NEW COMPLETE EMBEDDED MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. We construct three kinds of complete embedded minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. The first is a simply connected, singly periodic, infinite total curvature surface. The second is an annular finite total curvature surface. These two are conjugate surfaces just as the helicoid and the catenoid are in $\mathbb{R}^3$. The third one is a finite total curvature surface which is conformal to $S^2 \backslash \{p_1, ..., p_k\}, k \geq 3$.

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1. Introduction

During recent years the theory of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ has been rapidly developed by many mathematicians. They found some interesting complete minimal surfaces as follows: the catenoid that is a surface of revolution about the $\mathbb{R}$-axis; the helicoid that is ruled by the horizontal geodesic; the Riemann type minimal surface that is foliated by horizontal circles and lines; the Scherk type minimal surface that is a minimal graph over an ideal polygon and is asymptotic to vertical planes (see [3], [8], [13], [14]).

By Hauswirth and Rosenberg [4], some properties of complete minimal surfaces of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$ have been revealed. The vertical plane $\Gamma \times \mathbb{R}$, where $\Gamma$ is a complete geodesic in $\mathbb{H}^2$, is clearly a complete minimal surface of finite total curvature. Apart from the vertical plane, the only such surface known to exist is the Scherk type minimal surface. Both surfaces are simply connected. So Hauswirth and Rosenberg [4] raised a natural question: is there a non-simply connected complete minimal surface of finite total curvature in $\mathbb{H}^2 \times \mathbb{R}$? In particular, is there a minimal annulus of total curvature $-4\pi$? Note that the rotational catenoid has infinite total curvature.

In this paper, using the conjugate surface method in $\mathbb{H}^2 \times \mathbb{R}$ we construct complete embedded minimal surfaces with $k$ vertical planar ends and total curvature $-4(k - 1)\pi$, giving an affirmative answer to Hauswirth and Rosenberg’s question.

The conjugate surface construction is initiated by Smyth [17], who constructed an embedded minimal disk in a tetrahedron $T \subset \mathbb{R}^3$ which is perpendicular to $\partial T$. Then Karcher, Rossman and others made some complete minimal surfaces by adopting this conjugate surface method (see [9], [10], [12]).

Recently Daniel [2] and Hauswirth, Sa Earp and Toubiana [5] generalized the notion of conjugate surface to $\mathbb{H}^2 \times \mathbb{R}$. Our construction of new minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is based on their theory. We construct a minimal graph $\Delta^k$ over an infinite triangle in $\mathbb{H}^2$ such that $\Delta^k$ is asymptotic to a vertical plane and is bounded by two horizontal geodesics (one finite, the other infinite) making an angle of $\pi/k$ and one vertical infinite geodesic (see Figure 1). It turns out that the conjugate
surface of $\Delta^k$ is also a minimal graph which is perpendicular along its boundary to a horizontal plane and the two vertical planes making an angle of $\pi/k$ in $\mathbb{H}^2 \times \mathbb{R}$.

By reflecting the conjugate surface across these planes we can construct a non-simply connected, genus zero, complete embedded minimal surface $\Sigma_k$ with total curvature $-4(k-1)\pi$ which is asymptotic to $k$ vertical planes, $k > 1$ (Theorem 4.1). This is similar to the $k$-noid of $\mathbb{R}^3$, but a remarkable difference is that $\Sigma_k$ is embedded in $\mathbb{H}^2 \times \mathbb{R}$ whereas the $k$-noid has self intersection in $\mathbb{R}^3$ if $k \geq 3$.

If we extend the minimal graph $\Delta^2$ by $180^\circ$-rotations about the horizontal boundary geodesics, we obtain a minimal graph $\Delta_n$ which is bounded by two vertical geodesics (see Figure 2). Rotating $\Delta_n$ by $180^\circ$ about the vertical boundary geodesics repeatedly, we obtain a simply connected complete embedded minimal surface which is singly periodic. This surface is different from the ruled helicoid of $\mathbb{H}^2 \times \mathbb{R}$ because it is not ruled and because its fundamental piece has finite total curvature $-4\pi$ whereas the fundamental piece of the ruled helicoid has infinite total curvature (see Theorem 3.2).

2. Preliminaries

In $\mathbb{H}^2 \times \mathbb{R}$ we consider the disk model for the hyperbolic plane $\mathbb{H}^2$ and solid cylinder model for whole space. Let $x, y$ denote the coordinates in $\mathbb{H}^2$ and $t$ denote the coordinate in $\mathbb{R}$. Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain. In $\mathbb{H}^2 \times \{0\}$, we denote $\partial \Omega = \partial \Omega \cup \partial_\infty \Omega$, where the boundary part is $\partial \Omega \subset \mathbb{H}^2 \times \{0\}$ and the ideal boundary part is $\partial_\infty \Omega \subset \partial_\infty \mathbb{H}^2 \times \{0\}$. Consider a $C^2$ function $t = u(x, y)$. The vertical minimal surface equation in $\mathbb{H}^2 \times \mathbb{R}$ is the following:

\[
\text{div}_\mathbb{H} \left( \frac{\nabla u}{W_u} \right) = 0,
\]

where $\text{div}_\mathbb{H}$ and $\nabla u$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla u|^2_\mathbb{H}}$, $|\cdot|_\mathbb{H}$ being the norm in $\mathbb{H}^2$.

In the disk model for $\mathbb{H}^2$,

\[
\mathbb{H}^2 = \{ (x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1 \},
\]

with the metric $ds^2 = \left( \frac{2}{1-x^2-y^2} \right)^2 (dx^2 + dy^2)$, the vertical minimal surface equation (2.1) becomes as follows:

\[
(1 + D^2(x, y)u_y^2) u_{xx} + (1 + D^2(x, y)u_x^2) u_{yy} - 2D^2(x, y)u_x u_y u_{xy} + D(x, y)(xu_x + yu_y)(u_x^2 + u_y^2) = 0,
\]

where $D(x, y) = \frac{1-x^2-y^2}{2}$.

We refer to the existence theorem of minimal surfaces.

**Theorem 2.1.** (Corollary 4.1 of [15])

Let $\Omega \subset \mathbb{H}^2 \times \{0\}$ be a domain and let $g : \partial \Omega \cup \partial_\infty \Omega \to \mathbb{R}$ be a bounded function everywhere continuous except perhaps at a finite set $S \subset \partial \Omega \cup \partial_\infty \Omega$. Assume that the finite boundary $\partial \Omega$ is convex. Then $g$ admits an extension $u : \overline{\mathbb{H}} \setminus S \to \mathbb{R}$ satisfying the vertical minimal surface equation (2.1). Furthermore, the total boundary of the graph of $u$ (that is the finite and ideal boundary) is the union of the graph of $g$ on $(\partial \Omega \cup \partial_\infty \Omega \setminus S)$ with the vertical segments

\[
\{ (q, t) | t \in [A := \liminf_{x \to q} g(x), B := \limsup_{x \to q} g(x)], x \in \partial \Omega \cup \partial_\infty \Omega \}
\]

at any $q \in S$.

**Theorem 2.2.** (Monotone convergence theorem of [1])

Let $\{u_n\}$ be a monotone sequence of solutions of (2.1) in $\Omega$. If the sequence $\{|u_n|\}$
is bounded at one point of $\Omega$, then there is a non-empty open set $U \subset \Omega$ (the convergence set) such that $\{u_n\}$ converges to a solution of (2.7) in $U$. The convergence is uniform on compact subsets of $U$ and the divergence is uniform on compact subsets of $\Omega - U = V$. $V$ is called the divergence set.

The following well-known theorems are the maximum principle for minimal surfaces. It is a special case of a lemma by Schoen [16], and is proven there.

**Theorem 2.3. (Maximum principle)**

1. (Interior maximum principle) Let $\Sigma_1$ and $\Sigma_2$ be minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Suppose $p$ is an interior point of both $\Sigma_1$ and $\Sigma_2$, and suppose $T_p(\Sigma_1) = T_p(\Sigma_2)$. If $\Sigma_1$ lies on one side of $\Sigma_2$ near $p$, then $\Sigma_1 = \Sigma_2$.

2. (Boundary point maximum principle) Suppose $\Sigma_1, \Sigma_2$ have $C^2$-boundaries $C_1, C_2$. Furthermore, suppose the tangent planes of both $\Sigma_1, \Sigma_2$ agree at $p$, i.e. suppose $T_p(\Sigma_1) = T_p(\Sigma_2), T_p(C_1) = T_p(C_2)$. If, near $p$, $\Sigma_1$ lies to one side of $\Sigma_2$, then $\Sigma_1 = \Sigma_2$.

By Daniel [2] and by Hauswirth, Sa Earp and Toubiana [5], we have the following two equivalent concepts of associate and conjugate surfaces. Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a surface equipped with a connection $\nabla$. Let $N$ denote its unit normal vector field, $J$ denote the rotation by angle $\frac{\pi}{2}$ on $T\Sigma$ and $S$ denote a field of symmetric operator $S_p : T_p\Sigma \to T_p\Sigma$ for each $p \in \Sigma$. Let $T$ be the projection of the vertical vector $\frac{\partial}{\partial t}$ onto the tangent space $T\Sigma$ of $\Sigma$ and $\nu = \langle N, \frac{\partial}{\partial t} \rangle$. We have $|T|^2 + \nu^2 = 1$. Let $TC(\Sigma)$ denote the total curvature of $\Sigma$, $TC(\Sigma) = \int_{\Sigma} K dA$ where $K(p) = \det S_p - (1 - |T_p|^2)$ (see, for instance, Daniel [2] or Hauswirth and Rosenberg [4] for details). We set

$$S_\theta = e^{\theta J}S = (\cos \theta)S + (\sin \theta)JS,$$

$$T_\theta = e^{\theta J}T = (\cos \theta)T + (\sin \theta)JT.$$

**Theorem 2.4. (Conjugate minimal surface I, [2])**

Let $\Sigma$ be a simply connected surface and $X : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ a conformal minimal immersion. Let $N$ be the normal, $S$ be the symmetric operator on $\Sigma$ induced by the shape operator of $X(\Sigma)$. Let $T$ and $\nu$ be defined as above. Let $z_0 \in \Sigma$. Then there exists a unique family $(X_\theta)_{\theta \in \mathbb{R}}$ of conformal minimal immersions $X_\theta : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ such that

1. $X_\theta(z_0) = X(z_0)$ and $dX_\theta(z_0) = dX(z_0)$,
2. the metrics induced on $\Sigma$ by $X$ and $X_\theta$ are the same,
3. the symmetric operator on $\Sigma$ induced by the shape operator of $X_\theta$ is $S_\theta$,
4. $\frac{\partial}{\partial \theta} = dX_\theta(T_0) + \nu N_0$, where $N_0$ is the unit normal to $X_\theta$.

Moreover the family $X_\theta$ is continuous with respect to $\theta$, and $X_0 = X$. The family of immersions $(X_\theta)_{\theta \in \mathbb{R}}$ is called the associate family of the immersion $X$. In particular the immersion $X_{\frac{\pi}{2}}$ is called the conjugate immersion of the immersion $X$.

Let $X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion. Then $\varphi$ is a harmonic map to $\mathbb{H}^2$ and $h$ is a harmonic function. The Hopf differential of $\varphi$ is the following holomorphic 2-form:

$$Q\varphi = 4\left(\frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial \bar{z}}\right)dz^2.$$

Because of conformality of $X$, $Q\varphi = -4(\frac{\partial h}{\partial z})^2 dz^2$, where $z = x + iy$ is a local coordinate on $\Sigma$ and $h = \pm \text{Re} \int 2i\sqrt{Q\varphi}dz$.

**Theorem 2.5. (Conjugate minimal surface II, [2], [5])**

Let $X = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a conformal minimal immersion, and $X_\theta = (\varphi_\theta, h_\theta)$ its associate family of conformal minimal immersions. In particular the immersion
Figure 1. Left: the graph of $u_n$; Right: the graph of $u$.

$X_\pi$ is called the conjugate immersion of the immersion $X$. Let $h_\pi$ be the harmonic conjugate of $h$. Then we have

$$Q\varphi_\theta = e^{-2\sqrt{\pi\theta}}Q\varphi, \quad h_\theta = (\cos \theta)h + (\sin \theta)h_\pi.$$  

Now, we refer to Krust’s type theorem for minimal vertical graphs and associate family of surfaces in $\mathbb{H}^2 \times \mathbb{R}$. We call that $G$ is a vertical graph in $\mathbb{H}^2 \times \mathbb{R}$ if $G$ is graph of $g$, where $g : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$.

Theorem 2.6. (Krust’s type theorem, Theorem 14 of [5])

Let $X(\Omega)$ be a minimal vertical graph on a convex domain $\Omega \subset \mathbb{H}^2$. Then the associate surface $X_\theta(\Omega)$, $\theta \in \mathbb{R}$ is also a vertical graph.

We can extend a minimal surface across its special boundary.

Theorem 2.7. (Schwarz reflection principle, [11])

Suppose a minimal surface $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ containing a curve $\Upsilon$ as its boundary.

1. $\Upsilon$ is a horizontal or vertical geodesic line then $\Sigma$ can be extended smoothly across $\Upsilon$ by $180^\circ$-rotation about $\Upsilon$.
2. $\Upsilon$ lies in a plane. $\Upsilon$ is a geodesic of $\Sigma$ and it is not a horizontal or vertical geodesic line, and $\Sigma$ meets orthogonal to the plane along $\Upsilon$ then $\Sigma$ can be extended smoothly across $\Upsilon$ by reflection through the plane containing $\Upsilon$.

3. Simply connected complete embedded minimal surface

Lemma 3.1. Let $0 < \alpha < 1$ and integer $k \geq 2$ be given. Let $D$ be a domain in $\mathbb{H}^2$ with $2k$ vertices $p_{2m-1} = \alpha e^{\sqrt{-1} \frac{(2m-1)\pi}{k}}$, $p_{2m} = e^{\sqrt{-1} \frac{(2m-1)\pi}{k}}$, $m = 1, \ldots, k$ and $2k$ sides $A_m$ be a geodesic from $p_{2m-1}$ to $p_{2m}$, $B_m$ be a geodesic from $p_{2m}$ to $p_{2m+1}$, $m = 1, \ldots, k$ and $p_1 = p_{2k+1}$.

Then there exists a unique (up to a vertical translation) embedded minimal surface $\Sigma(\alpha, k)$ which has vertical geodesic lines $V_m$ through the $p_{2m-1}$, $m = 1, \ldots, k$, as its boundary and the surface is of finite absolute total curvature at most $-\int KdA \leq (2k - 2)\pi$. More precisely, $\Sigma(\alpha, k)$ is the graph of a function $u : D \to \mathbb{R}$ with $u|_{A_m} = +\infty$ and $u|_{B_m} = -\infty$, $m = 1, \ldots, k$. 
Proof. Let $L_1$ be a geodesic segment from the origin 0 of $\mathbb{H}^2$ to $p_1$, $L_2$ a geodesic ray from 0 to $p_2$ and $\Gamma$ a geodesic ray from $p_1$ to $p_2$. Let $\Omega$ be a convex domain bounded by $L_1$, $L_2$ and $\Gamma$. Let $\tilde{\Gamma}$ be the complete geodesic containing $\Gamma$. For each $n \in \mathbb{N}$, let $g$ be a function on $\partial \Omega$ such that $g = 0$ on $L_1 \cup L_2$ and $g = n$ on $\Gamma$. By Theorem 2.1 there is a unique function $u_n : \Omega \to \mathbb{R}$ satisfying $u_n|_{L_i} = 0$, $i = 1, 2$ and $u_n|_{\Gamma} = n$ and the minimal surface equation (2.1). By the maximum principle, $\{u_n\}$ is a monotone increasing sequence with respect to $n$.

To show that the limit of the sequence $\{u_n\}$ exists, we need to find a suitable barrier. Let $E$ be the component of $\mathbb{H}^2 \setminus \tilde{\Gamma}$ which contains the domain $\Omega$. There exists a function $v \geq 0$ defined on $E$, asymptotic to $+\infty$ on $\tilde{\Gamma}$ and to zero on $\partial\infty(E)$ and $v$ satisfies the minimal surface equation (2.1) (see [1], [13]). So $v$ is a suitable barrier for the sequence $\{u_n\}$.

By the monotone convergence theorem we can find the limit function $u$ over $\Omega$ of the sequence $\{u_n\}$ such that $u|_{\Gamma} = +\infty$, $u|_{L_i} = 0$, $i = 1, 2$ and at $p_1$: $\Delta^k$, the graph of $u$, has a vertical geodesic ray as its boundary (see Figure 1).

Since $\Delta^k$ lies between the graph of $v$ and the vertical plane $\tilde{\Gamma} \times \mathbb{R}$, outside of a compact part of $\Delta^k$ uniformly converges to a vertical plane. We prove in the following that $\Delta^k$ has finite total curvature.

More precisely, let $\{q_j|q_j \in \tilde{\Gamma}\}$ be a sequence such that $|q_j - q_2|_{\mathbb{R}^2}$, the Euclidean distance between $q_j$ and $p_2$ on the disk, is monotone decreasing to zero. Let $\Omega(j)$ be a compact domain of the convex domain $\Omega$ with $\partial \Omega(j) = \overline{p_1q_j} \cup \overline{p_2q_j} \cup \overline{\Gamma q_j}$, where $\overline{ab}$ indicates the geodesic line segment in $\mathbb{H}^2$ from a to b. For $n, j \in \mathbb{N}$ fixed, we denote by $u_n(j)$ the function on $\Omega(j)$ which equals $n$ on $\overline{p_1q_j}$ and zero on $\overline{p_2q_j}$ and satisfies (2.1). Let $P$ be a point in $\mathbb{H}^2 \times \mathbb{R}$, we denote $P = (p, t)$, where $p$ is the complex coordinate of $\mathbb{H}^2$ and $t$ is the coordinate of $\mathbb{R}$. Denote $v_1(j) = (0, 0)$, $v_2(j) = (p_1, 0)$, $v_3(j) = (p_1, n)$, $v_4(j) = (q_j, n)$ and $v_5(j) = (q_j, 0)$, and $\gamma_i(j)$ be the geodesic segment in $\mathbb{H}^2 \times \mathbb{R}$ from $v_i(j)$ to $v_{i+1}(j)$, $i = 1, \ldots, 5$ and $v_6(j) = v_1(j)$. The graph of $u_n(j)$ denote by $\Delta^k_n(j)$, is bounded by $\gamma_i(j)$, $i = 1, \ldots, 5$. Applying the Gauss-Bonnet formula to $\Delta^k_n(j)$, we have:

$$\int_{\Delta^k_n(j)} K_n(j) + \sum_{i=1}^5 \int_{\gamma_i(j)} k_{g,i}(j) + \sum_{i=1}^5 \theta_i(j) = 2\pi,$$

where $K_n(j)$ is the Gaussian curvature function of $\Delta^k_n(j)$ on the domain $\Omega(j)$ and 0 on $\Omega - \Omega(j)$, and $k_{g,i}(j)$ is a geodesic curvature function of $\gamma_i(j)$, $i = 1, \ldots, 5$, and $\theta_i(j)$ is the exterior angle at $v_i(j)$, $i = 1, \ldots, 5$. By the Gauss equation, the Gaussian curvature function $K$ is nonpositive for any minimal surfaces in $\mathbb{H} \times \mathbb{R}$ (see [4]). Since $k_{g,i}(j)$ is identically zero and $\theta_i(j) = (k-1)/k \pi - \angle p_2q_i$, $\theta_i(j) = \pi$, $i = 2, \ldots, 5$, the total curvature of $\Delta^k_n(j)$ is $((1-k)/k) \pi - \angle p_2q_i$. As $j$ goes to infinity, the sequence of $\{u_n(j)\}$ converges monotonically to the previous function $u_n$ on $\Omega$ and the sequence of $\{\angle p_2q_i\}$ converges to zero. By theorem 2.2, the sequence of $\{u_n(j)\}$ converges uniformly on compact sets of $\Omega$ to $u_n$. By Fatou’s lemma, the absolute value of total curvature of $\Delta^k_n(j)$ is at most $|(1-k)/k|\pi$ for any $n$.

Similarly, as $n$ goes to infinity the absolute value of total curvature of $\Delta^k$ is at most $|(1-k)/k|\pi$.

Using the Schwarz reflection principle, we extend $\Delta^k$ about the geodesic $L_2$, extend again about the image of $L_1$, again about the image of $L_2$, and so forth. After $2k$ extensions we get $\Sigma(\alpha, k)$ which is an embedded minimal surface with $2k$ congruent pieces and $k$ vertical geodesics passing $p_{2m-1}$, $m = 1, \ldots, k$. So the absolute total curvature of $\Sigma(\alpha, k)$ is at most $(2k - 2)\pi$. By denoting $u$ a extended function defined on $D$ of the previous function $u$, the proof is completed. \qed
In case of $k = 2$, the $\Delta_v = \Sigma(\alpha, 2)$ has two vertical geodesic lines, $V_1$ and $V_2$. By the Schwarz reflection principle, we can extend the $\Delta_v$ to the complete minimal surface $\Sigma(\alpha)$ which is singly periodic. Hence the following theorem holds (see Figure 2).

**Theorem 3.2.** In $\mathbb{H}^2 \times \mathbb{R}$, there is a simply connected complete embedded minimal surface $\Sigma(\alpha)$ which is singly periodic under the horizontal hyperbolic translation $T_\alpha$, $|T_\alpha|_{\mathbb{H}} = \frac{4}{\alpha}$. And the fundamental piece $\Sigma(\alpha)/T_\alpha$ has finite total curvature $-4\pi$.

**Remark 3.3.**

1. We will compute the total curvature of $\Sigma(\alpha)/T_\alpha$ in the next section.
2. Let $\Delta^k = \Delta(\alpha, k)$ be a fundamental piece of $\Sigma(\alpha, k)$. If $\alpha_1 \neq \alpha_2$, $\Delta(\alpha_1, k)$ and $\Delta(\alpha_2, k)$ can not be conformally equivalent. So we have one-parameter family of $\Sigma(\alpha)$ for $\alpha$.

## 4. Nonsimply Connected Complete Embedded Minimal Surface

**Theorem 4.1.** For each integer $k \geq 2$, there exists a nonsimply connected complete embedded minimal surface $\Sigma(k) \subset \mathbb{H}^2 \times \mathbb{R}$ satisfying the following:

1. $\Sigma(k)$ has finite total curvature $-4(k - 1)\pi$;
2. $\Sigma(k)$ is conformal to a $k$-punctured 2-dimensional sphere;
3. $\Sigma(k)$ is symmetric about $k$ vertical planes and one horizontal plane.

**Proof.** We take the fundamental piece $\Delta = \Delta^k$ of $\Sigma(\alpha, k)$ in Lemma 3.1 if $k \geq 3$ we assume that $\alpha \geq \alpha(k)$, where $\alpha(k)$ is the value that $\overrightarrow{op_1}$ is perpendicular to $\overrightarrow{p_1p_2}$ at $p_1$. The $\Delta$ is the graph of $u$ over $\Omega$ and is bounded by two geodesic rays and one geodesic segment. Let $R_1$ be a vertical geodesic ray from $(p_1, 0)$, $R_2$ a horizontal geodesic segment from $(p_1, 0)$ to the origin $(0, 0)$ of $\mathbb{H}^2 \times \mathbb{R}$, and $R_3$ a horizontal geodesic ray from $(0, 0)$ to $(p_2, 0)$. Here we use the same notations as in Lemma 3.1 except $L_i, i = 1, 2$. Since we consider geodesics in $\mathbb{H}^2 \times \mathbb{R}$, we write $R_{i+1}, i = 1, 2$ instead of $L_i, i = 1, 2$.

Let $N$ be a normal vector field of $\Delta$. Since $\Delta$ is a simply connected, we write its immersion by $X = (\varphi, h) : D^S = D \setminus \{S, \text{segment}\} \to \Delta$, where $D$ is a closed unit.
Let $\Delta_\Sigma$ be a geodesic surface of $\Delta$ with its immersion denoted by $X_\Sigma = (\phi_\Sigma, h_\Sigma) : D^\Sigma \to \Delta_\Sigma$. Let $\gamma_1$ be an arclength parametrization of $R_1$. Note that the tangent vector field $\gamma'_1$ is identically $c_3$ along $R_1$ and $h_\Sigma$ is a harmonic conjugate of $h$. By Theorem 2.5 on $c_3$ we have

$$1 = dh \left( \frac{\partial}{\partial s} \right) = dh_\Sigma \left( J \frac{\partial}{\partial s} \right).$$

This means that the conormal vector field of $\Delta_\Sigma$ along $\tilde{R}_1$, the conjugate image of $R_1$, is identically $c_3$. So $\tilde{R}_1$ lies on a horizontal plane $\mathbb{H}^2 \times \{t\}$ and $\Delta_\Sigma$ is orthogonal to the horizontal plane along $\tilde{R}_1$. Without loss of generality we assume $t = 0$.

Let $\gamma_2$ be a parametrization of the horizontal geodesic segment $R_2$ and $\tilde{\gamma}_2$ a parametrization of curve $\tilde{R}_2$, the conjugate image of $R_2$. Since the conjugate transformation preserves the metric, $\tilde{R}_2$ is also a geodesic curve on $\Delta_\Sigma$. By (3) of Theorem 2.5 we have

$$0 = \langle \nabla_{\gamma'_2} \gamma'_2, N \rangle = \langle S \gamma'_2, \gamma'_2 \rangle = \langle S_\Sigma \gamma'_2, J \gamma'_2 \rangle,$$

where $\nabla$ is the connection in $\mathbb{H}^2 \times \mathbb{R}$. This means that $\tilde{\gamma}_2$ is a line of curvature. On $c_2$ we have

$$0 = dh \left( \frac{\partial}{\partial s} \right) = dh_\Sigma \left( J \frac{\partial}{\partial s} \right).$$

So the conormal vector field of $\Delta_\Sigma$ along $\tilde{R}_2$ is orthogonal to $\frac{\partial}{\partial s}$ and $T$, the tangential part of $\frac{\partial}{\partial t}$. As a result, the curve $\tilde{\gamma}_2$ is a line of curvature associated to the field $T$.

**Lemma 4.2.** (See the proof of Proposition 15 of [18])

Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a surface transversal to each slice $\mathbb{H}^2 \times \{t\}$. Let $N$ be a normal field of $\Sigma$, $T$ a vector field on $\Sigma$ such that $dX(T)$ is the projection of $\frac{\partial}{\partial s}$ onto tangent plane $TX(\Sigma)$ and $\nu = \langle N, \frac{\partial}{\partial s} \rangle$. Let $c : \tau \in I \subset \mathbb{R} \to c(\tau) \in \Sigma$ be a line of curvature associated to the vector field $T$. Then $c(I)$ is contained in a vertical totally geodesic plane.

By Lemma 4.2, $\tilde{R}_2$ is contained in a vertical plane $\Pi_1$. Using an isometry in $\mathbb{H}^2 \times \mathbb{R}$ we can assume that $\Pi_1 = \Gamma_1 \times \mathbb{R}$, where $\Gamma_1$ is a geodesic in $\mathbb{H}^2$ through zero. Let $N_\Sigma$ be a normal vector field of $\Delta_\Sigma$ and $N_1$ the unit normal vector of $\Pi_1$. Since $\tilde{\gamma}_2$ is a line of curvature and $\tilde{\gamma}'_2 \subset T_{\tilde{\gamma}_2} \Pi_1$, we have

$$\frac{d}{ds} \langle N_\Sigma, N_1 \rangle = \langle \nabla_{\tilde{\gamma}'_2} N_\Sigma, N_1 \rangle = \langle \mu \tilde{\gamma}'_2, N_1 \rangle = 0,$$

where $\mu$ is a real valued function. So $\langle N_\Sigma, N_1 \rangle = C_0$, where $C_0$ is a constant. Since $\nu = \langle N, \frac{\partial}{\partial s} \rangle = 1$ at $(0, 0)$ and is preserved by the conjugate transformation, $\nu = 1$ at $0$, the conjugate image of $(0, 0)$, i.e. $N_\Sigma = \frac{\partial}{\partial s}$. So $C_0 = \langle \frac{\partial}{\partial s}, N_1 \rangle = 0$. Hence $\Delta_\Sigma$ meets $\Pi_1$ orthogonally. By the same argument $\tilde{R}_3$, the conjugate image of $R_3$, is also contained in a vertical plane $\Pi_2$ and $\Delta_\Sigma$ meets $\Pi_2$ orthogonally. Since $\tilde{0} \in \tilde{R}_2$ and $0 \in \tilde{R}_3$, $\Pi_1 \cap \Pi_2 \neq \emptyset$. So we can assume that $\Pi_2 = \Gamma_2 \times \mathbb{R}$, where $\Gamma_2$ is a geodesic in $\mathbb{H}^2$ through zero.

Because $\nu = 1$ at $0$ and the angle between $R_2$ and $R_3$ is $\frac{\pi}{3}$, the angle between $\tilde{R}_2$ and $\tilde{R}_3$ is also $\frac{\pi}{3}$. This implies that the angle between $\Pi_1$ and $\Pi_2$ is $\frac{\pi}{3}$. Because
Figure 3. In case of $m = 2$, the boundary behavior of $\Delta_{\tilde{z}}$.

$\Delta$ is a graph over the convex domain $\Omega$, $\Delta_{\tilde{z}}$ is also a graph over $\Lambda \subset \mathbb{H}^2 \times \{0\}$ by Krust's type theorem. This theorem implies that $\Delta_{\tilde{z}}$ is embedded.

We claim that $\Delta_{\tilde{z}}$ is bounded by $\Pi_1, \Pi_2$ and $\mathbb{H}^2 \times \{0\}$.

We first focus on $\tilde{R}_2$. Define $d_i(q) = \text{dist}_{R_2}(q, \Gamma_i)$, the Euclidean distance from $q$ to $\Gamma_i$, $i = 1, 2$ on $\Pi_1, \Pi_2$. Since $\Delta_{\tilde{z}}$ is a graph over $\Lambda$, $\tilde{R}_2$ is also graph over $\Lambda \cap \Gamma_1$. We claim that $d_1(q)$ on $\Pi_1$ cannot have any interior critical point. Suppose $d_1(q)$ has a local maximum or minimum at $q_1 \in \tilde{R}_2$. We have $\nu(q_1) = 1$. Let $Q_1 \in R_2$ be a preimage of $q_1$. Since $\nu$ is preserved by the conjugate transformation, $\nu(Q_1)$ is also one. That is, at $Q_1$ the normal vector of $\Delta$ is $e_3$. Extend the $\Delta$ along $R_2$, the $Q_1$ is an interior point of the extended minimal surface. The normal vector of the extended minimal surface at $Q_1$ coincides with the one of the horizontal plane $\mathbb{H}^2 \times \{0\}$ and the intersection curve between the extended minimal surface and $\mathbb{H}^2 \times \{0\}$ is just a line. This contradicts to the interior maximum principle.

Since $\nu$ of $\Delta$ varies from 0 to 1 as $Q$ varies from $(p_1, 0)$ to $(0, 0)$, $\nu$ of $\Delta_{\tilde{z}}$ varies from 0 to 1 as $q$ varies from $\tilde{p}_1$ to $\tilde{0}$ along $\tilde{R}_2$. Here $\tilde{p}_1$ is the conjugate image of $(p_1, 0)$.

Similarly, $\tilde{R}_3$ is also a graph over $\Lambda \cap \Gamma_2$ and also $d_2(q)$ on $\Pi_2$ cannot have any interior local maximum or minimum. Since the $\nu$ converges to 0 as $Q \in R_3$ moves toward $(p_2, 0)$, the $\nu$ also converges to 0 as $q \in \tilde{R}_3$ varies far away from $\tilde{0}$. So the $\tilde{R}_3$ asymptotically approaches to a vertical geodesic which is orthogonal to $\Gamma_2$.

Now we consider the behavior of $\tilde{R}_1$. The curve $\tilde{R}_1$ cannot intersect with $\Gamma_1$ and $\Gamma_2$. Suppose not $\tilde{R}_1$ intersects $\Gamma_1$ at $s_1$. We extend $\Delta_{\tilde{z}}$ with respect to $\Pi_1$. Then the extended surface $\tilde{\Delta}_{\tilde{z}} = \Delta_{\tilde{z}} \cup \Delta^*_{\tilde{z}}$ has a self-intersection, where $\Delta^*_{\tilde{z}}$ is the mirror image of $\Delta_{\tilde{z}}$ with respect to $\Pi_1$. But $\tilde{\Delta}_{\tilde{z}}$ is a graph on the convex domain $\Omega \cup \Omega^*_{L_1}$, where $\Omega^*_{L_1}$ is the $180^\circ$-rotated domain about the $L_1$. This contradicts to the Krust’s type theorem. Similarly, $\tilde{R}_1$ cannot intersect with $\Gamma_2$. So $\Delta_{\tilde{z}}$ is bounded by $\Pi_1, \Pi_2$ and $\mathbb{H}^2 \times \{0\}$.
We claim that $\tilde{R}_1$ is not convex with respect to $\Lambda$ at any point. Suppose $\tilde{R}_1$ is convex at $q_0$. Because $\Delta_\tilde{z}$ is a graph over the domain $\Lambda \subset \mathbb{H}^2 \times \{0\}$, near $q_0$, $\Delta_\tilde{z}$ lies on one side of a vertical plane or a vertical strip of a vertical plane. We extend $\Delta_\tilde{z}$ with respect to $\mathbb{H}^2 \times \{0\}$, then the extended surface intersects with the vertical plane or the vertical strip of a vertical plane along a point or a line. This contradicts to the interior maximum principle. Since the $\tilde{R}_1$ is not convex with respect to $\Lambda$ and the length of $\tilde{R}_1$ is infinite, the only option is that $\tilde{R}_1$ goes to the ideal boundary $\partial_\infty \mathbb{H}^2 \times \{0\}$ (see Figure 3).

By the Schwarz reflection principle, we extend $\Delta_\tilde{z}$ which is of finite absolute total curvature at most $\frac{k-1}{k}\pi$ inductively about $\mathbb{H}^2 \times \{0\}$, $\Pi_1$ and its rotation around $\{0\} \times \mathbb{R}$-axis by $\frac{m}{k}\pi$ degrees, $m = 1, ..., k - 1$. In particular, $\Pi_2$ is the rotation of $\Pi_1$ around $\{0\} \times \mathbb{R}$-axis by $\frac{\pi}{k}$ degrees. Finally we get $\Sigma(k)$, a complete embedded surface of finite total curvature with $4k$ congruent fundamental pieces.

First, by Huber’s theorem $\Sigma(k)$ is conformal to a $k$-punctured 2-dimensional sphere (see [4] [6]). This implies that the segment $S$ is nothing but a point. Second, we apply Hauswirth and Rosenberg’s curvature estimation [4] to say that the Hopf map extends meromorphically to each puncture. Moreover, the degree of each pole depends in the number of curves are intersecting horizontal section at infinity. Since this number is one, the degree is zero. So the total curvature of $\Sigma(\alpha)$ is $-4(k-1)\pi$. Because the fundamental piece of $\Sigma(\alpha)$ is $-\frac{k-1}{k}\pi$, the total curvature of $\Sigma(\alpha)/T_\alpha$ is $-4\pi$. Third, $\mathbb{H}^2 \times \{0\}$, $\Pi_1$ and its rotation around $\{0\} \times \mathbb{R}$-axis by $\frac{m}{k}\pi$ degrees, $m = 1, ..., k - 1$ are symmetric planes. By [4] each end, a conformal parametrization of the punctured disk, is asymptotic to a vertical plane.

\textbf{Remark 4.3.}  
(1) The $\Sigma(k)$ is similar to the Jorge-Meeks $k$-noid in $\mathbb{R}^3$ [7]. And as in Remark 5.3, we have one-parameter family of minimal surfaces $\Sigma(k)$ with respect to $\alpha$.

(2) The $\Sigma(\alpha)$ (resp. $\Sigma(2)$) is quite similar to the Euclidean helicoid (resp. Euclidean catenoid). The $\Sigma(2)$ and the period $\Sigma(\alpha)/T_\alpha$ are conjugate minimal surfaces, in the sense of Theorem 2.4 or Theorem 5.5.

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