dy \right| \]

\[ + \left| \int_{B_{\delta}(x)} g_{\alpha}(\bar{z} - y - z + x)[h^\beta(y) - h^\beta(y + z - x)]dy \right| \]

\[ \equiv II_1(x, z) + II_2(x, z). \]

Since \(|h| \leq \|h\|_{\infty} \leq \|h\|_{C^{0,1}}\) and \(|(\bar{x} - y) - (\bar{z} - y - z + x)| = 2|x_n - z_n| \leq 2|x - z|\), we get

\[ II_1(x, z) \leq 2\|h\|_{\infty}^{\beta - 1}\|h\|_{C^{0,1}}|x - z| \int_{B_{\delta}(0)} \left| \frac{\partial g_{\alpha}(y)}{\partial y_n} \right| dy \]

\[ \leq C\|u\|_{\infty}^{\beta - 1}\|h\|_{C^{0,1}}|x - z| \int_{B_{\delta}(0)} [g_{\alpha}(y) + g_{\alpha - 1}(y)]dy. \]

Note that here we have applied the fact of [1] that for \(\alpha > 1\),

\[ \left| \frac{\partial g_{\alpha}(x)}{\partial x_i} \right| \leq C[g_{\alpha}(x) + g_{\alpha - 1}(x)], \quad \forall \ 1 \leq i \leq n. \]

Same as the argument of (11) we can also choose sufficiently small positive number \(\delta\) such that

\[ II_1(x, z) \leq \frac{1}{16}\|h\|_{C^{0,1}}|x - z|. \]

(12)

Now it is not too hard to show that for sufficiently small \(\delta\),

\[ II_2(x, z) \leq \frac{1}{16}\|h\|_{C^{0,1}}|x - z|. \]

(13)

Taking into account (10)-(13) we can deduce

\[ \|Fh\|_{C^{0,1}} \leq \frac{1}{2}\|h\|_{C^{0,1}}, \quad \forall h \in Y_0. \]

Therefore \(F\) is a shrinking operator from \(Y_0\) to \(C^{0,1}(\mathbb{R}^n_+)\) for sufficiently small \(\delta\).

(iii) It follows immediately from the definition of \(U(x)\) that

\[ \|U\|_{\infty} \leq \|u\|_{\infty}. \]

(14)
and $U(x',0) = 0$. Hence we have $U \in X_0$. In the following we show $U(x)$ is Lipschitz continuous. For any $x, z \in \mathbb{R}_+^n$, we see that

$$\left| U(x) - U(z) \right| = \left| \int_{\mathbb{R}_+^n \setminus B^+_x(z)} K_\alpha(x,y) u^\beta(y) dy - \int_{\mathbb{R}_+^n \setminus B^+_z(z)} K_\alpha(z,y) u^\beta(y) dy \right|$$

$$\leq \left| \int_{\mathbb{R}_+^n \setminus (B^+_x(z) \cup B^+_z(z))} [K_\alpha(x,y) - K_\alpha(z,y)] u^\beta(y) dy \right| + \int_{B^+_x(z) \setminus B^+_z(z)} K_\alpha(x,y) u^\beta(y) dy + \int_{B^+_z(z) \setminus B^+_x(z)} K_\alpha(z,y) u^\beta(y) dy$$

$$\leq \int_{\mathbb{R}_+^n \setminus (B^+_x(z) \cup B^+_z(z))} |g_\alpha(x - y) - g_\alpha(z - y)| u^\beta(y) dy$$

$$+ \int_{\mathbb{R}_+^n \setminus (B^+_x(z) \cup B^+_z(z))} |g_\alpha(x - y) - g_\alpha(z - y)| u^\beta(y) dy$$

$$+ \left| \int_{B^+_x(z) \setminus B^+_z(z)} K_\alpha(x,y) u^\beta(y) dy + \int_{B^+_z(z) \setminus B^+_x(z)} K_\alpha(z,y) u^\beta(y) dy \right|$$

$$= J_1(x,z) + J_2(x,z) + J_3(x,z).$$

For the first two part, we only estimate $J_1(x,z)$ since $J_2(x,z)$ can be deduced in a similar manner. One can easily verify that

$$J_1(x,z) = \int_{\mathbb{R}_+^n \setminus (B^+_x(z) \cup B^+_z(z))} |g'_\alpha(\theta x + (1 - \theta)z - y)||x - z| u^\beta(y) dy$$

$$\leq \|u\|_\infty^\beta |x - z| \int_{\mathbb{R}_+^n \setminus (B^+_x(z) \cup B^+_z(z))} |g'_\alpha(\theta x + (1 - \theta)z - y)| dy$$

$$\leq C \|u\|_\infty^\beta |x - z|.$$}

Noting that $K_\alpha(x)$ is bounded for $|x| > \delta$, we can derive

$$J_3(x,z) \leq C \int_{B^+_x(z) \setminus B^+_z(z)} u^\beta(y) dy + \int_{B^+_z(z) \setminus B^+_x(z)} u^\beta(y) dy$$

$$\leq C \|u\|_\infty^\beta \int_{(B_r(x) \setminus B_r(z)) \cup (B_r(z) \setminus B_r(x))} dy$$

$$\leq C \|u\|_\infty^\beta \delta^{n-1} |x - z|,$$

where we have applied the fact that the volume

$$|(B_r(x) \setminus B_r(z)) \cup (B_r(z) \setminus B_r(x))| \leq C r^{n-1} |x - z|.$$  

Hence

$$\sup_{x \neq z} \frac{|U(x) - U(z)|}{|x - z|} \leq C \|u\|_\infty^\beta.$$

Combining this and (14) yields the desired result, i.e., $U \in X_0 \cap Y_0$. 

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(iv) For any given \( g \in X_0 \cap Y_0 \), \( \|g\|_{\infty} \leq 2\|u\|_{\infty} \). It is obvious that \( Fg \in L^{\infty}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n_+) \), since \( F \) is a contracting operator from \( X_0 \) to \( L^{\infty}(\mathbb{R}^n) \) and a shrinking operator from \( Y_0 \) to \( C^{0,1}(\mathbb{R}^n_+) \). It then follows that \( Sg = Fg + U \in L^{\infty}(\mathbb{R}^n_+) \cap C^{0,1}(\mathbb{R}^n_+) \). According to (10) and (14), we derive
\[
\|Sg\|_{\infty} \leq \|Fg\|_{\infty} + \|U\|_{\infty} \leq \frac{1}{4}\|g\|_{\infty} + \|u\|_{\infty} \leq 2\|u\|_{\infty},
\]
which results in \( Sg \in X_0 \cap Y_0 \).

Up to now we have demonstrated (i)-(iv). Taking advantage of the Regularity Lifting II (Proposition 6) we can conclude that \( u(x) \) is Lipschitz continuous.

The proof of Theorem 1.3 is completed. \( \square \)

3. Nonexistence of solutions.

3.1. Two Lemmas. Before carrying on the method of moving planes, we should do some preparation work. Define a family of moving planes parameterized by \( \kappa \in \mathbb{R}_+ \), i.e., \( T_\kappa = \{ x \in \mathbb{R}^n_+ | x_n = \kappa \} \). For any \( x \in \mathbb{R}^n_+ \), we denote by \( x_\kappa = (x', 2\kappa - x_n) \) its reflection point through the plane \( T_\kappa \). Let \( \Sigma_\kappa \) be the region between the two planes \( x_n = 0 \) and \( x_n = \kappa \), i.e.,
\[
\Sigma_\kappa = \{ x \in \mathbb{R}^n_+ | 0 < x_n < \kappa \},
\]
and \( \Sigma_\kappa^C = \mathbb{R}^n_+ \setminus \Sigma_\kappa \) be the complement of \( \Sigma_\kappa \). Set
\[
K_\alpha(x, y) = g_\alpha(x - y) - g_\alpha(\bar{x} - y).
\]

**Lemma 3.1.** (i) For any \( x, y \in \Sigma_\kappa \), \( x \neq y \), we have
\[
K_\alpha(x_\kappa, y_\kappa) > \max\{K_\alpha(x_\kappa, y), K_\alpha(x, y_\kappa)\} \tag{15}
\]
and
\[
K_\alpha(x_\kappa, y_\kappa) - K_\alpha(x, y) > |K_\alpha(x_\kappa, y) - K_\alpha(x, y_\kappa)|. \tag{16}
\]
(ii) For any \( x \in \Sigma_\kappa \), \( y \in \Sigma_\kappa^C \), it holds
\[
K_\alpha(x, y) < K_\alpha(x_\kappa, y_\kappa). \tag{17}
\]

**Proof.** It is obvious that, for any \( x, y \in \Sigma_\kappa \),
\[
\rho(x_\kappa, y_\kappa) = \rho(x, y) < \rho(x_\kappa, y) = \rho(x, y_\kappa), \tag{18}
\]
where \( \rho(x, y) = |x - y| \) denotes the distance between the two points \( x \) and \( y \).

Let \( \tau(x, y) = 4x_ny_n \). It is not difficult to verify that
\[
\tau(x_\kappa, y_\kappa) \geq \max\{\tau(x_\kappa, y), \tau(x, y_\kappa)\} \geq \min\{\tau(x_\kappa, y), \tau(x, y_\kappa)\} \geq \tau(x, y) \tag{19}
\]
for any \( x, y \in \Sigma_\kappa \). Indeed,
\[
\tau(x_\kappa, y_\kappa) - \tau(x, y_\kappa) = 8(2\kappa - x_n)(\kappa - y_n) > 0,
\]
\[
\tau(x_\kappa, y) - \tau(x, y) = 4(2\kappa - x_n)y_n - 4x_ny_n = 8(\kappa - x_n)y_n > 0.
\]

Define
\[
f(\rho, \tau; t) = e^{-\frac{\tau}{\rho}} - e^{-\frac{\tau(x_\kappa, y_\kappa)}{t}}.
\]
Then direct calculations yield, for any \( \rho, \tau, t > 0 \), \( f(\rho, \tau; t) > 0 \),
\[
\frac{\partial f}{\partial \rho} = -\frac{\pi}{t}e^{-\frac{\tau}{\rho}} - e^{-\frac{\tau(x_\kappa, y_\kappa)}{t}} < 0, \tag{20}
\]
\[
\frac{\partial f}{\partial \tau} = \frac{\pi}{t}e^{-\frac{\tau(x_\kappa, y_\kappa)}{t}} > 0, \tag{21}
\]
\[
\frac{\partial f}{\partial \tau \partial \rho} = -\frac{\pi^2}{t^2}e^{-\frac{\tau(x_\kappa, y_\kappa)}{t}} < 0. \tag{22}
\]
And then we can conclude (16).

Lemma 3.2. Assume that

\[ K_\alpha(x_\kappa, y_\kappa) - K_\alpha(x, y) \]
\[ = g_\alpha(x_\kappa - y_\kappa) - g_\alpha(x - y_\kappa) - [g_\alpha(x_\kappa - y) - g_\alpha(x - y)] \]
\[ = \frac{1}{\gamma(\alpha)} \int_0^\infty [f(\rho(x_\kappa, y_\kappa), \tau(x_\kappa, y_\kappa); t) - f(\rho(x, y), \tau(x, y); t)] e^{\frac{-\tau}{t^{p(n-a)/2+1}}} dt > 0, \]

where \( \gamma(\alpha) = (4\pi)^{\frac{\sqrt{2}}{2}} \Gamma\left( \frac{\alpha}{2} \right) \). Similarly, we have \( K_\alpha(x_\kappa, y_\kappa) > K_\alpha(x, y_\kappa) \). Thus (15) holds.

By (18), (19) and (22), it follows that

\[ f(\rho(x_\kappa, y_\kappa), \tau(x_\kappa, y_\kappa); t) - f(\rho(x, y), \tau(x, y); t) \]
\[ = \int_{\tau(x,y)}^{\tau(x_\kappa, y_\kappa)} \frac{\partial f(\rho(x, y), s; t)}{\partial s} ds > \int_{\tau(x,y)}^{\tau(x_\kappa, y_\kappa)} \frac{\partial f(\rho(x, y), s; t)}{\partial s} ds \]
\[ \geq \left| \int_{\tau(x,y)}^{\tau(x_\kappa, y_\kappa)} \frac{\partial f(\rho(x, y), s; t)}{\partial s} ds \right| \]
\[ = |f(\rho(x_\kappa, y), \tau(x_\kappa, y)); t) - f(\rho(x, y_\kappa), \tau(x, y_\kappa); t)|. \]

And then we can conclude (16).

(ii) Since \( x \in \Sigma_\kappa, y \in \Sigma_\kappa^C \), we easily get

\[ \rho(x_\kappa, y) < \rho(x, y), \text{ and } \tau(x_\kappa, y) > \tau(x, y). \]

Same as the argument of (15), inequality (17) follows immediately from (20) and (21).

The proof is complete.

The following lemma is a key ingredient in our integral estimates.

**Lemma 3.2.** Assume that \( u(x) \) is a positive solution of integral equation (3), then it holds

\[ u(x) - u_\kappa(x) \leq \int_{\Sigma_\kappa} [K_\alpha(x_\kappa, y_\kappa) - K_\alpha(x, y_\kappa)] [u^\beta(y) - u_\kappa^\beta(y)] dy \]

for any \( x \in \Sigma_\kappa \), where \( u_\kappa(x) = u(x_\kappa) \).

**Proof.** It is easy to establish that

\[ u(x) = \int_{\Sigma_\kappa} K_\alpha(x, y) u^\beta(y) dy + \int_{\Sigma_\kappa} K_\alpha(x, y) u_\kappa^\beta(y) dy + \int_{\Sigma_\kappa^C \setminus \Sigma_\kappa} K_\alpha(x, y) u^\beta(y) dy \]
\[ = \int_{\Sigma_\kappa} K_\alpha(x, y) u^\beta(y) dy + \int_{\Sigma_\kappa} K_\alpha(x, y_\kappa) u_\kappa^\beta(y) dy + \int_{\Sigma_\kappa^C \setminus \Sigma_\kappa} K_\alpha(x, y) u^\beta(y) dy, \]

where \( \Sigma_\kappa = \{ x_\kappa | x \in \Sigma_\kappa \} \). Substituting \( x \) by \( x_\kappa \) yields

\[ u(x_\kappa) = \int_{\Sigma_\kappa} K_\alpha(x_\kappa, y) u^\beta(y) dy + \int_{\Sigma_\kappa} K_\alpha(x_\kappa, y_\kappa) u_\kappa^\beta(y) dy + \int_{\Sigma_\kappa^C \setminus \Sigma_\kappa} K_\alpha(x_\kappa, y) u^\beta(y) dy. \]
Evidently,

\[ u(x) - u(x_\kappa) = \int_{\Sigma_\kappa} [K_\alpha(x, y) - K_\alpha(x_\kappa, y)]u^\beta(y)dy \]

\[ + \int_{\Sigma_\kappa} [K_\alpha(x, y_\kappa) - K_\alpha(x_\kappa, y_\kappa)]u^\beta_\kappa(y)dy \]

\[ + \int_{\Sigma_\kappa^c \setminus \Sigma_\kappa} [K_\alpha(x, y) - K_\alpha(x_\kappa, y)]u^\beta(y)dy. \]  

(23)

By means of Lemma 3.1, it can be deduced that

\[ u(x) - u(x_\kappa) \leq \int_{\Sigma_\kappa} [K_\alpha(x, y) - K_\alpha(x_\kappa, y)]u^\beta(y)dy \]

\[ - \int_{\Sigma_\kappa} [K_\alpha(x_\kappa, y_\kappa) - K_\alpha(x, y_\kappa)]u^\beta_\kappa(y)dy \]

\[ \leq \int_{\Sigma_\kappa} [K_\alpha(x_\kappa, y_\kappa) - K_\alpha(x, y_\kappa)]u^\beta(y)dy \]

\[ - \int_{\Sigma_\kappa} [K_\alpha(x, y_\kappa) - K_\alpha(x_\kappa, y)]u^\beta_\kappa(y)dy \]

\[ = \int_{\Sigma_\kappa} [K_\alpha(x_\kappa, y_\kappa) - K_\alpha(x, y_\kappa)] [u^\beta(y) - u^\beta_\kappa(y)]dy. \]

The proof is finished. \qed

3.2. Proofs of Theorems 1.4 and 1.5. In this subsection we first show that the solution \( u(x) \) of (3) must be strictly monotone increasing with respect to the variable \( x_n \) under the assumption \( u \in L^q(\mathbb{R}^n) \) with \( q > \max\{\beta, (\beta - 1)/\alpha\} \). Then, based on the monotonicity of \( u(x) \) we derive the nonexistence of positive solutions to the integral equation (3) on a half space. That is, Theorems 1.4 and 1.5 are proved successively.

Proof of Theorem 1.4. The proof is divided into two steps. In the first step it will be showed that, for sufficiently small positive constant \( \kappa \),

\[ w_\kappa(x) := u_\kappa(x) - u(x) \geq 0, \quad \forall x \in \Sigma_\kappa. \]  

(24)

In other words, we are able to start moving the plane \( T_\kappa = \{ x \in \mathbb{R}^n_+ \mid x_n = \kappa \} \) from the very below end of our region \( \mathbb{R}^n_+ \), i.e., near \( x_n = 0 \). In the second step we will continuously move our plane \( T_\kappa \) along the positive direction of \( x_n \)-axis as long as inequality (24) holds.

Step 1. Define the set

\[ \Sigma_\kappa^- = \{ x \in \Sigma_\kappa \mid w_\kappa(x) < 0 \}. \]
We manifest that, for \( \kappa > 0 \) sufficiently small, the measure of \( \Sigma_\kappa^- \) must be zero. Actually, by virtue of Lemma 3.2, we easily acquire, for any \( x \in \Sigma_\kappa^- \),

\[
0 < u(x) - u_\kappa(x) \leq \int_{\Sigma_\kappa^-} [K_\alpha(x, y) - K_\alpha(x, y_\kappa)] [u^\beta(y) - u_\kappa^\beta(y)] dy
\]

\[
= \int_{\Sigma_\kappa^-} [K_\alpha(x, y) - K_\alpha(x, y_\kappa)] [u^\beta(y) - u_\kappa^\beta(y)] dy
\]

\[
+ \int_{\Sigma_\kappa \setminus \Sigma_\kappa^-} [K_\alpha(x, y) - K_\alpha(x, y_\kappa)] [u^\beta(y) - u_\kappa^\beta(y)] dy.
\]  

(25)

Since \( u^\beta(y) \leq u_\kappa^\beta(y) \) on \( \Sigma_\kappa \setminus \Sigma_\kappa^- \) and \( K_\alpha(x, y_\kappa) \geq K_\alpha(x, y) \) for \( y \in \Sigma_\kappa \setminus \Sigma_\kappa^- \) according to Lemma 3.1(i), the last integral in (25) is negative. Hence we get

\[
u(x) - u_\kappa(x) \leq \int_{\Sigma_\kappa^-} [K_\alpha(x, y) - K_\alpha(x, y_\kappa)] [u^\beta(y) - u_\kappa^\beta(y)] dy
\]

\[
\leq \int_{\Sigma_\kappa^-} K_\alpha(x, y) [u^\beta(y) - u_\kappa^\beta(y)] dy
\]

\[
\leq \int_{\Sigma_\kappa^-} g_\alpha(x - y_\kappa) |u^\beta(y) - u_\kappa^\beta(y)| dy
\]

\[
= \beta \int_{\Sigma_\kappa^-} g_\alpha(x - y) \psi_\kappa^{\beta - 1}(y) |u(y) - u_\kappa(y)| dy
\]

\[
\leq \beta \int_{\Sigma_\kappa^-} g_\alpha(x - y) u^{\beta - 1}(y) |u(y) - u_\kappa(y)| dy,
\]  

(26)

where we have used the mean value theorem with \( \psi_\kappa(y) \) valued between \( u(y) \) and \( u_\kappa(y) \), and the fact that

\[
0 \leq u_\kappa(y) \leq \psi_\kappa(y) \leq u(y) \quad \text{on} \quad \Sigma_\kappa^-.
\]

When Proposition 7 and Hölder’s inequality are applied successively, inequality (26) yields

\[
\|u_\kappa\|_{L^\beta(\Sigma_\kappa^-)} \leq C \|u^{\beta - 1} u_\kappa\|_{L^{\frac{3n}{n - 3\beta}}(\Sigma_\kappa^-)}
\]

\[
\leq C \left( \int_{\Sigma_\kappa^-} |u^{\beta - 1}(y)|^{\frac{3n}{3\beta - n}} dy \right)^{\frac{3\beta - n}{3n}} \left( \int_{\Sigma_\kappa^-} |u_\kappa(y)|^{\frac{3\beta}{2}} dy \right)^{\frac{\beta}{3n}}
\]

\[
= C \|u\|_{L^\beta(\Sigma_\kappa^-)} \|u_\kappa\|_{L^\beta(\Sigma_\kappa^-)}
\]

\[
\leq C \|u\|_{L^\beta(\Sigma_\kappa^-)} \|u_\kappa\|_{L^\beta(\Sigma_\kappa^-)}.
\]  

(27)

By the condition that \( u \in L^\beta(\mathbb{R}_+^n) \), we can take \( \kappa \) to be sufficiently small such that

\[
C \|u\|_{L^\beta(\Sigma_\kappa^-)}^{\beta - 1} \leq 1/2.
\]

Now it follows from inequality (27) that

\[
\|u_\kappa\|_{L^\beta(\Sigma_\kappa^-)} = 0,
\]

which implies \( \Sigma_\kappa^- \) must be empty. This demonstrates (24) and hence completes Step 1.

Step 2. Move the plane to the limiting position.
Inequality (24) illustrates that the plane $T_\kappa = \{ x \in \mathbb{R}^n_+ \mid x_n = \kappa \}$ can begin to move from the neighborhood of $x_n = 0$. We now drive $T_\kappa$ along the positive direction of $x_n$-axis as long as inequality (24) holds to the limiting position. Define
\[ \kappa_0 = \sup \{ \kappa \mid w_\rho(x) \geq 0, \forall \rho \leq \kappa \text{ and } x \in \Sigma_\rho \}. \]
It will be proved that the solution $u(x)$ must be monotone increasing with respect to the variable $x_n$ and be symmetric about $T_{\kappa_0}$, that is,
\[ w_{\kappa_0}(x) \equiv 0 \text{ on } \Sigma_{\kappa_0}. \quad (28) \]
Here we take advantage of the contradiction argument. Assume that $w_{\kappa_0} \geq 0$ for such a $\kappa_0$, but $w_{\kappa_0} \neq 0$ on $\Sigma_{\kappa_0}$. We prove that the plane $T_{\kappa_0}$ can be moved further toward the positive direction of $x_n$-axis. Precisely speaking, there exists an $\varepsilon > 0$ such that, for any $\kappa_0 \leq \kappa < \kappa_0 + \varepsilon$,
\[ u(x) \leq u_\kappa(x) \text{ on } \Sigma_\kappa. \]
We use the same argument as (27) to deduce that
\[ \| w_\kappa \|_{L^q(\Sigma^-)} \leq C \| u \|_{L^q(\Sigma^-)}^{\beta-1} \| w_\kappa \|_{L^q(\Sigma^-)}; \quad (29) \]
Notice the assumption $u \in L^q(\mathbb{R}^n_0)$, we can take $\varepsilon$ to be sufficiently small such that, for any $\kappa_0 \leq \kappa < \kappa_0 + \varepsilon$,
\[ C \| u \|_{L^q(\Sigma^-)}^{\beta-1} \leq 1/4. \quad (30) \]
Let us postpone the proof of (30) for a moment and continue. It now follows from (29) and (30) that $\| w_\kappa \|_{L^q(\Sigma^-)} = 0$ and hence $\Sigma^-_\kappa$ must be measure zero. Consequently, one can achieve, for values of $\kappa$ in the internal $[\kappa_0, \kappa_0 + \varepsilon]$,
\[ w_\kappa(x) \geq 0 \text{ for any } x \in \Sigma_\kappa. \]
This results in a contradiction to the definition of $\kappa_0$. Accordingly equality (28) is verified.

To complete the proof of this theorem, it now suffices to demonstrate inequality (30). For any small $\sigma > 0$, we can select $M$ large enough such that
\[ \left( \int_{\mathbb{R}^n_+ \setminus B_M(0)} u^q(y)dy \right)^{\frac{1}{q-1}} < \sigma. \quad (31) \]
And we then show that, for fixed $M$, the measure of $\Sigma^-_\kappa \cap B_M(0)$ is sufficiently small for $\kappa$ close to $\kappa_0$. First of all we affirm that $w_{\kappa_0}(x) > 0$ in $\Sigma_{\kappa_0}$. As a matter of fact, by means of (23) and Lemma 3.1(i), it can be deduced that, for any $x \in \Sigma_\kappa$,
\[ u_\kappa(x) - u(x) \geq \int_{\Sigma_\kappa} [K_\alpha(x, \kappa) - K_\alpha(x, y)] [u_\kappa^\beta(y) - u^\beta(y)]dy \]
\[ + \int_{\Sigma^-_\kappa \setminus \Sigma_\kappa} [K_\alpha(x, \kappa) - K_\alpha(x, y)] u^\beta(y)dy. \quad (32) \]
If our affirmation does not hold, we can find $x_0 \in \Sigma_{\kappa_0}$ such that $w_{\kappa_0}(x_0) = 0$. And then according to Lemma 3.1(i) and (32), we can achieve
\[ 0 = w_{\kappa_0}(x_0) - u(x_0) \geq \int_{\Sigma^-_{\kappa_0} \setminus \Sigma_{\kappa_0}} [K_\alpha(x, \kappa_0, y) - K_\alpha(x, y)] u^\beta(y)dy. \]
Making use of Lemma 3.1(ii) yields \( u(y) = 0 \) for all \( y \in \Sigma_{\kappa_0}^C \setminus \tilde{\Sigma}_{\kappa_0} \), which implies a contradiction with our condition that \( u(x) \) is positive. As a consequence, \( w_{\kappa_0}(x) > 0 \) in \( \Sigma_{\kappa_0} \).

For any \( \lambda > 0 \), we define
\[
B_1^\lambda = \{ x \in \Sigma_{\kappa_0} \cap B_M(0) | w_{\kappa_0}(x) > \lambda \}, \quad B_2^\lambda = (\Sigma_{\kappa_0} \cap B_M(0)) \setminus B_1^\lambda.
\]
It is easy to verify that \( \lim_{\lambda \to 0} \mu(B_2^\lambda) = 0 \). For \( \kappa > \kappa_0 \), denote
\[
B_\kappa = (\Sigma_\kappa \setminus \Sigma_{\kappa_0}) \cap B_M(0).
\]
Then we have
\[
\Sigma_\kappa \cap B_M(0) \subset (\Sigma_\kappa \cap B_1^\lambda) \cup B_2^\lambda \cup B_\kappa.
\]
It is obvious that, when \( \kappa \) is close to \( \kappa_0 \), the measure of \( B_\kappa \) is small.

We claim that the measure of \( \Sigma_{\kappa} \cap B_1^\lambda \) can be sufficiently small as \( \kappa \) close to \( \kappa_0 \). Actually, since \( w_\kappa(x) = u_\kappa(x) - u(x) = u_\kappa(x) - u_{\kappa_0}(x) + u_{\kappa_0}(x) - u(x) < 0 \) for any \( x \in \Sigma_{\kappa} \cap B_1^\lambda \), we can obtain \( u_{\kappa_0}(x) - u_\kappa(x) > w_{\kappa_0}(x) > \lambda \). Therefore we deduce
\[
\Sigma_{\kappa} \cap B_1^\lambda \subset \Omega_\lambda \equiv \{ x \in B_M(0) | u_{\kappa_0}(x) - u_\kappa(x) > \lambda \}.
\]
Applying the Chebyshev inequality generates
\[
\begin{align*}
\mu(\Omega_\lambda) &\leq \frac{1}{\lambda^{\beta+1}} \int_{\Omega_\lambda} |u_{\kappa_0}(x) - u_\kappa(x)|^{\beta+1} dx \\
&\leq \frac{1}{\lambda^{\beta+1}} \int_{B_M(0)} |u_{\kappa_0}(x) - u_\kappa(x)|^{\beta+1} dx.
\end{align*}
\]
For given \( \lambda \), when \( \kappa \) is close to \( \kappa_0 \), the above integral can be made as small as we desire. And hence our claim is true.

Now combining (33) and (34), we know that the measure of \( \Sigma_{\kappa} \cap B_M(0) \) can also be made sufficiently small. Consequently, inequality (30) follows immediately from this and (31).

This completes the proof of Theorem 1.4.

\( \square \)

**Proof of Theorem 1.5.** Assume the nonnegative solution \( u(x) \neq 0 \), then there exists some point \( y_0 \in \mathbb{R}^n_+ \) such that \( u(y_0) > 0 \). By the continuity of \( u(x) \), there exists a neighbor \( N(y_0) \) of \( y_0 \) such that \( u(y) > 0 \) for any \( y \in N(y_0) \). Noticing that \( K_\alpha(x, \cdot) \) is positive for any \( y \in \mathbb{R}^n_+ \), we can derive, for all \( x \in \mathbb{R}^n_+ \),
\[
u(x) = \int_{\mathbb{R}^n_+} K_\alpha(x, y)u^\beta(y)dy \geq \int_{N(y_0)} K_\alpha(x, y)u^\beta(y)dy > 0.
\]

By virtue of Theorem 1.4, we know that the plane \( T_\kappa \) can be moved to the limiting position \( T_{\kappa_0} \). We assert that
\[
\kappa_0 = +\infty,
\]
which can be seen by the simple contradiction argument. Assume \( \kappa_0 < +\infty \), then the symmetric image of the boundary \( \partial \mathbb{R}^n_+ \) through the plane \( T_{\kappa_0} \) is the plane \( x_n = 2\kappa_0 \). And therefore \( u(x) = 0 \) for any \( x \in T_{2\kappa_0} \), which is a contradiction to (35). This demonstrates our assertion.

Again making use of the theorem 1.4, it can be derived that \( u(x) \) is monotone increasing with respect to \( x_n \). This leads to a contradiction with the assumption \( u \in L^q(\mathbb{R}^n_+) \). As a result, the positive solution of (3) does not exist.
3.3. Proof of Theorem 1.6. For the convenience to readers, we repeat it here.

Theorem 3.3. Let \( q > \max\{\beta, \frac{\beta-1}{\alpha}\} \). Suppose that \( u(x) \in L^q(\mathbb{R}^n_+) \) is the non-negative solution of integral equation

\[
u(x) = \int_{\mathbb{R}^n_+} K_\alpha(x,y)f(y,u(y))dy, \quad x \in \mathbb{R}^n_+. \tag{36}
\]

If the nonnegative function \( f(x,u) \) satisfies the following conditions:

(i) \( f(x,u) \) is non-decreasing in the variable \( x_\alpha \) and non-decreasing with respect to \( u \),

(ii) \( \frac{\partial f}{\partial u} \in L^{\frac{n}{n-\beta}}(\mathbb{R}^n_+) \) is non-decreasing with respect to \( u \),

then \( u \) is identically equal to zero.

The proof is essentially the same as that of Theorems 1.4 plus 1.5. That is, we first employ the method of moving planes in integral forms to prove that the solution \( u(x) \) of (36) does not exist. In order to avoid repetition, in this proof we only show that the plane \( T_\kappa \) can be moved from the very below end of \( \mathbb{R}^n_+ \), i.e., for \( \kappa > 0 \) sufficiently small,

\[ w_\kappa(x) = u_\kappa(x) - u(x) \geq 0, \quad \forall x \in \Sigma_\kappa. \]

Here, and in the remainder of this section, when we write an inequality or equality for an \( L^p \) function, we mean it holds almost everywhere in the given set.

Applying the same arguments as in Lemma 3.2, it is not hard to verify that, for any \( x \in \Sigma_\kappa \),

\[
u(x) - u_\kappa(x) \leq \int_{\Sigma_\kappa} [K_\alpha(x_\kappa,y_\kappa) - K_\alpha(x,y_\kappa)] [f(y,u(y)) - f(y_\kappa,u_\kappa(y))] dy
= \int_{\Sigma_\kappa} [K_\alpha(x_\kappa,y_\kappa) - K_\alpha(x,y_\kappa)] [f(y,u(y)) - f(y,u_\kappa(y))] dy
+ \int_{\Sigma_\kappa} [K_\alpha(x_\kappa,y_\kappa) - K_\alpha(x,y_\kappa)] [f(y,u_\kappa(y)) - f(y,u_\kappa(y))] dy.
\]

Note that \( K_\alpha(x_\kappa,y_\kappa) \geq K_\alpha(x,y_\kappa) \) for any \( x, y \in \Sigma_\kappa \) according to Lemma 3.1 (i), and that \( f(y,u_\kappa(y)) \leq f(y_\kappa,u_\kappa(y)) \) for \( y \in \Sigma_\kappa \) due to the monotonicity of \( f(x,u(x)) \) with respect to \( x_n \). Thus we derive, for any \( x \in \Sigma_\kappa \),

\[
u(x) - u_\kappa(x) \leq \int_{\Sigma_\kappa} [K_\alpha(x_\kappa,y_\kappa) - K_\alpha(x,y_\kappa)] [f(y,u(y)) - f(y,u_\kappa(y))] dy.
\]

We divide the above integral into a sum of two integrals \( \int_{\Sigma_+} + \int_{\Sigma_- \setminus \Sigma_+} \). Since \( f(x,u(x)) \) is non-decreasing in \( u \), we have \( f(y,u(y)) \leq f(y,u_\kappa(y)) \) for \( y \in \Sigma_\kappa \setminus \Sigma_-^\kappa \).
Hence the second integral is negative, and we then obtain
\[ u(x) - u_\kappa(x) \leq \int_{\Sigma^-} [K_\alpha(x,y) - K_\alpha(x,y_\kappa)] [f(y,u(y)) - f(y,u_\kappa(y))] \, dy \]
\[ \leq \int_{\Sigma^-} K_\alpha(x,y) [f(y,u(y)) - f(y,u_\kappa(y))] \, dy \]
\[ \leq \int_{\Sigma^-} g_\alpha(x-y) [f(y,u(y)) - f(y,u_\kappa(y))] \, dy \]
\[ = \int_{\Sigma^-} g_\alpha(x-y) \frac{\partial f}{\partial u}(y,\psi_\kappa(y))[u(y) - u_\kappa(y)] \, dy \]
\[ \leq \int_{\Sigma^-} g_\alpha(x-y) \frac{\partial f}{\partial u}(y,u(y))[u(y) - u_\kappa(y)] \, dy, \quad (37) \]
where we have used the mean value theorem with \( \psi_\kappa(y) \) valued between \( u(y) \) and \( u_\kappa(y) \), the monotonicity of \( \frac{\partial f}{\partial u} \) in \( u \) and the fact that \( 0 \leq u_\kappa(y) \leq \psi_\kappa(y) \leq u(y) \) on \( \Sigma^- \). By making use of Proposition 7 and Hölder’s inequality to (37), it can be deduced that
\[ \|w_\kappa\|_{L^s(\Sigma^-)} \leq C \left\| \frac{\partial f}{\partial u} w_\kappa \right\|_{L^{\frac{n}{n-\alpha}}(\Sigma^-)} \leq C \left\| \frac{\partial f}{\partial u} \right\|_{L^{\frac{n}{n-\alpha}}(\Sigma^-)} \|w_\kappa\|_{L^s(\Sigma^-)} \]
\[ \leq C \|\partial f/\partial u\|_{L^{\frac{n}{n-\alpha}}(\Sigma^-)} \|w_\kappa\|_{L^s(\Sigma^-)}. \quad (38) \]
Noticing that \( \frac{\partial f}{\partial u} \in L^{\frac{n}{n-\alpha}}(\mathbb{R}^n_+) \), we can take \( \kappa \) to be sufficiently small such that
\[ C \left( \int_{\Sigma^-} \left| \frac{\partial f}{\partial u}(y,u(y)) \right|^{\frac{n}{n-\alpha}} \, dy \right)^{\frac{\alpha-1}{\alpha}} \leq \frac{1}{3}. \]
Then combining this with (38), we can derive
\[ \|w_\kappa\|_{L^s(\Sigma^-)} = 0, \]
which implies the measure of \( \Sigma^- \) must be zero. \( \square \)

4. The relation between PDEs and integral equations. In this section we prove

**Theorem 4.1.** Assume that \( \alpha \) is a positive even number. If \( u(x) \) is a positive smooth solution of integral equation
\[ u(x) = \int_{\mathbb{R}^n_+} K_\alpha(x,y)u^\beta(y) \, dy, \quad x \in \mathbb{R}^n_+, \]
then \( u(x) \) satisfies
\[ \begin{cases} (I - \Delta)^{\frac{n}{2}} u = u^\beta & \text{in } \mathbb{R}^n_+, \\ u = (-\Delta) u = \cdots = (-\Delta)^{\frac{n}{2}-1} u = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases} \]

**Proof.** A direct computation yields
\[ (-\Delta)^k g_\alpha(x-y) = (-\Delta)^k g_\alpha(\bar{x} - y) \]
for \( x = \bar{x} \), where \( k = 0, 1, \ldots, \frac{n}{2} - 1 \). Thus we deduce
\[ u = (-\Delta) u = \cdots = (-\Delta)^{\frac{n}{2}-1} u = 0 \quad \text{on } \partial \mathbb{R}^n_+. \]
Due to (ii) and (iii) of Proposition 3, we can obtain, for any \( x \in \mathbb{R}^n_+ \),
\[
(I - \Delta)^{\frac{1}{2}} K_\alpha(x, y) = (I - \Delta)^{\frac{1}{2}} g_\alpha(x - y) - (I - \Delta)^{\frac{1}{2}} g_\alpha(\bar{x} - y) = \delta(x - y).
\]
Therefore it follows that
\[
(I - \Delta)^{\frac{1}{2}} u(x) = \int_{\mathbb{R}^n_+} (I - \Delta)^{\frac{1}{2}} K_\alpha(x, y) u^\beta(y) dy
\]
\[
= \int_{\mathbb{R}^n_+} \delta(x - y) u^\beta(y) dy
\]
\[
= u^\beta(x).
\]
This finishes the proof.

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