CIRCLE ACTIONS ON SIX DIMENSIONAL ORIENTED
MANIFOLDS WITH ISOLATED FIXED POINTS

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Abstract. Let the circle group act on a 6-dimensional compact oriented manifold \( M \) with isolated fixed points. The fixed point data of \( M \) is the collection of signs and weights at the fixed points. To classify such a manifold \( M \), the fixed point data is an essential information. Under an assumption that each isotropy submanifold is orientable, we classify the fixed point data of \( M \), by showing that we can convert it to the empty collection by performing a combination of a number of types of operations. We do so by proving that we can successively take equivariant connected sums of \( M \) at fixed points with \( S^6 \), \( \mathbb{CP}^3 \), and 6-dimensional analogue \( Z_1 \) and \( Z_2 \) of the Hirzebruch surfaces (and these with opposite orientations), to a fixed point free action on a compact oriented 6-manifold.

1. Introduction

Torus actions on compact manifolds have been studied in low dimensions. In dimensions 1 and 2 the classification results are simple. Torus actions and circle actions on 3- and 4-manifolds have been studied and classified in 1960’s and 1970’s. Raymond classified circle actions on compact 3-manifolds \([R]\). Orlik and Raymond proved that a 2-torus action on a simply connected closed orientable 4-manifold is \( S^4 \), or a connected sum of \( \mathbb{CP}^2 \), \( \overline{\mathbb{CP}^2} \), and the Hirzebruch surfaces \([OR1]\). Fintushel proved an analogous result that a circle action on a simply connected closed orientable 4-manifold is a connected sum of \( S^4 \), \( \mathbb{CP}^2 \), \( \overline{\mathbb{CP}^2} \), and \( S^2 \times S^2 \) \([F2]\). For non-simply connected case, Orlik and Raymond showed that a 2-torus action on a closed orientable 4-manifold is determined by its orbit data \([OR2]\). Fintushel showed that a circle action on a closed oriented 4-manifold is also determined by its orbit data \([F3]\). Also see \([F1]\), \([J1]\), \([P]\) for some classification results on certain 4-dimensional oriented \( S^1 \)-manifolds.

Circle actions on different types of 4-manifolds with fixed points have been also studied. Carrell, Howard, and Kosniowski studied complex surfaces \([CHK]\), Ahara and Hattori \([AH]\), Audin \([Au]\), and Karshon \([Ka]\) studied symplectic 4-manifolds. For both complex 4-manifolds and symplectic 4-manifolds, there is a common phenomenon; we can successively blow down...
any such manifold to one of minimal manifolds, which are $\mathbb{CP}^2$, the Hirzebruch surfaces, and ruled surfaces. Also see a generalization of these results to almost complex 4-manifolds when there are finite fixed points [J2].

We consider a torus action on a manifold that has a non-empty finite fixed point set. Because the dimension of a manifold and the dimension of its fixed point set have the same parity, the next dimension to consider is dimension 6. For any type of manifold, classifying torus actions on 6-manifolds is more difficult and known results for torus actions on 6-manifolds assume more restrictions, on actions and/or manifolds. For instance, McGavran and Oh considered $T^3$-actions on closed orientable 6-manifolds that have orbit space $D^3$ [MO], Kuroki considered $T^3$-actions on closed oriented 6-manifolds with fixed points and with vanishing odd cohomology groups [Ku], Tolman considered Hamiltonian $S^1$-actions on symplectic manifolds with minimal cohomology groups [T], and Ahara and the author considered $S^1$-actions on almost complex 6-manifolds with small number of fixed points [Ah], [J3].

In this paper, we study circle actions on 6-dimensional compact oriented manifolds with isolated fixed points. A geometric statement of our result is that for a 6-dimensional compact oriented $S^1$-manifold with isolated fixed points, we can successively take equivariant connected sums to a fixed point free action, where we take each connected sum with $S^6$, $\mathbb{CP}^3$, $Z_1$, or $Z_2$.

**Theorem 1.1.** Let the circle group $S^1$ act effectively on a 6-dimensional compact connected oriented manifold $M$ with a discrete fixed point set. Suppose that for an even integer $w \geq 4$, each component of the isotropy submanifold $M^{Z_w}$ containing an $S^1$-fixed point is orientable. Then we can successively take equivariant connected sums at fixed points of $M$ and at fixed points of circle actions on $S^6$, $\mathbb{CP}^3$, and 6-dimensional analogue $Z_1$ and $Z_2$ of the Hirzebruch surfaces (and these with opposite orientations) to construct another 6-dimensional compact connected oriented manifold, which is equipped with a fixed point free circle action. There is a definite procedure that this ends in a finite number of steps. The circle actions on $S^6$, $\mathbb{CP}^3$, $Z_1$, and $Z_2$ all have non-empty finite fixed point sets.

Let the circle group $S^1$ act on a $2n$-dimensional compact oriented manifold $M$ with a discrete fixed point set. Let $p$ be an isolated fixed point. The tangent space at $p$ decomposes into real 2-dimensional irreducible $S^1$-equivariant vector spaces

$$T_p M = \bigoplus_{i=1}^n L_{p,i},$$

where each $L_{p,i}$ is isomorphic to a complex 1-dimensional $S^1$-equivariant complex space on which the circle acts as multiplication by $g^{w_{p,i}}$ for all $g \in S^1 \subset \mathbb{C}$, where $w_{p,i}$ is a non-zero integer. For each $i$, we give an orientation of $L_{p,i}$ so that $w_{p,i}$ is positive. We call the positive integers $w_{p,i}$ **weights** at $p$. Let $\epsilon_M(p) = +1$ if the orientation on $M$ agrees with the orientation on the representation space $L_{p,1} \oplus \cdots \oplus L_{p,n}$, and $\epsilon_M(p) = -1$
otherwise. We call \( \epsilon_M(p) \) the sign of \( p \). If there is no confusion we shall omit the subscript \( M \) and use \( \epsilon(p) \). We define the fixed point data of \( p \) to be

\[
\Sigma_p := \{ \epsilon(p), w_{p,1}, \ldots, w_{p,n} \}.
\]

We will always write the sign of \( p \) first, and write the weights at \( p \) next. More precisely, the fixed point \( p \) must be an ordered pair \((\epsilon(p), \{w_{p,1}, \ldots, w_{p,n}\})\), where \( \{w_{p,1}, \ldots, w_{p,n}\} \) is the multiset of the weights at \( p \). For simplicity of notation, we will use \( \{\epsilon(p), w_{p,1}, \ldots, w_{p,n}\} \) instead of \((\epsilon(p), \{w_{p,1}, \ldots, w_{p,n}\})\).

When we write \( \epsilon(p) \) inside the fixed point data of \( p \), we shall omit 1 and only write its sign. We define the fixed point data of \( M \) to be a collection

\[
\Sigma_M := \bigcup_{p \in M^{S^1}} \{\epsilon(p), w_{p,1}, \ldots, w_{p,n}\}
\]

of fixed point datum of all fixed points of \( M \).

For a classification of a torus action on a manifold, the fixed point data is a necessary information. A combinatorial statement of our result classifies the fixed point data of any 6-dimensional compact oriented \( S^1 \)-manifold with isolated fixed points.

**Theorem 1.2.** Let the circle group \( S^1 \) act effectively on a 6-dimensional compact oriented manifold \( M \) with a discrete fixed point set. Suppose that for an even integer \( w \geq 4 \), each component of the isotropy submanifold \( M^{S^1w} \) containing an \( S^1 \)-fixed point is orientable. Then we can successively apply a combination of 5 types of operations to convert the fixed point data of \( M \) to the empty collection. There is a definite procedure that ends in a finite number of steps.

More precisely, to the fixed point data \( \Sigma_M \) of \( M \), we can apply a combination of the following operations to convert \( \Sigma_M \) to the empty set.

1. Remove \( \{+, A, B, C\} \) and \( \{-, A, B, C\} \) together.
2. Remove \( \{\pm, A, B, C\} \) and \( \{\mp, C - A, C - B, C\} \) where \( A < B < C \), and add \( \{\pm, A, B - A, C - A\} \) and \( \{\mp, B - A, C - B\} \).
3. Remove \( \{\pm, A, B, C\} \) and \( \{\pm, A, C - B, C\} \) where \( A < B < C \), and add \( \{\pm, C - B, C - A, A\} \), \( \{\pm, C - B, B, A\} \), \( \{\mp, C - B, B - A, A\} \), \( \{\pm, C - A, B - A, A\} \).
4. Remove \( \{\pm, A, A, C\} \) and \( \{\pm, A, C - A, C\} \) where \( 2A < C \), and add \( \{\pm, C - A, C - 2A, A\} \), \( \{\pm, C - A, A, A\} \), \( \{\pm, C - A, A, A\} \), \( \{\mp, C - 2A, A, A\} \).
5. Remove \( \{\pm, A, A, C\} \) and \( \{\pm, A, C - A, C\} \) where \( A < C < 2A \), and add \( \{\mp, C - A, 2A - C, A\} \), \( \{\pm, C - A, A, A\} \), \( \{\pm, C - A, A, A\} \), \( \{\mp, 2A - C, A, A\} \).
6. Remove \( \{\pm, C, A, A\} \) and \( \{\mp, C, C - A, C - A\} \) where \( 2A < C \), and add \( \{\pm, C - A, C - 2A, A\} \), \( \{\pm, C - A, A, A\} \), \( \{\pm, C - A, A, A\} \),
Theorem 1.2 means that given any such manifold $M$, we must be able to perform one of the operations in Theorem 1.2 to the fixed point data of $M$. After performing one of those, again we can perform one of those (unless we already have the empty collection), and so on until we get the empty set. The sign convention in Theorem 1.2 means the following: suppose that the fixed point data of $M$ contains $\{-, A, B, C\}$ and $\{+, C - B, A - B, A\}$. Then we can perform Operation (2) to remove these, and add $\{-, A, B - A, C - A\}$ and $\{+, B, B - A, C - B\}$. If the fixed point data of $M$ contains instead $\{+, A, B, C\}$ and $\{-, C - A, C - B, C\}$ (same fixed point data as above but with opposite signs), then we can perform Operation (2) to remove these and add $\{+, A, B - A, C - A\}$ and $\{-, B, B - A, C - B\}$.

The operations in Theorem 1.2 work in a way that they remove the biggest weight $C$; every weight in what we add is strictly smaller than $C$. Therefore, the process of Theorem 1.2 stops in a finite number of steps. Operation (1) corresponds to a connected sum with $S^6$, Operation (2) corresponds to a connected sum with $\mathbb{CP}^3$ (or $\overline{\mathbb{CP}^3}$), Operations (3) and (3') correspond to a connected sum with the manifold $Z_1$ (or $\overline{Z_1}$), Operations (4) and (4') correspond to a connected sum with the manifold $Z_2$ (or $\overline{Z_2}$), and Operation (5) corresponds to a connected sum with the manifold $Z_2^g \sharp Z_2$ in Example 3.8, which is a connected sum at fixed points of two copies of $Z_2$, one with reversed orientation. Unlike a tradition, we take each connected sum at two fixed points of $M$ and at two fixed points of an $S^1$-action on another manifold. The only difference between Operations (3) and (3') is some sign issue that occurs at two fixed points with fixed point data $\{\mp, C - B, B - A, A\}$ and $\{\pm, C - A, B - A, A\}$ in Operation (3) and two fixed points with fixed point data $\{\pm, C - B, A - B, A\}$ and $\{\mp, C - A, A - B, A\}$ in Operation (3'). In the former $A < B$ and in the latter $B < A$; since $B < A$ in the latter case, the fixed point data $\{\pm, C - B, B - A, A\}$ of the former becomes $\{\mp, C - B, A - B, A\}$ of the latter, if we allow a weight to be negative. Similarly, Operations (4) and (4') only differ by such a sign issue at two fixed points.

It is natural to ask if we can drop the assumption on orientability of isotropy submanifolds.

**Question 1.3.** Let the circle group $S^1$ act effectively on a 6-dimensional compact oriented manifold $M$ with a discrete fixed point set. Let $w \geq 4$. Then is each component of the isotropy submanifold $M^2w$ orientable if it contains an $S^1$-fixed point?

If a 6-dimensional compact oriented $S^1$-manifold has exactly 2 fixed points, the two fixed points have the same multiset of weights and have different signs (see Theorem 2.7); by performing Operation (1) of Theorem 1.2 once to the fixed point data of the manifold, we reach the empty set.
We illustrate how Theorem 1.2 works with an example.

**Example 1.4.** Let the circle act on \( \mathbb{C}P^3 \) by
\[
g \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : g^a z_1 : g^b z_2 : g^c z_3]
\]
for some positive integers \( a < b < c \), for all \( g \in S^1 \subset \mathbb{C} \) and for all \( [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \). The action has 4 fixed points \( [1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0] \), and \( [0 : 0 : 0 : 1] \). The fixed point data of \( \mathbb{C}P^3 \) is
\[
\{+, a, b, c\}, \{-, a, b - a, c - a\}, \{+, b - a, c - b\}, \{-, c - a, c - b\},
\]
see Example 3.2. Because the fixed points \( [1 : 0 : 0 : 0] \) and \( [0 : 0 : 0 : 1] \) have the biggest weight \( c \), we perform Operation (2) (which amounts to taking an equivariant connected sum with \( \mathbb{C}P^3 \) at two fixed points \( [1 : 0 : 0 : 0] \) and \( [0 : 0 : 0 : 1] \) for both manifolds) to remove \( \{+, a, b, c\} \) and \( \{-, c - a, c - b\} \) and add \( \{+, a, b - a, c - a\} \) and \( \{-, b - a, c - b\} \), to have a collection
\[
\{-, a, b - a, c - a\}, \{+, b - a, c - b\}, \{+, a, b - a, c - a\}, \{-, b - a, c - b\}.
\]
Next, on this collection we perform Operation (1) twice, first to remove \( \{-, a, b - a, c - a\} \) and \( \{+, a, b - a, c - a\} \) and second to remove \( \{+, b - a, c - b\} \) and \( \{-, b - a, c - b\} \) to reach the empty set.

Of course for the action on \( \mathbb{C}P^3 \) one can take an equivariant connected sum with the same action on \( \mathbb{C}P^3 \) at all fixed points, to get a fixed point free \( S^1 \)-manifold. However, for a general 6-dimensional compact oriented \( S^1 \)-manifold \( M \) with a discrete fixed point set, when the fixed point data of \( M \) contains \( \{+, a, b, c\} \) and \( \{-, c - a, c - b\} \), it need not contain \( \{+, b - a, c - b\} \) and \( \{+, a, b - a, c - a\} \).

The structure of this paper is as follows. In Section 2 we review necessary background on oriented \( S^1 \)-manifolds with discrete fixed points. In Section 3, we describe \( S^1 \)-actions on \( S^6 \), \( \mathbb{C}P^3 \), \( \mathbb{Z}_n \), and \( \mathbb{Z}_2 \mathbb{Z}_2 \) that we need for proving Theorems 1.1 and 1.2. In Section 4, we prove relationships between the fixed point datum of fixed points in an isotropy submanifold. Finally, in Section 5 we prove Theorems 1.1 and 1.2.

### 2. Background

Let the circle group \( S^1 \) act on a manifold \( M \). The **equivariant cohomology** of \( M \) is
\[
H^*_S^i (M) = H^i (M \times_{S^1} S^\infty).
\]
Suppose that \( M \) is oriented and compact. The projection map \( \pi : M \times_{S^1} S^\infty \to \mathbb{C}P^\infty \) induces a natural push-forward map
\[
\pi_* : H^*_S^i (M; \mathbb{Z}) \to H^{i - \dim M}(\mathbb{C}P^\infty; \mathbb{Z})
\]
for all \( i \in \mathbb{Z} \). This map is given by integration over the fiber \( M \), and also denoted by \( \int_M \).

**Theorem 2.1.** [ABBV localization theorem] [AB] *Let the circle group \( S^1 \) act on a compact oriented manifold \( M \). Let \( \alpha \in H^*_S^i (M; \mathbb{Q}) \). As an element of \( \mathbb{Q}(t) \),*
\[
\int_M \alpha = \sum_{F \subset M^{S}} \int_F \alpha |e_{S}(N_F)|
\]

where the sum is taken over all fixed components, and \(e_{S}(N_F)\) denotes the equivariant Euler class of the normal bundle to \(F\) in \(M\).

For a compact oriented manifold \(M\), the Atiyah-Singer index theorem states that the analytical index of the signature operator on \(M\) is equal to the topological index of the operator. As an application to a compact oriented \(S^1\)-manifold with a discrete fixed point set, it yields the following formula.

**Theorem 2.2.** [Atiyah-Singer index theorem] [AS] Let the circle group \(S^1\) act on a \(2n\)-dimensional compact oriented manifold \(M\) with a discrete fixed point set. Then the signature of \(M\) satisfies

\[
\text{sign}(M) = \sum_{p \in M^{S}} \epsilon(p) \prod_{i=1}^{n} \frac{1 + t^{w_{p,i}}}{1 - t^{w_{p,i}}}
\]

for all indeterminate \(t\), and is a constant.

As an application of Theorem 2.2, we obtain the following result.

**Proposition 2.3.** Let the circle act on a compact oriented manifold \(M\) with a discrete fixed point set. Suppose that \(\dim M \equiv 2 \mod 4\). Then the signature of \(M\) vanishes, and the number of fixed points \(p\) with \(\epsilon(p) = +1\) and the number of fixed point \(p\) with \(\epsilon(p) = -1\) are equal.

**Proof.** By the Hirzebruch signature theorem [H], the signature of \(M\) is equal to the \(L\)-genus of \(M\). Since \(\dim M \neq 0 \mod 4\), the \(L\)-genus of \(M\) vanishes; thus \(\text{sign}(M) = 0\). Taking \(t = 0\) in Theorem 2.2,

\[
\text{sign}(M) = \sum_{p \in M^{S}} \epsilon(p)
\]

and this proposition follows. \(\square\)

Let \(a\) be the smallest positive weight, that is, \(a = \min\{w_{p,i} \mid 1 \leq i \leq n, p \in M^{S}\}\). Manipulating the index formula of Theorem 2.2 as

\[
\text{sign}(M) = \sum_{p \in M^{S}} \epsilon(p) \prod_{i=1}^{n} \frac{1 + t^{w_{p,i}}}{1 - t^{w_{p,i}}} = \sum_{p \in M^{S}} \epsilon(p) \prod_{i=1}^{n} [(1 + t^{w_{p,i}}) \sum_{j=0}^{n} t^{jw_{p,i}}] =
\]

\[
\sum_{p \in M^{S}} \epsilon(p) \prod_{i=1}^{n} (1 + 2 \sum_{j=0}^{n} t^{jw_{p,i}})
\]

and comparing the coefficients of the \(t^a\)-terms, the below lemma holds.

**Lemma 2.4.** [J1, Mu] Let the circle act on a compact oriented manifold \(M\) with a discrete fixed point set. Let \(a\) be the smallest positive weight. Then
Consider an effective circle action on a compact oriented manifold $M$. Let $w > 2$ be an integer. As a subgroup of $S^1$, the group $\mathbb{Z}_w$ acts on $M$. The set $M^{\mathbb{Z}_w}$ of points in $M$ that are fixed by the $\mathbb{Z}_w$-action is a union of smaller dimensional closed submanifolds.

Lemma 2.5. Let the circle act effectively on an orientable manifold $M$. Let $w \geq 3$ be an odd integer. Then the $\mathbb{Z}_w$-fixed point set $M^{\mathbb{Z}_w}$ is orientable.

It is well-known that if the number of fixed points is odd, the dimension of the manifold is a multiple of 4.

Corollary 2.6. [J1] Let the circle act on a compact oriented manifold $M$. If the number of fixed points is odd, the dimension of $M$ is divisible by four.

If a circle action on a compact oriented manifold has two fixed points, the two fixed points have the same weights and have opposite signs; this easily follows from Theorem 2.2.

Theorem 2.7. [Kos] Let the circle act on a compact oriented manifold with two fixed points $p$ and $q$. Then the weights at $p$ and $q$ are equal and $\epsilon(p) = -\epsilon(q)$.

If the multiset of weights at every fixed point are the same, the number of fixed points must be even and the signature of the manifold vanishes.

Theorem 2.8. [J1] Let the circle act on a $2n$-dimensional compact oriented manifold $M$ with a discrete fixed point set. Suppose that the weights at each fixed point are $\{a_1, \ldots, a_n\}$ for some positive integers $a_1, \ldots, a_n$. Then the number of fixed points $p$ with $\epsilon(p) = +1$ and that with $\epsilon(p) = -1$ are equal. Moreover, the signature of $M$ vanishes.

Proof. By Theorem 2.2,

$$\text{sign}(M) = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \frac{1 + t^{w_{p,i}}}{1 - t^{w_{p,i}}} = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \frac{1 + t^{a_i}}{1 - t^{a_i}} = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \left[1 + t^{a_i} \sum_{j=0}^{\infty} t^{ja_i}\right] = \sum_{p \in M^{S^1}} \epsilon(p) \prod_{i=1}^{n} \left(1 + 2 \sum_{j=0}^{\infty} t^{ja_i}\right)$$

for all indeterminate $t$. Since the signature of $M$ is a constant, this theorem follows. □

We describe an equivariant connected sum of two oriented $S^1$-manifolds $M$ and $N$ at fixed points. Unlike a tradition, we will take equivariant connected sums at several fixed points $p_1, \ldots, p_k$ of $M$ and $q_1, \ldots, q_k$ of $N$,
where for each $i$ the two fixed points $p_i$ and $q_i$ have the same multiset of weights and different signs.

Let $M$ and $N$ be two $2n$-dimensional connected oriented $S^1$-manifolds with discrete fixed point sets. Suppose that for $i = 1, \cdots, k$, $p_i \in M^{S^1}$ and $q_i \in N^{S^1}$ satisfy $\epsilon_M(p_i) = -\epsilon_N(q_i)$ and \{w_{p,1}, \cdots, w_{p,n}\} = \{w_{q,1}, \cdots, w_{q,n}\}.

For each $i$, there is an equivariant diffeomorphism $f_i$ from a unit disk $D_{2n}$ in $\mathbb{C}^n$ to a neighborhood of $p_i (q_i)$, where the circle acts on $\mathbb{C}^n$ by
\[
g \cdot (z_1, \cdots, z_n) = (g^{w_{p,1}}z_1, \cdots, g^{w_{p,n}}z_n)
\]
for all $g \in S^1 \subset \mathbb{C}$ and for all $(z_1, \cdots, z_n) \in \mathbb{C}^n$.

**Definition 2.9.** The equivariant connected sum of $M$ and $N$ (at $p_1, \cdots, p_k \in M$ and $q_1, \cdots, q_k \in N$) is the quotient
\[
(M \setminus \bigcup_{i=1}^k f_i(0)) \cup (N \setminus \bigcup_{i=1}^k g_i(0)) / \sim,
\]
where we identify $f_i(tu)$ with $g_i((1-t)u)$ for each $u \in \partial D_{2n}$ and each $0 < t < 1$, for each $i$.

If we take the equivariant connected sum of $M$ and $N$ at $p_i$ of $M$ and $q_i$ of $M$ for all $i$, because $\epsilon_M(p_i) = -\epsilon_N(q_i)$ for $1 \leq i \leq k$ and each gluing map reverses orientation, we get an oriented $S^1$-manifold $P$ with fixed points $(M^{S^1} \setminus \{p_1, \cdots, p_k\}) \cup (N^{S^1} \setminus \{q_1, \cdots, q_k\})$, which is also connected.

Consequently, the fixed point data of $P$ is $(\Sigma_M \setminus \bigcup_{i=1}^k \Sigma_{p_i}) \cup (\Sigma_N \setminus \bigcup_{i=1}^k \Sigma_{q_i})$.

**Lemma 2.10.** Let $M$ and $N$ be two $2n$-dimensional connected oriented $S^1$-manifolds with discrete fixed point sets. Suppose that for $i = 1, \cdots, k$, $p_i \in M^{S^1}$ and $q_i \in N^{S^1}$ satisfy $\epsilon_M(p_i) = -\epsilon_N(q_i)$ and \{w_{p,1}, \cdots, w_{p,n}\} = \{w_{q,1}, \cdots, w_{q,n}\}. The equivariant connected sum of $M$ and $N$ at $p_1, \cdots, p_k$ and $q_1, \cdots, q_k$ is a $2n$-dimensional connected oriented $S^1$-manifold $P$ with a discrete fixed point set, whose fixed point set is $(M^{S^1} \setminus \{p_1, \cdots, p_k\}) \cup (N^{S^1} \setminus \{q_1, \cdots, q_k\})$ and fixed point data is $(\Sigma_M \setminus \bigcup_{i=1}^k \Sigma_{p_i}) \cup (\Sigma_N \setminus \bigcup_{i=1}^k \Sigma_{q_i})$. If $M$ and $N$ are compact, so is $P$.

For a manifold $M$, let $\chi(M)$ denote the Euler number of $M$. Kobayashi proved that for a circle action on a compact manifold, its Euler number is equal to the sum of the Euler numbers of its fixed components.

**Theorem 2.11.** [Kob] Let the circle act on a compact oriented manifold $M$. Then
\[
\chi(M) = \sum_{F \subset M^{S^1}} \chi(F),
\]
where the sum is taken over all fixed components of $M^{S^1}$.

The Euler number of a compact oriented surface of genus $g$ is $2 - 2g$ and the Euler number of a point is 1. Therefore, Theorem 2.11 has the following consequence.

**Lemma 2.12.** [J1] Let $M$ be a compact connected oriented surface of genus $g$. 

(1) If $g = 0$, i.e., $M$ is the 2-sphere $S^2$, then any non-trivial circle action on it has two fixed points.

(2) If $g = 1$, i.e., $M$ is the 2-torus $T^2$, then any non-trivial circle action on it is fixed point free.

(3) If $g > 1$, then $M$ does not admit a non-trivial circle action.

3. $S^6$, $\mathbb{C}P^3$, and 6-dimensional analogue $Z_n$ of the Hirzebruch surfaces

In this section, we describe $S^1$-actions on $S^6$, $\mathbb{C}P^3$, 6-dimensional analogue $Z_n$ of the Hirzebruch surfaces, and $Z_2 \# Z_2$ (a connected sum at fixed points of two copies of $Z_2$.) We will need these actions to prove Theorems 1.1 and 1.2.

Example 3.1 (The 6-sphere $S^6$). Let $a$, $b$, and $c$ be positive integers. Let the circle act on $S^6$ by

$$g \cdot (z_1, z_2, z_3, x) = (g^az_1, g^bz_2, g^cz_3, x)$$

for all $g \in S^1 \subset \mathbb{C}$ and for all $(z_1, z_2, z_3, x) \in S^6$, where $S^6 = \{(z_1, z_2, z_3, x) \in \mathbb{C}^3 \times \mathbb{R} : x^2 + \sum_{i=1}^3 |z_i|^2 = 1\}$. The action has two fixed points $q_1 = (0, 0, 0, 1)$ and $q_2 = (0, 0, 0, -1)$. The weights at $q_i$ are $\{a, b, c\}$, and $\epsilon(q_1) = -\epsilon(q_2) = 1$. The fixed point data of this action on $S^6$ is hence

$$\{+, a, b, c\}, \{-, a, b, c\}.$$

Example 3.2 (The complex projective space $\mathbb{C}P^3$). Let $0 < a < b < c$ be positive integers. Let the circle act on $\mathbb{C}P^3$ by

$$g \cdot [z_0 : z_1 : z_2 : z_3] = [g^az_0 : g^bz_1 : g^cz_2 : g^dz_3]$$

for all $g \in S^1 \subset \mathbb{C}$ and for all $[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3$. The action has 4 fixed points $q_1 = [1 : 0 : 0 : 0], q_2 = [0 : 1 : 0 : 0], q_3 = [0 : 0 : 1 : 0], q_4 = [0 : 0 : 0 : 1], \text{ that have weights } \{a, b, c\}, \{-a, b-a, c-a\}, \{-b, a-b, c-b\}, \text{ and } \{-c, a-c, b-c\} \text{ as complex } S^1\text{-representations, respectively. The fixed point data of } \mathbb{C}P^3 \text{ is}

$$\{+, a, b, c\}, \{-, a, b-a, c-a\}, \{+, b-b-a, c-b\}, \{-, c, a-c, b-c\}.$$

Example 3.3 (The manifold $Z_n$, 6-dimensional analogue of the Hirzebruch surfaces). Fix an integer $n$. By the 6-dimensional analogue $Z_n$ of Hirzebruch surfaces we mean a compact complex manifold

$$Z_n = \{(z_0 : z_1 : z_2 : z_3, [w_2 : w_3]) \in \mathbb{C}P^3 \times \mathbb{C}P^1 : z_2w_3^n = z_3w_2^n\}.$$

Let $a$, $b$, and $c$ be positive integers such that $b-a \neq 0, nc-a \neq 0$, and $nc-b \neq 0$. Let the circle act on $Z_n$ by

$$g \cdot ([z_0 : z_1 : z_2 : z_3, [w_2 : w_3]) = ([g^az_0 : g^bz_1 : g^cz_2 : g^nzc_3, [w_2 : g^cw_3])$$

for all $g \in S^1 \subset \mathbb{C}$ and for all $([z_0 : z_1 : z_2 : z_3], [w_2 : w_3]) \in Z_n$. We denote by $Z_n(a,b,c)$ the manifold with this action. The action has 6 fixed points; at each fixed point, we exhibit local coordinates and the weights at the fixed point as complex $S^1$-representations.
(1) $q_1 = ([1 : 0 : 0 : 0], [1 : 0]):$ local coordinates $(z_1/z_0, z_2/z_0, w_3/w_2)$, weights $\{b - a, -a, c\}$

(2) $q_2 = ([1 : 0 : 0 : 0], [0 : 1]):$ local coordinates $(z_1/z_0, z_3/z_0, w_2/w_3)$, weights $\{b - a, nc - a, -c\}$

(3) $q_3 = ([0 : 1 : 0 : 0], [1 : 0]):$ local coordinates $(z_0/z_1, z_2/z_1, w_3/w_2)$, weights $\{a - b, -b, c\}$

(4) $q_4 = ([0 : 1 : 0 : 0], [0 : 1]):$ local coordinates $(z_0/z_1, z_3/z_1, w_2/w_3)$, weights $\{a - b, nc - b, -c\}$

(5) $q_5 = ([0 : 0 : 1 : 0], [1 : 0]):$ local coordinates $(z_0/z_2, z_1/z_2, w_3/w_2)$, weights $\{a, b, c\}$

(6) $q_6 = ([0 : 0 : 0 : 1], [0 : 1]):$ local coordinates $(z_0/z_3, z_1/z_3, w_2/w_3)$, weights $\{a - nc, b - nc, -c\}$

For instance, local coordinates of $q_2$ are $(z_1/z_0, z_3/z_0, w_2/w_3)$, and the circle acts near $q_2$ by

$$g \cdot \left(\frac{z_1}{z_0}, \frac{z_3}{z_0}, \frac{w_2}{w_3}\right) = \left(\frac{g^b z_1}{g^a z_0}, \frac{g^nc z_3}{g^a z_0}, \frac{w_2}{w_3}\right) = \left(\frac{g^{a - b} z_1}{z_0}, \frac{g^{nc - a} z_3}{z_0}, \frac{g^{-c} w_2}{w_3}\right).$$

Thus the complex $S^1$-weights at $q_2$ are $\{b - a, nc - a, -c\}$.

To simplify the proof of Theorems 1.1 and 1.2, in Example 3.3 we will take specific values of $a$, $b$, $c$, and $n$ and record it as a separate example.

**Example 3.4** (The manifold $Z_1(a, b, c)$ with $a > b > c > 0$). Suppose that $a > b > c > 0$. Take the manifold $Z_1(a, b, c)$ in Example 3.3. The weights at the fixed points as complex $S^1$-representations are

$$\{b - a, -a, c\}, \{b - a, c - a, -c\}, \{a - b, -b, c\}, \{a - b, c - b, -c\}, \{a, b, c\}, \{a - c, b - c, -c\},$$

respectively. Therefore, as real $S^1$-representations, the fixed point data of $Z_1(a, b, c)$ is

$$\{+, a - b, a, c\}, \{-, a - b, a - c, c\}, \{-, a - b, b, c\}, \{+, a - b, b - c, c\}, \{+, a, b, c\}, \{-, a - c, b - c, c\}.$$

**Example 3.5** (The manifold $Z_1(a, b, c)$ with $a > c > b > 0$). Suppose that $a > c > b > 0$. Take the manifold $Z_1(a, b, c)$ in Example 3.3. The weights at the fixed points as complex $S^1$-representations are

$$\{b - a, -a, c\}, \{b - a, c - a, -c\}, \{a - b, -b, c\}, \{a - b, c - b, -c\}, \{a, b, c\}, \{a - c, b - c, -c\}.$$

As real $S^1$-representations, the fixed point data of $Z_1(a, b, c)$ is

$$\{+, a - b, a, c\}, \{-, a - b, a - c, c\}, \{-, a - b, b, c\}, \{-, a - b, c - b, c\}, \{+, a, b, c\}, \{+, a - c, c - b, c\}.$$

The only difference between Examples 3.4 and 3.5 is some sign issue on the fixed point data of $q_4$ and $q_6$.

**Example 3.6** (The manifold $Z_2(a, d, d)$ with $0 < 2d < a$). Let $a$ and $d$ be positive integers such that $2d < a$. In Example 3.3, take $n = 2$ and $b = c = d$. Complex $S^1$-weights at the fixed points are
\{d - a, -a, d\}, \{d - a, 2d - a, -d\}, \{a - d, -d, -d\}, \{a - d, d, -d\}, \\
\{a - 2d, -d, -d\}.

As real $S^1$-representations, the fixed point data of $Z_2(a, d, d)$ is 
\{+, a - d, a, d\}, \{-, a - d, a - 2d, d\}, \{-, a - d, d, d\}, \\
\{-, a - d, d, d\}, \{+, a - d, d, d\}.

Example 3.7 (The manifold $Z_2(a, d, d)$ with $2d > a > 0$). Let $a$ and $d$ be positive integers such that $2d > a$. In Example 3.3, take $n = 2$ and $b = c = d$. Complex $S^1$-weights at the fixed points are 
\{d - a, -a, d\}, \{d - a, 2d - a, -d\}, \{a - d, -d, -d\}, \{a - d, d, -d\}, \\
\{a - 2d, -d, -d\}.

As real $S^1$-representations, the fixed point data of $Z_2(a, d, d)$ is 
\{+, a - d, a, d\}, \{+, a - d, 2d - a, d\}, \{-, a - d, d, d\}, \\
\{+, a - d, d, d\}, \{-, 2d - a, d, d\}.

As for Examples 3.4 and 3.5, the only difference between Examples 3.6 and 3.7 is some sign issue on the fixed point data of $q_2$ and $q_6$.

Example 3.8 (The manifold $Z_2(a, e, e)\#\overline{Z_2(a, a - e, a - e)}$). Let $a$ and $e$ be positive integers such that $2e < a$. Take $Z_2(a, e, e)$ of Example 3.6 that has fixed point data 
\{+, a - e, a, e\}, \{-, a - e, a - 2e, e\}, \{-, a - e, e, e\}, \{-, a - e, e, e\}, \\
\{+, a - 2e, e, e\}, \{+, a - 2e, a, e\}.

Denote the fixed points by $q_1', \cdots, q_6'$, respectively. Since $2e < a$, it follows $a < 2(a - e)$. We take $Z_2(a, a - e, a - e)$ (take $d = a - e$) of Example 3.7 that has fixed point data 
\{+, e, a, a - e\}, \{+, e, a - 2e, a - e\}, \{-, e, a - e, a - e\}, \{-, e, a - e, a - e\}, \\
\{+, a - e, a - e\}, \{-, a - 2e, a - e, a - e\}.

We reverse the orientation of $Z_2(a, a - e, a - e)$ to get a manifold $\overline{Z_2(a, a - e, a - e)}$ that has fixed point data 
\{-, e, a, a - e\}, \{-, e, a - 2e, a - e\}, \{+, e, a - e, a - e\}, \{+, e, a - e, a - e\}, \\
\{-, a - e, a - e\}, \{+, a - 2e, a - e, a - e\}.

Denote the fixed points by $q''_1, \cdots, q''_6$, respectively. Now, $q''_1$ and $q''_6$ have the same weights (as real $S^1$-representations) and satisfy $\epsilon(q''_1) = -\epsilon(q''_6)$. Therefore, we can take an equivariant connected sum at $q''_1$ of $Z_2(a, e, e)$ and at $q''_6$ of $\overline{Z_2(a, a - e, a - e)}$ to construct another 6-dimensional compact connected oriented $S^1$-manifold $Z_2(a, e, e)\#\overline{Z_2(a, a - e, a - e)}$ with 10 fixed points $\hat{q}_1, \cdots, \hat{q}_{10}$, that has fixed point data 
\{-, a - e, a - 2e, e\}, \{-, a - e, a - 2e, e\}, \{-, a - e, e, e\}, \{+, a, e, e\}, \\
\{+, a - 2e, e, e\}, \{-, e, a - 2e, a - e\}, \{+, e, a - e, a - e\}, \\
\{+, e, a - e, a - e\}, \{-, a, a - e, a - e\}, \{+, a - 2e, a - e, a - e\}. 
4. Relation between fixed point datum of fixed points in isotropy submanifold

To prove Theorems 1.1 and 1.2, for a 6-dimensional oriented $S^1$-manifold with isolated fixed points we need some technical lemmas on relationships between fixed point datum of fixed points, which lie in the same component of an isotropy submanifold $M^{Z_l}$, where $l$ is the biggest weight. To prove these lemmas, we need more terminologies.

Let the circle act effectively on a $2n$-dimensional compact oriented manifold $M$ with a discrete fixed point set. Let $w > 2$ be a positive integer. Let $F$ be a component of $M^{Z_w}$ such that $F \cap M^S \neq \emptyset$. By Lemma 2.5, $F$ is orientable; hence the normal bundle $NF$ is also orientable. Choose an orientation of $F$ and that of $NF$, so that the induced orientation on $TF \oplus NF$ agrees with the orientation of $M$. Let $p \in F \cap M^S$ be an $S^1$-fixed point. By permuting $L_{p,i}$'s, we may let

$$T_pM = L_{p,1} \oplus \cdots \oplus L_{p,m} \oplus L_{p,m+1} \oplus \cdots \oplus L_{p,n},$$

where $T_pF = L_{p,1} \oplus \cdots \oplus L_{p,m}$ and $N_pF = L_{p,m+1} \oplus \cdots \oplus L_{p,n}$, and the circle acts on each $L_{p,i}$ with weight $w_{p,i}$. As before, we orient each $L_{p,i}$ so that $w_{p,i}$ is positive.

**Definition 4.1.** (1) $\epsilon_F(p) = +1$ if the orientation on $F$ agrees with the orientation on $L_{p,1} \oplus \cdots \oplus L_{p,m}$, and $\epsilon_F(p) = -1$ otherwise.

(2) $\epsilon_N(p) = +1$ if the orientation on $NF$ agrees with the orientation on $L_{p,m+1} \oplus \cdots \oplus L_{p,n}$, and $\epsilon_N(p) = -1$ otherwise.

By definition, $\epsilon(p) = \epsilon_F(p) \cdot \epsilon_N(p)$.

Now suppose that $\dim M = 6$. Suppose that the biggest weight $l$ is bigger than 2. Assume that $M^{Z_l}$ has a 2-dimensional component $F$ that contains an $S^1$-fixed point $q$. Then $F$ is the 2-sphere, and $F$ contains another fixed point $q'$. A relationship between the fixed point data of $q$ and that of $q'$ is as follows.

**Lemma 4.2.** Let the circle act effectively on a 6-dimensional compact oriented manifold $M$ with a discrete fixed point set. Suppose that the biggest weight $l = \max\{w_{p,i} | 1 \leq i \leq n, p \in M^S\}$ is bigger than 2. Suppose that there is a fixed point $q$ that has weights $\{l, a, b\}$ for some positive integers $a$ and $b$ such that $a, b < l$. If $l$ is even, assume furthermore that the component $M^{Z_l}$ containing $q$ is orientable. Then there exists another fixed point $q'$ so that one of the following holds:

1. $\epsilon(q') = -\epsilon(q)$ and the weights at $q'$ are $\{l, a, b\}$.
2. $\epsilon(q') = -\epsilon(q)$ and the weights at $q'$ are $\{l, l - a, l - b\}$.
3. $\epsilon(q') = \epsilon(q)$ and the weights at $q'$ are $\{l, a, l - b\}$.
4. $\epsilon(q') = \epsilon(q)$ and the weights at $q'$ are $\{l, l - a, b\}$.

The fixed points $q$ and $q'$ are in the same component of $M^{Z_l}$, which is the 2-sphere.
Proof. Let $F$ be a component of $M^{Z_l}$ that contains $q$, which is a smaller dimensional closed submanifold of $M$. Since $q$ has only one weight divisible by $l$, $\dim F = 2$. Then $F$ is orientable, by Lemma 2.5 for $l$ odd and by assumption for $l$ even. We choose an orientation of $F$ and that of $NF$ so that the induced orientation on $TF \oplus NF$ agrees with the orientation of $M$. The circle action on $M$ restricts to act on $F$, and this induced action on $F$ has $q$ as a fixed point. By Lemma 2.12, $F$ is the 2-sphere and has another fixed point $q'$. Applying Theorem 2.7 to the induced action on $F$, $\epsilon_F(q) = -\epsilon_F(q')$. Since $NF$ is an oriented $Z_w$-bundle over $F$ and $F$ is connected, the $Z_w$-representations of $N_F$ and $N_{q'}F$ are isomorphic.

Let $N_qF = L_{q,2} \oplus L_{q,3}$, where the circle acts on $L_{q,2}$ with weight $a$ and on $L_{q,3}$ with weight $b$. Similarly, let $N_{q'}F = L_{q',2} \oplus L_{q',3}$, where the circle acts on $L_{q',2}$ with weight $c$ and on $L_{q',3}$ with weight $d$ for some positive integers $c$ and $d$. Note that $c, d < l$.

First, suppose that $\epsilon(q) = \epsilon(q')$. Since $\epsilon_F(q) = -\epsilon_F(q')$, with $\epsilon(q) = \epsilon_F(q) \cdot \epsilon_N(q)$ and $\epsilon(q') = \epsilon_F(q') \cdot \epsilon_N(q')$, this implies that $\epsilon_N(q) = -\epsilon_N(q')$. Therefore, there is an orientation reversing isomorphism $\phi$ from $L_{q,2} \oplus L_{q,3}$ to $L_{q',2} \oplus L_{q',3}$ as $Z_l$-representations. Without loss of generality, by permuting $L_{q',2}$ and $L_{q',3}$ if necessary, we may assume that this isomorphism takes $L_{q,2}$ to $L_{q',2}$ and $L_{q,3}$ to $L_{q',3}$. Then one of the following holds.

(a) The isomorphism $\phi$ is orientation preserving from $L_{q,2}$ to $L_{q',2}$, and orientation reversing from $L_{q,3}$ to $L_{q',3}$.

(b) The isomorphism $\phi$ is orientation reversing from $L_{q,2}$ to $L_{q',2}$, and orientation preserving from $L_{q,3}$ to $L_{q',3}$.

Assume that Case (a) holds. Since $\phi$ is an orientation preserving isomorphism from $L_{q,2}$ to $L_{q',2}$ as $Z_l$-representations, this implies that $a \equiv c \mod l$. Since $a, c < l$, it follows that $a = c$. Next, since $\phi$ is an orientation preserving isomorphism from $L_{q,3}$ to $L_{q',3}$ as $Z_l$-representations, this implies that $b \equiv -d \mod l$. With $b, d < l$ this means that $d = l - b$. This is Case (3) of this lemma. Similarly, if Case (b) holds, then $c = l - a$ and $b = d$; this is Case (4) of this lemma.

Second, suppose that $\epsilon(q) = -\epsilon(q')$. Since $\epsilon_F(q) = -\epsilon_F(q')$, it follows that $\epsilon_N(q) = \epsilon_N(q')$. Hence there is an orientation preserving isomorphism $\phi$ from $L_{q,2} \oplus L_{q,3}$ to $L_{q',2} \oplus L_{q',3}$ as $Z_l$-representations. We may assume that this isomorphism takes $L_{q,2}$ to $L_{q',2}$ and $L_{q,3}$ to $L_{q',3}$. There are two possibilities.

(i) The isomorphism $\phi$ is orientation preserving from $L_{q,i}$ to $L_{q',i}$ for both $i \in \{2, 3\}$.

(ii) The isomorphism $\phi$ is orientation reversing from $L_{q,i}$ to $L_{q',i}$ for both $i \in \{2, 3\}$.

Case (i) means that $a = c$ and $b = d$; this is Case (1) of this lemma. Case (ii) means that $c = l - a$ and $d = l - b$; this is Case (2) of this lemma. □
Suppose now that $M^{\mathbb{Z}_l}$ has a 4-dimensional component that contains an $S^1$-fixed point $q$. Equivalently, suppose that there is a fixed point $q$ that has weight $l$ twice. Then there must exist another fixed point $q'$ that has the same weights as $q$, and has the opposite sign; $\epsilon(q') = -\epsilon(q)$.

**Lemma 4.3.** Let the circle act effectively on a 6-dimensional compact oriented manifold $M$ with a discrete fixed point set. Suppose that the biggest weight $l = \max \{ w_{p,i} \mid 1 \leq i \leq n, p \in M^{S^1} \}$ is bigger than 2. If $l$ is even, assume furthermore that each component of the isotropy submanifold $M^{\mathbb{Z}_l}$ containing an $S^1$-fixed point is orientable.

1. Suppose that there is a fixed point $q$ that has weights $\{l, l, a\}$ for some positive integer $a$. Then there exists another fixed point $q'$ that has $\epsilon(q') = -\epsilon(q)$ and weights $\{l, l, a\}$. Moreover, $q$ and $q'$ are in the same component of $M^{\mathbb{Z}_l}$.

2. For each positive integer $a$, the number of fixed points that have fixed point data $\{+, l, l, a\}$ is equal to the number of fixed points that have fixed point data $\{-, l, l, a\}$.

**Proof.** Suppose that a fixed point $p_1 := q$ has weights $\{l, l, a\}$ for some positive integer $a$. Since the action is effective and $l > 2$ is the biggest weight, $a$ is strictly smaller than $l$. Let $F$ be a component of $M^{\mathbb{Z}_l}$ that contains $p_1$; $F$ is a smaller dimensional closed submanifold of $M$. Then $F$ is orientable, by Lemma 2.5 for $l$ odd and by assumption for $l$ even. We choose an orientation of $F$, so that $\epsilon_F(p_1) = +1$ for simplicity of the proof. Also choose an orientation of $NF$ so that the induced orientation on $TF \oplus NF$ agrees with the orientation of $M$. The circle action on $M$ restricts to a circle action on $F$, and the fixed point set $F^{S^1}$ of this action on $F$ is equal to $F \cap M^{S^1}$ so it is non-empty and finite. For any $p \in F^{S^1}$, the weights in $T_pF$ are multiples of $l$, and hence they are all equal to $l$ as $l$ is the biggest weight. Applying Theorem 2.8 to the induced $S^1$-action on $F$ by taking $a_1 = a_2 = l$, the number of fixed points $p \in F^{S^1}$ with $\epsilon_F(p) = +1$ and the number of fixed points $p \in F^{S^1}$ with $\epsilon_F(p) = -1$ are equal. Let $p_1, \ldots, p_k \in F^{S^1}$ have $\epsilon_F(p_i) = +1$ and let $q_1, \ldots, q_k \in F^{S^1}$ have $\epsilon_F(q_i) = -1$.

By permuting $p_i$'s if necessary, let $\epsilon(p_1) = \cdots = \epsilon(p_s)$ and $\epsilon(p_{s+1}) = \cdots = \epsilon(p_k) = -\epsilon(p_1)$. Similarly, by permuting $q_j$'s if necessary, let $\epsilon(q_1) = \cdots = \epsilon(q_t) = -\epsilon(p_1)$ and $\epsilon(q_{t+1}) = \cdots = \epsilon(q_k) = \epsilon(p_1)$.

Let $\{l, l, a\}$ be the weights at $p_i$ and let $\{l, l, b\}$ be the weights at $q_j$; then $a_i, b_j < l$. Since $NF$ is an oriented $\mathbb{Z}_l$-bundle over $F$ and $F$ is connected, the $\mathbb{Z}_l$-representations of $N_pF$ and $N_{p'}F$ are isomorphic for any two fixed points $p$ and $p'$ in $F^{S^1}$.

Consider $p_i$ for $1 \leq i \leq s \ (s + 1 \leq i \leq k)$. Since $\epsilon(p_i) = \epsilon(p_1)$ ($\epsilon(p_i) = -\epsilon(p_1)$) and $\epsilon_F(p_i) = \epsilon_F(p_1)$ ($\epsilon_F(p_i) = -\epsilon_F(p_1)$), it follows that $\epsilon_N(p_i) = \epsilon_N(p_1)$ ($\epsilon_N(p_i) = -\epsilon_N(p_1)$). Thus, there is an orientation preserving (reversing) isomorphism from the representation $N_{p_1}F$ with weight $a$ to
that $N_{p_i}F$ with weight $a_i$ as $\mathbb{Z}_l$-representations. This implies that $a \equiv a_i \mod l$ ($a \equiv -a_i \mod l$) and hence $a = a_i$ ($a_i = l - a$, respectively.)

Similarly, for $q_j$ with $1 \leq j \leq t$ ($t + 1 \leq j \leq k$), $\epsilon(q_j) = -\epsilon(p_l)$ ($\epsilon(q_j) = \epsilon(p_l)$) and $\epsilon_F(q_j) = -\epsilon_F(p_l)$ imply that $\epsilon_N(q_j) = \epsilon_N(p_l)$ ($\epsilon_N(q_j) = -\epsilon_N(p_l)$) and hence there is an orientation preserving (reversing) isomorphism from $N_{p_i}F$ with weight $a$ to $N_{q_j}F$ with weight $b_j$ as $\mathbb{Z}_l$-representations, and thus $a \equiv b_j \mod l$ ($a \equiv -b_j \mod l$), i.e., $b_j = a$ ($b_j = l - a$, respectively.)

Let $e_{S^1}(NF)$ denote the equivariant Euler class of the normal bundle to $F$ in $M$. Let $u$ be a degree two generator of $H^4(\mathbb{C}P^\infty; \mathbb{Z})$. The restriction of $e_{S^1}(NF)$ at $p_i$ is

(1) $e_{S^1}(NF)(p_i) = \epsilon_N(p_i) \cdot au = \epsilon_N(p_l) \cdot au$ if $1 \leq i \leq s$.

(2) $e_{S^1}(NF)(p_i) = \epsilon_N(p_i) \cdot (l - a)u = -\epsilon_N(p_l) \cdot (l - a)u$ if $s + 1 \leq i \leq k$.

Similarly, the restriction of $e_{S^1}(NF)$ at $q_j$ is

(1) $e_{S^1}(NF)(q_j) = \epsilon_N(q_j) \cdot au = \epsilon_N(p_l) \cdot au$ if $1 \leq j \leq t$.

(2) $e_{S^1}(NF)(q_j) = \epsilon_N(q_j) \cdot (l - a)u = -\epsilon_N(p_l) \cdot (l - a)u$ if $t + 1 \leq j \leq k$.

For the induced action on $F$, $\int_F: = \pi_*$ is a map from $H^i_{S^1}(F; \mathbb{Z})$ to $H^{i - \dim F}(\mathbb{C}P^\infty; \mathbb{Z})$ for all $i \in \mathbb{Z}$. Since $\dim F = 4$ and $e_{S^1}(NF)$ has degree 2, the image under $\int_F$ of $e_{S^1}(NF)$ vanishes;

$$\int_F e_{S^1}(NF) = 0.$$ 

On the other hand, applying the ABBV localization formula (Theorem 2.1) to the induced action on $F$ with taking $\alpha = e_{S^1}(NF),$

$$\int_F e_{S^1}(NF) = \sum_{p \in F} \int_p e_{S^1}(NF)_{\big|T_pF},$$

$$\int_F e_{S^1}(NF) = \sum_{i=1}^{k} \left\{ \epsilon_F(p_i) \frac{e_{S^1}(NF)(p_i)}{e_{S^1}(T_pF)} \right\} + \sum_{j=1}^{k} \left\{ \epsilon_F(q_j) \frac{e_{S^1}(NF)(q_j)}{e_{S^1}(T_qF)} \right\}$$

$$= \sum_{i=1}^{s} \left\{ \epsilon_F(p_i) \frac{e_{S^1}(NF)(p_i)}{e_{S^1}(T_pF)} \right\} + \sum_{i=s+1}^{k} \left\{ \epsilon_F(p_i) \frac{e_{S^1}(NF)(p_i)}{e_{S^1}(T_pF)} \right\}$$

$$+ \sum_{j=1}^{t} \left\{ \epsilon_F(q_j) \frac{e_{S^1}(NF)(q_j)}{e_{S^1}(T_qF)} \right\} + \sum_{j=t+1}^{k} \left\{ \epsilon_F(q_j) \frac{e_{S^1}(NF)(q_j)}{e_{S^1}(T_qF)} \right\}$$

$$= \sum_{i=1}^{s} \left\{ \frac{+\epsilon_N(p_i) \cdot au}{l^2u^2} \right\} + \sum_{i=s+1}^{k} \left\{ \frac{-\epsilon_N(p_i) \cdot (l - a)u}{l^2u^2} \right\}$$

$$+ \sum_{j=1}^{t} \left\{ \frac{-\epsilon_N(p_i) \cdot au}{l^2u^2} \right\} + \sum_{j=t+1}^{k} \left\{ \frac{-\epsilon_N(p_i) \cdot (l - a)u}{l^2u^2} \right\}.$$ 

Note that the bundle in the denominator is the normal bundle of $p$ in $F$, which is therefore the tangent space $T_pF$ of $p$ in $F$. 


\[ \frac{\epsilon_N(p_1)}{l^2w} \{ sa - (k - s)(l - a) - ta + (k - t)(l - a) \} = \frac{\epsilon_N(p_1)}{lw} (s - t). \]

Therefore, \( s = t \). Then \( p_1, \ldots, p_s \) and \( q_1, \ldots, q_s \) have weights \( \{ l, l, a \} \) and \( \epsilon(p_1) = \cdots = \epsilon(p_s) = -\epsilon(q_1) = \cdots = -\epsilon(q_s) \). Also, \( p_{s+1}, \ldots, p_k \) and \( q_{s+1}, \ldots, q_k \) have weights \( \{ l, l, l - a \} \) and \( \epsilon(p_{s+1}) = \cdots = \epsilon(p_k) = -\epsilon(q_{s+1}) = \cdots = -\epsilon(q_k) = -\epsilon(p_1) \). Thus this lemma holds. \( \square \)

5. Proof of main results

In this section, we prove Theorem 1.1 and Theorem 1.2 together.

**Proof of Theorems 1.1 and 1.2.** We prove by showing that we can successively remove fixed points of \( M \) that have the biggest weight, by taking connected sums at fixed points of \( S^1 \)-actions on \( S^6, \mathbb{CP}^3, Z_1, \) and \( Z_2 \) (and these with opposite orientations.)

Quotienting out by the subgroup that acts trivially, we may assume that the action is effective. Let \( l \) be the biggest weight; \( l = \max \{ w_{p,j} | 1 \leq i \leq 3, p \in MS^1 \} \). If \( l = 1 \) or \( l = 2 \), then both of Theorems 1.1 and 1.2 follow from Lemma 5.1, in which we use \( S^6 \) for an equivariant connected sum; in terms of the operations in Theorem 1.2, this corresponds to Operation (1).

Therefore, from now on, we suppose that \( l > 2 \). Suppose that a fixed point \( p_1 \) has weight \( l \). Without loss of generality, by reversing the orientation of \( M \) if necessary, we may assume that \( \epsilon(p_1) = +1 \). This is to simplify the proof, but there is a caution. In the proof below we will take a connected sum of \( M \) and \( N \in \{ S^6, \mathbb{CP}^3, \mathbb{CP}^3, Z_1, Z_2, Z_2, Z_2, Z_2, Z_2 \} \). If instead we have \( \epsilon(p_1) = -1 \), then we will need to take a connected sum of \( M \) and \( \overline{N} \).

Let \( F \) be a component of \( M^2 \) that contains \( p_1 \). Then \( F \) is orientable, by Lemma 2.5 for \( l \) odd and by assumption for \( l \) even. Since the action is effective, there are two possibilities.

1. The multiplicity of \( l \) in \( T_{p_1}M \) is 2.
2. The multiplicity of \( l \) in \( T_{p_1}M \) is 1.

Assume Case (1) holds. Let \( \{ l, l, x \} \) be the weights at \( p_1 \) for some positive integer \( x \). Then \( x < l \). By Lemma 4.3, there exists another fixed point \( p_2 \) that has \( \epsilon(p_2) = -\epsilon(p_1) = -1 \) and has weights \( \{ l, l, x \} \). Then we can take an equivariant connected sum at \( p_1 \) and \( p_2 \) of \( M \) and \( q_2 \) and \( q_1 \) of \( S^6 \) in Example 3.1 with \( \{ a, b, c \} = \{ l, l, x \} \) to construct another 6-dimensional compact connected oriented \( S^1 \)-manifold \( M' \) that has fixed point data \( \Sigma_M \setminus (\{ +, l, l, x \} \cup \{ -, l, l, x \}) \). This corresponds to Operation (1) of Theorem 1.2; \( \{ l, l, x \} \) here corresponds to \( \{ A, B, C \} \) of Operation (1).

Assume Case (2) holds. Let \( \{ l, x, y \} \) be the weights at \( p_1 \) for some positive integers \( x \) and \( y \). Then \( x, y < l \). By Lemma 4.2, there exists another fixed point \( p_2 \) so that one of the following holds:
(a) $\epsilon(p_2) = -\epsilon(p_1) = -1$ and the weights at $p_2$ are $\{l, x, y\}$.
(b) $\epsilon(p_2) = -\epsilon(p_1) = -1$ and the weights at $p_2$ are $\{l, l-x, l-y\}$.
(c) $\epsilon(p_2) = \epsilon(p_1) = 1$ and the weights at $p_2$ are $\{l, x, l-y\}$.
(d) $\epsilon(p_2) = \epsilon(p_1) = 1$ and the weights at $p_2$ are $\{l, l-x, y\}$.

Up to permuting $x$ and $y$, Case (c) and Case (d) are equivalent; we only need to consider Cases (a-c).

Assume that Case (2-a) holds. The fixed points $p_1$ and $p_2$ have fixed point data $\{+, l, x, y\}$ and $\{-, l, x, y\}$. As in Case (1), we can take an equivariant connected sum at $p_1$ and $p_2$ of $M$ and $q_2$ and $q_1$ of $S^6$ in Example 3.1 with $\{a, b, c\} = \{l, x, y\}$ to construct another 6-dimensional compact connected oriented $S^1$-manifold $M'$ that has fixed point data $\Sigma_M \setminus \{\{+, l, x, y\} \cup \{-, l, x, y\}\}$; this corresponds to Operation (1) of Theorem 1.2.

Assume that Case (2-b) holds. We have two cases.

(i) $x \neq y$.
(ii) $x = y$.

Suppose that Case (2-b-i) holds. Permuting $x$ and $y$ if necessary, we may assume that $x > y$. The fixed points $p_1$ and $p_2$ have fixed point data $\{+, l, x, y\}$ and $\{-, l-x, l-y\}$. In Example 3.2 of the action on $\mathbb{C}P^3$, take $c = l$, $b = x$, and $a = y$, and reverse its orientation; its fixed point data is

$$\Sigma_{q_1} = \{-, y, x, l\}, \Sigma_{q_2} = \{+, y, x-y, l-y\}, \Sigma_{q_3} = \{-, x, x-y, l-x\}, \Sigma_{q_4} = \{+, l, l-y, l-x\}.$$

Then we can take an equivariant connected sum at $p_1$ and $p_2$ of $M$ and $q_1$ and $q_4$ of $\mathbb{C}P^3$ (with this action) to construct another oriented $S^1$-manifold $M'$ that has fixed point data

$$\{\Sigma_M \setminus \{\{+, l, x, y\} \cup \{-, l, l-y, l-x\}\} \cup \{\{+, y, x-y, l-y\} \cup \{-, x, x-y, l-x\}\}.$$

This corresponds to Operation (2) of Theorem 1.2 with $l = C$, $x = B$, and $y = A$.

Suppose that Case (2-b-ii) holds. The fixed points $p_1$ and $p_2$ have fixed point data $\{+, l, x, x\}$ and $\{-, l-x, l-x\}$. First, suppose that $2x < l$. In Example 3.8 of the action on $Z := Z_2(a, e, e)$, we take $a = l$ and $e = x$ and reverse its orientation; its fixed point data is

$$\{+, l-x, l-2x, x\}, \{+, l-x, x, x\}, \{+, l-x, x, x\}, \{-, l-x, x\}, \{-, l-2x, x, x\}, \{+, l-x, l-x\}, \{-, l-x, l-x\}, \{-, l-2x, l-x, l-x\}.$$

We can take a connected sum at $p_1$ and $p_2$ of $M$ and $\hat{q}_4$ and $\hat{q}_9$ of $\overline{Z}$ to construct another $S^1$-manifold $M'$ with fixed point data

$$\Sigma_{M'} = \{\Sigma_M \setminus \{\{+, l, x, x\} \cup \{-, l-x, l-x\}\} \cup \{\{+, l-x, l-2x, x\} \cup \{+, l-x, x, x\} \cup \{-, l-2x, x, x\} \cup \{+, l-x, l-2x, l-x\} \cup \{-, x, l-x, l-x\} \cup \{-, l-2x, l-x, l-x\}\}.$$
This corresponds to Operation (5) of Theorem 1.2 with \( l = C \) and \( x = A \).

Second, assume Case (2-b-ii) and \( 2x > l \). In Example 3.8 of the action on \( Z_2(a,e,e) \), \( Z_2(a, -e, a - e) \) we take \( a = l \) and \( e = l - x \); its fixed point data is

\[
\{-, x, 2x - l, l - x\}, \{-, x, l - x, l - x\}, \{-, x, l - x, -x\},
\{+, l, l - x, l - x\}, \{+, 2x - l, l - x, l - x\}, \{-, l - x, 2x - l, x\},
\{+, l - x, x, x\}, \{-, l, x, x\}, \{+, 2x - l, x, x\}.
\]

We can take a connected sum at \( p_1 \) and \( p_2 \) of \( M \) and \( \hat{q}_0 \) and \( \hat{q}_4 \) of \( Z_2(l, l - x, l - x) \) to construct another \( S^1 \)-manifold \( M' \) with fixed point data

\[
\Sigma_{M'} = \{ \Sigma_M \setminus \{ (+, l, x, x) \cup \{-, l, l - x, l - x\} \} \} \cup \{\{-, x, 2x - l, l - x\} \cup \{-, x, l - x, l - x\}, \{+, 2x - l, l - x, l - x\} \cup \{-, l - x, 2x - l, x\} \cup \{+, l - x, x, x\} \cup \{+, 2x - l, x, x\} \}.
\]

This also corresponds to Operation (5) of Theorem 1.2 with \( l = C \) and \( x = C - A \).

Third, assume Case (2-b-ii) and \( 2x = l \). Since \( p_1 \) has weights \( \{2x(= l), x, x\} \) and the action is effective, this implies that \( x = 1 \). Then \( p_1 \) and \( p_2 \) have fixed point data \( \{+, 2, 1, 1\} \) and \( \{-, 2, 1, 1\} \) and this case is Case (2-a); thus proceed as in Case (2-a).

Assume that Case (2-c) holds. We have two cases.

(i) \( x \neq y \).

(ii) \( x = y \).

Suppose that Case (2-c-i) holds. The fixed points \( p_1 \) and \( p_2 \) have fixed point data \( \{+, l, x, y\} \) and \( \{+, l, x, l - y\} \). First, suppose that \( x < y \). In Example 3.4 of the action on \( Z_1(a, b, c) \) we take \( a = l \), \( b = y \), and \( c = x \), and reverse its orientation; its fixed point data is

\[
\{-, l - y, y, x\}, \{+, l - y, l - x, x\}, \{+, l - y, y - x, x\},
\{-, l - y, y - x, x\}, \{-, l, y, x\}, \{+, l - x, y - x, x\}.
\]

We can take a connected sum at \( p_1 \) and \( p_2 \) of \( M \) and \( q_5 \) and \( q_1 \) of \( Z_1(l, y, x) \) to construct another \( S^1 \)-manifold \( M' \) with fixed point data

\[
\Sigma_{M'} = \{ \Sigma_M \setminus \{ (+, l, x, y) \cup \{+, l, x, l - y\} \} \} \cup \{\{-, l - y, y, x\} \cup \{+, l - y, l - x, x\} \cup \{|+, l - y, y - x, x| \cup \{+, l - x, y - x, x| \} \}.
\]

This corresponds to Operation (3) of Theorem 1.2 with \( x = A \), \( y = B \), and \( l = C \).

Next, suppose Case (2-c-i) with \( x > y \). In Example 3.5 of the action on \( Z_1(a, b, c) \) we take \( a = l \), \( b = y \), and \( c = x \) and reverse its orientation; its fixed point data is

\[
\{-, l - y, l, x\}, \{+, l - y, l - x, x\}, \{+, l - y, y, x\}, \{+, l - y, x - y, x\},
\{-, l, y, x\}, \{-, l - x, x - y, x\}.
\]
We can take a connected sum at $p_1$ and $p_2$ of $M$ and $q_5$ and $q_1$ of $Z_1(l, y, x)$ to construct another $S^1$-manifold $M'$ with fixed point data

$$\Sigma_{M'} = \{\Sigma_M \setminus \{+l, x, y\} \cup \{+l, x, l - y\}\} \cup \{+l - y, l - x, x\} \cup \{+l - y, x - y, x\} \cup \{-l - x, x - y, x\}\}.$$ 

This corresponds to Operation (3') of Theorem 1.2 with $x = A, y = B$, and $l = C$.

Suppose that Case (2-c-ii) holds. The fixed points $p_1$ and $p_2$ have fixed point data $\{+l, x, x\}$ and $\{+l, x, l - x\}$. First, suppose that $2x < l$. In Example 3.6 we take $a = l$ and $d = x$ and reverse its orientation; its fixed point data is

$$\{+l - x, l - 2x, x\}, \{-l - x, x, x\}, \{+l - x, x\}.$$ 

We can take a connected sum at $p_1$ and $p_2$ of $M$ and $q_5$ and $q_1$ of $Z_2(l, x, x)$ to construct another $S^1$-manifold $M'$ with fixed point data

$$\Sigma_{M'} = \{\Sigma_M \setminus \{+l, x, x\} \cup \{+l, x, l - x\}\} \cup \{+l - x, l - 2x, x\} \cup \{-l - x, x, x\}.$$ 

This corresponds to Operation (4) of Theorem 1.2 with $x = A$ and $l = C$.

Second, suppose Case (2-c-ii) with $2x > l$. In Example 3.7 we take $a = l$ and $d = x$ and reverse its orientation; its fixed point data is

$$\{-l - x, l, x\}, \{-l - x, 2x - l, x\}, \{+l - x, x, x\}, \{+l - x, x\}.$$ 

We can take a connected sum at $p_1$ and $p_2$ of $M$ and $q_5$ and $q_1$ of $Z_2(l, x, x)$ to construct another $S^1$-manifold $M'$ with fixed point data

$$\Sigma_{M'} = \{\Sigma_M \setminus \{+l, x, x\} \cup \{+l, x, l - x\}\} \cup \{-l - x, 2x - l, x\} \cup \{+l - x, x, x\} \cup \{+l - x, l - x, x\}.$$ 

This corresponds to Operation (4') of Theorem 1.2 with $x = A$ and $l = C$.

Third, suppose Case (2-c-ii) with $2x = l$. Since $p_1$ has weights $\{2x, x, x\}$ and the action is effective, this implies that $x = 1$, that is, the biggest weight $l$ is 2. By Lemma 5.1 below, we can take an equivariant connected sum at fixed points of $M$ and fixed points of rotations of $S^6$'s to construct a fixed point free $S^1$-action on a compact connected oriented manifold $M'$, which corresponds to taking Operation (1) of Theorem 1.2 many times.

To sum up, by repeatedly applying the above steps, we can successively take equivariant connected sums at two fixed points of $M$ and at two fixed points of $S^6$-actions on $S^6$, $CP^3$, $Z_1$, $Z_2$, and $Z_2 \times Z_2$ (and these with opposite orientations) to construct another 6-dimensional compact connected oriented $S^1$-manifold $M''$ with a discrete fixed point set, in which the biggest weight is strictly smaller than $l$; this is because every weight of a fixed point that we add is smaller than $l$. Now, on $M''$, take the biggest weight $l'' = \max\{w_{p, i} | 1 \leq i \leq 3, p \in (M'')^S\} < l$ and repeat the above argument.
In the end, by successively taking equivariant connected sums with the above manifolds, we get a 6-dimensional compact connected oriented $S^1$-manifold $\hat{M}$ with a discrete fixed point set, in which every weight at any fixed point is 1 or 2 if the fixed point set is non-empty. As in the beginning of this proof, by Lemma 5.1 we can take an equivariant connected sum at fixed points of $\hat{M}$ and at fixed points of rotations of $S^6$’s to construct a fixed point free $S^1$-action on another 6-dimensional compact connected oriented manifold. This last step corresponds to Operation (1) of Theorem 1.2. □

In the proof of Theorems 1.1 and 1.2 above, we used the following lemma.

**Lemma 5.1.** Let the circle act effectively on a 6-dimensional compact connected oriented manifold $M$ with a discrete fixed point set. Suppose that the biggest weight is 2, that is, $\max\{w_{p,i} \mid 1 \leq i \leq 3, p \in M^{S^1}\} = 2$. Then the number of fixed points with fixed point data $\{+, 1, 1, 1\}$ and $\{+, 1, 1, 2\}$ and the number of fixed points with fixed point data $\{-, 1, 1, 1\}$ and $\{-, 1, 2, 2\}$ are equal, respectively. Consequently, we can take an equivariant connected sum at fixed points of $M$ and at fixed points of rotations of $S^6$’s to construct a fixed point free $S^1$-action on a 6-dimensional compact connected oriented manifold.

**Proof.** Since the action is effective, possible multisets of weights at a fixed point are $\{1, 1, 1\}$, $\{1, 1, 2\}$, and $\{1, 2, 2\}$. Let $k_1, \cdots, k_6$ be the numbers of fixed points with fixed point data $\{+, 1, 1, 1\}$, $\{+, 1, 1, 2\}$, $\{+, 1, 2, 2\}$, $\{-, 1, 1, 1\}$, $\{-, 1, 1, 2\}$, and $\{-, 1, 2, 2\}$, respectively. By Proposition 2.3, the number $k_1 + k_2 + k_3$ of fixed points with sign $+1$ is equal to the number $k_4 + k_5 + k_6$ of fixed points with sign $-1$, that is, (1) $k_1 + k_2 + k_3 = k_4 + k_5 + k_6$.

Applying Lemma 2.4 to $M$ (take $a = 1$), it follows that (2) $3k_1 + 2k_2 + k_3 = 3k_4 + 2k_5 + k_6$.

Taking $\alpha = 1$ in Theorem 2.1,

$$\int_M 1 = \sum_{p \in M^{S^1}} \frac{1}{\prod_{i=1}^3 w_{p,i}} = k_1 + k_2 \frac{1}{2} + k_3 \frac{1}{4} - k_4 - k_5 \frac{1}{2} - k_6 \frac{1}{4}. $$

On the other hand, since the equivariant cohomology class 1 has degree 0, by a dimensional reason that $\int_M$ is a map from $H^i_{S^1}(M)$ to $H^{i-6}(\mathbb{C}P^\infty)$, its image under $\int_M$ vanishes, that is, $\int_M 1 = 0$. Thus, $0 = k_1 + k_2 \frac{1}{2} + k_3 \frac{1}{4} - k_4 - k_5 \frac{1}{2} - k_6 \frac{1}{4}$, that is, (3) $4k_1 + 2k_2 + k_3 = 4k_4 + 2k_5 + k_6$. Then (1-3) imply that $k_1 = k_4$, $k_2 = k_5$, and $k_3 = k_6$.

Therefore, for each pair of fixed points $(p_i, q_i)$ that have the same multiset of weights $\{a, b, c\}$ and $e(p_i) = -e(q_i) = 1$, we can take an equivariant connected sum at $p_i$ and $q_i$ of $M$ and at fixed points of a rotation of $S^6$ in Example 3.1 to construct a fixed point free $S^1$-action on a 6-dimensional compact connected oriented manifold. □
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