ON THE DISCRETE SPECTRUM OF TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATORS

Z. I. MUMINOV

ABSTRACT. In the present paper our aim is to explore some spectral properties of the family two-particle discrete Schrödinger operators $h_d(k) = h^0_d(k) + v$, $k \in \mathbb{T}^d$, on the $d$ dimensional lattice $\mathbb{Z}^d$, $d \geq 1$, $k$ being the two-particle quasi-momentum. Under some condition in the case $k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$, we establish necessary and sufficient conditions for existence of infinite discrete spectrum of the operator $h_d(k)$.

Subject Classification: Primary: 81Q10, Secondary: 35P20, 47N50

Key words and phrases: discrete Schrödinger operator, Friedrichs model, infinitely many eigenvalues, discrete spectrum.

1. INTRODUCTION.

Some spectral properties of the two-particle Schrödinger operators are studied in [1, 3, 2, 4, 5, 7].

The fundamental difference between multiparticle discrete and continuous Schrödinger operators is that in the discrete case the analogue of the Laplasian $-\Delta$ is not rotationally invariant.

Due to this fact, the Hamiltonian of a system does not separate into two parts, one relating to the center-of-motion and other one to the internal degrees of freedom. In particular, such a handy characteristics of inertia as mass is not available. Moreover, such a natural local substituter as the effective mass-tensor (of a ground state) depends on the quasi-momentum of the system and, in addition, it is only semi-additive (with respect to the partial order on the set of positive definite matrices). This is the so-called excess mass phenomenon for lattice systems (see, e.g., [10] and [11]): the effective mass of the bound state of an $N$-particle system is greater than (but, in general, not equal to) the sum of the effective masses of the constituent quasi-particles.

The two-particle problem on lattices can be reduced to an effective one-particle problem by using the Gelfand transform instead: the underlying Hilbert space $\ell^2((\mathbb{Z}^d)^2)$ is decomposed to a direct von Neumann integral, associated with the representation of the discrete group $\mathbb{Z}^d$ by the shift operators on the lattice and then the total two-body Hamiltonian appears to be decomposable as well. In contrast to the continuous case, the corresponding fiber Hamiltonians $h_d(k)$ associated with the direct decomposition depend parametrically on the internal binding $k$, the quasi-momentum, which ranges over a cell of the dual lattice. As a consequence, due to the loss of the spherical symmetry of the problem, the spectra of the family

Date: January 15, 2013.
$h^d(k)$ turn out to be rather sensitive to the variation of the quasi-momentum $k$, $k \in \mathbb{T}^d \equiv (-\pi, \pi]^d$.

For example, in [1, 5, 7] it is established that the discrete Schrödinger operator $h^3(k)$, with a zero-range attractive potential, for all non-zero values of the quasi-momentum $0 \neq k \in \mathbb{T}^3$, has a unique eigenvalue below the essential spectrum if $h^d(0)$ has a virtual level (see [1, 5]) or an eigenvalue below its essential spectrum (see [7]).

The similar result is obtained in [4], which is the (variational) proof of existence of the discrete spectrum below the bottom of the essential spectrum of the fiber Hamiltonians $h^d(k)$ for all non-zero values of the quasi-momentum $0 \neq k \in \mathbb{T}^d$, provided that the Hamiltonian $h^d(0)$ has either a virtual level (in dimensions of three and four) or a threshold eigenvalue (in all dimensions $d \geq 5$). We emphasize that the authors of [4] considered more general class of two-particle discrete Hamiltonians interacting via a short-range pair potentials.

It is also worth to mention that two-particle discrete operator $h^3(k)$, for some values of the quasi-momentum $k \neq 0$ may generate a “rich” infinite discrete spectrum outside its essential spectrum and these eigenvalues accumulate at the edge of its essential spectrum (see [3]).

In this paper, we explore the some spectral properties of some $d$-dimensional two-particle discrete Schrödinger operator $h^d(k) = h^d_0(k) + v, k \in \mathbb{T}^d$, $k$ being the two-particle quasi-momentum (see (2.2)), corresponding to the energy operator $H^d$ of the system of two quantum particles moving on the $d$-dimensional lattice $\mathbb{Z}^d$ with the short-range potential $v$. In the case $k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$, we establish necessary and sufficient conditions for existence of infinite number eigenvalues of the operator $h^d(k)$ (Theorem 3.4).

We now describe the organization of the present paper. In Sec. 2 describe the two-particle Hamiltonians in the coordinate representation, introduce the two-particle quasi-momentum, and decompose the energy operator into the von Neumann direct integral of the fiber Hamiltonians $h^d(k)$, thus providing the reduction to the effective one-particle case.

In Section 3 in case the particles have (no) equal masses we obtain the finiteness of the discrete spectrum of $h^d(k)$ for all $(k \in \mathbb{T}^d) k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$, provided that the Hamiltonian $h^d(0)$ has finite discrete spectrum (Theorem 3.2) and prove the main result of this paper, Theorem 3.4 in the case $k \in \mathbb{T}^d \setminus (-\pi, \pi)^d$.

In Appendix 3.2, for readers convenience, in the case $k \in M = \mathbb{T}^d \setminus (-\pi, \pi)^d$ we give an example that the infiniteness of the discrete spectrum of $h^d(k)$ depends on $k \in M$.

In order to facilitate a description of the content of this paper, we briefly introduce the notation used throughout this manuscript. Let $\mathbb{Z}^d$ be the $d$-dimensional lattice and $\mathbb{T}^d$ be the $d$-dimensional torus, the cube $(-\pi, \pi]^d$ with appropriately identified sides, $d \geq 1$. We remark that the torus $\mathbb{T}^d$ will always be considered as an abelian group with respect to the addition and multiplication by the real numbers regarded as operations on $\mathbb{R}^d$ modulo $(2\pi\mathbb{Z})^d$. Denote by $L^2((\mathbb{T}^d)^m)$ the Hilbert space of square-integrable functions defined on $(\mathbb{T}^d)^m$. 

We remark that the torus $\mathbb{T}^d$ will always be considered as an abelian group with respect to the addition and multiplication by the real numbers regarded as operations on $\mathbb{R}^d$ modulo $(2\pi\mathbb{Z})^d$. Denote by $L^2((\mathbb{T}^d)^m)$ the Hilbert space of square-integrable functions defined on $(\mathbb{T}^d)^m$. 


2. THE DESCRIPTION OF THE TWO-PARTICLE OPERATOR

The free Hamiltonian $H_0^d$ of two quantum particles on the $d$-dimensional lattice $\mathbb{Z}^d$ usually associated with the following self-adjoint operator in the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$ of $\ell^2$-sequences $f(x), x \in (\mathbb{Z}^d)^2$:

$$H_0^d = -\frac{1}{2m_1}\Delta_{x_1} - \frac{1}{2m_2}\Delta_{x_2},$$

with

$$\Delta_{x_1} = \Delta \otimes I_d, \quad \Delta_{x_2} = I_d \otimes \Delta,$$

where $I_d$ is the identical operator on $\ell^2(\mathbb{Z}^d)$, $m_1, m_2 > 0$ are meaning of masses of particles and $\Delta$ is standard discrete Laplasian:

$$(\Delta f)(x) = \frac{1}{2}\sum_{j=1}^{d} (2f(x) - f(x + \vec{e}_j) - f(x - \vec{e}_j)), \quad f \in \ell_2(\mathbb{Z}^d),$$

that is,

$$\Delta = \frac{1}{2}\sum_{j=1}^{d} (2I_d - T(\vec{e}_j) - T^*(\vec{e}_j)).$$

Here $T(y)$ is the shift operator by $y \in \mathbb{Z}^d$:

$$(T(y)f)(x) = f(x + y), \quad f \in \ell_2(\mathbb{Z}^d),$$

and $\vec{e}_j, j = 1, \ldots, d$, is the unit vector along the $j$-th direction of $\mathbb{Z}^d$.

Remark 2.1. For more general definitions of discrete Laplasians see [4], [15].

Remark 2.2. Note that the free Hamiltonian $H_0^d$ is a multi-dimensional Laurent-Toeplitz type operator defined by the function $E(\cdot, \cdot) : (\mathbb{T}^d)^2 \to \mathbb{R}^1$:

$$E(k_1, k_2) = \frac{1}{m_1}\varepsilon(k_1) + \frac{1}{m_2}\varepsilon(k_2), \quad k_1, k_2 \in \mathbb{T}^d,$$

where

$$\varepsilon(p) = \sum_{i=1}^{d} (1 - \cos p^{(i)}), \quad p = (p^{(1)}, \ldots, p^{(d)}) \in \mathbb{T}^d.$$  \hfill (2.1)

The last claim can be obtained by the Fourier transform $F : \ell^2((\mathbb{Z}^d)^2) \to L^2(\mathbb{T}^d)^2)$ which gives the unitarity of $H_0^d$ and the multiplication operator on $L^2((\mathbb{T}^d)^2)$ by the function $E(\cdot, \cdot)$. It is easy to see that the free Hamiltonian $H_0^d$ is a bounded operator and its spectrum coincides with the interval $[0, d(m_1^{-1} + m_2^{-1})]$.

The total Hamiltonian $H^d$ (in the coordinate representation) of the system of the two quantum particles moving on $d$-dimensional lattice $\mathbb{Z}^d$ with the real-valued pair interaction $V$ is a self-adjoint bounded operator in the Hilbert space $\ell^2((\mathbb{Z}^d)^2)$ of the form

$$H^d = H_0^d + V,$$

where

$$(Vf)(x_1, x_2) = v(x_1 - x_2)f(x_1, x_2), \quad f \in \ell^2((\mathbb{Z}^d)^2),$$
with $\nu: \mathbb{Z}^d \to \mathbb{R}$ a bounded function.

As we note above the Hamiltonian $H^d$ does not split into a sum of a center of mass and relative kinetic energy as in the continuous case using the center of mass and relative coordinates.

Using the direct von Neumann integral decomposition the two-particle problem on lattices can be reduced to an effective one-particle problem. Here we shortly recall some results of \cite{15} related to direct integral decomposition.

The energy operator $H^d$ is obviously commutable with the group of translations $U_s, s \in \mathbb{Z}^d$:

$$(U_sf)(x_1, x_2) = f(x_1 + s, x_2 + s), \quad f \in L^2((\mathbb{Z}^d)^2), \quad x_1, x_2 \in \mathbb{Z}^d,$$

for any $s \in \mathbb{Z}^d$.

One can check that the group $\{U_s\}_{s \in \mathbb{Z}^d}$ is the unitary representation of the abelian group $\mathbb{Z}^d$ in the Hilbert space $L^2((\mathbb{Z}^d)^2)$.

Consequently, we can decompose the Hilbert space $L^2((\mathbb{Z}^d)^2)$ in a diagonal for the operator $H^d$ direct integral whose fiber is parameterized by $k \in \mathbb{T}^d$ and consists of functions on $L^2((\mathbb{Z}^d)^2)$ satisfying the condition

$$(U_sf)(x_1, x_2) = \exp(-i(s, k))f(x_1, x_2), \quad s \in \mathbb{Z}^d, k \in \mathbb{T}^d.$$

Here the parameter $k$ is naturally interpreted as the total quasi-momentum of the two-particles and is called the two-particle quasi-momentum.

More precisely, let us introduce the mapping $F: L^2((\mathbb{Z}^d)^2) \to L^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ by the equality

$$(Ff)(x_1, k) = \frac{1}{(2\pi)^\frac{d}{2}} \sum_{s \in \mathbb{Z}^d} e^{-i(s, k)}f(x_1 + s, s), \quad f \in L^2((\mathbb{Z}^d)^2).$$

Remark that $F$ is the unitary mapping of the space $L^2((\mathbb{Z}^d)^2)$ onto $L^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ and its adjoint $F^* : L^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d) \to L^2((\mathbb{Z}^d)^2)$ defined as

$$(F^*g)(x_1, x_2) = \frac{1}{(2\pi)^\frac{d}{2}} \int_{k' \in \mathbb{T}^d} e^{-i(x_2, k')}g(x_1 - x_2, k')dk', \quad g \in L^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d).$$

It follows that $H^d_0 = FH^d_0F^*$ acts in $L^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ as a multiplication by the operator function $h^d_0(k), k \in \mathbb{T}^d$, acting in the Hilbert space $L^2(\mathbb{Z}^d)$ by

$$h^d_0(k) = \frac{1}{2} \sum_{j=1}^d \left(2\mu(0)I_d - \mu(k^{(j)}T(\vec{e}_j) - \mu(k^{(j)})T^*(\vec{e}_j))\right),$$

where

$$\mu(y) = m_1^{-1} + m_2^{-1}e^{-iy}, \quad y \in \mathbb{T}^d.$$

Remark 2.3. Note that the operator $h^d_0(k), k \in \mathbb{T}^d$, also is a multi-dimensional Laurent-Toeplitz type operator defined by

$$E_k(p) = \frac{1}{m_1}\epsilon(p) + \frac{1}{m_2}\epsilon(k - p), \quad p \in \mathbb{T}^d,$$

where $\epsilon(\cdot)$ is defined by (2.1).
Clearly, the operator $V$ commutes with $\{U_s\}_{s \in \mathbb{Z}^d}$. As a result $FVF^*$ acts in $\ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ as a multiplication operator by the form

$$(FVF^*)g(x,k) = v(x)g(x,k), \quad g \in \ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d).$$

Consequently, the operator $\hat{H}^d = FH^dF^*$ acts in $\ell^2(\mathbb{Z}^d) \otimes L^2(\mathbb{T}^d)$ as a multiplication by the operator-function $h^d(k), k \in \mathbb{T}^d$, acting in the Hilbert space $\ell^2(\mathbb{Z}^d)$ in the form

$$(2.2) \quad h^d(k) = h^d_0(k) + v,$$

where $v$ is the multiplication operator by the function $v(\cdot)$ in $\ell^2(\mathbb{Z}^d)$.

This procedure is quite similar to the ”separation” of motion of the center of mass for dispersive Hamiltonian in the continuous case. However, in this case the role of $\mathbb{Z}^d$ and $\mathbb{T}^d$ was played by $\mathbb{R}^d$.

The form $(2.2)$ for $h^d(k)$ shows that the two-particle problem to be reduced to an effective one-particle problem.

For any $l, l \in N_d = \{1, \ldots, d\}$, let $A^d_l$ be the family of all ordered set $\alpha = \{\alpha_1, \ldots, \alpha_l\} \subset N_d$, and $\bar{\alpha}$ be the complement of $\alpha \in A^d_l$ with respect to the set $N_d$.

We denote by $M^d_l(\alpha), \alpha \in A^d_l$, the set of all $k \in M = \mathbb{T}^d \setminus (-\pi, \pi)^d$ such that only $\alpha_1$-th \ldots $\alpha_l$-th coordinates are equal to $\pi$, i.e

$$M^d_l(\alpha) \equiv \{k \in \mathbb{T}^d : k^{(j)} = \pi, j \in \alpha \quad \text{and} \quad k^{(j)} \neq \pi, j \in \bar{\alpha}\}. $$

Let be $\mathbb{Z}^l(\alpha) \subset \mathbb{Z}^d$ be the lattice with the basis $\{\vec{e}_j\}, j \in \alpha$. Of course $\mathbb{Z}^l(\alpha)$ is isomorphic to the abelian group $\mathbb{Z}^l$ and $\mathbb{Z}^l(\alpha) \oplus \mathbb{Z}^{d-l}(\bar{\alpha}) = \mathbb{Z}^d$.

By $\mathbb{Z}^{d-l}(\bar{\alpha})$ we denote the shift of $\mathbb{Z}^{d-l}(\bar{\alpha})$ by $x \in \mathbb{Z}^l(\alpha)$, i.e

$$\mathbb{Z}_{x}^{d-l}(\bar{\alpha}) = \mathbb{Z}^{d-l}(\bar{\alpha}) + x.$$

In order to get the main results of the paper we assume the following technical hypotheses that guarantee the finiteness of the discrete spectrum of the operator $h^d(0)$, corresponding to the zero value of the quasi-momentum $k$, and the compactness of the interaction $v$.

**Hypothesis 2.4.** Here and further with regard to the function $v(\cdot)$ it is assumed that:

(A) $\lim_{x \to \infty} v(x) = 0$;

(B) for any $l \in N_{d-1}, \alpha \in A^d_l$, $x \in \mathbb{Z}^l(\alpha)$ the potential $\vec{v}$, corresponding to the restriction $\vec{v}(\cdot)$ of the function $v(\cdot)$ on the set $\mathbb{Z}_{x}^{d-l}(\bar{\alpha})$, do not produce infinite discrete spectrum for the operator

$$h^{d-l}(0) = h^{d-l}_0(0) + \vec{v}.$$

**Remark 2.5.** The real valued function $v(\cdot)$, being exponentially decreasing function, satisfies Hypothesis 2.4.
3. THE FORMULATION AND PROOF OF THE MAIN RESULTS

Under the Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{T}^d)$ the operator $h^{\delta}(k), k \in \mathbb{T}^d$, turns into the operator $\hat{h}^{\delta}(k), k \in \mathbb{T}^d$, (the Friederichs model) acting in $L^2(\mathbb{T}^d)$ as

$$\hat{h}^{\delta}(k) = \hat{h}_0^{\delta}(k) + \hat{v}.$$  

where $\hat{h}_0^{\delta}(k)$ is the multiplication operator by the function $E_k(q)$:

$$(\hat{h}_0^{\delta}(k)f)(q) = E_k(q)f(q), \quad f \in L^2(\mathbb{T}^d),$$

and $\hat{v}$ is the integral operator of convolution type:

$$(\hat{v}f)(q) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} \hat{v}(q - s)f(s)ds, \quad f \in L^2(\mathbb{T}^d).$$

Here $\hat{v}(\cdot)$ is the Fourier series with the Fourier coefficients $v(s), s \in \mathbb{Z}^d$.

Under assumption (A) of Hypothesis 2.4 the perturbation $v$ of the operator $h_0^{\delta}(k), k \in \mathbb{T}^d$, is a Hilbert-Schmidt operator and, therefore, according to the Weil theorem, the essential spectrum of the operator $h^{\delta}(k)$ fills in the following interval on the real axis:

$$\sigma_{\text{ess}}(h(k)) = [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)],$$

where

$$\mathcal{E}_{\min}(k) = \min_{q \in \mathbb{T}^d} E_k(q), \quad \mathcal{E}_{\max}(k) = \max_{q \in \mathbb{T}^d} E_k(q).$$

The function $E_k(p)$ can be rewritten in the form

$$E_k(p) = d\mu(0) - \sum_{j=1}^{d} r(k^{(j)}) \cos(p^{(j)} - p(k^{(j)})),$$  \hspace{1cm} (3.1)

where the coefficients $r(k^{(j)})$ and $p(k^{(j)})$ are given by

$$r(k^{(j)}) = |\mu(k^{(j)})|, \quad p(k^{(j)}) = \arcsin \frac{\text{Im}\mu(k^{(j)})}{r(k^{(j)})}, \quad k^{(j)} \in (-\pi, \pi].$$

The equality (3.1) implies the following representation for $E_k(\cdot)$:

$$E_k(p + p(k)) = d\mu(0) - \sum_{j=1}^{d} r(k^{(j)}) \cos p^{(j)},$$  \hspace{1cm} (3.2)

where $p(k)$ is the vector-function

$$p : \mathbb{T}^3 \rightarrow \mathbb{T}^3, \quad p(k) = (p(k^{(1)}), \ldots, p(k^{(d)})) \in \mathbb{T}^3.$$  

Then

$$\mathcal{E}_{\min}(k) = d\mu(0) - \sum_{j=1}^{d} r(k^{(j)}), \quad \mathcal{E}_{\max}(k) = d\mu(0) + \sum_{j=1}^{d} r(k^{(j)})$$

and

$$E_k(p + p(k)) - \mathcal{E}_{\min}(k) \leq E_0(p) \leq \frac{1}{A(k)}(E_k(p + p(k)) - \mathcal{E}_{\min}(k)), \hspace{1cm} (3.3)$$
since \( r(k^{(j)}) \leq r(0) \) and \( r(0) \leq r(k^{(j)})/A(k) \), where
\[
A(k) = \min_{j=1,...,d} r(k^{(j)})/r(0).
\]

**Remark 3.1.** Note that \( A(k) = 0 \) holds and only if \( k \in \mathbb{T}^d \setminus (-\pi, \pi)^d \) and \( m_1 = m_2 \).

3.1. Case \( A(k) \neq 0 \).

**Theorem 3.2.** Assume Hypothesis 2.4 and for \( k \in \mathbb{T}^d \) the inequality \( A(k) \neq 0 \) holds. Then the operator \( h^d(k) \) has only finite discrete spectrum.

**Proof.** Let be \( U_k f(p) = f(p + p(k)) \) is the shift operator on \( \mathbb{T}^d \) and \( A(k) \neq 0 \). Using the inequality (3.3) one can get the inequalities
\[
U_k^* (\hat{h}_0^d(k) - \varepsilon_{\min}(k)I_d) U_k \leq \hat{h}_0^d(0) \leq \frac{1}{A(k)} U_k^* (\hat{h}_0^d(k) - \varepsilon_{\min}(k)I_d) U_k.
\]
Consequently, we obtain (3.4)
\[
U_k^* (h^d(k) - \varepsilon_{\min}(k)I_d) U_k \leq \hat{h}_0^d(0) \leq \frac{1}{A(k)} U_k^* (\hat{h}_0^d(k) + A(k)\hat{v} - \varepsilon_{\min}(k)I_d) U_k,
\]
since \( U_k^* \hat{v} U_k = \hat{v} \).

Now we define a unitary staggering transformation \( U \) (see [7]), which plays a key role in understanding the relation between the parts of the discrete spectrum \( \sigma_{\text{disc}}(h^d(k)) \) lying below and above \( \sigma_{\text{ess}}(h^d(k)) \) of \( (h^d(k)) \)
\[
(U f)(x) = (-1)^{\sum_{j=1}^d x^{(j)}} f(x), \quad f \in \ell^2(\mathbb{Z}^d).
\]

The transformation \( U \) has the important intertwining property
\[
h_0^d(k) + \hat{v} = U^{-1} (-1 (4d(m_1^{-1} + m_2^{-1}) + h_0^d(k) - \hat{v})) U
\]
This equality shows that it is enough to consider the discrete spectrum below the essential spectrum of \( h_0^d(k) + \hat{v} \).

This facts, inequalities 3.4 Hypothesis 2.4 and the min-max principle yields the statement of the theorem.

To get oriented, let us recall some results and facts relevant to our discussion.

In one- and two- dimensional case the following slightly changed form of Theorem 4.5 in [6] hold and its proof is omitted.

**Theorem 3.3.** Assume Hypothesis 2.4 and \( A(k) \neq 0 \). Let \( d = 1 \) or \( 2 \). If \( \sigma(h^d(k)) \subset [\varepsilon_{\min}(k), \varepsilon_{\max}(k)] \), then \( \hat{v} = 0 \).

This is connected to the fact that the discrete and continuous one-particle Schrödinger operators in one and two dimensions always have a bound state for nontrivial potentials (see relevant discussions [2], [3] and [8], [9], [13]).

In three and upper dimensional case for the potentials satisfying Hypothesis 2.4 in \( d \geq 3 \)-dimensional case two-particle discrete Schrödinger operators \( h^d(k) \), has only finite number of eigenvalues outside the essential spectrum. Moreover,
for sufficiently small potentials, the corresponding operator need not have bound states (see [3]).

This fact follows from the uniformly boundedness of the norm of the Birman-Schwinger compact operator $G(k, z) = -(r^d_0(k, z))^{1/2} v(r^d_0(k, z))^{1/2}$, $r^d_0(k, z) = (h^d_0(k) - zI_d)^{-1}$, $z < E_{\text{min}}(k)$, near the bottom of the essential spectrum of $h^d(k)$. Moreover, in this case the discrete spectrum $\sigma_{\text{disc}}(h^d(k))$ of $h^d(k)$ is empty, since for sufficiently small potentials the norm of the Birman-Schwinger operator $G(k, z)$, $z < E_{\text{min}}(k)$, is small near the bottom of the essential spectrum of $h^d(k)$.

3.2. Case $A(k) = 0$. The equality $A(k) = 0$ implies this equality $m_1 = m_2$ and $k \in M = T^d \setminus (-\pi, \pi)^d$.

For any $\{j\} \in A^d_1$, $\alpha \in A^d_1$, $l \in N_d$, and $n \in \mathbb{N}$ we say the set

$$\Pi^{-1}_n(\{j\}) = \bigcup_{x \in \{-n, \ldots, n\}} Z^{-1}_x(\{j\}),$$

resp.

$$\Pi^{-1}_n(\alpha) = \bigcap_{\alpha_j \in \alpha} \Pi^{-1}_n(\{\alpha_j\}),$$

is the $d-1$ resp. $d-l$ dimensional lattice strip on $\mathbb{Z}^d$, along direction corresponding to $\{j\}$ resp. $\alpha$.

Define

$$X_n(\alpha) = \{x \in \mathbb{Z}^d(\alpha) : x^{(j)} \in \{-n, \ldots, n\} \text{ for all } j \in \alpha\}, \quad n \in \mathbb{N}^d,$$

the subset (the lattice cube with side $n$) of $\mathbb{Z}^d(\alpha)$.

One can check that for any $\alpha \in A^d_1$, $l \in N_d$,

$$\Pi^{-1}_n(\alpha) = \bigcup_{x \in X_n(\alpha)} Z^{-1}_x(\alpha) \quad \text{and} \quad \Pi^{-1}_n(\alpha) = \mathbb{Z}^{-1}_d(\alpha) \oplus X_n(\alpha)$$

and in particular case $l = d$ the equality $\Pi^{-1}_n(\alpha) = X_n(\alpha)$ holds and $\Pi^{-1}_n(\alpha)$ is the lattice cube with side $n$ in $\mathbb{Z}^d$.

Theorem 3.4. Assume Hypothesis 2.4. The for any $l \in N_d$, $\alpha \in A^d_1$, and for all $k \in M^d_1(\alpha)$ we have:

a) if $d - 2 \leq l \leq d$ then the operator $h^d(k)$ has an infinite number of eigenvalues outside of the essential spectrum if and only if for any $n \in \mathbb{N}$ the relation

$$\text{supp} v \not\subset \Pi^{-1}_n(\alpha)$$

is accomplished.

b) if $1 \leq l < d - 2$ then the operator $h^d(k)$ has a finite number of eigenvalues outside of the essential spectrum.

Proof. To be simplicity we take $\tilde{\alpha} = \{1, \ldots, d - l\}$, $\alpha = \{d - l + 1, \ldots, d\}$ and we use notations $k = (k^{(1)}, \ldots, k^{(d-l)}) \in T^{d-l}$ for $k = (k, \pi, \ldots, \pi) \in M^d_1(\alpha)$, and $x = (\tilde{x}, \hat{x}) \in \mathbb{Z}^d$ for $\tilde{x} \in \mathbb{Z}^{d-l}(\tilde{\alpha})$, $\hat{x} \in \mathbb{Z}^l(\alpha)$.
Clearly
\[ \ell^2(\mathbb{Z}^d) = \bigcup_{\vec{x} \in \mathbb{Z}^l} \oplus \ell^2(\mathbb{Z}^d_{\vec{x}}(\vec{\alpha})) \]
and \( \ell^2(\mathbb{Z}_\vec{x}^{d-l}(\vec{\alpha})) \), \( \vec{x} \in \mathbb{Z}^l \), is isomorphic to \( \ell^2(\mathbb{Z}^d) \). This isomorphism is established with the restriction \( P_{\vec{x}} \equiv P|_{\ell^2(\mathbb{Z}^{d-l}_\vec{x}(\vec{\alpha}))} \) of the isometric operator \( P : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}_\vec{x}^{d-l}(\vec{\alpha})) \):

\[ (Pf)(\vec{x}) = f(\vec{x}, \hat{x}), \quad f \in \ell^2(\mathbb{Z}^d), \]
on \( \ell^2(\mathbb{Z}_\vec{x}^{d-l}(\vec{\alpha})) \).

If \( k \in M_i^d(\alpha) \), then
\[ h^d_0(k) = \frac{1}{2} \sum_{j=1}^{d-l} (2\mu(0)I_d - \mu(k^j)T(\vec{e}_j) - \mu(\overline{k^j})\overline{T(\vec{e}_j)}) + \mu(0)I_d. \]

Using the last equality it is easy to see that the Hilbert space \( \ell^2(\mathbb{Z}_\vec{x}^{d-l}(\vec{\alpha})) \) is invariant under \( h^d_0(k) \) and \( v \).

One can check that \( h^d(k) \) is described as
\[ h^d(k) = \sum_{x \in \mathbb{Z}^l} \oplus P_x^\ast (I\mu(0)I_{d-l} + h^d_{\vec{x}}(\vec{k}))P_{\vec{x}}, \]
where \( h^d_{\vec{x}}(\vec{k}) \) is the \( d-l \)-dimensional discrete Schrödinger operator corresponding to the Hamiltonian of a system of two identical particles moving on \( \mathbb{Z}^{d-l} \) and interacting short-range attractive potential \( v_{\vec{x}}(\cdot) \), being multiplication operator by the restriction \( v_{\vec{x}}(\cdot) : \mathbb{Z}_\vec{x}^{d-l}(\vec{\alpha}) \to \mathbb{R}^1 \), of the function \( v(\cdot) \) on \( \mathbb{Z}_\vec{x}^{d-l}(\vec{\alpha}) \simeq \mathbb{Z}^{d-l} \).

That is we decompose \( h^d(k) \) a direct sum of \( d-l \) dimensional two-particle Schrödinger operators
\[ h^d_{\vec{x}}(\vec{k}) = h^d_{0}(\vec{k}) + v_{\vec{x}}, \quad \vec{k} \in \mathbb{T}^{d-l}, \quad \vec{x} \in \mathbb{Z}^l, \]
with the potential \( v_{\vec{x}}, \vec{x} \in \mathbb{Z}^l \):
\[ h^d(k) \simeq \sum_{\vec{x} \in \mathbb{Z}^l} \oplus (I\mu(0)I_{d-l} + h^d_{\vec{x}}(\vec{k}) + v_{\vec{x}}). \]

Since \((3.5)\) and \( h^d_{0}(\vec{k}) \) does not depend on \( x \) and \( v_{\vec{x}} \) is a compact operator for the essential spectrum of \( h^d(k) \) we get
\[ \sigma_{ess}(h^d(k)) = \mu(0) + \bigcup_{x \in \mathbb{Z}^l} \sigma_{ess}(h^d_{\vec{x}}(\vec{k})) = [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)] \]
and hence
\[ \sigma_{disc}(h^d(k)) = \mu(0) + \bigcup_{x \in \mathbb{Z}^l} \sigma_{disc}(h^d_{\vec{x}}(\vec{k})). \]

a) Case \( l = d \). Then \( M_i^d(\alpha) = \{ \vec{\pi} \}, \quad \vec{\pi} = (\pi, \ldots, \pi) \in \mathbb{T}^d \) and \( h^d(\vec{\pi}) = d\mu(0) + v \). Hence \( \sigma_{disc}(h^d(\vec{\pi})) = d\mu(0) + \cup_{x \in \mathbb{Z}^d} \{ v(x) \}. \)
Remark 3.5. In the case the operator dimensional ) two-particle discrete operators there exists and obviously manifold homeomorphic to spectrum for the perturbation of Hamiltonian whose Fouriersymbol attains its paper provided an elementary proof that negative potential lead to infinite discrete

Let \( \hat{\varepsilon}, \hat{\sigma} \in \mathbb{R} \) exist potential, such that, generates infinite discrete spectrum for the operator \( \hat{\varepsilon} \neq 0 \), that is supp \( \varepsilon \) does not belong in

\[
\Pi_{\hat{\varepsilon}}^{d-l}(\alpha) = \bigcup_{\hat{x} \in \hat{X}_n(\alpha)} \mathbb{Z}^{d-l}(\hat{x}).
\]

b) If \( d - l = 3 \) or \( d - l > 3 \) (that is \( 1 \leq l < d - 2 \)) then by Theorem 3.2 the operator \( \hat{\varepsilon} \neq 0 \), \( \hat{\sigma} \neq 0 \), \( \hat{\varepsilon} \in \mathbb{R} \) has no the infinite number of eigenvalues outside of the essential spectrum.

By virtue Hypothesis 2.4 for any \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that for all \( x \in \bigcup_{-n \leq x \leq n} \mathbb{Z}^d \), the inequality \(|v(x)| < \epsilon \) holds.

Then from the discussed facts relating to the emptiness of the (three and upper dimensional ) two-particle discrete operators there exists \( n \in \mathbb{N} \) such that for all \( \hat{x} \in \mathbb{Z}^d(\alpha) \setminus X_{n}^{d-l}(\alpha) \) the discrete spectrum \( \sigma_{\text{disc}}(\hat{\varepsilon}) \) is empty.

Hence

\[
\sigma_{\text{disc}}(\hat{\varepsilon}) = l\mu(0) + \bigcup_{x \in X_{n}^{d-l}(\alpha)} \sigma_{\text{disc}}(\hat{\varepsilon}(\hat{x}))
\]

since (3.8).

This fact and compactness of \( X_n^{d-l}(\alpha) \) ended the proof of Theorem 3.4 \( \square \)

Remark 3.5. In the case \( k \in M^d(\alpha_1, \ldots, \alpha_l) \) the Fourier symbol \( E_k(\cdot) \) of the free one-particle discrete operator \( h_0^d(k) \) attains its minimum along a \( l \) dimensional manifold homeomorphic to \( \mathbb{T}^l \). If this manifold is \( d - 1 \) or \( d \) dimensional then there exists potential, such that, generates infinite discrete spectrum for the operator \( h_0^d(k) \).

This result can resemble with the result established in [12]. The authors of this paper provided an elementary proof that negative potential lead to infinite discrete spectrum for the perturbation of Hamiltonian whose Fourier symbol attains its minimal value on \( n - 1 \) dimensional manifold of \( \mathbb{R}^n \).

Appendix A. Example of the infiniteness of the discrete spectrum

The main goal of this appendix is to show that in the case \( k \in \mathbb{T}^d \setminus (-\pi, \pi)^d \) the infiniteness of the discrete spectrum depends on the direction \( \alpha \in A_1^d \).

Example A.1. Let \( d = 2 \) and \( v(x) = \begin{cases} e^{-|x(1)|}, & x \in \mathbb{Z}^1 \times \{0\} \\ 0, & \text{otherwise.} \end{cases} \)

Then for \( \alpha = \{1\} \in A_1^2, n \in \mathbb{N} \), we have

\[
\Pi_{\alpha_1}^{d-l}(\{1\}) = \{x \in \mathbb{Z}^1 : |x(1)| \leq n\}, \quad \Pi_{\alpha_2}^{d-l}(\{2\}) = \{x \in \mathbb{Z}^1 : |x(2)| \leq n\},
\]

and obviously

\[
\text{supp}(\cdot) \notin \Pi_{\alpha_1}^{d-l}(\{1\}) \quad \text{and} \quad \text{supp}(\cdot) \subset \Pi_{\alpha_2}^{d-l}(\{2\}).
\]
Then by theorem for any $k \in M_2^2(\{1\})$ the operator $h_{\alpha}(k)$ has a infinite number of eigenvalues outside of the essential spectrum, but for $k \in M_2^2(\{2\})$ it has no infinite discrete spectrum.

Acknowledgments I thank Prof. S. N. Lakaev for useful discussions and gratefully acknowledge the hospitality of the The Abdus Salam International Centre for Theoretical Physics and I am also indebted to the the International Mathematical Union for the travel grant. This work was also partially supported by the Fundamental Science Foundation of Uzbekistan.

REFERENCES

[1] J. I. Abdullayev, S. N. Lakaev Asymptotics of the Discrete Spectrum of the Three-Particle Schrödinger Difference Operator on a Lattice, Theoretical and Mathematical Physics, 136, No. 2, (2003) 1096-1109.
[2] J. I. Abdullayev, S. N. Lakaev: On the Spectral Properties of the Matrix-Valued Friedrichs Model. Advances in soviet Mathematics. American Mathematical Society. 5 (1991) 1-37.
[3] S. Albeverio, S. N. Lakaev and J.I. Abdullayev. On the spectral properties of two-particle discrete Schrödinger operators. Preprint, Bonn, Bonn University, (2004) pp.14.
[4] S. Albeverio, S. N. Lakaev, K. A. Makarov, Z. I. Muminov: The Threshold Effects for the Two-particle Hamiltonians on Lattices, Comm.Math.Phys. 262(2006), 91-115.
[5] S. Albeverio, S. N. Lakaev, Z. I. Muminov: Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics. Ann. Henri Poincaré. 5 743–772 (2004).
[6] D. Damanik, D. Hundertmark, R. Killip, B. Simon: Variational Estimates for Discrete Schrodinger Operators with Potentials of Indefinite Sign. Comm.Math.Phys. 238(2003), 545–562.
[7] P. A. Faria da Viega, L. Ioriatti and M. O’Carrol: Energy-momentum spectrum of some two-particle lattice Schrödinger Hamiltonian. Phys. Rev. E(3) 66, 016130, 9 pp. (2002).
[8] M. Klaus: On the bound state of Schrodinger operators in one dimension. Ann. Phys. 108, (1977) 288300.
[9] L. D. Landau, E. M. Lifshitz: Quantum Mechanics: Non-relativistic Theory. Course of Theoretical Physics, Vol. 3. Reading, Mass: Addison-Wesley, 1958.
[10] D. C. Mattis: The few-body problem on a lattice. Rev. Modern Phys. 58, (1986) 361–379.
[11] A. Mogilner: Hamiltonians in solid state physics as multi-particle discrete Schrödinger operators: Problems and results. Advances in Soviet Mathematics 5, (1991) 139–194.
[12] K. Pankrashkin: Variational principle for hamiltonians with degenerate bottom. In the book I. Beltita, G. Nenciu, R. Purice (Eds.): Mathematical Results in Quantum Mechanics. Proceedings of the QMath10 Conference (World Scientific, 2008) 231-240 Preprint [arXiv:0710.4790].
[13] B. Simon, The bound state of weakly coupled Schrödinger operators in one and two-dimentional, Ann. Phys. 97, 279-288, (1976).
[14] M. Reed and B. Simon, Methods of modern mathematical physics. VI: Analysis of Operators, Academic Press, New York, 1979.
[15] D. R. Yafaev : Scattering theory: Some old and new problems, Lecture Notes in Mathematics, 1735. Springer-Verlag, Berlin, 2000, 169 pp.