Even- and Odd-Parity Charmed Meson Masses in Heavy Hadron Chiral Perturbation Theory

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Abstract

We derive mass formulae for the ground state, \(J^P = 0^-\) and \(1^-\), and first excited even-parity, \(J^P = 0^+\) and \(1^+\), charmed mesons including one loop chiral corrections and \(O(1/m_c)\) counterterms in heavy hadron chiral perturbation theory. We show a variety of fits to the current data. We find that certain parameter relations in the parity doubling model are not renormalized at one loop, providing a natural explanation for the equality of the hyperfine splittings of ground state and excited doublets.

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I. INTRODUCTION

Excited charmed mesons with angular momentum and parity $J^P = 0^+$ and $1^+$ have been observed in several experiments. The masses of the $J^P = 0^+$ and $1^+$ charmed strange mesons, $D_s(2317)$ and $D_s(2460)$ \cite{1, 2}, are below threshold for decays into ground state charmed mesons and kaons. The only strong decay modes are via isospin-violating $\pi^0$ emission, making the states quite narrow ($\Gamma < 5.5$ MeV). Other experiments \cite{3, 4, 5} claim to observe the nonstrange $J^P = 0^+$ and $1^+$ states. These states can decay to the ground states by $S$-wave pion emission and therefore are quite broad ($\Gamma \sim 300$ MeV).

The spectrum of the $J^P = 0^+$ and $1^+$ charmed mesons presents a number of puzzles for theory. Before their discovery, quark model and lattice calculations predicted that the masses of the $J^P = 0^+$ and $1^+$ charmed strange mesons would be significantly higher than observed \cite{6, 7, 8, 9, 10}. Further, the hyperfine splittings of all ground state charmed mesons and the hyperfine splitting of the $D_s(2317)$ and $D_s(2460)$ are all equal to within 2%. This is surprising because there is no obvious symmetry of quantum chromodynamics (QCD) which predicts these equalities. Finally, the $SU(3)$ splittings of the $J^P = 0^+$ and $1^+$ charmed mesons are much smaller than theoretical expectations.

In the heavy quark limit, the coupling of the heavy quark spin to the light degrees of freedom in the heavy meson vanishes and the angular momentum and parity of the light degrees of freedom, $j^p$, can be used to classify heavy meson states. The spectrum consists of degenerate heavy meson doublets with definite $j^p$. The $J^P = 0^-$ and $1^-$ heavy mesons are members of the $j^p = \frac{1}{2}^-$ ground state doublet. The lowest lying excited states, the $J^P = 0^+$ and $1^+$ heavy mesons, are members of the $j^p = \frac{1}{2}^+$ doublet. There is also an excited doublet of heavy mesons with $j^p = \frac{3}{2}^+$, whose members have $J^P = 1^+$ and $2^+$. These mesons decay to the ground state by $D$-wave pion emission, typically have widths $\Gamma \sim 20$ MeV, and therefore have well-measured masses. The hyperfine splittings of all of these heavy quark doublets are suppressed by $1/m_Q$, where $m_Q$ is the heavy quark mass.

The experimental data on the masses of the known charmed mesons is summarized in Table II. The lowest lying flavor $SU(3)$ anti-triplets are $J^P = 0^-$ ($c\bar{u}$, $c\bar{d}$, $c\bar{s}$) $\sim (D^0, D^+, D_s^+)$ and $J^P = 1^-$ ($D^{*0}$, $D_s^{*+}$, $D_s^{*+}$). The first excited states are $J^P = 0^+$ ($D_0^0$, $D_0^+$, $D_{0s}^+$) and $J^P = 1^+$ ($D_1^0$, $D_1^{*+}$, $D_{1s}^{*+}$). The members of the $j^p = \frac{3}{2}^+$ doublet are $J^P = 1^+$ ($D_1^0$, $D_1^+$, $D_{1s}^+$) and $J^P = 2^+$ ($D_2^0$, $D_2^+$, $D_{2s}^+$). Not shown is a narrow charmed strange meson,
TABLE I: The spectrum of charmed mesons. $j^P$ is the angular momentum and parity of the light degrees of freedom. $J^P$ is the angular momentum and parity of the meson.

$D_*(2632)$, recently observed by the SELEX collaboration \[11\]. The spin and parity of this meson and its place in the charmed meson spectrum is currently unknown. For all mesons except the nonstrange $j^P = \frac{1}{2}^+$ doublet, we use numbers from the Particle Data Group \[12\]. For nonstrange $j^P = \frac{1}{2}^-$ mesons, we use the Belle \[4\] measurement of the $D_0^-$ mass and average the CLEO \[3\] and Belle \[4\] measurements of the $D_0^+$ mass. \[1\]

As stated earlier, the hyperfine splittings of the $j^P = \frac{1}{2}^-$ and $j^P = \frac{1}{2}^+$ doublets are nearly equal. The known hyperfine splittings of the $j^P = \frac{1}{2}^-$ and $j^P = \frac{1}{2}^+$ charmed mesons are:

\[
\begin{align*}
    m_{D^*0} - m_{D^0} & = 142.1 \pm 0.07 \text{ MeV} \\
    m_{D^{*+}} - m_{D^+} & = 140.6 \pm 0.1 \text{ MeV} \\
    m_{D_{s1}^+} - m_{D_{s1}^+} & = 143.8 \pm 0.4 \text{ MeV} \\
    m_{D_{s1}^{++}} - m_{D_{s0}^{++}} & = 141.9 \pm 1.6 \text{ MeV} \\
    m_{D_{s1}^{*+}} - m_{D_{s0}^{*+}} & = 130 \pm 48 \text{ MeV}. \quad (1)
\end{align*}
\]

Here, the first three numbers are the hyperfine splittings quoted by the Particle Data Group \[12\]. The last two numbers are obtained by taking the difference of the masses

\[1\] The FOCUS collaboration reports structures in excess of background in the $D^+\pi^-$ and $D^0\pi^+$ invariant mass spectra which could be interpreted as scalar resonances \[6\]. However, if these resonances exist their masses are 99 MeV higher than the Belle measurement and 80 MeV higher than the mass of the $D_{s0}^+$. It is implausible that such resonances are related to the $D_{s0}^+$ by $SU(3)$ symmetry so we do not use this data to determine the $J^P = 0^+$ nonstrange meson masses.
in Table I. The error in the last two lines of Eq. (1) is obtained by adding the errors in
the individual masses in quadrature. All four hyperfine splittings which have been mea-
sured accurately are ≈ 142 MeV to within 2 MeV or less. Hyperfine splittings in different
heavy quark doublets are unrelated by heavy quark symmetry. For example, the hyperfine
splitting for \( j^p = \frac{3}{2}^+ \) doublets is \( \sim 40 \) MeV, which differs significantly from the \( j^p = \frac{1}{2}^- \)
and \( \frac{1}{2}^+ \) hyperfine splittings. In the \( SU(3) \) limit, the hyperfine splittings of nonstrange and
strange ground state mesons are the same. That this \( SU(3) \) prediction holds to within 2% is
surprising given the typical size of \( SU(3) \) breaking effects in QCD.

Another puzzling feature of the spectrum is the pattern of \( SU(3) \) violation in the splittings
between the even- and odd-parity doublets. Finite light quark (\( m_u, m_d, \) and \( m_s \)) masses
and electromagnetic effects cause flavor-splitting among the mesons. The isospin splitting
seen in the charmed meson mass spectrum is of expected size, but the splitting between the
strange and nonstrange sector is unexpected. The mass difference between strange and
nonstrange mesons whose other quantum numbers are identical is expected to be \( \sim 100 \)
MeV. For the ground state charmed mesons this is indeed the case. For the excited states,
however, the \( SU(3) \) breaking is

\[
m_{D^{*+}} - m_{D^{+}} = 21 \pm 31 \text{ MeV} \\
m_{D^{*0}} - m_{D^{0}} = 9 \pm 36 \text{ MeV}.
\]

(2)

Even allowing for the large errors due to the uncertainty in the masses of the nonstrange
\( j^p = \frac{1}{2}^+ \) charmed mesons, the \( SU(3) \) splitting is far below theoretical expectations.

The \( D_s(2317) \) and \( D_s(2460) \) are only 40 MeV below the \( DK \) and \( D^*K \) threshold, respec-
tively. This fact as well as the puzzles mentioned above have led to the hypothesis that they
are bound states of \( D^{(*)} \) and \( K \) \[13, 14, 15\]. An analysis of electromagnetic decay patterns
in Ref. \[16\] shows that this hypothesis disagrees with experiment. An interpretation of these
particles as exotic \( c\bar{s}q\bar{q} \) tetraquarks has also been proposed \[14, 17, 18, 19, 20, 21\].

In this paper we analyze the spectroscopy of charm mesons using heavy-hadron chiral
perturbation theory (HH\( \chi \)PT) \[22\]. This theory can be used to analyze the low energy
strong interactions of heavy mesons in a systematic expansion in light quark masses, \( m_q \),
and inverse heavy quark masses, \( 1/m_Q \). Nonanalytic corrections from loops with Goldstone
bosons can be calculated in this formalism. The masses of the ground state heavy mesons
have been studied in the heavy quark limit \[23, 24\], including leading corrections from finite
heavy quark masses and nonzero light quark masses \cite{25, 26, 27, 28, 29, 30}. These papers use a version of HH$\chi$PT which includes only the lowest lying $j^p = 1^-_2$ heavy quark doublets. Many recent studies of excited $J^P = 0^+$ and $1^+$ heavy mesons use Lagrangians that include only $j^p = 1^-_2$ and $j^p = 1^+_2$ heavy quark doublets as explicit degrees of freedom. However, the excited $j^p = 3^+_2$ doublets are only separated from the $j^p = 1^+_2$ doublets by $\lesssim 130$ MeV. Further, the $j^p = 3^+_2$ doublets couple to the $j^p = 1^+_2$ doublets at leading order in the chiral expansion, while the coupling of the $j^p = 3^+_2$ doublets to the ground state doublets is higher order in the chiral expansion \cite{42}. For these reasons, loops with virtual excited $j^p = 3^+_2$ could have important effects on the physics of $j^p = 1^+_2$ doublets. In this paper we will study the version of HH$\chi$PT containing only the $j^p = 1^-_2$ and $j^p = 1^+_2$ heavy quark doublets and leave investigation of loop effects from more highly excited states for future work.

A model of heavy mesons closely related to HH$\chi$PT is the parity doubling model of Refs. \cite{31, 32, 33, 34}. The parity doubling model is the analog of the linear sigma model for heavy mesons. Heavy meson doublets transforming linearly under $SU(3)_L \times SU(3)_R$ couple in a chirally invariant way to a field $\Sigma$ transforming in the $(\bar{3}, 3)$ of $SU(3)_L \times SU(3)_R$. The field $\Sigma$ develops a vacuum expectation value and the resulting theory of heavy mesons has the same form as HH$\chi$PT for the low lying odd- and even-parity doublets. Unlike HH$\chi$PT, the parity doubling model predicts relationships among otherwise independent parameters in the theory. One important prediction is that the hyperfine splittings of the $j^p = 1^-_2$ and $j^p = 1^+_2$ doublets are equal at tree level. This interesting prediction could partially explain the observed pattern of heavy meson hyperfine splittings, but it is not clear from Refs. \cite{31, 32, 33, 34} whether this prediction survives beyond tree level. This is a concern because loop corrections in HH$\chi$PT can be significant.

In this paper, we calculate the one loop HH$\chi$PT corrections to the masses of $j^p = 1^-_2$ and $j^p = 1^+_2$ heavy meson doublets. We include all $O(1/m_Q)$ heavy quark spin symmetry violating operators that appear to this order. A brief review of the HH$\chi$PT formalism is given in section II and explicit formulae for the masses at one loop appear in the Appendix. In section III, we attempt to fit the observed mass spectrum with our one-loop formulae. The large number of free parameters allows a reproduction of the masses of the $j^p = 1^+_2$ nonstrange charmed mesons. However, the parameter values obtained in such fits are cause for some concern, as we will discuss. In the $m_Q \to \infty$ limit our calculation of the difference of the $SU(3)$ splittings in HH$\chi$PT agrees with Ref. \cite{35}. Our analysis differs from that in
Ref. [35] in that we include $1/m_Q$ operators and perform a global fit to the spectrum with all counterterms treated as free parameters. In the approximation used in Ref. [35] there is a single counterterm constrained using lattice data.

In section IV, we examine corrections to the hyperfine splittings and discuss the naturalness of the parity doubling model. The parity doubling model predicts that the hyperfine splittings and the magnitudes of the axial couplings of the $j^p = \frac{1}{2}^-$ and $j^p = \frac{1}{2}^+$ doublets are equal at tree level. We find that these parameter relations are preserved by the one loop corrections so the model provides a natural explanation for the equality of hyperfine splittings. Finally, in section V, we use heavy quark effective theory (HQET) to estimate the masses of the $j^p = \frac{1}{2}^+ B$ mesons, which have not yet been observed. These predictions may be helpful to experimentalists looking for these states.

II. $\chi$PT FORMALISM

In $\chi$PT, the ground state doublet is represented by the field

$$H_a = \frac{1 + \gamma'}{2} (H_a^\mu \gamma_\mu - H_a \gamma_5) ,$$

(3)

where $a$ is an SU(3) index. In the charm sector, $H_a$ consists of the $(D^0, D^+, D_{s}^+)$ pseudoscalar mesons and $H^\mu$ are the $(D^{*0}, D^{*+}, D_{s}^{*+})$ vector mesons. The $j^p = \frac{1}{2}^+$ doublet is represented by the field

$$S_a = \frac{1 + \gamma'}{2} (S_a^\mu \gamma_\mu \gamma_5 - S_a) ,$$

(4)

where the scalar states in the charm sector are $S_a = D_{ba}$ and the axial vectors are $S_a^\mu = D_{1a}^\mu$. The kinetic terms of these fields are included in:

$$L_v^{\text{kinetic}} = -\text{Tr}[\overline{H}_a (iv \cdot D_{ba} - \delta_H \delta_{ab}) H_b] + \text{Tr}[\overline{S}_a (iv \cdot D_{ba} - \delta_S \delta_{ab}) S_b] ,$$

(5)

where $\delta_H$ and $\delta_S$ are the residual masses of the $H$ and $S$ fields, respectively, and $D_{ba}$ is the chirally covariant derivative. In the theory with only $H$ fields one is free to set $\delta_H = 0$. Since the only dimensionful parameters entering the loops in this theory are hyperfine splittings and meson masses, the UV divergences (in dimensional regularization) vanish in the $m_q \to 0$ and $m_Q \to \infty$ limit. Divergences in loop corrections are canceled by counterterms which are $O(m_q)$ or $O(1/m_Q)$. Once the $S$ fields are added to the theory, there
is another dimensionful quantity, $\delta_S - \delta_H$, which does not vanish as $m_q \to 0$ and $m_Q \to \infty$. The $H$ self-energy diagrams with virtual $S$ fields give a UV divergent contribution which survives in the $m_q \to 0$ and $m_Q \to \infty$ limit. Such a divergence must be canceled by a mass counterterm which respects $SU(3)$ and heavy-quark spin symmetry and the only available counterterm is $\delta_H \text{Tr} \overline{H}aH_a$. However, after one-loop divergences are canceled one is free to define the finite part of $\delta_H$ for convenience.

The fields have axial couplings to the pseudo–Goldstone bosons,

$$\mathcal{L}^{axial}_v = g \text{Tr} [\overline{H}aH_b \sigma_{a\gamma} \gamma_5] + g' \text{Tr} [\overline{S}_a S_b \sigma_{a\gamma} \gamma_5] + h \text{Tr} [\overline{H}aS_b \sigma_{a\gamma} \gamma_5 + h.c.] , \quad (6)$$

where $g$, $g'$, and $h$ are dimensionless constants to be determined from experiment. The other terms in the Lagrangian required are higher order mass counterterms. We use the notation of Ref. [29] and generalize it to include the $S$ field as well as the $H$ field.

$$\mathcal{L}^{mass}_v = -\frac{\Delta H}{8} \text{Tr} [\overline{H}a \sigma_{\mu\nu} H_a \sigma_{\mu\nu}] + \frac{\Delta S}{8} \text{Tr} [\overline{S}_a \sigma_{\mu\nu} S_a \sigma_{\mu\nu}]
+a_H \text{Tr} [\overline{H}_a H_b] m_b^\xi - a_S \text{Tr} [\overline{S}_a S_b] m_b^\xi + \sigma_H \text{Tr} [\overline{H}a H_a] m_b^\xi - \sigma_S \text{Tr} [\overline{S}_a S_a] m_b^\xi
-\frac{\Delta H}{8} \text{Tr} [\overline{H}a \sigma_{\mu\nu} H_b \sigma_{\mu\nu}] m_b^\xi + \frac{\Delta S}{8} \text{Tr} [\overline{S}_a \sigma_{\mu\nu} S_b \sigma_{\mu\nu}] m_b^\xi
-\frac{\Delta H}{8} \text{Tr} [\overline{H}a \sigma_{\mu\nu} H_b \sigma_{\mu\nu}] m_b^\xi + \frac{\Delta S}{8} \text{Tr} [\overline{S}_a \sigma_{\mu\nu} S_a \sigma_{\mu\nu}] m_b^\xi , \quad (7)$$

where $m_b^\xi = \frac{1}{2}(\xi m_q \xi + \xi^\dagger m_q \xi^\dagger)_{ba}$, $m_q = \text{diag}(m_u, m_d, m_s)$ and $\xi = \sqrt{\Sigma} = \exp(i\Pi/f)$, where $\Pi$ is the matrix of Goldstone bosons. The first line in Eq. (7) contains spin-symmetry violating operators which give rise to hyperfine splittings. The second line contains terms which are $O(m_q)$ and preserve heavy-quark spin symmetry. Finally, the last two lines contain terms which are $O(m_q)$ and violate heavy-quark spin symmetry.

At tree level the residual masses are

$$m_{H_a}^0 = \delta_H - \frac{3}{4} \Delta H + \sigma_H \overline{m} + a_H m_a - \frac{3}{4} \Delta_H(\overline{m}) + \frac{3}{4} \Delta_H(a) m_a$$
$$m_{H_s}^0 = \delta_H + \frac{1}{4} \Delta_H + \sigma_H \overline{m} + a_H m_a + \frac{1}{4} \Delta_H(\overline{m}) + \frac{1}{4} \Delta_H(a) m_a$$
$$m_{S_a}^0 = \delta_S - \frac{3}{4} \Delta_S + \sigma_S \overline{m} + a_S m_a - \frac{3}{4} \Delta_S(\overline{m}) + \frac{3}{4} \Delta_S(a) m_a$$
$$m_{S_s}^0 = \delta_S + \frac{1}{4} \Delta_S + \sigma_S \overline{m} + a_S m_a + \frac{1}{4} \Delta_S(\overline{m}) + \frac{1}{4} \Delta_S(a) m_a \quad (8)$$

where $m_a = (m_u, m_d, m_s)$ and $\overline{m} = m_u + m_d + m_s$. Here the asterisk denotes the spin-1 member of the heavy quark doublet. In the isospin limit $m_a = m_d$. HH$\chi$PT is a double
FIG. 1: One-loop self energy diagrams for the $H$ and $S$ fields. $H$ fields are single lines, $S$ fields are double lines and Goldstone bosons are dashed lines.

expansion in $\Lambda_{QCD}/m_Q$ and $Q/\Lambda_\chi$, where $Q \sim m_\pi, m_K, m_\eta$ and $\Lambda_\chi = 4\pi f \approx 1.5 \text{ GeV}$. The parameters $\delta_H$, $\delta_S$, $\Delta_H$, and $\Delta_S$ are treated as $O(Q)$ in the power counting \[16\]. Since $m_q \propto m_\pi^2 \sim Q^2$ the remaining terms in Eq. (8) are formally higher order in the power counting. The loop corrections to the masses are shown in Fig. 1. Single lines represent the $H$ field and double lines represent the $S$ fields. All diagrams are $O(Q^3)$. The loop corrections are regulated using dimensional regularization. Complete one loop expressions for the masses are given in the Appendix.

III. CHARMED MESON SPECTRUM

In this section we analyze the charmed meson spectrum using the one-loop mass formulae given in the Appendix. We will work in the isospin limit, where the masses of $H_1$ and $H_2$, for instance, are identical. Then there are eight different residual masses: $m_{H_1}$, $m_{H_3}$, $m_{H_1^*}$, $m_{H_3^*}$, $m_{S_1}$, $m_{S_3}$, $m_{S_1^*}$, and $m_{S_3^*}$. To determine the experimental values of $m_{H_1}$ and $m_{H_1^*}$, we average the masses of the two known isospin states. The residual masses are defined to be the difference between the real masses and an arbitrarily chosen reference mass of $O(m_Q)$. We will measure all masses relative to the nonstrange spin averaged $H$ mass, so $(m_{H_1} + 3m_{H_1^*})/4 = 0$. Therefore, the experimentally measured residual masses we will fit to are:

\[
m_{H_1} = -106.1 \text{ MeV} \quad m_{H_3} = -4.75 \text{ MeV} \quad m_{H_1^*} = 35.4 \text{ MeV} \quad m_{H_3^*} = 139.1 \text{ MeV} \\
m_{S_1} = 335.0 \text{ MeV} \quad m_{S_3} = 344.4 \text{ MeV} \quad m_{S_1^*} = 465.0 \text{ MeV} \quad m_{S_3^*} = 486.3 \text{ MeV} . \ (9)
\]
The tree level expressions in Eq. (8) reproduce these values with $\delta_S + \sigma_S \overline{m} - \delta_H - \sigma_H \overline{m} = 432 \pm 26$ MeV, $\Delta_H + \Delta^{(a)}_H \overline{m} = 146 \pm 1$ MeV, $\Delta_S + \Delta^{(a)}_S \overline{m} = 129 \pm 50$ MeV, $a_H = 1.14 \pm 0.06$, $a_S = 0.21 \pm 0.29$, $\Delta^{(a)}_H = -0.03 \pm 0.01$, and $\Delta^{(a)}_S = 0.14 \pm 0.55$. The errors used to obtain this fit are the experimental ones, dominated by the uncertainty in the nonstrange $0^+$ and $1^+$ masses. This gives rise to the large uncertainties seen in parameters in the Lagrangian involving the $S$ fields. The fits presented in this section use Mathematica [37] and/or Minuit. [38]

The loop corrections depend on eleven parameters: $g, g', h, a_H, a_S, \Delta^{(a)}_H, \Delta^{(a)}_S, \delta_H + \sigma_H \overline{m}, \delta_S + \sigma_S \overline{m}, \Delta_H + \Delta^{(a)}_H \overline{m},$ and $\Delta_S + \Delta^{(a)}_S \overline{m}$. The parameters $\sigma_H, \sigma_S, \Delta^{(a)}_H, \Delta^{(a)}_S$ cannot be separately determined because they always appear in linear combination with the parameters $\delta_H, \delta_S, \Delta_H,$ and $\Delta_S$, respectively. Below we will absorb the contribution of the parameters $\sigma_H, \sigma_S, \Delta^{(a)}_H,$ and $\Delta^{(a)}_S$ into the measured values of $\delta_H, \delta_S, \Delta_H,$ and $\Delta_S$, respectively.

An analysis of $D^*$ decays extracted $g = 0.27^{+0.06}_{-0.03}$ [39]. From the widths of the nonstrange resonances observed by Belle we have extracted $h = 0.69 \pm 0.09$ [16]. Both couplings are of order unity and therefore consistent with naive power counting. The remaining parameters are unknown.

We use $f = 120$ MeV, which is the value extracted in Ref. [39], using the one loop formulae for pion and kaon decay constants, first derived in Ref. [40]. We set $m_u = m_d = 4$ MeV and $m_s = 90$ MeV. Below we show several different fits. In the first case we fix $g$ and $h$ to the values (given above) extracted from previous analyses. This leaves nine remaining free parameters. Performing a least chi-squared fit to the meson spectrum, using experimental uncertainties, we obtain the following masses

$$m_{H_1} = -106 \text{ MeV} \quad m_{H_3} = -5 \text{ MeV} \quad m_{H'_1} = 35 \text{ MeV} \quad m_{H'_3} = 139 \text{ MeV}$$
$$m_{S_1} = 160 \text{ MeV} \quad m_{S_3} = 344 \text{ MeV} \quad m_{S'_1} = 296 \text{ MeV} \quad m_{S'_3} = 486 \text{ MeV}. \quad (10)$$

The parameters extracted from this fit are: $g' = 0.09 \pm 0.03$, $\delta_H = -83 \pm 3$ MeV, $\delta_S = 244 \pm 1$ MeV, $\Delta_H = 133 \pm 2$ MeV, $\Delta_S = 136 \pm 1$ MeV, $a_H = 1.70 \pm 0.01$, $a_S = 0.25 \pm 0.08$, $\Delta^{(a)}_H = -0.07 \pm 0.01$, and $\Delta^{(a)}_S = 0.04 \pm 0.03$. Six of the mass parameters are reproduced quite well while $m_{S_1}$ and $m_{S'_1}$ are lower than the central values of experiments by about 175 and 169 MeV, respectively. This qualitative picture persists without much sensitivity to the value of $g'$. However, these fits are not very good and such a procedure may not be very realistic. The values of $g$ and $h$ used above were extracted using a fit to a one-loop
calculation not including the $S$ fields, and a tree-level fit, respectively. There is no reason to believe that these values are the ones which are appropriate for a calculation that includes graphs with internal $S$ states. Note that large changes between tree- versus loop-extracted parameter values do not necessarily indicate poor convergence; what is important is that the observables do not suffer large changes between orders.

If we include $g$ and $h$ as free parameters in an 11-parameter fit, there are many solutions which yield central values identical to the experimental residual masses given in Eq. (9). In addition to the experimental errors we also include 20% “theoretical” errors to mimic the fact that our analysis is only accurate to $\mathcal{O}(Q^3)$. The masses obtained are then accompanied by errors at the $\pm$ 30 to 40 MeV level. Examples of parameter sets which give these results are:

(a) $g = 1.15 \pm 0.06$, $|g'| = 0.90 \pm 0.06$, $h = 2.3 \pm 0.2$, $\delta_H = 194 \pm 41$ MeV, $\delta_S = 332 \pm 30$ MeV, $\Delta_H = 465 \pm 24$ MeV, $\Delta_S = 597 \pm 28$ MeV, $a_H = 7 \pm 1$, $a_S = -4 \pm 1$, $\Delta^{(g)}_H = -4.4 \pm 0.7$, and $\Delta^{(a)}_S = -10 \pm 2$.

(b) $g = 0.65 \pm 0.06$, $|g'| = 0.89 \pm 0.08$, $h = -0.2 \pm 1$, $\delta_H = 117 \pm 21$ MeV, $\delta_S = 646 \pm 40$ MeV, $\Delta_H = 68 \pm 42$ MeV, $\Delta_S = 447 \pm 23$ MeV, $a_H = 3.8 \pm 0.7$, $a_S = -3.1 \pm 0.7$, $\Delta^{(a)}_H = -0.3 \pm 1$, and $\Delta^{(g)}_S = -2.8 \pm 1$.

(c) $g = 0.89 \pm 0.07$, $|g'| = 0.24 \pm 0.13$, $h = 0.98 \pm 0.11$, $\delta_H = 203 \pm 39$ MeV, $\delta_S = 399 \pm 26$ MeV, $\Delta_H = 242 \pm 25$ MeV, $\Delta_S = 116 \pm 59$ MeV, $a_H = 5.8 \pm 1.1$, $a_S = -1.4 \pm 1.5$, $\Delta^{(a)}_H = -1.7 \pm 0.8$, and $\Delta^{(g)}_S = 2.1 \pm 1.7$.

(d) $|g| = -|g'| = 0.70 \pm 0.03$, $h = 2.4 \pm 0.2$, $\delta_H = 114 \pm 64$ MeV, $\delta_S = 231 \pm 36$ MeV, $\Delta_H = 682 \pm 4$ MeV, $a_H = 4.3 \pm 0.7$, $a_S = -3.0 \pm 2.1$, $\Delta^{(a)}_H = -0.89 \pm 0.96$, and $\Delta^{(g)}_S = -2.7 \pm 0.8$. (In this fit, the constraint $\Delta_S = \Delta_H + 30$ MeV was used.)

There are clearly many local minima which Minuit may find. Fits (a)-(c) are shown to illustrate that the charmed meson spectrum favors a $g$ value larger than that found in the analysis of ref. [39], and that $h$ is poorly constrained. A concern about these fits is that they produce large values for the hyperfine coefficients. The operators which cause hyperfine splitting should be $1/m_Q$ suppressed compared to the leading order ones. If we include $g$ and $h$ as free parameters, we find that $|g|$ near $|g'|$ is favored, and that in such a case a
solution is possible (e.g., fit (d)) where \( \Delta_S \) is within 30 MeV of \( \Delta_H \). The relevance of that result will become apparent in the next section.

IV. HYPERFINE SPLITTINGS

In this section we study the one loop corrections to the hyperfine splittings to see if \( HH_\chi PT \) can provide insight into the observed near equality of the hyperfine splittings. Using the formulae in the Appendix we find that the next-to-leading order difference between even-parity and odd-parity hyperfine splittings in the strange sector is given by

\[
(m_{S^3} - m_{S_a}) - (m_{H^3} - m_{H_a}) = (m_{S^3}^0 - m_{S_a}^0) - (m_{H^3}^0 - m_{H_a}^0)
\]

\[
+ \frac{g'^2}{f^2} \left[ \frac{2}{3} K_1(m_{S^3}^0 - m_{S_a}^0, m_K) + \frac{2}{9} K_1(m_{S^3}^0 - m_{S_a}^0, m_\eta) + \frac{4}{3} K_1(m_{S^3}^0 - m_{S_a}^0, m_K) \right]
\]

\[
- \frac{g'^2}{f^2} \left[ \frac{2}{3} K_1(m_{H^3}^0 - m_{H_a}^0, m_K) + \frac{2}{9} K_1(m_{H^3}^0 - m_{H_a}^0, m_\eta) + \frac{4}{3} K_1(m_{H^3}^0 - m_{H_a}^0, m_K) \right]
\]

\[
+ \frac{\hbar^2}{f^2} \left[ 2K_2(m_{S^3}^0 - m_{S_a}^0, m_K) + \frac{2}{3} K_2(m_{S^3}^0 - m_{S_a}^0, m_\eta) - 2K_2(m_{H^3}^0 - m_{H_a}^0, m_K) - \frac{2}{3} K_2(m_{H^3}^0 - m_{H_a}^0, m_\eta) \right]
\]

Suppose one imposes at tree level the condition that all hyperfine splittings in each of the doublets are degenerate:

\[
m_{H^3}^0 - m_{H_a}^0 = m_{S^3}^0 - m_{S_a}^0 = \Delta
\]

(12)

This can be arranged by invoking the tree level prediction of the parity doubling model, \( \Delta_H = \Delta_S = \Delta \) and neglecting the terms proportional to \( m_\eta \) in Eq. (12), which are formally higher order in the power counting. Then \( m_{S_a}^0 - m_{H_a}^0 = m_{S_a}^0 - m_{H_a}^0 \) and it is easy to verify that all contributions proportional to \( \hbar^2 \) vanish, and the remaining terms are:

\[
(m_{S^3} - m_{S_a}) - (m_{H^3} - m_{H_a}) = \frac{g'^2}{f^2} \left[ \frac{2}{3} K_1(-\Delta, m_\pi) + \frac{2}{9} K_1(-\Delta, m_\eta) + \frac{16}{9} K_1(0, m_K) \right]
\]

\[
-2K_1(\Delta, m_K) - \frac{2}{3} K_1(\Delta, m_\eta) \right] - \frac{g'^2}{f^2} \left[ \frac{2}{3} K_1(-\Delta, m_\pi) + \frac{2}{9} K_1(-\Delta, m_\eta) \right]
\]

\[
+ \frac{16}{9} K_1(0, m_K) - 2K_1(\Delta, m_K) - \frac{2}{3} K_1(\Delta, m_\eta) \right]
\]

(13)
This vanishes if $g^2 = g'^2$, which is consistent with the parity doubling model prediction. A similar cancellation occurs for the nonstrange hyperfine splittings. So the parity doubling model explanation for the equality of the $j^p = \frac{1}{2}^-$ and $\frac{1}{2}^+$ hyperfine splittings is robust in the sense that one loop corrections do not spoil the prediction.

The parity doubling model prediction for the axial couplings and hyperfine splittings singles out a subspace of the parameter space of HHχPT that is preserved under renormalization group evolution. From our mass formulae it is easy to derive the following renormalization group equations for the renormalized parameters $\Delta_H$ and $\Delta_S$:

\[
\mu^2 \frac{d}{d\mu^2} \Delta_H = \frac{4g^2}{9\pi^2 f^2} \Delta_H^3 - \frac{h^2}{3\pi^2 f^2}(\Delta_S - \Delta_H) \left[ 3(\delta_S - \delta_H)^2 - \frac{3}{2}(\Delta_S - \Delta_H)(\delta_S - \delta_H) + \frac{7}{16}(\Delta_S - \Delta_H)^2 \right].
\]

\[
\mu^2 \frac{d}{d\mu^2} \Delta_S = \frac{4g'^2}{9\pi^2 f^2} \Delta_S^3 + \frac{h^2}{3\pi^2 f^2}(\Delta_S - \Delta_H) \left[ 3(\delta_S - \delta_H)^2 - \frac{3}{2}(\Delta_S - \Delta_H)(\delta_S - \delta_H) + \frac{7}{16}(\Delta_S - \Delta_H)^2 \right],
\]

which leads to

\[
\mu^2 \frac{d}{d\mu^2}(\Delta_S - \Delta_H) = \frac{4}{9\pi^2 f^2}(g'^2 \Delta_S^3 - g^2 \Delta_H^3) + \frac{2h^2}{3\pi^2 f^2}(\Delta_S - \Delta_H) \left[ 3(\delta_S - \delta_H)^2 - \frac{3}{2}(\Delta_S - \Delta_H)(\delta_S - \delta_H) + \frac{7}{16}(\Delta_S - \Delta_H)^2 \right].
\]

We also derive the one loop renormalization group equation for the couplings $g$ and $g'$. For this we need the wavefunction renormalization of the fields $H$ and $S$, which is obtained from the graphs in Fig. 1 and the one loop corrections to the axial couplings. The relevant graphs for the renormalization of $g$ are shown in Fig. 2 and the graphs for $g'$ can be obtained from those in Fig. 2 by interchanging $H$ and $S$ lines. Note that we only need the ultraviolet divergences of these graphs to obtain the renormalization group equation. Furthermore, the counterterms for the wavefunction renormalization and the axial couplings are defined to be independent of $m_q$ and $m_Q$. Ultraviolet divergences proportional to $m_q$ and $1/m_Q$ are absorbed into higher order counterterms. For example, a divergence proportional to $m_q$.
in the one-loop correction to the axial coupling of the $H$ field should be renormalized by counterterms with structures like $\text{Tr}[^{H}_a H_b A_{bc} \gamma_5] m_{ca}^\xi$, $\text{Tr}[^{H}_a H_b A_{ba} \gamma_5] m_{cc}^\xi$, etc. Therefore we can ignore Goldstone boson masses and hyperfine splittings in computing the ultraviolet divergences, which greatly simplifies the calculation. The graphs in Figs. 1a, 1c, 2a, and 2e vanish in this limit because the integrals are scaleless. Graphs in Figs. 2c and 2d do not contribute either. This is because the $H$-$S$-$\pi$ coupling in Figs. 2c and 2d gives a factor of $v \cdot k$, where $k^\mu$ is the four-momentum of the external Goldstone boson. Ultraviolet divergences in Figs. 2c and 2d are proportional to $v \cdot k$ and are canceled by counterterms with an additional covariant derivative acting on the field $A_{ab}^\mu$, such as $\text{Tr}[^{H}_a H_b i v \cdot D_{bc} A_{ca} \gamma_5]$. Therefore, all that is needed to obtain the running of $g$ are the ultraviolet divergent parts of Fig. 1b and 2b in the limit where Goldstone bosons and hyperfine splittings are neglected. The running of $g'$ is obtained from Fig. 1d and the analog Fig. 2b, with $S$ and $H$ lines interchanged. The result can be obtained from the corresponding graphs for the renormalization of $g$ by simply substituting $g \leftrightarrow g'$ and $\delta_S - \delta_H \rightarrow -(\delta_S - \delta_H)$. The renormalization group equations for $g$ and $g'$ are

$$
\mu \frac{d}{d\mu} g = -\frac{\hbar^2}{4\pi^2 f^2} (\delta_S - \delta_H)^2 (g' + 8g)
$$
\[ \mu \frac{d}{d\mu} g' = -\frac{h^2}{4\pi^2 f^2} (\delta_H - \delta_S)^2 (g + 8g') , \tag{17} \]

which can be rewritten as

\[ \mu \frac{d}{d\mu} (g + g') = -\frac{9h^2}{4\pi^2 f^2} (\delta_H - \delta_S)^2 (g + g') \]
\[ \mu \frac{d}{d\mu} (g - g') = -\frac{7h^2}{4\pi^2 f^2} (\delta_H - \delta_S)^2 (g - g') . \tag{18} \]

To understand the significance of this result, consider the naive quark model prediction
\[ g' = g/3 \] \[ 42 \]
From the renormalization group equations in Eq. (17) one sees that \( g \) and \( g' \) vary with changes of the renormalization scale in such a way that the condition \( g' = g/3 \) can only hold at one value of \( \mu \). The quark model prediction is meaningless beyond tree level without also specifying a particular renormalization scheme and scale at which the relation is expected to hold. However, if \( g = \pm g' \) holds at any \( \mu \), it will hold for all \( \mu \) (at least at one loop order). Also, if \( g^2 = g'^2 \) and \( \Delta_S = \Delta_H \) the right hand side of Eq. (16) vanishes. Thus the predictions of the parity doubling model, \( \Delta_H = \Delta_S \) and \( g = -g' \), are invariant under renormalization group flow in HH\( \chi \)PT to one loop order.

V. HQET AND PREDICTIONS FOR EXCITED B MESONS

In this section, we comment on the theoretical expectations for the spectrum of excited even-parity bottom mesons which have yet to be discovered. Our HH\( \chi \)PT results for the charm meson spectrum may be used, but there are unknown \( O(1/m_Q) \) effects which make it difficult to obtain precise predictions for the \( B \) meson. For finite quark masses, to obtain the bottom meson spectrum from the charm meson results, the hyperfine operators should be rescaled by \( m_c/m_b \), which is not very well determined. Other parameters can receive \( O(\Lambda_{QCD}/m_c - \Lambda_{QCD}/m_b) \) corrections. For instance, the reduced kinetic energy of the \( b \) quark significantly reduces the mass splitting of the \( H \) and \( S \) doublets in the \( b \) sector relative to what is observed in the charmed system. These \( O(1/m_Q) \) corrections introduce significant uncertainty in HH\( \chi \)PT predictions.

We will instead use the \( O(1/m_Q) \) HQET formulae for the mass of a heavy hadron \( X \) that contains a heavy quark \( Q \) \[ 44 \]:

\[ m_X^{(Q)} = m_Q + \bar{\Lambda} X - \frac{\lambda_1^X}{2m_Q} + n J \frac{\lambda_2^X}{2m_Q} , \tag{19} \]
where $\lambda_1^S$ and $\lambda_2^X$ are hadronic matrix elements of the HQET operators $\bar{h}(iD)^2h$ and $g_s\bar{h}\sigma^{\mu\nu}G_{\mu\nu}h/2$, respectively, and $n_J = +1$ for $J = 1$ states and $n_J = -3$ for $J = 0$ states. The first $1/m_Q$ correction, $-\lambda_1^S/(2m_Q)$, is the kinetic energy of the heavy quark. The second $1/m_Q$ correction contributes to hyperfine splittings. The difference between the spin averaged masses of the $j^p = \frac{1}{2}^-$ and $j^p = \frac{1}{2}^+$ mesons, $\overline{m}_H^{(Q)} = (3m_{H^0}^{(Q)} + m_{H^+}^{(Q)})/4$ and $\overline{m}_S^{(Q)} = (3m_{S^0}^{(Q)} + m_{S^+}^{(Q)})/4$, respectively, is given by

$$\overline{m}_S^{(Q)} - \overline{m}_H^{(Q)} = \bar{\lambda}^S - \bar{\lambda}^H - \frac{\lambda_1^S}{2m_Q} + \frac{\lambda_1^H}{2m_Q},$$

which leads to the following formulae for the splitting of the even- and odd-parity states in the bottom sector:

$$\overline{m}_S^{(b)} - \overline{m}_H^{(b)} = \overline{m}_S^{(c)} - \overline{m}_H^{(c)} + (\lambda_1^S - \lambda_1^H) \left( \frac{1}{2m_c} - \frac{1}{2m_b} \right).$$

A recent global fit to $B$ decays yields $\lambda_1^H = -0.20 \pm 0.06 \text{ GeV}^2$. The parameter $\lambda_1^S$ is unknown. From the spectroscopy of excited $j^p = \frac{3}{2}^+$ $D$ and $B$ mesons, Ref. [46] extracts $\lambda_1^{3/2} - \lambda_1^H = -0.23 \text{ GeV}^2$, where $\lambda_1^{3/2}$ is the $\lambda_1$ matrix element for the $j^p = \frac{3}{2}^+$ doublet. The sign here indicates that the kinetic energy of the heavy quark in the excited heavy meson is larger than that in the ground state, which agrees with intuition. We expect the kinetic energy of the heavy quark in the $j^p = \frac{1}{2}^+$ states to be comparable to that of $j^p = \frac{3}{2}^+$ states. To estimate $\overline{m}_S^{(b)}$ with conservative errors, we take $\lambda_1^S - \lambda_1^H = -0.2 \pm 0.1 \text{ GeV}^2$, $m_c = 1.4$ GeV, and $m_b = 4.8$ GeV to find

$$\overline{m}_S^{(b)} - \overline{m}_H^{(b)} = \overline{m}_S^{(c)} - \overline{m}_H^{(c)} - 50 \pm 25 \text{ MeV}$$

In the bottom nonstrange sector, $m_{H_1}^{(b)} = 5279$ MeV and $m_{H_1}^{(b)} = 5325$ MeV, which yields $\overline{m}_{H_1}^{(b)} = 5314$ and therefore Eq. (22) predicts $\overline{m}_{S_1}^{(b)} = 5696 \pm 30 \pm 25$ MeV. The first error comes from the uncertainty in the charm nonstrange $j^p = \frac{1}{2}^+$ masses and the second is the estimated uncertainty in $\lambda_1^S$. These states are well above the threshold for $S$-wave pion decays to the ground state and should be broad like their charm counterparts.

In the bottom strange sector, only the $0^-$ state with mass $m_{H_0}^{(b)} = 5370$ MeV has been observed. To proceed we need to estimate the mass of the bottom strange $1^-$ state. Note that

$$\frac{m_{H_0}^{(b)} - m_{H_0}^{(b)}}{m_{H_0}^{(c)} - m_{H_0}^{(c)}} = \frac{m_{S_0}^{(b)} - m_{S_0}^{(b)}}{m_{S_0}^{(c)} - m_{S_0}^{(c)}} = \frac{m_b}{m_c}.$$
up to $O(1/m_Q)$ corrections. Thus, all the hyperfine splittings in the bottom sector are related to those in the charm sector by a universal factor. Combining this with the measured value of $m_{H_1^+}^{(b)} - m_{H_1^+}^{(c)}$ leads to the prediction that $m_{H_3^+}^{(b)} - m_{H_3^+}^{(c)} = m_{S_3^+}^{(b)} - m_{S_3^+}^{(c)} = 46$ MeV, and $m_{S_1^+}^{(b)} - m_{S_1^+}^{(c)} = 42$ MeV. These predictions have approximately 25% uncertainty due to higher order $O(\Lambda_{QCD}/m_c - \Lambda_{QCD}/m_b)$ corrections and the prediction for $m_{S_3^+}^{(b)} - m_{S_3^+}^{(c)}$ estimate has an additional 20% uncertainty due to the poorly known $m_{S_1^+}^{(c)}$ and $m_{S_3^+}^{(c)}$ masses. Given these hyperfine splittings, one expects $m_{H_3^+}^{(b)} = 5424$ MeV and then Eq. (22) predicts $m_{S_3^+}^{(b)} = 5722 \pm 25$ MeV. Here the error is dominated by our ignorance of $\lambda_3^1$. Note that the excited bottom strange mesons are expected to lie well below the threshold for decays to ground state $B$ mesons and kaons and should be narrow like $j^p = \frac{1}{2}^+$ charmed strange mesons.

VI. CONCLUSIONS

We have enumerated the leading and subleading operators which describe the even-parity charmed meson masses in heavy hadron chiral perturbation theory. We performed a loop calculation to analyze the charmed meson masses to $O(Q^3)$. There are nominally eleven unknown parameters in the prediction, and only eight experimental masses. Two of the parameters, the axial coupling $g$ for the lowest doublet of charmed mesons, and the coupling $h$ which dominates the strong decay between the even-parity and ground state doublets, have been extracted from previous calculations. See Ref. [39] and Ref. [16], respectively. However, the even-parity states were not included in the extraction of $g$ in Ref. [39]. Also, the prediction of $h$ was only performed at tree level. Since these values for $g$ and $h$ were not obtained under the same conditions as the mass calculations performed in this paper, it is not clear that the values should be used in our fit. Indeed, if the values from Refs. [39] and [16] are used, it is not possible to obtain the nonstrange even-parity masses as large as they are observed to be.

If we allow all eleven parameters to be used in a fit to the spectrum of eight charmed meson masses, many solutions are possible, including ones whose values of $g$ and $h$ are not unreasonably far from their previously extracted values. However, the parameter values found for the hyperfine operators are of concern. These hyperfine operators should have coefficients which scale as $O(\Lambda_{QCD}^2/m_Q)$ relative to the $O(\Lambda_{QCD})$ coefficients of the leading
operators. The fact that global fits find coefficients which are sometimes larger for the hyperfine operators than for the leading order operators may signal a problem in the $1/m_Q$ expansion. On the other hand, the unexpectedly small SU(3) breaking in the even-parity doublet is going to require unusual parameter values.

Next we consider the parity doubling model introduced in Refs. \[31, 32, 33, 34\]. While the parity doubling model is not a result of QCD, but requires additional assumptions, it is interesting because it provides an explanation of the observed equality of the hyperfine splitting in the even-parity doublet and the hyperfine splitting in the odd-parity doublet. QCD symmetries alone do not dictate any relationship between these hyperfine splittings. While the parity doubling model provides an explanation for the equality of the hyperfine splittings, the question we address here is whether it is a natural explanation. That is, does it survive beyond tree level? Is it stable under RG flow? We find that there are “fixed lines” at $|g| = |g'|$. (These are axial operator coefficients from Eq. (6).) That is, if at any time in their evolution $g = g'$ or $g = -g'$, RG analysis shows that the relationship will be maintained. This in turn assures that if at tree level the parameters $\Delta_H$ and $\Delta_S$ in Eq. (7) are equal, they remain so to one loop. This lends credence to the parity doubling model.

Going back to the parameter fit, we do find that solutions with $|g|$ near $|g'|$ are favored. Also, fits are possible with $\Delta_H$ near $\Delta_S$. However, these values of $\Delta_H$ and $\Delta_S$ are much larger than expected by power counting.

Finally, we discuss how the charm meson spectrum results can be used to make predictions for the analog $B$ meson spectrum. It is necessary to know the charm and bottom quark masses in order to rescale the operators, which brings in significant uncertainty. Also, there are additional $1/m_Q$ operators with unknown parameters. However, it is possible to use heavy quark effective theory to estimate that the odd parity strange spin-zero $B$ meson has mass $\sim 5722$ MeV while its spin-one partner has mass $\sim 5768$ MeV. This places them below the threshold for decay to a kaon and the ground state $B$. Therefore, we expect narrow $B^*_s$ meson analogs to the narrow $D^*_s$ excited mesons.

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VII. APPENDIX

We express our results in terms of the functions

\[
K_1(\eta, M) = \frac{1}{16\pi^2} \left[ (-2\eta^3 + 3M^2\eta) \ln\left(\frac{M^2}{\mu^2}\right) + 2\eta(\eta^2 - M^2) F\left(\frac{\eta}{M}\right) + 4\eta^3 - 5\eta M^2 \right]
\]

\[
K_2(\eta, M) = \frac{1}{16\pi^2} \left[ (-2\eta^3 + M^2\eta) \ln\left(\frac{M^2}{\mu^2}\right) + 2\eta^3 F\left(\frac{\eta}{M}\right) + 4\eta^3 - \eta M^2 \right]
\]

(24)

where

\[
F(x) = 2\sqrt{1 - x^2} \left[ \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) \right] \quad |x| < 1
\]

(25)

\[
= -2\sqrt{x^2 - 1} \ln\left(x + \sqrt{x^2 - 1}\right) \quad |x| > 1
\]

(26)

The function \(K_1(\eta, M)\) appears whenever the virtual heavy meson inside the loop is in the same doublet as the external heavy meson, while \(K_2(\eta, M)\) appears when the virtual heavy meson is from the opposite parity doublet.

In the limit \(M << \eta\) these functions become

\[
K_1(\eta, M) = \frac{1}{16\pi^2} \left[ -2\eta^3 \ln\left(\frac{4\eta^2}{\mu^2}\right) + 3\eta M^2 \ln\left(\frac{4\eta^2}{\mu^2}\right) + \frac{3 M^4}{4\eta} \ln\left(\frac{M^2}{4\eta^2}\right) + \ldots \right]
\]

\[
K_2(\eta, M) = \frac{1}{16\pi^2} \left[ -2\eta^3 \ln\left(\frac{4\eta^2}{\mu^2}\right) + \eta M^2 \ln\left(\frac{4\eta^2}{\mu^2}\right) - \frac{M^4}{4\eta} \ln\left(\frac{M^2}{4\eta^2}\right) + \ldots \right].
\]

(27)

In these equations we have dropped polynomials of \(\eta, M\). The functions \(K_1(\eta, M)\) and \(K_2(\eta, M)\) have well-defined \(M \rightarrow 0\) limits. Furthermore, the dependence on \(M\) is analytic when \(M/\eta \rightarrow 0\), so in this limit the S fields can be integrated out and their effect on the chiral corrections can be absorbed into local counterterms as expected. This limit is not relevant to the real world as \(\eta \sim M\). In the opposite limit, \(\eta = 0\), which is relevant for loops in which external and virtual heavy mesons are the same,

\[
K_1(\eta, M) = -\frac{M^3}{8\pi} + \frac{3}{16\pi^2} \eta M^2 \ln\left(\frac{4\eta^2}{\mu^2}\right) + O(\eta^3)
\]

\[
K_2(\eta, M) = \frac{1}{16\pi^2} \eta M^2 \ln\left(\frac{4\eta^2}{\mu^2}\right) + O(\eta^3)
\]

(28)

Including the one loop diagrams we find:

\[
m_{H_1} = m_{H_1}^0 + \frac{g^2}{f^2} \left[ \frac{3}{2} K_1(m_{H_1}^0 - m_{H_1}^0, \pi) + \frac{1}{6} K_1(m_{H_1}^0 - m_{H_1}^0, \rho) + K_1(m_{H_1}^0 - m_{H_1}^0, \omega) + K_1(m_{H_3}^0 - m_{H_1}^0, m_K) \right]
\]

\[
+ \frac{h^2}{f^2} \left[ \frac{3}{2} K_2(m_{S_1}^0 - m_{H_1}^0, \pi) + \frac{1}{6} K_2(m_{S_1}^0 - m_{H_1}^0, \rho) + K_2(m_{S_1}^0 - m_{H_1}^0, \omega) + K_2(m_{S_3}^0 - m_{H_1}^0, m_K) \right].
\]

(29)
\[ m_{H_3} = m^0_{H_3} + \frac{g^2}{f^2} \left[ 2K_1(m^0_{H_1} - m^0_{H_3}, m_K) + \frac{2}{3}K_1(m^0_{H_3} - m^0_{H_5}, m_\eta) \right] \\
+ \frac{\hbar^2}{f^2} \left[ 2K_2(m^0_{S_1} - m^0_{H_3}, m_K) + \frac{2}{3}K_2(m^0_{S_3} - m^0_{H_3}, m_\eta) \right]. \tag{30} \]

\[ m_{H_1'} = m^0_{H_1'} + \frac{g^2}{f^2} \frac{1}{3} \left[ \frac{3}{2}K_1(m^0_{H_1} - m^0_{H_1'}, m_\pi) + \frac{1}{6}K_1(m^0_{H_1} - m^0_{H_1'}, m_\eta) + K_1(m^0_{H_3} - m^0_{H_1'}, m_K) \right] \\
+ \frac{g^2}{f^2} \left[ \frac{3}{2}K_1(0, m_\pi) + \frac{1}{6}K_1(0, m_\eta) + K_1(m^0_{H_3} - m^0_{H_1'}, m_K) \right] \\
+ \frac{\hbar^2}{f^2} \left[ \frac{3}{2}K_2(m^0_{S_1} - m^0_{H_1'}, m_\pi) + \frac{1}{6}K_2(m^0_{S_1} - m^0_{H_1'}, m_\eta) + K_2(m^0_{S_3} - m^0_{H_1'}, m_K) \right]. \tag{31} \]

\[ m_{H_3'} = m^0_{H_3'} + \frac{g^2}{f^2} \left[ 2K_1(m^0_{H_3} - m^0_{H_3'}, m_K) + \frac{2}{3}K_1(m^0_{H_3} - m^0_{H_5}, m_\eta) \right] \\
+ \frac{\hbar^2}{f^2} \left[ 2K_2(m^0_{S_1} - m^0_{H_3'}, m_\pi) + \frac{1}{6}K_2(0, m_\pi) + K_2(m^0_{S_3} - m^0_{H_3'}, m_K) \right]. \tag{32} \]

\[ m_{S_1} = m^0_{S_1} + \frac{g^2}{f^2} \left[ \frac{3}{2}K_1(m^0_{S_1} - m^0_{S_1'}, m_\pi) + \frac{1}{6}K_1(m^0_{S_1} - m^0_{S_1'}, m_\eta) + K_1(m^0_{S_3} - m^0_{S_1}, m_K) \right] \\
+ \frac{\hbar^2}{f^2} \left[ \frac{3}{2}K_2(m^0_{H_1} - m^0_{S_1}, m_\pi) + \frac{1}{6}K_2(0, m_\pi) + K_2(m^0_{H_3} - m^0_{S_1}, m_K) \right]. \tag{33} \]

\[ m_{S_3} = m^0_{S_3} + \frac{g^2}{f^2} \left[ 2K_1(m^0_{S_3} - m^0_{S_3}, m_K) + \frac{2}{3}K_1(m^0_{S_3} - m^0_{S_3'}, m_\eta) \right] \\
+ \frac{\hbar^2}{f^2} \left[ 2K_2(m^0_{H_1} - m^0_{S_3}, m_\pi) + \frac{1}{6}K_2(0, m_\pi) + K_2(m^0_{H_3} - m^0_{S_3}, m_K) \right]. \tag{34} \]

\[ m_{S_1'} = m^0_{S_1'} + \frac{g^2}{f^2} \frac{1}{3} \left[ \frac{3}{2}K_1(m^0_{S_1} - m^0_{S_1'}, m_\pi) + \frac{1}{6}K_1(m^0_{S_1} - m^0_{S_1'}, m_\eta) + K_1(m^0_{S_3} - m^0_{S_1'}, m_K) \right] \\
+ \frac{g^2}{f^2} \left[ \frac{3}{2}K_1(0, m_\pi) + \frac{1}{6}K_1(0, m_\eta) + K_1(m^0_{S_3} - m^0_{S_1'}, m_K) \right] \\
+ \frac{\hbar^2}{f^2} \left[ \frac{3}{2}K_2(m^0_{H_1} - m^0_{S_1'}, m_\pi) + \frac{1}{6}K_2(0, m_\pi) + K_2(m^0_{H_3} - m^0_{S_1'}, m_K) \right]. \tag{35} \]

\[ m_{S_3'} = m^0_{S_3'} + \frac{g^2}{f^2} \frac{1}{3} \left[ 2K_1(m^0_{S_1} - m^0_{S_3'}, m_K) + \frac{2}{3}K_1(m^0_{S_3} - m^0_{S_3'}, m_\eta) \right] \\
+ \frac{g^2}{f^2} \left[ 2K_1(m^0_{S_3} - m^0_{S_3'}, m_K) + \frac{2}{3}K_1(0, m_\eta) \right] \\
+ \frac{\hbar^2}{f^2} \left[ 2K_2(m^0_{H_1} - m^0_{S_3'}, m_\pi) + \frac{1}{6}K_2(0, m_\pi) + K_2(m^0_{H_3} - m^0_{S_3'}, m_K) \right]. \tag{36} \]
We agree with Ref. [29] for our $H$ superfield results in the limit where $m_{\pi} \to 0$, $m_{\eta}^2 \to \frac{4}{3} m_K^2$ and $\eta/M \ll 1$. Our answer also agrees with that of Ref. [35], which computes mass corrections to the $H$ and $S$ masses including $SU(3)$ breaking corrections but not hyperfine splittings.

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